

On the Existence and Stability of Rotating Wave Solutions to Lattice Dynamical Systems

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Abstract

Rotating wave solutions to evolution equations have been shown to govern many important biological and chemical processes. Much of the rigorous mathematical investigations of rotating waves rely on the model exhibiting a continuous Euclidean symmetry, which is only present in an idealized situation. Here we investigate the existence of rotationally propagating solutions in a discrete spatial setting, in which typical symmetry methods cannot be applied, thus presenting an unique perspective on rotating waves.

Our goal in this thesis is to demonstrate the existence and potential stability of rotating wave solutions to a spatially discretized infinite systems of coupled differential equations. This goal is achieved by considering so-called Lambda-Omega systems, which have frequently been used to model typical oscillatory dynamics. Our work is broken into three major components:

1. An infinite system of coupled phase equations is investigated and we demonstrate that under some mild assumptions the system exhibits a phase-locked rotating wave solution. The phase system is derived from a limiting case of the original Lambda-Omega system, and therefore solutions of the phase equation will be useful in finding rotating wave solutions to the full Lambda-Omega system.
2. We examine the stability of the rotating wave solution found in the coupled phase equations. This is achieved by providing a link with an underlying graph-theoretic geometry endowed by the spatially discretized system. We use results from random walks on infinite graphs to provide a general stability theorem for coupled phase equations.
3. We use the rotating wave solution of the phase equations to extend to a rotating wave solution of the full Lambda-Omega system. This result is achieved using a non-standard Implicit Function Theorem, since we show that typical implicit function arguments cannot be applied to our present situation.

Dedications

To my beautiful wife, my parents and my sister.

Acknowledgement

I would like to thank my supervisors Benoit Dionne and Victor LeBlanc for all their help through this long process. Without the two of them none of this work would be possible. Secondly, I would like to thank the University of Ottawa for providing me with this excellent opportunity to further my education in such a fantastic mathematics department. This work has also been generously supported throughout my time at the University of Ottawa by an Ontario Graduate Scholarship.

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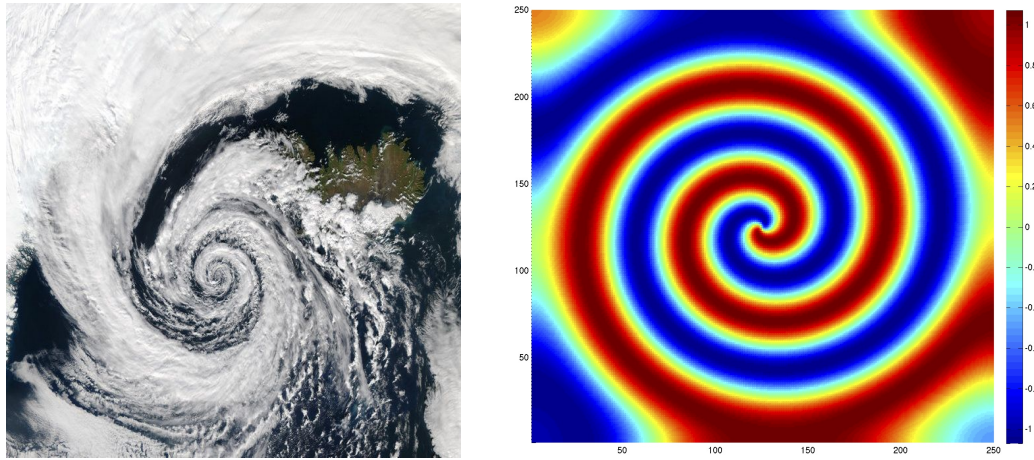
Chapter 1

Introduction

Non-equilibrium dynamics characterize much of the world around us, from the movement of microscopic bacteria to the motion of the planets. Such dynamics are also a central focus in modern analysis of dynamical systems and bifurcation theory. In particular, wave propagation through a given spatial medium remains an intense area of study to modern experts in partial differential equations and the physical sciences. Linearly propagating solutions to differential equations often arise as traveling waves or pulses, whose temporal evolution can be described by a linear translation in space. Solutions of this type often arise quite naturally when considering invading species in spatial ecology or electrical impulses propagating through a one-dimensional medium such as nerves. On the other hand, differential equations may exhibit rotationally propagating solutions, referred to as rotating waves, which come as time-periodic solutions whose temporal evolution can be described by a rotation in space.

Examples of rotating waves abound in nature and have been an intense area of rigorous mathematical investigation for many decades now. Spiral waves are a particularly important example of rotating waves which present themselves as striking visual patterns (see Figure 1.1), whose formal mathematical study dates back to the work of Winfree in chemical reaction theory [72, 73]. There are some obvious examples in nature of spiral waves, such as hurricanes and the shape of galaxies, but they have also been shown to arise in more subtle areas and are thought to be associated with many serious phenomena in electrophysiological pathologies. This includes, but is not limited to, cortical spreading depression, hallucinations and ventricular fibrillation [6, 34, 44, 46, 49, 64]. With spiral waves occurring in such circumstances, it follows that they remain an active and intense area of study both in mathematics and throughout the physical sciences.

The goal of this thesis is to expand the current knowledge of spiral waves from a mathematical perspective. We will achieve this goal by demonstrating the existence of spiral wave solutions to spatially discretized systems of infinitely-many coupled differential equations. This work is aimed at furthering the scientific community's present



(a) A hurricane taking the form of a spiral wave. Image courtesy of NASA's public inventory.

(b) Numerical integration of a differential equation resulting in a spiral wave solution.

Figure 1.1: Spiral waves both in physical reality and theoretical analysis. Temporal evolution is given by a continuous clockwise or counter-clockwise rotation of the basic spiral pattern.

understanding of the behaviour of spiral waves in the absence of key symmetries in the differential equation, thus providing an unique perspective on the dynamics of spiral waves.

1.1 Euclidean Symmetry

Mathematical investigations of rotating waves have highlighted that the underlying symmetry of a differential equation plays a critical role in understanding the dynamics and bifurcations of these waves [3, 4, 33, 63]. Typical investigations of spiral waves focus on reaction-diffusion equations for exactly this reason. That is, consider the partial differential equation (PDE) of the form

$$\frac{\partial u}{\partial t} = D \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) + \mathcal{F}(u), \quad (1.1.1)$$

where $u = u(x, y, t) : \mathbb{R}^2 \times \mathbb{R}^+ \rightarrow \mathbb{R}^n$, $D > 0$ is the diffusion coefficient and $\mathcal{F} : \mathbb{R}^n \rightarrow \mathbb{R}^n$, for some $n \geq 1$. Equation (1.1.1) possesses an important symmetry property: if $u(x, y, t)$ is a solution to (1.1.1) then so is

$$\tilde{u}(x, y, t) = u(x \cos \theta - y \sin \theta + p_1, x \sin \theta + y \cos \theta + p_2, t), \quad (1.1.2)$$

for any angle θ and translation $(p_1, p_2) \in \mathbb{R}^2$ (this can easily be checked using the chain rule). These rotations and translations together form the special Euclidean group, often denoted $SE(2)$, and equation (1.1.1) precisely is said to be invariant with respect to the action of this continuous group on suitable function spaces. In this way we see that the differential equation (1.1.1) belongs to a larger class of differential equations which are invariant with respect to the special Euclidean group.

Using the symmetry property (1.1.2) of the differential equation we can construct a centre-manifold reduction of the infinite-dimensional partial differential equation to a finite-dimensional system of ordinary differential equations near rotating wave solutions of (1.1.1) [61, 62]. The dynamics on this centre manifold have been shown to govern much of the dynamics and bifurcations of rotating waves, thus furthering the understanding of such waves in the continuous spatial medium.

Euclidean symmetry has been an excellent tool to describe the macroscopic behaviour of rotating waves, but one should note that in reality it is a modelling hypothesis which is, at best, an approximation. That is, bounded domains, heterogeneities and anisotropy are all important in physical models and violate a Euclidean symmetry assumption. In experiments, spiral waves have been shown to anchor around inhomogeneities [55], drift along boundaries [74], and exhibit phase-locking of two-frequency meandering spirals in the presence of anisotropy [59]. These marked differences in the qualitative dynamics of spiral waves when a Euclidean symmetry assumption is violated in the model have therefore motivated this work in an effort to understand how the dynamic behaviour of the spiral waves differ when the continuous Euclidean symmetry is absent.

An important and motivating example for this question comes from an investigation of spiral wave dynamics due to Ashwin, Melbourne and Nicol [1]. Their work details an important theoretical result in the presence of full Euclidean symmetry which we summarize here. Begin by considering a spiral wave with rotational frequency controlled by a parameter, ω . Their work details that upon letting $\omega \rightarrow 0$ the centre of rotation of the spiral wave should move in a single direction without bound and at the critical value $\omega = 0$ the spiral completely unwinds itself giving a solution which is linearly translating with nonzero speed. As the parameter ω is varied further beyond this critical value, a reverse wound spiral should form. This qualitative change in dynamics is termed a *drift bifurcation*. The authors remark that they had attempted many times to provide numerical simulations to confirm their results, although they were unsuccessful. One immediately infers that a possibility for their unsuccessful attempts was due to the fact that any numerical scheme requires a discretization of space, thus breaking the Euclidean symmetry assumption in the model.

Similar lines of questioning have led to some investigations of symmetry-breaking perturbations, which have demonstrated small, but measurable, discrepancies between systems with full symmetry and those with broken symmetry [10, 52, 53].

This work attempts to provide the necessary incremental steps toward addressing these questions from an alternative perspective, by considering only discrete spatial symmetries. The centre manifold reductions in the presence of Euclidean symmetry require continuous symmetry groups and therefore moving to discrete symmetry groups necessitates entirely different techniques to analyze solutions to such differential equations.

1.2 Lattice Dynamical Systems

As an alternative to considering symmetry breaking perturbations, investigations in the absence of Euclidean symmetry have led to considering a countably infinite system of coupled ordinary differential equations, termed a lattice dynamical system (LDS), of the form

$$\dot{u}_\eta = g_\eta(\{u_\xi\}_{\xi \in \Lambda}), \quad \eta \in \Lambda. \quad (1.2.1)$$

Here Λ is a discrete subset of \mathbb{R}^N , commonly referred to as a lattice, and the u_η are time-dependent functions indexed by the lattice. We consider the variables u_η for each $\eta \in \Lambda$ to be coordinates in the state vector $u = \{u_\eta\}_{\eta \in \Lambda}$ and g_η a function on these coordinates. The notation \dot{u}_η for $\eta \in \Lambda$ is taken to be differentiation with respect to time, i.e. $\dot{u}_\eta = du_\eta/dt$. Differential equations of the type (1.2.1) generally require that $u(t)$ belongs to a Banach space of sequences indexed by Λ , to ensure the solution is sufficiently well-behaved.

Throughout this thesis we will consider lattice dynamical systems posed on the two-dimensional integer lattice of the form

$$\dot{u}_{i,j} = \alpha(u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1} - 4u_{i,j}) + f(u_{i,j}), \quad (1.2.2)$$

where $u_{i,j} = u_{i,j}(t) : \mathbb{R} \rightarrow \mathbb{R}^n$, $n \geq 1$, for each $(i,j) \in \mathbb{Z}^2$. Here $\alpha > 0$ is regarded as the strength of coupling between neighbouring elements in the integer lattice and $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a general nonlinearity. One can see that in moving from the partial differential equation context of (1.1.1) to that of the lattice dynamical system (1.2.2) we have replaced the continuous two-dimensional spatial medium with a grid, or lattice, which moves our problem into a discrete spatial setting. The coupling in (1.2.2) is often referred to as *nearest-neighbour* coupling with respect to the lattice \mathbb{Z}^2 . For the ease of notation, throughout this thesis we will simply write

$$\sum_{i',j'} (u_{i',j'} - u_{i,j}) := u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1} - 4u_{i,j}, \quad (1.2.3)$$

where the sum is taken over the four nearest-neighbours of the index (i,j) . One may also consider more complicated connection topologies, as is being done in some

recent works on finite lattices [19, 69], but since the long term objective is to explore solutions to differential equations in discrete space versus continuous space, the nearest-neighbour connections of system (1.2.2) will suffice.

The nearest-neighbour coupling of system (1.2.2) can be derived as the leading order of a typical finite-difference approximation of the second order spatial derivatives of the reaction-diffusion equation (1.1.1). Indeed, for a spatial step size $h > 0$, one may use the approximation

$$\frac{\partial^2 u}{\partial x^2}(x, y) \approx \frac{u(x+h, y) + u(x-h, y) - 2u(x, y)}{h^2}, \quad (1.2.4)$$

and an analogous approximation for $\partial^2 u / \partial y^2$. Then moving to the spatial grid $x = i$ and $y = j$ for $i, j \in h\mathbb{Z}$, (1.1.1) gives the discrete spatial approximation

$$\frac{d}{dt}u(i, j, t) \approx \frac{D}{h^2} \sum_{i', j'} (u(i', j', t) - u(i, j, t)) + \mathcal{F}(u(i, j, t)) \quad (1.2.5)$$

for each $i, j \in h\mathbb{Z}$. Therefore, in this context the reader may interpret the coupling strength α as the quotient of the diffusion coefficient and the square of the spatial step size.

In addition to lattice systems being a prototype for spatial discretizations of PDEs, they have proven extremely useful in describing numerous phenomena irrespective of their continuous space counterparts. LDSs arise naturally in various physical settings such as material science, in particular metallurgy, where LDSs have been used to model solidification of alloys [8, 13], chemical reactions [29], optics [31] and biology; particularly with chains of coupled oscillators arising in models of neural networks [23, 25]. For these applications and many more, LDSs have therefore proven to be an interesting area of research in their own right.

The choice to study lattice dynamical systems as prototypes for the behaviour of waves in the absence of Euclidean symmetry comes from their overwhelming success in furthering our understanding of traveling wave dynamics. Let us begin by illustrating an important and motivating example. Consider the reaction-diffusion equation in one spatial dimension

$$\frac{\partial u}{\partial t} = D \frac{\partial^2 u}{\partial x^2} + u(1-u)(a-u), \quad (1.2.6)$$

where $D > 0$, $a \in (0, 1)$ and $u = u(x, t) : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$. Finding traveling wave solutions to (1.2.6) requires determining the existence of a solution of the form $u(x, t) = \phi(x - ct)$, where $c \in \mathbb{R}$ is the wave speed and $\phi : \mathbb{R} \rightarrow \mathbb{R}$ is the wave profile, satisfying appropriate boundary conditions. Much work has been done to demonstrate the existence of these desired solutions to equation (1.2.6), beginning with the pioneering work of Fife and McLeod [30]. One finds that the solutions ϕ not only exist, but further exhibits an explicit dependence between the wave speed, c , and the parameter

a. Particularly, at the critical parameter value $a = 1/2$ the waves have zero speed and thus fail to propagate through the continuous spatial medium. This phenomenon is often referred to as *propagation failure*.

The analogous lattice dynamical system to (1.2.6) is

$$\dot{u} = \alpha(u_{i+1} + u_{i-1} - 2u_i) + u_i(1 - u_i)(a - u_i), \quad i \in \mathbb{Z}. \quad (1.2.7)$$

Traveling wave solutions now take the form $u_i(t) = \phi(i - ct)$, again for a wave profile $\phi : \mathbb{R} \rightarrow \mathbb{R}$ and appropriate boundary conditions. Searching for traveling wave solutions to (1.2.7) requires considerably different techniques to that of the continuous spatial medium, with the existence of such solutions being demonstrated most notably by Zinner [75]. Here the wave speed has not been explicitly related to the parameter a , although Keener has demonstrated that when coupling is sufficiently weak there are entire open regions in parameter space which lead to propagation failure [48].

By moving from one spatial dimension to two spatial dimensions we arrive at further comparisons between the discrete and continuous spatial settings. For example, isotropy of (1.1.1) implies that the direction in which a traveling wave propagates does not effect the qualitative dynamics of the wave. This is not necessarily the case in the discrete spatial setting, since it has been shown that the direction of propagation can play a direct role in determining the waves ability to propagate through the discrete spatial medium [9].

To date there have been numerous studies on the existence and properties of traveling wave solutions to lattice dynamical systems, with a particular emphasis on fronts which fail to propagate through the discrete spatial medium [9, 22, 45, 48]. This thesis is therefore motivated by the many investigations of traveling waves demonstrating the slight, but measurable, differences in dynamics between continuous and discrete space. That is, it has become apparent that systems which do not satisfy a continuous Euclidean symmetry assumption can in some cases provide qualitatively different traveling wave solutions to the Euclidean invariant case, and hence one is naturally led to question how these investigations can be extended to the study of rotating waves. Therefore this thesis aims to further this line of investigation by considering rotational propagation, providing the necessary existence results, in a similar way to what Zinner has done for traveling waves [75].

1.3 Spatially Discrete Lambda-Omega Systems

Our investigation begins by considering so-called Lambda-Omega reaction-diffusion systems, typically written in terms of a single complex variable $z(x, y, t) : \mathbb{R}^2 \times \mathbb{R} \rightarrow \mathbb{C}$ of the form

$$\frac{\partial z}{\partial t} = D \left(\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} \right) + z[\lambda(|z|) + i\omega(|z|)], \quad (1.3.1)$$

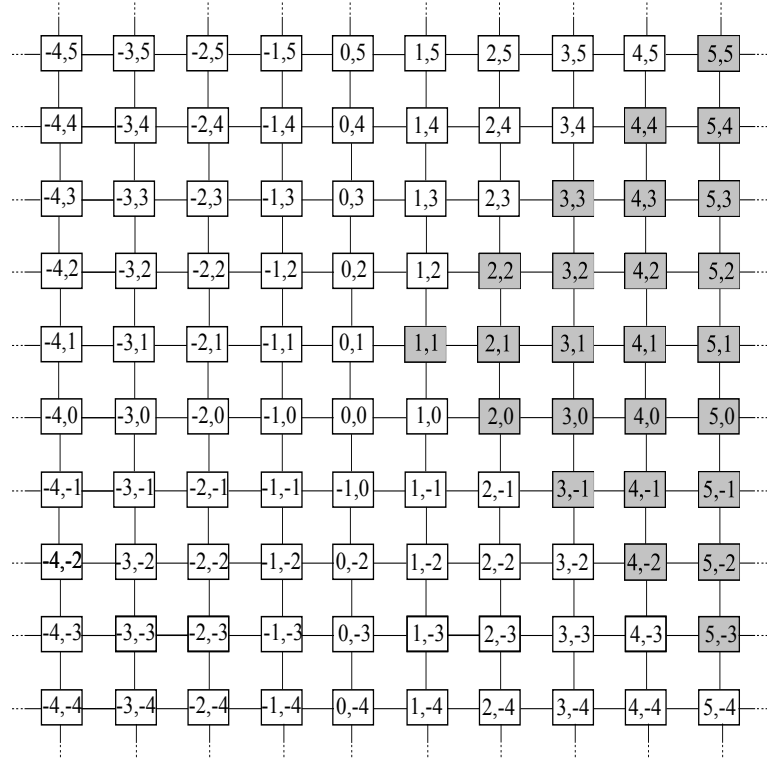


Figure 1.2: A 2-dimensional lattice. Connections between cells represent the nearest-neighbour coupling inferred by equation (1.3.2). The shaded cells represent the indices of the reduced system, as will be defined in Definition 5.2.1.

where $i = \sqrt{-1}$ is the imaginary constant. The specific forms of the functions λ and ω are typically taken as generalizations of the functions $\lambda(R) = \pm a^2 \mp R^2$ for some $a > 0$ and $\omega(R)$ a constant function. Since their inception by Howard and Kopell, Lambda-Omega systems have become an archetype for oscillatory behaviour in reaction-diffusion systems [42]. These reaction-diffusion equations come as generalizations of the complex Ginzburg-Landau equation, and are well-known to arise as the lowest order perturbation of any reaction-diffusion system near a Hopf bifurcation [12]. Most importantly is that PDEs of this type are well-known to exhibit spiral wave solutions [12, 35, 43, 27], and therefore provides a natural starting point for the mathematical investigation presented here.

In this work we will consider the spatially discrete LDS analogue of (1.3.1) given by

$$\dot{z}_{i,j} = \alpha \sum_{i',j'} (z_{i',j'} - z_{i,j}) + z_{i,j} [\lambda(|z_{i,j}|) + i\omega(|z_{i,j}|, \alpha)], \quad (i, j) \in \mathbb{Z}^2. \quad (1.3.2)$$

Figure 1.2 shows an interpretation of the lattice in which the boxes represent the distinct elements and the lines connected to the neighbouring boxes represent the nearest-neighbour coupling of system (1.3.2). Since Lambda-Omega systems provided the setting for the first rigorous inspection of spiral wave solutions to PDEs [12], it is therefore natural to consider (1.3.2) for the investigation of spiral waves in LDSs. Throughout this thesis we will make the following assumptions on the λ and ω functions.

Hypothesis 1.3.1. *The functions λ and ω in (1.3.2) satisfy the following:*

- (1) $\lambda : [0, \infty) \rightarrow \mathbb{R}$ is continuously differentiable and there exists some $a > 0$, with the property that $\lambda(a) = 0$ and $\lambda'(a) \neq 0$.
- (2) $\omega = \omega(R, \alpha) : [0, \infty) \times \mathbb{R} \rightarrow \mathbb{R}$ is continuously differentiable in both its arguments such that

$$\omega(R, \alpha) - \omega(a, \alpha) = \alpha \omega_1(R, \alpha), \quad (1.3.3)$$

for some function $\omega_1(R, \alpha)$ which is continuously differentiable on the same domain with $\omega_1(a, \alpha) = 0$ for all $\alpha \in \mathbb{R}$.

Conditions (1) and (2) of Hypothesis 1.3.1 arise as natural generalizations of the normal form of a Hopf bifurcation and are similar to those which were assumed by Cohen et. al. and Greenberg in their respective proofs of spiral wave solutions in spatially-continuous reaction-diffusion equations [12, 35]. As previously mentioned, typical investigations consider $\lambda(R) = \pm a^2 \mp R^2$, which clearly satisfies condition (1) and $\omega(R, \alpha)$ to be a constant function independent of its arguments. When extending to non-constant functions $\omega(R, \alpha)$, in similar works exploring rotating waves in Lambda-Omega systems, these functions are taken to be slight perturbations of constant functions. Typically one considers

$$\omega(R) = \beta + \varepsilon R^2, \quad (1.3.4)$$

where β is a real-valued constant and ε is a small parameter. Then writing $\varepsilon = \alpha \varepsilon'$ we recast (1.3.4) to satisfy the condition (2) by observing that

$$\begin{aligned} \omega(R) - \omega(a, \alpha) &= \beta + \alpha \varepsilon' R^2 \\ &= \beta + \alpha \varepsilon' a^2 + \alpha [2\varepsilon' a(R - a) + \varepsilon'(R - a)^2], \end{aligned} \quad (1.3.5)$$

where we have $\omega(a, \alpha) = \beta + \alpha \varepsilon' a^2$ and $\omega_1(R, \alpha) = 2\varepsilon' a(R - a) + \varepsilon'(R - a)^2$, fitting the requirements of condition (2). An important point to note is that as $\alpha \rightarrow 0^+$ the function ω reduces to a constant function, which is what has been found to be a necessary condition in related works and numerical integrations of the system on a finite lattice.

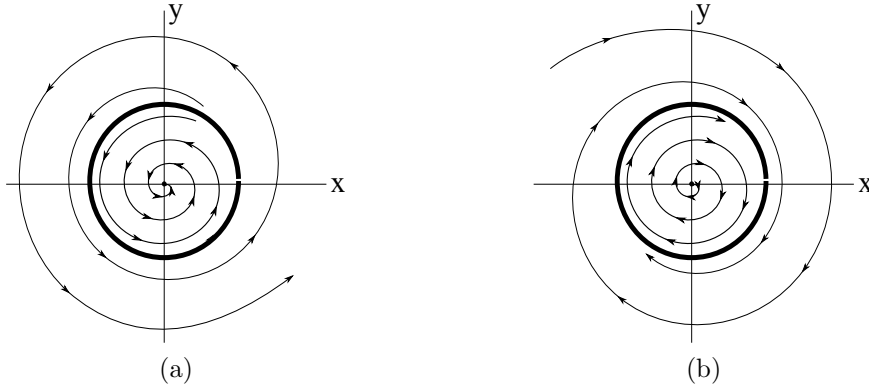


Figure 1.3: Typical phase portraits of the periodic solution (1.3.9) and some nearby trajectories. There are two cases: (a) locally repelling when $\lambda'(a) > 0$ and (b) locally attracting when $\lambda'(a) < 0$.

The most important characteristic of our hypotheses on the λ and ω functions is that in the absence of coupling ($\alpha = 0$) each component exhibits an identical periodic oscillation with amplitude a and frequency $2\pi/\omega(a, 0)$. Indeed, notice that when $\alpha = 0$ the elements of system (1.3.2) completely decouple and are therefore acting independently of each other, resulting in the system

$$\dot{z}_{i,j} = z_{i,j}[\lambda(|z_{i,j}|) + i\omega(|z_{i,j}|, 0)], \quad (1.3.6)$$

for each $(i, j) \in \mathbb{Z}^2$. Decomposing each $z_{i,j}$ into polar variables using the ansatz

$$z_{i,j}(t) = r_{i,j}(t)e^{i\theta_{i,j}(t)} \quad (1.3.7)$$

results in the set of ordinary differential equations

$$\begin{aligned} \dot{r}_{i,j} &= r_{i,j}\lambda(r_{i,j}), \\ \dot{\theta}_{i,j} &= \omega(a, 0), \end{aligned} \quad (1.3.8)$$

for each $(i, j) \in \mathbb{Z}^2$. Taking $r_{i,j} = a$ leads to a periodic solution of the form

$$z_{i,j}(t) = ae^{i(\omega(a,0)t + \theta_{i,j}^0)}, \quad (1.3.9)$$

where $\theta_{i,j}^0 \in S^1$ is an initial phase value for each $(i, j) \in \mathbb{Z}^2$. Furthermore, the non-degeneracy condition $\lambda'(a) \neq 0$ guarantees that this solution is either locally attracting or repelling. Figure 1.3 gives characteristic phase portraits of the system (1.3.8) in the Cartesian plane upon writing $z_{i,j} = x + iy$.

Therefore, when $\alpha = 0$ we see that our elements can fall into independent periodic oscillation patterns. Our goal in this thesis is to determine the existence of

synchronized periodic oscillation patterns as coupling is introduced to the system. In particular we are interested in finding synchronized spatial patterns which are analogous to rotating and spiral waves in the discrete spatial setting. In doing so we provide the necessary existence results which will lead to future investigations into rotational propagation through discrete spatial mediums.

1.4 Outline of the Thesis

To obtain the existence of rotating wave solutions to (1.3.2), our work here is broken down into incremental steps. In Chapter 2 we reduce the full Lambda-Omega LDS down to a system of coupled phase equations indexed by the two-dimensional integer lattice in the limit as $\alpha \rightarrow 0^+$. We prove a theorem demonstrating the existence of rotating wave solutions in this context for a wide class of these coupled phase equations by extending previous results on finite square lattices. We also detail how the study of this reduced phase model and our existence result provide rotating wave solutions to the full Lambda-Omega system (1.3.2) when coupling is absent.

Following our demonstration of rotating wave solutions to our coupled phase model, we review some relevant facts and definitions from the theory of infinite graphs. This overview is provided in Chapter 3, and in Chapter 4 we will demonstrate how these results can be applied to obtain local asymptotic stability of our rotating wave solution. Chapter 4 breaks down into two major components: spectral stability results and a more general stability theorem obtained using results from random walks on infinite graphs. The results pertaining to the former component give local exponential stability of the rotating wave in the Hilbert space setting, whereas the latter component addresses some shortcomings of the spectral stability section by extending these results to algebraic decay rates for slight perturbations off the rotating wave solution. Chapter 4 in essence addresses problems which are unique to infinite-dimensional differential equations such as (1.3.2) in that the Banach space setting of the problem greatly influences the stability of a given solution.

The existence of rotating wave solutions to (1.3.2) is stated and proved in Chapter 5. It is here that we will see that some technical hurdles prevent the direct application of the traditional Implicit Function Theorem to extend the solution at $\alpha = 0$ of Chapter 2 into sufficiently small $\alpha > 0$ values. This necessitates a non-standard Implicit Function Theorem which allows us to extend the rotating wave solution at $\alpha = 0$ into positive coupling coefficients under weaker assumptions than the standard Implicit Function Theorem. The application of this non-standard Implicit Function Theorem is in no way trivial and is applied to a new mapping whose roots lie in one-to-one correspondence with rotating wave solutions to our Lambda-Omega LDS. An application of this new technical tool eventually allows us to arrive at the desired existence results for rotating wave solutions to our Lambda-Omega LDS.

In Chapter 6 we discuss some avenues for future work and extensions of the results, along with some concluding remarks. In particular, we discuss the possibility of extending the work done throughout this thesis to multi-armed spiral wave solutions to (1.3.2) and how this undertaking would differ from the analysis herein. Another topic of discussion is the potential asymptotic stability of the rotating wave solution found in Chapter 5, and why we cannot apply the methods from Chapter 4 to this situation. This thesis concludes with a summary of the results and properly frames them as the necessary investigation for continuing work on rotational propagation through spatially discretized media.

Chapter 2

The Phase Components

To properly analyze system (1.3.2) we begin by introducing the polar ansatz

$$z_{i,j} = r_{i,j} e^{i(\omega(a,\alpha)t + \theta_{i,j})}, \quad (2.0.1)$$

for each (i, j) , where $r_{i,j} = r_{i,j}(t)$ and $\theta_{i,j} = \theta_{i,j}(t)$. System (1.3.2) can now be written as

$$\dot{r}_{i,j} = \alpha \sum_{i',j'} (r_{i',j'} \cos(\theta_{i',j'} - \theta_{i,j}) - r_{i,j}) + r_{i,j} \lambda(r_{i,j}), \quad (2.0.2a)$$

$$\dot{\theta}_{i,j} = \alpha \sum_{i',j'} \frac{r_{i',j'}}{r_{i,j}} \sin(\theta_{i',j'} - \theta_{i,j}) + \alpha \omega_1(r_{i,j}, \alpha), \quad (2.0.2b)$$

for each $(i, j) \in \mathbb{Z}^2$. Our work in this chapter aims at demonstrating the existence of phase-locked solutions to systems of coupled phase equations based upon (2.0.2b). In Section 2.1 we provide the exact coupled phase system we wish to study in this work, and demonstrate how it relates to (2.0.2b). Section 2.2 reviews relevant results from systems of coupled oscillators on finite square lattices, and in Section 2.3 we use these results to extend to solutions of systems of coupled oscillators indexed by the infinite lattice \mathbb{Z}^2 . In Section 2.4 we will return to the full system (2.0.2) and discuss how the work in this chapter has moved us closer to the goal of demonstrating the existence of rotating waves to (1.3.2).

2.1 Reduction to the Phase Model

Here we will introduce the model which forms the basis for our investigation in this chapter. We begin by considering the lattice dynamical system (1.3.2) and make the following assumption:

Hypothesis 2.1.1. *In the absence of coupling ($\alpha = 0$) each element of (1.3.2) is at the limit cycle solution of the system*

$$\dot{z}_{i,j} = z_{i,j}[\lambda(|z_{i,j}|) + i\omega(|z_{i,j}|, 0)], \quad (2.1.1)$$

which is guaranteed by Hypothesis 1.3.1. That is, for each index $(i, j) \in \mathbb{Z}^2$ the element $z_{i,j}(t)$ is oscillating with frequency $2\pi/\omega(a, 0)$ and amplitude $|z_{i,j}| = a > 0$.

We proceed by introducing the polar ansatz (2.0.1) to obtain the polar decomposition (2.0.2) of (1.3.2). We wish to find time-periodic rotating wave solutions to (1.3.2) and thus we reduce to searching for non-trivial steady-state solutions of the polar variables equations (2.0.2). This reduces to solving

$$\begin{aligned} 0 &= \alpha \sum_{i',j'} (r_{i',j'} \cos(\theta_{i',j'} - \theta_{i,j}) - r_{i,j}) + r_{i,j} \lambda(r_{i,j}), \\ 0 &= \sum_{i',j'} \frac{r_{i',j'}}{r_{i,j}} \sin(\theta_{i',j'} - \theta_{i,j}) + \omega_1(r_{i,j}, \alpha), \end{aligned} \quad (2.1.2)$$

for each $(i, j) \in \mathbb{Z}^2$. Notice that any steady-state solutions $\{(\bar{r}_{i,j}, \bar{\theta}_{i,j})\}_{(i,j) \in \mathbb{Z}^2}$ leads to solutions of the Lambda-Omega system (1.3.2) of the form

$$z_{i,j}(t) = \bar{r}_{i,j} e^{i(\omega(a,\alpha)t + \bar{\theta}_{i,j})}, \quad (2.1.3)$$

for all $(i, j) \in \mathbb{Z}^2$. That is, each $z_{i,j}(t)$ is oscillating with a frequency of $2\pi/\omega(a, \alpha)$ but differ through time-independent phase-lags $\bar{\theta}_{i,j}$ and magnitudes $\bar{r}_{i,j}$.

From Hypothesis 2.1.1, letting $\alpha \rightarrow 0^+$ gives that $r_{i,j} \rightarrow a$ for all (i, j) . This then reduces (2.1.2) to solving

$$0 = \sum_{i',j'} \sin(\theta_{i',j'} - \theta_{i,j}), \quad (2.1.4)$$

for each (i, j) , which correspond to the aforementioned time-independent phase-lags for a potential solution of the form (2.1.3). More precisely, solutions to system (2.1.4) provide solutions to the full Lambda-Omega system when $\alpha = 0$ of the form

$$z_{i,j}(t) = a e^{i(\omega(a,0)t + \bar{\theta}_{i,j})}. \quad (2.1.5)$$

In this chapter we solve (2.1.4), by considering a more general system of coupled phase equations whose coupling function only retains the necessary characteristics of the sine function. Specifically we consider the system of coupled phase equations of the form

$$\dot{\theta}_{i,j}(t) = \omega_0 + \sum_{i',j'} H(\theta_{i',j'}(t) - \theta_{i,j}(t)), \quad (i, j) \in \mathbb{Z}^2, \quad (2.1.6)$$

where $\theta_{i,j} : \mathbb{R}^+ \rightarrow S^1$ for each $(i, j) \in \mathbb{Z}^2$ and $\omega_0 \in \mathbb{R}$. By introducing the ansatz

$$\theta_{i,j}(t) = \omega_0 t + \bar{\theta}_{i,j}, \quad (2.1.7)$$

where each oscillator has the same frequency but differs through the time-independent phase-lag $\bar{\theta}_{i,j}$, we reduce (2.1.6) to solving

$$0 = \sum_{i',j'} H(\bar{\theta}_{i',j'} - \bar{\theta}_{i,j}), \quad (2.1.8)$$

a similar system to (2.1.4) above. Our interest in this work specifically is proving the existence of rotating wave solutions to system (2.1.6). We provide a formal definition.

Definition 2.1.2. *Using any four cell ring in the lattice as a core, a **rotating wave solution** of system (2.1.6) is defined so that the phase-lags over concentric rings around this core increase from 0 up to 2π .*

Throughout this work we will consider coupling functions $H : S^1 \rightarrow S^1$ satisfying the following hypothesis:

Hypothesis 2.1.3. *The coupling function $H : S^1 \rightarrow S^1$ is such that*

- $H \in C^\infty(S^1)$,
- $H(x + 2\pi) = H(x)$ for all $x \in S^1$,
- $H(-x) = -H(x)$ for all $x \in S^1$,
- $H'(x) > 0$ for all $x \in (-\frac{\pi}{2}, \frac{\pi}{2})$.

The reader should further note that the final two conditions of Hypothesis 2.1.3 can be combined to see that we necessarily have $H(0) = 0$ and

$$H(x) > 0, \quad x \in (0, \frac{\pi}{2}]. \quad (2.1.9)$$

In the coming sections we will see how this set of conditions is minimal in that each is necessary for the results obtained in this work.

The coupled phase model (2.1.6) also comes as a generalization of the celebrated Kuramoto model which has widespread applications, particularly in neuroscience [17, 51]. There has been an extensive body of work on one-dimensional lattices, or chains, of coupled systems of phase equations with similar coupling functions in both the finite and infinite settings [24, 25]. The study of two-dimensional lattices remains mostly unexplored, with the exception of some work on the finite square lattice [28, 56].

Finally, one should note that the lattice structure and nearest-neighbour connections give systems (1.3.2) and (2.1.6) a natural underlying graph theoretic geometry which we exploit throughout this work. We recall that nearest-neighbours are one step along the lattice from each other and then extend this notion inductively so that we say an element is k steps away from another element if the shortest path along

the lattice via nearest-neighbour connections requires us to move through exactly k lattice points. For example, the cells with indices $(1, 0)$ and $(2, 2)$ are said to be 3 steps away from each other with an example of a shortest path between these indices via nearest-neighbour connections given by $(1, 0) \rightarrow (2, 0) \rightarrow (2, 1) \rightarrow (2, 2)$. Aside from this geometric perspective, this can be quantified analytically by saying that two elements indexed by the lattice points (i_1, j_1) and (i_2, j_2) are k steps away from each other if

$$|i_1 - i_2| + |j_1 - j_2| = k. \quad (2.1.10)$$

This notion of distance along the lattice structure has important implications for our system. We see that the first derivative of any oscillator depends on the value of all nearest-neighbours as well as itself. It follows that the second derivative of any oscillator will depend on the derivative of each of its nearest-neighbours and the derivative of itself, implying that by the form of our LDS, we have the second derivative of any oscillator depending on the values of all oscillators two steps or less from it. More generally, the k th derivative of any oscillator depends on the value of all oscillators k steps or less from it. This interconnectivity between oscillators will become crucial throughout this work and provide the basis for much of the work carried out.

2.2 Rotating Waves on Finite Lattices

It was shown by Ermentrout and Paultet in [56] that there exists phase-locked rotating wave solutions on finite square lattices. Here we will review this work and in the following section demonstrate how it can be extended to give solutions on the infinite lattice. We begin by fixing an integer $N \geq 2$ and consider the truncated phase equations

$$\dot{\theta}_{i,j} = \omega_0 + \sum_{i',j'} H(\theta_{i',j'} - \theta_{i,j}), \quad (2.2.1)$$

where $1 - N \leq i, j \leq N$, $\omega_0 \in \mathbb{R}$ and the sum is over the nearest-neighbours of the cell (i, j) on the finite square integer lattice with side lengths $2N$. Note that not all cells have four neighbours in this case because of the truncation to a finite lattice. That is, those cells along the edges have at most three nearest-neighbours included in the sum, and if they are a corner then there will be only two nearest-neighbours in the sum.

We will follow the arguments laid out in [56] to show that there exists a phase-locked solution to (2.2.1) of the form $\theta_{i,j}(t) = \omega_0 t + \bar{\theta}_{i,j}$. Such solutions all have the same frequency and a fixed phase lag $\bar{\theta}_{i,j}$. We can further assume that $\omega_0 = 0$ with no loss of generality by applying the transformation $\theta_{i,j} \rightarrow \theta_{i,j} + \omega_0 t$ to eliminate ω_0 in (2.2.1), now reducing the search to steady-states corresponding to the phase lags. Following Definition 2.1.2, we seek a steady-state solution with the symmetry

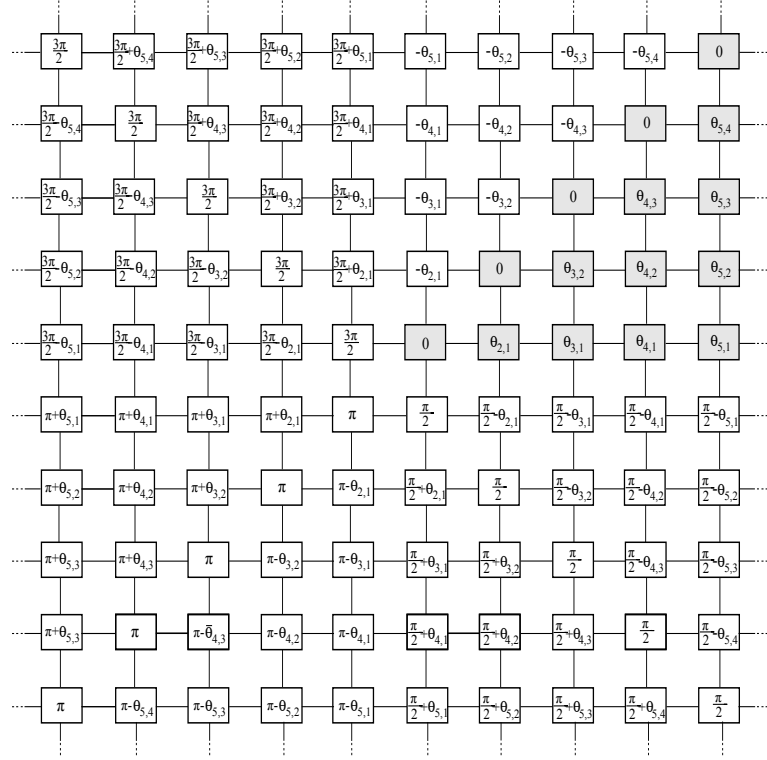


Figure 2.1: Symmetry of the phase-locked solution on the finite lattice. The shaded cells represent the reduced phase system.

shown in Figure 2.1. In this way, we may reduce the number of equations from $4N^2$ to $\frac{1}{2}N(N-1)$ by focusing on those whose indices lie in the range $1 \leq j < i \leq N$. We give a formal definition.

Definition 2.2.1. For every $N \geq 2$ the **reduced phase system** of (2.2.1) is given by restricting ourselves to elements whose indices lie in the range $1 \leq j < i \leq N$ together with the boundary conditions

$$\begin{aligned} \theta_{i,i} &= 0 \\ \theta_{i,0} &= \frac{\pi}{2} - \theta_{i,1}. \end{aligned} \tag{2.2.2}$$

We also refer to the **reduced phase system** of (2.1.6) given by restricting ourselves to the elements whose indices lie in the range $1 \leq j < i$ with the boundary conditions (2.2.2). When the context is obvious we will simply refer to either of these cases as the *reduced phase system*.

The elements of the reduced phase system are represented by the shaded cells in Figure 2.1, where the boundary conditions (2.2.2) are also represented. Upon applying

these reductions to the system we arrive at the problem of finding a steady-state to the system of equations given by

$$\dot{\theta}_{i,j} = \sum_{i',j'} H(\theta_{i',j'} - \theta_{i,j}), \quad 1 \leq j < i \leq N. \quad (2.2.3)$$

We will also consider the initial conditions $\theta_{i,j}(0) = \frac{\pi}{4}$ imposed upon the elements of (2.2.3).

Remark 2.2.2. One will notice that there is a slight difference in the initial conditions between the work in [56] and of that which is given here. In the former, the initial conditions are taken to be $\theta_{i,j}(0) = 0$, whereas here we will take them to be $\theta_{i,j}(0) = \frac{\pi}{4}$. This does not lead to any significantly different analysis, but will be useful in extending these results to the infinite lattice.

The boundary conditions (2.2.2) lead to certain properties which are used to show the existence of a steady-state solution. In particular, the cells directly below the diagonal ($j = i - 1$) are connected to two diagonal elements (above and to the left) which reduces their differential equations to

$$\begin{aligned} \dot{\theta}_{i,i-1} &= H(0 - \theta_{i,i-1}) + H(0 - \theta_{i,i-1}) + \cdots \\ &= -2H(\theta_{i,i-1}) + \cdots . \end{aligned} \quad (2.2.4)$$

Here we have used the odd symmetry of the coupling function and those terms indicated in the ellipsis are the coupling terms to the right and below. Also, the cells in the first row of the reduced phase system ($j = 1$) have a special term due to their connection with the boundary terms at $j = 0$ given by

$$\begin{aligned} \dot{\theta}_{i,1} &= H\left(\frac{\pi}{2} - \theta_{i,1} - \theta_{i,1}\right) + \cdots \\ &= H\left(\frac{\pi}{2} - 2\theta_{i,1}\right) + \cdots . \end{aligned} \quad (2.2.5)$$

The proof that a rotating wave solution exists on the finite lattice is broken down into two lemmas which together show that the trajectories of (2.2.3) are decreasing and bounded below for all $t > 0$. These lemmas together show that the trajectories therefore tend to an equilibrium as $t \rightarrow \infty$. The proof will only be briefly summarized here as it is nearly the same as that given by Ermentrout and Paullet, with the only distinction being that of a sign change due to the choice of initial conditions. Furthermore, an understanding of the methods employed in the proof on the finite lattice leads to a greater understanding of those used to extend to the infinite lattice.

Lemma 2.2.3. *Given $\theta_{i,j}(0) = \frac{\pi}{4}$, there exists a $t_0 > 0$ such that $\dot{\theta}_{i,j}(t) < 0$ for all $0 < t < t_0$ and $1 \leq j < i \leq N$.*

Proof: Let us observe that with the initial conditions $\theta_{i,j}(0) = \frac{\pi}{4}$

$$\dot{\theta}_{i,j}(0) = \sum_{i',j'} H(0) = 0, \quad (2.2.6)$$

for all $j \neq i - 1$ because $H(0) = 0$ by the odd symmetry of H . In the case when $j = i - 1$, from (2.2.4) we get

$$\dot{\theta}_{i,j}(0) = -2H\left(\frac{\pi}{4}\right) + 2H(0) < 0. \quad (2.2.7)$$

This implies that there is an interval of t values to the right of zero in which the value $\theta_{i,j}(t)$ indexed by an element directly below the diagonal of the reduced phase system is decreasing.

An inductive argument will show that when $j \neq i - 1$ we have

$$\left. \frac{d^k \theta_{i,j}}{dt^k} \right|_{t=0} = 0, \quad k = 1, \dots, i - j - 1 \quad (2.2.8)$$

and

$$\left. \frac{d^{(i-j)} \theta_{i,j}}{dt^{(i-j)}} \right|_{t=0} < 0. \quad (2.2.9)$$

That is, the number of steps an element of the reduced phase system is from the diagonal determines which order derivative will be nonzero first. Then conditions (2.2.8) and (2.2.9) together imply that upon expanding each $\theta_{i,j}(t)$ as a Taylor series about $t = 0$ we get

$$\theta_{i,j}(t) = \frac{\pi}{4} + \frac{a_{i,j}}{(i-j)!} t^{i-j} + \mathcal{O}(|t|^{i-j+1}), \quad (2.2.10)$$

where $a_{i,j} < 0$ is used to denote the term (2.2.9) for each (i, j) . Differentiating (2.2.10) with respect to t provides the Taylor series for $\dot{\theta}_{i,j}(t)$ about $t = 0$, given by

$$\dot{\theta}_{i,j}(t) = \frac{a_{i,j}}{(i-j-1)!} t^{i-j-1} + \mathcal{O}(|t|^{i-j}). \quad (2.2.11)$$

Since $a_{i,j} < 0$, we have that $\dot{\theta}_{i,j}(t) < 0$ for sufficiently small $t > 0$ for each $1 \leq j < i \leq N$. But then since there are only finitely many elements in the reduced phase system, it follows that there exists a $t_0 > 0$ small enough so that $\dot{\theta}_{i,j}(t) < 0$ for all $0 < t < t_0$ and $1 \leq j < i \leq N$, giving the desired result. \blacksquare

Lemma 2.2.4. *For all $t > 0$ and $1 \leq j < i \leq N$ we have $0 < \theta_{i,j}(t) < \frac{\pi}{4}$ and $\dot{\theta}_{i,j}(t) < 0$.*

Proof: Assume the contrary. That is, let $\hat{t} > 0$ be the first place where either $\theta_{\hat{i},\hat{j}}(\hat{t}) = 0$ or $\dot{\theta}_{\hat{i},\hat{j}}(\hat{t}) = 0$ for some index (\hat{i}, \hat{j}) . Notice from Lemma 2.2.3 that these conditions are minimal since we have that $\theta_{i,j}(0) = \frac{\pi}{4}$ and $\dot{\theta}_{i,j}(t) < 0$ for all $1 \leq j < i \leq N$ in a sufficiently small neighbourhood to the right of $t = 0$. We will break this proof up into two case: (1) to show that there cannot be an index which satisfies $\theta_{\hat{i},\hat{j}}(\hat{t}) = 0$ and (2) that there cannot be an index such that $\dot{\theta}_{\hat{i},\hat{j}}(\hat{t}) = 0$ for all $\hat{t} > 0$ and finite.

Case 1: Working with the first case we assume that $\theta_{\hat{i},\hat{j}}(\hat{t}) = 0$ for some (\hat{i}, \hat{j}) . Then we note that from the minimality of \hat{t} we necessarily have $0 \leq \theta_{i,j}(\hat{t}) < \frac{\pi}{4}$ and $\dot{\theta}_{i,j}(\hat{t}) \leq 0$ for all i, j . Using the special form (2.2.5) of those elements with $j = 1$ we see that $\hat{j} \neq 1$ since

$$\dot{\theta}_{i,1}(\hat{t}) = H\left(\frac{\pi}{2}\right) + H(\theta_{i,2}(\hat{t})) + H(\theta_{i-1,1}(\hat{t})) + \begin{cases} H(\theta_{i+1,1}(\hat{t})) & : \hat{i} \neq N \\ 0 & : \hat{i} = N \end{cases}, \quad (2.2.12)$$

which following (2.1.9) gives that each term in the sum is nonnegative, with $H(\frac{\pi}{2}) > 0$, thus giving that $\dot{\theta}_{i,1}(\hat{t}) > 0$, contradicting our assumption.

Moving to a pair (\hat{i}, \hat{j}) with $\hat{j} \neq 1$ we see that if $\theta_{\hat{i},\hat{j}}(\hat{t}) = 0$ we get

$$\dot{\theta}_{\hat{i},\hat{j}}(\hat{t}) = \sum_{i',j'} H(\theta_{i',j'}(\hat{t})). \quad (2.2.13)$$

Since all elements of the reduced phase system are nonnegative at $t = \hat{t}$ we have that $\dot{\theta}_{\hat{i},\hat{j}}(\hat{t}) \geq 0$, again from (2.1.9). But from the definition of \hat{t} we know that $\dot{\theta}_{\hat{i},\hat{j}}(\hat{t}) \leq 0$, implying that $\dot{\theta}_{\hat{i},\hat{j}}(\hat{t}) = 0$. The only way in which this is possible is if elements indexed by nearest-neighbours of (\hat{i}, \hat{j}) are such that $\theta_{i',j'}(\hat{t}) = 0$ as well. This allows us to move to the nearest-neighbour indexed by $(\hat{i}, \hat{j} - 1)$ and perform the same analysis to similarly find that $\dot{\theta}_{\hat{i},\hat{j}-1}(\hat{t}) \geq 0$. Again, the only way in which this is possible is if all elements indexed by nearest-neighbours of $(\hat{i}, \hat{j} - 1)$ take the value 0 at $t = \hat{t}$. This process continues by systematically moving down one cell at a time through the lattice until we find that $\theta_{\hat{i},2}(\hat{t}) = 0$. But then

$$\dot{\theta}_{\hat{i},2}(\hat{t}) = H(\theta_{\hat{i},1}(\hat{t})) + \dots, \quad (2.2.14)$$

where the neglected terms in the ellipsis are those elements to the left, above and to the right (if $\hat{i} \neq N$). As before, the neglected terms return a nonnegative value but one notices that since we have already shown that $\theta_{\hat{i},1}(\hat{t}) > 0$, it follows from (2.1.9) that $H(\theta_{\hat{i},1}(\hat{t})) > 0$. This then gives that $\dot{\theta}_{\hat{i},2}(\hat{t}) > 0$, which is a contradiction. Therefore, no $\theta_{i,j}(t)$ can reach 0 at $t = \hat{t}$.

Case 2: Turning to the second case, we suppose $\dot{\theta}_{\hat{i},\hat{j}}(\hat{t}) = 0$. Clearly we cannot have $\dot{\theta}_{i,j}(\hat{t}) = 0$ for all i, j since this would mean that we have reached an equilibrium

point in finite t , which is impossible. Therefore, it can be assumed that there exists some index (i_0, j_0) such that $\dot{\theta}_{i_0, j_0}(\hat{t}) < 0$ and again $0 \leq \theta_{i, j}(\hat{t}) < \frac{\pi}{4}$ and $\dot{\theta}_{i, j}(\hat{t}) \leq 0$ for all i, j .

Let θ_{i_0, j_0} be the closest indexed element to $\theta_{\hat{i}, \hat{j}}$ with $\dot{\theta}_{i_0, j_0}(\hat{t}) < 0$. Without loss of generality, we may assume that (\hat{i}, \hat{j}) and (i_0, j_0) are nearest-neighbours. Indeed, as previously noted, at $t = \hat{t}$ not all elements of the reduced phase system can have a derivative that vanishes, therefore there must be a pair of nearest-neighbours such that one of their derivatives vanishes at $t = \hat{t}$ and the other does not. If we can show that this cannot be possible, then necessarily we have that every element of the reduced phase system $\theta_{i, j}(t)$ is such that either $\dot{\theta}_{i, j}(\hat{t}) < 0$ or $\dot{\theta}_{i, j}(\hat{t}) = 0$. Since the later is impossible, we must have the former, completing the proof.

Now, we proceed under the assumption that (\hat{i}, \hat{j}) and (i_0, j_0) are nearest-neighbours. The assumption $\dot{\theta}_{\hat{i}, \hat{j}}(\hat{t}) = 0$ implies that

$$\ddot{\theta}_{\hat{i}, \hat{j}}(\hat{t}) = \sum_{\hat{i}', \hat{j}'} H'(\theta_{\hat{i}', \hat{j}'}(\hat{t}) - \theta_{\hat{i}, \hat{j}}(\hat{t})) \dot{\theta}_{\hat{i}', \hat{j}'}(\hat{t}), \quad (2.2.15)$$

where H' denotes the derivative of H with respect to its argument. By assumption, at $t = \hat{t}$ we have $0 \leq \theta_{i, j}(\hat{t}) < \frac{\pi}{4}$ for all $1 \leq j < i \leq N$, so that

$$|\theta_{\hat{i}', \hat{j}'}(\hat{t}) - \theta_{\hat{i}, \hat{j}}(\hat{t})| < \frac{\pi}{4}. \quad (2.2.16)$$

Since H is assumed to be strictly increasing on $(-\frac{\pi}{2}, \frac{\pi}{2})$, it follows that

$$H'(\theta_{\hat{i}', \hat{j}'}(\hat{t}) - \theta_{\hat{i}, \hat{j}}(\hat{t})) > 0. \quad (2.2.17)$$

Furthermore, every element of the reduced phase system satisfies $\dot{\theta}_{i, j}(\hat{t}) \leq 0$, with the additional assumption that $\dot{\theta}_{i_0, j_0}(\hat{t}) < 0$ giving

$$\ddot{\theta}_{\hat{i}, \hat{j}}(\hat{t}) = \sum_{\hat{i}', \hat{j}'} H'(\theta_{\hat{i}', \hat{j}'}(\hat{t}) - \theta_{\hat{i}, \hat{j}}(\hat{t})) \dot{\theta}_{\hat{i}', \hat{j}'}(\hat{t}) < 0, \quad (2.2.18)$$

since (i_0, j_0) and (\hat{i}, \hat{j}) are nearest-neighbours.

Now, expanding $\dot{\theta}_{\hat{i}, \hat{j}}(t)$ as a Taylor series about $t = \hat{t}$ gives

$$\dot{\theta}_{\hat{i}, \hat{j}}(t) = \ddot{\theta}_{\hat{i}, \hat{j}}(\hat{t})(t - \hat{t}) + \mathcal{O}(|t - \hat{t}|^2), \quad (2.2.19)$$

since $\dot{\theta}_{\hat{i}, \hat{j}}(\hat{t}) = 0$. Since $\ddot{\theta}_{\hat{i}, \hat{j}}(\hat{t}) < 0$, we have that there exists a sufficiently small nontrivial interval of t values to the left of \hat{t} such that $\dot{\theta}_{\hat{i}, \hat{j}}(t) > 0$. But this implies that there exists a positive $t' < \hat{t}$ such that $\dot{\theta}_{\hat{i}, \hat{j}}(t') = 0$ since Lemma 2.2.3 gave that $\dot{\theta}_{\hat{i}, \hat{j}}(t) < 0$ for sufficiently small $t > 0$. This therefore contradicts the minimality of \hat{t}

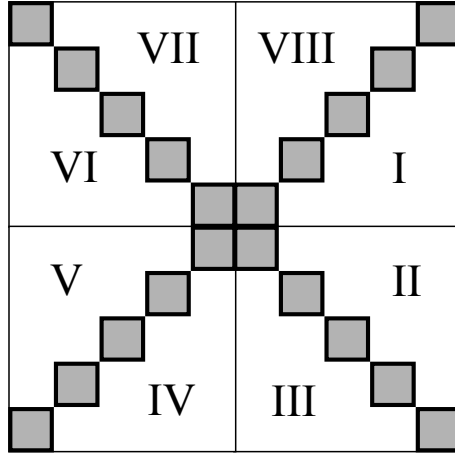


Figure 2.2: The eight distinct regions of the finite lattice defined by the reduced phase system.

and hence from our arguments above, no derivative of the $\theta_{i,j}(t)$ can vanish at a finite value of t , completing the proof. \blacksquare

In summary, it was shown that for any $N \geq 2$ each element of the reduced phase system is such that

$$0 < \theta_{i,j}(t) < \frac{\pi}{4} \quad \text{and} \quad \dot{\theta}_{i,j}(t) < 0 \quad (2.2.20)$$

for all $t > 0$ and $1 \leq j < i \leq N$. Hence, $\theta_{i,j}(t)$ is a decreasing function which is bounded below, so

$$\bar{\theta}_{i,j} = \lim_{t \rightarrow \infty} \theta_{i,j}(t) \quad (2.2.21)$$

exists and lies in the interval $[0, \frac{\pi}{4})$. Therefore $\{\bar{\theta}_{i,j}\}_{1 \leq j < i \leq N}$ gives an equilibrium for the reduced phase system. As a brief aside, one can further show via the same methods used in Lemma 2.2.4 that $\bar{\theta}_{i,j} > 0$ for all $1 \leq j < i \leq N$.

To extend the solution of the reduced phase system to the entire square lattice we refer to Figure 2.2 where the finite $2N \times 2N$ lattice has been partitioned into eight distinct regions. To begin, it has already been remarked that those elements along the diagonal between regions *I* and *VIII* are fixed at 0. Then those on the diagonal between regions *II* and *III* are fixed at $\frac{\pi}{2}$, those between regions *IV* and *V* are fixed at π and those between regions *VI* and *VII* are fixed at $\frac{3\pi}{2}$. If we write $\bar{\theta}$ to be the solution of the reduced phase system found above, the solutions in each of the regions

of Figure 2.2 are as follows:

$$\begin{aligned}
I : \bar{\theta} &\rightarrow \bar{\theta} \\
II : \bar{\theta} &\rightarrow \frac{\pi}{2} - \bar{\theta} \\
III : \bar{\theta} &\rightarrow \frac{\pi}{2} + \bar{\theta} \\
IV : \bar{\theta} &\rightarrow \pi - \bar{\theta} \\
V : \bar{\theta} &\rightarrow \pi + \bar{\theta} \\
VI : \bar{\theta} &\rightarrow \frac{3\pi}{2} - \bar{\theta} \\
VII : \bar{\theta} &\rightarrow \frac{3\pi}{2} + \bar{\theta} \\
VIII : \bar{\theta} &\rightarrow -\bar{\theta}.
\end{aligned} \tag{2.2.22}$$

Note that these extensions give exactly the symmetry of the solution shown in Figure 2.1.

2.3 Rotating Waves on an Infinite Lattice

We now consider the existence of rotating waves on an infinite lattice. Consider the infinite system of phase equations

$$\dot{\theta}_{i,j} = \omega_0 + \sum_{i',j'} H(\theta_{i',j'} - \theta_{i,j}), \tag{2.3.1}$$

where $(i, j) \in \mathbb{Z}^2$ and as before $\omega_0 \in \mathbb{R}$. We again seek rotating wave solutions to such a system of differential equations of the form $\theta_{i,j}(t) = \omega_0 t + \bar{\theta}_{i,j}$, where $\bar{\theta}_{i,j}$ is a time-independent phase-lag. We provide the following result.

Theorem 2.3.1. *The system of differential equations (2.3.1) has a nontrivial phase-locked rotating wave solution.*

As before, without loss of generality we can assume $\omega_0 = 0$ and we are therefore looking for equilibria of the phase system which correspond to the phase lags of the rotating wave solution. Throughout this section when referring to the solution on a finite lattice, we mean the one found in the previous section, and illustrated in Figure 2.1.

Lemma 2.3.2. *Let $N \geq 2$ and finite. If we denote $\bar{\theta}_{i,j}$ as the steady-state solutions on the finite $2N \times 2N$ lattice in the reduced phase system of (2.2.3), then $\bar{\theta}_{i+1,j} \geq \bar{\theta}_{i,j}$ for all $1 \leq j \leq i \leq N - 1$.*

Proof: This proof follows in a very similar way to how we proceeded in Lemmas 2.2.3 and 2.2.4 to show that the $\theta_{i,j}(t)$ of the reduced phase system are decreasing for all $t > 0$. We begin by using the conditions of the derivatives (2.2.8) and (2.2.9) to find that there exists a small interval to the right of zero for which $\theta_{i,j}(t) < \theta_{i+1,j}(t)$ for all $t > 0$ belonging to this interval. Then we assume that this interval is finite to arrive at a contradiction showing that these inequalities hold for all $t > 0$. Upon showing that these inequalities hold for all $t > 0$, we may extend them to the steady-state solution by having $t \rightarrow \infty$, giving the desired result.

Recall that we take our initial conditions to be $\theta_{i,j}(0) = \frac{\pi}{4}$ for all (i, j) in the reduced phase system and those on the diagonal fixed at 0. From Lemma 2.2.3 we have $\dot{\theta}_{i,i-1}(0) < 0$ and

$$\left. \frac{d^k \theta_{i,j}}{dt^k} \right|_{t=0} = 0, \quad k = 1, \dots, i - j - 1 \quad (2.3.2)$$

with

$$\left. \frac{d^{(i-j)} \theta_{i,j}}{dt^{(i-j)}} \right|_{t=0} < 0, \quad (2.3.3)$$

for each $j \neq i - 1$. Then from these facts we have that upon expanding the difference $\theta_{i,j}(t) - \theta_{i+1,j}(t)$ as a Taylor series about $t = 0$ the first $(i - j - 1)$ terms vanish leaving

$$\theta_{i,j}(t) - \theta_{i+1,j}(t) = \frac{b_{i,j}}{(i - j)!} t^{(i-j)} + \mathcal{O}(|t|^{i-j+1}), \quad (2.3.4)$$

where

$$b_{i,j} = \left. \frac{d^{(i-j)} \theta_{i,j}}{dt^{(i-j)}} \right|_{t=0} - \underbrace{\left. \frac{d^{(i-j)} \theta_{i+1,j}}{dt^{(i-j)}} \right|_{t=0}}_{=0} = \left. \frac{d^{(i-j)} \theta_{i,j}}{dt^{(i-j)}} \right|_{t=0} < 0. \quad (2.3.5)$$

Therefore, there exists small a $t > 0$ such that $\theta_{i,j}(t) - \theta_{i+1,j}(t) < 0$, thus implying that $\theta_{i,j}(t) < \theta_{i+1,j}(t)$ on this interval. Since this is true for all elements of the reduced phase system, which is finite, there exists a $t_0 > 0$ such that $\theta_{i,j}(t) < \theta_{i+1,j}(t)$ for all $t \in (0, t_0)$ and $1 \leq j < i \leq N - 1$.

We now assume that this ordering of the elements of the reduced phase system only persists for finite t . That is, let $t_0 > 0$ be the first value of t in which the inequality no longer holds for all elements of the reduced phase system. Then there exists at least one index of the reduced phase system, (i_0, j_0) , such that $\theta_{i_0+1,j_0}(t_0) = \theta_{i_0,j_0}(t_0)$, and $\theta_{i+1,j}(t_0) \geq \theta_{i,j}(t_0)$ for all $(i, j) \neq (i_0, j_0)$. We first note that $j_0 \neq i_0$. Indeed, by Lemma 2.2.4 the elements of the reduced phase system satisfy $0 < \theta_{i,j}(t) < \frac{\pi}{4}$ for all $t > 0$ and one sees that

$$\theta_{i_0+1,i_0}(t) > 0 = \theta_{i_0,i_0}(t). \quad (2.3.6)$$

Hence, along every row of the reduced phase system, there must be at least one strict inequality at $t = t_0$. Therefore, without loss of generality, we may assume that the index (i_0, j_0) is such that $\theta_{i_0+1,j_0}(t_0) = \theta_{i_0,j_0}(t_0) > \theta_{i_0-1,j_0}(t_0)$.

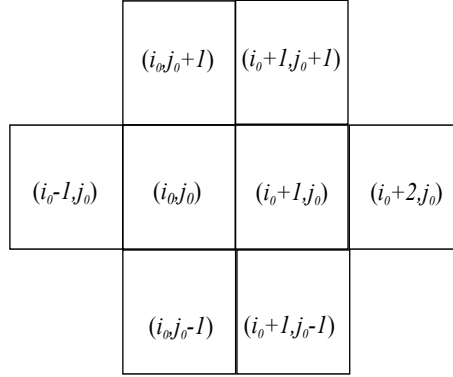


Figure 2.3: The lattice points indexed by (i_0, j_0) and $(i_0 + 1, j_0)$ along with their nearest-neighbours.

Now let us investigate $\dot{\theta}_{i_0+1, j_0}(t_0) - \dot{\theta}_{i_0, j_0}(t_0)$. Using the form of the differential equations given in (2.3.1) we see that

$$\begin{aligned} \dot{\theta}_{i_0+1, j_0}(t_0) - \dot{\theta}_{i_0, j_0}(t_0) &= H(\theta_{i_0+1, j_0+1}(t_0) - \theta_{i_0, j_0}(t_0)) - H(\theta_{i_0, j_0+1}(t_0) - \theta_{i_0, j_0}(t_0)) \\ &\quad + H(\theta_{i_0+1, j_0-1}(t_0) - \theta_{i_0, j_0}(t_0)) - H(\theta_{i_0, j_0-1}(t_0) - \theta_{i_0, j_0}(t_0)) \\ &\quad + \begin{cases} H(\theta_{i_0+2, j_0}(t_0) - \theta_{i_0, j_0}(t_0)) - H(\theta_{i_0-1, j_0}(t_0) - \theta_{i_0, j_0}(t_0)) & : i_0 \neq N - 1 \\ -H(\theta_{i_0-1, j_0+1}(t_0) - \theta_{i_0, j_0}(t_0)) & : i_0 = N - 1 \end{cases} \end{aligned} \quad (2.3.7)$$

where $\theta_{i_0+1, j_0}(t_0)$ has been replaced with $\theta_{i_0, j_0}(t_0)$ by our assumption. Furthermore, notice that the coupling between these cells does not appear in the difference since

$$H(\theta_{i_0+1, j_0}(t_0) - \theta_{i_0, j_0}(t_0)) = H(0) = 0, \quad (2.3.8)$$

because $\theta_{i_0+1, j_0}(t_0) = \theta_{i_0, j_0}(t_0)$. Figure 2.3 shows the location of these elements in relation to each other on the lattice for visual reference. Since $\theta_{i_0+1, j_0}(t_0) = \theta_{i_0, j_0}(t_0) > \theta_{i_0-1, j_0}(t_0)$ we see that upon using the odd symmetry of H we have

$$-H(\theta_{i_0-1, j_0}(t_0) - \theta_{i_0, j_0}(t_0)) = H(\theta_{i_0, j_0}(t_0) - \theta_{i_0-1, j_0}(t_0)) > 0. \quad (2.3.9)$$

Moreover, in the case when $i_0 \neq N - 1$ our assumption gives $\theta_{i_0+2, j_0}(t_0) \geq \theta_{i_0+1, j_0}(t_0) = \theta_{i_0, j_0}(t_0)$ which implies

$$H(\theta_{i_0+2, j_0}(t_0) - \theta_{i_0, j_0}(t_0)) \geq 0. \quad (2.3.10)$$

Again by the minimality of t_0 we get that $\theta_{i_0+1, j_0 \pm 1}(t_0) \geq \theta_{i_0, j_0 \pm 1}$, leading to the fact that

$$H(\theta_{i_0+1, j_0 \pm 1}(t_0) - \theta_{i_0, j_0}(t_0)) - H(\theta_{i_0, j_0 \pm 1}(t_0) - \theta_{i_0, j_0}(t_0)) \geq 0. \quad (2.3.11)$$

Putting this all together reveals that

$$\dot{\theta}_{i_0+1,j_0}(t_0) - \dot{\theta}_{i_0,j_0}(t_0) > 0. \quad (2.3.12)$$

Expanding $\theta_{i_0+1,j_0}(t) - \theta_{i_0,j_0}(t)$ as a Taylor series about $t = t_0$ gives

$$\begin{aligned} & \theta_{i_0+1,j_0}(t) - \theta_{i_0,j_0}(t) = \\ & \underbrace{[\theta_{i_0+1,j_0}(t_0) - \theta_{i_0,j_0}(t_0)]}_{=0} + \underbrace{[\dot{\theta}_{i_0+1,j_0}(t_0) - \dot{\theta}_{i_0,j_0}(t_0)]}_{>0} (t - t_0) + \mathcal{O}(|t - t_0|^2). \end{aligned} \quad (2.3.13)$$

Thus, there exists an $\varepsilon > 0$ such that $\theta_{i_0+1,j_0}(t) - \theta_{i_0,j_0}(t) < 0$ on $(t_0 - \varepsilon, t_0)$. But it was already shown that for $t > 0$ sufficiently small we have $\theta_{i_0+1,j_0}(t) - \theta_{i_0,j_0}(t) > 0$, which from the Intermediate Value Theorem implies that there is some positive $t' < t_0$ such that $\theta_{i_0+1,j_0}(t') = \theta_{i_0,j_0}(t')$. This contradicts the minimality of t_0 , thus giving that no such t_0 can exist.

Therefore, $\theta_{i+1,j}(t) > \theta_{i,j}(t)$ for all $t > 0$. Allowing $t \rightarrow \infty$ we see that the elements of the equilibrium must satisfy $\bar{\theta}_{i+1,j} \geq \bar{\theta}_{i,j}$ for all $1 \leq j < i \leq N - 1$, giving the desired result. \blacksquare

Lemma 2.3.3. *Let us denote $\bar{\theta}_{i,j}^{(N)}$ as the solutions of the finite $2N \times 2N$ lattice in the reduced phase system for any $N \geq 2$. Then $\bar{\theta}_{i,j}^{(N)} \leq \bar{\theta}_{i,j}^{(N+1)}$ for all $1 \leq j < i \leq N$. That is, the value of the equilibrium point at each index in the reduced phase system is increasing as a function of the size of the lattice.*

Proof: Let us fix $N \geq 2$ and assume there exists an index (i_0, j_0) of the reduced phase system (2.2.3) such that $\bar{\theta}_{i_0,j_0}^{(N)} > \bar{\theta}_{i_0,j_0}^{(N+1)}$. This proof is carried out by examining the case when (i_0, j_0) is an index from the last column (i.e. $i_0 = N$), deriving a contradiction and systematically decreasing the possible value of i_0 one step at a time to show that no such (i_0, j_0) can exist. We show that the contradiction which is derived for the last column ($i_0 = N$) is easily extended to derive a contradiction in the cases that $i_0 < N$. This in turn exhausts all possibilities of indices (i_0, j_0) in a finite number of steps, thus showing that $\bar{\theta}_{i,j}^{(N)} \leq \bar{\theta}_{i,j}^{(N+1)}$ for all $1 \leq j < i \leq N$.

Case $i_0 = N$: We begin by assuming that $i_0 = N$. Consider the difference of the differential equations $\dot{\theta}_{N,j_0}^{(N+1)} - \dot{\theta}_{N,j_0}^{(N)}$ evaluated at the respective equilibrium points for that size of lattice. This gives

$$\begin{aligned} 0 = & H(\bar{\theta}_{N,j_0+1}^{(N+1)} - \bar{\theta}_{N,j_0}^{(N+1)}) - H(\bar{\theta}_{N,j_0+1}^{(N)} - \bar{\theta}_{N,j_0}^{(N)}) \\ & + H(\bar{\theta}_{N,j_0-1}^{(N+1)} - \bar{\theta}_{N,j_0}^{(N+1)}) - H(\bar{\theta}_{N,j_0-1}^{(N)} - \bar{\theta}_{N,j_0}^{(N)}) \\ & + H(\bar{\theta}_{N-1,j_0}^{(N+1)} - \bar{\theta}_{N,j_0}^{(N+1)}) - H(\bar{\theta}_{N-1,j_0}^{(N)} - \bar{\theta}_{N,j_0}^{(N)}) \\ & + H(\bar{\theta}_{N+1,j_0}^{(N+1)} - \bar{\theta}_{N,j_0}^{(N+1)}), \end{aligned} \quad (2.3.14)$$

where there is an odd number of terms since the $2N \times 2N$ lattice does not have a right input at the index (N, j_0) . From Lemma 2.3.2 we have that $\bar{\theta}_{N+1, j_0}^{(N+1)} - \bar{\theta}_{N, j_0}^{(N+1)} \geq 0$, which from (2.1.9) implies that

$$H(\bar{\theta}_{N+1, j_0}^{(N+1)} - \bar{\theta}_{N, j_0}^{(N+1)}) \geq 0. \quad (2.3.15)$$

Then using (2.3.15) we may rearrange (2.3.14) to find that

$$\begin{aligned} 0 \geq & H(\bar{\theta}_{N, j_0+1}^{(N+1)} - \bar{\theta}_{N, j_0}^{(N+1)}) - H(\bar{\theta}_{N, j_0+1}^{(N)} - \bar{\theta}_{N, j_0}^{(N)}) \\ & + H(\bar{\theta}_{N, j_0-1}^{(N+1)} - \bar{\theta}_{N, j_0}^{(N+1)}) - H(\bar{\theta}_{N, j_0-1}^{(N)} - \bar{\theta}_{N, j_0}^{(N)}) \\ & + H(\bar{\theta}_{N-1, j_0}^{(N+1)} - \bar{\theta}_{N, j_0}^{(N+1)}) - H(\bar{\theta}_{N-1, j_0}^{(N)} - \bar{\theta}_{N, j_0}^{(N)}). \end{aligned} \quad (2.3.16)$$

This in turn implies that at least one of the following must be true:

- $H(\bar{\theta}_{N, j_0+1}^{(N+1)} - \bar{\theta}_{N, j_0}^{(N+1)}) - H(\bar{\theta}_{N, j_0+1}^{(N)} - \bar{\theta}_{N, j_0}^{(N)}) \leq 0$,
- $H(\bar{\theta}_{N, j_0-1}^{(N+1)} - \bar{\theta}_{N, j_0}^{(N+1)}) - H(\bar{\theta}_{N, j_0-1}^{(N)} - \bar{\theta}_{N, j_0}^{(N)}) \leq 0$,
- $H(\bar{\theta}_{N-1, j_0}^{(N+1)} - \bar{\theta}_{N, j_0}^{(N+1)}) - H(\bar{\theta}_{N-1, j_0}^{(N)} - \bar{\theta}_{N, j_0}^{(N)}) \leq 0$.

By definition of our coupling function H , we have that $H'(x) > 0$ for all $x \in (-\frac{\pi}{2}, \frac{\pi}{2})$, and recalling from Lemma 2.2.4 that the maximal difference between any two elements of the reduced phase system is strictly bounded by $\pi/2$, we find that the above conditions reduce to having at least one of the following being true

- $\bar{\theta}_{N, j_0+1}^{(N+1)} - \bar{\theta}_{N, j_0}^{(N+1)} \leq \bar{\theta}_{N, j_0+1}^{(N)} - \bar{\theta}_{N, j_0}^{(N)} \implies \bar{\theta}_{N, j_0+1}^{(N)} > \bar{\theta}_{N, j_0+1}^{(N+1)}$,
- $\bar{\theta}_{N, j_0-1}^{(N+1)} - \bar{\theta}_{N, j_0}^{(N+1)} \leq \bar{\theta}_{N, j_0-1}^{(N)} - \bar{\theta}_{N, j_0}^{(N)} \implies \bar{\theta}_{N, j_0-1}^{(N)} > \bar{\theta}_{N, j_0-1}^{(N+1)}$,
- $\bar{\theta}_{N-1, j_0}^{(N+1)} - \bar{\theta}_{N, j_0}^{(N+1)} \leq \bar{\theta}_{N-1, j_0}^{(N)} - \bar{\theta}_{N, j_0}^{(N)} \implies \bar{\theta}_{N-1, j_0}^{(N)} > \bar{\theta}_{N-1, j_0}^{(N+1)}$.

Here these conditions have been reduced by recalling that by assumption $\bar{\theta}_{i_0, j_0}^{(N)} > \bar{\theta}_{i_0, j_0}^{(N+1)}$. For simplicity we relabel the index (N, j_0) as η_1 and let η_2 to be the nearest-neighbour of η_1 with the property that $\bar{\theta}_{\eta_2}^{(N)} > \bar{\theta}_{\eta_2}^{(N+1)}$.

We note that there are restrictions on the choice of η_2 in certain cases. That is, if $j_0 = 1$ then by the form of the solutions on the finite lattice we necessarily have

$$\bar{\theta}_{i_0, 0}^{(N+1)} = \frac{\pi}{2} - \bar{\theta}_{i_0, 1}^{(N+1)} > \frac{\pi}{2} - \bar{\theta}_{i_0, 1}^{(N)} = \bar{\theta}_{i_0, 0}^{(N)}, \quad (2.3.17)$$

meaning that η_2 cannot be below η_1 when $j_0 = 1$. Furthermore, if $j_0 = i_0 - 1$ then $\bar{\theta}_{i_0, i_0}^{(N+1)} = 0 = \bar{\theta}_{i_0, i_0}^{(N)}$ showing that η_2 cannot be above or to the left of η_1 when $j_0 = i_0 - 1$. Since $N \geq 2$ we can always find an index η_2 with the prescribed properties in either situation.

We now apply a similar argument to the difference of the differential equations

$$(\dot{\theta}_{\eta_1}^{(N+1)} + \dot{\theta}_{\eta_2}^{(N+1)}) - (\dot{\theta}_{\eta_1}^{(N)} + \dot{\theta}_{\eta_2}^{(N)}) \quad (2.3.18)$$

evaluated at the respective equilibrium for that size of lattice. This gives

$$\begin{aligned} 0 = & \sum_{\eta'_1} \left[H(\bar{\theta}_{\eta'_1}^{(N+1)} - \bar{\theta}_{\eta_1}^{(N+1)}) - H(\bar{\theta}_{\eta'_1}^{(N)} - \bar{\theta}_{\eta_1}^{(N)}) \right] \\ & + \sum_{\eta'_2} \left[H(\bar{\theta}_{\eta'_2}^{(N+1)} - \bar{\theta}_{\eta_2}^{(N+1)}) - H(\bar{\theta}_{\eta'_2}^{(N)} - \bar{\theta}_{\eta_2}^{(N)}) \right], \end{aligned} \quad (2.3.19)$$

where we have paired the elements by their index. This expression can be simplified slightly by recalling that η_1 and η_2 are nearest-neighbours in the lattice. Therefore, the terms

$$H(\bar{\theta}_{\eta_1}^{(N+1)} - \bar{\theta}_{\eta_2}^{(N+1)}) \quad (2.3.20)$$

and

$$H(\bar{\theta}_{\eta_2}^{(N+1)} - \bar{\theta}_{\eta_1}^{(N+1)}) \quad (2.3.21)$$

both appear in this sum. Thus, using the odd symmetry of the coupling function, these terms eliminate themselves from the sum. Similarly, the terms $-H(\bar{\theta}_{\eta_1}^{(N)} - \bar{\theta}_{\eta_2}^{(N)})$ and $-H(\bar{\theta}_{\eta_2}^{(N)} - \bar{\theta}_{\eta_1}^{(N)})$ cancel each other in the sum by the odd symmetry of the coupling function H .

Then (2.3.19) again has the term $H(\bar{\theta}_{N+1,j_0}^{(N+1)} - \bar{\theta}_{N,j_0}^{(N+1)})$ being nonnegative, coming from the index η_1 . Rearranging (2.3.19) as above then shows that at least one of

$$H(\bar{\theta}_{\eta'_1}^{(N+1)} - \bar{\theta}_{\eta_1}^{(N+1)}) - H(\bar{\theta}_{\eta'_1}^{(N)} - \bar{\theta}_{\eta_1}^{(N)}) \leq 0 \quad (2.3.22)$$

or

$$H(\bar{\theta}_{\eta'_2}^{(N+1)} - \bar{\theta}_{\eta_2}^{(N+1)}) - H(\bar{\theta}_{\eta'_2}^{(N)} - \bar{\theta}_{\eta_2}^{(N)}) \leq 0 \quad (2.3.23)$$

must hold for a nearest-neighbour of either η_1 or η_2 . Let us denote η_3 to be this index. Notice that $\eta_3 \neq \eta_1, \eta_2$ since the coupling terms between these neighbouring cells has been eliminated by the odd symmetry of the coupling function. Now for $i = 1$ or 2 we have that

$$H(\bar{\theta}_{\eta_3}^{(N+1)} - \bar{\theta}_{\eta_i}^{(N+1)}) - H(\bar{\theta}_{\eta_3}^{(N)} - \bar{\theta}_{\eta_i}^{(N)}) \leq 0, \quad (2.3.24)$$

then from the argument laid out for η_2 above, we have that

$$\bar{\theta}_{\eta_3}^{(N+1)} - \bar{\theta}_{\eta_i}^{(N+1)} \leq \bar{\theta}_{\eta_3}^{(N)} - \bar{\theta}_{\eta_i}^{(N)} \implies \bar{\theta}_{\eta_3}^{(N+1)} < \bar{\theta}_{\eta_3}^{(N)}, \quad (2.3.25)$$

simply by recalling that $\bar{\theta}_{\eta_i}^{(N+1)} < \bar{\theta}_{\eta_i}^{(N)}$ by definition of η_1 and η_2 . Finally, the choice of η_3 is restricted to those indices (i, j) of the reduced phase system such that neither $j = 0$ nor $i = j$, by the previous discussion for the possibilities for η_2 .

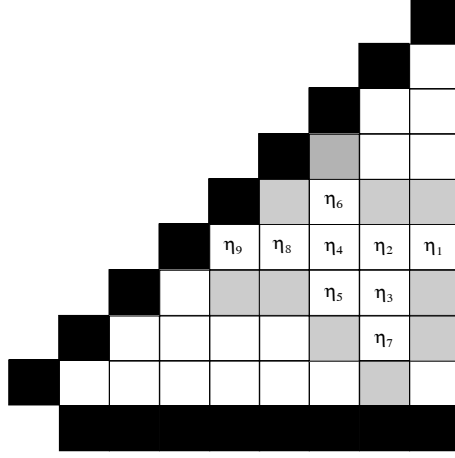


Figure 2.4: The reduced phase system with $N = 8$ and a possible collection of the indices $\{\eta_k\}_{k=1}^9$ with the eligible choices for η_{10} shaded in. The black cells represent the boundaries of the reduced phase system which cannot be included in the sequence $\{\eta_k\}_{k=1}^{N(N+2)/2}$.

We continue this process inductively by considering the differential equations

$$\sum_{k=1}^m \dot{\theta}_{\eta_k}^{(N+1)} - \sum_{k=1}^m \dot{\theta}_{\eta_k}^{(N)} \quad (2.3.26)$$

evaluated at the respective equilibrium points, for any $m \geq 1$. Since the differential equation of the element indexed by η_1 is always considered in this sum we will always have a nonnegative term in $H(\bar{\theta}_{N+1, j_0}^{(N+1)} - \bar{\theta}_{N, j_0}^{(N+1)})$, allowing us to determine that there is a nearest-neighbour, η_{m+1} , of one of the η_k such that $\bar{\theta}_{\eta_{m+1}}^{(N)} > \bar{\theta}_{\eta_{m+1}}^{(N+1)}$, via the same process outlined for $m = 1, 2$. As before, the sum will eliminate any coupling present between neighbouring indices, meaning that at each step the cardinality of the set of indices $\{\eta_k\}_{k=1}^m$ increases by one, and it can never be the case that $\eta_{m+1} = (i, j)$ is such that $j = 0$ or $i = j$. In this way we are restricted in our choices to those which are contained within the reduced phase system. This process is illustrated in Figure 2.4 for visual reference.

We have already noted that the choices of the η_k are restricted to those in the reduced phase system, which is finite. Therefore, this process eventually terminates showing that for every element of the reduced phase system we have $\bar{\theta}_{i,j}^{(N)} > \bar{\theta}_{i,j}^{(N+1)}$. At this final step we then consider

$$\sum_{k=1}^{N(N-1)/2} \dot{\theta}_{\eta_k}^{(N+1)} - \sum_{k=1}^{N(N-1)/2} \dot{\theta}_{\eta_k}^{(N)} \quad (2.3.27)$$

evaluated at the respective equilibria of that lattice size to see that only the interactions with the boundaries of the reduced phase system remain. That is, we obtain

$$\begin{aligned}
0 = & (H(\bar{\theta}_{2,1}^{(N)}) - H(\bar{\theta}_{2,1}^{(N+1)})) + 2 \sum_{k=2}^{N-1} \left(H(\bar{\theta}_{k,k-1}^{(N)}) - H(\bar{\theta}_{k,k-1}^{(N+1)}) \right) \\
& + (H(\bar{\theta}_{N,N-1}^{(N)}) - H(\bar{\theta}_{N,N-1}^{(N+1)})) + \sum_{k=1}^N \left(H\left(\frac{\pi}{2} - 2\bar{\theta}_{k,1}^{(N+1)}\right) - H\left(\frac{\pi}{2} - 2\bar{\theta}_{k,1}^{(N)}\right) \right) \quad (2.3.28) \\
& + \sum_{k=1}^N H(\bar{\theta}_{N+1,k}^{(N+1)} - \bar{\theta}_{N,k}^{(N+1)}),
\end{aligned}$$

where the first three groups of terms come from the coupling with the diagonal (i, i) terms, the fourth grouping coming from the coupling with the $j = 0$ row and the final grouping of terms is from the coupling to the right of the N th column. We have used the special forms (2.2.4) and (2.2.5) of the boundary interactions to obtain these reductions. Now from our inductive procedure we have shown that $\bar{\theta}_{i,j}^{(N)} > \bar{\theta}_{i,j}^{(N+1)}$ for every element of the reduced phase system, and from the fact that H is increasing on $(\frac{-\pi}{2}, \frac{\pi}{2})$ we have that

$$\begin{aligned}
H(\bar{\theta}_{2,1}^{(N)}) - H(\bar{\theta}_{2,1}^{(N+1)}) &> 0, \\
\sum_{k=2}^{N-1} \left(H(\bar{\theta}_{k,k-1}^{(N)}) - H(\bar{\theta}_{k,k-1}^{(N+1)}) \right) &> 0, \\
H(\bar{\theta}_{N,N-1}^{(N)}) - H(\bar{\theta}_{N,N-1}^{(N+1)}) &> 0, \\
\sum_{k=1}^N \left(H\left(\frac{\pi}{2} - 2\bar{\theta}_{k,1}^{(N+1)}\right) - H\left(\frac{\pi}{2} - 2\bar{\theta}_{k,1}^{(N)}\right) \right) &> 0.
\end{aligned} \quad (2.3.29)$$

Furthermore,

$$\sum_{k=1}^N H(\bar{\theta}_{N+1,k}^{(N+1)} - \bar{\theta}_{N,k}^{(N+1)}) \geq 0 \quad (2.3.30)$$

from our results in Lemma 2.3.2, showing that the right hand side of (2.3.28) is strictly positive, which is impossible. Therefore, $i_0 \neq N$.

Case $i_0 = N - 1$: Now that we have proven $i_0 \neq N$, we will move one column to the left and consider the case when $i_0 = N - 1$. We begin by noting that if $\bar{\theta}_{N-1,j_0}^{(N)} > \bar{\theta}_{N-1,j_0}^{(N+1)}$ then necessarily

$$H(\bar{\theta}_{N,j_0}^{(N+1)} - \bar{\theta}_{N-1,j_0}^{(N+1)}) - H(\bar{\theta}_{N,j_0}^{(N)} - \bar{\theta}_{N-1,j_0}^{(N)}) \geq 0. \quad (2.3.31)$$

Indeed, if we assume that this is not true then it is the case that

$$\bar{\theta}_{N,j_0}^{(N+1)} - \bar{\theta}_{N-1,j_0}^{(N+1)} < \bar{\theta}_{N,j_0}^{(N)} - \bar{\theta}_{N-1,j_0}^{(N)} \implies \bar{\theta}_{N,j_0}^{(N)} > \bar{\theta}_{N,j_0}^{(N+1)}, \quad (2.3.32)$$

which we have already shown to be impossible. With this in mind we can proceed as above by considering the differential equations

$$\dot{\theta}_{N-1,j_0}^{(N+1)} - \dot{\theta}_{N-1,j_0}^{(N)} \quad (2.3.33)$$

at their respective equilibria. This gives

$$\begin{aligned} 0 = & H(\bar{\theta}_{N-1,j_0+1}^{(N+1)} - \bar{\theta}_{N-1,j_0}^{(N+1)}) - H(\bar{\theta}_{N-1,j_0+1}^{(N)} - \bar{\theta}_{N-1,j_0}^{(N)}) \\ & + H(\bar{\theta}_{N-1,j_0-1}^{(N+1)} - \bar{\theta}_{N,j_0}^{(N+1)}) - H(\bar{\theta}_{N-1,j_0-1}^{(N)} - \bar{\theta}_{N-1,j_0}^{(N)}) \\ & + H(\bar{\theta}_{N-2,j_0}^{(N+1)} - \bar{\theta}_{N-1,j_0}^{(N+1)}) - H(\bar{\theta}_{N-2,j_0}^{(N)} - \bar{\theta}_{N-1,j_0}^{(N)}) \\ & + H(\bar{\theta}_{N,j_0}^{(N+1)} - \bar{\theta}_{N-1,j_0}^{(N+1)}) - H(\bar{\theta}_{N,j_0}^{(N)} - \bar{\theta}_{N-1,j_0}^{(N)}). \end{aligned} \quad (2.3.34)$$

One will notice a slight difference to the case when $i_0 = N$ in that now we have an even number of terms in the equation, but from (2.3.31) we may rearrange (2.3.34) to be handled in a similar way to the case when $i_0 = N$ by

$$\begin{aligned} 0 \geq & H(\bar{\theta}_{N-1,j_0+1}^{(N+1)} - \bar{\theta}_{N-1,j_0}^{(N+1)}) - H(\bar{\theta}_{N-1,j_0+1}^{(N)} - \bar{\theta}_{N-1,j_0}^{(N)}) \\ & + H(\bar{\theta}_{N-1,j_0-1}^{(N+1)} - \bar{\theta}_{N,j_0}^{(N+1)}) - H(\bar{\theta}_{N-1,j_0-1}^{(N)} - \bar{\theta}_{N-1,j_0}^{(N)}) \\ & + H(\bar{\theta}_{N-2,j_0}^{(N+1)} - \bar{\theta}_{N-1,j_0}^{(N+1)}) - H(\bar{\theta}_{N-2,j_0}^{(N)} - \bar{\theta}_{N-1,j_0}^{(N)}). \end{aligned} \quad (2.3.35)$$

Thus the term (2.3.31) acts as the single nonnegative term in the difference when $i_0 = N$. Proceeding as above we can find a nearest-neighbour of $(N-1, j_0)$ to which the element at that index for the $2N \times 2N$ lattice is greater than the element at that index for the $2(N+1) \times 2(N+1)$ lattice. This leads to the same chain of steps as in the case when $i_0 = N$, where we now find that the indices which give the desired conditions are limited to not only those in the reduced phase system, but to those not in the N th column.

Again this procedure terminates in a finite number of steps, showing that $\bar{\theta}_{i,j}^{(N+1)} < \bar{\theta}_{i,j}^{(N)}$ for all $1 \leq j < i \leq N-1$. Then as above, the differential equations

$$\sum_{1 \leq j < i \leq N-1} \dot{\theta}_{i,j}^{(N+1)} - \sum_{1 \leq j < i \leq N-1} \dot{\theta}_{i,j}^{(N)} \quad (2.3.36)$$

evaluated at their respective equilibria leads to a cancelling of all interior coupling

terms, thus leaving only connections with the boundary:

$$\begin{aligned}
0 = & (H(\bar{\theta}_{2,1}^{(N)}) - H(\bar{\theta}_{2,1}^{(N+1)})) + 2 \sum_{k=2}^{N-2} \left(H(\bar{\theta}_{k,k-1}^{(N)}) - H(\bar{\theta}_{k,k-1}^{(N+1)}) \right) \\
& + (H(\bar{\theta}_{N-1,N-2}^{(N)}) - H(\bar{\theta}_{N-1,N-2}^{(N+1)})) + \sum_{k=1}^{N-1} \left(H\left(\frac{\pi}{2} - 2\bar{\theta}_{k,1}^{(N+1)}\right) - H\left(\frac{\pi}{2} - 2\bar{\theta}_{k,1}^{(N)}\right) \right) \\
& + \sum_{k=1}^{N-1} \left(H(\bar{\theta}_{N,k}^{(N+1)} - \bar{\theta}_{N-1,k}^{(N+1)}) - H(\bar{\theta}_{N,k}^{(N)} - \bar{\theta}_{N-1,k}^{(N)}) \right).
\end{aligned} \tag{2.3.37}$$

In exactly the same way as the case $i_0 = N$ we have that

$$\begin{aligned}
H(\bar{\theta}_{2,1}^{(N)}) - H(\bar{\theta}_{2,1}^{(N+1)}) &> 0, \\
\sum_{k=2}^{N-2} \left(H(\bar{\theta}_{k,k-1}^{(N)}) - H(\bar{\theta}_{k,k-1}^{(N+1)}) \right) &> 0, \\
H(\bar{\theta}_{N-1,N-2}^{(N)}) - H(\bar{\theta}_{N-1,N-2}^{(N+1)}) &> 0, \\
\sum_{k=1}^{N-1} \left(H\left(\frac{\pi}{2} - 2\bar{\theta}_{k,1}^{(N+1)}\right) - H\left(\frac{\pi}{2} - 2\bar{\theta}_{k,1}^{(N)}\right) \right) &> 0.
\end{aligned} \tag{2.3.38}$$

Furthermore, following (2.3.31) one has that

$$\sum_{k=1}^{N-1} \left(H(\bar{\theta}_{N,k}^{(N+1)} - \bar{\theta}_{N-1,k}^{(N+1)}) - H(\bar{\theta}_{N,k}^{(N)} - \bar{\theta}_{N-1,k}^{(N)}) \right) \geq 0, \tag{2.3.39}$$

therefore, upon putting this all together, the right hand side of (2.3.37) is strictly positive. This gives a contradiction, thus showing that $i_0 \neq N - 1$.

Cases $i_0 < N - 1$: We may continue with this method by showing that if $i_0 \neq N - k$, for some $1 \leq k < N$, then $i_0 \neq N - k - 1$ by merely applying the same arguments which were used in proving that $i_0 \neq N - 1$ from the result that $i_0 \neq N$. This leads to a process of systematically decreasing i_0 by one each step and repeating a similar argument used in the cases $i_0 = N, N - 1$ to see that there is no such column in the reduced phase system which can contain an element that satisfies $\bar{\theta}_{i,j}^{(N)} > \bar{\theta}_{i,j}^{(N+1)}$. This then completes the proof of the lemma since there is only a finite number of columns to check. \blacksquare

This leads to the proof of Theorem 2.3.1.

Proof: (Proof of Theorem 2.3.1)

Lemma 2.3.3 shows that by observing the value of the equilibrium solution for each

lattice size at a single index in the reduced phase system we form an increasing sequence. By taking any $N \geq 2$ we can identify the equilibrium solution inside the reduced phase system of the $2N \times 2N$ lattice as an element of the reduced phase system in the infinite lattice by appending the elements

$$\bar{\theta}_{i,j}^{(N)} = 0 \quad (2.3.40)$$

for all $i > N$ and $1 \leq j < i$. We have now created a sequence of elements in the reduced phase system of the infinite lattice which is pointwise increasing and each element is bounded above by $\pi/4$, therefore this sequence converges pointwise as $N \rightarrow \infty$.

Let us write

$$\bar{\theta}_{i,j} := \lim_{N \rightarrow \infty} \bar{\theta}_{i,j}^{(N)}, \quad 1 \leq j < i. \quad (2.3.41)$$

By the continuity of the differential equations at each index of the reduced phase system we see that these elements are themselves an equilibrium of the reduced phase system of the infinite lattice and that $\bar{\theta}_{i,j} \in (0, \frac{\pi}{4}]$ for all $1 \leq j < i$. Moreover, we can apply the symmetries of the finite lattices shown in Figure 2.2 to extend this equilibrium in the reduced phase system to an equilibrium of the entire two-dimensional lattice via the same extensions outlined by Ermentrout and Paultet in (2.2.22). Then writing this equilibrium extended to the entire infinite lattice \mathbb{Z}^2 as $\bar{\theta} = \{\bar{\theta}_{i,j}\}_{i,j}$, the phase locked solution to the phase equation (2.3.1) is given by

$$\theta_{i,j}(t) = \omega_0 t + \bar{\theta}_{i,j} \quad (2.3.42)$$

for all $(i, j) \in \mathbb{Z}^2$. ■

Remark 2.3.4. It was noted previously that a rotating wave solution on the lattice is such that the phase-lags over any concentric ring about the centre four cell ring increase from 0 up to 2π . Although not explicitly stated in our result, this is indeed the case. Furthermore, such a result was implied in Ermentrout and Paultet's work on the finite lattice, although it is notably absent from their work. We state the following lemma without proof.

Lemma 2.3.5. *Let $N \geq 2$ and finite. If we denote $\bar{\theta}_{i,j}$ as the solutions on the finite $2N \times 2N$ lattice in the reduced phase system, then $\bar{\theta}_{i,j} \geq \bar{\theta}_{i,j+1}$ for all $1 \leq j < i \leq N$.*

The proof of Lemma 2.3.5 is carried out in an almost identical process to that of Lemma 2.3.2 and is therefore omitted from this work. Letting $N \rightarrow \infty$ we see that the inequalities along both the rows and columns given by Lemmas 2.3.2 and 2.3.5, respectively, remain true and once the extensions from the reduced phase system to the whole lattice are applied we obtain a true rotating wave solution.

2.4 Discussion

In this chapter we have demonstrated the existence of rotating wave solutions to systems of coupled oscillators of the form (2.1.6) indexed by an infinite lattice. The general hypotheses on our coupling function H allow us to provide a solution to the system of equations

$$\begin{aligned} 0 &= \alpha \sum_{i',j'} (r_{i',j'} \cos(\theta_{i',j'} - \theta_{i,j}) - r_{i,j}) + r_{i,j} \lambda(r_{i,j}), \\ 0 &= \sum_{i',j'} \frac{r_{i',j'}}{r_{i,j}} \sin(\theta_{i',j'} - \theta_{i,j}) + \omega_1(r_{i,j}, \alpha), \end{aligned} \tag{2.4.1}$$

for $\alpha = 0$. It was pointed out in Section 2.1 that solutions to (2.4.1) lead to solutions of the Lambda-Omega system (1.3.2) which are oscillating with a frequency of $2\pi/\omega(a, \alpha)$. Therefore, the solution to the phase equations (2.1.6) given in Section 2.3 provides a rotating wave solution when $\alpha = 0$ of the form

$$z_{i,j}(t) = a e^{i(\omega(a,0)t + \bar{\theta}_{i,j})} \tag{2.4.2}$$

for all $(i, j) \in \mathbb{Z}^2$, where $\{\bar{\theta}_{i,j}\}_{(i,j) \in \mathbb{Z}^2}$ is the rotating wave solution obtained by Theorem 2.3.1 for $H(x) = \sin(x)$. Our goal now becomes extending this solution into $\alpha > 0$. Typically this would be achieved via an Implicit Function Theorem argument, but in the following chapters we will see that there are some technical hurdles which prevent this. Later in this thesis we will overcome these technical hurdles and provide the existence result for the full Lambda-Omega system.

In conclusion, this chapter has been able to demonstrate that rotating wave solutions can be shown to exist in a class of lattice dynamical systems modelling infinitely many coupled phase oscillators. Here we have provided the necessary starting point for further investigations of both infinite arrays of coupled oscillators and rotating wave solutions to spatially discretized media.

Chapter 3

Denumerable Graph Networks

In this chapter we will provide all the relevant framework for infinite graphs. Our goal will be to take the theory that has been extensively developed for graphs and apply it to our phase model (2.3.1) to demonstrate some stability properties of the rotating wave solution found in Chapter 2. In Section 3.1 we provide the basic definitions and some preliminary results that will be used throughout. In Section 3.2 we will sharpen our focus to investigating discrete elliptic operators acting on Hilbert spaces. Sections 3.3 and 3.4 together provide an alternative view of infinite graphs by considering them as metric-measure spaces and showing how this framework can be used to understand random walks along the vertices and edges of a graph. We will show in the following chapter that it is this random walk theory that provides useful insight for understanding the stability of phase-locked solutions to our coupled phase model.

3.1 Preliminary Definitions

We consider a graph $G = (V, E)$ with a countably infinite collection of vertices, V , and a set of unoriented edges between these vertices, E , with the property that at most one edge can connect two vertices. We refer to a loop as an edge which initiates and terminates at the same vertex. If there exists an edge $e \in E$ connecting the two vertices $v, v' \in V$ then we write $v \sim v'$, and since the edges are unoriented this relation is naturally symmetric in that $v' \sim v$ as well. In this way we may equivalently consider the set of edges E as a subset of the product $V \times V$. A graph is called connected if for any two vertices $v, v' \in V$ there exists a finite sequence of vertices in V , $\{v_1, v_2, \dots, v_n\}$, such that $v \sim v_1, v_1 \sim v_2, \dots, v_n \sim v'$. Throughout this thesis we will only consider connected graphs.

We will also consider a weight function on the edges between vertices given by $w : V \times V \rightarrow [0, \infty)$ such that for all $v, v' \in V$ we have $w(v, v') = w(v', v)$ and $w(v, v') > 0$ if and only if $v \sim v'$. This then leads to the notion of a weighted graph,

written as the triple $G = (V, E, w)$. The weight function also naturally extends to the notion of the measure (or sometimes weight) of a vertex, $m : V \rightarrow [0, \infty]$, defined by

$$m(v) := \sum_{v' \in V} w(v, v') = \sum_{v \sim v'} w(v, v'). \quad (3.1.1)$$

Throughout this work we will only consider graphs and weight functions such that $m(v) < \infty$ for all $v \in V$. The weight function then leads to the definition of two important operators acting on the graph.

Definition 3.1.1. *For any function $f : V \rightarrow \mathbb{C}$ acting on the vertices of $G = (V, E, w)$, we define the **combinatorial graph Laplacian** to be the operator L_{comb} acting on these functions by*

$$L_{comb}f(v) = \sum_{v \sim v'} w(v, v')(f(v) - f(v')). \quad (3.1.2)$$

*Similarly, for any function $f : V \rightarrow \mathbb{C}$ acting on the vertices of $G = (V, E, w)$, we define the **normalized graph Laplacian** to be the operator L_{norm} acting on these functions by*

$$L_{norm}f(v) = \frac{1}{m(v)} \sum_{v \sim v'} w(v, v')(f(v) - f(v')). \quad (3.1.3)$$

We see that the only difference between the combinatorial and the normalized graph Laplacians is the factor $1/m(v)$. This factor helps to ensure that the normalized graph Laplacian is well-behaved in the case when the weight function can take on arbitrarily large values over the set of edges. Throughout this thesis we will see how both graph Laplacian operators play an important role in furthering our understanding of the phase system (2.1.6) and its rotating wave solution.

Remark 3.1.2. Notice that the coupling terms in (1.2.2) represent a combinatorial graph Laplacian on the graph with vertex set \mathbb{Z}^2 , edge set being the set of all nearest-neighbour connections and the weight of every edge being identically 1. In this case the measure of each vertex, $(i, j) \in \mathbb{Z}^2$, is identically 4 and therefore simply rescaling α or t can result in a normalized graph Laplacian.

Natural spatial settings for the graph Laplacian operators of Definition 3.1.1 are the sequence spaces

$$\ell^p(V) = \{f : V \rightarrow \mathbb{C} \mid \sum_{v \in V} |f(v)|^p < \infty\}, \quad (3.1.4)$$

for any $p \in [1, \infty)$. The vector space $\ell^p(V)$ become a Banach space when equipped with the norm

$$\|f\|_p := \left(\sum_{v \in V} |f(v)|^p \right)^{\frac{1}{p}}. \quad (3.1.5)$$

We may also consider the Banach space $\ell^\infty(V)$, the vector space of all uniformly bounded functions $f : V \rightarrow \mathbb{C}$ with norm given by

$$\|f\|_\infty := \sup_{v \in V} |f(v)|. \quad (3.1.6)$$

One often writes the elements of the sequence spaces in the alternate form as sequences indexed by the elements of V , $f = \{f_v\}_{v \in V}$, where $f_v := f(v)$. Also, it should be noted that these definitions extend to any countable index set V , independent of a respective graph.

Throughout this thesis we will also consider linear operators acting between these sequence spaces. That is, consider a linear operator $T : \ell^p(V) \rightarrow \ell^q(V)$ for some $1 \leq p, q \leq \infty$. We denote the norm of this operator as

$$\|T\|_{p \rightarrow q} := \sup_{0 \neq f \in \ell^p(V)} \frac{\|Tf\|_q}{\|f\|_p}. \quad (3.1.7)$$

One should note that there are many different, but equivalent, versions of this norm which one may work with. We now consider the graph Laplacians as operators acting upon these sequence spaces and present sufficient conditions under which the operator is not only well-defined, but also bounded. Before doing so, we remark that for a vertex $v \in V$ we define the degree to be

$$\text{Deg}(v) = \#\{v' \in V \mid v \sim v'\}, \quad (3.1.8)$$

where $\#\{\cdot\}$ represents the cardinality of the set. We use this definition to provide the following results.

Lemma 3.1.3. *Let $G = (V, E, w)$ be a weighted graph. If there exists finite $D, M > 0$ such that $\text{Deg}(v) \leq D$ and $w(v, v') \leq M$ for all $v, v' \in V$ then the combinatorial graph Laplacian (3.1.2) defines a bounded linear operator on $\ell^p(V)$ for all $p \in [1, \infty]$.*

Proof: Following the interpolation result of Exercise 12 of §2.6 from [2], showing that $L_{comb} : \ell^1(V) \rightarrow \ell^1(V)$ and $L_{comb} : \ell^\infty(V) \rightarrow \ell^\infty(V)$ are bounded operators implies that $L_{comb} : \ell^p(V) \rightarrow \ell^p(V)$ are uniformly bounded for all $1 < p < \infty$.

$L_{comb} : \ell^1(V) \rightarrow \ell^1(V)$: From the constants D, M from the statement of the Lemma, we get that

$$\begin{aligned} |L_{comb}f(v)| &\leq \sum_{v \sim v'} w(v, v') (|f(v)| + |f(v')|) \\ &\leq DM|f(v)| + M \sum_{v \sim v'} |f(v')|, \end{aligned} \quad (3.1.9)$$

for all $v \in V$. Then summing over all $v \in V$ gives

$$\begin{aligned} \sum_{v \in V} |L_{comb}f(v)| &\leq DM \sum_{v \in V} |f(v)| + M \sum_{v \in V} \sum_{v \sim v'} |f(v')| \\ &\leq 2DM \sum_{v \in V} |f(v)| \\ &= 2DM \|f\|_1, \end{aligned} \tag{3.1.10}$$

showing that $\|L_{comb}\|_{1 \rightarrow 1} \leq 2DM$.

$L_{comb} : \ell^\infty(V) \rightarrow \ell^\infty(V)$: We merely use (3.1.9) and extend it further to get that

$$|L_{comb}f(v)| \leq 2DM \|f\|_\infty, \tag{3.1.11}$$

for all $v \in V$. Taking the supremum over all $v \in V$ gives $\|L_{comb}f\|_\infty \leq 2DM \|f\|_\infty$ and therefore $\|L_{comb}\|_{\infty \rightarrow \infty} \leq 2DM$ as well. \blacksquare

The following proof gives sufficient conditions for boundedness of the normalized Laplacian operator. One should notice that we do not need uniform boundedness on the weights anymore. As previously mentioned, this is due to the fact that the $1/m(v)$ term acts to ensure the operator is well-behaved even when the weights are not.

Lemma 3.1.4. *Let $G = (V, E, w)$ be a weighted graph. If there exists finite $D > 0$ such that $\text{Deg}(v) \leq D$ then the normalized graph Laplacian (3.1.3) defines a bounded linear operator on $\ell^p(V)$ for all $p \in [1, \infty]$.*

Proof: This proof follows in a very similar way to that of Lemma 3.1.3. Therefore, we will only reduce the proof to a point where one can directly follow the proof of Lemma 3.1.3.

Begin by noticing that from the definition of the measure (3.1.1) we have

$$\frac{w(v, v')}{m(v)} \leq \sum_{v \sim v'} \frac{w(v, v')}{m(v)} = 1, \tag{3.1.12}$$

for all $v, v' \in V$. Hence,

$$\begin{aligned} |L_{norm}f(v)| &\leq \frac{1}{m(v)} \sum_{v \sim v'} w(v, v') (|f(v)| + |f(v')|) \\ &= |f(v)| + \sum_{v \sim v'} \frac{w(v, v')}{m(v)} |f(v')|, \end{aligned} \tag{3.1.13}$$

for all $v \in V$. One may apply Hölder's Inequality to find

$$\sum_{v \sim v'} \frac{w(v, v')}{m(v)} |f(v')| \leq \left(\sup_{v' \in V} \frac{w(v, v')}{m(v)} \right) \left(\sum_{v \sim v'} |f(v')| \right) \leq \sum_{v \sim v'} |f(v')|, \quad (3.1.14)$$

for all $v \in V$. Combining this with (3.1.13) gives

$$|L_{norm}f(v)| \leq |f(v)| + \sum_{v \sim v'} |f(v')|, \quad (3.1.15)$$

which is an analogous bound to (3.1.9) above. Using this bound we may simply follow the proof of Lemma 3.1.3 to obtain the result. \blacksquare

3.2 Discrete Elliptic Operators

In this section we continue the work of the previous section with a specific focus of the combinatorial graph Laplacian as an operator on $\ell^2(V)$. Recall that $\ell^2(V)$ is a Hilbert space equipped with the natural inner product

$$\langle f, g \rangle = \sum_{v \in V} f(v) \cdot \overline{g(v)}, \quad (3.2.1)$$

for all $f, g \in \ell^2(V)$. Then, by the symmetry of the weight function, L_{comb} as an operator on $\ell^2(V)$ is a symmetric operator, and therefore its spectrum is contained in the real line. Furthermore, the Min-Max Theorem for Hilbert spaces gives that the infimum of the spectrum of the combinatorial graph Laplacian, denoted $\mu_0(L_{comb})$, is given by

$$\mu_0(L_{comb}) = \inf_{\|f\|_2=1} \langle L_{comb}f, f \rangle. \quad (3.2.2)$$

One may simplify this to obtain

$$\langle L_{comb}f, f \rangle = \frac{1}{2} \sum_{v \in V} \sum_{v \sim v'} w(v, v') |f(v) - f(v')|^2. \quad (3.2.3)$$

To justify this notice that $w(v, v')$ contributes to the sum twice by the symmetry of the weight function. Then this gives

$$\begin{aligned} w(v, v')(f(v) - f(v'))\overline{f(v)} + w(v', v)(f(v') - f(v))\overline{f(v')} \\ = w(v, v')(f(v) - f(v'))(\overline{f(v)} - \overline{f(v')}) \\ = w(v, v')|f(v) - f(v')|^2. \end{aligned} \quad (3.2.4)$$

The factor of one half in front of the sum comes from the fact that everything is being summed twice, again by the symmetry of the weight function. Thus, L_{comb} is a positive operator since $\mu_0(L_{comb}) \geq 0$.

Definition 3.2.1. Let $G = (V, E, w)$ be a weighted graph with associated combinatorial graph Laplacian, denoted L_{comb} , and a function $P : V \rightarrow \mathbb{R}$. We call the operator $A = L_{comb} + P$ acting on the functions $f : V \rightarrow \mathbb{C}$ by

$$Af(v) = \sum_{v \sim v'} w(v, v')(f(v) - f(v')) + P(v)f(v) \quad (3.2.5)$$

a discrete elliptic operator.

One sees that P acts diagonally on the functions $f : V \rightarrow \mathbb{C}$, and therefore P is also a symmetric operator on $\ell^2(V)$. This in turn gives that A is a symmetric operator, and an extension of the combinatorial graph Laplacians. The operator P is often referred to as a potential operator. This alternative nomenclature comes from the fact that discrete elliptic operators resemble the right hand side of a spatially discretized Schrödinger equation since the combinatorial graph Laplacian acts similarly to a spatially discretized Laplacian operator.

Since $A = L_{comb} + P$ is a symmetric operator, the Min-Max Theorem for Hilbert Spaces again dictates that the minimum of its spectrum is given by

$$\mu_0(A) = \inf_{\|f\|_2=1} \langle Af, f \rangle. \quad (3.2.6)$$

One follows the same manipulations as above to find that

$$\langle Af, f \rangle = \frac{1}{2} \sum_{v \in V} \sum_{v \sim v'} w(v, v') |f(v) - f(v')|^2 + \sum_{v \in V} P(v) |f(v)|^2. \quad (3.2.7)$$

Now though, one sees that A is not necessarily a positive operator because of the added potential terms P . A sufficient condition for A to be a positive operator would be that $P(v) \geq 0$ for all $v \in V$.

Definition 3.2.2. A ground state of the discrete elliptic operator $A = L_{comb} + P$ is a real-valued function $\zeta : V \rightarrow \mathbb{R}$ such that $\zeta(v) > 0$ for all $v \in V$ and

$$A\zeta = \mu_0(A)\zeta. \quad (3.2.8)$$

It should be noted that even if ζ exists, it does not necessarily lie in $\ell^2(V)$. For a discussion of problems of this type in the continuous setting see, for example, Pinsky [57]. This now leads to the following result from Dodziuk [20] which will be used in the coming chapter to demonstrate the stability of the rotating wave solution found in the previous chapter.

Theorem 3.2.3 ([20], §2, Theorem 2.1). Consider a discrete elliptic operator $A = L_{comb} + P$ on a connected weighted graph $G = (V, E, w)$ with no loops where L_{comb} is positive and symmetric and there exists a constant $c \in \mathbb{R}$ such that $P(v) \geq c$ for all $v \in V$. There exists a ground state ζ for A .

Remark 3.2.4. The result of Theorem 3.2.3 is obtained under the assumption that the weighted graph does not contain any loops. This will be the only place where a graph is assumed not to have loops.

An important and motivating application for Theorem 3.2.3 comes from the case when $P(v) = 0$ for all $v \in V$. In this case the discrete elliptic operator A reduces down to the combinatorial graph Laplacian L_{comb} . Moreover, one sees that using (3.2.3) a ground state exists by taking $\zeta = \{C\}_{v \in V}$, where $C > 0$ is a real valued constant. The important thing to note here is that this ζ does not belong to $\ell^2(V)$, but Theorem 3.2.3 does give that 0 belongs to the spectrum of all combinatorial graph Laplacians when acting on $\ell^2(V)$. Hence, $L_{comb} : \ell^2(V) \rightarrow \ell^2(V)$ never has a bounded inverse.

3.3 Graphs as Metric-Measure Spaces

In this section we extend some of the notions introduced in Section 3.1 and provide the necessary nomenclature to introduce random walks on graphs. We saw that for a weighted graph, $G = (V, E, w)$, we define the measure of a vertex as in (3.1.1). This notion extends to the volume of a subset, $V_0 \subset V$, by defining

$$\text{Vol}(V_0) := \sum_{v \in V_0} m(v). \quad (3.3.1)$$

Hence, under this definition of volume, a weighted graph naturally becomes a measure space on the σ -algebra given by the power set of V .

Graphs also have a natural underlying metric, ρ , given by the function which returns the smallest number of edges to produce a path between two vertices $v, v' \in V$. First note that since G is assumed connected, the distance function is well-defined. Second, this graph distance is just an abstract generalization of the steps along the lattice used extensively in Chapter 2. This metric allows for the consideration of a ball of radius $r \geq 0$ centred at the vertex $v \in V$, given by

$$B(v, r) := \{v' \mid \rho(v, v') \leq r\}. \quad (3.3.2)$$

In this work we will write $\text{Vol}(v, r)$ to denote $\text{Vol}(B(v, r))$. The combination of the graph metric and the vertex measure allows one to interpret a weighted graph as a *metric-measure space*.

We provide a series of definitions to further our understanding of graphs as metric-measure spaces.

Definition 3.3.1. *The weighted graph $G = (V, E, w)$ satisfies a **uniform polynomial volume growth** condition of order d , abbreviated $VG(d)$, if there exists $d > 0$ and $c_{vol,1}, c_{vol,2} > 0$ such that*

$$c_{vol,1}r^d \leq \text{Vol}(v, r) \leq c_{vol,2}r^d, \quad (3.3.3)$$

for all $v \in V$ and $r \geq 0$.

In some cases one may also consider graphs with more general volume growth conditions, but for the purposes of this thesis we will restrict ourselves to polynomial growth conditions. The characteristic examples of graphs satisfying $VG(d)$ are the integer lattices \mathbb{Z}^d with all edges of weight 1 such that there exists an edge between two vertices $n, n' \in \mathbb{Z}^d$ if and only if $\|n - n'\|_1 = 1$. This is pointed out on, for example, page 10 of [5].

Definition 3.3.2. We say $G = (V, E, w)$ satisfies $\Delta(\kappa)$ if there exists a $\kappa > 0$ such that

$$w(v, v') \geq \kappa m(v) \quad (3.3.4)$$

for all $v, v' \in V$ such that $v' \sim v$.

This property is sometimes referred to as a local elliptic property. The following lemma points out an important set of sufficient conditions to satisfy $\Delta(\kappa)$.

Lemma 3.3.3. Let $G = (V, E, w)$ be a weighted graph. If there exists constants $D, w_{min}, w_{max} > 0$ such that

$$w_{min} \leq w(v, v') \leq w_{max}, \quad (3.3.5)$$

and $\deg(v) \leq D$ for all $v \in V$ and $v \sim v'$, then G satisfies $\Delta(\kappa)$.

Proof: For all $v \in V$ we have

$$m(v) = \sum_{v \sim v'} w(v, v') \leq Dw_{max}. \quad (3.3.6)$$

This gives

$$\frac{w(v, v')}{m(v)} \geq \frac{w_{min}}{Dw_{max}} > 0. \quad (3.3.7)$$

Thus, $G = (V, E, w)$ will satisfy $\Delta(\kappa)$ for any $\kappa > 0$ such that $w_{min} \geq \kappa Dw_{max}$. \blacksquare

Definition 3.3.4. The weighted graph $G = (V, E, w)$ satisfies the **Poincaré inequality**, abbreviated **PI**, if there exists a $C_{PI} > 0$ such that

$$\sum_{v \in B(v_0, r)} m(v) |f(v) - f_B(v_0)|^2 \leq C_{PI} r^2 \left(\sum_{v, v' \in B(v_0, 2r)} w(v, v') (f(v) - f(v'))^2 \right), \quad (3.3.8)$$

for all functions $f : V \rightarrow \mathbb{R}$, all $v_0 \in V$, all $r > 0$, where

$$f_B(v_0) = \frac{1}{\text{Vol}(v_0, r)} \sum_{v \in B(v_0, r)} m(v) f(v). \quad (3.3.9)$$

The Poincaré Inequality is certainly the most difficult of the three definitions to work with. In practice it can be quite difficult to confirm whether or not a weighted graph satisfies *PI*, although some methods to obtain this inequality are given in [14]. Next we will introduce an important definition and result from [40] that can aid in determining if a graph satisfies *PI*.

Definition 3.3.5. *Let $G = (V, E, w)$ and $G' = (V', E', w')$ be two infinite weighted graphs satisfying $\Delta(\kappa)$ with respective graph metrics given by ρ and ρ' . A map $T : V \rightarrow V'$ is called a **rough isometry** if there exists $a, c > 1$, $b > 0$ and $M > 0$ such that*

$$a^{-1}\rho(v_1, v_2) - b \leq \rho'(T(v_1), T(v_2)) \leq a\rho(v_1, v_2) + b, \quad \forall v_1, v_2 \in V, \quad (3.3.10a)$$

$$\rho'(T(V), v') \leq M, \quad \forall v' \in V', \quad (3.3.10b)$$

$$c^{-1}m(v) \leq m'(T(v)) \leq cm(v), \quad \forall v \in V, \quad (3.3.10c)$$

where m and m' are the vertex measures associated to the graph G and G' , respectively. If $T : G \rightarrow G'$ is a rough isometry, G and G' are said to be **rough isometric**.

Proposition 3.3.6 ([40], §5.3, Proposition 5.15(2)). *Let G and G' be two infinite weighted graphs satisfying $\Delta(\kappa)$ that are rough isometric. Then if there exists a $d > 0$ such that G satisfies $VG(d)$ and *PI*, then G' satisfies $VG(d)$ and *PI* as well.*

Hambly and Kumagai's original statement of Proposition 3.3.6 refers to our *PI* as a *weak* Poincaré inequality since they sometimes use a stronger inequality in their work. Hambly and Kumagai also originally provide their statement in terms of a more general volume growth condition, but we will work with the weaker version stated here since we are only interested in polynomial volume growth. In fact, using property (3.3.10c) one can show that property $VG(d)$ is preserved under rough isometries in a straightforward way.

3.4 Random Walks on Infinite Weighted Graphs

Throughout this section we will introduce some important results for random walks on infinite weighted graphs. We will use similar nomenclature and notation to that of Delmotte [18], which appears to now be quite standard. Another excellent source which summarizes much of the relevant results and more is Telcs' textbook [67].

Our interest here will be in continuous time random walks on the vertices of a weighted graph $G = (V, E, w)$. One can interpret this as being at a single vertex on the graph, and then waiting an exponentially distributed amount of time to move along an edge to another vertex of the graph. Upon arriving at the next vertex, this process begins again by waiting an exponentially distributed amount of time to move

along an edge to another vertex of the graph. The weights in this scenario act as preferences to moving along a certain edge; the greater the weight, the greater the preference. More precisely, if we are at the vertex $v \in V$, then the probability we move to $v' \sim v$ is given by $w(v, v')/m(v)$. The important thing to note here is that there are exactly two probabilities involved: the probability of when to move from one vertex to the next and the probability of choosing the vertex to move to. We use the notation $p_t(v, v')$ to denote the probability that at time $t \geq 0$ we have arrived at the vertex v' having started at vertex v . By definition we have

$$\sum_{v' \in V} p_t(v, v') = 1 \quad (3.4.1)$$

for all $v \in V$. Delmotte points out that the $p_t(\cdot, \cdot)$ are not necessarily symmetric in their arguments due to the weights on the graph, but it has been shown that

$$\frac{p_t(v, v')}{m(v')} = \frac{p_t(v', v)}{m(v)}. \quad (3.4.2)$$

This has prompted some authors [7, 40, 41] to instead study the symmetric transition densities

$$q_t(v, v') := \frac{p_t(v, v')}{m(v')} \quad (3.4.3)$$

for all $v, v' \in V$.

Much work has been done to understand the long-time behaviour of the probabilities $p_t(\cdot, \cdot)$, notably the pioneering work of Delmotte [18]. Most applicable to our present situation is that these probabilities are used to understand the solution to the spatially discrete heat equation

$$\dot{x}_v = \frac{1}{m(v)} \sum_{v' \in V} w(v, v')(x_{v'} - x_v), \quad v \in V. \quad (3.4.4)$$

When considering all elements $\{x_v\}_{v \in V}$, the right hand side of (3.4.4) is the negative of a normalized graph Laplacian operator, again denoted L_{norm} . Then as stated in Theorem 23 of [50], $-L_{norm}$ is the infinitesimal generator of the semigroup $P_t = e^{-L_{norm}t}$. For an initial condition $x_0 = \{x_{v,0}\}_{v \in V}$ the fundamental solution to (3.4.4) with this initial condition is given by

$$x_v(t) = [P_t x_0]_v = \sum_{v' \in V} p_t(v, v') x_{v',0} \quad (3.4.5)$$

for each $v \in V$, thus showing the connection between the probabilities $p_t(\cdot, \cdot)$ and the semigroup P_t . The fact that (3.4.5) solves (3.4.4) was pointed out by Delmotte, and other sources include, but are not limited to, [41, 71] for continuous time transitions

and [36, 37] for discrete time transitions. One also can see using the identity (3.4.3), the solution (3.4.5) now can be interpreted as the Lebesgue integral

$$[P_t x_0]_v = \sum_{v' \in V} q_t(v, v') x_{v', 0} m(v'), \quad (3.4.6)$$

for a symmetric kernel defined with the q_t 's and the initial condition over a discrete space with respect to the measure $m : V \rightarrow [0, \infty)$.

Proposition 3.4.1 ([18], §3.1, Proposition 3.1). *Assume there exists $d, \kappa > 0$ such that the weighted graph $G = (V, E, w)$ satisfies $VG(d)$, PI and $\Delta(\kappa)$. Then for all $v, v' \in V$ and $t \geq 0$ there exists a constant $C_0 > 0$ independent of v, v' and t such that*

$$p_t(v, v') \leq C_0 m(v') t^{-\frac{d}{2}}. \quad (3.4.7)$$

Delmotte proves a much stronger version of Proposition 3.4.1 under more general volume growth conditions that applies to a more diverse range of graphs (such as fractal graphs and trees), but for our purposes we work with Proposition 3.4.1 as it is stated here. Delmotte also goes further to prove that the assumptions of Proposition 3.4.1 are equivalent to a Parabolic Harnack Inequality, which we do not explicitly state here because it will not be necessary to our result. What is important to note though is that Theorem 2.32 of [38] dictates that any graph (or more generally metric space) satisfying the Parabolic Harnack Inequality further satisfies the estimate

$$|p_t(v_1, v_3) - p_t(v_2, v_3)| \leq C m(v_3) \left(\frac{\rho(v_1, v_2)}{\sqrt{t}} \right)^\eta p_{2t}(v_1, v_3) \quad (3.4.8)$$

for all $v_1, v_2, v_3 \in V$ and some independent $C, \eta > 0$. Thus, we may assume that when the conditions of Proposition 3.4.1 are satisfied, then so must be (3.4.8).

Corollary 3.4.2. *Let $G = (V, E, w)$ be a weighted graph satisfying the assumptions of Proposition 3.4.1. If there exists an $M > 0$ such that $m(v) \leq M$ for all $v \in V$, then for any $x_0 \in \ell^1(V)$ there exists a constant $C > 0$ such that*

$$\|P_t\|_{1 \rightarrow \infty} \leq C t^{-\frac{d}{2}} \|x_0\|_1. \quad (3.4.9)$$

Proof: Since the conditions of Proposition 3.4.1 are satisfied for some $d > 0$, there exists a $C_0 > 0$ such that (3.4.7) holds. Then apply Hölder's Inequality to the general solution (3.4.5) to find that

$$|[P_t x_0]_v| \leq C_0 M t^{-\frac{d}{2}} \|x_0\|_1. \quad (3.4.10)$$

Taking the supremum over all $v \in V$ gives the desired result. ■

Ultracontractive properties such as that stated in Corollary 3.4.2 of one parameter semigroups have been intensely studied, notably in the seminal work of Varopoulos who provided similar results to (3.4.9) in a much more general setting, but nonetheless arrived at a similar conclusion [70].

It has been demonstrated (see, for example, page 219 of [50]) that if $0 \leq x_{v,0} \leq 1$ for all $v \in V$, then $0 \leq [P_t x_0]_v \leq 1$ for all $v \in V$. The lower bound follows immediately from the positivity of the probabilities $p_t(\cdot, \cdot)$, whereas the upper bound follows from a direct application of Hölder's Inequality and the identity (3.4.1). Following the comments at the beginning of Section 1.2 of [7], this implies that there exists a $C_{op} > 0$ for which

$$\|P_t\|_{p \rightarrow p} \leq C_{op} \quad (3.4.11)$$

for all $1 \leq p \leq \infty$. These uniform bounds and the ultracontractivity property (3.4.10) can be extended further by the following lemma.

Lemma 3.4.3. *Assume there exists constants $C, C_{op} > 0$ such that $\|P_t\|_{1 \rightarrow \infty} \leq Ct^{-\frac{d}{2}}$ and $\|P_t\|_{1 \rightarrow 1} \leq C_{op}$. Then for all $1 \leq p \leq \infty$ there exists a constant $C_p > 0$ (depending on p) such that*

$$\|P_t\|_{1 \rightarrow p} \leq C_p t^{-\frac{d}{2}(1-\frac{1}{p})}. \quad (3.4.12)$$

Proof: We begin by recalling the log-convexity property of ℓ^p norms. For any $1 \leq p_0 \leq p_1 \leq \infty$ and $0 < \theta < 1$ we define

$$\frac{1}{p_\theta} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}. \quad (3.4.13)$$

Then for all $x \in \ell^{p_0}(V)$ we have

$$\|x\|_{p_\theta} \leq \|x\|_{p_0}^{1-\theta} \|x\|_{p_1}^\theta. \quad (3.4.14)$$

To apply this log-convexity property to our present situation we take $p_0 = 1$ and $p_1 = \infty$. Then $p_\theta = \frac{1}{1-\theta}$ and for any $x \in \ell^1(V)$ we have

$$\begin{aligned} \|P_t x\|_{p_\theta} &\leq \|P_t x\|_1^{\frac{1}{p_\theta}} \|P_t x\|_\infty^{1-\frac{1}{p_\theta}} \\ &\leq \|P_t\|_{1 \rightarrow 1}^{\frac{1}{p_\theta}} \|x\|_1^{\frac{1}{p_\theta}} \|P_t\|_{1 \rightarrow \infty}^{1-\frac{1}{p_\theta}} \|x\|_1^{1-\frac{1}{p_\theta}} \\ &\leq C_{op}^{\frac{1}{p_\theta}} C^{1-\frac{1}{p_\theta}} t^{-\frac{d}{2}(1-\frac{1}{p_\theta})} \|x\|_1. \end{aligned} \quad (3.4.15)$$

Thus, taking $\|x\|_1 = 1$ shows $\|P_t\|_{1 \rightarrow p_\theta} \leq C_{op}^{\frac{1}{p_\theta}} C^{1-\frac{1}{p_\theta}} t^{-\frac{d}{2}(1-\frac{1}{p_\theta})}$. By varying $\theta \in (0, 1)$ we obtain the result for $1 < p < \infty$ and the endpoints $p = 1, \infty$ are taken care of by assumption. \blacksquare

We will see in the coming chapter that our investigations will greatly utilize the case $p = 2$ from Lemma 3.4.3, which gives

$$\|P_t\|_{1 \rightarrow 2} \leq C_2 t^{-\frac{d}{4}}. \quad (3.4.16)$$

Finally, to avoid the singularity at $t = 0$ in (3.4.10) and (3.4.16), we will use the alternative upper bounds:

$$\|P_t\|_{1 \rightarrow \infty} \leq \tilde{C}(1+t)^{-\frac{d}{2}}, \quad (3.4.17a)$$

$$\|P_t\|_{1 \rightarrow 2} \leq \tilde{C}(1+t)^{-\frac{d}{4}}, \quad (3.4.17b)$$

with a new constant $\tilde{C} > 0$. Note that such an alternative upper bound is possible since for large t these new upper bounds decay at the same rate as the bounds in (3.4.10) and (3.4.16) and for small $t \geq 0$ the operator P_t is well-behaved and finite. We will also use such an alternative upper bound of $(1+t)^{-\frac{d}{2}}$ in (3.4.8) for the same reason.

This concludes our very brief exploration of the rich and diverse area of random walks on graphs. The results stated in this section and the others of this chapter will be applied to our phase system (2.1.6) to understand the stability of the rotating wave solution demonstrated to exist in Chapter 2.

Chapter 4

Stability Results for the Phase Solution

In this chapter we work to apply the results of denumerable graph networks from the previous chapter to demonstrate some stability aspects of the coupled phase equation examined in Chapter 2. In particular, our interest lies in investigating phase-locked solutions to (2.1.6) of the form

$$\theta_{i,j}(t) = \omega_0 t + \bar{\theta}_{i,j}, \quad (4.0.1)$$

where $\bar{\theta}_{i,j}$ is a time-independent phase-lag for all $(i, j) \in \mathbb{Z}^2$. Recall that ansatz (4.0.1) reduces system (2.1.6) to solving

$$\sum_{i',j'} H(\bar{\theta}_{i',j'} - \bar{\theta}_{i,j}) = 0, \quad (4.0.2)$$

which is how we obtained the rotating wave solution of Theorem 2.3.1.

Assuming we have a solution to (4.0.2), our interest turns to applying a slight perturbation to the ansatz (4.0.1), written

$$\theta_{i,j}(t) = \omega_0 t + \bar{\theta}_{i,j} + \psi_{i,j}(t) \quad (4.0.3)$$

and inspecting conditions to which $\psi_{i,j}(t) \rightarrow 0$ as $t \rightarrow \infty$ for all $(i, j) \in \mathbb{Z}^2$. Notice that the perturbed ansatz (4.0.3) leads to the system of differential equations

$$\dot{\psi}_{i,j} = \sum_{i',j'} H(\bar{\theta}_{i',j'} + \psi_{i',j'} - \bar{\theta}_{i,j} - \psi_{i,j}), \quad (i, j) \in \mathbb{Z}^2, \quad (4.0.4)$$

which has a steady-state solution given by $\psi = 0$. System (4.0.4) forms the basis for our investigation in this chapter.

This chapter is laid out as follows. We begin by investigating the spectrum of the linearization about the equilibrium solution $\psi = 0$ to (4.0.4). In Section 4.1 we will

demonstrate how the spectral stability of the solution depends on the Banach space in which the ψ 's belong to. These spectral results motivate Section 4.2 where we use the results from Section 3.4 to provide algebraic decay rates for the ψ 's (as opposed to exponential) under a mild collection of hypotheses. The proof of our main stability theorem is left to Section 4.3. This chapter concludes with a return to the framework of the phase system (2.1.6) where we will apply the result of our stability theorem for coupled networks to this situation to demonstrate the asymptotic stability of the rotating wave solution.

4.1 Spectral Stability

To begin, we notice that system (2.1.6) has an important symmetry property in that if $\{\theta_{i,j}(t)\}_{(i,j) \in \mathbb{Z}^2}$ is a solution then so is $\{\theta_{i,j}(t) + C\}_{(i,j) \in \mathbb{Z}^2}$ for any constant $C \in \mathbb{R}$. This comes from the fact that the system of differential equations (2.1.6) is only dependent on the difference between neighbouring elements. In this section we will work around this by taking any single index, say (i_0, j_0) , and fixing the value of the phase-lag $\bar{\theta}_{i_0, j_0}$ such that $\psi_{i_0, j_0} = 0$. Then introducing the transformation

$$\bar{\phi}_{i,j} = \bar{\theta}_{i,j} - \bar{\theta}_{i_0, j_0} \quad (4.1.1)$$

for all (i, j) will remove the translational invariance of the solution and pose the problem on the infinite lattice $\tilde{\mathbb{Z}}^2 := \mathbb{Z}^2 \setminus \{(i_0, j_0)\}$. The perturbed ansatz (4.0.3) will now be considered of the form

$$\theta_{i,j}(t) = \omega_0 t + \bar{\phi}_{i,j} + \psi_{i,j}(t), \quad (4.1.2)$$

for all $(i, j) \in \tilde{\mathbb{Z}}^2$. Essentially the perturbation ψ is now restricted to the flow-invariant subspace $\psi_{i_0, j_0} = 0$.

Remark 4.1.1. Notice that (4.1.1) does not effect the difference between neighbouring elements which are both indexed by elements of $\tilde{\mathbb{Z}}^2$ since we have

$$\bar{\phi}_{i',j'} - \bar{\phi}_{i,j} = (\bar{\theta}_{i',j'} - \bar{\theta}_{i_0, j_0}) - (\bar{\theta}_{i,j} - \bar{\theta}_{i_0, j_0}) = \bar{\theta}_{i',j'} - \bar{\theta}_{i,j}. \quad (4.1.3)$$

This contrasts with the case of an index which is a nearest-neighbour of (i_0, j_0) since we now have

$$\bar{\theta}_{i'_0, j'_0} - \bar{\theta}_{i_0, j_0} = \bar{\phi}_{i'_0, j'_0}. \quad (4.1.4)$$

Throughout this section we will have $\{\bar{\theta}_{i,j}\}_{(i,j) \in \mathbb{Z}^2}$ represent the phase-lags corresponding to the rotating wave solution found in Theorem 2.3.1 with the exact symmetries of Figure 2.1. The linearization of (4.0.4) about the steady-state $\psi = 0$, which we denote A , allows us to decompose this operator into the sum of two distinct

operators acting on a yet to be specified sequence space with elements indexed by $\tilde{\mathbb{Z}}^2$. That is, for $x = \{x_{i,j}\}_{(i,j) \in \tilde{\mathbb{Z}}^2}$ we write $A = L + P$ where

$$(Lx)_{i,j} = \sum_{(i',j') \in \tilde{\mathbb{Z}}^2} H'(\bar{\phi}_{i',j'} - \bar{\phi}_{i,j})(x_{i',j'} - x_{i,j}), \quad (4.1.5)$$

and

$$(Px)_{i,j} = \begin{cases} -H'(-\bar{\phi}_{i,j})x_{i,j} & : (i,j) = (i_0 \pm 1, j_0), (i_0, j_0 \pm 1) \\ 0 & : \text{otherwise} \end{cases} \quad (4.1.6)$$

Hence, one sees that $-A$ is a discrete elliptic operator over a graph whose vertices lie in one-to-one correspondence with the elements of $\tilde{\mathbb{Z}}^2$. The reason we say that $-A$ is a discrete elliptic operator and not A is that the order of the differences in L is the reverse of a standard combinatorial graph Laplacian operator. In this case the weight function $w : \tilde{\mathbb{Z}}^2 \times \tilde{\mathbb{Z}}^2 \rightarrow [0, \infty)$ is given by

$$w((i,j), (k,l)) = \begin{cases} H'(\bar{\phi}_{k,l} - \bar{\phi}_{i,j}) & : (k,l) \text{ and } (i,j) \text{ are nearest - neighbours} \\ 0 & : \text{otherwise} \end{cases} \quad (4.1.7)$$

This is indeed well-defined, since all local interactions are such that $|\bar{\theta}_{i',j'} - \bar{\theta}_{i,j}| \leq \frac{\pi}{2}$. More precisely, with the exception of the 'centre' four cells at $(0,0), (0,1), (1,0), (1,1)$ all local interactions are such that $|\bar{\theta}_{i',j'} - \bar{\theta}_{i,j}| < \frac{\pi}{2}$, whereas the coupling between any two of the four centre cells is exactly $\pi/2$. Furthermore, since H is an odd function, it follows that H' is an even function, and therefore we see that the weight function (4.1.7) is both symmetric and has nonnegative values when H satisfies Hypothesis 2.1.3. Hence, we consider the connected graph with vertex set that lies in one-to-one correspondence with the elements of $\tilde{\mathbb{Z}}^2$ and an edge set which at least contains all nearest-neighbour connections less those connections between any two of the centre four cells.

Remark 4.1.2. In order for the edge set to contain all nearest-neighbour interactions we require $H'(\pm\frac{\pi}{2}) \neq 0$, although it is not necessary to our work since the graph remains connected in either situation.

From the transformation (4.1.1) we have shown in (4.1.4) that the elements $\bar{\phi}_{i'_0, j'_0}$ are given by $\bar{\theta}_{i'_0, j'_0} - \bar{\theta}_{i_0, j_0}$. Again, since we have remarked that each element $\bar{\theta}_{i,j}$ can have at most two nearest-neighbours such that $|\bar{\theta}_{i',j'} - \bar{\theta}_{i,j}| = \frac{\pi}{2}$, from our transformation (4.1.1) we have that the operator P must act nontrivially on at least two elements indexed by nearest-neighbours of (i_0, j_0) . Indeed, term

$$-H'(-\bar{\phi}_{i'_0, j'_0}) = -H'(\bar{\theta}_{i'_0, j'_0} - \bar{\theta}_{i_0, j_0}) \quad (4.1.8)$$

can only vanish at a maximum of two indexed nearest-neighbours of (i_0, j_0) , meaning that at least two indexed nearest-neighbours of (i_0, j_0) are such that $-H'(\bar{\phi}_{i'_0, j'_0}) \neq 0$. This gives that P acts diagonally and nontrivially.

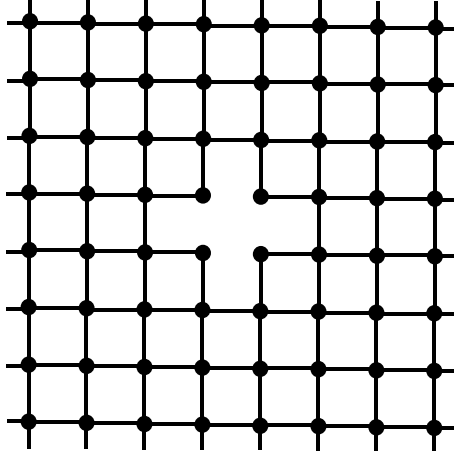


Figure 4.1: A possible representation of the underlying graph of the linearization about the rotating wave solution. The vertex set lies in one-to-one correspondence with the indices of the lattice.

Furthermore, one should understand that prior to applying the transformation (4.1.1) the resulting linearization would merely be a combinatorial graph Laplacian operator. By applying (4.1.1) we have effectively reduced this graph Laplacian operator to acting on the subspace of the sequences $x = \{x_{i,j}\}_{(i,j) \in \mathbb{Z}^2}$ with the property that $x_{i_0, j_0} = 0$. An illustration of this graph with full vertex set \mathbb{Z}^2 in the case that $H'(\pm \frac{\pi}{2}) = 0$ is given in Figure 4.1 for visualization. In the case that $H'(\pm \frac{\pi}{2}) \neq 0$ the only amendment to Figure 4.1 is that the edges connecting the centre four vertices will be present.

Proposition 4.1.3. $A : \ell^2(\tilde{\mathbb{Z}}^2) \rightarrow \ell^2(\tilde{\mathbb{Z}}^2)$ has a bounded inverse.

Proof: First we show that A is a bounded operator on $\ell^2(\tilde{\mathbb{Z}}^2)$. Indeed, since we can decompose $A = L + P$, proving each component, L and P , are bounded operators will show that A is a bounded operator. In our present situation $-L$ is a combinatorial graph Laplacian such that each vertex is connected to at most four other vertices (its nearest-neighbours) and the fact that the weight function (4.1.7) is uniformly bounded follows from a direct application of the Extreme Value Theorem since $|\bar{\theta}_{i',j'} - \bar{\theta}_{i,j}| \leq \frac{\pi}{2}$ for every local interaction over the lattice. Thus, Lemma 3.1.3 gives that $-L$ is a bounded operator, and therefore so is L . Finally, P is a diagonal operator which effects a maximum of four indices of the lattice, therefore making it a bounded operator acting on $\ell^2(\tilde{\mathbb{Z}}^2)$.

Using (3.2.7) we get

$$\langle -Ax, x \rangle = \sum_{i'_0, j'_0} H'(\bar{\phi}_{i'_0, j'_0}) |x_{i'_0, j'_0}|^2 + \frac{1}{2} \sum_{(i, j) \in \tilde{\mathbb{Z}}^2} \sum_{(i', j') \cap \tilde{\mathbb{Z}}^2} H'(\bar{\phi}_{i', j'} - \bar{\phi}_{i, j}) |x_{i', j'} - x_{i, j}|^2 \geq 0, \quad (4.1.9)$$

for every $x \in \ell^2(\tilde{\mathbb{Z}}^2)$ since H' is positive for all such interactions. Taking the infimum of (4.1.9) over all elements $x \in \ell^2(\tilde{\mathbb{Z}}^2)$ such that $\langle x, x \rangle = 1$ shows that $\mu_0(-A) \geq 0$, using the notation of (3.2.2). That is, the spectrum of the operator $-A$ lies in the nonnegative real numbers and therefore the spectrum of A lies in the nonpositive real numbers.

Let us now assume that 0 belongs to the spectrum of A . Since $-A$ is a discrete elliptic operator (with $-P \geq 0$) and its spectrum lies in one-to-one correspondence with that of A , Theorem 3.2.3 implies that there exists an element $\zeta = \{\zeta_{i, j}\}_{(i, j) \in \tilde{\mathbb{Z}}^2}$ such that $\zeta_{i, j} > 0$ and $-A\zeta = 0$. Again, we note that ζ does not necessarily belong to $\ell^2(\tilde{\mathbb{Z}}^2)$, but using (4.1.9) above we get

$$\langle -A\zeta, \zeta \rangle = \langle 0, \zeta \rangle = 0. \quad (4.1.10)$$

From (4.1.9) the only way in which this can happen is when ζ is such that $L\zeta = 0$ and $P\zeta = 0$ independently. Since our graph is connected, we see that the only way to have $L\zeta = 0$ is when ζ is a constant function on the vertices. Furthermore, since P is a nontrivial, diagonal operator it follows that in order to have $P\zeta = 0$ at least one $\zeta_{i, j}$ must be zero. Thus, $\zeta = 0$, a contradiction. ■

Corollary 4.1.4. *There exists a $\sigma_1, \sigma_2 < 0$ such that the spectrum of the operator $L : \ell^2(\tilde{\mathbb{Z}}^2) \rightarrow \ell^2(\tilde{\mathbb{Z}}^2)$ is contained in the interval $[\sigma_1, \sigma_2]$.*

Proof: Recall that the spectrum of a bounded operator is closed and bounded. Denote σ_1 and σ_2 to be the infimum and supremum of the spectrum of L , respectively. Then, from Proposition 4.1.3 and its proof we have that

$$\sigma_2 := \sup_{x \in \ell^2(V), \langle x, x \rangle = 1} \langle Lx, x \rangle < 0. \quad (4.1.11)$$

Therefore, the spectrum of L lies in the interval $[\sigma_1, \sigma_2]$. ■

Hence, we see that using the underlying graph structure $A : \ell^2(\tilde{\mathbb{Z}}^2) \rightarrow \ell^2(\tilde{\mathbb{Z}}^2)$ possesses a spectral gap, which in turn implies that the rotating wave solution to system (2.1.6) persists under sufficiently small perturbations in $\ell^2(\tilde{\mathbb{Z}}^2)$. Hence, when $\psi(0) \in \ell^2(\tilde{\mathbb{Z}}^2)$ is chosen sufficiently small, we find that $\phi(t) \rightarrow 0$ as $t \rightarrow \infty$ at an exponential rate. This approach directly parallels that which is undertaken by Ermentrout

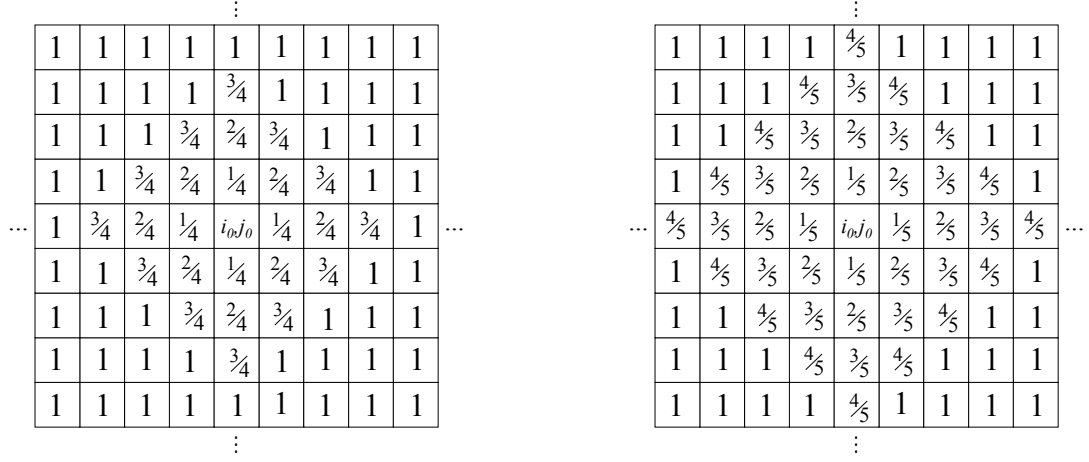


Figure 4.2: A visualization of two vectors from the sequence $\{x^{(n)}\}_{n=1}^{\infty}$ from the proof of Proposition 4.1.5. Here we have $n = 4, 5$ centred at the deleted index (i_0, j_0) .

in [23] where a general theorem on the stability of solutions to finite coupled networks is given. It is apparent that the underlying graph structure of coupled networks, be it finite or infinite, is crucial for both understanding how solutions can be found and their stability. An important distinction that is unique to having an infinite number of oscillators is that stability is dependent on the underlying space in which the problem is posed. To illustrate this fact the following proposition is presented.

Proposition 4.1.5. $A : \ell^{\infty}(\tilde{\mathbb{Z}}^2) \rightarrow \ell^{\infty}(\tilde{\mathbb{Z}}^2)$ does not have a bounded inverse.

Proof: To show that A does not have a bounded inverse, we show that it is not bounded below. That is, we show that there does not exist a $\delta > 0$ such that $\|Ax\|_{\infty} \geq \delta\|x\|_{\infty}$ for all $x \in \ell^{\infty}(\tilde{\mathbb{Z}}^2)$. We do this by constructing a sequence $\{x^{(n)}\}_{n=1}^{\infty} \subset \ell^{\infty}(\tilde{\mathbb{Z}}^2)$ such that $\|x^{(n)}\|_{\infty} = 1$ for all $n \geq 1$ but $\|Ax^{(n)}\|_{\infty} \rightarrow 0$ as $n \rightarrow \infty$.

The sequence of vectors is constructed in the following way: Begin by fixing $n \geq 1$. For those indices which are one step along the integer lattice \mathbb{Z}^2 (nearest-neighbours) to (i_0, j_0) we set the elements of the vector with these indices to $1/n$. Then we set the eight elements which are two steps from the index (i_0, j_0) (nearest-neighbours of the nearest-neighbours) to $2/n$. Next we set the twelve elements which are three steps from the index (i_0, j_0) to $3/n$. We continue this pattern so that for any $k \leq n$ we set those elements which are exactly k steps from the index (i_0, j_0) to k/n . For the remaining elements of whose indices lie at more than n steps from the index (i_0, j_0) we set to 1. Two vectors of this form are shown in Figure 4.2 for $n = 4, 5$ to visualize the form and demonstrate how the vectors change as n increases.

Then clearly for each n this vector has norm 1 in $\ell^{\infty}(\tilde{\mathbb{Z}}^2)$, but one should notice

that by construction we have

$$|x_{i',j'} - x_{i,j}| \leq \frac{1}{n} \quad (4.1.12)$$

for any (i, j) and a nearest-neighbour (i', j') in $\tilde{\mathbb{Z}}^2$. Furthermore,

$$(Px^{(n)})_{i,j} = \begin{cases} \frac{-H'(\bar{\phi}_{i,j})}{n} & : (i, j) = (i_0 \pm 1, j_0), (i_0, j_0 \pm 1) \\ 0 & : \text{otherwise} \end{cases} \quad (4.1.13)$$

which approaches the zero operator as $n \rightarrow \infty$. Similarly,

$$|(Lx^{(n)})_{i,j}| \leq \sum_{(i',j') \cap \tilde{\mathbb{Z}}^2} H'(\bar{\phi}_{i',j'} - \bar{\phi}_{i,j}) |x_{i',j'}^{(n)} - x_{i,j}^{(n)}| \leq \frac{4}{n} \cdot \max_{z \in [-\frac{\pi}{2}, \frac{\pi}{2}]} H'(z) \quad (4.1.14)$$

for all $(i, j) \in \tilde{\mathbb{Z}}^2$. Therefore $\|Ax^{(n)}\|_\infty \rightarrow 0$ as $n \rightarrow \infty$, showing that L is not bounded below and completing the proof. \blacksquare

The results of Proposition 4.1.5 show that vectors with sufficiently large norm in $\ell^\infty(\tilde{\mathbb{Z}}^2)$ can be mapped by A to vectors with arbitrarily small norms in $\ell^\infty(\tilde{\mathbb{Z}}^2)$ since the linearization A only takes into account the difference between nearest-neighbours. Thus, one sees that even by eliminating the translational symmetry of the solution on the full lattice \mathbb{Z}^2 , remnants of this eigenvalue remain on the reduced lattice $\tilde{\mathbb{Z}}^2$. This leads one to say that 0 belongs to the *essential spectrum* of the linearization of (2.1.6) about the rotating wave solution guaranteed by Theorem 2.3.1 since 0 is not an isolated eigenvalue in the spectrum. Hence, simply attempting to quotient out the translational symmetry (and hence the 0 eigenvalue) does not make the linearization invertible. In the following section we will show how to overcome this 0 element in the spectrum, to give local stability results for the rotating wave solutions which exhibit algebraic (as opposed to exponential) decay rates.

4.2 A General Stability Theorem for Coupled Networks

In the interest of generality, we will provide a stability theorem which is applicable to a wide variety of coupled lattice systems, including our own phase model (2.1.6). Let V be a countable collection of indices, or vertices. Throughout this work we will consider the system

$$\dot{u}_v = \sum_{v' \in N(v)} H(u_{v'} - u_v), \quad (4.2.1)$$

for each $v \in V$. Here $H : \mathbb{R} \rightarrow \mathbb{R}$ is a twice-differentiable general nonlinearity and $N(v) \subseteq V \setminus \{v\}$. We may also write (4.2.1) abstractly as an ordinary differential equation in the variable $u = \{u_v(t)\}_{v \in V}$ as

$$\dot{u} = f(u), \quad (4.2.2)$$

where $f : \mathbb{R}^V \rightarrow \mathbb{R}^V$. If $v' \in N(v)$ we say that the state $x_{v'}$ influences the state x_v . We make the following assumption.

Hypothesis 4.2.1. *For each $v \in V$, $v' \in N(v)$ if and only if $v \in N(v')$. Furthermore, there exists a finite $D \geq 1$ such that $1 \leq \#N(v) \leq D$ for every $v \in V$.*

Hypothesis 4.2.1 says that each element u_v is influenced by a finite number of other elements, and that the number of influences on any single element is uniformly bounded. Moreover, we see that Hypothesis 4.2.1 details that if $u_{v'}$ influences u_v , then u_v influences $u_{v'}$. That is, the influence topology is symmetric.

Remark 4.2.2. The abstract notation of (4.2.1) is entirely based upon our perturbation system (4.0.4). In the situation of (4.0.4) we have $V = \mathbb{Z}^2$ and for each $(i, j) \in \mathbb{Z}^2$ we have $N((i, j)) = \{(i+1, j), (i-1, j), (i, j+1), (i, j-1)\}$, the four nearest-neighbours of the index (i, j) .

As stated in the introduction to this chapter, our interest is in determining the stability of steady-state solutions to systems of the form (4.2.1). Let us assume that $\bar{u} = \{\bar{u}_v\}_{v \in V}$ is a steady-state solution to system (4.2.1). That is,

$$0 = \sum_{v' \in N(v)} H(\bar{u}_{v'} - \bar{u}_v), \quad (4.2.3)$$

for every $v \in V$, or equivalently

$$f(\bar{u}) = 0. \quad (4.2.4)$$

Linearizing (4.2.1) about the steady-state \bar{u} results in a differential equation whose linear part is governed by the operator $L := Df(\bar{u})$ acting on the elements $x = \{x_v\}_{v \in V}$ by

$$[Lx]_v = \sum_{v' \in N(v)} H'(\bar{u}_{v'} - \bar{u}_v)(x_{v'} - x_v), \quad (4.2.5)$$

for all $v \in V$. This leads to the next hypothesis.

Hypothesis 4.2.3. *The linearization L presented in (4.2.5) is such that*

$$H'(\bar{u}_{v'} - \bar{u}_v) = H'(\bar{u}_v - \bar{u}_{v'}) \geq 0 \quad (4.2.6)$$

for all $v \in V$ and $v' \in N(v)$. Furthermore, we have that

$$\sup_{v \in V, v' \in N(v)} H'(\bar{u}_{v'} - \bar{u}_v) < \infty. \quad (4.2.7)$$

As in the previous section, we see that Hypothesis 4.2.3 implies that $-L$ can be interpreted as a combinatorial graph Laplacian operator acting upon a graph $G_L = (V, E, w)$ with vertex set V and edge set, E , contained in the set of all possible influences. Indeed, the symmetric weight function is given by

$$w(v, v') = \begin{cases} H'(\bar{u}_{v'} - \bar{u}_v) & : v' \in N(v) \\ 0 & : v' \notin N(v). \end{cases} \quad (4.2.8)$$

The condition (4.2.7) guarantees a uniform bound on the weights of the graph Laplacian. Combining this with the fact that $\#N(v)$ is uniformly bounded, Lemma 3.1.3 gives that $L : \ell^p(V) \rightarrow \ell^p(V)$ is a bounded linear operator for all $1 \leq p \leq \infty$.

Notice that a necessary condition for there to be an edge between vertices $v, v' \in V$ is that $v' \in N(v)$ (or equivalently $v \in N(v')$). This condition is not sufficient since it could be the case that for some $v \in V$ and $v' \in N(v)$ we have $H'(\bar{u}_{v'} - \bar{u}_v) = 0$, and therefore there is no edge between v and v' by definition of a weight function on a graph. This implies that even if $v' \in N(v)$, the distance between these vertices on the graph G_L is not guaranteed to be 1 since there may not be an edge connecting these vertices. This necessitates the following hypothesis.

Hypothesis 4.2.4. *G_L is a connected graph. Furthermore the associated graph metric, $\rho_L : V \times V \rightarrow [0, \infty)$, satisfies*

$$\sup_{v \in V, v' \in N(v)} \rho_L(v, v') < \infty. \quad (4.2.9)$$

Before we are able to apply the results of random walks on infinite graphs, we must point out that the linearization L given in (4.2.5) is not in the form that was investigated through random walks. That is, we are missing the $1/m(v)$ term from (3.4.4). If $m(v)$ is positive and independent of v we may simply rescale $t \rightarrow m(v)t$, which will apply the appropriate $1/m(v)$ term to obtain the normalized Laplacian. Then the operator $[1/m(v)]L$ is in the appropriate form to apply the theory from random walks on graphs.

We will now describe how to overcome this problem when $m(v)$ is not independent of v . To begin, note that Hypotheses 4.2.1 and 4.2.3 together give that there exists an $M > 0$ such that

$$m(v) \leq M \quad (4.2.10)$$

for all $v \in V$. Letting $t \rightarrow (M + 1)t$ scales (4.2.2) to the equivalent differential equation

$$\dot{u} = \frac{1}{M + 1} f(u). \quad (4.2.11)$$

Furthermore, linearizing about the steady-state \bar{u} now results in the linearization

$$\tilde{L} := \frac{1}{M + 1} L. \quad (4.2.12)$$

We have now normalized the operator L , and wish to consider a new graph, \tilde{G} , so that the measure of the vertices is given by $\tilde{m}(v) = M + 1$ for all $v \in V$. In doing so we will have that \tilde{L} is of the proper form to apply the results of the previous section. First, notice that

$$\sum_{v' \in V} \frac{w(v, v')}{M + 1} = \frac{m(v)}{M + 1} \leq \frac{M}{M + 1} < 1. \quad (4.2.13)$$

Let us extend the graph G to \tilde{G} by adding a loop at every vertex (an edge which originates and terminates at the same vertex) and augment to a new weight function $\tilde{w} : V \times V \rightarrow [0, \infty)$ given by

$$\tilde{w}(v, v') = \begin{cases} H'(\bar{x}_{v'} - \bar{x}_v) & : v' \in N(v), v' \neq v \\ 1 + M - \sum_{v''} H'(\bar{x}_{v''} - \bar{x}_v) & : v' = v \\ 0 & : v' \notin N(v). \end{cases} \quad (4.2.14)$$

That is, the missing weight for the measure \tilde{m} to be identically $M + 1$ for each $v \in V$ is made up for by the new loop connecting each vertex to itself. Notice that adding loops to a graph does not change the form of the Laplacian. Indeed, for each $v \in V$ we have

$$\begin{aligned} \sum_{v' \in N(v)} w(v, v')(x_{v'} - x_v) &= \sum_{v' \in N(v)} \underbrace{\tilde{w}(v, v')}_{=w(v, v')}(x_{v'} - x_v) \\ &= \sum_{v' \in N(v)} \tilde{w}(v, v')(x_{v'} - x_v) + \underbrace{\tilde{w}(v, v)(x_v - x_v)}_{=0} \\ &= \sum_{v' \in N(v) \cup \{v\}} \tilde{w}(v, v')(x_{v'} - x_v). \end{aligned} \quad (4.2.15)$$

The underlying graph will be denoted $\tilde{G}_L = (V, \tilde{E}, \tilde{w})$. Notice that if G_L is connected then \tilde{G}_L is also connected since we have not eliminated any edges from G_L to form \tilde{G}_L . Furthermore, we again have a uniform bound on the weight function \tilde{w} given by $M + 1$.

Hypothesis 4.2.5. *Assume that one of the following is true:*

- *If $m(v)$ is independent of $v \in V$, assume there exists a $d \geq 2$ such that the graph $G_L = (V, E, w)$ satisfies $VG(d)$, PI and $\Delta(\kappa)$.*
- *If $m(v)$ is not independent of $v \in V$, assume there exists a $d \geq 2$ such that the graph $\tilde{G}_L = (V, \tilde{E}, \tilde{w})$ (as constructed above) satisfies $VG(d)$, PI and $\Delta(\kappa)$.*

Notice that the assumptions of this hypothesis imply that the graph satisfies the assumptions of Proposition 3.4.1, and therefore we obtain the algebraic decay rates on the transition probabilities of a random walk on the vertices of the graph [18]. This hypothesis then in turn allows one to infer the results of Corollary 3.4.2 and Lemma 3.4.3. This leads to the following stability theorem whose proof is left to Section 4.3.

Theorem 4.2.6. *Consider the system (4.2.1) for a function $H \in C^2(\mathbb{R})$ satisfying the Hypothesis 4.2.1. Assume further that there is a steady-state solution \bar{u} such that linearizing about this steady-state leads to a linear operator, L , satisfying Hypotheses 4.2.3, 4.2.4 and 4.2.5. Then, there exists an $\epsilon > 0$ for which any $u_0 = \{u_{v,0}\}_{v \in V}$ with the property that*

$$\|\bar{u} - u_0\|_1 \leq \epsilon, \quad (4.2.16)$$

leads to a solution of (4.2.1), $u(t)$ for all $t \geq 0$, satisfying the following properties:

1. $u(0) = u_0$.
2. $\bar{u} - u(t) \in \ell^p(V)$ for all $1 \leq p \leq \infty$.
3. There exists a $C > 0$ such that

$$\|\bar{u} - u(t)\|_2 \leq C(1+t)^{-\frac{d}{4}} \|\bar{u} - u_0\|_1, \quad (4.2.17a)$$

$$\|\bar{u} - u(t)\|_\infty \leq C(1+t)^{-\frac{d}{2}} \|\bar{u} - u_0\|_1, \quad (4.2.17b)$$

for all $t \geq 0$.

Remark 4.2.7. One expects that further decay rates in Theorem 4.2.6 can be obtained for the various ℓ^p norms in a straightforward way using the interpolation result (3.4.12). What is important to note though is that the most restrictive assumption is the symmetry condition (4.2.6) of Hypothesis 4.2.3. When (4.2.6) is broken for even a single index, the graph becomes a directed graph (or digraph) and therefore all of the theory from Chapter 3 can no longer be applied. This situation would require the development of comparable techniques to obtain similar decay rates for random walks on digraphs. This appears to be a budding area of interest for researchers, but is not relevant to the present situation and therefore we will see that our assumptions are well-suited to our phase system (2.1.6).

4.3 Proof of Theorem 4.2.6

Throughout this entire section we will assume that Hypothesis 4.2.1, 4.2.3, 4.2.4 and 4.2.5 hold. We will work through this proof under the assumption that the second case of Hypothesis 4.2.5 holds, although the proof using the first case is nearly identical. Following the discussion prior to stating Hypothesis 4.2.5, we will apply the appropriate re-parametrization of t . We also apply the change of variable $\tilde{u} = u - \bar{u}$. Upon dropping the tilde for ease of notation, this allows us to write (4.2.2) in the equivalent form

$$\dot{u} = \tilde{L}u + g(u), \quad (4.3.1)$$

where $g(u) = \frac{1}{M+1}f(\bar{u} + u) - \tilde{L}u$ and we remind the reader of the dependence of u on the independent variable t . Notice that $g(0) = 0$ and $Dg(0) = 0$. Denoting $P_t = e^{\tilde{L}t}$

to be the semigroup generated by the linearization \tilde{L} for all $t \geq 0$, we arrive at the equivalent formulation of (4.3.1) given by

$$u(t) = P_t u(0) + \int_0^t P_{t-s} g(u(s)) ds \quad (4.3.2)$$

for any $t \geq 0$. Then any function $u(t)$ which satisfies (4.3.2) satisfies the differential equation (4.3.1) for $t \geq 0$.

Let $Q : \mathbb{R}^V \rightarrow \mathbb{R}$ be the operator acting upon the elements $u = \{u_v\}_{v \in V}$ by

$$Q(u) = \sum_{v \in V} \sum_{v' \in N(v)} |u_{v'} - u_v|^2. \quad (4.3.3)$$

Clearly $Q(u) \geq 0$ for all u , and furthermore using the Parallelogram Law one can see that for any $u \in \ell^2(V)$ we have

$$0 \leq Q(u) \leq 2D \|u\|_2^2, \quad (4.3.4)$$

where we recall that from Hypothesis 4.2.1 we have that $\#N(v) \leq D$ for all $v \in V$. $\sqrt{Q(\cdot)}$ defines a seminorm, and therefore satisfies the triangle inequality (for example, see Lemma 4.3 of [47]). This leads to the first result.

Lemma 4.3.1. *For any $u \in \ell^2(V)$, there exists a $K > 0$, depending on $\|u\|_2$, such that*

$$\|g(u)\|_1 \leq KQ(u). \quad (4.3.5)$$

Proof: Let us write $\delta := \|u\|_2$. We then have that $\|u\|_\infty \leq \delta$ and hence $|u_{v'} - u_v| \leq 2\delta$ for all $v, v' \in V$. Since $H \in C^2(\mathbb{R})$ we can define

$$K_1(\delta) := \sup_{|x| \leq 2\delta} |H''(x)| < \infty. \quad (4.3.6)$$

By Taylor's Theorem, for all $v \in V$ and $v' \in N(v)$ we have

$$|H(\bar{u}_{v'} - \bar{u}_v + u_{v'} - u_v) - H(\bar{u}_{v'} - \bar{u}_v) - H'(\bar{u}_{v'} - \bar{u}_v)(u_{v'} - u_v)| \leq \frac{K_1(\delta)}{2} |u_{v'} - u_v|^2. \quad (4.3.7)$$

Then recalling $g(0) = \frac{1}{M+1} f(\bar{u}) = 0$ and using the previous inequality we get

$$\begin{aligned} \|g(u)\|_1 &= \|g(u) - g(0)\|_1 \\ &= \frac{1}{M+1} \sum_v \left| \sum_{v' \in N(v)} H(\bar{u}_{v'} - \bar{u}_v + u_{v'} - u_v) - H(\bar{u}_{v'} - \bar{u}_v) \right. \\ &\quad \left. - H'(\bar{u}_{v'} - \bar{u}_v)(u_{v'} - u_v) \right| \\ &\leq \frac{K_1(\delta)}{2(M+1)} \sum_v \sum_{v' \in N(v)} |u_{v'} - u_v|^2 \\ &= \frac{K_1(\delta)}{2(M+1)} Q(u), \end{aligned} \quad (4.3.8)$$

completing the proof of the lemma. ■

Now from the decay rates in Section 3.4, for all $t \geq 0$ we have the following decay estimates for the semigroup P_t :

$$\|P_t\|_{p \rightarrow p} \leq C_{op}, \quad \text{for all } 1 \leq p \leq \infty, \quad (4.3.9a)$$

$$\|P_t\|_{1 \rightarrow 2} \leq C_1(1+t)^{-\frac{d}{4}}, \quad (4.3.9b)$$

$$\|P_t\|_{1 \rightarrow \infty} \leq C_1(1+t)^{-\frac{d}{2}}, \quad (4.3.9c)$$

for some $C_{op}, C_1 > 0$. There is also one more important estimate which must be established in the following lemma.

Lemma 4.3.2. *Let $\eta > 0$ be the associated value to P_t that satisfies the estimate (3.4.8). For all $u \in \ell^2(V)$, there exists a constant $C_Q > 0$ independent of u such that*

$$\sqrt{Q(P_t u)} \leq C_Q(1+t)^{-\frac{\eta}{2}} \|P_t |u|\|_2, \quad (4.3.10)$$

where $|u| = \{|u_v|\}_{v \in V}$.

Proof: To begin, since Hypothesis 4.2.5 guarantees that the measure of each vertex is uniformly bounded, we combine this statement with (3.4.8) and the uniform boundedness of the metric given in Hypothesis 4.2.4 to find that there exists a $C > 0$ such that

$$|p_t(v, v'') - p_t(v', v'')| \leq C(1+t)^{-\frac{\eta}{2}} p_{2t}(v, v''), \quad (4.3.11)$$

for all $v, v'' \in V, v' \in N(v)$. Then for any $x \in \ell^2(V)$ we have

$$\begin{aligned} |[P_t u]_v - [P_t u]_{v'}| &\leq \sum_{v'' \in V} |p_t(v, v'') - p_t(v', v'')| |u_{v''}| \\ &\leq C(1+t)^{-\frac{\eta}{2}} \sum_{v'' \in V} p_{2t}(v, v'') |u_{v''}| \\ &= C(1+t)^{-\frac{\eta}{2}} [P_{2t} |u|]_v. \end{aligned} \quad (4.3.12)$$

This in turn gives

$$\begin{aligned} \sqrt{Q(P_t u)} &= \sqrt{\sum_{v \in V} \sum_{v' \in N(v)} |[P_t u]_{v'} - [P_t u]_v|^2} \\ &\leq C(1+t)^{-\frac{\eta}{2}} \sqrt{\sum_{v \in V} \sum_{v' \in N(v)} |[P_{2t} |u|]_v|^2} \\ &\leq CD(1+t)^{-\frac{\eta}{2}} \sqrt{\sum_{v \in V} |[P_{2t} |u|]_v|^2} \\ &\leq CD(1+t)^{-\frac{\eta}{2}} \|P_{2t} |u|\|_2. \end{aligned} \quad (4.3.13)$$

Finally, using the fact that $P_{2t} = P_t P_t$ and the decay estimate $\|P_t\|_{2 \rightarrow 2} \leq C_{op}$ from (4.3.9a) we arrive at the final result

$$\sqrt{Q(P_t u)} \leq CC_{op} D(1+t)^{-\frac{\eta}{2}} \|P_t u\|_2. \quad (4.3.14)$$

■

Let us now consider an initial condition $u_0 \in \ell^1(V)$. We want to prove that if $\|u_0\|_1$ is chosen small enough, there exists a solution $u(t)$ to (4.2.1) with $u(0) = u_0$ belonging to the space

$$\mathcal{U} = \left\{ u(t) \left| \begin{array}{l} u(0) = u_0, \|u(t)\|_2 \leq 2C_1(1+t)^{-\frac{d}{4}} \|u_0\|_1 \text{ and} \\ \sqrt{Q(u(t))} \leq 2C_1 C_Q (1+t)^{-\frac{d}{4} - \frac{\eta}{2}} \|u_0\|_1, \forall t \geq 0 \end{array} \right. \right\} \quad (4.3.15)$$

where $C_1 > 0$ is the constant taken from the decay estimates (4.3.9) and $C_Q > 0$ is the constant from Lemma 4.3.2. Prior of showing the existence of a solution to 4.2.1, we show that functions belonging to \mathcal{U} indeed satisfy the additional statements of Theorem 4.2.6. We require the following lemma, which has been repurposed from [11].

Lemma 4.3.3 ([11], §3, Lemma 3.2). **(Restated)** *Let γ_1, γ_2 be positive real numbers. If $\gamma_1, \gamma_2 \neq 1$ or if $\gamma_1 = 1 < \gamma_2$ then there exists a $C_{\gamma_1, \gamma_2} > 0$ such that*

$$\int_0^t (1+t-s)^{-\gamma_1} (1+s)^{-\gamma_2} ds \leq C_{\gamma_1, \gamma_2} (1+t)^{-\min\{\gamma_1 + \gamma_2 - 1, \gamma_1, \gamma_2\}}, \quad (4.3.16)$$

Proposition 4.3.4. *Let $u(t) \in \mathcal{U}$ be a solution to (4.2.1). Then for all $t \geq 0$ we have the following:*

1. $u(t) \in \ell^p(V)$ for all $1 \leq p \leq \infty$.
2. There exists a $C_\infty > 0$ such that

$$\|u(t)\|_\infty \leq C_\infty (1+t)^{-\frac{d}{2}} \|u_0\|_1. \quad (4.3.17)$$

Proof: First recall that $\ell^1(V) \subsetneq \ell^p(V)$ for all $1 < p \leq \infty$. Thus if we show that $u(t) \in \ell^1(V)$ for all $t \geq 0$, we have therefore proven the first statement. Note that (4.3.9a) details that $\|P_t\|_{1 \rightarrow 1} \leq C_{op}$ for all $t \geq 0$, and since $u(t) \in \mathcal{U}$ there exists a $\delta > 0$ such that $\|u(t)\|_2 \leq \delta$ for all $t \geq 0$, thus allowing for a uniform $K > 0$ satisfying

the result of Lemma 4.3.1 for all $t \geq 0$. Using the fact that $u(t)$ is a solution to (4.2.1) for all $t \geq 0$ we now have

$$\begin{aligned}
\|u(t)\|_1 &\leq \|P_t u_0\|_1 + \int_0^t \|P_{t-s} g(u(s))\|_1 ds \\
&\leq C_{op} \|u_0\|_1 + C_{op} \int_0^t \|g(u(s))\|_1 ds \\
&\leq C_{op} \|u_0\|_1 + C_{op} K \int_0^t Q(u(s)) ds \\
&\leq C_{op} \|u_0\|_1 + 4C_{op} C_1^2 C_Q^2 K \int_0^t (1+s)^{-\frac{d}{2}-\eta} \|u_0\|_1^2 ds \\
&= C_{op} \|u_0\|_1 + \frac{4C_{op} C_1^2 C_Q^2 K}{\frac{d}{2} + \eta} [1 - (1+t)^{1-\frac{d}{2}-\eta}] \|u_0\|_1^2.
\end{aligned} \tag{4.3.18}$$

Since $d \geq 2$ and $\eta > 0$ we have that $[1 - (1+t)^{1-\frac{d}{2}-\eta}] \leq 1$ for all $t \geq 0$. Thus, $\|u(t)\|_1 < \infty$ for all $t \geq 0$, proving the first statement.

Proving the second statement follows in a similar manner, although we now use the decay estimate (4.3.9c). In this case we now have

$$\begin{aligned}
\|u(t)\|_\infty &\leq \|P_t u_0\|_\infty + \int_0^t \|P_{t-s} g(u(s))\|_\infty ds \\
&\leq C_1 (1+t)^{-\frac{d}{2}} \|u_0\|_1 + C_1 \int_0^t (1+t-s)^{-\frac{d}{2}} \|g(u(s))\|_1 ds \\
&\leq C_1 (1+t)^{-\frac{d}{2}} \|u_0\|_1 + 4C_1^3 C_Q^2 K \int_0^t (1+t-s)^{-\frac{d}{2}} (1+s)^{-\frac{d}{2}-\eta} \|u_0\|_1^2 ds.
\end{aligned} \tag{4.3.19}$$

Applying Lemma 4.3.3 with $\gamma_1 = \frac{d}{2} \geq 1$ and $\gamma_2 = \frac{d}{2} + \eta > 1$ gives that there exists $C_{\frac{d}{2}, \frac{d}{2} + \eta} > 0$ such that

$$\int_0^t (1+t-s)^{-\frac{d}{2}} (1+s)^{-\frac{d}{2}-\eta} ds \leq C_{\frac{d}{2}, \frac{d}{2} + \eta} (1+t)^{-\frac{d}{2}}. \tag{4.3.20}$$

Therefore, we have the decay estimate

$$\|u(t)\|_\infty \leq [C_1 + 4C_1^3 C_{\frac{d}{2}, \frac{d}{2} + \eta} C_Q^2 K \|u_0\|_1] (1+t)^{-\frac{d}{2}} \|u_0\|_1, \tag{4.3.21}$$

giving the desired estimate. ■

We now provide the necessary results to proving Theorem 4.2.6. We first define the mapping

$$Tu(t) = P_t u_0 + \int_0^t P_{t-s} g(u(s)) ds, \quad (4.3.22)$$

where $t \geq 0$. Notice that a fixed point of this mapping belonging to \mathcal{U} for all $t \geq 0$ will satisfy the differential equation (4.2.1).

Lemma 4.3.5. *Let $t_0 > 0$ such that $u(t)$ satisfies*

$$\|u(t)\|_2 \leq 2C_1(1+t)^{-\frac{d}{4}} \|u_0\|_1 \quad (4.3.23)$$

and

$$\sqrt{Q(u(t))} \leq 2C_1 C_Q (1+t)^{-\frac{d}{4} - \frac{\eta}{2}} \|u_0\|_1 \quad (4.3.24)$$

for all $0 \leq t \leq t_0$. Then there exists an $\varepsilon_1 > 0$, independent of t_0 , such that if $\|u_0\|_1 \leq \varepsilon_1$ we have

$$\|Tu(t)\|_2 \leq 2C_1(1+t)^{-\frac{d}{4}} \|u_0\|_1 \quad (4.3.25)$$

for all $0 \leq t \leq t_0$.

Proof: This proof follows in a very similar way to the proof of Proposition 4.3.4. Begin by considering $\|u_0\|_1 \leq \frac{1}{2C_1}$ so that $\|u(t)\|_2 \leq 1$ for all $0 \leq t \leq t_0$. This in turn guarantees the existence of a uniform $K > 0$ such that the results of Lemma 4.3.1 hold. Then using the decay estimate (4.3.9b) we now have

$$\begin{aligned} \|Tu(t)\|_2 &\leq \|P_t u_0\|_2 + \int_0^t \|P_{t-s} g(u(s))\|_2 ds \\ &\leq C_1(1+t)^{-\frac{d}{4}} \|u_0\|_1 + C_1 \int_0^t (1+t-s)^{-\frac{d}{4}} \|g(u(s))\|_2 ds \\ &\leq C_1(1+t)^{-\frac{d}{4}} \|u_0\|_1 + 4C_1^3 C_Q^2 K \int_0^t (1+t-s)^{-\frac{d}{4}} (1+s)^{-\frac{d}{2}-\eta} \|u_0\|_1^2 ds \end{aligned} \quad (4.3.26)$$

for all $0 \leq t \leq t_0$. From Lemma 4.3.3, there exists $C_{\frac{d}{4}, \frac{d}{2}+\eta} > 0$ such that

$$\int_0^t (1+t-s)^{-\frac{d}{4}} (1+t)^{-\frac{d}{2}-\eta} ds \leq C_{\frac{d}{4}, \frac{d}{2}+\eta} (1+t)^{-\frac{d}{4}}. \quad (4.3.27)$$

Thus,

$$\|Tu(t)\|_2 \leq [C_1 + 4C_1^3 C_Q^2 C_{\frac{d}{4}, \frac{d}{2}+\eta} K \|u_0\|_1] (1+t)^{-\frac{d}{4}} \|u_0\|_1. \quad (4.3.28)$$

Taking

$$\varepsilon_1 := \min \left\{ \frac{1}{2C_1}, \frac{1}{4C_1^3 C_Q^2 C_{\frac{d}{4}, \frac{d}{2}+\eta} K} \right\} \quad (4.3.29)$$

gives the desired result. ■

Lemma 4.3.6. *Let $t_0 > 0$ such that $u(t)$ satisfies*

$$\|u(t)\|_2 \leq 2C_1(1+t)^{-\frac{d}{4}}\|u_0\|_1 \quad (4.3.30)$$

and

$$\sqrt{Q(u(t))} \leq 2C_1C_Q(1+t)^{-\frac{d}{4}-\frac{\eta}{2}}\|u_0\|_1 \quad (4.3.31)$$

for all $0 \leq t \leq t_0$. Then there exists an $\varepsilon_2 > 0$, independent of t_0 , such that if $\|u_0\|_1 \leq \varepsilon_2$ we have

$$\sqrt{Q(Tu(t))} \leq 2C_1C_Q(1+t)^{-\frac{d}{4}-\frac{\eta}{2}}\|u_0\|_1 \quad (4.3.32)$$

for all $0 \leq t \leq t_0$.

Proof: We again take $\|u_0\|_1 \leq \frac{1}{2C_1}$ so that $\|u(t)\|_2 \leq 1$ for all $0 \leq t \leq t_0$, and therefore we have a uniform $K > 0$ such that the results of Lemma 4.3.1 hold for these values of t . Then since $\sqrt{Q(\cdot)}$ satisfies the triangle inequality, we proceed as in the previous proof. We use the bounds given in Lemma 4.3.2 to obtain

$$\begin{aligned} \sqrt{Q(Tu(t))} &\leq \sqrt{Q(P_t u_0)} + \int_0^t \sqrt{Q(P_{t-s}g(u(s)))} ds \\ &\leq C_Q(1+t)^{-\frac{\eta}{2}}\|P_t u_0\|_2 + C_Q \int_0^t (1+t-s)^{-\frac{\eta}{2}}\|P_{t-s}|g(u(s))\|_2 ds \\ &\leq C_1C_Q(1+t)^{-\frac{d}{4}-\frac{\eta}{2}}\|u_0\|_1 \\ &\quad + 4C_1^3C_Q^3K \int_0^t (1+t-s)^{-\frac{d}{4}-\frac{\eta}{2}}(1+s)^{-\frac{d}{2}-\eta}\|u_0\|_1^2 ds. \end{aligned} \quad (4.3.33)$$

From Lemma 4.3.3, there exists a $C_{\frac{d}{4}+\frac{\eta}{2}, \frac{d}{2}+\eta} > 0$ such that

$$\int_0^t (1+t-s)^{-\frac{d}{4}-\frac{\eta}{2}}(1+s)^{-\frac{d}{2}-\eta} ds \leq C_{\frac{d}{4}+\frac{\eta}{2}, \frac{d}{2}+\eta}(1+t)^{-\frac{d}{4}-\frac{\eta}{2}}. \quad (4.3.34)$$

Therefore, taking

$$\varepsilon_2 := \min \left\{ \frac{1}{2C_1}, \frac{1}{4C_1^3C_Q^2C_{\frac{d}{4}+\frac{\eta}{2}, \frac{d}{2}+\eta}K} \right\} \quad (4.3.35)$$

gives

$$\sqrt{Q(Tu(t))} \leq 2C_1C_Q(1+t)^{-\frac{d}{4}-\frac{\eta}{2}}\|u_0\|_1, \quad (4.3.36)$$

as desired. ■

One can see that Lemmas 4.3.5 and 4.3.6 combine to give us that

$$T : \mathcal{U} \rightarrow \mathcal{U} \quad (4.3.37)$$

when $\|u_0\|_1 \leq \min\{\varepsilon_1, \varepsilon_2\}$. We now present the proof of Theorem 4.2.6.

Proof: (Proof of Theorem 4.2.6)

First, since $Dg(0) = 0$, we may consider a $\delta > 0$ such that

$$\sup_{\|u\|_2 \leq \delta} \|Dg(u)\|_{2 \rightarrow 2} \leq \frac{1}{2C_{op}}. \quad (4.3.38)$$

Notice that such a uniform bound is guaranteed by Hypotheses 4.2.1, 4.2.3 and the smoothness of the function H . Then when $\|u_0\|_1 \leq \frac{\delta}{2C_1}$, we have that $2C_1(1+t)^{-\frac{d}{4}}\|u_0\|_1 \leq \delta$. Therefore, let

$$\varepsilon := \min \left\{ \varepsilon_1, \varepsilon_2, \frac{\delta}{2C_1} \right\} > 0, \quad (4.3.39)$$

where ε_1 is the constant required by Lemma 4.3.5 and ε_2 is the constant required by Lemma 4.3.6. Take $\|u_0\|_1 \leq \varepsilon$.

Now consider the space

$$\mathcal{U}_1 = \left\{ u(t), t \in [0, 1] \mid \begin{array}{l} u(0) = u_0, \|u(t)\|_2 \leq 2C_1(1+t)^{-\frac{d}{4}}\|u_0\|_1 \text{ and} \\ \sqrt{Q(u(t))} \leq 2C_1C_Q(1+t)^{-\frac{d}{4}-\frac{n}{2}}\|u_0\|_1, 0 \leq t \leq 1 \end{array} \right\}. \quad (4.3.40)$$

Then by our choice of ε in (4.3.39) we have that

$$T : \mathcal{U}_1 \rightarrow \mathcal{U}_1 \quad (4.3.41)$$

is well-defined for $0 \leq t \leq 1$. Let us consider the metric on the space \mathcal{U}_1 given by

$$\rho_{\mathcal{U}_1}(u_1(t), u_2(t)) := \sup_{t \in [0, 1]} \|u_1(t) - u_2(t)\|_2, \quad (4.3.42)$$

for any $u_1(t), u_2(t) \in \mathcal{U}_1$. Using the fact that $\|P_t\|_{2 \rightarrow 2} \leq C_{op}$ for all $t \geq 0$, for any

$u_1(t), u_2(t) \in \mathcal{U}_1$ this metric gives

$$\begin{aligned}
\rho_{\mathcal{U}_1}(Tu_1(t), Tu_2(t)) &\leq \sup_{t \in [0,1]} \int_0^t \|P_t[g(u_1(s)) - g(u_2(s))]\|_2 ds \\
&\leq \sup_{t \in [0,1]} C_{op} \int_0^t \|g(u_1(s)) - g(u_2(s))\|_2 ds \\
&\leq \sup_{t \in [0,1]} \frac{1}{2} \int_0^t \|u_1(s) - u_2(s)\|_2 ds \\
&\leq \sup_{t \in [0,1]} \frac{t}{2} \rho_{\mathcal{U}_1}(u_1(t), u_2(t)) \\
&\leq \frac{1}{2} \rho_{\mathcal{U}_1}(u_1(t), u_2(t)),
\end{aligned} \tag{4.3.43}$$

where we have used the fact that $\|g(u_1(t)) - g(u_2(t))\|_2 \leq \frac{1}{2C_{op}} \|u_1(t) - u_2(t)\|_2$ which follows from (4.3.38) and our choice of $\varepsilon > 0$. Thus, $T : \mathcal{U}_1 \rightarrow \mathcal{U}_1$ is a contraction. We provide the following claim, which will be proved after we have completed this proof.

Claim 4.3.7. \mathcal{U}_1 is complete with respect to the metric (4.3.42).

Therefore by the contraction mapping principle there exists an unique fixed point, $u_1^*(t) \in \mathcal{U}_1$. That is, we have identified an unique solution to the differential equation (4.2.1) satisfying the decay rates of the space \mathcal{U} for $t \in [0, 1]$.

We now proceed by induction. Let us assume that for some positive integer $n \geq 1$ there exists an unique solution to the differential equation (4.2.1) satisfying the decay rates of the space \mathcal{U} for $t \in [0, n]$. That is, there exists an unique fixed point $u_n^*(t)$ to T on the interval $[0, n]$ with the properties that $u_n^*(0) = u_0$, $\|u_n^*(t)\|_2 \leq 2C_1(1+t)^{-\frac{d}{4}} \|u_0\|_1$ and $\sqrt{Q(u_n^*(t))} \leq 2C_1C_Q(1+t)^{-\frac{d}{4}-\frac{\eta}{2}} \|u_0\|_1$ for all $t \in [0, n]$. We wish to use this function to extend to a solution on $[0, n+1]$.

Begin by defining the space

$$\begin{aligned}
\mathcal{U}_{n+1} = \left\{ u(t), t \in [0, n+1] \mid u(t) = u_n^*(t) \ t \in [0, n], \|u(t)\|_2 \leq 2C_1(1+t)^{-\frac{d}{4}} \|u_0\|_1 \right. \\
\left. \text{and } \sqrt{Q(u(t))} \leq 2C_1C_Q(1+t)^{-\frac{d}{4}-\frac{\eta}{2}} \|u_0\|_1, n \leq t \leq n+1 \right\}.
\end{aligned} \tag{4.3.44}$$

Again, from our choice of $\varepsilon > 0$, Lemmas 4.3.5 and 4.3.6 guarantee that

$$T : \mathcal{U}_{n+1} \rightarrow \mathcal{U}_{n+1} \tag{4.3.45}$$

is well-defined. Let us consider the metric on \mathcal{U}_{n+1} given by

$$\rho_{\mathcal{U}_{n+1}}(u_1(t), u_2(t)) := \sup_{t \in [0, n+1]} \|u_1(t) - u_2(t)\|_2. \tag{4.3.46}$$

for any $u_1(t), u_2(t) \in \mathcal{U}_{n+1}$. A proof nearly identical to that of the proof of Claim 4.3.7 shows that \mathcal{U}_{n+1} is complete with respect to the metric (4.3.46).

Now, for any $u_1(t), u_2(t) \in \mathcal{U}_{n+1}$ we have that $u_1(t) = u_2(t)$ for all $t \in [0, n]$. This then gives

$$\begin{aligned}
\rho_{\mathcal{U}_{n+1}}(Tu_1(t), Tu_2(t)) &\leq \sup_{t \in [n, n+1]} \int_n^t \|P_t[g(u_1(s)) - g(u_2(s))]\|_2 ds \\
&\leq \sup_{t \in [n, n+1]} \frac{1}{2} \int_n^t \|u_1(s) - u_2(s)\|_2 ds \\
&\leq \sup_{t \in [n, n+1]} \frac{t-n}{2} \rho_{\mathcal{U}_{n+1}}(u_1(t), u_2(t)) \\
&\leq \frac{1}{2} \rho_{\mathcal{U}_{n+1}}(u_1(t), u_2(t)),
\end{aligned} \tag{4.3.47}$$

showing that $T : \mathcal{U}_{n+1} \rightarrow \mathcal{U}_{n+1}$ is a contraction. By the contraction mapping principle, there exists a unique solution $u_{n+1}^*(t)$ which extends the solution $u_n^*(t)$ onto the interval $[0, n+1]$ and satisfying the required decay rates on this interval. Therefore, there exists a solution to the differential equation (4.2.1) for arbitrarily large values of t and satisfies the decay rates of the space \mathcal{U} . Coupling these results with the results of Proposition 4.3.4 gives the results of Theorem 4.2.6. \blacksquare

We conclude with the proof of Claim 4.3.7.

Proof: (Proof of Claim 4.3.7)

Let $\{u_n(t)\}_{n=1}^\infty$ be a Cauchy sequence in \mathcal{U}_1 . For each fixed $t \in [0, 1]$, $\{u_n(t)\}_{n=1}^\infty$ forms a Cauchy sequence in the complete space $\ell^2(V)$. Hence, there exists a pointwise limit to the sequence, denoted $u(t)$ which belongs to $\ell^2(V)$ for all $t \in [0, 1]$. Furthermore, the uniformity of the metric $\rho_{\mathcal{U}_1}$ further implies that

$$\lim_{n \rightarrow \infty} \rho_{\mathcal{U}_1}(u_n(t), u(t)) = 0. \tag{4.3.48}$$

It therefore only remains to show that $u(t) \in \mathcal{U}_1$.

Let $\varepsilon > 0$ be arbitrary. From (4.3.48) there exists $N \geq 0$ such that for all $n \geq N$ and $t \in [0, 1]$ we have

$$\|u_n(t) - u(t)\|_2 < \varepsilon. \tag{4.3.49}$$

Then for all $t \in [0, 1]$ we have

$$\|u(t)\|_2 \leq \|u_n(t) - u(t)\|_2 + \|u_n(t)\|_2 < \varepsilon + 2C_1(1+t)^{-\frac{d}{4}}\|u_0\|_1. \tag{4.3.50}$$

Letting $\varepsilon \rightarrow 0^+$ gives that $\|u(t)\|_2 \leq 2C_1(1+t)^{-\frac{d}{4}}\|u_0\|_1$ for all $t \in [0, 1]$.

Now for the final condition. Notice that (4.3.4) dictates that since $\rho_{\mathcal{U}_1}(u_n(t), u(t)) \rightarrow 0$ as $n \rightarrow \infty$, we have that

$$\lim_{n \rightarrow \infty} \sup_{t \in [0,1]} \sqrt{Q(u_n(t) - u(t))} \leq 2D \lim_{n \rightarrow \infty} \rho_{\mathcal{U}_1}(u_n(t), u(t)) = 0. \quad (4.3.51)$$

Thus, we merely repeat the previous arguments showing the bounds on $\|u(t)\|_2$ to obtain the appropriate bound on $\sqrt{Q(u(t))}$. This completes the proof of the claim. ■

4.4 Stability of Phase-Locked Patterns in the Coupled Phase Model

In this section will be use Theorem 4.2.6 to analyze some stable states of system (2.1.6), with a particular emphasis on the rotating wave solution found in Chapter 2. Returning to the system (4.0.4), we see that we have a system of the form (4.2.1). We will further assume that the coupling function H in (4.0.4) again satisfies Hypothesis 2.1.3.

Hypothesis 2.1.3 is in many ways stronger than those assumed for the statement of Theorem 4.2.6. That is, periodicity and infinite differentiability of the coupling function H immediately implies that condition (4.2.7) will be satisfied. Also, the sum in (4.0.4) is taken over the four nearest-neighbours of the index (i, j) , immediately giving that Hypothesis 4.2.1 is satisfied. Finally, we have already seen how the odd symmetry of the coupling function H will give the necessary symmetry requirement (4.2.6), although the positivity requirement will need to be checked on a case-by-case basis.

We will begin by illustrating the application of Theorem 4.2.6 to the coupled phase model (2.1.6) by considering the simplest possible phase-lag solution, the trivial solution. From here we will move to demonstrate the stability of the rotating wave solution.

4.4.1 The Trivial Solution

We begin by illustrating an application of Theorem 4.2.6 with the simplest solution to (4.0.2), the trivial solution. One should immediately note that since H is assumed to be odd, that $H(0) = 0$ and therefore taking $\bar{\theta}_{i,j} = 0$ for all $(i, j) \in \mathbb{Z}^2$ leads to a solution to (4.0.2).

Taking $\bar{\theta}_{i,j} = 0$ and linearizing (4.0.4) about $\psi = 0$ results in the linear operator

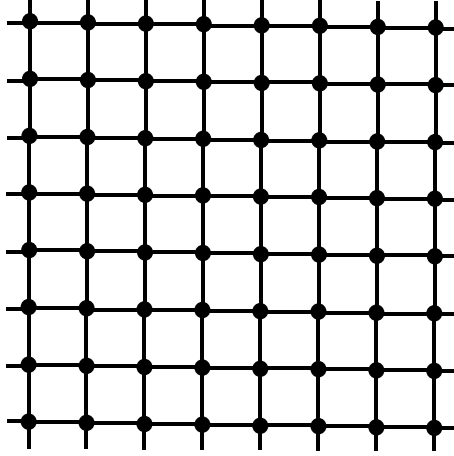


Figure 4.3: A representation of the graph G_1 associated with the linearization about the trivial phase-locked solution. Dots represent vertices of the graph and the lines connecting these vertices represent the edges.

acting on the sequences $x = \{x_{i,j}\}_{(i,j) \in \mathbb{Z}^2}$ by

$$[L_1 x]_{i,j} = \sum_{i',j'} H'(0)(x_{i',j'} - x_{i,j}), \quad (4.4.1)$$

for every $(i, j) \in \mathbb{Z}^2$. Since H is strictly increasing on the interval $(-\frac{\pi}{2}, \frac{\pi}{2})$, it follows that $H'(0) > 0$. Here we interpret the underlying graph, G_1 , to have vertex set \mathbb{Z}^2 and an edge set, denoted NN , containing all nearest-neighbour interactions between vertices. The weights of each edge are identically given by $H'(0)$, thus giving that the weight of each vertex is identically $4H'(0)$. An illustration of G_1 is given in Figure 4.3 for visual reference. It is clear that G_1 is connected and satisfies (4.2.9) since all influential interactions lead to edges in the graph.

Following the discussion prior to Hypothesis (4.2.5), we apply the linear time re-parametrization $t \rightarrow 4H'(0)t$ to system (4.0.4). Our re-parametrized system now results in the linearization about $\psi = 0$ given by

$$[\tilde{L}_1 x_{i,j}] = \sum_{i',j'} \frac{1}{4}(x_{i',j'} - x_{i,j}), \quad (4.4.2)$$

for every $(i, j) \in \mathbb{Z}^2$. What we have done is rescaled all edge weights to be exactly 1, and one should note that due to the fact that the weight of each vertex is identical, we have not added any loops to the original underlying graph G_1 . What is important to note though is now one sees that $-\tilde{L}_1$ is in the form of a normalized Laplacian. Let us denote the normalized graph $\tilde{G}_1 = (\mathbb{Z}^2, NN, \tilde{w}_1)$. Notice that the visual representation of \tilde{G}_1 remains identical to that of Figure 4.3.

The operator $-\tilde{L}_1$ and its underlying graph \tilde{G}_1 are a well-studied example in the theory of random walks on infinite graphs. Following Definition 3.3.1 it was pointed out that this graph \tilde{G}_1 satisfies $VG(2)$, and Telcs points out in the proof of Theorem 3 in [68] that this graph satisfies PI and $\Delta(\kappa)$ as well. Therefore, Hypothesis 4.2.5 holds for the trivial phase-locked solution, along with Hypotheses 4.2.1, 4.2.4 and 4.2.3, thus allowing us to apply Theorem 4.2.6 to this situation.

4.4.2 The Rotating Wave Solution

We now turn our attention back to the rotating wave solution to (2.1.6) proven to exist in Theorem 2.3.1. Let us simply denote the phase-lags of this solution by $\bar{\theta} = \{\bar{\theta}_{i,j}\}_{(i,j) \in \mathbb{Z}^2}$. Recall that a visual representation of this solution is given in Figure 2.1.

Linearizing about this rotating wave solution leads to the linear operator, L_2 , acting upon the sequences $x = \{x_{i,j}\}_{(i,j) \in \mathbb{Z}^2}$ by

$$[L_2x]_{i,j} = \sum_{i',j'} H'(\bar{\theta}_{i',j'} - \bar{\theta}_{i,j})(x_{i',j'} - x_{i,j}), \tag{4.4.3}$$

for all $(i, j) \in \mathbb{Z}^2$. Recall that with the exception of the 'centre' four cells at $(i, j) = (0, 0), (0, 1), (1, 0), (1, 1)$ all local interactions are such that $|\bar{\theta}_{i',j'} - \bar{\theta}_{i,j}| < \frac{\pi}{2}$, whereas the coupling between any two of the four centre cells is exactly $\pi/2$. Hence, our weighted and connected graph, denoted $G_2 = (\mathbb{Z}^2, E_2, w_2)$, has an edge set, E_2 , which at least contains all nearest-neighbour connections less those connections between any two of the centre four cells.

The graph G_2 comes in exactly two varieties depending of the value of $H'(\frac{\pi}{2})$. If $H'(\frac{\pi}{2}) \neq 0$, then all nearest-neighbour edge connections are present and G_2 has a visual representation given by that of G_1 in Figure 4.3 (i.e. $E_2 = NN$). The second case is when $H'(\frac{\pi}{2}) = 0$, for which there are now no edges connecting the 'centre' four cells. A visual representation of G_2 when $H'(\frac{\pi}{2}) = 0$ is given in Figure 4.1. For the duration of this investigation we will restrict ourselves to the case when $H'(\frac{\pi}{2}) = 0$, since it is most relevant to the motivating example $H(x) = \sin(x)$. In fact, the case when $H'(\frac{\pi}{2}) \neq 0$ can be undertaken via a slightly easier application of the following analysis. Most importantly, both cases lead to G_2 being connected.

Let us recall some important properties of this rotating wave solution. First, the solution is obtained via phase advances and phase delays of a solution obtained on the indices $1 \leq j \leq i$, which is represented by the shaded cells in Figure 2.1. Furthermore, we have that

$$\begin{aligned} \bar{\theta}_{j,j} &= 0, \\ 0 &< \bar{\theta}_{i,j} \leq \frac{\pi}{4} \\ \bar{\theta}_{j,0} &= \frac{\pi}{2} - \bar{\theta}_{j,1}, \end{aligned} \tag{4.4.4}$$

for all $1 \leq j < i$. Then one uses these facts to see that

$$|\bar{\theta}_{i',j'} - \bar{\theta}_{i,j}| \leq \frac{\pi}{4}, \quad (4.4.5)$$

for all $1 \leq j \leq i$ and $1 \leq j' \leq i'$. Another property of the solution is that

$$\bar{\theta}_{i,j} \leq \bar{\theta}_{i+1,j} \quad (4.4.6)$$

for all $1 \leq j \leq i$. This now gives that

$$\frac{\pi}{2} > \bar{\theta}_{2,0} - \bar{\theta}_{2,1} = \frac{\pi}{2} - 2\bar{\theta}_{2,1} \geq \frac{\pi}{2} - 2\bar{\theta}_{i,1} = \bar{\theta}_{i,0} - \bar{\theta}_{i,1} \geq 0, \quad (4.4.7)$$

for all $i \geq 2$. Equations (4.4.5) and (4.4.7) therefore combine to show that all nearest-neighbour interactions within the indices $1 \leq j \leq i$ remain bounded away from $\pm \frac{\pi}{2}$. This therefore gives that

$$0 < \inf_{1 \leq j \leq i} H'(\bar{\theta}_{i',j'} - \bar{\theta}_{i,j}) \leq \sup_{1 \leq j \leq i} H'(\bar{\theta}_{i',j'} - \bar{\theta}_{i,j}) < \infty. \quad (4.4.8)$$

Since the elements at the indices $1 \leq j \leq i$ are used to define the solution over all the indices, one has that the edge weights are uniformly bounded above and away from 0.

Now one should note that G_2 is significantly different from G_1 in that not all vertices have the same number of edges attached to it and that the weights of each edge are not identical. Let $w_{min} > 0$ and $w_{max} > 0$ be uniform lower and upper bounds on the edge weights, respectively. This allows one to apply the time reparametrization given by $t \rightarrow (4w_{max} + 1)t$ (since each vertex has degree at most 4), resulting in the linear operator $\tilde{L}_2 := 1/(4w_{max} + 1)L_2$ and resulting graph \tilde{G}_2 . Recall from our work in Section 4.2 that the graph \tilde{G}_2 is merely the graph G_2 with added edges connecting each vertex to itself (loops). Moreover, by the construction (4.2.14), the weight of each loop is bounded above by $4w_{max} + 1$ and below by 1. Hence, the edge weights of \tilde{G}_2 are uniformly bounded above by an $\tilde{w}_{max} > 0$ and below away by an $\tilde{w}_{min} > 0$. Therefore, Lemma 3.3.3 implies that there exists a $\kappa > 0$ such that \tilde{G}_2 satisfies $\Delta(\kappa)$.

Proposition 4.4.1. *The graphs \tilde{G}_1 and \tilde{G}_2 are rough isomorphic.*

Proof: Since \tilde{G}_1 and \tilde{G}_2 have the same vertex set, let us consider the identity mapping $\mathcal{I} : \mathbb{Z}^2 \rightarrow \mathbb{Z}^2$ which acts by $\mathcal{I}((i, j)) = (i, j)$ for all $(i, j) \in \mathbb{Z}^2$. We will systematically verify the three rough isometry properties (3.3.10a), (3.3.10b) and (3.3.10c) to show that \mathcal{I} is a rough isometry between the graphs \tilde{G}_1 and \tilde{G}_2 . We will let ρ_1, m_1 be the distance and vertex weight functions on the graph \tilde{G}_1 and ρ_2, m_2 be the distance and vertex weight functions on the graph \tilde{G}_2 .

Property (3.3.10a): Recall that the edge sets of \tilde{G}_1 and \tilde{G}_2 differ only by the centre four edges which are present in the former and absent in the latter. Let $n_1 = (i_1, j_1), n_2 = (i_2, j_2) \in \mathbb{Z}^2$. Then since \tilde{G}_1 has more connections between vertices than \tilde{G}_2 , one immediately has

$$\rho_1(n_1, n_2) \leq \rho_2(n_1, n_2), \quad (4.4.9)$$

since any path between the vertices n_1 and n_2 in \tilde{G}_2 could potentially be shortened by the addition of edges. Conversely, any shortest path connecting vertices in \tilde{G}_1 potentially traverses an edge which is absent in \tilde{G}_2 . Following along this path in \tilde{G}_2 requires one to replace those steps across the missing edges with two additional steps to circumvent the missing edge. Since there are a maximum of four missing edges to circumvent, we obtain

$$\rho_2(n_1, n_2) \leq \rho_1(n_1, n_2) + 8. \quad (4.4.10)$$

Therefore, to obtain the bounds (3.3.10a) we use (4.4.9) and (4.4.10) and take, for example, $a = 2$ and $b = 8$ to have

$$\frac{1}{2}\rho_1(n_1, n_2) - 8 \leq \rho_2(n_1, n_2) \leq 2\rho_1(n_1, n_2) + 8. \quad (4.4.11)$$

Property (3.3.10b): This property is trivially satisfied for any $M > 0$ since for all $n = (i, j) \in \mathbb{Z}^2$ we have $\rho_2(\mathbb{Z}^2, n) = 0$.

Property (3.3.10c): Recall that for all $n = (i, j) \in \mathbb{Z}^2$ we have $m_1(n) = 4$. Furthermore, we keep with the notation above to denote $\tilde{w}_{min} > 0$ and $\tilde{w}_{max} > 0$ as uniform lower and upper bounds, respectively, on the weight function of \tilde{G}_2 . Therefore, since each vertex has at least one edge connected to it and at most five (four nearest-neighbours and one loop) we have

$$\tilde{w}_{min} \leq m_2(n) \leq 5\tilde{w}_{max}, \quad (4.4.12)$$

for all $n = (i, j) \in \mathbb{Z}^2$. Hence,

$$\frac{\tilde{w}_{min}}{4}m_1(n) = \tilde{w}_{min} \leq m_2(n) \leq 5\tilde{w}_{max} = \frac{5\tilde{w}_{max}}{4}m_1(n). \quad (4.4.13)$$

Taking $c := \max\{\frac{5\tilde{w}_{max}}{4}, \frac{4}{\tilde{w}_{min}}, 2\} > 1$ gives

$$c^{-1}m_1(n) \leq m_2(n) \leq cm_1(n) \quad (4.4.14)$$

for all $n = (i, j) \in \mathbb{Z}^2$. This completes the proof since we have shown that $\mathcal{I} : \mathbb{Z}^2 \rightarrow \mathbb{Z}^2$ satisfies all three conditions to be a rough isometry. \blacksquare

Corollary 4.4.2. \tilde{G}_2 satisfies $VG(2)$, PI and $\Delta(\kappa)$.

Proof: Since \tilde{G}_1 and \tilde{G}_2 are rough isometric and both satisfy $\Delta(\kappa)$, Proposition 3.3.6 details that \tilde{G}_2 satisfies $VG(2)$ and PI since \tilde{G}_1 does. ■

Corollary 4.4.2 gives that Hypothesis 4.2.5 is satisfied. We have already discussed that Hypotheses 4.2.1 and 4.2.3 are satisfied. Finally, it is very easy to check that (4.2.9) holds, thus giving that Hypothesis 4.2.4 is satisfied. Therefore, we are now in a position to apply Theorem 4.2.6 and conclude the local stability of our rotating wave solution.

Chapter 5

The Full System

In this chapter we will extend the rotating wave solution of Chapter 2 at $\alpha = 0$ into positive values of α . To properly define a rotating wave solution in this context we will make use of the rotation operator acting on the indices of the lattice given by

$$R(z_{i,j}) = z_{j,1-i}, \quad (5.0.1)$$

where we rotate the lattice clockwise through an angle of $\pi/2$ about a theoretical centre cell at $i = j = 1/2$. This theoretical centre will act as the centre of rotation for our rotating wave solution, although due to the translational invariance of the lattice this centre can be chosen to lie between any square arrangement of cells and still give a rotating wave solution. The effect this operator has on the closest cells to its centre of rotation is shown in Figure 5.1. For rotations through the angle π we merely apply R to itself, denoted R^2 . Similarly, for clockwise rotations through the angle $3\pi/2$ (or counterclockwise rotations through the angle $\pi/2$) we apply R to itself three times, denoted R^3 . This rotation operator works only at the lattice level, and therefore does not alter the internal dynamics of the individual cells. That is, we merely move cells around in the lattice with this operator, but never alter their time-dependent dynamics.

We will return our study to the full Lambda-Omega system, which we restate here for reference:

$$\dot{z}_{i,j} = \alpha \sum_{i',j'} (z_{i',j'} - z_{i,j}) + z_{i,j} [\lambda(|z_{i,j}|) + i\omega(|z_{i,j}|, \alpha)], \quad (i, j) \in \mathbb{Z}^2. \quad (5.0.2)$$

One should further recall that the specifics of the λ and ω functions are detailed in Hypothesis 1.3.1. This leads to the central definition of this chapter.

Definition 5.0.1. *A rotating wave solution, $\{z_{i,j}(t)\}_{(i,j) \in \mathbb{Z}^2}$, is a time-periodic solution of system (5.0.2) such that for all (i, j) we have*

$$R(z_{i,j}(t)) = e^{i\frac{\pi}{2}} \cdot z_{i,j}(t). \quad (5.0.3)$$

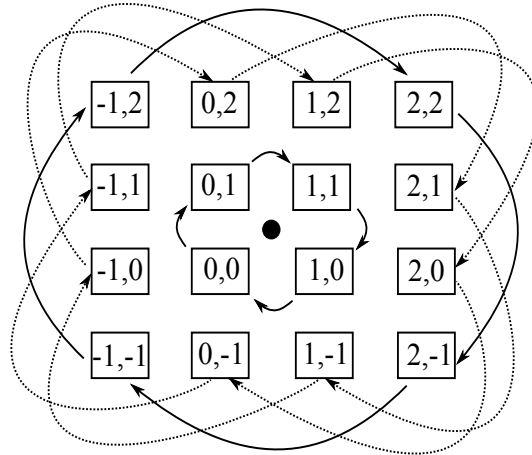


Figure 5.1: A diagram showing how the rotation operator defined in (5.0.1) effects elements of the lattice. The operator rotates lattice points by $\pi/2$ about a theoretical centre cell at $i = j = 1/2$, represented by the dot in the centre of the diagram.

Definition 5.0.1 details that rotating elements of the lattice about the centre of rotation leads to a phase advance of exactly $\frac{\pi}{2}$. Hence, we work to find a time-periodic solution defined by this rotational symmetry.

This chapter is broken down as follows. In Section 5.1 we review some important facts and definitions as well as provide an alternative Implicit Function Theorem which will be an integral part of our work. Section 5.2 provides an appropriate reduction of system (1.3.2) along with a statement of the main result, which relies heavily on an important theorem whose proof is left to Section 5.3. In Section 5.4 we follow our formal analysis with possible connections of the results provided here with those proven for finite lattices in order to gain further intuition into the behaviour of the solution as α varies.

5.1 Overview of Relevant Results

In this section we will provide a brief overview of the relevant facts from infinite-dimensional Banach space theory. The results in this section are not original to this work and therefore will be stated without proof. For a more complete introduction to these topics see for example [60], and for a more in-depth inspection of linear operators acting between Banach spaces see [32].

Throughout this chapter we will denote the operator norm of a linear operator

$T : X \rightarrow Y$ acting between Banach spaces as

$$\|T\|_{op} = \sup_{0 \neq x \in X} \frac{\|Tx\|_Y}{\|x\|_X}, \quad (5.1.1)$$

where $\|\cdot\|_X$ is the norm on X and $\|\cdot\|_Y$ is the norm on Y . Although we will consider linear operators acting between many different Banach spaces, we will always use the notation (5.1.1) to denote the norm of the operator. We now use this notation here as opposed to the notation (3.1.7) since the spaces X and Y are not necessarily going to be sequence spaces.

The *dual* space of the Banach space X is the set of all bounded linear operators $f : X \rightarrow \mathbb{R}$ equipped with the usual operator norm. The dual space is complete with respect to the operator norm, and is therefore a Banach space, here denoted X^* . Then, for a linear operator $T : X \rightarrow Y$, we define the *adjoint* of T , denoted T^* , to be the operator $T^* : Y^* \rightarrow X^*$ acting by

$$T^*(f)(x) = f(Tx) \quad (5.1.2)$$

for all $x \in X$ and $f \in Y^*$. In this way one can show that $\|T\|_{op} = \|T^*\|_{op}$ and if T is invertible then $(T^{-1})^* = (T^*)^{-1}$. We present the following lemma which becomes important in a later section of this chapter.

Lemma 5.1.1 ([32], §II.3, Theorem II.3.7). *A linear operator T has dense range if and only if T^* is one-to-one.*

Our particular interest in this work will be the Banach space $\ell^\infty(\mathcal{I})$, defined in Section 3.1, and its closed subspace

$$c_0(\mathcal{I}) := \{x \in \ell^\infty \mid \forall \varepsilon > 0, \#\{n \in \mathcal{I} \mid |x_n| \geq \varepsilon\} < \infty\}, \quad (5.1.3)$$

where $\#\{\cdot\}$ has been used to denote the cardinality of the set. When $\mathcal{I} = \mathbb{N}$ then $c_0(\mathbb{N})$ is merely the set of all sequences which converge to 0. The definition provided in (5.1.3) is a natural extension of this space to more diverse countable index sets. Moreover, since $c_0(\mathcal{I})$ is a closed subspace of a Banach space, it is itself a Banach space with respect to the $\|\cdot\|_\infty$ norm. An important characteristic of the space $c_0(\mathcal{I})$ is that its dual space, denoted $(c_0(\mathcal{I}))^*$, is isometrically isomorphic to $\ell^1(\mathcal{I})$, and can therefore be identified with this space. Hence, if $T : c_0(\mathcal{I}) \rightarrow c_0(\mathcal{I})$ is a bounded linear operator, then the corresponding dual operator T^* is a bounded linear operator on $\ell^1(\mathcal{I})$.

Furthermore, the spaces $c_0(\mathcal{I})$ and $\ell^1(\mathcal{I})$ are both separable and exhibit a Schauder basis. In both spaces a Schauder basis is given by the canonical basis $\{\delta_n\}_{n \in \mathcal{I}}$, where δ_n is the sequence indexed by the elements of \mathcal{I} with a 1 at index n and 0's everywhere

else. Hence, for $X = c_0(\mathcal{I})$ or $\ell^1(\mathcal{I})$ and a linear operator $T : X \rightarrow X$ we can represent T as an infinite matrix $T = [t_{nm}]_{n,m \in \mathcal{I}}$ by

$$t_{nm} = \langle T\delta_n, \delta_m \rangle, \quad (5.1.4)$$

where $\langle \cdot, \cdot \rangle$ represents the usual dot product given by the sum of component-wise multiplications. Hence, one can see that t_{nm} is merely the element at the m th index of $T\delta_n$. Then using this notation one can show that $T^* : X^* \rightarrow X^*$ is given by the transpose of the infinite matrix $[t_{nm}]_{n,m \in \mathcal{I}}$.

In this chapter we will be concerned with a slightly stronger version of Fréchet differentiation. We say that $F : X \rightarrow Y$ is *strongly Fréchet differentiable* at the point x_0 if

$$\lim_{x_1, x_2 \rightarrow x_0} \frac{\|F(x_1) - F(x_2) - F'(x_0)(x_1 - x_2)\|_Y}{\|x_1 - x_2\|_X} = 0, \quad (5.1.5)$$

where $F'(x_0)$ again denotes the Fréchet derivative of F at $x = x_0$. The following theorem relates Fréchet differentiability to strong Fréchet differentiability.

Theorem 5.1.2 ([65], §25, Theorem 25.23). *If $F : X \rightarrow Y$ is differentiable on an open set, then F is continuously differentiable if and only if F is strongly differentiable on the same set.*

Again take X and Y to be Banach spaces with respective norms $\|\cdot\|_X$ and $\|\cdot\|_Y$, and D a dense subset of X . A (not necessarily bounded) linear mapping $M : D \rightarrow Y$ is called *approximately right invertible* if, for each $\mu \in (0, 1)$, there exists a norm $\|\cdot\|_\mu$ on X , a bounded mapping $B_\mu : Y \rightarrow X$, and a bound $\Gamma(\mu)$, depending on μ , such that for all $y \in Y$ we have

$$\|MB_\mu y - y\|_Y \leq \mu \|y\|_Y \quad (5.1.6)$$

and

$$\|B_\mu y\|_\mu \leq \Gamma(\mu) \|y\|_Y, \quad (5.1.7)$$

with the property that for all $x \in X$ we have

$$\{\|x\|_\mu\} \nearrow \|x\|_X \text{ as } \mu \searrow 0. \quad (5.1.8)$$

Here we use the notation \nearrow to denote monotonically increasing convergence and \searrow as monotonically decreasing convergence. Then each B_μ is called an *approximate right inverse* of M . We denote X_μ to be the completion of X with respect to the norm $\|\cdot\|_\mu$. Note that B_μ need not be linear, and particularly in the present situation it will not be.

Our work centres around applying the following theorem due to Craven and Nashed to our lattice dynamical system.

Theorem 5.1.3 ([16], §3, Theorem 2). *Let X and Y be real Banach spaces, with $a \in X$. Let S be a closed convex cone in Y . Let the function $G : X \rightarrow Y$ be strongly Fréchet differentiable at a . Let $b := G(a)$ and assume $b \in S$. Let the Fréchet derivative $M = G'(a) : X \rightarrow Y$ be bounded linear with approximate right inverses B_μ and bound function $\Gamma(\mu) = k_0\mu^{-\gamma}$, with $\gamma < 1$. Then for sufficiently small μ , whenever c satisfies $-[G(a) + G'(a)c] \in S$, and $\|c\|_\mu = 1$, there exists a solution $x = a + tc + \eta(t) \in X_\mu$ to $-G(x) \in S$, valid for all sufficiently small $t > 0$, with $x \neq a$. With an appropriate choice of $\mu = \mu(t) \rightarrow 0$ as $t \rightarrow 0^+$, $\|\eta(t)\|_{\mu(t)} = o(t)$ as $t \rightarrow 0^+$.*

The original statement of Craven and Nashed's Implicit Function Theorem uses a weaker form of the derivative called the Hadamard derivative [16]. Since we will only be concerned with the stronger Fréchet derivative in this work, we merely restate the theorem with this form of differentiability. Since an appropriate reference was unable to be found, a proof that strong Fréchet differentiability at a point implies restricted strong Hadamard differentiability at the same point is provided in Appendix A. This therefore poses no problem with our restatement of the theorem here.

5.2 Existence of Rotating Wave Solutions

In this section we will present the main result of this thesis by demonstrating how solutions in a reduction of the lattice can be extended via symmetries to a solution over the full lattice. Begin by introducing the ansatz

$$z_{i,j} = r_{i,j} e^{i(\Omega t + \theta_{i,j})}, \quad (5.2.1)$$

with $r_{i,j} = r_{i,j}(t)$ and $\theta_{i,j} = \theta_{i,j}(t)$ for each $(i, j) \in \mathbb{Z}^2$. Then the LDS (1.3.2) can now be written in polar form as

$$\begin{aligned} \dot{r}_{i,j} &= \alpha \sum_{i',j'} (r_{i',j'} \cos(\theta_{i',j'} - \theta_{i,j}) - r_{i,j}) + r_{i,j} \lambda(r_{i,j}), \\ \dot{\theta}_{i,j} &= \alpha \sum_{i',j'} \frac{r_{i',j'}}{r_{i,j}} \sin(\theta_{i',j'} - \theta_{i,j}) + (\omega(r_{i,j}, \alpha) - \Omega), \quad (i, j) \in \mathbb{Z}^2. \end{aligned} \quad (5.2.2)$$

Under Hypothesis 1.3.1, without loss of generality we may assume $\Omega := \Omega(\alpha) = \omega(a, \alpha)$ for each α . Indeed, if $\Omega \neq \omega(a, \alpha)$ then we may apply the linear change of variable

$$\hat{\theta}_{i,j}(t) = \theta_{i,j}(t) - (\omega(a, \alpha) - \Omega)t \quad (5.2.3)$$

thereby reducing equations (5.2.2) to

$$\begin{aligned} \dot{r}_{i,j} &= \alpha \sum_{i',j'} (r_{i',j'} \cos(\hat{\theta}_{i',j'} - \hat{\theta}_{i,j}) - r_{i,j}) + r_{i,j} \lambda(r_{i,j}), \\ \dot{\hat{\theta}}_{i,j} &= \alpha \sum_{i',j'} \frac{r_{i',j'}}{r_{i,j}} \sin(\hat{\theta}_{i',j'} - \hat{\theta}_{i,j}) + \alpha \omega_1(r_{i,j}, \alpha), \quad (i, j) \in \mathbb{Z}^2. \end{aligned} \quad (5.2.4)$$

Moreover, a solution (if it exists) becomes

$$z_{i,j}(t) = r_{i,j}(t) e^{i(\Omega t + \theta_{i,j}(t))} = r_{i,j}(t) e^{i(\Omega t + \hat{\theta}_{i,j}(t) + (\omega(a, \alpha) - \Omega)t)} = r_{i,j}(t) e^{i(\omega(a, \alpha)t + \hat{\theta}_{i,j}(t))}, \quad (5.2.5)$$

showing that we can take $\Omega = \Omega(\alpha) = \omega(a, \alpha)$ without any loss of generality in the lattice system. This argument also retroactively justifies the ansatz (2.0.1) introduced to decompose to the previously studied phase equations.

It is system (5.2.4) which will be of interest throughout this chapter. As pointed out in Chapter 2, searching for nontrivial steady-state solutions to the system (5.2.4) with $\alpha \geq 0$ requires solving the nonlinear equations

$$\begin{aligned} 0 &= \alpha \sum_{i',j'} (r_{i',j'} \cos(\theta_{i',j'} - \theta_{i,j}) - r_{i,j}) + r_{i,j} \lambda(r_{i,j}), \\ 0 &= \sum_{i',j'} \frac{r_{i',j'}}{r_{i,j}} \sin(\theta_{i',j'} - \theta_{i,j}) + \omega_1(r_{i,j}, \alpha), \end{aligned} \quad (5.2.6)$$

upon dropping the hats, for all $(i, j) \in \mathbb{Z}^2$. Then one sees that solving these nonlinear equations for nontrivial steady-states results in a periodic solution $\{z_{i,j}(t)\}_{(i,j) \in \mathbb{Z}^2}$ of the form (5.2.1) where each element of the lattice is oscillating with a frequency of $2\pi/\omega(a, \alpha)$. In order to solve these equations we present the following definition.

Definition 5.2.1. *We refer to the **reduced system** as the lattice dynamical system (5.2.4) restricted to the indices $\Lambda \subset \mathbb{Z}^2$ given by*

$$\Lambda = \{(i, j) \in \mathbb{Z}^2 \mid i \geq 1 \text{ and } 2 - i \leq j \leq i\}. \quad (5.2.7)$$

along with the boundary conditions

$$\begin{aligned} &\bullet r_{i,1-i} = r_{i,i}, \\ &\bullet \theta_{i,1-i} = \theta_{i,i} + \frac{\pi}{2}, \\ &\bullet r_{i,i+1} = r_{1+i,1-i}, \\ &\bullet \theta_{i,i+1} = \theta_{1+i,1-i} + \frac{3\pi}{2}. \end{aligned} \quad (5.2.8)$$

Remark 5.2.2. The indices of the reduced system, denoted by Λ , are represented in Figure 1.2 by the shaded cells for visual reference. The key point here is that

$$\mathbb{Z}^2 = \Lambda \sqcup R(\Lambda) \sqcup R^2(\Lambda) \sqcup R^3(\Lambda), \quad (5.2.9)$$

and hence we have partitioned the full lattice into four mutually disjoint subsets. A steady-state solution in the reduced system will then be used to determine a solution over the entire lattice by forcing the rotational symmetry condition (5.0.3), thus obtaining a solution to the full system with the required symmetry. The boundary conditions (5.2.8) give the rotational symmetry requirement $R(\theta_{i,i}) = \theta_{i,i} + \frac{\pi}{2}$ along with the requirement that $R^3(\theta_{1+i,1-i}) = \theta_{1+i,1-i} + \frac{3\pi}{2}$. Moreover, when viewing a single column in Λ we find that these boundary conditions impose that the top and bottom of each column are linked. For example, the differential equation at the index $(1, 1)$ becomes before canceling terms

$$\begin{aligned} \dot{\theta}_{1,1} = & \underbrace{\sin(\theta_{2,1} - \theta_{1,1})}_{\text{Right}} + \underbrace{\sin(\theta_{1,1} + \frac{\pi}{2} - \theta_{1,1})}_{\text{Down}} + \underbrace{\sin(\theta_{1,1} + \frac{3\pi}{2} - \theta_{1,1})}_{\text{Left}} \\ & + \underbrace{\sin(\theta_{2,0} + \frac{3\pi}{2} - \theta_{1,1})}_{\text{Up}}. \end{aligned} \quad (5.2.10)$$

The boundary conditions on the radial components follow in a similar manner in that $R(r_{i,i}) = r_{i,i}$ and $R^3(r_{1+i,1-i}) = r_{1+i,1-i}$.

Abstractly, solving for steady-states to the reduced system requires obtaining zeros of the mappings

$$\begin{aligned} F_{i,j}^1(\alpha, r, \theta) &= \alpha \sum_{i',j'} [r_{i',j'} \cos(\theta_{i',j'} - \theta_{i,j}) - r_{i,j}] + r_{i,j} \lambda(r_{i,j}), \\ F_{i,j}^2(\alpha, r, \theta) &= \sum_{i',j'} \frac{r_{i',j'}}{r_{i,j}} \sin(\theta_{i',j'} - \theta_{i,j}) + \omega_1(r_{i,j}, \alpha), \end{aligned} \quad (5.2.11)$$

where $(i, j) \in \Lambda$ along with the boundary conditions (5.2.8). When $\alpha = 0$ we obtain

$$F_{i,j}^1(0, r, \theta) = r_{i,j} \lambda(r_{i,j}), \quad (5.2.12)$$

which from condition (1) of Hypothesis 1.3.1 we have that a nontrivial solution exists given by $r_{i,j} = a > 0$ for all $(i, j) \in \Lambda$. Clearly this solution satisfies the boundary conditions of the reduced system since all elements are identical. We denote the element $\mathbf{a} = \{a\}_{(i,j) \in \Lambda}$.

Evaluating the second components of (5.2.11) at $\alpha = 0$ and $r = \mathbf{a}$ results in

$$F_{i,j}^2(0, \mathbf{a}, \theta) = \sum_{i,j} \sin(\theta_{i',j'} - \theta_{i,j}), \quad (5.2.13)$$

since condition (2) of Hypothesis 1.3.1 implies that $\omega_1(a, 0) = 0$. From our work in Chapter 2, we have already obtained a solution to systems of equations of this type. We now detail that this solution is also a solution to $F_{i,j}^2(0, \mathbf{a}, \theta) = 0$ for all $(i, j) \in \Lambda$ by demonstrating that it satisfies the boundary conditions of the reduced system.

Let us denote $\bar{\theta} = \{\bar{\theta}_{i,j}\}_{(i,j) \in \mathbb{Z}^2}$ to be this steady-state solution corresponding to the phase-lags of the solution guaranteed by Theorem 2.3.1. Then those elements satisfying $1 \leq j < i$ were shown to be such that $\bar{\theta}_{i,j} \in (0, \frac{\pi}{4}]$ and $\theta_{i,i} = 0$ for all $i \geq 1$. Recalling the extensions (2.2.22), those elements whose indices satisfy $1 \leq i < j$ are such that $\bar{\theta}_{i,j} = -\bar{\theta}_{j,i}$ and those elements whose indices satisfy $1 - i < j \leq 0$ are such that $\bar{\theta}_{i,j} = \frac{\pi}{2} - \bar{\theta}_{i,1-j}$. For reference these symmetries are reflected in Figure 2.1. The most important point to note is that solutions in regions other than the reduced phase system are merely phase advances and delays from those elements indexed by $1 \leq j \leq i$. It should also be noted that system (5.2.13) possesses a translational symmetry in that simply adding a constant to all elements results in the same differential equation. Therefore roots will come as a one-dimensional family of solutions, but when referring to the steady-state solution $\bar{\theta}$ we will be referring to the particular solution which is described here and whose elements are reflected in Figure 2.1.

It is a straightforward exercise to see that $\bar{\theta}$ satisfies the boundary conditions of the reduced system. For each $i \geq 1$ we have

$$\begin{aligned} \bar{\theta}_{i,1-i} &= \frac{\pi}{2} \\ &= \bar{\theta}_{i,i} + \frac{\pi}{2}, \end{aligned} \tag{5.2.14}$$

since $\bar{\theta}_{i,i} = 0$ and

$$\begin{aligned} \bar{\theta}_{i,i+1} &= -\bar{\theta}_{i+1,i} \\ &= 2\pi - \bar{\theta}_{i+1,i} \\ &= \frac{3\pi}{2} + \left(\frac{\pi}{2} - \bar{\theta}_{i+1,i} \right) \\ &= \frac{3\pi}{2} + \bar{\theta}_{1+i,1-i} \end{aligned} \tag{5.2.15}$$

since all phases are equivalent modulo 2π . Therefore, $\bar{\theta} = \{\bar{\theta}_{i,j}\}_{(i,j) \in \Lambda}$ satisfies the boundary conditions of the reduced system, and hence $F_{i,j}^2(0, \mathbf{a}, \bar{\theta}) = 0$ for all $(i, j) \in \Lambda$. Moreover, one should note that we have $0 \leq \bar{\theta}_{i,j} \leq \frac{\pi}{2}$ for all $(i, j) \in \Lambda$. This now leads to the following Theorem.

Theorem 5.2.3. *Let λ and ω be functions satisfying Hypothesis 1.3.1. For $\alpha > 0$ and sufficiently small, there exists uniformly bounded $r(\alpha) = \{r_{i,j}(\alpha)\}_{(i,j) \in \Lambda}$ and $\theta(\alpha) = \{\theta_{i,j}(\alpha)\}_{(i,j) \in \Lambda}$ steady-state solutions to the reduced system. That is, these functions*

satisfy $F_{i,j}^1(\alpha, r(\alpha), \theta(\alpha)) = F_{i,j}^2(\alpha, r(\alpha), \theta(\alpha)) = 0$ for all $(i, j) \in \Lambda$ and the boundary conditions (5.2.8). As $\alpha \rightarrow 0^+$ we have $r(\alpha) \rightarrow \mathbf{a}$ uniformly and $\theta(\alpha) \rightarrow \bar{\theta}$.

The proof of Theorem 5.2.3 is a meticulous application of the Theorem 5.1.3 and will be undertaken in the following section. Prior to proving Theorem 5.2.3 we provide the main result of the thesis, which comes as a direct consequence of this result.

Corollary 5.2.4. *Let λ and ω be functions satisfying Hypothesis 1.3.1. Then for $\alpha > 0$ sufficiently small there exists a nontrivial uniformly bounded rotating wave solution to the lattice dynamical system (1.3.2) rotating with frequency $2\pi/\omega(a, \alpha)$.*

Proof: The solution of the reduced system from Theorem 5.2.3 extends to a solution of the entire lattice via a set of transformations which follow from Definition 2.1.2 of a rotating wave in this framework. That is, using the rotation operator defined in (5.0.1) we apply the following transformations over the other three distinct regions of the lattice:

<u>Region</u>	<u>Radial Transformation</u>	<u>Phase Transformation</u>	
$R(\Lambda)$	$R(r_{i,j}(\alpha)) = r_{i,j}(\alpha)$	$R(\theta_{i,j}(\alpha)) = \theta_{i,j}(\alpha) + \frac{\pi}{2}$	(5.2.16)
$R^2(\Lambda)$	$R^2(r_{i,j}(\alpha)) = r_{i,j}(\alpha)$	$R^2(\theta_{i,j}(\alpha)) = \theta_{i,j}(\alpha) + \pi$	
$R^3(\Lambda)$	$R^3(r_{i,j}(\alpha)) = r_{i,j}(\alpha)$	$R^3(\theta_{i,j}(\alpha)) = \theta_{i,j}(\alpha) + \frac{3\pi}{2}$	

It is straightforward to see that these transformations give a solution over the entire lattice since for any elements in the interior of one of the three regions we have that the radial components remain the same as those in the reduced system and the phase components are all translated by exactly the same value, thus leaving their difference unchanged. The interactions with the boundary are taken care of by the boundary conditions imposed on the reduced system. Figure 5.2 shows the four distinct regions of the lattice and how the solution of the reduced system can be extended to a solution over the entire lattice. Writing

$$z_{i,j}(t, \alpha) = r_{i,j}(\alpha) e^{i(\omega(a, \alpha)t + \theta_{i,j}(\alpha))} \quad (5.2.17)$$

for each (i, j) gives that the symmetry requirement (5.0.3) has been met by definition of the extensions, thus giving a rotating wave solution to the lattice dynamical system (1.3.2). Uniform boundedness follows from Theorem 5.2.3 and by the extensions over the entire lattice, thus completing the proof of the theorem. ■

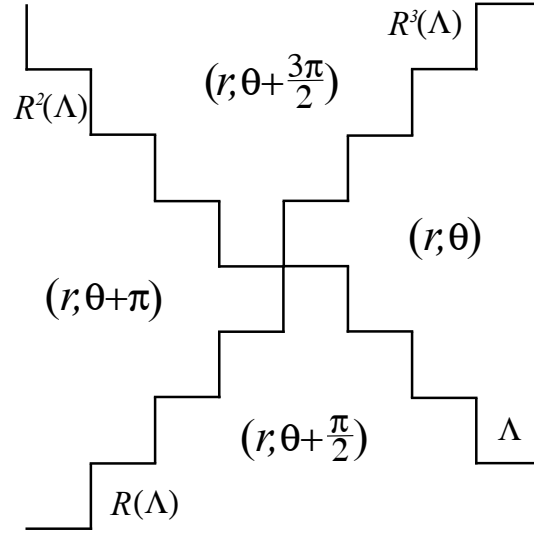


Figure 5.2: Extension of the solution in the reduced system over the entire lattice using the rotation operator.

Remark 5.2.5. In the case that $\omega(\cdot, \alpha)$ is a constant function (i.e. $\omega_1 \equiv 0$ in (5.2.6)) the reduced system Λ can be further partitioned to give an extra symmetry in the system. This method was employed in Chapter 2 where the odd symmetry of the sine function could be exploited. In this case the phase solutions of system (1.3.2) would exhibit the same symmetry as that given for the solution $\bar{\theta}$ in Figure 2.1. In this way we find that the effect a non-constant $\omega(\cdot, \alpha)$ has on the system is to break the odd symmetry in the differential equations of the phase components. It was shown by Ermentrout and Paullet in [26] that these non-constant $\omega(\cdot, \alpha)$'s can induce the familiar spiral wave spatial pattern, thus breaking the symmetry of the solution $\bar{\theta}$.

Remark 5.2.6. One should also notice that by the minimal assumptions on the function λ given in Hypothesis 1.3.1 that our results apply to both the normal forms of super- and subcritical Hopf bifurcations. This provides a more general result than those typically undertaken when investigating Lambda-Omega systems. Although subcritical Hopf bifurcations typically lead to unstable periodic solutions which are difficult to observe in a physical setting, this does provide our analysis with a more robust result allowing for further studies to compare the super- and subcritical variations.

5.3 Proof of Theorem 5.2.3

Our goal in this section is to prove Theorem 5.2.3 through an application of Theorem 5.1.3. We seek to define a proper mapping for use in Theorem 5.1.3 whose roots lie

in correspondence with roots of the mapping (5.2.11). Our interest in this section is the product of Banach spaces

$$X := \mathbb{R} \times \ell^\infty(\Lambda) \times c_0(\Lambda) \quad (5.3.1)$$

equipped with the norm

$$\|(\alpha, s, \psi)\|_X := \max\{|\alpha|, \|s\|_\infty, \|\psi\|_\infty\}. \quad (5.3.2)$$

The fact that X is complete with respect to this norm follows from the completion of each of the components \mathbb{R} , $\ell^\infty(\Lambda)$ and $c_0(\Lambda)$ (defined in (5.1.3)) with respect to their norms, thus giving that X is a Banach space.

Recalling from Hypothesis 1.3.1 that $a > 0$ is a root of λ , we denote $B_{\frac{a}{2}}(0)$ to be the ball of radius $a/2$ centred at 0 in $\ell^\infty(\Lambda)$. Then consider the mapping

$$G = (\alpha, G^1, G^2)^T : \mathbb{R} \times B_{\frac{a}{2}}(0) \times c_0(\Lambda) \subset X \rightarrow X \quad (5.3.3)$$

with components G^1 and G^2 given by

$$\begin{aligned} G_{i,j}^1(\alpha, s, \psi) &= \alpha \sum_{i',j'} [(a + s_{i',j'}) \cos(\bar{\theta}_{i',j'} + \psi_{i',j'} - \bar{\theta}_{i,j} - \psi_{i,j}) - (a + s_{i,j})] \\ &\quad + (a + s_{i,j})\lambda(a + s_{i,j}), \\ G_{i,j}^2(\alpha, s, \psi) &= i^{-1} \left[\sum_{i',j'} \frac{(a + s_{i',j'})}{(a + s_{i,j})} \sin(\bar{\theta}_{i',j'} + \psi_{i',j'} - \bar{\theta}_{i,j} - \psi_{i,j}) \right. \\ &\quad \left. + \omega_1(a + s_{i,j}, \alpha) \right]. \end{aligned} \quad (5.3.4)$$

The elements s and ψ can be seen as the deviation from the solution when $\alpha = 0$, and from here it is easy to see that since $i \geq 1$ the roots of G^1 and G^2 lie in one-to-one correspondence with those of (5.2.11). Indeed, notice that for all $(i, j) \in \Lambda$ we have

$$\begin{aligned} G_{i,j}^1(\alpha, s, \psi) &= F_{i,j}^1(\alpha, \mathbf{a} + s, \bar{\theta} + \psi), \\ G_{i,j}^2(\alpha, s, \psi) &= i^{-1} F_{i,j}^2(\alpha, \mathbf{a} + s, \bar{\theta} + \psi) \end{aligned} \quad (5.3.5)$$

In this way one should notice that the only difference between (5.2.11) and G^1, G^2 is that now G^2 has a decay term added to it, which we will prove acts to guarantee that this mapping is well-defined.

We define the closed convex cone

$$S := \mathbb{R} \times \{0\} \times \{0\} \subset X. \quad (5.3.6)$$

One sees from our work in the previous section that $G^1(0, 0, 0) = 0$ and $G^2(0, 0, 0) = 0$, giving that $G(0, 0, 0) = (0, 0, 0) \in S$. Therefore, proving Theorem 5.2.3 is reduced to finding elements (α, s, ψ) with $\alpha \neq 0$ such that $G(\alpha, s, \psi) \in S$ since this will imply that $G^1(\alpha, s, \psi) = 0$ and $G^2(\alpha, s, \psi) = 0$.

Proposition 5.3.1. *The mapping G as defined in (5.3.4) is a well-defined operator from its domain into X .*

Proof: Clearly the first component of G is well-defined, and therefore we need only check that the components G^1 and G^2 map into $\ell^\infty(\Lambda)$ and $c_0(\Lambda)$, respectively.

We begin by showing that G^1 maps into $\ell^\infty(\Lambda)$. Let $(\alpha, s, \psi) \in X$ be fixed with $\|s\|_\infty \leq \frac{a}{2}$. Let

$$C_\lambda := \max_{R \in [-a/2, a/2]} |\lambda(a + R)|. \quad (5.3.7)$$

Then for each $(i, j) \in \Lambda$ we have

$$\begin{aligned} |G_{i,j}^1(\alpha, s, \psi)| &\leq |\alpha| \sum_{i', j'} (|a + s_{i', j'}| + |a + s_{i, j}|) + |a + s_{i, j}| |\lambda(a + s_{i, j})| \\ &\leq 8|\alpha|a + 8|\alpha| \|s\|_\infty + (a + \|s\|_\infty) C_\lambda \\ &\leq 12|\alpha|a + \frac{3}{2}a C_\lambda, \end{aligned} \quad (5.3.8)$$

where we have used the fact that each element has at most four nearest-neighbours. Taking the supremum over all $(i, j) \in \Lambda$ gives that

$$\|G^1(\alpha, s, \psi)\|_\infty = \sup_{(i, j) \in \Lambda} |G_{i,j}^1(\alpha, s, \psi)| \leq 12|\alpha|a + \frac{3}{2}a C_\lambda < \infty, \quad (5.3.9)$$

thus showing that G^1 is a well-defined mapping.

We now turn to G^2 . Again let $(\alpha, s, \psi) \in X$ with $\|s\|_\infty \leq a/2$. Notice that

$$\left| \frac{(a + s_{i', j'})}{(a + s_{i, j})} \sin(\bar{\theta}_{i', j'} + \psi_{i', j'} - \bar{\theta}_{i, j} - \psi_{i, j}) \right| \leq 3 \quad (5.3.10)$$

for all $(i, j) \in \Lambda$ since $|s_{i, j}|, |s_{i', j'}| < \frac{a}{2}$. For the specific α in question, let us denote

$$C_\omega := \max_{R \in [-a/2, a/2]} |\omega_1(a + R, \alpha)|. \quad (5.3.11)$$

Then, putting this all together gives that

$$\begin{aligned} |G_{i,j}^2(\alpha, s, \psi)| &\leq i^{-1} \left[\sum_{i', j'} \left| \frac{(a + s_{i', j'})}{(a + s_{i, j})} \sin(\bar{\theta}_{i', j'} + \psi_{i', j'} - \bar{\theta}_{i, j} - \psi_{i, j}) \right| + |\omega_1(a + s_{i, j}, \alpha)| \right] \\ &\leq i^{-1} [12 + C_\omega]. \end{aligned} \quad (5.3.12)$$

Now let $\varepsilon > 0$ be arbitrary and let $i_0 \geq 1$ be the smallest integer such that

$$i_0 > \varepsilon^{-1} [12 + C_\omega]. \quad (5.3.13)$$

Therefore, for all $i \geq i_0$ we have that

$$|G_{i,j}^2(\alpha, s, \psi)| \leq i^{-1}[12 + C_\omega] < \varepsilon, \quad (5.3.14)$$

showing that the only indices which can exceed ε are those with $i < i_0$, which is a finite subset of Λ . Hence, G^2 is well-defined. \blacksquare

We now turn to analyzing the strong Fréchet differentiability of the mapping G at $(\alpha, s, \psi) = (0, 0, 0)$.

Proposition 5.3.2. *$G(\alpha, s, \psi)$ is strongly Fréchet differentiable at $(\alpha, s, \psi) = (0, 0, 0)$. That is, the Fréchet derivative at this point exists, is a bounded linear operator and can be written in the block matrix form as*

$$G'(0, 0, 0) := M = \begin{bmatrix} 1 & \mathbf{0} & \mathbf{0} \\ M_{21} & M_{22} & \mathbf{0} \\ \mathbf{0} & M_{32} & M_{33} \end{bmatrix}, \quad (5.3.15)$$

where $\mathbf{0}$ represents the trivial operator which sends every element to the 0 of the appropriate space. The operators have the following specific forms:

- $M_{21} : \mathbb{R} \rightarrow \ell^\infty(\Lambda)$ is the bounded linear operator acting by

$$[M_{21}\alpha]_{i,j} = a\alpha \sum_{i',j'} [\cos(\bar{\theta}_{i',j'} - \bar{\theta}_{i,j}) - 1], \quad (5.3.16)$$

for all $(i, j) \in \Lambda$.

- $M_{22} : \ell^\infty(\Lambda) \rightarrow \ell^\infty(\Lambda)$ is the bounded linear operator acting by

$$[M_{22}s]_{i,j} = a\lambda'(a)s_{i,j} \quad (5.3.17)$$

- $M_{32} : \ell^\infty(\Lambda) \rightarrow c_0(\Lambda)$ is the bounded linear operator acting by

$$[M_{32}s]_{i,j} = i^{-1} \left[\frac{1}{a} \sum_{i',j'} s_{i',j'} \sin(\bar{\theta}_{i',j'} - \bar{\theta}_{i,j}) + s_{i,j} \partial_1 \omega_1(a + R, \alpha) \Big|_{(R,\alpha)=(0,0)} \right], \quad (5.3.18)$$

for all $(i, j) \in \Lambda$, where ∂_1 denotes the partial derivative with respect to the first argument.

- $M_{33} : c_0(\Lambda) \rightarrow c_0(\Lambda)$ is the bounded linear operator acting by

$$[M_{33}\psi]_{i,j} = i^{-1} \sum_{i',j'} \cos(\bar{\theta}_{i',j'} - \bar{\theta}_{i,j}) (\psi_{i',j'} - \psi_{i,j}) \quad (5.3.19)$$

for all $(i, j) \in \Lambda$.

The proof of Proposition 5.3.2 is left to Appendix B.

Lemma 5.3.3. *The operator $M_{22} : \ell^\infty(\Lambda) \rightarrow \ell^\infty(\Lambda)$ is injective, surjective and has a bounded inverse.*

Proof: Injectivity: Now from Hypothesis 1.3.1 we have that $\lambda'(a) \neq 0$, and therefore M_{22} is a nontrivial diagonal operator. Then one sees that $M_{22}s = 0$ gives for each $(i, j) \in \Lambda$ that

$$[M_{22}s]_{i,j} = a\lambda'(a)s_{i,j} = 0 \implies s_{i,j} = 0, \quad (5.3.20)$$

thus giving that $s = 0$ and hence that M_{22} is injective.

Surjectivity: Consider the element $y = \{y_{i,j}\}_{(i,j) \in \Lambda} \in \ell^\infty(\Lambda)$. Then clearly the element $s = \{[a\lambda'(a)]^{-1}y_{i,j}\}_{(i,j) \in \Lambda}$ belongs to $\ell^\infty(\Lambda)$ with norm given by $\|s\|_\infty = [a\lambda'(a)]^{-1}\|y\|_\infty < \infty$ and satisfies $M_{22}s = y$.

Bounded Inverse: This is an immediate consequence of the fact that $M_{22} : \ell^\infty(\Lambda) \rightarrow \ell^\infty(\Lambda)$ is a bounded bijection. ■

To obtain comparable results for M_{33} we first provide the following lemma.

Lemma 5.3.4. *Consider the operator $T : c_0(\Lambda) \rightarrow c_0(\Lambda)$ acting by*

$$[T_{\bar{\theta}}\psi]_{i,j} = \sum_{i',j'} \cos(\bar{\theta}_{i',j'} - \bar{\theta}_{i,j})(\psi_{i',j'} - \psi_{i,j}) \quad (5.3.21)$$

for all $(i, j) \in \Lambda$. Then $T_{\bar{\theta}}$ is injective and has dense range.

Proof: Injectivity: It was remarked in Chapter 4 on this element $\bar{\theta}$ that all nearest-neighbour interactions in Λ are such that $|\bar{\theta}_{i',j'} - \bar{\theta}_{i,j}| < \frac{\pi}{2}$ with the exceptions of $|\bar{\theta}_{1,0} - \bar{\theta}_{1,1}|$ and $|\bar{\theta}_{0,1} - \bar{\theta}_{1,1}|$ both taking the value $\frac{\pi}{2}$, and so $\cos(\bar{\theta}_{i',j'} - \bar{\theta}_{i,j}) > 0$ for all but two neighbouring interactions.

Now let us assume that $\psi = \{\psi_{i,j}\}_{(i,j) \in \Lambda}$ is such that $T_{\bar{\theta}}\psi = 0$. Then $[T_{\bar{\theta}}\psi]_{i,j} = 0$ for each $(i, j) \in \Lambda$ and we have

$$0 = \sum_{(i,j) \in \Lambda} 0 \cdot \psi_{i,j} = \sum_{(i,j) \in \Lambda} [T_{\bar{\theta}}\psi]_{i,j} \cdot \psi_{i,j} = \frac{1}{2} \sum_{(i,j) \in \Lambda} \sum_{i',j'} -\cos(\bar{\theta}_{i',j'} - \bar{\theta}_{i,j})(\psi_{i',j'} - \psi_{i,j})^2. \quad (5.3.22)$$

To justify this notice that $\cos(\bar{\theta}_{i',j'} - \bar{\theta}_{i,j})$ contributes to the sum twice by the even symmetry of the cosine function. This gives for any index $(i, j) \in \Lambda$

$$\begin{aligned} \cos(\bar{\theta}_{i',j'} - \bar{\theta}_{i,j})(\psi_{i',j'} - \psi_{i,j})\psi_{i,j} + \cos(\bar{\theta}_{i,j} - \bar{\theta}_{i',j'})(\psi_{i,j} - \psi_{i',j'})\psi_{i',j'} \\ = \cos(\bar{\theta}_{i',j'} - \bar{\theta}_{i,j})(\psi_{i',j'} - \psi_{i,j})(\psi_{i,j} - \psi_{i',j'}) \\ = -\cos(\bar{\theta}_{i',j'} - \bar{\theta}_{i,j})(\psi_{i',j'} - \psi_{i,j})^2. \end{aligned} \quad (5.3.23)$$

Furthermore, the factor of one half in front of the sum comes from the fact that everything is being summed twice, again by the symmetry of the cosine function. Therefore (5.3.22) returns nonpositive values for any sequence indexed by the elements of Λ since $\cos(\bar{\theta}_{i',j'} - \bar{\theta}_{i,j}) \geq 0$ for all $(i, j) \in \Lambda$ and its nearest-neighbours, (i', j') .

Thus, in order for (5.3.22) to hold, it must be the case that each $\psi_{i,j}$ is equal to its nearest-neighbours in Λ . This is because the only case when $\cos(\bar{\theta}_{i',j'} - \bar{\theta}_{i,j}) = 0$ is at the index $(1, 1)$ and these represent connections with the boundary. Since the element at $(1, 1)$ is connected to the element indexed by $(1, 2)$, which is in Λ , we have that every index is connected to at least one other index in Λ by some non-zero weight $\cos(\bar{\theta}_{i',j'} - \bar{\theta}_{i,j}) > 0$. This allows one to see that $T_{\bar{\theta}}\psi = 0$ if and only if there exists a $C \in \mathbb{R}$ such that $\psi_{i,j} = C$ for all $(i, j) \in \Lambda$. By definition these constant sequences are not elements of $c_0(\Lambda)$, thus giving that $T_{\bar{\theta}} : c_0(\Lambda) \rightarrow c_0(\Lambda)$ is injective.

Dense Range: Now to see that $T_{\bar{\theta}} : c_0(\Lambda) \rightarrow c_0(\Lambda)$ has dense range, one merely applies Lemma 5.1.1 again. We may equivalently write $T_{\bar{\theta}}$ as an infinite matrix, say B , indexed by the elements of $\Lambda \times \Lambda$ such that

$$B = (b_{(i_1,j_1),(i_2,j_2)})_{\Lambda \times \Lambda} = \begin{cases} -\sum_{(i'_1,j'_1)} \cos(\bar{\theta}_{i'_1,j'_1} - \bar{\theta}_{i_1,j_1}) & : (i_1, j_1) = (i_2, j_2) \\ \cos(\bar{\theta}_{i_2,j_2} - \bar{\theta}_{i_1,j_1}) & : (i_1, j_1) \sim (i_2, j_2) \\ 0 & : \textit{otherwise} \end{cases} \quad (5.3.24)$$

where we have used the notation $(i_1, j_1) \sim (i_2, j_2)$ to denote that (i_1, j_1) and (i_2, j_2) are nearest-neighbours. Now again using the even symmetry of the cosine function we see that this matrix is clearly symmetric since for $(i_1, j_1) \neq (i_2, j_2)$ we have

$$\begin{aligned} b_{(i_1,j_1),(i_2,j_2)} &= \begin{cases} \cos(\bar{\theta}_{i_2,j_2} - \bar{\theta}_{i_1,j_1}) & : (i_1, j_1) \sim (i_2, j_2) \\ 0 & : \textit{otherwise} \end{cases} \\ &= \begin{cases} \cos(\bar{\theta}_{i_1,j_1} - \bar{\theta}_{i_2,j_2}) & : (i_1, j_1) \sim (i_2, j_2) \\ 0 & : \textit{otherwise} \end{cases} \\ &= b_{(i_2,j_2),(i_1,j_1)}. \end{aligned} \quad (5.3.25)$$

This implies that $T_{\bar{\theta}}^* : \ell^1(\Lambda) \rightarrow \ell^1(\Lambda)$ merely acts as $T_{\bar{\theta}}$. Repeating the above arguments we see that $T_{\bar{\theta}}^*$ must be injective since the constant sequences are not in $\ell^1(\Lambda)$. This therefore gives that $T_{\bar{\theta}} : c_0(\Lambda) \rightarrow c_0(\Lambda)$ has dense range. \blacksquare

Remark 5.3.5. Notice that operator T defined in Lemma 5.3.4 can be interpreted as the negative of a combinatorial graph Laplacian in the same way we interpreted the linearization about the rotating wave solution as one in Section 4.1. Here the vertex set lies in one-to-one correspondence with the subset of the lattice Λ and the edges come from the nearest-neighbour connections along with the connections induced by the boundary conditions of Definition 5.2.1. Figure 5.3 gives a visual representation of

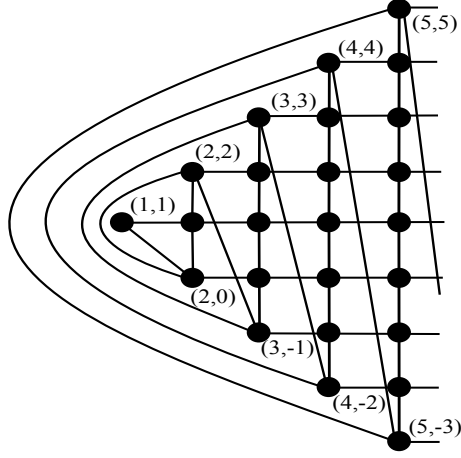


Figure 5.3: Underlying graph structure of the operator T defined in Lemma 5.3.4. Edge connections reflect the boundary conditions of the reduced system presented in Definition 5.2.1

this underlying graph and also allows the reader to visualize the boundary conditions. Knowing this, one sees that the argument of injectivity of T is identical to how we previously simplified (3.2.3).

Corollary 5.3.6. *The operator $M_{33} : c_0(\Lambda) \rightarrow c_0(\Lambda)$ is injective and has dense range,*

Proof: *Injectivity:* Let us assume that $\psi = \{\psi_{i,j}\}_{(i,j) \in \Lambda} \in c_0(\Lambda)$ is such that $M_{33}\psi = 0$. This gives that for all $(i,j) \in \Lambda$ we have

$$i^{-1} \sum_{i',j'} \cos(\bar{\theta}_{i',j'} - \bar{\theta}_{i,j}) (\psi_{i',j'} - \psi_{i,j}) = 0 \iff \sum_{i',j'} \cos(\bar{\theta}_{i',j'} - \bar{\theta}_{i,j}) (\psi_{i',j'} - \psi_{i,j}) = 0. \quad (5.3.26)$$

since $i^{-1} > 0$ for all $i \geq 1$. Therefore, $M_{33}\psi = 0$ if and only if $T_{\bar{\theta}}\psi = 0$, as defined in the statement of Lemma 5.3.4. Since $T_{\bar{\theta}}$ was shown to be injective, it follows that $\psi = 0$, thus showing that M_{33} is injective.

Dense Range: Let $y = \{y_{i,j}\}_{(i,j) \in \Lambda} \in c_0(\Lambda)$ and $\varepsilon > 0$ be arbitrary. Then, since $y \in c_0(\Lambda)$ there exists an $i_0 \geq 1$ such that $|y_{i,j}| < \varepsilon/2$ for all $i > i_0$. Define the element $\tilde{y} = \{\tilde{y}_{i,j}\}_{(i,j) \in \Lambda} \in c_0(\Lambda)$ by

$$\tilde{y}_{i,j} = \begin{cases} y_{i,j} & \text{if } i \leq i_0 \\ 0 & \text{if } i > i_0 \end{cases}. \quad (5.3.27)$$

Then by construction we have that

$$\|y - \tilde{y}\|_{\infty} < \frac{\varepsilon}{2}. \quad (5.3.28)$$

Now consider the element $y' = \{y'_{i,j}\}_{(i,j) \in \Lambda} \in c_0(\Lambda)$ defined by $y'_{i,j} = i \cdot \tilde{y}_{i,j}$ for all $(i,j) \in \Lambda$. Note that since y' only has finitely many nonzero components, it therefore trivially belongs to $c_0(\Lambda)$. Then since $T_{\bar{\theta}} : c_0(\Lambda) \rightarrow c_0(\Lambda)$ (as defined in Lemma 5.3.4) has dense range, there exists a $\psi \in c_0(\Lambda)$ such that

$$\|T_{\bar{\theta}}\psi - y'\|_{\infty} = \sup_{(i,j) \in \Lambda} \left| \sum_{i',j'} \cos(\bar{\theta}_{i',j'} - \bar{\theta}_{i,j})(\psi_{i',j'} - \psi_{i,j}) - y'_{i,j} \right| < \frac{\varepsilon}{2}. \quad (5.3.29)$$

Hence,

$$\begin{aligned} \|M_{33}\psi - \tilde{y}\|_{\infty} &= \sup_{(i,j) \in \Lambda} \left| i^{-1} \sum_{i',j'} \cos(\bar{\theta}_{i',j'} - \bar{\theta}_{i,j})(\psi_{i',j'} - \psi_{i,j}) - \tilde{y}_{i,j} \right| \\ &= \sup_{(i,j) \in \Lambda} i^{-1} \left| \sum_{i',j'} \cos(\bar{\theta}_{i',j'} - \bar{\theta}_{i,j})(\psi_{i',j'} - \psi_{i,j}) - y'_{i,j} \right| \\ &\leq \sup_{(i,j) \in \Lambda} \left| \sum_{i',j'} \cos(\bar{\theta}_{i',j'} - \bar{\theta}_{i,j})(\psi_{i',j'} - \psi_{i,j}) - y'_{i,j} \right| \\ &< \frac{\varepsilon}{2}. \end{aligned} \quad (5.3.30)$$

Therefore, putting this all together gives

$$\|M_{33}\psi - y\|_{\infty} \leq \|M_{33}\psi - \tilde{y}\|_{\infty} + \|y - \tilde{y}\|_{\infty} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon, \quad (5.3.31)$$

showing that M_{33} has dense range since $\varepsilon > 0$ and $y \in c_0(\Lambda)$ were arbitrary. \blacksquare

One notices from Proposition 5.3.2 that M is a block lower-triangular matrix. This greatly simplifies much of our analysis and implies that many of the properties of the diagonal elements carry over to the full block matrix M . In particular, Lemma 5.3.3 and Corollary 5.3.6 lead to the following corollary regarding the density of the range of the operator M in (5.3.15).

Corollary 5.3.7. $M : X \rightarrow X$ is injective and has dense range.

Proof: Injectivity: Let us assume that $Mx = 0$, where $x = (\alpha, s, \psi)$. This gives

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & \mathbf{0} & \mathbf{0} \\ M_{21} & M_{22} & \mathbf{0} \\ \mathbf{0} & M_{32} & M_{33} \end{bmatrix} \cdot \begin{bmatrix} \alpha \\ s \\ \psi \end{bmatrix} = \begin{bmatrix} \alpha \\ M_{21}\alpha + M_{22}s \\ M_{32}s + M_{33}\psi \end{bmatrix}. \quad (5.3.32)$$

One can immediately see that this implies that $\alpha = 0$. This in turn yields

$$0 = M_{21}\alpha + M_{22}s = M_{22}s, \quad (5.3.33)$$

since $M_{21}0 = 0$. From Lemma 5.3.3 we have that M_{22} is injective, thus giving that $s = 0$.

Having $s = 0$ gives

$$0 = M_{32}s + M_{33}\psi = M_{33}\psi, \quad (5.3.34)$$

since $M_{32}0 = 0$. From Lemma 5.3.6 we have that M_{33} is injective, thus giving that $\psi = 0$. Hence, $Mx = 0$ implies that $x = 0$, and therefore M is injective.

Density of Range: Let $\varepsilon > 0$. We wish to show that for any $x = (\alpha, s, \psi) \in X$ that there exists $x_\varepsilon = (\alpha_\varepsilon, s_\varepsilon, \psi_\varepsilon) \in X$ such that

$$\|Mx_\varepsilon - x\|_X < \varepsilon. \quad (5.3.35)$$

Using (5.3.32) above we have

$$Mx_\varepsilon - x = \begin{bmatrix} \alpha_\varepsilon - \alpha \\ M_{21}\alpha_\varepsilon + M_{22}s_\varepsilon - s \\ M_{32}s_\varepsilon + M_{33}\psi_\varepsilon - \psi \end{bmatrix}. \quad (5.3.36)$$

Immediately one sees that we may take $\alpha_\varepsilon = \alpha$ to get that $|\alpha_\varepsilon - \alpha| = 0 < \varepsilon$. Then denoting the bounded inverse of M_{22} by $M_{22}^{-1} : \ell^\infty(\Lambda) \rightarrow \ell^\infty(\Lambda)$, we will take $s_\varepsilon = M_{22}^{-1}[s - M_{21}\alpha]$ to see that

$$\|M_{22}s_\varepsilon - (s - M_{21}\alpha)\|_\infty = 0 < \varepsilon. \quad (5.3.37)$$

Then using the density of the range of M_{33} we have that there exists a $\psi_\varepsilon \in c_0(\Lambda)$ such that

$$\|M_{33}\psi_\varepsilon - (\psi - M_{32}s_\varepsilon)\|_\infty < \varepsilon, \quad (5.3.38)$$

since $(\psi - M_{32}s_\varepsilon) \in c_0(\Lambda)$, by definition. Putting this all together gives that if $x_\varepsilon = (\alpha_\varepsilon, s_\varepsilon, \psi_\varepsilon)$ as above we obtain

$$\begin{aligned} & \|Mx_\varepsilon - x\|_X \\ &= \max\{|\alpha_\varepsilon - \alpha|, \|M_{21}\alpha_\varepsilon + M_{22}s_\varepsilon - s\|_\infty, \|M_{32}s_\varepsilon + M_{33}\psi_\varepsilon - \psi\|_\infty\} \\ &< \varepsilon, \end{aligned} \quad (5.3.39)$$

completing the proof. ■

It should be noted that the density of the range of $M : X \rightarrow X$ is all that we can conclude about the image of this operator. Following from the fact that M_{33} is not necessarily surjective, we find that $M : X \rightarrow X$ is not necessarily surjective. Furthermore, the spectral results of Chapter 4 guarantee that M_{33} cannot have a closed image, and therefore cannot have a bounded inverse. This is a result of the fact M_{33} does have a nontrivial kernel when acting on $\ell^\infty(\Lambda)$, the second dual of $c_0(\Lambda)$.

Now since $M : X \rightarrow X$ is not surjective, traditional Implicit Function Theorem arguments cannot be applied to the situation. This necessitates the application of a non-standard Implicit Function Theorem such as Theorem 5.1.3 to obtain solutions to $G \in S$ for $\alpha > 0$. The great benefit of Theorem 5.1.3 is that it can be applied to both the situation when $M : X \rightarrow X$ is surjective and when it is not, with no change in the analysis here.

We will now work to define an approximate right inverse. Let us fix $\mu \in (0, 1)$. To begin, let $y \in X$ be an element of the image of M . Then there exists a unique $x \in X$ such that $Mx = y$. We will define $B_\mu y := x$. In this way one sees that

$$\|MB_\mu y - y\|_X = 0 \leq \mu\|y\|_X \quad (5.3.40)$$

for each y in the image of M . For any $y' = (\alpha_y, s_y, \psi_y) \in X$ not in the image of M , we proceed in much the same way as the proof of Corollary 5.3.7. That is, we will define $B_\mu y' = x' = (\alpha_x, s_x, \psi_x) \in X$ to be such that

$$\begin{aligned} \alpha_x &:= \alpha_y, \\ s_x &:= M_{22}^{-1}[s_y - M_{21}\alpha_y], \end{aligned} \quad (5.3.41)$$

and $\psi_x \in c_0(\Lambda)$ to be an element such that

$$\|M_{33}\psi_x - (\psi_y - M_{32}s_x)\|_\infty \leq \mu\|y\|_X, \quad (5.3.42)$$

which is guaranteed to exist since M_{33} has dense range. Hence, one sees that as in the proof of Corollary 5.3.7, we have that

$$\|Mx' - y'\|_X \leq \mu\|y'\|_X. \quad (5.3.43)$$

Therefore the mapping $B_\mu : X \rightarrow X$ satisfies the first condition for an approximate right inverse. The reader should note that our mapping $B_\mu : X \rightarrow X$ is neither linear nor injective nor unique for any $\mu \in (0, 1)$ since $M : X \rightarrow X$ is not surjective.

We now require the appropriate space to bound our approximate right inverse B_μ . To begin, recall that $c_0(\Lambda)$ exhibits a Schauder basis given by the canonical basis, $\{\delta_{i,j}\}_{(i,j) \in \Lambda} \subset c_0(\Lambda)$, where the element $\delta_{i,j}$ has zeros at every index but for a 1 at the index (i, j) . Then writing

$$\delta_{i,j}^\psi = (0, 0, \delta_{i,j}) \in X, \quad (5.3.44)$$

we can uniquely write any $x = (0, 0, \psi) \in X$ as

$$x = \sum_{(i,j) \in \Lambda} \psi_{i,j} \delta_{i,j}^\psi. \quad (5.3.45)$$

For any $n \geq 1$, consider the finite-dimensional closed subspace of X given by

$$E_n = \{(0, 0, \psi) \in X \mid \psi_{i,j} = 0 \ \forall i > n\}. \quad (5.3.46)$$

One can see that for each $n \geq 1$ the subspace E_n is merely those elements of X in which $\alpha = 0$, $s = 0$ and the only nonzero components of $\psi = \{\psi_{i,j}\}_{(i,j) \in \Lambda}$ belong to the first n columns of Λ . Furthermore, the set of vectors in X given by

$$B_n := \{\delta_{i,j}^\psi\}_{i \leq n} \quad (5.3.47)$$

forms a basis of E_n , for each $n \geq 1$.

Lemma 5.3.8. *For each $n \geq 1$, $M : E_n \rightarrow X$ has a bounded inverse from its range to E_n .*

Proof: Fix $n \geq 1$ and finite. Then E_n is a finite-dimensional subspace and the image of the space under the mapping M , denoted $M(E_n)$, is a subspace of X . From the Rank-Nullity Theorem one has that $M(E_n)$ is finite-dimensional since E_n is finite-dimensional. Hence, $M : E_n \rightarrow M(E_n)$ is an operator acting between finite-dimensional spaces. Since invertibility is equivalent to injectivity in finite-dimensions, $M : E_n \rightarrow M(E_n)$ has a bounded inverse since we have shown in Corollary 5.3.7 that it is injective. \blacksquare

Denoting $M(E_n)$ to be the image of E_n under the action of M , we find that $M(E_n)$ is a finite-dimensional subspace of X and therefore is closed. Moreover, since $M : E_n \rightarrow M(E_n)$ is invertible, it follows that the set $M(B_n) = \{M\delta_{i,j}^\psi\}_{i \leq n}$ is a linearly independent spanning set of $M(E_n)$. A well-known consequence of the Hahn-Banach theorem is that for each element of the spanning set $M(B_n)$, say v , there exists a linear functional $\phi_v \in X^*$ such that $\phi_v(v) = 1$ and $\phi_v(v') = 0$ for all $v' \in M(B_n) \setminus \{v\}$. This gives a natural continuous projection, $P_{M(E_n)} : X \rightarrow M(E_n)$ by

$$P_{M(E_n)}x = \sum_{i \leq n} \phi_{i,j}^\psi(x) M\delta_{i,j}^\psi, \quad (5.3.48)$$

where $\phi_{i,j}^\psi \in X^*$ is the linear functional corresponding as above to $M\delta_{i,j}^\psi \in M(B_n)$. Linearity and continuity immediately follow from the fact that the elements of X^* used to define $P_{M(E_n)}$ are linear and continuous.

Now, since $P_{M(E_n)} : X \rightarrow M(E_n)$ is continuous for all $n \geq 1$, this implies that we may write $X = M(E_n) \oplus V_n$, where V_n is the kernel of $P_{M(E_n)}$, which is closed. Let us denote $M(X)$ to be the image of X under the action of $M : X \rightarrow X$. Consider the linear subspaces of X :

$$U_n := M^{-1}(V_n \cap M(X)). \quad (5.3.49)$$

That is, U_n is the inverse image of the complement of $M(E_n)$ in $M(X)$.

Claim 5.3.9. *For each $n \geq 1$ we have $E_n \cap U_n = \{0\}$.*

Proof: Let us fix $n \geq 1$ and assume that $x \in E_n \cap U_n$. Then, since $x \in E_n$ we get that $Mx \in M(E_n)$. Similarly, since $x \in U_n$, this implies that $Mx \in V_n \cap M(X) \subsetneq V_n$. So, $Mx \in M(E_n) \cap V_n$. But $M(E_n) \cap V_n = \{0\}$ by definition, giving that $Mx = 0$. Injectivity of M implies that $x = 0$. ■

Claim 5.3.10. U_n is closed as a subspace of X for every $n \geq 1$.

Proof: Fix $n \geq 1$ and let $\{x_k\}_{k=1}^\infty \subset U_n$ be a sequence converging in the norm of X to some $x \in X$. We wish to show that $x \in U_n$.

For each $k \geq 1$, let us define $y_k := Mx_k$. By definition we have that $y_k \in V_n \cap M(X) \subset V_n$. Let us further define $y := Mx$ to see that

$$\|y_k - y\|_X = \|Mx_k - Mx\|_X \leq \|M\|_{op} \|x_k - x\|_X, \quad (5.3.50)$$

where $\|M\|_{op} < \infty$ is the operator norm of $M : X \rightarrow X$. Then since $\|x_k - x\|_X \rightarrow 0$ as $k \rightarrow \infty$ it follows that $y_k \rightarrow y$ as $k \rightarrow \infty$ in the norm of X . Since V_n is closed, we have that $y \in V_n$. Furthermore, $y = Mx \in M(X)$, so we see that $y \in V_n \cap M(X)$. Therefore, $x = M^{-1}y \in M^{-1}(V_n \cap M(X)) = U_n$, giving that U_n is closed. ■

Corollary 5.3.11. For each $n \geq 1$ we have $X = E_n \oplus U_n$.

Proof: From Claims 5.3.9 and 5.3.10, we have that for every $n \geq 1$, the direct sum $E_n \oplus U_n$ is well-defined since both components are closed and mutually disjoint as subspaces of X . Furthermore, $E_n \oplus U_n \subseteq X$ by definition. It therefore only remains to show that $X \subseteq E_n \oplus U_n$.

Now fix $n \geq 1$ and take $x \in X$. Then $Mx \in M(X) \subset X = M(E_n) \oplus V_n$. Hence, there exists $y_E \in M(E_n)$ and $y_V \in V_n$ such that $Mx = y_E + y_V$. Now since $y_E \in M(E_n)$ and $M : E_n \rightarrow M(E_n)$ is invertible, there exists a unique $x_E \in E_n$ such that $Mx_E = y_E$. So,

$$M(x - x_E) = Mx - y_E = y_V, \quad (5.3.51)$$

and therefore y_V belongs to $M(X)$. Hence, $y_V \in V_n \cap M(X)$, which implies there exists an element $x_U \in U_n$ such that $Mx_U = y_V$. Putting this all together shows that

$$M(x - x_E - x_U) = y - y_E - y_V = 0. \quad (5.3.52)$$

From injectivity of M we get that $x = x_E + x_U \in E_n \oplus U_n$, completing the proof. ■

Following the result of Corollary 5.3.11 we see that for every $n \geq 1$ we have two decompositions of X as the direct sum of closed subspaces: $E_n \oplus U_n$ and $M(E_n) \oplus V_n$.

Moreover, one sees that $M(U_n) \subset V_n$, showing that M preserves this splitting. Now, since $X = E_n \oplus U_n$, there exists a continuous projection $P_n : X \rightarrow E_n$ which acts on every $x = x_E + x_U \in E_n \oplus U_n$ by $P_n x = x_E$. By definition we have that the range of P_n is given by E_n and its kernel is exactly U_n .

Claim 5.3.12. *For every $x \in X$ and $n \geq 1$, $MP_n x = P_{M(E_n)} Mx$.*

Proof: Let $x \in X$ and $n \geq 1$. Then we may write

$$x = P_n x + (I - P_n)x, \quad (5.3.53)$$

where $I : X \rightarrow X$ is the identity operator on X , so that $P_n x \in E_n$ and $(I - P_n)x \in U_n$. Applying M to this equations gives

$$Mx = MP_n x + M(I - P_n)x, \quad (5.3.54)$$

so that by definition we have $MP_n x \in M(E_n)$ and $M(I - P_n)x \in V_n$. Applying $P_{M(E_n)}$ now gives that

$$P_{M(E_n)} Mx = MP_n x, \quad (5.3.55)$$

since V_n is the kernel of $P_{M(E_n)}$ by definition. This completes the proof of the claim. ■

Claim 5.3.12 gives the following commutative diagram:

$$\begin{array}{ccc} E_n \oplus U_n & \xrightarrow{M} & M(E_n) \oplus V_n \\ \downarrow P_n & & \downarrow P_{M(E_n)} \\ E_n & \xrightarrow{M} & M(E_n) \end{array}$$

Let us denote $M_n^{-1} : M(E_n) \rightarrow E_n$ the bounded inverse of $M : E_n \rightarrow M(E_n)$ for each $n \geq 1$. Moreover, let us write $C(n) := \|M_n^{-1}\|_{op}$, which is finite for each finite $n \geq 1$. One can see that $C(n)$ is an increasing function of n such that $C(n) \rightarrow \infty$ as $n \rightarrow \infty$. This follows from the fact that $E_n \subseteq E_{n'}$ for each $1 \leq n \leq n'$ and that as $n \rightarrow \infty$ we are approaching an inverse of M_{33} , which cannot be bounded since $M : X \rightarrow X$ is not surjective.

Lemma 5.3.13. *For each $n \geq 1$ and $x = (\alpha, s, \psi) \in X$, we have that*

$$P_n x = \sum_{i \leq n} \psi_{i,j} \delta_{i,j}^\psi. \quad (5.3.56)$$

Proof: Fix $n \geq 1$ and consider some $x = (\alpha, s, \psi) \in X$. Then $y = Mx \in M(X) \subset X$. Using (5.3.48) gives

$$P_{M(E_n)}y = \sum_{i \leq n} \phi_{i,j}^\psi(y) M \delta_{i,j}^\psi. \quad (5.3.57)$$

Since $P_{M(E_n)}y \in M(E_n)$ we may apply M_n^{-1} to see that

$$M_n^{-1}P_{M(E_n)}y = M_n^{-1}MP_nx = P_nx, \quad (5.3.58)$$

where we have applied the identity from Claim 5.3.12. Then linearity of M_n^{-1} gives that

$$P_nx = M_n^{-1}P_{M(E_n)}y = \sum_{i \leq n} \phi_{i,j}^\psi(y) \delta_{i,j}^\psi. \quad (5.3.59)$$

Since the set B_n is a linearly independent spanning set of E_n , it follows that

$$\phi_{i,j}^\psi(y) = \psi_{i,j} \quad (5.3.60)$$

for all (i, j) . This proves the lemma. ■

We are now in a position to define an appropriate norm to bound our approximate right inverse.

Proposition 5.3.14. *Let $n \geq 1$ and define the function $\|\cdot\|_n : X \rightarrow [0, \infty)$ by*

$$\|x\|_n = \|(\alpha, s, \psi)\|_n := \max \left\{ |\alpha|, \|s\|_\infty, \|P_nx\|_X, \frac{1}{\|M\|_{op}} \|Mx\|_X \right\} \quad (5.3.61)$$

for all $x \in X$, and $\|M\|_{op}$ denotes the operator norm of M . Then $\|\cdot\|_n$ defines a norm on X . Moreover, for each $x \in X$ we have that $\|x\|_n \leq \|x\|_{n'} \leq \|x\|_X$ for any $1 \leq n \leq n'$ and $\|x\|_n \rightarrow \|x\|_X$ as $n \rightarrow \infty$.

Remark 5.3.15. For the duration of this section we will let X_n denote the completion of X with respect to the norm $\|\cdot\|_n$.

Proof: To see that (5.3.61) defines a norm is a straightforward checking of the axioms which primarily follows from the injectivity of M along with the linearity of both P_n and M .

Now to see that such an ordering of the norms holds one observes that for any $n \geq 1$ and $x = (\alpha, s, \psi) \in X$, we have from Lemma 5.3.13 that

$$\|P_nx\|_X = \max_{(i,j) \in \Lambda, i \leq n} |\psi_{i,j}|. \quad (5.3.62)$$

Hence, one can see that $\|P_n x\|_X \leq \|P_{n'} x\|_X$ for any $1 \leq n \leq n'$ and $x \in X$. This also leads to the fact that $\|P_n x\|_X \leq \|x\|_X$ for all $n \geq 1$. Hence,

$$\begin{aligned} \|x\|_n &= \max \left\{ |\alpha|, \|s\|_\infty, \|P_n x\|_X, \frac{1}{\|M\|_{op}} \|Mx\|_X \right\} \\ &\leq \max \left\{ |\alpha|, \|s\|_\infty, \|P_{n'} x\|_X, \frac{1}{\|M\|_{op}} \|Mx\|_X \right\} \\ &= \|x\|_{n'} \end{aligned} \quad (5.3.63)$$

and since $\|Mx\|_X \leq \|M\|_{op} \|x\|_X$ we clearly have that $\|x\|_n \leq \|x\|_X$.

Finally, if $x = (\alpha, s, \psi) \in X$, then $\psi \in c_0(\Lambda)$, implying that there exists $(i_0, j_0) \in \Lambda$ such that $|\psi_{i_0, j_0}| = \|\psi\|_\infty$. Then for every $n \geq i_0$ we have

$$\|P_n x\|_X = \max_{(i,j) \in \Lambda, i \leq n} |\psi_{i,j}| = \|\psi\|_\infty, \quad (5.3.64)$$

thus implying that

$$\|x\|_n = \max \left\{ |\alpha|, \|s\|_\infty, \|\psi\|_\infty, \frac{1}{\|M\|_{op}} \|Mx\|_X \right\} = \|x\|_X. \quad (5.3.65)$$

Hence, for every $n \geq i_0$ we have $\|x\|_n = \|x\|_X$, giving convergence as $n \rightarrow \infty$. \blacksquare

Lemma 5.3.16. *For every $\mu \in (0, 1)$ and $n \geq 1$ there exists a bound $\Gamma(n)$, independent of μ , such that $\|B_\mu y\|_n \leq \Gamma(n) \|y\|_X$ for every $y \in X$. Moreover, $\Gamma(n)$ is an increasing function of n such that $\Gamma(n) \rightarrow \infty$ as $n \rightarrow \infty$.*

Proof: Fix $\mu \in (0, 1)$ and $n \geq 1$. Let us begin by considering $y = (\alpha_y, s_y, \psi_y) \in M(X)$. Then there exists a unique $x = (\alpha_x, s_x, \psi_x) \in X$ such that $Mx = y$. Immediately from the identity stated in Claim 5.3.12 we have that

$$MP_n x = P_{M(E_n)} y \quad (5.3.66)$$

and since $MP_n x \in M(E_n)$, we may apply $M_n^{-1} : M(E_n) \rightarrow E_n$ to find that

$$P_n x = M_n^{-1} MP_n x = M_n^{-1} P_{M(E_n)} y. \quad (5.3.67)$$

Hence,

$$\|P_n x\|_X = \|M_n^{-1} P_{M(E_n)} y\|_X \leq C(n) \|P_{M(E_n)} y\|_X \leq C(n) \cdot \|P_{M(E_n)}\|_{op} \|y\|_X, \quad (5.3.68)$$

where $\|P_{M(E_n)}\|_{op} < \infty$ denotes the operator norm of the bounded projection $P_{M(E_n)} : X \rightarrow M(E_n)$.

Now, since $Mx = y$, we may use the form of M from Proposition 5.3.2 to find that we necessarily have

$$\begin{aligned}\alpha_x &:= \alpha_y, \\ s_x &:= M_{22}^{-1}[s_y - M_{21}\alpha_y],\end{aligned}\tag{5.3.69}$$

where $M_{22}^{-1} : \ell^\infty(\Lambda) \rightarrow \ell^\infty(\Lambda)$ denotes the bounded inverse of M_{22} . Therefore,

$$|\alpha_x| = |\alpha_y| \leq \max\{|\alpha_y|, \|s_y\|_\infty, \|\psi_y\|_\infty\} = \|y\|_X\tag{5.3.70}$$

and

$$\|s_x\|_\infty \leq \|M_{22}^{-1}\|_{op}[\|s_y\|_\infty + \|M_{21}\|_{op}|\alpha_y|] \leq \|M_{22}^{-1}\|_{op}[1 + \|M_{21}\|_{op}]\|y\|_X,\tag{5.3.71}$$

where all operator norms are finite.

Putting this all together shows that for any $y = Mx \in M(X)$ we get that $B_\mu y = x$ and

$$\begin{aligned}\|B_\mu y\|_n &= \|x\|_n \\ &= \max\{|\alpha_x|, \|s_x\|_\infty, \|P_n x\|_X, \frac{1}{\|M\|_{op}}\|y\|_X\} \\ &\leq \max\left\{1, \|M_{22}^{-1}\|_{op}[1 + \|M_{21}\|_{op}], C(n) \cdot \|P_{M(E_n)}\|_{op}, \frac{1}{\|M\|_{op}}\right\}\|y\|_X.\end{aligned}\tag{5.3.72}$$

Now for any $y \in X \setminus M(X)$, recall that we have defined $x := B_\mu y \in X$ to be so that

$$\|Mx - y\|_X \leq \mu\|y\|_X.\tag{5.3.73}$$

But since $y' := Mx$ belongs to the image of M , it follows that $B_\mu y' = x$ as well. Hence, from above we get

$$\|B_\mu y'\|_n \leq \max\left\{1, \|M_{22}^{-1}\|_{op}[1 + \|M_{21}\|_{op}], C(n) \cdot \|P_{M(E_n)}\|_{op}, \frac{1}{\|M\|_{op}}\right\}\|y'\|_X.\tag{5.3.74}$$

Then rearranging (5.3.73) with the reverse triangle inequality gives that

$$\|y'\|_X = \|Mx\|_X \leq (1 + \mu)\|y\|_X.\tag{5.3.75}$$

Therefore, putting all of this together gives

$$\begin{aligned}\|B_\mu y\|_n &= \|B_\mu y'\|_n \\ &\leq \max\left\{1, \|M_{22}^{-1}\|_{op}[1 + \|M_{21}\|_{op}], C(n) \cdot \|P_{M(E_n)}\|_{op}, \frac{1}{\|M\|_{op}}\right\}\|y'\|_X \\ &\leq (1 + \mu) \cdot \max\left\{1, \|M_{22}^{-1}\|_{op}[1 + \|M_{21}\|_{op}], C(n) \cdot \|P_{M(E_n)}\|_{op}, \frac{1}{\|M\|_{op}}\right\}\|y\|_X \\ &\leq 2 \cdot \max\left\{1, \|M_{22}^{-1}\|_{op}[1 + \|M_{21}\|_{op}], C(n) \cdot \|P_{M(E_n)}\|_{op}, \frac{1}{\|M\|_{op}}\right\}\|y\|_X,\end{aligned}\tag{5.3.76}$$

since $\mu < 1$.

Now, we have that $1 \leq \|P_{M(E_n)}\|_{op} < \infty$ for every n so that $C(n) \cdot \|P_{M(E_n)}\|_{op} \rightarrow \infty$ as $n \rightarrow \infty$ from the properties of $C(n)$. Therefore, we will define

$$\Gamma(n) := 2 \cdot \max \left\{ 1, \|M_{22}^{-1}\|_{op}[1 + \|M_{21}\|_{op}], C(n) \cdot \|P_{M(E_n)}\|_{op}, \frac{1}{\|M\|_{op}} \right\} \quad (5.3.77)$$

for each $n \geq 1$ to find that $\Gamma(n) \rightarrow \infty$ as $n \rightarrow \infty$. Since $n \geq 1$ we also have that $\|B_\mu y\|_n \leq \Gamma(n)\|y\|_X$ for all $\mu \in (0, 1)$ and $y \in X$. \blacksquare

Now that we have a bound on the norm of $B_\mu : X \rightarrow X$ as a function of n , we wish to translate this into an appropriate bound in μ to successfully apply Theorem 5.1.3. That is, we wish to obtain a function $n(\mu) : (0, 1) \rightarrow \mathbb{N}$ such that $\Gamma(n(\mu)) \leq k_0\mu^{-\frac{1}{2}}$ for all $\mu \in (0, 1)$. We obtain this through an inductive definition of the function $n(\mu)$. To begin, take $k_0 > 0$ such that

$$\Gamma(1) \leq k_0. \quad (5.3.78)$$

Then, let $\mu_1 \in (0, 1)$ be such that

$$\Gamma(2) \leq k_0\mu_1^{-\frac{1}{2}}. \quad (5.3.79)$$

One should note that this value μ_1 can always be found since $k_0\mu^{-\frac{1}{2}}$ is unbounded as $\mu \rightarrow 0^+$ and $\Gamma(n)$ is finite for every $n \geq 1$. Now we define $n(\mu) = 1$ on $(\mu_1, 1)$. This clearly gives $\Gamma(n(\mu)) \leq k_0\mu^{-\frac{1}{2}}$ on $(\mu_1, 1)$. Next, let $\mu_2 \in (0, \mu_1)$ such that

$$\Gamma(3) \leq k_0\mu_2^{-\frac{1}{2}}. \quad (5.3.80)$$

Again, this can be found since $k_0\mu^{-\frac{1}{2}}$ is unbounded as $\mu \rightarrow 0^+$. We define $n(\mu) = 2$ on $(\mu_2, \mu_1]$, now giving that $\Gamma(n(\mu)) \leq k_0\mu^{-\frac{1}{2}}$ on $(\mu_2, 1)$.

This process continues inductively in that for any $n = d \geq 2$, we define $\mu_d \in (0, \mu_{d-1})$ so that

$$\Gamma(d+1) \leq k_0\mu_d^{-\frac{1}{2}}. \quad (5.3.81)$$

Then $n(\mu) = d$ on $\mu \in (\mu_d, \mu_{d-1}]$ so that $\Gamma(n(\mu)) \leq k_0\mu^{-\frac{1}{2}}$ on $(\mu_d, 1)$. This process is illustrated in Figure 5.4.

It follows from the fact that $C(n) \rightarrow \infty$ as $n \rightarrow \infty$ that we have $\mu_d \rightarrow 0$ as $d \rightarrow \infty$. Hence, we have a function $n(\mu)$ which gives us that for any $\mu \in (0, 1)$

$$\|MB_\mu y - y\|_X \leq \mu\|y\|_X \text{ and } \|B_\mu y\|_{n(\mu)} \leq k_0\mu^{-\frac{1}{2}}\|y\|_X \quad (5.3.82)$$

for every $y \in X$. Therefore, $B_\mu : X \rightarrow X$ is an approximate right inverse of M for each $\mu \in (0, 1)$ satisfying the hypothesis of Theorem 5.1.3.

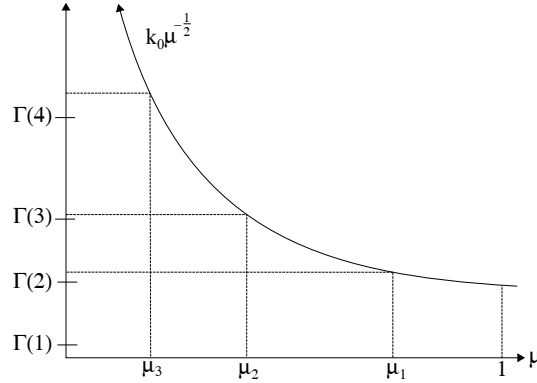


Figure 5.4: Visualization of how we inductively define $n(\mu)$ on $(0, 1)$.

The final step to applying Theorem 5.1.3 to our situation is determining the existence of the element c such that $-[G(0, 0, 0) + Mc] \in S$ and $\|c\|_{n(\mu)} = 1$. One should note that by definition X is a dense subset of X_n for every $n \geq 1$. Then we extend $M : X \rightarrow X$ to a linear operator $\overline{M}^n : X_n \rightarrow X$ with the following lemma.

Lemma 5.3.17. *For each $n \geq 1$, $M : X \rightarrow X$ extends to a unique bounded linear operator $\overline{M}^n : X_n \rightarrow X$ such that $\overline{M}^n|_X = M$.*

Proof: By definition X is a dense subset of X_n , for every $n \geq 1$. Then for any $n \geq 1$ and an $x \in X \subset X_n$ we have

$$\begin{aligned} \|Mx\|_X &= \frac{\|M\|_{op}}{\|M\|_{op}} \|Mx\|_X \\ &\leq \|M\|_{op} \cdot \max\{|\alpha|, \|s\|_\infty, \|P_n x\|_X, \frac{1}{\|M\|_{op}} \|Mx\|_X\} \\ &= \|M\|_{op} \|x\|_n. \end{aligned} \tag{5.3.83}$$

That is, $M : X \subset X_n \rightarrow X$ is a densely defined, bounded linear operator. The existence of a unique bounded linear extension, denoted $\overline{M}^n : X_n \rightarrow X$, directly follows from the Bounded Linear Transformation (B.L.T.) Theorem, a proof of which can be found in [58]. ■

The operator \overline{M}^n is often referred to as the closure of the operator M with respect to the space X_n . For ease of notation we will simply write $M : X_n \rightarrow X$ to denote this closure with respect to the space X_n , although one should always keep in mind that the closure entirely depends on the value of n . We now prove the following lemma.

Lemma 5.3.18. *$M : X_n \rightarrow X$ is surjective for every $n \geq 1$.*

Proof: To show that $M : X_n \rightarrow X$ is surjective, we show that its range is closed and dense. First, since $X \subset X_n$, from Corollary 5.3.7 we have that $M : X_n \rightarrow X$ has dense range. Now let $y \in X$. From the density of the range of $M : X \rightarrow X$, there exists a sequence $\{y_k\}_{k=1}^{\infty}$ in the range of M such that $\|y_k - y\|_X \rightarrow 0$ as $k \rightarrow \infty$. Then by definition, for each y_k , there exists $x_k \in X$ such that $Mx_k = y_k$. Notice that for any $\mu \in (0, 1)$ we have $B_\mu y_k = x_k$ since y_k belongs to the range of $M : X \rightarrow X$. Furthermore, since $y_k - y_{k'}$ belongs to the range of $M : X \rightarrow X$, we have that $B_\mu(y_k - y_{k'}) = x_k - x_{k'}$ for any $k, k' \geq 1$. Therefore, from Lemma 5.3.16 we have that

$$\|x_k - x_{k'}\|_n = \|B_\mu(y_k - y_{k'})\|_n \leq \Gamma(n)\|y_k - y_{k'}\|_X. \quad (5.3.84)$$

Since $\Gamma(n) < \infty$ and $\{y_k\}_{k=1}^{\infty}$ is a Cauchy sequence, it follows that $\{x_k\}_{k=1}^{\infty}$ is a Cauchy sequence in X_n . Since X_n is complete, there exists an element $x \in X_n$ such that $x_k \rightarrow x$ as $k \rightarrow \infty$. Then by the continuity of $M : X_n \rightarrow X$, it follows that $Mx = y$. Hence, $M : X_n \rightarrow X$ is closed. ■

Finally, we are now in a position to apply Theorem 5.1.3 to our mapping G in (5.3.4).

Proof: (Proof of Theorem 5.2.3)

Recall that our interest lies in the closed convex cone $S = \mathbb{R} \times \{0\} \times \{0\}$ defined in (5.3.6). For each $\mu \in (0, 1)$ we have the relation $n(\mu)$ defined above, and an approximate right inverse $B_\mu : X \rightarrow X$ which has a bound function given by $k_0\mu^{-\frac{1}{2}}$, for some $k_0 > 0$, thus satisfying the hypothesis of Theorem 5.1.3.

Then for any μ small and fixed, from Lemma 5.3.18 there exists a $c' \in X_{n(\mu)}$ such that

$$Mc' = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \in S. \quad (5.3.85)$$

Taking $c = \frac{1}{\|c'\|_{n(\mu)}}c' \in X_{n(\mu)}$ gives $-Mc \in S$ and $\|c\|_{n(\mu)} = 1$. Hence,

$$-[G(0, 0, 0) + Mc] = \frac{1}{\|c'\|_{n(\mu)}} \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix} \in S, \quad (5.3.86)$$

from $G(0, 0, 0) = (0, 0, 0)^T$ and the linearity of M . Therefore, from Theorem 5.1.3 we have that upon writing $c = (c_1, c_2, c_3)$, there exists a solution

$$\begin{pmatrix} \alpha(t) \\ s(t) \\ \psi(t) \end{pmatrix} = t \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} + \begin{pmatrix} \eta_\alpha(t) \\ \eta_s(t) \\ \eta_\psi(t) \end{pmatrix} \in X_{n(\mu)}, \quad (5.3.87)$$

to $-G(\alpha, s, \psi) \in S$ for some sufficiently small $t > 0$. This implies that both $G^1(\alpha, s, \psi) = 0$ and $G^2(\alpha, s, \psi) = 0$ at these solutions.

A close inspection of the proof of Theorem 5.1.3 in [16] shows that this solution is obtained by solving

$$G(\alpha(t), s(t), \psi(t)) - G(0, 0, 0) - tMc = 0 \tag{5.3.88}$$

for $t > 0$. Then, using the fact that $Mc = \frac{1}{\|c'\|_{n(\mu)}}(1, 0, 0)^T$ and the form of G , we have that the first component of (5.3.88) gives

$$\alpha(t) = \frac{1}{\|c'\|_{n(\mu)}}t. \tag{5.3.89}$$

Since, $\frac{1}{\|c'\|_{n(\mu)}} > 0$ we have that $\alpha(t) > 0$ for all values of $t > 0$ for which it is defined. Furthermore, $\eta_\alpha(t) \equiv 0$ for all $t > 0$ in its domain. The function $\alpha(t)$ is locally invertible allowing one to consider the function

$$t(\alpha) = \|c'\|_{n(\mu)}\alpha, \tag{5.3.90}$$

for sufficiently small $\alpha > 0$. Therefore solutions $(\alpha(t), s(t), \psi(t))$ can be re-parametrized in terms of small values of $\alpha > 0$.

An a priori check reveals that since $s(t)$ (or equivalently $s(\alpha)$) represents the second component of an element in $X_{n(\mu)}$ and that $s(t) \rightarrow 0$ as $t \rightarrow 0^+$, we notice that for sufficiently small $t > 0$ we do indeed have $\|s(t)\|_\infty < \frac{a}{2}$, and therefore belonging to the domain of G . Furthermore, as long as this solution exists we have that $s(t) \in \ell^\infty(\Lambda)$ by definition of the norm on $X_{n(\mu)}$, thus showing that the radial perturbation s is uniformly bounded and can be made uniformly as small as necessary.

Then as noted at the beginning of this section, the zeros of G^1 and G^2 lie in one-to-one correspondence with those of (5.2.11). This gives solutions parametrized by sufficiently small $\alpha > 0$ given by

$$\begin{pmatrix} r(\alpha) \\ \theta(\alpha) \end{pmatrix} = \begin{pmatrix} \mathbf{a} \\ \bar{\theta} \end{pmatrix} + t(\alpha) \begin{pmatrix} c_2 \\ c_3 \end{pmatrix} + \begin{pmatrix} \eta_s(t(\alpha)) \\ \eta_\psi(t(\alpha)) \end{pmatrix} \tag{5.3.91}$$

to (5.2.11). Furthermore, letting $\alpha \rightarrow 0^+$ gives $t \rightarrow 0^+$, thus giving that $r(\alpha) \rightarrow \mathbf{a}$ and $\theta(\alpha) \rightarrow \bar{\theta}$, finishing the proof of Theorem 5.2.3. ■

Remark 5.3.19. It should be noted that our choice of c' in the above proof may not be unique since Lemma 5.3.18 merely proved that $M : X_n \rightarrow X$ is surjective, but says nothing to the injectivity of the operator. The reason for this is that in the proof of Lemma 5.3.6 it was pointed out that the operator M_{33} does have a nontrivial

kernel which is spanned by the constant functions (i.e. $\psi_{i,j} = C \in \mathbb{R}$ for all $(i, j) \in \Lambda$) but this element cannot belong to X . When completing the space X with respect to the norms $\|\cdot\|_n$ there is a potential that these constant ψ elements have been added to the space, and therefore $M : X_n \rightarrow X$ will fail to be injective. This possibly nontrivial kernel is fairly intuitive since our system (5.3.4) only takes into account the differences between neighbouring phases. Therefore, any one solution found via Theorem 5.2.3, written (α, s, ψ_1) , can be trivially extended to an entire continuum of solutions (α, s, ψ_2) such that $\psi_1 - \psi_2 = \{C\}_{(i,j) \in \Lambda}$ for some $C \in \mathbb{R}$.

5.4 Comparison with the Finite Lattice

In [26] the existence of rotating wave solutions to systems of type (1.3.2) on finite square lattices are proven for $\lambda(r_{i,j}) = 1 - r_{i,j}^2$ and $\omega(\cdot, \alpha)$ constant. In this case the interactions with boundary elements are absent from the sum notation containing nearest-neighbour interactions, similar to the finite phase system (2.2.1). In this case the boundary conditions are said to reflect Neumann boundary conditions coming from the model prior to spatial discretization. The authors show that the solution originating at $\alpha = 0$ cannot persist for all $\alpha > 0$ and must meet the trivial solution at a bifurcation point. Their methods are easily generalized to those λ functions satisfying condition (1) of Hypothesis 1.3.1 with the added stipulation that $\lambda'(a) < 0$ and hence can be used to provide insight into the expected behaviour of the full system on an infinite lattice when $\omega(\cdot, \alpha)$ is constant. We briefly summarize how these results apply to our own situation here.

Let us consider a $\lambda(\cdot)$ satisfying condition (1) of Hypothesis 1.3.1 with $\lambda'(a) < 0$ and $\omega(\cdot, \alpha)$ constant and real-valued, which as previously remarked, clearly satisfies condition (2) of Hypothesis 1.3.1. We will assume that $a > 0$ is the leftmost root of λ in $[0, \infty)$ so that $\lambda(0) > 0$. Linearizing a finite lattice with these Neumann boundary conditions about the trivial equilibrium leads to eigenvalues with real parts given by $\lambda(0) + \alpha\nu$, where $\nu \leq 0$ is an eigenvalue of the discretized Laplacian operator on the finite lattice. Clearly any bifurcation must take place when $\lambda(0) + \alpha\nu = 0$, thus we can solve for the values of α which lead to bifurcations from the trivial equilibrium. One always has that $\nu = 0$ is an eigenvalue, but in this case we cannot trigger a bifurcation by varying α . The second smallest eigenvalue is given by

$$\nu = 2 \left[\cos \left(\frac{\pi}{N} \right) - 1 \right], \quad (5.4.1)$$

where we assume the square lattice has $N \times N$ elements. Hence, the minimal value of α which can lead to a bifurcation from the trivial equilibrium is given as a function

of the size of the lattice by

$$\alpha^*(N) = \frac{\lambda(0)}{2 \left[1 - \cos \left(\frac{\pi}{N} \right) \right]}. \quad (5.4.2)$$

This value $\alpha^*(N)$ guarantees a minimal range of existence of a rotating wave solution given by $\alpha \in [0, \alpha^*(N)]$. One can further see that $\alpha^*(N) \rightarrow \infty$ monotonically as $N \rightarrow \infty$, and naturally leads one to conjecture that when $\omega(\cdot, \alpha)$ is a constant function, the limiting case $N = \infty$ undertaken in Corollary 5.2.4 gives existence for all $\alpha > 0$.

The situation when $\lambda'(a) > 0$ is considerably less explored since solutions of this type on the finite lattice are unstable in a neighbourhood of $\alpha = 0$. This has prevented any thorough investigations of the behaviour of such solutions as α varies, and therefore aside from local existence results one has little intuition as to what to expect from such solutions. Although some investigations along this line would be interesting from a mathematical point of view, there has been little to no motivation to further study such solutions since they are unstable.

Solutions on the finite lattice can be obtained in a similar manner to our investigation herein using the standard Implicit Function Theorem, which also guarantees local uniqueness of the implicit function solution. A notable shortcoming of Theorem 5.1.3 is that we have no guarantee that the solution is unique. Although it would be desirable to obtain uniqueness results (up to a translation in the phase components), changing norms to create an approximate right inverse prohibits typical methods of demonstrating results of this type. That is, the notion of ‘‘closeness’’ inferred by the $\|\cdot\|_n$ norm differs significantly to that of the norm $\|\cdot\|_X$. This difference manifests itself as the component of (5.3.61) weighted by the operator M , and therefore differentiability of the mapping of G in a neighbourhood of $(0, \mathbf{a}, \bar{\theta})$ is no longer guaranteed with respect to these new norms.

Using results on the finite lattice as a basis, one may formulate many conjectures into the nature of rotating wave solutions to Lambda-Omega systems. These conjectures provide effective starting points for future investigations which are motivated by the work in this thesis. Having now demonstrated their existence, further explorations of rotating waves in the discrete spatial setting can be initiated.

Chapter 6

Conclusion and Future Work

In this thesis we have demonstrated the existence of rotating wave solutions to lambda-omega lattice dynamical systems, thus providing a study of rotating waves without the use of continuous Euclidean symmetry. We saw in Chapter 2 that our lambda-omega system (1.3.2) can be reduced to an infinite system of coupled phase equations which in turn can be shown to possess a rotating wave solution. This rotating wave solution to the coupled phase equations forms the basis for much of the investigation in the following chapters.

In Chapter 3 we presented a review of some relevant graph-theoretic results in an effort to understand some stability aspects of the rotating wave solution to the coupled phase equations. Most interestingly we saw that the theory of random walks on infinite graphs has a direct and practical application to our work in this thesis, thus providing a link between seemingly unrelated areas of mathematics. In Section 4.2 of Chapter 4 a stability theorem is presented in a completely general context in an effort to be applied to further systems of coupled oscillators with potentially more complex connection topologies. This therefore allows for the future study and identification of a diverse range of stable phase-locked solutions to infinite coupled phase equations.

Finally, in Chapter 5 we returned to the full lattice dynamical system (1.3.2) where the existence of rotating wave solutions to this system are presented under mild technical assumptions. This result is achieved through a nontrivial application of an alternative Implicit Function Theorem, where we extend a solution at $\alpha = 0$ into small $\alpha > 0$. We create the solution at $\alpha = 0$ using the rotating wave solution to the coupled phase equations found in Chapter 2. The application of this alternative Implicit Function Theorem required the creation of a new mapping whose roots lie in one-to-one correspondence with rotating wave solutions to (1.3.2) and an exploitation of the square symmetry of the lattice \mathbb{Z}^2 .

6.1 Multi-Armed Spirals

Although multi-armed spirals are not well-documented in the typical motivating application of cardiac electrophysiology, they are known to exist in nature nonetheless. For example, Belousov-Zhabotinskii chemical reactions are light sensitive chemical oscillators known to self-organize into spiral concentration patterns [72]. Moreover, studies have shown that not only single armed spirals can be found in these chemical reactions, but also spirals with up to six arms [54]. Therefore the presence of multi-armed spirals in such natural settings could potentially motivate the study of these objects in the setting of system (1.3.2).

Unlike continuous spatial models where much of the analysis of single and multi-armed rotating waves is handled in a similar way, the analysis undertaken here does not seem to lend itself to the existence of multi-armed rotating waves. That is, for a positive integer m , an m -armed rotating wave solution to (1.3.2) would satisfy

$$z_{i,j}(t) = \bar{r}_{i,j} e^{\Omega(\alpha)t + m\bar{\theta}_{i,j}}, \quad (6.1.1)$$

where $(\bar{r}_{i,j}, \bar{\theta}_{i,j})$ are steady-state solutions to

$$\begin{aligned} \dot{r}_{i,j} &= \alpha \sum_{i',j'} [r_{i',j'} \cos(m(\theta_{i',j'} - \theta_{i,j})) - r_{i,j}] + r_{i,j} \lambda(r_{i,j}), \\ \dot{\theta}_{i,j} &= \alpha \sum_{i',j'} \frac{r_{i',j'}}{r_{i,j}} \sin(m(\theta_{i',j'} - \theta_{i,j})) + \alpha \omega_1(r_{i,j}, \alpha). \end{aligned} \quad (6.1.2)$$

Despite the cases for $m > 1$ and $m = 1$ appearing quite similar, we remark that a steady-state solution to the phase components as $\alpha \rightarrow 0^+$ must satisfy

$$0 = \sum_{i',j'} \sin(m(\theta_{i',j'} - \theta_{i,j})) \quad (6.1.3)$$

where we again use Hypothesis 2.1.1 to have that $r_{i,j} \rightarrow a$ as $\alpha \rightarrow 0^+$. When $m > 1$ this coupling function does not satisfy Hypothesis 2.1.3, and even in the case when a solution is obtained for a specific $m > 1$ we may not necessarily have

$$\cos(m(\theta_{i',j'} - \theta_{i,j})) \geq 0 \quad (6.1.4)$$

for all nearest-neighbour interactions. Thus linearizing about this solutions will not necessarily lead to a combinatorial graph Laplacian, and hence much of the theory from Chapter 3 cannot be applied to this situation. Particularly, methods used in the proof of Lemma 5.3.6 to show injectivity becomes significantly more complicated, if they can be undertaken at all.

Numerical investigations undertaken in [21] concluded that a 6×6 lattice can exhibit a two-armed rotating wave solution, thus leading one to conjecture that there

exists an extension of this work to at least two-armed waves. It appears that one may undertake a similar line of arguments to that for single-armed rotating waves detailed in [56] to show that on any finite square lattice there exists a two-armed rotating wave solution as governed by the truncation of the system (6.1.3) from infinite to finite with Neumann boundary conditions. Although this would still not address the problem of injectivity mentioned previously.

It is intuitive to think that spirals with 2 and 4 arms could at least be found in our coupled phase system (2.1.6) since these number of arms are commensurate with the symmetries of the lattice. Therefore, we could potentially work with a similar reduced phase system and exploit the symmetries of the square lattice to extend this solution over the whole lattice, much like what was done in (2.2.22). On the other, values such as $m = 3$ will be incommensurate with the symmetries of the lattice, thus presenting a technical hurdle in the analysis which at present has no way of being overcome.

6.2 Stability of the Rotating Wave Solutions

From the work in this thesis, the natural follow-up investigation becomes determining the stability properties of the rotating wave solution found in Chapter 5. One expects that the case when $\lambda'(a) < 0$ leads to asymptotically stable rotating wave solutions in a neighbourhood of $\alpha = 0$. The reason for this being that when $\alpha = 0$ and $\lambda'(a) < 0$, the uncoupled system exhibits independent locally asymptotically stable limit cycle solutions at each $(i, j) \in \mathbb{Z}^2$, as was pointed out in Section 1.3. Along the same line of investigation, one expects that the rotating wave solutions in the case $\lambda'(a) > 0$ are unstable in a neighbourhood of $\alpha = 0$ for the very same reason.

Of course both of these statements require rigorous analytic confirmation, which is left to future work. We take this time to briefly comment on why these statements cannot be confirmed through similar methods to those for the phase system outlined in Chapter 4. Notice that the coupling functions in our phase components (2.0.2b) exhibit an odd symmetry when $\alpha = 0$, since we have imposed $r_{i,j} = a$ for all $(i, j) \in \mathbb{Z}^2$. It was exactly this odd symmetry when $\alpha = 0$ that has allowed us to utilize results for infinite graphs since we know that the derivative of an odd coupling function (sine) is even, thus providing the necessary symmetry requirements of a weight function for a weighted graph. When $\alpha \neq 0$, this symmetry is broken since we do not necessarily have the $r_{i,j}$'s identical anymore. In theory, linearizing the phase components about the rotating wave solution could lead to a directed graph Laplacian, i.e. the weight function is no longer symmetric. As was detailed in Remark 4.2.7, the theory of random walks on directed graphs is significantly less studied and therefore may not provide all the necessary results to obtain algebraic decay rates as was done in Theorem 4.2.6.

Stability of rotating and spiral wave solutions in the continuous spatial setting has also been quite a difficult problem to tackle. For example, Scheel proves the existence of arbitrary armed spiral waves forming via Hopf bifurcations in reaction-diffusion equations but notes that linearizing about these spiral waves leads to an operator with zero being (at least) a triple eigenvalue, due to the euclidean symmetry of the original reaction-diffusion equation [66]. In the supercritical Hopf bifurcation case, linearization further leads to an operator with zero in its essential spectrum and thus exponential stability cannot be determined. Hagan gives evidence in [39] that one-armed spirals should be stable whereas their multi-armed counterparts should not be. It therefore appears that the question of stability of rotating wave solutions in both the continuous and discrete spatial settings is one which is deserving of an intense mathematical investigation which would extend far beyond the scope of this thesis.

6.3 Dynamics of Rotating Waves in Discrete Space

In this thesis we have demonstrated the existence of rotating wave solutions to infinite systems of coupled ordinary differential equations. This therefore lays the foundation for subsequent formal analyses of the dynamics of these waves, and in particular, how they differ from the continuous spatial setting. This task is ambitious and long-term and therefore currently has ill-defined short-term goals. It is the intention of this work to generate a foundation in which further studies can be expounded upon.

This work very much parallels that of Zinner's which demonstrated the existence of traveling wave solutions to one-dimensional lattice dynamical systems. Zinner's existence proof provided a necessary framework for later studies into the dynamics of these travel waves solutions to build upon. The study of traveling waves in spatially discretized media continues to grow as many researchers add their unique perspectives to this diverse subtopic of applied mathematics. This thesis helps to develop the study of rotating waves in spatially discretized media in much the same way as Zinner did for traveling waves.

What has become apparent throughout this work is that the link between lattice dynamical systems and graph theory cannot be ignored. We have seen that the connection topology of a lattice dynamical system endows an important geometric underlying to these systems, which can be best understood from the perspective of graph networks. This tempts the reader to consider the question of more complex connection topologies in our lattice dynamical system or even different lattice structures, such as the hexagonal or rhombic lattices. It is probable that much of the same work from this thesis could be applied to these problems, but one expects that stronger connection topologies could potentially lead to faster decay rates upon applying Theorem 4.2.6.

It has also been well-documented that continuous Euclidean symmetry plays an important role in the bifurcations from spiral waves [61, 62]. For example, when a spiral wave undergoes a Hopf bifurcation, the interaction of the Hopf frequency and the rotational frequency of the spiral wave can produce a spiral wave which is linearly meandering through the continuous two-dimensional spatial medium [33]. All of this work makes great use of continuous Euclidean symmetry in the model equations, and therefore does not lend itself to the work herein. Thus an interesting line of inquiry would be to examine Hopf bifurcations from spiral wave solutions to lattice dynamical systems in an effort to understand how the loss of Euclidean symmetry effects such bifurcations. All of these questions and more leave the door open to many future investigations.

Appendix A

Strongly Fréchet Differentiable Implies Restricted Strongly Hadamard Differentiable

To begin, let X and Y be Banach spaces. The function $F : X \rightarrow Y$ is *strongly Hadamard differentiable* at the point $x_0 \in X$ if there exists a bounded linear operator $A : X \rightarrow Y$ such that, for any continuous function $u : [0, \infty) \rightarrow X$ for which $u'(0^+)$ exists and $u(0) = x_0$, the composition $F \circ u$ is strongly differentiable at 0^+ with derivative $(F \circ u)'(0^+) = Au'(0^+)$. Furthermore, $F : X \rightarrow Y$ is called *restrictedly strongly Hadamard differentiable* at the point $x_0 \in X$ if the strongly Hadamard differentiable property holds when u is restricted to being strongly differentiable at 0^+ .

Proposition A.1. *Assume $F : X \rightarrow Y$ is strongly Fréchet differentiable at the point $x_0 \in X$. Then F is restricted strongly Hadamard differentiable at x_0 .*

Proof: Let $\|\cdot\|_X$ and $\|\cdot\|_Y$ denote the norms of the Banach spaces X and Y , respectively. Fix $\varepsilon > 0$. Then since $F : X \rightarrow Y$ strongly Fréchet differentiable at the point $x_0 \in X$, there exists a bounded linear operator $A : X \rightarrow Y$ and $\delta_1 > 0$ such that

$$\|F(x_1) - F(x_2) - A(x_1 - x_2)\|_Y < \varepsilon\|v_1 - v_2\|_X, \quad (\text{A.1})$$

for all $\|x_1 - x_0\|_X, \|x_2 - x_0\|_X < \delta_1$.

Now let $u : [0, \infty) \rightarrow X$ be a continuous function which is strongly differentiable at 0^+ and satisfies $u(0) = x_0$. Then since u is continuous and satisfies $u(0) = x_0$, there exists a $\delta_2 > 0$ such that $\|u(t) - x_0\|_X < \delta_1$ for all $t \in [0, \delta_2)$. Furthermore, since u is strongly differentiable at 0^+ there exists a $\delta_3 > 0$ such that

$$\|u(t_1) - u(t_2) - u'(0^+)(t_1 - t_2)\|_X < \varepsilon|t_1 - t_2|, \quad (\text{A.2})$$

for all $0 < t_1, t_2 < \delta_3$.

Now, let $\delta := \min\{\delta_2, \delta_3\} > 0$ and consider any $t_1, t_2 \in [0, \delta)$. Then, from (A.1) and (A.2) we have

$$\begin{aligned}
 & \|F(u(t_1)) - F(u(t_2)) - A(u(t_1) - u(t_2))\|_Y \\
 & \quad < \varepsilon \|u(t_1) - u(t_2)\|_X \\
 & \quad \leq \varepsilon \|u(t_1) - u(t_2) - u'(0^+)(t_1 - t_2)\|_X + \varepsilon \|u'(0^+)(t_1 - t_2)\|_X \\
 & \quad < \varepsilon^2 |t_1 - t_2| + \varepsilon |u'(0^+)| |t_1 - t_2|.
 \end{aligned} \tag{A.3}$$

Similarly, using (A.2) and the fact that A is bounded linear, denoting its operator norm $\|A\|_{op}$, we obtain

$$\begin{aligned}
 \|Au(t_1) - Au(t_2) - Au'(0^+)(t_1 - t_2)\|_Y & \leq \|A\|_{op} \|u(t_1) - u(t_2) - u'(0^+)(t_1 - t_2)\|_X \\
 & < \varepsilon \|A\|_{op} |t_1 - t_2|.
 \end{aligned} \tag{A.4}$$

Putting this all together shows that

$$\begin{aligned}
 & \|F(u(t_1)) - F(u(t_2)) - Au'(0^+)(t_1 - t_2)\|_Y \\
 & \quad \leq \|F(u(t_1)) - F(u(t_2)) - A(u(t_1) - u(t_2))\|_Y \\
 & \quad \quad + \|Au(t_1) - Au(t_2) - Au'(0^+)(t_1 - t_2)\|_Y \\
 & \quad < \varepsilon^2 |t_1 - t_2| + \varepsilon |u'(0^+)| |t_1 - t_2| + \varepsilon \|A\|_{op} |t_1 - t_2|.
 \end{aligned} \tag{A.5}$$

Dividing by $|t_1 - t_2|$ gives

$$\frac{\|F(u(t_1)) - F(u(t_2)) - Au'(0^+)(t_1 - t_2)\|_Y}{|t_1 - t_2|} < \varepsilon^2 + \varepsilon(\|A\|_{op} + |u'(0^+)|). \tag{A.6}$$

Since $\varepsilon > 0$ was arbitrary, this quotient can be made arbitrarily small, thus completing the proof. ■

Appendix B

Proof of Proposition 5.3.2

Here we will prove Proposition 5.3.2. This proof is broken down into two lemmas which lead to the proof of the Proposition. Recall that $B_{\frac{a}{2}}(0)$ denotes the ball of radius $\frac{a}{2}$ centred at 0 in the space $\ell^\infty(\Lambda)$.

Lemma B.1. $G^1(\alpha, s, \psi) : \mathbb{R} \times B_{\frac{a}{2}}(0) \times c_0(\Lambda) \rightarrow \ell^\infty(\Lambda)$ is strongly Fréchet differentiable at $(\alpha, s, \psi) = (0, 0, 0)$. The Fréchet derivative at this point is the bounded linear operator which acts by $(\alpha, s, \psi) \mapsto M_{21}\alpha + M_{22}s$, where M_{21} and M_{22} are as defined in (5.3.16) and (5.3.17), respectively.

Proof: For ease of notation let us return to the variables $r_{i,j} = a + s_{i,j}$ and $\theta_{i,j} = \bar{\theta}_{i,j} + \psi_{i,j}$ for all $(i, j) \in \Lambda$. We will write $r = \{r_{i,j}\}_{(i,j) \in \Lambda}$ and $\theta = \{\theta_{i,j}\}_{(i,j) \in \Lambda}$. Note that $G^1(\alpha, s, \psi)$ strongly Fréchet differentiable at $(\alpha, s, \psi) = (0, 0, 0)$ is equivalent to $G^1(\alpha, r, \theta)$ strongly Fréchet differentiable at $(\alpha, r, \theta) = (0, \mathbf{a}, \bar{\theta})$.

Begin by taking any $\varepsilon > 0$ and consider $(\alpha^1, s^1, \psi^1), (\alpha^2, s^2, \psi^2) \in \mathbb{R} \times B_{\frac{a}{2}}(0) \times c_0(\Lambda)$. We denote r^1 and r^2 to correspond to s^1 and s^2 , respectively. Similarly θ^1 and θ^2 correspond to ψ^1 and ψ^2 .

Since cosine is an infinitely differentiable function on \mathbb{R} , from Theorem 5.1.2¹ we have that there exists $\delta_1 > 0$ such that

$$\begin{aligned}
 & |\alpha^1 r_{i',j'}^1 \cos(\theta_{i',j'}^1 - \theta_{i,j}^1) - \alpha^2 r_{i',j'}^2 \cos(\theta_{i',j'}^2 - \theta_{i,j}^2) - (\alpha^1 - \alpha^2) a \cos(\bar{\theta}_{i',j'} - \bar{\theta}_{i,j})| \\
 & \quad < \varepsilon \max\{|\alpha^1 - \alpha^2|, |r_{i',j'}^1 - r_{i',j'}^2|, |(\psi_{i',j'}^1 - \psi_{i,j}^1) - (\psi_{i',j'}^2 - \psi_{i,j}^2)|\} \\
 & \quad \leq \varepsilon \max\{|\alpha^1 - \alpha^2|, \|r^1 - r^2\|_\infty, |\psi_{i',j'}^1 - \psi_{i',j'}^2| + |\psi_{i,j}^1 - \psi_{i,j}^2|\} \\
 & \quad \leq 2\varepsilon \max\{|\alpha^1 - \alpha^2|, \|r^1 - r^2\|_\infty, \|\psi^1 - \psi^2\|_\infty\} \\
 & \quad = 2\varepsilon \|(\alpha^1, s^1, \psi^1) - (\alpha^2, s^2, \psi^2)\|_X,
 \end{aligned} \tag{B.1}$$

¹Here we are applying Theorem 5.1.2 to the function $f : \mathbb{R}^4 \rightarrow \mathbb{R}$ acting by $f(x_1, x_2, x_3, x_4) = x_1 x_2 \cos(x_3 - x_4)$ at the point $(x_1, x_2, x_3, x_4) = (0, a, \bar{\theta}_{i',j'}, \bar{\theta}_{i,j})$ for each $(i, j) \in \Lambda$ and (i', j') .

provided that $\max\{|\alpha^1|, |r_{i',j'}^1|, |\psi_{i',j'}^1|, |\psi_{i,j}^1|\}, \max\{|\alpha^2|, |r_{i',j'}^2|, |\psi_{i',j'}^2|, |\psi_{i,j}^2|\} < \delta_1$. Then (B.1) holds for all $(i, j) \in \Lambda$ when $\|(\alpha^1, s^1, \psi^1)\|_X, \|(\alpha^2, s^2, \psi^2)\|_X < \min\{\delta_1, \frac{a}{2}\}$. Similarly, from condition (1) of Hypothesis 1.3.1 we have assumed that λ is continuously differentiable on $[0, \infty)$, and since $a > 0$ we may apply Theorem 5.1.2² to obtain a $\delta_2 > 0$ such that

$$\begin{aligned} |r_{i,j}^1 \lambda(r_{i,j}^1) - r_{i,j}^2 \lambda(r_{i,j}^2) - a \lambda'(a)(s_{i,j}^1 - s_{i,j}^2)| &< \varepsilon |s_{i,j}^1 - s_{i,j}^2| \\ &\leq \varepsilon \|s^1 - s^2\|_\infty \\ &\leq \varepsilon \|(\alpha^1, s^1, \psi^1) - (\alpha^2, s^2, \psi^2)\|_X, \end{aligned} \quad (\text{B.2})$$

provided $|r_{i,j}^1|, |r_{i,j}^2| < \delta_2$. This in turn shows that (B.2) holds for all $(i, j) \in \Lambda$ when $\|s^1\|_\infty, \|s^2\|_\infty < \min\{\frac{a}{2}, \delta_2\}$.

As a final aside, again from Theorem 5.1.2³ there exists a $\delta_3 > 0$ such that

$$\begin{aligned} |\alpha^1 r_{i,j}^1 - \alpha^2 r_{i,j}^2 - (\alpha^1 - \alpha^2)a| &< \varepsilon \max\{|\alpha^1 - \alpha^2|, |r_{i,j}^1 - r_{i,j}^2|\} \\ &\leq \varepsilon \max\{|\alpha^1 - \alpha^2|, \|r^1 - r^2\|_\infty\} \\ &\leq \varepsilon \|(\alpha^1, s^1, \psi^1) - (\alpha^2, s^2, \psi^2)\|_X, \end{aligned} \quad (\text{B.3})$$

provided $\max\{|\alpha^1 - \alpha^2|, |r_{i,j}^1 - r_{i,j}^2|\} < \delta_3$. Then we have that (B.3) holds for all $(i, j) \in \Lambda$ when $\max\{|\alpha^1 - \alpha^2|, \|r^1 - r^2\|_\infty\} < \delta_3$.

Therefore, we take $\delta = \min\{\frac{a}{2}, \delta_1, \delta_2, \delta_3\} > 0$ and assume $\|(\alpha^1, s^1, \psi^1)\|_X, \|(\alpha^2, s^2, \psi^2)\|_X < \delta$. Then this gives

$$\begin{aligned} &|[G^1(\alpha^1, r^1, \theta^1) - G^1(\alpha^2, r^2, \theta^2) - M_{21}(\alpha^1 - \alpha^2) - M_{22}(s^1 - s^2)]_{i,j}| \\ &\leq \left| \sum_{i',j'} [\alpha^1 r_{i',j'}^1 \cos(\theta_{i',j'}^1 - \theta_{i,j}^1) - \alpha^2 r_{i',j'}^2 \cos(\theta_{i',j'}^2 - \theta_{i,j}^2) - (\alpha^1 - \alpha^2)a \cos(\bar{\theta}_{i',j'} - \bar{\theta}_{i,j})] \right| \\ &\quad + \left| \sum_{i',j'} [\alpha^1 r_{i,j}^1 - \alpha^2 r_{i,j}^2 - (\alpha^1 - \alpha^2)a] + |r_{i,j}^1 \lambda(r_{i,j}^1) - r_{i,j}^2 \lambda(r_{i,j}^2) - a \lambda'(a)(s_{i,j}^1 - s_{i,j}^2)| \right| \\ &< 2\varepsilon \sum_{i',j'} \|(\alpha^1, s^1, \psi^1) - (\alpha^2, s^2, \psi^2)\|_X \\ &\quad + \varepsilon \sum_{i',j'} \|(\alpha^1, s^1, \psi^1) - (\alpha^2, s^2, \psi^2)\|_X + \varepsilon \|(\alpha^1, s^1, \psi^1) - (\alpha^2, s^2, \psi^2)\|_X \\ &\leq 8\varepsilon \|(\alpha^1, s^1, \psi^1) - (\alpha^2, s^2, \psi^2)\|_X + 4\varepsilon \|(\alpha^1, s^1, \psi^1) - (\alpha^2, s^2, \psi^2)\|_X \\ &\quad + \varepsilon \|(\alpha^1, s^1, \psi^1) - (\alpha^2, s^2, \psi^2)\|_X \\ &= 13\varepsilon \|(\alpha^1, s^1, \psi^1) - (\alpha^2, s^2, \psi^2)\|_X, \end{aligned} \quad (\text{B.4})$$

²Here we are applying Theorem 5.1.2 to the function $f : \mathbb{R} \rightarrow \mathbb{R}$ acting by $f(x) = x\lambda(x)$ at the point $x = a$.

³Here we are applying Theorem 5.1.2 to the function $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ acting by $f(x_1, x_2) = x_1 x_2$ at the point $(x_1, x_2) = (0, a)$.

where we have used the fact that each element has at most four nearest neighbours. Therefore taking the supremum over all $(i, j) \in \Lambda$ shows that

$$\frac{\|G^1(\alpha^1, r^1, \theta^1) - G^1(\alpha^2, r^2, \theta^2) - M_{21}(\alpha^1 - \alpha^2) - M_{22}(s^1 - s^2)\|_\infty}{\|(\alpha^1, s^1, \psi^1) - (\alpha^2, s^2, \psi^2)\|_X} < 13\varepsilon. \quad (\text{B.5})$$

Since $\varepsilon > 0$ was arbitrary, this quotient can be made arbitrarily small, showing that G^1 is strongly Fréchet differentiable at $(\alpha, s, \psi) = (0, 0, 0)$.

Now to show that this Fréchet derivative is bounded, first we see that for any $\alpha \in \mathbb{R}$ and $(i, j) \in \Lambda$ we have

$$|[M_{21}\alpha]_{i,j}| = |\alpha| \cdot |a \sum_{i',j'} [\cos(\bar{\theta}_{i',j'} - \bar{\theta}_{i,j}) - 1]| \leq 8a|\alpha|, \quad (\text{B.6})$$

where we have again used the fact that each element has at most four nearest-neighbours. Similarly, for M_{22} we have that

$$|[M_{22}s]_{i,j}| = a|\lambda'(a)|s_{i,j} \leq a|\lambda'(a)| \cdot \|s\|_\infty \quad (\text{B.7})$$

for any $s \in \ell^\infty(\Lambda)$ and $(i, j) \in \Lambda$. Putting this all together gives that for any $x = (\alpha, s, \psi) \in X$ we have

$$|[M_{21}\alpha + M_{22}s]_{i,j}| \leq 8a|\alpha| + a|\lambda'(a)| \cdot \|s\|_\infty \leq (8a + a|\lambda'(a)|)\|x\|_X. \quad (\text{B.8})$$

Taking the supremum over $(i, j) \in \Lambda$ gives

$$\|M_{21}\alpha + M_{22}s\|_X \leq a(8 + |\lambda'(a)|)\|x\|_X, \quad (\text{B.9})$$

showing that the Fréchet derivative is bounded. ■

Lemma B.2. $G^2(\alpha, s, \psi) : \mathbb{R} \times B_{\frac{a}{2}}(0) \times c_0(\Lambda) \rightarrow c_0(\Lambda)$ is strongly Fréchet differentiable at $(\alpha, s, \psi) = (0, 0, 0)$. The Fréchet derivative at this point is the bounded linear operator which acts by $(\alpha, s, \psi) \mapsto M_{32}\alpha + M_{33}s$, where M_{32} and M_{33} are as defined in (5.3.18) and (5.3.19), respectively.

Proof: As in the proof of Lemma B.1, we will use the variables $r_{i,j} = a + s_{i,j}$ and $\theta_{i,j} = \bar{\theta}_{i,j} + \psi_{i,j}$ for all $(i, j) \in \Lambda$. We will write $r = \{r_{i,j}\}_{(i,j) \in \Lambda}$ and $\theta = \{\theta_{i,j}\}_{(i,j) \in \Lambda}$ and note that $G^2(\alpha, s, \psi)$ strongly Fréchet differentiable at $(\alpha, s, \psi) = (0, 0, 0)$ is equivalent to $G^2(\alpha, r, \theta)$ strongly Fréchet differentiable at $(\alpha, r, \theta) = (0, \mathbf{a}, \bar{\theta})$.

Begin by taking any $\varepsilon > 0$ and consider $(\alpha^1, s^1, \psi^1), (\alpha^2, s^2, \psi^2) \in \mathbb{R} \times B_{\frac{a}{2}}(0) \times c_0(\Lambda)$. Here r^1 and r^2 correspond to s^1 and s^2 , respectively. Similarly θ^1 and θ^2 correspond to ψ^1 and ψ^2 .

Using the fact that sine is infinitely differentiable on \mathbb{R} and that $a > 0$ to apply Theorem 5.1.2 and find that there exists a $\delta_1 > 0$ such that

$$\begin{aligned}
& \left| \frac{r_{i',j'}^1}{r_{i,j}^1} \sin(\theta_{i',j'}^1 - \theta_{i,j}^1) - \frac{r_{i',j'}^2}{r_{i,j}^2} \sin(\theta_{i',j'}^2 - \theta_{i,j}^2) - \frac{1}{a} (s_{i',j'}^1 - s_{i',j'}^2) \sin(\bar{\theta}_{i',j'} - \bar{\theta}_{i,j}) \right. \\
& \quad \left. - \cos(\bar{\theta}_{i',j'} - \bar{\theta}_{i,j}) [(\psi_{i',j'}^1 - \psi_{i,j}^1) - (\psi_{i',j'}^2 - \psi_{i,j}^2)] + \frac{1}{a} (s_{i,j}^1 - s_{i,j}^2) \sin(\bar{\theta}_{i',j'} - \bar{\theta}_{i,j}) \right| \\
& \leq \varepsilon \max\{|s_{i',j'}^1 - s_{i',j'}^2|, |s_{i,j}^1 - s_{i,j}^2|, |(\psi_{i',j'}^1 - \psi_{i,j}^1) - (\psi_{i',j'}^2 - \psi_{i,j}^2)|\} \\
& \leq \varepsilon \max\{\|s^1 - s^2\|_\infty, |\psi_{i',j'}^1 - \psi_{i',j'}^2| + |\psi_{i,j}^1 - \psi_{i,j}^2|\} \\
& \leq 2\varepsilon \max\{\|s^1 - s^2\|_\infty, \|\psi^1 - \psi^2\|_\infty\} \\
& \leq 2\varepsilon \|(\alpha^1, s^1, \psi^1) - (\alpha^2, s^2, \psi^2)\|_X,
\end{aligned} \tag{B.10}$$

provided $|s_{i',j'}^1|, |s_{i',j'}^2|, |s_{i,j}^1|, |s_{i,j}^2|, |\psi_{i',j'}^1|, |\psi_{i',j'}^2|, |\psi_{i,j}^1|, |\psi_{i,j}^2| < \delta_1$. Therefore, (B.10) holds for every $(i, j) \in \Lambda$ whenever $\|s^1\|_\infty, \|s^2\|_\infty, \|\psi^1\|_\infty, \|\psi^2\|_\infty < \min\{\frac{a}{2}, \delta_1\}$.

Now, recalling that

$$\sum_{i',j'} \sin(\bar{\theta}_{i',j'} - \bar{\theta}_{i,j}) = 0 \tag{B.11}$$

for every $(i, j) \in \Lambda$, we trivially have that

$$\frac{1}{a} (s_{i,j}^1 - s_{i,j}^2) \sum_{i',j'} \sin(\bar{\theta}_{i',j'} - \bar{\theta}_{i,j}) = 0 \tag{B.12}$$

for every $(i, j) \in \Lambda$. Therefore, combining this result with (B.10) above gives that whenever $\|s^1\|_\infty, \|s^2\|_\infty, \|\psi^1\|_\infty, \|\psi^2\|_\infty < \min\{\frac{a}{2}, \delta_1\}$ we have

$$\begin{aligned}
& \left| \sum_{i',j'} \frac{r_{i',j'}^1}{r_{i,j}^1} \sin(\theta_{i',j'}^1 - \theta_{i,j}^1) - \sum_{i',j'} \frac{r_{i',j'}^2}{r_{i,j}^2} \sin(\theta_{i',j'}^2 - \theta_{i,j}^2) - \frac{1}{a} \sum_{i',j'} (s_{i',j'}^1 - s_{i',j'}^2) \sin(\bar{\theta}_{i',j'} - \bar{\theta}_{i,j}) \right. \\
& \quad \left. - \sum_{i',j'} \cos(\bar{\theta}_{i',j'} - \bar{\theta}_{i,j}) [(\psi_{i',j'}^1 - \psi_{i,j}^1) - (\psi_{i',j'}^2 - \psi_{i,j}^2)] \right| \\
& \stackrel{(B.12)}{\leq} \sum_{i',j'} \left| \frac{r_{i',j'}^1}{r_{i,j}^1} \sin(\theta_{i',j'}^1 - \theta_{i,j}^1) - \frac{r_{i',j'}^2}{r_{i,j}^2} \sin(\theta_{i',j'}^2 - \theta_{i,j}^2) - \frac{1}{a} (s_{i',j'}^1 - s_{i',j'}^2) \sin(\bar{\theta}_{i',j'} - \bar{\theta}_{i,j}) \right. \\
& \quad \left. - \cos(\bar{\theta}_{i',j'} - \bar{\theta}_{i,j}) [(\psi_{i',j'}^1 - \psi_{i,j}^1) - (\psi_{i',j'}^2 - \psi_{i,j}^2)] + \frac{1}{a} (s_{i,j}^1 - s_{i,j}^2) \sin(\bar{\theta}_{i',j'} - \bar{\theta}_{i,j}) \right| \\
& \stackrel{(B.10)}{<} 2\varepsilon \sum_{i',j'} \|(\alpha^1, s^1, \psi^1) - (\alpha^2, s^2, \psi^2)\|_X \\
& \leq 8\varepsilon \|(\alpha^1, s^1, \psi^1) - (\alpha^2, s^2, \psi^2)\|_X,
\end{aligned} \tag{B.13}$$

where we have used the fact that each element has at most four nearest-neighbours.

In a similar fashion, we recall that from condition (2) of Hypothesis 1.3.1 we have that ω_1 is continuously differentiable in both its arguments on $[0, \infty) \times \mathbb{R}$. Let us denote

$$K := \partial_1 \omega_1(a + \rho, \alpha) \Big|_{(\rho, \alpha) = (0, 0)}, \quad (\text{B.14})$$

where ∂_k denotes the partial derivative with respect to the k th argument. One should also note that since $\omega_1(a, \alpha) = 0$ for all $\alpha > 0$, we have that

$$\partial_2 \omega_1(a + \rho, \alpha) \Big|_{(\rho, \alpha) = (0, 0)} = 0. \quad (\text{B.15})$$

Now since $a > 0$, Theorem 5.1.2 gives that there exists a $\delta_2 > 0$ such that

$$\begin{aligned} |\omega_1(r_{i,j}^1, \alpha^1) - \omega_1(r_{i,j}^2, \alpha) - K(s_{i,j}^1 - s_{i,j}^2)| &< \varepsilon \max\{|\alpha^1 - \alpha^2|, |r_{i,j}^1 - r_{i,j}^2|\} \\ &\leq \varepsilon \max\{|\alpha^1 - \alpha^2|, \|r^1 - r^2\|_\infty\} \\ &\leq \varepsilon \|(\alpha^1, s^1, \psi^1) - (\alpha^2, s^2, \psi^2)\|_X, \end{aligned} \quad (\text{B.16})$$

for any $|\alpha^1|, |\alpha^2|, |s_{i,j}^1|, |s_{i,j}^2| < \delta_2$. Therefore (B.16) holds for all $(i, j) \in \Lambda$ provided $|\alpha^1|, |\alpha^2|, \|s^1\|_\infty, \|s^2\|_\infty < \min\{\frac{a}{2}, \delta_2\}$.

Let $\delta = \min\{\frac{a}{2}, \delta_1, \delta_2\} > 0$. Then for all $\|(\alpha^1, s^1, \psi^1)\|_X, \|(\alpha^2, s^2, \psi^2)\|_X < \delta$ we get that

$$\begin{aligned} &|[G^2(\alpha^1, r^1, \theta^1) - G^2(\alpha^2, r^2, \theta^2) - M_{32}(s^1 - s^2) - M_{22}(\psi^1 - \psi^2)]_{i,j}| \\ &\leq i^{-1} \left| \sum_{i',j'} \frac{r_{i',j'}^1}{r_{i,j}^1} \sin(\theta_{i',j'}^1 - \theta_{i,j}^1) - \sum_{i',j'} \frac{r_{i',j'}^2}{r_{i,j}^2} \sin(\theta_{i',j'}^2 - \theta_{i,j}^2) \right. \\ &\quad \left. - \frac{1}{a} \sum_{i',j'} (s_{i',j'}^1 - s_{i',j'}^2) \sin(\bar{\theta}_{i',j'} - \bar{\theta}_{i,j}) \right. \\ &\quad \left. - \sum_{i',j'} \cos(\bar{\theta}_{i',j'} - \bar{\theta}_{i,j}) [(\psi_{i',j'}^1 - \psi_{i,j}^1) - (\psi_{i',j'}^2 - \psi_{i,j}^2)] \right| \\ &\quad + i^{-1} |\omega_1(r_{i,j}^1, \alpha^1) - \omega_1(r_{i,j}^2, \alpha) - K(s_{i,j}^1 - s_{i,j}^2)| \\ &\stackrel{(\text{B.13}), (\text{B.16})}{<} 8i^{-1} \varepsilon \|(\alpha^1, s^1, \psi^1) - (\alpha^2, s^2, \psi^2)\|_X + i^{-1} \varepsilon \|(\alpha^1, s^1, \psi^1) - (\alpha^2, s^2, \psi^2)\|_X \\ &= 9 \|(\alpha^1, s^1, \psi^1) - (\alpha^2, s^2, \psi^2)\|_X, \end{aligned} \quad (\text{B.17})$$

since $i \geq 1$. Therefore, taking the supremum over $(i, j) \in \Lambda$ and dividing by $\|(\alpha^1, s^1, \psi^1) - (\alpha^2, s^2, \psi^2)\|_X$ gives the quotient

$$\frac{\|G^2(\alpha^1, r^1, \theta^1) - G^2(\alpha^2, r^2, \theta^2) - M_{32}(s^1 - s^2) - M_{22}(\psi^1 - \psi^2)\|_\infty}{\|(\alpha^1, s^1, \psi^1) - (\alpha^2, s^2, \psi^2)\|_X} < 9\varepsilon. \quad (\text{B.18})$$

Since $\varepsilon > 0$ was arbitrary, this quotient can be made as small as necessary, thus showing that G^2 is strongly Fréchet differentiable at $(\alpha, s, \psi) = (0, 0, 0)$.

To see that this Fréchet derivative is bounded, we first observe that for any $s \in c_0(\Lambda)$ and $(i, j) \in \Lambda$ we have

$$|[M_{32}s]_{i,j}| \leq \frac{1}{a} i^{-1} \sum_{i',j'} |s_{i',j'}| \sin(\bar{\theta}_{i',j'} - \bar{\theta}_{i,j}) \leq \frac{4}{a} \|s\|_\infty, \quad (\text{B.19})$$

where we have used the fact that each element has at most four nearest-neighbours. Similarly, for any $\psi \in c_0(\Lambda)$ and $(i, j) \in \Lambda$ we get

$$|[M_{33}\psi]_{i,j}| \leq i^{-1} \sum_{i',j'} |\cos(\bar{\theta}_{i',j'} - \bar{\theta}_{i,j})| |\psi_{i',j'} - \psi_{i,j}| \leq 2 \sum_{i',j'} \|\psi\|_\infty \leq 8 \|\psi\|_\infty. \quad (\text{B.20})$$

Therefore, for $(\alpha, s, \psi) \in X$ we have

$$|[M_{32}s + M_{33}\psi]_{i,j}| \leq \frac{4}{a} \|s\|_\infty + 8 \|\psi\|_\infty \leq \left(\frac{4}{a} + 8 \right) \|x\|_X. \quad (\text{B.21})$$

Taking the supremum over $(i, j) \in \Lambda$ gives that this Fréchet derivative is bounded. ■

We can now prove Proposition 5.3.2.

Proof: (Proof of Proposition 5.3.2)

Let us fix $\varepsilon > 0$. Then to begin let us note that for any $(\alpha^1, s^1, \psi^1), (\alpha^2, s^2, \psi^2) \in X$ and M as defined in (5.3.15) we have

$$\begin{aligned} & \|G(\alpha^1, s^1, \psi^1) - G(\alpha^2, s^2, \psi^2) - M[(\alpha^1, s^1, \psi^1) - (\alpha^2, s^2, \psi^2)]\|_X \\ &= \max\{0, \|G^1(\alpha^1, s^1, \psi^1) - G^1(\alpha^2, s^2, \psi^2) - M_{21}(\alpha^1 - \alpha^2) - M_{22}(s^1 - s^2)\|_\infty, \\ & \quad \|G^2(\alpha^1, s^1, \psi^1) - G^2(\alpha^2, s^2, \psi^2) - M_{32}(s^1 - s^2) - M_{33}(\psi^1 - \psi^2)\|_\infty\}. \end{aligned} \quad (\text{B.22})$$

Now from Lemma B.1, we have that there exists $\delta_1 > 0$ such that

$$\begin{aligned} & \|G^1(\alpha^1, s^1, \psi^1) - G^1(\alpha^2, s^2, \psi^2) - M_{21}(\alpha^1 - \alpha^2) - M_{22}(s^1 - s^2)\|_\infty \\ & < \varepsilon \|(\alpha^1, s^1, \psi^1) - (\alpha^2, s^2, \psi^2)\|_X \end{aligned} \quad (\text{B.23})$$

for any elements of X satisfying $\|(\alpha^1, s^1, \psi^1)\|, \|(\alpha^2, s^2, \psi^2)\| < \min\{\frac{a}{2}, \delta_1\}$. Similarly, from Lemma B.2 we have that there exists a $\delta_2 > 0$ such that

$$\begin{aligned} & \|G^2(\alpha^1, s^1, \psi^1) - G^2(\alpha^2, s^2, \psi^2) - M_{32}(s^1 - s^2) - M_{33}(\psi^1 - \psi^2)\|_\infty \\ & < \varepsilon \|(\alpha^1, s^1, \psi^1) - (\alpha^2, s^2, \psi^2)\|_X \end{aligned} \quad (\text{B.24})$$

for any elements of X satisfying $\|(\alpha^1, s^1, \psi^1)\|, \|(\alpha^2, s^2, \psi^2)\| < \min\{\frac{a}{2}, \delta_2\}$. Therefore, any pair of elements of X satisfying $\|(\alpha^1, s^1, \psi^1)\|, \|(\alpha^2, s^2, \psi^2)\| < \min\{\frac{a}{2}, \delta_1, \delta_2\}$ we get

$$\begin{aligned} & \|G(\alpha^1, s^1, \psi^1) - G(\alpha^2, s^2, \psi^2) - M[(\alpha^1, s^1, \psi^1) - (\alpha^2, s^2, \psi^2)]\|_X \\ & < \varepsilon \max\{0, \|(\alpha^1, s^1, \psi^1) - (\alpha^2, s^2, \psi^2)\|_X\} \\ & \leq \varepsilon \|(\alpha^1, s^1, \psi^1) - (\alpha^2, s^2, \psi^2)\|_X. \end{aligned} \tag{B.25}$$

Since $\varepsilon > 0$ was arbitrary, we have shown that G is strongly Fréchet differentiable at $(\alpha, s, \psi) = (0, 0, 0)$. Boundedness of $M : X \rightarrow X$ is a trivial consequence of Lemmas B.1 and B.2, thus completing the proof. \blacksquare

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