Free Vibration Analysis Of Continuous Orthotropic Plates And Bridge Decks

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Vibhu Kaul

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ABSTRACT

Numerical methods are widely used for the solution of complex differential equations for which exact solutions are not possible. The series solution, although known to be very rapidly converging was applied in the past only for the free vibration analysis of rectangular plates, satisfying the requirements that at least two opposite sides of the plate are simply-supported. Although Gorman [14] extended the application of the series solution to cases where this constraint no longer applied, his treatment was restricted to the the free vibration analysis of single span isotropic rectangular plates.

In the method of superposition as it applies to free vibration problems of plates, the plate element under consideration is broken into more basic units, referred to as the building blocks, for which the solution can be either directly obtained or inferred from a Levy type solution.

In the present study, a series solution is formulated for the free vibration analysis of continuous orthotropic plates with "bridge type" boundary conditions. The solution is tested for convergence by varying the number of terms used in the solution and the convergence is found to be very rapid. The equations obtained in the eigenvalue problem are solved using the IMSL routine LINV3F. Tables covering a wide range of values in terms of dimensionless parameters

have been generated for ready reference and for a quick estimation of the natural frequencies of vibration of continuous rectangular plates. Whenever possible results of the present method are compared with available data in technical literature and close agreements are found.

A very rapidly converging solution, the generality of the present method and the ability to use the solutions of the building blocks developed for other problems are its obvious advantages over other numerical methods generally restricted to problems with specific boundary conditions.

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NOMENCLATURE

2a	Dimension of the plate in the x direction.
Am, Bm, Cm, Dm	Constants to be determined from the boundary conditions.
2b	Dimension of the plate in the y direction.
Ca	Rigidity paramater $b/a(Dx/Dy)^{1/4}$
D_1	Coupling rigidity of orthotropic plate
Dx, Dy	Flexural rigidity of the orthotropic plate in the x and y directions.
Dxy	Torsional rigidity of the orthotropic plate.
E_x, E_y	Elastic modulus of the plate in the x and y direction.
Gxy	Shear modulus.
H_1	Torsional Rigidity
m, n	No. of terms in series solution.
Ra	Frequency ratio.
Rb	Frequency ratio.
t	Time
W	Deflection function of the plate.
w_1	Deflection function of the first building block.
w_2	Deflection function of the second building block.
x, y	Rectangular cartesian coordinates.
λ_b^2	Plate eigenvalue = $\omega b^2\sqrt{(\rho_0/D_y)}$
ρ_0	Mass density per unit area of plate.
ψ_m ψ_n	Frequency coefficients

ϕ_m ϕ_n	Frequency coefficients
ω	Circular frequency of plate vibration.
ϕ	Plate aspect ratio = b/a
ϕ_1	Inverse aspect ratio = a/b.
$\nu_x \nu_y$	Poissons ratio of orthotropic plate in x and y directions.
ξ	Dimensionless plate coordinate = x/a
η	Dimensionless plate coordinate = y/b.
μ	Torsion paramater $H_1/\sqrt{D_x D_y}$

Chapter I

INTRODUCTION

Structurally orthotropic plates are increasingly being utilized as components of large scale structures such as ship bottoms, auditoriums and building floor systems. Also orthotropic materials such as fibre reinforced plates and shells are now frequently being employed in the aerospace and nuclear industry. This has been primarily due to attempts by designers to distribute structural rigidities in different directions, according to functional requirements, resulting in a better utilization of the material.

The orthotropic plate theory also finds important applications in the static and dynamic analysis of highway bridges. Often, not only must the designer ensure that structures withstand the applied static loading but also ensure that excessive vibrations due to resonance caused by the matching of the frequency of the driving periodic force and the plate's natural frequency does not occur. Hence in order to evaluate the dynamic response, a study of the vibration characteristics of the structure is essential.

Modern trends towards slender aesthetic bridges, resulting from higher material strength and more precise computations are increasingly yielding lighter bridge slabs with little reserve strength, which in turn demand a precise determination of the natural frequencies for the dynamic analysis of such slabs.

While free vibration characteristics of square and rectangular plates with various boundary conditions have been investigated by a number of authors, [44] [30] [24] [18] the problem of free vibration of continuous plates has not received nearly as much attention. This is especially true of continuous orthotropic plates with "bridge type" boundary conditions (Figure 5). Analytical solutions to the governing differential equation are difficult to obtain and hence researchers have to frequently resort to numerical techniques and indirect solutions. Literature on free vibration of continuous plates has been generally limited to simply supported and clamped boundary conditions. This is probably due to the relative ease of assumed functions satisfying the boundary conditions for these types of supports, for the various numerical techniques adopted by previous researchers.

In the present study a series solution is formulated for the free vibration analysis of continuous orthotropic plates with "bridge type" boundary conditions. Levy [26] proposed a single Fourier series solution to problem of bending of rectangular plates, having at least two opposite edges simply supported and various boundary conditions along the remaining edges. The series is capable of representing the plate displacement function to any desired degree of accuracy by taking up sufficient terms. Gorman [14] introduced the method of superposition for obtaining accurate solutions to classical plate vibration problems. Levy's solution required that at least two opposite sides be simply supported. In his method Gorman analyzed single span isotropic plates with any prescribed boundary conditions by exploiting the method of superposition. The method of superposition consists essentially of judiciously choosing forced vibration problems, for which solutions of the Levy type can be obtained. The known solutions are then superimposed and parameters appearing in their boundary formulations are so adjusted that the combined solutions satisfy the prescribed boundary conditions of the plate. The solution exactly satisfies the differential equation and the boundary conditions are subsequently satisfied to any degree of accuracy by superimposition.

The solution presented here extends the Levy type series solution to solve continuous orthotropic plate problems, where the requirement of at least two opposite sides being simply supported need not be satisfied. The treatment in this thesis is restricted to equal span continuous orthotropic plates with "bridge type" boundary conditions.

The symmetry of the problem is taken into account in formulating the solution.

1.1 OBJECT AND SCOPE

The main objective of this thesis is to formulate a general series solution for the problem of free vibration of continuous orthotropic plates. The convergence characteristics of the developed solution are examined and after the validity of the method has been established, tables for eigenvalues of continuous orthotropic plates covering a large range of parameters in dimensionless form are generated for ready reference.

1.2 Outline of the thesis

Since the main objective of this thesis is to formulate the series solution to free vibration of continuous orthotropic plates, existing literature on vibration of simple span and continuous plates is briefly reviewed in chapter II. Chapter III is devoted to listing of the assumptions and to formulate equations in non-dimensional form for various types of boundary conditions, edge moments and vertical edge reactions for such plates. A Levy type solution is developed to the governing differential equation for two cases; case 1 corresponds to a Levy type solution with trigonometric functions running in the ξ direction, while in case 2 the solution is developed for trigonometric functions running in the η direction. These solutions are later utilized to obtain solutions to the building blocks as developed in subsequent chapters.

Chapter IV outlines the analysis for doubly symmeteric modes of vibration. The two building blocks, representing components of the desired solution are developed and the formulation of the eigenvalue matrix leading to the extraction of the required eigenvalues is discussed. The equations are solved using the IMSL routine LINV3F.

In Chapter V, the antisymmetric - symmetric modes of vibration are discussed. The formulation follows steps similar to those in chapter IV and the final solution is obtained by superimposing the building blocks.

Chapter VI is devoted to doubly symmetric and symmetric - antisymmetric modes of vibration and the governing eigenvalue equation is developed in each case.

Finally in the last chapter, results of the present study are discussed and the conclusions drawn from them are summarized.

All computations are programmed in FORTRAN IV for the Amdahl 470. The program listings are included in the appendix.

Chapter II

LITERATURE REVIEW.

Investigations on the dynamic behaviour of single span bridge decks were made by many authors including Inglis [19] , Hillerborg [17] and Biggs [3]. Generally the bridge deck was assumed to behave as a beam and obviously such an assumption is valid only for relatively long span bridges.

Louw [27] proposed an approximate method by which the frequencies and corresponding peak dynamic deflections of simple span highway bridges under the action of a two axle vehicle could be predicted. In the analysis, again the bridge was idealized as a prismatic beam.

The governing differential equation of orthotropic plates subject to pure bending is attributed to Huber and its development was presented in detail by Leissa [24] in 1969. The problem of free vibrations of rectangular plates was investigated by Warburton [44] who obtained approximate frequency equations for isotropic plates with various boundary conditions by using beam functions in two orthogonal directions. Laura [23] proposed a method of determining the fundamental frequency of orthotropic plates of polygonal boundary shape, employing the Rayleigh - Ritz method, while Knazawa and Kawai [20] studied the orthotropic plate problem for several edge conditions by the integral equation method.

Huffington and Hoppmann [18] determined the exact frequencies and modal eigenfunctions for single span rectangular plates with "bridge type" boundary conditions by direct solution of the governing differential equation of motion using the Levy approach.

Hearmon [15] and Rajappa [30] gave frequency equations for orthotropic plates with simply supported or clamped edges. Yamada and Veletsos [43] studied the vibration of single span I beam and slab bridge decks treating the slab as continuous over a series of flexible beams using the Rayleigh - Ritz method. The free vibration problem idealizing the structure as an equivalent orthotropic plate was also solved by them using the same method.

The natural frequencies of continuous plates hinged along two opposite edges were determined by Veletsos and Newmark [42]. The formulation is, however, only valid for isotropic panels and is restricted by the fact that the available tables of stiffness and carry - over factors needed for the analysis cover only a small range of values.

Elishakoff [13] solved the eigenvalues of rectangular isotropic plates continuous over rigid supports at regular intervals with an arbitrary number of spans, using an approximate analytical technique. He solved for all possible combinations of simply supported and clamped boundary conditions.

Durvasula [12] applied the partition method to the vibration problem of plates, by expressing the solution of the governing differential equation in terms of functions satisfying the boundary conditions and setting to zero the error in the differential equation over a partitioned domain. The application of the method was discussed with special reference to clamped boundary conditions.

Nagaya [28] [29] studied the transient response of a elastically supported continuous plate. The solution was obtained from the equation of motion in terms of "unknown" external forces, equivalent to the actual restoring forces of the elastic support, using the Laplace transform method.

Vibration characteristics of rectangular plates continuous over intermediate rigid supports and simply supported along two opposite edges with simply supported and or clamped end conditions were calculated by Azimi [1] using the Receptance method.

Sakata [33] [34] [35] [36] demonstrated that the natural frequency of an orthotropic continuous plate could be estimated from that of an isotropic continuous plate with the same boundary conditions by using the reduction formulae derived by him. He also used this method to solve for the buckling force in the plate. His treatment was restricted to simply supported and elastically restrained edges.

Other methods used in the free vibration analysis of orthotropic plates are listed below:

2.1 Series Solution

The Fourier series method proves to be extremely powerful for the case of rectangular plates with simply supported edges. In 1820 Navier presented a paper to the French Academy Of Sciences on the solution of small deflection problem of simply supported rectangular plates by the double Fourier series method. Levy [26] suggested a single Fourier series solution to problems of rectangular plates that have at least two opposite edges simply supported. The proposed solution is very rapidly converging with a one term solution yielding deflection correct to three significant places [39]. Gorman [14] undertook a comprehensive free vibration analysis of single span isotropic rectangular plates with various boundary conditions by exploiting the method of superposition. He also solved problems of point supported and diagonally supported rectangular plates.

2.2 Ritz Method.

Lord Rayleigh [31] in his classic work gave a method for approximating the frequencies of dynamic systems. Later W. Ritz [32] gave what is called the Rayleigh - Ritz method for approximating frequencies in dynamic systems. The Rayleigh - Ritz solution to vibration problems was obtained by Young [45] for square plates with combinations of free and clamped edges, and by Hearmon [16] for orthotropic rectangular plates with simply supported and clamped conditions.

2.3 Finite Strip method

The finite strip method pioneered by Cheung [5] was also applied by Cheung and Reddy [4] to the solution of the problem of frequency analysis of single and continuous span bridges.

In the finite strip method the bridge deck is assumed as an assemblage of parallel orthotropic strips each of which may have orthotropic properties which differ between strips but are constant within each strip. A displacement function of the form X(x) Y(y) is chosen, where Y(y) is a characteristic beam function that satisfies the end conditions of the strip and X(x) is a simple polynomial given in terms of the displacement parameters at the two adjacent node lines. Compatibility between two adjacent strips is ensured as the function defines the displacements and its first derivatives uniquely along the interface by the common nodal displacements parameters. By using a function of this type two dimensional problems are reduced to one dimensional ones. Consequently the size of resulting matrices is considerably reduced and this is advantageous for programming on a small computer. The finite strip method was used by Cheung [5] to solve vibration problems of continuous plates using single span beam functions while plates continuous in one or two directions were studied by Wu and Cheung [6] using the finite strip method with continuous beam functions. Smith [37] used this method to analyze the dynamic response of beam and slab type highway bridges.

2.4 Finite Element method

The advent of high speed computers during the last two decades has led to a growing popularity of the method. The versatility and generality of this form of solution, obtained without solving the governing differential equation directly are some of its main advantages, although the required computer storage may sometimes be a limiting factor. Dawe [11] solved the case of vibration of rectangular plates having various edge conditions, using the finite element method.

Chapter III

SOLUTIONS TO THE GOVERNING DIFFERENTIAL EQUATION OF AN ORTHOTROPIC PLATE

3.1 Introduction

The differential equation governing the behaviour of orthotropic plates is attributed to Huber and a detailed treatment is given by Triotsky [41]

For ease of computation the governing differential equation is first transformed to dimensionless coordinates and is then solved for two cases ; Case 1 represents the solution to the governing differential equation when the opposite edges corresponding to $\xi=0$ and $\xi=1$ are simply supported and case 2 represents the case of two opposite edges $\eta=0$ and $\eta=1$, being simply supported. These solutions are utilized in the chapters to follow while formulating the building blocks leading to the solution of the free vibration problem.

3.2 Basic assumptions

Assumptions: The following are the assumptions made about the equivalent orthotropic plate.

1. The deflection of the plate is small and does not exceed one fifth of the plate thickness.
2. The plate material is perfectly elastic, continuous, homogeneous, obeys Hooke's law and possesses different elastic properties in orthogonal directions.
3. Thickness of the plate is small in comparison to the dimensions of the plate.
4. There is no deformation of the middle plane of the plate; i.e; the middle plane of the plate remains unstretched during bending.
5. Linear elements perpendicular to the middle plane of the plate before bending remain straight and normal to the deflected surface of the plate after bending.
6. The normal stresses transverse to the plane of the plate are neglected.

3.3 Boundary Conditions

Equations for boundary conditions, distributed bending moments and vertical edge reactions are developed in this section. The equations are presented in the cartesian and dimensionless coordinate systems.

3.3.1 Formulation of boundary conditions

Mathematical formulation of the plate boundary conditions is carried out in reference to Figures 2,3 and 4.

Simply supported edges

Cartesian Coordinates

$$w(x,y) = \frac{\partial^2 w(x,y)}{\partial x^2} = 0 \qquad 3.3.1$$

Dimensionless Coordinates

$$w(\xi,\eta) = \frac{\partial^2 w(\xi,\eta)}{\partial \xi^2} = 0 \qquad 3.3.2$$

Clamped Edges

Cartesian Coordinates

$$w(x,y) = \frac{\partial w(x,y)}{\partial x} = 0 \qquad 3.3.3$$

Dimensionless Coordinates

$$w(\xi,\eta) = \frac{\partial w(\xi,\eta)}{\partial \xi} = 0 \qquad 3.3.4$$

Free Edges

Cartesian Coordinates ; edge x=a

$$D_x\left[\frac{\partial^2 w(x,y)}{\partial x^2}+\nu_y\frac{\partial^2 w(x,y)}{\partial y^2}\right]=0 \qquad 3.3.5$$

$$D_x\left[\frac{\partial^3 w(x,y)}{\partial x^3}+\frac{(2H_1-D_1)}{D_x}\frac{\partial^3 w(x,y)}{\partial x\partial y^2}\right]=0 \qquad 3.3.6$$

Cartesian Coordinates ; edge y=b

$$D_y\left[\frac{\partial^2 w(x,y)}{\partial y^2}+\nu_x\frac{\partial^2 w(x,y)}{\partial x^2}\right]=0 \qquad 3.3.7$$

$$D_y\left[\frac{\partial^3 w(\xi,\eta)}{\partial y^3}+\frac{(2H_1-D_1)}{D_y}\frac{\partial^3 w}{\partial y\partial x^2}\right]=0 \qquad 3.3.8$$

Dimensionless Coordinates edge $\eta=1$

$$D_x\left[\frac{\partial^2 w(\xi,\eta)}{\partial \xi^2}+\frac{\nu_y}{\phi^2}\frac{\partial^2 w(\xi,\eta)}{\partial \eta^2}\right]=0 \qquad 3.3.9$$

$$D_x\left[\frac{\partial^3 w(\xi,\eta)}{\partial \xi^3}+\frac{(2H_1-D_1)}{D_x\phi^2}\frac{\partial^3 w(\xi,\eta)}{\partial \xi\partial \eta^2}\right]=0 \qquad 3.3.10$$

Dimensionless Coordinates edge $\xi=1$

$$D_y\left[\frac{\partial^2 w(\xi,\eta)}{\partial \eta^2}+\frac{\nu_x}{\phi^2}\frac{\partial^2 w(\xi,\eta)}{\partial \xi^2}\right]=0 \qquad 3.3.11$$

$$D_y\left[\frac{\partial^3 w(\xi,\eta)}{\partial \eta^3}+\frac{(2H_1-D_1)\phi^2}{D_y}\frac{\partial^3 w(\xi,\eta)}{\partial \xi^2\partial \eta}\right]=0 \qquad 3.3.12$$

3.3.2 Formulation of Distributed Bending Moment

Cartesian Coordinates

$$M_x = -D_x \left[\frac{\partial^2 w}{\partial x^2} + \nu_y \frac{\partial^2 w}{\partial y^2} \right] \qquad 3.3.13$$

$$M_y = -D_y \left[\frac{\partial^2 w}{\partial y^2} + \nu_x \frac{\partial^2 w}{\partial x^2} \right] \qquad 3.3.14$$

Dimensionless Coordinates

$$\frac{M_\xi a}{D_x} = -\left[\frac{\partial^2 w(\xi,\eta)}{\partial \xi^2} + \frac{\nu_y}{\phi^2} \frac{\partial^2 w(\xi,\eta)}{\partial \eta^2} \right] \qquad 3.3.15$$

$$\frac{M_\eta \phi b}{D_y} = -\left[\frac{\partial^2 w(\xi,\eta)}{\partial \eta^2} + \nu_x \phi^2 \frac{\partial^2 w(\xi,\eta)}{\partial \xi^2} \right] \qquad 3.3.16$$

3.3.3 Formulation of Vertical edge reaction.

Cartesian Coordinates

$$V_x = -D_x \left[\frac{\partial^3 w(x,y)}{\partial x^3} + \frac{(2H_1 - D_1)}{D_x} \frac{\partial^3 w}{\partial x \partial y^2} \right] \qquad 3.3.17$$

$$V_y = -D_y \left[\frac{\partial^3 w(x,y)}{\partial y^3} + \frac{(2H_1 - D_1)}{D_y} \frac{\partial^3 w}{\partial y \partial x^2} \right] \qquad 3.3.18$$

Dimensionless Coordinates

$$\frac{V_\xi a^2}{D_x} = -\left[\frac{\partial^3 w(\xi,\eta)}{\partial \xi^3} + \frac{(2H_1 - D_1)}{D_x}\frac{\partial^3 w(\xi,\eta)}{\partial \xi \partial \eta^2}\right] \qquad 3.3.19$$

$$\frac{V_\eta \phi b^2}{D_y} = -\left[\frac{\partial^3 w(\xi,\eta)}{\partial \eta^3} + \frac{(2H_1 - D_1)}{D_y}\phi^2 \frac{\partial^3 w(\xi,\eta)}{\partial \eta \partial \xi^2}\right] \qquad 3.3.20$$

3.4 Solutions to the differential equation

The governing differential equation for the transverse free vibrations of a homogeneous orthotropic plate is obtained by replacing the load term in the differential equation for applied static loading by the inertia force term $\rho_0 \partial w/\partial t^2$ wherein the term ρ_0 is the mass per unit area of the plate, and is given by.

$$\frac{D_x \partial^4 w}{\partial x^4} + \frac{2H_1 \partial^4 w}{\partial x^2 \partial y^2} + \frac{D_y \partial^4 w}{\partial y^4} + \frac{\rho_0 \partial^2 w}{\partial t^2} = 0 \qquad 3.4.1$$

where

$$D_x = \frac{E_x h^3}{12(1 - \nu_x \nu_y)}$$

$$D_y = \frac{E_y h^3}{12(1 - \nu_x \nu_y)}$$

$$D_{xy} = \frac{G_{xy} h^3}{G}$$

$$2H_1 = D_x \nu_y + \nu_x D_y + 4D_{xy}$$

Using w(x,y,t)=w(x,y).T(t) in Equation 3.4.1 above yields the following equation.

$$\frac{D_x\partial^4 w}{\partial x^4}+\frac{2H_1\partial^4 w}{\partial x^2\partial y^2}+\frac{D_y\partial^4 w}{\partial y^4}-\omega^2\rho_0 w=0 \qquad 3.4.2$$

where ω represents the circular frequency of the system.

Equation 3.4.2 is made dimensionless with respect to side b and is transformed to dimensionless coordinates ξ and η corresponding to the cartesian coordinates x and y yielding

$$\frac{bD_x}{a^4}\frac{\partial^4 w(x,y)/b}{\partial(x/a)^4}+\frac{2H_1 b}{a^2b^2}\frac{\partial^4 w(x,y)/b}{\partial(x/a)^2\partial(y/b)^2}$$
$$+\frac{D_y b}{b^4}\frac{\partial^4 w(x,y)/b}{\partial(y/b)^4}-b\omega^2\rho_0 w(x,y)/b=0 \qquad 3.4.3$$

Introducing the dimensionless coordinates $\xi=x/a$ and $\eta=y/b$ we get

$$D_y\frac{\partial^4 w(\xi,\eta)}{\partial\eta^4}+\frac{2H_1}{\phi_1^2}\frac{\partial^4 w(\xi,\eta)}{\partial\xi^2\partial\eta^2}+\frac{D_x}{\phi_1^4}\frac{\partial^4 w(\xi,\eta)}{\partial\xi^4}-\frac{a^4}{\phi_1^4}\rho_0\omega^2 w(\xi,\eta)=0$$

Substituting $\phi=1/\phi_1$ in the above equation yields

$$\frac{\partial^4 w(\xi,\eta)}{\partial\eta^4}+\frac{2H_1\phi^2}{D_y}\frac{\partial^4 w(\xi,\eta)}{\partial\xi^2\partial\eta^2}+\frac{D_x\phi^4}{D_y}\frac{\partial^4 w(\xi,\eta)}{\partial\xi^4}-\frac{\phi^4a^4\rho_0\omega^2 w(\xi,\eta)}{D_y}=0 \qquad 3.4.4$$

The above equation is solved for two cases corresponding to two opposite sides being simply supported ; Case 1 representing the case when the two opposite edges corresponding to ξ=0 and ξ=1 are simply supported Case 2 being the case when the two opposite edges corresponding to η=0 and η=1 are simply supported.

Case 1. Two opposite sides ξ =0 and ξ=1 being simply supported.

The deflection function as suggested by Levy [26] is chosen to be of the form

$$w(\xi,\eta) = \sum_{m=1}^{\infty} Y_m(\eta) Sin(m\pi\xi) \quad 3.4.5$$

This series solution involves the Fourier trigonometric function Sin($m\pi\xi$) and the coefficients Y_m are functions of the variable η. The series is capable of representing the plate displacement function to any desired degree of accuracy, if the number of terms in the series is made sufficiently large. Substitution of the above equation into the governing differential equation (3.4.4) yields the following characteristic equation.

$$Y_m(\eta)^{IV} - \frac{2H_1\phi^2}{D_y}(m\pi)^2 Y_m(\eta)^{II} + \phi^4\left[D_x/D_y(m\pi)^4 - a^4\omega^2\rho_0/D_y\right]Y_m(\eta) = 0 \quad 3.4.6$$

Roots of the above ordinary fourth order partial differential equation with constant coefficients are

$$\pm\phi(m\pi)\left(\frac{D_x}{D_y}\right)^{\frac{1}{4}}\left[\frac{H_1}{\sqrt{D_xD_y}} \pm \left[\frac{H_1^2}{D_xD_y} - 1 + \frac{\omega^2}{\frac{(m\pi)^4 D_x}{a^4\rho_0}}\right]^{\frac{1}{2}}\right]^{\frac{1}{2}} \quad 3.4.7$$

Introducing

$$R_a^2 = \frac{\omega^2}{\frac{(m\pi)^4}{a^4}\frac{D_x}{\rho_0}} \qquad 3.4.8$$

$$= \frac{\lambda_b^4}{(m\pi)^4}\frac{1}{D_x/D_y}\phi_1^4 \quad \text{where} \quad \lambda_b^4 = \omega^2 b^4 \frac{\rho_0}{D_y}$$

The roots of equation 3.4.6 can now be written as

$$\pm m\pi C_a\left[\mu \pm \left[\mu^2 - 1 + R_a^2\right]^{1/2}\right]^{1/2}$$

where

$$\mu = \frac{H_1}{\sqrt{D_x D_y}} \qquad 3.4.10$$

and

$$C_a = \phi\left(\frac{D_x}{D_y}\right)^{1/4}$$

Depending upon the sign of the last term $D_x/D_y(m\pi)^4 - a^4\omega^2\rho_0/D_y$ in equation 3.4.6, the general solution of the governing differential equation 3.4.4 is given by

for $\quad \frac{D_x}{D_y}(m\pi^4) > \frac{a^4\omega^2\rho_0}{D_y}$

$$Y_m(\eta) = A_m Cosh\psi_m\eta + B_m Sinh\psi_m\eta + C_m Sinh\phi_m\eta + D_m Cosh\phi_m\eta \qquad 3.4.11$$

and $\quad \frac{D_x}{D_y}(m\pi^4) < \frac{a^4\omega^2\rho_0}{D_y}$

$$Y_m(\eta) = A_m Cosh_m \ \eta + B_m Sinh\psi_m\eta + C_m Sin\phi_m\eta + D_m Cos\phi_m\eta \qquad 3.4.12$$

where

$$\psi_m^2 = (m\pi C_a)^2\left[\left[\mu^2 - 1 + R_a^2\right]^{1/2} + \mu\right]$$

$$\phi_m^2 = (m\pi C_a)^2\left[\left[\mu^2 - 1 + R_a^2\right]^{1/2} - \mu\right]$$

or $= (m\pi C_a)^2\left[\mu - \left[\mu^2 - 1 + R_a^2\right]^{1/2}\right]$ whichever is positive.

A_m , B_m , C_m , D_m are constants depending upon the boundary conditions to be imposed on the solution.

Case 2. Two opposite sides η=0 and η=1, being simply supported.

For solutions with trigonometric functions running in the η direction, w is of the form

$$w(\xi,\eta) = \sum_{n=1}^{\infty} Y_n(\eta) Sin(n\pi\eta)$$

yielding the characteristic equation

$$Y_n(\xi)^{IV} - \frac{2H_1}{\phi^2}\frac{(n\pi)^2}{D_x} Y_n(\xi)^{II} + \frac{1}{\phi^4}\left[D_x/D_y(n\pi)^4 - \phi^4 a^4 \rho_0 \omega^2/D_x\right] Y_n(\xi) = 0 \qquad 3.4.13$$

Roots of the above characteristic equation are

$$\frac{\pm n\pi}{\phi}\left(\frac{D_x}{D_y}\right)^{\frac{1}{4}}\left[\frac{H_1}{\sqrt{D_x D_y}} \pm \left[\frac{H_1^2}{D_x D_y} - 1 + \frac{\phi^4\omega^2}{\frac{(n\pi)^4 D_y}{a^4\rho_0}}\right]^{\frac{1}{2}}\right]^{\frac{1}{2}} \qquad 3.4.14$$

Introducing

$$R_b^2 = \frac{\phi^4\omega^2}{\frac{(n\pi)^4 D_y}{a^4\rho_0}} \qquad 3.4.15$$

$$= \lambda_b^4 \frac{1}{(n\pi)^4} \qquad \text{where} \quad \lambda_b^4 = \omega^2 b^4 \frac{\rho_0}{D_y}$$

Equation 3.4.14 giving the roots of the characteristic equation can now be written as:

$$\frac{\pm n\pi}{\phi} C_a \left[\mu \pm \left[\mu^2 - 1 + R_b^2\right]^{1/2}\right]^{1/2} \qquad 3.4.16$$

where

$$\mu = \frac{H_1}{\sqrt{D_x D_y}}$$

and

$$C_a = \phi\left(\frac{D_x}{D_y}\right)^{\frac{1}{4}}$$

The general solution in this case depends upon the sign of the last term $D_y/D_x(n\pi)^4 - \phi^4 a^4 \rho_0 \omega^2/D_x$. and is given as

$$Y_n(\xi) = A_n Cosh\psi_n\eta + B_n Sinh\psi_n\eta + C_n Sin\ \phi_n\eta + D_n Cos\phi_n\eta \qquad 3.4.17$$

for $\frac{D_y}{D_x}(n\pi)^4 < \frac{\phi^4 a^4 \rho_0 \omega^2}{D_x}$

and as

$$Y_n(\xi) = A_n Cosh\psi_n\eta + B_n Sinh\psi_n\eta + C_n Sinh\phi_n\eta + D_n Cosh\phi_n\eta \qquad 3.4.18$$

for

$$\frac{D_y}{D_x}(n\pi)^4 > \frac{\phi^4 a^4 \rho_0 \omega^2}{D_x}$$

where

$$\psi_n^2 = \frac{(n\pi)^2}{C_a^2}\left[\left[\mu^2 - 1 + R_b^2\right]^{1/2} + \mu\right]$$

$$\phi_n^2 = \frac{(n\pi)^2}{C_a^2}\left[\left[\mu^2 - 1 + R_b^2\right]^{1/2} - \mu\right]$$

or $= \frac{(n\pi)^2}{C_a^2}\left[\mu - \left[\mu^2 - 1 + R_b^2\right]^{1/2}\right]$ whichever is positive.

The formulation of the theoretical model is simplified by taking advantage of symmetry. Symmetry dictates that all modes of vibration must be either symmetric or anti-symmetric about the co-ordinate axes, if they are suitably chosen. The study of free vibration modes hence consists of the following modes of vibration.

1. Doubly symmetric modes
2. Modes anti-symmetric about both the axes.
3. Modes symmetric about the ξ axis and anti-symmetric about the η axis.

4. Modes anti-symmetric about the ξ axis and symmetric about the η axis.

Chapter IV

DOUBLY SYMMETRIC MODES

In order to determine the doubly symmetric modes only a quarter segment of the continuous plate need be analyzed, as depicted in Figure 6. In order to analyze the plate segment shown we have to superimpose the solutions corresponding to the building blocks (1) and (2) and then adjust the constants appearing in the boundary formulation so that their combination provides boundary conditions that satisfy the requirements of the plate under consideration. Slip shear conditions exist along the edges of the plate undergoing symmetric vibration modes thereby implying that the plate has zero vertical edge reaction and the slope of the plate normal to the edge is zero.

4.1 Solution for the first building block

For the first building block, using a Levy type solution

$$w_1(\xi,\eta) = \sum_{m=1,3,5}^{k^*} [A_m Cosh\,\psi_m\eta + D_m Cos\phi_m\eta]Sin(m\pi\xi) \quad +$$

$$\sum_{m=k^*+2}^{\infty} [A_m Cosh\psi_m\eta + D_m Cosh\phi_m\eta] Sin(m\pi\xi) \qquad 4.1.1$$

Antisymmetric terms about the origin are deleted in order to satisfy the boundary conditions along the edge ξ=0. The first summation ($m=k^*$) includes terms for which $D_x/D_y(m\pi^4) > a^4\omega^2\rho_0/D_y$ and the second summation includes terms for which $D_x/D_y(m\pi^4) < a^4\omega^2\rho_0/D_y$. Henceforth these two cases are referred to as case a and case b respectively and so are the corresponding equations. Slip shear conditions are to be enforced on the edge η=0 and we have simply supported conditions at the edges ξ=0 and ξ=1. The remaining edge η=1 is to have zero vertical edge reaction but is acted upon by a distributed moment of magnitude $M_1(\xi)$ which varies cyclically with time and has the same frequency as the plate. Enforcing zero vertical edge reaction along the edge η=1 we have

$$D_y \frac{\partial^3 w_1(\xi,\eta)}{\partial\eta^3} + \frac{\phi^2(2H_1 - D_1)\partial^3 w_1(\xi,\eta)}{\partial\xi^2\partial\eta} = 0 \qquad 4.1.2$$

yielding

$$D_y\left[A_m\psi_m^3 Sinh\psi_m\eta + D_m\phi_m^2 Sin\phi_m\eta\right]$$

$$+\phi^2[2H_1 - D_1][-A_m\psi_m Sinh\psi_m\eta + D_m\phi_m Sin\phi_m\eta](m\pi)^2 = 0 \qquad (4.1.3)$$

Hence

$$A_m\psi_m Sinh\psi_m\left[\psi_m^2 - \phi^2\frac{(2H_1 - D_1)}{D_y}(m\pi)^2\right] +$$

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$$D_m \phi_m Sin\phi_m \left[\phi_m^2 - \phi^2 \frac{(2H_1 - D_1)}{D_y} (m\pi)^2 \right] = 0 \qquad 4.1.4a$$

and using the companion equation developed from the second summation of equation 4.1 we get

$$A_m \phi_m Sinh\psi_m \left[\psi_m^2 - \phi^2 \frac{(2H_1 - D_1)}{D_y} (m\pi)^2 \right]$$

$$+ D_m \phi_m Sinh\phi_m \left[\phi_m^2 + \phi^2 \frac{(2H_1 - D_1)}{D_y} (m\pi)^2 \right] = 0 \qquad 4.1.4b$$

Hence the bracketed quantities of the summation of equation 4.1.1 can be written as

$$Y_m(\eta) = A_m \left[Cosh\psi_m \eta + \theta_{1m} Cos\phi_m \eta \right] \qquad 4.1.5a$$

or

$$Y_m(\eta) = A_m \left[Cosh\psi_m \eta + \theta_{2m} Cosh\phi_m \eta \right] \qquad 4.1.5b$$

respectively, where

$$\theta_{1m} = \frac{\left[\phi^2 (2H_1 - D_1)/D_y (m\pi)^2 - \psi_m^2 \right] \psi_m Sinh \;\; _m}{\left[\phi^2 (2H_1 - D_1)/D_y (m\pi)^2 - \phi_m^2 \right] \phi_m Sinh\phi_m} \qquad 4.1.6a$$

$$\theta_{2m} = \frac{\left[\phi^2 (2H_1 - D_1)/D_y (m\pi)^2 - \psi_m^2 \right] \psi_m Sinh \;\; _m}{\left[\phi_m^2 - \phi^2 (2H_1 - D_1)/D_y (m\pi)^2 \right] \phi_m Sinh\phi_m} \qquad 4.1.6b$$

Enforcing moment equilibrium along the edge η=1 we have

$$\frac{\phi b M_1(\xi)}{D_y} = - \left[\frac{\partial^2 w_1(\xi,\eta)}{\partial \eta^2} + \frac{D_1}{D_y} \phi^2 \frac{\partial^2 w_1(\xi,\eta)}{\partial \xi^2} \right] \qquad 4.1.7$$

Using equations 4.1.1 and 4.1.5 we get

$$w_1(\xi,\eta) = \sum_{m=1,3,5}^{k} A_m[Cosh\psi_m\eta + \theta_{1m}Cos\phi_m\eta]Sin(m\pi\xi)$$

$$+ \sum_{m=k^*+2}^{\infty} A_m[Cosh\psi_m\eta + \theta_{2m}Cosh\phi_m\eta]Sin(m\pi\xi) \qquad 4.1.8$$

Using equation 4.1.7 and differentiating one obtains

$$E_m = -A_m\left[\left[\psi_m^2 - \frac{D_1}{D_y}\phi^2(m\pi)^2\right]Cos\psi_m - \theta_{1m}\left[\phi_m^2 + \frac{D_1}{D_y}\phi^2(m\pi)^2\right]Cos\phi_m\right] \qquad 4.1.9a$$

or

$$E_m = -A_m\left[\left[\psi_m^2 - \frac{D_1}{D_y}\phi^2(m\pi)^2\right]Cos\psi_m + \theta_{2m}\left[\phi_m^2 - \frac{D_1}{D_y}\phi^2(m\pi)^2\right]Cosh\phi_m\right] \qquad 4.1.9b$$

Hence

$$A_m = \frac{E_m}{\theta_{11m}} \qquad 4.1.10a$$

or

$$A_m = \frac{E_m}{\theta_{22m}} \qquad 4.1.10b$$

where

$$\theta_{11m} = \left[\frac{D_1}{D_y}\phi^2(m\pi)^2 - \psi_m^2\right]Cosh\psi_m + \theta_{1m}\left[\phi_m^2 + \frac{D_1}{D_y}\phi^2(m\pi)^2\right]Cos\phi_m \qquad 4.1.11a$$

and

$$\theta_{22m} = \left[\frac{D_1}{D_y}\phi^2(m\pi)^2 - \psi_m^2\right]Cosh\psi_m + \theta_{2m}\left[\frac{D_1}{D_y}\phi^2(m\pi)^2 - \phi_m^2\right]Cosh\phi_m \qquad 4.1.11b$$

Hence the expression for w_1 is given by

$$w_1(\xi,\eta) = \sum_{m=1,3,5}^{k^*} \frac{E_m}{\theta_{11m}}\left[Cosh\psi_m\eta + \theta_{1m}Cos\phi_m\eta\right]Sin(m\pi\xi)$$

$$+\sum_{k^*+2}^{\infty} \frac{E_m}{\theta_{22m}} \left[Cosh\psi_m\eta + \theta_{2m}Cosh\phi_m\eta\right] Sin(m\pi\xi) \qquad 4.1.12$$

4.2 Solutions for the second building block.

Referring to the second building block of figure 6, the Levy type solution can be written as:

$$w_2(\xi,\eta) = \sum_{n=1,2}^{\infty} Y_n(\xi)Cos(n\pi\eta) \qquad 4.2.1$$

Expanding the dimensionless moments in series form we get

$$a\frac{M_2(\eta)}{D_x} = \sum_{n=1,2}^{\infty} E_n Cos(n\pi\eta) \qquad 4.2.2$$

Substituting into the governing differential equation yields

$$Y_n(\xi) = A_m Cosh\psi_n\xi + B_m Sinh\psi_n\xi + C_m Sin\phi_n\xi + D_m Cos\phi_n\xi \qquad 4.2.3a$$

or

$$Y_n(\xi) = A_m Cosh\psi_n\xi + B_m Sinh\psi_n\xi + C_m Sinh\phi_n\xi + D_m Cosh\phi_n\xi \qquad 4.2.3b$$

In view of the simple support conditions along the edge $\xi=0$ the symmetric terms are deleted from the above two equations yielding

$$Y_n(\xi) = B_m Sinh\psi_n\xi + C_m Sin\phi_n\xi \qquad 4.2.4a$$

or

$$Y_n(\xi) = B_m Sinh\psi_n\xi + C_m Sinh\phi_n\xi \qquad 4.2.4b$$

Enforcing the condition $Y_n(\xi=1)=0$ we get

$$B_n = -C_n \frac{Sin\phi_n}{Sinh\psi_n} \qquad 4.2.5a$$

or

$$B_n = -C_n \frac{Sinh\phi_n}{Sinh\psi_n} \qquad 4.2.5b$$

Hence

$$Y_n(\xi) = C_n \left[Sin\phi_n \xi - \frac{Sin\phi_n}{Sinh\psi_n} Sinh\psi_n \xi \right] \qquad 4.2.6a$$

or

$$Y_n(\xi) = C_n \left[Sinh\phi_n \xi - \frac{Sinh\phi_n}{Sinh\psi_n} Sinh\psi_n \xi \right] \qquad 4.2.6b$$

The boundary condition of equation 4.2.2 must now be enforced.

$$\frac{aM_2(\eta)}{D_x} = \left[\frac{\partial^2 w_2(\xi,\eta)}{\partial \xi^2} + \frac{D_1}{D_x} \frac{1}{\phi^2} \frac{\partial^2 w_2(\xi,\eta)}{\partial \eta^2} \right] \qquad 4.2.7$$

Hence

$$C_n = \frac{E_n}{(\phi_n^2 + \psi_n^2) Sin\phi_n} \qquad 4.2.8a$$

or

$$C_n = \frac{E_n}{(\psi_n^2 - \phi_n^2) Sinh\phi_n} \qquad 4.2.8b$$

Using equation 4.2.8 to express C_n in terms of E_n we get

$$Y_n(\xi) = \frac{E_n}{(\psi_n^2 + \phi_n^2)} \left[\frac{Sin\phi_n \xi}{Sin\phi_n} - \frac{Sinh\psi_n \xi}{Sinh\psi_n} \right] \qquad 4.2.9a$$

or

$$Y_n(\xi) = \frac{E_n}{(\psi_n^2 - \phi_n^2)}\left[\frac{Sin\phi_n\xi}{Sin\phi_n} - \frac{Sinh\psi_n\xi}{Sinh\psi_n}\right] \qquad 4.2.9b$$

From Equation 4.1.1 one obtains the general solution as

$$w_2(\xi,\eta) = \sum_{n=1}^{k^*} \frac{E_n}{(\psi_n^2 + \phi_n^2)}\left[\frac{Sin\phi_n\xi}{Sin\phi_n} - \frac{Sinh\psi_n\xi}{Sinh\psi_n}\right]Cos(n\pi\eta)$$

$$+ \sum_{k^*+1}^{\infty} \frac{E_n}{(\psi_n^2 - \phi_n^2)}\left[\frac{Sinh\phi_n\xi}{Sinh\phi_n} - \frac{Sinh\psi_n\xi}{Sinh\psi_n}\right]Cos(n\pi\eta) \qquad 4.2.10$$

4.3 The solution for doubly symmetric modes

The superposition of the solutions w_1 and w_2 yield the the sought eigenvalues. The Fourier coefficients in the moment expansion must be constrained so that the following edge conditions are satisfied by the superimposed solution.

1. There must be no residual moment along the edge $\eta=1$
2. There must be no residual slope along the edge $\xi=1$

The contribution of the first building block along the edge $\eta=1$ can be expanded as a sine series and is given by.

$$\frac{\phi b M_1(\xi)}{D_y} = -\left[\frac{\partial^2 w_1(\xi,\eta)}{\partial\eta^2} + \frac{D_1}{D_y}\phi^2\frac{\partial^2 w_1(\xi,\eta)}{\partial\xi^2}\right] \qquad 4.3.1$$

The contribution to the bending moment along the edge $\eta=1$ can be expanded as a sine series and the contribution to the bending moment along the same edge of the building block is given by

$$-\left[\frac{\partial^2 w_2(\xi,\eta)}{\partial\eta^2} + \frac{D_1}{D_y}\phi^2\frac{\partial^2 w_2(\xi,\eta)}{\partial\xi^2}\right] \qquad 4.3.2$$

Substituting the appropriate derivatives of the function w_2 in equation 4.3.2, we obtain contribution of each term to the moment as

$$-\frac{E_n Cos(n\pi)}{\psi_n^2+\phi_n^2}\left[\left[(n\pi)^2+\frac{D_1}{D_y}\phi^2\phi_n^2\right]\frac{Sin\phi_n\xi}{Sin\phi_n} - \left[(n\pi)^2-\frac{D_1}{D_y}\phi^2\phi_n^2\right]\frac{Sinh\psi_n\xi}{Sinh\psi_n}\right] \qquad 4.3.3a$$

or

$$\frac{E_n Cos(n\pi)}{\psi_n^2+\phi_n^2}\left[\left[(n\pi)^2-\frac{D_1}{D_y}\phi^2\phi_n^2\right]\frac{Sinh\phi_n\xi}{Sinh\phi_n} - \left[(n\pi)^2-\frac{D_1}{D_y}\phi^2\psi_n^2\right]\frac{Sinh_n\ \xi}{Sinh\psi_n}\right] \qquad 4.3.3b$$

Next applying the condition of zero slope along the edge $\xi=1$

$$\frac{\partial w_2(\xi,\eta)}{\partial\xi} + \frac{\partial w_1(\xi,\eta)}{\partial\xi} = 0 \qquad 4.3.4$$

and

$$\frac{\partial w_1}{\partial\xi} = \sum_{m=1,3}^{k^*}(m\pi)\frac{E_m}{\theta_{11m}}[Cosh\psi_m\eta + \theta_{1m}Cos\phi_m\eta]Cos(m\pi)$$

$$+\sum_{m=k^*+2}^{\infty}(m\pi)\frac{E_m}{\theta_{22m}}[Cosh\psi_m\eta + \theta_{2m}Cos(m\pi\eta)]Cos(m\pi) \qquad 4.3.5$$

$$\frac{\partial w_2}{\partial\xi} = \sum_{n=1}^{k^*}\frac{E_n}{(\psi_n^2+\phi_n^2)}\left[\frac{\phi_n Cos\phi_n}{Sin\phi_n} - \frac{\psi_n Cosh\psi_n}{Sinh\psi_n}\right]Cos(n\pi\eta)$$

$$+\sum_{k^*+1}^{\infty}\frac{E_n}{(\psi_n^2-\phi_n^2)}\left[\frac{\phi_n Cosh\phi_n}{Sinh\phi_n}-\frac{\psi_n Cosh\psi_n}{Sinh\psi_n}\right]Cos(n\pi\eta) \qquad 4.3.6$$

Contribution of the first building block to the slope along $\xi=1$ is

$$\frac{\partial w_1}{\partial\xi}=\frac{E_m}{\theta_{11m}}(m\pi)Cos(m\pi)[Cosh\psi_m\eta+\theta_{1m}Cos\phi_m\eta] \qquad 4.3.7a$$

or

$$\frac{\partial w_1}{\partial\xi}=\frac{E_m}{\theta_{22m}}(m\pi)Cos(m\pi)[Cosh\psi_m\eta+\theta_{2m}Cosh\phi_m\eta] \qquad 4.3.7b$$

Contribution of the second building block to the slope along $\xi=1$ will be

$$\frac{\partial w_2}{\partial\xi}=\frac{E_n}{(\psi_n^2+\phi_n^2)}\left[\frac{\phi_n Cos\phi_n}{Sin\phi_n}-\frac{\psi_n Cosh\psi_n}{Sinh\psi_n}\right]Cos(n\pi\eta) \qquad 4.3.8a$$

or

$$\frac{\partial w_2}{\partial\xi}=\frac{E_n}{(\psi_n^2+\phi_n^2)}\left[\frac{\phi_n Cosh\phi_n}{Sinh\phi_n}-\frac{\psi_n Cosh\psi_n}{Sinh\psi_n}\right]Cos(n\pi\eta) \qquad 4.3.8b$$

If we use t terms in each solution, we shall have 2t unknowns and 2t equations available to relate these unknowns. This would lead to a matrix 2t x 2t, the first t rows resulting from enforcing the condition of zero net slope along the edge $\eta=1$ the second t rows resulting from enforcing the condition of zero net bending moment along the edge $\xi=1$. A schematic representation of the matrix obtained is shown in Figure 10. Expressions for each segment corresponding to case a and case b are formulated below:

1. Segment(1,1)

It is a diagonal segment with each element equal to unity.

2. Segment (1,2)

Elements of this segment, evaluated by multiplying the appropriate expressions of equation 4.3.3 by $2Sin(m\pi\xi)$ and integrating from 0 to 1 are given by.

$$\frac{2(m\pi)Cos(n\pi)Cos(m\pi)}{\psi_n^2 + \phi_n^2} x_1 \qquad 4.3.9$$

where

$$x_1 = \frac{(n\pi)^2 + \frac{D_1}{D_y}\phi^2\phi_n^2}{\phi_n^2 - (m\pi)^2} + \frac{(n\pi)^2 - \frac{D_1}{D_y}\phi^2\psi_n^2}{\psi_n^2 + (m\pi)^2} \qquad 4.3.10$$

or

$$\frac{2(m\pi)Cos(n\pi)Cos(m\pi)}{\psi_n^2 - \phi_n^2} x_1 \qquad 4.3.11$$

where

$$x_1 = \frac{(n\pi)^2 - \frac{D_1}{D_y}\phi^2\psi_n^2}{\psi_n^2 + (m\pi)^2} - \frac{(n\pi)^2 - \frac{D_1}{D_y}\phi^2\phi_n^2}{\phi_n^2 + (m\pi)^2} \qquad 4.3.12$$

3. Segment (2,1)

Elements of this segment are evaluated by multiplying the appropriate expression of equation 4.3.7 by 2 $Sin(n\pi\eta/2)$ and integrating from 0 to 1 The integrals are obtained as

$$2(m\pi)Cos(m\pi)Cos(n\pi)\frac{x_1}{\theta_{11m}} \qquad 4.3.13$$

where

$$x_1 = \frac{\psi_m Sinh\psi_m}{\psi_m^2 + (n\pi)^2} + \frac{\theta_{1m}\phi_m Sin\phi_m}{\phi_m^2 - (n\pi)^2} \qquad 4.3.14$$

or

$$2(m\pi)Cos(m\pi)Cos(n\pi)\frac{x_1}{\theta_{22m}} \qquad 4.3.15$$

where

$$x_1 = \frac{{}_mSinh\psi_m}{\psi_m^2 + (n\pi)^2} + \frac{\theta_{2m}\phi_m Sinh\phi_m}{\phi_m^2 + (n\pi)^2} \qquad 4.3.16$$

4. Segment(2,2)

This segment is diagonal with each element obtained from the appropriate expression of Equation 4.3.8 and the elements of this segment are given by

$$\frac{1}{\psi_n^2 + \phi_n^2}\left[\frac{\phi_n Cos\phi_n}{Sin\phi_n} - \frac{\psi_n Cosh\psi_n}{Sinh\psi_n}\right] \qquad 4.3.17$$

or

$$\frac{1}{\psi_n^2 - \phi_n^2}\left[\frac{\phi_n Cosh\phi_n}{Sinh\phi_n} - \frac{\psi_n Cosh\psi_n}{Sinh\psi_n}\right] \qquad 4.3.18$$

The eigenvalue is determined using the IMSL routine LINV3F using the steps outlined in the Appendix.

Chapter V

ANTISYMMETRIC - SYMMETRIC MODES

The quarter segment of the plate that needs to be analyzed is shown in Figure 7. In order to analyze the plate segment shown we have to superimpose the solutions corresponding to the building blocks (1) and (2) and adjust the constants appearing in the boundary formulation so that their combination satisfies the requirement of the continuous plate.

5.1 Solution for the first building block.

For the first building block taking the solution in Levy form we have

$$w_1(\xi,\eta) = \sum_{m=1,2,3}^{\infty} Y_m(\eta) Sin(m\pi\xi) \qquad 5.1.1$$

Expanding the moment as a sine series

$$M_1(\xi) = \sum_{m=1}^{\infty} E_m Sin(m\pi\xi) \qquad 5.1.2$$

Since we have simply supported conditions along the edge η=0 odd terms in the function $Y_m(\eta)$ must be deleted.

Hence

$$Y_m(\eta) = B_m Sinh\psi_m\eta + C_m Sin\phi_m\eta \qquad 5.1.3a$$

or

$$Y_m(\eta) = B_m Sinh\psi_m\eta + C_m Sinh\phi_m\eta \qquad 5.1.3b$$

The condition of zero vertical edge reaction along the edge η=1 requires

$$\frac{\partial^3 w_1(\xi,\eta)}{\partial\eta^3} + \frac{\phi^2(2H_1 - D_1)}{D_y}\frac{\partial^3 w_1(\xi,\eta)}{\partial\xi^2\partial\eta} = 0 \qquad 5.1.4$$

Taking the derivatives and enforcing the above boundary condition yields

$$B_m\psi_m\left[\psi_m^2 - \frac{\phi^2(2H_1 - D_1)}{D_y}(m\pi)^2\right]Cosh\psi_m$$

$$-C_m\phi_m\left[\phi_m^2 + \frac{\phi^2(2H_1 - D_1)}{D_y}(m\pi)^2\right]Cos\phi_m = 0 \qquad 5.1.5a$$

or

$$B_m\psi_m\left[\psi_m^2 - \frac{\phi^2(2H_1 - D_1)}{D_y}(m\pi)^2\right]Cosh\psi_m$$

$$+C_m\phi_m\left[\phi_m^2 - \frac{\phi^2(2H_1 - D_1)}{D_y}(m\pi)^2\right]Cosh\phi_m = 0 \qquad 5.1.5b$$

Eliminating the constant B_m in equation 5.1.3 yields.

$$Y_m = C_m\left[\theta_{1m} Sinh\psi_m\eta + Sin\phi_m\eta\right] \qquad 5.1.6$$

where

$$\theta_{1m} = \frac{\phi_m\left[\phi_m^2 + \phi^2(2H_1 - D_1)/D_y(m\pi)^2\right]Cos\phi_m}{\psi_m[\psi_m^2 - \phi^2(2H_1 - D_1)/D_y(m\pi)^2]Cosh\psi_m} \tag{5.1.7}$$

or

$$Y_m = C_m\left[\theta_{2m}Sinh\psi_m\eta + Sinh\phi_m\eta\right] \tag{5.1.8}$$

where

$$\theta_{2m} = \frac{\phi_m\left[\phi_m^2 - \phi^2(2H_1 - D_1)/D_y(m\pi)^2\right]Cosh\phi_m}{\psi_m[\psi_m^2 - \phi^2(2H_1 - D_1)/D_y(m\pi)^2]Cosh\psi_m} \tag{5.1.9}$$

Moment equilibrium along the edge $\eta=1$ requires

$$\frac{\phi b M_1(\xi)}{D_y} = -\left[\frac{\partial^2 w_1(\xi,\eta)}{\partial\eta^2} + \frac{D_1}{D_y}\phi^2\frac{\partial^2 w_1(\xi,\eta)}{\partial\xi^2}\right] \tag{5.1.10}$$

Expressing the necessary derivatives in series form we get

$$\sum_{m=1,3,5}^{\infty} E_m Sin(m\pi\xi) = \sum_{m=1,3,5}^{k^*} C_m\theta_{11m}Sin(m\pi\xi) + \sum_{m=k^*+1}^{\infty} C_m\theta_{22m}Sin(m\pi\xi) \tag{5.1.11}$$

where

$$\theta_{11m} = \left[\phi_m^2 + \frac{D_1}{D_y}\phi^2(m\pi)^2\right]Sin\phi_m - \theta_{1m}\left[\psi_m^2 - \frac{D_1}{D_y}\phi^2(m\pi)^2\right]Sinh\phi_m \tag{5.1.12a}$$

and

$$\theta_{22m} = \theta_{2m}\left[\frac{D_1}{D_y}\phi^2(m\pi)^2 - \psi_m^2\right]Sinh\psi_m + \left[\frac{D_1}{D_y}\phi^2(m\pi)^2 - \phi_m^2\right]Sinh\phi_m \tag{5.1.12b}$$

Eliminating the constant C_m from the equation we obtain

$$Y_m(\eta) = \frac{E_m}{\theta_{11m}}\left[\theta_{1m}Sinh\psi_m\eta + Sin\phi_m\eta\right] \tag{5.1.13a}$$

or

$$Y_m(\eta) = \frac{E_m}{\theta_{22m}}[\theta_{2m} Sinh\psi_m\eta + Sinh\phi_m\eta] \qquad 5.1.13b$$

Referring to the second building block of figure 7 the Levy type of solution is written as

$$w_2(\xi,\eta) = \sum_{n=1,2}^{\infty} Y_n(\xi) Sin\left(\frac{n\pi\eta}{2}\right) \qquad 5.2.1$$

and

$$\frac{aM_2(\eta)}{D_x} = \sum_{n=1}^{\infty} Sin\left(\frac{n\pi\eta}{2}\right) \qquad 5.2.2$$

Since we have simple support conditions along the edge $\xi=0$ symmetric terms are deleted from the expressions for $Y_n(\xi)$ giving

$$Y_n(\xi) = B_m Sinh\psi_n\xi + C_m Sin\phi_n\xi \qquad 5.2.3a$$

or

$$Y_n(\xi) = B_m Sinh\psi_n\xi + C_m Sinh\phi_n\xi \qquad 5.2.3b$$

5.2 Solution for the second building block

The required solution can be inferred from the second building block for the doubly symmetric modes. The required expression is given by.

$$w_2(\xi,\eta) = \sum_{n=1,3,5}^{k^*} \frac{E_n}{(\psi_n^2 + \phi_n^2)} \left[\frac{Sin\phi_n\xi}{Sin\phi_n} - \frac{Sinh\psi_n\xi}{Sinh\psi_n} \right] Sin\left(\frac{n\pi\eta}{2}\right)$$

$$+ \sum_{k^*+2}^{\infty} \frac{E_n}{(\psi_n^2 - \phi_n^2)} \left[\frac{Sinh\phi_n\xi}{Sinh\phi_n} - \frac{Sinh\psi_n\xi}{Sinh\psi_n} \right] Sin\left(\frac{n\pi\eta}{2}\right) \qquad 5.2.4$$

5.3 The solution for antisymmetric-symmetric modes.

The two edge conditions that must be satisfied by the superimposed solutions are

1. There must be no net residual moment along the edge $\eta=1$
2. There must be no net residual slope along the edge $\xi=1$

The contribution of the first building block can be expanded as a sine series

$$\sum_{n=1,2}^{\infty} Y_n(\xi) Sin(\frac{n\pi\eta}{2}) = \frac{\phi b M_1(\xi)}{D_y} = -\left[\frac{\partial^2 w_1(\xi,\eta)}{\partial\eta^2} + \frac{D_1}{D_y}\phi^2 \frac{\partial^2 w_1(\xi,\eta)}{\partial\xi^2}\right] \qquad 5.3.1$$

Contribution to the bending moment along the same edge of the second building block is given by

$$-\left[\frac{\partial^2 w_2(\xi,\eta)}{\partial\eta^2} + \frac{D_1}{D_y}\phi^2 \frac{\partial^2 w_2(\xi,\eta)}{\partial\xi^2}\right] \qquad 5.3.2$$

Substituting the appropriate derivatives of the function w_2 in equation 5.3.2, the contribution of each term to the moment is found to be

$$-\frac{E_n Sin(\frac{n\pi}{2})}{(\psi_n^2+\phi_n^2)}\left[\left[+\left(\frac{n\pi}{2}\right)^2+\frac{D_1}{D_y}\phi^2\phi_n^2\right]\frac{Sin\phi_n\xi}{Sin\phi_n}-\left[\left(\frac{n\pi}{2}\right)^2-\frac{D_1}{D_y}\phi^2\psi_n^2\right]\frac{Sinh\psi_n\xi}{Sinh\psi_n}\right] \quad 5.3.3a$$

or

$$-\frac{E_n Sin(\frac{n\pi}{2})}{(\phi_n^2+\psi_n^2)}\left[\left[+\left(\frac{n\pi}{2}\right)^2-\frac{D_1}{D_y}\phi^2\phi_n^2\right]\frac{Sin\phi_n\xi}{Sin\phi_n}-\left[\left(\frac{n\pi}{2}\right)^2-\frac{D_1}{D_y}\phi^2\psi_n^2\right]\frac{Sinh\psi_n\xi}{Sinh\psi_n}\right] \quad 5.3.3b$$

Condition of zero slope along the edge ξ=1 implies that

$$\frac{\partial w_2(\xi,\eta)}{\partial\xi}+\frac{\partial w_1(\xi,\eta)}{\partial\xi}=0 \quad 5.3.4$$

The contribution of the first building block to the slope along the edge ξ=1 is obtained for each term by taking appropriate derivatives yielding

$$\frac{\partial w_1}{\partial\xi}=\frac{E_m}{\theta_{11m}}[\theta_{1m}Sinh\psi_m\eta+Sin\phi_m\eta]m\pi Cos(m\pi) \quad 5.3.5a$$

or

$$\frac{\partial w_1}{\partial\xi}=\frac{E_m}{\theta_{22m}}[\theta_{2m}Sinh\psi_m\eta+Sinh\phi_m\eta]m\pi Cos(m\pi) \quad 5.3.5b$$

Contribution from the solution w_2 to the slope along the edge ξ=1 becomes

$$\frac{E_n}{\psi_n^2+\phi_n^2}\left[\frac{\phi_n Cos\phi_n\xi}{Sin\phi_n}-\frac{\psi_n Cosh\psi_n}{Sinh\psi_n}\right]Sin\left(\frac{n\pi\eta}{2}\right) \quad 5.3.6a$$

or

$$\frac{E_n}{\psi_n^2-\phi_n^2}\left[\frac{\phi_n Cosh\phi_n}{Sinh\phi_n}-\frac{\psi_n Cosh\psi_n}{Sinh\psi_n}\right]Sin\left(\frac{n\pi\eta}{2}\right) \quad 5.3.6b$$

Following an approach similar to the one used for forming the coefficient for the doubly symmetric modes we obtain expressions for each segment of the matrix.

1. Segment (1,1)

   This is a diagonal segment with each element equal to unity

2. Segment (1,2)

   Elements of this segment obtained by multiplying appropriate expressions of equation 5.3.3 by 2 Sin($m\pi\xi$) and integrating between 0 and 1 are given by the expression

$$\frac{2(m\pi)Sin(\frac{n\pi}{2})Cos(m\pi)x_1}{\psi_n^2+\phi_n^2} \qquad 5.3.7$$

where

$$x_1=\frac{(\frac{n\pi}{2})^2+\frac{D_1}{D_y}\phi^2\phi_n^2}{\phi_n^2-(m\pi)^2}+\frac{(\frac{n\pi}{2})^2-\frac{D_1}{D_y}\phi^2\psi_n^2}{\psi_n^2+(m\pi)^2} \qquad 5.3.8$$

or

$$\frac{2(m\pi)Sin(\frac{n\pi}{2})Cos(m\pi)x_1}{\psi_n^2-\phi_n^2} \qquad 5.3.9$$

where

$$x_1=\frac{\frac{D_1}{D_y}\phi^2\phi_n^2-(\frac{n\pi}{2})^2}{\phi_n^2-(m\pi)^2}+\frac{(\frac{n\pi}{2})^2-\frac{D_1}{D_y}\phi^2\psi_n^2}{\psi_n^2-(m\pi)^2} \qquad 5.3.10$$

3. Segment(2,1)

Elements are obtained by multiplying the appropriate expression of equation 5.3.5 by 2 Sin(nπη/2) and integrating from 0 to 1 giving

$$2(m\pi)Cos(m\pi)Sin\left(\frac{n\pi}{2}\right)\frac{x_1}{\theta_{11m}} \qquad 5.3.11$$

where

$$x_1 = \frac{\theta_{1m}\psi_m Cosh\psi_m}{\psi_m^2 + \left(\frac{n\pi}{2}\right)^2} - \frac{\phi_m Cos\phi_m}{\phi_m^2 - \left(\frac{n\pi}{2}\right)^2} \qquad 5.3.12$$

or

$$2(m\pi)Cos(m\pi)Sin\left(\frac{n\pi}{2}\right)\frac{x_1}{\theta_{22m}} \qquad 5.3.13$$

where

$$x_1 = \frac{\theta_{2m}\psi_m Cosh\psi_m}{\psi_m^2 + \left(\frac{n\pi}{2}\right)^2} + \frac{\phi_m Cosh\phi_m}{\phi_m^2 + \left(\frac{n\pi}{2}\right)^2} \qquad 5.3.14$$

4. Segment (2,2)

This is a diagonal segment with each element obtained from the appropriate expression of 5.3.6 is given by

$$\frac{\phi_n Cos\phi_n/Sin\phi_n - \psi_n Cosh\psi_n/Sinh\psi_n}{\psi_n^2 + \phi_n^2} \qquad 5.3.15$$

or

$$\frac{\phi_n Cos\phi_n/Sinh\phi_n - \psi_n Cosh\psi_n/Sinh\psi_n}{\psi_n^2 - \phi_n^2} \qquad 5.3.16$$

The eigenvalue matrix having been established eigenvalues are obtained in a way similar to that for the doubly symmetric modes.

Chapter VI

DOUBLY ANTISYMMETRIC AND SYMMETRIC - ANTISYMMETRIC MODES

6.1 Doubly Antisymmetric modes

The general form of the solution is given by

$$w(\xi,\eta) = \sum_{m=1}^{\infty} [A_m Cosh\psi_m\eta + B_m Sinh\psi_m\eta + C_m Sin\phi_m\eta + D_m Cos\phi_m\eta]\, Sin(m\pi\xi) \qquad 6.1.1a$$

or

$$w(\xi,\eta) = \sum_{m=1}^{\infty} [A_m Cosh\psi_m\eta + B_m Sinh\psi_m\eta + C_m Sinh\phi_m\eta + D_m Cosh\phi_m\eta]\, Sin(m\pi\xi) \qquad 6.1.1b$$

6.1.1 Case a; Solutions for $D_x/D_y(m\pi^4) < a^4\omega^2\rho_0/D_y$

Enforcing the condition of zero displacement and zero bending moment for the edge η=0 yields

$$Y(\eta) = B_m Sinh\psi_m\eta + C_m Sin\phi_m\eta \qquad 6.1.2$$

The condition of zero bending moment and zero vertical edge reaction at the edge η=1 yields the solution to the eigenvalue problem. Condition of zero bending moment requires

$$\left[\frac{\partial^2 w}{\partial \eta^2} + \frac{D_1}{D_y}\phi^2\frac{\partial^2 w}{\partial \xi^2}\right] = 0 \qquad 6.1.3$$

Taking appropriate derivatives and substituting them in equation 6.1.3 yields

$$B_m\left[\psi_m^2 - \frac{D_1}{D_y}\phi^2(m\pi)^2\right]Sinh\psi_m - C_m\left[\phi_m^2 + \frac{D_1}{D_y}\phi^2(m\pi)^2\right]Sin\phi_m = 0 \qquad 6.1.4$$

Next applying the condition of zero vertical edge reaction

$$\frac{\partial^3 w}{\partial \eta^3} + \frac{\phi^2(2H_1 - D_1)}{D_y}\frac{\partial^3 w}{\partial \xi^2 \partial \eta} = 0 \qquad 6.1.5$$

and substituting the appropriate derivatives into the equation we get

$$B_m\psi_m\left[\psi_m^2 - \frac{\phi^2(2H_1 - D_1)}{D_y}(m\pi)^2\right]Cosh\psi_m$$

$$-C_m\phi_m\left[\phi_m^2 - \frac{\phi^2(2H_1 - D_1)}{D_y}(m\pi)^2\right]Cos\phi_m = 0 \qquad 6.1.6$$

A non trivial solution exists for B_m and C_m of equations 6.1.4 and 6.1.6 only if the determinant of their coefficient matrix vanishes. The eigenvalue equation is hence given by

$$\phi_m\left[\phi_m^2+\frac{\phi^2(2H_1-D_1)}{D_y}(m\pi)^2\right]\left[\psi_m^2-\frac{D_1}{D_y}\phi^2(m\pi)^2\right]Sinh\psi_m Cos\phi_m$$

$$-\psi_m\left[\psi_m^2-\frac{\phi^2(2H_1-D_1)}{D_y}(m\pi)^2\right]\left[\phi_m^2+\frac{D_1}{D_y}\phi^2(m\pi)^2\right]Cosh\psi_m Sin\phi_m=0 \qquad 6.1.7$$

6.1.2 Case b; Solutions for $D_x/D_y(m\pi^4) > a^4\omega^2\rho_0/D_y$

Following the steps outlined for case a and using equation 6.1.1b we get the solution to the eigenvalue equation as:

$$\phi_m\left[\phi_m^2-\frac{\phi^2(2H_1-D_1)}{D_y}(m\pi)^2\right]\left[\psi_m^2-\frac{D_1}{D_y}\phi^2(m\pi)^2\right]Sinh\psi_m Cosh\phi_m$$

$$-\psi_m\left[\psi_m^2-\frac{\phi^2(2H_1-D_1)}{D_y}(m\pi)^2\right]\left[\phi_m^2-\frac{D_1}{D_y}\phi^2(m\pi)^2\right]Cosh\psi_m Sinh\phi_m=0 \qquad 6.1.8$$

6.2 Symmetric - Antisymmetric modes

The general form of the solution is

$$w(\xi,\eta)=\sum_{m=1}^{\infty}\left[A_m Cosh\psi_m\eta+B_m Sinh\psi_m\eta+C_m Sin\phi_m\eta+D_m Cos\phi_m\eta\right]Sin(m\pi\xi) \qquad 6.2.1a$$

or

$$w(\xi,\eta)=\sum_{m=1}^{\infty}\left[A_m Cosh\psi_m\eta+B_m Sinh\psi_m\eta+C_m Sinh\phi_m\eta+D_m Cosh\phi_m\eta\right]Sin(m\pi\xi) \qquad 6.2.1b$$

6.2.1 Case a; Solutions for $D_x/D_y(m\pi^4) < a^4\omega^2\rho_0/D_y$.

In order to satisfy the boundary conditions at the edge $\eta=0$ Antisymmetric terms have to be deleted. Hence the expression for w is given by.

$$w(\xi,\eta) = \sum_{m=1}^{\infty} \left[A_m Cosh\psi_m\eta + D_m Cos\phi_m\eta\right] Sin(m\pi\xi) \qquad 6.2.2$$

Imposing the condition of zero vertical edge reaction at the edge $\eta=1$, we have

$$\frac{\partial^3 w}{\partial\eta^3} + \frac{\phi^2(2H_1 - D_1)}{D_y}\frac{\partial^3 w}{\partial\xi^2\partial\eta} = 0 \qquad 6.2.3$$

Substituting appropriate derivatives into equation 6.2.3 yields

$$A_m\psi_m\left[\psi_m^2 - \frac{\phi^2(2H_1 - D_1)}{D_y}(m\pi)^2\right] Sinh\psi_m$$

$$+D_m\phi_m\left[\phi_m^2 - \frac{\phi^2(2H_1 - D_1)}{D_y}(m\pi)^2\right] Sin\phi_m \qquad 6.2.4$$

Applying the condition of zero bending moment at the edge $\eta=1$

$$\left[\frac{\partial^2 w}{\partial\eta^2} + \frac{D_1}{D_y}\phi^2\frac{\partial^2 w}{\partial\xi^2}\right] = 0 \qquad 6.2.5$$

Substituting the required derivatives of w into equation 6.2.5 yields the following

$$A_m\left[\psi_m^2 - \frac{D_1}{D_y}\phi^2(m\pi)^2\right] Cosh\psi_m\eta - D_m\left[\phi_m^2 - \frac{D_1}{D_y}\phi^2(m\pi)^2\right] Cos\phi_m\eta = 0 \qquad 6.2.6$$

A non trivial solution for the unknowns B_m and C_m of equations 6.2.4 and 6.2.6 requires the determinant of the coefficient matrix to vanish yielding the eigenvalue equation as

$$\phi_m\left[\phi_m^2 + \frac{\phi^2(2H_1 - D_1)}{D_y}(m\pi)^2\right]\left[\psi_m^2 - \frac{D_1}{D_y}\phi^2(m\pi)^2\right]\left[Cosh\psi_m Sin\phi_m\right]$$

$$-\psi_m \left[\psi_m^2 - \frac{\phi^2(2H_1 - D_1)}{D_y}(m\pi)^2\right] \left[\phi_m^2 + \frac{D_1}{D_y}\phi^2(m\pi)^2\right] [Sinh\psi_m Cos\phi_m] = 0 \quad 6.2.7$$

6.2.2 Case b;· Solution for $D_x/D_y(m\pi^4) > a^4\omega^2\rho_0/D_y$

Following the steps outlined for case a and using equation 6.1.1b we get the solution to the eigenvalue equation as

$$\phi_m \left[\phi_m^2 + \frac{\phi^2(2H_1 - D_1)}{D_y}(m\pi)^2\right] \left[\psi_m^2 - \frac{D_1}{D_y}\phi^2(m\pi)^2\right] [Cosh\psi_m Sinh\phi_m]$$

$$-\psi_m \left[\psi_m^2 - \frac{\phi^2(2H_1 - D_1)}{D_y}(m\pi)^2\right] \left[\phi_m^2 + \frac{D_1}{D_y}\phi^2(m\pi)^2\right] [Sinh\psi_m Cosh\phi_m] = 0 \quad 6.2.8$$

Chapter VII

EMPIRICAL FORMULATION OF EIGENVALUES.

7.1 Formulation Of Equations

A general solution to the problem of determining eigenvalues for various vibration modes was discussed in the previous chapters. The input parameters $(D_x/D_y)^{1/4}$, μ and the aspect ratio ϕ are combined to produce a closed form solution giving sufficiently accurate values for all practical purposes. The method is essentially a trial and error approach and consists of studying the variation of the eigenvalue by treating one parameter at a time, for constant values of the remaining parameters. Individual algebraic expressions governing the nature of variations are developed by using curve fitting techniques and the final expression is written by superimposing the individual expressions to give a closed form solution. To study the variation of the eigenvalues Tables (Nos 3 to 22) covering a wide range of values $(D_x/D_y)^{1/4}$ = 1 to 20; μ=0.1 to 1.0 and ϕ=1 to 4 were generated. The lateral contraction of the plate was neglected. Hence, D_1 is taken equal to zero. This is true for Beam-Slab type bridge decks. [21]

7.2 Empirical Equations

The following Empirical equations have been proposed, based upon curve fitting techniques.

1. Doubly Antisymmetric mode.

$$\lambda_b^2 = \omega b^2 \sqrt{\rho_0 / D_y}$$
$$= \pi^2 p^2 \frac{1}{\phi^2}$$

where

$$p = \left[\frac{D_x}{D_y}\right]^{1/4}$$

2. Doubly Symmetric mode.

$$\lambda_b^2 = \frac{\pi^3}{2.011} p^2 \frac{1}{\phi^2}$$

3. Symmetric-Antisymmetric mode.

For $1 < p \leq 2$

$$\lambda_b^2 = E_1 \left[\frac{2}{\phi}\right]^{E_2} + [E_3 - E_4][\mu - 0.5]/0.9$$

where

$$E_1 = 3.5714 + 0.4296p + 2.4215p^2 + 0.001259p^3$$

$$E_2 = 1.28 + 0.3(p-1)$$
$$E_3 = 4.54 + 2.35(p-1)$$
$$E_4 = 1.4(\phi - 2) - 0.8$$

For 2<p<20

$$\lambda_b^2 = E_1\left[\frac{2}{\phi}\right]^{E_2} + [E_3 - E_4(\phi - 2)/3]\,(\mu - 0.5)/0.9$$

where

$$E1 = 0.0871 + 0.00428p + 3.8544p^2$$
$$E2 = (2-\phi)\left[0.212 - 0.081p^2\right] + 0.274p^2 + 1.342$$
$$E3 = \left[1.88 - \frac{0.74}{3}p^{-1.73}(\phi - 1)\right](\mu - 0.5) * 1.29 - 0.15$$

4. Antisymmetric-Symmetric mode.

For 1≤p<2

$$\lambda_b^2 = E_1\left[\frac{2}{\phi}\right]^{E_2} + E_3$$

where

$$E1 = 0.0871 + 0.00428p + 3.8544p^2$$
$$E2 = (2-\phi)\left[0.212 - 0.081p^2\right] + 0.274p^2 + 1.342$$
$$E3 = \left[1.88 - \frac{0.74}{3}p^{-1.73}(\phi - 1)\right](\mu - 0.5) * 1.29 - 0.15$$

For 2<p<20

$$\lambda_b^2 = \frac{4}{\phi^2} E_1 + 1.976(\mu - 0.5)$$

where

$$E1 = 0.0871 + 0.00428p + 3.8544p^2$$

$$E2 = (2 - \phi)\left[0.212 - 0.081p^2\right] + 0.274p^2 + 1.342$$

$$E3 = \left[1.88 - \frac{0.74}{3} p^{-1.73}(\phi - 1)\right](\mu - 0.5) * 1.29 - 0.15$$

The variation between theoretical and empirical values is shown typically in Figures 11 to 20 for the symmetric-antisymmetric mode. Using the empirical equations a maximum error of 15.93% was observed in the first symmetric-antisymmetric mode for p=1, μ=0.2 and ϕ=3.5. The error decreased to a maximum of 10.5% for p=2 and became less than 5% for p≥8. For the anti-symmetric-symmetric mode the maximum observed error was 11.15% for p=1 and decreased to less than 5% for p≥4. The error for the doubly symmetric and doubly anti-symmetric mode was noted to be less than 0.1%.

Chapter VIII

RESULTS AND CONCLUSIONS

The Fourier series solution was used to obtain the solution for the free vibration analysis of continuous orthotropic plates. The results were obtained for aspect ratios ranging from 1.0 to 4.0 and the ratio of the rigidity parameter $(D_x/D_y)^{1/4}$ ranging from 1 to 20. The eigenvalues for the tables were obtained using ten terms solutions, thus yielding eigenvalues for a wide range of interest in practical problems.

To determine the convergence characteristics of this method of free vibration analysis, numerical experimentation was carried out by varying the number of terms in the displacement function w. Results thus obtained are shown in Table 1. It can be seen that the convergence of the solution is excellent, the variation in the answer for a three term solution and the ten term solution being only in the fifth decimal digit. The convergence of the solution does not deteriorate with higher modes.

Table 2 shows the frequencies for the first 7 modes of vibration of an isotropic continuous plate for aspect ratio of 1. The results are compared with those obtained by Cheung [4] and Warburton [44]. The agreement has been found to be very good (maximum variation less than 2.6%). For orthotropic plates, no results are available in technical literature, hence no comparison can be made.

Tables 3 to 22 show the eigenvalues for doubly symmetric, antisymmetric - symmetric, doubly antisymmetric and symmetric - antisymmetric modes of vibration for different aspect and rigidity ratios. The value of D_1 was assumed to be zero, since the lateral contraction of the plate can be neglected for a beam-slab type of bridge [21].

Eigenvalues for the first doubly anti-symmetric and the doubly symmetric mode are found to be independent of the torsion parameter μ. The eigenvalues vary directly as $(D_x/D_y)^{1/4}$ for constant values of the aspect ratio ϕ and inversely as the square of the aspect ratio ϕ for constant values of $(D_x/D_y)^{1/4}$

For all values of the plate aspect ratio and the rigidity ratios, the fundamental frequency is found to reside in the first doubly antisymmetric mode and is hence independent of the torsion parameter μ.

For the symmetric-antisymmetric and the antisymmetric-symmetric mode, the eigenvalues exhibit a strong dependence on the torsion parameter μ. This is especially true for low values of the rigidity parameter $(D_x/D_y)^{1/4}$. Hence, eigenvalues for isotropic plates are sensitive to the values of μ. This dependence on μ is more pronounced in the symmetric-antisymmetric mode than for the antisymmetric symmetric mode. The variation of eigenvalues, which decrease with decreasing values of the torsion parameter is approximately of the first order. The significance of the torsion parameter is found to decrease with increasing values of the rigidity ratio.

The following conclusions can be drawn for all modes of vibration.

1. For given values of the rigidity ratio and the torsion parameter μ , greater the aspect ratio, smaller the frequency.
2. Under constant values of the aspect and rigidity ratio, the frequency decreases or remains unchanged, with decreasing values of the torsion parameter.
3. An increase in the rigidity ratio for constant values of all other parameters leads to an increase in the frequency of vibration.

Empirical equations were developed in Chapter VII using curve fitting techniques and the equations obtained give eigenvalues sufficiently accurate for practical design. The empirical formulation enables a rapid computation of the frequencies of vibration without going through a detailed analysis.

Finally from the present study it can be concluded that the present method is a very accurate and general method for free vibration analysis. All combinations of the classical boundary conditions;i.e simply supported, free and clamped can be readily solved using this method. Once solutions for different types of building blocks have been developed, they can be used to formulate solutions to other free vibration problems.

The tables cover the practical range of values in terms of dimensionless parameters enable a quick estimation of the natural frequency of vibration. Computing time and memory requirements being small the method is ideally suited for computers with relatively smaller capacity and speed.

It may further be added that the solution technique adopted in this thesis is quite general in its application and further studies could be directed towards obtaining solutions for free or forced vibration problems of continuous bridges of unequal span or orthotropic plates continuous in both directions.

BIBLIOGRAPHY

1. Azimi S, Hamilton J.F, Soedel W. The Receptance Method Applied to The Free Vibration Of Continuous Rectangular Plates. Journal Of Sound And Vibration 93(1), 1984, pp9-29.

2. Bauer, L. and Reiss E.L. Flexural Vibration Of Clamped Orthotropic Plates Journal Acoustic Soc. Am. Vol 53, No 5, 1973, pp 1360-64.

3. Biggs J.M Vibration And Stresses In Girder Bridges Highway Research Board Bull. 124 Washington D.C, 1956.

4. Cheung M.S , Cheung Y.K, Reddy D.Y. Frequency Analysis Of Certain Single And Continuous Span Bridges, Developments In Bridge Design And Construction Paper presented at Conference At Univ. College, Cardiff 1971.

5. Cheung Y.K. and M.S. Cheung Flexural Vibrations Of Rectangular And Other Polygonal Plates. A.S.C.E,Of Engineering Mechanics Division Vol 97, 1971 pp391-411

6. Cheung Y.K and Wu C. I. Frequency Analysis Of Rectangular Plates Continuous In One Or Two Directions. Journal Of Earthquake Engineering And Structural Dynamics. Vol III, 1974. pp3-14.

7. Cheung Y.K Analysis Of Box Girder Bridges By Finite Strip Method. Proc. second International Symposium on concrete Bridge design, Chicago, April, ACI Publication SP26.

8. Chu K.H and Krishnamoorty G. Use of Orthotropic Plate Theory In Bridge Design. A.S.C.E Vol 88, No.ST3 June 1962.

9. Classen, R.W. and Thorne C.J. Vibration Of Thin Rectangular Isotropic Plates Journal Of Appl. Mech., Vol 28, No.2, June 1961, pp304-305.

10. Crandall S.H. Engineering Analysis McGraw-Hill, New York, 1956.

11. Dawe D.J. A Finite Element Approach To Plate Vibration Problems. Journal Mech. Engr. Sci Vol 7,No 1, 1956, pp 28-32.

12. Durvasula S. and Nair P.S. Application Of Partition Method To Vibration Problems Of Plates. Journal Of Sound And Vibrations, 37 (3), 1974, pp 429-445.

13. Elishakoff I. and Sternberg A. Eigenfrequencies Of Continuous Plates With Arbitrary Number Of Equal Spans. Trans. of the A.S.M.E Vol 46, Sept. 1979.

14. Gorman D.J Free Vibration Analysis Of Rectangular Plates 1982.

15. Hearmon R.F.S The Frequency Of Flexural Vibrations Of Rectangular Orthotropic Plates With Clamped Or Simply Supported Edges. Journal Of Applied Mechanics Vol 26, Nos 3-4, Dec 1959, pp 537-540.

16. Hearmon R.F.S and Adams E.H The Bending And Twisting Of Anisotropic Plates British Jour. Of Appl. Physics Vol 3, 1952, pp150-156.

17. Hillerborg A. Dynamic influences of smoothly running bodies on simply supported girders. Institute of Structural Engineering and Bridge Building of the Royal Institute Of Technology Stockholm, 1951.

18. Huffington N.J Jr and Hoppmann W. H On The Transverse Vibrations Of Rectangular Orthotropic Plates Jour. Of Appl. Mech., Trans ASME 1958, 25 pp 389-395.

19. Inglis C.E. A Mathematical treatise on vibration in railway bridges Cambridge Univ. Press, London, 1934.

20. Kanazawa T. and Kawai T. On The Lateral Vibration Of Anisotropic rectangular Plates Proc. Of the Second Japan

National Congress for Applied Mechanics, Vol2 1952, pp 333-338.

21. Kulkarni G.G and Ng S.F Optimum Design Of Longitudinally Stiffened Simply Supported Orthotropic Bridge decks Journal Of Sound And Vibration Vol 40, No.2 pp 273-284, 1975.

22. Kulkarni G.G and Ng S.F Dynamic Analysis Of Two Dimensional Simply Supported Orthotropic Bridge Decks. Canadian Journal Of Civil Engineering Vol 5, No.1, pp58-69, 1978.

23. Laura P.A.A A Method For The determination Of The Fundamental Frequency Of Orthotropic Plates Of Polygonal Boundary Shape. Journal Of Sound And Vibration , 70 (1), 1980, pp 77-84.

24. Leissa A.W. Vibration Of Plates Washington DC, 1969.

25. Lekhnitskii S.G Anisotropic Plates Translated from the nd Russian Ed. by S.W Tsai and T.Cheron, Gordon and Breach Science, 1956.

26. Levy M. Sur L'Equilbre Elastique d'Un Plaque rectangulaire C.R Acd. Sci. Vol 129, 1899, pp 535-539. ; as cited in Timoshenko S.P , Theory Of Plates And Shells.

27. Louw J.M. The Vibration Of Two Span And Simple Span Highway Bridges.Civil Engineer In South Africa 1963.

28. Nagaya K. Transient response Of A Continuous Plate On Elastic Support.Journal Of Sound And Vibration, 47 (3), 1976, pp 351-370.

29. Nagaya K. Vibration Of A Plate With Straight Line Boundaries Journal Of Sound And Vibration 49(1), 1976, pp45-52.

30. Rajappa N.R Free Vibration Of Rectangular and Circular Orthotropic Plates Journal of AIAA 1963, 1, pp 1194-1195.

31. Rayleigh The Theory Of Sound Dover Publications NY, 1945.

32. Ritz W. Theoire der Transversalchwingungen einer quadratischen platte mit freien Randem Annalen der Physic Vol 28, 1909, pp 737-786, cited in Ref No.

33. Sakata T. A Reduction Method For Vibrating And Buckling Problems Of Orthotropic Continuous Plates Journal Of Sound and Vibration 49 (1), 1976, pp45-52.

34. Sakata T. Eigenvalues Of Orthotropic Continuous Plates With Two Opposite Edges Simply Supported Journal Of Sound And Vibration 1976, 47(4), pp 577-583.

35. Sakata T. Reduction Methods For The Problem Of Vibration Of Orthotropic Plates. Part II Generalized Reduction Method For Generally Orthotropic Plates with Arbitrary Shape Shock & Vibration Digest Vol 11 n6 June 1979 pp19-22.

36. Sakata T. Natural Frequencies Of Orthotropic Continuous Plates Reprinted From Memoirs Of Chubu Institute Of Technology, Vol 13-A December 1977.

37. Smith J.W. Finite Strip Analysis Of The Dynamic Response Of Beam And Slab Highway Bridges. Journal Of Earthquake Engineering And Structural Dynamics,Vol I , 1973, pp 357-370.

38. Szilard R. Theory And Analysis Of Plates, Classical and Numerical Methods Prentice Hall, New Jersey, 1973.

39. Timoshenko S.P and Kreiger S.W Theory Of Plates And Shells. McGraw Hill, New York 1959.

40. Timoshenko S. Vibration Problems In Engineering 3rd Ed. Mcgraw Hill, New York, 1961.

41. Troitsky M. S. Orthotropic Bridges, theory and design The James F. Lincoln Arc Welding Foundation, Cleveland Ohio, 1968.

42. Veletsos A.S. and Newmark M. Determination Of Natural Frequencies Of Continuous Plates Hinged Along Two Opposite Edges Journal Of Appl. Mech., Trans ASME 1956, 23, 97-102.

43. Yamada Y. and Veletsos A.S Free Vibration Of Simple Span I BridgesPartB, 8th Progress Report, Highway Bridge Impact Investigation, Univ. Of Illinois, 1958.

44. Warburton G.B The Vibration Of Rectangular Plates. Proceedings Of The Institution Of Mechanical Engineers, Vol 168, 1954, pp 371-384.

45. Young D. Vibration Of Rectangular Plates By The Ritz Method Jour. Of Appl. Mech., Vol. 24, No.4, Dec 1950, pp 448-453.

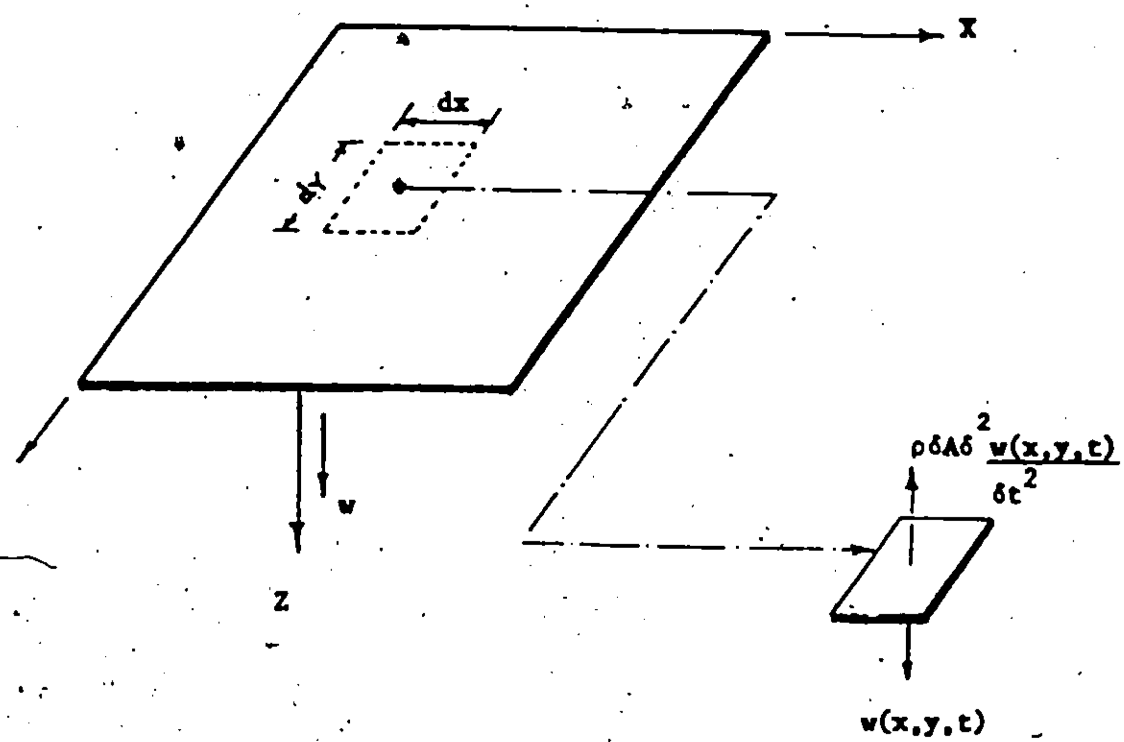

Figure 1: Rectangular plate showing forces acting on a differential element

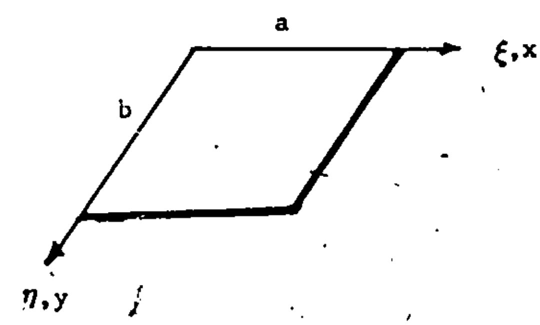

Figure 2: Rectangular plate dimensionless coordinate system.

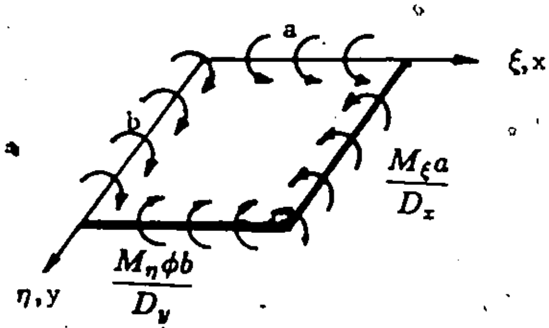

Figure 3: Distributed bending moments along edges of a rectangular plate.

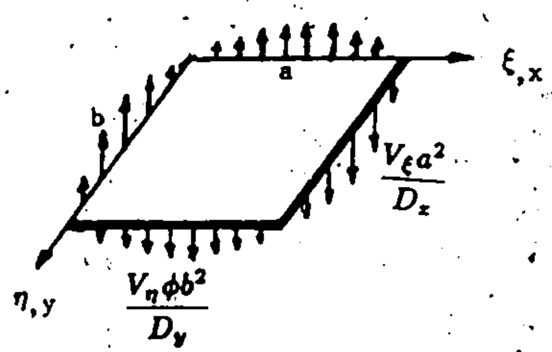

Figure 4: Distributed vertical edge reactions along the edges of a rectangular plate.

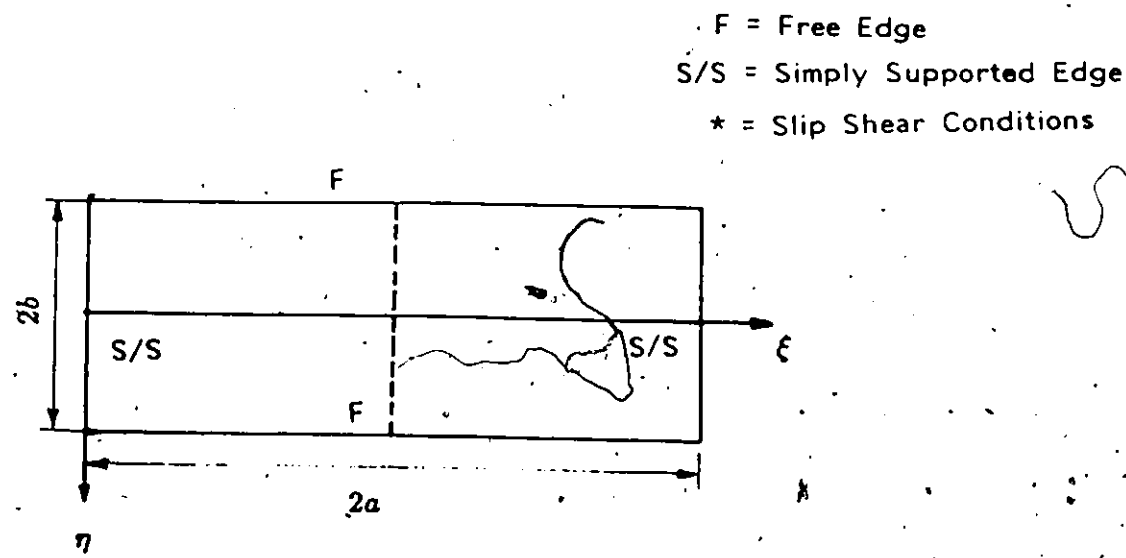

Figure 5: Continuous orthotropic plate with bridge type boundary conditions

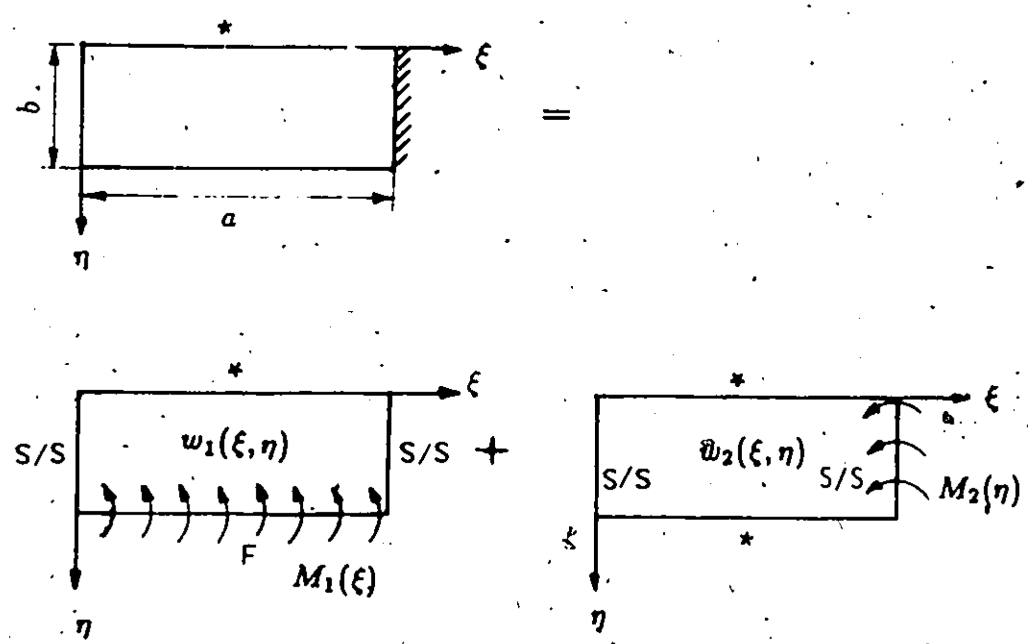

Figure 6: Building blocks for analysis of doubly symmetric vibration modes.

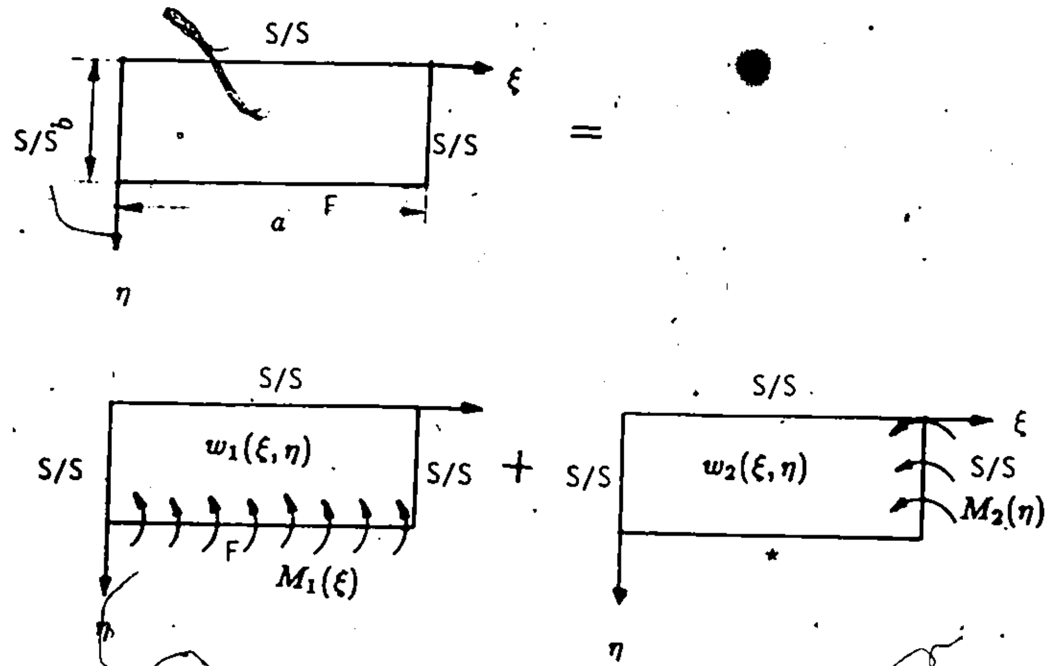

Figure 7: Building blocks for analysis of antisymmetric - symmetric modes.

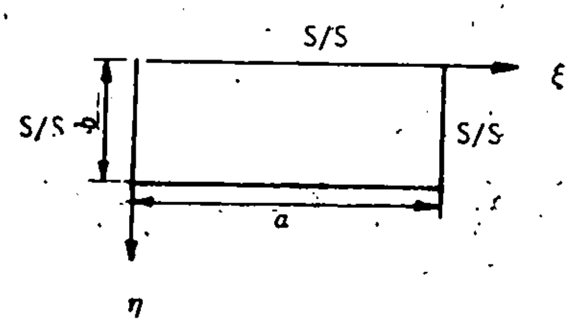

Figure 8: Building block for analysis of doubly antisymmetric modes.

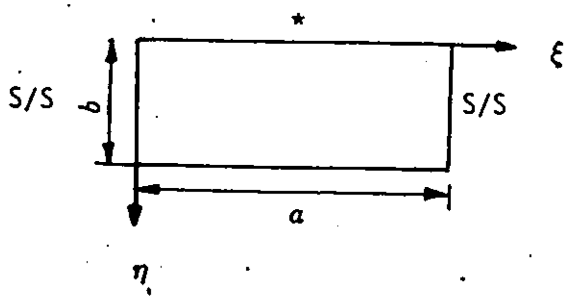

Figure 9: Building block for analysis of symmetric - antisymmetric modes.

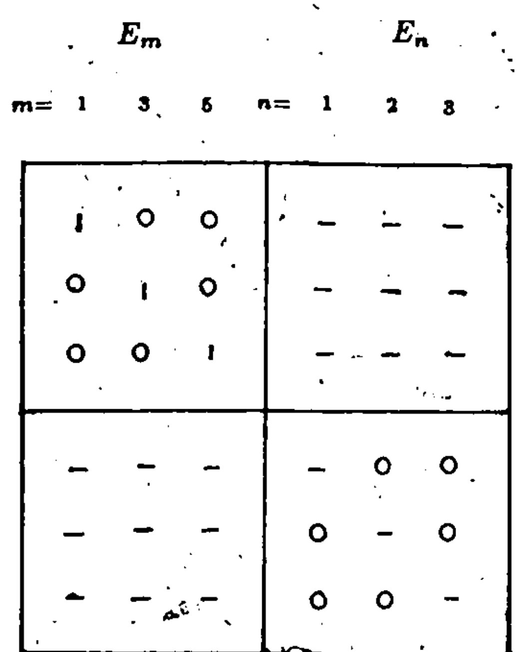

Figure 10: Schematic representation of the coefficient matrix for a three term solution.

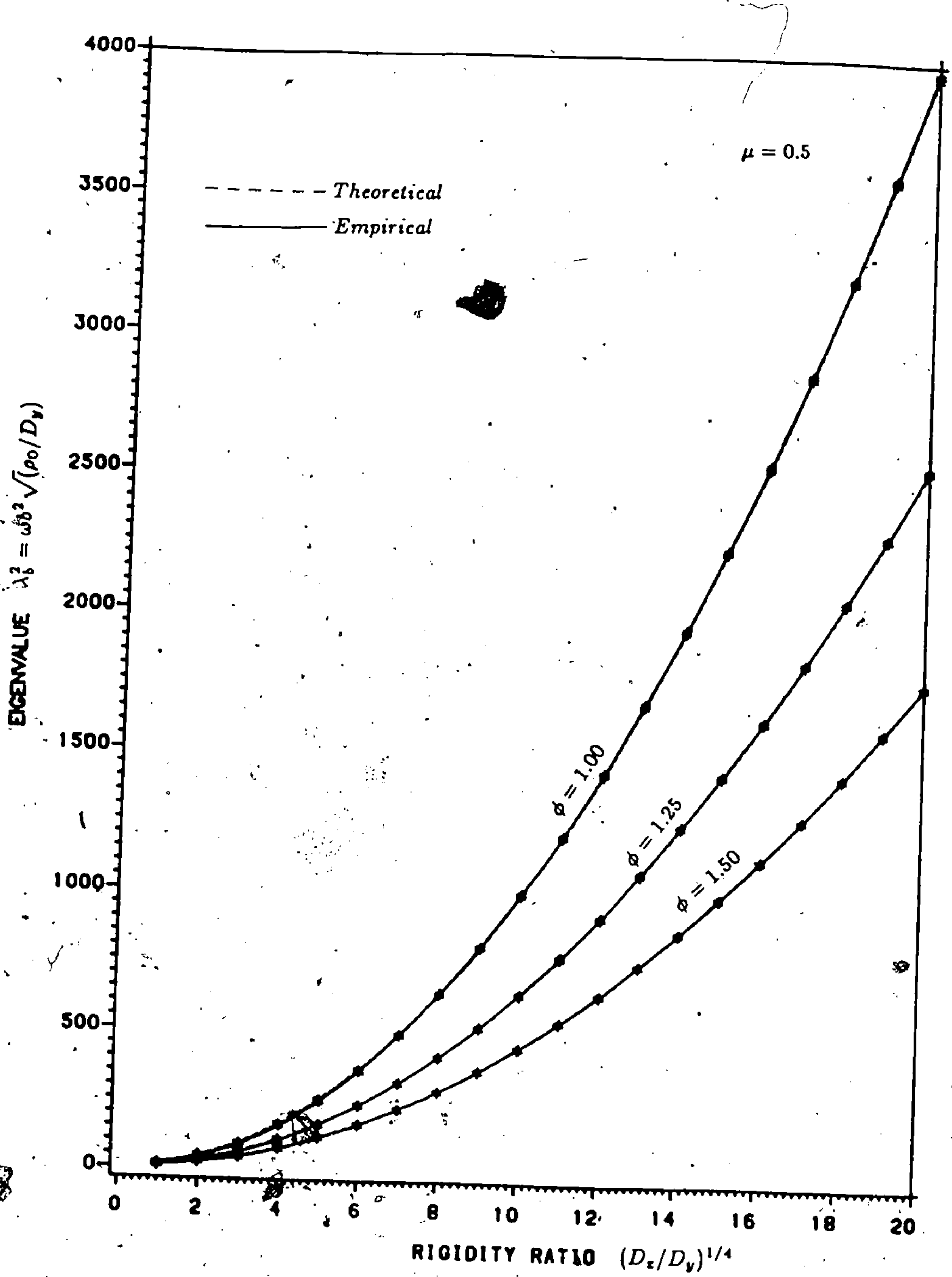

Figure 11: Variation of Eigenvalues with rigidity ratio for symmetric-antisymmetric modes.

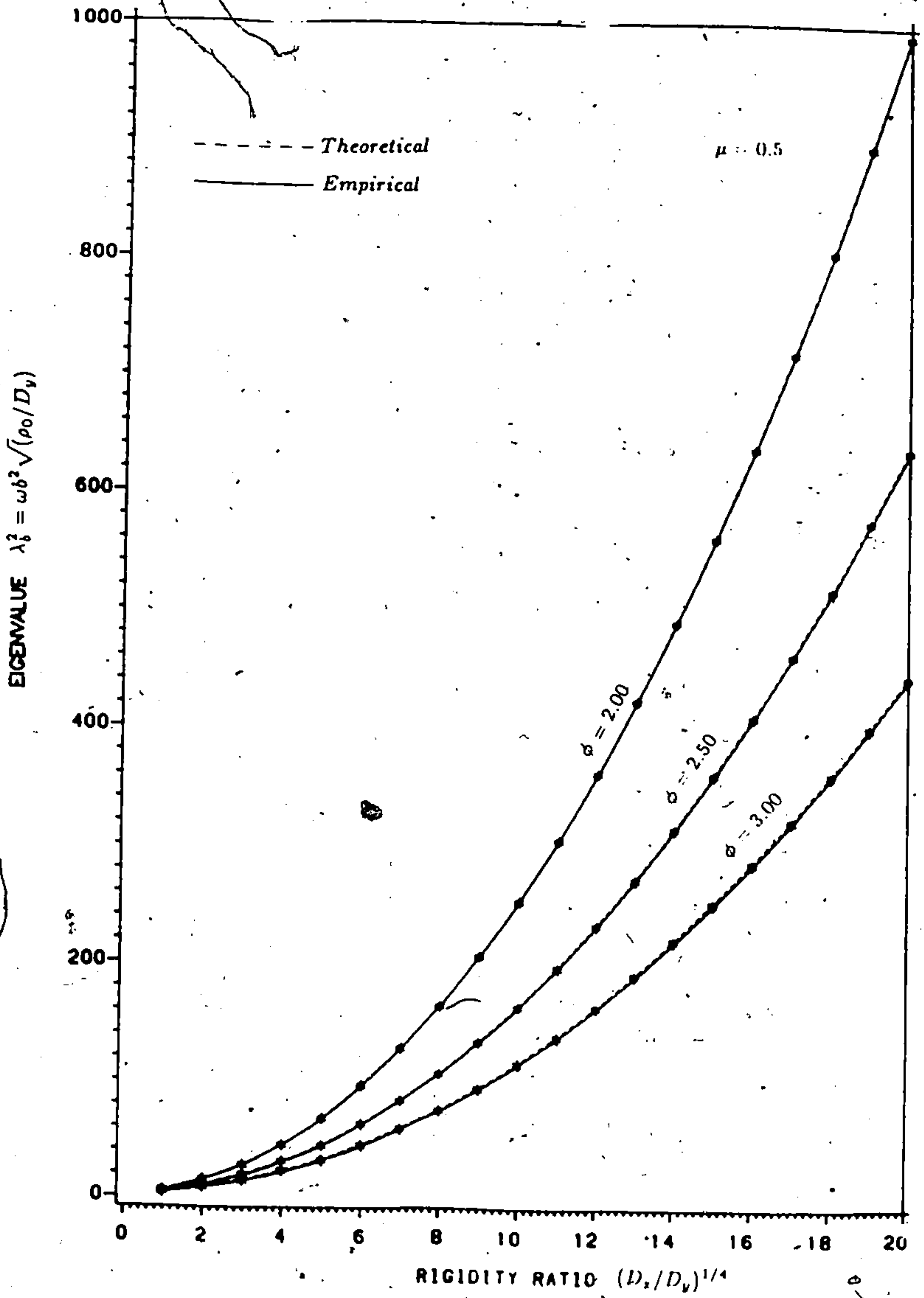

Figure 12: Variation of Eigenvalues with rigidity ratio for symmetric antisymmetric modes.

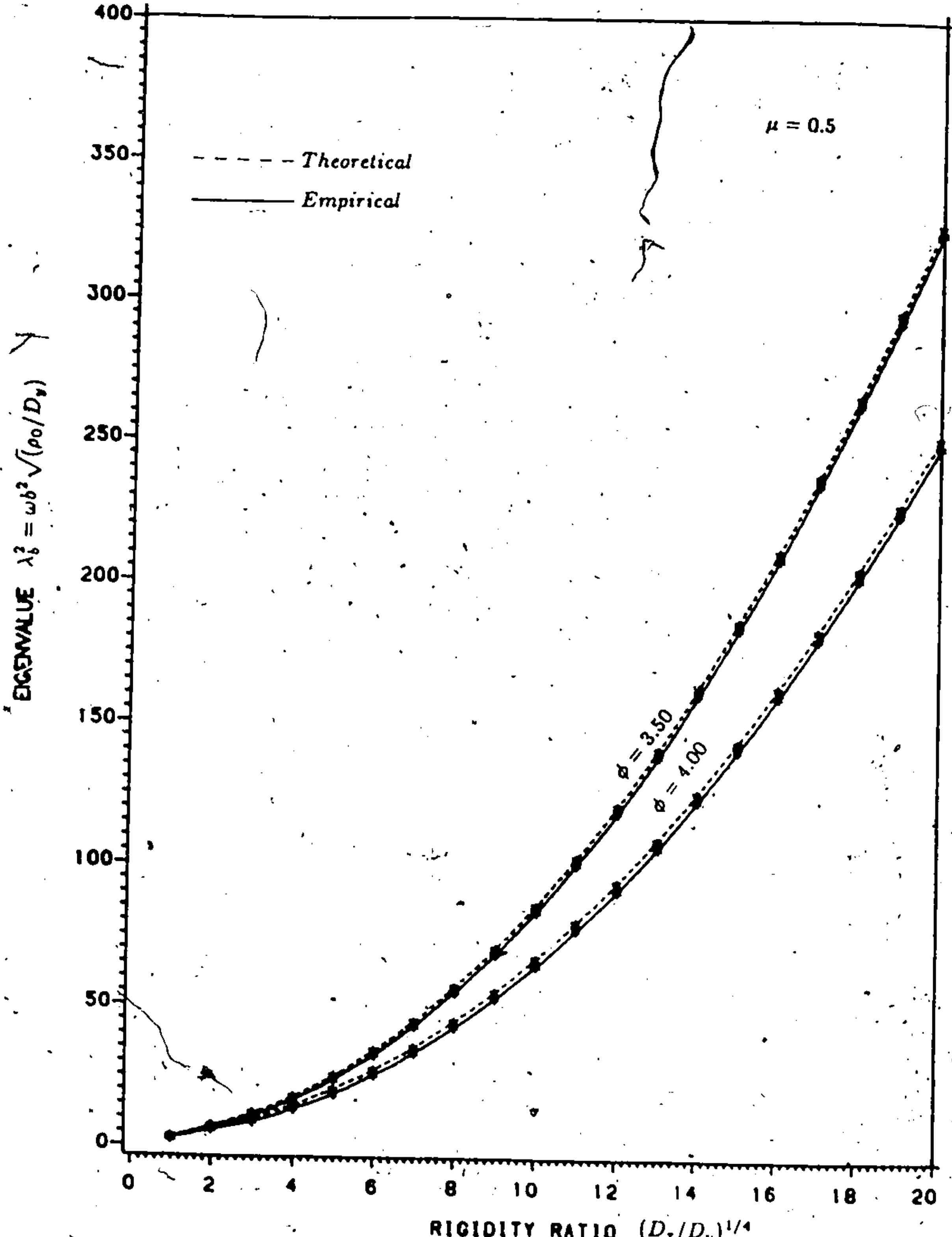

Figure 13: Variation of Eigenvalues with rigidity ratio for symmetric-antisymmetric modes.

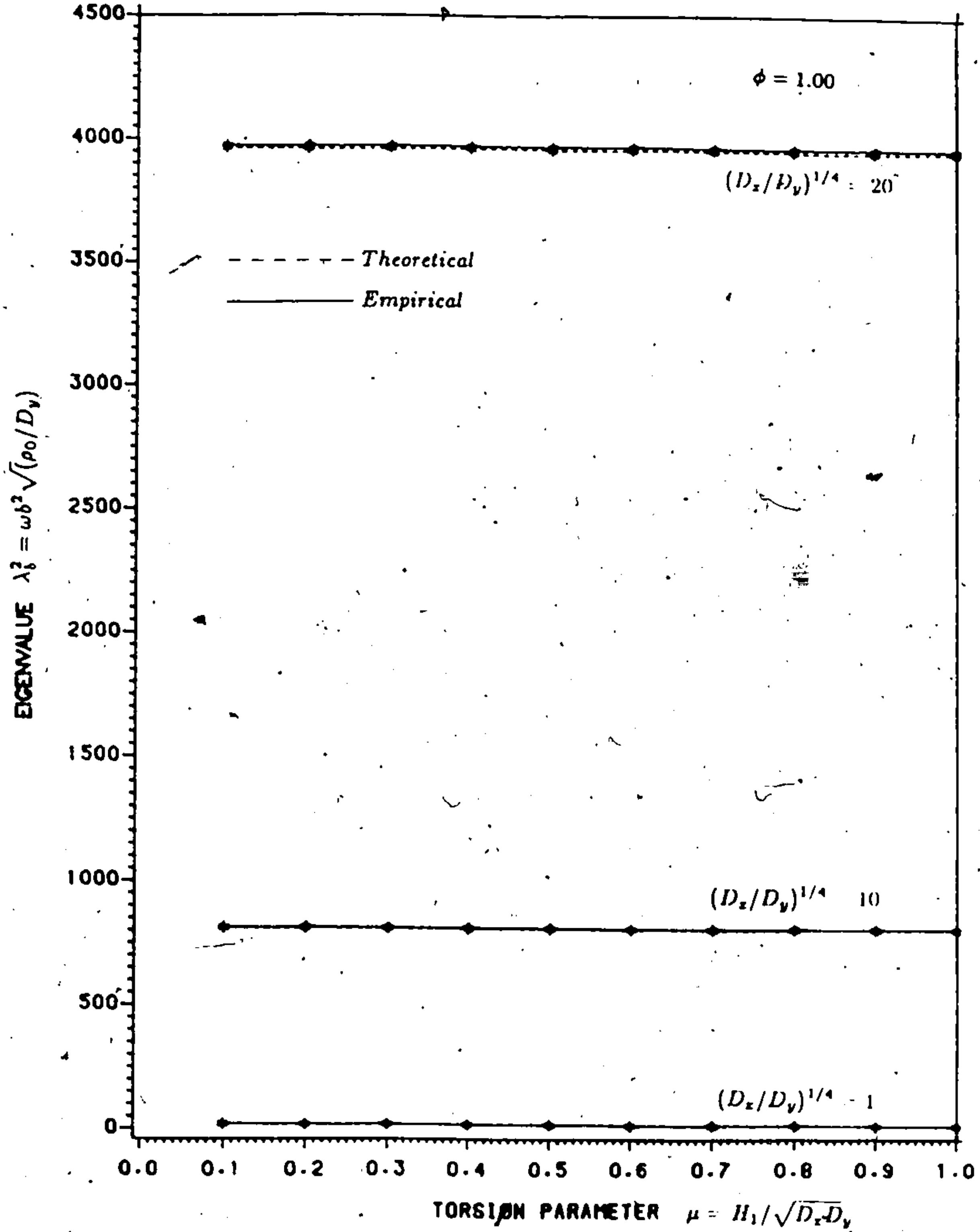

Figure 14: Variation of Eigenvalues with torsion parameter for symmetric-antisymmetric modes.

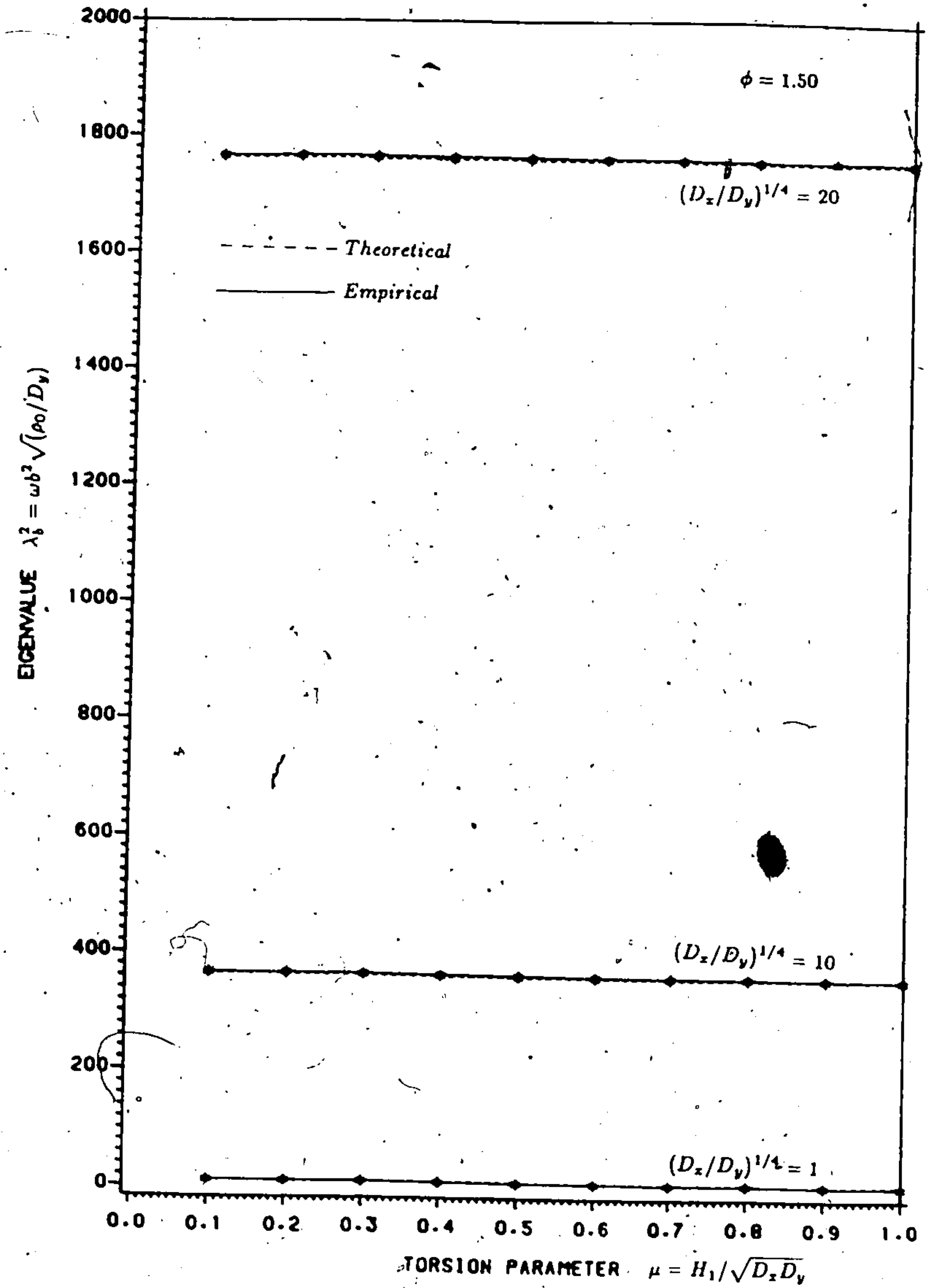

Figure 15: Variation of Eigenvalues with torsion parameter for symmetric-antisymmetric modes.

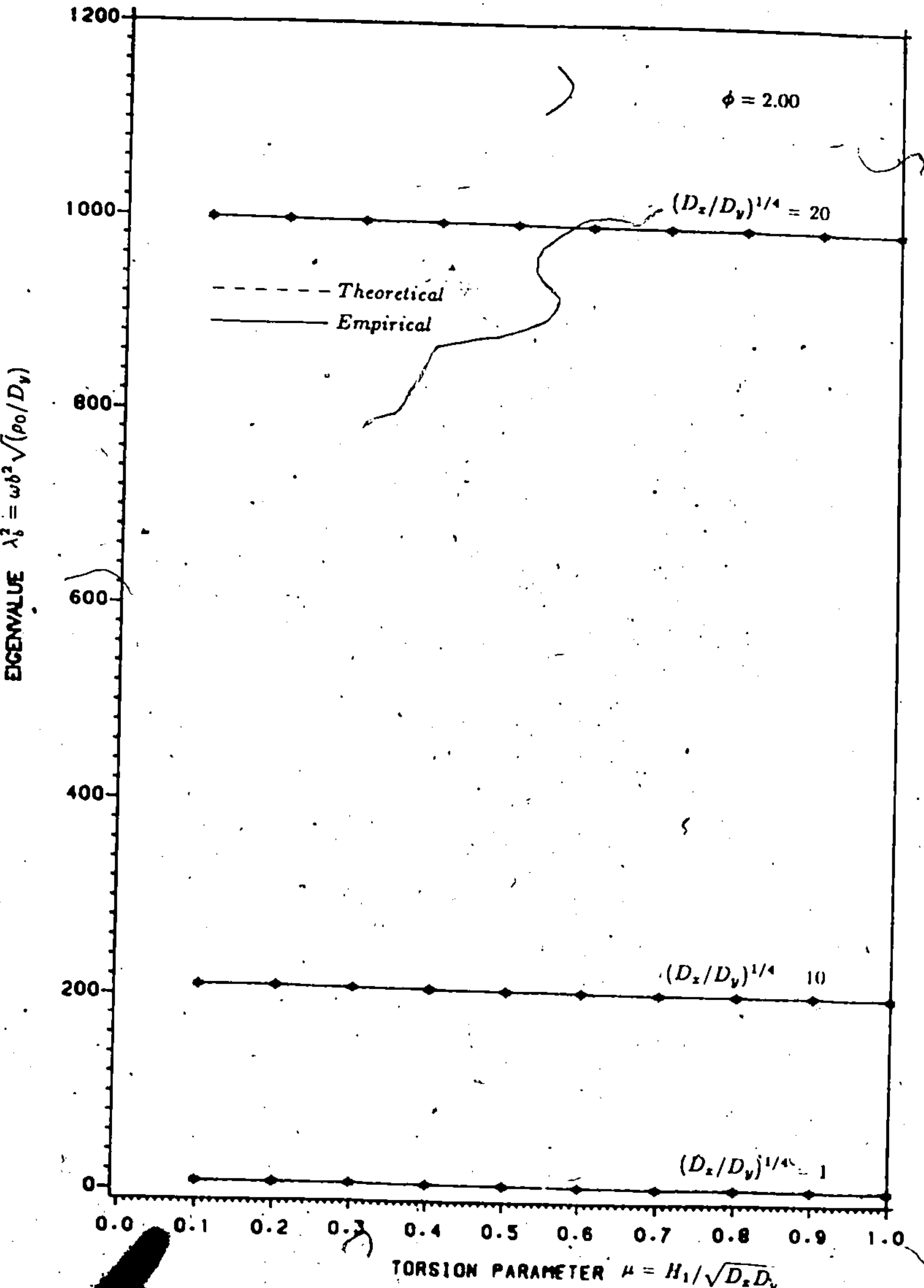

Figure 16: Variation of Eigenvalues with torsion parameter for symmetric-antisymmetric modes.

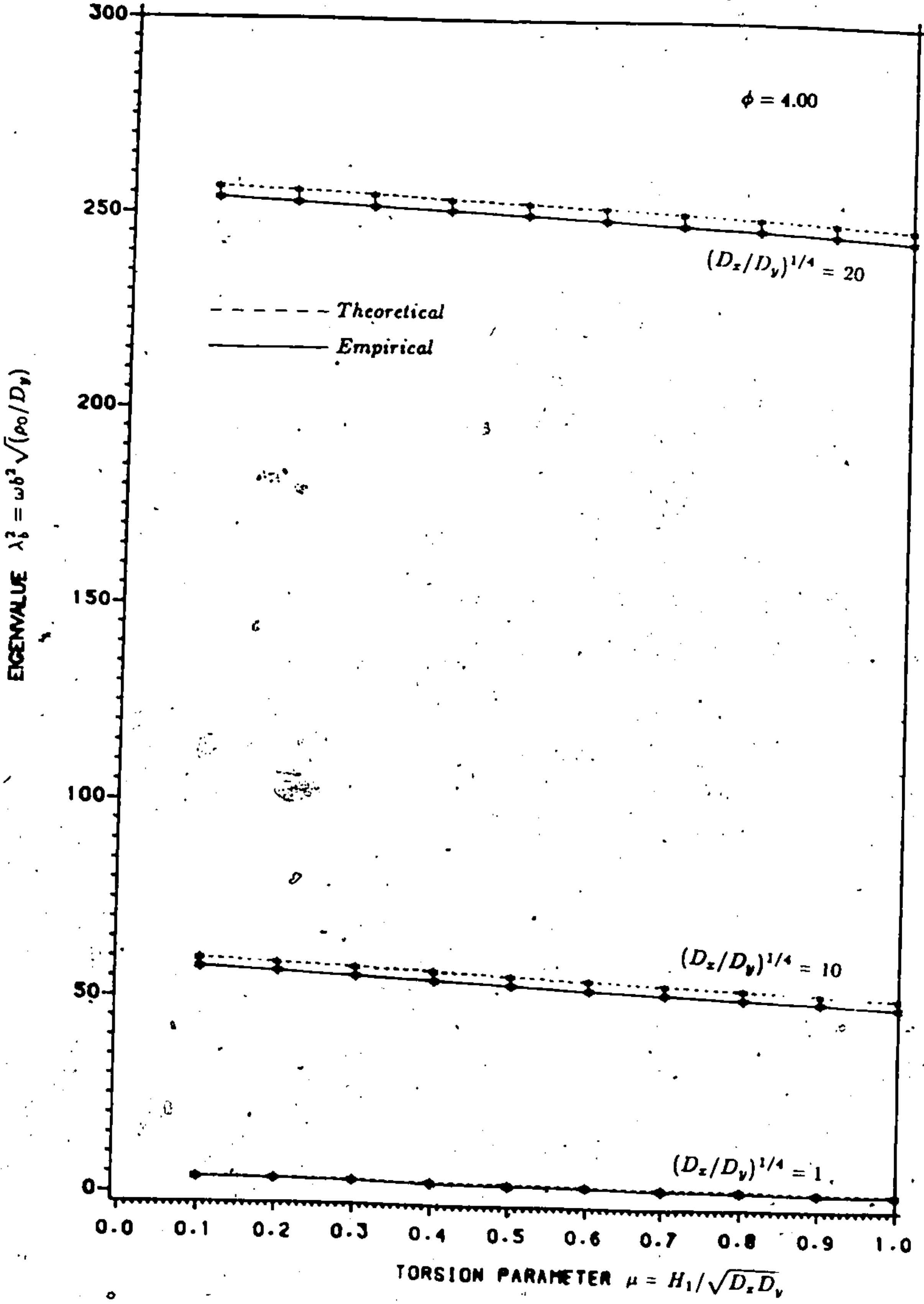

Figure 17: Variation of Eigenvalues with torsion parameter for [illegible]ymmetric-antisymmetric modes.

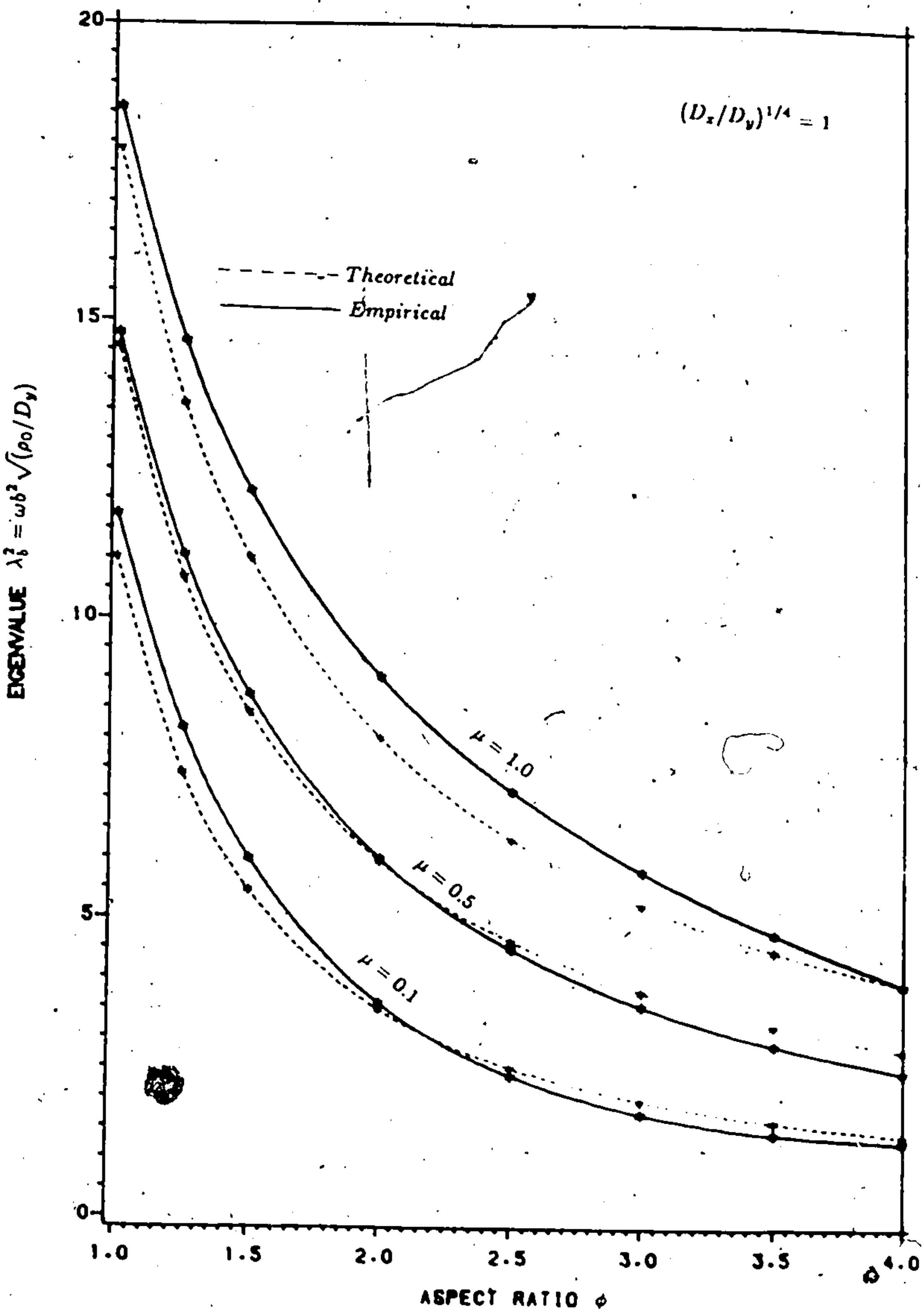

Figure 18: Variation of Eigenvalues with aspect ratio for symmetric-antisymmetric modes.

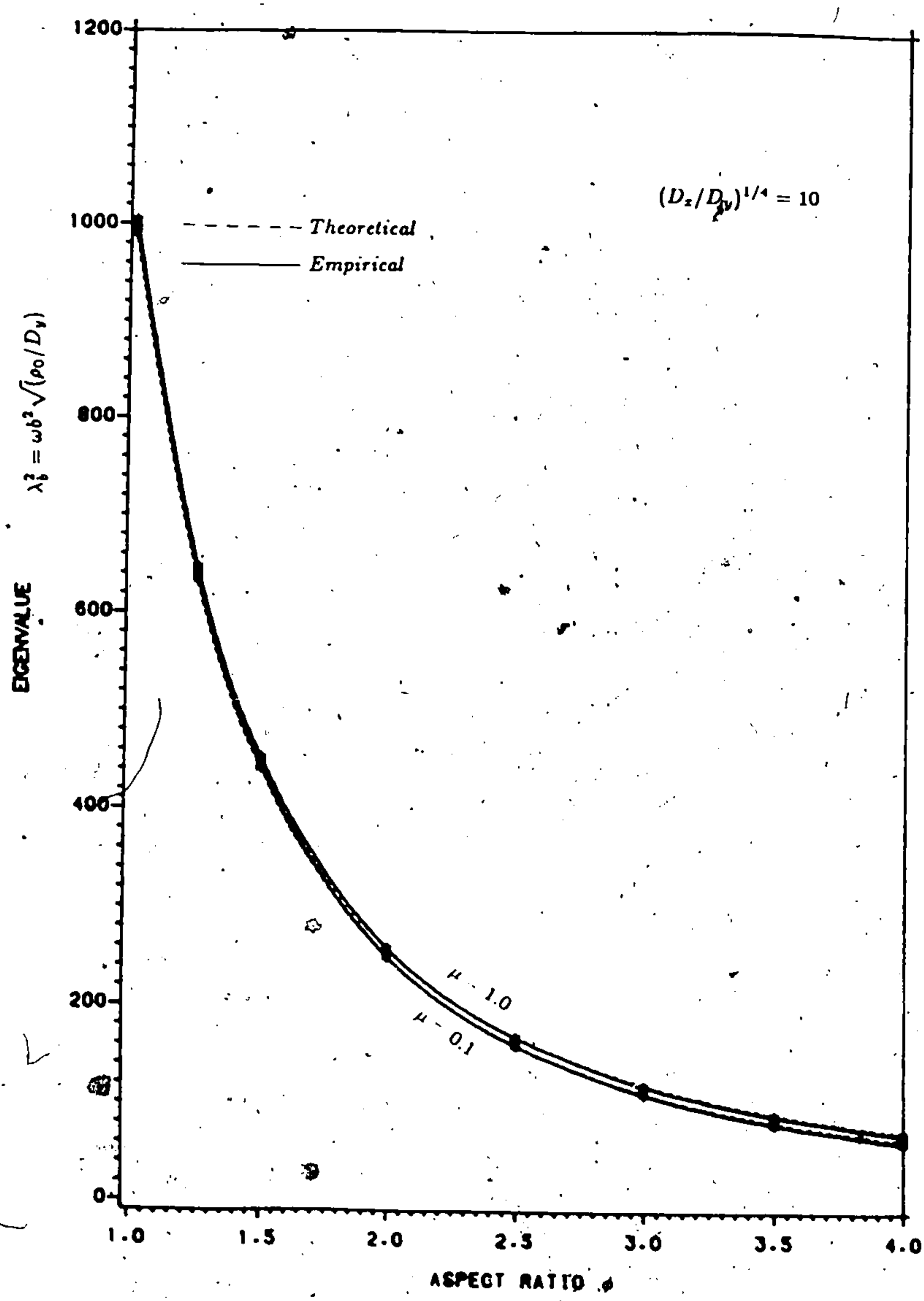

Figure. 19: Variation of Eigenvalues with aspect ratio for symmetric-antisymmetric modes

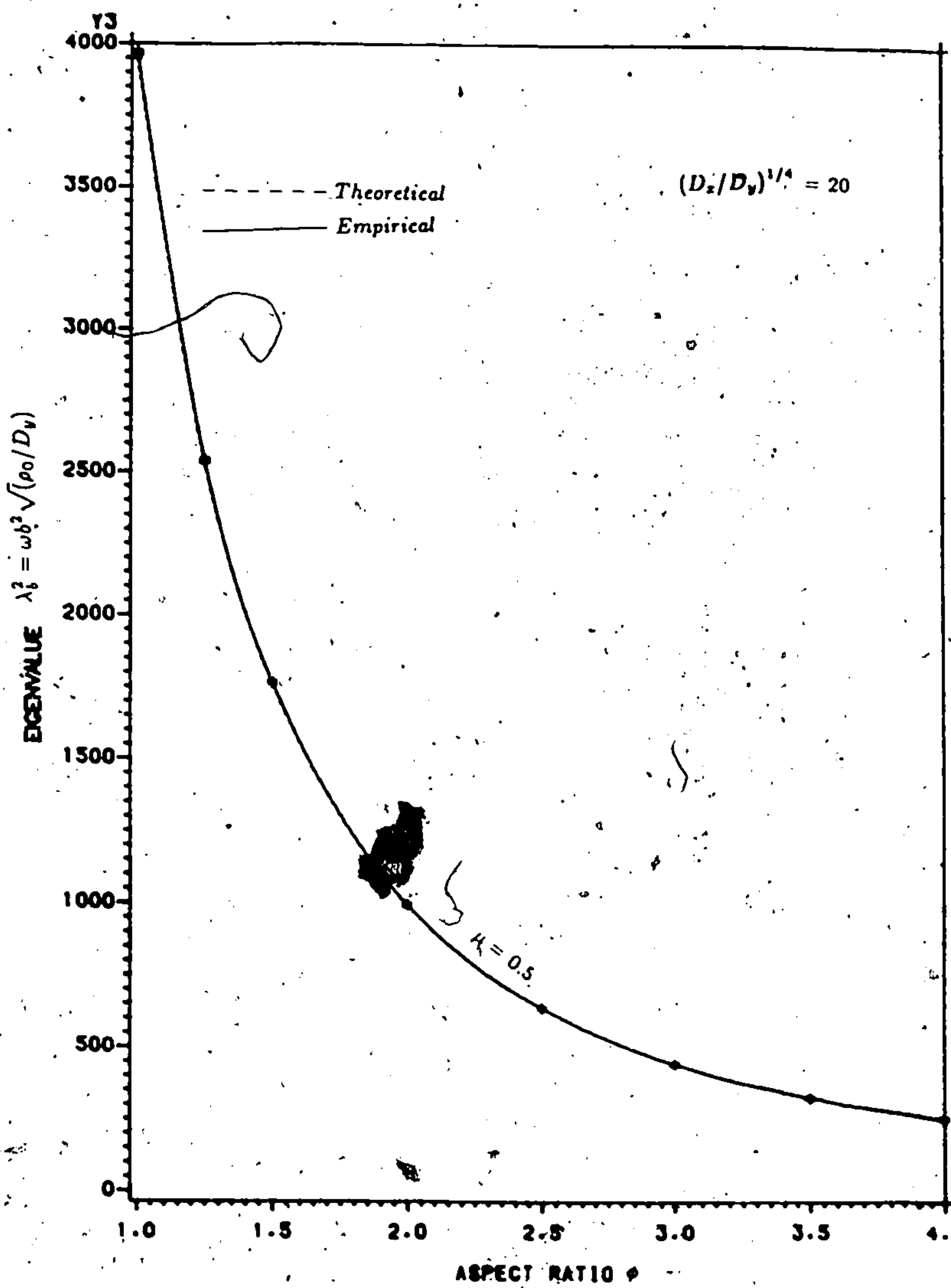

Figure 20: Variation of Eigenvalues with aspect ratio for symmetric antisymmetric modes

Table 1: Convergence table for free vibration of continuous orthotropic plate.

Doubly symmetric mode $\phi=1.0$ $D_x = D_y = 1.0$ $D_1 = 0.0$; $D_{xy} = 0.5$

No of Terms	Eigenvalue
3	15.418291
5	15.418290
7	15.418289
10	15.418289

Table 2: Eigenvalues of a two span continuous orthotropic plate.

Aspect ratio $\phi=1.0$ $D_x = D_y = 1.0$ $D_1 = 0.0$; $D_{xy} = 0.5$

Freq.	Trans. Strip(4)	Long. Strip(4)	Ref. (44)	Present Solution
	9.87	9.63	9.86	9.8707
	15.43	15.41	.	15.4183
	.	.	.	17.2584
	17.92	16.83	17.17	17.8821
	22.21	22.16	.	22.8130
	.	.	.	32.3612
	39.65	38.25	39.48	39.4785

Table 3: Eigenvalues for doubly symmetric mode $(D_x/D_y)^{1/4} = 1,2,3,4.$

ϕ \ μ	1.0	0.9	0.8	0.7	0.6	0.5	0.4	0.3	0.2	0.1
1.00	15.4183	15.4182	15.4183	15.4182	15.4183	15.4183	15.4182	15.4181	15.4182	15.4182
1.25	9.8677	9.8677	9.8677	9.8677	9.8677	9.8677	9.8677	9.8677	9.8677	9.8676
1.50	6.8525	6.8525	6.8526	6.8526	6.8526	6.8526	6.8526	6.8526	6.8526	6.8526
2.00	3.8546	3.8546	3.8546	3.8546	3.8546	3.8546	3.8546	3.8546	3.8546	3.8546
2.50	2.4670	2.4670	2.4669	2.4669	2.4670	2.4670	2.4669	2.4669	2.4669	2.4670
3.00	1.7131	1.7131	1.7131	1.7131	1.7131	1.7131	1.7131	1.7131	1.7131	1.7131
3.50	1.2586	1.2586	1.2586	1.2586	1.2586	1.2586	1.2586	1.2586	1.2586	1.2586
4.00	0.9636	0.9636	0.9636	0.9636	0.9636	0.9636	0.9636	0.9636	0.9636	0.9636
1.00	61.6729	61.6729	61.6729	61.6729	61.6728	61.6728	61.6728	61.6729	61.6729	61.6727
1.25	39.4707	39.4706	39.4707	39.4707	39.4707	39.4706	39.4707	39.4706	39.4706	39.4706
1.50	27.4101	27.4101	27.4101	27.4101	27.4101	27.4102	27.4102	27.4101	27.4101	27.4101
2.00	15.4183	15.4182	15.4183	15.4182	15.4183	15.4183	15.4182	15.4181	15.4182	15.4182
2.50	9.8677	9.8677	9.8677	9.8677	9.8677	9.8677	9.8677	9.8677	9.8677	9.8676
3.00	6.8525	6.8525	6.8525	6.8525	6.8525	6.8525	6.8525	6.8525	6.8525	6.8525
3.50	5.0345	5.0345	5.0345	5.0345	5.0345	5.0345	5.0345	5.0345	5.0345	5.0345
4.00	3.8546	3.8546	3.8546	3.8546	3.8546	3.8546	3.8546	3.8546	3.8546	3.8546
1.00	138.7639	138.7639	138.7639	138.7639	138.7639	138.7639	138.7639	138.7639	138.7639	138.7638
1.25	88.8089	88.8089	88.8089	88.8088	88.8089	88.8089	88.8089	88.8088	88.8088	88.8088
1.50	61.6729	61.6729	61.6729	61.6729	[illegible].6728	61.6728	61.6728	61.6729	61.6729	61.6727
2.00	34.6910	34.6910	34.6911	34.6910	34.6910	34.6910	34.6910	34.6910	34.6910	34.6910
2.50	22.2023	22.2022	22.2022	22.2021	22.2022	22.2022	22.2022	22.2022	22.2022	22.2022
3.00	15.4182	15.4182	15.4182	15.4182	15.4182	15.4182	15.4182	15.4182	15.4182	15.4182
3.50	11.3277	11.3277	11.3277	11.3277	11.3277	11.3277	11.3277	11.3277	11.3277	11.3277
4.00	8.6727	8.6727	8.6727	8.6727	8.6727	8.6727	8.6727	8.6727	8.6727	8.6727
1.00	246.6913	246.6913	246.6914	246.6914	246.6914	246.6913	246.6913	246.6913	246.6914	246.6914
1.25	157.8825	157.8824	157.8824	157.8824	157.8825	157.8825	157.8825	157.8824	157.8825	157.8825
1.50	109.6405	109.6407	109.6407	109.6406	109.6406	109.6407	109.6406	109.6406	109.6406	109.6405
2.00	61.6729	61.6729	61.6729	61.6729	61.6728	61.6728	61.6728	61.6729	61.6729	61.6727
2.50	39.4707	39.4706	39.4707	39.4707	39.4707	39.4706	39.4707	39.4706	39.4706	39.4706
3.00	27.4101	27.4101	27.4101	27.4101	27.4101	27.4101	27.4101	27.4101	27.4101	27.4101
3.50	20.1381	20.1381	20.1381	20.1381	20.1381	20.1381	20.1381	20.1381	20.1381	20.1381
4.00	15.4182	15.4182	15.4182	15.4182	15.4182	15.4182	15.4182	15.4182	15.4182	15.4182

Table 4: Eigenvalues for doubly symmetric mode $(D_x/D_y)^{1/4} = 5,6,7,8.$

ϕ \ μ	1.0	0.9	0.8	0.7	0.6	0.5	0.4	0.3	0.2	0.1
1.00	385.4552	385.4552	385.4552	385.4552	385.4552	385.4552	385.4552	385.4552	385.4552	385.4550
1.25	246.6913	246.6913	246.6914	246.6914	246.6914	246.6913	246.6913	246.6913	246.6914	246.6914
1.50	171.3135	171.3135	171.3135	171.3134	171.3135	171.3134	171.3135	171.3134	171.3133	171.3133
2.00	96.3638	96.3638	96.3639	96.3638	96.3639	96.3639	96.3639	96.3638	96.3639	96.3637
2.50	61.6729	61.6729	61.6729	61.6729	61.6728	61.6728	61.6728	61.6729	61.6729	61.6727
3.00	42.8283	42.8284	42.8284	42.8284	42.8284	42.6283	42.8283	42.8283	42.8284	42.8284
3.50	31.4657	31.4657	31.4657	31.4657	31.4657	31.4657	31.4657	31.4657	31.4657	31.4657
4.00	24.0909	24.0909	24.0909	24.0909	24.0909	24.0909	24.0909	24.0909	24.0909	24.0909
1.00	555.0554	555.0554	555.0555	555.0554	555.0555	555.0554	555.0555	555.0555	555.0555	555.0554
1.25	355.2355	355.2355	355.2355	355.2355	355.2355	355.2355	355.2355	355.2355	355.2354	355.2354
1.50	246.6913	246.6913	246.6914	246.6914	246.6914	246.6913	246.6913	246.6913	246.6914	246.6914
2.00	138.7639	138.7639	138.7639	138.7639	138.7639	138.7639	138.7639	138.7639	138.7639	138.7638
2.50	88.8089	88.8089	88.8089	88.8088	88.8089	88.8089	88.8089	88.8088	88.8088	88.8088
3.00	61.6728	61.6728	61.6728	61.6728	61.6728	61.6728	61.6728	61.6728	61.6728	61.6728
3.50	45.3106	45.3106	45.3106	45.3106	45.3106	45.3106	45.3106	45.3106	45.3106	45.3106
4.00	34.6910	34.6910	34.6910	34.6910	34.6910	34.6910	34.6910	34.6910	34.6910	34.6910
1.00	755.4921	755.4921	755.4921	755.4921	755.4921	755.4922	755.4921	755.4922	755.4922	755.4921
1.25	483.5150	483.5150	483.5150	483.5150	483.5150	483.5149	483.5149	483.5149	483.5149	483.5149
1.50	335.7743	335.7743	335.7743	335.7743	335.7744	335.7744	335.7744	335.7743	335.7743	335.7742
2.00	188.8731	188.8731	188.8730	188.8730	188.8730	188.8731	188.8731	188.8730	188.8730	188.8729
2.50	120.8788	120.8787	120.8788	120.8788	120.8787	120.8788	120.8787	120.8788	120.8788	120.8787
3.00	83.9436	83.9436	83.9436	83.9436	83.9436	83.9436	83.9436	83.9436	83.9436	83.9436
3.50	61.6728	61.6728	61.6728	61.6728	61.6728	61.6728	61.6728	61.6728	61.6728	61.6728
4.00	47.2183	47.2183	47.2183	47.2183	47.2183	47.2183	47.2183	47.2183	47.2183	47.2183
1.00	986.7652	986.7652	986.7652	986.7652	986.7653	986.7652	986.7652	986.7652	986.7653	986.7651
1.25	631.5297	631.5298	631.5298	631.5298	631.5297	631.5298	631.5298	631.5297	631.5297	631.5296
1.50	438.5624	438.5624	438.5623	438.5624	438.5623	438.5624	438.5624	438.5623	438.5624	438.5622
2.00	246.6913	246.6913	246.6914	246.6914	246.6914	246.6913	246.6913	246.6913	246.6914	246.6914
2.50	157.8825	157.8824	157.8824	157.8824	157.8825	157.8825	157.8825	157.8824	157.8825	157.8825
3.00	109.6406	109.6406	109.6406	109.6406	109.6406	109.6406	109.6406	109.6406	109.6406	109.6406
3.50	80.5523	80.5523	80.5523	80.5523	80.5523	80.5523	80.5523	80.5523	80.5523	80.5523
4.00	61.6728	61.6728	61.6728	61.6728	61.6728	61.6728	61.6728	61.6728	61.6728	61.6728

Table 5: Eigenvalues for doubly symmetric mode (Dx/Dy) = 9,10,11,12.

ϕ \ μ	1.0	0.9	0.8	0.7	0.6	0.5	0.4	0.3	0.2	0.1
1.00	1248.8747	1248.8747	1248.8748	1248.8747	1248.8747	1248.8747	1248.8747	1248.8747	1248.8747	1248.8747
1.25	799.2798	799.2798	799.2798	799.2798	799.2799	799.2798	799.2798	799.2798	799.2798	799.2797
1.50	555.0554	555.0554	555.0555	555.0554	555.0555	555.0554	555.0555	555.0555	555.0555	555.0554
2.00	312.2187	312.2187	312.2187	312.2187	312.2187	312.2187	312.2187	312.2186	312.2186	312.2187
2.50	199.8200	199.8200	199.8200	199.8199	199.8200	199.8200	199.8200	199.8200	199.8199	199.8199
3.00	138.7639	138.7639	138.7639	138.7639	138.7639	138.7639	138.7639	138.7639	138.7639	138.7639
3.50	101.9490	101.9490	101.9490	101.9490	101.9490	101.9490	101.9490	101.9490	101.9490	101.9490
4.00	78.0547	78.0547	78.0547	78.0547	78.0547	78.0547	78.0547	78.0547	78.0547	78.0547
1.00	1541.8206	1541.8206	1541.8206	1541.8207	1541.8206	1541.8206	1541.8206	1541.8206	1541.8205	1541.8205
1.25	986.7652	986.7652	986.7652	986.7652	986.7653	986.7652	986.7652	986.7652	986.7653	986.7651
1.50	685.2536	685.2537	685.2537	685.2536	685.2536	685.2536	685.2537	685.2537	685.2536	685.2535
2.00	385.4552	385.4552	385.4552	385.4552	385.4552	385.4552	385.4552	385.4552	385.4552	385.4550
2.50	246.6913	246.6913	246.6914	246.6914	246.6914	246.6913	246.6913	246.6913	246.6914	246.6914
3.00	171.3134	171.3134	171.3134	171.3134	171.3134	171.3134	171.3134	171.3134	171.3134	171.3134
3.50	125.8629	125.8629	125.8629	125.8629	125.8629	125.8629	125.8629	125.8629	125.8629	125.8629
4.00	96.3638	96.3638	96.3638	96.3638	96.3638	96.3638	96.3638	96.3638	96.3638	96.3638
1.00	1865.6029	1865.6029	1865.6029	1865.6029	1865.6029	1865.6030	1865.6028	1865.6028	1865.6028	1865.6029
1.25	1193.9859	1193.9859	1193.9859	1193.9859	1193.9859	1193.9859	1193.9859	1193.9859	1193.9859	1193.9859
1.50	829.1569	829.1569	829.1569	829.1568	829.1569	829.1568	829.1569	829.1569	829.1569	829.1568
2.00	466.4008	466.4008	466.4008	466.4008	466.4008	466.4007	466.4006	466.4006	466.4007	466.4006
2.50	298.4965	298.4965	298.4965	298.4965	298.4965	298.4965	298.4965	298.4965	298.4965	298.4965
3.00	207.2892	207.2892	207.2892	207.2892	207.2892	207.2892	207.2892	207.2892	207.2892	207.2892
3.50	152.2941	152.2941	152.2941	152.2941	152.2941	152.2941	152.2941	152.2941	152.2941	152.2941
4.00	116.6002	116.6002	116.6002	116.6002	116.6002	116.6002	116.6002	116.6002	116.6002	116.6002
1.00	2220.2217	2220.2217	2220.2217	2220.2217	2220.2216	2220.2216	2220.2217	2220.2216	2220.2216	2220.2216
1.25	1420.9419	1420.9419	1420.9419	1420.9419	1420.9419	1420.9419	1420.9419	1420.9419	1420.9419	1420.9418
1.50	986.7652	986.7652	986.7652	986.7652	986.7653	986.7652	986.7652	986.7652	986.7653	986.7651
2.00	555.0554	555.0554	555.0555	555.0554	555.0555	555.0554	555.0555	555.0555	555.0555	555.0554
2.50	355.2355	355.2355	355.2355	355.2355	355.2355	355.2355	355.2355	355.2355	355.2354	355.2354
3.00	246.6913	246.6913	246.6913	246.6913	246.6913	246.6913	246.6913	246.6913	246.6913	246.6913
3.50	181.2426	181.2426	181.2426	181.2426	181.2426	181.2426	181.2426	181.2426	181.2426	181.2426
4.00	138.7639	138.7639	138.7639	138.7639	138.7639	138.7639	138.7639	138.7639	138.7639	138.7639

Table 6: Eigenvalues for doubly symmetric mode $(D_x/D_y)^{1/4}$ = 13,14,15,16.

ϕ \ μ	1.0	0.9	0.8	0.7	0.6	0.5	0.4	0.3	0.2	0.1
1.00	2605.6768	2605.6768	2605.6768	2605.6768	2605.6768	2605.6768	2605.6768	2605.6768	2605.6768	2605.6767
1.25	1667.6332	1667.6331	1667.6332	1667.6332	1667.6332	1667.6332	1667.6331	1667.6332	1667.6330	1667.6330
1.50	1158.0786	1158.0786	1158.0786	1158.0787	1158.0786	1158.0786	1158.0786	1158.0786	1158.0786	1158.0786
2.00	651.4192	651.4192	651.4193	651.4192	651.4192	651.4192	651.4192	651.4192	651.4192	651.4192
2.50	416.9083	416.9083	416.9083	416.9083	416.9083	416.9083	416.9083	416.9082	416.9082	416.9082
3.00	289.5196	289.5196	289.5196	289.5196	289.5196	289.5196	289.5196	289.5196	289.5196	289.5196
3.50	212.7083	212.7083	212.7083	212.7083	212.7083	212.7083	212.7083	212.7083	212.7083	212.7083
4.00	162.8548	162.8548	162.8548	162.8548	162.8548	162.8548	162.8548	162.8548	162.8548	162.8548
1.00	3021.9684	3021.9684	3021.9684	3021.9683	3021.9683	3021.9684	3021.9684	3021.9684	3021.9684	3021.9683
1.25	1934.0598	1934.0598	1934.0597	1934.0598	1934.0598	1934.0597	1934.0597	1934.0598	1934.0598	1934.0596
1.50	1343.0971	1343.0971	1343.0971	1343.0971	1343.0971	1343.0971	1343.0971	1343.0971	1343.0971	1343.0971
2.00	755.4921	755.4921	755.4922	755.4921	755.4921	755.4921	755.4922	755.4921	755.4921	755.4920
2.50	483.5150	483.5150	483.5150	483.5150	483.5150	483.5149	483.5149	483.5149	483.5149	483.5149
3.00	335.7743	335.7743	335.7743	335.7743	335.7743	335.7743	335.7743	335.7743	335.7743	335.7743
3.50	246.6913	246.6913	246.6913	246.6913	246.6913	246.6913	246.6913	246.6913	246.6913	246.6913
4.00	188.8730	188.8730	188.8730	188.8730	188.8730	188.8730	188.8730	188.8730	188.8730	188.8730
1.00	3469.0963	3469.0964	3469.0964	3469.0964	3469.0964	3469.0963	3469.0963	3469.0963	3469.0963	3469.0963
1.25	2220.2217	2220.2217	2220.2217	2220.2217	2220.2216	2220.2216	2220.2217	2220.2216	2220.2216	2220.2216
1.50	1541.8206	1541.8206	1541.8206	1541.8207	1541.8206	1541.8206	1541.8206	1541.8206	1541.8205	1541.8205
2.00	867.2741	867.2741	867.2741	867.2742	867.2741	867.2742	867.2741	867.2741	867.2741	867.2740
2.50	555.0554	555.0554	555.0555	555.0554	555.0555	555.0554	555.0555	555.0555	555.0555	555.0554
3.00	385.4551	385.4551	385.4551	385.4551	385.4551	385.4551	385.4551	385.4551	385.4551	385.4551
3.50	283.1915	283.1915	283.1915	283.1915	283.1915	283.1915	283.1915	283.1915	283.1915	283.1915
4.00	216.8185	216.8185	216.8185	216.8185	216.8185	216.8185	216.8185	216.8185	216.8185	216.8185
1.00	3947.0607	3947.0607	3947.0607	3947.0607	3947.0607	3947.0607	3947.0607	3947.0607	3947.0607	3947.0606
1.25	2526.1189	2526.1189	2526.1189	2526.1188	2526.1189	2526.1189	2526.1187	2526.1187	2526.1188	2526.1188
1.50	1754.2493	1754.2492	1754.2493	1754.2492	1754.2493	1754.2493	1754.2492	1754.2492	1754.2493	1754.2492
2.00	986.7652	986.7652	986.7652	986.7652	986.7653	986.7652	986.7652	986.7652	986.7653	986.7651
2.50	631.5297	631.5298	631.5298	631.5298	631.5297	631.5298	631.5298	631.5297	631.5297	631.5296
3.00	438.5623	438.5623	438.5623	438.5623	438.5623	438.5623	438.5623	438.5623	438.5623	438.5623
3.50	322.2090	322.2090	322.2090	322.2090	322.2090	322.2090	322.2090	322.2090	322.2090	322.2090
4.00	246.6913	246.6913	246.6913	246.6913	246.6913	246.6913	246.6913	246.6913	246.6913	246.6913

Table 7: Eigenvalues for doubly symmetric mode $(D_x/D_y)^{1/4}$ = 17, 18, 19, 20.

ϕ \ μ	1.0	0.9	0.8	0.7	0.6	0.5	0.4	0.3	0.2	0.1
1.00	4455.8615	4455.8615	4455.8615	4455.8615	4455.8615	4455.8615	4455.8615	4455.8615	4455.8615	4455.8614
1.25	2851.7514	2851.7513	2851.7514	2851.7514	2851.7514	2851.7514	2851.7514	2851.7514	2851.7514	2851.7512
1.50	1980.3829	1980.3829	1980.3830	1980.3830	1980.3829	1980.3829	1980.3829	1980.3829	1980.3830	1980.3829
2.00	1113.9654	1113.9654	1113.9655	1113.9654	1113.9654	1113.9654	1113.9654	1113.9654	1113.9655	1115.8087
2.50	712.9379	712.9378	712.9378	712.9379	712.9379	712.9379	712.9379	712.9379	712.9378	712.9377
3.00	495.0957	495.0957	495.0957	495.0957	495.0957	495.0957	495.0957	495.0957	495.0957	495.0957
3.50	363.7438	363.7438	363.7438	363.7438	363.7438	363.7438	363.7438	363.7438	363.7438	363.7438
4.00	278.4913	278.4913	278.4913	278.4913	278.4913	278.4913	278.4913	278.4913	278.4913	278.4913
1.00	4995.4987	4995.4988	4995.4987	4995.4987	4995.4987	4995.4987	4995.4987	4995.4987	4995.4987	4995.4987
1.25	3197.1192	3197.1191	3197.1192	3197.1192	3197.1192	3197.1192	3197.1192	3197.1190	3197.1191	3197.1191
1.50	2220.2217	2220.2217	2220.2217	2220.2217	2220.2216	2220.2216	2220.2217	2220.2216	2220.2216	2220.2216
2.00	1248.8747	1248.8747	1248.8748	1248.8747	1248.8747	1248.8747	1248.8747	1248.8747	1248.8747	1248.8747
2.50	799.2798	799.2798	799.2798	799.2798	799.2799	799.2798	799.2798	799.2798	799.2798	799.2797
3.00	555.0554	555.0554	555.0554	555.0554	555.0554	555.0554	555.0554	555.0554	555.0554	555.0554
3.50	407.7958	407.7958	407.7958	407.7958	407.7958	407.7958	407.7958	407.7958	407.7958	407.7958
4.00	312.2187	312.2187	312.2187	312.2187	312.2187	312.2187	312.2187	312.2187	312.2187	312.2187
1.00	5565.9724	5565.9723	5565.9724	5565.9723	5565.9723	5565.9723	5565.9723	5565.9723	5565.9724	5565.9723
1.25	3562.2223	3562.2223	3562.2223	3562.2223	3562.2223	3562.2223	3562.2223	3562.2222	3562.2222	3562.2222
1.50	2473.7655	2473.7655	2473.7655	2473.7655	2473.7655	2473.7655	2473.7655	2473.7655	2473.7655	2473.7654
2.00	1391.4931	1391.4932	1391.4931	1391.4931	1391.4931	1391.4931	1391.4931	1391.4931	1391.4931	1391.4931
2.50	890.5556	890.5556	890.5556	890.5556	890.5556	890.5556	890.5556	890.5556	890.5556	890.5555
3.00	618.4414	618.4414	618.4414	618.4414	618.4414	618.4414	618.4414	618.4414	618.4414	618.4414
3.50	454.3651	454.3651	454.3651	454.3651	454.3651	454.3651	454.3651	454.3651	454.3651	454.3651
4.00	347.8733	347.8733	347.8733	347.8733	347.8733	347.8733	347.8733	347.8733	347.8733	347.8733
1.00	6167.2824	6167.2823	6167.2823	6167.2824	6167.2824	6167.2824	6167.2824	6167.2823	6167.2823	6167.2823
1.25	3947.0607	3947.0607	3947.0607	3947.0607	3947.0607	3947.0607	3947.0607	3947.0607	3947.0607	3947.0606
1.50	2741.0144	2741.0144	2741.0144	2741.0144	2741.0144	2741.0144	2741.0144	2741.0143	2741.0143	2741.0143
2.00	1541.8206	1541.8206	1541.8206	1541.8207	1541.8206	1541.8206	1541.8206	1541.8206	1541.8205	1541.8205
2.50	986.7652	986.7652	986.7652	986.7652	986.7653	986.7652	986.7652	986.7652	986.7653	986.7651
3.00	685.2536	685.2536	685.2536	685.2536	685.2536	685.2536	685.2536	685.2536	685.2536	685.2536
3.50	503.4516	503.4516	503.4516	503.4516	503.4516	503.4516	503.4516	503.4516	503.4516	503.4516
4.00	385.4551	385.4551	385.4551	385.4551	385.4551	385.4551	385.4551	385.4551	385.4551	385.4551

Table 8: Eigenvalues for antisymmetric - symmetric mode $(D_x/D_y)^{1/4}$ = 1,2,3,4.

φ \ μ	1.0	0.9	0.8	0.7	0.6	0.5	0.4	0.3	0.2	0.1
1.00	17.2584	17.0913	16.9209	16.7428	16.5085	16.3131	16.1093	15.8932	15.6586	15.3914
1.25	11.6840	11.5241	11.3609	11.1919	10.9641	10.7598	10.5570	10.3445	10.1164	9.8644
1.50	8.6331	8.4814	8.3267	8.1691	7.9424	7.7403	7.5408	7.3321	7.1083	6.8624
2.00	5.5392	5.4041	5.2663	5.1340	4.9173	4.7255	4.5367	4.3382	4.1243	3.8886
2.50	4.0407	3.9206	3.7987	3.6930	3.4887	3.3122	3.1374	2.9524	2.7509	2.5261
3.00	3.1759	3.0685	2.9588	2.8766	2.6853	2.5270	2.3671	2.2754	1.9613	1.7158
3.50	2.6169	2.5204	2.4212	2.3528	2.1790	2.0713	2.0567	1.7570	1.5502	1.3419
4.00	2.2266	2.1394	2.0492	1.9890	1.8685	1.8618	1.6503	1.4561	1.2797	1.0911
1.00	63.5243	63.3450	63.1650	62.9809	62.7601	62.5669	62.3702	62.1642	61.9374	61.6458
1.25	41.3230	41.1463	40.9683	40.7848	40.5578	40.3636	40.1641	39.9546	39.7239	39.4408
1.50	29.2614	29.0875	28.9119	28.7296	28.4987	28.3011	28.1004	27.8892	27.6583	27.3868
2.00	17.2584	17.0913	16.9209	16.7428	16.5085	16.3131	16.1093	15.8932	15.6586	15.3914
2.50	11.6840	11.5241	11.3609	11.1919	10.9641	10.7598	10.5570	10.3445	10.1164	9.8644
3.00	8.6331	8.4814	8.3266	8.1690	7.9423	7.7403	7.5408	7.3320	7.1082	6.8624
3.50	6.7702	6.6271	6.4806	6.3353	6.1130	5.9152	5.7205	5.5163	5.2971	5.0563
4.00	5.5392	5.4042	5.2662	5.1340	4.9174	4.7256	4.5367	4.3383	4.1244	3.8885
1.00	140.6118	140.4299	140.2477	140.0634	139.8538	139.6642	139.4713	139.2714	139.0528	138.7401
1.25	90.6588	90.4781	90.2968	90.1126	89.8969	89.7059	89.5110	89.3075	89.0832	88.7768
1.50	63.5243	63.3450	63.1650	62.9809	62.7601	62.5669	62.3702	62.1642	61.9374	61.6458
2.00	36.5433	36.3675	36.1901	36.0070	35.7788	35.5831	35.3832	35.1730	34.9425	34.6643
2.50	24.0511	23.8794	23.7055	23.5242	23.2910	23.0923	22.8910	22.6789	22.4479	22.1805
3.00	17.2583	17.0912	16.9209	16.7428	16.5084	16.3131	16.1093	15.8933	15.6587	15.3914
3.50	13.1532	12.9910	12.8248	12.6527	12.4237	12.2209	12.0171	11.8037	11.5740	11.3187
4.00	10.4783	10.3210	10.1606	9.9953	9.7674	9.5636	9.3618	9.1505	8.9238	8.6742
1.00	248.5374	248.3544	248.1712	247.9870	247.7841	247.5959	247.4057	247.2099	246.9987	246.6784
1.25	159.7299	159.5478	159.3653	159.1809	158.9730	158.7836	158.5917	158.3926	158.1753	157.8566
1.50	111.4895	111.3082	111.1264	110.9423	110.7295	110.5387	110.3449	110.1443	109.9216	109.6006
2.00	63.5243	63.3450	63.1650	62.9809	62.7601	62.5669	62.3702	62.1642	61.9374	61.6458
2.50	41.3230	41.1463	40.9683	40.7848	40.5578	40.3636	40.1641	39.9546	39.7239	39.4408
3.00	29.2613	29.0875	28.9119	28.7295	28.4986	28.3011	28.1004	27.8892	27.6584	27.3876
3.50	21.9852	21.8146	21.6416	21.4610	21.2268	21.0278	20.8262	20.6174	20.3803	20.0894
4.00	17.2583	17.0912	16.9209	16.7428	16.5084	16.3131	16.1093	15.8933	15.6587	15.3914

Table 9: Eigenvalues for antisymmetric - symmetric mode $(D_x/D_y)^{1/4}$ = 5,6,7,8.

ϕ \ μ	1.0	0.9	0.8	0.7	0.6	0.5	0.4	0.3	0.2	0.1
1.00	387.3000	387.1167	386.9332	386.7489	386.5502	386.3630	386.1740	385.9812	385.7761	385.4610
1.25	248.5374	248.3544	248.1712	247.9870	247.7841	247.5959	247.4057	247.2099	246.9987	246.6784
1.50	173.1606	172.9782	172.7956	172.6113	172.4044	172.2152	172.0235	171.8253	171.6088	171.2854
2.00	98.2134	98.0324	97.8509	97.6666	97.4521	97.2611	97.0672	96.8642	96.6403	96.3286
2.50	63.5243	63.3450	63.1650	62.9809	62.7601	62.5669	62.3702	62.1642	61.9374	61.6458
3.00	44.6807	44.5034	44.3250	44.1412	43.9152	43.7220	43.5228	43.3135	43.0828	42.7956
3.50	33.3177	33.1427	32.9660	32.7832	32.5542	32.3573	32.1571	31.9465	31.7159	31.4407
4.00	25.9409	25.7683	25.5937	25.4120	25.1795	24.9813	24.7802	24.5685	24.3375	24.0686
1.00	556.8996	556.7160	556.5323	556.3479	556.1521	555.9655	555.7776	555.5866	555.3859	555.0921
1.25	357.0806	356.8973	356.7137	356.5295	356.3301	356.1428	355.9536	355.7602	355.5540	355.2395
1.50	248.5374	248.3544	248.1712	247.9870	247.7841	247.5959	247.4057	247.2099	246.9987	246.6784
2.00	140.6118	140.4299	140.2477	140.0634	139.8538	139.6642	139.4713	139.2714	139.0528	138.7401
2.50	90.6588	90.4781	90.2969	90.1126	89.8968	89.7059	89.5110	89.3075	89.0831	88.7767
3.00	63.5242	63.3450	63.1649	62.9809	62.7600	62.5669	62.3702	62.1642	61.9374	61.6457
3.50	47.1629	46.9853	46.8064	46.6228	46.3973	46.2040	46.0058	45.7967	45.5659	45.2754
4.00	36.5432	36.3673	36.1901	36.0069	35.7787	35.5830	35.3832	35.1730	34.9425	34.6643
1.00	757.3358	757.1521	756.9682	756.7839	756.5899	756.4038	756.2166	756.0270	755.8294	755.5456
1.25	485.3594	485.1758	484.9922	484.8079	484.6110	484.4243	484.2360	484.0443	483.8419	483.5375
1.50	337.6195	337.4362	337.2528	337.0685	336.8686	336.6812	336.4919	336.2981	336.0911	335.7779
2.00	190.7199	190.5374	190.3546	190.1703	189.9645	189.7757	189.5840	189.3866	189.1726	188.8460
2.50	122.7274	122.5457	122.3637	122.1795	121.9679	121.7778	121.5843	121.3835	121.1635	120.8554
3.00	85.7937	85.6132	85.4321	85.2479	85.0313	84.8403	84.6449	84.4410	84.2164	83.9130
3.50	63.5242	63.3450	63.1649	62.9809	62.7600	62.5669	62.3702	62.1642	61.9374	61.6457
4.00	49.0705	48.8926	48.7137	48.5298	48.3048	48.1115	47.9141	47.7051	47.4742	47.1809
1.00	988.6087	988.4248	988.2409	988.0566	987.8640	987.6781	987.4914	987.3027	987.1073	986.8242
1.25	633.3738	633.1900	633.0063	632.8219	632.6268	632.4406	632.2529	632.0626	631.8631	631.5698
1.50	440.4069	440.2234	440.0399	439.8555	439.6579	439.4710	439.2825	439.0902	438.8867	438.5829
2.00	248.5374	248.3544	248.1712	247.9870	247.7841	247.5959	247.4057	247.2099	246.9987	246.6784
2.50	159.7299	159.5478	159.3653	159.1809	158.9730	158.7836	158.5917	158.3926	158.1753	157.8566
3.00	111.4895	111.3082	111.1264	110.9422	110.7294	110.5387	110.3448	110.1443	109.9215	109.6007
3.50	82.4026	82.2222	82.0412	81.8570	81.6399	81.4486	81.2530	81.0489	80.8240	80.5226
4.00	63.5242	63.3450	63.1649	62.9809	62.7600	62.5669	62.3702	62.1642	61.9374	61.6457

Table 10: Eigenvalues for antisymmetric - symmetric mode $(D_x/D_y)^{1/4}$ = 9,10,11,12.

ϕ \ μ	1.0	0.9	0.8	0.7	0.6	0.5	0.4	0.3	0.2	0.1
1.00	1250.7179	1250.5339	1250.3499	1250.1657	1249.9741	1249.7886	1249.6023	1249.4141	1249.2192	1248.6136
1.25	801.1235	800.9397	800.7558	800.5715	800.3779	800.1918	800.0046	799.8154	799.6182	799.3342
1.50	556.8996	556.7160	556.5323	556.3479	556.1521	555.9655	555.7776	555.5866	555.3859	555.0921
2.00	314.0641	313.8809	313.6976	313.5133	313.3127	313.1251	312.9355	312.7411	312.5335	312.2213
2.50	201.6666	201.4840	201.3012	201.1169	200.9117	200.7231	200.5317	200.3346	200.1210	199.8066
3.00	140.6118	140.4299	140.2476	140.0634	139.8537	139.6642	139.4712	139.2713	139.0528	138.7401
3.50	103.7982	103.6171	103.4355	103.2512	103.0375	102.8465	102.6529	102.4506	102.2271	101.9113
4.00	79.9051	79.7249	79.5440	79.3598	79.1423	78.9507	78.7550	78.5506	78.3256	78.0265
1.00	1543.6637	1543.4797	1543.2957	1543.1113	1542.9206	1542.7353	1542.5491	1542.3607	1542.1076	1541.9862
1.25	988.6087	988.4248	988.2409	988.0566	987.8640	987.6781	987.4914	987.3027	987.1073	986.8242
1.50	687.0975	686.9137	686.7299	686.5457	686.3511	686.1649	685.9774	685.7873	685.5891	685.2984
2.00	387.3000	387.1167	386.9332	386.7489	386.5502	386.3630	386.1740	385.9812	385.7761	385.4610
2.50	248.5374	248.3544	248.1712	247.9870	247.7841	247.5959	247.4057	247.2099	246.9987	246.6784
3.00	173.1606	172.9782	172.7956	172.6113	172.4043	172.2151	172.0235	171.8253	171.6088	171.2854
3.50	127.7113	127.5296	127.3475	127.1632	126.9523	126.7623	126.5689	126.3684	126.1489	125.8396
4.00	98.2133	98.0323	97.8509	97.6666	97.4520	97.2611	97.0671	96.8641	96.6402	96.3286
1.00	1867.4459	1867.2618	1867.0777	1866.8934	1866.7034	1866.5181	1866.3303	1866.1468	1865.9519	1865.7681
1.25	1195.8291	1195.6452	1195.4612	1195.2769	1195.0852	1194.8996	1194.7132	1194.5249	1194.3302	1193.9408
1.50	831.0006	830.8166	830.6327	830.4485	830.2550	830.0690	829.8820	829.6927	829.4960	829.2121
2.00	468.2453	468.0618	467.8781	467.6938	467.4966	467.3098	467.1215	466.9295	466.7267	466.4226
2.50	300.3420	300.1588	299.9755	299.7912	299.5903	299.4026	299.2127	299.0183	298.8099	298.4983
3.00	209.1357	208.9530	208.7702	208.5859	208.3811	208.1926	208.0014	207.8045	207.5913	207.2763
3.50	154.1417	153.9596	153.7772	153.5929	153.3845	153.1951	153.0027	152.8035	152.5859	152.2690
4.00	118.4489	118.2674	118.0854	117.9012	117.6893	117.4989	117.3052	117.1041	116.8848	116.5651
1.00	2222.0646	2221.8805	2221.6963	2221.5120	2221.3220	2221.1377	2220.9515	2220.7665	2220.5751	2220.3874
1.25	1422.7851	1422.6010	1422.4169	1422.2327	1422.0417	1421.8562	1421.6700	1421.4819	1421.2843	1421.1391
1.50	988.6087	988.4248	988.2409	988.0566	987.8640	987.6781	987.4914	987.3027	987.1073	986.8242
2.00	556.8996	556.7160	556.5323	556.3479	556.1521	555.9655	555.7776	555.5866	555.3859	555.0921
2.50	357.0806	356.8973	356.7137	356.5295	356.3301	356.1428	355.9536	355.7602	355.5540	355.2395
3.00	248.5373	248.3543	248.1712	247.9870	247.7841	247.5959	247.4057	247.2098	246.9986	246.678[illegible]
3.50	183.0895	182.9071	182.7244	182.5401	182.3338	182.1448	181.9531	181.7559	181.5401	181.21[illegible]8
4.00	140.6118	140.4299	140.2476	140.0634	139.8537	139.6642	139.4712	139.2713	139.0528	138.7401

Table 11: Eigenvalues for antisymmetric - symmetric mode $(D_x/D_y)^{1/4}$ = 13,14,15,16.

ϕ \ μ	1.0	0.9	0.8	0.7	0.6	0.5	0.4	0.3	0.2	0.1
1.00	2607.5196	2607.3355	2607.1513	2606.9671	2606.7780	2606.5933	2606.4076	2606.2225	2606.0326	2605.8434
1.25	1669.4762	1669.2922	1669.1081	1668.9238	1668.7334	1668.5480	1668.3619	1668.1716	1667.9912	1667.7656
1.50	1159.9219	1159.7380	1159.5540	1159.3697	1159.1778	1158.9922	1158.8058	1158.6174	1158.4226	1158.0801
2.00	653.2632	653.0794	652.8957	652.7114	652.5164	652.3302	652.1426	651.9525	651.7534	651.4603
2.50	418.7530	418.5696	418.3860	418.2017	418.0037	417.8167	417.6280	417.4354	417.2314	416.9274
3.00	291.3652	291.1821	290.9988	290.8145	290.6133	290.4255	290.2355	290.0409	289.8326	289.5129
3.50	214.5547	214.3720	214.1891	214.0048	213.8004	213.6119	213.4208	213.2240	213.0112	212.6956
4.00	164.7022	164.5199	164.3373	164.1531	163.9455	163.7562	163.5644	163.3656	163.1486	162.8281
1.00	3023.8112	3023.6270	3023.4428	3023.2586	3023.0699	3022.8853	3022.6997	3022.5146	3022.3257	3022.1362
1.25	1935.9027	1935.7186	1935.5345	1935.3503	1935.1603	1934.9749	1934.7921	1934.6031	1934.4334	1934.2181
1.50	1344.9403	1344.7563	1344.5722	1344.3879	1344.1967	1344.0112	1343.8249	1343.6368	1343.4412	1343.3535
2.00	757.3358	757.1521	756.9682	756.7839	756.5899	756.4038	756.2166	756.0270	755.8294	755.5456
2.50	485.3594	485.1758	484.9922	484.8080	484.6110	484.4244	484.2360	484.0443	483.8420	483.5375
3.00	337.6194	337.4362	337.2528	337.0685	336.8686	336.6811	336.4919	336.2980	336.0911	335.7779
3.50	248.5373	248.3543	248.1712	247.9870	247.7841	247.5959	247.4057	247.2098	246.9986	246.6783
4.00	190.7198	190.5373	190.3545	190.1703	189.9645	189.7756	189.5840	189.3866	189.1727	188.8460
1.00	3470.9390	3470.7548	3470.5708	3470.3865	3470.1981	3470.0146	3469.8282	3469.6431	3469.4549	3469.2654
1.25	2222.0646	2221.8805	2221.6963	2221.5120	2221.3220	2221.1377	2220.9515	2220.7665	2220.5751	2220.3874
1.50	1543.6637	1543.4797	1543.2957	1543.1113	1542.9206	1542.7353	1542.5491	1542.3607	1542.1076	1541.9862
2.00	869.1177	868.9338	868.7500	868.5656	868.3724	868.1865	867.9995	867.8103	867.6142	867.3332
2.50	556.8996	556.7160	556.5323	556.3479	556.1521	555.9655	555.7776	555.5866	555.3859	555.0921
3.00	387.3000	387.1166	386.9331	386.7488	386.5502	386.3630	386.1740	385.9812	385.7761	385.4610
3.50	285.0371	284.8540	284.6708	284.4865	284.2851	284.0972	283.9072	283.7124	283.5039	283.1809
4.00	218.6649	218.4821	218.2992	218.1149	217.9107	217.7223	217.5312	217.3346	217.1220	216.8059
1.00	3948.9034	3948.7192	3948.5351	3948.3508	3948.1627	3947.9777	3947.7929	3947.6080	3947.4201	3947.2307
1.25	2527.9616	2527.7776	2527.5934	2527.4092	2527.2200	2527.0359	2526.8497	2526.6668	2526.4759	2526.2901
1.50	1756.0923	1755.9082	1755.7241	1755.5398	1755.3496	1755.1642	1754.9779	1754.8027	1754.6034	1754.4466
2.00	988.6087	988.4248	988.2409	988.0566	987.8640	987.6781	987.4914	987.3027	987.1073	986.8242
2.50	633.3738	633.1900	633.0063	632.8219	632.6268	632.4406	632.2529	632.0626	631.8631	631.5698
3.00	440.4069	440.2234	440.0398	439.8555	439.6579	439.4710	439.2824	439.0901	438.8867	438.5828
3.50	324.0543	323.8711	323.6877	323.5034	323.3032	323.1157	322.9262	322.7321	322.5247	322.2122
4.00	248.5373	248.3543	248.1712	247.9870	247.7841	247.5959	247.4057	247.2098	246.9986	246.6783

Table 12: Eigenvalues for antisymmetric - symmetric mode $(D_x/D_y)^{1/4}$ = 17,18,19,20.

ϕ \ μ	1.0	0.9	0.8	0.7	0.6	0.5	0.4	0.3	0.2	0.1
1.00	4457.7041	4457.5199	4457.3358	4457.1515	4456.9638	4456.7789	4456.5940	4456.4091	4456.2216	4456.0325
1.25	2853.5942	2853.4100	2853.2259	2853.0416	2852.8529	2852.6677	2852.4829	2852.2961	2852.1100	2851.9347
1.50	1982.2258	1982.0418	1981.8577	1981.6734	1981.4835	1981.2980	1981.1136	1980.9259	1980.7429	1980.5349
2.00	1115.8088	1115.6248	1115.4408	1115.2566	1115.0645	1114.8788	1114.6924	1114.5039	1114.3089	1113.9957
2.50	714.7817	714.5979	714.4140	714.2298	714.0355	713.8493	713.6620	713.4720	713.2740	712.9901
3.00	496.9401	496.7565	496.5728	496.3886	496.1918	496.0051	495.8169	495.6254	495.4232	495.1188
3.50	365.5888	365.4054	365.2220	365.0377	364.8385	364.6512	364.4621	364.2689	364.0629	363.7481
4.00	280.3370	280.1539	279.9707	279.7864	279.5848	279.3969	279.2068	279.0121	278.8029	278.4792
1.00	4997.3413	4997.1572	4996.9729	4996.7887	4996.6012	4996.4163	4996.2315	4996.0467	4995.8596	4995.6706
1.25	3198.9619	3198.7778	3198.5936	3198.4094	3198.2196	3198.0360	3197.8534	3197.6650	3197.4794	3197.2757
1.50	2222.0646	2221.8805	2221.6963	2221.5120	2221.3220	2221.1377	2220.9515	2220.7665	2220.5751	2220.3874
2.00	1250.7179	1250.5339	1250.3499	1250.1657	1249.9741	1249.7886	1249.6023	1249.4141	1249.2192	1248.6136
2.50	801.1235	800.9397	800.7558	800.5715	800.3779	800.1918	800.0046	799.8164	799.6182	799.3342
3.00	556.8996	556.7159	556.5322	556.3479	556.1520	555.9655	555.7776	555.5865	555.3858	555.0920
3.50	409.6405	409.4571	409.2736	409.0893	408.8911	408.7040	408.5152	408.3225	408.1183	407.8097
4.00	314.0640	313.8808	313.6975	313.5132	313.3127	313.1251	312.9354	312.7412	312.5335	312.2213
1.00	5567.8149	5567.6307	5567.4465	5567.2623	5567.0747	5566.8901	5566.7053	5566.5209	5566.3336	5566.1449
1.25	3564.0650	3563.8809	3563.6967	3563.5124	3563.3241	3563.1395	3562.9540	3562.7688	3562.5902	3562.3893
1.50	2475.6084	2475.4242	2475.2401	2475.0559	2474.8666	2474.6808	2474.4964	2474.3074	2474.1231	2473.9456
2.00	1393.3363	1393.1523	1392.9682	1392.7839	1392.5929	1392.4074	1392.2211	1392.0330	1391.8365	1391.7054
2.50	892.3992	892.2153	892.0314	891.8471	891.6540	891.4680	891.2812	891.0921	890.8960	890.6188
3.00	620.2854	620.1017	619.9179	619.7336	619.5384	619.3520	619.1644	618.9739	618.7742	618.4809
3.50	456.2096	456.0261	455.8425	455.6582	455.4608	455.2740	455.0856	454.8935	454.6904	454.3864
4.00	349.7183	349.5351	349.3516	349.1674	348.9678	348.7804	348.5912	348.3976	348.1911	347.8772
1.00	6169.1249	6168.9408	6168.7565	6168.5722	6168.3849	6168.2004	6168.0155	6164.8878	6167.6441	6167.4557
1.25	3948.9034	3948.7192	3948.5351	3948.3508	3948.1627	3947.9777	3947.7929	3947.6080	3947.4201	3947.2307
1.50	2742.8572	2742.6730	2742.4889	2742.3046	2742.1161	2741.9308	2741.7478	2741.5596	2741.3749	2741.1742
2.00	1543.6637	1543.4797	1543.2957	1543.1113	1542.9206	1542.7353	1542.5491	1542.3607	1542.1076	1541.9862
2.50	988.6087	988.4248	988.2409	988.0566	987.8640	987.6781	987.4914	987.3027	987.1073	986.8242
3.00	687.0974	686.9137	686.7299	686.5456	686.3510	686.1648	685.9774	685.7872	685.5891	685.2983
3.50	505.2959	505.1124	504.9287	504.7444	504.5478	504.3611	504.1729	503.9816	503.7795	503.4753
4.00	387.3000	387.1166	386.9331	386.7488	386.5502	386.3630	386.1740	385.9812	385.7761	385.4609

Table 13: Eigenvalues for doubly antisymmetric mode $(D_x/D_y)^{1/4} = 1,2,3,4.$

φ \ μ	1.0	0.9	0.8	0.7	0.6	0.5	0.4	0.3	0.2	0.1
1.00	9.8697	9.8697	9.8697	9.8697	9.8697	9.8697	9.8697	9.8697	9.8697	9.8697
1.25	6.3166	6.3166	6.3166	6.3166	6.3166	6.3166	6.3166	6.3166	6.3166	6.3166
1.50	4.3865	4.3865	4.3865	4.3865	4.3865	4.3865	4.3865	4.3865	4.3865	4.3865
2.00	2.4675	2.4675	2.4675	2.4675	2.4675	2.4675	2.4675	2.4675	2.4675	2.4675
2.50	1.5792	1.5792	1.5792	1.5792	1.5792	1.5792	1.5792	1.5792	1.5792	1.5792
3.00	1.0966	1.0966	1.0966	1.0966	1.0966	1.0966	1.0966	1.0966	1.0966	1.0966
3.50	0.8057	0.8057	0.8057	0.8057	0.8057	0.8057	0.8057	0.8057	0.8057	0.8057
4.00	0.6169	0.6169	0.6169	0.6169	0.6169	0.6169	0.6169	0.6169	0.6169	0.6169
1.00	39.4785	39.4785	39.4785	39.4785	39.4785	39.4785	39.4785	39.4785	39.4785	39.4785
1.25	25.2662	25.2662	25.2662	25.2662	25.2662	25.2662	25.2662	25.2662	25.2662	25.2662
1.50	17.5460	17.5460	17.5460	17.5460	17.5460	17.5460	17.5460	17.5460	17.5460	17.5460
2.00	9.8697	9.8697	9.8697	9.8697	9.8697	9.8697	9.8697	9.8697	9.8697	9.8697
2.50	6.3166	6.3166	6.3166	6.3166	6.3166	6.3166	6.3166	6.3166	6.3166	6.3166
3.00	4.3865	4.3865	4.3865	4.3865	4.3865	4.3865	4.3865	4.3865	4.3865	4.3865
3.50	3.2227	3.2227	3.2227	3.2227	3.2227	3.2227	3.2227	3.2227	3.2227	3.2227
4.00	8.0049	7.6405	7.2565	6.8496	6.4152	5.9473	5.4373	4.8716	4.2282	3.4635
1.00	88.8265	88.8265	88.8265	88.8265	88.8265	88.8265	88.8265	88.8265	88.8265	88.8265
1.25	56.8490	56.8490	56.8490	56.8490	56.8490	56.8490	56.8490	56.8490	56.8490	56.8490
1.50	39.4785	39.4785	39.4785	39.4785	39.4785	39.4785	39.4785	39.4785	39.4785	39.4785
2.00	22.2067	22.2067	22.2067	22.2067	22.2067	22.2067	22.2067	22.2067	22.2067	22.2067
2.50	14.2123	14.2123	14.2123	14.2123	14.2123	14.2123	14.2123	14.2123	14.2123	14.2123
3.00	9.8696	9.8696	9.8696	9.8696	9.8696	9.8696	9.8696	9.8696	9.8696	9.8696
3.50	7.2511	7.2511	7.2511	7.2511	7.2511	7.2511	7.2511	7.2511	7.2511	7.2511
4.00	12.5953	12.0931	11.5662	11.0107	10.4219	9.7933	9.1165	8.3792	7.5633	6.6390
1.00	157.9137	157.9137	157.9137	157.9137	157.9137	157.9137	157.9137	157.9137	157.9137	157.9137
1.25	101.0648	101.0648	101.0648	101.0648	101.0648	101.0648	101.0648	101.0648	101.0648	101.0648
1.50	70.1839	70.1839	70.1839	70.1839	70.1839	70.1839	70.1839	70.1839	70.1839	70.1839
2.00	39.4785	39.4785	39.4785	39.4785	39.4785	39.4785	39.4785	39.4785	39.4785	39.4785
2.50	25.2662	25.2662	25.2662	25.2662	25.2662	25.2662	25.2662	25.2662	25.2662	25.2662
3.00	17.5460	17.5460	17.5460	17.5460	17.5460	17.5460	17.5460	17.5460	17.5460	17.5460
3.50	12.8909	12.8909	12.8909	12.8909	12.8909	12.8909	12.8909	12.8909	12.8909	12.8909
4.00	17.8821	17.2747	16.6410	15.9774	15.2794	14.5413	13.7558	12.9129	11.9991	10.9948

Table 14: Eigenvalues for doubly antisymmetric mode $(D_x/D_y)^{1/4} = 5,6,7,8.$

ϕ \ μ	1.0	0.9	0.8	0.7	0.6	0.5	0.4	0.3	0.2	0.1
1.00	246.7402	246.7402	246.7402	246.7402	246.7402	246.7402	246.7402	246.7402	246.7402	246.7402
1.25	157.9137	157.9137	157.9137	157.9137	157.9137	157.9137	157.9137	157.9137	157.9137	157.9137
1.50	109.6623	109.6623	109.6623	109.6623	109.6623	109.6623	109.6623	109.6623	109.6623	109.6623
2.00	61.6851	61.6851	61.6851	61.6851	61.6851	61.6851	61.6851	61.6851	61.6851	61.6851
2.50	39.4785	39.4785	39.4785	39.4785	39.4785	39.4785	39.4785	39.4785	39.4785	39.4785
3.00	27.4156	27.4156	27.4156	27.4156	27.4156	27.4156	27.4156	27.4156	27.4156	27.4156
3.50	20.1420	20.1420	20.1420	20.1420	20.1420	20.1420	20.1420	20.1420	20.1420	20.1420
4.00	24.0482	23.3620	22.6506	21.9108	21.1387	20.3293	19.4766	18.5724	17.6061	16.5627
1.00	355.3058	355.3058	355.3058	355.3058	355.3058	355.3058	355.3058	355.3058	355.3058	355.3058
1.25	227.3957	227.3957	227.3957	227.3957	227.3957	227.3957	227.3957	227.3957	227.3957	227.3957
1.50	157.9137	157.9137	157.9137	157.9137	157.9137	157.9137	157.9137	157.9137	157.9137	157.9137
2.00	88.8265	88.8265	88.8265	88.8265	88.8265	88.8265	88.8265	88.8265	88.8265	88.8265
2.50	56.8490	56.8490	56.8490	56.8490	56.8490	56.8490	56.8490	56.8490	56.8490	56.8490
3.00	39.4784	39.4784	39.4784	39.4784	39.4784	39.4784	39.4784	39.4784	39.4784	39.4784
3.50	29.0046	29.0046	29.0046	29.0046	29.0046	29.0046	29.0046	29.0046	29.0046	29.0046
4.00	31.2244	30.4792	29.7112	28.9176	28.0953	27.2400	26.3465	25.4080	24.4149	23.3543
1.00	483.6107	483.6107	483.6107	483.6107	483.6107	483.6107	483.6107	483.6107	483.6107	483.6107
1.25	309.5108	309.5108	309.5108	309.5108	309.5108	309.5108	309.5108	309.5108	309.5108	309.5108
1.50	214.9381	214.9381	214.9381	214.9381	214.9381	214.9381	214.9381	214.9381	214.9381	214.9381
2.00	120.9027	120.9027	120.9027	120.9027	120.9027	120.9027	120.9027	120.9027	120.9027	120.9027
2.50	77.3777	77.3777	77.3777	77.3777	77.3777	77.3777	77.3777	77.3777	77.3777	77.3777
3.00	53.7345	53.7345	53.7345	53.7345	53.7345	53.7345	53.7345	53.7345	53.7345	53.7345
3.50	39.4784	39.4784	39.4784	39.4784	39.4784	39.4784	39.4784	39.4784	39.4784	39.4784
4.00	39.4956	38.7059	37.8960	37.0639	36.2069	35.3214	34.4030	33.4456	32.4404	31.3741
1.00	631.6547	631.6547	631.6547	631.6547	631.6547	631.6547	631.6547	631.6547	631.6547	631.6547
1.25	404.2590	404.2590	404.2590	404.2590	404.2590	404.2590	404.2590	404.2590	404.2590	404.2590
1.50	280.7355	280.7355	280.7355	280.7355	280.7355	280.7355	280.7355	280.7355	280.7355	280.7355
2.00	157.9137	157.9137	157.9137	157.9137	157.9137	157.9137	157.9137	157.9137	157.9137	157.9137
2.50	101.0648	101.0648	101.0648	101.0648	101.0648	101.0648	101.0648	101.0648	101.0648	101.0648
3.00	70.1839	70.1839	70.1839	70.1839	70.1839	70.1839	70.1839	70.1839	70.1839	70.1839
3.50	51.5636	51.5636	51.5636	51.5636	51.5636	51.5636	51.5636	51.5636	51.5636	51.5636
4.00	48.9147	48.0908	47.2495	46.3890	45.5072	44.6012	43.6672	42.6996	41.6900	40.6246

Table 15: **Eigenvalues for doubly antisymmetric mode** $(D_x/D_y)^{1/4}$ = 9,10,11,12.

ϕ \ μ	1.0	0.9	0.8	0.7	0.6	0.5	0.4	0.3	0.2	0.1
1.00	799.4380	799.4380	799.4380	799.4380	799.4380	799.4380	799.4380	799.4380	799.4380	799.4380
1.25	511.6403	511.6403	511.6403	511.6403	511.6403	511.6403	511.6403	511.6403	511.6403	511.6403
1.50	355.3058	355.3058	355.3058	355.3058	355.3058	355.3058	355.3058	355.3058	355.3058	355.3058
2.00	199.8595	199.8595	199.8595	199.8595	199.8595	199.8595	199.8595	199.8595	199.8595	199.8595
2.50	127.9101	127.9101	127.9101	127.9101	127.9101	127.9101	127.9101	127.9101	127.9101	127.9101
3.00	88.8264	88.8264	88.8264	88.8264	88.8264	88.8264	88.8264	88.8264	88.8264	88.8264
3.50	65.2602	65.2602	65.2602	65.2602	65.2602	65.2602	65.2602	65.2602	65.2602	65.2602
4.00	59.5135	58.6630	57.7976	56.9158	56.0158	55.0953	54.1511	53.1780	52.1681	51.1069
1.00	986.9605	986.9605	986.9605	986.9605	986.9605	986.9605	986.9605	986.9605	986.9605	986.9605
1.25	631.6547	631.6547	631.6547	631.6547	631.6547	631.6547	631.6547	631.6547	631.6547	631.6547
1.50	438.6491	438.6491	438.6491	438.6491	438.6491	438.6491	438.6491	438.6491	438.6491	438.6491
2.00	246.7402	246.7402	246.7402	246.7402	246.7402	246.7402	246.7402	246.7402	246.7402	246.7402
2.50	157.9137	157.9137	157.9137	157.9137	157.9137	157.9137	157.9137	157.9137	157.9137	157.9137
3.00	109.6623	109.6623	109.6623	109.6623	109.6623	109.6623	109.6623	109.6623	109.6623	109.6623
3.50	80.5682	80.5682	80.5682	80.5682	80.5682	80.5682	80.5682	80.5682	80.5682	80.5682
4.00	71.3115	70.4401	69.5558	68.6575	67.7438	66.8127	65.8613	64.8853	63.8771	62.8217
1.00	1194.2222	1194.2222	1194.2222	1194.2222	1194.2222	1194.2222	1194.2222	1194.2222	1194.2222	1194.2222
1.25	764.3022	764.3022	764.3022	764.3022	764.3022	764.3022	764.3022	764.3022	764.3022	764.3022
1.50	530.7654	530.7654	530.7654	530.7654	530.7654	530.7654	530.7654	530.7654	530.7654	530.7654
2.00	298.5556	298.5556	298.5556	298.5556	298.5556	298.5556	298.5556	298.5556	298.5556	298.5556
2.50	191.0756	191.0756	191.0756	191.0756	191.0756	191.0756	191.0756	191.0756	191.0756	191.0756
3.00	132.6913	132.6913	132.6913	132.6913	132.6913	132.6913	132.6913	132.6913	132.6913	132.6913
3.50	97.4875	97.4875	97.4875	97.4875	97.4875	97.4875	97.4875	97.4875	97.4875	97.4875
4.00	84.3205	83.4325	82.5333	81.6221	80.6976	79.7583	78.8018	77.8242	76.8185	75.7696
1.00	1421.2231	1421.2231	1421.2231	1421.2231	1421.2231	1421.2231	1421.2231	1421.2231	1421.2231	1421.2231
1.25	909.5828	909.5828	909.5828	909.5828	909.5828	909.5828	909.5828	909.5828	909.5828	909.5828
1.50	631.6547	631.6547	631.6547	631.6547	631.6547	631.6547	631.6547	631.6547	631.6547	631.6547
2.00	355.3058	355.3058	355.3058	355.3058	355.3058	355.3058	355.3058	355.3058	355.3058	355.3058
2.50	227.3957	227.3957	227.3957	227.3957	227.3957	227.3957	227.3957	227.3957	227.3957	227.3957
3.00	157.9137	157.9137	157.9137	157.9137	157.9137	157.9137	157.9137	157.9137	157.9137	157.9137
3.50	116.0182	116.0182	116.0182	116.0182	116.0182	116.0182	116.0182	116.0182	116.0182	116.0182
4.00	98.5481	97.6466	96.7355	95.8138	94.8808	93.9352	92.9748	91.9963	90.9932	89.9508

Table 16: Eigenvalues for doubly antisymmetric mode $(D_x/D_y)^{1/4}$ = 13,14,15,16.

φ \ μ	1.0	0.9	0.8	0.7	0.6	0.5	0.4	0.3	0.2	0.1
1.00	1667.9632	1667.9632	1667.9632	1667.9632	1667.9632	1667.9632	1667.9632	1667.9632	1667.9632	1667.9632
1.25	1067.4965	1067.4965	1067.4965	1067.4965	1067.4965	1067.4965	1067.4965	1067.4965	1067.4965	1067.4965
1.50	741.3170	741.3170	741.3170	741.3170	741.3170	741.3170	741.3170	741.3170	741.3170	741.3170
2.00	416.9908	416.9908	416.9908	416.9908	416.9908	416.9908	416.9908	416.9908	416.9908	416.9908
2.50	266.8742	266.8742	266.8742	266.8742	266.8742	266.8742	266.8742	266.8742	266.8742	266.8742
3.00	185.3292	185.3292	185.3292	185.3292	185.3292	185.3292	185.3292	185.3292	185.3292	185.3292
3.50	136.1603	136.1603	136.1603	136.1603	136.1603	136.1603	136.1603	136.1603	136.1603	136.1603
4.00	113.9990	113.0865	112.1656	111.2356	110.2957	109.3449	108.3814	107.4024	106.4018	105.3656
1.00	1934.4425	1934.4425	1934.4425	1934.4425	1934.4425	1934.4425	1934.4425	1934.4425	1934.4425	1934.4425
1.25	1238.0432	1238.0432	1238.0432	1238.0432	1238.0432	1238.0432	1238.0432	1238.0432	1238.0432	1238.0432
1.50	859.7523	859.7523	859.7523	859.7523	859.7523	859.7523	859.7523	859.7523	859.7523	859.7523
2.00	483.6107	483.6107	483.6107	483.6107	483.6107	483.6107	483.6107	483.6107	483.6107	483.6107
2.50	309.5108	309.5108	309.5108	309.5108	309.5108	309.5108	309.5108	309.5108	309.5108	309.5108
3.00	214.9381	214.9381	214.9381	214.9381	214.9381	214.9381	214.9381	214.9381	214.9381	214.9381
3.50	157.9137	157.9137	157.9137	157.9137	157.9137	157.9137	157.9137	157.9137	157.9137	157.9137
4.00	130.6763	129.7548	128.8258	127.8888	126.9433	125.9883	125.0223	124.0429	123.0446	122.0141
1.00	2220.6610	2220.6610	2220.6610	2220.6610	2220.6610	2220.6610	2220.6610	2220.6610	2220.6610	2220.6610
1.25	1421.2231	1421.2231	1421.2231	1421.2231	1421.2231	1421.2231	1421.2231	1421.2231	1421.2231	1421.2231
1.50	986.9605	986.9605	986.9605	986.9605	986.9605	986.9605	986.9605	986.9605	986.9605	986.9605
2.00	555.1653	555.1653	555.1653	555.1653	555.1653	555.1653	555.1653	555.1653	555.1653	555.1653
2.50	355.3058	355.3058	355.3058	355.3058	355.3058	355.3058	355.3058	355.3058	355.3058	355.3058
3.00	246.7401	246.7401	246.7401	246.7401	246.7401	246.7401	246.7401	246.7401	246.7401	246.7401
3.50	181.2784	181.2784	181.2784	181.2784	181.2784	181.2784	181.2784	181.2784	181.2784	181.2784
4.00	148.5821	147.6530	146.7173	145.7746	144.8244	143.8659	142.8979	141.9181	140.9218	139.8964
1.00	2526.6188	2526.6188	2526.6188	2526.6188	2526.6188	2526.6188	2526.6188	2526.6188	2526.6188	2526.6188
1.25	1617.0360	1617.0360	1617.0360	1617.0360	1617.0360	1617.0360	1617.0360	1617.0360	1617.0360	1617.0360
1.50	1122.9417	1122.9417	1122.9417	1122.9417	1122.9417	1122.9417	1122.9417	1122.9417	1122.9417	1122.9417
2.00	631.6547	631.6547	631.6547	631.6547	631.6547	631.6547	631.6547	631.6547	631.6547	631.6547
2.50	404.2590	404.2590	404.2590	404.2590	404.2590	404.2590	404.2590	404.2590	404.2590	404.2590
3.00	280.7354	280.7354	280.7354	280.7354	280.7354	280.7354	280.7354	280.7354	280.7354	280.7354
3.50	206.2546	206.2546	206.2546	206.2546	206.2546	206.2546	206.2546	206.2546	206.2546	206.2546
4.00	167.7177	166.7823	165.8410	164.8935	163.9394	162.9779	162.0081	161.0280	160.0334	159.0126

Table 17: Eigenvalues for doubly antisymmetric mode $(D_x/D_y)^{1/4}$ = 17,18,19,20.

ϕ \ μ	1.0	0.9	0.8	0.7	0.6	0.5	0.4	0.3	0.2	0.1
1.00	2852.3157	2852.3157	2852.3157	2852.3157	2852.3157	2852.3157	2852.3157	2852.3157	2852.3157	2852.3157
1.25	1825.4821	1825.4821	1825.4821	1825.4821	1825.4821	1825.4821	1825.4821	1825.4821	1825.4821	1825.4821
1.50	1267.6959	1267.6959	1267.6959	1267.6959	1267.6959	1267.6959	1267.6959	1267.6959	1267.6959	1267.6959
2.00	713.0790	713.0790	713.0790	713.0790	713.0790	713.0790	713.0790	713.0790	713.0790	713.0790
2.50	456.3706	456.3706	456.3706	456.3706	456.3706	456.3706	456.3706	456.3706	456.3706	456.3706
3.00	316.9240	316.9240	316.9240	316.9240	316.9240	316.9240	316.9240	316.9240	316.9240	316.9240
3.50	232.8421	232.8421	232.8421	232.8421	232.8421	232.8421	232.8421	232.8421	232.8421	232.8421
4.00	188.0843	187.1435	186.1975	185.2459	184.2883	183.3244	182.3530	181.3727	180.3795	179.3628
1.00	3197.7519	3197.7519	3197.7519	3197.7519	3197.7519	3197.7519	3197.7519	3197.7519	3197.7519	3197.7519
1.25	2046.5612	2046.5612	2046.5612	2046.5612	2046.5612	2046.5612	2046.5612	2046.5612	2046.5612	2046.5612
1.50	1421.2231	1421.2231	1421.2231	1421.2231	1421.2231	1421.2231	1421.2231	1421.2231	1421.2231	1421.2231
2.00	799.4380	799.4380	799.4380	799.4380	799.4380	799.4380	799.4380	799.4380	799.4380	799.4380
2.50	511.6403	511.6403	511.6403	511.6403	511.6403	511.6403	511.6403	511.6403	511.6403	511.6403
3.00	355.3058	355.3058	355.3058	355.3058	355.3058	355.3058	355.3058	355.3058	355.3058	355.3058
3.50	261.0410	261.0410	261.0410	261.0410	261.0410	261.0410	261.0410	261.0410	261.0410	261.0410
4.00	209.6825	208.7371	207.7870	206.8319	205.8715	204.9054	203.9327	202.9521	201.9601	200.9469
1.00	3562.9272	3562.9272	3562.9272	3562.9272	3562.9272	3562.9272	3562.9272	3562.9272	3562.9272	3562.9272
1.25	2280.2735	2280.2735	2280.2735	2280.2735	2280.2735	2280.2735	2280.2735	2280.2735	2280.2735	2280.2735
1.50	1583.5232	1583.5232	1583.5232	1583.5232	1583.5232	1583.5232	1583.5232	1583.5232	1583.5232	1583.5232
2.00	890.7318	890.7318	890.7318	890.7318	890.7318	890.7318	890.7318	890.7318	890.7318	890.7318
2.50	570.0684	570.0684	570.0684	570.0684	570.0684	570.0684	570.0684	570.0684	570.0684	570.0684
3.00	395.8808	395.8808	395.8808	395.8808	395.8808	395.8808	395.8808	395.8808	395.8808	395.8808
3.50	290.8512	290.8512	290.8512	290.8512	290.8512	290.8512	290.8512	290.8512	290.8512	290.8512
4.00	232.5127	231.5634	230.6098	229.6517	228.6888	227.7208	226.7470	225.7661	224.7751	223.7651
1.00	3947.8418	3947.8418	3947.8418	3947.8418	3947.8418	3947.8418	3947.8418	3947.8418	3947.8418	3947.8418
1.25	2526.6188	2526.6188	2526.6188	2526.6188	2526.6188	2526.6188	2526.6188	2526.6188	2526.6188	2526.6188
1.50	1754.5964	1754.5964	1754.5964	1754.5964	1754.5964	1754.5964	1754.5964	1754.5964	1754.5964	1754.5964
2.00	986.9605	986.9605	986.9605	986.9605	986.9605	986.9605	986.9605	986.9605	986.9605	986.9605
2.50	631.6547	631.6547	631.6547	631.6547	631.6547	631.6547	631.6547	631.6547	631.6547	631.6547
3.00	438.6491	438.6491	438.6491	438.6491	438.6491	438.6491	438.6491	438.6491	438.6491	438.6491
3.50	322.2728	322.2728	322.2728	322.2728	322.2728	322.2728	322.2728	322.2728	322.2728	322.2728
4.00	256.5754	255.6227	254.6660	253.7053	252.7403	251.7707	250.7958	249.8147	248.8245	247.8172

Table 18: Eigenvalues for symmetric - antisymmetric mode $(D_x/D_y)^{1/4} = 1,2,3,4.$

ϕ \ μ	1.0	0.9	0.8	0.7	0.6	0.5	0.4	0.3	0.2	0.1
1.00	17.8821	17.2747	16.6410	15.9774	15.2795	14.5414	13.7558	12.9129	11.9991	10.9948
1.25	13.5904	13.0647	12.5137	11.9336	11.3196	10.6654	9.9627	9.1997	8.3593	7.4142
1.50	10.9983	10.5382	10.0547	9.5439	9.0012	8.4200	7.7916	7.1032	6.3348	5.4522
2.00	8.0049	7.6406	7.2566	6.8496	6.4152	5.9474	5.4373	4.8717	4.2282	3.4636
2.50	6.3141	6.0146	5.6984	5.3628	5.0039	4.6160	4.1911	3.7167	3.1704	2.5056
3.00	5.2213	4.9677	4.7000	4.4155	4.1108	3.7809	3.4187	3.0123	2.5408	1.9575
3.50	4.4546	4.2351	4.0033	3.7569	3.4927	3.2064	2.8914	2.5371	2.1238	1.6066
4.00	3.8860	3.6927	3.4885	3.2714	3.0385	2.7859	2.5077	2.1941	1.8270	1.3638
1.00	48.9147	48.0908	47.2495	46.3890	45.5073	44.6013	43.6673	42.6997	41.6901	40.6246
1.25	34.3980	33.6335	32.8473	32.0368	31.1990	30.3302	29.4253	28.4778	27.4787	26.4147
1.50	26.3231	25.6153	24.8831	24.1233	23.3322	22.5053	21.6366	20.7187	19.7413	18.6904
2.00	17.8821	17.2747	16.6410	15.9774	15.2795	14.5414	13.7558	12.9129	11.9991	10.9948
2.50	13.5904	13.0647	12.5137	11.9336	11.3196	10.6654	9.9627	9.1997	8.3593	7.4142
3.00	10.9983	10.5382	10.0546	9.5439	9.0011	8.4200	7.7916	7.1031	6.3347	5.4521
3.50	9.2579	8.8507	8.4219	7.9681	7.4846	6.9651	6.4007	5.7781	5.0758	4.2541
4.00	8.0049	7.6405	7.2565	6.8496	6.4152	5.9473	5.4373	4.8716	4.2282	3.4635
1.00	98.5481	97.6467	96.7355	95.8139	94.8809	93.9352	92.9748	91.9964	90.9932	89.9508
1.25	66.4475	65.5840	64.7067	63.8145	62.9058	61.9786	61.0298	60.0548	59.0457	57.9879
1.50	48.9147	48.0908	47.2495	46.3890	45.5073	44.6013	43.6673	42.6997	41.6901	40.6246
2.00	31.2244	30.4793	29.7112	28.9177	28.0953	27.2400	26.3466	25.4080	24.4150	23.3543
2.50	22.7377	22.0655	21.3677	20.6410	19.8815	19.0840	18.2422	17.3476	16.3892	15.3515
3.00	17.8821	17.2747	16.6410	15.9774	15.2794	14.5413	13.7558	12.9129	11.9991	10.9948
3.50	14.7640	14.2130	13.6362	13.0299	12.3892	11.7081	10.9785	10.1892	9.3241	8.3586
4.00	12.5953	12.0931	11.5662	11.0107	10.4219	9.7933	9.1165	8.3792	7.5633	6.6390
1.00	167.7178	166.7824	165.8411	164.8936	163.9394	162.9779	162.0081	161.0280	160.0335	159.0127
1.25	110.8108	109.9004	108.9813	108.0528	107.1142	106.1643	105.2014	104.2225	103.2214	102.1840
1.50	79.8491	78.9663	78.0717	77.1644	76.2432	75.3064	74.3515	73.3744	72.3677	71.3166
2.00	48.9147	48.0908	47.2495	46.3890	45.5073	44.6013	43.6673	42.6997	41.6901	40.6246
2.50	34.3980	33.6335	32.8473	32.0368	31.1990	30.3302	29.4253	28.4778	27.4787	26.4147
3.00	26.3230	25.6153	24.8830	24.1232	23.3321	22.5052	21.6366	20.7186	19.7412	18.6903
3.50	21.2884	20.6331	19.9520	19.2416	18.4977	17.7151	16.8872	16.0051	15.0572	14.0271
4.00	17.8821	17.2747	16.6410	15.9774	15.2794	14.5413	13.7558	12.9129	11.9991	10.9948

Table 19: **Eigenvalues for symmetric - antisymmetric mode $(D_x/D_y)^{1/4}$ = 5,6,7,8.**

ϕ \ μ	1.0	0.9	0.8	0.7	0.6	0.5	0.4	0.3	0.2	0.1
1.00	256.5755	255.6227	254.6661	253.7053	252.7404	251.7707	250.7959	249.8147	248.8246	247.8172
1.25	167.7178	166.7824	165.8411	164.8936	163.9394	162.9779	162.0081	161.0280	160.0335	159.0127
1.50	119.4217	118.5061	117.5823	116.6498	115.7079	114.7556	113.7913	112.8121	111.8123	110.7780
2.00	71.3115	70.4402	69.5559	68.6576	67.7438	66.8127	65.8614	64.8854	63.8771	62.8217
2.50	48.9147	48.0908	47.2495	46.3890	45.5073	44.6013	43.6673	42.6997	41.6901	40.6246
3.00	36.6131	35.8369	35.0396	34.2190	33.3722	32.4955	31.5841	30.6318	29.6295	28.5642
3.50	29.0653	28.3353	27.5817	26.8017	25.9918	25.1477	24.2638	23.3331	22.3459	21.2887
4.00	24.0482	23.3620	22.6506	21.9108	21.1387	20.3293	19.4766	18.5724	17.6061	16.5627
1.00	365.1553	364.1926	363.2272	362.2589	361.2876	360.3132	359.3353	358.3531	357.3651	356.3655
1.25	237.2267	236.2767	235.3224	234.3637	233.4004	232.4321	231.4580	230.4771	229.4863	228.4768
1.50	167.7178	166.7824	165.8411	164.8936	163.9394	162.9779	162.0081	161.0280	160.0335	159.0127
2.00	98.5481	97.6467	96.7355	95.8139	94.8809	93.9352	92.9748	91.9964	90.9932	89.9508
2.50	66.4475	65.5840	64.7067	63.8145	62.9058	61.9786	61.0298	60.0548	59.0457	57.9879
3.00	48.9147	48.0908	47.2495	46.3890	45.5072	44.6012	43.6672	42.6996	41.6900	40.6246
3.50	38.2447	37.4606	36.6560	35.8287	34.9759	34.0940	33.1785	32.2232	31.2191	30.1531
4.00	31.2244	30.4792	29.7112	28.9176	28.0953	27.2400	26.3465	25.4080	24.4149	23.3543
1.00	493.4675	492.4987	491.5277	490.5547	489.5795	488.6020	487.6220	486.6391	485.6520	484.6570
1.25	319.3558	318.3965	317.4341	316.4684	315.4993	314.5265	313.5496	312.5679	311.5792	310.5770
1.50	224.7658	223.8177	222.8652	221.9080	220.9459	219.9785	219.0051	218.0243	217.0330	216.0219
2.00	130.6763	129.7548	128.8258	127.8889	126.9434	125.9883	125.0224	124.0429	123.0446	122.0141
2.50	87.0684	86.1775	85.2757	84.3622	83.4358	82.4952	81.5378	80.5600	79.5548	78.5072
3.00	63.3122	62.4542	61.5820	60.6942	59.7893	58.8649	57.9180	56.9437	55.9342	54.8748
3.50	48.9147	48.0908	47.2495	46.3890	45.5072	44.6012	43.6672	42.6996	41.6900	40.6246
4.00	39.4956	38.7059	37.8960	37.0639	36.2069	35.3214	34.4030	33.4456	32.4404	31.3741
1.00	641.5158	640.5428	639.5682	638.5921	637.6143	636.6347	635.6533	634.6696	633.6831	632.6908
1.25	414.1121	413.1466	412.1786	411.2082	410.2352	409.2594	408.2805	407.2980	406.3105	405.3130
1.50	290.5767	289.6200	288.6599	287.6963	286.7288	285.7573	284.7813	283.7998	282.8106	281.8063
2.00	167.7178	166.7824	165.8411	164.8936	163.9394	162.9779	162.0081	161.0280	160.0335	159.0127
2.50	110.8108	109.9004	108.9813	108.0528	107.1142	106.1643	105.2014	104.2225	103.2214	102.1840
3.00	79.8491	78.9662	78.0716	77.1644	76.2432	75.3064	74.3515	73.3743	72.3677	71.3166
3.50	61.1252	60.2714	59.4030	58.5185	57.6164	56.6942	55.7488	54.7752	53.7654	52.7049
4.00	48.9147	48.0908	47.2495	46.3890	45.5072	44.6012	43.6672	42.6996	41.6900	40.6246

Table 20: Eigenvalues for symmetric - antisymmetric mode $(D_x/D_y)^{1/4}$ = 9,10,11,12.

ϕ \ μ	1.0	0.9	0.8	0.7	0.6	0.5	0.4	0.3	0.2	0.1
1.00	809.3016	808.3257	807.3486	806.3703	805.3906	804.4096	803.4272	802.4430	801.4566	800.4660
1.25	521.4982	520.5284	519.5566	518.5829	517.6071	516.6291	515.6488	514.6657	513.6788	512.6844
1.50	365.1553	364.1926	363.2272	362.2589	361.2876	360.3132	359.3353	358.3531	357.3651	356.3655
2.00	209.6825	208.7371	207.7870	206.8319	205.8715	204.9054	203.9327	202.9521	201.9602	200.9470
2.50	137.6911	136.7664	135.8346	134.8953	133.9478	132.9913	132.0244	131.0448	130.0474	129.0190
3.00	98.5481	97.6466	96.7355	95.8138	94.8808	93.9352	92.9748	91.9963	90.9932	89.9508
3.50	74.9044	74.0279	73.1390	72.2367	71.3196	70.3859	69.4329	68.4564	67.4488	66.3952
4.00	59.5135	58.6630	57.7976	56.9158	56.0158	55.0953	54.1511	53.1780	52.1681	51.1069
1.00	996.8257	995.8478	994.8689	993.8889	992.9079	991.9258	990.9426	989.9580	988.9717	987.9822
1.25	641.5158	640.5428	639.5682	638.5921	637.6143	636.6347	635.6533	634.6696	633.6831	632.6908
1.50	448.5040	447.5369	446.5676	445.5959	444.6218	443.6452	442.6658	441.6831	440.6958	439.6996
2.00	256.5755	255.6227	254.6661	253.7053	252.7404	251.7707	250.7959	249.8147	248.8246	247.8172
2.50	167.7178	166.7824	165.8411	164.8936	163.9394	162.9779	162.0081	161.0280	160.0335	159.0127
3.00	119.4217	118.5060	117.5822	116.6498	115.7079	114.7556	113.7912	112.8120	111.8122	110.7780
3.50	90.2685	89.3744	88.4698	87.5538	86.6254	85.6832	84.7250	83.7469	82.7423	81.6962
4.00	71.3115	70.4401	69.5558	68.6575	67.7438	66.8127	65.8613	64.8853	63.8771	62.8217
1.00	1204.0885	1203.1090	1202.1287	1201.1476	1200.1656	1199.1827	1198.1989	1197.2140	1196.2277	1195.2389
1.25	774.1654	773.1900	772.2134	771.2354	770.2561	769.2753	768.2930	767.3090	766.3226	765.3317
1.50	540.6240	539.6535	538.6812	537.7070	536.7309	535.7526	534.7721	533.7888	532.8020	531.8080
2.00	308.3993	307.4409	306.4793	305.5143	304.5458	303.5735	302.5969	301.6152	300.6264	299.6234
2.50	200.8954	199.9518	199.0032	198.0495	197.0902	196.1249	195.1527	194.1722	193.1798	192.1652
3.00	142.4769	141.5502	140.6166	139.6757	138.7270	137.7695	136.8022	135.8225	134.8256	133.7985
3.50	107.2272	106.3192	105.4022	104.4755	103.5384	102.5897	101.6275	100.6486	99.6470	98.6082
4.00	84.3205	83.4325	82.5333	81.6221	80.6976	79.7583	78.8018	77.8242	76.8185	75.7696
1.00	1431.0901	1430.1095	1429.1282	1428.1461	1427.1634	1426.1799	1425.1956	1424.2104	1423.2241	1422.2358
1.25	919.4474	918.4703	917.4920	916.5126	915.5321	914.5505	913.5675	912.5831	911.5968	910.6069
1.50	641.5158	640.5428	639.5682	638.5921	637.6143	636.6347	635.6533	634.6696	633.6831	632.6908
2.00	365.1553	364.1926	363.2272	362.2589	361.2876	360.3132	359.3353	358.3531	357.3651	356.3655
2.50	237.2267	236.2767	235.3224	234.3637	233.4004	232.4321	231.4580	230.4771	229.4863	228.4768
3.00	167.7177	166.7823	165.8410	164.8935	163.9394	162.9779	162.0081	161.0280	160.0334	159.0126
3.50	125.7861	124.8670	123.9401	123.0050	122.0610	121.1071	120.1418	119.1625	118.1635	117.1314
4.00	98.5481	97.6466	96.7355	95.8138	94.8808	93.9352	92.9748	91.9963	90.9932	89.9508

Table 21: Eigenvalues for symmetric - antisymmetric mode $(D_x/D_y)^{1/4}$ = 13,14,15,16.

ϕ \ μ	1.0	0.9	0.8	0.7	0.6	0.5	0.4	0.3	0.2	0.1
1.00	1677.8308	1676.8492	1675.8671	1674.8843	1673.9010	1672.9170	1671.9323	1670.9469	1669.9606	1668.9726
1.25	1077.3621	1076.3836	1075.4041	1074.4236	1073.4422	1072.4598	1071.4762	1070.4915	1069.5053	1068.5161
1.50	751.1799	750.2049	749.2285	748.2508	747.2717	746.2911	745.3090	744.3250	743.3386	742.3475
2.00	426.8446	425.8785	424.9100	423.9391	422.9656	421.9895	421.0104	420.0279	419.0404	418.0434
2.50	276.7132	275.7580	274.7992	273.8367	272.8701	271.8994	270.9238	269.9424	268.9529	267.9474
3.00	195.1468	194.2044	193.2570	192.3041	191.3456	190.3809	189.4091	188.4286	187.4359	186.4204
3.50	145.9488	145.0208	144.0860	143.1440	142.1944	141.2364	140.2686	139.2889	138.2923	137.2662
4.00	113.9990	113.0865	112.1656	111.2356	110.2957	109.3449	108.3814	107.4024	106.4018	105.3656
1.00	1944.3105	1943.3282	1942.3454	1941.3621	1940.3783	1939.3939	1938.4089	1937.4233	1936.4370	1935.4493
1.25	1247.9097	1246.9300	1245.9495	1244.9681	1243.9860	1243.0030	1242.0190	1241.0340	1240.0478	1239.0591
1.50	869.6164	868.6398	867.6621	866.6831	865.7030	864.7216	863.7388	862.7545	861.7682	860.7780
2.00	493.4675	492.4987	491.5277	490.5547	489.5795	488.6020	487.6220	486.6391	485.6520	484.6570
2.50	319.3558	318.3966	317.4342	316.4684	315.4994	314.5266	313.5496	312.5680	311.5792	310.5770
3.00	224.7657	223.8176	222.8651	221.9080	220.9459	219.9785	219.0050	218.0242	217.0330	216.0219
3.50	167.7177	166.7823	165.8410	164.8935	163.9394	162.9779	162.0081	161.0280	160.0334	159.0126
4.00	130.6763	129.7548	128.8258	127.8888	126.9433	125.9883	125.0223	124.0429	123.0446	122.0141
1.00	2230.5293	2229.5465	2228.5631	2227.5793	2226.5951	2225.6104	2224.6252	2223.6394	2222.6530	2221.6655
1.25	1431.0901	1430.1095	1429.1282	1428.1461	1427.1634	1426.1799	1425.1956	1424.2104	1423.2241	1422.2358
1.50	996.8257	995.8478	994.8689	993.8889	992.9079	991.9258	990.9426	989.9580	988.9717	987.9822
2.00	565.0245	564.0534	563.0805	562.1057	561.1291	560.1505	559.1697	558.1864	557.1997	556.2061
2.50	365.1553	364.1926	363.2272	362.2589	361.2876	360.3132	359.3353	358.3531	357.3651	356.3655
3.00	256.5754	255.6227	254.6660	253.7053	252.7403	251.7707	250.7958	249.8147	248.8245	247.8172
3.50	191.0943	190.1529	189.2062	188.2540	187.2961	186.3318	185.3603	184.3799	183.3869	182.3707
4.00	148.5821	147.6530	146.7173	145.7746	144.8244	143.8659	142.8979	141.9181	140.9218	139.8964
1.00	2536.4873	2535.5039	2534.5202	2533.5360	2532.5514	2531.5665	2530.5811	2529.5952	2528.6088	2527.6214
1.25	1626.9035	1625.9221	1624.9401	1623.9575	1622.9743	1621.9904	1621.0058	1620.0204	1619.0341	1618.0461
1.50	1132.8077	1131.8287	1130.8488	1129.8681	1128.8864	1127.9037	1126.9201	1125.9353	1124.9490	1123.9600
2.00	641.5158	640.5428	639.5682	638.5921	637.6143	636.6347	635.6533	634.6696	633.6831	632.6908
2.50	414.1121	413.1466	412.1786	411.2082	410.2352	409.2594	408.2805	407.2980	406.3105	405.3130
3.00	290.5766	289.6200	288.6599	287.6962	286.7288	285.7573	284.7812	283.7997	282.8105	281.8062
3.50	216.0797	215.1331	214.1819	213.2259	212.2648	211.2981	210.3251	209.3444	208.3527	207.3405
4.00	167.7177	166.7823	165.8410	164.8935	163.9394	162.9779	162.0081	161.0280	160.0334	159.0126

Table 22: Eigenvalues for symmetric - antisymmetric mode $(D_x/D_y)^{1/4}$ = 17,18,19,20.

ϕ \ μ	1.0	0.9	0.8	0.7	0.6	0.5	0.4	0.3	0.2	0.1
1.00	2862.1844	2861.2006	2860.2165	2859.2321	2858.2472	2857.2620	2856.2765	2855.2905	2854.3040	2853.3167
1.25	1835.3499	1834.3679	1833.3854	1832.4023	1831.4186	1830.4343	1829.4495	1828.4640	1827.4776	1826.4898
1.50	1277.5625	1276.5826	1275.6019	1274.6205	1273.6382	1272.6551	1271.6711	1270.6861	1269.6998	1268.7112
2.00	722.9414	721.9669	720.9910	720.0136	719.0348	718.0545	717.0725	716.0885	715.1021	714.1107
2.50	466.2263	465.2584	464.2884	463.3162	462.3417	461.3647	460.3850	459.4022	458.4151	457.4193
3.00	326.7698	325.8099	324.8469	323.8807	322.9112	321.9381	320.9611	319.9793	318.9907	317.9890
3.50	242.6744	241.7235	240.7685	239.8093	238.8455	237.8767	236.9025	235.9215	234.9309	233.9220
4.00	188.0843	187.1435	186.1975	185.2459	184.2883	183.3244	182.3530	181.3727	180.3795	179.3628
1.00	3207.6207	3206.6366	3205.6522	3204.6674	3203.6824	3202.6970	3201.7113	3200.7252	3199.7387	3198.7515
1.25	2056.4293	2055.4468	2054.4638	2053.4803	2052.4963	2051.5117	2050.5267	2049.5410	2048.5546	2047.5670
1.50	1431.0901	1430.1095	1429.1282	1428.1461	1427.1634	1426.1799	1425.1956	1424.2104	1423.2241	1422.2358
2.00	809.3016	808.3257	807.3486	806.3703	805.3906	804.4096	803.4272	802.4430	801.4566	800.4660
2.50	521.4982	520.5284	519.5566	518.5829	517.6071	516.6291	515.6488	514.6657	513.6788	512.6844
3.00	365.1553	364.1926	363.2271	362.2589	361.2876	360.3132	359.3352	358.3531	357.3651	356.3655
3.50	270.8790	269.9245	268.9663	268.0042	267.0381	266.0677	265.0923	264.1110	263.1213	262.1153
4.00	209.6825	208.7371	207.7870	206.8319	205.8715	204.9054	203.9327	202.9521	201.9601	200.9469
1.00	3572.7962	3571.8118	3570.8271	3569.8421	3568.8569	3567.8714	3566.8855	3565.8994	3564.9128	3563.9257
1.25	2290.1418	2289.1588	2288.1754	2287.1915	2286.2072	2285.2224	2284.2372	2283.2514	2282.2650	2281.2775
1.50	1593.3907	1592.4094	1591.4275	1590.4450	1589.4618	1588.4780	1587.4934	1586.5081	1585.5218	1584.5337
2.00	900.5963	899.6194	898.6413	897.6621	896.6817	895.7001	894.7172	893.7329	892.7465	891.7565
2.50	579.9280	578.9565	577.9832	577.0082	576.0313	575.0525	574.0716	573.0882	572.1015	571.1082
3.00	405.7333	404.7682	403.8007	402.8306	401.8578	400.8822	399.9035	398.9211	397.9335	396.9357
3.50	300.6939	299.7362	298.7752	297.8108	296.8427	295.8708	294.8944	293.9128	292.9238	291.9203
4.00	232.5127	231.5634	230.6098	229.6517	228.6888	227.7208	226.7470	225.7661	224.7751	223.7651
1.00	3957.7109	3956.7262	3955.7413	3954.7562	3953.7707	3952.7851	3951.7991	3950.8129	3949.8263	3948.8392
1.25	2536.4873	2535.5039	2534.5202	2533.5360	2532.5514	2531.5665	2530.5811	2529.5952	2528.6088	2527.6214
1.50	1764.4641	1763.4823	1762.4999	1761.5170	1760.5334	1759.5493	1758.5645	1757.5791	1756.5927	1755.6049
2.00	996.8257	995.8478	994.8689	993.8889	992.9079	991.9258	990.9426	989.9580	988.9717	987.9822
2.50	641.5158	640.5428	639.5682	638.5921	637.6143	636.6347	635.6533	634.6696	633.6831	632.6908
3.00	448.5040	447.5369	446.5675	445.5959	444.6218	443.6452	442.6658	441.6831	440.6958	439.6995
3.50	332.1192	331.1589	330.1955	329.2290	328.2592	327.2860	326.3088	325.3269	324.3384	323.3370
4.00	256.5754	255.6227	254.6660	253.7053	252.7403	251.7707	250.7958	249.8147	248.8245	247.8172

Appendix A

PROGRAM LISTINGS

```
C                               **A**
C          COMPUTER PROGRAM  FOR ANALYSING DOUBLY SYMMETRIC FREE
C                            VIBRATION MODES
C       INPUT VARIABLES
C       R(I)=TITLE OF DATA SET
C       KT=NUMBER OF TERMS CONSIDERED
C       KL=NUMBER OF EIGENVALUES REQUIRED
C       ACC=NUMBER OF DECIMAL DIGITS REQUIRED IN
C           THE EIGENVALUE SEARCH
C       NRD=NUMBER OF VALUES OF OF D1 FOR WHICH SOLUTION IS REQUIRED
C       NF=NUMBER OF VALUES OF THE ASPECT RATIO
C       ND=NUMBER OF VALUES OF DX/DY
C       NMU=NUMBER OF VALUES OF MU
C       ARD=VALUES OF D1 INPUT
C       APHI=VALUES OF ASPECT RATIO INPUT
C       ARDXY=VALUES OF DX/DY
C       AMU=VALUES OF MU
C       ** NOTE SAME NOTATIONS USED IN ALL THE PROGRAMS **
C
        IMPLICIT REAL*8(A-H,O-Z)
        REAL*8 MU,MUS,LAMS,LAMF,NP,NPS,NPF,MP,MPS,MPF
        DIMENSION A(20,20),B(20),WKAREA(20),R(200)
        DIMENSION APHI(20),ARDXY(20),AMU(20),ARD(20),ALAMS(10,10)
        DIMENSION ADEL(30),AFAD(30),AFL(30)
        IR=5
        IP=6
        READ(IR,69) (R(I),I=1,15)
69      FORMAT(15A4)
        WRITE (IP,69) (R(I),I=1,15)
        WRITE (IP,12)
        WRITE(IP,77)
77      FORMAT (1H ,3X,'**A*** D.SYMM.MODES')
        WRITE(IP,12)
        READ(IR,10) KT,KL,ACC,DEN
10      FORMAT (2I10,2F10.3)
        IF(KT.LT.1) KT=15
        WRITE (IP,11) KT,KL,ACC,DEN
11      FORMAT (1H ,3X,'NO OF TERMS KT =',I5,/,3X,
       1'NO OF EIGENVALUES KL=',I5,/,1X,
       1'ACC=',D16.5,/,1X,'DEN=',F10.4)
        WRITE(IP,12)
```

```
12    FORMAT(/)
C
      KT1=2*KT-1
      KT2=2.0*KT
      DPA=1.0
      PI=DATAN(DPA)*4
      READ (IR,41) NRD,NF,ND,NMU
      WRITE (IP,41)NRD,NF,ND,NMU
41    FORMAT (4I10)
      READ(IR,2) (ARD(I),I=1,NRD)
      READ(IR,2) (APHI(J),J=1,NF)
      READ(IR,2) (ARDXY(K),K=1,ND)
      READ(IR,2) (AMU(L),L=1,NMU)
2     FORMAT(6F10.2)
      WRITE (IP,3) (ARD(I),I=1,NRD)
      WRITE (IP,3) (APHI(J),J=1,NF)
      WRITE (IP,3) (ARDXY(K),K=1,ND)
      WRITE (IP,3) (AMU(L),L=1,NMU)
3     FORMAT(1H ,/,6F10.2)
      READ (IR,90) (ADEL(I),AFAD(I),I=1,NF)
      READ (IR,90) (AFL(I),I=1,NF)
90    FORMAT (6F10.5)
      WRITE (IP,12)
      WRITE (IP,90) (ADEL(I),AFAD(I),I=1,NF)
      WRITE (IP,12)
      WRITE (IP,91) (AFL(I),I=1,NF)
91    FORMAT(6F10.4)
C
      DO 4 IX=1,NRD
      DO 4 JX=1,NF
      DEL=ADEL(JX)
      FAD=AFAD(JX)
      FLAMS=AFL(JX)
      DO 4 KX=1,ND
      LAMS=FLAMS
      DO 4 LX=1,NMU
      RD=ARD(IX)
      PHI=1.0/APHI(JX)
      RDXY=ARDXY(KX)**4
      MU=AMU(LX)
C
      NJ=0
      KNT=0
      JN=1
      IC=0
      ICC=0
      CLAMS=0
      FA=1.0
14    FORMAT (4F10.2,2D10.4)
18    FORMAT (1H ,3X,'LAMS=',F8.3,5X,'DLIM=',F8.3,5X,'DEL=',F8.3
     1,/,3X,'FAD=',F8.3,5X,'ACC=',D15.5,5X,'DEN=',F10.4)
      WRITE (IP,12)
C
64    CONTINUE
```

```
      PHIS=PHI*PHI
      PHIF=PHIS*PHIS
      CA=PHI*RDXY**0.25
      CAS=CA*CA
      MUS=MU*MU
      LAMF=LAMS*LAMS
      DO 15 I=1,KT2
      DO 15 J=1,KT2
15    A(I,J)=0.0
C
      DO 100  M=1,KT
      DO 100 NI=1,KT
      N=NI-1
      J=KT+N+1
      I=M
      NP=N*PI
      NPS=NP*NP
      NPF=NPS*NPS
      MP=M*PI
      MPS=MP*MP
      MPF=MPS*MPS
      RAS=LAMF/MPF/RDXY/PHIF
      RA=DSQRT(RAS)
      EXP=MUS-1.0+RAS
      IF(EXP.LE.0.0) EXP=-1.0*EXP
      EXPR=DSQRT(EXP)
      TE=MPS*CAS
      SIMS=TE*(EXPR+MU)
      SIM=DSQRT(SIMS)
      T1=MU-EXPR
      FIMS=TE*DABS(T1)
      FIM=DSQRT(FIMS)
      ARG=RDXY*MPF-LAMF/PHIF
C
      IF (ARG.GT.0.0) GO TO 20
      IF (N.GT.0) GO TO 32
      A(I,I)=1.0
32    CONTINUE
      HSDY=MU*DSQRT(RDXY)
      TEA=PHIS*(2.0*HSDY-RD)*MPS
      TEB=RD*PHIS*MPS
      THM1=(TEA-SIMS)*SIM*DS(SIM)/(TEA + FIMS)/FIM/DSIN(FIM)
      THM11=(TEB-SIMS)*DC(SIM) + THM1*(FIMS+TEB)*DCOS(FIM)
      PA=SIM*DS(SIM)/(SIMS+NPS)
      P1=PA + THM1*FIM*DSIN(FIM)/(FIMS-NPS)
      A(J,I) =2.0*MP*DCOS(MP)*DCOS(NP)/THM11*P1
      IF(N.EQ.0) A(J,I)=A(J,I)/2.0
      GO TO 99
C
20    CONTINUE
      HSDY=MU*DSQRT(RDXY)
      IF(N.GT.0) GO TO 35
      A(I,I)=1.0
35    CONTINUE
```

```
C
      TEA=PHIS*(2.0*HSDY-RD)*MPS
      THM2=(TEA-SIMS)* SIM *DS(SIM)/(FIMS-TEA)/FIM/DS(FIM)
      TEB=RD*PHIS*MPS
      THM22=(TEB-SIMS)*DC(SIM) + THM2*(TEB-FIMS)*DC(FIM)
      PB= SIM*DS(SIM)/(SIMS+NPS)
      P1=PB + THM2*FIM*DS(FIM)/(FIMS+NPS)
      A(J,I)=2.0*MP*DCOS(MP)*DCOS(NP)/THM22*P1
      IF(N.EQ.0) A(J,I)=A(J,I)/2.0
C
99    CONTINUE
100   CONTINUE
C
      DO 200 M=1,KT
      DO 200 NI=1,KT
      N=NI-1
      J=KT+NI
      I=M
C
      NP=N*PI
      NPS=NP*NP
      NPF=NPS*NPS
      MP=M*PI
      MPS=MP*MP
      MPF=MPS*MPS
      IF(N.EQ.0) GO TO 140
C
      RBS=LAMF/NPF
      RB=DSQRT(RBS)
      EXP=(MUS-1.0+RBS)
      EXP=DABS(EXP)
      EXPR=DSQRT(EXP)
      TE=NPS/CAS
      SYNS=TE*(EXPR+MU)
      SYN=DSQRT(SYNS)
      T2=EXPR-MU
      FINS=TE*DABS(T2)
      FIN=DSQRT(FINS)
      GO TO 141
140   SYNS=LAMS/CAS
      SYN=DSQRT(SYNS)
      FINS=SYNS
      FIN=SYN
141   CONTINUE
      ARG=1.0/RDXY*NPF-LAMF/RDXY
C
      IF(ARG.GT.0.0) GO TO 120
      IF(M.GT.1) GO TO 125
      VAL=FIN*DCOS(FIN)/DSIN(FIN) - SYN*DC(SYN)/DS(SYN)
      A(J,J)=VAL/(SYNS+FINS)
C
125   CONTINUE
      E1=(NPS+RD*PHIS*FINS)/(FINS-MPS) + (NPS-RD*PHIS
     1*SYNS)/(SYNS+MPS)
```

```
      A(I,J)=2.0*MP*DCOS(NP)*DCOS(MP)/(SYNS+FINS)*E1
      GO TO 199
120   IF(M.GT.1)GO TO 137
C
      VAL=FIN*DC(FIN)/DS(FIN)-SYN*DC(SYN)/DS(SYN)
      A(J,J)=VAL/(SYNS-FINS)
137   CONTINUE
C
      E2=(NPS-RD*PHIS*SYNS)/(SYNS+MPS)-(NPS-
     1RD*PHIS*FINS)/(FINS+MPS)
      A(I,J)=2.0*MP*DCOS(NP)*DCOS(MP)/(SYNS-FINS)*E2
199   CONTINUE
200   CONTINUE
C
      IJOB=4
      N=KT2
      IA=KT2
      D1=1
      IC=IC+1
      CALL LINV3F(A,B,IJOB,N,IA,D1,D2,WKAREA,IER)
      D=D1*2**D2
      IF(IC.EQ.1) GO TO 302
C
      IF(NJ.LE.0) GO TO 402
      IF(DABS(D).GT.DABS(SSD)/1.5) GO TO 401
      GO TO 402
401   IF(NJ.EQ.0) GO TO 302
      NJ=0
      FA=1.0
      D=RSD
      LAMS=RSL
      GO TO 302
402   IF((FA.LT.1.0).AND.(LAMS.GT.RSL)) FA=1.0
      IF(D/SD.GT.0.0) GO TO 302
      IF(DABS(D). GT.ACC) GO TO 302
      IF(DABS(SD).GT.ACC) GO TO 302
C
303   FORMAT (1H ,3X,'LAMS=',F10.4,5X,'D=',E12.4,5X,'SD=',E12.4,
     13X,'IC=',I5)
      IF(IC.EQ.ICC) GO TO 321
      IF(FA.LT.0.50) GO TO 320
      ICS=IC
      SSD=SD
      SSL=SLAMS
      RSD=D
      RSL=LAMS
      FA=FAD
      LAMS=SLAMS +0.95*DABS(SD)/(DABS(SD)+DABS(D))*(LAMS-SLAMS)
      ICC=IC+1
      NJ=1
C     WRITE (IP,113)LAMS,SLAMS,D,SD,IC
C113  FORMAT(1H ,'*11 LAM,SLA,D,SD',2F9.3,2X,2E12.4,1X,I6)
      GO TO 64
C
```

```
321    SD=D
       SLAMS=LAMS
       FA=-1.0*FAD
       LAMS=LAMS+FA*DEL
       NJ=2
       GO TO 64
C
320    IF(DABS(D).GT.ACC/DEN/10.) GO TO 407
       GO TO 409
407    NJ=0
       FA=1.0
       LAMS=RSL
       D=RD
       GO TO 302
409    CONTINUE
       FA=1.0
       WRITE(IP,303) LAMS,D,SD,IC
       ALAMS(LX,JX)=LAMS
       IF(LAMS-CLAMS.GT.DEL) GO TO 117
       WRITE(IP,118)
118    FORMAT(1H ,'CHANGE ACC')
       STOP
117    CLAMS=LAMS
       LAMS=LAMS+DEL*FA
       SD=D
       IF(NJ.EQ.2)SD=-1.0*D
       KNT=KNT+1
       NJ=0
       IF(KNT.LT.KL) GO TO 64
       GO TO 1
302    CONTINUE
C      WRITE (IP,999) LAMS,SLAMS,D,SD,IC,FA
       SLAMS=LAMS
       SD=D
       LAMS=LAMS+DEL*FA
999    FORMAT(1H ,'LAM SLA D SD IC',2F10.4,2E12.4,I5,E9.2)
88     FORMAT (1H ,'NO OF ITERATIONS > 9000')
33     CONTINUE
       GO TO 64
1      CONTINUE
       LAMS=FLAMS
4      CONTINUE
       WRITE (IP,44) ((ALAMS(MY,JY),MY=1,NMU),JY=1,NF)
44     FORMAT (10F10.4)
       STOP
       END
       FUNCTION DS(X)
       IMPLICIT REAL *8(A-H,O-Z)
       IF(X.GT.125) GO TO 10
       DS=DSINH(X)
       RETURN
10     DS=1.0
       RETURN
       END
```

```
      FUNCTION DC(X)
      IMPLICIT REAL *8(A-H,O-Z)
      IF(X.GT.125) GO TO 10
      DC=DCOSH(X)
      RETURN
10    DC=1.0
      RETURN
      END
```

```
C                   PGM *** BC ****
C          COMPUTER PROGRAM VIB FOR ANALYSING DOUBLY ANTI-SYMMETRIC
C          AND
C          SYMMETRIC-ANTISYMMETRIC FREE VIBRATION  MODES
      IMPLICIT REAL*8(A-H,O-Z)
      REAL*8 MU,MUS,LAMS,LAMF,NP,NPS,NPF,MP,MPS,MPF
      DIMENSION A(2,2),B(30),WKAREA(30),R(200),BLAMS(10,10)
      DIMENSION APHI(20),ARDXY(20),AMU(20),ARD(20),ALAMS(10,10)
      DIMENSION ADEL(30),AFAD(30),AFL(30)
      IR=5
      IP=6
      READ(IR,69) (R(I),I=1,15)
69    FORMAT(15A4)
      WRITE(IP,69) (R(I),I=1,15)
      WRITE(IP,12)
      WRITE (IP,77)
77    FORMAT(1H ,3X,'** B ** SYMM.ASYMM.MODES')
      KL=2
      MM=1
      ACC=0.01
      DEN=1.0
      READ(IR,69) (R(I),I=1,15)
      WRITE(IP,11) KL,MM,ACC,DEN
11    FORMAT (1H ,3X,' KL =',I5,/3X,'MM=',I5,3X,'ACC=',D12.5,5X,
     1'DEN=',D12.4)
12    FORMAT (/)
      KT=1
      KT2=2*KT
      DPA=1.0
      PI=DATAN(DPA)*4
C
      READ (IR,41) NRD,NF,ND,NMU
      WRITE (IP,41)NRD,NF,ND,NMU
41    FORMAT (4I10)
      READ(IR,2) (ARD(I),I=1,NRD)
      READ(IR,2) (APHI(J),J=1,NF)
      READ(IR,2) (ARDXY(K),K=1,ND)
      READ(IR,2) (AMU(L),L=1,NMU)
2     FORMAT(6F10.2)
      WRITE (IP,3) (ARD(I),I=1,NRD)
      WRITE (IP,3) (APHI(J),J=1,NF)
      WRITE (IP,3) (ARDXY(K),K=1,ND)
      WRITE (IP,3) (AMU(L),L=1,NMU)
3     FORMAT(1H ,/,6F10.2)
      READ (IR,90) (ADEL(I),AFAD(I),I=1,NF)
      READ (IR,90) (AFL(I),I=1,ND)
90    FORMAT (6F10.5)
      WRITE (IP,12)
      WRITE (IP,90) (ADEL(I),AFAD(I),I=1,NF)
      WRITE (IP,12)
      WRITE (IP,91) (AFL(I),I=1,ND)
91    FORMAT(6F10.4)
C
      DO 4 IX=1,NRD
      DO 4 JX=1,NF
```

```
      DEL=ADEL(JX)
      FAD=AFAD(JX)
      DO 4 KX=1,ND
      FLAMS=AFL(KX)
      LAMS=FLAMS
      DO 4 LX=1,NMU
      RD=ARD(IX)
      PHI=1.0/APHI(JX)
      RDXY=ARDXY(KX)**4
      MU=AMU(LX)
C
      KNT=1
      IC=0
      FA=1.0
      M=1
14    FORMAT (4F10.2,2D10.4)
18    FORMAT (1H ,3X,'LAMS=',F8.3,5X,'DLIM=',F8.3,5X,'DEL=',F8.3
     1,/,3X,'FAD=',F8.3,5X,'ACC=',D15.5,5X,'DEN=',F10.4)
      WRITE (IP,12)
64    CONTINUE
C
      PHIS=PHI*PHI
      PHIF=PHIS*PHIS
      CA=PHI*RDXY**0.25
      CAS=CA*CA
      MUS=MU*MU
      LAMF=LAMS*LAMS
      HSDY=MU*DSQRT(RDXY)
      DO 15 I=1,KT2
      DO 15 J=1,KT2
15    A(I,J)=0.0
C
      MP=M*PI
      MPS=MP*MP
      MPF=MPS*MPS
      RAS=LAMF/MPF/RDXY/PHIF
      RA=DSQRT(RAS)
      EXP=MUS-1.0+RAS
      EXP=DABS(EXP)
      EXPR=DSQRT(EXP)
      TE=MPS*CAS
      SIMS=TE*(EXPR+MU)
      SIM=DSQRT(SIMS)
      FIMS=TE*(MU-EXPR)
      FIMS=DABS(FIMS)
      FIM=DSQRT(FIMS)
      SIM2=SIM/2.0
      FIM2=FIM/2.0
      ARG=RDXY*MPF-LAMF/PHIF
      IF(KNT/2*2.EQ.KNT) GO TO 51
      GO TO 52
51    CONTINUE
      IBC=2
      IF (ARG.GT.0.0) GO TO 20
```

```
      FS=PHIS*MPS
      A(1,1)=(SIMS-RD*FS)*DS(SIM2)
      A(1,2)=-(FIMS+RD*FS)*DSIN(FIM2)
      A(2,1)=SIM*(SIMS-(2.0*HSDY-RD)*FS)*DC(SIM2)
      A(2,2)=-FIM*(FIMS+(2.0*HSDY-RD)*FS)*DCOS(FIM2)
      GO TO 99
C
20    CONTINUE
      FS=PHIS*MPS
      A(1,1)=(SIMS-RD*FS)*DS(SIM2)
      A(1,2)=(FIMS-RD*FS)*DS(FIM2)
      A(2,1)=SIM*(SIMS-(2.0*HSDY-RD)*FS)*DC(SIM2)
      A(2,2)=FIM*(FIMS-(2.0*HSDY-RD)*FS)*DC(FIM2)
      GO TO 99
C
52    IBC=1
      IF (ARG.GT.0.0) GO TO 22
      FS=PHIS*MPS
      A(1,1)=(SIMS-RD*FS)*DC(SIM2)
      A(1,2)=-1.0*(FIMS+RD*FS)*DCOS(FIM2)
      A(2,1)=SIM*(SIMS-(2.0*HSDY-RD)*FS)*DS(SIM2)
      A(2,2)=FIM*(FIMS+(2.0*HSDY-RD)*FS)*DSIN(FIM2)
      GO TO 99
C
22    CONTINUE
      FS=PHIS*MPS
      A(1,1)=(SIMS-RD*FS)*DC(SIM2)
      A(1,2)=(FIMS-RD*FS)*DC(FIM2)
      A(2,1)=SIM*(SIMS-(2.0*HSDY-RD)*FS)*DS(SIM2)
      A(2,2)=FIM*(FIMS-(2.0*HSDY-RD)*FS)*DS(FIM2)
C
99    CONTINUE
      IC=IC+1
      CALL SR(A,D)
      IF (IC.EQ.1)GO TO 302
      IF(D/SD.GT.0.0) GO TO 302
C
303   FORMAT (1H ,3X,'LAMS=',F10.4,5X,'D=',E18.6,5X,'SD=',E18.6)
      IF(FA.LT.0.20) GO TO 320
      FA=FAD
      LAMS=SLAMS+DEL*FA
      GO TO 64
320   FA=1.0
C     WRITE(IP,303) LAMS,D,SD
      IF (IBC.EQ.1) ALAMS(LX,JX)=LAMS
      IF (IBC.EQ.2) BLAMS(LX,JX)=LAMS
      LAMS=LAMS+DEL*FA
      SD=D
      KNT=KNT+1
      IC=0
      FA=1.0
      IF(KNT.LE.KL) GO TO 64
      KNT=1
      M=M+1
```

```
      LAMS=FLAMS
      FA=1.0
      IC=0
      IF(M.LE.MM) GO TO 64
      GO TO 1
302   SLAMS=LAMS
      SD=D
      LAMS=LAMS+DEL*FA
      GO TO 64
  1   CONTINUE
      LAMS=FLAMS
4     CONTINUE
      WRITE (IP,44)  ((ALAMS(MY,JY),MY=1,NMU),JY=1,NF)
      WRITE (IP,17)
17    FORMAT (1H ,///)
      WRITE (IP,44)  ((BLAMS(MY,JY),MY=1,NMU),JY=1,NF)
44    FORMAT (10F10.4)
      STOP
      END
      FUNCTION DS(X)
      IMPLICIT REAL *8(A-H,O-Z)
      DS=DSINH(X)
      RETURN
      RETURN
      END
      FUNCTION DC(X)
      IMPLICIT REAL *8(A-H,O-Z)
      DC=DCOSH(X)
      RETURN
      RETURN
      END
      SUBROUTINE SR(A,D)
      IMPLICIT REAL *8(A-H,O-Z)
      DIMENSION A(2,2)
      D=A(1,1)*A(2,2) - A(1,2)*A(2,1)
      RETURN
      END
```

```
C                      **PGM-D**
C          COMPUTER   PROGRAM  FOR   ANALYSING    THE
C          ANTI-SYMMETRIC SYMMETRIC FREE VIBRATION MODES
      IMPLICIT REAL*8(A-H,O-Z)
      REAL*8 MU,MUS,LAMS,LAMF,NP,NPS,NPF,MP,MPS,MPF,NPT,NPTS,NPTF
      DIMENSION A(20,20),B(20),WKAREA(20),R(200)
C     DIMENSION A(30,30),B(30),WKAREA(30),R(200)
      DIMENSION APHI(20),ARDXY(20),AMU(20),ARD(20),ALAMS(10,10)
      DIMENSION ADEL(30),AFAD(30),AFL(30)
      IR=5
      IP=6
      READ(IR,69) (R(I),I=1,15)
69    FORMAT(15A4)
      WRITE(IP,69) (R(I),I=1,15)
      WRITE(IP,12)
      WRITE(IP,77)
77    FORMAT(1H ,3X,'** D ** ASYM.ASYM.MODES')
      WRITE(IP,12)
      READ(IR,10) KT,KL,ACC,DEN
10    FORMAT (2I10,2F10.3)
      IF(KT.LT.1) KT=15
      WRITE(IP,11) KT,KL,ACC,DEN
11    FORMAT (1H ,3X,'NO OF TERMS KT =',I5,//,3X,
     1'NO OF EIGEN VALUES  =',I5,/,3X,'ACC='
     1,D16.5,/,3X,'DEN=',F10.4)
      WRITE(IP,12)
12    FORMAT (/)
C
      KT1=2*KT-1
      KT2=2.0*KT
      DPA=1.0
      PI=DATAN(DPA)*4
      READ(IR,41) NRD,NFI,ND,NMU
      WRITE(IP,41)NRD,NFI,ND,NMU
41    FORMAT(4I10)
      READ(IR,2) (ARD(I),I=1,NRD)
      READ(IR,2) (APHI(J),J=1,NFI)
      READ(IR,2) (ARDXY(K),K=1,ND)
      READ(IR,2) (AMU(L),L=1,NMU)
2     FORMAT(6F10.2)
      WRITE(IP,3) (ARD(I),I=1,NRD)
      WRITE(IP,3) (APHI(J),J=1,NFI)
      WRITE(IP,3) (ARDXY(K),K=1,ND)
      WRITE(IP,3) (AMU(L),L=1,NMU)
3     FORMAT(1H ,/,6F10.2)
      READ(IR,90) (ADEL(I),AFAD(I),I=1,NFI)
      READ(IR,90) (AFL(I),I=1,NFI)
90    FORMAT(6F10.2)
      WRITE(IP,12)
      WRITE(IP,90) (ADEL(I),AFAD(I),I=1,NFI)
      WRITE(IP,12)
      WRITE(IP,91) (AFL(I),I=1,NFI)
91    FORMAT(6F10.4)
C
      DO 4 IX=1,NRD
```

```
      DO 4 JX=1,NFI
      DEL=ADEL(JX)
      FAD=AFAD(JX)
      FLAMS=AFL(JX)
      DO 4 KX=1,ND
C     FLAMS=AFL(KX)
      LAMS=FLAMS
      DO 4 LX=1,NMU
      RD=ARD(IX)
      PHI=1.0/APHI(JX)
      RDXY=ARDXY(KX)**4
      MU=AMU(LX)
C
      KNT=0
      JN=1
      NJ=0
      IC=0
      ICC=0
      FA=1.0
      CLAMS=0
14    FORMAT (4F10.2,2D10.4)
18    FORMAT (1H ,'LAMS=',F8.3,5X,'DLIM=',F8.3,5X,'DEL=',F8.3,
     1/,1X,'FAD=',F8.3,5X,'ACC=',D12.3,5X,'DEN=',F10.4)
      WRITE(IP,12)
C
64    CONTINUE
      PHIS=PHI*PHI
      PHIF=PHIS*PHIS
      CA=PHI*RDXY**0.25
      CAS=CA*CA
      MUS=MU*MU
      LAMF=LAMS*LAMS
      DO 15 I=1,KT2
      DO 15 J=1,KT2
15    A(I,J)=0.0
C
      DO 100  M=1,KT
      DO 100 NI=1,KT1,2
      N=NI
      J=KT+(NI+1)/2
      I=M
      NP=N*PI
      NPS=NP*NP
      NPF=NPS*NPS
      NPT=NP/2.0
      NPTS=NPT*NPT
      MP=M*PI
      MPS=MP*MP
      MPF=MPS*MPS
      RAS=LAMF/MPF/RDXY/PHIF
      RA=DSQRT(RAS)
      EXP=MUS-1.0+RAS
      EXP=DABS(EXP)
      EXPR=DSQRT(EXP)
```

```
      TE=MPS*CAS
      SIMS=TE*(EXPR+MU)
      SIM=DSQRT(SIMS)
      T1=MU-EXPR
      FIMS=TE*DABS(T1)
      FIM=DSQRT(FIMS)
      ARG=RDXY*MPF-LAMF/PHIF
      IF (ARG.GT.0.0) GO TO 20
C
      IF (N.GT.1) GO TO 32
      A(I,I)=1.0
32    CONTINUE
      HSDY=MU*DSQRT(RDXY)
      TEA=PHIS*(2.0*HSDY-RD)*MPS
      THM1=FIM*(FIMS+TEA)*DCOS(FIM)/SIM/(SIMS-TEA)/DC(SIM)
C
C
      TEB=RD*PHIS*MPS
      THM11=(FIMS+TEB)*DSIN(FIM)-THM1*(SIMS-TEB)*DS(SIM)
      PA=FIM*DCOS(FIM)/(FIMS-NPTS)
      P1=THM1*SIM*DC(SIM)/(SIMS+NPTS)-PA
      A(J,I) =2.0*MP*DCOS(MP)*DSIN(NPT)/THM11*P1
      GO TO 99
C
20    CONTINUE
      HSDY=MU*DSQRT(RDXY)
      IF(N.GT.1) GO TO 35
      A(I,I)=1.0
35    CONTINUE
C
      TEA=PHIS*(2.0*HSDY-RD)*MPS
      THM2=FIM*(TEA-FIMS)*DC(FIM)/SIM/(SIMS-TEA)/DC(SIM)
      TEB=RD*PHIS*MPS
      THM22=THM2*(TEB-SIMS)*DS(SIM)+(TEB-FIMS)*DS(FIM)
      PB= FIM*DC(FIM)/(FIMS+NPTS)
      P1=THM2*SIM*DC(SIM)/(SIMS+NPTS)+PB
      A(J,I)=2.0*MP*DCOS(MP)*DSIN(NPT)/THM22*P1
C
99    CONTINUE
100   CONTINUE
C
      DO 200 M=1,KT
      DO 200 NI=1,KT1,2
      N=NI
      J=KT+(NI+1)/2
      I=M
      NP=N*PI
      NPS=NP*NP
      NPF=NPS*NPS
      NPT=NP/2.0
      NPTS=NPT*NPT
      NPTF=NPTS*NPTS
      MP=M*PI
      MPS=MP*MP
```

```
      MPF=MPS*MPS
C
      RCS=LAMF/NPTF
      RC=DSQRT(RCS)
      EXP=(MUS-1.0+RCS)
      EXP=DABS(EXP)
      EXPR=DSQRT(EXP)
      TE=NPTS/CAS
      SYNS=TE*(EXPR+MU)
      SYN=DSQRT(SYNS)
      T2=EXPR-MU
      FINS=TE*DABS(T2)
      FIN=DSQRT(FINS)
      ARG=1.0/RDXY*NPTF-LAMF/RDXY
      IF(ARG.GT.0.0) GO TO 120
C
      IF(M.GT.1) GO TO 125
      VAL=FIN*DCOS(FIN)/DSIN(FIN) - SYN*DC(SYN)/DS(SYN)
      A(J,J)=VAL/(SYNS+FINS)
C
125   CONTINUE
      E1=(NPTS +RD*PHIS*FINS)/(FINS-MPS) +(NPTS-RD*PHIS*
     1SYNS)/(SYNS+MPS)
      A(I,J)=2.0*MP*DSIN(NPT)*DCOS(MP)/(SYNS+FINS)*E1
      GO TO 199
C
120   IF(M.GT.1)GO TO 137
      VAL=FIN*DC(FIN)/DS(FIN)-SYN*DC(SYN)/DS(SYN)
      A(J,J)=VAL/(SYNS-FINS)
C
137   CONTINUE
      E2=(RD*PHIS*FINS-NPTS)/(FINS+MPS)+(NPTS-
     1RD*PHIS*SYNS)/(SYNS+MPS)
      A(I,J)=2.0*MP*DSIN(NPT)*DCOS(MP)/(SYNS-FINS)*E2
C
199   CONTINUE
200   CONTINUE
C
C
      IJOB=4
      N=KT2
      IA=KT2
      D1=1
      IC=IC+1
C
      IF(IC.LT.1000) GO TO 87
      WRITE (IP,88)
88    FORMAT (1H ,'ERROR IN LAMS')
      GO TO 89
87    CONTINUE
C
      CALL LINV3F(A,B,IJOB,N,IA,D1,D2,WKAREA,IER)
      D=D1*2**D2
      IF (IC.EQ.1)GO TO 302
```

```
C
      IF(NJ.LE.0) GO TO 402
      IF(DABS(D).GT.DABS(SSD)/1.5) GO TO 401
      GO TO 402
 401  IF(NJ.EQ.0) GO TO 302
      NJ=0
      FA=1.0
      D=RSD
      LAMS=RSL
      GO TO 302
 402  IF((FA.LT.1.0).AND.(LAMS.GT.RSL)) FA=1.0
      IF(D/SD.GT.0.0) GO TO 302
      IF(DABS(D).GT.ACC) GO TO 302
      IF(DABS(SD).GT.ACC) GO TO 302
C
 303  FORMAT (1H ,3X,'LAMS=',F10.4,4X,'D=',E12.4,4X,'SD=',E12.4,
     13X,'IC=',I5)
      IF(IC.EQ.ICC) GO TO 321
      IF(FA.LT.0.50) GO TO 320
      ICS=IC
      SSD=SD
      SSL=SLAMS
      RSD=D
      RSL=LAMS
      FA=FAD
      LAMS=SLAMS+0.95*DABS(SD)/(DABS(SD)+DABS(D))*(LAMS-SLAMS)
      ICC=IC+1
C     WRITE(IP,113) LAMS,SLAMS,D,SD,IC
      NJ=1
 113  FORMAT(1H ,'*11 LAM,SLA,D,SD',2F9.3,2E12.4,1X,I6)
      GO TO 64
C
 321  SD=D
      SLAMS=LAMS
      FA=-1.0*FAD
      LAMS=LAMS+FA*DEL
C     WRITE(IP,1112) LAMS,SLAMS,D,SD,IC
1112  FORMAT(1H ,'*22 LAM SLA D SD',2F9.3,2X,2E12.4,1X,I6)
      NJ=2
      GO TO 64
C
 320  IF(DABS(D).GT.ACC/DEN/10.) GO TO 407
      GO TO 409
 407  NJ=0
      FA=1.0
      LAMS=RSL
      D=RD
      GO TO 302
 409  CONTINUE
      FA=1.0
 89   WRITE(IP,303) LAMS,D,SD,IC
      ALAMS(LX,JX)=LAMS
      IF(LAMS-CLAMS.GT.DEL) GO TO 117
      WRITE(IP,118)
```

```
 118    FORMAT (1H ,3X,'CHANGE ACC VALUE')
        STOP
 117    CLAMS=LAMS
        LAMS=LAMS+DEL*FA
        SD=D
        IF(NJ.EQ.2) SD=-1.0*D
        KNT=KNT+1
        NJ=0
        IF(KNT.LT.KL) GO TO 64
        GO TO 1
 302    CONTINUE
        SLAMS=LAMS
        SD=D
        LAMS=LAMS+DEL*FA
C
        GO TO 64
 1      CONTINUE
        LAMS=FLAMS
 4      CONTINUE
C
        WRITE(IP,44) ((ALAMS(MY,JY),MY=1,NMU),JY=1,NFI)
 44     FORMAT (10F10.4)
        STOP
        END
        FUNCTION DS(X)
        IMPLICIT REAL *8(A-H,O-Z)
        IF(X.GT.125) GO TO 10
        DS=DSINH(X)
        RETURN
 10     DS=1.0
        RETURN
        END
        FUNCTION DC(X)
        IMPLICIT REAL *8(A-H,O-Z)
        IF(X.GT.125) GO TO 10
        DC=DCOSH(X)
        RETURN
 10     DC=1.0
        RETURN
        END
```

Appendix B

SOLUTION ROUTINE

IMSLD ROUTINE LINV3F

Purpose - in place inverse, equation solution and or determinant evalulation

Usage -CALL LINV3F(A,B,IJOB,N,IA,D1,D2,WKAREA,IER)

Arguments

A - Input/Output matrix of dimension N by N. See parameter IJOB

B - Input/Output Vector of length N when IJOB = 2 or 3, otherwise B is not used. On input B contains the right hand side of the equation AX=B. On the Output the solution X replaces B.

IJOB - INPUT OPTION PARAMETER.

IJOB=1 ; Invert matrix A. A is replaced by its inverse.

IJOB=2 ; Solve the equation AX=B. A is replaced by a LU decomposition of a rowwise permutation of A, where U is the upper triangular and L is the lower triangular, matrix with unit diagonal.

IJOB=3 ; Solve AX=B and invert matrix A. A is replaced by its inverse.

IJOB=4 ; Compute the determinant of A. A is replaced by the LU decomposition of a rowwise permutation of A.

N - Order of A (Input)

IA - Row dimension of matrix A

D1 Input/Output. If the D1 and D2 components of the determinanat of A =D1*2**D2 are desired input D1 GE 0, otherwise input D1 LT 0. D2 is never input.

WKAREA Work area of length atleast 2*N for IJOB=1 or IJOB=3. Work area of input length atleast N for IJOB=2 or IJOB=4.

IER Error parameter.

Once the eigenvalue matrix is established, the eigenvalues are searched using the following steps.

1. After selecting the number of terms in the series and a trial value for the eigenvalue establish the coefficient matrix (Figure 10).
2. Evaluate and store the determinant.
3. The trial value of the eigenvalue is incremented and after repeating steps 1 and 2 a value of the eigenvelue causing the determinant of the coefficient matrix to vanish is established. A vanishing determinant indicates that a non trivial solution exists for E_m and E_n and hence the associated value is an eigenvalue.