Theoretical and Experimental Studies on the Minimum Size 2-edge-connected Spanning Subgraph Problem

Yu Sun

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Ottawa-Carleton Institute for Computer Science
University of Ottawa

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Abstract

A graph is said to be 2-edge-connected if it remains connected after the deletion of any single edge. Given an unweighted bridgeless graph G with n vertices, the minimum size 2-edge-connected spanning subgraph problem (2EC) is that of finding a 2-edge-connected spanning subgraph of G with the minimum number of edges. This problem has important applications in the design of survivable networks. However, because the problem is $\mathcal{NP}$-hard, it is unlikely that efficient methods exist for solving it. Thus efficient methods that find solutions that are provably close to optimal are sought. In this thesis, an approximation algorithm is presented for 2EC on bridgeless cubic graphs which guarantees to be within $\frac{5}{4}$ of the optimal solution value, improving on the previous best proven approximation guarantee of $\frac{5}{4}+\epsilon$ for this problem. We also focus on the linear programming (LP) relaxation of 2EC, which provides important lower bounds for 2EC in useful solution techniques like branch and bound. The "goodness" of this lower bound is measured by the integrality gap of the LP relaxation for 2EC, denoted by α^{2EC}. Through a computational study, we find the exact value of α^{2EC} for graphs with small n. Moreover, a significant improvement is found for the lower bound on the value of α^{2EC} for bridgeless subcubic graphs, which improves the known best lower bound on α^{2EC} from $\frac{9}{8}$ to $\frac{8}{7}$.

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Dedication

For my parents, who gave me my life, as well as much of theirs.

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Notations

$\alpha(G)$ the ratio between $OPT(G)$ and $OPT_{LP}(G)$ for some graph G

α^{2EC} integrality gap of the LP relaxation for 2EC, i.e. maximum $\alpha(G)$ over all G

$\delta(X)$ subgraph of $G = (V, E)$ induced from the vertex subset $X \subset V$

$\overline{V'}$ compliment of a vertex subset V', i.e. $V \setminus V'$

e an edge in a graph

E or $E(G)$ edge set of graph G

$G - v$ deletion of a vertex

$G - V'$ deletion of a set of vertices

$G = (V, E)$ graph G with the vertex set V and edge set E

$G \setminus e$ deletion of an edge

$G \setminus E'$ deletion of a set of edges

G graph

$G[X]$ subgraph of $G = (V, E)$ induced from the vertex subset $X \subset V$

$ILP(G)$ the integer linear program for 2EC

K_n a complete graph with n vertices

$LP(G)$ the linear programming relaxation on $ILP(G)$

n number of vertices in a graph

$OPT(G)$ the optimal objective value of $ILP(G)$

$OPT_{LP}(G)$ the optimal objective value of $LP(G)$

$S \subset G$ a proper subgraph of G

$S \subseteq G$ a subgraph of G

v a vertex in a graph

V or $V(G)$ vertex set of graph G

$V' \subseteq V$ a vertex subset V' of the vertex set V

$x(F)$ $\sum_{e \in F} x_e$ where $F \subseteq E$, and x_e is a decision variable defined on e

2EC minimum size 2-edge-connected spanning subgraph problem

C2EC minimum size 2-edge-connected spanning (multi-)subgraph problem

graph TSP graphic traveling salesman problem

ILP integer linear program

LP linear program

M2EC minimum size 2-edge-connected spanning multi-subgraph problem

TSP traveling salesman problem

Chapter 1

Introduction

Given an unweighted[1] bridgeless graph $G = (V, E)$, the *minimum size 2-edge-connected spanning subgraph problem* (henceforth *2EC*) consists of finding a 2-edge-connected spanning subgraph [2] H of G with minimum number of edges. Note that a *2-edge-connected* graph $G = (V, E)$ is a graph that remains connected with the removal of any edge $e \in E$. An edge in a graph whose removal disconnects the graph into two components is called a *bridge*, so sometimes a 2-edge-connected graph is referred to as a *bridgeless graph*. In a solution for $2EC$, multiple copies of any edge $e \in E$ are not allowed; that is to say, each edge can only be used at most once in the subgraph.

With reliability being emphasized more and more in today's network design, 2-edge-connectivity is becoming a critical property of a robust network. A network featuring 2-edge-connectivity between every two terminals guarantees the network connectivity and thus the network functionality in case of a main transmission link failure. It follows that 2EC, as an important network design problem, has many applications, such as the design of survivable communication networks, and power lines or railways that survive the loss of one link. In this thesis, we provide a comprehensive

[1] A *weighted graph* $G = (V, E)$ has a *cost (or weight) function* that associate a *cost* (or *weight*) with every edge $e \in E$; while an unweighted graph does not have such a function.

[2] A *subgraph* of a graph $G = (V, E)$ is a graph H whose vertices and edges are subsets of V and E respectively. A subgraph of G is called a *spanning subgraph* if it covers all vertices of G.

study on 2EC, both theoretically and experimentally.

With multiple copies of any edge $e \in E$ not allowed in any feasible solution (and thus the optimal solution) of 2EC, it follows that the *integer linear program* [3] of 2EC is a 0-1 integer program, where the decision variables can only be of value 0 or 1. Denoted by $ILP(G)$, the integer linear program of 2EC for some graph $G = (V, E)$ is presented as follows. Note that for any $S \subset V$, $\delta(S)$ is the set of edges with one end in S , and the other end not in S.

$$\begin{aligned} \text{Minimize} \quad & \textstyle\sum_{e \in E} x_e \quad , & & (1.1) \\ \text{subject to} \quad & x(\delta(S)) \geq 2 \quad \text{for all} \quad \emptyset \subset S \subset V, & & (1.2) \\ & x_e \in \{0, 1\} \quad \text{for all} \quad e \in E. & & (1.3) \end{aligned}$$

In $ILP(G)$, the objective function shown in 1.1 is defined on the set of decision variables x_e, in which every one is associated with an edges $e \in E$ indicating e is contained in the solution by $x_e = 1$ and otherwise by $x_e = 0$. We refer to the set of constraints shown in 1.2 as the *cut constraints*, and the constraints shown in 1.3 are known as the *0-1 constraints*. Note that many of the LP and ILP formulations in this thesis are related to edge sets of a graph $G = (V, E)$. For such formulations, for any $F \subseteq E$, we use the notation $x(F)$ to denote $\sum_{e \in F} x_e$.

By relaxing the 0-1 constraints in 1.3 with its integrality constraints removed, the so-called *linear programming relaxation* (henceforth *LP relaxation*) of $ILP(G)$, denoted by $LP(G)$, is obtained and presented as follows, with the objective function shown in 1.4, a set of cut constraints shown in 1.5, a set of non-negativity constraints

[3] A *linear program* (henceforth *LP*) is a program of "maximizing or minimizing a linear function of real variables that are subject to linear equality or inequality constraints"[BM08]. A linear program consists of a objective function and a series of constraints, where both are defined on a set of decision variables. A linear program whose decision variables are all constrained to be only integers is called an *integer linear program* (henceforth *ILP*).

shown in 1.6 and a set of upper bound constraints shown in 1.7.

$$\begin{aligned} \text{Minimize} \quad & \textstyle\sum_{e \in E} x_e \quad , && (1.4) \\ \text{subject to} \quad & x(\delta(S)) \geq 2 \quad \text{for all} \quad \emptyset \subset S \subset V, && (1.5) \\ & x_e \geq 0 \quad \text{for all} \quad e \in E, && (1.6) \\ & x_e \leq 1 \quad \text{for all} \quad e \in E. && (1.7) \end{aligned}$$

We use the notation $OPT(G)$ and $OPT_{LP}(G)$ to denote the optimal objective value [4] for $ILP(G)$ and $LP(G)$ respectively.

It is known that 2EC is NP-hard[5] [JRV03] and also MAX SNP-hard[6] (even for cubic graphs) [CKK02, BIT13], where a cubic graph is a graph in which every vertex has degree[7] 3. This does not only imply that 2EC is very unlikely to be solved in polynomial time, but also suggests that the existence of a *polynomial time approximation scheme*[8] is highly doubtful. For this reason, looking for reasonably good solutions, which are not too far from optimal and obtainable efficiently enough becomes a more promising approach for obtaining solutions for 2EC that are provably close to optimal. Note that since 2EC on cubic graphs is the simplest among all 2EC problems which remains $\mathcal{NP}$-hard, we concentrate on solving 2EC for cubic graphs in order to learn more about 2EC. For some constant $\rho \geq 1$, which is called the *approximation ratio*, a *ρ-approximation algorithm* for 2EC produces a feasible solution whose objective value is no more than ρ times the optimal objective value [BM08] (i.e. $OPT(G)$). It follows that a 1-approximation algorithm produces an optimal solution. Due to the difficulty of finding $OPT(G)$, it seems impossible to prove that

[4] Any set of values assigned to the decision variables that satisfies all constraints is referred to as a *feasible solution*. A feasible solution at which the objective function achieves its optimum is an *optimal solution*, and the optimum is called the *optimal objective value*.

[5] Please refer to Section 2.1.2 for definitions.

[6] Please refer to Section 2.1.2 for definitions.

[7] The degree of a vertex is the number of edges incident with it.

[8] Please refer to Section 2.1.2 for definitions.

such an algorithm always yields a solution within ρ times $OPT(G)$. It follows that a lower bound for $OPT(G)$ is often a necessary element in designing an approximation algorithm. For any graph G with n vertices, if there exists one, a Hamilton cycle[9] of G is naturally a minimum 2-edge-connected spanning subgraph of G. It leads to an obvious lower bound for $OPT(G)$, which is the number of vertices in G, denoted by n. Note that n is the lowest possible bound for $OPT(G)$, since a Hamilton cycle, if there exists one, is the minimum-possible size 2-edge-connected spanning subgraph for any graph. The lower bound of n is employed in [BIT13] and [Huh04], as well as for the approximation algorithm presented in this thesis.

On the other hand, since $LP(G)$ is a relaxation of $ILP(G)$, the optimal objective value of $LP(G)$, denoted by $OPT_{LP}(G)$, serves as a relatively tighter lower bound for $OPT(G)$, i.e., $OPT_{LP}(G) \geq n$. For a graph $G = (V, E)$, let $\alpha(G)$ denote the ratio between $OPT(G)$ and $OPT_{LP}(G)$. This leads to the concept of the *integrality gap* of the LP relaxation for 2EC, denoted by α^{2EC}, which is the worst-case ratio between $OPT(G)$ and $OPT_{LP}(G)$. That is to say, $\alpha^{2EC} = max\frac{OPT(G)}{OPT_{LP}(G)}$ over all G. As a critical topic throughout this thesis, we studied the integrality gap of the LP relaxation for 2EC intensively. There are two main reasons this is useful. First, the integrality gap itself serves as an indicator of the quality of the lower bound given by $LP(G)$. This is important for methods, such as branch and bound, that depend on good lower bounds for their success. Secondly, an algorithmic proof for $\alpha^{2EC} = k$ yields a k-approximation algorithm for 2EC [ABEM06]. In this thesis, we give an upper bound on the value of α^{2EC} on bridgeless cubic graphs with an algorithmic proof, while the lower bounds on the integrality gap of 2EC are investigated through computational studies.

[9] A Hamilton cycle in a graph G is a cycle that covers all vertices in G. Please refer to Section 2.1.1 for its formal definitions.

1.1 Contributions

The major contributions of this thesis are as follows:

1. We develop a new $\frac{5}{4}$-approximation algorithm for 2EC on bridgeless cubic graphs, which constructs a feasible solution of 2EC on a given bridgeless cubic graph $G = (V, E)$ that contains at most $\frac{5}{4}|V| - 1$ edges with two specified edges included in $O(n^3)$ time. It improves upon the previous best approximation ratio of $\frac{5}{4} + \epsilon$ given by Csaba, Karpinski and Krysta [CKK02] for 2EC on maximum degree 3 graphs[10]. Along with the introduction of this algorithm, a theorem indicating that the integrality gap of 2EC is at most $\frac{5}{4}$ for bridgeless cubic graphs is proved, which defines a tighter upper bound than the integrality gap of the LP relaxation for 2EC on maximum degree 3 graphs as $\frac{5}{4} + \epsilon$ for any fixed $\epsilon > 0$ [CKK02].

2. Focusing on the integrality gap of the LP relaxation for 2EC, we conduct a computational study by designing a program that calculates $OPT(G)$ and $OPT_{LP}(G)$ and thus gives $\alpha(G)$ exactly for all $G \in \mathcal{G}$, where $\mathcal{G}$ contains all test cases in three categories:

    - General bridgeless graphs for $3 \leq n \leq 10$;
    - Cubic bridgeless graphs for $6 \leq n \leq 16$; and
    - Subcubic bridgeless graphs for $3 \leq n \leq 16$.

    In our experiments, one subcubic graph, G_{16}, stands out. It has 16 vertices and gives $\alpha(G_{16}) = \frac{9}{8}$, which is the same value as the previous known best lower bound on the integrality gap of the LP relaxation for 2EC [SV12], with G_{16} different from the example used in [SV12].

[10] A maximum degree 3 graph has degree at most 3 everywhere. If the given graph is bridgeless, a maximum degree 3 graph is equivalent to a subcubic graph, in which the degree for any vertex is either 2 or 3.

3. With the knowledge gained from the bottlenecks we faced during the design of the approximation algorithm mentioned in 1, we provide a family of subcubic graphs G for which $\frac{OPT(G)}{n} = \frac{4}{3}$ asymptotically, as well as a family of cubic graphs G for which $\frac{OPT(G)}{n} = \frac{7}{6}$ asymptotically. These two families of graphs thus suggest that for any approximation algorithm with a performance guarantee of $k \cdot n$, $k = \frac{4}{3}$ would be the best possible for subcubic bridgeless graphs, and $k = \frac{7}{6}$ would be the best possible for cubic bridgeless graphs.

4. Using the knowledge gained through the data analysis for the computational study, we provide two different families of subcubic graphs G for which $\frac{OPT(G)}{OPT_{LP}(G)} = \frac{8}{7}$ asymptotically, which further tightens the lower bound of the integrality gap to $\frac{8}{7}$. It significantly improves the previous known best lower bound of α^{2EC} for subcubic graphs from $\frac{9}{8}$ [SV12].

1.2 Outlines

With a literature review on 2EC given in the remainder of this chapter, an overview of the structure of this thesis is as follows:

In Chapter 2, general definitions and notations that are utilized throughout this thesis are given in Section 2.1. In Section 2.2, followed by a discussion on their relationship with 2EC, several problems which are highly related to 2EC are introduced.

In Chapter 3, we first conduct a detailed review in Section 3.1 on a $\frac{6}{5}$-approximation algorithm for 2EC on 3-edge-connected cubic graphs, which is referred to as Apx2EC and designed by Boyd, Iwata and Takazawa [BIT13]. As the cornerstone of our algorithmic studies, a deep analysis on the bottlenecks for generalizing the above algorithm is demonstrated in Section 3.2. This is followed by our design of a $\frac{5}{4}$-approximation algorithm for 2EC on bridgeless cubic graphs, which is an extension of Algorithm Apx2EC, by overcoming the bottlenecks discussed in the previous section.

Following this algorithm, a theorem indicating that the integrality gap of 2EC is at most $\frac{5}{4}$ for bridgeless cubic graphs is proved.

In Chapter 4, with the desire to learn more about the integrality gap of 2EC, we investigate the worst-case ratio between $ILP(G)$ and $LP(G)$ computationally for graphs with a small number of vertices. By designing and realizing a program as presented in Section 4.1, that generates requested experimental objects with the help from `nauty`[McK], and gives optimal solutions to $ILP(G)$ and $LP(G)$ with the aid of Gurobi™Optimizer [Gur12b], we thus obtained $\alpha(G)$, as the ratio between $OPT(G)$ and $OPT_{LP}(G)$, for every experimental object G. Following an explicit analysis on all data given by our program, several discoveries are discussed in Section 4.2.

Finally, by combining the knowledge we gained through both the algorithmic and computational studies from Chapters 3 and 4, a further study is conducted concerning the lower bounds on the integrality gap of 2EC by demonstrating four families of graphs. The latter two families of graphs yields a significant improvement on the known best lower bound for α_{2EC} on subcubic graphs.

1.3 Literature Review on 2EC

Constant factor approximation algorithms[11] for 2EC have been intensively studied.

For an unweighted bridgeless graph $G = (V, E)$ on n vertices, a 2-edge-connected spanning subgraph of G can be found based on a spanning subtree T of G resulted from a *depth-first-search* (henceforth *DFS*), which is a widely-used graph traversing algorithm that starts from an arbitrary vertex as the root and explores as deep as possible along each branch in prior to backtracking. All non-tree edges are referred to as *back edges*, i.e., edges whose one end is an ancestor of the other end in T. By taking all edges in T and the deepest backward edge from every non-root vertex, a 2-edge-

[11]A *constant factor approximation algorithm* is an approximation algorithm whose approximation ratio is of a fixed value.

connected spanning subgraph of G is obtained with at most $2n-2$ edges [VV00]. This simply gives a 2-approximation algorithm for $2EC$ with n serving as the lower bound. Another easy 2-approximation algorithm can be found in [CSS98]. In 1994, Khuller and Vishkin [KV94] first improved this approximation ratio of 2 to $\frac{3}{2}$ with a better lower bound obtained from the idea of "tree carving", by presenting an algorithm also based on depth-first search but with a stricter scheme for picking back edges. Around the same time, Garg, Santosh and Singla [GSS93] claimed that with a better lower bound, which is obtained from an extended idea of "tree carving", a $\frac{5}{4}$-approximation algorithm can be achieved, but no complete proof or details were ever provided. In 1998, Cheriyan, Sebö and Szigeti [CSS98] improved the approximation ratio from $\frac{3}{2}$ to $\frac{17}{12}$ by using an "ear decomposition" [CSS98, SV12] to obtain a feasible solution. The ratio was later improved to $\frac{4}{3}$ in 2000 by Vempala and Vetta [VV00]. One year later, Krysta and Kumar [KK01] improved the approximation ratio to $\frac{4}{3} - \epsilon$ where $\epsilon = \frac{1}{1344}$ by extending the technique of Vempala and Vetta. In 2003, Jothi, Raghavachari and Varadarajan claimed to have a $\frac{5}{4}$-approximation algorithm, but their proof was incomplete, and a complete proof was never provided. Very recently, Sebö and Vygen [SV12] designed a simpler and more elegant $\frac{4}{3}$-approximation algorithm for 2EC, along with other results for related problems, by introducing the technique called "nicer ears" based on the idea of ear decomposition.

In the meantime, research on 2EC has also been conducted for special classes of graphs, especially on cubic bridgeless graphs, on which 2EC still remains $\mathcal{NP}$-hard. In 2001, along with their $(\frac{4}{3} - \epsilon)$-approximation algorithm for 2EC on general graphs, Krysta and Kumar [KK01] also presented an approximation algorithm for 2EC on cubic graphs with the approximation ratio of $\frac{21}{16} + \epsilon$. One year later, Csaba, Karpinski and Krysta [CKK02] designed a $(\frac{5}{4} + \epsilon)$-approximation algorithm for 2EC on *maximum degree 3 graphs*, which are also subcubic graphs considering the input graph is bridgeless. In 2004, Huh [Huh04] presented an algorithm yielding a solution

H to the 2EC on the r-edge-connected ($r \geq 2$) graphs $G = (V, E)$ with at most $|V| + \frac{|E|-|V|}{r-1}$ edges in H. It follows that for r-regular[12], r-edge-connected graphs, the approximation ratio is $\frac{5}{4}, \frac{4}{3}, \frac{11}{8}, \frac{7}{5}$ for $r = 3, 4, 5, 6$ respectively. A more recent significant improvement came from Boyd, Iwata and Takazawa [BIT13] with a $\frac{6}{5}$-approximation algorithm for 2EC on cubic 3-edge-connected graphs.

Concerning the integrality gap of 2EC, α^{2EC}, on unweighted graphs, Csaba, Karpinski and Krysta [CKK02] proved that for maximum degree 3 graphs, the integrality gap of the LP relaxation for 2EC is at most $\frac{5}{4} + \epsilon$ for any fixed $\epsilon > 0$. It was also stated in [CKK02] that the known best lower bound on 2EC is $\frac{10}{9}$ for maximum degree 3 graphs (and thus subcubic graphs). In 2012, by giving a $\frac{6}{5}$-approximation algorithm, Boyd, Iwata and Takazawa defined an upper bound as $\frac{6}{5}$ on the value of α^{2EC} for 3-edge-connected cubic graphs. Around the same time, Sebö and Vygen [SV12] proved that $\alpha^{2EC} \leq \frac{4}{3}$ by providing a $\frac{4}{3}$-approximation algorithm for 2EC on all bridgeless graphs; and $\alpha^{2EC} \leq \frac{9}{8}$ by using the same example as shown in Figure 1 in [ABEM06] with unit weights, which improved the known best lower bound on α^{2EC} defined by [CKK02]. Most other studies concerning the integrality gap of the LP relaxation for 2EC are based on 2EC with general cost functions, which are introduced in detail in Sections 2.2.2 and 2.2.4.

[12]A *r-regular graph* is a graph where every vertex is incident to r edges, i.e., every vertex has the same degree of r.

Chapter 2

Preliminaries

This chapter provides definitions and notations that are utilized throughout this thesis. This is followed by an introduction to three problems, i.e., *minimum size 2-edge-connected spanning multi-subgraph problem*, *minimum cost 2-edge-connected spanning (multi-)subgraph problem* and *graph traveling salesman problem*, that are highly related to 2EC. Literature reviews on these three problems are also included to discuss some important studies which motivated and inspired our work.

2.1 General Definitions and Notations

2.1.1 Graphs, Subgraphs and Trees

As defined by Bondy and Murty in *Graph Theory*[BM08, p. 2]:

> A *graph* G is an ordered pair $(V(G), E(G))$ consisting of a set $V(G)$ of *vertices* and a set $E(G)$, disjoint from $V(G)$, of *edges*, together with an *incidence function* ψ_G that associates with each edge of G an unordered pair of (not necessarily distinct) vertices of G. If e is an edge and u and v are vertices such that $\psi_G(e) = \{u, v\}$, then e is said to *join* u and v, and vertices u and v are called the *ends* of e.

In this thesis, we denote a graph by $G = (V, E)$, where V represents the *vertex set*, E is the *edge set*, and the incidence function is implied. Unless stated otherwise, n is used to denote the number of vertices in G, i.e. $n = |V|$. Usually a lowercase letter, such as u or v, is employed to denoted a vertex in a graph. For an edge in a graph with two ends as u and v, an unordered pair (u, v) is used to denote the edge, or sometimes it is simply denoted by uv. The following list presents some commonly used definitions and notations in graph theory, which are utilized in this thesis.

1. An edge is said to be *incident* with its ends. Two vertices which are incident with an identical edge are said to be *adjacent* to each other. Also, two edges which share an identical end are said to be *adjacent* to each other.

2. For a pair of adjacent vertices u and v, u is said to be a *neighbour* of v and vice versa.

3. For a vertex $v \in V$, $\delta(v)$ is used to denote the set of edges incident to v. For any vertex $v \in V$, the number of edges in $\delta(v)$ is referred to as the *degree* of v, denoted by d_v.

4. For any subset of vertices $V' \subseteq V$, $\overline{V'}$ is used to denote $V \setminus V'$; and $\delta(V')$ is used to denote the set of edges with exactly one end in V'.

5. An edge with identical ends is called a *loop*; two or more edges ending at the same pair of vertices are referred to as *parallel edges*. Unless stated otherwise, a graph in this thesis contains no loops.

Consider the graph $G = (V, E)$ as shown in Figure 2.1 as an example. This graph consists of 5 vertices (i.e. $n = 5$) and 10 edges, where the vertex set $V = \{0, 1, 2, 3, 4\}$, the edge set $E = \{a, b, c, d, e, f, g, h, i, j\}$, and the incidence function is defined as:

$$\psi_G(a) = \{1, 2\}, \psi_G(b) = \{2, 3\}, \psi_G(c) = \{3, 4\}, \psi_G(d) = \{4, 0\}, \psi_G(e) = \{0, 1\},$$
$$\psi_G(f) = \{2, 0\}, \psi_G(g) = \{1, 3\}, \psi_G(h) = \{2, 4\}, \psi_G(i) = \{3, 0\}, \psi_G(j) = \{1, 4\}.$$

Considering vertex 1 as an example, it is adjacent to vertices 2, 3, 4 and 0, and incident with edges a, e, g and j. That is also to say, vertices 2, 3, 4 and 0 are the neighbors of vertex 1. It also can be observed from the graph that $\delta(1) = \{a, e, g, j\}$ and $d_1 = 4$. Take the vertex subset $V' = \{1, 2\} \subset V$ as an example, we have $\overline{V'} = \{3, 4, 0\}$ and $\delta(V') = \{b, e, f, g, h, j\}$.

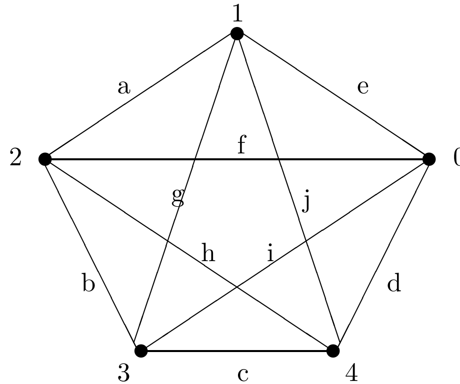


FIGURE 2.1: An example of a graph $G = (V, E)$.

In this thesis, we only study *finite graphs* and *nontrivial graphs*, which have finite vertex and edge sets and contain more than one vertex. For any finite nontrivial graph, a *simple graph* is a graph that has no loops or parallel edges. In Chapters 3, 4 and 5, we consider only simple graphs as the study objects. Belonging to simple graphs, the following families of graphs are related to this thesis:

1. A *complete graph*, usually denoted by K_n on n vertices, is a simple graph in which every two vertices are adjacent.

2. A *k-regular graph* is a graph where every vertex has the same degree of k.

3. A *cubic graph*, also referred to as a *3-regular graph*, is a graph where every vertex has degree 3.

4. A graph G is called a *subcubic graph* if every vertex in G has either degree 2 or degree 3.

For example, the graph in Figure 2.1 is a complete graph, and it is a 4-regular graph as well. Other examples are shown in Figure 2.2. Figure 2.2(a) illustrates a simple graph, however, it is not a complete graph. In contrast, Figure 2.2 (b) and (c) shows two graphs which are not simple since (b) contains a loop, while (c) contains parallel edges.

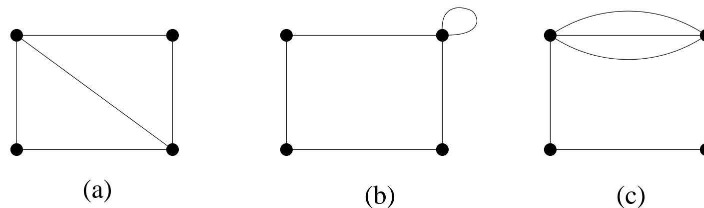


FIGURE 2.2: Graph examples. (a) A simple (but not complete) graph. (b) A graph with a loop. (c) A graph with parallel edges.

For the representation of graphs, various data structures are applied in practice. In this thesis, Chapter 4 in particular, we represent a graph $G = (V, E)$ as an adjacency matrix. Assuming the vertices of graph $G = (V, E)$ are numbered $0, 1, \ldots, n-1$ where $n = |V|$, the adjacency matrix representation of G consists of an $n \times n$ matrix $A = (a_{ij})$ such that

$$a_{ij} = \begin{cases} 1 & \text{if } (i,j) \in E \\ 0 & \text{otherwise.} \end{cases}$$

Other representations include adjacency list, incidence list and incidence matrix. Among the four representations listed, adjacency list and adjacency matrix are considered to be more commonly used. Since only the adjacency matrix is applied in this thesis, details of other representations are not given.

A graph $G = (V, E)$ is *connected* if, for any partition of V into two nonempty sets $X \subset V$ and $Y \subset V$ such that $X \cap Y = \emptyset$ and $X \cup Y = V$, there exists at least one edge with one end in X and the other end in Y; otherwise, G is *disconnected* [BM08].

A graph $G = (V, E)$ is referred to as a *weighted graph* if every edge $e \in E$ has an associated *cost* or *weight*, which is a real number denoted by c_e. Weighted graphs are

very useful for solving practical problems. The adjacency list can be easily adapted to represent a weighted graph by storing the weight c_e of the edge $e = (i, j) \in E$ as the entry in row i and column j of the adjacency matrix. However, different from the adjacency matrix for an *unweighted graph* which has no cost on any edge, for an edge $e \notin E$, it is represented by a NIL value in the corresponding matrix entry; although in practice, it is convenient to use a value such as 0, ∞, or $\sum_{e \in E} c_e$ if there exists $c_e = 0$ for any $e \in E$. In this thesis, unless stated otherwise, our research is based on unweighted graphs, especially in Chapters 3, 4 and 5. Previous research conducted on weighted graphs are reviewed in Subsection 2.2.2 for a better understanding of this problem.

In a graph, a *path* is "a simple graph whose vertices can be arranged in a linear sequence in such a way that two vertices are adjacent if they are consecutive in the sequence and are nonadjacent otherwise"[BM08, p. 4]. A path in a graph G is called a *Hamilton path* if it contains all vertices in G. Likewise, a *cycle* on three or more vertices is "a simple graph whose vertices can be arranged in a cyclic sequence in such a way that two vertices are adjacent if they are consecutive in the sequence, and nonadjacent otherwise"[BM08, p. 4]. Similarly, a *Hamilton cycle* in a graph $G = (V, E)$ is a cycle that covers all vertices in G. In this thesis, k-cycle is used to denote a cycle which contains k vertices.

A graph which does not contain any cycle is referred to as an *acyclic graph*, which is also known as a *forest*. A connected acyclic graph is called a *tree*. The term *node* is used to represent a vertex in a tree. When defining a tree, one of the nodes is usually selected as the root of the tree, which is denoted by r. Since a tree is acyclic, any two nodes are connected by exactly one simple path, where no loops or parallel edges are allowed. For any node v, except the root r, in a tree T, let $P(v)$ denote such a simple path connecting r and v. If the last edge on $P(V)$ is (u, v), u is called the *parent* of v, and v is called a *child* of u. Note that because there is exactly one simple

path between the root and every other node, every node except the root has exactly one parent; however, the number of children of a node can be zero or any positive value. Any node in a tree that has zero children is called a *leaf* of the tree; and a non-leaf node is called an *internal node*. For any node v in T other than the root r, the length of the simple path $P(v)$ connecting r and v is referred to as the *depth* of v in T; on the other hand, the *height* of v in T is measured by the number of edges in the longest simple downward path from v to a leaf. The *height* of a tree T, denoted by h, is the height of the root of T [CLRS09].

Following the basic knowledge on graphs presented above, we introduce the concept of *subgraphs*. A graph H is referred to as a *subgraph* of the graph G denoted by $H \subseteq G$ or $G \supseteq H$, if $V(H) \subseteq V(G), E(H) \subseteq E(G)$, and ψ_H is the restriction of ψ_G to $E(H)$ [BM08, p. 40]. Moreover, a *subtree* of a graph is a subgraph which is a tree. Note that any graph is a subgraph of itself.

Generally speaking, there are two operations used to obtain a subgraph of a graph G, edge deletions and vertex deletions. For any graph G, edge deletion on an edge $e \in E(G)$, denoted by $G \setminus e$, leads to a subgraph $H \subset G$, where $H = (V(G), E(G) \setminus \{e\})$; while vertex deletion on a vertex $v \in V(G)$, denoted by $G - v$, leads to a subgraph $H \subset G$, where $H = (V(G) \setminus \{v\}, E(G) \setminus \delta(v))$. In this thesis, $G - V'$ where $V' \subseteq V$ is used to denote the deletion of a set of vertices from G, while $G \setminus E'$ where $E' \subseteq E$ is used to denote the deletion of a set of edges from G. Figure 2.3 illustrates three subgraphs of the complete graph G shown in Figure 2.1. Figure 2.3 (a) and (c) are obtained from edge deletions and Figure 2.3 (b) is obtained from vertex deletions.

A *spanning subgraph* H of a graph G is a subgraph obtained by edge deletions only. That is to say, a subgraph H of G is referred to as a spanning subgraph of G if $V(H) = V(G)$. A *spanning tree* of a graph G is a subtree of G that contains all vertices in G. A graph is connected if and only if it has a spanning tree [BM08, p. 106]; while a disconnected graph G has *a spanning forest*, consisting of a set of spanning

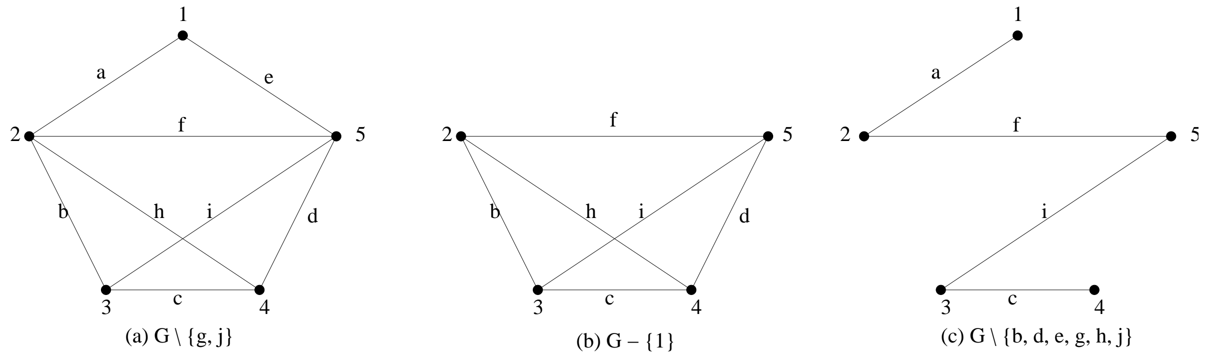


FIGURE 2.3: Examples of subgraphs of the graph G in Figure 2.1. (a) A subgraph obtained from the edge deletions. (b) A subgraph obtained from the vertex deletion. (c) A tree obtained from the edge deletions.

trees for each connected components[1]. Consider the graph G shown in Figure 2.1 as an example, graphs shown in Figures 2.3 (a) and (c) and G itself are all spanning subgraphs of G, whereas the graph shown in Figure 2.3 (c) is a spanning tree of G as well. However, the graph shown in Figure 2.3 (b) is a subgraph of G but not its spanning subgraph.

As two of the very important and fundamental algorithms for graphs, *breadth-first search* (*BFS* henceforth) and *depth-first search* (*DFS* henceforth) are two major strategies for traversals throughout a graph. For any graph $G = (V, E)$, both BFS and DFS result in a spanning tree of G if G is connected or a spanning forest of G otherwise.

In graph theory, a spanning graph of a graph G can also be referred to as a *factor* of G. A *k-factor* of a graph is a spanning k-regular subgraph. For a graph $G = (V, E)$, a *1-factor* of the graph G is a spanning subgraph of G consisting of a set of vertex-disjoint edges; and a *2-factor* is a collection of cycles in G that spans all vertices of G. A 1-factor is also referred to as a *perfect matching*, where a *matching* is a set of vertex-disjoint edges. A matching of a graph G does not necessarily cover all vertices in G. Note that for cubic graphs, the complement of a 2-factor is a perfect matching.

[1]For a disconnected graph G, the *connected components*, or simply the *components*, of G denote the vertex set of the maximal connected subgraphs.

For a graph $G = (V, E)$, a *cut*, usually defined by $\delta(V') = \{(u, v) : (u, v) \in E, u \in V', v \notin V'\}$ for some $V' \subset V$ [CCPS98], consists of a set of edges $E' \subset E$ such that every edge $e \in E'$ has exactly one end in V'. The removal of a cut $\delta(V')$ for some $V' \subset V$ in a connected graph $G = (V, E)$ disconnects G into two components, $G[V']$ and $G[\overline{V'}]$. Note that $G[V']$ denotes the subgraph of G *induced* by $V' \subset V$, i.e., $G[V'] = (V', \gamma(V'))$, where $\gamma(V')$ is the set of edges which have both ends in V'. A cut that contains k edges is called a *k-edge cut* if it does not contain any other cut as a proper subset. If a cut contains only one edge, this particular edge is called a *cut edge* (or a *bridge*). A graph is called *bridgeless* if it does not contain any cut edge. A graph G is said to be *k-edge-connected* if every cut in G contains at least k edges. Another way of stating this is that a graph G is k-edge-connected if and only if G remains connected after the removal of any set of $(k - 1)$ edges. Trivially, a graph that is k-edge-connected is also $(k - 1)$-edge-connected. Note that a graph is 1-edge-connected if and only if it is connected, and a bridgeless graph is also a 2-edge-connected graph. Consider the graphs shown in Figure 2.3 as examples, Figure 2.3 (a) is 2-edge-connected, Figure 2.3 (b) is 3-edge-connected, and Figure 2.3 (c) is 1-edge-connected.

Different from a cut of some graph $G = (V, E)$ that consists of a set of edges $E' \in E$, a *vertex cut* consists of a subset of V that separates some pair of non-adjacent vertices of G. In another word, a vertex cut of some graph $G = (V, E)$ is defined by a proper subset $R \subset V$ whose removal disconnects G, i.e. $G - R$ contains more than one component. A vertex cut R which contains k vertices is referred to as a *k-vertex cut*; when $k = 1$, this particular vertex that forms the 1-vertex cut is called a *cut vertex*. Note that a complete graph has no vertex cut. We say a graph G is *k-vertex-connected* if every vertex cut in G contains at least k vertices. Consider the graphs shown in Figure 2.3 as examples, Figure 2.3 (a) is 2-vertex-connected where $\{2, 5\}$ forms a 2-vertex cut, Figure 2.3 (b) has no vertex cut, and Figure 2.3 (c) is

1-vertex-connected where vertices 2, 3 an 5 are all cut vertices.

2.1.2 NP-Hardness and Approximation Algorithms

For an algorithm, the *computational complexity* is used to measure the number of basic computational steps required for the execution of the algorithm [BM08]. For any algorithm, we usually use *O-notation* to give an asymptotic upper bound on its computational complexity, in which the lower order terms and coefficients become irrelevant [CLRS09]. Given an input of size n, an algorithm is called a *polynomial-time algorithm* if the number of steps required to complete the algorithm for a given input is $O(n^k)$ for some non-negative k, which does not depend on n. For example, both BFS and DFS are polynomial-time algorithms with the computational complexity of $O(|V| + |E|)$, for an input $G = (V, E)$ with n vertices. Polynomial-time algorithms are usually considered to be computationally feasible, even for large inputs [BM08]. A problem is said to be solvable in polynomial time if there is a polynomial-time algorithm for it.

A *decision problem* is a problem whose answer is either positive or negative. The class of decision problems that can be solved in polynomial time is usually denoted by $\mathcal{P}$. On the other hand, the class of $\mathcal{NP}$, where $\mathcal{NP}$ denotes "nondeterministic polynomial", consist of decision problems "with the property that for any input that has a positive answer, there is a *certificate* from which the correctness of this answer can be derived in polynomial time" [CCPS98, p. 309]. One of the extensively studied problem in $\mathcal{NP}$ is the *Hamilton cycle problem* (*HAM-CYCLE* henceforth) [CLRS09], which questions "Does G have a Hamilton cycle?" for some graph G. Considering HAM-CYCLE as an example, if the answer to HAM-CYCLE is positive, it can be verified by giving a Hamilton cycle of G as a certificate, which can be checked in polynomial time. Therefore it provides a way to check the correctness of the answer in polynomial time. Note that the class $\mathcal{NP}$ includes all problems in the class $\mathcal{P}$, i.e.,

$\mathcal{P} \subseteq \mathcal{NP}$.

A decision problem Π is called $\mathcal{NP}$*-complete* if it is in $\mathcal{NP}$ and every problem in $\mathcal{NP}$ can be transformed to Π within polynomial time. The procedure of the transformation is usually referred to as *polynomial-time reduction*. Irrelevant to this thesis, the details of the polynomial-time reduction will not be given here. In other words, a problem Π is $\mathcal{NP}$-complete if $\Pi \in \mathcal{NP}$ and an algorithm for solving Π can be converted into one for solving any other problem in the class of $\mathcal{NP}$. It implies that if any $\mathcal{NP}$-complete problem can be proved to be solvable in polynomial time, then all problems in $\mathcal{NP}$ can be solved in polynomial time, which leads to $\mathcal{P} = \mathcal{NP}$. However, no polynomial-time algorithm has been found for any $\mathcal{NP}$-complete problem until now, which leaves the $\mathcal{P}$ *versus* $\mathcal{NP}$ *problem* a major unsolved problem in computer science. Note that HAM-CYCLE $\in \mathcal{NP}$-complete holds as well.

In real life, not all problems are decision problems. As a matter of fact, many problems of interest are *optimization problems*, in which "each feasible solution has an associated value, and we wish to find a feasible solution with the best value" [CLRS09]. For an optimization problem Π, we use $\Pi \in \mathcal{NP}$-hard to indicate Π is at least as hard as an $\mathcal{NP}$-complete problem. An optimization problem cannot be referred to as an $\mathcal{NP}$-complete problem, because this term applies only to decision problems [KT05]. Note that an $\mathcal{NP}$-hard problem does not necessarily belong to $\mathcal{NP}$, since it might not be a decision problem. An $\mathcal{NP}$-complete problem belongs to both $\mathcal{NP}$ and $\mathcal{NP}$-hard [BW05].

A famous example for $\mathcal{NP}$-hardness is the *symmetric traveling salesman problem*. Given the complete graph $K_n = (V, E)$ on n vertices with non-negative real cost c_e associated with every $e \in E$, the *Symmetric Traveling Salesman Problem (STSP)* consists of finding a Hamiltonian cycle in K_n with minimum cost. Symmetric traveling salesman problems on some graph K_n are referred to as *metric STSP* if the costs on edges in K_n satisfy the triangle inequality. It is known that the STSP, includ-

ing the metric STSP, is NP-hard [BB08]. In Subsection 2.1.4, we will introduce a mathematical formulation of the problem and give more details.

Since many problems of practical significance are $\mathcal{NP}$-complete or $\mathcal{NP}$-hard, which are unlikely to be solved efficiently enough, *approximation algorithms* are commonly applied to solve such problems. For any NP-hard problem, a *ρ-approximation algorithm* is a polynomial algorithm which delivers a solution whose cost is at most ρ times the optimum. The factor ρ is usually called the *approximation ratio*. Since the cost of the optimal solution is normally unknown, a bound for the optimal solution is often involved in an approximation algorithm. Let x denotes the value of the bound utilized in a ρ-approximation algorithm, $\rho \cdot x$ is referred to as the performance guarantee of this approximation algorithm.

Based on the property of the approximation ratio, there are three types of approximation algorithms: *full polynomial time approximation scheme*, *polynomial time approximation scheme* and *constant-factor approximation algorithm*. Quoted from [ALM+98], we give the formal definitions for the above three approximation schemes.

> A *full polynomial time approximation scheme* is an algorithm that, for any given $\epsilon > 0$, approximates the problem within a factor $1 + \epsilon$ in time that is polynomial in the input size and $1/\epsilon$.
>
> A *polynomial time approximation scheme* is an algorithm that, for any given $\epsilon > 0$, approximates the problem within a factor $1 + \epsilon$ in time that is polynomial in the input size (and could depend arbitrarily upon $1/\epsilon$).
>
> A *constant-factor approximation algorithm* are algorithms which, for some fixed constant $c > 1$, are able to approximate the optimal solution to within a factor c in polynomial time.
>
> [ALM+98]

According to the existence of the different types of approximation algorithms, a prob-

lem can be divided in different classes. A problem with only constant-factor approximation algorithms belongs to the class APX, a problem with a polynomial time approximation scheme falls into the class $PTAS$, and a problem with a full polynomial time approximation scheme is of the class $FPTAS$. It follows from the definition that $FPTAS \subset PTAS \subset APX$. Note that $PTAS = APX$ if and only if $\mathcal{P} = \mathcal{NP}$ [ALM+98]. In simpler terms, a problem $\Pi \in APX$ only has polynomial-time algorithms with the approximation ratio as some fixed factor; a problem $\Pi \in PTAS$ has polynomial-time algorithms delivering solutions within every fixed factor greater than 1 of the optimum. A problem is said to be *MAX SNP-hard* if it has no polynomial time approximation scheme unless $\mathcal{P} = \mathcal{NP}$, which is considered very unlikely so far.

2.1.3 Linear Program and Integer Linear Program

Many problems in real life are in the form of maximizing or minimizing an objective by searching for an optimum allocation of the limited resources under competing constraints. *Linear program* came to the public attention with a variety of such practical applications. A *linear program* (henceforth *LP*) is "a problem of maximizing or minimizing a linear function of real variables that are subject to linear equality or inequality constraints"[BM08, p. 197]. Generally, an LP consists of a *linear function* which is referred to as the *objective function*, a set of real *decision variables* and a set of *constraints* presented in the form of *linear equalities* or *linear inequalities*. We call a problem a *maximization linear program* if we are to maximize the objective function, and a *minimization linear program* if we are to minimize the objective function. Mathematically, any LP with n variables and $(m + n)$ constraints, either a maximization linear program or a minimization linear program, can be described in the following *standard form* (Expression 2.1) by simple substitutions. We denote an

LP in the standard form by a tuple $(\mathbf{A}, \mathbf{b}, \mathbf{c})$.

$$\begin{aligned} \text{maximize } & \mathbf{c}^T\mathbf{x} \\ \text{subject to } & \mathbf{A}\mathbf{x} \leq \mathbf{b} \\ & \mathbf{x} \geq \mathbf{0}, \end{aligned} \tag{2.1}$$

where $\mathbf{A} = (a_{ij})$ is an $m \times n$ matrix, $\mathbf{b} = (b_i)$ is a m-vector, and $\mathbf{c}^T = (c_j)$ and $\mathbf{x} = (x_j)$ are both n-vectors for $i = 1, 2, ..., m$ and $j = 1, 2, ..., n$. In the standard form presented above, the first line, as the inner product of two vectors, is the *objective function*, and the second and the third lines are the *constraints*. More specifically, the third line contains n *non-negativity constraints*, which means that each entry of the vector $\mathbf{x}$ must be non-negative in the standard form. Since it is irrelevant to this thesis, the method of converting an LP to its standard form will not be introduced here. However, it can be found in [CLRS09, p. 852]. Note that in this thesis, for the ease of understanding 2EC and its related problems, which are all minimization problems, it is not necessary to have their LPs or ILPs written in the standard form.

A setting of the variables $\overline{\mathbf{x}}$ to values which satisfy all the constraints is referred to as a *feasible solution*, while a setting of the variables $\overline{\mathbf{x}}$ to values which fail to satisfy some constraint is referred to as an *infeasible solution*. A solution $\overline{\mathbf{x}}$ leads to an *objective value* $\mathbf{c}^T\overline{\mathbf{x}}$. A feasible solution, denoted by $\overline{\mathbf{x}}^*$ is referred to as an *optimal solution* if its objective value is maximum (or minimum, for a minimization problem) over all feasible solutions, and we call its objective value $\mathbf{c}^T\overline{\mathbf{x}}^*$ the *optimal objective value*.

Associated with every LP, there is another LP, which is referred to as the *dual* of the original LP. The original LP is also known as the *primal*. For example, for the

LP in the standard form shown in Expression 2.1, its dual is presented below.

$$\begin{aligned} &\text{minimize } \mathbf{b}^T\mathbf{y} \\ &\text{subject to } \mathbf{A}\mathbf{y} \geq \mathbf{c} \\ &\qquad\qquad\quad \mathbf{y} \geq \mathbf{0}, \end{aligned} \tag{2.2}$$

where $\mathbf{A}$, $\mathbf{b}$ and $\mathbf{c}$ represents the same thing as they do in the primal LP, and $\mathbf{y}$ is an m-vector. It follows that for a primal LP consisting of n variables and $(m+n)$ constraints, its dual consists of m variables and $(n+m)$ constraints, among which m constraints are non-negativity constraints. Consider $LP(G)$ given in 1.4 - 1.7 as an example, which is the LP relaxation for $ILP(G)$ as introduced in Chapter 1. Its corresponding dual LP, denoted by $LP_{dual}(G)$, is presented as follows:

$$\text{maximize} \qquad 2\textstyle\sum_{\emptyset \subset S \subset V} y_S, \tag{2.3}$$

$$\text{subject to} \qquad \textstyle\sum_{e \in \delta(S)} y_S \geq 1 \quad \text{for all } e \in E, \tag{2.4}$$

$$y_s \geq 0 \qquad \text{for all } \emptyset \subset S \subset V. \tag{2.5}$$

$LP_{dual}(G)$ takes all proper non-empty subset $\emptyset \subset S \subset V$ as decision variables, and consists of the maximization objective function shown in 2.3, a set of constraints associated with every $e \in G$ shown in 2.4, and a set of non-negativity constraints shown in 2.5.

The concept of *duality* provides a way to prove that a solution to the primal LP is indeed optimal. Thanks to von Neumann (1928)[BM08], Theorem 2.1.1 guarantees that one can always certify optimality of a primal solution by providing a dual solution which has the same objective value as the objective value of the primal solution.

Theorem 2.1.1 (**Duality Theorem** [BM08])**.** *If an LP has an optimal solution, then its dual also has an optimal solution, and the optimal values of these two LPs are equal.*

The *simplex algorithm* (also known as the *simplex method*), invented by George Dantzig in 1947, is one of most commonly-used algorithms for solving the LP. In 1972, Klee and Minty [KM72] have provided an example demonstrating that the worst-case complexity of the simplex algorithm is exponential, which leads to the fact that the simplex method is not a polynomial algorithm. However, despite the worst-case performance, the simplex algorithm performs very well in practice. [IC94].

Later in 1979, LP was proved to be possible to solve in polynomial time in the worst case by L.G. Khachian [Kha79] using the *ellipsoid algorithm*. To run the ellipsoid algorithm on the LP relaxation of 2EC, $LP(G)$, on some graph $G = (V, E)$, we need to be able to decide, given $x \in \mathbb{R}^E$, whether x satisfies all constraints of $LP(G)$. This decision problem is also known as the *separation problem*. It is known [BP90] that, since the separation problem for the cut constraints can be solved in polynomial time using the procedure given by Gomory and Hu [GH61], $LP(G)$ can be optimized in polynomial time by using the ellipsoid algorithm[2]. However, it is very slow in practice. Even though the ellipsoid algorithm is of significant theoretical importance, it does not appear to be competitive with the simplex algorithm in practice [CLRS09]. It also follows that there does not exist a polynomial-time algorithm for solving $LP(G)$ very efficiently.

A larger breakthrough was brought by N. Karmarkar [Kar84] in 1984 with the introduction of his revolutionary *interior-point method*. Even though Karmarkar's interior-point method was proved to be a polynomial-time algorithm and seemed to perform faster than ellipsoid algorithm, it still cannot beat the simplex method practically.

Similar to the LP, an *integer linear program* (henceforth *ILP*) has the same form as an LP but with the added restriction that the decision variables must be integer-valued. An ILP is called *0-1 integer program* or *binary integer program* (henceforth

[2]For a description of the general approach, please refer to [GLS81].

BIP), if all its variables are restricted to be either 0 or 1. Presented below is an ILP and a BIP in their standard forms respectively.

$$\begin{aligned} \text{maximize } & \mathbf{c}^T\mathbf{x} \\ \text{subject to } & \mathbf{A}\mathbf{x} \leq \mathbf{b} \\ & \mathbf{x} \geq \mathbf{0} \\ & \mathbf{x} \text{ integer}, \end{aligned} \tag{2.6}$$

$$\begin{aligned} \text{maximize } & \mathbf{c}^T\mathbf{x} \\ \text{subject to } & \mathbf{A}\mathbf{x} \leq \mathbf{b} \\ & \mathbf{x} \geq \mathbf{0} \\ & \mathbf{x} \in \{0,1\}. \end{aligned} \tag{2.7}$$

However, different from the LP, it is known that solving an ILP is NP-hard in general.[BW05] As a special case, the 0-1 integer program is one of *Karp's 21 NP-complete problems* [Kar72].

2.1.4 Integrality Gap

For an ILP, by dropping its integrality constraints we can obtain a related linear program for the ILP, which is called *linear programming relaxation* (henceforth *LP relaxation*). In particular, for a 0-1 integer program, one can obtain its *LP relaxation* by removing the constraints of $\mathbf{x} \in \{0,1\}$ and replacing them with the constraints of $\mathbf{x} \geq \mathbf{0}$ and $\mathbf{x} \leq \mathbf{1}$. The maximum ratio between the optimal objective value of the integer program and that of its LP relaxation is called the *integrality gap* of the LP relaxation.

The concept of LP relaxation is of significant importance for the reason that it transforms an NP-complete problem into a related problem that is often solvable in

polynomial time. The solution to the relaxed LP often gives an upper (or a lower) bound for the original maximization (or minimization) integer linear program.

2.2 Related Problems

Considering their particularly close connections to 2EC, the following three 2EC-related problems are introduced in this section: the *minimum size 2-edge-connected spanning multi-subgraph problem (M2EC)*, the *minimum cost 2-edge-connected spanning (multi-)subgraph problem (C2EC)*, and the *graph traveling salesman problem (graph TSP)*. A discussion on the relationship between 2EC along with M2EC, C2EC and graph TSP is given in Subsection 2.2.4.

2.2.1 The Minimum Size 2-edge-connected Spanning Multi-Subgraph Problem

Given an unweighted connected graph $G = (V, E)$, the *minimum size 2-edge-connected spanning multi-subgraph problem* (henceforth *M2EC*) consists of finding a 2-edge-connected spanning multi-subgraph H of G with minimum number of edges, where the *multi-subgraph* refers to a subgraph of G which allows multiple copies of an edge to be chosen. Formulated below is the ILP for the M2EC, which is denoted by $ILP^{M2EC}(G)$ for some graph $G = (V, E)$.

$$\begin{aligned}
&\text{minimize} && \textstyle\sum_{e \in E} x_e && , && (2.8)\\
&\text{subject to} && x(\delta(S)) \geq 2 && \text{for all} \quad \emptyset \subset S \subset V, && (2.9)\\
& && x_e \geq 0 && \text{for all} \quad e \in E, && (2.10)\\
& && x_e \quad \text{integer} && \text{for all} \quad e \in E. && (2.11)
\end{aligned}$$

Note that x_e is the decision variable of $ILP^{M2EC}(G)$ associated with every edge $e \in E$, which indicates the number of copies of edge e included in the solution. Similar to $ILP(G)$, the set of constraints shown in 2.9 is referred to as the *cut constraints*; differently, the set of constraints shown in 2.10 is known as the *non-negativity constraints*, and the set of constraints shown in 2.11 is called the *integrality constraints*. The LP relaxation of M2EC, denoted by $LP^{M2EC}(G)$ for a graph $G = (V, E)$, can be obtained by dropping the set of constraints shown in 2.11. We use $OPT^{M2EC}(G)$ and $OPT^{M2EC}_{LP}(G)$ to denote the optimal objective value of $ILP^{M2EC}(G)$ and $LP^{M2EC}(G)$ respectively. The integrality gap of M2EC, i.e., maximum $\frac{OPT^{M2EC}(G)}{OPT^{M2EC}_{LP}(G)}$ over all G, is denoted by α^{M2EC}.

It is obvious that a spanning subgraph H as the solution to M2EC contains two copies of each cut edge in G. However, for an edge $e \in E$ which is not a cut edge, it is not necessary to have it presented as parallel edges in the solution. That is to say, for M2EC, if the graph is bridgeless, there is always an optimal solution that does not use multiple edges. Lemma 2.2.1 proves that such an optimal solution of M2EC on a bridgeless graph always exists.

Lemma 2.2.1. *For any bridgeless graph $G = (V, E)$ and edge $e \in E$, there always exists an optimal solution to M2EC that does not contain multiple copies of e.*

Proof. Suppose, for contradiction, an optimal solution H to M2EC on $G = (V, E)$ contains two copies of some edge $e \in E$, denoted by e and e'. Then the following two cases shall be considered:

CASE 1. e and e' do not form a cut of H.

In this case, since e and e' do not form a cut of H, removing either of them will not change the 2-edge-connectivity of H. Hence the second copy of e can simply be deleted and H remains 2-edge-connected. It follows that $H \setminus e$ becomes a different solution but with one fewer edge than H, which contradicts the fact that H is an optimal solution.

CASE 2. e and e' form a cut of H.

Let $\delta(R) = \{e, e'\}, R \subset V$ be the cut that e and e' form, where R contains exactly one end of e and e'. Since G is bridgeless, there must exist another edge $f \in E$ that is not identical with e and has exactly one end belonging to R and the other one belonging to $\overline{R}$; because otherwise e will be a cut edge whose removal disconnects R and $\overline{R}$. By simply replacing e' by f, we can avoid the existence of double copies of e. The newly-generated spanning subgraph remains 2-edge-connected and has the same number of edges as H, which means it is also an optimal solution.

It follows that in a minimum size 2-edge-connected spanning subgraph of some bridgeless graph $G = (V, E)$, multiple copies of any edge $e \in E$ is always avoidable. Hence there always exists an optimal solution to M2EC that does not contain parallel edges for any bridgeless graph. □

It follows from Lemma 2.2.1 that for bridgeless graphs, 2EC and M2EC are equivalent, and thus $OPT(M2EC) = OPT(2EC)$.

M2EC is also known to be both NP-hard [SV12] and also MAX SNP-hard (even for cubic graphs) [CKK02]. The previous research and the obtained results reviewed in Section 1.3 for 2EC also apply to M2EC.

2.2.2 The Minimum Cost 2-edge-connected Spanning (Multi-)Subgraph Problem

Given a weighted complete graph $K_n = (V, E)$ on n vertices, let $c_e \in \mathcal{R}, c_e \neq 0$ be a set of costs associated with every edge $e \in E$. The *minimum cost 2-edge-connected spanning (multi-)subgraph problem* (henceforth *C2EC*) is that of finding a 2-edge-connected spanning multi-subgraph of K_n of minimum cost with respect to the costs on edges [ABEM06]. The problem C2EC on some graph K_n is referred to as *metric*

C2EC if the costs on edges in K_n satisfy the following triangle inequality:

$$c_{(x,y)} + c_{(y,z)} \geq c_{(x,z)} \text{ for any three vertices } x, y, z \in V. \tag{2.12}$$

For a solution to metric C2EC on some complete graph $K_n = (V, E)$, multiple copies of edges become unnecessary. This is because for every edge $e = (u, v) \in E$, c_e is no greater than the cost of the shortest path between u and v. We formulate C2EC with an ILP as follows, denoted by $ILP^{C2EC}(K_n)$ for some complete graph $K_n = (V, E)$ with n vertices.

$$\begin{aligned}
\text{minimize} \quad & \textstyle\sum_{e\in E} c_e x_e \quad , && (2.13)\\
\text{subject to} \quad & x(\delta(S)) \geq 2 \quad \text{for all} \quad \emptyset \subset S \subset V, && (2.14)\\
& x_e \geq 0 \quad \text{for all} \quad e \in E, && (2.15)\\
& x_e \quad \text{integer} \quad \text{for all} \quad e \in E. && (2.16)
\end{aligned}$$

In the above formulations, with 2.13 given as the objective function of C2EC, 2.14 represents all cut constraints, 2.15 gives the set of all non-negativity constraints, and 2.16 gives all integer constraints. By replacing constraints shown in 2.15 and 2.16 with

$$x_e \in \{1, 0\} \quad \text{for all} \quad e \in E,$$

the ILP formulation for metric C2EC is obtained, which is a binary integer program.

Similarly to $ILP(G)$, by dropping constraints in 2.16, the corresponding LP relaxation of $ILP^{C2EC}(K_n)$, denoted by $LP^{C2EC}(K_n)$, is obtained. For a complete graph $K_n = (V, E)$ with n vertices, $OPT^{C2EC}(K_n)$ and $OPT_{LP}^{C2EC}(K_n)$ are used to denote the optimal objective value of $ILP^{C2EC}(K_n)$ and $LP^{C2EC}(K_n)$ respectively. The integrality gap of the LP relaxation for C2EC, denoted by α^{C2EC}, is defined as

follows, which is over all K_n and over all cost functions.

$$\alpha^{C2EC} = \max_{\mathbf{c}>0} \frac{OPT(G)}{OPT_{LP}(G)} \text{ over all } G.$$

C2EC is also known to be NP-complete and MAX SNP-hard [CL99]. In 1982, Frederickson and Ja Ja [FJ82] first designed a $\frac{3}{2}$-approximation algorithm for the metric C2EC. Later in 1998, Jain [Jai01] presented a 2-approximation algorithm for generalized Steiner network problems for finding a minimum-cost spanning subgraph with "at least a specified number of edges in each cut" [Jai01], which includes the problem of finding a minimum cost k-edge-connected spanning subgraph, and thus includes C2EC.

Meanwhile, more research has been conducted considering the integrality gap of C2EC, i.e. α^{C2EC}. However, except that $\frac{6}{5} \leq \alpha^{C2EC} \leq \frac{3}{2}$ [ABEM06], not much is known so far. As integrality gap is defined for minimum 2-edge-connected spanning subgraph problem over all cost functions, α^{C2EC} also applies for $2EC$, which is a special case of C2EC with cost 1 on every edge in the graph G, and infinity on edges not in G. As mentioned in [CR98] and [ABEM06], a very useful and interesting result follows from a result of Cunningham [MMP90] and the *Parsimonious Property* presented by Goemans and Bertsimas [GB90] in 1990, indicating that when the costs on the edges satisfy the triangle inequality, there exists an optimal solution for $LP^{C2EC}(K_n)$[3] which is also feasible and hence optimal for $SEP(K_n)$[4] [ABEM06]. In addition, they also conjectured with supportive explanation that the α^{C2EC}, as the integrality gap of the C2EC, is $\frac{4}{3}$. It was also mentioned that a family of cost functions that asymptotically shows $\alpha^{C2EC} \geq \frac{6}{5}$ can be demonstrated in [CR98]. Later in 2002, Alexander, Boyd and Elliott-Magwood [ABEM06] confirmed $\alpha^{C2EC} \geq \frac{6}{5}$ asymptotically by illustrating a different family of cost functions. Exact value for the ratios

[3]refers to the LP relaxation defined by 2.13 - 2.15.

[4]refers to the LP relaxation for the traveling salesman problem defined by 2.18 - 2.21 in Subsection 2.2.3.

between $OPT^{C2EC}(K_n)$ and $OPT_{LP}^{C2EC}(K_n)$ for small values of n up to 10 and a tight lower bound for α^{C2EC} are also presented in [ABEM06].

2.2.3 The Graph Traveling Salesman Problem

Given a complete graph $K_n = (V, E)$ with a cost c_e associated with every edge $e \in E$, the *traveling salesman problem* (henceforth *TSP*) consists of finding a Hamilton cycle of K_n of minimum cost. A systematic and precise introduction and literature reviews on this problem and its variations can be found in [LLKS85]. Note that in this thesis only symmetric traveling salesman is considered, where no order for the two ends of any edge in the underlying graph is defined.

As one of the most famous and fundamental problems in combinatorial optimization, TSP has been intensively studied for decades, especially for a special case called the *metric traveling salesman problem* (henceforth *metric TSP*), in which the triangle inequality 2.12 is added as an additional condition on the input graphs. That is to say, given a weighted complete graph $K_n = (V, E)$ with costs associated with all edges which satisfy inequality 2.12, metric TSP is to find a minimum-cost Hamilton cycle of K_n [BM08].

As a relaxation of metric TSP, the graph traveling salesman problem is considerably related to 2EC. Given an unweighted connected graph $G = (V, E)$, defining the cost between two vertices as the number of edges on the shortest path between them yields a complete weighted graph on V, which is called the *the metric completion of* G [BSSS12]. The *graph traveling salesman problem* (henceforth *graph TSP*), is to find a Hamilton cycle of minimum cost in the metric completion of a given unweighted connected graph G. It is equivalent to formulate this problem as finding a spanning Eulerian[5] multi-subgraph $H = (V, E')$ in a given unweighted graph $G = (V, E)$ with the minimum number of edges.

[5] An *Eulerian* graph is a connected graph $G = (V, E)$ in which the degree of every vertex $v \in V$ is even.

As Metric TSP is known to be $\mathcal{NP}$-hard and MAX SNP-hard [LLKS85], even for graph TSP [PY93], a huge amount of research has been done, striving for better approximation algorithms for this problem [6]. Although no improvements have been discovered for more than three decades on the $\frac{3}{2}$-approximation algorithm described by Christofides [Chr76] for the metric, some progress for graph TSP has been given very recently. In 2005, focusing on graph TSP on 3-edge-connected cubic graphs, Gamarnik, Lewenstein and Maxim [GLS05] first improved the approximation ratio from $\frac{3}{2}$ to $(\frac{3}{2}-\frac{5}{389})$ (approximately 1.487). It was followed by Boyd, Sitters, van der Ster and Stougie [BSSS12] with their $\frac{4}{3}$-approximation algorithm for all cubic graphs by using polyhedral techniques. Other previous studies conducted for graph TSP on special classes of graphs can also be found in [BSSS12]. Around the same time, Mömke and Svensson [MS11] improved the approximation ratio for graph-TSP on general graphs to 1.461 based on a polyhedral idea, along with a 4/3-approximation algorithm for this problem on subcubic graphs as a side result. In 2012, based on an improved analysis of the approach presented in [MS11], Mucha [Muc12] gave a better lower bound on the approximation ratio to $\frac{13}{9}$ (approximately 1.444) for graph TSP on general graphs. This ratio was later improved by Sebö and Vygen [SV12] with a $\frac{7}{5}$-approximation algorithm.

2.2.4 Relationship between M2EC, C2EC, Graph TSP and 2EC

The relationship between 2EC, M2EC and C2EC is pretty straight-forward. Among all these three problems, C2EC is considered as the generalized form of the former two. The problem M2EC can be considered as a special case of C2EC in a way that an input graph $G = (V, E)$ for M2EC holds the cost 1 for all edges shown in G, while

[6] In general, the TSP cannot be approximated in polynomial time to within any constant unless P = NP [BSSS12]

the cost ∞ for all edges not shown in G. As an even more relaxed version, 2EC requires that an input graph does not only hold such cost functions, but also features 2-edge-connectivity (i.e. an input graph is required to be bridgeless). It leads to

$$2EC \subseteq M2EC \subseteq C2EC, \tag{2.17}$$

which indicates that any algorithms for either C2EC or M2EC applies for 2EC. This relationship also has significant meaning on the investigation of the integrality gap of the LP relaxation for 2EC, to which the integrality gap of the LP relaxation for C2EC applies.

On the other hand, with 2EC being a relaxation of the graph TSP, any optimal solution of graph TSP in a bridgeless graph G leads to a feasible solution to 2EC on G with at most the same number of edges [SV12]. Consider the LP relaxation for C2EC given in 2.13 - 2.15. By changing those cut constraints in 1.5 to equality for all $S \subset V$ where S contains exactly one vertex, we obtain the following LP relaxation for TSP, which is also known as the *subtour relaxation* [CR98] or *subtour elimination problem*.

$$\begin{aligned}
\text{minimize} \quad & \textstyle\sum_{e\in E} c_e x_e \quad , && \qquad (2.18)\\
\text{subject to} \quad & x(\delta(v)) = 2 \quad \text{for all} \quad v \in V, && \qquad (2.19)\\
& x(\delta(S)) \geq 2 \quad \text{for all} \quad \emptyset \subset S \subset V, && \qquad (2.20)\\
& x_e \geq 0 \quad \text{for all} \quad e \in E. && \qquad (2.21)
\end{aligned}$$

It leads to the other important connection between graph TSP and 2EC that lies on the well-known $\frac{4}{3}$ *TSP conjecture* presented as follows:

Conjecture 2.2.2. *The integrality gap of the subtour relaxation for metric TSP is at most* $\frac{4}{3}$.

Following from Conjecture 2.2.2, in 1998, Carr and Ravi [CR98] gives a related conjecture based on Conjecture 2.2.2 as follows:

Conjecture 2.2.3. *[CR98] The minimum cost of a 2-edge connected subgraph is within $\frac{4}{3}$ times the cost of the optimal subtour solution for the TSP.*

Conjecture 2.2.3 suggests that the integrality gap of the LP relaxation for metric C2EC is at most $\frac{4}{3}$. With Conjecture 2.2.2 implying Conjecture 2.2.3, both remain unsolved. However, it is suggested [CSS98] that if Conjecture 2.2.3 holds, the integrality gap of the LP relaxation for 2EC is at most $\frac{4}{3}$. Therefore, the $\frac{4}{3}$-approximation algorithm given by Vampala and Vetta [VV00], and the $\frac{4}{3}$-approximation algorithm given by Sebö and Vygen [SV12], are of significant importance for lending support to Conjecture 2.2.3, and thus to the $\frac{4}{3}$ TSP conjecture.

In this chapter, with basic knowledge given for graphs, computational complexity, linear program and integer linear program, and integrality gap for this thesis, we presented the formal definition for the 2-edge-connected spanning subgraph problem on both unweighted and weighted graphs. The following cognition can be summarized with the knowledge gained through Chapters 1 and 2.

1. For 2EC, as well as M2EC, the best known approximation ratio with detailed proof is $\frac{4}{3}$ [VV00, SV12] for general graphs, $\frac{6}{5}$ for 3-edge-connected cubic graphs [BIT13], and $\frac{5}{4} + \epsilon$ for any fixed $\epsilon > 0$ for maximum degree 3 graphs. The other interesting finding worth our attention is that Csaba et al. stated that the worst known lower bound on the integrality gap for 2EC is $\frac{10}{9}$ on subcubic graphs, which is later improved to $\frac{9}{8}$ by Sebö and Vygen [SV12]. Thus the special family of graphs, which is derived from our computational study and gives the ratio of $\frac{8}{7}$ between $OPT(G)$ and $OPT_{LP}(G)$, is of some significance.

2. For C2EC, the best known approximation algorithms gives approximation ratio

of 2 [KV94, Jai01] for general and $\frac{3}{2}$ for the metric case [FJ82]. It is also known that the integrality gap for C2EC falls in the range between $\frac{6}{5}$ and $\frac{3}{2}$.

3. For graph TSP, the best-known approximation algorithm has the approximation ratio of $\frac{7}{5}$, due to Sebö and Vygen [SV12].

Chapter 3

A 5/4-Approximation Algorithm for 2EC on Bridgeless Cubic Graphs

In this chapter, our algorithm is based on the $\frac{6}{5}$ Algorithm Apx2EC for 2EC on 3-edge-connected cubic graphs [BIT13] due to Boyd, Iwata and Takazawa. In Section 3.1, we give an overview of the algorithm Apx2EC. In Section 3.2, we discuss the reasons that Apx2EC cannot be used to obtain a $\frac{6}{5}$-approximation algorithm for 2EC for cubic graphs with 2-edge cuts. In section 3.3, we show it is possible to overcome these problems and obtain a $\frac{5}{4}$-approximation. We present a $\frac{5}{4}$-approximation algorithm for 2EC on any bridgeless cubic graphs. The previous best approximation ratio known for 2EC for this class of graph was $\frac{5}{4} + \epsilon$ for any $\epsilon > 0$, given by Csaba, Karpinski and Krysta.

Following the $\frac{5}{4}$-approximation algorithm demonstrated in Section 3.3 for bridgeless cubic graphs, we prove that the integrality gap of the LP relaxation for 2EC on bridgeless cubic graphs is at most $\frac{5}{4}$.
Given below are some common definitions widely used in this chapter.

Definition Recall that in a graph $G = (V, E)$, a *Hamilton path* is a path in G that covers all vertices in G; whereas a *Hamilton cycle* is a cycle that contains all vertices in G. For a Hamilton path, its two ends are referred to as the *initial vertex* and the *terminal vertex* respectively. A *chord* of a cycle C in a graph $G = (V, E)$ is an edge in $E \setminus E(C)$, where both of its ends lie on C.

3.1 Review of a 6/5-approximation Algorithm for 2EC on Cubic 3-edge-connected Graphs [BIT13]

Taking a cubic 3-edge-connected graph G with n vertices as an input, S. Boyd, S. Iwata and K. Takazawa designed an approximation algorithm for 2EC on G with a performance guarantee of $\frac{6}{5}n$, which is referred to as Algorithm APX2EC. In this section, we present a detailed review on this algorithm prior to the introduction of our $\frac{5}{4}$-approximation algorithm adapted from APX2EC for solving 2EC on bridgeless cubic graphs, i.e. cubic 2-edge-connected graphs.

For a cubic 3-edge-connected graph $G = (V, E)$ with n vertices, the first step of Algorithm APX2EC is to use an algorithm called 34CUT [BIT13] to obtain a 2-factor F of G covering all the 3- and 4-edge cuts. Note that a 2-factor F of a graph G that covers all 3-edge cuts in G includes a pair of edges of every 3-edge cut in one of the cycles of F. That is to say, such a 2-factor of any bridgeless cubic graph does not contain any cycle consisting of three vertices, i.e. triangles. A 2-factor F of a graph G that covers all 4-edge cuts in G contains a pair of edges of every 4-edge cut in one of the cycles of F and the other pair of edges in a different cycle of F. This suggests such a 2-factor of any 3-edge-connected cubic graph does not contain any cycle consisting of 4 vertices, i.e. squares. Theorem 3.1.1[BIT13] proves that such a 2-factor exists in any bridgeless cubic graphs.

Theorem 3.1.1 (THEOREM 5.4 [BIT13]). *Given any bridgeless cubic graph G, Al-*

gorithm 34CUT *finds a 2-factor in* G *covering all the 3- and 4-edge cuts and not containing a specified edge* $e^* \in E$ *in* $O(n^3)$ *time.*

The edges in $E \setminus F$ are called the *matching edges.* Since G is 3-edge-connected, which means it does not contain any 2-edge cut, the structures shown in Figure 3.1 (a) and (b) do not exist in G. Moreover, with all 3- and 4-edge cuts covered, triangles and squares obtained from the structures shown in Figure 3.1 (c) and (d) are avoided in F. Therefore any cycle in F must contain at least 5 vertices. Denote the family of

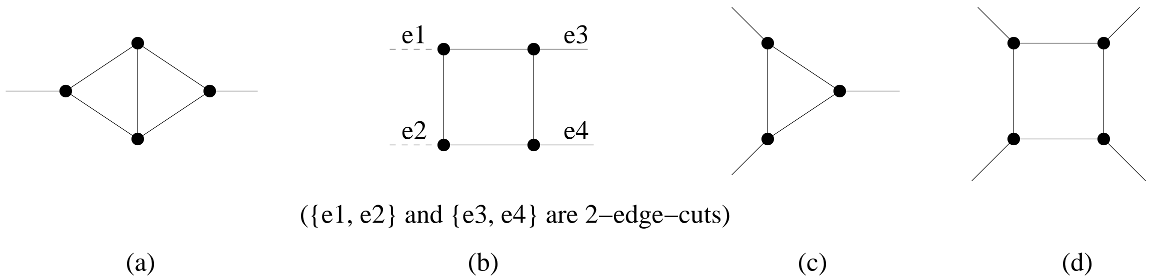


FIGURE 3.1: Eliminated structures in Algorithm APX2EC.

cycles in F by $\mathcal{C}$, if any cycle $C \in \mathcal{C}$ contains at most 9 vertices, C is called a *small cycle*, otherwise it is called a *large cycle*. Observe that all small cycles C have at most two chords in $G[V(C)]$ unless C contains all vertices in G, otherwise $\delta(V(C))$ is a 3- or 4-edge cut covered by F. With Lemma 3.1.2 below applied, this first step guarantees that any small cycle in F is traversable by a Hamilton path starting with any vertex that is not incident to any chord and terminating at another vertex which is also not incident to any chord.

Lemma 3.1.2 (Lemma 6.1 [BIT13])**.** *Let* $G = (V, E)$ *be a 2-edge-connected graph and* C *be a cycle in* G *with at most two chords. Let* $V^* \subseteq V(C)$ *be the set of vertices not incident with the chords. For a vertex* $v^* \in V^*$*, there is a Hamilton path in* $G[V(C)]$ *starting at* v^* *and ending at some vertex* $u^* \in V^*$*.*

With a 2-edge-connected subgraph H with the property $|E(H)| \leq \frac{6}{5}|V(H)|$ maintained all through the algorithm, the construction of the 2-edge-connected spanning

subgraph of G starts with initializing H with an arbitrary cycle $C_0 \in \mathcal{C}$, and terminates when $V(H) = V$, which is the result of a series of expansion operations in which H is augmented by adding another subgraph H'. Every expansion starts with choosing an arbitrary matching edge $e = (u, v) \in \delta(H)$, where $u \in V(H)$ and $v \in V \setminus V(H)$, and initializing $H' := (u, \emptyset)$. Let $C \in \mathcal{C}$ denotes the cycle containing v. Note that initially C is not contained in H. Given $e = (u, v)$ and C, repeat the following different procedures for enlarging H' apply according to the following four cases until $v \in H$ is reached. Once $v \in H$ is reached, the current expansion is finished and H is augmented by H' and e, i.e., $H := (V(H) \cup V(H'), E(H) \cup E(H') \cup \{e\})$.

Case 1. If C is a large cycle that is not contained in H' yet, e and C are added to H'. Let $f \in \delta(H') \setminus e$ and let C_{new} be the cycle containing the other end of f. Update $e := f$ and $C := C_{new}$.

Case 2. If C is a small cycle that is not contained in H' yet, edge e and the Hamilton path P of $G[(V)]$, that starts at v and ends at some other vertex w which is not incident to any chord, are added to H'. The existence of such a Hamilton path is proved by Lemma 3.1.2. We refer to the matching edge which is incident to w as the *leaving edge* of P. Update e to be the leaving edge $(w, z) \in \delta(C)$, and C to be the cycle containing z.

Case 3. If C is a large cycle that has already been included in H', a *lollipop* L, which is defined as "a subgraph consisting of the large cycle C and the elements of H' added after C was added to H'" [BIT13], is constructed, see Figure 3.2 for an illustration. It is followed by updates to both edge e and cycle C, as $e := f \in \delta(L) \setminus E(H')$ and $C := C_{new}$, where C_{new} is the cycle containing the other end of f.

Case 4. If C is a small cycle that has already been included in H', a *tadpole*, which is defined as "a subgraph consisting of the Hamilton path P derived from C plus

the elements of H' added after P was added to H'" [BIT13], is constructed, see Figure 3.3 for an illustration. Let v_C and w_C be the initial and terminal vertices of P. The *head of a tadpole* T "consists of the subgraph of P connecting v and w_C, and the elements of T added to H' after P was added to H'" [BIT13]; while the *tail of a tadpole* H' is "a subgraph of P, the path connecting v_C and V" [BIT13]. The growth of H' is continued from a matching edge connecting the head of T to $V \setminus V(T)$. The existence of such a matching edge is proved in the Lemma 3.1.3 [BIT13].

Lemma 3.1.3 (Lemma 6.2 [BIT13])**.** *For* Case *4, there exists an edge connecting the head of the tadpole T and $V \setminus V(T)$.*

Proof. Suppose, for contradiction, there is no edge connecting the head of the tadpole T and $V \setminus V(T)$. Let $C \in \mathcal{C}$ be the cycle that provides the tail of T. It follows from the assumption that there is no edge between $V \setminus V(T)$ and $V(T) \setminus V(C)$. Given that G is a 3-edge-connected cubic graph and the 2-factor F of G resulted from Algorithm 34Cut covers all 3- and 4-edge cuts, therefore there are at least five matching edges between $V(C)$ and $V \setminus V(T)$ and at least another five matching edges between $V(C)$ and $V(T) \setminus V(C)$. It follows that C contains at least 10 vertices, which contradicts the fact that C is a small cycle. □

Note that in the Case 3 and Case 4, it is possible that C is already included in a lollipop or a tadpole in H'. In this algorithm, a cycle in $\mathcal{C}$ which does not belong to any lollipop and tadpole is referred to as an *independent cycle* [BIT13]. If C is neither in H nor independent, a larger lollipop $\hat{L}$ is constructed if C is contained in a lollipop L, which consists of L and all the elements of H' added after the construction of L; while a larger tadpole $\hat{T}$ is constructed if C is contained in a tadpole T.

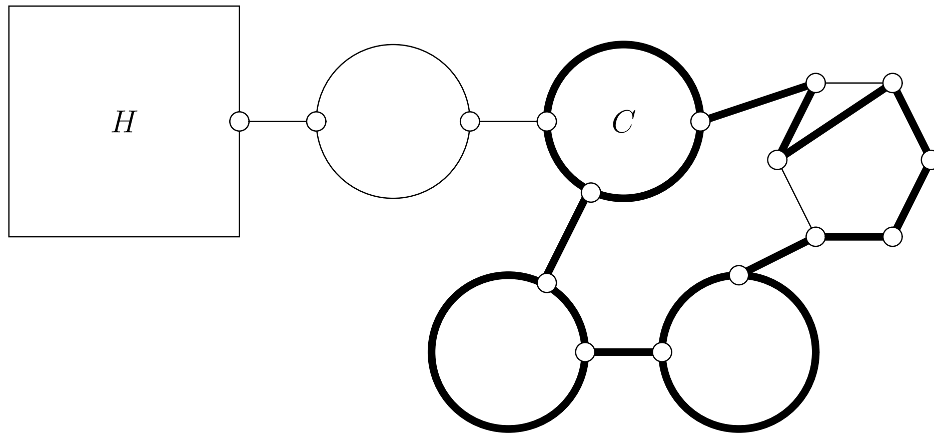


FIGURE 3.2: Construction of a lollipop L. The thick edges are the edges in L. The hexagon is a small cycle of size 6, and the four circles indicate large cycles. (Some vertices and edges are omitted.) (FIG. 6.1. [BIT13]).

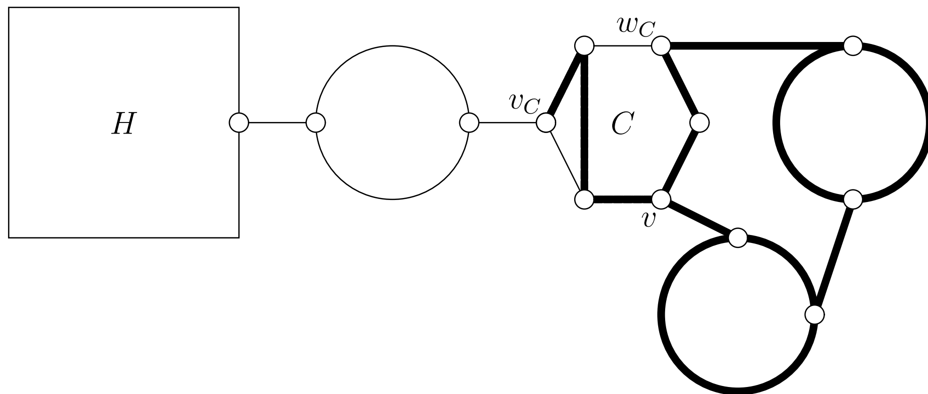


FIGURE 3.3: Construction of a tadpole T. The thick edges are the edges in T. The hexagon is a small cycle of size 6, and the four circles indicate large cycles. (Some vertices and edges are omitted.) The thick edges connecting v_c and v form the tail of T, and other thick edges form the head of T. (FIG. 6.2. [BIT13]).

Since in Algorithm Apx2EC, only inclusion-wise maximal lollipops and tadpoles are handled, a list of lollipops and tadpoles is kept along the process of constructing the 2-edge-connected spanning subgraph. Whenever a larger lollipop $\hat{L}$ (or a larger tadpole $\hat{T}$) is built, all lollipops and tadpoles contained in $\hat{L}$ (or $\hat{T}$) are removed from the list.

Theorem 3.1.4 (Lemma 6.4 [BIT13])**.** *Algorithm* Apx2EC *is a* $\frac{6}{5}$*-approximation algorithm for* 2*EC in 3-edge-connected cubic graphs. More precisely, for a 3-edge-connected cubic graph* $G = (V, E)$*, Algorithm* Apx2EC *finds a 2-edge-connected subgraph* H *of* G *with* $E(H) \leq \frac{6}{5}n - 1$ *in* $O(n^3)$ *time.*

Corollary 3.1.5 (Corollary 6.5 [BIT13])**.** *For a 3-edge-connected cubic graph* $G = (V, E)$ *and two adjacent edges* $e_1, e_2 \in E$*, one can find a 2-edge-connected subgraph* H *of* G *with* $E(H) \leq \frac{6}{5}n - 1$ *and* $\{e_1, e_2\} \subseteq E(H)$ *in* $O(n^3)$ *time.*

From Theorem 3.1.4 and Corollary 3.1.5, it follows that Algorithm Apx2EC produces a 2-edge-connected spanning subgraph for 3-edge-connected cubic graphs with no more than $\frac{6}{5}n - 1$ edges, which contains a specified pair of adjacent edges.

3.2 Why Apx2EC Does Not Work for Bridgeless Cubic Graphs

With the knowledge that Algorithm Apx2EC only applies to 3-edge-connected cubic graphs, we investigate the problems encountered when applying Apx2EC to bridgeless cubic graphs, i.e., 2-edge-connected cubic graphs.

An obvious difference between 3-edge-connected cubic graphs and bridgeless cubic graphs is that there exist 2-edge cuts in the latter. For any bridgeless cubic graph $G = (V, E)$, after Algorithm 34Cut is applied to G, the existence of the 2-edge cuts in G may lead to the follow two problems.

1. **Existence of small cut cycles in $\mathcal{C}$.**

   Recall that $\mathcal{C}$ is used to denote the family of cycles in the 2-factor F, where F is the result of applying Algorithm 34CUT to G. A cycle $C \subset \mathcal{C}$ is referred to as a *cut cycle* if the removal of C leaves the rest of G with more than one component. More precisely, for any cycle $C \subset \mathcal{C}$, if $G - G[V(C)]$ has more than one component, C is considered as a cut cycle. Figure 3.4 illustrates an example of a small cut cycle that contains 6 vertices and 1 chord.

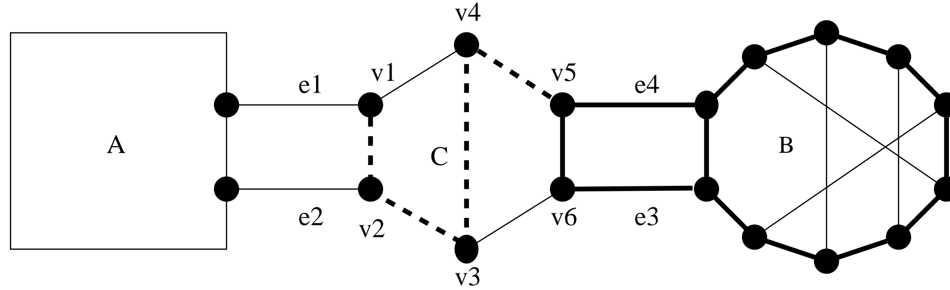


FIGURE 3.4: A small cut cycle C of size 6, whose removal separate the graph into two components A and B, where B is a large cycle with 10 vertices.

   The problem with cut cycles for graphs with 2-edge cuts arises when a cut cycle in $C \in \mathcal{C}$ is a small one. The proof of Lemma 3.1.3 depends on the fact that all cuts arising from a cut cycle must have size at least 5 (i.e. contain at least five edges), and thus no small cycle can be a cut cycle in a 3-edge-connected graph in Algorithm APX2EC. However, if G is only 2-edge-connected, small cut cycles can exist, which may invalidate Lemma 3.1.3 and the construction process of some subgraph H'. Consider Figure 3.4 as an example, in which C is a small cut cycle with 6 vertices whose removal disconnects G into two components, A and B. Suppose A is H, after edge $e1$ is chosen to start the construction of H', Algorithm APX2EC chooses "$v1 - v2 - v3 - v4 - v5 - v6$" as the Hamilton path to traverse the small cut cycle C that consists of six vertices, and continues the construction of H' from the matching edge $e3$. Denote the cycle reached from $e3$ by C_B. Since C_B contains 10 vertices, which means C_B is considered as a

large cycle, it will be added into H' according to Case 1 in Algorithm Apx2EC. At this point, $e4$ becomes the only choice to continue the construction of H', from which C is reached again. It follows with the construction of a tadpole T with its head shown with thick solid edges and its tail shown with thick dotted edges. It becomes easy to see that there does not exist any edge connecting the head of T and $V \setminus V(T)$. In this case, Algorithm Apx2EC cannot be applied.

2. **Existence of 4-cycles in $\mathcal{C}$**

   The existence of 2-edge cuts in a bridgeless cubic graph G may also lead to the existence of 4-cycles in $\mathcal{C}$, which is the family of cycles in a 2-factor covering all 3- and 4-edge cuts in G, which is avoided in Algorithm Apx2EC with all 4-edge cuts covered. However, with 2-edge cuts existing in G, the existence of structures shown in Figure 3.1 (a) and (b) becomes possible. The structure shown in Figure 3.1 (a), which is referred to as a *diamond structure*, may lead to a 4-cycle in $\mathcal{C}$, while the structure shown in Figure 3.1 (b), which is a 4-cycle joined by two 2-edge cuts, may not only cause a 4-cycle but may also cause a small cut cycle in $\mathcal{C}$. Note that a diamond structure contains a cycle of four vertices with one chord, and the two matching edges in $\delta(V(C))$ form a 2-edge cut.

   The reason that we are concerned about the existence of 4-cycles is that in the proof for Theorem 3.1.4, the definition of small and large cycles was applied to obtain the relationship between numbers of small cycles as well as large cycles in $\mathcal{C}$ and the number of vertices in the input 3-edge-connected cubic graph $G = (V, E)$, i.e. $|V| \geq 5|\mathcal{S}| + 10|\mathcal{L}|$, where $\mathcal{S} \subseteq \mathcal{C}$ denotes the set of small cycles and $\mathcal{L} \subseteq \mathcal{C}$ denotes the set of large cycles. However, this relationship does not apply any longer with the existence of 4-cycles, and thus the definition of small cycles is required to be modified.

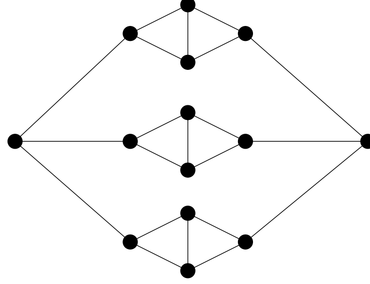


FIGURE 3.5: A bridgeless cubic graph G illustrating that it is not always possible to find a 2-factor that covers all 2-edge cuts.There are three 2-edge cuts in G, while only two of them can be covered by any 2-factor of G to a maximum extent.

Considering the above two problems in applying Algorithm APX2EC to bridgeless cubic graphs, as well as the fact that it is not always possible to find a 2-factor that covers all 2-edge cuts (see Figure 3.5 for a counter example), in extending Algorithm APX2EC to bridgeless cubic graphs, the concept of small cycles and large cycles is redefined to compromise with 4-cycles, and two new strategies are applied during the construction of tadpoles to make our algorithm compatible with the existence of small cut cycles. With these ideas realized, we present a $\frac{5}{4}$-approximation algorithm for 2EC on bridgeless cubic graphs in the upcoming section.

3.3 A $5/4$-approximation Algorithm for 2EC on Bridgeless Cubic Graphs

As inspired by Algorithm APX2EC[BIT13], the general idea of our algorithm for 2EC on a bridgeless cubic graph $G = (V, E)$ is also to build the 2-edge-connected spanning subgraph of G based on a 2-factor of the graph G covering all 3- and 4-edge cuts obtained by Algorithm 34CUT [BIT13], which applies to all bridgeless cubic graphs. As before, we use F to denote the 2-factor of G that covers all 3- and 4-edge cuts obtained from Algorithm 34CUT [BIT13], and $\mathcal{C}$ is used to denote the family of cycles in F. Note that since F covers all 3-edge cuts in G, it follows that triangles do not

exist in $\mathcal{C}$, and also considering that loops and parallel edges are not considered in this chapter, the minimum-possible cycles in $\mathcal{C}$ contain 4 vertices.

In order to solve the two problems stated in Section 3.2, the following modifications are applied in the design of our algorithm, which is referred to as $\frac{5}{4}$-APX2EC.

CHANGE 1. In Algorithm $\frac{5}{4}$-APX2EC, a cycle $C \subset \mathcal{C}$ is referred to as a *small cycle* if $4 \leq |V(C)| \leq 7$; and it is called as a *large cycle* if $|V(C)| \geq 8$.

CHANGE 2. When a small cut cycle is first encountered during the construction of some subgraph H', certain constraints for selecting a proper Hamilton path for a certain initial vertex are added in Algorithm $\frac{5}{4}$-APX2EC.

CHANGE 3. When a small cut cycle $C \in \mathcal{C}$ is reached again during the construction of some subgraph H', instead of certainly building a tadpole as in Algorithm APX2EC, Algorithm $\frac{5}{4}$-APX2EC may choose to build a lollipop rather than a tadpole if the vertex, from which the small cycle is reached again, is incident to the terminal vertex of the Hamilton path of $G[V(C)]$ in the subgraph H' under construction.

With a 2-factor obtained which covers all the 3- and 4-edge cuts while excluding a specified edge $e^* \in E$, the construction of the 2-edge-connected spanning subgraph in Algorithm $\frac{5}{4}$-APX2EC starts with an arbitrary cycle $C_0 \in \mathcal{C}$. During the whole process, we maintain a subgraph H satisfying the following properties:

1. H is 2-edge-connected;
2. V(H) is the union of the vertex set of a subfamily of the cycles in $\mathcal{C}$;
3. $|E(H)| \leq \frac{5}{4}|V(H)|$.

Initially, let $H := C_0$, and then H is repeatedly augmented by adding another subgraph H'. Algorithm $\frac{5}{4}$-APX2EC terminates when $V(H) = V$.

We start to build H' by initializing $H' := (u, \emptyset)$, where $u \in V(H)$ is an arbitrary vertex on H. Since G is a cubic graph, there exists an edge $e = uv \in \delta(H)$, where

$u \in V(H)$ is the arbitrary vertex we chose to initialize H' and $v \in V \setminus V(H)$. Apparently, v belongs to a cycle, denoted by C, where $C \subset \mathcal{C}$ is not contained in H. At this point, according to whether C is a large cycle or a small one, we handle C differently.

1. **C is A Large Cycle.**

   If C is a large cycle, then we add e and C into H', i.e. we let $H' := (V(H') \cup V(C), V(H') \cup e \cup E(C))$. Afterwards, redefine $e := f$, where f is an arbitrary matching edge other than $e \in \delta(C)$, and redefine C as the cycle we reach at the other end of f. We continue to grow H' from the newly-defined e and C.

2. **C is A Small Cycle.**

   If C is a small cycle, things become more complicated. With the definition for small cycles and large cycles modified, it becomes necessary to verify whether Lemma 3.1.2 still applies for any small cycle $C \in \mathcal{C}$ such that a Hamilton path does exist in $G[V(C)]$ starting at any vertex on C.

   Lemma 3.3.1. *For any small cycle $C \in \mathcal{C}$, C has at most two chords.*

   Proof. Following the definition of small cycles, a small cycle $C \in \mathcal{C}$ can only be of size 4, 5, 6 and 7.

   If $|V(C)| = 4$, C can have at most one chord, otherwise G is not connected.

   If $|V(C)| = 5$, C cannot have any chord, otherwise either F doesn't cover a 3-edge cut or G has a cut cycle.

   If $|V(C)| = 6$, C can have at most two chords, otherwise G is not connected.

   If $|V(C)| = 7$, C can have at most one chord, otherwise either F doesn't cover a 3-edge cut or G has a cut edge.

   In conclusion, any small cycle $C \in \mathcal{C}$ can have at most two chords. □

   Lemma 3.3.1 implies that any small cycle in a 2-factor covering all 3- and 4-edge cuts in graph G has at most two chords. According to Lemma 3.1.2[BIT13],

for any small cycle C in the 2-factor F of G, no matter from which vertex v we enter the cycle C, there is always a Hamilton path of $G[V(C)]$ that takes v as the initial vertex and terminates at a vertex incident to no chord.

With the existence of such a Hamilton path of $G[V(C)]$ proved, different operations are applied to C, which is a small cycle, depending on whether C is a cut cycle or not.

If C is not a cut cycle, an arbitrary Hamilton path P of $G[V(C)]$, that takes v as the initial vertex and terminates at some vertex not incident to any chord, is chosen. Suppose the terminal vertex of P is w, and the matching edge incident to w is f. Add e and P into H', and continue the growth of H' using $e := f$ and using the cycle containing the other end of f as C.

If C is a cut cycle, instead of choosing any Hamilton cycle starting from v and ending at a vertex incident to no chord, it becomes necessary to select a certain Hamilton cycle if multiple options do exist, which is the second modification stated above. For this purpose, concepts of *backward edges* and *forward edges* are introduced concerning a small cut cycle, denoted by C_{cut}^S. Let the component in $G - C_{cut}^S$ that contains C_0 be denoted by S_B, while the rest of $G - C_{cut}^S$ is denoted as S_F. An edge $e \in \delta(C_{cut}^S)$ is labeled as a *backward edge* if e has exactly one end in C_{cut}^S and the other end in S_B, and a *forward edge* if e has exactly one end in C_{cut}^S and the other end in S_F. Lemma 3.3.2 below proves some important characteristics of a small cut cycle in $C_{cut}^S \in \mathcal{C}$.

Lemma 3.3.2. *A small cut cycle $C \in \mathcal{C}$ has the following properties:*

(a) $G - C$ consists of either 2 or 3 components;

(b) $|V(C)| = 7$ or $|V(C)| = 6$ or $|V(C)| = 4$;

(c) Only when $|V(C)| = 6$, C can have one chord, otherwise, C has no chord.

(d) The number of forwarded edges for C is 2 or 4 or 5.

Proof. Since G is 2-edge-connected and F covers all the 3- and 4-edge cuts, each component in $G - C$ is connected to C by either 2 edges or at least 5 edges in G.

As C is a small cycle where $|V(C)| \leq 7$, the number of components in $G - C$ is at most 3. Considering C is also a cut cycle, the number of components in $G - C$ is either 2 or 3. Next, we list all possible conditions for such a small cycle in G.

- 7-cycles:

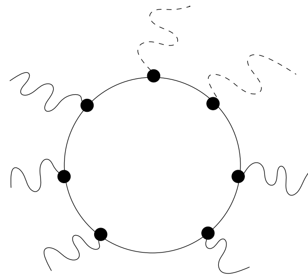


FIGURE 3.6: A cut cycle of order 7 (Case I).

  When $|V(C)| = 7$, $G - C$ can only have 2 components and no chord. The number of the forwarded edges is either 2 or 5. (As shown in Figure 3.6)

- 6-cycles:

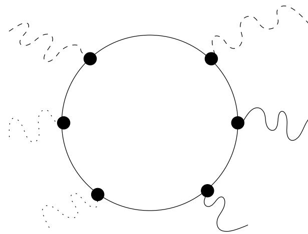


FIGURE 3.7: A cut cycle of order 6 with no chord (Case II)

  When $|V(C)| = 6$ and C has no chord, $G - C$ has 3 components. The number of its forwarded edges is either 2 or 4. (As shown in Figure 3.7)

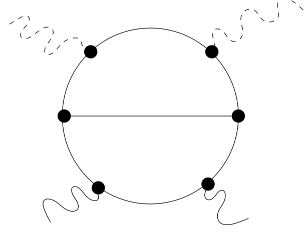


FIGURE 3.8: A cut cycle of order 6 with one chord (Case III).

When $|V(C)| = 6$ and C has one chord, $G - C$ has 2 components. The number of the forwarded edges is 2. (As shown in Figure 3.8)

- 5-cycles:

  Since the 2-factor F we obtained in Step 1 covers all 3- and 4-edge cuts, a cycle with the order of 5 cannot be a cut cycle. Otherwise G will have a cut edge, which contradicts the fact that G is a bridgeless cubic graph.

- 4-cycles:

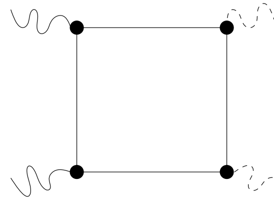


FIGURE 3.9: A cut cycle of order 4 (Case IV).

When $|V(C)| = 4$, $G - C$ can only have 2 components and no chord. The number of its forwarded edges is 2. (As shown in Figure 3.9)

In Conclusion, Lemma 3.3.2 is proved. □

With the knowledge of Lemma 3.3.2, a Hamilton path on $G[V(C)]$ when C is a small cut cycle is chosen in the following way. If there is only one Hamilton path of $G[V(C)]$ that starts at v and ends at a vertex not incident to any chord, this particular Hamilton path, which is referred to as P, is selected. Let w denote terminal vertex of P, and f is used to denote the matching edge

incident to w. Otherwise, if multiple choices do exist for such a Hamilton path, the chosen Hamilton path P must be selected as follows. If there is a Hamilton path P whose leaving edge is a backward edge, this Hamilton path P with a backward leaving edge is chosen. Otherwise, we will choose the Hamilton path P of $G[V(C)]$ whose leaving edge is the closest to a backward edge of C other than e.

With the Hamilton path P chosen, let $H' := (V(H') \cup V(C), E(H') \cup e \cup E(P))$, and continue the growth of H' with the leaving edge f of P. Redefine $e := f$ and C as the cycle containing the other end of f.

Note that in order to identify a small cut cycle in linear time, a list of small cut cycles is generated once the 2-factor F is obtained. This can be done simply by temporarily removing every small cycle and see whether its removal disconnects the graph G. For each small cut cycle in the list, the Hamilton path chosen for it is recorded and maintained in the list as well.

We shall only face the above two situations at the very beginning of the construction of H'. More accurately, the above two procedures are only applied when C is not contained in H or H'.

If we reach a cycle C which is already contained in H, the growth of H' is completed and we augment H with H', i.e. $H := (V(H) \cup V(H'), E(H) \cup E(H') \cup \{e\})$. After this, we will check whether the equation $V(H) = V$ holds. If it holds, which means we have a subgraph which covers all vertices in G and has the property of 2-edge-connectivity, the algorithm terminates. Otherwise, we start to construct a new H' from an arbitrary edge $e \in \delta(H)$ all over again.

If we reach a cycle C which is already contained in H', we need to build either a *lollipop* or a *tadpole*, which are already defined in Section 3.1. Similarly, a *lollipop* is built if C is a large cycle while a *tadpole* is constructed if C is a small cycle. However, special strategies apply when C is a small cut cycle, which will be discussed in detail

later. Before we get to the details on the construction of a lollipop or a tadpole, it shall be noted that it is possible that the cycle C we reached has already been contained in a lollipop or a tadpole. Recall that an *independent cycle* refers to a cycle in $\mathcal{C}$ which does not belong to any lollipop or tadpole. For the purpose of identifying whether a cycle is an independent cycle or not, Algorithm $\frac{5}{4}$-APX2EC maintains a list of lollipops and a list of tadpoles all along our algorithm. With two such lists, the following cases are handled respectively.

1. If C is an independent large cycle in H', we construct a lollipop, denoted by L, which consists of the large cycle C and the elements in H' which were added after C was added to H'. Add L to the list of lollipops.

2. If C is an independent small cycle in H', we will not simply construct either a lollipop or a tadpole. Let P be the Hamilton path in $G[V(C)]$ contained in H'. Let v_C and w_C be the initial and the terminal vertex of P. Let the matching edge $e = (u, v)$ be the edge which led us back to the cycle C. Then we have the following two sub-cases.

   (a) If w_C and v are adjacent and C is a small cut cycle, we will build a lollipop L instead of a tadpole. As demonstrated in the example shown in Figure 3.10, we will replace the original Hamilton path P of C by a new Hamilton path P' whose initial and terminal vertices at w_C and v and whose edges are $E(C) \setminus (v, w_c)$. Update the Hamilton path associated with C in the list of small cut cycles. After adding L to the list of lollipops, we continue the growth of H' by redefining $e := f$ where $f \in \delta L \setminus E(H')$ and redefining C as the cycle reached from the other end of f.

   (b) Otherwise, we build a tadpole, denoted by T. The tail of T consists of the subgraph of P connecting v_C and v. The head of T consists of the subgraph of P connecting v and w_C and the elements of H' added after P

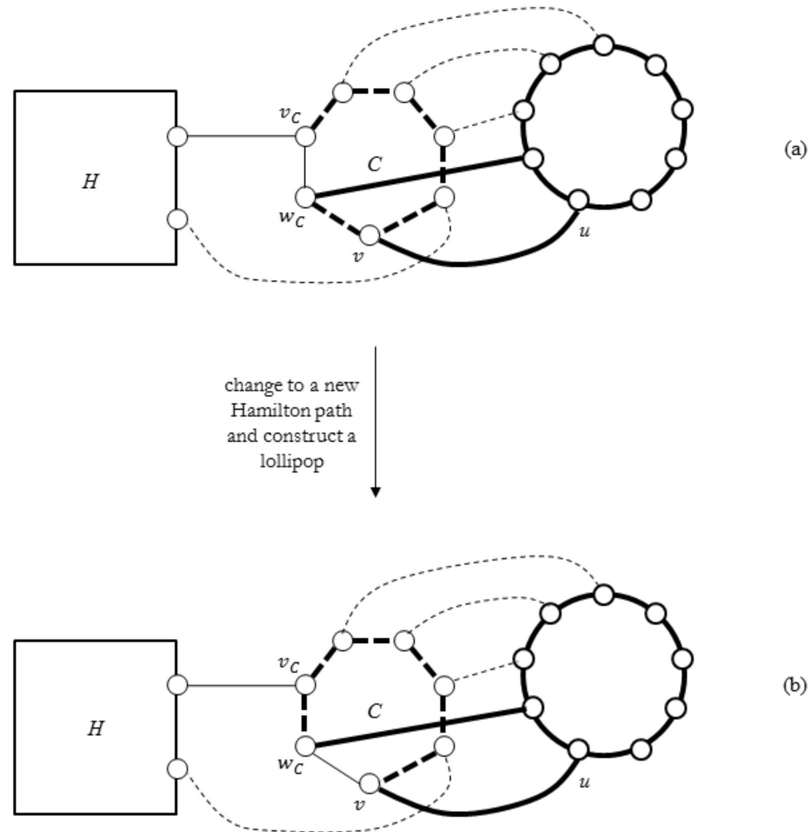


FIGURE 3.10: Construct a lollipop L when returning back to an independent small cycle C from an adjacent vertex v to the terminal vertex w_c of the original Hamilton path in C, as explained in Case 2(a). The thick dotted edges in (a) are edges in the original Hamilton path on C, while the thick dotted edges in (b) are edges in the modified Hamilton path on C. All thick edges in (c) are edges in the lollipop.

was added to H'. Add T to the list of tadpoles.

3. If C is not independent and contained in a lollipop, say L, in H', we construct a larger lollipop denoted by $\hat{L}$, where $\hat{L}$ consists of L and all the elements in H' which were added after L was built. Delete L from the list of lollipops, and add $\hat{L}$ to the list of lollipops.

4. If C is contained in a tadpole, say T, we have the following two sub-cases depending on whether C is a small cut cycle or not.

   (a) C satisfies the following three conditions: (1) C is a small cut cycle; (2) v is adjacent to w_C; and (3) part of P, as the Hamilton path on G[V(C)], is the tail of T. In this case, we modify T into a lollipop $\hat{L}$. This will be done by replacing P by a different Hamilton path P' whose initial and terminal vertices are v and w_C and whose edges are the edges of $E(C) \setminus (v, w_c)$. Figure 3.11 shows an example of the transformation. Update the Hamilton path associated with the small cut cycle C in the list of small cut cycles. Delete T from the list of tadpoles and add $\hat{L}$ to the list of lollipops. Note that we proved case-by-case that this situation is not possible to exist in our algorithm with the restriction on the size of small cycles. However, in order to complete this algorithm and, more importantly, to leave the chance to improve or generalize our algorithm, the method for dealing with this situation is still documented here. Also, the example shown in Figure 3.11 is only for the purpose of illustration and is not possible to appear in the process of Algorithm $\frac{5}{4}$-APX2EC.

   (b) If C does not satisfy the three conditions from (a), we construct a larger tadpole $\hat{T}$, consisting of T and all elements in H' added after T was added. If vertex $v \in V(C)$ belongs to the tail of T, the tail $\hat{T}$ is the subgraph of P connecting v_C and v, and the head is the rest of $\hat{T}$ without the edges

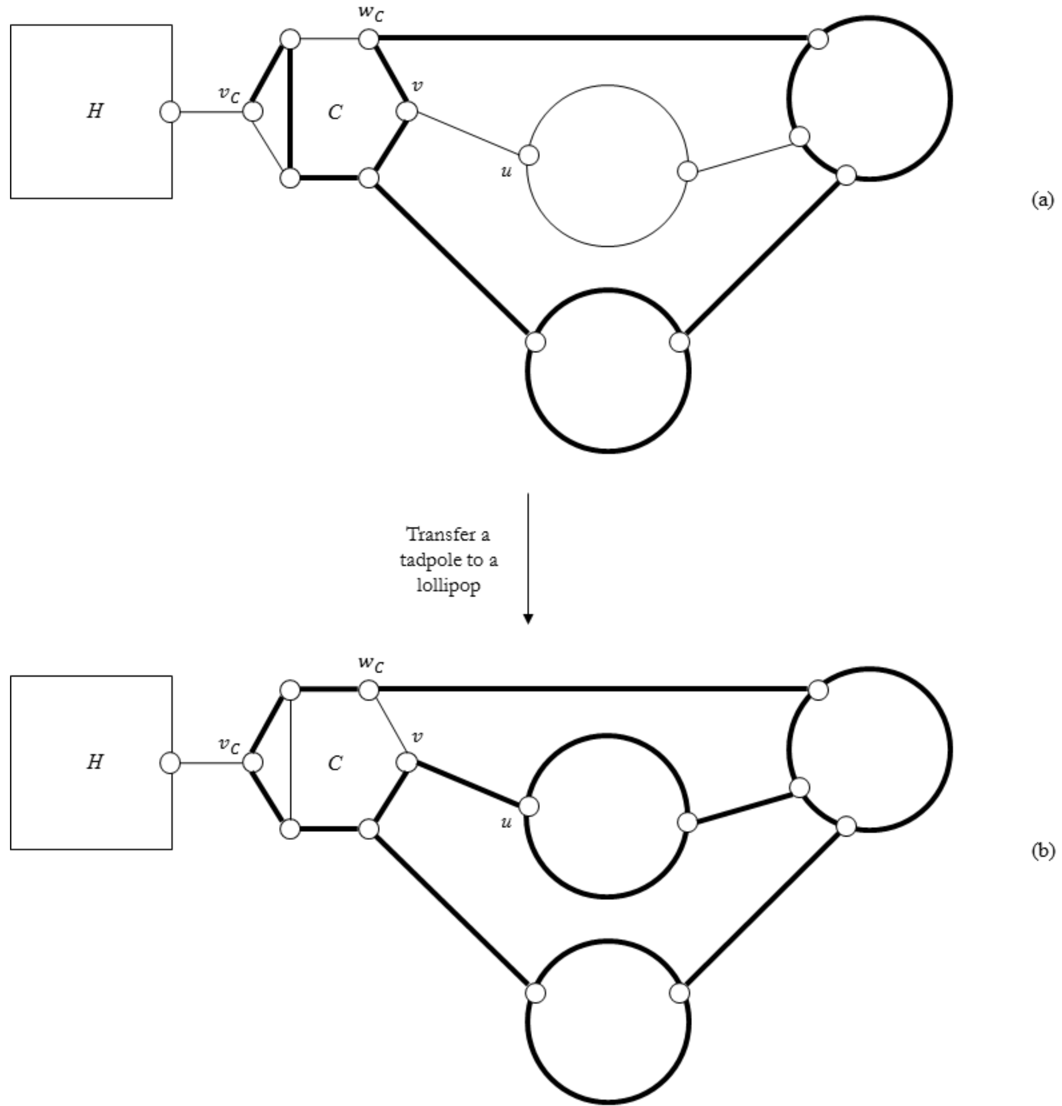


FIGURE 3.11: Transform a tadpole to a lollipop when returning back to a small cycle C that satisfies all three conditions presented in Case 4(a). The thick edges in (a) are edges in the original tadpole T, while the thick edges in (b) are edges in the lollipop transferred from T.

in its tail. If vertex $v \in V(C)$ belongs to the head of T, then the tail of $\hat{T}$ remains the same as the tail of T and its head consists of the head of T and the elements of H' which were added after the construction of T. After removing T from the list of tadpoles, add $\hat{T}$ into the list of tadpoles.

After the above procedures, if we constructed a lollipop L, then we continue the growth of H' by redefining $e := f$ where $f \in \delta(L) \setminus E(H')$ and redefining C as the cycle reached at the other end of f. On the contrary, if a tadpole was built, we continue the growth of H' by redefining $e := f$ where f is a matching edge connecting the head of T to $V \setminus V(T)$ and redefining C as the cycle reached at the other end of f.

Note that by maintaining the list of lollipops and the one of tadpoles, we can ensure that we only deal with inclusion-wise maximal lollipops and tadpoles. This ensures that whenever we reach a cycle which belongs to a lollipop or a tadpole, it does not belong to any other lollipops or tadpoles.

Below we presents a more systematical description of the algorithm.

Algorithm 3.1. $\frac{5}{4}$-APX2EC

Input. a 2-edge-connected cubic graph $G = (V, E)$ and two adjacent edge e_1 and e_2.

Output. a 2-edge-connected subgraph H of G that has at most $\frac{5|V|}{4}$ edges and contains e_1 and e_2.

STEP 1. Using Algorithm 34CUT to find a 2-factor F of G that covers all the 3- and 4-edge cuts but excludes the edge $e_3 = \delta(x) \setminus \{e_1, e_2\}$, where x is the common vertex that e_1 and e_2 are both incident to. For every small cycle in F, add it to the list of small cut cycles if its removal disconnects the graph G. Choose the cycle C_0 in F that contains e_1 and e_2, set $H := C_0$, and then go to Step 2.

STEP 2. If $V(H) = V$, then return H. Otherwise, choose an arbitrary edge $e = uv \in V(H)$ and $v \in V \setminus V(H)$, and let H' be the graph consisting of the single vertex

u, and go to Step 3.

STEP 3. Denote by C the cycle in F containing v. If C is not contained in H, then go to Step 4. Otherwise, go to Step 5.

STEP 4. Here we have the following cases.

Case 1. If C is a large cycle not contained in $H \cup H'$, then go to Step 4-1.

Case 2. If C is a small cycle not contained in $H \cup H'$, then go to Step 4-2.

Case 3. If C is an independent large cycle in H' or contained in a lollipop in H', then go to Step 4-3.

Case 4. If C is an independent small cycle in H' or contained in a tadpole in H', then go to Step 4-4.

STEP 4.1. Add e and C to H'. That is, set $V(H') := V(H') \cup V(C)$ and $E(H') := E(H') \cup E(C) \cup e$. Then choose an arbitrary edge f from $\delta(C) \setminus e$, set the vertex that is the end of f not in C to v, set $e := f$, and then go to Step 3.

STEP 4.2. (Modification 1)

If C is not a cut cycle, choose a Hamilton path of $G[V(C)]$, starting at v and ending at a vertex w not incident with a chord of C, denoted by P.

If C is a cut cycle and there is more than one such Hamilton path, then choose the Hamilton path as follows. Let S be the vertex set of the component of $G - V(C)$ containing H. Label the two different sets of edges which are connecting $V(C)$ with S and $V(C)$ with $V \setminus (S \cup C)$ as backward edges and forward edges respectively. If among all the Hamilton cycles there exists one whose leaving edge is a backward edge, then choose this one as P. Otherwise, choose the Hamilton path whose leaving edge is closest to

a backward edge. Define the Hamilton path associated with C in the list of small cut cycles as P.

Add e and P to H', that is, set $V(H') := V(H') \cup V(C)$ and $E(H') := E(H') \cup E(C) \cup e$. Then let f be the matching edge incident with w, set the vertex that is the other end of f to v, set $e := f$, and then go to Step 3.

STEP 4.3. Add e to H' and construct a lollipop L. Remove any lollipops or tadpoles contained in L from the list of lollipops and tadpoles. Then choose an arbitrary edge f from $\delta(L) \setminus E(H')$, set the vertex that is the end of f not in L to v, set $e := f$, and then go to Step 3.

STEP 4.4. (Modification 2)

If C is one of the following conditions:

(a). C is not a cut cycle;

(b). C is an independent cut cycle and v is not adjacent to the terminal vertex of the Hamilton path P of $G[V(C)]$;

(c). C is a cut cycle which is completely contained in the head of some tadpole; or

(d). C is a cut cycle in which part of the Hamilton path P of $G[V(C)]$ forms the tail of some tadpole, but v is not adjacent to the terminal vertex of P.

Add e to H' and construct a tadpole T. Remove any lollipops or tadpoles that are contained in T from the list of lollipops and tadpoles. Then, choose an arbitrary edge f connecting the head of T to V $V(T)$, set the vertex that is the end of f not in T to v, set $e := f$, and go to Step 3.

If C does not satisfy the above conditions, C is a small cut cycle, that is reached again from v which is adjacent to the terminal vertex w_c of

the Hamilton path P of $G[V(C)]$, which will form the tail of a tadpole. Replace P on $G[V(C)]$ with a new Hamilton path whose initial vertex and terminal vertex are v and w_c and whose edges are $E(C) \setminus (v, w_c)$. Update the Hamilton path P associated with C in the list of small cut cycles. By doing this, instead of a tadpole, a lollipop L is constructed and thus added to H' along with e. Then, choose an arbitrary edge f from $\delta(L) \setminus E(H')$, set the vertex that is the end of f not in L to v, set $e := f$, and go to Step 3.

STEP 5. Augment H by H' and e. That is, set $V(H) := V(H) \cup V(H')$ and $E(H) := E(H) \cup E(H') \cup e$. Then go to Step 2.

Before we finished the algorithm completely, there is still one point remaining to be proved. One may question that after the construction of a tadpole T in the algorithm, is there another edge connecting the head of the tadpole T to $V \setminus V(T)$? Because if such an edge does not exist, any edge belonging to the tail of T can be a cut edge for the solution, in which case our solution will not be 2-edge-connected. Lemma 3.3.3 presented below solves this question with no doubt.

Lemma 3.3.3. *For a tadpole T, there exists an edge connecting the head of the tadpole T and $V \setminus V(T)$.*

Proof. Suppose, to the contrary, that there exists no edge connecting the head of a tadpole T and $V \setminus V(T)$. Denote the small cycle providing the tail of T by C_T, which must be a small cut cycle. Denote the chosen Hamilton path of $G[C_T]$ by P, and the matching edge used to come to C_T the first time by $m_T = (u, v)$, where $u \in V(C_T)$. Denote the component in $G - C_T$ containing the initial cycle C_0 by H_0, and the other part by H_1. Note that $G - C_T$ has only two components (as both H_0 and H_1 are connected).

We can prove this lemma considering the following different cases, which are in accordance with the four cases presented in the previous proof for Lemma 3.3.2.

Case I: $|V(C)| = 7$, $G - C$ can only have 2 components and no chord.

Case II: $|V(C)| = 6$ and C has no chord, $G - C$ has 3 components.

Case III: $|V(C)| = 6$ and C has one chord, $G - C$ has 2 components.

Case IV: $|V(C)| = 4$, $G - C$ can only have 2 components and no chord.

- For Case II

  As $G - C_T$ has only two components, this case is eliminated.

- For Cases I and IV

  Denote the forward edge which is used to leave C_T by (w_c, w_x) and the forward edge which is used to come back to C_T by (u, v), where $w_c, v \in V(C_T)$ and $w_x, u \in V(H_1)$. According to Step 4-4 in our algorithm, there must be at least one vertex between w_c and v on P; otherwise we form a lollipop, not a tadpole, according to Modification 2. Let p denote one such vertex.

  Since C has no chord in Cases I and IV, p must be incident with a forward edge (otherwise we have an edge from the head of T to $V \setminus V(T)$). Also, C consists of the Hamilton path P plus edge (v_0, w_c), where v_0 is the node where we first enter C in the algorithm.

  Thus there exist at least two vertices on P between (w_c, w_x) and a backward edge. According to Step 4-2, the matching edges incident with three vertices closest to v on P are forward edges. Therefore, C_T has at least six forward edges, which eliminates Cases I and IV.

- For Case III

  For Case III, where we have one chord in the 6-cycle, there are six possibilities for the set of forward and backward edges, as shown in Figure 3.12.

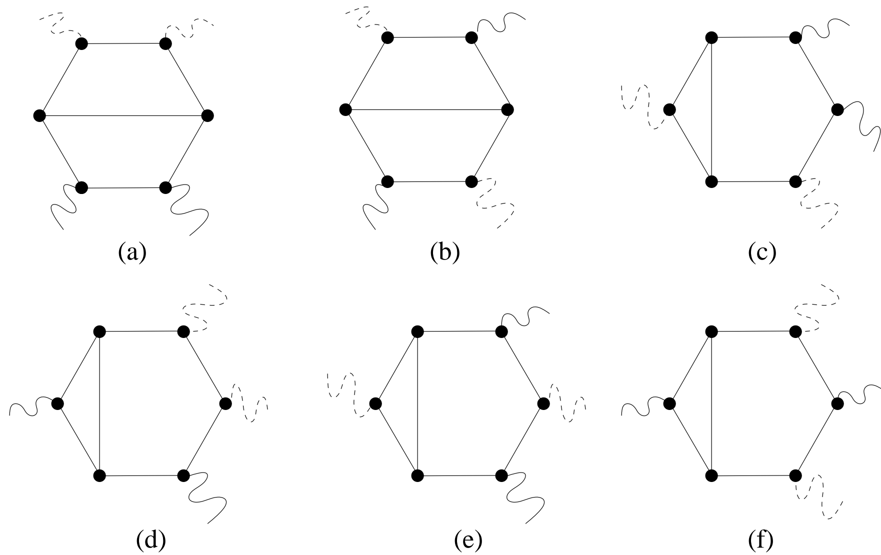


FIGURE 3.12: Six combinations of backward (dashed) and forward (solid) edges in a 6-cycle with one chord.

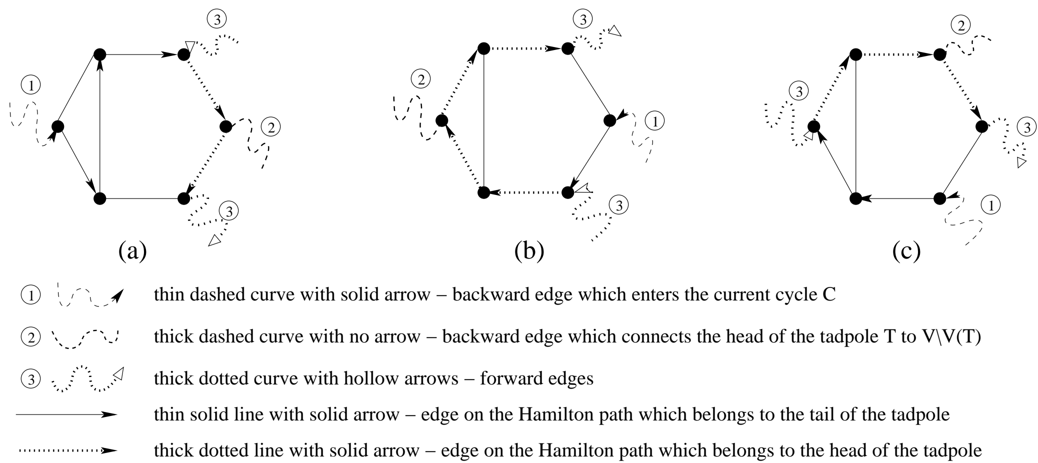


FIGURE 3.13: Different layout of backward edges and forward edges for Figure 3.12 (e) and (f).

For Figures 3.12 (a), (b), (c) and (d), it is clear that no matter from which backward matching edge of the two we entered the cycle, i.e. the first backward edge, we will always have a Hamilton path in which the leaving edge is also a backward edge. Thus, according to Modification 1, we choose the Hamilton path which ends with a backward leaving edge, and the cycle will not be a tadpole.

For Figure 3.12 (e) and (f), no matter from which backward edge we entered, we will get out with a Hamilton path with a leaving forward edge. After we come back from another forward edge, it will form a tadpole. However, the head of the tadpole contains a vertex which is incident to the unused backward edge. See Figure 3.13. Note that the other possibility for (c) in Figure 3.13 does not occur because of Step 4.2, in which a Hamilton path whose leaving edge is the closest to a backward edge is chosen. Hence there exists an edge connecting the head of the tadpole T and $V(T)$.

In conclusion, we have Lemma 3.3.3 is proved. □

Lemma 3.3.4. *For an output H of Algorithm 3.1, it holds that $|E(H)| \leq \frac{5}{4}n - 1$.*

Proof. For $n \leq 7$, the initial 2-factor which covers all the 3- and 4-edge cuts must consist of just one cycle and thus $|E(H)| = n$.

For $n \geq 8$, partition the family $\mathcal{C}$ of cycles in F into two sub-families, $\mathcal{L}$ and $\mathcal{S}$, where $\mathcal{L}$ is the family of large cycles, while $\mathcal{S}$ is that of small cycles. By construction, it is clear that H contains at most $2(|\mathcal{C}| - 1)$ edges from the matching edges $E \setminus F$. Moreover, we use $|\mathcal{C}|$ edges from $G[(V)]$ where $C \in \mathcal{L}$, and $(|\mathcal{C}| - 1)$ edges from $G[(V)]$ where $C \in \mathcal{S}$, except possibly the initial cycle C_0. If the initial cycle C_0 is a small cycle, then $|\mathcal{C}|$ edges from C_0 will be used. Thus, using $|F| = n$, it follows that:

$$|E(H)| \leq n - (|\mathcal{S}| - 1) + 2(|\mathcal{S}| + |\mathcal{L}| - 1) = n + |\mathcal{S}| + 2|\mathcal{L}| - 1 \leq \frac{5}{4}n - 1 \qquad (3.1)$$

Note that the last inequality holds because we took small cycles starting from 4 and large cycles starting from 8, hence $n \geq 4|\mathcal{S}| + 8|\mathcal{L}|$, which means, $|\mathcal{S}| + 2|\mathcal{L}| \leq \frac{1}{4}n$. It follows that $|E(H)| \leq max\{n, \frac{5}{4}n - 1\}$. However, because for any cubic graphs, $n \geq 4$, which leads to $n \leq \frac{5}{4}n - 1$. In conclusion, $|E(H)| \leq \frac{5}{4}n - 1$. □

Corollary 3.3.5. *For a 2-edge-connected cubic graph $G = (V, E)$ and any two adjacent edges $e_1, e_2 \in E$, there exists a 2-edge-connected spanning subgraph H of G with $|E(H)| \leq \frac{5}{4}n - 1$ and $\{e_1, e_2\} \subseteq E(H)$ which can be found in $O(n^3)$ time.*

Proof. Let $e_1, e_2 \in E$ be adjacent at some vertex $x \in V$, and let $e_3 = \delta(v) \setminus \{e_1, e_2\}$. It has been stated that the Algorithm 34Cut we applied at Step 1 can obtain a 2-factor in a 2-edge-connected cubic graph covering all the 3- and 4-edge cut in G and not containing a certain edge e^*. Let e_3 be the certain edge that we exclude from the 2-factor we found in Step 1 using the algorithm 34Cut. It follows that e_1 and e_2 must be contained in one of the cycles in F, say C^*. According to the Step 1 of Algorithm 3.1, by choosing C^* to be the initial cycle to start the algorithm, e_1 and e_2 are certainly contained in the resulted spanning subgraph H.

Considering the complexity of Algorithm 3.1, we first give a list of differences between Algorithm 3.1 and Algorithm Apx2EC as follows:

1. After a 2-factor F covering all 3- and 4-edge cuts in the graph G is obtained, there is an additional operation in Step 1, which is to detect all small cut cycles in the family of cycles in F. For the simplicity of the algorithm, this operation can be done by temporarily removing the small cycle C under detection and see whether the rest of G has more than one component. We have at most $\frac{n}{4}$ small cycles, and for each of them the checking procedure takes $O(n^2)$ in the worst case. Hence this operation takes $O(n^3)$ in the worst case.

2. In Step 4.2, instead of choosing an arbitrary Hamilton path, we need to choose a certain one if the underlying cycle is a small cut cycle. However, since the

number of Hamilton paths in a small cut cycle is constant, it would not change the complexity of this step.

3. In Step 4.4, instead of building a tadpole, we only construct a tadpole if the underlying cycle C can pass a verification procedure by satisfying the following conditions:

   (a). C is not a cut cycle;

   (b). C is an independent cut cycle and v is not adjacent to the terminal vertex of the Hamilton path P of $G[V(C)]$;

   (c). C is a cut cycle which is completely contained in the head of some tadpole; and

   (d). C is a cut cycle that part of the Hamilton path P of $G[V(C)]$ forms the tail of some tadpole, but v is not adjacent to the terminal vertex of P.

   Otherwise, we construct a lollipop by updating the original Hamilton path on C to a new one. This operation is completed in constant time, with the aid of the list of small cut cycles where the associated Hamilton path for each of them is maintained.

   With a list of small cut cycles, including the information on their associated Hamilton paths, a list of tadpoles and a list of lollipops maintained, the above verification procedure is completed in linear time. And the rest of the procedure for building a tadpole stays the same as that for Algorithm Apx2EC.

Given that the computational complexity for Algorithm Apx2EC is $O(n^3)$, it follows from the above discussions that the computational complexity for Algorithm 3.1 is also $O(n^3)$.

Hence Corollary 3.3.5 is proved. □

Theorem 3.3.6. *The integrality gap of 2EC is at most $\frac{5}{4}$ for bridgeless cubic graphs.*

Proof. From Lemma 3.3.4, it is known that for an output H of Algorithm 3.1, it holds that $|E(H)| \leq \frac{5}{4}n - 1$, i.e.

$$OPT(G) \leq \frac{5}{4}n - 1.$$

The integrality gap of $2EC$ for bridgeless cubic graphs, denoted as $\alpha^{2EC}(G)$, equals the ratio between $OPT(G)$ and $OPT_{LP}(G)$ for any bridgeless cubic graph G with any cost function. Considering that for any bridgeless cubic graph G with n vertices,

$$OPT_{LP}(G) \geq n,$$

it follows that

$$\begin{aligned} \alpha^{2EC}(G) &= \frac{OPT(G)}{OPT_{LP(G)}} \\ &\leq \frac{5n/4 - 1}{n} \\ &= \frac{5}{4}. \end{aligned} \tag{3.2}$$

Hence the integrality gap of 2EC is at most $\frac{5}{4}$ for bridgeless cubic graphs. □

With Section 3.1 given as a fundamental introduction on Algorithm Apx2EC [BIT13], which is the cornerstone of Algorithm 3.1 we presented in Sections 3.3, we explained the problems of applying Algorithm 3.1 to obtain solution of 2EC on bridgeless cubic graphs in Section 3.2. By giving methods to overcome the above barriers, we gave a $\frac{5}{4}$-approximation algorithm for bridgeless cubic graphs with the computational complexity of $O(n^3)$. Furthermore, Algorithm 3.1 provides an upper bound on the integrality gap of the LP relaxation on 2EC for bridgeless cubic graphs as $\frac{5}{4}$.

Chapter 4

Computational Study on the Integrality Gap of 2EC

Having tried different approaches for designing an approximation algorithm with better approximation ratios, special families of graphs with some structures, such as the diamond structures, were found to cause a high ratio between $OPT(G)$ and the number of vertices n for graphs G in such families. However, they do not necessarily cause a high ratio between $OPT(G)$ and $OPT_{LP}(G)$. This finding led to further studies on the integrality gap of 2EC, which was the motivation for the computational study presented in this chapter.

In this chapter, we investigate the worst-case ratio between $ILP(G)$ and $LP(G)$ computationally for graphs with a small number of vertices. More specifically, we do this for the following classes of graphs:

- All bridgeless graphs with k vertices, where $3 \leq k \leq 10$.
- All bridgeless cubic graphs with k vertices, where $6 \leq k \leq 16$.
- All bridgeless subcubic graphs with k vertices, where $3 \leq k \leq 16$.

This chapter includes two sections representing the two major stages of this ex-

perimental study. The first stage, program design and results acquisition, consists of designing a program that generates all graphs in the different classes with a specified number of vertices as the test cases, and conducts a series of calculations on all of them afterwards. The second stage, results analysis, consists of a detailed analysis based on all results obtained in the first stage.

The program designed for our experiments is developed using the C programming language via Mircosoft®Visual Studio 2010. It is later installed on a 64-bit system running Mircosoft®Windows 7 Professional, with a Lenovo®Thinkpad X201 laptop equipped with Intel®Core™i5 M480 @ 2.67GHz, and 4.00 GB installed memory (RAM).

4.1 Program Design and Results Acquisition

With the purpose to find out more about the lower bound for the integrality gap of the LP relaxation for 2EC, α^{2EC}, the principle behind the design of this program is very straight-forward. It contains three major steps: data generation, data modeling and data solution. With a set of graphs $\mathcal{G}$ generated in the first data generation step as the pool of test cases, the program constructs the integer linear program $ILP(G)$ and the corresponding LP relaxation $LP(G)$ for every $G \in \mathcal{G}$ in the second step. For every $G \in \mathcal{G}$, in the third and last step we obtain optimal solutions to $ILP(G)$ and $LP(G)$ with the optimal objective value $OPT(G)$ and $OPT_{LP}(G)$ respectively, and find the ratio between $OPT(G)$ and $OPT_{LP}(G)$. For any graph G, the ratio between $OPT(G)$ and $OPT_{LP}(G)$ is denoted by $\alpha(G)$.

In our experiments, $\mathcal{G}$ is a set containing types of graphs (i.e., general, cubic and subcubic graphs) with constraints on their sizes. The maximum ratio $\alpha(G)$ among all $G \in \mathcal{G}$ provides a lower bound for the value of α^{2EC} for graphs of that type. Details of every step in this program are given in the following three subsections.

4.1.1 Data Generation

It is known that the computational complexity for solving $ILP(G)$ is NP-hard, however, it is practically possible to solve $ILP(G)$ in reasonable time for graphs G of small size. Here "small" is considered in terms of the number of vertices n in G. For this reason, all graphs studied in the experiment are of a restricted number of vertices. More specifically, the research objects in this experimental study are limited to the following three categories:

- General graphs $\mathbb{G}_k$, where $3 \leq k \leq 10$ denotes the number of vertices in the graphs;
- Cubic graphs $\mathbb{C}_k$, where $6 \leq k \leq 16$ denotes the number of vertices in the graphs; and
- Subcubic graphs $\mathbb{S}_k$, where $3 \leq k \leq 16$ denotes the number of vertices in the graphs.

Note that even though $\mathbb{C}_k \subset \mathbb{S}_k \subset \mathbb{G}_k$, $\mathbb{C}_k$ and $\mathbb{S}_k$ are studied separately for their uniqueness and importance in the research for a better approximation algorithm. Besides, due to the stronger constraints, the number of graphs in $\mathbb{C}_k$ or $\mathbb{S}_k$ are much smaller than that in $\mathbb{G}_k$, which allowed us to extend the experimental studies to graphs with a higher number of vertices.

In addition, with our focus on 2EC, we will require that all graphs under consideration must be bridgeless to ensure there is a feasible solution, as well as simple[1]. Prior to generating the data for all graphs to be studied, another concept called *isomorphism* on graphs is introduced beforehand.

As presented in Chapter 2, a graph G is defined by the vertex set $V(G)$, the edge set $E(G)$, and the incidence function ψ_G. Two graphs G and H are considered to

[1] A simple solution means the resulting 2-edge-connected spanning subgraph does not contain multiple copies of an edge.

be *identical graphs* if $V(G) = V(H)$, $E(G) = E(H)$ and $\psi_G = \psi_H$. For some pair of graphs G and H which are not identical, however, as long as they share the same structures, G and H are said to be *isomorphic*. For demonstrating that two graphs are isomorphic, a pair of mappings, which is called an *isomorphism*, needs to be provided. Formally defined in [BM08], an *isomorphism* between two isomorphic graphs G and H consists of a pair of bijections $\theta : V(G) \to V(H)$ and $\phi : E(G) \to E(H)$, such that $\psi_G(e) = \{u, v\}$ if and only if $\psi_H(\phi(e)) = \{\theta(u), \theta(v)\}$. Consider the pair of graphs G

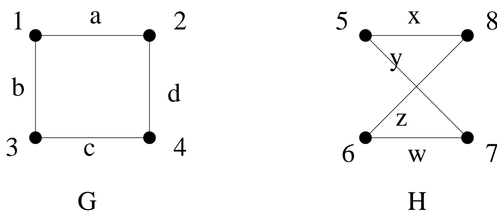


FIGURE 4.1: An example of two graphs G and H which are isomorphic.

and H, shown in Figure 4.1, for example. G and H are isomorphic because the follow isomorphism can be provided:

$$\theta := \begin{pmatrix} 1 & 2 & 3 & 4 \\ 5 & 8 & 7 & 6 \end{pmatrix}, \phi := \begin{pmatrix} a & b & c & d \\ x & y & w & z \end{pmatrix}.$$

Since isomorphic graphs share the identical graph structures, it becomes unnecessary to conduct the experiment on every graph in an equivalence class of isomorphic graphs. It leads to another requirement on the generation of the input data in the experiment. Up to isomorphism, it shall be guaranteed that every pair of graphs in any class under study are non-isomorphic.

In order to generate graph data satisfying the conditions above, a program called `geng` was applied. This program is one of the suite of programs called `gtools`, which is part of the `nauty` package (version 2.4) developed by Brendan D. McKay [McK81]. Note that `nauty`, representing "*no automorphisms, yes?*", is a program for computing

automorphism groups of graphs [McK]. In Graph Theory, an *automorphism* of a graph is an isomorphism of the graph to itself. Since it is not closely related to the experimental study, details on automorphism will not be introduced.

The package `gtools` contains a series of programs for handling graphs in bulk. In particular, `geng` can generate non-isomorphic graphs satisfying certain requirements very quickly. Executing the command "`./geng --help`" gives the usage of `geng` as follows:

External Program 4.1.1. `geng`: Generate all non-isomorphic graphs [McK].

```
Usage:  geng [-cCmtfbd#D#] [-uygsnh] [-lvq]
                     [-x#X#] n [mine[:maxe]] [res/mod] [file]

n :  the number of vertices
mine:maxe :  a range for the number of edges
#:0 means '# or more' except in the case 0:0
res/mod :  only generate subset res out of subsets 0..mod-1
-c :  only write connected graphs
-C : only write biconnected graphs
-t :  only generate triangle-free graphs
-f :  only generate 4-cycle-free graphs
-b :  only generate bipartite graphs
(-t, -f and -b can be used in any combination)
-m :  save memory at the expense of time
(only makes a difference in the absence of -b, -t, -f and n <= 28).
-d# :  a lower bound for the minimum degree
-D# :  a upper bound for the maximum degree
-v :  display counts by number of edges
-l :  canonically label output graphs
-u :  do not output any graphs, just generate and count them
-g :  use graph6 output (default)
-s :  use sparse6 output
-y :  use the obsolete y-format instead of graph6 format
-h :  for graph6 or sparse6 format, write a header too
-q :  suppress auxiliary output (except from -v)
```

Note that in the option "`-C : only write biconnected graphs`", *biconnected*, which means 2-vertex connected, is not equivalent to "2-edge-connected". The two counter examples shown in Figure 4.2 demonstrate that biconnected and 2-edge-connected are not equivalent. It is obvious that $G' - v$ contains two components, and $G'' \setminus e$ contains two components; while G' is 2-edge-connected, and G'' is 2-vertex-connected. Other terminology for `geng` irrelevant to this thesis is not given here, however, their

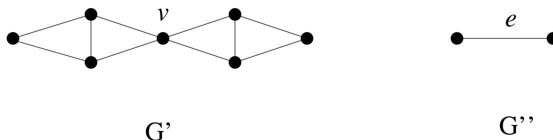


FIGURE 4.2: G' is 2-edge-connected but contains a cut vertex v; while G'' is biconnected but contains a cut edge e.

definitions can be found in [BM08], [McK81] and [McK].

Since biconnected is not equivalent to 2-edge-connected, the constraints of 2-edge-connectivity (i.e. bridgeless) on the input data cannot be directly satisfied by using `geng`. However, it is feasible to generate connected graphs using `geng` with the option "`-c`" in prior to obtain the bridgeless graphs.

In our experiments, the following command is used to generate all non-isomorphic connected general graphs, $\mathbb{G}_k$, where $3 \leq k \leq 10$:

```
./geng -c [k] [filename]
```

For example, in order to generate all non-isomorphic connected graphs with 10 vertices, the command, `./geng -c 10 General10.Graph6`, shall be executed, where `General10.Graph6` is the name of the file that will receive the results. Note that this generates all non-isomorphic connected graphs with 10 vertices in the *graph6* format by default. Optionally, one can also choose to store the graph data in *sparse6* format with the option "`-s`" offered by `geng`. Both graph6 and sparse6 formats are defined in [McK]. Because they are not essentially related to this experimental study, the

details on the decoding of graph6 and sparse6 encodings are not given, but they can be found in [McK, p. 74].

Similarly, the generation of all non-isomorphic connected cubic graphs, $\mathbb{C}_k$, where $6 \leq k \leq 16$, can be accomplished by the following command:

```
./geng -cd3D3 [k] [filename]
```

Also, the generation of all non-isomorphic connected subcubic graphs, $\mathbb{S}_k$, where $3 \leq k \leq 16$, can be accomplished by the following command:

```
./geng -cd2D3 [k] [filename]
```

For easier handling for the rest of this experiment, another program in `gtools` called `showg` is applied. The usage of `showg` is given in the block External Program 4.1.1.

External Program 4.1.1. `showg`: Write graphs in human-readable format [McK].

```
Usage:  showg [-p#:#l#o#Ftq] [-a|-A|-c|-d|-e] [infile [outfile]]

infile is the input file in graph6 or sparse6 format
outfile is the output file
Defaults are standard input and standard output.

-p#, -p#:#, -p#-# :  only display one graph or a sequence of
graphs.  The first graph is number 1.  A second number
which is empty or zero means infinity.
-a :  write the adjacency matrix
-A : same as -a with a space between entries
-d :  write output to satisfy dreadnaut
-c :  write compact dreadnaut form with minimal line-breaks
-e :  write a list of edges, preceded by the order and the number of edges
-o# :  specify number of first vertex (default is 0)
-t :  write upper triangle only (affects -a, -A, -d and default)
-F : write a form-feed after each graph except the last
-l# :  specify screen width limit (default 78, 0 means no limit)
This is not currently implemented with -a or -A.
-q :  suppress auxiliary output

-a, -A, -c, -d and -e are incompatible.
```

It follows that by executing the following command, all graphs obtained in graph6 format can be transformed to the format of an adjacency matrix:

`./showg -A [infile [outfile]]`.

For example, for transforming every non-isomorphic connected graphs with 10 vertices stored in the file `General10.Graph6` into its adjacency matrix, command "`./showg -A General10.Graph6 General10.AdjMx`" is used, where `General10.AdjMx` is the file storing the adjacency matrices of all the graphs.

Until now, we have obtained all non-isomorphic connected graphs with required size in the three categories: general, cubic and subcubic. However, as required in this experiment, any input graph must be 2-edge-connected. This leads to the following

design of the experiment. For every graph under study, we first check whether it is 2-edge-connected using the algorithm 4.1 given below.

Algorithm 4.1. IsBridgeless(G)

Input. A connected graph $G = (V, E)$.

Output. A boolean value indicating G to be bridgeless with the value of `true`, and otherwise with the value of `false`.

for all $e \in E$ **do**
  if IsConnected($G \setminus e$) **then**
    continue;
  else
    return false
  end if
end for
return true

Due to the restriction on graph size studied in this experiment, the idea behind this algorithm is very simple. For any graph $G = (V, E)$ to be checked, we examine whether $G \setminus e$ is connected for every $e \in E$. If for every $e \in E$, $G \setminus e$ remains connected, one can safely reach the conclusion that G is 2-edge-connected where the removal of any edge in G cannot disconnect G. We are aware of the existence of faster algorithms which detect all cut edges in one run of depth-first search; however, as the sizes of graphs in our experiments are relatively small, the amount of time spent on deciding whether some graph G is 2-edge-connected or not is nothing compared to the amount of time for solving $ILP(G)$ and $LP(G)$. Therefore we choose the algorithm below for its simplicity.

In Algorithm 4.1, function `IsConnected(`G`)` is a procedure used to examine whether the input graph G is connected. It ensures the connectivity of G through traversing G using breadth-first-search to see whether all vertices can be reached

from an arbitrary vertex. The details of `IsConnected(`G`)` are given in Algorithm 4.2 below. Note that during the initialization stage, an array `visited` of size n, a first-in-first-out queue `vis_queue` of size n, and an integer `count` shall be defined, where n is the number of vertices in G. Array `visited` is used to indicate the state of vertices during the traversing process. Every entry in `visited` corresponds to every vertex $v \in V$, where each entry has three possible values: unvisited (`unvis`), visited but with unvisited neighbors (`partvis`), and visited with all neighbors visited as well (`fullvis`). Initially, all entries in `visited` has the value `unvis`. Array `vis_queue` is used to keep record of the indices all vertices to be visited and their neighbors. Initially, an arbitrary vertex v_0 is pushed into the queue as the first vertex to be visited. This is followed by the pushing operation for every vertex that is adjacent to v_0. Afterwards, the vertex to be visited next is always the one at the head of `vis_queue`, and all its neighbors will be pushed to the tail of `vis_queue` as the vertex being visited. The variable `count` is the number of vertices already visited. That is to say, whenever a vertex is popped from `vis_queue`, `count` is incremented by 1.

Algorithm 4.2. IsConnected(G)

Input. a graph $G = (V, E)$ with n vertices.

Output. a boolean value indicating G is connected with the value of `true`, and otherwise with the value of `false`.

```
/* Initialization */
declare and define visited, vis_queue, count;
for i = 0 → n − 1 do
    visited[v] = unvis;
end for
vis_queue = empty;
count = 0;
/* Choose vertex v_0 as the first vertex to visit */
visited[0] = partvis;
vis_queue.push(v_0);
/* Start breadth-first-search from v_0 */
while (count ≤ n) and (vis_queue != empty) do
    v_C = vis_queue.pop();
    count = count + 1;
    for all v_i adjacent to v_C do
        visited[i] = partvis;
        vis_queue.push(v_i);
    end for
    visited[C] = fullvis;
end while
/* All vertices which can be reached from v_0 have been visited */
if count == n then
```

return true

else

return false

end if

So far, the procedures for generating the required data of all graphs for the experimental study have been presented. The subsequent data processing, which yields the results to be analyzed in Section 4.2, is introduced in the next two subsections.

4.1.2 Data Modeling

With the set $\mathcal{G}$ containing all graphs to be studied obtained, the $ILP(C)$ and the corresponding $LP(G)$ for every graph $G \in \mathcal{G}$ are constructed in this step.

Recall that for a graph $G = (V, E)$, $ILP(G)$ is given as follows.

$$\begin{aligned} \text{minimize} \quad & \textstyle\sum_{e \in E} x_e & & \quad (4.1) \\ \text{subject to} \quad & x(\delta(S)) \geq 2 \quad \text{for all} \quad \emptyset \subset S \subset V & & \quad (4.2) \\ & x_e \in \{0, 1\} \quad \text{for all} \quad e \in E & & \quad (4.3) \end{aligned}$$

Given that all graphs generated in Subsection 4.1.1 are in the format of an adjacency matrix, the details on constructing all components in the above ILP, including the objective function (Expression 4.1), the cut constraints (Expression 4.2) and the 0-1 constraints (Expression 4.3), are given below. Note that since the experimental objects are restricted to simple graphs without any parallel edges, the adjacency matrix of each one of the graphs in this experiment is symmetric along its diagonal.

1. Objective Function

   As defined in Section 2.2, for any graph $G = (V, E)$, the decision variables for $ILP(G)$ include $|E|$ variables x_e assigned on every $e \in E$, whose values are either 0 or 1. It follows that the objective function, as minimizing the number

of edges in a 2-edge-connected spanning subgraph of G, is to minimize the summation of x_e on all $e \in E$. For this purpose, a traverse through the upper right triangle of the adjacency matrix of G is conducted to search for all edges in G. For any edge $e = (u, v) \in E$, the decision variable for this edge is stored as "$x[u]$-$[v]$", where $[u]$ and $[v]$ represent the indices of the two ends u and v of e. For example, the decision variable for edge $(1, 5)$ is written as "x1-5". Algorithm 4.3 below takes the adjacency matrix of a graph G as input, and produces a string representing the objective function of $ILP(G)$.

Algorithm 4.3. GenerateObjFunction(G)

Input. a bridgeless graph $G = (V, E)$ with n vertices in the adjacency matrix format.

Output. a string `obj` containing the objective function of $ILP(G)$.

```
String obj = "Minimize ";
Queue edge_queue = null;
for i = 0 → n − 1 do
  for j = i + 1 → n − 1 do
    if G[i][j] == 1 then
      /* the edge (i, j) ∈ E */
      edge_queue.push((i, j));
    end if
  end for
end for
/* for any (u, v) in edge_queue, define its decision variable as "x[u]-[v]" */
/* append the first "x[u]-[v]" to the end of the string obj */
(u, v) = edge_queue.pop();
strcat(obj, "x[u]-[v]")[2];
```

[2]char *strcat(char *restrict s1, const char *restrict s2) is a C library function for appending a

while $((u, v) = \text{edge_queue.pop}())$!= null **do**

/* append the plus symbol and "$x[u]$-$[v]$" to the end of the string obj */

strcat(obj, " + ");

strcat(obj, "$x[u]$-$[v]$");

end while

---

2. Cut Constraints

For some graph $G = (V, E)$ with n vertices, the Expression 4.2 shows that every nonempty proper subset $S \subset V$ where $S \neq \emptyset$ is associated with one cut constraint. Therefore, the prerequisite for the construction on the cut constraints becomes generating all non-empty proper subsets of V.

In this program, a *0-1 sequence* is used to generate a subset of V. For any graph $G = (V, E)$ with n vertices, a *0-1 sequence* consists of n elements whose value are either 0 or 1. Therefore, every vertex $v \in V$ corresponds to one bit in the 0-1 sequence. Note that since we are searching for a non-empty subset, the 0-1 sequence that has all entries valued 0 is not valid. Similarly, the condition that the subset must be a proper subset denied the validity of the 0-1 sequence in which all entries have value 1. For example, the fifth vertex corresponds to the fifth entry in the 0-1 sequence. By applying a valid 0-1 sequence to V, a valid subset of V is generated, which contains all vertices whose corresponding bit in the 0-1 sequence has the value 1. Take the graph $G = (V, E)$ shown in Figure 2.1 as an example, for $V = \{0, 1, 2, 3, 4\}$, a valid 0-1 sequence would be "01011", which yields the subset $\{1, 3, 4\}$ consisting of vertices $1, 3$ and 4. Since the number of all possible valid 0-1 sequences with a size of n is $2^n - 2$, the number of non-empty proper subsets of V where $|V| = n$ is $2^n - 2$. It follows that the number of cut constraints is also $2^n - 2$. It follows that the problem

copy of the string pointed to by s2 (including the terminating null byte) to the end of the string pointed to by s1 [KR88].

for generating the cut constraints of G with n vertices has become the problem of generating all valid 0-1 sequences with a size of n. For the reason that every valid 0-1 sequence corresponds to some integer between 0 and 2^n, the generation of all valid 0-1 sequences in this program is achieved by the so-called *sequence increment operation* described below, which imitates the principle of increment operation on an integer. Starting from the 0-1 sequence corresponding to a empty set with all entries valued 0, this program keeps searching for the right-most entry of the current 0-1 sequence whose value is 0. By setting this entry to 1 and set all entries on the right of it to 0, the *sequence increment operation* is completed, and thus the next 0-1 sequence is obtained. Algorithm 4.4 gives the details of the sequence increment operation and produces the next valid 0-1 sequence.

Algorithm 4.4. GenerateSubset($SEQ[n]$)

Input. the current 0-1 sequence $SEQ[n]$. In the initial step, $SEQ[n] = \{0, 0, \ldots, 0\}$.

Output. the updated $SEQ[n]$ indicating the next valid 0-1 sequence.

/* define `mark` to indicate the index of the right-most entry whose value is 0 */

int mark = -1;

/* Searching for the right-most entry whose value is 0 */

for $i = (n-1) \rightarrow 0$ **do**

  if SEQ[i] == 0 **then**

    mark = i;

    break;

  end if

end for

/* Set the value of the right-most entry valued 0 to be 1 */

SEQ[mark] = 1;

/* Set the value of all entries at the right of SEQ[mark] to be 0 */

for $i = (n-1) \rightarrow (mark+1)$ **do**

SEQ[i] = 0;

end for

In the program, based on an initial n-array, denoted by `SEQ[n]`, in which all entries have the value 0, all valid 0-1 sequences and thus all subsets are obtained by keep applying `GenerateSubset(SEQ[n])`. For example, all 0-1 sequence and thus all non-empty proper subset for some $V = \{0, 1, 2\}$ is generated as follows.

valid $0-1$ sequence	...	corresponding subset
000(initial sequence)	...	$\emptyset$(invalid subset)
	$\downarrow$ sequence increment operation	
001	...	$\{2\}$
	$\downarrow$ sequence increment operation	
010	...	$\{1\}$
	$\downarrow$ sequence increment operation	
011	...	$\{1, 2\}$
	$\downarrow$ sequence increment operation	
100	...	$\{0\}$
	$\downarrow$ sequence increment operation	
101	...	$\{0, 2\}$
	$\downarrow$ sequence increment operation	
110	...	$\{0, 1\}$

Since the number of all non-empty proper subsets for $V = \{0, 1, 2\}$ is $2^3 - 2 = 6$, the procedure stops after 6 sequence increment operations. In addition, because the cut constraints are based on the cut edges between a non-empty proper subset $S \subset V$ and its complement $\overline{S}$, the constraint $\delta(S) \geq 2$ and the constraint $\delta(\overline{S}) \geq 2$ are equivalent. Note that for a 0-1 sequence SEQ for a non-empty proper subset $S \subset V$, the 0-1 sequence for $\overline{S}$ has the value 1 at every position i that $SEQ[i] = 0$, and the value 0 at every position j that $SEQ[j] = 1$. Hence the procedure in this program actually stops after $2^{n-1} - 1$ sequence increment operations. Considering the above example, the procedure stops after 3 sequence increment operations.

After an non-empty proper subset $S \subset V$ is obtained, the cut constraint $\delta(S) \geq 2$ shall be generated as follows. Let $G[n][n]$ denotes the adjacency matrix of the graph $G = (V, E)$. For a vertex $u \in S$ indexed by i, scan through the ith row in the adjacency matrix $G[i][j]$ where $0 \leq j \leq n - 1$. If $G[i][j] = 1$ and the corresponding vertex v with the index j does not belong to S, the variable $x[u]$-$[v]$ is added as an addend to the summation on left-hand side of the inequality, where $[u] = i$ and $[v] = j$. Repeat this procedure until all vertices in S have been covered, and by then the cut constraint $\delta(S) \geq 2$ has been completely generated. Applying the process to all the $2^{n-1} - 1$ subsets produces all cut constraints for the graph $G = (V, E)$ with n vertices.

3. 0-1 Constraints

The generation for the 0-1 constraints is relatively easy, thanks to the queue containing all edges, `edge_queue`, which we obtained in the process for generating the objective function. For each item (u, v) of `edge_queue` that represents an edge $e = (u, v) \in E$ in a graph $G = (V, E)$, a corresponding 0-1 constraint

$$x[u]\text{-}[v] \in \{0, 1\}$$

is generated, where $[u]$ and $[v]$ are the indices of the two ends u and v of e.

For a graph $G = (V, E)$, $ILP(G)$ is generated with all the above three steps. Correspondingly, for generating the LP relaxation, $LP(G)$, of $ILP(G)$, the procedures for generating the objective function and generating the cut constraints shall remain the same, while the procedure for generating the non-negative constraints for $LP(G)$ is also accomplished by using `edge_queue` obtained from Algorithm 4.3, which contains the edge set of G. For every item (u, v) in `edge_queue`, that represents an edge $e = (u, v) \in E$ in G, a pair of the corresponding non-negative and upper bound constraints

$$\begin{aligned} x[u]\text{-}[v] &\geq 0 \\ x[u]\text{-}[v] &\leq 1 \end{aligned}$$

is generated, where $[u]$ and $[v]$ are the indices of the two ends u and v of e. Sample files for $ILP(G)$ and $LP(G)$ are given in Appendix A.

4.1.3 Data Solution

For solving $ILP(G)$ and its LP relaxation $LP(G)$ for a graph G, Gurobi™ Optimizer (Version 5.0) (henceforth *Gurobi*), a commercial optimization software developed by Gurobi Optimization, Inc., is applied to obtain solutions to $ILP(G)$ and $LP(G)$. Gurobi is an advanced solver for a wide range of optimization problems, including linear programs and integer linear programs, which has been proven to give solutions to large and difficult models with a relatively high performance [Gur12a]. For the purpose of research, Gurobi's academic license applies.

As the C programming language is used, the C interface of Gurobi optimization libraries is employed in this experimental study. Thanks to the strong support from Gurobi Documentations [Gur12a], the process for solving $ILP(G)$ and $LP(G)$ is

designed as follows.

Step 1 Environment Construction

Using the function `GRBloadenv(GRBenv **env, const char *logfilename)`, an environment, served as "a container for all data associated with a set of optimization runs" [Gur12b], is constructed. In this program, only one optimization model, either $ILP(G)$ or $LP(G)$, is to be solved in each run; therefore only one environment is required.

Step 2 Optimization Model Construction

In this program, according to the format readable by Gurobi, the optimization models for both $ILP(G)$ and $LP(G)$ are written into files with the suffix of ".lp", which is the valid suffix for model files in Gurobi. Appendix A provides a sample model for each of $ILP(G)$ and $LP(G)$. During the construction of an optimization model to be solved by Gurobi, `GRBreadmodel(GRBenv *env, const char *input, GRBmodel **model)` is called to read the model from the assigned `input` file and build the corresponding `model` under the environment `env` created in the last step.

For $ILP(G)$ (or $LP(G)$) for some graph $G = (V, E)$, `model` consists of a set of variables x_e for all $e \in E$, a minimization linear objective function on $\sum_{e \in E} x_e$, a set of constraints corresponding to the cut constraints and 0-1 constraints (or non-negativity and upper bound constraints). Specifically, for 0-1 constraints, `model` associate a type as `binary` with each variable x_e for all $e \in E$; for non-negativity and upper bound constraints, `model` associates a lower `bound` of 0 and an upper `bound` of 1 with each variable x_e for all $e \in E$.

Step 3 Model Solution

After `model` for $ILP(G)$ (or $LP(G)$) is built, `GRBoptimize(GRBmodel *model)` is applied to obtain the optimal solution to the `model`. As explained in [Gur12b],

by default, `GRBoptimize(GRBmodel *model)` uses the concurrent optimizer to solve LP models, and the branch-and-cut algorithm to solve ILP models. Considering the test cases pool $\mathcal{G}$ for this experiment contains only 2-edge-connected graphs, and thereby any $G \in \mathcal{G}$ has at least one 2-edge-connected subgraph, `model` built in the last step is always feasible. Once obtained, the optimal solution is stored as a set of attributes of `model`.

STEP 4 SOLUTION INFORMATION LOGGING

With all information concerning the solution to the optimization model stored as a set of attributes of `model`, `GRBgetdblattr (GRBmodel *model, const char *attrname, double *valueP)` is used to query the value of a double-valued model attribute, where `model` is the model loaded in Step 2 and optimized in Step 3, `attrname` is the name of a double-valued model attribute defined by Gurobi, and `valueP` refers to the location where the current value of the requested attribute shall be saved [Gur12b].

In this program particularly, `GRBgetdblattr(model, GRB_DBL_ATTR_OBJVAL, &objval)` is used to obtain the objective value of `model` and save the value in `objval`. A full list of the names of all available attributes can be found in [Gur12b].

In addition, Gurobi library also provides the function `GRBwrite (GRBmodel *model, const char *output)` to write a variety of optimization data to a file according to the suffixes of the output file. For example, a Gurobi solution file with the suffix ".sol" is used to output a solution vector. A full list of the valid suffixes corresponding to all available types of optimization data can be found in [Gur12b], including "optimization models, solutions vectors, basis vectors, start vectors, or parameter settings". In this program, we use `GRBwrite(model, ‘‘GRAPH_NAME.sol’’)` to log the complete solution vector of `model`, including the objective value and the solution vector for values assigned to the set of

decision variables.

Note that at the end of solving $ILP(G)$ and $LP(G)$, functions `GRBfreemodel()` and `GRBfreeenv()` are called to release associated resources of the optimization model along with the environment built during the solution process.

Sample solutions obtained by Gurobi for $ILP(G)$ and $LP(G)$ are presented in Appendix B.

Note that for every $(u, v) \in E$ for a given graph $G = (V, E)$, its corresponding decision variable in the solution vector recorded by function `GRBwrite(model, ‘‘GRAPH_NAME.sol’’)` is in the format of

`X[u]-[v] VAR_VAL`,

where `VAR_VAL` denotes the value of the decision variable `X[u]-[v]` in the solution. For the ease of result analysis, the result file is translated to a so-called *solution adjacency matrix* defined as follows. For some graph $G = (V, E)$ with n vertices, we denote the *solution adjacency matrix* for $ILP(G)$ or $LP(G)$ by $SOL[n][n]$. For some line, written as "`X[u]-[v]` 1", in a solution file for the $ILP(G)$ for some $G = (V, E)$, which indicates that the edge $(u, v) \in E$ is included in the resulting 2-edge-connected spanning subgraph, $SOL[i][j]$ and $SOL[j][i]$ shall be set to be 1 where $[u] = i$ and $[v] = j$; if "`X[u]-[v]` 0" is in the solution file, which indicates that the edge $(u, v) \in E$ is excluded in the resulted 2-edge-connected spanning subgraph, $SOL[i][j]$ and $SOL[j][i]$ shall be set to be -1 where $[u] = i$ and $[v] = j$. On the other hand, for some line, written as "`X[u]-[v]` val" where $0 < val < 1$, in a solution file for the $LP(G)$ for some $G = (V, E)$, $SOL[i][j]$ and $SOL[j][i]$ shall be set to the fraction format of val where $[u] = i$ and $[v] = j$; if "`X[u]-[v]` 1" is in the solution file, $SOL[i][j]$ and $SOL[j][i]$ shall be set to be 1 where $[u] = i$ and $[v] = j$; if "`X[u]-[v]` 0" is in the solution file, $SOL[i][j]$ and $SOL[j][i]$ shall be set to be -1 where $[u] = i$ and $[v] = j$.

4.2 Results Analysis

Recall that in the pool of all test cases for this experimental study, $\mathcal{G}$, all research objects are limited to three categories of connected graphs: general graphs, cubic graphs and subcubic graphs. Tables 4.1, 4.2 and 4.3 present the information concerning the number of research objects under each categories respectively.

TABLE 4.1: Number of research objects for general graphs $\mathbb{G}_k$ where $3 \leq k \leq 10$

Graph Size (k)	# All Non-isomorphic Graphs	# Bridgeless Graphs
3	2	1
4	6	3
5	21	11
6	112	60
7	853	502
8	11,117	7403
9	261,080	197,442
10	11,716,571	9,804,368

TABLE 4.2: Number of research objects for cubic graphs $\mathbb{C}_k$ where $6 \leq k \leq 16$

Graph Size (k)	# All Non-isomorphic Graphs	# Bridgeless Graphs
6	2	2
8	5	5
10	19	18
12	85	81
14	509	480
16	4,060	3,874

Facing such a large amount of data, it becomes difficult for us to analyze all the results with the limited resources. For this reason, more attention is given to the data that resulted in a higher ratio between $OPT(G)$ and $OPT_{LP}(G)$, in order to learn more about the lower bound for the value of α^{2EC} in general, and the upper bound for particular classes and sizes of graphs. Tables 4.4, 4.5 and 4.6 present the maximum ratio between $OPT(G)$ and $OPT_{LP}(G)$ for all $G \in \mathbb{G}_k (3 \leq k \leq 10)$, $G \in \mathbb{C}_k (6 \leq k \leq 16)$ and $G \in \mathbb{S}_k (3 \leq k \leq 16)$, respectively.

[1] "approx." is the abbreviation of "approximately" hence forth.

TABLE 4.3: Number of research objects for subcubic graphs $\mathbb{S}_k$ where $3 \leq k \leq 16$

Graph Size (k)	# All Non-isomorphic Graphs	# Bridgeless Graphs
3	1	1
4	3	3
5	4	4
6	11	10
7	21	18
8	60	49
9	148	115
10	458	349
11	1,353	1,011
12	4,566	3,421
13	15,530	11,679
14	56,973	43,418
15	214,763	165,993
16	848,895	666,854

Based on all data given in Table 4.4, 4.5 and 4.6, Figure 4.3 demonstrates the trend on the changes of the maximum ratios between $OPT(G)$ and $OPT_{LP}(G)$ for $G \in \mathbb{G}_k$, $G \in \mathbb{C}_k$ and $G \in \mathbb{S}_k$ along the increase on the size of graphs, i.e., the value of k. Let $\alpha(\mathbb{G}_k)$, $\alpha(\mathbb{C}_k)$ and $\alpha(\mathbb{S}_k)$ denote the maximum ratios between $OPT(G)$ and $OPT_{LP}(G)$ for $G \in \mathbb{G}_k$, $G \in \mathbb{C}_k$ and $G \in \mathbb{S}_k$ respectively. Some interesting findings follow:

1. Previous research showed that the known best lower bound on the integrality gap of the LP relaxation for 2EC is $\frac{9}{8}$ [SV12]. Our result on $\alpha(\mathbb{S}_{16})$ reaches this lower bound with only 16 vertices.

2. The changes on the values of $\alpha(\mathbb{G}_k)$, $\alpha(\mathbb{C}_k)$ and $\alpha(\mathbb{S}_k)$ along the value of k do not follow a general trend. Figure 4.3 shows that high fluctuations exist between neighbors, where the difference on the size of graphs is only 1, such as $\alpha(\mathbb{G}_8)$ and $\alpha(\mathbb{G}_9)$. However, with the data we have for now, no certain pattern of the fluctuations can be traced.

3. Based on the eight classes $\mathbb{G}_k$ where $3 \leq k \leq 10$ and the corresponding eight

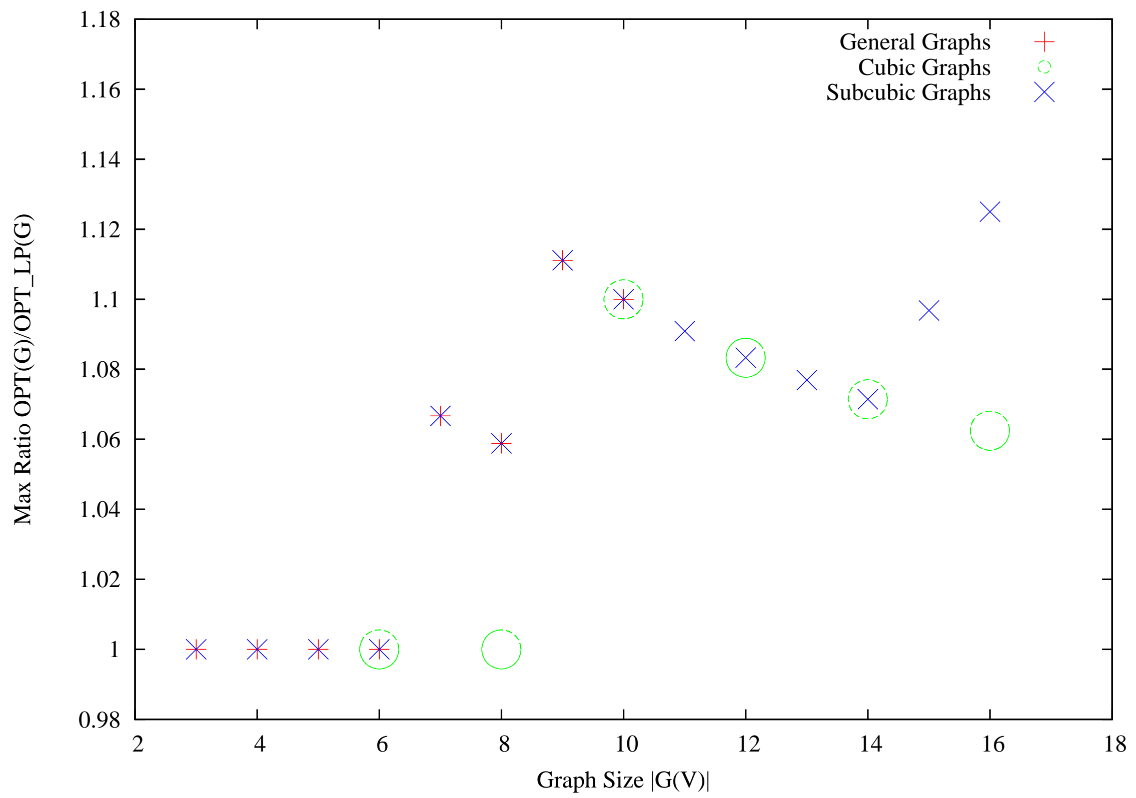


FIGURE 4.3: Data Analysis.

TABLE 4.4: Maximum Ratio of $OPT(G)$ and $OPT_{LP}(G)$ for $\mathbb{G}_k$ where $3 \leq k \leq 10$

Graph Size (k)	Maximum Ratio		Max-ratio Objective Values		Running Time
	Ratio	# Cases	$OPT(G)$	$OPT_{LP}(G)$	
3	1	(all)	3.0	3.0	0.03 seconds
4	1	(all)	4.0	4.0	0.14 seconds
5	1	(all)	5.0 or 6.0	5.0 or 6.0	0.50 seconds
6	1	(all)	6.0 or 7.0 or 8.0	6.0 or 7.0 or 8.0	2.81 seconds
7	16/15	4	8.0	7.5	24.78 seconds
8	18/17	44	9.0	8.5	439.15 seconds
9	10/9	4	10.0	9.0	11,188 seconds
10	11/10	48	11.0	10.0	approx.[1] 11 days

TABLE 4.5: Maximum Ratio of $OPT(G)$ and $OPT_{LP}(G)$ for $\mathbb{C}_k$ where $6 \leq k \leq 16$

Graph Size (k)	Maximum Ratio		Max-ratio Objective Values		Running Time
	Ratio	# Cases	$OPT(G)$	$OPT_{LP}(G)$	
6	1	(all)	6.0	6.0	0.10 seconds
8	1	(all)	8.0	8.0	0.26 seconds
10	11/10	1	11.0	10.0	1.18 seconds
12	13/12	1	13.0	12.0	10.66 seconds
14	15/14	5	15.0	14.0	163.49 seconds
16	17/16	27	17.0	16.0	6,907 seconds

classes $\mathbb{S}_k$ where $3 \leq k \leq 10$, it shall be noted that for each value of k ($3 \leq k \leq 10$), the set of graphs with k vertices that gives the worst ratio among all graphs with the same size, must include subcubic graphs.

The above discoveries naturally raise the following three questions:

QUESTION 1. Is there a better lower bound for the integrality gap of the LP relaxation for 2EC which is even smaller than $\frac{9}{8}$? How can we find it?

QUESTION 2. What could possibly be the reason that causes the sharp fluctuation?

QUESTION 3. Is it possible that the equation $\alpha(\mathbb{G}_k) = \alpha(\mathbb{S}_k)$ holds for any k and thus the worst case ratio between $OPT(G)$ and $OPT_{LP}(G)$ is always given by subcubic graphs?

Table 4.6: Maximum Ratio of $OPT(G)$ and $OPT_{LP}(G)$ for $\mathbb{S}_k$ where $3 \leq k \leq 16$

Graph Size (k)	Maximum Ratio		Max-ratio Objective Values		Running Time
	Ratio	# Cases	$OPT(G)$	$OPT_{LP}(G)$	
3	1	(all)	3.0	3.0	0.03 seconds
4	1	(all)	4.0	4.0	0.13 seconds
5	1	(all)	5.0 or 6.0	5.0 or 6.0	0.19 seconds
6	1	(all)	6.0 or 7.0	6.0 or 7.0	0.44 seconds
7	16/15	1	8.0	7.5	0.83 seconds
8	18/17	1	9.0	8.5	2.22 seconds
9	10/9	1	10.0	9.0	6.71 seconds
10	11/10	3	11.0	10.0	28.56 seconds
11	12/11	9	12.0	11.0	88.61 seconds
12	13/12	33	13.0	12.0	439.66 seconds
13	14/13	120	14.0	13.0	2,169 seconds
14	15/14	8	15.0	14.0	10,601 seconds
15	34/31	5	17.0	15.5	26 hours
16	9/8	1	18.0	16.0	approx. 9 days

Because the reason behind the sharp fluctuation might lead us to a better lower bound smaller than $\frac{9}{8}$, it is beneficial to study the first two questions together. Exploring more on these two questions, more investigations on the actual structures of graphs, which gives the highest $\alpha(\mathbb{G}_k)$, $\alpha(\mathbb{C}_k)$ and $\alpha(\mathbb{S}_k)$, become necessary.

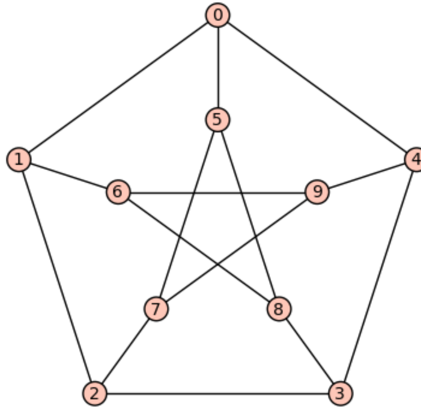
For this purpose, assistance on drawing graphs from their adjacency matrices has been obtained from *Sage Mathematical Software System* (*Sage* henceforth). Licensed under the GPL, Sage is a free open-source mathematics software system [sag].

In this experimental study, Sage is used to draw all graphs which give the highest value of $\alpha(\mathbb{G}_k)$, $\alpha(\mathbb{C}_k)$ and $\alpha(\mathbb{S}_k)$. Since for some classes of graphs, such as $\mathbb{S}_{13}$, there are many graphs which gives the same value on the worst case ratio between $OPT(G)$ and $OPT_{LP}(G)$, it is not necessary to put all of them in the thesis if they do not give clues leading to a better lower bound on the value of α^{2EC} for those three classes of graphs. In addition, considering $\alpha(\mathbb{G}_k) = \alpha(\mathbb{S}_k)$ holds for $3 \leq k \leq 10$ and $\mathbb{S}_k \subset \mathbb{G}_k$, as well as $\alpha(\mathbb{S}_k) = \alpha(\mathbb{C}_k)$ holds for $k = 10, 12$ and 14 and $\mathbb{C}_k \subset \mathbb{S}_k$, to avoid repetition, lists of all drawings on graphs yielding $\alpha(\mathbb{G}_k)(3 \leq k \leq 10)$ are not given here, and

lists of all drawings on graphs yielding $\alpha(\mathbb{C}_k)$ are included in the corresponding lists of graphs yielding $\alpha(\mathbb{S}_k)$. Therefore, listed below are only the classes of graphs which are deemed of significant importance for either yielding a very high ratio between $OPT(G)$ and $OPT_{LP}(G)$ or cause a increased fluctuation in Figure 4.3, including $\alpha(\mathbb{C}_{10})$, $\alpha(\mathbb{C}_{16})$, $\alpha(\mathbb{S}_9)$ and $\alpha(\mathbb{S}_{16})$. Studies and analysis on them are also provided. Note that graphs which give the ratio valued 1 were not studied.

1. Bridgeless Cubic Graphs of Size 10 Yielding $\alpha(\mathbb{C}_{10}) = \frac{11}{10}$

   There is only one graph that gives $\alpha(\mathbb{C}_{10}) = \frac{11}{10}$, and it is not surprising that this graphs turns out to be the Petersen graph, which is the only non-Hamiltonian cubic 2-edge-connected graph of size 10. The Petersen graph becomes a candidate for our studies for the structures which may lead to better lower bounds on the value of α^{2EC} naturally. More investigation on such structures is given in Section 5.4.



C10–14

FIGURE 4.4: Bridgeless cubic graphs in $\mathbb{C}_{10}$ which give $\alpha(\mathbb{C}_{10})$.

2. Bridgeless Cubic Graphs of Size 16 Yielding $\alpha(\mathbb{C}_{16}) = \frac{17}{16}$

   Figure 4.5 and Figure 4.6 list all twenty-seven cubic graphs $G \in \mathbb{C}_{16}$ that give $\alpha(\mathbb{C}_{16})$. An interesting discovery happened after the comparison of our graphs,

with the thirty-three non-Hamiltonian cubic 2-edge-connected graphs of size 16 given by David A. Pike in [Pik97]. We use "(Pike `id`)", where `id` corresponds to the graph index used in [Pik97], beside the index of each graph listed to mark the correspondence. Except for the graphs labeled 16.2, 16.3, 16.5, 16.6, 16.7 which give the ratio of 1 between $OPT(G)$ and OPT_{LP}, and the graph labeled 16.4 that gives the ratio of $\frac{34}{33}$, a one-to-one correspondence is found between all the left twenty-seven non-Hamiltonian cubic 2-edge-connected graphs given by Pike and the graphs we list here. Such correspondence is also found for $\mathbb{C}_{14}$, $\mathbb{C}_{12}$ and $\mathbb{C}_{10}$.

Another finding worth attention is that the pattern of the Petersen graph can be seen in every one of the listed twenty-seven graphs here, which strengthens the guess we have for the Petersen graph and gives a clue that it is very possible the variations on the structure of the Petersen graph may lead us to families of graphs which gives a better lower bound of the value of α^{2EC}.

3. Bridgeless Subcubic Graphs of Size 16 Yielding $\alpha(\mathbb{S}_9) = \frac{10}{9}$

   This class of graphs grabbed our attention easily, for reaching the same value previously known as the best lower bound for α^{2EC} on maximum degree 3 graphs. This graph consists of two triangles connected by three subdivided edges, which yield three pairs of 2-edge cuts. With intensive attention paid on this particular graph, it is learned that the three pairs of 2-edge cuts forces $OPT(S9-99)$ to go up to 10, while the two triangles cause the gap between $OPT(S9-99)$ and $OPT_{LP}(S9-99)$. More studies towards the structure given by this graph is given in Section 5.3.

4. Bridgeless Subcubic Graphs of Size 16 Yielding $\alpha(\mathbb{S}_{16}) = \frac{9}{8}$

   There is only one bridgeless subcubic graph of size 16 that gives $\alpha(\mathbb{S}_{16}) = \frac{9}{8}$. This graph becomes critical because it gives the same ratio between $OPT(G)$

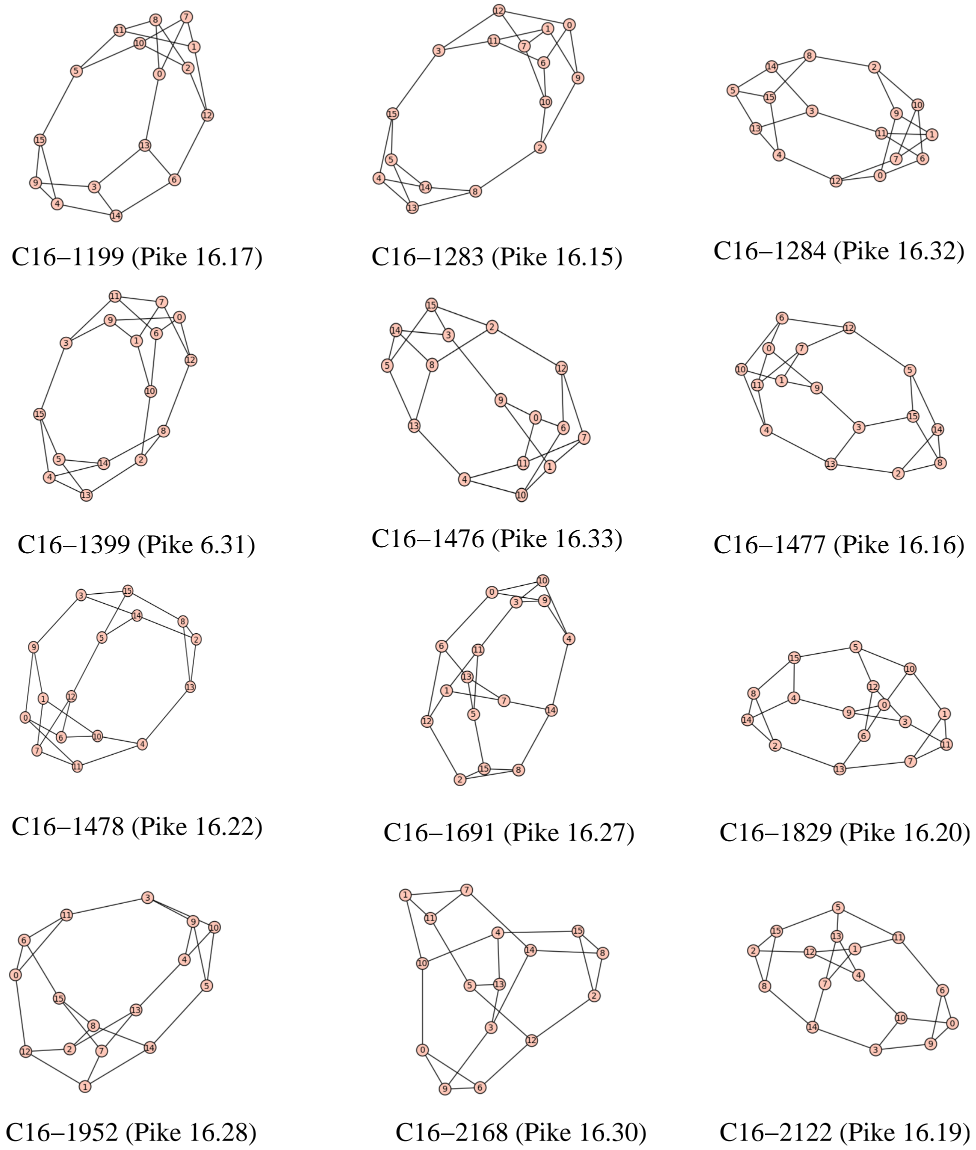
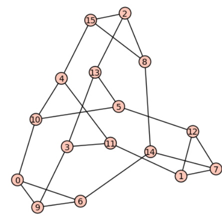
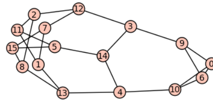


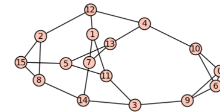
FIGURE 4.5: Bridgeless cubic graphs in $\mathbb{C}_{16}$ which give $\alpha(\mathbb{C}_{16})$.



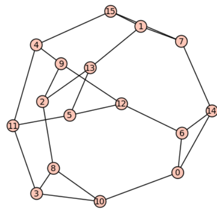
C16–2224 (Pike 16.29)



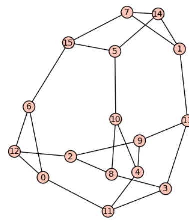
C16–2329 (Pike 16.8)



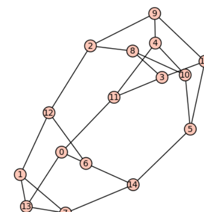
C16–2337 (Pike 16.11)



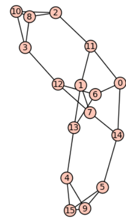
C16–3091 (Pike 16.25)



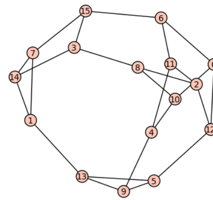
C16–3270 (Pike 16.18)



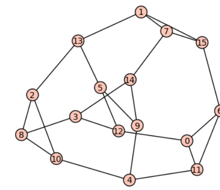
C16–3271 (Pike 16.24)



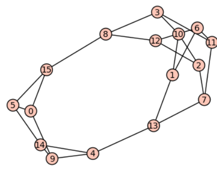
C16–3445 (Pike 16.1)



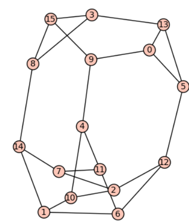
C16–3499 (Pike 16.21)



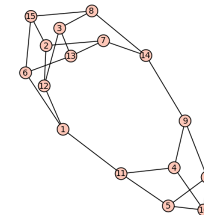
C16–3555 (Pike 16.26)



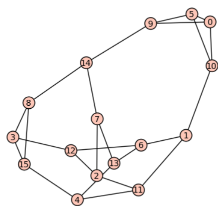
C16–3715 (PIke 16.14)



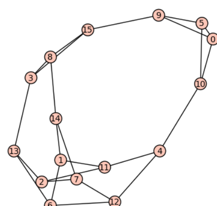
C16–3763 (Pike 16.23)



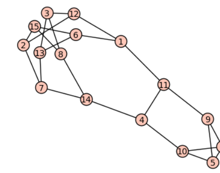
C16–3816 (Pike 16.13)



C16–3961 (Pike 16.12)

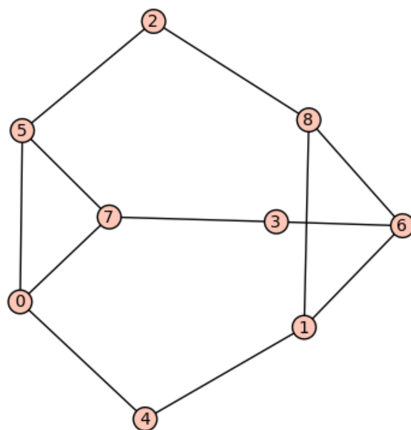


C16–4022 (Pike 16.9)



C16–4042 (Pike 16.10)

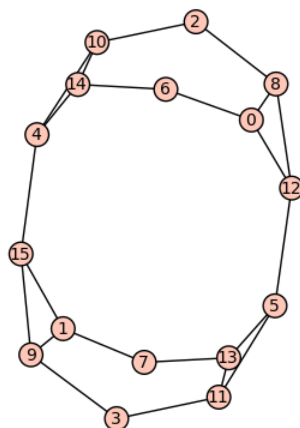
FIGURE 4.6: Bridgeless cubic graphs in $\mathbb{C}_{16}$ which give $\alpha(\mathbb{C}_{16})$.



S9–99

FIGURE 4.7: The bridgeless subcubic graph in $\mathbb{S}_9$ that gives $\alpha(\mathbb{S}_9)$.

and $OPT_{LP}(G)$ as the previous known best lower bound for α^{2EC}. It is interesting to find that this particular graph consists of two identical components connected by a pair of 2-edge cuts. The similarity between these components and the graph S9-99 naturally brings up a possible method to generate large graphs based on the structure of S9-99, which might lead us to families of graphs yielding tighter lower bound for the value of α^{2EC}. This conjecture is later verified in Section 5.3 and Section 5.4.



S16 – 820604

FIGURE 4.8: The bridgeless subcubic graph in $\mathbb{S}_{16}$ that gives $\alpha(\mathbb{S}_{16})$.

From the above discussion, the Petersen graph with $OPT(G)/OPT_{LP}(G) = \frac{11}{10}$, as well as graph S9-99 with $OPT(G)/OPT_{LP}(G) = \frac{10}{9}$, which is the source of the structure appeared in S16-820604 shown in Figure 4.8, stand out as the reason that causes a high ratio between $OPT(G)$ and $OPT_{LP}(G)$, and thus give answers to Question 2. More studies on these two structures are given in Section 5.3 and Section 5.4. The breakthrough we have in these two sections later answers Question 1 with a better lower bound of $\frac{8}{7}$ on the value of α^{2EC} for subcubic graphs.

However, we do not have enough evidence to support the conjecture in Question 3. In order to have a better knowledge on it, more experiments are required with expansion on the size of graphs.

In this chapter, we have provided details on the design and execution of the experimental study, in order to learn more about the bound on the value of α^{2EC} for three categories, with our findings summarized in Table 4.7.

A detailed analysis conducted on the results yields two special structures worth

TABLE 4.7: Summary of the Experimental Study

Graph Category	Max $\alpha(G)$	Corresponding $\|V(G)\|$
$\mathbb{G}_k$ $(3 \le k \le 10)$	10/9	9
$\mathbb{C}_k$ $(6 \le k \le 16)$	11/10	10
$\mathbb{S}_k$ $(3 \le k \le 16)$	9/8	16

studying on more deeply, which will be demonstrated in the upcoming chapter.

Chapter 5

Lower Bounds for the Integrality Gap for 2EC

In Chapter 3, we provided a $\frac{5}{4}$-approximation algorithm for the bridgeless cubic graph $G = (V, E)$ based on the lower bound of n, which is the number of vertices in G. Furthermore, Theorem 3.3.6 followed to state that both approximation ratios presented above yield upper bounds of $\alpha(2EC)$ for bridgeless cubic graphs, which is $\frac{5}{4}$. However, since $OPT_{LP}(G) \geq n$, the actual value of $\alpha(2EC)$ is not necessarily as high as the upper bounds proved in the above theorems.

In this chapter, we investigate lower bounds on the integrality gap of 2EC (henceforth $\alpha^2 EC$), for cubic and subcubic graphs, as well as the worst-case ratio between $OPT(G)$ and n. We achieve our findings by providing families of graphs for which the ratios achieve certain values asymptotically. These families were, for the most part, found through extrapolating our results from Chapter 4.

In Section 5.1, we provide a family of subcubic graphs G for which $\frac{OPT(G)}{n} = \frac{4}{3}$ asymptotically; and in Section 5.2 we provide a family of cubic graphs G for which $\frac{OPT(G)}{n} = \frac{7}{6}$ asymptotically. Note that this indicates that for any approximation algorithm with a performance guarantee of $k \cdot n$, $k = \frac{4}{3}$ would be the best possible for

subcubic bridgeless graphs, and $k = \frac{7}{6}$ would be the best possible for cubic bridgeless graphs.

In Sections 5.3 and 5.4, we provide two different families of subcubic graphs G for which $\frac{OPT(G)}{OPT_{LP}(G)} = \frac{8}{7}$ asymptotically, which shows that α^2EC on subcubic bridgeless graphs is at least $\frac{8}{7}$. The findings of these two families are of significant importance, for updating the known best lower bound of α^2EC from $\frac{9}{8}$ [SV12] to $\frac{8}{7}$.

5.1 Special Subcubic Family G with $OPT(G)/n = 4/3$

As discussed in Section 3.2, the diamond structure, as demonstrated in Figure 3.1 (a) and Figure 5.1 (a), is one of the reasons that Algorithm Apx2EC cannot be applied to bridgeless cubic graphs. Similar to diamond structures, the *square structures*, as shown in Figure 5.1 (b), can also cause a high ratio, considering that the 2-edge-connected spanning subgraph on a square structure contains six edges and four vertices, leading to the ratio based on number of vertices being $\frac{3}{2}$. Since the diamond structures and square structures are likely to be the reasons for large ratios between $OPT(G)$ and n, we present studies on a special family of graphs containing square structures in this section, and another family of graphs with diamond structures in the next section.

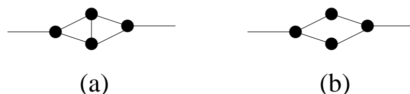


Figure 5.1: (a)Diamond structure; (b) square structure.

Consider a *complete binary tree*[CLRS09] of height h, denoted by BT_h, in which all leaves share the same depth h and all internal nodes have two children. Recall that a *leaf* refers to a node in the tree with no child, and all non-leaf nodes are referred to as *internal nodes* In the following calculations in this chapter. Figure 5.2 illustrates such a complete binary tree, BT_3, of height 3, where all leaves share the same depth 3 and all internal nodes have two children.

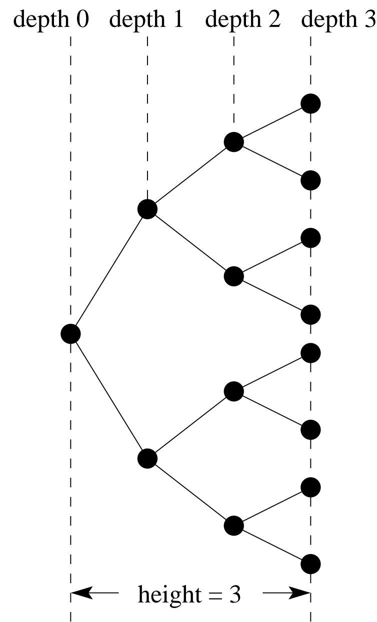


FIGURE 5.2: A complete binary tree BT_3 of height 3.

It is well known that for any binary tree BT_h with height h, the number of nodes at depth d is 2^d[CLRS09, p. 1179]. Thus

$$|Leaves(BT_h)| = 2^h, \tag{5.1}$$

and

$$\begin{aligned}|Internal(BT_h)| &= 1 + 2^1 + \cdots + 2^{h-1} \\ &= \sum_{i=0}^{h-1} 2^i \qquad (5.2) \\ &= 2^h - 1.\end{aligned}$$

Thus, the total number of nodes in BT_h is

$$\begin{aligned}|V(BT_h)| &= (2^h - 1) + 2^h \\ &= 2^{h+1} - 1 \qquad (5.3)\end{aligned}$$

.

In addition, since every internal node has 2 children, the total number of edges in T_h is

$$\begin{aligned}|E(BT_h)| &= 1 \times 2 + 2^1 \times 2 + \cdots + 2^{h-1} \times 2 \\ &= 2 \times \sum_{i=0}^{h-1} 2^i \qquad (5.4) \\ &= 2^{h+1} - 2.\end{aligned}$$

With this knowledge of complete binary trees, we follow the following steps to generate a family of graphs containing the square structures.

Step 1. Take three complete binary trees BT_{k-1} of height $k - 1$.

Step 2. Introduce a degree-3 vertex, say s. Add three edges connecting s and the corresponding degree-2 root nodes on the three complete binary threes, which leads to a tree rooted on s with height k.

Step 3. Mirror the tree resulting from Step 2 along its leaves.

Note that for a tree resulted from Step 2, all its internal nodes have degree 3, while

the leaves have degree 1. For the convenience of description, we refer to such a tree as a *cubic tree*. In a cubic tree of height k, denoted by T_k , all leaves of T_k share the same depth of k. Except for the root s which holds three children, all other internal nodes have two children. The number of nodes and edges in T_k can be calculated by using Equation 5.3 and Equation 5.4:

$$\begin{aligned} |V(T_k)| &= 3 \times |V(BT_{k-1})| + 1 \\ &= 3 \times (2^k - 1) + 1 \\ &= 3 \times 2^k - 2, \end{aligned} \tag{5.5}$$

$$\begin{aligned} |E(T_k)| &= 3 \times |E(BT_{k-1})| + 3 \\ &= 3 \times (2^k - 2) + 3 \\ &= 3 \times 2^k - 3. \end{aligned} \tag{5.6}$$

In addition, the number of leaves in T_k is three times the number of leaves in BT_{k-1}, which is

$$\begin{aligned} |Leaves(T_k)| &= 3 \times |Leaves(BT_{k-1})| \\ &= 3 \times 2^{k-1}, \end{aligned} \tag{5.7}$$

and the number of internal nodes in T_k is three times the number of internal nodes in BT_{k-1} plus the root s, which is

$$\begin{aligned} |Internal(T_k)| &= 3 \times |Internal(BT_{k-1})| + 1 \\ &= 3 \times (2^{k-1} - 1) + 1 \\ &= 3 \times 2^{k-1} - 2. \end{aligned} \tag{5.8}$$

The above procedures produce a family of graphs, denoted by $\mathcal{F}^S$, where the superscript S represents that this family of graphs contains the square structures. This

is because the mirror operation generates $\frac{|Leaves(T_k)|}{2}$, i.e. $3 * 2^{k-2}$, square structures in the resulting graph. For any graph $G_k^S \in \mathcal{F}^S$, $k \geq 1$ is referred to as the *diameter* of G_k^S, where the value of k equals the height of the cubic tree generated by Step 2.

Figure 5.3 illustrates the construction process of the graph $G_4^S \in \mathcal{F}^S$, for which the diameter is 4 and thus the height of the embedded complete binary trees is 3.

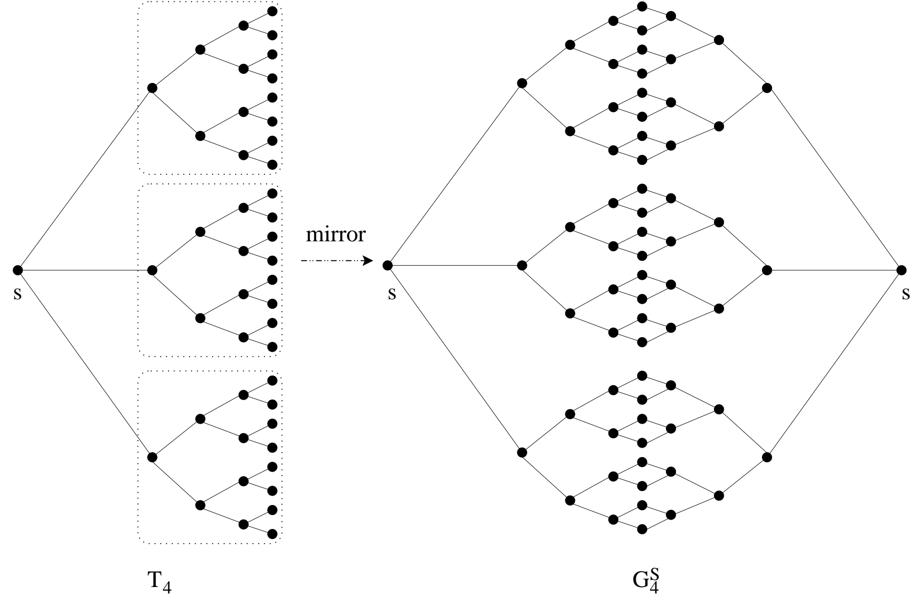


FIGURE 5.3: Construction of a graph G_4^S of diameter 4 in the family $\mathcal{F}^S$ which contains square structures.

Theorem 5.1.1. *For any graph $G_k^S \in \mathcal{F}^S$, the following equations hold:*

$$\lim_{k\to\infty} \frac{OPT(G_k^S)}{|V(G_k^S)|} = \frac{4}{3}, \tag{5.9}$$

$$\frac{OPT(G_k^S)}{OPT_{LP}(G_k^S)} = 1. \tag{5.10}$$

Proof. According to our constructing procedure of $\mathcal{F}^S$, for any graph $G_k^S \in \mathcal{F}^S$ with the diameter of k, we have three complete binary trees of height $k-1$ in G_k^S. Since the mirror operation was done along the leaves of the cubic tree T_k, the number of vertices and edges in G_k^S can be calculated by using the results we obtained in Equation 5.5,

Equation 5.6 and Equation 5.7.

$$\begin{aligned}|V(G_k^S)| &= 2 \times |V(T_k)| - |Leaves(T_k)| \\ &= 2 \times 3 \times 2^k - 2 - 3 \times 2^{k-1} \\ &= 9 \times 2^{k-1} - 4,\end{aligned} \tag{5.11}$$

$$\begin{aligned}|E(G_k^S)| &= 2 \times |E(T_k)| \\ &= 2 \times (3 \times 2^k - 3) \\ &= 12 \times 2^{k-1} - 6.\end{aligned} \tag{5.12}$$

For any graph $G_k^S \in \mathcal{F}^S$, every edge in G_k^S can be paired with another edge to form a 2-edge-cut. Therefore in a feasible solution to both $ILP(G_k^S)$ and $LP(G_k^S)$, $x_e = 1 \ \forall \ e \in E(G_k^S)$. It follows that

$$\begin{aligned}OPT(G_k^S) &= |E(G_k^S)| \\ &= 12 \times 2^{k-1} - 6,\end{aligned} \tag{5.13}$$

$$\begin{aligned}OPT_{LP}(G_k^S) &= |E(G_k^S)| \\ &= 12 \times 2^{k-1} - 6.\end{aligned} \tag{5.14}$$

Forcing k towards infinity, we have

$$\begin{aligned}\lim_{k\to\infty} \frac{OPT(G_k^S)}{|V(G_k^S)|} &= \lim_{k\to\infty} \frac{12 \times 2^{k-1} - 6}{9 \times 2^{k-1} - 4} \\ &= \frac{4}{3}.\end{aligned} \tag{5.15}$$

Moreover,

$$\begin{aligned}\frac{OPT(G_k^S)}{OPT_{LP}(G_k^S)} &= \frac{12 \times 2^{k-1} - 6}{12 \times 2^{k-1} - 6} \\ &= 1.\end{aligned} \tag{5.16}$$

Therefore, Equation 5.9 and Equation 5.10 hold. □

In conclusion, for the family of graphs $G_k^S \in \mathcal{F}^S$, the ratio between $OPT(G_k^S)$ and n approaches $\frac{4}{3}$, while the ratio between $OPT(G_k^S)$ and $OPT_{LP}(G_k^S)$ is 1. This shows that for bridgeless subcubic graphs G, approximation algorithms with the performance guarantee lower, and thus better, than $\frac{4}{3}n$ do not exist.

5.2 Special Cubic Family G with $OPT(G)/n = 7/6$

With the knowledge we gained in Section 5.1, it becomes straight-forward to see that a family of graphs containing the diamond structures can be obtained by replacing all the square structures in any graph $G_k^S \in \mathcal{F}^S$ with the diamond structures. The obtained family of graphs is denoted by $\mathcal{F}^D$, where the superscript D represents that this family of graphs contains the diamond structures. More specifically, to obtain a graph $G_k^D \in \mathcal{F}^D$, where $k \geq 1$ is the *diameter* of G_k^D, we first construct a graph $G_k^S \in \mathcal{F}^S$ as described in Section 5.1. Then we replace any square structure in G_k^S with a diamond structure. Figure 5.4 illustrates the procedure of obtaining the graph $G_4^D \in \mathcal{F}^D$ from the graph $G_4^S \in \mathcal{F}^S$.

Theorem 5.2.1. *For any graph $G_k^D \in \mathcal{F}^D$, the following equations hold.*

$$\lim_{k\to\infty} \frac{OPT(G_k^D)}{|V(G_k^D)|} = \frac{7}{6}, \tag{5.17}$$

$$\frac{OPT(G_k^D)}{OPT_{LP}(G_k^D)} = 1. \tag{5.18}$$

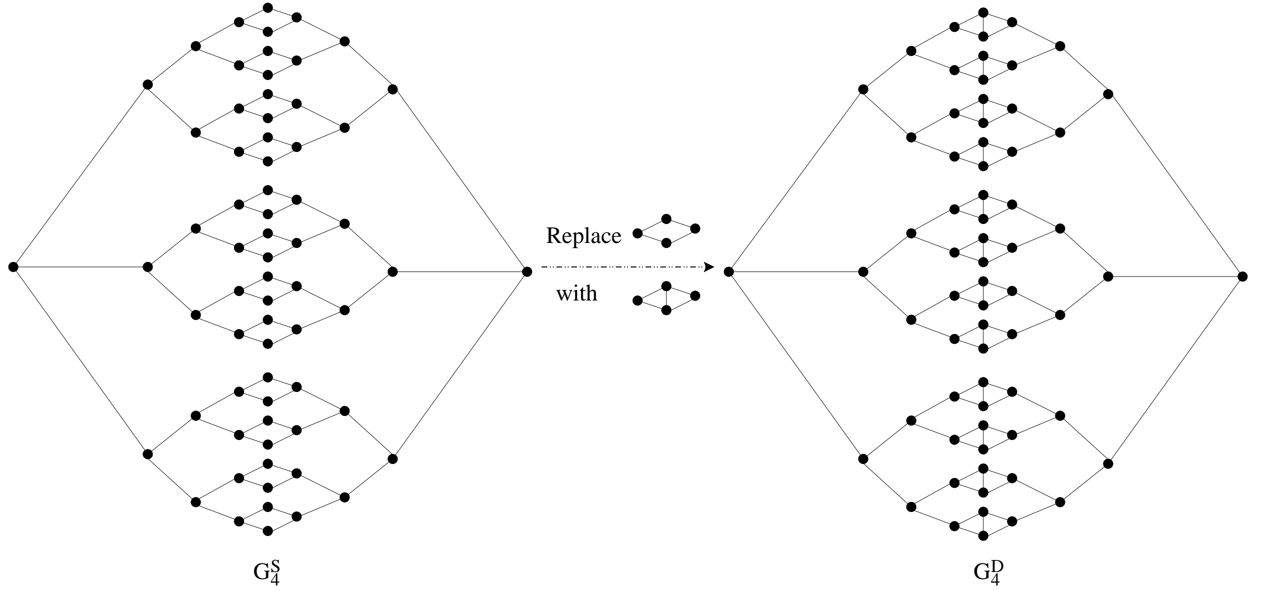


FIGURE 5.4: Construction of a graph G_4^D of diameter 4 in the family $\mathcal{F}^D$ which contains diamond structures.

Proof. From the construction procedure of a graph $G_k^D \in \mathcal{F}^D$ as illustrated above, it is obvious that the number of vertices in G_k^D is the same as G_k^S, and the number of edges is increased by 1 for each diamond (or square) structure. As mentioned in Section 5.1, the number of squares equals half the number of leaves of the cubic tree T_k. Similarly, we have

$$\begin{aligned} |diamonds| &= \frac{1}{2}|Leaves(T_k)| \\ &= \frac{1}{2} \times (3 \times 2^{k-1}) \\ &= 3 \times 2^{k-2}. \end{aligned} \tag{5.19}$$

Hence with the aid of Equations 5.11, 5.12 and 5.19, the number of vertices and edges in G_k^D shall be calculated as follows.

$$\begin{aligned} |V(G_k^D)| &= |V(G_k^S)| \\ &= 9 \times 2^{k-1} - 4, \end{aligned} \tag{5.20}$$

$$\begin{aligned} |E(G_k^D)| &= |E(G_k^S)| + |diamonds| \\ &= (12 \times 2^{k-1} - 6) + 3 \times 2^{k-2} \\ &= 27 \times 2^{k-2} - 6. \end{aligned} \tag{5.21}$$

For any graph $G_k^D \in \mathcal{F}^D$, except for all edges belonging to the diamond structures in G_k^D, every other edge belongs to a 2-edge cut in the graph. For the convenience of calculation, we refer to the edges belonging to the diamond structures as “diamond edges”, and the rest of the edges as “2-cut edges”. Any 2-cut edge contributes 1 to the objective value of any feasible solution of either $ILP(G_k^D)$ or $LP(G_k^D)$. However, since there exists a Hamilton path traversing through the diamond structure, each diamond structure contributes exactly 3 in total to $OPT(G_k^D)$. Take Figure 5.5 as an example, the following assignment on x_e for e as a diamond edge gives a feasible solution to $ILP(G_k^D)$ with the total cost of 3 on the diamond structure:

$$x_{ab} = 1, x_{ac} = 1, x_{ad} = 0, x_{bc} = 0, x_{cd} = 1.$$

On the other hand, with Figure 5.5 as an example, we have the following constraints for $LP(G_k^D)$ on the diamond part:

$$\begin{aligned} x_{ab} + x_{ac} + x_{ad} &\geq 2 \text{ for } \{a\} \subset V, \\ x_{ab} + x_{bc} &\geq 1 \text{ for } \{b\} \subset V, \\ x_{ac} + x_{bc} + x_{cd} &\geq 2 \text{ for } \{c\} \subset V, \\ x_{ad} + x_{cd} &\geq 1 \text{ for } \{d\} \subset V. \end{aligned}$$

Adding all the above constraints together leads to the following inequality;

$$\begin{aligned} & 2x_{ab} + 2x_{ac} + 2x_{ad} + 2x_{bc} + 2x_{cd} \geq 6 \\ \implies & x_{ab} + x_{ac} + x_{ad} + x_{bc} + x_{cd} \geq 3. \end{aligned} \tag{5.22}$$

The inequality 5.22 implies that for any feasible solution to $LP(G_k^D)$, a diamond structure contributes at least 3 to the objective value.

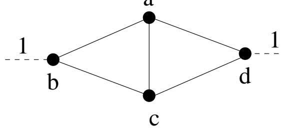


FIGURE 5.5: Contribution of a diamond structure to $OPT(G_k^D)$ and $OPT_{LP}(G_k^D)$.

It follows that

$$\begin{aligned} OPT(G_k^D) &= 1 \times (|E(G_k^D)| - 5 \times |diamonds|) + 3 \times |diamonds| \\ &= 1 \times [27 \times 2^{k-2} - 6 - 5 \times (3 \times 2^{k-2})] + 3 \times (3 \times 2^{k-2}) \\ &= 21 \times 2^{k-2} - 6, \end{aligned} \tag{5.23}$$

and

$$\begin{aligned} OPT_{LP}(G_k^D) &= 1 \times (|E(G_k^D)| - 5 \times |diamonds|) + 3 \times |diamonds| \\ &= 21 \times 2^{k-2} - 6. \end{aligned} \tag{5.24}$$

Since $|V(G_k^D)| = 9 \times 2^{k-1} - 4$, by forcing k towards infinity, we have

$$\begin{aligned} \lim_{k\to\infty} \frac{OPT(G_k^D)}{|V(G_k^D)|} &= \lim_{k\to\infty} \frac{21 \times 2^{k-2} - 6}{9 \times 2^{k-1} - 4} \\ &= \lim_{k\to\infty} \frac{21 \times 2^{k-2} - 6}{18 \times 2^{k-2} - 4} \\ &= \frac{7}{6}, \end{aligned} \tag{5.25}$$

and

$$\begin{aligned} \frac{OPT(G_k^D)}{OPT_{LP}(G_k^D)} &= \frac{21 \times 2^{k-2} - 6}{21 \times 2^{k-2} - 6} \\ &= 1 \end{aligned} \tag{5.26}$$

Therefore, Equation 5.17 and Equation 5.18 hold. □

In conclusion, for the family of graphs $G_k^D \in \mathcal{F}^D$, the ratio between $OPT(G_k^D)$ and n approaches $\frac{7}{6}$, while the ratio between $OPT(G_k^S)$ and $OPT_{LP}(G_k^D)$ is 1. This shows that for cubic bridgeless graphs G, approximation algorithms with the performance guarantee lower, and thus better, than $\frac{7}{6}n$ do not exist.

5.3 Special Subcubic Family G Showing $\alpha^2 EC \geq 8/7$

During our experiment, as described in Chapter 4, the following pattern as shown in Figure 5.6 (a) came to our attention. The original pattern is a subcubic graph of order 9, denoted by G_9. It gives a relatively high ratio, $\frac{10}{9}$, between $OPT(G_9)$ and $OPT_{LP}(G_9)$ in the early stage of our experimental test period for subcubic graphs.

For this reason, we started to study G_9, which we refer to as a *9-pattern*. We

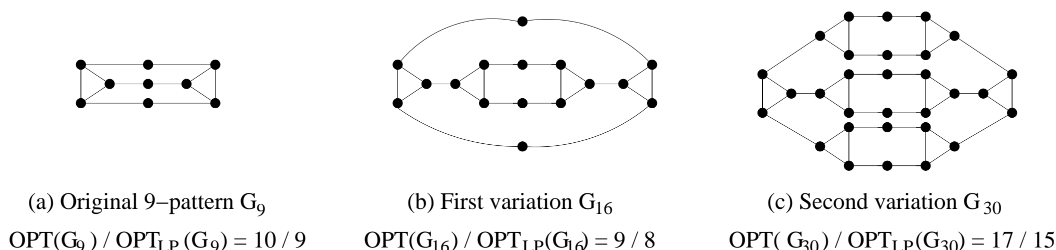


FIGURE 5.6: 9-pattern and its derivation.

tried replacing one of the degree-2 vertex with a variation of the 9-pattern, which we referred to as the *9-pattern gadget*, as shown in Figure 5.7. This replacement leads to a new graph with 16 vertices (see Figure 5.6 (b)), G_{16}, for which the ratio between $OPT(G_{16})$ and $OPT_{LP}(G_{16})$ turned out to be even higher than the ratio for the 9-pattern itself, which was also verified later in the experimental test for subcubic graphs with 16 vertices. Following this trend, we replaced all three degree-2 vertices in the nine pattern with the 9-pattern gadget. This resulted in a graph with 30 vertices, G_{30}, shown in Figure 5.6 (c), for which the ratio between $OPT(G_{30})$ and

$OPT_{LP}(G_{30})$ became higher still.

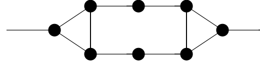


FIGURE 5.7: The 9-pattern gadget.

The fact that $\frac{OPT(G_{30})}{OPT_{LP}(G_{30})} > \frac{OPT(G_{16})}{OPT_{LP}(G_{16})} > \frac{OPT(G_9)}{OPT_{LP}(G_9)}$ inspired us to define a new family of graphs, which is obtained by continuously replacing all degree-2 vertices with the 9-pattern gadget, starting from G_9. We refer to such a family of graphs as $\mathcal{F}^N$, where the superscript N represents that this family of graphs contains the 9-pattern structures. Executing the replacement for t times, based on the original 9-pattern G_9, gives us the graph $G_t^N \in \mathcal{F}^N (t \geq 0)$. Figure 5.8 presents the graph $G_3^N \in \mathcal{F}^N$, which was obtained by repeating the replacement for 3 times.

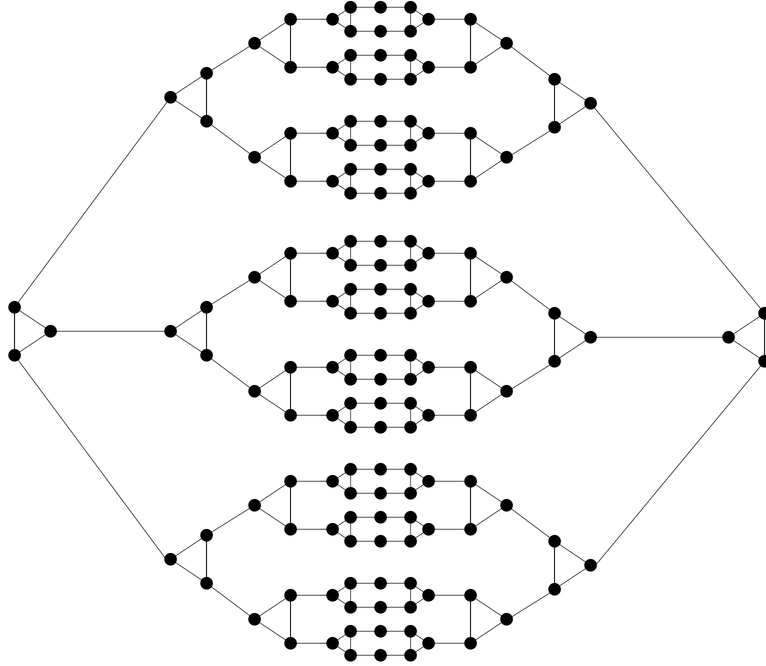


FIGURE 5.8: A special subcubic graph G_3^N derived from the 9-pattern gadget.

Theorem 5.3.1. *For any graph $G_t^N \in \mathcal{F}^N$, the following equations hold.*

$$\lim_{t\to\infty} \frac{OPT(G_t^N)}{|V(G_t^N)|} = \frac{8}{7}, \tag{5.27}$$

$$\lim_{t\to\infty} \frac{OPT(G_t^N)}{OPT_{LP}(G_t^N)} = \frac{8}{7}. \tag{5.28}$$

Proof. For the convenience of calculation, we can simplify the above family by considering every triangle as a virtual vertex, as shown in Figure 5.9.

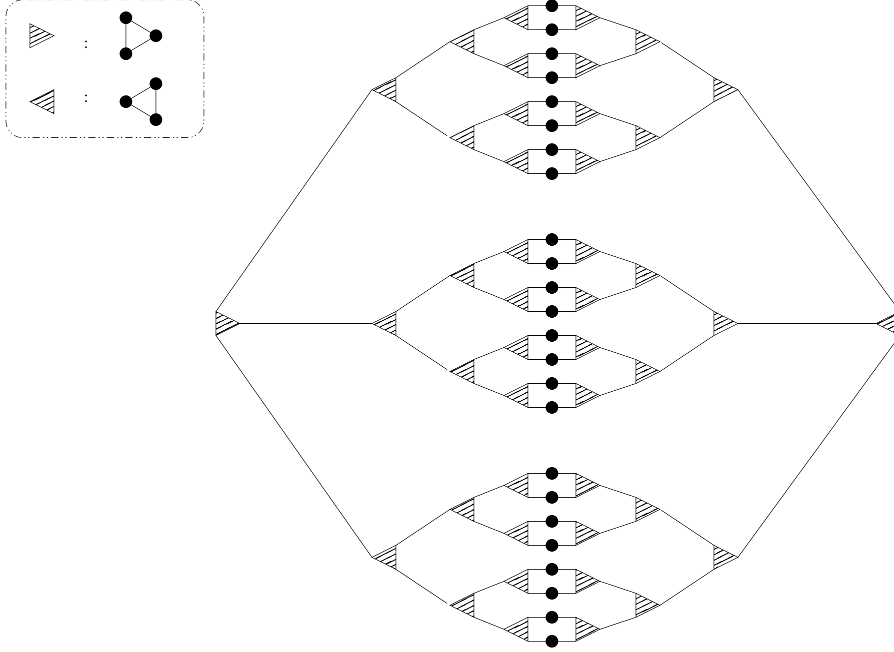


FIGURE 5.9: Simplified graph family from Figure 5.8.

The similarity between Figure 5.3 and Figure 5.9 is easy to identify. Any graph $G_t^N \in \mathcal{F}^N$ corresponds to the graph $G_{t+1}^S \in \mathcal{F}^S$, where every virtual vertex in G_t^N corresponds to an internal node in G_{t+1}^S and every actual vertex corresponds to a leaf in T_{t+1} for the construction of G_{t+1}^S.

Benefiting from the calculation we have already done in Section 5.1, it is known that for any graph $G_t^N \in \mathcal{F}^N$, there are totally $(6 \times 2^t - 4)$ virtual vertices[1], and (3×2^t) actual vertices[2]. Hence the number of vertices for any graph $G_t^N \in \mathcal{F}^N$ can

[1]Please refer to Equation 5.8.

[2]Please refer to Equation 5.7.

be calculated as follows.

$$
\begin{aligned}
|V(G_t^N)| &= 3 \times (6 \times 2^t - 4) + 3 \times 2^t \\
&= 18 \times 2^t - 12 + 3 \times 2^t \\
&= 21 \times 2^t - 12.
\end{aligned} \tag{5.29}
$$

For obtaining $OPT(G_t^N)$ and $OPT_{LP}(G_t^N)$, we can also take advantage of what is known in Section 5.1. Since all edges in the simplified model belong to 2-edge cuts, a feasible ILP solution must include such edges. According to Equation 5.12 and the correspondence between $G_t^N \in \mathcal{F}^N$ and $G_{t+1}^S \in \mathcal{F}^S$, it can be deduced that we have $(12 \times 2^t - 6)$ such edges.

Returning back to the original graph, we need to consider the values of variables on the edges belonging to the virtual vertex. As shown in Figure 5.10, a virtual vertex contributes 2 to $OPT(G_t^N)$, and $\frac{3}{2}$ to $OPT_{LP}(G_t^N)$. Hence for any graph $G_t^N \in \mathcal{F}^N$,

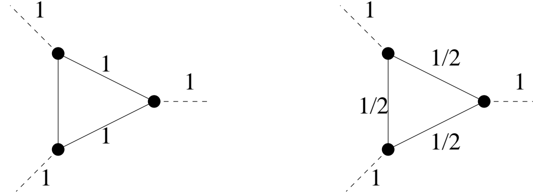


FIGURE 5.10: Feasible solutions to $ILP(G_t^N)$ (left) and $LP(G_t^N)$ (right) on a virtual vertex.

$OPT(G_t^N)$ and $OPT_{LP}(G_t^N)$ can be calculated as follows:

$$
\begin{aligned}
OPT(G_t^N) &= (12 \times 2^t - 6) + 2 \times (6 \times 2^t - 4) \\
&= 24 \times 2^k - 14,
\end{aligned} \tag{5.30}
$$

$$
\begin{aligned}
OPT_{LP}(G_t^N) &\geq (12 \times 2^t - 6) + \frac{3}{2} \times (6 \times 2^t - 4) \\
&= 12 \times 2^t - 6 + 9 \times 2^t - 6 \\
&= 21 \times 2^t - 12.
\end{aligned} \tag{5.31}
$$

Note that Equations 5.29 and 5.31 indicate that $OPT_{LP}(G_t^N) = |V(G_t^N)|$. Since $OPT_{LP}(G_t^N)$ must be at least $|V(G_t^N)|$, the above LP solution is optimal.

Forcing t towards infinity, we have

$$\begin{aligned}\lim_{t\to\infty} \frac{OPT(G_t^N)}{|V(G_t^N)|} &= \lim_{t\to\infty} \frac{24\times 2^t - 14}{21\times 2^t - 12} \\ &= \frac{8}{7}\end{aligned} \tag{5.32}$$

$$\begin{aligned}\lim_{t\to\infty} \frac{OPT(G_t^N)}{OPT_{LP}(G_t^N)} &= \lim_{t\to\infty} \frac{24\times 2^t - 14}{21\times 2^t - 12} \\ &= \frac{8}{7}\end{aligned} \tag{5.33}$$

Therefore, Equation 5.27 and Equation 5.28 hold. □

In conclusion, for any graph $G_t^N \in \mathcal{F}^N$, the ratio between $OPT(G_t^N)$ and n approaches $\frac{8}{7}$, and the ratio between $OPT(G_t^N)$ and $OPT_{LP}(G_t^N)$ is also $\frac{8}{7}$. In addition, for this particular family of graphs, $OPT_{LP}(G_t^N)$ equals the number of vertices in G_t^N. From these findings, we can conclude that α^2EC must be at least $\frac{8}{7}$, even for subcubic graphs.

5.4 Combining the Petersen Graph and the 9-Pattern

Considering that the Petersen graph gives the worst ratio of $\frac{11}{10}$ between $OPT(G)$ and $OPT_{LP}(G)$ where G is a cubic graph in the experimental study, it becomes interesting to us to see what ratio we can reach by growing the Petersen graph using the 9-pattern, which leads to the worst-so-far ratio between $OPT(G)$ and $OPT_{LP}(G)$ for any graph G. In this section, we show that such a family also achieves a ratio of $\frac{8}{7}$ between $OPT(G)$ and $OPT_{LP}(G)$.

Following the procedure presented below, we embed the 9-patterns in the Petersen graph $G_P = (V_P, E_P)$ where $|V_P| = 10$ and $|E_P| = 15$ in order to obtain a new family of graphs. For a more clear description, a vertex in the Petersen graph is referred to

as a *P-vertex*, and an edge in thr Petersen graph is referred to as a *P-edge*.

STEP 1 For every P-vertex $v \in V_P$, replace it with a triangle as shown in Figure 5.11. For the convenience of calculation presented later, we illustrate the triangle by a virtual vertex, as applied in the proof for Theorem 5.3.1 in Section 5.3.

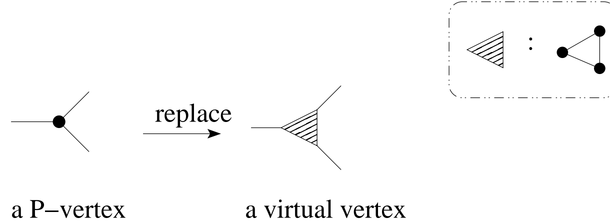


FIGURE 5.11: Replace every P-vertex in the Petersen graph with a triangle.

STEP 2 For every P-edge $e \in E_P$, embed a 9-pattern gadget (5.7) on e.

STEP 3 Extend the 9-pattern gadget to infinity by continuously applying the *replacement operation*, which is to replace all degree-2 vertices in the graph with a new 9-pattern gadget.

Figure 5.12 illustrates the transformation explained in Steps 2 and 3 obtained by conducting the replacement operation 2 times. Similarly, we represent a triangle consisting of three vertices and three edges by a virtual vertex.

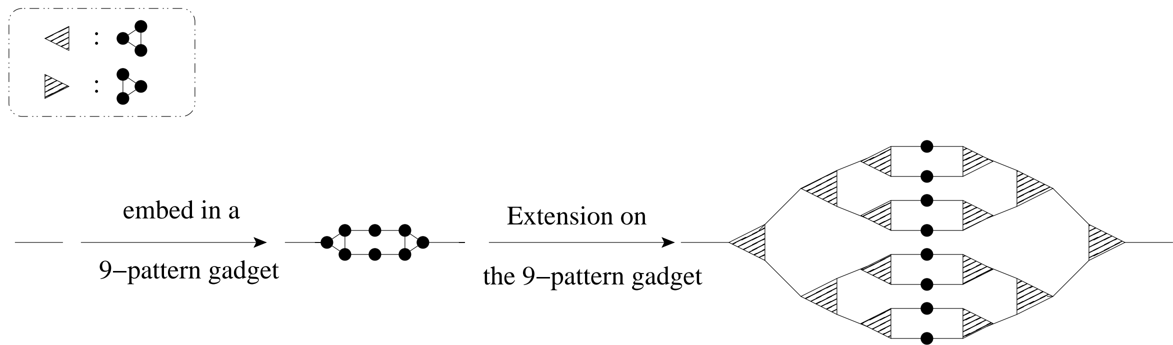


FIGURE 5.12: Embedding 9-pattern in every P-edge in the Petersen graph.

By processing the Petersen graph with the above three steps, a new family of graphs is generated, denoted by $\mathcal{F}^P$, where P denotes that the construction of this

family is based on the Petersen graph. For any graph $G_t^P \in \mathcal{F}^P$, the subscript t denotes the number of times the replacement operation is conducted in Step 3. Note that in Figure 5.12, if we treat a virtual vertex as an actual vertex, the half of the extended 9-gadget embedded in every P-edge is in the form of a binary tree of height 3, with all virtual vertices as the internal vertices and all actual vertices in the middle as leaves. We refer to the half of an extended 9-gadget, obtained by replacing all degree-2 vertices with a new 9-gadget for t times, as a *pseudo binary tree* (henceforth PBT_t), where the height of the pseudo binary tree is $t+1$. This observation simplifies the calculation on the ratio between $OPT(G_t^P)$ and $OPT_{LP}(G_t^P)$ for any graph $G_t^P \in \mathcal{F}^P$.

Figure 5.13 demonstrates $G_0^P \in \mathcal{F}^P$, where no replacement operation was conducted during the construction. In this case, the height of the pseudo binary tree in G_0^P is 1.

Theorem 5.4.1. *For any graph $G_t^P \in \mathcal{F}^P$, the following equations hold.*

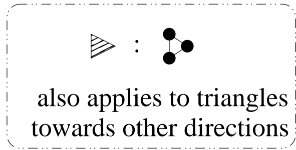
$$\lim_{t\to\infty} \frac{OPT(G_t^P)}{|V(G_t^P)|} = \frac{8}{7}, \tag{5.34}$$

$$\lim_{t\to\infty} \frac{OPT(G_t^P)}{OPT_{LP}(G_t^P)} = \frac{8}{7}. \tag{5.35}$$

Proof. It is known that the Petersen graph has 10 vertices (P-vertices) and 15 edges (P-edges). Let $G_P = (V_P, E_P)$ denote the original Petersen graph, on which the graph G_t^p is derived. We first give an analysis of the result of the transformation on P-vertices and P-edges respectively.

- P-vertices

  Transformed into a triangle, every P-vertex $v \in V_P$ is replaced by a virtual vertex, consisting of 3 vertices and 3 edges, in G_k^P. Therefore the number of



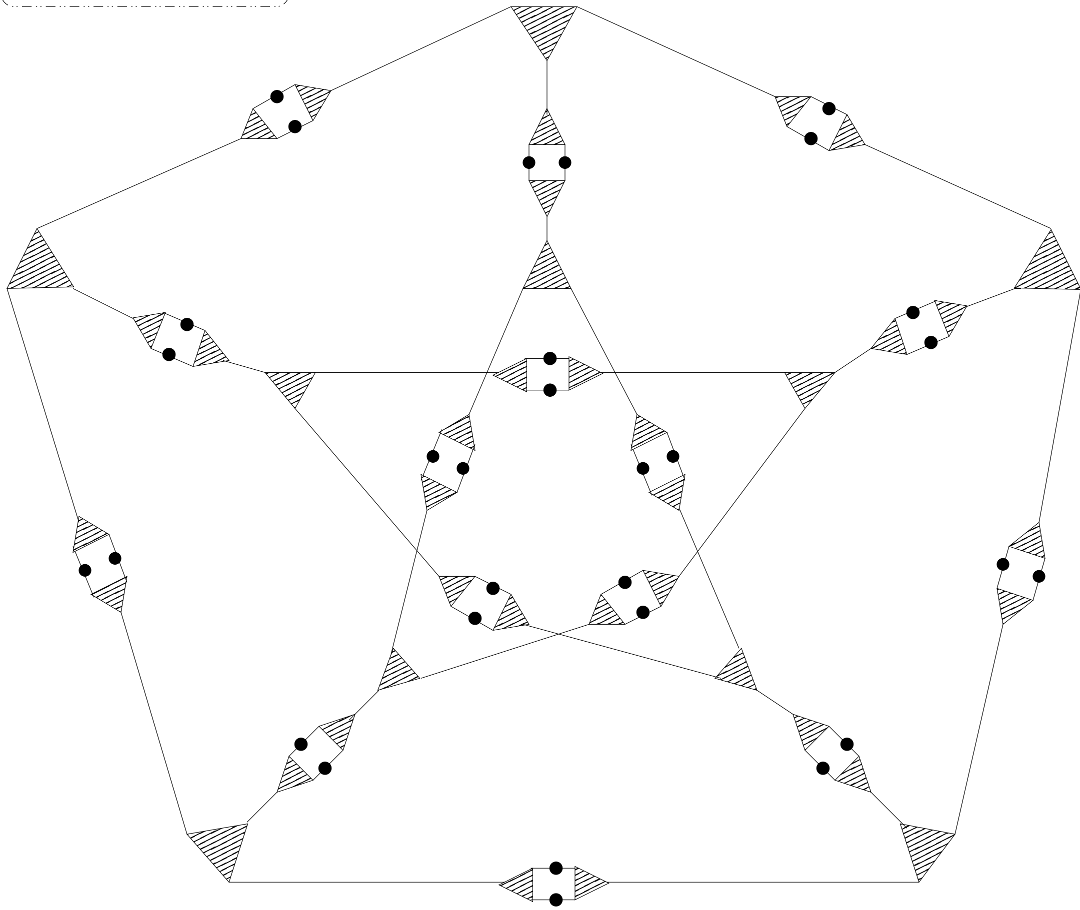


FIGURE 5.13: A special cubic graph G_0^P derived from the combination of the Petersen graph and 9-pattern gadgets.

virtual vertices in G_t^P produced by a P-vertex is

$$|Virtual(v)| = 1 \tag{5.36}$$

- P-edges

  For any P-edge $e \in E_P$, as discussed above, the embedded 9-pattern and its extension transformed e into a 2-edge cut connecting a mirrored pair of pseudo binary trees (PBT henceforth) with the height of $t+1$ in G_t^P. Benefiting from Equations 5.1, 5.2 and 5.4 on binary trees, the total number of virual vertices, actual vertices and 2-edge cuts produced by the transformation on a P-edge $e \in E_P$ in the Petersen graph are as follows:

$$\begin{aligned} |Virtual(e)| &= 2 \times |Internal(PBT_{t+1})| \\ &= 2 \times (2^{t+1} - 1) \\ &= 2^{t+2} - 2, \end{aligned} \tag{5.37}$$

$$\begin{aligned} |Actual(e)| &= |Leaves(PBT_{t+1})| \\ &= 2^{t+1}, \end{aligned} \tag{5.38}$$

$$\begin{aligned} |\textit{2-EdgeCuts}(e)| &= |E(PBT_{t+1})| + 1 \\ &= 2^{t+2} - 1. \end{aligned} \tag{5.39}$$

Therefore, in G_t^P, the total number of virtual vertices, actual vertices and 2-edge cuts are given as follows:

$$\begin{aligned} |Virtual(G_t^P)| &= 10 \times |Virtual(v)| + 15 \times |Virtual(e)| \\ &= 10 \times 1 + 15 \times (2^{t+2} - 2) \\ &= 30 \times 2^{t+1} - 20, \end{aligned} \tag{5.40}$$

$$\begin{aligned} |Actual(G_t^P)| &= 15 \times |Actual(e)| \\ &= 15 \times 2^{t+1}, \end{aligned} \tag{5.41}$$

$$\begin{aligned} |\mathit{2\text{-}EdgeCuts}(G_t^P)| &= 15 \times |\mathit{2\text{-}EdgeCuts}(e)| \\ &= 30 \times 2^{t+1} - 15. \end{aligned} \tag{5.42}$$

These lead to the number of vertices in G_t^P as:

$$\begin{aligned} |V(G_t^P)| &= 3 \times (30 \times 2^{t+1} - 20) + 15 \times 2^{t+1} \\ &= 105 \times 2^{t+1} - 60. \end{aligned} \tag{5.43}$$

It is easily noted that in the simplified model of G_t^P with virtual vertices replacing all triangles, every edge belongs to a 2-edge cut. For solutions to $ILP(G_t^P)$ and $LP(G_t^P)$, every 2-edge cut contributes 2 to both $OPT(G_t^P)$ and $OPT_{LP}(G_t^P)$. As for the case for the special family derived from the 9-pattern gadgets presented in Section 5.3, a virtual vertex contributes 2 to $OPT(G_t^P)$ while it contributes $\frac{3}{2}$ to $OPT_{LP}(G_t^P)$ (see Figure 5.10). This leads to the calculations for $OPT(G_t^P)$ and $OPT_{LP}(G_t^P)$ presented below.

$$\begin{aligned} OPT(G_t^P) &= 2 \times (30 \times 2^{t+1} - 15) + 2 \times (30 \times 2^{t+1} - 20) \\ &= 120 \times 2^{t+1} - 70, \end{aligned} \tag{5.44}$$

$$\begin{aligned} OPT_{LP}(G_t^P) &= 2 \times (30 \times 2^{t+1} - 15) + \frac{3}{2} \times (30 \times 2^{t+1} - 20) \\ &= 105 \times 2^{t+1} - 60. \end{aligned} \tag{5.45}$$

Note that Equations 5.43 and 5.45 suggest that $OPT_{LP}(G_t^P) = n$. Because $OPT_{LP}(G_t^P) \geq n$, it follows that the above solution and corresponding objective value for $LP(G_t^P)$ is optimal.

Forcing t towards infinity, we have

$$\lim_{t\to\infty} \frac{OPT(G_t^P)}{|V(G_t^P)|} = \lim_{t\to\infty} \frac{120 \times 2^{t+1} - 70}{105 \times 2^{t+1} - 60} = \frac{8}{7}, \tag{5.46}$$

$$\lim_{t\to\infty} \frac{OPT(G_t^P)}{OPT_{LP}(G_t^P)} = \lim_{t\to\infty} \frac{120 \times 2^{t+1} - 70}{105 \times 2^{t+1} - 60} = \frac{8}{7}. \tag{5.47}$$

Therefore, Equation 5.34 and Equation 5.35 hold. □

In conclusion, for any graph $G_t^P \in \mathcal{F}^P$, the ratio between $OPT(G_t^P)$ and n as well as the ratio between $OPT(G_t^P)$ and $OPT_{LP}(G_t^P)$ are both $\frac{8}{7}$. In addition, for any $G_t^P \in \mathcal{F}^P$, $OPT_{LP}(G_t^P)$ equals to the number of vertices in G_t^P.

5.5 Conclusion on Gaps

From what has been discussed in the above four sections in this chapter, Corollaries 5.5.1, 5.5.2 and 5.5.3 can be proved easily from Theorems 5.1.1, 5.2.1, 5.3.1 and 5.4.1.

Corollary 5.5.1. *For any approximation algorithm for 2EC on subcubic graphs with a performance guarantee of* $k \cdot n$, $k \geq \frac{4}{3}$.

Corollary 5.5.2. *For any approximation algorithm for 2EC on cubic graphs with a performance guarantee of* $k \cdot n$, $k \geq \frac{7}{6}$.

Corollary 5.5.3. *The integrality gap for 2EC is at least* $\frac{8}{7}$*, even when restricted to subcubic bridgeless graphs.*

In this chapter, four different special families were studied, which drew our attention for different reasons. The first two families of graphs, with square structures

and diamond structures embedded respectively, came from the theoretical studies presented in Chapter 3. They are shown to be barriers to both better approximation algorithms with smaller approximation ratios, as well as tighter upper bound for the integrality gap of $2EC$. These two family of graphs give big ratios, $\frac{4}{3}$ and $\frac{7}{6}$ respectively, between $OPT(G)$ and the number of vertices in G; however, their ratios between $OPT(G)$ and OPT_{LP} are 1. On the other hand, the 9-pattern, as discovered from the experimental studies conducted on millions of graphs, inspires the latter two families of graphs, which both gave a ratio of $\frac{8}{7}$ between $OPT(G)$ and $OPT_{LP}(G)$. This ratio of $\frac{8}{7}$ defines a new lower bound for α^2EC, even when restricted to subcubic graphs, which represents a great improvement on the known lower bound of α^2EC [SV12].

Chapter 6

Conclusion and Future Work

Focusing on the integrality gap of the LP relaxation for 2EC, α^{2EC}, a comprehensive study is presented throughout this thesis.

Theoretically, as a derived result from the $\frac{5}{4}$-approximation algorithm for 2EC on bridgeless cubic graphs, we proved that $\alpha^{2EC} \leq \frac{5}{4}$ for all bridgeless cubic graphs. This finding, along with the $\frac{5}{4}$-approximation algorithm, updated both the approximation ratio and the integrality gap obtained by Csaba, Karpinski and Krysta [CKK02] as $\frac{5}{4} + \epsilon$ for any $\epsilon > 0$. Following from a discussion on the diamond structure and the square structure, which are considered to be the bottleneck for improving our algorithm, we constructed a special family of cubic graphs based on the diamond structure and a special family of subcubic graphs based on the square structure. The former family proved that the performance guarantee for an approximation algorithm for 2EC on a bridgeless cubic graph $G = (V, E)$ is at least $\frac{7}{6}|V|$, while the latter proved that for a bridgeless subcubic graph $G = (V, E)$, the performance guarantee for an approximation algorithm for 2EC on G is at least $\frac{4}{3}|V|$.

On the other hand, with a computational study conducted on millions of graphs divided into three categories (i.e., general, cubic and subcubic), a subcubic graph with 16 vertices stood out by giving the same ratio between $OPT(G)$ and $OPT_{LP}(G)$

as the known best lower bound on α^{2EC}, $\frac{9}{8}$, as given in [SV12]. In addition, with studies on the graphs G which yielded high ratios between $OPT(G)$ and $OPT_{LP}(G)$ in our experiments, the 9-pattern gadget was discovered, with which we raised the lower bound of α^{2EC} to $\frac{8}{7}$ by demonstrating two different families of graphs that give this ratio asymptotically.

There are a few directions for continuing our research.

1. The following conjecture given in [ABEM06] remains open.

   Conjecture 6.0.4. *The integrality gap of the LP relaxation for C2EC is* $\frac{6}{5}$.

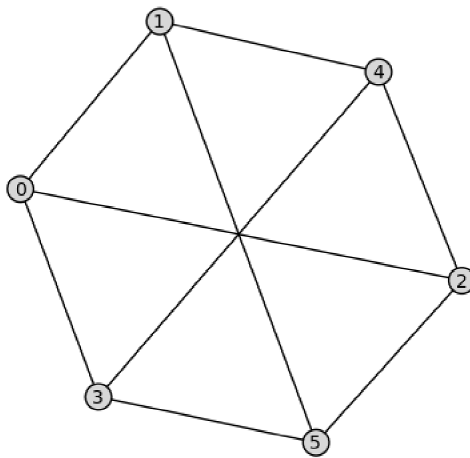
   Considering that the existence of the diamond structure is the major barrier that prevents us from pushing the upper bound of the value of α^{2EC} on all bridgeless cubic graphs to $\frac{6}{5}$, it appears promising to study these structures further and design a proper handling for them in order to find a $\frac{6}{5}$-approximation algorithm for 2EC on all bridgeless cubic graphs, and thus obtain a new upper bound, $\frac{6}{5}$, on the value of α^{2EC}. This will definitely lend some support to Conjecture 6.0.4, which we believe to be true. It also remains possible that more research on the diamond structure, as well as the square structure, may lead us to a family of graphs which contradicts Conjecture 6.0.4.

2. With more time and resources, we can continue our computational study by expanding the set of research objects, which may lead to a tighter lower bound on the value of α^{2EC}. In addition, investigating every graph G which gives a large ratio between $OPT(G)$ and $OPT_{LP}(G)$ may shed light on other structures which may also lead to a tighter lower bound on the value of α^{2EC}.

Appendix A

Sample Models for ILP(G) and LP(G)

In both Appendix A and Appendix B, the sample models and sample solutions are given for the following cubic graph $G = (V, E)$ with 6 vertices. The adjacency matrix for G is also provided.



cubic graph $G = (V, E)$ with 6 vertices.

$ILP(G)$: `Cubic6.ILP.1.lp`

Minimize

$$X0-1+X0-2+X0-3+X1-4+X1-5+X2-4+X2-5+X3-4+X3-5$$

Subject To

$$X1-5+X2-5+X3-5 \quad >= \quad 2$$
$$X1-4+X2-4+X3-4 \quad >= \quad 2$$
$$X1-4+X2-4+X3-4+X1-5+X2-5+X3-5 \quad >= \quad 2$$
$$X0-3+X3-4+X3-5 \quad >= \quad 2$$
$$X0-3+X3-4+X1-5+X2-5 \quad >= \quad 2$$
$$X0-3+X3-5+X1-4+X2-4 \quad >= \quad 2$$
$$X0-3+X1-4+X2-4+X1-5+X2-5 \quad >= \quad 2$$
$$X0-2+X2-4+X2-5 \quad >= \quad 2$$
$$X0-2+X2-4+X1-5+X3-5 \quad >= \quad 2$$
$$X0-2+X2-5+X1-4+X3-4 \quad >= \quad 2$$
$$X0-2+X1-4+X3-4+X1-5+X3-5 \quad >= \quad 2$$
$$X0-2+X2-4+X2-5+X0-3+X3-4+X3-5 \quad >= \quad 2$$
$$X0-2+X2-4+X0-3+X3-4+X1-5 \quad >= \quad 2$$
$$X0-2+X2-5+X0-3+X3-5+X1-4 \quad >= \quad 2$$
$$X0-2+X0-3+X1-4+X1-5 \quad >= \quad 2$$
$$X0-1+X1-4+X1-5 \quad >= \quad 2$$
$$X0-1+X1-4+X2-5+X3-5 \quad >= \quad 2$$
$$X0-1+X1-5+X2-4+X3-4 \quad >= \quad 2$$
$$X0-1+X2-4+X3-4+X2-5+X3-5 \quad >= \quad 2$$
$$X0-1+X1-4+X1-5+X0-3+X3-4+X3-5 \quad >= \quad 2$$
$$X0-1+X1-4+X0-3+X3-4+X2-5 \quad >= \quad 2$$
$$X0-1+X1-5+X0-3+X3-5+X2-4 \quad >= \quad 2$$
$$X0-1+X0-3+X2-4+X2-5 \quad >= \quad 2$$
$$X0-1+X1-4+X1-5+X0-2+X2-4+X2-5 \quad >= \quad 2$$
$$X0-1+X1-4+X0-2+X2-4+X3-5 \quad >= \quad 2$$
$$X0-1+X1-5+X0-2+X2-5+X3-4 \quad >= \quad 2$$
$$X0-1+X0-2+X3-4+X3-5 \quad >= \quad 2$$
$$X0-1+X1-4+X1-5+X0-2+X2-4+X2-5+X0-3+X3-4+X3-5 \quad >= \quad 2$$
$$X0-1+X1-4+X0-2+X2-4+X0-3+X3-4 \quad >= \quad 2$$
$$X0-1+X1-5+X0-2+X2-5+X0-3+X3-5 \quad >= \quad 2$$
$$X0-1+X0-2+X0-3 \quad >= \quad 2$$

Binaries

$$X0-1X0-2X0-3X1-4X1-5X2-4X2-5X3-4X3-5$$

End

$LP(G)$: `Cubic6.LP.1.lp`

Minimize

$$X0-1+X0-2+X0-3+X1-4+X1-5+X2-4+X2-5+X3-4+X3-5$$

Subject To

$$
\begin{aligned}
X1-5+X2-5+X3-5 &>= 2\\
X1-4+X2-4+X3-4 &>= 2\\
X1-4+X2-4+X3-4+X1-5+X2-5+X3-5 &>= 2\\
X0-3+X3-4+X3-5 &>= 2\\
X0-3+X3-4+X1-5+X2-5 &>= 2\\
X0-3+X3-5+X1-4+X2-4 &>= 2\\
X0-3+X1-4+X2-4+X1-5+X2-5 &>= 2\\
X0-2+X2-4+X2-5 &>= 2\\
X0-2+X2-4+X1-5+X3-5 &>= 2\\
X0-2+X2-5+X1-4+X3-4 &>= 2\\
X0-2+X1-4+X3-4+X1-5+X3-5 &>= 2\\
X0-2+X2-4+X2-5+X0-3+X3-4+X3-5 &>= 2\\
X0-2+X2-4+X0-3+X3-4+X1-5 &>= 2\\
X0-2+X2-5+X0-3+X3-5+X1-4 &>= 2\\
X0-2+X0-3+X1-4+X1-5 &>= 2\\
X0-1+X1-4+X1-5 &>= 2\\
X0-1+X1-4+X2-5+X3-5 &>= 2\\
X0-1+X1-5+X2-4+X3-4 &>= 2\\
X0-1+X2-4+X3-4+X2-5+X3-5 &>= 2\\
X0-1+X1-4+X1-5+X0-3+X3-4+X3-5 &>= 2\\
X0-1+X1-4+X0-3+X3-4+X2-5 &>= 2\\
X0-1+X1-5+X0-3+X3-5+X2-4 &>= 2\\
X0-1+X0-3+X2-4+X2-5 &>= 2\\
X0-1+X1-4+X1-5+X0-2+X2-4+X2-5 &>= 2\\
X0-1+X1-4+X0-2+X2-4+X3-5 &>= 2\\
X0-1+X1-5+X0-2+X2-5+X3-4 &>= 2\\
X0-1+X0-2+X3-4+X3-5 &>= 2\\
X0-1+X1-4+X1-5+X0-2+X2-4+X2-5+X0-3+X3-4+X3-5 &>= 2\\
X0-1+X1-4+X0-2+X2-4+X0-3+X3-4 &>= 2\\
X0-1+X1-5+X0-2+X2-5+X0-3+X3-5 &>= 2\\
X0-1+X0-2+X0-3 &>= 2
\end{aligned}
$$

Continues...

Bounds

$X0 - 1 \quad >= \quad 0$
$X0 - 2 \quad >= \quad 0$
$X0 - 3 \quad >= \quad 0$
$X1 - 4 \quad >= \quad 0$
$X1 - 5 \quad >= \quad 0$
$X2 - 4 \quad >= \quad 0$
$X2 - 5 \quad >= \quad 0$
$X3 - 4 \quad >= \quad 0$
$X3 - 5 \quad >= \quad 0$
$X0 - 1 \quad <= \quad 1$
$X0 - 2 \quad <= \quad 1$
$X0 - 3 \quad <= \quad 1$
$X1 - 4 \quad <= \quad 1$
$X1 - 5 \quad <= \quad 1$
$X2 - 4 \quad <= \quad 1$
$X2 - 5 \quad <= \quad 1$
$X3 - 4 \quad <= \quad 1$
$X3 - 5 \quad <= \quad 1$

End

Appendix B

Sample Gurobi^{TM} Solutions to ILP(G) and LP(G)

- **Solving $ILP(G)$ stored in Cubic6.ILP.1.lp**

```
Read LP format model from file Cubic6.ILP.1.lp
Reading time = 0.00 seconds
(null):  31 rows, 9 columns, 144 nonzeros
Optimize a model with 31 rows, 9 columns and 144 nonzeros
Presolve removed 25 rows and 0 columns
Presolve time:  0.00s
Presolved:  6 rows, 9 columns, 18 nonzeros
Variable types:  0 continuous, 9 integer (9 binary)
Found heuristic solution:  objective 6.0000000
```

Giving solution:

```
# Objective value = 6
X0-1 0          X0-2 1          X0-3 1
X1-4 1          X1-5 1          X2-4 0
X2-5 1          X3-4 1          X3-5 0
```

- **Solving $LP(G)$ stored in Cubic6.LP.1.lp**

```
Read LP format model from file Cubic6.LP.1.lp
Reading time = 0.00 seconds
(null):  31 rows, 9 columns, 144 nonzeros
Optimize a model with 31 rows, 9 columns and 144 nonzeros
Presolve time:  0.00s
Presolved:  31 rows, 9 columns, 144 nonzeros
 Iteration Objective Primal Inf.  Dual Inf.  Time
0 0.0000000e+00 6.200000e+01 0.000000e+00 0s
8 6.0000000e+00 0.000000e+00 0.000000e+00 0s
 Solved in 8 iterations and 0.00 seconds
Optimal objective 6.000000000e+00
```

Giving solution:

```
# Objective value = 6
X0-1 0          X0-2 1          X0-3 1
X1-4 1          X1-5 1          X2-4 0
X2-5 1          X3-4 1          X3-5 0
```

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