Hecke type algebras, triple Schubert calculus and their applications to equivariant positivity

Rui Xiong

Thesis submitted to the University of Ottawa in partial fulfillment of the requirements for the degree of

Doctorate in Philosophy Mathematics and Statistics [1]

Department of Mathematics and Statistics

Faculty of Science

University of Ottawa

[1]The PhD program is a joint program with Carleton University, administered by the Ottawa-Carleton Institute of Mathematics and Statistics

Abstract

This thesis introduces a triple variable framework for Schubert calculus that generalizes traditional single and double versions. The primary focus of this work is the computation of structure constants for the equivariant cohomology of flag varieties. First, I provide a new proof of Triple Pieri rules, which offer explicit formulas for the product of certain geometric classes. These rules are derived using skew divided difference operators and identities within Hecke-type algebras. Second, I develop pipe puzzles, a visual and combinatorial model for computing these structure constants. This model is specifically designed for permutations with separated descents. It generalizes earlier structures, including bumpless pipe dreams and the puzzle model of Knutson and Zinn-Justin. In the proof of this model, I utilize lattice models and the Yang–Baxter equation. I also provide the code used to compute R-matrices for specific representations. A key finding of this research is that triple Schubert structure constants satisfy two types of recurrence relations. This suggests that although the triple framework is broader than its predecessors, its underlying proofs can be simplified. Third, I present the major geometric contribution of this work: a new interpretation of triple variables as equivariant parameters within higher-dimensional flag varieties. This perspective allows for the proof of equivariant positivity conjectures originally proposed by Samuel and Kirillov. The proof relies on a refined version of Graham's positivity theorem regarding effective cycles.

Finally, there are several potential applications of these results. The Hecke-type algebra identities may extend to generalized cohomology theories. The simplified proof for puzzle rules can likely be applied to other combinatorial models. Additionally, the R-matrix code can be adapted for further representations. The geometric interpretation offers a path to explaining positivity in K-theory and beyond.

Contents

List of Figures

List of Tables

CHAPTER 1

Introduction

This thesis is primarily devoted to introducing the author's research work on **triple Schubert calculus**. Combinatorially, we study the expansion of the product of two double Grothendieck polynomials in different secondary variables

$$\mathfrak{G}_u(x,y)\cdot\mathfrak{G}_v(x,t)=\sum_w c^w_{u,v}(t,y)\cdot\mathfrak{G}_w(x,t), \tag{1.0.1}$$

Setting $\beta=0$, one obtains the Schubert polynomial version

$$\mathfrak{S}_u(x,y)\cdot\mathfrak{S}_v(x,t)=\sum_w \overline{c}^w_{u,v}(t,y)\cdot\mathfrak{S}_w(x,t), \tag{1.0.2}$$

In this thesis, we will introduce results in three directions.

Triple Pieri rules: In [**44**, Section 7], a triple Pieri rule was stated for Schubert polynomials. In this thesis, we will give a simpler proof via the methods developed by the author and his collaborators [**16**].

Pipe puzzles: In [**17**], the author and his collaborators gave a combinatorial model when u,v are permutations with separated descents. In this thesis, we will give a more detailed treatment with its representation-theoretic background.

Graham positivity: In [**18**], the author and his collaborators proved the Graham positivity of triple Schubert calculus in the cohomology case. In this thesis, we will explain the geometric meanings of triple Schubert calculus.

It is worth mentioning that the proof of the triple Pieri rules relies in part on Hecke-type (nil-Hecke algebras and 0-Hecke algebras) — the formulas for which were generalized by the author and collaborators in [**37**].

1.1. Triple Schubert Calculus

Classical Schubert calculus studies the structure constants of the Schubert basis of the (equivariant) cohomology ring of flag varieties. Combinatorially, this can be reformulated in terms of **Schubert polynomials**:

$\mathfrak{S}_u(x)\mathfrak{S}_v(x)$ $=\sum_w c^w_{uv}\mathfrak{S}_w(x)$	$\mathfrak{S}_u(x,t)\mathfrak{S}_v(x,t)$ $=\sum_w c^w_{uv}(t)\mathfrak{S}_w(x,t)$	$\mathfrak{S}_u(x,y)\mathfrak{S}_v(x,t)$ $=\sum_w c^w_{uv}(t,y)\mathfrak{S}_w(x,t)$
(single) Schubert calculus	double Schubert calculus	triple Schubert calculus

The geometric meaning of triple Schubert calculus is not immediately apparent from its definition. One interpretation involves treating y as a variable that will be substituted by $y=f(t)$ for some $f\in S_n$. Then the coefficients will be the equivariant Schubert expansion for non-transversal intersections. Following the idea that y could be viewed as the parameter of the transversality of the intersection, we can generalize the idea to an intersection problem over the product of two flag varieties $G/B \times G/B$; see Knutson and Tao [**25**]. We will explain another geometric meaning and prove positivity using it in Chapter 6.

Combinatorially, the relation between single/double/triple Schubert calculus can be summarized as follows

$$\boxed{\text{single}} \quad \xLeftarrow{t=0} \quad \boxed{\text{double}} \quad \xLeftarrow{y=t} \quad \boxed{\text{triple}}$$

From the geometric perspective, the requirement that a flag variety admits a torus action is a strong condition. In some cases, double Schubert calculus could be more approachable than single Schubert calculus in some problems. The extra variable t will provide extra information and is usually helpful in the proofs. However, note that the double Schubert structure constants are "single", i.e. only one family of variables $t = (t_1, t_2, \ldots)$ is involved. Triple Schubert calculus could be viewed as a model that doubles the Schubert constants. Following a similar philosophy, under certain situations, while the problem of computing triple Schubert structure constants is broader, its proof could be simpler. This is what we will see in Chapter 5.

The case that both u and v are k-Grassmannian permutations was solved by Wheeler and Zinn-Justin [**49**, Theorems 2" and 3"]. After setting $\beta = 0$, $\mathfrak{S}_u$ and $\mathfrak{S}_v$ are both double Schur polynomials, and the question was considered by Molev–Sagan [**39**]; see also Knutson and Tao [**25**, Section 6] and Zinn-Justin [**50**].

There is another branch of such questions, concerning the coefficients of a product of a single Schubert polynomial and a double Schubert polynomial.

$$\boxed{\begin{aligned} &\mathfrak{S}_u(x)\mathfrak{S}_v(x,t) \\ &= \textstyle\sum_w c_{uv}^w(t,0)\mathfrak{S}_w(x,t) \end{aligned}}$$

single times double (**SxD**)

This question also specializes to single Schubert calculus

$$\boxed{\text{single}} \quad \xLeftarrow{t=0} \quad \boxed{\textbf{SxD}} \quad \xLeftarrow{y=0} \quad \boxed{\text{triple}}$$

Moreover, this **SxD** turns out to be equivalent to a question of skew operators applied on single Schubert polynomials, considered by Kirillov [**22**].

Now let us switch to the detailed introduction of the three aforementioned results.

1.2. Triple Pieri rules

In this section, we restrict ourselves to Schubert polynomials (1.0.2). A triple Pieri rule is a formula of $\overline{c}_{u,v}^w(t,y)$ when

$$v, w \in S_n \text{ are arbitrary}, \qquad u = s_{k-r+1} \cdots s_{k-1} s_k$$

for some $0 \le r \le k$. In this case, the single Schubert polynomial $\mathfrak{S}_u(x)$ is the elementary symmetric polynomial in $x_1, \ldots, x_k$. Here is a brief history of Pieri rules in Schubert calculus.

(i) In the case where $y = t = 0$, we are restricted to single Schubert polynomials. When v is a k-Grassmannian permutationi, i.e.

$$v(1) < \cdots < v(k), \qquad v(k+1) < \cdots < v(n),$$

the single Schubert polynomial $\mathfrak{S}_v(x)$ is a Schur polynomial in $x_1, \ldots, x_k$. As a result, if we specialize $y = t = 0$, the formula reduces to the classical Pieri rule for Schur polynomials [**36**]. When $y = t = 0$ and $r = 1$, i.e. $u = s_k$, the expansion is classically obtained by Monk [**40**] and Chevalley [**13**]. For general r in the case $y = t = 0$, the single Pieri rule was obtained by Sottile [**45**] (see also Lascoux and Schützenberger [**30**]).

(ii) When $y = t$, i.e. we are restricted to double Schubert polynomials in the same secondary variables. The double Pieri rule was obtained by Robinson [**43**]. A geometric proof is established by Li, Ravikumar, Sottile and Yang [**33**].

(iii) When $y = 0$, as explained above, the formula is equivalent to a formula of skew divided difference operators applied on Schubert polynomials. The case $r = 1$ was established by Kirillov [**22**], and the case for general r is due to Liu [**34**].

(iv) The triple Pieri rule was obtained by Samuel [**44**, Section 7]. In this thesis, we will present a simple proof via the approach in [**16**].

The key combinatorial structure in establishing Pieri rule is the k-Bruhat order [**31, 4, 5, 32, 45**], i.e. the partial order generated by the relation

$$u \lessdot ut_{ab} \text{ if } a \le k < b \text{ with } u < w \text{ in Bruhat order.}$$

It was generalized to any parabolic subgroups by Knutson, Lam and Speyer [**24**]. Its extended versions were considered by the author and collaborators in [**16**] and [**15**] respectively.

Now let us state the triple Pieri rule.

Theorem A (Triple Pieri rule). *Let $1 \le r \le k \le n$ and $u = s_{k-r+1} \cdots s_{k-1} s_k$. For any $v \in S_n$, we have*

$$\mathfrak{S}_u(x, y) \cdot \mathfrak{S}_v(x, t) = \sum_w \overline{c}^w_{u,v}(t, y) \cdot \mathfrak{S}_w(x, t)$$

where the sum ranges over all $w \in S_n$ such that $w = vt_{a_1b_1} \cdots t_{a_{r'}b_{r'}}$ with (0) $r' \le r$; (1) $a_i \le k < b_i$; (2) a_i's are distinct; (3) $\ell(vt_{a_1b_1} \cdots t_{a_ib_i}) = \ell(v) + i$ and the coefficients

$$\overline{c}^w_{uv}(t, y) = \mathfrak{S}_{s_{k-r+1} \cdots s_{k-r'}}(t, y)|_{t_i \mapsto t_{d(i)}}$$

for any permutation $d \in S_n$ such that the image set of $[k - r']$ is exactly $\Delta_k(v, w) := \{u(i) : 1 \le i \le k, u(i) \ne w(i)\}$..

Note that the formula presented here is different from the formula in [**44**]. See also Theorem 3.3.1 for another formulation.

The idea of the proof can be summarized by reversing the arrows in the previous section by

$$\boxed{\text{single}} \xrightarrow{\text{Rigidity}} \boxed{\textbf{SxD}} \xrightarrow{\text{Expansion}} \boxed{\text{triple}}.$$

The first step is to prove the Rigidity Theorem (Theorem 3.2.1), which means that the **SxD** structure constants can be controlled by single coefficients. The second step is to utilize the known expansion of double Schubert polynomials to single Schubert polynomials.

1.3. Pipe Puzzles

In this section, we will give a formula for $c_{u,w}^{w}(y,t)$ in (1.0.1) for two permutations $u, v \in S_n$ with separated descents at position k, that is,

$$\text{maxdes}(u) \leq k \leq \text{mindes}(v), \tag{1.3.1}$$

where $\text{maxdes}(u) = \max\{i \colon u(i) > u(i+1)\}$ and $\text{mindes}(v) = \min\{i \colon v(i) > v(i+1)\}$. Here, for the identity permutation $\text{id} = 12\cdots n$, we use the convention that $\text{maxdes}(\text{id}) = 0$ and $\text{mindes}(\text{id}) = +\infty$.

Our formula for $c_{u,v}^{w}(t,y)$, see Theorem C, is described in terms of "**pipe puzzles**". The formula includes the following specializations and applications.

(i) The case $y = t$. Theorem C recovers the puzzle rule for permutations with separated descents by Knutson and Zinn-Justin [**28**, Theorem 1], which is manifestly positive in the sense of Anderson, Griffeth and Miller [**2**] (an equivariant K-theory extension of Graham's positivity theorem [**19**]).

(ii) The case $\beta = 0$. This corresponds to the limit from K-theory to cohomology. Theorem C becomes a combinatorial rule for the expansion of the product $\mathfrak{S}_u(x,y) \cdot \mathfrak{S}_v(x,t)$ of two Schubert polynomials in different secondary variables, see Theorem B. We point out that in the case $y = t = 0$, Huang [**20**] derived a tableau formula for the product $\mathfrak{S}_u(x) \cdot \mathfrak{S}_v(x)$ of two single Schubert polynomials for u, v with separated descents.

(iii) The case that both u and v are k-Grassmannian permutations. Theorem C extends the puzzle formula for the product $\mathfrak{G}_\lambda(x,t) \cdot \mathfrak{G}_\mu(x,t)$ by Wheeler and Zinn-Justin [**49**, Theorem 2] (The latter formula on the one hand is an equivariant extension of Vakil's puzzle formula [**47**] for the product $\mathfrak{G}_\lambda(x) \cdot \mathfrak{G}_\mu(x)$ of two single Grothendieck polynomials, and on the other hand is a K-theory extension of the Knutson–Tao puzzle formula [**25**] for the product $s_\lambda(x,t) \cdot s_\mu(x,t)$ of two double Schur polynomials).

We remark that (1) an alternative puzzle formula (different from the one in [**49**]) for $\mathfrak{G}_\lambda(x,t) \cdot \mathfrak{G}_\mu(x,t)$ was conjectured by Knutson and Vakil, and proved by Pechenik and Yong [**42**] (after a modification), (2) Wheeler and Zinn-Justin [**49**, Theorems 2" and 3"] gave puzzle formulas for the product of two **dual** Grothendieck polynomials in different secondary variables, and (3) puzzle formulations of the Molev–Sagan tableau formula [**39**] for the product $s_\lambda(x,y) \cdot s_\mu(x,t)$ of two double Schur polynomials in different secondary variables were given by Knutson and Tao [**25**, Section 6] and Zinn-Justin [**50**].

(iv) The case that $k = n$ (this means u may be any permutation of $\{1, 2, \ldots, n\}$), $v = \text{id}$, and $x = t$. Theorem C reduces to the bumpless pipe dream model of double Grothendieck polynomials by Weigandt [**48**], which, by setting $\beta = 0$, leads to the bumpless pipe dream model of double Schubert polynomials by Lam, Lee and Shimozono [**29**]. An alternative proof of Weigandt's model was given by Buciumas and Scrimshaw [**11**] based on colored lattice models.

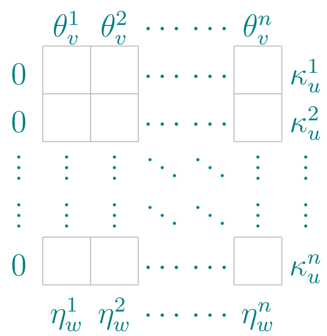
An innovation in our approach is finding that $c_{u,v}^{w}(t,y)$ satisfies two kinds of recurrence relations, as given in Section 5.2. Such recurrence relations work well when u and v are restricted to permutations with separated descents. This could essentially simplify the proof of Theorem C. Specifically, to prove Theorem C, it suffices to show that our pipe puzzle formula obeys the same recurrence relations (together with an initial condition).

This could be achieved (without too much effort) by realizing pipe puzzles as an integrable lattice model.

We remark that the above mentioned recurrence relations are no longer available in the case $y = t$. The proof of the $y = t$ case in [**28**, Theorem 1] is achieved by studying the geometric representation of quantized loop algebras and quiver varieties [**26, 27**]. It turns out that while the problem of computing triple Schubert structure constants is broader, its proof could be simpler. From this point of view, our approach may provide new insights into the study of Schubert calculus for flag manifolds.

In the remainder of this section, we assume that u and v are permutations of S_n with separated descents at position k (1.3.1). We are going to describe our pipe puzzle formula for $c^w_{u,v}(t, y)$. To begin, consider an n by n grid with labeled boundary:

(1.3.2)



$$\kappa_u^i = \begin{cases} u^{-1}(i), & u^{-1}(i) \le k, \\ 0, & u^{-1}(i) > k. \end{cases}$$

$$\theta_v^i = \begin{cases} 0, & v^{-1}(i) \le k, \\ v^{-1}(i), & v^{-1}(i) > k. \end{cases}$$

$$\eta_w^i = w^{-1}(i).$$

We see that the nonzero labels on the right side are $1, \ldots, k$, and the nonzero labels on the top side are $k+1, \ldots, n$. There is no obstruction to rebuilding u, v and w from the boundary labeling, because of the separated-descent assumption. For the sake of brevity, the label 0 on the boundary will often be omitted. See Example 1.3.2 for the boundary labeling for $u = 42135$, $v = 14532$, $w = 53412$, and $k = 2$.

Our formula is a weighted counting of tilings of the n by n grid by unit tiles (with pipes), subject to certain conditions. To warm up, we first give the formula for double Schubert polynomials.

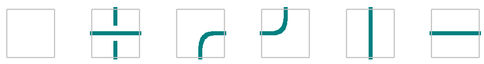
1.3.1. Statement for double Schubert polynomials.

Assume that

$$\mathfrak{S}_u(x, y) \cdot \mathfrak{S}_v(x, t) = \sum_w \bar{c}^w_{u,v}(t, y) \cdot \mathfrak{S}_w(x, t). \tag{1.3.3}$$

The admissible tiles are

(1.3.4)

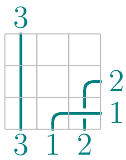


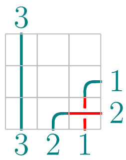
The curves drawn on the tiles are referred to as **pipes**. A tiling of (1.3.2) built upon the tiles in (1.3.4) is a network of pipes such that

(1) there are a total of n pipes, among which k pipes enter horizontally from rows on the right side labeled $1, \ldots, k$, and $n-k$ pipes enter vertically from columns on the top side labeled $k+1, \ldots, n$. The pipes inherit the labels of the corresponding rows and columns.
(2) the n pipes end vertically on the bottom side, such that the label of each pipe matches the label of the column where it ends.

A **Schubert pipe puzzle** for u, v, w is a tiling of (1.3.2) with the tiles in (1.3.4), subject to the following restriction on the tiles :

(1.3.5) The horizontal pipe in  must receive a smaller label. For example,



is allowed, while


is not allowed.

Denote by $\mathrm{PP}_0(u, v, w)$ the set of Schubert pipe puzzles for u, v, w. For each $\pi \in \mathrm{PP}_0(u, v, w)$, define its **Schubert weight** by

$$\mathrm{wt}_0(\pi) = \prod_{(i,j)} (t_j - y_i),$$

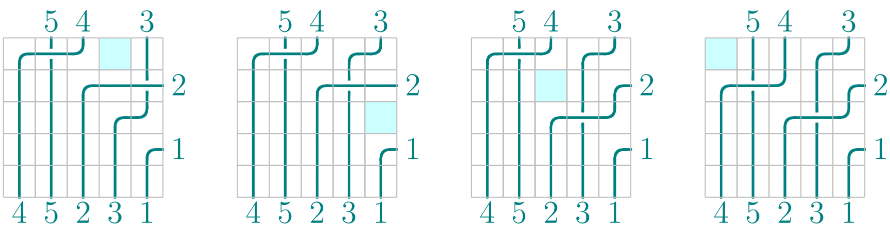
where the sum is over empty tiles  at the (i, j)-positions (in the matrix coordinate).

Theorem B. *Let $u, v \in S_n$ be permutations with separated descents at position k. For $w \in S_n$, we have*

$$\overline{c}_{u,v}^{w}(t, y) = \sum_{\pi \in \mathrm{PP}_0(u,v,w)} \mathrm{wt}_0(\pi). \tag{1.3.6}$$

Remark 1.3.1. It may happen that $\mathfrak{S}_w(x, t)$, $w \in S_{n'}$ with $n < n'$, appears in the expansion of $\mathfrak{S}_u(x, y) \cdot \mathfrak{S}_v(x, t)$. In such a case, to compute $\overline{c}_{u,v}^{w}(t, y)$, one needs only to embed naturally S_n into $S_{n'}$, and then apply Theorem B (u and v are now viewed as permutations in $S_{n'}$).

Example 1.3.2. *Let $u = 42135$, $v = 14532$, and set $k = 2$. For $w = 53412$, there are four Schubert pipe puzzles in $\mathrm{PP}_0(u, v, w)$:*



Here, the empty tiles are colored. So it follows from (1.3.6) *that*

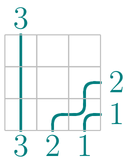
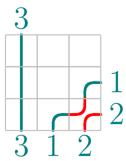
$$\overline{c}_{42135,14532}^{53412} = (t_4 - y_1) + (t_5 - y_3) + (t_3 - y_2) + (t_1 - y_1).$$

1.3.2. Statement for double Grothendieck polynomials. In addition to the tiles in (1.3.4), one more admissible tile than (1.3.4):

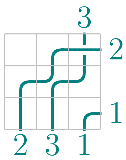
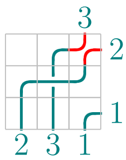
(1.3.7)

The extra tile in (1.3.7) is a "bumping" tile. The following restrictions apply to the use of the bumping tile:

(1.3.8)

> If the two pipes in the bumping tile are from the same side, then the northwest pipe must receive a greater label. For example,
>
> 
>
> is allowed, while
> 
>
> is not allowed.

(1.3.9)

> If the two pipes in the bumping tile are from different sides, then the northwest pipe must enter from the right side (equivalently, it receives a smaller label). For example,
>
> 
>
> is allowed, while
> 
>
> is not allowed.

A **(Grothendieck) pipe puzzle** for u, v, w is a tiling of (1.3.2) with the tiles in (1.3.7) obeying the restriction (1.3.5) on the crossing tile, as well as the restrictions (1.3.8) and (1.3.9) on the bumping tile.

Let $\mathrm{PP}(u, v, w)$ be the set of pipe puzzles for u, v, w. For $\pi \in \mathrm{PP}(u, v, w)$, its **weight** $\mathrm{wt}(\pi)$ is the product of factors contributed by all tiles of π: at the (i, j)-position,

(1) an empty tile contributes $t_j \ominus y_i$;
(2) an elbow tile, in which the pipe is from the right side, contributes $1+\beta(t_j \ominus y_i)$;
(3) an elbow tile, in which the pipe is from the top side, contributes $1+\beta(t_j \ominus y_i)$;
(4) a bumping tile, in which the two pipes are from the same side, contributes β;
(5) a bumping tile, in which the two pipes are from different sides, contributes $\beta(1 + \beta(t_j \ominus y_i))$;
(6) any other tile except for the above cases contributes 1.

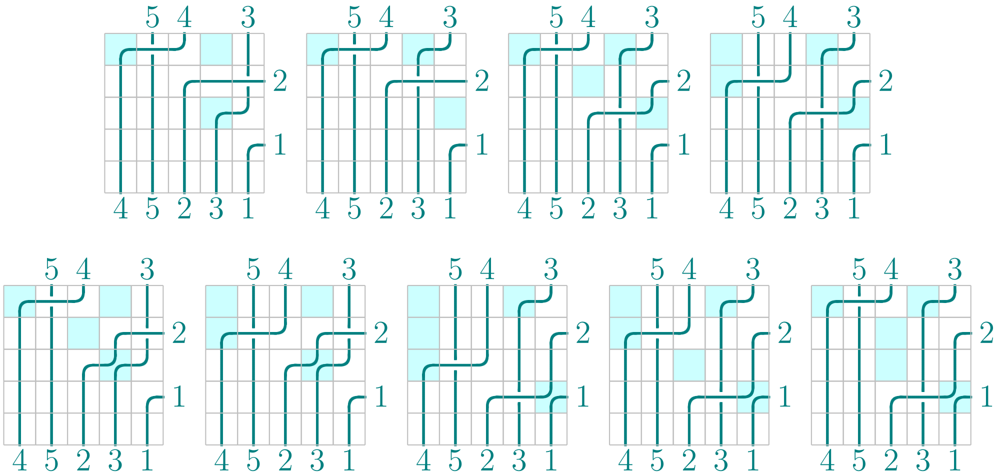
Here $x \ominus y = \frac{x-y}{1+\beta y}$, whose geometric meaning will be explained in Section 2.3.3.

Theorem C. *Let $u, v \in S_n$ be permutations with separated descents at position k. For $w \in S_n$, we have*

$$c^w_{u,v}(t, y) = \sum_{\pi \in \mathrm{PP}(u,v,w)} \mathrm{wt}(\pi). \tag{1.3.10}$$

Note that Remark 1.3.1 is still valid for Theorem C. We also remark that Theorem C specializes to Theorem B in the case $\beta = 0$ by noting that $\mathrm{wt}(\pi)|_{\beta=0} = 0$ whenever $\pi \notin \mathrm{PP}_0(u, v, w)$, and $\mathrm{wt}(\pi)|_{\beta=0} = \mathrm{wt}_0(\pi)$ for $\pi \in \mathrm{PP}_0(u, v, w)$.

Example 1.3.3. *Take the same setting as in Example 1.3.2. There are nine pipe puzzles in* $\mathrm{PP}(u, v, w)$*, among which the pipe puzzles in the top row are those appearing* *in Example 1.3.2. Here, the tiles with weights not equal to* 1 *are colored.*



As a result,

$$
\begin{aligned}
c^{53412}_{42135,14532} = {} & (t_4 \ominus y_1)(1+\beta(t_1 \ominus y_1))(1+\beta(t_4 \ominus y_3)) \\
& + (t_5 \ominus y_3)(1+\beta(t_1 \ominus y_1))(1+\beta(t_4 \ominus y_1)) \\
& + (t_3 \ominus y_2)(1+\beta(t_1 \ominus y_1))(1+\beta(t_4 \ominus y_1))(1+\beta(t_5 \ominus y_3)) \\
& + (t_1 \ominus y_1)(1+\beta(t_4 \ominus y_1))(1+\beta(t_1 \ominus y_2))(1+\beta(t_5 \ominus y_3)) \\
& + \beta(t_4 \ominus y_1)(t_3 \ominus y_2)(1+\beta(t_1 \ominus y_1))(1+\beta(t_4 \ominus y_3)) \\
& + \beta(t_1 \ominus y_1)(t_4 \ominus y_1)(1+\beta(t_1 \ominus y_2))(1+\beta(t_4 \ominus y_3)) \\
& + \beta(t_1 \ominus y_1)(t_1 \ominus y_2)(1+\beta(t_4 \ominus y_1))(1+\beta(t_1 \ominus y_3)) \\
& + \beta(t_1 \ominus y_1)(t_3 \ominus y_3)(1+\beta(t_4 \ominus y_1))(1+\beta(t_1 \ominus y_2)) \\
& + \beta(t_3 \ominus y_2)(t_3 \ominus y_3)(1+\beta(t_1 \ominus y_1))(1+\beta(t_4 \ominus y_1)).
\end{aligned}
$$

1.4. Equivariant positivity

In this section, we will explain a geometric meaning of triple Schuber calculus of (1.0.2) and the Graham positivity.

In [**18**], we proved the following conjecture of Samuel [**44**, Conjecture 1.1] on $\overline{c}^w_{u,v}(y,t)$

THEOREM D. *For* $u, v, w \in S_\infty$, $\overline{c}^w_{u,v}(y,t) \in \mathbb{N}[t_i - y_j]_{i,j\geq 1}$.

Setting $y = 0$, we obtain Kirillov's conjecture [**22**]:

THEOREM E. *For* $u, v, w \in S_\infty$, $\partial_{w/v}\mathfrak{S}_u(x) \in \mathbb{N}[x]$.

The following are some special cases:

(i) In the special case $w = v = \mathsf{id}$, we have $\overline{c}^w_{u,v}(y,t) = \mathfrak{S}_u(t,y)$ and it implies the positivity of double Schubert polynomials $\mathfrak{S}_u(x,y) \in \mathbb{N}[x_i - y_j]_{i,j\geq 0}$. Correspondently, in this case the skew divided difference operator $\partial_{w/v}$ is the identity operator, and it implies the positivity of double Schubert polynomials $\mathfrak{S}_u(x) \in \mathbb{N}[x]$.

(ii) In the special case $\ell(w) = \ell(u) + \ell(v)$, $\overline{c}^w_{u,v}(y,t)$ has degree 0, and thus $\overline{c}^w_{u,v}(y,t) = \overline{c}^w_{u,v}(0,0)$ is a constant. As a result, it implies the positivity of single structure constants.

(3) When $v \lessdot w$ in Bruhat order, the divided difference operator $\partial_{w/v}$ is a difference operator ∂_{ab} for some $a < b$, and in this case the conjecture was established by Kirillov himself [**22**, Section 8].

Our proof relies on a refined version of Graham's positivity theorem [**19**] (Theorem 6.2.3). To apply this result, we establish a new geometric interpretation (Section 6.4) for the coefficients $c_{u,v}^{w}(\mathbf{y}, \mathbf{t})$, distinct from the one given by Knutson–Tao [**25**].

Acknowledgment

I would like to express my sincere gratitude to my advisor, Kirill Zainoulline, for his guidance and supervision. I also thank Yuly Billig, Leonardo Mihalcea, Monica Nevins, and Hadi Salmasian for carefully reading an earlier version of this thesis and for their helpful comments. I am grateful to Neil Fan, Yibo Gao, Peter Guo, Martina Lanini and Kirill Zainoulline for their collaboration [**16, 37, 17, 18**].

CHAPTER 2

Geometric and Combinatorial Background

2.1. Flag varieties and Schubert varieties

Consider $G = GL_n$ the general linear group over complex numbers $\mathbb{C}$. We define $B = B^+$ (resp., B^-) the subgroup of the upper (lower) triangular matrices in G, and $T = B \cap B^-$ the subgroup of diagonal matrices. That is,

$$T = \begin{bmatrix} * & & \\ & \ddots & \\ & & * \end{bmatrix}, \qquad B = B^+ = \begin{bmatrix} * & \cdots & * \\ & \ddots & \vdots \\ & & * \end{bmatrix}, \qquad B^- = \begin{bmatrix} * & & \\ \vdots & \ddots & \\ * & \cdots & * \end{bmatrix}.$$

The **(full) flag variety** is the variety of (full) flags in $\mathbb{C}^n$, i.e.

$$\mathcal{F}\ell_n = \{0 = V_0 \subsetneq V_1 \subsetneq \cdots \subsetneq V_{n-1} \subsetneq V_n = \mathbb{C}^n : \dim V_i = i\}.$$

Let $\phi^+ \in \mathcal{F}\ell_n$ be the **standard flag**, i.e.

$$\phi_i^+ := \operatorname{span}(\mathbf{e}_1, \dots, \mathbf{e}_i)$$

where $\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$ is the standard basis of $\mathbb{C}^n$. The group G acts on $\mathcal{F}\ell_n$ by changing basis, and it induces an isomorphism[1]

$$G/B \xrightarrow{\sim} \mathcal{F}\ell_n, \qquad gB \longmapsto g\phi_i^+. \tag{2.1.1}$$

Let $W = S_n$ be the symmetric group. For each $w \in W$, we can define $\phi_w \in \mathcal{F}\ell_n$ by

$$\phi_{w,i} = \operatorname{span}(\mathbf{e}_{w(1)}, \dots, \mathbf{e}_{w(i)}).$$

For example, $\phi^+ = \phi_{\mathsf{id}}$. Moreover, the T-fixed points

$$(\mathcal{F}\ell_n)^T = \{\phi_w\}_{w \in W}.$$

Precisely, a flag $V_\bullet \in \mathcal{F}\ell_n$ is fixed by T if each V_i is T-equivariant, i.e. is a coordinate subspace. Then $V_\bullet$ must take the form ϕ_w for some $w \in S_n$. Under the isormorphism (2.1.1), ϕ_w corresponds to $wB \in G/B$, where w is considered as a permutation matrix.

2.1.1. Schubert cells.

We define the **Schubert cell and opposite Schubert cell** of $w \in S_n$ to be

$$X^\circ(w) = B \cdot \phi_w \subset \mathcal{F}\ell_n, \qquad Y^\circ(w) = B^- \cdot \phi_w \subset \mathcal{F}\ell_n.$$

Under the isomorphism (2.1.1), they correspond to BwB/B and B^-wB/B respectively. We can describe the flags in $X^\circ(w)$ or $Y^\circ(w)$ in terms of linear algebra. To describe this, we need the concept of relative position. Let $V_\bullet, U_\bullet$ be two flags. We say they are of **relative position** $w \in S_n$ if there exists a basis $\{v_1, \dots, v_n\}$ of $\mathbb{C}^n$ such that

$$V_i = \operatorname{span}(v_{w(1)}, \dots, v_{w(i)}), \qquad U_j = \operatorname{span}(v_1, \dots, v_j).$$

[1]Strictly speaking, the algebraic variety structure of $\mathcal{F}\ell_n$ is actually defined such that this bijection is an isomorphism of algebraic variety.

Note that there always exists a unique $w \in S_n$ such that the above condition holds. The existence is a standard linear algebra exercise. The permutation w is unique since w can be recovered from the dimensions

$$\dim(V_i \cap U_j) = r_{ji} := \#\{a \leq j : w(a) \leq i\}.$$

Note that r_{ji} is the number of 1's in the first $j \times i$ submatrix of the permutation matrix of w. For example

$$w = 231 = \begin{bmatrix} & & 1 \\ 1 & & \\ & 1 & \end{bmatrix}, \qquad r = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & 2 \\ 1 & 2 & 3 \end{bmatrix}.$$

For example, if $V_\bullet$ and $U_\bullet$ are of relative position $\mathsf{id} \in S_n$, then we have $V_\bullet = U_\bullet$. It follows from definition that

$$V_\bullet \in X^\circ(w) \iff V_\bullet \text{ and } \phi^+ \text{ are of relative position } w.$$

Let us denote $w_0 = n \cdots 21 \in S_n$ the longest element. We denote $\phi^- = \phi_{w_0} \in \mathcal{F}\ell_n$ the **standard opposite flag**, i.e.

$$\phi_i^- = \operatorname{span}(\mathbf{e}_n, \ldots, \mathbf{e}_{n-i+1}).$$

Then we have

$$V_\bullet \in Y^\circ(w) \iff V_\bullet \text{ and } \phi^- \text{ are of relative position } w_0 w.$$

From the discussion above, we can decompose $\mathcal{F}\ell_n$ into disjoint union, called **Bruhat decomposition**

$$\mathcal{F}\ell_n = \bigsqcup_{w \in W} X^\circ(w) = \bigsqcup_{w \in W} Y^\circ(w). \tag{2.1.2}$$

We define the **(opposite) Schubert varieties** of $w \in S_n$ to be the closure

$$X(w) = \overline{X^\circ(w)} \subset \mathcal{F}\ell_n, \qquad Y(w) = \overline{Y^\circ(w)} \subset \mathcal{F}\ell_n.$$

Let us denote the Bruhat order over S_n by $\leq$. Then we have

$$X(w) = \bigcup_{u \leq w} X^\circ(u), \qquad Y(w) = \bigcup_{u \geq w} Y^\circ(u).$$

Thus the Bruhat decomposition defines a stratification of $\mathcal{F}\ell_n$.

2.1.2. Coordinates of Schubert cells. Let us describe the structure of $X^\circ(w)$ and $Y^\circ(w)$. It will be useful in the later sections. First, the stabilizer of ϕ_w in G is

$$\{b \in G : b\phi_w = \phi_w\} = \{b \in G : bwB/B = wB/B\} = wBw^{-1}.$$

Thus

$$\begin{aligned} X^\circ(w) &= B/(B \cap wBw^{-1}) && \text{as } B\text{-varieties}, \\ Y^\circ(w) &= B^-/(B^- \cap wBw^{-1}) && \text{as } B^-\text{-varieties}. \end{aligned}$$

Let us define subgroups

$$N = N^+ = \begin{bmatrix} 1 & \cdots & * \\ & \ddots & \vdots \\ & & 1 \end{bmatrix}, \qquad N^- = \begin{bmatrix} 1 & & \\ \vdots & \ddots & \\ * & \cdots & 1 \end{bmatrix}.$$

Then we can rewrite the above identification

$$X^\circ(w) = N/(N \cap wNw^{-1}) \qquad \text{as } B\text{-varieties},$$
$$Y^\circ(w) = N^-/(N^- \cap wNw^{-1}) \qquad \text{as } B^-\text{-varieties}.$$

Let $\ell : S_n \to \mathbb{Z}_{\geq 0}$ be the length function, i.e.

$$\ell(w) = \#\{(i,j) : i < j, w(i) > w(j)\}.$$

We have

$$N^\pm \cap wN^\pm w^{-1} \cong \mathbb{A}^{\ell(w_0)-\ell(w)}, \qquad N^\pm \cap wN^\mp w^{-1} \cong \mathbb{A}^{\ell(w)}$$

and the multiplication induces an isomorphism of varieties

$$\left(N^\pm \cap wN^\mp w^{-1}\right) \times \left(N^\pm \cap wN^\pm w^{-1}\right) \xrightarrow{\sim} N^\pm.$$

For example, for $w = 231 \in S_3$, we have

$$wNw^{-1} = \begin{bmatrix} 1 & & \\ * & 1 & * \\ * & & 1 \end{bmatrix}, \qquad wN^-w^{-1} = \begin{bmatrix} 1 & * & * \\ & 1 & \\ & * & 1 \end{bmatrix}.$$

Then

$$N \cap wNw^{-1} = \begin{bmatrix} 1 & & \\ & 1 & * \\ & & 1 \end{bmatrix} \cong \mathbb{A}^1, \qquad N \cap wN^-w^{-1} = \begin{bmatrix} 1 & * & * \\ & 1 & \\ & & 1 \end{bmatrix} \cong \mathbb{A}^2.$$

As a result, we have

$$X^\circ(w) \cong N \cap wN^-w^{-1} \cong \mathbb{A}^{\ell(w)}, \tag{2.1.3}$$
$$Y^\circ(w) \cong N \cap wNw^{-1} \cong \mathbb{A}^{\ell(w_0)-\ell(w)}. \tag{2.1.4}$$

2.1.3. Kleiman's transversality theorem. For two subvarieties A, B of $\mathcal{F}\ell_n$, we say the intersection $A \cap B$ is proper if

$$\dim \mathcal{F}\ell_n + \dim A \cap B = \dim A + \dim B$$

and is transverse at the generic point if

$$\operatorname{Tan}_x A \cap \operatorname{Tan}_x B = \operatorname{Tan}_x(A \cap B)$$

at each generic point x of each components of $A \cap B$. Note that the second condition does not imply the first condition. For example, when $A = B \neq Fl_n$, the second condition is satisfied, while the first condition is not.

THEOREM 2.1.1 (Kleiman [**23**]). *Let $A, B \subset \mathcal{F}\ell_n$ be two subvarieties. There exists an open dense subset of $g \in G$ such that the intersection $gA \cap B$ is proper and transverse at the generic point.*

COROLLARY 2.1.2. *For all $u, v \in W$, the intersection $X(u) \cap Y(v)$ is proper and transverse at the generic point.*

PROOF. By Kleiman theorem, there exists an open dense subset of $g \in G$ such that $gX(u) \cap Y(v)$ is proper and transverse at the generic point. Note that $B^-B \subset G$ is an open dense subset. This follows from dimension counting

$$\dim B^- \cdot B = \dim(B^- \cdot B/B) + \dim B = \dim(Y^\circ(\mathsf{id})) + \dim B = \dim G.$$

Since the intersection of two open dense subsets is nonempty, we can find an element $g = xy$ for $x \in B^-$ and $y \in B$. Since G acts on $\mathcal{F}\ell_n$, so the intersection $yX(u) \cap x^{-1}Y(v) =$

$x^{-1}(gX(u) \cap Y(v))$ is also propert and transversal. Since $X(u)$ is B-invariant, and $Y(v)$ is B^--invariant, we have $yX(u) = X(u)$ and $xY(v) = Y(v)$. This proves the Corollary. $\square$

2.1.4. Flag varieties for other types. Let us briefly mention the flag variety for other types. Now assume G is a reductive group. Let B and B^- be two opposite Borel subgroups, and $T = B \cap B^-$ be a maximal torus. The flag variety of G is

$$\mathcal{B} = \{\text{Borel subgroups of } G\}.$$

We can identify

$$G/B \xrightarrow{\sim} \mathcal{B}, \qquad gB \longmapsto gBg^{-1}.$$

Let $W = N_G(T)/T$ be the Weyl group of G. By an abuse of notation, we will also denote $w \in N_G(T)$ a representative for $w \in W$. Then the fixed points

$$(G/B)^T = \{wB/B\}_{w \in W}.$$

We can similarly define **(opposite) Schubert cells** as in type A

$$X^\circ(w) = BwB/B \subset G/B, \qquad Y^\circ(w) = B^-wB/B \subset G/B$$

and we also have the **Bruhat decomposition**

$$G/B = \bigsqcup_{w \in W} X^\circ(w) = \bigsqcup_{w \in W} Y^\circ(w).$$

We define the **(opposite) Schubert varieties** to be the closure

$$X^\circ(w) = \overline{X^\circ(w)} \subset G/B, \qquad Y^\circ(w) = \overline{Y^\circ(w)} \subset G/B.$$

2.2. Examples of Schubert varieties

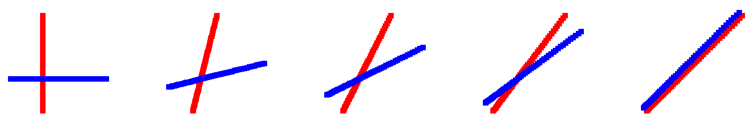
In this paragraph, let us give examples of Schubert varieties, which will appear in the later sections. Recall that

$$V_\bullet \in X^\circ(w) \iff \dim(V_i \cap \phi_j^+) = r_{ji}.$$

From the combinatorics of permutations, we have

$$V_\bullet \in X(w) \iff \dim(V_i \cap \phi_j^+) \geq r_{ji}.$$

This can be explained by the semi-continuity of the dimension of intersection, i.e. when two subspaces degenerate to special position, the dimension of the intersection will increase.



Let $s_i \in S_n$ be the simple transposition of i and $i+1$.

2.2.1. Example: $n = 2$. When $n = 2$, the flag variety can be identified with the projective line

$$\mathcal{F}\ell_2 = \{0 \subsetneq V_1 \subsetneq \mathbb{C}^2 : \dim V_1 = 1\} = \mathbb{P}^1.$$

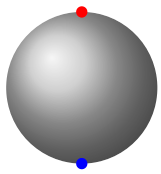
Note that $W = \{\mathsf{id}, w_0\}$. Using the classical identification of

$$\mathbb{P}^1 = \mathbb{C} \cup \{\infty\}$$

we have

$$\begin{aligned} X^\circ(\mathsf{id}) &= X(\mathsf{id}) = \{0\}, & Y^\circ(\mathsf{id}) &= \mathbb{C},\ Y(\mathsf{id}) = \mathbb{P}^1, \\ X^\circ(w_0) &= \mathbb{P}^1 \setminus \{0\},\ X(w_0) = \mathbb{P}^1, & Y^\circ(w_0) &= Y(w_0) = \{\infty\}. \end{aligned}$$

Here is a picture



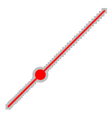
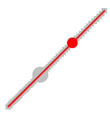
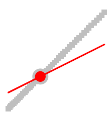
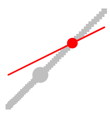
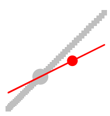
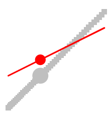
2.2.2. Example: $n = 3$. When $n = 3$,

$$\mathcal{F}\ell_2 = \{0 \subsetneq V_1 \subsetneq V_2 \subsetneq \mathbb{C}^3 : \dim V_i = i\}$$

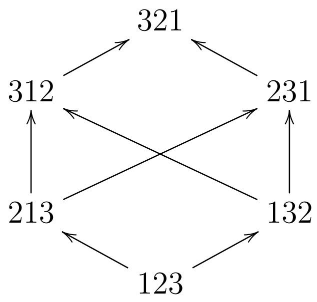
is the incidence variety, i.e. it can be identified with the space of incidence relations over the projective plane $\mathbb{P}^2$, i.e.

$$\left\{ (P \in L) : \begin{array}{l} P \text{ is a point in } \mathbb{P}^2 \\ L \text{ is a line in } \mathbb{P}^2 \end{array} \right\}.$$

Let (P_0, L_0) be the incidence relation corresponding to ϕ^+. The description of $(P, L) \in X(w)$ can be described as follows

$w = 123$		$\begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix}$	$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}$	$\begin{array}{ccc} P & = & P_0 \\ \Cap & & \Cap \\ L & = & L_0 \end{array}$
$w = 213$		$\begin{bmatrix} & 1 & \\ 1 & & \\ & & 1 \end{bmatrix}$	$\begin{bmatrix} 0 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}$	$\begin{array}{ccc} P & & P_0 \\ \Cap & & \Cap \\ L & = & L_0 \end{array}$
$w = 132$		$\begin{bmatrix} 1 & & \\ & & 1 \\ & 1 & \end{bmatrix}$	$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 2 \\ 1 & 2 & 3 \end{bmatrix}$	$\begin{array}{ccc} P & = & P_0 \\ \Cap & & \Cap \\ L & & L_0 \end{array}$
$w = 231$		$\begin{bmatrix} & & 1 \\ 1 & & \\ & 1 & \end{bmatrix}$	$\begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & 2 \\ 1 & 2 & 3 \end{bmatrix}$	$\begin{array}{ccc} P & & P_0 \\ \Cap & \rotatebox{45}{$\Cap$} & \Cap \\ L & & L_0 \end{array}$
$w = 312$		$\begin{bmatrix} & 1 & \\ & & 1 \\ 1 & & \end{bmatrix}$	$\begin{bmatrix} 0 & 1 & 1 \\ 0 & 1 & 2 \\ 1 & 2 & 3 \end{bmatrix}$	$\begin{array}{ccc} P & & P_0 \\ \Cap & \rotatebox{-45}{$\Cap$} & \Cap \\ L & & L_0 \end{array}$
$w = 321$		$\begin{bmatrix} & & 1 \\ & 1 & \\ 1 & & \end{bmatrix}$	$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 2 \\ 1 & 1 & 3 \end{bmatrix}$	$\begin{array}{ccc} P & & P_0 \\ \Cap & & \Cap \\ L & & L_0 \end{array}$

The following diagram represents the Bruhat order



2.2.3. Simple reflections. Let us describe the Schubert variety $X(s_k)$ for a simple reflection s_k with $1 \le k \le n-1$. Note that $\dim X(s_k) = 1$, so $X(s_k)$ is a curve. By definition, it can be described as

$$X(s_k) = \left\{ V_\bullet \in \mathcal{F}\ell_n : j \neq k \Rightarrow V_j = \phi_j^+ \right\} \subset \mathcal{F}\ell_n .$$

As a result, we can identify $X(s_i)$ as a projective line

$$X(s_k) \cong \{ V_k : \phi_{k-1}^+ \subsetneq V_k \subsetneq \phi_{k+1}^+ \} \cong \mathbb{P}(\phi_{k+1}^+/\phi_{k-1}^+) = \mathbb{P}^1.$$

The condition can be described in a diagram

$$\begin{array}{ccccccccccccccc} 0 = \phi_0^+ & \subset & \phi_1^+ & \subset & \cdots & \subset \phi_{k-1}^+ \subset & V_k & \subset \phi_{k+1}^+ \subset & \cdots & \subset \phi_{n-1}^+ \subset & \phi_n^+ = \mathbb{C}^n \\ & & & & & \subset & & \subset & & & \\ & & & & & & \phi_k^+ & & & & \end{array}$$

This example has two generalizations.

- For $a \ge k$,

$$X(s_a s_{a-1} \cdots s_k) = \left\{ V_\bullet \in \mathcal{F}\ell_n : \begin{array}{c} k \le j \le a \Rightarrow V_j \subset \phi_{j+1}^+ \\ j < k \text{ or } j > a \Rightarrow V_j = \phi_j^+ \end{array} \right\}.$$

  The conditions can be described in a diagram

$$\begin{array}{ccccccccc} \cdots \subset \phi_{k-1}^+ \subset & V_k & \subset \cdots \subset & V_{a-1} \subset & V_a & \subset \phi_{a+1}^+ \subset \cdots \\ \subset & \subset & \cdots & \subset & \subset & \\ & \phi_k^+ & \subset \phi_{k+1}^+ \subset \cdots \subset & \phi_a^+ & & \end{array}$$

- Similarly, for $b \le k$,

$$X(s_b s_{b+1} \cdots s_k) = \left\{ V_\bullet \in \mathcal{F}\ell_n : \begin{array}{c} b \le j \le k \Rightarrow V_j \supset \phi_{j-1}^+ \\ j < k \text{ or } j > a \Rightarrow V_j = \phi_j^+ \end{array} \right\}.$$

  The conditions can be pictured as follows.

$$\begin{array}{ccccccccc} \cdots \subset \phi_{b-1}^+ \subset & V_b & \subset V_{b+1} \subset & \cdots & \subset & V_k & \subset \phi_{k+1}^+ \subset \cdots \\ \subset & \subset & & \cdots & \subset & \subset & \\ & \phi_b^+ & \subset \cdots \subset & \phi_{k-1}^+ & \subset & \phi_k^+ & \end{array}$$

2.2.4. Schubert divisors. Let us describe the Schubert variety $Y(s_k)$ for a simple reflection s_k with $1 \leq k \leq n-1$. Note that the codimension of $Y(s_k)$ is 1, so $Y(s_k)$ is a divisor. It can be described as

$$Y(s_k) = \left\{V_\bullet \in \mathcal{F}\ell_n : \dim(V_k \cap \phi^-_{n-k}) \geq 1\right\}.$$

Equivalently, the complement is

$$\mathcal{F}\ell_n \backslash Y(s_k) = \left\{V_\bullet \in \mathcal{F}\ell_n : V_k \cap \phi^-_{n-k} = 0\right\}.$$

This example has two generalizations.

- For $a \geq k$,
  $$Y(s_a s_{a-1} \cdots s_k) = \left\{V_\bullet \in \mathcal{F}\ell_n : \dim(V_k \cap \phi^-_{n-a}) \geq 1\right\}$$
  whose complement is
  $$\mathcal{F}\ell_n \backslash Y(s_a s_{a-1} \cdots s_k) = \left\{V_\bullet \in \mathcal{F}\ell_n : V_k \cap \phi^-_{n-a} = 0\right\}.$$
  For example, when $a = n-1$,
  $$Y(s_{n-1} \cdots s_k) = \{V_\bullet \in \mathcal{F}\ell_n : \phi^-_1 \subset V_k\}.$$
- More generally, for $b \leq k$,
  $$Y(s_b s_{b+1} \cdots s_k) = \left\{V_\bullet \in \mathcal{F}\ell_n : \dim(V_k \cap \phi^-_{n-b}) \geq k-b+1\right\}$$
  whose complement is
  $$\mathcal{F}\ell_n \backslash Y(s_a s_{a-1} \cdots s_k) = \left\{V_\bullet \in \mathcal{F}\ell_n : \dim(V_k \cap \phi^-_{n-b}) = k-b\right\}.$$
  For example, when $b = 1$,
  $$Y(s_1 \cdots s_k) = \{V_\bullet \in \mathcal{F}\ell_n : V_k \subset \phi^-_{n-1}\}.$$

The two families of Schubert classes introduced above are called **special Schubert classes**.

2.2.5. Dominant permutations. Recall a **Hessenberg function** is a function $h : [n] \to [n]$ such that

$$i \leq h(i) \leq n, \qquad i \leq j \Rightarrow h(i) \leq h(j).$$

We have an associated permutation $w_h \in S_n$ such that

$$w(i) = \max\left(\{1, \ldots, h(i)\} \setminus \{w(1), \ldots, w(i-1)\}\right).$$

That is, $w(i)$ is the lexicographically maximal permutation such that $w(i) \leq h(i)$. Then

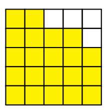
$$X(w_h) = \left\{V_\bullet \in \mathcal{F}\ell_n : V_i \subseteq \phi^+_{h(i)}\right\}.$$

The permutation of the form w_h is called a **codominant permutation**. Let $\overline{w}_h = w_0 w_h \in S_n$. Then

$$Y(\overline{w}_h) = \left\{V_\bullet \in \mathcal{F}\ell_n : V_i \subseteq \phi^-_{h(i)}\right\}.$$

The permutation of the form $\overline{w}_h$ is called a **dominant permutation** or a **132-avoiding permutation**. For example, for $n = 5$ and Hessenberg function

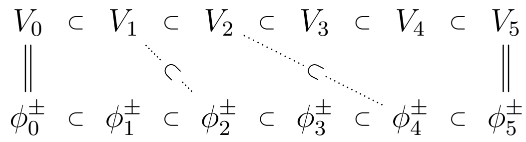
$$h(1) = 2, \quad h(2) = 4,$$
$$h(3) = h(4) = h(5) = 5$$



Then

$$w_h = 24531, \qquad \overline{w}_h = 42135.$$

The condition for $X(w_h)$ or $Y(\overline{w}_h)$ can be pictured as follows.



2.3. Equivariant cohomology and K-theory

2.3.1. Equivariant cohomology. Let $\mathbb{E}_N = (\mathbb{C}^N \setminus 0)^n$ equipped with the natural action of the torus $T = (\mathbb{C}^\times)^n$. For a nonsingular T-variety X, the **equivariant cohomology** of X is a graded ring whose i-th component is

$$H^i_T(X) := H^i(\mathbb{E} \times^T X),$$

where $\mathbb{E} = \mathbb{E}_N$ for $N \gg 0$. For a T-equivariant vector bundle $\mathcal{V}$ over X, we can construct an induced bundle $\mathcal{V}' = \mathbb{E} \times^T \mathcal{V}$ over $\mathbb{E} \times^T X$. For $i \geq 0$, **the i-th equivariant Chern class** is defined by

$$c_i(\mathcal{V}) := c_i(\mathcal{V}') \in H^{2i}(\mathbb{E} \times^T X) = H^{2i}_T(X).$$

For a T-equivariant subvariety $Y \subset X$ of codimension d, we have a subvariety $Y' = \mathbb{E} \times^T Y \subset \mathbb{E} \times^T X$ of the same codimension, its **equivariant fundamental class** is

$$[Y] := [Y'] \in H^{2d}(\mathbb{E} \times^T X) = H^{2d}_T(X).$$

Let $\mathsf{pt} = \operatorname{Spec} \mathbb{C}$ be a single $\mathbb{C}$-point. Following the historical convention in Schubert calculus, we write

$$t_i = c_1(\mathbb{C}_{-z_i}) \in H^2_T(\mathsf{pt}),$$

where $\mathbb{C}_{-z_i}$ is the bundle corresponding to the character

$$T \ni \operatorname{diag}(z_1, \dots, z_n) \longmapsto z_i^{-1} \in \mathbb{C}^\times.$$

Then $H^\bullet_T(\mathsf{pt})$ is naturally isomorphic to the ring of polynomials in $t_1, \dots, t_n$:

$$H^*_T(\mathsf{pt}) = \mathbb{Q}[t_1, \dots, t_n]. \tag{2.3.1}$$

Over the flag variety $\mathcal{F}\ell_n$, we have the **tautological flag**

$$0 = \mathcal{V}_0 \subsetneq \mathcal{V}_1 \subsetneq \cdots \subsetneq \mathcal{V}_n = \mathcal{O}_{\mathcal{F}\ell_n}^{\oplus n},$$

where the total space of $\mathcal{V}_i$ is

$$\{(V_\bullet, v) \in \mathcal{F}\ell(n) \times \mathbb{C}^n \colon v \in V_i\}.$$

For $i = 1, \dots, n$, let

$$x_i = c_1\big((\mathcal{V}_i/\mathcal{V}_{i-1})^\vee\big) \in H^2_T(G/B)$$

be the first equivariant Chern class of the line bundle $(\mathcal{V}_i/\mathcal{V}_{i-1})^\vee$.

We have the following **Borel presentation** [**8**]; see also [**1**, Section 15.6]

$$H^*_T(\mathcal{F}\ell_n) = \frac{\mathbb{Q}[t_1, \dots, t_n, x_1, \dots, x_n]}{\langle e_k(\mathbf{x}) - e_k(\mathbf{t}) : 1 \leq k \leq n \rangle} \tag{2.3.2}$$

where $\mathbf{t} = (t_1, \dots, t_n)$ and $\mathbf{x} = (x_1, \dots, x_n)$ and e_k is the k-th elementary symmetric function in n variables.

Since Schubert varieties are T-equivariant, we could consider their equivariant fundamental classes

$$[X(w)],\ [Y(w)] \in H^*_T(G/B), \qquad w \in W.$$

They are known as **equivariant Schubert classes**. Since the Bruhat decomposition defines an affine stratification of $\mathcal{F}\ell_n$, we have the following isomorphisms of $H_T^*(\mathsf{pt})$-modules

$$H_T^*(G/B) = \bigoplus_{w\in W} H_T^*(\mathsf{pt})\cdot[X(w)] = \bigoplus_{w\in W} H_T^*(\mathsf{pt})\cdot[Y(w)].$$

We will explain the polynomial representation of them in the next section.

Since $\mathcal{F}\ell_n$ is a projective variety, we have an $H_T^*(\mathsf{pt})$-linear map

$$\int_{\mathcal{F}\ell_n} : H_T^*(\mathcal{F}\ell_n) \longrightarrow H_T^*(\mathsf{pt}),$$

the push-forward map induced by the unique morphism $\mathcal{F}\ell_n \to \mathsf{pt}$. The **Poincaré pairing** of two classes $\gamma_1,\gamma_2 \in H_T^*(\mathcal{F}\ell_n)$ is

$$\langle\gamma_1,\gamma_2\rangle = \langle\gamma_1,\gamma_2\rangle_{\mathcal{F}\ell_n} := \int_{\mathcal{F}\ell_n}\gamma_1\cup\gamma_2 \in H_T^*(\mathsf{pt}).$$

The classes $[Y(w)]$ form the dual basis of the classes $[X(w)]$ under the Poincaré pairing, i.e. for $w,u\in W$, we have

$$\big\langle[X(w)],[Y(u)]\big\rangle = \delta_{u,v}.$$

Actually by Corollary 2.1.2, $\big\langle[X(w)],[Y(u)]\big\rangle$ is the number of points of the intersection $X(w)\cap Y(u)$ when it is nonempty of dimension 0, and is zero otherwise. Thus only when $w=u$, it is a nonzero number and in this case $X(w)\cap Y(w)=\{\phi_w\}$ is a single point.

2.3.2. Equivariant K-theory. Let X be a nonsingular T-variety. The T-equivariant K-group $K_T(X)$ of X is the Grothendieck group of the category of T-equivariant coherent sheaves on X. To be specific,

$$K_T(X) = \bigoplus_{\mathcal{F}}\mathbb{Q}[\mathcal{F}]/\langle[\mathcal{F}]=[\mathcal{F}_1]+[\mathcal{F}_2]\rangle$$

where the sum is over all T-equivariant coherent sheaves, and the relation is spanned over all short exact sequences $0\to\mathcal{F}_1\to\mathcal{F}\to\mathcal{F}_2\to 0$ of equivariant coherent sheaves. Since X is non-singular, there is a well-defined product structure such that

$$[\mathcal{F}]\otimes[\mathcal{G}] := [\mathcal{F}\otimes\mathcal{G}]$$

when $\mathcal{F}$ is flat. For a reference of the equivariant K-theory, see for example [**14**].

Following the classical notation of Schubert calculus, we define

$$T_i = [\mathbb{C}_{-z_i}]\in K_T(\mathsf{pt}), \qquad t_i = 1-[\mathbb{C}_{z_i}]\in K_T(\mathsf{pt}),$$

where, as before, $\mathbb{C}_{-z_i}$ is the bundle corresponding to the character

$$T\ni \operatorname{diag}(z_1,\dots,z_n)\longmapsto z_i^{-1}\in\mathbb{C}^\times.$$

Then $K_T(\mathsf{pt})$ is isomorphic to

$$K_T(\mathsf{pt}) = \mathbb{Q}[T_1^{\pm1},\dots,T_n^{\pm1}] = \mathbb{Q}\big[t_1,\tfrac{-t_1}{1-t_1},\dots,t_n,\tfrac{-t_n}{1-t_n}\big].$$

Over the flag variety $\mathcal{F}\ell_n$, for $i=1,\dots,n$, we write

$$X_i = [(\mathcal{V}_i/\mathcal{V}_{i-1})^\vee]\in K_T(G/B), \qquad x_i = 1-[\mathcal{V}_i/\mathcal{V}_{i-1}]\in K_T(G/B).$$

Since Schubert varieties are T-equivariant, we can consider their structure sheaves

$$[\mathcal{O}_{X(w)}],\ [\mathcal{O}_{Y(w)}]\in K_T(G/B), \qquad w\in W.$$

We similarly have

$$K_T(G/B) = \bigoplus_{w\in W} K_T(\mathsf{pt})\cdot[\mathcal{O}_{X(w)}] = \bigoplus_{w\in W} K_T(\mathsf{pt})\cdot[\mathcal{O}_{Y(w)}].$$

2.3.3. Connective K-theory. It would be useful to consider connective K-theory. In general, a formal group law F determines a generalized cohomology theory $\mathtt{h}_F(-)$ (in the topological sense). For example,

$$\begin{aligned} F(x,y) &= x+y, & \mathtt{h}_F(-) &= H^{\text{even}}(-),\\ F(x,y) &= x+y-xy, & \mathtt{h}_F(-) &= K(-). \end{aligned}$$

A **formal group law** is a formal power series

$$F(x,y) = x+y+\cdots$$

satisfying commutative group axioms:

(1) $F(x,F(y,z)) = F(F(x,y),z)$;
(2) $F(x,0) = x$ and $F(0,x) = x$;
(3) $F(x,\mathsf{inv}(x)) = F(\mathsf{inv}(x),x) = x$ for some $\mathsf{inv} = -x+o(x)$;
(4) $F(x,y) = F(y,x)$.

A **logarithm** of F is a formal power series $\lambda_F(x) = x+\cdots$ such that

$$\lambda_F(F(x,y)) = \lambda_F(x)+\lambda_F(y). \tag{2.3.3}$$

An **exponential** of F is a formal power series $\epsilon_F(x) = x+\cdots$ such that

$$F(\epsilon_F(x),\epsilon_F(y))) = \epsilon(x+y).$$

When the base ring is a field of characteristic zero, we can solve for λ_F as follows. Applying $\frac{\partial f}{\partial y}$ to both sides of (2.3.3), we get

$$\lambda_F'(F(x,y))F_y(x,y) = \lambda_F'(y). \tag{2.3.4}$$

Setting $y=0$, we have $\lambda_F'(x)F_y(x,0)=1$. Thus

$$\lambda_F(x) = \int_0^x \frac{dx}{F_y(x,0)} = x+\cdots, \tag{2.3.5}$$

where the integral is formally defined to be

$$\int_0^x \sum_{k\geq 0} a_k x^k := \sum_{k\geq 0} a_k \frac{1}{k+1} x^{k+1}.$$

We still need to check $\lambda_F(x)$ satisfies (2.3.3). It suffices to check the derivative (2.3.4). Substituting λ_F into (2.3.4), we need to check

$$\frac{1}{F_y(F(x,y),0)}F_y(x,y) = \frac{1}{F_y(y,0)}. \tag{2.3.6}$$

Taking $\frac{\partial}{\partial z}$ on both sides of

$$F(F(x,y),z) = F(x,F(y,z)),$$

we will get

$$F_y(F(x,y),z) = F_y(x,F(y,z))F_y(y,z).$$

Setting $z=0$, we will get (2.3.6).

Assume the exponential ϵ_F exists. We have a linear map called the **Chern character** commutative with the pull-back

$$\mathrm{ch} : \mathtt{h}(X) \longrightarrow H^{\mathrm{even}}(X)$$

such that when $X = \mathbb{P}^n$ for $n \geq 1$

$$\mathrm{ch} : \mathtt{h}(\mathbb{P}^n) \to H^{\mathrm{even}}(\mathbb{P}^n), \qquad i_{H*}(1) \longmapsto \epsilon_F([\mathbb{P}^{n-1}]),$$

where $i_H : H = \mathbb{P}^{n-1} \to \mathbb{P}^n$ is the natural inclusion. Since it commutes with pull-back, the Chern character ch is an algebra homomorphism.

EXAMPLE 2.3.1. *For example, consider the case $F(x, y) = x + y - xy$. Then*

$$F_y(x, 0) = 1 - x, \qquad \lambda_F(x) = \int_0^x \frac{dx}{1-x} = -\ln(1-x).$$

Thus $\epsilon_F(x) = 1 - e^{-x}$. It is easy to check that

$$\begin{aligned} F(\epsilon_F(x), \epsilon_F(y)) &= (1 - e^{-x}) + (1 - e^{-y}) - (1 - e^{-x})(1 - e^{-y}) \\ &= 1 - e^{-x-y} = \epsilon_F(x + y). \end{aligned}$$

Now the Chern character is the usual Chern character for K-theory. More precisely, it suffices to check for finite dimensional projective space $\mathbb{P}^n$, the hyperplane $H = \mathbb{P}^{n-1}$ is cut by a section of $\mathcal{O}(1)$, i.e. we have

$$0 \longrightarrow \mathcal{O}(-1) \longrightarrow \mathcal{O} \longrightarrow \mathcal{O}_H \longrightarrow 0.$$

This proves $[\mathcal{O}_{\mathbb{P}^{n-1}}] = 1 - \mathcal{O}(-1)$. By definition,

$$\mathrm{ch}([\mathcal{O}_{\mathbb{P}^{n-1}}]) = \epsilon_F([\mathbb{P}^{n-1}]) = 1 - e^{-[\mathbb{P}^{n-1}]} \in H^*(\mathbb{P}^n).$$

That is, $\mathrm{ch}(\mathcal{O}(-1)) = e^{c_1(\mathcal{O}(-1))}$, *which agrees with the classical Chern character.*

Now let us consider the formal group law with coefficients in $\mathbb{Z}[\beta]$

$$F(x, y) = x + y + \beta xy.$$

It has exponential

$$\epsilon_F = \beta^{-1}(1 - e^{\beta x}) = x + \beta \frac{x^2}{2!} + \beta^2 \frac{x^3}{3!} + \cdots .$$

The corresponding oriented cohomology is called the **connective K-theory**, since it connects cohomology ($\beta = 0$) and K-theory ($\beta = -1$):

We will take connective K-theory as an intermediate step of taking the limit from K-theory to cohomology. This will serve as an algebraic trick in the next section. We will denote

$$\begin{aligned} x \oplus y &= F(x, y) = x + y + \beta xy \\ x \ominus y &= F(x, \mathsf{inv}(y)) = \frac{x - y}{1 + \beta y}. \end{aligned}$$

2.4. Schubert and Grothendieck polynomials

2.4.1. Schubert polynomials. Let ∂_i be the **BGG Demazure operator**:

$$\partial_i f = \frac{f - f|_{x_i \leftrightarrow x_{i+1}}}{x_i - x_{i+1}}.$$

They satisfy the relations of the nil-Hecke algebra, that is,

$$\begin{aligned} \partial_i^2 &= 0 \\ \partial_i \partial_j &= \partial_j \partial_i \qquad |i-j| > 2 \\ \partial_i \partial_{i+1} \partial_i &= \partial_{i+1} \partial_i \partial_{i+1}. \end{aligned}$$

The **double Schubert polynomial** $\mathfrak{S}_w(x,t)$ for $w \in S_n$ is determined by the following two properties:

$$\mathfrak{S}_{w_0}(x,t) = \prod_{i+j \le n} (x_i - t_j);$$

$$\partial_i \mathfrak{S}_w(x,t) = \mathfrak{S}_{ws_i}(x,t), \qquad \text{if } ws_i < w.$$

Here $w_0 = n \cdots 21 \in S_n$ is the longest element. Since $\partial_i^2 = 0$, it follows that

$$\partial_i \mathfrak{S}_w(x,t) = \begin{cases} \mathfrak{S}_{ws_i}(x,t), & \text{if } ws_i < w, \\ 0, & \text{if } ws_i > w. \end{cases} \tag{2.4.1}$$

Setting $t_i = 0$ defines the **single Schubert polynomial**

$$\mathfrak{S}_w(x) = \mathfrak{S}_w(x,0).$$

PROPOSITION 2.4.1. *Under the convention of Section 2.3.1, the double Schubert polynomial*

$$\mathfrak{S}_w(x,t) = [Y(w)] \in H_T^*(G/B).$$

PROOF. We give a sketch of the proof, the proof is essential due to [**6**], and the detail can be found in [**1**]. Let us consider a partial flag variety

$$\mathcal{P} = \left\{ 0 = V_0 \subsetneq \cdots \subsetneq V_{i-1} \subsetneq V_{i+1} \cdots \subsetneq V_n = \mathbb{C}^n : \dim V_k = k \right\}.$$

We have an obvious forgetful map

$$\pi : \mathcal{F}\ell_n \to \mathcal{P}$$

by forgetting the i-th subspace V_i in $V_\bullet \in \mathcal{F}\ell_n$. Note that π is a $\mathbb{P}^1$-bundle. We can consider the composition, known as **push-pull operators**

$$\Delta : H_T^*(\mathcal{F}\ell_n) \xrightarrow{\pi_*} H_T^{*-2}(\mathcal{P}) \xrightarrow{\pi^*} H_T^{*-2}(\mathcal{F}\ell_n).$$

On one hand, one can check ∂_i coincides with Δ, under the convention of Section 2.3.1. Using spectral sequence, we have

$$H_T^*(\mathcal{F}\ell_n) \cong H_T^*(\mathcal{P}) \oplus H_T^*(\mathcal{P}) \cdot x_i$$

and the Δ corresponds to the projection to the first component. Since $H_T^*(\mathcal{P})$ could be viewed as the subalgebra which is invariant under the change of variables $x_i \leftrightarrow x_{i+1}$, it is direct to check that the operator ∂_i does the same job. This can also be seen from localization.

On the other hand, we have

$$\pi^{-1}(\pi(Y(w))) = \begin{cases} Y(ws_i), & ws_i < w, \\ Y(w), & ws_i > w. \end{cases}$$

Moreover, $Y(w)$ is birational to $\pi(Y(w))$ only if $ws_i < w$. This proves

$$\Delta([Y(w)]) = \begin{cases} [Y(ws_i)], & ws_i < w, \\ 0, & ws_i > w. \end{cases}$$

It remains to show the Proposition when $w = w_0$. The variety $Y(w_0)$ is the fixed point w_0. The class is characterized by

$$[Y(w_0)]|_w = \begin{cases} \prod_{i<j}(t_j - t_i), & w = w_0, \\ 0, & w \neq w_0. \end{cases}$$

It is direct to check that

$$\mathfrak{S}_{w_0}(wt, t) = \prod_{i+j\leq n} (t_{w(i)} - t_j) = \begin{cases} \prod_{i<j}(t_j - t_i), & w = w_0, \\ 0, & w \neq w_0. \end{cases}$$

So we have $\mathfrak{S}_{w_0}(x, y) = [Y(w_0)] \in H_T^*(G/B)$. ☐

Definition 2.4.1. For $w \in S_n$, we define

$$\partial_w = \partial_{i_1} \cdots \partial_{i_\ell}$$

for any reduced word $w = s_{i_1} \cdots s_{i_\ell}$.

This is well-defined since the ∂_i's satisfy braid relations. Then we have

$$\mathfrak{S}_w(x, t) = \partial_{w^{-1}w_0} \mathfrak{S}_{w_0}(x, t).$$

2.4.2. Stability of Schubert polynomials. Let us consider the sequence

$$S_1 \subset S_2 \subset S_3 \subset \cdots \subset S_n \subset \cdots$$

where we identify $S_n \subset S_{n+1}$ as the subgroup preserving the element $n+1$. The union of them

$$S_\infty = \bigcup_{n\geq 1} S_n = \{\text{bijective } \mathbb{Z} \xrightarrow{f} \mathbb{Z} : m \gg 0 \Rightarrow f(m) = m\}.$$

We will show $\mathfrak{S}_w(x, t)$ is defined for $w \in S_\infty$.

Proposition 2.4.2. *The Schubert polynomial $\mathfrak{S}_w(x, t)$ does not depend on the choice of n with $w \in S_n$.*

Proof. Let $w_0^{(n)}$ be the longest element of S_n. It suffices to show

$$\mathfrak{S}_{w_0^{(n-1)}}(x, t) = \prod_{i+j\leq n-1} (x_i - t_j).$$

By definition, we have

$$\mathfrak{S}_{w_0^{(n-1)}}(x, t) = \partial_{n-1} \cdots \partial_1 \prod_{i+j\leq n} (x_i - t_j).$$

Applying the following identity

$$\begin{aligned} \partial_i \left(\prod_{j=1}^{k+1}(x_i - t_j) \prod_{j=1}^{k}(x_{i+1} - t_j) \right) &= \left(\prod_{j=1}^{k}(x_i - t_j) \prod_{j=1}^{k}(x_{i+1} - t_j) \right) \pi_i(x_i - t_j) \\ &= \prod_{j=1}^{k}(x_i - t_j) \prod_{j=1}^{k}(x_{i+1} - t_j), \end{aligned}$$

we get the result. □

The following Proposition is well-known, and it was first pointed out in [**10**].

PROPOSITION 2.4.3. *We have*

$$\mathfrak{S}_w|_{x_i \mapsto t_i} = \begin{cases} 1, & w = \mathsf{id}, \\ 0, & w \neq \mathsf{id}. \end{cases}$$

PROOF. For any $f \in \mathbb{Q}[t_1, \ldots, t_n, x_1, \ldots, x_n]$, let us introduce

$$\operatorname{supp} f = \{w \in S_n : f|_{x_i \mapsto t_{w(i)}} \neq 0\}.$$

Then direct computation shows

$$\operatorname{supp}(\partial_i f) \subseteq \operatorname{supp}(\partial_i f) \cup \operatorname{supp}(\partial_i f) s_i.$$

Since $\operatorname{supp}(\mathfrak{S}_{w_0}) = \{w_0\}$, we can obtain by induction that

$$\operatorname{supp}(\mathfrak{S}_w) \subseteq \{u \in S_n : u \geq w\}.$$

In particular, $\mathfrak{S}_w|_{x_i \mapsto t_i} = 0$ if $w \neq \mathsf{id}$.

When $w = \mathsf{id}$, we need to notice that id is the longest element of S_1. By the Proposition above, we have $\mathfrak{S}_{\mathsf{id}} = 1$. □

PROPOSITION 2.4.4. *We have*

$$(\partial_w \mathfrak{S}_u)|_{x_i \mapsto t_i} = \begin{cases} 1, & u = w, \\ 0, & u \neq w. \end{cases}$$

PROOF. Note that

$$\partial_w \mathfrak{S}_u = 0 \text{ or } \mathfrak{S}_{uw^{-1}}.$$

Applying Proposition 2.4.3, we get the assertion. □

PROPOSITION 2.4.5. *The double Schubert polynomials $\mathfrak{S}_w$ for $w \in S_\infty$ form a $\mathbb{Q}[t_1, t_2, \ldots]$-basis of $\mathbb{Q}[t_1, t_2, \ldots][x_1, x_2, \ldots]$.*

PROOF. Let us define

$$c_w : \mathbb{Q}[t_1, t_2, \ldots][x_1, x_2, \ldots] \to \mathbb{Q}[t_1, t_2, \ldots], \qquad f \mapsto (\partial_w f)|_{x_i \mapsto t_i}.$$

Note that

$$c_w(\mathfrak{S}_u) = (\partial_w \mathfrak{S}_u)_{x_i \mapsto t_i} = \delta_{u,w}.$$

This proves the linear independence.

Claim. For any $f \in \mathbb{Q}[t_1, t_2, \ldots][x_1, x_2, \ldots]$, there are only finitely many $w \in S_\infty$ with $c_w(f) \neq 0$.

Note that f involves finitely many variables, say $f \in \mathbb{Q}[t_1, t_2, \ldots][x_1, \ldots, x_m]$. We have $\partial_i f = 0$ if f does not contain x_i and x_{i+1}. So $c_w(f) \neq 0$ only when $ws_i > w$ for $i > m$. Note that $c_w(f) = 0$ if $\ell(w) > \deg(f)$. Now the claim follows from

$$\#\{w \in S_\infty : \ell(w) < \ell, ws_i > w \text{ for } i > m\} < \infty$$

for a given ℓ and m.

Claim. For any $f \in \mathbb{Q}[t_1, t_2, \ldots][x_1, x_2, \ldots]$,

$$c_w(f) = 0 \text{ for all } w \in S_\infty \Rightarrow f = 0.$$

By induction it is easy to see if w is minimal in $\operatorname{supp}(f)$, then $\mathsf{id} \in \operatorname{supp}(\partial_w f)$. As a result, $f|_{x_i \mapsto t_{w(j)}} = 0$ for any $w \in S_\infty$. Pick an $n \gg 0$ such that f only involves $x_1, \ldots, x_n, t_1, \ldots, t_n$. Consider the involution $\tau \in S_{2n}$ with $\tau(i) = n + i$ and $\tau(n+i) = i$. Then $f|_{x_i \mapsto t_{w(j)}} = 0$ implies $f = 0$.

The two claims imply double Schubert polynomials $\mathfrak{S}_w$ for $w \in S_\infty$ span the polynomial ring $\mathbb{Q}[t_1, t_2, \ldots][x_1, x_2, \ldots]$. □

Lemma 2.4.6. *For $w \in S_\infty \setminus S_n$, we have*

$$\mathfrak{S}_w|_{x_i \mapsto t_{u(i)}} = 0$$

for all $u \in S_n$.

Proof. Let us define

$$\operatorname{supp}(f) = \{x \in S_\infty : f|_{x_i \mapsto t_{u(i)}} \neq 0\}.$$

We know from the proof above that for $m \geq 0$

$$\operatorname{supp}(\mathfrak{S}_w) \cap S_m \subseteq \{x \in S_m : x \geq w\}.$$

By taking $m \to \infty$, we get

$$\operatorname{supp}(\mathfrak{S}_w) \subseteq \{x \in S_\infty : x \geq w\}.$$

Note that $\operatorname{supp}(\mathfrak{S}_w)$ contains no element of S_n. The proposition follows. □

Proposition 2.4.7. *For $w \in S_\infty \setminus S_n$, the class*

$$\mathfrak{S}_w|_{x_{n+1} = t_{n+1} = \cdots = 0} = 0 \in H_T^*(\mathcal{F}\ell_n).$$

Proof. It follows immediately from the localization theorem that

$$H_T^*(\mathcal{F}\ell_n) \longrightarrow H_T^*(W) = \bigoplus_{w \in W} H_T^*(wB/B)$$

is injective. The map sends x_i to $(t_{w(i)})_{w \in W}$. □

Corollary 2.4.8. *If*

$$\mathfrak{S}_u(x,t)\mathfrak{S}_v(x,t) = \sum_{w \in S_\infty} c_{u,v}^w(t)\mathfrak{S}_w(x,t),$$

then in $H_T^*(\mathcal{F}\ell_n)$,

$$[Y(u)] \cdot [Y(v)] = \sum_{w \in S_n} c_{u,v}^w(t)[Y(w)].$$

2.4.3. Skew operators. Let us explain the skew operators introduced by Macdonald [**35**, Chapter II].

Definition 2.4.2. Let us define skew operators $\partial_{w/u}$ for $u, w \in S_n$ by

$$\partial_w(fg) = \sum_{u \in S_n} \partial_u(f)\partial_{w/u}(g).$$

Example 2.4.9. *For example,*

$$\partial_i(fg) = \partial_i(f)s_i(g) + f\partial_i(g).$$

So

$$\partial_{s_i/\,\mathsf{id}} = \partial_i, \qquad \partial_{s_i/s_i} = s_i.$$

By induction, it is easy to see

$$\partial_{w/\,\mathsf{id}} = \partial_w, \qquad \partial_{w/w} = w.$$

Proposition 2.4.10. *Let us fix a reduced word $w = s_{i_1}\cdots s_{i_\ell}$. We have the following formula for the skew operator*

$$\partial_{w/u} = \sum_J \prod_{j=1}^{\ell} \begin{cases} s_{i_j}, & j \in J, \\ \partial_{i_j}, & j \notin J, \end{cases}$$

where the sum is over all reduced subwords J for u.

Example 2.4.11. *Assume $u < ut_{ij} = w$ with $\ell(w) = \ell(u)$ where t_{ij} is the transposition of $i < j$. Then we have $\partial_{u/w} = \partial_{ij}$ the operator with*

$$\partial_{ij} f = \frac{f - f|_{x_i \leftrightarrow x_j}}{x_i - x_j}.$$

Proposition 2.4.12. *If*

$$\mathfrak{S}_u(x,t)\mathfrak{S}_v(x,t) = \sum_{w \in S_\infty} c^w_{u,v}(t)\mathfrak{S}_w(x,t),$$

then

$$c^w_{u,v}(t) = \partial_{w/u}(\mathfrak{S}_v)|_{x_i \mapsto t_i}.$$

Proof. Recall that $c_w : f \mapsto (\partial_w f)|_{x_i \mapsto t_i}$. We thus have

$$c_w(fg) = \sum_{u \in S_n} c_u(f)(\partial_{w/u} g)|_{x_i \mapsto t_i}.$$

Apply this formula to $\mathfrak{S}_u(x,t)\mathfrak{S}_v(x,t) = \sum_{w \in S_\infty} c^w_{u,v}(t)\mathfrak{S}_w(x,t)$, we get immediately the assertion. ☐

Corollary 2.4.13. *We have*

$$\partial_w(fg) = \sum_{u,v} c^w_{u,v}(x)\partial_u(f)\partial_v(g)$$

Proof. We can expand $\partial_{w/u} = \sum_v f^w_{u,v}(x)\partial_v$. Applying this to $\mathfrak{S}_v$, we get $f^w_{u,v}(x) = c^w_{u,v}(x)$ from the Proposition above. ☐

2.4.4. Grothendieck polynomials. In this section, we will fix β as the parameter of the connective K-theory in Section 2.3.3. Recall the notation introduced in Section 2.3.3

$$x \ominus y = \frac{x - y}{1 + \beta y}.$$

Let π_i be the **Demazure operator**:

$$\pi_i f = \frac{(1 + \beta x_{i+1})f - (1 + \beta x_i)f|_{x_i \leftrightarrow x_{i+1}}}{x_i - x_{i+1}}.$$

They satisfy the relations of 0-Hecke algebra

$$\pi_i^2 = -\beta\pi$$
$$\pi_i\pi_j = \pi_j\pi_i \qquad |i-j| > 2$$
$$\pi_i\pi_{i+1}\pi_i = \pi_{i+1}\pi_i\pi_{i+1}.$$

The **double Grothendieck polynomial** $\mathfrak{G}_w(x,t)$ for $w \in S_n$ is determined by the following two properties:

$$\mathfrak{G}_{w_0}(x,t) = \prod_{i+j\le n}(x_i \ominus t_j);$$

$$\pi_i\mathfrak{G}_w(x,t) = \mathfrak{G}_{ws_i}(x,t), \qquad \text{if } ws_i < w.$$

Here $w_0 = n\cdots 21 \in S_n$ is the longest element. Since $\pi_i^2 = -\beta\pi_i$, it follows that

$$\pi_i\mathfrak{G}_w(x,t) = \begin{cases} \mathfrak{G}_{ws_i}(x,t), & \text{if } ws_i > w, \\ -\beta\mathfrak{G}_w(x,t), & \text{if } ws_i < w. \end{cases} \tag{2.4.2}$$

Letting $t_i = 0$ defines the **single Grothendieck polynomial**

$$\mathfrak{G}_w(x) = \mathfrak{G}_w(x,0).$$

Setting $\beta = 0$, we get the **double (resp., single) Schubert polynomial**

$$\mathfrak{S}_w(x,t) = \mathfrak{G}_w(x,t)|_{\beta=0}, \qquad (\text{resp., } \mathfrak{S}_w(x) = \mathfrak{G}_w(x)|_{\beta=0}).$$

PROPOSITION 2.4.14. *Similarly, under the convention of Section 2.3.2, the double Grothendieck polynomial satisfies*

$$\mathfrak{G}_w(x,t)|_{\beta=-1} = [\mathcal{O}_{Y(w)}] \in K_T(G/B).$$

All the properties on double Schubert polynomials in the previous section can be generalized to double Grothendieck polynomials.

PROPOSITION 2.4.15. *We have*

$$\mathfrak{G}_w|_{x_i\mapsto t_i} = \begin{cases} 1, & w = \mathsf{id}, \\ 0, & w \neq \mathsf{id}. \end{cases}$$

PROOF. The proof is the same as Proposition 2.4.3. □

Let us consider

$$\hat{\pi}_i = \pi + \beta.$$

Then

$$\hat{\pi}_i^2 = \beta\hat{\pi}$$
$$\hat{\pi}_i\hat{\pi}_j = \hat{\pi}_j\hat{\pi}_i \qquad |i-j| > 2$$
$$\hat{\pi}_i\hat{\pi}_{i+1}\pi_i = \hat{\pi}_{i+1}\hat{\pi}_i\hat{\pi}_{i+1}.$$

PROPOSITION 2.4.16. *We have*

$$(\hat{\pi}_w\mathfrak{G}_u)|_{x_i\mapsto t_i} = \begin{cases} 1, & u = w, \\ 0, & u \neq w. \end{cases}$$

Proof. If $\ell(w) \leq \ell(u)$, the proof is the same as the case of Schubert polynomials, since

$$\hat{\pi}_w = \pi_w + \mathrm{span}(\pi_u : u < w).$$

The remaining case follows by induction on w on the following statement

$$\forall u \in S_n, \qquad \ell(w) > \ell(u) \Rightarrow (\hat{\pi}_w \mathfrak{G}_u)|_{x_i \mapsto t_i} = 0.$$

Assume $ws_i < w$. Then

$$\begin{aligned} \hat{\pi}_w \pi_{u^{-1}w_0} &= \hat{\pi}_{ws_i}(\pi_i + \beta)\pi_{u^{-1}w_0} \\ &= \hat{\pi}_{ws_i} \begin{cases} \pi_{s_i u^{-1} w_0} + \beta \pi_{u^{-1}w_0}, & us_i < u, \\ 0, & us_i > u. \end{cases} \end{aligned}$$

Thus

$$\hat{\pi}_w \mathfrak{G}_u = \hat{\pi}_{ws_i} \begin{cases} \mathfrak{G}_{us_i} + \beta \mathfrak{G}_u, & us_i < u, \\ 0, & us_i > u. \end{cases}$$

If $us_i > u$, there is nothing to prove, so let us assume $us_i < u$. We have

$$\hat{\pi}_{ws_i}(\mathfrak{G}_{us_i})|_{x_i \mapsto t_i} = 0$$

by induction. If $\hat{\pi}_{ws_i}\beta\mathfrak{G}_u \neq 0$, then by induction, this only happens when $u = ws_i$. But this is impossible since $ws_i < w$ and $us_i < u$. □

We can define similarly the skew operator $\hat{\pi}_{w/u}$ for $u, w \in S_n$ by

$$\hat{\pi}_w(fg) = \sum_{u \in S_n} \hat{\pi}_u(f) \hat{\pi}_{w/u}(g).$$

Let us fix a reduced word $w = s_{i_1} \cdots s_{i_\ell}$. We have the following formula for skew operator

$$\hat{\pi}_{w/u} = \sum_J \beta^{|J| - \ell(u)} \prod_{j=1}^{\ell} \begin{cases} s_{i_j}, & j \in J, \\ (1 + \beta x_i)\partial_{i_j}, & j \notin J, \end{cases}$$

where the sum over all subwords such that $\prod_{j \in J} \pi_j = \beta^{|J| - \ell(u)} \pi_u$.

Proposition 2.4.17. *If*

$$\mathfrak{G}_u(x,t)\mathfrak{G}_v(x,t) = \sum_{w \in S_\infty} c_{u,v}^w(t) \mathfrak{G}_w(x,t),$$

then

$$c_{u,v}^w(t) = \hat{\pi}_{w/u}(\mathfrak{G}_v)|_{x_i \mapsto t_i}.$$

Remark 2.4.18. The algebra generated by ∂_i and π_i's are all Hecke type algebras. In [**37**], we study its generalization to all formal group laws. The skew operators, combinatorially defined above can be lifted to the existence of coproduct structure. The formulas in Proposition 2.4.12 and Proposition 2.4.17 are known as reconstruction formula loc.cit..

2.5. Examples of Grothendieck polynomials

This section should be compared with Section 2.2.

2.5.1. Example: $n = 2$. When $n = 2$, we can compute directly

$$\mathfrak{G}_{w_0} = x_1 \ominus y_1 = \frac{x_1 - y_1}{1 + \beta y_1}, \qquad \mathfrak{G}_{\mathsf{id}} = 1.$$

In particular,

$$\mathfrak{S}_{w_0} = x_1 - y_1, \qquad \mathfrak{S}_{\mathsf{id}} = 1.$$

Let us explain the geometric meaning of them. Recall that

$$\mathcal{F}\ell_2 = \mathbb{P}^1 = \mathbb{C} \cup \{\infty\}, \qquad Y(w_0) = \{\infty\}.$$

The infinity point $\{\infty\}$ is the zero of a global section of $\mathcal{O}(1) = \mathcal{V}_1^\vee$. More precisely, we can identify $H^0(\mathbb{P}^1, \mathcal{V}_1^\vee) = (\mathbb{C}^2)^*$, and $\infty = [0 : 1]$ is the zero of the first projection $\mathbb{C}^2 \to \mathbb{C}$. Equivariantly, we need to twist $\mathcal{V}_1^\vee$ by the trivial bundle $\mathbb{C}_{-z_1}$ with nontrivial torus weight to get a equivariant section, so

$$[\infty] = x_1 - y_1, \qquad [\mathcal{O}_\infty] = 1 - \mathcal{V}_1 \otimes [\mathbb{C}_{-z_1}] = x_1 \ominus y_1.$$

2.5.2. Example: $n = 3$. When $n = 3$, we can compute

$$\begin{aligned}
\mathfrak{G}_{321} &= (x_1 \ominus y_1)(x_1 \ominus y_2)(x_2 \ominus y_1), \\
\mathfrak{G}_{231} &= (x_1 \ominus y_1)(x_2 \ominus y_1), \\
\mathfrak{G}_{312} &= (x_1 \ominus y_1)(x_1 \ominus y_2), \\
\mathfrak{G}_{213} &= (x_1 \ominus y_1), \\
\mathfrak{G}_{132} &= (x_1 \oplus x_2) \ominus (y_1 \oplus y_2), \\
\mathfrak{G}_{123} &= 1.
\end{aligned}$$

Let us explain the geometry. Recall

$$\mathcal{F}\ell_3 = \{0 \subset V_1 \subset V_2 \subset \mathbb{C}^3 : \dim V_i = i\}.$$

We take $Y(132)$ and $Y(312)$ as an example. Recall $Y(132)$ can be described as the variety of flags $V_\bullet$ with $\mathbb{C}e_3 \subset \mathcal{V}_2$. This condition can be described as the vanishing of the morphism

$$\mathcal{O}e_3 \xrightarrow{\subset} \mathcal{O}_{\mathcal{F}\ell_3}^{\oplus 3} \longrightarrow \mathcal{O}_{\mathcal{F}\ell_3}^{\oplus 3}/\mathcal{V}_2.$$

As a result,

$$\begin{aligned}
[Y(132)] &= c_1(\mathcal{O}_{\mathcal{F}\ell_3}^{\oplus 3}/\mathcal{V}_2) - c_1(\mathcal{O}e_3) = -x_3 + y_3 = x_1 + x_2 - y_1 - y_2 \\
[\mathcal{O}_{Y(132)}] &= 1 - \mathcal{O}_{\mathcal{F}\ell_3}^{\oplus 3}/\mathcal{V}_2 \otimes (\mathcal{O}e_3)^\vee = y_3 \ominus x_3 = (x_1 \oplus x_2) \ominus (y_1 \oplus y_2).
\end{aligned}$$

Similarly, $Y(312)$ can be described as the variety of flags $V_\bullet$ with $\mathcal{V}_1 = \mathbb{C}e_3$. This condition is equivalent to the vanishing of

$$\mathcal{V}_1 \xrightarrow{\subset} \mathcal{O}_{\mathcal{F}\ell_3}^{\oplus 3} \longrightarrow \mathcal{O}_{\mathcal{F}\ell_3}^{\oplus 3}/\mathcal{O}e_3.$$

As a result,

$$\begin{aligned}
[Y(312)] &= \text{Euler class of } \mathcal{O}_{\mathcal{F}\ell_3}^{\oplus 3}/\mathcal{O}e_3 \otimes \mathcal{V}_1^\vee \\
&= (x_1 - y_1)(x_1 - y_2) \\
[\mathcal{O}_{Y(312)}] &= \text{Koszul complex of } \mathcal{O}_{\mathcal{F}\ell_3}^{\oplus 3}/\mathcal{O}e_3 \otimes \mathcal{V}_1^\vee \\
&= (x_1 \ominus y_1)(x_1 \ominus y_2).
\end{aligned}$$

2.5.3. Schubert divisor. We have

$$\mathfrak{G}_{s_k} = (x_1 \oplus \cdots \oplus x_k) \ominus (y_1 \oplus \cdots \oplus y_k).$$

Recall

$$Y(s_k) = \{V_\bullet \in \mathcal{F}\ell_n : \dim(V_k \cap \phi^-_{n-k}) \geq 1\}.$$

The condition can be restated as the degeneration of

$$\mathcal{V}_k \longrightarrow \mathcal{O}^{\oplus n}_{\mathcal{F}\ell_n} \longrightarrow \mathcal{O}^{\oplus n}_{\mathcal{F}\ell_n}/\phi^-_{n-k}.$$

That is, the vanishing of

$$\det \mathcal{V}_k \longrightarrow \det(\mathcal{O}^{\oplus n}_{\mathcal{F}\ell_n}/\phi^-_{n-k}).$$

As a result,

$$\begin{aligned}
[Y(s_k)] &= c_1(\det(\mathcal{O}^{\oplus n}_{\mathcal{F}\ell_n}/\phi^-_{n-k})) - c_1(\det \mathcal{V}_k) \\
&= x_1 + \cdots + x_k - y_1 - \cdots - y_k, \\
[\mathcal{O}_{Y(s_k)}] &= 1 - \det \mathcal{V}_k \otimes \det(\mathcal{O}^{\oplus n}_{\mathcal{F}\ell_n}/\phi^-_{n-k}) \\
&= (x_1 \oplus \cdots \oplus x_k) \ominus (y_1 \oplus \cdots \oplus y_k).
\end{aligned}$$

2.5.4. Dominant permutations. Let $w = \overline{w}_h$ be the dominant permutation defined by a Hessenberg function h. We have

$$\mathfrak{G}_w = \prod_{i=1}^{n} \prod_{j=1}^{n-h(i)} (x_i \ominus y_j).$$

Let us explain its geometric meaning. Recall that

$$Y(\overline{w}_h) = \{V_\bullet \in \mathcal{F}\ell_n : V_i \subseteq \phi^-_{h(i)}\}.$$

Let us construct

$$Y_k = \{V_\bullet \in \mathcal{F}\ell_n : V_i \subseteq \phi^-_{h(i)} \text{ for all } i \leq k\}.$$

We prove by induction that

$$\mathcal{O}_{Y_j} = \prod_{i=1}^{k} \prod_{j=1}^{n-h(i)} (x_i \ominus y_j).$$

Note that $Y_k \subseteq Y_{k-1}$ is the subvariety with $V_k \subseteq \phi^-_{h(k)}$, i.e.

$$\mathcal{V}_k \xrightarrow{\subset} \mathcal{O}^{\oplus n} \longrightarrow \mathcal{O}^{\oplus n}/\phi^-_{h(k)}$$

should vanish. Note that the restriction of this morphism to $\mathcal{V}_{k-1}$ already vanishes. So $\mathcal{O}_{Y_k}$ is the zero locus of

$$\mathcal{V}_k/\mathcal{V}_{k-1} \longrightarrow \mathcal{O}^{\oplus n}/\phi^-_{h(k)}.$$

Thus

$$\begin{aligned}
[\mathcal{O}_{Y_k}] &= [\mathcal{O}_{Y_{k-1}}] \cdot \Big(\text{Koszul complex of } (\mathcal{V}_k/\mathcal{V}_{k-1})^\vee \otimes (\mathcal{O}^{\oplus n}/\phi^-_{h(k)})\Big) \\
&= [\mathcal{O}_{Y_{k-1}}] \cdot \prod_{j=1}^{n-h(j)} (x_k \ominus y_j).
\end{aligned}$$

CHAPTER 3

Triple Pieri Rule

In this chapter, we will compute the coefficients

$$\mathfrak{S}_u(x,y)\cdot\mathfrak{S}_v(x,t)=\sum_w \bar{c}^w_{u,v}(t,y)\cdot\mathfrak{S}_w(x,t), \tag{3.0.1}$$

when u takes the form $s_a s_{a+1}\cdots s_{b-1}s_b$. We will call the formula the **triple Pieri rule**. We will show how to establish the triple Pieri rule from the single Pieri rule. That is,

$y=t=0$		$y=0$		arbitrary
classical Pieri rule	$\Rightarrow$	single times double	$\Rightarrow$	triple Pieri rule

In Section 3.1, we will review the classical Pieri rule, where we recall k-Bruhat order. In Section 3.2, we will establish the first implication, where the rigidity theorem in [**16**] plays an essential role. In Section 3.3, we will establish the second implication, by expanding double Schubert polynomial into single Schubert polynomials.

3.1. Classical Pieri Rule

Let us fix an k with $1\le r\le k\le n-1$. Let $u=s_{k-r+1}\cdots s_{k-1}s_k$. Then the single Schubert polynomial is

$$\mathfrak{S}_u(x)=e_r(x_1,\dots,x_k)=e_r(x_{[k]})=\sum_{1\le i_1<\cdots<i_r\le k} x_{i_1}\cdots x_{i_r}.$$

Let us recall the following classical result. In the following, we will use t_{ab} as the transposition of a and b.

THEOREM 3.1.1 (Pieri rule [**45**], [**30**]). *Let $v\in S_n$. We have*

$$e_r(x_{[k]})\cdot\mathfrak{S}_v(x)=\sum_w \mathfrak{S}_w(x),$$

where the sum ranges over all $w\in S_n$ such that $w=vt_{a_1b_1}\cdots t_{a_rb_r}$ with (1) $a_i\le k<b_i$; (2) a_i's are distinct; (3) $\ell(vt_{a_1b_1}\cdots t_{a_ib_i})=\ell(v)+i$.

This rule can be equivalently stated using k-Bruhat order on S_n. Let us define

$$u\lessdot_k w \iff w=ut_{ab} \text{ for some } a\le k<b \text{ and } \ell(w)=\ell(u)+1.$$

The k-Bruhat order the partial order generated by $\lessdot_k$, i.e.

$$u\le_k w \iff \text{there is a chain } u\lessdot_k u'\lessdot_k\cdots\lessdot_k w.$$

Figure 1 shows the order over S_3 where $k=1,2$. Figures 2 and 3 show the the order over S_4 where $k=1,2$.

This was implemented using the following SageMath code

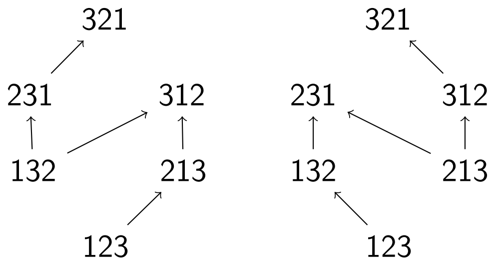


FIGURE 1. k-Bruhat order of S_3

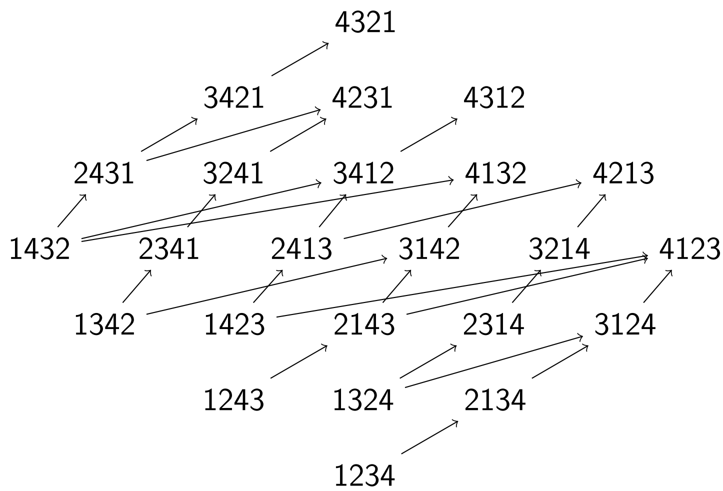


FIGURE 2. 1-Bruhat order of S_4

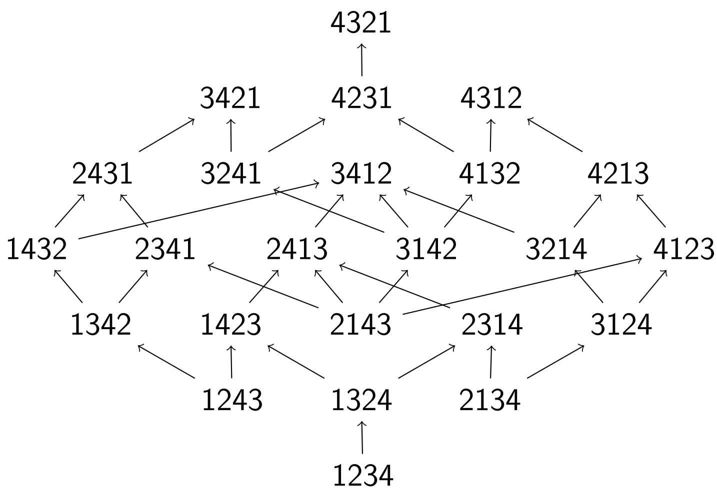


FIGURE 3. 2-Bruhat order of S_4

```
n = 4; k = 2; W = Permutations(n)
def t(a,b):
    res = [1..n]
```

```
    res[a-1],res[b-1]=b,a
    return W(res)
def k_cover(u):
    res= {}
    for a in [1..k]:
        for b in [k+1..n]:
            w=t(a,b)*u
            if w.length()==u.length()+1:
                res[w]=u(a)
    return res
def show(w):
    return "".join("%s"%w(i) for i in [1..n])
H = {}
for w in W:
    ell = w.length()
    if ell not in H.keys(): H[ell]=[]
    H[ell].append(w)
# H = {i:sorted(H[i]) for i in H}

D = DiGraph({w:k_cover(w) for w in W})
D.layout(heights=H,layout="ranked",save_pos=True)

res = ""
Nodes = D.get_pos()
for w in Nodes:
    res+="\\node (%s) at (%.2f,%.2f) {\\(\\sf%s\\)};"%(show(w),Nodes[w][0]*2,Nodes[w][1],show(w))
res+="\n"
for u in W:
    for w in k_cover(u):
        res+= "\\draw[->] (%s) to (%s);"%(show(u),show(w))
res = "$$\\begin{tikzpicture}\n"+res+"\n\\end{tikzpicture}$$"
print(res)

D.plot(vertex_labels=show,vertex_size=200,arrowsize=1,vertex_shape="")
```

THEOREM 3.1.2 (Pieri rule). *Let $v \in S_n$. We have*

$$e_r(x_{[k]}) \cdot \mathfrak{S}_v(x) = \sum_w \mathfrak{S}_w(x), \tag{3.1.1}$$

where the sum ranges over all $w \in S_n$ such that there is path

$$v \longrightarrow vt_{a_1b_1} \longrightarrow vt_{a_1b_1}t_{a_2b_2} \longrightarrow \cdots \longrightarrow vt_{a_1b_1}\cdots t_{a_rb_r} = w$$

from v to w in the k-Bruhat order, and $a_1, a_2, \ldots, a_r$ are distinct.

In [**41**], a new formulation of Pieri rule was found. Let us introduce

$$u \xrightarrow{\tau} w \iff u <_k w = ut_{ab} \text{ and } \tau = u(a) \text{ for some } a < b.$$

We call a chain

$$\gamma : u_0 \xrightarrow{\tau_1} u_1 \xrightarrow{\tau_2} \cdots \xrightarrow{\tau_\ell} u_\ell$$

is decreasing if $\tau_1 > \tau_2 > \cdots > \tau_\ell$. We denote $\mathrm{end}(\gamma) = u_\ell$ and $\ell(\gamma) = \ell$.

THEOREM 3.1.3 (Pieri rule). *Let $v \in S_n$. We have*

$$e_r(x_{[k]}) \cdot \mathfrak{S}_v(x) = \sum_\gamma \mathfrak{S}_{\mathrm{end}(\gamma)}(x), \tag{3.1.2}$$

where the sum ranges over all decreasing chains γ all decreasing paths starting at u of length $\ell(\gamma) = r$.

COROLLARY 3.1.4. *For $v, w \in S_n$, there exists at most one decreasing path starting from v to w, and it exists if and only if the condition in Theorem 3.1.1 is satisfied.*

3.2. Rigidity Theorem

In this section, we will review the main method of [**16**]. The key step is the following Rigidity Theorem stating that the coefficients of the equivariant Pieri rule are controlled by the single Pieri rule.

THEOREM 3.2.1 (Rigidity Theorem [**16**, Section 4]). *Let $v \in S_n$ and a k-subset $A \subset [n]$. Suppose that*

$$e_r(x_A)\mathfrak{S}_v(x,t) = \sum_{w \in S_n} c^w_{u,r}(t) \cdot \mathfrak{S}_w(x,t). \tag{3.2.1}$$

Then we have

$$c^w_{u,r}(t) = \sum_{r' \le r} c^w_{u,r'}(0) \cdot e_{r-r'}(t_{\Delta_A(u,w)}) \tag{3.2.2}$$

where $\Delta_A(u,w) := \{u(i) \colon i \in A \text{ and } u(i) \neq w(i)\}$.

Let us denote

$$\Delta_k(u,w) = \Delta_{[k]}(u,w) = \{u(i) : 1 \le i \le k, u(i) \neq w(i)\}.$$

Applying this Theorem, we obtain the following "single times double" Pieri rule.

COROLLARY 3.2.2. *Let $v \in S_n$. We have*

$$e_r(x_{[k]}) \cdot \mathfrak{S}_v(x,t) = \sum_{\gamma} e_{r-\ell(\gamma)}(t_{\Delta_k(v,\mathrm{end}(\gamma))}) \cdot \mathfrak{S}_{\mathrm{end}(\gamma)}(x,t),$$

where γ runs over all decreasing paths starting at u in the extended k-Bruhat order. Here, $\ell(\gamma)$ denotes the length of γ, namely, the nunber of edges in γ.

The remaining of this section is devoted to a proof of Theorem 3.2.1. The key step is a translation of the coefficients in term of skew operators in Section 2.4.3.

LEMMA 3.2.3. *We have*

$$c^w_{u,r}(x) = \partial_{w/v} e_r(x_{[k]}).$$

PROOF. Recall that $c_w : f \mapsto (\partial_w f)|_{x_i \mapsto t_i}$ gives the coefficient of $\mathfrak{S}_w$ in the double Schubert expansion of f. We thus have

$$c_w(fg) = \sum_{v \in S_n} c_v(f)(\partial_{w/v} g)|_{x_i \mapsto t_i}.$$

Apply this formula to the left-hand side of (3.2.1), we will get

$$c^w_{u,r}(t) = \partial_{w/v} e_r(x_{[k]})|_{x_i \mapsto t_i}.$$

Since $\partial_{w/v} e_r(x_{[k]})$ only involves variable $x = (x_1, x_2, \ldots)$, the assertion is proved. $\square$

Let us introduce a generating function for a subset A of $\{1,\ldots,n\}$

$$Q(x_A)=\prod_{a\in A}(1+qx_a)=\sum_{r\geq 0}q^r e_r(x_A).$$

The following lemma follows from direct computation.

LEMMA 3.2.4. *For $a\neq b$, we have*

$$\partial_{ab}Q(x_A)=(\delta_{a\in A}q-\delta_{b\in A}q)Q(x_{A\setminus\{a,b\}}). \tag{3.2.3}$$

PROOF. By definition

$$\partial_{ab}Q_{x_A}=\frac{1}{x_a-x_b}\big(Q(x_A)-Q(x_{A'})\big)$$

where

$$A'=\begin{cases}A, & a,b\in A \text{ or } a,b\notin A\\ A\setminus\{a\}\cup\{b\} & a\in A, b\notin A,\\ A\setminus\{b\}\cup\{a\} & a\notin A, b\in A.\end{cases}$$

In the first case, we have $\partial_{ab}Q(x_A)=0$. In the second case,

$$\partial_{ab}Q(x_A)=Q(x_{A\setminus\{a\}})\frac{(1+qx_a)-(1+qx_b)}{x_a-x_b}=qQ(x_{A\setminus\{a\}})=qQ(x_{A\setminus\{a,b\}}).$$

Similarly, in the third case,

$$\partial_{ab}Q(x_A)=Q(x_{A\setminus\{b\}})\frac{(1+qx_b)-(1+qx_a)}{x_a-x_b}=-qQ(x_{A\setminus\{b\}})=-qQ(x_{A\setminus\{a,b\}}).$$

The three cases can be summarized as $(\delta_{a\in A}q-\delta_{b\in A}q)Q(x_{A\setminus\{a,b\}})$. □

LEMMA 3.2.5. *Assume $w=t_{a_1b_1}\cdots t_{a_mb_m}\in S_n$. If*

$$\partial_{a_1b_1}\cdots\partial_{a_mb_m}Q(x_A)\neq 0 \tag{3.2.4}$$

for some subset A, then the non-fixed points

$$M(w):=\{1\leq i\leq n: w(i)\neq i\}=\{a_1,b_1,\ldots,a_m,b_m\}. \tag{3.2.5}$$

PROOF. It is obvious that $M(w)\subseteq\{a_1,b_1,\ldots,a_m,b_m\}$. We next prove the reverse inclusion by induction on m. If $m=0$, then w is the identity permutation, and both sides of (3.2.5) are empty.

Now consider the case $m>0$. Suppose to the contrary that $w(x)=x$ for some $x\in\{a_1,b_1,\ldots,a_m,b_m\}$. For $0\leq j\leq m-1$, let $w_j=t_{a_{j+1}b_{j+1}}\cdots t_{a_mb_m}$. As $x\in\{a_1,b_1,\ldots,a_m,b_m\}$, there must exist j such that $w_j(x)\neq x$. Let i be the smallest such j. Since $w_0(x)=w(x)=x$, we have $i>0$. Noticing that $w_{i-1}(x)=t_{a_ib_i}w_i(x)=x$ and $w_i(x)\neq x$, we obtain

$$\{a_i,b_i\}=\{x,w_i(x)\}. \tag{3.2.6}$$

On the other hand, by (3.2.4), we have

$$\partial_{a_{i+1}b_{i+1}}\cdots\partial_{a_mb_m}Q(x_A)\neq 0,$$

and so it follows by induction that

$$M(w_i)=\{a_{i+1},b_{i+1},\ldots,a_m,b_m\},$$

which along with Lemma 3.2.4 implies that

$$\partial_{a_{i+1}b_{i+1}}\cdots\partial_{a_mb_m}Q(x_A)\in\mathbb{Z}[q]Q(x_{A\setminus M(w_i)}). \tag{3.2.7}$$

Since $w_i(x) \neq x$, we see that $w_i(w_i(x)) \neq w_i(x)$. Thus both x and $w_i(x)$ belong to $M(w_i)$. In view of (3.2.6), both a_i and b_i belong to $M(w_i)$. Consequently,

$$\partial_{a_ib_i} Q(x_{A\setminus M(w_i)}) = 0,$$

which together with (3.2.7) would yield

$$\partial_{a_ib_i}\partial_{a_{i+1}b_{i+1}}\cdots\partial_{a_mb_m} Q(x_A) = 0,$$

contrary to the assumption in (3.2.4). This completes the proof. □

LEMMA 3.2.6. *For $v, w \in S_n$, we have*

$$\partial_{w/v} Q(x_A) \in \mathbb{Z}[q]\cdot Q(x_{\Delta_A(u,w)}). \tag{3.2.8}$$

PROOF. By Proposition 2.4.10, we can write

$$\partial_{w/v} = \sum_J \nabla_J, \qquad w_J = \prod_{j\in J} s_{i_j} \qquad \nabla_J = \prod_{j=1}^{\ell} \begin{cases} s_{i_j}, & j\in J,\\ \partial_{i_j}, & j\notin J,\end{cases} \tag{3.2.9}$$

where the sum over all reduced subword J for u.

For any $v \in \mathfrak{S}_n$, it is easy to check that

$$\partial_{ab}\, v = v\, \partial_{v^{-1}(a)v^{-1}(b)}.$$

Hence, for $J \subseteq [\ell]$ such that $w_J = u$, the Demazure operators appearing in ∇_J defined in (3.2.9) can be moved one by one to the rightmost side of w_J, and so we may assume that ∇_J takes the form

$$\nabla_J = u\,\partial_{a_1b_1}\cdots\partial_{a_mb_m},$$

where $m = \ell - \#J$. On the other hand, since $t_{ab}\, v = v\, t_{v^{-1}(a)v^{-1}(b)}$ for any $v \in \mathfrak{S}_n$, we can use exactly the same procedure with ∇_J to deduce that

$$w = u\, t_{a_1b_1}\cdots t_{a_mb_m},$$

or equivalently,

$$u^{-1}w = t_{a_1b_1}\cdots t_{a_mb_m}.$$

Combining Lemma 3.2.4 and Lemma 3.2.5, we obtain that

$$\partial_{a_1b_1}\cdots\partial_{a_mb_m} Q(x_A) \in \mathbb{Z}[q]\cdot Q\left(x_{A\setminus M(u^{-1}w)}\right). \tag{3.2.10}$$

Therefore,

$$\begin{aligned}\nabla_J\, Q(x_A) &= u\partial_{a_1b_1}\cdots\partial_{a_mb_m} Q(x_A)\\ &\in \mathbb{Z}[q]\cdot Q\left(x_{u(A\setminus M(u^{-1}w))}\right) = \mathbb{Z}[q]\cdot Q\left(x_{\Delta_A(u,w)}\right),\end{aligned}$$

yielding (3.2.8). □

PROOF OF THEOREM 3.2.1. Let

$$c_v^w(q,t) = \sum_{r\geq 0} q^r c_{v,r}^w(t). \tag{3.2.11}$$

Then

$$Q(x_A)\cdot \mathfrak{S}_v(x,t) = \sum_{r\geq 0} q^r e_r(x_A)\cdot \mathfrak{S}_v(x,t) = \sum_{w\in\mathfrak{S}_n} c_u^w(q,t)\cdot \mathfrak{S}_w(x,t).$$

By Lemma 3.2.3, we obtain that

$$c_v^w(q,t) = \partial_{w/v} Q(x_A)\big|_{x_i\mapsto t_i}. \tag{3.2.12}$$

By (3.2.12) and Lemma 3.2.6, we see that

$$c_u^w(q,t) = f(q)Q(t_{\Delta_A(u,w)}),$$

where $f(q) \in \mathbb{Z}[q]$. Setting all $t_i = 0$ on both sides, we obtain that $c_u^w(q,0) = f(q,0)$, and hence,

$$c_u^w(q,t) = c_u^w(q,0) \cdot Q(t_{\Delta_A(u,w)}) \tag{3.2.13}$$

$$= c_u^w(q,0) \cdot \sum_{r\geq 0} e_r(t_{\Delta_A(u,w)})q^r \tag{3.2.14}$$

Comparing the coefficients of q^r in (3.2.11) and (3.2.14), we are led to (3.2.2). □

3.3. Triple Pieri rule

Now let us prove the triple Pieri rule. Recall $u = s_{k-a+1}\cdots s_{k-1}s_k$.

THEOREM 3.3.1 (Triple Pieri rule). *Let $v \in S_n$. We have*

$$\mathfrak{S}_u(x,y) \cdot \mathfrak{S}_v(x,t) = \sum_w \overline{c}_{u,v}^w(t,y) \cdot \mathfrak{S}_w(x,t)$$

where the sum ranges over all $w \in S_n$ such that there exists a decreasing path of length $r' \leq r$ from u to w, and

$$\overline{c}_{uv}^w(t,y) = \mathfrak{S}_{s_{k-r+1}\cdots s_{k-r'}}(t,y)|_{t_i\mapsto t_{d(i)}}. \tag{3.3.1}$$

Here, $d \in S_n$ can be taken as any permutation in S_n such that the image of $[k-r']$ is exactly $\Delta_k(v,w)$.

The proof of Theorem 3.3.1 is very straightforward — it is obtained by expanding $\mathfrak{S}_u(x,y)$ into single $e_r(x_{[k]})$. To do this, we need to introduce the complete symmetric polynomial

$$h_r(x_1,\ldots,x_k) = \sum_{1\leq i_1\leq\cdots\leq i_r\leq k} x_{i_1}\cdots x_{i_r}. \tag{3.3.2}$$

The following expansion is well-known.

LEMMA 3.3.2. *We have*

$$\mathfrak{S}_u(x,y) = \sum_{i+j=a} e_i(x_{[k]}) \cdot h_j(-y_{[k-r+1]}).$$

PROOF. This is a special case of so-called Giambelli formula [**35**]

$$\mathfrak{S}_u(x,t) = \sum_{\substack{u=u_2^{-1}u_1 \\ \ell(u)=\ell(u_1)+\ell(u_2)}} \mathfrak{S}_{u_1}(x)\mathfrak{S}_{u_2}(-t).$$

Since $u = s_{k-r+1}\cdots s_{k-1}s_k$ has only one reduced word, the only possible decomposition is given by

$$u = \underbrace{s_{k-a+1}\cdots s_{k-i}}_{u_2^{-1}} \cdot \underbrace{s_{k-i+1}\cdots s_{k-1}s_k}_{u_1}.$$

The Lemma follows from the fact $\mathfrak{S}_{s_ks_{k+1}\cdots s_{k+r-1}}(x) = h_r(x_1,\ldots,x_k)$ in (3.3.2). □

Proof of Theorem 3.3.1. Let w be a permutation admitting a decreasing path from u to w of length $r' \le r$. By Corollary 3.2.2

$$\begin{aligned}
\overline{c}_{u,v}^{w}(t,y) &= \sum_{i+j=r} e_{i-r'}(t_{\Delta_k(v,w)}) \cdot h_j(-y_{[k-r+1]}) \\
&= \sum_{i+j=r-r'} e_i(t_{\Delta_k(v,w)}) h_j(-y_{[k-r+1]}) \\
&= \sum_{i+j=r-r'} e_i(t_{\Delta_k(v,w)}) h_j(-y_{[(k-r')-(r-r')+1]}) \\
&= \mathfrak{S}_{s_{k-r+1}\cdots s_{k-r'}}(t,y)|_{t_i \mapsto t_{d(i)}}.
\end{aligned}$$

In the first equality, we used Corollary 3.1.4. □

As a summary, using Corollary 3.1.4 we can restate the triple Pieri rule in terms of condition of classical Pieri rule Theorem 3.1.1.

Theorem 3.3.3 (Triple Pieri rule). *Let $v \in S_n$. We have*

$$\mathfrak{S}_u(x,y) \cdot \mathfrak{S}_v(x,t) = \sum_{w} \overline{c}_{u,v}^{w}(t,y) \cdot \mathfrak{S}_w(x,t)$$

where the sum ranges over all $w \in S_n$ such that $w = v t_{a_1 b_1} \cdots t_{a_{r'} b_{r'}}$ with (0) $r' \le r$; (1) $a_i \le k < b_i$; (2) a_i's are distinct; (3) $\ell(v t_{a_1 b_1} \cdots t_{a_i b_i}) = \ell(v) + i$ and the coefficients

$$\overline{c}_{uv}^{w}(t,y) = \mathfrak{S}_{s_{k-r+1}\cdots s_{k-r'}}(t,y)|_{t_i \mapsto t_{d(i)}}.$$

Remark 3.3.4. When $r = k$, the formula above can be simplified further. In this case,

$$\mathfrak{S}_u(x,y) = \prod_{j=1}^{k} (x_i - y_1).$$

Moreover, for $w \in S_n$ in the sum, we have

$$\overline{c}_{u,v}^{w}(t,y) = \mathfrak{S}_{s_{k-r+1}\cdots s_{k-r'}}(t,y)|_{t_i \mapsto t_{d(i)}} = \prod_{a \in \Delta_k(u,w)} (t_a - y_1).$$

CHAPTER 4

Yang–Baxter Equations

This chapter provides the representation-theoretic background required for the subsequent chapter. The most important trick in the proof of Theorem C is the application of the **Yang–Baxter equations**. In this part, we will give a general, self-contained, computational treatment of this topic.

We will overview the representation theory of quantum groups in Section 4.1, which serves as a primary source for solutions to the Yang–Baxter equation. We compute many examples arising from representation theory in Section 4.2. The code and detailed steps are provided. There is another classical source of solutions from the vertex model. We illustrate how to solve it with the assistance of a computer in Section 4.3.

4.1. Quantum groups

4.1.1. Finite quantum groups. Recall the Lie algebra $\mathfrak{sl}_n \subset \mathfrak{gl}_n$ is the set of traceless $n \times n$ matrices. The Chevalley generators of $\mathfrak{sl}_n$ are

$$E_i = \begin{bmatrix} \ddots & & & \\ & 0 & 1 & \\ & & 0 & \\ & & & \ddots \end{bmatrix}, \qquad F_i = \begin{bmatrix} \ddots & & & \\ & 0 & & \\ & 1 & 0 & \\ & & & \ddots \end{bmatrix}, \qquad H_i = \begin{bmatrix} \ddots & & & \\ & 1 & & \\ & & -1 & \\ & & & \ddots \end{bmatrix}$$

for $1 \le i \le n-1$, where in the above formulas the shown block is the 2×2 submatrix corresponding to the indices i and i+1. The Dynkin diagram and the Cartan matrix is

$$\underset{1}{\circ} - \underset{2}{\circ} - \cdots - \underset{n-2}{\circ} - \underset{n-1}{\circ} \qquad\qquad C = \begin{bmatrix} 2 & -1 & & & & \\ -1 & 2 & -1 & & & \\ & -1 & 2 & \ddots & & \\ & & \ddots & \ddots & -1 & \\ & & & -1 & 2 & -1 \\ & & & & -1 & 2 \end{bmatrix}.$$

That is,

$$c_{ij} = \begin{cases} 2, & i = j, \\ -1, & |i-j| = 1, \\ 0, & \text{otherwise.} \end{cases}$$

The enveloping algebra $U(\mathfrak{sl}_n)$ admits the following **Serre presentation**

$$[H_i, H_j] = 0 \tag{4.1.1}$$

$$[H_i, E_j] = c_{ij} \cdot E_j, \qquad [H_i, F_j] = -c_{ij} \cdot F_j, \qquad [E_i, F_j] = \delta_{ij} H_i, \tag{4.1.2}$$

$$\sum_{k=0}^{1-c_{ij}} (-1)^k \binom{1-c_{ij}}{k} E_i^k E_j E_i^{1-c_{ij}-k} = 0 \quad (i \neq j), \tag{4.1.3}$$

$$\sum_{k=0}^{1-c_{ij}} (-1)^k \binom{1-c_{ij}}{k} F_i^k F_j F_i^{1-c_{ij}-k} = 0 \quad (i \neq j). \tag{4.1.4}$$

The last two relations (4.1.3) and (4.1.4) are called the **Serre relations**. They are obtained from

$$\underbrace{[E_i, [E_i, \dots [E_i,}_{1-c_{ij}} E_j] \cdots]] = \underbrace{[F_i, [F_i, \dots [F_i,}_{1-c_{ij}} F_j] \cdots]] = 0$$

The algebra $U(\mathfrak{sl}_n)$ is a Hopf algebra, whose coproduct comultiplication is given by

$$\Delta : U(\mathfrak{sl}_n) \longrightarrow U(\mathfrak{sl}_n) \otimes U(\mathfrak{sl}_n), \qquad \begin{aligned} H_i &\longmapsto H_i \otimes 1 + 1 \otimes H_i, \\ E_i &\longmapsto E_i \otimes 1 + 1 \otimes E_i, \\ F_i &\longmapsto F_i \otimes 1 + 1 \otimes F_i. \end{aligned} \tag{4.1.5}$$

The **quantum group** $U_q(\mathfrak{sl}_n)$ is a quantum analogue of the enveloping algebra $U(\mathfrak{sl}_n)$, generalized by quantizing the above presentation. More precisely, $U_q(\mathfrak{sl}_n)$ is the $\mathbb{C}(q)$-algebra generated by $E_i, F_i, K_i^{\pm 1}$ for $1 \leq i \leq n-1$ with relations

$$K_i K_j = K_j K_i, \qquad K_i K_i^{-1} = 1 \tag{4.1.6}$$

$$K_i E_j K_i^{-1} = q^{c_{ij}} \cdot E_j, \qquad K_i F_j K_i^{-1} = q^{-c_{ij}} \cdot F_j, \qquad [E_i, F_j] = \delta_{ij} \frac{K_i - K_i^{-1}}{q - q^{-1}}, \tag{4.1.7}$$

$$\sum_{k=0}^{1-c_{ij}} (-1)^k \begin{bmatrix} 1-c_{ij} \\ k \end{bmatrix} E_i^k E_j E_i^{1-c_{ij}-k} = 0 \quad (i \neq j), \tag{4.1.8}$$

$$\sum_{k=0}^{1-c_{ij}} (-1)^k \begin{bmatrix} 1-c_{ij} \\ k \end{bmatrix} F_i^k F_j F_i^{1-c_{ij}-k} = 0 \quad (i \neq j). \tag{4.1.9}$$

Here we use the notation of **quantum numbers**

$$[m] = \frac{q^m - q^{-m}}{q - q^{-1}}, \qquad [m]! = [m] \cdots [1], \qquad \begin{bmatrix} n \\ k \end{bmatrix} = \frac{[n]!}{[k]![n-k]!}.$$

The last two relations (4.1.8) and (4.1.9) are also called the **Serre relations**.

The quantum group $U_q(\mathfrak{sl}_n)$ also forms a Hopf algebra, whose comultiplication is given by

$$\Delta : U_q(\mathfrak{sl}_n) \longrightarrow U_q(\mathfrak{sl}_n) \otimes U_q(\mathfrak{sl}_n), \quad \begin{aligned} K_i^{\pm 1} &\longmapsto K_i^{\pm 1} \otimes K_i^{\pm 1}, \\ F_i &\longmapsto F_i \otimes 1 + K_i \otimes F_i, \\ E_i &\longmapsto E_i \otimes K_i^{-1} + 1 \otimes E_i. \end{aligned} \tag{4.1.10}$$

We remark that there are actually many different choices of comultiplications, and they all differ from each other by automorphisms of $U_q(\mathfrak{sl}_n)$.

4.1.1.1. *Representation theory.* The category of finite-dimensional representations of $U(\mathfrak{sl}_n)$ is a symmetric tensor category. This follows from the fact the comultiplication of $U(\mathfrak{sl}_n)$ is cocommutative, i.e. the following diagram is commutative

$$\begin{array}{ccc} U(\mathfrak{sl}_n) & \xrightarrow{\Delta} & U(\mathfrak{sl}_n) \otimes U(\mathfrak{sl}_n) \\ \| & & \downarrow{\scriptstyle \tau: X\otimes Y \mapsto Y \otimes X} \\ U(\mathfrak{sl}_n) & \xrightarrow{\Delta} & U(\mathfrak{sl}_n) \otimes U(\mathfrak{sl}_n). \end{array}$$

However, the category $\mathcal{C}$ of finite-dimensional (over $\mathbb{C}(q)$) representations of $U(\mathfrak{sl}_n)$ is only a tensor category, since the comultiplication of $U_q(\mathfrak{sl}_n)$ fails to be cocommutative. However, the general representation theory of $U_q(\mathfrak{sl}_n)$ implies (1) the category $\mathcal{C}$ is semisimple; (2) the Grothendieck group of $\mathcal{C}$ forms a commutative ring; see [**12**]. In particular, for two representations U and V in $\mathcal{C}$, there exists an isomorphism of $U_q(\mathfrak{sl}_n)$-modules

$$U \otimes V \cong V \otimes U. \tag{4.1.11}$$

Note that the naïve map $\tau : U \otimes V \to V \otimes U$ given by $\tau(x \otimes y) = y \otimes x$ is not a $U_q(\mathfrak{sl}_n)$-homomorphism. These isomorphisms (4.1.11) can actually be made "functorial", or in fancy language, the category forms a **braided tensor category**. That is, for any representations U and V, we could fix a choice of an isomorphism

$$R = R_{U,V} : U \otimes V \xrightarrow{\sim} V \otimes U$$

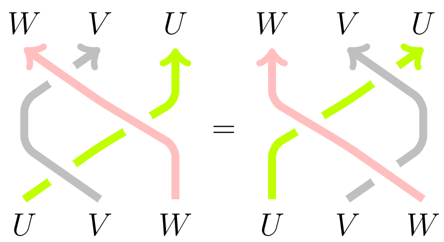
which is functorial in both U and V. Moreover, R_{UV} satisfies the **Yang–Baxter equation**

$$(R_{VW} \otimes 1)(1 \otimes R_{UW})(R_{UV} \otimes 1) = (1 \otimes R_{UV})(R_{UW} \otimes 1)(1 \otimes R_{VW})$$

as isomorphisms between

$$U \otimes V \otimes W \xrightarrow{\sim} W \otimes V \otimes U.$$

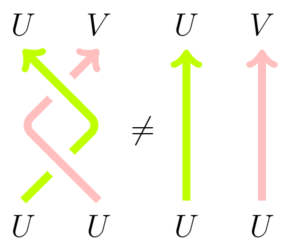
The following illustrates the Yang–Baxter equation



We remark that in general

$$R_{V,U} \circ R_{U,V} \neq \mathsf{id}_{U\otimes V}\,.$$

In terms of diagrams, it says the "braid" cannot resolved in general:



Let us briefly explain the construction of R for completeness. First observe that

$$\begin{aligned}\mathrm{Hom}_{\mathbb{C}}(U\otimes V, V\otimes U) &= \mathrm{Hom}_{\mathbb{C}}(U\otimes V, U\otimes V)\circ\tau\\ &= \mathrm{End}_{\mathbb{C}}(U)\otimes\mathrm{End}_{\mathbb{C}}(V)\circ\tau.\end{aligned}$$

Note that the representation map $U_q(\mathfrak{sl}_n)\to\mathrm{End}_{\mathbb{C}}(U)$ is surjective when U is irreducible. Actually, this is known as the density theorem in ring theory. Moreover, two elements $x, y\in U_q(\mathfrak{sl}_n)$ are distinct if and only if they act differently on some representation U in $\mathcal{C}$. In particular, the question is almost equivalent to the construction of a universal element (called the **universal R-matrix**)

$$\Theta\in U_q(\mathfrak{sl}_n)\otimes U_q(\mathfrak{sl}_n)$$

such that

$$\Delta(X)\cdot\Theta = \Theta\cdot\tau\Delta(X),\qquad \forall X\in U_q(\mathfrak{sl}_n).$$

However, such Θ only exists in a completion of

$$U_q^+\otimes U_q^-\subseteq U_q(\mathfrak{sl}_n)\otimes U_q(\mathfrak{sl}_n),$$

where U_q^+ (resp., U_q^-) is the subalgebra generated by H_i and E_i (resp., H_i and F_i) for $1\le i\le n-1$. These are quantum versions of the upper (resp., lower) triangular Lie algebra. Moreover, the element Θ is unique, and can be constructed explicitly; see [**12**].

4.1.2. Quantum Loop groups. The loop algebra $L\mathfrak{sl}_n$ is a base-change

$$L\mathfrak{sl}_n = \mathfrak{sl}_n\otimes\mathbb{C}[z^{\pm1}],\qquad [X\otimes z^n, Y\otimes z^m] = [X,Y]\otimes z^{m+n}.$$

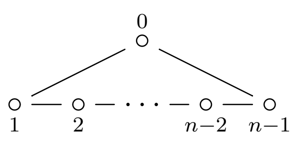
Note that $L\mathfrak{sl}_n$ is **NOT** a Kac–Moody algebra. To obtain a Kac–Moody algebra, we still need to introduce the differential operator $d=\frac{1}{z}\frac{d}{dz}$ and take a central extension. We have a natural inclusion

$$\mathfrak{sl}_n\cong\mathfrak{sl}_n\otimes 1\subset L\mathfrak{sl}_n.$$

Let us denote

$$E_0=\begin{bmatrix}0 & & \\ & \ddots & \\ z & & 0\end{bmatrix},\qquad F_0=\begin{bmatrix}0 & & z^{-1}\\ & \ddots & \\ & & 0\end{bmatrix},\qquad H_0=\begin{bmatrix}-1 & & \\ & \ddots & \\ & & 1\end{bmatrix}.$$

We also define the Dynkin diagram and the Cartan matrix ($n\ge 3$)



$$C=\begin{bmatrix}2 & -1 & & & & -1\\ -1 & 2 & -1 & & & \\ & -1 & 2 & \ddots & & \\ & & \ddots & \ddots & -1 & \\ & & & -1 & 2 & -1\\ -1 & & & & -1 & 2\end{bmatrix}.$$

That is,

$$c_{ij}=\begin{cases}2, & i=j,\\ -1, & i\equiv j\pm1 \bmod n,\\ 0, & \text{otherwise.}\end{cases}$$

The case $n = 2$ is special, its Dynkin diagram and Cartan matrix are defined separately to be

$$\underset{1}{\circ} \Longleftrightarrow \underset{2}{\circ}, \qquad C = \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix}, \quad \text{i.e.} \begin{cases} c_{11} = c_{00} = 2, \\ c_{10} = c_{01} = -2. \end{cases}$$

The enveloping algebra $U(L\mathfrak{sl}_n)$ admits the same Serre presentation (4.1.1) to (4.1.4), but now the index 0 is included. Similarly, $U(L\mathfrak{sl}_n)$ is a Hopf algebra whose coproduct is given by the same formula (4.1.5), again, including the case $i = 0$.

Similarly, we can define the **quantum loop group** $U_q(L\mathfrak{sl}_n)$ to be the $\mathbb{C}(q)$-algebra generated by $E_i, F_i, K_i^{\pm 1}$ for $0 \leq i \leq n-1$ (0 is included) with the same relations (4.1.6) to (4.1.9). Moreover, $U_q(L\mathfrak{sl}_n)$ is a Hopf algebra whose comultiplication is given by the same formula (4.1.10).

The quantum loop group also admits a **Drinfeld presentation**. That is, $U_q(L\mathfrak{sl}_n)$ is the $\mathbb{C}(q)$-algebra generated by $E_{i,r}, F_{k,r}, K_i^{\pm 1}, \psi_{i,s}$ ($1 \leq i \leq n-1$, $r \in \mathbb{Z}$, $s \in \mathbb{Z} \setminus \{0\}$) with relations

(4.1.12) $$K_i K_j = K_j K_i, \qquad K_i K_i^{-1} = 1$$

(4.1.13) $$\psi_i^{\pm}(z)\psi_j^{\pm'}(w) = \psi_j^{\pm'}(w)\psi_i^{\pm}(z)$$

(4.1.14) $$K_i x_j^{\pm}(z) K_i^{-1} = q^{\pm c_{ij}} E_j(z)$$

(4.1.15) $$[x_i^+(z), x_j^-(w)] = \frac{\delta_{ij}}{q - q^{-1}}\big(\delta(\tfrac{w}{z})\psi_i^+(w) - \delta(\tfrac{z}{w})\psi_j^-(w)\big),$$

(4.1.16) $$\psi_i^+(z) x_j^{\pm}(w) = \frac{q^{\pm c_{ij}} z - w}{z - q^{\pm c_{ij}} w} x_j^{\pm'}(w)\psi_i^+(z),$$

(4.1.17) $$\psi_i^-(z) x_j^{\pm}(w) = \frac{q^{\pm c_{ij}} w - z}{w - q^{\pm c_{ij}} z} x_j^{\pm'}(w)\psi_i^-(z),$$

(4.1.18) $$x_i^{\pm}(z) x_j^{\pm}(w) = \frac{q^{\pm c_{ij}} z - w}{z - q^{\pm c_{ij}} w} x_j^{\pm}(w) x_i^{\pm}(z),$$

(4.1.19)
$$\operatorname*{Sym}_{z_1, z_2, \dots} \sum_{k=0}^{1-c_{ij}} (-1)^k \begin{bmatrix} 1 - c_{ij} \\ k \end{bmatrix} x_i^{\pm}(z_1) \cdots x_i^{\pm}(z_i) x_j^{\pm}(w) x_i^{\pm}(z_{i+1}) \cdots x_i^{\pm}(z_{1-c_{ij}}) = 0 \quad (i \neq j)$$

where

$$\delta(z) = \sum_{r=-\infty}^{\infty} z^r, \qquad x_i^+(z) = \sum_{r=-\infty}^{\infty} E_{i,r} z^{-r}, \qquad x_i^-(z) = \sum_{r=-\infty}^{\infty} F_{i,r} z^{-r},$$
$$\psi_i^+(z) = \sum_{r>0} \psi_{i,s} z^{-s}, \qquad \psi_i^-(z) = \sum_{r<0} \psi_{i,s} z^{-s}.$$

This presentation is crucial when describing its representation theory.

4.1.2.1. *Reprensetation theory.* The category of finite-dimensional representations of $L\mathfrak{sl}_n$ (i.e. of $U(L\mathfrak{sl}_n)$) is a commutative tensor category. For a representation V of $\mathfrak{sl}_n$, we denote the representation of $L\mathfrak{sl}_n = \mathfrak{sl}_n \otimes \mathbb{C}[z^{\pm 1}]$ by

$$V(z) = V \otimes \mathbb{C}[z^{\pm 1}].$$

For $z_0 \in \mathbb{C}^\times$, we similarly denote

$$V(z_0) = V \otimes \mathbb{C}_{z_0}$$

where $\mathbb{C}_{z_0} = \mathbb{C}[z^{\pm 1}]/(z - z_0)$. The finite-dimensional irreducible representations of $U(L\mathfrak{sl}_n)$ are of the form

$$\bigotimes_{i=1}^{m} V_i(z_i)$$

where $z_1, \ldots, z_m$ are distinct points over $\mathbb{C}^\times$ and each V_i is a nontrivial finite-dimensional irreducible representation of $\mathfrak{sl}_n$. Note that the category of finite-dimensional representations of $U(L\mathfrak{sl}_n)$ is not semisimple, because of the existence of representations such as

$$V(z)/(z - z_0)^2 = V \otimes \mathbb{C}[z^{\pm 1}]/(z - z_0)^2$$

for some $z_0 \in \mathbb{C}^\times$.

We warn readers that the enveloping algebra does not commute with base change:

$$U(L\mathfrak{sl}_n) = U(\mathfrak{sl}_n \otimes \mathbb{C}[z^{\pm 1}]) \neq U(\mathfrak{sl}_n) \otimes \mathbb{C}[z^{\pm 1}].$$

Actually a finite dimensional representation of $U(L\mathfrak{sl}_n)$ is equivalent to a quasi-coherent sheaf of $\mathfrak{sl}_n$-representations over $\mathbb{C}^\times = \operatorname{Spec} \mathbb{C}[z^{\pm 1}]$ such that the points whose stack is a non-trivial representation are finite.

Now let us switch to the representation theory of the quantum loop group $U_q(L\mathfrak{sl}_n)$. Note that the relations in the Drinfeld presentation (4.1.12) to (4.1.19) are written in terms of generating functions. In particular, one can check directly that we can define an automorphism

$$\rho_{z_0} : U_q(L\mathfrak{sl}_n) \longrightarrow U_q(L\mathfrak{sl}_n)$$

for any $z_0 \in \mathbb{C}^\times$ by

$$x_i^\pm(z) \longmapsto x^\pm(z \cdot z_0), \qquad \psi^\pm(z) \longmapsto \psi^\pm(z \cdot z_0), \qquad K_i \longmapsto K_i.$$

For any representation V of $U_q(L\mathfrak{sl}_n)$, we can define a representation $V(z_0)$ the representation obtained by twisting with the automorphism ρ_{z_0}. We can similarly define an $\mathbb{C}[z^\pm]$-linear representation $V[z^\pm] = V \otimes \mathbb{C}[z^\pm]$ such that $V(z_0) = V[z^\pm]/(z - z_0)$. We denote $V(z) = V[z^\pm] \otimes_{\mathbb{C}[z^\pm]} \mathbb{C}(z)$.

The category $\mathcal{C}$ of finite-dimensional (over $\mathbb{C}(q)$) representations of $U_q(L\mathfrak{sl}_n)$ is a tensor category, which is not symmetric in general. But, in contrast of the case of finite quantum groups, $\mathcal{C}$ is **NOT** braided. Actually for two representations U, V in $\mathcal{C}$, the tensor product $U \otimes V$ is not isomorphic to $V \otimes U$ in general. This is because the universal R-matrix Θ of $U_q(L\mathfrak{sl}_n)$ does not always converge on the tensor product $U \otimes V$.

But this does not happen very often. Actually, for two representations U, V in $\mathcal{C}$, the tensor products are isomorphic

$$U(x_0) \otimes V(y_0) \cong V(y_0) \otimes U(x_0)$$

for generic $x_0, y_0 \in \mathbb{C}^\times$. Actually, the image of the universal R-matrix

$$\mathcal{R} : U(x) \otimes V(y) \cong V(y) \otimes U(x)$$

is well-defined as a product of a matrix of rational function in x/y and a (meromorphic) function in x/y. So the matrix $\mathcal{R}$ is well-defined when x_0/y_0 avoids the zeros and the poles of its determinant. Usually, we normalize the R-matrix such that it is identity when applied to the highest weight vectors (of $U_q(\mathfrak{sl}_n)$), so it will be a rational function in x/y, and we denote it by

$$R_{UV}(x/y) : U(x) \otimes V(y) \longrightarrow V(y) \otimes U(x)$$

to emphasize its dependence on x/y. To distinguish it from the finite R-matrix, we call $R_{UV}(z)$ the R-matrix with **spectral parameters**.

Moreover, when U, V are both irreducible, the tensor product $U(x_0) \otimes V(y_0)$ is irreducible for generic $x_0, y_0 \in \mathbb{C}^\times$. Note that the same phenomenon appears in the representation theory of $L\mathfrak{sl}_n$. As a result, for generic $x_0, y_0 \in \mathbb{C}^\times$, we have

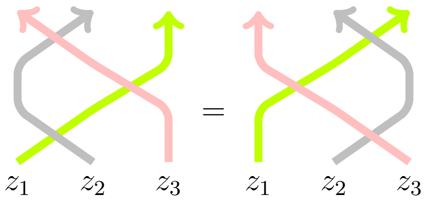
$$\mathrm{Hom}_{\mathcal{C}}\big(U(x_0) \otimes V(y_0), V(y_0) \otimes U(x_0)\big) = \mathrm{span}\big(R_{UV}(x_0/y_0)\big).$$

This is the uniqueness of R-matrices with spectral parameters. Note that the finite R-matrix is unique considered as a functor, while the spectral R-matrix is unique for each single pair of irreducible representations. This fact will be useful in the computation.

The spectral R-matrices satisfy the **Yang–Baxter equations**, i.e. as an operator $U(z_1) \otimes V(z_2) \otimes W(z_3) \to W(z_3) \otimes V(z_2) \otimes U(z_1)$,

$$\begin{aligned}&(R_{VW}(z_2/z_3) \otimes 1)(1 \otimes R_{UW}(z_1/z_3))(R_{UV}(z_1/z_2) \otimes 1)\\ &= (1 \otimes R_{UV}(z_1/z_2))(R_{UW}(z_1/z_3) \otimes 1)(1 \otimes R_{VW}(z_2/z_3)).\end{aligned}$$

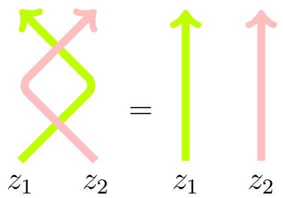
The following is a diagram for the equation.



As a feature of spectral R-matrices, we also have the **unitary equation**, i.e. as an operator $U(z_1) \otimes V(z_2) \to V(z_2) \otimes U(z_1)$,

$$R_{VU}(z_2/z_1) \circ R_{UV}(z_1/z_2) = \mathsf{id}_{U\otimes V}\,.$$

The following is a diagram for it:



4.2. Examples of R-matrices

To author's knowledge, there is no existing algorithm for computing R-matrices in the literature or in any mathematics software system. In standard textbooks of quantum groups [**12**], the only known example of R-matrices is for natural representations of $U_q(L\mathfrak{sl}_n)$. Moreover, it is actually not from a computation but a proof that a given matrix is the R-matrix. In this section, we illustrate how to compute the spectral R-matrices.

4.2.1. Examples of $\mathfrak{sl}_2$. Let us consider the natural representation of $\mathfrak{sl}_2$.

$$\mathfrak{sl}_2 \curvearrowright \mathbb{C}^2, \qquad H_1 = \begin{bmatrix} 1 & \\ & -1 \end{bmatrix}, E_1 = \begin{bmatrix} 0 & 1 \\ & 0 \end{bmatrix}, F_1 = \begin{bmatrix} 0 & \\ -1 & 0 \end{bmatrix}.$$

Its quantum version is given by

$$U_q(\mathfrak{sl}_2) \curvearrowright \mathbb{C}(q)^3, \qquad K_1 = \begin{bmatrix} q & \\ & q^{-1} \end{bmatrix}, E_1 = \begin{bmatrix} 0 & 1 \\ & 0 \end{bmatrix}, F_1 = \begin{bmatrix} 0 & \\ -1 & 0 \end{bmatrix}.$$

Next, let us consider the adjoint representation of $\mathfrak{sl}_2$.

$$\mathfrak{sl}_2 \curvearrowright \mathbb{C}^3, \qquad H_1 = \begin{bmatrix} 2 & & \\ & 0 & \\ & & -2 \end{bmatrix}, E_1 = \begin{bmatrix} 0 & -2 & \\ & 0 & 1 \\ & & 0 \end{bmatrix}, F_1 = \begin{bmatrix} 0 & & \\ -1 & 0 & \\ & 2 & 0 \end{bmatrix}.$$

Its quantum version is given by

$$U_q(\mathfrak{sl}_2) \curvearrowright \mathbb{C}(q)^3, \qquad K_1 = \begin{bmatrix} q^2 & & \\ & 1 & \\ & & q^{-2} \end{bmatrix}, E_1 = \begin{bmatrix} 0 & -[2] & \\ & 0 & 1 \\ & & 0 \end{bmatrix}, F_1 = \begin{bmatrix} 0 & & \\ -1 & 0 & \\ & [2] & 0 \end{bmatrix}.$$

Recall that $[2] = q^{-1} + q$.

We can extend $\mathbb{C}^2$ and $\mathbb{C}^3$ to representations of the loop algebra $L\mathfrak{sl}_2$ by

$$H_0 = -H_1, \qquad E_0 = zF_1, \qquad F_0 = z^{-1}E_1.$$

Its quantum version is given by

$$K_0 = K_1^{-1}, \qquad E_0 = zF_1, \qquad F_0 = z^{-1}E_1.$$

Let $U = \mathbb{C}(q,z)^2$ and $V = \mathbb{C}(q,z)^3$. Let us compute the R-matrices

$$\begin{aligned} R_{UU}(x/y) &: U(x) \otimes U(y) \longrightarrow U(y) \otimes U(x), \\ R_{VV}(x/y) &: V(x) \otimes V(y) \longrightarrow V(y) \otimes V(x), \\ R_{UV}(x/y) &: U(x) \otimes V(y) \longrightarrow V(y) \otimes U(x), \\ R_{VU}(x/y) &: V(x) \otimes U(y) \longrightarrow U(y) \otimes V(x). \end{aligned}$$

It turns out that we only need the action of $K_1, F_1, F_0 \in U_q(L\mathfrak{sl}_n)$ to solve the R-matrix. Let us illustrate the computation of $U(x) \otimes V(y)$ as an example. By definition the representation

$$\begin{aligned} U_q(L\mathfrak{sl}_2) &\curvearrowright U(x) \otimes V(y), \\ U_q(L\mathfrak{sl}_2) &\curvearrowright V(y) \otimes U(x), \end{aligned} \qquad \begin{cases} K_1 \mapsto K_1 \otimes K_1, \\ F_1 \mapsto F_1 \otimes 1 + K_1 \otimes F_1, \\ F_0 \mapsto F_0 \otimes 1 + K_0 \otimes F_0. \end{cases}$$

The actions of K_1 are given by

$$\left[\begin{array}{ccc|ccc} q^3 & 0 & 0 & 0 & 0 & 0 \\ 0 & q & 0 & 0 & 0 & 0 \\ 0 & 0 & q^{-1} & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & q & 0 & 0 \\ 0 & 0 & 0 & 0 & q^{-1} & 0 \\ 0 & 0 & 0 & 0 & 0 & q^{-3} \end{array}\right], \qquad \left[\begin{array}{cc|cc|cc} q^3 & 0 & 0 & 0 & 0 & 0 \\ 0 & q & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & q & 0 & 0 & 0 \\ 0 & 0 & 0 & q^{-1} & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & q^{-1} & 0 \\ 0 & 0 & 0 & 0 & 0 & q^{-2} \end{array}\right];$$

the actions of F_1 are

$$\left[\begin{array}{ccc|ccc} 0 & 0 & 0 & 0 & 0 & 0 \\ -q & 0 & 0 & 0 & 0 & 0 \\ 0 & q^2+1 & 0 & 0 & 0 & 0 \\ \hline 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -q^{-1} & 0 & 0 \\ 0 & 0 & 1 & 0 & 1+q^{-2} & 0 \end{array}\right], \qquad \left[\begin{array}{cc|cc|cc} 0 & 0 & 0 & 0 & 0 & 0 \\ q^2 & 0 & 0 & 0 & 0 & 0 \\ \hline -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & q+q^{-1} & 0 & 0 & 0 \\ 0 & 0 & 0 & q+q^{-1} & q^{-2} & 0 \end{array}\right];$$

the actions of F_0 are

$$\left[\begin{array}{ccc|ccc} 0 & -(1+q^{-2})y^{-1} & 0 & x^{-1} & 0 & 0 \\ 0 & 0 & q^{-1}y^{-1} & 0 & x^{-1} & 0 \\ 0 & 0 & 0 & 0 & 0 & x^{-1} \\ \hline 0 & 0 & 0 & 0 & -(q^2+1)y^{-1} & 0 \\ 0 & 0 & 0 & 0 & 0 & qy^{-1} \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right], \qquad \left[\begin{array}{cc|cc|cc} 0 & q^{-2}x^{-1} & -(q+q^{-1})y^{-1} & 0 & 0 & 0 \\ 0 & 0 & 0 & -(q+q^{-1})y^{-1} & 0 & 0 \\ \hline 0 & 0 & 0 & x^{-1} & y^{-1} & 0 \\ 0 & 0 & 0 & 0 & 0 & y^{-1} \\ \hline 0 & 0 & 0 & 0 & 0 & q^2x^{-1} \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

Assume the R-matrix is

$$R = R_{UV}(x/y) = \left[\begin{array}{cc|cc|cc} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 \\ x_7 & x_8 & x_9 & x_{10} & x_{11} & x_{12} \\ x_{13} & x_{14} & x_{15} & x_{16} & x_{17} & x_{18} \\ \hline x_{19} & x_{20} & x_{21} & x_{22} & x_{23} & x_{24} \\ x_{25} & x_{26} & x_{27} & x_{28} & x_{29} & x_{30} \\ x_{31} & x_{32} & x_{33} & x_{34} & x_{35} & x_{36} \end{array}\right].$$

The equation $K_1 \circ R = R \circ K_1$ implies

$$x_i = 0, \qquad \text{unless } i \in \{1, 8, 10, 14, 16, 21, 23, 27, 29, 36\}.$$

That is,

$$R = R_{UV}(x/y) = \left[\begin{array}{cc|cc|cc} x_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & x_8 & 0 & x_{10} & 0 & 0 \\ 0 & x_{14} & 0 & x_{16} & 0 & 0 \\ \hline 0 & 0 & x_{21} & 0 & x_{23} & 0 \\ 0 & 0 & x_{27} & 0 & x_{29} & 0 \\ 0 & 0 & 0 & 0 & 0 & x_{36} \end{array}\right].$$

The equation $F_1 \circ R = R \circ F_1$ implies the following equations

$$\begin{aligned} q^2x_1 &= -qx_8 + x_{10} \\ -x_1 &= -qx_{14} + x_{16} \\ -x_8 + x_{14} &= \left(q^2+1\right)x_{21} + x_{23} \\ -x_{10} + x_{16} &= -q^{-1}x_{23} \\ (q+q^{-1})x_{14} &= (q^2+1)x_{27} + x_{29} \\ (q+q^{-1})x_{16} &= -q^{-1}x_{29} \\ (q+q^{-1})x_{21} + q^{-2}x_{27} &= x_{36} \\ (q+q^{-1})x_{23} + q^{-2}x_{29} &= (1+q^{-2})x_{36}. \end{aligned}$$

The equation $R \circ F_0 = F_0 \circ R$ implies the following

$$\begin{aligned}
q^{-2}x^{-1}x_8 - (q+q^{-1})y^{-1}x_{14} &= -(1+q^2)y^{-1}x_1 \\
q^{-2}x^{-1}x_{10} - (q+q^{-1})y^{-1}x_{16} &= x^{-1}x_1 \\
-(q+q^{-1})y^{-1}x_{21} &= q^{-1}y^{-1}x_8 \\
-(q+q^{-1})y^{-1}x_{23} &= x^{-1}x_8 - (1+q^2)y^{-1}x_{10} \\
x^{-1}x_{21} + y^{-1}x_{27} &= q^{-1}y^{-1}x_{14} \\
x^{-1}x_{23} + y^{-1}x_{29} &= x^{-1}x_{14} - (1+q^2)y^{-1}x_{16} \\
y^{-1}x_{36} &= x^{-1}x_{21} + qy^{-1}x_{23} \\
q^2x^{-1}x_{36} &= x^{-1}x_{27} + qy^{-1}x_{29}.
\end{aligned}$$

With the normalization $x_1 = 1$, we can solve

$$\begin{aligned}
&x_8 = \frac{-q^4x+x}{q^3x-y} \qquad x_{10} = \frac{-q^2y+qx}{q^3x-y} \qquad x_{14} = \frac{q^2x-qy}{q^3x-y} \qquad x_{16} = \frac{-q^2y+y}{q^3x-y} \\
&x_{21} = \frac{q^2x-x}{q^3x-y} \qquad x_{23} = \frac{q^2x-qy}{q^3x-y} \qquad x_{27} = \frac{-q^2y+qx}{q^3x-y} \qquad x_{29} = \frac{q^4y-y}{q^3x-y} \\
&x_{36} = x_1 = 1.
\end{aligned}$$

As a result, we have

$$(4.2.1) \qquad R_{UV}(x/y) = \left[\begin{array}{cc|cc|cc}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{-q^4x+x}{q^3x-y} & 0 & \frac{-q^2y+qx}{q^3x-y} & 0 & 0 \\
0 & \frac{q^2x-qy}{q^3x-y} & 0 & \frac{-q^2y+y}{q^3x-y} & 0 & 0 \\
\hline
0 & 0 & \frac{q^2x-x}{q^3x-y} & 0 & \frac{q^2x-qy}{q^3x-y} & 0 \\
0 & 0 & \frac{-q^2y+qx}{q^3x-y} & 0 & \frac{q^4y-y}{q^3x-y} & 0 \\
0 & 0 & 0 & 0 & 0 & 1
\end{array}\right].$$

By a similar manner, we can compute the R-matrices

$$(4.2.2) \qquad R_{VU}(x/y) = \left[\begin{array}{ccc|ccc}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{-q^2x+x}{q^3x-y} & \frac{q^2x-qy}{q^3x-y} & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & \frac{q^4x-x}{q^3x-y} & \frac{q^2y-qx}{-q^3x+y} & 0 \\
0 & \frac{q^2y-qx}{-q^3x+y} & \frac{-q^4y+y}{q^3x-y} & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & \frac{q^2x-qy}{q^3x-y} & \frac{q^2y-y}{q^3x-y} & 0 \\
0 & 0 & 0 & 0 & 0 & 1
\end{array}\right]$$

$$
R_{UU}(x/y) = \left[\begin{array}{cc|cc}
1 & 0 & 0 & 0 \\
0 & \dfrac{q^2x - x}{q^2x - y} & \dfrac{-qx + qy}{-q^2x + y} & 0 \\
\hline
0 & \dfrac{-qx + qy}{-q^2x + y} & \dfrac{q^2y - y}{q^2x - y} & 0 \\
0 & 0 & 0 & 1
\end{array}\right] \tag{4.2.3}
$$

$$
R_{VV}(x/y) = \left[\begin{array}{ccc|ccc|ccc}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \blacktriangle & 0 & \blacklozenge & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \bigstar & 0 & \clubsuit & 0 & \spadesuit & 0 & 0 \\
\hline
0 & \blacklozenge & 0 & \blacktriangledown & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \blacktriangleleft & 0 & \blacksquare & 0 & \blacktriangleright & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \blacktriangle & 0 & \blacklozenge & 0 \\
\hline
0 & 0 & \spadesuit & 0 & \varclubsuit & 0 & \varstar & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \blacklozenge & 0 & \blacktriangledown & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{array}\right] \tag{4.2.4}
$$

where

$$
\blacktriangle = \frac{q^4x - x}{q^4x - y}, \qquad \blacklozenge = \frac{-q^2x + q^2y}{-q^4x + y}, \qquad \blacktriangledown = \frac{q^4y - y}{q^4x - y},
$$

$$
\bigstar = \frac{(q^2-1)^2(q^2+1)x^2}{(-q^2x+y)(-q^4x+y)}, \qquad \varstar = \frac{(q^2-1)^2(q^2+1)y^2}{(-q^2x+y)(-q^4x+y)},
$$

$$
\blacktriangleleft = \frac{(q^3-q)(-x^2+xy)}{(-q^2x+y)(-q^4x+y)}, \qquad \blacktriangleright = \frac{(q^3-q)(-xy+y^2)}{(-q^2x+y)(-q^4x+y)},
$$

$$
\clubsuit = \frac{(q^3-q)(q^2+1)^2(-x^2+xy)}{(-q^2x+y)(-q^4x+y)}, \qquad \varclubsuit = \frac{(q^3-q)(q^2+1)^2(-xy+y^2)}{(-q^2x+y)(-q^4x+y)},
$$

$$
\blacksquare = \frac{q^6xy + q^4x^2 - 2q^4xy - 2q^2xy + q^2y^2 + xy}{(-q^2x+y)(-q^4x+y)}, \qquad \spadesuit = \frac{q^2(-x+y)(q^2y-x)}{(-q^2x+y)(-q^4x+y)}.
$$

4.2.1.1. *Codes.* The above computation can be done via SageMath, and here is the code.

```
R = PolynomialRing(QQ,["q","x","y"]); q,x,y = R.gens()
class myrep(list): pass
def qnum(m,k=1): return gaussian_binomial(m,k)(q^2)*q^(-k*(m-k))

# define the representations Uq(sl2)
U = myrep([2]);
U.E = matrix([[0,1],[0,0]])
U.F = matrix([[0,0],[1,0]])
U.K = matrix([[q,0],[0,q^(-1)]])
```

```

V = myrep([3]);
V.E = matrix([[0,-q-q^(-1),0],[0,0,1],[0,0,0]])
V.F = matrix([[0,0,0],[-1,0,0],[0,q+q^(-1),0]])
V.K = matrix([[q^2,0,0],[0,1,0],[0,0,q^(-2)]])
```

```
def checkrel(V):
    K = V.K; E = V.E; F = V.F;
    comm = lambda X,Y: X*Y-Y*X
    if K*E*K^(-1) != q^2*E: return False
    if K*F*K^(-1) != q^(-2)*F: return False
    if comm(E,F)!= (K-K^(-1))/(q-q^(-1)): return False
    return True
print(checkrel(U))
print(checkrel(V))
```

```
# define the representation of Uq(Lsl2) via evaluation
def rep(prerep,para=1):
    res = myrep(list(prerep)+[para])
    res.E1 = prerep.E
    res.F1 = prerep.F
    res.K1 = prerep.K
    res.E0 = para*(prerep.F)
    res.F0 = para^(-1)*(prerep.E)
    res.K0 = (prerep.K)^(-1)
    res.I  = res.K0*res.K0^(-1)
    return res

# define tensor product
def ten(U,V):
    tp = lambda X,Y: X.tensor_product(Y)
    res = myrep(list([(U[i],V[i]) for i in range(len(U))]))
    res.E1 = tp(U.E1,V.K1^(-1)) + tp(U.I,V.E1)
    res.E0 = tp(U.E0,V.K0^(-1)) + tp(U.I,V.E0)
    res.F1 = tp(U.F1,V.I) + tp(U.K1,V.F1)
    res.F0 = tp(U.F0,V.I) + tp(U.K0,V.F0)
    res.K1 = tp(U.K1,V.K1)
    res.K0 = tp(U.K0,V.K0)
    return res
```

```
# construct the tensor representation
X = rep(U,x)
Y = rep(V,y)
XY = ten(X,Y)
YX = ten(Y,X)
# ten(X,Y).E1

S = PolynomialRing(R,["x"+str(i) for i in range(X[0]^2*Y[0]^2)]);
x = S.gens();
Rmat = matrix([[x[i*X[0]*Y[0]+j] for j in range(X[0]*Y[0])] for i in range(X[0]*Y[0])
    ])

# find all relations
Relations = [x[0]] # x[0] = 1 by normalization
```

```

def add_rel(mat):
    for row in mat:
        for ent in row:
            if ent!=0:
                Relations.append(ent)
add_rel(YX.K1 * Rmat - Rmat * XY.K1)
add_rel(YX.F1 * Rmat - Rmat * XY.F1)
add_rel(YX.F0 * Rmat - Rmat * XY.F0)
# add_rel(YX.E1 * Rmat - Rmat * XY.E1)
# add_rel(YX.E0 * Rmat - Rmat * XY.E0)

# linearize the relations
def linear_rel(poly):
    res = [0 for xx in x]
    my_dict = poly.dict()
    for ind in my_dict:
        i = 0
        while ind[i]==0:
            i+=1
        res[i] = my_dict[ind]
    return res

LHS = Matrix([linear_rel(rel) for rel in Relations])
RHS = [(1 if i==0 else 0) for i in range(len(Relations))]
x_0 = Matrix(LHS).solve_right(vector(RHS))
Rmat = matrix([[x_0[i*X[0]*Y[0]+j] for j in range(X[0]*Y[0])] for i in range(X[0]*Y
    [0])])
print(Rmat)
```

```
# double check:
print(YX.K1 * Rmat == Rmat * XY.K1)
print(YX.K0 * Rmat == Rmat * XY.K0)
print(YX.E1 * Rmat == Rmat * XY.E1)
print(YX.F1 * Rmat == Rmat * XY.F1)
print(YX.E0 * Rmat == Rmat * XY.E0)
print(YX.F0 * Rmat == Rmat * XY.F0)
```

There is some room to optimize the algorithm.

4.2.2. Natural representations. Consider the natural representation of the Lie algebra $\mathfrak{sl}_n$

$$\mathfrak{sl}_n \curvearrowright \mathbb{C}^n = \bigoplus_{i=1}^n \mathbb{C}\mathbf{e}_i \qquad \text{by} \qquad \begin{aligned} H_i\mathbf{e}_j &= (\delta_{ij} - \delta_{i+1,j})\mathbf{e}_j, \\ F_i\mathbf{e}_j &= \delta_{ij}\mathbf{e}_{i+1}, \\ E_i\mathbf{e}_{j+1} &= \delta_{ij}\mathbf{e}_i. \end{aligned}$$

Its quantum version is given by

$$U_q(\mathfrak{sl}_n) \curvearrowright \mathbb{V} := \bigoplus_{i=1}^n \mathbb{C}(q)\mathbf{e}_i \qquad \text{by} \qquad \begin{aligned} K_i\mathbf{e}_j &= q^{\delta_{ij} - \delta_{i+1,j}}\mathbf{e}_j, \\ F_i\mathbf{e}_j &= \delta_{ij}\mathbf{e}_{i+1}, \\ E_i\mathbf{e}_{j+1} &= \delta_{ij}\mathbf{e}_i. \end{aligned}$$

We can represent the actions by the following diagram:

$$\mathbf{e}_n \underset{F_{n\text{-}1}}{\overset{E_{n\text{-}1}}{\rightleftarrows}} \cdots \rightleftarrows \cdots \underset{F_2}{\overset{E_2}{\rightleftarrows}} \mathbf{e}_2 \underset{F_1}{\overset{E_1}{\rightleftarrows}} \mathbf{e}_1.$$

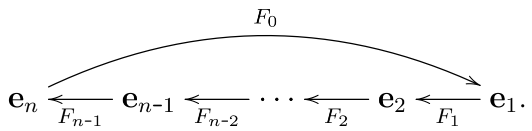
We can extend $\mathbb{C}^n$ to a representation of the loop algebra $L\mathfrak{sl}_n$

$$L\mathfrak{sl}_n \curvearrowright \mathbb{C}^n(z) \qquad \text{by} \qquad \begin{aligned} H_0\mathbf{e}_j &= (\delta_{nj} - \delta_{1j})\mathbf{e}_j, \\ F_0\mathbf{e}_j &= \delta_{nj} z^{-1}\mathbf{e}_1, \\ E_0\mathbf{e}_{j+1} &= \delta_{nj} z\mathbf{e}_n. \end{aligned}$$

Similarly, its quantum version is given by

$$U_q(L\mathfrak{sl}_n) \curvearrowright \mathbb{V}(z) \qquad \text{by} \qquad \begin{aligned} K_0\mathbf{e}_j &= q^{\delta_{nj}-\delta_{1j}}\mathbf{e}_j, \\ F_0\mathbf{e}_j &= \delta_{nj} z^{-1}\mathbf{e}_1, \\ E_0\mathbf{e}_{j+1} &= \delta_{nj} z\mathbf{e}_n. \end{aligned}$$

We can represent the actions by the following diagram (for simplicity, we only represent the action of F_i, and ignore the scalar)



Let us compute the spectral R-matrix for

$$\mathcal{R} = R(x/y) : \mathbb{V}(x) \otimes \mathbb{V}(y) \longrightarrow \mathbb{V}(y) \otimes \mathbb{V}(x).$$

The following is our strategies:

(1) Note that the R-matrix preserves weights, since it commutes with the action of K_i for $1 \leq i \leq n-1$. The weight of $\mathbf{e}_i \otimes \mathbf{e}_j$ is $\epsilon_i + \epsilon_j \in \mathbb{Z}^n/\mathbb{Z}(1, \ldots, 1)$, so we have

$$\mathcal{R}(\mathbf{e}_i \otimes \mathbf{e}_j) \in \operatorname{span}(\mathbf{e}_i \otimes \mathbf{e}_j, \mathbf{e}_j \otimes \mathbf{e}_i).$$

This includes the case $i = j$.

(2) The representation $\mathbb{V} \otimes \mathbb{V}$ is generated by $\mathbf{e}_1 \otimes \mathbf{e}_n$ as a $U_q(\mathfrak{sl}_n)$-representation over $\mathbb{Q}(q)$. This fact will also be illustrated in the computation. As a result, any $U_q(L\mathfrak{sl}_n)$-homomorphism is determined by the image

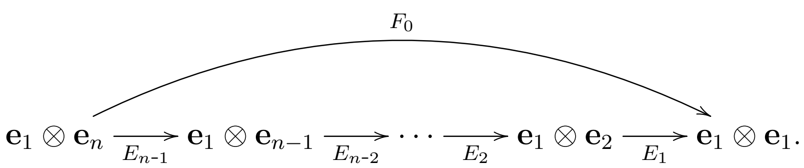
$$\mathcal{R}(\mathbf{e}_1 \otimes \mathbf{e}_n) \in \mathbb{V}(y) \otimes \mathbb{V}(x).$$

So we first describe $\mathcal{R}(\mathbf{e}_i \otimes \mathbf{e}_j)$ in terms of the coefficients f and g defined below and taking advantage of the fact that $\mathcal{R}$ is a $U_q(\mathfrak{sl}_n)$-homomorphism.

(3) Assume

$$\mathcal{R}(\mathbf{e}_1 \otimes \mathbf{e}_n) = f \cdot \mathbf{e}_1 \otimes \mathbf{e}_n + g \cdot \mathbf{e}_n \otimes \mathbf{e}_1$$

for some $f, g \in \mathbb{C}(q)(x, y)$. To determine f and g, we could equalize the image $\mathcal{R}(\mathbf{e}_1 \otimes \mathbf{e}_1)$ in two ways:



Let us describe $\mathcal{R}(\mathbf{e}_i \otimes \mathbf{e}_j)$ in terms of f and g.

Lemma 4.2.1. *If $i < j$, we have*

$$\mathcal{R}(\mathbf{e}_i \otimes \mathbf{e}_j) = f \cdot \mathbf{e}_i \otimes \mathbf{e}_j + g \cdot \mathbf{e}_j \otimes \mathbf{e}_i.$$

Proof. We can compute

$$E_j(\mathbf{e}_i \otimes \mathbf{e}_{j+1}) = E_j\mathbf{e}_i \otimes K_j^{-1}\mathbf{e}_{j+1} + \mathbf{e}_i \otimes E_j\mathbf{e}_{j+1} = 0 + \mathbf{e}_i \otimes \mathbf{e}_j,$$
$$E_j(\mathbf{e}_{j+1} \otimes \mathbf{e}_i) = E_j\mathbf{e}_{j+1} \otimes K_j^{-1}\mathbf{e}_i + \mathbf{e}_{i+1} \otimes E_j\mathbf{e}_i = \mathbf{e}_j \otimes \mathbf{e}_i + 0$$

when $i < j < n$; and

$$F_{i\text{-}1}(\mathbf{e}_{i\text{-}1} \otimes \mathbf{e}_j) = F_{i-1}\mathbf{e}_{i\text{-}1} \otimes \mathbf{e}_j + K_{i-1}\mathbf{e}_{i\text{-}1} \otimes F_{i-1}\mathbf{e}_j = \mathbf{e}_i \otimes \mathbf{e}_j + 0,$$
$$F_{i\text{-}1}(\mathbf{e}_j \otimes \mathbf{e}_{i\text{-}1}) = F_{i-1}\mathbf{e}_j \otimes \mathbf{e}_{i\text{-}1} + K_{i-1}\mathbf{e}_j \otimes F_{i-1}\mathbf{e}_{i\text{-}1} = 0 + \mathbf{e}_i \otimes \mathbf{e}_j$$

when $1 < i < j$. The claim follows from these identities and induction. □

Lemma 4.2.2. *When $i = j$, we have*

$$\mathcal{R}(\mathbf{e}_i \otimes \mathbf{e}_i) = (f + q^{-1} \cdot g) \cdot \mathbf{e}_i \otimes \mathbf{e}_i.$$

Proof. This follows from the similar computation above,

$$E_i(\mathbf{e}_i \otimes \mathbf{e}_{i+1}) = E_i\mathbf{e}_i \otimes K_i^{-1}\mathbf{e}_{i+1} + \mathbf{e}_i \otimes E_i\mathbf{e}_{i+1} = 0 + \mathbf{e}_i \otimes \mathbf{e}_i,$$
$$E_i(\mathbf{e}_{i+1} \otimes \mathbf{e}_i) = E_i\mathbf{e}_{i+1} \otimes K_i^{-1}\mathbf{e}_i + \mathbf{e}_{i+1} \otimes E_i\mathbf{e}_i = \mathbf{e}_i \otimes q^{-1}\mathbf{e}_i + 0$$

when $i < n$; and

$$F_{i\text{-}1}(\mathbf{e}_{i\text{-}1} \otimes \mathbf{e}_i) = F_{i-1}\mathbf{e}_{i\text{-}1} \otimes \mathbf{e}_i + K_{i-1}\mathbf{e}_{i\text{-}1} \otimes F_{i-1}\mathbf{e}_i = \mathbf{e}_i \otimes \mathbf{e}_i + 0,$$
$$F_{i\text{-}1}(\mathbf{e}_i \otimes \mathbf{e}_{i\text{-}1}) = F_{i-1}\mathbf{e}_i \otimes \mathbf{e}_{i\text{-}1} + K_{i-1}\mathbf{e}_i \otimes F_{i-1}\mathbf{e}_{i\text{-}1} = 0 + q^{-1}\mathbf{e}_i \otimes \mathbf{e}_i$$

when $i > 1$. □

Lemma 4.2.3. *If $i > j$, we have*

$$\mathcal{R}(\mathbf{e}_i \otimes \mathbf{e}_j) = (f + (q^{-1} - q)g) \cdot \mathbf{e}_i \otimes \mathbf{e}_j + g \cdot \mathbf{e}_j \otimes \mathbf{e}_i.$$

Proof. We first compute

$$F_j(\mathbf{e}_j \otimes \mathbf{e}_j) = F_j\mathbf{e}_j \otimes \mathbf{e}_i + K_j\mathbf{e}_j \otimes F_j\mathbf{e}_j = \mathbf{e}_{j+1} \otimes \mathbf{e}_j + q\mathbf{e}_j \otimes \mathbf{e}_{j+1}.$$

As a result,

$$\mathcal{R}(\mathbf{e}_{j+1} \otimes \mathbf{e}_j + q\mathbf{e}_j \otimes \mathbf{e}_{j+1}) = (f + q^{-1} \cdot g)\big(\mathbf{e}_{j+1} \otimes \mathbf{e}_i + q\mathbf{e}_j \otimes \mathbf{e}_{j+1}\big)$$

We know from Lemma 4.2.1 that

$$\mathcal{R}(\mathbf{e}_j \otimes \mathbf{e}_{j+1}) = f \cdot \mathbf{e}_j \otimes \mathbf{e}_{j+1} + g \cdot \mathbf{e}_{j+1} \otimes \mathbf{e}_j.$$

Thus

$$\begin{aligned}\mathcal{R}(\mathbf{e}_{\mathrm{j}+1} \otimes \mathbf{e}_j) &= (f + q^{-1} \cdot g)\big(\mathbf{e}_{j+1} \otimes \mathbf{e}_i + q\mathbf{e}_j \otimes \mathbf{e}_{j+1}\big) - q\mathcal{R}(\mathbf{e}_j \otimes \mathbf{e}_{j+1}) \\ &= (f - qg + q^{-1}g) \cdot \mathbf{e}_{j+1} \otimes \mathbf{e}_j + g \cdot \mathbf{e}_j \otimes \mathbf{e}_{j+1}.\end{aligned}$$

For general $i > j$, it follows from identities similar to those used in the proof of Lemma 4.2.1. □

Note that all of the above computations do not involve any spectral parameters x, y, since we only use the finite quantum group $U_q(\mathfrak{sl}_n)$-actions. Actually, since

$$\dim \mathrm{Hom}_{U_q(\mathfrak{sl}_n)}(\mathbb{V} \otimes \mathbb{V}, \mathbb{V} \otimes \mathbb{V}) = 2,$$

any choice of f, g determines a morphism of $U_q(\mathfrak{sl}_n)$. Now let us use the action of F_0 to determine f and g.

LEMMA 4.2.4. *Denote $z = x/y$. We have*

$$f = \frac{(1-q^2)z}{1-q^2 z}, \qquad g = q\frac{1-z}{1-q^2 z}.$$

PROOF. Let us compute in $\mathbb{V}(x) \otimes \mathbb{V}(y)$

$$F_0(\mathbf{e}_1 \otimes \mathbf{e}_n) = F_0\mathbf{e}_1 \otimes \mathbf{e}_n + K_0\mathbf{e}_1 \otimes F_0\mathbf{e}_n = 0 + q^{-1}\mathbf{e}_1 \otimes y^{-1}\mathbf{e}_1,$$
$$F_0(\mathbf{e}_n \otimes \mathbf{e}_1) = F_0\mathbf{e}_n \otimes \mathbf{e}_1 + K_0\mathbf{e}_n \otimes F_0\mathbf{e}_1 = x^{-1}\mathbf{e}_1 \otimes \mathbf{e}_1 + 0.$$

The computation in $\mathbb{V}(y) \otimes \mathbb{V}(x)$ is obtained by swapping the roles of x and y. As a result,

$$\mathcal{R}(q^{-1}\mathbf{e}_1 \otimes y^{-1}\mathbf{e}_1) = f \cdot q^{-1}\mathbf{e}_1 \otimes x^{-1}\mathbf{e}_1 + g \cdot y^{-1}\mathbf{e}_1 \otimes \mathbf{e}_1.$$

That is,

$$\mathcal{R}(\mathbf{e}_1 \otimes \mathbf{e}_1) = (x^{-1}y \cdot f + q \cdot g)\mathbf{e}_1 \otimes \mathbf{e}_1.$$

By the normalization, we can assume $\mathcal{R}(\mathbf{e}_1 \otimes \mathbf{e}_1) = \mathbf{e}_1 \otimes \mathbf{e}_1$, so we have

$$f + q^{-1} \cdot g = 1, \qquad x^{-1}y \cdot f + q \cdot g = 1.$$

So we can solve for f and g, as asserted. □

Substituting the f and g above, we have

LEMMA 4.2.5. *The spectral R-matrix is given by*

$$R(z)(\mathbf{e}_i \otimes \mathbf{e}_j) = \begin{cases} \frac{(1-q^2)z}{1-q^2 z}\mathbf{e}_i \otimes \mathbf{e}_j + q\frac{1-z}{1-q^2 z}\mathbf{e}_j \otimes \mathbf{e}_i, & i < j, \\ \mathbf{e}_i \otimes \mathbf{e}_i, & i = j, \\ \frac{1-q^2}{1-q^2 z}\mathbf{e}_i \otimes \mathbf{e}_j + q\frac{1-z}{1-q^2 z}\mathbf{e}_j \otimes \mathbf{e}_i, & i > j. \end{cases} \tag{4.2.5}$$

4.3. Vertex models

We have seen the Yang–Baxter equation as a property of R-matrices. But the existence of R-matrices with Yang–Baxter equations does not always follow from the representation theory of a known algebra. Actually, new R-matrices usually lead to new interesting quantum algebras.

4.3.1. Six-vertex model. Historically, the R-matrix was first found in statistical physics in the study of the **six-vertex model**. The mathematical formulation is the following. Let $V = \Bbbk \cdot \mathbf{e}_1 \oplus \Bbbk \cdot \mathbf{e}_2$ be a two-dimensional space over some field $\Bbbk$. We want to find some matrix (also called an R-matrix)

$$R(z) \in \mathrm{Hom}(V \otimes V, V \otimes V)(z)$$

satisfying the **Yang–Baxter equation**

$$(R(x) \otimes 1)(1 \otimes R(xy))(R(y) \otimes 1)$$
$$= (1 \otimes R(y))(R(xy) \otimes 1)(1 \otimes R(x)).$$

From the computation of the previous section, we see the assumption that

$$R(x)(\mathbf{e}_i \otimes \mathbf{e}_j) \in \mathrm{span}(\mathbf{e}_i \otimes \mathbf{e}_j, \mathbf{e}_j \otimes \mathbf{e}_i)$$

could make the computation simpler. This is exactly what the **six-vertex model** refers to. More precisely, the **six-vertex model** is the case of the R-matrix taking the form

$$R(z) = \left[\begin{array}{cc|cc} a_1(z) & 0 & 0 & 0 \\ 0 & b_2(z) & c_1(z) & 0 \\ \hline 0 & c_2(z) & b_1(z) & 0 \\ 0 & 0 & 0 & a_2(z) \end{array}\right] \qquad \begin{aligned} R(z)(\mathbf{e}_1 \otimes \mathbf{e}_1) &= a_1(z) \cdot \mathbf{e}_1 \otimes \mathbf{e}_1 \\ R(z)(\mathbf{e}_2 \otimes \mathbf{e}_2) &= a_2(z) \cdot \mathbf{e}_2 \otimes \mathbf{e}_2 \\ R(z)(\mathbf{e}_1 \otimes \mathbf{e}_2) &= b_1(z) \cdot \mathbf{e}_1 \otimes \mathbf{e}_2 + c_1(z) \cdot \mathbf{e}_2 \otimes \mathbf{e}_1 \\ R(z)(\mathbf{e}_2 \otimes \mathbf{e}_1) &= b_2(z) \cdot \mathbf{e}_1 \otimes \mathbf{e}_2 + c_2(z) \cdot \mathbf{e}_2 \otimes \mathbf{e}_1. \end{aligned}$$

We will further assume

$$a_1(z), a_2(z), b_1(z), b_2(z), c_1(z), c_2(z)$$

are all non-zero functions. The Yang–Baxter equation is equivalent to

$$\begin{aligned}
a_1(x)b_2(xy)a_1(y) &= b_2(x)a_1(xy)b_2(y) + c_2(x)b_2(xy)c_1(y) \\
a_1(x)c_1(xy)b_2(y) &= c_1(x)a_1(xy)b_2(y) + b_1(x)b_2(xy)c_1(y) \\
b_2(x)c_2(xy)a_1(y) &= c_2(x)b_2(xy)b_1(y) + b_2(x)a_1(xy)c_2(y) \\
b_2(x)b_1(xy)b_2(y) &= b_1(x)b_2(xy)b_1(y) \\
c_1(x)a_1(xy)b_1(y) + b_2(x)b_1(xy)c_1(y) &= a_1(x)c_1(xy)b_1(y) \\
b_2(x)a_2(xy)b_2(y) + c_1(x)b_2(xy)c_2(y) &= a_2(x)b_2(xy)a_2(y) \\
c_1(x)b_2(xy)b_1(y) + b_2(x)a_2(xy)c_1(y) &= b_2(x)c_1(xy)a_2(y) \\
c_2(x)b_1(xy)b_2(y) + b_1(x)a_1(xy)c_2(y) &= b_1(x)c_2(xy)a_1(y) \\
b_1(x)a_1(xy)b_1(y) + c_2(x)b_1(xy)c_1(y) &= a_1(x)b_1(xy)a_1(y) \\
c_2(x)a_2(xy)b_2(y) + b_1(x)b_2(xy)c_2(y) &= a_2(x)c_2(xy)b_2(y) \\
b_1(x)b_2(xy)b_1(y) &= b_2(x)b_1(xy)b_2(y) \\
b_1(x)c_1(xy)a_2(y) &= c_1(x)b_1(xy)b_2(y) + b_1(x)a_2(xy)c_1(y) \\
a_2(x)c_2(xy)b_1(y) &= c_2(x)a_2(xy)b_1(y) + b_2(x)b_1(xy)c_2(y) \\
a_2(x)b_1(xy)a_2(y) &= b_1(x)a_2(xy)b_1(y) + c_1(x)b_1(xy)c_2(y)
\end{aligned}$$

The purpose of this section is to partially solve these functional equations.

LEMMA 4.3.1. *To have a solution of the Yang–Baxter equation, it is necessary to have*

$$\frac{a_1(z)a_2(z) - b_1(z)b_2(z) + c_1(z)c_2(z)}{a_i(z)c_j(z)} \in \mathbb{k}, \qquad \forall i, j \in \{1, 2\}.$$

EXAMPLE 4.3.2. *Recall the R-matrix in* (4.2.3) *or* (4.2.5) *from the previous sections:*

$$\left[\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & \dfrac{q^2z - z}{q^2z - 1} & \dfrac{-qz + q}{-q^2z + 1} & 0 \\ \hline 0 & \dfrac{-qz + q}{-q^2z + 1} & \dfrac{q^2 - 1}{q^2z - 1} & 0 \\ 0 & 0 & 0 & 1 \end{array}\right] \qquad \begin{aligned} a_1(z) &= a_2(z) = 1, \\ b_1(z) &= \frac{q^2 - 1}{q^2z - 1}, \\ b_2(z) &= \frac{(q^2 - 1)z}{q^2z - 1}, \\ c_1(z) &= c_2(z) = q\frac{z - 1}{q^2z - 1}. \end{aligned}$$

We have

$$\frac{a_1(z)a_2(z) - b_1(z)b_2(z) + c_1(z)c_2(z)}{a_i(z)c_j(z)} = q + q^{-1},$$

a constant.

The proof of Lemma 4.3.1 will be given in Section 4.3.2 below. Our next step is to divide the solution into two families, depending on whether

$$a_1(z)a_2(z) - b_1(z)b_2(z) + c_1(z)c_2(z) = 0.$$

Before that, we mention some "freedom" of $R(z)$-matrices, which could be used to normalize our solution.

LEMMA 4.3.3. *If $R(z)$ satisfies the Yang–Baxter equation, then so does the matrix*

$$\tilde{R}(z) = f \left[\begin{array}{cc|cc} a_1(z^m) & 0 & 0 & 0 \\ 0 & b_2(z^m) & g \cdot c_1(z^m) & 0 \\ \hline 0 & g^{-1}c_2(z^m) & b_1(z^m) & 0 \\ 0 & 0 & 0 & a_2(z^m) \end{array}\right], \qquad \begin{array}{c} f \in \Bbbk(z)^\times, \\ g \in \Bbbk^\times, \\ m \in \mathbb{Z}. \end{array}$$

PROOF. It suffices to deal with three cases

(i) $f = 1$ and $g = 1$, (ii) $f = 1$ and $m = 1$, (iii) $g = 1$ and $m = 1$.

The cases (i) and (iii) are both trivial. The case (ii) can be seen directly from the equations since the change preserves the Yang–Baxter equation. More precisely, either $c_1(?)c_2(?)$ comes in pair, or $c_i(?)$ appears singly in each factor, e.g.

$$\begin{aligned} \cdots &= \cdots \\ b_1(x)a_1(xy)b_1(y) + c_2(x)b_1(xy)c_1(y) &= a_1(x)b_1(xy)a_1(y) \\ c_2(x)a_2(xy)b_2(y) + b_1(x)b_2(xy)c_2(y) &= a_2(x)c_2(xy)b_2(y) \\ \cdots &= \cdots \end{aligned}$$

There is another, more conceptual way of seeing (ii). We can construct a matrix

$$Q(\mathbf{e}_i \otimes \mathbf{e}_j) = g^{\delta_{i>j}} \mathbf{e}_i \otimes \mathbf{e}_j.$$

Then in case (ii), we have

$$\tilde{R}(z) = Q \circ R(z) \circ Q^{-1}.$$

also satisfies the Yang–Baxter equation. The trick is, we can construct matrix

$$\widehat{Q}(\mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k) = q^{\delta_{i>j}+\delta_{i>k}+\delta_{j>k}} \mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k$$

such that

$$\begin{aligned} \widehat{Q}(R(z) \otimes 1)\widehat{Q}^{-1} &= (QR(z)Q^{-1} \otimes 1), \\ \widehat{Q}(1 \otimes R(z))\widehat{Q}^{-1} &= (1 \otimes QR(z)Q^{-1}). \end{aligned}$$

So

$$\begin{aligned} &(\tilde{R}(x) \otimes 1)(1 \otimes \tilde{R}(xy))(\tilde{R}(y) \otimes 1) \\ &= \widehat{Q}(R(x) \otimes 1)(1 \otimes R(xy))(R(y) \otimes 1)\widehat{Q}^{-1} \\ &= \widehat{Q}(1 \otimes R(y))(R(xy) \otimes 1)(1 \otimes R(x))\widehat{Q}^{-1} \\ &= (1 \otimes \tilde{R}(y))(\tilde{R}(xy) \otimes 1)(1 \otimes \tilde{R}(x)). \end{aligned}$$ □

Case A. When $a_1(z)a_2(z) - b_1(z)b_2 + c_1(z)c_2(z) \neq 0$, we have

$$a_1(z)/a_2(z) \in \Bbbk, \qquad c_1(z)/c_2(z) \in \Bbbk$$

are both constants. By Lemma 4.3.3, we can normalize $R(z)$ such that

$$a_1(z) = 1, \qquad a_2(z) = a_2 \in \Bbbk, \qquad c(z) := c_1(z) = c_2(z).$$

Assume

$$\Delta = \frac{a_2 - b_1(z)b_2(z) + c(z)^2}{c(z)} \in \Bbbk.$$

Then we have

$$a_2 = 1 \tag{4.3.1}$$

$$b_1(y/x) = \frac{b_1(y)b_2(x)}{-\Delta c(x) + c(x)c(y) + 1} \tag{4.3.2}$$

$$b_2(y/x) = \frac{b_1(x)b_2(y)}{-\Delta c(x) + c(x)c(y) + 1} \tag{4.3.3}$$

$$c(y/x) = \frac{c(y) - c(x)}{-\Delta c(x) + c(x)c(y) + 1}. \tag{4.3.4}$$

This is done using computer, and the following is the code. Here we used the fact

$$b_1(z) = \frac{1 - \Delta c_1(z) + c_1(z)^2}{b_2(z)}$$

to reduce the number of variables.

```
# 6-vertex model
S = PolynomialRing(QQ,["D","a2","b2x","cx","b2y","cy"]);
x = S.gens(); exec("; ".join("%s = x[%s]"%(x[i],i) for i in range(len(x))))
a1x = 1; a2x = a2; c1x = c2x = cx
b1x = (a2x-D*c1x+c1x^2)/b2x
a1y = 1; a2y = a2;c1y = c2y = cy
b1y = (a2y-D*c1y+c1y^2)/b2y
Poly = PolynomialRing(S,[abc_name+i+"z" for abc_name in ["a","b","c"] for i in ["1","2
    "]]);
x = Poly.gens(); exec("; ".join("%s = x[%s]"%(x[i],i) for i in range(len(x))))
# a1x = a1, etc; a1y = a1', etc; a1z = a1'', etc;

K_Poly = FractionField(Poly)
Rx = matrix(K_Poly,[[a1x ,0,0,0],[0,b2x ,c1x ,0],[0,c2x ,b1x ,0],[0,0,0,a2x ]])
Ry = matrix(K_Poly,[[a1y ,0,0,0],[0,b2y ,c1y ,0],[0,c2y ,b1y ,0],[0,0,0,a2y ]])
Rz = matrix(K_Poly,[[a1z ,0,0,0],[0,b2z ,c1z ,0],[0,c2z ,b1z ,0],[0,0,0,a2z ]])
I  = matrix(K_Poly,[[1,0],[0,1]])
def ten(A,B): return A.tensor_product(B)

LHS = ten(Rx,I)*ten(I,Ry)*ten(Rz,I);
RHS = ten(I,Rz)*ten(Ry,I)*ten(I,Rx)

Rel = []
for (i,j) in [(i,j) for i in range(8) for j in range(8)]:
    lhs = LHS[i][j]; rhs = RHS[i][j]
    if lhs != rhs:
        Rel.append(lhs-rhs)

def linear(f): # f is a linear function in S, return the corresponding vector
```

```
    f = Poly(K_Poly(f).numerator())
    res = [0 for i in range(6)]
    my_dict = f.dict(); # print(f.dict())
    for ind in my_dict:
        i = 0
        while ind[i]==0 :i+=1
        res[i] = S(my_dict[ind])
    return res
Rel = [linear(rel) for rel in Rel]; # print(matrix(Rel))
Rel = matrix(FractionField(S),Rel)
sol = Rel.transpose().kernel().basis()[0]
[a1z,a2z,b1z,b2z,c1z,c2z] = sol
for ind in sol:
    print(ind.factor())
print(a2z == 1)
print(b1z == b1y*b2x/(-D*cx + cx*cy + a2))
print(b2z == b1x*b2y/(-D*cx + cx*cy +  a2))
print(c1z == (c1y-c1x)/(-D*cx + cx*cy +  a2))
```

Now let us solve the function equation (4.3.2), (4.3.3) and (4.3.4).

- Let us assume $\Delta = q + q^{-1}$ for some q in some field extension of $\Bbbk$. The trick is, we can write
$$\frac{y-x}{-(q+q^{-1})x + xy + 1} = e^{-1}\left(\frac{e(y)}{e(x)}\right)$$
where $e(z) = \frac{qz-q}{q^2z-1}$ with $e^{-1}(z) = \frac{qz-q}{q^2z-1}$. As a result, (4.3.4) is equivalent to
$$e(c(y/x)) = \frac{e(c(y))}{e(c(x))}.$$
The rational function solutions are of the form $e(c(z)) = z^n$ for some $n \in \mathbb{Z}$. That is,
$$c(z) = q\frac{z^n - 1}{q^2z^n - 1}.$$
- By (4.3.2) and (4.3.3), we have
$$\frac{b_1(y/x)}{b_2(y/x)} = \frac{b_1(y)}{b_2(y)}\Big/\frac{b_1(x)}{b_2(x)}.$$
This implies $\frac{b_2(z)}{b_1(z)} = z^m$ for some $m \in \mathbb{Z}$.
- It remains to solve
$$b_1(y/x) = \frac{x^m \cdot b_1(y)b_1(x)(q^2y^n - 1)(q^2x^n - 1)}{(q^2y^n - x^n)(q^2 - 1)}.$$
We can write it as
$$f(y/x) = f(y)f(x), \qquad \text{where} \quad f(z) = \frac{b_1(z)(q^2z^n - 1)}{q^2 - 1}z^{\frac{m-n}{2}}.$$
This implies $f(z) = f(z^{-1})$ and thus $f(y/x) = f(y)f(1/x)$. Thus $f = 1$. So
$$b_1(z) = z^{\frac{n-m}{2}}\frac{q^2-1}{q^2z^n-1}, \qquad b_2(z) = z^{\frac{n+m}{2}}\frac{q^2-1}{q^2z^n-1}.$$

In conclusion, in this case, we get the following family of solutions of the Yang–Baxter equation

$$R(z) = f \left[\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & z^{\frac{n+m}{2}}\dfrac{q^2-1}{q^2z^n-1} & g\cdot\dfrac{z^n-1}{q^2z^n-1} & 0 \\ \hline 0 & g^{-1}\dfrac{z^n-1}{q^2z^n-1} & z^{\frac{n-m}{2}}\dfrac{q^2-1}{q^2z^n-1} & 0 \\ 0 & 0 & 0 & 1 \end{array}\right], \qquad \begin{array}{c} f\in \Bbbk(z)^\times, \\ g\in\Bbbk^\times, \\ q\in\Bbbk^\times, \\ n\in\mathbb{Z}\setminus\{0\} \\ m\in n+2\mathbb{Z}. \end{array}$$

This is a generalization of Example 4.3.2.

Case B. When $a_1(z)a_2(z)-b_1(z)b_2+c_1(z)c_2(z)=0$. By a "rational group", we mean a group object in the category of varieties with rational morphisms (morphism defined on a nonempty open subset). Let

$$\tilde{\mathbb{G}} = \left\{(a_1,a_2,b_1,b_2,c_1,c_2)\in\mathbb{C}^\times : a_1a_2-b_1b_2+c_1c_2=0\right\}$$

Then $\mathbb{G}=\tilde{\mathbb{G}}/\mathbb{C}^\times$ is rational group with product determined by

$$(a_1,\dots,c_1)\cdot(a_1',\dots,c_1')^{-1} = (a_1'',\dots,c_1'')$$

the right-hand side $(a_1'',\dots,c_1'')$ is the unique vector satisfying the Yang–Baxter equation up to constant, under the convention of Section 4.3.2. Note that the product is defined over an open subset since the right-hand side is not necessarily in $\mathbb{G}$.

The group structure of $\mathbb{G}$ was described by Brubaker, Bump and Friedberg [**9**]. We consider a birational group homomorphism

$$\Delta:\tilde{\mathbb{G}}\longrightarrow H\subset GL_4, \qquad H=\left\{\begin{bmatrix} b_1 & & \\ & A & \\ & & b_2\end{bmatrix} : \det A = b_1b_2\right\}$$

by

$$(a_1,a_2,b_1,b_2,c_1,c_2)\longmapsto \begin{bmatrix} b_1 & 0 & 0 & 0 \\ 0 & a_1 & -c_1 & 0 \\ 0 & c_2 & a_2 & 0 \\ 0 & 0 & 0 & b_2\end{bmatrix}^{-1}.$$

Direct computation shows that the rational group structure over $\tilde{\mathbb{G}}$ obtained by Δ is a lifting of $\mathbb{G}$. That is,

$$\begin{aligned} a_1'' &= (a_2a_1'+c_2c_1')/(a_1a_2+c_1c_2) \\ a_2'' &= (a_1a_2'+c_1c_2')/(a_1a_2+c_1c_2) \\ b_1'' &= b_1'/b_1 \\ b_2'' &= b_2'/b_2 \\ c_1'' &= -(c_1a_1'-a_1c_1')/(a_1a_2+c_1c_2) \\ c_2'' &= (-c_2a_2'+a_2c_2')/(a_1a_2+c_1c_2) \end{aligned}$$

is the unique non-zero solution of Yang–Baxter equation up to a constant. As a result, $\mathbb{G}$ has a birational group homomorphism to $H/\mathbb{C}^\times\cong GL_2$.

Note that a solution of $R(z)$ is equivalent to a rational group homomorphism $\mathbb{C}^\times\to\mathbb{G}$. By the discussion above, it is equivalent to a rational group homomorphism $\chi:\mathbb{C}^\times\to GL_2$.

Example 4.3.4. *Consider the following R-matrices from quantum supgroups $U_q(L\mathfrak{sl}(1|1))$*

$$\left[\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & \frac{q^2z-z}{q^2z-1} & \frac{-qz+q}{-q^2z+1} & 0 \\ \hline 0 & \frac{-qz+q}{-q^2z+1} & \frac{q^2-1}{q^2z-1} & 0 \\ 0 & 0 & 0 & \frac{q^2-z}{q^2-1} \end{array}\right] \qquad \begin{aligned} a_1(z) &= 1, \\ a_2(z) &= \frac{q^2-z}{q^z-1} \\ b_1(z) &= \frac{q^2-1}{q^2z-1}, \\ b_2(z) &= \frac{(q^2-1)z}{q^2z-1}, \\ c_1(z) = c_2(z) &= q\frac{z-1}{q^2z-1}. \end{aligned}$$

Note that the only difference comparing with te Example 4.3.2 is $a_2(z)$. We have

$$a_1(z)a_2(z) - b_1(z)b_2(z) + c_1(z)c_2(z) = 0$$

The corresponding rational group homomorphism is

$$\mathbb{C}^\times \to GL_2, \qquad z \mapsto \frac{1}{q^2-1}\begin{bmatrix} q^2z-1 & -qz+q \\ qz-q & q^2-z \end{bmatrix}.$$

4.3.2. Proof of Lemma 4.3.1. In the following computation, we will denote

$$\begin{array}{llllll} a_1 = a_1(x) & a_2 = a_2(x) & b_1 = b_1(x) & b_1 = b_2(x) & c_1 = c_1(x) & c_2 = c_2(x), \\ a'_1 = a_1(xy) & a'_2 = a_2(xy) & b'_1 = b_1(xy) & b'_1 = b_2(xy) & c'_1 = c_1(xy) & c'_2 = c_2(xy), \\ a''_1 = a_1(y) & a''_2 = a_2(y) & b''_1 = b_1(y) & b''_1 = b_2(y) & c''_1 = c_1(y) & c''_2 = c_2(y). \end{array}$$

By taking generic x, y, z, we can assume

$$a_1, \ldots, c_2, \qquad a'_1, \ldots, c'_2, \qquad a''_1, \ldots, c''_2$$

are all nonzero.

We can view the Yang–Baxter equation as a linear function in $a''_1, \ldots, c''_2$ with coefficients in $a_1, \ldots, c_1$ and $a'_1, \ldots, c'_2$. That is,

$$\begin{bmatrix} a_1b'_2 & 0 & 0 & -b_2a'_1 & -c_2b'_2 & 0 \\ 0 & 0 & 0 & -c_1a'_1 + a_1c'_1 & -b_1b'_2 & 0 \\ b_2c'_2 & 0 & -c_2b'_2 & 0 & 0 & -b_2a'_1 \\ 0 & 0 & -b_1b'_2 & b_2b'_1 & 0 & 0 \\ 0 & 0 & c_1a'_1 - a_1c'_1 & 0 & b_2b'_1 & 0 \\ 0 & -a_2b'_2 & 0 & b_2a'_2 & 0 & c_1b'_2 \\ 0 & -b_2c'_1 & c_1b'_2 & 0 & b_2a'_2 & 0 \\ -b_1c'_2 & 0 & 0 & c_2b'_1 & 0 & b_1a'_1 \\ -a_1b'_1 & 0 & b_1a'_1 & 0 & c_2b'_1 & 0 \\ 0 & 0 & 0 & c_2a'_2 - a_2c'_2 & 0 & b_1b'_2 \\ 0 & 0 & b_1b'_2 & -b_2b'_1 & 0 & 0 \\ 0 & b_1c'_1 & 0 & -c_1b'_1 & -b_1a'_2 & 0 \\ 0 & 0 & -c_2a'_2 + a_2c'_2 & 0 & 0 & -b_2b'_1 \\ 0 & a_2b'_1 & -b_1a'_2 & 0 & 0 & -c_1b'_1 \end{bmatrix} \cdot \begin{bmatrix} a''_1 \\ a''_2 \\ b''_1 \\ b''_2 \\ c''_1 \\ c''_2 \end{bmatrix} = 0.$$

To have a non-trivial solution, we need to assume this matrix has rank < 6, i.e. the minor of size 6 has to vanish. Let us consider the ideal generated by the minors in the Laurent polynomial ring

$$\mathbb{Q}[a_1^{\pm 1}, \ldots, c_2^{\pm 1}, a_1'^{\pm 1}, \ldots, c_2'^{\pm 1}].$$

With the assistance of computer, we are able to describe these relations:

$$\frac{a_1'a_2' - b_1'b_2' + c_1'c_2'}{a_1'c_2'} = \frac{a_1a_2 - b_1b_2 + c_1c_2}{a_1c_2} \tag{4.3.5}$$

$$\frac{a_1'a_2' - b_1'b_2' + c_1'c_2'}{a_2'c_1'} = \frac{a_1a_2 - b_1b_2 + c_1c_2}{a_2c_1} \tag{4.3.6}$$

Note that (4.3.5) and (4.3.6) imply

$$\frac{a_2'c_1'}{a_1'c_2'} = \frac{a_2c_1}{a_1c_2}$$

if $a_1a_2 - b_1b_2 + c_1c_2 \neq 0$. The following is the code. More precisely, we first compute the list of the determinants of those minors, and normalize them by dividing each of them by a monomials if necessary. In the polynomial ring $\mathbb{Q}[a_1, \ldots, c_2, a_1', \ldots, c_2']$, the ideal generated by the relation has two associated primes. It is easy to see that in the Laurent polynomial ring, they reduce to one prime.

```
# 6-vertex model
S = PolynomialRing(QQ,[abc_name+i+var for var in ["x","y"] for abc_name in ["a","b","c"] for i in ["1","2"]]);
x = S.gens(); exec("; ".join("%s = x[%s]"%(x[i],i) for i in range(len(x))))
Poly = PolynomialRing(S,[abc_name+i+"z" for abc_name in ["a","b","c"] for i in ["1","2"]]);
x = Poly.gens(); exec("; ".join("%s = x[%s]"%(x[i],i) for i in range(len(x))))
# a1x = a1, etc; a1y = a1', etc; a1z = a1'', etc;

def my_print(f):
    res = str(f)
#     for xx,yy in [(abc_name+i+var,"%s_%s(%s)"%(abc_name,i,{"x":"x","y":"y","z":"xy"}[var])) for var in ["x","y","z"] for abc_name in ["a","b","c"] for i in ["1","2"]]:
    for xx,yy in [(abc_name+i+var,"%s_%s%s"%(abc_name,i,{"x":"","y":"'","z":"''"}[var])) for var in ["x","y","z"] for abc_name in ["a","b","c"] for i in ["1","2"]]:
        res = res.replace(xx,yy)
    return res.replace("*","")

Rx = matrix(Poly,[[a1x ,0,0,0],[0,b2x ,c1x ,0],[0,c2x ,b1x ,0],[0,0,0,a2x ]])
Ry = matrix(Poly,[[a1y ,0,0,0],[0,b2y ,c1y ,0],[0,c2y ,b1y ,0],[0,0,0,a2y ]])
Rz = matrix(Poly,[[a1z ,0,0,0],[0,b2z ,c1z ,0],[0,c2z ,b1z ,0],[0,0,0,a2z ]])
I  = matrix(Poly,[[1,0],[0,1]])
def ten(A,B): return A.tensor_product(B)

LHS = ten(Rx,I)*ten(I,Ry)*ten(Rz,I);
RHS = ten(I,Rz)*ten(Ry,I)*ten(I,Rx)

Rel = []
for (i,j) in [(i,j) for i in range(8) for j in range(8)]:
    lhs = LHS[i][j]; rhs = RHS[i][j]
    if lhs != rhs:
        print("%s & = %s \\\\"%(my_print(lhs),my_print(rhs)))
        Rel.append(lhs-rhs)
```

```
def linear(f): # f is a linear function in S, return the corresponding vector
    f = Poly(f)
    res = [0 for i in range(6)]
```

```
    my_dict = f.dict(); # print(f.dict())
    for ind in my_dict:
        i = 0
        while ind[i]==0 :i+=1
        res[i] = S(my_dict[ind])
    return res
Rel = [linear(rel) for rel in Rel]; # print(Rel)

# Rel.sort(key = lambda ind: tuple(i==0 for i in ind))
print(("\\left[\\begin{matrix}\n%s\n\\end{matrix}\\right]"%("\\\\\n".join(" & ".join(
    my_print(f) for f in row) for row in Rel))).replace("-","-"))
```

```
# assume ai, bi, ci! = 0
def red(f): # return the polynomial g such that f = g * (a monomial)
    f = S(f); my_dict = f.dict()
    expo = list(my_dict.keys())[0]
    for ind in my_dict:
        # print(f,ind)
        expo = [min(ind[i],expo[i]) for i in range(len(expo))]
    return S(f/S.monomial(*expo))

Dets = []
for A in Subsets(len(Rel),6):
    det = matrix([Rel[s-1] for s in A]).determinant()
    if det!=0: Dets.append(red(S(det)))
print("relations computed")

I = S.ideal(Dets);
ap = I.associated_primes()
print("prime ideal computed")

for prime in ap:
    B = prime.gens() # B = prime.groebner_basis()
    for b in B:
        print(my_print(b))
    print()
```

We can also view the Yang–Baxter equation as linear functions in $a_1, \ldots, c_2$ with coefficients in $a'_1, \ldots, c'_2$ and $a''_1, \ldots, c''_1$. Using the above algorithm, we can get the following relations

$$\frac{a''_1 a''_2 - b''_1 b''_2 + c''_1 c''_2}{a''_1 c''_1} = \frac{a'_1 a'_2 - b'_1 b'_2 + c'_1 c'_2}{a'_1 c'_1}, \tag{4.3.7}$$

$$\frac{a''_1 a''_2 - b''_1 b''_2 + c''_1 c''_2}{a''_2 c''_2} = \frac{a'_1 a'_2 - b'_1 b'_2 + c'_1 c'_2}{a'_2 c'_2}. \tag{4.3.8}$$

Note that the right-hand sides are different from the left-hand sides of (4.3.5) and (4.3.6).

There is a more direct way of doing this, by solving the Yang–Baxter equation under the assumption (4.3.5) and (4.3.6). That is, we are working over the fraction field of

$$\mathbb{Q}[a_1^{\pm 1}, \ldots, c_2^{\pm 1}, a_1'^{\pm 1}, \ldots, c_2'^{\pm 1}] / \langle (4.3.5), (4.3.6) \rangle \,.$$

Now the kernel of the matrix has dimension 1 and we can compute explicitly a non-zero vector

$$\begin{aligned}
a_1'' &= -b_1 b_2 c_2 a_2' b_2' c_1' + c_1 c_2^2 a_2' b_2' c_1' - a_2 c_1 c_2 b_2' c_1' c_2' \\
a_2'' &= -b_1 b_2 c_1 a_2' b_2' c_2' + c_1^2 c_2 a_2' b_2' c_2' - a_2 c_1^2 b_2' c_2'^2 \\
b_1'' &= -a_1 b_2 c_2 a_2'^2 c_1' + a_1 a_2 b_2 a_2' c_1' c_2' - b_1 b_2^2 a_2' c_1' c_2' + b_2 c_1 c_2 a_2' c_1' c_2' - a_2 b_2 c_1 c_1' c_2'^2 \\
b_2'' &= -a_2 b_1 c_1 b_2'^2 c_2' \\
c_1'' &= a_1 c_1 c_2 a_2' b_2' c_1' - a_1 a_2 c_1 b_2' c_1' c_2' \\
c_2'' &= a_2 c_1 c_2 a_2' b_2' c_2' - a_2^2 c_1 b_2' c_2'^2
\end{aligned}$$

Direct computation shows (4.3.7) and (4.3.8).

The following is the code. More precisely, we denote $\Delta_1 = (4.3.5)$ and $\Delta_2 = (4.3.6)$. Then

$$a_1 = \frac{\Delta_2 c_1 a_2}{\Delta_1 c_2}, \qquad a_1' = \frac{\Delta_2 c_1' a_2'}{\Delta_1 c_2'},$$
$$b_1 = -\frac{\Delta_2 c_1 a_2 - a_1 a_2 - c_1 c_2}{b_2} \qquad b_1' = -\frac{\Delta_2 c_1' a_2' - a_1' a_2' - c_1' c_2'}{b_2'}.$$

Then the fraction field is isomorphic to

$$\mathbb{Q}(\Delta_1, \Delta_2,\ a_2, b_2, c_1, c_2,\ a_2', b_2', c_1', c_2').$$

```
# 6-vertex model
S = PolynomialRing(QQ,["D1","D2"]+[abc_name+i+var for var in ["x","y"] for abc_name in ["a","b","c"] for i in ["1","2"] if abc_name+i not in ["a1","b1"]]);
x = S.gens(); exec("; ".join("%s = x[%s]"%(x[i],i) for i in range(len(x))))
a1x = (D2*c1x*a2x)/(D1*c2x)
b1x = -(D2*c1x*a2x - a1x*a2x - c1x*c2x)/b2x
a1y = (D2*c1y*a2y)/(D1*c2y)
b1y = -(D2*c1y*a2y - a1y*a2y - c1y*c2y)/b2y
Poly = PolynomialRing(S,[abc_name+i+"z" for abc_name in ["a","b","c"] for i in ["1","2"]]);
x = Poly.gens(); exec("; ".join("%s = x[%s]"%(x[i],i) for i in range(len(x))))
# a1x = a1, etc; a1y = a1', etc; a1z = a1'', etc;

K_Poly = FractionField(Poly)
Rx = matrix(K_Poly,[[a1x ,0,0,0],[0,b2x ,c1x ,0],[0,c2x ,b1x ,0],[0,0,0,a2x ]])
Ry = matrix(K_Poly,[[a1y ,0,0,0],[0,b2y ,c1y ,0],[0,c2y ,b1y ,0],[0,0,0,a2y ]])
Rz = matrix(K_Poly,[[a1z ,0,0,0],[0,b2z ,c1z ,0],[0,c2z ,b1z ,0],[0,0,0,a2z ]])
I  = matrix(K_Poly,[[1,0],[0,1]])
def ten(A,B): return A.tensor_product(B)

LHS = ten(Rx,I)*ten(I,Ry)*ten(Rz,I);
RHS = ten(I,Rz)*ten(Ry,I)*ten(I,Rx)

Rel = []
for (i,j) in [(i,j) for i in range(8) for j in range(8)]:
    lhs = LHS[i][j]; rhs = RHS[i][j]
    if lhs != rhs:
        Rel.append(lhs-rhs)

def linear(f): # f is a linear function in S, return the corresponding vector
    f = Poly(K_Poly(f).numerator())
    res = [0 for i in range(6)]
```

```
    my_dict = f.dict(); # print(f.dict())
    for ind in my_dict:
        i = 0
        while ind[i]==0 :i+=1
        res[i] = S(my_dict[ind])
    return res
Rel = [linear(rel) for rel in Rel]; # print(matrix(Rel))
Rel = matrix(FractionField(S),Rel)
sol = Rel.transpose().kernel().basis()[0]
[a1z,a2z,b1z,b2z,c1z,c2z] = sol
Dx = a1x*a2x - b1x*b2x + c1x*c2x
Dy = a1y*a2y - b1y*b2y + c1y*c2y
Dz = a1z*a2z - b1z*b2z + c1z*c2z
# print(Dx/(a1x*c2x) == Dy/(a1y*c2y))
# print(Dx/(a2x*c1x) == Dy/(a2y*c1y))
print(Dz/(a1z*c1z) == Dy/(a1y*c1y))
print(Dz/(a2z*c2z) == Dy/(a2y*c2y))
# print(Dx, Dz)

K_S = FractionField(S);
Rx = matrix(K_S,[[a1x ,0,0,0],[0,b2x ,c1x ,0],[0,c2x ,b1x ,0],[0,0,0,a2x ]])
Ry = matrix(K_S,[[a1y ,0,0,0],[0,b2y ,c1y ,0],[0,c2y ,b1y ,0],[0,0,0,a2y ]])
Rz = matrix(K_S,[[a1z ,0,0,0],[0,b2z ,c1z ,0],[0,c2z ,b1z ,0],[0,0,0,a2z ]])
I  = matrix(K_S,[[1,0],[0,1]])
LHS = ten(Rx,I)*ten(I,Ry)*ten(Rz,I);
RHS = ten(I,Rz)*ten(Ry,I)*ten(I,Rx)
# LHS - RHS
```

```
oldS = PolynomialRing(QQ,[abc_name+i+var for var in ["x","y"] for abc_name in ["a","b","c"] for i in ["1","2"]]);
x = oldS.gens(); exec("; ".join("%s = x[%s]"%(x[i],i) for i in range(len(x))))
D1 = (a1x*a2x-b1x*b2x+c1x*c2x)/(a1x*c2x)
D2 = (a1x*a2x-b1x*b2x+c1x*c2x)/(a2x*c1x)
# D1 = (a1y*a2y-b1y*b2y+c1y*c2y)/(a1y*c2y)
# D2 = (a1y*a2y-b1y*b2y+c1y*c2y)/(a2y*c1y)
oldvar_name = ["D1","D2"]+[abc_name+i+var for var in ["x","y"] for abc_name in ["a","b","c"] for i in ["1","2"] if abc_name+i not in ["a1","b1"]]
oldvar = []; exec(";".join("oldvar.append(%s)"%var_name for var_name in oldvar_name))

def my_print(f):
    res = str(f)
    for xx,yy in [(abc_name+i+var,"%s_%s%s"%(abc_name,i,{"x":"","y":"'","z":"''"}[var])) for var in ["x","y","z"] for abc_name in ["a","b","c"] for i in ["1","2"]]:
        res = res.replace(xx,yy)
    return res.replace("*","")

for ind in sol:
    res = ind(*oldvar)*(-b1x*b2x*c2x*a2y*c1y + c1x*c2x^2*a2y*c1y - a2x*c1x*c2x*c1y*c2y)*b2y
    print(my_print(res)+"\\\\")
```

By (4.3.5), (4.3.6), (4.3.7) and (4.3.8), we get the equation in Lemma 4.3.1.

4.3.3. Five-vertex model. There is another model slightly simpler than the six-vertex model. Similar approach of this section also works.

Let us assume in the six-vertex model

$$c_1(z) = 0, \qquad \text{other entries} \neq 0.$$

That is, the matrix $R(z)$ is lower triangular. This is known as the **five-vertex model** in the literature. Now the Yang–Baxter equation can be written as

$$\begin{bmatrix} a_1b_2' & 0 & 0 & -b_2a_1' & 0 \\ b_2c_2' & 0 & -c_2b_2' & 0 & -b_2a_1' \\ 0 & 0 & -b_1b_2' & b_2b_1' & 0 \\ 0 & -a_2b_2' & 0 & b_2a_2' & 0 \\ -b_1c_2' & 0 & 0 & c_2b_1' & b_1a_1' \\ -a_1b_1' & 0 & b_1a_1' & 0 & 0 \\ 0 & 0 & 0 & c_2a_2'-a_2c_2' & b_1b_2' \\ 0 & 0 & b_1b_2' & -b_2b_1' & 0 \\ 0 & 0 & -c_2a_2'+a_2c_2' & 0 & -b_2b_1' \\ 0 & a_2b_1' & -b_1a_2' & 0 & 0 \end{bmatrix} \begin{bmatrix} a_1'' \\ a_2'' \\ b_1'' \\ b_2'' \\ c_2'' \end{bmatrix} = 0.$$

In a similar manner, we can compute

$$\frac{a_1'a_2' - b_1'b_2'}{a_1'c_2'} = \frac{a_1a_2 - b_1b_2}{a_1c_2}, \qquad \frac{a_1'a_2' - b_1'b_2'}{a_2'c_2'} = \frac{a_1''a_2'' - b_1''b_2''}{a_2c_2}.$$

Example 4.3.5. *Consider the following R-matrix:*

$$\lim_{q\to 0} \left[\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & \dfrac{q^2z-z}{q^2z-1} & \dfrac{-q^2z+q^2}{-q^2z+1} & 0 \\ \hline 0 & \dfrac{-z+1}{-q^2z+1} & \dfrac{q^2-1}{q^2z-1} & 0 \\ 0 & 0 & 0 & 1 \end{array}\right] = \left[\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & z & 0 & 0 \\ \hline 0 & -z+1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right] \qquad \begin{aligned} a_1(z) &= a_2(z) = 1, \\ b_1(z) &= 1, \\ b_2(z) &= z, \\ c_2(z) &= 1-z. \end{aligned}$$

We have

$$\frac{a_1(z)a_2(z) - b_1(z)b_2(z)}{a_1(z)c_2(z)} = 1.$$

Case A. If $a_1(z)a_2(z) - b_1(z)b_2(z) \neq 0$. Then similarly as above, we have $a_1(z)/a(z) \in \Bbbk$, and we can assume $a(z) := a_1(z) = a_2(z) = 1$. Assume

$$\Delta = \frac{a_1(z)a_2(z) - b_1(z)b_2(z)}{a_1(z)c_2(z)} \in \Bbbk.$$

We can solve

$$b_1(y/x) = \frac{b_1(y)}{b_1(x)}, \qquad b_2(y/x) = \frac{b_2(y)}{b_2(x)},$$
$$c_2(y/x) = \frac{c_2(y) - c_2(x)}{1 - \Delta c_2(x)}.$$

Then we have

$$b_1(z) = z^m, \qquad b_2(z) = z^n, \qquad c_2(z) = \frac{1-z^k}{\Delta}, \qquad m, n, k \in \mathbb{Z}.$$

We make sure $\Delta \in \Bbbk$, we need to assume $k = m + n$. In conclusion, in this case, we get the following family of solutions of the Yang–Baxter equation

$$R(z) = f \left[\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & z^m & 0 & 0 \\ \hline 0 & \frac{1-z^{m+n}}{\Delta} & z^n & 0 \\ 0 & 0 & 0 & 1 \end{array}\right], \qquad \begin{array}{r} f \in \Bbbk(z)^\times, \\ \Delta \in \Bbbk^\times, \\ m, n \in \mathbb{Z}. \end{array}$$

This is a generalization of Example 4.3.5 above.

Case B. If $a_1(z)a_2(z) - b_1(z)b_2(z) = 0$. Similarly, it reduces to a rational group homomorphism $\chi : \mathbb{C}^\times \to B$ where B is the upper triangular matrices.

Example 4.3.6. *The following is another family of R-matrices*

$$R(z) = f \left[\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & z^m & 0 & 0 \\ \hline 0 & \frac{1-z^{m+n}}{\Delta} & z^n & 0 \\ 0 & 0 & 0 & z^{m+n} \end{array}\right], \qquad \begin{array}{r} f \in \Bbbk(z)^\times, \\ \Delta \in \Bbbk^\times, \\ m, n \in \mathbb{Z}. \end{array}$$

The corresponding rational group homomorphism is

$$\mathbb{C}^\times \to B, \qquad z \mapsto \begin{bmatrix} 1 & 0 \\ 1 - z & z \end{bmatrix}.$$

CHAPTER 5

Pipe Puzzles

In this chapter, we will prove Theorem C. The proof consists of three parts. We will first degenerate the R-matrix obtained in the previous Chapter. Then we get the recursion formulas of the coefficients. Lastly, we prove the Theorem C via lattice models. We will also discuss some applications of Theorem C.

5.1. Degeneration of R-matrices

There are three steps in the degeneration.

5.1.1. Step 1. Let us consider an infinite dimensional vector space

$$\mathbb{V} = \bigoplus_{i\in\mathbb{Z}} \mathbb{C}(q)\mathbf{e}_i = \cdots \oplus \mathbb{C}(q)\mathbf{e}_{-1} \oplus \mathbb{C}(q)\mathbf{e}_0 \oplus \mathbb{C}(q)\mathbf{e}_1 \oplus \cdots .$$

Let us define

$$R(z) \in \mathrm{Hom}(\mathbb{V}\otimes\mathbb{V}, \mathbb{V}\otimes\mathbb{V})(z)$$

by

$$R(z)(\mathbf{e}_i\otimes\mathbf{e}_j) = \begin{cases} \dfrac{1-q}{1-qz}\mathbf{e}_i\otimes\mathbf{e}_j + \dfrac{1-z}{1-qz}\mathbf{e}_j\otimes\mathbf{e}_i & i<j, \\ \mathbf{e}_i\otimes\mathbf{e}_j & i=j, \\ \dfrac{(1-q)z}{1-qz}\mathbf{e}_i\otimes\mathbf{e}_j + \dfrac{q(1-z)}{1-qz}\mathbf{e}_j\otimes\mathbf{e}_i & i>j. \end{cases}$$

PROPOSITION 5.1.1. *The matrix $R(z)$ satisfies the Yang–Baxter equation*

$$\begin{aligned} &(R(x)\otimes 1)(1\otimes R(xy))(R(y)\otimes 1) \\ &= (1\otimes R(y))(R(xy)\otimes 1)(1\otimes R(x)). \end{aligned}$$

PROOF. The R-matrix $R(z)$ is essentially the R-matrix obtained in (4.2.5). Let us denote the R-matrix in (4.2.5) by $R'(z)$. Let us define $Q\in\mathrm{End}(\mathbb{V},\mathbb{V})$ by

$$Q(\mathbf{e}_i\otimes\mathbf{e}_j) = q^{\delta_{i>j}}\mathbf{e}_i\otimes\mathbf{e}_j.$$

We claim that

$$R''(z) = Q\circ R'(z)\circ Q^{-1}$$

also satisfies the Yang–Baxter equation. The trick is, we can construct matrix

$$\widehat{Q}(\mathbf{e}_i\otimes\mathbf{e}_j\otimes\mathbf{e}_k) = q^{\delta_{i>j}+\delta_{i>k}+\delta_{j>k}}\mathbf{e}_i\otimes\mathbf{e}_j\otimes\mathbf{e}_k$$

such that

$$\begin{aligned} \widehat{Q}(R'(z)\otimes 1)\widehat{Q}^{-1} &= (QR'(z)Q^{-1}\otimes 1), \\ \widehat{Q}(1\otimes R'(z))\widehat{Q}^{-1} &= (1\otimes QR'(z)Q^{-1}). \end{aligned}$$

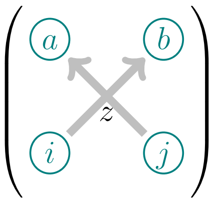
So

$$
\begin{aligned}
&(R''(x)\otimes 1)(1\otimes R''(xy))(R''(y)\otimes 1)\\
&=\widehat{Q}(R'(x)\otimes 1)(1\otimes R'(xy))(R'(y)\otimes 1)\widehat{Q}^{-1}\\
&=\widehat{Q}(1\otimes R'(y))(R'(xy)\otimes 1)(1\otimes R'(x))\widehat{Q}^{-1}\\
&=(1\otimes R''(y))(R''(xy)\otimes 1)(1\otimes R''(x)).
\end{aligned}
$$

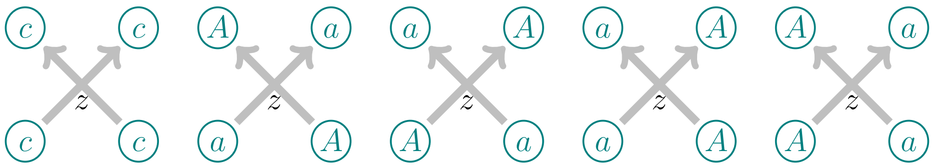
Note that $R(z)$ can be written as $R''(1/z)|_{q\mapsto q^{-1/2}}$, which is also a solution of Yang–Baxter equation. $\square$

There is a diagrammatic way of presenting R-matrix coefficients. Let us denote

$$\begin{matrix}\text{coefficient of } \mathbf{e}_a\otimes\mathbf{e}_b\\ \text{in } R(z)(\mathbf{e}_i\otimes\mathbf{e}_j)\end{matrix} = R_{ij}^{ab}(z) =$$

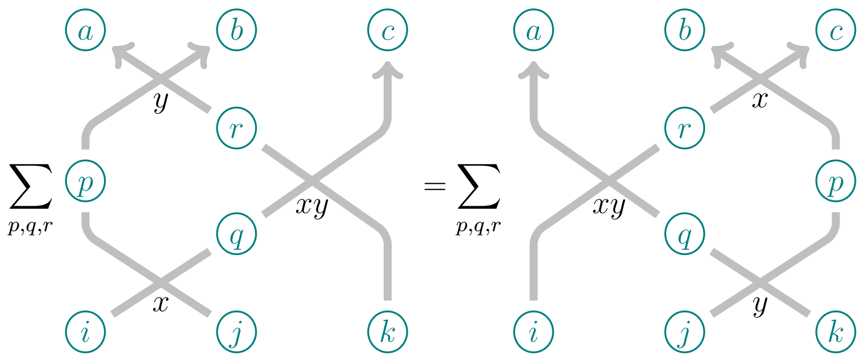


By our definition, the coefficient is zero unless

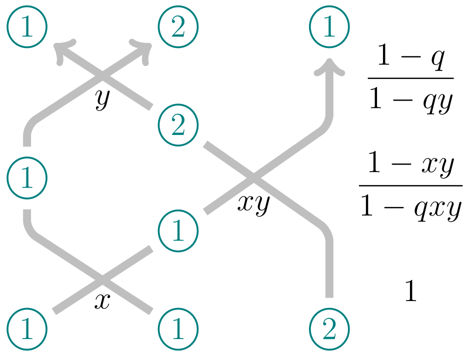


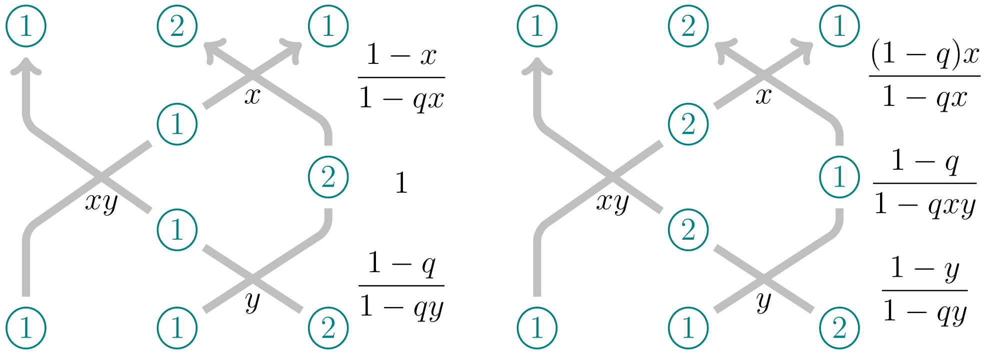
$$1 \qquad \frac{1-z}{1-qz} \qquad \frac{q(1-z)}{1-qz} \qquad \frac{1-q}{1-qz} \qquad \frac{(1-q)z}{1-qz}$$

where $a<A$ and c arbitrary. The Yang–Baxter equation can be reformulated as



for any fixed i,j,k,a,b,c. Actually, the left/right-hand side is the coefficient of $\mathbf{e}_a\otimes\mathbf{e}_b\otimes\mathbf{e}_c$ of the left/right-hand side of Yang–Baxter equation applied to $\mathbf{e}_i\otimes\mathbf{e}_j\otimes\mathbf{e}_k$. For example, let $(i,j,k)=(1,1,2)$ and $(a,b,c)=(1,2,1)$. Then the nonzero contribution of the two sides is





We have

$$\frac{1-q}{1-qy}\cdot\frac{1-xy}{1-qxy}\cdot 1=\frac{1-x}{1-qx}\cdot 1\cdot\frac{1-q}{1-qy}+\frac{(1-q)x}{1-qx}\cdot\frac{1-q}{1-qxy}\cdot\frac{1-y}{1-qy}.$$

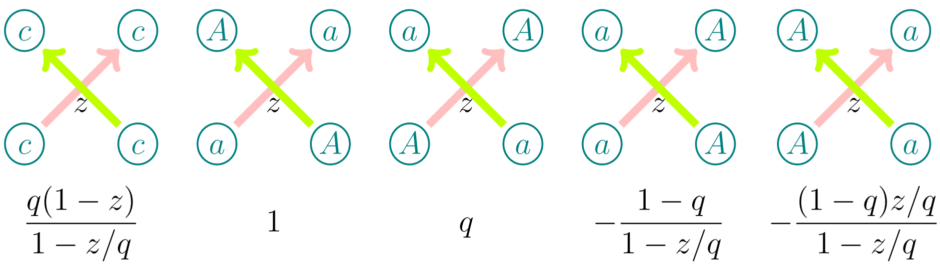
5.1.2. Step 2. Let us denote $\mathbb{V}(z)=\mathbb{V}\otimes\mathbb{C}(z)$. Let us denote $V_1(z)=\mathbb{V}(z)$ and $V_2(z)=\mathbb{V}(z/q)$. Though they are isomorphic as vector spaces, we will treat them as different spaces. We denote $R_1(z)=R_2(z)=R(z)$ and

$$L(z)=\frac{1-z}{1-z/q}R(z/q).$$

We view

$$R_i(z_1/z_2):V_i(z_1)\otimes V_i(z_2)\longrightarrow V_i(z_2)\otimes V_i(z_1)\qquad i=1,2$$
$$L(z_1/z_2):V_1(z_1)\otimes V_2(z_2)\longrightarrow V_2(z_2)\otimes V_1(z_1)$$

The coefficient of $L(z)$ is given by

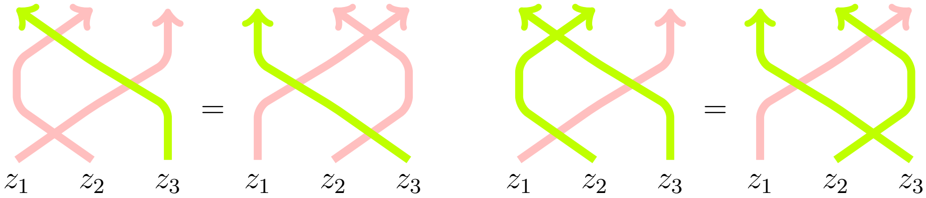


where $a < A$ and c arbitrary. It is clear that we have the Yang–Baxter equations

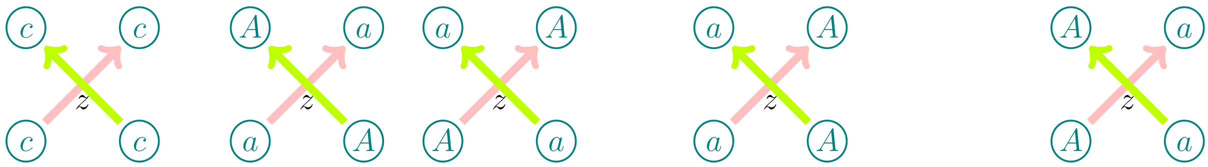
$$(5.1.1)\qquad \begin{aligned}&(L(z_2/z_3)\otimes 1)(1\otimes L(z_1/z_3))(R_1(z_1/z_2)\otimes 1)\\ =\ &(1\otimes R_1(z_1/z_2))(L(z_1/z_3)\otimes)(1\otimes L(z_2/z_3));\end{aligned}$$

$$(5.1.2)\qquad \begin{aligned}&(R_2(z_2/z_3)\otimes 1)(1\otimes L(z_1/z_3))(L(z_1/z_2)\otimes 1)\\ =\ &(1\otimes L(z_1/z_2))(L(z_1/z_3)\otimes)(1\otimes R_2(z_2/z_3)).\end{aligned}$$

They can be pictured as follows



Since V_1 and V_2 are viewed as two representations, the space of solutions of the equations (5.1.1) and (5.1.2) has an extra degree of freedom, leaving us some room to modify. For our purpose, we can consider another matrix $\tilde{L}(z)$ defined by



$\frac{q(1-z)}{1-z/q}$ $\quad c \neq 0$	1	q	$-\frac{1-q}{1-z/q}$ $\quad a<A<0$	$-\frac{(1-q)z/q}{1-z/q}$ $\quad a<A<0$
$\frac{(1-z)/q}{1-z/q}$ $\quad c = 0$			$-\frac{1-q}{1-z/q}$ $\quad 0<a<A$	$-\frac{(1-q)z/q}{1-z/q}$ $\quad 0<a<A$
			$-\frac{(1-q)/q}{1-z/q}$ $\quad a\leq 0\leq A$	$-\frac{(1-q)z}{1-z/q}$ $\quad a\leq 0\leq A$

where $a < A$ and c arbitrary.

PROPOSITION 5.1.2. *The matrix* $\tilde{L}(z)$ *also satisfies the Yang–Baxter equation* (5.1.1) *and* (5.1.2).

PROOF. The proof is very similar to that of Proposition 5.1.1. Let us introduce $Q_1 \in \mathrm{End}(V_1 \otimes V_2)$ and $Q_2 \in \mathrm{End}(V_2 \otimes V_1)$ by

$$Q_1(\mathbf{e}_a\otimes\mathbf{e}_b) = q_1(a,b)\mathbf{e}_a\otimes\mathbf{e}_b, \qquad q_1(a,b) = \begin{cases} q, & a<0<b \text{ or } a=b=0,\\ 1, & \text{otherwise.}\end{cases}$$

$$Q_2(\mathbf{e}_a\otimes\mathbf{e}_b) = q_2(a,b)\mathbf{e}_a\otimes\mathbf{e}_b, \qquad q_2(a,b) = \begin{cases} q^{-1}, & a\leq 0\leq b,\\ 1, & \text{otherwise.}\end{cases}$$

Then we have

$$\tilde{L}(z) = Q_2^{-1}L(z)Q_1.$$

We can construct $\widehat{Q}_1 \in \mathrm{End}(V_1 \otimes V_1 \otimes V_2)$, $\widehat{Q}_{1.5} \in \mathrm{End}(V_1 \otimes V_2 \otimes V_1)$, $\widehat{Q}_2 \in \mathrm{End}(V_2 \otimes V_1 \otimes V_1)$ by

$$\begin{aligned}\widehat{Q}_1(\mathbf{e}_a \otimes \mathbf{e}_b \otimes \mathbf{e}_c) &= q_1(a,c)q_1(b,c)\mathbf{e}_1 \otimes \mathbf{e}_b \otimes \mathbf{e}_c\\ \widehat{Q}_{1.5}(\mathbf{e}_a \otimes \mathbf{e}_b \otimes \mathbf{e}_c) &= q_1(a,b)q_2(b,c)\mathbf{e}_1 \otimes \mathbf{e}_b \otimes \mathbf{e}_c\\ \widehat{Q}_2(\mathbf{e}_a \otimes \mathbf{e}_b \otimes \mathbf{e}_c) &= q_2(a,b)q_2(a,c)\mathbf{e}_1 \otimes \mathbf{e}_b \otimes \mathbf{e}_c.\end{aligned}$$

Then it is not hard to check

$$\begin{aligned}(R_1(z) \otimes 1)\widehat{Q}_1 &= \widehat{Q}_1(R_1(z) \otimes 1),\\ \widehat{Q}_2(1 \otimes R_2(z)) &= (1 \otimes R_2(z))\widehat{Q}_2,\\ \widehat{Q}_2^{-1}(L(z) \otimes 1)\widehat{Q}_{1.5} &= (\tilde{L}(z) \otimes 1),\\ \widehat{Q}_{1.5}^{-1}(1 \otimes L(z))\widehat{Q}_1 &= (1 \otimes \tilde{L}(z)).\end{aligned}$$

This implies (5.1.1). A similar argument works for (5.1.2). □

5.1.3. Step 3. Our last step is to specialize the R-matrices $R_1(z), R_2(z), L(z)$ we will use. It is obtained by

(1) setting $q = 0$;
(2) shifting the indices

old	$-k$	$\cdots$	-1	0	1	2	$\cdots$
new	1	$\cdots$	k	0	$k+1$	$k+2$	$\cdots$

(3) changing of the variables from z to x

$$\beta x = 1 - \frac{1}{z}.$$

Correspondently, $z_1 z_2$ are replaced by $x_1 \oplus x_2 := x_1 + x_2 + \beta x_1 x_2$, and z_1/z_2 are replaced by

$$x_1 \ominus x_2 := \frac{x_1 - x_2}{1 + \beta x_2}.$$

The resulting matrices are listed as follows. The matrix $\tilde{L}(z)$ becomes Table 1 and the matrix $R_i(z)$ becomes (rotated for later use) Table 2 and Table 3.

Remark 5.1.3. Using the motivic Segre class, it suffices to build the degeneration on matrix $\tilde{L}(z)$; see [**28**].

5.1.4. Examination. Since the matrices are obtained by degeneration, they automatically satisfy the Yang–Baxter equations. But it would be more straightforward to check the Yang–Baxter equation directly. Here we provide the SageMath code.

```
Qbeta.<beta> = PolynomialRing(QQ); ominus = lambda x,y: (x-y)/(1+beta*y);
Pol = PolynomialRing(Qbeta,["x","y"]);  x, y = Pol.gens()
```

```
def R(N, E, W, S, var = x):
    if N == E == W == S == 0: return var
    if E == W == 0 < N == S: return 1
    if N == S == 0 < E == W: return 1
    if 0 < E == W < N == S: return 1
```

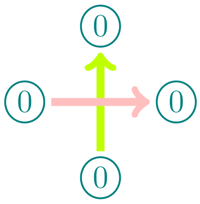
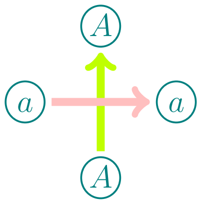
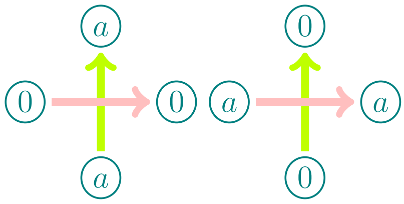
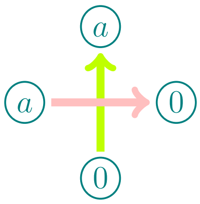
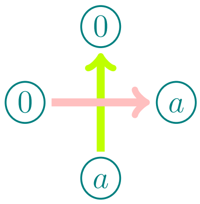
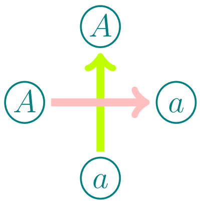
		
x	$1 \quad (0 < a < A)$	$1 \quad (0 < a)$
		
$1+\beta x \quad (0 < a \leq k)$ $1 \quad (k < a)$	$1 \quad (0 < a \leq k)$ $1+\beta x \quad (k < a)$	$\beta \quad (0 < a < A \leq k)$ $\beta \quad (k < a < A)$ $\beta(1+\beta x) \quad (0 < A \leq k < a)$

TABLE 1. The matrix $\tilde{L}(z)$

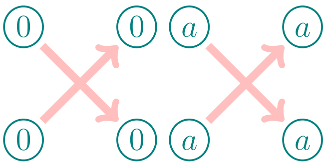
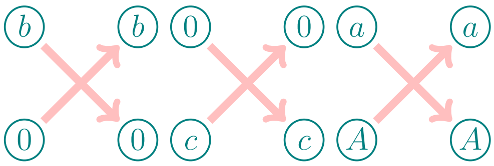
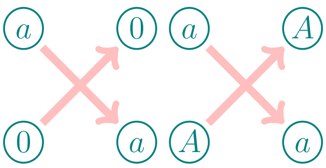
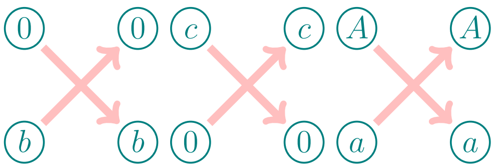
A	B_1
	
$1 \quad (0 < a)$	$1 \quad (0 < b \leq k < c$ and $0 < a < A)$
C	B_2
	
$x \quad (0 < a < A)$	$1+\beta x \quad (0 < b \leq k < c$ and $0 < a < A)$

TABLE 2. The R-matrix R_{row}.

```
    if E == S == 0 < N == W <= k: return 1+beta*var
    if E == S == 0 and k < N == W: return 1
    if N == W == 0 < E == S <= k: return 1
    if N == W == 0 and k < E == S: return 1+beta*var
    if 0 < E == S < N == W <= k: return beta
    if k < E == S < N == W: return beta
    if 0 < N == W <= k < E == S: return beta*(1+beta*var)
    return 0
```

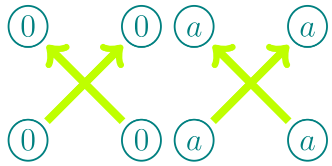
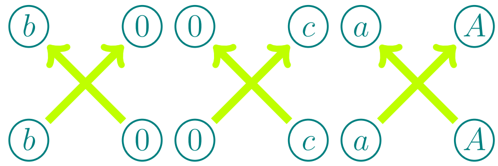
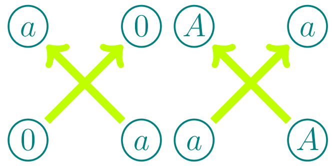
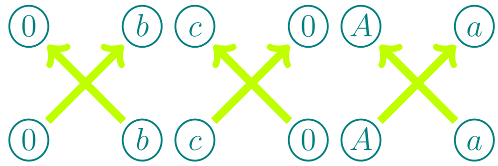
A	B_1
	
$1 \quad (0 < a)$	$1 \quad (0 < b \leq k < c \text{ and } 0 < a < A)$
C	B_2
	
$x \quad (0 < a < A)$	$1 + \beta x \quad (0 < b \leq k < c \text{ and } 0 < a < A)$

TABLE 3. The R-matrix R_{col}.

```
def R_row(NE,NW,SE,SW, var = x):
    if NE == NW == SE == SW: return 1 # case A
    if NE == NW and SE == SW: # case B
        if (0 == SE and NE <= k) or (0 == NE and k < SE) or (0 < NE < SE): return 1 # case B1
        if (0 == NE and SE <= k) or (0 == SE and k < NE) or (0 < SE < NE): return 1+beta*var # case B2
    if (NE == SW == 0 < NW == SE) or (0 < NW == SE < NE == SW): return var # case C
    return 0
def R_col(NE,NW,SE,SW, var = x):
    if NE == NW == SE == SW: return 1 # case A
    if NE == SE and NW == SW: # case B
        if (0 == NE and NW <= k) or (0 == NW and k < NE) or (0 < NW < NE): return 1 # case B1
        if (0 == NW and NE <= k) or (0 == NE and k < NW) or (0 < NE < NW): return 1+beta*var # case B2
    if (NE == SW == 0 < NW == SE) or (0 < NE == SW < NW == SE): return var # case C
    return 0
```

```
def Check_row_YBE(Alph): # check thm
    Tri = [(i,j,k) for i in Alph for j in Alph for k in Alph]
    for (a1,a2,a3) in Tri:
        for (b1,b2,b3) in Tri:
            LHS = sum(R(a1,a2,p,q,x)*R(q,a3,r,b3,y)*R_row(p,b1,r,b2,ominus(x,y)) for (p,q,r) in Tri)
            RHS = sum(R(a1,p,b1,q,y)*R(q,r,b2,b3,x)*R_row(a2,p,a3,r,ominus(x,y)) for (p,q,r) in Tri)
            if LHS!= RHS: return False
    return True
def Check_col_YBE(Alph):  # check thm
    Tri = [(i,j,k) for i in Alph for j in Alph for k in Alph]
    for (a1,a2,a3) in Tri:
        for (b1,b2,b3) in Tri:
```

```
        LHS = sum(R(p,r,b1,b2,x)*R(q,a3,r,b3,y)*R_col(a2,a1,q,p,ominus(x,y)) for (
p,q,r) in Tri)
        RHS = sum(R(a1,r,b1,p,y)*R(a2,a3,r,q,x)*R_col(q,p,b3,b2,ominus(x,y)) for (
p,q,r) in Tri)
        if LHS!= RHS: return False
    return True
```

```
k = 3;
for A in Subsets([0..6],3): # check thm 4.3
    print(A,Check_row_YBE(A))
for A in Subsets([0..6],3): # check thm 4.5
    print(A,Check_col_YBE(A))
```

5.2. Recursion formulas

The purpose of this section is to give recursion formulas for the coefficients $c_{u,v}^w(t,y)$.

PROPOSITION 5.2.1. *If $s_iu < u$, then*

$$c_{s_iu,v}^w = -\frac{1+\beta y_i}{y_i - y_{i+1}} c_{u,v}^w + \frac{1+\beta y_{i+1}}{y_i - y_{i+1}} c_{u,v}^w|_{y_i \leftrightarrow y_{i+1}}. \tag{5.2.1}$$

PROPOSITION 5.2.2. *If $s_iw > w$, then*

$$c_{u,v}^{s_iw} = \begin{cases} -\dfrac{1+\beta t_{i+1}}{t_i - t_{i+1}} c_{u,v}^w|_{t_i\leftrightarrow t_{i+1}} + \dfrac{1+\beta t_i}{t_i - t_{i+1}} c_{u,v}^w + c_{u,s_iv}^w|_{t_i\leftrightarrow t_{i+1}}, & s_iv < v, \\ -\dfrac{1+\beta t_i}{t_i - t_{i+1}} c_{u,v}^w|_{t_i\leftrightarrow t_{i+1}} + \dfrac{1+\beta t_i}{t_i - t_{i+1}} c_{u,v}^w, & s_iv > v. \end{cases} \tag{5.2.2}$$

Since when t_i and t_{i+1} are exchanged, y_i and y_{i+1} are invariant, we should keep t and y independent in the induction.

LEMMA 5.2.3. *We have*

$$c_{u,v}^{\mathsf{id}}(t,y) = \begin{cases} \mathfrak{G}_u(t,y), & \text{if } v = \mathsf{id}, \\ 0, & \text{otherwise.} \end{cases}$$

PROOF. Taking $x = t$ in (1.0.1) and then applying Proposition 2.4.15, we obtain the following relationship. □

As a result, Proposition 5.2.1 and Proposition 5.2.2 determine $c_{u,v}^w(t,y)$ with the initial value

$$c_{w_0,v}^{\mathsf{id}}(t,y) = \begin{cases} \mathfrak{G}_{w_0}(t,y), & \text{if } v = \mathsf{id}, \\ 0, & \text{otherwise.} \end{cases}$$

5.2.1. Left operators. Define **the (left) Demazure operator** by

$$\varpi_i f = -\frac{(1+\beta t_i)f - (1+\beta t_{i+1}) f|_{t_i\leftrightarrow t_{i+1}}}{t_i - t_{i+1}}.$$

PROPOSITION 5.2.4. *We have*

$$\varpi_i \mathfrak{G}_w(x,t) = \begin{cases} \mathfrak{G}_{s_iw}(x,t), & \text{if } s_iw < w, \\ -\beta \mathfrak{G}_w(x,t), & \text{if } s_iw > w. \end{cases} \tag{5.2.3}$$

A geometric proof of Proposition 5.2.4 can be found in [**38**]. Here, we provide an algebraic proof. To this end, we need the **Hecke product** on permutations:

$$s_i * w = \begin{cases} s_i w, & \text{if } s_i w > w, \\ w, & \text{if } s_i w < w, \end{cases} \quad \text{and} \quad w * s_i = \begin{cases} w s_i, & \text{if } w s_i > w, \\ w, & \text{if } w s_i < w. \end{cases}$$

This defines a monoid structure over S_∞ called the **0-Hecke monoid**.

Proof of Proposition 5.2.4. Without loss of generality, we may assume $\beta = -1$. Suppose that $w \in S_n$. Let $w_0 = n \cdots 21$ be the longest element in S_n. Define

$$\mathfrak{G}^w(x,t) = \mathfrak{G}_{w_0 w}(x,t).$$

Then (2.4.2) can be rewritten as

$$\pi_i \mathfrak{G}^w = \mathfrak{G}^{w * s_i}. \tag{5.2.4}$$

Note that the identity in (5.2.3) can be restated as

$$\varpi_i \mathfrak{G}^w = \mathfrak{G}^{s_{n-i} * w}. \tag{5.2.5}$$

We prove (5.2.5) by induction on length. When $w = \mathrm{id}$, it follows from direct computation that

$$\varpi_i \mathfrak{G}^{\mathrm{id}}(x,t) = \varpi_i \mathfrak{G}_{w_0}(x,t) = \prod_{\substack{a+b \le n \\ (a,b) \ne (n-i,i)}} (x_a \ominus t_b),$$

which coincides with $\mathfrak{G}^{s_{n-i}}(x,t) = \mathfrak{G}_{w_0 s_{n-i}}(x,t) = \pi_{n-i} \mathfrak{G}_{w_0}(x,t)$.

For $\ell(w) > 0$, one can find an index j such that $w s_j < w$, and so by induction,

$$\varpi_i \mathfrak{G}^w = \varpi_i \pi_j \mathfrak{G}^{w s_j} = \pi_j \varpi_i \mathfrak{G}^{w s_j} = \pi_j \mathfrak{G}^{s_{n-i} * w s_j} = \mathfrak{G}^{s_{n-i} * w s_j * s_j} = \mathfrak{G}^{s_{n-i} * w}.$$

Here, we used the fact that the operators π_j and ϖ_j commute in the second equality, and (5.2.4) in the fourth equality. □

5.2.2. The proofs. Now, we can give proofs of Propositions 5.2.1 and 5.2.2.

Proof of Proposition 5.2.1. We introduce another operator

$$\varphi_i f = -\frac{(1+\beta y_i) f - (1+\beta y_{i+1}) f|_{y_i \leftrightarrow y_{i+1}}}{y_i - y_{i+1}},$$

which is the same as the operator ϖ_i, but acts on the variable y. Assume that $s_i u < u$. Applying φ_i to (1.0.1), by Proposition 5.2.4, the left-hand side is

$$\mathfrak{G}_{s_i u}(x,y) \cdot \mathfrak{G}_v(x,t) = \sum_w c^w_{s_i u, v}(t,y) \cdot \mathfrak{G}_w(x,t).$$

While the right-hand side is

$$\sum_w \varphi_i c^w_{u,v}(t,y) \cdot \mathfrak{G}_w(x,t).$$

Comparing the coefficients of $\mathfrak{G}_w(x,t)$, we are given $c^w_{s_i u, v} = \varphi_i c^w_{u,v}$, as desired. □

Proof of Proposition 5.2.2. Apply ϖ_i to (1.0.1). By Proposition 5.2.4, the left-hand side is

$$\begin{cases} \mathfrak{G}_u(x,y)\cdot\mathfrak{G}_{s_iv}(x,t) = \sum_w c^w_{u,s_iv}(t,y)\cdot\mathfrak{G}_w(x,t), & s_iv<v,\\ -\beta\mathfrak{G}_u(x,y)\cdot\mathfrak{G}_v(x,t) = \sum_w -\beta c^w_{u,v}(t,y)\cdot\mathfrak{G}_w(x,t), & s_iv>v.\end{cases}$$

To compute the right-hand side, we use the following property of ϖ_i:

$$\varpi_i(fg) = (f|_{t_i\leftrightarrow t_{i+1}})(\varpi_i g) - \frac{1+\beta t_i}{t_i-t_{i+1}}(f - f|_{t_i\leftrightarrow t_{i+1}})g. \tag{5.2.6}$$

By (5.2.6) and Proposition 5.2.4, the right-hand side is

$$\begin{aligned}
&\sum_w \varpi_i\big(c^w_{u,v}\cdot\mathfrak{G}_w(x,t)\big)\\
&=\sum_w\left((c^w_{u,v}|_{t_i\leftrightarrow t_{i+1}})\varpi_i\mathfrak{G}_w - \frac{1+\beta t_i}{t_i-t_{i+1}}(c^w_{u,v}-c^w_{u,v}|_{t_i\leftrightarrow t_{i+1}})\mathfrak{G}_w\right)\\
&=\sum_{s_iw<w}\left((c^w_{u,v}|_{t_i\leftrightarrow t_{i+1}})\mathfrak{G}_{s_iw} - \frac{1+\beta t_i}{t_i-t_{i+1}}(c^w_{u,v}-c^w_{u,v}|_{t_i\leftrightarrow t_{i+1}})\mathfrak{G}_w\right)\\
&\quad+\sum_{s_iw>w}\left(-\beta(c^w_{u,v}|_{t_i\leftrightarrow t_{i+1}})\mathfrak{G}_w - \frac{1+\beta t_i}{t_i-t_{i+1}}(c^w_{u,v}-c^w_{u,v}|_{t_i\leftrightarrow t_{i+1}})\mathfrak{G}_w\right)\\
&=-\sum_{s_iw<w}\frac{1+\beta t_i}{t_i-t_{i+1}}(c^w_{u,v}-c^w_{u,v}|_{t_i\leftrightarrow t_{i+1}})\mathfrak{G}_w\\
&\quad+\sum_{s_iw>w}\left(c^{s_iw}_{u,v}|_{t_i\leftrightarrow t_{i+1}} - \beta c^w_{u,v}|_{t_i\leftrightarrow t_{i+1}} - \frac{1+\beta t_i}{t_i-t_{i+1}}(c^w_{u,v}-c^w_{u,v}|_{t_i\leftrightarrow t_{i+1}})\right)\mathfrak{G}_w\\
&=-\sum_{s_iw<w}\frac{1+\beta t_i}{t_i-t_{i+1}}(c^w_{u,v}-c^w_{u,v}|_{t_i\leftrightarrow t_{i+1}})\mathfrak{G}_w\\
&\quad+\sum_{s_iw>w}\left(c^{s_iw}_{u,v}|_{t_i\leftrightarrow t_{i+1}} - \frac{1+\beta t_i}{t_i-t_{i+1}}c^w_{u,v} + \frac{1+\beta t_{i+1}}{t_i-t_{i+1}}c^w_{u,v}|_{t_i\leftrightarrow t_{i+1}}\right)\mathfrak{G}_w.
\end{aligned}$$

Extracting the coefficients of $\mathfrak{G}(w)$ with $s_iw>w$ on both sides, we deduce that

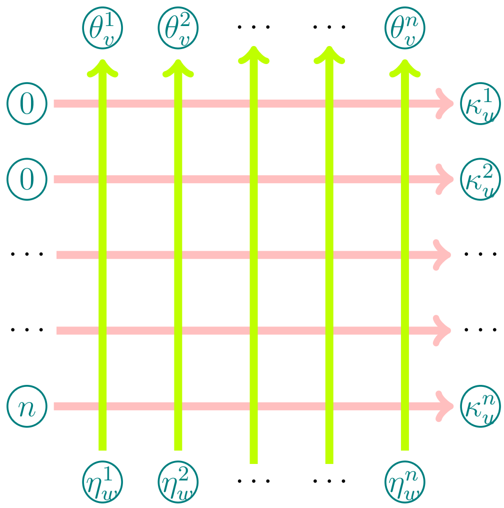
$$c^{s_iw}_{u,v}|_{t_i\leftrightarrow t_{i+1}} = \frac{1+\beta t_i}{t_i-t_{i+1}}c^w_{u,v} - \frac{1+\beta t_{i+1}}{t_i-t_{i+1}}c^w_{u,v}|_{t_i\leftrightarrow t_{i+1}} + \begin{cases} c^w_{u,s_iv}, & s_iv<v,\\ -\beta c^w_{u,v}, & s_iv>v,\end{cases}$$

which coincides with (5.2.2) after the variable exchange $t_i\leftrightarrow t_{i+1}$. □

5.3. Proof of the results

5.3.1. Lattice models. Consider a square grid with n horizontal lines and n vertical lines. The intersection point of two lines will be a vertex (so there are a total of n^2 vertices). The lines between two vertices are called edges. We shall also attach additional half edges to the vertices on the boundary, so that there are four half edges around each vertex.

Let us consider a square grid as follows



$$\kappa_u^i = \begin{cases} u^{-1}(i), & u^{-1}(i) \le k, \\ 0, & u^{-1}(i) > k. \end{cases}$$

$$\theta_v^i = \begin{cases} 0, & v^{-1}(i) \le k, \\ v^{-1}(i), & v^{-1}(i) > k. \end{cases}$$

$$\eta_w^i = w^{-1}(i).$$

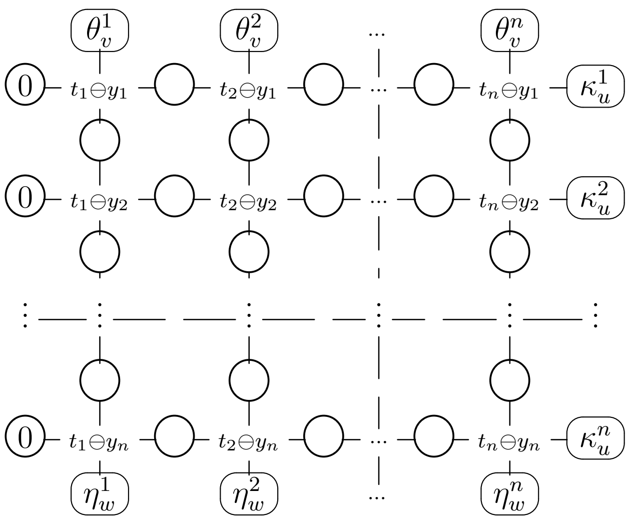
A **state** is a labeling of all the (half) edges with labels from $\{0, 1, 2 \ldots, n\}$, with a fixed boundary condition which is consistent with that in (1.3.2): the left half edges are all labeled 0, the right half edges are labeled $\kappa_u^1, \ldots, \kappa_u^n$ from top to bottom, the top (resp., bottom) half edges are labeled $\theta_v^1, \ldots, \theta_v^n$ (resp., $\eta_w^1, \ldots, \eta_w^n$) from left to right. The label of each (half) edge will be marked with a circle, and a vertex will be formally assigned a parameter x. A state is **admissible** if the local configurations around each vertex (namely, the labeled half edges adjacent to each vertex) satisfy exactly one of the conditions as listed in the middle column of Table 4. Moreover, each allowable local configuration is assigned a weight as given in the first column of Table 4.

The lattice model $\mathrm{L}(u, v, w)$ we are considering is defined as the set of all admissible states ($\mathrm{L}(u, v, w)$ can be regarded as a colored lattice model if the labels $1, 2, \ldots, n$ are viewed as n colors). The weight $\mathrm{wt}(S)$ of a state S in $\mathrm{L}(u, v, w)$ is the product of all the weights of vertices with $x = t_j \ominus y_i$ in row i and column j. The **partition function** of $\mathrm{L}(u, v, w)$ is defined by

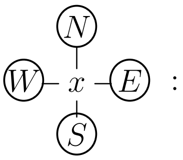
$$Z_{u,v}^w(t, y) = \sum_{S \in \mathrm{L}(u,v,w)} \mathrm{wt}(S).$$

That is,

$Z_{u,v}^w(t, y) =$



Each configuration around a vertex naturally corresponds to a tile that is used to define a pipe puzzle, as illustrated in the last column of Table 4, with pipes inheriting the labels of edges. We display the information in Table 4 more intuitively in Table 5. Therefore, each admissible state generates a pipe puzzle, and vice versa. See Figure 1 for an admissible state and its corresponding pipe puzzle.



weights	conditions	tiles
x	$N = E = W = S = 0$	
1	$E = W = 0 < N = S$	
1	$N = S = 0 < E = W$	
1	$0 < E = W < N = S$	
$1+\beta x$	$E = S = 0 < N = W \leq k$	
1	$E = S = 0$ and $k < N = W$	
1	$N = W = 0 < E = S \leq k$	
$1+\beta x$	$N = W = 0$ and $k < E = S$	
β	$0 < E = S < N = W \leq k$	
β	$k < E = S < N = W$	
$\beta(1+\beta x)$	$0 < N = W \leq k < E = S$	

Table 4. Weights, local configurations, and tiles.

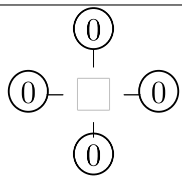
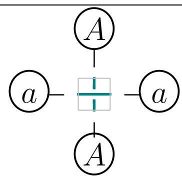
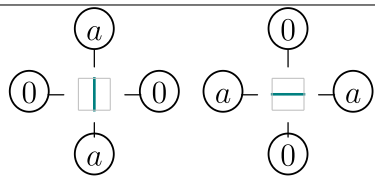
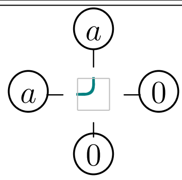
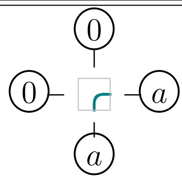
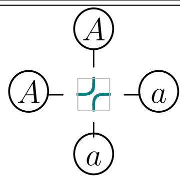
		
x	$1 \quad (0 < a < A)$	$1 \quad (0 < a)$
		
$1+\beta x \quad (0 < a \leq k)$ $1 \quad (k < a)$	$1 \quad (0 < a \leq k)$ $1+\beta x \quad (k < a)$	$\beta \quad (0 < a < A \leq k)$ $\beta \quad (k < a < A)$ $\beta(1+\beta x) \quad (0 < A \leq k < a)$

Table 5. Diagram illustration of Table 4.

Collecting the above observations, we summarize the following facts.

Proposition 5.3.1. *Let $u, v \in S_n$ be permutations with separated descents at position k. Then, for $w \in S_n$,*

(1) *The set $L(u, v, w)$ of admissible states are in bijection with the set $\mathrm{PP}(u, v, w)$ of pipe puzzles.*

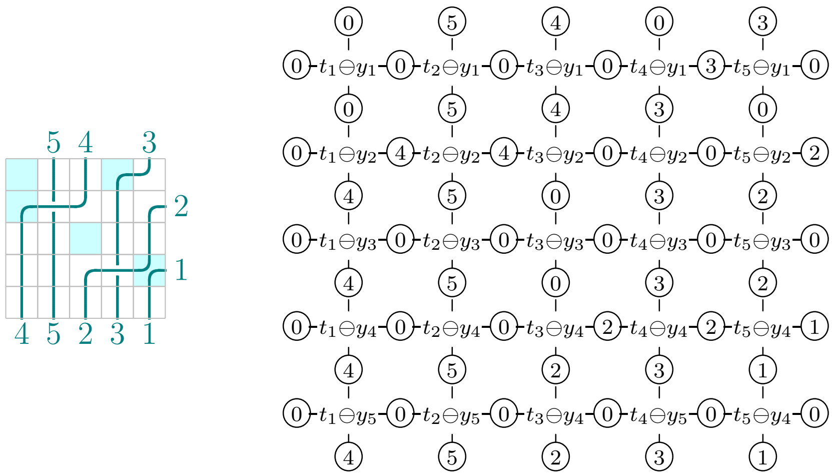


FIGURE 1. Correspondence between a pipe puzzle and an admissible state.

(2) *We have*

$$Z^w_{u,v}(t,y) = \sum_{\pi \in \mathrm{PP}(u,v,w)} \mathrm{wt}(\pi).$$

5.3.2. Initial cases. Denote by $u_0 = n(n-1)\cdots(n-k+1)\,12\cdots(n-k) \in S_n$ the unique longest permutation among those $u \in S_n$ with $\mathrm{maxdes}(u) \le k$:

$$u_0(i) = \begin{cases} n+1-i, & i \le k, \\ i-k, & k < i \le n. \end{cases} \tag{5.3.1}$$

By direct computation, we have

$$\mathfrak{G}_{u_0}(x,t) = \prod_{i=1}^{k}\prod_{j=1}^{n-i}(x_i \ominus t_j).$$

Actually u_0 is a dominant permutation corresponds to Hessenberg function h with

$$h(i) = \min(i,k).$$

The Grothendieck polynomial for dominant permutation was explained in Section 2.5.4. This can also be verified by induction on n. This is clearly true for $k = n$. If the statement is true for k, then applying operators $\pi_1 \cdots \pi_k$, we can compute the case of $k-1$. This, along with Lemma 5.2.3, leads to the initial condition.

PROPOSITION 5.3.2. *For* $v \in S_n$,

$$c^{\mathrm{id}}_{u_0,v} = \begin{cases} \displaystyle\prod_{i=1}^{k}\prod_{j=1}^{n-i}(t_i \ominus y_j), & \text{if } v = \mathrm{id}, \\ 0, & \text{otherwise.} \end{cases}$$

Propositions 5.2.1 and 5.2.2 are valid for any $u, v, w \in S_n$. We explain that such recurrences are closed when restricting $u, v \in S_n$ to permutations with separated descents at k. In other words, we could use Propositions 5.2.1 and 5.2.2 (only applied to permutations with separated descents at k), along with the initial condition in Proposition 5.3.2, to compute $c^w_{u,v}(t,y)$ for any $u, v \in S_n$ with separated descents at k.

- First, compute $c^{\mathrm{id}}_{u,v}(t,y)$ for $w=\mathrm{id}$. The initial case is for the longest permutation $u=u_0$, as done in Proposition 5.3.2. We next consider $c^{\mathrm{id}}_{u,v}(t,y)$ with $\ell(u)<\ell(u_0)$. Since $u\neq u_0$, one can always choose an integer i among the first k values $u(1),\ldots,u(k)$, such that i appears before $i+1$ in u. For example, given $u=7423156$ and $k=4$, we may choose $i=4$ or $i=2$.

  Now we have $u<s_iu\in S_n$. It is easily checked that $\mathrm{maxdes}(s_iu)\leq k$. Set $u'=s_iu$. By backwards induction on the length of u, the value of $c^{\mathrm{id}}_{u',v}(t,y)$ is known, which allows us to compute $c^{\mathrm{id}}_{u,v}(t,y)=c^{\mathrm{id}}_{s_iu',v}(t,y)$ from $c^{\mathrm{id}}_{u',v}(t,y)$ by means of Proposition 5.2.1.
- Second, compute $c^{w}_{u,v}(t,y)$ for $\ell(w)>0$. In this case, choose any s_i such that $s_iw<w$. It is also easily checked that if $s_iv<v$, then we still have $\mathrm{mindes}(s_iv)\geq k$. Set $w'=s_iw$. By induction on the length of w, the values of $c^{w'}_{u,v}(t,y)$ and $c^{w'}_{u,s_iv}(t,y)$ are known. Applying Proposition 5.2.2, we may deduce $c^{w}_{u,v}(t,y)=c^{s_iw'}_{u,v}(t,y)$ from $c^{w'}_{u,v}(t,y)$ and $c^{w'}_{u,s_iv}(t,y)$.

5.3.3. Induction on u. Suppose that $s_iu<u$. It is easily checked that $\mathrm{maxdes}(s_iu)\leq k$. Recalling the definition in (1.3.2), we see that $0<\kappa_u^{i+1}<\kappa_u^i\leq k$ or $\kappa_u^i=0<\kappa_u^{i+1}\leq k$, depending on the positions where i and $i+1$ lie. Clearly, κ_{s_iu} is obtained from κ_u by interchanging κ_u^i and κ_u^{i+1}. For example, for $n=7$ and $k=3$, we list a descending chain as follows:

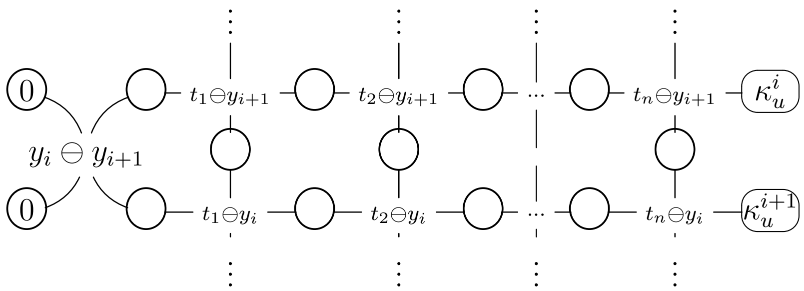
$$\begin{aligned} u &= 5431267\cdots > 5341267\cdots > 5241367\cdots > 4251367\cdots \\ \kappa_u &= 0032100\cdots \to 0023100\cdots \to 0203100\cdots \to 0201300\cdots \end{aligned}$$

Proposition 5.3.3. *If $s_iu<u$, then*

$$Z^w_{s_iu,v} = -\frac{1+\beta y_i}{y_i-y_{i+1}}Z^w_{u,v}+\frac{1+\beta y_{i+1}}{y_i-y_{i+1}}Z^w_{u,v}|_{y_i\leftrightarrow y_{i+1}}. \tag{5.3.2}$$

Proof. Consider the lattice model $L(u,v,w)$. We attach an R_{row} to the left boundary of row i and row $i+1$ (Meanwhile, we make the variable exchange $y_i\leftrightarrow y_{i+1}$ in the states of $L(u,v,w)$), as illustrated in (5.3.3).

(5.3.3)

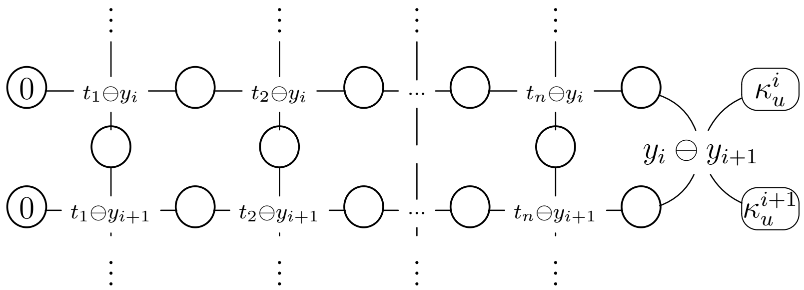


By Table 2, there is exactly one admissible configuration for the R-matrix R_{row} (from A in Table 2). So the partition function of (5.3.3) reads as

$$Z^w_{u,v}|_{y_i\leftrightarrow y_{i+1}}. \tag{5.3.4}$$

Noticing that $(t_j\ominus y_{i+1})\ominus(t_j\ominus y_i)=y_i\ominus y_{i+1}$, we may apply repeatedly the Yang–Baxter equation in Section 5.1 to (5.3.3), resulting in a model depicted in (5.3.5), with an R-matrix R_{row} attached on the right boundary.

(5.3.5)



Consider the partition function of (5.3.5). Keep in mind that $0 < \kappa_u^{i+1} < \kappa_u^i \leq k$ or $\kappa_u^i = 0 < \kappa_u^{i+1} \leq k$. For each situation, there are two admissible configurations for the R-matrix R_{row} respectively from B_2 and C in Table 2, corresponding respectively to the models $L(u, v, w)$ and $L(s_i u, v, w)$. Thus, the partition function of (5.3.5) is

$$(1 + \beta(y_i \ominus y_{i+1}))Z_{u,v}^w + (y_i \ominus y_{i+1})Z_{s_i u,v}^w. \tag{5.3.6}$$

Equating (5.3.4) and (5.3.6), we get the desired formula in (5.3.2). $\square$

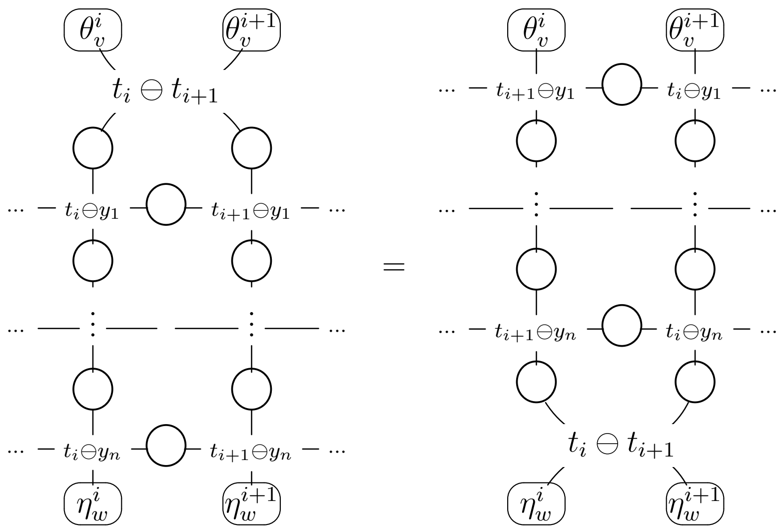
5.3.4. Induction on w. We now establish the recurrence relation for $Z_{u,v}^w$, which is parallel to Proposition 5.2.2.

PROPOSITION 5.3.4. *If $s_i w > w$, then*

$$Z_{u,v}^{s_i w} = \begin{cases} -\dfrac{1+\beta t_{i+1}}{t_i - t_{i+1}} Z_{u,v}^w|_{t_i \leftrightarrow t_{i+1}} + \dfrac{1+\beta t_i}{t_i - t_{i+1}} Z_{u,v}^w + Z_{u,s_i v}^w|_{t_i \leftrightarrow t_{i+1}}, & s_i v < v, \\ -\dfrac{1+\beta t_i}{t_i - t_{i+1}} Z_{u,v}^w|_{t_i \leftrightarrow t_{i+1}} + \dfrac{1+\beta t_i}{t_i - t_{i+1}} Z_{u,v}^w, & s_i v > v. \end{cases} \tag{5.3.7}$$

PROOF. This time we attach an R_{col} to the top boundary of $L(u, v, w)$. Applying the Yang–Baxter equation in Section 5.1, we obtain equivalent models given in (5.3.8).

(5.3.8)



We first consider the partition function of the right model in (5.3.8). The assumption $s_i w > w$ implies $0 < \eta_w^i < \eta_w^{i+1}$. Notice also that $\eta_{s_i w}$ is obtained from η_w by interchanging η_w^i and η_w^{i+1}. In view of Table 3, there are two admissible configurations for the R-matrix R_{col} (one is from B_1 in Table 3, and the other is from C in Table 3), corresponding respectively to the models $L(u, v, w)$ and $L(u, v, s_i w)$. So the partition function of the right model in (5.3.8) is

$$Z^{w}_{u,v}|_{t_i \leftrightarrow t_{i+1}} + (t_i \ominus t_{i+1}) Z^{s_i w}_{u,v}|_{t_i \leftrightarrow t_{i+1}}. \tag{5.3.9}$$

We next consider the partition function of the left model in (5.3.8). There are two cases.

Case 1. $s_i v < v$. In this case, notice that $k < \theta_u^{i+1} < \theta_u^i$ or $0 = \theta_u^{i+1} < \theta_u^i$, and that $\theta_{s_i v}$ is obtained from θ_v by interchanging θ_v^i and θ_v^{i+1}. By Table 3, for either $k < \theta_u^{i+1} < \theta_u^i$ or $0 = \theta_u^{i+1} < \theta_u^i$, there are two choices for the configurations of R_{col} (one is from B_2, and the other is from C), corresponding respectively to the models $L(u, v, w)$ and $L(u, s_i v, w)$. So, the partition function of the left model in (5.3.8) is

$$(1 + \beta(t_i \ominus t_{i+1})) Z^{w}_{u,v} + (t_i \ominus t_{i+1}) Z^{w}_{u,s_i v}. \tag{5.3.10}$$

Equating (5.3.9) and (5.3.10), we deduce that

$$\begin{aligned} Z^{s_i w}_{u,v}|_{t_i \leftrightarrow t_{i+1}} &= \frac{1 + \beta(t_i \ominus t_{i+1})}{t_i \ominus t_{i+1}} Z^{w}_{u,v} + Z^{w}_{u,s_i v} - \frac{1}{t_i \ominus t_{i+1}} Z^{w}_{u,v}|_{t_i \leftrightarrow t_{i+1}} \\ &= \frac{1 + \beta t_i}{t_i - t_{i+1}} Z^{w}_{u,v} + Z^{w}_{u,s_i v} - \frac{1 + \beta t_{i+1}}{t_i - t_{i+1}} Z^{w}_{u,v}|_{t_i \leftrightarrow t_{i+1}}, \end{aligned}$$

which, after the variable exchange $t_i \leftrightarrow t_{i+1}$, becomes the first equality in (5.3.7).

Case 2. $s_i v > v$. In this case, i appears before $i+1$ in v. So we have $0 = \theta_v^i = \theta_v^{i+1}$, or $0 = \theta_v^i$ and $k < \theta_v^{i+1}$, or $k < \theta_v^i < \theta_v^{i+1}$. By Table 3, for each of these situations, there is exactly one admissible configuration (from A or B_1) of R_{col}, and we see that the partition function of the left model in (5.3.8) is precisely equal to $Z^{w}_{u,v}$. By equating with (5.3.9), we obtain that

$$\begin{aligned} Z^{s_i w}_{u,v}|_{t_i \leftrightarrow t_{i+1}} &= \frac{1}{t_i \ominus t_{i+1}} Z^{w}_{u,v} - \frac{1}{t_i \ominus t_{i+1}} Z^{w}_{u,v}|_{t_i \leftrightarrow t_{i+1}} \\ &= \frac{1 + \beta t_{i+1}}{t_i - t_{i+1}} Z^{w}_{u,v} - \frac{1 + \beta t_{i+1}}{t_i - t_{i+1}} Z^{w}_{u,v}|_{t_i \leftrightarrow t_{i+1}}. \end{aligned}$$

After the variable exchange $t_i \leftrightarrow t_{i+1}$ on both sides, we reach the second equality in (5.3.7). ☐

5.3.5. Initial condition. We finally verify the initial case for u_0 (as defined in (5.3.1)) and $w = \text{id}$.

THEOREM 5.3.5. *For $v \in S_n$, we have*

$$Z^{\text{id}}_{u_0,v} = \begin{cases} \prod_{i=1}^{k} \prod_{j=1}^{n-i} (t_i \ominus y_j), & \textit{if } v = \text{id}, \\ 0, & \textit{otherwise.} \end{cases} \tag{5.3.11}$$

PROOF. Here, we go back to the pipe puzzle model $\text{PP}(u_0, v, \text{id})$ for the computation of $Z^{\text{id}}_{u_0,v}$. The boundary condition is illustrated in the left diagram in Figure 2. Evidently, the pipes labeled $k+1, \ldots, n$ must go vertically from the top side down to the bottom side. So we have $Z^{\text{id}}_{u_0,v} = 0$ whenever $v \neq \text{id}$. It remains to check the case $v = \text{id}$. It is easily checked that there is exactly one pipe puzzle in $\text{PP}(u_0, \text{id}, \text{id})$, see the right diagram of Figure 2. This pipe puzzle contributes a weight as displayed in (5.3.11). ☐

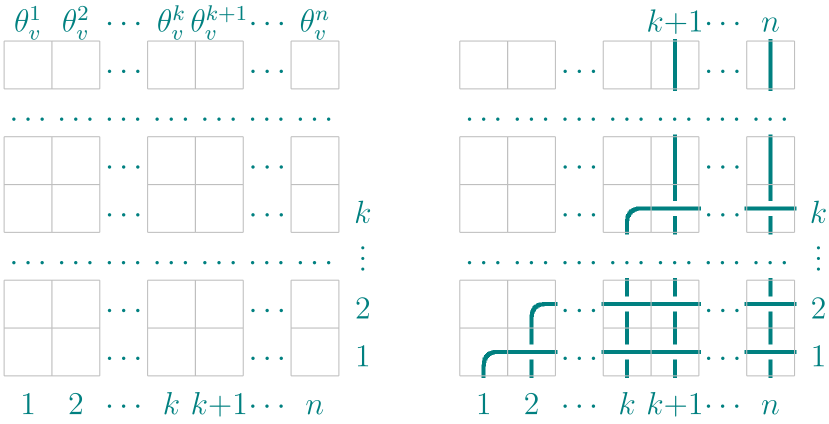


FIGURE 2. Boundary condition and the unique pipe puzzle in $\mathrm{PP}(u_0, \mathrm{id}, \mathrm{id})$.

5.4. Applications

We list two main applications of Theorem C. The first application is to recover the puzzle formula discovered by Knutson and Zinn-Justin [**28**, Theorem 1].

5.4.1. Separated-descent puzzles. Consider (1.0.1) by setting $y = t$:

$$\mathfrak{G}_u(x,t) \cdot \mathfrak{G}_v(x,t) = \sum_w c_{u,v}^w(t,t) \cdot \mathfrak{G}_w(x,t).$$

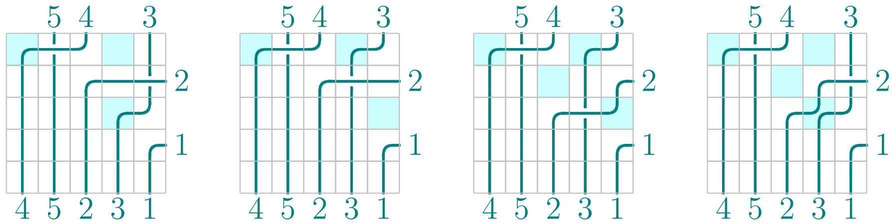
Assume that $u, v \in S_n$ have separated descents at k. For $w \in S_n$, a pipe puzzle $\pi \in \mathrm{PP}(u,v,w)$ has weight zero if and only if π has (at least) one empty tile on the diagonal. This implies that $c_{u,v}^w(t,t)$ is a weighted counting of pipe puzzles $\pi \in \mathrm{PP}(u,v,w)$ such that π has no empty tile on the diagonal. For such pipe puzzles, we have the following observation:

- Each position on the diagonal is tiled with either or , and each position lying strictly to the southwest of the diagonal is tiled with .

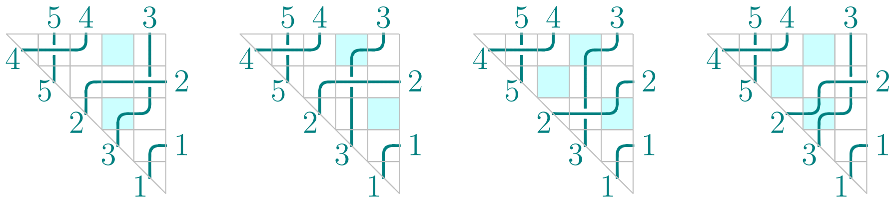
This can be checked as follows. First, the tile at the position $(1,1)$ must be tiled with either or since (1) the tile cannot be empty, and (2) the labels on the left boundary are all 0. Therefore, all positions below $(1,1)$ in the first column must be tiled with . The same analysis applies to the remaining positions $(2,2), \ldots, (n,n)$.

Let $\pi \in \mathrm{PP}(u,v,w)$ be a pipe puzzle without empty tile on the diagonal. Cut π along its diagonal into two triangles, and denote by $P(\pi)$ the upper-right triangle. By the above observation, π can be recovered from $P(\pi)$. To get the puzzle visualization of Knutson and Zinn-Justin [**28**, Theorem 1], we rotate $P(\pi)$ counterclockwise by 45 degrees, and then warp it into an equilateral triangle. If further assuming that u and v are both k-Grassmannian, there is a bijection to the classical Grassmannian puzzles, see [**28**, §5.1] for more details.

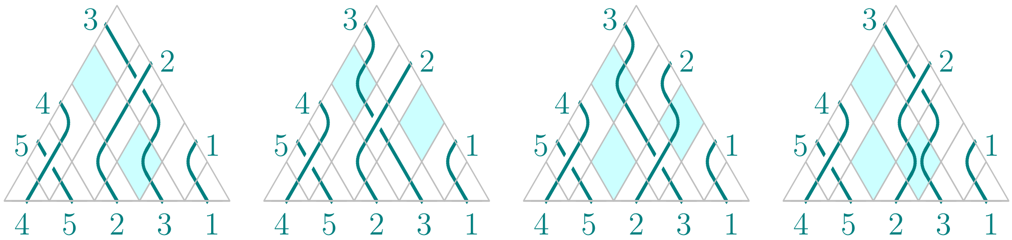
EXAMPLE 5.4.1. *Consider the pipe puzzles in Example 1.3.3. The following four puzzles survive after setting* $y = t$.



Their upper-right triangular regions are



After rotation and warping, the corresponding puzzles are



In the second application, we explain that Theorem C could be used to recover the bumpless pipe dream model of double Grothendieck polynomials by Weigandt [**48**].

5.4.2. Bumpless pipe dreams. Let $k = n$ and $v = \mathrm{id}$. In this case, arbitrary $u \in S_n$ satisfies the separated-descent condition in (1.3.1). By Lemma 5.2.3,

$$c_{u,\mathrm{id}}^{\mathrm{id}}(t, y) = \mathfrak{G}_u(t, y).$$

Let $\pi \in \mathrm{PP}(u, \mathrm{id}, \mathrm{id})$. Then all pipes enter into π from the right side. Apply the following operations to π:

- reflecting π across the diagonal;
- replacing $\kappa_u^i = u^{-1}(i)$ by i, and $\eta_w^i = i$ by $u(i)$.

The resulting diagram is denoted as $B(\pi)$. Write

$$\mathrm{BP}(u) = \{B(\pi) \colon \pi \in \mathrm{PP}(u, \mathrm{id}, \mathrm{id})\}.$$

By the restriction (1.3.5) on along with the restriction (1.3.8) on , it can be checked that for a diagram in $\mathrm{BP}(u)$: (1) two pipes cross at most once, and (2) if two pipes have a "bumping" at position (i, j), then they must cross at a position to the northeast of (i, j). This implies that the set $\mathrm{BP}(u)$ is precisely the set of bumpless pipe dreams of u, as defined in [**48**].

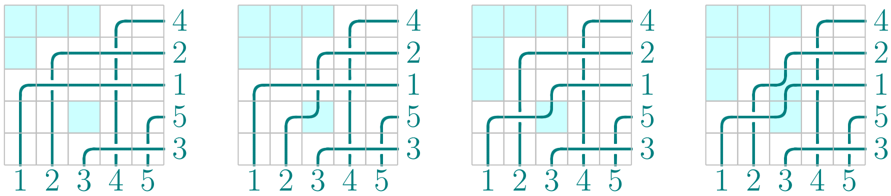
Remark 5.4.2. As bumpless pipe dreams in $\mathrm{BP}(u)$ are obtained from pipe puzzles in $\mathrm{PP}(u, \mathrm{id}, \mathrm{id})$ after a reflection, a tile at position (i, j) is assigned a wight in the following way:

(1) an empty tile contributes $t_i \ominus y_j$;
(2) an elbow tile contributes $1 + \beta(t_i \ominus y_j)$;
(3) a bumping tile contributes β;

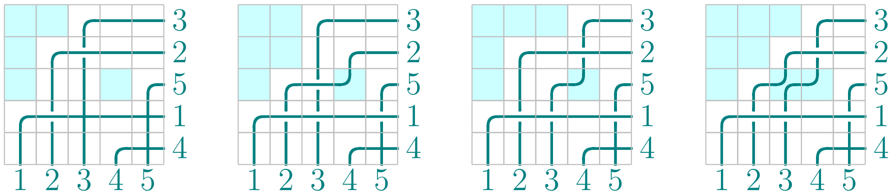
(4) any other tile contributes 1.

The weights described above are slightly different from the weights adopted in [**48**]. It seems that when setting $\beta = 0$, the weights we used imply more explicitly the bumpless pipe dream model of double Schubert polynomials due to Lam, Lee and Shimozono [**29**].

EXAMPLE 5.4.3. *Let* $u = 32514$. *Below are pipe puzzles in* $\mathrm{PP}(u, \mathrm{id}, \mathrm{id})$.



After reflection and relabeling, the resulting bumpless pipe dreams of u *are*



CHAPTER 6

Graham positivity

In this section, we will establish Theorem D by providing a new geometric meaning of triple Schubert calculus.

6.1. An example

Let us give a motivating example. We will need the following double Schubert polynomials

$$\mathfrak{S}_{s_1}(x,y) = x_1 - y_1, \qquad \mathfrak{S}_{s_2s_1}(x,y) = (x_1-y_1)(x_1-y_2).$$

We can compute

$$\begin{aligned}\mathfrak{S}_{s_1}(x,y)\cdot\mathfrak{S}_{s_1}(x,t) &= (x_1-y_1)(x_1-t_1) = (x_1-t_2+t_2-y_1)(x_1-t_1)\\ &= (t_2-y_1)\cdot\mathfrak{S}_{s_1}(x,t) + \mathfrak{S}_{s_2s_1}(x,t).\end{aligned}$$

To explain the geometric meaning, we will work with the flag variety Fl_4, and regard y_1, y_2 as equivariant parameter t_3, t_4. We have

$$\begin{aligned}\mathfrak{S}_{s_1}(x,t) \text{ represents } &\{V_\bullet \in Fl_4 : V_1 \subseteq \operatorname{span}(e_2,e_3,e_4)\},\\ \mathfrak{S}_{s_1}(x,y) \text{ represents } &\{V_\bullet \in Fl_4 : V_1 \subseteq \operatorname{span}(e_1,e_2,e_4)\},\\ \mathfrak{S}_{s_2s_1}(x,t) \text{ represents } &\{V_\bullet \in Fl_4 : V_1 \subseteq \operatorname{span}(e_3,e_4)\}.\end{aligned}$$

As a result,

$$\mathfrak{S}_{s_1}(x,t)\cdot\mathfrak{S}_{s_1}(x,y) \text{ represents } \{V_\bullet \in Fl_4 : V_1 \subseteq \operatorname{span}(e_2,e_4)\}.$$

Let $W = \operatorname{span}(e_2,e_3)$. Let us consider

$$\begin{array}{ccc} & X & \\ {}^{\pi_1}\swarrow & & \searrow^{\pi_2} \\ Fl_4 & & \mathbb{P}(W) \end{array} \qquad X = \{(V_\bullet,[v]) \in Fl_4 \times \mathbb{P}(W) : V_1 \subseteq \operatorname{span}(v,e_4)\}.$$

Then

$$\begin{aligned}\mathfrak{S}_{s_1}(x,t) &\text{ represents the image of } X \text{ under } \pi_1,\\ \mathfrak{S}_{s_1}(x,t)\cdot\mathfrak{S}_{s_1}(x,y) &\text{ represents the image of } X_2 \text{ under } \pi_1,\\ \mathfrak{S}_{s_2s_1}(x,t) &\text{ represents the image of } X_3 \text{ under } \pi_1\end{aligned}$$

where X_i is the fiber at $[v]=[e_i]$ along π_2 for $i=2,3$. Over $\mathbb{P}(W)$,

$$\left.\begin{aligned} x-t_2 \text{ represents the point } [e_2]\in\mathbb{P}(W)\\ x-y_1 \text{ represents the point } [e_3]\in\mathbb{P}(W)\end{aligned}\right\} \Longrightarrow \begin{gathered}\text{in } H_T(\mathbb{P}(W)), \text{ we have}\\ [e_2] = (t_2-y_1)[\mathbb{P}(W)] + [e_3].\end{gathered}$$

Applying $\pi_{1*}\pi_2^*$, we get

$$\mathfrak{S}_{s_1}(x,y)\cdot\mathfrak{S}_{s_1}(x,t) = (t_2-y_1)\cdot\mathfrak{S}_{s_1}(x,t) + \mathfrak{S}_{s_2s_1}(x,t).$$

6.2. A refined Graham positivity

Let G be a connected, complex, reductive algebraic group, $B \subset G$ be a Borel subgroup, B^- be its opposite Borel subgroup, $T = B \cap B^-$ be a maximal torus, N (resp., N^-) be the unipotent radical of B (resp., B^-), $W = N_G(T)/T$ be the Weyl group and Φ be the associated root system, with positive roots Φ^+ and simple roots $\Delta = \{\alpha_1, \ldots, \alpha_r\}$. For $\alpha \in \Phi$, write s_α for the corresponding reflection, and write s_i for s_{α_i} for simplicity. For $w \in W$, its (left) **inversion set** is $I(w) := \{\alpha \in \Phi^+ \,|\, w^{-1}\alpha \in \Phi^-\}$ and its **non-inversion set** is $J(w) := \{\alpha \in \Phi^+ \mid w^{-1}\alpha \in \Phi^+\} = I(ww_0)$, where w_0 is the longest element in W. For each $w \in W$, we pick a representative of it in $N_G(T)$, and also denote it by w, slightly abusing notation.

The **flag variety** G/B admits a **Bruhat decomposition** $\bigsqcup_{w\in W} B^-wB/B$ into **Schubert cells**. Their closures $\overline{B^-wB/B} \subseteq G/B$ are the **Schubert varieties**. The **Schubert classes** $\{[\overline{B^-wB/B}]_T \mid w \in W\}$ form an $H_T^*(\mathsf{pt})$-basis of $H_T^*(\mathcal{F}\ell_m)$.

We define the following closed subgroups of G.

Definition 6.2.1. For $w \in W$, define $N^-(w) := N^- \cap wN^-w^{-1}$ and $B^-(w) = T \cdot N^-(w)$.

By [**21**, Section 28], as a variety, $N^-(w)$ is isomorphic to the Schubert cell $B^-wB/B \cong \mathbb{C}^{\ell(w_0)-\ell(w)}$ via $x \mapsto xwB/B$. In particular, $N^-(w)$ is connected. Its Lie algebra is

$$\mathfrak{n}^-(w) = \operatorname{span}(F_\alpha \mid \alpha \in J(w)) \subseteq \mathfrak{g} := \operatorname{Lie} G$$

where F_α is a root vector of weight $-\alpha$.

Lemma 6.2.1. *If $ws_i > w$, then $N^-(ws_i)$ is a normal subgroup of $N^-(w)$.*

Proof. Recall that $[F_\alpha, F_\beta] \in \mathbb{C}^\times F_{\alpha+\beta}$ if $\alpha+\beta$ is a root, and $[F_\alpha, F_\beta] = 0$ otherwise. As $ws_i > w$, we have $J(w) = J(ws_i) \cup \{w\alpha_i\}$ and thus $\mathfrak{n}^-(ws_i) \subset \mathfrak{n}^-(w)$, $N^-(ws_i)$ is a closed subgroup of $N^-(w)$ by [**21**, Theorem 13.1].

In fact, $\mathfrak{n}^-(ws_i)$ is an ideal of $\mathfrak{n}^-(w)$. To check this, take any $\alpha \in J(ws_i)$ and $\beta \in J(w)$, and it suffices to show $[F_\alpha, F_\beta] \in \mathfrak{n}^-(ws_i)$. If $\alpha + \beta \notin \Phi$, $[F_\alpha, F_\beta] = 0$. Thus, consider $\gamma = \alpha+\beta \in \Phi^+$. If $\beta \in J(ws_i)$, then $\gamma \in J(ws_i)$ as well; and if $\beta = w\alpha_i$, $\gamma \neq w\alpha_i \in J(w)$, so $\gamma \in J(ws_i)$. Indeed, $\mathfrak{n}^-(ws_i)$ is an ideal of $\mathfrak{n}^-(w)$ and by [**21**, Theorem 13.3], $N^-(ws_i)$ is a normal subgroup of $N^-(w)$. $\square$

We are now in a position to state a refined version of Graham positivity theorem. We need the following Lemma.

Lemma 6.2.2 (Grahma [**19**]; see [**1**, Proposition 19.4.4]). *Assume we are given*

$$\begin{array}{ccc} & & T \\ & & \cap \\ B' & \subset & B \\ \cup & & \cup \\ U' & \underset{\chi}{\subset} & U \end{array}$$

- *$B = T \ltimes U$ is a solvable group with maximal torus T and unipotent radical U;*
- *$U' \subset U$ is a normal subgroup of B with $\dim U/U' = 1$, and $B' = T \ltimes U'$;*
- *the T-weight of $\operatorname{Lie} U/\operatorname{Lie} U'$ is $\chi \in X^*(T)$.*

Let B act on a non-singular variety X. If Y is a B'-invariant effective cycle in X, then there exist B-invariant effective cycles Z_1, Z_2 such that

$$[Y]_T = [Z_1]_T + \chi[Z_2]_T$$

in $H_T^*(X)$.

THEOREM 6.2.3. *Let B^- act on a non-singular variety X, and let Y be a $B^-(w)$-invariant effective cycle in X. Then there exist B^--invariant effective cycles $Z_1, \dots, Z_m$ such that*

$$[Y]_T \in \sum_{i=1}^{m} \mathbb{N}[-\alpha]_{\alpha\in I(w)} \cdot [Z_i]_T$$

in $H_T^*(X)$.

PROOF. We use induction on $\ell(w)$. When $w = \mathsf{id}$, $B^-(w) = B^-$, so there is nothing to prove. Assume the theorem holds for w. Let us prove the statement for $ws_i > w$. By Lemma 6.2.1, the pair $B^-(ws_i) \subset B^-(w)$ satisfies the condition of Lemma 6.2.2 above with $\chi = -w\alpha_i$. So there exist $B^-(w)$-invariant effective cycles Z_1 and Z_2 such that $[Y]_T = [Z_1]_T + \chi[Z_2]_T$. Since $I(ws_i) = I(w) \cup \{w\alpha_i\}$, the inductive step is now established. □

COROLLARY 6.2.4. *Let $X = G/B$ be the flag variety. Under the above setting, we have*

$$[Y]_T \in \sum_{w\in W} \mathbb{N}[-\alpha]_{\alpha\in I(w)} \cdot [\overline{B^-wB/B}]_T$$

in $H_T^*(G/B)$.

PROOF. Since the flag variety has finitely many B^--orbits, any B^--invariant effective cycle over G/B must be a non-negative combination of Schubert classes $[\overline{B^-wB/B}]_T$. □

REMARK 6.2.5. When $w = w_0$ is the longest element, $B^-(w_0) = T$ and Theorem 6.2.3 gives Graham's positivity theorem [**19**, Theorem 3.2]. Our assumption is stronger than that of Graham's, since a $B^-(w)$-invariant cycle is necessarily T-invariant, yielding a stronger positivity result where roots are restricted to $I(w)$ rather than all positive roots. A similar generalization in this direction appeared in the proof of [**3**, Theorem 8.7 and Corollary 8.11].

6.3. A geometric explanation of positivity in Billey's formula

As an application of Theorem 6.2.3, we give a geometric explanation of positivity in Billey's formula [**7**]. See [**46**] for a great survey on this subject.

The T-fixed points of G/B are $(G/B)^T = \{wB/B \mid w \in W\}$. Define the localization to be the restriction map $\cdot|_w : H_T^*(G/B) \to H_T^*(wB/B) \simeq H_T^*(\mathsf{pt})$. Billey's formula [**7**] is a combinatorial formula of the localization of Schubert classes $[\overline{B^-uB/B}]_T|_w$ for any $u, w \in W$. From its explicit form, which we do not provide here, we see that

$$[\overline{B^-uB/B}]_T|_w \in \mathbb{N}[\alpha]_{\alpha\in I(w)}. \tag{6.3.1}$$

We provide a geometric explanation of this positivity using Corollary 6.2.4.

Recall that the point $w_0B/B = B^-w_0B/B$ is invariant under B^-. So the torus fixed point ww_0B/B is invariant under wB^-w^{-1}. This implies that ww_0B/B is invariant under $B^-(w)$. By Corollary 6.2.4, we have

$$[ww_0B/B]_T \in \sum_{u\in W} \mathbb{N}[-\alpha]_{\alpha\in I(w)} \cdot [\overline{B^-uB/B}]_T.$$

Following [**38**, Section 3] and [**1**, Section 16.5], for a Weyl group element $w \in W$, the automorphism $gB \mapsto wgB/B$ induces a left action $w^L : H_T^*(G/B) \to H_T^*(G/B)$. We remark that w^L is not $H_T^*(\mathsf{pt})$-linear unless $w = \mathsf{id}$, but it is semilinear with respect to the automorphism of $H_T^*(\mathsf{pt})$ induced by w. Applying w_0^L to the above, we get

$$[w_0ww_0B/B]_T \in \sum_{u\in W} \mathbb{N}[-w_0\alpha]_{\alpha\in I(w)} \cdot [\overline{Bw_0uB/B}]_T.$$

As $I(w_0ww_0) = \{-w_0\alpha \mid \alpha \in I(w)\}$, replacing w_0ww_0 by w and w_0u by u, we can rewrite

$$[wB/B]_T \in \sum_{u\in W} \mathbb{N}[\alpha]_{\alpha\in I(w)} \cdot [\overline{BuB/B}]_T. \tag{6.3.2}$$

Now let us take the Poincaré pairing with $[\overline{B^-uB/B}]_T$ on both sides. The left-hand side is $[\overline{B^-uB/B}]_T|_w$, and the right-hand side is the coefficient of $[\overline{BuB/B}]_T$ in (6.3.2) by [**1**, Proposition 7.3]. This gives (6.3.1).

6.4. Application to triple Schubert calculus

We restrict to the case of $G = GL_m$. By [**1**, Theorem 10.6.4], the class $[\overline{B^-\pi B/B}]_T$ is represented by the **double Schubert polynomial** $\mathfrak{S}_\pi(x;t)$. We will not be working with the algebraic definition of $\mathfrak{S}_\pi(x;t)$'s. For any permutation $\pi \in S_m$, we naturally identify it as $\pi \in S_m \hookrightarrow S_\infty$ via $\pi(k) = k$ for all $k > m$.

Fix permutations u, v. By picking $n \gg 0$, we can assume (1.0.2) only involves those $w \in S_n$. Let $G := GL_{2n}$ and B, B^-, T be as above. We identify $H_T^*(\mathsf{pt}) = \mathbb{Q}[t_1, \dots, t_{2n}]$ with $t_i = -\epsilon_i := -c_1(\mathbb{C}_{\epsilon_i})$, where ϵ_i is the character of T corresponding to the i-th diagonal entry. We rename the variables $t_{n+i} = y_i$ for all $i \in [n] := \{1, 2, \dots, n\}$. Consider a special permutation $\tau \in S_{2n}$ such that $\tau(i) = n + i$, $\tau(n + i) = i$ for all $i \in [n]$. By [**1**, Section 16.5], $[\tau\overline{B^-uB/B}]_T$ is represented by $\mathfrak{S}_u(x;\tau t) = \mathfrak{S}_u(x;y)$.

Lemma 6.4.1. *The intersection $\tau\overline{B^-uB/B} \cap \overline{B^-vB/B}$ is proper and transverse at the generic point.*

Proof. Let $w_0^{(m)} \in S_m$ be the longest permutation $m\ m{-}1 \cdots 1$. For permutations $w, \pi \in S_n$, write $w \times \pi \in S_{2n}$ as the direct sum of w and π; that is, $w \times \pi(i)$ is $w(i)$ if $i \le n$, and is $\pi(i - n) + n$ if $i > n$. Since $v \in S_n$, $s_iv > v$ for $n \le i < 2n$, and thus $\overline{B^-vB/B}$ is invariant under s_i, where $s_i = (i\ i{+}1)$ is the simple transposition. Therefore, we have $\overline{B^-vB/B} = u_0\overline{B^-vB/B}$ where $u_0 = 1 \times w_0^{(n)}$. Similarly, $\tau\overline{B^-uB/B} = \tau u_0\overline{B^-uB/B}$. As $u_0^{-1}\tau u_0 = w_0^{(2n)}$, the lemma follows from [**1**, Section 19.3]. □

We are now ready to prove our main theorem.

Proof of Theorem D. By Lemma 6.4.1, we can rewrite the coefficients of interest via

$$[\tau\overline{B^-uB/B} \cap \overline{B^-vB/B}]_T = \sum_{w\in S_n} c_{u,v}^w(y,t) \cdot [\overline{B^-wB/B}]_T.$$

Since $\tau\overline{B^-uB/B}$ is closed under $\tau N^-\tau^{-1}$ and $\overline{B^-vB/B}$ is closed under N^-, the intersection is closed under $N^- \cap \tau N^-\tau^{-1} =: N^-(\tau)$. By Corollary 6.2.4, we conclude that $c_{u,v}^w(y,t) \in \mathbb{N}[-\alpha]_{\alpha\in I(\tau)}$, where we compute $I(\tau) = \{y_j - t_i \mid 1 \le i, j \le n\}$. □

Bibliography

[1] D. Anderson and W. Fulton, Equivariant Cohomology in Algebraic Geometry, Cambridge Studies in Advanced Mathematics, Cambridge Univ. Press, Cambridge, 2023 (A preliminary version is available at https://people.math.osu.edu/anderson.2804/ecag/index.html).

[2] D. Anderson, S. Griffeth and E. Miller, Positivity and Kleiman transversality in equivariant K-theory of homogeneous spaces, J. Eur. Math. Soc. (JEMS) 13 (2011), 57–84.

[3] David Anderson. Infinite flags and Schubert polynomials. Forum of Mathematics, Sigma, 2025.

[4] N. Bergeron and F. Sottile, Schubert polynomials, the Bruhat order, and the geometry of flag manifolds, Duke Math. J. 95 (1998), 373–423.

[5] N. Bergeron and F. Sottile, A monoid for the Grassmannian Bruhat order, European J. Combin. 20 (1999), 197–211.

[6] I. N. Bernšteĭn, I. M. Gel'fand and S. I. Gel'fand, Schubert cells, and the cohomology of the spaces G/P, Uspekhi Mat. Nauk 28 (1973), 3–26.

[7] Sara C. Billey. Kostant polynomials and the cohomology ring for G/B. Duke Math. J., 96(1):205–224, 1999.

[8] A. Borel, Sur la cohomologie des espaces fibrés principaux et des espaces homogénes de groupes de Lie compacts, Ann. Math. 57 (1953), 115–207

[9] B. Brubaker, D. Bump, and S. Friedberg. Schur polynomials and the Yang-Baxter equation. Comm. Math. Phys., 308(2):281–301, 2011.

[10] A. Buch and R. Rimányi, Specializations of Grothendieck polynomials, C. R. Acad. Sci. Paris, Ser. I 339 (2004), 1–4.

[11] V. Buciumas and T. Scrimshaw, Double Grothendieck polynomials and colored lattice models, Int. Math. Res. Not. IMRN, (2022), DOI:10.1093/imrn/rnaa327.

[12] V. Chari, A. Pressley, A Guide To Quantum Groups, Cambridge Univ. Press (1994)

[13] C. Chevalley, Sur les decompositions cellulaires des espaces G / B , in Algebraic Groups and their Generalizations: Classical Methods, American Mathematical Society, (1994), 1-23. Proceedings and Symposia in Pure Mathematics, vol. 56, Part 1.

[14] N. Chriss and V. Ginzburg, Representation theory and complex geometry, Boston: Birkhäuser, 1997.

[15] Neil J.Y. Fan, Peter L. Guo, Changjian Su and Rui Xiong, Chern Classes of Open Projected Richardson Varieties and of Affine Schubert Cells. IMRN, Volume 2025, Issue 19, October.

[16] Neil J.Y. Fan, Peter L. Guo and Rui Xiong, Pieri and Murnaghan–Nakayama type Rules for Chern classes of Schubert Cells. Selecta Math. (N.S.), accepted.

[17] Neil J.Y. Fan, Peter L. Guo and Rui Xiong, Bumpless pipe dreams meet puzzles. Adv. Math., Volume 463, March 2025, 110113.

[18] Yibo Gao and Rui Xiong Graham positivity of triple Schubert calculus. arXiv:2506.09421

[19] W. Graham, Positivity in equivariant Schubert calculus, Duke Math. J. 109 (2001), 599–614.

[20] D. Huang, Schubert products for permutations with separated descents, Int. Math. Res. Not. IMRN (2022), rnac299, https://doi.org/10.1093/imrn/rnac299.

[21] James E. Humphreys. Linear algebraic groups, volume No. 21 of Graduate Texts in Mathematics. Springer-Verlag, New York-Heidelberg, 1975.

[22] A. Kirillov, Skew divided difference operators and Schubert polynomials, SIGMA Symmetry Integrability Geom. Methods Appl. 3 (2007), Paper 072, 14.

[23] S. Kleiman, The transversality of a general translate, Compositio Math. 28 (1974), 287–297.

[24] Knutson, A., T. Lam, and D. Speyer. Projections of Richardson varieties. J. Reine Angew. Math. 687 (2014): 133–57.

[25] A. Knutson and T. Tao, Puzzles and (equivariant) cohomology of Grassmannians, Duke Math. J. 119 (2003), 221–260.

[26] A. Knutson and P. Zinn-Justin, Schubert puzzles and integrability I: invariant trilinear forms, Comm. Amer. Math. Soc. 6 (2026), 1-67.
[27] A. Knutson and P. Zinn-Justin, Schubert puzzles and integrability II: multiplying motivic Segre classes forms, Comm. Amer. Math. Soc. 6 (2026), 68-152.
[28] A. Knutson and P. Zinn-Justin, Schubert puzzles and integrability III: separated descents, 2023, arXiv:2306.13855.
[29] T. Lam, S.J. Lee and M. Shimozono, Back stable Schubert calculus, Compos. Math. 157 (2021), 883–962.
[30] A. Lascoux and M. Schützenberger, Polynômes de Schubert, C. R. Math. Acad. Sci. Paris, Sér. I Math. 294(1982), 447–450.
[31] C. Lenart, S. Robinson and F. Sottile, Grothendieck polynomials via permutation patterns and chains in the Bruhat order, Amer. J. Math. 128 (2006), 805–848.
[32] C. Lenart and F. Sottile, A Pieri-type formula for the K-theory of a flag manifold, Trans. Amer. Math. Soc. 359 (2007), 2317–2342.
[33] C. Li, V. Ravikumar, F. Sottile and M. Yang, A geometric proof of an equivariant Pieri rule for flag manifolds, Forum Math. 31 (2019), 779–783.
[34] R.I. Liu, Twisted Schubert polynomials, Selecta Math. (2022).
[35] I.G. Macdonald, Notes on Schubert Polynomials, Laboratoire de combinatoire et d'informatique mathématique (LACIM), Université du Québec á Montréal, Montreal, 1991.
[36] Ian G. Macdonald. Symmetric functions and Hall polynomials. Oxford Mathematical Monographs, The Clarendon Press, Oxford University Press, Second edition, 1995. With contributions by A. Zelevinsky, Oxford Science Publications
[37] Martina Lanini, Rui Xiong, and Kirill Zainoulline. Structure algebras, Hopf algebroids and oriented cohomology of a group. 2023.
[38] L. Mihalcea, H. Naruse and C. Su, Left Demazure–Lusztig operators on equivariant (quantum) cohomology and K-theory, Int. Math. Res. Not. IMRN (2022), 12096–12147.
[39] A.I. Molev and B.E. Sagan, A Littlewood–Richardson rule for factorial Schur functions, Trans. Amer. Math. Soc. 351 (1999), 4429–4443.
[40] D. Monk, The geometry of flag manifolds, Proc. London Math. Soc., 9 (1959), 253-286.
[41] A. Morrison and F. Sottile, Two Murnaghan–Nakayama rules in Schubert calculus, Ann. Combin. 22 (2018), 363–375.
[42] O. Pechenik and A. Yong, Equivariant K-theory of Grassmannians II: The Knutson–Vakil conjecture, Compos. Math. 153 (2017), 667-–677.
[43] S. Robinson, A Pieri-type formula for $H_T^*(SL_n(\mathbb{C})/B)$, J. Algebra 249 (2002), 38–58.
[44] M.J. Samuel, Molev–Sagan type formula for double Schubert polynomials, J. Pure Appl. Algebra 228 (2024) 107636, 39 pp.
[45] F. Sottile, Pieri's formula for flag manifolds and Schubert polynomials, Ann. de l'Institut Fourier 46 (1996), 89–110.
[46] Julianna Tymoczko. Billey's formula in combinatorics, geometry, and topology. In Schubert calculus— Osaka 2012, volume 71 of Adv. Stud. Pure Math., pages 499–518. Math. Soc. Japan, [Tokyo], 2016.
[47] R. Vakil, A geometric Littlewood–Richardson rule, Ann. of Math. 164 (2006), 371–421.
[48] A. Weigandt, Bumpless pipe dreams and alternating sign matrices, J. Combin. Theory Ser. A 182 (2021), 105470.
[49] M. Wheeler and P. Zinn-Justin, Littlewood–Richardson coefficients for Grothendieck polynomials from integrability, J. Reine Angew. Math. 757 (2019), 159–195.
[50] P. Zinn-Justin, Littlewood–Richardson coefficients and integrable tilings, Electron. J. Combin. 16 (2009), Research Paper 12.