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LA THÈSE A ÉTÉ  
MICROFILMÉE TELLE QUE  
NOUS L'AVONS REÇUE

ON THE MAXIMUM EIGENVALUE CONJECTURE  
FOR THE OSCILLATION OF  
SECOND ORDER ORDINARY DIFFERENTIAL SYSTEMS

by

KAZUO AKIYAMA

A thesis presented  
to  
the School of Graduate Studies  
of the  
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To my parents

Shigeo and Nobu AKIYAMA

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## ABSTRACT

For the vector differential equation

$$y'' + Q(t)y = 0, \quad t \in [0, \infty)$$

where  $Q(t)$  is a real continuous  $n$  by  $n$  symmetric matrix function, we present here some results pertaining to the oscillatory behavior at  $\infty$  in terms of the "maximum eigen-

value" of  $P(t) = \int_0^t Q(s) ds$ .

## INTRODUCTION

The oscillation theory for the second-order scalar differential equation

$$(A) \quad u'' + q(t)u = 0, \quad t \in [0, \infty), \quad \left( " = \frac{d^2}{dt^2} \right)$$

where  $q(t) \in C^0[0, \infty)$ , has been studied by so many authors. The equation (A) is said to be *oscillatory at  $\infty$*  or *non-oscillatory at  $\infty$*  according as one (hence every) solution  $u = u(t) \not\equiv 0$  of (A) has or does not have an infinity of zeros on  $0 \leq t < \infty$ . Probably the first such qualitative study originated in the classic paper of C. Sturm [S-3;1836]. After him, A. Kneser [K-1;1893] gave the following criteria for equation (A),

$$(B) \quad \begin{cases} q(t) \leq \frac{1}{4t^2} & \rightarrow \text{non-oscillation} \\ q(t) \geq \frac{(1 + \delta)}{4t^2} & \rightarrow \text{oscillation, } \delta > 0 \end{cases}$$

Later, W. B. Fite [F-1;1918] proved that  $q(t) > 0$  and

$$\int_0^{\infty} q(s) ds = \infty \quad \text{together imply oscillation. (see [W-2], [W-3])}$$

The condition  $q(t) > 0$  in Fite's result was later removed

by A. Wintner [W-4;1949], who proved that

$$(C) \quad \lim_{t \rightarrow \infty} \frac{1}{t} \int_T^t \left( \int_T^s q(u) du \right) ds = \infty \rightarrow \text{oscillation}$$

This is probably the first criterion using a "mean". Later, P. Hartman [H-1;1952] proved that the non-existence (as a single finite number) of the limit in (C) and the condition

$$(D) \quad \liminf_{t \rightarrow \infty} \frac{1}{t} \int_T^t \left( \int_T^s q(u) du \right) ds > -\infty$$

imply oscillation.

A generalization of (D) was given by W. J. Coles and D. Willett [CW-1;1968] where the arithmetical mean was replaced by a more general mean

$$(E) \quad (KQ)(t) = \int_0^t K(t,s)Q(s) ds$$

where  $Q(t) = \int_0^t q(s) ds$ . Here  $K(t,s)$  is a kernel which will be described in detail later on (see Chapter III). P. Hartman [H-3;1977] then gave a more general necessary condition for equation (A) to be non-oscillatory at  $\infty$ : namely that either

$$(F) \quad \liminf_{t \rightarrow \infty} \int_0^t K(t,s)Q(s) \, ds = -\infty$$

or that there exists a constant  $C$  such that

$$(G) \quad \int_0^t K(t,s) |C - Q(s)|^\tau \, ds \rightarrow 0 \text{ as } t \rightarrow \infty,$$

for fixed  $\tau$ ,  $1 \leq \tau \leq 2$ . (F) and (G) are basically analogous to (C) and (D).

On the other hand, G. J. Etgen and J. F. Pawlowski [EP-1;1976] gave an oscillation criterion ((I) below) for the matrix differential equation

$$(H) \quad Y'' + Q(t)Y = 0,$$

where  $Q(t)$  is a real continuous  $n$  by  $n$  symmetric matrix function, by introducing a positive linear functional  $g$  on symmetric matrices. The condition

$$(I) \quad \lim_{t \rightarrow \infty} g \left( \int_0^t Q(s) \, ds \right) = \infty$$

implies that (H) is oscillatory.

Note The definition of oscillatory in the matrix case is similar to that of the scalar case. Details will be given later on (see Chapter I).

It has been conjectured (see e.g., [HL-1;1980]) that, for a vector differential equation

$$(J) \quad y'' + Q(t)y = 0,$$

where  $y$  is in  $\mathbb{R}^n$  and  $Q(t)$  is as in (H),

$$(K) \quad \lim_{t \rightarrow \infty} \lambda_1 \left[ \int_0^t Q(s) ds \right] = \infty$$

implies that equation (J) is oscillatory at  $\infty$ . (Here  $\lambda_1[\cdot]$  denotes the largest eigenvalue of the matrix under consideration.)

A. B. Mingarelli [M-1;1980] combined Hartman's result (D) and a particular positive linear functional, in fact, the "trace" functional, to obtain sufficient conditions for equation (J) to be oscillatory at  $\infty$ : specifically,

$$(L) \quad \liminf_{t \rightarrow \infty} \frac{1}{t} \int_0^t \text{tr} \left[ \int_0^s Q(u) du \right] ds > -\infty$$

and (K).

In this thesis, we are going to exploit a more general sufficient condition for equation (J) to be oscillatory at  $\infty$ .

# CHAPTER I

## §1. Definitions.

The present work deals mainly with the formally self-adjoint differential system

$$(1.1) \quad y'' + Q(t)y = 0, \quad t \in J, \quad \left( " = \frac{d^2}{dt^2} \right)$$

for a vector  $y$  in  $\mathbb{R}^n$  where  $J$  is a real interval and  $Q(t)$  is a real  $n$  by  $n$  symmetric continuous matrix function on some interval  $J$ .

The trivial solution  $y(t) \equiv 0$  will not be referred to as a solution. We follow the notation introduced in [W-5] in the scalar case ( $n = 1$ ).

**DEFINITION 1.1.** The two distinct points  $t_1, t_2 \in J = [a, b]$  will be called (*mutually*) *conjugate* on the interval  $a \leq t \leq b$  if there exists a solution  $y(t)$  of (1.1) satisfying  $y(t_1) = y(t_2) = 0$  (the zero vector) on  $[a, b]$ .

**DEFINITION 1.2.** The equation (1.1) will be called *disconjugate* on  $[a, b]$ , if every solution of (1.1) vanishes at most once in  $[a, b]$ .

The term "conjugate" has its origin in a fixed end point problem of Lagrange in the Calculus of Variations ([S-2, p. 311]).

DEFINITION 1.3. When  $J = [0, \infty)$  the equation (1.1) will be called *oscillatory at  $\infty$*  if for every  $a > 0$ , (1.1) is not disconjugate on  $[a, \infty)$ . It will be *non-oscillatory* otherwise.

Associated with equation (1.1) is the matrix differential equation

$$(1.2) \quad Y'' + Q(t)Y = 0, \quad t \in [0, \infty);$$

for a  $n$  by  $n$  matrix  $Y$  where  $Q$  is as in equation (1.1). It is clear that such equations possess solutions for  $t, 0 \leq t < \infty$ , since one can reduce them to a first-order system where  $Q$  is now of size  $2n$  by  $2n$  and then apply a known existence theorem (e.g., [CL-1, Chapter I]).

DEFINITION 1.4. A solution  $Y = Y(t)$  of equation (1.2) is called *non-trivial* if  $\det Y(t) \neq 0$  for at least one  $t \in [0, \infty)$ , ( $\det Y(t)$  will denote the determinant of  $Y(t)$ ). Also, a solution  $Y(t)$  of equation (1.2) is *non-singular* at  $t = a$ ,  $a \in [0, \infty)$ , if and only if  $\det Y(a) \neq 0$ .

DEFINITION 1.5. A non-trivial solution  $Y = Y(t)$  of equation (1.2) will be called *oscillatory at  $\infty$*  if and only if  $\det Y(t)$  has infinitely many zeros in  $[0, \infty)$ . It will be *non-oscillatory* otherwise.

A simple differentiation shows that if  $Y = Y(t)$  and  $Y = X(t)$  are two solutions of equation (1.2), then

$$(1.3) \quad Y^*(t)X'(t) - Y^{*'}(t)X(t) = C$$

where  $C$  is a constant  $n$  by  $n$  matrix and the asterisk (\*) will denote the conjugate transpose of a matrix, e.g., if  $A$  is a Hermitian matrix,  $A^* = A$ . (1.3) defines  $W(X, Y)(t)$  which may be called the *Wronskian* of two solutions  $Y$  and  $X$  of equation (1.2) (cf., [B-1]). Note that if  $Y = Y(t)$  and  $Y = X(t)$  are solutions of equation (1.2), then  $W(X, Y)$  is a constant matrix and  $W(Y, Y)$  is a skew-symmetric matrix. In particular, there corresponds to every solution  $Y = Y(t)$  of equation (1.2) a constant matrix  $C$  for which (1.3) is satisfied with  $x = y$ .

DEFINITION 1.6. A solution  $Y = Y(t)$  of equation (1.2) is *prepared* (or *self-conjugate*, or *conjoined*) if

$$(1.4) \quad H^* = H, \quad \text{where } H = Y^*Y'$$

or equivalently if for  $X = Y$  in (1.3),  $C = 0$ .

The following example, given by E. S. Noussair and C. A. Swanson [NS-1], shows that the "prepared" hypothesis on a solution  $Y = Y(t)$  of equation (1.2) is needed in order that an analog of the classical theory of oscillation ( $n = 1$ ) can be developed for linear systems. Consider the matrix differential equation

$$Y''(t) + Y(t) = 0 \quad t \in [0, \infty)$$

This is the analog of the scalar equation  $v'' + v = 0$ , all of whose solutions are of the form  $v(t) = c_1 \sin t + c_2 \cos t$ ,  $c_1$  and  $c_2$  being arbitrary constants. These solutions are oscillatory.

On the other hand, one solution of the matrix differential equation above is given by

$$Y(t) = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix},$$

which is non-oscillatory at  $\infty$  since  $\det Y \equiv 1$  on  $[0, \infty)$ . Note that  $Y(t)$  is not a prepared solution.

It is known (see e.g. [H-2, p. 388 Theorem 10.2]) that whenever equation (1.1) is disconjugate on  $J = [a, \infty)$ ,  $a \geq 0$ , there exists a non-trivial prepared solution of the associated differential matrix equation (1.2) such that  $Y(t)$  is non-singular in  $(a, \infty)$ . In this case

$$(1.5) \quad V(t) \equiv Y'(t)Y^{-1}(t) \quad , \quad t \in (a, \infty) \quad ,$$

will be a symmetric matrix and  $V^2(t)$  will be a non-negative definite matrix (i.e., for any vector  $\hat{y}$  in  $\mathbb{R}^n$ ,  $\hat{y}^*V^2\hat{y} \geq 0$ ).

The standard notations of matrix algebra and calculus are used throughout. In so far as is possible, capital letters (A, B, Q, ...) are used to denote matrices and most of the matrices to be considered are n by n, consequently if the term "matrix" is used it is assumed to be a n by n, square matrix. Lower case letters (x, y, ...) will denote vectors in  $\mathbb{R}^n$ , while the Greek letters ( $\alpha$ ,  $\beta$ , ...) will denote real scalars. The symbols I (resp. 0) will denote the identity matrix (resp. zero matrix) regardless of dimension. If A is a matrix, then,  $\text{tr } A$  shall denote the trace of A, and in the event  $\det A \neq 0$ ,  $A^{-1}$  shall denote the inverse of A. If  $A(t)$  is a matrix each of whose entries is a function, then  $A(t)$  will be termed continuous, differentiable, etc., when each of its entries has the indicated property. Let S be the real linear vector space of n by n symmetric matrices. Whenever A is in S, its eigenvalues (all necessarily real) will be denoted by  $\lambda_k[A]$ ,  $1 \leq k \leq n$ , where we take it that  $\lambda_1[A]$

(resp.  $\lambda_n[A]$ ) indicates the largest (resp. the smallest) eigenvalue of  $A$ . Finally, for  $A$  and  $B$  in  $S$ ,  $A > B$  (resp.  $A \geq B$ ) will mean that  $A - B$  is a positive definite matrix (resp. a non-negative or semi-positive definite matrix), i.e., for any vector  $y$  in  $\mathbb{R}^n$ ,  $\langle (A - B)y, y \rangle > 0$  (resp.  $\geq 0$ ) where  $\langle \cdot, \cdot \rangle$  is the usual inner product on  $\mathbb{R}^n$ . The symbol  $\square$  is used to indicate the end of a proof.

A non-linear functional (see [H-4])  $q : S \rightarrow \mathbb{R}$  is called *superadditive* and (*positively*) *superhomogeneous* if

$$(1.6) \quad q(A + B) \geq q(A) + q(B) \quad \text{and}$$

$$(1.7) \quad q(\alpha A) \geq \alpha q(A) \quad \text{for } \alpha \geq 0 .$$

Correspondingly, a functional  $p : S \rightarrow \mathbb{R}$  is called *subadditive* and (*positively*) *subhomogeneous* if

$$(1.8) \quad p(A + B) \leq p(A) + p(B) \quad \text{and}$$

$$(1.9) \quad p(\alpha A) \leq \alpha p(A) \quad \text{for } \alpha \geq 0 .$$

*Note (1):* Conditions (1.6) and (1.7) (or (1.8) and (1.9)) imply that  $q$  (or  $p$ ) is concave (or convex) and hence continuous on  $S$ . ([H-4])

The study of oscillation criteria for equation (1.1) using linear functionals on  $S$  (e.g., the "trace" functional)

has its beginnings in a paper of G. J. Etgen and J. F. Pawlowski [EP-1] (see theorem 2.1 below). This was followed by [EL-1] wherein a comparison theorem was formulated relating the oscillatory behavior of equation (1.1) as a special case, to the oscillatory behavior of a scalar differential equation. Subsequently P. Hartman [H-4] obtained a result in the same vein as the latter using the non-linear functionals ((1.6)—(1.9)) on  $S$ . Recently D. Hinton and R. T. Lewis [HL-1] have extended some of the results of [EL-1] to generalized differential systems (essentially "Stieltjes integrodifferential systems") and new oscillation criteria have been obtained in this more general setting.

A real linear functional  $g : S \rightarrow \mathbb{R}$  (i.e., satisfying

$$(1.10) \quad g(A + B) = g(A) + g(B) \quad , \quad g(\alpha A) = \alpha g(A)$$

for  $A$  and  $B$  in  $S$ ,  $\alpha \in \mathbb{R}$ ) is called *positive* (resp. *non-negative*) if

$$g(A) > 0 \quad (\text{resp. } \geq 0)$$

whenever  $A > 0$  (resp.  $A \geq 0$ ).

Since a positive linear functional is continuous,  
we have

$$g\left(\int_a^t A(s) ds\right) = \int_a^t g[A(s)] ds, \quad a, t \in [0, \infty),$$

whenever  $A$  is integrable, and

$$g[B'(t)] = \{g[B(t)]\}' \quad , \quad t \in [0, \infty) \quad ,$$

whenever  $B$  is differentiable. (see [EP-1])

## §2. Preliminary lemmas.

The following lemmas will play an important role in proving our main theorem.

### LEMMA 1.1. [M-2, Lemma 1.1]

Let  $A$  and  $B$  be in  $S$ . Then both  $\lambda_1$  and  $\lambda_n : S \rightarrow \mathbb{R}$  are non-linear functionals with the following properties:

- (i)  $\lambda_1[A + B] \leq \lambda_1[A] + \lambda_1[B]$  and  
 $\lambda_1[\alpha A] \leq \alpha \lambda_1[A]$  , whenever  $\alpha \geq 0$  is real;
- (ii)  $\lambda_n[A + B] \geq \lambda_n[A] + \lambda_n[B]$  and  
 $\lambda_n[\alpha A] \geq \alpha \lambda_n[A]$  , whenever  $\alpha \geq 0$  is real.

(iii) Whenever  $A$  is integrable on a compact set  $J = [a, b]$ , then  $\lambda_1$  and  $\lambda_n$  satisfy the Jensen-type inequalities,

$$(1) \quad \lambda_1 \left[ \int_a^b A(s) ds \right] \leq \int_a^b \lambda_1[A(s)] ds ,$$

$$(2) \quad \lambda_n \left[ \int_a^b A(s) ds \right] \geq \int_a^b \lambda_n[A(s)] ds .$$

(iv) If  $A > B$  (i.e.,  $A - B$  is a positive definite matrix), we have

$$\lambda_k[A] > \lambda_k[B] , \text{ for each } k, 1 \leq k \leq n .$$

(v) For  $A$  in  $S$ , we have

$$\lambda_1[A^2] \geq (\lambda_1[A])^2 .$$

### LEMMA 1.2.

Let  $A(t) \in S$  be a non-negative definite matrix,  $A(t) \geq 0$ , locally integrable on  $J = [0, \infty)$ , then there holds the inequality

$$(1.11) \quad \int_0^t \lambda_1[A(s)] ds \geq \lambda_1 \left[ \int_0^t A(s) ds \right] \geq \frac{1}{n} \int_0^t \lambda_1[A(s)] ds$$

for  $t$  in  $J$ .

PROOF.

For  $t > 0$ , we have  $\text{tr } A(t) = \sum_{i=1}^n \lambda_i[A(t)]$ .

Therefore,

$$\begin{aligned} \sum_{i=1}^n \left( \int_0^t \lambda_i[A(s)] \, ds \right) &= \int_0^t \text{tr } A(s) \, ds \\ &= \text{tr} \left( \int_0^t A(s) \, ds \right) \\ &= \sum_{i=1}^n \lambda_i \left( \int_0^t A(s) \, ds \right) \end{aligned}$$

Since  $\int_0^t A(s) \, ds$  is a non-negative definite matrix,

$$\begin{aligned} n\lambda_1 \left( \int_0^t A(s) \, ds \right) &\geq \sum_{i=1}^n \int_0^t \lambda_i[A(s)] \, ds \\ &\geq \int_0^t \lambda_1[A(s)] \, ds \end{aligned}$$

This proves the right-hand side of inequality (1.11) and the left-hand side inequality is the Jensen-type inequality in lemma 1.1 (iii) (1).  $\square$

## COROLLARY 1.3.

If  $A(t) \in S$  is a non-negative definite matrix which is locally integrable on  $J = [0, \infty)$  and is such that

$$\lambda_1[A(t)] \rightarrow \infty, \text{ as } t \rightarrow \infty,$$

then

$$\lambda_1 \left[ \int_0^t A(s) ds \right] \rightarrow \infty, \text{ as } t \rightarrow \infty.$$

PROOF.

Since  $\lambda_1[A(t)] \rightarrow \infty$  as  $t \rightarrow \infty$ , there exists  $T > 0$  such that

$$\lambda_1[A(t)] > 1, \text{ for all } t > T.$$

For such  $t > T$ ,

$$\begin{aligned} \lambda_1 \left[ \int_0^t A(s) ds \right] &\geq \frac{1}{n} \left[ \int_0^T \lambda_1[A(s)] ds + \int_T^t \lambda_1[A(s)] ds \right] \\ &\geq \frac{1}{n} [c(A, T) + t - T], \end{aligned}$$

where  $c(A, T)$  is some constant depending on  $A(t)$  and  $T$ .

If we let  $t \rightarrow \infty$ , then we have

$$\lambda_1 \left[ \int_0^t A(s) ds \right] \rightarrow \infty \quad \square$$

Note (2): The above lemma 1.2 implies that if  $A(t)$  is a non-negative definite matrix, then  $\lim_{t \rightarrow \infty} \lambda_1 \left[ \int_0^t A(s) ds \right] = \infty$  if and only if  $\lim_{t \rightarrow \infty} \int_0^t \lambda_1 [A(s)] ds = \infty$ .

LEMMA 1.4. (see [R-2]).

If  $g$  is a positive linear functional on  $S$ , then there holds the generalized Cauchy-Schwarz inequality:

$$(1.12) \quad |g(A)|^2 \leq g(I)g(A^2)$$

for all  $A$  in  $S$ .

PROOF.

The inequality is clearly valid when  $g(A) = 0$ . If  $g(A) \neq 0$ , let set  $U = \alpha A + \beta I$  where  $\beta = g(A)$ ,  $\alpha, \beta \in \mathbb{R}$ . Then

$$\begin{aligned} (1.13) \quad g(U^*U) &= g[(\alpha A^* + \beta I)(\alpha A + \beta I)] \\ &= \alpha^2 g(A^2) + 2\alpha\beta g(A) + \beta^2 g(I) \\ &= \alpha^2 g(A^2) + 2\alpha [g(A)]^2 + [g(A)]^2 g(I) \end{aligned}$$

Since  $U^*U$  is a non-negative definite matrix, then  $g(U^*U)$  is non-negative. Therefore, the discriminant of the quadratic in (1.13) is negative or zero,

i.e.,

$$D = [g(A)]^4 - [g(A)]^2 g(A^2) g(I) \leq 0 .$$

This proves the generalized Cauchy-Schwarz inequality.  $\square$

COROLLARY 1.5. [M-2]

If  $A$  is in  $S$ , then the trace of  $A$  satisfies the Schwarz-type inequality:

$$(1.14) \quad (\operatorname{tr} A)^2 \leq n(\operatorname{tr} A^2) .$$

Let  $A(t)$  be a matrix function in  $S$  defined on  $[0, \infty)$ , and suppose that

$$(1.15) \quad A(t_2) > A(t_1) , \quad T \leq t_1 < t_2 < \infty ,$$

then  $\lambda_k[A(t_2)] > \lambda_k[A(t_1)]$ ,  $1 \leq k \leq n$ , i.e., the real valued function  $\lambda_k[A(t)]$  is increasing for large  $t$ .

We therefore obtain

LEMMA 1.6. [M-2, Corollary 1.1]

Let  $A$  be in  $S$  and satisfy (1.15). Then

$$(1.16) \quad \lim_{t \rightarrow \infty} \lambda_k[A(t)]$$

exists for each  $k$ ,  $k = 1, 2, \dots, n$ , (but may be infinite).

Finally, if  $A$  is a symmetric and non-negative definite matrix, the following bound may be readily verified:

$$(1.17) \quad \lambda_1[A] \leq \text{tr } A \leq n\lambda_1[A] .$$

In fact for any matrix  $A$  in  $S$ , one always has

$$(1.18) \quad \lambda_n[A] \leq \frac{\text{tr } A}{n} \leq \lambda_1[A] .$$

Also, T. Walters [W-1] obtained the same kind of inequality for a positive linear functional. Before stating that inequality, we introduce the concept of the norm of a positive linear functional. Actually, he showed that the norm of a positive linear functional is equal to its value at the identity matrix [W-1, Theorem 3.2.]. i.e.,

$$(1.19) \quad \|g\| = \sup_{\|A\|_M=1} g(A) = g(I)$$

where  $\|\cdot\|_M$  is the operator norm of a matrix and  $A$  is in  $S$ , i.e.,

$$\|A\|_M = \sup_{\|v\|=1} \|Av\|_V ,$$

where  $v$  ranges over all column  $n$ -vectors and  $\|\cdot\|_V$  is any vector norm which is consistent with the matrix norm. Now, we may state the inequality [W-1] regarding positive linear functionals.

THEOREM 1.7. [W-1, Théorem 3.3]

Let  $g$  be a positive linear functional with norm 1. Then for every symmetric matrix  $A$ , we have

$$(1.20) \quad \lambda_n[A] \leq g(A) \leq \lambda_1[A] .$$

LEMMA 1.8.

Let  $A$  be a real  $n$  by  $n$  matrix and  $x$  be a column vector in  $\mathbb{R}^n$ , then

$$(1.21) \quad \text{tr} (xx^*A) = \langle Ax , x \rangle ,$$

where  $\langle , \rangle$  is the usual inner product on  $\mathbb{R}^n$ .

The proof of this lemma is straightforward and so will be omitted.

## CHAPTER II

§1. Conjectures.

In 1976, G. J. Etgen and J. F. Pawlowski [EP-1] gave oscillation criteria for equation (1.1) using positive linear functionals  $g : S \rightarrow \mathbb{R}$ .

## THEOREM 2.1. [EP-1, Theorem 1]

Let  $g$  be a positive linear functional on  $S$  such that

$$(2.1) \quad \lim_{t \rightarrow \infty} g \left( \int_0^t Q(s) ds \right) = \infty .$$

Then equation (1.1) is oscillatory at  $\infty$ .

An immediate consequence of the above theorem is that equation (1.1) is oscillatory at  $\infty$  whenever

$$(2.2) \quad \lim_{t \rightarrow \infty} \text{tr} \left( \int_0^t Q(s) ds \right) = \infty .$$

It is clear that the "trace" functional is a positive linear functional on  $S$ . For if  $e_i$  is the  $n$ -component vector

with a 1 in the  $i$ -th position and 0's elsewhere, then we can define a (positive) linear functional  $g_{e_i} : S \rightarrow \mathbb{R}$  by

$$(2.3) \quad g_{e_i}(A) = \langle Ae_i, e_i \rangle = a_{ii}, \quad 1 \leq i \leq n,$$

where  $\langle \cdot, \cdot \rangle$  is the usual inner product on  $\mathbb{R}^n$  and  $a_{ii}$  is a diagonal entry of  $A$ . Therefore, for any  $n$  by  $n$  matrix  $A$ ,

$$(2.4) \quad \text{tr } A = \sum_{i=1}^n g_{e_i}(A).$$

We see that (2.3) along with theorem 2.1 imply that equation (1.1) is oscillatory whenever

$$(2.5) \quad \lim_{t \rightarrow \infty} \int_0^t a_{ii}(s) \, ds = \infty$$

for some diagonal entry  $a_{ii}$ ,  $1 \leq i \leq n$ . This result was obtained by E. S. Noussair and C. A. Swanson [NS-1]. The latter extended a result of G. J. Etgen [E-1, Theorem 2] which stated that equation (1.2) is oscillatory if (2.2) holds.

On the other hand, A. B. Mingarelli [M-2] noted that

$$(2.6) \quad \liminf_{t \rightarrow \infty} \text{tr} \left[ \int_0^t Q(s) \, ds \right] = -\infty$$

may or may not, by itself, imply oscillation. For example,  $Q(t) = \text{diag}\{t, -2t\}$  satisfies (2.6) and, in fact, equation (1.1) is oscillatory. To see this, note that  $y$  satisfies equation (1.1) if and only if

$$(2.7) \quad \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}'' + \begin{pmatrix} t & 0 \\ 0 & -2t \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

that is

$$(2.8) \quad \begin{cases} y_1'' + ty_1 = 0 \\ y_2'' - 2ty_2 = 0 \end{cases}$$

(2.9)

Applying the Fite-Wintner oscillation criterion (p. vii (C), or [W-4]) to equation (2.8), we find that equation (2.8) is oscillatory and we can choose a solution of equation (2.9) to be the trivial solution. Then a solution of system (2.7) is  $y = \begin{pmatrix} y_1(t) \\ 0 \end{pmatrix}$ , i.e.,  $y(t)$  has infinitely many zeros.

On the other hand, if we let  $Q(t) = \text{diag}\{-t, -2t\}$  it once again satisfies (2.6). Also

$$(2.10) \quad \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}'' + \begin{pmatrix} -t & 0 \\ 0 & -2t \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

that is,

$$(2.11) \quad \begin{cases} y_1'' - ty_1 = 0 & , \\ y_2'' - 2ty_2 = 0 & . \end{cases}$$

Since we know that for  $q(t) \leq 0$ ,  $y'' + q(t)y = 0$  is non-oscillatory (see e.g., [S-1, p. 119, Theorem B]), both equations (2.11) and (2.12) have only non-oscillatory solutions, i.e.,  $y(t)$  has only a finite number of zeros. It is, in fact, disconjugate on  $[0, \infty)$ .

It was conjectured, ([HL-1]) that if

$Q^*(t) = Q(t)$  is continuous in  $[0, \infty)$ , then

$$(2.13) \quad \lim_{t \rightarrow \infty} \lambda_1 \left( \int_0^t Q(s) ds \right) = \infty$$

implies that equation (1.1) is oscillatory at  $\infty$ .

A. B. Mingarelli obtained a positive answer to this conjecture when  $Q(t)$  is a non-negative definite matrix.

THEOREM 2.2. [M-3, Theorem 2.1]

Let  $Q^*(t) = Q(t)$  be a non-negative definite matrix and be continuous on  $[0, \infty)$ . If (2.13) holds, then equation (1.1) is oscillatory at  $\infty$ .

PROOF.

Since  $Q(t) \geq 0$  for  $t \geq 0$ , then we must also have that  $\int_0^t Q(s) ds \geq 0$  for  $t \geq 0$ . However the estimate (1.17) implies that

$$\lambda_1 \left( \int_0^t Q(s) ds \right) \leq \text{tr} \left( \int_0^t Q(s) ds \right) \quad \text{for } t \geq 0 ;$$

thus (2.13) implies (2.2).

Now it follows from [E-1, Theorem 2] that equation (1.1) is oscillatory at  $\infty$ .  $\square$

The next lemma is straightforward but of interest, because this result shows that the conjecture is also verified when  $Q$  is a constant matrix.

LEMMA 2.3. [M-3, Lemma 2.1]

Let  $Q(t) \equiv Q$  be in  $S$  and a constant matrix.

- (a) If  $Q$  possesses at least one positive eigenvalue, then equation (1.1) is oscillatory at  $\infty$ .

(b) If  $\lambda_1[Q] \leq 0$ , equation (1.1) is non-oscillatory, and in fact, disconjugate on  $[0, \infty)$ .

Note (3): It is easily verified that (2.13) with

$$\liminf_{t \rightarrow \infty} \lambda_n \left( \int_0^t Q(s) ds \right) > -\infty$$

implies that equation (1.1) is oscillatory at  $\infty$  (see [M-2]). Also, if  $Q(t)$  is a diagonal matrix whose entries are all monotone, then the conjecture is true.

As we saw above, the conjecture has been completely verified in the special cases (i); when  $Q(t)$  is a non-negative definite matrix and (ii); when  $Q(t) \equiv Q$  is a constant matrix. In the latter case (2.13) is equivalent to  $\lambda_1[Q] > 0$  and this, in turn, implies that equation (1.1) is oscillatory at  $\infty$ , (by lemma 2.3).

Recently A. B. Mingarelli [M-2] also obtained a positive answer to the conjecture under the assumption that

$$(2.14) \quad \liminf_{t \rightarrow \infty} \frac{1}{t} \operatorname{tr} \left( \int_0^t Q(s) ds \right) > -\infty$$

Later, he [M-1] replaced (2.14) by

$$(2.15) \quad \liminf_{t \rightarrow \infty} \frac{1}{t} \int_0^t \operatorname{tr} \left[ \int_0^s Q(u) \, du \right] \, ds > -\infty,$$

see theorem 2.4 below.

**THEOREM 2.4.** [M-1, Theorem 1.]

Let  $Q(t)$  be in  $S$  and be continuous on  $[0, \infty)$ .

Assume that (2.15) holds. Then (2.13) implies that equation (1.1) is oscillatory at  $\infty$ .

Since  $\lambda_1$  satisfies the Jensen-type inequality (see lemma 1.1 (iii) (1)), then (2.13) implies

$$(2.16) \quad \lim_{t \rightarrow \infty} \int_0^t \lambda_1 [Q(s)] \, ds = \infty.$$

We can therefore formulate the following weaker question.

#### WEAKER CONJECTURE

Let  $Q(t)$  be in  $S$  and be continuous in  $[0, \infty)$ , then (2.16) implies that (1.1) is oscillatory at  $\infty$ .

Since, from note (2), (2.13) is equivalent to (2.16) when  $Q(t)$  is a non-negative definite matrix, this weaker conjecture is verified when  $Q^*(t) = Q(t)$  is a non-negative definite matrix. Moreover it will be clearly satisfied for all those  $Q$  in  $S$  which allow the estimate

$$\lambda_1[Q(t)] \leq C[\text{tr } Q(t)] \quad , \quad t \geq T$$

where  $C$  is bounded, uniformly in  $t$ . (When  $Q(t)$  is a non-negative definite matrix, we can choose  $C = 1$ .)

## CHAPTER III

§1. Introduction.

In 1952, Philip Hartman [H-1] published necessary conditions for the non-oscillation of a second-order scalar differential equation of the form

$$(3.1) \quad v'' + q(t)v = 0 \quad ; \quad q \in C^0[0, \infty) \quad , \quad t \in [0, \infty) \quad .$$

Also some of the results of the above paper [H-1] can be found in [H-2, Chapter XI, §7]. Those results were given by using the (arithmetic) mean

$$(3.2) \quad \frac{1}{t} \int_0^t Q(s) ds \quad , \quad \text{where} \quad Q(t) = \int_0^t q(s) ds \quad .$$

A generalization of [H-1] was given by W. J. Coles and D. Willett [CW-1], in which the arithmetical mean was replaced by a more general mean

$$(3.3) \quad (KQ)(t) = \int_0^t K(t,s)Q(s) ds \quad ,$$

where  $K(t,s)$  is the kernel  $K_n(t,s) = K_n(t,s;\phi_1, \dots, \phi_n)$ ,

$$K_n(t,s) = \phi_1(s) \int_{s \leq s_2 \leq \dots \leq s_n \leq t} \dots \int \phi_2(s_2) \dots \phi_n(s_n) ds_2 \dots ds_n / k_n(t) ,$$

(3.4)

$$\int_0^t K_n(t,s) ds = 1 ,$$

depending on non-negative functions  $\phi_1, \dots, \phi_n$ ; see [CW-1], [W-3], and the references therein. In [H-3], P. Hartman showed that the methods of [H-1] could be used for very general "means" (3.3) with suitable conditions on the kernel  $K(t,s) \geq 0$  for which the arguments of [H-1] are applicable.

On the other hand, G. J. Etgen and J. F. Pawlowski [EP-1] obtained oscillation criteria for matrix differential equations by using positive linear functionals. A. B. Mingarelli [M-1] also obtained a criterion for second order self-adjoint differential systems by adapting Hartman's [H-1] technique to systems. That theorem has already been stated above, i.e., theorem 2.4, which improved on the theorem of Etgen and Pawlowski [EP-1, Theorem 1]. Therefore we can expect a more general criterion for second order self-adjoint differential systems (i.e., more general than theorem 2.4) by adapting Hartman's [H-3] general "means" to systems and using Etgen and Pawlowski's [EP-1] concept of positive linear functional instead of the arithmetical mean and the "trace" functional.

§2. Representations of positive linear functionals.

Let  $S$  be the real linear vector space of  $n$  by  $n$  symmetric matrices. Define an inner product  $\langle \cdot, \cdot \rangle$  on  $S$  in the following manner:

For  $A$  and  $B$  in  $S$ ,

$$(3.5) \quad \langle A, B \rangle = \text{tr} (AB) = \sum_{i=1}^n \sum_{j=1}^n a_{ij} b_{ij} ,$$

where  $A = [a_{ij}]$  and  $B = [b_{ij}]$ .

It is easy to check that, in fact,  $\langle A, B \rangle = \text{tr} (AB)$  is an inner product on  $S$ .

Let  $C$  be some fixed matrix in  $S$ . We define a function  $g_C$  from  $S$  into the real field by

$$(3.6) \quad g_C(A) = \langle A, C \rangle = \text{tr} (AC) .$$

This function  $g_C$  is a linear functional on  $S$ , because,

$$\begin{aligned} g_C(\alpha A + B) &= \text{tr} [(\alpha A + B)C] \\ &= \alpha \text{tr} (AC) + \text{tr} (BC) \\ &= \alpha g_C(A) + g_C(B) , \end{aligned}$$

for  $A$  and  $B$  in  $S$  and  $\alpha$  in  $\mathbb{R}$ .

Now, we prove a theorem which deals with the representation of linear functionals by using the inner product defined by the "trace".

**THEOREM 3.1.** (cf. analogue to [HK-1, p. 236, Theorem 5])

Let  $S$  be the real linear vector space of  $n$  by  $n$  symmetric matrices with the inner product  $\langle A, B \rangle = \text{tr}(AB)$  for  $A$  and  $B$  in  $S$  and let  $g$  be a linear functional on  $S$ . Then there exists a unique matrix  $C$  in  $S$  such that  $g(A) = \langle A, C \rangle$  for all  $A$  in  $S$ .

In order to construct such a unique matrix  $C$  depending only on the linear functional  $g : S \rightarrow \mathbb{R}$  proceed as follows:  
(cf., [HK-1])

Let us define a matrix  $E^{ij}$  as follows:

$E^{ij}$ : only non-zero entry is a 1 in row  $i$  and column  $j$  if  $i = j$ .

: only non-zero entries are  $\frac{1}{\sqrt{2}}$  in row  $i$ ,

column  $j$  and row  $j$ , column  $i$  if  $i \neq j$ .

The matrices  $\{E^{ij}\}$ ,  $i \leq j$ ,  $i, j = 1, 2, \dots, n$ , form an orthonormal basis for  $S$  with respect to the inner product given by (3.5).

The inner product of

$$A = \sum_{i=1}^n \sum_{\substack{j=1 \\ i \leq j}}^n \alpha_{ij} E^{ij} \quad \text{and} \quad C = \sum_{i=1}^n \sum_{\substack{j=1 \\ i \leq j}}^n \beta_{ij} E^{ij},$$

where  $\alpha_{ij}$  and  $\beta_{ij}$  are in  $\mathbb{R}$ , is

$$\begin{aligned} \langle A, C \rangle &= \left\langle \sum_{i=1}^n \sum_{\substack{j=1 \\ i \leq j}}^n \alpha_{ij} E^{ij}, \sum_{i=1}^n \sum_{\substack{j=1 \\ i \leq j}}^n \beta_{ij} E^{ij} \right\rangle \\ &= \sum_{i=1}^n \sum_{\substack{j=1 \\ i \leq j}}^n \alpha_{ij} \beta_{ij} \langle E^{ij}, E^{ij} \rangle \\ &= \sum_{i=1}^n \sum_{\substack{j=1 \\ i \leq j}}^n \alpha_{ij} \beta_{ij}. \end{aligned}$$

If  $g$  is any linear functional on  $S$ , then  $g$  has the form

$$g(A) = g \left( \sum_{i=1}^n \sum_{\substack{j=1 \\ i \leq j}}^n \alpha_{ij} E^{ij} \right) = \sum_{i=1}^n \sum_{\substack{j=1 \\ i \leq j}}^n \alpha_{ij} g(E^{ij}).$$

If we wish to find a matrix  $C$  in  $S$  such that  $\langle A, C \rangle = g(A)$  for all  $A$  in  $S$ , then the components  $\beta_{ij}$  of  $C$  must satisfy  $\beta_{ij} = g(E^{ij})$ .

Accordingly,

$$(3.7) \quad C = \sum_{i=1}^n \sum_{\substack{j=1 \\ i \leq j}}^n g(E^{ij}) E^{ij}$$

is the desired matrix.

The next theorem gives us a characterization of positive linear functionals on  $S$ .

**THEOREM 3.2.** (cf., [BB-1, pp. 87])

A necessary and sufficient condition that a symmetric matrix  $C$  be positive definite is that the inequality

$$(3.8) \quad g_c(A) = \text{tr}(AC) > 0$$

holds for all non-negative definite matrices  $A$  in  $S$ . ( $A \neq 0$ )

T. Walters [W-1] also obtained the same kind of characterization of positive linear functionals on  $M_n$  which is the linear space of all  $n$  by  $n$  real matrices over the reals. He has used a slightly different representation of a positive linear functional, however, we will see that these representations are identical on  $S$ . We let  $A \otimes B$  denote the coordinatewise product  $[a_{ij}b_{ij}]$ ,  $i, j = 1, \dots, n$ , for  $A$

and  $B$  in  $S$  and  $u$  will denote the column vector in  $\mathbb{R}^n$  such that  $u_i = 1, i = 1, 2, \dots, n$ .

**THEOREM 3.3.** [W-1, Theorem 3.1]

Suppose that  $g$  is a positive linear functional on  $M_n$ .

Then there exists a nonnegative definite matrix  $[c_{ij}]$  in  $M_n$  such that

$$(3.9) \quad g(A) = \langle ([c_{ij}] \otimes A)u, u \rangle,$$

for every  $A$  in  $M_n$ . Conversely, if  $[c_{ij}]$  is a nonnegative definite matrix, then the linear functional  $g$ , defined by (3.9) for  $u$  as above, is a positive linear functional.

As we said above, the two representations (3.6) and (3.9) are equivalent for every symmetric matrix.

**COROLLARY 3.4.**

Let  $A$  and  $C$  be in  $S$  and  $C$  be a positive definite matrix. Then the two representations of positive linear functionals  $g : S \rightarrow \mathbb{R}$ ,

$$(3.6) \quad g(A) = \text{tr}(AC) \quad \text{and}$$

$$(3.9) \quad g(A) = \langle ([c_{ij}] \otimes A)u, u \rangle, \quad C = [c_{ij}], \quad i, j = 1, 2, \dots, n,$$

are equivalent for every symmetric matrix  $A$  and a unique symmetric matrix  $C > 0$  depending on  $g$ .

PROOF.

$$g(A) = \text{tr} (AC)$$

$$= \sum_{i=1}^n \sum_{j=1}^n a_{ij} c_{ij} \quad , \quad A = [a_{ij}] \quad , \quad .$$

$$= \sum_{i=1}^n (a_{i1} c_{i1} + a_{i2} c_{i2} + \dots + a_{in} c_{in})$$

$$= \left\langle \begin{pmatrix} a_{11}c_{11} + a_{12}c_{12} + \dots + a_{1n}c_{1n} \\ a_{21}c_{21} + a_{22}c_{22} + \dots + a_{2n}c_{2n} \\ \cdot \\ \cdot \\ \cdot \\ a_{n1}c_{n1} + a_{n2}c_{n2} + \dots + a_{nn}c_{nn} \end{pmatrix} , \begin{pmatrix} 1 \\ 1 \\ \cdot \\ \cdot \\ \cdot \\ 1 \end{pmatrix} \right\rangle$$

$$= \left\langle \begin{pmatrix} a_{11}c_{11} & a_{12}c_{12} & \dots & a_{1n}c_{1n} \\ a_{21}c_{21} & a_{22}c_{22} & \dots & a_{2n}c_{2n} \\ \cdot & \cdot & \dots & \cdot \\ \cdot & \cdot & \dots & \cdot \\ \cdot & \cdot & \dots & \cdot \\ a_{n1}c_{n1} & a_{n2}c_{n2} & \dots & a_{nn}c_{nn} \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ \cdot \\ \cdot \\ \cdot \\ 1 \end{pmatrix} , \begin{pmatrix} 1 \\ 1 \\ \cdot \\ \cdot \\ \cdot \\ 1 \end{pmatrix} \right\rangle$$

$$= \langle ([c_{ij}] \otimes A)u , u \rangle$$

which is the desired result.  $\square$

The next theorem shows that the values of a positive linear functional admit the following bounds in terms of the trace functional.

**THEOREM 3.5.**

Let  $g$  be a positive linear functional on  $S$ . Then for every positive definite matrix  $A$  in  $S$ , we have

$$(3.10) \quad \gamma_1 \operatorname{tr} A \leq g(A) \leq \gamma_2 \operatorname{tr} A,$$

where  $\gamma_1$  and  $\gamma_2$  ( $0 < \gamma_1 < \gamma_2$ ) are some constants in  $\mathbb{R}^+$ , depending on  $g$  only.

PROOF.

Every positive linear functional  $g$  on  $S$  can be represented by

$$(3.11) \quad g_c(A) = \operatorname{tr}(AC)$$

where  $C > 0$  is a symmetric matrix which depends on  $g$ .

By using the spectral decomposition of  $C$ , we have

$$(3.12) \quad C = \sum_{i=1}^n \lambda_i [C] T_i T_i^* = \sum_{i=1}^n \lambda_i [C] E_i$$

where  $T_i$  is the  $i$ -th column vector with norm 1 of the orthogonal matrix  $T$  such that  $C = TDT^*$  ( $D$  diagonal),

and  $E_i = T_i T_i^*$  satisfies  $E_i \in S$ ,  $E_i E_j = E_j E_i = 0$  if  $i \neq j$ ,  $E_i^2 = E_i$ ,  $E_i \geq 0$ , and  $E_1 + E_2 + \dots + E_n = I$ .  
 Multiplying (3.12) by a matrix  $A > 0$  and applying the "trace" to both sides, we have

$$\text{tr} (AC) = \text{tr} \left( \sum_{i=1}^n \lambda_i [C] AE_i \right)$$

$$= \sum_{i=1}^n \{ \lambda_i [C] \text{tr} (AE_i) \}$$

by lemma 1.8,

$$= \sum_{i=1}^n \{ \lambda_i [C] (AT_i, T_i) \}$$

where  $(AT_i, T_i) > 0$ ,  $i = 1, 2, \dots, n$ , since  $A > 0$ .

Hence 
$$g_C(A) = \text{tr} (AC) = \sum_{i=1}^n \{ \lambda_i [C] \text{tr} (AE_i) \}$$

Now,

$$\begin{aligned} g_C(A) &= \sum_{i=1}^n \{ \lambda_i [C] \text{tr} (AE_i) \} \\ &\geq \lambda_n [C] \left( \sum_{i=1}^n \text{tr} (AE_i) \right) \end{aligned}$$

since  $\text{tr} (AE_i) \geq 0$  for each  $i$ ,  $i = 1, 2, \dots, n$  and

$\lambda_i [C] > 0$ . Hence,

$$\begin{aligned} g_C(A) &\geq \lambda_n [C] \text{tr} [A(E_1 + E_2 + \dots + E_n)] \\ (3.13) \quad &= \lambda_n [C] \text{tr} A \end{aligned}$$

Since  $C$  depends on  $g$  only, therefore each positive linear functional satisfies the above inequality (3.13) for all  $A > 0$ .

A similar argument covers the assertion on the right-hand side of the inequality (3.10) where now  $\gamma_2 = \lambda_1[C]$ .  $\square$

### §3. Properties of the kernel $K(t,s)$ .

Consider (1.1) for  $Q$  in  $S$  and  $t \in [0, \infty)$ .

Define

$$(*) \quad P(t) = \int_0^t Q(s) \, ds .$$

Then  $P$  is an element of  $S$ .

Let  $g : S \rightarrow \mathbb{R}$  be some positive linear functional on  $S$ . With  $P$  as in (\*) let

$$(**) \quad \bar{P}(t) = g[P(t)]$$

$$\begin{aligned}
 (***) \quad (K\bar{P})(t) &= \int_0^t K(t,s)\bar{P}(s) \, ds \\
 &= \int_0^t K(t,s)g[P(s)] \, ds \\
 &= \int_0^t K(t,s)g\left[\int_0^s Q(u) \, du\right] \, ds .
 \end{aligned}$$

We use the letter  $K$  for both the kernel  $K(t,s)$  and the corresponding integral operator  $\bar{P} \mapsto K\bar{P}$  in (\*\*\*) . In what follows, we always assume that  $K(t,s) \geq 0$  is continuous for  $0 \leq s \leq t$ ,  $0 < t < \infty$ . We list some other conditions on  $K$  which will be assumed from time to time: The following conditions on  $K$  are analogous to Hartman's [H-3]. We summarize these briefly for reference purposes.

$$(a) \quad (KR)(t) \rightarrow 0, \quad t \rightarrow \infty, \quad \text{if } R \in (L_0^\infty \cap C^0)[0, \infty),$$

i.e., the operator  $K$  is a bounded map on the Banach space  $(L_0^\infty \cap C^0)[0, \infty)$ .

$$(b) \quad (KR)(t) \rightarrow \infty, \quad t \rightarrow \infty, \quad \text{if } C^0[0, \infty) \ni R(t) \rightarrow \infty, \quad t \rightarrow \infty.$$

$$(c_\sigma) \quad (KR)(t) \rightarrow 0, \quad t \rightarrow \infty, \quad \text{if } R \in (L^{2/\sigma} \cap C^0)[0, \infty)$$

for some fixed  $\sigma$ ,  $1 \leq \sigma \leq 2$ ; i.e., the map  $K$  is a bounded operator from the Banach space  $(L^{2/\sigma} \cap C^0)[0, \infty)$  to the Banach space  $(L_0^\infty \cap C^0)[0, \infty)$  for some  $\sigma$ ,  $1 \leq \sigma \leq 2$ .

(d) Assume that there exist functions  $0 < k(t) \in C^0[0, \infty)$ ,  $0 \leq m(t) \in C^0[0, \infty)$ , a kernel  $K_0(t,s) \geq 0$  continuous for  $0 \leq s \leq t$ ,  $0 < t < \infty$ , and a constant  $\alpha$ ,  $0 \leq \alpha \leq 2$ , with the following properties:

$$(3.14) \quad \kappa(t,s) = \frac{\partial}{\partial t} \int_s^t k(t)K(t,s) ds \geq 0$$

exists and is continuous, for  $0 \leq s \leq t$ ,  $0 < t < \infty$ ;

$$(3.15) \quad \frac{\kappa^{2-\alpha}(t,s)}{K_0(t,s)}$$

is continuous for  $0 \leq s \leq t$ ,  $0 < t < \infty$ ;

$$\int_0^\infty m(t) dt = \infty$$

$$(3.16) \quad \text{or } (\infty \geq) \quad \limsup_{t \rightarrow \infty} k(t) \int_t^\infty m(s) ds > 0,$$

and, finally (3.14) satisfies for  $s \leq t < \infty$ , and  $s$  sufficiently large,

$$(3.17) \quad \kappa(t,s) \geq k^2(t)m(t) \left( \int_0^t \frac{\kappa^{2-\alpha}(t,u)}{K_0(t,u)} du \right) K^\alpha(t,s) K_0(t,s).$$

Let us explain the notation  $o, O$  (Landau's symbol) here. Let  $f$  and  $g$  be real-valued functions defined on some interval  $J \subset \mathbb{R}$ ; then  $f = o(g)$  as  $t \rightarrow \infty$  means that  $\lim_{t \rightarrow \infty} |f(t) / g(t)| = 0$ . Similarly,  $f = O(g)$  as  $t \rightarrow \infty$  means that there exist constants  $M > 0$  and  $t_0 > 0$  such that  $|f(t)| \leq M|g(t)|$  for every  $t > t_0$  ( $M$  not depending on  $t$ ).

The following properties of condition (a) will be used in the proofs of the lemmas below (see [H-3]). We refer to [H-3] for the other properties of conditions (b) to (d) and details.

Condition (a) holds if and only if  $K$  satisfies the two conditions

$$(a_1) \quad \int_0^t K(t,s) ds = O(1) \quad \text{as } t \rightarrow \infty, \quad \dots (3.18)$$

and

$$(a_2) \quad \int_0^T K(t,s) ds = o(1), \quad t \rightarrow \infty, \quad \text{for fixed } T, \quad \dots (3.19)$$

$$0 < T < \infty.$$

§4. Main theorem.

The main result is

## THEOREM 3.6.

Let  $Q^*(t) = Q(t)$  be a continuous real matrix function on  $[0, \infty)$ . Assume that  $K$  satisfies (a), (b),  $(c_0)$  and (d). If

$$(3.A) \quad \liminf_{t \rightarrow \infty} \int_0^t K(t,s) g \left( \int_0^s Q(u) du \right) ds > -\infty,$$

for some positive linear functional  $g : S \rightarrow \mathbb{R}$ , then

$$(3.B) \quad \lim_{t \rightarrow \infty} \lambda_1 \left( \int_0^t Q(s) ds \right) = \infty$$

implies that equation (1.1) is oscillatory at  $\infty$ .

## LEMMA 3.7.

Let  $Q(t)$  be as in theorem 3.6 with the property that equation (1.1) is non-oscillatory at  $\infty$ , (This means that there exists an  $a > 0$  such that (1.1) is disconjugate on  $[a, \infty)$ ).

If  $K$  satisfies (a) and  $(c_0)$ , and

$$(3.C) \quad \lim_{t \rightarrow \infty} \lambda_1 \left( \int_a^t V^2(s) ds \right) < \infty, \quad a > 0,$$

( $a$  depends on  $V$ ), for every non-trivial prepared solution  $Y(t)$  of equation (1.2) such that  $\det Y(t) \neq 0$ ,  $a \leq t < \infty$ , and  $V$  is defined by  $V(t) \equiv Y'(t)Y^{-1}(t)$ , then there holds

$$(3.D) \quad \int_0^t K(t,s) |g[C - P(s)]|^\sigma ds \rightarrow 0, \text{ as } t \rightarrow \infty,$$

for  $\sigma$ ,  $1 \leq \sigma \leq 2$  and all positive linear functionals  $g : S \rightarrow \mathbb{R}$ , where  $C$  is a constant matrix and  $P(t)$  is defined as in (\*). In particular, (3.A) holds if  $\sigma = 1$ .

### LEMMA 3.8.

Let  $Q(t)$  be as in the theorem 3.6 with the property that equation (1.1) is non-oscillatory at  $\infty$ . Assume that  $K$  satisfies (a), (b) and (d). If (3.D) holds for a positive linear functional  $g : S \rightarrow \mathbb{R}$ , then (3.C) holds for every non-trivial prepared solution  $Y(t)$  of equation (1.2) and  $V$  as in lemma 3.7.

Note (3): These two lemmas (i.e., lemma 3.7 and lemma 3.8) improve A. B. Mingarelli's results [M-1, Lemma 1 and Corollary 1].

REMARK.

For the application of lemma 3.8, it is important to note that the proof of this lemma will show that condition (3.D) can be relaxed to (3.A). Therefore we have

## COROLLARY 3.9.

Let  $Q(t)$  be as in the theorem 3.6 with the property that equation (1.1) is non-oscillatory at  $\infty$ . Assume that  $K$  satisfies (a), (b) and (d). If (3.A) holds for a positive linear functional  $g : S \rightarrow \mathbb{R}$ , then this will imply that (3.C) holds for every non-trivial prepared solution  $Y(t)$  of equation (1.2) and  $V$  as in lemma 3.7.

The next lemma is actually in [M-1] but we will present a different proof from that therein.

## LEMMA 3.10. (see [M-1, Lemma 2])

Suppose that equation (1.1) is non-oscillatory at  $\infty$  and let  $Q(t)$  and  $V(t)$  be as in lemma 3.7. Then

$$(3.B) \quad \lim_{t \rightarrow \infty} \lambda_1 \left( \int_0^t Q(s) ds \right) = \infty$$

will imply that

$$(3.E) \quad \lim_{t \rightarrow \infty} \lambda_1 \left( \int_a^t v^2(s) ds \right) = \infty .$$

### PROOF OF LEMMA 3.7.

Since equation (1.1) is non-oscillatory at  $\infty$ , then there exists a  $a > 0$  such that equation (1.1) is disconjugate on  $[a, \infty)$ . Hence, by [H-2, p. 388, Theorem 10.2] there exists a non-trivial prepared solution  $Y(t)$  of equation (1.2) such that  $\det Y(t) \neq 0$  on  $(a, \infty)$ , or, without loss of generality, on  $[a, \infty)$ .  $V(t) = Y'(t)Y^{-1}(t)$  exists for  $t \geq a$  and so equation (1.2) shows that  $V(t)$  satisfies the matrix Riccati equation,

$$(3.20) \quad V'(t) + V^2(t) + Q(t) = 0 \quad t \geq a .$$

A quadrature gives

$$(3.21) \quad V(s) + \int_a^s v^2(u) du = V(a) - \int_a^s Q(u) du , \quad 0 < a \leq s < \infty .$$



Because of (3.C), we can rewrite (3.21) as

$$(3.22) \quad V(s) + \int_a^\infty V^2(u) \, du - \int_s^\infty V^2(u) \, du = V(a) - P(s) + P(a) .$$

or,

$$(3.23) \quad V(s) - \int_s^\infty V^2(u) \, du = C - P(s) ,$$

where  $C = V(a) + P(a) - \int_a^\infty V^2(u) \, du$ .

Now let  $g : S \rightarrow \mathbb{R}$  be some positive linear functional.

Then

$$g\left[V(s) - \int_s^\infty V^2(u) \, du\right] = g[C - P(s)] ,$$

or

$$(3.24) \quad g[V(s)] - g\left[\int_s^\infty V^2(u) \, du\right] = g[C - P(s)] .$$

Consequently,

$$(3.25) \quad |g[C - P(s)]|^\sigma \leq 2^\sigma |g[V(s)]|^\sigma + 2^\sigma \left(g\left[\int_s^\infty V^2(u) \, du\right]\right)^\sigma ,$$

where  $1 \leq \sigma \leq 2$  and we have used the inequality

$$|x + y|^\beta \leq 2^\beta (|x|^\beta + |y|^\beta) \quad \text{valid for } \beta \geq 1, \quad x, y \in \mathbb{R} .$$

Since  $K(t,s) \geq 0$ , we have

$$(3.26) \quad \int_a^t K(t,s) |g[C - P(s)]|^\sigma ds \\ \leq 2^\sigma \int_a^t K(t,s) |g[V(s)]|^\sigma ds + 2^\sigma \int_a^t K(t,s) \left( g \left[ \int_s^\infty v^2(u) du \right] \right)^\sigma ds.$$

Since the assumption (3.C) implies that

$$\lim_{t \rightarrow \infty} g \left[ \int_a^t v^2(s) ds \right] = \lim_{t \rightarrow \infty} \int_a^t g[v^2(s)] ds < \infty$$

by theorem 1.7, and

$$\int_a^\infty \left( |g[V(s)]|^\sigma \right)^{2/\sigma} ds = \int_a^\infty |g[V(s)]|^2 ds,$$

by lemma 1.4,

$$\leq \int_a^\infty g(I) g[v^2(s)] ds \\ = g(I) \left[ \int_a^\infty g[v^2(s)] ds \right] < \infty.$$

Therefore  $|g[V(s)]|^\sigma \in (L^{2/\sigma} \cap C^0)[a, \infty)$ .

Also;  $g \left[ \int_s^\infty v^2(u) du \right] \leq c \lambda_1 \left[ \int_s^\infty v^2(u) du \right] < \infty, \quad a \leq s < \infty,$

for some constant  $c = g(I) \in \mathbb{R}^+$ .

Therefore

$$g\left(\int_s^\infty v^2(u) du\right) \in (L_0^\infty \cap C^0)[a, \infty)$$

Let us define a function  $f(t)$  on  $[0, \infty)$  by

$$f(t) \equiv \begin{cases} |g[V(t)]|, & \text{if } t \geq a, \\ \zeta = |g[V(a)]|, & \text{if } 0 \leq t \leq a. \end{cases}$$

It is clear that  $f(t)$  is continuous on  $[0, \infty)$ , and

$$\begin{aligned} \int_0^\infty (|f(s)|^\sigma)^{2/\sigma} ds &= \int_0^a |f(s)|^2 ds + \int_a^\infty |f(s)|^2 ds \\ &= a\zeta^2 + \int_a^\infty |g[V(s)]|^2 ds \\ &< \infty. \end{aligned}$$

Therefore  $|f(s)|^\sigma \in (L^{2/\sigma} \cap C^0)[0, \infty)$  and for  $t \geq a$

$$0 \leq \int_a^t K(t,s) |g[V(s)]|^\sigma ds \leq \int_0^t K(t,s) |f(s)|^\sigma ds$$

Also let us define a function  $h(t)$  on  $[0, \infty)$  by

$$h(t) \equiv \begin{cases} g\left(\int_t^\infty v^2(u) du\right), & \text{if } t \geq a, \\ \eta = g\left(\int_a^\infty v^2(u) du\right), & \text{if } 0 \leq t \leq a. \end{cases}$$

It is clear that  $h(t)$  is continuous and bounded on  $[0, \infty)$ .

Therefore  $h(t) \in (L_0^\infty \cap C^0)[0, \infty)$  and for  $t \geq a$

$$0 \leq \int_a^t K(t,s) \left[ g \left( \int_s^\infty v^2(u) du \right) \right]^\sigma ds \leq \int_0^t K(t,s) [h(s)]^\sigma ds.$$

Hence, the first and last terms of the right-hand side of (3.26) tend to 0 as  $t \rightarrow \infty$ , by  $(c_\sigma)$  and (a).

Finally,

$$\begin{aligned} & \int_0^t K(t,s) |g[C - P(s)]|^\sigma ds \\ &= \int_0^a K(t,s) |g[C - P(s)]|^\sigma ds + \int_a^t K(t,s) |g[C - P(s)]|^\sigma ds \\ &\leq M \int_0^a K(t,s) ds + \int_a^t K(t,s) |g[C - P(s)]|^\sigma ds \end{aligned}$$

where  $M = \sup_{0 \leq t \leq a} |g[C - P(t)]|^\sigma$ .

Then by  $(a_2)$  and the fact that the right-hand side of (3.26) tends to zero as  $t \rightarrow \infty$ , we have

$$\int_0^t K(t,s) |g[C - P(s)]|^\sigma ds \rightarrow 0, \text{ as } t \rightarrow \infty.$$

Now (3.D) holds for  $\sigma$ ,  $1 \leq \sigma \leq 2$ ; so put  $\sigma = 1$ , then we have

$$\lim_{t \rightarrow \infty} \int_0^t K(t,s) g \left[ C - \int_0^s Q(u) du \right] ds = 0 .$$

This implies that

$$\begin{aligned} \lim_{t \rightarrow \infty} \int_0^t K(t,s) g \left( \int_0^s Q(u) du \right) ds &= \lim_{t \rightarrow \infty} \int_0^t K(t,s) g(C) ds \\ &= g(C) \left[ \lim_{t \rightarrow \infty} \int_0^t K(t,s) ds \right] , \end{aligned}$$

and by (a<sub>1</sub>), the right-hand side of the above is bounded.

In particular,

$$\liminf_{t \rightarrow \infty} \int_0^t K(t,s) g \left( \int_0^s Q(u) du \right) ds > -\infty .$$

This proves lemma 3.7.  $\square$ .

### PROOF OF LEMMA 3.8.

First of all we show that (3.C) exists (but may be infinite). Since  $v^2 \geq 0$ ,  $t \in [a, \infty)$ , then  $\int_a^t v^2(s) ds \geq 0$ ,  $t \in [a, \infty)$ . Also,

$$\int_a^{t_2} V^2(s) ds - \int_a^{t_1} V^2(s) ds = \int_{t_1}^{t_2} V^2(s) ds \geq 0$$

for  $a \leq t_1 \leq t_2 < \infty$ . By applying lemma 1.6, then

$\lim_{t \rightarrow \infty} \lambda_1 \left[ \int_a^t V^2(s) ds \right]$  exists (but may be infinite).

Suppose the conclusion is false, so that (3.C) fails for some prepared solution  $Y(t) \neq 0$  of equation (1.2). Let  $t = a$  be fixed, and sufficiently large, so that in particular,  $\det Y(t) \neq 0$  for  $t \geq a$ . Using equation (3.20), we have

$$(3.27) \quad V(s) + \int_a^s V^2(u) du = V(a) - P(s) + P(a), \quad 0 < a \leq s < \infty.$$

Let  $g : S \rightarrow \mathbb{R}$  be some positive linear functional.

Applying  $g$  to both sides of (3.27), we have

$$(3.28) \quad g[V(s)] + g\left[\int_a^s V^2(u) du\right] = g[V(a)] - g[P(s)] + g[P(a)].$$

Then, multiplying (3.28) by  $K(t,s)$  and integrating from  $a$  to  $t$ , we have

$$\begin{aligned}
 & \int_a^t K(t,s)g[V(s)] ds + \int_a^t K(t,s)g\left(\int_a^s V^2(u) du\right) ds \\
 &= \int_a^t K(t,s)\{g[V(a)] + g[P(a)]\} ds - \int_a^t K(t,s)g[P(s)] ds \\
 (3.29) \quad &= \int_a^t K(t,s)\{g[V(a)] + g[P(a)]\} ds + \int_0^a K(t,s)g[P(s)] ds \\
 &\quad - \int_0^t K(t,s)g[P(s)] ds .
 \end{aligned}$$

Combining the latter equalities we find that

$$(3.30) \quad \int_a^t K(t,s)g[V(s)] ds + S(t) = c(t,a) - \int_0^t K(t,s)g[P(s)] ds ,$$

where

$$(3.31) \quad S(t) = \int_a^t K(t,s)g\left(\int_a^s V^2(u) du\right) ds ,$$

$$(3.32) \quad c(t,a) = \int_a^t K(t,s)\{g[V(a)] + g[P(a)]\} ds + \int_0^a K(t,s)g[P(s)] ds .$$

Since (3.C) does not hold, theorem 3.5 and (b) and an extension argument similar to that presented in the proof of lemma 3.7 imply that

$$(3.33) \quad S(t) \rightarrow \infty , \text{ as } t \rightarrow \infty .$$

Moreover,  $c(t, a)$  is bounded as  $t \rightarrow \infty$  by assumptions  $(a_1)$  and  $(a_2)$ . Hence (3.D) implies that the right-hand side of (3.30) is bounded as  $t \rightarrow \infty$ , so that

$$(3.34) \quad \limsup_{t \rightarrow \infty} \left( \int_a^t K(t, s) g[V(s)] ds + S(t) \right) < \infty,$$

or

$$(3.35) \quad - \int_a^t K(t, s) g[V(s)] ds \geq \frac{S(t)}{2}, \quad \text{for } t \text{ large}$$

on account of (3.33).

For a fixed constant  $\alpha$ ,  $0 \leq \alpha \leq 2$ , Schwarz's inequality gives

$$(3.36) \quad \begin{aligned} & \left| \int_a^t K(t, s) g[V(s)] ds \right|^2 \\ &= \left| \int_a^t \sqrt{\frac{K^{2-\alpha}(t, s)}{K_0(t, s)}} \sqrt{K^\alpha(t, s)} \sqrt{K_0(t, s)} \sqrt{\{g[V(s)]\}^2} ds \right|^2 \\ &\leq \int_a^t \frac{K^{2-\alpha}(t, u)}{K_0(t, u)} du \int_a^t K^\alpha(t, s) K_0(t, s) \{g[V(s)]\}^2 ds \end{aligned}$$

whenever the right-hand side is meaningful.

From (3.35), we have

$$\left| \int_a^t K(t,s)g[V(s)] ds \right|^2 \geq \frac{S^2(t)}{4}$$

if  $t$  is sufficiently large so that  $S(t) > 0$ . This along with (3.36) gives

(3.37)

$$4k^2(t) \int_a^t \left( \frac{K^{2-\alpha}(t,u)}{K_0(t,u)} \right) du \times \int_a^t K^\alpha(t,s)K_0(t,s)\{g[V(s)]\}^2 ds \geq k^2(t)S^2(t).$$

By Fubini's theorem, (3.31) can be rewritten as

$$(3.38) \quad S(t) = \int_a^t \left[ \int_a^t K(t,u) du \right] g[V^2(s)] ds.$$

Hence, by (3.14), differentiation under the integral sign yields

$$(3.39) \quad [k(t)S(t)]' = \int_a^t \kappa(t,s)g[V^2(s)] ds$$

so that (3.17), lemma 1.4, and (3.37) imply

$$(3.40) \quad 4[k(t)S(t)]' \geq \frac{m(t)}{g(I)}[k(t)S(t)]^2$$

for  $t$  sufficiently large.

By a quadrature, for  $t$  large, we have

$$4 \int_t^u \frac{[k(x)S(x)]^2}{[k(x)S(x)]^2} dx \geq \frac{1}{g(I)} \int_t^u m(x) dx ,$$

therefore

$$\frac{4}{S(t)} \geq \frac{k(t)}{g(I)} \int_t^u m(x) dx$$

or

$$(3.41) \quad \frac{4g(I)}{S(t)} \geq k(t) \int_t^u m(x) dx$$

if  $t < u < \infty$ .

This contradicts the first alternative in (3.16) if  $u \rightarrow \infty$ ; and by (3.33), contradicts the second alternative if  $u \rightarrow \infty$  and then  $t \rightarrow \infty$ . This proves lemma 3.8.  $\square$

### PROOF OF COROLLARY 3.9.

The method of proof is almost identical to that of lemma 3.8, and so we shall omit most of the details.

Starting from (3.30) in the proof of lemma 3.8,

$$(3.30) \quad \int_a^t K(t,s)g[V(s)] ds + S(t) = c(t,a) - \int_0^t K(t,s)g[P(s)] ds ,$$

where

$$(3.31) \quad S(t) = \int_a^t K(t,s) g \left[ \int_a^s v^2(u) du \right] ds ,$$

$$(3.32) \quad c(t,a) = \int_a^t K(t,s) \{g[V(a)] + g[P(a)]\} ds + \int_0^a K(t,s) g[P(s)] ds .$$

Now  $c(t,a)$  is bounded as  $t \rightarrow \infty$  by assumptions  $(a_1)$  and  $(a_2)$  and so (3.A) implies that the right-hand side of (3.30) is bounded from above as  $t \rightarrow \infty$ , hence

$$(3.34) \quad \limsup_{t \rightarrow \infty} \left[ \int_a^t K(t,s) g[V(s)] ds + S(t) \right] < \infty .$$

The rest of argument follows the proof of lemma 3.8.  $\square$

### PROOF OF LEMMA 3.10.

From the proof of lemma 3.7, using equation (3.20), we have

$$(3.42) \quad -V(t) + V(a) = \int_a^t Q(s) ds + \int_a^t v^2(s) ds .$$

Since the last integral on the right-hand side of (3.42) is a non-negative definite matrix, we have

$$(3.43) \quad -V(t) + V(a) \geq \int_a^t Q(s) ds, \quad t \geq a.$$

Taking the maximum eigenvalue  $\lambda_1$  of both sides of (3.43) and using a combination of lemma 1.1, (iv) and (i) we have

$$(3.44) \quad \lambda_1 \left[ \int_a^t Q(s) ds \right] \leq \lambda_1 [-V(t)] + \lambda_1 [V(a)]$$

for  $t \geq a$ . Passing to the limit in (3.44) as  $t \rightarrow \infty$  we obtain

$$(3.45) \quad \lambda_1 [-V(t)] = \infty, \quad \text{as } t \rightarrow \infty.$$

We now apply lemma 1.1, (v) with  $A = -V(t)$  and use (3.45) to find that

$$(3.46) \quad \lambda_1 [V^2(t)] = \infty, \quad \text{as } t \rightarrow \infty.$$

Therefore

$$\int_a^t \lambda_1 [V^2(s)] ds = \infty, \quad \text{as } t \rightarrow \infty.$$

Hence, by using lemma 1.2, we have

$$\lambda_1 \left[ \int_a^t V^2(s) ds \right] = \infty, \quad \text{as } t \rightarrow \infty. \quad \square$$

PROOF OF THEOREM 3.6.

Assume that (3.A), (3.B) with  $K$  satisfying (a), (b), (c<sub>0</sub>) and (d) holds and assume on the contrary that equation (1.1) is non-oscillatory at  $\infty$ . Then corollary 3.9 implies

(3.C).

On the other hand, by lemma 3.10, (3.B) implies (3.E) which is the negation of (3.C). This is a contradiction. This proves theorem 3.6.  $\square$

THEOREM 3.11.

Let  $Q(t)$  be as in theorem 3.6. Assume that  $K$  satisfies (a), (b), (c<sub>0</sub>) and (d) and  $g$  is a positive linear functional. A necessary condition for equation (1.1) to be non-oscillatory at  $\infty$  is that

$$(3.F) \quad \liminf_{t \rightarrow \infty} \int_0^t K(t,s) g \left( \int_0^s Q(u) du \right) ds = +\infty$$

or that condition (3.D) of lemma 3.7 holds. \*

PROOF.

Suppose that equation (1.1) is non-oscillatory at  $\infty$  and (3.F) fails to hold. The negation of (3.F) is (3.A) and from corollary 3.9, (3.A) implies (3.C), so we have (3.D) by lemma 3.7.  $\square$

§5. Example.

The kernel  $K(t,s) = 1/t$  for  $0 \leq s \leq t$ ,  $t > 0$ , satisfies (a), (b), (c), (d) with  $k(t) = t$ ,  $m(t) = 1/t$ ,  $K_0 \equiv 1$ , and  $\alpha = 1$ , (see [H-3]). This example certainly includes the results of A. B. Mingarelli [M-1] and A. Wintner [W-4]. For other examples we refer the reader to [H-3]. Also, there is an interesting application for our main theorem in [H-5, p. 658].

§6. Conclusion.

We presented here an oscillation criterion for differential systems using some techniques of P. Hartman and A. B. Mingarelli. Certainly, more can be done in this direction. i.e., more general kernels can be used. Dr. M. K. Kwong (Northern Illinois University) mentioned to me that the kernel  $K(t,s) = 1/t^2$  can cause oscillation. but our class of kernels does not include the latter. Up to now the conjecture (p. 19) is still unsolved.

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## Résumé

Let us consider the system of differential equation

$$(1) \quad y'' + Q(t)y = 0, \quad t \in J$$

for a vector  $y$  in  $\mathbb{R}^n$  where  $J$  is a real interval and  $Q(t)$  is a real continuous  $n$  by  $n$  symmetric matrix function on  $J$ .

The equation (1) is called disconjugate on  $[a,b]$ , if every solution of (1) vanishes at most once on  $[a,b]$ .

When  $J = [0, \infty)$ , the equation (1) is called oscillatory at  $\infty$  if, for every  $a > 0$ , (1) is not disconjugate on  $[a, \infty)$ . It is non-oscillatory otherwise.

Let  $S$  be the real linear vector space of  $n$  by  $n$  symmetric matrices.

A real linear functional  $g : S \rightarrow \mathbb{R}$ , (i.e., satisfying

$$(2) \quad g(A + B) = g(A) + g(B), \quad g(\alpha A) = \alpha g(A)$$

for  $A$  and  $B$  in  $S$ ,  $\alpha$  in  $\mathbb{R}$ ) is called positive if

$$g(A) > 0$$

for every  $A > 0$  [ $A > 0$  means that  $A$  is a positive definite matrix, i.e., for any vector  $y$  in  $\mathbb{R}^n$ ,  $y^t A y > 0$ ].

Consider (1) for  $Q$  in  $S$  and  $t \in [0, \infty)$  and define

$$(*) \quad P(t) = \int_0^t Q(s) ds$$

and, a real linear functional  $g : S \rightarrow \mathbb{R}$ , and let

$$(**) \quad \bar{P}(t) = g[P(t)]$$

and

$$(***) \quad (K\bar{P})(t) = \int_0^t K(t,s) g \left[ \int_0^s Q(u) du \right] ds$$

The following conditions on the kernel  $K(t,s)$  were given by Hartman<sup>1</sup>,

$$(a) \quad (KR)(t) \rightarrow 0 \text{ as } t \rightarrow \infty, \text{ if } R \in (L_0^\infty \cap C^0)[0, \infty)$$

$$(b) \quad (KR)(t) \rightarrow \infty \text{ as } t \rightarrow \infty, \text{ if } C^0[0, \infty) \ni R(t) \rightarrow \infty \text{ as } t \rightarrow \infty$$

$$(c_\sigma) \quad (KR)(t) \rightarrow 0 \text{ as } t \rightarrow \infty, \text{ if } R \in (L^{2/\sigma} \cap C^0)[0, \infty)$$

$$(d) \quad \text{Assume that there exist functions } 0 < k(t) \in C^0[0, \infty), \text{ and}$$

$$0 \leq m(t) \in C^0[0, \infty), \text{ a kernel } K_0(t,s) \geq 0 \text{ continuous for}$$

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<sup>1</sup>P. Hartman, "On nonoscillatory linear differential equations of second order", Proc. Amer. Math. Soc. 64(1977), 251-259. MR 57#3510.

$0 \leq s \leq t$ ,  $0 < t < \infty$ , and a constant  $\alpha$ ,  $0 \leq \alpha \leq 2$ , with the following properties:

$$(2) \quad \kappa(t,s) \equiv \frac{\partial}{\partial t} \int_s^t k(t)K(t,u) du \geq 0$$

exists and is continuous for  $0 \leq s \leq t$ ,  $0 < t < \infty$ ;

$$(3) \quad \frac{K^{2-\alpha}(t,s)}{K_0(t,s)}$$

is continuous for  $0 \leq s \leq t$ ,  $0 < t < \infty$ ;

$$\int_0^\infty m(t) dt = \infty$$

(4)

$$\text{or } (\infty \geq) \quad \limsup_{t \rightarrow \infty} k(t) \int_t^\infty m(s) ds \geq 0,$$

and, finally, (2) satisfies for sufficiently large  $s$ ,  $s \leq t < \infty$ ,

$$(5) \quad \kappa(t,s) \geq k^2(t)m(t) \left( \int_0^t \frac{K^{2-\alpha}(t,u)}{K_0(t,u)} du \right) K^\alpha(t,s) K_0(t,s).$$

Then our main theorem is as follows.

Main Theorem.

Let  $Q^*(t) = Q(t)$  be a continuous real matrix function on  $[0, \infty)$ . Assume that  $K$  satisfies (a), (b), (c<sub>σ</sub>) and (d). If the following inequality

$$(A) \quad \liminf_{t \rightarrow \infty} \int_0^t K(t,s) g \left( \int_0^s Q(u) du \right) ds > -\infty$$

holds for some positive linear functional  $g : S \rightarrow \mathbb{R}$ , then

$$(B) \quad \lim_{t \rightarrow \infty} \lambda_1 \left( \int_0^t Q(s) ds \right) = \infty$$

implies that equation (1) is oscillatory at  $\infty$ . (Here  $\lambda_1[\cdot]$  denotes the largest eigenvalue of the matrix under consideration.)



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