ON CW-COMPLEXES

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ABSTRACT

This thesis is a study of cell complexes, of the conditions C and W of Whitehead and of CW-complexes. The topological properties of these spaces are considered in some detail and their relevance in algebraic topology examined. Finally the category of CW-complexes is compared to the category of compactly generated spaces (k-spaces) in the spirit of the conditions suggested by Steenrod in "A convenient category of topological spaces".

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INTRODUCTION

In this work we study CW-complexes (introduced first by J.H.C. Whitehead in [17] which is the basic reference for the subject) and in the process we see why most of their interesting properties depend on both conditions C and W, and finally, we compare the category of CW-complexes and continuous maps with Steenrod's category of compactly generated spaces and continuous maps.

In Chapter I, a cell complex is defined, and it is seen that the category of cell complexes is closed under the formation of subcomplexes, finite Cartesian products and quotients by closed subcomplexes if the quotients are Hausdorff.

In Chapter II, we consider cell complexes which have the weak topology with respect to their cells, i.e. W-complexes, and see that this category is not closed under the formation of subcomplexes or Cartesian products but quotients under the same restrictions as above are W-complexes. Also, subcomplexes need not be closed.

Chapter III deals with cell complexes which are closure finite, i.e. C-complexes. Subcomplexes and finite Cartesian products of C-complexes are C-complexes, quotients with the above restrictions are C-complexes but as above subcomplexes need not be closed.

In Chapter IV, we see that the category of CW-complexes (that is, cell complexes which satisfy both conditions C and W), is closed under the formation of subcomplexes and quotient complexes. Condition C is not strong enough to insure that the Cartesian product of two CW-complexes be a CW-complex. Several other properties of CW-complexes are investigated.

It is shown that the definition of a cell complex given in Chapter I together with conditions C and W is equivalent to an inductive definition which is useful in constructing and in recognizing CW-complexes. Then some (well known) examples are described and finally we look at locally finite CW-complexes and their properties.

In Chapter V, it is seen that CW-complexes are particularly useful in homotopy theory and for computing the homology groups.

Chapter VI contains a comparison of CW-complexes with compactly generated spaces. The category of CW-complexes is a full subcategory of Steenrod's category of compactly generated spaces and has the same categorical product.

We show that the category of pairs (K, L) and cellular maps, where K is a CW-complex and L is any subcomplex of K is closed under adjunction and product (if we use the categorical product). Finally we get some results analogous to those of Steenrod for neighbourhood retracts and filtered spaces.

We find that CW-complexes satisfy most of Steenrod's conditions for a convenient category. However there seem to be two difficulties. We can say almost nothing about function spaces, and as Steenrod mentions [15], some important spaces such as Cantor sets cannot be given a CW-decomposition.

Chapter I

CELL COMPLEXES

A cell *decomposition* of a non-empty Hausdorff space K, is a partition $\{e_\lambda\}_{\lambda \in \Lambda}$ of K such that for every λ, there is an integer $|\lambda| \geq 0$ (called the *dimension* of e_λ) and a mapping $f_\lambda\colon (D^{|\lambda|}, S^{|\lambda|-1}) \to (K, K^{|\lambda|-1})$, such that $K^n = \bigcup_{|\mu| \leq n} e_\mu$, $K^{-1} = \phi$, $K^0 \neq \phi$, and $f_\lambda \big| B^{|\lambda|}$ is a homeomorphism $B^{|\lambda|} \approx e_\lambda$.

D^n is the closed unit n-disk $\{x \in R^n \big| |x| \leq 1\}$. D^0 is a one point space.

B^n is the open unit n-disk $\{x \in R^n \big| |x| < 1\}$.

S^{n-1} is the boundary of D^n, $\partial D^n = \{x \in R \big| |x| = 1\}$, $S^{-1} = \phi$.

f_λ is the *characteristic map* of e_λ.

K^n is the *n-skeleton* of K and if $|\lambda| = n$, then e_λ is called an (open) *n-cell* of K, $\bar{e}_\lambda$ a *closed* n-cell of K.

The space K together with a cell decomposition is called a *cell complex*.

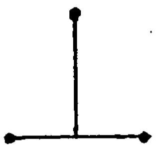
If $K = K^n$ for some n, then the *dimension of K*, $\dim K = n$. Otherwise $\dim K = \infty$.

Examples

1.1. Any Hausdorff space K can be made into a cell complex by defining every point to be a 0-cell.

1.2a. Let $K \subset R^2$ be composed of three 0-cells $(0,0)$, $(1,0)$ and $(\frac{1}{2}, 1)$ and two 1-cells, the intervals $((0,0),(1,0))$ and $((\frac{1}{2}, 0), (\frac{1}{2}, 1))$, with topology induced from R^2. K is not a cell complex since it does not satisfy the boundary condition.

1.2b. However, if the same set K has an additional 0-cell $(\frac{1}{2}, 0)$ and has as 1-cells the intervals $((0,0), (\frac{1}{2}, 0))$, $((\frac{1}{2}, 0), (1,0))$ and $((\frac{1}{2}, 0), (\frac{1}{2}, 1))$ then it is a cell complex.



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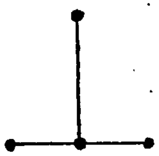


b

Remarks

1.3. For every λ, $f_{\lambda}(D^{|\lambda|}) = \bar{e}_{\lambda}$.

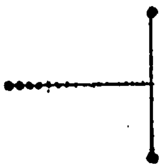
Since $D^{|\lambda|}$ is compact, f_{λ} continuous and K Hausdorff, f_{λ} is a closed mapping. Also since f_{λ} is continuous and closed, $\overline{f_{\lambda}(B)} = f_{\lambda}\overline{(B)}$. Thus $f_{\lambda}(D) = f_{\lambda}\overline{(B)} = \overline{f_{\lambda}(B)} = \bar{e}_{\lambda}$.

1.4. By 1.3, $\bar{e}_{\lambda}$ is compact for every $\lambda \in \Lambda$.

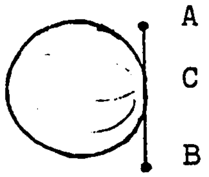
1.5. K^n need not be closed in K.

For example, let $K \subset R^2$ have as 0-cells those points of the form $(x, 0)$, $0 \leq x < 1$ and points $(1, \frac{1}{2})$ and $(1, -\frac{1}{2})$

and have one 1-cell, the interval $((1, -\frac{1}{2}), (1, \frac{1}{2}))$, with topology induced from the plane. K is a cell complex but K^0 is not closed in K.



1.6. The following example (cf. [4]) shows that $\bar{e}_\lambda$ need not be a union of cells. Let $K \subset R^3$ be the union of S^2 and the segment I joining the points $(1,1,0) = A$ and $(1,-1,0) = B$ (with topology induced from R^3), having two 0-cells $\{A\}$ and $\{B\}$, one 1-cell $I - \{A, B\}$ and one 2-cell $S^2 - C$, C being the point $(1,0,0)$. K is a cell complex although S^2 is not a union of cells.



Subcomplexes

$L \subset K$ is a subcomplex of K if L is a union of cells of K such that if $e_\lambda \subset L$ then $\bar{e}_\lambda \subset L$. If $L = \bigcup e_\lambda$ is a union of cells of K and L is closed in K, then L is a subcomplex of K, since if $e_\lambda \subset L$, then $\bar{e}_\lambda \subset \bar{L} \subset L$. However example 1.1 above shows that a subcomplex need not be a closed set. The boundary condition implies that K^n is a subcomplex for every n. Also the union and non-empty intersection of any set of subcomplexes are subcomplexes.

If $X \subset K$ is any subset of K, let $K(X)$ denote the smallest subcomplex of K which contains X. Clearly

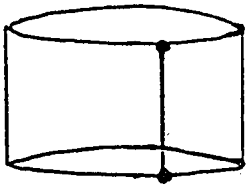
$K(p) = K(e) = K(\bar{e})$ where p is any point in K and $e \subset K$ is the cell containing p.

Any finite subcomplex (i.e. one containing a finite number of cells) is compact and closed since it is a finite union of compact sets.

Products

The topological product $K_1 \times K_2$ of cell complexes K_1 and K_2 is a cell complex if we take as its cells all products $e_\lambda \times e_\mu = e_{\lambda+\mu}$ of cells of K_1 and K_2. Let $h_{\lambda,\mu}: D^{|\lambda|+|\mu|} \to D^{|\lambda|} \times D^{|\mu|}$ be a homeomorphism for each pair of characteristic maps $f_\lambda: D^{|\lambda|} \to \bar{e}_\lambda$ and $g_\mu: D^{|\mu|} \to \bar{e}_\mu$ of K_1 and K_2 respectively. Then $(f_\lambda \times g_\mu)\circ h_{\lambda,\mu}: D^{|\lambda|+|\mu|} \to \bar{e}_\lambda \times \bar{e}_\mu$ is a characteristic map of the cell $e_\lambda \times e_\mu$ of $K_1 \times K_2$.

1.7. For example, if S^1 is considered as a union of one 0-cell and one 1-cell and I as a union of the two 0-cells $\{0\}$ and $\{1\}$ and one 1-cell $(0,1)$, then the product $S^1 \times I$ consists of two 0-cells, three 1-cells and one 2-cell.



Since $\overline{e_\lambda \times e_\mu} = \overline{e}_\lambda \times \overline{e}_\mu$, the product of two subcomplexes is a subcomplex of the product.

The collection of all cell complexes and continuous maps is a full subcategory, say CC of the category of Hausdorff spaces and continuous maps. The topological product together with the cell decomposition defined above is the product in this category. CC contains arbitrary coproducts (i.e. disjoint unions) but only finite products.

Quotients

Let A be a closed subcomplex of a cell complex K and f the identification mapping $f : K \to K/A$ identifying the points of A. Also let K/A be Hausdorff. This will be the case if K is regular. Then the cell decomposition of K/A induced by f consists of those cells of K which do not belong to A plus one 0-cell which is f(A). K/A is a union of cells satisfying the boundary condition and thus is a cell complex.

1.8. For example, let D^2 be considered as a union of one 2-cell B^2, one 1-cell $S^1 - p$, and one 0-cell the point p, with topology induced from the plane. Then if the points of the closed subcomplex S^1 are identified, the quotient space D^2/S^1 is the cell complex S^2 consisting of one 2-cell and one 0-cell.

Chapter II

W-COMPLEXES

The *weak topology* on a cell complex K coinduced by the family of characteristic maps $\{f_\lambda\}_{\lambda \in \Lambda}$ is defined as the finest topology on K such that the maps f_λ are continuous. Therefore U is open (closed) in K iff $f_\lambda^{-1}(U)$ is open (closed) in $D^{|\lambda|}$ for every $\lambda \in \Lambda$.

A cell complex K with the weak topology defined above is called a *W-complex*.

THEOREM 2.1. (cf. [1]) *Let* K *be a cell complex*. *Equivalent*:

(1) K *is a W-complex*

(2) $F \subset K$ *is closed in* K *iff* $F \cap \bar{e}$ *is closed in* K, *for each cell* $e \in K$

(2)' $F \subset K$ *is closed (open) in* K *iff* $F \cap \bar{e}$ *is closed (open) in* $\bar{e}$, *for every* $e \in K$. i.e. K *has the weak topology with respect to the closed cells*.

(3) *For every subset* $A \subset K$, A *closed or open in* K, *for all topological spaces* Y, *and for every* $g : A \to Y$, g *is continuous iff* $g|A \cap \bar{e}$ *is continuous for each cell* $e \in K$.

Proof

(1) $\Longrightarrow$ (2) If F is closed in K then $F \cap \bar{e}$ is closed

in $K, \forall\, e \in K$ since $\bar{e}$ is closed. If $F \cap \bar{e}$ is closed in $K, \forall\, e \in K$, then $f_\lambda^{-1}(F) = f_\lambda^{-1}(F \cap \bar{e}_\lambda)$ is closed since f_λ is continuous. Thus F is closed by (1).

(2) $\Longrightarrow$ (2)' F closed in K implies that $F \cap \bar{e}$ is closed in $\bar{e}$. Now if $F \cap \bar{e}$ is closed in $\bar{e}$, $\overline{F \cap \bar{e}}^{\,\bar{e}} = \overline{F \cap \bar{e}} \cap \bar{e} = F \cap \bar{e}$. But $\overline{F \cap \bar{e}} = \overline{F \cap \bar{e} \cap \bar{e}} \subset \overline{F \cap \bar{e}} \cap \bar{e}$. That is, $F \cap \bar{e}$ is closed in K and by (2) F is closed in K. Also, U open in K implies that $U \cap \bar{e}$ is open in $\bar{e}$. Conversely, if $U \cap \bar{e}$ is open in $\bar{e}$, then $\bar{e} - (U \cap \bar{e}) = (K - U) \cap \bar{e}$ is closed in $\bar{e}$ and by preceding result, $K - U$ is closed in K. That is U is open in K.

(2)' $\Longrightarrow$ (2) If $F \cap \bar{e}$ is closed in K, then $F \cap \bar{e}$ is closed in $\bar{e}$ and by (2)' F is closed in K.

(2) $\Longrightarrow$ (3) If g is continuous then all its restrictions are continuous. Therefore only the converse must be shown. Suppose that $A \subset K$ is closed in K, that $g|A \cap \bar{e}$ is continuous for each $e \in K$ and that F is any closed set of Y. Then $g^{-1}(F) \cap \bar{e} = (g|A \cap \bar{e})^{-1}(F)$ is a closed set of $A \cap \bar{e}, \forall\, e \in K$. But since A is closed, $A \cap \bar{e}$ is closed in K. Therefore $g^{-1}(F) \cap \bar{e}$ is closed in $K, \forall\, e \in K$. By (2) $g^{-1}(F)$ is closed and so g is continuous. If A is open a similar argument using (2)' gives the result.

(3) $\Longrightarrow$ (1) Let τ be the given topology on K and τ_W the weak one. The identity map $(K,\tau) \to (K,\tau_W)$ is continuous by (3) since for every τ_W-closed set F, $f_\lambda^{-1}(F)$ is closed in $D^{|\lambda|}$, $f_\lambda f_\lambda^{-1}(F) = F \cap \bar{e}_\lambda$ is τ-closed (f_λ being τ-closed). Then $\tau > \tau_W$ and therefore $\tau = \tau_W$ by definition of τ_W.

Remarks

2.2. Any finite cell complex is a W-complex, since, if $F \subset K$ then $F = F \cap K = F \cap (\bigcup_{i=1}^{n} \bar{e}_i) = \bigcup_{i=1}^{n} (F \cap \bar{e}_i)$ is closed iff $F \cap \bar{e}_i$ is closed in K for every i, $1 \leq i \leq n$.

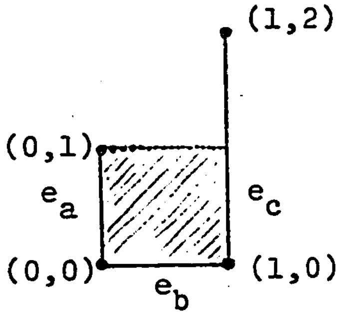
2.3. If K is a finite dimensional W-complex, i.e. if $K = K^n$, then each n-cell is open in K. (This is not true for arbitrary cell-complexes, e.g. [0,1] composed of 0-cells only with induced topology). For, let e_λ be an n-cell. Then, for $|\mu| \leq n$,

$$(K - e_\lambda) \cap \bar{e}_\mu = \begin{cases} \bar{e}_\mu & \text{if } \mu \neq \lambda \\ f_\lambda(S^{|\lambda|-1}) & \text{if } \mu = \lambda \end{cases}$$

Since f_λ is closed, $(K - e_\lambda) \cap \bar{e}_\mu$ is closed in K, $\forall e \in K$. Thus $(K - e_\lambda)$ is closed in K and e_λ is open in K.

2.4. A subcomplex of a W-complex need not be a W-complex. For example, if $K = D^2$ is the union of one 2-cell B^2 and S^1 composed of 0-cells only with topology induced from the plane, then K is a W-complex, since if $F \subset K$ is not closed, $F \cap D^2 = F$ is not closed. S^1 is a subcomplex of K but is not a W-complex, since if it were, then by 2.3 above, every point of $S^1 = K^0$, would be open. From this example it is seen that a subcomplex of a W-complex need not be closed (i.e. $S^1 - p$ is a subcomplex of K), and also that an n-skeleton of a W-complex need not be a W-complex ($S^1 = K^0$).

2.5. As in the case of arbitrary cell complexes, an n-skeleton K^n of a W-complex K need not be closed in K. For example let $K \subset R^2$ (with the induced topology) be the union of one 2-cell (the interior of I^2), three 1-cells, e_a, e_b, e_c, the three 0-cells (0,0), (1,0), and (1,2) and the interval [(0,1), (1,1)) composed of 0-cells only. Then K is a W-complex, since if $F \subset K$ and F is not closed, then as $F = F \cap K = (F \cap I^2) \cup (F \cap \bar{e}_c)$, either $F \cap I^2$ or $F \cap \bar{e}_c$ is not



closed in K. However K^0 is not closed in K. ($K^0 \cap I^2$ is not closed in I^2).

2.6. If K is finite dimensional and if every skeleton is a W-complex, then every subcomplex is closed.

Proof

Since every n-cell e_λ is open in K^n (cf. Remark 2.3), $K^n - K^{n-1} = \bigcup_{|\lambda|=n} e_\lambda$ is open in K^n. Thus K^{n-1} is closed in $K^n, \forall n \geq 0$, and therefore K^n is closed in K. Let L be a subcomplex of K. L^0 is closed in K^0 and thus in K. Suppose that L^{n-1} is closed in K^{n-1} and thus in K. L^n is the union of L^{n-1} and some n-cells. Therefore

$$L^n \cap \bar{e}_\lambda = \begin{cases} L^{n-1} \cap \bar{e}_\lambda & \text{if} \quad |\lambda| < n \\ \bar{e}_\lambda & \text{if} \quad |\lambda| = n \quad \text{and} \quad e_\lambda \in L^n \\ L^{n-1} \cap \bar{e}_\lambda & \text{if} \quad |\lambda| = n \quad \text{and} \quad e_\lambda \notin L^n \end{cases}$$

and thus L^n is closed in K^n and in K. Hence L is closed in K.

2.7. If K is finite dimensional and if every skeleton is a W-complex, then every subcomplex is a W-complex.

Proof

Let L be a subcomplex of K. Then L is closed in K (by 2.6), L^n is closed in K^n, K^n is closed in K^{n+1}, so L^n is closed in K^{n+1} and therefore in L^{n+1}. K^0 is a W-complex, therefore is discrete and L^0 is a W-complex. Suppose L^m is a W-complex for all $m \leq n$. Consider L^{n+1}. Let $F \subset L^{n+1}$ such that $F \cap \bar{e}_\lambda$ is closed in L^{n+1} for all $e_\lambda \in L^{n+1}$. If $e_\lambda \in L^n$, $F \cap \bar{e}_\lambda = (F \cap L^n) \cap \bar{e}_\lambda$ is closed in L^{n+1} and thus in L^n implying that $F \cap L^n$ is closed in L^n and thus in K^n and K^{n+1}. If $e_\lambda \in K^{n+1} - L^{n+1}$, $F \cap \bar{e}_\lambda = (F \cap L^n) \cap \bar{e}_\lambda$ is closed in K^{n+1}. Therefore $F \cap \bar{e}_\lambda$ is closed in K^{n+1} for all $e_\lambda \in K^{n+1}$, implying that F is closed in K^{n+1} and thus in L^{n+1} since $F \subset L^{n+1}$. Consequently L^{n+1} is a W-complex and so is L.

2.8. An example will be given later to show that the product of two W-complexes need not be a W-complex. However, if K_1 and K_2 are W-complexes and one of them is locally compact, then their product is a W-complex. This is a consequence of a theorem of J.H.C. Whitehead (cf. [5] p. 262) and the proof goes as follows. The characteristic maps $(f_\lambda)_{\lambda \in \Lambda}$ $((g_\gamma)_{\gamma \in \Gamma})$ of K_1 (K_2) induce a map $f(g)$ of the topological sum $D_1 = \bigsqcup_{\lambda \in \Lambda} D^{|\lambda|}$ $(D_2 = \bigsqcup_{\gamma \in \Gamma} D^{|\gamma|})$ onto K_1 (K_2). Then f and g are identifications (this is equivalent to saying that K_1 and K_2 are W-complexes). Let $f_\lambda \times g_\gamma$ be the characteristic maps of $K_1 \times K_2$. $D_1 \times D_2 \approx \bigsqcup_{\lambda,\gamma} (D^{|\lambda|} \times D^{|\gamma|})$. $K_1 \times K_2$ is a W-complex iff $f \times g$ is an identification. Since D_2 is locally compact, $f \times 1 : D_1 \times D_2 \to K_1 \times D_2$ is an identification by Whitehead's theorem. By the same theorem, if we assume K_1 to be locally compact, $1 \times g : K_1 \times D_2 \to K_1 \times K_2$ is also an identification. Finally since the composition of two identifications is an identification, $f \times g = (1 \times g) \circ (f \times 1)$ is an identification.

2.9. A W-complex is a CG space of Steenrod [15]. For let $F \cap A$ be closed in K for all A compact in K. $\bar{e}_\lambda$ is compact for every $\lambda \in \Lambda$. Thus $F \cap \bar{e}_\lambda$ is closed in K for every $e_\lambda \in K$. Therefore F is closed in K. It is not true in general that a CG cell complex is a W-complex as is seen by $K = S^1$ consisting of 0-cells only with the induced topology from the plane. We shall see later that this is true when the family $\{\bar{e}_\lambda\}$ is locally finite (cf. p. 36).

Quotients

PROPOSITION 2.10. The quotient K/A (we assume it to be Hausdorff) of a W-complex K by a closed subcomplex A is a W-complex.

Proof

We show that a subset F of K/A is closed if $(pf_\lambda)^{-1}$ (F) is closed for every characteristic map pf_λ of K/A, where $p:K \to K/A$ is the canonical projection and f_λ a characteristic map of K mapping into the complement of A. But F is closed iff $p^{-1}(F)$ is closed and in turn $p^{-1}(F)$ is closed iff $f_\lambda^{-1}\ p^{-1}(F)$ is closed for every characteristic map of K, since K is a W-complex. It is therefore sufficient to check that $f_\lambda^{-1}\ p^{-1}(F)$ is closed for the f_λ's mapping into A. In such a case, assuming first that F does not contain $p(A)$, A and $p^{-1}(F)$ are disjoint, $f_\lambda^{-1}\ p^{-1}(F) = \phi$ and is closed. Finally if F does contain $p(A)$, $f_\lambda^{-1}\ p^{-1}(F) = D^{|\lambda|}$ and is also closed, proving the proposition.

Chapter III

C-COMPLEXES

A cell complex K is called a C-complex (closure finite) if for every point p of K, $K(p)$ is a finite complex.

PROPOSITION 3.1. (cf. [1]) Let K be a cell complex. Equivalent:

(1) K is a C-complex.

(2) For every cell $e \in K$, $K(\overline{e})$ is finite.

(3) The adherence of each cell e of K meets only finitely many cells of K.

Proof

(1) $\Longleftrightarrow$ (2) $K(p) = K(\overline{e})$ for $p \in e \in K$ (cf. p. 4)

(2) $\Longrightarrow$ (3) $e_\mu \cap \overline{e}_\lambda \neq \phi$ implies that $e_\mu \subset K(\overline{e}_\lambda)$ since $K(\overline{e}_\lambda)$ is a union of cells.

(3) $\Longrightarrow$ (2) Remark first that if $e_\lambda \cap \overline{e}_\mu \neq \phi$, then $|\lambda| < |\mu|$. $K(\overline{e})$ can be described as the following union.

$K(\overline{e}) = e \cup \{e' | e' \cap \overline{e} \neq \phi\} \cup \{e'' | e'' \cap \overline{e}' \neq \phi\} \cup \ldots$.

At each step only finitely many cells are added by assumption and the process ends after finitely many steps by the previous remark.

Remarks

3.2. Any cell complex consisting entirely of 0-cells is a C-complex. Thus a C-complex need not be a W-complex.

3.3. The example of $K = D^n (n \geq 2)$ considered as the union of one n-cell B^n and S^{n-1} composed entirely of 0-cells shows that a W-complex need not be a C-complex. ($K(B^n)$ is not finite). (cf. Remark 2.4).

3.4. A subcomplex of a C-complex is a C-complex, for if $p \in L \subset K$, then $L(p) = K(p)$.

3.5. The product of two C-complexes is a C-complex, for $K_1(p_1) \times K_2(p_2) = (K_1 \times K_2)(p_1, p_2)$.

3.6. The quotient complex of a C-complex by a closed subcomplex is a C-complex, for $K/L\ (x) \subset pK(x')$, $x' \in p^{-1}(x)$.

3.7. Also any finite complex is clearly a C-complex.

3.8. As with W-complexes, neither a subcomplex nor an n-skeleton of a C-complex need be closed. (See example in Remark 1.5 page 2). Note however in this example that if K is given the weak topology then every subcomplex is closed.

PROPOSITION 3.9. [17] <u>If</u> K <u>is a C-complex</u>, <u>then</u> K <u>is a W-complex iff the following condition holds</u>: $F \subset K$ <u>is closed iff</u> $F \cap L$ <u>is closed in</u> K <u>for every finite subcomplex</u> L <u>of</u> K.

<u>Proof</u>

By Theorem 2.1, it is enough to show that if K is a C-complex then $F \cap \overline{e}$ is closed in K iff $F \cap L$ is closed in K for every cell of K and every finite subcomplex L of K. But the fact that $F \cap L = \bigcup_{e \in L} (F \cap \overline{e})$ is a finite union, proves the necessity. Since for every cell $e \in K$, $K(\overline{e})$ is finite by assumption, then $F \cap \overline{e} = (F \cap K(\overline{e})) \cap \overline{e}$ shows the sufficiency of the statement.

Condition C is too weak to insure that each compact set is contained in a finite subcomplex, and condition W is too weak to insure that every subcomplex is a W-complex. Also, neither C nor W imply that subcomplexes are closed. It will be seen that these results are desirable. Since C is a property of the cell structure and not topological, any C-complex can be made into a W-complex and remain a C-complex. We shall see that these two conditions on a cell complex imply the desired properties.

Chapter IV

CW-COMPLEXES

A cell complex is called a <u>CW-complex</u> if it is both a C-complex and a W-complex.

PROPOSITION 4.1. (cf. [1] or [17]) *If* $X \subset K$ *is a compact set in a CW-complex* K, *then* $K(X)$ *is finite*.

Proof

If X meets only finitely many cells e_λ, $\lambda=1,2, \ldots n$, then $K(X) \subset \bigcup_{\lambda=1}^{n} K(e_\lambda)$ is finite. Suppose X meets an infinite number of cells e_λ of K. Choose a point $p_\lambda \in X \cap e_\lambda$ for each λ and let $P = \{p_\lambda\}$. Since K is a C-complex each $\bar{e}_\mu$ of K meets only a finite number of cells of K. Thus $P \cap \bar{e}_\mu$ being finite, is closed for every $e_\mu \in K$, implying that P is closed since K is a W-complex. The same is true for every subset of P. Hence P is discrete. But P is a closed subset of a compact set X and thus is compact, which is impossible.

THEOREM 4.2. *Let* K *be a cell complex*.

Equivalent:

(1) K *is a CW-complex*.

(2) $F \subset K$ *is closed if* $F \cap L$ *is closed in* K *for every finite subcomplex* L *of* K.

(2)' $F \subset K$ *is closed (open) in* K *iff* $F \cap L$ *is closed (open) in* L, *for every finite subcomplex* L *of* K, i.e. K *has the weak topology with respect to the finite subcomplexes*.

(3) K *and every n-skeleton* K^n *of* K *are W-complexes*.

(4) *For every* $A \subset K$, A *closed or open in* K, *every topological space* Y, *and every* $g : A \to Y$, g *is continuous iff* $g|A \cap L$ *is continuous for every finite subcomplex* L *of* K.

(5) K *is the direct limit of the family of finite subcomplexes*.

Proof

(1) $\Longrightarrow$ (2) cf. Proposition 3.2.

(2) $\Longrightarrow$ (2)' F closed in K implies that $F \cap L$ is closed in L, $\forall L \subset K$. Now if $F \cap L$ is closed in L, $\overline{F \cap L}^L = \overline{F \cap L} \cap L = F \cap L$. But $\overline{F \cap L} = \overline{F \cap L \cap L} \subset \overline{F \cap L} \cap L$. That is, $F \cap L$ is closed in K and by (2) F is closed in K.

U open in K implies that $U \cap L$ is open in L. Conversely, if $U \cap L$ is open in L, for every $L \subset K$, then $L - (U \cap L) = (K - U) \cap L$ is closed in L, $\forall L \subset K$, and by preceding result, $K - U$ is closed in K. That is, U is open in K.

(2)' $\Longrightarrow$ (2) If $F \cap L$ is closed in K, then by (2)' $F \cap L$ is closed in L and also by (2)', F is closed in K.

(2) $\Longrightarrow$ (3) From the proof of Proposition 3.9, we see that

K is a W-complex. Also, (2) implies that each K^n is closed in K since if L is a finite subcomplex of K, $K^n \cap L$ is a finite subcomplex of K and is thus closed in K. Now suppose $F \subset K^n$ such that $F \cap \overline{e}_\lambda$ is closed in K^n for every $e_\lambda \in K^n$. For every finite subcomplex L of K, $L \cap K^n$ is a finite subcomplex of K^n. Thus $L \cap K^n$ is a finite union of closed cells of K^n, say $\cup\, \overline{e}_{\lambda'}$, and thus $F \cap L = F \cap L \cap K^n = F \cap (\cup \overline{e}_{\lambda'}) = \cup (F \cap \overline{e}_{\lambda'})$ is closed in K^n by assumption, and hence in K. Therefore F is closed in K by (2), and thus in K^n.

$(3) \Longrightarrow (1)$ (cf. [1] or [17]) It suffices to show that K is a C-complex. Clearly this is true for K^0. Assume K^{n-1} is a C-complex for some $n > 0$. Let e_λ be an n-cell. $\overline{e}_\lambda - e_\lambda$ is compact and thus by Proposition 4.1, $K(\overline{e}_\lambda - e_\lambda)$ is finite. Hence $K(\overline{e}_\lambda) = K(\overline{e}_\lambda - e_\lambda) \cup e_\lambda$ is finite. By induction K is a C-complex, since if $e_\lambda \in K$, $e_\lambda \in K^{|\lambda|}$.

$(2) \Longrightarrow (4)$ If g is continuous then every restriction of g is continuous. Therefore we need to show only the converse. Suppose that $A \subset K$ is closed in K, that $g|A \cap L$ is continuous for all finite subcomplexes L of K, and that F is any closed set of Y. Then $g^{-1}(F) \cap L = (g|A \cap L)^{-1}(F)$

is closed in $A \cap L$ for all $L \subset K$. But $A \cap L$ is closed in K since A and L are closed in K. Hence $g^{-1}(F) \cap L$ is closed in K for all $L \subset K$. By ′2) $g^{-1}(F)$ is closed in K and so g is continuous.

If A is open in K, and U is any open set of Y, $g^{-1}(U) \cap L = (g|A \cap L)^{-1}(U)$ is open in $A \cap L$ (since restriction is continuous) $\forall L \subset K$. Also $A \cap L$ is open in L (since A is open in K). Therefore $g^{-1}(U) \cap L$ is open in L, $\forall L \subset K$. By (2)' $g^{-1}(U)$ is open and thus g is continuous.

(4) $\Longrightarrow$ (2)' Let τ be the topology of K and τ_W the weak topology with respect to the finite subcomplexes. Then the identity map $(K, \tau) \to (K, \tau_W)$ is continuous, since the inclusion $(L, \tau) \to (K, \tau_W)$ is continuous by definition of τ_W for every finite subcomplex L of K. Hence τ is finer than τ_W and therefore $\tau = \tau_W$.

To prove that (5) is equivalent to the other conditions, we recall first that if X is a topological space and $\{A_\lambda\}$ a family of subspaces ordered by inclusion, $A = \underrightarrow{\lim}\ A_\lambda$ iff (i) $A = \bigcup\limits_\lambda A_\lambda$ and (ii) A has the weak topology with respect to the subspaces A_λ. For a cell complex K and the family of finite subcomplexes, the condition (i) is equivalent to C and (ii) is equivalent to (2)'. Since (1) implies C and (2)', (1) implies (5) and (5) implies (2)'.

Remarks

4.3. For an arbitrary cell complex K, one sees from the above proof that if every K^n is a W-complex, then every K^n is a C-complex and consequently K is a C-complex.

4.4. (cf. [17] p. 96) Every subcomplex of a CW-complex is closed and is a CW-complex. (This is proved in the same way as the fourth part of 4.2). Also K^0 is discrete. In fact, if P is a point of K^0, P is a 0-cell and thus open in K^0 (by Remark 2.3).

4.5. A cell complex is a compact CW-complex iff it is finite. The sufficiency is clear and the necessity follows from Proposition 4.1.

4.6. (cf. [1]). If K is a CW-complex, every n-cell e has an open neighbourhood U which meets no cell of dimension less than or equal to n, except e itself. It suffices to let $U = K - (K^n - e)$ which is open since $K^n - e$ is closed.

4.7. (cf. [17], [1]). Every CW-complex K is:

(a) normal,

(b) perfectly normal (i.e. normal and such that every closed set is a countable intersection of open sets) and thus (cf. [5]),

(c) completely normal (i.e. any two sets A and B of K such that $(A \cap \bar{B}) \cup (\bar{A} \cap B) = \phi$ can be separated.

4.8. Every CW-complex K is paracompact. The first proof of this is due to H. Miyazaki [12]. A proof by R. Mather can be found in [1].

4.9. Every CW-complex K is locally path connected [1]. Since the reference [1] is not easily obtainable, we reproduce the proof given there. Let x be a point of K, e the n-cell which contains x and U any open neighbourhood of x in K. It is necessary to show that U contains a path connected open set containing x. Let f be the characteristic map of e. $f^{-1}(U)$ is open in D^n and contains $f^{-1}(x)$ which is a point of B^n. Let V_0 be the image by f of an open ball of B^n centered at $f^{-1}(x)$ and contained in $f^{-1}(U)$. Then V_0 is open in e and $x \in V_0$. Consider the set of (n+1)-cells (e_λ) whose closures meet V_0. Let (f_λ) be their characteristic maps. $f_\lambda^{-1}(U)$ is open in $D^{|\lambda|}$ and contains $f_\lambda^{-1}(V_0) \subset S^{|\lambda|-1}$. Consider the cone with vertex $(0,0 \dots 0)$ and base $f_\lambda^{-1}(V_0)$. By taking the points of the cone sufficiently near $S^{|\lambda|-1}$, a path connected open set in $D^{|\lambda|}$, contained in $f_\lambda^{-1}(U)$, whose image by f_λ meets V_0, can be found. Let the image of this set be A_λ and let V_1 be the union of the sets A_λ and V_0. Then V_1 is path connected and open in K^{n+1}. Repeating this process with all (n+2)-cells (e_λ) the closures of which meet V_1, etc., ... , a sequence of sets $\{V_m\}_{m \geq n}$ is obtained for which $V_n \subset V_{n+1}$, $V_n \subset U$, $x \in V_n$, V_n is path connected and open in K^n. Let $V = \bigcup_{m=n}^{\infty} V_m$. V is a path connected and $x \in V \subset U$. $K^n \cap V = V_n$ is open in each K^n implying that V is open in K.

<u>Remark</u>

Whitehead has shown [17] that K is locally contractible, which implies that K is locally path connected [8].

4.10. The component and path-component of a point in a CW-complex K coincide. This is a corollary of 4.9 but we give an independent proof: Let $x \in K$, C the component of x and P the path component of x. C and P are subcomplexes of K: if $y \in C$ (P), $y \in e$ for some cell e and since $\bar{e}$ is connected (path-connected), $\bar{e} \subset C$ (P). We know that $P \subset C$, and we show by contradiction that $C - P = \phi$. Let $z \in C - P$, $z \in e$ for some cell e. Then $\bar{e} \subset C$ and therefore $\bar{e} \subset C - P$. Hence P and $C - P$ being subcomplexes are closed, disjoint, contradicting the fact that C is connected.

4.11. If a CW-complex K is connected then so is K^n, for every $n > 0$, (cf. [17], p. 96). Conversely, if K^n is connected for some $n > 0$, then so is K: If K is not connected, then $K = A \cup B$ is a disjoint union of non-empty closed sets A and B. Thus $K^n = (K^n \cap A) \cup (K^n \cap B)$ is a disjoint union of closed sets. If P is any point in A, then $K(P)$ being connected, belongs to A. But $K(P) \cap K^n \neq \phi$ since $K(P)$ must contain some 0-cell. Similarly $K^n \cap B \neq \phi$, so that K^n is not connected.

4.12. Equivalent: (i) K is connected, (ii) K^1 is connected, (iii) K is path connected. (Follows from 4.10 and 4.11).

4.13. A CW-complex need not be locally compact. CZ, the cone over Z (example 4.21) is not. Later we will define a property ensuring local compactness.

4.14. A CW-complex is not metrizable in general. For example the cone CZ is a simplicial complex which is not locally finite and therefore not metrizable (cf. [7]). We will look at this property later on.

4.15. For a CW-complex, countable, separable and Lindelöf are equivalent properties.

Proof

If K is countable then K is the continuous image of the separable space $\coprod_{\lambda \in \Lambda} D^{|\lambda|}$ (Λ indexes the set of all cells), and thus separable [5]. Since K is then separable and paracompact, it is also Lindelöf [5]. Conversely, suppose K is not countable. For each $\lambda \in \Lambda$, we can choose a point $p_\lambda \in e_\lambda$. Let $P = \{p_\lambda\}_{\lambda \in \Lambda}$. P is discrete and uncountable and thus cannot be Lindelöf. But this means that K is not Lindelöf, since if K is Lindelöf, P being a closed subset of K would also be Lindelöf [5].

4.16. The product of two CW-complexes is not a CW-complex in general. An example due to Dowker can be found in [5]. This example is equivalent to the product of the two cones CZ and CI, cf. 4.21. By 2.8 and 3.5, we know that if one of the two CW-complexes is locally-compact then their product is a CW-complex.

4.17. The quotient K/A of a CW-complex K by a subcomplex A is a CW-complex (see 2.10 and 3.6). In this case A being a subcomplex is closed and K/A is Hausdorff since K is normal.

Note also that the quotient map $p : K \to K/A$ is

(1) closed, since if F is closed in K,

$$p^{-1}p(F) = \begin{cases} F & \text{if } F \cap A = \phi \\ F \cup A & \text{if } F \cap A \neq \phi \end{cases}$$

(2) perfect if A is finite, since A is then compact.

4.18. (cf. [17] p. 97). If $f : K \to L$ is an identification from a W-complex K onto any cell complex L such that $L(f(\overline{e}))$ is finite for each cell e of K, then L is a CW-complex. (This is a little stronger than the reference because of 4.2 (2), but the proof is identical).

Other properties of CW-complexes, in particular homotopical ones can be found in [17].

The Inductive Approach

We would like to show that our definition of a CW-complex is equivalent to an inductive definition such as that given in [14].

Given a space X and a closed subspace A, we shall say that X is obtained from A by adjoining n-cells if there is a family of maps $\{\psi_\lambda : S^{n-1} \to A\}_{\lambda \in \Lambda}$ (Λ may be empty) and X is the adjunction space $(\bigsqcup_\lambda D^n) \cup_\psi A$ (cf. [5] p. 127) where $\bigsqcup$ denotes the topological sum and $\psi : \bigsqcup_\lambda S^{n-1} \to A$ is the unique map induced by $\{\psi_\lambda\}$. In other words X is the pushout of $(\bigsqcup_\lambda S^{n-1} \hookrightarrow \bigsqcup_\lambda D^n, \psi)$ (cf. [2] for instance).

THEOREM 4.19. A non-empty topological space K is a

CW-complex iff there is a sequence of closed subspaces $K^0 \subset K' \subset \dots \subset K^n \subset \dots$ *such that*

(1) K^0 *is discrete and non-empty*,

(2) *for* $n \geq 1$, K^n *is obtained from* K^{n-1} *by adjoining n-cells*.

(3) $K = \bigcup K^n$,

(4) *the topology of* K *is coherent with* $\{K^n\}_{n \geq 0}$, i.e.

K *has the weak topology with respect to* $\{K^n\}_{n \geq 0}$.

(*A space* K *with such a sequence of closed subspaces satisfying* (3) *and* (4) *is called a filtered space*).

Proof

Let us assume first that K is a CW-complex. Taking for K^n, $n \geq 0$, the n-skeleton of K, by remark (4.4) K^n is closed in K, K^0 is discrete and (3) holds. To prove (2), we first define $\overline{f} : \coprod_{|\lambda|=n} D^{|\lambda|} \to K^n$ as the unique map induced by the characteristic maps $\{f_\lambda\}_{|\lambda|=n}$, $f = \overline{f}| \coprod_{|\lambda|=n} S^{|\lambda|-1}$ and remark that $h = \overline{f}| \coprod_{|\lambda|=n} B^{|\lambda|}$ is a bijection $\coprod_{|\lambda|=n} B^{|\lambda|} \to K^n - K^{n-1}$. We prove that

$$\begin{tikzcd}
\coprod_{|\lambda|=n} S^{|\lambda|-1} \arrow[r, "f"] \arrow[d, hook, "j"'] & K^{n-1} \arrow[d, hook, "i"] \\
\coprod_{|\lambda|=n} D^{|\lambda|} \arrow[r, "\overline{f}"] & K^n
\end{tikzcd}$$

is a pushout. If Y is an arbitrary set, $u : K^{n-1} \to Y$ and $v : \coprod_{|\lambda|=n} D^{|\lambda|} \to Y$ are such that $uf = vj$ then there is a unique $g : K^n \to Y$ with $gi = u$ and $g\bar{f} = v$, g being defined by

$$g(x) = \begin{cases} u(x) & , x \in K^{n-1} \\ vh^{-1}(x), & x \in K^n - K^{n-1} \end{cases} .$$

So the above square is a pushout of sets. It remains to show that the topology of K^n is coinduced by $\{i, \bar{f}\}$. Taking $F \subset K^n$ such that $i^{-1}F = F \cap K^{n-1}$ and $\bar{f}^{-1}F$ are closed in K^{n-1} and $\coprod_{|\lambda|=n} D^{|\lambda|}$ respectively, we show that F is closed in K^n, i.e. that $F \cap \bar{e}$ is closed in $\bar{e}$ for every cell of dimension $\leq n$, since K^n is a W-complex. $F \cap K^{n-1}$ closed in K^{n-1} implies $F \cap \bar{e}$ closed in $\bar{e}$ for every cell of dimension $\leq n-1$. $\bar{f}^{-1}F$ closed in $\coprod_{|\lambda|=n} D^{|\lambda|}$ implies $\bar{f}^{-1}F \cap D^{|\lambda|} = f_\lambda^{-1}(F \cap \bar{e}_\lambda)$ closed in $D^{|\lambda|}$ for every λ with $|\lambda| = n$. Since $f_\lambda : D^{|\lambda|} \to \bar{e}$ is a closed surjection, $f_\lambda f_\lambda^{-1}(F \cap \bar{e}_\lambda) = F \cap \bar{e}_\lambda$ is closed in e_λ for every n-cell e_λ.

It remains to show that K has the weak topology with respect to its skeletons. Let $F \subset K$ be such that $F \cap K^n$

is closed in K^n for every skeleton K^n. If e is any cell of K, e belongs to K^n for some n and therefore $F \cap \bar{e} = (F \cap K^n) \cap \bar{e}$ is closed in $\bar{e}$ since $F \cap K^n$ is closed in K^n and K^n is a W-complex. K having the weak topology with respect to its cells, F is closed in K.

Conversely, we now assume that K possesses a sequence of nested closed subspaces K^n satisfying the conditions (1) to (4). First we show that K is a cell complex. Consider a fixed n. By (2), K^n is an adjunction space $(\bigsqcup_\lambda D^n) \cup_\psi K^{n-1}$, where $\psi : \bigsqcup_\lambda S^{n-1} \to K^{n-1}$ is induced by some family of maps $\{\psi_\lambda : S^n \to K^n\}_{\lambda \in \Lambda}$. For every $\lambda \in \Lambda$ we define a map $f_\lambda : (D^n, S^{n-1}) \to (K^n, K^{n-1})$ as the composite

$$(D^n, S^{n-1}) \hookrightarrow ((\bigsqcup_\lambda D^n) \sqcup K^{n-1}, (\bigsqcup_\lambda S^{n-1}) \sqcup K^{n-1}) \xrightarrow{p} (K^n, K^{n-1}),$$

where p is the canonical projection; we define also $e_\lambda = f_\lambda(B^n)$. We know (cf. [5]) that $\bigsqcup_\lambda B^n$ is homeomorphic to its image $K^n - K^{n-1}$ by p and consequently $f_\lambda | B^n$ is a homeomorphism $B^n \to e_\lambda$ for every λ; put $|\lambda| = n$. By induction we show that K^n is a cell complex for every n.

This is true for K^0 since it is discrete by (1); assuming that K^{n-1} is a cell complex, $K^n = K^{n-1} \cup (K^n - K^{n-1})$ and $K^n - K^{n-1} = \bigcup_{|\lambda|=n} e_\lambda$ (disjoint) imply that K^n has a cell decomposition with characteristic maps f_λ. The fact that K^n is Hausdorff follows from a general result about adjunction spaces (cf. [2] or [6]), since K^{n-1} is Hausdorff by the induction hypothesis, $\bigsqcup_\lambda D^n$ is regular and $\bigsqcup_\lambda S^{n-1}$ is a neighborhood retract of $\bigsqcup_\lambda D^n$.

To prove that K is Hausdorff, consider two distinct points $x, y \in K$. Then $x, y \in K^m$ for some $m \geq 0$ and thus for all $n \geq m$. We construct a sequence $\{U_n, V_n\}_{n \geq m}$ of sets such that U_n, V_n are open in K^n, $x \in U_n$, $y \in V_n$, $U_n \cap V_n = \phi$ and $U_{n+1} \cap K^n = U_n$, $V_{n+1} \cap K^n = V_n$. Since K^m is Hausdorff, we can find open sets U_m, V_m separating x and y in K^m. Suppose we have $\{U_n, V_n\}$ for $m \leq n \leq p$. K^p is a retract of an open neighbourhood $W = \bigsqcup_\lambda (D^{p+1} - 0) \cup_f K^p$ in K^{p+1}, the retraction $r : W \to K^p$ being induced by the radial retraction $D^{p+1} - 0 \to S^p$. Let $U_{p+1} = r^{-1}(U_p)$ and $V_{p+1} = r^{-1}(V_p)$. Since W is open in K^{p+1}, U_{p+1} and V_{p+1}

are open in K^{p+1} and separate x and y. Let $U = \bigcup_{n=m}^{\infty} U_n$ and $V = \bigcup_{n=m}^{\infty} V_n$. U and V are open since they meet each K^n in open sets of K^n and separate x and y. Therefore K is a cell complex.

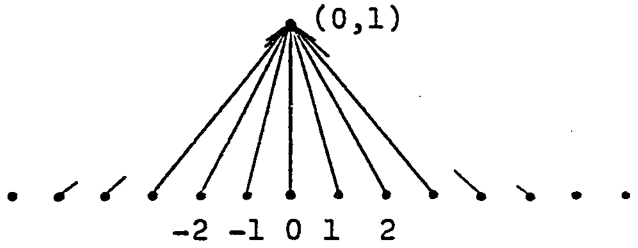
We now show that every K^n is a W-complex, again by induction. K^0 being discrete is clearly a W-complex. Assume that K^{n-1} is a W-complex. Then K^n being an adjunction space, if $F \subset K^n$, F is closed in K^n iff $F \cap K^{n-1}$ is closed in K^{n-1} and $f_\lambda^{-1}(F \cap \bar{e}_\lambda)$ is closed in D^n for all λ with $|\lambda| = n$. But this means that F is closed in K^n iff $F \cap \bar{e}_\lambda$ is closed in $\bar{e}_\lambda$ for all λ with $|\lambda| \leq n-1$ (since K^{n-1} is a W-complex), and $F \cap \bar{e}_\lambda = f_\lambda f_\lambda^{-1}(F \cap \bar{e}_\lambda)$ is closed in $\bar{e}_\lambda$ for all λ with $|\lambda| = n$ (since $f_\lambda : D^n \to \bar{e}_\lambda$ is a closed surjection), that is K^n has the weak topology with respect to its cells.

By Theorem 4.2, it remains to show that K is a W-complex. Consider $F \subset K$ such that $F \cap \bar{e}$ is closed in K for every cell e of K. Then $F \cap \bar{e} \cap K^n$ is closed in K^n for every e and n. Since K^n is a W-complex it follows that $F \cap K^n$ is closed in K^n for all n and by (4) F is closed in K since K has the weak topology with respect to $\{K^n\}$.

Examples

4.20. Any C-complex can be given the weak topology with respect to its cells, making it a CW-complex. In particular this is true for all 0 and 1-dimensional cell complexes which are automatically C-complexes. Every finite cell complex is a CW-complex.

4.21. [1]. Let K be the 1-dimensional CW-complex defined as the following subset of R^2: the 0-cells are the points (0,1) and (n,0), $n \in Z$, the 1-cells are the open segments joining (0,1) to (n,0) for all $n \in Z$.



K is homeomorphic to the cone CZ, i.e. the quotient complex $Z \times I/Z \times \{1\}$, where Z is considered as a 0-dimensional CW-complex and $I = [0,1]$ as a 1-dimensional CW-complex with the 0-cells {0} and {1} and the 1-cell (0,1). (cf. [5]).

Giving the underlying set of K the induced topology from R^2 and the same cell decomposition we get a new cell complex K', which is not a W-complex since the induced topology is strictly coarser than the weak one. In fact, the set $\{p_n | n \in N\}$, where p_n lies on the segment joining $(n,0)$ to $(0,1)$ at the distance $1/n$ from this latter point, is closed in the weak topology, but is not closed in the induced topology since it does not contain its cluster point $(0,1)$.

Another interesting description of K' can be given as follows. We consider the subspace of the ℓ^2-space consisting of the sequences (x_n) with all but one term equal to zero and $0 \leq x_n \leq 1$. Let the 0-cells be the sequence (x_n) with the non zero term equal to 1 and (0), the origin of ℓ^2, and the i'th 1-cell $\{(x_n) | 0 < x_i < 1$, $i = 1, 2, \ldots$. Given a point (x_n) in this space, the assignment $x_{2n} \rightarrow (n, 1 - x_{2n})$, $x_{2n-1} \rightarrow (1 - n, 1 - x_{2n-1})$ and $(0) \rightarrow (0,1)$ defines a bijection with the subset of R^2 above. One sees that this space, with the topology induced by the ℓ^2-norm is homeomorphic to K', but not isometric.

The cone CI, where I is considered as 0-dimensional CW-complex provides a similar example.

4.22. (cf. [4]). Let $I^{(N)} \subset I^N$ be the set of sequences $t = (t_n)_{n \in N}$, $t_n \in [0,1]$, for which at most a finite number of terms are different from zero. Then $\sum_1^\infty t_n$ makes sense. Let $K = \{t \in I^{(N)} | \sum_{n=1}^\infty t_n = 1\}$, with the following cell decomposition. Let I_n be the set of $n+1$ natural integers $\underline{i} = (i_0 < i_1 < \dots < i_n)$. For each $n > 0$ and $\underline{i} \in I_n$, define an n-cell $e_{\underline{i}}^n$ by

$e_{\underline{i}}^n = \{(t_m) \in K | 0 < t_k < 1$ if $k \in \underline{i}$, $t_k = 0$ if $k \notin \underline{i}\}$.

For $n = 0$ and each $\underline{i} \in I_0 = N$, define a 0-cell $e_{\underline{i}}^0$ by

$$e_i^0 = \{(t_k) \in K | t_i = 1, t_k = 0 \text{ if } k \neq i\}.$$

K may be given the following topologies:

(a) the product topology, i.e. that induced by I^N,

(b) the ℓ^2-topology, i.e. that induced by the norm

$$||(t_n)|| = \sqrt{\sum_{n=1}^\infty t_n^2},$$

(c) the weak topology with respect to the cells, making K a CW-complex. With this topology, K is homeomorphic to $\Delta^\infty = \bigcup_{n \geq 0} \Delta^n$ with the weak topology with respect to the Δ^n's, where Δ^n is the standard n-simplex in R^{n+1}, i.e. the convex hull of the unit points on the axis.

The ℓ^2-topology is strictly finer than the product topology. Any open set in the product topology contains a subset U of the form

$U = (a_{i_1}, b_{i_1}) \times \ldots \times (a_{i_k}, b_{i_k}) \times \Pi\{I_j | I_j = I, j \neq i_1, \ldots, i_k\}$. Then any point $(t_n) \in U$ is the center of an ℓ^2-open ball entirely contained in U. By taking $r = \min_{1 \leq j \leq k} \min\{|t_{i_j} - a_{i_j}|, |t_{i_j} - b_{i_j}|\}$, the ball $\{(s_n) \in K | d((s_n), (t_n)) < r\} \subset U$. Moreover, the sequence of 0-cells $\{e_n^0\}_{n \in N}$ converges to $(0, \ldots, 0, \ldots)$ in the product topology but not in the ℓ^2-topology, showing that this latter topology is strictly finer than the product topology.

The weak topology is strictly finer than the ℓ^2-topology. This latter induces the Euclidean topology on any standard simplex Δ^n and since K has the weak topology with respect to these Δ^n, the weak topology is finer than the ℓ^2 one. On the other hand the set $B = \{b_n\}_{n \geq 0}$ of barycenters b_n of Δ^n (i.e. $b_n = (t_k)$, $t_k = \frac{1}{n+1}$, for $k = 1, 2, \ldots, n+1$ and $t_k = 0$ for $k > n + 1$) is not ℓ^2-closed since $0 = (0, \ldots, 0, \ldots)$ is a cluster point but it is closed in the weak topology since $B \cap \Delta^n$ consists of $n+1$ points.

4.23. The infinite sphere $S^\infty = \bigcup_{n \geq 0} S^n$ with the weak topology with respect to the S^n's is a CW-complex. S^0 is discrete, by the standard embedding S^n is closed in S^{n+1}, S^{n+1} can be obtained from S^n by adjoining two (n+1)-cells, the two hemispheres, and the result follows by Theorem 4.19.

4.24. Let F denote one of the fields R, C or H and $d = \dim_R F$, i.e. d = 1, 2, or 4 respectively.

The projective n-space P^nF is obtained from $P^{n-1}F$ by adjoining a single dn-cell via the canonical projection $S^{dn-1} \to P^{n-1}F$, [6]. Therefore P^nF is a finite cell-complex (and thus a CW-complex) of dimension dn. Since $P^{n-1}F$ can be considered as a closed subspace of P^nF, the <u>infinite projective space</u> $P^\infty F = \bigcup_{n\geq 0} P^nF$ with the weak topology with respect to the P^nF's is a CW-complex by Theorem 4.19.

4.25. A less trivial example is given by a cell decomposition of the Stiefel manifolds, cf. [16].

<u>Locally Finite Cell Complexes</u>

A cell complex K is <u>locally finite</u> if every point of K has a neighbourhood which is a finite subcomplex of K.

One can easily show that this definition is equivalent to the following one. (cf. [1]): a cell complex is locally finite if it is a C-complex and the family of open (or closed) cells is locally finite.

PROPOSITION 4.26. A cell complex is locally finite iff it is a locally compact CW-complex.

Proof

Suppose the cell complex K locally finite and let $F \subset K$ be such that $F \cap L$ is closed for every finite subcomplex L of K. We show that $K - F$ is open and therefore K is a CW-complex by 4.2. Let $x \in K - F$. By assumption there exists a finite subcomplex L_0 which is a neighbourhood of x. Then $L_0 - (F \cap L_0)$ is a neighbourhood of x contained in $K - F$, which is therefore open. Moreover since a finite complex is compact, K is locally compact.

Conversely, if K is a locally compact CW-complex, every point has a compact neighbourhood which is contained in a finite subcomplex by 4.1 and therefore K is locally finite.

For the sake of completeness we include the following results (cf. [1] and [5]).

4.27. A connected locally finite complex is second countable and therefore separable and σ-compact.

4.28. A CW-complex is first countable iff it is locally finite.

PROPOSITION 4.29. A CW-complex is metrizable iff it is locally finite.

Proof

Let K be a metrizable CW-complex and suppose that K is not locally finite. Then there exists $p \in K$ such that every neighbourhood of p meets infinitely many cells. Since K is metrizable it is first countable and let $\{V_n\}_{n \geq 0}$ be a local base at p, such that $V_n \supset V_{n+1}$ for all n. In each V_n we can choose a point $x_n \neq p$, such that if $n \neq m$, x_n and x_m belong to different (open) cells. Now the sequence (x_n) converges to p and therefore the set $E = \{x_n\}$ is not closed. On the other hand K being a C-complex $E \cap \bar{e}$ is finite for every cell and therefore E is closed since K is a W-complex. Thus we get a contradiction.

(We remark that this proves half of 4.28).

Conversely, suppose that K is locally finite. Then the identification $f : \bigsqcup D^{|\lambda|} \to K$ induced by the characteristic maps is closed. The result follows from the following theorem by Morita and Hanai [13], which says that if $f : X \to Y$ if a closed surjection of a metrizable space X, then Y is metrizable iff $f^{-1}(y)$ has a compact boundary for every $y \in Y$.

From these results it follows that for CW-complexes the concepts of local finiteness, first countability, local compactness and metrizability are equivalent.

Chapter V

CW-COMPLEXES AND ALGEBRAIC TOPOLOGY

The category of CW-complexes and continuous maps or cellular maps seems to be the best adapted to homotopy theory. We shall mention only three results to support this claim, and refer to [14] for their proof and further details.

That it is sufficient to consider cellular maps follows from the cellular-approximation theorem:

THEOREM 5.1. *Any map between CW-complexes is homotopic to a cellular map. Moreover if there are two such cellular maps they are cellular homotopic.*

THEOREM 5.2. *For CW-complexes, the notions of weak homotopy equivalence and homotopy equivalence coincide.*

THEOREM 5.3. *Any space* X *has a CW-approximation, i.e. a CW-complex* K *and a weak homotopy equivalence* $f : K \to X$. *Moreover* K *is unique up to homotopy type.*

Another advantage of a CW-decomposition of a space is that it provides a method of computing the (singular) homology of this space. This method can be obtained by using a spectral sequence (cf. [14]) or directly (cf. [3]). We shall offer a proof of this result based on notes of a course given by A. Dold and using only elementary properties of homology. We introduce a few definitions first.

A skeletal decomposition of a space X is a filtration $X^{-1} \subset X^0 \subset X^1 \subset \ldots \subset X^n \subset \ldots \subset X$ by subspaces (called skeletons) satisfying the conditions

(S1) the singular chain complex $SX = \bigcup_n SX^n$.

(S2) $H_q(X^n, X^{n-1}) = 0$ for $q \neq n$.

We shall call a space X together with a skeletal decomposition a skeletal space.

If X and Y are two skeletal spaces, a map $f:X \to Y$ is skeletal if $f(X^n) \subset Y^n$ for all n.

Skeletal spaces together with skeletal maps obviously form a category.

THEOREM 5.4. If X is a skeletal space, there is a non-negative chain complex CX, (the skeletal chain complex of X) defined by $C_nX = H_n(X^n, X^{n-1})$, $n \geq 0$, and $\partial_n : C_nX \to C_{n-1}X$ is the connecting homomorphism of the homology sequence of the triple (X^n, X^{n-1}, X^{n-2}). The skeletal homology $H(CX)$ is naturally isomorphic to the homology $H(X, X^{-1})$.

Proof

We begin by a few remarks. First $H_q(X^{n+k}, X^n) = 0$ for $k \geq 0$ and $n + k < q$, $q \leq n$. This follows by induction on K and by using the homology sequence of the triple

$(X^{n+k}, X^{n+k-1}, X^n)$ with the condition (S2).

Next, $H_q(X, X^n) = 0$ for $q \leq n$, as follows from the condition (S1) and the preceding remark. Finally $H_n(X^{n+1}, X^{-1}) \to H_n(X, X^{-1})$ is an isomorphism for all $n \geq 0$ as follows from the homology sequence of the triple (X, X^{n+1}, X^{-1}) and the preceding remark.

We consider now the following diagram defining the boundary homomorphisms ∂_n and where the rows and columns are homology sequences of triples.

$$\begin{tikzcd}
& H_{n+1}(X^{n+1}, X^n) \arrow[d, "\phi"'] \arrow[dr, dashed, "\partial_{n+1}"] & & H_{n-1}(X^{n-2},X^{-1})=0 \arrow[d] \\
0=H_n(X^{n-1},X^{-1}) \arrow[r] & H_n(X^n,X^{-1}) \arrow[r] \arrow[d, "\psi"'] & H_n(X^n,X^{n-1}) \arrow[r] \arrow[dr, dashed, "\partial_n"'] & H_{n-1}(X^{n-1}, X^{-1}) \arrow[d] \\
& H_n(X^{n+1}, X^{-1}) \arrow[d] & & H_{n-1}(X^{n-1}, X^{n-2}) \\
& H_n(X^{n+1}, X^n) = 0 & &
\end{tikzcd}$$

Obviously, $\partial_n \partial_{n+1} = 0$ since the middle row is exact.

$\ker \partial_n \cong H_n(X^n, X^{-1})$, $\operatorname{im} \partial_{n+1} \cong \operatorname{im} \phi$,

$H_n(CX) \cong \operatorname{coker} \phi \cong \operatorname{im} \psi \cong H_n(X^{n+1},X^{-1}) \cong H_n(X,X^{-1})$.

It remains to show that ∂_n is actually the connecting homomorphism of the sequence of the triple (X^n,X^{n-1},X^{n-2}).

$H_n(K^n, K^{n-1})$, $n \geq 0$ with the free abelian group on the set of n-cells.

Let e be an n-cell of K and f its characteristic map. From the proof of the preceding theorem, one can see that f_* maps the generator (say 1) of $H_n(D^n, S^{n-1})$ onto the generator e of $H_n(K^n, K^{n-1})$. Therefore, considering the diagram

$$\begin{tikzcd}
H_n(D^n, S^{n-1}) \arrow[r, "\partial_*", "\cong"'] \arrow[d, "f_*"'] & H_{n-1}S^{n-1} \arrow[d, "(f|S^{n-1})_*"] \arrow[dr, dashed, "j_*"] & \\
H_n(K^n, K^{n-1}) \arrow[r, "\partial_*"] \arrow[rr, bend right, "\partial_n"'] & H_{n-1}K^{n-1} \arrow[r] & H_{n-1}(K^{n-1}, K^{n-2})
\end{tikzcd}$$

it follows that $\partial_n(e) = j_*(1) = \sum_{|\lambda|=n-1} a_\lambda e_\lambda$, where a_λ is the degree of $f|S^{n-1}$ on e_λ. More precisely, a_λ is the degree of the composite map

$$S^{n-1} \xrightarrow{f} K^{n-1} \to K^{n-1}/K^{n-2} \to S^{n-1},$$

since $H_{n-1}(K^{n-1}, K^{n-2}) \cong \tilde{H}_{n-1}(K^{n-1}/K^{n-2})$, K^{n-1}/K^{n-2} is the wedge of (n-1)-spheres (the closure of all (n-1) cells being identified to a point), and the last map identifies all spheres but the λ-th one to a point.

As an application of this remark, we shall show that there is always a CW-complex with preassigned homology. This illustrates the fact that CW-complexes form a reasonably large category.

PROPOSITION 5.7. Given a sequence $\{A_n\}$ $n \geq 0$ of abelian groups, with A_0 free, there exists a CW-complex K, such that $H_n K = A_n$, $n \geq 0$.

Proof

Given a free abelian group F with basis $\{x_\lambda\}_{\lambda \in \Lambda}$ the wedge of spheres $\bigvee_{\lambda \in \Lambda} S^n$ with the weak topology with respect to the spheres is a CW-complex, and $H_q(\bigvee_\lambda S^n_\lambda) = F$ if $q = n$ and 0 otherwise. Now, any abelian group A_n can be represented as a quotient of a free abelian group F, i.e. we have a short exact sequence

$$0 \to N \xrightarrow{\phi} F \xrightarrow{\psi} A_n \to 0$$

where F has a basis, say $\{x_\lambda\}_{\lambda \in \Lambda}$ and N being also free has a basis $\{y_\mu\}_{\mu \in M}$. Therefore, by 5.4, if we can find a family of maps $\{f_\mu : S^n \to \bigvee_{\lambda \in \Lambda} S^n_\lambda\}_{\mu \in M}$ such that the boundary homomorphism ∂_{n+1} of the skeletal chain complex of $K = \left(\bigsqcup_{|\mu|=n+1} D^{|\mu|} \right) \cup_{(f_\mu)} \left(\bigvee_{\lambda \in \Lambda} S^n_\lambda \right)$ is equal to ϕ, then

$\tilde{H}_q K = A_n$ if $q = n$ and 0 otherwise.

Since $\phi(y_\mu) = \sum_\lambda a^\lambda_\mu x$, f_μ must be chosen so that its degree on x_λ is a^λ_μ, and the problem is reduced to the existence of such maps.

For a given μ, $a^\lambda_\mu = 0$ for all but finitely many λ's, say $\lambda_1, \ldots, \lambda_r$. Construct $f_\mu : S^n \to \bigvee_{\lambda \in \Lambda} S^n_\lambda$ by composition of the following maps. Divide S^n into r lunes by r half meridians and identify these meridians to a point. This gives a map $S^n \to \bigvee_{i=1}^{r} S^n_{\lambda_i}$ which is of degree 1 on every $S^n_{\lambda_i}$. Construct $\bigvee_i S^n_{\lambda_i} \to \bigvee S^n_{\lambda_i}$ by taking $S^n_{\lambda_i} \to S^n_{\lambda_i}$ of degree $a^{\lambda_i}_\mu$ for $i = 1, \ldots, r$, and finally include $\bigvee_{i=1}^{r} S^n_{\lambda_i}$ into $\bigvee_{\lambda \in \Lambda} S^n_\lambda$. Then the composite f_μ will have the degree a^μ_λ on S^n_λ.

Chapter VI

COMPARISON WITH STEENROD'S CATEGORY OF COMPACTLY GENERATED SPACES

We would like to look at the category CW of CW-complexes and continuous maps in the light of Steenrod's paper [15] on the category CG of compactly generated spaces (k-spaces) and continuous maps, that is we would like to see just how "convenient" CW is (cf. with Steenrod's own result at the end of his paper [15]).

A compactly generated space is a Hausdorff space which has the weak topology with respect to its compact sets, or equivalently a Hausdorff space which is the direct limit of the family of its compact subspaces.

A compactly generated space is characterized by the following universal property: if $X \in CG$, Y is Hausdorff and $f : X \to Y$ a function which is continuous on each compact set, then f is continuous.

CG contains all locally compact and all first countable (in particular all metrizable) spaces. With the usual definitions of subspace, product and function spaces, CG does not satisfy the conditions given by Steenrod for a convenient (for algebraic topology) category of topological spaces. Examples are given to show that a subset of a CG space, with induced topology, which is not closed or open, need not belong to CG, that the Cartesian product of two CG spaces need not belong to CG and that if X and Y are in CG, the function space C(X, Y) with compact-open topology need not be in CG. To overcome these difficulties, Steenrod defines a functor k (the retraction functor) from the category H of Hausdorff spaces and continuous maps, onto CG. If $X \in H$, then k(X) is the set X with the weak topology with respect to the family of compact subsets of the original space X. If $f : X \to Y$ is a mapping in H, then k(f) is the same function $k(X) \to k(Y)$. k is the coadjoint to the inclusion (i.e. if $X \in CG$ and $Y \in H$ and i is the inclusion $CG \to H$, then $\hom_H(i\,X, Y) \approx \hom_{CG}(X, kY)$). k refines the topology on a Hausdorff space but does not change the family of compact sets, and consequently does not change the homology.

CW is a full subcategory of CG. We have

$$\begin{array}{ccccc} C & & \subset & & H \\ \cup & & & & \cup \\ CW & \subset & W & \subset & CG \end{array}$$

(where C(W) is the category of C(W)-complexes and continuous maps) and these inclusions are proper.

We shall now consider separately the conditions proposed by Steenrod.

Subcomplexes and Quotient Complexes

There is no difficulty here. Every subcomplex and every quotient complex of a CW-complex is a CW-complex.

Products

The Cartesian product of two compactly generated spaces need not be compactly generated. Steenrod defines then the product of two CG spaces X and Y by $X \times_k Y = k(X \times Y)$. This turns out to be the categorical product in CG.

As in CG, we have seen that the Cartesian product of two CW-complexes need not be a CW-complex: One can see that the functor k maps the Cartesian product of two CW-complexes into CW.

PROPOSITION 6.1. (cf. [3] p. 98) If K and L are CW-complexes then $K \times_k L$ is a CW-complex.

We shall prove this a little later.

COROLLARY 6.2. [3]. The Cartesian product $K \times L$ of two CW-complexes K and L is a CW-complex iff $K \times L \in CG$.

Note that the categorical product in CG is the categorical product in CW.

The Cartesian product of two CG spaces X and Y is compactly generated if X or Y is locally compact. To compare with this in CW:

PROPOSITION 6.3. If K and L are CW-complexes and either K or L is locally finite or if K and L are locally countable* then the Cartesian product $K \times L$ is a CW-complex.

---

*) A CW-complex is locally countable if every point has a neighbourhood which is contained in a countable subcomplex.

Proof

The first part is due to Whitehead [17]. J. Milnor [10] shows that the Cartesian product of two countable CW-complexes is compactly generated and thus a CW-complex. This result is extended to locally countable CW-complexes by Cooke and Finney [3].

As in CG, the first part of the above result shows that the concept of homotopy is unchanged in CW.

If $f : X \to Y$ is an identification, $X \in CG$ and Y is Hausdorff, then $Y \in CG$. In CW we have a similar result: If $f : K \to L$ is an identification, $K \in CW$ and L is any cell complex, then L is a CW-complex iff $L(f(\overline{e}))$ is finite for every $e \in K$ (cf. 4.18).

The Cartesian product of two identifications need not be an identification. An example due to Dowker can be found in [5]. We have observed (cf. 2.8) that a cell complex is a W-complex iff the map induced by its characteristic maps is an identification. However Steenrod shows [15] that the categorical product in CG of two identifications is an identification. By 6.1, the same proposition holds for CW, that is:

PROPOSITION 6.4. If $f : K_1 \to L_1$ and $g : K_2 \to L_2$ are identifications in CW, then $f \times_k g : K_1 \times_k K_2 \to L_1 \times_k L_2$ is an identification.

Steenrod's result about products of identification in CG mentioned above also yields a proof of 6.1:

Proof of 6.1.

Suppose K and L are CW-complexes, and f and g the identifications induced by their characteristic maps, (cf. 2.8). Then we know by 2.8 that $K \times_k L$ is a CW-complex iff $f \times_k g$ is an identification, and it is one by Steenrod.

Function Spaces

Since the function space C(X, Y) with compact-open topology is Hausdorff iff Y is Hausdorff, we see that if X and Y are CG spaces, $Y^X = k(C(X, Y))$ is compactly generated. Thus the functor k works well in CG, and Steenrod shows that for any spaces X, Y, Z in CG,

(1) $(Y \times_k Z)^X = Y^X \times_k Z^X$

(2) $Z^{Y \times_k X} = (Z^Y)^X$

(3) the composition of mappings $X \to Y \to Z$ is a continuous function $Z^Y \times_k Y^X \to Z^X$.

As in CG, if K and L $\in$ CG, C(K, L) need not be in CW. C({0,1}, CI) gives us an example, since this is homeomorphic to CI × CI. Moreover if K, L $\in$ CW, C(K, L) does not seem to have a natural cell decomposition, so that CW is not closed with respect to the function spaces.

To our knowledge, the best results in this direction are the following:

6.5. (Milnor [11]). If Y has the homotopy type of a CW-complex and X is compact, then $C(X, Y)$ has the homotopy type of a CW-complex.

6.6. (Hyman [9]). If K is a CW-complex and X is compact metric, then K^X is an M-space (cf. Hyman's paper for this definition).

Neighbourhood Deformation Retracts

Steenrod calls a closed subspace A of a CG space X a neighbourhood deformation retract (NDR) if there exists a mapping $u : X \to I$ and a homotopy $h : X \times I \to X$ such that $A = u^{-1}(0)$, $h(x, 0) = x$ for all $x \in X$, $h(x, t) = x$ for all $x \in A$ and $t \in I$ and $h(x, 1) \subset A$ for all x such that $u(x) < 1$.

If A is an NDR of X, (X, A) is called an NDR pair.

Steenrod considers then the category of NDR pairs, shows that every such pair has the homotopy extension property and that this category is closed under the formation of products and adjunction spaces.

We shall now show that the category of CW-pairs (K, L), where K is a CW-complex and L a subcomplex of K (with cellular maps of pairs, i.e. mapping n-skeletons into n-skeletons for all n) is actually a subcategory of the category of NDR pairs, and then derive for this category of CW pairs a few results analogous to those of Steenrod.

PROPOSITION 6.7. Every CW pair is an NDR pair.

Proof

Whitehead [17] proved that the inclusion of subcomplex into a CW-complex is a cofibration. The result follows by Theorem 7.1 of [15].

COROLLARY 6.8. A CW pair (K, L) has the following equivalent properties

(i) $K \times 0 \cup L \times I$ is a deformation retract of $K \times I$.

(ii) $K \times 0 \cup L \times I$ is a retract of $K \times I$.

(iii) (X, A) has the homotopy extension property.

Proof

Follows from 6.7 and [15] Theorem 7.1.

THEOREM 6.9. Let (K, L) be a CW pair, Y a CW-complex and $h : L \to Y$ a cellular map. Then $(Y \cup_h K, Y)$ is a CW pair.

Proof

We first show that $Z = Y \cup_h K$ is a cell complex. The (open) cells of Z are those of Y and those of K - L with the following characteristic maps. If $\{f_\lambda\}_{\lambda \in \Lambda}$ ($\{g_\mu\}_{\mu \in M}$) are the characteristic maps of the cells of Y (K - L) then the characteristic maps of Z are $\{\bar{i}\, f_\lambda\}_{\lambda \in \Lambda}$ and $\{\bar{h}\, g_\mu\}_{\mu \in M}$, where $\bar{i}$ and $\bar{h}$ are the maps of the pushout diagram

$$\begin{tikzcd}
L \arrow[r, "h"] \arrow[d, hook] & Y \arrow[d, hook, "\bar{i}"] \\
K \arrow[r, "\bar{h}"] & Z
\end{tikzcd}$$

Since h is cellular, the boundary condition for the cells of $K - L$ is satisfied.

Let $p : Y \sqcup K \to Z$ be the canonical projection. Since $Y \sqcup K$ is a CW-complex, Z a cell complex and p an identification, by 4.18, K is a CW-complex if $Z(p\bar{e})$ is finite for every cell of $Y \sqcup K$. If $e \subset Y$, then $Z(p\bar{e}) = Y(\bar{e})$ is finite, since Y is a CW-complex which can be identified to a subcomplex of Z. If $e \subset L$, $p(\bar{e})$ is a compact subset of Y and thus contained in a finite subcomplex. If $e \subset K - L$, $\bar{e}$ meets only finitely many cells of K and therefore finitely many of $K - L$ and finitely many of L. Since $p(\bar{e} \cap L)$ meets only finitely many cells of Y, we see that $p(\bar{e})$ will meet only finitely many cells of Z, which is equivalent to saying that $Z(p\bar{e})$ is finite by 3.1.

Therefore we see that this category of CW pairs is closed under adjunction and product, provided that we use the categorical product (cf. below 6.2).

REMARK 6.10. For CW-complexes, the NDR property as defined above is actually equivalent to the ordinary notion of a strong deformation retract of a neighbourhood, as defined for instance in [5]. This follows from Theorems 7.1 of [15] and XV. 7.4 of [5].

A CW pair is also a collared pair in the sense of Greenberg [6].

Filtered Spaces

Let $X = \bigcup_{n=0}^{\infty} X_n$ be a filtered space (cf. Theorem 4.19 for the definition) such that every X_n is a CW-complex. Since each (X_{n+1}, X_n) is a CW pair, it is an NDR pair and by Theorem 9.4 [15], X is Hausdorff and therefore a cell complex with the obvious cell decomposition. Actually X is a CW-complex by 4.2 (2): Let $F \subset X$ be such that $F \cap L$ is closed in K for every finite subcomplex L of X. For any n, let M be a finite subcomplex of X_n. Since X_n is imbedded in X, M is a finite subcomplex of X. Thus $F \cap M = (F \cap X_n) \cap M$ is closed in X and hence in X_n. Since X_n is a CW-complex, $F \cap X_n$ is closed in X_n. This is true for every n. Consequently F is closed in X since X has the weak topology with respect to the family $\{X_n\}_{n \geq 0}$. We have therefore proved:

THEOREM 6.11. <u>A space</u> X <u>filtered by CW-complexes</u> X_n <u>is a CW-complex and</u> (X, X_n) <u>is a CW pair for all</u> $n \geq 0$.

In Hyman's terminology [9], we see that the category of CW-complexes is closed under adjunction and weak union. Hyman's category of M-spaces is the smallest such category containing all metric spaces. It contains all CW-complexes and is contained in the category CG of Steenrod.

Our last result will be concerned with filtrations of an identification space. In particular, if X is filtered by CW-complexes X_n and if L is a subcomplex of X, then one can see that X/L is filtered by the CW-complexes $X_n/X_n \cap L$. This will follow from the following more general result.

THEOREM 6.12. Let X be a space filtered by CW-complexes X_n, K a cell complex and $f : X \to K$ an identification such that $K(f\bar{e})$ is finite for every cell e of X. Then K is filtered by the CW-complexes $K(fX_n)$.

Proof

First, X is a CW-complex by 6.11, K and every $K_n =$ $= K(f\,X_n)$ are CW-complexes by 4.18. The K_n's form an expanding sequence of closed subspaces of K and it is therefore sufficient to prove that the topology of K is coherent with $\{K_n\}_{n\geq 0}$. Let $F \subset K$ be such that $F \cap K_n$ is closed in K_n for all $n \geq 0$. Let L be a finite subcomplex of K. Then $L \subset K_n$ for some n and $F \cap L = F \cap K_n \cap L$ is closed in K_n and thus in K. This implies that F is closed in K since K is a CW-complex, proving that the topology of K is the weak one with respect to the K_n's.

APPENDIX

Remarks

1. A CW-complex K, being completely regular, can be embedded in some cube, that is, in some Cartesian product of the unit interval $I = [0, 1]$, (see [5]).

2. A CW-complex K can be embedded in the Hilbert cube I^∞ iff K is locally finite and countable.

Proof

If K is locally finite and countable, then $K = \bigsqcup_{\alpha \in A} K_\alpha$, where K_α is locally finite and connected and thus 2° countable, and A is a countable set. Thus K is 2° countable and so is homeomorphic to a subspace of I^∞, (see [5], p. 195). Conversely, I^∞ is metrizable and separable and so is every subspace of I^∞, [5]. But a CW-complex which is metrizable and separable is locally finite and countable by 4.29 and 4.15.

We thank Dr. H. Helfenstein for suggesting the above remarks.

3. If a cell complex $K \subset R^n$ is a subset of R^n with the induced topology, then K is a CW-complex iff K is locally finite.

Proof

The sufficiency follows from Proposition 4.26. Proposition 4.29 and the fact that R^n and every subspace of R^n is metrizable proves the necessity.

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