

ON THE DUAL OF A  $C^*$ -ALGEBRA

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## INTRODUCTION

In this thesis, we are mainly concerned with the properties of the dual  $\hat{A}$  of a  $C^*$ -algebra  $A$ . We give three equivalent definitions of the topology on the dual and show how each of these is used in obtaining various known properties of  $\hat{A}$ . We also show how the dual is used in characterizing some particular classes of  $C^*$ -algebras.

Our work is divided into three chapters. In Chapter I, we state definitions and basic concepts which are used throughout the thesis. In Chapter II, we give the three equivalent definitions of the topology on the dual  $\hat{A}$  of a  $C^*$ -algebra  $A$ . The first definition is given in terms of the Jacobson structure space  $\text{Prim}(A)$  of  $A$ ; the second is based on the space of pure states  $P(A)$  of  $A$ ; and the third is defined in terms of the set of non-zero irreducible representations  $\text{Irr}(A)$  of  $A$ .

In Chapter III, we investigate the duals of special  $C^*$ -algebras. These algebras are : the algebra  $LC(H)$  of compact operators on a Hilbert space  $H$ , the dual  $C^*$ -algebras, CCR-algebras and finally GCR-algebras. We show that for separable  $C^*$ -algebras, these algebras can be completely characterized by means of their duals. We also discuss some specific examples of these algebras. The last section of this chapter is devoted to the study of Mackey Borel structure on the dual of a separable  $C^*$ -algebra in general, and of GCR-algebra in

particular. We show that for separable  $C^*$ -algebras, the Mackey Borel structure on  $\hat{A}$  is finer than the Borel structure arising from the topology on  $\hat{A}$ , and that for GCR-algebras they are identical.

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## Chapter I

### Preliminaries

#### § 1. Definitions and terminology.

Let  $A$  be an algebra over the field  $\mathbb{C}$  of complex numbers.

A mapping  $x \longrightarrow x^*$  of  $A$  onto  $A$  such that

$$(i) \quad (x^*)^* = x;$$

$$(ii) \quad (x + y)^* = x^* + y^*;$$

$$(iii) \quad (\alpha x)^* = \bar{\alpha} x^*;$$

$$(iv) \quad (xy)^* = y^*x^*,$$

for all  $x, y$  in  $A$  and for all  $\alpha \in \mathbb{C}$ , is called an involution on  $A$ . An algebra  $A$  with an involution is called an involution algebra. A subalgebra  $B$  of an involutive algebra  $A$  is called an involution subalgebra of  $A$  if  $B^* = B$ , where  $B^* = \{ x^* : x \in B \}$ . A normed algebra  $A$  with an involution such that  $\|x\| = \|x^*\|$  for all  $x$  in  $A$ , is called an involution normed algebra. If  $A$  is complete, then  $A$  is called an involution Banach algebra. An involutive Banach algebra  $A$  such that

$$(v) \quad \|x\|^2 = \|x^*x\|,$$

for all  $x$  in  $A$ , is called a C\*-algebra (or a B\*-algebra in [15]). If  $A$  is a C\*-algebra, then every closed involutive subalgebra of  $A$  is a C\*-algebra, called the sub-C\*-algebra of  $A$ .

Let  $A$  be a C\*-algebra. If  $I$  is a closed two-sided ideal of  $A$ , then  $I^* = I$ . Let  $A/I = \{ x' = x + I : x \in A \}$ . Define in  $A/I$  the operations of addition, multiplication, multiplication by scalars,

involution and norm as follows :

$$x' + y' = (x + y)';$$

$$x'y' = (xy)';$$

$$\alpha x' = (\alpha x)';$$

$$(x')^* = (x^*)';$$

$$\|x'\| = \inf_{m \in I} \|x + m\|,$$

for all  $x', y'$  in  $A/I$  and for all  $\alpha \in \mathbb{C}$ . Then with these operations,  $A/I$  becomes a  $C^*$ -algebra, called the quotient  $C^*$ -algebra of  $A$  (with respect to  $I$ ) (cf. 1.8.2. in [3]).

Let  $A$  be a  $C^*$ -algebra. An element  $x$  in  $A$  is called hermitian if  $x^* = x$ ; normal if  $xx^* = x^*x$ ; positive if  $x$  is of the form  $yy^*$  for some  $y$  in  $A$ . If  $A$  has an identity element  $1$ , then the element  $x$  in  $A$  is called unitary if  $xx^* = x^*x = 1$ .

Let  $A$  be an algebra and let  $\tilde{A} = \mathbb{C} \times A$  be the Cartesian product of  $\mathbb{C}$  and  $A$ . Define in  $\tilde{A}$  the operations of addition, multiplication and multiplication by scalars as follows :

$$(\alpha, x) + (\beta, y) = (\alpha + \beta, x + y);$$

$$(\alpha, x)(\beta, y) = (\alpha\beta, \beta x + \alpha y + xy);$$

$$\beta(\alpha, x) = (\beta\alpha, \beta x),$$

for all  $x, y$  in  $A$  and for all  $\alpha, \beta \in \mathbb{C}$ . Then with these operations,  $\tilde{A}$  becomes an associative algebra. Moreover, if  $A$  is an involutive normed algebra, then  $\tilde{A}$  is an involutive normed algebra with the involution given by  $(\alpha, x)^* = (\bar{\alpha}, x^*)$  and the norm, for example, by

$$\|(\alpha, x)\| = |\alpha| + \|x\|.$$

(We observe that  $\tilde{A}$  is also an involutive normed algebra. under the norm  $\|(\alpha, x)\| = \sqrt{|\alpha|^2 + \|x\|^2}$  or the norm  $\|(\alpha, x)\| = \max(|\alpha|, \|x\|)$ .) It is clear that if  $A$  is complete, then  $\tilde{A}$  is complete and that  $A$  can be considered as a maximal closed two-sided ideal of  $\tilde{A}$ . Furthermore, if  $A$  is a  $C^*$ -algebra, then there exists a unique norm under which  $\tilde{A}$  is a  $C^*$ -algebra. In fact, if  $A$  has an identity element, then  $\tilde{A}$  is a  $C^*$ -algebra under the norm  $\|(\alpha, x)\| = \max(|\alpha|, \|x\|)$ . If  $A$  has no identity, we consider the left regular representation  $x \longrightarrow L_x$  of  $\tilde{A}$ . Then under the norm  $\|x\| = \|L_x\|$ ,  $\tilde{A}$  is a  $C^*$ -algebra (cf. 1.3.8. in [3]). The identity element of  $\tilde{A}$  is  $(1, 0)$ .  $\tilde{A}$  is called the  $C^*$ -algebra deduced from  $A$  by the adjunction of an identity element.

Let  $H$  be a Hilbert space and  $L(H)$  the algebra of all bounded linear operators on  $H$ . If  $T^*$  denote the adjoint operator of  $T$ , then  $T \longrightarrow T^*$  is an involution on  $L(H)$  and  $L(H)$  together with this involution and the operator norm, is a  $C^*$ -algebra.

Let  $A$  and  $B$  be two  $C^*$ -algebras over the same field  $\mathbb{C}$  of complex numbers. A homomorphism  $\varphi$  from  $A$  into  $B$  such that for every  $x$  in  $A$ ,  $\varphi(x^*) = \varphi(x)^*$ , is called a  $*$ -homomorphism. If  $\varphi$  is one-to-one, then it is called a  $*$ -isomorphism. If there is a  $*$ -isomorphism from  $A$  onto  $B$ , then  $A$  and  $B$  are said to be  $*$ -isomorphic. If  $\varphi$  is a  $*$ -homomorphism from  $A$  into  $B$ , then  $\varphi(A)$  is a sub- $C^*$ -algebra of  $B$  and  $\|\varphi(x)\| \leq \|x\|$  for all  $x$  in  $A$ . If  $\varphi$  is a  $*$ -isomorphism, then it is also an isometry (cf. 1.3.7, 1.8.1, & 1.8.3. in [3]).

§ 2. Representations and positive linear functionals.

Let  $A$  be a  $C^*$ -algebra and  $H$  a Hilbert space. A  $*$ -homomorphism  $\pi$  from  $A$  into  $L(H)$  is called a representation of  $A$  in  $H$ . If  $\pi$  is a  $*$ -isomorphism, then it is called a faithful representation. The dimension of  $H$  is called the dimension of  $\pi$ . The space  $H$  is called the space of  $\pi$  and is denoted by  $H_\pi$ . A subspace  $K$  of  $H$  is said to be invariant under  $\pi(A)$  if  $\pi(A)K \subseteq K$ . Let  $\xi \in H$ . Then the closure of  $\pi(A)\xi$  is a closed subspace of  $H$  invariant under  $\pi(A)$ . If  $\overline{\pi(A)\xi} = H$ , then  $\xi$  is called a topologically cyclic vector for  $\pi(A)$ . The representation  $\pi$  of  $A$  in  $H$  is said to be topologically irreducible if  $H \neq (0)$  and if the closed subspaces of  $H$  which are invariant under  $\pi(A)$  are  $(0)$  and  $H$  only. It is clear that if  $\pi$  is topologically irreducible, then every non-zero vector in  $H$  is topologically cyclic. It has been established by Kadison [10] that every topologically irreducible representation of a  $C^*$ -algebra is also algebraically irreducible.

Let  $A$  be a  $C^*$ -algebra and  $H_1, H_2$  two Hilbert spaces. Let  $\pi_1$  and  $\pi_2$  be the representations of  $A$  in  $H_1$  and  $H_2$  respectively. Then  $\pi_1$  and  $\pi_2$  are said to be equivalent if there exists an isometric isomorphism  $\Phi$  from  $H_1$  onto  $H_2$  which transforms  $\pi_1(x)$  into  $\pi_2(x)$  for all  $x$  in  $A$ , i.e., such that for all  $x$  in  $A$ ,

$$\Phi \pi_1(x) = \pi_2(x) \Phi.$$

The collection of all equivalence classes of non-zero irreducible representations of  $A$  is denoted by  $\hat{A}$ .

Let  $\{H_\lambda\}_{\lambda \in \Lambda}$  be a family of Hilbert spaces and let  $H$  be the family of all functions  $(\xi_\lambda)$  defined on  $\Lambda$  such that

(i) for each  $\lambda \in \Lambda$ ,  $\xi_\lambda \in H_\lambda$ ;

(ii)  $(\xi_\lambda)$  contains at most a countable number of elements which are different from zero;

(iii) 
$$\sum_{\lambda} \|\xi_\lambda\|^2 < \infty.$$

We define addition, multiplication by scalars and inner product in  $H$  by the following formulae :

$$(\xi_\lambda) + (\eta_\lambda) = (\xi_\lambda + \eta_\lambda);$$

$$\alpha (\xi_\lambda) = (\alpha \xi_\lambda);$$

$$((\xi_\lambda) | (\eta_\lambda)) = \sum_{\lambda} (\xi_\lambda | \eta_\lambda).$$

Then with these operations,  $H$  becomes a Hilbert space, called the Hilbert sum (or direct sum) of the Hilbert spaces  $H_\lambda$  and we denote it by

$$H = \bigoplus_{\lambda \in \Lambda} H_\lambda.$$

Let  $A$  be a  $C^*$ -algebra,  $\{H_\lambda\}_{\lambda \in \Lambda}$  a family of Hilbert spaces and  $\{\pi_\lambda\}_{\lambda \in \Lambda}$  a family of representations of  $A$  such that  $\pi_\lambda$  is a representation of  $A$  in  $H_\lambda$  for each  $\lambda \in \Lambda$ . Let  $H$  be the Hilbert sum of  $H_\lambda$ . Let  $(\xi_\lambda) \in H$  and  $x \in A$ . Then, since for each  $x$  in  $A$  and each  $\lambda \in \Lambda$ ,  $\|\pi_\lambda(x)\| \leq \|x\|$ , it follows that  $(\pi_\lambda(x)\xi_\lambda) \in H$ ; in fact

$$\begin{aligned} \|(\pi_\lambda(x)\xi_\lambda)\|^2 &= ((\pi_\lambda(x)\xi_\lambda), (\pi_\lambda(x)\xi_\lambda)) \\ &= \sum_{\lambda} (\pi_\lambda(x)\xi_\lambda, \pi_\lambda(x)\xi_\lambda) \\ &= \sum_{\lambda} \|\pi_\lambda(x)\xi_\lambda\|^2. \end{aligned}$$

$$\leq \sum_{\lambda} \|\pi_{\lambda}(x)\|^2 \|\xi_{\lambda}\|^2 \leq \|x\|^2 \|(\xi_{\lambda})\|^2.$$

Now let  $\pi(x)$  be the operator on  $H$  defined by

$$\pi(x)(\xi_{\lambda}) = (\pi_{\lambda}(x)\xi_{\lambda}).$$

Then it is easy to see that  $\pi(x)$  is a bounded linear operator on  $H$  for every  $x$  in  $A$ , with  $\|\pi(x)\| \leq \|x\|$ . Since each  $\pi_{\lambda}$  is a representation of  $A$  in  $H_{\lambda}$ , it is easily seen that  $\pi$  is a representation of  $A$  in  $H$  which we call the Hilbert sum (or direct sum) of the representations  $\pi_{\lambda}$  and denote it by

$$\pi = \bigoplus_{\lambda \in \Lambda} \pi_{\lambda}.$$

(cf. 2.2.3. in [3]).

Let  $A$  be a  $C^*$ -algebra. A linear functional  $f$  on  $A$  is called positive if  $f(x^*x) \geq 0$  for all  $x$  in  $A$ . A positive continuous linear functional  $f$  on  $A$  such that  $\|f\| = 1$  is called a state of  $A$ . Let  $f$  and  $g$  be two linear functionals on  $A$ . Then  $f$  is said to majorize  $g$  or  $g$  is said to be majorized by  $f$  if  $f - g$  is positive and we write  $f \geq g$  or  $g \leq f$ . A positive continuous linear functional  $f$  on  $A$  is said to be pure if  $f \neq 0$  and if every positive continuous linear functional on  $A$  majorized by  $f$  is of the form  $\alpha f$  where  $0 \leq \alpha \leq 1$ . The set of all pure states on  $A$  is denoted by  $P(A)$ .

Let  $A$  be a  $C^*$ -algebra,  $f$  a positive continuous linear functional on  $A$  and  $\tilde{A}$  the  $C^*$ -algebra deduced from  $A$  by the adjunction of an identity element. Define  $\tilde{f}$  on  $\tilde{A}$  by

$$\tilde{f}((\alpha, x)) = \alpha \|f\| + f(x),$$

for all  $(\alpha, x)$  in  $\tilde{A}$ . Then it is easily seen that  $\tilde{f}$  is a positive continuous linear functional on  $\tilde{A}$ . We call  $\tilde{f}$  the canonical extension of  $f$  to  $\tilde{A}$  (cf. 2.1.5. in [3]).

Let  $A$  be a  $C^*$ -algebra and  $\pi$  a representation of  $A$  in a Hilbert space  $H$ . Let  $\xi \in H$ . Then  $x \longrightarrow (\pi(x)\xi | \xi)$  is a positive continuous linear functional on  $A$ , called the linear functional defined by  $\pi$  and  $\xi$ . For  $\pi$  fixed and  $\xi$  varying in  $H$ , we get linear functionals associated to  $\pi$ . If  $S$  is a set of representations of  $A$ , then a linear functional  $f$  on  $A$  is said to be associated to  $S$  if  $f$  is associated to at least one element of  $S$ . Let  $B$  be a sub- $C^*$ -algebra of  $L(H)$  and  $\xi$  an element in  $H$ . Then the positive linear functional on  $B$  defined by the identity representation on  $B$  and  $\xi$ , i.e., the linear functional  $x \longrightarrow (x\xi | \xi)$ , is usually denoted by  $\omega_\xi$ .

Let  $A$  be a  $C^*$ -algebra and  $f$  a positive linear functional on  $A$ . For every  $x, y$  in  $A$ , we define  $(x | y) = f(y^*x)$ . Then it is clear that this functional is linear in  $x$  and semi-linear in  $y$  and  $(x | x) \geq 0$  for all  $x$  in  $A$ . We have the following Cauchy-Schwartz inequality :

$$(1) \quad |f(y^*x)|^2 \leq f(x^*x)f(y^*y),$$

for all  $x, y$  in  $A$  (cf. p.213 in [15]). Let

$$N = \{ x \in A : f(x^*x) = 0 \}.$$

Then  $N$  is a left ideal of  $A$  and by (1),

$$N = \{ x \in A : f(y*x) = 0 \text{ for all } y \text{ in } A \}.$$

Let  $H' = A/N$  and for  $x', y'$  in  $H'$ , define

$$(x' | y') = f(y*x).$$

Then  $(x' | y')$  is an inner product of  $H'$  for which  $H'$  is a prehilbert space (or inner product space). In fact, using this construction we can show that every positive continuous linear functional  $f$  on  $A$  gives rise to a unique representation  $\pi_f$  of  $A$  in  $H_f$ , the Hilbert space completion of  $H'$ . Moreover, there exists a unique vector  $\xi_f$  in  $H_f$  such that

$$f(x) = (\pi_f(x)\xi_f | \xi_f),$$

for all  $x$  in  $A$ . The representation  $\pi_f$  and the vector  $\xi_f$  are said to be defined by  $f$  (cf. 2.4.4. in [3]).

## Chapter II

### The dual of a C\*-algebra

#### § 1. The Jacobson structure space and the dual of a C\*-algebra.

Let  $A$  be a C\*-algebra,  $H$  a Hilbert space and  $\pi$  a non-zero irreducible representation of  $A$  in  $H$ . Then the kernel  $\text{Ker } \pi$  of  $\pi$  is called a primitive ideal of  $A$ . It is clear that every primitive ideal is a closed two-sided ideal of  $A$ . By 2.9.7. in [3], it follows that every closed two-sided ideal of  $A$  is the intersection of all primitive ideals containing it. The C\*-algebra  $A$  is said to be primitive if the zero ideal is a primitive ideal of  $A$ . The collection of all primitive ideals of  $A$  is denoted by  $\text{Prim}(A)$ . It is clear that if two representations are equivalent, then they have the same kernel. But, in general, the converse is not true. (cf. [9]) Thus  $\pi \longrightarrow \text{Ker } \pi$  ( $\pi \in \hat{A}$ ) is a mapping from  $\hat{A}$  onto  $\text{Prim}(A)$ , but not one-to-one. We call this mapping the canonical mapping of  $\hat{A}$  onto  $\text{Prim}(A)$  and denote it by

$$\varphi : \hat{A} \longrightarrow \text{Prim}(A).$$

Lemma (2.1.1). Let  $A$  be a C\*-algebra,  $H$  a Hilbert space and  $\pi$  a non-zero irreducible representation of  $A$  in  $H$ .

(i) If  $I$  is a closed two-sided ideal of  $A$  and if  $\pi(I) \neq 0$ , then the restriction  $\pi|_I$  of  $\pi$  to  $I$  is irreducible.

(ii) If  $I_1$  and  $I_2$  are two closed two-sided ideals of  $A$  such that  $\pi(I_1) \neq 0$  and  $\pi(I_2) \neq 0$ , then  $\pi(I_1 \cdot I_2) \neq 0$ .

Proof: [3] The set  $H'$  of elements  $\xi$  of  $H$  such that  $\pi(I)\xi = 0$

is an invariant subspace with respect to  $\pi(A)$ . In fact, suppose  $\xi \in H'$ . Since  $\pi(I)\pi(A)\xi = \pi(I \cdot A)\xi \subset \pi(I)\xi = 0$ , it follows that  $\pi(A)\xi \in H'$  and so  $\pi(A)H' \subset H'$ . Hence  $H'$  is an invariant subspace. Since  $\pi(I) \neq 0$ ,  $H'$  is distinct from  $H$ . But  $\pi$  is irreducible, therefore  $H' = (0)$ . Thus, if  $\xi \neq 0$  in  $H$ , then  $\pi(I)\xi \neq 0$ . Since  $I$  is a two-sided ideal, it follows that  $\pi(I)\xi$  is invariant with respect to  $\pi(A)$ . The irreducibility of  $\pi$  implies that  $\pi(I)\xi = H$  and the proof of (i) is complete. Similarly, it follows that  $\pi(I_1)H = H$  and  $\pi(I_2)H = H$ . Thus  $\pi(I_1)(\pi(I_2)H) = H$ . But  $\pi(I_1)\pi(I_2) = \pi(I_1 \cdot I_2)$ . Therefore  $\pi(I_1 \cdot I_2) \neq 0$  and this proves (ii).

Lemma (2.1.2). Let  $I_1$  and  $I_2$  be two closed two-sided ideals of a  $C^*$ -algebra  $A$  and  $P$  a primitive ideal of  $A$ . If  $P \supset I_1 \cdot I_2$  (in particular, if  $P \supset I_1 \cap I_2$ ), then either  $P \supset I_1$  or  $P \supset I_2$ .

Proof: [ 3 ] Suppose  $P \not\supset I_1$  and  $P \not\supset I_2$ . Let  $\pi$  be the irreducible representation of  $A$  such that  $\text{Ker } \pi = P$ . Then  $\pi(I_1) \neq 0$  and  $\pi(I_2) \neq 0$ . Therefore, by Lemma (2.1.1)(ii),  $\pi(I_1 \cdot I_2) \neq 0$  and hence  $P \not\supset I_1 \cdot I_2$ ; a contradiction. Thus either  $P \supset I_1$  or  $P \supset I_2$ . This completes the proof.

Let  $A$  be a  $C^*$ -algebra,  $\text{Prim } (A)$  the collection of all primitive ideals of  $A$  and  $T$  a subcollection of  $\text{Prim } (A)$ . Define

$$I(T) = \bigcap_{P' \in T} P' .$$

Then  $I(T)$  is a closed two-sided ideal of  $A$  since every  $P' \in T$  is a closed two-sided ideal of  $A$ . Now define  $\overline{T}$  to be the set of all

primitive ideals of  $A$  containing  $I(T)$ , i.e.,

$$(1) \quad \bar{T} = \{ P : P \in \text{Prim}(A) \text{ and } P \supset I(T) \}.$$

If  $T = \emptyset$ , we define  $\bar{T} = T$ , i.e.,  $\bar{\emptyset} = \emptyset$ . Then we have:

Lemma (2.1.3). Let  $A$  be a  $C^*$ -algebra and let  $\bar{T}$  be defined as (1).

Then (i)  $\bar{\emptyset} = \emptyset$ ;

(ii)  $T \subset \bar{T}$  for every  $T \subset \text{Prim}(A)$ ;

(iii)  $\overline{\bar{T}} = T$  for every  $T \subset \text{Prim}(A)$ ;

(iv)  $\overline{T_1 \cup T_2} = \bar{T}_1 \cup \bar{T}_2$  for every  $T_1 \subset \text{Prim}(A)$  and  $T_2 \subset \text{Prim}(A)$ .

Proof: [3] (i) By definition.

(ii) It is clear that for every  $P' \in T$ ,  $P' \supset I(T)$  and thus  $P' \in \bar{T}$ . Hence  $T \subset \bar{T}$  for every  $T \subset \text{Prim}(A)$ .

(iii) We have  $I(\bar{T}) = \bigcap_{P \in \bar{T}} P \subset \bigcap_{P' \in T} P' = I(T)$  since  $T \subset \bar{T}$ .

Since, for every  $P \in \bar{T}$ ,  $P \supset I(T)$ , hence  $I(\bar{T}) = \bigcap_{P \in \bar{T}} P \supset I(T)$ .

Hence  $I(\bar{T}) = I(T)$  and therefore  $\overline{\bar{T}} = T$  for every  $T \subset \text{Prim}(A)$ .

(iv) Let  $I_1 = I(T_1)$  and  $I_2 = I(T_2)$ . Then

$$\bar{T}_1 \cup \bar{T}_2 = \{ P : P \in \text{Prim}(A) \text{ and either } P \supset I_1 \text{ or } P \supset I_2 \}.$$

Now, since

$$I(T_1 \cup T_2) = \bigcap_{P' \in T_1 \cup T_2} P' = \left( \bigcap_{P' \in T_1} P' \right) \cap \left( \bigcap_{P' \in T_2} P' \right) = I_1 \cap I_2,$$

we have

$$\overline{\bar{T}_1 \cup \bar{T}_2} = \{ P : P \in \text{Prim}(A) \text{ and } P \supset I_1 \cap I_2 \}.$$

It is clear that  $\bar{T}_1 \cup \bar{T}_2 \subset \overline{\bar{T}_1 \cup \bar{T}_2}$ . Next, if  $P \in \overline{\bar{T}_1 \cup \bar{T}_2}$ ,

then  $P \supset I_1 \cap I_2$  and therefore, by Lemma (2.1.2), either  $P \supset I_1$

or  $P \supset I_2$ , i.e., either  $P \in \overline{T}_1$  or  $P \in \overline{T}_2$ . Hence  $P \in \overline{T}_1 \cup \overline{T}_2$  and so  $\overline{T_1 \cup T_2} \subset \overline{T}_1 \cup \overline{T}_2$ . Consequently,  $\overline{T_1 \cup T_2} = \overline{T}_1 \cup \overline{T}_2$  for every  $T_1 \subset \text{Prim}(A)$  and  $T_2 \subset \text{Prim}(A)$ . This completes the proof.

We remark that the statements contained in Lemma (2.1.3) are just the Kuratowski closure axioms for a topology. Hence it follows from this lemma that there exists a unique topology on  $\text{Prim}(A)$  such that for every  $T \subset \text{Prim}(A)$ ,  $\overline{T}$  is the closure of  $T$  under this topology. This topology is called the Jacobson topology on  $\text{Prim}(A)$  (in [3]) or hull-kernel topology on  $\text{Prim}(A)$  (in [15]).  $\text{Prim}(A)$  with this topology is called the Jacobson structure space of  $A$  (cf. [12]).

Theorem (2.1.4). Let  $A$  be a  $C^*$ -algebra and  $\text{Prim}(A)$  its Jacobson structure space. Let  $T$  be a subset of  $\text{Prim}(A)$ . Then  $T$  is closed if and only if  $T$  is the set of all primitive ideals containing a subset of  $A$ .

Proof: [3] If  $T$  is closed, then  $\overline{T} = T$  and hence  $T$  is the set of all primitive ideals of  $A$  containing  $I(T)$ , a subset of  $A$ . Conversely, let  $M$  be a subset of  $A$  and let  $T$  be the set of all primitive ideals of  $A$  containing  $M$ . Then we have

$$I(T) = \bigcap_{P \in T} P \supset M.$$

Thus, if  $P \in \overline{T}$ , then  $P \supset I(T) \supset M$ . Hence  $P \in T$  and so  $\overline{T} \subset T$ . Therefore  $T = \overline{T}$  and this completes the proof.

Theorem (2.1.5). Let  $A$  be a primitive  $C^*$ -algebra and  $\text{Prim}(A)$  its

Jacobson structure space. Then there exists a set in  $\text{Prim}(A)$  consisting of a single point which is dense in  $\text{Prim}(A)$ .

Proof: Since  $A$  is a primitive  $C^*$ -algebra, the zero ideal  $(0)$  is in  $\text{Prim}(A)$ . Let  $T = \{(0)\}$  be the subset of  $\text{Prim}(A)$  containing the zero ideal  $(0)$  only. Then  $I(T) = (0)$  and so  $\overline{T} = \text{Prim}(A)$ .

Theorem (2.1.6). Let  $A$  be a  $C^*$ -algebra. Then its Jacobson structure space  $\text{Prim}(A)$  is a  $T_0$ -space.

Proof: [3] Let  $P_1$  and  $P_2$  be two distinct points in  $\text{Prim}(A)$ . Then either  $P_1 \not\subset P_2$  or  $P_2 \not\subset P_1$ . Suppose  $P_1 \subset P_2$ . Then, by Theorem (2.1.4), the set  $T$  of all  $P \in \text{Prim}(A)$  such that  $P \supset P_1$  is a closed subset of  $\text{Prim}(A)$ . We have:  $P_1 \in T$ . But  $P_2 \notin T$ . In fact, if  $P_2 \in T$ , then  $P_1 \subset P_2$ ; a contradiction. Hence the space  $\text{Prim}(A)$  is a  $T_0$ -space.

Theorem (2.1.7). Let  $A$  be a  $C^*$ -algebra and  $\text{Prim}(A)$  its Jacobson structure space. Let  $P \in \text{Prim}(A)$ . Then the set  $\{P\}$  is closed if and only if  $P$  is a maximal primitive ideal.

Proof: [3] This follows from the fact that the closure  $\overline{\{P\}}$  of the set  $\{P\}$  is the set of all primitive ideals of  $A$  containing  $P$ .

Let  $A$  be a  $C^*$ -algebra and  $\hat{A}$  the set of all equivalence classes of non-zero irreducible representations of  $A$ . We shall now define a topology on  $\hat{A}$ .

Let  $S$  be a subset of  $\hat{A}$ . We say that  $S$  is open if there is an open set  $T$  in  $\text{Prim}(A)$  such that

$$\varphi^{-1}(\pi) = S,$$

where  $\varphi$  is the canonical mapping  $\varphi : \hat{A} \longrightarrow \text{Prim}(A)$ . It is clear that the collection of all such open sets is a topology on  $\hat{A}$ , called the inverse image topology and  $\hat{A}$  with this topology is called the dual of  $A$ . (In [3],  $\hat{A}$  is called the "spectre" of  $A$ ). Thus, if  $S$  is a subset of  $\hat{A}$ , then the closure  $\bar{S}$  of  $S$  is the set of all  $\pi \in \hat{A}$  such that

$$\text{Ker } \pi \supset \bigcap_{\rho \in S} \text{Ker } \rho .$$

Remark: It is easily seen that with this topology on  $\hat{A}$  the canonical mapping  $\varphi : \hat{A} \longrightarrow \text{Prim}(A)$  is open and continuous.

Theorem (2.1.8). Let  $A$  be a primitive  $C^*$ -algebra. Then there exists a point  $\pi$  in  $\hat{A}$  which is dense in  $\hat{A}$ ; in particular, every open set of  $\hat{A}$  is dense in  $\hat{A}$ .

Proof: Since  $A$  is primitive, by Theorem (2.1.5), the set  $T = \{(0)\}$  containing the zero ideal  $(0)$  only, is dense in  $\text{Prim}(A)$ . Let  $\pi \in \hat{A}$  be such that  $\text{Ker } \pi = (0)$  and let  $S = \{\pi\}$ . Then by the definition of the topology on the dual  $\hat{A}$ , it follows immediately that  $\bar{S} = \hat{A}$ .

Theorem (2.1.9). Let  $A$  be a  $C^*$ -algebra and  $\hat{A}$  its dual. Then the following conditions are equivalent:

- (i)  $\hat{A}$  is a  $T_0$ -space.
- (ii) The canonical mapping  $\varphi : \hat{A} \longrightarrow \text{Prim}(A)$  is a homeomorphism.
- (iii) Two non-zero irreducible representations of  $A$  having

the same kernel are equivalent.

Proof: [3] We prove (iii)  $\implies$  (ii)  $\implies$  (i)  $\implies$  (iii).

(iii)  $\implies$  (ii) : Suppose (iii) holds. Then the canonical mapping  $\varphi : \hat{A} \longrightarrow \text{Prim}(A)$  is one-to-one and onto. Since this mapping is open and continuous by the Remark above, it follows that  $\varphi$  is a homeomorphism.

(ii)  $\implies$  (i) : Since  $\text{Prim}(A)$  is a  $T_0$ -space, it follows that if the canonical mapping  $\varphi : \hat{A} \longrightarrow \text{Prim}(A)$  is a homeomorphism, then  $\hat{A}$  is a  $T_0$ -space.

(i)  $\implies$  (iii) : Let  $\pi_1$  and  $\pi_2 \in \hat{A}$  be two representations having the same kernel. Then every open set in  $\hat{A}$  containing  $\pi_1$  also contains  $\pi_2$ . Since  $\hat{A}$  is a  $T_0$ -space, it follows that  $\pi_1 = \pi_2$  and this completes the proof.

Let  $A$  be a  $C^*$ -algebra,  $I$  a closed two-sided ideal of  $A$ ,  $\hat{A}_I$  (resp.  $\hat{A}^I$ ) the set of all  $\pi \in \hat{A}$  such that  $\pi(I) = 0$  (resp.  $\pi(I) \neq 0$ ) and  $\text{Prim}_I(A)$  (resp.  $\text{Prim}^I(A)$ ) the set of all  $P \in \text{Prim}(A)$  such that  $P \supset I$  (resp.  $P \not\supset I$ ). Then

$$\hat{A} = \hat{A}_I \cup \hat{A}^I \quad \text{and} \quad \text{Prim}(A) = \text{Prim}_I(A) \cup \text{Prim}^I(A).$$

For each  $\pi \in \hat{A}_I$ , there corresponds the representation  $\pi' \in (A/I)^\wedge$  given by  $\pi'(x') = \pi(x)$ , where  $x' = x + I \in A/I$ , and conversely, for each  $\pi' \in (A/I)^\wedge$ , there corresponds the representation  $\pi \in \hat{A}_I$  defined by  $\pi(x) = \pi'(x')$  for all  $x$  in  $A$ . It is easily seen that the mapping  $\pi \longrightarrow \pi'$  of  $\hat{A}_I$  into  $(A/I)^\wedge$  is one-to-one and onto. Also we can show that  $P \longrightarrow P/I$

is a one-to-one mapping of  $\text{Prim}_I(A)$  onto  $\text{Prim}(A/I)$  (cf. 2.11.2. and 2.11.5. in [3]). Furthermore, we have

Theorem (2.1.10). (i) The mapping  $P \longrightarrow P/I$  is a homeomorphism of  $\text{Prim}_I(A)$  onto  $\text{Prim}(A/I)$  and is denoted by

$$\text{Prim}_I(A) \longrightarrow \text{Prim}(A/I).$$

(ii) The mapping  $\pi \longrightarrow \pi'$  is a homeomorphism of  $\hat{A}_I$  onto  $(A/I)^\wedge$  and is denoted by

$$\hat{A}_I \longrightarrow (A/I)^\wedge .$$

Proof: [3] We observe first that the mapping  $P \longrightarrow P/I$  is one-to-one and onto. Next let  $T$  be a subset of  $\text{Prim}_I(A)$  and

$$T' = \{ P/I : P \in T \} ,$$

the image of  $T$  in  $\text{Prim}(A/I)$  under the given mapping. Since, for every  $P'$  in  $\text{Prim}_I(A)$ ,

$$P' \supset \bigcap_{P \in T} P \text{ if and only if } P'/I \supset \bigcap_{P \in T} P/I,$$

it follows that  $T$  is closed in  $\text{Prim}_I(A)$  if and only if  $T'$  is closed in  $\text{Prim}(A/I)$ . Hence the mapping  $P \longrightarrow P/I$  is a homeomorphism. This proves (i). The second part of the theorem follows from (i) and the definition of the dual. This completes the proof.

Next, for each  $\pi \in \hat{A}_I$ , there corresponds the representation  $\pi|_I \in \hat{I}$ , and conversely, for each  $\pi' \in \hat{I}$ , there corresponds the representation  $\pi \in \hat{A}_I$  such that  $\pi' = \pi|_I$ . It is easily seen that the mapping  $\pi \longrightarrow \pi|_I$  of  $\hat{A}_I$  into  $\hat{I}$  is one-to-one and onto. Also we can show that the mapping

$P \longrightarrow P \cap I$  is a one-to-one mapping of  $\text{Prim}^I(A)$  onto  $\text{Prim}(I)$  (cf. 2.11.2. and 2.11.5. in [3]). Furthermore, we have

Theorem (2.1.11). (i) The mapping  $P \longrightarrow P \cap I$  is a homeomorphism of  $\text{Prim}^I(A)$  onto  $\text{Prim}(I)$  and is denoted by

$$\text{Prim}^I(A) \longrightarrow \text{Prim}(I).$$

(ii) The mapping  $\Pi \longrightarrow \Pi|_I$  is a homeomorphism of  $\hat{A}^I$  onto  $\hat{I}$  and is denoted by

$$\hat{A}^I \longrightarrow \hat{I}.$$

Proof: [3] We observe first that the mapping  $P \longrightarrow P \cap I$  is one-to-one and onto. Next let  $T$  be a subset of  $\text{Prim}^I(A)$  and

$$T' = \{ P \cap I : P \in T \},$$

the image of  $T$  in  $\text{Prim}(I)$  under the given mapping. Suppose  $P' \in \bar{T}$ .

Then  $P' \supset \bigcap_{P \in T} P$  and so  $P' \cap I \supset \bigcap_{P \in T} (P \cap I)$ . Hence

$P' \cap I \in \bar{T}'$ . Conversely, suppose

$$P' \cap I \supset \bigcap_{P \in T} (P \cap I).$$

Then  $P' \supset (\bigcap_{P \in T} P) \cap I$  and  $P' \not\supset I$  so that  $P' \supset \bigcap_{P \in T} P$

by Lemma (2.1.2), and hence  $P' \in \bar{T}$ . Therefore  $T$  is closed in

$\text{Prim}^I(A)$  if and only if  $T'$  is closed in  $\text{Prim}(I)$ . Hence the

mapping  $P \longrightarrow P \cap I$  is a homeomorphism. This proves (i). The

second part of the theorem follows from (i) and the definition of

the dual. This completes the proof.

Corollary (2.1.12).  $\text{Prim}_I(A)$  is a closed set in  $\text{Prim}(A)$  and  $\text{Prim}^I(A)$  is an open set in  $\text{Prim}(A)$ . Also  $\hat{A}_I$  is a closed set in

$\hat{A}$  and  $\hat{A}^I$  is an open set in  $\hat{A}$ .

Proof: By Theorem (2.1.4),  $\text{Prim}_I(A)$  is a closed set in  $\text{Prim}(A)$  and, since  $\text{Prim}^I(A)$  is the complement of  $\text{Prim}_I(A)$ ,  $\text{Prim}^I(A)$  is an open set in  $\text{Prim}(A)$ . It follows now that  $\hat{A}^I$  is a closed set in  $\hat{A}$  and  $\hat{A}^I$  is an open set in  $\hat{A}$ . This completes the proof.

Corollary (2.1.13). Let  $A$  be a  $C^*$ -algebra and  $I$  a closed two-sided ideal of  $A$ . Then we can identify  $(A/I)^\wedge$  with the closed subset  $\hat{A}^I$  of  $\hat{A}$  and  $\hat{I}$  with the open subset  $\hat{A}^I$ , which is the complement of  $\hat{A}^I$ ; we have  $(A/I)^\wedge = \hat{A} - \hat{I}$ .

Theorem (2.1.14). Let  $A$  be a  $C^*$ -algebra. Then  $I \longrightarrow \hat{I}$  is a one-to-one mapping from the collection of all closed two-sided ideals of  $A$  onto the collection of all open sets in  $\hat{A}$ . We have  $I_1 \subset I_2$  if and only if  $\hat{I}_1 \subset \hat{I}_2$ .

Proof: [3] By Corollary (2.1.13),  $\hat{I}$  is an open subset of  $\hat{A}$  for every closed two-sided ideal  $I$  of  $A$ . Now let  $U$  be an open set in  $\hat{A}$  and  $F = \hat{A} - U$ . Then  $F$  is a closed set in  $\hat{A}$ . Let  $I = \bigcap_{\rho \in F} \text{Ker } \rho$ . Then  $I$  is a closed two-sided ideal of  $A$ . Since  $F$  is closed in  $\hat{A}$ , every  $\pi \in \hat{A}$  such that  $\text{Ker } \pi \supset I$  is in  $F$  so that  $F = (A/I)^\wedge$ . Therefore  $\hat{I} = \hat{A} - F = U$ . Hence the mapping  $I \longrightarrow \hat{I}$  is onto.

Next suppose  $I_1$  and  $I_2$  are two closed two-sided ideals of  $A$  such that  $\hat{I}_1 = \hat{I}_2$ . Then every primitive ideal of  $A$  containing  $I_1$  also contains  $I_2$ , and conversely. Now, since every closed two-sided ideal of  $A$  is the intersection of all primitive ideals containing it, it follows that  $I_1 = I_2$ . Hence the mapping  $I \longrightarrow \hat{I}$  is one-to-one. The final assertion of the theorem is clear.

§ 2. The second definition of the topology on the dual.

Let  $A$  be a  $C^*$ -algebra and  $A'$  the conjugate space of all continuous linear functionals on  $A$ . Let  $P(A)$  be the set of all pure states on  $A$ . Then  $P(A)$  becomes a topological space with the relative topology induced by the weak topology  $\sigma(A', A)$  on  $A'$ . Furthermore  $P(A)$  is a Baire space (cf. p. 355 in [3]).

Let  $A$  be a  $C^*$ -algebra. If  $\pi$  is a non-zero irreducible representation of  $A$  in a Hilbert space  $H$  and  $\xi$  is a non-zero vector in  $H$ , then the linear functional  $f$  defined by  $f(x) = (\pi(x)\xi | \xi)$  is a pure positive continuous linear functional on  $A$ . If we choose  $\xi$  such that  $\|f\| = 1$ , then  $f$  is a pure state on  $A$ . Conversely, if  $f$  is a pure positive continuous linear functional on  $A$ , then the representation  $\pi_f$  defined by  $f$  is a non-zero irreducible representation of  $A$  (cf. 2.5.4 in [3]). Thus the mapping of  $P(A)$  into  $\hat{A}$  which associates to each element  $f$  in  $P(A)$  the equivalence class of irreducible representation defined by  $f$ , is well-defined and is onto  $\hat{A}$ . We denote this mapping by

$$P(A) \longrightarrow \hat{A},$$

and call it the canonical mapping of  $P(A)$  onto  $\hat{A}$ . If  $\pi \in \hat{A}$ , its inverse image in  $P(A)$  under this canonical mapping is the set of all pure states associated to  $\pi$ . In fact, let  $f'$  be in the inverse image of  $\pi$  and  $\pi'$  the representation defined by  $f'$  in the equivalence class determined by  $\pi$ . Since  $\pi$  and  $\pi'$  are equivalent, there is an isometric isomorphism  $\Phi$  from  $H_{\pi}$  onto

$H_{\pi'}$  such that  $\Phi \pi(x) = \pi'(x) \Phi$  for all  $x$  in  $A$ , i.e.,

$$\Phi \pi(x) \Phi^{-1} = \pi'(x) \quad \text{for all } x \text{ in } A. \quad \text{Let } \xi' \text{ be the vector}$$

defined by  $f'$ . Then

$$\begin{aligned} f'(x) &= (\pi'(x)\xi' \mid \xi') = (\Phi \pi(x)\Phi^{-1}\xi' \mid \xi') \\ &= (\pi(x)\Phi^{-1}\xi' \mid \Phi^{-1}\xi') = (\pi(x)\eta \mid \eta), \end{aligned}$$

where  $\eta = \Phi^{-1}\xi' \in H_{\pi}$ , and so  $f'$  is associated to  $\pi$ . Thus

the inverse image of  $\pi \in \hat{A}$  consists of all pure states associated to  $\pi$ . It follows that the canonical mapping  $P(A) \longrightarrow \hat{A}$  is one-to-one if and only if every irreducible representation of  $A$  is of dimension 1 and this condition is satisfied if and only if  $A$  is commutative (cf. 2.5.7. in [3]).

Theorem (2.2.1). Let  $A$  be a  $C^*$ -algebra,  $\pi$  a representation of  $A$  and  $S$  a set of representations of  $A$ . Then the following conditions are equivalent :

$$(i) \quad \text{Ker } \pi \supset \bigcap_{\rho \in S} \text{Ker } \rho .$$

(ii) Every positive linear functional on  $A$  associated to  $\pi$  is the weak limit of linear combinations of positive linear functionals associated to  $S$ .

(iii) Every state on  $A$  associated to  $\pi$  is the weak limit of states of the form

$$\sum_{j=1}^n \omega_{\xi_j} \circ \rho_j ,$$

where  $\xi_j \in H_{\rho_j}$ ,  $\rho_j \in S$  and the linear functional  $\omega_{\xi_j} \circ \rho_j$  defined by the relation

$$\omega_{\xi_j}(\rho_j(x)) = (\rho_j(x)\xi_j | \xi_j),$$

$j = 1, 2, \dots, n.$

If the representation  $\pi$  admits a topologically cyclic vector  $\xi$ , then the above conditions are equivalent to the following:

(iv) The positive linear functional  $x \rightarrow f(x) = (\pi(x)\xi | \xi)$  on  $A$  is the weak limit of linear combinations of positive linear functionals associated to  $S$ .

Proof: [3] We prove (i)  $\implies$  (iii)  $\implies$  (ii)  $\implies$  (i) and (ii)  $\iff$  (iv).

(i)  $\implies$  (iii): Suppose  $\text{Ker } \pi \supset \bigcap_{\rho \in S} \text{Ker } \rho$ . Then every state on  $A$  associated to  $\pi$  is zero on  $\bigcap_{\rho \in S} \text{Ker } \rho$  and hence, by 3.4.2. in [3], we get (iii).

(iii)  $\implies$  (ii): Obvious.

(ii)  $\implies$  (i): Let  $\rho \in S$ ,  $\xi_\rho \in H_\rho$  and  $f$  the positive linear functional defined by  $\rho$  and  $\xi_\rho$ . Suppose  $a \in \bigcap_{\rho \in S} \text{Ker } \rho$ . Then

$$f(a^*a) = (\rho(a^*a)\xi_\rho | \xi_\rho) = \|\rho(a)\xi_\rho\|^2 = 0,$$

i.e.,  $f$  is zero at  $a^*a$ . Thus every positive linear functional associated to  $S$  is zero at  $a^*a$ . Since (ii) holds, the positive linear functional  $g$  associated to  $\pi$  is zero at  $a^*a$ . But

$$g(a^*a) = (\pi(a^*a)\xi | \xi) = \|\pi(a)\xi\|^2$$

for some non-zero  $\xi \in H_\pi$ , and so  $\pi(a)\xi = 0$ . Now if  $\xi$  varies over  $H_\pi$ ,  $g$  varies over the set of positive linear functionals associated to  $\pi$ . Hence  $\pi(a)\xi = 0$  for all  $\xi \in H_\pi$  and therefore  $\pi(a) = 0$ , i.e.;  $a \in \text{Ker } \pi$ . Hence  $\text{Ker } \pi \supset \bigcap_{\rho \in S} \text{Ker } \rho$ .

(ii)  $\iff$  (iv): It is clear that (ii) implies (iv). Now suppose

(iv) holds. We observe first that if  $f$  is a positive linear functional associated to  $\pi$ , then for each  $y$  in  $A$ ,  $x \longrightarrow g(x) = f(y^*xy)$  is a positive linear functional associated to  $\pi$ ; in fact,

$$g(x) = f(y^*xy) = (\pi(y^*xy)\xi \mid \xi) = (\pi(x) \pi(y)\xi \mid \pi(y)\xi),$$

for some non-zero  $\xi \in H$ , and so  $g$  is associated to  $\pi$ . To complete the proof, we need to show that for every  $y$  in  $A$ , the linear functional  $x \longrightarrow g(x) = f(y^*xy)$  is the weak limit of linear combinations of positive linear functionals associated to  $S$ . Since (iv) holds, it follows that  $f$  is the weak limit of linear functionals  $f_i$ , where each  $f_i$  is the linear combination of positive linear functionals associated to  $S$ . Hence  $g$  is the weak limit of linear functionals

$$x \longrightarrow g_i(x) = f_i(y^*xy).$$

Since each  $g_i$  is a linear combination of positive linear functionals associated to  $S$ , hence (ii) follows. This completes the proof.

It is clear that if  $\pi \in S$ , then the conditions (i), (ii) and (iii) of Theorem (2.2.1) hold, but not conversely. Hence we give the following definition:

Definition. Let  $A$  be a  $C^*$ -algebra,  $\pi$  a representation of  $A$  and  $S$  a set of representations of  $A$ . If  $\pi$  and  $S$  satisfy any one of the equivalent conditions (i), (ii) and (iii) in the Theorem (2.2.1), then  $\pi$  is said to be weakly contained in  $S$ .

Let  $A$  be a  $C^*$ -algebra with the dual  $\hat{A}$  and  $\pi$  a representation of  $A$ . Then the support  $S$  of  $\pi$  is defined to be the set of all  $\rho \in \hat{A}$  which is weakly contained in  $\pi$ , i.e.,

$$S = \{ \rho \in \hat{A} : \text{Ker } \rho \supseteq \text{Ker } \pi \}.$$

Theorem (2.2.2). Let  $A$  be a  $C^*$ -algebra with the dual  $\hat{A}$  and  $\pi$  a representation of  $A$ . Then the support  $S$  of  $\pi$  is a closed subset of  $\hat{A}$ .

Proof: By definition, the closure  $\bar{S}$  of  $S$  is the set of all  $\rho \in \hat{A}$  such that  $\text{Ker } \rho \supset \bigcap_{\sigma \in S} \text{Ker } \sigma$ . Hence, it follows that if  $\rho \in \bar{S}$ , then  $\text{Ker } \rho \supset \bigcap_{\sigma \in S} \text{Ker } \sigma \supset \text{Ker } \pi$ , so that  $\rho \in S$ . Hence  $\bar{S} = S$  and this completes the proof.

Theorem (2.2.3). Let  $A$  be a  $C^*$ -algebra with the dual  $\hat{A}$ ,  $\pi \in \hat{A}$  and  $S \subset \hat{A}$ . Then the following conditions are equivalent:

- (i)  $\pi \in \bar{S}$ .
- (ii)  $\pi$  is weakly contained in  $S$ .
- (iii) Some of the non-zero positive linear functionals associated to  $\pi$  is the weak limit of positive linear functionals associated to  $S$ .
- (iv) Every state associated to  $\pi$  is the weak limit of the states associated to  $S$ .

Proof: [3] We prove (i)  $\iff$  (ii) and (iv)  $\implies$  (iii)  $\implies$  (ii)  $\implies$  (iv).

(i)  $\iff$  (ii): We have  $\pi \in \bar{S}$  if and only if  $\text{Ker } \pi \supset \bigcap_{\rho \in S} \text{Ker } \rho$  if and only if  $\pi$  is weakly contained in  $S$ .

(iv)  $\implies$  (iii): Obvious.

(iii)  $\implies$  (ii): Since  $\pi$  is irreducible, every non-zero vector  $\xi \in H_\pi$  is a cyclic vector. Since some of the non-zero positive linear functionals associated to  $\pi$  is the weak limit of positive linear functionals associated to  $S$ , there is  $\xi \neq 0$  in  $H_\pi$  such that

the positive linear functional  $x \rightarrow (\pi(x)\xi | \xi)$  is the weak limit of positive linear functionals associated to  $S$ . Hence, by Theorem (2.2.1), it follows that  $\pi$  is weakly contained in  $S$ .

(ii)  $\Rightarrow$  (iv): Suppose (ii) holds. Then  $\text{Ker } \pi \supset \bigcap_{\rho \in S} \text{Ker } \rho$  so that every state associated to  $\pi$  is zero on  $\bigcap_{\rho \in S} \text{Ker } \rho$  and hence, by 3.4.2. in [3], every state associated to  $\pi$  is the weak limit of states associated to  $S$ . This completes the proof.

Theorem (2.2.4). Let  $A$  be a  $C^*$ -algebra,  $P(A)$  the space of all pure states on  $A$  and  $\hat{A}$  the dual of  $A$ . Then the canonical mapping

$$P(A) \longrightarrow \hat{A}$$

is continuous and open.

Proof: [3] Let  $S$  be a set in  $\hat{A}$  and  $Q$  be its inverse image in  $P(A)$  under the canonical mapping. Let  $f \in P(A)$  and  $\pi$  be its image in  $\hat{A}$ . By Theorem (2.2.3), it follows that  $f \in \bar{Q}$  if and only if  $\pi \in \bar{S}$ . Hence  $S$  is closed in  $\hat{A}$  if and only if  $Q$  is closed in  $P(A)$  and so the canonical mapping  $P(A) \longrightarrow \hat{A}$  is continuous. Next let  $U$  be an open set in  $P(A)$  and  $V$  the image of  $U$  in  $\hat{A}$  under the canonical mapping. For every unitary element  $u$  of  $\tilde{A}$ , let  $u(U)$  be the set of the linear functionals  $x \rightarrow g(u^*xu)$  where  $g$  runs over  $U$ . Then  $u(U)$  is an open set in  $P(A)$ . Let

$$U' = \bigcup_{u \in \tilde{A}} u(U),$$

where  $u$  is an unitary element. Then  $U'$  is an open set in  $P(A)$ .

Since, by 2.8.6. in [3], the irreducible representations  $\pi_f$  and  $\pi_g$  defined by the pure states  $f$  and  $g$  are equivalent if and

only if there exists a unitary element  $u$  in  $\tilde{A}$  such that  $g(x) = f(u^*xu)$  for all  $x$  in  $A$ , it follows that  $U'$  is the inverse image of  $V$  in  $P(A)$ . But we know that a set  $S \subseteq A$  is closed if and only if its inverse image  $Q$  in  $P(A)$  is closed. Hence  $V$  is open and so the canonical mapping  $P(A) \longrightarrow \hat{A}$  is open. This completes the proof.

Hence we can state the second definition of the topology on  $\hat{A}$  as follows: It is the quotient topology of that of  $P(A)$  under the canonical mapping  $P(A) \longrightarrow \hat{A}$ .

Remark. We see that if  $A$  is commutative, then  $\hat{A}$  is exactly the carrier space of  $A$  (cf. p.110 in [15]).

Theorem (2.2.5). Let  $A$  be a  $C^*$ -algebra with the dual  $\hat{A}$ . Then  $\hat{A}$  is a Baire space.

Proof: [3] Let  $(V_1, V_2, \dots)$  be a countable family of open dense sets in  $\hat{A}$ . For each  $n$ , let  $U_n$  be the inverse image of  $V_n$  under the canonical mapping  $P(A) \longrightarrow \hat{A}$ . Then, by Theorem (2.2.4), it follows that each  $U_n$  is an open set in  $P(A)$  and, since the canonical mapping is onto, each  $U_n$  is dense in  $P(A)$ . Hence  $(U_1, U_2, \dots)$  is a countable family of open dense sets in  $P(A)$ . Since  $P(A)$  is a Baire space,  $\bigcap_n U_n$  is a dense set in  $P(A)$  and, since  $\bigcap_n V_n$  is the image of  $\bigcap_n U_n$ , Theorem (2.2.4) implies that  $\bigcap_n V_n$  is dense in  $\hat{A}$ . Hence  $\hat{A}$  is a Baire space and this completes the proof.

By an arguments similar to that given in the proof of Theorem (2.2.5), we can show that the Jacobson structure space  $\text{Prim}(A)$  of a  $C^*$ -algebra  $A$  is a Baire space.

§ 3. The third definition of the topology on the dual.

For each cardinal number  $n$ , let  $H_n$  be the  $n$ -dimensional Hilbert space of sequences  $\{\xi_i\}$  of complex numbers such that

$$\sum_i |\xi_i|^2 < \infty,$$

indexed by a set of cardinal  $n$ .

Let  $A$  be a  $C^*$ -algebra with the dual  $\hat{A}$ . For each cardinal  $n$ , let  $\text{Rep}_n(A)$  be the set of all representations of  $A$  in  $H_n$  and  $\text{Irr}_n(A)$  the set of all non-zero irreducible representations of  $A$  in  $H_n$ . Let  $\psi$  be a mapping of  $\text{Irr}_n(A)$  into  $\hat{A}$  which associates with each element in  $\text{Irr}_n(A)$  its equivalence class in  $\hat{A}$ . We call  $\psi$  the canonical mapping of  $\text{Irr}_n(A)$  into  $\hat{A}$  and denote it by

$$\psi : \text{Irr}_n(A) \longrightarrow \hat{A}.$$

Let  $\hat{A}_n$  be the image of  $\text{Irr}_n(A)$  in  $\hat{A}$  under the mapping  $\psi$ . It is clear that  $\hat{A}_n$  is the set of all equivalence classes of non-zero irreducible representations of dimension  $n$ .

For each cardinal number  $n$ , let  $L(H_n)$  be the algebra of all bounded linear operators on  $H_n$ . Then the topology on  $L(H_n)$  defined by the semi-norms

$$T \longrightarrow |(T\xi | \eta)|,$$

where  $\xi, \eta \in H_n$ , is called the weak topology on  $L(H_n)$ , and  $L(H_n)$  with this topology is denoted by  $L_w(H_n)$ . The topology on  $L(H_n)$  defined by the semi-norms

$$T \longrightarrow \|T\xi\|,$$

where  $\xi \in H_n$ , is called the strong topology on  $L(H_n)$ , and  $L(H_n)$

with this topology is denoted by  $L_S(H_n)$ . It is easy to show that  $L_W(H_n)$  and  $L_S(H_n)$  are topological vector spaces whose topologies are Hausdorff. If  $V_W$  is a neighborhood of zero in  $L_W(H_n)$ , then there exist  $\varepsilon > 0$  and elements  $\xi_1, \dots, \xi_m, \eta_1, \dots, \eta_m$  in  $H_n$  such that  $V_W$  is of the form

$$(1) \quad V_W = \{ T : |(T\xi_i | \eta_i)| < \varepsilon \quad ; i = 1, \dots, m \};$$

and if  $V_S$  is a neighborhood of zero in  $L_S(H_n)$ , then there exist  $\varepsilon > 0$  and  $\zeta_1, \dots, \zeta_k$  in  $H_n$  such that  $V_S$  is of the form

$$(2) \quad V_S = \{ T : \|T\zeta_i\| < \varepsilon \quad ; i = 1, \dots, k \}.$$

Now let  $L(A, L_W(H_n))$  be the set of all continuous linear mappings of  $A$  into  $L_W(H_n)$ . Then  $\text{Rep}_n(A) \subset L(A, L_W(H_n))$ . In fact, let  $\pi$  be any element in  $\text{Rep}_n(A)$  and  $V_W$  a neighborhood of zero in  $L_W(H_n)$ . We need to prove that there is a neighborhood  $W$  of zero in  $A$  such that  $\pi(W) \subset V_W$ . We observe that  $V_W$  is given in the form (1) and for every  $x$  in  $A$ ,  $\|\pi(x)\| \leq \|x\|$ . Then, since

$$|(\pi(x)\xi_i | \eta_i)| \leq \|\pi(x)\xi_i\| \|\eta_i\| \leq \|x\| \|\xi_i\| \|\eta_i\|,$$

the set

$$W = \left\{ x \in A : \|x\| < \frac{\varepsilon}{\|\xi_i\| \|\eta_i\|} ; i = 1, \dots, m \right\}$$

is the required neighborhood in  $A$ . Similarly, we can show that  $\text{Rep}_n(A) \subset L(A, L_S(H_n))$ , where  $L(A, L_S(H_n))$  is the set of all continuous linear mappings of  $A$  into  $L_S(H_n)$ .

Let  $E$  and  $F$  be two topological vector spaces and  $L(E, F)$  the set of all continuous linear mappings of  $E$  into  $F$ . Let  $\mathcal{F}$  be the family of all finite subsets of  $E$ . For each  $M \in \mathcal{F}$  and each neighborhood  $V$  of zero in  $F$ , let

$$U_o(M, V) = \{ u \in L(E, F) : u(M) \subset V \}.$$

For each  $u_o \in L(E, F)$ , form the set

$$(3) \quad u_o + U_o(M, V).$$

Then finite intersections of sets of the form (3) define a fundamental system of neighborhoods of the point  $u_o$  for a topology on  $L(E, F)$ . We call this topology on  $L(E, F)$  the topology of simple convergence. (cf. p.18 in [1]).

If  $E = A$  and  $F = L_w(H_n)$ , the the topology of simple convergence on  $L(A, L_w(H_n))$  is called the topology of weak simple convergence, and  $L(A, L_w(H_n))$  with this topology is denoted by  $L_w(A, L_w(H_n))$ . If  $E = A$  and  $F = L_s(H_n)$ , then the topology of simple convergence on  $L(A, L_s(H_n))$  is called the topology of strong simple convergence, and  $L(A, L_s(H_n))$  with this topology is denoted by  $L_s(A, L_s(H_n))$ .

The induced topology on  $\text{Rep}_n(A)$  as a subset of  $L_w(A, L_w(H_n))$  (resp.  $L_s(A, L_s(H_n))$ ) is called the topology of the weak simple convergence in A. (resp. the topology of the strong simple convergence in A).

Take  $E = A$ . For each  $M \in \mathcal{F}$  and for every neighborhood  $V_w$  of zero in  $L_w(H_n)$ , let

$$U_o'(M, V_w) = U_o(M, V_w) \cap \text{Rep}_n(A) = \{ \pi \in \text{Rep}_n(A) : \pi(M) \subset V_w \}.$$

Hence if  $V_w$  is given by (1) and if  $M = \{ a_1, \dots, a_p \}$ , then

$$U_o'(M, V_w) = \{ \pi \in \text{Rep}_n(A) : |(\pi(a_i)\xi_j | \eta_j)| < \varepsilon ; \begin{matrix} i=1, \dots, p \\ j=1, \dots, m \end{matrix} \}$$

and thus for each  $\pi_o \in \text{Rep}_n(A)$ ,

$$N_w(\pi_0) = \pi_0 + U_0'(M, v_w)$$

is given by

$$(4) \quad N_w(\pi_0) = \left\{ \pi \in \text{Rep}_n(A) : \left| (\pi(a_i)\xi_j | \eta_j) - (\pi_0(a_i)\xi_j | \eta_j) \right| < \varepsilon ; \quad i = 1, \dots, p; \quad j = 1, \dots, m \right\}$$

Thus the fundamental system of neighborhoods at each point  $\pi_0$  of  $\text{Rep}_n(A)$  with the topology of the weak simple convergence in  $A$  is given by the finite intersections of the sets of the form (4).

Similarly we can show that the fundamental system of neighborhoods at the point  $\pi_0$  of  $\text{Rep}_n(A)$  with the topology of the strong simple convergence in  $A$  is given by the finite intersections of the sets of the form

$$(5) \quad N_s(\pi_0) = \left\{ \pi \in \text{Rep}_n(A) : \left\| \pi(a_i)\zeta_j - \pi_0(a_i)\zeta_j \right\| < \varepsilon ; \quad i = 1, \dots, p; \quad j = 1, \dots, k \right\}$$

It is easy to see that relative to the topology of the weak simple convergence in  $A$ , a net  $\{\pi_\lambda\}_{\lambda \in \Lambda}$  in  $\text{Rep}_n(A)$  converges to  $\pi \in \text{Rep}_n(A)$  if and only if  $(\pi_\lambda(a)\xi | \eta) \rightarrow (\pi(a)\xi | \eta)$  for all  $a$  in  $A$  and for all  $\xi, \eta \in H_n$ , and that relative to the topology of the strong simple convergence in  $A$ , a net  $\{\pi_\lambda\}_{\lambda \in \Lambda}$  converges to  $\pi$  if and only if  $\|\pi_\lambda(a)\xi - \pi(a)\xi\| \rightarrow 0$  for all  $a$  in  $A$  and for all  $\xi \in H_n$ .

We now show that the above two topologies defined on  $\text{Rep}_n(A)$  coincide. For this it suffices to show that the identity mapping of  $\text{Rep}_n(A)$  with weak simple convergence topology onto  $\text{Rep}_n(A)$  with strong simple convergence topology, preserves closure, i.e., it suffices to show that  $(\pi_\lambda(a)\xi | \eta) \rightarrow (\pi(a)\xi | \eta)$  for all  $a$  in  $A$

and for all  $\xi, \eta \in H_n$ , if and only if  $\|\pi_\lambda(a)\xi - \pi(a)\xi\| \rightarrow 0$  for all  $a$  in  $A$  and for all  $\xi \in H_n$ . In fact, suppose that  $(\pi_\lambda(a)\xi | \eta) \rightarrow (\pi(a)\xi | \eta)$  for all  $a$  in  $A$  and for all  $\xi, \eta \in H_n$ . Then

$$\begin{aligned} \|\pi_\lambda(a)\xi - \pi(a)\xi\|^2 &= (\pi_\lambda(a)\xi - \pi(a)\xi | \pi_\lambda(a)\xi - \pi(a)\xi) \\ &= (\pi_\lambda(a)\xi | \pi_\lambda(a)\xi) - 2\operatorname{Re}(\pi_\lambda(a)\xi | \pi(a)\xi) + (\pi(a)\xi | \pi(a)\xi) \\ &= (\pi_\lambda(a^*a)\xi | \xi) - 2\operatorname{Re}(\pi_\lambda(a)\xi | \pi(a)\xi) + (\pi(a^*a)\xi | \xi) \\ &\rightarrow (\pi(a^*a)\xi | \xi) - 2\operatorname{Re}(\pi(a)\xi | \pi(a)\xi) + (\pi(a^*a)\xi | \xi) = 0. \end{aligned}$$

The converse follows from the following inequality:

$$\begin{aligned} |(\pi_\lambda(a)\xi | \eta) - (\pi(a)\xi | \eta)| &= |(\pi_\lambda(a)\xi - \pi(a)\xi | \eta)| \\ &\leq \|(\pi_\lambda(a)\xi - \pi(a)\xi)\| \|\eta\|. \end{aligned}$$

Since  $\operatorname{Irr}_n(A)$  is a subset of  $\operatorname{Rep}_n(A)$ , the topology on  $\operatorname{Rep}_n(A)$  induces the relative topology on  $\operatorname{Irr}_n(A)$ . Let  $\pi_0 \in \operatorname{Irr}_n(A)$ . Then every neighborhood of  $\pi_0$  in  $\operatorname{Irr}_n(A)$  contains a neighborhood  $N(\pi_0)$  of  $\pi_0$  constructed in the following way: let  $a_1, \dots, a_p \in A$ ,  $\xi_1, \dots, \xi_m \in H_n$  and  $\varepsilon > 0$ ; take

$$(6) \quad N(\pi_0) = \left\{ \pi \in \operatorname{Irr}_n(A) : \|\pi(a_i)\xi_j - \pi_0(a_i)\xi_j\| < \varepsilon; \right. \\ \left. i = 1, \dots, p; j = 1, \dots, m \right\}.$$

We recall that  $\hat{A}_n$  is a subspace of  $\hat{A}$  with the relative topology induced by the topology on  $\hat{A}$  and that  $\operatorname{Irr}_n(A)$  has the topology of the strong simple convergence in  $A$ .

Theorem (2.3.1). Let  $A$  be a  $C^*$ -algebra and  $n$  any cardinal. Then

the canonical mapping

$$\psi : \operatorname{Irr}_n(A) \rightarrow \hat{A}_n$$

of  $\text{Irr}_n(A)$  onto  $\hat{A}_n$  is continuous and open.

Proof: [3] Let  $Q$  be a subset of  $\text{Irr}_n(A)$  and  $\pi_0$  a point of  $\text{Irr}_n(A)$ . Let  $S$  be the image of  $Q$  in  $\hat{A}_n$  and  $\tau_0$  the image of  $\pi_0$  in  $\hat{A}_n$ , under the canonical mapping  $\psi : \text{Irr}_n(A) \longrightarrow \hat{A}_n$ . Now suppose  $\pi_0 \in \bar{Q}$ . Then, by Theorem (2.2.1), every positive linear functional associated to  $\pi_0$  is the weak limit of the positive linear functionals associated to  $Q$ . Hence, by Theorem (2.2.3),  $\tau_0 \in \bar{S}$  so that  $\psi(\bar{Q}) \subseteq \overline{\psi(Q)}$ . Consequently  $\psi$  is continuous,

Next, let  $\pi_0$  be any point of  $\text{Irr}_n(A)$ . Then every neighborhood of  $\pi_0$  in  $\text{Irr}_n(A)$  contains a neighborhood  $N(\pi_0)$  given by (6). Let  $\tau_0$  be the image of  $\pi_0$  in  $\hat{A}_n$ . By 3.5.7. in [3], it follows that the image of  $N(\pi_0)$  in  $\hat{A}_n$  is neighborhood of  $\tau_0$  in  $\hat{A}_n$ . Hence  $\psi$  is open and this completes the proof.

It follows then that the topology on  $\hat{A}_n$  can be defined to be the quotient topology of that of  $\text{Irr}_n(A)$  under the canonical mapping

$$\psi : \text{Irr}_n(A) \longrightarrow \hat{A}_n.$$

Let  $A$  be a  $C^*$ -algebra. Then

$$\hat{A} = \bigcup_n \hat{A}_n,$$

and it is clear that if  $m \neq n$ , then  $\hat{A}_m \cap \hat{A}_n = \emptyset$ . We observe that the dimension of every non-zero irreducible representation of  $A$  is bounded by a fixed cardinal. (cf. 2.3.3. in [3]). In particular, if  $A$  is a separable  $C^*$ -algebra, then the dimension of every non-zero irreducible representation of  $A$  is bounded by  $\aleph_0$  and we have

$$\hat{A} = \bigcup_{n=1}^{\infty} \hat{A}_n.$$

Let  $\text{Irr}(A) = \bigcup_n \text{Irr}_n(A)$  and let  $\psi : \text{Irr}(A) \longrightarrow \hat{A}$  be the mapping of  $\text{Irr}(A)$  into  $\hat{A}$  which associates with each element of  $\text{Irr}(A)$  its equivalence class in  $\hat{A}$ . Then, since  $\hat{A} = \bigcup_n \hat{A}_n$ , the mapping  $\psi$  is onto. Let  $\hat{A}_n$  be the topological space with the quotient topology of that of  $\text{Irr}_n(A)$  for each  $n$ . Then, by Theorem (2.3.1), this topology coincides with the relative topology on  $\hat{A}_n$  induced by the topology on  $\hat{A}$ . So the third definition of the topology on  $\hat{A}$  is given as follows: it is the topology on  $\hat{A}$  whose open sets  $S$  are the subsets  $S$  of  $\hat{A}$  such that  $S \cap \hat{A}_n$  is open in  $\hat{A}_n$  for all  $n$ , where each  $\hat{A}_n$  has the quotient topology of that of  $\text{Irr}_n(A)$ .

§ 4. Some properties of the dual.

Theorem (2.4.1). Let  $A$  be a  $C^*$ -algebra,  $\hat{A}$  the dual of  $A$  and

$\{x_\lambda\}_{\lambda \in \Lambda}$  a family of elements dense in  $A$ . Let

$$Z_\lambda = \{ \pi : \pi \in \hat{A} \text{ and } \|\pi(x_\lambda)\| > 1 \}.$$

Then the family  $\{Z_\lambda\}_{\lambda \in \Lambda}$  forms a base for the topology on  $\hat{A}$ .

Proof: [3] Since, by 3.3.2. in [3], the function  $\pi \longrightarrow \|\pi(x)\|$  is lower semi-continuous, each  $Z_\lambda$  is an open set in  $\hat{A}$ . Now let  $U$  be an open set in  $\hat{A}$  and let  $\pi \in U$ . We need to prove that there exists  $Z_\lambda$  for some  $\lambda \in \Lambda$  such that  $\pi \in Z_\lambda \subset U$ . Let

$$I = \bigcap_{\rho \in \hat{A} - U} \text{Ker } \rho.$$

Since  $\pi \in U$ , there exists  $x$  in  $I$  such that  $\pi(x) \neq 0$ . Thus there exists  $x$  in  $I$  such that  $\|\pi(x)\| = 2$ . But for  $\rho \in \hat{A} - U$ , we have  $\rho(x) = 0$ . Hence  $\|\rho(x)\| = 0$  for all  $\rho \in \hat{A} - U$ . Since  $\{x_\lambda\}_{\lambda \in \Lambda}$

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is dense in  $A$ , there exists a  $\lambda \in \Lambda$  such that  $\|x - x_\lambda\| < 1$ , and since for every representation  $\rho$  of  $A$ , we have

$$\begin{aligned} |\|\rho(x)\| - \|\rho(x_\lambda)\|| &\leq \|\rho(x) - \rho(x_\lambda)\| \\ &= \|\rho(x - x_\lambda)\| \leq \|x - x_\lambda\| < 1, \end{aligned}$$

it follows that  $\|\pi(x_\lambda)\| > 1$ . Therefore  $\pi \in Z_\lambda$ . To show that  $Z_\lambda \subset U$ , we observe that  $\|\rho(x_\lambda)\| < 1$  for all  $\rho \in \hat{A} - U$ . Hence if  $\pi \in Z_\lambda$  and  $\pi \notin U$ , then  $\pi \in \hat{A} - U$  and so  $\|\pi(x_\lambda)\| < 1$ , a contradiction. Hence  $Z_\lambda \subset U$  and this completes the proof.

Corollary (2.4.2). Let  $A$  be a separable  $C^*$ -algebra and  $\hat{A}$  the dual of  $A$ . Then  $\hat{A}$  satisfies the second axiom of countability.

Proof: Since  $A$  is separable, there is a countable dense subset  $\{x_i\}$  in  $A$ . By Theorem (2.4.1), it follows that the sets

$$Z_i = \{ \pi : \pi \in \hat{A} \text{ and } \|\pi(x_i)\| > 1 \}$$

form a countable base for the topology on  $\hat{A}$ .

Since every second countable space is separable, we have:

Corollary (2.4.3). If  $A$  is a separable  $C^*$ -algebra, then  $\hat{A}$  is separable.

Theorem (2.4.4). Let  $A$  be a  $C^*$ -algebra and  $\{x_\lambda\}_{\lambda \in \Lambda}$  a family of elements dense in  $A$ . Let

$$T_\lambda = \{ \text{Ker } \pi : \pi \in \hat{A} \text{ and } \|\pi(x_\lambda)\| > 1 \}.$$

Then the family  $\{T_\lambda\}_{\lambda \in \Lambda}$  forms a base for the topology on  $\text{Prim}(A)$ .

Proof: Since the canonical mapping

$$\varphi : \hat{A} \longrightarrow \text{Prim}(A)$$

is open,  $T_\lambda$ , being the image of the open set  $Z_\lambda$ , is open for each  $\lambda$ , where  $Z_\lambda$  is defined in Theorem (2.4.1). Now let  $V$  be a open set in  $\text{Prim}(A)$  and  $P \in V$ . Then there is a  $\pi \in \hat{A}$  such that  $\text{Ker } \pi = P$  and  $\pi \in \varphi^{-1}(V)$ . Since the canonical mapping  $\varphi$  is continuous,  $\varphi^{-1}(V)$  is open in  $\hat{A}$ , so that by Theorem (2.4.1), there exists  $Z_\lambda$  for some  $\lambda$  such that  $\pi \in Z \subset \varphi^{-1}(V)$ . Hence  $P \in T_\lambda \subset V$ , so that the family  $\{T_\lambda\}_{\lambda \in \Lambda}$  forms a base for the topology on  $\text{Prim}(A)$ . This completes the proof.

Corollary (2.4.5). If  $A$  is a separable  $C^*$ -algebra, then  $\text{Prim}(A)$  is a second countable space and hence separable.

Theorem (2.4.6). Let  $A$  be a  $C^*$ -algebra. If the dual  $\hat{A}$  of  $A$  is a Hausdroff space, then the function  $\pi \rightarrow \|\pi(x)\|$  is continuous on  $\hat{A}$  for all  $x$  in  $A$ .

Proof: [3] Let  $x \in A$ . Then, by 3.3.7. in [3], for every scalar  $\alpha > 0$ ,  $Z = \{ \pi : \pi \in \hat{A} \text{ and } \|\pi(x)\| \geq \alpha \}$  is a compact subset of  $\hat{A}$ . If  $\hat{A}$  is Hausdroff, then  $Z$  is a closed subset of  $\hat{A}$  and so the function  $\pi \rightarrow \|\pi(x)\|$  is upper semi-continuous. But, by 3.3.2. in [3],  $\pi \rightarrow \|\pi(x)\|$  is lower semi-continuous. Hence  $\pi \rightarrow \|\pi(x)\|$  is continuous on  $\hat{A}$  and this completes the proof.

Theorem (2.4.7). Let  $A$  be a  $C^*$ -algebra. If the Jacobson structure space  $\text{Prim}(A)$  is Hausdroff, then the function  $\pi \rightarrow \|\pi(x)\|$  is continuous on  $\hat{A}$  for all  $x$  in  $A$ .

Proof: Let  $x \in A$ . Then, by 3.3.7. in [3], for every scalar  $\alpha > 0$ ,  $Z = \{ \pi : \pi \in \hat{A} \text{ and } \|\pi(x)\| \geq \alpha \}$  is a compact subset of  $\hat{A}$ . Let

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$$T = \{ \text{Ker } \pi : \pi \in Z \}.$$

Since the canonical mapping  $\varphi : \hat{A} \longrightarrow \text{Prim}(A)$  is continuous, it follows that  $T$  is a compact subset of  $\text{Prim}(A)$ . Thus if  $\text{Prim}(A)$  is Hausdorff, then  $T$  is closed. Since the mapping  $\varphi$  is continuous, it follows that  $Z$  is closed so that the function  $\pi \longrightarrow \pi(x)$  is upper semi-continuous on  $\hat{A}$  and hence is continuous on  $\hat{A}$ . This completes the proof.

Theorem (2.4.8). Let  $A$  be a non-primitive separable  $C^*$ -algebra and  $\hat{A}$  the dual of  $A$ . Then there exist in  $\hat{A}$  two non-empty disjoint open sets.

Proof: It is clear that if any two non-empty open sets in  $\hat{A}$  meet each other, then every open set in  $\hat{A}$  is dense in  $\hat{A}$ . Thus, to prove the theorem, we need to prove that there is a non-empty open set in  $\hat{A}$  which is not dense in  $\hat{A}$ . Suppose that every non-empty open set in  $\hat{A}$  is dense in  $\hat{A}$ . Let  $x \in A$ . Since, by 3.3.2, in [3], the function  $\pi \longrightarrow \pi(x)$  is lower semi-continuous on  $\hat{A}$ , it follows that for each  $\varepsilon > 0$ ,

$$Z = \{ \pi : \pi \in \hat{A} \text{ and } \|\pi(x)\| > \|x\| - \varepsilon \}$$

is a non-empty open set in  $\hat{A}$  and thus is dense in  $\hat{A}$  by assumption. Since  $A$  is separable, there is a countable dense subset  $\{x_i\}$  in  $A$ . For  $i > 0$  and  $j > 0$ , let

$$Z_{ij} = \{ \pi : \pi \in \hat{A} \text{ and } \|\pi(x_i)\| > \|x_i\| - \frac{1}{j} \}.$$

Then  $Z_{ij}$  is a non-empty open set which is dense in  $\hat{A}$  by the above argument. Since  $\hat{A}$  is a Baire space by Theorem (2.2.5), it follows that  $U = \bigcap_{i,j} Z_{ij}$  is an open set which is dense in  $\hat{A}$ . Let  $\pi \in U$ .

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Then for every  $i, j$ , we have

$$\|x_i\| - \frac{1}{j} < \|\pi(x_i)\| \leq \|x_i\|.$$

Since  $j$  is arbitrary, hence  $\|\pi(x_i)\| = \|x_i\|$  for every  $x_i$ .

Since  $\{x_i\}$  is dense in  $A$ , it follows that

$$\|\pi(x)\| = \|x\|,$$

for every  $x$  in  $A$ . Hence  $\text{Ker } \pi = (0)$  so that  $(0)$  is a primitive ideal of  $A$  which contradicts the fact that  $A$  is non-primitive. Hence there is a non-empty open set in  $\hat{A}$  which is not dense in  $\hat{A}$ . This completes the proof.

Theorem (2.4.9). Let  $A$  be a non-primitive separable  $C^*$ -algebra and  $\text{Prim}(A)$  the Jacobson structure space of  $A$ . Then there exist in  $\text{Prim}(A)$  two non-empty disjoint open sets; i.e., there is a non-empty open set in  $\text{Prim}(A)$  which is not dense in  $\text{Prim}(A)$ .

Proof: Suppose that every non-empty open set in  $\text{Prim}(A)$  is dense in  $\text{Prim}(A)$ . Then, since every open set in  $\hat{A}$  is the inverse image of some open set in  $\text{Prim}(A)$  under the canonical mapping

$$\varphi : \hat{A} \longrightarrow \text{Prim}(A),$$

which is onto, it follows that every open set in  $\hat{A}$  is dense in  $\hat{A}$ . But this contradicts Theorem (2.4.8). Hence there is a non-empty open set in  $\text{Prim}(A)$  which is not dense in  $\text{Prim}(A)$ . This completes the proof.

## Chapter III

### The duals of special C\*-algebras

#### § 1. The algebra of compact operators and its dual.

Let  $H$  be a Hilbert space and  $T$  a linear operator on  $H$ . Then  $T$  is said to be compact (or completely continuous) if it maps every bounded set into relatively compact set; equivalently,  $T$  is said to be compact if for every bounded sequence  $\{\xi_n\}$  in  $H$ , the sequence  $\{T\xi_n\}$  contains a convergent subsequence. The collection  $LC(H)$  of all compact operators on  $H$  forms a closed two-sided ideal of  $L(H)$  and is itself a C\*-algebra. It is clear that if  $H$  is finite dimensional, then  $LC(H) = L(H)$ .

Theorem (3.1.1). The algebra  $LC(H)$  is an irreducible Banach algebra of linear operators in  $H$ .

Proof: We need to prove that for any  $\xi, \eta \in H$ , there exists  $T$  in  $LC(H)$  such that  $T\xi = \eta$ . In fact, there exists a continuous linear functional  $f$  on  $H$  such that  $f(\xi) = 1$ . Now, define  $T : H \rightarrow H$  by  $T = \eta \otimes f$ , i.e.,

$$T\zeta = f(\zeta)\eta,$$

for every  $\zeta \in H$ . Then  $T$  is of rank one and so  $T \in LC(H)$ , and

$$T\xi = f(\xi)\eta = \eta.$$

This completes the proof.

Corollary (3.1.2). The identity representation of  $LC(H)$  is irreducible.

Theorem (3.1.3). Every pure positive linear functional on  $LC(H)$  is of the form

$$x \longrightarrow (x\xi | \xi),$$

where  $\xi$  is a non-zero vector in  $H$ .

Proof: [3] Since the identity representation of  $LC(H)$  is irreducible, by 2.5.4. in [3], it follows that  $x \longrightarrow (x\xi | \xi)$ , where  $\xi \neq 0$  in  $H$ , is a pure positive linear functional on  $LC(H)$ . Now, suppose that  $f$  is a pure positive linear functional on  $LC(H)$ . Then, by 4.1.3. in [3],  $f$  is of the form

$$f(x) = \sum_{i=1}^{\infty} \alpha_i (x\xi_i | \xi_i),$$

where  $(\xi_1, \xi_2, \dots)$  is an orthonormal system in  $H$  and  $\alpha_i \geq 0$  and such that  $\sum_i \alpha_i < \infty$ . Next, let

$$g(x) = \alpha_1 (x\xi_1 | \xi_1).$$

Then it is clear that  $g$  is majorized by  $f$ . Since  $f$  is pure,  $g$  is of the form  $\alpha f$ , where  $0 \leq \alpha \leq 1$ . Thus

$$\alpha f(x) = \alpha_1 (x\xi_1 | \xi_1),$$

and so  $f$  is of the form  $x \longrightarrow (x\xi | \xi)$ , where  $\xi$  is a non-zero vector in  $H$ . This completes the proof.

Theorem (3.1.4). Every non-zero irreducible representation of  $LC(H)$  is equivalent to the identity representation.

Proof: Let  $\pi$  be a non-zero irreducible representation of  $LC(H)$  in  $H_\pi$ . Since  $\pi$  is irreducible, every non-zero vector  $\xi' \in H_\pi$  is cyclic for  $\pi$ , and  $x \longrightarrow (\pi(x)\xi' | \xi')$  is a pure positive

linear functional on  $LC(H)$ . Thus, by Theorem (3.1.3), it follows that

$$(\pi(x)\xi' | \xi') = (x\xi | \xi),$$

for some  $\xi \neq 0$  in  $H$ . Consider the mapping  $\underline{\Psi}$  given by

$$\underline{\Psi}(x\xi) = \pi(x)\xi'.$$

Then  $\underline{\Psi}$  is linear and, since

$$\begin{aligned} (x\xi | y\xi) &= (y^*x\xi | \xi) = (\pi(y^*x)\xi' | \xi') \\ &= (\pi(x)\xi' | \pi(y)\xi'), \end{aligned}$$

$\underline{\Psi}$  is an isometry. As  $\{x\xi : x \in LC(H)\}$  and  $\{\pi(x)\xi' : x \in LC(H)\}$  are dense in  $H$  and  $H_{\pi}$  respectively,  $\underline{\Psi}$  can be extended uniquely to an isometric isomorphism of  $H$  onto  $H_{\pi}$ . We denote this isomorphism by  $\underline{\Phi}$ . It remains to show that  $\underline{\Phi}x = \pi(x)\underline{\Phi}$  for every  $x \in LC(H)$ . Since for every  $y$  in  $LC(H)$ , we have

$$\begin{aligned} (\underline{\Phi}x)(y\xi) &= \underline{\Phi}(xy\xi) = \pi(xy)\xi' = \pi(x)(\pi(y)\xi') \\ &= \pi(x)(\underline{\Phi}(y\xi)) = (\pi(x)\underline{\Phi})(y\xi) \end{aligned}$$

and, since  $\{y\xi : y \in LC(H)\}$  is dense in  $H$ , it follows that  $\underline{\Phi}x = \pi(x)\underline{\Phi}$ . Hence  $\pi$  is equivalent to the identity representation. This completes the proof.

Corollary (3.1.5). (i) The  $C^*$ -algebra  $LC(H)$  is primitive.

(ii) The zero ideal  $(0)$  is the only primitive ideal.

(iii) The dual  $\widehat{LC(H)}$  consists of only one point.

Corollary (3.1.6). The  $C^*$ -algebra  $LC(H)$  is simple; i.e., the only closed two-sided ideals of  $LC(H)$  are  $(0)$  and  $LC(H)$ .

Proof: [3] Follows from Corollary (3.1.5)(ii) and the fact that every closed two-sided ideal of  $LC(H)$  is the intersection of all primitive ideals containing it.

Theorem (3.1.7). Every separable simple  $C^*$ -algebra  $A$  is  $*$ -isomorphic to  $LC(H)$  for some Hilbert space  $H$ .

Proof: Let  $\pi$  be a non-zero irreducible representation of  $A$  in  $H_\pi$ . Since  $A$  is simple, zero ideal  $(0)$  is the only primitive ideal of  $A$ . Thus  $\pi$  is a faithful representation. Since  $\pi(A)$  is an irreducible complex algebra of operators in  $H_\pi$ ,  $\pi(A)$  is strictly dense in  $H_\pi$ . (cf. pp. 60-61 in [15]) Let  $I$  be a closed right ideal in  $\pi(A)$ . Then, by Theorem (2.4.18) in [15],  $I$  contains every element of  $\pi(A)$  with finite rank whose range is contained in  $M$ , where  $M$  is the smallest closed linear subspace of  $H_\pi$  containing the range of each operator in  $I$ . Hence

$$\pi(A) \cap LC(H_\pi) \neq (0).$$

Therefore, by 4.1.10. in [3],  $\pi(A) \supseteq LC(H_\pi)$  and hence every irreducible representation of  $A$  having the same kernel as  $\pi$  is equivalent to  $\pi$ . Since  $(0)$  is the only primitive ideal in  $A$ , it follows now that any two non-zero irreducible representations of  $A$  are equivalent. Thus the dual  $\hat{A}$  of  $A$  consists of only one point. Finally, by Theorem 4 in [16], it follows that  $A$  is  $*$ -isomorphic to  $LC(H)$  for some Hilbert space  $H$ .

§ 2. Dual C\*-algebra and its dual.

Let  $A$  be a C\*-algebra. For any subset  $M$  of  $A$ , let

$$L(M) = \{ x \in A : xM = 0 \},$$

$$R(M) = \{ x \in A : Mx = 0 \}.$$

Then  $L(M)$  is called the left annihilator and  $R(M)$  is called the right annihilator of  $M$ . It is clear that  $L(M)$  is a closed left ideal and  $R(M)$  is a closed right ideal of  $A$ . Also we have

$$M \subset L(R(M)) \quad \text{and} \quad M \subset R(L(M)),$$

and if  $M_1 \subset M_2$ , then  $L(M_2) \subset L(M_1)$  and  $R(M_2) \subset R(M_1)$ . If for every closed right ideal  $I$  of  $A$ ,  $R(L(I)) = I$ , then for every closed left ideal  $J$  of  $A$ ,  $L(R(J)) = J$ . A C\*-algebra  $A$  is called a dual C\*-algebra if for every closed right ideal  $I$  of  $A$ , we have  $R(L(I)) = I$ .

Theorem (3.2.1). The algebra  $LC(H)$  of all compact operators on a Hilbert space  $H$  is a dual C\*-algebra.

Proof: Let  $I$  be a closed right ideal of  $LC(H)$  and let  $M$  be the smallest closed linear subspace of  $H$  containing the range of each operator in  $I$ . Then  $I$  contains every element of  $LC(H)$  whose range is contained in  $M$ . In fact, let

$$I_0 = \{ T \in LC(H) : T(H) \subset M \}.$$

Then it is clear that  $I \subset I_0$ . Now let  $T \in I_0$ . Then there exist operators  $T_n$  of finite rank such that  $T_n \rightarrow T$ . Let  $E$  be the projection on  $M$ . Since  $T(H) \subset M$ , it follows that  $ET = T$  and so  $ET_n \rightarrow ET = T$ . Since  $ET_n$  is of finite rank and whose range

is contained in  $M$  for each  $n$  and since  $LC(H)$  is strictly dense in  $H$ , by Theorem (2.4.18) in [15],  $ET_n \in I$  for each  $n$ . Since  $I$  is closed,  $ET = T \in I$  and so  $I = I_0$ . Next let

$$N = \{ T \in LC(H) : T(M) = 0 \},$$

and we show that  $L(I) = N$ . Let  $T' \in N$ . Then since  $T(H) \subset M$  for each  $T \in I$ , it follows that  $T'T(H) \subset T'(M)$  so that  $T'T(H) = 0$ ; i.e.,  $T'T = 0$  for each  $T \in I$ . Hence  $T' \in L(I)$  and so  $N \subset L(I)$ . Now let  $\xi$  be any element of  $M$ . Since  $LC(H)$  is irreducible, there exists  $T \in LC(H)$  such that  $T\xi = \xi$ . Let  $E$  be the projection on  $M$ . Then  $ET \in I$  and  $ET\xi = \xi$ . Hence  $T'\xi = T'ET\xi = 0$  for every  $T' \in L(I)$ . Therefore  $L(I) \subset N$  and so  $L(I) = N$ . To show that  $I = R(L(I))$ , we need only to show that  $I \supset R(L(I))$  since  $I \subset R(L(I))$ . Let  $T \in R(L(I))$  and suppose that  $T \notin I$ . Then there exists  $\xi_0 \in H$  such that  $T\xi_0 \notin M$ . Since  $M$  is closed, there exists a continuous linear functional  $f$  on  $H$  such that  $f(M) = 0$  and  $f(T\xi_0) = 1$ . Define  $T' : H \rightarrow H$  by  $T' = \xi_0 \otimes f$ ; i.e., for every  $\eta \in H$ ,

$$T'(\eta) = f(\eta)\xi_0.$$

Since  $T'$  is of rank one and  $f(M) = 0$ , it follows that  $T' \in LC(H)$  and  $T'(M) = 0$  and so  $T' \in L(I)$ . But then  $T'T = 0$  and  $0 = T'T\xi_0 = f(T\xi_0)\xi_0 = \xi_0$ , which contradicts the fact that  $T\xi_0 \notin M$ . Hence  $R(L(I)) \subset I$  and consequently  $R(L(I)) = I$  so that  $LC(H)$  is a dual  $C^*$ -algebra.

Theorem (3.2.2). Let  $B$  be a sub- $C^*$ -algebra of  $LC(H)$ . Then  $B$  is a dual  $C^*$ -algebra.

Proof: Let  $\rho$  be a non-zero irreducible representation of  $B$  in  $H_\rho$ . Then, by 2.10.2. in [3],  $\rho$  can be extended to an irreducible representation  $\pi$  of  $LC(H)$  in  $H_\pi$  such that  $H_\rho$  is a closed subspace of  $H_\pi$  and  $\rho(x) = \pi(x)|_{H_\rho}$  for all  $x$  in  $B$ . In fact,  $H_\rho$  is given by  $H_\rho = \overline{\pi(B)\eta}$  for some  $\eta \in H_\pi$ . (cf. proof of 2.10.2. in [3]) Since  $\pi$  is equivalent to the identity representation,  $\pi(B) \subset LC(H_\pi)$  and so  $\rho(B) \subset LC(H_\rho)$ . Since  $\rho$  is non-zero, by 4.1.11. in [3],  $\rho(B) = LC(H_\rho)$ . We show next that  $\rho$  is faithful. Let  $x \in B; x \neq 0$ . Since  $B$  is semi-simple, there exists  $y$  in  $B$  such that  $xy \neq 0$ . Therefore  $\pi(xy)\eta \neq 0$ . But  $\xi = \pi(y)\eta$  is a non-zero element of  $H_\rho$ . Therefore  $\rho(x)\xi = \pi(x)\xi = \pi(x)\pi(y)\eta = \pi(xy)\eta \neq 0$ . Hence  $\rho$  is faithful. Thus  $B$  is  $*$ -isomorphic to a dual  $C^*$ -algebra and consequently  $B$  is a dual  $C^*$ -algebra. This completes the proof.

Let  $(A_\lambda)_{\lambda \in \Lambda}$  be a family of  $C^*$ -algebras. Let  $\sum_\lambda A_\lambda$  be the class of all functions  $(x_\lambda)$  defined on  $\Lambda$  such that  $x_\lambda \in A_\lambda$  for each  $\lambda$  and  $\sup_\lambda \|x_\lambda\| < \infty$ . Then  $\sum_\lambda A_\lambda$  is closed under the following algebraic operations:

$$(x_\lambda) + (y_\lambda) = (x_\lambda + y_\lambda);$$

$$\alpha(x_\lambda) = (\alpha x_\lambda);$$

$$(x_\lambda)(y_\lambda) = (x_\lambda y_\lambda);$$

$$(x_\lambda)^* = (x_\lambda^*),$$

for all  $(x_\lambda), (y_\lambda) \in \sum_\lambda A_\lambda$  and for all  $\alpha \in \mathbb{C}$ . It is easy

to see that with these operations,  $\sum_{\lambda} A_{\lambda}$  is a C\*-algebra under the norm given by

$$\| (x_{\lambda}) \| = \sup_{\lambda} \| x_{\lambda} \| .$$

We call  $\sum_{\lambda} A_{\lambda}$  , the product C\*-algebra of  $A_{\lambda}$  or C\*-sum of  $A_{\lambda}$  .

If we take all functions  $(x_{\lambda})$  such that for any  $\varepsilon > 0$ , the set

$\{ \lambda : \| x_{\lambda} \| \geq \varepsilon \}$  is finite, then this collection is a sub-C\*-algebra of  $\sum_{\lambda} A_{\lambda}$  which is called the C\*-( $\infty$ ) sum of  $A_{\lambda}$  and is denoted by  $(\sum_{\lambda} A_{\lambda})_0$ . The set of all functions  $(x_{\lambda})$  which are zero except at a finite number of the indices  $\lambda$  , is dense in  $(\sum_{\lambda} A_{\lambda})_0$ .

Lemma (3.2.3). Let  $A = (\sum_{\lambda} LC(H_{\lambda}))_0$  where  $H_{\lambda}$  are Hilbert spaces and let  $H = \bigoplus_{\lambda \in \Lambda} H_{\lambda}$  be the Hilbert sum of  $H_{\lambda}$  . Then every element  $(T_{\lambda})$  in  $A$  can be identified uniquely with an operator  $T \in L(H)$  such that  $T\xi = (T_{\lambda} \xi_{\lambda})$ , (i.e.,  $T|_{H_{\lambda}} = T_{\lambda}$  ) where  $\xi = (\xi_{\lambda}) \in H$  and  $T_{\lambda} \in LC(H_{\lambda})$ , and the collection  $A'$  of all such operators  $T$  in  $L(H)$  forms a sub-C\*-algebra of the algebra  $LC(H)$ . The mapping  $(T_{\lambda}) \longrightarrow T$  is an isometric \*-isomorphism of  $A$  into  $LC(H)$ .

Proof: Since each  $T_{\lambda}$  in  $(T_{\lambda}) \in A$  is bounded, the operator  $T$  defined on  $H$  by

$$T\xi = (T_{\lambda} \xi_{\lambda}) ,$$

for every  $\xi = (\xi_{\lambda}) \in H$ , is a bounded linear operator and

$\| T \| = \sup_{\lambda} \| T_{\lambda} \|$  . It is clear that  $(T_{\lambda}) \longrightarrow T$  is an isometric isomorphism and to show that it is a \*-isomorphism, we need to show that  $(T_{\lambda}^*) \longrightarrow T^*$ , since  $(T_{\lambda}^*)$  is the adjoint of  $(T_{\lambda})$

in  $A$ . That is, if  $T_\lambda = T|_{H_\lambda}$ , we need to show that  $T_\lambda^* = T^*|_{H_\lambda}$ . Suppose that  $T^*|_{H_\lambda} = T_\lambda'$ . Then for any  $\xi = (\xi_\lambda)$  and  $\eta = (\eta_\lambda) \in H$ , we have

$$(T\xi | \eta) = \sum_\lambda (T_\lambda \xi_\lambda | \eta_\lambda) = \sum_\lambda (\xi_\lambda | T_\lambda^* \eta_\lambda)$$

and

$$(\xi | T^*\eta) = \sum_\lambda (\xi_\lambda | T_\lambda' \eta_\lambda).$$

Hence

$$\sum_\lambda (\xi_\lambda | T_\lambda^* \eta_\lambda) = \sum_\lambda (\xi_\lambda | T_\lambda' \eta_\lambda)$$

or

$$\sum_\lambda (\xi_\lambda | (T_\lambda^* - T_\lambda') \eta_\lambda) = 0$$

for all  $(\xi_\lambda)$  and  $(\eta_\lambda)$  in  $H$ . Hence  $T_\lambda^* - T_\lambda' = 0$  for all  $\lambda \in \Lambda$ , and so  $(T_\lambda) \rightarrow T$  is a \*-isomorphism. In particular,  $A'$  is a sub-C\*-algebra of  $L(H)$ . (cf. 1.3.7. and 1.8.3. in [3.] )

We observe next that the set  $B$  of all elements  $(T_\lambda)$  which are zero except at a finite number of the indices  $\lambda$ , is dense in  $A$ . Let  $B'$  be the corresponding collection of  $B$  in  $A'$ . Then  $B'$  is dense in  $A'$ . Hence to show that  $A' \subset LC(H)$ , we need to prove every element  $T$  in  $B'$  is compact. Let

$$\{ \xi^{(n)} \} = \{ (\xi_\lambda^{(n)}) \}$$

be a bounded sequence of elements in  $H$  and let  $T \in B'$ . We shall prove that  $\{ T\xi^{(n)} \}$  contains a convergent subsequence. We have

$T\xi^{(n)} = (T_\lambda \xi_\lambda^{(n)})$  with  $T_\lambda \in LC(H_\lambda)$  and  $T_\lambda \xi_\lambda^{(n)} = 0$  except at a finite number of the indices  $\lambda$ . Let  $\Gamma$  be the subset of  $\Lambda$

for which  $T_\lambda \neq 0$  and let the elements of  $\Gamma$  be  $\lambda_1, \lambda_2, \dots,$

$\lambda_m$ ; i.e.,  $\Gamma = \{ \lambda_1, \lambda_2, \dots, \lambda_m \}$ . We arrange the elements of  $\{ T\xi^{(n)} \}$  as follows:

$$\mathbb{T}\xi^{(1)} = (\mathbb{T}_{\lambda_1} \xi_{\lambda_1}^{(1)}, \mathbb{T}_{\lambda_2} \xi_{\lambda_2}^{(1)}, \dots, \mathbb{T}_{\lambda_m} \xi_{\lambda_m}^{(1)}, 0, 0, \dots)$$

$$\mathbb{T}\xi^{(2)} = (\mathbb{T}_{\lambda_1} \xi_{\lambda_1}^{(2)}, \mathbb{T}_{\lambda_2} \xi_{\lambda_2}^{(2)}, \dots, \mathbb{T}_{\lambda_m} \xi_{\lambda_m}^{(2)}, 0, 0, \dots)$$

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$$\mathbb{T}\xi^{(n)} = (\mathbb{T}_{\lambda_1} \xi_{\lambda_1}^{(n)}, \mathbb{T}_{\lambda_2} \xi_{\lambda_2}^{(n)}, \dots, \mathbb{T}_{\lambda_m} \xi_{\lambda_m}^{(n)}, 0, 0, \dots)$$

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where some of the  $\mathbb{T}_{\lambda_i} \xi_{\lambda_i}^{(n)}$  may be zero ( $i = 1, \dots, m; n = 1, 2, \dots$ ). We then have :

$$(1) \quad \mathbb{T}_{\lambda_1} \xi_{\lambda_1}^{(1)}, \mathbb{T}_{\lambda_1} \xi_{\lambda_1}^{(2)}, \dots, \mathbb{T}_{\lambda_1} \xi_{\lambda_1}^{(n)}, \dots \in H_{\lambda_1};$$

$$(2) \quad \mathbb{T}_{\lambda_2} \xi_{\lambda_2}^{(1)}, \mathbb{T}_{\lambda_2} \xi_{\lambda_2}^{(2)}, \dots, \mathbb{T}_{\lambda_2} \xi_{\lambda_2}^{(n)}, \dots \in H_{\lambda_2};$$

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$$(m) \quad \mathbb{T}_{\lambda_m} \xi_{\lambda_m}^{(1)}, \mathbb{T}_{\lambda_m} \xi_{\lambda_m}^{(2)}, \dots, \mathbb{T}_{\lambda_m} \xi_{\lambda_m}^{(n)}, \dots \in H_{\lambda_m}.$$

Since  $\mathbb{T}_{\lambda_1} \in LC(H_{\lambda_1})$  and  $\{ \xi_{\lambda_1}^{(n)} \}$  is a bounded sequence of elements in  $H_{\lambda_1}$ ,  $\{ \mathbb{T}_{\lambda_1} \xi_{\lambda_1}^{(n)} \}$  contains a convergent subsequence, say

$$\left\{ \mathbb{T}_{\lambda_1} \xi_{\lambda_1}^{(n_k^{(1)})} \right\},$$

where  $\{ n_k^{(1)} \}$  is a subset of  $\{ 1, 2, \dots \}$ . Let the limit of this sequence be  $\gamma_{\lambda_1}$ . Now choose the corresponding elements from (2), i.e., consider the sequence

$$\left\{ \mathbb{T}_{\lambda_2} \xi_{\lambda_2}^{(n_k^{(1)})} \right\}.$$

Since  $T_{\lambda_2} \in LC(H_{\lambda_2})$  and  $\left\{ \bar{\xi}_{\lambda_2}^{(n_k^{(1)})} \right\}$  is a bounded sequence of elements in  $H_{\lambda_2}$ ,  $\left\{ T_{\lambda_2} \bar{\xi}_{\lambda_2}^{(n_k^{(1)})} \right\}$  contains a convergent subsequence, say

$$\left\{ T_{\lambda_2} \bar{\xi}_{\lambda_2}^{(n_k^{(2)})} \right\},$$

where  $\left\{ n_k^{(2)} \right\}$  is a subset of  $\left\{ n_k^{(1)} \right\}$ . Let the limit of this sequence be  $\eta_{\lambda_2}$ . By repeating the same process, we arrive at the  $m^{\text{th}}$  convergent subsequence of the sequence given by (m), say

$$\left\{ T_{\lambda_m} \bar{\xi}_{\lambda_m}^{(n_k^{(m)})} \right\},$$

where  $\left\{ n_k^{(m)} \right\}$  is a subset of  $\left\{ n_k^{(m-1)} \right\}$ . Let the limit of this sequence be  $\eta_{\lambda_m}$ . Then the sequence

$$\left\{ T \bar{\xi}^{(n_k^{(m)})} \right\} = \left\{ (T_{\lambda_1} \bar{\xi}_{\lambda_1}^{(n_k^{(m)})}, \dots, T_{\lambda_m} \bar{\xi}_{\lambda_m}^{(n_k^{(m)})}, 0, 0, \dots) \right\}$$

is a convergent subsequence of  $\left\{ T \bar{\xi}^{(n)} \right\}$  with the limit given by

$$\eta = (\eta_{\lambda_1}, \eta_{\lambda_2}, \dots, \eta_{\lambda_m}, 0, 0, \dots),$$

since

$$\left\| T \bar{\xi}^{(n_k^{(m)})} - \eta \right\| = \sum_{\lambda \in \Gamma} \left\| T_{\lambda} \bar{\xi}_{\lambda}^{(n_k^{(m)})} - \eta_{\lambda} \right\|.$$

Hence  $T$  is a compact operator and this completes the proof.

Theorem (3.2.4). Let  $A$  be a  $C^*$ -algebra. Then  $A$  is dual if and only if  $A$  is  $*$ -isomorphic to a sub- $C^*$ -algebra of the algebra  $LC(H)$  of compact operators on a Hilbert space  $H$ .

Proof: Suppose  $A$  is dual. Then, by Theorem 8.4. in [11],  $A$  is of the form  $\left( \sum_{\lambda \in \Lambda} A_{\lambda} \right)_0$  where each  $A_{\lambda}$  is  $*$ -isomorphic to the

algebra  $LC(H_\lambda)$  on some Hilbert space  $H_\lambda$ . Hence it follows from Lemma (3.2.3) that  $A$  is  $*$ -isomorphic to a sub- $C^*$ -algebra of the algebra  $LC(H)$  on a Hilbert space  $H$ . The converse follows from Theorem (3.2.2).

It follows immediately from Theorem (3.2.4) that :

Corollary (3.2.5). Every sub- $C^*$ -algebra of a dual  $C^*$ -algebra is dual.

Theorem (3.2.6). If  $A$  is a dual  $C^*$ -algebra, then the Jacobson structure space  $\text{Prim}(A)$  of  $A$  is discrete.

Proof: Let  $\{I_\lambda : \lambda \in \Lambda\}$  be a family of minimal-closed two-sided ideals of  $A$ . Then, by Theorem (4.10.14) in [15],  $A = (\sum_{\lambda} I_\lambda)_0$ . Since each  $I_\lambda$  is simple, it follows from Theorem (4.9.24) in [15] that the Jacobson structure space  $\text{Prim}(A)$  is discrete.

Corollary (3.2.7). The dual  $\hat{A}$  of a dual  $C^*$ -algebra  $A$  is discrete.

Theorem (3.2.8). Let  $A$  be a separable  $C^*$ -algebra. If its Jacobson structure space  $\text{Prim}(A)$  is discrete, then  $A$  is dual.

Proof: Suppose  $\text{Prim}(A)$  is discrete. Then, by Theorem(4.9.24) in [15],  $A$  is of the form  $(\sum_{\lambda \in \Lambda} A_\lambda)_0$  where each  $A_\lambda$  is a simple  $C^*$ -algebra. Since  $A$  is separable and since each  $A_\lambda$  can be identified with a closed two-sided ideal of  $A$ ,  $A_\lambda$  is separable for each  $\lambda$ . Hence, by Theorem (3.1.7),  $A_\lambda$  is  $*$ -isomorphic to  $LC(H)$  for some Hilbert space  $H$  for each  $\lambda$ , i.e., each  $A_\lambda$  is dual. Since  $A = (\sum_{\lambda \in \Lambda} A_\lambda)_0$ , by Theorem (4.10.25) in [15], it follows that  $A$  is dual.

§ 3. CCR-algebra and its dual.

A  $C^*$ -algebra  $A$  is called a CCR-algebra if for every irreducible representation  $\pi$  of  $A$  and for every  $x$  in  $A$ ,  $\pi(x)$  is compact; equivalently, a  $C^*$ -algebra  $A$  is called a CCR-algebra if for every primitive ideal  $P$  of  $A$ ,  $A/P$  is  $*$ -isomorphic to the algebra of all compact operators on a Hilbert space. Every commutative  $C^*$ -algebra is a CCR-algebra since the irreducible representations are of dimension one. By Theorem (3.2.4), it follows that every dual  $C^*$ -algebra is a CCR-algebra; in particular, the algebra  $LC(H)$  of all compact operators on a Hilbert space  $H$  is a CCR-algebra which is primitive by Theorem (3.1.5)(i). Later on, we shall give an example of a CCR-algebra which is not dual.

Theorem (3.3.1). Let  $A$  be a CCR-algebra and  $\pi$  a non-zero irreducible representation of  $A$  in  $H_\pi$ .

(i)  $\pi(A) = LC(H_\pi)$  and every non-zero irreducible representation of  $A$  having the same kernel as  $\pi$  is equivalent to  $\pi$ .

(ii) The kernel of  $\pi$  is a maximal closed two-sided ideal of  $A$ .

Proof: (i) Since  $A$  is a CCR-algebra,  $\pi(A) \subset LC(H_\pi)$  and since  $\pi(A) \neq 0$ , (i) follows from 4.1.10. in [3].

(ii) Since  $\pi(A) = LC(H_\pi)$  and since  $LC(H_\pi)$  is simple, it follows that the kernel of  $\pi$  is a maximal closed two-sided ideal of  $A$ .

Corollary (3.3.2). Every primitive ideal of a CCR-algebra  $A$  is a maximal closed two-sided ideal of  $A$ .

Theorem (3.3.3). If  $A$  is a primitive CCR-algebra, then  $A$  is  $*$ -isomorphic to  $LC(H)$  for some Hilbert space  $H$ , i.e., a primitive CCR-algebra is dual.

Proof: Since  $A$  is primitive, there is a non-zero faithful irreducible representation  $\pi$  of  $A$  in  $H_\pi$ . By Theorem (3.3.1)(i), it follows that  $\pi$  is a one-to-one mapping of  $A$  onto  $LC(H_\pi)$ . Hence  $A$  is  $*$ -isomorphic to  $LC(H_\pi)$  and this completes the proof.

Theorem (3.3.4). Let  $A$  be a CCR-algebra.

(i) The Jacobson structure space  $\text{Prim}(A)$  is a  $T_1$ -space.

(ii) The dual  $\hat{A}$  is a  $T_1$ -space.

Proof: Since  $A$  is a CCR-algebra, by Corollary (3.3.2), every primitive ideal of  $A$  is maximal. Hence, by Theorem (2.1.7), it follows that every subset  $\{P\}$  of  $\text{Prim}(A)$  consisting only one point is closed so that  $\text{Prim}(A)$  is a  $T_1$ -space. This proves (i). Now, by Theorem (3.3.1)(i), any two non-zero irreducible representations of  $A$  having the same kernel are equivalent. Hence, by Theorem (2.1.9), the canonical mapping  $\varphi : \hat{A} \longrightarrow \text{Prim}(A)$  is a homeomorphism. Since  $\text{Prim}(A)$  is a  $T_1$ -space,  $\hat{A}$  is a  $T_1$ -space. This completes the proof.

It has been proved by Kaplansky [12] that the Jacobson structure space  $\text{Prim}(A)$  of a CCR-algebra is not only a  $T_1$ -space, but it is also of second category.

Let  $A$  be a  $C^*$ -algebra. An ascending sequence  $(I_\mu)_{0 \leq \mu \leq \lambda}$  of closed two-sided ideals of  $A$ , indexed by the ordinals between 0

and a certain ordinal  $\lambda$ , is called a composition series of  $A$  if

(i)  $I_0 = (0)$  and  $I_\lambda = A$ ; and (ii) if  $\mu \leq \lambda$  is a limiting ordinal, then  $I_\mu = \bigcup_{\mu' < \mu} I_{\mu'}$ .

In [12], Kaplansky shows that every CCR-algebra  $A$  has a composition series  $(I_\mu)_{0 \leq \mu \leq \lambda}$  such that each  $I_{\mu+1}/I_\mu$  has a Hausdorff Jacobson structure space.

We end this section with an example of CCR-algebra which is not dual.

Let  $Q$  be a non-discrete compact Hausdorff space and  $C(Q)$  the algebra of all continuous complex-valued functions on  $Q$ . Then the mapping  $f \longrightarrow \bar{f}$  is an involution on  $C(Q)$ . Under this involution and the norm  $\|f\|$  defined by  $\|f\| = \sup_{x \in Q} |f(x)|$ ,  $C(Q)$  becomes a commutative  $C^*$ -algebra. Hence  $C(Q)$  is a CCR-algebra. In particular, every primitive ideal of  $C(Q)$  is maximal. Since  $C(Q)$  has an identity, by Theorem 3.1.7. in [3], every maximal closed two-sided ideal of  $C(Q)$  is primitive. Therefore  $\text{Prim}(C(Q))$  contains all the maximal closed two-sided ideals of  $C(Q)$ . Then, by Theorem 19C. in [13], there is a one-to-one correspondence between the elements of  $\text{Prim}(C(Q))$  and the points of  $Q$ . Since  $Q$  is a compact Hausdorff space, it follows that for every closed subset  $F$  of  $Q$  and for every point  $p$  not in  $F$ , there exists  $f \in C(Q)$  such that  $f(F) = 0$  and  $f(p) \neq 0$ . Hence, by Theorem 19F. in [13],  $Q$  and  $\text{Prim}(C(Q))$  are homeomorphic. If  $C(Q)$  is dual, then, by Theorem (3.2.6), the Jacobson structure space  $\text{Prim}(C(Q))$  is discrete and so  $Q$  is discrete. But  $Q$  is not discrete, hence  $C(Q)$  is not a dual  $C^*$ -algebra.

§ 4. GCR-algebra and its dual.

A C\*-algebra  $A$  is called a GCR-algebra if every non-zero quotient C\*-algebra of  $A$  (by a closed two-sided ideal of  $A$ ) has a non-zero CCR-closed two-sided ideal; equivalently, a C\*-algebra  $A$  is called a GCR-algebra if  $A$  has a composition series  $(I_\mu)_{0 \leq \mu \leq \lambda}$  such that every  $I_{\mu+1}/I_\mu$  is a CCR-algebra. Every sub-C\*-algebra and every quotient C\*-algebra of a GCR-algebra is a GCR-algebra. (cf. 4.3.5. in [3] )

Theorem (3.4.1): Every CCR-algebra  $A$  is a GCR-algebra.

Proof: [3] Let  $I$  be any closed two-sided ideal of  $A$ ,  $I \neq A$ . It suffices to show that the quotient C\*-algebra  $A/I$  is a CCR-algebra. To do this, let  $\pi'$  be any non-zero irreducible representation of  $A/I$  and let  $x' = x + I$  be any element of  $A/I$ . Then, by Theorem (2.1.10)(ii), there is a non-zero irreducible representation  $\pi$  of  $A$  such that  $\pi'(x') = \pi(x)$ . Since  $A$  is a CCR-algebra,  $\pi(x)$  is compact and so  $\pi'(x')$  is compact for every  $x'$  in  $A/I$ . Hence  $A/I$  is a CCR-algebra; and since this is true for every closed two-sided ideal  $I$  of  $A$ ,  $I \neq A$ , it follows that  $A$  is a GCR-algebra. This completes the proof.

The converse of this theorem is not true. We give the following counter example:

Let  $H$  be an infinite dimensional Hilbert space and let  $A$  be a commutative von Neumann algebra in  $H$  such that  $LC(H) \cap A = (0)$ . Then  $B = LC(H) + A$  is a GCR-algebra. In fact, let  $I_0 = (0)$ ,

$I_1 = LC(H)$  and  $I_2 = B$ . Then  $(I_\mu)_0 \leq \mu \leq 2$  is a composition series of  $B$  such that  $I_1/I_0$  and  $I_2/I_1$  are CCR-algebras, since

$$I_1/I_0 = LC(H)/(0) \cong LC(H)$$

and

$$I_2/I_1 = B/LC(H) = (LC(H) + A)/LC(H) \cong A$$

Hence  $B$  is a GCR-algebra. But  $B$  is not a CCR-algebra. In fact, since the identity representation of  $LC(H)$  is irreducible, it follows that the identity representation  $\pi_1$  of  $B$  is irreducible. Thus, if  $x \in A$ , then  $\pi_1(x) = x \notin LC(H)$  and so  $B$  is not a CCR-algebra.

A  $C^*$ -algebra  $A$  is called an NGCR-algebra if it does not have any non-zero GCR-closed two-sided ideals; equivalently, a  $C^*$ -algebra  $A$  is called an NGCR-algebra if it does not have any non-zero CCR-closed two-sided ideals.

Let  $H$  be a separable Hilbert space of infinite dimension and let  $A = L(H)/LC(H)$ . We shall show that  $A$  is an NGCR-algebra. We observe first that  $L(H)$  contains an uncountable number of orthogonal projections each of which is not a compact operator. Thus  $A$  contains an uncountable number of projections (i.e., elements  $x$  such that  $x^* = x$  and  $x^2 = x$ ). Using this fact, we show that  $A$  is not separable. We know that  $A$  is a simple  $C^*$ -algebra (with identity). Hence, if  $A$  is separable, then, by Theorem (3.1.7),  $A$  is isometrically  $*$ -isomorphic to  $LC(H')$  for some Hilbert space  $H'$ . Since  $A$  contains an identity, it follows that  $H'$  is finite dimensional (because the identity operator is compact). But this means that  $A$  contains at

most a finite numbers of projections, a contradiction. Hence  $A$  is not separable. Now let  $\pi$  be a non-zero irreducible representation of  $A$ . Since  $A$  is simple,  $\pi$  is faithful. We show that

$\pi(A) \cap LC(H_\pi) = (0)$ . In fact, if  $\pi(A) \cap LC(H_\pi) \neq (0)$ , then, by 4.1.10. in [3],  $\pi(A) \supset LC(H_\pi)$ , and, by the simplicity of  $A$  and hence of  $\pi(A)$ ,  $\pi(A) = LC(H_\pi)$ . Since  $\pi(A)$  contains the identity operator,  $H_\pi$  is finite dimensional. Hence  $\pi(A)$  is separable, hence  $A$  is separable, a contradiction. Thus  $A$  is an NGCR-algebra. In particular,  $L(H)$  is neither a GCR-algebra nor an NGCR-algebra.

It was shown by J. Glimm [9] that if  $A$  is an NGCR-algebra, then there exist families  $\{\pi_\lambda\}_{\lambda \in \Lambda}$  and  $\{\rho_\lambda\}_{\lambda \in \Lambda}$  of irreducible representations of  $A$  such that the direct sums  $\bigoplus_\lambda \pi_\lambda$  and  $\bigoplus_\lambda \rho_\lambda$  are faithful and  $\text{Ker } \pi_\lambda = \text{Ker } \rho_\lambda$  but  $\pi_\lambda$  is not equivalent to  $\rho_\lambda$  for each  $\lambda \in \Lambda$ . Thus for an NGCR-algebra  $A$ , the canonical mapping  $\varphi : \hat{A} \longrightarrow \text{Prim}(A)$  is not one-to-one.

Theorem (3.4.2). Let  $A$  be a GCR-algebra. Then

(i) for every non-zero irreducible representation  $\pi$  of  $A$ ,  
 $\pi(A) \supset LC(H_\pi)$ ;

(ii) any two non-zero irreducible representations of  $A$  having the same kernel are equivalent.

Proof: [3] Let  $\pi$  be a non-zero irreducible representation of  $A$ . Since  $A$  is a GCR-algebra,  $A/\text{Ker } \pi$  contains a non-zero CCR-closed two-sided ideal  $J$ . Let  $I = \{x \in A : x' = x + \text{Ker } \pi, \pi \in J\}$ . Then  $I$  is a closed two-sided ideal of  $A$ . In fact, if  $x \in I$  and

$y \in A$ , then, since  $(xy)' = x'y'$  and  $(yx)' = y'x'$  are in  $J$ ,  $xy$  and  $yx$  are in  $I$  so that  $I$  is a two-sided ideal. Next suppose that  $x_n \in I$  and let  $x_n \rightarrow x$ . Then  $x_n' \in J$  and  $x_n' \rightarrow x'$ . Since  $J$  is closed,  $x' \in J$  and therefore  $x \in I$ . Hence  $I$  is closed. We have  $J = I/\text{Ker } \pi$ . Since  $I \neq \text{Ker } \pi$ ,  $\pi|_I \neq 0$  and, by Theorem (2.1.11)(ii),  $\pi|_I$  is a non-zero irreducible representation of  $I$ . Moreover, the Hilbert space of the representation  $\pi|_I$  is  $H_\pi$  (cf. proof of Lemma (2.1.1)). Then, by Theorem (2.1.12)(ii), the representation  $\pi'$  of  $J = I/\text{Ker } \pi$  defined by  $\pi'(x') = \pi(x)$  for every  $x' = x + \text{Ker } \pi$  with  $x \in I$ , is irreducible and has  $H_\pi$  as its representation space. Since  $J$  is a CCR-algebra,  $\pi'(J) = \text{LC}(H_\pi)$  so that  $\pi(I) = \text{LC}(H_\pi)$ . But  $\pi(A) \supseteq \pi(I)$ , hence  $\pi(A) \supseteq \text{LC}(H_\pi)$  and this proves (i). The second part of the theorem follows from (i) and 4.1.10. in [3].

Theorem (3.4.3). Let  $A$  be a GCR-algebra. Then the dual  $\hat{A}$  of  $A$  is a  $T_0$ -space.

Proof: By Theorem (3.4.2)(ii), it follows that any two non-zero irreducible representations of  $A$  having the same kernel are equivalent. Hence by Theorem (2.1.9), the dual  $\hat{A}$  of  $A$  is a  $T_0$ -space.

Theorem (3.4.4). Let  $A$  be a GCR-algebra. Then  $A$  is a CCR-algebra if and only if the dual  $\hat{A}$  is a  $T_1$ -space.

Proof: If  $A$  is a CCR-algebra, then, by Theorem (3.3.3)(ii), the dual  $\hat{A}$  is a  $T_1$ -space. Conversely, suppose that  $\hat{A}$  is a  $T_1$ -space. Then, by Theorem (2.1.9),  $\text{Prim}(A)$  is a  $T_1$ -space and so each primitive ideal of  $A$  is maximal. Therefore  $\pi(A)$  is a simple  $C^*$ -algebra

for every  $\pi \in A$ . But, by Theorem (3.4.2)(i),  $\pi(A) \supseteq \text{LC}(H_\pi)$ . Hence  $\pi(A) = \text{LC}(H_\pi)$  for every  $\pi \in \hat{A}$ . Thus  $A$  is a CCR-algebra.

Theorem (3.4.5). Let  $A$  be a separable  $C^*$ -algebra. Then the following statements are equivalent:

- (i)  $A$  is a GCR-algebra.
- (ii)  $\hat{A}$  is a  $T_0$ -space.
- (iii) Any two non-zero irreducible representations of  $A$  having the same kernel are equivalent.

Proof: (i)  $\iff$  (iii) : This follows from Theorem 5 in [6].

(ii)  $\iff$  (iii) : This is given by Theorem (2.1.9).

Theorem (3.4.6). Let  $A$  be a separable  $C^*$ -algebra. Then  $A$  is a CCR-algebra if and only if the dual  $\hat{A}$  is a  $T_1$ -space.

Proof: If  $A$  is a CCR-algebra, then, by Theorem (3.3.3)(ii), the dual  $\hat{A}$  is a  $T_1$ -space. Conversely, suppose that the dual  $\hat{A}$  is a  $T_1$ -space. Then, by Theorem (3.4.5),  $A$  is a GCR-algebra and so, by Theorem (3.4.4),  $A$  is a CCR-algebra.

Theorem (3.4.7). Let  $A$  be a non-zero separable  $C^*$ -algebra and  $(I_\mu)_{0 \leq \mu \leq \lambda}$  be a composition series of  $A$ . If  $I_{\mu+1}/I_\mu$  is not zero for all  $\mu$ , then the family  $(I_\mu)_{0 \leq \mu \leq \lambda}$  is countable. Moreover, if  $A$  is a GCR-algebra, then there exists a composition series  $(I_n)_{0 \leq n \leq \infty}$  such that the dual  $(I_{n+1}/I_n)^\wedge$  of the quotient  $I_{n+1}/I_n$  is Hausdorff for every  $n$ .

Proof: [3] Let  $(I_\mu)_{0 \leq \mu \leq \lambda}$  be a composition series of  $A$  such that  $I_{\mu+1}/I_\mu \neq (0)$  for all  $\mu < \lambda$ . For each ordinal  $\mu < \lambda$ ,

let  $x_{\mu+1} \in I_{\mu+1}$  be such that

$$\|x'_{\mu+1}\| = \inf_{y \in I_{\mu}} \|x_{\mu+1} + y\| = 1,$$

where  $x'_{\mu+1} = x_{\mu+1} + I_{\mu}$ ; i.e., such that the distance  $d(x_{\mu+1}, I_{\mu})$  from  $I_{\mu+1}$  to  $I_{\mu}$  is 1. Since  $\|x'_{\mu+1}\| \leq \|x_{\mu+1}\|$ , it follows that  $\|x_{\mu+1}\| \geq 1$ . Furthermore, for each  $\mu$ ,  $\|x_{\mu+1} - x_{\mu}\| \geq 1$ . To show that the family  $\{x_{\mu}\}$  is countable, let

$$S_{\mu} = \{x \in A : \|x - x_{\mu}\| < \frac{1}{2}\},$$

for each  $\mu < \lambda$ . Then  $S_{\mu} \cap S_{\mu+1} = \emptyset$ . Now, since  $A$  is separable, there is a countable dense subset, say  $\{y_n\}_{n=1}^{\infty}$ , in  $A$ . Then every  $S_{\mu}$  contains at least one  $y_n$  and every  $y_n$  belongs to exactly one  $S_{\mu}$ . Therefore  $\{S_{\mu}\}$  is countable. Hence  $\{x_{\mu}\}$  is countable and consequently the composition series  $(I_{\mu})_{0 \leq \mu \leq \lambda}$  is countable.

Now suppose that  $A$  is a GCR-algebra. Then there is a closed two-sided ideal  $I_1$  of  $A$  such that  $\hat{I}_1$  is a Hausdorff space. (cf. 4.4.4. and 4.5.3. in [3]) If  $I_1 = A$ , then the theorem is proved. If  $I_1 \neq A$ , then  $A/I_1$  is a non-zero GCR-algebra so that there is a non-zero closed two-sided ideal  $I_2'$  of  $A/I_1$  such that  $\hat{I}_2'$  is a Hausdorff space. But then there exists a closed two-sided ideal  $I_2$  of  $A$  such that  $I_2' = I_2/I_1$ . Continuing by induction, we obtain a composition series  $(I_n)_{0 \leq n \leq \infty}$  with the property that each  $(I_{n+1}/I_n)^{\wedge}$  of  $I_{n+1}/I_n$  is Hausdorff. This completes the proof.

Theorem (3.4.8). Let  $A$  be a non-zero separable GCR-algebra. Then the dual  $\hat{A}$  is the union of a countable family of mutually disjoint subsets  $B_n$  of  $A$  ( $n = 1, 2, \dots$ ) such that each  $B_n$  with the

relative topology induced by the topology of  $\hat{A}$  is a locally compact Hausdroff space satisfying the second axiom of countability.

Proof: By Theorem (3.4.7),  $A$  has a countable composition series  $(I_n)_{0 \leq n \leq \infty}$  such that  $I_{n+1}/I_n \neq (0)$  and  $(I_{n+1}/I_n)^\wedge$  is Hausdroff for all  $n$ . Moreover, by Theorem (2.1.14), we have

$$\hat{I}_0 \subset \hat{I}_1 \subset \hat{I}_2 \subset \dots$$

and therefore

$$\hat{A} = \bigcup_{n=0}^{\infty} \hat{I}_n.$$

In fact, it is clear that  $\hat{A} \supset \bigcup_{n=0}^{\infty} \hat{I}_n$ . We show that

$$\hat{A} \subset \bigcup_{n=0}^{\infty} \hat{I}_n. \text{ Suppose there is a } \pi \in \hat{A} \text{ such that } \pi \notin \bigcup_{n=0}^{\infty} \hat{I}_n.$$

Then  $\pi \notin \hat{I}_n$  for all  $n$ , i.e.,  $\pi \mid I_n = 0$  for all  $n$ . Since  $\bigcup_{n=0}^{\infty} I_n$  is dense in  $A$ , we have  $\pi(A) = 0$ , a contradiction. Hence

$$\hat{A} \subset \bigcup_{n=0}^{\infty} \hat{I}_n. \text{ Since } (I_n) \text{ is an increasing sequence of sets, we}$$

have

$$\bigcup_{n=0}^{\infty} \hat{I}_n = \bigcup_{n=0}^{\infty} (\hat{I}_{n+1} - \hat{I}_n),$$

so that  $\hat{A}$  is the union of a countable family of mutually disjoint subsets  $B_n = \hat{I}_{n+1} - \hat{I}_n$  ( $n = 1, 2, \dots$ ). That  $B_n$  is the intersection of a closed set and an open set of  $\hat{A}$  follows from the

fact that

$$B_n = \hat{I}_{n+1} - \hat{I}_n = (\hat{A} - \hat{I}_n) \cap (\hat{I}_{n+1}).$$

Now again, by Theorem (3.4.7),  $B_n$  is Hausdroff because

$$\hat{I}_{n+1} - \hat{I}_n = (I_{n+1}/I_n)^\wedge,$$

and, by 3.3.8. in [3],  $B_n$  is locally compact. Finally, since  $I_{n+1}/I_n$  is a separable  $C^*$ -algebra, by Theorem (2.4.2),  $B_n$  is second countable. It remains to show that the relative topology on  $B_n$  induced by the topology of  $\hat{A}$  coincides with the given topology of the dual of  $I_{n+1}/I_n$ . We know that  $B_n$  is a closed subset of  $\hat{I}_{n+1}$ . Since by Theorem (2.1.10)(ii),  $B_n$  is homeomorphic to  $(\hat{I}_{n+1})_{I_n}$ , it follows that the relative topology on  $B_n$  induced by the topology on  $\hat{I}_{n+1}$  is the same as the given topology on  $B_n$ . Thus every open set  $U$  in  $B_n$  is of the form  $U' \cap B_n$  where  $U'$  is open in  $\hat{I}_{n+1}$ . But, by Theorem (2.1.14), to each open set  $U'$  in  $\hat{I}_{n+1}$  there corresponds uniquely a closed two-sided ideal  $I'$  in the  $C^*$ -algebra  $I_{n+1}$ . Since  $I_{n+1}$  is a closed two-sided ideal of  $A$ , by 1.8.5. in [3],  $I'$  is also a closed two-sided ideal of  $A$ . Therefore  $U'$  is an open set in  $\hat{A}$ . Hence the relative topology on  $B_n$  induced by the topology of  $\hat{A}$  is finer than the given topology on  $B_n$ . But if  $U'$  is an open set in  $\hat{A}$ , then  $U' \cap \hat{I}_{n+1}$  is also open in  $\hat{A}$ , since  $\hat{I}_{n+1}$  is open in  $\hat{A}$ . Thus  $U' \cap B_n = U' \cap \hat{I}_{n+1} \cap B_n$  is an open set in  $B_n$  with the given topology. Hence the given topology on  $B_n$  is finer than the relative topology induced by the topology of  $\hat{A}$ . Therefore the two topologies coincide. This completes the proof.

Corollary (3.4.9). Let  $A$  be a non-zero separable GCR-algebra. Then there exists a countable compositive series  $(I_n)_{0 \leq n \leq \infty}$  such that for all  $n$ ,  $I_{n+1}/I_n$  is not zero and  $(I_{n+1}/I_n)^\wedge = \hat{I}_{n+1} - \hat{I}_n$  is a locally compact Hausdorff space satisfying the second axiom of countability. Each  $\hat{I}_{n+1} - \hat{I}_n$  is the intersection of an open set

and a closed set of  $\hat{A}$  and the topology on  $\hat{I}_{n+1} - \hat{I}_n$  coincides with the relative topology on  $\hat{I}_{n+1} - \hat{I}_n$  induced by the topology of  $\hat{A}$ . Furthermore, we have

$$\hat{A} = \bigcup_{n=1}^{\infty} (\hat{I}_{n+1} - \hat{I}_n).$$

§ 5. Borel structure on the dual of a GCR-algebra.

A set  $X$ , with a family  $\beta$  of subsets of  $X$  such that

(i)  $\emptyset, X \in \beta$  ;

(ii)  $\bigcup_{n=1}^{\infty} S_n, \bigcap_{n=1}^{\infty} S_n \in \beta$  whenever  $S_n \in \beta$   
( $n=1, 2, \dots$ );

(iii) the complement of  $S$  is in  $\beta$  whenever  $S \in \beta$  ,

is called a Borel space and is denoted by  $(X, \beta)$ . We shall often suppress the  $\beta$  and simply speak of the Borel space  $X$ . The family

$\beta$  is called a Borel structure on  $X$  and we call the elements of

$\beta$  the Borel subsets of  $X$ . Clearly for each family  $\mathcal{F}$  of subsets of a set  $X$  there is a unique smallest Borel structure for  $X$  which contains  $\mathcal{F}$ . We call it the Borel structure generated by  $\mathcal{F}$  and we call  $\mathcal{F}$  a generating family for the structure. In particular, the family of open subsets of  $X$  with respect to a topology  $\mathcal{J}$  on  $X$  generate a Borel structure which we call the Borel structure generated by the topology. The corresponding Borel space is said to be associated with or defined by the given topological space.

Let  $(X, \beta_1)$  and  $(X, \beta_2)$  be two Borel spaces. If  $\beta_2 \subset \beta_1$ , (i.e., if every subset of  $X$  belonging to  $\beta_2$  also belongs to  $\beta_1$ ) then we say that the Borel structure of  $(X, \beta_1)$

is finer than the Borel structure of  $(X, \mathcal{B}_2)$ .

A topological space  $X$  is called a polonais space if the topology on  $X$  admits a countable basis and if there exists a metric on  $X$  which gives  $X$  the same topology and under which  $X$  is a complete metric space.

A Borel space  $X$  is called standard if its Borel structure is generated by a topology of a polonais space.

Let  $X$  and  $Y$  be two Borel spaces and  $f: X \longrightarrow Y$  a mapping from  $X$  into  $Y$ . If  $f^{-1}(S)$  is a Borel subset of  $X$  whenever  $S$  is a Borel subset of  $Y$ , then  $f$  is called a Borel mapping. If  $f$  is one-to-one and onto and if  $f$  and  $f^{-1}$  are Borel mappings, then  $f$  is called a Borel isomorphism of  $X$  onto  $Y$  and  $X, Y$  are called isomorphic as Borel spaces.

Let  $X$  be a Borel space and  $X_1$  a subset of  $X$ . Then  $X_1$  becomes a Borel space on defining the Borel subsets of  $X_1$  to be the sets  $X_1 \cap S$ , where  $S$  runs over the Borel subsets of  $X$ . We call  $X_1$  a Borel subspace of  $X$ . The subspace of a standard Borel space is a standard Borel space.

Let  $X$  be a Borel space and let  $r$  be an equivalence relation in  $X$ . Let  $X'$  denote the set of all equivalence classes and let  $r(x)$  ( $x \in X$ ) denote the equivalence class to which  $x$  belongs. Let  $\mathcal{B}$  be the family of all  $S \subset X'$  such that  $r^{-1}(S)$  is a Borel set in  $X$ . Then it is clear that  $\mathcal{B}$  is a Borel structure for  $X'$ , called the quotient structure of that of  $X$  under the mapping

$r : X \longrightarrow X'$ , and  $X'$  with this structure  $\beta$  is called a quotient Borel space of  $X$  (with respect to  $r$ ).

Let  $\{X_\lambda\}_{\lambda \in \Lambda}$  be a family of Borel spaces such that  $X_\lambda \cap X_\mu = \emptyset$  for all  $\lambda$  and  $\mu$  with  $\lambda \neq \mu$ . We define a Borel structure on  $X = \bigcup_{\lambda \in \Lambda} X_\lambda$  by defining a subset  $S$  of  $X$  to be a Borel subset of  $X$  if and only if  $X_\lambda \cap S$  is a Borel subset in  $X_\lambda$  for each  $\lambda \in \Lambda$ . We call  $X$  the union of the Borel spaces  $X_\lambda$ . The union of the standard Borel spaces is a standard Borel space. (cf. Theorem 3.1. in [14] )

Let  $A$  be a separable  $C^*$ -algebra. For each cardinal  $n \leq \aleph_0$ , we equip  $\text{Irr}_n(A)$  and  $\text{Rep}_n(A)$  with the Borel structures generated by their topologies respectively. By 3.7.1. and 3.7.4. in [3],  $\text{Irr}_n(A)$  and  $\text{Rep}_n(A)$  are standard Borel spaces. Let

$$\text{Irr}(A) = \bigcup_n \text{Irr}_n(A).$$

Since for  $m \neq n$ ,  $\text{Irr}_m(A) \cap \text{Irr}_n(A) = \emptyset$ , we can thus define a Borel structure on  $\text{Irr}(A)$ . We give  $\text{Irr}(A)$  the Borel structure which is the union of the Borel structures on  $\text{Irr}_n(A)$ . Since the union of the standard Borel spaces is standard, it follows that  $\text{Irr}(A)$  is a standard Borel space.

We recall that the canonical mapping

$$\psi : \text{Irr}(A) \longrightarrow \hat{A}$$

from  $\text{Irr}(A)$  into  $\hat{A}$ , which corresponds to each element in  $\text{Irr}(A)$  its equivalence class, is an onto mapping.

Let  $A$  be a separable  $C^*$ -algebra. Then the quotient structure of that of  $\text{Irr}(A)$  under the canonical mapping  $\psi : \text{Irr}(A) \longrightarrow \hat{A}$  is called the Mackey Borel structure on  $\hat{A}$ .

Theorem (3.5.1). Let  $A$  be a separable  $C^*$ -algebra. Then the Mackey Borel structure on  $\hat{A}$  is finer than the Borel structure on  $\hat{A}$  generated by the topology on  $\hat{A}$ .

Proof: [3] Let  $\beta_1$  be the Mackey Borel structure on  $\hat{A}$  and  $\beta_2$  the Borel structure on  $\hat{A}$  generated by its topology. Let  $S$  be an open subset of  $\hat{A}$  (with respect to the given topology on  $\hat{A}$ ). Since the relative topology on  $\hat{A}_n$  induced by that of  $\hat{A}$  and the given topology on  $\hat{A}_n$  coincide,  $S \cap \hat{A}_n$  is open in  $\hat{A}_n$  for each  $n$ . By Theorem (2.3.1), the inverse image of  $S \cap \hat{A}_n$  under the canonical mapping

$$\psi : \text{Irr}(A) \longrightarrow \hat{A}$$

is open in  $\text{Irr}_n(A)$ , so that  $\psi^{-1}(S)$  is an open set in  $\text{Irr}(A)$ .

Thus  $\psi^{-1}(S)$  is a Borel subset of  $\text{Irr}(A)$ , i.e.,  $S \in \beta_1$ .

Since every element in  $\beta_2$  is generated by open subsets of  $\hat{A}$ , it follows that if  $S \in \beta_2$  then  $S \in \beta_1$ . Hence  $\beta_2 \subset \beta_1$  and therefore  $\beta_1$  is finer than  $\beta_2$ .

Theorem (3.5.2). Let  $A$  be a separable GCR-algebra. Then the Mackey Borel structure on  $\hat{A}$  coincides with the Borel structure generated by the topology on  $\hat{A}$  and, with this Borel structure,  $\hat{A}$  is a standard Borel space.

Proof: [3] Let  $\beta_1$  be the Mackey Borel structure on  $\hat{A}$  and  $\beta_2$  the Borel structure on  $\hat{A}$  generated by the topology on  $\hat{A}$ .

Let  $\hat{A}_1$  and  $\hat{A}_2$  be the corresponding Borel spaces. Then  $\hat{A}_1$  is the quotient of the standard Borel space  $\text{Irr}(\Lambda)$ . By Corollary (3.4.9),  $\hat{A}_2$  is the union of a sequence  $\{S_n\}$  of mutually disjoint subsets of  $\hat{A}$ , each of which is a locally compact Hausdorff space with a countable base. Since a locally compact Hausdorff space with a countable base is metrizable with a complete metric (cf. [7]), it follows that each  $S_n$  is a standard Borel space with the Borel structure induced by  $\beta_2$ . Since  $\hat{A}_2$  is the union of  $S_n$ ,  $\hat{A}_2$  is a standard Borel space. Now let  $f$  be the identity mapping of  $\hat{A}_1$  onto  $\hat{A}_2$ . By Theorem (3.5.1), we know that  $\beta_1$  is finer than  $\beta_2$  and therefore  $f$  is a one-to-one Borel mapping of  $\hat{A}_1$  onto  $\hat{A}_2$ . Since  $\hat{A}_2$  is standard and  $\hat{A}_1$  is the quotient of a standard Borel space, Theorem 4.2. in [14] implies that  $f$  is a Borel isomorphism of  $\hat{A}_1$  onto  $\hat{A}_2$ . Hence  $\beta_1 = \beta_2$  and this completes the proof.

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