A Free Boundary Problem Modeling the Spread of Ecosystem Engineers

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Abstract

Most models for the spread of an invasive species into a new environment are based on Fisher's reaction-diffusion equation. They assume that habitat quality is independent of the presence or absence of the invading population. Ecosystem engineers are species that modify their environment to make it (more) suitable for them. A potentially more appropriate modeling approach for such an invasive species is to adapt the well-known Stefan problem of melting ice. Ahead of the front, the habitat is unsuitable for the species (the ice); behind the front, the habitat is suitable (the open water). The engineering action of the population moves the boundary ahead (the melting). This approach leads to a free boundary problem.

In this thesis, we mathematically analyze a novel free-boundary model for the spread of ecosystem engineers that was recently derived from an individual random walk model. The Stefan condition for the moving boundary is replaced by a biologically derived two-sided condition that models the movement behavior of individuals at the boundary as well as the process by which the population moves the boundary to expand their territory.

We first consider the model with logistic growth and study its well-posedness.We assign a convex functional to this problem so that the evolution system governed by this convex potential is exactly the system of evolution equations describing the above model. We then apply variational and fixed-point methods to deal with this free boundary problem and prove the existence of local in-time solutions.

We next study traveling wave solutions of the model with the strong Allee growth function. We use phase plane analysis to find traveling wave solutions of different types and their corresponding existence range of speed for the model with an imposed speed of the moving boundary. We then find the speeds in those ranges at which the corresponding traveling wave follows the speed of the free boundary.

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Chapter 1

Introduction

Modeling the spatial spread of organisms in the environment continues to be a highly active area of research, in particular for invasive species [44, 65]. The rigorous mathematical analysis of the resulting models has been a rich source of challenging mathematical problems for many years, e.g., [41]. Here, we use a specific approach, namely a reaction-diffusion equation with a free boundary, to study the spatial spread of a certain type of biological species, known as ecosystem engineers.

1.1 Reaction-Diffusion Equation for Spatial Spread

Reaction-diffusion equations are the framework of choice for many models for species' spread, dating back to the well-known Fisher-KPP model [26, 40]. Let u denote the density of the species at location x in space and time t. The most basic model reads

$$u_t - Du_{xx} = f(u), \qquad x \in \mathbb{R}, \quad t \in \mathbb{R}^+, \tag{1.1.1}$$

where $f(u) = u(r - qu)$ is the logistic growth function, with $r > 0$ the maximum per capita growth rate and $q > 0$ the intra-specific competition rate, and $D > 0$ represents the diffusion coefficient. Indices denote partial derivatives. Equation (1.1.1) describes the simultaneous random movement of individuals and population dynamics such as birth and death. The diffusive term can be derived from the assumption of individuals following an unbiased random walk [70]. The logistic growth term models the bounded growth of a population and is frequently used in biology to describe the growth of a population with limited resources. The environment is assumed to be independent of space and time, and also of the presence or absence of the invading species.

An important property of equation (1.1.1), which makes it a strong tool in modeling species spread, is that it supports traveling waves, i.e., solutions of the form $u(t, x) = U(x - ct) = U(z)$. Fisher proved the existence of a traveling wave profile for (1.1.1), for all speeds c greater than or equal to the minimal traveling wave speed

$\hat{c} := 2\sqrt{rD}$ [26]. Later authors proved that solutions with compactly supported initial conditions develop into two-sided traveling waves that expand at that minimal speed $\hat{c}$ [3, 75]. Traveling wave solutions are interpreted as the density of a species that invades novel territory [44], and the speed at which this invasion happens is one of the most important quantities in the study of invasive species [31].

Over the past decades, the original model has been expanded to include a number of biotic and abiotic interactions [41,44]. Biotic interactions refer to interactions with other species, such as competition or predation. The resulting models are systems of reaction-diffusion equations with or without monotonicity properties. Abiotic interactions describe how movement and population growth depends on landscape features, such as resource availability or movement barriers. These landscape features are encoded in the model by considering diffusion or growth in equation (1.1.1) to depend on space. Most recently, in an attempt to model the effects of climate change, several models considered the growth function to depend on $x - \tilde{c}t$, where $\tilde{c}$ is the speed with which temperature isoclines move to higher latitudes or altitudes, e.g., [62].

1.2 Ecosystem Engineers

One aspect that has so far not been included in models for spatial spread, but that could significantly affect the spreading process [30], is that some species modify the physical habitat. In fact, some ecologists argue that all species modify some abiotic aspects of their habitat, but some species clearly have a huge impact on their environment. Examples include beavers, woodpeckers, ants, and even some trees [35]. In the words of Jones *et al.* [34]: *"The ecological literature is rich in examples of habitat modification by organisms, some of which have been extensively studied [...]. However, in general, population and community ecology have neither defined nor systematically identified and studied the role of organisms in the creation and maintenance of habitats."* To address species with this habitat modification behavior, Jones and coauthors coined the term "ecosystem engineers."

Hence, ecosystem engineers are species that significantly modify their habitat, typically in order to make it more suitable for them. For example, beavers cut down trees to build dams and create ponds in which they can build their lodges, and be protected from predators. The physical changes caused by their engineering activity can have positive or negative effects on the environment, other species, and even the engineer itself [35]. It turns out that ecosystem engineers are among the most successful invasive species and many of them have quite severe impacts on the environment and other species [13]. As Cuddington and Hastings write: *"The importance of incorporating habitat modification is that it may both qualitatively and quantitatively change our predictions about the probability, speed, and multi-species effects of such an invasion"* [13]. Therefore, including engineering activity is important when making predictions about invasion success. Yet, most models for invasive species do not consider this aspect. Our work here is one step in this direction.

We focus on so-called obligate engineers, such as the European bee-eater (as a burrowing fauna) and coastal redwoods [14]. Obligate engineers are species that need to modify their habitat in order to survive there. In other words, without their engineering activity, their net growth rate will be negative and they can not survive. In a companion paper [48], we presented, to the best of our knowledge, the first model for the spatial spread of an obligate ecosystem engineer (but see the work of Du *et al.* [19] below). In the model, the state of the habitat is binary: either engineered or non-engineered. The engineered habitat is suitable for population growth, but the non-engineered habitat is not. The engineering activity transforms non-engineered habitat into engineered habitat and thereby moves the boundary between them from the engineered habitat into non-engineered habitat. This approach leads to a novel parabolic, two-sided free boundary problem. We briefly review some aspects of free-boundary problems here before we give a detailed description of our model in the next section.

1.3 Free Boundary Problems: The Stefan Problem

Many free boundary problems stem from the so-called Stefan problem, which describes, among other things, the phase transition between water and ice [68]. Modeling phase transitions goes back to earlier work by Lamé and Clapeyron [42]. The first proof of the well-posedness of the Stefan problem was given by Rubinstein; see [64] for the first comprehensive historical development and mathematical treatment in English.

To briefly recall the Stefan problem, consider the phase transition between water and ice on a one-dimensional interval $[0,\infty)$. Assume that at any time t, the interface between water and ice is at location $x = L(t)$, where we have water in the region $[0, L(t))$, and ice in $(L(t),\infty)$. Let $u(t, x)$ denote the temperature of the water at time t and location x. Then u satisfies the heat equation in the region $[0, L(t))$ and drops to zero at the interface with the ice, $x = L(t)$. The boundary between water and ice is unknown and is changing with time. However, its movement can be described by the so-called Stefan condition, which represents the physical fact that the heat flux from water to ice melts the ice. Hence, it expresses the movement of the boundary as proportional to the negative of the heat flux at the interface, $u_x(t, L(t)^-)$. In the simplest form, the model can be expressed as follows:

$$\begin{cases} u_t - du_{xx} = 0, & t > 0, 0 < x < L(t), \\ u(t, L(t)) = 0, & t > 0, \\ L'(t) = -\mu u_x(t, L(t)), & t > 0, \end{cases} \tag{1.3.1}$$

with d, μ given positive constants.

A one-sided (and one-dimensional) free boundary problem is typically posed on an interval with the quantity of interest present only on one side of the boundary, and the velocity of the boundary is usually determined by the gradient of this quantity. Examples include the one-phase Stefan problem. In contrast, two-sided free boundary problems track the densities of two substances, one on each side of the boundary. Examples include the two-phase Stefan problem, Muskat's model [56], and Verigin's model [71]; see [59] for a summary of well-posedness results for these models. These modeling approaches and their extensions have since been used to model various phenomena and processes in different fields, including physics and engineering [59,72], and biology [27]. In the following, we concentrate on ecological applications.

One of the pioneering works in applying free boundary problems in modeling an ecological phenomenon was [52], where they used a two-sided free boundary problem to describe the competition of two species in a habitat. In a later section, we discuss this model in more detail. Since then, there have been various applications of free boundary problems to different aspects of ecology. However, the application of free boundary problems to the spatial spread of invasive species is more recent. In the next section, we present a summary of prominent works on the spatial spread of species based on the Stefan problem.

1.4 The Stefan Problem Applied to Species Spread

The Stefan problem and the moving boundary idea behind it were first used for modeling the spatial spread of an invasive species by Du *et al.* [19]. They coupled the Fisher-KPP equation with a special kind of Stefan condition to describe the spread of an invading species into a new environment by using the free boundary to represent the location of the spreading front. To be more precise, they considered a one-dimensional domain $[0, \infty)$, where a new species with an initial density of u_0 is introduced and occupies an initial region $[0, l_0]$. The species then starts to spread in the habitat according to a reaction-diffusion equation on an expanding domain $[0, L(t)]$, where $L(t)$ represents the location of the propagation front. There are no individuals ahead of the front. The movement of the front is proportional to the gradient of the density at the moving boundary. Hence, the model can be expressed mathematically as follows:

$$\begin{cases} u_t - Du_{xx} = u(r - qu), & t > 0, 0 < x < L(t), \\ u_x(t, 0) = 0, u\left(t, L(t)\right) = 0, & t > 0, \\ L'(t) = -\mu u_x\left(t, L(t)\right), & t > 0, \\ L(0) = l_0, u(0, x) = u_0(x), & 0 \le x \le l_0, \end{cases} \tag{1.4.1}$$

where r, q and D are as in (1.1.1), and μ is a given positive constant.

Du *et al.* argue that this approach is more realistic than the Fisher-KPP approach in Eq. (1.1.1), which has a positive population density everywhere for any $t > 0$ because of its parabolic nature. The authors proved the well-posedness of the model and showed that the model has a spreading-vanishing dichotomy, i.e. we get one of the following scenarios in the long term, as $t \to \infty$:

- $L(t) \to \infty$ and $u(t,x) \to \frac{r}{q}$, the species can spread into the whole environment and approach a stable positive density.
- $L \to L_\infty \leq \frac{\pi}{2}\sqrt{\frac{D}{r}}$ and $u(t,x) \to 0$, the front location remains bounded, and the species dies out and vanishes.

The authors discussed conditions for either of the two scenarios to happen in terms of parameters l_0, u_0, and μ.

In particular, they showed that if the population density is initially large enough it will spread and form a solution akin to a traveling wave solution. This solution can be found from an auxiliary elliptic equation. Then they used this profile to estimate the asymptotic speed of spread and proved that the spreading front $x = L(t)$ has a constant asymptotic speed k_0, that is,

$$L(t) = (k_0 + o(1))t \quad \text{as} \quad t \to \infty. \tag{1.4.2}$$

They further investigate the dependence of k_0 on different parameters in (1.4.1).

Since the work of Du *et al.* [19], the idea of using the free boundary problems to represent a spreading front has been adopted in other papers. Some of them extended the results of Du *et al.* [4, 10, 17, 18]. Most of these works are based on a one-sided free boundary problem, in the sense that they considered the density of a species, whose range is changing, to be on one side of the free boundary and let the boundary represent the spreading front. In all of these works, the density is considered to be continuous and equal to zero at the interface. In a review paper [16], Du presents a comprehensive discussion of these models and their results.

Some important features of Du *et al.*'s model are as follows. They considered a clearly defined range boundary for the presence of the species, which results in having the density of the species drop to zero at the moving boundary. In addition, they described the movement of the boundary using a Stefan-type condition without biologically deriving it. However, they justified their use of this condition in [10].

1.5 Related Free Boundary Problems in Ecology

As mentioned before, a well-known example of a two-sided free boundary problem in ecological modeling is the early work of Mimura *et al.* [52]. They considered a bounded domain occupied by two species that are separated by a moving boundary.

On either side of the boundary, both species are constantly struggling to expand their habitat toward the other's, and hence try to move the boundary. The authors presented the following model:

$$\begin{cases} u_{1t} - d_1 u_{1xx} = u_1 f(u_1), & t > 0, 0 < x < L(t), \\ u_{2t} - d_2 u_{2xx} = u_2 f(u_2), & t > 0, L(t) < x < l, \\ u_1\left(t, L(t)\right) = u_2\left(t, L(t)\right) = 0, & t > 0, \\ L'(t) = -\alpha_1 u_{1x}\left(t, L(t)\right) - \alpha_2 u_{2x}\left(t, L(t)\right), & t > 0, \\ u_1(t,0) = m_1, \quad u_2(t,0) = m_2, & t > 0, \\ u_1(0,x) = u_{01}(x) \geq 0, & 0 \leq x \leq l_0, \\ u_2(0,x) = u_{02}(x) \geq 0, & l_0 \leq x \leq l, \\ L(0) = l_0, \end{cases}$$

where d_i, m_i, α_i, l_0, with $i = 1, 2$, are given positive constants and u_{0i} are initial conditions. They proved the well-posedness of the problem and discussed the asymptotic behaviour of its solution [52–54].

It is worth noting that although they considered a two-sided free boundary model, they have a different species on each side of the boundary. The species interaction at the free boundary is through the direction of the movement of the boundary not through density flux across the boundary. The density of both species drops to zero at the free boundary. More recent studies that build on similar ideas include [15, 32, 33, 74]. To describe the competition between species, other authors also used two-phase free-boundary problems of Verigin type to capture the changing ranges of the species in a bounded domain with continuous flux and density on the interface [39].

Another type of free boundary model with densities of two species on both sides of the boundary is introduced by Lin in [46]. He combined the Lotka-Volterra model for the dynamics of prey and predator with a free boundary for the extent of the prey. The predator initially occupies a limited region and starts to spread. The precise model is as below:

$$\begin{cases} u_{1t} - d_1 u_{1xx} = u_1\big(a_1 - b_{11}u_1 + c_{12}u_2\big), & t > 0, \quad 0 < x < L(t), \\ u_{2t} - d_2 u_{2xx} = u_2\big(a_2 - b_{21}u_1 - c_{22}u_2\big), & t > 0, \quad 0 < x < l, \\ u_1\left(t, L(t)\right) = 0, & t > 0, \\ u_1\left(t, x\right) \equiv 0, & t > 0, \quad L(t) < x < l, \\ L'(t) = -\mu u_{1x}\left(t, L(t)\right), & t > 0, \\ u_{1x}(t,0) = u_{2x}(t,0) = u_{2x}(t,l) = 0, & t > 0, \\ u_1(0,x) = u_{10}(x) \geq 0, & 0 \leq x \leq l_0, \\ u_2(0,x) = u_{20}(x) \geq 0, & l_0 \leq x \leq l, \\ L(0) = l_0, \quad (0 < l_0 < l) \end{cases}$$

where $d_1, b_{ij}, a_i, c_{ij}, \mu$ are positive constants. Here, u_1 and u_2 represent the densities of predator and prey, respectively. In [46], the author showed the well-posedness of the model and proved that the predator spreads to the entire domain. In this model, the prey is present in the whole domain and so on both sides of the moving boundary. However, the predator, whose spread in the domain is the main subject, is just on one side of the moving boundary, which represents the spreading front of the predator. The two species are not interacting through the free boundary in any way. Therefore, in many respects this work is similar to the idea of Du *et al.* [19].

The mentioned restrictions of these two well-studied models highlights the importance of our model in bringing a more realistic and applicable approach to the study of an ecological phenomenon. It is often difficult in practice to establish exactly where organisms are present and where they are absent. Typically, the density declines gradually and some organisms venture outside their existing range. For that reason, we model the spread of an ecosystem engineer as a two-sided free boundary problem more similar in spirit to the models in [56, 71].

In the next section, we present our model, originally introduced by Lutscher *et al.* in [48], which is derived from basic principles via an uncorrelated random walk [70].

1.6 Our Model

We give a fairly concise derivation of our model and some explanation of its mathematical challenges. For more ecological details of the derivation, we refer to [48]. We study the spatial spread of the ecosystem engineer on a bounded one-dimensional domain. The environment consists of two habitat types: engineered and non-engineered. The engineered habitat is suitable for population growth, but the non-engineered habitat is not. The engineering activity transforms non-engineered habitat into engineered habitat and thereby moves the boundary between these two types of habitat.

We let $u(t, x)$ denote the density of the species at time $t > 0$ in location $x \in [a, b]$ for $a, b \in \mathbb{R}$. We assume that the domain is large enough for the species to spread in the long term. So the two ends of the boundary are far from each other, that is $a \ll b$. We let $x = L(t)$ represent the location of the boundary between engineered and non-engineered habitats at time t, which is moving between the two ends of the boundary, $a < L(t) < b$. Therefore, at any time t, the region $[a, L(t))$ is the engineered habitat and $(L(t), b]$ is the non-engineered habitat.

The model tracks the population dynamics of the species in the engineered and non-engineered habitats. Matching conditions relate the population density and flux across the moving boundary, which is the interface between the two types of habitat. Equations for the movement of the boundary by the engineering activity complete the model. We discuss each aspect separately.

Population Dynamics

To describe the population dynamics of the species in each habitat, we assume that the environmental conditions inside the engineered habitat are homogeneously favorable. We use a reaction-diffusion equation with a reaction term to represent the growth function of the species. So, in the engineered habitat we have

$$u_t = D_1 u_{xx} + f_1(u), \qquad (t,x) \in (0,T) \times (a, L(t)), \tag{1.6.1}$$

where D_1 is the diffusion coefficient in the engineered habitat and f_1 is the growth function. Depending on the assumptions on the growth of the species, we consider either a logistic growth function or a growth function with a strong Allee effect. For the logistic growth function, we have

$$f_1(u) = u(r - qu), \tag{1.6.2}$$

where $r > 0$ is the maximum per capita growth rate and q is the intra-specific competition rate in the engineered habitat due to limited resources. For a strong Allee growth function, we have

$$f_1(u) = r\left(\frac{u - k_0}{k_1}\right)\left(1 - \frac{u}{k_1}\right)u, \tag{1.6.3}$$

where r is the intrinsic growth rate, $k_1 > 0$ is the carrying capacity in the engineered habitat, and $0 \leq k_0 < k_1$ is the Allee threshold.

These two growth functions capture different biological scenarios in population dynamics. According to the logistic function, a species has positive (net) growth at low densities; in fact, the maximum per capita growth rate occurs at zero density. However, in a population with a strong Allee effect, the per capita (net) growth rate is negative at low densities so that the species cannot actually grow while their density is below this Allee threshold density. The maximum per capita growth rate occurs at intermediate densities between the Allee threshold and the carrying capacity [2, 11].

The non-engineered habitat is not favorable for the species, so it can not survive there and just dies with a constant rate. We again use a reaction-diffusion equation to model the movement far from the moving boundary, and the reaction part is just a linear function with a constant death rate. The equation is

$$u_t = D_2 u_{xx} + f_2(u), \qquad (t,x) \in (0,T) \times (L(t), b), \tag{1.6.4}$$

where D_2 is the diffusion coefficient in non-engineered habitat and $f_2(u) = -mu$ with m being the death rate.

Matching Conditions

We next need to impose matching conditions for the density and the flux at the moving interface. The continuous matching that many authors used turns out to yield highly unrealistic outcomes when applied to ecological models [51]. Instead, certain discontinuous matching conditions can be derived from basic random-walk principles [61] and those lead to realistic ecological outcomes [51]. To formulate these conditions, Maciel and Lutscher [51] derived the following matching condition for the population density from an uncorrelated random walk at the interface between the two types of habitat. They assume that an individual at the range boundary moves to the engineered habitat with probability α_1 and to the non-engineered habitat with probability α_2; it stays at the boundary with probability $\alpha = 1 - (\alpha_1 + \alpha_2)$. They end up with the following condition for the density at the interface

$$u(t, L(t)^-) = ku(t, L(t)^+), \tag{1.6.5}$$

where $k = \frac{\alpha_1 D_2}{\alpha_2 D_1}$, and superscripts $\pm$ denote the one-sided limits. In particular, if $k \neq 1$, then the density is discontinuous at the interface. Since the movement in the favorable habitat is typically slower than the movement in the unfavorable habitat [12], we can assume that $D_1 \leq D_2$. In addition, as α_1, α_2 are measures for habitat preference, we can also assume that $\alpha_1 \geq \alpha_2$. Therefore, we assume that

$$k = \frac{\alpha_1 D_2}{\alpha_2 D_1} > 1. \tag{1.6.6}$$

To match the flux across the interface, we assume that no individual is added or removed at the interface. To impose this condition, we argue by conservation of mass for the diffusion equation in the absence of population dynamics as in [50] that the total density must remain constant in the domain. We arrive at the following condition at the interface:

$$D_1 u_x(t, L(t)^-) + L'(t)u(t, L(t)^-) = D_2 u_x(t, L(t)^+) + L'(t)u(t, L(t)^+). \tag{1.6.7}$$

At the two ends of the interval, we consider no-flux boundary conditions, biologically representing that no individual can leave the domain from those ends:

$$u_x(t, a) = u_x(t, b) = 0. \tag{1.6.8}$$

Moving condition

For the condition needed to describe the movement of the boundary, Lutscher *et al.* [48] consider three different scenarios derived from uncorrelated random walks as well.

- **Scenario 1:** The main assumption, in this case, is that individuals at the interface of two types of habitat will engineer the neighbouring non-engineered habitat and thereby extend their range and move the boundary. Applying this assumption, Lutscher *et al.* derive the movement of the boundary to be proportional to the density of the species at the boundary, i.e.

$$L'(t) = 2\eta_1 D_1 \alpha u(t, L(t)^-), \tag{1.6.9}$$

where η_1 is a proportionality constant. We note here that if $\alpha > 0$, then the boundary is expanding and so $L'(t) > 0$ on $[0, T]$. However, if $\alpha = 0$, i.e. no individual stays at the boundary, then the boundary does not move and $L'(t) = 0$ on $[0, T]$.

- **Scenario 2:** This Scenario will specifically consider cases in which the boundary can retract. To derive this case, Lutscher *et al.* suppose that the engineered structure may need maintenance and will decay otherwise. This requires that a certain population density be present at the boundary at all times. Population densities above this required threshold at the boundary will result in extending the boundary and below it will result in retracting the boundary. The expression is

$$L'(t) = 2\eta_1 D_1 \alpha \big(u(t, L(t)^-) - \bar{u}\big), \tag{1.6.10}$$

where $\bar{u}$ is the threshold density, and D_1 and η_1 are as in Scenario 1.

- **Scenario 3:** In this case, the assumption is that individuals at the boundary who leave the habitat due to pressure from higher density behind them will move the boundary. Inserting this assumption in the random walk derivation of the equation for the movement of the boundary leads to a Stefan-type condition of the following form

$$L'(t) = -2\eta_2 D_1 u_x(t, L(t)^-). \tag{1.6.11}$$

Hence, formulating all aspects of the model, we can combine the three scenarios into a single general formulation of our model as follows: for $(t, x) \in (0, T) \times (a, b)$, we have the system

$$\begin{aligned} u_t &= D_1 u_{xx} + f_1(u), && (t, x) \in (0, T) \times (a, L(t)), \\ u_t &= D_2 u_{xx} + f_2(u), && (t, x) \in (0, T) \times (L(t), b), \end{aligned}$$

with matching and boundary conditions

$$\begin{aligned} u(t, L(t)^-) &= k u(t, L(t)^+), && t \in (0, T], \\ \Big(D_1 u_x + L'(t) u\Big)\Big|_{(t, L(t)^-)} &= \Big(D_2 u_x + L'(t) u\Big)\Big|_{(t, L(t)^+)}, && t \in (0, T], \\ u_x(t, a) &= u_x(t, b) = 0, && t \in (0, T], \end{aligned}$$

and initial values

$$\begin{aligned} L(0) = l_0, \quad a < L(t) < b, \quad & t \in [0, T], \\ u(0, x) = u_0(x), \qquad\quad & x \in [a, b]. \end{aligned}$$

and moving condition

$$L'(t) = B(u, u_x)|_{(t, L(t)^-)}, \qquad t \in [0, T],$$

where B is linear in both arguments.

Most existing free boundary models in ecology have a given species confined to one side of the free boundary, either in the form of a one-sided free boundary that describes the abrupt edge of the range of the species or as a two-sided free boundary that separates two species. While fairly abrupt edges of species' ranges can occur, in particular for territorial species [55], the range edge of a spreading species is typically more gradual. On the other hand, in these models, individuals interact only through the movement of the boundary but not through the exchange of density. In our model, the same species occupies both sides of the free boundary and individuals can move back and forth across the boundary, which is more realistic. Similar considerations appear in fluid dynamics [28] and thermodynamics [76]. Another novel aspect is the derivation of both, the matching condition and the free boundary equation, from random walks. Previous ecological applications have argued more heuristically, e.g., [10] for a one-sided free boundary or [39] by conservation laws for a two-sided problem, and none include the discontinuity in the density matching at the free boundary.

1.7 Overview

In this thesis, we study two different extensions of the model introduced in [48]. We first study the well-posedness of the model with a logistic growth function. As mentioned in Section 1.6, we have three scenarios for the moving boundary in the model. We concentrate on proving the existence of a solution for scenario I since this scenario is the most different from what had been studied in the literature before. Second, we study the existence of traveling wave solutions when the population dynamics have a strong Allee effect. We organize this thesis as follows:

In Chapter 2, we consider the model with the logistic growth function and prove the existence of a local solution for the first scenario. We first review some preliminary results and basic definitions for each aspect of our work. In particular, we recall important results about the theory of nonlinear evolution equation generated by a subdifferential operator from [60], along with some standard results about fixed point theory, which are the main tools that we use in the proof of the existence result. We then discuss the main result that we have for the model. We divide the proof of the main result into two major parts: In the first part, we consider the model for any given curve as a moving boundary. To do that, we carefully formulate the model as an abstract nonlinear evolution equation involving the subdifferential of an appropriate time-dependent functional on a Hilbert space. Then we can apply the results from [60] to prove the existence of solutions for the reduced problem. In the second part, we show the existence of solutions for the model with a free boundary that satisfies the moving boundary condition in the model. In this part of the proof, we follow the idea of [22] and use Schauder's fixed point theorem on an appropriate operator.

In Chapter 3, we consider our model with a strong Allee effect and study the existence of traveling wave solutions for it. Similar to Chapter 2, we first recall some background results about traveling solutions for reaction-diffusion equations from [29], and the corresponding results for free boundary problems, in particular, for our model with logistic growth function [48]. We then present our results, again in two parts. Using phase-plane analysis, we first study the reduced model with a given speed and prove the existence of traveling wave solutions of different types for an appropriate range of speeds. Second, we consider the full model for each scenario and show the existence of traveling wave solutions of different types with desired speeds in the corresponding range.

Finally, in Chapter 4, we discuss the contribution of our work to the literature and several possible future directions in studying our model.

Chapter 2

Existence of Solutions

In this chapter, we consider our model with logistic growth function and study its well-posedness with the first scenario of the movement of the free boundary. Most of the content of this chapter is published in [6].

2.1 Preliminaries

In this section, we recall some important definitions and theorems that we are going to use in the proof of our main result.

2.1.1 Sobolev Spaces

In this section, we introduce some Banach spaces that are essential in the study of solutions of partial differential equations.

Definition 2.1.1 (L^p **Spaces**). *Assume Ω is an open subset of $\mathbb{R}^n$ and $1 \le p \le \infty$. For a measurable function $u : \Omega \to \mathbb{R}$, we set*

$$\|u\|_{\mathrm{L}^p(\Omega)} := \begin{cases} \Big(\displaystyle\int_\Omega |u|^p dx \Big)^{1/p} & \text{if } 1 \le p < \infty, \\ \operatorname{ess\,sup}_\Omega |u| & \text{if } p = \infty, \end{cases} \tag{2.1.1}$$

and define $\mathrm{L}^p(\Omega)$ to be the linear space of all measurable functions $u : \Omega \to \mathbb{R}$ for which $\|u\|_{\mathrm{L}^p(\Omega)}$ is finite (see [23] for more details).

Recall that $\mathrm{C}_0^\infty(\Omega)$ represents the space of smooth functions $\eta : \Omega \to \mathbb{R}$, which are infinitely many times continuously differentiable and have compact support in Ω, and $\mathrm{L}^1_{loc}(\Omega)$ is the space of locally summable functions $u : \Omega \to \mathbb{R}$, which are integrable on any compact subset of Ω. We define a weak derivative of a function u as follows.

Definition 2.1.2 (Weak derivative). *Suppose $u, v \in \mathrm{L}^1_{loc}(\Omega)$ and α is a multi-index. We say that v is the α^{th}-weak partial derivative of u, written*

$$D^\alpha u = v,$$

provided

$$\int_\Omega u D^\alpha \eta dx = (-1)^{|\alpha|} \int_\Omega v \eta dx,$$

for all test functions $\eta \in \mathrm{C}_0^\infty(\Omega)$ (see [23, p.256]).

Definition 2.1.3 (Sobolev Spaces). *The Sobolev space $\mathrm{W}^{k,p}(\Omega)$ consists of all locally summable functions $u : \Omega \to \mathbb{R}$ such that for each multi-index α with $|\alpha| \leq k$, $D^\alpha u$ exists in the weak sense and belongs to $\mathrm{L}^p(\Omega)$ (see [23, p.258]).*

Definition 2.1.4. *If $u \in \mathrm{W}^{k,p}(\Omega)$, we define its norm to be*

$$\|u\|_{\mathrm{W}^{k,p}(\Omega)} := \begin{cases} \Big(\sum\limits_{|\alpha| \leq k} \int_\Omega |D^\alpha u|^p dx \Big)^{\frac{1}{p}} & \text{if } 1 \leq p < \infty, \\ \\ \sum\limits_{|\alpha| \leq k} \operatorname{ess\,sup}_\Omega |D^\alpha u| & \text{if } p = \infty. \end{cases} \tag{2.1.2}$$

It is worth noting that the Sobolev spaces with the above norm are Banach Spaces (see [23] for details). For $p = 2$, the Sobolev spaces are Hilbert spaces. We denote

$$\mathrm{H}^k(\Omega) = \mathrm{W}^{k,2}(\Omega), \qquad k = 0, 1, \ldots.$$

We introduce the following notation: for any function space X and domain $\Omega = \Omega_1 \cup \Omega_2$, we write

$$u \in X(\Omega_1) \cap X(\Omega_2) \tag{2.1.3}$$

for the set of functions u on Ω such that the restrictions of u to Ω_i belong to $X(\Omega_i)$. In general, we do not have $u \in X(\Omega)$, but this might be true in special cases. For example, for $a < c < b$ we have

$$\big\{ u \in \mathrm{H}^1(a, c) \cap \mathrm{H}^1(c, b); u(c^-) = u(c^+) \big\} = \mathrm{H}^1(a, b),$$

where the superscripts $\pm$ denote one-sided limits.

Embedding

We devote this section to an important embedding result in Sobolev spaces, which we will use in the sequel. We use the notation $\hookrightarrow$ and $\hookrightarrow\!\!\!\!\!\hookrightarrow$ to denote continuous and compact embeddings, respectively.

Lemma 2.1.5 (Sobobev Embedding). *If* $(\alpha,\beta)\subset\mathbb{R}$*, then* $\mathrm{H}^1(\alpha,\beta)\hookrightarrow \mathrm{L}^p(\alpha,\beta)$ *for* $p\in[1,\infty)$*. Indeed,*

$$\|u\|_{\mathrm{L}^p(\alpha,\beta)}\leq C_{p,\beta-\alpha}\|u\|_{\mathrm{H}^1(\alpha,\beta)},$$

where

$$C_{p,\beta-\alpha}=(\beta-\alpha)^{\frac{1}{p}-\frac{1}{2}}+(\beta-\alpha)^{\frac{1}{p}+\frac{1}{2}}. \tag{2.1.4}$$

Proof: Let $u(\cdot)\in\mathrm{H}^1(\alpha,\beta)$. Then we can calculate

$$|u(y)-u(z)|=\left|\int_z^y u_x(x)dx\right| \quad\text{for}\quad y,z\in(\alpha,\beta),$$

and we have

$$|u(y)|\leq|u(z)|+\int_\alpha^\beta|u_x(x)|dx \quad\text{for}\quad y,z\in(\alpha,\beta).$$

Integrating with respect to z, we obtain

$$(\beta-\alpha)|u(y)|\leq\int_\alpha^\beta|u(z)|dz+(\beta-\alpha)\int_\alpha^\beta|u_x(x)|dx \quad\text{for}\quad y\in(\alpha,\beta).$$

Using Hölder's inequality, we find for $y\in(\alpha,\beta)$,

$$(\beta-\alpha)|u(y)|\leq(\beta-\alpha)^{\frac{1}{2}}\left(\int_\alpha^\beta|u(z)|^2dz\right)^{\frac{1}{2}}+(\beta-\alpha)^{\frac{3}{2}}\left(\int_\alpha^\beta|u_x(x)|^2dx\right)^{\frac{1}{2}}.$$

Hence, we have

$$|u(y)|\leq\left(\frac{1}{(\beta-\alpha)^{\frac{1}{2}}}+(\beta-\alpha)^{\frac{1}{2}}\right)\|u\|_{\mathrm{H}^1(\alpha,\beta)} \quad\text{for}\quad y\in(\alpha,\beta).$$

Finally, we find the inequality for the L^p-norm of u to be

$$\begin{aligned}\|u\|_{\mathrm{L}^p(\alpha,\beta)} &\leq \left(\frac{1}{(\beta-\alpha)^{\frac{1}{2}}}+(\beta-\alpha)^{\frac{1}{2}}\right)(\beta-\alpha)^{\frac{1}{p}}\|u\|_{\mathrm{H}^1(\alpha,\beta)}\\ &= \left((\beta-\alpha)^{\frac{1}{p}-\frac{1}{2}}+(\beta-\alpha)^{\frac{1}{p}+\frac{1}{2}}\right)\|u\|_{\mathrm{H}^1(\alpha,\beta)}.\end{aligned}$$

Hence, we obtain

$$\|u\|_{\mathrm{L}^p(\alpha,\beta)} \leq C_{p,\beta-\alpha}\|u\|_{\mathrm{H}^1(\alpha,\beta)},$$

where $C_{p,\beta-\alpha}$ is a constant which depends on $(\beta-\alpha)$ and p. For all $p \in [1,\infty)$, this constant remains bounded as $\beta-\alpha$ ranges over any compact subset of $(0,\infty)$. ∎

2.1.2 Banach-Valued Functions

In this section, we introduce the spaces in which we are looking for solutions for evolution equations. Assume X is a real Banach space, with norm $\|\cdot\|$. The notions of measurability and integrability are extended to mapping $u : [0,T] \to X$.

Definition 2.1.6. *A function $s : [0,T] \to X$ is called simple if it has the form*

$$s(t) = \sum_{i=1}^{m} \chi_{E_i}(t) c_i \quad \text{for } 0 \leq t \leq T,$$

where for $i = 1, \ldots, m$, $c_i \in X$, each E_i is a Lebesgue measurable subset of $[0,T]$. The characteristic function χ_A is defined as

$$\chi_A(x) = \begin{cases} 1 & \text{if} \quad x \in A, \\ 0 & \text{if} \quad x \notin A. \end{cases}$$

A function $u : [0,T] \to X$ is strongly measurable if there exists a sequence of simple functions $\{s_n\}_n$ such that

$$s_n(t) \to u(t) \quad \text{strongly in } X \text{ and for a.e. } \; 0 \leq t \leq T$$

(see [23, p.733]).

Definition 2.1.7. *If $s(t) = \sum_{i=1}^{m} \chi_{E_i}(t) c_i$ is simple, we define*

$$\int_0^T s(t)dt := \sum_{i=1}^{m} |E_i| c_i,$$

where $|E_i|$ is the measure of the set E_i for each i. We say the strongly measurable function $u : [0,T] \to X$ is summable, or Bochner integrable, if there exists a sequence $\{s_n\}_n$ of simple functions such that

$$\int_0^T \|s_n(t) - u(t)\| dt \to 0 \text{ as } n \to \infty.$$

If u is summable, we define

$$\int_0^T u(t)dt = \lim_{n\to\infty} \int_0^T s_n(t)dt$$

(see [23, p.733]).

We define L^p spaces involving time as follows.

Definition 2.1.8. *The space*

$$L^p(0,T;X)$$

consists of all strongly measurable functions $u : [0,T] \to X$ with

$$\|u\|_{L^p(0,T;X)} := \left(\int_0^T \|u(t)\|^p dt \right)^{1/p} < \infty,$$

for $1 \leq p < \infty$ and

$$\|u\|_{L^\infty(0,T;X)} = \operatorname{ess\,sup}_{0\leq t\leq T} \|u(t)\| < \infty$$

(see [23, p.301]).

Definition 2.1.9. *Let $u \in L^1(0,T;X)$. We say $v \in L^1(0,T;X)$ is the weak derivative of u, written*

$$u' = \frac{du}{dt} = v,$$

provided

$$\int_0^T \eta'(t)u(t)dt = -\int_0^T \eta(t)v(t)dt,$$

for all scalar test functions $\eta \in \mathrm{C}_0^\infty(0,T)$ (see [23, p.301]).

Now we can define Sobolev spaces involving time as follows.

Definition 2.1.10. *The Sobolev space*

$$\mathrm{W}^{1,p}(0,T;X)$$

consists of all functions $u \in \mathrm{L}^p(0,T;X)$ such that u' exists in the weak sense and belongs to $\mathrm{L}^p(0,T;X)$. Furthermore,

$$\|u\|_{W^{1,p}(0,T;X)} := \begin{cases} \left(\int_0^T \|u(t)\|^p + \|u'(t)\|^p dt \right)^{1/p} & \text{if } \quad 1 \leq p < \infty, \\ \\ \operatorname{ess\,sup}_{0\leq t\leq T} (\|u(t)\| + \|u'(t)\|) & \text{if } \quad p = \infty \end{cases}$$

(see [23, p.303]).

Embedding

Here, we bring a fundamental result regarding compact embeddings in vector-valued function spaces. As above, $\hookrightarrow$ and $\hookrightarrow\!\!\!\!\!\!\hookrightarrow$ denote continuous and compact embeddings, respectively.

Theorem 2.1.11 (**Aubin-Lions-Simon**). *Let $B_0 \hookrightarrow\!\!\!\!\!\!\hookrightarrow B_1 \hookrightarrow B_2$ be three Banach spaces with the embedding $B_0 \hookrightarrow\!\!\!\!\!\!\hookrightarrow B_1$ compact and the embedding $B_1 \hookrightarrow B_2$ continuous. Let p, r be such that $1 \leq p, r \leq \infty$. For $T > 0$ we define*

$$E_{p,r} = \left\{v \in \mathrm{L}^p(0, T; B_0), \frac{dv}{dt} \in \mathrm{L}^r(0, T; B_2)\right\}.$$

Then,

i) If $p < \infty$, the embedding of $E_{p,r}$ in $L^p(0, T; B_1)$ is compact.

ii) If $p = \infty$ and if $r > 1$, the embedding of $E_{p,r}$ in $\mathrm{C}^0([0, T], B_1)$ is compact.

(See Theorem II.3.10 in [8, p. 83] for a proof.)

2.1.3 Convex Analysis

Consider a real Banach space X and its dual X^*. The duality pairing $(X, X^*, \langle\cdot,\cdot\rangle)$ is given by a bilinear map $(x, x^*) \to \langle x, x^*\rangle$ into $\mathbb{R}$ which separates points. We endow X and X^* with the corresponding weak topology $\sigma(X, X^*)$ [21].

A function $F : X \to (-\infty, +\infty]$ is proper if $F \not\equiv +\infty$. The effective domain of F is the set $\mathcal{D}(F) = \{u \in X; F(u) < \infty\}$.

Definition 2.1.12 (**Lower semi-continuity**). *A function $F : X \to (-\infty, +\infty]$ is called lower semi-continuous if*

$$F(x) \leq \liminf_{n\to\infty} F(x_n),$$

for each $x \in X$ and each sequence $\{x_n\}_n$ converging to x in X.

Definition 2.1.13 (**The sub-differential of a convex function**). *Consider a proper convex function $F : X \to (-\infty, +\infty]$. Let $\mathcal{P}(X^*)$ be the power set of X^*. The sub-differential of F is the mapping $\partial F : X \to \mathcal{P}(X^*)$ defined by*

$$\partial F(x) = \left\{x^* \in X^*; F(u) - F(x) \geq \langle u - x, x^*\rangle, \forall u \in X\right\}.$$

The elements $x^* \in \partial F(x)$ are called sub-gradients of F at x. The set of those x for which $\partial F(x) \neq \emptyset$ is called the domain of ∂F and is denoted by $\mathcal{D}(\partial F)$. Clearly, $\mathcal{D}(\partial F)$ is a subset of $\mathcal{D}(F)$. The function F is said to be sub-differentiable at x if $x \in \mathcal{D}(\partial F)$ (see [5, p.82] for more details).

The following definitions and results are from [20, 21] where the proofs can be found.

Definition 2.1.14 (Gâteaux-differentiability). *A function $F : X \to (-\infty, +\infty]$, finite in a neighborhood of x, is Gáteaux-differentiable at x if there exists some $x^* \in X^*$ such that*

$$\lim_{h\to 0} \frac{F(x+hy) - F(x)}{h} = \langle x^*, y\rangle \quad \textit{for all} \quad y \in X \ .$$

We then write $DF(x) = x^*$.

When F is lower semi-continuous, convex and finite on some neighborhood of $x \in X$, it must be sub-differentiable at x [21], but it need not be differentiable in any sense. So, sub-differentiability is a strict extension of differentiability, as the following result shows:

Proposition 2.1.15. *If $F : X \to (-\infty, +\infty]$ is a convex and lower semi-continuous function and if its sub-differential at x is a singleton, $\partial F(x) = \{x^*\}$, then F is Gâteaux-differentiable at x and $x^* = DF(x)$. If $F : X \to (-\infty, +\infty]$ is Gâteaux-differentiable and continuous at x, then $\partial F(x) = \{DF(x)\}$ (see Proposition 9, [20, p.92] for more details).*

2.1.4 Fixed Point Theorems

Theorem 2.1.16 (Schauder's Fixed Point Theorem). *Suppose $B \subset X$ is compact and convex, and assume also that*

$$A : B \to B$$

is continuous. Then A has a fixed point in B(see Theorem 3 in [23, p. 502] for a proof).

Definition 2.1.17 (Compact Mapping). *A nonlinear mapping $A : X \to X$ is called compact provided for each bounded sequence $\{u_n\}_n$ the sequence $\{A[u_n]\}_n$ is precompact; that is, there exists a subsequence, denoted again by $\{u_n\}_n$ such that $\{A[u_n]\}_n$ converges in X (see [23, p. 503]).*

Next, we state another version of Schauder's Fixed Point Theorem.

Theorem 2.1.18 (Schauder's Fixed Point Theorem II). *Let X be a Banach space and let B be a convex, closed and bounded region of X. If A is a compact and continuous (nonlinear) function from B to B, then it has at least one fixed point in B (see Theorem II.3.10 in [8, p. 83] for a proof).*

2.1.5 Nonlinear Evolution Equations Generated by Subdifferential Operators

In this section, we present the theory of nonlinear evolution equations generated by subdifferential of convex functionals as we use it in our approach to the problem. We refer the reader to [60] for proofs and more details. All of the theorems and definitions in this section can be found in [60].

Consider the following Cauchy problem in a real Hilbert space H:

$$\begin{cases} \dfrac{du(t)}{dt} + \partial\varphi^t(u(t)) + \mathrm{B}(t, u(t)) \ni f(t), \quad 0 < t < T, \\ u(0) = u_0, \end{cases} \tag{2.1.5}$$

where H is equipped with the norm $\|\cdot\|_H$ and the inner product $(\cdot,\cdot)_H$, $\partial\varphi^t$ is the subdifferential of a time-dependent lower semicontinuous convex function φ^t from H into $[0, +\infty]$ with $\varphi^t \not\equiv +\infty$ and $\mathrm{B}(t,\cdot)$ is a possibly nonmonotone multivalued nonlinear operator from $\mathcal{D}(\mathrm{B}(t,\cdot)) \subset H$ into $\mathcal{P}(X^*)$ such that $\mathcal{D}(\partial\varphi^t) \subset \mathcal{D}(\mathrm{B}(t,\cdot))$ for all $t \in [0,T]$. We define a strong solution of (2.1.5) as follows.

Definition 2.1.19. *A function $u(t) \in \mathrm{C}([0,S];H)$ is said to be a strong solution of* (2.1.5) *in $[0,S]$, if the following conditions are satisfied.*

i) $u(t)$ is an H-valued absolutely continuous function (see [63, p. 119]) on $[\delta, S]$ for all $\delta > 0$ and $u(t) \to u_0$, as $t \to 0$.

ii) $u \in \mathcal{D}(\partial\varphi^t)$ for a.e. $t \in (0,S)$ and there exist functions g and $b \in \mathrm{L}^2_{loc}(0,S;H)$ such that $g(t) \in \partial\varphi^t(u(t))$, $b(t) \in \mathrm{B}(t,u(t))$ and

$$\frac{du(t)}{dt} + g(t) + b(t) = f(t),$$

hold for a.e. $t \in (0,S)$.

In addition, if $du/dt, g$ and b belong to $\mathrm{L}^2(0,S;H)$, then $u(t)$ is said to be an s-strong solution of (2.1.5) *in* $[0,S]$.

Let $\mathcal{M}$ be the family of all positive monotone increasing functions on $[0,+\infty)$. Consider the following four conditions on $\partial\varphi^t$ and $\mathrm{B}(t,\cdot)$:

($A.\varphi^t$) For each $t \in [0,T]$, φ^t is a proper lower semicontinuous convex function from H into $[0,+\infty]$. Furthermore, there exists a positive constant δ_0 such that for each $t_0 \in [0,T]$ and $x_0 \in \mathcal{D}\varphi^{t_0}$, one can find an H-valued function $x(t)$ on $I(t_0) := [\max(0, t_0 - \delta_0), \min(t_0 + \delta_0, T)]$ satisfying

$$\|x(t) - x_0\|_H \le m_1(\|x_0\|_H)|t - t_0| \left(\varphi^{t_0}(x_0) + m_2(\|x_0\|_H)\right)^\beta \tag{2.1.6}$$

and

$$\varphi^t(x(t)) \le \varphi^{t_0}(x_0) + m_1(\|x_0\|_H)|t-t_0| \left(\varphi^{t_0}(x_0) + m_2(\|x_0\|_H)\right) \tag{2.1.7}$$

for all $t \in I(t_0)$, where β is a constant in $[0,1]$ and m_i, for $i=1,2$, are continuous functions belonging to $\mathcal{M}$.

(A.1) For each $t \in [0,T]$ and $P \in (0,+\infty)$, the set $\{u \in H; \varphi^t(u) + \|u\|_H^2 \le P\}$ is compact in H.

(A.2) For each interval $[s_1,s_2]$ in $[0,T]$, the following conditions hold.

(i) $\mathrm{B}(t,u)$ is a convex subset of H for all $t \in [s_1,s_2]$ and $u \in \mathcal{D}(\partial\varphi^t)$.

(ii) $\mathrm{B}(t,\cdot)$ is measurable in the following sense:
For each function $u \in \mathrm{C}([s_1,s_2];H)$ such that $du/dt \in \mathrm{L}^2(s_1,s_2;H)$ and there exists a function $g \in \mathrm{L}^2(s_1,s_2;H)$ with $g(t) \in \partial\varphi^t(u(t))$ for a.e. $t \in [s_1,s_2]$, there exists an H-valued measurable function b such that $b(t) \in \mathrm{B}(t,u(t))$ for a.e. $t \in [s_1,s_2]$.

(iii) $\mathrm{B}(t,\cdot)$ is demiclosed in the following sense: if $u^n \to u$ in $\mathrm{C}([s_1,s_2];H)$, $g^n \rightharpoonup g$ weakly in $\mathrm{L}^2(s_1,s_2;H)$ with $g^n(t) \in \partial\varphi^t(u^n(t))$, and $g(t) \in \partial\varphi^t(u(t))$ for a.e. $t \in [s_1,s_2]$, and $b^n \rightharpoonup b$ weakly in $\mathrm{L}^2(s_1,s_2;H)$ with $b^n(t) \in \mathrm{B}(t,u^n(t))$ for a.e. $t \in [s_1,s_2]$, then $b(t) \in \mathrm{B}(t,u(t))$ holds for a.e. $t \in [s_1,s_2]$.

(A.3) There exist functions $l \in \mathcal{M}$, $C \in \mathrm{L}^1(0,T)$ and a constant $d \in (0,1)$ such that

$$\begin{aligned} |||\mathrm{B}(t,u)|||_H^2 \le d\|\partial^0\varphi^t(u)\|_H^2 + l\left(\varphi^t(u) + \|u\|_H\right)|C(t)| \\ \text{for all } t \in [0,T],\ u \in \mathcal{D}(\partial\varphi^t), \end{aligned} \tag{2.1.8}$$

where $|||\mathrm{B}(t,u)|||_H = \sup\{\|b\|_H; b \in \mathrm{B}(t,u)\}$, and $\partial^0\varphi^t$ denotes the minimal section of $\partial\varphi^t$, i.e., $\partial^0\varphi^t(t)$ is the unique element of least norm in $\partial\varphi^t$.

We can now state the following existence result for (2.1.5).

Theorem 2.1.20. *Let* $(A.\varphi^t)$ *and* $(A.1), (A.2), (A.3)$ *be satisfied. Let* $u_0 \in \mathcal{D}(\varphi^0)$ *and* $f \in \mathrm{L}^2(0,T;H)$. *Then, there exists a positive number* $S \in (0,T]$ *depending on* $\|u_0\|_H$ *and* $\varphi^0(u_0)$ *such that* (2.1.5) *has an s-strong solution* $u(t)$ *in* $[0,S]$.

In order to prove the existence of a global solution for (2.1.5), in addition to $(A.\varphi^t)$, $(A.1)$–$(A.3)$, we need the following assumptions:

(A.4) There exist functions $\ell \in \mathcal{M}$, $C \in \mathrm{L}^1(0,T)$ and a constant $d \in (0,1)$ such that

$$\begin{aligned} |||\mathrm{B}(t,u)|||_H^2 \le d\|\partial^0\varphi^t(u)\|_H^2 + \ell(\|u\|_H)\left(\{\varphi^t(u)\}^2 + |C(t)|\right) \\ \text{for all } t \in [0,T] \quad \text{and} \quad u \in \mathcal{D}(\partial\varphi^t). \end{aligned} \tag{2.1.9}$$

(A.5) There exists a positive constant α and a positive function $D \in \mathrm{L}^1(0,T)$ such that

$$\langle -\partial\varphi^t(u) - \mathrm{B}(t,u), u\rangle_H + \alpha\varphi^t(u) \le D(t)(\|u\|_H^2 + 1)$$

$$\text{for a.e.} \quad t \in [0,T] \quad \text{and} \quad u \in \mathcal{D}(\partial\varphi^t), \tag{2.1.10}$$

where $\langle -\partial\varphi^t(u) - \mathrm{B}(t,u), u\rangle_H = \sup\big\{(-g(t) - b(t), u(t))_H;\ g \in \partial\varphi^t(u), b \in \mathrm{B}(t,u)\big\}$. Now, we state the following global existence result from [60].

Theorem 2.1.21. *Assume that $(A.\varphi^t)$ and $(A.1), (A.2), (A.4)$ and $(A.5)$ are satisfied. Let $f \in \mathrm{L}^2(0,T;H)$. Then every local s-strong solution of* (2.1.5) *can be continued globally as an s-strong solution of* (2.1.5) *in $[0,T]$. In particular, the assertion of Theorem 2.1.20 holds true with $T = S$.*

We have the following regularity result for an s-strong solution of (2.1.5).

Proposition 2.1.22. *Let $(A.\varphi^t)$ be satisfied and $u(t)$ be an H-valued absolutely continuous function on $[0,S]$ with $\varphi^t(u(t))$ absolutely continuous on $[0,S]$. Define $\mathcal{L} = \{t \in [0,S]; du(t)/dt \quad \text{and} \quad d\varphi^t(u(t))/dt \quad \text{exist}, \quad u(t) \in \mathcal{D}(\partial\varphi^t)\}$. Then*

$$\begin{aligned}\left|\frac{d\varphi^t}{dt}(u(t)) - \big\langle g, \frac{du(t)}{dt}\big\rangle_H\right| \quad &\leq \quad m_1(\|u\|_H)\|g\|_H\big(\varphi^t(u(t)) + m_2(\|u(t)\|_H)\big)^\beta \\ &\quad +m_1(\|u(t)\|_H)\big(\varphi^t(u(t)) + m_2(\|u\|_H)\big)\end{aligned}$$

holds for all $t \in \mathcal{L}$ and $g \in \partial\varphi^t(u(t))$.

Remark 2.1.23. *In the application of the results of this section to our problem, we can verify assumption $(A.\varphi^t)$ for $I = [0,T]$, $m_1(\cdot) = m$, a positive constant, $m_2(\cdot) = 1$, and $\beta = 1/2$. Therefore, in the remainder of this section, we state the results for this specific case.*

Lemma 2.1.24. *Let u be an s-strong solution to* (2.1.5)*. If $g(t) \in \partial\varphi^t(u), b(t) \in B(t,u)$ in $[0,S]$ for $0 < S \leq T$, then*

$$\frac{3d(1-d)}{2(1+3d)}\|g(t)\|_H^2 + \frac{d}{dt}\varphi^t(u(t)) \leq M(C_0)\left(\{\varphi^t(u(t))\}^2 + 1 + |C(t)| + \|f(t)\|_H\right)$$
$$\text{for a.e. } t \in (0,S),$$

where d is the number given in $(A.4)$, C_0 is a constant depending on $\|u_0\|_H$ and $\|f(t)\|_{L^2(0,T;H)}$, and M depends on $\ell(C_0)$ in $(A.4)$ and $m_1 = m$ in Proposition 2.1.22.

Proof: As u is an s-strong solution to (2.1.5), by definition 2.1.19, there exist $g(t)$ and $b(t) \in L^2(0,S;H)$ such that $g(t) \in \partial\varphi^t(u(t))$, $b(t) \in \mathrm{B}(t,u(t))$ and

$$\frac{du(t)}{dt} + g(t) + b(t) = f(t), \tag{2.1.11}$$

for a.e. $t \in (0, S)$. Consider Proposition 2.1.22 with $m_1(\cdot) = m$, $m_2(\cdot) = 1$, and $\beta = 1/2$. Multiplying (2.1.11) by $g(t) \in \partial\varphi^t(u(t))$, and then applying the special case of Proposition 2.1.22 along with Young's inequality, we find

$$\begin{aligned} \|g(t)\|_H^2 + \frac{d}{dt}\varphi^t(u(t)) \le \big(\frac{1}{2} + \varepsilon_d\big)\|g\|_H^2 + \frac{1}{2}\|b(t)\|_H^2 \\ + \frac{1}{2\varepsilon_d}\Big(\|f(t)\|_H^2 + m^2\big(\varphi^t(u(t)) + 1\big)\Big) \\ + m\big(\varphi^t(u(t)) + 1\big), \end{aligned} \tag{2.1.12}$$

where $\varepsilon_d = (1-d)/2(1+3d)$. On the other hand, the scalar product of (2.1.11) and $u(t)$ yields

$$\left\langle \frac{du(t)}{dt}, u(t) \right\rangle = \langle -g(t) - b(t), u(t)\rangle + \langle f(t), u(t)\rangle,$$

which is equivalent to

$$\frac{1}{2}\frac{d}{dt}\left(\|u(t)\|_H^2\right) = \langle -g(t) - b(t), u(t)\rangle + \langle f(t), u(t)\rangle. \tag{2.1.13}$$

Adding $\alpha\varphi^t(u(t))$ to equation (2.1.13) and recalling $(A.5)$ and Hölder's inequality, we find

$$\frac{1}{2}\frac{d}{dt}\left(\|u(t)\|_H^2\right) + \alpha\varphi^t(u(t)) \le D(t)\big(\|u(t)\|_H^2 + 1\big) + \|f(t)\|_H\|u(t)\|_H \quad \text{for a.e. } t \in (0, S). \tag{2.1.14}$$

From (2.1.14), we can conclude that there exists a constant C_0, depending on $\|u_0\|_H$ and $\|f(t)\|_{L^2(0,T;H)}$, but not on S, such that

$$\max_{0\le t\le S}\|u(t)\|_H + \int_0^S \varphi^t(u(t))dt \le C_0. \tag{2.1.15}$$

Therefore, considering $(A.4)$, i.e. (2.1.9), (2.1.12), (2.1.14) and (2.1.15), after rearranging the terms, we can conclude that there exists a function $M \in \mathcal{M}$ such that

$$\frac{3d(1-d)}{2(1+3d)}\|g(t)\|_H^2 + \frac{d}{dt}\varphi^t(u(t)) \le M(C_0)\left(\{\varphi^t(u(t))\}^2 + 1 + |C(t)| + \|f(t)\|_H\right) \quad \text{for a.e. } \ t \in (0, S).$$

∎

The following result is conjectured but not proved in [60].

Lemma 2.1.25. *Let u be the s-strong solution to* (2.1.5) *in* $[0, S]$ *for* $0 < S \le T$*. Then there exists a bound independent of u on $sup_{0\le t\le T}\varphi^t(u)$.*

Proof: For $S \in [0,T]$, we have from Lemma 2.1.24

$$\frac{3d(1-d)}{2(1+3d)}\|g(t)\|_H^2 + \frac{d}{dt}\varphi^t(u(t)) \leq M(C_0)\Big(\{\varphi^t(u(t))\}^2 + 1 + |C(t)| + \|f(t)\|_H\Big) \quad \text{for a.e. } t \in (0,S).$$

As $d \in (0,1)$ and $\|g(t)\|_H^2 > 0$, we find

$$\frac{d}{dt}\varphi^t(u(t)) \leq \xi_1(t)\varphi^t(u(t)) + \xi_2(t),$$

with $\xi_1 = M(C_0)\varphi^t(u(t))$ and $\xi_2 = M(C_0)\{1 + |C(t)| + |f(t)|\}$. Therefore, by Gronwall's inequality, we find for all $t \in [0,T]$ that

$$\begin{aligned} \varphi^t(u(t)) &\leq e^{\int_0^t \xi_1(s)ds}\left[\varphi^0(u(0)) + \int_0^t \xi_2(s)ds\right] \\ &= e^{M(C_0)\int_0^t \varphi^t(u(s))ds} \\ &\quad \times \left[\varphi^0(u(0)) + M(C_0)\int_0^t \left(1 + |C(s)| + \|f(s)\|_H^2\right) ds\right]. \end{aligned} \tag{2.1.16}$$

As seen in Lemma 2.1.24, we know that there exists a constant C_0 depending on $\|u_0\|_H$ and $\|f(t)\|_H$, but not on S such that

$$\int_0^S \varphi^t(u(t))dt \leq C_0.$$

Hence, from (2.1.16), we find

$$\varphi^t(u(t)) \leq e^{M(C_0)C_0}\left[\varphi^0(u(0)) + M(C_0)\int_0^t (1 + |C(s)|)\ ds + \|f\|^2_{L^2(0,T;H)}\right] \quad \text{for a.e. } t \in (0,S)$$

Therefore, since $C(t) \in \mathrm{L}^1(0,T)$, we can conclude that

$$\sup_{0\leq t\leq T} \varphi^t(u(t)) \leq C_p,$$

where C_p is a constant independent of S. ∎

Finally, we present the following result for the integral of a functional φ^t satisfying assumptions $(A.\varphi)$ and $(A.1)$ from [37].

Lemma 2.1.26. *Assume that a functional φ^t satisfies assumptions $(A.\varphi)$ and $(A.1)$, for $t \in [0,T]$. Then the functional $\Phi : \mathrm{L}^2(0,T;H) \to \mathbb{R}^+$, given by*

$$\Phi(u(t)) = \int_0^T \varphi^t(u)dt \quad \textit{for} \quad u \in \mathrm{L}^2(0,T;H),$$

is proper, lower semicontinuous and convex on $\mathrm{L}^2(0,T;H)$. In addition, for $u, u^* \in \mathrm{L}^2(0,T;H)$, we have $u^* \in \partial\Phi(u)$ if and only if

$$u^*(t) \in \partial\varphi^t(u(t)) \quad \textit{for a.e.} \quad t \in [0,T].$$

2.2 Rescaling the Model

We rescale our model and formulate it in such a way that the theory from the preceding section can be applied. To restate our model in the region $[0,T]\times[a,b]$ with $a<b$, $a,b\in\mathbb{R}$ and $T\in\mathbb{R}^+$, we have the following system of reaction–diffusion equations for a function $u:[0,T]\times[a,b]\to\mathbb{R}$ with a free boundary $L:[0,T]\to\mathbb{R}$:

$$\begin{cases} u_t = D_1 u_{xx} + f_1(u), \quad (t,x)\in(0,T]\times(a,L(t)), & (2.2.1)\\ u_t = D_2 u_{xx} + f_2(u), \quad (t,x)\in(0,T]\times(L(t),b), & (2.2.2)\end{cases}$$

where $D_i>0$, for $i=1,2$, are the diffusion rates, and $f_1(u)=u(r-qu)$ and $f_2(u)=-mu$ are the reaction terms with r,q,m given positive real numbers.

We have the following boundary and initial conditions for the system

$$\begin{cases} u(t,L(t)^-) = ku(t,L(t)^+), & t\in(0,T], \quad (2.2.3)\\ \Big(D_1u_x + L'(t)u\Big)\Big|_{(t,L(t)^-)} = \Big(D_2u_x + L'(t)u\Big)\Big|_{(t,L(t)^+)}, & t\in(0,T], \quad (2.2.4)\\ u_x(t,a) = u_x(t,b) = 0, & t\in(0,T], \quad (2.2.5)\\ L(0)=l_0, \quad a<L(t)<b, & t\in[0,T], \quad (2.2.6)\\ u(0,x) = u_0(x), & x\in[a,b]. \quad (2.2.7)\end{cases}$$

The moving boundary for the first scenario follows the equation

$$L'(t) = 2\eta_1 D_1 \alpha u\left(t,L(t)^-\right), \qquad t\in[0,T]. \tag{2.2.8}$$

Because of the density matching condition at the moving boundary, a solution of our model is generally multi-valued at $x=L(t)$. The following rescaling eliminates this difficulty. We set

$$w(t,x) = \begin{cases} u(t,x), & (t,x)\in[0,T]\times[a,L(t)],\\ ku(t,x), & (t,x)\in[0,T]\times[L(t),b].\end{cases} \tag{2.2.9}$$

We then rewrite problem (2.2.1)–(2.2.8) as

$$\textbf{\textit{(S)}}\begin{cases} w_t = D_1 w_{xx} + f_1(w), & (t,x)\in(0,T]\times(a,L(t)), \quad (2.2.10)\\ w_t = D_2 w_{xx} + kf_2(\frac{w}{k}), & (t,x)\in(0,T]\times(L(t),b), \quad (2.2.11)\end{cases}$$

with matching and initial conditions

$$\textbf{\textit{(BIC)}}\begin{cases} w(t,L(t)^-) = w(t,L(t)^+), & t\in(0,T], \ (2.2.12)\\ \Big(D_1w_x + L'(t)w\Big)\Big|_{(t,L(t)^-)} = \Big(D_2\frac{w_x}{k} + L'(t)\frac{w}{k}\Big)\Big|_{(t,L(t)^+)}, & t\in(0,T], \ (2.2.13)\\ w_x(t,a) = w_x(t,b) = 0, & t\in(0,T], \ (2.2.14)\\ L(0)=l_0, \quad a<L(t)<b, & t\in[0,T], \ (2.2.15)\\ w(0,x) = w_0(x) := u_0(x), & x\in[a,l_0], \ (2.2.16)\\ w(0,x) = w_0(x) := ku_0(x), & x\in(l_0,b], \ (2.2.17)\end{cases}$$

and moving boundary condition

$$\textbf{\textit{(MC)}} \quad L'(t) = 2\eta_1 D_1 \alpha w\left(t, L(t)^-\right), \quad t \in (0,T). \tag{2.2.18}$$

In this setting, w is indeed a single-valued function. We shall work with the scaled system from hereon.
Our main result in this work is the existence of a local solution for the above free-boundary problem. In general, nonlinear evolution equations are of the form

$$\frac{du(t)}{dt} = A(t, u(t)) + b(t),$$

where A is a possibly nonlinear operator defined from a Banach space V of Bochner type to its dual V^* for which the triple $V \hookrightarrow H \hookrightarrow V^*$ forms a Gelfand triple for an appropriate Hilbert space H. There are many existence and uniqueness results where the operator A enjoys certain monotonicity or pseudo-monotonicity properties [66]. However, the evolution system considered in the present paper does not fall within the standard categories as a possible solution lands on a triple of the form $V \hookrightarrow H \hookrightarrow W$, where W is a Banach space much bigger than V^*. This issue is due to the differential equation defined on the moving boundary. To be more precise, in our problem $V = \mathrm{L}^2(0,T;\mathrm{H}^1(a,b))$ and $W = \mathrm{L}^{4/3}\big(0,T;(\mathrm{H}^1(a,b))^*\big)$, for the Sobolev space $\mathrm{H}^1(a,b)$. This setting resembles the three-dimensional Navier-Stokes equations for which solutions are proved to exist for the setting of the form $\mathrm{L}^2(0,T;\mathrm{H}^1(\Omega)) \hookrightarrow \mathrm{L}^2(0,T;\mathrm{L}^2(\Omega)) \hookrightarrow \mathrm{L}^{4/3}(0,T;(\mathrm{H}^1(\Omega))^*)$, where Ω is a bounded domain in $\mathbb{R}^3$ [8].

2.3 Existence of Solutions: Definitions and Main Results

In this section, we state the local existence theorem for our model. We present a detailed proof of this main result at the end of Section 2.5. In order to show the existence of a local solution for this model, we prove the existence of a curve L and a function w that belong to appropriate functional spaces and satisfy the model in an appropriate sense.

We begin by defining the notion of a solution of our system and the appropriate function spaces. Let $L \in \mathrm{H}^1(0,T)$ with $a < L(t) < b$ for all t. For each $t \in [0,T]$, we equip the real Hilbert space $H = \mathrm{L}^2(a,b)$ with the inner product

$$(z,v)_H = \int_a^{L(t)^-} zv\ dx + \frac{1}{k}\int_{L(t)^+}^b zv\ dx \qquad \text{for all } \ z, v \in H,$$

and the norm

$$\|z\|_H^2 = \|z\|^2_{\mathrm{L}^2(a,L(t))} + \frac{1}{k}\|z\|^2_{\mathrm{L}^2(L(t),b)}, \tag{2.3.1}$$

where $k > 1$ is the parameter defined in (1.6.6), appearing in (2.2.12).

Remark 2.3.1. *The norm introduced in* (2.3.1) *is equivalent to the standard* L^2*-norm,* $\|z\|_{L^2(a,b)} = \left(\int_a^b z^2 dx\right)^{1/2}$*, since*

$$\frac{1}{\sqrt{k}}\|z\|_{\mathrm{L}^2(a,b)} \leq \|z\|_H \leq \|z\|_{\mathrm{L}^2(a,b)}.$$

Hence, we may use these two norms on H *interchangeably.*

When we consider a function $w = w(t,x)$ on $[0,T]\times[a,b]$ as an H-valued function $w : [0,T] \to H$, denoted by $w(t,\cdot)$, we can apply the norm in (2.3.1) to $w(t,\cdot)$ for each $t \in [0,T]$.

Definition 2.3.2. *For each* $L \in \mathrm{H}^1(0,T)$ *with* $L'(t) \geq 0$ *on* $[0,T]$*, we define* X_L *to be the reflexive Banach space*

$$X_L = \left\{ z \in \mathrm{L}^2(0,T;\mathrm{H}^1(a,b)) : \int_0^T L'(t)\Big(z(t,L(t))\Big)^2 dt + \int_0^T \int_a^{L(t)} |z|^3 dxdt \leq \infty \right\},$$

equipped with the norm

$$\|z\|_{X_L} = \|z\|_{\mathrm{L}^2(0,T;\mathrm{H}^1(a,b))} + \left(\int_0^T L'(t)\Big(z(t,L(t))\Big)^2 dt\right)^{\frac{1}{2}} + \left(\int_0^T \int_a^{L(t)} |z|^3 dxdt\right)^{\frac{1}{3}}.$$

We denote the dual space of X_L *with* X_L^* *and the pairing between* X_L *and* X_L^* *by* $\langle\langle\cdot,\cdot\rangle\rangle$ *(or* $\int_0^T \langle\cdot,\cdot\rangle dt$*). In particular, we let*

$$\begin{aligned} \langle\langle z, z^*\rangle\rangle &:= \int_0^T \langle z, z^*\rangle \; dt \\ &:= \int_0^T \int_a^{L(t)} zz^* \; dxdt + \frac{1}{k}\int_{L(t)}^b zz^* \; dxdt. \end{aligned} \tag{2.3.2}$$

It can be shown that $X_L \hookrightarrow \mathrm{L}^2\big((0,T)\times(a,b)\big) \hookrightarrow X_L^*$ *are dense and continuous. If we identify* $\mathrm{L}^2\big((0,T)\times(a,b)\big) \approx \mathrm{L}^2(0,T;H)$*, the scalar product* $\int_0^T (\cdot,\cdot)_H \; dt$ *and the pairing* $\langle\langle\cdot,\cdot\rangle\rangle$ *coincide whenever they both make sense (see Remark 3 [9, p.136]), that is,*

$$\int_0^T \langle z, v\rangle \; dt = \int_0^T (z,v)_H \; dt \qquad \text{for all} \quad z \in X_L \text{ and } v \in \mathrm{L}^2(0,T;H). \tag{2.3.3}$$

Remark 2.3.3. *The spaces X_L and X_L^* depend on the chosen curve L. We use the fact that for a curve $L \in \mathrm{H}^1(0,\mathrm{T})$, by Definition 2.3.2, we have $\mathrm{L}^4(0,T;\mathrm{H}^1(a,b)) \hookrightarrow X_L \hookrightarrow \mathrm{L}^2(0,T;\mathrm{H}^1(a,b))$, and hence $X_L^* \hookrightarrow \mathrm{L}^{4/3}\big(0,T;(\mathrm{H}^1(a,b))^*\big)$. Indeed,*

$$\|z\|_{X_L} \leq C\big(1+\|L\|_{\mathrm{H}^1(0,T)}\big)\|z\|_{\mathrm{L}^4(0,T;\mathrm{H}^1(a,b))}, \qquad \text{for all } z \in X_L,$$

where the constant C is independent of L.

Remark 2.3.4. *For the space X_L with this particular norm, one can prove (see for example Theorem II.5.13 [8, p. 101]) that if $z \in X_L$ with $z_t \in X_L^*$, then $z(t,\cdot) \in \mathrm{C}([0,T];H)$. In addition, we have the following integration by part formula*

$$\begin{aligned}&\big(z(T,\cdot),v(T,\cdot)\big)_H - \big(z(0,\cdot),v(0,\cdot)\big)_H = \\ &\quad\int_0^T \langle v_t, z\rangle\ dt + \int_0^T \langle z_t, v\rangle\ dt + \frac{k-1}{k}\int_0^T L'(t)z(t,L(t))v(t,L(t))\ dt \\ &\qquad\qquad\qquad\qquad \text{for} \quad v \in X_L \quad \text{with} \quad v_t \in X_L^*. \end{aligned} \tag{2.3.4}$$

In particular,

$$\int_0^T \frac{d}{dt}\|z(t,\cdot)\|_H^2 dt = 2\int_0^T \langle z, z_t\rangle\ dt + \frac{k-1}{k}\int_0^T L'(t)\big(z(t,L(t))\big)^2\ dt.$$

Having the appropriate space, we can now define the notion of a solution for our system.

Definition 2.3.5. *The pair $(L,w) \in \mathrm{H}^1(0,T) \times \mathrm{C}\big([0,T];\mathrm{L}^2(a,b)\big)$ is a weak solution of system **(S)**, (2.2.10)–(2.2.11), with conditions **(BIC)**, (2.2.12)–(2.2.16), and **(MC)**, (2.2.18), on [0,T] if*

i) The function L defined on $[0,T]$ satisfies:

i.1) $L(0) = l_0$ and $a < L(t) < b$ for $t \in [0,T]$.

i.2) $L'(t)$ satisfies the moving boundary condition (2.2.18) for a.e $t \in [0,T]$.

ii) For the function $w \in X_L$ defined on $[0,T]\times[a,b]$ with $w_t \in X_L^*$ we have:

*ii.1) w satisfies the system **(S)** in the weak sense, that is*

$$\begin{aligned}&\int_0^T \int_a^{L(t)} w_t\eta\ dx + \frac{1}{k}\int_{L(t)}^b w_t\eta\ dx\ dt + \int_0^T\int_a^{L(t)} D_1 w_x\eta_x\ dx \\ &+ \frac{1}{k}\int_{L(t)}^b D_2 w_x\eta_x\ dx\ dt + \frac{k-1}{k}\int_0^T L'(t)w(t,L(t))\eta(t,L(t))\ dt \\ &= \int_0^T\int_a^{L(t)} f_1(w)\eta\ dx + \frac{1}{k}\int_{L(t)}^b kf_2(\frac{w}{k})\eta\ dx\ dt \\ &\qquad\qquad\qquad\qquad\qquad\qquad \text{for all} \quad \eta \in X_L,\end{aligned}$$

ii.2) $w(0,x)$ *satisfies the initial conditions* (2.2.16) *and* (2.2.17) *for a.e.* $x \in [a,b]$.

We can now state the main result of this section.

Theorem 2.3.6. *Suppose that* $a < l_0 < b$, $w_0 \in \mathrm{H}^1(a,b)$ *with* $w_0 \geq 0$ *for all* $x \in [a,b]$. *Then, there exists a real number* $T > 0$ *such that the system* ***(S)***, (2.2.10)–(2.2.11), *with conditions* ***(BIC)***, (2.2.12)–(2.2.16), *and* ***(MC)***, (2.2.18), *has a weak solution* (L,w) *over* $[0,T]$.

We break the proof of this theorem into two parts: in the first part in Section 2.4, we assume that $T > 0$ and a curve L with appropriate properties are given and we prove the existence of a solution to the moving boundary problem, namely system ***(S)*** with boundary and initial conditions ***(BIC)***. For the second part, in Section 2.5, we prove the existence of a curve L for which the solution w of the first part corresponding to this curve along with the curve L itself satisfy the whole free boundary problem ***(S)*** with conditions ***(BIC)***, and the moving boundary condition ***(MC)***.

2.4 Existence of a Local Solution for a Given Curve as a Moving Boundary

Throughout this Section, we assume that a curve L with appropriate properties is given. We first solve the problem of existence of solutions for $L \in \mathrm{H}^3(0,T)$ (Section 2.4.1). Then we provide regularity estimates (Section 2.4.3) that allow us to extend the result to $L \in \mathrm{H}^1(0,T)$ (Section 2.4.6), using an approximating sequence (Section 2.4.4).

Let $T > 0$ be given and consider a curve L on $[0,T]$ satisfying the following assumptions:

$$L(0) = l_0, \quad L(t) \in (a+\delta, b-\delta), \quad \text{for some } \delta > 0, \quad \text{and } L'(t) \geq 0 \quad \text{for } t \in [0,T]. \tag{2.4.1}$$

We assume that the initial condition satisfies

$$w_0 \in \mathrm{H}^1(a,b), \quad \text{and} \quad w_0(x) \geq 0 \quad \text{for} \quad x \in [a,b]. \tag{2.4.2}$$

The positivity assumption is natural since the function represents a (scaled) biological population. The continuity and regularity assumptions imply, via (2.2.16) and (2.2.17), that the initial function u_0 satisfies the jump condition at the interface, $x = l_0$, and is sufficiently regular.

We define the notion of a strong and a weak solution for our reduced system ***(S)*** and ***(BIC)***.

Definition 2.4.1. *We define strong and weak solutions for system **(S)**,* (2.2.10)–(2.2.11)*, with conditions **(BIC)**,* (2.2.12)–(2.2.16)*, as follows:*

i) *The function $w \in \mathrm{C}\big([0,T];H\big)$ is a strong solution of the system **(S)**,* (2.2.10)–(2.2.11)*, with conditions **(BIC)**,* (2.2.12)–(2.2.16)*, on [0,T] corresponding to a curve $L \in \mathrm{H}^1(0,T)$ satisfying assumption* (2.4.1) *if it satisfies:*

i.1) *equations* (2.2.10) *for a.e. $(t,x) \in (0,T) \times (a, L(t))$, and* (2.2.11) *for a.e. $(t,x) \in (0,T) \times (L(t), b)$.*

i.2) $w(t,\cdot) \in \mathrm{H}^2(a,L(t)) \cap \mathrm{H}^2(L(t),b)$ *for a.e. $t \in [0,T]$, and $w_t \in L^2\big(0,T;H\big)$.*

i.3) $w(t,L(t)^{\pm})$ *and* $w_x(t,L(t)^{\pm})$ *satisfy the interface conditions* (2.2.12) *and* (2.2.13) *for almost every $t \in [0,T]$.*

i.4) $w_x(t,a) = w_x(t,b) = 0$ *for a.e. $t \in [0,T]$.*

i.5) $w(0,x)$ *satisfies the initial conditions* (2.2.16) *and* (2.2.17) *for a.e. $x \in [a,b]$.*

ii) *The function $w \in \mathrm{C}\big([0,T];H\big)$ is a weak solution to system **(S)**,* (2.2.10)–(2.2.11)*, with conditions **(BIC)**,* (2.2.12)–(2.2.16)*, on [0,T] corresponding to a curve $L \in \mathrm{H}^1(0,T)$ satisfying assumption* (2.4.1) *if it satisfies:*

ii.1) *System **(S)** with boundary conditions* (2.2.12) *to* (2.2.15) *in the weak sense, that is*

$$\begin{aligned}
&\int_0^T \int_a^{L(t)} w_t \eta \; dx + \frac{1}{k}\int_{L(t)}^b w_t \eta \; dx \; dt + \int_0^T \int_a^{L(t)} D_1 w_x \eta_x \; dx \\
&+ \frac{1}{k}\int_{L(t)}^b D_2 w_x \eta_x \; dx \; dt + \frac{k-1}{k}\int_0^T L'(t) w(t,L(t))\eta(t,L(t)) \; dt \\
&= \int_0^T \int_a^{L(t)} f_1(w)\eta \; dx + \frac{1}{k}\int_{L(t)}^b k f_2(\frac{w}{k})\eta \; dx \; dt \\
&\qquad\qquad\qquad\qquad\qquad\qquad \textit{for all} \quad \eta \in X_L,
\end{aligned}$$

ii.2) $w(0,x)$ *satisfies the initial conditions* (2.2.16) *and* (2.2.17) *for a.e. $x \in [a,b]$.*

Remark 2.4.2. *A strong solution is also a weak solution. A weak solution w with $w_t \in L^2\big(0,T;H\big)$ is also a strong solution. A weak solution to the moving boundary problem in the sense of Definition 2.4.1 satisfies properties $ii.1$–$ii.2$ of Definition 2.3.5. Hence, the solution for the free boundary problem that we find later is of weak type as well.*

2.4.1 Existence of a Local Solution for a Given Curve $L \in \mathrm{H}^3(0,T)$.

We apply the theory introduced in Section 2.1.5, in particular Theorems 2.1.20 and 2.1.21, to get our first existence result under the higher regularity assumption $L \in \mathrm{H}^3(0,T)$. We begin by rewriting system ***(S)***, (2.2.10)–(2.2.11), as

$$\begin{cases} w_t + w = D_1 w_{xx} + F_1(w), & (t,x) \in (0,T) \times (a, L(t)), \\ w_t + w = D_2 w_{xx} + F_2(w), & (t,x) \in (0,T) \times (L(t), b), \end{cases} \tag{2.4.3}$$

where $F_1(w) = f_1(w) + w$ and $F_2(w) = kf_2(\frac{w}{k}) + w$. We define the time-dependent functional $\varphi^t : H \to [0,\infty]$ for $t \in [0,T]$ by setting

$$\varphi^t(z) = \begin{cases} \dfrac{1}{2}\displaystyle\int_a^{L(t)} \left(D_1 z_x^2 + z^2\right)\, dx + \dfrac{1}{2k}\displaystyle\int_{L(t)}^{b} \left(D_2 z_x^2 + z^2\right)\, dx & \\ + \dfrac{k}{k-1}\dfrac{L'(t)}{2}\left(\dfrac{1}{k} z(L(t)^+) - z(L(t)^-)\right)^2, & \text{if } z \in \mathcal{D}(\varphi^t), \\ \infty & \text{if } z \notin \mathcal{D}(\varphi^t), \end{cases} \tag{2.4.4}$$

with effective domain $\mathcal{D}(\varphi^t) = \left\{z \in H; z \in \mathrm{H}^1(a, L(t)) \cap \mathrm{H}^1(L(t), b), z(L(t)^-) = z(L(t)^+)\right\}$. Equivalently, $\mathcal{D}(\varphi^t) = \mathrm{H}^1(a,b)$; see Section 2.1. We also have $\mathcal{D}(\partial\varphi^t) = \left\{z \in \mathcal{D}(\varphi^t); z \in \mathrm{H}^2(a, L(t)) \cap \mathrm{H}^2(L(t), b)\right\}$. Since $L'(t) \geq 0$ for all $t \in [0,T]$ and $k > 1$, it can be shown that φ^t is proper, convex and lower semicontinuous for all $t \in [0,T]$.

Next, for $t \in [0,T]$ we define the operator B with $\mathcal{D}(\mathrm{B}) = \mathcal{D}(\varphi^t)$ as

$$\mathrm{B}(t,z) := \begin{cases} -F_1(z_+) = -f_1(z_+) - z_+ & \text{if} \quad x \in (a, L(t)), \\ -F_2(z_+) = -kf_2(\dfrac{z_+}{k}) - z_+ & \text{if} \quad x \in (L(t), b), \end{cases} \tag{2.4.5}$$

where $z_+(x) = \max\{z(x), 0\}$.

In this setting, system ***(S)***, (2.2.10)–(2.2.11), with boundary and initial conditions ***(BIC)***, (2.2.12)–(2.2.17), is equivalent to the following abstract Cauchy problem generated by the subdifferential of the functional φ^t,

$$\begin{cases} w_t(t,\cdot) + \partial\varphi^t(w(t,\cdot)) + \mathrm{B}(t, w(t,\cdot)) \ni 0, & 0 < t < T, \\ w(0,\cdot) = w_0(\cdot). \end{cases} \tag{2.4.6}$$

Remark 2.4.3. *An s-strong solution of system* (2.4.6) *in the sense of Definition 2.1.19 is equivalent to a strong solution of system **(S)** with condition **(BIC)**, satisfying properties* $i.1)$ *to* $i.5)$ *in Definition 2.4.1. We provide a detailed proof in Proposition 2.4.5.*

Remark 2.4.4. *Since any $z \in \mathcal{D}\varphi^t(z)$ satisfies $z \in C[a,b]$ and $z(L(t)^-) = z(L(t)^+) = z(L(t))$, we can write $\varphi^t(z)$ equivalently as*

$$\varphi^t(z) = \begin{cases} \dfrac{1}{2}\displaystyle\int_a^{L(t)} \left(D_1 z_x^2 + z^2\right)\,dx + \dfrac{1}{2k}\displaystyle\int_{L(t)}^b \left(D_2 z_x^2 + z^2\right)\,dx & \\ \quad + \dfrac{k-1}{k}\dfrac{L'(t)}{2}\Big(z(L(t))\Big)^2, & \text{if } z \in \mathrm{H}^1(a,b), \\ \infty & \text{if } z \notin \mathrm{H}^1(a,b). \end{cases} \tag{2.4.7}$$

We shall use this version in the following. However, the equivalence of the s-strong solution of the abstract Cauchy problem (2.4.6) *and the solution of system **(S)** and conditions **(BIC)** is easier to see from* (2.4.4).

Proposition 2.4.5. *Let $T > 0$ and consider a given curve $L \in \mathrm{H}^3(0,T)$, satisfying (2.4.1), and initial condition $w_0 \in \mathrm{H}^1(a,b)$, satisfying (2.4.2). Then there exists a non-negative strong solution for **(S)**,* (2.2.10)–(2.2.11)*, with boundary and initial conditions **(BIC)**,* (2.2.12)–(2.2.17)*, satisfying properties $i.1$) to $i.5$) in Definition 2.4.1.*

We first verify assumptions $(A.\varphi^t)$, $(A.1)$–$(A.5)$ of Theorems 2.1.20 and 2.1.21 in the next two lemmas.

Lemma 2.4.6. *The functional φ^t defined in* (2.4.4) *satisfies assumptions $(A.\varphi^t)$ and $(A.1)$ for $t \in [0,T]$ and $w(t,\cdot) \in \mathcal{D}(\varphi^t)$.*

Proof:

Verifying $(A.\varphi^t)$**:** From the construction in (2.4.4), $\varphi^t(w) : H \to [0,+\infty]$ is proper, lower semicontinuous and convex for each $t \in [0,T]$. Now, consider $s \in [0,T]$ and $w^s(\cdot) \in \mathcal{D}(\varphi^s)$, that is $w^s(\cdot) \in \mathrm{H}^1(a,b)$ and $w^s(L(s)^-) = w^s(L(s)^+)$. We can extend $w^s(\cdot) : [a,b] \to \mathbb{R}$ to $w^s(\cdot) : \mathbb{R} \to \mathbb{R}$ by setting

$$w^s(x) := \begin{cases} w^s(a), & x \le a, \\ w^s(x), & a < x < b, \\ w^s(b), & b \le x. \end{cases}$$

For any $\delta_0 > T$, we have $I(s) := [\max(0, s-\delta_0), \min(s+\delta_0, T)] = [0,T]$.

For $t \in [0,T]$, we can define the linear map

$$\mathcal{T}(x) := \mathcal{T}(t,x) = \mu(x-a) + a, \qquad \text{where} \quad \mu = \mu(t) = \frac{L(s)-a}{L(t)-a}.$$

Then $\mathcal{T}(L(t)) = L(s)$ and $\mathcal{T}(a) = a$. For $t \in [0,T]$, we now introduce

$$w(t,x) = w^s(\mathcal{T}(x)) \quad \text{for} \quad x \in [a,b].$$

As $w^s(\cdot) \in \mathcal{D}(\varphi^s)$, by construction, we also have $w(t,\cdot) \in \mathcal{D}(\varphi^t)$, since we have

$$\begin{aligned} w(t, L(t)^-) &= w^s(\mathcal{T}(L(t)^-)) = w^s(L(s)^-) \\ &= w^s(L(s)^+) = w^s(\mathcal{T}(L(t)^+)) = w(t, L(t)^+). \end{aligned}$$

We now calculate

$$\begin{aligned} \left\|w(t,\cdot) - w^s(\cdot)\right\|_H^2 &= \int_a^b \Big(w(t,x) - w^s(x)\Big)^2 dx \\ &= \int_a^b \Big(w^s(\mathcal{T}(x)) - w^s(x)\Big)^2 dx \\ &= \int_a^b \left(\int_x^{\mathcal{T}(x)} w_y^s(y)dy\right)^2 dx \\ &\le \int_a^b \left(\int_x^{\mathcal{T}(x)} \big(w_y^s(y)\big)^2 dy\right) |\mathcal{T}(x) - x|\, dx, \end{aligned}$$

where we applied Hölder's inequality. We can now change the order of integration. To do this, we note

$$|\mathcal{T}(x) - x| = |\mu - 1|(x - a) \quad \text{ with } \quad \mu - 1 = \frac{L(s) - L(t)}{L(t) - a}.$$

First, assume that $L(s) \ge L(t)$. Then we have $\mathcal{T}(x) = (\mu-1)(x-a) + x \ge x$, and so $\mathcal{T}(b) \ge b$. Therefore, we obtain

$$\begin{aligned} \left\|w(t,\cdot) - w^s(\cdot)\right\|_H^2 \le\ & \int_a^b \int_{\mathcal{T}^{-1}(y)}^y \big(w_y^s(y)\big)^2 |\mathcal{T}(x) - x|\ dxdy \\ &+ \int_b^{\mathcal{T}(b)} \int_{\mathcal{T}^{-1}(y)}^b \big(w_y^s(y)\big)^2 |\mathcal{T}(x) - x|\ dxdy, \end{aligned}$$

where $\mathcal{T}^{-1}(y) = a + \frac{y-a}{\mu}$. Since $w^s(x) = w^s(b)$, we have $w_x^s(x) = 0$ for $b \le x \le \mathcal{T}(b)$, and we can simplify further to find

$$\begin{aligned} \left\|w(t,\cdot) - w^s(\cdot)\right\|_H^2 &\le \int_a^b \big(w_y^s(y)\big)^2 |\mu - 1| \int_{\mathcal{T}^{-1}(y)}^y (x-a)\ dxdy \\ &\quad + \int_b^{\mathcal{T}(b)} \big(w_y^s(b)\big)^2 |\mu - 1| \int_{\mathcal{T}^{-1}(y)}^b (x-a)\ dxdy \\ &= \int_a^b \big(w_y^s(y)\big)^2 |\mu - 1|\ \frac{(x-a)^2}{2}\bigg|_{\mathcal{T}^{-1}(y)}^y dy + 0 \\ &= |\mu - 1| \left(\int_a^b \big(w_y^s(y)\big)^2 \left[\frac{(y-a)^2}{2} - \frac{(y-a)^2}{2\mu^2}\right] dy\right) \end{aligned}$$

$$\begin{aligned}
&\leq \frac{(\mu-1)^2(\mu+1)}{2\mu^2}(b-a)^2\int_a^b \big(w_y^s(y)\big)^2 dy\\
&\leq C_1\frac{(\mu-1)^2(\mu+1)}{2\mu^2}(b-a)^2\varphi^s(w^s(.)),
\end{aligned}$$

for an appropriate constant $C_1 = C_1(k)$. As $\mu - 1 = \frac{L(s)-L(t)}{L(t)-a}$, we have

$$\begin{aligned}
&\big\|w(t,\cdot) - w^s(\cdot)\big\|_H^2\\
\leq\ & C_1\left(\frac{L(s)-L(t)}{L(t)-a}\right)^2\left|\frac{(L(s)+L(t)-2a)(L(t)-a)}{2(L(s)-a)^2}\right|(b-a)^2\varphi^s(w^s(.))\\
\leq\ & C_1\|L'\|^2_{\mathrm{L}^\infty(0,T)}(t-s)^2\left(\frac{1}{L(t)-a}\right)^2\left(\frac{1}{L(s)-a}\right)^2(b-a)^4\varphi^s(w^s(.)).
\end{aligned}$$

Using the assumption that $L(t) \in (a+\delta, b-\delta)$ for all $t \in [0,T]$ and some $\delta > 0$, we get $\left|\frac{1}{L(t)-a}\right| \leq \frac{1}{\delta}$. In addition, $L \in \mathrm{H}^3(0,T)$ implies that $\|L'\|_{\mathrm{L}^\infty(0,T)}$ is bounded by some $C_{L'} > 0$. Hence, we obtain

$$\begin{aligned}
\big\|w(t,\cdot) - w^s(\cdot)\big\|_H^2 &\leq C_1\Big(\frac{C_{L'}}{\delta^2}\Big)^2(t-s)^2(b-a)^4\varphi^s(w^s(\cdot))\\
&\leq C_3^2(t-s)^2\varphi^s(w^s(\cdot))
\end{aligned}$$

for an appropriate constant $C_3 = C_3(\|L'\|_{\mathrm{L}^\infty(0,T)}, k, b-a, \frac{1}{\delta})$. Hence, we have

$$\big\|w(t,\cdot) - w^s(\cdot)\big\|_H \leq C_3|t-s|\big(\varphi^s(w^s(\cdot))\big)^{1/2},$$

which is (2.1.6) in $(A.\varphi^t)$, with $m_1 = C_3, m_2 = 0$ and $\beta = \frac{1}{2}$.

Next, assume that $L(s) \leq L(t)$. Thus, $\mathcal{T}(x) = (\mu-1)(x-a) + x \leq x$, and so $\mathcal{T}(b) \leq b$. Following the same steps as the previous case, we can verify that (2.1.6) in $(A.\varphi^t)$ holds with $m_1 = C_4$ for some constant $C_4 = C_4(\|L'\|_{\mathrm{L}^\infty(0,T)}, k, b-a, \frac{1}{\delta})$, $m_2 = 0$, and $\beta = \frac{1}{2}$.

To verify (2.1.7), we apply the change of variable $y = \mathcal{T}(x) = \mu(x-a)+a$ to obtain

$$w(t,x) = w^s(\mathcal{T}(x)) = w(s,y), \quad \text{and} \quad w_x(t,x) = \frac{d}{dx}\big(w^s(\mu(x-a)+a)\big) = \mu w_y^s(y).$$

We note that $\mathcal{T}(L(t)^\pm) = L(s)^\pm$ and $\mathcal{T}(a) = a$. Therefore, from definition 2.4.7, we find

$$\begin{aligned}
&\varphi^t(w(t,\cdot)) - \varphi^s(w^s(\cdot))\\
=\ & \frac{D_1}{2}\bigg(\int_a^{L(s)}\big(\mu w_y^s(y)\big)^2\,\frac{dy}{\mu} - \int_a^{L(s)}\big(w_x^s(x)\big)^2\,dx\bigg)
\end{aligned}$$

$$
\begin{aligned}
&+\frac{D_2}{2k}\Big(\int_{L(s)}^{\mathcal{T}(b)} \big(\mu w_y^s(y)\big)^2 \, \frac{dy}{\mu} - \int_{L(s)}^{b} \big(w_x^s(x)\big)^2 \, dx\Big) \\
&+\frac{1}{2}\Big(\int_a^{L(s)} \big(w^s(y)\big)^2 \frac{dy}{\mu} - \int_a^{L(s)} \big(w^s(x)\big)^2 dx\Big) \\
&+\frac{1}{2k}\Big(\int_{L(s)}^{\mathcal{T}(b)} \big(w^s(y)\big)^2 \frac{dy}{\mu} - \frac{1}{k}\int_{L(s)}^{b} \big(w^s(x)\big)^2 dx\Big) \\
&+\frac{k-1}{2k}\Bigg(L'(t)\Big(w^s\big(\mathcal{T}\big(L(t)\big)\big)\Big)^2 - L'(s)\Big(w^s\big(L(s)\big)\Big)^2\Bigg). \qquad (2.4.8)
\end{aligned}
$$

First, we consider $\mathcal{T}(b) \geq b$. Since $w^s(x) = w^s(b)$ is constant for $b < x < \mathcal{T}(b)$, we can simplify (2.4.8) to

$$
\begin{aligned}
\varphi^t(w(t,\cdot)) - \varphi^s(w^s(\cdot)) \;=\; & (\mu-1)\Big(\frac{D_1}{2}\int_a^{L(s)} \big(w_x^s(x)\big)^2 dx + \frac{D_2}{2k}\int_{L(s)}^{b} \big(w_x^s(x)\big)^2 dx\Big) \\
&+\frac{1}{2}(\frac{1}{\mu}-1)\Big(\int_a^{L(s)} \big(w^s(x)\big)^2 dx + \frac{1}{k}\int_{L(s)}^{b} \big(w^s(x)\big)^2 dx\Big) \\
&+\frac{1}{2\mu k}\big(w^s(b)\big)^2 \left(\mathcal{T}(b) - b\right) \\
&+\frac{k-1}{2k}\big(L'(t) - L'(s)\big)\Big(w^s(L(s))\Big)^2.
\end{aligned}
$$

Since $\mu = \frac{L(s)-a}{L(t)-a}$ and $\mathcal{T}(b) - b = \frac{L(s)-L(t)}{L(t)-a}(b-a)$, we obtain

$$
\begin{aligned}
&\varphi^t(w(t,\cdot)) - \varphi^s(w^s(\cdot)) \\
&\leq \left|\frac{L(s)-L(t)}{L(t)-a}\right| \Big(\frac{D_1}{2}\int_a^{L(s)} \big(w_x^s(x)\big)^2 dx + \frac{D_2}{2k}\int_{L(s)}^{b} \big(w_x^s(x)\big)^2 dx\Big) \\
&\quad+\left|\frac{L(t)-L(s)}{L(s)-a}\right| \Big(\frac{1}{2}\int_a^{L(s)} \big(w^s(x)\big)^2 dx + \frac{1}{2k}\int_{L(s)}^{b} \big(w^s(x)\big)^2 dx\Big) \\
&\quad+\frac{1}{2k}\left|\frac{L(s)-L(t)}{L(s)-a}\right| (b-a)\big(w^s(b)\big)^2 \\
&\quad+\frac{k-1}{2k}\left|L'(t) - L'(s)\right| \Big(w^s(L(s))\Big)^2.
\end{aligned}
$$

For $L \in \mathrm{H}^3(0,T)$, we have some constant $C_{L''}$, such that $\|L'\|_{\mathrm{L}^\infty(0,T)}, \|L''\|_{\mathrm{L}^\infty(0,T)} \leq C_{L''}$. Therefore, since $a+\delta \leq L(t) \leq b-\delta$ for all $t \in [0,T]$ and some $\delta > 0$, we can use the definition of $\varphi^t(w^s(.))$ to obtain

$$
\varphi^t(w(t,\cdot)) - \varphi^s(w^s(\cdot)) \;\leq\; C_5\Big(\frac{C_{L''}}{\delta}\Big)|t-s| \left(\varphi^s(w^s(.)) + \frac{1}{2k}(b-a)|w^s(b)|^2 \right),
$$

for some constant C_5. Since the constant of the embedding $\mathrm{H}^1(a,b) \hookrightarrow \mathrm{C}([a,b])$ depends only on $(b-a)$ (see Theorem 6.3 [1, p.168]), we can also bound $w^s(b)$ by

$\varphi^s(w^s(\cdot))$. Hence, for an appropriate constant $C_6 = C_6(\|L'\|_{\mathrm{L}^\infty(0,T)}, \|L''\|_{\mathrm{L}^\infty(0,T)}, k, (b-a), \frac{1}{\delta})$, we have

$$\varphi^t(w(t,\cdot)) \quad \leq \quad \varphi^s(w^s(\cdot)) + C_6|t-s|\varphi^s(w^s(\cdot)),$$

which is (2.1.7) in $(A.\varphi^t)$, with $m_1 = C_6$ and $m_2 = 0$.

For $\mathcal{T}(b) \leq b$, we can verify (2.1.7) in $(A.\varphi^t)$ by similar arguments, with $m_1 = C_7 = C_7(\|L'\|_{\mathrm{L}^\infty(0,T)}, \|L''\|_{\mathrm{L}^\infty(0,T)}, k, (b-a), \frac{1}{\delta})$ and $m_2 = 0$. Hence, condition $(A.\varphi^t)$ is satisfied with $m_1 = m_1\big(\|L'\|_{\mathrm{L}^\infty(0,T)}, \|L''\|_{\mathrm{L}^\infty(0,T)}, \frac{1}{\delta}, (b-a)\big)$ and $m_2 = 0$.

Verifying $(A.1)$**:** The compact embedding of $\mathrm{H}^1(a,b) \hookrightarrow \mathrm{L}^2(a,b)$ implies the compactness of the set $\{w \in H; \varphi^t(w) + \|w\|_H^2 \leq P\}$ in H, for all $t \in [0,T]$ and $P \in (0,\infty)$. Hence, $(A.1)$ is verified. ∎

Lemma 2.4.7. *For all $t \in [0,T]$ and $w \in \mathcal{D}(\mathrm{B})$, the nonlinear operator $\mathrm{B}(t,w)$ defined in (2.4.5) satisfies assumptions $(A.2)$ and $(A.3)$ of Theorem 2.1.20, and $(A.4)$ and $(A.5)$ of Theorem 2.1.21.*

Proof:

Verifying $(A.2)$**:** As $\mathrm{B}(t,w)$ is a single-valued function in H for all $t \in [0,T]$, we have $i), ii)$ satisfied.

To see $iii)$, consider a sequence $\{w^n\}$ such that $w^n \to w$ in $\mathrm{C}([0,T];H)$, and the corresponding sequence $b^n(t) := \mathrm{B}(t,w^n)$ for all $t \in (0,T)$. Assume that $b^n \rightharpoonup b$ weakly in $\mathrm{L}^2(0,T;H)$, which by considering the equivalent norm $\|\cdot\|_{\mathrm{L}^2(a,b)}$ in H implies,

$$\lim_{n\to\infty} \int_0^T \int_a^b b^n \eta \; dxdt = \int_0^T \int_a^b b\eta \; dxdt \qquad \text{for all } \eta \in \mathrm{L}^2(0,T;H). \tag{2.4.9}$$

We need to show that $\mathrm{B}(t,w) = b(t)$ for a.e. $t \in [0,T]$. To do so, we recall the equation of $\mathrm{B}(t,w)$ in (2.4.5), and for all $\bar{\eta} \in \mathrm{L}^\infty\big((0,T)\times(a,b)\big)$ we calculate

$$\begin{aligned}&\int_0^T \int_a^b b^n \bar{\eta} \; dxdt \\ &\quad = \int_0^T \int_a^{L(t)} \big((r+1)w_+^n - q(w_+^n)^2\big)\bar{\eta} \; dxdt + \int_0^T \int_{L(t)}^b (1-m)w_+^n \bar{\eta} \; dxdt.\end{aligned} \tag{2.4.10}$$

Fix $\bar{\eta} \in \mathrm{L}^\infty\big((0,T)\times(a,b)\big)$. Considering the terms on the right-hand side of (2.4.10) separately, we first note that having $w^n \to w$ strongly in $\mathrm{C}(0,T;H)$ implies

$$\lim_{n\to\infty} \int_0^T \int_a^{L(t)} \big((r+1)w_+^n\big)\bar{\eta} \; dxdt = \int_0^T \int_a^{L(t)} \big((r+1)w_+\big)\bar{\eta} \; dxdt,$$

$$\lim_{n\to\infty}\int_0^T\int_{L(t)}^b (1-m)w_+^n\bar{\eta}\;dxdt = \int_0^T\int_{L(t)}^b (1-m)w_+\bar{\eta}\;dxdt. \tag{2.4.11}$$

Now, to calculate the limit of the remaining integral in (2.4.10), we rewrite it as follows:

$$\lim_{n\to\infty}\int_0^T\int_a^{L(t)} \big(-q(w_+^n)^2\big)\bar{\eta}\;dxdt = \lim_{n\to\infty}\int_0^T\int_a^{L(t)} -q\big(w_+^n\big)\big(w_+^n\;\bar{\eta}\big)\;dxdt. \tag{2.4.12}$$

Note that as $w^n \to w$ strongly in $\mathrm{C}(0,T;H)$, we have $w^n \to w$ strongly in $\mathrm{L}^2(0,T;H)$ as well. On the other hand, we have $w_+^n\;\bar{\eta} \rightharpoonup w_+\;\bar{\eta}$ weakly in $\mathrm{L}^2(0,T;H)$. Combining the two, we obtain

$$\lim_{n\to\infty}\int_0^T\int_a^{L(t)} \big(-q(w_+^n)^2\big)\bar{\eta}\;dxdt = \int_0^T\int_a^{L(t)} \big(-qw_+^2\big)\bar{\eta}\;dxdt. \tag{2.4.13}$$

Then from (2.4.11) and (2.4.13), we can conclude that

$$\lim_{n\to\infty}\int_0^T\int_a^b b^n\bar{\eta}\;dxdt = \int_0^T\int_a^b \mathrm{B}(t,w)\bar{\eta}\;dxdt. \tag{2.4.14}$$

Therefore, from (2.4.14) and (2.4.9) with $\bar{\eta}$, we can assert that

$$\int_0^T\int_a^b \big(b-\mathrm{B}(t,w)\big)\bar{\eta}\;dxdt = 0 \qquad \text{for all } \bar{\eta}\in \mathrm{L}^\infty\big((0,T)\times(a,b)\big). \tag{2.4.15}$$

Since (2.4.15) is true for all $\bar{\eta} \in \mathrm{L}^\infty\big((0,T)\times(a,b)\big)$, and since $\big(b-\mathrm{B}(t,w)\big) \in \mathrm{L}^1\big((0,T)\times(a,b)\big)$, we can conclude that

$$b(t) = \mathrm{B}(t, w(t,\cdot)) \qquad \text{for a.e. } t\in[0,T],$$

as desired.

Verifying $(A.3)$**:** For all $t\in[0,T]$ and $w(t,\cdot)\in\mathcal{D}(\partial\varphi^t)$, we can calculate

$$\begin{aligned}\|\mathrm{B}(t,w)\|_H^2 &= \int_a^{L(t)} \big((r+1)w_+ - q(w_+)^2\big)^2\,dx + \frac{1}{k}\int_{L(t)}^b ((1-m)w_+)^2\,dx \\ &\le \Big(2(r+1)^2 + \frac{1}{k}(1-m)^2\Big)\int_a^b w^2\;dx + 2q^2\int_a^b w^4\;dx. \end{aligned}\tag{2.4.16}$$

We know that the constant of the compact embedding of $\mathrm{H}^1(a,b) \hookrightarrow \mathrm{L}^p(a,b)$ depends only on $(b-a)$ and p (see Theorem 6.3 [1, p.168]). Hence, applying this embedding along with the definition of $\varphi^t(w)$, we find

$$\|\mathrm{B}(t,w)\|_H^2 \;\le\; \Big(2(r+1)^2 + \frac{(1-m)^2}{k}\Big)C_1\|w\|_{\mathrm{H}^1(a,b)}^2 + 2q^2C_2\|w\|_{\mathrm{H}^1(a,b)}^4$$

$$
\begin{aligned}
&\leq C\big(\|w\|^2_{\mathrm{H}^1(a,b)} + \|w\|^4_{\mathrm{H}^1(a,b)}\big)\\
&\leq C\ell\left(\varphi^t(w)\right),
\end{aligned}
$$

for an appropriate constant $C = C\big(r, m, q, k, (b-a)\big)$ and the function $\ell(x) = x + x^2$. Hence, $\mathrm{B}(t, w)$ satisfies assumption $(A.3)$, equation (2.1.8), for any $d \in (0, 1)$.

Verifying $(A.4)$**:** Recalling $\mathrm{B}(t, w)$ defined by (2.4.5) for all $t \in [0, T]$ and $w(t, \cdot) \in \partial\varphi^t(w)$, and applying Hölder's inequality, we find

$$
\begin{aligned}
&\|\mathrm{B}(t,w)\|^2_H\\
\leq\;& 2(r+1)^2 \int_a^{L(t)} (w_+)^2\ dx + 2q^2 \int_a^{L(t)} (w_+)(w_+)^3\ dx + \frac{1}{k}(1-m)^2 \int_{L(t)}^b (w_+)^2\ dx\\
\leq\;& \Big(2(r+1)^2 + \frac{1}{k}(1-m)^2\Big)\|w\|^2_{\mathrm{L}^2(a,b)} + 2q^2\|w\|_{\mathrm{L}^2(a,b)} \left(\int_a^b w^6 dx\right)^{\frac{1}{2}}\\
\leq\;& \|w\|_H\bigg(\Big(2k(r+1)^2 + (1-m)^2\Big)\big(1 + \|w\|^4_{\mathrm{L}^2(a,b)}\big) + 2kq^2\big(1 + \|w\|^4_{\mathrm{L}^6(a,b)}\big)\bigg).
\end{aligned}
$$

As before, from the embedding $\mathrm{H}^1(a,b) \hookrightarrow \mathrm{L}^p(a,b)$ we get

$$
\begin{aligned}
\|\mathrm{B}(t,w)\|^2_H \;&\leq\; \|w\|_H\Bigg(\Big(2k(r+1)^2 + (1-m)^2\Big)\big(1 + C_1\|w\|^4_{\mathrm{H}^1(a,b)}\big)\\
&\qquad\qquad + 2kq^2\big(1 + C_2\|w\|^4_{\mathrm{H}^1(a,b)}\big)\Bigg)\\
&\leq\; \|w\|_H \left(C_3 + C_4\{\varphi^t(w)\}^2\right) \qquad (2.4.17)
\end{aligned}
$$

for appropriate constants $C_3 = C_3(r, m, k, q)$ and $C_4 = C_4\big(r, m, k, q, (b-a)\big)$. Therefore, we can further simplify (2.4.17) to find

$$
\|\mathrm{B}(t,w)\|^2_H \;\leq\; \ell(\|w\|_H)\left(c + \{\varphi^t(w)\}^2\right),
$$

where $\ell(x) = C_4 x$ and $c = \frac{C_3}{C_4}$. Hence, for any $d \in (0, 1)$, we have verified that B satisfies $(A.4)$, equation (2.1.9).

Verifying $(A.5)$**:** Let α be a positive constant. We would like to find a value for α such that $(A.5)$, (2.1.10), holds. To do that, for any $w \in \mathcal{D}(\partial\varphi^t)$, we calculate

$$
\begin{aligned}
&\big\langle -\partial\varphi^t(w) - \mathrm{B}(t,w), w\big\rangle_H + \alpha\varphi^t(w)\\
=\;& \big(\frac{\alpha}{2} - 1\big)\int_a^{L(t)} D_1 w_x^2 + w^2\ dx + \frac{1}{k}\big(\frac{\alpha}{2} - 1\big)\int_{L(t)}^b D_2 w_x^2 + w^2\ dx\\
&+ \big(\frac{\alpha}{2} - 1\big)\frac{k-1}{k} L'(t)\big(w(t, L(t))\big)^2
\end{aligned}
$$

$$+(r+1)\int_a^{L(t)} w_+^2 dx + \frac{1}{k}(1-m)\int_{L(t)}^b w_+^2 dx - \int_a^{L(t)} q w_+^3 dx.$$

Now, choosing $0<\alpha<2$, we have $\frac{\alpha}{2}-1<0$, and since $w_+\geq 0$, we obtain

$$\begin{aligned}\langle -\partial\varphi^t(w) - \mathrm{B}(t,w), w\rangle_H + \alpha\varphi^t(w) &\leq \big(|r+1|+|1-m|\big)\|w\|_H^2 \\ &\quad \text{for all} \quad t\in[0,T] \quad \text{all} \quad w(t,\cdot)\in\mathcal{D}(\partial\varphi^t),\end{aligned}$$

which implies that $\mathrm{B}(t,w(t))$ satisfies $(A.5)$, equation (2.1.10), for all $t\in[0,T]$ and $w(t,\cdot)\in\mathcal{D}(\partial\varphi^t)$ with $D=\big(|r+1|+|1-m|\big)$.

∎

Before stating and proving the existence result, we state two lemmas regarding the properties of the solution.

Lemma 2.4.8. *Let $T>0$. Consider a function w_0 satisfying assumption* (2.4.2) *and a curve $L\in\mathrm{H}^3(0,T)$ satisfying assumption* (2.4.1)*. Assume that w is a strong solution to system **(S)**,* (2.2.10)–(2.2.11)*, with boundary and initial condition **(BIC)**,* (2.2.12)–(2.2.16)*. Then w satisfies the following inequality*

$$\frac{1}{2}\frac{d}{dt}\|w(t,\cdot)\|_H^2 + 2\varphi^t(w) - \frac{k-1}{2k}L'(t)\big(w(t,L(t))\big)^2 + q\int_a^{L(t)}\big(w(t,x)\big)^3 dx \leq C_s\|w(t,\cdot)\|_H^2$$

for a.e. $t\in[0,T]$, where $C_s=C_s(r,m)$ is a positive constant that depends on r,m.

Proof: As w solves system ***(S)*** with boundary and initial conditions ***(BIC)***, it satisfies

$$\begin{cases} w_t + w = D_1 w_{xx} + f_1(w) + w & \text{for} \quad (t,x)\in(0,T)\times(a,L(t)),\\ w_t + w = D_2 w_{xx} + k f_2(\dfrac{w}{k}) + w & \text{for} \quad (t,x)\in(0,T)\times(L(t),b).\end{cases}$$

Multiplying each equation with w and integrating, we obtain:

$$\begin{aligned}\int_a^{L(t)} w_t w\ dx + \int_a^{L(t)}\big(-D_1 w_{xx}+w\big)w\ dx &= \int_a^{L(t)}(r+1)w^2 - qw^3\ dx,\\ \int_{L(t)}^b w_t w\ dx + \int_{L(t)}^b\big(-D_2 w_{xx}+w\big)w\ dx &= \int_{L(t)}^b (1-m)w^2\ dx\end{aligned} \tag{2.4.18}$$

for a.e. $t\in[0,T]$. Dividing the second equation by k, integrating by parts and using the boundary conditions in (2.2.14), we can write

$$\begin{aligned}
&\frac{1}{2}\int_a^{L(t)} \frac{d(w^2)}{dt}\,dx - D_1 w_x(t,L(t)^-)w(t,L(t)^-) + \int_a^{L(t)} \left(D_1 w_x^2 + w^2\right)\,dx \\
&= (r+1)\|w(t,\cdot)\|^2_{\mathrm{L}^2(a,L(t))} - q\int_a^{L(t)} w^3 dx, \\
&\frac{1}{2k}\int_{L(t)}^{b} \frac{d(w^2)}{dt}\,dx + \frac{D_2}{k}w_x(t,L(t)^+)w(t,L(t)^+) + \frac{1}{k}\int_{L(t)}^{b} \left(D_2 w_x^2 + w^2\right)\,dx \\
&= \frac{(1-m)}{k}\|w(t,\cdot)\|^2_{\mathrm{L}^2(L(t),b)}
\end{aligned} \tag{2.4.19}$$

for a.e. $t \in [0,T]$. As w is the solution, by definition of the norm in H, we can calculate

$$\begin{aligned}
&\frac{d}{dt}\left(\|w(t,\cdot)\|^2_H\right) = \\
&L'(t)\big(w(t,L(t)^-)\big)^2 + \int_a^{L(t)} \frac{d(w^2)}{dt}\,dx - \frac{L'(t)}{k}\big(w(t,L(t)^+)\big)^2 + \frac{1}{k}\int_{L(t)}^{b} \frac{d(w^2)}{dt}\,dx.
\end{aligned} \tag{2.4.20}$$

Then, noting that $w(t,L(t)^-) = w(t,L(t)^+)$ by (2.2.12), we add the two equations in (2.4.19) and simplify to obtain for a.e. $t \in [0,T]$:

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|w(t,\cdot)\|^2_H - \frac{k-1}{2k}L'(t)\big(w(t,L(t))\big)^2 \\
&\qquad + \Big(\frac{D_2}{k}w_x(t,L(t)^+) - D_1 w_x(t,L(t)^-)\Big)w(t,L(t)) \\
&\qquad + \int_a^{L(t)} \left(D_1 w_x^2 + w^2\right)\,dx + \frac{1}{k}\int_{L(t)}^{b} \left(D_2 w_x^2 + w^2\right)\,dx \\
&= (r+1)\|w(t,\cdot)\|^2_{\mathrm{L}^2(a,L(t))} - q\int_a^{L(t)} w^3 dx + \frac{(1-m)}{k}\|w(t,\cdot)\|^2_{\mathrm{L}^2(L(t),b)}.
\end{aligned} \tag{2.4.21}$$

Since w solves system ***(S)*** and conditions ***(BIC)***, it satisfies conditions (2.2.12) and (2.2.13), from which we have

$$\frac{D_2}{k}w_x(t,L(t)^+) - D_1 w_x(t,L(t)^-) = \frac{k-1}{k}L'(t)w(t,L(t)).$$

So, we can simplify (2.4.21) to find for a.e. $t \in [0,T]$:

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|w(t,\cdot)\|_H^2 &+ \frac{k-1}{2k}L'(t)\big(w(t,L(t))\big)^2 \\ &+ \int_a^{L(t)} \left(D_1 w_x^2 + w^2\right)\, dx + \frac{1}{k}\int_{L(t)}^b \left(D_2 w_x^2 + w^2\right)\, dx \\ &= (r+1)\|w(t,\cdot)\|^2_{\mathrm{L}^2(a,L(t))} - q\int_a^{L(t)} w^3 dx + \frac{(1-m)}{k}\|w(t,\cdot)\|^2_{\mathrm{L}^2(L(t),b)}.\end{aligned}$$

Therefore, using the definition of $\varphi^t(w)$ and rearranging terms, we find

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|w(t,\cdot)\|_H^2 + 2\varphi^t(w) - \frac{k-1}{2k}L'(t)\big(w(t,L(t))\big)^2 + q\int_a^{L(t)} \big(w(t,x)\big)^3\, dx& \\ = (r+1)\|w(t,\cdot)\|^2_{\mathrm{L}^2(a,L(t))} + \frac{(1-m)}{k}\|w(t,\cdot)\|^2_{\mathrm{L}^2(L(t),b)}& \\ \le C_s\|w(t,\cdot)\|_H^2 \quad \text{for a.e.} \quad t \in [0,T],&\end{aligned} \tag{2.4.22}$$

which proves the lemma. ∎

In the next lemma, we claim that if system ***(S)*** with conditions ***(BIC)*** has a solution, it should be non-negative.

Lemma 2.4.9. *Consider a function w_0 satisfying assumption* (2.4.2)*, that is $w_0 \in \mathrm{H}^1(a,b)$ and $w_0(x) \ge 0$ for $x \in [a,b]$, and a curve $L \in \mathrm{H}^3(0,T)$ satisfying assumption* (2.4.1)*. Then, if w is a strong solution to system **(S)**,* (2.2.10)–(2.2.11)*, and conditions **(BIC)**,* (2.2.12)–(2.2.17)*, it satisfies*

$$w(t,x) \ge 0 \qquad \text{for } (t,x) \in [0,T]\times[a,b]. \tag{2.4.23}$$

Proof: Let w be a solution to system ***(S)*** and conditions ***(BIC)*** corresponding to the function w_0 satisfying (2.4.2) and the curve $L \in \mathrm{H}^3(0,T)$ satisfying assumption (2.4.1). Multiplying each equation in system (2.4.3) with $w_- = \max\{0,-w\}$ and proceeding as in the proof of Lemma 2.4.8, we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|w_-(t,\cdot)\|_H^2 + 2\varphi^t(w_-) - \frac{k-1}{2k}L'(t)\big(w_-(t,L(t))\big)^2& \\ \le C_s\|w_-(t,\cdot)\|_H^2 + q\int_a^{L(t)} (w_-(t,x))^3\, dx \qquad \text{for a.e. } t \in [0,T].&\end{aligned} \tag{2.4.24}$$

Now, recalling the definition of $\varphi^t(w)$, we have $2\varphi^t(w_-) - \frac{k-1}{2k}L'(t)\big(w_-(t,L(t))\big)^2 \ge 0$ for $t \in [0,T]$. Hence, we can drop the non-negative term on the left-hand side of

(2.4.24) and simplify to obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|w_-(t,\cdot)\|_H^2 &\leq C_s\|w_-(t,\cdot)\|_H^2 + q\int_a^{L(t)} (w_-(t,x))^3 dx \\ &\leq C_s\|w_-(t,)\cdot\|_H^2 + q\|w_-(t,\cdot)\|_{\mathrm{C}[a,L(t)]}\|w_-(t,\cdot)\|^2_{\mathrm{L}^2(a,L(t))} \\ &\leq \big(C_s + q\|w_-(t,\cdot)\|_{\mathrm{C}[a,b]}\big)\|w_-(t,\cdot)\|_H^2. \end{aligned} \tag{2.4.25}$$

It is clear that $\big(C_s + q\|w(t,\cdot)\|_{\mathrm{C}[a,b]}\big)$ is positive and integrable on $[0,T]$. Indeed, as w is the strong solution, we have $w \in X_L$, which implies $w \in \mathrm{L}^2(0,T;\mathrm{H}^1(a,b))$, and $w_t \in \mathrm{L}^2(0,T;H)$. Then, using the fact that the embedding $\mathrm{H}^1(a,b) \hookrightarrow \mathrm{C}[a,b]$ is compact and $\mathrm{C}[a,b] \hookrightarrow \mathrm{L}^2(a,b)$ is continuous, we can use the Aubin-Lions-Simon Theorem (see Theorem II.5.16 [8, p. 102]) to conclude that $w \in \mathrm{L}^2(0,T;\mathrm{C}[a,b])$ as required. Therefore, we apply Gronwall's inequality to (2.4.25) and find

$$\|w_-(t,\cdot)\|_H^2 \leq e^{2\int_0^T \big(C_s + q\|w(t,\cdot)\|_{\mathrm{C}[a,b]}\big)dt}\|w_-(0,\cdot)\|_H^2. \tag{2.4.26}$$

But $w(0,\cdot) = w_0(\cdot)$ on $[a,b]$, and by assumption (2.4.2), w_0 is non-negative. Thus, $w_-(0,\cdot) = 0$, and so we have

$$\|w_-(t,\cdot)\|_H^2 = 0 \quad \text{for a.e.} \quad t \in [0,T],$$

from which we can conclude that $w_-(t,x) = 0$ for $(t,x) \in [0,T]\times[a,b]$, and hence the solution is non-negative. ∎

Now, we prove the existence result for the system ***(S)***, (2.2.10)–(2.2.11), with boundary and initial conditions ***(BIC)***, (2.2.12)–(2.2.17) for a given curve L.

Proof of Proposition 2.4.5

By Lemma 2.4.6 and Lemma 2.4.7, we know that $\varphi^t(w)$ satisfies assumptions $(A.\varphi^t)$ and $(A.1)$ and that $\mathrm{B}(t,w)$ satisfies assumptions $(A.2)$–$(A.5)$ of Theorems 2.1.20 and 2.1.21. Hence, there exists $w(t,\cdot) \in \mathcal{D}(\partial\varphi^t)$ for a.e. $t \in (0,T)$ satisfying (2.4.6), which is equivalent to

$$\partial\varphi^t(w(t,\cdot)) \ni -w_t(t,\cdot) - \mathrm{B}(t,w(t,\cdot)) \qquad \text{for a.e.} \quad t \in [0,T], \tag{2.4.27}$$

and

$$w(0,\cdot) = w_0(\cdot). \tag{2.4.28}$$

Since $w_0 \geq 0$ by assumption, we can show that this solution is non-negative. Indeed, as w is a strong solution to (2.1.5), by definition 2.1.19, there exist $g(t)$ and $b(t) \in \mathrm{L}^2(0,S;H)$ such that $g(t) \in \partial\varphi^t(w)$, $b(t) \in \mathrm{B}(t,w(t))$ and

$$\frac{\partial w(t,\cdot)}{\partial t} + g(t) + b(t) = 0, \tag{2.4.29}$$

for a.e. $t \in (0,T)$. Multiplying (2.4.29) by w_-, we find

$$\Big(\frac{\partial w(t,\cdot)}{\partial t}, w_-\Big)_H = \big(-g(t) - b(t), w_-(t,\cdot)\big)_H. \tag{2.4.30}$$

Having $\big(\frac{\partial w(t,\cdot)}{\partial t}, w_-\big)_H = -\frac{1}{2}\frac{d}{dt}\left(\|w_-(t,\cdot)\|_H^2\right) + \frac{k-1}{2k}L'(t)\big(w_-(t,L(t)\big)^2$, and using the definition of $\varphi^t(w_-(t,\cdot))$, its subdifferential and $\mathrm{B}(t,w)$, and the fact that $w_-w_+ = 0$, we can simplify (2.4.30) to obtain

$$\begin{aligned}
-\frac{1}{2}\frac{d}{dt}\left(\|w_-(t,\cdot)\|_H^2\right) \quad = \quad & \int_a^{L(t)} D_1(w_-)_x^2 + w_-^2 \; dx + \frac{1}{k}\int_{L(t)}^b D_2(w_-)_x^2 + w_-^2 \; dx \\
& + \frac{k-1}{2k}L'(t)\big(w_-(t,L(t))\big)^2 \\
& = 2\varphi(w_-(t,\cdot)) - \frac{k-1}{2k}L'(t)\big(w_-(t,L(t))\big)^2 \\
& \geq 0.
\end{aligned} \tag{2.4.31}$$

We then have $\frac{1}{2}\frac{d}{dt}\left(\|w_-(t,\cdot)\|_H^2\right) \leq 0$, for a.e. $t \in [0,T]$. Therefore, we can again use Gronwall's inequality and the fact that the initial condition is non-negative to obtain $w_-(t,x) = 0$ for $(t,x) \in [0,T] \times [a,b]$. Hence, the solution $w(t,\cdot)$ is non-negative.

It remains to show that the non-negative function w satisfying (2.4.27) and (2.4.28) is equivalently a strong solution to the original system ***(S)***, (2.2.10)–(2.2.11) and satisfies conditions ***(BIC)***, (2.2.12)–(2.2.15).

By the definition of the subdifferential, we can equivalently write (2.4.27) as

$$\varphi^t(z) - \varphi^t(w) \geq \big(-w_t - \mathrm{B}(t,w), z - w\big)_H \tag{2.4.32}$$

for all $z \in \mathcal{D}(\varphi^t(w))$ and a.e. $t \in (0,T)$. Let $z = w + \lambda\eta$ for $\eta \in C^\infty\Big([0,T] \times [a,b]\Big)$ and for some small $\lambda > 0$. Note that we have $z(t,\cdot) \in \mathcal{D}(\varphi^t)$ for each $t \in [0,T]$. From (2.4.32), we find

$$\varphi^t(w + \lambda\eta) - \varphi^t(w) \geq \big(-w_t - \mathrm{B}(t,w), \lambda\eta\big)_H \;\text{ for a.e.} \quad t \in (0,T). \tag{2.4.33}$$

Now recalling the definition of $\varphi^t(w)$ from (2.4.7), we can extend (2.4.33), divide it by λ and take the limit as $\lambda \to 0$ to obtain after some simplification that

$$\begin{aligned}
&\int_a^{L(t)} D_1 w_x \eta_x + w\eta \; dx + \frac{1}{k}\int_{L(t)}^b D_2 w_x \eta_x + w\eta \; dx + \frac{k-1}{k}L'(t)w(t,L(t))\eta(t,L(t)) \\
&= \int_a^{L(t)} \left(-w_t + F_1(w)\right)\eta \; dx + \frac{1}{k}\int_{L(t)}^b \big(-w_t + F_2(w)\big)\eta \; dx,
\end{aligned} \tag{2.4.34}$$

where we get the equality from the fact that we can do the same process for $-\eta$ and hence have the reverse inequality. Having $w(t,\cdot) \in \mathcal{D}(\partial\varphi^t) = \mathrm{H}^2(a,L(t)) \cap \mathrm{H}^2(L(t),b)$

for $t \in [0,T]$, implies that $w_x(t,\cdot)$ is an absolutely continuous function on $(a, L(t))$ and $(L(t), b)$. Therefore, $w_x(t,a), w_x(t,b), w_x(t,L(t)^-)$ and $w(t,L(t)^+)$ exist for $t \in [0,T]$, and so we can apply integration by part to find

$$\begin{aligned}
&\int_a^{L(t)} \big(- D_1 w_{xx} + w\big)\eta \; dx + D_1 w_x(t,L(t)^-)\eta(t,L(t)^-) - D_1 w_x(t,a)\eta(t,a) \\
&+\frac{1}{k}\int_{L(t)}^b \big(- D_2 w_{xx} + w\big)\eta \; dx + \frac{1}{k} D_2 w_x(t,b)\eta(t,b) - \frac{1}{k} D_2 w_x(t,L(t)^+)\eta(t,L(t)^+) \\
&+\frac{k-1}{k} L'(t) w(t,L(t))\eta(t,L(t)) \\
&\qquad\qquad = \int_a^{L(t)} \big(- w_t + F_1(w)\big)\eta \; dx + \frac{1}{k}\int_{L(t)}^b \left(-w_t + F_2(w)\right)\eta \; dx
\end{aligned} \tag{2.4.35}$$

for a.e. $t \in [0,T]$. Now, by choosing $\eta \in C_0^\infty\Big((0,T) \times (a, L(t))\Big)$ arbitrary, we can deduce from (2.4.35) that

$$-D_1 w_{xx} + w = -w_t + F_1(w), \qquad \text{for a.e. } t \in (0,T),\ x \in (a, L(t)), \tag{2.4.36}$$

which is equivalent to equation (2.2.10) in system ***(S)***.

Similarly, if we choose $\eta \in C_0^\infty\Big((0,T) \times (L(t), b)\Big)$, we obtain

$$-D_2 w_{xx} + w = -w_t + F_2(w), \qquad \text{for a.e. } t \in (0,T),\ x \in (L(t), b), \tag{2.4.37}$$

which is equivalent to equation (2.2.11) in system ***(S)***.

To see that w also satisfies the boundary conditions ***(BIC)***, (2.2.12)–(2.2.15), we proceed as follows. First note that $w(t,\cdot) \in \mathcal{D}(\varphi^t) = \mathrm{H}^1(a,b)$ is continuous. Therefore, the condition (2.2.12) is already satisfied by w. Then from (2.4.36) and (2.4.37), we can see that for any $\eta \in C^\infty\Big([0,T] \times [a,b]\Big)$, (2.4.35) reduces to

$$\begin{aligned}
&D_1 w_x(t,L(t)^-)\eta(t,L(t)^-) - D_1 w_x(a)\eta(t,a) + \frac{1}{k} D_2 w_x(t,b)\eta(t,b) \\
&-\frac{1}{k} D_2 w_x(t,L(t)^+)\eta(t,L(t)^+) + \frac{k-1}{k} L'(t) w(t,L(t))\eta(t,L(t)) = 0.
\end{aligned} \tag{2.4.38}$$

Now, we let $\eta \in C_0^\infty\Big((0,T); C^1[a,b]\Big)$, such that $\eta(t,x) = 0$ if $x > a + \epsilon$ for some small $\epsilon > 0$. Hence, from (2.4.38) we find $-D_1 w_x(t,a)\eta(t,a) = 0$, which is equivalent to

$$w_x(t,a) = 0, \qquad \text{for } t \in (0,T). \tag{2.4.39}$$

This is the boundary condition at $x = a$, in (2.2.14). Similarly, by choosing $\eta \in C_0^\infty\Big((0,T); C^1[a,b]\Big)$ with $\eta(t,x) = 0$ if $x < b - \epsilon$ for some small $\epsilon > 0$, from (2.4.38) we find

$$w_x(t,b) = 0, \qquad \text{for } t \in (0,T), \tag{2.4.40}$$

which is the boundary condition at $x = b$, in (2.2.14). Now, having (2.4.39) and (2.4.40), for any $\eta \in C^\infty\Big([0,T]\times[a,b]\Big)$, (2.4.38) reduces to

$$\begin{aligned}&D_1 w_x(t,L(t)^-)\eta(t,L(t)^-) - \frac{1}{k}D_2 w_x(t,L(t)^+)\eta(t,L(t)^+)\\&+\frac{k-1}{k}L'(t)w(t,L(t))\eta(t,L(t)) = 0.\end{aligned} \tag{2.4.41}$$

Since $w(t,L(t)^-) = w(t,L(t)^+)$, we can further simplify (2.4.41) to find

$$D_1 w_x(t,L(t)^-) - \frac{D_2}{k}w_x(t,L(t)^+) = L'(t)\left(\frac{1}{k}w(t,L(t)^+) - w(t,L(t)^-)\right) \tag{2.4.42}$$

for a.e. $t \in (0,T)$, which is the interface condition (2.2.13). As a result, from equations (2.4.36), (2.4.37), (2.4.39), (2.4.40) and (2.4.42), we can deduce that for a fixed curve L, the function $w(t,\cdot) \in \mathcal{D}(\varphi^t)$ satisfying (2.4.6), is a solution for ***(S)***, (2.2.10)–(2.2.11) with conditions ***(BIC)***, (2.2.12)–(2.2.17) satisfying properties $ii.1$) to $ii.5$) in Definition 2.4.1.

To see the converse, first note that as in (2.4.34), we can use definition of the subdifferential for $w(t,\cdot) \in \mathcal{D}(\varphi^t)$ to rewrite the equation in (2.4.27) in the following form:

$$\begin{aligned}&\int_a^{L(t)}\big(D_1 w_x z_x + wz\big)dx + \frac{1}{k}\int_{L(t)}^b\big(D_2 w_x z_x + wz\big)dx + \frac{k-1}{k}L'(t)w(t,L(t))z(t,L(t))\\&= \int_a^{L(t)}\left(-w_t + F_1(w)\right)z\;dx + \frac{1}{k}\int_{L(t)}^b\big(-w_t + F_2(w)\big)z\;dx\\&\qquad\qquad\text{for a.e}\quad t\in[0,T],\ \text{and for all}\quad z(t,\cdot)\in\mathcal{D}(\varphi^t).\end{aligned} \tag{2.4.43}$$

If w solves system ***(S)*** with condition ***(BIC)*** and is non-negative, by Lemma 2.4.9, we can apply integration by part and recall properties $ii.1$) to $ii.5$) in Definition 2.4.1, to see that the equality in (2.4.43) holds. In particular, for $g(t) \in \partial\varphi^t(w)$ we get

$$\begin{aligned}g(t) &= -D_1 w_{xx}(t,x) + w(t,x) \quad \text{for } (t,x)\in(0,T)\times(a,L(t)),\\ g(t) &= -D_2 w_{xx}(t,x) + w(t,x) \quad \text{for } (t,x)\in(0,T)\times(L(t),b).\end{aligned}$$

This completed the proof. ∎

Remark 2.4.10. *The solution w found in Proposition 2.4.5 is an s-strong solution for the Cauchy problem (2.4.6). Hence, by Definition 2.1.19, we have $w_t, B(t,w), w_{xx} \in$ $\mathrm{L}^2(0,T;H)$.*

2.4.2 Uniqueness of the Local Solution

Next, we prove the uniqueness of the solution in the following lemma.

Lemma 2.4.11. *For a given curve $L \in \mathrm{H}^3(0,T)$, satisfying assumption* (2.4.1)*, and a function w_0, satisfying assumption* (2.4.2)*, the solution to system **(S)**,* (2.2.10)–(2.2.11)*, with conditions **(BIC)**,* (2.2.12)–(2.2.17)*, is unique.*

Proof: We prove this result by way of contradiction. Assume that w_1 and w_2 are two solutions of system ***(S)***, (2.2.10)–(2.2.11), with conditions ***(BIC)***, (2.2.12)–(2.2.17). Then, we have

$$\begin{cases} (w_1 - w_2)_t + (w_1 - w_2) = D_1(w_1 - w_2)_{xx} + f_1(w_1) - f_1(w_2) + (w_1 - w_2), \\ (w_1 - w_2)_t + (w_1 - w_2) = D_2(w_1 - w_2)_{xx} + kf_2(\dfrac{w_1}{k}) - kf_2(\dfrac{w_2}{k}) + (w_1 - w_2), \end{cases}$$

where the first equation holds for $(t,x) \in (0,T) \times (a, L(t))$ and the second for $(t,x) \in (0,T) \times (L(t), b)$. Applying Lemma 2.4.8, we obtain

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|w_1 - w_2\|_H^2 + 2\varphi^t(w_1 - w_2) - \frac{k-1}{2k}L'(t)\Big(w_1(t, L(t)) - w_2(t, L(t))\Big)^2 & \\ + q\int_a^{L(t)} \big(w_1^2 - w_2^2\big)\big(w_1 - w_2\big)dx & \\ \leq C_s\|w_1 - w_2\|_H^2 \quad \text{for a.e.} \quad t \in [0,T]. & \end{aligned} \tag{2.4.44}$$

We again recall that

$$2\varphi^t(w_1 - w_2) - \frac{k-1}{2k}L'(t)\Big(w_1(t, L(t)) - w_2(t, L(t))\Big)^2 \geq 0$$

and

$$q\int_a^{L(t)} \big(w_1^2 - w_2^2\big)\big(w_1 - w_2\big)\ dx = q\int_a^{L(t)} \big(w_1 - w_2\big)^2\big(w_1 + w_2\big)\ dx \geq 0$$

since, by Lemma 2.4.9, w_1 and w_2 are non-negative solutions of system ***(S)*** with conditions ***(BIC)***. Hence, simplifying (2.4.44) and applying Gronwall's inequality yields

$$\|w_1(t,\cdot) - w_2(t,\cdot)\|_H^2 \leq e^{2\int_0^t C_s d\tau}\|w_1(0,\cdot) - w_2(0,\cdot)\|_H^2 \quad \text{for} \quad t \in [0,T].$$

But both w_1 and w_2 satisfy initial conditions (2.2.16) and (2.2.17), so we have $w_1(0,x) = w_2(0,x) = w_0(x)$, for $x \in [a,b]$. Therefore, we can assert $\|w_1(t,\cdot) - w_2(t,\cdot)\|_H^2 = 0$ for a.e. $t \in [0,T]$ and hence $w_1(t,x) = w_2(t,x)$ for a.e. $(t,x) \in [0,T] \times [a,b]$. ∎

2.4.3 Regularity of the Strong Solution

Having proved the existence and uniqueness of the solution, we now discuss the regularity of the solution that we found for a given curve $L \in \mathrm{H}^3(0,T)$ and find different estimates for it. As mentioned in Remark 2.4.10, we already know that $w_t, B(t,w)$ and therefore also w_{xx} belong to $\mathrm{L}^2(0,T;H)$. However, to be able to reduce the regularity required for the curve L, we need to find bounds independent of the curve L on the corresponding solution w.

Lemma 2.4.12. *If w is the strong solution to system **(S)**,* (2.2.10)–(2.2.11), *with conditions **(BIC)**,* (2.2.12)–(2.2.17), *corresponding to a given curve $L \in \mathrm{H}^3(0,T)$, satisfying assumption* (2.4.1), *and initial condition $w_0 \in \mathrm{H}^1(a,b)$ with $w_0(x) \geq 0$ on $[a,b]$, then we have*

i) $\sup_{0\leq t\leq T} \|w(t,\cdot)\|_H^2 \leq C_h$,

ii) $\int_0^T \varphi^t(w)dt + \int_0^T \int_a^{L(t)} |w|^3 dxdt \leq C_p$,

iii) $\int_0^T \left|\frac{d}{dt}\|w(t,\cdot)\|_H^2\right| dt \leq C_0$,

for appropriate constants C_h, C_0, C_p that depend on r, m, T and $\|w_0\|_H$.

Proof: As w is the solution, we can apply Lemma 2.4.8 to obtain

$$\frac{1}{2}\frac{d}{dt}\|w(t,\cdot)\|_H^2 + 2\varphi^t(w) - \frac{k-1}{2k}L'(t)\big(w(t,L(t))\big)^2 + q\int_a^{L(t)}\big(w(t,x)\big)^3 dx \leq C_s\|w(t,\cdot)\|_H^2 \tag{2.4.45}$$

for a.e. $t \in [0,T]$. We recall the formulation of $\varphi^t(w)$ in (2.4.7) and note that w is non-negative to justify dropping the non-negative terms on the left-hand side of (2.4.45) and conclude that

$$\frac{1}{2}\frac{d}{dt}\|w(t,\cdot)\|_H^2 \leq C_s\|w(t,\cdot)\|_H^2, \quad \text{for a.e.} \quad t \in [0,T]. \tag{2.4.46}$$

Hence, Gronwall's inequality implies

$$\sup_{0\leq t\leq T} \|w(t,\cdot)\|_H^2 \leq C_h, \tag{2.4.47}$$

where $C_h = C_h(r,m,T,\|w_0\|_H)$ is a constant that depends on r, m, T and $\|w_0\|_H$. This proves $i)$.

To prove $ii)$, we first integrate (2.4.45) over $[0,T]$ and use $i)$ to obtain

$$\begin{aligned}\frac{1}{2}\int_0^T \frac{d}{dt}\|w(t,\cdot)\|_H^2\, dt + 2\int_0^T \varphi^t(w)\, dt - \frac{k-1}{2k}\int_0^T L'(t)\big(w(t,L(t))\big)^2\, dt&\\ + q\int_0^T\int_a^{L(t)} \big(w(t,x)\big)^3\, dxdt&\\ \leq C_s\int_0^T \|w(t,\cdot)\|_H^2\, dt \leq C_sC_hT.&\end{aligned} \tag{2.4.48}$$

But we can simplify (2.4.48) to find

$$\frac{1}{2}\|w(T,\cdot)\|_H^2 + \int_0^T \varphi^t(w)\ dt + \int_0^T \int_a^{L(t)} \left(D_1 w_x^2 + w^2\right)\ dx + \int_{L(t)}^b \left(D_2 w_x^2 + w^2\right)\ dxdt \\ + q\int_0^T \int_a^{L(t)} |w|^3\ dxdt \\ \leq C_s C_h T + \|w_0\|_H^2. \tag{2.4.49}$$

Therefore, by dropping the positive terms on the left-hand side of (2.4.49), we have ii), that is,

$$\int_0^T \varphi^t(w)\ dt + q\int_0^T \int_a^{L(t)} |w|^3\ dxdt \leq C_p,$$

where $C_p = C_p(r, m, T, \|w_0\|_H)$ is a positive constant.

To prove iii), we first note that from the equality in (2.4.22), non-negativity of the solution and the assumptions on L and K, we can conclude that

$$\frac{1}{2}\left|\frac{d}{dt}\|w(t,\cdot)\|_H^2\right| \leq \left|(r+1)\|w(t,\cdot)\|^2_{\mathrm{L}^2(a,L(t))} + \frac{(1-m)}{k}\|w(t,\cdot)\|^2_{\mathrm{L}^2(L(t),b)}\right| \\ + 2\varphi^t(w) + \frac{k-1}{2k}L'(t)\big(w(t,L(t))\big)^2 + q\int_a^{L(t)} \big(w(t,x)\big)^3\ dx \\ \leq C_s\|w(t,\cdot)\|_H^2 + 3\varphi^t(w) + 3q\int_a^{L(t)}\big(w(t,x)\big)^3 dx \quad \text{for a.e. } t\in[0,T], \tag{2.4.50}$$

where C_s is as in Lemma 2.4.8. Now we can integrate both sides of (2.4.50) and use the result of part ii) to obtain

$$\int_0^T \left|\frac{d}{dt}\|w(t,\cdot)\|_H^2\right| dt \leq 2C_s\int_0^T \|w(t,\cdot)\|_H^2 dt + 6\int_0^T \varphi^t(w) dt + 6q\int_0^T\int_a^{L(t)}\big(w(t,x)\big)^3 dxdt \\ \leq C_0,$$

where $C_0 = C_0(r, m, T, \|w_0\|_H)$ is a constant that depends on r, m, T and $\|w_0\|_H$. ∎

Remark 2.4.13. *Consider the space X_L and its dual X_L^* from Definition 2.3.2. The strong solution to system **(S)** and conditions **(BIC)**, corresponding to a curve $L \in \mathrm{H}^3(0,T)$ satisfying assumption* (2.4.1)*, belongs to the space X_L. This is a direct result of part ii) of Lemma 2.4.12 and formulation of the functional φ^t,* (2.4.7)*. In addition, due to Remark 2.4.10 and the embedding $\mathrm{L}^2(0,T;H) \hookrightarrow X_L^*$, we have $w_t \in X_L^*$.*

In the next lemma we prove that the boundedness of w_t in X_L^* is independent of the choice of the curve L.

Lemma 2.4.14. *Let $L \in \mathrm{H}^3(0,T)$ satisfy assumption* (2.4.1) *and assume that w is the strong solution corresponding to the curve L of system **(S)** with conditions **(BIC)** found in Proposition 2.4.5. Then there exists a bound independent of L on $\|w_t\|_{X_L^*}$.*

Proof: We first note that as a solution to system ***(S)*** with conditions ***(BIC)***, w satisfies the Cauchy problem (2.4.6) as well. Thus, as seen in equation (2.4.43), equivalently for each $\eta \in \mathrm{L}^2(0,T,\mathrm{H}^1(a,b))$ we can write

$$\begin{aligned}\int_0^T \big(w_t,\eta\big)_H\,dt + \int_0^T\int_a^{L(t)} \big(D_1 w_x\eta_x + w\eta\big)\,dxdt + \frac{1}{k}\int_0^T\int_{L(t)}^b \big(w_x\eta_x + D_2 w\eta\big)\,dxdt \\ + \frac{k-1}{k}\int_0^T L'(t)w(t,L(t))\eta(t,L(t))\,dt \\ = \int_0^T\int_a^{L(t)} \big((r+1)w - q(w)^2\big)\,\eta\,dx + \frac{1}{k}\int_{L(t)}^b \big((1-m)w\big)\eta\,dx\,dt.\end{aligned} \tag{2.4.51}$$

Applying Young's inequality and rearranging terms, we find

$$\begin{aligned}\left|\int_0^T \big(w_t,\eta\big)_H\,dt\right| \;\le\; & \int_0^T \varphi^t(w)\,dt + \int_0^T \varphi^t(\eta)\,dt \\ & + \frac{(r+1)^2}{2}\int_0^T \|w\|^2_{\mathrm{L}^2(a,L(t))}dt + \frac{1}{2}\int_0^T \|\eta\|^2_{\mathrm{L}^2(a,L(t))}\,dt \\ & + \frac{2}{3}q^{\frac{3}{2}}\int_0^T\int_a^{L(t)} |w|^3 dxdt + \frac{1}{3}\int_0^T\int_a^{L(t)} |\eta|^3\,dxdt \\ & + \frac{(1-m)^2}{2k}\int_0^T \|w\|^2_{\mathrm{L}^2(L(t),b)}\,dt + \frac{1}{2k}\int_0^T \|\eta\|^2_{\mathrm{L}^2(L(t),b)}\,dt \\ & \le C_1\int_0^T \varphi^t(w)\,dt + C_2\int_0^T \varphi^t(\eta)\,dt \\ & + \frac{2}{3}q^{\frac{3}{2}}\int_0^T\int_a^{L(t)} |w|^3 dxdt + \frac{1}{3}\int_0^T\int_a^{L(t)} |\eta|^3\,dxdt\end{aligned} \tag{2.4.52}$$

for appropriate constants C_1 and C_2. We now take $\eta \in X_L$ with $\|\eta\|_{X_L} \le 1$. As mentioned in Remark 2.4.10, as w is the solution corresponding to the curve $L \in \mathrm{H}^3(0,T)$ then $w_t \in \mathrm{L}^2(0,T;H)$. By Definition 2.3.2, this allows us to identify the pairing $\langle\langle\cdot,\cdot\rangle\rangle$ between X_L and X_L^* with $\int_0^T(\cdot,\cdot)_H dt$, the inner product in $\mathrm{L}^2(0,T;H)$. Hence, we can recall the bound in part *ii)* from Lemma 2.4.12 to conclude that for an appropriate constant $M = M(r,m,T,\|w_0\|_H)$, independent of L, we have

$$|\langle\langle w_t,\eta\rangle\rangle| \le M \quad \text{for all} \quad \eta \in X_L, \quad \|\eta\|_{X_L} \le 1. \tag{2.4.53}$$

Therefore, by definition of the norm in a dual space, we can conclude the desired result from (2.4.53). ∎

2.4.4 Relaxing the Regularity Assumption on L

In Section 2.4.1, we showed the existence of a strong solution to system ***(S)***, with conditions ***(BIC)*** for a given curve $L \in \mathrm{H}^3(0,T)$. However, using the estimates on the solution found in the previous section, we can extend our existence result for a curve $L \in \mathrm{H}^1(0,T)$. To do that, we use an approximating sequence of solutions and corresponding estimates for it, with which we can reduce the regularity needed for the curve L.

We first recall the following standard result, which is a direct consequence of Theorem 8.8 [9, p. 212].

Lemma 2.4.15. *Let $\{L^n\}$ be a bounded sequence in $\mathrm{H}^1(0,T)$. Then there exists a function $L \in \mathrm{H}^1(0,T)$ such that $L^n \to L$ uniformly on $[0,T]$.*

To find an approximating sequence, we start by finding a sequence of curves that satisfy the required assumptions and converge to the desired curve with reduced regularity. In the next lemma, we prove that for any curve $L \in \mathrm{H}^1(0,T)$ such a sequence exists.

Lemma 2.4.16. *Consider a curve $L \in \mathrm{H}^1(0,T)$ satisfying assumption* (2.4.1)*, that is,*

$$L(0) = l_0, \quad L(t) \in (a+\delta, b-\delta), \quad \textit{for some } \delta > 0, \quad \textit{and } L'(t) \geq 0 \quad \textit{for } t \in [0,T].$$

Then, there exists a sequence $\{L^n\} \in \mathrm{H}^3(0,T)$ that also satisfies assumption (2.4.1) *and*

$$\|L^n - L\|_{\mathrm{H}^1(0,T)} \to 0 \textit{ as } n \to \infty.$$

Proof: For any $n = 1, 2, \ldots$, consider the following differential equation on $[0,T]$:

$$\begin{cases} -\dfrac{1}{n} {h_n}''(t) + h_n(t) = L(t), \\ {h_n}'(0) = 0, \ {h_n}'(T) = 0. \end{cases} \tag{2.4.54}$$

As $L \in \mathrm{H}^1(0,T)$, by a standard result (see for example Proposition 8.17 [9, p.225]), we know that for each n, there exists a unique solution $h_n \in \mathrm{H}^3(0,T)$ satisfying (2.4.54) in $[0,T]$. In addition, for all n, we can define the functional $\psi_n : \mathrm{H}^1(0,T) \to [0,+\infty)$ as follows

$$\psi_n(h) = \frac{1}{2}\int_0^T h'(t)^2 dt + \frac{n}{2}\int_0^T (h(t) - L(t))^2 dt.$$

Then ψ_n is proper, coersive, convex and lower semicontinuous, so it has a minimizer. We can see that for each n we have $\min_{h \in H^1(0,T)} \psi_n(h) = \psi_n(h_n)$. In particular, we have

$$\psi_n(h_n) \leq \psi_n(L) = \frac{1}{2}\int_0^T L'(t)^2 dt, \quad \text{for all } n. \tag{2.4.55}$$

As $L \in \mathrm{H}^1(0,T)$ and ψ_n is coersive for all n, we can deduce from (2.4.55) that $\{h_n\}$ is bounded in $\mathrm{H}^1(0,T)$. We claim that $h_n \to L$ strongly in $\mathrm{H}^1(0,T)$. Indeed, we first conclude from (2.4.55) that $\frac{1}{2}\int_0^T (h_n - L)^2 dt \leq \frac{M}{n}$, for an appropriate constant M. Thus, we have $h_n \to L$ strongly in $\mathrm{L}^2(0,T)$. As $\{h_n\}$ is bounded in $\mathrm{H}^1(0,T)$, up to a subsequence, again denoted by $\{h_n\}$, we have that $\{h_n\}$ is weakly convergent in $\mathrm{H}^1(0,T)$. Hence, having $h_n \to L$ strongly in $\mathrm{L}^2(0,T)$, we obtain $h_n \rightharpoonup L$ weakly in $\mathrm{H}^1(0,T)$. Now, we note that from weak lower semicontinuity of the norm we have $\int_0^T (L')^2 dt \leq \liminf_{n\to\infty} \int_0^T (h_n')^2 dt$. On the other hand, from (2.4.55), we find that $\limsup_{n\to\infty} \frac{1}{2}\int_0^T h_n'(t)^2 dt \leq \frac{1}{2}\int_0^T L'(t)^2 dt$. Therefore, we have $h_n \to L$ in $\mathrm{H}^1(0,T)$, as claimed.

Next, to show that for all n we have $h_n'(t) \geq 0$ on $[0,T]$, we first note that by regularity we have $h_n \in \mathrm{H}^3(0,T)$ for all n. Therefore, we can consider the following system for each h_n',

$$\begin{cases} -\dfrac{1}{n}(h_n')''(t) + h_n'(t) = L'(t), \\ h_n'(0) = 0, \; h_n'(T) = 0. \end{cases}$$

Since $L' \geq 0$ for $t \in [0,T]$, by the maximum principle (see for example Thoerem 8.19 [9, p. 229]), we can conclude that $h_n'(t) \geq \min\left\{h_n'(0), h_n'(T)\right\} \geq 0$ on $[0,T]$.

In addition, as $h_n(t) \to L(t)$ uniformly and $a + \delta < L(t) < b - \delta$ for all t, there exists $N > 0$ such that up to a subsequence we have $a + \delta < h_n(t) < b - \delta$ for all t and $n \geq N$.

Now, for an appropriate subsequence $\{h_n\}$, we consider the sequence $\{L^n(t)\}$ as follows

$$L^n(t) = h_n(t) + (l_0 - h_n(0)) \quad \text{for} \quad t \in [0,T] \quad \text{and for all } n. \tag{2.4.56}$$

Then, we have $L^n(0) = l_0$ for all n and, using the properties that we proved for $\{h_n\}$, we can see that $\{L^n\}$ is the desired sequence. ∎

Based on the sequence $\{L^n\}$ found in Lemma 2.4.16, we can now define the approximating sequence.

Remark 2.4.17. *Consider a function $w_0 \in \mathrm{H}^1(a,b)$ with $w_0(x) \geq 0$ for $x \in [a,b]$, and a curve $L \in \mathrm{H}^1(0,T)$ that satisfies assumption (2.4.1). Let $\{L^n\}$ be a sequence in $\mathrm{H}^3(0,T)$, found in Lemma (2.4.16), that converges strongly to L in $\mathrm{H}^1(0,T)$. For each curve in the sequence $\{L^n\}$, we consider the space $H_n = \mathrm{L}^2(a,b)$ with the inner product $(\cdot,\cdot)_{H_n}$ and the norm $\|\cdot\|_{H_n}$ analogous to the inner product and norm of H in (2.3.1). We also define the corresponding functional, denoted by φ_n^t, in the same way as in equation (2.4.7).*

*By Proposition 2.4.5, for each n one can find the strong solution w^n corresponding to the curve L^n and initial condition w_0 for the system **(S)** and conditions **(BIC)**. This sequence $\{w^n\}$ is the approximating sequence.*

Recalling the Banach space introduced in Definition 2.3.2, we let $X_n = X_{L^n}$ *and* $X = X_L$ *with* X_n^* *and* X^* *the dual spaces of* X_n *and* X*, respectively. We denote the pairing between* X_n *and* X_n^* *by* $\int_0^T \langle \cdot, \cdot \rangle_n dt$ *and the one between* X *and* X^* *by* $\int_0^T \langle \cdot, \cdot \rangle dt$*. We already know from Remark 2.4.13 and Lemma 2.4.14 that for each* n *we have* $w^n \in X_n$ *and* $w_t^n \in X_n^*$*.*

Using the regularity of the strong solutions w^n in the sequence together with the appropriate bounds from Section 2.4.3, we analyze the convergence of the sequence $\{w^n\}$ in the following proposition.

Proposition 2.4.18. *Consider the curve* $L \in \mathrm{H}^1(0,T)$*, satisfying assumption* (2.4.1)*, and let* $\{L^n\}$ *be the sequence, found in Lemma 2.4.16, such that* $L^n \to L$ *strongly in* $\mathrm{H}^1(0,T)$*. Let* $\{w^n\}$ *be the approximating sequence introduced in Remark 2.4.17. Then there exists an appropriate subsequence, still denoted by* $\{w^n\}$*, and a function* w *for which we have the following results:*

i) $w^n \rightharpoonup w$ *weakly in* $\mathrm{L}^2\big(0,T;\mathrm{H}^1(a,b)\big)$.

ii) $w^n \to w$ *strongly in* $\mathrm{L}^2(0,T;\mathrm{L}^2(a,b))$.

iii) $w^n(t,x) \to w(t,x)$ *a.e. in* $[0,T] \times [a,b]$.

iv) $\|w^n(t,\cdot)\|^2_{\mathrm{L}^2(a,b)} \to \|w(t,\cdot)\|^2_{\mathrm{L}^2(a,b)}$ *strongly in* $\mathrm{L}^p(0,T)$ *for all* $1 \le p < \infty$.

v) $\|w(t,\cdot)\|^2_{\mathrm{L}^2(a,b)} \overset{\star}{\rightharpoonup} \|w(t,\cdot)\|^2_{\mathrm{L}^2(a,b)}$*, in weak** $\mathrm{L}^\infty(0,T)$ *for all* $\eta \in L^1(0,T)$.

vi) $w^n(t,x)\chi_{[a,L^n(t)]}(x) \to w(t,x)\chi_{[a,L(t)]}(x)$ *a.e. in* $[0,T] \times [a,b]$,

$w^n(t,x)\chi_{[L^n(t),b]}(x) \to w(t,x)\chi_{[L(t),b]}(x)$ *a.e. in* $[0,T] \times [a,b]$.

Proof: For all n, we can apply Lemma 2.4.12 to see that w^n satisfies

$$\sup_{0 \le t \le T} \|w^n(t,\cdot)\|^2_{H_n} \le C_h, \tag{2.4.57}$$

$$\int_0^T \varphi_n^t(w^n)dt + \int_0^T \int_a^{L^n(t)} |w^n|^3 dxdt \le C_p, \tag{2.4.58}$$

$$\int_0^T \left| \frac{d}{dt}\|w^n(t,\cdot)\|^2_{H_n} \right| dt \le C_0, \tag{2.4.59}$$

where the constants C_h, C_0 and C_p may depend on n through $\|w^n(0,\cdot)\|_{H_n}$. However, by assumption (2.4.2) we have $w^n(0,\cdot) = w_0(\cdot)$ for all n, and $w_0 \in \mathrm{H}^1(a,b)$. In addition, by assumption (2.4.1) we also have $L^n(0) = l_0$ for all n. Hence, $\|w_0\|_{H_n} =$

$\|w_0\|_H$ for all n. This implies that the bounds in (2.4.57)–(2.4.58) are independent of n and are uniform.

First, we note that from (2.4.58) and definition of functional φ_n^t, we have

$$\int_0^T \|w^n\|^2_{\mathrm{H}^1(a,b)}\,dt \le k\int_0^T \|w^n\|^2_{\mathrm{H}^1(a,L^n(t))} + \frac{1}{k}\|w^n\|^2_{\mathrm{H}^1(L^n(t),b)}\,dt \le kC_p. \qquad (2.4.60)$$

Thus, we can assert that $\{w^n\}$ is bounded in $\mathrm{L}^2\big(0,T;\mathrm{H}^1(a,b)\big)$, which is a reflexive Banach space, so there exists a subsequence, still denoted by $\{w^n\}$, and a function $w \in \mathrm{L}^2\big(0,T;\mathrm{H}^1(a,b)\big)$ such that

$$w^n \rightharpoonup w \quad \text{weakly in} \quad \mathrm{L}^2\big(0,T;\mathrm{H}^1(a,b)\big),$$

which is part $i)$.

To prove $ii)$ and $iii)$, we first note that from Lemma 2.4.14 we can conclude for each n that we have $w_t^n \in X_n^*$ and $\|w_t^n\|_{X_n^*} \le M$ for all n. Considering the fact that the constant M does not depend on n, we can recall the embedding of $X_n^* \hookrightarrow \mathrm{L}^{4/3}\big(0,T;(\mathrm{H}^1(a,b))^*\big)$ from Remark 2.3.3, to be able to deduce that $\{w_t^n\}$ is uniformly bounded in $\mathrm{L}^{4/3}\big(0,T;(\mathrm{H}^1(a,b))^*\big)$, that is,

$$\|w_t^n\|_{\mathrm{L}^{4/3}\left(0,T;(\mathrm{H}^1(a,b))^*\right)} \le CM \qquad (2.4.61)$$

for an appropriate constant C independent of n. Now, consider the set

$$E_{2,\frac{4}{3}} = \Big\{u : u \in \mathrm{L}^2(0,T;\mathrm{H}^1(a,b)), u_t \in \mathrm{L}^{4/3}\big(0,T;(\mathrm{H}^1(a,b))^*\big)\Big\}. \qquad (2.4.62)$$

We can see from (2.4.60) and (2.4.61) that the sequence $\{w^n\}$ is uniformly bounded in $E_{2,\frac{4}{3}}$. Thus, we have $w \in E_{2,\frac{4}{3}}$ as well and due to the compact embedding of $E_{2,\frac{4}{3}} \hookrightarrow \mathrm{L}^2(0,T;\mathrm{L}^2(a,b))$ (see Theorem II.5.15 [8, p. 102]), up to a subsequence we have

$$w^n \to w \qquad \text{strongly in} \qquad \mathrm{L}^2(0,T;\mathrm{L}^2(a,b)),$$

which implies $w^n(t,\cdot) \to w(t,\cdot)$ in $\mathrm{L}^2(a,b)$ for a.e. $t \in [0,T]$ and hence (see Theorem 4.9 [9, p. 94]),

$$w^n(t,x) \to w(t,x) \qquad \text{a.e. in} \quad [0,T]\times[a,b],$$

for an appropriate subsequence, as desired.

Next, we note that $\|w^n\|^2_{\mathrm{L}^2(a,b)} \le k\|w^n\|^2_{H_n}$ and $\left|\frac{d}{dt}\|w^n\|^2_{\mathrm{L}^2(a,b)}\right| \le \left|\frac{d}{dt}\|w^n\|^2_{H_n}\right|$. Then from (2.4.57) and (2.4.59), we can assert that

$$\Big\|\|w^n\|^2_{L^2(a,b)}\Big\|_{\mathrm{W}^{1,1}(0,T)} \le C_1 \qquad (2.4.63)$$

for an appropriate constant $C_1 = C_1(r,m,k,\|w_0\|_H,T)$ independent of n. We can use (2.4.63) and the compact embedding $\mathrm{W}^{1,1}(0,T) \hookrightarrow \mathrm{L}^p(0,T)$, for all $1 \le p < \infty$

(see [1, p. 168]), to conclude that there exists a subsequence, still denoted by $\{w^n\}$, such that $\{\|w^n(t,\cdot)\|^2_{\mathrm{L}^2(a,b)}\}$ converges strongly in $\mathrm{L}^p(0,T)$ for $1 \leq p < \infty$. But the pointwise convergence in part *iii*) yields that the limit must be $\|w(t,\cdot)\|^2_{\mathrm{L}^2(a,b)}$, and hence we obtain part *iv*).

Furthermore, note that from (2.4.57), we can also assert that $\|w^n(t,\cdot)\|^2_{\mathrm{L}^2(a,b)}$ is bounded in $\mathrm{L}^\infty(0,T)$. Hence, up to a subsequence, it converges in the weak* topology to some function $j(t) \in \mathrm{L}^\infty(0,T)$, i.e.,

$$\lim_{n\to\infty} \int_0^T \|w^n(t,\cdot)\|^2_{\mathrm{L}^2(a,b)} \eta(t)\ dt = \int_0^T j(t)\eta(t)\ dt \quad \text{for all} \quad \eta \in \mathrm{L}^1(0,T).$$

But we have from part *iv*) that $\|w^n(t,\cdot)\|^2_{\mathrm{L}^2(a,b)} \to \|w(t,\cdot)\|^2_{\mathrm{L}^2(a,b)}$ strongly in $\mathrm{L}^2(0,T)$, so we must have that $j(t) = \|w(t,\cdot)\|^2_{\mathrm{L}^2(a,b)}$ for $t \in [0,T]$. This proves part *v*).

Finally, the pointwise convergence of $\{w^n\}$ in part *iii*) and the uniform convergence of $L^n \to L$ from Lemma 2.4.15 yields part *vi*) ∎

Remark 2.4.19. *Based on the bounds on $\{w^n\}$ and the convergence results in Proposition 2.4.18, one can verify that $w \in X$.*

In the next step, we need to prove that the function w, the limit of the approximating sequence $\{w^n\}$, is the weak solution, defined in Definition 2.4.1, to the system ***(S)*** with conditions ***(BIC)*** corresponding to the curve $L \in \mathrm{H}^1(0,T)$. In the next section, we state a sequence of lemmas required for this argument.

2.4.5 Convergence Results for the Approximating Sequence

In the following lemmas, we use the above convergence result for the sequence $\{w^n\}$ to prove the required arguments for extending the existence of solution for a curve $L \in H^1(0,T)$.

Lemma 2.4.20. *Let L, $\{L^n\}$, w and $\{w^n\}$ be as in Remark 2.4.17. For $\eta \in \mathrm{L}^2(0,T;\mathrm{L}^2(a,b))$, we have*

$$\lim_{n\to\infty} \int_0^T \big(w^n,\eta\big)_{H_n}\ dt = \int_0^T \big(w,\eta\big)_H\ dt. \tag{2.4.64}$$

In particular,

$$w^n\chi_{[a,L^n]} \to w\chi_{[a,L]} \quad \textit{weakly in} \quad \mathrm{L}^2(0,T;\mathrm{L}^2(a,b)),$$
$$w^n\chi_{[L^n,b]} \to w\chi_{[L,b]} \quad \textit{weakly in} \quad \mathrm{L}^2(0,T;\mathrm{L}^2(a,b)).$$

Proof: Considering the characteristic function on $[a, L^n]$, we first look at

$$\lim_{n\to\infty}\int_0^T\int_a^{L^n(t)} w^n\eta\ dxdt = \lim_{n\to\infty}\int_0^T\int_a^b w^n\chi_{[a,L^n]}\eta\ dxdt. \tag{2.4.65}$$

We now recall that by part *i)* in Proposition 2.4.18, we have $w^n \rightharpoonup w$, weakly in $\mathrm{L}^2\big(0,T;\mathrm{L}^2(a,b)\big)$. In addition, having the uniform convergence of $L^n \to L$ from Lemma 2.4.15, we can see that $\chi_{[a,L^n]}\eta \to \chi_{[a,L]}\eta$, strongly in $\mathrm{L}^2\big(0,T;\mathrm{L}^2(a,b)\big)$. Hence, we can apply a standard result (see Proposition 3.5 [9, p. 58]) to obtain

$$\lim_{n\to\infty}\int_0^T\int_a^b w^n\chi_{[a,L^n]}\eta\ dxdt = \int_0^T\int_a^b w\chi_{[a,L]}\eta\ dxdt. \tag{2.4.66}$$

Repeating the same approach for $w^n\chi_{[L^n,b]}$, we will have the desired result. ∎

Remark 2.4.21. *Following the same argument as in the proof of the previous lemma, we can also prove*

$$\lim_{n\to\infty}\int_0^T (w_x^n,\eta)_{H_n}dt \ = \ \int_0^T (w_x,\eta)_H\ dt \quad \textit{for} \quad \eta\in\mathrm{L}^2(0,T;\mathrm{L}^2(a,b)),$$

which implies

$$\begin{aligned} w_x^n\chi_{[a,L^n]} &\to w_x\chi_{[a,L]} \quad \textit{weakly in} \quad \mathrm{L}^2(0,T;\mathrm{L}^2(a,b)),\\ w_x^n\chi_{[L^n,b]} &\to w_x\chi_{[L,b]} \quad \textit{weakly in} \quad \mathrm{L}^2(0,T;\mathrm{L}^2(a,b)). \end{aligned}$$

Lemma 2.4.22. *Let L, $\{L^n\}$, w and $\{w^n\}$ be as in Remark 2.4.17. We have*

$$\int_0^T \|w^n\|_{H_n}^2 dt \to \int_0^T \|w\|_H^2 dt. \tag{2.4.67}$$

In particular,

$$\begin{aligned} w^n\chi_{[a,L^n]} &\to w\chi_{[a,L]} \quad \textit{strongly in} \quad \mathrm{L}^2(0,T;\mathrm{L}^2(a,b)),\\ w^n\chi_{[L^n,b]} &\to w\chi_{[L,b]} \quad \textit{strongly in} \quad \mathrm{L}^2(0,T;\mathrm{L}^2(a,b)). \end{aligned} \tag{2.4.68}$$

Proof: We follow the same idea as in the proof of Lemma 2.4.20. Considering the characteristic function on $[a, L^n(t)]$, we first note that

$$\lim_{n\to\infty}\int_0^T\int_a^{L^n(t)} (w^n)^2\ dxdt = \lim_{n\to\infty}\int_0^T\int_a^b (w^n)^2\chi_{[a,L^n]}\ dxdt.$$

Hence, we just need to recall that $w^n \to w$ strongly in $\mathrm{L}^2(0,T;\mathrm{L}^2(a,b))$, from part $ii)$ in Proposition 2.4.18, and $w^n\chi_{[a,L^n]} \to w\chi_{[a,L]}$ weakly in $\mathrm{L}^2(0,T;\mathrm{L}^2(a,b))$, from Lemma (2.4.20), to be able to again apply a standard result (see [9, p. 58]) and obtain

$$\lim_{n\to\infty}\int_0^T\int_a^b (w^n)^2\chi_{[a,L^n]}\ dxdt = \int_0^T\int_a^b w^2\chi_{[a,L]}\ dxdt.$$

Arguing in the same way for $w^n\chi_{[L^n,b]}$, we will have (2.4.67). Finally (2.4.67) along with the weak convergence of $w^n\chi_{[a,L^n]}$ and $w^n\chi_{[L^n,b]}$ in $\mathrm{L}^2(0,T;\mathrm{L}^2(a,b))$ from Lemma 2.4.20 implies (2.4.68). ∎

Lemma 2.4.23. *Let L, $\{L^n\}$, w, $\{w^n\}$, X_n, and X be as in Remark 2.4.17. For $\eta \in \mathrm{C}\big([0,T];\mathrm{C}[a,b]\big)$, we have*

$$\lim_{n\to\infty}\int_0^T \langle (w^n)^2,\eta\rangle_n\ dt = \int_0^T \langle w^2,\eta\rangle\ dt.$$

Proof: We first observe that $w^n\eta \rightharpoonup w\eta$, weakly in $\mathrm{L}^2\big(0,T;\mathrm{L}^2(a,b)\big)$ since $\eta \in \mathrm{C}\big([0,T];\mathrm{C}[a,b]\big)$, and by part $i)$ in Proposition 2.4.18, we have $w^n \rightharpoonup w$ weakly in $\mathrm{L}^2(0,T;\mathrm{L}^2(a,b))$. Hence, recalling the strong convergence of $w^n\chi_{[a,L^n]}$ and $w^n\chi_{[L^n,b]}$ in $\mathrm{L}^2(0,T;\mathrm{L}^2(a,b))$ from Lemma 2.4.22, we can proceed as in the proofs of the previous lemmas to conclude the desired result. ∎

Lemma 2.4.24. *Let L, $\{L^n\}$, w and $\{w^n\}$ be as in Remark 2.4.17. We have*

$$\liminf_{n\to\infty}\int_0^T\int_a^{L^n(t)} |w^n|^3\ dxdt \geq \int_0^T\int_a^{L(t)} |w|^3\ dxdt.$$

Proof: As seen in part $vi)$ of Proposition 2.4.18, we have $\chi_{[a,L^n(t)]}(x)w^n(t,x) \to \chi_{[a,L(t)]}(x)w(t,x)$ a.e. in $[0,T]\times[a,b]$. Therefore, we can apply Fatou's Lemma to obtain

$$\liminf_{n\to\infty}\int_0^T\int_a^b \chi_{[a,L^n]}|w^n|^3\ dxdt \geq \int_0^T\int_a^b \chi_{[a,L]}|w|^3 dxdt,$$

which proves the lemma. ∎

Lemma 2.4.25. *Let L, $\{L^n\}$, w and $\{w^n\}$ be as in Remark 2.4.17. We have*

$$w^n(t,L^n(t)) \to w(t,L(t)) \quad \textit{strongly in} \quad \mathrm{L}^2(0,T).$$

Proof: We first prove that $w^n(t, L^n(t)) \rightharpoonup w^n(t, L(t))$ weakly in $L^2(0,T)$. Having $w \in \mathrm{L}^2(0, T, \mathrm{L}^2(a,b))$, for all $z \in [a,b]$, we can write

$$w^n(t, L^n(t)) - w^n(t,z) = \int_z^{L^n(t)} w^n_x(t,x)\ dx.$$

Integrating with respect to z and multiplying by $\eta(t) \in L^2(0,T)$, we find

$$(b-a)w^n(t, L^n(t))\eta(t) - \int_a^b w^n(t,z)\ dz\eta(t) = \int_a^b \int_z^{L^n(t)} w^n_x(t,x)\eta(t)\ dxdz.$$

Integrating with respect to t over $[0,T]$, and using the characteristic function on $[z, L^n(t)]$, along with Fubini's theorem (Theorem 4.5 [9, p. 91]), we obtain

$$\begin{aligned}(b-a)\int_0^T w^n(t, L^n(t))\eta(t)\ dt - \int_0^T \int_a^b w^n(t,z)\eta(t)\ dzdt \\ = \int_a^b \int_0^T \int_a^b \chi_{[z,L^n]}(x)\eta(t)w^n_x(t,x)\ dxdtdz.\end{aligned} \tag{2.4.69}$$

We now consider the right-hand side in (2.4.69). Recalling the weak convergence of $w^n_x\chi_{[a,L^n]}$ from Remark 2.4.21, we can conclude that for all $z \in [a,b]$, we have

$$\lim_{n\to\infty} \int_0^T \int_a^b \chi_{[z,L^n]}(x)\eta(t)w^n_x(t,x)\ dxdt = \int_0^T \int_a^b \chi_{[z,L]}(x)\eta(t)w_x(t,x)\ dxdt. \tag{2.4.70}$$

On the other hand, we have $\eta \in \mathrm{L}^2(0,T)$ and $\|w^n_x\|_{\mathrm{L}^2(0,T;\mathrm{L}^2(a,b))} \leq C_p$, by (2.4.58). Hence, we can use Hölder's inequality to see that $\int_0^T \int_a^b \chi_{[z,L^n]}\eta w^n_x\ dxdt$ is bounded for all n and for all $z \in [a,b]$. We can now apply Lebesgue's dominated convergence theorem to deduce

$$\lim_{n\to\infty} \int_a^b \int_0^T \int_a^b \chi_{[z,L^n]}(x)\eta(t)w^n_x(t,x)dxdtdz = \int_a^b \int_0^T \int_a^b \chi_{[z,L]}(x)\eta(t)w_x(t,x)dxdtdz. \tag{2.4.71}$$

Hence, we can take the limit in (2.4.69) and use (2.4.71) along with fact that $w^n \rightharpoonup w$ weakly in $\mathrm{L}^2(0,T;\mathrm{L}^2(a,b))$, by part $i)$ in Proposition 2.4.18, to obtain

$$\begin{aligned}\lim_{n\to\infty}(b-a)\int_0^T w^n(t, L^n(t))\eta(t)\ dt = \int_a^b \int_0^T w(t,z)\eta(t)\ dzdt \\ + \int_a^b \int_0^T \int_a^b \chi_{[z,L]}(x)\eta(t)w_x(t,x)\ dxdtdz.\end{aligned} \tag{2.4.72}$$

The right-hand side in (2.4.72) is nothing but $(b-a)\int_0^T w(t,L(t))\eta(t)dt$. Therefore, from (2.4.72) we obtain

$$\lim_{n\to\infty}\int_0^T w^n(t,L^n(t))\eta(t)\ dt = \int_0^T w(t,L(t))\eta(t)\ dt, \tag{2.4.73}$$

which implies that $w^n(t,L^n(t)) \rightharpoonup w(t,L(t))$ weakly in $\mathrm{L}^2(0,T)$, as claimed.

In the next step, we show that $\lim_{n\to\infty}\|w^n(t,L^n(t))\|_{\mathrm{L}^2(0,T)} = \|w(t,L(t))\|_{\mathrm{L}^2(0,T)}$. To do so, we follow a similar idea to the previous part and write, for all $z\in[a,b]$,

$$\begin{aligned}(b-a)\int_0^T \big(w^n(t,L^n(t))\big)^2\ dt - \int_0^T\int_a^b \big(w^n(t,z)\big)^2\ dzdt \\ = 2\int_a^b\int_0^T\int_a^b \chi_{[z,L^n]}(x)w^n(t,x)w^n_x(t,x)\ dxdtdz.\end{aligned} \tag{2.4.74}$$

Looking at the right-hand side in (2.4.74), we note that for each z, we have $\chi_{[z,L^n]}w^n \to \chi_{[z,L]}w$ strongly in $\mathrm{L}^2(0,T;\mathrm{L}^2(a,b))$, by Lemma 2.4.22, and $w^n_x \rightharpoonup w_x$ weakly in $\mathrm{L}^2(0,T;\mathrm{L}^2(a,b))$, by part *i)* of Proposition 2.4.18. Hence, we can take the point-wise limit of $\int_0^T\int_a^b \chi_{[z,L^n]}w^n w^n_x\ dxdt$ for each $z\in[a,b]$. In addition, using Young's inequality and recalling (2.4.58) in the proof of Proposition 2.4.18, we can see that $\int_0^T\int_z^{L^n} w^n w^n_x\ dxdt$ is uniformly bounded for all z and for all n. Hence, we can apply Lebesgue's dominated convergence theorem for the right-hand side of (2.4.74) and use the strong convergence of $w^n\to w$ in $\mathrm{L}^2(0,T;\mathrm{L}^2(a,b))$, from part *ii)* of Proposition 2.4.18, for the second term on the left-hand side of (2.4.74), to obtain

$$\begin{aligned}\lim_{n\to\infty}&(b-a)\int_0^T \big(w^n(t,L^n(t))\big)^2\ dt \\ &= \int_0^T\int_a^b \big(w(t,z)\big)^2\ dzdt + 2\int_a^b\int_0^T\int_a^b \chi_{[z,L]}(x)w(t,x)w_x(t,x)\ dxdtdz \\ &= \int_0^T (b-a)\big(w(t,L(t))\big)^2\ dt,\end{aligned}$$

which implies

$$\lim_{n\to\infty}\|w^n(t,L^n(t))\|_{\mathrm{L}^2(0,T)} = \|w(t,L(t))\|_{\mathrm{L}^2(0,T)}. \tag{2.4.75}$$

As a result, applying a standard result (see [9, p.124]) from the weak convergence (2.4.73) and (2.4.75), we can deduce that $w^n(t,L^n(t)) \to w(t,L(t))$ strongly in $\mathrm{L}^2(0,T)$. ∎

Lemma 2.4.26. *Let L, $\{L^n\}$, w and $\{w^n\}$ be as in Remark 2.4.17. We have*

$$\int_0^T L^{n\prime}(t)\big(w^n(t,L^n(t))\big)^2 dt \to \int_0^T L'(t)\big(w(t,L(t))\big)^2 dt.$$

Proof: We proceed analogously to the proof of the previous lemma. For $z \in [a,b]$, we write

$$L^{n\prime}(t)\big(w^n(t,L^n(t))\big)^2 - L^{n\prime}(t)\big(w^n(t,z)\big)^2 = L^{n\prime}(t)\int_z^{L^n(t)} \frac{\partial}{\partial x}\Big(\big(w^n(t,x)\big)^2\Big)\,dx.$$

Integrating with respect to z and t, we find

$$\begin{aligned}(b-a)\int_0^T L^{n\prime}(t)\big(w^n(t,L^n(t))\big)^2\,dt - \int_0^T\int_a^b \big(w^n(t,z)\big)^2\,dz L^{n\prime}(t)dt \\ = 2\int_a^b\int_0^T\int_a^b \chi_{[z,L^n]}(x)L^{n\prime}(t)w^n(t,x)w^n_x(t,x)\,dxdtdz.\end{aligned} \tag{2.4.76}$$

Considering the right-hand side in (2.4.76), we claim that

$$L^{n\prime}(t)w^n(t,x) \to L'(t)w(t,x) \qquad \text{strongly in} \quad \mathrm{L}^2(0,T;\mathrm{L}^2(a,b)). \tag{2.4.77}$$

Indeed, we can first write

$$\begin{aligned}\int_0^T\int_a^b \Big(L^{n\prime}(t)\big(w^n(t,x)\Big)^2\,dxdt &= \int_0^T \big(L^{n\prime}(t)\big)^2\|w^n(t,\cdot)\|^2_{\mathrm{L}^2(a,b)}dt \\ = \int_0^T \Big(\big(L^{n\prime}(t)\big)^2 - \big(L'(t)\big)^2\Big)\|w^n(t,\cdot)\|^2_{\mathrm{L}^2(a,b)} &+ \big(L'(t)\big)^2\|w^n(t,\cdot)\|^2_{\mathrm{L}^2(a,b)}\,dt.\end{aligned} \tag{2.4.78}$$

Taking the limit in (2.4.78), we can see that the limit of the first term on the right-hand side of (2.4.78) is zero as $L^{n\prime} \to L'$ strongly in $\mathrm{L}^2(0,T)$ and $\{w^n\}$ is bounded in $\mathrm{L}^\infty(0,T;\mathrm{L}^2(a,b))$. For the second term on the right-hand side of (2.4.78), we note that by part $v)$ in Proposition 2.4.18, we have that $\|w^n(t,\cdot)\|^2_{\mathrm{L}^2(a,b)}$ converges in the weak* topology to $\|w(t,\cdot)\|^2_{\mathrm{L}^2(a,b)}$. Then, having $L'(t) \in \mathrm{L}^2(0,T)$, we can conclude

$$\begin{aligned}\lim_{n\to\infty}\int_0^T\int_a^b \Big(L^{n\prime}(t)w^n(t,x)\Big)^2\,dxdt &= \int_0^T \big(L'(t)\big)^2\|w(t,\cdot)\|^2_{\mathrm{L}^2(a,b)}\,dt \\ &= \int_0^T\int_a^b \Big(L'(t)w(t,x)\Big)^2\,dxdt,\end{aligned} \tag{2.4.79}$$

which yields (2.4.77), as we have $L^{n\prime}(t)w^n(t,x) \to L'(t)w(t,x)$ a.e. in $[0,T]\times[a,b]$ as well. We now obtain from the strong convergence in (2.4.77) and the weak convergence of $\chi_{[a,L^n]}w^n_x \rightharpoonup \chi_{[a,L]}w_x$ in $\mathrm{L}^2(0,T;\mathrm{L}^2(a,b))$, from part $vi)$ of Proposition 2.4.18, that $z \in [a,b]$, we have

$$\begin{aligned}\lim_{n\to\infty}\int_0^T\int_a^b \chi_{[z,L^n]}(x)w^n(t,x)L^{n\prime}(t)w^n_x(t,x)\,dxdt \\ = \int_0^T\int_a^b \chi_{[z,L]}(x)w(t,x)L'(t)w_x(t,x)\,dxdt.\end{aligned} \tag{2.4.80}$$

Moreover, we can use the bounds (2.4.57) and (2.4.58) on $\{w^n\}$ and the fact that $\{L^n\}$ is bounded in $\mathrm{H}^1(0,T)$ to deduce from Hölder's inequality that

$$\int_0^T \int_a^b \chi_{[z,L^n]}(x) w^n(t,x) L^{n\prime}(t) w_x^n(t,x)\ dxdt \ \leq \ \int_0^T L^{n\prime}(t) \|w^n\|_{\mathrm{L}^2(a,b)} \|w_x^n\|_{\mathrm{L}^2(a,b)} dt$$
$$\leq \|w^n\|_{\mathrm{L}^\infty(0,T;\mathrm{L}^2(a,b))} \|L^{n\prime}\|_{\mathrm{L}^2(0,T)} \|w_x^n\|_{\mathrm{L}^2(0,T;\mathrm{L}^2(a,b))} \leq C \tag{2.4.81}$$

for all n, $z \in [a,b]$ and some appropriate constant C. Therefore, we can now apply Lebesgue's dominated convergence theorem to conclude from (2.4.80) and (2.4.81) that for all $z \in [a,b]$, we have

$$\lim_{n\to\infty} \int_a^b \int_0^T \int_a^b \chi_{[z,L^n]}(x) w^n(t,x) L^{n\prime}(t) w_x^n(t,x)\ dxdtdz$$
$$= \int_a^b \int_0^T \int_a^b \chi_{[z,L]}(x) w(t,x) L'(t) w_x(t,x)\ dxdtdz. \tag{2.4.82}$$

On the other hand, we can calculate

$$\lim_{n\to\infty} \int_0^T \int_a^b \big(w^n(t,z)\big)^2 dz L^{n\prime}(t)\ dt = \lim_{n\to\infty} \int_0^T \|w^n(t,\cdot)\|^2_{\mathrm{L}^2(a,b)} L^{n\prime}(t)\ dt$$
$$= \int_0^T \|w(t,\cdot)\|^2_{\mathrm{L}^2(a,b)} L'(t)\ dt, \tag{2.4.83}$$

since $\|w^n(t,\cdot)\|^2_{\mathrm{L}^2(a,b)} \to \|w(t,\cdot)\|^2_{\mathrm{L}^2(a,b)}$ strongly in $\mathrm{L}^2(0,T)$ by part *iv)* of Proposition 2.4.18, and $L^{n\prime} \to L'$ in $\mathrm{L}^2(0,T)$. As a result, we can take the limit in (2.4.76) and use (2.4.82) and (2.4.83) to obtain

$$(b-a) \lim_{n\to\infty} \int_0^T L^{n\prime}(t) \big(w^n(t, L^n(t))\big)^2\ dt = \int_0^T \int_a^b \big(w(t,z)\big)^2\ dz L'(t) dt$$
$$+ 2 \int_a^b \int_0^T \int_a^b \chi_{[z,L]}(x) L'(t) w(t,x) w_x(t,x)\ dxdtdz$$
$$= (b-a) \int_0^T L'(t) \big(w(t,L(t))\big)^2\ dt,$$

which proves the lemma. ∎

Lemma 2.4.27. *Let L, $\{L^n\}$, w and $\{w^n\}$ be as in Remark 2.4.17. We have*

$$\liminf_{n\to\infty} \int_0^T \varphi_n^t(w^n)\ dt \geq \int_0^T \varphi^t(w)\ dt. \tag{2.4.84}$$

Proof: Recalling the definition of $\varphi_n^t(w^n)$ from (2.4.7) with L^n, we can write

$$\begin{aligned}\liminf_{n\to\infty}\int_0^T \varphi_n^t(w^n)\ dt = \liminf_{n\to\infty}\int_0^T \Bigg(&\frac{1}{2}\big(D_1\|w_x^n\chi_{[a,L^n]}\|^2_{\mathrm{L}^2(a,b)} + \|w^n\chi_{[a,L^n]}\|^2_{\mathrm{L}^2(a,b)}\big)\\ &+\frac{1}{2k}\big(D_2\|w_x^n\chi_{[L^n,b]}\|^2_{\mathrm{L}^2(a,b)} + \|w^n\chi_{[L^n,b]}\|^2_{\mathrm{L}^2(a,b)}\big)\\ &+\frac{k-1}{k}\frac{L^{n\prime}(t)}{2}\big(w^n(t,L^n(t))\big)^2\Bigg)\ dt.\end{aligned} \tag{2.4.85}$$

Having the convergence of $\int_0^T L^{n\prime}(t)\big(w^n(t,L^n(t))\big)^2 dt$ from Lemma 2.4.26 and the weak convergence of $w^n\chi_{[a,L^n]}$, $w^n\chi_{[a,L^n]}$ and $w_x^n\chi_{[a,L^n]}$, $w_x^n\chi_{[a,L^n]}$ in $\mathrm{L}^2(0,T;\mathrm{L}^2(a,b))$ from Lemma 2.4.20, we can take the limit in (2.4.85) and apply weak lower semicontinuity of the norm to obtain

$$\begin{aligned}\liminf_{n\to\infty}\int_0^T \varphi_n^t(w^n)dt \geq \int_0^T \Bigg(&\frac{1}{2}\big(D_1\|w_x\chi_{[a,L]}\|^2_{\mathrm{L}^2(a,b)} + \|w\chi_{[a,L]}\|^2_{\mathrm{L}^2(a,b)}\big)\\ +\frac{1}{2k}\big(D_2\|w_x\chi_{[L,b]}\|^2_{\mathrm{L}^2(a,b)} + \|w\chi_{[L,b]}\|^2_{\mathrm{L}^2(a,b)}\big) &+ \frac{k-1}{k}\frac{L'(t)}{2}\big(w(t,L(t))\big)^2\Bigg)dt\\ &= \int_0^T \varphi^t(w)dt,\end{aligned}$$

as desired. ∎

Lemma 2.4.28. *Let w_0, L, $\{L^n\}$, w and $\{w^n\}$ be as in Remark 2.4.17. For $t\in[0,T]$, we have*

$$\liminf_{n\to\infty}\|w^n(t,\cdot)\|^2_{H_n} \geq \|w(t,\cdot)\|^2_H.$$

Proof: As mentioned in the proof of Proposition 2.4.18, for all n, we have $\|w_0\|_{H_n} = \|w_0\|_H$, and by (2.4.57) for all $t\in[0,T]$ and for all n, we have

$$\|w^n(t,\cdot)\|^2_{H_n} = \|w^n(t,\cdot)\chi_{[a,L^n(t)]}\|^2_{\mathrm{L}^2(a,b)} + \frac{1}{k}\|w^n(t,\cdot)\chi_{[L^n(t),b]}\|^2_{\mathrm{L}^2(a,b)} \leq C_h. \tag{2.4.86}$$

Then by reflexivity of $\mathrm{L}^2(a,b)$ along with part $vi)$ in Proposition 2.4.18, we can assert that up to a subsequence, again denoted by w^n, and for each $t\in[0,T]$ we have $w^n(t,\cdot)\chi_{[a,L^n(t)]} \rightharpoonup w(t,\cdot)\chi_{[a,L(t)]}$ in $\mathrm{L}^2(a,b)$. Consequently, similar to the proof of the previous lemma, we can apply weak lower semicontinuity of the norm in (2.4.86) to complete the proof. ∎

Lemma 2.4.29. *Let L, $\{L^n\}$, w, $\{w^n\}$, X_n, and X be as in Remark 2.4.17. Then we have $w_t \in X^*$, and*

$$\lim_{n\to\infty} \int_0^T \langle w_t^n, \eta\rangle_n \, dt = \int_0^T \langle w_t, \eta\rangle \, dt \qquad \textit{for} \quad \eta \in \mathrm{C}_0^1([0,T];\mathrm{C}^1[a,b]).$$

Proof: We first note that for all n, w^n is the strong solution to system ***(S)*** with conditions ***(BIC)*** (see Definition 2.4.1) and it satisfies the Cauchy problem (2.4.6) as well. Thus, for each $\eta \in \mathrm{L}^2(0,T,\mathrm{H}^1(a,b))$ we can write

$$\begin{aligned}\int_0^T \big(w_t^n, \eta\big)_H dt + \int_0^T \int_a^{L(t)} \big(D_1 w_x^n \eta_x + w^n \eta\big) dx dt + \frac{1}{k}\int_0^T \int_{L(t)}^b \big(w_x^n \eta_x + D_2 w^n \eta\big) dx dt \\ + \frac{k-1}{k}\int_0^T L'(t) w^n(t,L(t))\eta(t,L(t))\, dt \\ = \int_0^T \int_a^{L(t)} \big((r+1)w^n - q(w^n)^2\big)\,\eta\, dx + \frac{1}{k}\int_{L(t)}^b \big((1-m)w^n\big)\eta\, dx dt.\end{aligned} \tag{2.4.87}$$

Now, consider (2.4.87) with $\eta \in \mathrm{C}_0^1([0,T];\mathrm{C}^1[a,b])$. Having $w_t^n, w^n \in \mathrm{L}^2(0,T;H_n)$, we can use the integration by part formula (2.3.4) along with Lemma 2.4.20 and Lemma 2.4.22 to take the limit of equation (2.4.87) and obtain

$$\begin{aligned}\int_0^T -\big(w, \eta_t\big)_H dt + \int_0^T \int_a^{L(t)} \big(D_1 w_x \eta_x + w\eta\big)\, dx dt + \frac{1}{k}\int_0^T \int_{L(t)}^b \big(D_2 w_x \eta_x + w\eta\big)\, dx dt \\ = \int_0^T \int_a^{L(t)} \big((r+1)w - q(w)^2\big)\,\eta\, dx + \frac{1}{k}\int_{L(t)}^b \big((1-m)w\big)\eta\, dx dt.\end{aligned} \tag{2.4.88}$$

From Remark 2.4.19 we have $w \in X$. Hence, we can identify $\int_0^T -\langle w, \eta_t\rangle dt = \int_0^T -\big(w,\eta_t\big)_H\, dt$. As seen in the proof of Proposition 2.4.18, we have $w \in E_{2,\frac{4}{3}}$, that is $w \in \mathrm{L}^2(0,T;\mathrm{H}^1(a,b))$, and $w_t \in \mathrm{L}^{4/3}\big(0,T;(\mathrm{H}^1(a,b))^*\big)$. Hence, we can again use integration by parts in (2.4.88) for $\eta \in \mathrm{C}_0^1([0,T];\mathrm{C}^1[a,b])$ in the same way as in Remark 2.3.4 (see Theorem II.5.12 [8, p.99]), but this time for w and obtain

$$\begin{aligned}\int_0^T \langle w_t, \eta\rangle\, dt + \int_0^T \int_a^{L(t)} \big(D_1 w_x \eta_x + w\eta\big)\, dx dt + \frac{1}{k}\int_0^T \int_{L(t)}^b \big(w_x \eta_x + D_2 w\eta\big)\, dx dt \\ + \frac{k-1}{k}\int_0^T L'(t) w(t,L(t))\eta(t,L(t))\, dt \\ = \int_0^T \int_a^{L(t)} \big((r+1)w - q(w)^2\big)\,\eta\, dx + \frac{1}{k}\int_{L(t)}^b \big((1-m)w\big)\eta\, dx\, dt.\end{aligned} \tag{2.4.89}$$

Due to a density argument, equation (2.4.89) also holds for all $\eta \in X$. Then having $w \in X$, we can proceed as in Lemma 2.4.14 to conclude that $w_t \in X^*$ as well.

Now, for $\eta \in \mathrm{C}_0^1([0,T];\mathrm{C}^1[a,b])$, we can apply integration by part (2.3.4) in Remark 2.3.4 and use Lemma 2.4.20 and similar argument as in the proof of Lemma (2.4.26) to obtain

$$\begin{aligned}
&\lim_{n\to\infty} \int_0^T \langle w_t^n, \eta\rangle_n \; dt \\
&\qquad = \lim_{n\to\infty} \left(-\int_0^T \langle w^n, \eta_t\rangle_n \; dt - \frac{k-1}{k} \int_0^T L^{n\prime}(t) w(t, L^n(t)) \eta(t, L^n(t)) \; dt \right) \\
&\qquad = -\int_0^T \langle w, \eta_t\rangle \; dt - \frac{k-1}{k} \int_0^T L'(t) w(t, L(t)) \eta(t, L(t)) \; dt \\
&\qquad = \int_0^T \langle w_t, \eta\rangle \; dt \qquad \text{for} \quad \eta \in \mathrm{C}_0^1([0,T];\mathrm{C}^1[a,b]),
\end{aligned}$$

as desired. ∎

Lemma 2.4.30. *Let L, $\{L^n\}$, w, $\{w^n\}$, X_n, and X be as in Remark 2.4.17. We have*

$$\liminf_{n\to\infty} \int_0^T \langle w^n, w_t^n\rangle_n \; dt \geq \int_0^T \langle w, w_t\rangle \; dt.$$

Proof: Since $w^n \in X$ and $w_t^n \in X_n^*$, we can apply the integration by part formula in Remark 2.3.4 along with Lemma 2.4.28 and Lemma 2.4.26 to obtain

$$\begin{aligned}
&\liminf_{n\to\infty} \int_0^T \langle w^n, w_t^n\rangle_n dt \\
&\qquad = \liminf_{n\to\infty} \left(\frac{1}{2}\|w^n(T,\cdot)\|_{H_n}^2 - \frac{1}{2}\|w_0\|_{H_n}^2 - \frac{k-1}{2k} \int_0^T L^{n\prime}(t) \big(w(t, L^n(t))\big)^2 \right) \\
&\qquad \geq \frac{1}{2}\|w(T,\cdot)\|_H^2 - \frac{1}{2}\|w_0\|_H^2 - \frac{k-1}{2k} \int_0^T L'(t)\big(w(t, L(t))\big)^2 \\
&\qquad = \int_0^T \langle w, w_t\rangle \; dt.
\end{aligned}$$

The last term makes sense as $w \in X$ and $w_t \in X^*$ by the previous lemma. ∎

2.4.6 Existence of the Local Solution for a Given Curve $L \in \mathrm{H}^1(0,T)$

To prove the existence of a local solution for a curve $L \in \mathrm{H}^1(0,T)$, it remains to show that the function w, found in Proposition 2.4.18, is actually the weak solution to system ***(S)*** and conditions ***(BIC)*** corresponding to the curve L.

Proposition 2.4.31. *Let $T > 0$ and consider a curve $L \in \mathrm{H}^1(0,T)$ satisfying assumption* (2.4.1)*, and initial condition $w_0 \in \mathrm{H}^1(a,b)$, with $w_0(x) \geq 0$ for $x \in [a,b]$. Then there exists a weak solution w, corresponding to the curve L, to system **(S)**,* (2.2.10)–(2.2.11)*, with conditions **(BIC)**,* (2.2.12)–(2.2.17)*, in $[0,T]$, that satisfies properties $ii.1$) to $ii.2$) in Definition 2.4.1.*

Proof: First note that by Lemma 2.4.16, there exists a sequence $\{L^n\}$ in $\mathrm{H}^3(0,T)$, satisfying assumption (2.4.1) such that $L^n \to L$ strongly in $\mathrm{H}^1(0,T)$. For all n, let w^n be the unique strong solution corresponding to L^n for system ***(S)***, (2.2.10)–(2.2.11) with conditions ***(BIC)***, (2.2.12)–(2.2.17). As seen in Proposition 2.4.18, there exists a function w such that $w^n \to w$ for a.e. $(t,x) \in [0,T] \times [a,b]$.

We need to show that w is the weak solution to system ***(S)***, (2.2.10)–(2.2.11) with conditions ***(BIC)***, (2.2.12)–(2.2.17) corresponding to the curve L in the sense of part ii) of Definition 2.4.1.

Note that for each n, the solution w^n has to satisfy the equivalent abstract nonlinear evolution equation (2.4.6) as well. Therefore, using the definition of the subdifferential for φ_n^t for each n we find for a.e. $t \in [0,T]$

$$\varphi_n^t(z) - \varphi_n^t(w^n) \geq \big(- w_t^n - B(t,w^n), z - w^n \big)_{H_n}, \quad \text{for all } z(t,.) \in H^1(a,b),$$

which is equivalent to (see [36, Proposition 1.1]),

$$\int_0^T \big(\varphi_n^t(z) - \varphi_n^t(w^n)\big)\, dt \geq \int_0^T \left(-w_t^n - B(t,w^n), z - w^n\right)_{H_n} dt \tag{2.4.90}$$

for all $z \in \mathrm{L}^2(0,T;\mathrm{H}^1(a,b))$. We note that as $X_n \hookrightarrow \mathrm{L}^2(0,T,\mathrm{H}^1(a,b))$, (2.4.90) is valid for $z \in X_n$ as well. Now, using the bounds (2.4.57), (2.4.59) and (2.4.58) on $\{w^n\}$, and the fact that these bounds do not depend on n, we can take the limit as $n \to \infty$ and apply Proposition 2.4.5 to conclude that w satisfies

$$\int_0^T \big(\varphi^t(z) - \varphi^t(w)\big)\, dt \geq \int_0^T \langle -w_t - B(t,w), z - w \rangle\, dt, \quad \text{for all } z \in X. \tag{2.4.91}$$

Indeed, consider functions $z \in \mathrm{C}_0^1([0,T];\mathrm{C}^1[a,b])$. Note that for each $t \in [0,T]$, we have $z(t,.) \in \mathcal{D}(\varphi^t(w)) = \mathrm{H}^1(a,b)$. With this choice of z, showing that (2.4.90), for $z \in X_n$, approaches (2.4.91) as n approaches ∞, is equivalent to showing that w

satisfies property $ii.1$) of a weak solution in Definition 2.4.1. We start by calculating $\lim_{n\to\infty}\int_0^T \varphi_n^t(z)$ in (2.4.90). Since $z \in \mathrm{C}_0^1([0,T];\mathrm{C}^1[a,b])$, we can apply similar ideas as in the proofs of Lemmas 2.4.20 and 2.4.26 to conclude

$$\lim_{n\to\infty}\int_0^T \varphi_n^t(z)dt \;=\; \int_0^T \varphi^t(z)dt. \tag{2.4.92}$$

In addition, considering the nonlinear terms on the right-hand side of (2.4.90), we calculate

$$\begin{aligned}\liminf_{n\to\infty}\int_0^T \langle -B(t,w^n),(z-w^n)\rangle_n \;&= \liminf_{n\to\infty}\int_0^T \Bigg(\int_a^{L^n(t)} (r+1)w^n z - q(w^n)^2 z\; dxdt \\ &\qquad +\frac{1}{k}\int_{L^n(t)}^b (1-m)w^n z\; dx\Bigg)dt \\ &\quad +\int_0^T \Bigg(\int_a^{L^n(t)} -(r+1)(w^n)^2 + q(w^n)^3\; dx \\ &\qquad -\frac{1}{k}\int_{L^n(t)}^b (1-m)(w^n)^2\; dx\Bigg)dt.\end{aligned} \tag{2.4.93}$$

Therefore, we can apply Lemmas 2.4.20, 2.4.22 and 2.4.24 to deduce from (2.4.93) that

$$\liminf_{n\to\infty}\int_0^T \langle -B(t,w^n),(z-w^n)\rangle_n dt \ge \int_0^T \langle -B(t,w),(z-w)\rangle\, dt. \tag{2.4.94}$$

Now, looking at the limit of the term in (2.4.90), we obtain from (2.4.92) and (2.4.94) along with the results in Lemmas 2.4.27, 2.4.29 and 2.4.30 that the function w satisfies

$$\begin{aligned}\int_0^T \varphi_n^t(z) - \varphi^t(w)dt \;&\ge\; \liminf_{n\to\infty}\int_0^T \varphi_n^t(z) - \varphi_n^t(w^n)dt \\ &\ge\; \liminf_{n\to\infty}\int_0^T \langle -w_t^n - B(t,w^n), z-w^n\rangle_n\, dt \\ &\ge\; \int_0^T \langle -w_t - B(t,w), z-w\rangle\, dt\end{aligned} \tag{2.4.95}$$

for $z \in \mathrm{C}_0^1(0,T;\mathrm{C}^1[a,b])$. Finally, since $\mathrm{C}_0^1(0,T;\mathrm{C}^1[a,b])$ is dense in X, we obtain (2.4.91). Similarly to the proof of Proposition 2.4.5, from (2.4.91) we can see that w satisfies the property $ii.1$) of the weak solution in Definition 2.4.1. In addition, by Remark 2.3.4, we have $w \in \mathrm{C}([0,T];H)$.

In remains to show that w satisfies the initial conditions, property $ii.2$) of a weak solution in Definition 2.4.1. As shown in the proof of Proposition 2.4.5, in (2.4.90) we can choose $z = w^n + \lambda\eta$ with $\lambda > 0$ small and $\eta \in \mathrm{C}^1\big([0,T];\times[a,b]\big)$ with $\eta(T,\cdot) = 0$.

Since $w_t^n \in \mathrm{L}^2(0,T;\mathrm{L}^2(a,b))$, we can apply integration by part as in Remark 2.3.4, to show that (2.4.90) is equivalent to

$$\begin{aligned} -\big(w_0, \eta(0,.)\big)_H - \int_0^T \langle w^n, \eta_t\rangle_n \; dt + \int_0^T \int_a^{L^n(t)} \big(D_1 w_x^n \eta_x + w^n \eta\big) \; dxdt \\ + \frac{1}{k}\int_0^T \int_{L^n(t)}^b \big(D_2 w_x^n \eta_x + w^n \eta\big) \; dxdt \\ = \int_0^T \int_a^{L^n(t)} F_1(w^n)\eta \; dx + \frac{1}{k}\int_{L^n(t)}^b F_2(w^n)\eta \; dxdt \end{aligned} \tag{2.4.96}$$

for all n, as we have $w^n(0,\cdot) = w_0$ for all n. Similarly, recalling Remark 2.3.4 and the fact that $w_t \in X^*$, we can apply integration by part to (2.4.91) with $z = w + \lambda\eta$ and η as above to get

$$\begin{aligned} -\big(w(0,\cdot), \eta(0,\cdot)\big)_H - \int_0^T \langle w, \eta_t\rangle_H \; dt + \int_0^T \int_a^{L(t)} \big(D_1 w_x \eta_x + w\eta\big) dxdt \\ + \frac{1}{k}\int_0^T \int_{L(t)}^b \big(D_2 w_x \eta_x + w \eta\big) \; dxdt \\ = \int_0^T \int_a^{L(t)} F_1(w)\eta \; dx + \frac{1}{k}\int_{L(t)}^b F_2(w)\eta \; dxdt. \end{aligned} \tag{2.4.97}$$

Similarly to the proof of Lemma 2.4.29, we can take the limit in (2.4.96), using Lemmas 2.4.20 and 2.4.22, and compare the outcome with (2.4.97) to obtain that $\big(w_0, \eta(0,\cdot)\big)_H = \big(w, \eta(0,\cdot)\big)_H$, which implies $w(0,x) = w_0(x)$ for $x \in [a,b]$, as $\eta(0,\cdot)$ was arbitrary. Therefore, we have shown that w is the weak solution to the system ***(S)*** with initial and boundary condition ***(BIC)*** corresponding to curve $L \in \mathrm{H}^1(0,T)$ in the sense of Definition 2.4.1. ∎

Remark 2.4.32. *Non-negativity and uniqueness of the weak solution can be established similar to the proofs of Lemmas 2.4.9 and 2.4.11.*

Remark 2.4.33. *Using the the convergence result in Proposition 2.4.18 and the definition of the weak solution in Definition 2.4.1, it is clear that any weak solution corresponding to the curve $L \in \mathrm{H}^1(0,T)$ also satisfies the estimates of Lemma 2.4.12.*

2.5 Existence of a Solution for the Free Boundary Problem

In this part, we prove that there exists a function $L(t)$ for which the solution $w(t,x)$ found in Proposition 2.4.31 satisfies the condition on the moving interface ***(MC)***

(2.2.18), along with the the rest of boundary and interface conditions ***(BIC)***, (2.2.12)–(2.2.17). To do so, we apply the method analogous to Section 3 in [22, p. 923] and use Schauder's fixed point theorem to an appropriate operator.

Recall that the condition on the moving boundary is

$$L'(t) = 2\eta_1 D_1 \alpha w(t, L(t)), \quad \text{for } t \in [0, T],$$

where η_1, D_1 and α are given constants described in Section 1.6. Now, let R be some positive constant and define

$$T = \min\left\{\frac{\delta^2}{R^2}, \frac{R^2(b-a)}{8\eta_1^2 D_1^2 \alpha^2 C_h^2}, \frac{R^4}{256\eta_1^4 D_1^4 \alpha^4 C_h^2 C_p^2}\right\} > 0,$$
$$\delta = \min\left\{\frac{l_0 - a}{2}, \frac{b - l_0}{2}\right\}, \tag{2.5.1}$$

where $C_h = C_h(r, m, \|w_0\|_H, T^*)$ and $C_p = C_p(r, m, \|w_0\|_H, T^*)$ are the bounds found in Lemma 2.4.12 on $\|w\|_{\mathrm{L}^\infty(0,T^*;H)}$ and $\varphi^t(w)$, respectively, for some fixed $T^* > T$. Let $\mathcal{B} = \{r \in L^2(0,T);\ \|r\|_{L^2(0,T)} \le R\}$ be a closed ball in $L^2(0,T)$. Consider the set

$$\mathcal{B}^+ = \{r \in \mathcal{B};\ r(t) \ge 0, \quad \text{for} \quad t \in [0,T]\}. \tag{2.5.2}$$

We now introduce the operator $\mathcal{H}$ accordingly.

Definition 2.5.1. *For any $r(\cdot) \in \mathcal{B}^+$, we define the operator $\mathcal{H}(r)(\cdot)$ as follows:*

$$\mathcal{H}(r)(t) = 2\eta_1 D_1 \alpha w(t, L(t)) \qquad \text{for a.e.} \quad t \in [0,T],$$

*where $L(t) = l_0 + \int_0^t r(\tau)d\tau$ and w is the weak solution to system **(S)**,* (2.2.10)–(2.2.11)*, with conditions **(BIC)**,* (2.2.12)–(2.2.17) *corresponding to the curve L and initial condition w_0, satisfying* (2.4.2)*.*

Remark 2.5.2. *The curve L in Definition 2.5.1 satisfies assumption* (2.4.1)*. Indeed, we have $L(0) = l_0$, and since $T > 0$ is small enough, that is $T < \delta^2/R^2$, we get $L(t) \in \mathrm{H}^1(0,T)$ and $a + \delta < L(t) < b - \delta$ for all $t \in [0,T]$, as we can calculate*

$$|L(t) - l_0| = \int_0^t r(\tau)d\tau \le T^{\frac{1}{2}}\|r\|_{\mathrm{L}^2(0,T)} \le T^{\frac{1}{2}}R \le \delta.$$

*On the other hand, for $r \in \mathcal{B}^+$ we have $r(t) \ge 0$ on $[0,T]$, which implies $L'(t) \ge 0$ as required. Therefore, by Proposition 2.4.31, there exists a solution w to system **(S)** with conditions **(BIC)** corresponding to the curve $L = l_0 + \int_0^t r(\tau)d\tau$. As a result, the operator $\mathcal{H}$ in Definition 2.5.1 is well-defined.*

To show there exists a curve L, satisfying boundary condition (2.2.18), we can equivalently show that the operator $\mathcal{H}$, defined in Definition 2.5.1, has a fixed point. In order to do that, we will apply Schauder's fixed point theorem, for which we need the following lemmas about the properties of the operator $\mathcal{H}(r)(\cdot)$.

Lemma 2.5.3. *i)* $\mathcal{H}(r)(\cdot)$ *is measurable on* $[0,T]$.
ii) $\mathcal{H}(\mathcal{B}^+) \subset \mathcal{B}^+$.

Proof: The measurability of $\mathcal{H}$ follows from measurability of w and continuity of L. It remains to show $ii)$. Let $r \in \mathcal{B}^+$ and set $L(t) = l_0 + \int_0^t r(\tau)d\tau$. Then,

$$\int_0^T \Big(\mathcal{H}(r)(t)\Big)^2 dt \;\; \leq \;\; 4\eta_1^2 D_1^2 \alpha^2 \int_0^T \Big(w(t, L(t)\Big)^2 dt. \tag{2.5.3}$$

Note that for any $z \in [a,b]$ we can write

$$\Big(w(t, L(t))\Big)^2 = \Big(w(t,z)\Big)^2 + \int_z^{L(t)} 2w_x(t,x)w(t,x)dx.$$

Integrating with respect to z and t and applying Hölder's inequality, we obtain

$$\begin{aligned}
(b-a)\int_0^T \Big(w(t,L(t)\Big)^2 dt &= \int_0^T \int_a^{L(t)} \Big(w(t,z)\Big)^2 \, dzdt \\
&\quad + \int_0^T \int_a^b \int_z^{L(t)} 2w_x(t,x)w(t,x)\, dxdzdt \\
&\leq \int_0^T \|w\|_H^2 \, dt + 2\int_0^T \int_a^b \|w_x\|_H \|w\|_H dzdt \qquad (2.5.4)\\
&\leq \|w\|^2_{\mathrm{L}^\infty(0,T;H)} T + 2(b-a)\|w\|_{\mathrm{L}^\infty(0,T;H)} \int_0^T \|w_x\|_H \, dt \\
&\leq \|w\|^2_{\mathrm{L}^\infty(0,T;H)} T + 2(b-a)\|w\|_{\mathrm{L}^\infty(0,T;H)} \|w_x\|_{\mathrm{L}^2(0,T;H)} T^{\frac{1}{2}} \\
&\leq C_h^2 T + 2(b-a)C_h C_p T^{\frac{1}{2}}.
\end{aligned}$$

Hence, substituting (2.5.4) in (2.5.3), we find

$$\int_0^T \Big(\mathcal{H}(r)(t)\Big)^2 dt \;\; \leq \;\; 4\eta_1^2 D_1^2 \alpha^2 \Big(\frac{C_h^2}{b-a} T + 2C_h C_p T^{\frac{1}{2}}\Big). \tag{2.5.5}$$

As a result, recalling T from (2.5.1), we obtain

$$\int_0^T \Big(\mathcal{H}(r)(t)\Big)^2 dt \;\; \leq \;\; \frac{R^2}{2} + \frac{R^2}{2} \leq R^2,$$

and hence $\|\mathcal{H}(r)\|_{L^2(0,T)} \leq R$ for all $r \in \mathcal{B}^+$. On the other hand, as w is the solution to system ***(S)*** with boundary conditions ***(BIC)***, by Remark 2.4.32 we have that $w(t, L(t)) \geq 0$ for $t \in [0,T]$. Then, recalling that η_1, D_1 and α are non-negative constants, we can conclude that $\mathcal{H}(r)(\cdot) \geq 0$ for $r \in \mathcal{B}^+$. Therefore, we have that $\mathcal{H}(r)(\cdot) \in \mathcal{B}^+$ for all $r \in \mathcal{B}^+$ as desired. ∎

The next requirement for applying Schauder's fixed point theorem is that the operator $\mathcal{H}(r)(\cdot)$ be continuous and compact.

Lemma 2.5.4. *$\mathcal{H} : \mathcal{B}^+ \to \mathcal{B}^+$ is continuous and $H(\mathcal{B}^+) \subset \mathcal{B}^+$ is precompact in $\mathrm{L}^2(0,T)$.*

Proof: We first prove the compactness. For all n, let $r^n(t) \in \mathcal{B}^+$ and define $L^n(t) = l_0 + \int_0^t r^n(\tau)d\tau$. Therefore, for all n and for $t \in [0,T]$, we have $a+\delta \leq L^n(t) \leq b-\delta$ and $L^{n\prime}(t) \geq 0$. Since $\mathrm{L}^2(0,T)$ is reflexive and $\{r^n\}$ is a sequence in $\mathcal{B}^+$, there exists a subsequence, denoted again by $\{r^n\}$, such that $r^n(t)$ converges weakly to some $r(t) \in \mathcal{B}^+$ (see Corollary 3.22 [9, p. 71]). Set $L(t) = l_0 + \int_0^t r(\tau)d\tau$. As seen in Lemma 2.4.15, we can prove that $L^n \to L$ uniformly on $[0,T]$. For all n, let w^n be the unique weak solution of system ***(S)***, (2.2.10)–(2.2.11), with conditions ***(BIC)***, (2.2.12)–(2.2.17), corresponding to the curve L^n and initial condition w_0 satisfying (2.4.2). Applying Remark 2.4.33 and Proposition 2.4.18, we can show that there exists a function w such that $w^n \to w$ a.e. in $[0,T] \times [a,b]$. To show that $\mathcal{H}(r^n)$ converges strongly in $\mathcal{B}^+$, we can apply Lemma 2.4.25 and use the strong convergence of $w^n(t, L^n(t)) \to w(t, L(t))$ in $\mathrm{L}^2(0,T)$ to obtain

$$\mathcal{H}(r^n) \to 2\eta_1 D_1 \alpha w(t, L(t)) \quad \text{strongly in} \quad \mathrm{L}^2(0,T). \tag{2.5.6}$$

Hence, by the properties of w as the limit of $\{w^n\}$, we can conclude that the operator $\mathcal{H}$ is compact.

To prove continuity, we consider a sequence $\{r^n\}$ in $\mathcal{B}^+$ that converges strongly in $\mathrm{L}^2(0,T)$ to $r \in \mathcal{B}^+$. We follow a similar argument as above to conclude that the corresponding sequence of weak solutions $\{w^n\}$ converges to some function w a.e. in $[0,T] \times [a.b]$. In addition, having the strong convergence of $r^n \to r$ in $\mathrm{L}^2(0,T)$, which implies the strong convergence of the corresponding curves $L^{n\prime} \to L'$ in $\mathrm{L}^2(0,T)$, we can apply a similar argument as in the proof of Proposition 2.4.31 to conclude that w is the unique weak solution corresponding to L. It remains to show that $\mathcal{H}(r^n) \to \mathcal{H}(r)$ in $\mathrm{L}^2(0,T)$ But we can again use Lemma 2.4.25 as in (2.5.6) to conclude that

$$\lim_{n\to\infty} \int_0^T \Big(\mathcal{H}(r^n) - \mathcal{H}(r)\Big)^2 dt = 4\eta_1^2 D_1^2 \alpha^2 \lim_{n\to\infty} \int_0^T \Big(w^n(t, L^n(t)) - w(t, L(t))\Big)^2 dt = 0,$$

which implies the continuity of the map $\mathcal{H}$ as desired. ∎

Having all the required results, we proceed to prove Theorem 2.3.6, which is the main result of this chapter.

Proof of Theorem 2.3.6. By Lemma 2.5.4 we have that $\mathcal{H} : \mathcal{B}^+ \to \mathcal{B}^+$, is a continuous and compact mapping on the closed convex set $\mathcal{B}^+$. Hence, by Schauder's fixed point theorem (Theorem II.3.10 [8, p. 83]), we can conclude that there exists a function $r \in \mathcal{B}^+$ such that $\mathcal{H}(r)(t) = r(t)$ for $t \in [0,T]$ with T being small enough, as in (2.5.1). Therefore, the pair (L, w) with the curve $L(t) = l_0 + \int_0^t r(\tau)d\tau$ and the function $w(t,x)$, corresponding to the curve $L(t)$, which is given by Proposition 2.4.31, is the solution to system ***(S)***, (2.2.10)–(2.2.11), with boundary and initial conditions ***(BIC)***, (2.2.12)–(2.2.17), along with the moving boundary condition ***(MC)***, (2.2.18), in the sense of Definition 2.3.5. ∎

Chapter 3

Traveling Wave Solutions

In the previous chapter, we studied the existence of local in-time solutions for our model. In this chapter, we discuss the long-term behaviour of solutions for the model by studying traveling waves. Traveling wave solutions are classic tools for studying the behaviour of the system in the long term and good measures for predicting the spreading speed of a species in its environment [67]. The existence of traveling wave solutions for our model without Allee effect has been studied in [48]. Here, we study the existence of traveling wave solutions for a growth function with Allee effect. We start with a short survey of classic results regarding traveling wave solutions.

3.1 Preliminary

We first review some standard results on the subject of traveling wave solutions for reaction–diffusion equations from the classical papers of Fischer [26] and Kolmogorov, Petrovskii and Piskunov [40] as well as Hadeler and Rothe [29]. We then discuss the results in [48] for our model with a logistic growth function.

3.1.1 Traveling Waves for Reaction–Diffusion Equations

Consider a reaction–diffusion equation in one-dimensional space, namely

$$u_t - Du_{xx} = f(u), \qquad x \in \mathbb{R}, \quad t \in \mathbb{R}^+. \tag{3.1.1}$$

Traveling waves are nonconstant solutions of the form $u(t,x) = U(x-ct) = U(z)$ that connect two steady states of the ordinary differential equation $\dot{u} = f(u)$, i.e., two zeros of the function f [73]. For example, with $f(u) = u(1-u)$, a traveling wave has to satisfy

$$-cU' - DU'' = f(U), \tag{3.1.2}$$

and the conditions

$$U'(z) < 0 \quad \text{for} \quad z \in \mathbb{R}, \quad U(-\infty) = 1, \quad U(\infty) = 0, \tag{3.1.3}$$

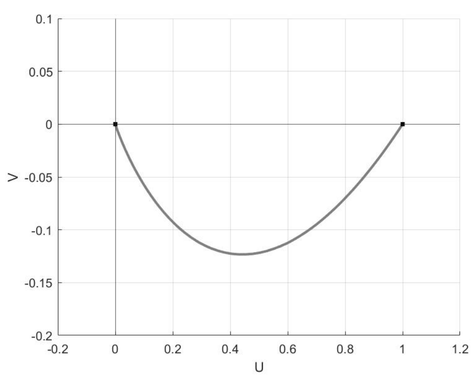
where $c \geq 0$ is the speed of the propagation front. The front here is the fixed spatial profile which is shifted in space by a factor of c for each time t. We can analyze this ODE in the phase plane by using the new variable $V = U'$. The system becomes

$$\begin{cases} U' = V, \\ V' = -\frac{c}{D}V - \frac{f(U)}{D}. \end{cases} \tag{3.1.4}$$

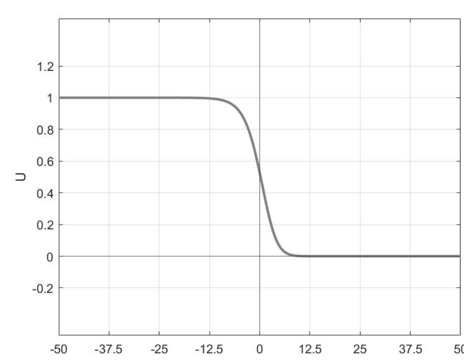
The system has two steady states: $(0,0)$ and $(1,0)$. Using linearization, we find the eigenvalues of the Jacobian of system (3.1.4) at these steady states to be

$$\lambda_i^{\pm}(c) = \frac{1}{2D}\left(-c \pm \sqrt{c^2 - 4Df'(i)}\right) \quad \text{for} \quad i = 0, 1. \tag{3.1.5}$$

For all values of c, the steady state $(1,0)$ is a saddle point. Depending on the value of c, $(0,0)$ can be an unstable node ($c \leq -2\sqrt{f'(0)D}$), an unstable focus ($-2\sqrt{f'(0)D} < c < 0$), a center ($c = 0$), a stable focus ($0 < c < 2\sqrt{f'(0)D}$), or a stable node ($c \geq 2\sqrt{f'(0)D}$). A traveling wave solution of (3.1.1) corresponds to a heteroclinic connection from the saddle $(1,0)$ to the stable point $(0,0)$ in (3.1.4), which can be either saddle-focus (for $0 < c < 2\sqrt{f'(0)D}$) or saddle-node (for $c \geq 2\sqrt{f'(0)D}$); see Figure 3.1.



(a)



(b)

Figure 3.1: Traveling wave solution for the reaction-diffusion model (3.1.1) with logistic growth: phase plane (a) and density function (b).

In [26], Fisher proved the existence of a traveling wave profile for (1.1.1), for all speeds $c \geq \hat{c}$, where $\hat{c} := 2\sqrt{rD} > 0$ is called the minimal speed. Note that $\hat{c}$ is the speed at which the behaviour of the steady state $(0,0)$ changes from a stable focus to a stable node. To have non-negative traveling wave solutions, which are biologically relevant, the saddle-focus connections are not accepted since they oscillate about $U = 0$. Therefore, the condition $c \geq \hat{c}$ is a necessary condition for the existence of nonnegative monotone traveling wave solutions for the Fisher-KPP equation (3.1.1). To see that it is also a sufficient condition, one can consider a trapping region (see Figure 3.1) and study the direction of the vector field on the edges of this region for $c > \hat{c}$. Then it can be shown that the unstable manifold of the saddle point $(1, 0)$ in the fourth quadrant cannot leave this region. Since there are no other steady states or periodic orbits in that region, the unstable manifold must converge to the stable steady state $(0, 0)$, see [41].

In the study of spatial spread, traveling wave solutions have been interpreted as the density of a species that occupies a considerable part of the environment and spreads with speed c. These solutions have been mainly applied to estimate the speed of invasion of a species, and $\hat{c}$ has been proven to be an appropriate approximation by experimental studies [44]. An interesting fact about $\hat{c}$ is that it does not depend on the intraspecific competition rate (denoted by q in the logistic nonlinearity) as long as there is no Allee effect.

As mentioned before, in the logistic growth function, the maximum per capita growth rate occurs at zero. Hence, the corresponding species does well at low densities. This results in traveling wave solutions of (3.1.1), known as pulled waves [69], with the spread being driven by the growth and dispersal occurring at the leading edge of the wave, where the density is lowest and the per capita growth rate is highest [43].

Equation (3.1.1) can be applied even more realistically to the phenomenon of a biological invasion. The typical scenario for the introduction of an invasive species is that it initiates locally and then spreads, which is not described completely by a traveling profile. To better describe this phenomenon, Aronson and Weinberger defined the notion of an asymptotic spreading speed [3]: a new species introduced to a limited part of an environment with population dynamics satisfying (3.1.1) has the asymptotic speed of spread $\hat{c}$ if an observer moving with speed c, eventually sees zero density of the species when $c > \hat{c}$, and eventually sees the population density at its carrying capacity when $c < \hat{c}$. Therefore, $\hat{c}$ for (3.1.1) can be expressed mathematically as follows:

$$\lim_{t\to\infty}\Big(\sup_{|x|\geq(\hat{c}+\epsilon)t} u(t,x)\Big) = 0 \quad \text{and} \quad \lim_{t\to\infty}\Big(\sup_{|x|\leq(\hat{c}+\epsilon)t} \Big|u(t,x) - \frac{r}{q}\Big|\Big) = 0, \qquad (3.1.6)$$

for all sufficiently small $\varepsilon > 0$ [44]. The interesting result here is that the asymptotic speed of spread of (3.1.1) is the same as the minimal speed of a traveling front, $\hat{c} = 2\sqrt{rD}$. One can show that the distribution of a spreading species starting from a local initial condition converges to a two-sided traveling front profile [75].

3.1.2 Traveling Wave Solution with Strong Allee Effect

As mentioned before, the logistic growth function is not realistic for many species, namely for those whose reproduction is reduced at low densities [2]. This effect, which is called an Allee effect (see Section 1.6), can result from difficulties in finding mates at low population densities and is quite common in many species [2, 11]. We can include this effect into model (3.1.1) by using a nonlinearity of the form (1.6).

In this subsection, we consider reaction–diffusion equation (3.1.1) when the nonlinearity is bistable, or in other words, has a strong Allee effect.

Consider a scaled version of (3.1.1) where $f(u) = u(u-a)(1-u)$ and $0 < a < 1$. A traveling wave solution $U(x - ct) = U(z)$ satisfies the following scaled system of first-order differential equations

$$\begin{cases} U' = V, \\ V' = -f(U) - cV. \end{cases} \tag{3.1.7}$$

This system has three steady states: $(0, 0), (1, 0)$, and $(a, 0)$. The eigenvalues of the Jacobian of system (3.1.7) at these steady states are

$$\lambda_i^{\pm}(c) = \frac{1}{2}\left(-c \pm \sqrt{c^2 - 4f'(i)}\right) \quad \text{for} \quad i = 0, a, 1. \tag{3.1.8}$$

For all values of c, the steady states $(0, 0)$ and $(1, 0)$ are saddle points. Depending on the value of c, $(a, 0)$ can be an unstable node ($c \leq -2\sqrt{f'(a)}$), an unstable focus ($-2\sqrt{f'(a)} < c < 0$), a center ($c = 0$), a stable focus ($0 < c < 2\sqrt{f'(a)}$), or a stable node ($c \geq 2\sqrt{f'(a)}$).

To find a nonnegative monotone traveling wave solution with properties defined in (3.1.3), we again need to find a heteroclinic connection between the steady states $(1, 0)$ and $(0, 0)$, both of which are saddle points. Saddle-saddle connections are rare, however, for this system, we can actually find an equation of the desired heteroclinic connection, which appears exactly for $c_0 := \sqrt{2}(1/2 - a)$ [41]. One can show that this traveling wave is unique (up to translation) [29].

As one can expect, the speed of this traveling wave solution c_0 depends on the Allee threshold, a, and it can be either positive, zero, or negative, which respectively corresponds to advancing, standing or retreating waves.

Traveling waves with a strong Allee effect are not pulled waves. For this growth type, the per capita growth rate at low densities is negative, and the highest per capita growth rate occurs for higher densities above the Allee threshold. Hence, the resulting traveling waves are driven by growth and dispersal behind the front, where the density is the higher. Since the population density and dispersal behind the front pushes the wave forward, such a wave is called pushed wave [69].

Aside from this unique traveling wave solution that connects the zero state with the carrying capacity at speed c_0, several other types of traveling wave solutions with different asymptotic densities can appear in a model with strong Allee effect. Hadeler and Rothe studied their existence when $0 < a \leq 1/2$ [29]. They define the following critical values of speed:

$$\begin{aligned} c_0 &:= \sqrt{2}\left(\frac{1}{2} - a\right), \\ c_1 &:= 2\sqrt{f'(a)} = 2\sqrt{a(1-a)}, \\ c_2 &:= \frac{(1+a)}{\sqrt{2}}, \\ c_m &:= \begin{cases} c_2 & \text{for} \quad 0 < a \leq \frac{1}{3}, \\ c_1 & \text{for} \quad \frac{1}{3} < a < \frac{1}{2}. \end{cases} \end{aligned} \tag{3.1.9}$$

Then we have

$$\begin{array}{lll} c_1 < c_0 < c_m & \text{for} & 0 < a < \bar{a}, \\ c_0 < c_1 < c_m & \text{for} & \bar{a} < a < \dfrac{1}{3}, \\ c_0 < c_1 = c_m & \text{for} & \dfrac{1}{3} < a < \dfrac{1}{2}, \end{array}$$

where $\bar{a} = (1 - \sqrt{2/3})/2$.

With this notation, Hadeler and Rothe proved the existence of different traveling wave solutions that appear for different ranges of c as summarized in the following theorem.

Theorem 3.1.1 (Hadeler and Rothe (1975)). *Let c_0, c_1, and c_m be as defined above, Then:*

i) For $c \geq c_m$, there is a monotonely decreasing front with boundary conditions $u(-\infty) = 1$, $u(+\infty) = a$.

ii) For $c > c_1$, there is a monotonely increasing front with boundary conditions $u(-\infty) = 0$, $u(+\infty) = a$.

iii) For $c = c_0$, there is a unique monotone front with boundary conditions $u(-\infty) = 1$, $u(+\infty) = 0$.

iv) For $0 < c < c_1$, there is an oscillating wave with boundary conditions $u(-\infty) = 0$, $u(+\infty) = a$.

v) For $c_0 < c < c_1$ (impossible for $0 < a < \bar{a}$), there is an oscillating front with $u(-\infty) = 1$, $u(+\infty) = a$.

vi) *For* $\max\{c_0, c_1\} < c < c_m$ *(impossible for* $a > \frac{1}{3}$*), there is a front with boundary conditions* $u(-\infty) = 1$ *and* $u(+\infty) = a$*, which decreases monotonely to some value* $u < a$*, then increases towards* a *for* $t \to +\infty$*.*

vii) *For* $c = 0$*, there are periodic solutions and a unique nonvanishing front with* $u(-\infty) = u(+\infty) = 0$*.*

We will use the above theorem to determine the types of waves that appear in our model.

3.1.3 Traveling Wave Solutions of a Free Boundary Problem for Ecosystem Engineers with Logistic Growth

To study traveling wave solutions, we formulate the model from Section 1.6 on the whole real line. For a function $u : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$ with a free boundary $L : \mathbb{R}^+ \to \mathbb{R}$, we have

$$\begin{cases} u_t = D_1 u_{xx} + f(u), \quad t > 0, \quad x < L(t), & (3.1.10) \\ u_t = D_2 u_{xx} - mu, \quad t > 0, \quad x > L(t), & (3.1.11) \end{cases}$$

where $D_i > 0$ are the diffusion rates, $f(u)$ and $-mu$ are reaction terms, and the matching conditions

$$\begin{cases} \alpha_2 D_1 u(t, L(t)^-) = \alpha_1 D_2 u(t, L(t)^+), & (3.1.12) \\ \Big(D_1 u_x + L'(t) u\Big)\Big|_{(t,L(t)^-)} = \Big(D_2 u_x + L'(t) u\Big)\Big|_{(t,L(t)^+)}. & (3.1.13) \end{cases}$$

The moving boundary follows the equation

$$L'(t) = B(u, u_x)\Big|_{(t,L(t)^-)}, \tag{3.1.14}$$

where B can be one of the equations (1.6.9), (1.6.10) or (1.6.11). Only scenario 2 (1.6.10) allows $L'(t) < 0$.

Since movement in favorable habitat is typically slower than in unfavorable habitat [12], we can assume that $D_1 \leq D_2$. In addition, as α_1, α_2 are measures for habitat preference, we can also assume that $\alpha_1 \geq \alpha_2$. Hence, as before, we assume that k, defined in (1.6.6), is greater than one. To simplify notation, we work with the inverse of k and define

$$\Delta := \frac{1}{k} = \frac{\alpha_2 D_1}{\alpha_1 D_2} < 1. \tag{3.1.15}$$

The existence of solutions for this model with logistic growth function was studied in [48]. We give a short summary of their results.

Consider a traveling wave solution of the form

$$u(t,x) = U(x-ct) = U(z), \quad L(t) = l_0 + ct, \tag{3.1.16}$$

with asymptotic boundary conditions $U(-\infty) = 1$ and $U(\infty) = 0$. Without loss of generality, we may assume that $l_0 = 0$. Therefore, a traveling wave solution of our model satisfies

$$\begin{cases} -cU' = D_1U'' + f(U), & z < 0, \quad (3.1.17) \\ -cU' = D_2U'' - mU, & z > 0, \quad (3.1.18) \\ D_1U'(0^-) + cU(0^-) = D_2U'(0^+) + cU(0^+), & (3.1.19) \\ \alpha_2 D_1 U(0^-) = \alpha_1 D_2 U(0^+), & (3.1.20) \\ U(-\infty) = 1, & (3.1.21) \\ U(\infty) = 0, & (3.1.22) \\ c = B(U(0^-), U'(0^-)), & (3.1.23) \end{cases}$$

where $f(U) = U(r - qU)$, and B is the given equation for movement of the boundary in the three scenarios (1.6.9), (1.6.10) and (1.6.11).

Lutscher *et al.* first considered the speed c to be given and studied a reduced version of the model, that is (3.1.17)–(3.1.22). They determined the range of speeds for which the reduced model has a solution. In the second step, they studied the complete model (3.1.17)–(3.1.23) and proved the existence of a speed c, in the obtained range of speed in the first step, that satisfies the equation for the speed of the wave (3.1.23) for different scenarios.

To simplify the analysis of the reduced model, Lutscher *et al.* considered the following equivalent model and proved their equivalence (Lemma 3.2 [48]). For the convenience of the reader, we bring Lemma 3.2 of [48] and its proof here.

Lemma 3.1.2 (Lutscher *et al.* (2020))**.** *Equations* (3.1.17)–(3.1.22) *are equivalent to the following system:*

$$\begin{cases} D_1U'' + cU' + f(U) = 0, \quad z < 0, & (3.1.24) \\ D_1U'(0^-) = bU(0^-), & (3.1.25) \\ U(-\infty) = 1, & (3.1.26) \end{cases}$$

with

$$b = \Delta\left(\frac{c}{2} - \sqrt{\frac{c^2}{4} + D_2 m}\right) - c, \tag{3.1.27}$$

where $0 < \Delta = \frac{\alpha_2 D_1}{\alpha_1 D_2} < 1$ *is as in* (3.1.15).

Proof: The following transformation is similar to the one in [47] and [51]. Rewrite (3.1.18) as

$$D_2U'' + cU' - mU = 0, \quad z > 0.$$

The ansatz $U(z) = \exp(\nu z)$ leads to a quadratic equation that has the roots

$$\nu^{\pm} = \frac{1}{2D_2}(-c \pm \sqrt{c^2 + 4D_2 m}),$$

with $\nu^+ > 0 > \nu^-$. By the asymptotic boundary condition $U(\infty) = 0$, the solution for $z > 0$ is

$$U(z) = U(0^+)\exp(\nu^- z).$$

Hence, we also have

$$U'(0^+) = \nu^- U(0^+).$$

We use this relation to obtain boundary conditions from the matching conditions (3.1.19) and (3.1.20) at $z = 0$. We substitute and write

$$D_1 U'(0^-) + cU(0^-) = (D_2\nu^- + c)U(0^+) = (D_2\nu^- + c)\Delta U(0^-).$$

Hence,

$$D_1 U'(0^-) = [-c + (D_2\nu^- + c)\Delta]U(0^-) = bU(0^-)$$

as defined in (3.1.27). ∎

In order to study the reduced system (3.1.24)–(3.1.26), it is essential to know the properties of the function $b(\cdot)$ in (3.1.27). In addition to the monotonicity of $b(\cdot)$ as a function of c, we are also interested in its sign for different values of c, as b determines the slope of the boundary condition (3.1.25).

Lemma 3.1.3. *For $0 < \Delta < 1$, the expression $b = b(c)$ in (3.1.27) has the following properties:*

i) $\lim_{c\to\infty} b(c) = -\infty$.

ii) b is monotone decreasing in c.

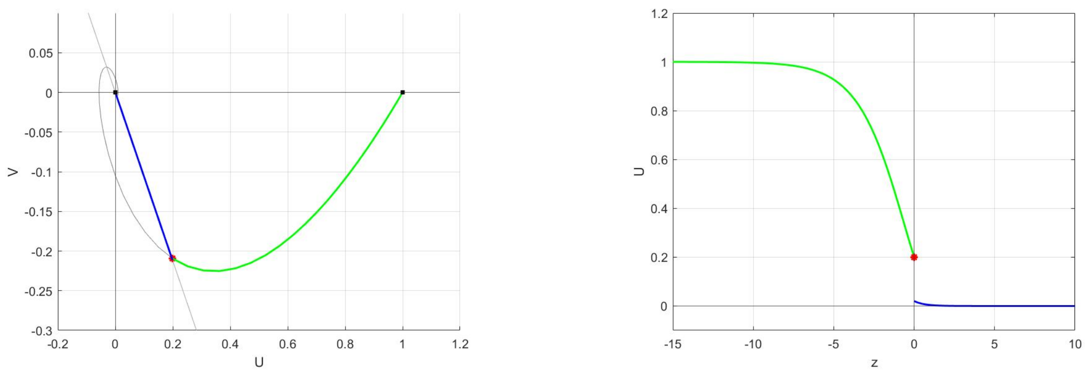
iii) There exists a threshold value

$$c_* := -\sqrt{\frac{\Delta^2}{1-\Delta} D_2 m} < 0, \tag{3.1.28}$$

at which b changes sign, i.e., $b(c_*) = 0$ and $b(c) < 0$ for $c > c_*$.

Proof: The claims in part *i*) and part *ii*) follow directly from the definition of b in (3.1.27).

To prove part *iii*), we set $b(c) = 0$ and simplify to find $c^2(1-\Delta) = \Delta^2 D_2 m$. For $0 < \Delta < 1$, this equation has real roots, among which the negative one is c_*. For $c > c_*$, we have $b < 0$. ∎



(a) (b)

Figure 3.2: Traveling wave solution (TWS) for the model with logistic growth: phase plane (a) and density function (b). The green curve in (a) is the TWO corresponding to a TWS for the reduced system (3.1.24)–(3.1.26).

A *traveling wave solution (TWS)* for reduced system (3.1.24)–(3.1.26) corresponds to an orbit in the phase plane of (3.1.24) that leaves the saddle point $(1, 0)$ along its unstable manifold and ends at the line $V = b(c)U$, which corresponds to boundary condition (3.1.25). We call such an orbit a *Traveling Wave Orbit (TWO)*, see Fig 3.2. To prove that such a TWO exists for a specific value of c, one needs to show that there is (at least) one intersection between the boundary line $V = b(c)U$ and the unstable manifold of $(1, 0)$, denoted by $M_1^+(c)$, in the phase plane of (3.1.24).

In [48], Lutscher *et al.* proved that for $c_* < c < \hat{c}$, with c_* defined in (3.1.28) and $\hat{c}$ Fischer's minimal speed, there exists a solution for the reduced system (3.1.24)–(3.1.25). Then they considered the full system (3.1.17)–(3.1.23) with different scenarios. For scenarios 1 and 3, they showed that there exists a unique speed in the above range that satisfies equation (3.1.23) for speed. For scenario 2, they determined the boundary between advancing and retreating waves. They showed that there exists a threshold value U^* for $\bar{u}$ in the equation for the speed of scenario 2, i.e. (3.1.23) with B defined in (1.6.10), for which the full system has a standing wave.

3.2 A Free Boundary Problem for Ecosystem Engineers with Allee Growth

We now consider the free boundary system (3.1.10)–(3.1.14) with a growth function that includes a strong Allee effect (see Section 1.6), that is

$$f(u) = r\left(\frac{u-k_0}{k_1}\right)\left(1-\frac{u}{k_1}\right)u,$$

where r is the intrinsic growth rate, $k_1 > 0$ is the carrying capacity in the engineered habitat, and $0 \le k_0 < k_1$ is the Allee threshold.

To reduce the number of parameters, we nondimensionalize the system by scaling time, space, and density as follows:

$$\begin{aligned} \hat{t} &= rt, & \hat{x} &= \sqrt{\frac{r}{D_1}}x, \\ \hat{u}(\hat{x},\hat{t}) &= \frac{u(x,t)}{k_1}, & \hat{L}(\hat{t}) &= \sqrt{\frac{r}{D_1}}L(t). \end{aligned} \tag{3.2.1}$$

After rescaling, if we rename

$$\begin{aligned} D &:= \frac{D_2}{D_1}, & \hat{\bar{u}} &= \frac{\bar{u}}{k_1}, & a &:= \frac{k_0}{k_1}, \\ \hat{m} &= \frac{m}{r}, & \hat{\eta}_1 &:= 2k_1\eta_1\sqrt{\frac{D_1}{r}}, & \hat{\eta}_2 &:= 2\eta_2 k_1, \end{aligned} \tag{3.2.2}$$

and drop the hats for simplicity of notation, we find the scaled system to be

$$\begin{cases} u_t = u_{xx} + u(u-a)(1-u), & x < L(t), \quad t > 0, \\ u_t = Du_{xx} - mu, & x > L(t), \quad t > 0, \\ \alpha_2 u(t, L(t)^-) = \alpha_1 D u(t, L(t)^+), & t > 0, \\ \Big(u_x + L'(t)u\Big)\Big|_{(t,L(t)^-)} = \Big(Du_x + L'(t)u\Big)\Big|_{(t,L(t)^+)}, & t > 0, \\ L'(t) = \widetilde{B}(u, u_x), & t > 0, \end{cases} \tag{3.2.3}$$

where $\widetilde{B}$ is the scaled form of one of the equations (1.6.9), (1.6.10) or (1.6.11), that is,

$$L'(t) = \eta_1 \alpha u(t, L(t)^-), \tag{3.2.4}$$

$$L'(t) = \eta_1 \alpha \left(u(t, L(t)^-) - \bar{u}\right), \tag{3.2.5}$$

$$L'(t) = -\eta_2 u_x(t, L(t)^-). \tag{3.2.6}$$

Based on the biological application, the ranges of parameters in the scaled model are as follows

$$0 < a < 1, \quad 0 < \alpha_1, \alpha_2 < 1, \quad 0 \leq \alpha < 1 \quad \text{and} \quad D, m, \eta_1, \eta_2 > 0. \tag{3.2.7}$$

We are going to study the existence of traveling wave solutions for system (3.2.3). As mentioned in the preceding section, a traveling wave is a solution of the form

$$u(t, x) = U(x - ct) = U(z), \quad L(t) = l_0 + ct, \tag{3.2.8}$$

with asymptotic boundary conditions $U(-\infty) = 1$ and $U(\infty) = 0$. Without loss of generality, we may assume that $l_0 = 0$. As in the preceding section, a traveling wave solution of our model satisfies

$$\begin{cases} -cU' = U'' + f(U), & z < 0, \quad (3.2.9) \\ -cU' = DU'' - mU, & z > 0, \quad (3.2.10) \\ U'(0^-) + cU(0^-) = DU'(0^+) + cU(0^+), & (3.2.11) \\ \alpha_2 U(0^-) = \alpha_1 D U(0^+), & (3.2.12) \\ U(-\infty) = 1, & (3.2.13) \\ U(\infty) = 0, & (3.2.14) \\ c = \widetilde{B}(U(0^-), U'(0^-)), & (3.2.15) \end{cases}$$

where $\widetilde{B}$ is the given equation for movement of the boundary in the three scenarios (3.2.4), (3.2.5), and (3.2.6). In contrast to the preceding section, the reaction term has the form $f(U) = U(U - a)(1 - U)$, which includes the strong Allee effect.

Similar to Lemma 3.1.2, we can equivalently write the above system as a nondimensionalized version of system (3.1.24)–(3.1.26), in which we have $D_1 = 1$ in (3.1.24) and (3.1.25), the expression for b in (3.1.25) is as in (3.1.27) with D_2 and m replaced by the scaled parameters D and m, respectively. The same substitutions need to be made in the expression for c_* in (3.1.28).

To study the existence of a traveling wave solution for our model, we first consider the speed c to be given and consider a reduced version of the model, that is (3.2.9)–(3.2.14). We determine the range of speeds for which the reduced model has a solution in Section 3.3. In the second step, we consider the complete model (3.2.9)–(3.2.15) in Section 3.4. Then we prove the existence of a speed c in the obtained range from the first step that satisfies the full model (3.2.9)–(3.2.15) for different scenarios.

3.3 Traveling Waves in the Reduced Model

In this section, we analyze the reduced model (3.2.9)–(3.2.14) with speed c as a free parameter.

The reduced system is an interesting system to study in its own right as it corresponds to a moving habitat model in one dimension. These models may describe the population dynamics of a species in a habitat that is moving on the real line due to climate change [49]. There are two kinds of moving habitat models: those with one interface [45] and those with two [50]. While in both types the interface(s) may shift in different directions, in the one-interface model the shift corresponds to either contraction or extension of the suitable habitat [7].

The reduced system is also closely related to a version of the well-known Fisher's model [26] with Allee effect, for which Hadeler and Rothe proved the existence of different traveling fronts and their speed [29].

To prove the existence of solutions for our system, we study the phase plane of (3.1.7) with speed c as a parameter. For a given speed c, we denote the unstable manifold corresponding to each steady state by $M_i^+(c)$, $(i = 0, a, 1)$. In the following lemma, we discuss the behaviour of $M_1^+(c)$ with respect to c.

Lemma 3.3.1. *The unstable manifold of* $(1,0)$*,* $M_1^+(c)$*, changes monotonically with respect to* c*, in the following sense:*

i) the part of $M_1^+(c)$ *leaving* $(1,0)$ *to the first quadrant is moving downward as* c *increases. This means that* $M_1^+(c')$ *lies above* $M_1^+(c'')$ *in the first quadrant if* $c' < c''$*.*

ii) the part of $M_1^+(c)$ *in the forth quadrant between* $(1,0)$ *and its first intersection with the positive part of the* U*-axis or negative part of the* V*-axis is moving upward as* c *increases. This means that* $M_1^+(c'')$ *lies above* $M_1^+(c')$ *in the fourth quadrant if* $c' < c''$*, at least until the first intersection with the* U*-axis or* V*-axis.*

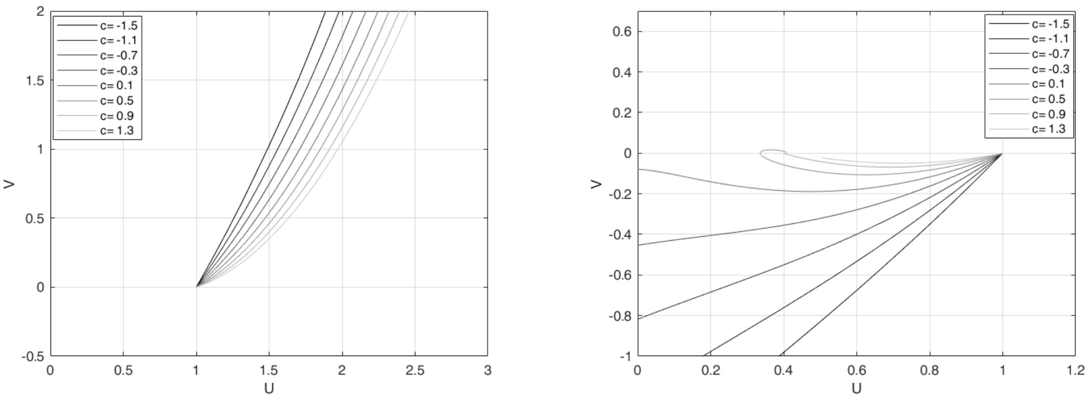
Proof: First, we consider the positive eigenvalue of the Jacobian at $(1,0)$, λ_1^+, defined in (3.1.8).

The unstable manifold of the saddle point $(1,0)$, $M_1^+(c)$, is tangent to the line $V = \lambda_1^+(U-1)$ at $(1,0)$. We can see that $\frac{d\lambda_1^+}{dc} < 0$.

To prove i), we consider the branch of $M_1^+(c)$ in the first quadrant. For $c' < c''$, $M_1^+(c'')$ lies below $M_1^+(c')$ near $(1,0)$. We show that there is no intersection between $M_1^+(c')$ and $M_1^+(c'')$ in the first quadrant. Suppose the contrary and assume that $(\bar{U}, \bar{V})$ is the point with the smallest value of $\bar{U} > 1$ and $\bar{V} > 0$, where the two unstable manifolds $M_1^+(c')$ and $M_1^+(c'')$ intersect. The slope of the vector field at that point is

$$\frac{dV}{dU} = \frac{V'}{U'} = \frac{-f(\bar{U}) - c\bar{V}}{\bar{V}} = -\frac{f(\bar{U})}{\bar{V}} - c. \tag{3.3.1}$$

Since $c' < c''$, the slope of $M_1^+(c')$ is steeper than that of $M_1^+(c'')$ at the intersection point. But this implies that for $U < \bar{U}$, $M_1^+(c'')$ lies above $M_1^+(c')$. This contradicts the above observation based on the linearization near $(1,0)$. Hence, there is no such



(a) (b)

Figure 3.3: An illustration of Lemma 3.3.1. The unstable manifold of $(1,0)$ for different values of speed c in the first quadrant (a) and in the fourth quadrant (b).

intersection. Therefore, $M_1^+(c')$ lies above $M_1^+(c'')$ in the first quadrant; see Figure 3.3.

Similar reasoning can be applied in the fourth quadrant to prove part *ii*). ∎

3.3.1 Classifying Traveling Wave Solutions for the Reduced System

As mentioned before, a traveling wave solution for system (3.1.24)–(3.1.26) corresponds to a TWO in the phase plane of (3.1.7) that leaves the saddle point $(1,0)$ along its unstable manifold and ends at the line $V = bU$, which corresponds to boundary condition (3.1.25). Depending on parameter values, there are four qualitatively different types of TWOs.

Definition 3.3.2 (Types of Traveling Wave Solutions)**.** *We define four types of traveling wave solutions for our system:*

i) If the TWO is entirely in the fourth quadrant, we say that the traveling wave is of type I (TWI); see Figure 3.4.

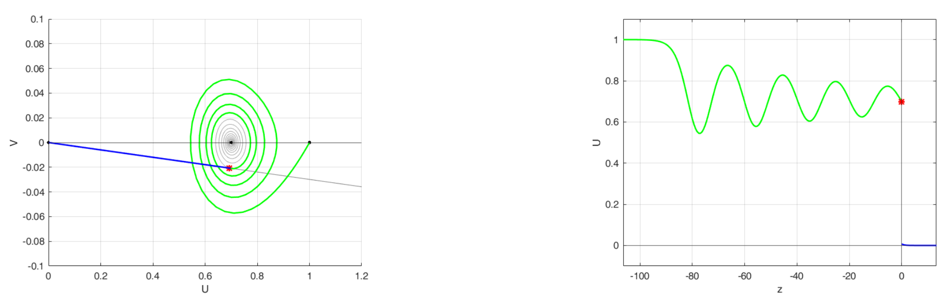
ii) If the TWO begins and ends in the fourth quadrant but crosses into the first quadrant in between, we say that the traveling wave is of type II (TWII); see Figure 3.5.

iii) If the TWO begins in the fourth quadrant and ends in the first, we say that the traveling wave is of type III (TWIII); see Figure 3.6.

iv) If the TWO is entirely in the first quadrant, we say that the TW is of type IV (TWIV); see Figure 3.7.

Remark 3.3.3. *One requirement for TWI and TWII is that the slope of the boundary line $V = bU$ be non-positive. By contrast, for TWIII and TWIV, we need the slope of the boundary line to be positive.*

Remark 3.3.4. *At $c = c_*$, by definition, the slope of the boundary line is zero, i.e. $b(c_*) = 0$. Hence, the intersection between the boundary line and the unstable manifold $M_1^+(c_*)$ occurs on the U-axis in the phase plane. If this occurs for $0 < a < 1/2 + \sqrt{(\Delta^2/2(1-\Delta))\,Dm}$, and $c_* < c_0$, there is only one intersection between the boundary line $V = 0$ and the unstable manifold $M_1^+(c_*)$ at $(1, 0)$, see Figure 3.4–(e). If this occurs when $1/2 + \sqrt{(\Delta^2/2(1-\Delta))\,Dm} < a < 1$, and $c_* > c_0$, then there are two intersections between the boundary line $V = 0$ and the unstable manifold $M_1^+(c_*)$ in the phase plane; one at $(1, 0)$ and one at $(U_*, 0)$ for some $0 < U_* < a$, see Figure 3.4–(g). When the intersection is at $(U_*, 0)$, the TWO between $(1, 0)$ and $(U_*, 0)$ is entirely in forth quadrant, the resulting TWS is a special case of TWI. However, when the intersection occurs at $(1, 0)$, the resulting TWS, which is equal to constant $U = 1$ for $z < 0$, see Figure 3.4–(f), can be considered as either TWI or TWIV. To have a clear definition for different possible TWS, we put this specific case in the category of TWI. We also note that when $c_0 = c_*$ at $a = 1/2 + \sqrt{(\Delta^2/2(1-\Delta))\,Dm}$, the second intersection is at $U = 0$. This intersection does not correspond to any of the TWS types as the orbit is the heteroclinic connection between $(1, 0)$ and $(0, 0)$.*



(a) (b)

Figure 3.5: Traveling Wave Type II: phase plane (a) and density profile (b).

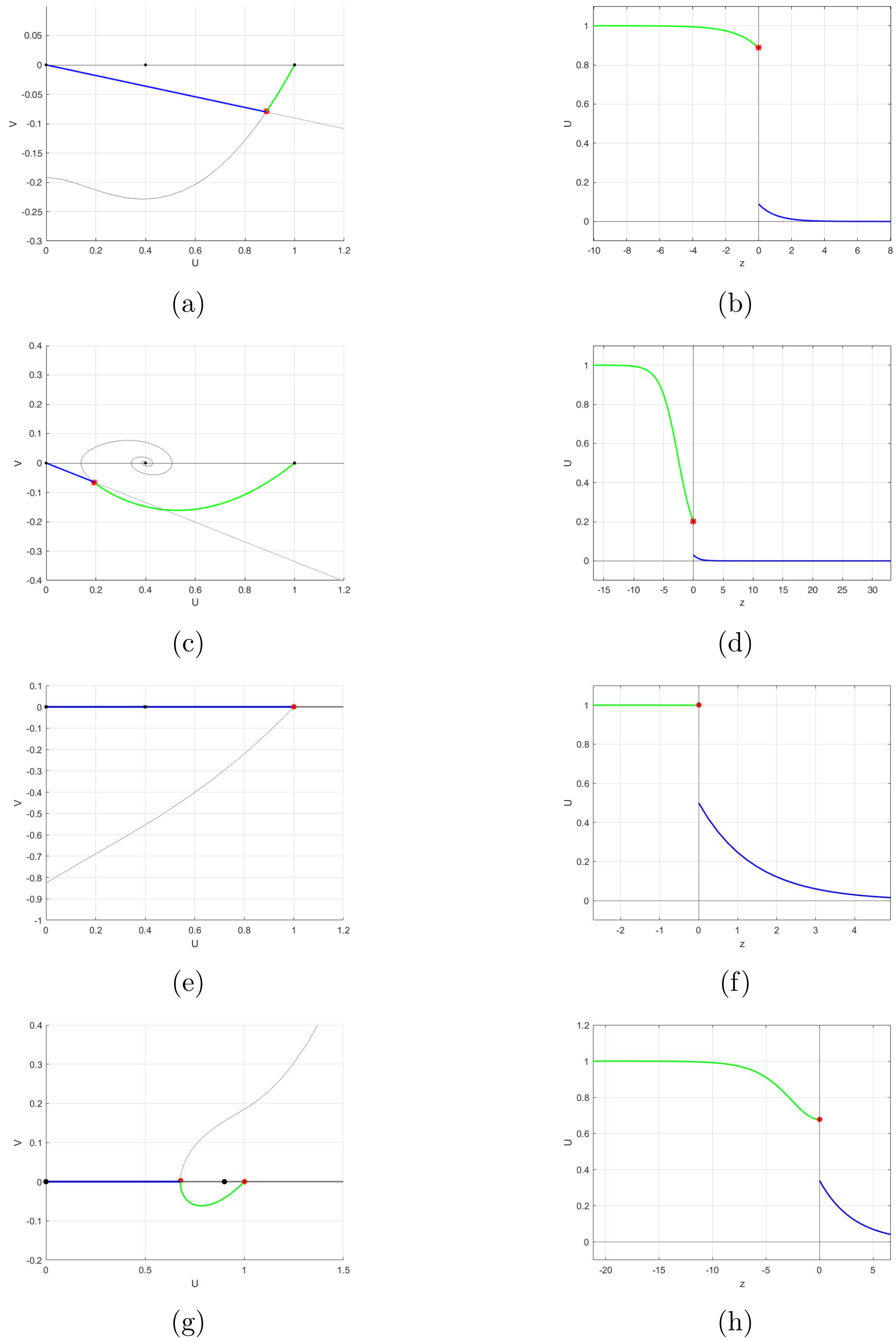
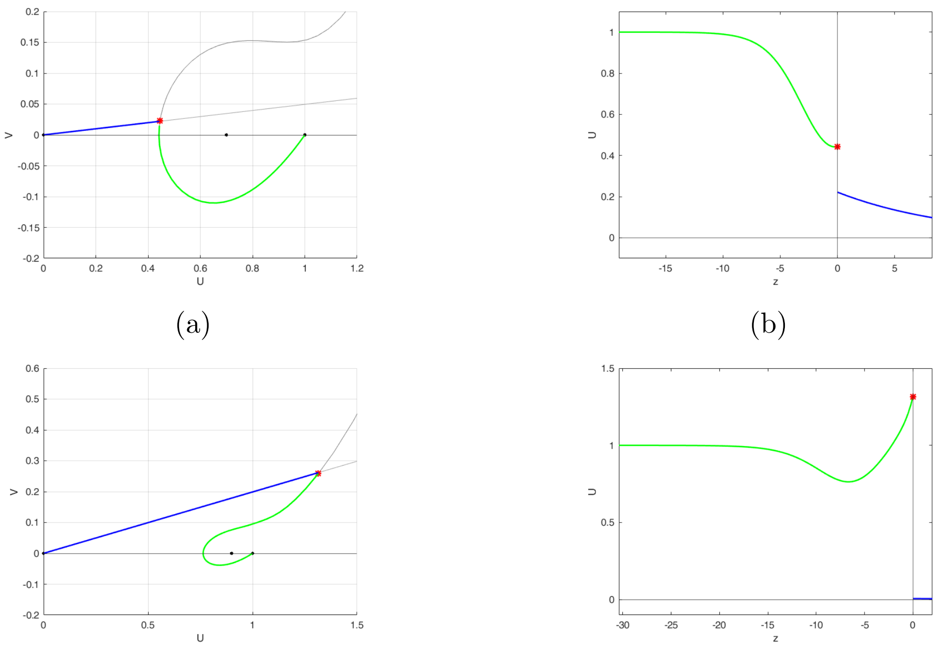


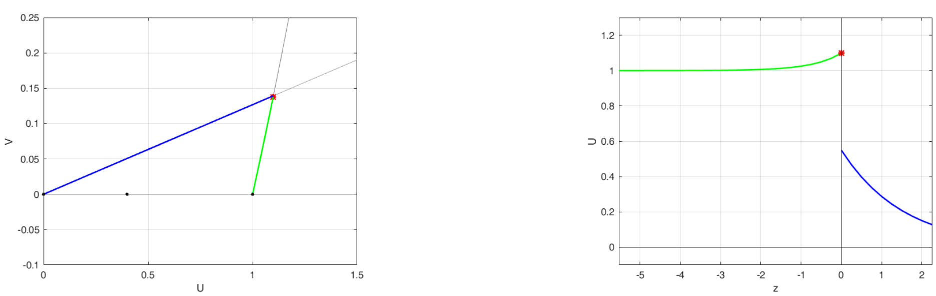
Figure 3.4: The four qualitatively different possible Traveling Wave Type I: (a), (c),(e), and (g) demonstrate the traveling wave orbit (TWO) in the phase plane, (b), (d), (f), and (h) show the corresponding density function. The green curve is the solution for (3.1.24)–(3.1.26) in the engineered habitat for $z < 0$, and the blue curve/line corresponds to the solution in non-engineered habitat for $z > 0$.



(a) (b)

(c) (d)

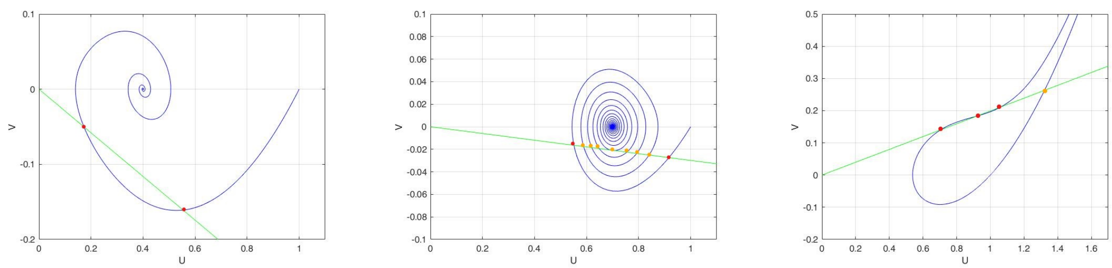
Figure 3.6: The two qualitatively different possible Traveling Waves of Type III: phase planes (a) and (c), and density functions (b) and (d).



(a) (b)

Figure 3.7: Traveling Wave Type IV: phase plane (a) and density function (b).

Remark 3.3.5. *For all types of traveling waves, except type IV, more than one traveling wave of the same type may exist for a given speed. It is also possible that we have traveling waves of different types for the same speed; see Figure 3.8.*



(a) (b) (c)

Figure 3.8: Visualisation of Remark 3.3.5: in (a) we have two TWI for the same speed. In (b), we have two TWI and seven TWII. In (c), we have one TWIV and three TWIII for the same speed.

3.3.2 Speed Range for Existence of Traveling Waves

In this section, we analyze the existence of traveling waves of different types according to the value of parameter $0 < a < 1$, and find the corresponding range of speeds for which each wave exists. In order to do that, we study the behavior of the unstable manifold $M_1^+(c)$ for different values of c and a. For different values of a, the part of $M_1^+(c)$ that enters the first quadrant behaves similarly for all c due to the underlying vector field. However, this is not the case for the part of $M_1^+(c)$ that enters the fourth quadrant. Following [29], we divide the a–c plane based on the behavior of the part of $M_1^+(c)$ that enters the fourth quadrant; see Figure 3.9.

In region 1, $M_1^+(c)$ leaves the fourth quadrant through the negative vertical axis. Hence, according to Definition 3.3.2, only TWI may appear here. In region 2, $M_1^+(c)$ is a heteroclinic connection from the saddle at $(1, 0)$ to the stable focus at $(a, 0)$. TWI and TWII may exist here. In region 3, $M_1^+(c)$ leaves the fourth quadrant through the horizontal axis and approaches the part of $M_1^+(c)$ that enters the first quadrant. Only TWI and TWIII may appear here. In region 0 and 4, $M_1^+(c)$ is a heteroclinic connection from $(1, 0)$ to $(a, 0)$, where the latter is a stable node. From the definition, only TWI may exist in these regions, but we will show below that no TWI can exist in region 0.

All of these regions only indicate where certain waves are possible. For their actual occurrence, we begin by determining threshold values of the speed and then discuss each wave type separately.

Maximal Speed for the Existence of Traveling Waves

Theorem 3.3.6. *There exists a threshold value of speed above which there is no traveling wave solution of any type; for any speed below the threshold, there exists at least one TWS of some type.*

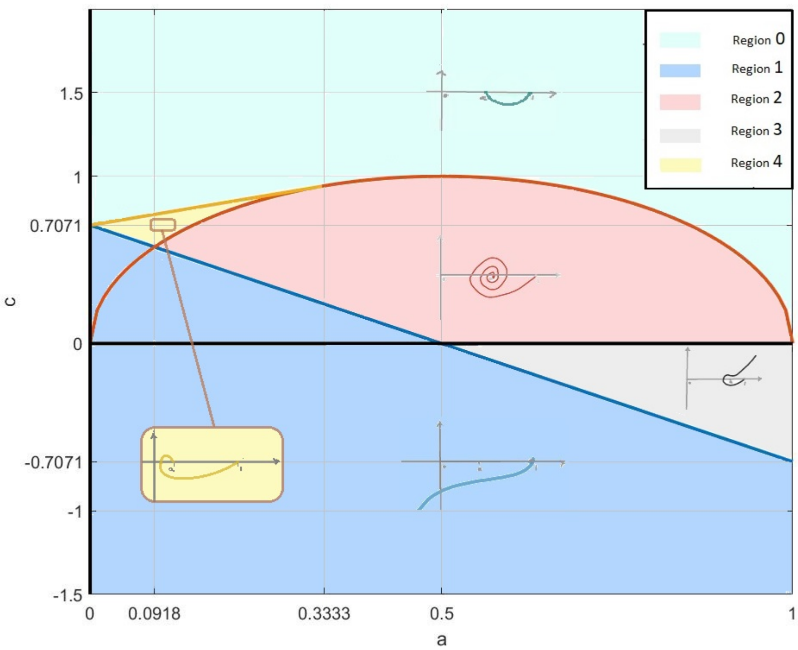


Figure 3.9: Possible shapes of the unstable manifold of (1,0) in the a-c plane. See the text for detailed explanations. The red curve is c_1, the yellow curve is c_2, and the blue curve is c_0, as defined in equation (3.1.9), with domain extended to $0 < a < 1$.

Proof: We need to show that for all speeds c above some threshold there is no intersection between the boundary line, $V = b(c)U$, and the unstable manifold, $M_1^+(c)$. For $c = c_0$, we have a heteroclinic connection between $(1, 0)$ and $(0, 0)$. The boundary line at this speed, $V = b(c_0)U$, may be above or below this connection. Accordingly, we divide our proof into the following cases.

Case 1: $b(c_0) \leq \lambda_0^-(c_0)$, i.e. the slope of the boundary line is less than or equal to the slope of the tangent line to the heteroclinic connection at $(0, 0)$ when $c = c_0$. In this case, the boundary line and the heteroclinic connection have no intersection at any point other than $(0, 0)$, and we must have $c_* < c_0$ as $b(c_0) < 0$. By Lemmas 3.1.3 and 3.3.1, we know that as c increases, $M_1^+(c)$ and $V = b(c)U$ move in different directions in the fourth quadrant. Hence, they will not intersect for $c > c_0$. However, we can follow the idea of proof of Lemma 3.6 in [48] to see that, for all $c < c_0$, the part of $M_1^+(c)$ that goes into the fourth quadrant has a bounded slope and so it hits the negative part of the V-axis. Hence, it must intersect the boundary line as long as the boundary line has a negative slope, i.e. $c > c_*$. Hence, we will have a TWS, which is of type TWI, for $c_* \leq c < c_0$. On the other hand, as the slope of the boundary line becomes positive for $c < c_*$, the boundary line will intersect the part of $M_1^+(c)$ that goes into the first quadrant and hence a TWIV appears, see Theorem 3.3.17 below for the proof. Therefore, in this case, the threshold value of c is c_0.

Case 2: $b(c_0) > \lambda_0^-(c_0)$, i.e. the slope of the boundary line is greater than the slope of the tangent line to the heteroclinic connection at $(0,0)$ when $c = c_0$. In this case, it may be possible to have $c_0 \leq c_*$ as $b(c_0)$ may be positive. Hence, we divide our proof for this case into the following sub-cases:

Case 2.1: $c_* < c_0$. Similar to Case 1, we can show that at least one TWS of type TWI or TWIV exists for $c \leq c_0$. We choose $\bar{c} > c_0$ such that $b(\bar{c}) < \lambda_0^-(c_0)$. The unstable manifold $M_1^+(\bar{c})$ lies above the heteroclinic connection, hits the U-axis between 0 and a and enters the first quadrant. Therefore, regardless of the value of $\bar{c}$, there is no intersection between the boundary line $V = b(\bar{c})U$ and $M_1^+(\bar{c})$. Hence, by continuity of the phase plane with respect to c, there is a maximal speed above which there is no traveling wave solution. For speeds less than this maximal speed and greater than c_0, we may have TWII in addition to TWI.

Case 2.2: $c_* \geq c_0$. In this case, for all values of $c > c_0$, we have $b(c) < b(c_0)$ so that the part of $M_1^+(c)$ in the fourth quadrant moves to the first quadrant. Therefore, we can argue as in the proof of Case 2.1, and use the continuity of the phase plane to show the existence of the maximal speed. The type of TWS that appears in this case is different from the other cases. For speeds between c_* and the maximal speed, we have TWI and for those positive, we may have TWII as well. For $c_0 < c < c_*$, we have TWIII and TWIV. Finally for $c \leq c_0$ we only have TWIV; see Figure 3.9 for illustration.

Hence, by continuity of the phase plane with respect to c, there is a maximal speed c above which there is no traveling wave solution.

∎

Remark 3.3.7. *We can characterize the threshold speed as the speed for which the unstable manifold at* $(1,0)$ *is tangent to the boundary line; see Fig 3.10. As c increases, the boundary line moves down whereas the portion of the unstable manifold between* $(1,0)$ *and its first intersection with the U axis moves up. Hence, there cannot be a TWO for any larger speed. We call this maximal speed* c^**. It is clear that* c^* *is also the threshold value in Theorem 3.3.6 above which there is no connection between the boundary line and the unstable manifold on* $(1,0)$ *and hence no solution for the system* (3.1.24)-(3.1.26).

Next, we discuss how c^* depends on the model parameters.

Corollary 3.3.8. *i) If*

$$Dm \geq 2\frac{a}{\Delta}\left(\frac{a}{\Delta} + \left(\frac{1}{2} - a\right)\right), \tag{3.3.2}$$

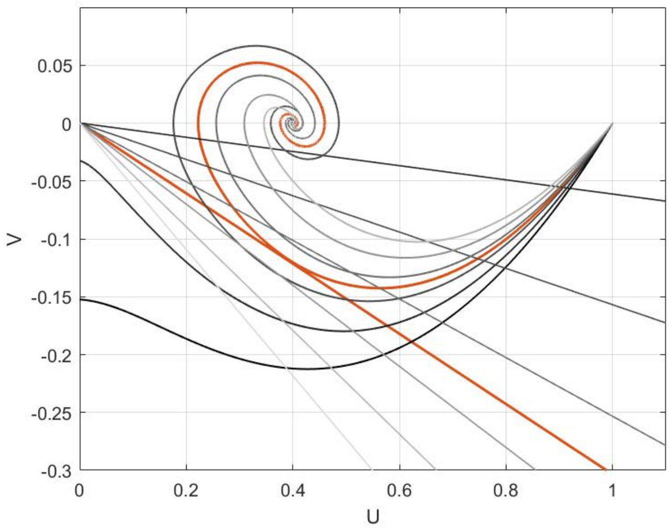


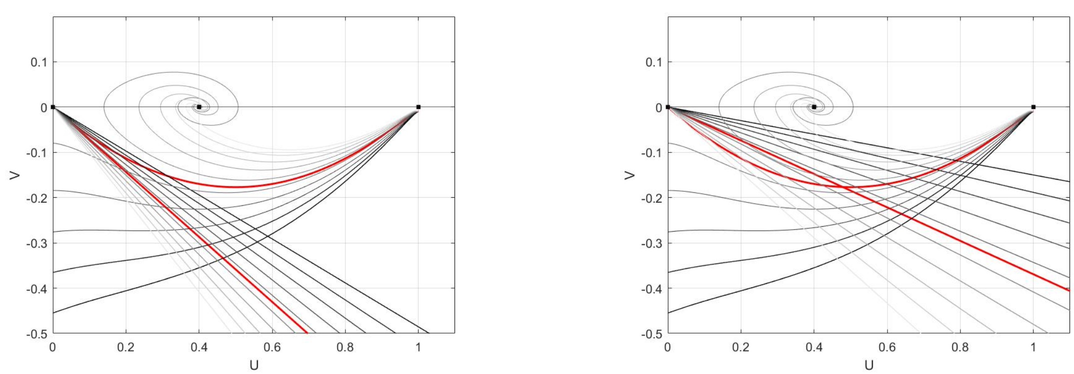
Figure 3.10: Visualization for Remark 3.3.7. We plot the unstable manifold of $(1,0)$, $M_1^+(c)$, and the boundary line, (3.1.25), for different values of c. Lighter colors correspond to higher values c. The orange curve shows an example of the situation for the maximal speed c^*. At the maximal speed c^* the manifold and the boundary line are tangent.

then $c^* = c_0$ is the supremum of speeds for which a TWS exists.

ii) If

$$Dm < 2\frac{a}{\Delta}\left(\frac{a}{\Delta} + (\frac{1}{2} - a)\right), \tag{3.3.3}$$

then c^* is the maximum of speeds for which a TWS exists.



(a) (b)

Figure 3.11: Visualization of Corollary 3.3.8. The red orbit and line correspond to the orbit connecting $(0,0)$ and $(1,0)$ for $c = c_0 = 0.1414$ and the boundary line with the slope $b(c_0)$, respectively. The grey orbits and boundary lines are varying for different values of $0 < c < 1$. The lighter the curve/line the higher the value of c. The value of the parameters for graph (a) are $a = 0.4, m = 0.1, \Delta = 0.9$ and $D = 1$, and for graph (b) are $a = 0.4, m = 0.1, \Delta = 0.9$ and $D = 5$.

Proof: After simplification, one can see that (3.3.2) is equivalent to $b(c_0) \leq \lambda_0^-(c_0) = -\sqrt{2}/2$. As seen in the proof of Theorem 3.3.6, in this case, the threshold value of speed for the existence of TWS is $c^* = c_0$. However, as the boundary line $V = b(c_0)U$ only intersects $M_1^+(0)$ at $(0,0)$, there exists no TWS of any type for system (3.1.24)–(3.1.26) at $c = c_0$. In the other case, when $b(c_0) > \lambda_0^-(c_0) = -\sqrt{2}/2$, the threshold value of speed for the existence of TWS is c^* with $c^* > c_0$. In this case $V = b(c^*)U$ is tangent to $M_1^+(c^*)$ at some point between $(0,0)$ and $(1,0)$, and so the TWO connecting $(1,0)$ and the boundary line corresponds to a TWI at c^*. Hence, c^* is the maximum value of speed for the existence of TWSs; see Figure 3.11. ∎

Unfortunately, there is no explicit formula for c^* when $c^* > c_0$. Using the results of [29], we can find a range for this maximal speed and study its properties, see Figure 3.12. In the following lemma, we use critical values of speed defined in (3.1.9), in which we extend the domain from $0 < a < 1/2$ to $0 < a < 1$.

Lemma 3.3.9. *For any values of the parameters, $D, m > 0$, $0 < \Delta < 1$, and $0 < a < 1$, we have $c_0 \leq c^* < c_m$.*

Proof: As seen in the proof of Theorem 3.3.6, we have either a TWI or TWIV for all $c < c_0$. Hence, the lower bound is clear. It remains to show that the upper bound is c_m. By part i) of Theorem 3.1.1, for $c \geq c_m$, there exists a heteroclinic connection between the steady states $(1,0)$ and $(a,0)$. This heteroclinic connection is tangent to $V = \lambda_a^+(c)(U-a)$ at $(a,0)$ [73]. If we compare the slope of the boundary line and this tangent line, we see

$$b(c) = -c - \frac{\Delta}{2}\left(\sqrt{c^2+4Dm} - c\right) < -c < \frac{1}{2}\left(-c + \sqrt{c^2 - 4a(1-a)}\right) = \lambda_a^+(c),$$

for all $c \geq c_m$. This implies that there cannot be any intersection between the boundary line and the heteroclinic connection for $c \geq c_m$. Hence, we must have $c^* < c_m$. ∎

The value and sign of c^* change with the nature of the phase plane, which in turn depends on the parameters in the system and in particular on a.

Recall that for $0 < a \leq 1/2$, we have $c_0 \geq 0$. By Lemma 3.3.9, we know that $c^* \geq c_0$. Therefore, in this case, we always have $c^* \geq 0$. However, for $a > 1/2$, the situation is not as straightforward.

Lemma 3.3.10. *For $a > 1/2$, if the boundary line $V = b(0)U$ lies below the homoclinic connection of $(1,0)$ at $c = 0$ (resp. is tangent to it), then $c^* < 0$ (resp. $c^* = 0$). Otherwise $c^* > 0$.*

Proof: As seen in Lemmas 3.1.3 and 3.3.1, as c increases, the boundary line $V = b(c)U$ and the unstable manifold $M_1^+(c)$ move in different directions. Hence,

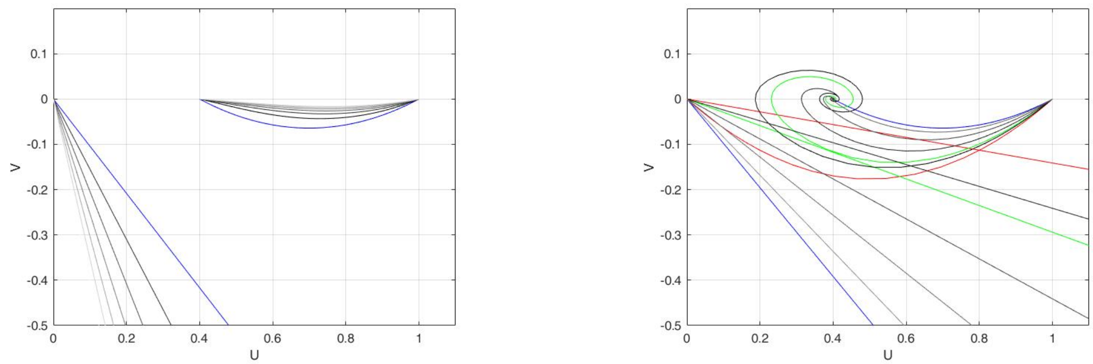


Figure 3.12: Visualization for Lemma 3.3.9. We chose $(1-\sqrt{2/3})/2 < a < 1/2$, so that that $c_1 > c_0$. The left-hand plot shows that there exists no orbit connecting $(1,0)$ to the boundary line for $c \geq c_1$. The blue curve and line are the orbit and the boundary line corresponding to $c = c_1$. The right-hand plot shows an example of parameter values for which we have $c_0 < c^* < c_1$ (the green curves correspond to c^*).

knowing the location of the boundary line and the unstable manifold, which is part of the homoclinic connection at $c = 0$, will determine the sign of c^*. If at $c = 0$ the boundary line is tangent to the homoclinic connection, then, as c increases, the boundary line moves downward and the part of $M_1^+(c)$ in the fourth quadrant moves upward. Therefore, there will be no intersection for $c > 0$ and so $c^* = 0$. Similarly, if the boundary line lies below the homoclinic connection at $c = 0$, then there is no intersection for $c \geq 0$. Therefore, the tangency condition for c^* must happen for some negative value of c.

We can find the critical parameter relation at which $c^* = 0$. To do that, we need to find slope μ of a line $V = \mu U$, tangent to the homoclinic connection of $(1,0)$ for $c = 0$ at some point $U = U_{\mathrm{x}}$. We then compare the slope of the boundary line and this tangent line.

At $c = 0$, we can find the equation of the homoclinic connection. The standing wave for $c = 0$ satisfies the special case of (3.1.24);

$$U'' + f(U) = 0, \quad z < 0. \tag{3.3.4}$$

Equation (3.3.4) has a first integral. Multiplying (3.3.4) by U' and integrating with respect to z, we find the energy function

$$E(U, U') = \frac{1}{2}(U')^2 + F(U), \tag{3.3.5}$$

where $F(U)$ is the primitive of $f(U)$. The level curves of this energy function correspond to the orbits in the phase plane of (3.4.4). Hence, along any orbit in the phase plane the energy function E remains constant. Therefore, on the homoclinic orbit that we are looking for, E satisfies

$$E(U,V) = E(1,0),$$

for all (U,V) on the homoclinic connection of $(1,0)$. By (3.3.5), this is equivalent to

$$\frac{1}{2}(V)^2 + F(U) = F(1).$$

Hence, the equation of the homoclinic connection in the phase plane is

$$V_{\alpha\pm} = \pm\sqrt{\frac{1}{6} - \frac{a}{3} - 2F(U)} = \pm\sqrt{\frac{1}{6} - \frac{a}{3} + \frac{U^4}{2} - \frac{2(1+a)U^3}{3} + aU^2}. \tag{3.3.6}$$

As we are looking for the tangent line to the homoclinic connection in the fourth quadrant, we consider the negative part in (3.3.6). Now we find μ such that $V = \mu U$ is tangent to $V_{\alpha-}$ at some point U_{x}, i.e. $V_{\alpha-}(U_{\text{x}}) = \mu U_{\text{x}}$ and $V'_{\alpha-}(U_{\text{x}}) = \mu$. This leads to solving the following system of equations:

$$\begin{cases} \mu U_{\text{x}} = -\sqrt{\frac{1}{6} - \frac{a}{3} - 2F(U_{\text{x}})}, & (3.3.7) \\ \mu = \dfrac{2F'(U_{\text{x}})}{2\sqrt{\frac{1}{6} - \frac{a}{3} - 2F(U_{\text{x}})}}. & (3.3.8) \end{cases}$$

By simplifying, we find a fourth degree equation for U_{x}, which we solve to find

$$\begin{aligned} U_{\text{x}} = {} & \frac{1}{9}\sqrt[3]{8a^3 + 42a^2 + 27\sqrt{(2a-1)^2(a^2+6a+17)} + 195a - 109} \\ & + \frac{4a^2 + 14a - 8}{9\sqrt[3]{8a^3 + 42a^2 + 27\sqrt{(2a-1)^2(a^2+6a+17)} + 195a - 109}} \\ & + \frac{1}{9}(2a-1). \end{aligned} \tag{3.3.9}$$

Hence, we can find the slope of the desired tangent line μ by substituting (3.3.9) in (3.3.8). As the equation for μ is long and does not give any insight about the relations between the parameters later, we will not bring the expression for μ here. From (3.3.9), one can see that μ also depends on a, but not on the other parameters D, m and Δ.

If the slope of the boundary line at $c = 0$ is equal to μ, that is $b(0) = -\Delta\sqrt{DM} = \mu$, then $c^* = 0$. Similarly, if $b(0) < -\Delta\sqrt{DM} = \mu$, i.e. the boundary line at $c = 0$ lies below the homoclinic connection, then $c^* < 0$. Otherwise, we have $c^* > 0$ for $a > \frac{1}{2}$. ∎

Remark 3.3.11. *We defined the maximal speed c^* such that for $c > c^*$, there exists no TW solution of any type. As we will see in the following section, c^* is the maximal speed for the existence of TWI. However, we have different maximal speeds of existence for the other three traveling wave types. Following similar arguments as above, based on the opposite directions of the movement of the boundary line and the unstable manifold of $(1,0)$ with respect to c, we can see that there exists a maximal speed, different from c^* for the existence of TWII. The range for this maximal speed, which we call c^{**}, is $0 < c^{**} < c^*$. Also, we will prove in the following sections that the maximal speed for the existence of TWIII and TWIV is c_*.*

Having clearly defined the threshold speed c^*, we determine the speed range for the existence of each wave type in the following sub-sections.

Speed Range for Existence of TWI

As seen in Figure 3.9, TWI can occur in all regions 1–4. We will also see in Section 3.4 that TWI can be a solution to the full system for all three scenarios.

Theorem 3.3.12. *For $0 < a < 1$ and $c_* \leq c < c^*$, there exists at least one TWI for the reduced system (3.1.24)–(3.1.26). When $c^* > c_0$, we have a TWI at $c = c^*$ as well. When $c_0 > c_*$, the TWI is unique for $c_* \leq c \leq c_0$.*

Proof: The proof of existence is similar to the proof of Theorem 3.3.6 and Corollary 3.3.8. It remains to prove uniqueness for $c_* \leq c \leq c_0$, when $c_0 > c_*$. Let's assume that there is more than one intersection between the boundary line $V = b(c)U$ and the unstable manifold $M_1^+(c)$ in the fourth quadrant, for $c_* < c < c_0$. Then, at one of these intersections, the slope of the unstable manifold must be less than that of the boundary line (negative and steeper). In the following, we prove that such an intersection point does not exist in the fourth quadrant, when $c_* < c < c_0$.

For any $c_* < c < c_0$, the boundary line $V = b(c)U$ is either above, on or under the tangent line to the heteroclinic connection $V = \lambda_0^-(c_0)U = \frac{-\sqrt{2}}{2}U$ at $(0,0)$. Hence, we study the following two possible cases separately:

First consider $b(c) > \lambda_0^-(c_0)$. In this case, for any $c < c_0$, there exists only one intersection between the boundary line $V = b(c)U$ and the heteroclinic connection of $(1,0)$ and $(0,0)$ at some point $(\bar{U}, \bar{V})$ with $\bar{U} > 0$. There can not be any intersection between the boundary line $V = b(c)U$ and the unstable manifold $M_1^+(c)$ for $0 < U < \bar{U}$. This is due to the fact that in this region the boundary line is above the heteroclinic connection, but the unstable manifold is below it for $c < c_0$. Hence, $V = b(c)U$ and $M_1^+(c)$ cannot approach each other in this region.

On the other hand, we note that by definition of $b(c)$ in (3.1.27), we have $b(c) < -c$ for all c. At $U = a$, we have $f(U) = 0$, and hence $dV/dU = -c$. So, at any point

(U, V) in the fourth quadrant with $a < U < 1$, we have

$$\frac{dV}{dU} = -c + \frac{f(U)}{-V} = -c + \frac{f(U)}{|V|} > -c > b(c). \tag{3.3.10}$$

as $f(U) > 0$ for $a < U < 1$. Hence, for $a < U < 1$, the slope of the unstable manifold is always greater than the boundary line.

When $\bar{U} > a$, the slope of the vector field, and hence the slope of the unstable manifold, is always bigger than the slope of the boundary line for all points (U, V) with $a < \bar{U} < U < 1$ and $V < 0$, due to the above argument.

When $\bar{U} < a$, we need to check the slope of the vector field in the region with $\bar{U} < U < a$. In the region with $0 < U < a$, the heteroclinic is decreasing and concave up and $f(U) < 0$. In addition, for $V_2 < V_1 < 0$ we have $\frac{1}{V_2} > \frac{1}{V_1}$, and so for fixed $0 < U < a$, we have

$$-c - \frac{f(U)}{V_2} = -c + \frac{|f(U)|}{V_2} > -c + \frac{|f(U)|}{V_1} = -c - \frac{f(U)}{V_1}. \tag{3.3.11}$$

Hence, for fixed U, if we decrease $V < 0$, the slope of the vector field increases. Now we study the vector field in the region with $\bar{U} < U < a$. At $(\bar{U}, \bar{V})$, we have

$$\left.\frac{dV}{dU}\right|_{(\bar{U},\bar{V})} > b(c). \tag{3.3.12}$$

At any point (U_1, V_1) on the heteroclinic connection with $\bar{U} < U_1 < a$ and so $V_1 < \bar{V}$, we have

$$\left.\frac{dV}{dU}\right|_{(U_1,V_1)} > b(c), \tag{3.3.13}$$

because the heteroclinic connection is concave up. At any point (U_1, V_2) on the boundary line with $V_2 < V_1 < \bar{V}$, we have

$$\left.\frac{dV}{dU}\right|_{(U_1,V_2)} > b(c), \tag{3.3.14}$$

because of (3.3.11).

Hence, for $\bar{U} < U < a$, the slope of the vector field is bigger than the slope of the boundary line for $c_* < c < c_0$.

We have shown above that regardless of the value of $\bar{U}$, the slope of the unstable manifold $M_1^+(c)$ is always greater than the slope of the boundary line $V = b(c)U$ for all points (U, V) with $\bar{U} < U$ and $V < 0$. So, there cannot be any intersection point at which the slope of the manifold is smaller than the slope of the boundary line in this region. Therefore, the intersection between the boundary line $V = b(c)U$ and the unstable manifold $M_1^+(c)$ is unique for $c_* < c < c_0$, when $b(c) > \lambda_0^-(c_0)$.

When $b(c) \leq \lambda_0^-(c_0)$, we can not use the argument in the previous case. However, for these values of c, the intersection between the boundary line and $M_1^+(c)$ must occur at some point below or on the tangent line $V = \lambda_0^-(c_0)U$. So, to show that only one intersection between the boundary line and the corresponding unstable manifold is possible for these values of c, we only need to show that the slope of the vector field is bigger than $\lambda_0^-(c_0) = -\sqrt{2}/2$ for all points below or on the tangent line $V = \lambda_0^-(c_0)U$. Indeed, for the vector field, we have

$$\frac{dV}{dU} = -c - \frac{U(U-a)(1-U)}{V}. \tag{3.3.15}$$

To estimate the right-hand side of (3.3.15), we first note that for any point below or on the tangent line $V = \lambda_0^-(c_0)U = -\sqrt{2}/2U$, we have

$$0 > U/V \geq -\sqrt{2}. \tag{3.3.16}$$

Now, the expression $-(1-U)(U-a)$ is an upward parabola, which, for $0 \leq U \leq 1$, has a maximum value of a and minimum of $-(a-1)^2/4$. Hence, for $0 < a < 1$, we have

$$-1/4 < -(1-U)(U-a) < a. \tag{3.3.17}$$

From (3.3.16) and (3.3.17), we find

$$-\frac{U(U-a)(1-U)}{V} \geq -\sqrt{2}a. \tag{3.3.18}$$

Finally, from (3.3.18) and equation for c_0 in (3.1.9), we find that

$$\frac{dV}{dU} > -c_0 - \frac{U(U-a)(1-U)}{V} \geq -\frac{\sqrt{2}}{2}, \tag{3.3.19}$$

for all $c < c_0$, as claimed. ∎

Corollary 3.3.13. *If $c^* > c_0$, then there exist at least two intersections between the unstable manifold of $(1,0)$ and the boundary line $V = bU$ for $c_0 < c < c^*$. In other words, there exist at least two traveling wave solutions that satisfy (3.1.24)–(3.1.26) for each c between c_0 and c^*. If the number of intersections is two, we have two TWI. If there are more than two intersections, there are two TWI, and the rest are TWII.*

Speed Range for the Existence of TWII

The speed range for TWII depends on the range of a as well. However, it is not as straightforward as in the case of TWI to find the speed range for TWII and it involves a few different cases. In the following theorem, we summarize our observation in the phase plane regarding the speed range and the maximal speed for the existence of TWII for different values of a. The proof follows from a similar geometric consideration as in the proof of previous results.

Theorem 3.3.14. *Based on the value of a and the location of the boundary line at specific values of speed, we can describe the possible speed range for the existence of TWII in the following cases:*

i) If $b(c_0) \leq \lambda_0^-(c_0)$, there exists no TWII for the reduced system, (3.1.24)–(3.1.26).

ii) If $b(c_0) > \lambda_0^-(c_0)$, we have the following three cases:

ii-a) for $0 < a \leq (1-\sqrt{2/3})/2$, there exists no TWII.

ii-b) for $(1-\sqrt{2/3})/2 < a < 1/2$, we have

ii-b-1) if the number of intersections between the line $V = b(c_0)U$ and $M_1^+(c)$ is at most three for $c > c_0$, then there exists no TWII for $c_0 < c$.

ii-b-2) if the number of intersections between the line $V = b(c_0)U$ and $M_1^+(c)$ for some $c > c_0$ is more than three, then there exists a maximal speed c^{**} with $c^{**} < c^*$ such that for each $c_0 < c \leq c^{**}$ there exists at least one TWII.

ii-c) for $1/2 \leq a < 1$, we have

ii-c-1) if the number of intersections between the line $V = b(0)U$ and $M_1^+(c)$ is at most three for $c > 0$, then there exists no TWII for $c > c_0$.

ii-c-2) if the number of intersections between the line $V = b(0)U$ and $M_1^+(c)$ for some $c > c_0$ is more than three, then there exists a maximal speed c^{**} with $c^{**} < c^*$ such that for each $0 < c \leq c^{**}$, there exists at least one TWII.

Speed Range for the Existence of TWIII

This type of wave appears for a very specific range of speeds, in which the horizontal line $c = c_*$, corresponding to the zero of the slope of the boundary line b, passes through region 3 in Figure 3.9. Before determining the range for the existence of this wave type, we explore the position of the unstable manifold $M_1^+(c)$ with respect to the linearization $V = \lambda_1^+(c)(U-1)$. This information will be useful in the proof of existence of TWIII (and TWIV as well).

Lemma 3.3.15. *We have the following results regarding the position of the linearization of the unstable manifold of $(1, 0)$:*

i) The part of unstable manifold of $(1,0)$*,* $M_1^+(c)$*, that enters the first quadrant, either directly or after passing through the fourth quadrant, always lies above the linearization* $V = \lambda_1^+(c)(U-1)$*.*

ii) For $c < c_*$*, there exists an intersection between the boundary line* $V = b(c)U$ *and the linearization* $V = \lambda_1^+(c)(U-1)$ *at some* $U > 1$*.*

Proof: To prove part i), we first consider the part of unstable manifold $M_1^+(c)$ that goes directly to the first quadrant, which is tangent to its linearization $V = \lambda_1^+(c)(U-1)$ at $(1,0)$. If we take the inner product between the normal to the linearization and the vector field, we find

$$\begin{aligned}(-\lambda_1^+,1)\cdot(V,-cV-f(U)) &= -\lambda_1^+V - cV - U(U-a)(1-U)\\ &= -\lambda_1^+\left(\lambda_1^+(U-1)\right) - c\lambda_1^+(U-1) - U(U-a)(1-U)\\ &= (U-1)\left(-(\lambda_1^+)^2 - c\lambda_1^+ + U(U-a)\right),\end{aligned}$$

where we used the fact that we are on the line $V = \lambda_1^+(c)(U-1)$. We claim that $\left(-(\lambda_1^+)^2 - c\lambda_1^+ + U(U-a)\right) > 0$. Indeed, recalling that $\lambda_1^+ = \frac{-c}{2} + \frac{1}{2}\sqrt{c^2+4(1-a)}$, we can see that $(\lambda_1^+)^2 = -c\lambda_1^+ + (1-a)$. Since $U > 1$, we have

$$(\lambda_1^+)^2 + c\lambda_1^+ - U(U-a) < (\lambda_1^+)^2 + c\lambda_1^+ - (1-a) = 0.$$

Therefore, we have proved the claim. This proves that the angle between the normal to the line $V = \lambda_1^+(c)(U-1)$ and the vector field is acute. Hence, the part of the unstable manifold $M_1^+(c)$ that goes directly to the first quadrant cannot pass through its linearization at $(1,0)$, and will always stay above it for all c. This also proves that the part of $M_1^+(c)$ that goes to the fourth quadrant and then enters the first quadrant for all $c_0 < c < 0$ and $1/2 < a < 1$, when $(a,0)$ is unstable, is also lying above the line $V = \lambda_1^+(c)(U-1)$, see Figure 3.13.

To prove part ii), we note that the slope of the boundary line $V = b(c)U$ is positive for $c < c_*$, see Lemma 3.1.3. If it is smaller than $\lambda_1^+(c)$, there is an intersection between the boundary line and $V = \lambda_1^+(c)(U-1)$. Recalling the definition of $b(c)$ in (3.1.27) and $\lambda_1^+(c)$ in (3.1.8), we must show that

$$\left(\frac{\Delta}{2}-1\right)c - \Delta\sqrt{\frac{c^2}{4}+Dm} < -\frac{c}{2} + \sqrt{\frac{c^2}{4}+(1-a)},$$

which is equivalent to

$$-(1-\Delta)\,c < \sqrt{c^2+4(1-a)} + \Delta\sqrt{c^2+4Dm}. \tag{3.3.20}$$

But equation (3.3.20) is obvious for all c including $c \le c_*$. Therefore, there is an intersection between the boundary line $V = b(c)U$ and the line $V = \lambda_1^+(c)(U-1)$. ∎

We now prove the existence of TWIII for a specific range of c.

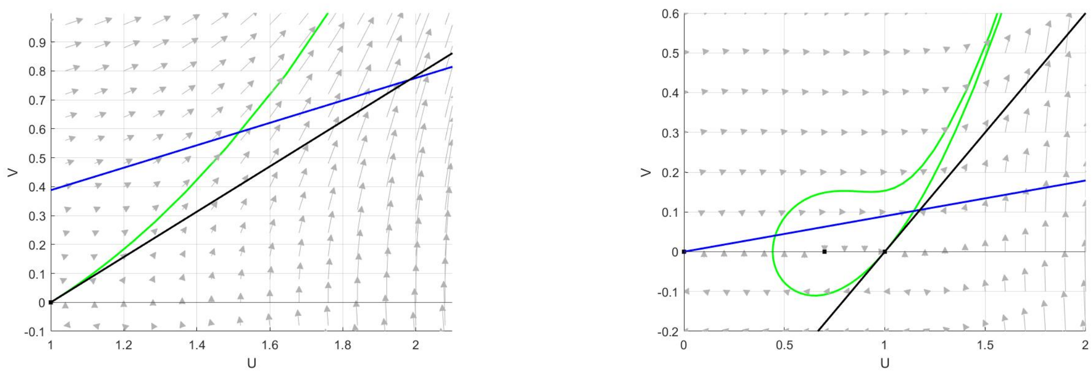


Figure 3.13: Visualization for Lemma (3.3.15). The green curve is $M_1^+(c)$, the black line is its linearization, $V = \lambda_1^+(c)(U-1)$, and the blue line is the boundary line. The figures show the angle between the vector field and the linearization, the position of $M_1^+(c)$ w.r.t its linearization, and the position of the intersections between the boundary line $V = b(c)U$, the line $V = \lambda_1^+(c)(U-1)$, and the curve $M_1^+(c)$. The picture on the left is the magnified version of the picture on the right around the intersection points. The value of the parameters for the graphs are $a = 0.7, m = 1, \Delta = 0.01$, $D = 1$, and $c = -0.1$.

Theorem 3.3.16. *Let $1/2 < a < 1$ and assume that $c_0 < c_*$. Then there exists a TWIII for all $c_0 < c < c_*$.*

Proof: For $1/2 < a < 1$, we have $c_0 < 0$. By Lemma 3.1.3, the slope of the boundary line $V = b(c)U$ is positive for $c < c_*$. Note that for $c < 0$, the steady state $(a, 0)$ is unstable. Following the vector field for $c_0 < c < c_*$, the part of $M_1^+(c)$ that goes into the fourth quadrant, moves above the heteroclinic connection corresponding to c_0 in the fourth quadrant intersects the U-axis between 0 and a, and enters the first quadrant. There, it approaches the part of $M_1^+(c)$ that leaves $(1, 0)$ into the first quadrant. Hence, by Lemma 3.3.15, this part of $M_1^+(c)$ also stays above the line $V = \lambda_1^+(c)(U-1)$ and so must intersect the boundary line. Recalling Definition 3.3.2, this results in a TWIII. ∎

Speed Range for the Existence of TWIV

As mentioned before, TWIV is different from the other three types of TWS in the sense that we look for an intersection between the boundary line and the part of the unstable manifold that leaves $(1, 0)$ to go directly to the first quadrant. This part of $M_1^+(c)$ does not depend on the stability of $(a, 0)$ and its behaviour is similar for all values of c. Hence, as long as the boundary line is in the first quadrant, we can prove the existence of TWIV using Lemma 3.3.15.

Theorem 3.3.17. *For $0 < a < 1$ and for all $c < c_*$ there exists a unique traveling wave solution of type IV.*

Proof: By part *i*) of Lemma 3.3.15, we know that the part of the unstable manifold $M_1^+(c)$ that goes to the first quadrant and is tangent to its linearization near $(1, 0)$ cannot cross this linearization and will always stay above it. In addition, by part *ii*) Lemma 3.3.15, there is an intersection between the boundary line $V = b(c)U$ and the line $V = \lambda_1^+(c)(U - 1)$ for all $c < c_*$. Hence, as the unstable manifold $M_1^+(c)$ lies above the line $V = \lambda_1^+(c)(U - 1)$ in the first quadrant, there is an intersection between the boundary line and $M_1^+(c)$ resulting in a TWIV. Similar to the proof of Lemma 3.3.15, we can follow the direction of the vector field to see that $M_1^+(c)$ always stays above the boundary line after their intersection. Hence, the intersection point is unique. ∎

Remark 3.3.18. *When $1/2 < a < 1$ and $c_0 < c_*$, we have TWSs of both types III and IV for all $c_0 < c < c_*$.*

3.4 Traveling Wave Solution for the Model with a Desired Speed in the Three Scenarios

In the previous section, we have shown that, for different ranges of c, we can find different types of TWS for reduced system (3.2.9)–(3.2.14), or equivalently (3.1.24)–(3.1.26). Now, to show that the full system (3.2.9)–(3.2.15) has a traveling wave solution of a specific type, we need to show that there exists at least one value of c in the corresponding range of speeds for the wave type at which the solution to reduced system (3.2.9)–(3.2.14) satisfies the equation for the speed of the wave, (3.2.15), as well. We discuss this for each of the scenarios for the movement of the boundary individually.

As seen in the previous section, the position of the boundary line at c_0 with respect to the tangent line at $(0, 0)$ to $M_1^+(c_0)$, the heteroclinic connection between $(1, 0)$ and $(0, 0)$ at c_0, determines the behaviour of the system. In other words, depending on whether $b(c_0) \leq \lambda_0^-(c_0)$, which is equivalent to

$$Dm \geq 2\frac{a}{\Delta}\left(\frac{a}{\Delta} + (\frac{1}{2} - a)\right),$$

or $b(c_0) > \lambda_0^-(c_0)$, which is equivalent to

$$Dm < 2\frac{a}{\Delta}\left(\frac{a}{\Delta} + (\frac{1}{2} - a)\right),$$

different TWS with different speeds may appear. The above relation between the parameters plays a significant role in the existence of TWS for the full system as well.

3.4.1 Scenario 1

In the first scenario, we rewrite condition (3.2.15) for TWS as

$$c = \eta_1 \alpha U(0^-). \tag{3.4.1}$$

Since η_1, α, and density $U(0^-)$ are positive, we need to look for a positive speed in the corresponding range for each type of TWS such that (3.4.1) is satisfied.

Remark 3.4.1. *We recall that on the right-hand side of* (3.4.1)*, the density* $U(0^-)$ *depends on* c*. Therefore, we write*

$$U_0(c) := U(0^-).$$

In addition, we know from Corollaries 3.3.13 and 3.3.12 that for some values of c*, namely* $c_0 < c$*, we may have more than one TWS for system* (3.2.9)–(3.2.14)*. When we need to denote different values of* $U(0^-)$ *corresponding to the same speed* c*, we write*

$$U_{0_i}(c) := U_i(0^-) \quad \textit{for} \quad i = 1, ..., k,$$

where k *is the number of traveling wave solutions of* (3.2.9)–(3.2.14) *corresponding to the speed* c*.*

Based on equation (3.4.1), the speed of the solution for the full system should be positive, whereas by Lemmas 3.3.16 and 3.3.17, we know that TWIV and TWIII appear only for negative speeds. Therefore, we cannot have any TWIII or TWIV satisfying system (3.2.9)–(3.2.14) and equation (3.4.1) at the same time. For TWI and TWII of the reduced system, we know from Section 3.3.2 that we may have $c^* > 0$. Therefore, we may have a solution of these types for system (3.2.9)–(3.2.15) in this scenario.

We start by studying the existence of TWI for the full system. In Corollary 3.3.13, we saw that for the same value of speed, we may have more than one TWI and TWII for the reduced system. In the following lemma, we look at the behavior of the $U_{0_i}(c)$ for TWI as a function of c.

Lemma 3.4.2. *Consider TWI for the reduced system* (3.1.24)–(3.1.26) *with* $0 < a < 1$ *and for different values of* c*. The functions* $U_{0_i}(c)$*, with* $i = 1, 2$*, have the fallowing properties:*

i) If $b(c_0) \leq \lambda_0(c_0)$*, then* $U_0(c)$ *is a continuous, monotone decreasing function of* c *on* $c_* \leq c < c_0$ *with* $\lim_{c \nearrow c_0} U_0(c) = 0$ *and* $\lim_{c \searrow c_*} U_0(c) = 1$*.*

ii) If $b(c_0) > \lambda_0(c_0)$, then for $U_{0_1}(c)$ and $U_{0_2}(c)$ with $0 < U_{0_2}(c) < U_0(c^*) < U_{0_1}(c) \le 1$, we have the following properties:

ii-a) $U_{0_1}(c)$ is monotone decreasing on $c_* \le c \le c^*$, $\lim_{c\searrow c_*} U_{0_1}(c) = 1$ and $\lim_{c\nearrow c^*} U_{0_1}(c) = U_0(c^*)$.

ii-b) If $c_0 \ge c_*$, then $U_{0_2}(c)$ is monotone increasing on $c_0 < c \le c^*$. Furthermore, $\lim_{c\searrow c_0} U_{0_2}(c) = 0$ and $\lim_{c\nearrow c^*} U_{0_2}(c) = U_0(c^*)$.

ii-c) If $c_* > c_0$, then $U_{0_2}(c)$ is monotone increasing on $c_* \le c \le c^*$. Furthermore $\lim_{c\searrow c_*} U_{0_2}(c) = U_{0_2}(c_*)$ and $\lim_{c\nearrow c^*} U_{0_2}(c) = U_0(c^*)$.

Proof: For part *i*), we first note that when $b(c_0) \le \lambda_0(c_0)$, we have $c^* = c_0$. On the other hand, Theorem 3.3.12 implies the existence of a unique TWI for the reduced system (3.2.9)–(3.2.14) for each $c < c_0$. Thus, in this case we have $k = 1$, and as in Remark 3.4.1, we let $U_{0_1}(c) = U_0(c)$. Continuity of $U_0(c)$ can be deduced from the continuity of the vector field and $b(c)$ with respect to c. Next, we show that $U_0(c)$ is monotone decreasing. By Lemma 3.3.1, we know that for $c' < c'' < c_0$, the unstable manifold $M_1^+(c'')$ lies above $M_1^+(c')$ in the fourth quadrant and before their first intersection with the U- or V-axis. On the other hand, we know from Lemma 3.1.3 that $db/dc < 0$, so that the boundary line $V = b(c')U$ lies above $V = b(c'')U$. This opposite movement of the boundary line and the unstable manifold and their continuity with respect to c imply that the corresponding intersection point, $U_0(c)$, is monotone decreasing in c.

We next prove that $\lim_{c\nearrow c_0} U_0(c) = 0$. Consider the heteroclinic connection of $(1, 0)$ and $(0, 0)$ for $c = c_0$. The slope of the tangent line to this orbit at $(0, 0)$ is $\lambda_0^-(c_0) = -\sqrt{2}/2$. By assumption, we have $b(c_0) \le \lambda_0^-(c_0)$, and by Lemma 3.1.3, we have $b(c_0) < b(c)$ for all $c < c_0$. We pick a point $(\bar{U}, \bar{V})$ on the heteroclinic connection of $c = c_0$ with $\bar{U} > 0$ small. We also choose some small $\varepsilon > 0$. By continuity of the vector field and $b(c)$ with respect to c, we can choose c large enough such that $M_1^+(c)$ passes through the ε–neighborhood of $(\bar{U}, \bar{V})$, and at the same time $b(c_0) < b(c) < \lambda_0^-(c_0)$. Hence, the boundary line $V = b(c)V$ stays below the tangent line $V = \lambda_0^-(c_0)U$. As a result of this choice of c, the unstable manifold $M_1^+(c)$ must pass through the ε–neighborhood of $(\bar{U}, \bar{V})$ and intersect the tangent line $V = \lambda_0^-(c_0)U$ before $V = b(c)U$. Therefore, the intersection of $M_1^+(c)$ and $V = b(c)U$ must occur at some point with $U_0(c) < \bar{U}$. Since $\bar{U}$ was arbitrary we can conclude that $\lim_{c\nearrow c_0} U_0(c) = 0$.

To show $\lim_{c\searrow c_*} U_0(c) = 1$, we recall that $b(c_*) = 0$. Therefore, the intersection of the boundary line and the unstable manifold can only happen near $(1, 0)$ for c close to c_*. Hence, for the limit as $c \searrow c_*$, we approximate $U_0(c)$ by the intersection of the boundary line $V = b(c)U$ and the linearization of $M_1^+(c)$ near $(1, 0)$, the line

$V = \lambda_1^+(c)(U-1)$. We find

$$U_0(c) \approx \frac{\lambda_1^+(c)}{\lambda_1^+(c) - b(c)} \quad \text{for } c_* < c.$$

By taking the right limit, we get

$$\lim_{c \searrow c_*} \frac{\lambda_1^+(c)}{\lambda_1^+(c) - b(c)} = 1,$$

as we have $b(c_*) = 0$. So we have proved i).

For part ii), we have $c^* > c_0$, and so for $c_0 < c < c^*$, the unstable manifold $M_1^+(c)$ hits the U-axis and leaves the forth quadrant. Hence, there are two intersection points between the part of the unstable manifold $M_1^+(c)$ in the fourth quadrant and the boundary line $b(c)$, namely $U_{0_i}(c)$, $i = 1, 2$. Considering the direction of the vector field in the fourth quadrant and the opposite movement of $M_1^+(c)$ and $b(c)$ with respect to c, one can see that $U_{0_2}(c) < U_0(c^*) < U_{0_1}(c)$.

All claims in part $ii{-}a$) can be proved as in part i) except $\lim_{c \nearrow c^*} U_{0_1}(c) = U_0(c^*)$, which can be deduced from continuity of $U_0(c)$.

For part $ii{-}b$), to show that $U_{0_2}(c)$ is increasing, we first note that $U_{0_2}(c)$ appears for $c > c_0$, where the corresponding $M_1^+(c)$ moves above the heteroclinic connection of $(1,0)$ and $(0,0)$ at c_0, and that $U_{0_2}(c) < U_{0_1}(c)$ is the second intersection of $M_1^+(c)$ and $b(c)$, which happens at values of U closer to 0. Therefore, by the movement of $M_1^+(c)$ and $b(c)$ with respect to c, we can conclude the result. Monotonicity and the fact that $\lim_{c \nearrow c^*} U_{0_2}(c) = U_0(c^*)$ can also be proved as in part i) and $ii - a$).

To show $\lim_{c \searrow c_0} U_{0_2}(c) = 0$, we follow an argument analogous to the one in part i). We again consider a point $(\bar{U}, \bar{V})$ on the heteroclinic connection of $c = c_0$, with $\bar{U} > 0$ small and choose some small $\varepsilon > 0$. By continuity, we can choose $c_0 < c < c^*$ close enough to c_0 such that $M_1^+(c)$ passes through the ε–neighborhood of $(\bar{U}, \bar{V})$. Due to the direction of the vector field and the fact that the line $V = b(c^*)U$ stays above the tangent line $V = \lambda_0^-(c_0)U$, the intersection of $M_1^+(c)$ with $b(c^*)$ happens and some U_X with $U_X < \bar{U} + \varepsilon$. As $c < c^*$, and so $b(c) > b(c^*)$, the intersection of $M_1^+(c)$ and $b(c)$ must occur at some $U_{0_2}(c) < U_X < \bar{U} + \varepsilon$. Since $\bar{U}$ and ε were arbitrary and small, we can conclude that $\lim_{c \searrow c_0} U_{0_2}(c) = 0$.

To prove part $ii{-}c$), one can show similar to the previous parts of the lemma that $U_{0_2}(c)$ is continuous and monotone increasing. Also, the claim that $\lim_{c \to c_*} U_{0_2}(c) = U_{0_2}(c_*)$ and $\lim_{c \to c^*} U_{0_2}(c) = U_0(c^*)$ are the results of continuity of $U_{0_2}(c)$ on $c_* \leq c \leq c^*$. ∎

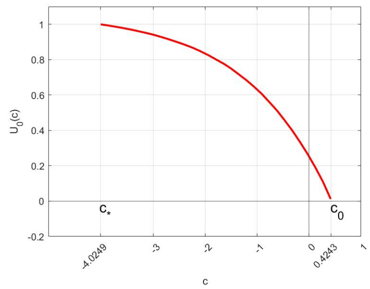
Remark 3.4.3. *We know that for* $0 < a < 1/2$ *and* $a = 1/2$*, we have* $c_0 > 0$ *and* $c_0 = 0$*, respectively. Hence, we always have* $c_0 > c_*$ *and only part* i)*,* $ii{-}a$) *and* $ii{-}b$)

describe the behaviour of $U_{0_i}(c)$. *In addition, we will only have two possible graphs of* $U_0(c)$ *(or* $U_{0_i}(c)$*) for each case* i*) and* ii*), see Fig 3.14. However, when* $1/2 < a < 1$*, we have* $c_0 < 0$ *and so* c^* *may be positive, negative or even zero and also we may have* $c_0 < c_*$*. Therefore, all three parts of Lemma 3.4.2 apply. In addition, for part* ii*) we have six possible graphs for* $U_{0_i}(c)$ *depending on the sign of* c^* *and the relation between* c_0 *and* c_**, see Fig 3.15.*

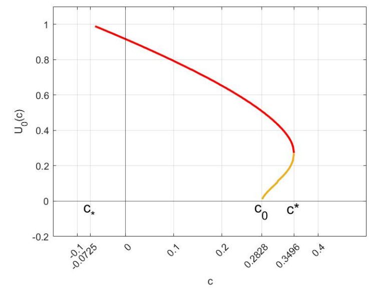
Having determined the behavior of $U_0(c)$, we now bring the results regarding the existence of TWI solutions for the full system for different ranges of parameters. To prove the existence of a solution for the full system, we need to show that the solution to the reduced system (3.1.24)–(3.1.26) satisfies (3.4.1) for a specific value of c in $[0, c^*]$, i.e. $c = \eta_1 \alpha U_0(c)$ has a solution in $[0, c^*]$. Equivalently, it suffices to show that there exists at least one intersection between the line $y_1 := c/\alpha\eta_1$ and the curve $y_2 := U_0(c)$, depicted in Figures 3.14 and 3.15 for different ranges of parameters. We use this idea to prove the existence results in the following lemmas. In the next lemma, we can see that at least for some ranges of parameters there exists a unique speed c satisfying (3.4.1).

Lemma 3.4.4. *If* $0 < a < 1/2$ *and* $b(c_0) \leq \lambda_0^-(c_0)$*, then there exists a unique speed* $c \in (0, c^*)$*, for which the corresponding TWI of* (3.2.9)–(3.2.14) *satisfies equation* (3.4.1) *for scenario 1 as well.*

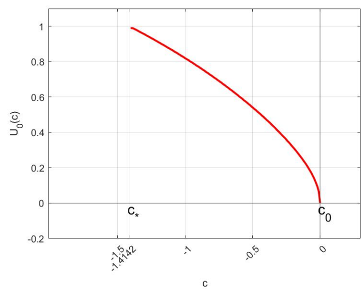
Proof: Here, we follow the proof of Theorem 3.7 in [48]. By Corollary 3.3.8, if $b(c_0) \leq \lambda_0^-(c_0)$, we have $c^* = c_0$. On the other hand, we know that in this scenario, the speed is positive. Hence, based on the result in Section 3.3.2, for any $c \in (0, c_0)$ there exists a unique TWI for the reduced system (3.2.9)–(3.2.14). It remains to show that this solution satisfies (3.4.1) for a specific value of c in this interval. At $c = 0$, we have $y_1 = 0$ and $y_2 = U_0(0) > 0$. By part i) of Lemma 3.4.2, we know that $y_2 = U_0(c)$ is continuous and monotone decreasing on $[0, c^*]$ and $\lim_{c \to c_0} U_0(c) = 0$, see Figure 3.14 ($a$). Then by the intermediate value theorem, there exists a unique intersection between the line $y_1 = c/\eta_1\alpha$ and the curve $y_2 = U_0(c)$, and so (3.4.1) has a unique solution in $(0, c_0)$. This solution corresponds to a TWI for the system (3.2.9)–(3.2.14) with (3.4.1). ∎



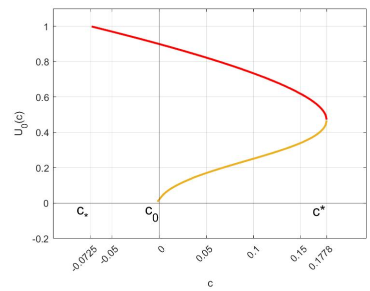
(a) $0 < a < \frac{1}{2}$, $b(c_0) \leq \lambda_0^-(c_0)$, $c^* = c_0 > 0$



(b) $0 < a < \frac{1}{2}$, $b(c_0) > \lambda_0^-(c_0)$, $c^* > c_0 > 0$



(c) $a = \frac{1}{2}$, $b(c_0) \leq \lambda_0^-(c_0)$, $c^* = c_0 = 0$

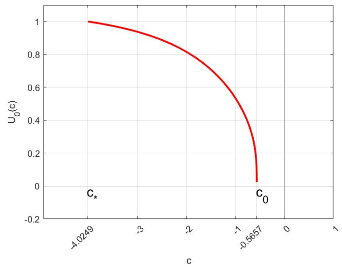


(d) $a = \frac{1}{2}$, $b(c_0) > \lambda_0^-(c_0)$, $c^* > c_0 = 0$

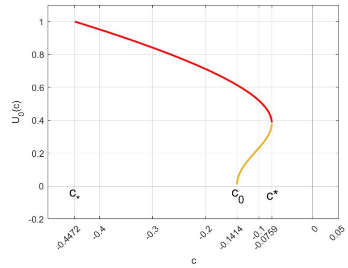
Figure 3.14: The graph of $U_{0_i}(c)$, $i = 1, 2$, w.r.t. c for $0 < a < \frac{1}{2}$. The red curve is $U_{0_1}(c)$ and the orange one is $U_{0_2}(c)$, when it exists. The graphs in (a) and (c) correspond to part i) of Lemma 3.4.2, and the ones in (b) and (d) correspond to parts ii-a) and ii-b) of Lemma 3.4.2. In part (a), we have $a = 0.2$, $\Delta = 0.9$, $D = 1$, $m = 2$. In part (b), we have $a = 0.3$, $\Delta = 0.5$, $D = 1$ and $m = 1$. In part (c), we have $a = 0.5$, $\Delta = 0.5$, $D = 2$, $m = 2$. In part (d), we have $a = 0.5$, $\Delta = 0.05$, $D = 2$ and $m = 1$.

Lemma 3.4.5. *If $0 < a < 1/2$ and $b(c_0) > \lambda_0^-(c_0)$, then there exists at least one speed $c \in (0, c^*]$, for which the corresponding TWI of* (3.2.9)–(3.2.14) *satisfies equation* (3.4.1) *for scenario 1 as well.*

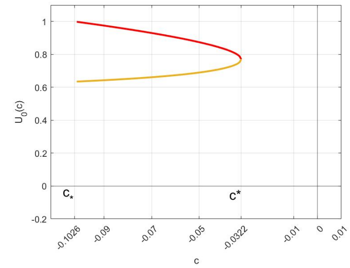
Proof: We note that for this range of parameters, we have $c_0 < c^*$. Also, we know from Lemma 3.4.2 that we have two branches for the intersection point, namely $U_{0_1}(c)$ and $U_{0_2}(c)$ with $U_{0_1}(c^*) = U_{0_2}(c^*)$ and $U_{0_1}(0) = 0$, see Figure 3.14, (b). Therefore, there will be an intersection between one of the curves $y_2 = U_{0_i}(c)$, $i = 1, 2$ and the line $y_1 = c/\eta_1\alpha$, regardless of its slope. However, if the line intersects $U_{0_2}(c)$, there may be more than one intersection between them depending on the shape of the $U_{0_2}(c)$ and the slope of the line. ∎



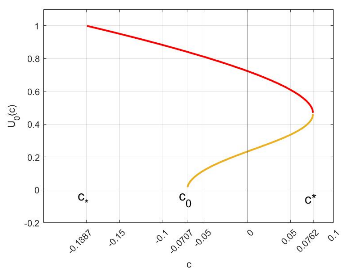
(a) $\frac{1}{2} < a < 1$, $b(c_0) \leq \lambda_0^-(c_0)$ $c^* = c_0 < 0$



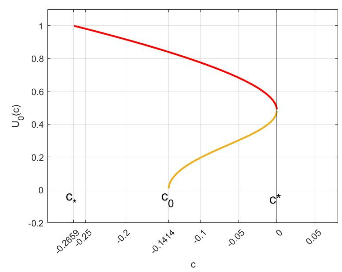
(b) $\frac{1}{2} < a < 1$, $b(c_0) > \lambda_0^-(c_0)$, $c^* < 0$, $c_* < c_0$



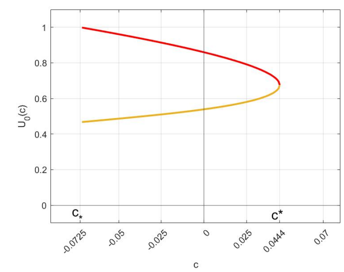
(c) $\frac{1}{2} < a < 1$, $b(c_0) > \lambda_0^-(c_0)$, $c^* < 0$, $c_0 < c_*$



(d) $\frac{1}{2} < a < 1$, $b(c_0) > \lambda_0^-(c_0)$, $c^* > 0$, $c_* < c_0$



(e) $\frac{1}{2} < a < 1$, $b(c_0) > \lambda_0^-(c_0)$, $c^* = 0$, $c_* < c_0$



(f) $\frac{1}{2} < a < 1$, $b(c_0) > \lambda_0^-(c_0)$, $c^* > 0$, $c_0 < c_*$

Figure 3.15: The graph $U_{0_i}(c)$, $i = 1, 2$, w.r.t. c for $\frac{1}{2} < a < 1$. The red curve is $U_{0_1}(c)$ and the orange one is $U_{0_2}(c)$, when it exists. The graph in (a) corresponds to part i) of Lemma 3.4.2, and the ones in (b), (d), and (e) correspond to parts ii-a) and ii-b) of Lemma 3.4.2. The graphs in (c) and (f) correspond to parts ii-a) and ii-c) of Lemma 3.4.2. In part (a), we have $a = 0.9$, $\Delta = 0.9$, $D = 1$, $m = 2$. In part (b), we have $a = 0.6$, $\Delta = 0.2$, $D = 2$ and $m = 2$. In part (c), we have $a = 0.9$, $\Delta = 0.05$, $D = 2$, $m = 2$. In part (d), we have $a = 0.55$, $\Delta = 0.09$, $D = 2$ and $m = 2$. In part (e), we have $a = 0.8$, $\Delta = 0.04$, $D = 1$ and $m = 1$. In part (f), we have $a = 0.7$, $\Delta = 0.05$, $D = 2$, and $m = 1$.

When $a = 1/2$, we have $c_0 = 0$ and we obtain the following result regarding the existence of solutions for the full system.

Lemma 3.4.6. *For $a = 1/2$, we have the following two cases:*

i) If $b(c_0) \leq \lambda_0^-(c_0)$, or equivalently $\Delta\sqrt{Dm} \geq \frac{\sqrt{2}}{2}$, then there exists no solution for the full system.

ii) If $b(c_0) > \lambda_0^-(c_0)$, or equivalently $\Delta\sqrt{Dm} < \frac{\sqrt{2}}{2}$, then one can choose η_1 such that there exists at least one solution for the full system.

Proof: To prove part i), we note that for this range of parameters, the boundary line is below the tangent line to the heteroclinic connection at $c_0 = 0$. Hence, $U_0(c)$ is monotone decreasing on $[c_*, 0]$. Therefore, there cannot be an intersection between the graph of $y_2 = U_0(c)$ and the line $y_1 = c/\eta_1\alpha$. Hence, there is no solution for the full system.

To prove part *ii*), we again note that for this range of parameters, we have $c_0 = 0 < c^*$. Also, we know by Lemma 3.4.2 that we have two branches for the intersection point, namely $U_{0_1}(c)$ and $U_{0_2}(c)$ with $U_{0_1}(c^*) = U_{0_2}(c^*)$ and $U_{0_1}(0) = 0$, see Figure 3.14 (d). Therefore, choosing η_1 appropriately results in at least one intersection between $y_2 = U_{0_i}(c)$, $i = 1, 2$ and the line $y_1 = c/\eta_1\alpha$. This intersection corresponds to a solution to the full system with scenario 1 as its moving boundary condition. ∎

When $1/2 < a < 1$, we have $c_0 < 0$, so that the maximal speed c^* can be negative. Obviously, for $c^* < 0$, we can not find a speed c that satisfies condition (3.4.1). As a result, we can discuss the existence of solutions for system (3.2.9)–(3.2.15) only when the maximal speed c^* for the reduced system is positive. We prove, similar to Lemma 3.4.4, that the desired speed $c \in (0, c^*)$ exists for different choices of η_1 and may be unique. In the following lemma, we discuss these conditions in detail. We note here that the condition for the sign of c^* can be checked as in Lemma 3.3.10.

Lemma 3.4.7. *For* $1/2 < a < 1$,

i) if $b(c_0) \leq \lambda_0^-(c_0)$*, then there exists no solution for the full system.*

ii) if $b(c_0) > \lambda_0^-(c_0)$ *and* $c^* \leq 0$*, then there exists no solution for the full system.*

iii) if $b(c_0) > \lambda_0^-(c_0)$ *and* $c^* > 0$*, then one can choose* η_1 *such that there exists at least one solution for the full system.*

Proof: The result can be proved by following the same ideas as in the proofs of Lemmas 3.4.4–3.4.6. ∎

In the following corollary, we present a sufficient condition for the existence of solutions for the full system when $1/2 \leq a < 1$.

Corollary 3.4.8. *Let* $a = 1/2$ *(resp.* $1/2 < a < 1$*). If*

$$c^* > \eta_1 \alpha U_0(c^*), \tag{3.4.2}$$

there is at least one (resp. two) speed(s) in $(0, c^*)$ *for which the traveling wave solution of* (3.2.9)–(3.2.14) *satisfies* (3.4.1) *as well.*

Proof: Satisfying condition (3.4.2) yields the line $y = c/\eta_1\alpha$ to stay above the curves $U_{0_i}(c)$ at c^*, which in turn results in at least one intersection between the two and hence a solution to the full system. ∎

In the following remark, we discuss the existence of TWII for different values of parameters.

Remark 3.4.9. *If a TWII exists for the reduced system* (3.1.24)–(3.1.26) *(see Theorem 3.3.14 for the conditions), it appears for* $c_0 < c \le c^{**}$*, when* $0 < a \le 1/2$*, and for* $0 < c \le c^{**}$*, when* $1/2 < a < 1$*. Then, one can choose* η_1 *such that there exists at least one TWII for the full system* (3.1.24)–(3.1.26) *with* (3.4.1)*. Otherwise, there exists no TWII for the full system with scenario 1.*

3.4.2 Scenario 2

For this scenario, condition (3.2.15) for the speed of the traveling wave can be written as

$$c = \eta_1 \alpha (U(0^-) - \bar{u}), \tag{3.4.3}$$

where $\bar{u}$ is threshold density at the front. Based on the value of $\bar{u}$, the speed of the traveling wave can be negative, positive or even zero. Therefore, the existence of all four types of TWS is possible for this scenario.

Existence of TWI for the full system with Scenario 2

Among the four types of waves, the only one that can appear for any sign of c is TWI. The boundary between advancing and retreating waves are standing waves with $c = 0$. We will see that there exist either one or two standing waves depending on the value of a. Therefore, the relative value of $\bar{u}$ with respect to these standing waves determines the sign of the speed.

Hence, we first follow [48] and study the existence of a standing wave for the reduced system. For $c = 0$, a standing wave satisfies the following special case of (3.1.24)–(3.1.26),

$$U'' + f(U) = 0, \quad z < 0, \tag{3.4.4}$$

$$U'(0^-) = -\Delta\sqrt{Dm}U(0^-), \tag{3.4.5}$$

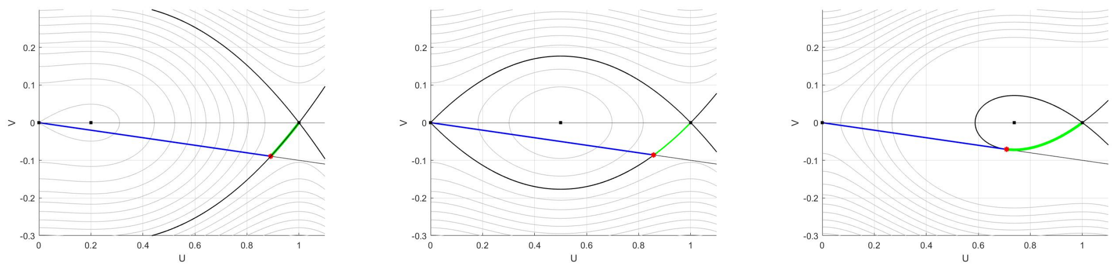
$$U(-\infty) = 1. \tag{3.4.6}$$

To show the existence of solutions for (3.4.4)–(3.4.6), we again need to show that the unstable manifold of $(1, 0)$ intersects the line $V = -\Delta\sqrt{Dm}U$ corresponding to the boundary condition (3.4.5). Equivalently, we can use the energy function from (3.3.5). The orbit that we are looking for should satisfy $E(U(0^-), U'(0^-)) = E(1, 0)$, which, recalling (3.3.5), is equivalent to

$$\frac{1}{2}\left(U'(0^-)\right)^2 + F(U(0^-)) = F(1). \tag{3.4.7}$$

If we substitute (3.4.5) in (3.4.7), we can further simplify to find

$$\frac{m}{2}\left(\frac{\alpha_1}{\alpha_2}\right)^2 \frac{D_1}{D_2}\left(U'(0^-)\right)^2 + F(U(0^-)) = F(1). \tag{3.4.8}$$



(a) $a = 0.25$ (b) $a = 0.5$ (c) $a = 0.7382$

Figure 3.16: Visualization of Lemma 3.4.10. The black curve corresponds to the level curve passing through the steady state $(1, 0)$, which includes $M_1^+(0)$. Depending on the slope of the line $V = b(0)U$, we may have only one standing wave in (a), either one or no standing waves in (b), and two, one or no standing waves in (c). Parameter values are $\Delta = 0.1,\ D = 1,\ m = 1$.

We show that at least one specific value of $U(0^-) := U_0(0)$ satisfies equation (3.4.8).

Lemma 3.4.10. *Depending on the value of a, we have the following cases:*

i) *For $0 < a < 1/2$, there exists a unique value $0 < U(0^-) = U_0(0) < 1$ that satisfies condition* (3.4.8) *and produces a standing wave.*

ii) *For $a = 1/2$, if $-\Delta\sqrt{Dm} \geq \frac{-1}{\sqrt{2}}$, then there exists a unique value $0 < U(0^-) = U_0(0) < 1$ that satisfies* (3.4.8).

iii) *For $1/2 < a < 1$, if the boundary line $V = b(0)U$ at $c = 0$ lies above the homoclinic connection of $(1, 0)$ at $c = 0$ (resp. is tangent to it), then there exist two values $0 < U_{0_2}(0), U_{0_1}(0) < 1$, (resp. a unique value $0 < U_0(0) < 1$) that satisfies* (3.4.8).

Proof: We follow the idea in Section 3.3.2. For part *i*), when $0 < a < 1/2$, we have $c_0 > 0$. Hence, $M_1^+(0)$ lies below the heteroclinic connection of $(1, 0)$ and $(0, 0)$ at c_0 and hits the negative part of the $V-$axis. On the other hand, as the boundary line $V = b(0)U$, corresponding to (3.4.5), has a negative slope, there must be an intersection between $M_1^+(0)$ and $V = b(0)U$ that satisfies (3.4.8). One can follow the proof of Theorem 3.3.12 and conclude from the direction of the vector field that the intersection is unique.

For part *ii*), when $a = 1/2$, we have the heteroclinic connection between $(1, 0)$ and $(0, 0)$ for $c_0 = 0$. To have a unique intersection point $U_0(0) > 0$ between the heteroclinic connection and the boundary line $V = b(0)U$, we follow the idea of Lemma 3.3.8 and choose parameters such that the slope of the boundary line is larger than the slope of the heteroclinic connection at $(0, 0)$. This yields the condition $-\Delta\sqrt{Dm} \geq \frac{-1}{\sqrt{2}}$.

For part *iii*), we follow the proof of Lemma 3.3.10. There, we showed that the slope of line that passes through the origin and is tangent to the homoclinic connection of $(1, 0)$ for $c = 0$ is μ, defined in (3.3.8). Therefore, if $b(0) < \mu$, there exists no intersection point between the boundary line and the homoclinic connection satisfying (3.4.8), and hence no standing wave for (3.4.4)–(3.4.6). However, if $b(0) > \mu$ (resp. $b(0) = \mu$), there exist(s) two (resp. one) intersection(s) between the boundary line and the homoclinic connection satisfying (3.4.8), and hence two (resp. one) standing wave(s) for (3.4.4)–(3.4.6). ∎

To study the existence of solutions for system (3.1.24)–(3.1.26) with the moving boundary condition (3.4.3), we need to study the graph $U_0(c) - \bar{u}$ with respect to c, which can be found by shifting the graphs in Figures 3.14 and 3.15 downward $\bar{u}$ units. The intersections of the graphs with the vertical axis in these two figures correspond to $U_i(0)$ of the standing waves of (3.4.4)–(3.4.6), which satisfy (3.4.8). In the following lemma, we discuss the existence of TWI with positive, zero or negative speed for the full system with different values for a. We will see how the existence and the speed of the waves depend on the value of $\bar{u}$ relative to $U_{0_i}(0)$.

Lemma 3.4.11. *For $0 < a < 1/2$, we have the following cases for the existence of TWI for system (3.1.24)–(3.1.26) that satisfies equation (3.4.3) for scenario 2 as well:*

i) If $b(c_0) \leq \lambda_0(c_0)$:

- *i-a) for $0 < \bar{u} < U_0(0)$, there exists a unique TWI with speed $c > 0$ for the full system (3.1.24)–(3.1.26) with (3.4.3).*
- *i-b) for $\bar{u} = U_0(0)$, there exists a unique TWI with speed $c = 0$ for the full system (3.1.24)–(3.1.26) with (3.4.3).*
- *i-c) for $\bar{u} > U_0(0)$, either there exists a unique TWI with speed $c < 0$ regardless of the choice of η_1 (when $U_0(0) < \bar{u} \leq U_0(c_*)$), or one can choose η_1 such that there exists a unique TWI with speed $c < 0$ (when $\bar{u} > U_0(c_*)$), for system (3.1.24)–(3.1.26) with (3.4.3).*

ii) If $b(c_0) > \lambda_0(c_0)$:

- *ii-a) for $0 < \bar{u} < U_0(0)$, either there exists at least one TWI with speed $c > 0$ (when $0 \leq \bar{u} < U_0(c^*)$), or there exists a unique TWI with speed $c > 0$ (when $U_0(c^*) \leq \bar{u} < U_0(0)$) for system (3.1.24)–(3.1.26) with (3.4.3).*
- *ii-b) if $\bar{u} = U_0(0)$, there exists a unique TWI with speed $c = 0$ for the full system (3.1.24)–(3.1.26) with (3.4.3).*
- *ii-c) if $\bar{u} > U_0(0)$, either there exists a unique TWI with speed $c < 0$ regardless of choice of η_1 (when $U_0(0) < \bar{u} \leq U_0(c_*)$), or one can choose η_1 such that*

there exists a unique TWI with speed $c < 0$ (when $\bar{u} > U_0(c_*)$) for the full system (3.1.24)–(3.1.26) *with* (3.4.3).

The lemma can be proved by using the ideas of the lemmas in Section 3.4.1 to study the shifted graph of $U_0(c)$. We can show analogous results for the case $a = 1/2$ and $a > 1/2$. However, as the statements are long and do not show any new insight we do not state them here. For completeness, we state these lemmas in an Appendix.

Existence of TWII for the full system with Scenario 2

Similar to the results regarding TWII, we briefly comment on existence results for TWII for the full system with scenario 2.

Remark 3.4.12. *If a TWII exists for the reduced system* (3.1.24)–(3.1.26) *(see Theorem 3.3.14 for the conditions), we have the following cases depending on the value of a:*

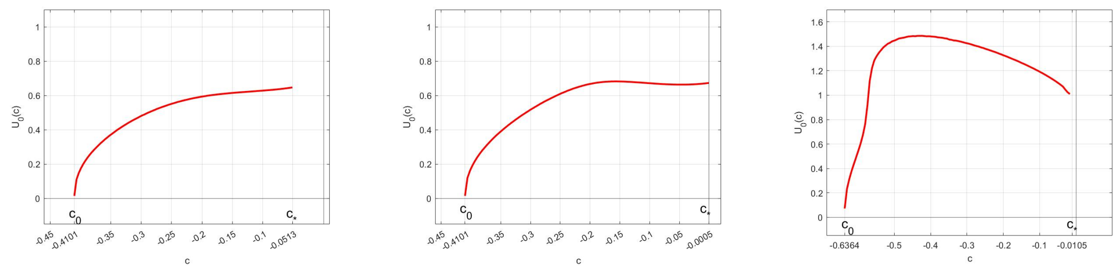
- *for $0 < a < 1/2$, if $0 < \bar{u} \leq U_{0_1}(c_0)$, then one can choose η_1 such that there exists at least one TWII with speed $c \in (c_0, c^{**}]$ for the full system* (3.1.24)–(3.1.26) *with* (3.4.3). *Otherwise, there exists no TWII for the full system with scenario 2.*

- *for $1/2 \leq a < 1$, if $0 < \bar{u} \leq U_{1_0}(0)$, then one can choose η_1 such that there exists at least one TWII with speed $c \in (0, c^{**}]$ for the full system* (3.1.24)–(3.1.26) *with* (3.4.3). *Otherwise, there exists no TWII for the full system with scenario 2.*

Existence of TWIII for the full system with Scenario 2

As seen in Section 3.3.2, a TWIII appears for a specific range of parameters when $c \in (c_0, c_*)$, and for a given speed, we may have more than one TWIII for the reduced system. One can see that the behaviour of the resulting $U_{0_i}(c)$ does not necessarily follow a general pattern. It is shown in Figure 3.18 that even the behaviour of the first intersection function with the smallest value of U, i.e. $U_{0_1}(c)$, may change considerably with different sets of parameters. However, with the right choice of parameters, we can prove the existence of TWIII for the full system. The next lemma can be proved by following the same ideas as in the proofs of the lemmas in Section 3.4.1.

Lemma 3.4.13. *If $c_0 \leq c_*$, then for $\bar{u} > 0$, one can choose η_1 such that there exists at least one value of speed $c \in (c_0, c_*)$ for which the corresponding TWIII of* (3.2.9)–(3.2.14) *satisfies equation* (3.4.3) *for scenario 2 as well.*



(a) $c_0 = -0.4101$, $c_* = -0.0513$ (b) $c_0 = -0.4101$, $c_* = -0.0005$ (c) $c_0 = -0.6364$, $c_* = 0.0105$

Figure 3.17: Possible profiles for the graph of $U_{0_1}(c)$ for TWIII. The parameter values in (a) are $a = 0.79,\ \Delta = 0.05,\ D = 1,\ m = 1$, those in (b) are $a = 0.79,\ \Delta = 0.05,\ D = 0.01,\ m = 0.01$, and those in (c) are $a = 0.95,\ \Delta = 0.1,\ D = 0.01,\ m = 1$.

Existence of TWIV for the full system with Scenario 2

As seen in Section 3.3.2, there exists a unique TWIV for the reduced system (3.1.24)–(3.1.26) for all $c \in (-\infty, c_*)$. Similar to the previous section, we can study the intersection point $U_0(c)$ for this wave type to see that it is a continuous, single-hump function of c on $(-\infty, c_*]$, with $\lim_{c \nearrow c_*} U_0(c) = 1$, and $\lim_{c \searrow -\infty} U_{0_1}(c) = 1$. Due to the range and shape of $U_0(c)$ for this case, we can only expect to obtain the existence of at least one TWIV for the full system, which can also be proved as in Section 3.4.1.

Lemma 3.4.14. *For $0 < a < 1$ and $\bar{u} > 1$, one can choose η_1 such that there exists at least one value of speed $c \in (-\infty, c_*)$ for which the corresponding TWIV of* (3.2.9)–(3.2.14) *satisfies equation* (3.4.3) *for scenario 2 as well.*

We now note that if we consider $\bar{u}$ in (3.4.3) to be a free parameter for system (3.2.9)–(3.2.14) with (3.4.3), then one can choose $\bar{u}$ and c such that we can get the desired type of traveling wave solution to the full system. For example, if we consider reduced system (3.2.9)–(3.2.14) for $c = 0$ and $0 < a < 1/2$, then, as shown above, we can choose $\bar{u} = U_0(0)$, so the standing wave solution of the reduced system (3.2.9)–(3.2.10) satisfies (3.4.3) as well. Hence, it is the solution to the complete free boundary system.

We can follow the same approach when $c \neq 0$, solve the reduced system first for a given speed c in a range for a specific type of traveling wave solution and then choose $\bar{u} = U_{0_i}(c)$ in (3.4.3) to obtain the solution of system (3.2.9)–(3.2.14) with (3.4.3). Similarly, if we have more than one traveling wave solution for the reduced system corresponding to the same speed, we will have different traveling wave solutions with the same speed corresponding to the complete system (3.2.9)–(3.2.14) with (3.4.3) for different values of $\bar{u}$. In the next remark, we discuss when more than one TWS type can happen for the full system.

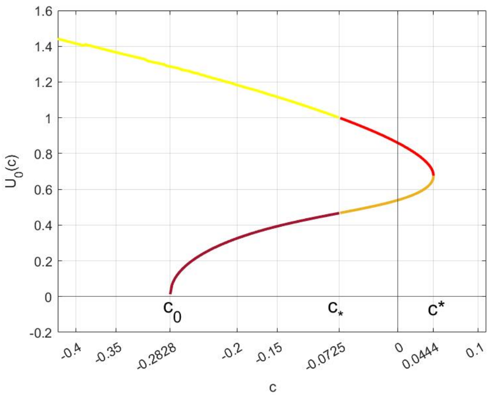


Figure 3.18: Visualization of Remark 3.4.15. The graph of $U_0(c)$. Here, we have $c^* = 0.0444$, $c_0 = 0.2828$, and $c_* = -0.0725$. The red and orange curves are $U_0(c)$ for TWI, the dark-red (currant) curve is $U_0(c)$ for TWIII, and the yellow curve corresponds to $U_0(c)$ for TWIV. One can choose η_1 and $\bar{u}$ such that there exist TWI, TWII, and TWIII at the same time for the full system with scenario 2. The parameters are $a = 0.7,\ \Delta = 0.05,\ D = 2$, and $m = 1$.

Remark 3.4.15. *As we saw in Lemmas 3.4.11, 3.5.1 and 3.5.2, we may have more than one TWS of the same type for the full system* (3.2.9)–(3.2.14) *and* (3.4.3) *for a given set of parameters. We can see in Figure 3.18 that it is also possible to have different TWS of different types for the full system with scenario 2, depending on the values of $\bar{u}$ or η_1.*

3.4.3 Scenario 3

In this scenario, $B = -\eta_2 U'(0^-)$ and the equation for the speed of the traveling wave is

$$c = -\eta_2 U'(0^-). \tag{3.4.9}$$

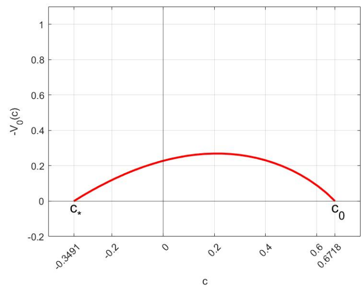
Based on the assumption for scenario 3, we know that for this scenario we have $U'(0^-) < 0$. This implies that for the traveling wave solution satisfying (3.4.9), we can only have positive speeds. Therefore, among the four types of waves seen in section 3.3.1, only those of type I and II may satisfy the equations for scenario 3.

In what follows, we discuss the existence of waves of types I and II for the system (3.2.9)–(3.2.14) with (3.4.9) for different ranges of a separately.

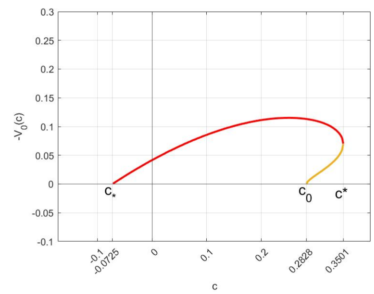
Remark 3.4.16. *Similar to Remark 3.4.1, we define $U_0'(c) := U'(0^-) = V(0^-)$ to denote the dependence of $U'(0^-)$ to c and use*

$$U_{0_i}'(c) := U_i'(0^-) \quad \textit{for} \quad i = 1, ..., k,$$

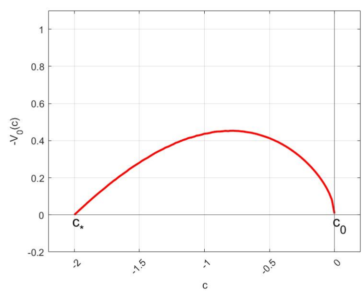
when we want to denote the k traveling wave solutions of (3.2.9)–(3.2.14) *corresponding to the same speed c.*



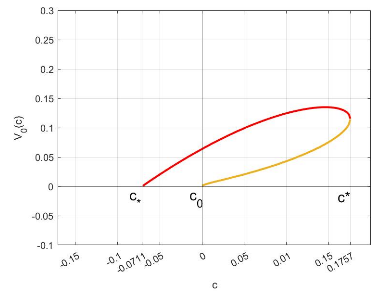
(a) $0 < a < \frac{1}{2}$, $b(c_0) \leq \lambda_0^-(c_0)$, $c^* = c_0 > 0$



(b) $0 < a < \frac{1}{2}$, $b(c_0) > \lambda_0^-(c_0)$, $c^* > c_0 > 0$



(c) $a = \frac{1}{2}$, $b(c_0) \leq \lambda_0^-(c_0)$, $c^* = c_0 = 0$



(d) $a = \frac{1}{2}$, $b(c_0) > \lambda_0^-(c_0)$, $c^* > c_0 = 0$

Figure 3.19: The graph of $-U'_{0_i}(c)$, $i = 1, 2$, w.r.t. c for $0 < a < \frac{1}{2}$. The red curve is $-U'_{0_1}(c)$ and the orange one is $-U'_{0_2}(c)$, when it exists. The graphs in (a) and (c) correspond to part i) of Lemma 3.4.17, and the ones in (b) and (d) correspond to parts ii-a) and ii-b) of Lemma 3.4.17. In part (a), we have $a = 0.025$, $\Delta = 0.16$, $D = 2$ and $m = 2$. In part (b), we have $a = 0.3$, $\Delta = 0.05$, $D = 2$, $m = 1$. In part (c), we have $a = 0.5$, $\Delta = 0.5$, $D = 2$ and $m = 4$. In part (d), we have $a = 0.5$, $\Delta = 0.1$, $D = 5$, $m = 10$.

Following the outline of Section 3.4.1, we start by studying the behaviour of $U'_0(c)$.

Lemma 3.4.17. *Consider TWI for the reduced system* (3.1.24)–(3.1.26) *with* $0 < a < 1$ *and different values of* c. *The functions* $-U'_{0_i}(c)$, *with* $i = 1, 2$, *have the fallowing properties:*

i) If $b(c_0) \leq \lambda_0(c_0)$, *then* $-U'_0(c)$ *is a continuous function of* c *on* $c_* \leq c \leq c_0$, $\lim_{c \nearrow c_0} U'_0(c) = 0$, *and* $\lim_{c \searrow c_*} U'_0(c) = 0$.

ii) If $b(c_0) > \lambda_0(c)$, *then we have the following properties for* $-U'_{0_1}(c)$ *and* $-U'_{0_2}(c)$:

ii-a) $-U'_{0_1}(c)$ *is a continuous function on* $c_* \leq c \leq c^*$, *and* $\lim_{c \searrow c_*} U'_{0_1}(c) = 0$, *and* $\lim_{c \nearrow c^*} U'_{0_1}(c) = U'_0(c^*)$.

ii-b) If $c_0 \geq c_*$, *then* $-U'_{0_2}(c)$ *is monotone increasing on* $c_0 < c \leq c^*$, *and* $\lim_{c \searrow c_0} U'_{0_2}(c) = 0$, *and* $\lim_{c \nearrow c^*} U'_{0_2}(c) = U'_0(c^*)$.

ii-c) If $c_* > c_0$, then $-U'_{0_2}(c)$ is monotone increasing on $c_* \leq c \leq c^*$, and $\lim_{c \searrow c_*} U'_{0_2}(c) = 0$, and $\lim_{c \nearrow c^*} U'_{0_2}(c) = U'_0(c^*)$.

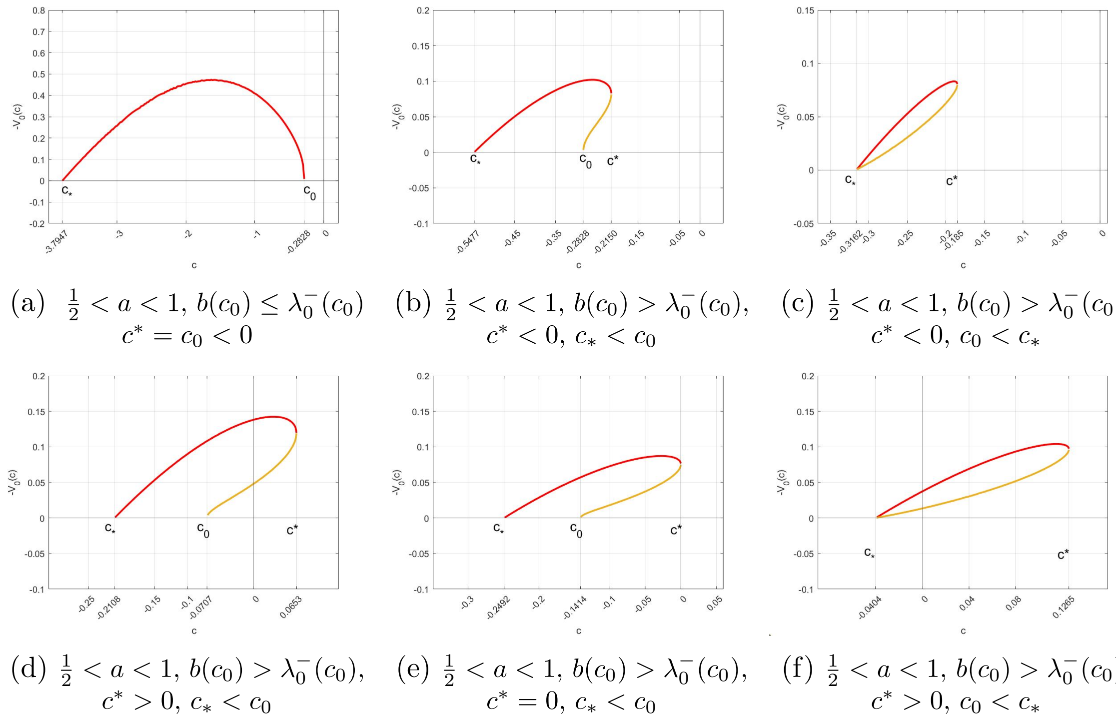
Proof: The proof of this lemma follows the ideas in the proof of Lemma 3.4.2. Continuity follows from the continuity of the vector field and the boundary line w.r.t. c. The limit values follow from the limits of $U_{0_i}(c)$, $i = 1, 2$, in Lemma 3.4.2 and the boundary condition $U'_{0_i}(c) = b(c)U_{0_i}(c)$, (3.1.25), along with the property of $b(c)$ in Lemma (3.1.3), in particular its continuity with respect to c. It remains to prove the different shapes of $U'_{0_2}(c)$ in part *ii*). In this part, we have two intersections between the boundary line and the unstable manifold. As seen before, $U'_{0_2}(c)$ emerge for $c > c_0$, when $c_0 \geq c_*$, or for $c > c_*$, when $c_* > c_0$. In both cases, as c increases from c_0 or from c_*, the V-coordinates of the intersection points, $U'_{0_2}(c)$, decrease from 0. This is due to the downward movement of the boundary line and the fact that $U_{0_2}(c)$ is increasing in c (parts *ii* of Lemma 3.4.2). At $c = c^*$, which is the maximal speed, the boundary line $V = b(c^*)U$ is tangent to $M_1^+(c^*)$. As $b(c^*) < 0$, this tangency must occur at some point with $U_0(c^*)$ less than the U-coordinate of the minimum points of $M_1^+(c^*)$ in the forth quadrant. We can conclude by the continuity of the phase plane with respect to c that the second intersection point for $c < c^*$ can not pass the corresponding minimum point of its unstable manifold and so stays above the tangent point at c^*. Hence, for $c_* \leq c < c^*$ (or $c_0 < c < c^*$), the second intersection point continues its monotone downward movement while it remains above the tangent point at c^*. Therefore, the V-coordinate of the second intersection point, $U'_{0_2}(c)$, decreases with c, which is equivalent to saying that $-U'_{0_2}(c)$ is a monotone increasing function of c. ∎

We next state the results for the existence of TWI solutions for the full system. Similar to Subsection 3.4.1, to prove the following lemmas, one can equivalently prove the existence of at least one intersection between the line $y = c/\alpha\eta_1$ and the curves for $-U'_0(c)$, which are depicted in Figures 3.19 and 3.20. One can verify the results by following the proofs of the analogous lemma in Subsection 3.4.1, for scenario 1. Although the graphs of $-U'_0(c)$ are not exactly the same as the analogous ones for $U_0(c)$, for the matter of proof the steps are quite similar.

Following the outline of Subsection 3.4.1, we start by stating the existence result for the full system when $0 < a < 1/2$.

Lemma 3.4.18. *If $0 < a < 1/2$ and $b(c_0) \leq \lambda_0(c_0)$ (resp. $b(c_0) > \lambda_0(c_0)$), then there exists at least one speed $c \in (0, c^*)$ (resp. $c \in (0, c^*]$) for which the corresponding TWI of* (3.2.9)–(3.2.14) *satisfies equation* (3.4.9) *for scenario 3 as well.*

When $a = 1/2$, $c_0 = 0$ and we have the following result regarding the existence of solutions for the full system.



(a) $\frac{1}{2} < a < 1$, $b(c_0) \leq \lambda_0^-(c_0)$ $c^* = c_0 < 0$

(b) $\frac{1}{2} < a < 1$, $b(c_0) > \lambda_0^-(c_0)$, $c^* < 0$, $c_* < c_0$

(c) $\frac{1}{2} < a < 1$, $b(c_0) > \lambda_0^-(c_0)$, $c^* < 0$, $c_0 < c_*$

(d) $\frac{1}{2} < a < 1$, $b(c_0) > \lambda_0^-(c_0)$, $c^* > 0$, $c_* < c_0$

(e) $\frac{1}{2} < a < 1$, $b(c_0) > \lambda_0^-(c_0)$, $c^* = 0$, $c_* < c_0$

(f) $\frac{1}{2} < a < 1$, $b(c_0) > \lambda_0^-(c_0)$, $c^* > 0$, $c_0 < c_*$

Figure 3.20: The graphs of $U_{0_i}(c)$, $i = 1, 2$, w.r.t. c for $\frac{1}{2} < a < 1$. The red curve is $U_{0_1}(c)$ and the orange one is $U_{0_2}(c)$, when it exists. The graph in (a) corresponds to part i) of Lemma 3.4.17, and the ones in (b), (d), and (e) correspond to parts ii-a) and ii-b) of Lemma 3.4.17. The graphs in (c) and (f) correspond to part ii-a) and ii-c) of Lemma 3.4.17. In part (a), we have $a = 0.7$, $\Delta = 0.6$, $D = 4$, $m = 4$. In part (b), we have $a = 0.7$, $\Delta = 0.2$, $D = 2$ and $m = 3$. In part (c), we have $a = 0.8$, $\Delta = 0.2$, $D = 1$, $m = 2$. In part (d), we have $a = 0.55$, $\Delta = 0.1$, $D = 1$ and $m = 4$. In part (e), we have $a = 0.6$, $\Delta = 0.05703$, $D = 6$, $m = 3$. In part (f), we have $a = 0.6$, $\Delta = 0.02$, $D = 2$, $m = 2$.

Lemma 3.4.19. *For $a = 1/2$ we have the following two cases:*

i) If $b(c_0) \leq \lambda_0(c_0)$, or equivalently $\Delta\sqrt{Dm} \geq \frac{\sqrt{2}}{2}$, then there exists no solution for the full system.

ii) If $b(c_0) \leq \lambda_0(c_0)$, or equivalently $\Delta\sqrt{Dm} < \frac{\sqrt{2}}{2}$, then one can choose η_2 such that there exists at least one solution for the full system.

As mentioned in Subsection 3.4.1, for $1/2 < a < 1$, we can only discuss the existence of solutions for the full system, when the maximal speed c^* for the reduced system is positive. We remind the reader that the condition for the sign of c^* can be checked in Lemma 3.3.10.

Lemma 3.4.20. *For $1/2 < a < 1$, we have the following three cases:*

i) *If* $b(c_0) \leq \lambda_0^-(c_0)$, *then there exists no solution for the full system.*

ii) *If* $b(c_0) > \lambda_0^-(c_0)$ *but* $c^* \leq 0$, *then there exists no solution for the full system.*

iii) *If* $b(c_0) > \lambda_0^-(c_0)$ *and* $c^* > 0$, *then one can choose* η_2 *such that there exists at least one solution for the full system.*

In the following corollary, we present a sufficient condition for the existence of solutions for the full system when $1/2 \leq a < 1$.

Corollary 3.4.21. *Let* $a = 1/2$ *(resp.* $1/2 < a < 1$*). If*

$$c^* > -\eta_2 U_0'(c^*),$$

there is at least one (resp. two) value(s) of speed in $(0, c^*)$ *for which the traveling wave solution of* (3.2.9)–(3.2.14) *satisfies* (3.4.9) *as well.*

In the following remark, we discuss the existence of TWII for different values of parameters.

Remark 3.4.22. *If a TWII exists for the reduced system* (3.1.24)–(3.1.26) *(see Theorem 3.3.14 for the conditions), it appears for* $c_0 < c \leq c^{**}$, *when* $0 < a \leq 1/2$, *and for* $0 < c \leq c^{**}$, *when* $1/2 < a < 1$. *Then, one can choose* η_1 *such that there exists at least one TWII for the full system* (3.1.24)–(3.1.26) *with* (3.4.9). *Otherwise, there exists no TWII for the full system with scenario 3.*

3.5 Appendix

Here, we state the existence results of TWI for the full system with scenario (3.4.3), which were left out from Section 3.4.2.

Lemma 3.5.1. *For* $a = \frac{1}{2}$, *we have the following cases for the existence of TWI for system* (3.1.24)–(3.1.26) *that satisfies equation* (3.4.3) *for scenario 2 as well:*

i) *If* $b(c_0) \leq \lambda_0(c_0)$, *for* $\bar{u} > 0$ *there exists either a unique TWI with speed* $c < 0$ *regardless of the choice of* η_1 *(when* $0 < \bar{u} \leq U_0(c_*)$*), or one can choose* η_1 *such that there exists a unique TWI with speed* $c < 0$ *(when* $\bar{u} > U_0(c_*)$*) for the full system* (3.1.24)–(3.1.26) *with* (3.4.3).

ii) *If* $b(c_0) > \lambda_0(c_0)$, *and*

ii-a) *if* $0 \leq \bar{u} < U_0(0)$, *either one can choose* η_1 *such that there exists at least one TWI with speed* $c > 0$ *(when* $0 \leq \bar{u} < U_0(c^*)$*), or there exists a unique TWI with speed* $c > 0$ *regardless of choice of* η_1 *(when* $U_0(c^*) \leq \bar{u} < U_0(0)$*) for the full system* (3.1.24)-(3.1.26) *with* (3.4.3).

ii-b) *if $\bar{u} = U_0(0)$, there exists a unique TWI with speed $c = 0$ for the full system* (3.1.24)–(3.1.26) *with* (3.4.3).

ii-c) *if $\bar{u} > U_0(0)$, either there exists a unique TWI with speed $c < 0$ regardless of the choice of η_1 (when $U_0(0) < \bar{u} \leq U_0(c_*)$), or one can choose η_1 such that there exists a unique TWI with speed $c < 0$ (when $\bar{u} > U_0(c_*)$) for the full system* (3.1.24)–(3.1.26) *with* (3.4.3).

Lemma 3.5.2. *For $\frac{1}{2} < a < 1$, we have the following cases for the existence of TWI for system* (3.1.24)–(3.1.26) *that satisfies equation* (3.4.3) *for scenario 2 as well:*

i) *If $b(c_0) \leq \lambda_0(c_0)$ and $\bar{u} > 0$, one can choose η_1 such that there exists a unique TWI with speed $c < 0$ for the full system* (3.1.24)-(3.1.26) *with* (3.4.3).

ii) *If $b(c_0) > \lambda_0(c_0)$, and*

ii-a) *if $c^* < 0$, then for $\bar{u} > 0$, there exists either no TWI for full system* (3.1.24)–(3.1.26) *with* (3.4.3) *(when $c_0 < c_*$ and $0 < \bar{u} \leq U_{0_2}(0)$), or one can choose η_1 such that there exists at least one TWI with speed $c < 0$ for full system* (3.1.24)–(3.1.26) *with* (3.4.3).

ii-b) *if $c^* = 0$,*

ii-b-1) *for $0 < \bar{u} < U_0(0)$, there exists either no TWI (when $c_0 < c_*$ and $0 < \bar{u} \leq U_{0_2}(0)$), or one can choose η_1 such that there exists at least one TWI with speed $c < 0$ for full system* (3.1.24)–(3.1.26) *with* (3.4.3).

ii-b-2) *for $\bar{u} = U_0(0)$, there exist one TWI with $c = 0$ and one can choose η_1 such that there exists at least one TWI with speed $c < 0$ for full system* (3.1.24)–(3.1.26) *with* (3.4.3).

ii-b-3) *for $\bar{u} > U_0(0)$, there exists either at least one TWI with $c < 0$ regardless of the choice of η_1 (when $U_0(0) < \bar{u} \leq U_{0_i}(c_*)$ for $i = 1, 2$), or one can choose η_1 such that there exists at least one TWI with speed $c < 0$ for full system* (3.1.24)–(3.1.26) *with* (3.4.3).

ii-c) *If $c^* > 0$,*

ii-c-1) *for $0 < \bar{u} < U_{0_2}(0)$, either one can choose η_1 such that there exists at least one TWI with speed $c > 0$, and there exists no TWI with $c < 0$ (when $c_0 < c_*$ and $0 < \bar{u} \leq U_{0_2}(c_*)$), or one can choose η_1 such that there exists at least one TWI with speed $c > 0$, or at least one TWI with speed $c < 0$, or both for full system* (3.1.24)–(3.1.26) *with* (3.4.3).

ii-c-2) *for $\bar{u} = U_{0_2}(0)$, There exists one TWI with $c = 0$ and one can choose η_1 such that there exists at least one TWI with speed $c < 0$, or at least one TWI with speed $c > 0$, or both for full system* (3.1.24)–(3.1.26) *with* (3.4.3).

ii-c-3) *for* $U_{0_2}(0) < \bar{u} < U_{0_1}(0)$*, there exists a unique TWI with speed* $c > 0$ *and one can choose* η_1 *such that there exists at least one TWI with* $c < 0$ *for the full system* (3.1.24)–(3.1.26) *with* (3.4.3).

ii-c-4) *for* $\bar{u} = U_{0_1}(0)$*, there exists a unique TWI with speed* $c = 0$ *and one can choose* η_1 *such that there exists at least one TWI with* $c < 0$ *for the full system* (3.1.24)–(3.1.26) *with* (3.4.3).

ii-c-5) *for* $\bar{u} > U_{0_1}(0)$*, there exists either a unique TWI with speed* $c < 0$ *(when* $U_{0_1}(0) < \bar{u} \leq U_0(c_*)$*), or one can choose* η_1 *such that there exist at least one TWI with* $c < 0$*, when* $\bar{u} > U_0(c_*)$*, for the full system* (3.1.24)–(3.1.26) *with* (3.4.3).

Chapter 4

Conclusion and Discussion

4.1 Summary and the Contribution

Modeling the movement and spread of invasive species into a new environment has been the subject of many publications in population ecology and applied mathematics. The overwhelming majority of models for population spread are based on Fisher's reaction–diffusion equation [26]. An important property of his model, which makes it a strong tool in modeling species spread, is that it supports traveling wave solutions, i.e., solution profiles that move in a given direction at a constant speed. Fisher proved the existence of a traveling wave profile for his model for all speeds greater than or equal to a minimal traveling wave speed [26]. However, Fisher's model and all the models that build on it assume that the invading species does not affect the environment in which it spreads.

Ecosystem engineers are organisms that modify their environment in order to make it more suitable for them. For example, beavers build dams and create ponds in which they can build their lodges and be protected from predators. Other examples include woodpeckers, ants, seagrass, and even certain trees. The physical changes caused by their engineering activity can have positive or negative effects on the environment, on other species, and on the engineer itself. In addition, ecosystem engineers are among the most successful invasive species [13] and sometimes the most destructive. Therefore, the study of this type of species is important when making predictions about future ecosystem dynamics. However, the impact of the engineering activity on spread cannot be examined with existing models.

In this thesis, we presented a model, originally derived by Lutscher *et al.* [48] for the population density of an ecosystem engineer. In this model, the engineering process was conceptualized as follows. Habitat can be engineered or non-engineered. The engineered habitat is suitable for population growth, but the non-engineered habitat is not. The engineering activity transforms non-engineered habitat into engineered habitat and thereby moves the boundary between them. Then, using the framework of uncorrelated random walks, they derived a two-sided free boundary problem of reaction–diffusion equations, where the boundary corresponds to the edge between engineered and non-engineered habitat. The model contains non-standard matching conditions across the free boundary. While a few one-sided free boundary models for population spread exist [19], the model derived in [48] differs from these in a number of essential mathematical aspects and is much harder to analyze. The purpose of this thesis was to mathematically analyze this free boundary problem.

In chapter 2, we proved the well-posedness of the model, i.e., we proved the local existence of solutions for the model with this two-sided moving boundary and the novel matching conditions. We divided the proof of the main result into two parts. First, we proved the existence of solutions for the model with a given curve as a moving boundary. To do that, we assigned a convex functional to this problem, so that the evolution equation governed by the subdifferential of this convex potential is exactly the system of reaction-diffusion equation describing the model. Then we applied the results in [60] to prove the existence of solutions for the reduced problem. Second, we showed the existence of solutions for the model with a free boundary that satisfies the moving boundary condition in the model. In this part of the proof, we followed the idea of [22] and used Schauder's fixed point theorem on an appropriate operator. Our proofs contain several novel ideas that can be used in broader contexts. These results have already been published in [6].

In chapter 3, we extended the model by including a strong Allee effect in population dynamics, whereby reproduction is reduced at low densities. This effect can result from difficulties in finding mates at low population densities and is quite common in many species, but it is often not considered in mathematical models because it complicates their analysis. We focused on the qualitative analysis of traveling wave solutions. We proved the existence of different types of traveling waves using phase-plane analysis and other techniques. Similar to the previous chapter, we broke our proof into two parts. We first considered the speed of the wave to be a free parameter. We recognized four types of traveling wave solutions that appear for this reduced model and found the corresponding range of speed for the existence of each. In the second part, we studied the full problem and proved the existence of waves of appropriate type with the desired speed that satisfies the equation of speed for each scenario. A similar analysis without Allee effect was carried out in [48].

The contribution of this thesis is twofold.

We developed a solution theory for our two-sided free boundary problem with discontinuity in the density at the interface. In previous applications of free boundary problems to biology, the authors usually consider continuous or even zero density at the interface so they can use appropriate transformations to fix the boundary. In this way, they can use the classic well-posedness and regularity results in that context for their model, and hence continue to study other aspects of the model [16].

Second, we studied traveling wave solutions for the model with a strong Allee effect. Our model with specific interface conditions allowed us to capture some new dynamics when studying TWS, which can not be captured in models similar to [19], where they have continuity of density and zero density at the interface. In addition, our result regarding the existence of different types of TWS for the reduced model can be considered as an extension of the results of Hadeler and Rothe [29] for the moving boundary problem with an imposed boundary. Models of this kind have been used to explore the effects of climate change and a moving climate envelope on population dynamics [45].

4.2 Future Directions and Challenges

Future research directions span from continuing the mathematical analysis of the model to using the numerical tools to study its outcome and interpreting them in its original application in ecology.

For the mathematical analysis, we would like to extend our proof for the local solution for scenario one to a global in-time solution and verify its uniqueness. For that, we need uniform estimates with respect to t. The next step is to study the well-posedness of the model with the other two scenarios for the moving boundary as described in the introduction, Chapter 1. The current analysis will not work with these cases and the existence result will require new approaches. For scenario 2, due to the definition of the moving boundary condition, L' may change signs. So the current analysis for scenario 1, in which we assume $L' > 0$, cannot be applied. However, for small $\bar{u}$, we expect that we can use a perturbation argument along with the current analysis to prove the existence of local in-time solutions as well. For scenario 3, we do not have the required regularity on u_x at the moving boundary. Hence, the fixed point argument in the current analysis can not be applied.

The next step in the analysis would be to study the asymptomatic behavior of the solution. We would hope to prove some properties of the solution similar to the spreading-vanishing dichotomy described by Du *et al.* in [19]. The main mechanism behind this dichotomy in Du *et al.*'s model is as follows. For one, in their model, they have a hostile environment, i.e., zero density beyond the free boundary, which is monotone increasing, $L' > 0$. At the same time, the Dirichlet condition on the free boundary that sets the density to zero models the loss of individuals through the free boundary. Hence, the critical patch size (first studied by Skellem [67] and Kierstead and Slobodkin [38] and further studied by Ludwig *et al.* [47], Okubo and Levin [58] and others) appears in their model, $L^* := (\pi/2)\sqrt{D/r}$ in (1.4.1), and "*serves as a barrier for the spreading*". Namely, if the length l_0 of the domain initially occupied by the species, as defined in (1.4.1), is bigger than this critical patch size, L^*, then, with the increasing free boundary, species spread to the whole environment regardless of the initial size u_0 of the population in (1.4.1). However, for species that start their spread on a domain smaller than the critical patch size, L^*, if they have a large enough initial population density and can push the boundary above the critical patch size barrier, i.e. depending on u_0 and μ in (1.4.1), then they can spread to the whole environment and survive. But if they cannot push the boundary beyond the barrier L^*, their domain just approaches a constant patch size, $L_\infty < L^*$, and the population will die out eventually, i.e., $\lim_{t\to\infty} ||u(t,\cdot)||_C = 0$.

In our model, the prescribed conditions on the free boundary, (1.6.5) and (1.6.7), causes a negative net flux through the free boundary and hence "*the boundary is leaky*", see [50]. Instead of zero density beyond the free boundary, we have an exponential decrease and death. The concept of critical patch size is defined for our model and has been calculated for analogous models with fixed boundary, see [51], and for a moving boundary model with an imposed boundary equation, see [50]. As we have these essential ingredients in our model, we expect that the corresponding critical patch size leads to the emergence of properties similar to the spreading-vanishing dichotomy. As in our model we have loss through the boundary and have exponential decay outside the engineered habitat, we believe that we should be able to prove such a dynamic for our system.

Given a spreading-vanishing dichotomy, we would then hope to be able to find the asymptotic spreading speed of a spreading solution and maybe even prove the convergence to a traveling wave. For traveling waves, we would like to study their asymptotic stability. This is important as for many parameter sets, we do not have the uniqueness of waves and may have more than one type of wave for the system. Hence, to know which of these waves are actually stable, even locally, would be the next thing to look at. Stability for Fisher- and other types of traveling waves have been studied for a long time; see e.g. [24, 25].

To verify the asymptotic behavior of the model and its dependence on parameters, we would like to consider numerical simulations. There are several numerical schemes in the literature that have been modeling different free boundary problems, see [28, 57]. For our model, these numerical simulations will be difficult in their own ways because of the free boundary and the matching condition on it. We have started this study by deriving a scheme for a fixed boundary that can capture the interface condition. For the analogous model with an imposed moving boundary, MacDonald *et al.* recently constructed and analyzed numerical methods using a finite difference scheme. To capture the conditions on the moving interface, including the jump in density, they construct a reference frame in which the interfaces are fixed in time [49]. They capture the jump in density with a specific placing of the nodes in the finite difference scheme. We would use these ideas to construct a scheme for our free boundary model to gain insight into how the system behaves, to study how model parameters affect the ability of an ecosystem engineer species to invade new environments, and to determine the speed at which such an invasion occurs.

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