Multiplicative tensor product of matrix factorizations and some applications

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Dedication

This work is dedicated to my family.

Acknowledgments

This thesis was written under the supervision of Prof. Dr Richard Blute. I would like to really thank him for putting in a lot of time to supervise me. During the preparation of this thesis, I was a recipient of the outstanding Queen Elizabeth II Diamond Jubilee Scholarship. My sincere gratitude goes to the administrators of this scholarship. I would also like to thank NSERC (Natural Sciences and Engineering Research Council) for their financial support. Many thanks also go to the prestigious Bank of Montreal Financial Group award I received during the last months of my PhD program.

Abstract

An $n \times n$ matrix factorization of a polynomial f is a pair of $n \times n$ matrices (P, Q) such that $PQ = fI_n$, where I_n is the $n \times n$ identity matrix. In this dissertation, we study matrix factorizations of an arbitrary element in a given unital ring. This study is motivated on the one hand by the construction of the unit object in the bicategory $\mathcal{LG}_K$ of Landau-Ginzburg models (of great utility in quantum physics) whose $1-$cells are matrix factorizations of polynomials over a commutative ring K, and on the other hand by the existing tensor product of matrix factorizations $\widehat{\otimes}$.

We observe that the pair of $n \times n$ matrices that appear in the matrix factorization of an element in a unital ring is not unique. Next, we propose a new operation on matrix factorizations denoted $\widetilde{\otimes}$ which is such that if X is a matrix factorization of an element f in a unital ring (e.g. the power series ring $K[[x_1, ..., x_r]] \ni f$) and Y is a matrix factorization of an element g in a unital ring (e.g. $g \in K[[y_1, ..., y_s]]$), then $X\widetilde{\otimes}Y$ is a matrix factorization of fg in a certain unital ring (e.g. in case $f \in K[[x_1, ..., x_r]]$ and $g \in K[[y_1, ..., y_s]]$, then $fg \in K[[x_1, ..., x_r, y_1, ..., y_s]]$). $\widetilde{\otimes}$ is called *the multiplicative tensor product* of X and Y. After proving that this product is bifunctorial, many of its properties are also stated and proved.

Furthermore, if $MF(1)$ denotes the category of matrix factorizations of the constant power series 1, we define the concept of one-step connected category and prove that there is a one-step connected subcategory of $(MF(1), \widetilde{\otimes})$ which is semi-unital semi-monoidal.

We also define the concept of right pseudo-monoidal category which generalizes the notion of monoidal category and we prove that $(MF(1), \widetilde{\otimes})$ is an example of this concept.

Furthermore, we define a summand-reducible polynomial to be one that can be written in the form $f = t_1 + \cdots + t_s + g_{11} \cdots g_{1m_1} + \cdots + g_{l1} \cdots g_{lm_l}$ under some specified conditions where each t_k is a monomial and each g_{ji} is a sum of monomials. We then use $\widehat{\otimes}$ and $\widetilde{\otimes}$ to improve the standard method for matrix factorization of polynomials on this class and we prove that if p_{ji} is the number of monomials in g_{ji}, then there is an improved version of the standard method for factoring f which produces factorizations of size $2^{\prod_{i=1}^{m_1} p_{1i}+\cdots+\prod_{i=1}^{m_l} p_{li}-(\sum_{i=1}^{m_1} p_{1i}+\cdots+\sum_{i=1}^{m_l} p_{li})}$ times smaller than the size one would normally obtain with the standard method.

Moreover, details are given to elucidate the intricate construction of the unit object of $\mathcal{LG}_K$. Thereafter, a proof of the naturality of the right and left unit maps of $\mathcal{LG}_K$ with respect to $2-$morphisms is presented. We also prove that there is no direct inverse for these (right and left) unit maps, thereby justifying the fact that their inverses are found only up to homotopy. Finally, some properties of matrix factorizations are exploited to state and prove a necessary condition to obtain a Morita context in $\mathcal{LG}_K$.

Introduction

A glimpse at matrix factorizations:
In 1980, Eisenbud introduced the concept of matrix factorization (cf. [25]). He came up with an approach to factorize both reducible and irreducible polynomials using matrices. For example the irreducible polynomial $f(x) = x^2+1$ over $\mathbb{R}[x]$ can be factored as follows:

$$\begin{bmatrix} x & -1 \\ 1 & x \end{bmatrix}\begin{bmatrix} x & 1 \\ -1 & x \end{bmatrix} = (x^2+1)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = fI_2$$

Thus; we say that $(\begin{bmatrix} x & -1 \\ 1 & x \end{bmatrix}, \begin{bmatrix} x & 1 \\ -1 & x \end{bmatrix})$ is a 2×2 matrix factorization of f.
Eisenbud also found out that matrix factorizations of $f \in K[[x]]$ are closely related to the homological properties of modules over quotient rings $K[[x]]/(f)$. In fact, he proved that matrix factorizations describe all maximal Cohen-Macaulay modules (MCM modules) without free summands (See [44] for notes on MCM modules).
Recall that if R is a commutative Noetherian ring, then (R, m, K) is said to be a local ring, if R has a unique maximal ideal m and $K = R/m$, the residue field. An $R-$module M is Cohen-Macaulay (CM) provided M is finitely generated and $depth(M) = dim(M)$. The ring R is CM provided R is CM as an $R-$module. The module M is maximal Cohen-Macaulay (MCM) provided M is CM and $depth(M) = dim(R)$ (cf. [33]).
Let K be a field and $K[[x]]$ be the formal power series ring in the variables $x = x_1, \cdots, x_r$ and $K[[y]]$ be the formal power series ring in the variables $y = y_1, \cdots, y_s$. Let $f \in K[[x]]$ and $g \in K[[y]]$ be nonzero noninvertible[1] elements. In 1998, Yoshino [67] considered the problem of how one could relate MCM modules over $K[[x]]/(f)$ and over $K[[y]]/(g)$ with MCM modules over $K[[x,y]]/(f+g)$. He presented a method of tensor product to relate those objects. In fact, he constructed a tensor product denoted $\widehat{\otimes}$ which is such that if X is a matrix factorizations of $f \in K[[x]]$ and Y is a matrix factorizations of $g \in K[[y]]$, then $X\widehat{\otimes}Y$ is a matrix factorization of $f+g \in K[[x,y]]$. In 2002 and 2003, Kapustin and Li in their papers [37] and [36], used matrix factorizations in string theory to study boundary conditions for strings in Landau-Ginzburg models. In 2013, Yu [69] in his PhD dissertation studied the geometry of the category of matrix factorizations. In 2014, Camacho [15] in chapter 4 of her PhD dissertation recalled the notion of graded matrix factorizations with special emphasis on $\mathbb{C}-$graded matrix factorizations. More recently in 2016, Carqueville and Murfet in their paper [17], briefly presented the construction of the bicategory $\mathcal{LG}_K$ of Landau-Ginzburg models whose 1-cells are matrix factorizations.

[1] Yoshino [67] requires an element $f \in K[[x]]$ to be nonzero noninvertible because if $f = 0$ then $K[[x]]/(f) = K[[x]]$ and if f is a unit, then $K[[x]]/(f)$ =K[[x]]/K[[x]]= {1}. But in this dissertation we will not bother about such restrictions because we will not deal with the homological methods used in [67].

In the same year, Crisler and Diveris [19] examined matrix factorizations of polynomials in the ring $\mathbb{R}[x_1, \cdots, x_n]$, using only techniques from elementary linear algebra. They focused mostly on factorizations of sums of squares of polynomials. They improved the standard method for factoring polynomials for this class of polynomials.
The category of matrix factorizations of a power series f is denoted by $MF(f)$. The objects of this category are matrix factorizations of f and the morphisms are pairs of matrices such that a certain diagram commutes (cf. subsection 2.1.2).

Importance of matrix factorizations:
One interesting reason for studying matrix factorizations is that irreducible polynomials can be factorized using matrices. For instance, the polynomial $f = x^2 + 1$ is irreducible over the real numbers but can be factorized as:

$$x^2 + 1 = \begin{bmatrix} x & -1 \\ 1 & x \end{bmatrix} \begin{bmatrix} x & 1 \\ -1 & x \end{bmatrix} = (x^2 + 1) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = fI_2$$

In a sense, this notion of factorizing polynomials using matrices can be seen as a generalization of the classical notion of polynomial factorization where a polynomial $p(x) = q(x)r(x)$ can be considered as the product of two 1×1 matrices.
Moreover, matrix factorizations play an important role in many areas of pure mathematics and physics. The notion of matrix factorization is one of the key tools used in representation theory of hypersurface rings. It is a classical tool in the study of hypersurface singularity algebras (cf. [25]). In [17], matrix factorizations were used to build the bicategory $\mathcal{LG}_K$ of Landau-Ginzburg models. These models are very important in physics. The initial model was used to describe superconductivity [54]. Some of these models are also used in the field of research of mirror theory [36]. A major advance was made by Orlov ([50], [51], [52], [53]), who showed that matrix factorizations could be used to study Kontsevich's homological mirror symmetry by giving a new description of singularity categories. Matrix factorizations have also proven useful for the study of cluster tilting [20], Cohen-Macaulay modules and singularity theory ([31], [14]), Khovanov-Rozansky homology ([40], [40]), moduli of curves [56], quiver and group representations ([4], [5]).
Moreover, Matrix factorizations were also used in string theory as boundary conditions for strings in Landau-Ginzburg models [37].
A word on Landau-Ginzburg models:
A Landau-Ginzburg model is a model in solid state physics for superconductivity. Superconductivity is a (quantum mechanical) phenomenon of exactly zero electrical resistance and expulsion of magnetic flux field occurring in superconductors when cooled below their critical temperature (i.e., a temperature at and above which their vapor cannot be liquefied no matter how much pressure is applied). Some excellent text books and papers giving a detailed account on superconductivity are [65] and [49].
Carqueville and Murfet studied the bicategory of Landau-Ginzburg models and showed that the bicategorical perspective offers a certain unified approach to Landau-Ginzburg models. This bicategory has polynomials as objects and matrix factorizations as $1-$morphisms.

Our contributions:
This dissertation contains several original contributions. We enumerate most of them below:

1. In chapter 2, we observe that an $n \times n$ matrix factorization of an element f in a

unital ring (e.g. the ring of power series) is not unique, for any natural number $n > 1$. Inspired by the definition of $\widehat{\otimes}$, we define another product $\widehat{\otimes}'$ different from the Yoshino's tensor product $\widehat{\otimes}$ such that if X and Y are respectively matrix factorizations of power series f and g, then $X\widehat{\otimes}'X'$ just like $X\widehat{\otimes}X'$ is a matrix factorization of $f + g$.

2. In particular in this dissertation, we make a contribution to the study of matrix factorizations by developing interesting new material from chapter 3 onwards. In fact, we propose a new operation on matrix factorizations denoted $\widetilde{\otimes}$ which is such that if X is a matrix factorization of the power series $f \in K[[x_1, ..., x_r]]$ and Y is a matrix factorization of the power series $g \in K[[y_1, ..., y_s]]$, then $X\widetilde{\otimes}Y$ is a matrix factorization of fg over $K[[x_1, ..., x_r, y_1, ..., y_s]]$. Our result also holds for any unital ring and not just the ring of power series. We call $\widetilde{\otimes}$, *the multiplicative tensor product* of X and Y.

3. After showing that $\widetilde{\otimes}$ is functorial in each argument and in both arguments (i.e., it is a bifunctor), we state and prove many of its properties among which are associativity, commutativity and distributivity.

4. Moreover, we give some applications of $\widetilde{\otimes}$ in chapter 3. First, we observe that $\widetilde{\otimes}$ is a binary operation on the category of matrix factorizations of the constant power series 1 (denoted $(MF(1), \widetilde{\otimes})$ and so we investigate if $(MF(1), \widetilde{\otimes})$ is a monoidal category. It turns out that it is not monoidal but is right pseudo-monoidal, which is a concept defined in this dissertation (cf. definition 3.8). This gives an application of $\widetilde{\otimes}$.
   Next, the concept of semi-unital semi-monoidal category was recently defined in [1] and an example was provided in that paper. But, this example required a considerable amount of set-up. We provide another example of this concept with an easy-to-understand small amount of set-up by extracting a one-step connected (cf. Chapter 3) subcategory of $(MF(1), \widetilde{\otimes})$ which is a semi-unital semi-monoidal category. This provides us with another application of $\widetilde{\otimes}$.

5. Furthermore, we provide yet another application of $\widetilde{\otimes}$ in chapter 4. In fact, we define a large class of polynomials and improve the standard method for factoring polynomials on this class thanks to the operations $\widetilde{\otimes}$ and $\widehat{\otimes}$. The standard method for factorizing polynomials builds matrix factorizations of sums of polynomials from "factorizations" of their summands. One conspicuous downside of this method is that the factorizations double in size for each new summand. We define a summand-reducible polynomial to be one that can be written in the form $f = t_1 + \cdots + t_s + g_{11} \cdots g_{1m_1} + \cdots + g_{l1} \cdots g_{lm_l}$ under some specified conditions where each t_k is a monomial and each g_{ji} is a sum of monomials. We then use tools which were not available in the days the standard method was developed namely; $\widehat{\otimes}$ and $\widetilde{\otimes}$, to improve the standard method for matrix factorization of polynomials on this class and we prove that if p_{ji} is the number of monomials in g_{ji}, then there is an improved version of the standard method for factoring f which produces factorizations of size $2^{\prod_{i=1}^{m_1} p_{1i}+\cdots+\prod_{i=1}^{m_l} p_{li}-(\sum_{i=1}^{m_1} p_{1i}+\cdots+\sum_{i=1}^{m_l} p_{li})}$ times smaller than the size one would normally obtain with the standard method.

6. In chapter 5, we spend quite some time developing a sophisticated machinery to explain the intricate construction (cf. [17]) of the unit object of $\mathcal{LG}_K$ providing a reasonable amount of proofs and details that are omitted in its original presentation (cf. [17]). Next, a proof of the naturality of the right and left unit maps with respect to 2−morphisms is presented. We also prove that there is no direct inverse for the right and left unit maps (also called unitors in [17]) of the $\mathcal{LG}_K$ bicategory, thereby justifying the fact that the inverses of these unitors in [17] are found only up to homotopy.

7. Finally, in chapter 6, properties of matrix factorizations are used to understand what a *Morita context* is in the bicategory $\mathcal{LG}_K$. In particular, we state and prove a necessary condition to obtain a *Morita context* in $\mathcal{LG}_K$. This gives us an application of matrix factorizations. We also state a trivial sufficient condition to obtain a Morita context in $\mathcal{LG}_K$.

Structure of the thesis: This thesis is organized as follows: In chapter 1, we give the needed preliminaries on bicategories, chain complexes and graded modules. We start the study of matrix factorizations in Chapter 2. We recall the definition of a matrix factorization of a power series and Yoshino's tensor product of matrix factorizations of power series. Our first original contributions also start in this chapter: we give a straightforward algorithm that helps obtain a bigger matrix factorization from a smaller one. Next, we prove that there is another product of matrix factorizations different from the Yoshino's tensor product but having the same effect on two matrix factorizations. Finally, we prove that an $n \times n$ matrix factorization of an element f in a given unital ring is not unique.
In chapter 3, we propose a new operation on matrix factorizations denoted by $\widetilde{\otimes}$ and we call it *the multiplicative tensor product* of matrix factorizations. After proving that this product is functorial in each argument, we prove that it is a bifunctor. Many of its properties are also stated and proved.
Furthermore, after comparing the tensor product defined by Yoshino and the one defined in this dissertation, we extract a one-step connected subcategory of $(MF(1), \widetilde{\otimes})$ which is semi-unital semi-monoidal.
Moreover, we define what a *right pseudo-monoidal category* is and prove that $(MF(1), \widetilde{\otimes})$ is a right pseudo-monoidal category.
In chapter 4, we discuss the standard method for factoring polynomial and improve it on the class of *summand-reducible polynomials* by using the tensor product of matrix factorizations $\widehat{\otimes}$ and the multiplicative tensor product $\widetilde{\otimes}$ defined in chapter 3.
We review the bicategory of Landau-Ginzburg models in chapter 5. After showing a link between linear factorizations and $\mathbb{Z}_2$-complexes, we briefly discuss matrix factorizations of finite rank. Next, we explain the intricate construction of the unit objet of $\mathcal{LG}_K$. A proof of the naturality of the unitors with respect to 2−morphisms and a proof of the nonexistence of a direct inverse for the unitors are presented. Finally, in chapter 6, we use properties of matrix factorizations to state and prove lemmas and theorems that give a necessary condition to obtain a Morita context in $\mathcal{LG}_K$. We also state a trivial sufficient condition.
We wrap up this dissertation with a conclusion and some questions generated by our study.

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PRELIMINARIES

In this Chapter, we introduce preliminaries on bicategories as well as the coherence equations. A few examples will also be provided. We will assume some familiarity with categories (cf. chap. 1-4 of [45]), modules and their tensor products (Chap. 10 of [22]), therefore we will not give preliminaries on these. We also give the definition of a semi-unital semi-monoidal category which is a concept defined recently in [1]. Finally, we recall some facts about permutation matrices.

> *The material in Section 1.1 essentially follows the work done in [6] and [10]. The slight difference is at the level of details and sometimes notation. Moreover, we show in complete detail that the category of bimodules is a bicategory. If the reader is familiar with these topics, this section can be skipped.*

1.1 Bicategories and more

Roughly speaking, a bicategory is a category equipped with two types of morphisms: the ordinary morphisms (arrows between objects) and morphisms between these ordinary morphisms. Furthermore, a number of coherence laws have to hold. For a more detailed discussion, see [6]. we now give a formal definition.

Definition 1.1. *[6]*
*A **bicategory** $\mathcal{B}$ is made up of the following data:*

1. *A class of objects $A, B, C, ...$*

2. *For each pair $\langle A, B\rangle$ of objects, a small (hom-)category $\mathcal{B}(A, B)$ with arrows $p, q, r, ...$ from A to B as objects and arrows $\alpha, \beta, \gamma, ...$ between them, referred to as 2-cells. The arrows $p, q, r, ...$ are called 1-cells.*
   An example of a 2-cell is denoted as follows: $\alpha : p \Longrightarrow q$. Composition in $\mathcal{B}(A, B)$ is denoted by a dot "$\cdot$" and the identity on p for each $p : A \longrightarrow B$, by $1_p : p \Longrightarrow p$.

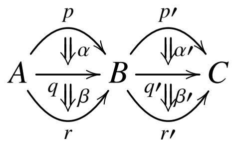
3. *For each object A, a functor that returns the identity on A.*

   $I_A : 1 \to \mathcal{B}(A, A)$, where 1 stands for the final object in the category Cat of small categories.

   If we write $1 = \{*\}$, that is the unique object of 1 is denoted $*$, then $\mathbf{I}_A(*)$ is the 1-cell $1_A : A \longrightarrow A$.

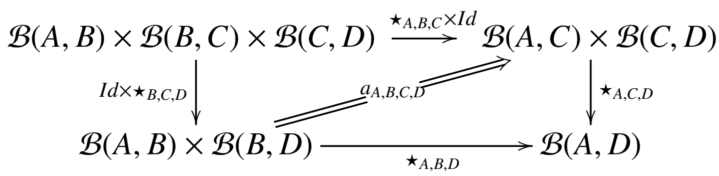
4. *For each triple $\langle A, B, C\rangle$ of objects, a composition law given by a functor* $\star_{A,B,C} : \mathcal{B}(A, B) \times \mathcal{B}(B, C) \longrightarrow \mathcal{B}(A, C)$.

   Note: Following the diagrammatic presentations (see fig.1 below), 2-cells in $\mathcal{B}$ come equipped with two forms of compositions called respectively vertical and horizontal. Composition in each hom category is said to be vertical whereas horizontal composition corresponds to the action of the functor $\star$ on 2-cells. The fact that $\star$ is a bifunctor gives rise to the following equation:
   $(\beta \star \beta') \cdot (\alpha \star \alpha') = (\beta \cdot \alpha) \star (\beta' \cdot \alpha')$ *which is referred to as the interchange law. Diagrammatically, we have:*



$fig.1$

5. *For each quadruple $\langle A, B, C, D\rangle$ of objects of $\mathcal{B}$, a natural isomorphism $a_{A,B,C,D}$ called associativity isomorphism, between the two composite functors bounding the diagram:*



i.e.,

$$a_{A,B,C,D} : \star_{A,B,D} \circ (Id \times \star_{B,C,D}) \Longrightarrow \star_{A,C,D} \circ (\star_{A,B,C} \times Id)$$

where Id is the identity functor.
The components of this isomorphism are the invertible 2-cells defined as follows: For $p : A \longrightarrow B$, $q : B \longrightarrow C$, $r : C \longrightarrow D$ and $t : D \longrightarrow E$, we have

$$a_{p,q,r} := a_{A,B,C,D}(p, q, r) : \star_{A,B,D} \circ (Id \times \star_{B,C,D})(p, q, r) \overset{\cong}{\Longrightarrow} \star_{A,C,D} \circ (\star_{A,B,C} \times Id)(p, q, r)$$

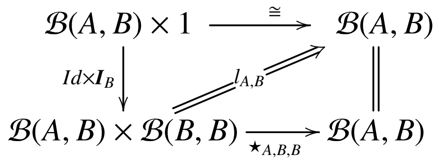
$$i.e., \quad a_{p,q,r} : \star_{A,B,D}((p, (q \star r)) \overset{\cong}{\Longrightarrow} \star_{A,C,D}((p \star q), r)$$

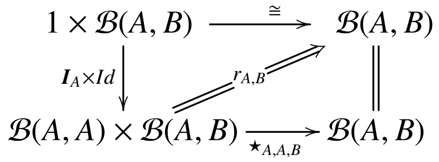
$$i.e., \quad a_{p,q,r} : p \star (q \star r) \overset{\cong}{\Longrightarrow} (p \star q) \star r$$

6. *For each pair $\langle A, B\rangle$ of objects of $\mathcal{B}$, two natural isomorphisms $l(A, B)$ and $r(A, B)$, called left and right isomorphisms or identities or unit laws (or unitors in [17]). They are given respectively by:*

$$l_{A,B} : \star_{A,B,B} \circ (Id \times I_B) \Longrightarrow Id$$
$$r_{A,B} : \star_{A,A,B} \circ (I_A \times Id) \Longrightarrow Id^1$$

Diagrammatically, we respectively have:





The components of these isomorphisms are the invertible 2-cells defined as follows: For $p : A \longrightarrow B$, we have

$$r_p := r_{A,B}(*, p) : \star_{A,A,B} \circ (I_A \times Id)(*, p) \overset{\cong}{\Longrightarrow} Id(*, p)$$

$$i.e., \quad r_p : \star_{A,A,B}(I_A(*), p) \overset{\cong}{\Longrightarrow} p$$

$$i.e., \quad r_p : 1_A \star p \overset{\cong}{\Longrightarrow} p$$

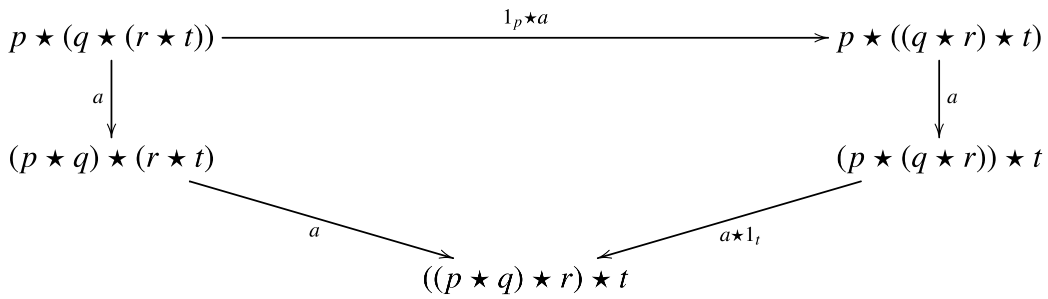
$$l_p := l_{A,B}(p, *) : \star_{A,B,B} \circ (Id \times I_B)(p, *) \overset{\cong}{\Longrightarrow} Id(p, *)$$

$$i.e., \quad l_p : \star_{A,B,B}(p, I_B(*)) \overset{\cong}{\Longrightarrow} p$$

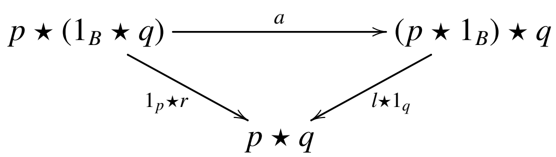
$$l_p : p \star 1_B \overset{\cong}{\Longrightarrow} p$$

The families of natural isomorphisms $a_{A,B,C,D}$, $r_{A,B}$ and $l_{A,B}$ are furthermore required to satisfy the coherence laws expressed by the commutativity of the following diagrams:
Associativity Coherence (A.C):

[1] this Id is actually: $Id : 1 \times \mathcal{B}(A, B) \longrightarrow \mathcal{B}(A, B)$ So it is not exactly the identity but can be identified with $\mathcal{B}(A, B) \xrightarrow{Id} \mathcal{B}(A, B)$.



and
Identity coherence (I.C):



Definition 1.2. *A **2-category** is a bicategory in which the families of natural isomorphisms a, l, r are identities.*

In this stricter setting, the arrows in the 2 diagrams above (A.C and I.C) will all be equalities. Hence, the coherence laws will hold trivially.

Example 1.1. *A prototype of 2-category is Cat, with small categories, functors and natural transformations as 0, 1 and 2-cells respectively. The proof is lengthy but not difficult. It uses the fact that the horizontal composition of natural transformations is given by the Godement product which is defined in [46].*

Example 1.2. *Any category C can be seen as a special case of a 2-category by regarding each homset $C(A, B)$ as a discrete category, that is a category whose only morphisms are the identity morphisms.*

Two interesting examples of bicategories (with generalizations [17], [6]) and applications ([17]) in quantum physics are the bicategories of Spans (see [6]) and that of bimodules. We will now prove in detail that the category of bimodules is a bicategory.
N.B.: It is necessary to prove it because we are soon going to generalize this construction to get another bicategory (cf. section 5.2) which is of great importance in this thesis. We reserve a full subsection for that.

1.1.1 The bicategory of bimodules BiMod

In this section, we prove that the category of associative algebras over a commutative ring, with bimodules as 1-cells is a bicategory which is not a 2-category.
To set the stage, we first recall some definitions that are well-known in the literature (cf. [22]).

Definition 1.3. *Bilinear map*
*Let U, V, W be three vectors spaces over a field F. A **bilinear map** B is a function $B : U \times V \longrightarrow W$ s.t.,*

$$\begin{array}{rcl} B(u+u',v) & = & B(u,v)+B(u',v) \\ B(u,v+v') & = & B(u,v)+B(u,v') \\ B(\alpha u,v) & = & \alpha B(u,v) \\ B(u,\alpha v) & = & \alpha B(u,v) \end{array}$$

When $W = F$, we say we have a bilinear form.
The definition works without any changes if instead of vector spaces over a field F, we use modules over a commutative ring R.

Definition 1.4. *Associative $R-$algebra*
Let R be a fixed commutative ring (possibly a field). An ***associative $R-$algebra*** *or simply an $R-$algebra is an additive abelian group A which has the structure of both a ring and an $R-$module such that for all $r \in R$ and $x, y \in A$, the scalar multiplication "$\cdot$" satisfies $r \cdot (xy) = (r \cdot x)y = x(r \cdot y)$. Furthermore, if A is assumed to be unital with unit 1_R then $1_R \cdot x = x = x \cdot 1_R$, $\forall x \in A$.*
In other words, A is an $R-$module together with:

- *An $R-$bilinear map $A \times A \longrightarrow A$*
- *A multiplicative identity,*

such that the multiplication is associative.
If A itself is commutative as a ring, then A is a commutative $R-$algebra. The "$\cdot$" representing scalar multiplication will most often be omitted when there would be no risk of confusion.

Definition 1.5. *$R-S-$bimodule*
If R and S are two rings, then an ***R-S-bimodule*** *is an abelian group M such that:*

1. *M is a left $R-$module and a right $S-$module*
2. *For all $r \in R$, $s \in S$ and $m \in M$: $(rm)s = r(ms)$*

An *$R-R-$bimodule* is also known as an $R-$bimodule or simply bimodule if there is no risk of confusion.

Example 1.3. *Any module over a commutative ring is automatically a bimodule.*

Example 1.4. *If R is a ring, then R itself is an $R-R-$bimodule.*

Definition 1.6. *Module homomorphism*
A ***module homomorphism*** *is a function between modules that preserves module structure. Explicitly:*
If M and N are left $R-$modules, where R is a ring, then $f : M \longrightarrow N$ is called a module homomorphism or an $R-$linear map if for any $x, y \in M$ and $r \in R$,

(i) $f(x+y) = f(x) + f(y)$

(ii) $f(rx) = rf(x)$

If M and N are instead right $R-$modules, then the last equality above is replaced by $f(xr) = f(x)r$

If M and N are both $R-S-$bimodule, then $f : M \longrightarrow N$ is a bimodule homomorphism if it is both a homomorphism of left $R-$modules and of right $S-$modules, i.e., for any $x, y \in M$ and $r', r \in R$

(i) $f(x + y) = f(x) + f(y)$

(ii) $f(rxr') = rf(x)r'$

We now prove the following theorem.

Theorem 1.1. *The category of associative algebras over a commutative ring, with bimodules as* $1-$*cells and homomorphisms as* $2-$*cells is a bicategory which is not a 2-category.*

Proof. In the sequel, we will denote this bicategory by $\mathcal{B}$, and R will denote a commutative ring.

(1) The objects of $\mathcal{B}$ are associative algebras over a commutative ring R.

(2) For each pair of objects $\langle A, B\rangle$, there is a small (hom-)category $Hom(A, B)$ which is the class of all $A - B-$bimodules, that is the class of all abelian groups on which A acts on the left and B acts on the right.
If M and N are objects of $Hom(A, B)$ (i.e., M and N are $A - B-$bimodules) then an arrow between them is a bimodule homomorphism between M and N, i.e., $2Hom(M, N)$ =class of all bimodule homomorphisms between the two bimodules M and N.

(3) For each triple $\langle A, B, C\rangle$ of objects of $\mathcal{B}$, we have a composition law given by a bifunctor $\star$.

$$\star_{A,B,C} : \quad Hom(A, B) \times Hom(B, C) \longrightarrow Hom(A, C)$$

$$\begin{array}{ccccc}
(M & , & N) & \longrightarrow & M \star N \\
\downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f\star f' := f\otimes_B f'} \\
(M' & , & N') & \longrightarrow & M' \star N' \\
\downarrow{\scriptstyle g} & & \downarrow{\scriptstyle g'} & & \downarrow{\scriptstyle g\star g' := g\otimes_B g'} \\
(M" & , & N") & \longrightarrow & M" \star N"
\end{array}$$

where $M \star N := M \otimes_B N$ is the tensor product of M and N over B. This is well defined since M is a right $B-$module and N is a left $B-$module.

We now show that $\star$ is a (bi)functor. It is easy to see that $M \star N := M \otimes_B N$ is an $A - C-$bimodule. In fact, given that M is a left $A-$module and N is a right $C-$module, we have respectively that:

- The left module action is given by $a(m \otimes_B n) := (am \otimes_B n)$
- The right module action is given by $(m \otimes_B n)c := (m \otimes_B nc)$

Hence, $M \star N := M \otimes_B N$ is an object of $Hom(A, C)$. Next, we show that the identity and composition axioms hold. Let M be an $A-B-$bimodule and N a $B-C-$bimodule, consider $1_{M,N} : (M, N) \longrightarrow (M, N)$.

$$\begin{aligned} \star(1_{(M,N)}) &= \star(1_M, 1_N) \\ &= 1_M \otimes_B 1_N \\ &= 1_{M\otimes_B N} \\ &= 1_{\star(M,N)} \end{aligned}$$

The third equality above holds since $1_M \otimes_B 1_N : M \otimes_B N \longrightarrow M \otimes_B N$ is a linear map which fixes every elementary tensors and so fixes all tensors. The other equalities are trivial.
As for the composition axiom, let f, f', g, g' be as in the diagram above. we show that $\star(g \circ f, g' \circ f') = \star(g, g') \circ \star(f, f')$ i.e., $g \circ f \star g' \circ f' = g \star g' \circ f \star f'$ i.e., $g \circ f \otimes_B g' \circ f' = g \otimes_B g' \circ f \otimes_B f'$ by definition of $\star$.

Now, since $g \circ f \otimes_B g' \circ f'$ and $g \otimes_B g' \circ f \otimes_B f'$ are linear maps, to prove their equality it suffices to check they have the same value at any elementary tensor $m \otimes_B n$, it is easy to see that they both have the value $g(f(m)) \otimes_B g'(f'(n))$. This completes the proof that the composition axiom holds. Hence $\star$ is a (bi)functor.

(4) For each object A (i.e., A is an associative algebra over R), we have $I_A : 1 = \{*\} \to \mathcal{B}(A, A)$ defined by $I_A(*) = 1_A := A$.
I_A is a functor.
In fact, I_A is well defined since A is an $A - A-$bimodule. Now, since there is only one possibility for an arrow in 1, it is easy to see that the identity and composition axioms for a category are verified for I_A. Thus I_A is a functor.

(5) For each quadruple $\langle A, B, C, D\rangle$ of objects of $\mathcal{B}$ and arrows M, N, P such that $A \xrightarrow{M} B \xrightarrow{N} C \xrightarrow{P} D$, I need to show that

$$a_{M,N,P} : M \star (N \star P) \overset{\cong}{\Longrightarrow} (M \star N) \star P$$

$$i.e., \quad a_{M,N,P} : M \otimes_B (N \otimes_C P) \overset{\cong}{\Longrightarrow} (M \otimes_B N) \otimes_C P$$

but this is true by associativity of tensor products (cf. theorem 14 of chapter 10, [22]). Since M is an $A - B-$bimodule, this is an isomorphism of $A-$modules (cf. theorem 14 of chapter 10, [22]).

(6) For each pair of objects $\langle A, B\rangle$, we show that we have two natural isomorphisms $r(A, B)$ and $l(A, B)$. Consider again $M : A \longrightarrow B$ i.e., M is an $A - B-$bimodule.
For $r(A, B)$, we need to show that we have the following isomorphism:

$$r_M : 1_A \star M \overset{\cong}{\Longrightarrow} M$$

$$i.e., \quad r_M : 1_A \otimes_A M \overset{\cong}{\Longrightarrow} M$$

But this is true since $A \otimes_A M \cong M$ given that M is an $A - B-$bimodule and A is an $A - A-$bimodule. It is easy to see that the assignment given by $a \otimes_A m \mapsto am$ where

$a \in A$ and $m \in M$; defines an isomorphism. For a more elaborate proof of the isomorphism $A \otimes_A M \cong M$, see Example 1. P.363 of [22].

For $l(A, B)$, we need to show:

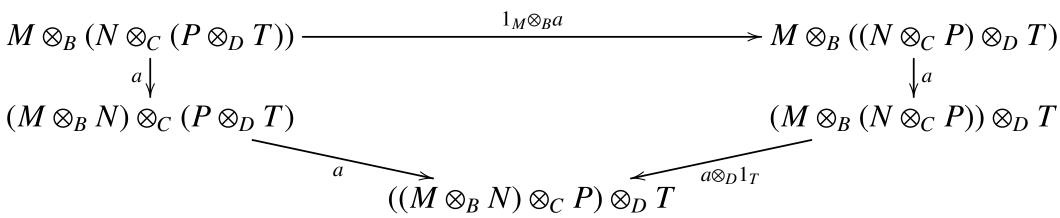
$$l_M : M \star 1_B \overset{\cong}{\Longrightarrow} M$$

$$i.e., \quad l_M : M \otimes_B 1_B \overset{\cong}{\Longrightarrow} M$$

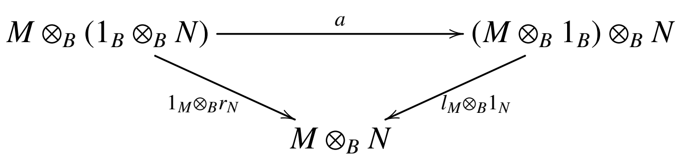
This is true since we have $M \otimes_B B \cong M$ given that M is an $A - B-$bimodule and B is a $B - B-$bimodule. It is easy to see that the assignment given by $m \otimes_B b \mapsto mb$ where $b \in B$ and $m \in M$ defines an isomorphism.

(7) we now verify the coherence laws, that is the commutativity of the diagrams below: For all objects $\langle A, B, C, D, E\rangle$ of $\mathcal{B}$ and arrows M, N, P, T such that
$A \xrightarrow{M} B \xrightarrow{N} C \xrightarrow{P} D \xrightarrow{T} E$

Associativity Coherence (A.C):



and
Identity coherence (I.C):



For (A.C) we need to show that $a \otimes_D 1_T \circ a \circ 1_M \otimes_B a = a \circ a$, i.e., for m, n, p, t respectively in $M, N, P,$ *and* T;
$(a \otimes_D 1_T \circ a \circ 1_M \otimes_B a)(m \otimes_B (n \otimes_C (p \otimes_D t))) = (a \circ a)(m \otimes_B (n \otimes_C (p \otimes_D t)))$.

Now, we have:

$(a \otimes_D 1_T \circ a \circ 1_M \otimes_B a)(m \otimes_B (n \otimes_C (p \otimes_D t)))$
$= a \otimes_D 1_T \circ a((1_M \otimes_B a)(m \otimes_B (n \otimes_C (p \otimes_D t))))$
$= a \otimes_D 1_T \circ a(m \otimes_B ((n \otimes_C p) \otimes_D t))$
$= a \otimes_D 1_T((m \otimes_B (n \otimes_C p)) \otimes_D t)$
$= a(m \otimes_B (n \otimes_C p)) \otimes_D 1_T(t)$
$= ((m \otimes_B n) \otimes_C p) \otimes_D t \quad ...\dagger$

$$
\begin{aligned}
a \circ a(m \otimes_B (n \otimes_C (p \otimes_D t))) &= a((m \otimes_B n) \otimes_C (p \otimes_D t)) \\
&= ((m \otimes_B n) \otimes_C p) \otimes_D t \quad \ldots\ddagger
\end{aligned}
$$

† and ‡ show that the desired equality holds, thus proving the (A.C).

For (I.C) we drop the subscripts of l and r for ease of notation and we show that $l \otimes_B 1_N \circ a = 1_M \otimes_B r$. Next, Recall that $1_B := B$. Let m, b, n be respectively in M, B, N then:

$$
\begin{aligned}
(l \otimes_B 1_N \circ a)(m \otimes_B (b \otimes_B n)) &= l \otimes_B 1_N((m \otimes_B b) \otimes_B n) \\
&= l(m \otimes_B b) \otimes_B 1_N(n) \\
&= mb \otimes_B n \\
&= m \otimes_B bn \quad \sharp
\end{aligned}
$$

the last equality holds because the tensor product is taken over B. (cf. page 364 of [22]).

$$
\begin{aligned}
(1_M \otimes_B r)(m \otimes_B (b \otimes_B n)) &= 1_M(m) \otimes_B r(b \otimes_B n) \\
&= m \otimes_B bn \quad \natural
\end{aligned}
$$

$\sharp$ and $\natural$ show that the desired equality holds.

Conclusion: $\mathcal{B}$ is a bicategory. But $\mathcal{B}$ is clearly not a 2-category because (it is obvious that) the families of natural isomorphisms a, l, r are not mere identities. □

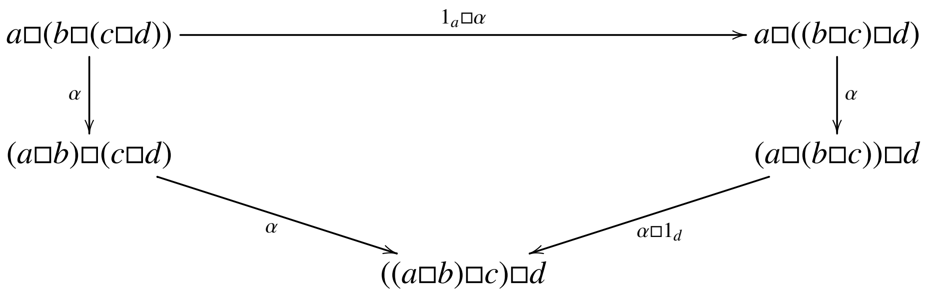
We now recall the definitions of several types of categories. Most of which will be useful in chapters ahead.

1.2 Special classes of categories

Here, we recall the definitions of some interesting types of categories.

Definition 1.7. *[45] A **monoidal category** $\mathcal{C} =< \mathcal{C}, \Box, e, \alpha, \lambda, \rho >$ is a category $\mathcal{C}$, a bifunctor $\Box : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$, an object $e \in \mathcal{C}$, and three natural isomorphisms α, λ, and ρ; such that:*

• *$\alpha = \alpha_{a,b,c} : a\Box(b\Box c) \cong (a\Box b)\Box c$ is natural for all $a, b, c \in \mathcal{C}$ and the pentagonal diagram*

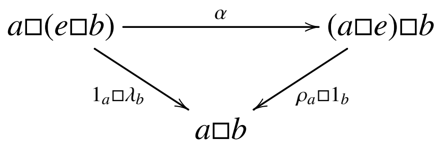


commutes for all $a, b, c, d \in \mathcal{C}$.

(α is also called associator (p.11 [17])).

• *λ and ρ are natural. On p.10 of [17], λ and ρ are also called left and right unit actions (or unitors).*

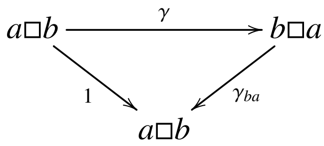
$\lambda_a : e\Box a \cong a$, $\rho_a : a\Box e \cong a$, for all objects $a \in C$.
Moreover, the triangular diagram

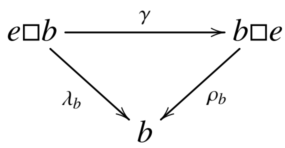


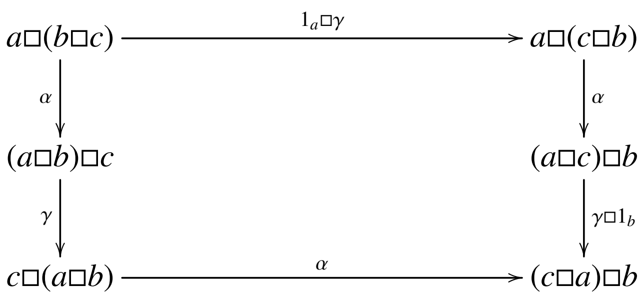
commutes for all $a, b \in C$ and $\lambda_e = \rho_e : e\Box e \to e$.

Example 1.5. *A monoidal category C with tensor product $\otimes$ and unit object I can be regarded as a bicategory B with a single 0-cell, the 1-cells of B being the objects of C with $\otimes$ as their composition, and the 2-cells of B being the morphisms of C.*

Definition 1.8. *[45] A **symmetric monoidal category** is a monoidal category together with a symmetry. A symmetry γ for a monoidal category $C =< C, \Box, e, \alpha, \lambda, \rho >$ is a natural isomorphism $\gamma = \gamma_{ab} : a\Box b \to b\Box a$ such that the following three diagrams commute for all $a, b, c \in C$:*

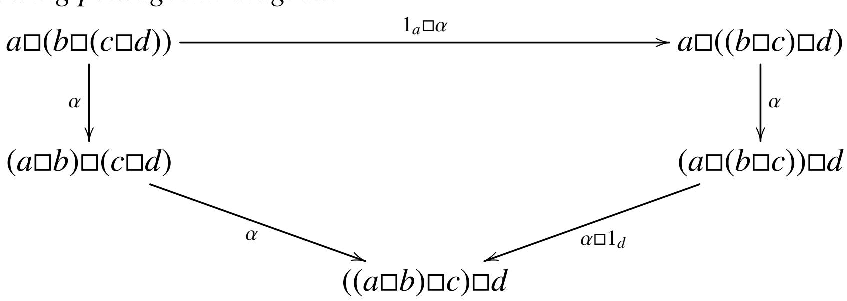






An example of a symmetric monoidal category is the category of vector spaces over some fixed field K, using the ordinary tensor product of vector spaces.

Definition 1.9. *[42] A **semi-monoidal category** $C =< C, \Box, \alpha >$ is a category C, a bifunctor $\Box : C \times C \to C$ and a natural transformation α, satisfying the following condition:*
• α is a natural isomorphism with components $\alpha_{a,b,c} : (a\Box b)\Box c \to a\Box(b\Box c)$ such that the following pentagonal diagram



commutes for all $a, b, c, d \in C$. (α is also called associator (p.11 [17])).

Definition 1.10. *(Defn 3.1 [59]) Given a category C and an object A of C, an* ***idempotent*** *of C is an endomorphism $e : A \to A$ with $e \circ e = e$. An idempotent $e : A \to A$ is said to* ***split*** *if there is an object B and morphisms $r : A \to B$, $s : B \to A$ such that $e = s \circ r$ and $id_B = r \circ s$.*

Remark 1.1. *(cf. Remark 3.4 [59]) The splitting of an idempotent is a special case of a categorical limit and colimit. More precisely, if $e : A \to A$ is an idempotent, then $r : A \to B$, $s : B \to A$ is a splitting of e if and only if r is a colimit and s is a limit of the diagram $e : A \to A$.*

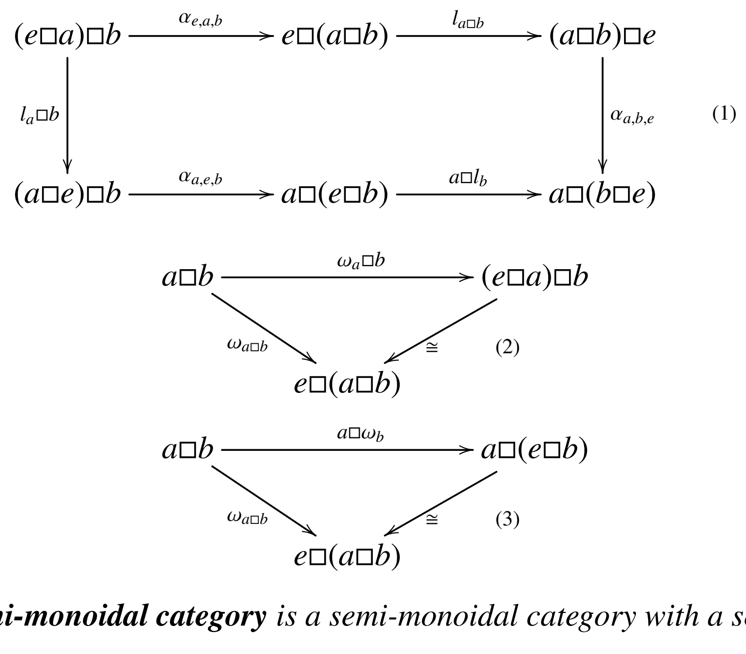
Definition 1.11. *[12] An ordinary category is* ***idempotent complete****, aka* ***Karoubi complete*** *or* ***Cauchy complete****, if every idempotent splits.*

We used the phrase "ordinary category" as opposed to the phrase "Higher category". Higher category theory is the generalization of category theory to a context where there are not only morphisms between objects, but generally n-morphisms between $(n-1)$-morphisms, for all $n \in \mathbb{N}$.

Definition 1.12. *[1] Let $(C, \Box)$ be a semi-monoidal category with natural isomorphism $\alpha_{X,Y,Z} : (X \Box Y) \Box Z \to X \Box (Y \Box Z)$ for all $X, Y, Z \in C$. Let $\mathbb{I}$ stand for the identity endofunctor on any given category. We say that $\mathbf{I} \in C$ is a* ***semi-unit*** *if the following conditions hold:*

1. *There is a natural transformation $\omega : \mathbb{I} \to (\mathbf{I} \Box -)$;*

2. *There exists an isomorphism of functors $\mathbf{I} \Box - \cong - \Box \mathbf{I}$, i.e., there is a natural isomorphism $l_X : \mathbf{I} \Box X \cong X \Box \mathbf{I}$ in C with inverse q_X, for each object X of C, such that $l_{\mathbf{I}} = q_{\mathbf{I}}$ and the following diagrams are commutative for all $X, Y \in C$:*



A ***semi-unital semi-monoidal category*** *is a semi-monoidal category with a semi-unit.*

Remark 1.2. *[1] If $X \cong \mathbf{I} \Box X (\cong X \Box \mathbf{I})$, then we say that X* ***firm*** *and set $\lambda_X := \omega_X^{-1} : \mathbf{I} \Box X \to X$.*
If $\mathbf{I}$ is firm (also called a ***pseudo-idempotent****) and $\omega_{\mathbf{I}}^{-1} \Box \mathbf{I} = \mathbf{I} \Box \omega_{\mathbf{I}}^{-1}$, then $\mathbf{I}$ is an* ***idempotent****.*
A semi-monoidal category becomes a monoidal category if it has a unit, i.e., $\mathbf{I}$ is such that $\lambda_{\mathbf{I}} = \kappa_{\mathbf{I}}$, $\lambda_{X \Box Y} = \lambda_X \Box Y$ and $\kappa_{X \Box Y} = X \Box \kappa_Y$ for all $X, Y \in C$, where $\kappa_X = \omega_X^{-1} \circ q_X : X \Box \mathbf{I} \cong \mathbf{I} \Box X \to X$.

Remark 1.3. *(cf. remark 3.3 of [1], [42]) Kock in [42] called an object I a* ***Saavedra unit*** *or* ***reduced unit*** *if and only if it is pseudo-idempotent and cancellable in the sense that the endofunctors* $I\Box-$ *and* $-\Box I$ *are full and faithful (equivalently, I is idempotent and the endofunctors* $I\Box-$ *and* $-\Box I$ *are equivalences of categories). Kock also showed that I is a unit just in case it is a Saavedra unit.*
Since every unit is a semiunit, the notion of semi-unital semi-monoidal categories generalizes the classical notion of monoidal categories.

Semi-rings were studied by many algebraists beginning with Dedekind [Dedekind]. They have significant applications in several areas, for instance Automata Theory and Optimization Theory (see [27] for applications).
The theory of semi-modules over semi-rings was developed by many authors including Takahashi [63]. In 2008, Jawad used the so called Takahashi's tensor-like product $\boxtimes$ of semi-modules over an associative semi-ring A [63], to introduce notions of semi-unital semi-rings and semi-counital semi-corings (cf. [1]). However, these could not be realized as monoids (comonoids) in the category ${}_AS_A$ of (A, A)-bisemi-modules. This is mainly due to the fact that the category $({}_AS_A, \boxtimes, A)$ is not monoidal in general. Motivated by the desire to fix this problem, Jawad [1] introduced and investigated a notion of semi-unital semi-monoidal categories with prototype $({}_AS_A, \boxtimes, A)$ and investigated semi-monoids (semi-comonoids) in such categories as well as their categories of semi-modules (semi-comodules). He realized that although the base semi-algebra A is not a unit in ${}_AS_A$, A nevertheless has properties of what he called a *semi-unit*. This motivated also introducing a more generalized notion of monads (comonads) in arbitrary categories (for more on this, see [1]).

Example 1.6. *An example of semi-unital semi-monoidal category as given by Jawad in [1] is the category of bisemi-modules over a semi-algebra A with the Takahashi tensor product* $({}_AS_A, \boxtimes, A)$*. That is the only example we found in the literature. unfortunately, it requires a great amount of set-up and so we refer the reader to theorem 5.11 of [1].*

Another (less involved) example of semi-unital semi-monoidal category will be given in this dissertation (cf. theorem 3.2).

There is also a notion of skew-monoidal category ([62]). We have the following definition.

Definition 1.13. *(cf. [62]) A* ***right-monoidal category*** $(\mathcal{M}, *, e, \alpha, \gamma, \rho)$ *consists of a category* $\mathcal{M}$*, a functor* $(-)*(-) : \mathcal{M}\times\mathcal{M} \to \mathcal{M}$*, an object e of* $\mathcal{M}$ *and natural transformations*
$\alpha_{L,M,N} : L * (M * N) \to (L * M) * N$
$\gamma_M : M \to e * M$
$\rho_M : M * e \to M$
subject to the following axioms: For all objects K, L, M, N

1. $(\alpha_{K,L,M} * N) \circ \alpha_{K,L*M,N} \circ (K * \alpha_{L,M,N}) = \alpha_{K*L,M,N} \circ \alpha_{K,L,M*N}$
2. $\alpha_{e,M,N} \circ \gamma_{M*N} = \gamma_M * N$
3. $\rho_{M*N} \circ \alpha_{M,N,e} = M * \rho_N$
4. $(\rho_M * N) \circ \alpha_{M,e,N} \circ (M * \gamma_N) = id_{M*N}$

5. $\rho_e \circ \gamma_e = id_e$.

Remark 1.4. *(cf. [62]) If $\mathcal{M}$ is replaced with $\mathcal{M}^{op}$, we obtain what is called a* ***left-monoidal category****.*
If α, γ and ρ are isomorphisms, we recover the notion of monoidal category. So a right-monoidal category is a generalization of a monoidal category.

1.3 Chain complexes and graded modules

In this section, we recall the notions of chain complex which will be useful in this thesis (e.g. subsection 5.1.1) and of graded modules.

> The material in this section is simply an elaboration of part of the work done in [66], [38]. Only those parts indispensable to this thesis will be reproduced here.

1.3.1 Chain Complexes

This notion will be a little bit used in chapter 5. In this section, R will stand for an arbitrary ring (i.e., an associative ring in the sense of [66].) with identity.

Definition 1.14. *[66] Complexes of R-Modules*
A ***chain complex*** $C_\bullet$ *of $R-$modules is a family $\{C_n\}_{n\in\mathbb{Z}}$ of $R-$modules together with $R-$module maps $d = d_n : C_n \longrightarrow C_{n-1}$ such that each composite $d \circ d : C_n \longrightarrow C_{n-2}$ is zero. The maps d_n are called the differentials of $C_\bullet$. The kernel of d_n is the module of $n-$cycles of $C_\bullet$ denoted $Z_n = Z_n(C_\bullet)$. The image of $d_{n+1} : C_{n+1} \longrightarrow C_n$ is the module of $n-$boundaries of $C_\bullet$ denoted $B_n = B_n(C_\bullet)$. Since $d \circ d = 0$, we have $0 \subseteq B_n \subseteq Z_n \subseteq C_n$ for all n. The n^{th} homology module of $C_\bullet$ is the subquotient $H_n(C_\bullet) = Z_n/B_n$ of $C_\bullet$.*

Remark 1.5. *We can also talk of chain complexes when instead of $R-$modules in the definition above, we have abelian groups with no module structure on them.*

We denote by Ch(Mod$-R$) the category whose objects are chain complexes of (right) $R-$modules and whose morphisms are defined as follows: let $C_\bullet$ and $D_\bullet$ be complexes of right $R-$modules; that is:

$$C_\bullet : \cdots \xrightarrow{d_{n+2}} C_{n+1} \xrightarrow{d_{n+1}} C_n \xrightarrow{d_n} C_{n-1} \xrightarrow{d_{n-1}} \cdots \text{ and}$$
$$D_\bullet : \cdots \xrightarrow{d'_{n+2}} D_{n+1} \xrightarrow{d'_{n+1}} D_n \xrightarrow{d'_n} D_{n-1} \xrightarrow{d'_{n-1}} \cdots$$

A morphism $u : C_\bullet \longrightarrow D_\bullet$ is a chain complex map, that is a family of $R-$module homomorphisms $u_n : C_n \longrightarrow D_n$ commuting with the differentials in the sense that $u_{n-1}d_n = d'_n u_n$. That is, such that the following diagram commutes for all n.

$$\begin{array}{ccc} C_n & \xrightarrow{d_n} & C_{n-1} \\ {\scriptstyle u_n}\downarrow & & \downarrow{\scriptstyle u_{n-1}} \\ D_n & \xrightarrow[d'_n]{} & D_{n-1} \end{array}$$

By reindexing with superscripts $C^n = C_{-n}$, we can define the notion of cochain complex (cf. [66]). For examples of chain complexes and operations on chain complexes, see [66]. We spend the rest of this subsection providing ingredients that suffice to state that Ch (the category of chain complexes and chain maps between them) is an additive category. My point of interest is mostly the definitions because they will be useful later.

Definition 1.15. *([45] and [66]) A category $\mathcal{A}$ is an **Ab-category** if every hom-set $\mathcal{A}(A, B)$ in $\mathcal{A}$ is equipped with the structure of an additive abelian group and composition of arrows is bilinear relative to this addition (i.e., composition distributes over addition). In particular given a diagram of the form*

$$A \xrightarrow{f} B \underset{g}{\overset{g'}{\rightrightarrows}} C \xrightarrow{h} D$$

in Hom(A, D); $h(g + g')f = hgf + hg'f$. If $\{f_n\}$ and $\{g_n\}$ are chain maps from a chain complex $C_\bullet$ to a chain complex $D_\bullet$, then their sum is the family of maps $\{f_n + g_n\}$. Thanks to this degreewise addition, Ch is an Ab-category.

Definition 1.16. *([45] and [66]) Additive category*
*An **additive category** is an Ab-category with a zero object (i.e., an object that is initial and terminal) and a biproduct for each pair of its objects.*

Example 1.7. *Ch is an additive category. (The zero object in Ch is the complex* ”0” *of zero modules and maps.) For more details, see page 5 of [66].*

1.3.2 A reminder on graded modules

References for this section are [38] and [64].

Here, we recall the definitions of graded modules and their morphisms. We also recall the definitions of a graded algebra and of differential graded module.
In the sequel, Γ is an indexing set such that $(\Gamma, +)$ forms an abelian group. We will mostly use $\mathbb{Z}$ and $\mathbb{Z}_2$.

Definition 1.17. *(section 2.1.1 of [64]) Graded ring*
*A **ring** R is said to be $\Gamma-$**graded** if there exists a family of additive abelian groups $\{R_\gamma\}_{\gamma\in\Gamma}$ of R such that:*

1. $R = \bigoplus_{\gamma\in\Gamma} R_\gamma$
2. $R_\gamma \cdot R_\delta \subseteq R_{\gamma+\delta}$, *where* $\gamma, \delta \in \Gamma$

Thus, every $r \in R$ is uniquely expressible as $r = \sum_{\gamma\in\Gamma} r_\gamma$ with each $r_\gamma \in R_\gamma$ and at most finitely many r_γ are nonzero. The r_γ are called the homogeneous components of r. The grade set of R is the subset of Γ defined by

$$\Gamma_R = \{\gamma \in \Gamma; R_\gamma \neq \{0\}\}.$$

The homogeneous elements of R are the elements of $\bigcup_{\gamma\in\Gamma} R_\gamma$. If $a \in R_\gamma$ and $a \neq 0$, we say that γ is the degree of a and write $\gamma = deg(a)$. As stated above, we always assume that a ring has an identity 1.

Remark 1.6. *For the graded ring R, we have R_0 is a subring of R, as $R_0 \cdot R_0 \subseteq R_0$, and it is easy to check that $1 \in R_0$.*

Notations 1.1. *[64] The graded ring R with the grading forgotten is denoted by $R^\natural$. In other words, $R^\natural$ is the underlying ungraded ring of R.*

Definition 1.18. *[64] Graded Modules*
*Let R be a graded ring. A right module M over $R^\natural$ equipped with a decomposition $M = \bigoplus_{\gamma\in\Gamma} M_\gamma$, where each M_γ is an additive subgroup of M and $M_\gamma \cdot R_\delta \subseteq M_{\gamma+\delta}$ for $\gamma, \delta \in \Gamma$, is called a right **graded R-module**. Left graded R-modules are defined likewise.*

Example 1.8. • *A graded vector space is an example of a graded module over a field (with the field having trivial grading).*
• *A graded ring is a graded module over itself.*

Example 1.9. *Let K be a commutative ring. Consider the ring $K[x]$ of polynomials in the indeterminate x. It is clearly a $K-$module. We give K the trivial grading, i.e., $K = K + 0$. If for $i \geq 0$, we let Kx^i be the additive subgroup of $K[x]$ made up of polynomials of degree i, then we have the following graded module $K[x] = \bigoplus_{n\in\mathbb{Z}} Kx^n$ where $Kx^n = 0$ if $n \leq 0$.*

We will refer to the following example in example 5.1.

Example 1.10. *Let $\mathbb{C}[x]$ be the ring of polynomials in the indeterminate x with coefficients in the field of complex numbers. We endow $\mathbb{C}[x]$ with the trivial grading, i.e., $\mathbb{C}[x] = \mathbb{C}[x] \oplus 0$. we show that $S = \mathbb{C}[x] \oplus \mathbb{C}[x]$ is a $\mathbb{Z}_2$-graded $\mathbb{C}[x]$-module. First of all, S is an abelian group as a direct sum of abelian groups. The left and right actions of $\mathbb{C}[x]$ on S are the obvious ones. By writing $S = S_0 \oplus S_1$ where $S_0 = \mathbb{C}[x] \oplus 0$ and $S_1 = 0 \oplus \mathbb{C}[x]$, it is easy to see that S_0 and S_1 are additive subgroups of S. It is also not difficult to verify the conditions stated in definition 1.18 to conclude that S is a $\mathbb{Z}_2$-graded $\mathbb{C}[x]$-module.*

In the sequel, we omit the $\natural$ on a ring R, and simply write R for graded and ungraded rings since it will be contextually clear.

Definition 1.19. *[38] morphism of graded $K-$modules*
*A **morphism** $f : M \longrightarrow L$ **of graded K-modules** of degree n is a $K-$linear morphism such that $f(M^p) \subset L^{p+n}$ for all $p \in \mathbb{Z}$.*

The tensor product $M \otimes N$ of two graded $K-$modules M and N is the graded $K-$module with components:

$$(M \otimes N)^p = \bigoplus_{m+n=p} M^m \otimes N^n, \quad p \in \mathbb{Z}$$

The tensor product $f \otimes g$ of two maps $f : M \longrightarrow M'$ and $g : N \longrightarrow N'$ of graded $K-$modules is defined using the Koszul sign rule (cf. [38]):

$$(f \otimes g)(m \otimes n) = (-1)^{rs} f(m) \otimes g(n)$$

if g is of degree r and m belongs to M^s.

Definition 1.20. *[38] Graded $K-$algebra*
*A **graded K-algebra** is a graded $K-$module $\mathcal{A}$ endowed with a multiplication morphism $\mathcal{A} \otimes \mathcal{A} \longrightarrow \mathcal{A}$ which is graded of degree 0, associative and admits a unit $1 \in \mathcal{A}^0$.*

Ordinary $K-$algebras are identified with graded $K-$algebras concentrated in degree 0. The category of graded $K-$modules is denoted by $\mathcal{G}(K)$.

Definition 1.21. *([38]) dg $K-$module*
*A **differential graded (dg) K-module** is a $\mathbb{Z}$-graded $K-$module X endowed with a differential d_X, i.e., a map $d_X : X \longrightarrow X$ of degree 1 such that $d_X^2 = 0$. Equivalently, X is a complex of $K-$modules.*

The tensor product of two dg graded $K-$modules is the graded module $M \otimes N$ endowed with the differential $d_M \otimes 1_N + 1_M \otimes d_N$.

1.4 Permutation matrices and $(1, 0)-$matrices

In this section, after defining $(1, 0)-$matrices, we recall some definitions and properties of permutation matrices. This will help in chapter 3 when we will be proving that there is a subcategory of the category of matrix factorizations of the constant power series 1 which is semi-unital semi-monoidal.
Most of the material of this section is taken from [26] and [32].

Definition 1.22. *A $(1, 0)-$matrix is a matrix whose entries are either 1 or 0.*

Definition 1.23. *[26] A permutation matrix is a square matrix obtained from the same size identity matrix by a permutation of rows.*
Equivalently, a permutation matrix is a matrix obtained by permuting the rows of an $n \times n$ identity matrix according to some permutation of the numbers 1 to n.

Some facts about permutation matrices:
• Every row and column of a permutation matrix contains precisely a single 1 with 0s everywhere else, and every permutation corresponds to a unique permutation matrix. There are therefore $n! = n(n-1)\cdots 3 \cdot 2 \cdot 1$ permutation matrices of size n.
• A product of permutation matrices is again a permutation matrix and the inverse of a permutation matrix is again a permutation matrix.
• A permutation matrix is nonsingular, and the determinant is always ± 1. In addition, a permutation matrix A satisfies $AA^T = I$ entailing that $A^{-1} = A^T$, where A^T is the transpose of A and I is the identity matrix.

Example 1.11. *Some examples of permutation matrices of order 2 and 3:*

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

and

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Notation and convention (for this dissertation):
We choose to write:

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, I_{(2,3)} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, I_{(1,2)} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

where $(2,3)$ and $(1,2)$ are transpositions or better still in this context, permutation vectors. $I_{(2,3)}$ is the identity matrix in which the second and third rows have been interchanged. $I_{(1,2)}$ has a similar meaning. The size of the identity matrix will always be clear in context. We will sometimes just write A_p where p will stand for a specified permutation vector. Applied to a matrix M, I_pM gives M with rows interchanged according to the permutation vector p, and MI_q gives M with the columns interchanged according to the permutation vector q.

Example 1.12.

$$Let\ M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

$$I_{(1,2)}M = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} c & d \\ a & b \end{bmatrix}.$$

Observe that the permutation used here is the identity matrix where the first and second rows have been interchanged. Consequently, $I_{(1,2)}M$ yields M in which the first and second rows have been interchanged.

$$Let\ N = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}.$$

$$I_{(2,3)}N = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = \begin{bmatrix} a & b & c \\ g & h & i \\ d & e & f \end{bmatrix}.$$

Observe that the permutation used here is the identity matrix where the second and third rows have been interchanged. Consequently, $I_{(2,3)}N$ yields N in which the second and third rows have been interchanged.

$$Similarly,\ if\ Q = \begin{bmatrix} a & b \\ d & e \\ g & h \end{bmatrix},\ then\ I_{(2,3)}N = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} a & b \\ d & e \\ g & h \end{bmatrix} = \begin{bmatrix} a & b \\ g & h \\ d & e \end{bmatrix}.$$

MATRIX FACTORIZATIONS

In this chapter, we briefly study matrix factorizations. After recalling the definition of a matrix factorization, we describe the category of matrix factorizations of a power series f. Next, we give a straightforward algorithm that helps to produce a bigger matrix factorization from a smaller one. Thereafter, inspired by the Yoshino's tensor product of matrix factorizations, we prove that there exists another product of matrix factorizations different from the Yoshino's tensor product but with the same effect. That is, just like with the Yoshino's tensor product, given matrix factorizations of two power series f and g, this new product produces a matrix factorization of the sum $f + g$. Furthermore, properties of matrix factorizations will be presented. Finally, we observe that an $n \times n$ matrix factorization of an element in a unital ring is not unique.
In all the chapter, except otherwise stated, the ring of power series in the indeterminates $x = x_1, \cdots, x_n$ will be denoted by $R = K[[x_1, \cdots, x_n]] = K[[x]]$ where K is a field. $K(x)$ will denote the field of rational functions[1] in x.

Remark 2.1. *The original definition of a matrix factorization was given by Eisenbud [25] as follows: a matrix factorization of an element x in a ring R (with unity) is an ordered pair of maps of free R−modules* $\phi : F \to G$ *and* $\psi : G \to F$ *s.t.,* $\phi\psi = x \cdot 1_G$ *and* $\psi\phi = x \cdot 1_F$. *Though this definition is valid for any arbitrary ring (with unity), in order to effectively study matrix factorizations, it is important to restrict oneself to specific rings. Working with specific rings makes it possible to easily give examples and it also allows one to carry out computations in a well-defined framework. Yoshino [67] restricted himself to matrix factorizations of power series. In this chapter, we will restrict ourselves to matrix factorizations of a power series (particularly of a polynomial) and also allude to the category of matrix factorizations of a rational function. In fact, the results we will present (in this chapter and in the next two chapters) hold not only for matrix factorizations of a power series, but also for elements of an arbitrary ring with unity.*

2.1 A word on matrix factorizations

In 1980, Eisenbud came up with an approach of factoring both reducible and irreducible polynomials in R using matrices. Furthermore, he found out that matrix factorizations of $f \in R$ are closely related to homological properties of modules over the quotient ring

[1]That is, elements of $K(x)$ are quotients of polynomials in x, with the denominator being different from zero.

$R/(f)$. These quotient rings are called *hypersurface rings*[2].
As mentioned in the introduction of this dissertation, one interesting reason why the study of matrix factorizations is important is that irreducible polynomials can be factorized using matrices. For instance, the polynomial $f = x^2+1$ is irreducible over the real numbers but can be factorized as follows:

$$x^2 + 1 = \begin{bmatrix} x & -1 \\ 1 & x \end{bmatrix} \begin{bmatrix} x & 1 \\ -1 & x \end{bmatrix} = (x^2 + 1) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = fI_2$$

2.1.1 Definition and some Examples

The notion of matrix factorization is defined in [67] for nonzero non-invertible[3] $f \in K[[x_1, x_2, \cdots, x_n]]$. We define it as in [19] slightly generalizing the one given in [67] by including elements like $1 \in K$. Defining it this way is important to us because, it will help in constructing an example of a semi-unital semi-monoidal category in chapter 3.

Definition 2.1. *[67], [19]*
*An $n \times n$ **matrix factorization** of a power series $f \in R$ is a pair of $n \times n$ matrices (P, Q) such that $PQ = fI_n$, where I_n is the $n \times n$ identity matrix and the coefficients of P and of Q are taken from R.*

Remark 2.2. *When $n = 1$, we get a 1×1 matrix factorization of f, i.e., $f = [g][h]$ which is simply a factorization of f in the classical sense. But in case f is not reducible, this is not interesting, that's why we will mostly consider $n > 1$.*

Example 2.1. *Let $f = x^2 + y^2$.*
We give a 2×2 matrix factorization of f:

$$\begin{bmatrix} x & -y \\ y & x \end{bmatrix} \begin{bmatrix} x & y \\ -y & x \end{bmatrix} = (x^2 + y^2) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = fI_2$$

Thus; $(\begin{bmatrix} x & -y \\ y & x \end{bmatrix}, \begin{bmatrix} x & y \\ -y & x \end{bmatrix})$ *is a 2×2 matrix factorization of f.*

Example 2.2. *Let $g = x^3 + 1$.*
We give a 2×2 matrix factorization of g:

$$\begin{bmatrix} x^2 & -1 \\ 1 & x \end{bmatrix} \begin{bmatrix} x & 1 \\ -1 & x^2 \end{bmatrix} = (x^3 + 1) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = gI_2$$

Thus;
$(\begin{bmatrix} x^2 & -1 \\ 1 & x \end{bmatrix}, \begin{bmatrix} x & 1 \\ -1 & x^2 \end{bmatrix})$
is a 2×2 matrix factorization of g.

[2]As mentioned in [19], these quotient rings encode geometric properties of the zero-locus of f, $Z(f) = \{x \in \mathbb{R}^n; f(x) = 0\}$. For more connection between matrix factorizations and algebraic geometry, see [68].
[3]Yoshino [67] requires an element $f \in K[[x]]$ to be nonzero non-invertible because if $f = 0$ then $K[[x]]/(f) = K[[x]]$ and if f is a unit, then $K[[x]]/(f)$ =K[[x]]/K[[x]]= {1}. But in this dissertation, we will not bother about such restrictions because we will not deal with the homological methods used in [67].

Example 2.3. *Let* $h = xy + xz^2 + yz^2$.
We give a 2×2 *matrix factorization of g:*

$$\begin{bmatrix} z^2 & y \\ x & -x-y \end{bmatrix} \begin{bmatrix} x+y & y \\ x & -z^2 \end{bmatrix} = (xy + xz^2 + yz^2) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = hI_2$$

Thus;
$(\begin{bmatrix} z^2 & y \\ x & -x-y \end{bmatrix}, \begin{bmatrix} x+y & y \\ x & -z^2 \end{bmatrix})$
is a 2×2 *matrix factorization of h.*

The standard method for factoring polynomials through matrices was given in [41], where Knörrer exploited it to prove his famous Periodicity theorem (cf. theorem 2.1). Section 3 of [19] also recalls that standard method in details and proposes and improvement for sums of square polynomials. We will also recall the standard method in chapter 4 and propose an improvement on a special class of polynomials.

Theorem 2.1. *Knörrer periodicity theorem [13]*
Suppose K is algebraically closed and char(K)≠ 2. If $f \in (x_1, \cdots, x_n) \subseteq K[[x_1, \cdots, x_n]]$, *then there is an equivalence between the categories* $MF(K[[x_1, \cdots, x_n]], f)$ *and* $MF(K[[x_1, \cdots, x_n, u, v]], f + u^2 + v^2)$.

2.1.2 The category of matrix factorizations of $f \in R$

In this subsection, we describe a category that is of interest to us in this thesis: the category of matrix factorizations of $f \in R$.
The category of matrix factorizations of a power series $f \in R = K[[x]] := K[[x_1, \cdots, x_n]]$ denoted by $MF(R, f)$ or $MF_R(f)$, (or even $MF(f)$ when there is no risk of confusion) is defined as follows:
• The objects are the matrix factorizations of f.
• Given two matrix factorizations of f; (ϕ_1, ψ_1) and (ϕ_2, ψ_2) respectively of sizes n_1 and n_2, a morphism from (ϕ_1, ψ_1) to (ϕ_2, ψ_2) is a pair of matrices (α, β) each of size $n_2 \times n_1$ which makes the following diagram commute [67]:

$$\begin{array}{ccccc} K[[x]]^{n_1} & \xrightarrow{\psi_1} & K[[x]]^{n_1} & \xrightarrow{\phi_1} & K[[x]]^{n_1} \\ \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\alpha} \\ K[[x]]^{n_2} & \xrightarrow{\psi_2} & K[[x]]^{n_2} & \xrightarrow{\phi_2} & K[[x]]^{n_2} \end{array} \qquad (\bigstar)$$

That is,

$$\begin{cases} \alpha\phi_1 = \phi_2\beta \\ \psi_2\alpha = \beta\psi_1 \end{cases}$$

• Given three matrix factorizations of f: (ϕ_1, ψ_1), (ϕ_2, ψ_2) and (ϕ_3, ψ_3) respectively of sizes n_1, n_2 and n_3, the composition

$$(\alpha_2, \beta_2) \circ (\alpha_1, \beta_1) : (\phi_1, \psi_1) \xrightarrow{(\alpha_1, \beta_1)} (\phi_2, \psi_2) \xrightarrow{(\alpha_2, \beta_2)} (\phi_3, \psi_3)$$

is the pair of matrices $(\alpha_2\alpha_1, \beta_2\beta_1)$ such that the following diagram commutes:

$$\begin{array}{ccccc}
K[[x]]^{n_1} & \xrightarrow{\psi_1} & K[[x]]^{n_1} & \xrightarrow{\phi_1} & K[[x]]^{n_1} \\
\downarrow{\scriptstyle \alpha_1} & & \downarrow{\scriptstyle \beta_1} & & \downarrow{\scriptstyle \alpha_1} \\
K[[x]]^{n_2} & \xrightarrow{\psi_2} & K[[x]]^{n_2} & \xrightarrow{\phi_2} & K[[x]]^{n_2} \\
\downarrow{\scriptstyle \alpha_2} & & \downarrow{\scriptstyle \beta_2} & & \downarrow{\scriptstyle \alpha_2} \\
K[[x]]^{n_3} & \xrightarrow{\psi_3} & K[[x]]^{n_3} & \xrightarrow{\phi_3} & K[[x]]^{n_3}
\end{array}$$

That is,

$$\begin{cases} (\alpha_2\alpha_1)\phi_1 = \phi_3(\beta_2\beta_1) \\ \psi_3(\alpha_2\alpha_1) = (\beta_2\beta_1)\psi_1 \end{cases}$$

• It is easy to see that *associativity of composition of maps of matrix factorizations of f* is a consequence of the fact that matrix multiplication is associative.
• For any $n \times n$ matrix factorization (ϕ, ψ) of f, there is a map $1_{(\phi,\psi)} : (\phi, \psi) \to (\phi, \psi)$ which is actually the pair of identity $n \times n$ matrices (I_n, I_n).
• It is also clear that composing any map of matrix factorizations of f with $1_{(\phi,\psi)}$ from the left or the right (whenever the composition is possible) leaves the given map unchanged.
This ends the definition of the category of matrix factorizations of $f \in R = K[[x]]$.

In definition 2.1, we defined the matrix factorization of a power series. As we said at the beginning of this chapter, the results we will develop (in this chapter and in the next two chapters) hold not only for power series but also for any element in a given unital ring with unity. Let us for instance, in the following definition, define the matrix factorization of a rational function.

Definition 2.2. *An $n \times n$ **matrix factorization** of a rational function $F \in K(x)$ is a pair of $n \times n$ matrices (P, Q) such that $PQ = FI_n$, where I_n is the $n \times n$ identity matrix and the coefficients of P and of Q are taken from $K(x)$.*

Again as discussed under definition 2.1, in practice, we would like to consider only $n > 1$.

Example 2.4. *Let $F = \dfrac{x}{x+1}$.*
We give a simple 2×2 matrix factorization of F:

$$\begin{bmatrix} \frac{1}{(x+1)^3} & 0 \\ 0 & \frac{1}{(x+1)^3} \end{bmatrix} \begin{bmatrix} x(x+1)^2 & 0 \\ 0 & x(x+1)^2 \end{bmatrix} = \frac{x}{x+1} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = FI_2$$

Thus; $\left(\begin{bmatrix} \frac{1}{(x+1)^3} & 0 \\ 0 & \frac{1}{(x+1)^3} \end{bmatrix}, \begin{bmatrix} x(x+1)^2 & 0 \\ 0 & x(x+1)^2 \end{bmatrix} \right)$ *is a 2×2 matrix factorization of F.*

There is a category of matrix factorizations of a rational function $F \in K(x)$ which is defined in a manner similar to the category of matrix factorizations of a polynomial f.

Remark 2.3. *Observe that in the foregoing example, the difficulty in evaluating a given nonzero entry of any of the matrices in the pair* $\left(\begin{bmatrix}\frac{1}{(x+1)^3} & 0\\ 0 & \frac{1}{(x+1)^3}\end{bmatrix}, \begin{bmatrix}x(x+1)^2 & 0\\ 0 & x(x+1)^2\end{bmatrix}\right)$, *seems to be greater than evaluating F itself. This is not very nice. The scenario was different in the examples we had above when factoring polynomials (see examples 2.4 and 2.3). Perhaps, this is one of the reasons why we could not find in the literature anyone interested in studying matrix factorizations of rational functions. We will mostly focus on power series.*

We will now like to answer the following question:
Given an $n{\times}n$ matrix factorization of $f \in R$, is there a procedure to obtain an $(n{+}r){\times}(n{+}r)$, $(r > 0)$, matrix factorization of $f \in R$?
It is clear that obtaining a 2×2 matrix factorization of a polynomial is not too demanding like obtaining a 3×3 or even a bigger matrix factorization of the same polynomial. We show that once we have a matrix factorization of f, it is easy to find one that is bigger.
We now describe a simple straightforward algorithm to obtain an $n \times n$ matrix factorization from one that is $m \times m$, where $m < n$.

Simple algorithm: Let P and Q be the $m \times m$ matrix factorizations of a power series f. Suppose we want an $n \times n$ matrix factorization of f, where $n > m$.
Let (i, j) stand for the entry in the i^{th} row and j^{th} column.

- Turn P and Q into $n \times n$ matrices by filling them with zeroes everywhere except at entries (i, j) where $1 \leq i \leq m$ and $1 \leq j \leq m$.
  Then for all entries (k, k) with $k > m$, Either:

  1. In P, replace the diagonal elements (which are zeroes) with f
     And
  2. In Q, replace the diagonal elements (which are zeroes) with 1

  Or:

  1. In Q, replace the diagonal elements (which are zeroes) with f
     And
  2. In P, replace the diagonal elements (which are zeroes) with 1

Remark that in the Either/Or part of the algorithm, the roles of P and Q have been interchanged.

It is easy to see that this algorithm works not only for power series but also for any element in a unital ring.

2.1.3 Yoshino's Tensor Product of Matrix Factorizations

In this subsection, we recall the definition of tensor product Yoshino[67] gave in terms of matrices. This will be useful when we will be describing the notion of *Morita Context* in $\mathcal{LG}_K$ (cf. subsection 6.2). In the sequel, except otherwise stated $R = K[[x_1, \cdots, x_n]] = K[[x]]$ and let $S = K[[y_1, \cdots, y_r]] = K[[y]]$, where K is a commutative ring with unity and $r \geq 1$.

Definition 2.3. *Let $X = (\phi, \psi)$ be an $n \times n$ matrix factorization of $f \in R$ and $X' = (\phi', \psi')$ an $m \times m$ matrix factorization of $g \in S$. These matrices can be considered as matrices over $L = K[[x, y]]$ and the* ***tensor product*** *$X \widehat{\otimes} X'$ is given by*
$$\left(\begin{bmatrix} \phi \otimes 1_m & 1_n \otimes \phi' \\ -1_n \otimes \psi' & \psi \otimes 1_m \end{bmatrix}, \begin{bmatrix} \psi \otimes 1_m & -1_n \otimes \phi' \\ 1_n \otimes \psi' & \phi \otimes 1_m \end{bmatrix}\right)$$
where each component is an endomorphism on $L^n \otimes L^m$.

It is also mentioned in [67], that $X \widehat{\otimes} X'$ is a matrix factorization of $f + g$ of size $2nm$. We give a proof of that statement here in the form of a lemma.

Lemma 2.1. *$X \widehat{\otimes} X'$ is a matrix factorization of $f + g$ of size $2nm$.*

Proof. Calculations are as follows:

$$\begin{bmatrix} \phi \otimes 1_m & 1_n \otimes \phi' \\ -1_n \otimes \psi' & \psi \otimes 1_m \end{bmatrix} \begin{bmatrix} \psi \otimes 1_m & -1_n \otimes \phi' \\ 1_n \otimes \psi' & \phi \otimes 1_m \end{bmatrix}$$
$$= \begin{bmatrix} (\phi \otimes 1_m)(\psi \otimes 1_m) + (1_n \otimes \phi')(1_n \otimes \psi') & (\phi \otimes 1_m)(-1_n \otimes \phi') + (1_n \otimes \phi')(\phi \otimes 1_m) \\ (-1_n \otimes \psi')(\psi \otimes 1_m) + (\psi \otimes 1_m)(1_n \otimes \psi') & (-1_n \otimes \psi')(-1_n \otimes \phi') + (\psi \otimes 1_m)(\phi \otimes 1_m) \end{bmatrix}$$
$$= \begin{bmatrix} \phi\psi \otimes 1_m 1_m + 1_n 1_n \phi' \psi' & \phi(-1_n) \otimes 1_m \phi' + 1_n \phi \otimes \phi' 1_m \\ -1_n \psi \otimes \psi' 1_m + \psi 1_n \otimes 1_m \psi' & (-1_n)(-1_n) \otimes \psi' \phi' + \psi\phi \otimes 1_m 1_m \end{bmatrix}$$
$$= \begin{bmatrix} f 1_n \otimes 1_m + 1_n \otimes g 1_m & 0 \\ 0 & 1_n \otimes g 1_m + f 1_n \otimes 1_m \end{bmatrix}, \ \phi\psi = f 1_n = \psi\phi \text{ and } \phi'\psi' = g 1_m = \psi'\phi'$$
$$= (f + g) \begin{bmatrix} 1_n \otimes 1_m & 0 \\ 0 & 1_n \otimes 1_m \end{bmatrix}$$
$$= (f + g) \begin{bmatrix} 1_{nm} & 0 \\ 0 & 1_{nm} \end{bmatrix} \ \text{since } 1_n \otimes 1_m = 1_{nm}$$
$$= (f + g) \cdot 1_{2nm}$$
This shows that $X \widehat{\otimes} X'$ is an object in $MF(f + g)$ of size $2nm$. QED □

We will drop the wide hat of the $\widehat{\otimes}$ and simply write $\otimes$. We give some examples to show how this tensor product works.
In the following example, we consider matrix factorizations in one variable.

Example 2.5. *Let $X = (x^2, x^2)$ and $X' = (y^2, y^4)$ be 1×1 matrix factorizations of $f = x^4$ and $g = y^6$ respectively. Then*
$$X \otimes X' = \left(\begin{bmatrix} x^2 \otimes 1 & 1 \otimes y^2 \\ -1 \otimes y^4 & x^2 \otimes 1 \end{bmatrix}, \begin{bmatrix} x^2 \otimes 1 & -1 \otimes y^2 \\ 1 \otimes y^4 & x^2 \otimes 1 \end{bmatrix}\right) = \left(\begin{bmatrix} x^2 & y^2 \\ -y^4 & x^2 \end{bmatrix}, \begin{bmatrix} x^2 & -y^2 \\ y^4 & x^2 \end{bmatrix}\right)$$
And

$$\begin{bmatrix} x^2 & y^2 \\ -y^4 & x^2 \end{bmatrix} \begin{bmatrix} x^2 & -y^2 \\ y^4 & x^2 \end{bmatrix} = \begin{bmatrix} x^2 + y^6 & 0 \\ 0 & x^2 + y^6 \end{bmatrix} = (x^2 + y^6) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$
This shows that $X \otimes X'$ is a matrix factorization of $f + g = x^2 + y^6$ and it is of size $2(1)(1) = 2$.

In the next example, we consider matrix factorizations in two variables.

Example 2.6. *Let*

X= $(\begin{bmatrix} x & -y \\ y & x \end{bmatrix}, \begin{bmatrix} x & y \\ -y & x \end{bmatrix})$, *and X'*= $(\begin{bmatrix} -x & y \\ -y & -x \end{bmatrix}, \begin{bmatrix} x & y \\ -y & x \end{bmatrix})$
be 2×2 *matrix factorizations of* $f = x^2 + y^2$ *and* $g = -x^2 - y^2$ *respectively. Let*

$$A= \begin{bmatrix} x & -y \\ y & x \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, B= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} -x & y \\ -y & -x \end{bmatrix}, C= \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \otimes \begin{bmatrix} x & y \\ -y & x \end{bmatrix},$$
$$D= \begin{bmatrix} x & y \\ -y & x \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, A'= \begin{bmatrix} x & y \\ -y & x \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, B'= \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \otimes \begin{bmatrix} -x & y \\ -y & -x \end{bmatrix},$$
$$C'= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} x & y \\ -y & x \end{bmatrix}, \text{ and } D'= \begin{bmatrix} x & -y \\ y & x \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Then
$$X \otimes X' = (\begin{bmatrix} A & B \\ C & D \end{bmatrix}, \begin{bmatrix} A' & B' \\ C' & D' \end{bmatrix})$$

$$= (\begin{pmatrix} x & 0 & -y & 0 & -x & y & 0 & 0 \\ 0 & x & 0 & -y & -y & -x & 0 & 0 \\ y & 0 & x & 0 & 0 & 0 & -x & y \\ 0 & y & 0 & x & 0 & 0 & -y & -x \\ -x & -y & 0 & 0 & x & 0 & y & 0 \\ y & -x & 0 & 0 & 0 & x & 0 & y \\ 0 & 0 & -x & -y & -y & 0 & x & 0 \\ 0 & 0 & y & -x & 0 & -y & 0 & x \end{pmatrix}, \begin{pmatrix} x & 0 & y & 0 & x & -y & 0 & 0 \\ 0 & x & 0 & y & y & x & 0 & 0 \\ -y & 0 & x & 0 & 0 & 0 & x & -y \\ 0 & -y & 0 & x & 0 & 0 & y & x \\ x & y & 0 & 0 & x & 0 & -y & 0 \\ -y & x & 0 & 0 & 0 & x & 0 & -y \\ 0 & 0 & x & y & y & 0 & x & 0 \\ 0 & 0 & -y & x & 0 & y & 0 & x \end{pmatrix})$$

If we call these two 8×8 *matrices P and Q respectively, then*

$$PQ = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} = 0 \cdot \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

From this we also see that $X \otimes X'$ *is also a matrix factorization of* $f+g = x^2+y^2-x^2-y^2 = 0$ *of size* $2(2)(2) = 8$.

Inspired by the definition of the Yoshino's tensor product of matrix factorizations, we now state and prove the following lemma which actually shows that there is more than one product of matrix factorizations which produces a matrix factorization of the sum of two polynomials from the matrix factorizations of each of those polynomials.

Lemma 2.2. *There is a product of matrix factorizations* $\widehat{\otimes}'$ *different from the Yoshino's tensor product* $\widehat{\otimes}$ *which is such that if X and X' are respectively matrix factorizations of power series f and g, then* $X \widehat{\otimes}' X'$ *is a matrix factorization of the sum* $f + g$.
Moreover, $X \widehat{\otimes}' X'$ *and* $X \widehat{\otimes} X'$ *are of the same size.*

Proof. Recall that from definition 2.3 if $X = (\phi, \psi)$ is an $n \times n$ matrix factorization of $f \in R$ and $X' = (\phi', \psi')$ is an $m \times m$ matrix factorization of $g \in S$, then these matrices can be considered as matrices over $L = K[[x, y]]$ and the tensor product $X\widehat{\otimes}X'$ is given by
$(\begin{bmatrix} \phi \otimes 1_m & 1_n \otimes \phi' \\ -1_n \otimes \psi' & \psi \otimes 1_m \end{bmatrix}, \begin{bmatrix} \psi \otimes 1_m & -1_n \otimes \phi' \\ 1_n \otimes \psi' & \phi \otimes 1_m \end{bmatrix})$
where each component is an endomorphism on $L^n \otimes L^m$. It is also mentioned in [67], that $X\widehat{\otimes}X'$ is a matrix factorization of $f + g$ of size $2nm$.
Now, let X and X' be as in the foregoing definition, we define $X\widehat{\otimes}'X'$ by
$(\begin{bmatrix} 1_n \otimes \phi' & \psi \otimes 1_m \\ \phi \otimes 1_m & -1_n \otimes \psi' \end{bmatrix}, \begin{bmatrix} 1_n \otimes \psi' & \psi \otimes 1_m \\ \phi \otimes 1_m & -1_n \otimes \phi' \end{bmatrix})$

We have:

$$\begin{bmatrix} 1_n \otimes \phi' & \psi \otimes 1_m \\ \phi \otimes 1_m & -1_n \otimes \psi' \end{bmatrix} \begin{bmatrix} 1_n \otimes \psi' & \psi \otimes 1_m \\ \phi \otimes 1_m & -1_n \otimes \phi' \end{bmatrix} = \begin{bmatrix} 1_n 1_n \otimes \phi'\psi' + \psi\phi \otimes 1_m 1_m & 1_n\psi \otimes \phi' 1_m - \psi 1_n \otimes 1_m \phi' \\ \phi 1_n \otimes 1_m \psi' - 1_n \phi \otimes \psi' 1_m & \phi\psi \otimes 1_m 1_m + 1_n 1_n \psi'\phi' \end{bmatrix}$$
$$= \begin{bmatrix} 1_n \otimes g \cdot 1_m + f \cdot 1_n \otimes 1_m & \psi \otimes \phi' - \psi \otimes \phi' \\ \phi \otimes \psi' - \phi \otimes \psi' & f \cdot 1_n \otimes 1_m + 1_n \otimes g \cdot 1_m \end{bmatrix} = \begin{bmatrix} (g + f) \cdot 1_{nm} & 0 \\ 0 & (f + g) \cdot 1_{nm} \end{bmatrix}$$
$$= (f + g) \begin{bmatrix} 1_{nm} & 0 \\ 0 & 1_{nm} \end{bmatrix} = (f + g) 1_{2nm}.$$

It is now clear from the work done so far for the two products that $X\widehat{\otimes}'X'$ and $X\widehat{\otimes}X'$ are of the same size which is $2nm$. QED. □

2.2 Properties of Matrix Factorizations

We will now state and prove some properties of matrix factorizations of polynomials. Some of them will be used when studying the notion of Morita Context between two objects in the bicategory $\mathcal{LG}_K$. Almost all of them are taken from [19]. We also state and prove a property of transpose of matrix factorizations thanks to which we conclude that an $n \times n$ matrix factorization of a polynomial f is not unique.
In this section, except otherwise stated $R = K[x_1, \cdots, x_n] = K[x]$.

2.2.1 Commonly used properties

All statements and proofs in this section are taken from [19]. They are reproduced here (sometimes with slight modifications) for the purposes of completeness.
The following lemma states that the determinant of a matrix that may appear in a matrix factorization of f must divide a power of f.

Lemma 2.3. *If $PQ = fI_n$, then det(P) divides f^n. If in addition, f is irreducible in R, then det(P) is a power of f.*

Proof. $PQ = fI_n$ implies $det(P)det(Q) = det(PQ) = det(fI_n) = f^n$. Hence, $det(P)$ is a factor of f^n as desired.
Next, if f is irreducible, then its only factors are powers of f. □

As remarked in [19], this lemma helps to see that to build an $n \times n$ matrix factorization of a nonzero polynomial $f \in R$, a first step could be to construct an $n \times n$ matrix P whose determinant divides f^n. Thereafter, we pass to $\mathcal{F}$, the fraction field of R, to find a matrix

Q such that $PQ = fI_n$. Anyway, as mentioned earlier, the *standard method* for factoring polynomials through matrices can be found in [41] and [19]. It will also be presented and improved in chapter 4.
When studying Morita contexts in chapter 6, we will make use of the following consequence of the above lemma.

Corollary 2.1. *If $0 \neq f \in R$ and $PQ = fI_n$, then over the field of fractions $\mathcal{F}$ of R, P is invertible.*

Proof. In fact, the above lemma guarantees that $det(P)$ is nonzero because it divides f^n. Whence, P is invertible over $\mathcal{F}$. □

Since P is invertible over $\mathcal{F}$, the unique Q such that $PQ = fI_n$ is $Q = P^{-1}fI_n = \frac{f}{det(P)}adj(P)$, where $adj(P)$ is the adjoint of the matrix P.

Now, $adj(P)$ is a matrix over $R = K[x_1, \cdots, x_n]$. So, if $det(P)$ divides f, then Q will also be a matrix over R. However, it is possible for Q not to have entries in R, and therefore P will not appear in any matrix factorization of f over R.

We state the following well known facts on determinants of block matrices due to Schur. (e.g [57], [60] (theorem 3)).

Fact 2.1. *If A, B, C and D are $n \times n$ matrices, and*

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix},$$

Then the following hold:

1. $det(M) = det(AD - CB)$ *if* $AC = CA$
2. $det(M) = det(AD - BC)$ *if* $CD = DC$
3. $det(M) = det(DA - BC)$ *if* $BD = DB$
4. $det(M) = det(DA - CB)$ *if* $AB = BA$

One of the things the following example illustrates is the fact that $det(P)$ divides f^n as stated in the foregoing lemma.

Example 2.7. *We saw in example 2.4 that a 2×2 matrix factorization of $f = x^2 + 1$ is:* $\left(\begin{bmatrix} x & -1 \\ 1 & x \end{bmatrix}, \begin{bmatrix} x & 1 \\ -1 & x \end{bmatrix}\right)$

We now use the easy algorithm we gave above, to produce a 4×4 matrix factorization (P, Q) of f. We obtain:

$$\begin{bmatrix} x & -1 & 0 & 0 \\ 1 & x & 0 & 0 \\ 0 & 0 & 1+x^2 & 0 \\ 0 & 0 & 0 & 1+x^2 \end{bmatrix} \begin{bmatrix} x & 1 & 0 & 0 \\ -1 & x & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= f \cdot \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} x & 1 & 0 & 0 \\ -1 & x & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x & -1 & 0 & 0 \\ 1 & x & 0 & 0 \\ 0 & 0 & 1+x^2 & 0 \\ 0 & 0 & 0 & 1+x^2 \end{bmatrix}$$

If we let

$$P = \begin{bmatrix} x & -1 & 0 & 0 \\ 1 & x & 0 & 0 \\ 0 & 0 & 1+x^2 & 0 \\ 0 & 0 & 0 & 1+x^2 \end{bmatrix}$$

Then letting

$$A = \begin{bmatrix} x & -1 \\ 1 & x \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad D = \begin{bmatrix} 1+x^2 & 0 \\ 0 & 1+x^2 \end{bmatrix}$$

We have

$$\begin{aligned} det(P) &= det \begin{bmatrix} A & B \\ C & D \end{bmatrix} \\ &= det(AD - CB) \; since \; AC = CA \; (see \; fact 2.1) \\ &= det[\begin{bmatrix} x & -1 \\ 1 & x \end{bmatrix} \begin{bmatrix} 1+x^2 & 0 \\ 0 & 1+x^2 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}] \\ &= det[\begin{bmatrix} x(1+x^2) & -(1+x^2) \\ 1+x^2 & x(1+x^2) \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}] \\ &= det \begin{bmatrix} x(1+x^2) & -(1+x^2) \\ 1+x^2 & x(1+x^2) \end{bmatrix} \\ &= (x^2+1)^3 \\ &= f^3 \end{aligned}$$

Clearly f^3 *is a factor of* f^4.
It is also very easy to verify that should we choose

$$P = \begin{bmatrix} x & 1 & 0 & 0 \\ -1 & x & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

we will get $det(P) = 1 + x^2 = f$ *and* f *is clearly a factor of* f^4.

In this example, we obtained $det(P) = f^3$ or $det(P) = f^1 = f$ depending on the matrix we choose for P. In any case, it is not difficult to see that $Q = fP^{-1}$ contains rational functions that are not polynomials. This illustrates the point that entries in Q may not always be in R.

We now prove that matrices appearing in a matrix factorization commute with each other.

Proposition 2.1. *[19]*
If $0 \neq f \in R$ and $P,\ Q$ are $n \times n$ matrices so that $PQ = fI_n$, then $QP = fI_n$.

Proof. Since P is invertible, we get $Q = P^{-1}fI_n$. Hence, given that fI_n commutes with all $n \times n$ matrices, we have:

$$QP = (P^{-1}fI_n)P = (fI_nP^{-1})P = fI_n(P^{-1}P) = fI_n$$

as desired. □

Corollary 2.2. *If $0 \neq f \in R$ and $PQ = fI_n$, then over the field of fractions $\mathcal{F}$ of R, P and Q are invertible.*

Proof. We saw in corollary 2.1 that If $PQ = fI_n$, then over the field of fraction $\mathcal{F}$ of R, P is invertible. So all we need show now is that Q is invertible too. Now, from the hypothesis of this corollary ($PQ = fI_n$) and proposition 2.1, we have $QP = fI_n$. The result now follows from corollary2.1. □

Remark 2.4. *It is important to note that corollary 2.1 and corollary 2.2 actually say that matrices that appear in a matrix factorization of a nonzero polynomial are invertible over the field of fractions $\mathcal{F}$ of R. This result will soon be very useful when describing the notion of Morita Context in $\mathcal{LG}_K$.*

2.2.2 Another property of matrix factorizations

We state and prove another property of matrix factorizations. It says that the transposes of the matrices appearing in a matrix factorizations of $f \in R$ also constitute a pair of matrix factorization for f.

Proposition 2.2. *If $0 \neq f \in R$ and $P,\ Q$ are $n \times n$ matrices such that $PQ = fI_n$, then $Q^tP^t = fI_n$, where Q^t (respectively P^t) stands for the transpose of Q (respectively P).*

Proof. Assume $0 \neq f \in R$ and $P,\ Q$ are $n \times n$ matrices such that $PQ = fI_n$. then:

$$\begin{aligned} PQ = fI_n &\Rightarrow (PQ)^t = (fI_n)^t \\ &\Rightarrow Q^tP^t = (f)^t(I_n^t) \\ &\Rightarrow Q^tP^t = fI_n \ \textit{since}\ (f)^t = f \ \textit{viewed as a}\ 1 \times 1 \ \textit{matrix} \end{aligned}$$

This shows that if (P, Q) is a matrix factorization of f, then (Q^t, P^t) is also a matrix factorization of f. □

Corollary 2.3. *If $0 \neq f \in R$ and $P,\ Q$ are $n \times n$ matrices such that $PQ = fI_n$, then $P^tQ^t = fI_n$, where Q^t (respectively P^t) stands for the transpose of Q (respectively of P).*

Proof. This follows from 2.2 and 2.1. □

Proposition 2.3. *Matrices that appear in an $n \times n$ matrix factorization of a polynomial $f \in R$ are not unique*

Proof. Case 1: f is nonzero
Let (P, Q) be a matrix factorization of a nonzero polynomial f. Then from the preceding proposition and corollary, we have that (P^t, Q^t) is also a matrix factorization of f.
Case 2: $f = 0$
Then of course, every pair (P, Q) of matrices in which either P or Q is zero will work. □

Example 2.8. *We earlier computed a* 2×2 *matrix factorization of* $f = x^2 + 1$*:*

$$\begin{bmatrix} x & -1 \\ 1 & x \end{bmatrix} \begin{bmatrix} x & 1 \\ -1 & x \end{bmatrix} = (x^2 + 1) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = fI_2$$

If we take the transpose of the two matrices involved, we have:

$$\begin{bmatrix} x & 1 \\ -1 & x \end{bmatrix} \begin{bmatrix} x & -1 \\ 1 & x \end{bmatrix} = (x^2 + 1) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = fI_2$$

This shows that the transpose matrices also constitute a pair of matrix factorization for $f = x^2 + 1$.

MULTIPLICATIVE TENSOR PRODUCT OF MATRIX FACTORIZATIONS

Yoshino in [67] presented a notion of tensor product of matrix factorizations to relate the matrix factorizations of the power series f and g with the matrix factorization of their sum $f + g$. More precisely, given two power series $f \in K[[x_1, ..., x_r]]$ and $g \in K[[y_1, ..., y_s]]$, whose matrix factorizations are respectively X and Y, he constructed by means of tensor product a new matrix factorization $X\widehat{\otimes}Y$ of $f + g$. A natural question arises: Is there an operation, call it $\widetilde{\otimes}$, on X and Y such that $X\widetilde{\otimes}Y$ is a matrix factorization of the product fg? In this chapter, we construct such a functorial operation using the standard operations of tensor product and direct sum of matrices. Though in our examples we mostly deal with power series and rational functions, it is easy to see from our definitions that our results also hold for any elements in a unital ring and not just the rings of power series and rational functions. We call $\widetilde{\otimes}$ *multiplicative tensor product* of matrix factorizations. Such a name is given to it because first and foremost, it has many nice properties of a tensor product, secondly because, it yields the product of two power series (or of two rational functions in case we are dealing with rational functions). After proving that the operation $\widetilde{\otimes}$ is functorial in each of its arguments and in both arguments, we prove many of its properties.

Furthermore, we give some applications of $\widetilde{\otimes}$ in this chapter. The concept of semi-unital semi-monoidal category was recently conceived in [1] and an example (cf. theorem 5.11 of [1]) was provided with a considerable amount of set-up. We provide another example (theorem 3.2) here with a smaller amount of set-up. In fact, if $MF(1)$ denotes the category of matrix factorizations of the constant power series 1, we define the concept of one-step connected category and prove that there is a one-step connected subcategory of $(MF(1), \widetilde{\otimes})$ which is semi-unital semi-monoidal.

Finally, we define the concept of right pseudo-monoidal category which generalizes the notion of monoidal category and we prove that $(MF(1), \widetilde{\otimes})$ is an example.

> *The material proposed in this chapter is entirely new to the best of our knowledge.*

3.1 A new product of matrix factorizations

In this section, we use the usual (standard) operations of tensor product and direct sum of matrices to construct a new product of matrix factorizations that we call *multiplicative*

tensor product of matrix factorization. After defining it, we give several examples.
First, we make a remark in the same line of thought with remark 2.1.

Remark 3.1. *It is worth noting that the new operation we are about to define as well as its properties (see next section) hold not only for matrix factorizations of power series and of rational functions (as our presentation seems to suggest throughout this chapter), but also for elements of an arbitrary ring (with unity).*

3.1.1 Motivation

We assume K is a field throughout this chapter. We'll denote by $K[x] = K[x_1, x_2, \cdots, x_r]$ (respectively $K[[x]] = K[[x_1, x_2, \cdots, x_r]]$) the polynomial ring (respectively the formal power series ring) in the variables $x_1, x_2, \cdots, x_r$. Let $f \in K[[x]]$ and $g \in K[[y]] = K[[y_1, y_2, \cdots, y_s]]$ be nonzero and non-invertible elements. Yoshino [67] considered the problem of relating MCM modules[1] (Maximal Cohen-Macaulay modules) over $K[[x]]/(f)$ and over $K[[y]]/(g)$ with MCM modules over $K[[x, y]]/(f+g)$. He presented a method[2] of tensor product of matrix factorizations to relate those objects. To be more precise, given X and Y which are matrix factorizations of f and of g respectively, he defined a tensor product of X and Y denoted $X\widehat{\otimes}Y$ which turned out to be a matrix factorization of $f + g$. So in a sense, he related the matrix factorizations of f, g and $f + g$. A natural question which arises after looking at his work is: Can we define an operation, call it $\widetilde{\otimes}$, on matrix factorizations in such a way that if X and Y are matrix factorizations of f and of g, then $X\widetilde{\otimes}Y$ would be a matrix factorization of fg?
In this section, we answer this question in the affirmative. We construct such an operation using the standard operations of tensor product and direct sum of matrices. It turns out that this operation also works for rational functions and in general, for any elements in a unital ring. For reasons mentioned at the beginning of this chapter, we call the newly defined operation *multiplicative tensor product* of matrix factorization.
In our presentation, we will limit ourselves to polynomials and we will not bother about MCM modules. After defining our operation and giving some examples, we'll prove that it is functorial in each of its arguments and in both arguments.

3.1.2 Definition and examples

Before defining a new operation on matrix factorizations, we recall that a pair (ϕ, ψ) of $n \times n$ matrices over $K[[x]]$ is called a matrix factorization of f of size n if $\phi\psi = \psi\phi = f \cdot I_n$. Also recall notation 5.2 that the category of matrix factorizations of $f \in R$ is denoted by $MF(R, f)$ or $MF(f)$. Moreover, recall that if A (resp. B) is an $m \times n$ (resp. $p \times q$) matrix, then their direct sum $A \oplus B = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$, where the 0 in the first line is a $p \times q$ matrix and the one in the second line is a $m \times n$ matrix. Finally, recall that if A (resp. B) is an $m \times n$ (resp. $p \times q$) matrix, then their tensor product $A \otimes B$ is the matrix obtained by replacing each entry a_{ij} of A with the matrix $a_{ij}B$. So, $A \otimes B$ is a $mp \times nq$ matrix.

Definition 3.1. *Let $X = (\phi, \psi)$ be a matrix factorization of $f \in K[[x]]$ of size n and let $X' = (\phi', \psi')$ be a matrix factorization of $g \in K[[y]]$ of size m. Thus, ϕ, ψ, ϕ' and ψ' can be*

[1]for a definition of MCM module and more on that, see [44].
[2]We presented his definition in subsection 2.1.3

considered as matrices over $S = K[[x, y]]$ *and the* ***multiplicative tensor product*** $X\widetilde{\otimes}X'$ *is given by*

$$((\phi \otimes \phi') \oplus (\phi \otimes \phi'), (\psi \otimes \psi') \oplus (\psi \otimes \psi')) = (\begin{bmatrix} \phi \otimes \phi' & 0 \\ 0 & \phi \otimes \phi' \end{bmatrix}, \begin{bmatrix} \psi \otimes \psi' & 0 \\ 0 & \psi \otimes \psi' \end{bmatrix})$$

where each component is an endomorphism on $S^n \otimes_S S^m$.

NB. We could not just define $X\widetilde{\otimes}X'$ to be $(\phi\otimes\phi', \psi\otimes\psi')$ because in order to multiply the two matrices involved, we need to make sure the number of columns of the first matrix is equal to the number of rows of the second matrix. This explains the presence of the zeroes in the definition given above.

Remark 3.2. *It is obvious that the foregoing definition is valid for matrix factorizations of elements of any unital ring and not only the ring of power series. Our results will be stated for power series and we will not bother to state or prove them for elements in arbitrary unital rings because both the statements and the proofs will be conspicuously similar and consequently will not be mentioned.*

Fact 3.1. *Let a and b be two elements of the ring* $K[[x_1, \cdots, x_n]]$.
Then, $aI_n \otimes bI_m = ab(I_n \otimes I_m)$ *where* $\otimes$ *is the tensor product of matrices.*

Lemma 3.1. *Let* $X = (\phi, \psi)$ *be an* $n \times n$ *matrix factorization of* $f \in K[[x]]$ *and let* $X' = (\phi', \psi')$ *be an* $m \times m$ *matrix factorization of* $g \in K[[y]]$. *Then,* $X\widetilde{\otimes}X'$ *is an object of* $MF(K[[x, y]], fg)$ *of size* $2nm$.

Proof. We have:

$$\begin{bmatrix} \phi \otimes \phi' & 0 \\ 0 & \phi \otimes \phi' \end{bmatrix} \begin{bmatrix} \psi \otimes \psi' & 0 \\ 0 & \psi \otimes \psi' \end{bmatrix}$$

$$= \begin{bmatrix} \phi\psi \otimes \phi'\psi' & 0 \\ 0 & \phi\psi \otimes \phi'\psi' \end{bmatrix}$$

$$= \begin{bmatrix} f1_n \otimes g1_m & 0 \\ 0 & f1_n \otimes g1_m \end{bmatrix}, \quad since\ \phi\psi = f1_n\ and\ \phi'\psi' = g1_n$$

$$= fg \begin{bmatrix} 1_n \otimes 1_m & 0 \\ 0 & 1_n \otimes 1_m \end{bmatrix}, by\ fact\ 3.1.$$

$$= fg \begin{bmatrix} 1_{nm} & 0 \\ 0 & 1_{nm} \end{bmatrix}, \quad since\ 1_n \otimes 1_m = 1_{nm}$$

$$= fg \cdot 1_{2nm}$$

□

This shows that $X\widetilde{\otimes}X'$ is an object of $MF(fg)$ of size $2nm$ as desired.

Example 3.1. *Consider the following* 1×1 *matrix factorizations of* $f = x^3$, $g = y^5$ *and* $h = z^7$*:*
$X = (x, x^2) \in MF(x^3)$, $X' = (y^2, y^3) \in MF(y^5)$, $X'' = (z^3, z^4) \in MF(z^7)$. *We compute* $X\widetilde{\otimes}X'$ *and* $(X\widetilde{\otimes}X')\widetilde{\otimes}X''$.

$$Y = X\widetilde{\otimes}X' = (\begin{bmatrix} x\otimes y^2 & 0 \\ 0 & x\otimes y^2 \end{bmatrix}, \begin{bmatrix} x^2\otimes y^3 & 0 \\ 0 & x^2\otimes y^3 \end{bmatrix}) = (\begin{bmatrix} xy^2 & 0 \\ 0 & xy^2 \end{bmatrix}, \begin{bmatrix} x^2y^3 & 0 \\ 0 & x^2y^3 \end{bmatrix}) = (\phi_Y, \psi_Y)$$

Y is a matrix factorization of $fg = x^3y^5$ *of size* $2(1)(1) = 2$. *In fact,*

$$\begin{bmatrix} xy^2 & 0 \\ 0 & xy^2 \end{bmatrix}\begin{bmatrix} x^2y^3 & 0 \\ 0 & x^2y^3 \end{bmatrix} = \begin{bmatrix} xy^2x^2y^3 & 0 \\ 0 & xy^2x^2y^3 \end{bmatrix} = \begin{bmatrix} x^3y^5 & 0 \\ 0 & x^3y^5 \end{bmatrix} = x^3y^5 \cdot I_2 = fg \cdot I_2$$

As desired.
Next, let $X'' = (\phi_{X''}, \psi_{X''}) = (z^3, z^4)$, *we compute:*

$$Y\widetilde{\otimes}X'' = (\begin{bmatrix} \phi_Y \otimes \phi_{X''} & 0 \\ 0 & \phi_Y \otimes \phi_{X''} \end{bmatrix}, \begin{bmatrix} \psi_Y \otimes \psi_{X''} & 0 \\ 0 & \psi_Y \otimes \psi_{X''} \end{bmatrix})$$

$$= (\begin{bmatrix} \begin{bmatrix} xy^2 & 0 \\ 0 & xy^2 \end{bmatrix}\otimes z^3 & 0 \\ 0 & \begin{bmatrix} xy^2 & 0 \\ 0 & xy^2 \end{bmatrix}\otimes z^3 \end{bmatrix}, \begin{bmatrix} \begin{bmatrix} x^2y^3 & 0 \\ 0 & x^2y^3 \end{bmatrix}\otimes z^4 & 0 \\ 0 & \begin{bmatrix} x^2y^3 & 0 \\ 0 & x^2y^3 \end{bmatrix}\otimes z^4 \end{bmatrix})$$

$$= (\begin{bmatrix} xy^2z^3 & 0 & 0 & 0 \\ 0 & xy^2z^3 & 0 & 0 \\ 0 & 0 & xy^2z^3 & 0 \\ 0 & 0 & 0 & xy^2z^3 \end{bmatrix}, \begin{bmatrix} x^2y^3z^4 & 0 & 0 & 0 \\ 0 & x^2y^3z^4 & 0 & 0 \\ 0 & 0 & x^2y^3z^4 & 0 \\ 0 & 0 & 0 & x^2y^3z^4 \end{bmatrix})$$

and we have:

$$\begin{bmatrix} xy^2z^3 & 0 & 0 & 0 \\ 0 & xy^2z^3 & 0 & 0 \\ 0 & 0 & xy^2z^3 & 0 \\ 0 & 0 & 0 & xy^2z^3 \end{bmatrix}\begin{bmatrix} x^2y^3z^4 & 0 & 0 & 0 \\ 0 & x^2y^3z^4 & 0 & 0 \\ 0 & 0 & x^2y^3z^4 & 0 \\ 0 & 0 & 0 & x^2y^3z^4 \end{bmatrix}$$

$$= x^3y^5z^7 \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

That is $Y\widetilde{\otimes}X'' \in MF(x^3y^5z^7) = MF((fg)h)$. *Moreover, we verify that* X'' *is of size* 1 *and Y is of size* 2, *and clearly* $Y\widetilde{\otimes}X''$ *is of size* $2(1)(2) = 4$. *QED*

Example 3.2. *We now give an example with 2×2 matrix factorizations.*
We saw in example 2.4 that a 2×2 matrix factorization of $f = x^2 + 1$ is:
$X = (\begin{bmatrix} x & -1 \\ 1 & x \end{bmatrix}, \begin{bmatrix} x & 1 \\ -1 & x \end{bmatrix}) = (\phi_X, \psi_X)$

Hence a matrix factorization of $g = -f = -(x^2+1)$ is obviously: $Y = (\begin{bmatrix} -x & 1 \\ -1 & -x \end{bmatrix}, \begin{bmatrix} x & 1 \\ -1 & x \end{bmatrix}) = (\phi_Y, \psi_Y)$

Let's find $X\widetilde{\otimes}Y$.

$$X\widetilde{\otimes}Y = (\begin{bmatrix} \phi_X \otimes \phi_Y & 0 \\ 0 & \phi_X \otimes \phi_Y \end{bmatrix}, \begin{bmatrix} \psi_X \otimes \psi_Y & 0 \\ 0 & \psi_X \otimes \psi_Y \end{bmatrix})$$

$$= (\begin{bmatrix} \begin{bmatrix} x & -1 \\ 1 & x \end{bmatrix} \otimes \begin{bmatrix} -x & 1 \\ -1 & -x \end{bmatrix} & 0 \\ 0 & \begin{bmatrix} x & -1 \\ 1 & x \end{bmatrix} \otimes \begin{bmatrix} -x & 1 \\ -1 & -x \end{bmatrix} \end{bmatrix}, \begin{bmatrix} \begin{bmatrix} x & 1 \\ -1 & x \end{bmatrix} \otimes \begin{bmatrix} x & 1 \\ -1 & x \end{bmatrix} & 0 \\ 0 & \begin{bmatrix} x & 1 \\ -1 & x \end{bmatrix} \otimes \begin{bmatrix} x & 1 \\ -1 & x \end{bmatrix} \end{bmatrix})$$

$$= (\begin{bmatrix} \begin{bmatrix} x\begin{bmatrix} -x & 1 \\ -1 & -x \end{bmatrix} & -1\begin{bmatrix} -x & 1 \\ -1 & -x \end{bmatrix} \\ 1\begin{bmatrix} -x & 1 \\ -1 & -x \end{bmatrix} & x\begin{bmatrix} -x & 1 \\ -1 & -x \end{bmatrix} \end{bmatrix} & 0 \\ 0 & \begin{bmatrix} x\begin{bmatrix} -x & 1 \\ -1 & -x \end{bmatrix} & -1\begin{bmatrix} -x & 1 \\ -1 & -x \end{bmatrix} \\ 1\begin{bmatrix} -x & 1 \\ -1 & -x \end{bmatrix} & x\begin{bmatrix} -x & 1 \\ -1 & -x \end{bmatrix} \end{bmatrix} \end{bmatrix},$$

$$\begin{bmatrix} \begin{bmatrix} x\begin{bmatrix} x & 1 \\ -1 & x \end{bmatrix} & 1\begin{bmatrix} x & 1 \\ -1 & x \end{bmatrix} \\ -1\begin{bmatrix} x & 1 \\ -1 & x \end{bmatrix} & x\begin{bmatrix} x & 1 \\ -1 & x \end{bmatrix} \end{bmatrix} & 0 \\ 0 & \begin{bmatrix} x\begin{bmatrix} x & 1 \\ -1 & x \end{bmatrix} & 1\begin{bmatrix} x & 1 \\ -1 & x \end{bmatrix} \\ -1\begin{bmatrix} x & 1 \\ -1 & x \end{bmatrix} & x\begin{bmatrix} x & 1 \\ -1 & x \end{bmatrix} \end{bmatrix} \end{bmatrix})$$

$$= (\begin{bmatrix} \begin{bmatrix} -x^2 & x & x & -1 \\ -x & -x^2 & 1 & x \\ -x & 1 & -x^2 & x \\ -1 & -x & -x & -x^2 \end{bmatrix} & 0 \\ 0 & \begin{bmatrix} -x^2 & x & x & -1 \\ -x & -x^2 & 1 & x \\ -x & 1 & -x^2 & x \\ -1 & -x & -x & -x^2 \end{bmatrix} \end{bmatrix},$$

$$\begin{bmatrix} \begin{bmatrix} x^2 & x & x & 1 \\ -x & x^2 & -1 & x \\ -x & -1 & x^2 & x \\ 1 & -x & -x & x^2 \end{bmatrix} & 0 \\ 0 & \begin{bmatrix} x^2 & x & x & 1 \\ -x & x^2 & -1 & x \\ -x & -1 & x^2 & x \\ 1 & -x & -x & x^2 \end{bmatrix} \end{bmatrix})$$

If we let $r = fg = (x^2+1)(-x^2-1) = -x^4 - 2x^2 - 1$ *and we call the two last matrices P and Q respectively, then we obtain:*

$$PQ = \begin{pmatrix} r & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & r & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & r & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & r & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & r & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & r & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & r & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & r \end{pmatrix} = (-x^4 - 2x^2 - 1) \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

So $X\widetilde{\otimes}Y \in MF(fg)$. *Moreover, we verify that X and Y are both of size* 2, *and clearly* $X\widetilde{\otimes}Y$ *is of size* $2(2)(2) = 8$. *QED*

We now give an example with rational functions.

Example 3.3. *Consider the following matrix factorizations of* $F = x^{-1}$, $G = y^{-5}$:
$X = (x, x^{-2}) \in MF(x^{-1})$, $X' = (y^{-2}, y^{-3}) \in MF(y^{-5})$. *We compute* $X\widetilde{\otimes}X'$.

$$Y = X\widetilde{\otimes}X' = (\begin{bmatrix} x \otimes y^{-2} & 0 \\ 0 & x \otimes y^{-2} \end{bmatrix}, \begin{bmatrix} x^{-2} \otimes y^{-3} & 0 \\ 0 & x^{-2} \otimes y^{-3} \end{bmatrix}) = (\begin{bmatrix} xy^{-2} & 0 \\ 0 & xy^{-2} \end{bmatrix}, \begin{bmatrix} x^{-2}y^{-3} & 0 \\ 0 & x^{-2}y^{-3} \end{bmatrix}) = (\phi_Y, \psi_Y)$$

Y is a matrix factorization of $FG = x^{-1}y^{-5}$. *In fact,*

$$\begin{bmatrix} xy^{-2} & 0 \\ 0 & xy^{-2} \end{bmatrix} \begin{bmatrix} x^{-2}y^{-3} & 0 \\ 0 & x^{-2}y^{-3} \end{bmatrix} = \begin{bmatrix} xy^{-2}x^{-2}y^{-3} & 0 \\ 0 & xy^{-2}x^{-2}y^{-3} \end{bmatrix} = \begin{bmatrix} x^{-1}y^{-5} & 0 \\ 0 & x^{-1}y^{-5} \end{bmatrix} = x^{-1}y^{-5} \cdot I_2 = FG \cdot I_2$$

As desired.

3.1.3 Functoriality of the operation $\widetilde{\otimes}$

This subsection is entirely devoted to the discussion of the functoriality of $\widetilde{\otimes}$. In our presentation, we will use power series. The functoriality of $\widetilde{\otimes}$ using elements of any unital ring (e.g. ring of rational functions) is similar and will be omitted.

Setting the stage:

In the following, except otherwise stated, $X = (\phi, \psi)$ (respectively $X' = (\phi', \psi')$) is always an object in $MF(K[[x]], f)$ (resp. $MF(K[[y]], g)$) of size n (resp. m). Moreover, except

otherwise stated, for $i = 1, 2$; $X_i = (\phi_i, \psi_i)$ (respectively $X'_i = (\phi'_i, \psi'_i)$) is always an object in $MF(K[[x]], f)$ (resp. $MF(K[[y]], g)$) of size n_i (resp. m_i). We assume throughout this subsection and the remainder of this chapter that $x = x_1, \cdots, x_r$ and $y = y_1, \cdots, y_s$ are disjoint.
Let (ϕ_1, ψ_1) (respectively (ϕ_2, ψ_2)) be an $n_1 \times n_1$ (respectively $n_2 \times n_2$) matrix factorization of $f \in K[[x]]$. Recall that we denote by $MF(K[[x]], f)$ (or simply $MF(f)$) the category of matrix factorizations of f. Objects are matrix factorizations of f, and a morphism from (ϕ_1, ψ_1) to (ϕ_2, ψ_2) is a pair of matrices (α,β) which makes the following diagram (cf. [67]) commute:

$$\begin{array}{ccccc} K[[x]]^{n_1} & \xrightarrow{\psi_1} & K[[x]]^{n_1} & \xrightarrow{\phi_1} & K[[x]]^{n_1} \\ \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \alpha} \\ K[[x]]^{n_2} & \xrightarrow{\psi_2} & K[[x]]^{n_2} & \xrightarrow{\phi_2} & K[[x]]^{n_2} \end{array} \qquad (\bigstar)$$

That is,

$$\begin{cases} \alpha\phi_1 = \phi_2\beta \\ \psi_2\alpha = \beta\psi_1 \end{cases}$$

Definition 3.2. *For a morphism $\zeta = (\alpha,\beta) : X_1 = (\phi_1, \psi_1) \to X_2 = (\phi_2, \psi_2)$ in $MF(K[[x]], f)$ and for any $m \times m$ matrix factorization $X' = (\phi', \psi')$ in $MF(K[[y]], g)$, we define $\zeta\widetilde{\otimes}X'$ by*

$$(\begin{bmatrix} \alpha \otimes 1_m & 0 \\ 0 & \alpha \otimes 1_m \end{bmatrix}, \begin{bmatrix} \beta \otimes 1_m & 0 \\ 0 & \beta \otimes 1_m \end{bmatrix})$$

Lemma 3.2. *$\zeta\widetilde{\otimes}X'$ is a morphism $X_1\widetilde{\otimes}X' \to X_2\widetilde{\otimes}X'$ in $MF(K[[x, y]], fg)$.*

Proof. It is true that for $i = 1, 2$; $X_i\widetilde{\otimes}X'$ is an object of $MF(K[[x, y]], fg)$ by definition of $\widetilde{\otimes}$, since $X_1, X_2 \in MF(f)$ and $X' \in MF(g)$. We know that:

$$X_i\widetilde{\otimes}X' = (\begin{bmatrix} \phi_i \otimes \phi' & 0 \\ 0 & \phi_i \otimes \phi' \end{bmatrix}, \begin{bmatrix} \psi_i \otimes \psi' & 0 \\ 0 & \psi_i \otimes \psi' \end{bmatrix})$$

where each component is an endomorphism on $S^{n_i} \otimes_S S^m$. $X_i\widetilde{\otimes}X'$ is an object of $MF(fg)$ of size $2n_im$.
To show that $\zeta\widetilde{\otimes}X'$ is a morphism, we need to show that the following diagram commutes:

$$\begin{array}{ccccc} K[[x,y]]^{2n_1m} & \xrightarrow{\begin{bmatrix} \psi_1 \otimes \psi' & 0 \\ 0 & \psi_1 \otimes \psi' \end{bmatrix}} & K[[x,y]]^{2n_1m} & \xrightarrow{\begin{bmatrix} \phi_1 \otimes \phi' & 0 \\ 0 & \phi_1 \otimes \phi' \end{bmatrix}} & K[[x,y]]^{2n_1m} \\ \downarrow{\scriptstyle \begin{bmatrix} \alpha \otimes 1_m & 0 \\ 0 & \alpha \otimes 1_m \end{bmatrix}} & & \downarrow{\scriptstyle \begin{bmatrix} \beta \otimes 1_m & 0 \\ 0 & \beta \otimes 1_m \end{bmatrix}} & & \downarrow{\scriptstyle \begin{bmatrix} \alpha \otimes 1_m & 0 \\ 0 & \alpha \otimes 1_m \end{bmatrix}} \\ K[[x,y]]^{2n_2m} & \xrightarrow[\begin{bmatrix} \psi_2 \otimes \psi' & 0 \\ 0 & \psi_2 \otimes \psi' \end{bmatrix}]{} & K[[x,y]]^{2n_2m} & \xrightarrow[\begin{bmatrix} \phi_2 \otimes \phi' & 0 \\ 0 & \phi_2 \otimes \phi' \end{bmatrix}]{} & K[[x,y]]^{2n_2m} \end{array}$$

viz. both the left and the right squares in the foregoing diagram commute.

• As for the right square, its commutativity is expressed by the following equality:

$$\begin{bmatrix}\alpha\otimes 1_m & 0\\ 0 & \alpha\otimes 1_m\end{bmatrix}\begin{bmatrix}\phi_1\otimes\phi' & 0\\ 0 & \phi_1\otimes\phi'\end{bmatrix}=\begin{bmatrix}\phi_2\otimes\phi' & 0\\ 0 & \phi_2\otimes\phi'\end{bmatrix}\begin{bmatrix}\beta\otimes 1_m & 0\\ 0 & \beta\otimes 1_m\end{bmatrix}$$

i.e.,

$$\begin{bmatrix}\alpha\phi_1\otimes 1_m\phi' & 0\\ 0 & \alpha\phi_1\otimes 1_m\phi'\end{bmatrix}=\begin{bmatrix}\phi_2\beta\otimes\phi' 1_m & 0\\ 0 & \phi_2\beta\otimes\phi' 1_m\end{bmatrix}$$

i.e.,

$$\begin{bmatrix}\alpha\phi_1\otimes\phi' & 0\\ 0 & \alpha\phi_1\otimes\phi'\end{bmatrix}=\begin{bmatrix}\phi_2\beta\otimes\phi' & 0\\ 0 & \phi_2\beta\otimes\phi'\end{bmatrix}$$

i.e.,

$$\begin{cases}\alpha\phi_1\otimes\phi' = \phi_2\beta\otimes\phi'\\ \alpha\phi_1\otimes\phi' = \phi_2\beta\otimes\phi'\end{cases}$$

i.e., all we need to show is the equality:

$$\alpha\phi_1\otimes\phi' = \phi_2\beta\otimes\phi' \qquad \sharp$$

Now by hypothesis, $\zeta = (\alpha,\beta) : X_1 = (\phi_1,\psi_1) \to X_2 = (\phi_2,\psi_2)$ is a morphism, meaning that diagram (★) above commutes. That is,

$$\begin{cases}\alpha\phi_1 = \phi_2\beta\cdots(i)\\ \psi_2\alpha = \beta\psi_1\cdots(ii)\end{cases}$$

Now considering (i), we immediately see that equality $\sharp$ holds, as desired.

• As for the left square, its commutativity is expressed by the following equality:

$$\begin{bmatrix}\beta\otimes 1_m & 0\\ 0 & \beta\otimes 1_m\end{bmatrix}\begin{bmatrix}\psi_1\otimes\psi' & 0\\ 0 & \psi_1\otimes\psi'\end{bmatrix}=\begin{bmatrix}\psi_2\otimes\psi' & 0\\ 0 & \psi_2\otimes\psi'\end{bmatrix}\begin{bmatrix}\alpha\otimes 1_m & 0\\ 0 & \alpha\otimes 1_m\end{bmatrix}$$

i.e.,

$$\begin{bmatrix}\beta\psi_1\otimes 1_m\psi' & 0\\ 0 & \beta\psi_1\otimes 1_m\psi'\end{bmatrix}=\begin{bmatrix}\psi_2\alpha\otimes\psi' 1_m & 0\\ 0 & \psi_2\alpha\otimes\psi' 1_m\end{bmatrix}$$

i.e.,

$$\begin{bmatrix}\beta\psi_1\otimes\psi' & 0\\ 0 & \beta\psi_1\otimes\psi'\end{bmatrix}=\begin{bmatrix}\psi_2\alpha\otimes\psi' & 0\\ 0 & \psi_2\alpha\otimes\psi'\end{bmatrix}$$

i.e.,

$$\begin{cases}\beta\psi_1\otimes\psi' = \psi_2\alpha\otimes\psi'\\ \beta\psi_1\otimes\psi' = \psi_2\alpha\otimes\psi'\end{cases}$$

i.e., all we need to show is the equality:

$$\beta\psi_1\otimes\psi' = \psi_2\alpha\otimes\psi' \qquad \sharp\sharp$$

Now by hypothesis, $\zeta = (\alpha,\beta) : X_1 = (\phi_1,\psi_1) \to X_2 = (\phi_2,\psi_2)$ is a morphism, so we can use the identity $\beta\psi_1 = \psi_2\alpha$ (i.e., (ii) above) to see that equality $\sharp\sharp$ holds. So, $\zeta\widetilde{\otimes}X'$ is a morphism in $MF(K[[x,y]], fg)$. QED □

Definition 3.3. *For a morphism $\zeta' = (\alpha', \beta') : X'_1 = (\phi'_1, \psi'_1) \to X'_2 = (\phi'_2, \psi'_2)$ in $MF(K[[y]], g)$ and for any $n \times n$ matrix factorization $X = (\phi, \psi)$ in $MF(K[[x]], f)$, we define $X\widetilde{\otimes}\zeta'$ by*

$$(\begin{bmatrix} 1_n \otimes \alpha' & 0 \\ 0 & 1_n \otimes \alpha' \end{bmatrix}, \begin{bmatrix} 1_n \otimes \beta' & 0 \\ 0 & 1_n \otimes \beta' \end{bmatrix})$$

Lemma 3.3. *$X\widetilde{\otimes}\zeta'$ is a morphism $X\widetilde{\otimes}X'_1 \to X\widetilde{\otimes}X'_2$ in $MF(K[[x, y]], fg)$.*

Proof. This proof is very similar to the proof given for lemma 3.2 and consequently is omitted. □

We can now state the following propositions which are consequences of lemma 3.2 and lemma 3.3.

Proposition 3.1. *For a fixed $n \times n$ matrix factorization of $f \in K[[x]]$, the multiplicative tensor product $X\widetilde{\otimes}(-)$ defines a functor $MF(K[[y]], g) \to MF(K[[x, y]], fg)$.*

Proof. We show that $F = X\widetilde{\otimes}(-)$ is a functor.

$$X\widetilde{\otimes}(-) : MF(K[[y]], g) \to MF(K[[x, y]], fg)$$

$$\begin{array}{ccc} X'_1 & \longrightarrow & X\widetilde{\otimes}X'_1 \\ {\scriptstyle \zeta'}\downarrow & & \downarrow{\scriptstyle X\widetilde{\otimes}\zeta'} \\ X'_2 & \longrightarrow & X\widetilde{\otimes}X'_2 \end{array}$$

We already showed in lemma 3.3 that $X\widetilde{\otimes}\zeta'$ is a morphism in $MF(K[[x, y]], fg)$. It now remains to show the composition and the identity axioms.

Identity Axiom:

Let $X' = (\phi', \psi')$ be an $m \times m$ matrix factorization in $MF(g)$ and let $1_{X'} : X' \to X'$, we show that $F(1_{X'}) = 1_{F(X')}$.

It is obvious that $1_{X'} = (1_m, 1_m)$.

Now that $1_{X'} = (1_m, 1_m)$, we can compute $X\widetilde{\otimes}1_{X'}$ using definition 3.3 to obtain:

$$X\widetilde{\otimes}1_{X'} = (\begin{bmatrix} 1_n \otimes 1_m & 0 \\ 0 & 1_n \otimes 1_m \end{bmatrix}, \begin{bmatrix} 1_n \otimes 1_m & 0 \\ 0 & 1_n \otimes 1_m \end{bmatrix}) = (\begin{bmatrix} 1_{nm} & 0 \\ 0 & 1_{nm} \end{bmatrix}, \begin{bmatrix} 1_{nm} & 0 \\ 0 & 1_{nm} \end{bmatrix}) \cdots \dagger$$

It is now time to compute $1_{F(X')} : F(X') \to F(X')$. i.e., $1_{F(X')} : X\widetilde{\otimes}X' \to X\widetilde{\otimes}X'$. It is obvious that $1_{F(X')} = (1_{2nm}, 1_{2nm})$ which is the same as

$$(\begin{bmatrix} 1_{nm} & 0 \\ 0 & 1_{nm} \end{bmatrix}, \begin{bmatrix} 1_{nm} & 0 \\ 0 & 1_{nm} \end{bmatrix}) \cdots \ddagger$$

† and ‡ show that $F(1_{X'}) = 1_{F(X')}$ as desired.

Composition Axiom:

Consider the situation:

$$X'_1 \xrightarrow{\zeta'_1} X'_2 \xrightarrow{\zeta'_2} X'_3 \ , \ X\widetilde{\otimes}X'_1 \xrightarrow{F(\zeta'_1)} X\widetilde{\otimes}X'_2 \xrightarrow{F(\zeta'_2)} X\widetilde{\otimes}X'_3$$

We need to show $F(\zeta_2' \circ \zeta_1') = F(\zeta_2') \circ F(\zeta_1')$.
Let $\zeta_1' = (\alpha_1', \beta_1')$ and $\zeta_2' = (\alpha_2', \beta_2')$. Now, for $i \in \{1,2\}$, $F(\zeta_i')$ is by definition

$$(\begin{bmatrix} 1_n \otimes \alpha_i' & 0 \\ 0 & 1_n \otimes \alpha_i' \end{bmatrix}, \begin{bmatrix} 1_n \otimes \beta_i' & 0 \\ 0 & 1_n \otimes \beta_i' \end{bmatrix})$$

The composition $F(\zeta_2') \circ F(\zeta_1')$ is then the product of pair of matrices:

$$(\begin{bmatrix} 1_n \otimes \alpha_2' & 0 \\ 0 & 1_n \otimes \alpha_2' \end{bmatrix}, \begin{bmatrix} 1_n \otimes \beta_2' & 0 \\ 0 & 1_n \otimes \beta_2' \end{bmatrix})(\begin{bmatrix} 1_n \otimes \alpha_1' & 0 \\ 0 & 1_n \otimes \alpha_1' \end{bmatrix}, \begin{bmatrix} 1_n \otimes \beta_1' & 0 \\ 0 & 1_n \otimes \beta_1' \end{bmatrix})$$

$$(\begin{bmatrix} 1_n \otimes \alpha_2' & 0 \\ 0 & 1_n \otimes \alpha_2' \end{bmatrix}\begin{bmatrix} 1_n \otimes \alpha_1' & 0 \\ 0 & 1_n \otimes \alpha_1' \end{bmatrix}, \begin{bmatrix} 1_n \otimes \beta_2' & 0 \\ 0 & 1_n \otimes \beta_2' \end{bmatrix}\begin{bmatrix} 1_n \otimes \beta_1' & 0 \\ 0 & 1_n \otimes \beta_1' \end{bmatrix})$$

$$= (\begin{bmatrix} 1_n 1_n \otimes \alpha_2'\alpha_1' & 0 \\ 0 & 1_n 1_n \otimes \alpha_2'\alpha_1' \end{bmatrix}, \begin{bmatrix} 1_n 1_n \otimes \beta_2'\beta_1' & 0 \\ 0 & 1_n 1_n \otimes \beta_2'\beta_1' \end{bmatrix})$$

$$= (\begin{bmatrix} 1_n \otimes \alpha_2'\alpha_1' & 0 \\ 0 & 1_n \otimes \alpha_2'\alpha_1' \end{bmatrix}, \begin{bmatrix} 1_n \otimes \beta_2'\beta_1' & 0 \\ 0 & 1_n \otimes \beta_2'\beta_1' \end{bmatrix}) \quad \cdots \natural'$$

Since $\zeta_1' = (\alpha_1', \beta_1')$ and $\zeta_2' = (\alpha_2', \beta_2')$, we have $\zeta_2' \circ \zeta_1' = (\alpha_2', \beta_2')(\alpha_1', \beta_1') = (\alpha_2'\alpha_1', \beta_2'\beta_1')$
Hence, $F(\zeta_2' \circ \zeta_1')$ is by definition

$$(\begin{bmatrix} 1_n \otimes \alpha_2'\alpha_1' & 0 \\ 0 & 1_n \otimes \alpha_2'\alpha_1' \end{bmatrix}, \begin{bmatrix} 1_n \otimes \beta_2'\beta_1' & 0 \\ 0 & 1_n \otimes \beta_2'\beta_1' \end{bmatrix}) \quad \cdots \natural'\natural'$$

From $\natural'$ and $\natural'\natural'$, we obtain $F(\zeta_2' \circ \zeta_1') = F(\zeta_2')F(\zeta_1')$ as desired. So, $X\widetilde{\otimes}(-)$ is a functor. QED □

Proposition 3.2. *For a fixed $m \times m$ factorization of $g \in K[[y]]$, the multiplicative tensor product $(-)\widetilde{\otimes}X'$ defines a functor $MF(K[[x]], f) \to MF(K[[x,y]], fg)$.*

Proof. This proof is very similar to the proof of proposition 3.1. □

So far, we have proved that $\widetilde{\otimes}$ is a functor in each of its arguments. We now want to prove that $\widetilde{\otimes}$ is a bifunctor.
Setting the stage: Let $X_f = (\phi, \psi)$, $X_f' = (\phi', \psi')$ and $X_f" = (\phi", \psi")$ be objects of $MF(K[[x]], f)$ respectively of sizes n, n' and $n"$. Let $X_g = (\sigma, \rho)$, $X_g' = (\sigma', \rho')$ and $X_g" = (\sigma", \rho")$ be objects of $MF(K[[y]], g)$ respectively of sizes m, m' and $m"$.

Definition 3.4. *For morphisms $\zeta_f = (\alpha_f, \beta_f) : X_f = (\phi, \psi) \to X_f' = (\phi', \psi')$ and $\zeta_g = (\alpha_g, \beta_g) : X_g = (\sigma, \rho) \to X_g' = (\sigma', \rho')$ respectively in $MF(K[[x]], f)$ and $MF(K[[y]], g)$, we define $\zeta_f \widetilde{\otimes} \zeta_g : X_f \widetilde{\otimes} X_g = (\phi, \psi)\widetilde{\otimes}(\sigma, \rho) \to X_f' \widetilde{\otimes} X_g' = (\phi', \psi')\widetilde{\otimes}(\sigma', \rho')$ by*

$$(\begin{bmatrix} \alpha_f \otimes \alpha_g & 0 \\ 0 & \alpha_f \otimes \alpha_g \end{bmatrix}, \begin{bmatrix} \beta_f \otimes \beta_g & 0 \\ 0 & \beta_f \otimes \beta_g \end{bmatrix})$$

Lemma 3.4. *$\zeta_f \widetilde{\otimes} \zeta_g$ is a morphism $X_f \widetilde{\otimes} X_g = (\phi,\psi)\widetilde{\otimes}(\sigma,\rho) \to X'_f \widetilde{\otimes} X'_g = (\phi',\psi')\widetilde{\otimes}(\sigma',\rho')$ in $MF(K[[x,y]], fg)$.*

Proof. It is true that $X_f \widetilde{\otimes} X_g$ and $X'_f \widetilde{\otimes} X'_g$ are objects in $MF(K[[x,y]], fg)$ by definition of $\widetilde{\otimes}$, since $X_f, X'_f \in MF(f)$ and $X_g, X'_g \in MF(g)$. We know that:

$$X_f \widetilde{\otimes} X_g = (\begin{bmatrix} \phi\otimes\sigma & 0 \\ 0 & \phi\otimes\sigma \end{bmatrix}, \begin{bmatrix} \psi\otimes\rho & 0 \\ 0 & \psi\otimes\rho \end{bmatrix})$$

where each component is an endomorphism on $S^n \otimes_S S^m$. $X_f \widetilde{\otimes} X_g$ is an object of $MF(fg)$ of size $2nm$. We also have

$$X'_f \widetilde{\otimes} X'_g = (\begin{bmatrix} \phi'\otimes\sigma' & 0 \\ 0 & \phi'\otimes\sigma' \end{bmatrix}, \begin{bmatrix} \psi'\otimes\rho' & 0 \\ 0 & \psi'\otimes\rho' \end{bmatrix})$$

where each component is an endomorphism on $S^{n'} \otimes_S S^{m'}$. $X'_f \widetilde{\otimes} X'_g$ is an object of $MF(fg)$ of size $2n'm'$.
To show that $\zeta_f \widetilde{\otimes} \zeta_g$ is a morphism, we need to show that the following diagram commutes:

$$\begin{array}{ccccc}
K[[x,y]]^{2nm} & \xrightarrow{\begin{bmatrix} \psi\otimes\rho & 0 \\ 0 & \psi\otimes\rho \end{bmatrix}} & K[[x,y]]^{2nm} & \xrightarrow{\begin{bmatrix} \phi\otimes\sigma & 0 \\ 0 & \phi\otimes\sigma \end{bmatrix}} & K[[x,y]]^{2nm} \\
\begin{bmatrix} \alpha_f\otimes\alpha_g & 0 \\ 0 & \alpha_f\otimes\alpha_g \end{bmatrix}\Big\downarrow & & \Big\downarrow\begin{bmatrix} \beta_f\otimes\beta_g & 0 \\ 0 & \beta_f\otimes\beta_g \end{bmatrix} & & \Big\downarrow\begin{bmatrix} \alpha_f\otimes\alpha_g & 0 \\ 0 & \alpha_f\otimes\alpha_g \end{bmatrix} \\
K[[x,y]]^{2n'm'} & \xrightarrow[\begin{bmatrix} \psi'\otimes\rho' & 0 \\ 0 & \psi'\otimes\rho' \end{bmatrix}]{} & K[[x,y]]^{2n'm'} & \xrightarrow[\begin{bmatrix} \phi'\otimes\sigma' & 0 \\ 0 & \phi'\otimes\sigma' \end{bmatrix}]{} & K[[x,y]]^{2n'm'}
\end{array}$$

viz. both the left and the right squares in the foregoing diagram commute.
• The commutativity of the right square and the left square are respectively expressed by the following equalities:

$$\begin{bmatrix} \alpha_f\otimes\alpha_g & 0 \\ 0 & \alpha_f\otimes\alpha_g \end{bmatrix}\begin{bmatrix} \phi\otimes\sigma & 0 \\ 0 & \phi\otimes\sigma \end{bmatrix} = \begin{bmatrix} \phi'\otimes\sigma' & 0 \\ 0 & \phi'\otimes\sigma' \end{bmatrix}\begin{bmatrix} \beta_f\otimes\beta_g & 0 \\ 0 & \beta_f\otimes\beta_g \end{bmatrix}$$

and

$$\begin{bmatrix} \beta_f\otimes\beta_g & 0 \\ 0 & \beta_f\otimes\beta_g \end{bmatrix}\begin{bmatrix} \psi\otimes\rho & 0 \\ 0 & \psi\otimes\rho \end{bmatrix} = \begin{bmatrix} \psi'\otimes\rho' & 0 \\ 0 & \psi'\otimes\rho' \end{bmatrix}\begin{bmatrix} \alpha_f\otimes\alpha_g & 0 \\ 0 & \alpha_f\otimes\alpha_g \end{bmatrix}$$

i.e.,

$$\begin{bmatrix} \alpha_f\phi\otimes\alpha_g\sigma & 0 \\ 0 & \alpha_f\phi\otimes\alpha_g\sigma \end{bmatrix} = \begin{bmatrix} \phi'\beta_f\otimes\sigma'\beta_g & 0 \\ 0 & \phi'\beta_f\otimes\sigma'\beta_g \end{bmatrix}$$

and

$$\begin{bmatrix} \beta_f\psi\otimes\beta_g\rho & 0 \\ 0 & \beta_f\psi\otimes\beta_g\rho \end{bmatrix} = \begin{bmatrix} \psi'\alpha_f\otimes\rho'\alpha_g & 0 \\ 0 & \psi'\alpha_f\otimes\rho'\alpha_g \end{bmatrix}$$

i.e.,

$$\begin{cases} \alpha_f\phi \otimes \alpha_g\sigma = \phi'\beta_f \otimes \sigma'\beta_g \\ \alpha_f\phi \otimes \alpha_g\sigma = \phi'\beta_f \otimes \sigma'\beta_g \end{cases}$$

and

$$\begin{cases} \beta_f\psi \otimes \beta_g\rho = \psi'\alpha_f \otimes \rho'\alpha_g \\ \beta_f\psi \otimes \beta_g\rho = \psi'\alpha_f \otimes \rho'\alpha_g \end{cases}$$

i.e., all we need to show is the pair of equalities:

$$\begin{cases} \alpha_f\phi \otimes \alpha_g\sigma = \phi'\beta_f \otimes \sigma'\beta_g \cdots (1) \\ \beta_f\psi \otimes \beta_g\rho = \psi'\alpha_f \otimes \rho'\alpha_g \cdots (2) \end{cases}$$

Now by hypothesis, $\zeta_f = (\alpha_f, \beta_f) : X_f = (\phi, \psi) \to X'_f = (\phi', \psi')$ and $\zeta_g = (\alpha_g, \beta_g) : X_g = (\sigma, \rho) \to X'_g = (\sigma', \rho')$ are morphisms, meaning that the following diagrams commute

$$\begin{array}{ccccc} K[[x]]^n & \xrightarrow{\psi} & K[[x]]^n & \xrightarrow{\phi} & K[[x]]^n \\ \downarrow{\scriptstyle \alpha_f} & & \downarrow{\scriptstyle \beta_f} & & \downarrow{\scriptstyle \alpha_f} \\ K[[x]]^{n'} & \xrightarrow{\psi'} & K[[x]]^{n'} & \xrightarrow{\phi'} & K[[x]]^{n'} \end{array}$$

and

$$\begin{array}{ccccc} K[[y]]^m & \xrightarrow{\rho} & K[[y]]^m & \xrightarrow{\sigma} & K[[y]]^m \\ \downarrow{\scriptstyle \alpha_g} & & \downarrow{\scriptstyle \beta_g} & & \downarrow{\scriptstyle \alpha_g} \\ K[[y]]^{m'} & \xrightarrow{\rho'} & K[[y]]^{m'} & \xrightarrow{\sigma'} & K[[y]]^{m'} \end{array}$$

That is,

$$\begin{cases} \alpha_f\phi = \phi'\beta_f \cdots (i) \\ \psi'\alpha_f = \beta_f\psi \cdots (ii) \end{cases}$$

and

$$\begin{cases} \alpha_g\sigma = \sigma'\beta_g \cdots (i') \\ \rho'\alpha_g = \beta_g\rho \cdots (ii') \end{cases}$$

Now considering (i) and (i'), we immediately see that equality (1) holds. Similarly, (ii) and (ii') yield (2).

So, $\zeta_f \widetilde{\otimes} \zeta_g$ is a morphism in $MF(K[[x,y]], fg)$. QED

□

We can now state the following result.

Theorem 3.1. *The multiplicative tensor product* $(-)\widetilde{\otimes}(-) : MF(K[[x]], f) \times MF(K[[y]], g) \to MF(K[[x,y]], fg)$ *is a bifunctor.*

Proof. In order to ease our computations, let's write $F = (-)\widetilde{\otimes}(-)$. We show that F is a bifunctor.
We have:

$$\begin{array}{ccccccc}
(-)\widetilde{\otimes}(-): & & MF(f)\times MF(g) & \longrightarrow & MF(fg) \\
 & (X_f & , & X_g) & \longrightarrow & X_f\widetilde{\otimes}X_g \\
 & \downarrow \zeta_f & & \downarrow \zeta_g & & \downarrow \zeta_f\widetilde{\otimes}\zeta_g:=(\alpha,\beta) \\
 & (X'_f & , & X'_g) & \longrightarrow & X'_f\widetilde{\otimes}X'_g \\
 & \downarrow \zeta'_f & & \downarrow \zeta'_g & & \downarrow \zeta'_f\widetilde{\otimes}\zeta'_g:=(\alpha',\beta') \\
 & (X_f" & , & X_g") & \longrightarrow & X_f"\widetilde{\otimes}X_g"
\end{array}$$

It is clear (as we saw at the beginning of this section) that the multiplicative tensor product of a pair in $MF(f)\times MF(g)$ is an object of $MF(fg)$. We also showed in lemma 3.4 that $\zeta_f\widetilde{\otimes}\zeta_g := (\alpha,\beta)$ is a morphism in $MF(K[[x,y]], fg)$, where

$$(\alpha,\beta) = (\begin{bmatrix}\alpha_f\otimes\alpha_g & 0\\ 0 & \alpha_f\otimes\alpha_g\end{bmatrix}, \begin{bmatrix}\beta_f\otimes\beta_g & 0\\ 0 & \beta_f\otimes\beta_g\end{bmatrix}).$$

Similarly, if $\zeta'_f := (\alpha'_f, \beta'_f)$ and $\zeta'_g := (\alpha'_g, \beta'_g)$ then we have

$$(\alpha',\beta') = (\begin{bmatrix}\alpha'_f\otimes\alpha'_g & 0\\ 0 & \alpha'_f\otimes\alpha'_g\end{bmatrix}, \begin{bmatrix}\beta'_f\otimes\beta'_g & 0\\ 0 & \beta'_f\otimes\beta'_g\end{bmatrix}).$$

It now remains to show the composition and the identity axioms.

Identity Axiom:
We show that $F(id_{(X_f,X_g)}) = id_{F(X_f,X_g)}$.
Now, $F(id_{(X_f,X_g)}) = F(id_{X_f}, id_{X_g}) := id_{X_f}\widetilde{\otimes}id_{X_g} : X_f\widetilde{\otimes}X_g \to X_f\widetilde{\otimes}X_g$.
And by definition 3.4, $id_{X_f}\widetilde{\otimes}id_{X_g}$ is the pair of matrices $(\begin{bmatrix}I_n\otimes I_m & 0\\ 0 & I_n\otimes I_m\end{bmatrix}, \begin{bmatrix}I_n\otimes I_m & 0\\ 0 & I_n\otimes I_m\end{bmatrix})$ †
Next, we compute $id_{F(X_f,X_g)} = id_{X_f\widetilde{\otimes}X_g} : X_f\widetilde{\otimes}X_g \to X_f\widetilde{\otimes}X_g$.
Now $MF(fg)$ is a category and $X_f\widetilde{\otimes}X_g$ is an object in $MF(fg)$, thus by definition of a morphism in such a category (cf. subsection 2.1.2), we know that $id_{X_f\widetilde{\otimes}X_g} := (\begin{bmatrix}I_{nm} & 0\\ 0 & I_{nm}\end{bmatrix}, \begin{bmatrix}I_{nm} & 0\\ 0 & I_{nm}\end{bmatrix})$ ††
Since $I_n\otimes I_m = I_{nm}$, we see that † and †† are the same, therefore $F(id_{(X_f,X_g)}) = id_{F(X_f,X_g)}$ as desired.

Composition Axiom:
Consider the situation:

$$X_f \xrightarrow{\zeta_f} X'_f \xrightarrow{\zeta'_f} X_f"$$

$$X_g \xrightarrow{\zeta_g} X'_g \xrightarrow{\zeta'_g} X_g"$$

$$X_f\widetilde{\otimes}X_g \xrightarrow{F(\zeta_f,\zeta_g)} X'_f\widetilde{\otimes}X'_g \xrightarrow{F(\zeta'_f,\zeta'_g)} X_f"\widetilde{\otimes}X_g"$$

We need to show $F(\zeta'_f \circ \zeta_f, \zeta'_g \circ \zeta_g) = F(\zeta'_f, \zeta'_g) \circ F(\zeta_f, \zeta_g)$.
By definition of composition of maps (cf. subsection 2.1.2), we have: $\zeta'_f \circ \zeta_f = (\alpha'_f \alpha_f, \beta'_f \beta_f)$ and $\zeta'_g \circ \zeta_g = (\alpha'_g \alpha_g, \beta'_g \beta_g)$.

Now using definition 3.4, we get:

$$(\zeta'_f \circ \zeta_f)\widetilde{\otimes}(\zeta'_g \circ \zeta_g) = (\begin{bmatrix} \alpha'_f\alpha_f \otimes \alpha'_g\alpha_g & 0 \\ 0 & \alpha'_f\alpha_f \otimes \alpha'_g\alpha_g \end{bmatrix}, \begin{bmatrix} \beta'_f\beta_f \otimes \beta'_g\beta_g & 0 \\ 0 & \beta'_f\beta_f \otimes \beta'_g\beta_g \end{bmatrix}) \qquad \dagger'$$

Next,

$$(\zeta'_f \widetilde{\otimes} \zeta'_g) \circ (\zeta_f \widetilde{\otimes} \zeta_g)$$
$$= (\begin{bmatrix} \alpha'_f \otimes \alpha'_g & 0 \\ 0 & \alpha'_f \otimes \alpha'_g \end{bmatrix}, \begin{bmatrix} \beta'_f \otimes \beta'_g & 0 \\ 0 & \beta'_f \otimes \beta'_g \end{bmatrix}) \circ (\begin{bmatrix} \alpha_f \otimes \alpha_g & 0 \\ 0 & \alpha_f \otimes \alpha_g \end{bmatrix}, \begin{bmatrix} \beta_f \otimes \beta_g & 0 \\ 0 & \beta_f \otimes \beta_g \end{bmatrix})$$
$$= (\begin{bmatrix} \alpha'_f\alpha_f \otimes \alpha'_g\alpha_g & 0 \\ 0 & \alpha'_f\alpha_f \otimes \alpha'_g\alpha_g \end{bmatrix}, \begin{bmatrix} \beta'_f\beta_f \otimes \beta'_g\beta_g & 0 \\ 0 & \beta'_f\beta_f \otimes \beta'_g\beta_g \end{bmatrix}) \qquad \dagger\dagger'$$

From † and ††′, we see that $F(\zeta'_f \circ \zeta_f, \zeta'_g \circ \zeta_g) = F(\zeta'_f, \zeta'_g) \circ F(\zeta_f, \zeta_g)$ as desired. So, $(-)\widetilde{\otimes}(-)$ is a bifunctor. QED □

3.2 Properties of the multiplicative tensor product of matrix factorizations

In this section, we give many properties of the multiplicative tensor product of matrix factorizations. Moreover, after proposing a definition for nilpotent (respectively idempotent) matrix factorization, we prove that given two matrix factorizations X and Y which are nilpotent (resp. idempotent), their multiplication $X\widetilde{\otimes}Y$ is nilpotent (respectively idempotent).
In this section, we denote by $X_1 = (\phi_1, \psi_1)$ (resp. $X_2 = (\phi_2, \psi_2)$) an $(n_1 \times n_1)$ (resp. $(n_2 \times n_2)$) matrix factorization of $f \in K[[x]]$. We also let $X' = (\phi', \psi')$ (resp. $X'' = (\phi'', \psi'')$) denotes a $(p \times p)$ (resp. $(m \times m)$) matrix factorization of $g \in K[[y]]$. $X = (\phi, \psi)$ will also be an $r \times r$ matrix factorization of $f \in K[[x]]$.
Our results are presented only for the case of power series, but it is easy to see that they also hold for elements in any unital ring (e.g. the ring of rational functions).

3.2.1 Associativity, commutativity and distributivity of $\widetilde{\otimes}$

Here, it is shown that $\widetilde{\otimes}$ is associative, commutative (up to isomorphism) and distributive.

Proposition 3.3. *(Associativity)*
There is an identity $(X\widetilde{\otimes}X')\widetilde{\otimes}X'' = X\widetilde{\otimes}(X'\widetilde{\otimes}X'')$.

Proof. We exploit the fact that the standard tensor product for matrices is associative to show the desired identity. We know that $X = (\phi, \psi)$, $X' = (\phi', \psi')$ and $X'' = (\phi'', \psi'')$, so we have

$$(X\widetilde{\otimes}X')\widetilde{\otimes}X'' = (\begin{bmatrix} \phi \otimes \phi' & 0 \\ 0 & \phi \otimes \phi' \end{bmatrix}, \begin{bmatrix} \psi \otimes \psi' & 0 \\ 0 & \psi \otimes \psi' \end{bmatrix})\widetilde{\otimes}(\phi'', \psi'')$$

$$= (\begin{bmatrix} \begin{bmatrix} \phi\otimes\phi' & 0 \\ 0 & \phi\otimes\phi' \end{bmatrix}\otimes\phi'' & 0 \\ 0 & \begin{bmatrix} \phi\otimes\phi' & 0 \\ 0 & \phi\otimes\phi' \end{bmatrix}\otimes\phi'' \end{bmatrix}, \begin{bmatrix} \begin{bmatrix} \psi\otimes\psi' & 0 \\ 0 & \psi\otimes\psi' \end{bmatrix}\otimes\psi'' & 0 \\ 0 & \begin{bmatrix} \psi\otimes\psi' & 0 \\ 0 & \psi\otimes\psi' \end{bmatrix}\otimes\psi'' \end{bmatrix})$$

$$= (\begin{bmatrix} (\phi\otimes\phi')\otimes\phi'' & 0 & 0 & 0 \\ 0 & (\phi\otimes\phi')\otimes\phi'' & 0 & 0 \\ 0 & 0 & (\phi\otimes\phi')\otimes\phi'' & 0 \\ 0 & 0 & 0 & (\phi\otimes\phi')\otimes\phi'' \end{bmatrix},$$

$$\begin{bmatrix} (\psi\otimes\psi')\otimes\psi'' & 0 & 0 & 0 \\ 0 & (\psi\otimes\psi')\otimes\psi'' & 0 & 0 \\ 0 & 0 & (\psi\otimes\psi')\otimes\psi'' & 0 \\ 0 & 0 & 0 & (\psi\otimes\psi')\otimes\psi'' \end{bmatrix})$$

$$= (\begin{bmatrix} \phi\otimes(\phi'\otimes\phi'') & 0 & 0 & 0 \\ 0 & \phi\otimes(\phi'\otimes\phi'') & 0 & 0 \\ 0 & 0 & \phi\otimes(\phi'\otimes\phi'') & 0 \\ 0 & 0 & 0 & \phi\otimes(\phi'\otimes\phi'') \end{bmatrix},$$

$$\begin{bmatrix} \psi\otimes(\psi'\otimes\psi'') & 0 & 0 & 0 \\ 0 & \psi\otimes(\psi'\otimes\psi'') & 0 & 0 \\ 0 & 0 & \psi\otimes(\psi'\otimes\psi'') & 0 \\ 0 & 0 & 0 & \psi\otimes(\psi'\otimes\psi'') \end{bmatrix})$$

$$= (\begin{bmatrix} \phi\otimes\begin{bmatrix} \phi'\otimes\phi'' & 0 \\ 0 & \phi'\otimes\phi'' \end{bmatrix} & 0 \\ 0 & \phi\otimes\begin{bmatrix} \phi'\otimes\phi'' & 0 \\ 0 & \phi'\otimes\phi'' \end{bmatrix} \end{bmatrix}, \begin{bmatrix} \psi\otimes\begin{bmatrix} \psi'\otimes\psi'' & 0 \\ 0 & \psi'\otimes\psi'' \end{bmatrix} & 0 \\ 0 & \psi\otimes\begin{bmatrix} \psi'\otimes\psi'' & 0 \\ 0 & \psi'\otimes\psi'' \end{bmatrix} \end{bmatrix})$$

$$= (\phi,\psi)\widetilde{\otimes}(\begin{bmatrix} \phi'\otimes\phi'' & 0 \\ 0 & \phi'\otimes\phi'' \end{bmatrix}, \begin{bmatrix} \psi'\otimes\psi'' & 0 \\ 0 & \psi'\otimes\psi'' \end{bmatrix})$$

$$\begin{aligned} &= (\phi,\psi)\widetilde{\otimes}((\phi',\psi')\widetilde{\otimes}(\phi'',\psi'')) \\ &= X\widetilde{\otimes}(X'\widetilde{\otimes}X'') \;\; as\, desired. \end{aligned}$$

The fourth equality above is by associativity of the standard tensor product "⊗" of matrices. □

Before stating the next proposition, it is worthwhile recalling (cf. section 3.1 [30]) that given two matrices A and B, the tensor products $A\otimes B$ and $B\otimes A$ are **permutation equivalent**. That is, there exist permutation matrices P and Q (so called commutation matrices) such that: $A\otimes B = P(B\otimes A)Q$. If A and B are square matrices, then $A\otimes B$ and $B\otimes A$ are even **permutation similar**, meaning we can take $P = Q^T$.
To be more precise [30], if A is a $p\times q$ matrix and B is an $r\times s$ matrix, then

$$B\otimes A = S_{p,r}(A\otimes B)S_{q,s}^T$$

where,

$$S_{m,n} = \sum_{i=1}^{m}(e_i^T \otimes I_n \otimes e_i) = \sum_{j=1}^{n}(e_j \otimes I_m \otimes e_j^T)$$

I_n is the $n \times n$ identity matrix and e_i is the i^{th} unit vector. $S_{m,n}$ is the **perfect shuffle** permutation matrix.
We use 2×2 matrices to illustrate the fact that $A \otimes B$ and $B \otimes A$ are permutation similar.

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $B = \begin{bmatrix} e & f \\ g & h \end{bmatrix}$. Then $A \otimes B = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \otimes \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} ae & af & be & bf \\ ag & ah & bg & bh \\ ce & cf & de & df \\ cg & ch & dg & dh \end{bmatrix}$ and

$B \otimes A = \begin{bmatrix} e & f \\ g & h \end{bmatrix} \otimes \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} ea & eb & fa & fb \\ ec & ed & fc & fd \\ ga & gb & ha & hb \\ gc & gd & hc & hd \end{bmatrix}$ and we have:

$$B\otimes A = \begin{bmatrix} ea & eb & fa & fb \\ ec & ed & fc & fd \\ ga & gb & ha & hb \\ gc & gd & hc & hd \end{bmatrix} \leftrightsquigarrow_{c_2 \leftrightarrow c_3} \begin{bmatrix} ea & fa & eb & fb \\ ec & fc & ed & fd \\ ga & ha & gb & hb \\ gc & hc & gd & hd \end{bmatrix} \leftrightsquigarrow_{r_2 \leftrightarrow r_3} \begin{bmatrix} ea & fa & eb & fb \\ ga & ha & gb & hb \\ ec & fc & ed & fd \\ gc & hc & gd & hd \end{bmatrix} = A\otimes B$$

The $\leftrightsquigarrow_{c_2 \leftrightarrow c_3}$ (respectively $\leftrightsquigarrow_{r_2 \leftrightarrow r_3}$) mean that the second and third column (respectively the second and third row) have been interchanged. The commutativity of $\widetilde{\otimes}$ is up to isomorphism. This isomorphism comes from the permutation similarity[3] of the matrices involved.

Proposition 3.4. *(commutativity)*
For matrix factorizations $X \in MF(f)$ and $X' \in MF(g)$, we have

$$X\widetilde{\otimes}X' \cong X'\widetilde{\otimes}X \text{ in } MF(fg).$$

Proof. We first prove that there is a morphism from the matrix factorization $X\widetilde{\otimes}X'$ to the matrix factorization $X'\widetilde{\otimes}X$. We know that:

$X\widetilde{\otimes}X' = (\begin{bmatrix} \phi\otimes\phi' & 0 \\ 0 & \phi\otimes\phi' \end{bmatrix}, \begin{bmatrix} \psi\otimes\psi' & 0 \\ 0 & \psi\otimes\psi' \end{bmatrix}); X'\widetilde{\otimes}X = (\begin{bmatrix} \phi'\otimes\phi & 0 \\ 0 & \phi'\otimes\phi \end{bmatrix}, \begin{bmatrix} \psi'\otimes\psi & 0 \\ 0 & \psi'\otimes\psi \end{bmatrix})$

Recall that X and X' are respectively of sizes r and p. By definition (cf. section 2.1.2) of a morphism in $MF(fg)$, we find the pair of matrices (δ, β) such that the following diagram commutes:

$$\begin{array}{ccccc} K[[x,y]]^{2rp} & \xrightarrow{\begin{bmatrix} \psi\otimes\psi' & 0 \\ 0 & \psi\otimes\psi' \end{bmatrix}} & K[[x,y]]^{2rp} & \xrightarrow{\begin{bmatrix} \phi\otimes\phi' & 0 \\ 0 & \phi\otimes\phi' \end{bmatrix}} & K[[x,y]]^{2rp} \\ \downarrow{\scriptstyle\delta} & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\delta} \\ K[[x,y]]^{2rp} & \xrightarrow[\begin{bmatrix} \psi'\otimes\psi & 0 \\ 0 & \psi'\otimes\psi \end{bmatrix}]{} & K[[x,y]]^{2rp} & \xrightarrow[\begin{bmatrix} \phi'\otimes\phi & 0 \\ 0 & \phi'\otimes\phi \end{bmatrix}]{} & K[[x,y]]^{2rp} \end{array}$$

[3]Recall that all the matrices involved in a matrix factorization are square matrices by definition, this justifies the fact that we talk of permutation similarity instead of permutation equivalence.

We claim that if we choose ($\delta = \begin{bmatrix} \phi'\otimes\phi & 0 \\ 0 & \phi'\otimes\phi \end{bmatrix}, \beta = \begin{bmatrix} \phi\otimes\phi' & 0 \\ 0 & \phi\otimes\phi' \end{bmatrix}$), then the above diagram will commute. That is,

$$\begin{cases} \begin{bmatrix} \phi'\otimes\phi & 0 \\ 0 & \phi'\otimes\phi \end{bmatrix}\begin{bmatrix} \phi\otimes\phi' & 0 \\ 0 & \phi\otimes\phi' \end{bmatrix} = \begin{bmatrix} \phi'\otimes\phi & 0 \\ 0 & \phi'\otimes\phi \end{bmatrix}\begin{bmatrix} \phi\otimes\phi' & 0 \\ 0 & \phi\otimes\phi' \end{bmatrix} \\ \begin{bmatrix} \psi'\otimes\psi & 0 \\ 0 & \psi'\otimes\psi \end{bmatrix}\begin{bmatrix} \phi'\otimes\phi & 0 \\ 0 & \phi'\otimes\phi \end{bmatrix} = \begin{bmatrix} \phi\otimes\phi' & 0 \\ 0 & \phi\otimes\phi' \end{bmatrix}\begin{bmatrix} \psi\otimes\psi' & 0 \\ 0 & \psi\otimes\psi' \end{bmatrix} \end{cases}$$

The first equation clearly holds. As for the second, it would hold if:
$(\psi'\otimes\psi)(\phi'\otimes\phi) = (\phi\otimes\phi')(\psi\otimes\psi')$ viz. $\psi'\phi'\otimes\psi\phi = \phi\psi\otimes\phi'\psi'$, i.e., $g\cdot I_p\otimes f\cdot I_r = f\cdot I_r\otimes g\cdot I_p$, i.e., $gfI_p\otimes I_r = fgI_r\otimes I_p$. Now, this last equality is true since $fg = gf$ and $I_r\otimes I_p = I_{rp} = I_{pr} = I_p\otimes I_r$.
Therefore the above diagram commutes meaning that there is a map from $X\widetilde{\otimes}X'$ to $X'\widetilde{\otimes}X$.
Secondly, we prove the isomorphism:
In fact,

$$\begin{aligned} X\widetilde{\otimes}X' &= (\begin{bmatrix} \phi\otimes\phi' & 0 \\ 0 & \phi\otimes\phi' \end{bmatrix}, \begin{bmatrix} \psi\otimes\psi' & 0 \\ 0 & \psi\otimes\psi' \end{bmatrix}) \\ &\cong (\begin{bmatrix} \phi'\otimes\phi & 0 \\ 0 & \phi'\otimes\phi \end{bmatrix}, \begin{bmatrix} \psi'\otimes\psi & 0 \\ 0 & \psi'\otimes\psi \end{bmatrix}) = X'\widetilde{\otimes}X \end{aligned}$$

The "$\cong$" in this proof is due to the fact that $\phi\otimes\phi'$ (respectively $\psi\otimes\psi'$) and $\phi'\otimes\phi$ (respectively $\psi'\otimes\psi$) are permutation similar. □

Proposition 3.5. *(Distributivity)*
If X_1 and X_2 are matrix factorizations (of $f \in K[[x]]$) of the same size, then there are natural isomorphisms

1. $(X_1\oplus X_2)\widetilde{\otimes}X' \cong (X_1\widetilde{\otimes}X')\oplus(X_2\widetilde{\otimes}X')$.
2. $X'\widetilde{\otimes}(X_1\oplus X_2) \cong (X'\widetilde{\otimes}X_1)\oplus(X'\widetilde{\otimes}X_2)$.

Proof. 1.

$$(X_1\widetilde{\otimes}X')\oplus(X_2\widetilde{\otimes}X')$$

$$= (\begin{bmatrix} \phi_1\otimes\phi' & 0 \\ 0 & \phi_1\otimes\phi' \end{bmatrix}, \begin{bmatrix} \psi_1\otimes\psi' & 0 \\ 0 & \psi_1\otimes\psi' \end{bmatrix})\oplus(\begin{bmatrix} \phi_2\otimes\phi' & 0 \\ 0 & \phi_2\otimes\phi' \end{bmatrix}, \begin{bmatrix} \psi_2\otimes\psi' & 0 \\ 0 & \psi_2\otimes\psi' \end{bmatrix})$$

$$= (\begin{bmatrix} \phi_1\otimes\phi' & 0 \\ 0 & \phi_1\otimes\phi' \end{bmatrix}\oplus\begin{bmatrix} \phi_2\otimes\phi' & 0 \\ 0 & \phi_2\otimes\phi' \end{bmatrix}), (\begin{bmatrix} \psi_1\otimes\psi' & 0 \\ 0 & \psi_1\otimes\psi' \end{bmatrix}\oplus\begin{bmatrix} \psi_2\otimes\psi' & 0 \\ 0 & \psi_2\otimes\psi' \end{bmatrix})$$

$$= (\begin{bmatrix} \phi_1\otimes\phi' & 0 & 0 & 0 \\ 0 & \phi_1\otimes\phi' & 0 & 0 \\ 0 & 0 & \phi_2\otimes\phi' & 0 \\ 0 & 0 & 0 & \phi_2\otimes\phi' \end{bmatrix}, \begin{bmatrix} \psi_1\otimes\psi' & 0 & 0 & 0 \\ 0 & \psi_1\otimes\psi' & 0 & 0 \\ 0 & 0 & \psi_2\otimes\psi' & 0 \\ 0 & 0 & 0 & \psi_2\otimes\psi' \end{bmatrix}) \quad \cdots(\sharp)$$

Let $P = \begin{bmatrix} \phi_1 \otimes \phi' & 0 & 0 & 0 \\ 0 & \phi_1 \otimes \phi' & 0 & 0 \\ 0 & 0 & \phi_2 \otimes \phi' & 0 \\ 0 & 0 & 0 & \phi_2 \otimes \phi' \end{bmatrix}$ and $Q = \begin{bmatrix} \psi_1 \otimes \psi' & 0 & 0 & 0 \\ 0 & \psi_1 \otimes \psi' & 0 & 0 \\ 0 & 0 & \psi_2 \otimes \psi' & 0 \\ 0 & 0 & 0 & \psi_2 \otimes \psi' \end{bmatrix}$

Next,

$$\begin{aligned}(X_1 \oplus X_2)\widetilde{\otimes}X' &= ((\phi_1, \psi_1) \oplus (\phi_2, \psi_2))\widetilde{\otimes}(\phi', \psi') \\ &= (\begin{bmatrix} \phi_1 & 0 \\ 0 & \phi_2 \end{bmatrix}, \begin{bmatrix} \psi_1 & 0 \\ 0 & \psi_2 \end{bmatrix})\widetilde{\otimes}(\phi', \psi')\end{aligned}$$

$$= (\begin{bmatrix} \begin{bmatrix} \phi_1 & 0 \\ 0 & \phi_2 \end{bmatrix} \otimes \phi' & 0 \\ 0 & \begin{bmatrix} \phi_1 & 0 \\ 0 & \phi_2 \end{bmatrix} \otimes \phi' \end{bmatrix}, \begin{bmatrix} \begin{bmatrix} \psi_1 & 0 \\ 0 & \psi_2 \end{bmatrix} \otimes \psi' & 0 \\ 0 & \begin{bmatrix} \psi_1 & 0 \\ 0 & \psi_2 \end{bmatrix} \otimes \psi' \end{bmatrix})$$

$$= (\begin{bmatrix} \phi_1 \otimes \phi' & 0 & 0 & 0 \\ 0 & \phi_2 \otimes \phi' & 0 & 0 \\ 0 & 0 & \phi_1 \otimes \phi' & 0 \\ 0 & 0 & 0 & \phi_2 \otimes \phi' \end{bmatrix}, \begin{bmatrix} \psi_1 \otimes \psi' & 0 & 0 & 0 \\ 0 & \psi_2 \otimes \psi' & 0 & 0 \\ 0 & 0 & \psi_1 \otimes \psi' & 0 \\ 0 & 0 & 0 & \psi_2 \otimes \psi' \end{bmatrix}) \quad \cdots (\sharp')$$

Let $M = \begin{bmatrix} \phi_1 \otimes \phi' & 0 & 0 & 0 \\ 0 & \phi_2 \otimes \phi' & 0 & 0 \\ 0 & 0 & \phi_1 \otimes \phi' & 0 \\ 0 & 0 & 0 & \phi_2 \otimes \phi' \end{bmatrix}$ and $N = \begin{bmatrix} \psi_1 \otimes \psi' & 0 & 0 & 0 \\ 0 & \psi_2 \otimes \psi' & 0 & 0 \\ 0 & 0 & \psi_1 \otimes \psi' & 0 \\ 0 & 0 & 0 & \psi_2 \otimes \psi' \end{bmatrix}$

Recall that X_1, X_2 and X' are respectively of sizes n_1, n_2 and p. By definition (cf. section 2.1.2) of a morphism in $MF(fg)$, we find the pair of matrices (δ,β) such that the following diagram commutes:

$$\begin{array}{ccccc} K[[x,y]]^{2(n_1+n_2)p} & \xrightarrow{Q} & K[[x,y]]^{2(n_1+n_2)p} & \xrightarrow{P} & K[[x,y]]^{2(n_1+n_2)p} \\ \downarrow{\scriptstyle \delta} & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \delta} \\ K[[x,y]]^{2(n_1+n_2)p} & \xrightarrow[N]{} & K[[x,y]]^{2(n_1+n_2)p} & \xrightarrow[M]{} & K[[x,y]]^{2(n_1+n_2)p} \end{array}$$

We claim that if we choose $(\delta = M, \beta = P)$, then the above diagram will commute viz. $\delta P = M\beta$ and $N\delta = \beta Q$. It is clear that with our choice of δ and β, the equation $\delta P = M\beta$ obviously holds. As for the equality $N\delta = \beta Q$, it is true just in case the following matrix equation holds:

$$\begin{bmatrix} \psi_1 \otimes \psi' & 0 & 0 & 0 \\ 0 & \psi_2 \otimes \psi' & 0 & 0 \\ 0 & 0 & \psi_1 \otimes \psi' & 0 \\ 0 & 0 & 0 & \psi_2 \otimes \psi' \end{bmatrix} \begin{bmatrix} \phi_1 \otimes \phi' & 0 & 0 & 0 \\ 0 & \phi_2 \otimes \phi' & 0 & 0 \\ 0 & 0 & \phi_1 \otimes \phi' & 0 \\ 0 & 0 & 0 & \phi_2 \otimes \phi' \end{bmatrix}$$

$$= \begin{bmatrix} \phi_1 \otimes \phi' & 0 & 0 & 0 \\ 0 & \phi_1 \otimes \phi' & 0 & 0 \\ 0 & 0 & \phi_2 \otimes \phi' & 0 \\ 0 & 0 & 0 & \phi_2 \otimes \phi' \end{bmatrix} \begin{bmatrix} \psi_1 \otimes \psi' & 0 & 0 & 0 \\ 0 & \psi_1 \otimes \psi' & 0 & 0 \\ 0 & 0 & \psi_2 \otimes \psi' & 0 \\ 0 & 0 & 0 & \psi_2 \otimes \psi' \end{bmatrix}$$

It is easy to see that this equality holds if each of the following smaller equalities hold.

a. $(\psi_1 \otimes \psi')(\phi_1 \otimes \phi') = (\phi_1 \otimes \phi')(\psi_1 \otimes \psi')$; i.e., $\psi_1\phi_1 \otimes \psi'\phi' = \phi_1\psi_1 \otimes \phi'\psi'$
(This clearly holds since $\psi_1\phi_1 = \phi_1\psi_1$ and $\phi'\psi' = \psi'\phi'$ because matrices involved in a matrix factorization commute (cf. prop. 2.1)).

b. $(\psi_2 \otimes \psi')(\phi_2 \otimes \phi') = (\phi_1 \otimes \phi')(\psi_1 \otimes \psi')$; i.e., $\psi_2\phi_2 \otimes \psi'\phi' = \phi_1\psi_1 \otimes \phi'\psi'$; i.e., $f \cdot I_{n_2} \otimes g \cdot I_p = f \cdot I_{n_1} \otimes g \cdot I_p$ since $X_i = (\phi_i, \psi_i)$, for $i = 1, 2$, is a matrix factorization of f.
i.e., $fg \cdot I_{n_2} \otimes I_p = fg \cdot I_{n_1} \otimes I_p$
(This clearly holds since $n_1 = n_2$ as X_1 and X_2 are matrix factorizations of the same size by assumption.)

c. $(\psi_1 \otimes \psi')(\phi_1 \otimes \phi') = (\phi_2 \otimes \phi')(\psi_2 \otimes \psi')$; i.e., $\psi_1\phi_1 \otimes \psi'\phi' = \phi_2\psi_2 \otimes \phi'\psi'$
(A similar argument as the previous one shows that this identity hold.)

d. $(\psi_2 \otimes \psi')(\phi_2 \otimes \phi') = (\phi_2 \otimes \phi')(\psi_2 \otimes \psi')$; i.e., $\psi_2\phi_2 \otimes \psi'\phi' = \phi_2\psi_2 \otimes \phi'\psi'$
(It is easy to see that this equality holds.)

Since the four foregoing equalities hold, the above diagram commutes.
The desired isomorphism now easily follows from ($\sharp$) and ($\sharp'$). In fact, observe that if $c_i \leftrightarrow c_j$ (resp. $r_i \leftrightarrow r_j$) stands for the operation of interchanging column i and column j (resp. row i and row j) then $\sharp$ and $\sharp'$ can be obtained from one another first by applying $c_2 \leftrightarrow c_3$ and then $r_2 \leftrightarrow r_3$ as follows:

$$\left(\begin{bmatrix} \phi_1\otimes\phi' & 0 & 0 & 0 \\ 0 & \phi_1\otimes\phi' & 0 & 0 \\ 0 & 0 & \phi_2\otimes\phi' & 0 \\ 0 & 0 & 0 & \phi_2\otimes\phi' \end{bmatrix}, \begin{bmatrix} \psi_1\otimes\psi' & 0 & 0 & 0 \\ 0 & \psi_1\otimes\psi' & 0 & 0 \\ 0 & 0 & \psi_2\otimes\psi' & 0 \\ 0 & 0 & 0 & \psi_2\otimes\psi' \end{bmatrix}\right) \quad \cdots(\sharp)$$

$$c_2 \leftrightarrow c_3\left(\begin{bmatrix} \phi_1\otimes\phi' & 0 & 0 & 0 \\ 0 & 0 & \phi_1\otimes\phi' & 0 \\ 0 & \phi_2\otimes\phi' & 0 & 0 \\ 0 & 0 & 0 & \phi_2\otimes\phi' \end{bmatrix}, \begin{bmatrix} \psi_1\otimes\psi' & 0 & 0 & 0 \\ 0 & 0 & \psi_1\otimes\psi' & 0 \\ 0 & \psi_2\otimes\psi' & 0 & 0 \\ 0 & 0 & 0 & \psi_2\otimes\psi' \end{bmatrix}\right)$$

$$r_2 \leftrightarrow r_3\left(\begin{bmatrix} \phi_1\otimes\phi' & 0 & 0 & 0 \\ 0 & \phi_2\otimes\phi' & 0 & 0 \\ 0 & 0 & \phi_1\otimes\phi' & 0 \\ 0 & 0 & 0 & \phi_2\otimes\phi' \end{bmatrix}, \begin{bmatrix} \psi_1\otimes\psi' & 0 & 0 & 0 \\ 0 & \psi_2\otimes\psi' & 0 & 0 \\ 0 & 0 & \psi_1\otimes\psi' & 0 \\ 0 & 0 & 0 & \psi_2\otimes\psi' \end{bmatrix}\right) \quad \cdots(\sharp')$$

2. The proof showing that there is a map from $X'\widetilde{\otimes}(X_1 \oplus X_2)$ to $(X'\widetilde{\otimes}X_1) \oplus (X'\widetilde{\otimes}X_2)$ is similar to the proof we gave in part 1. The isomorphism follows from the commutativity of $\widetilde{\otimes}$.

□

3.2.2 Duality, Nilpotence, Idempotence and Syzygy properties

In this subsection, we define the notions of nilpotence and idempotence for a matrix factorization. Next, we prove that the property of duality holds for $\widetilde{\otimes}$. We also make some

observations about the properties of nilpotence, idempotence and syzygy concerning $\widetilde{\otimes}$. The duality and syzygy properties were proved for the tensor product of matrix factorizations in [67] and were used in proving some results. We think it is important to verify if these properties also hold for the multiplicative tensor product $\widetilde{\otimes}$. The properties we prove for $\widetilde{\otimes}$ are not followed by applications, but we hope applications will be found in the future. For now in this dissertation, we simply explore some properties of this new product.
Recall that for a matrix factorization $X = (\phi, \psi) \in MF(f)$, the *Syzygy* (cf. [67]) $\Omega(\phi, \psi) = (\psi, \phi)$. Notice that Ω is an endo-functor on the category of matrix factorizations and $\Omega^2 = 1$. In fact, $\Omega(\Omega(\phi, \psi)) = \Omega(\psi, \phi) = (\phi, \psi)$.
Also recall (cf. [67]) that for a matrix factorization $X = (\phi, \psi)$, the dual X^* of X is defined as follows; $X^* = (\phi^t, \psi^t)$.

Proposition 3.6. *(Duality)*
For matrix factorizations $X \in MF(f)$ and $X' \in MF(g)$, we have the following identity: $(X\widetilde{\otimes}X')^* = X^*\widetilde{\otimes}X'^*$.

Proof. We know that

$$X\widetilde{\otimes}X' = (\begin{bmatrix} \phi\otimes\phi' & 0 \\ 0 & \phi\otimes\phi' \end{bmatrix}, \begin{bmatrix} \psi\otimes\psi' & 0 \\ 0 & \psi\otimes\psi' \end{bmatrix})$$

Hence,

$$(X\widetilde{\otimes}X')^* = (\begin{bmatrix} \phi\otimes\phi' & 0 \\ 0 & \phi\otimes\phi' \end{bmatrix}^t, \begin{bmatrix} \psi\otimes\psi' & 0 \\ 0 & \psi\otimes\psi' \end{bmatrix}^t)$$

$$= (\begin{bmatrix} (\phi\otimes\phi')^t & 0 \\ 0 & (\phi\otimes\phi')^t \end{bmatrix}, \begin{bmatrix} (\psi\otimes\psi')^t & 0 \\ 0 & (\psi\otimes\psi')^t \end{bmatrix}) \quad (\dagger)$$

The last equality holds because the blocks on the antidiagonal of each matrix in the pair of matrices above is zero.
Next, we have

$$X^*\widetilde{\otimes}X'^* = (\phi^t, \psi^t)\widetilde{\otimes}(\phi'^t, \psi'^t)$$

$$= (\begin{bmatrix} (\phi)^t\otimes(\phi')^t & 0 \\ 0 & (\phi)^t\otimes(\phi')^t \end{bmatrix}, \begin{bmatrix} (\psi)^t\otimes(\psi')^t & 0 \\ 0 & (\psi)^t\otimes(\psi')^t \end{bmatrix})$$

$$= (\begin{bmatrix} (\phi\otimes\phi')^t & 0 \\ 0 & (\phi\otimes\phi')^t \end{bmatrix}, \begin{bmatrix} (\psi\otimes\psi')^t & 0 \\ 0 & (\psi\otimes\psi')^t \end{bmatrix}) \quad (\ddagger)$$

The second to the last equality holds because as it is well known, for any matrices A and B, $(A\otimes B)^t = A^t\otimes B^t$.
From † and ‡, we get the identity $(X\widetilde{\otimes}X')^* = X^*\widetilde{\otimes}X'^*$. □

Definition 3.5. *A matrix factorization $X = (\phi, \psi)$ of $f \in K[[x]]$ is said to be* ***nilpotent*** *if $\exists n \in \mathbb{N}$ such that $\phi^n = 0 = \psi^n$. (here 0 is the zero matrix.)*

The following proposition states that for the multiplicative tensor product of two matrix factorizations to be nilpotent, it suffices for one of them to be nilpotent.

Proposition 3.7. *(Nilpotence)*
Let $X = (\phi, \psi)$ (resp. $X' = (\phi', \psi')$) be a matrix factorization of $f \in K[[x]]$ (resp. $g \in K[[y]]$). If X or X' is nilpotent, then $X\widetilde{\otimes}X'$ is also nilpotent.

The mixed product property will be used in proving this proposition. It states that if $A, B, C,$ *and* D are matrices such that one can form the products AC and BD, then $(A \otimes B)(C \otimes D) = AC \otimes BD$. Thus, for example, $(\phi \otimes \phi')(\phi \otimes \phi') = (\phi^2, \phi'^2)$ and $(\phi \otimes \phi')^n = (\phi^n \otimes \phi'^n)$. Similarly, $(\psi \otimes \psi')^n = (\psi^n \otimes \psi'^n)$.

Proof. Suppose X is nilpotent, then $\exists\, n \in \mathbb{N}$ such that $\phi^n = 0 = \psi^n$. Now,

$$(X\widetilde{\otimes}X')^n = (\begin{bmatrix} \phi \otimes \phi' & 0 \\ 0 & \phi \otimes \phi' \end{bmatrix}, \begin{bmatrix} \psi \otimes \psi' & 0 \\ 0 & \psi \otimes \psi' \end{bmatrix})^n$$

$$(X\widetilde{\otimes}X')^n = (\begin{bmatrix} \phi \otimes \phi' & 0 \\ 0 & \phi \otimes \phi' \end{bmatrix}^n, \begin{bmatrix} \psi \otimes \psi' & 0 \\ 0 & \psi \otimes \psi' \end{bmatrix}^n)$$

$$= (\begin{bmatrix} (\phi \otimes \phi')^n & 0 \\ 0 & (\phi \otimes \phi')^n \end{bmatrix}, \begin{bmatrix} (\psi \otimes \psi')^n & 0 \\ 0 & (\psi \otimes \psi')^n \end{bmatrix})$$

$$= (\begin{bmatrix} \phi^n \otimes \phi'^n & 0 \\ 0 & \phi^n \otimes \phi'^n \end{bmatrix}, \begin{bmatrix} \psi^n \otimes \psi'^n & 0 \\ 0 & \psi^n \otimes \psi'^n \end{bmatrix})$$

$$= (\begin{bmatrix} 0 \otimes \phi'^n & 0 \\ 0 & 0 \otimes \phi'^n \end{bmatrix}, \begin{bmatrix} 0 \otimes \psi'^n & 0 \\ 0 & 0 \otimes \psi'^n \end{bmatrix})$$

$$= (\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix})$$

The fourth equality above holds by the mixed product property. So we found an n s.t.

$$\begin{bmatrix} \phi \otimes \phi' & 0 \\ 0 & \phi \otimes \phi' \end{bmatrix}^n = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} \psi \otimes \psi' & 0 \\ 0 & \psi \otimes \psi' \end{bmatrix}^n$$

i.e., $(X\widetilde{\otimes}X')$ is nilpotent as claimed.
Similarly, one can easily prove that if X' is nilpotent, then the result holds. So, $(X\widetilde{\otimes}X')$ is nilpotent as claimed.

□

We define what an idempotent matrix factorization is. Thereafter, we prove that if two matrix factorizations are idempotent, then their multiplicative tensor product is also idempotent.

Definition 3.6. *A matrix factorization $X = (\phi, \psi)$ of $f \in K[[x]]$ is said to be* ***idempotent*** *if $X^2 = X$, i.e., $\phi^2 = \phi$ and $\psi^2 = \psi$.*

The following proposition states that for the multiplicative tensor product of two matrix factorizations to be idempotent, both of them have to idempotent.

Proposition 3.8. *(Idempotence)*
Let $X = (\phi, \psi)$ *(resp.* $X' = (\phi', \psi')$ *) be a matrix factorization of* $f \in K[[x]]$ *(resp.* $g \in K[[y]]$*). If* X *and* X' *are idempotent, then* $X\widetilde{\otimes}X'$ *is also idempotent.*

Proof. Suppose X and X' are idempotent.

$$(X\widetilde{\otimes}X')^2 = (\begin{bmatrix}\phi\otimes\phi' & 0\\ 0 & \phi\otimes\phi'\end{bmatrix}, \begin{bmatrix}\psi\otimes\psi' & 0\\ 0 & \psi\otimes\psi'\end{bmatrix})^2$$

$$(X\widetilde{\otimes}X')^2 = (\begin{bmatrix}\phi\otimes\phi' & 0\\ 0 & \phi\otimes\phi'\end{bmatrix}^2, \begin{bmatrix}\psi\otimes\psi' & 0\\ 0 & \psi\otimes\psi'\end{bmatrix}^2)$$

$$= (\begin{bmatrix}(\phi\otimes\phi')^2 & 0\\ 0 & (\phi\otimes\phi')^2\end{bmatrix}, \begin{bmatrix}(\psi\otimes\psi')^2 & 0\\ 0 & (\psi\otimes\psi')^2\end{bmatrix})$$

$$= (\begin{bmatrix}\phi^2\otimes\phi'^2 & 0\\ 0 & \phi^2\otimes\phi'^2\end{bmatrix}, \begin{bmatrix}\psi^2\otimes\psi'^2 & 0\\ 0 & \psi^2\otimes\psi'^2\end{bmatrix})$$

$$= (\begin{bmatrix}\phi\otimes\phi' & 0\\ 0 & \phi\otimes\phi'\end{bmatrix}, \begin{bmatrix}\psi\otimes\psi' & 0\\ 0 & \psi\otimes\psi'\end{bmatrix})$$

The fourth equality holds by the mixed product property. The last equality holds thanks to the hypothesis. Hence $X\widetilde{\otimes}X'$ is idempotent. □

We now want to state and prove a Syzygy[4] property for $\widetilde{\otimes}$. Recall that $\Omega(\phi, \psi) := (\psi, \phi)$. In [67], a Syzygy property was proved for $\widehat{\otimes}$, the *tensor product of matrix factorization* (cf. subsection 2.1.3). It was proved that $X\widehat{\otimes}X' = \Omega(X)\widehat{\otimes}\Omega(X')$ and $X\widehat{\otimes}\Omega(X') \cong \Omega(X\widehat{\otimes}X') \cong \Omega(X)\widehat{\otimes}X'$. But the Syzygy property that holds for $\widetilde{\otimes}$ is totally different. It shows that the functor Ω is "linear" with respect to the operation $\widetilde{\otimes}$.

Proposition 3.9. *(Syzygy property)*
There is an identity $\Omega(X\widetilde{\otimes}X') = \Omega(X)\widetilde{\otimes}\Omega(X')$.

Proof.

$$X\widetilde{\otimes}X' = (\begin{bmatrix}\phi\otimes\phi' & 0\\ 0 & \phi\otimes\phi'\end{bmatrix}, \begin{bmatrix}\psi\otimes\psi' & 0\\ 0 & \psi\otimes\psi'\end{bmatrix})$$

$$\Omega(X\widetilde{\otimes}X') = \Omega(\begin{bmatrix}\phi\otimes\phi' & 0\\ 0 & \phi\otimes\phi'\end{bmatrix}, \begin{bmatrix}\psi\otimes\psi' & 0\\ 0 & \psi\otimes\psi'\end{bmatrix})$$

[4]We use this word because that is the name (cf. [67]) given to the operator Ω we are going to use in prop 3.9.

$$= (\begin{bmatrix} \psi \otimes \psi' & 0 \\ 0 & \psi \otimes \psi' \end{bmatrix}, \begin{bmatrix} \phi \otimes \phi' & 0 \\ 0 & \phi \otimes \phi' \end{bmatrix}) \quad (\sharp)$$

Next,

$$\Omega(X)\widetilde{\otimes}\Omega(X') = (\psi, \phi)\widetilde{\otimes}(\psi', \phi')$$
$$= (\begin{bmatrix} \psi \otimes \psi' & 0 \\ 0 & \psi \otimes \psi' \end{bmatrix}, \begin{bmatrix} \phi \otimes \phi' & 0 \\ 0 & \phi \otimes \phi' \end{bmatrix}) \quad (\sharp')$$

$\sharp$ and $\sharp'$ yield the desired result. □

In the next section, we will compare $\widehat{\otimes}$ and $\widetilde{\otimes}$ and study the category of matrix factorizations of the constant power series 1.

3.3 A comparison of $\widehat{\otimes}$ and $\widetilde{\otimes}$, and a study of the category $(MF(1), \widetilde{\otimes})$

We compare $\widehat{\otimes}$ and $\widetilde{\otimes}$. Some applications of $\widetilde{\otimes}$ are also given in this section.
The Syzygy property will help to find some differences between these two operations. Moreover, we will observe that the multiplicative tensor product of two objects of $MF(1)$ is still an object of $MF(1)$ whereas the tensor product $\widehat{\otimes}$ of any two matrix factorizations of a power series f is not a matrix factorization of f (even for $f = 1$). This will motivate the study of $(MF(1), \widetilde{\otimes})$. Is it a monoidal category? or a generalization of this notion?
We will define the concept of *one-step connected category* and prove that there is a one-step connected subcategory $(\mathcal{T}, \widetilde{\otimes})$ of $(MF(1), \widetilde{\otimes})$ which is a semi-unital semi-monoidal category. This is particularly interesting because the concept of semi-unital semi-monoidal category was recently conceived in [1] and the example provided in that paper (cf. theorem 5.11 of [1]) required a considerable amount of set-up. But in this section, the example (cf. theorem 3.2) we give requires a smaller amount of set-up.
Furthermore, we will define the concept of *right pseudo-monoidal* category and prove that the category $(MF(1), \widetilde{\otimes})$ is an example of this concept.
First, we give some definitions. Recall definition 1.22: A $(1, 0)$-matrix is a matrix whose entries belong to the set $\{0, 1\}$.

We chose the terminology $(1, 0)-$matrix instead of $(0, 1)-$matrix because online (e.g. wikipedia), some authors use the terminology $(0, 1)-$matrix to refer to what we call here $(1, 0)-$matrix with some additional conditions.

Definition 3.7. *A category is said to be a* ***one-step connected category*** *if for every two objects of the category, there exists a nonzero morphism between them.*

3.3.1 A comparison of $\widehat{\otimes}$ and $\widetilde{\otimes}$

$\widehat{\otimes}$ and $\widetilde{\otimes}$ are different at several levels. First of all, the Syzygy Ω helps in pointing out some differences between these two operations. For instance, observe that in general, unlike with $\widehat{\otimes}$; $\Omega(X)\widetilde{\otimes}X' \not\cong X\widetilde{\otimes}\Omega(X')$ as can be seen from the result of the following computations:

$$\Omega(X)\widetilde{\otimes}X' = (\psi,\phi)\widetilde{\otimes}(\phi',\psi') = (\begin{bmatrix}\psi\otimes\phi' & 0\\ 0 & \psi\otimes\phi'\end{bmatrix}, \begin{bmatrix}\phi\otimes\psi' & 0\\ 0 & \phi\otimes\psi'\end{bmatrix}) \quad (\dagger)$$

$$X\widetilde{\otimes}\Omega(X') = (\phi,\psi)\widetilde{\otimes}(\psi',\phi') = (\begin{bmatrix}\phi\otimes\psi' & 0\\ 0 & \phi\otimes\psi'\end{bmatrix}, \begin{bmatrix}\psi\otimes\phi' & 0\\ 0 & \psi\otimes\phi'\end{bmatrix}) \quad (\dagger')$$

From $\dagger$ and $\dagger'$, we see that in general, $\Omega(X)\widetilde{\otimes}X' \not\cong X\widetilde{\otimes}\Omega(X')$. It is also easy to see that unlike with $\widehat{\otimes}$, $X\widetilde{\otimes}X' \neq \Omega(X)\widetilde{\otimes}\Omega(X')$.
Moreover, from the definitions of $\widehat{\otimes}$ (cf. subsection 2.1.3) and $\widetilde{\otimes}$ (cf. definition 3.1), we immediately see some similarities and differences. For example, given two matrix factorizations X_f of $f \in K[[x]]$ of size n and X_g of $g \in K[[y]]$ of size m, though $X_f\widehat{\otimes}X_g$ and $X_f\widetilde{\otimes}X_g$ are both of size $2nm$, they are objects of two different categories namely $MF(f+g)$ and $MF(fg)$. Even if we consider two objects of the same category, say X_f and Y_f of a nonzero power series $f \in K[[x]]$, $X_f\widehat{\otimes}Y_f$ (respectively $X_f\widetilde{\otimes}Y_f$) will be an object not in $MF(f)$ but instead of a different category namely $MF(f+f)$ (respectively $MF(f\cdot f) = MF(f^2)$). Now, there is a striking difference between the two tensor products when $f = 1$. In fact, if $f = 1$, then $X_f\widehat{\otimes}Y_f \in MF(1+1) \neq MF(1)$ but $X_f\widetilde{\otimes}Y_f \in MF(1\cdot 1) = MF(1)$. That is, the multiplicative tensor product of two objects of $MF(1)$ is still an object of $MF(1)$. This motivates the study of $(MF(1), \widetilde{\otimes})$ to know whether it is a monoidal category or a generalization of this notion.

3.3.2 An application of $\widetilde{\otimes}$: A semi-unital semi-monoidal subcategory of $MF(1)$

We study the category $MF(1)$ and prove that it has a one-step connected subcategory which is a semi-unital semi-monoidal category.
First recall that an $n\times n$ matrix factorization of an element f in a ring with unity is a pair of $n\times n$ matrices (ϕ,ψ) such that $\phi\psi = f\cdot I_n$.
Objects of $MF(1)$ are of the form (M_n, M_n^{-1}) where M_n is an $n\times n$ matrix, $n \in \mathbb{N}-\{0\}$.
Morphisms are pairs of matrices such that a certain diagram commutes (cf. subsection 2.1.2).

Theorem 3.2. *There is a one-step connected subcategory of $MF(1)$ which is a semi-unital semi-monoidal category.*

Proof. We extract a one-step connected subcategory of $MF(1)$ which is a semi-unital semi-monoidal category. We will call it $\mathcal{T}$.
• Objects of $\mathcal{T}$ are of the form e^n where $n \in \mathbb{N}-\{0\}$. We characterize these objects.

$e^1 = e = (1,1)$, $e^2 = e\widetilde{\otimes}e = \left(\begin{pmatrix}1\otimes 1 & 0\\ 0 & 1\otimes 1\end{pmatrix}, \begin{pmatrix}1\otimes 1 & 0\\ 0 & 1\otimes 1\end{pmatrix}\right) = (I_2, I_2)$

$$e^3 = e\widetilde{\otimes}e^2 = (1,1)\widetilde{\otimes}\left(\begin{pmatrix}1\otimes 1 & 0\\ 0 & 1\otimes 1\end{pmatrix}, \begin{pmatrix}1\otimes 1 & 0\\ 0 & 1\otimes 1\end{pmatrix}\right)$$

$$= \left(\begin{pmatrix}1\otimes\begin{pmatrix}1\otimes 1 & 0\\ 0 & 1\otimes 1\end{pmatrix} & 0\\ 0 & 1\otimes\begin{pmatrix}1\otimes 1 & 0\\ 0 & 1\otimes 1\end{pmatrix}\end{pmatrix}, \begin{pmatrix}1\otimes\begin{pmatrix}1\otimes 1 & 0\\ 0 & 1\otimes 1\end{pmatrix} & 0\\ 0 & 1\otimes\begin{pmatrix}1\otimes 1 & 0\\ 0 & 1\otimes 1\end{pmatrix}\end{pmatrix}\right) = (I_4, I_4)$$

It is easy to see that in general, $e^n = (I_{2^{n-1}}, I_{2^{n-1}})$.
We give a notation before defining morphisms between two objects of $\mathcal{T}$.

Notations 3.1. *We will denote by* $0_{n,m}$ *the zero matrix of size* $n \times m$ *whenever there would be a risk of confusion on the size of the zero matrix in the context under consideration. Otherwise, we will simply write* 0*. We will also sometimes write* 0_n *for the square matrix* $0_{n,n}$*.*

• Morphisms of $\mathcal{T}$ are defined as in $MF(1)$ (cf. subsection 2.1.2), but with some restrictions. Recall that morphisms in $MF(1)$ are pairs of matrices such that a certain diagram commutes. We now define what a morphism is in $\mathcal{T}$.
Discussion ♯:
First recall that a permutation matrix is a square matrix obtained from the same size identity matrix by a permutation of rows.
For $m, p \in \mathbb{N} - \{0\}$, recall that a morphism $e^m \to e^p$ in $MF(1)$ is a pair of matrices (δ,β) such that the following diagram commutes:

$$\begin{array}{ccccc} K[[x]]^{2^{m-1}} & \xrightarrow{I_{2^{m-1}}} & K[[x]]^{2^{m-1}} & \xrightarrow{I_{2^{m-1}}} & K[[x]]^{2^{m-1}} \\ \downarrow{\scriptstyle \delta} & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \delta} \\ K[[x]]^{2^{p-1}} & \xrightarrow{I_{2^{p-1}}} & K[[x]]^{2^{p-1}} & \xrightarrow{I_{2^{p-1}}} & K[[x]]^{2^{p-1}} \end{array} \qquad (\bigstar")$$

That is,

$$(S)\begin{cases} \delta I_{2^{m-1}} = I_{2^{p-1}}\beta \\ I_{2^{p-1}}\delta = \beta I_{2^{m-1}} \end{cases}$$

It follows from (S), that a morphism $e^m \to e^p$ in $MF(1)$ is a pair of matrices (δ,β) with $\delta = \beta$. This does not impose any restrictions on the entries of δ or β, the entries could be anything provided we have the equality $\delta = \beta$.

But, we define a morphism $e^m \to e^p$ in $\mathcal{T}$ to be a pair of matrices (δ,β) such that $\delta = \beta$ is a $(1, 0)-$matrix of size $2^{p-1} \times 2^{m-1}$ *with at most one nonzero entry in each row and each column*. This restriction will ensure that the composition of two morphisms in $\mathcal{T}$ is again a morphism in $\mathcal{T}$.
Thus, for example we could have the following values of δ and β for the pair (δ,β) to be a morphism in $\mathcal{T}$:

$$\delta = \beta = \begin{cases} (I_{2^{p-1}}, 0_{(2^{p-1},2^{m-1}-2^{p-1})}),\ if\, m > p \\ \begin{pmatrix} I_{2^{m-1}} \\ 0_{(2^{p-1}-2^{m-1},2^{m-1})} \end{pmatrix},\ if\, m < p \\ Z\ if\, m = p\ where\, Z\, is\, a\, 2^{m-1} \times 2^{m-1}\ permutation\, matrix \end{cases}$$

From the twin square diagram $\bigstar$", it is clear that δ and β should both be of size $2^{p-1}\times2^{m-1}$. The fact that we actually have a morphism from e^m to e^p for the above values of $\delta = \beta$ is obvious from diagram $\bigstar$".

Discussion $\sharp$ actually gives us a sufficient condition on a pair (δ,β) to be a nonzero morphism in $\mathcal{T}$.

It is not difficult to see that $\mathcal{T}$ is a subcategory of $MF(1)$.
In fact:
• For every $X \in Ob(\mathcal{T})$, the identity morphism id_X is in $hom(\mathcal{T})$,
• For every morphism $X \to Y$ *in* $hom(\mathcal{T})$, both the source X and the target Y are in $Ob(\mathcal{T})$,
• For every pair of morphisms ζ and ζ' in $hom(\mathcal{T})$, the composite $\zeta \circ \zeta'$ is in $hom(\mathcal{T})$ whenever it is defined. In fact, ζ and ζ' by definition are pairs of $(1,0)-$matrices such that in each matrix; each column and each row has at most one nonzero entry. It then follows that the composition of such matrices will yield another $(1,0)-$matrix in which each column and each row would have at most one nonzero entry, whence we will still have a morphism of $\mathcal{T}$.

Moreover, $\mathcal{T}$ is a one-step connected category because between any two objects of $\mathcal{T}$ say e^m and e^p, there exists a nonzero morphism as can be seen from discussion $\sharp$.

We now proceed to prove that $\mathcal{T}$ is a semi-unital semi-monoidal category.
• We first prove that $(\mathcal{T}, \widetilde{\otimes})$ is a semi-monoidal category (cf. definition 1.9):
- The fact that $\widetilde{\otimes} : MF(1) \times MF(1) \to MF(1)$ is a bifunctor follows from theorem 3.1 by replacing f and g by the constant power series 1 and by letting $x = y$.
- There is a natural isomorphism α from the functor $((-)\widetilde{\otimes}(-))\widetilde{\otimes}(-) : MF(1) \times MF(1) \times MF(1) \to (MF(1) \times MF(1)) \times MF(1)$ to the functor $(-)\widetilde{\otimes}((-)\widetilde{\otimes}(-)) : MF(1) \times MF(1) \times MF(1) \to MF(1) \times (MF(1) \times MF(1))$ with components $\alpha_{a,b,c} : (a\widetilde{\otimes}b)\widetilde{\otimes}c \to a\widetilde{\otimes}(b\widetilde{\otimes}c)$, where a, b *and* c are matrix factorizations of 1 in $\mathcal{T}$.
Let $a, b\, c, a', b',$ *and* c' be objects of $\mathcal{T}$. Let $f : a \to a'$, $g : b \to b'$ and $h : c \to c'$ be maps in $\mathcal{T}$. We show that the following diagram commutes:

$$\begin{array}{ccc} (a\widetilde{\otimes}b)\widetilde{\otimes}c & \xrightarrow{\alpha_{a,b,c}} & a\widetilde{\otimes}(b\widetilde{\otimes}c) \\ \Big\downarrow{\scriptstyle (f\widetilde{\otimes}g)\widetilde{\otimes}h} & & \Big\downarrow{\scriptstyle f\widetilde{\otimes}(g\widetilde{\otimes}h)} \\ (a'\widetilde{\otimes}b')\widetilde{\otimes}c' & \xrightarrow[\alpha_{a',b',c'}]{} & a'\widetilde{\otimes}(b'\widetilde{\otimes}c') \end{array}$$

i.e., $f\widetilde{\otimes}(g\widetilde{\otimes}h) \circ \alpha_{a,b,c} = \alpha_{a',b',c'} \circ (f\widetilde{\otimes}g)\widetilde{\otimes}h \cdots (E')$
(E') clearly holds: In fact, the matrices representing $\alpha_{a,b,c}$ and $\alpha_{a',b',c'}$ are identity matrices. Besides, the tensor product of maps is associative. Thus, (E') holds. That is α is a natural transformation. Moreover, for all a, b and c; $\alpha_{a,b,c}$ is an equality and so, it is an isomorphism. Hence, α is a natural isomorphism.
Next, let us show that the pentagonal diagram of definition 1.9 commutes for all $a, b, c, d \in MF(1)$

$$\begin{array}{ccccc} a\widetilde{\otimes}(b\widetilde{\otimes}(c\widetilde{\otimes}d)) & & \xrightarrow{1_a\widetilde{\otimes}\alpha} & & a\widetilde{\otimes}((b\widetilde{\otimes}c)\widetilde{\otimes}d) \\ \Big\downarrow{\scriptstyle \alpha} & & & & \Big\downarrow{\scriptstyle \alpha} \\ (a\widetilde{\otimes}b)\widetilde{\otimes}(c\widetilde{\otimes}d) & & & & (a\widetilde{\otimes}(b\widetilde{\otimes}c))\widetilde{\otimes}d \\ & \searrow^{\alpha} & & \swarrow^{\alpha\widetilde{\otimes}1_d} & \\ & & ((a\widetilde{\otimes}b)\widetilde{\otimes}c)\widetilde{\otimes}d & & \end{array}$$

Since all the maps linking the vertices of the pentagon are identity maps, this diagram must commute. In fact, we know that α (the associator) is an identity map. Moreover, since the pair of matrices making up α are identity matrices, it follows from the definition 3.4 (of the multiplicative tensor product of two maps) that $1_a\widetilde{\otimes}\alpha$ and $\alpha\widetilde{\otimes}1_d$ are also identity maps.

Therefore, $\mathcal{T}$ is a semi-monoidal category.

• Next, we prove that $\mathcal{T}$ is a semi-unital semi-monoidal category. To that end, we need to find a semi-unit in the semi-monoidal category $\mathcal{T}$.
Claim: $e = (1, 1)$ is a semi-unit in $\mathcal{T}$.
From the definition of a semi-unit (cf. definition 1.12), we need to find a natural transformation $\gamma : (-) = G \to F = e\widetilde{\otimes}(-)$, where G is the identity endofunctor on $\mathcal{T}$ and F is an endofunctor on $\mathcal{T}$, such that $F(a) = e\widetilde{\otimes}a$. Components of γ are:
$\gamma_a : a \to e\widetilde{\otimes}a$, where $a = e^p = (\phi, \psi)$ is an object of $\mathcal{T}$ of size n_1, that is $\phi = I_{2^{p-1}} = \psi$ with $n_1 = 2^{p-1}$. We have: $e\widetilde{\otimes}a = \left(\begin{bmatrix} I_{2^{p-1}} & 0 \\ 0 & I_{2^{p-1}} \end{bmatrix}, \begin{bmatrix} I_{2^{p-1}} & 0 \\ 0 & I_{2^{p-1}} \end{bmatrix}\right) = (I_{2^p}, I_{2^p})$ is of size $2^p = 2n_1$.
The family of morphisms γ should satisfy the following two requirements:

1. For each $a \in Ob(\mathcal{T})$, γ_a should be a morphism in $\mathcal{T}$.
   Since a and $e\widetilde{\otimes}a$ are objects of $\mathcal{T}$ which is a one-step connected category, we let γ_a be the nonzero morphism between a and $e\widetilde{\otimes}a$ such that: $\gamma_a = (\delta',\beta') = (\delta',\delta') = ((I_{n_1}, 0)^t, (I_{n_1}, 0)^t)$, where t is the operation of taking the transpose, 0 is the zero $n_1 \times n_1$ matrix. γ_a is clearly a morphism in $\mathcal{T}$ as discussed under discussion $\sharp$.

2. Naturality of γ:
   Let $b = (\phi', \psi')$ be a matrix factorization in $\mathcal{T}$ of size n_2 and let $\mu = (\alpha_\mu,\beta_\mu) : a \to b$ be a map of matrix factorizations. It is easy to see that α_μ and β_μ are each of size $n_2 \times n_1$. The following diagram should commute:

$$\begin{array}{ccc} a & \xrightarrow{\gamma_a} & e\widetilde{\otimes}a \\ {\scriptstyle\mu}\downarrow & & \downarrow{\scriptstyle e\widetilde{\otimes}\mu} \\ b & \xrightarrow[\gamma_b]{} & e\widetilde{\otimes}b \end{array}$$

   i.e., $e\widetilde{\otimes}\mu \circ \gamma_a = \gamma_b \circ \mu \cdots (E'')$
   We know that $e\widetilde{\otimes}a$ is of size $2n_1$ since a is of size n_1. We also know that $\gamma_b = [(I_{n_2}, 0)^t, (I_{n_2}, 0)^t]$. Now, by definition of composition of two morphisms in $\mathcal{T}$, the right hand side of equality (E'') becomes:
   $\gamma_b \circ \mu = [(I_{n_2}, 0)^t, (I_{n_2}, 0)^t] \circ (\alpha_\mu,\beta_\mu) = [(I_{n_2}, 0)^t\alpha_\mu, (I_{n_2}, 0)^t\beta_\mu] = [(\alpha_\mu, 0)^t, (\beta_\mu, 0)^t] \cdots \natural'$
   0 in $[(\alpha_\mu, 0)^t, (\beta_\mu, 0)^t]$ is the $n_2 \times n_1$ zero matrix.
   As for the left hand side of (E''), first recall that $\gamma_a = [(I_{n_1}, 0)^t, (I_{n_1}, 0)^t]$, (where 0 is the zero $n_1 \times n_1$ matrix) and by definition 3.3 of the multiplicative tensor product, we know that $e\widetilde{\otimes}\mu = (1, 1)\widetilde{\otimes}(\alpha_\mu,\beta_\mu) = (\begin{bmatrix} 1 \otimes \alpha_\mu & 0 \\ 0 & 1 \otimes \alpha_\mu \end{bmatrix}, \begin{bmatrix} 1 \otimes \beta_\mu & 0 \\ 0 & 1 \otimes \beta_\mu \end{bmatrix}) =$
   $(\begin{bmatrix} \alpha_\mu & 0 \\ 0 & \alpha_\mu \end{bmatrix}, \begin{bmatrix} \beta_\mu & 0 \\ 0 & \beta_\mu \end{bmatrix})$
   So, $e\widetilde{\otimes}\mu \circ \gamma_a = (\begin{bmatrix} \alpha_\mu & 0 \\ 0 & \alpha_\mu \end{bmatrix}, \begin{bmatrix} \beta_\mu & 0 \\ 0 & \beta_\mu \end{bmatrix}) \circ [(I_{n_1}, 0)^t, (I_{n_1}, 0)^t]$

$= (\begin{bmatrix}\alpha_\mu & 0\\ 0 & \alpha_\mu\end{bmatrix}(I_{n_1},0)^t, \begin{bmatrix}\beta_\mu & 0\\ 0 & \beta_\mu\end{bmatrix}(I_{n_1},0)^t) = [(\alpha_\mu,0)^t,(\beta_\mu,0)^t] \cdots \natural\natural'.$
From $\natural'$ and $\natural\natural'$, we see that equality (E'') holds.

Hence γ is a natural transformation.

The next step towards proving that $\mathcal{T}$ is a semi-unital semi-monoidal category is to prove that there is an isomorphism of functors $e\widetilde{\otimes}(-) \cong (-)\widetilde{\otimes}e$, i.e., there is a natural isomorphism $l_a : e\widetilde{\otimes}(a) \cong (a)\widetilde{\otimes}e$ with inverse q_a, for each object a of $\mathcal{T}$ such that $l_e = q_e$ and the following diagrams are commutative for all objects a and b of $\mathcal{T}$.

$$\begin{array}{ccccc} (e\widetilde{\otimes}a)\widetilde{\otimes}b & \xrightarrow{\alpha_{e,a,b}} & e\widetilde{\otimes}(a\widetilde{\otimes}b) & \xrightarrow{l_{a\widetilde{\otimes}b}} & (a\widetilde{\otimes}b)\widetilde{\otimes}e \\ \Big\downarrow {\scriptstyle l_a\widetilde{\otimes}b} & & & & \Big\downarrow {\scriptstyle \alpha_{a,b,e}} \\ (a\widetilde{\otimes}e)\widetilde{\otimes}b & \xrightarrow{\alpha_{a,e,b}} & a\widetilde{\otimes}(e\widetilde{\otimes}b) & \xrightarrow{a\widetilde{\otimes}l_b} & a\widetilde{\otimes}(b\widetilde{\otimes}e) \end{array} \qquad (1)$$

$$\begin{array}{ccc} a\widetilde{\otimes}b & \xrightarrow{\gamma_a\widetilde{\otimes}b} & (e\widetilde{\otimes}a)\widetilde{\otimes}b \\ & {\scriptstyle \gamma_{a\widetilde{\otimes}b}}\searrow \quad \swarrow{\scriptstyle \cong} & \\ & e\widetilde{\otimes}(a\widetilde{\otimes}b) & \end{array} \qquad (2)$$

$$\begin{array}{ccc} a\widetilde{\otimes}b & \xrightarrow{a\widetilde{\otimes}\gamma_b} & a\widetilde{\otimes}(e\widetilde{\otimes}b) \\ & {\scriptstyle \gamma_{a\widetilde{\otimes}b}}\searrow \quad \swarrow{\scriptstyle \cong} & \\ & e\widetilde{\otimes}(a\widetilde{\otimes}b) & \end{array} \qquad (3)$$

Before we define l_a, observe that $e\widetilde{\otimes}(a) = (a)\widetilde{\otimes}e$.
♣ We define $l_a : e\widetilde{\otimes}(a) \to (a)\widetilde{\otimes}e$ to be the pair of matrices $(I_{2n_1}, I_{2n_1}) = (I_{2^p}, I_{2^p})$ where $a = e^p$ is of size n_1. From discussion $\sharp$, it follows that l_a is a morphism in $\mathcal{T}$.
♣ Naturality of l:

Let $b = (\phi', \psi')$ be a matrix factorization of size n_2 and let $\mu' = (\alpha_{\mu'}, \beta_{\mu'}) : a \to b$ be a map of matrix factorizations. It is easy[5] to see that $\alpha_{\mu'}$ and $\beta_{\mu'}$ are each of size $n_2 \times n_1$. The following diagram should commute:

$$\begin{array}{ccc} e\widetilde{\otimes}a & \xrightarrow{l_a} & a\widetilde{\otimes}e \\ {\scriptstyle e\widetilde{\otimes}\mu'}\Big\downarrow & & \Big\downarrow{\scriptstyle \mu'\widetilde{\otimes}e} \\ e\widetilde{\otimes}b & \xrightarrow[l_b]{} & b\widetilde{\otimes}e \end{array}$$

i.e., $\mu'\widetilde{\otimes}e \circ l_a = l_b \circ e\widetilde{\otimes}\mu' \cdots (E''')$
Since l_a and l_b are just pairs of identity matrices, it suffices to show that $\mu'\widetilde{\otimes}e = e\widetilde{\otimes}\mu'$.
By definition 3.3 of the multiplicative tensor product, we know that $e\widetilde{\otimes}\mu' = (1,1)\widetilde{\otimes}(\alpha_{\mu'}, \beta_{\mu'}) =$

[5]By drawing the twin diagram that has to commute with $(\alpha_{\mu'}, \beta_{\mu'})$, we see the sizes of $\alpha_{\mu'}$ and $\beta_{\mu'}$.

$(\begin{bmatrix} 1\otimes\alpha_{\mu'} & 0 \\ 0 & 1\otimes\alpha_{\mu'} \end{bmatrix}, \begin{bmatrix} 1\otimes\beta_{\mu'} & 0 \\ 0 & 1\otimes\beta_{\mu'} \end{bmatrix}) = (\begin{bmatrix} \alpha_{\mu'} & 0 \\ 0 & \alpha_{\mu'} \end{bmatrix}, \begin{bmatrix} \beta_{\mu'} & 0 \\ 0 & \beta_{\mu'} \end{bmatrix})$
And we also have by definition 3.2 of the multiplicative tensor product, that
$\mu'\widetilde{\otimes}e = (\alpha_{\mu'}, \beta_{\mu'})\widetilde{\otimes}(1,1) = (\begin{bmatrix} \alpha_{\mu'}\otimes 1 & 0 \\ 0 & \alpha_{\mu'}\otimes 1 \end{bmatrix}, \begin{bmatrix} \beta_{\mu'}\otimes 1 & 0 \\ 0 & \beta_{\mu'}\otimes 1 \end{bmatrix}) = (\begin{bmatrix} \alpha_{\mu'} & 0 \\ 0 & \alpha_{\mu'} \end{bmatrix}, \begin{bmatrix} \beta_{\mu'} & 0 \\ 0 & \beta_{\mu'} \end{bmatrix})$.
Thus $e\widetilde{\otimes}\mu' = \mu'\widetilde{\otimes}e$, so l is a natural transformation.
♣ $l_a : e\widetilde{\otimes}(a) \cong (a)\widetilde{\otimes}e$ is a natural isomorphism. In fact, $l : e\widetilde{\otimes}(a) = (a)\widetilde{\otimes}e$ and its inverse $q : (a)\widetilde{\otimes}e = e\widetilde{\otimes}(a)$ is clearly such that $l_e = q_e$.

Commutativity of diagrams (1), (2) *and* (3)**:**
For diagram (1), it would commute if $\alpha_{a,b,e} \circ l_{a\widetilde{\otimes}b} \circ \alpha_{e,a,b} = a\widetilde{\otimes}l_b \circ \alpha_{a,e,b} \circ l_a\widetilde{\otimes}b \quad \cdots (\flat)$.
We show that all the maps involved in equality ($\flat$) are identities. For all objects x, y, z *in* $Ob(\mathcal{T})$, we clearly have by the definitions of $\alpha_{x,y,z}$ and l_x that they are identity maps. We now show that the other maps involved in diagram (1) are identity maps.
Since a is of size n_1, we have $l_a = (I_{2n_1}, I_{2n_1})$. Let $b = (\phi', \psi')$ be of size n_2. By definition 3.2, $l_a\widetilde{\otimes}b = (I_{2n_1}, I_{2n_1})\widetilde{\otimes}(\phi', \psi') = (\begin{bmatrix} I_{2n_1}\otimes I_{n_2} & 0 \\ 0 & I_{2n_1}\otimes I_{n_2} \end{bmatrix}, \begin{bmatrix} I_{2n_1}\otimes I_{n_2} & 0 \\ 0 & I_{2n_1}\otimes I_{n_2} \end{bmatrix}) =$
$(\begin{bmatrix} I_{2n_1n_2} & 0 \\ 0 & I_{2n_1n_2} \end{bmatrix}, \begin{bmatrix} I_{2n_1n_2} & 0 \\ 0 & I_{2n_1n_2} \end{bmatrix}) = (I_{4n_1n_2}, I_{4n_1n_2}) \cdots \ddagger$
Hence $l_a\widetilde{\otimes}b$ is an identity map as expected.

Similarly using definition 3.3, we prove that $a\widetilde{\otimes}l_b$ is an identity map. Let $a = (\phi, \psi)$ be of size n_1 and b be as above. Then, $a\widetilde{\otimes}l_b = (\phi, \psi)\widetilde{\otimes}(I_{2n_2}, I_{2n_2})$
$= (\begin{bmatrix} I_{n_1}\otimes I_{2n_2} & 0 \\ 0 & I_{n_1}\otimes I_{2n_2} \end{bmatrix}, \begin{bmatrix} I_{n_1}\otimes I_{2n_2} & 0 \\ 0 & I_{n_1}\otimes I_{2n_2} \end{bmatrix}) = (\begin{bmatrix} I_{2n_1n_2} & 0 \\ 0 & I_{2n_1n_2} \end{bmatrix}, \begin{bmatrix} I_{2n_1n_2} & 0 \\ 0 & I_{2n_1n_2} \end{bmatrix}) = (I_{4n_1n_2}, I_{4n_1n_2}) \cdots \ddagger'$

$\ddagger$ and $\ddagger'$ show that $a\widetilde{\otimes}l_b = l_a\widetilde{\otimes}b$.
It is easy to see that all the other maps involved in diagram (1) are equal to $(I_{4n_1n_2}, I_{4n_1n_2})$.
So diagram (1) is commutative.

Next, we show that diagram (2) commutes. To this end, we need to find an isomorphism $\zeta : (e\widetilde{\otimes}a)\widetilde{\otimes}b \to e\widetilde{\otimes}(a\widetilde{\otimes}b)$, such that $\zeta \circ \gamma_a\widetilde{\otimes}b = \gamma_{a\widetilde{\otimes}b}$.
Now, we know that $\gamma_a = [(I_{n_1}, 0)^t, (I_{n_1}, 0)^t]$, where 0 is the $n_1 \times n_1$ zero matrix and $b = (\phi', \psi')$ is of size n_2. Hence by definition 3.2 $\gamma_a\widetilde{\otimes}b = [(I_{n_1}, 0)^t, (I_{n_1}, 0)^t]\widetilde{\otimes}(\phi', \psi') =$
$(\begin{bmatrix} (I_{n_1}, 0_{n_1,n_1})^t\otimes I_{n_2} & 0_{2n_1n_2,n_1n_2} \\ 0_{2n_1n_2,n_1n_2} & (I_{n_1}, 0_{n_1,n_1})^t\otimes I_{n_2} \end{bmatrix}, \begin{bmatrix} (I_{n_1}, 0_{n_1,n_1})^t\otimes I_{n_2} & 0_{2n_1n_2,n_1n_2} \\ 0_{2n_1n_2,n_1n_2} & (I_{n_1}, 0_{n_1,n_1})^t\otimes I_{n_2} \end{bmatrix}) =$
$(\begin{bmatrix} (I_{n_1n_2}, 0_{n_1n_2})^t & 0_{2n_1n_2,n_1n_2} \\ 0_{2n_1n_2,n_1n_2} & (I_{n_1n_2}, 0_{n_1n_2})^t \end{bmatrix}, \begin{bmatrix} (I_{n_1n_2}, 0_{n_1n_2})^t & 0_{2n_1n_2,n_1n_2} \\ 0_{2n_1n_2,n_1n_2} & (I_{n_1n_2}, 0_{n_1n_2})^t \end{bmatrix}). \cdots \dagger$
Next, since $a\widetilde{\otimes}b$ is an object of size $2n_1n_2$, we obtain from the way γ is defined that $\gamma_{a\widetilde{\otimes}b} = ((I_{2n_1n_2}, 0_{2n_1n_2})^t, (I_{2n_1n_2}, 0_{2n_1n_2})^t) \cdots \dagger'$
From $\dagger$ and $\dagger'$, we see that $\gamma_{a\widetilde{\otimes}b}$ and $\gamma_a\widetilde{\otimes}b$ are both $(1,0)-$matrices with the same number of rows and columns. Moreover, they have the same number of 1s and each of these 1s is the only nonzero entry in its row and in its column. Simply put, the matrix from $\gamma_{a\widetilde{\otimes}b}$ is (row-)permutation equivalent to the matrix in $\gamma_a\widetilde{\otimes}b$. That is the rows have simply been interchanged.
Hence, from section 1.4, there exists a $4n_1n_2 \times 4n_1n_2$ permutation matrix P such that

$P\begin{bmatrix} (I_{n_1n_2}, 0_{n_1n_2})^t & 0_{2n_1n_2,n_1n_2} \\ 0_{2n_1n_2,n_1n_2} & (I_{n_1n_2}, 0_{n_1n_2})^t \end{bmatrix} = (I_{2n_1n_2}, 0_{2n_1n_2})^t.$

Still from section 1.4, we know that P is invertible and its inverse is P^t.

Now that we know P exists and is invertible, we need to check if the pair of matrices (P, P) is a map from $(e\widetilde{\otimes}a)\widetilde{\otimes}b$ to $e\widetilde{\otimes}(a\widetilde{\otimes}b)$ in $\mathcal{T}$ and if the pair of matrices (P^{-1}, P^{-1}) is a map from $e\widetilde{\otimes}(a\widetilde{\otimes}b)$ to $(e\widetilde{\otimes}a)\widetilde{\otimes}b$.

We do it for (P, P), the case of (P^{-1}, P^{-1}) is completely similar.

Once more, a and b are objects of $\mathcal{T}$. So, we can let $a = e^p$ and $b = e^m$. Observe that $e\widetilde{\otimes}(a\widetilde{\otimes}b) = (e\widetilde{\otimes}a)\widetilde{\otimes}b = (e\widetilde{\otimes}e^p)\widetilde{\otimes}e^m = e^{1+p+m} = (I_{2^{1+p+m-1}}, I_{2^{1+p+m-1}}) = (I_{2^{p+m}}, I_{2^{p+m}})$. Let $r = 2^{p+m}$.

All we need check now to conclude that (P, P) is a map in $\mathcal{T}$ is that the following diagram commutes:

$$\begin{array}{ccccc} K[[x]]^r & \xrightarrow{I_r} & K[[x]]^r & \xrightarrow{I_r} & K[[x]]^r \\ \downarrow P & & \downarrow P & & \downarrow P \\ K[[x]]^r & \xrightarrow{I_r} & K[[x]]^r & \xrightarrow{I_r} & K[[x]]^r \end{array} \qquad (\star\star)$$

Now, this diagram clearly commutes, so we can take $\zeta := (P, P)$ and $\zeta^{-1} := (P^{-1}, P^{-1}) = (P^t, P^t)$.

Therefore there exists an isomorphism namely ζ such that diagram (2) commutes.

A small remark: The foregoing proof for the commutativity of diagram (2) helps understand the motivation behind the choice of the objects of $\mathcal{T}$.

In fact, if objects were chosen arbitrarily say pairs of matrices (M, M^{-1}), as we showed in remark 3.3, the twin diagram $(\star\star)$ above will commute only if $PM = MP$. But as explained in remark 3.3, this is not possible as on the left side of the equality, the rows of M are permuted and on the right side the columns are permuted, since P is a permutation matrix.

Moreover, though diagram $(\star\star)$ commutes even if P is replaced with any matrix, what we need is a matrix that will make diagram (2) commute and that matrix should also be invertible because we need an isomorphism in diagram (2).

The commutativity of diagram (3) is proved in a manner similar to the proof given for the commutativity of diagram (2).

So e is a semi-unit in $(\mathcal{T}, \widetilde{\otimes})$.

Conclusion: $(\mathcal{T}, \widetilde{\otimes})$ is a one-step connected semi-unital semi-monoidal subcategory of $MF(1)$. □

The above proof works well for $\mathcal{T}$ because the objects of $\mathcal{T}$ are judiciously chosen so that the pair of matrices that make an object in $\mathcal{T}$ is not any kind of matrix and its inverse (in order for the product to yield 1 times the identity matrix of the right size), but they are identity matrices thanks to which diagrams will be commutative. In fact, diagrams (2) and (3) in definition 1.12, commute when a and b are objects in $\mathcal{T}$, i.e., of the form e^n for some $n \in \mathbb{N} - \{0\}$. But we will see in remark 3.3, that for arbitrary values of a and b, diagrams (2) and (3) are not always commutative. This implies that $(MF(1), \widetilde{\otimes})$ is not a semi-unital semi-monoidal category.

Remark 3.3. *We now explain why $(MF(1), \widetilde{\otimes})$ is not a semi-unital semi-monoidal category.*

1. *We explain that for $a \in Ob(MF(1))$ of size n_1, the only reasonable (nonzero) possible choice for $\gamma_a : a \to e\widetilde{\otimes}a$ is what we made for the subcategory $\mathcal{T}$, namely $\gamma_a = (\delta', \beta') = ((I_{n_1}, 0)^t, (I_{n_1}, 0)^t)$.*
   First of all, observe that considering the definition of morphisms in $\mathcal{T}$ (i.e., pairs of $(1,0)-$matrices s.t. each column and each row has at most one nonzero entry), the only possible choice for γ_a in $\mathcal{T}$ is the one we made above (cf. theorem 3.2), i.e., $\gamma_a = ((I_{n_1}, 0)^t, (I_{n_1}, 0)^t)$.
   It is clear that the only candidate to be a semi-unit in $\mathcal{T}$ was $e = (1, 1)$. Hence, it is also the only candidate for $(MF(1), \widetilde{\otimes})$ to be semi-unital. This entails that for a in $Ob(MF(1))$, the only possible way to define γ_a is $\gamma_a = ((I_{n_1}, 0)^t, (I_{n_1}, 0)^t)$. Otherwise, $e = (1, 1)$ would no more be a semi-unit in $\mathcal{T}$.

2. *Next, we prove that with this choice of $\gamma_a = ((I_{n_1}, 0)^t, (I_{n_1}, 0)^t)$, the diagram (2) above does not commute in general (i.e., for arbitrary values of a and b in $Ob(MF(1))$). That is,*

$$\begin{array}{ccc} a\widetilde{\otimes}b & \xrightarrow{\gamma_a\widetilde{\otimes}b} & (e\widetilde{\otimes}a)\widetilde{\otimes}b \\ & {\scriptstyle\gamma_{a\widetilde{\otimes}b}}\searrow \quad \swarrow{\scriptstyle\cong} & \quad (2) \\ & e\widetilde{\otimes}(a\widetilde{\otimes}b) & \end{array}$$

   does not commute. In fact, we showed in the proof of theorem 3.2 that with $\gamma = ((I_{n_1}, 0)^t, (I_{n_1}, 0)^t)$, the matrix constituting the map $\gamma_{a\widetilde{\otimes}b}$ is permutation equivalent to the matrix constituting the map $\gamma_a\widetilde{\otimes}b$. Hence, in order to find the desired isomorphism of diagram (2), all we need do is to find a permutation matrix as explained in the proof of theorem 3.2. Now, the catch is that we need to verify that this permutation matrix is actually the matrix of a map $e\widetilde{\otimes}(a\widetilde{\otimes}b) \to (e\widetilde{\otimes}a)\widetilde{\otimes}b$ in $MF(1)$. It turns out that it is not.
   Suppose we have already found the permutation matrix that enables us to move from the matrix of $\gamma_{a\widetilde{\otimes}b}$ to the matrix of $\gamma_a\widetilde{\otimes}b$, call it P'. Now by definition of $\widetilde{\otimes}$, we have $e\widetilde{\otimes}(a\widetilde{\otimes}b) = (e\widetilde{\otimes}a)\widetilde{\otimes}b$ which is an object of $MF(1)$, so there is a matrix M such that $e\widetilde{\otimes}(a\widetilde{\otimes}b) = (e\widetilde{\otimes}a)\widetilde{\otimes}b = (M, M^{-1})$. Our aim is to show that $(P', P') : e\widetilde{\otimes}(a\widetilde{\otimes}b) \to (e\widetilde{\otimes}a)\widetilde{\otimes}b$ is not a map in $MF(1)$ for arbitrary values of a and b, because the following diagram cannot commute:

$$\begin{array}{ccccc} K[[x]]^{n_1} & \xrightarrow{M^{-1}} & K[[x]]^{n_1} & \xrightarrow{M} & K[[x]]^{n_1} \\ \downarrow{\scriptstyle P'} & & \downarrow{\scriptstyle P'} & & \downarrow{\scriptstyle P'} \\ K[[x]]^{2n_1} & \xrightarrow{M^{-1}} & K[[x]]^{2n_1} & \xrightarrow{M} & K[[x]]^{2n_1} \end{array}$$

   For this diagram to commute, we need to have (from the second square) $P'M = MP'$. Now, as explained in section 1.4, $P'M$ is the matrix obtained from M by permuting the rows according to the permutation P' and MP' is the matrix obtained from M by permuting the columns according to the permutation P'. So, $P'M = MP'$

will be true just in case M is the identity matrix. Now, M is not necessarily the identity matrix, for instance if we take $a = (\begin{bmatrix} 4 & 3 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & -3 \\ -1 & 4 \end{bmatrix})$ *and* $b = (\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix})$

then $M = (1 \widetilde{\otimes} \begin{bmatrix} 4 & 3 \\ 1 & 1 \end{bmatrix}) \widetilde{\otimes} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ *is clearly not equal to the identity matrix.*

Remark 3.4. $(\mathcal{T}, \widetilde{\otimes})$ *is not a monoidal category because it has no unit. In fact, the only candidate to be a unit is e. Now, in order to be a unit, e needs to first of all be a pseudo-idempotent (cf. remark 1.3). But* $e = (1, 1)$ *is not even a pseudo-idempotent. We have* $e^2 = (1, 1)\widetilde{\otimes}(1, 1) = (I_2, I_2)$.
Let $\zeta_1 = (\delta_1, \beta_1) : e \to e^2$ *and* $\zeta_2 = (\delta_2, \beta_2) : e^2 \to e$. *Consider the following situation:*

$$\begin{array}{ccccc} K[[x]] & \xrightarrow{1} & K[[x]] & \xrightarrow{1} & K[[x]] \\ \downarrow{\scriptstyle \delta_1} & & \downarrow{\scriptstyle \beta_1} & & \downarrow{\scriptstyle \delta_1} \\ K[[x]]^2 & \xrightarrow{I_2} & K[[x]]^2 & \xrightarrow{I_2} & K[[x]]^2 \\ \downarrow{\scriptstyle \delta_2} & & \downarrow{\scriptstyle \beta_2} & & \downarrow{\scriptstyle \delta_2} \\ K[[x]] & \xrightarrow{1} & K[[x]] & \xrightarrow{1} & K[[x]] \end{array}$$

From the discussion we had about the choice of matrices constituting γ_a in remark 3.3, we have only one (nonzero) choice for ζ_1; namely $\zeta_1 = ((1,0)^t, (1,0)^t)$ and similarly we have only one way of defining ζ_2; $\zeta_2 = ((1,0),(1,0)) : e^2 \to e$. Hence, we clearly obtain $\zeta_2 \circ \zeta_1 = ((1,0)(1,0)^t, (1,0)(1,0)^t) = (1,1) = id_e$. Hence, $\zeta_2 \circ \zeta_1 = id_e$.
But, we do not obtain $\zeta_1 \circ \zeta_2 = id_{e^2}$.
In fact, $\zeta_1 \circ \zeta_2 = ((1,0)^t(1,0), (1,0)^t(1,0)) = (\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}) \neq (\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}) = id_{e^2}$

Therefore, there is no isomorphism between e and e^2. A consequence of remark 3.4 is that $(MF(1), \widetilde{\otimes})$ is not a monoidal category since the only candidate to be a unit, namely e is not even a pseudo-idempotent.

Remark 3.5. *Moreover, $(\mathcal{T}, \widetilde{\otimes})$ is not a right-monoidal category (cf. definition 1.13) because when trying to verify if the axioms of definition 1.13 hold for $\mathcal{T}$, instead of equalities we obtain maps which are not equal but whose representing matrices are row-permutation equivalent. Let us for example illustrate what we mean with the second axiom (cf. definition 1.13):*

$$\alpha_{e,a,b} \circ \gamma_{a\widetilde{\otimes}b} = \gamma_a \widetilde{\otimes} b \qquad \cdots \qquad (Ax.2)$$

where a, b are in $Ob(\mathcal{T})$, α is the associator and γ is the natural transformation defined in the proof of theorem 3.2. If a and b are respectively of sizes m and n, then by definition of γ, $\gamma_{a\widetilde{\otimes}b} = ((I_{2mn}, 0)^t, (I_{2mn}, 0)^t)$ and since $(e\widetilde{\otimes}a)\widetilde{\otimes}b = e\widetilde{\otimes}(a\widetilde{\otimes}b)$, $\alpha_{e,a,b} = (I_{4mn}, I_{4mn})$ and so the left hand side of $(Ax.2)$ becomes $\alpha_{e,a,b} \circ \gamma_{a\widetilde{\otimes}b} = (I_{4mn}(I_{2mn}, 0)^t, I_{4mn}(I_{2mn}, 0)^t) = ((I_{2mn}, 0)^t, (I_{2mn}, 0)^t) \quad \cdots (i)$.
Next, by definition 3.2, we compute the right hand side of $(Ax.2)$ as follows: $\gamma_a \widetilde{\otimes} b = ((I_m, 0)^t, (I_m, 0)^t)\widetilde{\otimes} b = (\begin{bmatrix} (I_m,0)^t \otimes I_n & 0 \\ 0 & (I_m,0)^t \otimes I_n \end{bmatrix}\begin{bmatrix} (I_m,0)^t \otimes I_n & 0 \\ 0 & (I_m,0)^t \otimes I_n \end{bmatrix})$
$= (\begin{bmatrix} (I_{mn},0)^t & 0 \\ 0 & (I_{mn},0)^t \end{bmatrix}\begin{bmatrix} (I_{mn},0)^t & 0 \\ 0 & (I_{mn},0)^t \end{bmatrix}) \quad \cdots (ii)$

The matrices we obtained in (i) and (ii) are row-permutation equivalent but not equal. This proves that $(Ax.2)$ does not hold in $(\mathcal{T}, \widetilde{\otimes})$, so it is not a right-monoidal category. A direct consequence of this result is that $(MF(1), \widetilde{\otimes})$ is not also a right-monoidal category.

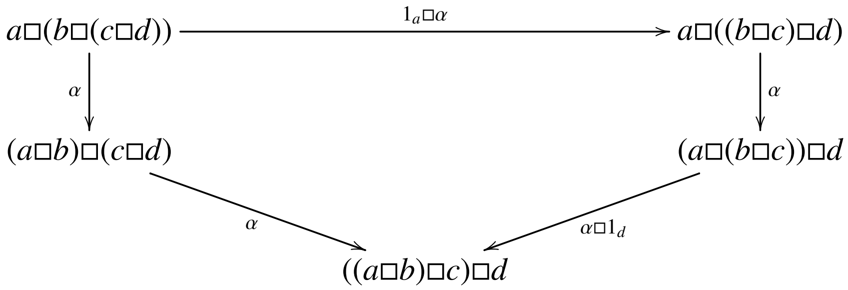
Nevertheless, $(MF(1), \widetilde{\otimes})$ is still a category which is close to being a monoidal category as we shall see (cf. subsection 3.3.3).

3.3.3 Another application of $\widetilde{\otimes}$: The category $(MF(1), \widetilde{\otimes})$ is a right pseudo-monoidal category

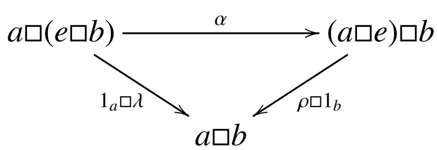
In this section, we first define what a *right pseudo-monoidal category* is. We observe that this notion is a generalization of the notion of monoidal category. We exploit the results obtained in the previous sections of this chapter to show that the category $MF(1)$ is a *right pseudo-monoidal category*.

First recall that a semi-monoidal category definition 1.9 is one endowed with a bifunctor and a natural isomorphism (called the associator [17]) such that the pentagon diagram (cf. definition 1.9) commutes.

Definition 3.8. *A* ***right pseudo-monoidal category*** $C =< C, \square, e, \alpha, \lambda, \rho >$ *is a category* C *which possesses a distinguished element* e*, a natural isomorphism* α *and two natural retractions* λ *and* ρ *s.t. the following hold:*
- *There exists a morphism* $\zeta : e^2 \to e$ *s.t.* ζ *has a right inverse.*
- α *is a natural isomorphism with components* $\alpha_{a,b,c} : (a \otimes b) \otimes c \to a \otimes (b \otimes c)$ *such that the following pentagonal diagram*



commutes for all $a, b, c, d \in C$.
- $\lambda : e\square(-) \to (-)$, $\rho : (-)\square e \to (-)$ *are natural*[6] *transformations.*
- *For all objects* $a \in C$, $\lambda_a : e\square a \to a$, $\rho_a : a\square e \to a$ *have right inverses but do not necessarily have left inverses.*
- $\lambda_e = \rho_e : e\square e \to e$.
- *For* $a = e$ *and for any object* $b \in C$*, the triangular diagram*



commutes.

Remark 3.6. *It is easy to see that every monoidal category (cf. definition 1.7) is a right pseudo-monoidal category. In fact, in the foregoing definition, if the triangular diagram commutes for all* $a \in C$*; and the maps* λ *and* ρ *are invertible, then we will recover the definition of a monoidal category. This shows that this notion is a generalization of the classical notion of monoidal category.*

Theorem 3.3. *The category* $(MF(1), \widetilde{\otimes})$ *is a right pseudo-monoidal category.*

[6]On p.10 of [17], λ and ρ are also called left and right unit actions (or unitors). The difference here is that in the definition of a right pseudo-monoidal category, we do not require these unitors to be natural isomorphisms but it is enough for them to have right inverses.

Proof. Following definition 3.8, we need to first of all show that $(MF(1), \widetilde{\otimes})$ is semi-monoidal (cf. definition 1.9). Thus we need to show that $\widetilde{\otimes}$ is a bifunctor, and the associator "α" in $(MF(1), \widetilde{\otimes})$ is a natural isomorphism such that the pentagon (cf. definition 3.8) diagram commutes.
Recall that an object of $MF(1)$ is of the form (M, N) where $M = N^{-1}$.
In the entire proof; $a, b\, and\, c$ stand for arbitrary objects of $MF(1)$, say $a = e^p = (\phi, \psi)$, $b = e^m = (\phi', \psi')$ and $c = e^r = (\phi'', \psi'')$.
• We proved in the first section of this chapter that $\widetilde{\otimes}$ is a bifunctor.
• We proved in proposition 3.3 that $\alpha = \alpha_{a,b,c} : a\widetilde{\otimes}(b\widetilde{\otimes}c) \xrightarrow{=} (a\widetilde{\otimes}b)\widetilde{\otimes}c$ is an identity map and hence it is an isomorphism. It is also easy to see that α is natural for all $a, b, c \in MF(1)$ and that the above pentagonal diagram commutes, in fact; we actually already proved it above when proving that $\mathcal{T}$ was a semi-monoidal category (cf. theorem 3.2) .

This shows that $(MF(1), \widetilde{\otimes})$ is a semi-monoidal category.

Next, we find the distinguished object "e" and the morphism $\zeta : e^2 \to e$ s.t. ζ has a right inverse.
• Take $e = (1, 1)$, the pair of 1×1 matrix factorization. We have $e^2 = (1, 1)\widetilde{\otimes}(1, 1) = (I_2, I_2)$. Consider the following situation:

$$\begin{array}{ccccc}
K[[x]] & \xrightarrow{1} & K[[x]] & \xrightarrow{1} & K[[x]] \\
\downarrow{\scriptstyle \delta_1} & & \downarrow{\scriptstyle \beta_1} & & \downarrow{\scriptstyle \delta_1} \\
K[[x]]^2 & \xrightarrow{I_2} & K[[x]]^2 & \xrightarrow{I_2} & K[[x]]^2 \\
\downarrow{\scriptstyle \delta_2} & & \downarrow{\scriptstyle \beta_2} & & \downarrow{\scriptstyle \delta_2} \\
K[[x]] & \xrightarrow{1} & K[[x]] & \xrightarrow{1} & K[[x]]
\end{array}$$

In order to find the map ζ and its right inverse, it suffices to take: $\zeta = (\delta_2 = (1, 0), \beta_2 = (1, 0)) : e^2 \to e$, let $\zeta' = (\delta_1 = (1, 0)^t, \beta_1 = (1, 0)^t) : e \to e^2$. Thus, $\zeta \circ \zeta' : e \to e$. Hence, we clearly obtain $\zeta \circ \zeta' = ((1, 0)(1, 0)^t, (1, 0)(1, 0)^t) = (1, 1) = id_e$ which proves that ζ' is a right inverse to ζ.
• We now show that the maps λ and ρ should be natural retractions satisfying $\lambda_e = \rho_e$. That is, for each $a\, in\, Ob(MF(1))$, λ_a and ρ_a have right inverses and $\lambda_e = \rho_e$.
λ is a natural transformation:
$\lambda : F = e\widetilde{\otimes}(-) \to (-) = G$ where G is the identity endofunctor on $MF(1)$ and F is an endofunctor[7] on $MF(1)$, such that $F(a) = e\widetilde{\otimes}a$.
The family of morphisms λ should satisfy the following two requirements:

1. For each $a\, in\, Ob(MF(1))$, λ_a should be a morphism between objects in $MF(1)$. Before we proceed, observe that, for any $a = (\phi, \psi)$ of size n_1 in $MF(1)$,

$$(1, 1)\widetilde{\otimes}(\phi, \psi) = (\begin{bmatrix} 1 \otimes \phi & 0 \\ 0 & 1 \otimes \phi \end{bmatrix}, \begin{bmatrix} 1 \otimes \psi & 0 \\ 0 & 1 \otimes \psi \end{bmatrix}) = (\begin{bmatrix} \phi & 0 \\ 0 & \phi \end{bmatrix}, \begin{bmatrix} \psi & 0 \\ 0 & \psi \end{bmatrix})$$

[7]Recall that we showed in lemma 3.2 that F was a functor.

To show that $\lambda_a : e\widetilde{\otimes}a \to a$ is a morphism, we need to find a pair of matrices (δ,β) such that the following diagram commute:

$$\begin{array}{ccccc} K[[x]]^{2n_1} & \xrightarrow{\begin{bmatrix}\psi & 0\\ 0 & \psi\end{bmatrix}} & K[[x]]^{2n_1} & \xrightarrow{\begin{bmatrix}\phi & 0\\ 0 & \phi\end{bmatrix}} & K[[x]]^{2n_1} \\ \Big\downarrow{\scriptstyle\delta} & & \Big\downarrow{\scriptstyle\beta} & & \Big\downarrow{\scriptstyle\delta} \\ K[[x]]^{n_1} & \xrightarrow{\psi} & K[[x]]^{n_1} & \xrightarrow{\phi} & K[[x]]^{n_1} \end{array} \qquad \star'$$

That is,

$$*\begin{cases} \delta\begin{bmatrix}\phi & 0\\ 0 & \phi\end{bmatrix} = \phi\beta \\ \psi\delta = \beta\begin{bmatrix}\psi & 0\\ 0 & \psi\end{bmatrix} \end{cases}$$

For $\delta = \beta = (I_{n_1}, 0)$, where 0 is the zero $n_1 \times n_1$ matrix, the equational system $*$ becomes

$$\begin{cases} (I_{n_1}, 0)\begin{bmatrix}\phi & 0\\ 0 & \phi\end{bmatrix} = \phi(I_{n_1}, 0) \\ \psi(I_{n_1}, 0) = (I_{n_1}, 0)\begin{bmatrix}\psi & 0\\ 0 & \psi\end{bmatrix} \end{cases}$$

That is;

$$\begin{cases} (\phi, 0) = (\phi, 0) \\ (\psi, 0) = (\psi, 0) \end{cases}$$

and this is clearly true. Therefore, we have found a pair of matrices (δ,β) such that diagram $\star'$ commutes, and this means that λ_a is a map of matrix factorizations.

2. Naturality of λ:
Let $b = (\phi', \psi')$ be a matrix factorization of size n_2 and let $\nu = (\alpha_\nu, \beta_\nu) : a \to b$ be a map of matrix factorizations. It is easy[8] to see that α_ν and β_ν are each of size $n_2 \times n_1$. The following diagram should commute:

$$\begin{array}{ccc} e\widetilde{\otimes}a & \xrightarrow{\lambda_a} & a \\ {\scriptstyle e\widetilde{\otimes}\nu}\Big\downarrow & & \Big\downarrow{\scriptstyle\nu} \\ e\widetilde{\otimes}b & \xrightarrow[\lambda_b]{} & b \end{array}$$

i.e., $\nu \circ \lambda_a = \lambda_b \circ e\widetilde{\otimes}\nu \cdots (E)$
We know that $e\widetilde{\otimes}a$ is of size $2n_1$ since a is of size n_1. We also know that $\lambda_a = [(I_{n_1}, 0), (I_{n_1}, 0)]$. Now by definition of composition of two morphisms in $MF(1)$, the left hand side of equality (E) becomes:
$\nu \circ \lambda_a = (\alpha_\nu, \beta_\nu) \circ [(I_{n_1}, 0), (I_{n_1}, 0)] = [\alpha_\nu(I_{n_1}, 0), \beta_\nu(I_{n_1}, 0)] = [(\alpha_\nu, 0), (\beta_\nu, 0)] \cdots \natural$

[8]By drawing the twin diagram that has to commute with (α_ν, β_ν), we see the sizes of α_ν and β_ν.

0 in $[(\alpha_v, 0), (\beta_v, 0)]$ is the $n_2 \times n_1$ zero matrix.
As for the right hand side of (E), first recall that $\lambda_b = [(I_{n_2}, 0), (I_{n_2}, 0)]$, (where 0 is the zero $n_2 \times n_2$ matrix) and by definition 3.3 of the multiplicative tensor product, we know that $e\widetilde{\otimes}v = (1, 1)\widetilde{\otimes}(\alpha_v, \beta_v) = (\begin{bmatrix} 1 \otimes \alpha_v & 0 \\ 0 & 1 \otimes \alpha_v \end{bmatrix}, \begin{bmatrix} 1 \otimes \beta_v & 0 \\ 0 & 1 \otimes \beta_v \end{bmatrix}) = (\begin{bmatrix} \alpha_v & 0 \\ 0 & \alpha_v \end{bmatrix}, \begin{bmatrix} \beta_v & 0 \\ 0 & \beta_v \end{bmatrix})$

So, $\lambda_b \circ e\widetilde{\otimes}v = [(I_{n_2}, 0), (I_{n_2}, 0)] \circ (\begin{bmatrix} \alpha_v & 0 \\ 0 & \alpha_v \end{bmatrix}, \begin{bmatrix} \beta_v & 0 \\ 0 & \beta_v \end{bmatrix})$
$= ((I_{n_2}, 0)\begin{bmatrix} \alpha_v & 0 \\ 0 & \alpha_v \end{bmatrix}, (I_{n_2}, 0)\begin{bmatrix} \beta_v & 0 \\ 0 & \beta_v \end{bmatrix}) = [(\alpha_v, 0), (\beta_v, 0)] \cdots \natural\natural.$
From $\natural$ and $\natural\natural$, we see that equality (E) holds. That is λ is a natural transformation.

• We find the right inverse of λ_a, for any $a = (\phi, \psi)$ of size n_1 in $MF(1)$. we denote it $\gamma_a : a \to e\widetilde{\otimes}a$. γ_a should be a member of the family of morphisms of a natural transformation $\gamma : (-) = G \to F = e\widetilde{\otimes}(-)$, where G is the identity endofunctor on $MF(1)$ and F is an endofunctor[9] on $MF(1)$, such that $F(a) = e\widetilde{\otimes}a$.
The family of morphisms γ should satisfy the following two requirements:

1. For each *a in Ob*$(MF(1))$, γ_a should be a morphism in $MF(1)$.
γ_a should be a pair of matrices (δ', β') such that the following diagram commutes:

$$\begin{array}{ccccc} K[[x]]^{n_1} & \xrightarrow{\psi} & K[[x]]^{n_1} & \xrightarrow{\phi} & K[[x]]^{n_1} \\ \downarrow{\scriptstyle \delta'} & & \downarrow{\scriptstyle \beta'} & & \downarrow{\scriptstyle \delta'} \\ K[[x]]^{2n_1} & \xrightarrow{\begin{bmatrix} \psi & 0 \\ 0 & \psi \end{bmatrix}} & K[[x]]^{2n_1} & \xrightarrow{\begin{bmatrix} \phi & 0 \\ 0 & \phi \end{bmatrix}} & K[[x]]^{2n_1} \end{array} \quad \star"$$

That is,

$$*' \begin{cases} \delta'\phi = \begin{bmatrix} \phi & 0 \\ 0 & \phi \end{bmatrix}\beta' \\ \begin{bmatrix} \psi & 0 \\ 0 & \psi \end{bmatrix}\delta' = \beta'\psi \end{cases}$$

For $\delta' = \beta' = (I_{n_1}, 0)^t$, where t is the operation of taking the transpose, 0 is the zero $n_1 \times n_1$ matrix, the equational system $*'$ becomes

$$\begin{cases} (I_{n_1}, 0)^t\phi = \begin{bmatrix} \phi & 0 \\ 0 & \phi \end{bmatrix}(I_{n_1}, 0)^t \\ \begin{bmatrix} \psi & 0 \\ 0 & \psi \end{bmatrix}(I_{n_1}, 0)^t = (I_{n_1}, 0)^t\psi \end{cases}$$

That is;

$$\begin{cases} (\phi, 0)^t = (\phi, 0)^t \\ (\psi, 0)^t = (\psi, 0)^t \end{cases}$$

and this is clearly true. Therefore, we have found a pair of matrices (δ', β') such that diagram $\star$" commutes, and this means that γ_a is a map of matrix factorizations.

[9]this was discussed above when dealing with λ

2. Naturality of γ:
Let $b = (\phi', \psi')$ be a matrix factorization of size n_2 and let $\mu = (\alpha_\mu, \beta_\mu) : a \to b$ be a map of matrix factorizations. It is easy[10] to see that α_μ and β_μ are each of size $n_2 \times n_1$. The following diagram should commute:

$$\begin{array}{ccc} a & \xrightarrow{\gamma_a} & e\widetilde{\otimes}a \\ {\scriptstyle \mu}\downarrow & & \downarrow{\scriptstyle e\widetilde{\otimes}\mu} \\ b & \xrightarrow[\gamma_b]{} & e\widetilde{\otimes}b \end{array}$$

i.e., $e\widetilde{\otimes}\mu \circ \gamma_a = \gamma_b \circ \mu \cdots (E')$
We know that $e\widetilde{\otimes}a$ is of size $2n_1$ since a is of size n_1. We also know that $\gamma_b = [(I_{n_2}, 0)^t, (I_{n_2}, 0)^t]$. Now by definition of composition of two morphisms in $MF(1)$, the right hand side of equality (E') becomes:
$\gamma_b \circ \mu = [(I_{n_2}, 0)^t, (I_{n_2}, 0)^t] \circ (\alpha_\mu, \beta_\mu) = [(I_{n_2}, 0)^t \alpha_\mu, (I_{n_2}, 0)^t \beta_\mu] = [(\alpha_\mu, 0)^t, (\beta_\mu, 0)^t] \cdots \natural'$
0 in $[(\alpha_\mu, 0)^t, (\beta_\mu, 0)^t]$ is the $n_2 \times n_1$ zero matrix.
As for the left hand side of (E'), first recall that $\gamma_a = [(I_{n_1}, 0)^t, (I_{n_1}, 0)^t]$, (where 0 is the zero $n_1 \times n_1$ matrix) and by definition 3.3 of the multiplicative tensor product, we know that $e\widetilde{\otimes}\mu = (1,1)\widetilde{\otimes}(\alpha_\mu, \beta_\mu) = (\begin{bmatrix} 1 \otimes \alpha_\mu & 0 \\ 0 & 1 \otimes \alpha_\mu \end{bmatrix}, \begin{bmatrix} 1 \otimes \beta_\mu & 0 \\ 0 & 1 \otimes \beta_\mu \end{bmatrix}) = (\begin{bmatrix} \alpha_\mu & 0 \\ 0 & \alpha_\mu \end{bmatrix}, \begin{bmatrix} \beta_\mu & 0 \\ 0 & \beta_\mu \end{bmatrix})$
So, $e\widetilde{\otimes}\mu \circ \gamma_a = (\begin{bmatrix} \alpha_\mu & 0 \\ 0 & \alpha_\mu \end{bmatrix}, \begin{bmatrix} \beta_\mu & 0 \\ 0 & \beta_\mu \end{bmatrix}) \circ [(I_{n_1}, 0)^t, (I_{n_1}, 0)^t]$
$= (\begin{bmatrix} \alpha_\mu & 0 \\ 0 & \alpha_\mu \end{bmatrix}(I_{n_1}, 0)^t, \begin{bmatrix} \beta_\mu & 0 \\ 0 & \beta_\mu \end{bmatrix}(I_{n_1}, 0)^t) = [(\alpha_\mu, 0)^t, (\beta_\mu, 0)^t] \cdots \natural\natural'$.
From $\natural'$ and $\natural\natural'$, we see that equality (E') holds. That is γ is a natural transformation.

Next, we show that γ_a is the right inverse of λ_a by computing the following: $\lambda_a \circ \gamma_a = [(I_{n_1}, 0), (I_{n_1}, 0)] \circ [(I_{n_1}, 0)^t, (I_{n_1}, 0)^t] = (I_{n_1}, I_{n_1}) = id_a$. So γ_a is the right inverse of λ_a.

• To see that ρ is a natural transformation and that for any objet a in $MF(1)$, ρ_a has a right inverse, it suffices to observe that both λ_a and ρ_a have the same domain and codomain since for any $a = (\phi, \psi)$ in $MF(1)$, we have:

$$(\phi, \psi)\widetilde{\otimes}(1,1) = (\begin{bmatrix} \phi \otimes 1 & 0 \\ 0 & \phi \otimes 1 \end{bmatrix}, \begin{bmatrix} \psi \otimes 1 & 0 \\ 0 & \psi \otimes 1 \end{bmatrix}) = (\begin{bmatrix} \phi & 0 \\ 0 & \phi \end{bmatrix}, \begin{bmatrix} \psi & 0 \\ 0 & \psi \end{bmatrix})$$

Similarly,

$$(1,1)\widetilde{\otimes}(\phi, \psi) = (\begin{bmatrix} 1 \otimes \phi & 0 \\ 0 & 1 \otimes \phi \end{bmatrix}, \begin{bmatrix} 1 \otimes \psi & 0 \\ 0 & 1 \otimes \psi \end{bmatrix}) = (\begin{bmatrix} \phi & 0 \\ 0 & \phi \end{bmatrix}, \begin{bmatrix} \psi & 0 \\ 0 & \psi \end{bmatrix})$$

So, we define $\rho_a = \lambda_a$ for any a in $MF(1)$.

We also clearly have $\rho_e = \lambda_e$.

[10]By drawing the twin diagram that has to commute with (α_μ, β_μ), we see the sizes of α_μ and β_μ.

• Finally, for any object $b \in MF(1)$ and for $a = e$, we prove that the following triangular diagram commutes:

$$\begin{array}{ccc} a\widetilde{\otimes}(e\widetilde{\otimes}b) & \xrightarrow[a,e,b]{\alpha} & (a\widetilde{\otimes}e)\widetilde{\otimes}b \\ & {\scriptstyle 1_a\widetilde{\otimes}\lambda_b}\searrow \quad \swarrow{\scriptstyle \rho_a\widetilde{\otimes}1_b} & \qquad (\star''') \\ & a\widetilde{\otimes}b & \end{array}$$

Our goal here is to show that the diagram $(\star''')$ commutes i.e., $\rho_a\widetilde{\otimes}1_b \circ \alpha_{a,e,b} = 1_a\widetilde{\otimes}\lambda_b$ i.e., $\rho_a\widetilde{\otimes}1_b = 1_a\widetilde{\otimes}\lambda_b$ since the associator α is the identity.
We use definition 3.4 to verify that this equality holds.

$$\rho_a\widetilde{\otimes}1_b = (\begin{bmatrix}(I_{n_1},0)\otimes I_{n_2} & 0\\ 0 & (I_{n_1},0)\otimes I_{n_2}\end{bmatrix}, \begin{bmatrix}(I_{n_1},0)\otimes I_{n_2} & 0\\ 0 & (I_{n_1},0)\otimes I_{n_2}\end{bmatrix})$$
$$= (\begin{bmatrix}(1,0)\otimes I_{n_2} & 0\\ 0 & (1,0)\otimes I_{n_2}\end{bmatrix}, \begin{bmatrix}(1,0)\otimes I_{n_2} & 0\\ 0 & (1,0)\otimes I_{n_2}\end{bmatrix})\ since\ n_1 = 1\ as\ a = e$$
$$= (\begin{bmatrix}(I_{n_2},0) & 0\\ 0 & (I_{n_2},0)\end{bmatrix}, \begin{bmatrix}(I_{n_2},0) & 0\\ 0 & (I_{n_2},0)\end{bmatrix}) \qquad \cdots \flat$$
$$1_a\widetilde{\otimes}\lambda_b = (\begin{bmatrix}I_{n_1}\otimes(I_{n_2},0) & 0\\ 0 & I_{n_1}\otimes(I_{n_2},0)\end{bmatrix}, \begin{bmatrix}I_{n_1}\otimes(I_{n_2},0) & 0\\ 0 & I_{n_1}\otimes(I_{n_2},0)\end{bmatrix})$$
$$= (\begin{bmatrix}1\otimes(I_{n_2},0) & 0\\ 0 & 1\otimes(I_{n_2},0)\end{bmatrix}, \begin{bmatrix}1\otimes(I_{n_2},0) & 0\\ 0 & 1\otimes(I_{n_2},0)\end{bmatrix})\ since\ n_1 = 1\ as\ a = e$$
$$= (\begin{bmatrix}(I_{n_2},0) & 0\\ 0 & (I_{n_2},0)\end{bmatrix}, \begin{bmatrix}(I_{n_2},0) & 0\\ 0 & (I_{n_2},0)\end{bmatrix}) \qquad \cdots \flat'$$

From $\flat$ and $\flat'$, it is clear that $\rho_a\widetilde{\otimes}1_b = 1_a\widetilde{\otimes}\lambda_b$.
Therefore $(MF(1), \widetilde{\otimes})$ is a right pseudo-monoidal category. QED

□

Remark 3.7. *When proving the commutativity of the triangular diagram in the foregoing proof, we kept writing a instead of directly writing e because we wanted to point out the fact that this diagram is simply the triangular diagram one has in the definition of a monoidal category, except that here, the diagram commutes only for $a = e$. It is easy to see that if $a \neq e$ (meaning $n_1 \neq 1$), then $\rho_a\widetilde{\otimes}1_b \neq 1_a\widetilde{\otimes}\lambda_b$. In fact, the pair of matrices representing these two maps $\rho_a\widetilde{\otimes}1_b$ and $1_a\widetilde{\otimes}\lambda_b$ will be permutation similar but not equal. So, $(MF(1), \widetilde{\otimes})$ resembles a monoidal category in many respects without being one. That is one of the motivations behind the appellation right pseudo-monoidal category.*

In this chapter, we gave two applications of the multiplicative tensor product $\widetilde{\otimes}$ by providing an example of a semi-unital semi-monoidal category and an example of right pseudo-monoidal category.
In the next chapter, we will give another application of $\widetilde{\otimes}$ by improving the algorithm for factoring polynomials. The improvement will be done for a large class of polynomials. In fact, we will prove that on this class, our algorithm produces matrix factorizations which are smaller in size than what the standard method produces.

CHAPTER FOUR

THE STANDARD METHOD IMPROVED ON THE CLASS OF SUMMAND-REDUCIBLE POLYNOMIALS

In this chapter, we first recall a standard technique for factoring polynomials which dates to the 1980s when Knörrer [13] exploited it to prove his celebrated periodicity theorem (cf. theorem 2.1). This standard technique, usually referred to as the standard method [19] for factoring polynomials, builds matrix factorizations of sums of polynomials from "factorizations" of their summands. One conspicuous downside of this algorithm is that for each new summand that is added to the polynomial being factorized, the size (i.e., the number of rows and columns) of the matrix factorization doubles. We define a large class of polynomials and improve the standard method (or algorithm) on this class. In fact, we define a *summand-reducible polynomial* to be one that can be written in the form $f = t_1+\cdots+t_s+g_{11}\cdots g_{1m_1}+\cdots+g_{l1}\cdots g_{lm_l}$ under some specified conditions where each t_k is a monomial and each g_{ji} is a sum of monomials. We then use tools which were not available in the days the standard method was developed, namely $\widehat{\otimes}$ and $\widetilde{\otimes}$, to improve the standard method for matrix factorization of polynomials on this class and we prove that if p_{ji} is the number of monomials in g_{ji}, then there is an improved version of the standard method for factoring f which produces factorizations of size $2^{\prod_{i=1}^{m_1} p_{1i}+\cdots+\prod_{i=1}^{m_l} p_{li}-(\sum_{i=1}^{m_1} p_{1i}+\cdots+\sum_{i=1}^{m_l} p_{li})}$ times smaller than the size one would normally obtain with the standard method.

> *All the results developed in this chapter are new (to the best of our knowledge) except for the ones on the standard method for factoring polynomials presented in the first section. In all sections, examples are original.*

In our presentation, we limit ourselves to polynomials in the ring $S = \mathbb{R}[x_1 \ldots, x_n]$ where $\mathbb{R}$ is the set of real numbers. Moreover, when dealing with polynomials in at most three variables, we will usually use letters from the set $\{x, y, z\}$ instead of using indexed variables from the set $\{x_1, \ldots, x_n\}$.

4.1 The standard method for factoring polynomials

The material in this section essentially follows the presentation in [19] except for the examples.

4.1.1 Introduction

Eisenbud [25] proved that both reducible and irreducible polynomials in S can be factorized using matrices. He showed that the matrix factorizations of the polynomial f are intimately related to homological properties of modules over the quotient ring $S/(f)$, known as the hypersurface ring. These hypersurface rings encode geometric properties of the zero-locus of f, $Z(f) = \{a \in \mathbb{R}^n / f(a) = 0\}$, which is a hypersurface in $\mathbb{R}^n$. [41] and [14] contain more background on the connection between matrix factorizations and algebraic geometry. These papers have details on the connection that exists between matrix factorizations and maximal Cohen-Macaulay Modules. In this section, we describe a way to construct matrix factorizations of a polynomial without resorting to the homological methods that Eisenbud introduced.

4.1.2 The standard method

Here, we recall the standard technique for factoring polynomials.

Proposition 4.1. *[19] For $i, j \in \{1, 2\}$, let (C_i, D_i) denote an $n \times n$ matrix factorization of the polynomial $f_i \in S$. In addition, assume that the matrices C_i and D_j commute when $i \neq j$. Then the matrices*

$$\left(\begin{array}{c|c} C_1 & -D_2 \\ \hline C_2 & D_1 \end{array}\right), \left(\begin{array}{c|c} D_1 & D_2 \\ \hline -C_2 & C_1 \end{array}\right)$$

give a $2n \times 2n$ matrix factorization of $f_1 + f_2$.

Proof. From the assumption, we have that $C_iD_j = D_jC_i$ for all i, j, $i \neq j$. Now using the fact that matrices that appear in a matrix factorization of a polynomial commute (cf. proposition 2.1), we obtain:

$$\left(\begin{array}{c|c} C_1 & -D_2 \\ \hline C_2 & D_1 \end{array}\right) \cdot \left(\begin{array}{c|c} D_1 & D_2 \\ \hline -C_2 & C_1 \end{array}\right)$$

$$= \left(\begin{array}{c|c} C_1D_1 + D_2C_2 & C_1D_2 - D_2C_1 \\ \hline C_2D_1 - D_1C_2 & C_2D_2 + D_1C_1 \end{array}\right)$$

$$= \left(\begin{array}{c|c} (f_1 + f_2)I_n & 0 \\ \hline 0 & (f_1 + f_2)I_n \end{array}\right) = (f_1 + f_2)I_{2n}$$

as desired. □

The following consequence of the foregoing result is actually the basis for the main construction of the standard algorithm for factoring polynomials.

Corollary 4.1. *If (C, D) is an $n \times n$ matrix factorization of f and g, h are two polynomials, then*

$$\left(\begin{array}{c|c} C & -gI_n \\ \hline hI_n & D \end{array}\right), \left(\begin{array}{c|c} D & gI_n \\ \hline -hI_n & C \end{array}\right)$$

give a $2n \times 2n$ matrix factorization of $f + gh$.

Proof. Since the matrices gI_n and hI_n commute with all $n \times n$ matrices, the proof follows from the previous proposition. □

Thanks to this corollary, one can inductively construct matrix factorizations of polynomials of the form:

$$f = f_k = g_1h_1 + g_2h_2 + \cdots + g_kh_k.$$

For $k = 1$, we have $f = g_1h_1$ and clearly $[g_1][h_1] = [g_1h_1] = [f_1]$ is a 1×1 matrix factorization. Next, assume that C and D are matrix factorizations of f_{k-1}, i.e., $CD = If_{k-1}$ where I is the identity matrix of the right size. Hence, using the foregoing corollary, we obtain a matrix factorization of f_k:

$$(\left(\begin{array}{c|c} C & -g_kI_n \\ \hline h_kI_n & D \end{array}\right), \left(\begin{array}{c|c} D & g_kI_n \\ \hline -h_kI_n & C \end{array}\right))$$

Definition 4.1. *[19] The foregoing algorithm is called* ***the standard method*** *for factoring polynomials.*

This algorithm can be used to produce matrix factorizations of any polynomial because any polynomial can be expressed as a sum of finitely many monomials.
Though this algorithm clearly works for any polynomial, it has a conspicuous downside. The sizes of factorizations grow very quickly due to the fact that for every new summand g_nh_n added to the polynomial, the factorizations double in size. It is easy to see that with this method, to factor a polynomial with k summands, say

$$f_k = g_1h_1 + g_2h_2 + \cdots + g_kh_k,$$

one obtains matrices of size 2^{k-1}. This entails that these factorizations can grow extremely large very quickly. For example if $k = 6$, we will obtain matrices of size $2^5 = 32$ and for $k = 10$, we will obtain matrices of size $2^9 = 512$

Diveris and Crisler in [19] improved this standard algorithm on a special class of polynomials: polynomials that are sums of squares i.e., $f_n = x_1^2 + \cdots + x_n^2$, for $n \leq 8$. The resulting factorizations they obtained have smaller matrices than one would obtain using the standard method. [14] studies matrix factorizations over quadratic hypersurfaces and also contains factorizations of $f_n = x_1^2 + \cdots + x_n^2$. The author of [14] first proves that there is an equivalence of categories between matrix factorizations of f_n and graded modules over a Clifford algebra associated to f_n. He then exploits this technique to generate matrix factorizations of f_n. [19] observes that this technique can be used to generate minimal matrix factorizations of polynomials f_n for all $n \geq 1$. In contrast, Diveris and Crisler use an elementary approach based on matrix algebra. They remark that their algorithm produces a factorization of f_8 with just 8×8 matrices whereas the standard method will produce a factorization of size 128×128. Moreover, they state that the results in [14] actually prove that their factorizations of f_n have the smallest possible size for $1 \leq n \leq 8$. In fact, [14] proves that for $n \geq 8$, the smallest possible matrix factorization for f_n is bounded below by $2^{\frac{n-2}{2}} \times 2^{\frac{n-2}{2}}$.
We now give some examples to illustrate the standard method.

Example 4.1. *(Example 1 of [19]) Let $h = x^2 + y^2$. then using the standard method, a matrix factorization of h is*

$$M = (\begin{bmatrix} x & -y \\ y & x \end{bmatrix}, \begin{bmatrix} x & y \\ -y & x \end{bmatrix})$$

Example 4.2. *Let $g = xy + x^2z + yz^2$. We use the standard method to find a matrix factorization of g. First a matrix factorization of $xy + x^2z$ is*

$$(\begin{bmatrix} x & -x^2 \\ z & y \end{bmatrix}, \begin{bmatrix} y & x^2 \\ -z & x \end{bmatrix})$$

Hence, a matrix factorization of $g = xy + x^2z + yz^2$ is then:

$$N = (\begin{bmatrix} x & -x^2 & -y & 0 \\ z & y & 0 & -y \\ z^2 & 0 & y & x^2 \\ 0 & z^2 & -z & x \end{bmatrix}, \begin{bmatrix} y & x^2 & y & 0 \\ -z & x & 0 & y \\ -z^2 & 0 & x & -x^2 \\ 0 & -z^2 & z & y \end{bmatrix})$$

Example 4.3. *Let $l = xy + (x^2 + yz)z$. Observe that $l = g$ where g is given in example 4.2. We use the standard method to find a matrix factorization of l and quickly find:*

$$M = (\begin{bmatrix} x & -(x^2 + yz) \\ z & y \end{bmatrix}, \begin{bmatrix} y & x^2 + yz \\ -z & x \end{bmatrix})$$

We observe that the factorization we obtain for l is not as nice as the one we obtain for g, in the sense that the complexity of some entries in the factorization of l is higher than what we have for g. For instance, in M the entry $-(x^2 + yz)$ is more complex than all the entries in N (cf. example 4.2). This shows that it is better to use the expanded version of a polynomial to find its matrix factorization.
Here is another more involved example to illustrate the point we just made.

Example 4.4. *Let $f = xy + (xy + x^2z + yz^2)(x^2 + y^2)$. We use the standard method to find a matrix factorization of f and quickly find it:*

$$P = (\begin{bmatrix} x & -(xy + x^2z + yz^2) \\ (x^2 + y^2) & y \end{bmatrix}, \begin{bmatrix} y & (xy + x^2z + yz^2) \\ -(x^2 + y^2) & x \end{bmatrix}) \cdots \dagger$$

Nota Bene: The matrices obtained in P are not satisfactory because the complexity of g and of h which are entries in P could be lower had we first expanded f as follows:

$$f = xy + (xy + x^2z + yz^2)(x^2 + y^2) = xy + xyx^2 + x^2zx^2 + yz^2x^2 + xy^3 + x^2zy^2 + yz^2y^2.$$

With this expanded version of f, if we apply the foregoing algorithm, it is easy to see that we will obtain factorizations of f which are better than the one in †. We say better in the sense that the entries in the matrices will not be sums of monomials (e.g. $(xy + x^2z + yz^2)$ in †) but simply monomials and thus providing a more interesting factorization.

> So, we make the important assumption that before applying the standard method to a given polynomial, it has to be written in its expanded form.

But this comes with a price! The size of the factorizations becomes big.
In fact, in order to obtain a matrix factorization of f in which the entries are monomials and not sums of monomials, since f has 7 summands, using the standard method would yield matrix factorizations of size $2^{7-1} = 2^6 = 64$!
Perhaps this is due to the fact that with the standard method, even if one knows the matrix

factorizations of some summands in a polynomial, there is no way to make use of them. Observe that $f = xy + (xy + x^2z + yz^2)(x^2 + y^2)$ and from the examples above, we have at hand the factorizations of each factor in the product $(xy + x^2z + yz^2)(x^2 + y^2)$. There is no way to exploit this information using the standard method.
In the following section, we will exploit this information using the new operation constructed in chapter 3 to obtain better results on the size of the factorizations.

4.2 The improved algorithm

Here, we use the tensor product of matrix factorizations and the multiplicative tensor product of matrix factorizations to improve the standard algorithm for factoring polynomials that are *(simple) summand-reducible* (cf. definition 4.2 & 4.4). As seen in the previous section, the standard method has a downside: For each new summand added to the polynomial we are factoring, the factorizations must double in size. We show that our approach produces factorizations that are of smaller sizes than the factorizations produced by the standard method.
We first define a class of polynomials which are such that they can be rewritten with less summands than what appears in their expanded version. This is achieved by factorizing some of the summands in a particular way.

Definition 4.2. *A polynomial f is said to be* ***simple summand-reducible*** *if it can be written in the form*

$$f = g_1h_1 + g_2h_2 + \cdots + g_kh_k, \ k \geq 2$$

where:

1. *For at least one $i \in \{1, 2, \cdots, k\}$, g_ih_i is the product of two polynomials (g_i and h_i), s.t. if n_{g_i} = number of monomials in g_i and n_{h_i} = number of monomials in h_i, then $n_{g_i}n_{h_i} \geq 6$.*
2. *If there is an $i \in \{1, 2, \cdots, k\}$ s.t. the expanded form of g_ih_i belongs to $\{a^n - b^n, a^n + b^n, (a^2 \pm 2ab + b^2); n \in \mathbb{N} - \{0\}\}$, (where a and b could be single variables or product of variables), then instead expand g_ih_i to have a smaller expression.*

The reason for denying that the two factors in g_ih_i should not be simultaneously $(a \pm b)$ is that when expanded, $(a \pm b)^2$ has three summands; thus, the standard method would yield a factorization of size $2^{3-1} = 4$ and as we shall see (cf. theorem 4.1), our new algorithm will produce factorizations of size $2(2^1)(2^1) = 8$. Indeed, we see that there will be no profit in having the products $(a \pm b)^2$ in our foregoing definition. Instead we need to expand them. Therefore, it is better to write the expression that has three summands. The explanation of the other part of the second point in the definition is that factorizing $\{a^n - b^n, a^n + b^n$ will instead cause the size of the matrices to grow big as we will have more monomials after factorizing.
The idea is that we want to be able to write polynomials with less summands than the ones that appear in their expanded form. Because, the fewer the number of summands, the smaller the size of the matrix factorizations as we will soon see with the help of the tensor products $\widehat{\otimes}$ and $\widetilde{\otimes}$. Consider the following polynomials:

1. $x^2 - y^2$ is not simple summand-reducible by the first condition of the definition. This easily factorizes as $(x-y)(x+y)$.

2. $xy + x^3 - y^3 = xy + (x-y)(x^2 + 2xy + y^2)$ is not simple summand-reducible. In fact if $g_i h_i = (x-y)(x^2 + 2xy + y^2)$, it is better to write it as $x^3 - y^3$ because this will help reduce the size of the matrix factorizations. It is not difficult to see that using the expanded form $x^3 - y^3$ yields a nice factorization of a size smaller than the one we would obtain should we use the factor form $(x-y)(x^2 + 2xy + y^2)$. This example explains why we need the second condition in definition 4.2.

3. $x^2 y + x^2 z^3$ is not simple summand-reducible. It is the expanded form of $x^2(y + z^3)$ which is a 1×1 matrix factorization of $x^2 y + x^2 z^3$.

In contrast:

Example 4.5. *The following are simple summand-reducible polynomials:*

1. $x^2 + yx^3 + zx^4 + yz^2x^2 + xy^3 + x^2zy^2 + xz^2y^2 = x^2 + (xy + x^2z + yz^2)(x^2 + y^2)$.

2. $f = x^2 + xyz + yx^3 + zx^4 + yz^2x^2 + xy^3 + x^2zy^2 + xz^2y^2 + xyz + x^2z^2 + yz^3 = x^2 + xyz + (xy + x^2z + yz^2)(x^2 + y^2 + z)$.

Remark 4.1. *Observe that our definition 4.2 mostly targets polynomials with more than six monomials because factorizations obtained with the standard method begin to be of considerable sizes such that it is even impossible to properly put them on an* A_4 *sheet of paper.*

Definition 4.3. *A polynomial f is said to be **simple summand-reduced** if it is in the form* $f = g_1h_1 + g_2h_2 + \cdots + g_kh_k$ *described in definition 4.2.*

Let us go back to the following problem:

Example 4.6. *Find a matrix factorization of* $f = xy + (xy + x^2z + yz^2)(x^2 + y^2)$.
Recall first of all that if X and Y are respectively matrix factorizations of polynomials f and g, then as seen in previous chapters, $X\widehat{\otimes}Y$ *and* $X\widetilde{\otimes}Y$ *are respectively matrix factorizations of the sum $f + g$ and the product fg.*
Hence, we can use $\widetilde{\otimes}$ *to find a matrix factorization of the product* $(xy+x^2z+yz^2)(x^2+y^2)$ *and then use* $\widehat{\otimes}$ *to obtain a matrix factorization of the sum* $f = xy+(xy+x^2z+yz^2)(x^2+y^2)$*. The only question that naturally arises is to know if the factorizations we obtain would be of smaller sizes than the ones we would obtain with the standard method. We already found M and N above which are respectively matrix factorizations of* (x^2+y^2) *and* $(xy+x^2z+yz^2)$*.* $N\widetilde{\otimes}M$ *will be of size* $2nm = 2(4)(2) = 16$ *where* $n = 4$ *is the size of N and* $m = 2$ *is the size of M. Now, let* $Q = ([x],[y])$ *be a* 1×1 *matrix factorization of xy. Then if* $L = N\widetilde{\otimes}M$; $Q\widehat{\otimes}L$ *would be a matrix factorization of* $f = xy + (xy + x^2z + yz^2)(x^2 + y^2)$ *of size* $2ql = 2(1)(16) = 32$*, where* $q = 1$ *is the size of the matrix factorization Q and* $l = 16$ *is the size of the matrix factorization L.*
So the size of the factorization we obtain for f using this method is $32 = \frac{64}{2}$*. That is one-half the size we obtain using the standard method!*
Finding a matrix factorization of f is now easy after all the foregoing explanation. Since it is space consuming, we will not be able to give it here.

Nota Bene: With the standard method, even if one knows matrix factorizations of polynomials f and g, one cannot derive from it a matrix factorization of the product fg or the sum $f + g$.
Recall that the multiplicative tensor product of matrix factorization $\widetilde{\otimes}$ developed in this dissertation produces a matrix factorization of the product of two polynomials from the matrix factorizations of each of these polynomials. In theorems 4.1 and 4.2, $\widetilde{\otimes}$ is one of the crucial ingredients in the improved algorithm used to obtain better matrix factorizations of polynomials; in the sense that, the size of the matrix factorizations we obtain is smaller than the size one would normally obtain with the standard method. In fact, $\widetilde{\otimes}$ helps in reducing the size of the factorizations as we shall notice in the proofs of theorems 4.1 and 4.2. We already saw the reduction power of $\widetilde{\otimes}$ in action in the foregoing example with $(xy + x^2z + yz^2)(x^2 + y^2)$. In fact, we saw that using $\widetilde{\otimes}$, we obtain a matrix factorization of size $2(4)(2) = 16$ but if we expand it and use the standard method, we would have 6 monomials and consequently a matrix factorization of size $2^{6-1} = 32$, this is twice the size we obtain with $\widetilde{\otimes}$.
Another crucial ingredient used in the improved algorithm is Yoshino's tensor product of matrix factorizations $\widehat{\otimes}$, because it produces a matrix factorization of the sum of two polynomials from the matrix factorizations of each of these polynomials.

Theorem 4.1. *There is an improved version of the standard method for factoring simple summand-reducible polynomials which produces factorizations which are at most one-half the size of the matrix factorizations one would obtain with the standard method.*

Proof. First, we construct the algorithm, then we prove that the size of the resulting matrix factorizations (for simple summand-reducible polynomials) is at most one-half the size one would obtain with the standard method.
We inductively construct the matrix factorizations of simple summand-reduced polynomials using tools ($\widehat{\otimes}$ and $\widetilde{\otimes}$) that were not existing in the 1980s when the standard method was constructed.
Let $f = f_k = g_1h_1 + g_2h_2 + \cdots + g_kh_k$ be a simple summand-reduced polynomial. Without loss of generality, we consider the last summand to be a product of sums of polynomials.
For $k = 1$:
If g_1h_1 is not a product of sums of monomials, then $f = [g_1][h_1]$ is a 1×1 matrix factorization of f.
If g_1h_1 is a product of sums of monomials, then:

1. Use the standard method to find a matrix factorization of g_1, call it $X_g = (A_g, B_g)$.
2. Use the standard method to find a matrix factorization of h_1, call it $X_h = (A_h, B_h)$.
3. Use the multiplicative tensor product $\widetilde{\otimes}$ to find $X_g\widetilde{\otimes}X_h$ which is a matrix factorization of g_1h_1.

Next, assume (A, B) is a matrix factorization of f_{k-1}:

a. Use the standard method to find a matrix factorization of g_k,

b. Use the standard method to find a matrix factorization of h_k,

c. Use the multiplicative tensor product $\widetilde{\otimes}$ to find a matrix factorization of $g_k h_k$, call it (C, D).

d. Finally, use the tensor product of matrix factorizations $\widehat{\otimes}$ to find $(A, B)\widehat{\otimes}(C, D)$ which is a matrix factorization of $f = f_{k-1} + g_k h_k$.

We now show that the size of the factorizations we obtain using the improved method is at most one-half the size one would obtain using the standard method.
Let f be a simple summand-reducible polynomial with n monomials in its expanded form. Then the standard method tells us that matrix factorizations of f will be of size 2^{n-1}.
Let $f = f_k = g_1 h_1 + g_2 h_2 + \cdots + g_k h_k$ be the simple summand-reduced form of f. Then at least one of the $g_i h_i$ should be the product of sums of monomials. Assume f has just one such summand. Without loss of generality (WLOG), let it be $g_k h_k$. We know by definition of simple summand-reducible polynomial that the number of monomials in g_k times the number of monomials in h_k should be at least 6. Suppose it is 6 (the reason why this assumption is sufficient is found in theorem 4.2 and the comment that follows it.) and suppose WLOG that g_k and h_k are respectively the sums of three and two monomials. Next, using the standard method, we can find factorizations of g_k and h_k respectively of sizes $2^{3-1} = 4$ and $2^{2-1} = 2$. Hence, using the multiplicative tensor product $(\widetilde{\otimes})$ of matrix factorizations, we find a matrix factorization of $g_k h_k$ of size $2(4)(2) = 16 = 2^4$.
Now f had n monomials in its expanded form, meaning that if we take away the 6 monomials that made up $g_k h_k$, we will be left with $n - 6$ monomials in f. Thus the standard method would yield a matrix factorization of size $2^{(n-6)-1}$ of $f - g_k h_k$. Finally, we use the tensor product of matrix factorizations $(\widehat{\otimes})$ to find a matrix factorization of $f = (f - g_k h_k) + g_k h_k$ since we have factorizations of each of these two summands. Thus, we obtain a factorization of size $2(2^{(n-6)-1})(2^4) = 2^{n-2}$ and this is clearly one-half the size we obtained when we solely used the standard method. □

Corollary 4.2. *Let f be a simple summand-reducible polynomial with $n + 1$ monomials in its expanded form. If f can be written as $t + gh$ where t is a monomial and gh is a product of sums of monomials, where p is the number of monomials in g and q is the number of monomials in h, then the improved algorithm of theorem 4.1 produces a matrix factorization of f which is $2^{n-(p+q)} = 2^{pq-(p+q)}$ times smaller in size than what one would obtain using the standard method.*

Proof. Let f be a simple summand-reducible polynomial with $n + 1$ monomials in its expanded form and such that $f = t + gh$ as in the assumption of the theorem. Let p be the number of monomials in g and q the number of monomials in h, then clearly $n = pq$ and the standard method would produce a factorization of f of size $2^{(n+1)-1} = 2^{(pq+1)-1} = 2^{pq}$.
Now, since $f = t + gh$, we use the standard method to find matrix factorizations of g and h and find respectively factorizations of size 2^{p-1} and 2^{q-1}. Next, we use the multiplicative tensor product of matrix factorizations $(\widetilde{\otimes})$ to find a factorization of size $2(2^{p-1})(2^{q-1}) = 2^{p+q-1}$ for the product gh.
Next, use the tensor product of matrix factorizations $(\widehat{\otimes})$ to find a matrix factorization of f of size $2(1)(2^{p+q-1}) = 2^{p+q}$.
Now, WLOG $p \geq 2$ and $q \geq 3$, so $pq > p + q$ and $2^{pq} > 2^{p+q}$.
Finally, $2^{pq} \div 2^{p+q} = 2^{(pq)-(p+q)}$ as desired. □

This theorem actually shows that the higher the product pq, the smaller the size of the factorization we obtain as compared to what one would obtain using the standard method.

We now generalize definition 4.2 and define the class of *summand-reducible* polynomials which is made up of polynomials in which some monomials can be factorized in a nice way, hence allowing the polynomial to be written with less summands.

Definition 4.4. *A polynomial f is said to be* ***summand-reducible*** *if*

$$f = t_1 + \cdots + t_s + g_{11} \cdots g_{1m_1} + \cdots + g_{l1} \cdots g_{lm_l},$$

where:

1. • *If $s = 0$, then there exist at least two products $g_{11} \cdots g_{1m_1}$ and $g_{21} \cdots g_{2m_2}$ in f.*
   • *If $s \neq 0$, then there exists at least one product $g_{11} \cdots g_{1m_1}$ in f.*
2. *Each t_i; $i = 1, \cdots, s$ is a monomial and so $t_i = h_{i1}h_{i2}$, where h_{i1} and h_{i2} are products of variables possibly raised to some power.*
3. *Each $g_{j1} \cdots g_{jm_j}$; $j = 1, \cdots, l$ is a product of sums of monomials, such that if it is expanded, $g_{j1} \cdots g_{jm_j}$ would have more monomials than the number that appears in the factor form $g_{j1} \cdots g_{jm_j}$.*
4. *At least one of the products $g_{j1} \cdots g_{jm_j}$ has at least two factors.*

Example 4.7. $f = xy + (xy + x^2z + yz^2)(x^2 + y^2) + (yz + xy^2 + x^2)(x^3z^2 + yx + y^2)$ *is a summand-reduced polynomial.*

Example 4.8. *Every simple summand-reducible polynomial is a summand-reducible polynomial. So, the examples (cf. example 4.5) we gave above for simple summand-reducible polynomial are also summand-reducible polynomials.*
It is easy to generate more examples.

We will now generalize the result given in theorem 4.1.

Theorem 4.2. *Let $f = t_1 + \cdots + t_s + g_{11} \cdots g_{1m_1} + \cdots + g_{l1} \cdots g_{lm_l}$ be a summand-reducible polynomial. Let p_{ji} be the number of monomials in g_{ji}. Then there is an improved version of the standard method for factoring f which produces factorizations of size*

$$2^{\prod_{i=1}^{m_1} p_{1i} + \cdots + \prod_{i=1}^{m_l} p_{li} - (\sum_{i=1}^{m_1} p_{1i} + \cdots + \sum_{i=1}^{m_l} p_{li})}$$

times smaller than the size one would normally obtain with the standard method.

Proof. First, we construct the algorithm, then we prove that the resulting matrix factorizations (for summand-reducible polynomials) are

$$2^{\prod_{i=1}^{m_1} p_{1i} + \cdots + \prod_{i=1}^{m_l} p_{li} - (\sum_{i=1}^{m_1} p_{1i} + \cdots + \sum_{i=1}^{m_l} p_{li})}$$

times smaller in size than what one would obtain with the standard method.
We inductively construct the matrix factorizations of summand-reduced polynomials using tools ($\widehat{\otimes}$ and $\widetilde{\otimes}$) that were not existing in the 1980s when the standard method was constructed.
The algorithm we propose here is just an improvement of the one we gave for simple summand-reducible polynomials (cf. proof of theorem 4.1).

Let $f = t_1 + \cdots + t_s + g_{11} \cdots g_{1m_1} + \cdots + g_{l1} \cdots g_{lm_l}$ be a summand-reducible polynomial. Let p_{ji} be the number of monomials in g_{ji}.
If $\forall k \in \{1, \cdots, s\}$, $t_k = 0$, then do:

1. For each $j \in \{1, \cdots, l\}$ and $i \in \{1, \cdots, m_j\}$, use the standard method to find a matrix factorization of g_{ji} of size $2^{p_{ji}-1}$.

2. Next, for each $j \in \{1, \cdots, l\}$; use the multiplicative tensor product of matrix factorizations $\widetilde{\otimes}$ to find a matrix factorization of $g_{j1} \cdots g_{jm_j}$ of size

$$(2^{m_j-1})(2^{\sum_{i=1}^{m_j} p_{ji}-m_j}) = 2^{\sum_{i=1}^{m_j} p_{ji}-1}$$

3. Now use the tensor product of matrix factorizations $\widehat{\otimes}$ to find a matrix factorization of $g_{11} \cdots g_{1m_1} + \cdots + g_{l1} \cdots g_{lm_l}$ of size

$$(2^{l-1})(\prod_{j=1}^{l} 2^{\sum_{i=1}^{m_j} p_{ji}-1}) = 2^{l-1+\sum_{i=1}^{m_1} p_{1i}+\cdots+\sum_{i=1}^{m_l} p_{li}-l} = 2^{\sum_{i=1}^{m_1} p_{1i}+\cdots+\sum_{i=1}^{m_l} p_{li}-1}.$$

   Let us find the size of matrices the standard method would produce for

$$g_{11} \cdots g_{1m_1} + \cdots + g_{l1} \cdots g_{lm_l}.$$

   Let n_j =number of monomials in the expanded form of the j^{th} product $g_{j1} \cdots g_{jm_j}$. Then $n_j = \prod_{i=1}^{m_j} p_{ji}$. Hence, the number of monomials in the expanded form of $g_{11} \cdots g_{1m_1} + \cdots + g_{l1} \cdots g_{lm_l}$ would be $\sum_{j=1}^{l} n_j = \sum_{j=1}^{l} \prod_{i=1}^{m_j} p_{ji}$.
   So the size of factorizations produced by the standard method would be $2^{(\sum_{j=1}^{l} \prod_{i=1}^{m_j} p_{ji})-1}$. Thus, the factorizations produced by our improved algorithm would be

$$2^{(\sum_{j=1}^{l} \prod_{i=1}^{m_j} p_{ji})-1} \div 2^{\sum_{i=1}^{m_1} p_{1i}+\cdots+\sum_{i=1}^{m_l} p_{li}-1} = 2^{(\sum_{j=1}^{l} \prod_{i=1}^{m_j} p_{ji})-(\sum_{i=1}^{m_1} p_{1i}+\cdots+\sum_{i=1}^{m_l} p_{li})}$$

   times smaller in size than the factorizations produced by the standard method.

4. If there exists $k \in \{1, \cdots, s\}$ such that $t_k \neq 0$, then use the standard method to inductively find a matrix factorization (A, B) of $t_1 + \cdots + t_s$ of size 2^{s-1}.

5. Then do steps 1), 2) and 3) above to find a matrix factorization (C, D) of $g_{11} \cdots g_{1m_1} + \cdots + g_{l1} \cdots g_{lm_l}$ of size $2^{\sum_{i=1}^{m_1} p_{1i}+\cdots+\sum_{i=1}^{m_l} p_{li}-1}$.

6. Now, use $\widehat{\otimes}$ to find a matrix factorization $(A, B)\widehat{\otimes}(C, D)$ of $f = t_1 + \cdots + t_s + g_{11} \cdots g_{1m_1} + \cdots + g_{l1} \cdots g_{lm_l}$ of size

$$2(2^{s-1})(2^{\sum_{i=1}^{m_1} p_{1i}+\cdots+\sum_{i=1}^{m_l} p_{li}-1}) = 2^{\sum_{i=1}^{m_1} p_{1i}+\cdots+\sum_{i=1}^{m_l} p_{li}+s-1}.$$

   Note that f in expanded form has

$$\sum_{j=1}^{l} n_j + s = (\sum_{j=1}^{l} \prod_{i=1}^{m_j} p_{ji}) + s$$

   monomials and so the standard method would produce factorizations of size $2^{(\sum_{j=1}^{l} \prod_{i=1}^{m_j} p_{ji})+s-1}$. Hence the factorizations our improved algorithm produce are

$$2^{(\sum_{j=1}^{l} \prod_{i=1}^{m_j} p_{ji})+s-1} \div 2^{\sum_{i=1}^{m_1} p_{1i}+\cdots+\sum_{i=1}^{m_l} p_{li}+s-1} = 2^{(\sum_{j=1}^{l} \prod_{i=1}^{m_j} p_{ji})-(\sum_{i=1}^{m_1} p_{1i}+\cdots+\sum_{i=1}^{m_l} p_{li})}$$

   times smaller in size than the factorizations produced by the standard method. QED.

□

Example 4.9. *Let* $f = xy + (xy + x^2z + yz^2)(x^2 + y^2) + (yz + xy^2 + x^2)(x^3z^2 + yx + y^2)$.
f *in expanded form has* $1 + 3 \times 2 + 3 \times 3 = 16$ *monomials and so the standard method will produce factorizations of size* $2^{16-1} = 2^{15}$.
The foregoing theorem gives us a way to find the size of the factorizations we would obtain using the improved algorithm.
$s = 1$, $l = 2$, $m_1 = 2$, $m_2 = 2$ $p_{11} = p_{21} = p_{22} = 3$ *and* $p_{12} = 2$, *so our algorithm would produce factorizations of size* $2^{p_{11}+p_{12}+p_{21}+p_{22}+s-1} = 2^{3+2+3+3+1-1} = 2^{11}$.
Hence, the theorem says that the improved algorithm produces factorizations of size $2^{15} \div 2^{11} = 2^4 = 16$ *times smaller than what the standard method produces!*

Remark 4.2. *Given that every polynomial is the sum of other polynomials, and that the operations* $\widetilde{\otimes}$ *and* $\widehat{\otimes}$ *produce matrix factorizations of polynomials* fg *and* $f + g$ *from the matrix factorizations of the polynomials* f *and* g, *it is not difficult to slightly modify our improved algorithm to be able to apply it to all polynomials. But we will not bother to do that since it is more interesting only in the case of summand-reducible polynomials as the size of factorizations is smaller than the one obtained with the standard method.*

CHAPTER FIVE

LINEAR FACTORIZATIONS AND THE BICATEGORY OF LANDAU-GINZBURG MODELS ($\mathcal{LG}_K$)

In this chapter, the notion of linear factorization is recalled. It is then used to review the bicategory $\mathcal{LG}_K$ of Landau-Ginzburg models (section 2.2 of [15]) over a commutative ring K whose construction is reminiscent of, but more complex than, that of BiMod (bicategory of associative algebras and bimodules). But, unlike [17], we do not restrict ourselves to *potentials* which are polynomials satisfying some conditions (definition 2.4 p.8 of [17])[1]. Potentials have many applications in quantum algebra [40] and mathematical physics [17]. They are used in [17] to construct the bicategory of Landau-Ginzburg models which is a bicategory with adjoints (also called duals, cf. [17]) and this helps in explaining a certain duality that exists in the setting of Landau-Ginzburg model in terms of some specified relations (cf. page 1 of [17]). It turns out that the authors of [17] used potentials to suit their purposes because even if we take the objects of $\mathcal{LG}_K$ to be polynomials rather than potentials and then apply the construction of $\mathcal{LG}_K$ given in [17], we obtain virtually the same bicategory except that we now have more objects. And in this new category whose objects are polynomials (without restrictions), the unit construction of the bicategory of Landau-Ginzburg models can still be explained. Therefore, in our presentation, the objects of $\mathcal{LG}_K$ are polynomials in general (without imposing any constraints on them). This is enough for what we desire to prove. In fact, we want to elucidate the construction of the unit of $\mathcal{LG}_K$, prove that the left and the right unit maps are natural with respect to $2-$morphisms in $\mathcal{LG}_K$ and finally prove that there is no direct inverse for these (left and right) unit maps.

> *Our contribution here is at several levels. First, we prove that $Hom_R(X, Y)$ is a $\mathbb{Z}_2-$graded complex as claimed in ([17] on p.9), where X and Y are linear factorizations of an element of R. Next, we give a detailed proof of the construction of the unit in $\mathcal{LG}_K$ from a new vantage point. Moreover, we prove the naturality of the unit maps with respect to $2-$morphisms in this bicategory. Finally, we show that there is no direct inverse for the left and right identities (also called unitors cf. [17]) of the $\mathcal{LG}_K$ bicategory, thereby justifying the fact that their construction in [17] is given only at the level of homotopy*[2].

In the sequel, except otherwise stated, K is a commutative ring with unity. Let $R =$

[1]For an earlier reference, see section 3 p.17-18 of [40].

[2][17] does not show why without homotopy the left and right unit maps would fail to be invertible.

$K[x_1, x_2, ..., x_n]$. We will write $K[x]$ for $K[x_1, x_2, ..., x_n]$.

5.1 A short note on linear factorizations

5.1.1 A link between linear factorizations and $\mathbb{Z}_2-$graded Complexes

In this section, we prove an observation made on p.9 of [17] that links linear factorizations and $\mathbb{Z}_2-$graded complexes.

Definition 5.1. *(p.8 of [17]) Linear factorization*
A ***linear factorization*** *of $f \in R$ is a $\mathbb{Z}_2-$graded $R-$module $X = X^0 \oplus X^1$ together with an odd (i.e., grade reversing) $R-$linear endomorphism $d : X \longrightarrow X$ such that $d^2 = f \cdot id_X$.*
$f \cdot id_X$ stands for the endomorphism $x \mapsto f \cdot x,\ \forall x \in X$.

(Since we are dealing with a $\mathbb{Z}_2-$grading and d is odd, we can also say d is a degree one map.) d is called a *twisted differential* in [16]. d is actually a pair of maps (d^0, d^1) that we may depict as follows:

$$X^0 \xrightarrow{d^0} X^1 \xrightarrow{d^1} X^0$$

and the stated condition on them is: $d^0 \circ d^1 = f \cdot id_{X^1}$ and $d^1 \circ d^0 = f \cdot id_{X^0}$.

Example 5.1. *Keeping to the notations of definition 5.1, let $R = \mathbb{C}[x]$ consider $f = x^3 \in R$. Then take X to be the $\mathbb{Z}_2$-graded $\mathbb{C}[x]$-module seen in example 1.10, namely $\mathbb{C}[x] \oplus \mathbb{C}[x]$. Then define $d : X \to X$ as follows:*

$$X^0 \xrightarrow{M} X^1 \xrightarrow{M} X^0$$

where some of the choices for M are the following:

$$M = \begin{bmatrix} 0 & 1 \\ x^3 & 0 \end{bmatrix}$$

or

$$M = \begin{bmatrix} 0 & x \\ x^2 & 0 \end{bmatrix}$$

Here, $d^0 = d^1$ and M is the matrix corresponding to the R-linear endomorphism d^0. In general, if $f = x^n$, then we can take M to be

$$M_q = \begin{bmatrix} 0 & x^q \\ x^{n-q} & 0 \end{bmatrix}$$

Clearly $d^2 = f \cdot id_X$.

Remark 5.1. *If X is a free $R-$module, then the pair (X, d) is called a matrix factorization. If M_0 and M_1 are respectively matrices of the $R-$linear endomorphisms d^0 and d^1, then the pair (M_0, M_1) would be a matrix factorization of f according to definition 2.1.*
The original definition of a matrix factorization was given by Eisenbud (p.15 of [25]) as follows: a matrix factorization of an element f in a ring R (with unity) is an ordered pair of maps of free $R-$modules $\phi : F \to G$ and $\psi : G \to F$ s.t., $\phi\psi = f \cdot 1_G$ and $\psi\phi = f \cdot 1_F$. From this definition of Eisenbud, it is clear that to obtain a linear factorization from a

matrix factorization, we need to have F and G represent respectively the even and the odd components of a $\mathbb{Z}_2-$graded $R-$module X such that ϕ and ψ are grade reversing maps. Just like in [17], in this chapter, we will often refer to a matrix factorization (X, d) by X without mentioning the differential d_X.

Given a basis for X, it is sometimes more convenient to identify d_X with its associated (block) matrix

$$d_X = \begin{pmatrix} 0 & d_X^1 \\ d_X^0 & 0 \end{pmatrix}$$

Two remarkable operations can be carried out on matrix factorizations, namely *direct sum* and *tensor product*. For the operation of tensor product; see subsection 2.1.3 of chapter 2, chapter 3 and subsection 5.1.3. As for the direct sum of two matrix factorizations X and Y it is defined in the obvious way:

$$(X \oplus Y)^i = X^i \oplus Y^i \ \ and \ \ d^i_{X \oplus Y} = d^i_X + d^i_Y$$

Matrix factorizations first appeared in the work of Eisenbud [25] who related them to maximal Cohen-Macaulay modules (cf. (Chapter 7 of [68]), [58], and references therein).

Definition 5.2. *(p.9 [17]) Morphism of linear factorization*
*A **morphism of linear factorizations** (X, d_X) and (Y, d_Y) is an even (i.e., a grade preserving) $R-$linear map $\phi : X \longrightarrow Y$ such that $d_Y\phi = \phi d_X$.*

Concretely (see page 19 of [40]), ϕ is a pair of maps $X^0 \xrightarrow{\phi^0} Y^0$ and $X^1 \xrightarrow{\phi^1} Y^1$ such that the following diagram commutes:

$$\begin{array}{ccccc}
X^0 & \xrightarrow{d_X^0} & X^1 & \xrightarrow{d_X^1} & X^0 \\
\downarrow{\scriptstyle \phi^0} & & \downarrow{\scriptstyle \phi^1} & & \downarrow{\scriptstyle \phi^0} \\
Y^0 & \xrightarrow{d_Y^0} & Y^1 & \xrightarrow{d_Y^1} & Y^0
\end{array}$$

Remark 5.2. *It is easy to see that the family of $R-$linear maps $Hom_R(X, Y)$ between two linear factorizations X and Y is an $R-$module with action $r(\phi^0, \phi^1) = (r\phi^0, r\phi^1)$, $r \in R$. Since X and Y are linear factorizations of $f \in R$, they are $\mathbb{Z}_2-$graded modules, thus we can write $X = X^0 \oplus X^1$ and $Y = Y^0 \oplus Y^1$. Hence maps in $Hom_R(X, Y)$ are of degree one or zero. Thus, we can write $Hom_R(X, Y) = Hom^1_R(X, Y) \oplus Hom^0_R(X, Y)$.*

As stated in [17] on page 9, $Hom_R(X, Y)$ is a $\mathbb{Z}_2-$graded complex. We now prove it here.
N.B. The letters d, d^0, *and* d^1 used in proposition 5.1 below are defined within this proposition and its proof. These same letters may mean something else once they are taken out of this context. In each context, their meaning is clearly stated.

Proposition 5.1. *Let X and Y be two linear factorizations of an element $f \in R$. Then $Hom_R(X, Y)$ is a $\mathbb{Z}_2-$graded complex with differential*

$$d : \phi \mapsto d_Y \circ \phi - (-1)^{|\phi|}\phi \circ d_X$$

where $|\phi|$ is the degree[3] *of the map ϕ.*

N.B. we will drop the R in $Hom_R(X, Y)$ for ease of notation.

Proof. Since X and Y are linear factorizations of $f \in R$, they are $\mathbb{Z}_2-$graded modules, thus we can write $X = X^0 \oplus X^1$ and $Y = Y^0 \oplus Y^1$. Let the components of d be d^0 and d^1. We have the diagram:

$$C_\bullet \quad \cdots \longrightarrow Hom^1(X, Y) \xrightarrow{d^1} Hom^0(X, Y) \xrightarrow{d^0} Hom^1(X, Y) \longrightarrow \cdots$$

where $Hom^1(X, Y) = Hom(X^0, Y^1) \oplus Hom(X^1, Y^0)$ (i.e., the degree one maps) and $Hom^0(X, Y) = Hom(X^0, Y^0) \oplus Hom(X^1, Y^1)$ (i.e., the degree zero maps).
To prove our proposition, we need to show that $C_\bullet$ is a complex, i.e., the composition of any two consecutive maps is the zero map. Explicitly, the differentials need to satisfy $d^0 \circ d^1 = 0$ and $d^1 \circ d^0 = 0$. First, we have the following fact.

Fact 5.1.

$$f \cdot id_Y \circ \phi = f \cdot \phi \ \ and \quad \phi \circ f \cdot id_{X^0} = f \cdot \phi$$

Proof. The first assertion is obvious. As for the second, it suffices to prove that for $x \in X^0$, $(\phi \circ f \cdot id_{X^0})(x) = (f \cdot \phi)(x)$.

$$\begin{aligned} (\phi \circ f \cdot id_{X^0})(x) &= (\phi(f \cdot id_{X^0}))(x) \\ &= f \cdot (\phi(id_{X^0}(x)))\ f \in R\, and\, \phi\, is\, R - linear. \\ &= f \cdot \phi(id_{X^0}(x)) \\ &= f \cdot \phi(x) \end{aligned}$$

□

We proceed with the proof of the proposition by showing that $d^0 \circ d^1 = 0$.
Let $\phi \in Hom^1(X, Y) = Hom(X^0, Y^1) \oplus Hom(X^1, Y^0)$.
We know that $d^i_Y : Y^i \longrightarrow Y^{i+1}$ and $d^i_X : X^i \longrightarrow X^{i+1}$ where $i \in \mathbb{Z}_2$ and $i + 1 = 0$ if $i = 1$.

Case 1. $\phi \in Hom(X^0, Y^1)$, so $|\phi| = 1$.
We have:

$$\begin{aligned} d^1(\phi) &= d^1_Y \circ \phi - (-1)^{|\phi|}\phi \circ d^1_X \\ &= d^1_Y \circ \phi - (-1)^1\phi \circ d^1_X \quad since\ |\phi| = 1 \\ &= d^1_Y \circ \phi + \phi \circ d^1_X \end{aligned}$$

Since $d^1_Y \circ \phi : X^0 \longrightarrow Y^0$ and $\phi \circ d^1_X : X^1 \longrightarrow Y^1$, we respectively have $|d^1_Y \circ \phi| = 0$ and $|\phi \circ d^1_X| = 0$. (We also see that $|d^1(\phi)| = 0$) .

[3]this is defined in definition1.19

$$\begin{aligned}
d^0 \circ d^1(\phi) &= d^0(d^1_Y \circ \phi + \phi \circ d^1_X) \quad by\ definition\ of\ d^1(\phi) \\
&= d^0(d^1_Y \circ \phi) + d^0(\phi \circ d^1_X) \\
&= d^0_Y \circ (d^1_Y \circ \phi) - (-1)^{|d^1_Y \circ \phi|}(d^1_Y \circ \phi) \circ d^1_X + d^1_Y \circ (\phi \circ d^1_X) - (-1)^{|\phi \circ d^1_X|}(\phi \circ d^1_X) \circ d^0_X \\
&= (d^0_Y \circ d^1_Y) \circ \phi - (-1)^0(d^1_Y \circ \phi) \circ d^1_X + (d^1_Y \circ \phi) \circ d^1_X - (-1)^0 \phi \circ (d^1_X \circ d^0_X) \\
&= (d^0_Y \circ d^1_Y) \circ \phi - \phi \circ (d^1_X \circ d^0_X) \quad since\ -(-1)^0(d^1_Y \circ \phi) \circ d^1_X + (d^1_Y \circ \phi) \circ d^1_X = 0 \\
&= f \cdot id_{Y^1} \circ \phi - \phi \circ f \cdot id_{X^0} \quad since\ d^0_Y \circ d^1_Y = f \cdot id_{Y^1}\ and\ d^1_X \circ d^0_X = f \cdot id_{X^0} \\
&= f \cdot \phi - f \cdot \phi \quad by\ fact\, 5.1 \\
&= 0
\end{aligned}$$

The third equality above is obtained by definition of d^0, the fourth is by associativity and the fact that $|d^1_Y \circ \phi| = 0$ and $|\phi \circ d^1_X| = 0$.

Case 2. $\phi \in Hom(X^1, Y^0)$, so $|\phi| = 1$.
We have:

$$\begin{aligned}
d^1(\phi) &= d^0_Y \circ \phi - (-1)^{|\phi|} \phi \circ d^0_X \\
&= d^0_Y \circ \phi - (-1)^1 \phi \circ d^0_X \quad since\ |\phi| = 1 \\
&= d^0_Y \circ \phi + \phi \circ d^0_X
\end{aligned}$$

Since $d^0_Y \circ \phi : X^1 \longrightarrow Y^1$ and $\phi \circ d^0_X : X^0 \longrightarrow Y^0$, we respectively have $|d^0_Y \circ \phi| = 0$ and $|\phi \circ d^0_X| = 0$. (We also see that $|d^1(\phi)| = 0$) .

$$\begin{aligned}
d^0 \circ d^1(\phi) &= d^0(d^0_Y \circ \phi + \phi \circ d^0_X) \quad by\ definition\ of\ d^1(\phi) \\
&= d^0(d^0_Y \circ \phi) + d^0(\phi \circ d^0_X) \\
&= d^1_Y \circ (d^0_Y \circ \phi) - (-1)^{|d^0_Y \circ \phi|}(d^0_Y \circ \phi) \circ d^0_X + d^0_Y \circ (\phi \circ d^0_X) - (-1)^{|\phi \circ d^0_X|}(\phi \circ d^0_X) \circ d^1_X \\
&= (d^1_Y \circ d^0_Y) \circ \phi - (-1)^0(d^0_Y \circ \phi) \circ d^0_X + (d^0_Y \circ \phi) \circ d^0_X - (-1)^0 \phi \circ (d^0_X \circ d^1_X) \\
&= (d^1_Y \circ d^0_Y) \circ \phi - \phi \circ (d^0_X \circ d^1_X) \quad since\ -(-1)^0(d^0_Y \circ \phi) \circ d^0_X + (d^0_Y \circ \phi) \circ d^0_X = 0 \\
&= f \cdot id_{Y^0} \circ \phi - \phi \circ f \cdot id_{X^1} \quad since\ d^1_Y \circ d^0_Y = f \cdot id_{Y^0}\ and\ d^0_X \circ d^1_X = f \cdot id_{X^1} \\
&= f \cdot \phi - f \cdot \phi \quad by\ fact\, 5.1 \\
&= 0
\end{aligned}$$

The third equality above is obtained by definition of d^0, the fourth is by associativity and the fact that $|d^0_Y \circ \phi| = 0$ and $|\phi \circ d^0_X| = 0$.
So $d^0 \circ d^1 = 0$ as desired.

Next, we show that $d^1 \circ d^0 = 0$.

Let $\phi \in Hom^0(X, Y) = Hom(X^0, Y^0) \oplus Hom(X^1, Y^1)$.

Case 1.

$\phi \in Hom(X^0, Y^0)$, so $|\phi| = 0$.

We have:

$$\begin{aligned} d^0(\phi) &= d_Y^0 \circ \phi - (-1)^{|\phi|}\phi \circ d_X^1 \\ &= d_Y^0 \circ \phi - (-1)^0\phi \circ d_X^1 \quad since\ |\phi| = 0 \\ &= d_Y^0 \circ \phi - \phi \circ d_X^1 \end{aligned}$$

Since $d_Y^0 \circ \phi : X^0 \longrightarrow Y^1$ and $\phi \circ d_X^1 : X^1 \longrightarrow Y^0$, we respectively have $|d_Y^0 \circ \phi| = 1$ and $|\phi \circ d_X^1| = 1$. (We also see that $|d^0(\phi)| = 1$) .

$$\begin{aligned} d^1 \circ d^0(\phi) &= d^1(d_Y^0 \circ \phi - \phi \circ d_X^1) \quad by\ definition\ of\ d^0(\phi) \\ &= d^1(d_Y^0 \circ \phi) - d^1(\phi \circ d_X^1) \\ &= d_Y^1 \circ (d_Y^0 \circ \phi) - (-1)^{|d_Y^0 \circ \phi|}(d_Y^0 \circ \phi) \circ d_X^1 - (d_Y^0 \circ (\phi \circ d_X^1) - (-1)^{|\phi \circ d_X^1|}(\phi \circ d_X^1) \circ d_X^0) \\ &= (d_Y^1 \circ d_Y^0) \circ \phi - (-1)^1 d_Y^0 \circ (\phi \circ d_X^1) - d_Y^0 \circ (\phi \circ d_X^1) + (-1)^1 \phi \circ (d_X^1 \circ d_X^0) \\ &= (d_Y^1 \circ d_Y^0) \circ \phi - \phi \circ (d_X^1 \circ d_X^0) \quad since\ -(-1)^1 d_Y^0 \circ (\phi \circ d_X^1) - d_Y^0 \circ (\phi \circ d_X^1) = 0 \\ &= f \cdot id_{Y^0} \circ \phi - \phi \circ f \cdot id_{X^0} \quad since\ d_Y^1 \circ d_Y^0 = f \cdot id_{Y^0}\ and\ d_X^1 \circ d_X^0 = f \cdot id_{X^0} \\ &= f \cdot \phi - f \cdot \phi \quad by\, fact\, 5.1 \\ &= 0 \end{aligned}$$

The third equality above is obtained by definition of d^1, the fourth is by associativity and the fact that $|d_Y^0 \circ \phi| = 1$ and $|\phi \circ d_X^1| = 1$.

Case 2. $\phi \in Hom(X^1, Y^1)$, so $|\phi| = 0$.

We have:

$$\begin{aligned} d^0(\phi) &= d_Y^1 \circ \phi - (-1)^{|\phi|}\phi \circ d_X^0 \\ &= d_Y^1 \circ \phi - (-1)^0\phi \circ d_X^0 \quad since\ |\phi| = 0 \\ &= d_Y^1 \circ \phi - \phi \circ d_X^0 \end{aligned}$$

Since $d_Y^1 \circ \phi : X^1 \longrightarrow Y^0$ and $\phi \circ d_X^0 : X^0 \longrightarrow Y^1$, we respectively have $|d_Y^1 \circ \phi| = 1$ and $|\phi \circ d_X^0| = 1$. (We also see that $|d^0(\phi)| = 1$) .

$$
\begin{aligned}
d^1 \circ d^0(\phi) &= d^1(d_Y^1 \circ \phi - \phi \circ d_X^0) \quad by\ definition\ of\ d^0(\phi) \\
&= d^1(d_Y^1 \circ \phi) - d^1(\phi \circ d_X^0) \\
&= d_Y^0 \circ (d_Y^1 \circ \phi) - (-1)^{|d_Y^1 \circ \phi|}(d_Y^1 \circ \phi) \circ d_X^0 - (d_Y^1 \circ (\phi \circ d_X^0) - (-1)^{|\phi \circ d_X^0|}(\phi \circ d_X^0) \circ d_X^1) \\
&= (d_Y^0 \circ d_Y^1) \circ \phi - (-1)^1(d_Y^1 \circ \phi) \circ d_X^0 - (d_Y^1 \circ \phi) \circ d_X^0 + (-1)^1 \phi \circ (d_X^0 \circ d_X^1) \\
&= (d_Y^1 \circ d_Y^0) \circ \phi - \phi \circ (d_X^0 \circ d_X^1) \ \ since\ \ -(-1)^1(d_Y^1 \circ \phi) \circ d_X^0 - (d_Y^1 \circ \phi) \circ d_X^0 = 0 \\
&= f \cdot id_{Y^1} \circ \phi - \phi \circ f \cdot id_{X^0} \ \ since\ d_Y^0 \circ d_Y^1 = f \cdot id_{Y^1}\ and\ d_X^1 \circ d_X^0 = f \cdot id_{X^0} \\
&= f \cdot \phi - f \cdot \phi \ \ by\ fact\ 5.1 \\
&= 0
\end{aligned}
$$

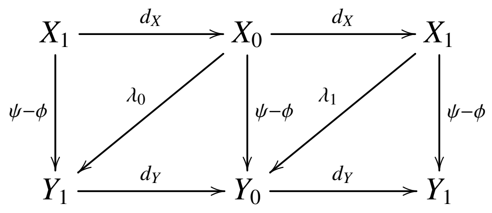
The third equality above is obtained by definition of d^1, the fourth is by associativity and the fact that $|d_Y^1 \circ \phi| = 1$ and $|\phi \circ d_X^0| = 1$.
So $d^1 \circ d^0 = 0$ as desired. This completes the proof of the proposition.

□

Recall that our main goal in this chapter is to review the bicategory $\mathcal{LG}_K$ and in particular to elucidate the intricate construction of the unit in this bicategory. To that end, we need more ingredients. We'll need the notion of homotopy between linear factorizations and we will also need the notion of tensor products of matrix factorizations.

Notations 5.1. *Let R be a commutative ring and $f \in R$. We will write $F(R, f)$ for the category whose objects are linear factorizations of f and morphisms are homomorphisms of linear factorization.*

Definition 5.3. *(p.9 [17]) homotopic linear factorizations*
*Let (X, d_X) and (Y, d_Y) be linear factorizations. Two morphisms $\varphi, \psi : X \longrightarrow Y$ are **homotopic** if there exists an odd $R-$linear map $\lambda : X \longrightarrow Y$ such that $d_Y\lambda + \lambda d_X = \psi - \varphi$.*
More precisely, the following diagram commutes:



†

i.e.,

$$d_Y \circ \lambda_0 + \lambda_1 \circ d_X = \psi - \phi$$

Recall that given a category C, a *congruence relation* R on C, is an equivalence relation R_{XY} on $Hom_C(X, Y)$ for objects X, Y s.t. the equivalence relations are compatible with composition (cf. definition **??**). Moreover, we know from definition **??**, that given a congruence relation R on C, we can define the quotient category C/R whose objects are those of C but morphisms are equivalence classes of morphisms in C. Composition of morphisms in C/R is well defined since R is a congruence relation.

It is easy to see that equality up to homotopy is an equivalence relation. This relation is compatible with composition and so one can form the quotient category denoted by $HF(R, f)$ in [17].

When defining a matrix factorization in the previous section, we did not talk about the rank of a matrix factorization, it is time to do so.

5.1.2 A glimpse of factorizations of finite rank

This subsection mostly relies on work done in [40] (p.18, 20) and [17]).
If X is a matrix factorization, then it is not always the case that the ranks of the $R-$modules X^0 and X^1 be finite.

Definition 5.4. *We say that a matrix factorization is of finite rank if its underlying free R-module is of* ***finite rank****.*

Given a basis for the R-module $X = X^0 \oplus X^1$, the differential d_X is sometimes identified with the matrix

$$d_X = \begin{pmatrix} 0 & d_X^1 \\ d_X^0 & 0 \end{pmatrix}$$

as mentioned in [17].

Thus,

$$d_X^2 = \begin{pmatrix} 0 & d_X^1 \\ d_X^0 & 0 \end{pmatrix}\begin{pmatrix} 0 & d_X^1 \\ d_X^0 & 0 \end{pmatrix} = \begin{pmatrix} d_X^1 d_X^0 & 0 \\ 0 & d_X^0 d_X^1 \end{pmatrix} = \begin{pmatrix} f \cdot id_{X^0} & 0 \\ 0 & f \cdot id_{X^1} \end{pmatrix} = f \begin{pmatrix} id_{X^0} & 0 \\ 0 & id_{X^1} \end{pmatrix}$$

So, $d_X^2 = fId$ where $Id = \begin{pmatrix} id_{X^0} & 0 \\ 0 & id_{X^1} \end{pmatrix}$.

Notations 5.2. *We keep the following notations used in [17]:*
- *We denote by $HF(R, f)$ the category of linear factorizations of $f \in R$ modulo homotopy.*
- *We also denote by $HMF(R, f)$ its full subcategory of matrix factorizations.*
- *Furthermore, we write $hmf(R, f)$ for the full subcategory of finite rank matrix factorizations, viz. the matrix factorizations whose underlying R-module is free of finite rank.*

Recall that the category of matrix factorizations of f is denoted $MF(R, f)$ (or sometimes simply denoted by $MF(f)$ when there is no risk of confusion).

$$Hom_{HMF(R,f)}(X, Y) = Hom_{MF(R,f)}(X, Y)/\{Null - homotopic\ maps\}$$

A null-homotopic map is a map that is homotopic to the zero map. $MF(R, f)$ is additive and $R-$linear (p.19 [40])[4].

If we choose bases in the free $R-$modules X^0 and X^1, then we can write the maps d_X^0 and d_X^1 as $m \times m$ matrices D_X^0 and D_X^1 with coefficients in R. These matrices satisfy the equalities:

$$D_X^0 D_X^1 = f \cdot Id, \quad D_X^1 D_X^0 = f \cdot Id$$

[4] In [40], the author writes MF_f^{all} instead of $MF(R, f)$

where Id stands for the identity $m \times m$ matrix. Alternatively, we can describe this factorization by a $2m \times 2m$ matrix with off-diagonal blocks D^0 and D^1:

$$D = \begin{pmatrix} 0 & D^1 \\ D^0 & 0 \end{pmatrix} \qquad D^2 = f \cdot Id, \text{ or written simply } D^2 = fId,$$

where we dropped the subscripts on D for ease of notation. The Id in the last equality above is the identity matrix of size $2m$.
Matrix description of objects in $hmf(R, f)$ extends to infinite rank factorizations. Matrices D^0 and D^1 then have infinite rank, but each of their columns has only finitely many nonzero entries. If factorizations X and Y are written in matrix form, $X = (D^0_X, D^1_X)$ and $Y = (D^0_Y, D^1_Y)$ a homomorphism $g : X \longrightarrow Y$ is a pair of matrices (G_0, G_1) such that $G_1 D^0_X = D^0_Y G_0$ and $G_0 D^1_X = D^1_Y G_1$.

Remark 5.3. *The two equations $G_1 D^0_X = D^0_Y G_0$ and $G_0 D^1_X = D^1_Y G_1$ are equivalent (p.20 [40], p.174 [44]). We give a proof of the equivalence here.*

$$\begin{aligned}
G_0 D^1_X = D^1_Y G_1 \quad &\Rightarrow \quad G_0 D^1_X D^0_X = D^1_Y G_1 D^0_X \\
&\Rightarrow \quad D^0_Y G_0 D^1_X D^0_X = D^0_Y D^1_Y G_1 D^0_X \\
&\Rightarrow \quad D^0_Y G_0 f \cdot Id = f \cdot Id G_1 D^0_X \\
&\Rightarrow \quad f \cdot D^0_Y G_0 Id = f \cdot Id G_1 D^0_X \\
&\Rightarrow \quad f \cdot Id D^0_Y G_0 = f \cdot Id G_1 D^0_X \quad \textit{since } D^0_Y G_0 Id = Id D^0_Y G_0 \\
&\Rightarrow \quad (f \cdot Id)(D^0_Y G_0 - G_1 D^0_X) = 0 \\
&\Rightarrow \quad D^0_Y G_0 = G_1 D^0_X \qquad \textit{as desired.}
\end{aligned}$$

The third implication above holds because $D^1_X D^0_X = f \cdot Id = D^0_Y D^1_Y$. The fourth holds because $f \in R$ and so can be moved across matrices. The last implication is true because here f is an arbitrary element of R and $f \cdot Id$ cannot be zero for all f. Similarly, one can show that $D^0_Y G_0 = G_1 D^0_X \Rightarrow G_0 D^1_X = D^1_Y G_1$. Hence the equivalence holds.

5.1.3 Tensor Products of Matrix Factorizations (Revisited)

The notion of tensor product of matrix factorizations first appeared in Yoshino's paper [67]. Recall that Yoshino [67] defines an $n \times n$ matrix factorization of a polynomial f to be a pair of matrices (P, Q) such that $PQ = f \cdot I_n$ (see definition 2.1). Also recall the definition of the Yoshino's tensor product of matrix factorizations (cf. definition 2.3):
Let $X = (\phi, \psi)$ be an $n \times n$ matrix factorization of $f \in R$ and $X' = (\phi', \psi')$ an $m \times m$ matrix factorization of $g \in S$. These matrices can be considered as matrices over $L = K[[x, y]]$ and the **tensor product** $X \widehat{\otimes} X'$ is given by
$\left(\begin{bmatrix} \phi \otimes 1_m & 1_n \otimes \phi' \\ -1_n \otimes \psi' & \psi \otimes 1_m \end{bmatrix}, \begin{bmatrix} \psi \otimes 1_m & -1_n \otimes \phi' \\ 1_n \otimes \psi' & \phi \otimes 1_m \end{bmatrix}\right)$
where each component is an endomorphism on $L^n \otimes L^m$.
N.B. Though Yoshino's tensor product is written with a hat, henceforth we will sometimes drop the hat of $\widehat{\otimes}$ and simply write $\otimes$ for ease of notation whenever there will be no risk of confusion.
Section 2 of [17] gives another definition of matrix factorizations in terms of $\mathbb{Z}_2$-graded

free modules. In fact, a matrix factorization of $f \in K[x_1, \cdots, x_n]$ can also be defined as a $\mathbb{Z}_2-$graded free $R-$module $X = X^0 \oplus X^1$ together with an odd $R-$linear endomorphism (also called differential map) d_X s.t. $d_X^2 = f \cdot 1_X$ (cf. section 2 of [17]). Thus, if we consider the pair of matrices representing the (even and odd) components of such a differential map, we recover the definition of a matrix factorization of f as earlier recalled (cf. definition 2.1).

Henceforth in this chapter, we will often simply write "factorization(s)" in place of "matrix factorization(s)".

We are soon going to see (cf. section 5.2) that a $1-$morphism in the bicategory $\mathcal{LG}_K$ is a factorization of the difference of two polynomials. It is important to recall that the definition of matrix factorization used in [17] is the one given in section 2 of [17], i.e., a pair (X, d_X) where X is a graded module and d_X is the differential map as explained above. We saw in definition 2.3 how to construct a factorization of the sum of two polynomials from the factorizations of each of them. But also remark that in definition 2.3, the factorizations are given as pairs of matrices and not in the form (X, d_X) as defined in [17]. So, the question is: how do we use the Yoshino's tensor product on say (X, d_X) (a factorization of f) and (Y, d_Y) (a factorization of g) to produce a factorization of $f + g$? Such details are not given in [17].

In this subsection, we explain how it is done. To that end, we need to prove lemma 5.1 which actually shows how to produce a factorization of the sum of two polynomials from the factorizations of each of them, when the factorizations are defined as pairs (X, d_X) as explained earlier.

Let R, T and S be commutative $K-$algebras with $f \in R = K[x]$ and $g \in S = K[y]$. Let (X, d_X) be a factorization of $f \in K[x]$ and (Y, d_Y) be a factorization of $g \in K[y]$, where we assume that X is an $R-S-$bimodule and Y is an $S-T-$bimodule. If we use the same notation for a map and the matrix representing it (precisely for d_X and d_Y), then, we can use the Yoshino's tensor product to define a new factorization $(X, d_X)\widehat{\otimes}(Y, d_Y) := (X\otimes Y, d_{X\otimes Y} = D)$ as follows:

$$(X \otimes Y)^0 = (X^0 \otimes Y^0) \oplus (X^1 \otimes Y^1) \qquad (X \otimes Y)^1 = (X^0 \otimes Y^1) \oplus (X^1 \otimes Y^0)$$

where the tensor product between the components of X and Y is taken over S. The matrix D of the differential is obtained as follows: Observe that looking at the matrices representing the Yoshino's tensor product of matrix factorizations (definition 2.3, recalled above), and mindful of the fact that (ϕ, ψ) and (ϕ', ψ') in that definition are actually represented here by (d_X^0, d_X^1) and (d_Y^0, d_Y^1), if we write $D = (D^0, D^1)$ the matrix representing the map $d_{X\otimes Y}$, then, we can write:

$$D^1 = \begin{pmatrix} d_X^1 \otimes id_{Y^1} & -id_{X^0} \otimes d_Y^0 \\ id_{X^1} \otimes d_Y^1 & d_X^0 \otimes id_{Y^0} \end{pmatrix} \qquad D^0 = \begin{pmatrix} d_X^0 \otimes id_{Y^1} & id_{X^1} \otimes d_Y^0 \\ -id_{X^0} \otimes d_Y^1 & d_X^1 \otimes id_{Y^0} \end{pmatrix}$$

where we used the same notation for a matrix and its map. In fact, for $i \in \{0, 1\}$, d_Y^i and d_X^i are respectively the matrices of the maps $d_Y^i : Y^i \to Y^{i+1}$ and $d_X^i : X^i \to X^{i+1}$. Likewise id_{Y^i} and id_{X^i} are the matrices of the corresponding identity maps.

In the sequel, we will denote the units of R and S by 1_R and 1_S respectively.

Lemma 5.1. *$(X\otimes Y, D)$ as defined above, determines an object of $MF(R\otimes S, f\otimes 1_S + 1_R\otimes g)$*

Proof. We need to verify that $(X \otimes Y, D)$ is a factorization of $f \otimes 1_S + 1_R \otimes g \in R \otimes S$. Let $h = f \otimes 1_S + 1_R \otimes g \in R \otimes S$.
Before proceeding with the proof of this lemma, recall what we saw immediately after definition 5.4, namely that for a matrix factorization $X = X^0 \oplus X^1$, its differential d_X could be viewed as a matrix (that we still denote by d_X) and the condition to be fulfilled should then be $d_X^2 = f \cdot Id$. Now that we have the module

$$X \otimes Y = (X^0 \otimes Y^0) \oplus (X^1 \otimes Y^1) \oplus (X^0 \otimes Y^1) \oplus (X^1 \otimes Y^0).$$

In order to prove the lemma, we will need to prove that $D^2 = h \cdot Id$.
We know that:

$$D^2 = \begin{pmatrix} 0 & D^1 \\ D^0 & 0 \end{pmatrix}\begin{pmatrix} 0 & D^1 \\ D^0 & 0 \end{pmatrix} = \begin{pmatrix} D^1 D^0 & 0 \\ 0 & D^0 D^1 \end{pmatrix}.$$

In the sequel, we are going to implicitly use the mixed-product property (cf. Lemma 4.2.10 in [32]) of the tensor product.

Lemma 5.2. *(cf. Lemma 4.2.10 in [32])*
If A, B, C and D are matrices of such size that one can form the matrix products AC and BD, then the product of two tensor products yields another tensor product $(A \otimes B)(C \otimes D) = AC \otimes BD$.

$$D^1 D^0 = \begin{pmatrix} d_X^1 \otimes id_{Y^1} & -id_{X^0} \otimes d_Y^0 \\ id_{X^1} \otimes d_Y^1 & d_X^0 \otimes id_{Y^0} \end{pmatrix}\begin{pmatrix} d_X^0 \otimes id_{Y^1} & id_{X^1} \otimes d_Y^0 \\ -id_{X^0} \otimes d_Y^1 & d_X^1 \otimes id_{Y^0} \end{pmatrix}$$

$$= \begin{pmatrix} (d_X^1 \otimes id_{Y^1})(d_X^0 \otimes id_{Y^1}) + (-id_{X^0} \otimes d_Y^0)(-id_{X^0} \otimes d_Y^1) & (d_X^1 \otimes id_{Y^1})(id_{X^1} \otimes d_Y^0) + (-id_{X^0} \otimes d_Y^0)(d_X^1 \otimes id_{Y^0}) \\ (id_{X^1} \otimes d_Y^1)(d_X^0 \otimes id_{Y^1}) + (d_X^0 \otimes id_{Y^0})(-id_{X^0} \otimes d_Y^1) & (id_{X^1} \otimes d_Y^1)(id_{X^1} \otimes d_Y^0) + (d_X^0 \otimes id_{Y^0})(d_X^1 \otimes id_{Y^0}) \end{pmatrix}$$

$$= \begin{pmatrix} d_X^1 d_X^0 \otimes id_{Y^1} id_{Y^1} + id_{X^0} id_{X^0} \otimes d_Y^0 d_Y^1 & d_X^1 id_{X^1} \otimes id_{Y^1} d_Y^0 - id_{X^0} d_X^1 \otimes d_Y^0 id_{Y^0} \\ id_{X^1} d_X^0 \otimes d_Y^1 id_{Y^1} - d_X^0 id_{X^0} \otimes id_{Y^0} d_Y^1 & id_{X^1} id_{X^1} \otimes d_Y^1 d_Y^0 + d_X^0 d_X^1 \otimes id_{Y^0} id_{Y^0} \end{pmatrix}$$

$$= \begin{pmatrix} f \cdot id_{X^0} \otimes id_{Y^1} + id_{X^0} \otimes g \cdot id_{Y^1} & d_X^1 \otimes d_Y^0 - d_X^1 \otimes d_Y^0 \\ d_X^0 \otimes d_Y^1 - d_X^0 \otimes d_Y^1 & id_{X^1} \otimes g \cdot id_{Y^0} + f \cdot id_{X^1} \otimes id_{Y^0} \end{pmatrix}$$

$$= \begin{pmatrix} f \cdot id_{X^0} \otimes id_{Y^1} + id_{X^0} \otimes g \cdot id_{Y^1} & 0 \\ 0 & id_{X^1} \otimes g \cdot id_{Y^0} + f \cdot id_{X^1} \otimes id_{Y^0} \end{pmatrix}$$

$$= \begin{pmatrix} (f \otimes 1_S) \cdot (id_{X^0} \otimes id_{Y^1}) + (1_R \otimes g) \cdot (id_{X^0} \otimes id_{Y^1}) & 0 \\ 0 & (1_R \otimes g) \cdot (id_{X^1} \otimes id_{Y^0}) + (f \otimes 1_S) \cdot (id_{X^1} \otimes id_{Y^0}) \end{pmatrix}$$

$$= \begin{pmatrix} (f \otimes 1_S + 1_R \otimes g) \cdot (id_{X^0} \otimes id_{Y^1}) & 0 \\ 0 & (f \otimes 1_S + 1_R \otimes g) \cdot (id_{X^1} \otimes id_{Y^0}) \end{pmatrix}$$

$$= (f \otimes 1_S + 1_R \otimes g) \cdot \begin{pmatrix} id_{X^0 \otimes Y^1} & 0 \\ 0 & id_{X^1 \otimes Y^0} \end{pmatrix}$$

$$= h \cdot \begin{pmatrix} id_{X^0 \otimes Y^1} & 0 \\ 0 & id_{X^1 \otimes Y^0} \end{pmatrix} \quad \ldots \quad (i)$$

$$D^0 D^1 = \begin{pmatrix} d_X^0 \otimes id_{Y^1} & id_{X^1} \otimes d_Y^0 \\ -id_{X^0} \otimes d_Y^1 & d_X^1 \otimes id_{Y^0} \end{pmatrix}\begin{pmatrix} d_X^1 \otimes id_{Y^1} & -id_{X^0} \otimes d_Y^0 \\ id_{X^1} \otimes d_Y^1 & d_X^0 \otimes id_{Y^0} \end{pmatrix}$$

$$= \begin{pmatrix} (d_X^0 \otimes id_{Y^1})(d_X^1 \otimes id_{Y^1}) + (id_{X^1} \otimes d_Y^0)(id_{X^1} \otimes d_Y^1) & (d_X^0 \otimes id_{Y^1})(-id_{X^0} \otimes d_Y^0) + (id_{X^1} \otimes d_Y^0)(d_X^0 \otimes id_{Y^0}) \\ (-id_{X^0} \otimes d_Y^1)(d_X^1 \otimes id_{Y^1}) + (d_X^1 \otimes id_{Y^0})(id_{X^1} \otimes d_Y^1) & (-id_{X^0} \otimes d_Y^1)(-id_{X^0} \otimes d_Y^0) + (d_X^1 \otimes id_{Y^0})(d_X^0 \otimes id_{Y^0}) \end{pmatrix}$$

$$= \begin{pmatrix} d_X^0 d_X^1 \otimes id_{Y^1} id_{Y^1} + id_{X^1} id_{X^1} \otimes d_Y^0 d_Y^1 & -d_X^0 id_{X^0} \otimes id_{Y^1} d_Y^0 + id_{X^1} d_X^0 \otimes d_Y^0 id_{Y^0} \\ -id_{X^0} d_X^1 \otimes d_Y^1 id_{Y^1} + d_X^1 id_{X^1} \otimes id_{Y^0} d_Y^1 & id_{X^0} id_{X^0} \otimes d_Y^1 d_Y^0 + d_X^1 d_X^0 \otimes id_{Y^0} id_{Y^0} \end{pmatrix}$$

$$= \begin{pmatrix} f \cdot id_{X^1} \otimes id_{Y^1} + id_{X^1} \otimes g \cdot id_{Y^1} & -d_X^0 \otimes d_Y^0 + d_X^0 \otimes d_Y^0 \\ -d_X^1 \otimes d_Y^1 + d_X^1 \otimes d_Y^1 & id_{X^0} \otimes g \cdot id_{Y^0} + f \cdot id_{X^0} \otimes id_{Y^0} \end{pmatrix}$$

$$= \begin{pmatrix} f \cdot id_{X^1} \otimes id_{Y^1} + id_{X^1} \otimes g \cdot id_{Y^1} & 0 \\ 0 & id_{X^0} \otimes g \cdot id_{Y^0} + f \cdot id_{X^0} \otimes id_{Y^0} \end{pmatrix}$$

$$= \begin{pmatrix} (f \otimes 1_S) \cdot (id_{X^1} \otimes id_{Y^1}) + (1_R \otimes g) \cdot (id_{X^1} \otimes id_{Y^1}) & 0 \\ 0 & (1_R \otimes g) \cdot (id_{X^0} \otimes id_{Y^0}) + (f \otimes 1_S) \cdot (id_{X^0} \otimes id_{Y^0}) \end{pmatrix}$$

$$= \begin{pmatrix} (f \otimes 1_S + 1_R \otimes g) \cdot (id_{X^1} \otimes id_{Y^1}) & 0 \\ 0 & (f \otimes 1_S + 1_R \otimes g) \cdot (id_{X^0} \otimes id_{Y^0}) \end{pmatrix}$$

$$= (f \otimes 1_S + 1_R \otimes g) \cdot \begin{pmatrix} id_{X^1 \otimes Y^1} & 0 \\ 0 & id_{X^0 \otimes Y^0} \end{pmatrix}$$

$$= h \cdot \begin{pmatrix} id_{X^1 \otimes Y^1} & 0 \\ 0 & id_{X^0 \otimes Y^0} \end{pmatrix} \quad \ldots \quad (ii)$$

Hence, putting (i) and (ii) together, we get the desired result, that is;

$$D^2 = \begin{pmatrix} 0 & D^1 \\ D^0 & 0 \end{pmatrix} \begin{pmatrix} 0 & D^1 \\ D^0 & 0 \end{pmatrix} = \begin{pmatrix} D^1 D^0 & 0 \\ 0 & D^0 D^1 \end{pmatrix} = h \cdot \begin{pmatrix} id_{X^0 \otimes Y^1} & 0 & 0 & 0 \\ 0 & id_{X^1 \otimes Y^0} & 0 & 0 \\ 0 & 0 & id_{X^1 \otimes Y^1} & \\ 0 & 0 & 0 & id_{X^0 \otimes Y^0} \end{pmatrix} = h \cdot Id$$

□

The following properties are proved in [67] and the tensor product here is the Yoshino's tensor product.

Lemma 5.3. *(section 2 of [67])*

- *If $h : (X, d) \longrightarrow (Y, d')$ is a morphism of factorizations of $f \in R$ and $(Z, d")$ is a factorization of $g \in S$, then there is an evident morphism of factorizations $h \otimes Z$. Likewise, if $r : (T, d_1) \longrightarrow (L, d_1')$ is a morphism of factorizations of $g \in S$ and $(Z_1, d_1")$ is a factorization of $f \in R$, then there is an evident morphism of factorizations $Z_1 \otimes r$.*

- *Let (X, d) be a factorization of f and (Y, d') be a factorization of g. Then there is an isomorphism $X \otimes Y \cong Y \otimes X$.*

- *There is an isomorphism of factorizations $(X \otimes Y) \otimes Z \cong X \otimes (Y \otimes Z)$*

- *The category $F(R, f)$ has biproducts which are obtained in the evident way, and tensor distributes over this biproduct:*

$$X \otimes (Y \oplus Z) \cong (X \otimes Y) \oplus (X \otimes Z)$$

We now review the bicategory $\mathcal{LG}_K$ constructed in [17]. We will elucidate the intricate construction of the unit 1−morphisms of this bicategory. As mentioned at the beginning of this chapter, in our presentation of $\mathcal{LG}_K$, objects are polynomials without the restrictions imposed on them in section 2.2 of [17]. We actually do not need those restrictions for what we want to do.

5.2 Construction of the bicategory ($\mathcal{LG}_K$)

In order to review the construction of $\mathcal{LG}_K$, we proceed in steps. We first construct a structure that has all the ingredients of a bicategory except for the existence of identity one-cells. Thereafter, we construct the entity that has to act as unit in the bicategory $\mathcal{LG}_K$. Finally, we prove that there is no direct inverse for the unitors (i.e., the right and left identities, see [17]) thereby explaining why in [17], their construction is done at the level of homotopy.

Before we proceed, it is worthwhile stating some well-known facts in the literature, see notes on tensor products by Conrad (for points 1. and 2. below see theorem 4.9, example 4.11 of [18], point 3. simply generalizes point 2.).

Lemma 5.4.
1. *Let $K[x_1, x_2, \cdots, x_n]$ and $K[x'_1, x'_2, \cdots, x'_m]$ be free $K-$modules with respective bases $\{e_i\}_{i=1}^n$ and $\{e'_j\}_{j=1}^m$. Then $\{e_i \otimes e'_j\}_{i=1,\cdots n; j=1,\cdots m}$ is a basis of $K[x_1, x_2, \cdots, x_n] \otimes K[x'_1, x'_2, \cdots, x'_m]$.*
2. *The $K-$modules $K[x_1, x_2, \cdots, x_n] \otimes K[x'_1, x'_2, \cdots, x'_m]$ and $K[x_1, x_2, \cdots, x_n, x'_1, x'_2, \cdots, x'_m]$ are isomorphic as $K-$modules.*
3. *If we let x stand for $x_1, x_2, \cdots, x_n$ and $x^{(l)}$ stand for $x_1^{(l)}, x_2^{(l)}, \cdots, x_{n_l}^{(l)}$, where $n, l, n_l \in \mathbb{N}$, $n = n_0$, and $\{x^{(l)}\}$ means x with l primes, e.g $x^{(2)} = x''$, $x^{(0)} := x$. Then more generally, we have:*

$$K[x, x^{(1)}, \cdots, x^{(l)}] \cong \bigotimes_{p=0,\cdots,l} K[x^{(p)}].$$

Proof.
1. See proof of theorem 4.8 of [18] which is a more general result.
2. It is easy to see that the assignment $\sum k_{ij}(e_i \otimes e'_j) \mapsto \sum k_{ij} e_i e'_j$ yields the required isomorphism of $K-$modules.
3. This goes by easy induction on l, the base case being the preceding point. Also observe that in this isomorphism, $x_i^{(p)}$ corresponds to the tensor $1 \otimes \cdots x_i \cdots \otimes 1$ where x_i is in the pth position, and $i \in \bigcup_{k=0}^{l} \{1, \cdots, n_k\}$.

□

As an aside, since power series are also of interest to us in this dissertation, it is important to mention that though $K[x] \otimes K[y] \cong K[x, y]$, it is not true ([7]) that $K[[x]] \otimes K[[y]] \cong K[[x, y]]$, where $K[[x]]$ is the ring of power series in the indeterminate x.

5.2.1 Bicategorical structure

In this subsection, we introduce the notion of $B-$category (cf. definition 5.5) in order to facilitate the discussion on the bicategorical structure of $\mathcal{LG}_K$.

Definition 5.5. *$B-$category*
*A $B-$**category** $\mathcal{B}$ is an algebraic structure that has all the structure of a bicategory as defined in definition 1.1 except for the existence of identity one-cells.*

That is, $\mathcal{B}$ is made up of the data given in definition 1.1 except point 3, and we should also stress in point 2 that identity one-cells are not required to exist.
Before we continue, we make the following useful remark about the homotopy category of matrix factorizations ($HMF(R, f)$) and one of its interesting full subcategories; namely the homotopy category of finite rank matrix factorizations that is denoted by $hmf(R, f)$.

Remark 5.4. *(p.9 of [17]) $HMF(R, f)$ is idempotent complete ([11], [48]). As earlier stated, we work with polynomials rather than power series, so $hmf(R, f)$ is not necessarily idempotent complete [39]. The idempotent closure of $hmf(R, f)$ (denoted by $hmf(R, f)^{\omega}$) is a full subcategory of $HMF(R, f)$ whose objects are those matrix factorizations which are direct summands of finite-rank matrix factorizations in the homotopy category. Moreover, $hmf(R, f)^{\omega}$ is an idempotent complete category.*
As explained in [17] (p.9), taking the idempotent completion is necessary because the composition of 1-morphisms in $\mathcal{LG}_K$ results in matrix factorisations which, while not finite-rank, are summands in the homotopy category of something finite-rank. There are two natural ways to resolve this: work throughout with power series rings and completed tensor products, or work with idempotent completions.

We construct a $B-$category which we call $B-Fac$. The objects of $B-Fac$ are polynomials f denoted by pairs (R, f) where $f \in R = K[x]$. Let $(R = K[x], f)$ and $(S = K[z], g)$ be elements of $B - Fac$. We then define the small category $B - Fac((R, f),(S, g))$ as follows:

$$B - Fac((R, f),(S, g)) := hmf(R \otimes S, 1_R \otimes g - f \otimes 1_S)^{\omega} = hmf(K[x, z], g - f)^{\omega}$$

viz. a 1-morphism between two polynomials f and g is a matrix factorization of $g - f$. (Recall that $hmf(K[x, z], g - f)^{\omega}$ is a subcategory of the category of matrix factorizations modulo homotopy denoted by HMF(K[x,z] , g - f)).

Then given two composable $1-$cells $X \in B-Fac((R, f),(S, g))$ and $Y \in B-Fac((S, g),(T, h))$, we define their composition using Yoshino's tensor product as discussed in 5.1.3. $Y \circ X := Y \otimes_S X \in HMF(R \otimes_K T, 1_R \otimes h - f \otimes 1_T)$ which is a $\mathbb{Z}_2-$graded module, where:

$$(Y \otimes_S X)^0 = (Y^0 \otimes_S X^0) \oplus (Y^1 \otimes_S X^1) \ \ and \ \ (Y \otimes_S X)^1 = (Y^0 \otimes_S X^1) \oplus (Y^1 \otimes_S X^0),$$

the differential ([17]) is defined like the one given above when discussing the tensor product of factorizations, except that here, one is taking the tensor product to be over S.

$$d_{Y \otimes X} = d_Y \otimes 1 + 1 \otimes d_X$$

where $1 \otimes d_X$ has the usual Koszul signs when applied to elements. That is;

$$d_{Y \otimes X}(y, x) = d_Y(y) \otimes x + (-1)^i y \otimes d_X(x)$$

(see p.28 [40]) where $y \in Y^i$.
By remark 2.1.8 on p.29 of [15], $Y \otimes_S X$ is a free module of infinite rank over $R \otimes_K T$. However, the argument of Section 12 of [24] shows that it is naturally isomorphic to a direct summand in the homotopy category of something finite-rank. Thus we may define $Y \circ X := Y \otimes_S X \in hmf(R \otimes_K T, 1_R \otimes h - f \otimes 1_T)^\omega = B - Fac((R, f), (T, h))$.

It is now time to define the tensor product of morphisms of matrix factorizations. Let X_1, X_2 be objects of $B - Fac((R, f), (S, g))$ and Y_1, Y_2 be objects of $B - Fac((S, g), (T, h))$. Let $\alpha : X_1 \to X_2$ and $\beta : Y_1 \to Y_2$ be two morphisms, then we define their tensor product in the obvious way $\beta \otimes \alpha : Y_1 \otimes X_1 \to Y_2 \otimes X_2$ in $B - Fac((R, f), (T, h))$.
With the above data, the composition (bi-)functor is entirely determined in our B-category:
$\star_{(R,f),(S,g),(T,h)} : B{-}Fac((R, f), (S, g))$ x $B{-}Fac((S, g), (T, h)) \to B{-}Fac((R, f), (T, h))$, $(X, Y) \mapsto Y \otimes_S X$.

The definition of the associativity morphism is easy to state. In fact, for $X \in B - Fac((R, f), (S, g))$, $Y \in B{-}Fac((S, g), (T, h))$ and $Z \in B{-}Fac((T, h), (P, r))$, the associator is the 2-isomorphism

$$a_{Z,Y,X} : Z \otimes (Y \otimes X) \to (Z \otimes Y) \otimes X$$

given by the usual formula

$$z \otimes (y \otimes x) \to (z \otimes y) \otimes x$$

where $x \in X$, $y \in Y$ and $z \in Z$.

Lemma 5.5. *The structure $B - Fac$ is a B-category.*

To prove this, we need to show that all the conditions required to obtain a bicategory are met except for the existence of units and the left and right unit actions. The proof though not difficult is lengthy and we would rather focus more on the units construction which is our main point of interest in the review of the bicategory of Landau-Ginzburg models.

For the remainder of the chapter, we will construct the unit $1-$morphisms of $\mathcal{LG}_K$ and prove the naturality of the right and left unit maps. We will prove that these unit actions do not possess direct inverses thereby accounting for the fact that their inverses are found only up to homotopy in [17].

In order to obtain the bicategory $\mathcal{LG}_K$ from the B-category $B - Fac$, we need to define the unit $1-$morphisms, the left and the right unitors.

5.2.2 Unit $1-$morphisms in $\mathcal{LG}_K$

Here, we will construct the identity 1-cells including all the intricacies involved. We let $R = K[x_1, x_2, \cdots, x_n]$.

From lemma 5.4, we have in particular that

$$R \otimes_K R \cong K[x_1, x_2, \cdots, x_n, x'_1, x'_2, \cdots, x'_n].$$

where $x_i = x_i \otimes 1$ and $x_i^{'} = 1 \otimes x_i$.
The subscript "K" in $\otimes_K$ will be very often omitted for ease of notation. We need an object $\Delta_f : (R, f) \to (R, f)$ in $hmf(R \otimes R, f \otimes id - id \otimes f)$ or equivalently,
$hmf(K[x_1, x_2, \cdots, x_n, x_1', x_2', \cdots, x_n'], h(x, x'))$, where $h(x, x') = f(x) - f(x')$, where id stands for 1_R. In the remainder of this chapter, we will often use id to denote the identity in the ring under consideration; even if we are dealing with more than one ring, the difference will always be clear in context.
Recall (cf. section 5.5 of [61]): The exterior algebra $\bigwedge(V)$ of a vector space V over a field K is defined as the quotient algebra of the tensor algebra; $T(V) = \bigoplus_{i=1}^{\infty} T^i(V) = K \bigoplus V \bigoplus (V \otimes V) \bigoplus (V \otimes V \otimes V) \bigoplus \cdots$, by the two-sided ideal I generated by all elements of the form $x \otimes x$ for $x \in V$. Symbolically, $\bigwedge(V) = T(V)/I$. The exterior product $\wedge$ of two elements of $\bigwedge(V)$ is the product induced by the tensor product $\otimes$ of $T(V)$. That is, if

$\pi : T(V) \to \bigwedge(V) = T(V)/I$

is the canonical surjection, and if a and b are in $\bigwedge(V)$, then there are α and β in $T(V)$ such that $a = \pi(\alpha)$ and $b = \pi(\beta)$ and $a \wedge b = \pi(\alpha \otimes \beta)$. Let $\theta_1, \theta_2, \cdots, \theta_n$ be formal symbols[5]
We consider the $R \otimes R-$module:

$$\Delta_f = \bigwedge\{\bigoplus_{i=1}^{n}(R \otimes R)\theta_i\}$$

This is an exterior algebra generated by n anti-commuting variables, the $\theta_i s$ modulo the relations that the θ's anti-commute, that is $\theta_i \wedge \theta_j = -\theta_j \wedge \theta_i$. Typically, we will omit the wedge product and write for instance $\theta_i \wedge \theta_j$ simply as $\theta_i \theta_j$. Here, the "$\wedge$" product is taken over K just like the tensor product. A typical element of Δ_f is $(r \otimes r')\theta_{i_1}\theta_{i_2} \cdots \theta_{i_k}$ or equivalently $h(x_1, x_2, \cdots, x_n, x_1', x_2', \cdots, x_n')\theta_{i_1}\theta_{i_2} \cdots \theta_{i_k}$ where $i_1, \cdots, i_k \in \{1, \cdots, n\}$ and $h(x_1, x_2, \cdots, x_n, x_1', x_2', \cdots, x_n') \in K[x_1, x_2, \cdots, x_n, x_1', x_2', \cdots, x_n']$.
Δ_f as an algebra is finitely generated by the set of formal symbols $\{\theta_1, \cdots, \theta_n\}$.
Δ_f as an $R \otimes R-$module is generated by the set containing the empty list and all products of the form $\theta_{i_1} \ldots \theta_{i_k}$ where $i_1, \cdots, i_k \in \{1, \cdots, n\}$. This set of generators must be finite because by definition of Δ_f, $\theta_{i_p}\theta_{i_q} = -\theta_{i_p}\theta_{i_q}$ and $\theta_{i_p}\theta_{i_q} = 0$ if $p = q$ $(i_p, i_q \in \{1, \cdots, n\})$. Moreover we know that since $\{\theta_1, \ldots, \theta_n\}$ is a set of formal symbols, it follows that the generating set of the $R \otimes R-$module Δ_f will be linearly independent and hence will form a finite basis. Whence, Δ_f is a finite rank $R \otimes R-$module. In fact:
The action of $R \otimes R$ is the obvious one. That is, for example, if $r_1 \otimes r_2 \in R \otimes R$, its left action on an element $(r \otimes r')\theta_{i_1}\theta_{i_2} \cdots \theta_{i_k}$ of Δ_f simply yields $(r_1 \otimes r_2)(r \otimes r')\theta_{i_1}\theta_{i_2} \cdots \theta_{i_k}$. The right action would yield $(r \otimes r')(r_1 \otimes r_2)\theta_{i_1}\theta_{i_2} \cdots \theta_{i_k}$ which is in fact the same as the left action since R is commutative (In fact; $(r \otimes r')(r_1 \otimes r_2) = (rr_1 \otimes r'r_2) = (r_1 r \otimes r_2 r') = (r_1 \otimes r_2)(r \otimes r')$).

Δ_f is endowed with the $\mathbb{Z}_2-$grading given by $\theta-$degree (where $\deg\theta_i = 1$ for each i). Thus $deg\theta_i^2 = 0$ *and* $deg\theta_i\theta_j = 0$.
Next, we define the differential as follows:

$$d : \Delta_f \longrightarrow \Delta_f$$

$$d(-) = \sum_{i=1}^{n}[(x_i - x_i^{'})\theta_i^*(-) + \partial_i(f)\theta_i \wedge (-)] \qquad \cdots \natural$$

[5] That is, we declare those symbols to be linearly independent by definition.

Where θ_i^* is the unique derivation extending the map $\theta_i^*(\theta_j) = \delta_{ij}$ and as mentioned in [17], it acts on an element $\theta_{i_1}\theta_{i_2}\cdots\theta_{i_k}$ of the exterior algebra by the Leibniz rule with Koszul signs. Here, we elucidate what this means. In fact,

$$\theta_i^*(\theta_{j_1}\theta_{j_2}\cdots\theta_{j_k}) = \begin{cases} 0 \; for \; i \neq j, \; \forall j \in \{j_1, j_2, \cdots, j_k\} \\ (-1)^{p+1}\theta_{j_1}\theta_{j_2}\cdots\hat{\theta}_i\cdots\theta_{j_k} \; otherwise \end{cases}$$

where $\hat{\theta}_i$ signifies that θ_i has been removed, and p is the position of θ_i in $\theta_{j_1}\theta_{j_2}\cdots\theta_i\cdots\theta_{j_k}$

N.B. As earlier mentioned ([17]), θ_i^* acts on an element $\theta_{i_1}\cdots\theta_{i_k}$ of the exterior algebra by the Leibniz rule with Koszul sign. So, $\theta_i^*(\theta_j\theta_k) = \theta_i^*(\theta_j)\theta_k + (-1)^{|\theta_j|}\theta_j\theta_i^*(\theta_k)$. Clearly, $\theta_i^*(\theta_j\theta_k) = \theta_i^*(-\theta_k\theta_j) = -\theta_i^*(\theta_k\theta_j)$. Moreover, observe that the following elements are in the same equivalence class $\theta_i\theta_j\theta_k$, $\theta_j\theta_k\theta_i$ and $\theta_k\theta_i\theta_j$. Hence, they have the same image under θ^*.

In order to better understand this, let's compute for example $\theta_4^*(\theta_2\theta_4\theta_7)$, $\theta_4^*(\theta_4\theta_2\theta_7)$ and $\theta_4^*(\theta_2\theta_3\theta_4\theta_7)$.
Since θ_i^* is a derivation, we have

$$\begin{aligned} \theta_4^*(\theta_2\theta_4\theta_7) &= \theta_4^*(\theta_2)\theta_4\theta_7 + (-1)^{|\theta_2|}\theta_2\theta_4^*(\theta_4\theta_7) \\ &= 0 - \theta_2[\theta_4^*(\theta_4)\theta_7 + (-1)^{|\theta_4|}\theta_4\theta_4^*(\theta_7)] \; since \; \theta_4^*(\theta_2) = 0 \\ &= -\theta_2\theta_7 \; since \; \theta_4^*(\theta_7) = 0, |\theta_4| = 1 \; and \; \theta_4^*(\theta_4) = 1 \end{aligned}$$

$$\begin{aligned} \theta_4^*(\theta_4\theta_2\theta_7) &= \theta_4^*(\theta_4)\theta_2\theta_7 + (-1)^{|\theta_4|}\theta_4\theta_4^*(\theta_2\theta_7) \\ &= \theta_2\theta_7 - \theta_4[\theta_4^*(\theta_2)\theta_7 + (-1)^{|\theta_2|}\theta_2\theta_4^*(\theta_7)] \\ &= \theta_2\theta_7 \; since \; \theta_4^*(\theta_7) = 0 = \theta_4^*(\theta_2), \; |\theta_4| = 1 \; and \; \theta_4^*(\theta_4) = 1 \end{aligned}$$

$$\begin{aligned} \theta_4^*(\theta_2\theta_3\theta_4\theta_7) &= \theta_4^*(\theta_2)\theta_3\theta_4\theta_7 + (-1)^{|\theta_2|}\theta_2\theta_4^*(\theta_3\theta_4\theta_7) \\ &= 0 - \theta_2[\theta_4^*(\theta_3)\theta_4\theta_7 + (-1)^{|\theta_3|}\theta_3\theta_4^*(\theta_4\theta_7)] \; since \; \theta_4^*(\theta_2) = 0 \\ &= -\theta_2[-\theta_3\theta_4^*(\theta_4\theta_7)] \; since \; \theta_4^*(\theta_3) = 0 \; and \; |\theta_3| = 1 \\ &= \theta_2\theta_3[\theta_4^*(\theta_4)\theta_7 + (-1)^{|\theta_4|}\theta_4\theta_4^*(\theta_7)] \\ &= \theta_2\theta_3\theta_7 \; since \; \theta_4^*(\theta_7) = 0, \; |\theta_4| = 1 \; and \; \theta_4^*(\theta_4) = 1 \end{aligned}$$

In order to complete the description of d, we need to say what ∂ is.

$$\partial_i : k[x_1, \cdots, x_n, x_1', \cdots, x_n'] \longrightarrow k[x_1, \cdots, x_n, x_1', \cdots, x_n']$$

is defined by,

$$\partial_i(h) = \frac{h(x_1', \cdots, x_{i-1}', x_i, \cdots, x_n, x_1', \cdots, x_n') - h(x_1', \cdots, x_i', x_{i+1}, \cdots, x_n, x_1', \cdots, x_n')}{x_i - x_i'}$$

where for ease of notation we wrote h as argument of ∂_i instead of the more cumbersome notation $\partial_i(h(x_1, \cdots, x_n, x_1', \cdots, x_n'))$, we will do same in what follows. But first, observe that ∂_i is well defined because its numerator will always have $(x_i - x_i^{'})$ as a factor. Thus, ensuring that ∂_i is a polynomial.

In the sequel, we will sometimes write x for $x_1, \cdots, x_n$ and $x^{'}$ for $x_1^{'}, \cdots, x_n^{'}$.

We can also like in [17] write:

$$\partial_i(h) = \frac{{}^{t_1 \cdots t_{i-1}}h(x_1, \cdots, x_n, x_1^{'}, \cdots, x_n^{'}) - {}^{t_1 \cdots t_i}\, h(x_1, \cdots, x_n, x_1^{'}, \cdots, x_n^{'})}{x_i - x_i^{'}}$$

where ${}^{t_i}(-) : K[x, x'] \to K[x, x']$, $h \mapsto h|_{x_i \mapsto x_i^{'}}$

is a variable changing map which in any polynomial, replaces the variable x_i by the variable x_i'.

So, in particular for the f in Δ_f, which is also the same f used in defining the differential map d (see ♮ above), we have $f \in K[x] \subseteq K[x, x']$,

$$\partial_i(f) = \frac{{}^{t_1 \cdots t_{i-1}} f(x_1, \cdots, x_n) - {}^{t_1 \cdots t_i}\, f(x_1, \cdots, x_n)}{x_i - x_i^{'}}$$

Example 5.2. *Let $f = x - y \in \mathbb{R}[x, y]$.*

$$\partial_1(f(x, y)) = \frac{f(x, y) - f(x', y)}{x - x'} = \frac{(x - y) - (x' - y)}{x - x'} = 1.$$

$$\partial_2(f(x, y)) = \frac{f(x', y) - f(x', y')}{y - y'} = \frac{(x' - y) - (x' - y')}{y - y'} = -1.$$

We now prove the following lemma which is stated without proof in [17].

Lemma 5.6. *[17] For*[6] *$f, g \in K[x, x']$, we have*

$$\partial_i(fg) = \partial_i(f)({}^{t_1 \cdots t_i} g) + ({}^{t_1 \cdots t_{i-1}} f)\partial_i(g).$$

Proof. First of all, observe that by definition of ${}^{t_i}(-)$, it is obvious that

$${}^{t_1 \cdots t_i}(fg) = ({}^{t_1 \cdots t_i} f)({}^{t_1 \cdots t_i} g).$$

[6] the f in this lemma should not be confused with the f before this lemma or after this lemma.

$(\partial_i f(x,x^{'}))({}^{t_1\cdots t_i}g(x,x^{'})) + ({}^{t_1\cdots t_{i-1}}f(x,x^{'}))(\partial_i g)$

$$= \frac{{}^{t_1\cdots t_{i-1}}f(x,x^{'})^{t_1\cdots t_i}g(x,x^{'}) -^{t_1\cdots t_i} f(x,x^{'})^{t_1\cdots t_i}g(x,x^{'})}{x_i - x_i^{'}}$$

$$+\frac{{}^{t_1\cdots t_{i-1}}f(x,x^{'})^{t_1\cdots t_{i-1}}g(x,x^{'}) -^{t_1\cdots t_{i-1}} f(x,x^{'})^{t_1\cdots t_i}g(x,x^{'})}{x_i - x_i^{'}}$$

$$= \frac{{}^{t_1\cdots t_{i-1}}f(x,x^{'})^{t_1\cdots t_{i-1}}g(x,x^{'}) -^{t_1\cdots t_i} f(x,x^{'})^{t_1\cdots t_i}g(x,x^{'})}{x_i - x_i^{'}}$$

$$= \frac{{}^{t_1\cdots t_{i-1}}(f(x,x^{'})g(x,x^{'})) -^{t_1\cdots t_i} (f(x,x^{'})g(x,x^{'}))}{x_i - x_i^{'}}$$

$= \partial_i(fg)$
as desired.

The first two equalities hold by definition of ∂_i . □

It is worth mentioning at this point in time that the authors in [17] state what the differential d for Δ_f is, but they do not prove that (Δ_f, d) determines a matrix factorization of $f \otimes id - id \otimes f$ (and this is important in order to see that (Δ_f, d) is the unit matrix factorization with respect to the tensor product of matrix factorizations).
We now have enough ingredients to produce such a proof. We state the following lemma which shows that (Δ_f, d) determines a finite rank matrix factorization of $f \otimes id - id \otimes f$. Its proof is lengthy and is one of our main contributions in this chapter.

Lemma 5.7. *The $R \otimes R-$module Δ_f together with the differential d defined above, determine a finite rank matrix factorization of $f \otimes id - id \otimes f$ (which is equivalent to both $h(x,x')$ and $f(x) - f(x')$).*

Proof. As already discussed under subsection 5.2.2, Δ_f is a finite rank $R \otimes R-$module. That is why we will conclude by the end of this proof that (Δ_f, d) is a finite rank matrix factorization given that its underlying module is of finite rank.
Next, we need to show that d is an odd degree map and that $d^2 = h(x,x') \cdot id$, where $h(x,x') = f(x) - f(x')$
It is easy to see that d is an odd degree map. In fact, since Δ_f is generated by the symbols $\theta_1, \cdots, \theta_n$ it suffices to consider the action of d on an arbitrary product of elements from the set $\{\theta_1, \cdots, \theta_n\}$.
<u>*Case 1*</u>:
Let $p = \theta_{j_1}\theta_{j_2}\cdots\theta_{j_k}$ be an odd degree element, i.e., k is odd. We also have $\theta_{j_l} \in \{\theta_1, \cdots, \theta_n\}$ with $l \in \{1, \cdots, k\}$

Claim 1: For each i, $(x_i - x_i')\theta_i^*(p)$ is either of even degree or is zero. This is easy to see as it is a direct consequence of the definition of θ_i^*. In fact, we know by definition of θ_i^* that when it is applied to p, if p contains no θ_i, then the result would be zero. Now, if p contains θ_i, then this θ_i will no longer appear among the θs in $\theta_i^*(p)$, but the other θs that were in p will remain. Thus, an even number of θs will remain [7] in $\theta_i^*(p)$.

Claim 2: For each i, $\partial_i(f)\theta_i \wedge (p)$ is of even degree. Observe that the degree of $\partial_i(f)\theta_i$ is deg$\theta_i = 1$ since $\partial_i(f)$ contains no θ. Thus, deg$[\partial_i(f)\theta_i \wedge (p)]$ =deg$[\theta_i \wedge (p)]$ =deg$(p) + 1$. So the claim is proved.
It now follows that when p is of odd degree, $d(p)$ is the summation of even degree elements and so is of even degree.
<u>*Case 2*</u>:
Let $p = \theta_{j_1}\theta_{j_2}\cdots\theta_{j_k}$ be an even degree element, i.e., k is even. We also have $\theta_{j_l} \in \{\theta_1, \cdots, \theta_n\}$ with $l \in \{1, \cdots, k\}$
Claim 3: For each i, $(x_i - x'_i)\theta_i^*(p)$ is of odd degree or zero.
A similar reasoning to that of claim 1 above can be applied to show that when p is of even degree, $(x_i - x'_i)\theta_i^*(p)$ is of odd degree or zero.

Claim 4: For each i, $\partial_i(h)\theta_i \wedge (p)$ is of odd degree. A similar reasoning to that of claim 2 can be applied to show that when p is of even degree, $\partial_i(h)\theta_i \wedge (p)$ is of odd degree.

Hence, when p is of even degree, $d(p)$ is the summation of odd degree elements and so is of odd degree.
Consequently, d is an odd degree map.
We will need the following definition is the sequel.

Definition 5.6. *If $i \in \{1, \cdots, n\}$ and $\theta_i \in \theta_1, \cdots, \theta_n$; we say that θ_i **is even** in $\theta_1, \cdots, \theta_n$ if θ_i occupies an even position in $\theta_1, \cdots, \theta_n$*

Example 5.3. *θ_5 is even in $\theta_2\theta_5\theta_6$ and θ_6 is odd in $\theta_2\theta_5\theta_6$.*

We now show that[8] $d^2 = h(x, x') \cdot id$ i.e., $d^2 = f(x) - f(x') \cdot id$.
To that end, define

$$A_i = (x_i - x'_i)\theta_i^* \textit{ and } B_i = \partial_i(f)\theta_i \wedge (-)$$

$$\begin{aligned}
&d(d(H))\\
&= \textstyle\sum_{i=1}^n (x_i - x'_i)\theta_i^*(\sum_{j=1}^n (x_j - x'_j)\theta_j^*(H) + \partial_j(f)\theta_j \wedge (H))+\\
&\partial_i(f)\theta_i \wedge (\textstyle\sum_{j=1}^n (x_j - x'_j)\theta_j^*(H) + \partial_j(f)\theta_j \wedge (H))\\
&\\
&= (x_1 - x'_1)\theta_1^*((x_1 - x'_1)\theta_1^*(H) + \partial_1(f)\theta_1 \wedge (H) + (x_2 - x'_2)\theta_2^*(H)+\\
&\partial_2(f)\theta_2 \wedge (H) + \cdots + (x_n - x'_n)\theta_n^*(H) + \partial_n(f)\theta_n \wedge (H))\\
&+\partial_1(f)\theta_1 \wedge ((x_1 - x'_1)\theta_1^*(H) + \partial_1(f)\theta_1 \wedge (H) + (x_2 - x'_2)\theta_2^*(H)+\\
&\partial_2(f)\theta_2 \wedge (H) + \cdots + (x_n - x'_n)\theta_n^*(H) + \partial_n(f)\theta_n \wedge (H))\\
&+(x_2 - x'_2)\theta_2^*((x_1 - x'_1)\theta_1^*(H) + \partial_1(f)\theta_1 \wedge (H) + (x_2 - x'_2)\theta_2^*(H)+\\
&\partial_2(f)\theta_2 \wedge (H) + \cdots + (x_n - x'_n)\theta_n^*(H) + \partial_n(f)\theta_n \wedge (H))\\
&+\partial_2(f)\theta_2 \wedge ((x_1 - x'_1)\theta_1^*(H) + \partial_1(f)\theta_1 \wedge (H) + (x_2 - x'_2)\theta_2^*(H)+
\end{aligned}$$

[7]For e.g. we saw after computation above that, $\theta_4^*(\theta_2\theta_4\theta_7) = -\theta_2\theta_7$ hence $(x_4 - x'_4)\theta_4^*(\theta_2\theta_4\theta_7) = -(x_4 - x'_4)\theta_2\theta_7$ which is of even degree. We also have for e.g. $\theta_4^*(\theta_2) = 0$ hence $(x_4 - x'_4)\theta_4^*(\theta_2) = 0$.
[8]the author of [40] simply writes $d^2 = w$, omitting the "$\cdot id$". So, according to this notation, we want to show $d^2 = h(x, x') = f(x) - f(x')$.

$\partial_2(f)\theta_2 \wedge (H) + \cdots + (x_n - x'_n)\theta_n^*(H) + \partial_n(f)\theta_n \wedge (H))$
$+ \cdots +$
$+(x_n - x'_n)\theta_n^*((x_1 - x'_1)\theta_1^*(H) + \partial_1(f)\theta_1 \wedge (H) + (x_2 - x'_2)\theta_2^*(H)+$
$\partial_2(f)\theta_2 \wedge (H) + \cdots + (x_n - x'_n)\theta_n^*(H) + \partial_n(f)\theta_n \wedge (H))$
$+\partial_n(f)\theta_n \wedge ((x_1 - x'_1)\theta_1^*(H) + \partial_1(f)\theta_1 \wedge (H) + (x_2 - x'_2)\theta_2^*(H)+$
$\partial_2(f)\theta_2 \wedge (H) + \cdots + (x_n - x'_n)\theta_n^*(H) + \partial_n(f)\theta_n \wedge (H))$

Recalling that

$$A_i = (x_i - x'_i)\theta_i^* \textit{ and } B_i = \partial_i(f)\theta_i \wedge (-)$$

we get:

$d(d(-))$
$= A_1 \circ A_1 + A_1 \circ B_1 + A_1 \circ A_2 + A_1 \circ B_2 + \cdots + A_1 \circ A_n + A_1 \circ B_n$
$+ B_1 \circ A_1 + B_1 \circ B_1 + B_1 \circ A_2 + B_1 \circ B_2 + \cdots + B_1 \circ A_n + B_1 \circ B_n$
$+ A_2 \circ A_1 + A_2 \circ B_1 + A_2 \circ A_2 + A_2 \circ B_2 + \cdots + A_2 \circ A_n + A_2 \circ B_n$
$+ B_2 \circ A_1 + B_2 \circ B_1 + B_2 \circ A_2 + B_2 \circ B_2 + \cdots + B_2 \circ A_n + B_2 \circ B_n$
$+ \cdots$
$+ A_n \circ A_1 + A_n \circ B_1 + A_n \circ A_2 + A_n \circ B_2 + \cdots + A_n \circ A_n + A_n \circ B_n$
$+ B_n \circ A_1 + B_n \circ B_1 + B_n \circ A_2 + B_n \circ B_2 + \cdots + B_n \circ A_n + B_n \circ B_n$

Thus we get:

$$d^2 = \sum_{i,j=1}^{n} [A_i \circ A_j + A_i \circ B_j + B_i \circ A_j + B_i \circ B_j]$$

We also make the following observations:

- For all i, $A_i \circ A_i = B_i \circ B_i = 0$.
  We first show that $A_i \circ A_i = 0$, i.e., $A_i \circ A_i(H) = 0$ for all $H \in \Delta_f$, where $H = h(x, x')\theta_{j_1}\theta_{j_2}\cdots\theta_{j_k} = h(x, x')p$, with $p = \theta_{j_1}\theta_{j_2}\cdots\theta_{j_k}$; and $\theta_{j_1}, \theta_{j_2}, \cdots, \theta_{j_k} \in \{\theta_1, \theta_2, \cdots, \theta_n\}$. We have $(A_i \circ A_i)(H) = h(x, x')(A_i \circ A_i)(p)$ since $h(x, x') \in K[x, x']$ and θ^* as a derivation, is $(K[x, x']-)$linear[9] i.e., a $K[x, x']$-module.
  We always have $(A_i \circ A_i)(p) = 0$. In fact, if p contains θ_i, then $A_i(p)$ will not contain any θ_i, and so $(A_i \circ A_i)(p) = 0$. Now, if p does not contain θ_i, then $A_i(p) = 0$, hence $(A_i \circ A_i)(p) = A_i(0) = 0$.

  Next, to prove $B_i \circ B_i = 0$, recall that $\theta_i \wedge \theta_i = \theta_i\theta_i = 0$.

  $B_i \circ B_i(H) = \partial_i(f)\theta_i \wedge (\partial_i(f)\theta_i \wedge (H)) = \partial_i(f)\partial_i(f)(\theta_i \wedge \theta_i) \wedge H = 0.$

- If $i \neq j$, then $A_i \circ B_j = -B_j \circ A_i$ Since Δ_f is generated by the θ_is, it suffices to verify that this equality holds for an arbitrary $p = \theta_{j_1}\theta_{j_2}\cdots\theta_{j_k} \in \Delta_f$. It is good to keep in mind the assumption $i \neq j$.

[9] θ^* is an endomorphism on Δ_f which is an $R \otimes R$-module

$$\begin{aligned} A_i \circ B_j(\theta_{j_1}\theta_{j_2}\cdots\theta_{j_k}) &= A_i(\partial_j(f)\theta_j \wedge (\theta_{j_1}\theta_{j_2}\cdots\theta_{j_k})) \\ &= (x_i - x_i')\theta_i^*(\partial_j(f)\theta_j \wedge (\theta_{j_1}\theta_{j_2}\cdots\theta_{j_k})) \\ &= (x_i - x_i')\partial_j(f)\theta_i^*(\theta_j \wedge (\theta_{j_1}\theta_{j_2}\cdots\theta_{j_k})) \\ &= \begin{cases} 0, \ i \notin \{j_1, \cdots, j_k\} \\ (x_i - x_i')\partial_j(f)(\theta_j(\theta_{j_1}\theta_{j_2}\cdots\hat{\theta}_i\cdots\theta_{j_k})), \ \theta_i \text{ is even in } \theta_{j_1}\cdots\theta_{j_k} \\ -(x_i - x_i')\partial_j(f)(\theta_j(\theta_{j_1}\theta_{j_2}\cdots\hat{\theta}_i\cdots\theta_{j_k})), \ \theta_i \text{ is odd in } \theta_{j_1}\cdots\theta_{j_k} \end{cases} \end{aligned}$$

Next, we compute:

$$\begin{aligned} B_j \circ A_i(\theta_{j_1}\theta_{j_2}\cdots\theta_{j_k}) &= \partial_j(f)\theta_j \wedge ((x_i - x_i')\theta_i^*(\theta_{j_1}\theta_{j_2}\cdots\theta_{j_k})) \\ &= (x_i - x_i')\partial_j(f)\theta_j \wedge (\theta_i^*(\theta_{j_1}\theta_{j_2}\cdots\theta_{j_k})) \\ &= \begin{cases} 0, \ i \notin \{j_1, \cdots, j_k\} \\ -(x_i - x_i')\partial_j(f)(\theta_j(\theta_{j_1}\theta_{j_2}\cdots\hat{\theta}_i\cdots\theta_{j_k})), \theta_i \text{ is even in } \theta_{j_1}\cdots\theta_{j_k} \\ (x_i - x_i')\partial_j(f)(\theta_j(\theta_{j_1}\theta_{j_2}\cdots\hat{\theta}_i\cdots\theta_{j_k})), \ \theta_i \text{ is odd in } \theta_{j_1}\cdots\theta_{j_k} \end{cases} \end{aligned}$$

Hence, $A_i \circ B_j(\theta_{j_1}\theta_{j_2}\cdots\theta_{j_k}) = -B_j \circ A_i(\theta_{j_1}\theta_{j_2}\cdots\theta_{j_k})$. So, $A_i \circ B_j = -B_j \circ A_i$ as desired.

- If $i \neq j$, then $A_i \circ A_j = -A_j \circ A_i$.
  We consider p as before and verify that $A_i \circ A_j(p) = -A_j \circ A_i(p)$.
  First of all, it is easy to see that if either i or j is not in $I = \{j_1, \cdots, j_k\}$, then $A_i \circ A_j = 0 = A_j \circ A_i$, and so $A_i \circ A_j = -A_j \circ A_i$ as desired.
  Next, suppose that $i, j \in I$ and $j \leq i$ without loss of generality. We distinguish four cases.
  <u>*Case 1*</u>: θ_i and θ_j are both even in $\{\theta_{j_1}\theta_{j_2}\cdots\theta_{j_k}\}$

$$\begin{aligned} A_i \circ A_j(\theta_{j_1}\theta_{j_2}\cdots\theta_{j_k}) &= (x_i - x_i')\theta_i^*((x_j - x_j')\theta_j^*(\theta_{j_1}\theta_{j_2}\cdots\theta_{j_k})) \\ &= -(x_i - x_i')(x_j - x_j')\theta_i^*(\theta_{j_1}\theta_{j_2}\cdots\hat{\theta}_j\cdots\theta_i\cdots\theta_{j_k}), \ \theta_j \text{ is even in } \theta_{j_1}\theta_{j_2}\cdots\theta_{j_k} \\ &= -(x_i - x_i')(x_j - x_j')(\theta_{j_1}\theta_{j_2}\cdots\hat{\theta}_j\cdots\hat{\theta}_i\cdots\theta_{j_k}), \ \theta_i \text{ is odd in } \theta_{j_1}\cdots\hat{\theta}_j\cdots\theta_i\cdots\theta_{j_k} \end{aligned}$$

Next, we compute:

$$\begin{aligned} A_j \circ A_i(\theta_{j_1}\theta_{j_2}\cdots\theta_{j_k}) &= (x_j - x_j')\theta_j^*((x_i - x_i')\theta_i^*(\theta_{j_1}\theta_{j_2}\cdots\theta_{j_k})) \\ &= -(x_j - x_j')\theta_j^*(x_i - x_i')(\theta_{j_1}\cdots\theta_j\cdots\hat{\theta}_i\cdots\theta_{j_k}), \ \theta_i \text{ is even in } \theta_{j_1}\theta_{j_2}\cdots\theta_{j_k} \\ &= (x_j - x_j')(x_i - x_i')(\theta_{j_1}\cdots\hat{\theta}_j\cdots\hat{\theta}_i\cdots\theta_{j_k}), \ \theta_j \text{ is even in } \theta_{j_1}\cdots\theta_j\cdots\hat{\theta}_i\cdots\theta_{j_k} \end{aligned}$$

It follows that $(A_j \circ A_i)(\theta_{j_1}\theta_{j_2}\cdots\theta_{j_k}) = -(A_i \circ A_j)(\theta_{j_1}\theta_{j_2}\cdots\theta_{j_k})$, thus $A_j \circ A_i = -A_i \circ A_j$ as desired.

Case 2: θ_i and θ_j are both odd.
It is easy to see that the result holds here by following a reasoning completely similar to that of case 1.

Case 3: θ_i and θ_j are respectively odd and even.

$$\begin{aligned} A_i \circ A_j(\theta_{j_1}\theta_{j_2}\cdots\theta_{j_k}) &= (x_i - x_i')\theta_i^*((x_j - x_j')\theta_j^*(\theta_{j_1}\theta_{j_2}\cdots\theta_{j_k})) \\ &= -(x_i - x_i')(x_j - x_j')\theta_i^*(\theta_{j_1}\theta_{j_2}\cdots\hat{\theta}_j\cdots\theta_i\cdots\theta_{j_k}),\ \theta_j\, is\, even\, in\, \theta_{j_1}\theta_{j_2}\cdots\theta_{j_k} \\ &= (x_i - x_i')(x_j - x_j')(\theta_{j_1}\theta_{j_2}\cdots\hat{\theta}_j\cdots\hat{\theta}_i\cdots\theta_{j_k}),\ \theta_i\, is\, even\, in\, \theta_{j_1}\cdots\hat{\theta}_j\cdots\theta_i\cdots\theta_{j_k} \end{aligned}$$

Next, we compute:

$$\begin{aligned} A_j \circ A_i(\theta_{j_1}\theta_{j_2}\cdots\theta_{j_k}) &= (x_j - x_j')\theta_j^*((x_i - x_i')\theta_i^*(\theta_{j_1}\theta_{j_2}\cdots\theta_{j_k})) \\ &= (x_j - x_j')\theta_j^*(x_i - x_i')(\theta_{j_1}\cdots\theta_j\cdots\hat{\theta}_i\cdots\theta_{j_k}),\ \theta_i\, is\, odd\, in\, \theta_{j_1}\theta_{j_2}\cdots\theta_{j_k} \\ &= -(x_j - x_j')(x_i - x_i')(\theta_{j_1}\cdots\hat{\theta}_j\cdots\hat{\theta}_i\cdots\theta_{j_k}),\ \theta_j\, is\, even\, in\, \theta_{j_1}\cdots\theta_j\cdots\hat{\theta}_i\cdots\theta_{j_k} \end{aligned}$$

Thus, under this case too, $A_j \circ A_i = -A_i \circ A_j$ as desired.

Case 4: θ_i and θ_j are respectively even and odd.
This case is analogous to case 3.
Hence, we always have $A_j \circ A_i = -A_i \circ A_j$.

- If $i \neq j$, then $B_i \circ B_j = -B_j \circ B_i$.
  In fact, since $\theta_i\theta_j = -\theta_j\theta_i$, we have:
  $B_i \circ B_j(-) = \partial_i(f)\theta_i \wedge (\partial_j(f)\theta_j \wedge (-)) = -\partial_i(f)\partial_j(f)\theta_j\theta_i(-) = -B_j \circ B_i(-)$

- If $H = r_1 \otimes r_2\theta_{i_1}\cdots\theta_{i_n} \in \Delta_f$, then either $A_i \circ B_i(H) = 0$ or $B_i \circ A_i(H) = 0$. In either case the result will be of the form $(x_i - x_i')\partial_i(f)H$.
  Case 1: $i \in \{i_1, \cdots, i_n\}$
  $A_i \circ B_i(H) = (x_i - x_i')\theta_i^*(\partial_i(f)\theta_i \wedge r_1 \otimes r_2\theta_{i_1}\cdots\theta_{i_n}) = 0$. Since we will have the product $\theta_i\theta_i$ in this expression, and this product is clearly 0 and $\theta_i^*(0) = 0$ as θ_i^* is a derivation. So the result follows.
  Still under this case,

$$B_i \circ A_i(r_1 \otimes r_2\theta_{i_1} \cdots \theta_{i_n}) = \partial_i(f)\theta_i \wedge ((x_i - x_i')\theta_i^*(r_1 \otimes r_2\theta_{i_1} \cdots \theta_{i_n}))$$

$$= \begin{cases} -(x_i - x_i')\partial_i(f)\theta_i \wedge (r_1 \otimes r_2\theta_{i_1} \cdots \hat{\theta}_i \cdots \theta_{i_n}), \theta_i \, is \, even \, in \, \theta_{i_1} \cdots \theta_{i_n} \\ (x_i - x_i')\partial_i(f)\theta_i \wedge (r_1 \otimes r_2\theta_{i_1} \cdots \hat{\theta}_i \cdots \theta_{i_n}), \theta_i \, is \, odd \, in \, \theta_{i_1} \cdots \theta_{i_n} \end{cases}$$

$$= \begin{cases} -(x_i - x_i')\partial_i(f) \wedge (-r_1 \otimes r_2\theta_{i_1} \cdots \theta_i \cdots \theta_{i_n}) \\ (x_i - x_i')\partial_i(f) \wedge (r_1 \otimes r_2\theta_{i_1} \cdots \theta_i \cdots \theta_{i_n}) \end{cases}$$

$$= (x_i - x_i')\partial_i(f)H$$

The sign in the first part (respectively second part) of the second to the last equality is due to the fact θ_i is moved an odd number of times (respectively an even number of times) to occupy its position, and each move is affected by a minus sign.

<u>Case 2</u>: $i \notin \{i_1, \cdots, i_n\}$
$B_i \circ A_i(r_1 \otimes r_2\theta_{i_1} \cdots \theta_{i_n}) = \partial_i(f)\theta_i \wedge (x_i - x_i')\theta_i^*(r_1 \otimes r_2\theta_{i_1} \cdots \theta_{i_n})$. This yields 0 since $\theta_i^*(r_1 \otimes r_2\theta_{i_1} \cdots \theta_{i_n}) = 0$ as $i \notin \{i_1, \cdots, i_n\}$.
Still under this case,

$$A_i \circ B_i(H) = (x_i - x_i')\theta_i^*(\partial_i(f)\theta_i \wedge r_1 \otimes r_2\theta_{i_1} \cdots \theta_{i_n})$$

$$= (x_i - x_i')\partial_i(f)r_1 \otimes r_2\theta_i^*(\theta_i\theta_{i_1} \cdots \theta_{i_n})$$

$$= (x_i - x_i')(\partial_i(f)r_1 \otimes r_2\theta_{i_1} \cdots \theta_{i_n}), \; by \, definition \, of \, \theta_i^*$$

$$= (x_i - x_i')\partial_i(f)H$$

So, $A_i \circ B_i(H) = 0$ or $B_i \circ A_i(H) = 0$, and in either case the result is in the desired form, $(x_i - x_i')\partial_i(f)H$.

Given all the above, we simplify the expression of d^2 as follows:

$d^2 = \sum_{i,j=1}^n [A_i \circ A_j + A_i \circ B_j + B_i \circ A_j + B_i \circ B_j]$

$= \sum_{j=1}^n A_1 \circ A_j + A_1 \circ B_j + B_1 \circ A_j + B_1 \circ B_j + A_2 \circ A_j + A_2 \circ B_j + B_2 \circ A_j + B_2 \circ B_j + \cdots + A_n \circ A_j + A_n \circ B_j + B_n \circ A_j + B_n \circ B_j$

$= A_1 \circ A_1 + A_1 \circ B_1 + B_1 \circ A_1 + B_1 \circ B_1 + A_1 \circ A_2 + A_1 \circ B_2 + B_1 \circ A_2 + B_1 \circ B_2 + \cdots + A_1 \circ A_n + A_1 \circ B_n + B_1 \circ A_n + B_1 \circ B_n + A_2 \circ A_1 + A_2 \circ B_1 + B_2 \circ A_1 + B_2 \circ B_1 + A_2 \circ A_2 + A_2 \circ B_2 + B_2 \circ A_2 + B_2 \circ B_2 + \cdots + A_2 \circ A_n + A_2 \circ B_n + B_2 \circ A_n + B_2 \circ B_n + \cdots + A_n \circ A_1 + A_n \circ B_1 + B_n \circ A_1 + B_n \circ B_1 + A_n \circ A_2 + A_n \circ B_2 + B_n \circ A_2 + B_n \circ B_2 + \cdots + A_n \circ A_n + A_n \circ B_n + B_n \circ A_n + B_n \circ B_n$

$= A_1 \circ B_1 + B_1 \circ A_1 + A_2 \circ B_2 + B_2 \circ A_2 + \cdots + A_n \circ B_n + B_n \circ A_n$

$= \sum_{i=1}^n [A_i \circ B_i + B_i \circ A_i]$

The second to the last equality above results from applying the identities obtained above namely $A_i \circ A_i = B_i \circ B_i = 0$, and if $i \neq j$ then $A_i \circ B_j = -B_j \circ A_i$, $A_i \circ A_j = -A_j \circ A_i$ and $B_i \circ B_j = -B_j \circ B_i$.

Thus the sum representation for d^2 reduces to:

$$d^2(H) = \sum_{i=1}^{n} [A_i \circ B_i(H) \ or \ B_i \circ A_i(H)]$$

Since $A_i \circ B_i = 0 \ or \ B_i \circ A_i = 0$.

This sum is telescoping, In fact:

$d^2 = \sum_{i=1}^{n} [A_i \circ B_i \ or \ B_i \circ A_i]$

$= \sum_{i=1}^{n} (x_i - x_i')\partial_i f$

$= (x_1 - x_1')\partial_1 f + (x_2 - x_2')\partial_2 f + \cdots + (x_{n-1} - x_{n-1}')\partial_{n-1} f + (x_n - x_n')\partial_n f$

$= f(x_1, \cdots, x_n) - f(x_1', x_2, \cdots, x_n)$
$+ f(x_1', x_2, \cdots, x_n) - f(x_1', x_2', x_3, \cdots, x_n)$
$+ \cdots$
$+ f(x_1', \cdots, x_{n-2}', x_{n-1}, x_n) - f(x_1', \cdots, x_{n-1}', x_n)$
$+ f(x_1', \cdots, x_{n-1}', x_n) - f(x_1', \cdots, x_n')$

$= f(x_1, \cdots, x_n) - f(x_1', \cdots, x_n')$

$= f(x) - f(x')$

So $d^2 = (f(x) - f(x')) \cdot id$ as desired.

QED. □

We call Δ_f the unit matrix factorization as it is the unit with respect to the tensor product of matrix factorizations. It is also referred to as the stabilised diagonal [23] or Koszul model of the diagonal [17]. The diagonal here refers to R as an $R \otimes R-$module (with multiplication giving the module structure), which is a matrix factorization of $f \otimes id - id \otimes f$ (or equivalently $f(x) - f(x')$) with differential zero. We give it the zero differential structure, which works, since $f(x) - f(x')$ acts as 0 on this module. Thus, we have a canonical object in $hmf(R \otimes R, f(x) - f(x'))$ namely $(R, 0)$.

Since we are constructing a bicategory, we would like to verify at this level that for $f \in R = K[x]$ and for each object (R, f) (of the bicategory $\mathcal{LG}_K$), there is a functor $I_{(R,f)}$ that returns the identity Δ_f on (R, f).

Define $I_{(R,f)} : 1 = \{*\} \longrightarrow hmf(R \otimes R, id \otimes f - f \otimes id)^{\omega}$, $* \mapsto \Delta_f$.

$I_{(R,f)}$ is a functor. In fact:

Clearly Δ_f is an object of $hmf(R \otimes R, id \otimes f - f \otimes id)$ as discussed above.

Next, let $1_* : * \longrightarrow *$, then clearly $I_{(R,f)}(1_*) = 1_{\Delta_f} = 1_{I_{(R,f)}(*)}$.

Now, consider two maps F and G in the small category $\{*\}$ (which is the final object in the category Cat of small categories), then clearly the only possibility for the two maps is $F = G = 1_*$. We clearly have:

$I_{(R,f)}(F \circ G) = I_{(R,f)}(1_* \circ 1_*) = I_{(R,f)}(1_*) = 1_{\Delta_f}$ ... (i)
$I_{(R,f)}(F) \circ I_{(R,f)}(G) = I_{(R,f)}(1_*) \circ I_{(R,f)}(1_*) = 1_{\Delta_f} \circ 1_{\Delta_f} = 1_{\Delta_f}$... (ii)
(i) and (ii) show that $I_{(R,f)}(F \circ G) = I_{(R,f)}(F) \circ I_{(R,f)}(G)$, and this completes the proof that $I_{(R,f)}$ is a functor.

Lemma 5.8. *There is a canonical map of factorizations* $\pi : \Delta_f \longrightarrow R$ *given by* $\pi[(r \otimes r')\theta_{i_1}\theta_{i_2} \cdots \theta_{i_k}] = \delta_{k,0} rr'$. π *is in fact the composition of the projection* $\pi^* : \Delta_f \longrightarrow R \otimes R$ *to* $\theta-$*degree* 0*, followed by the multiplication map* $m : R \otimes R \longrightarrow R = R_0 \oplus R_1$*, where we endow* R *with the trivial grading i.e.,* $R_0 = R$ *and* $R_1 = \{0\}$.

Proof. To show that π is a map of factorizations, we need to show that it is grade preserving, $R \otimes R-$linear and satisfies $d_R \pi = \pi d$, where d and d_R are respectively the differentials given to Δ_f and R, and $d_R = 0$ as explained above just before the statement of the lemma.

1. π is an even map, viz. a grade preserving map.
   We need to show that π sends elements of even (resp. odd) degree in Δ_f to elements of even (resp. odd) degree in R.
   In fact, since the degree of the empty word[10] is zero, we have that $r \otimes r'$ is of degree 0 and its image is also of degree 0 because $\pi(r \otimes r') = rr' \in R = R_0$.
   Next, we see by definition of π that all odd degree elements are mapped to $0 \in R_1 = \{0\}$.
   Moreover, if $k \neq 0$ and is even, then $(r \otimes r')\theta_{i_1}\theta_{i_2} \cdots \theta_{i_k}$ is of even degree and its image under π is also of even degree since $\pi[(r \otimes r')\theta_{i_1}\theta_{i_2} \cdots \theta_{i_k}] = 0 \in R = R_0$
   So, π is a degree preserving map.

2. π is $R \otimes R-$linear. Let $r_1 \otimes r_1' \in R \otimes R$.
   We need to show that $\pi((r_1 \otimes r_1')(r \otimes r')\theta_{i_1} \cdots \theta_{i_k}) = (r_1 \otimes r_1')\pi((r \otimes r')\theta_{i_1} \cdots \theta_{i_k})$:
   Now,
   if $k \geq 1$ then:
   $\pi((r_1 \otimes r_1')(r \otimes r')\theta_{i_1} \cdots \theta_{i_k}) = \pi((r_1 r \otimes r_1' r')\theta_{i_1} \cdots \theta_{i_k}) = 0 \quad ... \quad *$
   and if $k < 1$ we have:
   $\pi((r_1 \otimes r_1')(r \otimes r')\theta_{i_1} \cdots \theta_{i_k}) = \pi((r_1 r \otimes r_1' r')\theta_{i_1} \cdots \theta_{i_k}) = r_1 r r_1' r' \quad ... \quad **$
   Next,
   if $k \geq 1$ then:
   $(r_1 \otimes r_1')\pi((r \otimes r')\theta_{i_1} \cdots \theta_{i_k}) = (r_1 \otimes r_1')(0) = 0 \quad ... \quad *'$
   and for $k < 1$ we have:
   $(r_1 \otimes r_1')\pi((r \otimes r')\theta_{i_1} \cdots \theta_{i_k}) = (r_1 \otimes r_1')rr' = r_1 r_1' rr' = r_1 r r_1' r' \quad ... \quad **'$
   This last equality is by commutativity in R, and the second to the last equality is due to the fact that the $R \otimes R-$module structure on R is given by multiplication.
   Since $*$ and $*'$ are the same, and $**$ is same as $**'$, it follows that

   $$\pi((r_1 \otimes r_1')(r \otimes r')\theta_{i_1} \cdots \theta_{i_k}) = (r_1 \otimes r_1')\pi((r \otimes r')\theta_{i_1} \cdots \theta_{i_k})$$

   as desired.
   To complete the proof that π is $R \otimes R-$linear, we need to show that it is additive. Now, this is true by definition of π.

[10]By empty word here, we mean the word $\theta_{i_1}\theta_{i_2} \cdots \theta_{i_k}$, where $k < 1$.

3. Finally, we show that $d_R\pi = \pi d$.
   Recall that $d_R = 0$ and so $d_R\pi = 0$. Hence, it suffices to show that $\pi d = 0$.
   Let $H = (r \otimes r')\theta_{i_1} \cdots \theta_{i_k} \in \Delta_f$.
   $\pi d(H) = \pi(\sum_{i=1}^{n}[(x_i - x_i')\theta_i^*(r \otimes r')\theta_{i_1} \cdots \theta_{i_k} + \partial_i(h)\theta_i \wedge (r \otimes r')\theta_{i_1} \cdots \theta_{i_k}])$
   $= \pi((x_1 - x_1')(r \otimes r')\theta_{i_1} \cdots \hat{\theta}_1 \cdots \theta_{i_k} + \cdots + (x_n - x_n')(r \otimes r')\theta_{i_1} \cdots \hat{\theta}_n \cdots \theta_{i_k}$
   $+ \partial_1(h)(r \otimes r')\theta_1\theta_{i_1} \cdots \theta_{i_k} + \cdots + \partial_n(h)(r \otimes r')\theta_n\theta_{i_1} \cdots \theta_{i_k})$

The image under π of the expression in blue is zero because for each summand: Either $\theta_i \in \{\theta_{i_1} \cdots \theta_{i_k}\}$ and so we will have θ_i^2 in that summand which will cause the summand to boil down to zero or $\theta_i \notin \{\theta_{i_1} \cdots \theta_{i_k}\}$ and so the image under π of what is left will be zero by definition of π since $n \geq 1$.

The image under π of the expression in green is zero because for each summand: If in the summand there is a θ_{i_l}, ($l \in \{1, \cdots, k\}$), then by definition of π, the image would be zero.
If in the summand there is no θ_{i_l}, ($l \in \{1, \cdots, k\}$), then its image under π is $\pi((x_i - x_i')(r \otimes r')) = (r \otimes r')\pi(x_i - x_i')$ which is zero because $\pi(x_i - x_i') = \pi(x_i) - \pi(x_i') = 0$ in R. □

We now work out (Δ_f, d_{Δ_f}) for a specific f. We will simply write d for d_{Δ_f}.

Example 5.4. *Let $f = x \in \mathbb{R}[x]$ be a one variable polynomial over the polynomial ring $\mathbb{R}[x]$, where $\mathbb{R}$ is the set of real numbers. Let $R = \mathbb{R}[x]$. We have the $R \otimes R-$module:*

$$\Delta_f = \bigwedge\{\bigoplus(R \otimes R)\theta\}$$

This is an exterior algebra generated by the variable θ.

To be more precise, it is the free $R \otimes R-$module generated by the empty list and the symbol θ, modulo the relation that the $\theta^2 = 0$ that is $\theta \wedge \theta = 0$. Typically, we will omit the wedge product and write for instance $\theta\wedge\theta$ simply as $\theta\theta$. Here, the "$\wedge$" product is taken over K just like the tensor product. A typical element of Δ_f^1 is $(r \otimes r')\theta$ or equivalently $h(x, x')\theta$ where $h(x, x') \in \mathbb{R}[x, x']$. A typical element of Δ_f^0 is $(r \otimes r')$ or equivalently $h(x, x')$ where $h(x, x') \in \mathbb{R}[x, x']$.
The action of $R \otimes R$ is the obvious one. That is, for example, if $r_1 \otimes r_2 \in R \otimes R$, its left action on an element $(r \otimes r')\theta$ of Δ_f simply yields $(r_1 \otimes r_2)(r \otimes r')\theta$. The right action would yield $(r \otimes r')(r_1 \otimes r_2)\theta$ which is in fact the same as the left action since R is commutative. (In fact; $(r \otimes r')(r_1 \otimes r_2) = (rr_1 \otimes r'r_2) = (r_1 r \otimes r_2 r') = (r_1 \otimes r_2)(r \otimes r')$).

Δ_f is endowed with the $\mathbb{Z}_2-$grading given by $\theta-$degree (where $deg\theta = 1$). Thus $deg\theta^2 = deg0 = 0$.
Next, we define the differential as follows:

$$d : \Delta_f \longrightarrow \Delta_f$$
$$d(-) = [(x - x^{'})\theta^*(-) + \partial(f)\theta \wedge (-)]$$
$$\partial(f(x)) = \frac{f(x) - f(x')}{x - x'} = \frac{x - x'}{x - x'} = 1.$$

We now show that $d^0 \circ d^1 = (f(x) - f(x')) \cdot id_{\Delta^1}$ and $d^1 \circ d^0 = (f(x) - f(x')) \cdot id_{\Delta^0}$. But first, $d^0 : \Delta_f^0 \longrightarrow \Delta_f^1$ and $d^0(h(x, x')) = (x - x^{'})\theta^*(h(x, x')) + \partial(x)\theta \wedge (h(x, x')) = \theta h(x, x')$ since $\partial(x) = \partial f(x) = 1$ and $\theta^*(h(x, x')) = 0$ as θ^* is a derivation w.r.t θ.

Next, $d^1 : \Delta_f^1 \longrightarrow \Delta_f^0$ *and* $d^1(h(x,x')\theta) = (x-x^{'})\theta^*(h(x,x')\theta) + \partial(x)\theta \wedge (h(x,x')\theta) = (x-x^{'})h(x,x')$ *since* $\theta^*(\theta) = 1$ *and* $\theta \wedge (h(x,x')\theta) = 0$ *as* $\theta\theta = 0$.
We now compute

$$\begin{array}{rcl} d^0 \circ d^1(h(x,x')\theta) &=& d^0((x-x^{'})(h(x,x')) \\ &=& (x-x^{'})\theta^*(x-x^{'})(h(x,x')) + \partial(x)\theta \wedge (x-x^{'})(h(x,x')) \\ &=& \theta(x-x^{'})(h(x,x'))\ since\ \theta^*(x-x^{'})(h(x,x')) = 0\ and\ \partial(x) = 1 \\ &=& (x-x^{'})(h(x,x'))\theta \\ &=& (x-x^{'})\cdot id_{\Delta^1}[(h(x,x'))\theta] \end{array}$$

i.e., $d^0 \circ d^1 = (x-x^{'})\cdot id_{\Delta^1} = (f(x)-f(x'))\cdot id_{\Delta^1}$

and

$$\begin{array}{rcl} d^1 \circ d^0(h(x,x')) &=& d^1((h(x,x')\theta) \\ &=& (x-x^{'})\theta^*(h(x,x')\theta) + \partial(x)\theta \wedge (h(x,x')\theta) \\ &=& (x-x^{'})(h(x,x'))\ since\ \theta^*(\theta) = 1\ and\ \theta^2 = 0 \\ &=& (x-x^{'})(h(x,x')) \\ &=& (x-x^{'})\cdot id_{\Delta^0}[(h(x,x'))] \end{array}$$

i.e., $d^1 \circ d^0 = (x-x^{'})\cdot id_{\Delta^0} = (f(x)-f(x'))\cdot id_{\Delta^0}$

Example 5.5. *Let* $f = x - y \in \mathbb{R}[x,y]$ *be a two-variable polynomial over the polynomial ring* $\mathbb{R}[x,y]$. *Let* $R = \mathbb{R}[x,y]$. *We have the* $R\otimes R-$*module:*

$$\Delta_f = \bigwedge\{\bigoplus_{i=1}^{2}(R\otimes R)\theta_i\}$$

This is an exterior algebra generated by 2 *anti-commuting variables,* θ_1 *and* θ_2.

To be more precise, for $i,j \in \{1,2\}$*; it is the free* $R\otimes R-$*module generated by the set containing the empty list and the symbols* θ_i*, modulo the relations that the* θ*'s anti-commute, that is* $\theta_i \wedge \theta_j = -\theta_j \wedge \theta_i$. *Typically, we will omit the wedge product and write for instance* $\theta_i \wedge \theta_j$ *simply as* $\theta_i\theta_j$. *Here, the "*$\wedge$*" product is taken over* $\mathbb{R}$ *just like the tensor product. A typical element of* Δ_f^0 *is either of the form* $(r\otimes r')$ *(or equivalently* $h(x,y,x',y')$*) or* $(r\otimes r')\theta_i\theta_j$ *(or equivalently* $h(x,y,x',y')\theta_i\theta_j$*) where* $h(x,y,x',y') \in \mathbb{R}[x,y,x',y']$. *A typical element of* Δ_f^1 *is* $(r\otimes r')\theta_i$ *or equivalently* $h(x,y,x',y')\theta_i$ *where* $h(x,y,x',y') \in \mathbb{R}[x,y,x',y']$.
The action of $R\otimes R$ *is the obvious one. That is, for example, if* $r_1\otimes r_2 \in R\otimes R$*, its left action on an element* $(r\otimes r')\theta_i\theta_j$ *of* Δ_f *simply yields* $(r_1\otimes r_2)(r\otimes r')\theta_i\theta_j = (r_1r\otimes r_2r')\theta_i\theta_j$. *The right action would yield* $(r\otimes r')(r_1\otimes r_2)\theta_i\theta_j = (rr_1\otimes r'r_2)\theta_i\theta_j$ *which is in fact the same as the left action since R is commutative. (In fact;* $(r\otimes r')(r_1\otimes r_2) = (rr_1\otimes r'r_2) = (r_1r\otimes r_2r') = (r_1\otimes r_2)(r\otimes r')$*).*

Δ_f *is endowed with the* $\mathbb{Z}_2-$*grading given by* $\theta-$*degree (where* $deg\theta_i = 1$ *for each i). Thus* $deg\theta_i^2 = 0$ *and* $deg\theta_i\theta_j = 0$.
Next, we define the differential as follows:

$$d : \Delta_f \longrightarrow \Delta_f$$

$$d(-) = \sum_{i=1}^{2}[(x_i - x_i^{'})\theta_i^*(-) + \partial_i(f)\theta_i \wedge (-)]$$

$$\partial_1(f(x,y)) = \frac{f(x,y) - f(x',y)}{x - x'} = \frac{(x-y)-(x'-y)}{x-x'} = 1.$$
$$\partial_2(f(x,y)) = \frac{f(x',y) - f(x',y')}{y - y'} = \frac{(x'-y)-(x'-y')}{y-y'} = -1.$$

We now need to show that $d^0 \circ d^1 = [(x-y)-(x'-y')] \cdot id_{\Delta^1}$ *and* $d^1 \circ d^0 = [(x-y)-(x'-y')] \cdot id_{\Delta^0}$. *We will not show all the details because it is computed in a manner similar to the calculations in the above example. We only specify* d^0 *and* d^1.
$d^0 : \Delta_f^0 \longrightarrow \Delta_f^1$ *and* $d^0(-) = (x-x^{'})(\theta_1^*)^0(-) + \partial_1(f)\theta_1 \wedge (-) + (y-y^{'})(\theta_2^*)^0(-) + \partial_2(f)\theta_2 \wedge (-)$ *i.e.,* $d^0(-) = (x-x^{'})(\theta_1^*)^0(-) + \theta_1 \wedge (-) + (y-y^{'})(\theta_2^*)^0(-) - \theta_2 \wedge (-)$ *since* $\partial_1(f) = 1$ *and* $\partial_2(f) = -1$.
Next, $d^1 : \Delta_f^1 \longrightarrow \Delta_f^0$ *and* $d^1(-) = (x-x^{'})(\theta_1^*)^1(-) + \partial_1(f)\theta_1 \wedge (-) + (y-y^{'})(\theta_2^*)^1(-) + \partial_2(f)\theta_2 \wedge (-)$ *i.e.,* $d^0(-) = (x-x^{'})(\theta_1^*)^1(-) + \theta_1 \wedge (-) + (y-y^{'})(\theta_2^*)^1(-) - \theta_2 \wedge (-)$ *since* $\partial_1(f) = 1$ *and* $\partial_2(f) = -1$.

5.2.3 The left and the right units of $\mathcal{LG}_K$

In this subsection, we define the left and right identities for our bicategory $\mathcal{LG}_K$. After proving their naturality with respect to 2−morphisms, we prove that they do not have direct inverses thereby justifying the fact that their inverses in [17] are found only up to homotopy.
We will denote the right (respectively left) unit by ρ (respectively λ). In the whole of this subsection, we will be dealing only with the left unit ρ and we will omit the statements and proofs of the results for λ because they are similar to the ones presented for ρ.
Consider a 1-morphism $X \in hmf(R \otimes S, 1_R \otimes g - f \otimes 1_S)^{\omega} = hmf(R \otimes S, id \otimes g - f \otimes id)^{\omega}$.
Thus, X is a matrix factorization of $id \otimes g - f \otimes id$ and is also an $R \otimes S-$module.
Let $1_X : X \longrightarrow X$ be the identity map and π be the projection defined in lemma 5.8.

Remark 5.5. *Observe that any* $S-$*module* N *can be considered as an* $R-$*module by letting* $rn := f(r)n$ *where* $f : R \longrightarrow S$ *is a homomorphism of commutative rings. It is easy to see that the* $R \otimes S-$*module* X *can be considered as an* $R-$*module via the following* $K-$*homomorphism of commutative unitary rings* $f : R \longrightarrow R \otimes S$ *defined by* $f(r) = r \otimes 1_S$, *and hence one can also see* X *as an* $R \otimes R-$*module by means of the following (multiplicative map which is a)* $K-$*homomorphism of commutative unitary rings* $m : R \otimes R \longrightarrow R$ *defined by* $m(r \otimes r') = rr'$. *It is not difficult to see that the* $R \otimes R-$*module* Δ_f *can be considered as an* $R-$*module via the following homomorphism of commutative unitary rings* $f : R \longrightarrow R \otimes R$ *defined by* $f(r) = r \otimes 1_R$.

Thanks to this remark, it makes sense to form the following tensor product over R: $X \otimes_R R$ and $X \otimes_R \Delta_f$ since X and Δ_f can be viewed as $R-$modules. Consequently, we will simply write $X \otimes R$ and $X \otimes \Delta_f$ for ease of notation.
Similarly, since the $R \otimes S-$module X and the $S \otimes S-$module Δ_g can be viewed as $S-$modules, we can form the module $\Delta_g \otimes_S X$ that we simply write as $\Delta_g \otimes X$.
We also consider the map $u : X \otimes R \longrightarrow X$ defined by $u(x \otimes r) = xr$. This definition makes sense since X can be viewed as an $R-$module. u is an isomorphism (See example 1 page 363 of [22]).

Now, define $\rho_X : X \otimes \Delta_f \longrightarrow X$ by $\rho_X := u \circ (1_X \otimes \pi)$ and $\lambda_X : \Delta_g \otimes X \longrightarrow X$ by $\lambda_X := u \circ (\pi \otimes 1_X)$.
ρ_X and λ_X are clearly morphisms in $hmf(R \otimes S, id \otimes g - f \otimes id)^{\omega}$.
We show that ρ is natural w.r.t. 2−morphisms in the variable X and we explain why there is no direct inverse[11] for ρ_X, for each X. A similar result and a similar proof hold for the other unit map λ, and consequently is omitted.

Lemma 5.9. *ρ and λ are natural w.r.t. 2−morphisms in the variable X.*

Proof. The proof for ρ is given and the one for λ is omitted because it is similar.
In fact:

1. ρ is natural in X.
   In fact, we have the functor

$$F := (-) \otimes \Delta_f : hmf(R \otimes S, id \otimes g - f \otimes id)^{\omega} \longrightarrow hmf(R \otimes S, id \otimes g - f \otimes id)^{\omega},$$

   $X \mapsto X \otimes \Delta_f$.
   This is well defined because $X \otimes \Delta_f$ is an $R \otimes S$−module since X is an $R \otimes S$−module and Δ_f is an $R \otimes R$−module.
   We also have the identity functor:

$$G := Id : hmf(R \otimes S, id \otimes g - f \otimes id)^{\omega} \longrightarrow hmf(R \otimes S, id \otimes g - f \otimes id)^{\omega},$$

   $X \mapsto X$.
   For each object $X \in hmf(R \otimes S, id \otimes g - f \otimes id)^{\omega}$, $\rho_X : X \otimes \Delta_f \longrightarrow X$ is a map in $hmf(R \otimes S, id \otimes g - f \otimes id)^{\omega}$.
   Components ρ_X of ρ are such that for each $p : X \longrightarrow Y$, $\rho_Y \circ F(p) = G(p) \circ \rho_X$, viz. $\rho_Y \circ p \otimes id = p \circ \rho_X$ because the following diagram obviously commutes:

$$\begin{array}{ccc} X \otimes \Delta_f & \xrightarrow{p \otimes id} & Y \otimes \Delta_f \\ \rho_X \downarrow & \circlearrowleft & \downarrow \rho_Y \\ Id(X) = X & \xrightarrow[G(p)=p]{} & Id(Y) = Y \end{array}$$

   (we wrote id for id_{Δ_f}) In fact, let $r \otimes r'\theta_{i_1} \cdots \theta_{i_k} \in \Delta_f$, $i_1, \cdots, i_k \in \{1, \cdots, n\}$. Recalling that $\rho_X = u \circ (1_X \otimes \pi)$, $u : X \otimes R \xrightarrow{\cong} X$, $x \otimes r \mapsto xr$; and also that π is the θ−degree 0 map, we have on the one hand:

$$\begin{aligned} (i) \quad \rho_Y \circ (p \otimes id)(x \otimes (r \otimes_K r')\theta_{i_1} \cdots \theta_{i_k}) &= \rho_Y(p(x) \otimes (r \otimes_K r')\theta_{i_1} \cdots \theta_{i_k}) \\ &= u(1_Y \otimes \pi)(p(x) \otimes (r \otimes_K r')\theta_{i_1} \cdots \theta_{i_k}) \\ &= u(p(x) \otimes \pi((r \otimes_K r')\theta_{i_1} \cdots \theta_{i_k})) \\ &= \begin{cases} 0 \; if \; k \geq 1, \\ u(p(x) \otimes rr') = p(x)rr', \; otherwise. \end{cases} \end{aligned}$$

   On the other hand, we have:

[11] In [17], it is proved that the left and right unit maps are isomorphisms up to homotopy, but the reason why they are not direct isomorphisms is not given explicitely, so we give that reason at the end of this chapter.

$$
\begin{aligned}
(ii)\quad p\circ\rho_X(x\otimes(r\otimes_K r')\theta_{i_1}\cdots\theta_{i_k}) &= p\circ u(1_X\otimes\pi)(x\otimes(r\otimes_K r')\theta_{i_1}\cdots\theta_{i_k})\\
&= p\circ u(x\otimes\pi((r\otimes_K r')\theta_{i_1}\cdots\theta_{i_k}))\\
&= \begin{cases} 0 \; if\; k\geq 1,\\ p(u(x\otimes rr')) = p(xrr') = p(x)rr' \; otherwise. \end{cases}
\end{aligned}
$$

The very last equality above is obtained by $R-$linearity of p which is a map of factorization.
The work done in (i) and (ii) shows that $\rho_Y\circ(p\otimes id) = p\circ\rho_X$. So ρ is indeed a natural transformation.

2. ρ_X is a morphism of factorizations, viz an $R-$linear even map commuting with the differentials.
$\bullet\rho_X = u\circ(1_X\otimes\pi)$ is an $R-$linear even map.
In fact, The projection map π was shown to be linear and even, in the proof of lemma 5.8, and we know that the identity map 1_X is clearly linear and even, so $1_X\otimes\pi$ is a linear even map.
Moreover, u is a linear map (see example 1, page 363 of [22]). Hence ρ_X is linear as composition of linear maps.
It now remains to show that u is an even map. We give the trivial grading to R, i.e., $R = R_0\oplus\{0\}$. Now, $(X_0\oplus X_1)\otimes R = X_0\otimes R\oplus X_1\otimes R$. We have:
$u : (X_0\oplus X_1)\otimes R = X_0\otimes R\oplus X_1\otimes R \longrightarrow X_0\oplus X_1$ defined by: $u(x_0\otimes r_0) = x_0r_0\in X_0$ and $u(x_1\otimes r_0) = x_1r_0\in X_1$, since X is an R-module. This shows that u is an even map.

$\bullet$ ρ_X commutes with the differentials.
Let $Z = X\otimes\Delta_f$ and let's write each factorization with its pair of differentials as follows:
(X, d_X^0, d_X^1), (Z, d_Z^0, d_Z^1) and $(\Delta_f, d_\Delta^0, d_\Delta^1)$. Since X and $Z = X\otimes\Delta_f$ are factorizations of $h := id\otimes g - f\otimes id$, the following hold:

$$
\begin{cases}
d_X^1 d_X^0 = h\cdot id_{X^0}\\
d_X^0 d_X^1 = h\cdot id_{X^1}\\
d_Z^1 d_Z^0 = h\cdot id_{Z^0}\\
d_Z^0 d_Z^1 = h\cdot id_{Z^1}
\end{cases}
$$

We have the following pair of maps

$Z^0 \xrightarrow{\rho^0} X^0$ and $Z^1 \xrightarrow{\rho^1} X^1$.
To show that ρ_X commutes with the differentials, it suffices to show that the following diagram commutes:

$$
\begin{array}{ccccc}
Z^0 & \xrightarrow{d_Z^0} & Z^1 & \xrightarrow{d_Z^1} & Z^0\\
\downarrow{\scriptstyle\rho^0} & & \downarrow{\scriptstyle\rho^1} & & \downarrow{\scriptstyle\rho^0}\\
X^0 & \xrightarrow{d_X^0} & X^1 & \xrightarrow{d_X^1} & X^0
\end{array}
$$

i.e.,

$$\begin{cases} \rho^0 d_Z^1 = d_X^1 \rho^1 \; ... \; (i) \\ d_X^0 \rho^0 = \rho^1 d_Z^0 \; ... \; (ii) \end{cases}$$

It suffices to prove only one of these equalities because they are equivalent[12].
We briefly discuss (ii) without giving a full proof because of the amount of the details involved: First recall that $Z^0 = (X \otimes \Delta_f)^0 = (X^0 \otimes \Delta_f^0) \oplus (X^1 \otimes \Delta_f^1)$.
First recall that the $\mathbb{Z}_2-$graded $R \otimes S-$module $X \otimes \Delta_f$ is a finite rank matrix factorization of the polynomial $g - f$ and it comes with an odd $R \otimes S-$linear endomorphism $d_Z : Z \to Z$ s.t. $d_Z^2 = (g - f) \cdot id_Z$. We have

$$d_Z^0 : Z^0 = (X^0 \otimes \Delta_f^0) \oplus (X^1 \otimes \Delta_f^1) \to (X^0 \otimes \Delta_f^1) \oplus (X^1 \otimes \Delta_f^0) = Z^1.$$

Now if for ease of notation we write id_Δ for id_{Δ_f} then $d_Z^0 = (d_X \otimes id_\Delta + id_X \otimes d_\Delta)^0$ where $id_X \otimes d_\Delta$ has the usual Koszul signs when applied to elements. We have:
$\rho^0 = (u \circ (1_X \otimes \pi))^0 = u^0 \circ (1_X \otimes \pi)^0 + u^1 \circ (1_X \otimes \pi)^1$.
$\rho^1 = (u \circ (1_X \otimes \pi))^1 = u^1 \circ (1_X \otimes \pi)^0 + u^0 \circ (1_X \otimes \pi)^1$.
$(1_X \otimes \pi)^0 = (1_X^0 \otimes \pi^0) \oplus (1_X^1 \otimes \pi^1)$, $(1_X \otimes \pi)^1 = (1_X^0 \otimes \pi^1) \oplus (1_X^1 \otimes \pi^0)$.
With all these data, one can prove ii). Details are omitted because of their large quantity.

□

The authors of [17] give a difficult proof in which they exhibit the homotopy inverses of ρ_X and λ_X but they do not show why the direct inverses of ρ_X and λ_X do not exist.
In the following theorem, we show that ρ_X does not have a direct inverse. This helps to understand why ρ_X can only have an inverse up to homotopy. A similar job can be done for the other unit map λ_X.

Theorem 5.1. *ρ_X and λ_X do not have direct inverses.*

Proof. The proof for ρ_X is presented and the one for λ_X is omitted because it is similar.
We find a direct right inverse of ρ_X that we call ψ_X, next we prove that ψ_X is not a direct left inverse of ρ_X and then proceed by contradiction to show that ρ_X has no direct inverse.
• Define $\psi_X : X \longrightarrow X \otimes \Delta_f$ by $x \mapsto x \otimes 1_{R \otimes R}$.
First, it is important to show that ψ is natural in X. It is easy to see that $\psi_X : X \longrightarrow X \otimes \Delta_f$ is a map in $hmf(R \otimes S, id \otimes g - f \otimes id)^\omega$. Let X *and* $Y \in hmf(R \otimes S, id \otimes g - f \otimes id)^\omega$ and $j : X \mapsto Y$. We need to show that the following diagram commutes:

$$\begin{array}{ccc} Id(X) = X & \xrightarrow{\;j\;} & Id(Y) = Y \\ \downarrow{\scriptstyle \psi_X} & \circlearrowleft & \downarrow{\scriptstyle \psi_Y} \\ X \otimes \Delta_f & \xrightarrow[j \otimes id]{} & Y \otimes \Delta_f \end{array}$$

i.e., $\psi_Y \circ j = j \otimes id \circ \psi_X$.
Now, $\psi_Y \circ j(x) = j(x) \otimes 1_{R \otimes R}$ by definition of ψ. $\cdots$ †
$j \otimes id \circ \psi_X(x) = j \otimes id(x \otimes 1_{R \otimes R}) = j(x) \otimes id(1_{R \otimes R}) = j(x) \otimes 1_{R \otimes R} \cdots$ ‡.
† and ‡ yield the desired equality. Hence ψ is a natural transformation.

[12] We proved something quite similar using matrix representations in Remark 5.3.

Now, $\psi_X = \psi_X^0 \oplus \psi_X^1$, where for $x = x_0 \oplus x_1$, we have:
$\psi_X^0 : X^0 \longrightarrow (X \otimes \Delta_f)^0 = (X^0 \otimes \Delta_f^0) \oplus (X^1 \otimes \Delta_f^1))$, defined by $\psi_X^0(x_0) = x_0 \otimes 1_{R\otimes R} = x_0 \otimes (1_R \otimes 1_R)$
$\psi_X^1 : X^1 \longrightarrow (X \otimes \Delta_f)^1 = (X^0 \otimes \Delta_f^1) \oplus (X^1 \otimes \Delta_f^0)$, defined by $\psi_X^1(x_1) = (x_1 \otimes 1_{R\otimes R}) = x_1 \otimes (1_R \otimes 1_R)$

• It is easy to see that ψ_X is an $R-$linear map. ψ_X is an even map by construction.

• We now show that ψ_X commutes with the differentials. This condition is represented diagrammatically by the commutativity of the following diagram where $Z = X \otimes \Delta_f$.

$$\begin{array}{ccccc} X^0 & \xrightarrow{d_X^0} & X^1 & \xrightarrow{d_X^1} & X^0 \\ \downarrow{\scriptstyle \psi^0} & & \downarrow{\scriptstyle \psi^1} & & \downarrow{\scriptstyle \psi^0} \\ Z^0 & \xrightarrow{d_Z^0} & Z^1 & \xrightarrow{d_Z^1} & Z^0 \end{array}$$

That is, we need to show that:

$$\begin{cases} \psi^1 d_X^0 = d_Z^0 \psi^0 \; ... \; (i)' \\ \psi^0 d_X^1 = d_Z^1 \psi^1 \; ... \; (ii)' \end{cases}$$

It suffices to prove only one of these equalities because they are equivalent[13].

We prove $(i)'$:

$$\begin{array}{rcl} d_Z^0 \psi^0(x_0) & = & (d_X^0 \otimes id + id \otimes d_\Delta^0)(x_0 \otimes 1_{R\otimes R}) \\ & = & d_X^0(x_0) \otimes 1_{R\otimes R} + (-1)^{|x_0|} id(x_0) \otimes d_\Delta^0(1_{R\otimes R}) \\ & = & d_X^0(x_0) \otimes 1_{R\otimes R} + x_0 \otimes 0 \;\; since \; d_\Delta^0(1_{R\otimes R}) = 0 \; and \; | x_0 | = 0 \\ & = & \psi^1 d_X^0(x_0) \end{array}$$

Hence $(i)'$ holds. So ψ_X is a map of factorization.

• ψ_X is a direct right inverse of ρ_X.
We need to show that $\rho_X \circ \psi_X = id_X$.

$$\begin{array}{rcl} \rho_X \circ \psi_X(x_0 \oplus x_1) & = & \rho_X(\psi_X(x_0) \oplus \psi_X(x_1)), \; \psi_X \; is \; linear \\ & = & u(1_X \otimes \pi)(x_0 \otimes 1_{R\otimes R} \oplus x_1 \otimes 1_{R\otimes R}) \; by \; definition \; of \; \rho_X \; and \; \psi_X \\ & = & u((1_X \otimes \pi)(x_0 \otimes 1_{R\otimes R}) \oplus (1_X \otimes \pi)(x_1 \otimes 1_{R\otimes R})) \\ & = & u(x_0 \otimes 1_R 1_R \oplus x_1 \otimes 1_R 1_R) \; as \; \pi(1_{R\otimes R}) = \pi(1_R \otimes 1_R) = 1_R 1_R. \\ & = & u(x_0 \otimes 1_R \oplus x_1 \otimes 1_R) \\ & = & u(x_0 \otimes 1_R) \oplus u(x_1 \otimes 1_R), \; since \; u \; is \; linear \\ & = & x_0 1_R \oplus x_1 1_R \; by \; definition \; of \; u \\ & = & x_0 \oplus x_1 \; as \; desired \end{array}$$

So $\rho_X \circ \psi_X = id_X$.
i.e., ψ_X is a direct right inverse of ρ_X.

• We now show that ψ_X is not a (direct) left inverse to ρ_X. All we need show is that $\psi_X \circ \rho_X \neq id_Z$. Let $x \otimes (r \otimes r')\theta_1 \in Z = X \otimes \Delta_f$, then:

[13] We proved something quite similar using matrix representations in Remark 5.3.

$$\begin{aligned}
\psi_X \circ \rho_X(x \otimes (r \otimes r')\theta_1) &= \psi_X(u(1_X \otimes \pi)(x \otimes (r \otimes r')\theta_1)) \\
&= \psi_X(u(x \otimes \pi(r \otimes r')\theta_1)) \\
&= \psi_X(0), \ \textit{since } \pi((r \otimes r')\theta_1) = 0 \\
&= 0 \\
&\neq id_Z(x \otimes (r \otimes r')\theta_1)
\end{aligned}$$

• Finally, suppose towards a contradiction that ρ_X has an inverse (i.e., a map that is both a right and a left direct iverse of ρ_X), call it $\chi_X : X \mapsto X \otimes \Delta_f$.
Then: $\rho_X \circ \chi_X = id_X \cdots \sharp$
and $\chi_X \circ \rho_X = id_{X\otimes\Delta_f} \cdots \sharp'$ by definition of the inverse of a map.
And $\rho_X \circ \psi_X = id_X \cdots \sharp$" since ψ_X is a right inverse of ρ_X.
from $\sharp$ *and* $\sharp$" we have:

$$\begin{aligned}
\rho_X \circ \psi_X = \rho_X \circ \chi_X &\Rightarrow \chi_X \circ (\rho_X \circ \psi_X) = \chi_X \circ (\rho_X \circ \chi_X) \\
&\Rightarrow (\chi_X \circ \rho_X) \circ \psi_X = (\chi_X \circ \rho_X) \circ \chi_X, \ by\, associativity \\
&\Rightarrow (id_{X\otimes\Delta_f}) \circ \psi_X = (id_{X\otimes\Delta_f}) \circ \chi_X \ since\, \chi \circ \rho_X = id_{X\otimes\Delta_f} \\
&\Rightarrow \psi_X = \chi_X
\end{aligned}$$

But this last equality is a contradiction since ψ_X which is not a left inverse of ρ_X cannot be equal to χ_X which is a left inverse of ρ_X.
So ρ_X is not invertible as claimed. □

A similar work with the left unitor (or unit action) $\lambda_X : \Delta_g \otimes X \to X$ shows that it is not directly invertible.
We refer the reader to sections 3 and 4 of [17] for a proof showing that the unitors have inverses up to homotopy. The authors in [17] used *Atiyah classes* (cf. section 3 of [17]) in their proof. But they never explained why it does not work at a non-homotopic setting. We did it in the foregoing lemma.

Remark 5.6. *Observe that ψ_X is a right homotopy inverse of ρ_X. In fact, we need to verify that $\rho_X \circ \psi_X \sim id_X$ where $\sim$ stands for the homotopy relation.*
(Recall that: $\rho_X \circ \psi_X, id_X : X \longrightarrow X$).
We need to find an odd degree R-linear map $\alpha : X \longrightarrow X$ s.t., $d_X\alpha + \alpha d_X = \rho_X \circ \psi_X - id_X$.
It suffices to take α to be the zero map (which is clearly an R-linear odd degree map).
In fact, if $\alpha = 0$ then:
$d_X\alpha + \alpha d_X = \rho_X \circ \psi_X - id_X \Longleftrightarrow 0 = \rho_X \circ \psi_X - id_X$, i.e., $\rho_X \circ \psi_X = id_X$ which is true since ψ_X is the right direct inverse of ρ.
A similar work can be done for the other (left) unit map λ_X.

AN APPLICATION OF MATRIX FACTORIZATIONS: MORITA CONTEXTS IN $\mathcal{LG}_K$

In this chapter, we give an application of matrix factorizations using the concept of Morita context which is a weakening of the concept of Morita equivalence. The notion of *Morita context* we use here is the same as the one defined in ([55]). We study this concept in the bicategory $\mathcal{LG}_K$ on a particular class of objects. It turns out that a matrix approach is more suitable for this kind of study. We first observe that if X and X' are respectively two matrix factorizations of some polynomials r and s, then the determinants of the four matrices appearing in the Yoshino tensor products $X \otimes X' = (P, Q)$ and $X' \otimes X = (P', Q')$ are all equal. Next, when we translate this in $\mathcal{LG}_K$ where a $1-$morphism is a matrix factorizations of the difference of two polynomials, we find out that those four determinants are all equal to zero. This result together with some other properties (cf. chap. 2) of matrix factorizations help us to state and prove a necessary condition to obtain a Morita context between two arbitrary objects of $\mathcal{LG}_K$. We show that this condition is not sufficient and we also state a trivial sufficient condition to obtain a Morita context in $\mathcal{LG}_K$.
Except otherwise stated, $R = K[x_1, \cdots, x_n]$ where K is a commutative ring with unity.

The results developed and obtained for the notion of "Morita context" in $\mathcal{LG}_K$ in this chapter are original to the best of our knowledge.

6.1 The notion of Morita context

In this section, we define what a *Morita Context* is between two objects of a bicategory and we discuss why we think this notion is interesting.

6.1.1 Introduction and Definition

A *Morita context* also called pre-equivalence data ([9]), is a generalization of *Morita equivalence*[1] between categories of modules.
The following definition is a classical one and is due to Morita (cf. section 22 [3])

Definition 6.1. *Two rings R and S are called* ***Morita equivalent*** *if the categories of left $R-$Modules ($R-$Mod) and of left $S-$Modules ($S-$Mod) are equivalent.*

[1] For more on Morita Equivalence, see [3]

The prototype of Morita equivalent rings is provided by a ring R and the ring of $n \times n$ matrices over R (for details, see corollary 22.6 of [3]).
It is evident that if two rings are isomorphic, they are Morita equivalent. However, the converse in general is not true. On page 470 of [43], we find an example of two rings that are not isomorphic, yet are Morita equivalent. In fact, it suffices to take a ring R and the ring of $n \times n$ matrices over R. However, the following partial converse holds.

Theorem 6.1. *([43], P.494) If two rings are Morita equivalent, then their centers are isomorphic. In particular, if the rings are commutative, then they are isomorphic.*

Because of this theorem, Morita equivalence is interesting solely in the situation of noncommutative rings.
Classical Result (Morita): (cf. Section 22 of [3]) Given rings R and S, the f.a.e:
(1) The categories of left $S-$modules and left $R-$modules are equivalent;
(2) The categories of right $S-$modules and right $R-$modules are equivalent.
(3) There are R-S-bimodule M and S-R-bimodule N such that $(-) \otimes_R M$ and $(-) \otimes_S N$ form an adjoint equivalence between the category of right $S-$modules and the category of right $R-$modules.

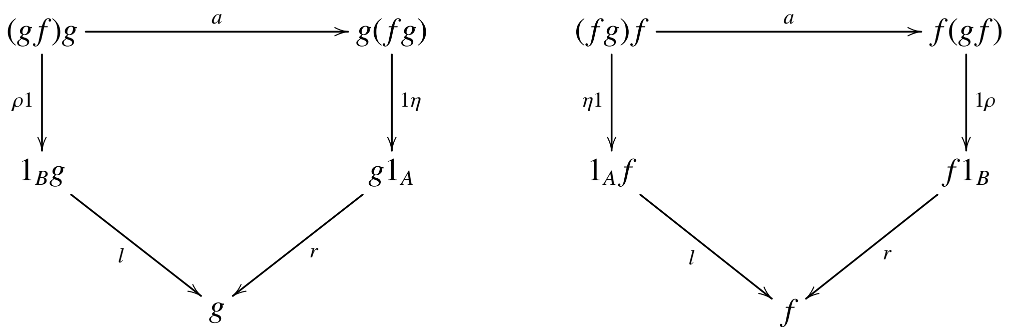
Morita contexts were first introduced in the bicategory ringBimod of unitary rings and bimodules[2] as 6-tuples (A, B, M, N, ϕ, ψ); where A and B are rings, M is an A-B-bimodule, N is a B-A-bimodule and $\phi : M \otimes N \to A$ and $\psi : N \otimes M \to B$ are homomorphisms satisfying $\phi \otimes M = M \otimes \psi$ and $N \otimes \phi = \psi \otimes N$.
Bass (cf. chap. 2, section 4.4 of [8]) proved that:
The Morita context (A, B, M, N, ϕ, ψ) is a Morita equivalence if and only if M is both projective and a generator (cf. definition 6.2) in the category of $A-$modules.
Recall (cf. [43]): Let A be a ring. An A-module P is projective if for every surjective A-linear map $f : M \to N$ and every A-linear map $g : P \to N$ there is a unique A-linear map $h : P \to M$ such that $g = fh$.

Definition 6.2. *A set S of objects of a category C is said to generate C, when to any parallel pair $h, h' : A \to B$ of arrows of C, $h \neq h'$ implies that there is $D \in S$ and an arrow $f : D \to A$ with $hf \neq h'f$. It is also possible for a single element to generate a category C.*

Example 6.1. *Any one point set generates the category of **Set** of sets. The set of integers $\mathbb{Z}$ generates the categories **Grp** of groups and **Ab** of abelian groups. A ring R generates the category $R-Mod$ of right $R-$modules.*

Definition 6.3. *[55]*
*Let $\mathcal{B}$ be a bicategory with natural isomorphisms a, r and l. Given two 0-cells A and B, we define a **Morita Context** between A and B as a four-tuple $\Gamma = (f, g, \eta, \rho)$ consisting of two 1-cells $f \in Hom_{\mathcal{B}}(A, B)$ and $g \in Hom_{\mathcal{B}}(B, A)$, and two 2-cells $\eta : fg \to 1_A$ and $\rho : gf \to 1_B$ such that the following diagrams commute*

[2] See theorem 5.4 of [55] for a characterization of Morita context in the bicategory whose objects are unitary rings and $1-$morphisms are bimodules.



Equationally, we have:

1. $r \circ 1\eta \circ a = l \circ \rho 1$

2. $r \circ 1\rho \circ a = l \circ \eta 1$

We immediately make some observations:

Remark 6.1. *[55]*

- *Observe that any adjunction* $< A \xrightarrow{f} B\ ,\ B \xrightarrow{g} A\ ,\ 1_A \xrightarrow{\varepsilon} fg\ ,\ gf \xrightarrow{\eta} 1_B >$ *becomes a Morita context as soon as its unit* ε *is invertible.*

- *A Morita context is* ***strict*** *if both* η *and* ρ *in the foregoing definition are isomorphisms. Strict Morita contexts and adjoint equivalences are basically the same. One can switch between them by inverting the unit* ε *of the adjunction.*

Nota Bene: It is perhaps good to mention that what is called *Morita context* in this dissertation is instead called *wide right Morita context* from B to A in [35]. In [55], it is also called an *abstract bridge*. Both authors declare that the notion of *left Morita context* is defined by reversing 2-cells. We will never deal with *left Morita context* in our work.

6.1.2 Why is the notion of Morita context interesting?

In this subsection, we discuss why the notion of Morita context is interesting.
The first reason is that it generalizes the very important notion of Morita equivalence as discussed above. Another reason is that it is used to prove some celebrated results; for example, the Morita context which has been introduced in [47] was used since to prove Wedderburn theorem on the structure of simple rings [8]. Morita contexts were also used in [2] to obtain various results: Goldie's theorem ([28], [29]) on the ring of quotients of semi-prime rings and as a specialization, Wedderburn's structure theorems of semi-simple Artinian rings were obtained.
Other applications, though sometimes not stated in an explicit form, can be found in various places (e.g. [34], p. 75]).

6.2 Morita contexts in $\mathcal{LG}_K$

We now discuss what a Morita context is in $\mathcal{LG}_K$. We will often write $K[x]$ for $K[x_1, x_2, \cdots, x_n]$, where x_i is an indeterminate for $i = \{1, 2, \cdots, n\}$.

Description of Morita context in $\mathcal{LG}_K$

Let (R, f) and (S, g) be two objects of $\mathcal{LG}_K$, that is polynomials such that $f \in R = K[x]$ and $g \in S = K[y]$. In all of this subsection, except otherwise stated, we want to keep the following remark and assumption in mind:

Remark 6.2. *Let X and Y be two* $1-$*morphisms of* $\mathcal{LG}_K$*, we normally have:*

1. $X \in \mathcal{LG}_K((R, f), (S, g)) = hmf(R \otimes_K S, g - f)^{\omega}$ *i.e.,* $X : (R, f) \to (S, g)$ *is a matrix factorization which is a direct summand of a finite rank matrix factorization of* $g-f$.
2. $Y \in \mathcal{LG}_K((S, g), (R, f)) = hmf(S \otimes_K R, f - g)^{\omega}$ *i.e.,* $Y : (S, g) \to (R, f)$ *is a matrix factorization which is a direct summand of a finite rank matrix factorization of* $f-g$.
3. $\Delta_f \in \mathcal{LG}_K((R, f), (R, f)) = hmf(R \otimes_K R, f \otimes id - id \otimes f)$ *i.e.,* $\Delta_f : (R, f) \to (R, f)$ *is a finite rank matrix factorization of* $f \otimes id - id \otimes f$.
4. $\Delta_g \in \mathcal{LG}_K((S, g), (S, g)) = hmf(S \otimes_K S, g \otimes id - id \otimes g)$ *i.e.,* $\Delta_g : (S, g) \to (S, g)$ *is a finite rank matrix factorization of* $g \otimes id - id \otimes g$.

Observe that the $1-$*morphisms X as defined in the previous remark can also be a finite rank matrix factorization i.e., an object of* $hmf(R \otimes_K S, g - f)$ *which is a subcategory of* $hmf(R \otimes_K S, g - f)^{\omega}$*. A similar observation applies to Y.*
We would like to use determinant of matrices to discuss the notion of Morita context in $\mathcal{LG}_K$*, therefore, it is important for us to deal with entities that are of finite rank. That is why we need the following assumption.*
Assumption*: We restrict our study of Morita context in* $\mathcal{LG}_K$ *to those objects* (R, f) *and* (S, g) *which are such that the following* $1-$*morphisms are all finite rank matrix factorizations: X, Y,* $X \otimes Y$ *and* $Y \otimes X$.
Under this assumption, all the matrix factorizations we will be dealing with will be of finite rank, so it will make sense to talk of the determinants of the pairs of matrices constituting them. In paragraphs ahead, we will exploit this fact to characterize the notion of Morita Context in $\mathcal{LG}_K$ *for the class of objects that satisfy the above assumption. We will sometimes intentionally not remember to add the phrase "finite rank" in front of the phrase "matrix factorization", because here, we are only dealing with finite rank matrix factorizations.*

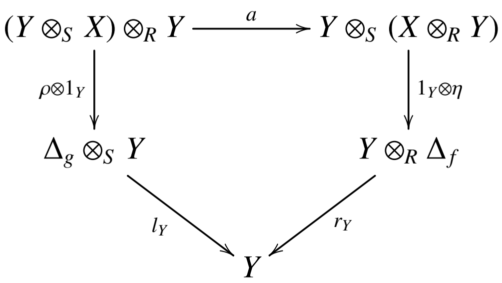
In the sequel, whenever we will talk about a Morita context between two objects of $\mathcal{LG}_K$, we will mean two objects such that the foregoing assumption holds.
A Morita Context between objects (R, f) and (S, g) of $\mathcal{LG}_K$ is a four-tuple $\Gamma = (X, Y, \eta, \rho)$ where:

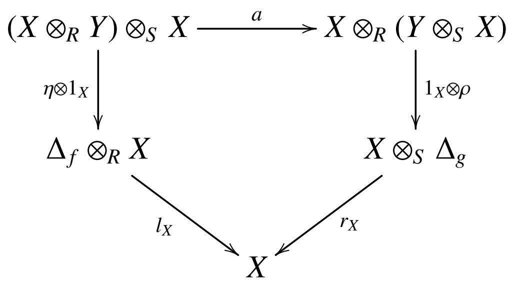
- $X : (R, f) \to (S, g)$ is a matrix factorization of $g - f$.
- $Y : (S, g) \to (R, f)$ is a matrix factorization of $f - g$.
- $\eta : X \otimes_R Y \to 1_{(R,f)} = \Delta_f$ where[3] Δ_f is the identity on (R, f) in $\mathcal{LG}_K$ i.e., a matrix factorization of $f \otimes id - id \otimes f$ as seen in the previous chapter.

[3]We will sometimes drop the subscript on the tensor product $\otimes$ for ease of notation. Thus instead of writing for instance $\otimes_R$, we may simply write $\otimes$.

- $\rho : Y \otimes_S X \to 1_{(S,g)} = \Delta_g$

  such that the following diagrams (thereafter referred to as the *M.C.LG$_K$* diagrams) commute up to homotopy:





  That is, the following two conditions hold:

  1. $r_Y \circ 1_Y \otimes \eta \circ a \approx l_Y \circ \rho \otimes 1_Y$
  2. $r_X \circ 1_X \otimes \rho \circ a \approx l_X \circ \eta \otimes 1_X$

  where $\approx$ stands for the homotopy relation.

equivalently:

1. $\exists \lambda : Z \to Y$ s.t. $d_Y \lambda + \lambda d_Z = \psi - \phi$, where $Z = (Y \otimes_S X) \otimes_R Y$, $\psi = r_Y \circ 1_Y \otimes \eta \circ a$, and $\phi = l_Y \circ \rho \otimes 1_Y$

2. $\exists \xi : Z' \to X$ s.t. $d_X \xi + \xi d_{Z'} = \psi' - \phi'$, where $Z' = (X \otimes_R Y) \otimes_S X$, $\psi' = r_X \circ 1_X \otimes \rho \circ a$, and $\phi' = l_X \circ \eta \otimes 1_X$

We now wish to find necessary and sufficient conditions on η and ρ for $\Gamma = (X, Y, \eta, \rho)$ to be a *Morita Context*.
First, we observe that two equal morphisms of linear factorizations are homotopic; since it would suffice to take $\lambda = 0$ in definition 5.3. We immediately have the following remark which gives a (trivial) sufficient condition to obtain a *Morita context* in $\mathcal{LG}_K$.

Remark 6.3. *Let (R, f) and (S, g) be two objects of $\mathcal{LG}_K$. Let $X : (R, f) \to (S, g)$ be a matrix factorization of $g - f$ and $Y : (S, g) \to (R, f)$ a matrix factorization of $f - g$. Then $\Gamma = (X, Y, 0, 0)$ is a Morita Context in $\mathcal{LG}_K$. That is, it suffices to take $\eta = 0$ and $\rho = 0$ in definition 6.3 to obtain a Morita Context in $\mathcal{LG}_K$ provided we already have X and Y.*

In fact:
First observe that the zero map in $\mathcal{LG}_K$ is a possible morphism between matrix factorizations because it satisfies definition 5.2.
In the *M.C.LG$_K$* diagrams, we know that the maps r and l are morphisms of linear factorization and so, they are linear. Consequently, the image of zero under these morphisms is zero.
Now, if $\eta = 0$ and $\rho = 0$, then:

1. $\psi = r_Y \circ 1_Y \otimes \eta \circ a = r_Y \circ 0 \circ a = 0$ and $\phi = l_Y \circ \rho \otimes 1_Y = l_Y \circ 0 = 0$

2. $\psi' = r_X \circ 1_X \otimes \rho \circ a = r_X \circ 0 \circ a = 0$, and $\phi' = l_X \circ \eta \otimes 1_X = l_X \circ 0 = 0$

This now means that the $M.C.LG_K$ diagrams commute up to homotopy if:
$\exists\lambda : Z \to Y$ s.t. $d_Y\lambda + \lambda d_Z = \psi - \phi = 0 - 0$, where $Z = (Y \otimes_S X) \otimes_R Y$ and $\exists\xi : Z' \to X$ s.t. $d_X\xi + \xi d_{Z'} = \psi' - \phi' = 0 - 0$, where $Z' = (X \otimes_R Y) \otimes_S X$

Hence, it now suffices to choose $\lambda = 0$ and $\xi = 0$ to see that $\Gamma = (X, Y, 0, 0)$ is a *Morita Context* in $\mathcal{LG}_K$.

An illustration of this result is what comes next.

Example 6.2. *Consider $f = -x^2 + 1 \in \mathbb{R}[x] = R$ and $g = y^2 + 1 \in \mathbb{R}[y] = S$. A Morita Context between (R, f) and (S, g) is a quadruple $\Gamma = (X, X', \eta, \rho)$ where:*

- *$X : (R, f) \to (S, g)$ is a matrix factorization of $g - f = y^2 + x^2 \in \mathbb{R}[x, y]$. We take* $X = (\begin{bmatrix} x & -y \\ y & x \end{bmatrix}, \begin{bmatrix} x & y \\ -y & x \end{bmatrix})$ *since* $\begin{bmatrix} x & -y \\ y & x \end{bmatrix}\begin{bmatrix} x & y \\ -y & x \end{bmatrix} = (g - f) \cdot I_2$
- *$X' : (S, g) \to (R, f)$ is a matrix factorization of $f - g = -y^2 - x^2 \in \mathbb{R}[x, y]$. We take* $X' = (\begin{bmatrix} -x & y \\ -y & -x \end{bmatrix}, \begin{bmatrix} x & y \\ -y & x \end{bmatrix})$ *since* $\begin{bmatrix} -x & y \\ -y & -x \end{bmatrix}\begin{bmatrix} x & y \\ -y & x \end{bmatrix} = (f - g) \cdot I_2$
- *$\eta = X \otimes X' \to \Delta_f$, $x \otimes x' \mapsto 0$ viz. η is the zero map.*
- *$\rho = X' \otimes X \to \Delta_g$, $x' \otimes x \mapsto 0$ viz. ρ is the zero map.*

So,

$$\Gamma = [(\begin{bmatrix} x & -y \\ y & x \end{bmatrix}, \begin{bmatrix} x & y \\ -y & x \end{bmatrix}), (\begin{bmatrix} -x & y \\ -y & -x \end{bmatrix}, \begin{bmatrix} x & y \\ -y & x \end{bmatrix}), 0, 0]$$

is a Morita context between (R, f) and (S, g).

Remark 6.4.

1. *It is good to mention that in such a setting (remark 6.3) not all Morita Contexts between two objects are identical. In fact, they differ at the level of the matrix factorizations X and Y. What we mean is that given four distinct objects $(R, f), (S, g), (R', h)$ and (S', k) of $\mathcal{LG}_K$, a Morita context between (R, f) and (S, g) could look like $\Gamma = (X, Y, 0, 0)$ and a Morita context between (R', h) and (S', k) could look like $\Gamma = (X', Y', 0, 0)$ with $X \neq X'$ and $Y \neq Y'$.*
2. *Intuitively, Morita contexts being pre-equivalences, it is not interesting to study cases where the two polynomials f and g are equal.*

We would now like to find interesting necessary and/or sufficient conditions on η and ρ to obtain a *Morita context* in $\mathcal{LG}_K$. We will use a matrix approach since it appears to be easier to use matrices than linear transformations. Thus we will have recourse to one of the properties of matrix transformations we studied earlier (cf. corollaries 2.1 and 2.2); namely that matrices that appear in a matrix factorization of a nonzero polynomial are invertible.

In the sequel, except otherwise stated, matrices that appear in a matrix factorization are of a fixed size $n \in \mathbb{N}$. So we will not bother to mention the sizes of pair of matrices constituting a matrix factorization.

Let (P, Q) and (R, T) be pair of matrices representing respectively the matrix factorizations $X \otimes_R Y$ and Δ_f. We assume f is not a constant polynomial, thus Δ_f is not the zero polynomial and so, by corollaries 2.1 and 2.2, we have that R (respectively T) is invertible

over the field of fractions of R (respectively over the field of fractions of T).
We know that
$\eta = (\eta^0, \eta^1) : X \otimes Y \to 1_{(R,f)} = \Delta_f$
is a morphism of matrix factorizations if the following diagram commutes:

$$\begin{array}{ccccc} (X\otimes Y)^0 & \xrightarrow{d^0_{X\otimes Y}} & (X\otimes Y)^1 & \xrightarrow{d^1_{X\otimes Y}} & (X\otimes Y)^0 \\ \downarrow{\scriptstyle \eta^0} & & \downarrow{\scriptstyle \eta^1} & & \downarrow{\scriptstyle \eta^0} \\ (\Delta_f)^0 & \xrightarrow{d^0_{\Delta_f}} & (\Delta_f)^1 & \xrightarrow{d^1_{\Delta_f}} & (\Delta_f)^0 \end{array}$$

That is:

$$\begin{cases} \eta^0 d^1_{X\otimes Y} = d^1_{\Delta_f}\eta^1 \\ \eta^1 d^0_{X\otimes Y} = d^0_{\Delta_f}\eta^0 \end{cases}$$

In matrix form:

$$\sharp\begin{cases} \eta^0 Q = T\eta^1 ...i) \\ \eta^1 P = R\eta^0 ...ii) \end{cases}$$

where for ease of notation we wrote η^i for the matrix of η^i, $i = 0, 1$.
Now, X (respectively Y) being a matrix factorization of $g(y)-f(x)$ (respectively $f(x)-g(y)$) we obtain from the Yoshino's tensor product of matrix factorizations (cf. definition 1.2 of [67]) that, $X \otimes Y$ is a factorization of $(g(y) - f(x)) + (f(x) - g(y)) = 0$. This means that $PQ = 0$.

from ii), since R is invertible, we have:

$$\begin{aligned} \eta^1 P = R\eta^0 &\Rightarrow R^{-1}\eta^1 P = \eta^0 \cdots \bigstar \\ &\Rightarrow R^{-1}\eta^1 PQ = \eta^0 Q \\ &\Rightarrow \eta^0 Q = 0 \cdots \bigstar\bigstar \end{aligned}$$

Putting this in i), since T is invertible, we obtain:

$$\begin{aligned} T\eta^1 = \eta^0 Q = 0 &\Rightarrow \eta^1 = T^{-1}0 = 0 \\ &\Rightarrow \eta^1 = 0 \end{aligned}$$

Hence, if we keep the above notation[4], then we have actually proved the following lemma:

Lemma 6.1. *A necessary condition on $\eta = (\eta^0, \eta^1)$ for $\Gamma = (X, Y, \eta, \rho)$ to be a Morita context in $\mathcal{LG}_K$ is $\eta^1 = 0$ and $\eta^0 Q = 0$.*

We now prove an auxiliary lemma that will help us arrive at the main result of this chapter. It states that the determinants of the matrices in the matrix factorizations $X \otimes X'$ and $X' \otimes X$ are all equal.

Lemma 6.2. *Let $X = (\phi, \psi)$ be an $n \times n$ matrix factorization of $f \in R$ and $X' = (\phi', \psi')$ be an $m \times m$ matrix factorizations of $g \in S$ where ϕ and ψ (respectively ϕ' and ψ') are matrices over $K[x]$ (respectively $K[y]$). These matrices can be considered as matrices over $L = K[x, y]$ and let*
$X\widehat{\otimes}X' = (P, Q) = (\begin{bmatrix} \phi\otimes 1_m & 1_n \otimes \phi' \\ -1_n\otimes\psi' & \psi\otimes 1_m \end{bmatrix}, \begin{bmatrix} \psi\otimes 1_m & -1_n \otimes \phi' \\ 1_n\otimes\psi' & \phi\otimes 1_m \end{bmatrix})$ *and*

[4] Recall that (P, Q) is a pair of matrices for the matrix factorization $X \otimes Y$ of $(g - f) + (f - g) = 0$.

$$X'\widehat{\otimes}X = (P', Q') = (\begin{bmatrix} \phi' \otimes 1_n & 1_m \otimes \phi \\ -1_m \otimes \psi & \psi' \otimes 1_n \end{bmatrix}, \begin{bmatrix} \psi' \otimes 1_n & -1_m \otimes \phi \\ 1_m \otimes \psi & \phi' \otimes 1_n \end{bmatrix})$$

where each component is an endomorphism on $L^n \otimes L^m$.
Then

$$\begin{cases} det(P) = det(P'), \\ det(Q) = det(Q') \end{cases}$$

Furthermore, all the four determinants are equal.

Proof. We will make use of fact 2.1(2), to compute the determinants of P, P', Q and Q' which are block matrices. Looking at P, in order to apply fact 2.1(2), we first have to observe that $(-1_n\otimes\psi')(\psi\otimes 1_m) = -1_n\psi\otimes\psi' 1_m = -\psi\otimes\psi' = \psi(-1_n)\otimes 1_m\psi' = (\psi\otimes 1_m)(-1_n\otimes\psi')$ and looking at P' we equally observe that $(-1_m \otimes \psi)(\psi' \otimes 1_n) = -1_m\psi' \otimes \psi 1_n = -\psi' \otimes \psi = \psi'(-1_m) \otimes 1_n\psi = (\psi' \otimes 1_n)(-1_m \otimes \psi)$, we can compute the determinants of P and P' as follows:

$$\begin{aligned} det(P) &= det[(\phi \otimes 1_m)(\psi \otimes 1_m) - (1_n \otimes \phi')(-1_n \otimes \psi')] \\ &= det[\phi\psi \otimes 1_m 1_m + 1_n 1_n \otimes \phi'\psi'] \\ &= det[\phi\psi \otimes 1_m + 1_n \otimes \phi'\psi'] \\ &= det[f 1_n \otimes 1_m + 1_n \otimes g 1_m] \; since \; \phi\psi = f 1_n \; and \; \phi'\psi' = g 1_m \\ &= det[f(1_n \otimes 1_m) + g(1_n \otimes 1_m)] \; by \; properties \; of \otimes \\ &= det[(f + g)(1_n \otimes 1_m)] \\ &= det[(f + g)1_{nm}] \; since \; 1_n \otimes 1_m = 1_{nm} \\ &= (f + g)^{nm} det(1_{nm}) \; by \; properties \; of \; determinants \\ &= (f + g)^{nm} \cdot 1 \\ &= (f + g)^{nm} \end{aligned}$$

$$\begin{aligned} det(P') &= det[(\phi' \otimes 1_n)(\psi' \otimes 1_n) - (1_m \otimes \phi)(-1_m \otimes \psi)] \\ &= det[\phi'\psi' \otimes 1_n 1_n + 1_m 1_m \otimes \phi\psi] \\ &= det[\phi'\psi' \otimes 1_n + 1_m \otimes \phi\psi] \\ &= det[g 1_m \otimes 1_n + 1_m \otimes f 1_n] \; since \; \phi\psi = f 1_n \; and \; \phi'\psi' = g 1_m \\ &= det[g(1_m \otimes 1_n) + f(1_m \otimes 1_n)] \\ &= det[(g + f)(1_m \otimes 1_n)] \\ &= det[(g + f)1_{mn}] \; since \; 1_m \otimes 1_n = 1_{mm} \\ &= (g + f)^{mn} det(1_{mn}) \; by \; properties \; of \; determinants \\ &= (g + f)^{mn} \cdot 1 \\ &= (f + g)^{mn} \; by \; commutativity \; in \; K[x, y] \\ &= det(P) \; as \; desired. \end{aligned}$$

In order to use fact 2.1(2) to compute $det(Q)$, from Q, we observe that $(1_n \otimes \psi')(\phi \otimes 1_m) = 1_n\phi\otimes\psi' 1_m = \phi\otimes\psi' = \phi 1_n\otimes 1_m\psi' = (\phi\otimes 1_m)(1_n\otimes\psi')$ and looking at Q' we equally observe that $(1_m \otimes \psi)(\phi' \otimes 1_n) = 1_m\phi' \otimes \psi 1_n = \phi' \otimes \psi = \phi' 1_m \otimes 1_n\psi = (\phi' \otimes 1_n)(1_m \otimes \psi)$ we can compute the determinants of Q and Q' as follows:

$$\begin{aligned}
det(Q) &= det[(\psi \otimes 1_m)(\phi \otimes 1_m) - (-1_n \otimes \phi')(1_n \otimes \psi')] \\
&= det[\psi\phi \otimes 1_m 1_m + 1_n 1_n \otimes \phi'\psi'] \\
&= det[\psi\phi \otimes 1_m + 1_n \otimes \phi'\psi'] \\
&= det[f 1_n \otimes 1_m + 1_n \otimes g 1_m] \; since\, \psi\phi = f1_n \, and\, \phi'\psi' = g1_m \\
&= det[f(1_n \otimes 1_m) + g(1_n \otimes 1_m)] \; by\, properties\, of \otimes \\
&= det[(f+g)(1_n \otimes 1_m)] \\
&= det[(f+g)1_{nm}] \; since\; 1_n \otimes 1_m = 1_{nm} \\
&= (f+g)^{nm} det(1_{nm}) \; by\; properties\; of\; determinants \\
&= (f+g)^{nm} \cdot 1 \\
&= (f+g)^{nm}
\end{aligned}$$

$$\begin{aligned}
det(Q') &= det[(\psi' \otimes 1_n)(\phi' \otimes 1_n) - (-1_m \otimes \phi)(1_m \otimes \psi)] \\
&= det[\psi'\phi' \otimes 1_n 1_n + 1_m 1_m \otimes \phi\psi] \\
&= det[\psi'\phi' \otimes 1_n + 1_m \otimes \phi\psi] \\
&= det[g 1_m \otimes 1_n + 1_m \otimes f 1_n] \; since\, \psi\phi = f1_n \, and\, \phi'\psi' = g1_m \\
&= det[g(1_m \otimes 1_n) + f(1_m \otimes 1_n)] \\
&= det[(g+f)(1_m \otimes 1_n)] \\
&= det[(g+f)1_{mn}] \; since\; 1_m \otimes 1_n = 1_{mn} \\
&= (g+f)^{mn} det(1_{mn}) \; by\; properties\; of\; determinants \\
&= (g+f)^{mn} \cdot 1 \\
&= (f+g)^{mn} \; by\; commutativity\; in\; K[x,y] \\
&= det(Q) \; as\; desired.
\end{aligned}$$

As we can see from the answers, all the four determinants are equal. □

Remark 6.5. *Observe that in the case of a Morita Context in $\mathcal{LG}_K$, the f and the g we have in the above auxiliary lemma are actually additive inverses of each other by definition of 1-morphisms in $\mathcal{LG}_K$. In fact, if X is a morphism from the polynomial f_1 to the polynomial f_2, then X is a matrix factorization of $f_2 - f_1 = f$. And if Y is a morphism from the polynomial f_2 to the polynomial f_1, then Y is a matrix factorization of $f_1 - f_2 = g$. We see that $f = -g$.*

Consequently, we have the following corollary.

Corollary 6.1. *Let (R, f) and (S, g) be two objects in $\mathcal{LG}_K$ and let $X : (R, f) \to (S, g)$ and $X' : (S, g) \to (R, f)$ be 1-morphisms in $\mathcal{LG}_K$. If $X \otimes X' = (P, Q)$ and $X' \otimes X = (P', Q')$, then $det(P) = det(Q) = det(P') = det(Q') = 0$*

Proof. The proof follows immediately from the remark and the foregoing auxiliary lemma. □

It follows from this corollary that Q in lemma 6.1 is not invertible and so, the equality $\eta^0 Q = 0$ will never have a unique solution for η^0. There would be several solutions from this equality, among which the one(s) that will be sufficient to obtain a *Morita context*. Keeping the notations of the foregoing auxiliary lemma and corollary, we give an example to show that the determinants in X and X' are all equal.

Example 6.3. *We use the X and X' given in example 6.2. We compute $X \otimes X'$. We already computed it in example 2.6 but our goal there was not the same with our aim here, so we recopy it here. Moreover we'll also compute $X' \otimes X$.*

Let $X = (\begin{bmatrix} x & -y \\ y & x \end{bmatrix}, \begin{bmatrix} x & y \\ -y & x \end{bmatrix})$, *and* $X' = (\begin{bmatrix} -x & y \\ -y & -x \end{bmatrix}, \begin{bmatrix} x & y \\ -y & x \end{bmatrix})$
be matrix factorizations of $g - f = x^2 + y^2$ *and* $f - g = -x^2 - y^2$ *respectively. Let*

$$A = \begin{bmatrix} x & -y \\ y & x \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} -x & y \\ -y & -x \end{bmatrix}, C = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \otimes \begin{bmatrix} x & y \\ -y & x \end{bmatrix},$$
$$D = \begin{bmatrix} x & y \\ -y & x \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, A' = \begin{bmatrix} x & y \\ -y & x \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, B' = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \otimes \begin{bmatrix} -x & y \\ -y & -x \end{bmatrix},$$
$$C' = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} x & y \\ -y & x \end{bmatrix}, \text{ and } D' = \begin{bmatrix} x & -y \\ y & x \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Then

$$X \otimes X' = (\begin{bmatrix} A & B \\ C & D \end{bmatrix}, \begin{bmatrix} A' & B' \\ C' & D' \end{bmatrix})$$

$$= (\begin{pmatrix} x & 0 & -y & 0 & -x & y & 0 & 0 \\ 0 & x & 0 & -y & -y & -x & 0 & 0 \\ y & 0 & x & 0 & 0 & 0 & -x & y \\ 0 & y & 0 & x & 0 & 0 & -y & -x \\ -x & -y & 0 & 0 & x & 0 & y & 0 \\ y & -x & 0 & 0 & 0 & x & 0 & y \\ 0 & 0 & -x & -y & -y & 0 & x & 0 \\ 0 & 0 & y & -x & 0 & -y & 0 & x \end{pmatrix}, \begin{pmatrix} x & 0 & y & 0 & x & -y & 0 & 0 \\ 0 & x & 0 & y & y & x & 0 & 0 \\ -y & 0 & x & 0 & 0 & 0 & x & -y \\ 0 & -y & 0 & x & 0 & 0 & y & x \\ x & y & 0 & 0 & x & 0 & -y & 0 \\ -y & x & 0 & 0 & 0 & x & 0 & -y \\ 0 & 0 & x & y & y & 0 & x & 0 \\ 0 & 0 & -y & x & 0 & y & 0 & x \end{pmatrix})$$

=(P,Q)

and let

$$A_1 = \begin{bmatrix} -x & y \\ -y & -x \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, B_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} x & -y \\ y & x \end{bmatrix}, C = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \otimes \begin{bmatrix} x & y \\ -y & x \end{bmatrix},$$
$$D = \begin{bmatrix} x & y \\ -y & x \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, A' = \begin{bmatrix} x & y \\ -y & x \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, B'_1 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \otimes \begin{bmatrix} x & -y \\ y & x \end{bmatrix},$$
$$C' = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} x & y \\ -y & x \end{bmatrix}, \text{ and } D'_1 = \begin{bmatrix} -x & y \\ -y & -x \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Then

$$X' \otimes X = (\begin{bmatrix} A_1 & B_1 \\ C & D \end{bmatrix}, \begin{bmatrix} A' & B'_1 \\ C' & D'_1 \end{bmatrix})$$

$$= (\begin{pmatrix} -x & 0 & y & 0 & x & -y & 0 & 0 \\ 0 & -x & 0 & y & y & x & 0 & 0 \\ -y & 0 & -x & 0 & 0 & 0 & x & -y \\ 0 & -y & 0 & -x & 0 & 0 & y & x \\ -x & -y & 0 & 0 & x & 0 & y & 0 \\ y & -x & 0 & 0 & 0 & x & 0 & y \\ 0 & 0 & -x & -y & -y & 0 & x & 0 \\ 0 & 0 & y & -x & 0 & -y & 0 & x \end{pmatrix}, \begin{pmatrix} x & 0 & y & 0 & -x & y & 0 & 0 \\ 0 & x & 0 & y & -y & -x & 0 & 0 \\ -y & 0 & x & 0 & 0 & 0 & -x & y \\ 0 & -y & 0 & x & 0 & 0 & -y & -x \\ x & y & 0 & 0 & -x & 0 & y & 0 \\ -y & x & 0 & 0 & 0 & -x & 0 & y \\ 0 & 0 & x & y & -y & 0 & -x & 0 \\ 0 & 0 & -y & x & 0 & -y & 0 & -x \end{pmatrix})$$

=(P',Q')

Using properties of block matrices (cf. fact 2.1), one can show that the determinants of P, Q, P' and Q' are all equal to 0. We do it for Q, the rest is done similarly. Write Q as a block matrix:

$$Q = \begin{bmatrix} E & F \\ G & H \end{bmatrix}$$

where

$$E = \begin{pmatrix} x & 0 & y & 0 \\ 0 & x & 0 & y \\ -y & 0 & x & 0 \\ 0 & -y & 0 & x \end{pmatrix}, G = \begin{pmatrix} x & y & 0 & 0 \\ -y & x & 0 & 0 \\ 0 & 0 & x & y \\ 0 & 0 & -y & x \end{pmatrix}, F = \begin{pmatrix} x & -y & 0 & 0 \\ y & x & 0 & 0 \\ 0 & 0 & x & -y \\ 0 & 0 & y & x \end{pmatrix}, H = \begin{pmatrix} x & 0 & -y & 0 \\ 0 & x & 0 & -y \\ y & 0 & x & 0 \\ 0 & y & 0 & x \end{pmatrix}$$

Observe that $GH = HG$ and so by fact 2.1, $det(Q) = det(EH - FG)$ Hence,

$$det(Q) = det(\begin{pmatrix} x & 0 & y & 0 \\ 0 & x & 0 & y \\ -y & 0 & x & 0 \\ 0 & -y & 0 & x \end{pmatrix}\begin{pmatrix} x & 0 & -y & 0 \\ 0 & x & 0 & -y \\ y & 0 & x & 0 \\ 0 & y & 0 & x \end{pmatrix} - \begin{pmatrix} x & -y & 0 & 0 \\ y & x & 0 & 0 \\ 0 & 0 & x & -y \\ 0 & 0 & y & x \end{pmatrix}\begin{pmatrix} x & y & 0 & 0 \\ -y & x & 0 & 0 \\ 0 & 0 & x & y \\ 0 & 0 & -y & x \end{pmatrix})$$

$$= det(\begin{pmatrix} x^2+y^2 & 0 & 0 & 0 \\ 0 & x^2+y^2 & 0 & 0 \\ 0 & 0 & x^2+y^2 & 0 \\ 0 & 0 & 0 & x^2+y^2 \end{pmatrix} - \begin{pmatrix} x^2+y^2 & 0 & 0 & 0 \\ 0 & x^2+y^2 & 0 & 0 \\ 0 & 0 & x^2+y^2 & 0 \\ 0 & 0 & 0 & x^2+y^2 \end{pmatrix})$$

$$= det\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$= 0$$

So $det(Q) = 0$ as claimed.

Remark 6.6. *The result we obtained above about Q not being invertible helps to see that the necessary condition given in lemma 6.1 is not sufficient. This is so because in the proof of that lemma, we cannot reverse the direction of the implication symbol from ★★ to ★. In fact, suppose $\eta^0 Q = 0$ then since $PQ = 0$, we have $\eta^0 Q = R^{-1}\eta^1 PQ$ implying $R\eta^0 Q = \eta^1 PQ$ which implies $(R\eta^0 - \eta^1 P)Q = 0$. Now, Q being noninvertible, we cannot obtain from here that $(R\eta^0 - \eta^1 P) = 0$ which is ★. So, the necessary condition is not a sufficient one.*

So far, apart from the trivial sufficient condition given in remark 6.3, we have also given a necessary condition on η. We would now like to give a necessary condition on ρ.

Let (P', Q') and (R', T') be pair of matrices representing respectively the matrix factorizations $Y \otimes_S X$ and Δ_g. We assume g is not a constant polynomial, thus Δ_g is not the zero polynomial and so, by corollaries 2.1 and 2.2, we have that R' and T' are invertible. We know that
$\rho = (\rho^0, \rho^1) : Y \otimes X \to 1_{(S,g)} = \Delta_g$
is a morphism of matrix factorizations if the following diagram commutes:

$$\begin{array}{ccccc}
(Y \otimes X)^0 & \xrightarrow{d^0_{Y\otimes X}} & (Y \otimes X)^1 & \xrightarrow{d^1_{Y\otimes X}} & (Y \otimes X)^0 \\
\downarrow{\scriptstyle \rho^0} & & \downarrow{\scriptstyle \rho^1} & & \downarrow{\scriptstyle \rho^0} \\
(\Delta_g)^0 & \xrightarrow{d^0_{\Delta_g}} & (\Delta_g)^1 & \xrightarrow{d^1_{\Delta_g}} & (\Delta_g)^0
\end{array}$$

That is:

$$\begin{cases} \rho^0 d^1_{Y\otimes X} = d^1_{\Delta_g} \rho^1 \\ \rho^1 d^0_{Y\otimes X} = d^0_{\Delta_g} \rho^0 \end{cases}$$

In matrix form:

$$\dagger \begin{cases} \rho^0 Q' = T' \rho^1 ...i)' \\ \rho^1 P' = R' \rho^0 ...ii)' \end{cases}$$

where for ease of notation we wrote ρ^i for the matrix of ρ^i, $i = 0, 1$.
Now, X (respectively Y) being a matrix factorization of $g(y) - f(x)$ (respectively $f(x) - g(y)$) we obtain from the Yoshino's tensor product of matrix factorization (cf. definition 1.2 of [67]) that $Y \otimes X$ is a factorization of $(f(x) - g(y)) + (g(y) - f(x)) = 0$. This means that $P'Q' = 0$.

from $ii)'$, since R' is invertible, we have:

$$\begin{aligned} \rho^1 P' = R' \rho^0 &\Rightarrow R'^{-1} \rho^1 P' = \rho^0 \\ &\Rightarrow R'^{-1} \rho^1 P' Q' = \rho^0 Q' \\ &\Rightarrow \rho^0 Q' = 0 \end{aligned}$$

Putting this in $i)'$, since T' is invertible, we get:

$$\begin{aligned} T' \rho^1 = \rho^0 Q' = 0 &\Rightarrow \rho^1 = T'^{-1} 0 = 0 \\ &\Rightarrow \rho^1 = 0 \end{aligned}$$

Hence, if we keep the above notation[5] then we have actually proved the following lemma:

Lemma 6.3. *A necessary condition on $\rho = (\rho^0, \rho^1)$ for $\Gamma = (X, Y, \eta, \rho)$ to be a Morita context in $\mathcal{LG}_K$ is $\rho^1 = 0$ and $\rho^0 Q' = 0$.*

It follows from corollary 6.1 that Q' is not invertible and so in lemma 6.3, the equality $\rho^0 Q = 0$ will never have a unique solution for ρ^0. There would be several solutions from this equality, among which the one(s) that will be sufficient to obtain a *Morita context*.

[5] Recall that (P', Q') is a pair of matrices for the matrix factorization $Y \otimes X$ of $(f - g) + (g - f) = 0$.

Remark 6.7. *Just like we observed in the case of η in remark 6.6 that the necessary condition of lemma 6.1 was not sufficient, we similarly have that the necessary condition for ρ of lemma 6.3 is not sufficient.*

We now state and prove the following theorem which is one of the main results of this chapter. It actually gives some conditions on η and ρ for $\Gamma = (X, Y, \eta, \rho)$ to be a *Morita context*. These conditions were obtained thanks to properties of matrix factorizations that is why the following result is an application of matrix factorizations.

Theorem 6.2. *Let*

- (R, f) *and* (S, g) *be two objects of* $\mathcal{LG}_K$.
- $X \in \mathcal{LG}_K((R, f), (S, g)) = hmf(R \otimes_K S, g - f)$ *i.e.,* $X : (R, f) \to (S, g)$ *is a finite rank matrix factorization of* $g - f$.
- $Y \in \mathcal{LG}_K((S, g), (R, f)) = hmf(S \otimes_K R, f - g)$ *i.e.,* $Y : (S, g) \to (R, f)$ *is a finite rank matrix factorization of* $f - g$.
  such that $X \otimes Y$ *and* $Y \otimes X$ *are finite rank matrix factorizations.*
- $\Delta_f \in \mathcal{LG}_K((R, f), (R, f)) = hmf(R \otimes_K R, f \otimes id - id \otimes f)$ *i.e.,* $\Delta_f : (R, f) \to (R, f)$ *is a finite rank matrix factorization of* $f \otimes id - id \otimes f$.
- $\Delta_g \in \mathcal{LG}_K((S, g), (S, g)) = hmf(R \otimes_K S, g \otimes id - id \otimes g)$ *i.e.,* $\Delta_g : (S, g) \to (S, g)$ *is a finite rank matrix factorization of* $g \otimes id - id \otimes g$.
- $\eta : X \otimes_R Y \to 1_{(R,f)} = \Delta_f$ *and let* (P, Q) *and* (R, T) *be pairs of matrices representing respectively the finite rank matrix factorizations* $X \otimes Y$ *and* Δ_f.
- $\rho : Y \otimes_S X \to 1_{(S,g)} = \Delta_g$ *and let* (P', Q') *and* (R', T') *be pairs of matrices representing respectively the finite rank matrix factorizations* $Y \otimes X$ *and* Δ_g.

Then:

1. *A trivial sufficient condition for* $\Gamma = (X, Y, \eta, \rho)$ *to be a Morita Context is* $\eta = 0 = \rho$ *i.e.,* $(\eta^0, \eta^1) = (0, 0) = (\rho^0, \rho^1)$.
2. *A necessary condition for* $\Gamma = (X, Y, \eta, \rho)$ *to be a Morita Context is*

$$\begin{cases} \eta^1 = 0, \text{ and } \eta^0 Q = 0, \\ \rho^1 = 0, \text{ and } \rho^0 Q' = 0. \end{cases}$$

Proof. The first part of this theorem is simply remark 6.3. The second part follows from lemma 6.1 and lemma 6.3. □

Keeping the notations of the foregoing theorem, we have the following:

Corollary 6.2. *A necessary and sufficient condition on* η^1 *and* ρ^1 *for* $\Gamma = (X, Y, \eta, \rho)$ *to be a Morita Context is* $\eta^1 = 0 = \rho^1$.

Proof. This is a direct consequence of theorem 6.2. □

Conclusion and further problems

In this dissertation, we elucidated the intricate construction of the unit object of the bicategory $\mathcal{LG}_K$ and we also studied matrix factorizations of an arbitrary element in a given unital ring. Our main focus was the ring of power series. We defined a new bifunctorial operation on matrix factorizations. We gave three applications of this operation: first, it was used to give an example of a semi-unital semi-monoidal category. Next, it was used to give an example of a right pseudo-monoidal category which is a concept we defined (cf. definition 3.8) in this dissertation. Finally, it was used to improve the existing standard algorithm for factoring polynomials using matrices. We were also able to give an application of matrix factorizations by using them to derive a necessary condition to obtain a Morita context in the bicategory $\mathcal{LG}_K$. This dissertation contains several original contributions. We enumerate most of them below:

1. In chapter 2, we observed that an $n \times n$ matrix factorization of an element f in a unital ring is not unique, for any natural number $n > 1$. Inspired by the definition of $\widehat{\otimes}$, we defined another product $\widehat{\otimes}'$ different from the Yoshino's tensor product $\widehat{\otimes}$ such that if X and Y are respectively matrix factorizations of power series f and g, then $X\widehat{\otimes}'X'$ just like $X\widehat{\otimes}X'$ is a matrix factorization of $f + g$.

2. In particular in this dissertation, we made a contribution to the study of matrix factorizations by developing interesting new material from chapter 3 onwards. In fact, we proposed a new operation on matrix factorizations denoted $\widetilde{\otimes}$ which is such that if X is a matrix factorization of the power series $f \in K[[x_1, ..., x_r]]$ and Y is a matrix factorization of the power series $g \in K[[y_1, ..., y_s]]$, then $X\widetilde{\otimes}Y$ is a matrix factorization of fg over $K[[x_1, ..., x_r, y_1, ..., y_s]]$. We observed that our result also holds for any unital ring and not just the ring of power series. We called $\widetilde{\otimes}$, *the multiplicative tensor product* of X and Y.

3. After showing that $\widetilde{\otimes}$ is a bifunctor, we stated and proved many of its properties among which are associativity, commutativity and distributivity.

4. Moreover, we observed that $\widetilde{\otimes}$ is a binary operation on the category of matrix factorizations of the constant power series 1 (denoted $(MF(1), \widetilde{\otimes})$ and so we investigated if $(MF(1), \widetilde{\otimes})$ was a monoidal category. It turned out that it was not monoidal but was right pseudo-monoidal, a concept we defined in this dissertation (cf. definition 3.8).
The concept of semi-unital semi-monoidal categoy was recently defined in [1] and an example was provided in that paper. But this example required a considerable

amount of set-up. We provided another example of this concept with an easy-to-understand small amount of set-up by extracting a one-step connected (cf. Chapter 3) subcategory of $(MF(1), \widetilde{\otimes})$ which is a semi-unital semi-monoidal category.

5. In chapter 4, we defined a large class of polynomials and improved the standard method for factoring polynomials on this class. The standard method for factorizing polynomials builds matrix factorizations of sums of polynomials from "factorizations" of their summands. One conspicuous downside of this method is that the factorizations double in size for each new summand. We defined a summand-reducible polynomial to be one that can be written in the form $f = t_1 + \cdots + t_s + g_{11} \cdots g_{1m_1} + \cdots + g_{l1} \cdots g_{lm_l}$ under some specified conditions where each t_k is a monomial and each g_{ji} is a sum of monomials. $\widehat{\otimes}$ and $\widetilde{\otimes}$ were then used to improve the standard method on this class and to prove that if p_{ji} is the number of monomials in g_{ji}, then there is an improved version of the standard method for factoring f which produces factorizations of size $2^{\prod_{i=1}^{m_1} p_{1i} + \cdots + \prod_{i=1}^{m_l} p_{li} - (\sum_{i=1}^{m_1} p_{1i} + \cdots + \sum_{i=1}^{m_l} p_{li})}$ times smaller than the size one would normally obtain with the standard method.

6. In chapter 5, we developed a sophisticated machinery to explain the intricate construction (cf. [17]) of the unit object of $\mathcal{LG}_K$ providing a reasonable amount of proofs and details that are omitted in its original presentation (cf. [17]). Next, a proof of the naturality of the right and left unit maps with respect to $2-$morphisms was presented. We also proved that there is no direct inverse for the right and left unit maps (also called unitors, cf. [17]) of the $\mathcal{LG}_K$ bicategory, thereby justifying the fact that the inverses of these unitors in [17] are found only up to homotopy.

7. In chapter 6, properties of matrix factorizations were used to understand what a *Morita context* is in the bicategory $\mathcal{LG}_K$. In particular, we stated and proved a necessary condition to obtain a *Morita context* in $\mathcal{LG}_K$ on a particular class of objects. This provided us with an application of matrix factorizations. We also stated a trivial sufficient condition to obtain a Morita context in $\mathcal{LG}_K$.

FURTHER PROBLEMS:

This research generated a number of other questions that we think are interesting.

1. Yoshino [67] defined the tensor product $(\widehat{\otimes})$ of matrix factorizations and proved that if X and Y are respectively the matrix factorizations of power series f and g, such that X and Y are indecomposable, it might still happen that $X\widehat{\otimes}Y$ be decomposable. He proved theorems (section 3 of [67]) which give the bounds for the number of indecomposable components in the direct decomposition of $X\widehat{\otimes}Y$. It would be interesting to know if similar results could be derived for the multiplicative tensor product proposed in this dissertation.

2. In chapter 4, we gave a necessary condition on η and ρ to obtain a *Morita context* in $\mathcal{LG}_K$. But the only sufficient condition we were able to give on a 4-tuple (X, Y, η, ρ) to be a *Morita context* was the trivial one, i.e., $\eta = \rho = 0$. An interesting question would be to find a nontrivial sufficient condition on η and ρ.

3. In this dissertation, we learned about factorization of polynomials using two $n \times n$ matrices ($n \geq 2$), it could be interesting to study how polynomials could be factorized using m matrices of size n where ($n \geq 2$) and $m \geq 3$. And one could also be interested in finding conditions under which a given polynomial has at most r matrix factors, where $r \geq 2$.

4. So far, we have tensor products $\widehat{\otimes}$ and $\widetilde{\otimes}$ which have the ability to produce matrix factorizations of the sum $f + g$ and the product fg respectively; from the matrix factorizations of the polynomials f and g. An interesting question would be to find a tensor product which can produce a matrix factorization of the composition $f \circ g$ whenever this composition makes sense.

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