INTER-TEMPORAL OPTIMAL CONSUMPTION - SAVING ALLOCATION UNDER UNCERTAINTY: AN ECONOMIC APPLICATION OF STOCHASTIC OPTIMAL CONTROL THEORY

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CURRICULUM STUDIORUM

Moses O. Odaro was born September 13, 1939, in Benin City, Nigeria. He received the Bachelor of Science degree in Economics from the University of Nigeria, Nsukka, Nigeria, in 1965, and the Master of Arts degree in Economics from McGill University, Montreal, Canada, in 1969. The title of his thesis was The Treatment of Capital Problem in Economic Development Literature.

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ABSTRACT OF [1]

The problem of intertemporal optimal consumption-saving allocation under uncertainty (a stochastic extension of Ramsey's original deterministic problem) is studied within the framework of stochastic control theory. First, existing literature on the problem is surveyed extensively. This survey unveils a number of important issues which must always be remembered in any study of this type, but which have so far not been integrated. These issues relate to the types, sources and "measurement" of uncertainty and their respective impacts on the optimality conditions of the models.

At the end of this extensive survey, argument is advanced concerning the best tool to employ in the analysis of the problem. First, stochastic control theory is seen as naturally suggesting itself. Next, among the tools of stochastic control theory, the stochastic maximum principle is seen as most suited, not only because of (1) its more general mathematical validity (e.g., as compared with the method of dynamic programming); and (2) its wealth of qualitative properties (which are always attractive to the economist); but also (3) because of the success already achieved at the deterministic level in employing and interpreting the Pontryagin maximum principle as a theorem in economics.

In the literature of stochastic optimal control, there have been two attempts to directly develop stochastic counterpart from the Pontryagin's maximum principle. One attempt made by Kushner was developed for fairly general transition equations except that only additive noise is admitted. The second which is due to Sworder takes care of multiplicative noise, but his transition equations are less

1 M.O. Odaro, Intertemporal Optimal Consumption-Saving Allocation Under Uncertainty: an Economic Application of Stochastic Optimal Control Theory, Unpublished Doctoral Dissertation, University of Ottawa, 1971.

general than those of Kushner. Compared with the generality of stochastic differential equations or the multiplicity of ways uncertainty may be admitted in practical problems, the existing two versions of the stochastic maximum principle constitute only a starting point for the development of a full fledged theory. However, of the two versions, Sworder's seems of special interest in view of the predominance of multiplicative noise not only in the subject-matter of this thesis, but also in several other economic problems.

The economic problem considered earlier in the study is therefore reformulated in terms of the Sworder stochastic maximum principle. The economic interpretations of the principle seem straightforward extensions of their deterministic counterparts, but the results are nevertheless quite interesting. For instance, by drawing on established relationships between dynamic programming and the maximum principle it becomes easy to see the adjoint function of the latter as a marginal indirect utility function and the maximum of the Hamiltonian as the change in the indirect utility with respect to time, holding all other variables constant. A stochastic counterpart of the famous Keynes-Ramsey rule derives almost trivially from the maximum principle independently of dynamic programming.

However, immense computational difficulties are encountered once the criterion functional is nonquadratic. This threatens to deny one the advantages of qualitative properties which closed form solutions generally afford. However, the Sworder stochastic maximum principle presents no computational difficulties when applied to economic problems with quadratic criteria or to nonquadratic criteria when the two step solution suggested in this thesis can be utilized. The advantage of the procedure is that it does not require the restrictive assumptions needed by the certainty equivalence principle.

CHAPTER I

INTRODUCTION

Economists have long recognized that any optimal growth theory which assumes away real-life factors of uncertainty is only a first approximation. Even in his seminal article of 1928, F.P. Ramsey [108, P. 549] did recognize that "the most serious factor" he was neglecting was "the possibility of future wars and earthquakes destroying our accumulation." Yet, writing forty-one years (1969) after Ramsey, Levhari and Srinivasan [81] could cite only two papers[1] as attempts that have been made to introduce uncertainty explicitly into the Ramsey-type model. This is rather surprising, especially when one realises that on one hand, there has been a great outburst of literature on the Ramsey-type problem since the 1950's and on the other hand, there have been important developments in the economics of uncertainty following the works of von Neumann and Morgenstern [140] and Arrow [3] .

An obvious need, therefore, has been that of integrating optimal growth theory with the economics of uncertainty. This need was recognized recently by Dobell [28, P. 47] when he emphasized the great importance that awaits stochastic control theory in future economic theory. The present work explores <u>one</u> aspect[2] of economic theory, namely, the problem of intertemporal optimal consumption-saving allocation under uncertainty, in which stochastic control theory

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1 Mirrlees [90] and Phelps [100] . However, Phelps refers to an unpublished paper by Beckmann [12] .

2 Of course, the present work is only one of several possible economic applications of stochastic control theory. Some other examples are mentioned briefly at the end of chapter VII. In principle, the tools of stochastic control theory are capable of handling <u>any</u> problem of dynamic optimization under uncertainty.

may be expected to play that future important role.[3]

The work may be visualized as consisting of two parts, the first of which presents the economic problem; and the second of which describes the technical tools by means of which the problem may be analysed. In this vein, part one is made up of chapters I through V, and chapters VI through IX constitute part two.

Part one is essentially a survey of relevant economic literature on the problem of intertemporal optimal consumption saving allocation models under uncertainty. The subject-matter covered logically belongs to the theory of personal saving.

Although the classic Ramsey model and what later became its offspring the theory of optimal economic growth-naturally belong to the field of macroeconomics, almost all the attempts so far made to generalize the theory by introducing uncertainty have been micro-economic in character. Phelps [100], Levhari-Srinivasan [81], and others to be discussed later, all take the planning agent as an individual consumer planning his consumption-saving allocation over a certain interval $[t_0, t_1]$ which may be finite or infinite.

The introduction of uncertainty into the model immediately brings to mind a number of questions such as those relating to the types, sources and measurement of uncertainty, and their respective

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3 There now exists a substantial amount of literature applying deterministic optimal control theory to a variety of economic problems, especially optimal growth theory. For example, see Arrow [6], Burmeister and Dobell [20], Dorfman [31] and Shell [120].

impacts on optimality. Each of these questions is the subject of an individual chapter.

In addition to Phelps and Mirrlees a number of important articles have been produced in the last few months. These include an unpublished paper by Dreze and Modigliani [35] and published papers by Samuelson [113] , Merton [88] , Fama [36] , Hakansson [57] and Hakansson and Liu [59] . Each of these papers has an important contribution of its own.

Samuelson [113] and Merton [88] , in a pair of companion papers treat the Ramsey-Phelps-Mirrlees-Levhari-Srinivasan problem showing the latter as special cases of their own more general model which simultaneously treats both optimal saving and portfolio problems. Since these models will be discussed in detail in chapter III, nothing more will be said about them at this point.

Nearly all of the above concentrate on the problem of "capital risk" as opposed to that of "income risk". "Capital risk" relates to uncertainty over the yield of capital investment; whereas "income risk" relates to uncertainty about future wage income. The effects on optimality of these two sources of uncertainty is the subject-matter of Sandmo [116] . Dreze and Modigliani also treat the two problems, but they as well as Sandmo deal only with two-period models.[4]

By assuming statistical independence for the rate of returns to capital in different periods, Samuelson [113] , Merton [88] , Phelps [100]

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However, according to Fama [36] , this is not necessarily restrictive, since under "fairly" general conditions, multi-period models can always be reduced to two-period models. See proposition V.4. (chapter V) below.

and others are able to demonstrate the so-called "separation theorem"[5] of portfolio theory which states in essence that savings and portfolio decisions can be taken independently of each other. Hakansson and Liu [59] address themselves to the problem of the derivation of optimal properties for consumption-investment paths as well as dynamic portfolio selection under uncertainty when statistical independence of yields cannot be assumed. By employing some powerful tools from the theory of stochastic processes, Hakansson and Liu are able to demonstrate that the "separation theorem" as well as "myopia"[6] can still be proved.

In terms of the types of uncertainty admissible in these models, the most complete work so far done is that of Dreze and Modigliani [35]. While Fama [36, P. 166] concerns himself with what may be called a once-and-for-all decision problem of "timeless" uncertainty, all of the other writers above cited deal with the problem of "temporal" or delayed uncertainty. But no where, except in Dreze and Modigliani [35] is the distinction between these two types of uncertainty carefully drawn out.[7] Since this is the subject of the next chapter no further discussion of the distinction is called for at this point.

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5 The " separation theorem" was first proved by Tobin [137] in a static framework.

6 "Myopia" refers to the conditions under which it is optimal to base each period's portfolio decision on that period's saving and probability distribution of yields only, disregarding the future completely.

7 See, however, Mossin [92].

An introduction to the problem of the required technical tools is given in the first part of chapter VI. The second part of chapter VI catalogues definitions, propositions, and theorems from the theory of probability and stochastic processes that are needed in various parts of this study. The problem of the classification of stochastic control systems is taken up in chapter VII, and chapter VIII ties the stochastic maximum principle of Kushner [72 , 73] and Sworder [126 - 130] with the well-known deterministic maximum principle of Pontryagin et. al. [104] .

Chapter IX is an economic application and interpretation of the tools of chapter VIII: the problem of chapter III is reformulated by means of the new tools.

The main results are discussed in sections 2 and 3 of chapter IX. Section 2 gives economic interpretations to the major relations which comprise the stochastic maximum principle. Utilisation of established relationships between dynamic programming and the maximum principle makes it particularly easy to derive interesting results. Section 3 deals with several of the computational difficulties involved in an attempt to obtain explicit solutions for the three equations representing optimal control, and the corresponding trajectories and adjoint function. The difficulties encountered tend to spring mainly from the nonquadracticity of the criterion function.

The main strands of the entire study are collected together in the concluding chapter X. Some suggestions for further research are also given.

CHAPTER II

TYPES OF UNCERTAINTY

As mentioned in chapter I, two different types of uncertainty situations may be distinguished from the economic literature dealing with the problem of intertemporal optimal consumption-saving decisions under uncertainty. These two types of situations have been labelled as "timeless" uncertainty and "temporal" or "delayed" uncertainty by Dreze and Modigliani [35] . Mossin [92, p. 172] , began his explanation of this distinction, in the following way. " When someone asks you which of a set of uncertain prospects you prefer you should answer: that depends upon when the outcomes will become known. "

One way to capture the import of this distinction is to imagine yourself in a situation in which you have to take a decision about some problem the result of which depends on some event taking place in the future. Naturally, until the outcome of that future event is known, you are uncertain what the result of your decision will be. If the uncertainty about the event is to be removed before you take your decision, the situation you are in is said to be that of "timeless" uncertainty. Whereas, if the uncertainty about the event is not to be removed until after you have taken your decision, your situation is that of "temporal" uncertainty. Mossin [92, p. 174] gives the example of a professor who is "indifferent between teaching course A and course B next fall, but (who) in view of the preparations that must be made in the meantime, would certainly not want the decision to be postponed until classes begin" .
As was stressed by Markowitz [86, chapters 10-11] an important feature distinguishing "temporal" from "timeless" uncertainty is the possibility of "intervening decisions" in the former case and the lack of it in the latter The intervening decision in the above example is to prepare

either course A or B or both.

Another way to appreciate the difference between the two types of uncertainty is to visualize the situation as that of a game between the decision maker (yourself) and "mother nature" (chance). The distinction then lies in who has the first move: if "chance" has the first move, the situation is that of timeless uncertainty (e.g., lottery); whereas, if you have the first move, the situation is that of temporal uncertainty (e.g. life insurance policy).

While the importance of the distinction between the two types of uncertainty was noted by Markowitz as early as 1959, it remained almost unnoticed in the literature on intertemporal optimal consumption-saving under uncertainty until the recent work of Dreze and Modigliani. For instance, Phelps [100], Levhari and Srinivasan [81], Samuelson and Merton [113,88] and others to be discussed in chapter III, never stated explicitly which of the two types of uncertainty they were interested in. Yet, as was emphasized by Mossin [92, p. 172] the importance of the distinction is not only because it affects the choice of a decision maker, "but more fundamentally because it has to do with the question of whether or not it is possible to represent such choices by means of a utility function."

The Dreze-Modigliani analysis as well as that of Mossin [92] was confined to a two-period framework. It is not clear whether the distinction carries over to the multi-period framework.[1] The remainder of this chapter will, therefore, be confined to a two-period analysis, and will rely to a large extent on Dreze and Modigliani [35] which is the most meticulous analysis of the distinction so far available.

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[1] See, however, Fama [36], and section 2 of chapter IX below.

1. Timeless and Temporal Uncertainty in a Two-period Consumption Model.

In order to understand the meaning and implication of the distinction for subsequent analysis, consider the problem faced by a consumer who must allocate his total wealth y,

$$y = y_1 + y_2 (1 + r)^{-1} \tag{2.1}$$

between a flow of current (or period one) consumption c_1 and a residual stock $y - c_1$ out of which future consumption c_2 (including bequests) will be financed. In equation (2.1)

y_1 is the net market value of the consumer's assets, plus his labour income in period one;

$y_2(1 + r)^{-1}$ is the present value of his future labour income and receipts from sources other than his period one assets (e.g. inheritance);

r is the real interest rate; and

c_2 is defined by

$$c_2 = (y \quad c_1)(1 + r) = (y_1 - c_1)(1 + r) + y_2 \tag{2.2}$$

It is assumed that in the period one y_1 is known with certainty, but y_2 and r are unknown. Temporal uncertainty is defined as the situation which results if the uncertainty about y_2 and r is removed only at the end of period one; i.e., after c_1 has been chosen. On the other hand, the situation is that of timeless uncertainty if the consumer knows that the uncertainty about y_2 and r will be completely removed before the choice of c_1 is made.

The first question that may be examined with regard to the two types of uncertainty is the consumer's expected utility in each case. This is important since for the problem at hand, optimality in uncertainty situations is often determined in terms of the maximisation of expected utility subject to given constraints. For the present purpose, assume $r = r^0$ is non-stochastic, so that the only random variable in the situation is y_2, having defined c_2 by equation (2.2). Having done this, define the expected utility associated with temporal uncertainty as J_1 and that associated with timeless uncertainty as J_2, where

$$J_1 = \max_{c_1} \int u\{c_1, (y_1-c_1)(1+r^0)+y_2\} \, d\Phi(y_2, r^0) \quad (2.3)$$

$$J_2 = \int \max_{c_1} u\{c_1, (y_1-c_1)(1+r^0)+y_2\} \, d\Phi(y_2, r^0)$$

$$= \int V\{y_1 + y_2(1+r^0)^{-1}, r^0\} \, d\Phi(y_2, r^0). \quad (2.4)$$

$V(\,.\,, r^0)$ is the cardinal indirect utility function associated with $u(c_1, c_2)$, and $\Phi(\,.\,, r^0)$ is the consumer's subjective probability distribution function for the uncertain prospect (y_2, r^0).

In words, what equations (2.3) and (2.4) are saying is that in the case of temporal uncertainty, the consumer maximizes over c_1, the expected value of the utility derivable from c_1 and c_2. In the case of timeless uncertainty, the consumer does approximately the reverse. He first maximizes his utility for c_1 and c_2 over c_1, and then takes the expected value of that utility.

Dreze and Modigliani invoked an important theorem of Marschak [87, p. 201, footnote 8] which shows that J_1 is at most equal to J_2,

$$J_1 \le J_2 \quad (2.5)$$

with equality holding if and only if

$$u_1\{c_1^*, (y_1-c_1^*)(1+r)+y_2\} \equiv (1+r)u_2\{c_1^*, (y_1-c_1^*)\} \quad (2.6)$$

i.e., $$u_1(c_1^*, c_2^*) \equiv (1+r)u_2(c_1^*, c_2^*)$$

over the smallest domain D for which

$$\int_D d\,\psi\,(c_2) = 1$$

where $\psi(c_2)$ is the distribution function of the random variable c_2. The only economically relevant situation in which the identity (2.6) holds is under complete certainty with $\frac{dc_1}{dr} = 0$.

Furthermore, it may be noted that $J_2 - J_1$ equals what is known in statistical decision theory as "the expected value of perfect information" (EVPI) ; i.e., in the present case, "the loss to the consumer from lacking information about y_2 when choosing c_1"[2]

The economic interpretation of inequality (2.5) yields

Proposition II.1

> A temporal uncertain prospect is never preferred to the timeless uncertain prospect described by the same mass or density function, no matter what the consumer's utility function may be.[3]

Equation (2.4) shows that under timeless uncertainty, the indirect function $V(y, r^0)$ summarizes all the information required for choice among uncertain prospects. Hence, under timeless uncertainty, both the direct and indirect utility functions are defined. However, under temporal uncertainty, the choice of a prospect and the consumption decision cannot be separated. Problem (2.3) must be solved first and both the ordinal and cardinal properties of $u(c_1, c_2)$ are relevant in

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2 See Dreze and Modigliani [35, pp. 13-14], Marschak [87, pp. 201-205].

3 Dreze and Modigliani [35, p. 12]; their footnote 15 should be noted.

selecting the optimal c_1. It has been stressed by Mossin [92] that under temporal uncertainty the indirect utility function is generally not defined.

2. Timeless, Temporal Uncertainty and the Arrow-Pratt Risk Aversion Measure

The Arrow-Pratt [4, 105] measure of risk aversion provides a convenient way of visualizing the implication of proposition II.1 for optimal behaviour under temporal vis-a-vis timeless uncertainty. The Pratt-Arrow measure corresponds to the case of timeless uncertainty about total wealth y. Hence, if r is given, the Pratt-Arrow absolute risk aversion function is given by

$$-\left(\frac{1}{1+r^0}\right)\frac{V_{yy}}{V_y}\Big|_{r^0} = -\frac{V_{y_2y_2}}{V_{y_2}}\Big|_{r^0} \qquad (2.7)$$

where subscripts denote partial differentiation; e.g.,

$$V_y = \frac{dv}{dy}, \quad V_{y_2y_2} = \frac{d^2V}{dy_2^2}.$$

Dreze and Modigliani showed that at any point in the (c_1, c_2) space, the risk aversion measure that corresponds to the case of temporal uncertainty is given by

$$-\frac{u_{22}}{u_2}\Big|_{c_1} = -\frac{V_{y_2y_2}}{V_{y_2}} + \left[\frac{dc_1}{dy_2}\right]^2 \left[\frac{d^2c_2}{dc_1^2}\right]\Big|_u \qquad (2.8)$$

where

$\frac{dc_1}{dy_2}$ and $V_{y_2y_2}/V_{y_2}$ are evaluated along the Engel curve going through that point; i.e. the Engel curve that corresponds to a rate of interest r such that $1 + r = \frac{u_1}{u_2}$ at that point.[4] If all partial derivatives of third and higher orders vanish, then the second term on the right hand side of equation (2.8) is the EVPI. In that case, the risk aversion index under temporal uncertainty would be the sum of the EVPI and risk aversion index if the same uncertainty were timeless.[5]

With the assumption that marginal propensity to consume lies in the open interval (0, 1) and that indifference curves are strictly convex, the second term on the right hand side of (2.8) is positive, and hence

$$- \frac{u_{22}}{u_2} > - \frac{V_{y_2 y_2}}{V_{y_2}} \qquad (2.9)$$

Since $V_{y_2 y_2}/V_{y_2}$ is the Pratt-Arrow measure of risk aversion in the case of timeless uncertainty, it means that using the latter as a measure of risk aversion under conditions of temporal uncertainty[6] under-estimates the degree of risk aversion. Dreze and Modigliani actually say that inequality (2.9) implies that one's risk aversion increases with time.

Equation (2.8) decomposes risk aversion under temporal uncertainty into a first term, $- V_{y_2 y_2}/V_{y_2}$, which measures purely

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4 See Dreze and Modigliani [35, p. 17].

5 See Dreze and Modigliani [35, p. 18, footnote 23].

6 As done by all writers (except Sandmo [116] and Dreze-Modigliani) to be discussed in chapter V below.

cardinal properties, and a second term $(dc_1/dy_2)^2(d^2c_2/dc_1^2)|_u$, "which measures purely ordinal properties". For "infinitesimal risks", this second term alone measures the EVPI. The second factor of the term, $(d^2c_2/dc_1^2)|_u$ $(= 1/\frac{dc}{dr})$, is a local measure of the curvature of the indifference curves. To see the relevance of the curvature of the indifference curves in comparing timeless and temporal uncertainty, consider the following two extreme situations:

(1) Indifference curves are approximately linear in the neighbourhood of the equilibrium point: in this case, c_1 and c_2 are almost perfect substitutes, and $(d^2c_2/dc_1^2)|_u = 0$. EVPI is therefore approximately zero.[7] For such a consumer, there is hardly any difference between temporal uncertainty and timeless uncertainty.

(2) Indifference curves approximate to right angles in the neighbourhood of the equilibrium: in this case, c_1 and c_2 are almost perfect complements. $(d^2c_2/dc_1^2)|_u$ is very large and, therefore, dc_1/dr is very small. For such a consumer, there is a big difference between temporal and timeless uncertainty.

Equation (2.8), therefore, shows that aversion to temporal uncertainty grows with the curvature of the indifference curves.

To understand the role of the second factor, dc_1/dy_2, also consider the two extreme cases: $dc_1/dy_2 = 0$, and $dc_1/dy_2 = 1$. $dc_1/dy_2 = 0$ means that optimal c_1 can be chosen without reference to y_2, so that perfect information is again worthless. $dc_1/dy_2 = 1$ means

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[7] In limit, the indifference curves are complete straight lines; therefore curvature is zero and EVPI is zero, implying that perfect information is completely worthless.

that the individual wants to consume all his wealth now (in period one) because he derives no utility from future (period two) consumption.[8]

Although the foregoing analysis has been based on the assumption of given r^0 and random y_2, a similar analysis could be undertaken with the assumption of given y_2^0 and random r.

However, Dreze and Modigliani showed that in this case, the "inferiority of temporal over timeless uncertainty" is not as great as in the case in which y_2 is the random variable. In view of the fact that most of the work dealt with in chapter III assume random rate of return rather than random income, this remark should be consolatory.[9] But with both y_2 and r assumed random, the deviation of $-{}^{u}22/u_2 \mid c_1^*$ from $-{}^{V}y_2y_2/{}^{V}y_2$ should be greater than shown by equation (2.8).[10]

The behaviour of $-{}^{u}22/{}^{u}2$ is summarized by

<u>Proposition II.2</u>

Risk aversion under temporal uncertainty will be lower:

(i) the lower is risk aversion under timeless uncertainty,

(ii) the larger is the marginal propensity to consume and/or the responsiveness of c_1 to a compensated change in r;

(iii) the lower is the substitution effect (in absolute term) of a change in interest rate on his current consumption.[11]

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8 An example of such an individual is a consumer who is sure to die in period one, and does not care to bequeath anything to his heirs.

9 See footnote 6 above.

10 Details of this are given in Dreze and Modigliani [35, footnote 33].

11 <u>ibid</u>., p. 24.

A corresponding measure to the Pratt-Arrow relative (proportional) risk aversion could also be derived for the foregoing case of temporal uncertainty. However, the difficulty here is that whereas the Pratt-Arrow utility function has only one argument, namely wealth (in the above case y_2), the utility function in the case of temporal uncertainty has at least two arguments, namely, c_1 and c_2, and so, there is the problem of which of them to use as weight for the risk aversion measure. One can however conclude that the relative risk aversion index for temporal uncertainty is also greater than that for timeless uncertainty provided $c_1, c_2 > 0$.

3. Summary and Conclusions

The foregoing discussion should serve as a reminder of the constant need for caution in employing the Arrow-Pratt measure of risk aversion in the explanation of the impact of uncertainty on optimal saving-consumption decisions. Inequality (2.9) shows that under temporal uncertainty, the Pratt-Arrow measure of risk aversion may be an underestimate. Yaari in a recent article [152] derives the risk aversion measure using the more general "states of nature" approach to decision making under uncertainty, and showed the Pratt-Arrow measure as a special case of his own more general measure.

As Dreze and Modigliani have warned, while part (ii) of proposition II.2, may be used to confirm empirical observation that people are high savers because they are exposed to uncertainty, the causal relation could equally well run in the other direction, namely: they are exposed to risk because they are high savers. Hakansson [56, p. 457] seems to have come close to this conclusion namely, that no simple relationship can be drawn between risk aversion index and optimal saving under uncertainty.

Having sounded the above warnings, the term "uncertainty" will be used in subsequent chapters to mean temporal uncertainty, with explicit qualification to be given only when the danger of misunderstanding makes it necessary. As Mossin [92, p. 174] put it, "in the real world temporal prospects, not timeless ones, are the rule rather than the exception."

CHAPTER III

SOURCES OF UNCERTAINTY

In chapter I, reference was made to the few attempts made so far to introduce uncertainty explicitly into the Ramsey-type model. A simple version of the deterministic Ramsey model may be written as

$$\text{Max} \int_0^T e^{-\rho t} u[c(t)]\, dt \qquad (3.1)$$

subject to

$$c(t) = ry(t) \quad \dot{y}(t) \qquad (3.2)$$

and

$$y(0) = y_0, \quad y(T) = y_T \qquad (3.3)$$

where c(t) and y(t) are given per capita interpretation, and

(1) $u[c(t)]$ is the utility of the rate of consumption at time t;

(2) y(t) is wealth (capital) at time t;

(3) $\dot{y}(t) = \frac{dy(t)}{dt}$;

(4) r is the real rate of interest, and

(5) ρ is the rate of time preference.

The utility function in equation (3.1) is assumed to be additive in the Pollak [103] sense.[1] Additivity in this sense implies

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[1] Denote a consumption path on the interval $[0,T]$ by $C = \{c_o^T(t)\}$, where c(t) denotes the rate of consumption at time t. Under certainty, the consumer is assumed to be maximizing an ordinal utility function u(c). Under uncertainty, he is assumed to be maximizing the expected value of a cardinal von Neumann-Morgenstern utility function V(c). Additivity in the Pollak sense means that there exists a twice differentiable function $F, F' > 0$ and function $u[c(t), t]$ such that

$$F[U\{c_o^T(t)\}] = \int_o^T u[c(t),t]\, dt.$$

For the von Neumann-Morgenstern case, there exists a function $v[c(t),t]$ such that

$$\check{V}\{c_o^T(t)\} = \int_o^T v[c(t),t]\, dt.$$

See Pollak [103, p. 493].

that at each time there is a return (utility) which depends only on the values of the consumption at time t, such that the utility of a whole history is the sum or integral of the value of the utilities at each moment of time.

A little reflection on the model of equations (3.1) (3.3) suggests a variety of ways through which uncertainty may be introduced into the problem. For instance, it may be admitted that, among other sources, uncertainty emanates from the following four, namely:

(1) the rate of return to capital,

(2) a. the production function (if the model is interpreted as a national output growth model), or

b. non-capital income (if the model is interpreted as an individual's planning problem),

(3) the utility function; and

(4) the time horizon.

It turns out that these four sources are actually the ones that have, in varying degree, attracted the greatest attention, and of the four, the most widely discussed so far is the first - the rate of return to capital. The latter which is also the earliest attempt at introducing uncertainty explicity into the Ramsey-type model, began with the work of Edmund Phelps [100] in 1962. This also forms the starting point of study in the present chapter

1. Uncertain Rate of Return to Capital

In the context of a finite horizon, discrete time-form, Phelps studied the problem of a consumer who aims at maximizing

$$J_T(c) = \underset{r}{E} \left[\sum_{t=1}^{T} \rho^{t-1} u(c_t) \right] \qquad 0 < \rho \leq 1 \qquad (3.4)$$

subject to

$$y_{t+1} = r_t(y_t - c_t) + x \qquad y_1 = k \tag{3.5}$$

where

the subscript t stands for time period;

c_t is consumption,

u is the individual's von Neumann-Morgenstern utility function in period t,

y_t is capital (wealth) on hand,

x is non-capital income assumed constant in every period,

$r_t - 1$ is the random rate of return to capital, and the r_t are independent and drawn from the same probability distribution.

In solving the above problem, Phelps employed the method of stochastic discrete-dynamic programming, and arrived at the following results. First, optimal consumption is an increasing function of both age and capital, thus confirming the result of the deterministic theory, and pointed out that without further restrictions on the utility function, it is not possible to make any more definite statement. Phelps then studied three special cases of positive constant relative risk aversion utility functions, which while they cannot yield general theorems, do help in "providing counter-examples to conjectures".[2] For instance, he shows that quite apart from reasons of time preference the classical phenomenon of

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2 Phelps [100, p. 730].

"hump saving"[3] need not occur if capital is risky. Instead, a low-capital "trap" region is possible in which it is optimal to maintain or decumulate capital no matter how distant the planning horizon.

The three special types of utility function studied are:

(1) $u(c_t) = \bar{u} - \lambda c_t^{-\gamma} \qquad \bar{u},\ \gamma > 0,\ \lambda > 1$ (3.6)

(2) $u(c_t) = \lambda c^{\gamma} \qquad \lambda > 0,\ 0 < \gamma < 1$ (3.7)

(3) $u(c_t) = \log c_t$ (3.8)

What these three utility functions have in common is that they are the only monotone increasing and strictly concave utility functions for which the proportional risk aversion index

$$q(c) = -cu''(c) / u'(c) \qquad (3.9)$$

is a positive constant.[4] They have been widely studied in the literature because of the convenience of analytical solution possible when they are assumed.[5]

Taking equation (3.6) it may be shown that optimal consumption out of uncertain income is smaller than out of certain income (the "structural" effect). Also, an increase in expected

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3 A consumer may have two different motives for saving: (1) saving for posterity - i.e. saving to provide for his heirs; (2) saving only to be able to spend at some later stage (e.g. retirement). This latter has been widely discussed in the literature under the name of "hump saving", so called because the saving path in this case is usually humped. The "hump saver" saves while young and dissaves as he grows older, e.g., see de Graaff [53] and Ramsey [108]. The conditions under which hump saving will take place when capital is risky have been carefully spelt out by Samuelson [113]. See pp 28-29 below.

4 See Hakansson [56, p. 450].

5 For instance, for this class of utility functions, it has been possible to prove the so called "separation theorem".

return encourages consumption, while an increase in uncertainty of capital lowers optimal consumption (the "marginal" effect). The reverse is true in the case of the utility function in (3.7). That is, the utility functions (3.6) and (3.7) yield opposite results. However, it is interesting to note that in all cases, "risk always opposes return". Also, it is clear from the foregoing that the "structural" and the marginal effects are in the same direction.[6]

To understand the meaning of the contrasting results given by equations (3.6) and (3.7) one only needs to realize that both of them have the same parameter γ for the elasticity with respect to consumption except that in the one case the parameter carries a negative sign and in the other a positive sign.

For the utility function in (3.8) i.e., the logarithmic utility function, the optimal consumption rate is independent both of the expected return and size and change in the uncertainty of capital.

The above results have been confirmed by more recent writers including Levhari and Srinivasan [81] Samuelson [113], Merton [88], Hakansson [56,57] and Hahn [55].

Levhari and Srinivasan followed Phelps in studying the same problem, with the little simplification that non-capital income equals zero. However, Levhari and Srinivasan studied the case of infinite time horizon instead of first analysing the model for finite time horizon, and studying its limiting properties as T tends to infinity as Phelps did. Levhari-Srinivasan derived a set of conditions characterizing an optimal policy and proved the sufficiency of concavity of utility function for a maximum. They also extended their model to

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6 The "structural" and the "marginal" effects will be defined precisely in Chapter V.

cover the problem of dynamic portfolio selection. Hahn's work is best regarded as a companion to that of Levhari-Srinivasan. Taking for granted the existence of solution, Hahn's objectives were to derive the properties of the optimal path "in a simpler way" and to reinterpret them.

Samuelson [113] and Merton [88] studied identical problems, one in discrete time form, and the other in continuous time form. Since they also reached the same conclusions, it is necessary to discuss only one of them. Samuelson's formulation will be employed here not only because it is simpler than that of Merton (which, or course, is a good enough reason), but also because it is closer to the Phelps' formulation presented above. Besides, although Merton presents the continuous version, he also begins with the simpler discrete case, and to solve his model he also resorts to dynamic programming formulation just as Samuelson does.

Several aspects of the Samuelson-Merton problem are also studied in Hakansson [56] which also covers some problems not touched upon by Samuelson and Merton. For instance, Hakansson explicity includes in his model the problem of random time horizon as well as the possibility for the decision maker to borrow or lend and buy insurance on his life or sell insurance on the lives of others. However, the remainder of this section will be devoted largely to the Samuelson-Merton study, and the main part of the Hakansson study will be taken up in section 4 below.

<u>The Samuelson-Merton Model</u>

Consider the simplified Ramsey model described by equations (3.1)-(3.3) which are repeated here for convenience :

$$\text{Max} \int_o^T e^{-\rho t} u\,[c(t)]\,dt \tag{3.10}$$

subject to

$$c(t) = r\,y(t) - \dot{y}(t) \tag{3.11}$$

and

$$y(o) = y_o, \quad y(T) = y_T \tag{3.12}$$

where the variables are as defined above.

Since there is no terminal wealth, the model may be reduced to a standard calculus-of-variations problem,

$$J = \max_{\{y(t)\}} \int_o^T e^{-\rho t} u[r\,y - \dot{y}]\,dt \tag{3.13}$$

which can be related to a discrete-time formulation

$$J = \max \sum_o^T (1 + \rho)^{-t} u\,[c_t] \tag{3.14}$$

subject to

$$c_t = y_t - \frac{y_{t+1}}{1 + r} \tag{3.15}$$

or

$$\max \sum_o^T (1 + \rho)^{-t} u[y_t - \frac{y_{t+1}}{1 + r}] \tag{3.16}$$

for prescribed y_o, y_{T+1}

Recursion conditions for regular interior maximum

$$\frac{1 + \rho}{1 + r} u' [y_{t-1} - \frac{y_t}{1 + r}] = u' [y_t \quad \frac{y_{t+1}}{1 + r}] \tag{3.17}$$

are derived from (3.16) by differentiating partially with respect to each y_t in turn. With a concave utility function u, solving the second order difference equation (3.17) with boundary conditions (y_o, y_{T+1}) suffices to give an optimal life-time consumption-investment programme.

So far, the model has been deterministic. To make it stochastic, suppose that part of the individual's wealth is now invested in a risky asset. To be specific, postulate that in addition to:

(i) a safe asset that makes \$1.00 invested in it at time t return to the investor at the end of the period \$1.00 $(1 + r)$ there is also

(ii) a risky asset that makes \$1.00 invested in it at time t, return to the investor at the end of the period \$1.00 Z_t;

where Z_t is a random variable subject to the probability distribution

$$P(Z_t \leq z) = P(Z_t), \quad z \geq o \tag{3.18}$$

Hence, Z_{t+1} 1 is the percentage "yield" of each outcome.

Suppose that at each instant of time, the optimal fraction of wealth invested in the risky asset is θ_t. Then, $1 - \theta_t$ is the optimal fraction going into the safe asset. Once these optimal fractions are known, the constraint (3.15) must be written as

$$c_t = [y_t - \frac{y_{t+1}}{[(1-\theta_t)(1+r) + \theta_t Z_t]}] . \tag{3.19}$$

Using (3.19) instead of (3.15), the stochastic generalization of (3.14) and (3.15) or (3.16) is written as

$$J_T(y_o) = \max \; E \sum_{o}^{T} (1+\rho)^{-t} \; u[c_t]$$

subject to

$$c_t = [y_t \quad \frac{y_{t+1}}{[(1-\theta_t)(1+r) + \theta_t Z_t} \quad] \tag{3.20}$$

y_o given, y_{T+1} prescribed.

The programme (3.20) is the basic stochastic programming problem that needs to be solved simultaneously for optimal saving-consumption and portfolio selection decisions over time.

To grasp the meaning of the problem represented by (3.20) imagine an individual who has to act now (period o) to select c_o, and

θ_o. Assume he knows y_o but does not know yet how Z_o will turn out. He must act now, knowing that one period later, knowledge of Z_o's outcome will make y_1 known.[7] Contingent on knowledge of y_1, he will again have to make a new decision, but for now the best he can do is guess what that decision will be.

Employing backward-optimization dynamic programming, Samuelson showed that the optimal conditions for the problem of (3.20) satisfy the following recursive optimality equations:

$$o = u'(c_o) - (1+\rho)^{-1} E J'_{T-1}(y_o) \{(1-\theta_o)(1+r) + \theta_o Z_o\}$$

$$o = EJ'_{T-1}(y_1)(y_o - c_o)(Z_o \quad 1 - r)$$

$$o = u'(c_{T-1}) \quad (1+\rho)^{-1} E J_{T-t}'(y_t)\{(1-\theta_{t-1})(1+r)+\theta_{t-1}Z_{t-1}\} \tag{3.21}$$

$$o = E J'_{T-t}[y_{t-1} \quad c_{t-1})(Z_{t-1} - 1 - r)] \tag{3.22}$$

$$t = 1, \dots T \quad 1$$

Solving (3.22) at any stage gives the optimal decision rules for consumption-saving and for portfolio selection, in the form:

$$c_t^* = f_{T-t}(y_t) \tag{3.23}$$

and

$$\theta_t^* = g_{T-t}(y_t) \tag{3.24}$$

since the Z's are independently distributed.

Having thus solved the problem for any finite T, what is left is the important case of infinite time horizon. There are two

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7 This is the definition of temporal uncertainty. See Chapter II, p. 6 above.

alternative procedures: (1) For well-behaved u (i.e., $u' > 0$, $u'' < 0$), simply let $T \to \infty$ in the above formulae or (2) as often happens, the infinite case may be the easiest of all to solve, since for it, $c_t = f(y_t)$ and $\theta_t = g(y_t)$, independently of time. Samuelson showed that both c_t and θ_t can be deduced as solutions to the functional equations:

$$o = u'[f(y)] - (1+\rho)^{-1} \int_o^\infty J'[y-f(y)] \{(1+r)-g(y)Z-1-r\} [(1+r)-g(y)Z-1-r)] \, dp(Z) \quad (3.25)$$

$$o = \int_o^\infty u'[\{y-f(y)\} \{1+r-g(y)(Z-1-r)\}] [Z-1-r] \quad (3.26)$$

He then applied the results to the special case of isoelastic marginal utility functions, $u'(c) = c^{\gamma-1}$, $\gamma < 1$ and obtained the result that "the optimal portfolio decision is independent of wealth at each stage and independent of all consumption-saving decisions".[8] This result was confirmed by Merton who adds that for the Bernoulli case (i.e. where $\gamma = o$) the "separation" goes both ways, i.e. consumption decision is independent of financial parameters, except the level of wealth.[9]

Hakansson [56, p. 458] goes even a step further. He proves that for utility functions possessing positive constant relative risk aversion, not only is optimal portfolio of risky assets independent of wealth, but in addition, it is independent of non-capital income stream, age and rate of impatience to consume. Moreover, the size of the total

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8 See Samuelson [113, p. 244].

9 Merton [88, p. 253].

investment commitment in risky opportunities in each period is proportional to the individual's wealth plus the present value of his potential non-capital income. With non-capital income equal zero, the ratio of risky portfolio to total portfolio is independent of wealth, age and impatience.

The optimal fraction of wealth invested in the risky asset θ^* is then a constant, which as, Samuelson shows, is a solution of

$$o = \int_o^\infty [(1-\theta)(1+r) + \theta Z]^{\gamma - 1} (Z - 1 \quad r)\, dP(Z) \tag{3.27}$$

For the no bequest case (i.e. $y_{T+1} = o$), Samuelson shows that the optimal consumption decision at each stage is of the form

$$c^*_{T-i} = c_i \, y_{T-i} \tag{3.28}$$

where one can deduce the recursion relations

$$c_1 = a_1 \;/\; (1 + \theta_1) \tag{3.29}$$

$$a_1 = [\,(1+\rho) \;/\; (1 + r^*)^\gamma]^{\frac{1}{1-\gamma}} \tag{3.30}$$

where $1-\gamma$ is the Pratt-Arrow measure of relative risk aversion.

$$(1+r^*)^\gamma = \int_o^\infty [(1-\theta^*)(1+r) + \theta^* Z]^\gamma \; dP(Z) \tag{3.31}$$

$$c_i = a_1 c_{i-1} / (1 + a_1 c_{i-1}) \tag{3.32}$$

$$= a_1^i / (1 + a_1 + a_1^2 + \ldots a_1^i) < c_{i-1} \tag{3.33}$$

$$= a_1^i (a_1 \quad 1) / (a_1^{i+1} - 1), \quad a_1 \neq 1 \tag{3.34}$$

$$= 1 / (1 + i) \; ; \quad a_1 = 1 \tag{3.35}$$

The limiting case where $\gamma \to o$ yields the Bernoulli logarithmic function,

$a_1 = 1 + \rho$ from (3.30)

independent of r^* and all saving propensities depend on subjective time preference ρ only, being independent of technological investment opportunites (except to the degree that y_t will itself definitely depend on those opportunities).

$1 + r^*$ can be interpreted as a kind of "risk corrected" mean yield; and the behaviour of a long-lived individual depends critically on whether:

$$(1+r^*)^\gamma \gtrless (1+\rho), \text{ corresponding to } a_1 \lessgtr 1.$$ [10]

where γ is the elasticity of the utility function, $u(c) = \frac{1}{\gamma} c^\gamma$.

(i) For $(1+r^*)^\gamma = (1+\rho)$, i.e., $a_1 = 1$, the individual plans always to consume at a uniform rate, dividing current y_{T-i} evenly by remaining life, $1/(1+i)$. If young enough, he saves on the average; in the so called "hump saving" fashion, he dissaves later as the end comes sufficiently close in sight.

(ii) For $(1+r^*)^\gamma > (1+\rho)$, $a_1 < 1$ and investment opportunities are so tempting that one consumes nothing at the beginning of a long-life. Again, hump saving must take place. With $T = \infty$, the perpetual life-time problem is divergent and ill-defined except for $\gamma \leq 0$ and $\rho > 0$.

(iii) For $(1 + \gamma^*) < (1 + \rho)$, $a_1 > 1$, and consumption at very early ages drops only to a limiting positive fraction, rather than zero. Whether or not there will be initial hump saving depends on the size of $r^* - c_\infty$, or whether

$$r^* - 1 - [(1+\gamma^*) / (1+\rho)]^{\frac{1}{1-\gamma}} > 0$$

All the above derivations depend on the no-bequest assumption, but the Merton companion paper shows how the analysis can be

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10 The conditions under which hump saving will take place when capital is risky are thus made more precise. See footnote 3 above.

generalized to cases where a bequest function $B_T(y_{T+1})$ is added to $\Sigma_o^T (1+\rho)^{-t} u(c_t)$.

Optimality under Serially Dependent Yields:

So far in this section, all the studies cited have assumed that investment yields in the various periods are statistically independent of yields in previous periods. The extension to cover the case in which yields in the various periods are serially correlated is the subject of Hakansson and Liu [59]. By utilizing a number of powerful assumptions and theorems in the theory of stochastic processes, Hakansson and Liu are able to prove that the "separation theorem", proven for the statistically independent case, still holds. The most important of the assumptions are:

(1) The economy obeys a (possibly) non-stationary Markov process.

(2) The Markov chain formed by the transition probability is irreducible and ergodic.[11]

The first assumption is not as restrictive as it might seem at first sight. Proposition VI.3 of Chapter VI states that a given stochastic process can always be converted into an equivalent Markov process by appropriate definition of the state space.

The validity of the second assumption is not entirely clear. Whereas the importance and usefulness of stationary processes are due to the fact that ergodicity theorems can be proved for this class, Hakansson-Liu do not require the economy to be stationary in any of

[11] See definitions VI.15 and VI.16 in Chapter VI.

the various senses of the term. The proof of the validity or invalidity of the ergodicity assumption in this case is beyond the scope of the present work, but it does seem that the separation theorem proved by Hakansson-Liu depends significantly on this assumption.

Other main results of the Hakansson-Liu study are the following:

1) Optimal investment policy at each decision point is myopic, i.e. it is independent of the behaviour of the economy and the available opportunities beyond the current period;

2) Optimal investment policy is the same as the optimal sequential policy when the utility of distant wealth is logarithmic.

3) With serially dependent yields, the logarithmic utility for distant wealth is the only utility function which gives an optimal (sequential) policy which is myopic;

4) The optimal policy is such that there is no possibility that the investor will be ruined. Even if he pays himself dividends so large that the expected growth rate of his capital is negative, he will survive indefinitely long !

5) While the optimal investment policy is independent of wealth, dividend, and returns beyond the current period, it does depend on everything else; the interest rate, the economy's transition probabilities, the distribution functions of the risky opportunities for the state concerned, and therefore, on the state itself.

2. Uncertain Production Function.

There is virtually no literature on the problem of introducing uncertainty through the production function. In fact, the only direct attempt is that of Mirrlees [90], and this is unpublished. Mirrlees assumes that labour is subject to Harrod-neutral technolo-change,[12] A_t, which is a random variable. In particular, log A_t is

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[12] Harrod-neutral technological change is defined in several standard texts on growth theory, e.g. Burmeister and Dobell [20, p.69].

assumed forced by a Wiener-Brownian motion process. The definition of the Wiener-Brownian motion process is given as definition VI. 18 in Chapter VI. X(t) in that definition corresponds to log A_t in Mirrlees. The main assumptions are as follows:

(A. 1) if $0 < t < t'$, then $A_{t'}/A_t$ is independent of $A_{t''}$, $(0 \leq t'' \leq t)$, this is the Markov property.

(A. 2) the distribution of A_{T+t} / A_T is independent of T for fixed t; this is the homogeneity property.

(A. 3) log A_t is normally distributed with mean $(\alpha \quad \beta)t$ and variance $2 \beta t$;

(A. 4) A_o takes some value with probability 1.

A few comments may be made on these assumptions. The Markov property, assumption (A. 1), may be supported by proposition VI. 3 of Chapter VI, and so needs no further comment. Assumption (A. 2), the homogeneity property is another way of saying that the Markov process has stationary transition probabilities. A Markov process is said to be homogeneous in time if the transition probability depends on t and T only through the difference (T-t). The third assumption (A. 3) requires that log A_t be normally distributed. In other words technological progress A_t is lognormally distributed, and hence the Brownian motion is for log A_t and not A_t itself. One possible reason for this assumption rather than the assumption of normality is the fact that the entire range of the lognormal distribution is positive whereas the normal distribution is symmetrical with respect to zero. The lognormal distribution is positively skewed. Frequency curves of the lognormal distribution for various values of mean and variance are pictured in figures I and II (p. 58 below). The mean m and variance V of a lognormal distribution are given by

$$m = e^{\mu + \frac{1}{2}\sigma^2} \quad \text{and} \quad V = e^{2\mu + \sigma^2}(e^{\sigma^2} \; 1)$$

where μ and σ^2 are the mean and variance of the corresponding normal distribution. In the case of assumption (A.3) these become

$$m = e^{\alpha t} \quad , \text{ and } \quad V = e^{2(\alpha + \beta)t} - e^{2\alpha t}$$

It need hardly be said that assumptions (A.1)- (A.3) together are indeed very strong. Mirrlees does recognize this. Whether or not the picture thus painted can be supported by emprical evidence will not be entered into here, but it is a worthwhile effort if it can be done.

Having stated and discussed Mirrlees' assumptions, his problem may be framed as that of maximizing

$$J = E \int_0^\infty e^{-\rho t} L_t \, u\left(\frac{c_t}{L_t}\right) dt \tag{3.36}$$

subject to

$$c_t + \dot{K}_t = A_t L_t \, f\left(\frac{K_t}{A_t L_t}\right) \quad , \quad K_t \geq 0, \; K_o \text{ given} \qquad L_t = L_o e^{nt} \tag{3.37}$$

where

L = labour

K = capital

c = consumption

ρ = discount factor

t = time subscript

A = the random Harrod-neutral technological change

Mirrlees derives a set of conditions characterizing an optimal consumption policy for this problem and he shows these conditions to be a function of per capita capital and the level of technology. His conditions may be regarded as the stochastic equivalent of the Keynes-Ramsey rule or the Euler equations and transversality condition for the optimal path under certainty. For the case in which the elasticity of marginal utility is a constant greater than unity and the production function is Cobb-Douglas, Mirrlees shows that optimal saving will increase with uncertainty.

Samuelson has shown that his problem is equivalent to that of Mirrlees provided

(a) equation (3.19) above is replaced by

$$A_t f(y_t/A_t) - ny_t - (y_{t+1} - y_t)$$

(b) technological change is allowed to be governed by the probability distribution

$$P\{A_t \leq A_{t+1} Z\} = P(Z)$$

(c) y_t is reinterpreted to be Mirrlees' per capita capital K_t/L_t, with L_t growing at the natural rate n, and

(d) it is posited that $A_t f(y_t / A_t)$ is a homogeneous first degree concave neoclassical production function in terms of efficiency units of labour.

The possibility of drawing this relationship gives support to the legitimacy of interpreting equation (3.2) as a transition equation either in the context of a growth model or in the context of an individual planning model as has been done on p. 18 above.

3. Uncertain Utility Function

In all of the foregoing analysis, a common practice is the assumption of a utility function which is static in one sense or the other. Thus write Dreze and Modigliani [35, p. 68] "consumption preferences are assumed to be independent of the events underlying the realizations of the random variable (y_2, r)". Also, Phelps' utility function, equation (3.4) implies separability, stationarity and independence.[13] On an individual planning level, a stationary utility function is restrictive enough; on a society planning level, the only defence for such a utility function is its simplicity. There is an obvious need to make allowance for the fact that future preferences both of the individual and the society are, if anything, uncertain.

On a macro-planning level, this need has been recognized by a few recent writers including Goldman [51, 52] , Phelps and Pollak [101] and Inagaki [64] . But they take a rather different viewpoint from the one of interest in the present work. Phelps and Pollak analyse the problem as one of n-person non co-operative game, in which the interests of the present generation are in conflict with those of the future. Goldman suggests the practice of "continued planning",[14] arguing that plans once formulated become immediately obsolete since the composi tion of the society is constantly changing.

These special issues will not be pursued here, since the interest is on the sense in which the utility function may be regarded as a source of uncertainty. There are two senses in which a utility function may be regarded as uncertain. In the first place, a utility function may be regarded as uncertain because its arguments are stochastic; e.g., the utility function u(c) is uncertain because c is

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13 These terms are explained in Phelps [100, p. 732] .

14 "Continued planning" requires that at each moment in time the current population be free to select whatever future plan it desires and to begin the immediate implementation of this plan. This should be compared with "sequential planning" under which new plans are instituted only at the expiration of old ones.

stochastic. In the second place, the utility function u(c) may be regarded as uncertain in the sense that the function u is itself stochastic. This second sense of uncertain utility function, is no doubt much more weird than the first and there is little or no literature on the subject.[15] Uncertainty of utility function in the first sense is the subject-matter of Stigum [123]. Although Stigum formulates a rigorous definition of an uncertain utility function in this sense, no theorems emanate from his study and one is left in the dark as to what impact this source of uncertainty has on optimal decisions. It is, therefore, best at this point to conclude that the analysis of intertemporal consumption-saving allocation with uncertain utility function is right now only at its descriptive stage.

4. Uncertain Time Horizon

Several writers, e.g., Fama [36] and Stigum [123] have included in their models the problem of uncertain time horizon, but Yaari [150], and Hakansson [56] are devoted entirely to this problem. In all these studies, the procedure is to first solve the optimality problem under the assumption that the horizon T is a known fixed number, and then proceed to investigate what happens to the optimal properties when T is no more fixed, but a random variable distributed, say, on an interval $[0, \bar{T}]$. This procedure was adopted, in particular by Yaari who analyzed the problem for four combinations of bequest/no bequest motive and insurance/no insurance, by means of classical variational calculus. The same problem has been analyzed by Hakansso for the special case in which the one-period utility functions have relative risk aversion indices which are positive constants. Hakansson employs discrete-time dynamic programming.

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15 See, however, Hildenbrand [61] and the literature therein cited.

Yaari formulates two alternative hypotheses to describe consumer behaviour in the face of uncertainty of life time. One sees the consumer as attempting to maximize the expected value of what he calls a "Fisher utility function", subject to the constraint that his net assets at time of death be non-negative with probability one. The other sees the consumer as attempting to maximize a "Marshall utility function". In the Fisher case, there is no bequest motive; in the Marshall case, there is. The analysis is carried out for each of the two utility functions for the case in which (i) insurance is not available, and (ii) insurance is available.

The Fisher Problem Under Certainty

For the purpose of comparison, the optimality problem is first analyzed for the case of certainty and in which there is no bequest motive. The model may be represented as

$$\text{Max } J_f(c) = \int_o^T \rho(t)\; u[c(t)]\,dt \tag{3.41}$$

subject to

(i) $y(T) = o$

(ii) $c(t) \geq o$ for all t in $[0, T]$

(iii) $y(t) = \int_o^T \{\exp \int_t^T r(\tau)\, d\tau\} \{x(t) - c(t)\}\, dt = 0$

The symbols are to be interpreted as before, namely:

(1) ρ is a subjective discount function satisfying $\rho(0) = 1$;

(2) u is a cardinal utility function - concave and twice differentiable;

(3) $y(t)$ is the consumer's net asset at time t;

(4) $r(t)$ is the rate of interest expected at time (t);

(5) $x(t)$ is the rate of earnings (other than interest)

(6) $c(t)$ is the stream of consumption;

(7) $y(T)$ is net asset at terminal time T.

(8) The subscript f on J is mnemonic for Fisher.

Yaari showed that the optimal plan c^* satisfies the differential equation:

$$\dot{c}^*(t) = -\left\{ r(t) \quad \frac{\dot{\rho}(t)}{\rho(t)} \right\} \frac{u'[c^*(t)]}{u''[c^*(t)]} \tag{3.42}$$

where $\dot{\rho}/\rho$ may be regarded as the consumer's subjective rate of discount. For instance, if $\rho(t) = e^{kt}$, then $\dot{\rho}(t) / \rho(t) = k$. Thus equation (3.42) says that c^* is increasing whenever $r(t) > \dot{\rho}(t)/\rho(t)$.

Uncertain Horizon

With random T, J(c) and y(T) are also random since these depend on T. To solve the problem one needs to find the maximum of the expected value of J, subject to the constraint, $y(T) = 0$, which is now probabilistic. The constraint problem may be solved either by replacing $y(T) = 0$ with a new constraint.

$P\{ y(T) = 0 \} = 1$ or by imposing a penalty $\varphi[y(T)]$ such that

$$\varphi(\tau) = 0 \quad \text{for } \tau \geq 0$$
$$< 0 \quad \text{for } \tau < 0$$

In this case a new utility function (a Marshall utility function) $J_m(c)$ is defined as

$$J_m(c) = \int_0^T \rho(t)\, u[c(t)]\, dt + \beta(T)\, \varphi[y(T)] \tag{3.43}$$

where φ is a concave utility function for bequests and β is a subjective discount function for bequests.

Case A:

Fisher Utility Function, Insurance Unavailable

$$\text{Max} \int_{0}^{T} P(t) \;\rho(t)\; u\,[\,c(t)\,]\,dt \tag{3.44}$$

subject to:

(i) $c(t) \geq 0$ for all t

(ii) $c(t) \leq x(t)$ whenever $y(t) = 0$

(iii) $y(\overline{T}) = 0$

where $P(t)$ is the probability that the consumer will be alive at time t, and T obeys the probability law specified by the density function π defined on $[\,0, \overline{T}\,]$.

Yaari showed that the solution c^* of this problem will in general, be composed of three types of segments: (1) segments in which $c^*(t) = 0$, the constraint (i) being effective; (2) segments in which $c^*(t) = x(t)$, the constraint (ii) being effective; and (3) interior segments in which neither constraint is effective. He also showed that whenever $c^*(t)$ is interior it satisfies

$$\dot{c}^*(t) = \left\{ r(t) + \frac{\dot{\rho}(t)}{\rho(t)} \quad \pi_t(t) \right\} \frac{u'\,[c^*(t)]}{u''\,[c^*(t)\,]} \tag{3.45}$$

which is the counterpart of equation (3.42) under uncertainty. The only difference between equations (3.42) and (3.45) is that $\dot{\rho}(t)/\rho(t)$ in the former is replaced by $\pi_t(t) \quad \dot{\rho}(t)/\rho(t)$ in the latter. Since $\pi_t(t) \quad \dot{\rho}(t)/\rho(t) \succ -\dot{\rho}(t)/\rho(t)$, it means that the future is discounted more heavily in the uncertainty case. This is the Fisher case in a world in which the consumer has no loved dependents. This result has been confirmed and expanded upon by Hakansson [56, pp 456-58] using positive constant relative risk aversion utility functions.

Case B:

Marshallian Utility Function, Insurance Unavailable

Let $\bar{J}_m$ be the expected value of J_m given by equation (3.43). After writing $\bar{J}$ and changing the order of integration, one obtains:

$$\bar{J}_m(c) = EJ_m(c) = \int_o^{\bar{T}} \{P(t)\ \rho(t)\ u[c(t)] + \pi_t \beta(t) \varphi [y(t)] dt \qquad (3.46)$$

Yaari showed that maximizing (3.46) subject to $c \geq o$ yields a pair of simultaneous equations for the optimal consumption plan c^* and the corresponding asset function y^* respectively, and that c^* and y^* satisfy

$$\dot{c}^*(t) = \{r(t) + \frac{\dot{\rho}(t)}{\rho(t)} \quad \pi_t(t)\ \} \frac{u'[c^*(t)]}{u''[c^*(t)]} - \frac{\pi_t(t)\beta(t)\varphi'[y(t)]}{\rho(t)\ u''[c^*(t)]},\ c > o \qquad (3.47)$$

$$\dot{y}^*(t) = x(t) \quad c^*(t) + r(t)\ y^*(t) \qquad (3.48)$$

Equation (3.47) may be rewritten as

$$\dot{c}^*(t) = -\{ r(t) + \frac{\dot{\rho}(t)}{\rho(t)} \} \frac{u'[c^*(t)]}{u''[c^*(t)]} + \frac{\pi_t(t)\rho(t)u'[c^*(t)] - \beta(t)\varphi'[y^*(t)]}{\rho(t)\ u''[c^*(t)]} \qquad (3.49)$$

The first term on the r.h.s. of (3.49) is the same as the entire r.h.s. of equation (3.42). It follows that the consumer is more impatient or less impatient according as the second term is negative or positive, i.e., according as $\rho(t)\ u'[c^*(t)] \lesseqgtr \beta(t)\ \varphi'[y^*(t)]$. Again, Hakansson has obtained explicit solution for the case in which $u'(\cdot) = \varphi'(\cdot)$ and $x(t) = 0$, but for his special class of utility functions. The optimal properties are the same as in Case A except that the propensity to consume out of "permanent income" is now lower at each point in time.

Case C:

Fisher Utility Function, Insurance Available

The problem is to maximize

$$\bar{J}_f = \int_0^T P(t)\,\rho(t)\,u\,[c(t)]\,dt \tag{3.50}$$

subject to

(i) $c(t) \geq 0$ for all t

(ii) $\int_0^T \{\exp [\int_0^t j(\tau)]\} \{x(t) - c(t)\}\,dt = 0$

where

$j(t) = r(t) + \pi_t(t)$ denotes the rate of interest on "actuarial notes", and equation 3.50 (ii) approximates

$$\int_0^{\bar{T}-\Delta} \{\exp [-\int_0^t j(\tau)\,d\tau]\} \{x(t) - c(t)\}\,dt = 0$$

where $\bar{T} - \Delta$, $\Delta > o$, is the time the consumer must settle his account with the insurance company.

The solution of (3.50) $c^* > 0$ which satisfies

$$\dot{c}^*(t) = -\{j(t) + \frac{\dot{P}(t)}{P(t)} + \frac{\dot{\rho}(t)}{\rho(t)}\} \frac{u'[c^*(t)]}{u''[c^*(t)]} \tag{3.51}$$

may be obtained from equation (3.42) by putting j in place of r and ρP in place of ρ. From the definition of P and $\pi_t(t)$ (in Section A) it can be shown that

$$\dot{P}(t) / P(t) = -\pi_t(t)$$

so that upon using $j(t) = r(t) + \pi_t(t)$, equation (3.51) becomes

$$\dot{c}^*(t) = -\{r(t) + \frac{\dot{\rho}(t)}{\rho(t)}\} \frac{u'[c^*(t)]}{u''[c^*(t)]},\ c > 0 \tag{3.52}$$

which is exactly equation (3.42). The constants of integration will, however, not be the same in both cases. For the special class of positive constant relative risk aversion utility functions, Hakansson [56, p. 461] shows that the consumer is better off without insurance, and that the only possible justification for insurance when there is no bequest motive is to meet the "solvency constraint".

Case D :

Marshall Utility Function, Insurance Available

The problem is to maximise

$$\bar{J}_m(c,y) = \int_o^{\bar{T}} \{ P(t)\ \rho(t)\ u[c(t)] + \pi_t(t)\beta(t)\varphi[y(t)\}dt \quad (3.53)$$

subject to

(i) $c(t) \geq 0$ for all t

(ii) $\int_o^T \{\exp[\int_o^t j(\tau)\,d\tau]\} \{x(t) - c(t) - \pi_t(t)\ y(t)\}\,dt = 0$

The consumer is now faced with a portfolio problem, namely, how much of his total assets $R(t)$ he should hold in regular stock $y(t)$ and how much in actuarial notes $Q(t)$. The actual maximisation is a routine matter and it can be shown that $c^*(t)$ satisfies equation (3.52) while $y^*(t)$ satisfies

$$\dot{y}(t) = -\{ r(t) + \frac{\dot{\beta}(t)}{\beta(t)} \} \frac{\varphi'[y^*(t)]}{\varphi''[y^*(t)]} \quad (3.54)$$

As can be seen from (3.52) and (3.54) $c^*(t)$ and $y^*(t)$ are symmetric in the sense that (3.54) gives the same rule of behaviour as (3.52) with β replacing ρ and φ replacing u. The most important feature of equations (3.52) and (3.54) is that the latter does not involve c^*. This means that when insurance is available, the consumer can separate his consumption decision from his bequest decision.[15]

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[15] Hakansson [56] has shown how the analysis can be extended to include the problem of optimally determining the amount of insurance to purchase.

5. Conclusion

A generalisation of the type of problem tackled in this chapter calls for designing a model which simultaneously takes account of all the sources of uncertainty. Such achievement will herald the maturity of stochastic control theory and of its application in economics. Without doubt, this is not an easy problem and needless to say none of the studies surveyed in this chapter aspires to that goal.

Obviously, simultaneous recognition of all possible sources of uncertainty will significantly complicate both the problem and its solution. It is not yet known if the complicated model that will thus arise is capable of solution by any existing tool. For instance, in his exposition of the stochastic maximum principle, Kushner [73, p. 14] noted that recognition of a random horizon involves an addition of several entirely new concepts to an already complicated problem.

However, one need not be discouraged by this fact. The quotation from Koopmans [68] which appears at the beginning of Chapter VI below is a plea for the interaction between tools and economic science. A problem well-posed is half solved. Once posed, the existence of the problem could stimulate research towards the development of tools necessary for its solution. In this vein, therefore, one could visualise the problem of this chapter as that of selecting an admissible control $[c^*(t), \theta^*(t)]$ such that the trajectory which transfers the phase point $y(o) = y_o$ to $y(T) = y_T$ yields a maximum to equation (3.1) in which u, c, θ, ρ, T, r, y, y_o, and y_T are all stochastic.

Before one can determine what the stochastic nature of the problem means for the optimal properties of the various models, it will be necessary to find out how one might "measure" the uncertainty that is involved in a whole system such as that represented by equations (3.1) - (3.3) in which uncertainty is admitted from all sources. This might well be an impossible problem. Attempts to "measure" uncertainty have so far been confined to the static framework. Even in this case, the task has been far from easy. In the dynamic framework, the task will, no doubt, be much more difficult. One might wonder if in the absence of analogous measure in the dynamic framework, one can employ the "static" measures, and if so, what that will imply for the stochastic processes that will be assumed over the relevant variables. These questions will be tackled in Chapter IV.

CHAPTER IV

MEASUREMENT OF UNCERTAINTY

Having discussed in the preceding chapters types as well as sources of uncertainty, it seems natural to ask how uncertainty may be "measured".[1] The problem of the measurement of uncertainty is an interesting one in its own right. The task of having to determine the impact of uncertainty on optimality makes the measurement problem all the more important.

As was mentioned in the last paragraph of chapter III, attempts at measuring uncertainty have so far been confined to the static framework. In this literature, three different approaches to the problem may be distinguished, namely:

(1) the mean-variance approach;

(2) the approach involving partial ordering of cumulative probability distributions: the concept of stochastic dominance; and

(3) the entropy approach.

In the first section of this chapter, the above three approaches will be discussed. The second section will be devoted to a critique of the uncertainty measures employed in the literature surveyed in chapter III. An extension to this critique will be undertaken in section 3 which deals with additional problems that are introduced when the foregoing "static" measures are employed in a dynamic framework.

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[1] The word "measure" is used in this chapter not in the rigorous sense of mathematical measure theory, but as an umbrella term which, for want of a better term, is supposed to cover intuitive notions of cardinal and ordinal measure.

1. Approaches to the Measurement of Uncertainty

(a) The Mean-variance Approach:

The commonest, and perhaps the earliest, approach to the measurement of uncertainty is the so-called mean-variance approach of Tobin [136] and Markowitz [86]. This approach simply identifies the variance of a distribution with the uncertainty (or riskiness) that is involved in the situation. It is now well-known that the mean-variance approach is generally invalid, and that the approach may be defended, only under two alternative highly restrictive assumptions. These two assumptions are:

(1) the utility function of interest is quadratic; and

(2) the random variable of interest has a distribution which belongs to the family of distributions differing only by "location of parameters".[2]

The quadratic utility function has been criticized on two main grounds:

(1) it is relevant only for the rising portion of the curve; and

(2) it violates the hypothesis of decreasing risk aversion which Arrow [4] and Pratt [105] believe to be a reasonable one.

The first criticism is not a serious one, since the objection can easily be met by imposing appropriate bounds to guarantee that the function is increasing over the relevant range. The second criticism is an empirical one.

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[2] See Rothschild and Stiglitz [108, p. 241]. Two distributions F and G are said to differ only by "location parameters" if $G(x) = F(ax + b)$ where $a > 0$. b is called the "centering" parameter and a the "scale" parameter. See Feller [40, p. 44].

In the context of the present chapter, a much more important objection to the quadratic utility function is the fact that with such a function, optimality is not affected by the degree of uncertainty.[3] Therefore, one interested in measuring uncertainty solely for the purpose of determining its impact on optimality need not worry about how uncertainty is measured if the relevant utility function is quadratic.

The question of the most appropriate probability distribution to assume for the random variable of interest is another subject for empirical verification. The normal and the rectangular distributions are well-known members of the family of distributions differing only by location parameters. So far, agreement is far from being reached as to whether the normal distribution, for instance, is an appropriate assumption for the rate of returns to capital.[4] The rectangular distribution has not even been suggested by any writer. Further comments on this issue is differed to section 2 of this chapter.

(b) Ordering of Cumulative Probability Distributions: the concept of Stochastic Dominance.

This approach to the '"measurement" or scaling of uncertainty is a product of the general dissatisfaction with the variance measure, and the literature owes the approach to the works of Hadar and Russell [54], Hanoch and Levy [60], Rothschild and Stiglitz [109] and Whitmore [141]. The essential idea behind the approach, which is largely an exercise in the manipulation of cumulative probability distributions, is that of "stochastic dominance". The concept of stochastic dominance relates to rules by which uncertain prospects can be ordered given different sets of admissible utility functions. Three different types

[3] See Leland [80, p. 467].

[4] The normality assumption has been questioned by several of the contributors in Cootner [23]. See also Fama [36, p. 168] and Samuelson [114, p. 537].

of stochastic dominance have been identified, namely first- , second- and third-degree stochastic dominance and these correspond respectively to (1) the set of unrestricted utility functions; (2) the set of risk-averse (concave) utility functions; and (3) the set of decreasing risk-averse utility functions.[5]

Let F(x) and G(x) be two cumulative probability distributions where x is a random variable (continuous or discrete) representing the outcome of a prospect. Then, the three rules are:

(1) <u>First-degree stochastic dominance</u> unrestricted utility functions:

The prospect F(x) is said to be preferred to the prospect G(x), if and only if,

$$F(x) - G(x) \leq 0 \text{ for all } x \text{ in } [a,b]. \qquad (4.1)$$

(2) <u>Second-degree stochastic dominance</u> - risk-averse (concave) utility functions:

The prospect F(x) is said to be preferred to the prospect G(x) if, and only if,

$$\int_a^x [F(y) - G(y)]\, dy \leq 0 \text{ for all } x \text{ in } [a,b] \qquad (4.2)$$[6]

(3) <u>Third-degree stochastic dominance</u> - decreasing risk averse utility functions:

The prospect F(x) is said to be preferred to the prospect G(x) if, and only if

$$\int_a^x \int_a^y [F(z) - G(z)]\, dz dy \leq 0 \text{ for all } x \text{ in } [a,b] \qquad (4.3)$$

and

$$\int_a^b [F(y) - G(y)]\, dy \leq 0 \qquad (4.5)$$

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[5] See Whitmore [14 1].

[6] All the integrals in this section are to be understood as Lebesgue-Stieltjes integrals. See Cramer [26, pp. 70-71].

From the above three rules, it is clear that from the point of view of distribution functions that can be ordered the most general is third-degree stochastic dominance, followed by second-degree and first-degree in that order. The converse is true from the point of view of admissible utility functions. The extension to third-degree stochastic dominance is due to Whitmore. For other writers, e.g. Hanoch and Levy [60] and Rothschild and Stiglitz [109], the investigation stops at second degree stochastic dominance. It is sufficient for present purposes also to stop at second degree stochastic dominance; and more so, as it can be shown that second-degree implies third-degree stochastic dominance.[7] Besides, there are many practical situations which cannot be ordered by third-degree stochastic dominance.[8]

Rothschild and Stiglitz have suggested three intuitive ways in which the prospect G, may be regarded as more uncertain than the prospect F, and shown that partial orderings derived in the three different ways are all equivalent. Let X and Y be two random variables, and F and G, the corresponding cumulative distribution functions. Then,

(1) Y is said to be more uncertain than X if Y has the same distribution as $X + Z$, and Z is a random variable with the property that

$$E(Z|X) = 0 \text{ for all } X \tag{4.6}$$

(2) Y may be said to be more uncertain than X, if X and Y have the same mean, and

$$EU(X) \geq EU(Y) \text{ for all concave } U. \tag{4.7}$$

(3) Y may be said to be more uncertain than X if X and Y have density functions f and g, and if g can be obtained from f by taking some of the probability weight from the centre of f and adding it to each tail of f in such a way as to leave the mean unchanged.

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7 See Whitmore [141, p. 458]

8 Ibid.

The following definitions and concepts are useful in defining the partial ordering that corresponds to each of the above three concepts of increasing uncertainty.

(1) Mean Preserving Spread:

A function s(x) will be called a mean preserving spread (MPS) if it satisfies

$$s(x) = \begin{cases} \alpha \geq 0 & \text{for } a < x < a+t \\ -\alpha \leq 0 & \text{for } a+d < x \leq a+d+t \\ -\beta \leq 0 & \text{for } b < x < b+t \\ \beta \geq 0 & \text{for } b+e < x < b+e+t \\ 0 & \text{otherwise} \end{cases} \quad (4.8a)$$

where

$$0 \leq a \leq a+t \leq a+d \leq a+d+t \quad (4.8b)$$

and

$$\beta e = \alpha d \quad (4.8c)$$

It can be shown that for such a function,

$$\int_0^1 s(x)\,dx = \int_0^1 xs(x)dx = 0 \quad (4.8d)$$

Therefore, if f is a density function then

$$\int_0^1 g(x)dx = \int_0^1 f(x)dx + \int_0^1 s(x)dx = 1 \quad (4.9)$$

and

$$\int_0^1 xg(x)dx = \int_0^1 x[f(x) + s(x)]\,dx = \int_0^1 xf(x)dx \quad (4.10)$$

It follows that if $g(x) \geq 0$ for all x, then g is a density function with the same mean as f. The effect of adding a MPS function s to f is to shift the probability weight from the centre to the tails of the distribution. The function g is said to differ from the function f by a single MPS if g and f are densities and $s = g - f$.

(2) Integral Conditions:

For two densities g and f differing by a single MPS, s, the difference of the corresponding cumulative distribution functions (c.d.f's) G and F is the indefinite integral of s. That is,

$$s = g \quad f \Rightarrow S = G - F \tag{4.11}$$

where

$$S(x) = \int_0^x s(u)du \tag{4.12}$$

The function $S(x)$ obeys the following properties:

(i) $S(0) = S(1) = 0$ (4.13)

(ii) There exists a z such that

$$S(x) \begin{cases} \geq 0 & \text{if } x \leq z \\ \leq 0 & \text{if } x > z \end{cases} \tag{4.14}$$

(iii) if $T(y) = \int_0^y S(x)dx$, then

$$T(1) = 0 \tag{4.15}$$

since $T(1) = \int_0^1 S(x)dx = x\,S(x) \Big|_0^1 - \int_0^1 xs(x)dx = 0$

(iv) Conditions (4.14) and (4.15) together imply that

$$T(y) \geq 0, \quad 0 \leq y \leq 1 \tag{4.16}$$

Conditions (iii) and (iv) shall be called "integral conditions".

Transitivity

A reasonable definition of "greater uncertainty" must obey the property of transitivity. This property is obeyed if one can show that given two random variables X and Y with c.d.f.'s F and G, G may be obtained from F by a sequence of MPS's. The integral conditions guarantee that this can be done.[9]

The partial orderings corresponding to the above three concepts of greater uncertainty are the following:

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9 See theorems (1a and 1b) and lemmas (1 and 2) in Rothschild-Stiglitz [10 9].

(1) The partial ordering $\leq_a$

$F \leq_a G$ if, and only if there exists a joint distribution function $H(x,z)$ of the random variables X and Z defined on $[0,1] \times [-1,1]$ such that if

$$J(y) = P(X + Z \leq y) \tag{4.17}$$

then

$$F(x) = H(x,1), \quad 0 \leq x \leq 1 \tag{4.18}$$

$$G(y) = J(y), \quad 0 \leq y \leq 1 \tag{4.19}$$

and

$$E(Z \mid X = x) = 0 \text{ for all } x.$$

In terms of the random variables X and Y, an equivalent definition is $X \leq_a Y$, provided there is a random variable Z satisfying (4.6) such that Y has the same distribution as $X + Z$.

(2) The partial ordering $\leq_u$.

$F \leq_u G$ if, and only if

$$\int_0^1 u(x)\, dF(x) \geq \int_0^1 u(x)\, dG(x) \tag{4.20}$$

This is familiar from the theory of expected utility maximisation.

(3) The partial ordering $\leq_I$

$F \leq_I G$ if, and only if

$G - F$ satisfies the integral conditions.[10]

The main result of the Rothschild-Stiglitz analysis is to prove that the three different partial orderings defined above are equivalent (their theorem 2). The proof of the theorem consists of demonstrating that

$$\leq_a \Rightarrow \leq_u \Rightarrow \leq_I \Rightarrow \leq_a \tag{4.21}$$

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10 Relations (4.15) and (4.16) above. The similarity between the integral conditions and the second degree stochastic dominance is clear. See relation (4.2) above.

The three different intuitive ways of visualizing greater uncertainty, therefore, lead to a single definition of greater uncertainty. This definition, however, differs from that resulting from the mean-variance approach.

Whereas each of the foregoing three orderings is a partial ordering, an ordering based on the mean-variance approach ($x \leq_v Y$, if $EX = EY$ and $EX^2 \leq EY^2$) is not a partial, but a complete ordering, and is, therefore, stronger than any of the above partial orderings.

(c) "Entropy" as a Measure of Uncertainty.

It is interesting to realize that information theorists also measure uncertainty in a way that approximates the discussion of the last section; i.e., in terms of probability distribution functions. In the literature of information theory,[11] the crucial concept is that of the "entropy" or the expected information of a probability distribution.

Consider a set n of events, $X_1 \ldots X_n$, with probabilities $p_1 \ldots p_n$, forming a complete set in the sense that exactly one of them will certainly occur. When one receives a reliable message that X_i has actually occurred, the information content of the message is defined as

$$h(p_i) = -\log p_i \tag{4.22}$$

The information content $h(p_i)$ of a message is a decreasing function of the probability p_i. Whereas, any decreasing function will do, it is conventional to take the log of the reciprocal of the probability p_i because this affords the convenience of additivity in the case of independent events.

The expected information (entropy) is therefore defined as

$$H(P) = \sum_{i=1}^{n} P_i h(P_i) = \sum_{i=1}^{n} P_i \log \frac{1}{P_i} = \sum_{i=1}^{n} P_i \log P_i \tag{4.23}$$

That is, H(P) is a weighted average of $h(P_i)$. H(P) is non-negative, and

[11] E.g. Shannon [118], Shannon and Weaver [119], Theil [133,134], Murphy [94] and Kullback [71].

is zero if and only if one of the probabilities P_i is 1 and all others 0. In this case, $P_i \log P_i$ is really not defined, being of the form zero times minus infinity. However, by convention, $P_i \log P_i$ is taking as equal zero if $P_i = 0$. Since $H(\cdot) \geq 0$, its minimum value is zero, and this is attained when $P_i = 1$ for some i, $P_j = 0$ for all $j \neq i$. In other words, no uncertainty exists in a situation when it is known before hand that some X_i has probability 1. This is the case in which the probability distribution is concentrated on a single point X_i.

On the other hand, the entropy (the uncertainty) is at a maximum when all probabilities P_i are equal. To see this, maximize $H(\cdot)$ with respect to P_i subject to $\Sigma P_i = 1$. The result is

$$-1 - \log P_i - \lambda = 0, \; i = 1, \ldots, n \tag{4.24}$$

where λ is the Lagrangian multiplier.

(4.24) implies

$$\log P_i = -1 \quad \lambda \tag{4.25}$$

Hence

$$P_i = e^{(-1 - \lambda)} = \frac{1}{e^{1+\lambda}} \tag{4.26}$$

implying that the probabilities are the same and hence equal 1/n. The case of equal probabilities yields what is known to statisticians as the uniform (or rectangular) distribution.

The foregoing analysis has been confined to the discrete case because it is simpler, and also because it is more general in the following sense.[12] A discrete distribution may be associated with qualitative characteristics as well as quantitative variables. However, continuous distributions are more closely related to discrete distributions of the quantitative-variable type. Furthermore, while a change

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12 See Theil [134, pp 391-92] .

in measurement does not affect the entropy in the discrete case,[13] the entropy in the continuous-distributions case is affected by changes in the unit and/or scale of measurement. However, once these reservations are borne in mind, the continuous extension of the discrete entropy measure is straightforward.

Suppose now that the distribution is continuous with density function f(·). The entropy of a continuous distribution is defined as

$$H = \int_{-\infty}^{\infty} f(x) \log f(x) dx \tag{4.27}$$

As an example, take the normal distribution with mean μ and variance σ^2:

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp \left[-\frac{1}{2} \frac{(x \quad \mu)^2}{\sigma^2} \right] \tag{4.28}$$

Hence

$$H = (\log \sigma\sqrt{2\pi}) \int_{-\infty}^{\infty} f(x) dx + \frac{1}{2\sigma^2} \int_{-\infty}^{\infty} (x - \mu)^2 f(x) dx \tag{4.29}$$

$$= \frac{1}{2} \log 2\pi e\sigma^2 \tag{4.30}$$

Equation (4.30) shows that entropy (and hence the uncertainty) depends on variance alone, and not on the mean. Hence in this case, the variance is an appropriate measure of uncertainty.

The following statements have been proved by Goldman [50, pp. 127-134]:

(1) Out of all distributions with a given variance, the normal distribution is the one with the largest entropy (uncertainty) provided the range of variation is $(-\infty, \infty)$.

(2) When the range of variation is finite the rectangular distribution has the largest entropy.[14]

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13 Observe from equation (4.23) that x does not enter into the definition of the entropy. For the continuous case (see equation 4.27), x does enter into the definition of entropy.

14 This is in agreement with the result given by equation (4.26) above.

(3) For a distribution over $(0, \infty)$, the maximum entropy case is that of the negative exponential distribution, the density function of which is

$$\alpha e^{-\alpha x}$$

where $\alpha = \frac{1}{\mu}$ is fixed, and μ is the mean of exponential distribution.

In the literature of statistical decision theory, the uniform distribution has also been associated with maximum uncertainty (or "total ignorance", to use the decision theoretic term), but this has sparked off a long-standing controversy. The conception by which the uniform distribution is associated with total ignorance or maximum uncertainty has been known in the literature as the "Bayes' Postulate" [15] or the "Principle of Insufficient Reason".[16]

One may conjecture a statistician's analog of the entropy measure of uncertainty by drawing attention to the fact that the uniform distribution (maximum entropy) is the limit of what statisticians call "platykurtosis", while the degenerate case with the entire probability mass (or density) concentrated on one point (minimum entropy) is the limit of what is called "leptokurtosis". One could therefore define a partial ordering similar to those of sub-section (b), provided one notes that kurtosis and uncertainty are inversely related, whereas entropy and uncertainty are positively related, at least for smooth distributions having fourth moments.

Perhaps the easiest avenue of carrying out the comparison between the measure of uncertainty discussed in this sub-section and that of the preceding one is via the third concept of greater uncertainty in that sub-section (p. 48). One could envisage that as the process of removing some probability weight from the centre of f to add to each tail of f

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15 See Schlaifer [117, p. 445].

16 Fellner [41, p. 27]. See also Pratt, Raiffa and Schlaifer [106, chapter II, section 4.4].

continues, the resulting density will approach that of the uniform distribution.

2. Uncertainty Measures in the Literature surveyed in Chapter III

A brief review of the measures of uncertainty employed by the studies surveyed in chapter III will now be undertaken. As mentioned in section 1 of this chapter, the commonest measure of uncertainty employed in the literature is the variance. A review of chapter III reveals that one form or the other of the variance measure actually figured prominently. Dreze-Modigliani, Sandmo, Leland, Merton, Hahn, Phelps, Levhari-Srinivasan, all employed the variance measure.

Levhari-Srinivasan [81, p. 16] identified uncertainty with the variance of a lognormal distribution. They then visualized a situation of increased uncertainty about the yield rate r as one in which the expectation of r, $\bar{r}$ remains constant, while the variance of r increases. Levhari-Srinivasan neither discussed nor gave any reason for their choice of the lognormal distribution. A brief diversion is, therefore, called for in order to bridge this gap.

For the simplest case of two parameter distribution, the mean α and var β^2 of a lognormal distribution are given by

$$\alpha = e^{\mu + \frac{1}{2}\sigma^2} \tag{4.31}$$

and

$$\beta^2 = e^{2\mu + \sigma^2}(e^{\sigma^2} - 1) \tag{4.32}$$

where

μ and σ^2 are the mean and variance of the corresponding normal distribution.[17]

From equation (4.32), it is easy to see that the variance of a lognormal distribution is a function of both the mean and the variance

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[17] See Aitchison and Brown [1, p. 8].

of the corresponding normal distribution. But by holding constant the mean of the lognormal distribution, the variance of the latter becomes a function of the variance of the normal distribution alone. It is interesting to note that the lognormal distribution is positively skewed, and that the greater σ^2, the greater the skewness. An examination of the frequency curves for various values of σ^2 reveals in what sense increased variance may be associated with increased uncertainty; namely the concentration of the density towards the right tail is greater, the greater is the variance σ^2. In this latter sense, the mean is at least as good a measure of uncertainty as the variance or any other moment of the distribution, and there is no better reason for holding the mean constant rather than the variance. Figures I and II illustrate. In figure I, μ is fixed and σ^2 is varied, while in fig II, the reverse is done.

Hahn [55] which is a companion to Levhari-Srinivasan, is simply contented with the investigation of cases "when uncertainty can be measured by variance".

Leland [80] and Sandmo [116] employed the variance as a measure of uncertainty, but in order to avoid the restrictions of this approach, each of them tried to derive the variance in a particular way Leland's approach, which is also adopted by Dreze-Modigliani [35], is to develop the decision-maker's first order condition in a Taylor's series up to quadratic terms.

The problem may be formulated as follows:

$$\text{Max } E[U(c_1, c_2)] \tag{4.34}$$

subject to

$$c_1 = (1 - s)Y_1 \tag{4.35}$$

$$c_2 = Y_2 + (1 + r)sY_1 \quad (4.36$$

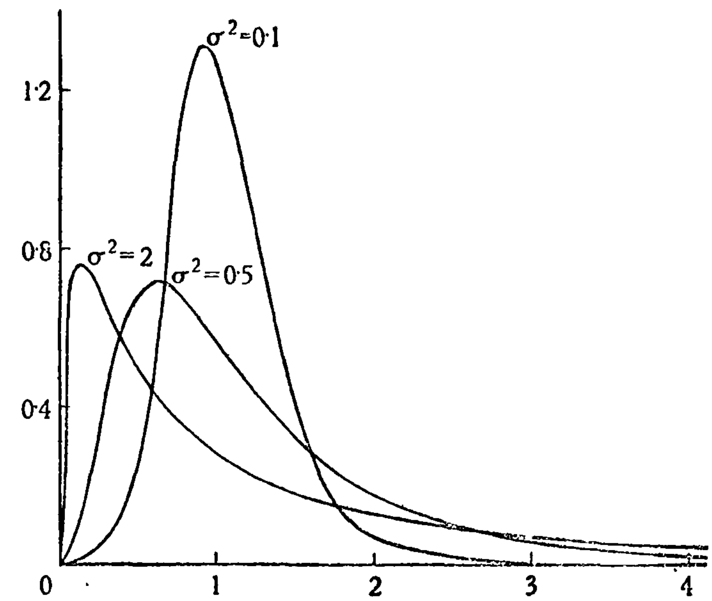

Fig. I. Frequency curves of the lognormal distribution for three values of σ^2.

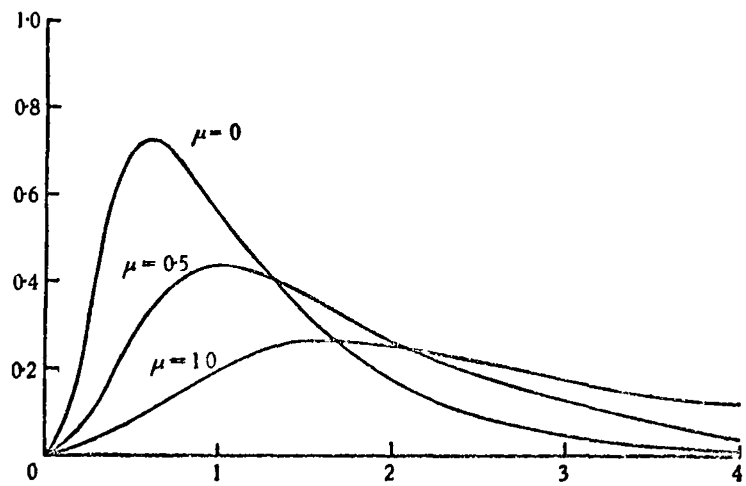

Fig. II. Frequency curves of the lognormal distribution for three values of μ.

Source: Aitchison and Brown [1,p.10].

$$E(Y_2) = \bar{Y}_2 \tag{4.37}$$

$$E(Y_2 \quad \bar{Y}_2)^2 = \sigma^2 \tag{4.38}$$

Y_1 is the fixed income in period one, and Y_2 is the random income in period two. The saving rate s is the control variable.

Problem (4.34) is a simple calculus problem yielding the first order condition:

$$E(U_1) = (1 + r)\, E(U_2) \tag{4.39}$$

where

$$U_1 = \frac{\partial U}{\partial c_1}, \text{ and } U_2 = \frac{\partial U}{\partial c_2}$$

Assuming that optimal s has been determined for the situation where $Y_2 = \bar{Y}_2$, from (4.35) and (4.36) the optimal s^0 will determine a (c_1^0, c_2^0) satisfying

$$U_1^0 = (1 + r) U_2^0 \tag{4.40}$$

where the superscript indicates that the appropriate partial derivatives were evaluated at (c_1^0, c_2^0). Equation (4.39) may be expanded in a Taylor series around (c_1^0, c_2^0). For fixed s, only c_2 is random, and so equation (4.39) may be expanded as a function of c_2 alone, and this yields

$$E(U_1)^0 = \int_{Y_2} [U_1^0 + U_{12}^0 (c_2 - c_2^0) + \tfrac{1}{2} U_{122}^0 (c_2 - c_2^0 + \ldots] \; f(Y_2)\, dY_2 \tag{4.41}$$

where

$$E(U_1)^0 = E(U_1) \text{ when } s = s^0$$

From equation (4.36), with fixed s,

$$(c_2 - c_2^0) = (Y_2 \quad \bar{Y}_2) \tag{4.42}$$

Substituting (4.42) in (4.41) and integrating term by term yields

$$E(U_1)^0 = U_1^0 + \tfrac{1}{2} U_{122}^0 \sigma^2 + o(\sigma^2) \tag{4.43}$$

which contains the variance of Y_2.

Sandmo [116] adopts a more direct approach. He considers two types of shift in the probability distribution of Y_2, namely:

(i) an additive shift which is equivalent to an increase in the mean with all other moments constant;

(ii) a multiplicative shift, by which the distribution is "stretched" around zero, or more precisely stretched on the right side of zero, since $Y_2 \geq 0$.

Sandmo then defines "a pure increase in dispersion" as "a stretching of the distribution around a constant mean". To illustrate, consider future income written as

$$\gamma Y_2 + \theta \tag{4.44}$$

with expectation

$$E[\gamma Y_2 + \theta]. \tag{4.45}$$

γ is the multiplicative shift parameter, and θ is the additive one. Since $Y_2 \geq 0$, a multiplicative shift around zero will increase the mean. The multiplicative shift must therefore be counteracted by an additive shift in the negative direction so that expectation is constant. Taking the differential of (4.45), the requirement is that

$$dE[\gamma Y_2 + \theta] = E[Y_2 d\gamma + d\theta] = 0 \tag{4.46}$$

implying

$$d\theta / d\gamma = -E(Y_2) \tag{4.47}$$

and the indifference curve between the additive and multiplicative shifts in the probability distribution of Y_2 are negatively sloped, the slope being equal to the negative of the expectation of Y_2. The implication of the analysis is that provided one remains on such indifference curve, increased variance may be interpreted as increased uncertainty.

Phelps [100, p. 739] simply takes a rectangular distribution and is thus enabled to measure uncertainty by the size of the variance.

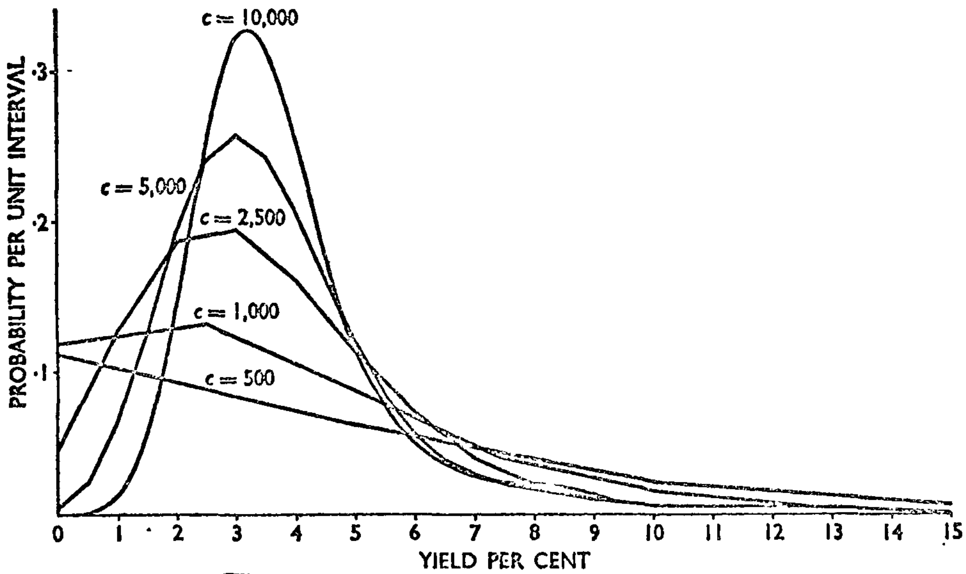

FIG. III. Comparison of yield distributions for various stakes.

Source: Plackett [102, p. 12].

The variance is a good measure of uncertainty when the probability distribution is uniform. However, it is important to ask how reasonable it is to assume a uniform distribution for the rate of returns on capital. While empirical evidence is as yet scanty on the most appropriate stochastic process to assume for investment returns, most of what exists suggests the general class of random walk which includes normality and lognormality as special cases.[18]

Perhaps, no one particular distribution or stochastic process is ideal. Plackett [102] suggests that yield distribution changes drastically with changes in c = number of bond (investment) units times the number of years the units are held. In figure III, yield per cent (or rate of return) is measured along the horizontal axis and probability per unit interval is measured along the vertical axis. The distribution is shown to vary from nearly normal (for c = 10,000) to nearly triangular (for c = 500).

3. Dynamics and the Measurement of Uncertainty.

From section 2, it could be seen that the literature of chapter III invariably employed the variance measure of uncertainty. Also, it has been seen that even within the static framework in which the variance measure was originally developed, the latter is valid only under highly restrictive conditions: the utility function is either quadratic or the relevant random variable has a normal or rectangular distribution. On the other hand, it has been seen that the two other approaches to the measurement of uncertainty stochastic dominance and entropy - permit a wider class both of utility functions and of distribution functions. Furthermore, the evidence of figure III suggests that no one particular type of probability distribution may be appropriately assumed for the random rate of return to capital over an arbitrary time interval. While

[18] See Cootner [23] and Samuelson [112].

this observation deals a devastating blow on the variance measure, it also gives a greater boost to the stochastic dominance and entropy measures, since these permit probability distributions to change with time. But this is not suggesting that the stochastic dominance and/or the entropy measure can be carried over to an explicit dynamic framework without serious undertones. What the stochastic dominance and the entropy measures permit is the freedom to re-order the distribution functions after every period. From period to period given distributions may change their parameters. Secondly, a given distribution may become completely inapplicable .

The implication of any uncertainty measure, such as the variance measure, which requires the distribution to remain unchanged from period to period is obvious: the stochastic process that has to be assumed for the random variable must be of the stationary type.[19] Limiting the admissible distributions in the static framework to the normal and the rectangular is restrictive enough. Adding the assumption of stationarity to ensure validity of the measure in the dynamic framework is still more restrictive, since the implications of stationarity, e.g., for the rate of return to capital do not even appeal to one's common sense, nor to the results of actual empirical research. Intuition would suggest that the uncertainty involved in situations in which only stationary processes are admitted is less than that involved if the more realistic non-stationary processes are admissible. This is because for non-stationary processes uncertainty affects not only the realisation of the random variables, but also the values of the parameters. If this intuition is correct it means that the uncertainty involved in dynamic optimisation problems, such as those of chapter III is actually greater than those studies suggest.[20] Unfortunately, it is beyond the

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[19] See definition VI. 14 in chapter VI

[20] Recall that in chapter II it was also found that these studies under estimate the degree of risk aversion.

scope of the present work to devise a truly rigorous measure of uncertainty that will be acceptable in a dynamic context. If anything, the purpose of the foregoing comment has been to put one on one's guard in interpreting the impact of uncertainty as set out in chapter V. In reality, it may be completely impossible to "measure" the amount of uncertainty involved in a whole system such as that represented by equations (3.1) (3.3) of chapter III in which uncertainty is admissible through every possible source.

CHAPTER V

THE IMPACT OF UNCERTAINTY ON OPTIMALITY: A SYNTHESIS

1. Introduction

The impact of uncertainty on optimal decisions has been referred to in several contexts in the preceding chapters. The commonest way this impact has been rationalized has been in terms of the Arrow-Pratt risk aversion indices,

$$-u''(c)/u'(c) \quad \text{and} \quad -u''(c)\,c/u'(c). \tag{5.1}$$

where the primes denote differentiation.

In the special case in which

$$u(c) = \frac{1}{1-\alpha}\, c^{1-\alpha} \tag{5.2}$$

it is straight-forward to show that

$$\alpha = -u''(c)\,c/u'(c) \tag{5.3}$$

is the elasticity of marginal utility of consumption with respect to consumption.

Mirrlees [90] considered the case in which $\alpha > 1$ and showed that optimal saving increases with increasing uncertainty. In addition to $\alpha \geqslant 1$, Phelps [100], Levhari-Srinivasan [81], Hahn [55], Samuelson-Merton [113, 88] and Sandmo [116] considered $\alpha < 1$ as well as the borderline case of $\alpha = 1$ which results if $u(c) = \log c$, and they were all unanimous in the conclusion that optimal saving:

(1) increases with increasing uncertainty if $\alpha > 1$;

(2) decreases with increasing uncertainty if $\alpha < 1$;

(3) remains unaffected by increasing uncertainty if $\alpha = 1$.

A most vivid analysis along these line is available in Hahn [55]. Hahn studied the Levhari-Srinivasan formulation of the problem; that is:

$$\max\ E\left[\sum_{t=o}^{\infty} \rho^{t}\, u\,(c(t))\right]; \quad \rho > 0 \tag{5.4}$$

subject to

$$y(t) = [y(t-1) - c(t-1)]\ r,\ y(0) > 0,\ y(t) \geq 0 \tag{5.5}$$

where:

u is a concave utility function of consumption c,

$r \geq 0$, is a random variable, distributed independently of t; and

y(t) is wealth at time t.

In the Hahn analysis, the function g(.) defined as

$$g\{r,\ c[y(0)]\} = u\{c[y(1))]\}\ r \tag{5.6}$$

is crucial. A similar approach has also been followed by Mirman [89]. g(.) may be regarded as the value of the consumption prospect at time zero. From (5.6) one may compute

$$g_r = u'\{c[y(1)]\}\quad (1 - \alpha) \tag{5.7}$$

and

$$g_{rr} = u'\{c[y(1)]\}\ \alpha(1 - \alpha)/r \tag{5.8}$$

where α has been defined by equation (5.3). By substituting $\alpha \lesseqgtr 1$ in (5.7) and (5.8), it can be readily shown that the function g(.) is

a decreasing convex function of r if $\alpha > 1$
an increasing concave function of r if $\alpha < 1$, and
linear in r if $\alpha = 1$.

Figures IV and V illustrate the effects of uncertainty on optimal saving for the case in which there are only two possible rates of return (r_1, r_2) and these are changed to (r_1', r_2') leaving the mean E (r) unchanged.

"What is relevant to the saving decision is the marginal gain, in any period, from consuming a little less in the preceding

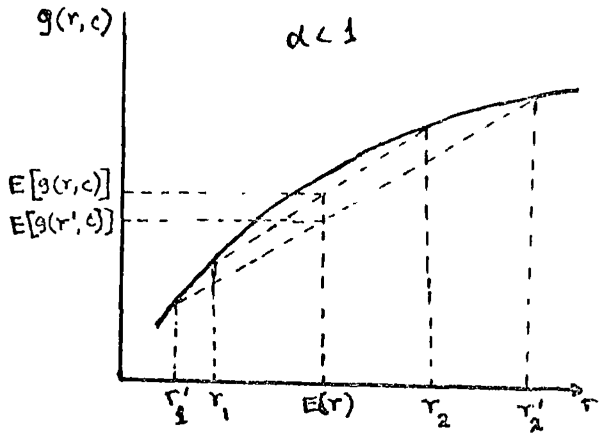

Fig. IV. Effects of Uncertainty on Optimal Saving, α<1

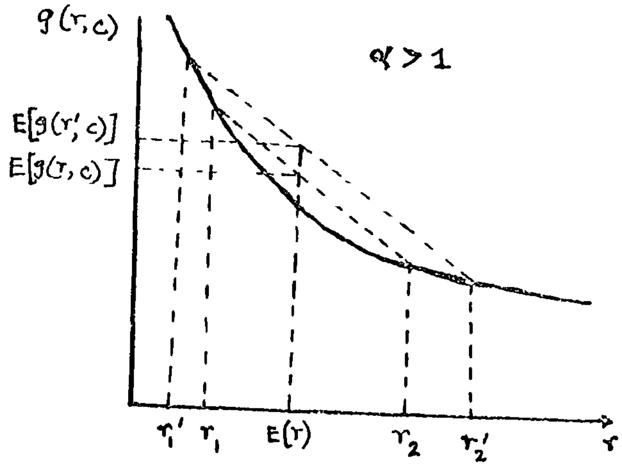

Fig. V. Effects of Uncertainty on Optimal Saving, α>1

Source: Hahn [55, p. 22]

period.'[1] By substituting $\alpha \gtreqless 1$ in (5.7) and (5.8) one can see what happens to marginal valuation as consumption prospect changes. The analysis may be rationalised in a number of ways:

(1) When $\alpha > 1$, the expected marginal valuation <u>increases</u> (see fig. V) when the rate of return changes from (r_1, r_2) to (r_1', r_2'). This causes the decision maker to "marginally prefer the "unsafe " course of action, which is future consumption. Hence in this case, optimal saving can increase with uncertainty. The converse is true when $\alpha < 1$. When $\alpha = 1$ there is no change in marginal valuation, and hence no change in saving.

(2) By analogy with the conventional gambling-insurance analysis, one may say that when $\alpha > 1$, the decision maker is "insuring" his future (risk-averter), whereas when $\alpha < 1$ he is "gambling" his future (risk-lover).[2]

(3) $\alpha > 1$ implies there may be "infinite disasters, but only finite gains", because in this case the utility function is bounded above but unbounded below; the reverse is implied when $\alpha < 1$. Hence, when $\alpha > 1$, increased uncertainty will cause the decision maker to increase his saving in order to avoid the possibility of "infinite disaster."[3]

(4) As is done in the Hicks-Slutsky analysis of price effects in ordinary consumer theory. the uncertainty effect on optimal saving may be broken into income effect and substitution effects. $\alpha > 1$ may, therefore, be interpreted as the case in which the positive income effect dominates the negative substitution effect, whereas $\alpha < 1$ corresponds to the case in which the substitution effect dominates.[4]

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1 Hahn [55, p. 23] .

2 Note, however, however that the utility function employed here is strictly concave and is, therefore, not a risk-lover's utility function.

3 For this interpretation, Hahn gives credit to David Gale.

4 Sandmo [116] and Diamond [27] give this interpretation.

(5) Finally, applying ordinary maximisation rule of equating margins also yields the same results. Equation (5.7) with $\alpha > 1$ implies the right-hand side is negative. As no maximiser ever operates at a point with a negative marginal value, the optimal course of action is to raise the marginal value by consuming less; i.e. by saving more. The reverse is true if $\alpha < 1$. In this case, the marginal valuation g_r is only a fraction of the marginal utility u'. Hence optimality calls for reducing the right hand side by consuming more; i.e. saving less.

In the remainder of the present chapter, a more detailed look at the impact of uncertainty on optimality will be undertaken in the light of the materials covered in chapters II, III and IV. In particular, there will be an examination of how the consideration of different types, sources, time dimension (multiperiod and two period) and "measurements" of uncertainty affects the impact of uncertainty on optimality.

2. Overall and Marginal Impacts of Uncertainty

To begin with, it is necessary to distinguish two different types of impacts, namely, the "overall" and the "marginal" impacts, of uncertainty on optimality. While this distinction is implicit in the works of other writers,[5] the precise distinction is due to Dreze and

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5 For instance, in the analysis of section 1, the decision maker was visualized as "marginally" preferring one course of action over another. Hahn [55] was quoted above to have said that "what is relevant to the saving decision is the marginal gain . . . " On the other hand, in the Yaari-Hakansson analysis of uncertain horizon, the procedure was to compare optimal saving under certainty (deterministic case) with optimal saving under uncertainty (stochastic case).

Modigliani [35].[6] In analysing the overall impact, interest lies in the relationship between optimal saving under certainty (the deterministic case) and optimal saving under uncertainty (the stochastic case), and not in the changes in optimal saving as uncertainty changes. The latter is the appropriate focus when one is interested in the marginal impact.

Following Dreze and Modigliani, the overall impact is defined as that impact which results when an uncertain prospect $\phi\,(y_2, r)$ is replaced by some specified "sure" prospect (y_2^+, r^+), whereas the marginal impact is that which results when a "little" more uncertainty is added to the uncertain prospect $\phi\,(y_2, r)$.

Dreze and Modigliani emphasized the point that in general one cannot ensure that the two impacts are well defined without specifying the reference "sure" prospect (y_2^+, r^+) for the overall impact and what is meant by "little more uncertainty" in the case of the marginal impact. According to them however, in the special case of "infinitesimal risks" à la Arrow and Pratt, the two impacts are well defined and are identical. In this special case, one starts from a given sure prospect (y_2^+, r^+) for which initial optimal consum-

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[6] It is fair to mention that Phelps [100, p. 739] did clearly understand this distinction; his distinction between "marginal" and "structural" effects is the same as the Dreze-Modigliani distinction between "overall" and "marginal". Also, Phelps noted that the overall and marginal impacts are in the same direction.

ption is c_1^+, and then moves to an uncertain pro pect $\phi(y_2, r)$ with $E(y_2) = y_2^+$, $E(r) = r^+$ and probability concentrated in a neighbourhood of (y_2^+, r^+).

In order to elicit the importance of the reference "sure" prospect for the impact of uncertainty, Dreze and Modigliani suggested three different criteria and formulated what they called a "certainty equivalence theorem" for each of the three.

The reference criteria:

(a) Expected value: $y_2^+ = \bar{y}_2$, $r^+ = \bar{r}$,

(b) Market value: y_2^+ and r^+ are such that:

- the present value of y_2^+ equals the present value of an uncertain future income with marginal density $\phi(y_2)$;
- the (stock market) price of an asset with sure yield r^+ equals the price of an asset whose yield has the marginal density $\phi(r)$.

(c) Expected utility: y_2^+ and r^+ are a solution of the implicit equation

$$\begin{aligned} \max \int & U[c_1, (y_1 - c_1)(1 + r) + y_2]\, d\phi(y_2, r) \\ &= \int U[\hat{c}_1, (y_1 - \hat{c}_1)(1 + r) + y_2]\, d\phi(y_2, r) \\ &= \max_{c_1} U[c_1, (y_1 - c_1)(1 + r^+) + y_2^+] \end{aligned}$$

The Theorems[7]

Theorem 5.1: Expected Value

Let D denote the smallest convex domain in (c_1, c_2) space defined

7 Only the results are presented here. Anyone interested in details should consult Dreze and Modigliani.

by

$$\{ c_1 = \hat{c}_1, \int_D d\,\psi\,(c_2) = 1 \}$$

Let r^x equal the rate of return on a perfectly safe asset, if such an asset exists and is traded on a perfect market.

Let $u_{122} - (1 + r^x)\, u_{222} \equiv \dfrac{du_{22}}{dc_1}$,*

$$\bar{\bar{y}}_2 \equiv E(y_2) + (y_1 - \hat{c}_1)\ [E(r) \quad r^x]$$, and

define $\bar{\bar{c}}_1 = \bar{\bar{c}}_1\,(r^x, \bar{\bar{y}}_2)$

If $\dfrac{dy_{22}}{dc_1}$ has a constant sign over D, then

$$\frac{du_{22}}{dc_1} \gtreqless 0 \text{ implies } \hat{c}_1 \gtreqless \bar{\bar{c}}_1$$

Theorem 5.2: Expected Utility

Let $\partial \ -\dfrac{u_{22}}{u_2} / \partial c_1 - (1 + r^x)\partial \quad \dfrac{u_{22}}{u_2} / \partial c_2 \equiv \dfrac{d\ \dfrac{u_{22}}{u_2}}{dc_1}$.

Let y_2^0 be such that

$$\max_{c_1}\ u\,[\,c_1, (y_1 \quad c_1)\,(1 + r^x) + y_2^0\,] = E\,u[\hat{c}_1, (y_1 - \hat{c}_1)(1 + r) + y_2] ,$$

and define

$$c_1^0 = c_1^{0=}\,(r^x, y_2^0).$$

Then, if $d \quad \dfrac{u_{22}}{u_2} / dc_1$, has a constant sign over D, $d\ -\dfrac{u_{22}}{u_2}/dc_1 \gtrless 0$

implies

$$\hat{c}_1 \lesseqgtr c_1^0 .$$

---

* The subscripts on u denote partial differentiation.

Theorem 5.3: Market Value

Let $y_2^x = \dfrac{E y_2 u_2 (\hat{c}_1, c_2)}{E u_2 (\hat{c}_1, c_2)}$, and define $c_1^x = c_1^x (r^x, y_2^x)$.

Then, if $\partial^2 \dfrac{u_1}{u_2} / \partial c_2^2$ has a constant sign over D,

$$\partial^2 \frac{u_1}{u_2} / \partial c_2^2 \gtreqless 0 \text{ implies } \hat{c}_1 \gtreqless c_1^x .$$

Corollary 5.1

If over D, $\partial^2 \dfrac{u_1}{u_2} / \partial c_2^2 \leq (=) 0$, and $d - \dfrac{u_{22}}{u_2} / dc_1$ has a constant sign, then $\hat{c}_1 \leq (=) c_1^x \leq c_1^0 \leq \bar{\bar{c}}_1$,

where

$\hat{c}_1$ is the optimal consumption given the uncertain prospect $\phi (y_2, r)$;

$c_1^x, c_1^0, \bar{\bar{c}}$, are the optimal consumption resulting from the market value, expected utility and expected value criteria respectively.

Discussion of the Theorems and Corollary

The first point to re-emphasize after stating the above theorems and corollary is that no substantive statement about optimality under uncertainty can be made on the basis of the risk aversion function above. Note that a different condition is needed for the completion of each of the three theorems. In theorem 5.1, the condition is on the sign of $d^{u}22/dc_1$. In 5.2 it is on the sign of $d - \dfrac{u_{22}}{u_2} / dc_1$. While in theorem 5.3 the condition is on the sign of $\partial^2 \dfrac{u_1}{u_2} / \partial c_2^2$. As Dreze and Modigliani have noted, these conditions are "rather refined properties"[8]

[8] Dreze and Modigliani, p. 58.

of the consumer's tastes.

$d^{u}22/dc_1$ is the second derivate of u as a function of c_2 alone differentiated with respect to c_1. It can be shown that $d^{u}22/dc_1$ includes the condition required for theorem 5.3, $\partial^2 \frac{u_1}{u_2} / \partial c_2^2$. *

u_1/u_2 is the consumer's marginal rate of substitution (MRS), and $\partial^2 \frac{u_1}{u_2} / \partial c_2^2$ is its second derivative with respect to c_2. MRS increases with c_2, and

$$\frac{\partial^2 \frac{u_1}{u_2}}{\partial c_2^2} \gtreqless 0 \Rightarrow \begin{array}{l} \text{MRS increases at an increasing rate} \\ \text{MRS increases at a constant rate} \\ \text{MRS increases at a decreasing rate.} \end{array}$$

$\dfrac{d - \frac{u_{22}}{u_2}}{dc_1}$ is the derivative of the absolute risk aversion with respect to c_1. If this derivative is positive, it means that absolute risk aversion about c_2 decreases, when c_2 increases as a result of a lower c_1. This situation has been labelled as one of "endogenously diminishing absolute risk aversion", in contrast to the "exogenous" case in which the increase in c_2 is not caused by a lower c_1.[9] It can be shown that $\partial^2 \frac{u_1}{u_2} / \partial c_2^2 \equiv 0$ implies endogenously diminishing

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* See Dreze and Modigliani, p. 38, equation (6.2.3). This equation brings out the respective roles of risk aversion and of purely ordinal properties of the consumer's tastes.

9 ibid., p. 41

absolute risk aversion. Also, it can be shown that $\frac{dc_1}{dr} = 0$ implies $\partial^2 \frac{u_1'}{u_2} / \partial c_2^2 = 0$. Thus $\partial^2 \frac{u_1}{u_2} / \partial c_2^2$ describes an ordinal property of u that is necessary for a zero interest elasticity of consumption at all r. This condition is also sufficient for there to exist a time-distribution of income bringing about such zero elasticity.[10] Dreze and Modigliani believe that empirical evidence supports $\partial^2 \frac{u_1}{u_2} / \partial c_2^2 = 0$ approximately, implying that the Arrow-Pratt hypothesis of decreasing absolute risk aversion is a reasonable one.

Apart from the above "rather refined properties", the impact of uncertainty on optimally also depends on the opportunities determining the consumer's prospect $\phi(y_2, r)$.

The following proposition may be stated.

Proposition V.1

> When there exists perfect insurance and asset markets and
>
> $\partial^2 \frac{u_1}{u_2} / \partial c_2^2 = 0$, then the consumption and portfolio decisions
>
> are separable with the former entirely determined by the market values z_2 and r_0, independently of the chosen portfolio and of the consumer's risk aversion.[11]

where

(1) z_2 is the present value of future earnings y_2.
(2) r_0 is the rate of return on the perfectly safe asset.

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10 ibid., p. 47

11 ibid., p. 60

In the present context. the "separation theorem" for the widely discussed special case of quadratic utility function can be made valid only by adding an extra restriction, namely: that future earnings y_{i_2} be <u>uncorrelated</u>[12] with the rates of return $\ell_j = r_j \quad r_0$, where r_j is the rate of return on the jth risky asset.

In the absence of perfect insurance for future earnings, proposition V.1 falls down completely $\hat{c}_1$ then varies with the consumer's risk aversion, and y_2^x is determined endogenously with $\hat{c}_1$ and x_j's given by $\phi\,(y_2, r)$,

where:

x_i is the amount invested in asset i, and

y_2^x is as defined by theorem 5.3.

However, the following two propositions may be deduced from corollary 5.1.

<u>Proposition V.2</u>

$\hat{c}_1$ is less than would be the case if the consumer could exchange his uncertain prospect for a sure prospect (y_2^0, r^0) yielding the same expected utility.[13]

<u>Proposition V.3</u>

$\hat{c}_1$ is less than would be the case if the consumer could exchange his uncertain prospect against a sure prospect (y_2, r^0) yielding a present value of total consumption equal to his current expectation.[13]

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12 cf. comments following equation (7.5), p. 104,below.

13 See Dreze and Modigliani, p. 65.

In the particular case where $r \equiv r^0$, the most general statement that can be made to the effect that uncertainty is apt to discourage consumption, is that whenever $\partial^2 \frac{u_1}{u_2} / \partial c_2^2 \leq 0$, then $\hat{c}_1 \leq c_1^x \leq c_1^0 \leq \bar{\bar{c}}_1$. When the consumer has access to risky investments, in addition to the perfectly safe one x_0, he is better off in the sense that his y_2^0 is higher than would be the case without risky investments. Also, when $(\bar{r} - r^x)(y_1 - c_1) > 0$ then $\bar{\bar{y}}_2 > \bar{y}_2$.

When neither earnings nor assets are traded on perfect markets, both y_2^x and r^x are determined endogenously, and the hypothesis of decreasing absolute risk aversion, $\partial^2 \frac{u_1}{u_2} / \partial c_2^2 = 0$, does not help much.

But if the hypothesis of $\partial^2 \frac{u_1}{u_2} / \partial c_2^2 = 0$ is abandoned, then the various situations are better classified on the basis of market opportunities rather than in terms of "refined properties" of consumer's tastes. Taking the case of $\partial^2 \frac{u_1}{u_2} / \partial c_2^2 < 0$, theorem 5.3 indicates that $\hat{c}_1 \leq c_1^x$. This implies that a consumer having access to the market for the sure prospect $(y_2^x = z_2, r^x = r_0)$ may choose instead an uncertain prospect (defined by $\alpha \neq 0$, $x \neq 0$)† of equal value, entailing a higher expected utility, but calling for a smaller current consumption. Such a consumer would thus accept a current sacrifice for the sake of an uncertain future gain.

If there do not exist perfect insurance and/or asset markets, the most useful result emanates from theorem 5.2, namely: "endogenously

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† α is the uninsured fraction of y_2.

diminishing absolute risk aversion" implies $\hat{c}_1 < c_1^0$. In other words, current consumption falls short of the level that would obtain under certainty with $r = r^x$ but no gain or loss in expected utility. If investment opportunities are constant, then it is possible for the consumer to shift from a perfectly safe to a risky occupation, at a gain of expected utility, while making downward adjustment in current consumption. One may conclude that consumers choosing to bear more uncertainty may well choose simultaneously to consume less.[14]

3. Types of Uncertainty and the Impact on Optimality

In this, section a brief comment is made concerning how the recognition of different types of uncertainty affect optimality. In chapter II, it was indicated that the only type of uncertainty that will be of interest is the temporal one. As Mossin [92, p. 174] has noted the case of temporal uncertainty is often the one that is of practical relevance. Also, Markowitz [86] followed by Mossin [92] demonstrated that whereas a direct and a corresponding indirect utility function may be defined in a situation of timeless uncertainty, an indirect utility function may not exist in the case of temporal uncertainty. Difference in the relevant utility functions in the two situations also leads to difference in the size of the risk aversion measure. The inequality (2.9) adds some weight to the strength of proposition II.1. However,

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14 cf. proposition II.1, P. 10 above. Consumers with a lower marginal propensity to consume are also better suited to bear (temporal) uncertainty.

the work of Fama [36] seems to imply that temporal uncertainty may be converted into timeless uncertainty by defining a utility function that is analogous to an indirect utility function in a multiperiod (dynamic) framework.

4. Sources of Uncertainty and the Impact on Optimality

In the $\alpha \lesseqgtr 1$ analysis of section 1, consideration was given to only one source of uncertainty, namely capital risk. An interpretation reducing the Mirrlees' model with stochastic rate of technological progress to the general stochastic saving-consumption model was suggested in chapter III. However, in that chapter, three other sources of uncertainty were identified, namely income risk, uncertain utility function and uncertain time horizon (stochastic stopping time).

For the uncertain utility function, no results are so far available. If the utility function is regarded as uncertain because its arguments are stochastic, then the impact of uncertainty resulting from this source may be inferred from the analysis of other sources of uncertainty. But if the utility function is regarded as uncertain in the sense that the function is itself unknown, then the analysis can become rather intractable.[15]

The impact of uncertainty resulting from income risk was covered by Sandmo [116] as well as Dreze and Modigliani. The result is definite: increased uncertainty always increases optimal saving provided one assumes decreasing temporal risk aversion.

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15 In fact, this has not been attempted by anyone.

The impact of uncertainty resulting from uncertain time horizon received detailed attention from Yaari [150] and Hakansson [56]. The results in this case depend on

(1) whether the utility function is Marshallian or Fisherian and

(2) whether insurance is available or unavailable.

Without insurance, the impact of uncertain time horizon is to increase optimal saving if the utility function is Fisherian (compare equation (3.42) with (3.45)), which agrees with what happens in the case of capital risk if $\alpha > 1$. With a Marshallian utility function, the relationship between optimal saving under certainty vis-à-vis uncertainty depends on the sign of

$$\rho(t)\, u'[c^*(t)] - \beta(t)\, \varphi'[y^*(t)] \,.$$ [16]

If the sign is positive, optimal saving under uncertainty will be greater than under certainty; if it is negative, it will be smaller. In order words, what counts is the relative size of the discounted marginal utility of consumption vis-à-vis the discounted marginal utility of bequest.

With insurance available, uncertainty brings no change to optimal saving, whether the utility function is Fisherian or Marshallian. Besides, with a Marshallian utility function, availability of insurance means that the decision maker can separate his consumption decision from his bequest decision.

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16 See equation (3.49) in chapter III.

5. Two-period Versus Multiperiod Models and the Impact of Uncertainty

The next issue to consider is whether results obtained in a two-period framework can be proved valid for multiperiod problems. For the multiperiod problem no general theorem has been stated except for the class of constant risk aversion utility functions. For the latter class, the "separation" theorem as well as "myopia" has been proved. The separation theorem says that optimal consumption-saving and portfolio decisions can be made independently of each other. "Myopia" refers to the situation under which the choice of optimal portfolio mix depends only on one-period utilities and returns. When myopia is optimal the investor's sequence of decisions is obtained as a series of single-period decisions (starting with the first period), with each period treated as if it were the only one. If in addition to constant relative risk aversion, one assumes identical yield distributions in all periods, then a fixed portfolio (i.e., a portfolio independent of wealth) is optimal. With serially dependent yields, myopia is optimal only if the utility of final (distant) wealth is logarithmic.[17] Theorems derived from two-period models may not be as restrictive as one might conjecture. An important proposition due to Fama [36] is the following:

Proposition V.4:

> If the consumer's utility function for lifetime consumption is strictly concave and markets for consumption goods and portfolio assets are perfect, then the consumer's observable behaviour in the market in any period is indistinguishable from that of a risk averse (concave utility) expected utility maximizer who has a one-period horizon.

17

See Hakansson and Liu [59] discussed in chapter III above.

In other words, one does not need the restrictive assumptions about utility function and/or yield distribution to prove "myopia" optimal. This is certainly an important proposition, but it needs to be taken with a grain of salt, for if it is true under "general conditions", there will be no more point in building complicated multiperiod models. Fama's "trick" in utilizing his proposition is to interpret the consumer's lifetime planning problem as equivalent to the consumption-saving problem of a risk averse consumer with state-dependent utilities and a one-period horizon.

Mirman [89] gives another interpretation which reduces a multiperiod planning problem to a two-period problem. In this interpretation, multiperiod (including infinite horizon) may be broken into two periods only in which the second (and last) period involves the (maximal discounted sum of all) future utility obtainable from the wealth left after the first period. In this case the second period utility function has the same form as a welfare function defined as the maximum expected sum of discounted utility over the horizon (possibly infinite).

Two-period theorists emphasize that the qualitative differences between optimality under uncertainty and optimality under certainty depend largely on the entire shape of the utility function, and not only risk aversion functions. In particular the third derivative must be considered.[18]

18

Recall that unrestricted risk aversion functions consider only the first and second derivatives.

This was also emphasized by Hahn even though he used the variance measure of uncertainty.[19] Of the multiperiod theorists, Hakansson emphasized that it may be too simple to rationalize the impact of uncertainty on optimality on the basis of risk aversion function alone.

Even within the framework of two-period models, different kinds of restrictions have to be imposed on the shape of the utility function if definite results are to be obtained.[20] The following theorem is due to Mirman [89].

Theorem 5.4

Let s^*_c be the optimal saving policy under certainty and s^*_u the optimal saving policy under uncertainty. Then, a sufficient condition for $s^*_c \geq s^*_u$ is that

$$g(E(Y)) \geq E(g(Y))$$

for all random variables Y, or equivalently that g be concave.[21] The function g employed here is similar to that utilized by Hahn [55], and the results are of course the same.

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19 It can be readily shown that differentiating the risk aversion function results in a term which contains the third derivative of the utility function. But Hahn did not do this. He emphasized the importance of the third derivative on the ground that changes in uncertainty (variance) must involve the third derivative.

20 See theorems 5.1 to 5.3 above, for instance.

21 This theorem is an application of Jensen's inequality; See Feller [40, pp. 151 - 52].

Let $g(c) = c\, u_2'(c)$

Then $g(c)$ is strictly concave or convex according as

$$2u_2''(c) + cu'''(c) \lesseqgtr 0 .$$

With $u_2''(c)$ negative by assumption, a negative u''' is sufficient for $g(c)$ to be concave and convexity requires $u_2'''(c) \geq \frac{-2u_2''(c)}{c}$, which means that $u_2'''(c)$ must be positive. According to Mirman [89] there is no a priori economic reason to expect either requirement to be satisfied. But he does not seem to be aware that the Arrow-Pratt hypothesis of decreasing absolute risk aversion implies u_2''' must be non-negative. By contrast, increasing absolute risk aversion is sufficient for $u'''(c)$ to be non-positive. Increasing absolute risk aversion is, therefore, sufficient for optimal saving under uncertainty to be at least as great as optimal saving under certainty.

6. Measurement of Uncertainty and the Impact on Optimality

Finally, this section takes up the question of how a change in the measurement of uncertainty affects the impact on optimality. From the discussion of section 2 above, it can be seen that if one is interested in the overall impact, the question of how uncertainty is measured may not be of direct concern, since the question here is that of comparing a situation in which there is uncertainty with one in which there is none, and not in how much uncertainty exists. For the purpose of the overall impact, therefore, the analysis can continue as soon as one can represent uncertainty in one form or the other. For instance, the existence of a non-degenerate probability distribution function over the variable of interest could be taken to indicate the presence of uncertainty. On the other hand, in analysing the marginal impact, there is a direct interest in how uncertainty is measured since the problem here involves what happens when uncertainty changes.

Various "measures" of uncertainty were discussed in detail in chapter IV, and the problems involved in employing these "static" measures in explicitly dynamic situations were noted. In particular the shortcomings of the well-known variance measure were noted, and the approach based on partial ordering of cumulative probability distributions was found to admit a number of intuitive interpretations all of which lead to the same (consistent) results. This approach was also found to be consistent with the entropy analysis of information theorists; whereas the variance measure was found to be generally inconsistent with the other measures and to lead to results which are not generally true.

However, as was discussed in chapter IV, section 2, virtually all the literature covered in chapter III employed the variance measure. Rothschild and Stiglitz in an article just published [110] illustrate how their stochastic dominance measure of uncertainty can be employed in analysing the marginal impact of uncertainty on optimal saving and portfolio decisions. Unfortunately, this article came out too late to receive adequate coverage in the present work. Fortunately, their results are consistent with those of Mirrlees, Phelps, Levhari-Srinivasan, Hahn etc., in section 1. However, the advantage of the Rothschild-Stiglitz stochastic dominance analysis is nontrivial. Whereas, the results of section 1 were derived for highly specialized utility function and distribution function, these restrictions are not required in the Rothschild-Stiglitz analysis.

While the Rothschild-Stiglitz analysis is no doubt, a step forward, it certainly cannot constitute the last word on the issue. The analyses done by Hahn, Mirman, Dreze-Modigliani and even Rothschild-Stiglitz all use two period problems as examples. Whether these results are supposed to carry over to multiperiod problems is not clear.

7. The Road Ahead

In spite of these questions, the ensuing survey of economic literature must terminate at this point. Beginning with the next chapter, effort will be concentrated on how the problem of this and the preceding chapters namely, the problem of intertemporal optimal consumption-saving allocation under uncertainty, may be analysed by means of the new tools to be introduced. These new tools belong to the theory of stochastic optimal control. It will be argued that these tools are the natural ones to use for the problem on hand since their deterministic counterpart have played a vital role in the analysis of the deterministic counterpart of the ensuing economic problem.

The new theory is introduced by chapter VI which also contains definitions of several technical terms that appeared in the earlier chapters as well as those that will be employed in subsequent chapters. Chapter VII is a rather detailed taxonomy of stochastic systems. In a sense, this chapter supplements chapter III since it elaborates on the problem of the various forms uncertainty may take in a dynamic problem and shows the sources (classes) identified in chapter III to be special cases of a more general structure. Chapter VII is, therefore, descriptive, but chapter VIII presents the actual analytical tools that may be employed in solving the problems, and chapter IX illustrates how this can be done.

CHAPTER VI

INTRODUCTION OF THE TOOLS

> In principle, tools have servant's status. The best choice of tools depends on the problem area selected and on the extent to which at least partial answers have been found... If we look with a historian's interest at the development of a science, however, we find that tools have a life of their own... The solution of important problems may be delayed because the requisite tools are not perceived. Or availability of certain tools may lead to an awareness of problems, important or not, that can be solved with their help ...
>
> Koopmans [68, p. 110] .

Dorfman's recent article [31] shows how optimal control theory has helped in the solution of problems that could not even be formulated before the tool was applied. The examples cited by Dorfman are the stationary state of the classical economists and the equilibrium of Bohm-Bawerk's theory of the period of production. Dorfman argued that both of these examples "describe the state of affairs in which further capital accumulation is not worthwhile", and that even though such analysis is "poorly suited to an understanding of capital accumulation and economic growth, no other technique seemed available for most of the history of capital theory". If the foregoing is true on the deterministic level, it is probably more true in stochastic situations.

Reference has already been made in Chapter I to the works of Arrow, Burmeister and Dobell, and Shell as economic applications of deterministic optimal control theory. Although there are still a number of unresolved issues [1] in the theory as well as in the economic applications of deterministic optimal control, the success so far achieved by the above

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1 See Dobell [28, pp. 46-47] and Dobell and Ho [29 p. 7] for some of these issues. Notable among the issues is the so-called "singular" problem, which is of keen interest to the economist.

named writers, among others, seems to indicate that it would be fruitful to investigate how the theory of stochastic optimal control might help in the solution of dynamic economic problems under uncertainty such as those surveyed in chapter III.

Stochastic control theory is a very broad subject. The problems with which it is concerned arise when there is a need to control dynamical systems involving uncertainty in one form or the other.[2] For analytical purposes, one assumes that the dynamical system can be described by systems of stochastic differential or difference equations. The theory then addresses itself to the following three problem areas:[3]

(a) analysis the problem of determining the statistical properties of the system variables.[4]

(b) parametric optimization - given a system and a control with a given structure, but with unknown parameter, how are the parameters to be adjusted in order to optimize the system with respect to a given criterion?[5]

(c) stochastic optimal control - given a system and a criterion, find a control law which optimizes the criterion.

The present work is limited only to the third of the above three problems areas, namely: the problem of stochastic optimal control.

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2 Some of the ways uncertainty may enter into the problem were discussed in chapter III which dealt with sources of uncertainty. More will be introduced in chapter VII.

3 See Astrom [9, p. 6].

4 This is the problem of smoothing, filtering and prediction. See definition VI. 11 below.

5 This is the problem of parameter adaptive control. See definition VI. 12.

Optimal control theory has traditionally employed two well-known tools: the maximum principle and dynamic programming. On the deterministic level both of these tools have been widely applied to a variety of economic problems. On the stochastic level, only dynamic programming has been applied.[6] There is legitimate interest in what additional insights, if any, can be obtained from the application of the stochastic maximum principle to problems of the type only stochastic dynamic programming has hitherto been applied.

So far, the appeal of dynamic programming (both deterministic and stochastic) to economists as well as to other scientists has been due more to its feasibility as a practical algorithm, than to its qualitative properties.[7] On the other hand, the appeal of the maximum principle derives, besides its more general mathematical validity,[8] from its qualitative properties which are invaluable to the economist. The maximum principle concepts such as the Hamiltonian and the transversality conditions which have been given meaningful economic interpretations[9] have either no straightforward counterpart in dynamic programming or if such counterparts exist, they have not been adequately studied. The main focus of the present work, therefore, will be the stochastic maximum principle as a direct counterpart of the Pontryagin's deterministic maximum principle. The relationship between the maximum principle and dynamic programming will be discussed in chap. VIII, and will be used to advantage in the economic interpretations of chapter IX. However, before describing the maximum principle, a few

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6 The works of Phelps, Levhari-Srinivasan, Samuelson-Merton, Hahn and others surveyed in chapter III above, all employ stochastic dynamic programming.

7 The only attempt (known to the present writer) at a qualitative economic interpretation of dynamic programming is implicit in Fama [36, p. 166]. In the context of multi-period consumption investment decision under uncertainty, the optimal value function of dynamic programming J^o is "the relevant utility function for timeless gambles taking place at period 1".

8 See Pontryagin [104, p. 73], Fel'dbaum [38, p. 85] and Wishart [143, p. 314].

9 E.g., the works of Arrow, Dorfman, Shell and others cited above.

preliminaries are in order. The next section catalogues required definitions, propositions and theorems from the theory of probability and stochastic processes. The section may be skipped by the initiated without loss of continuity. The importance of chapter VII should be obvious after reading chapters VIII and IX, namely the revelation that only a small class of stochastic systems has so far received any attention from economists.

Required definitions, propositions and theorems from the theory of probability and stochastic processes[10]

Definition VI. 1 Fields and Sigma (Borel) Fields

A class F of ω sets is called a field if it has the following properties:

(i) $\Omega \in F$

where Ω is an abstract space of points ω ;

(ii) if n is any natural number, and if

$A_1, \ldots, A_n \in F$, then

$\bigcup_1^n A_j \in F, \quad \bigcap_1^n A_j \in F.$

A field F is called a σ-field α if it satisfies the following additional property:

(iv) if $A_1, A_2, \ldots, \in F$, then $\bigcup_1^\infty A_j \in F$, and $\bigcap_1^\infty A_j \in F$.

Given a class α_m of ω sets, there is a unique σ-field of ω sets, $\sigma(\alpha_m)$ with the properties

(v) $\alpha_m \subset \sigma(\alpha_m)$

(vi) if α_1 is a σ-field of ω sets and if $\alpha_m \subset \alpha_1$, then $\sigma(\alpha_m) \subset \alpha_1$. $\sigma(\alpha_m)$ is the smallest σ-field of ω sets which contains all the sets of α_m.

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10 Complete treatment of topics mentioned here is available in standard texts' e.g. Loeve [83] and Doob [30].

It is also called the σ-field generated by a_m. A Borel field is a σ-field defined on the extended real line $\overline{R}$.

Definition VI.2 Random Variables

A random variable X is a mapping from Ω to the extended real line $\overline{R}$ (the real line including $\pm\infty$) such that

$$X^{-1} A \in a$$

for all $A \in$ Borel Field (σ-field on $\overline{R}$) where X^{-1} is the inverse mapping of X; i.e. $X^{-1} A = \{\omega | X(\omega) \in A, \omega \in \Omega\}$

Such an X is called a-measurable.

Definition VI.3 Expectation

Define an indicator function I_A as

$$I_A(\omega) = \begin{cases} 1, & \omega \in A \\ 0, & \omega \notin A \end{cases}$$

The expectation of I_A is defined to be

$$EI_A = PA$$

where P is a probability measure.

A finite linear combination of indicator functions is called a simple function. If

$$X = \sum_{i=1}^{m} a_i I_{A_i} = \sum_{j=1}^{n} b_j I_{B_j}$$

where A_i and B_j are measurable, i.e. $A_i, B_j \in a$, then the expectation of X is defined to be

$$EX = \sum_{1}^{m} a_i PA_i = \sum_{1}^{n} b_j PB_j$$

If X is a non-negative random variable, and if $\{X_n\}$ and $\{Y_n\}$ are two sequences of measurable simple functions such that

$$X_n \to X \quad \text{and} \quad Y_n \to X,$$

then

$$\lim EX_n = \lim EY_n$$

and this common value is defined to be EX. The expectation of a random variable X on (Ω, α, P) is defined to be

$$EX = EX^+ \quad EX^-$$

where

$$X^+ = \max (X, 0) \text{ , } X \quad = \max (0, -X)$$

EX is also written as

$$EX = \int X dP$$

<u>Definition VI.4</u> Absolute Continuity

Suppose two probabilities P and Q are available for the same (Ω, α). P is said to be absolutely continuous with respect to Q, (written $P << Q$) if $Q(A) = 0$ implies $P(\Lambda) = 0$.

<u>Proposition VI.1</u> Radon-Nikodym Theorem

$P << Q$ if and only if there exists a measurable[11] function f (written as dP/dQ) such that

(i) $f \geq 0$, Q almost everywhere;[12]

(ii) $PA = \int fdQ$ for arbitrary $A \in \alpha$, $\int fdQ < \infty$;

(iii) Q is unique almost everywhere.

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[11] The function f is a measurable function if for every open set G in the real number system, f^{-1} (G) is a measurable set. See Munroe [93, chapter IV] .

[12] That is, except on a set of probability zero.

<u>Definition VI.5</u> Conditional Expectation

Let (Ω, α, P) be a given probability triple, Σ a sub-sigma field of α and X a random variable such that

$$\int |X| \, dP < \infty$$

It can be shown by the Radon-Nikodym theorem that there exists a function h such that

(i) h is Σ measurable,

(ii) $\int_A h dP = \int_A X dP$ for all $A \in \Sigma$

(iii) h is unique, P almost everywhere.

This function h is written as $E(X|\Sigma)$ and is called the conditional expectation of X with respect to Σ .

<u>Definition VI.6</u> Stochastic Process

A stochastic process $\{X_t(\omega), t \in T\}$ may be defined intuitively as a family of random variables indexed by some set T, and which are all defined on the same probability triple (Ω, α, P). If X_t is discrete for each $t \in T$, then the problem is said to have a discrete state space. Similarly, if X_t is continuous valued, then the problem is said to have a continuous state space. The index set T itself may be discrete or continuous. If it is discrete (continuous) then the stochastic process is said to be a discrete (continuous) parameter stochastic process.

A formal definition is given as follows. Let a probability triple (Ω, α, P) and a parameter set T be given <u>a priori</u>. A stochastic process is then defined to be a <u>finite</u> real valued function $X(t,\omega)$ which for every fixed $t \in T$ is a α-measurable function of $\omega \in \Omega$.

Note: (1) The set T does not have to represent time;

(2) The stochastic process is not defined for $-\infty$ and $+\infty$.

Definition VI.7 Sample Function

The stochastic process $\{X(t),\ t \in T\}$ is in reality a function of two arguments $\{X(t,\omega),\ t \in T, \omega \in \Omega\}$. For fixed value of t, $X(t,\ \cdot)$ is a function of the sample space Ω, or equivalently, $X(t,\ \cdot)$ is a random variable. On the other hand, for fixed $\omega \in \Omega$, $X(\cdot\ ,\ \omega)$ is a function of t that represents a possible observation on the stochastic process $\{X(t),\ t \in T\}$. The function $X(\cdot\ ,\omega)$ is called a realization or sample function of the process.

Definition VI.8 Markov Process

A discrete parameter stochastic process x(k), k = 0,1, . . . , N is said to be Markovian if

$$P[x(k+1) \mid x(k), x(k-1), \ldots, x(0)] = P[x(k+1) \mid x(k)] \qquad (6.5)$$

for all k; that is, the probability density function of x (k + 1) depends only on knowledge of x(k) and not on x(k i), i = 1,2,... For the continuous case, a process is Markovian if it is completely specified by giving the joint density function

$$P[x(t), x(\tau)] \text{ for all } t,\ \tau \in (t_0,\ t_1)\ . \qquad (6.6)$$

Since

$$P[x(t), x(\tau)] = P[x(t) \mid x(\tau)]\ \ P[x(\tau)] \qquad (6.7)$$

a Markov process is also completely specified by giving the density functions

$$P[x(t) \mid x(\tau)] \text{ and } P[x(\tau)]\ \forall t,\ \tau \in (t_0, t_1) \qquad (6.8)$$

Proposition VI.2 The Markov Property

"The natural and perhaps the only stochastic extension of the deterministic concept of state is the Markov process."[13] For the

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13 Kushner [74, p. 334].

Markov process, knowledge of the present state eliminates all need for knowledge of the past.

Proposition VI.3 The Generality of the Markov Property

Given any discrete (continuous) stochastic process that depends on the finite past (involving a finite number of time derivatives), one can always convert it to an equivalent Markov process by properly redefining the state space.[14]

Proposition VI.4

A linear combination of Gaussian random vectors is also a Gaussian random vector.

Definition VI.9

A Gauss-Markov process is a Markov process with the added restriction that

$$P\,[x(k)] \text{ and } P\,[x(k+1) \mid x(k)]$$

or

$$P\,[x(k)] \text{ and } P[x(t) \mid x(\tau)]$$

are Gaussian probability density functions for all k, or for all t, in (t_0, t_1).

Proposition VI.5 The Gauss-Markov Property

A discrete (continuous) Gauss-Markov process can always be represented by the state vector of a multistage (continuous) linear dynanic system forced by a Gaussian purely stochastic process in which the initial state is Gaussian.[15]

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14 See Bryson and Ho [19. pp. 317 and 328], Hakansson and Liu [59, p. 386], Aoki [2, chap IV].

15 See Bryson and Ho [19. pp. 320, 328-9].

Proposition VI.6

The transition probabilities of Markov processes are governed by linear equations even if the original stochastic differential equations are highly non-linear.[16]

Definition VI.10 "Open" versus closed-looped systems

In the presence of stochastic elements, control is much more meaningful if it is of the "closed-loop" or feedback type than if it is of the "open loop". In open-loop control, the control u is stated only as a function of time t. This requires the system being programmed in advance to give the desired output (state variables x), which may vary with t. Under the closed-loop approach, the control (input) is stated as a function of both the state variable x and time t. According to Astrom [7, p. 174], this implies that the state variables must be quantifiable. One may liken the open loop approach to a once-and-for-all decision, whereas closed loop calls for sequential decision making. One has to compare input with output (or some function of the output) and use the difference to activate the control elements. One of the short-comings of deterministic optimal control theory is that it does not make the appropriate distinction between open loop and closed loop controls.[17] In fact, for deterministic systems, open loop and closed loop controls yield the same results, whereas for stochastic and adaptive systems this is not so.[18] The distinction between open loop versus closed loop systems on the one hand, and purely stochastic and parameter adaptive systems on the other hand, is to be carefully

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16 Astrom [7, p. 175].

17 See Astrom [9, pp. 2-3].

18 See Dreyfus [33, p. 209], Whittle [142, p. 325].

borne in mind.[19]

Definition VI.11

The process of estimating the most likely values of the state variables is called "smoothing", "filtering" or "prediction", depending on whether one is estimating the "past", the "present" or the "future" values of the state variables.

Definition VI.12 Dual (adaptive) Control

When the characteristics of the system noise (disturbance) or some other parameters of the system are unknown, one must use the control function to both control the system and to estimate the parameters. This has been called "dual control" by Fel'dbaum [38, p.31] and "adaptive control" by Aoki [2, p. 10].

Definition VI.13 "Purely" stochastic control system

A stochastic control system is said to be "purely stochastic" if all the variables involved have known probability distributions, or at least have known first, second, and possibly, higher moments.

Let

$$x_{t+1} = g_t(x_t, u_t, \omega_t), \; t = 0, 1, \ldots, T \tag{6.9}$$

where

ω is a random disturbance;

u is the action (control variable) of the decision maker;

x is the state variable;

g is a given transformation function; and

t is the time index.

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19 See definitions VI.12 and VI.13 below.

The stochastic case occurs when the joint probability density function (p.d.f) of $\omega_0, \omega_1, \ldots, \omega_T$ is known to be $P(\omega_0, \omega_1, \ldots, \omega_T)$ where $\omega_0, \omega_1, \ldots, \omega_T$ are not necessarily independent. In the adaptive case, the joint p.d.f. of $\omega_0, \omega_1, \ldots, \omega_T$ given θ is known to be $q(\omega_0, \ldots, \omega_T | \theta)$, where θ is an unknown (possibly vector) parameter, and the "prior" joint p.d.f. of θ is known to be $L(\theta)$.

Proposition VI.7

Adaptive stochastic systems are reducible to purely stochastic ones.

Proof:

The proof of the proposition is reproduced here because it is quite short. Define the function v_t by

$$J_t(x_t, u_t, x_{t+1}) = J_t(x_t, u_t, g_t(x_t, u_t, \omega_t)) \text{ from (6.9)}$$

$$\equiv v_t(x_t, u_t, \omega_t) \qquad (6.10)$$

In the adaptive case, the expectation EJ of (6.10) is given by

$$EJ \equiv \int \ldots \int \left[\sum_0^T v_t(x_t, u_t, \omega_t)\right] q(\omega_0, \ldots, \omega_T | \theta)\, L(\theta) d\omega_0 \ldots d\omega_T d\theta \qquad (6.11)$$

and the (marginal) joint p.d.f. of $\omega_0, \ldots, \omega_T$ is known, since

$$P(\omega_0, \ldots, \omega_T) = \int q(\omega_0, \ldots, \omega_T | \theta)\, L(\theta) d\theta \qquad (6.12)$$

Since the v_t are independent of θ, equation (6.11) can be re-written as

$$EJ = \int \ldots \int \left[\sum_0^T v_t(x_t, u_t, \omega_t)\right] P(\omega_0, \ldots, \omega_T) d\omega_0 \ldots d\omega_T \qquad (6.13)$$

which is simply the expression for the total output J for the stochastic case. Hence, the proposition is proved.[20]

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20 See Miyasawa [91, pp. 275-76].

Definition VI.14 Stationarity

Intuitively, a stationary process is one whose distribution remains the same as time progresses. In order to give a more formal definition of stationarity define a linear index set T as an index set with the property that the sum $t + h$ of any two members t and h of T also belongs to T. Examples of a linear index set are: (1) $T = \{1, 2, \ldots\}$, (2) $T = \{0, \pm 1, \pm 2, \ldots\}$ and (3) $T = \{t : t \geq 0\}$. Then a stochastic process whose index set T is linear, is said to be

(a) strictly stationary of order k, where k is a given positive integer, if for any k points $t_1, \ldots, t_k$ in T, and any h in T, the k-dimensional random vectors

$$[X(t_1), \ldots, X(t_k)] \quad \text{and} \quad [X(t_1 + h), \ldots, X(t_k + h)]$$

are identically distributed;

(b) strictly stationary if for any integer k it is strictly stationary of order k.

(c) covariance stationary[21] if it possesses finite second moments and its covariance kernel $K(s,t)$ is a function only of the absolute difference $|s - t|$, in the sense that there exists a function $R(v)$ such that for all s and t in T

$$K(s,t) = R(s - t);$$

or more precisely, $R(v)$ has the property that for every t and v in T

$$\text{cov}\,[X(t), X(t + v)] = R(v).$$

$R(v)$ is called the covariance function of the stationary process $\{X(t), t \in T\}$. For most practical applications, covariance

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21 Covariance stationary is also known as "weakly stationary", "stationary in the wide sense", or "second order stationary".

stationarity suffices, and the more mathematically involved concept of strict stationarity is not required.

The notion of stationarity derives its importance from the fact that the ergodic theorem was first proved for stationary processes.[22]

Definition VI. 15 Ergodicity

Ergodicity deals with the problem of determining the statistics of a process $X(\omega, t)$ from a single observation. $X(\omega, t)$ is ergodic in the most general form if (with probability 1) all its statistics can be determined from a single sample function $X(\omega, \cdot)$ of the process.

Definition VI. 16 Markov Chain and Irreducibility

A Markov chain is said to be irreducible if all pairs of states of the chain communicate, so that the chain consists of exactly one communicating class. Two states, j and k, are said to communicate if j is accessible from k, and k is accessible from j. A state k is said to be accessible from a state j if for some integer $N \geq 1$, $P_{jk}(N) > 0$.

Definition VI. 17 Separability

A process $\{X_t(\omega), t \in T\}$ is said to be separable if there exists a countable set $S \subset T$ and a fixed null set Λ such that for any closed set $K \subset (-\infty, \infty)$ and any open interval I the two sets $\{\omega : X_t(\omega) \in k, t \in I \cap T\}$ and $\{\omega : X_t(\omega) \in k, t \in I \cap S\}$ differ by at most a subset of Λ .

22

See Parzen [96, p. 72].

Proposition VI.8

For every stochastic process $\{X_t, t \in T\}$ there exists a process $\{\hat{X}_t, t \in T\}$ defined on the same probability space such that

(i) $\{\hat{X}_t, t \in T\}$ is separable

(ii) $P(X_t = \hat{X}_t) = 1$ for each $t \in T$.

Definition VI.18 Wiener-Brownian Motion Process

A stochastic process $\{X(t), t \geq 0\}$ is said to be a Wiener-Brownian motion process if

(i) $\{X(t), t \geq 0\}$ has stationary independent increments;

(ii) For every $t > 0$, $x(t)$ is normally distributed;

(iii) For all $t \geq 0$, $[E\ X(t)] = 0$

(iv) $X(0) = 0$

Because of properties (i) and (iv), to state the probability law of the stochastic process $X(t)$, it suffices to state the probability law of the increment $X(t) - X(s)$ for any $s < t$. Since $X(t) \quad X(s)$ is normal, its probability law is determined by its mean and variance. It can be shown that

(v) $E[X(t) \quad X(s)] = 0$, and

the variance of $X(t) - X(s)$ is proportional to $(t \quad s)$,

i.e.,

(vi) $\text{var.}\ [X(t) \quad X(s)] = \sigma^2 |t - s|$

The probability law of a Wiener process is, therefore, determined by properties (i) to (iv) up to a parameter σ^2 - an empirical constant which must be determined from observations.

CHAPTER VII

CLASSIFICATION OF STOCHASTIC SYSTEMS

One possible classification of stochastic systems, namely, "purely stochastic" and "parameter adaptive", is already obvious from the definitions VI. 12 and VI. 13 of the preceding chapter. Another approach is to formulate a very general model of stochastic systems, and then exemplify various types of control systems by varying the assumptions on the parameters and/or variables of the system. Using this approach Aoki [2] , has identified several different classes of control systems. While the sequel is rather close to the Aoki classification and presentation, a different criterion of classification will be emphasized, namely; whether or not the system is amenable to the "certainty equivalence principle" of Simon [121] and Theil [132]. This is, indeed, a useful classification since it is wellknown that for systems obeying the certainty equivalence principle, the functional form of the optimal control is unaffected by the presence of stochastic elements in the model.

The certainty equivalence principle has also been dubbed the "separation theorem"[1] because when the principle holds, the problem of optimal control divides into two separate parts:[2] the estimation of the state vectors from the observation data; and the determination

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1 E.g., see Bryson and Ho [19, p. 414]. Strictly speaking, the certainty equivalence principle is not the same thing as the separation theorem. The relationship between separability and certainty equivalence is the subject of a recent article by Patchell and Jacobs [98]. As has been shown there, separability is a necessary condition for certainty equivalence, whereas a condition known as "neutrality" may be sufficient, but has not yet been proved.

2 cf. the separation theorem of portfolio theory cited in several contexts, e.g., see pp 4 and 81 above.

of the feedback law from the corresponding deterministic system.[3] The study of systems which satisfy the certainty equivalence principle is the subject of a new book by Astrom [9].

1. The Certainty Equivalence Principle

The certainty equivalence principle is the name given to the procedure of obtaining control policies for stochastic systems by considering the optimal control policies for the related deterministic systems where the random variables are replaced by their expected values. When the optimal control policies for the deterministic system thus obtained are also optimal for the original stochastic systems, one says that the certainty equivalence principle holds; otherwise, the situation is that of "certainty difference principle".[4]

The conditions under which the certainty equivalence principle holds are given below without proof.[5] The conditions given here are for continuous systems. Analogous conditions hold for discrete time and are, therefore, omitted.

Conditions for Validity of the Certainty Equivalence Principle:

1. The criterion functional must be quadratic.
2. The transition equation,

$$\dot{x} = F(t)x + H(t)u + \omega(t) \tag{7.1}$$

and the observation equation

$$y(\tau) = G(\tau)x(\tau) + \eta(\tau) \tag{7.2}$$

must be linear.

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[3] Aoki [2, p. 51].

[4] Dreyfus [33, p. 224] .

[5] For proof, see Bryson and Ho [19, PP 414-16, 428-32].

3. The system noise $\omega(t)$ and the observation noise $\eta(t)$ must be white gaussian vector processes; that is, they must obey

$$E\begin{bmatrix}\omega(t)\\ \eta(t)\end{bmatrix}[\omega'(\tau),\ \eta'(\tau)] = \begin{bmatrix}Q(t) & N(t)\\ N'(t) & R(t)\end{bmatrix}\delta(t-\tau) \qquad (7.3)$$

where the prime denotes transposition and $\delta(t-\tau)$ is the Dirac delta function;

and

$$E[\omega(t)] = E[\eta(t)] = E(x(t_0)] = 0 \qquad (7.4)$$

4. $x(t_0)$ must be a gaussian random vector independent of $\omega(t)$ and $\eta(t)$· that is

$$E[x(t_0)x'(t_0)] = P_0,\ E[x(t_0)\eta'(t)] = E[x(t_0)\omega'(t)] = 0 \quad (7.5)$$

These are rather strong conditions. Note that conditions 3 and 4 together imply that the variance-covariance of $\eta(t)$ must be independent of $u(t)$. This is important, for without it, it is easy to construct examples obeying conditions 1 to 4, and for which the certainty equivalence principle does not hold[6]. Condition 1 constitutes a serious setback for the use of the certainty equivalence principle in the economic context when the criterion is a utility functional. In chapter IV the handicaps of the quadratic utility function were noted. In particular, it was noted that with a quadratic utility uncertainty is of no significance. Besides, the fact that the quadratic utility function alone does not guarantee the validity of certainty equivalence principle is demonstrated by the fact that in the examples given below, a quadratic criterion is taken in each case, but not every case satisfies the principle of certainty equivalence.

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6 For example, see Dreyfus [33, pp. 222-4] .

2. A General Model of Stochastic Optimal Control

(a) Discrete time form

Consider a control system described by

$$x_{t+1} = F_t(x_t, u_t, \omega_t, \alpha_t), \quad t = 0, 1, \ldots, T-1 \tag{7.6}$$

and observed by

$$y_t = H_t(x_t, \eta_t, \beta_t), \quad y_t \in Y_t \tag{7.7}$$

The criterion functional is taken as[7]

$$J = \sum_{1}^{T} W_t(x_t, u_{t-1}), \quad W_t \geq 0, \quad u_t \in U_t \tag{7.8}$$

Notation

1. x denotes state variables, and u control variables;
2. ω and η are random noises in the system dynamics (equation (7.6)) and observation device (equation(7.7)), respectively;
3. α and β are unknown parameter vectors

   $\alpha \in \text{Ⓟ}_\alpha$ and $\beta \in \text{Ⓟ}_\beta$

where Ⓟ_α and Ⓟ_β are parameter sub-spaces, α and β may be time-varying or time-invariant.

4. The subscript t denotes time period

(b) Continuous time form

Consider a stochastic control system described by

$$\dot{x}(t) = F[x(t), u(t), \omega(t), \alpha(t), t] \tag{7.9}$$

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7 Aoki [2, P.39] requires that the function $W_t > 0$, but it is not clear why this is necessary

and observed by

$$y(t) = H(x(t), \eta(t), \theta(t), t) ; \quad y(t) \in Y(t) \qquad (7.10)$$

The criterion is taken as

$$J = \int_{o}^{T} F(x(t), u(t), t) + \phi (x (T)); u(t) \in U(t), \ W = F + \phi \qquad (7.11)$$

The above formulation will now be specialized to a number of cases. For instance, the criterion function (7.8) or (7.11) can be reduced to terminal value problems by putting $W_t = 0$, $t = 1, \ldots, T-1$ and taking W_T to be a function of x_T only (equation 7.8) or by setting F equal to zero in (7.11). For the purpose of illustration, a number of one-dimensional examples are given below. For simplicity, only discrete time-form, terminal value problems are considered, and the constraint on the control u is ignored.[8] Needless to say that any combination of the classes in the sequel is itself a conceivable class.

3. Systems Obeying the Certainty Equivalence Principle

(1) A deterministic control system

This system is included in this section to facilitate comparison with non-deterministic systems to be considered later. Consider a special case of equation (7.6) written as

$$x_{t+1} = ax_t - bu_t \quad 0 \leq t \leq T - 1 \qquad (7.12)$$

and observed by

$$y_t = x_t, \quad 0 \leq t \leq T \quad 1 \qquad (7.13)$$

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8 The examples are all drawn from Aoki [2].

Take a criterion function of the form

$$J = x_T^2 = (ax_{T-1} - bu_{T-1})^2 \tag{7.14}$$

x, a, b, and u are taken to be scalar quantities, and the parameters a and b are assumed known. It is easy to show that the optimal control u_t^* which maximizes the criterion J subject to the constraint of equation (7.12) is

$$u_t^* = ax_t/b \tag{7.15}$$

(2) A Stochastic Control System with Random Time Constant.

The system is the same as that of example (1), except that the constant "a" is now replaced by a sequence $\{a_t\}$ of independent and identically distributed random variables with known mean θ and known variance σ^2. Also, the criterion function is still the same x_T^2, except that x_T is now a random variable. The optimal control must now maximize the expected value of J, EJ.

Consider the problem of choosing u_{T-1} at the (T-1)th control stage. Since

$$EJ = Ex_T^2 = E[E(x_T^2 \mid x_0, x_1, \ldots x_{T-1}, u_0, u_1, \ldots, u_{T-1})] \tag{7.16}$$

where the outer expectation is taken with respect to the random variables $x_0, x_1, \ldots, x_{T-1}$,[9] Ex_T^2 is maximized by maximizing the inner conditional expectation with respect to u_{T-1} for every possible collection of $x_0, \ldots, x_{T-1}, u_0, \ldots, u_{T-1}$.

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9 For non-randomized closed-loop control policies, $u_0, \ldots, u_{T-1}$ are some definite functions of $x_0, \ldots, x_{T-1}$ for any given control policy.

Substituting for x_T^2 from (7.14),

$$E(x_T^2 \mid x_0, x_1, \ldots, x_{T-1}, u_0, \ldots, u_{T-1}) = E(a_{T-1}x_{T-1} - bu_{T-1})^2$$

$$= (\theta x_{T-1} - bu_{T-1})^2 + \sigma^2 x_{T-1}^2 \qquad (7.17)$$

Hence

$$u_{T-1}^* = \theta x_{T-1}/b \qquad (7.18)$$

Comparison of equations (7.18) and (7.15) reveals that they are identical, except that "a" in (7.15) is replaced by θ in (7.18). Hence the certainty equivalence principle holds.

(3) Stochastic Control System with Noisy Observation

This represents another modification to the system of example (1). The only change is in the observation equation (7.13), which now becomes

$$y_t = x_t + \eta_t \qquad 0 \le t \le T-1 \qquad (7.19)$$

where η_t is the noise in the observation mechanism (observation error random variable of the system at time t). The first and second moments of η_t are assumed given. However, it is no longer possible to say (as was said for example (1)) that the control variable (7.15) is optimal since what is known at time T-1 is the collection $y_{T-1}, \ldots, y_0$ rather than of x_{T-1}; i.e. x_{T-1} is not available for the purpose of synthesizing control variable u_{T-1}. One must now consider closed-loop control policies where u_t is some deterministic function of the current and past observations on the system's state variables, and of past employed controls. That is, the control is now taken to be

$$u_t = \phi_t(y_0, y_1, \ldots, y_t, u_0, \ldots, u_{t-1}) \qquad (7.20)$$

and the function $\phi_0, \ldots, \phi_{T-1}$ must be chosen to maximize EJ.

Denote the conditional mean and variance of x_t by

$$E(x_t \mid y_0, \ldots, y_t) = \mu_t \tag{7.21}$$

and

$$\operatorname{var}(x_t \mid y_0, \ldots y_t) = \sigma_t^2 \quad o \leq t \leq T-1 \tag{7.22}$$

It is easy to verify that

$$E(x_T^2 \mid y_0, \ldots y_{T-1}, u_0, \ldots, u_{T-1}) = E\,[(ax_{T-1} - bu_{T-1})^2 \mid$$

$$y_0, \ldots, y_{T-1}, u_0, \ldots u_{T-1}] = (ax_{T-1} - bu_{T-1})^2 + a^2\,\sigma_{T-1}^2 \tag{7.23}$$

By choosing u_{T-1} to maximize (7.23) for given $y_0, \ldots, y_{T-1}, u_0, \ldots, u_{T-1}$, Ex_T^2 is maximized, since

$$Ex_T^2 = E[E(x_T^2 \mid y^{T-1}, u^{T-1})]^{10} \tag{7.24}$$

For the certainty equivalence principle to hold in this case, it is necessary to add an extra assumption, namely that σ_{T-1} is independent of u_{T-1}; i.e. the noise must not be control dependent. With the imposition of this extra assumption, it is straightforward to show that

$$u_{T-1}^* = au_{T-1}/b = aE(x_{T-1} \mid y^{T-1})/b \quad u_0, \ldots, u_{T-2} \tag{7.25}$$

is optimal in the sense that this control policy maximizes EJ, and

$$\max E(J \mid y^{T-1}, u^{T-1}) = a^2\,\sigma_{T-1}^2 \tag{7.26}$$

The problem of choosing u_{T-1} optimally is therefore reduced to that of estimating x_{T-1} given y^{T-1} by the conditional mean μ_{T-1}.

(4) Stochastic Control System with Additive Transition Noise

Again, the system is the same as example (1), except that

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10 The notation y^{T-1} is used for $y_o, \ldots, y_{T-1}$ and u^{T-1} for $u_o, \ldots, u_{T-1}$.

random disturbances are now added to the transition equation:

$$x_{t+1} = ax_t + bu_t + \omega_t \qquad (7.27)$$

x_0 given

$$y_t = x_t \quad 0 \le t \le T-1 \qquad (7.13)$$

where ω_t are independent with

$$E(\omega_t) = 0 \qquad (7.28)$$

$$E(\omega_t^2) = \sigma_t^2 \quad 0 \le t \le T-1 \qquad (7.29)$$

Proceeding as in example (2)

$$E(x_T^2 \mid x^{T-1}, u^{T-1}) = (ax_{T-1} - bu_{T-1})^2 + \sigma_{T-1}^2 \qquad (7.30)$$

since

$$E(x_T \mid x_{T-1}, u_{T-1}) = (ax_{T-1} \quad bu_{T-1})$$

and

$$\text{var}(x_T \mid x_{T-1}, u_{T-1}) = \sigma_{T-1}^2$$

because the conditional probability density $P(x_T \mid x_{T-1}, u_{T-1})$ is given by that of ω_{T-1} with $\omega_{T-1} = x_T \quad ax_{T-1} - bu_{T-1}$.

From (7.30) optimal policy is given by

$$u_{T-1}^* = ax_{T-1}/b \qquad (7.31)$$

since σ_{T-1} is a constant independent of u_{T-1}. Note that the effect on EJ is similar to the effect of noisy observation in example (3). In both cases, the maximum of $E(J \mid y^{T-1})$ is decreased by an amount proportional to the variance of the disturbances. Since the mean of ω_t is zero, the system of example (1) is the deterministic system obtained from example (4) by replacing ω_t by its mean, i.e. by applying the certainty equivalence principle to the system. Equation (7.31) is identical with (7.15).

4. Certainty Difference Principle Systems or Systems Violating the Certainty Equivalence Principle.

All of the above classes clearly fall within the class labelled in chapter VI as "purely stochastic". One may conjecture that systems obeying the certainty equivalence principle must be "purely stochastic", since none of the above problems has unknown distribution parameters. However, not all purely stochastic systems are certainty equivalent, as evidenced by the stringent conditions given in section 1 above. Each of the classes can also be discussed in the parameter adaptive context. It turns out that none of them will be certainty equivalent.

(5) Stochastic Control System with Unknown Time Constant

Consider the system described by

$$x_{t+1} = ax_t + bu_t \tag{7.32}$$

$$y_t = x_t + \eta_t \tag{7.33}$$

$$J = x_T^2 \tag{7.34}$$

This is the same system with that of example (1) except that the state variables x_t are no longer exactly observed. Assume that the noise in the observation (equation 7.33) prevents the determination of "a" exactly by measuring the state variables at two or more distinct time instants. Assume, therefore that "a" is now a random variable but with known mean α and variance σ_t^2. The η_t are assumed independent, and independent of "a". This problem is a simple example of plant parameter adaptive control systems. The certainty equivalence principle does not apply. It requires a much more involved computation to show that optimal control at time T-1 is given by

$$u_{T-1}^* = \widehat{ax}_{T-1}/b \tag{7.35}$$

where $\widehat{ax}_{T-1} = E\,(ax_{T-1} \,|\, y^{T-1}, u^{T-1})$[11]

6. Stochastic Control System with Unknown Gain

This is another example in which the certainty equivalence principle does not apply. Consider a system represented by

$$x_{t+1} = ax_t + bu_t + \omega_t \qquad (7.36)$$

x_0 given

$$y_t = x_t \qquad 0 \leq t \leq T-1 \qquad (7.37)$$

where a is a known constant, but b is now assumed to be a random variable, independent of ω_t with finite mean and variance. Assume ω_t are independently and identically distributed random variables with

$$E(\omega_t) = 0 \qquad (7.38)$$

$$\text{var}\,(\omega_t) = \Sigma_o^2 \qquad 0 \leq t \leq T-1 \qquad (7.39)$$

It can be shown that[12]

$$u^*_{T-1} = - \frac{b}{b^2_{T-1} + \sigma^2_{T-1}} (ax_{T-1}) \qquad (7.40)$$

where

$$b_{T-1} = E(b \,|\, x^{T-1}) \qquad (7.41)$$

7. Random Time-Constant System with Unknown Mean

The system is that of example (2) with the exception that the

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11 See Aoki [2, P. 15].

12 See Aoki [2, pp. 111 ff].

the mean θ of $\{a_t\}$ is now assumed unknown. The system is described by

$$x_{t+1} = a_t x_t - b_t u_t \tag{7.42}$$

$$y_t = x_t \tag{7.43}$$

where a_t's are independently and identically distributed gaussian random variables with unknown mean θ and known variance σ^2. Denoting the distribution of a random variable by $L(\cdot)$, it is assumed that

$$L(a_t) = N(\theta, \sigma^2) \tag{7.44}$$

where N(a,b) is a standard notation for a normal distribution with mean a and variance b. The unknown mean is assumed to have the a priori distribution

$$L_0(\theta) = N(\theta_0, \sigma_0^2) \tag{7.45}$$

with θ_0 and σ_0^2 given.

This is another example of parameter adaptive system. Again, the computation required to derive the optimal control is quite involved.[13]

(8) Control System with Unknown Initial Condition.

This could have been treated under example (3) above, since the presence of an observation equation in that example already admitted that the vector x including x_o could not be observed directly. However, in that example it was assumed that $x_o, \eta_o, \ldots, \eta_{T-1}$ were all independent normally distributed random variables with known mean μ and variance σ^2. If it is now assumed that not only is x_o unknown, but also its mean μ_o is unknown, one must proceed as in example (7) by assuming a prior probability distribution over x_o. The effect of the

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13 *Ibid.*, pp 17-18 .

unknown mean θ of x_o is to replace σ^2 by $\sigma^2 + \hat{\sigma}^2$ where $\hat{\sigma}^2$ is the variance of the unknown mean.[14]

(9) Systems with Stochastic Stopping Times

For problems belonging to this class of systems, the terminal of control depends on random events. For such problems, the criterion functional maybe modified by including the random stopping T among its arguments, so that instead of

$$J = \sum_{t=1}^{T} W_t (x_t, u_{t-1})$$

one has

$$J = \sum_{t=1}^{T} W_t (x_t, u_{t-1}, T)$$

where

T is a random variable.

Methods for solving the examples given above do not apply in this case. Some suggestions for tackling this problem are available in Aoki [2, pp 303-308].

5. Concluding Comments: economic interpretations of the Classified Stochastic Systems.

The simplicity of the illustrative stochastic systems employed in this chapter may becloud their usefulness in the economic context. However, with little modification in each case, it is possible to fit some interesting economic models within the framework of the systems described. The various classes of stochastic systems described may of course, be regarded as a classification of the various sources from which

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14 See Aoki [2 , pp 58-59, 93-94].

uncertainty may be introduced into a dynamic optimization problem. From this point of view, three of the four sources of uncertainty discussed in chapter III fit squarely within the foregoing classification. These are the rate of returns to capital and the rate of technological progress, both of which can be easily interpreted as special cases of stochastic systems with "random time constant" (example (2))[15] and the uncertain time horizon which is, of-course, a clear example of systems with stochastic stopping times.

Systems with uncertain utility (criterion) functions are problematic not only in economics, but in other disciplines as well. Recall from chapters III and V the two different senses in which a utility function may be regarded as uncertain. Aoki also takes the easier route of assuming the utility function as non-stochastic except to the extent that it is a function of random variables.

Systems with unknown initial conditions were not encountered in chapter III, but their relevance in economics is beyong doubt. For instance, they seem the most natural to employ for several interesting problems in the economics of conservation. For a decision maker, say, in the field of oil or gas exploration, it is only a first approximation to assume that the initial stock of resources x_o is known, as was assumed, for instance, by Burt and Cummings [21].

Systems with unknown initial condition also seem most realistic in the problem of general economic planning in the under-developed

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[15] Equation (3.5) of chapter III is a special case of the above classes of stochastic systems in which the random time constant and unknown gain appear simultaneously, even though a and b happens to be the same parameter in this case.

countries, since in general, the initial stock of resources is unknown.

Another of the above classes of stochastic systems which will be quite useful in the problems of mineral exploration and economic planning in the underdeveloped countries is the class with unknown gain (example (6)). The uncertainty of the effect of control action may be taken account of by permitting the coefficient b of the control variable u to be a random variable with known or unknown statistics.

The economic relevance of systems with random time constant, and unknown initial conditions is further justified by recent works by Kendrick [67] and Turnovsky [139]. Kendrick's is a macro planning model simultaneously involving the three classes as illustrated respectively by uncertain capital-output ratio b_t, unknown initial capital stock k_o, and unknown consumption-tax parameter β_t. Turnovsky's is a multiplier-accelerator model involving random time constant (the marginal propensity to save s) or unknown gain (the adjustment coefficient a), but not both simultaneously.

The problem represented by systems with noisy observation is analogous to what has long been known in econometrics by the name of "errors in variables" the problem posed by the fact that "most economic statistics contain errors of measurement, so that they are only approximations to the underlying true values"[16]. In equation (7.19) the variables x and y may be interpreted as the "true" and the observed values, respectively, e.g. of gross national product, levels of unemployment, money supply, capital stock or capital

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16 Johnston [66, P. 148].

output ratio.[17]

Systems with additive transition noise also have analogy in econometrics. The ω term (often called the stochastic term) has been used to take care of "error in equation"; i.e. an umbrella term for all variables which cannot otherwise be represented. In the original Ramsey [108] article, the additive transition noise would stand for "the possibility of future wars and earthquakes destroying our accumulation.[18]

Before concluding this chapter, it may be worthwhile to mention that neither the stochastic systems described above nor the interpretations attempted was intended to be really exhaustive. It is believed, however, that the exercise in this chapter has served the purpose of throwing some light on the question of how many factors are being assumed away by each of the models of chapter III. Some of the "open areas of research" in the general problem of chapter III should therefore, be obvious from this chapter. There should, however, be no illusions regarding the price at which the several factors left out by each of the models of chapter III can be simultaneously taken account of. As will be evident after the study of chapter VIII, it is not yet clear how much complication will be introduced into the stochastic maximum principle to be presented there, for instance, if unknown initial conditions and/or stochastic time horizon are permitted in the system.

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17 An example in which the observations of the last two variables are corrupted by uncorrelated noise is treated in section 3 of Kendrick [67].

18 Ramsey [108, P 549].

Finally, it may be observed that several of the examples of economic interpretations of stochastic systems given in this chapter have no direct relevance to the problem of chapter III. This sort of extended footnote was undertaken in order to justify the statement made earlier in chapter I that in principle the tools of stochastic optimal control theory are capable of handling any economic problem of dynamic optimisation under uncertainty.

CHAPTER VIII

THE MAXIMUM PRINCIPLE

The objective of the last chapter was taxonomy. In contrast, the present chapter concentrates on a description of tools by means of which the problems of chapter VII may be solved. In keeping with the argument of chapter VI, the maximum principle (both deterministic and stochastic) is presented as an alternative to dynamic programming.

For the purpose of applying the maximum principle, optimal control problems may be classified in a number of ways. Pontryagin et. al. [104] distinguished among:

1) autonomous and non-autonomous systems;
2) fixed time and free time problems; and
3) fixed end and free end problems.

Autonomous systems are those in which the time variable t does not enter explicitly into the right-hand side of the transition equation, e.g.,

$$\frac{dx^i}{dt} = f^i(x,u), \quad i = 1, \ldots n; \tag{8.1}$$

Nonautonomous systems are those in which t enters the right-hand side of (8.1); i.e.,

$$\frac{dx^i}{dt} = f^i(x,u,t) \; ; \quad i = 1, \ldots n \tag{8.2}$$

Fixed time problems are those in which the time interval for control $[t_0, t_1]$ is fixed, whereas for free time problems t_0 and/or t_1 are left free.

For fixed end problems, the state at which the trajectory x(t) begins x_0 and the state at which it ends x_1 are given. For free end problems, x_0 and x_1 are not given, but are merely restricted to take their values from given smooth manifolds S_0 and S_1 respectively. If S_0 and S_1 degenerate into points, the problem with free end points becomes the problem with fixed end points.

By altering assumptions on the above three classifications, one can come up with several different optimal control problems for each of which a maximum principle theorem can be stated and proved. Seven out of the first nine theorems of Pontryagin were formulated on this basis.[1]

In choosing the class of problems to study below a major consideration was not only the probable usefulness in the economic context, but also the availability of the stochastic counterpart with which to compare the deterministic version. Thus, theorems VIII.1 and VIII.2 below have been chosen because they correspond to Kushner [72 and 73] and Sworder [126-130].

The remaining part of this chapter will be organised as follows. Section 1 describes in detail the nature of the control problem. Section 2 develops the deterministic maximum principle stating two theorems, one for fixed time and the other for free time but both with variable right-hand end points. The stochastic maximum principle is introduced in section 3. Kushner's version of the stochastic maximum principle is covered in section 4, while section 5 is devoted to the Sworder version- the stochastic maximum principle when the noise structure is multiplicative rather than additive as in the case considered by Kushner Section 6 compares the Pontryagin

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[1] See also Bryson and Ho [19, chap. 2], Athans and Falb [10, sec. 5.] and Lee and Markus [79, chapters 4 and 5] for similar classifications.

maximum principle with the Kushner and Sworder stochastic versions, and section 7 briefly presents a dynamic programming approach to the Sworder problem of section 5. In the concluding section 8 some of the unanswered questions in the earlier sections are mentioned.

1. The Control Problem

Consider the fundamental system of transition equation

$$\frac{dx^i}{dt} = f^i(x, u), \; i = 1, \ldots n, \; x \in X, \; u \in U \tag{8.1}$$

where

$$x = (x^1, x^2, \ldots x^n) \text{ and } u = (u^1, u^2, \ldots u^r)$$

Assume f^i are continuous in the variables x, u, and continuously differentiable with respect to x.

With a given control law $u = u(t)$, equation (8.1) becomes

$$\frac{dx^i}{dt} = f^i(x, u(t)) \;\; i = 1, \ldots n, \; x \in X, \; u \in U \tag{8.3}$$

From (8.3) (for any initial conditions $x(t_0) = x_0$) the transition of the system $x = x(t)$ is uniquely determined, in a certain interval $[t_0, t_1]$. x(t) is called the solution of (8.1) corresponding to the control u(t) for the initial condition $x(t_0) = x_0$. The solution x(t) may not be defined for the entire interval $[t_0, t_1]$, on which u (t) is given, but if it is and, in addition, passes through the point x_1 at time t_1, then the admissible control u(t), $t_0 \le t \le t_1$, is said to have transferred the phase point from the position x_0 to the position x_1.

Now, in addition to (8.1) consider one more function $f^o(x,u)$. Assume this function is continuous together with its partial derivatives $\partial f^o / \partial x^i$, i = 1, ..., n, on the $X \cdot U$ space. The control problem is stated as follows:

In the phase space X, two points x_0 and x_1 are given. Among all admissible controls $u = u(t)$ which transfer the phase point from the position x_0 to the position x_1 (if such controls exist), find one for which the functional

$$J = \int_{t_0}^{t_1} f^o(x(t), u(t))\, dt \tag{8.4}$$

takes the least possible value.[2]

The control $u^*(t)$ which solves the problem is called an optimal control corresponding to a transition from x_o to x_1. The corresponding trajectory $x^*(t)$ is called the optimal trajectory.

In order to solve the problem some reformulation is useful To the phase vector x, adjoin one more element x^0, and let the latter vary according to

$$\frac{dx^0}{dt} = f^o(x, u), \tag{8.5}$$

where f^o is the function appearing in the definition of J in equation (8.4). In other words, instead of the system of differential equations (8.1) consider

$$\frac{dx^i}{dt} = f^i(x,u), \quad i = 0, 1, \ldots, n \tag{8.6}$$

whose right hand does not depend on x^0. Introduce the vector

$$\mathfrak{x} = (x^0, x^1, x^2, \ldots, x^n) = (x^0, x)$$

in the (n+1) dimensional vector space x. Re-write the system (8.6) in vector form as

$$\frac{d\mathfrak{x}}{dt} = \mathfrak{f}(x, u) \tag{8.7}$$

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2 Pontryagin et. al. [104, p. 13] . In the present work interest is in maximization rather than minimization. The difference between the two problems, is, however, a matter of sign,

where $\mathfrak{f} = (f^0, f)$ does not depend on the element x^0 of the vector $\mathfrak{x}$. The solution of (8.7) with initial condition $\mathfrak{x}(t_0) = \mathfrak{x}_0$, corresponding to the control u(t), has the form

$$x^0 = \int_{t_0}^{t} f^0 (x(\tau), u(\tau)] \, d\tau , \quad x = x(t).$$

In particular, when $t = t_1$

$$x^0 = \int_{t_0}^{t_1} f^0[x(t), u(t)] \, dt = J, \quad x = x_1$$

This means that the solution x(t) of (8.7) with initial condition $x(t_0) = x_0$ passes through the point $\mathfrak{x} = (J, x_1)$ at $t = t_1$. Another way to see what is going on here is the following. Let π be the line in X passing through the point $\mathfrak{x} = (0, x_1)$ and parallel to the x^0 axis. $\mathfrak{x}(t)$ may be regarded as passing through a point on π with coordinate $x^0 = J$, at time $t = t_1$. The control problem may, therefore, be reformulatedin the following way:

> In the (n+1) - dimensional phase space X the point $x_0 = (0,x)$ and the line π are given. The line π is assumed parallel to the x axis and to pass through the point $(0, x_1)$. Among all admissible controls $u = u(t)$, having the property that the corresponding solution $\mathfrak{x}(t)$ of (8.7) with initial condition $x(t_0) = x_0$ intersects π, find one whose point of intersection with π has the smallest coordinate x^0.[3]

2. The Deterministic Maximum Principle

In order to formulate the maximum principle, it is useful to consider in addition to the system (8.6),

$$\frac{dx^i}{dt} = f^i(x,u), \quad i = 0, 1, \ldots, n, \tag{8.6}$$

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3 ibid, p. 15.

another system of equations in auxiliary (adjoint) variables P_0, $P_1, \ldots, P_n$:

$$\frac{dP_i}{dt} = - \sum_{\alpha=0}^{n} \frac{\partial f^\alpha(x,u)}{\partial x^i} P_\alpha, \quad i = 0, 1, \ldots, n. \qquad (8.8)$$

If an admissible control has been chosen, and the corresponding trajectory $x(t)$ of (8.6) has been found with initial condition $x(t_0) = x_0$, the system (8.8) takes the form

$$\frac{dP_i}{dt} = - \sum_{\alpha=0}^{n} \frac{\partial f^\alpha(x(t), u(t))}{\partial x^i} P_\alpha, \; i = 0, 1, \ldots, n, \qquad (8.9)$$

Equation (8.9) is linear and homogeneous, and therefore, for any initial condition it admits the unique solution

$$P = (P_0, P_1, \ldots, P_n) .$$

Combining equations (8.6) and (8.8), one obtains the Hamiltonian H as

$$H(P, x, u) = P' f(x, u) = \sum_{\alpha=0}^{n} P_\alpha f^\alpha (x, u) \qquad (8.10)$$

where $f = (f^0, f)$ and the prime means transpose.

Equations (8.6) and (8.8) can be written with the aid of H as

$$\frac{dx^i}{dt} = \frac{\partial H}{\partial P_i}, \quad i = 0, 1, \ldots, n \qquad (8.11)$$

$$\frac{dP_i}{dt} = -\frac{\partial H}{\partial x_i}, \quad i = 0, 1, \ldots, n \qquad (8.12)$$

Taking an arbitrary (measurable)[4] control $u(t)$, $t_0 \leq t \leq t_1$, and the initial condition $x(t_0) = x_0$, one can find the corresponding (i.e. satisfying equation (8.11)) trajectory $x(t) = (x^0(t), x^1(t), \ldots, x^n(t))$.

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[4] See footnote 11 on P. 92 above.

After that one can find the solutions of system (8.12)

$$P(t) = (P_0(t), P_1(t), \ldots, P_n(t))$$

corresponding to the functions u(t) and x(t). For fixed P and x, the function H becomes a function of the parameter $u \in U$. Denote the least upper bound of the values of H by M(P,x):

$$M(P,x) = \sup_{u \in U} H(P,x,u) \tag{8.13}$$

If H attains its upper bound on U, then M(P,x) is the maximum of the values of H for fixed P and x. Hence the theorem is called the maximum principle.

Theorem VIII 1: Maximum Principle for Autonomous fixed time, free end problem[5]

Let u(t), $t_o \leq t \leq t_1$, be an admissible control which transfers the phase point from some initial position $x_o \in S_o$[6] to the position $x_1 \in S_1$,[6] and let x(t) be the corresponding trajectory starting at the point $\bar{x}_o = (o, x_o)$. In order that u*(t) and x*(t) yield the solution of the autonomous fixed time, free end problem (see equations (8.4) and (8.6)) it is necessary that there exist a nontrivial continuous vector function P*(t) such that

(1^o) for every t, $t_o \leq t \leq t_1$, the function H (P(t), x(t), u) of the variable $u \in U$ attains its maximum at the point u = u*(t):

$$H(P(t), x(t), u^*(t)) = M(P(t), x(t)) \tag{8.14}$$

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5 See Lee and Markus [79, pp. 315-16] .

6 S_o and S_1 are smooth manifolds (in the n-dimensional Euclidean space X) of arbitrary dimensions r_o, $r_1 < n$. See Pontryagin [104, pp 45-48] .

(2°) $M(P(t), x(t))$ is constant and $P_o^*(t) < 0$

(3°) $P^*(t_1)$ is orthogonal to T_1,[7] the tangent plane to S_1 at the point x_1.

<u>Theorem VIII</u> 2: Autonomous free time, free end problem[8]

Let $u(t)$, $t_0 \le t \le t_1$, be an admissible control which transfers the phase point from some initial position $x_o \in S_o$ to the position $x_1 \in S_1$, and let $x(t)$ be the corresponding trajectory starting at the point $x_o = (o, x_o)$. In order that $u^*(t)$ and $x^*(t)$ yield the solution of the optimal problem with free end points, it is necessary that there exist a nontrivial continuous vector function $P(t)$, such that

(1°) for every t, $t_o \le t \le t_1$, the function $H(P(t), x(t), u)$ of the variable $u \in U$ attains its maximum at the point $u = u^*(t)$:

$$H(P(t), x(t), u^*(t)) = M(P(t), x(t)) ; \qquad (8.15)$$

(2°) at the terminal time t_1 the relations

$$P_o(t_1) \le 0 \quad , \quad M(P(t_1), x(t_1)) = 0 \qquad (8.16)$$

(3°) and the transversality condition at t_1 are satisfied. Furthermore, if $P^*(t)$, $x^*(t)$ and $u^*(t)$ satisfy equations (8.11) and (8.12), and condition (1°), the time functions $P_o(t)$ and $M(P(t), x(t))$ are constant, and (8.16) may be verified, not just at t_1, but at any time t, $t_o \le t \le t_1$; i.e.,

$$M(P(t), x(t)) = 0 \text{ and } P_o^*(t) \le o \text{ everywhere on } t_o \le t \le t_1. \qquad (8.16a)$$

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7 Orthogonality of $P(t_1)$ and T_1 at x_1 means that the transversality condition is satisfied at that point.

8 See Pontryagin [104, p. 50, Theorem 3].

Remark 8.1

It should be emphasized that the maximum principle and the transversality conditions of theorems VIII.1 and VIII.2 constitute only "necessary" conditions for optimum. In general, the optimal problem cannot be considered solved until the "sufficient" conditions have been investigated. However, by analogy with ordinary calculus, one may ensure that the "necessary" conditions are also "sufficient" by assuming that the Hamiltonian is a concave function.[9]

The nature of the two theorems is qualitative, rather than quantitative. Whereas the theorems identified the differential equations that must be satisfied by $x^*(t)$ and $P(t)$ and the relation that must be satisfied by the optimal Hamiltonian as compared to other Hamiltonians, the theorems are mute on how the value of the optimal control may be computed. It should be added that, by analogy to ordinary calculus, if the maximum is attained at an interior point of the control region U, the maximum condition (equations (8.14) and (8.15) is satisfied only if

$$\frac{H(P(t),\, x(t),\, u)}{\partial u}\Big|_{u = u^*(t)} = o \tag{8.17}$$

and the value of the optimal control may be computed from equation (8.17). If the maximum is attained at the boundary of the control region, the Kuhn-Tucker theory must be applied.

3. The Stochastic Maximum Principle

Preliminary

Consider a system described by the general stochastic

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9 This has been proved by Mangasarian [85]. See also Arrow [6, p. 92] and Shell [120, pp. 252-53].

differential equations

$$dx = f(x,u,t)\,dt + g(x, u, z, t)\,dz \tag{8.18}$$

where

$\{z(t), t \in T\}$ is a Wiener process with incremental covariance Idt. Equation (8.18) may be rewritten as

$$x(t) = x(t_o) + \int_{t_o}^{t} f(x,u,t)\,dt + \int_{t_o}^{t} g(x,u,z,t)\,dz \tag{8.19}$$

where the integrals on the right hand side are interpeted, not as an ordinary Stieltjes integral, but as an Ito integral, a Stratonovich integral or the integral I_λ[10], $o \leq \lambda \leq 1$. It can be shown that if f and g satisfy certain conditions, then equation (8.19) has a unique solution.[10a]

The symbols x and u in equations (8.18) and (8.19) are to be interpreted as vectors of state and control variables, respectively. A more general interpretation of x and u is also possible. For instance, instead of interpreting them as vectors of state and control variables at time t, they may be interpreted as integrals of these variables up to time t. In this latter interpretation, the system is said to possess "memory" which may be finite or infinite depending

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[10] See Astrom [9, P. 71]. The Ito integral is obtained by setting λ equal to zero and the Stratonovich integral by setting λ equal to 0.5. The advantage of the Stratonovich integration is that it permits one to fall back on the ordinary formula for integration by parts, whereas the Ito integration does not. On the other hand, the Ito integral concept has the advantage that the mean value function of the process can, in this case, be intuitively justified. The reason the integrals in (8.19) cannot be interpreted as ordinary Stieltjes integrals is that almost all sample functions of a Wiener process have unbounded variation.

[10a] See Astrom [9, p. 71].

on the limits of integration. Conversely, systems in which x and u are interpreted simply as vectors are called systems without memory. Although systems with memory may be much more realistic to employ in certain situations, they are also much more difficult to handle and the sequel will, therefore, be limited to systems without memory.

The system described by equation (8.18) is general enough to cover cases in which the noise component appears either additively or multiplicatively or both. The cases considered by Kushner [72,73] and Sworder [127-130] can be easily obtained from equation (8.18).

Kushner 's equation

$$dx = f(x,u)\,dt + dz(\omega,t) \qquad (8.20)$$

is the special case obtained from equation (8.18) by setting f(x,u,t) equal to f(x, u) and g(x, u, z, t) dz equal to unity.[11] Note that the noise component $dz(\omega,t)$ in equation (8.20) appears additively.

Similary, Sworder's equation

$$\frac{dx}{dt} = A(\omega, t)\,x(t) + B(\omega, t)\,\nu(t) \qquad (8.21)$$

$$x(o) = x_o$$

is the special case that results from equation (8.18) when f is replaced by $A(\omega, t)\ x(t)$, g by $B(\omega,t)\ \nu(t)$ and dz by dt.
In (8.21), the coefficients A and B are stochastic matrices possessing appropriate dimensions. Hence in this case, the noise appears in the system multiplicatively. In view of the preponderance of multiplicative noise in several economic systems including those of

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11 In this case, the adjoint equation is identical to Pontryagin's [104]. That is, it is a system of ordinary (not stochastic) differential equations. See Fleming [43, p. 195] .

chapter III, the Sworder version of the stochastic maximum principle is of special interest. The problem with the Sworder version of the stochastic maximum principle, however, is that the differential equation for the adjoint variable is significantly different from that of the deterministic (Pontryagin) problem or the stochastic one obtained by Kushner. Because, $A(\omega,t)$ and $B(\omega,t)$ enter into the equation for the adjoint variable, the latter itself becomes a stochastic process. As will be seen later after assuming a particular form for the adjoint variable, the differential equation becomes a stochastic partial differential equation. This may be difficult to solve.

4. Kushner's Stochastic Maximum Principle

Preliminary :

Consider the system governed by the stochastic differential equation

$$dx(\omega,t) = f(x(\omega,t),\ u(\omega,t))dt + dz(\omega,t) \tag{8.22}$$

where

$x(\omega,t)$, $f(x(\omega,t)$, $u(\omega,t)$ and $z(\omega,t)$ are n-dimensional vectors; $x(\omega,t)$ is a stochastic process; and $u(\omega,t)$ called the control, a stochastic process with Lebesgue measurable sample functions, such that for all finite t_1, $\int_{t_0}^{t_1} \| u(\omega,t) \| \, dt < \infty$. For each $u(.,.)$ define a criterion functional

$$J(u) = Ec'\, x(\omega,t_1) = c' \int_{\Omega} x(\omega,t_1) d\omega \tag{8.23}$$

where

the prime denotes transpose;

c is any chosen vector; and Ω is the sample space with points ω ; Equation (8.22) is to be interpreted always as an integral. (See equation 8.18)). When one asserts that (8.22) has a unique solution for a given ω, $u(\omega,.)$ and initial condition $x(\omega,0)$ what is meant is that there exists a unique function $x(\omega,.)$ such that

$$x(\omega,t) = x(\omega,0) + \int_{t_0}^{t_1} f(x(\omega,\tau), u(\omega,\tau))\, d\tau + z(\omega,t) - z(\omega,0) \tag{8.24}$$

for all t, $t_0 \leq t \leq t_1$

The optimal control problem is to select an admissible $u(\omega,t)$, $t_0 \leq t \leq t_1$ such that the corresponding trajectory (see equation (8.22)) starting at $x(\omega,0)$ at time t_0, maximize (8.23)

<u>Notation</u>

$\mathcal{B}$ = the Borel Field over $[t_0,t_1]$;

$\Sigma(t)$ = the minimal σ-field over which $z(.,\tau)$, $\tau \leq t$ is measurable;

$\tilde{\Sigma}$ = $\Sigma(t_1) \times \mathcal{B}$ = the minimal σ-field over which $z(.,.)$ is measurable;

$\mu(d\omega)$ = probability measure on $\Sigma(t_1)$

dt = Lebesgue measure = the measure over the Borel field $[t_0,t_1]$.

$m(d\omega \times dt)$ = the measure on $\tilde{\Sigma}$.

$\Sigma_m(t)$ is the minimal σ-field over which the information available at t is measurable.[12]

Note: The argument ω will often be deleted to simplify notation; e.g., $x(\omega,.)$ and $x(\omega,t)$ may be written as $x(.)$ and $x(t)$.

The stochastic maximum principle will now be presented in the form of lemmas and two theorems corresponding to theorems VIII.1 and VIII.2 given above for the deterministic case. Neither

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12 In general $\Sigma_m(t) \subset \Sigma(t)$.

proofs nor the detailed assumptions necessary for the formulation of the lemmas and theorems will be given.[13]

(a) <u>Stochastic Maximum Principle for autonomous Fixed Time Problem with Variable Right-hand Endpoint</u>

<u>Lemma VIII.1</u>

Let u(.) be an admissible control. Then there exists a null set N, not depending on u(.), such that, for each ω in Ω N (where Ω is the sample space of points ω), the equation

$$x(t) = \int_{t_0}^{t_1} f(x(\tau), u(\tau))d\tau + z(t) - z(0) \tag{8.25}$$

has a unique Lebesgue integrable solution.

Also $E \; || \; x(t) \; || < \infty$, and

$$\int_{t_0}^{t_1} E \; || \; x(t)) \; || \; dt < \infty \tag{8.26}$$

Note: Equation (8.25) is equation (8.24) with ω dropped.

<u>Lemma VIII.2</u>

Assume the conditions of lemma VIII.1. Let x(.) be the solution of (8.25) corresponding to u(.) and let δu(.) be an admissible control. Then, for all ω in Ω N, there is a unique Lebesgue integrable solution x(.) + δ u(.); also δx(.) satisfies

$$\delta x(.) = \int_{t_0}^{t_1} f[(x(\tau) + \delta x(\tau), u(\tau) + \delta u(\tau)) - f(x(\tau), u(\tau))] dt \tag{8.27}$$

and the function defined by (8.28) is Lebesgue integrable in $[t_0, t_1]$.

$$\delta \dot{x}(t) = f(x(t) + \delta x(t), u(t) + \delta u(t)) - f(x(t), u(t)) \tag{8.28}$$

13

For these, see Kushner [72 and 73] .

Furthermore,

$$||\delta x(t)|| \le Kne^{knt_1} \int_{t_0}^{t_1} ||\delta u(t)|| \, d\tau \text{ [14]} \tag{8.29}$$

Lemma VIII.3

Assume an admissible optimal control u(.) exists. Let x(.) correspond to u(.) via (8.25) for each ω in Ω - N. Let P(.) be defined as the solution of the differential equation

$$\begin{aligned} dP(t) &= -f'_x(x(t), u(t))P(t)dt \\ P(t_1) &= c \end{aligned} \tag{8.30}$$

(where f'_x denotes the transpose of a square matrix f_x with its ijth element $f_{x_{ij}}$)

Then, for each ω in Ω N, the solution of (8.30) exists and is unique and uniformly bounded in ω and t, and each component of P(.) is Lebesgue integrable. Define for any admissible u(.),

$$H(x(t), u(t), P(t)) = P'(t) f(x(t), u(t)) \tag{8.31}$$

Then H is well defined and Lebesgue integrable for each ω in Ω - N. Also

$$\int_{t_0}^{t_1} E \, | H(x(t), u(t), P(t)) | \, dt < \infty$$

Theorem VIII.3

Assume the conditions of lemmas VIII.3. Then there exists a null ω set $\hat{N}$ such that for any arbitrary admissible control u(.) and each ω in $\Omega - \hat{N}$, one has

$$E\{H[x(t), u^*(t), P(t)] \mid \Sigma_m(t)\} \ge E\{H[x(t), u(t), P(t)] \mid \Sigma_m(t)\} \tag{8.32}$$

[14] From (8.27), $||\delta x(t)|| \le \int Kn(\delta x(\tau)|| + ||\delta u(\tau))d\tau$. Hence (8.29) follows.

except perhaps on a t set of Lebesgue measure zero.

Recall that $\Sigma_m(t)$ denotes the minimal σ-field over which the information available at t is measurable. Relation (8.32) is the conditional expectation defined by the Radon-Nikodym Theorem[15] as a $\Sigma_m(t)$ measurable function with the property

$$\int_A E\{H[x(t),u(t),P(t)] \mid \Sigma_m(t)\}\, \mu(d\omega) = \int_A H[x(t),u(t),P(t)]\, \mu(d\omega)$$

for any set A in $\Sigma_m(t)$.
The maximum value of (8.32) determines a $u^*(t)$ that is a function of the information available at t only and may intuitively be replaced by the maximization of

$$E\{H[x(t),\ u^*(t),\ P(t)] \mid \text{information available at } t\} \tag{8.33}$$

(b) Stochastic Maximum Principle for autonomous Variable Time Problem with Variable Right-hand Endpoint

Lemma VIII.4

Let $\Phi(t_0,t_1)$ be an n x n matrix such that, for almost all ω, it is defined and measurable with respect to $\tilde{\Sigma}$ and $||\Phi(t_1,t)|| \leq K_1$, for some K_1,
where

$$\tilde{\Sigma}(t_1) = \Sigma(t_1) \times \mathcal{B}$$

Define

$$\nu^i(t) = u(t) + \delta u^i(t) \tag{8.34}$$

where
$\delta u^i(.)$ is an admissible perturbation and u(.) is an admissible control.

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15 See proposition VI.1 in chapter VI.

Let x(.) be the solution of equation (8.22) corresponding to u(.). Then the vectors (8.35) and (8.36) are defined with probability 1 and are finite.

$$h(t,\nu^i) = E\,\Phi(t_0,t_1)\,\{f[x(t),\ \nu^i(t)] - f[x(t),\ u(t)]\} \tag{8.35}$$

$$\theta(\nu_a^i) = \int_{t_0}^{t_1} h(t,\ \nu^i)dt \tag{8.36}$$

Also, for any $\nu^1(.)$, $\nu^2(.)$ and $a, 1 \geq a \geq 0$ there exists an admissible perturbation $\delta u_a(.)$ and $\nu_a(.) = u(.) + \delta u_a(.)$ such that

$$\theta(\nu) = \int_{t_0}^{t_1} h(t,\nu_a)dt = a\theta(\nu^1) + (1-a)\theta\nu^2 \tag{8.37}$$

with probability 1.

In other words, the range of $\theta(.)$ over all admissible $\nu(.)$ is convex.

<u>Theorem VIII.4</u>

Let $u^*(.)$ be the optimal control. Define, for some d, a function P(.) of ω and t such that

$$dP(t) = f_x'\,[x(t),\ u(t)]\,P(t)\,dt \tag{8.38}$$

$$P(t_1) = d$$

Then there exists a null ω set N and a d, such that for any admissible perturbed control $\nu(.) = u^*(.) + \delta u(.)$ and $\omega \in \Omega - N$,

$$E\{P'(t)f[x^*(t),\ u^*(t)]\,|\,\Sigma_m(t)\} \geq E\{P'(t)f[x(t),\nu(t)]\,|\,\Sigma_m(t)\} \tag{8.39}$$

except perhaps on a t set of Lebesgue measure zero.

The transversality conditions are derived as

$$d\gamma = c + \lambda g_\beta(\beta) \tag{8.40}$$

where $\beta = Ex(\omega,t)$, $\beta^* = Ex^*(\omega,t_1)$ is the maximum value of equation (8.23) subject to $Ex(\omega,t)$ being given in a closed convex set G and γ

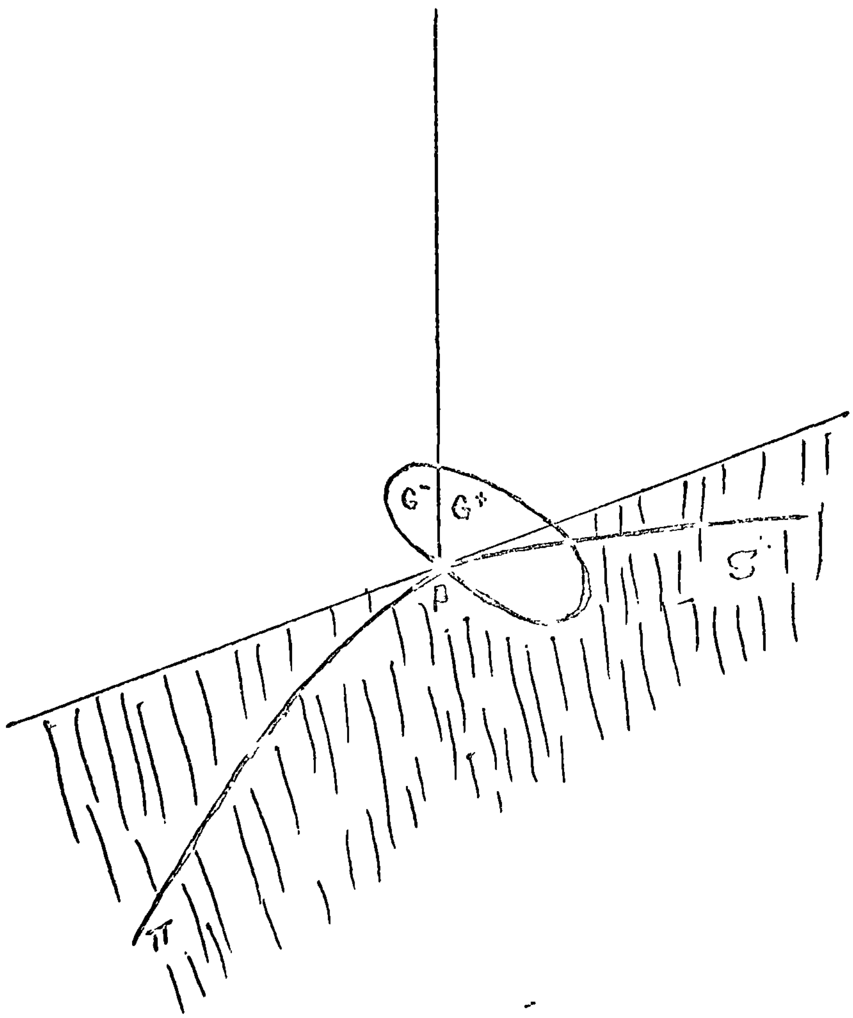

Fig. VI. A Geometric Illustration of the Transversality Condition

Source: Kushner [73, p. 91].

and λ are scalars.

The geometric interpretation of the transversality condition is given in figure VI, in which II is the convex range (over all admissible $\nu(.)$) of the translated function $E x^*(t_1) + \theta(.)$.

5. Multiplicative Noise and the Stochastic Maximum Principle.

Consider a system described by the first order linear stochastic differential equation

$$\frac{dx}{dt} = \dot{x}(t) = A(\omega,t)\, x(t) + B(\omega,t)\, \nu(t),$$

$$x(o) = x_o \tag{8.41}$$

which is reproduced here for ease of reference.

Let $x(t)$ be an n-dimensional state vector. $\nu(t)$, the control, is for convenience taken to be a scalar. Let $A(\omega,t)$ and $B(\omega,t)$ be stochastic matrices whose values determine the transition of the system dynamics. Assume that each element of $A(\omega,t)$ and $B(\omega,t)$ is a separable Markov process with a finite number of states.

Suppose the controller is capable of making use of two different kinds of information: one from the instantaneous state of the system $x(t)$ and the other from the coefficients $A(\omega,t)$ and $B(\omega,t)$. The observations on the coefficients $A(\omega,t)$ and $B(\omega,t)$ are assumed representable in the control law by a vector $\psi(t)$ where $\psi_i(t)$ is the posterior probability at time t that $[A(\omega,t), B(\omega,t)] = [A_i, B_i]$.

The feedback signal accessible to the controller may therefore be written as

$$G(t) = [t, x(t), \psi(t)]^T \tag{8.42}$$

where T denotes matrix transpose. A control $\nu(t)$ is to be selected on the basis of the observed value of G(t), i.e.

$$\nu(t) = u(G(t)) \tag{8.43}$$

where

u, the control rule, is a function from the observation space to the real line. Note that although G(t) and $\nu(t)$ are random, u is nonrandom.

Sworder simplified his problem by assuming that $\psi(t)$ satisfies a differential equation of the form

$$\dot{\psi}(t) = f(\psi(t), s(t), t) \tag{8.44}$$

implying that the system is passive. That is, the evolution of the posterior probability $\psi(t)$ is independent of the state x(t). In equation (8.44), s(t) stands for incoming data from the "plant sensors".[16] Assume also that the joint process given by $\psi(t)$ and $[A(\omega,t), B(\omega,t)]$ is Markovian.

Suppose the decision maker wants to maximize the functional

$$J(u) = E\left\{\int_o^T f^o(\nu(t))\,dt\right\} \tag{8.45}$$

subject to the constraint given by equation (8.41). Assume an optimal control $\nu^*(t)$ exists and that it is a piecewise twice differentiable function of its arguments. $\nu(t)$ is otherwise unconstrained. Suppose that corresponding to ν^* there exists a solution to equation (8.41)

$$\dot{x}^*(t) = A(\omega,t)\,x^*(t) + B(\omega,t)\,u^*(G^*(t)) \tag{8.46}$$

$$x^*(0) = x_o$$

---

16 A "plant sensor" is any equipment or device attached to a plant to provide information about the operation of the plant. In the economic application to be tackled in chapter IX, a plant sensor is to be interpreted as any mechanism by which the consumer can acquire experience regarding past behaviour of the rate of return. It is not too much to assume that he knows the past states of his capital stock.

where

$$G^*(t) = [t, x^*(t), \psi(t)]^T$$

and

$$\nu^*(t) = u^*[G^*(t)]$$

The adjoint variable P(t), itself now a stochastic process, is defined by

$$\dot{P}(t) = [-A(\omega, t)^T \quad (B(\omega, t) \frac{\partial u^*(G^*(t))}{\partial x^*}^T] \quad P(t)$$

$$+ \left(\frac{\partial f^o}{\partial x^*} \cdot \frac{\partial v^* [G^*(t)]}{\partial x^*} \right)^T \qquad (8.48)$$

$$P(T) = 0$$

The important difference between equation (8.48) and its deterministic counterpart (equation (8.9)) or the Kushner stochastic counterpart (equation (8.30)) is the appearance of extra terms (precisely $\frac{\partial u^*}{\partial x^*} (G^*(t))$ which result from the explicit feedback nature of the problem (see equations (8.43) and (8.44)). Deterministic systems require no feedback law. In the Kushner stochastic case, the feedback does not affect the optimal control because the control is not an argument of the criterion functional.

Theorem VIII.5

Denote the stochastic Hamiltonian by

$$H(x, P, u, G, t)$$
$$= P(t)^T [A(t)\, x(t) + B(t)\, u(G(t))]$$ [17]
$$+ f^o(x, u(G^*(t))) \qquad (8.49)$$

Then, if u^* is an optimal control

$$E \{H(x^*, P^*, u^*, G^*, t \mid G^*(t)\}$$
$$\geq E \{H(x^*, P, u, G^*, t \mid G^*(t)\} \qquad (8.50)$$

---

[17] The argument omega has been dropped from the coefficients $A(\omega, t)$ and $B(\omega, t)$ to simplify notation.

almost surely for all $t \in [0,T]$[18]

Using this theorem and the assumption that u is a twice differentiable piecewise continuous function of its arguments, the optimal control u* may be evaluated by setting the derivative of the left hand side of relation (8.50) with respect to u equal to zero. That is

$$E\{P(t)^T \quad B(t) \mid G^*(t)\} - \frac{\partial f^o}{\partial x} \cdot \frac{\partial u}{\partial x} = o, \text{ a.s.} \tag{8.51}$$

In earlier sections of this chapter, the development of the maximum principle (deterministic as well as stochastic) did not proceed beyond this point. See Remark 8.1. Without assuming special forms for P and f^o, it is not possible to obtain general analytic solutions. Hence, to illustrate the steps involved assume

$$f^o = x(t)^T R\, x(t) + \nu(t)^2 \tag{8.52}$$

and

$$P(t) = 2\, K(t)\, x^*(t) \tag{8.53}$$

The deterministic counterpart of this problem has been solved by Athans and Falb [10, chap. 9] who advance the case for the special form taken by f^o and P. The argument for (8.53) is that the canoni cal equations relating x(t) and P(t) is linear. (8.52) is justified on practical grounds. The contribution of Sworder to this development lies in treating K(t) as a stochastic matrix.

Utilizing (8.52) and (8.53) in (8.51) gives the optimal control rule as

$$u^*(G^*(t)) = E\ \{B(t)^T K(t) \mid G^*(t)\}\ x(t) \tag{8.54}$$

18

See Sworder [127, P. 182] .

Up to this point the development of the stochastic counterpart of the maximum principle has been quite straightforward. However, equation (8.54) introduces the first complication that must be dealt with as a result of the stochastic nature of the problem. In order to obtain an explicit expression for u^*, one must evaluate the conditional expectation in equation (8.54). Using equations (8.53), (8.48), (8.46) and (8.54) one can write a differential equation for $K(t)$ as

$$\dot{K}(t) = -A(t)^T K(t) \quad K(t)\,A(t) + R - 2K(t)\,B(t)\,\overline{B(t)^T K(t)} + \overline{K(t)\,B(t)} \quad \overline{B(t)^T K(t)} \tag{8.55}$$

where

$$\overline{B(t)^T K(t)} = E\,\{\,B(t)^T K(t) \mid G(t)\,\}$$

Because of the conditional expectations on the right hand side of (8.55) K cannot be obtained by direct integration as could be done in a deterministic problem. The conditional expectations depend not only on the present value of K but also on what one expects its future value to be. Written out in full, the conditional expectation of $\dot{K}(t)$ given $\psi(t)$ and the event $[A(t), B(t)] = [A_i, B_i]$, is a partial differential equation in $K(t)$.

$$E\,\{\,\dot{K}(t) \mid \psi(t), [i(t)]\,\} = \frac{\partial \bar{K}_i(\psi(t), t)}{\partial t} + \frac{\partial \bar{K}_i(\psi(t), t)}{\partial \psi(t)} E\,\{\,\dot{\psi}(t) \mid \psi(t), [i(t)]\,\} + \sum_{j=1}^{m} \bar{K}_j(\psi(t), t)\, q_{ij}(t) \tag{8.56}$$

where

$$\text{Prob}\,([j(t+\Delta)] \mid [i(t)]) = \begin{cases} q_{ij}(t)\,\Delta + o(\Delta), & i \neq j \\ 1 + q_{ij}\,\Delta + o(\Delta), & i = j \end{cases}$$ [19]

$$\bar{K}_i(\psi(t), t) = E\{K(t) \mid G^*(t), [i(t)]\}$$

$[i(t)]$ = the event $[A(t), B(t)] = [A_i, B_i]$

Substituting the right hand side of (8.56) for $\dot{K}(t)$ in (8.55) and dropping the arguments of $\bar{K}_i$, one obtains

$$\frac{\partial \bar{K}_i}{\partial t} + \frac{\partial \bar{K}_i}{\partial \psi(t)} E\{\dot{\psi}(t) \mid \psi(t), [i(t)]\} + \sum_{j=1}^{m} \bar{K}_j q_{ij}(t)$$

$$= \quad A_i^T \bar{K}_i - \bar{K}_i A_i + R \quad 2\bar{K}_i B_i \sum_{j=1}^{m} B_j^T K_j \psi_j(t)$$

$$+ \left(\sum_{j=1}^{m} \bar{K}_j B_j \psi_j(t)\right)\left(\sum_{j=1}^{m} B_j^T \bar{K}_j \psi_j(t)\right) \qquad (8.57)$$

$$\bar{K}_i(\psi(T), T) = o, \forall \; \psi(T), i$$

where

$$A_i^T \; \bar{K}_i = E\{A(t)^T K(t) \mid G^*(t), [i(t)]\},$$

and

$$\sum_{j=1}^{m} B_j \bar{K}_j (\psi_j(t)) = \overline{B(t)^T K(t)}$$

In general, the partial differential equation in (8.57) is difficult to solve. However, if "certain conditions" are met the partial differential equation may be reduced to an ordinary differential equation by employing the method of characteristics.[20] Suppose, in the second term

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[19] See Cox and Miller [25, sec. 4.5].

[20] See Garabedian [49, chap. 2] or any text on first order partial differential equations.

on the left hand side of (8.57)

$$E\{\dot{\psi}(t) \mid \psi(t), [i(t)]\} = E\{(f(\psi(t), s(t), t) \mid \psi(t), [i(t)]\}^{21}$$

$$= \bar{f}(\psi(t), t, i) \quad (8.58)$$

Then if

$$\dot{h}(\tau; i) = \bar{f}(h(\tau), \tau, i), \quad t \le \tau \le T$$

$$h(t; i) = \psi(t); \quad i = 1, \ldots, m \quad (8.59)$$

and

$$\begin{aligned}\dot{\hat{K}}_i(\tau) &= -A_i^T \hat{K}(\tau) \quad \hat{K}_i(\tau) A_i + R - 2\hat{K}_i(\tau) B_i \sum_{j=1}^{m} B_j^T \hat{K}_j(\tau) h(\tau, i) \\ &+ \left(\sum_{j=1}^{m} \hat{K}_j(\tau) B_j h_j(\tau, i)\right)\left(\sum_{j=1}^{m} B^T \hat{K}_j(\tau) h_j(\tau, i)\right) \\ &- \sum_{j=1}^{m} \hat{K}_j(\tau)\, q_{ij}(\tau)\end{aligned} \quad (8.60)$$

$$\hat{K}(T) = 0$$

where

$$\hat{K}_i(t) = \bar{K}_i(\psi(t), t)$$

The optimal control rule is

$$u^*(G^*(t)) = \sum_{j=1}^{m} B_j^T \hat{K}_j(t)\, \psi_j(t)\, x^*(t) \quad (8.61)$$

Equations (8.59), (8.60) and (8.61) give the solution to the problem posed by equations (8.45) and (8.46). The solution equations are rather complicated but not as much as they look. First note that (8.59) and (8.60) may be solved independently of each other and the values of ψ_j and $\hat{K}_j$ obtained, substituted in (8.61). This independence removes problems associated with two-point boundary problems.

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21 From equation (8.44).

The procedure for obtaining the solution is to first integrate equation (8.59) forward in time, and then use its results in (8.60) to integrate backward in time. The control rule, equation (8.61), involves only algebraic manipulations of the solutions to equations (8.59) and (8.60). Note, however, that since equation (8.60) is Riccati, closed-form solutions are in general not available, and one must resort to a digital computer. An algorithm for computing the deterministic counterpart of this equation is available in Athans and Falb [10, P. 767] .

The solution equations suggest some interesting qualitative properties implicit in the system. These are summarized in Note VIII.1.

Note VIII.1[22]

(a) Equation (8.59) indicates how fast the controller 'adapts' to changes in its dynamics. As the solution approaches the vector

$$\psi_j(t) = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}$$

equation (8.60) rapidly approaches the corresponding equation for observable [A(t), B(t)] .

(b) If in addition to (a), the transition probabilities of the process are small, then equation (8.60) converges to the deterministic Riccati equation.

(c) The certainty equivalence principle of chapter VII does not apply, since the differential equation for $\dot{h}(t)$ does not even give the expected value of $\psi(t)$.

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22 See Sworder [127, P. 185] .

6. The Pontryagin Kushner Sworder Maximum Principles Compared

For the purpose of comparison, it is instructive to gather together the main strands of the maximum principle as developed by Pontryagin, Kushner and Sworder. Table I serves this purpose. The pertinent equations are

(1) equations (8.7), (8.22) and (8.42) for the transition equation;

(2) equations (8.9), (8.30) and (8.48) for the adjoint function; and

(3) equations (8.10), (8.31) and (8.49) for the Hamiltonian.

From Table I it is clear that there is practically no difference between Pontryagin and Kushner. In particular, the equations for the adjoint variable and the Hamiltonian in both cases are identical. On the other hand, some complication is evident in the Sworder version. In particular, the equation for the adjoint variable in the Sworder version is written as a new stochastic process in terms of the gradient of the state variable vector x.[23] Whereas the Pontryagin as well as Kushner P(t) function is obtained from a set of ordinary differential equations, the Sworder equation for P(t) generally leads to a set of stochastic partial differential equations. Some reason for this phenomenon was advanced in the paragraph immediately following equation (8.55) above. In general, partial differential equations are more difficult to solve than ordinary differential equations. The attack suggested by Sworder above is to reduce the partial stochastic to an ordinary stochastic differential equation by using the method of

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23 See the paragraph immediately following equation (8.48), P.139 above.

TABLE I

THE PONTRYAGIN - KUSHNER - SWORDER MAXIMUM PRINCIPLE

Transition Equation	(1) Pontryagin:	$\frac{dx}{dt} = f(x,u)$	(8.7)
	(2) Kushner:	$dx(t,\omega) = f(x(t,\omega), u(t,\omega))dt + dz(t,\omega)$	(8.22)
	(3) Sworder:	$\frac{dx}{dt} = A(t,\omega)x(t) + B(t,\omega)\,\nu(t)$	(8.41)
		$G(t) = [t, x(t), \psi(t)]$	(8.42)
		$\nu(t) = u[G(t)]$	(8.43)
Adjoint Equation	(1) Pontryagin:	$\frac{dP}{dt}(t) = -\frac{\partial f}{\partial x}(x(t), u(t))\,P(t)$	(8.9)
	(2) Kushner:	$dP(t) = -\frac{\partial f}{\partial x}[x(t), u(t)]\,P(t)\,dt$	(8.30)
	(3) Sworder:	$\frac{dP}{dt} = -[A(t,\omega)+B(t,\omega)\frac{\partial u^*(G^*(t))}{\partial x^*}]^T P(t) + (\frac{\partial u^*}{\partial x^*} \cdot \frac{\partial u^*(G^*(t))}{\partial x^*})^T$	(8.48)
		$P(T) = 0$	
The Hamiltonian	(1) Pontryagin:	$H(P,x,u) = (P, f(x,u)) = P\,f(x,u)$	(8.10)
	(2) Kushner:	$H(P,x,u) = P\,f(x,u)$	(8.31)
	(3) Sworder:	$H(t,x,u,G) = P(t)^T[A(t,\omega)x+B(t,\omega)u[G(t)]+f^o(u(G(t)))$ a.s.	(8.49)

characteristics. However, this device does not remove the Riccati nature of the resulting ordinary differential equations. There are no generally known methods for solving Riccati equations analytically although they can be solved quite fast by modern digital computers.

7. Dynamic Programming Approach to the Sworder Problem

In chapter IX, the Sworder version of the stochastic maximum principle will be applied to the economic problem surveyed in chapter III. As was seen in chapter III and mentioned in the introductory part of chapter VI, dynamic programming has been utilized to advantage for these and a wide variety of similar problems. Dynamic programming is a valuable tool in its own right and how the method ties in with the maximum principle has been clearly demonstrated by Rozonoer [111] as far back as 1959. The advantage of this symmetry between the two tools has, however, not been adequately exploited by economists who have been applying these tools for nearly a decade now. A combination of ideas gathered from chapters II and III with the relationship between dynamic programming and the maximum principle to be highlighted later in this section could give a lead on the possible economic interpretations of these tools.

The usual argument of the proponents of the maximum principle is that dynamic programming does not have any rigorous logical basis in several of the cases where it may be successfully utilized as a valuable heuristic tool. Pontryagin [104, P. 73], Fel'dbaum [38, P.85] and Wishart [143, P. 314] may be cited in this regard. Precisely, the argument is that the optimal value function to be introduced below is arbitrarily assumed to possesss continuous partial derivatives with respect to x and t. While this weakness may be important to mathematicians, economists may prefer to gloss over it. Therefore, in order to exploit the advantage of the established symmetry between dynamic programming and the maximum principle, the remainder of this section will briefly present the basics of the dynamic programming approach to the Sworder problem of section 5.

To solve the problem of equations (8.45) and (8.41), dynamic programming calls for the introduction of an optimal value function $J^o(x(t), t)$, $o \leq t \leq T$, defined as

$$J^o(x(t)) = E\left\{ \max_{\substack{u(\tau) \in U \\ x(t) \in X}} \int_t^T f^o(\nu(t))\,dt \right\} \qquad (8.62)$$

where E is the expectation operator. The basis of dynamic programming rests on the assumption (in addition to the assumptions needed to formulate the maximum principle) that the function $J^o(x(t))$ has continuous partial derivatives with respect to x and t.

Having made the foregoing remark, the fundamental partial differential equation of dynamic programming is usually derived in the following manner. First, it is shown that the function $J^o(x(t))$ may be written as

$$J^o(x(t)) = E\left\{ \max_{\substack{u \in U \\ x(t) \in X}} \int^{t+\Delta t} f^o(\nu(t))dt + \int_{t+\Delta}^{T} f^o(\nu(t))\,dt \right\} \qquad (8.63)$$

or, for sufficiently small Δt,

$$J^o(x(t)) = E\left\{ \max_{\substack{u \in U \\ x \in X}} f^o(\nu(t))\,\Delta t + J^o[x(t+\Delta t), t+\Delta t] \right\} \qquad (8.64)$$

A Taylor series expansion of $x(t + \Delta t)$ gives

$$x(t + \Delta t) = x(t) + \dot{x}(t)\,\Delta t + \frac{1}{2}\,\ddot{x}\,(t + \theta\Delta t)(\Delta t)^2$$

where $0 \leq \theta \leq 1$. Substituting for $\dot{x}$ from equation (8.41)

$$x(t + \Delta t) = x(t) + [A(t)\, x(t) + B(t)\, \nu(t)]\, \Delta t + 0_1(\Delta t) \tag{8.65}$$

where $0_1(\Delta t)$ denotes a small quantity of small order higher than Δt.

Substituting (8.65) in the second term of the right hand side of (8.64) gives

$$J^o(x(t + \Delta t), t + \Delta t)$$

$$= J^o\{ x(t) + [A(t)\, x(t) + B(t)\, \nu(t)]\, \Delta t + 0_1(\Delta t),\ t + \Delta t\} \tag{8.66}$$

Applying a Taylor expansion on the function J^o of (8.66) in the neighbourhood of the point (x, t) yields

$$J^o[x(t + \Delta t), t + \Delta t]$$

$$= J^o[x(t), t] + \{ \nabla J\, [x(t), t]\, [A(t)\, x(t) + B(t)\, \nu(t)] \}\, \Delta t \tag{8.67}$$

$$+ \frac{\partial J^o}{\partial t} [x(t), t]\, \Delta t + 0_2(\Delta t)$$

where $0_2(t)$ is a quantity of higher small order compared to Δt, and ∇J^o is the gradient of J^o with respect to x.

Substituting (8.67) into (8.64), dividing by Δt and letting $\Delta t \rightarrow o$, the formula takes the form

$$\frac{\partial J^o}{\partial t} [x(t), t] = \max \{ f^o(\nu(t)) + \nabla J^o [x(t), t] .$$
$$[A(t)\, x(t) + B(t)\, \nu(t)] \} \tag{8.68}$$

which is known as the Bellman equation. (8.68) is a singular partial differential equation since as a result of the maximisation x(t) vanishes from the right hand side for all $t \in [0, T]$. In the general case this equation cannot be solved analytically and numerical methods must be used. However, for the purpose of comparison with the stochastic maximum principle, assume again that f^o takes the form given by equation (8.52). Substituting (8.52) in (8.68) yields

$$0 = \max E \left\{ [x(t)^T R\, x(t) + \nu(t)^2] + \frac{\partial J^o}{\partial x}(Ax + B\nu) + \frac{\partial J^o}{\partial t} \right\} \quad (8.69)$$

from which the optimal control is

$$u(G(t)) = -\frac{1}{2} E(B \mid G(t)) \frac{\partial J^o}{\partial x} \quad (8.70)$$

If $J^o(x, t)$ is of the form

$$J^o(x, t) = K(t)\, x^2(t) \quad (8.71)$$

then there is no difference between equations (8.70) and (8.54) once the interpretation of $\frac{\partial J^o}{\partial x}$ in observation VIII.1 below is recalled.

Observation VIII.1 Relationship between the Maximum Principle and Dynamic Programming.

Some interesting results can be obtained on closer examination of equation (8.68) and comparison of same with the expected value of equation (8.49), the stochastic Hamiltonian. The right hand sides of these two equations are identical once P(t) is defined equal to ∇J^o. It means that the maximum of the Hamiltonian of the maximum principle is the same thing as the partial derivative (with respect to time) of the optimal value function provided the partial derivative of the latter function (with respect to x) is defined equal to the adjoint variable P(t). This last statement is not arbitrary; rather, it is a condition that must hold on the optimal trajectory, and was demonstrated by Rozonoer [111, part 3] as far back as 1959.[24]

8. Concluding Comments

No doubt, several loose ends are evident in the discussion of the preceding sections. A brief comment in this regard will be useful before terminating this chapter. First, it is worth repeating that the

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[24] See also Bryson and Ho [19, pp 49 and 134] .

development of the stochastic counterpart of the maximum principle is just beginning.

Section 3 made it clear that in the attempt at developing the stochastic maximum principle attention has so far been concentrated on a restrictive class of stochastic differential equations. The class of problems to which theorems VIII.3 and VIII.5 may be applied are different and the proofs of the theorems are also different.[25] It is not to be construed from sections 4 and 5 that the respective maximum principle theorems can be upheld for all systems of additive and multiplicative disturbances. Other important assumptions employed in the development should be noted. In Kushner [72,73], the only constraint on the admissible control is that its integral over the control time exists. In Sworder, the control is virtually unconstrained. While Kushner's transition equation is nonlinear, that of Sworder is linear, and there is no assurance that his theorem carries over to nonlinear systems with multiplicative noise.

Also, both Kushner and Sworder treat systems in which initial conditions are known, the time horizon is non-stochastic and the states are perfectly observed. Sworder has no comment on what happens if the horizon is stochastic. Kushner notes that problems with stochastic horizon involve the addition of several new concepts to an already complicated problem. What those new concepts are, are not known. These circumstances dictate the type of illustrative problem attempted in the next chapter.

25

Sworder [130, p. 36] .

CHAPTER IX

ECONOMIC APPLICATION OF THE STOCHASTIC MAXIMUM PRINCIPLE

In the economic literature, the commonest example of the application of the maximum principle at the deterministic level is the derivation of optimal properties for one sector models of economic growth. In this connection, reference was made in preceding chapters to the works of Arrow [6], Dobell [28], Dorfman [31] and Shell [120] among others. This chapter is an attempt at providing a stochastic prarllel to these works. Alternatively, the present chapter may be regarded as a reinterpretation of the general model in chapter III above. The vehicle of analysis will be the tools developed in chapter VIII, section 5.

As in chapter III, the Samuelson-Merton Model will be used as the pivot of discussion. Section 1 reproduces this model and describes it in the language of chapter VIII, section 5. Section 2 attempts some economic interpretations of the major relations. A number of interesting results are derived which may be regarded as stochastic counterparts of the Keynes-Ramsey rule. Some of the results are particularly easy to derive when one exploits known relationships between dynamic programming and the maximum principle which were noted in observation VIII.1. An example of this is the interpretation of the adjoint function P(t) as a marginal indirect utility function. Section 3 is addressed to a number of computational problems involved in the derivation of the optimal control (i.e., the optimal consumption path) using the Sworder stochastic maximum principle. These problems are removed once the model can be treated as a "state regulator". The major findings of the chapter are summarized in section 4.

1. Model Formulation and Description

A stochastic version of the Samuelson-Merton model in continuous time may be written as

$$\dot{y}(t) = r(\omega,t)y(t) - c(t) \tag{3.11}$$

$$y(o) = y_o > 0 \tag{3.12}$$

where

$y(t)$ is capital stock at time t

$c(t)$ is rate of consumption at time t

$r(\omega,t)$ is a stochastic rate of return at time t, and

$$\dot{y}(t) = \frac{dy(t)}{dt}$$

The consumer is assumed according to this model to generate all his income from the yield of his capital investment. He has no wage income. He has initial capital y_o which he invests at an exogenous rate of return $r(\omega,t)$. The yield on his capital at time t is, therefore, $r(.,t)\,y(t)$, a random variable out of which he consumes $c(t)$.

The process described by equation (3.11) is Markovian since the equation is a first order linear stochastic differential equation; i.e., a state model.[1] The equation is a particular case of equation (8.41) in which $A(\omega,t) = r(\omega,t)$, $B(\omega,t) = -1$, $x(t) = y(t)$ and $\nu(t) = c(t)$. In the Samuelson-Merton model instead of the process $r(\omega,t)$ one has and ordinary random variable $r(t)$ independently distributed with respect to time; i.e. no serial correlation is permitted. For the

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1. See Astrom [9, P. 44] .

application of the Sworder maximum principle, $r(\omega,t)$ is assumed to be a separable[2] Markov process with a finite number of states. In other words, for each t, one only need to assume that the random rate of return $r(.,t)$ assumes a finite number of values m, each with a posterior probability $\psi_i(t)$. It is not necessary to assume away serial correlation. This is an advantage over the Samuelson-Merton analysis.

In practice, separability has the advantage of enabling one to reduce the index set of the process (which may not be countable) to a countable set. The assumption is not restrictive since, by proposition VI.8, for every stochastic process X_t, there exists a separable process $\hat{X}_t$ defined on the same probability space such that $P(X_t = \hat{X}_t) = 1$. The Markovian assumption can be justified by invoking propostion VI.3 the generality of the Markov process. The finiteness of the number of states m is not regarded as restrictive for the present problem since in principle m may be chosen as large as one pleases, short of infi nity.

Turning to the interpretation of the feedback control law, equation (8.43) is rewritten in the present context as

$$c(t) = \ell\,[G(t)] \tag{9.1}$$

where, by equation (8.42),

$$G(t) = [t,\ y(t),\ \psi(t)]' \tag{9.2}$$

$\psi_i(t)$ is the posterior probability at time t that $r(.,t) = r_i(.,t)$;

the prime denotes transpose;

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2 See definition VI.17 in chapter VI.

and, by equation (8.43),

$$\dot{\psi}_i(t) = f\,[\,\psi(t),\ s(t),\ t\,] \tag{9.3}$$

where

$\psi(t)$ is the entire vector $[\,\psi_1(t),\ \psi_2(t), \ldots, \psi_m(t)]$,

and

$s(t)$ is a scalar function representing data from the "plant sensors"[3]; i.e. the experience based on past behaviour of $r(\omega, t)$, thus giving the model a sort of Bayesian character.[4]

The explicit appearance of t on the right hand side of equation (9.3) could be given an interpretation analogous to technological progress in a production function. In this sense, t may symbolize the growth in the efficiency of the information gathering device (the plant sensor).

The feedback control law equation (9.1) makes consumption at time t a function of capital on hand at time t, the posterior probability distribution over $r(.,t)$, and the time the decision is being made. Observe that although $G(t)$ and thus $c(t)$ are random, the function ℓ is not random.

To complete the description of the control problem, it must be assumed that the consumer has a well behaved utility function, u defined over his consumption at time t, and that his objective is to maximize the criterion functional

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3 See footnote 16 in chapter VIII.

4 However, the subjective element inherent in the prior probability distribution of Bayesian analysis seems lacking in this model.

$$J(c(t)) = E \int_0^T e^{-\rho t} u[c(t)]\, dt \tag{9.4}$$

subject to the constraints imposed by equations (3.11) and (3.12).

Following the procedure of chapter VIII, section 5, the adjoint function, itself now a stochastic process, is defined by

$$\dot{P}(t) = -e^{-\rho t} \left\{ \frac{\partial u}{\partial y^*} \right\} \frac{\partial \ell^* [G^*(t)]}{\partial y^*} - \left\{ r(\cdot, t) - \frac{\partial \ell^* [G^*(t)]}{\partial y^*} \right\} P(t) \tag{9.5}$$

$$P(T) = 0$$

where

the stars denote functions evaluated at their optimal levels, and

$$G^*(t) = [t, y^*(t), \psi(t)]'$$

Note that it is not necessary to assume a priori that $P(t)$ is a stochastic process. It is by definition a stochastic process since $r(\omega, t)$ is a stochastic process.

2. Economic Interpretation of the stochastic maximum principle

The function $J(c(t))$ defined by equation (9.4) was interpreted in chapter II (see equation (2.3)) as the relevant utility function in the context of temporal uncertainty. The corresponding function under timeless uncertainty would be defined as

$$J^o(y(t)) = E \max_{\ell} \int_0^T e^{-\rho t} u\{\ell[G(t)]\}\, dt \tag{9.6}$$

Equation (9.6) can be recognized as the definition of the optimal expected value function in stochastic dynamic programming.[5] One may

5. See equation (8.62) in chapter VIII.

therefore, infer that the stochastic dynamic programming approach to dynamic optimisation under uncertainty always involves a first step in which the inherent "temporal" uncertainty is reduced to "timeless" uncertainty.[6] The interpretation may be regarded as a partial answer to the skepticism expressed in Crabbé and Odaro [25, p. 5] as to whether or not the distinction between "temporal" and "timeless" uncertainty can be usefully extended beyond the two-period analysis, when the Fama [36] proposition does not hold.[7] The right hand side of equation (9.6) requires that $y(t)$ be known at the time $c(t)$ is chosen. The explicit dependence of the left hand side on $y(t)$ suggests that the expected optimal value function of stochastic dynamic programming may be interpreted as a dynamic analogue of the usually "metastatic" indirect utility function[8] in conventional economic theory.

The $P(t)$ which solves the differential equation (9.5) satisfies the interpretation given in observation VIII. 1 (p.150). In other words $P(t)$ is equal to the gradient with respect to $y(t)$ of the expected optimal value function. Therefore, it follows that if the optimal value function is called an indirect utility function, the adjoint function $P(t)$ must be

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6 In view of proposition II. 1 namely that temporal uncertainty is never preferred to timeless uncertainty described by the same mass or density function, this two step approach of stochastic dynamic programming seems to make a lot of sense.

7 E.g., when T is infinite.

8 Fama [36 p. 166] has given this interpretation.

interpreted as a "marginal indirect utility function".

It is true that in the deterministic framework, a general "shadow price" type of interpretation for P(t) has been recognized by a number of writers,[9] but the specific interpretation suggested here has never before been recognized in the literature dealing with the subject matter of this thesis. This specific interpretation is, therefore, to be underlined. In the stochastic framework, the marginal indirect utility function (the "shadow price" function) depends on the state of the world. In other words, for each y(t), there are now as many shadow prices as there are states of the world; the gradient of the optimal value function is stochastic.

It would be interesting to be able to compare the motion of P(t) in the deterministic with that in the stochastic context. The difficulties on the way of such comparison will be discussed in section 3 below. For now, the task of interpretation turns on the stochastic Hamiltonian.

Following usual procedure, the stochastic Hamiltonian is obtained from equations (3.11) and (9.4) as

$$H(y, P, \ell, G) = e^{-\rho t} u \{ \ell [G(t)]\} + P(t) [r(\omega,t)y(t)-c(t)]$$
$$= e^{-\rho t} u [c(t)] + P(t) \dot{y}(t) \qquad (9.7)$$

where

the relationship (9.1) has been used, and G(t) is as defined by equation (9.2).

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9 See, e.g. Arrow [6,p.88] and Dorfman [31] for interpretation in optimal control context and Zangwill [153, pp. 62-68] in the context of mathematical programming.

The stochastic Hamiltonian is made up of two parts:

(1) the discounted utility of consumption at time t; and

(2) investment valued in units of "marginal indirect utility", P(t).

Hence, the Hamiltonian is the sum of instantaneous flows of utility from all sources, including a part enjoyed immediately, $e^{-\rho t} u$ and a part expected to be enjoyed in future $P(t)\,\dot{y}(t)$. This double-component interpretation of the Hamiltonian was recognized by Arrow [6, P.88] in the deterministic context, but in the present stochastic context, the interpretation is good only for one state of the world ω. A different flow of utility results for each state of the world.

Next, by invoking theorem VIII.1, sub-sections (1^o) and (2^o), and assuming that the left hand side of relation (8.50) (p.139above) does not violate the conditions of these two sub-sections, it requires no computations whatever to show that the famous Keynes-Ramsey rule must hold. The theorem says that on substituting the optimal control path $\ell^*(G(t))$ into the expected Hamiltonian, the latter equals M, a constant, which may be interpreted as Ramsey's Bliss point B. Therefore, on substituting ℓ^* on the right hand side of (9.7) and taking expectation, the Ramsey rule

$$B \quad e^{-\rho t} Eu = EP(t)\,\dot{y}(t) \tag{9.8}$$

must hold.

Observation VIII.1 yields another interesting economic interpretation in the context of the present problem. Recalling the relationship

$$\frac{\partial EJ^o}{\partial t} = \max EH = M,$$

one can rationalize the following:

(1) if the consumer has a "flexible" planning horizon, and if his investment $\dot{y}(t)$ depends only on his capital and consumption, and not on time, then his indirect utility will be a constant function of time, since M equals zero[10]; whereas

(2) if the consumer has a fixed planning horizon and/or if his investment depends on time as well as on his capital and consumption, then his indirect utility will not be a constant function, but will change through time at a constant rate M.[11]

It may be interesting to make a few comments here concerning the impact of uncertainty. Of course, the first point to note is that uncertainty introduces lack uniqueness into the system. Since the Hamiltonian is now defined only for one state of the world at a time, it means that there are as many different flows of utility as there are states of the world. Without prior knowledge of the resulting Hamiltonian for each ω, it is not possible to compare the flow of utility under uncertainty with the flow under certainty. This comparison if it were possible, would indicate the "overall" impact of uncertainty. For the "marginal" impact, one needs an acceptable measure of uncertainty to be explicitly included in the stochastic Hamiltonian and with respect to which the latter must be partially differentiated. It is not clear how one may go about this.

Back to equation (9.7), if the optimal consumption path $c^*(t)$

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[10] Provided the left hand side of relation (8.50) (see p.139 above) obeys relation (8.16a), p. 126 above .

[11] Provided the left hand side of relation (8.50) obeys condition 2^{o} of theorem VIII.1. See p.125 above.

is substituted for $c(t)$, then the relation

$$E\{e^{-\rho t} u[\ell^{*}(G^{*}(t))] + P(t)\dot{y}(t) \mid G(t)\}$$
$$\geq E\{e^{-\rho t} u[\ell(G(t))] + P(t)\dot{y}(t) \mid G(t)\} \tag{9.14}$$

must hold.

The left hand side Hamiltonian in (9.14) satisfies the relation

$$\frac{\partial E(H \mid G(t))}{\partial c} = 0 \tag{9.15}$$

which simply says that at the optimum, the expected total flow of utility must be irresponsive to consumption changes. The complete expression for equation (9.15) is

$$e^{-\rho t} \frac{\partial E\{u[\ell(G(t))]\}}{\partial \ell} - E[P(t) \mid G(t)] = 0 \tag{9.16}$$

where $c(t) = \ell(G(t))$ has been substituted from(9.1). Equation (9.16) shows that the optimal consumption is such that the discounted marginal expected utility of consumption equals expected marginal indirect utility of capital.

In general, it is not possible to say whether or not the $\ell^{*}(t)$ which solves (9.16) will increase or decrease as uncertainty increases since this depends on the explicit form of the right-hand side of the equation. However, following similar argument by Rothschild and Stiglitz [110, p. 69], it seems reasonable to say that increased uncertainty will increase optimal $\ell(t)$ if $E(P(t) \mid G(t))$ is concave and decrease it if $E(P(t) \mid G(t))$ is convex.

3. Computational Problems

Synopsis

Analytic solution for $\ell^{*}(t)$ is generally not possible without assuming specific forms of utility and adjoint functions. In fact the appeal of the maximum principle, even in the deterministic framework, has never been claimed to lie in its computational efficiency. As will be seen in this section considerable difficulties arise in the attempt to obtain explicit solution even when the particular utility function of chapter III is assumed. Closed-form solution follows immediately on assuming equation (9.17) below for the utility function. But that closed form solution is written in terms of a function which is itself unknown. Once an acceptable function is given for $E(P|G(t))$ the path of the optimal control can be easily described. But the difficulty in obtaining $P(t)$ lies in the fact that it is interdependent with $y(t)$, itself an unknown function. The efficiency of the techniques described in chapter VIII thus seems greatly reduced once a criterion other than a quadratic is employed. In order to elicit some of the main problems involved, the Samuelson-Merton Model will be used.

Samuelson-Merton Model Without Portfolio the Phelps-Ramsey Problem

The model formulated in section 1 and interpreted in section 2 does not involved any portfolio problem since it involves only one asset. In the language of chapter III this special case of the Samuelson-Merton model is called the Phelps-Ramsey problem.

If the specialized utility function,

$$u(c) = \frac{1}{\gamma} c^{\gamma} = \frac{1}{\gamma} \ell [G(t)]^{\gamma} \qquad (9.17)$$

where

γ, $0 < \gamma < 1$ is the elasticity of the utility function, employed in chapter III is substituted for $u(\ell)$ in equation (9.16), then it is easy to show that the optimal consumption path $\ell^*(t)$ is given by

$$c^*(t) = [e^{\rho t} E\{P(t) \mid G(t)\}]^{\frac{1}{\gamma-1}} \tag{9.18}$$

which would be identical to Merton [88, equation (18)] if the dependence of P(t) on G(t) were ignored, since by observation VIII. 1

$$E[P(t)] = E[\frac{\partial J^o}{\partial y}] .$$

The qualification is, however, an unpalatable pill to swallow. There is no a priori reason to expect that P(t) will be independent of t, y and ψ. If, however, this were true, it would mean that the "marginal indirect utility" P(t) is a constant, and this is a contradiction of the specialized utility function (9.17). If P(t) is not a constant, then there is the important problem of determining its behaviour, e.g., as compared with the corresponding P(t) in the deterministic framework. Before embarking on this task a few words can be said about the behaviour of the optimal consumption path as given by equation (9.18).

According to equation (9.18), three different variables are vital in the determination of the consumer's optimal consumption path: (1) the expected value of his marginal indirect utility conditional on his instantaneous capital stock, his posterior probability over the rate of return on his capital stock and the time the consumption decision is made; (2) his rate of time preference ρ : and (3) his relative risk aversion γ-1. Holding every other thing constant:

(1) increase (decrease) in his relative risk aversion results in a lower (higher) optimal consumption path;

(2) increase (decrease) in his rate of time preference results in a higher (lower) optimal consumption path;

(3) increase (decrease) in the conditional expected value of the marginal indirect utility function results in a higher (lower) optimal consumption path;

(4) the rate at which optimal consumption increases through time depends on the relationship between his rate of time preference ρ and the inverse of his relative risk aversion $\frac{1}{\gamma-1}$;
if ρ is greater than $\frac{1}{\gamma-1}$, the consumption that will be optimal to the consumer will increase steadily through time at a rate equal to $\frac{\rho}{\gamma-1}$;
if ρ is less than $\frac{1}{\gamma-1}$, the consumption that will be optimal to the consumer will decrease steadily through time at a rate equal to $\frac{\rho}{\gamma-1}$;
if ρ is equal to $\frac{1}{\gamma-1}$, his optimal consumption will increase at the rate $\rho^2 = 1/(\gamma-1)^2$.

All these seem interesting and straightforward, and they do confirm results obtained by other writers using other methods.[12] What is not so easy to determine is the behaviour of $P(t)$ which according to the canonical equations

$$\dot{P}(t) = \{-r(\omega,t) + \frac{\partial[e^{\rho t}E(P^*(t)|G(t))]^{\frac{1}{\gamma-1}}}{\partial y^*}\} P(t)$$

$$e^{-\rho t}[e^{\rho t}E(P^*(t)|G(t))]^{\frac{\gamma}{1-\gamma}} \cdot \frac{\partial[e^{\rho t}E(P^*(t)|G(t))]^{\frac{1}{\gamma-1}}}{\partial y^*} \qquad (9.19)$$

$$\dot{y}(t) = r(\omega,t)\, y(t) - [e^{\rho t}E(P(t)|G(t))]^{\frac{1}{\gamma-1}} \qquad (9.20)$$

depends, among other things, on the unspecified function $E(P(t)|G(t))$.

In a deterministic model this problem would not arise, since

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12 See e.g. Merton [88, p. 250].

there would be no need for the problematic terms involving $E(P(t)|G(t))$ in equation (9.19).[13]

Following the development of chapter VIII, the first step in determining the behaviour of P(t) is to assume that the solution P(t) of (9.19) takes the form

$$P(t) = K(t)\ y(t) \tag{9.21}$$

so that

$$\dot{P}(t) = \dot{K}(t)\ y(t) + K(t)\ \dot{y}(t)\ , \tag{9.22}$$

where K(t) is a matrix each element of which is a stochastic process conditionally independent of y(t), and then derive a solution for $\dot{K}(t)$ from a Riccati equation of the type given by equation (8.55) in chapter VIII. However, such a procedure does not seem very useful here since (9.20) is not linear. If the procedure were to be adopted, it would be immediately revealed that K(t) is not independent of y(t) since y(t) would appear explicitly on the right hand side of the differential equation for K(t). To see this, substitute equation (9.21) into (9.20); substitute the result in (9.22) to obtain an expression for $\dot{P}(t)$, and substitute (9.21) into (9.19) to obtain a second expression for $\dot{P}(t)$. Equalize the two expressions for $\dot{P}$ and solve for $\dot{K}(t)$. The result is

$$\begin{aligned}\dot{K}(t) &= -2r(\omega,t)K(t) + \frac{K(t)}{y(t)}\left[e^{\rho t}E(K(t)\,|\,G(t))y(t)\right]^{\frac{1}{\gamma-1}} \\ &\quad + \frac{K(t)}{\gamma-1}\left[e^{\rho t}\,E(K(t)\,|\,G(t))\,y(t)\right]^{\frac{\gamma}{1-\gamma}} \cdot \left[e^{\rho t}E(K(t)\,|\,G(t))\right] \\ &\quad - \frac{1}{(\gamma-1)y(t)}\left[e^{\rho t}E(K(t)\,|\,G(t))y(t)\right]^{\frac{2\gamma}{1-\gamma}} \cdot \left[E(K(t)\,|\,G(t))\right] \\ K(T) &= d = \text{constant}\end{aligned} \tag{9.23}$$

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13 Recall that the term $\dfrac{\partial\left[e^{\rho t}E(P^*(t)\,|\,G(t))\right]^{\frac{1}{\gamma-1}}}{\partial y^*}$ enters into (9.19 because of the feedback nature of the problem. In a deterministic model this feedback would be unnecessary since the optimal control would be the same with or without feedback. See Definition VI. 10 in chapter VI.

in which $\dot{K}(t)$ is explicitly dependent on y(t). Also, the Sworder condition $K(T) = 0$ cannot be imposed in this case because with this condition $\dot{K}(t)$ does not exist whenever $\gamma < 1$.

Equation (9.23) is a non-linear stochastic partial differential equation. By applying the method of characteristics suggested in chapter VIII, equation (9.23) reduces to the ordinary stochastic differential equation

$$\begin{aligned}\dot{\hat{K}}_i(\tau) = &- 2\hat{K}_i(\tau) r_i(\omega,t) + \frac{\hat{K}_i(\tau)}{y(t)} \left[e^{\rho t} \sum_{j=1}^{m} \hat{K}_j(\tau) h_j(\tau,i) y(t)\right]^{\frac{1}{\gamma-1}} \\ &+ \frac{\hat{K}_i(\tau)}{\gamma \quad 1} \left[e^{\rho t} \sum_{j=1}^{m} \hat{K}_j(\tau) h_j(\tau,i) y(t)\right]^{\frac{\gamma}{1-\gamma}} \cdot \left[e^{\rho t} \sum_{j=1}^{m} \hat{K}_j(\tau) h_j(\tau,i)\right] \\ &- \frac{1}{(\gamma-1)y(t)} \left[e^{\rho t} \sum_{j=1}^{m} \hat{K}_j(\tau) h_j(\tau,i) y(t)\right]^{\frac{2\gamma}{1 \quad \gamma}} \cdot \sum_{j=1}^{m} \hat{K}_j(\tau) h_j(\tau,i)\end{aligned} \qquad (9.24)$$

$$\hat{K}_i(T) = d$$

where

$$\hat{K}_i(\tau) = E\{K(t) \mid G(t), [i(t)]\}$$

$[i(t)]$ is the event $r(.,t) = r_i$, $i = 1, \ldots, m$; and

h_j is the solution of equation (8.59) in chapter VIII.

Unlike in chapter VIII, one now has to simultaneously solve a system of non-linear partial differential equation so that the technique described by Athans and Falb [10, p 767] breaks down. On substituting $P(t) = K(t)\,y(t)$ in (9.20) one has

$$\begin{aligned}\dot{y}(t) &= r(\omega,t)y(t) \quad \left[e^{\rho t} E(K(t) \mid G(t)) y(t)\right]^{\frac{1}{\gamma-1}} \\ &= r(\omega,t)y(t) \quad \left[e^{\rho t} \sum_{j=1}^{m} \hat{K}_j(\tau) h_j(\tau,i) y(t)\right]\end{aligned} \qquad (9.20a)$$

The problem is clear One cannot solve for y(t) until one knows $\hat{K}_j$, $j = 1, \ldots, m$; neither can one solve for $\hat{K}_j$ in equation (9.24) until one knows y(t).

Several attempts were made at obtaining a solution for (9.20) and (9.24) by assuming alternative functional forms for $E(P(t) \mid G(t))$. The problem was always the same. As long as $P(t)$ is interpreted as a marginal indirect utility function any plausible functional form for it must involve $y(t)$, so that one always comes back to the situation of having to find $y(t)$ in terms of $y(t)$! For instance, the function

$$E(P(t)|G(t)) = e^{-\lambda\, y(t)}$$ [14]

was tried, so that equation (9.20) became

$$\dot{y}(t) = r(.,t)\, y(t) - \left[e^{(\rho - \lambda)\, y(t)}\right]^{\frac{1}{\gamma-1}} \qquad (9.20b)$$

A steady state solution for $y(t)$ was sought for this expression, yielding

$$y(t) = \{\left[e^{(\rho-\lambda)\, t.y(t)}\right]^{\frac{1}{\gamma-1}} \Big/ \quad r(.,t) \qquad (9.20c)$$

or

$$\log y(t) = \left(\frac{1}{\gamma-1}\right)(\rho\, t - \lambda y(t)) - \log r(.,t) \qquad (9.20d)$$

which can be more easily solved for t than for $y(t)$. Attempt was made to estimate the function $y(t)$ from the equation

$$t = \left[(\gamma-1) \log y(t) + \log r(.,t) + \lambda\, y(t)\right] / \rho \qquad (9.20e)$$

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[14] This particular form was chosen on the ground of its plausibility as a marginal indirect utility function since it obeys

$$u'(y(t) = E\,(P(t) \mid G(t)) = e^{-\lambda\, y(t)} > 0, \quad \lambda > 0$$

and

$$u''(y(t)) = \frac{\partial E(P(t) \mid G(t))}{\partial y} = -\lambda e^{-\lambda y(t)} < 0 \;.$$

by using particular values for ρ, λ, γ, $r(.,t)$ and varying $y(t)$. By using this procedure, it is possible to obtain a particular (but by no means general) function $y(t)$ which is valid only for the particular set of parameter values used. For each different set of parameter values, a different function $y(t)$ results, and the possible combination of the parameters is infinite.[15]

A particular function $y(t)$ obtained in the above fashion could be fed into equation (9.24) and a numerical solution attempted for $K(t)$. Of course, a different $K(t)$ would result for different $y(t)$'s. In the end, so many arbitrary assumptions shall have been made about the parameters as to make the results worthless in general, although fully specified problems could be solved numerically.

No doubt, the foregoing computational problems are formidable, and tend to depreciate the great potential of the new tool. This handicap seems however due to the particular problem being solved here. The problem seems to arise from the fact that the original principle was developed from the so-called "state regulator" model[15a] in which the decision maker seeks to minimize a criterion functional of the form

$$J = \frac{1}{2} y(t)' F y(t) + \frac{1}{2} \int_0^T (y(t)' M y(t) + c' N c) \, dt \qquad (9.25)$$

subject to

$$\dot{y} = r(t) y(t) \quad c(t)$$

where

y and c are now deviations from equilibrium levels or deviations from levels specified by the decision maker.

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15. This is a general weakness of the state-of-the-world approach to decision making under uncertainty. See Kendrick [67, pp 1-2].

15a. See Athans and Falb [10, sect. 9.3]

M and N are positive definite matrixes; and F, a positive semi definite matrix representing the condition imposed on terminal capital stock.

The optimal control for this problem is known to exist and is given by

$$c^*(t) = N^{-1} K(t) y(t)^{16} = N^{-1} P(t) \qquad (9.26)$$

substituting from (9.21). Therefore, any problem that can be reduced to the state regulator formulation would seem to be easily solvable by means of this tool.

The most promising route of escape from the complications indicated in the preceding pages, therefore, is the following. If one is prepared to decompose the problem into the two steps suggested by Kendrick [67, p. 3], then its solution by means of the Sworder stochastic maximum principle is straightforward. Such a solution is however only an approximation, but it is better than no solution, or the solution, if any, that the complicated procedure of the preceeding pages may yield.

Consider again the stochastic Hamiltonian (equation (9.7)) with $u[c(t)] = \frac{1}{\gamma} c^{\gamma}$; i.e.,

$$H = e^{-\rho t} \frac{1}{\gamma} c^{\gamma} + P(t) \{ r(.,t) y(t) - c(t) \} \qquad (9.27)$$

The first step is to assume that the stochastic process $r(\omega, t)$ takes is expected value $\bar{r}(\omega, t)$. The Hamiltonian (9.27) is then deterministic,

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16 The corresponding equation in Athans and Falb [10, eqn. (9.60)] would be $c^*(t) = -N^{-1}(t) B(t)' K(t) y(t)$, but in the above example $B(t)' = -1$.

and so the feedback may be ignored. The optimal consumption path is given by

$$c^*(t) = [e^{-\rho t} P(t)]^{\frac{1}{\gamma-1}} \tag{9.28}$$

and the canonical equations are

$$\dot{P}(t) = -\frac{\partial H}{\partial y} = -P(t)\,\bar{r}(.,t) \tag{9.29}$$

and

$$\dot{y}(t) = \bar{r}(.,t)\; y(t) - c^*(t)$$
$$= \bar{r}(.,t)\; y(t) - [e^{-\rho t} P(t)]^{\frac{1}{\gamma-1}} \tag{9.30}$$

The solution P(t) from (9.29) is

$$P(t) = M e^{-\int \bar{r}(.,s)ds}, \quad M = \text{constant} > 0 \tag{9.31}$$

which is a decreasing function as is already well-known.

Substituting in (9.30)

$$\dot{y}(t) = \bar{r}(.,t)\, y(t) \left[M \exp\left\{-\rho t - \int_{t_o}^{t} r\,(.,s)\; ds\right\}\right]^{\frac{1}{\gamma-1}} \tag{9.32}$$

The solution of the homogeneous part is

$$\bar{y}(t) = M\, e^{\int \bar{r}(.,t)dt}, \quad M > 0$$

The complete solution is

$$\bar{y}(t) = e^{\int \bar{r}(.,t)dt} \left(\int Q(t)\, e^{-\int \bar{r}(.,t)dt}\, dt + M\right) \tag{9.33}$$

where

$$Q(t) = -\left[M \exp\left\{-\rho t - \int_{t_o}^{t} r\,(.,s)\; ds\right\}\right]^{\frac{1}{\gamma-1}}$$

In equation (9.33), the expression $\int \frac{Q(t)}{e^{\int \bar{r}(.,t)dt}}dt$ is the dampening factor and the path of y(t) depends on the relative magnitude of $\int \frac{Q(t)}{e^{\int \bar{r}(.,t)dt}}\, dt$ and the constant M. Equations (9.28), (9.31) and (9.33) may be regarded as the "nominal" solutions to the original problem.

The second step in the solution process is to determine what happens to equations (9.28), (9.31) and (9.33) whenever $r(.,t)$ does not assume its expected value. This second step lends itself directly to the application of the Sworder maximum principle, for as Kendrick [67, p 11] has observed, the criterion functional for the second step of the solution always turns out to be quadratic irrespective of the original criterion.[17] The second step, therefore, is to minimize

$$J = E\left\{ \int \frac{1}{2} \; c_d(t)^2 \, dt \right\} \tag{9.34}$$

subject to

$$\dot{y}_d(t) = r_d(t)\, y_d(t) - c_d(t) \tag{9.35}$$

$y_d(t)$, $c_d(t)$ and $r_d(t)$ are now to be interpreted as deviations from the nominal paths. e.g., $y_d(t) = \bar{y} - y_a$ i.e., nominal y minus actual y. The solution is

$$\ell_d^*[G(t)] = -\sum_{j=1}^{m} \hat{K}_j(\tau)\, \psi_j(t)\, y_d(t) \; ^{18} \tag{9.36}$$

where $\hat{K}_j$ is as defined on P. 166 above. The corresponding optimal trajectory and adjoint functions are given by

$$\dot{y}_d^*(t) = r_d(t)\, y_d^*(t) + \sum_{j=1}^{m} \hat{K}_j(\tau)\, \psi_j(t)\, y_d^*(t) \tag{9.37}$$

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17 The present procedure is not the same thing with the certainty equivalent principle. The restrictive assumptions of the latter are not required by the technique suggested here.

18 This is the special case of equation (8.61) in which B_i^T takes value -1 and $\psi_j(t) = p\{r(.,t) = r_j(t)$ and $B(.,t) = 1\}$

and

$$\left.\begin{aligned}\dot{\hat{K}}_i(t) &= -2 r_i \hat{K}_i(\tau) + 2 \hat{K}_i(\tau) \sum_{j=1}^{m} \hat{K}_j(\tau) h_j(\tau, i) + \\ &\left(\sum_{j=1}^{m} \hat{K}_j(\tau) h(\tau, i)\right)^2 - \sum_{j=1}^{m} \hat{K}_j(\tau) q_{ij}(\tau, i)\end{aligned}\right\} \quad (9.38)$$

[19]

$$\hat{K}(T) = 0$$

Equation (9.38) is now of the same form as equation (8.60) in chapter VIII and so, it may be solved by means of the algorithm described in Athans and Falb [10, P. 767].

Equations (9.36), (9.37) and (9.38) measure the differences between the deterministic and stochastic solutions of the optimal control, optimal trajectory and the corresponding adjoint function, respectively. For instance, whether the optimal consumption path will be higher or lower under stochastic (uncertainty) than under deterministic (certainty) depends on the sign of equation (9.36).[20] This statement boils down to saying that, other things being equal, optimal consumption path under uncertainty will be lower than under certainty if the deviation of capital path (given by equation (9.37)) from its nominal path (given by equation (9.33)) is negative. The converse is true if the deviation is positive.

Similar reasoning applies in comparing the other two sets of

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19. See equation (8.60) in chapter VIII.

20 Of course, all the variables and parameters of equation (9.28) have to be held constant for this statement to be true.

functions. If the solution $\hat{K}(t)$ of (9.38) is positive, the result will be an upward shift of the entire path of the marginal indirect utility function except at the terminal point; i.e., where $\hat{K}(T) = 0$. In other words, the path of the stochastic marginal indirect utility function lies above the corresponding deterministic path except at the terminal point where the two paths are equal.

The behaviour of equation (9.37) depends on the signs and relative magnitudes of $\hat{K}(t)$, $y_d^*(t)$ and $r_d(t)$.

4. Summary and Conclusions

The major findings of this chapter may now be summarized. The objective of the chapter was to apply the tools chapter VIII to the general economic problem of chapter III. At the qualitative level, straightforward stochastic counterparts of well-known results in the deterministic literature were derived without any difficulty. This was the case, for instance, with the classic Keynes-Ramsey rule which could almost trivially be inferred by applying theorems VIII.1 and VIII.5, chapter VIII. Established relationships between dynamic programming and the maximum principle (observation VIII.1) were used to advantage. The result of this relationship, applied to the problem studied here, was to yield the interesting interpretation of the adjoint function P(t) as a marginal indirect utility function and of the maximum of the Hamiltonian as the change in the indirect utility with respect to time, holding all other variables constant. These relationships also made it possible to rationalize the behaviour, through time, of the indirect utility function.

ECONOMIC APPLICATION OF THE STOCHASTIC MAXIMUM PRINCIPLE

Compared with the stochastic maximum principle, it was discovered that stochastic dynamic programming always involves a two step procedure in which the first step calls for reducing a given situation of temporal uncertainty to one of timeless uncertainty while the second step calls for optimizing the resulting indirect utility function. While this procedure is sensible in view of the proposition that a temporal uncertain prospect is never preferred to a timeless uncertain prospect described by the same probability mass or density function, the stochastic maximum principle has the advantage of simultaneously yielding the three important functions: the optimal consumption path $\ell(t)$, the corresponding capital path $y(t)$ and the path of utility function $P(t)$ associated with it.

However, in the attempt to obtain explicit solutions for these three functions, a number of computational difficulties were encountered and these threatened to break down the new tool which initially seemed theoretically very powerful. These difficulties were deliberately studied in detail in order to serve as a guide to future researchers in the area. The root of the difficulty was traced to the fact that the original tool was developed for the so-called "state regulator" model. It became obvious, therefore, that if the original problem could be converted to a state regulator formulation, the computational difficulties would immediately disappear A technique was suggested for effecting this conversion. The overall impact of uncertainty on the three functions was verbalized from equations (9.36), (9.37) and (9.38).

CHAPTER X

SUMMARY AND CONCLUSIONS

The problem of intertemporal optimal consumption-saving allocation under uncertainty has been investigated within the framework of stochastic optimal control theory. To begin with, an extensive survey of the relevant economic literature was undertaken. This survey unveiled a number of important issues which must always be remembered in any study of this type, but which have so far not been integrated. These issues always arise once one recognizes the fact that in a dynamic optimisation problem of the type studied in this thesis: (1) different types of uncertainty may be identified; (2) these uncertainties may emanate from a number of different sources; and (3) various "measures" of uncertainty may be employed.

The implications of these distinctions for the impact of uncertainty on optimal solutions were examined in detail. In particular, it was noted that all of the existing "measures" of uncertainty were developed in a static framework. While the development of a truly rigorous "measure" of uncertainty was not attempted in this thesis, sufficient warning was given in this regard concerning the interpretation of the impact of uncertainty on optimality.

Taking inspiration from the work of Koopmans [68, p. 110], argument was advanced concerning the best tool to employ in the analysis of the problem posed. First, stochastic control theory was seen as naturally suggesting itself. Next, among the tools of stochastic control theory, the stochastic maximum principle was seen as most suited, not only because of (1) its more general mathematical validity (e.g., as compared with the method of dynamic programming); and (2) its wealth of qualitative properties (which are always attractive to the economist); but also (3) because of the success already achieved at the deterministic level in employing and interpretating the Pontryagin maximum principle as a proposition in economics.

From a survey of the literature of stochastic control theory, it was discovered that two attempts have been made at directly developing a stochastic counterpart of the Pontryagin maximum principle. One attempt made by Kushner was developed for fairly general transition equations except that it handles only the case of additive noise. The second which is due to Sworder permits multiplicative noise, but the transition equations are less general than those of Kushner. Compared with the generality of stochastic differential equations or the multiplicity of ways in which uncertainty may be admitted in practical problems, the existing two versions of the stochastic maximum principle constitute only a starting point for the development of a future, full-fledged theory of the stochastic maximum principle. However, of the two versions, Sworder's was recognized to be of special interest because of the prevalence of multiplicative noise not only in the subject-matter of this thesis, but also in several other economic problems.

A comprehensive exposition of the two stochastic maximum principles and how they relate to the well-known Pontryagin maximum principle and the method of dynamic programming was given. Juxtaposed with each other (Table I, p. 146), the three versions of the maximum principle did not reveal any significant differences in principle. In fact, the Kushner version was found to be almost identical with that of Pontryagin. However, some extra computational complexity was evident in Sworder's equation for the adjoint function $P(t)$. Whereas the

Pontryagin as well as the Kushner P(t) function is obtained from a set of ordinary differential equations, the Sworder equation for P(t) generally leads to a set of stochastic partial differential equations of the Riccati type.

Important implications for the future role of stochastic optimal control theory in economics were noted after a detailed study of classifications of stochastic control systems. When the various classes of stochastic systems were interpreted as a classification of the various sources from which uncertainty may be introduced into a dynamic problem, several "areas of open research" became evident: in the literature of stochastic optimal control theory, eight different classes of stochastic systems were exemplified. Of these eight, only two classes - random time constant and stochastic stopping time were seen to have so far received attention in the economic literature dealing with the subject-matter of this thesis. There is no doubt that increased realism can be attained by admitting uncertainty from several sources at the same time. However, there should be no illusions regarding the price at which this can be done. It is not yet clear how much complication will be introduced into the stochastic maximum principle, for instance, if unknown initial conditions and/or stochastic time horizon are simultaneously permitted in the model. It might, therefore, be wise for economists to wait until mathematicians and control theorists are able to sort out these issues. These were the considerations that influenced the type of illustrative problem attempted in this thesis.

Perhaps the most significant contribution of this thesis is its pioneering effort in employing and interpreting the Sworder stochastic maximum principle (so far as it has been developed) in the analysis of the Phelps - Ramsey problem. A number of interesting results were derived some of which may be regarded as stochastic counterparts of the classic Keynes-Ramsey rule. For instance, the latter was inferred without any difficulty by applying theorems VIII.I and VIII.5. Established relationships between dynamic programming and the maximum principle (observation VIII.I) were used to advantage. Applied to the problem studied in this thesis, the result of these established relationships was to yield the interesting interpretation of the adjoint function P(t) as a marginal indirect utility function and of the maximum of the Hamiltonian as the change in the indirect utility with respect to time, holding all other variables constant. These relationships also made it possible to rationalize the behaviour, through time, of the indirect utility function.

Compared with the stochastic maximum principle, it was discovered that stochastic dynamic programming always involves a two step procedure in which: (1) the first step calls for reducing a given problem involving temporal uncertainty to one involving timeless uncertainty; while (2) the second step calls for optimizing the resulting indirect utility function. This procedure was recognized as sensible in view of the proposition that a temporal uncertain prospect is never preferred to a timeless uncertain prospect described by the same mass or density function. On the other hand, the stochastic maximum principle has the advantage of simultaneously yielding the three important functions: the optimal control (the optimal consumption path), the corresponding optimal trajectory (the optimal capital path) and the adjoint function (the utility function) associated with it.

However, in the attempt to obtain explicit solutions for these three functions, a number of computational difficulties were encountered. These difficulties initially threatened to suppress the might of the new tool, and they were deliberately studied in detail in order to serve as a guide to future researchers in the area. The root of the difficulties was traced to the fact that the original tool was developed on the basis of the socalled "state regulator" model. It became obvious, therefore, that if the original problem could be converted to a state regulator formulation, the computational difficulties would immediately disappear. A technique was suggested for effecting this conversion. The advantage of the procedure is that it does not require the restrictive assumptions needed for the application of the certainty equivalence principle.

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1 Abbreviations used are taken from the American Economic Association index except for the Journal of Mathematical Analysis and Applications and the International Journal of Control which are denoted by JOMAA and IJC respectively.

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