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SELINGER'S CPM CONSTRUCTION AND NUCLEAR IDEALS

By
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Abstract

Monoidal categories have recently been seen to be appropriate structures for establishing abstract axioms for quantum physics. The tensor product in a monoidal category corresponds to the creation of a composite system obtained by joining two independent quantum systems. This correspondence is formalized in the work of Abramsky-Coecke and Selinger. From this point of view, the tensor product in a monoidal category is abstracting the tensor product of Hilbert spaces. The notion of a dagger compact closed category axiomatizes further structure in the Hilbert space category. In particular, the dagger is an abstraction of the familiar Hilbert adjoint. Peter Selinger associates to each dagger compact closed category $\mathcal{C}$ its category of completely positive maps, denoted $\mathbf{CPM}(\mathcal{C})$. He proves that the resulting category is again a dagger compact closed category. We seek a similar result for tensored $\dagger$-categories equipped with nuclear ideals.

We establish an appropriate notion of completely positive maps in a nuclear ideal. We then define a CPM construction for tensored $\dagger$-categories equipped with nuclear ideals. Analogous to Selinger's construction, given a nuclear ideal $\mathcal{N}$ for a tensored $\dagger$-category $\mathcal{C}$, our construction yields its category of completely positive maps, $\mathbf{CPM}_{\mathcal{N}}(\mathcal{C})$. We prove that $\mathbf{CPM}_{\mathcal{N}}(\mathcal{C})$ is again a tensored $\dagger$-category. In the process, we also verify that our completely positive maps properly extend Selinger's notion of CPMs. In particular, they preserve *nuclear-positive matrices*, which are generalizations of von Neumann's positive density operators. As well, we characterize the completely positive maps for the nuclear ideal of Hilbert-Schmidt operators for **Hilb** and the nuclear ideal of finite relations for **LFR**. Our construction abstracts the passage from finite-dimensional Hilbert spaces to the category of all Hilbert spaces.

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Dedication

I dedicate this work to the memory of my friend and step-father Roy.

Contents

Chapter 1

Introduction

1.1 Historical Overview

Monoidal categories play diverse roles in mathematics and evidence of their importance abounds. For example, categories of games have recently been of fundamental importance in a number of branches of logic and theoretical computer science. For instance, they provide complete models of various fragments of Girard's linear logic [15, 25], as well as providing fully abstract models of numerous programming languages. The fundamental connective in such categories is a tensor, which allows one to play two games in parallel. See [4]. The monoidal category of sets and relations has applications to concurrency theory as well [1]. The category of linear representations of a compact group possesses a tensor structure that is central to the Tannaka-Krein duality theorems [14, 21]. A similar result holds in the more general case of Hopf algebras [19, 17]. The Tannaka-Krein theorems, from today's point of view, give a method of recovering an algebraic structure from its category of representations. In such theorems, one has a forgetful functor to the category of vector spaces which preserves the monoidal structure. One then recovers the group or Hopf algebra as the set of structure-preserving endomorphisms of this forgetful functor.

Compact closed categories [20] form an interesting class of symmetric monoidal categories. These categories are in particular closed, i.e., $A \otimes (-)$ has a right adjoint, which allows for internal Homs. In a compact closed category, each object A has an assigned dual object A^* together with a unit map $\eta_A : I \to A^* \otimes A$ and a counit map $\epsilon_A : A \otimes A^* \to I$. These maps interact in a way that is captured by the *compact closed equations.* Examples of compact closed categories include the category of finite-dimensional vector spaces, the category of finitely-generated projective modules over a commutative ring, the category **Rel** of sets and relations, and the category of finite-dimensional representations of a group. An important property of a compact closed category is that each morphism has a *transpose.* More precisely this means that for each morphism $f : A \to B$ there is a morphism $\ulcorner f \urcorner : I \to A^* \otimes B$, which can be viewed as an element of $A^* \otimes B$.

A special class of compact closed categories is obtained by adding the notion of an *adjoint* of a morphism. This yields Abramsky and Coecke's *strongly compact closed categories* [3] or Selinger's *dagger compact closed categories* [26]. In such a category, we can define unitarity and positivity abstractly. A compact closed category $\mathcal{C}$ is dagger compact closed if it is equipped with an involutive contravariant functor $(-)^\dagger : \mathcal{C} \to \mathcal{C}$ that is the identity on objects. Moreover, this dagger functor must preserve the compact closed structure of $\mathcal{C}$ in the sense that $(-)^\dagger$ is a strict symmetric monoidal functor, and the structural isomorphisms are all unitary. As well, for all objects A, the dagger of the counit map at A coincides with the unit map at A followed by a twist.

Doplicher and Roberts, in [12], also develop a theory of monoidal categories with an adjoint structure of this sort. They use it to define a duality theory for compact groups. They were especially interested in these structures for physical reasons, as explained in [12]. This has led to much further work, as in [22].

This adjoint structure appears in many different guises. The standard example is the Hilbert adjoint in the category $\mathbf{Hilb}_{fd}$ of finite-dimensional Hilbert spaces. The category of sets and relations has a rather simple adjoint structure; given a binary relation $R \subseteq A \times B$, its adjoint $R^\dagger \subseteq B \times A$ is obtained by swapping componentwise the elements in R. The category **nCob** of closed oriented (n-1)-manifolds and n-*cobordisms* possesses an inherent adjoint structure. Here, a morphism $\Sigma_1 \to \Sigma_2$ is an equivalence class of compact oriented n-manifolds M each equipped with an orientation-preserving diffeomorphism $g_M : \Sigma_1 \amalg \Sigma_2^* \to \partial M$, where the dual object

Σ_2^* is Σ_2 with orientation reversed. Two such n-cobordisms M and M' represent the same morphism if there is an orientation-preserving diffeomorphism $\Psi : M \to M'$ with $g_{M'} = \Psi g_M$. If M represents a morphism, M with reverse orientation represents the adjoint morphism. This category is essential to topological quantum field theories (TQFTs).

Although there are many examples of compact closed categories equipped with an adjoint structure, this structure need not be restricted to these categories. This observation led Selinger to the idea of a *dagger category* to describe a general category equipped with an adjoint structure [26]. With Selinger's terminology, Abramsky and Coecke's strongly compact closed categories become dagger compact closed categories. Of interest to us is the dagger category **Hilb** of arbitrary Hilbert spaces. Here the dagger structure is again the usual Hilbert adjoint. In addition we investigate the category **LFR** of sets and *locally finite relations*. By a locally finite relation $R : A \to B$, we mean a subset $R \subseteq A \times B$ such that the set $\{ b \in B \mid (a, b) \in R \}$ is finite for all $a \in A$, and the set $\{ a \in A \mid (a, b) \in R \}$ is finite for all $b \in B$. The category **LFR** inherits a dagger structure from **Rel**. As we study the dagger structures for these categories, we would also like to consider any transposes that may be present, and how these structures interact with the dagger structure. For this purpose, the language of nuclear ideals is very useful.

The goal of nuclear ideal theory [2] is to extend the theory of compact closed categories to include the category of arbitrary Hilbert spaces. To achieve this, we consider categories that, like **Hilb**, possess an adjoint structure as well as a great deal of the compact closed structure, but in which there may be morphisms lacking transposes. These categories were introduced by Abramsky, Blute, and Panangaden as tensored $*$-categories [2]. However, we reserve the star notation for compact closed structures, so we refer to these categories as tensored $\dagger$-categories. In such a category $\mathcal{C}$, we consider a class of transposable morphisms called nuclear morphisms. In particular, this class of maps is required to be closed under composition with arbitrary morphisms in $\mathcal{C}$, and closed under the operations in $\mathcal{C}$. Abramsky et. al. coined the term *nuclear ideal* to describe such a situation. For example, the class of finite relations form a nuclear ideal for the category of locally finite relations. A more interesting example is the class of Hilbert-Schmidt operators in the category **Hilb**. A Hilbert-Schmidt operator $T : H \to K$ is a bounded linear map satisfying

$$\sum_{b_h \in B_H} \sum_{b_k \in B_K} |\langle T b_h, b_k \rangle|^2 < \infty,$$

where B_H and B_K are orthonormal bases for H and K, respectively.

Compact closed categories allow for canonical traces, which in the case of vector spaces correspond to the usual notion. In the category of arbitrary Hilbert spaces, one has a partial notion of trace. Traces can be taken on a certain class of Hilbert-Schmidt operators called *trace class maps*. See [28]. The notion of a nuclear ideal allows one to abstract this structure. Categories equipped with a nuclear ideal have a collection of trace class maps.

In [26], Selinger describes a CPM construction for dagger compact closed categories. Completely positive maps, or CPMs, arise in quantum theory in the density operator characterization of quantum mechanics. As an alternative to the traditional state space axiomatization of quantum mechanics, there is an equivalent description due to von Neumann in which a system is completely described by its *density operators* or *density matrices.* Details of this perspective can be found in Nielsen and Chuang's *Quantum Computation and Quantum Information* [24]. One advantage of this approach is its ability to describe the behaviour of individual subsystems within a composite quantum system. Roughly speaking, a CPM is a map from the set of density operators of the input space Q_1 to the set of density operators of the output space Q_2 which maps positive operators on Q_1 to positive operators on Q_2. Additionally, a CPM must retain this positivity property when tensored with arbitrary identity maps. Positive operators are important, as they can be used to encode probabilities in the quantum setting.

Selinger's CPM construction associates to each dagger compact closed category $\mathcal{C}$ its category of completely positive maps, denoted $\mathbf{CPM}(\mathcal{C})$. He proves that the resulting category is again a dagger compact closed category. We seek a similar result for tensored †-categories equipped with nuclear ideals.

1.2 Thesis Summary

Our ultimate goal is to extend Selinger's CPM construction to the setting of nuclear ideals. We first provide relevant background material regarding Hilbert spaces and category theory in Chapters 2 and 3 respectively. We begin with an in depth look at infinite-dimensional Hilbert spaces and bounded linear maps. We discuss the Hilbert adjoint and, following [18], we give a detailed description of the tensor product for **Hilb**. This leads to the notion of a *Hilbert-Schmidt operator.* We then describe *trace class* maps in **Hilb** and their relation to Hilbert-Schmidt operators. Following that, we discuss general monoidal categories and compact closed categories. In the final section of Chapter 3 we describe Selinger's *dagger categories.*

In Chapter 4 we provide a detailed review of Selinger's work with completely positive maps. We begin with the concepts of positive maps and positive matrices in an arbitrary dagger compact closed category. Next we define Selinger's notion of a completely positive map (CPM) and establish several properties of these maps. We consider also the relationship between CPMs and positive matrices. Next we describe Selinger's CPM construction for a dagger compact closed category $\mathcal{C}$, and outline the dagger compact closed structure for $\mathbf{CPM}(\mathcal{C})$. We close the chapter with examples of the CPM construction applied to the categories $\mathbf{Hilb}_{fd}$ and **Rel**.

Following that, we set up the nuclear ideal framework in Chapter 5. We describe *tensored †-categories* and discuss two key examples: **Hilb** and **LFR**. After that we define the notion of a *nuclear ideal* for a tensored †-category. We look at the nuclear ideal of Hilbert-Schmidt operators for **Hilb**, and the nuclear ideal of finite relations for **LFR**. At this point we have the tools necessary to construct our generalized CPMs.

In Chapter 6, we first establish an appropriate notion of a completely positive map in a nuclear ideal. We then prove numerous intermediate results that facilitate our CPM construction. Next we define a CPM construction for tensored †-categories equipped with nuclear ideals. Analogous to Selinger's construction, given a nuclear ideal $\mathcal{N}$ for a tensored †-category $\mathcal{C}$, our construction yields its category of completely positive maps, $\mathbf{CPM}_{\mathcal{N}}(\mathcal{C})$. We prove that the resulting category is again a tensored †-category. In the process, we also verify that our completely positive maps behave correctly. That is to say, they properly extend Selinger's notion of CPMs. For example, they preserve *nuclear-positive matrices*, which are generalizations of von Neumann's positive density operators. As well, we characterize the completely positive maps for the nuclear ideal of Hilbert-Schmidt operators for **Hilb** and the nuclear ideal of finite relations for **LFR**.

Chapter 2

Infinite-dimensional Hilbert Spaces

The structure of **Hilb**, the category of arbitrary Hilbert spaces and bounded linear maps, is central to this paper. Unlike its finite-dimensional counterpart $\mathbf{Hilb}_{fd}$, this category fails to be *compact closed*. However, **Hilb** does possess the majority of the structure that makes $\mathbf{Hilb}_{fd}$ an ideal framework for the axiomatization of quantum mechanics. For this reason, we begin with a careful investigation of this category. We assume the reader knows what a Hilbert space is. See [18].

For the duration of this section, we fix Hilbert spaces H_i with respective orthonormal bases B_i, $i = 1, \ldots, n$.

First recall what we mean by a bounded linear map of Hilbert spaces.

Definition 2.0.1. A linear operator $T : H_1 \to H_2$ is ***bounded*** if there exists a non-negative real number c such that $\|Tx\| \leq c\|x\|$, for all $x \in H_1$. When T is bounded, the smallest real number with this property is called the ***bound*** of T and is denoted $\|T\|$.

Theorem 1.5.5 of [18] provides some conditions equivalent to the operator T being bounded. We state this result in the following proposition.

Proposition 2.0.2. *For a linear operator $T : H_1 \to H_2$, the following conditions are equivalent:*
(a) T is bounded.
(b) $sup\{\, \|Tx\|/\|x\| \mid x \in H_1, x \neq 0 \,\} < \infty$.

(c) $sup\{ \|Tx\| \mid x \in H_1, \|x\| = 1 \} < \infty$.
(d) T *is continuous.*

Remark 2.0.3. In the case of finite-dimensional Hilbert spaces, all linear operators are bounded. To see this, let H_1 be finite-dimensional with $B_1 = \{ e_i \mid i = 1 \ldots m \}$, and let $x \in H_1$ with $\|x\| = 1$. Write

$$x = \sum_{i=1}^{m} \alpha_i e_i.$$

Then

$$1 = \|x\|^2 = \langle \Sigma_{i=1}^m \alpha_i e_i, \Sigma_{j=1}^m \alpha_j e_j \rangle = \Sigma_{i=1}^m \Sigma_{j=1}^m \alpha_i \overline{\alpha_j} \langle e_i, e_j \rangle = \Sigma_{i=1}^m \alpha_i \overline{\alpha_i} = \Sigma_{i=1}^m |\alpha_i|.$$

Hence $|\alpha_i| \leq 1$, for $i = 1 \ldots m$. Now,

$$\|Tx\| = \|T(\Sigma_{i=1}^m \alpha_i e_i)\| = \|\Sigma_{i=1}^m \alpha_i T(e_i)\| \leq \Sigma_{i=1}^m |\alpha_i| \|T(e_i)\| \leq \Sigma_{i=1}^m \|T(e_i)\|.$$

Therefore

$$sup\{ \|Tx\| \mid x \in H_1, \|x\| = 1 \} \leq \Sigma_{i=1}^m \|T(e_i)\| < \infty,$$

so T is bounded.

In the category **Hilb** it is well-known that, in general, Hom-sets cannot be realized as Hilbert spaces. Expressed categorically, **Hilb** is not (monoidal) *closed*. It is the case, however, that these Hom-sets can be viewed as vector spaces. We describe this linear structure below as it will come into play shortly.

Remark 2.0.4. For all objects $H_1, H_2 \in$ **Hilb**, the set $Hom_{\mathbf{Hilb}}(H_1, H_2)$ is a vector space over $\mathbb{C}$. For $S, T \in Hom(H_1, H_2)$ and $\alpha \in \mathbb{C}$, we define

$$(\alpha S + T)(x) = \alpha S(x) + T(x).$$

Then $\alpha S + T \in Hom_{\mathbf{Hilb}}(H_1, H_2)$ since

$$\|(\alpha S + T)(x)\| = \|\alpha S(x) + T(x)\| \leq |\alpha| \|S(x)\| + \|T(x)\| \leq |\alpha| \|S\| + \|T\|,$$

for all $x \in H_1$ with $\|x\| = 1$.

2.1 The Hilbert Adjoint

In this section we describe an *adjoint* structure possessed by both $\mathbf{Hilb}_{fd}$ and **Hilb**. This structure will simplify our construction of the tensor product for **Hilb** in the next section, and will be central to the notion of a *trace class operator* in Section 2.3. As well, we aim to abstract this adjoint structure in Section 3.3 on dagger categories.

We will need the following well-known theorem which can be found in [11], for example.

Theorem 2.1.1. ***(Riesz Representation Theorem)*** *Let $\varphi : H \to \mathbb{C}$ be a bounded linear functional on a Hilbert space H. There exists a unique $x_0 \in H$ such that $\varphi(x) = \langle x, x_0 \rangle$ for all $x \in H$. Moreover, we have $\|x_0\| = \|\varphi\|$.*

A brief description of the Hilbert adjoint can be found in [11]; we provide the details in the proposition below.

Proposition 2.1.2. *Let $T : H \to K$ be a bounded linear map of Hilbert spaces. Then there exists a bounded linear map $T^\dagger : K \to H$ satisfying*

$$\langle Tx, y \rangle = \langle x, T^\dagger y \rangle \qquad \textit{for all } x \in H \textit{ and } y \in K.$$

Proof. Let $T : H \to K$ be a bounded linear map. For each fixed $y_0 \in K$ the map $\varphi : H \to \mathbb{C}$ defined by

$$\varphi(x) = \langle Tx, y_0 \rangle$$

is a bounded linear functional on H. By Theorem 2.1.1 there is a unique $x_0 \in H$ such that

$$\varphi(x) = \langle Tx, y_0 \rangle = \langle x, x_0 \rangle \qquad \text{for all } x \in H.$$

Now let $T^\dagger : K \to H$ be the map sending each $y_0 \in K$ to that unique $x_0 \in H$. Then we have

$$\langle Tx, y \rangle = \langle x, T^\dagger y \rangle \qquad \text{for all } x \in H \text{ and } y \in K.$$

To see that $T^\dagger$ is linear, let $y, y' \in K$ and $\alpha \in \mathbb{C}$. Since

$$\begin{aligned} \langle x, \alpha T^\dagger y + T^\dagger y' \rangle &= \overline{\alpha} \langle x, T^\dagger y \rangle + \langle x, T^\dagger y' \rangle \\ &= \overline{\alpha} \langle Tx, y \rangle + \langle Tx, y' \rangle \\ &= \langle Tx, \alpha y + y' \rangle \end{aligned}$$

by uniqueness we must have that $T^\dagger(\alpha y + y') = \alpha T^\dagger y + T^\dagger y'$. Finally, we show that $T^\dagger$ is bounded. By Schwarz's inequality we have

$$|\langle x, T^\dagger y\rangle| = |\langle Tx, y\rangle| \leq \|Tx\|\|y\| \leq \|T\|\|x\|\|y\|.$$

So for $x = T^\dagger y$ we have

$$\|T^\dagger y\|^2 = \langle T^\dagger y, T^\dagger y\rangle \leq \|T\|\|T^\dagger y\|\|y\|.$$

Hence

$$\|T^\dagger y\| \leq \|T\|\|y\|$$

$\forall\, y \in K$, i.e., $T^\dagger$ is bounded with $\|T^\dagger\| \leq \|T\|$. □

Definition 2.1.3. The map $T^\dagger$ in Proposition 2.1.2 is called the ***adjoint*** of T.

This is typically denoted T^* but we will reserve this notation for compact closed structures. This definition yields some immediate consequences:

$$\begin{aligned} id^\dagger &= id, \\ (ST)^\dagger &= T^\dagger S^\dagger, \\ T^{\dagger\dagger} &= T. \end{aligned}$$

Categorically speaking, $(-)^\dagger$ is a contravariant functor on **Hilb**. From the fact that $\dagger$ is involutive it follows (by replacing T with $T^\dagger$ in the proof of Proposition 2.1.2) that $\|T^\dagger\| = \|T\|$.

Remark 2.1.4. Let $T : H_1 \longrightarrow H_2$ be a bounded linear map. If a bounded linear map $S : H_2 \longrightarrow H_1$ satisfies

$$\langle Tb_1, b_2\rangle = \langle b_1, Sb_2\rangle,$$

$\forall\, b_1 \in B_1$, $\forall\, b_2 \in B_2$, then $S = T^\dagger$. This means that it suffices to consider bases when computing adjoints.

There are situations when the adjoint of an operator T is related to T in an interesting way. The following definitions capture such situations.

Definition 2.1.5. If $T^\dagger = T : H \to H$, i.e., $\langle Tx, y\rangle = \langle x, Ty\rangle$ for all $x, y \in H$, then T is called ***self-adjoint***.

Definition 2.1.6. A map $T : H \to H$ is called ***unitary*** if $T^\dagger T = id = TT^\dagger$, i.e., $T^\dagger = T^{-1}$.

Remark 2.1.7. In $\mathbf{Hilb}_{fd}$ and **Hilb**, the unitary maps are precisely the isomorphisms. If $T : H_1 \to H_2$ is unitary, then

$$\langle Tx, Tx'\rangle = \langle x, T^\dagger Tx'\rangle = \langle x, x'\rangle \qquad \forall\, x, x' \in H_1,$$

hence T is an isomorphism. Conversely, if $T : H_1 \to H_2$ is an isomorphism, then

$$\langle Tx, y\rangle = \langle Tx, TT^{-1}y\rangle = \langle x, T^{-1}y\rangle \qquad \forall\, x \in H_1, \forall\, y \in H_2,$$

so we must have $T^\dagger = T^{-1}$. Thus T is unitary.

Definition 2.1.8. A map $T : H \to H$ is called ***positive*** if it is self-adjoint and $\langle Tx, x\rangle \geq 0$ for all $x \in H$.

This next theorem which is part of Theorem 4.6.7 in [11] leads to an equivalent characterization of positive maps.

Theorem 2.1.9. *Let $T : H \to H$ be positive. Then T has a unique positive square root, i.e., there exists a unique positive map S such that $T = S^2 = S^\dagger S$.*

Conversely, if T factors as $S^\dagger S$ for some map S then T is clearly self-adjoint with

$$\langle Tx, x\rangle = \langle S^\dagger Sx, x\rangle = \langle Sx, Sx\rangle \geq 0,$$

$\forall\, x \in H$. Hence T is positive. Therefore, a map $T : H \to H$ is positive if and only if T factors as $S^\dagger S$ for some map S.

2.2 The Tensor Product and Hilbert-Schmidt Operators

We are interested in a tensor structure for **Hilb**. Following [18], we give a definition of the tensor product H of two Hilbert spaces H_1 and H_2 which emphasizes the universal property of H. We then identify H with the completion of H_0, the algebraic tensor product of H_1 and H_2, relative to a specified inner product on H_0. Finally, we show how H can be viewed as the Hilbert space of *Hilbert-Schmidt operators* from the *conjugate* space $\overline{H_1}$ into H_2.

Definition 2.2.1. A mapping $\varphi : H_1 \times \ldots \times H_n \to \mathbb{C}$ is a ***bounded multilinear functional*** on $H_1 \times \ldots \times H_n$ if φ is linear in each component and there exists a non-negative real number c such that

$$|\varphi(x_1, \ldots, x_n)| \leq c\|x_1\| \cdots \|x_n\|,$$

for all $x_1 \in H_1, \ldots x_n \in H_n$. When φ is bounded, the smallest real number with this property is called the ***bound*** of φ and is denoted $\|\varphi\|$.

Definition 2.2.2. A bounded multilinear functional $\varphi : H_1 \times \ldots \times H_n \to \mathbb{C}$ is a ***Hilbert-Schmidt functional*** on $H_1 \times \ldots \times H_n$ if the sum

$$\sum_{b_1 \in B_1} \cdots \sum_{b_n \in B_n} |\varphi(b_1, \ldots, b_n)|^2 \tag{1}$$

is finite.

Although it appears that the above sum depends on our choice of bases $B_1, \ldots, B_n$, Proposition 2.6.1 of [18] asserts that the value of the sum is in fact independent of the choice of orthonormal bases for $H_1, \ldots, H_n$. We denote the set of Hilbert-Schmidt functionals on $H_1 \times \cdots \times H_n$ by $HSF(H_1 \times \cdots \times H_n)$.

The set of Hilbert-Schmidt functionals on $H_1 \times \cdots \times H_n$ is a Hilbert space. This is established in Proposition 2.6.2 [18]. The details are as follows.

Proposition 2.2.3. *The set $HSF(H_1 \times \cdots \times H_n)$ is a Hilbert space when the linear structure and inner product are defined by:*

$$(\alpha\varphi + \beta\psi)(x_1, \ldots, x_n) = \alpha\varphi(x_1, \ldots, x_n) + \beta\psi(x_1, \ldots, x_n), \tag{2}$$

$$\langle \varphi, \psi \rangle = \sum_{b_1 \in B_1} \cdots \sum_{b_n \in B_n} \varphi(b_1, \ldots, b_n)\overline{\psi(b_1, \ldots, b_n)}. \tag{3}$$

The sum in (3) *is absolutely convergent and independent of the choice of bases* $B_1, \ldots, B_n$. *For each* $v_1 \in H_1, \ldots, v_n \in H_n$, *the equation*

$$\varphi_{v_1,\ldots,v_n}(x_1, \ldots, x_n) = \langle x_1, v_1 \rangle \cdots \langle x_n, v_n \rangle \qquad (x_1 \in H_1, \ldots, x_n \in H_n)$$

defines an element $\varphi_{v_1,\ldots,v_n} \in HSF(H_1 \times \cdots \times H_n)$, *and*

$$\langle \varphi_{v_1,\ldots,v_n}, \varphi_{w_1,\ldots,w_n} \rangle = \langle w_1, v_1 \rangle \cdots \langle w_n, v_n \rangle.$$

The set

$$\{ \varphi_{b_1,\ldots,b_n} \mid b_1 \in B_1, \ldots, b_n \in B_n \}$$

is an orthonormal basis for $HSF(H_1 \times \cdots \times H_n)$. *Moreover, there is a unitary map* $U : HSF(H_1 \times \cdots \times H_n) \to l_2(B_1 \times \cdots \times B_n)$ *such that* $U\varphi$ *is the restriction* $\varphi|_{B_1 \times \cdots \times B_n}$ *for* $\varphi \in HSF(H_1 \times \cdots \times H_n)$.

The norm for $HSF(H_1 \times \cdots \times H_n)$ is induced by the inner product, and is denoted $\|\cdot\|_2$. This is not to be confused with the bound of an operator, for which we have established the notation $\|\cdot\|$.

The next definition will simplify what follows.

Definition 2.2.4. For a Hilbert space $K = (K, +, \cdot, \langle -, - \rangle)$, the ***conjugate Hilbert space*** $\overline{K} = (K, +, \bar{\cdot}, \langle -, - \rangle^{-})$ is the Hilbert space with the same underlying set as K, addition as in K, but with scalar multiplication and inner product given by:

$$\alpha \bar{\cdot} x = \overline{\alpha} x \quad \text{and} \quad \langle x, y \rangle^{-} = \overline{\langle x, y \rangle} = \langle y, x \rangle.$$

This definition eliminates the need to discuss conjugate-linear maps. By considering conjugate Hilbert spaces when necessary, all maps can be described as multilinear. Note that if B is an orthonormal basis for the Hilbert space K, then B is also an orthonormal basis for $\overline{K}$. We note also that for any Hilbert space K, $\overline{K} \cong K^\sharp$, the bounded dual space.

We can extend this construction to a *conjugate* functor $\overline{(-)}$ on **Hilb**. For a bounded linear operator $T : H \to K$ we define $\overline{T}$ to be the map T viewed as a map from $\overline{H}$ to $\overline{K}$.

Next we require the following definitions.

Definition 2.2.5. A map $L : H_1 \times \cdots \times H_n \to K$ is a ***bounded multilinear operator*** if it is linear in each component and there exists a nonnegative real number c such that

$$\|L(x_1, \ldots, x_n)\| \leq c\|x_1\| \cdots \|x_n\|$$

for all $x_1 \in H_1, \ldots, x_n \in H_n$. When L is a bounded multilinear operator, the least such c is denoted $\|L\|$.

Definition 2.2.6. A bounded multilinear map $L : H_1 \times \cdots \times H_n \to K$ is a ***weak Hilbert-Schmidt operator*** if

(i) for each $u \in K$ the map L_u defined by

$$L_u(x_1, \ldots, x_n) = \langle L(x_1, \ldots, x_n), u\rangle$$

is a Hilbert-Schmidt functional on $H_1 \times \cdots \times H_n$;

(ii) there exists a nonnegative real number d such that

$$\|L_u\|_2 \leq d\|u\|$$

for each $u \in K$.
When these conditions are satisfied, the least such d is denoted $\|L\|_2$.

Theorem 2.6.4 [18] provides the final step in the tensor product construction for **Hilb**.

Theorem 2.2.7. *Let $H_1, \ldots, H_n$ be Hilbert spaces with respective orthonormal bases $B_1, \ldots, B_n$.*

(i) There exists a Hilbert space H and a weak Hilbert-Schmidt operator $p : H_1 \times \cdots \times H_n \to H$ with the following property: given any weak Hilbert-Schmidt operator L from $H_1 \times \cdots \times H_n$ into a Hilbert space K, there exists a unique bounded linear operator $T : H \to K$ such that $L = Tp$:

$$\begin{tikzcd} H_1 \times \cdots \times H_n \arrow[r, "L"] \arrow[d, "p"'] & K \\ H \arrow[ur, dashed, "T"'] & \end{tikzcd}$$

Moreover, $\|T\| = \|L\|_2$.

(ii) If H' *and* p' *have the properties attributed in (i) to* H *and* p*, there is a unitary map* $U : H \to H'$ *such that* $p' = Up$*:*

$$\begin{tikzcd} H_1 \times \cdots \times H_n \arrow[r, "p'"] \arrow[d, "p"'] & H' \\ H \arrow[ur, dashed, "U"'] & \end{tikzcd}$$

(iii) If $v_i, w_i \in H_i$ $(i = 1, \ldots, n)$*, then*

$$\langle p(v_1, \ldots, v_n), p(w_1, \ldots, w_n)\rangle = \langle v_1, w_1\rangle \cdots \langle v_n, w_n\rangle,$$

the set $\{\, p(b_1, \ldots, b_n) \mid b_1 \in B_1, \ldots, b_n \in B_n \,\}$ *is an orthonormal basis for* H*, and* $\|p\|_2 = 1$.

In order to understand the tensor product for **Hilb**, we require some details from the proof. In particular the Hilbert space H and the operator $p : H_1 \times \cdots \times H_n \to H$ in (i) are constructed as follows. First, $H \stackrel{\text{def}}{=} HSF(\overline{H_1} \times \cdots \times \overline{H_n})$ with Hilbert space structure given by Proposition 2.2.3. Then for $v_1 \in H_1, \ldots, v_n \in H_n$,

$$p(v_1, \ldots, v_n) \stackrel{\text{def}}{=} \varphi_{v_1,\ldots,v_n} \in HSF(\overline{H_1} \times \cdots \times \overline{H_n}),$$

where

$$\varphi_{v_1,\ldots,v_n}(x_1, \ldots, x_n) = \langle x_1, v_1\rangle^- \cdots \langle x_n, v_n\rangle^- = \langle v_1, x_1\rangle \cdots \langle v_n, x_n\rangle.$$

Part (ii) of Theorem 2.2.7 asserts that the Hilbert space H, together with the weak Hilbert-Schmidt operator p, is uniquely determined up to isomorphism by the universal property in part (i). Thus we can take this to be our definition.

Definition 2.2.8. The Hilbert space H constructed above is the ***tensor product*** of $H_1, \ldots, H_n$ and is denoted by $H_1 \otimes \cdots \otimes H_n$. The map p is the ***canonical product mapping*** from $H_1 \times \cdots \times H_n$ to $H_1 \otimes \cdots \otimes H_n$.

The vector $p(x_1, \ldots, x_n) \in H_1 \otimes \cdots \otimes H_n$ is denoted by $x_1 \otimes \cdots \otimes x_n$. Finite linear combinations of these simple tensors form an everywhere-dense subspace H_0 of

$H_1 \otimes \cdots \otimes H_n$. Furthermore, the set $\{ b_1 \otimes \cdots \otimes b_n \mid b_1 \in B_1, \ldots, b_n \in B_n \}$ is an orthonormal basis for $H_1 \otimes \cdots \otimes H_n$.

As one would expect, the vector $x_1 \otimes \cdots \otimes x_n$ behaves like a formal product of $x_1, \ldots, x_n$. In particular, the following properties are a consequence of the multilinearity of p and part (iii) of Theorem 2.2.7:

$$\begin{aligned} x_1 \otimes \cdots \otimes x_{m-1} \otimes (\alpha x'_m + \beta x''_m) \otimes x_{m+1} \otimes \cdots \otimes x_n \\ = \alpha(x_1 \otimes \cdots \otimes x_{m-1} \otimes x'_m \otimes x_{m+1} \otimes \cdots \otimes x_n) \\ + \beta(x_1 \otimes \cdots \otimes x_{m-1} \otimes x''_m \otimes x_{m+1} \cdots \otimes x_n); \end{aligned} \tag{4}$$

$$\langle x_1 \otimes \cdots \otimes x_n, y_1 \otimes \cdots \otimes y_n \rangle = \langle x_1, y_1 \rangle \cdots \langle x_n, y_n \rangle; \tag{5}$$

$$\|x_1 \otimes \cdots \otimes x_n\| = \|x_1\| \cdots \|x_n\|. \tag{6}$$

Note that

$$\alpha(x_1 \otimes \cdots \otimes x_n) = (\alpha x_1) \otimes x_2 \otimes \cdots \otimes x_n,$$

so H_0 consists of all finite sums of simple tensors.

The tensor product for **Hilb** is associative (see Proposition 2.6.5 [18]), so it suffices to consider the case $n = 2$ in what follows.

Proposition 2.6.6 [18] investigates linear dependence in H_0.

Proposition 2.2.9. *Let H_0 be the everywhere-dense subspace of $H_1 \otimes H_2$ generated by the simple tensors.*

(i) If $x_1, \ldots, x_n \in H_1$ and $y_1, \ldots, y_n \in H_2$, then $\Sigma_{j=1}^n x_j \otimes y_j = 0$ if and only if there is an $n \times n$ complex matrix $[c_{ik}]$ such that

$$\sum_{j=1}^n c_{jk} x_j = 0 \quad (k = 1, \ldots, n),$$

$$\sum_{k=1}^n c_{jk} y_k = y_j \quad (k = 1, \ldots, n).$$

(ii) If L is a bilinear mapping from $H_1 \otimes H_2$ into a complex vector space V, there exists a unique linear mapping $T : H_0 \to V$ such that $T = L|_{H_0}$.

Part (i) ensures that any linear dependence relations that occur in H_0 result from the bilinearity of the canonical product mapping p. Thus H_0 can be identified with the quotient of the linear space of all formal finite sums of simple tensors by the subspace of those finite sums that must vanish for p to be bilinear. This is the traditional algebraic tensor product of H_1 and H_2, and part (ii) of the proposition gives its characteristic universal property. This means that we can identify $H = H_1 \otimes H_2$ with the completion of the algebraic tensor product H_0, relative to the unique inner product on H_0 satisfying

$$\langle x_1 \otimes y_1, x_2 \otimes y_2 \rangle = \langle x_1, x_2 \rangle \langle y_1, y_2 \rangle \qquad (x_1, x_2 \in H_1),\ (y_1, y_2 \in H_2).$$

In this paper, we will take this as our tensor product for **Hilb** unless otherwise stated. Even so, we show how the Hilbert space $H_1 \otimes H_2$ can be identified with a special class of operators from $\overline{H_1}$ to H_2. This alternate presentation will serve to motivate the concept of a *nuclear ideal* in Chapter 5.

Definition 2.2.10. A bounded linear operator $T : H_1 \to H_2$ is a ***Hilbert-Schmidt operator*** if the following sum is finite

$$\sum_{b_1 \in B_1} \sum_{b_2 \in B_2} |\langle Tb_1, b_2 \rangle|^2 = \sum_{b_2 \in B_2} \sum_{b_1 \in B_1} |\langle T^\dagger b_2, b_1 \rangle|^2. \tag{7}$$

Again, Proposition 2.6.1 guarantees that the value of the above sum is independent of the orthonormal bases chosen. Moreover, the sum (7) can be written in the alternate forms:

$$\sum_{b_1 \in B_1} \|Tb_1\|^2, \quad \sum_{b_2 \in B_2} \|T^\dagger b_2\|^2$$

using Parseval's equation. We denote the set of Hilbert-Schmidt operators from H_1 to H_2 by $HSO(H_1, H_2)$.

Remark 2.2.11. Hilbert-Schmidt operators from a two-sided ideal in **Hilb**. Let $S : H_2 \to H_3$ be a Hilbert-Schmidt operator, and let $R : H_1 \to H_2$ and $T : H_3 \to H_4$

be arbitrary. Then TSR is a Hilbert-Schmidt operator:

$$\begin{aligned}
\sum_{b_1\in B_1} \|TSR(b_1)\|^2 &\leq \|T\|^2 \sum_{b_1\in B_1} \|SR(b_1)\|^2 \\
&= \|T\|^2 \sum_{b_1\in B_1}\sum_{b_3\in B_3} |\langle SR(b_1), b_3\rangle|^2 && \text{(P.E.)} \\
&= \|T\|^2 \sum_{b_1\in B_1}\sum_{b_3\in B_3} |\langle b_1, R^\dagger S^\dagger(b_3)\rangle|^2 \\
&= \|T\|^2 \sum_{b_3\in B_3}\sum_{b_1\in B_1} |\langle R^\dagger S^\dagger(b_3), b_1\rangle|^2 \\
&= \|T\|^2 \sum_{b_3\in B_3} \|R^\dagger S^\dagger(b_3)\|^2 && \text{(P.E.)} \\
&\leq \|T\|^2\|R^\dagger\|^2 \sum_{b_3\in B_3} \|S^\dagger(b_3)\|^2 \\
&= \|T\|^2\|R^\dagger\|^2 \sum_{b_3\in B_3}\sum_{b_2\in B_2} |\langle S^\dagger(b_3), b_2\rangle|^2 && \text{(P.E.)} \\
&= \|T\|^2\|R^\dagger\|^2 \sum_{b_2\in B_2}\sum_{b_3\in B_3} |\langle b_3, S(b_2)\rangle|^2 \\
&= \|T\|^2\|R\|^2 \sum_{b_2\in B_2}\sum_{b_3\in B_3} |\langle S(b_2), b_3\rangle|^2 < \infty,
\end{aligned}$$

where P.E. indicates that we have applied Parseval's equation.

Next we consider the relationship between Hilbert-Schmidt functionals and Hilbert-Schmidt operators. For this we will need the following lemma, which is a restatement of Theorem 2.4.1 in [18].

Lemma 2.2.12. *If H_1 and H_2 are Hilbert spaces and $T \in Hom_{\mathbf{Hilb}}(H_1, H_2)$, the equation*

$$f_T(x,y) = \langle Tx, y\rangle \qquad (x \in H_1,\ y \in \overline{H_2})$$

defines a bounded bilinear functional f_T on $H_1 \times \overline{H_2}$, and $\|f_T\| = \|T\|$. Each bounded bilinear functional on $H_1 \times \overline{H_2}$ arises in this way from a unique element of $Hom_{\mathbf{Hilb}}(H_1, H_2)$.

The above correspondence restricts to give a bijective correspondence between $HSO(H_1, H_2)$ and $HSF(H_1 \times \overline{H_2})$. In other words, $T \in Hom_{\mathbf{Hilb}}(H_1, H_2)$ is a Hilbert-Schmidt operator if and only if f_T is a Hilbert-Schmidt functional on $H_1 \times \overline{H_2}$. This is clear from the definition of f_T. Furthermore, the set $HSO(H_1, H_2)$ of Hilbert-Schmidt operators from H_1 into H_2 forms a linear subspace of the vector space $Hom_{\mathbf{Hilb}}(H_1, H_2)$. Thus, via the (linear) mapping $T \mapsto f_T$, $HSO(H_1, H_2)$ inherits the Hilbert space structure of $HSF(H_1 \times \overline{H_2})$ outlined in Proposition 2.2.3. The inner product for $HSO(H_1, H_2)$ is then given as follows:

$$\langle S, T \rangle = \sum_{b_1 \in B_1} \sum_{b_2 \in B_2} \langle Sb_1, b_2 \rangle \langle b_2, Tb_1 \rangle,$$

where $S, T \in HSO(H_1, H_2)$. Once again, this inner product is basis-independent. Accordingly, we use the notation $\| \cdot \|_2$ to denote the norm for $HSO(H_1, H_2)$ as well.

We now view $H_1 \otimes H_2$ as the Hilbert space $HSO(\overline{H_1}, H_2)$ of Hilbert-Schmidt operators from $\overline{H_1}$ into H_2. The following is Proposition 2.6.9 in [18].

Theorem 2.2.13. *If H_1 and H_2 are Hilbert spaces, then for each $x \in H_1$ and each $y \in H_2$ the equation*

$$T_{x,y} u = \langle u, x \rangle^{-} y = \langle x, u \rangle y \qquad u \in \overline{H_1}$$

defines a Hilbert-Schmidt operator $T_{x,y} : \overline{H_1} \longrightarrow H_2$. Moreover, the linear map $U : H_1 \otimes H_2 \longrightarrow HSO(\overline{H_1}, H_2)$ satisfying $U(x \otimes y) = T_{x,y}$ is an isomorphism of $H_1 \otimes H_2$ onto $HSO(\overline{H_1}, H_2)$.

It follows that the set $\{ T_{b_1, b_2} \mid b_1 \in B_1, b_2 \in B_2 \}$ is an orthonormal basis for $HSO(\overline{H_1}, H_2)$.

Remark 2.2.14. Note that we also have a linear bijection from $H_1 \otimes H_2$ onto the vector space $Hom(\mathbb{C}, H_1 \otimes H_2)$ sending $x \otimes y \in H_1 \otimes H_2$ to the map defined by $[1 \mapsto x \otimes y]$. In total we have a linear bijection $\theta' : HSO(\overline{H_1}, H_2) \longrightarrow Hom(\mathbb{C}, H_1 \otimes H_2)$ with $\theta'(T_{b_1, b_2}) = [1 \mapsto b_1 \otimes b_2]$. Or, equivalently, we have a linear bijection $\theta : HSO(H_1, H_2) \longrightarrow Hom(\mathbb{C}, \overline{H_1} \otimes H_2)$.

2.3 Trace Class Maps

There is a special class of Hilbert-Schmidt operators called trace class operators.

Definition 2.3.1. A bounded linear operator $T : H_1 \to H_1$ is ***trace class*** if the following sum converges

$$\sum_{b_1 \in B_1} \langle T b_1, b_1 \rangle. \tag{8}$$

Note that the sum is independent of the choice of orthonormal basis. When T is trace class we denote the sum (8) by $tr(T)$ and call it the ***trace*** of T.

Note that all maps of the form $T : H_1 \to H_1$ in the category $\mathbf{Hilb}_{fd}$ are trace class.

Theorem 2.3.2. *The set of trace class maps on a Hilbert space H_1 is a two-sided ideal in $Hom_{\mathbf{Hilb}}(H_1, H_1)$.*

Trace class operators can be characterized as follows.

Theorem 2.3.3. *An operator $T : H \to H$ is trace class if and only if there exists a Hilbert space K and Hilbert-Schmidt operators $S : H \to K$ and $R : K \to H$ such that $T = RS$.*

Hence trace class maps are Hilbert-Schmidt maps by Remark 2.2.11. However, not all Hilbert-Schmidt maps $T : H \to H$ are trace class as the following example illustrates.

Example 2.3.4.
Let H be a Hilbert space with orthonormal basis $\{ e_n \mid n \in \mathbb{N} \}$ and $T : H \to H$ be the map sending e_n to $\frac{1}{n} e_n$ for each n. Then

$$\sum_{n \in \mathbb{N}} \|T e_n\|^2 = \sum_{n \in \mathbb{N}} \|\frac{1}{n} e_n\|^2 = \sum_{n \in \mathbb{N}} \frac{1}{n^2} < \infty,$$

while

$$\sum_{n \in \mathbb{N}} \langle T e_n, e_n \rangle = \sum_{n \in \mathbb{N}} \langle \frac{1}{n} e_n, e_n \rangle = \sum_{n \in \mathbb{N}} \frac{1}{n} \longrightarrow \infty.$$

Therefore T is a Hilbert-Schmidt operator that is not trace class.

If $T : H \to H$ is trace class with $T = RS$ as prescribed in Theorem 2.3.3, then the map $\theta : HSO(H, K) \to Hom(\mathbb{C}, \overline{H} \otimes K)$ provides an alternate way to calculate the trace of T. We have the following map

$$\mathbb{C} \xrightarrow{\theta(S)} \overline{H} \otimes K \xrightarrow{\overline{\theta(R)}^{\dagger}} \overline{\mathbb{C}} \cong \mathbb{C}.$$

Via the bijective correspondence $Hom(\mathbb{C}, \mathbb{C}) \cong \mathbb{C}$ this defines an element $\alpha \in \mathbb{C}$. Moreover, we have $\alpha = tr(T)$. In Chapter 6, we generalize the notion of trace class maps to the setting of *nuclear ideals.*

See [28] for details of the results of this section.

Chapter 3

Categorical Preliminaries

Here we give all the category theory necessary to understand the thesis. We assume the reader is familiar with the basics of category theory. See [23].

3.1 Symmetric Monoidal Categories

Definition 3.1.1. A ***monoidal category*** $\mathcal{C} = \langle \mathcal{C}, \otimes, I, \alpha, \lambda, \rho \rangle$ is a category $\mathcal{C}$ equipped with a bifunctor $\otimes : \mathcal{C} \times \mathcal{C} \longrightarrow \mathcal{C}$, a distinguished object $I \in \mathcal{C}$ called the tensor unit, and three natural isomorphisms α, λ, and ρ. More precisely, the maps

$$\alpha_{A,B,C} : (A \otimes B) \otimes C \xrightarrow{\cong} A \otimes (B \otimes C)$$

$$\lambda_A : I \otimes A \xrightarrow{\cong} A$$

$$\rho_A : A \otimes I \xrightarrow{\cong} A$$

are natural $\forall\, A,\, B,\, C \in \mathcal{C}$,

$$\lambda_I = \rho_I : I \otimes I \longrightarrow I, \tag{9}$$

and the following two diagrams commute $\forall\, A,\, B,\, C,\, D \in \mathcal{C}$:

$$\begin{tikzcd}
& (A \otimes B) \otimes (C \otimes D) \arrow[dr, "\alpha_{A,B,C \otimes D}"] & \\
((A \otimes B) \otimes C) \otimes D \arrow[ur, "\alpha_{A \otimes B,C,D}"] \arrow[d, "\alpha_{A,B,C} \otimes D"'] & & A \otimes (B \otimes (C \otimes D)) \\
(A \otimes (B \otimes C)) \otimes D \arrow[rr, "\alpha_{A,B \otimes C,D}"'] & & A \otimes ((B \otimes C) \otimes D) \arrow[u, "A \otimes \alpha_{B,C,D}"']
\end{tikzcd} \tag{10}$$

$$\begin{tikzcd}
(A \otimes I) \otimes B \arrow[rr, "\alpha_{A,I,B}"] \arrow[dr, "\rho_A \otimes B"'] & & A \otimes (I \otimes B) \arrow[dl, "A \otimes \lambda_B"] \\
& A \otimes B &
\end{tikzcd} \tag{11}$$

Equation (10) is called the Mac Lane pentagon.

Definition 3.1.2. A monoidal category $\mathcal{C} = \langle \mathcal{C}, \otimes, I, \alpha, \lambda, \rho \rangle$ is ***symmetric*** if it is equipped with natural isomorphisms

$$\sigma_{A,B} : A \otimes B \xrightarrow{\cong} B \otimes A$$

for A, $B \in \mathcal{C}$, such that the following three diagrams commute:

$$\begin{tikzcd}
A \otimes B \arrow[rr, "\sigma_{A,B}"] \arrow[dr, "A \otimes B"'] & & B \otimes A \arrow[dl, "\sigma_{B,A}"] \\
& A \otimes B &
\end{tikzcd} \tag{12}$$

$$\begin{tikzcd}
A \otimes I \arrow[rr, "\sigma_{A,I}"] \arrow[dr, "\rho_A"'] & & I \otimes A \arrow[dl, "\lambda_A"] \\
& A &
\end{tikzcd} \tag{13}$$

$$
\begin{tikzcd}
 & (B \otimes A) \otimes C \arrow[r, "\alpha_{B,A,C}"] & B \otimes (A \otimes C) \arrow[dr, "B \otimes \sigma_{A,C}"] & \\
(A \otimes B) \otimes C \arrow[ur, "\sigma_{A,B} \otimes C"] \arrow[dr, "\alpha_{A,B,C}"'] & & & B \otimes (C \otimes A) \\
 & A \otimes (B \otimes C) \arrow[r, "\sigma_{A,B\otimes C}"'] & (B \otimes C) \otimes A \arrow[ur, "\alpha_{B,C,A}"'] &
\end{tikzcd}
\tag{14}
$$

We usually omit the subscripts on the isomorphisms α, λ, ρ, and σ when they are clear from the context. This is especially handy for presenting tidy commutative diagrams.

Example 3.1.3.
The category **Rel** of sets and relations consists of the following data. The objects are sets and a morphism $R : A \to B$ is a subset of $A \times B$. For a set A we have $id_A = \{ (a, a) \mid a \in A \}$, and for relations $R : A \to B$ and $S : B \to C$ the composite $SR = \{ (a, c) \mid \exists b \in B \text{ with } (a, b) \in R \text{ and } (b, c) \in S \} \subseteq A \times C$. **Rel** has the following symmetric monoidal structure. $A \otimes B = A \times B$, the cartesian product of sets. Given maps $R : A \to C$ and $S : B \to D$ we have

$$R \otimes S = \{ ((a, b), (c, d)) \mid (a, c) \in R \text{ and } (b, d) \in S \} \subseteq (A \times B) \times (C \times D).$$

The tensor unit $I = \{*\}$ is any one element set. For sets A, B, and C we have natural isomorphisms

$$
\begin{aligned}
\alpha_{A,B,C} &= \{ (((a, b), c), (a, (b, c))) \mid a \in A, b \in B, c \in C \} \\
\lambda_A &= \{ ((*, a), a) \mid a \in A \} \\
\rho_A &= \{ ((a, *), a) \mid a \in A \} \\
\sigma_{A,B} &= \{ ((a, b), (b, a)) \mid a \in A, b \in B \}.
\end{aligned}
$$

Example 3.1.4.
LFR is the subcategory of **Rel** consisting of sets and *locally finite relations.* Let

$R : A \to B$ in **Rel** and for each $a \in A$ denote ${}_aR = \{ b \in B \mid (a,b) \in R \}$. Similarly, for each $b \in B$, denote $R_b = \{ a \in A \mid (a,b) \in R \}$. Then R is locally finite if $\forall\, a \in A$ and $\forall\, b \in B$ the sets ${}_aR$ and R_b are finite. It is clear that the identity morphisms are locally finite, and a short calculation shows that the composition of two locally finite relations is again locally finite. Moreover, **LFR** inherits a symmetric monoidal structure from **Rel**. One must check that the tensor product preserves local finiteness and the isomorphisms α, λ, ρ, and σ are locally finite. Let $R : A \to C$ and $S : B \to D$ be locally finite. Then

$$R \otimes S = \{ ((a,b),(c,d)) \mid (a,c) \in R \text{ and } (b,d) \in S \}$$

is locally finite since

$$\left|{}_{(a,b)}R \otimes S\right| = |{}_aR| \cdot |{}_bS| < \infty$$

and

$$\left|R \otimes S_{(c,d)}\right| = |R_c| \cdot |S_d| < \infty.$$

For the isomorphisms we have

$$\begin{aligned} \left|{}_{((a,b),c)}\alpha\right| &= 1 = \left|\alpha_{(a,(b,c))}\right| \\ \left|{}_{(*,a)}\lambda\right| &= 1 = |\lambda_a| \\ \left|{}_{(a,*)}\rho\right| &= 1 = |\rho_a| \\ \left|{}_{(a,b)}\sigma\right| &= 1 = \left|\sigma_{(b,a)}\right|. \end{aligned}$$

Example 3.1.5.

The category $\mathbf{Vec}_{fd}$ of finite-dimensional vector spaces (over a fixed field $\Bbbk$) is symmetric monoidal. Let U and V be vector spaces with respective bases $\{ e_i \mid i = 1, \dots, n \}$ and $\{ f_j \mid j = 1, \dots, m \}$. Then $U \otimes V$ is the (nm)-dimensional vector space with basis $\{ e_i \otimes f_j \mid i = 1, \dots, n \text{ and } j = 1, \dots, m \}$ and $I = \Bbbk$. Moreover, if $S : U \to U'$ and $T : V \to V'$ then $S \otimes T$ is the linear map satisfying $S \otimes T(e_i \otimes f_j) = Se_i \otimes Tf_j$. Let W be a vector space with basis $\{ g_k \mid k = 1, \dots, p \}$. Then in this case, the isomorphisms (specified on bases) are

$$\alpha_{U,V,W} : (U \otimes V) \otimes W \longrightarrow U \otimes (V \otimes W) \qquad (e_i \otimes f_j) \otimes g_k \mapsto e_i \otimes (f_j \otimes g_k)$$

$$\lambda_V : \Bbbk \otimes V \longrightarrow V \qquad 1 \otimes e_i \mapsto e_i$$
$$\rho_V : V \otimes \Bbbk \longrightarrow V \qquad e_i \otimes 1 \mapsto e_i$$
$$\sigma_{U,V} : U \otimes V \longrightarrow V \otimes U \qquad e_i \otimes f_j \mapsto f_j \otimes e_i.$$

Example 3.1.6.
The category $\mathbf{Hilb}_{fd}$ of finite-dimensional Hilbert spaces and (bounded) linear operators is symmetric monoidal. Let H and K be Hilbert spaces with respective orthonormal bases $\{ e_i \mid i = 1, \ldots, n \}$ and $\{ f_j \mid j = 1, \ldots, m \}$. Then, as in $\mathbf{Vec}_{fd}$, $I = \mathbb{C}$ and $H \otimes K$ is the Hilbert space with orthonormal basis $\{ e_i \otimes f_j \mid i = 1, \ldots, n$ and $j = 1, \ldots, m \}$ and with inner product given by

$$\langle e_i \otimes f_j, e_k \otimes f_l \rangle = \langle e_i, e_k \rangle \langle f_j, f_l \rangle.$$

On morphisms, the tensor product is defined as in $\mathbf{Vec}_{fd}$. This is the finite-dimensional version of the tensor product constructed in Section 2.2. Recall that in the finite-dimensional setting, all linear operators are bounded. This was shown in Remark 2.0.3. Hence the isomorphisms from $\mathbf{Vec}_{fd}$ are valid in $\mathbf{Hilb}_{fd}$ as well.

Example 3.1.7.
The category **Hilb** of arbitrary Hilbert spaces and bounded linear operators is also symmetric monoidal. We established its monoidal structure in Section 2.2. Recall that if H and K are Hilbert spaces with respective orthonormal bases $\{ e_i \mid i \in I \}$ and $\{ f_j \mid j \in J \}$, then $H \otimes K$ has an orthonormal basis $\{ e_i \otimes f_j \mid i \in I,\ j \in J \}$. Here, the tensor unit is again $I = \mathbb{C}$. We have already dealt with associativity, and the monoidal equation (11) follows from equation (4) of Section 2.2. It remains to consider symmetry. The isomorphism

$$\sigma_{H,K} : H \otimes K \longrightarrow K \otimes H \qquad e_i \otimes f_j \mapsto f_j \otimes e_i$$

is clearly bounded and satisfies the symmetric monoidal equations (12) - (14).

There are examples of monoidal categories that are not symmetric. For example, if $\mathcal{C}$ is a fixed (small) category, then we can form a monoidal category $Func(\mathcal{C})$. The objects are endofunctors on $\mathcal{C}$ and the morphisms are natural transformations.

The tensor product here is given by composition, which is not symmetric. A second example of a non-symmetric monoidal category is the category of representations of a Hopf algebra.

The following theorem, which is due to Mac Lane, will be assumed throughout this paper. This will reduce the work necessary to establish the commutativity of diagrams.

Theorem 3.1.8. *(Coherence) In a monoidal category, all diagrams built from α, λ, and ρ, $\otimes$, and identities commute.*

An analogous theorem holds in the symmetric case, however the symmetry map σ makes the theorem slightly more difficult to state. Note for example, the following two maps are not equal:

$$\sigma_{A,A} : A \otimes A \longrightarrow A \otimes A$$

and

$$id_{A,A} : A \otimes A \longrightarrow A \otimes A.$$

To handle this, one can associate to each morphism in the free symmetric monoidal category a permutation of the atoms appearing in the domain (and hence also the codomain). Then two morphisms are equal if and only if they have the same underlying permutation.

Definition 3.1.9. Let $\mathcal{C} = \langle \mathcal{C}, \otimes, I, \alpha, \lambda, \rho \rangle$ and $\mathcal{C}' = \langle \mathcal{C}', \otimes, I', \alpha', \lambda', \rho' \rangle$ be monoidal categories. A ***monoidal functor*** $F = \langle F, m, m_I \rangle : \mathcal{C} \to \mathcal{C}'$ consists of the following:

- A functor $F : \mathcal{C} \to \mathcal{C}'$;
- $\forall A, B \in \mathcal{C}$ a morphism $m_{A,B} : F(A) \otimes F(B) \to F(A \otimes B)$ in $\mathcal{C}'$ which is natural in A and B;
- for the tensor units I and I' a morphism $m_I : I' \to F(I)$.

In addition the following three diagrams must commute:

$$\begin{tikzcd}
(F(A) \otimes F(B)) \otimes F(C) \arrow[r, "\alpha'"] \arrow[d, "m_{A,B} \otimes F(C)"'] & F(A) \otimes (F(B) \otimes F(C)) \arrow[d, "F(A) \otimes m_{B,C}"] \\
F(A \otimes B) \otimes F(C) \arrow[d, "m_{A \otimes B,C}"'] & F(A) \otimes F(B \otimes C) \arrow[d, "m_{A,B \otimes C}"] \\
F((A \otimes B) \otimes C) \arrow[r, "F(\alpha)"'] & F(A \otimes (B \otimes C))
\end{tikzcd} \tag{15}$$

$$\begin{tikzcd}
F(A) \otimes I' \arrow[r, "\rho'"] \arrow[d, "F(A) \otimes m_I"'] & F(A) \\
F(A) \otimes F(I) \arrow[r, "m_{A,I}"'] & F(A \otimes I) \arrow[u, "F(\rho)"']
\end{tikzcd} \tag{16}$$

$$\begin{tikzcd}
I' \otimes F(A) \arrow[r, "\lambda'"] \arrow[d, "m_I \otimes F(A)"'] & F(A) \\
F(I) \otimes F(A) \arrow[r, "m_{I,A}"'] & F(I \otimes A) \arrow[u, "F(\lambda)"']
\end{tikzcd} \tag{17}$$

Definition 3.1.10. A monoidal functor F is ***strong*** if m_I and all $m_{A,B}$ are isomorphisms, and ***strict*** if m_I and all $m_{A,B}$ are identities.

Remark 3.1.11. For a (covariant) strict monoidal functor $F = \langle F, m, m_I \rangle : \mathcal{C} \to \mathcal{C}'$, we have the following data:

- $\forall A, B \in \mathcal{C}$, $F(A \otimes B) = F(A) \otimes F(B)$;
- $F(I) = I'$;
- $\forall f : A \to C$ and $g : B \to D$, $F(f \otimes g) = F(f) \otimes F(g) : A \otimes B \to C \otimes D$;
- $F(\alpha) = \alpha'$, $F(\lambda) = \lambda'$, and $F(\rho) = \rho'$.

In the case that F is contravariant, the last two points read:

- $\forall f : A \to C$ and $g : B \to D$, $F(f \otimes g) = F(f) \otimes F(g) : C \otimes D \to A \otimes B$;
- $F(\alpha) = (\alpha')^{-1}$, $F(\lambda) = (\lambda')^{-1}$, and $F(\rho) = (\rho')^{-1}$.

Definition 3.1.12. Let $\mathcal{C} = \langle \mathcal{C}, \otimes, I, \alpha, \lambda, \rho, \sigma \rangle$ and $\mathcal{C}' = \langle \mathcal{C}', \otimes, I', \alpha', \lambda', \rho', \sigma' \rangle$ be symmetric monoidal categories. A ***symmetric monoidal functor*** $F = \langle F, m, m_I \rangle : \mathcal{C} \to \mathcal{C}'$ is a monoidal functor satisfying one additional equation:

$$\begin{tikzcd} F(A) \otimes F(B) \arrow[r, "\sigma'"] \arrow[d, "m_{A,B}"'] & F(B) \otimes F(A) \arrow[d, "m_{B,A}"] \\ F(A \otimes B) \arrow[r, "F(\sigma)"'] & F(B \otimes A) \end{tikzcd} \tag{18}$$

Definition 3.1.13. A symmetric monoidal category $\mathcal{C}$ is ***closed*** if $\forall A \in \mathcal{C}$ the functor $A \otimes (-)$ has a right adjoint $A \multimap (-)$. More precisely, there is a bijection $\varphi : Hom_{\mathcal{C}}(A \otimes B, C) \cong Hom_{\mathcal{C}}(B, A \multimap C)$, which is natural in B and C. Or, equivalently [23], we can specify the unit and counit maps, which are natural transformations of the form:

$$\eta : Id \longrightarrow A \multimap (A \otimes (-)) \qquad \text{(unit)}$$
$$\epsilon : A \otimes (A \multimap (-)) \longrightarrow Id \qquad \text{(counit)}.$$

Example 3.1.14.
Recall that **Rel** is symmetric monoidal with tensor product on objects given by the cartesian product of sets. **Rel** is closed with $A \multimap (-) = A \otimes (-)$. The bijection $\varphi : Hom_{\mathcal{R}el}(A \otimes B, C) \cong Hom_{\mathcal{R}el}(B, A \multimap C)$ is given by

$$R \subseteq (A \times B) \times C \mapsto \{\, (b, (a, c)) \mid ((a, b), c) \in R \,\} \subseteq B \times (A \times C).$$

For naturality, we must check the following two diagrams for relations $S : C \to C'$ and $T : B \to B'$:

$$\begin{tikzcd} Hom(A \otimes B, C) \arrow[r, "\varphi"] \arrow[d, "Hom(A \otimes B, S)"'] & Hom(B, A \multimap C) \arrow[d, "Hom(B, A \multimap S)"] \\ Hom(A \otimes B, C') \arrow[r, "\varphi"'] & Hom(B, A \multimap C') \end{tikzcd}$$

and

$$\begin{tikzcd}
Hom(A \otimes B', C) \arrow[r, "\varphi"] \arrow[d, "Hom(A \otimes T, C)"'] & Hom(B', A \multimap C) \arrow[d, "Hom(T, A \multimap C)"] \\
Hom(A \otimes B, C) \arrow[r, "\varphi"'] & Hom(B, A \multimap C)
\end{tikzcd}$$

We will check the first one. Let $R \in Hom(A \otimes B, C)$. Then

$$\begin{aligned}
&(b,(a,c')) \in (Hom(B, A \multimap S) \circ \varphi)(R) = (id_A \otimes S) \circ \varphi(R) \\
&\iff \exists\,(a,c) \in A \times C \text{ with } (b,(a,c)) \in \varphi(R) \text{ and } ((a,c)(a,c')) \in (id_A \otimes S) \\
&\iff \exists\, c \in C \text{ with } (b,(a,c)) \in \varphi(R) \text{ and } (c,c') \in S \\
&\iff \exists\, c \in C \text{ with } ((a,b),c) \in R \text{ and } (c,c') \in S \\
&\iff ((a,b),c') \in R \circ S \\
&\iff (b,(a,c')) \in \varphi(R \circ S) = (\varphi \circ Hom(A \otimes B, S))(R).
\end{aligned}$$

The second equation is similar. In this case, the unit and counit maps are given by the following:

$$\begin{aligned}
&\eta : Id \longrightarrow A \otimes (A \otimes (-)) \\
&\eta_B = \{\, (b,(a,(a,b))) \mid a \in A,\, b \in B \,\}; \\
&\epsilon : A \otimes (A \otimes (-)) \longrightarrow Id \\
&\epsilon_B = \{\, ((a,(a,b)),b) \mid a \in A,\, b \in B \,\}.
\end{aligned}$$

Example 3.1.15.
Both $\mathbf{Vec}_{fd}$ and $\mathbf{Hilb}_{fd}$ are closed. Rather than work out the details here, we will see in Section 3.2 that these categories satisfy a stronger condition.

There are some important categories equipped with monoidal structures under which they fail to be monoidal closed. We will revisit the following examples often. In particular, we will see that both possess a *nuclear ideal* structure.

Example 3.1.16.
LFR does not inherit the closed structure from **Rel**. Suppose the adjunction $A \otimes$

$(-) \dashv A \multimap (-)$ holds in **LFR** for all sets A, where $A \multimap (-) = A \otimes (-)$. Consider the unit map $\eta : Id \longrightarrow A \otimes (A \otimes (-))$. Its component at I would be $\eta_I = \{ (*, (a, (a, *))) \mid a \in A \}$. But then $|_* \eta_I| = |A|$, which is not necessarily finite.

Example 3.1.17.
It is well-known that the category **Hilb** fails to be monoidal closed. In particular, the closed structure for $\mathbf{Hilb}_{fd}$ cannot be extended to this category.

3.2 Compact Closed Categories

In this section we consider a special class of symmetric monoidal categories, namely those equipped with a *compact closed* structure.

Definition 3.2.1. A ***compact closed category*** is a symmetric monoidal category for which each object A has a dual object A^* as well as a unit map $\eta_A : I \to A^* \otimes A$ and a counit map $\epsilon_A : A \otimes A^* \to I$ satisfying the following two equations:

$$\begin{tikzcd}
A \arrow[r, "\rho_A^{-1}"] \arrow[rrddd, "A"'] & A \otimes I \arrow[r, "A \otimes \eta_A"] & A \otimes (A^* \otimes A) \arrow[d, "\alpha^{-1}_{A,A^*,A}"] \\
 & & (A \otimes A^*) \otimes A \arrow[d, "\epsilon_A \otimes A"] \\
 & & I \otimes A \arrow[d, "\lambda_A"] \\
 & & A
\end{tikzcd} \tag{19}$$

$$
\begin{tikzcd}
A^* \arrow[r, "\lambda^{-1}_{A^*}"] \arrow[rrdddd, "A^*"'] & I \otimes A^* \arrow[r, "\eta_A \otimes A^*"] & (A^* \otimes A) \otimes A^* \arrow[d, "\alpha_{A^*,A,A^*}"] \\
 & & A^* \otimes (A \otimes A^*) \arrow[d, "A^* \otimes \epsilon_A"] \\
 & & A^* \otimes I \arrow[d, "\rho_{A^*}"] \\
 & & A^*
\end{tikzcd}
\tag{20}
$$

It is worth noting that the use of the terms unit and counit is appropriate as the maps η and ϵ do in fact establish an adjunction $F \dashv F^*$ via (19) and (20) in the monoidal category $Func(\mathcal{C})$ of endofunctors on a fixed category $\mathcal{C}$. In this case tensor is composition, $I = id_{\mathcal{C}}$, and $F = F^*$.

Proposition 3.2.2. *In a compact closed category $\mathcal{C}$, the operation $(-)^*$ extends to a contravariant functor on $\mathcal{C}$.*

This result appears in [20]. We show how to define $(-)^*$ on morphisms and omit the proof of functoriality. For a morphism $f : A \to B$ we define $f^* : B^* \to A^*$ as follows:

$$
\begin{tikzcd}
B^* \arrow[r, "\lambda^{-1}_{B^*}"] \arrow[rrdddddd, "f^*"'] & I \otimes B^* \arrow[r, "\eta_A \otimes B^*"] & (A^* \otimes A) \otimes B^* \arrow[d, "(A^* \otimes f) \otimes B^*"] \\
 & & (A^* \otimes B) \otimes B^* \arrow[d, "\alpha_{A^*,B,B^*}"] \\
 & & A^* \otimes (B \otimes B^*) \arrow[d, "A^* \otimes \epsilon_B"] \\
 & & A^* \otimes I \arrow[d, "\rho_{A^*}"] \\
 & & A^*
\end{tikzcd}
$$

Proposition 3.2.3. *In a compact closed category $\mathcal{C}$, the functor $(-)^*$ is strong and yields natural isomorphisms $A \cong A^{**}$.*

Again, we will not need the details of the proof so we just provide the required isomorphisms:

$$\begin{aligned} (A \otimes B)^* &\cong A^* \otimes B^* \quad \text{(natural in } A \text{ and } B\text{);} \\ I^* &\cong I; \\ A &\cong A^{**} \quad \text{(natural in } A\text{).} \end{aligned}$$

First we have:

$$(A \otimes B)^* \xrightarrow{\lambda^{-1} \circ \lambda^{-1}} I \otimes I \otimes (A \otimes B)^* \xrightarrow{\eta_A \otimes \eta_B \otimes id} A^* \otimes A \otimes B^* \otimes B \otimes (A \otimes B)^*$$
$$\xrightarrow{id \otimes \sigma \otimes id} A^* \otimes B^* \otimes A \otimes B \otimes (A \otimes B)^* \xrightarrow{id \otimes \epsilon_{A \otimes B}} A^* \otimes B^* \otimes I$$
$$\xrightarrow{\rho} A^* \otimes B^*$$

with inverse

$$A^* \otimes B^* \xrightarrow{\rho^{-1}} A^* \otimes B^* \otimes I \xrightarrow{id \otimes \eta_{A \otimes B}} A^* \otimes B^* \otimes (A \otimes B)^* \otimes A \otimes B$$
$$\xrightarrow{id \otimes \sigma} A^* \otimes B^* \otimes A \otimes B \otimes (A \otimes B)^* \xrightarrow{id \otimes \sigma \otimes id} A^* \otimes A \otimes B^* \otimes B \otimes (A \otimes B)^*$$
$$\xrightarrow{\sigma \otimes \sigma \otimes id} A \otimes A^* \otimes B \otimes B^* \otimes (A \otimes B)^* \xrightarrow{\epsilon_A \otimes \epsilon_B \otimes id} I \otimes I \otimes (A \otimes B)^*$$
$$\xrightarrow{\lambda \circ \lambda} (A \otimes B)^*.$$

Next, we have the following maps which are inverse:

$$I \xrightarrow{\eta_I} I^* \otimes I \xrightarrow{\rho} I^*$$
$$I^* \xrightarrow{\lambda^{-1}} I \otimes I^* \xrightarrow{\epsilon_I} I.$$

Finally, we have

$$A \xrightarrow{\lambda^{-1}} I \otimes A \xrightarrow{\eta_{A^*} \otimes A} A^{**} \otimes A^* \otimes A \xrightarrow{A^{**} \otimes \sigma} A^{**} \otimes A \otimes A^*$$
$$\xrightarrow{A^{**} \otimes \epsilon_A} A^{**} \otimes I \xrightarrow{\rho} A^{**}$$

and

$$A^{**} \xrightarrow{\rho^{-1}} A^{**} \otimes I \xrightarrow{A^{**} \otimes \eta_A} A^{**} \otimes A^* \otimes A \xrightarrow{\sigma \otimes A} A^* \otimes A^{**} \otimes A$$
$$\xrightarrow{\epsilon_{A^*} \otimes A} I \otimes A \xrightarrow{\lambda} A.$$

Remark 3.2.4. Since our compact closed categories are symmetric, we get a natural isomorphism

$$(A \otimes B)^* \cong B^* \otimes A^*. \tag{21}$$

This isomorphism appears extensively in [26].

Example 3.2.5.
Rel is compact closed with $X^* = X$ and

$$\eta_X = \{\, (*, (x, x)) \mid x \in X \,\} \subseteq I \times (X \times X)$$

$$\epsilon_X = \{\, ((x, x), *) \mid x \in X \,\} \subseteq (X \times X) \times I.$$

It follows from the definition of relational composition that

$$id_X = \lambda_X \circ (\epsilon_X \otimes id_X) \circ \alpha^{-1} \circ (id_X \otimes \eta_X) \circ \rho_X^{-1}$$

and

$$id_{X^*} = \rho_{X^*} \circ (id_{X^*} \otimes \epsilon_X) \circ \alpha \circ (\eta_X \otimes id_{X^*}) \circ \lambda_{X^*}^{-1}$$

as required. By Proposition 3.2.2, $(-)^*$ extends to a functor. For a morphism $R : A \to B$, the map $R^* : B \to A$ is the converse relation, i.e., $R^* = \{\, (b, a) \mid (a, b) \in R\}$.

Example 3.2.6.
$\mathbf{Vec}_{fd}$ is compact closed. Let V be a vector space with basis $\{\, e_i \mid i = 1, \ldots, n \,\}$. Then V^* is the usual dual space with basis $\{\, e_i{}^* \mid i = 1, \ldots, n \,\}$ and the unit and counit maps are given by (the linear extensions of):

$$\begin{aligned} \eta_V &: \Bbbk \to V^* \otimes V & 1 &\mapsto \sum_{i=1}^{n} e_i{}^* \otimes e_i \\ \epsilon_V &: V \otimes V^* \to \Bbbk & e_i \otimes e_j{}^* &\mapsto e_j{}^*(e_i) = \delta_{ij}. \end{aligned}$$

We check equations (19) and (20) on basis elements.

$$\begin{tikzcd}
e_i \arrow[r, mapsto, "\rho^{-1}"] \arrow[ddddrr, mapsto, "V"'] & e_i \otimes 1 \arrow[r, mapsto, "V \otimes \eta_V"] & e_i \otimes \left(\sum_{j=1}^n {e_j}^* \otimes e_j\right) = \sum_{j=1}^n e_i \otimes ({e_j}^* \otimes e_j) \arrow[d, mapsto, "\alpha^{-1}"] \\
 & & \sum_{j=1}^n (e_i \otimes {e_j}^*) \otimes e_j \arrow[d, mapsto, "\epsilon_V \otimes V"] \\
 & & 1 \otimes e_i \arrow[d, mapsto, "\lambda"] \\
 & & \\
 & & e_i
\end{tikzcd}$$

$$\begin{tikzcd}
{e_i}^* \arrow[r, mapsto, "\lambda^{-1}"] \arrow[ddddrr, mapsto, "V^*"'] & 1 \otimes {e_i}^* \arrow[r, mapsto, "\eta_V \otimes V^*"] & \left(\sum_{j=1}^n {e_j}^* \otimes e_j\right) \otimes {e_i}^* = \sum_{j=1}^n ({e_j}^* \otimes e_j) \otimes {e_i}^* \arrow[d, mapsto, "\alpha"] \\
 & & \sum_{j=1}^n {e_j}^* \otimes (e_j \otimes {e_i}^*) \arrow[d, mapsto, "V^* \otimes \epsilon_V"] \\
 & & {e_i}^* \otimes 1 \arrow[d, mapsto, "\rho"] \\
 & & \\
 & & {e_i}^*
\end{tikzcd}$$

Again, we can use Proposition 3.2.2 to determine the action of $(-)^*$ on a map $T : V \to W$. If $f \in W^*$, then $T^*(f) = \sum_{i=1}^n f(Te_i){e_i}^* \in V^*$.

Example 3.2.7.

This structure makes $\mathbf{Hilb}_{fd}$ compact closed as well. The dual space of a finite-dimensional Hilbert space H, i.e., the set of (bounded) linear functionals on H, is itself a Hilbert space. The inner product is given by:

$$\langle f, g \rangle = \sum_{i=1}^n f(e_i) \cdot \overline{g(e_i)}$$

where $\{ e_i \mid i = 1, \ldots, n \}$ is an orthonormal basis for H. Moreover, the analogues of the maps η and ϵ in $\mathbf{Vec}_{fd}$ are valid here as well.

As the above examples suggest, a compact closed category is indeed symmetric monoidal closed.

Theorem 3.2.8. *If $\mathcal{C}$ is compact closed, then $\mathcal{C}$ is monoidal closed.*

Proof. Let $A \in \mathcal{C}$. Define $A \multimap B \stackrel{\text{def}}{=} A^* \otimes B$. We prove the adjunction by specifying the unit $u_B : B \longrightarrow A^* \otimes (A \otimes B)$ and counit $c_B : A \otimes (A^* \otimes B) \longrightarrow B$. These maps are given by:

$$u_B : B \xrightarrow{\lambda_B^{-1}} I \otimes B \xrightarrow{\eta_A \otimes B} (A^* \otimes A) \otimes B \xrightarrow{\alpha_{A^*,A,B}} A^* \otimes (A \otimes B)$$

$$c_B : A \otimes (A^* \otimes B) \xrightarrow{\alpha^{-1}_{A,A^*,B}} (A \otimes A^*) \otimes B \xrightarrow{\epsilon_A \otimes B} I \otimes B \xrightarrow{\lambda_B} B.$$

We first show that u is natural. Let $f : B \to C$, then we need:

$$\begin{tikzcd}
B \arrow[r, "u_B"] \arrow[d, "f"'] & A^* \otimes (A \otimes B) \arrow[d, "A^* \otimes (A \otimes f)"] \\
C \arrow[r, "u_C"'] & A^* \otimes (A \otimes C)
\end{tikzcd}$$

But replacing u_B and u_C gives

$$\begin{tikzcd}
B \arrow[r, "\lambda^{-1}"] \arrow[d, "f"'] \arrow[dr, phantom, "\lambda \text{ nat.}"] & I \otimes B \arrow[r, "\eta_A \otimes B"] \arrow[d, "I \otimes f"] \arrow[dr, "\eta_A \otimes f"] \arrow[dr, phantom, "\otimes \text{ func.}", shift left=4ex] \arrow[dr, phantom, "\otimes \text{ func.}"', shift right=4ex] & (A^* \otimes A) \otimes B \arrow[r, "\alpha"] \arrow[d, "(A^* \otimes A) \otimes f"] \arrow[dr, phantom, "\alpha \text{ nat.}"] & A^* \otimes (A \otimes B) \arrow[d, "A^* \otimes (A \otimes f)"] \\
C \arrow[r, "\lambda^{-1}"'] & I \otimes C \arrow[r, "\eta_A \otimes C"'] & (A^* \otimes A) \otimes C \arrow[r, "\alpha"'] & A^* \otimes (A \otimes C)
\end{tikzcd}$$

Naturality of c is similar. It remains to show that the following two adjunction triangles commute:

$$
\begin{tikzcd}
A^* \otimes B \arrow[r, "u_{A^*\otimes B}"] \arrow[dr, swap, "A^* \otimes B"] & A^* \otimes (A \otimes (A^* \otimes B)) \arrow[d, "A^* \otimes c_B"] \\
& A^* \otimes B
\end{tikzcd}
\qquad
\begin{tikzcd}
A \otimes B \arrow[r, "A \otimes u_B"] \arrow[dr, swap, "A \otimes B"] & A \otimes (A^* \otimes (A \otimes B)) \arrow[d, "c_{A\otimes B}"] \\
& A \otimes B
\end{tikzcd}
$$

For the first triangle, we replace id_{A^*} using (20). The diagram is then filled in as shown below.

$$
\begin{tikzcd}
A^* \otimes B \arrow[r, "\lambda^{-1}"] \arrow[d, swap, "\lambda^{-1} \otimes B"] \arrow[dr, phantom, "\text{coh.}"] & I \otimes (A^* \otimes B) \arrow[r, "\eta_A \otimes (A^* \otimes B)"] \arrow[d, phantom, "\alpha \text{ nat.}"] & (A^* \otimes A) \otimes (A^* \otimes B) \arrow[d, "\alpha"] \\
(I \otimes A^*) \otimes B \arrow[ur, "\alpha"] \arrow[d, swap, "(\eta_A \otimes A^*) \otimes B"] & {} \arrow[d, phantom, "\text{Mac Lane pentagon}"] & A^* \otimes (A \otimes (A^* \otimes B)) \arrow[d, "A^* \otimes \alpha^{-1}"] \\
((A^* \otimes A) \otimes A^*) \otimes B \arrow[uurr, "\alpha"] \arrow[d, swap, "\alpha \otimes B"] & {} & A^* \otimes ((A \otimes A^*) \otimes B) \arrow[d, "A^* \otimes (\epsilon_A \otimes B)"] \\
(A^* \otimes (A \otimes A^*)) \otimes B \arrow[urr, "\alpha"] \arrow[d, swap, "(A^* \otimes \epsilon_A) \otimes B"] \arrow[r, phantom, "\alpha \text{ nat.}"] & {} \arrow[d, phantom, "\text{coh.}"] & A^* \otimes (I \otimes B) \arrow[d, "A^* \otimes \lambda"] \\
(A^* \otimes I) \otimes B \arrow[urr, "\alpha"] \arrow[rr, swap, "\rho \otimes B"] & & A^* \otimes B
\end{tikzcd}
$$

Similarly, we use (19) to replace id_A to establish the second triangle. □

This means in particular that **LFR** does not inherit the compact closed structure from **Rel** since we have already seen that **LFR** is not closed with this structure. Likewise, the category **Hilb** fails to be compact closed.

3.3 Dagger Categories

Abramsky and Coecke introduced dagger compact closed categories to capture the structure of compact closed categories that, like $\mathbf{Hilb}_{fd}$, possess an adjoint structure [3]. They used the term *strongly compact closed* to describe such categories. However, Selinger observed that a dagger structure can be added to almost any type of category. In [26] he introduces the notion of a general dagger category and then proceeds to dagger categories with additional structure. We follow Selinger's approach below.

Definition 3.3.1. A ***dagger category*** is a category $\mathcal{C}$ equipped with an involutive contravariant functor $\dagger : \mathcal{C} \to \mathcal{C}$ which is the identity on objects. More precisely, for each object $A \in \mathcal{C}$ we have

$$id_A^\dagger = id_A : A \to A \tag{22}$$

and for morphisms $f : A \to B$ and $g : B \to C$ in $\mathcal{C}$ we have

$$(g \circ f)^\dagger = f^\dagger \circ g^\dagger : C \to A \quad \text{and} \tag{23}$$

$$f^{\dagger\dagger} = f : A \to B. \tag{24}$$

The map $f^\dagger$ is called the ***adjoint*** of f.

The notions of self-adjoint and unitary maps on Hilbert spaces generalize to the setting of dagger categories.

Definition 3.3.2. In a dagger category, a morphism $f : A \to A$ is called ***self-adjoint*** if $f^\dagger = f$. A morphism $f : A \to B$ is called ***unitary*** if $f^\dagger = f^{-1}$.

Example 3.3.3.
We have seen in Section 2.1 that the Hilbert adjoint makes $\mathbf{Hilb}_{fd}$ and **Hilb** dagger categories.

Example 3.3.4.
Rel is a dagger category. For a map $R : A \to B$, we define $R^\dagger : B \to A$ to be the converse relation. In this case, $(-)^*$ and $(-)^\dagger$ coincide.

Example 3.3.5.
The functor $\dagger$ on **Rel** restricts to **LFR**, and thus **LFR** is a dagger category as well.

The dagger categories that we will deal with will also be monoidal. We will require that the dagger and monoidal structures interact correctly, in the following sense.

Definition 3.3.6. A ***dagger symmetric monoidal category*** is a symmetric monoidal category $\mathcal{C}$ equipped with a strict monoidal dagger functor. As outlined in Remark 3.1.11, this means that $\dagger : \mathcal{C}^{op} \to \mathcal{C}$ preserves the monoidal structure as follows. For morphisms $f : A \to B$, and $g : C \to D$ we require:

$$(f \otimes g)^\dagger = f^\dagger \otimes g^\dagger : B \otimes D \to A \otimes C; \tag{25}$$

$$\alpha^\dagger_{A,B,C} = \alpha^{-1}_{A,B,C} : A \otimes (B \otimes C) \to (A \otimes B) \otimes C; \tag{26}$$

$$\lambda^\dagger_A = \lambda^{-1}_A : A \to I \otimes A; \tag{27}$$

$$\rho^\dagger_A = \rho^{-1}_A : A \to A \otimes I; \tag{28}$$

$$\sigma^\dagger_{A,B} = \sigma^{-1}_{A,B} : B \otimes A \to A \otimes B \tag{29}$$

i.e., the maps α, λ, ρ and σ are all unitary.

Example 3.3.7.
Both $\mathbf{Hilb}_{fd}$ and **Hilb** are dagger symmetric monoidal categories. Let H, K, M, and N be Hilbert spaces with respective bases $\{h_i\}$, $\{k_i\}$, $\{m_i\}$, and $\{n_i\}$. Let $S : H \to K$ and $T : M \to N$. We have

$$\begin{aligned}\langle S \otimes T(h_i \otimes m_j), k_s \otimes n_t\rangle &= \langle S(h_i), k_s\rangle \cdot \langle T(m_j), n_t\rangle \\ &= \langle h_i, S^\dagger k_s\rangle \cdot \langle m_j, T^\dagger n_t\rangle \\ &= \langle h_i \otimes m_j, S^\dagger \otimes T^\dagger(k_s \otimes n_t)\rangle.\end{aligned}$$

Hence $(S \otimes T)^\dagger = S^\dagger \otimes T^\dagger$. It follows from Remark 2.1.7 that the isomorphisms α, λ, ρ and σ are unitary.

Example 3.3.8.
Rel is also dagger symmetric monoidal. Let $R : A \to C$ and $S : B \to D$ be maps in **Rel**. Then

$$\begin{aligned}(R \otimes S)^\dagger &= \{\, ((a,b),(c,d)) \mid (a,c) \in R \text{ and } (b,d) \in S \,\}^\dagger \\ &= \{\, ((c,d),(a,b)) \mid (a,c) \in R \text{ and } (b,d) \in S \,\} \\ &= \{\, ((c,d),(a,b)) \mid (c,a) \in R^\dagger \text{ and } (d,b) \in S^\dagger \,\} \\ &= R^\dagger \otimes S^\dagger.\end{aligned}$$

Again, the isomorphisms α, λ, ρ and σ are clearly unitary.

Example 3.3.9.
The argument above shows that **LFR** is dagger symmetric monoidal as well.

In [26], the dagger symmetric monoidal categories are compact closed. Selinger calls such categories dagger compact closed rather than strongly compact closed as they were first introduced by Abramsky and Coecke. We will follow Selinger's terminology.

Definition 3.3.10. A ***dagger compact closed category*** $\mathcal{C}$ is a dagger symmetric monoidal category that is also compact closed, and such that the following diagram commutes $\forall\, A \in \mathcal{C}$:

$$\begin{tikzcd} I \arrow[rr, "\epsilon_A^\dagger"] \arrow[dr, "\eta_A"'] & & A \otimes A^* \arrow[dl, "\sigma_{A,A^*}"] \\ & A^* \otimes A & \end{tikzcd} \tag{30}$$

Remark 3.3.11. In a dagger compact closed category, the isomorphisms from Proposition 3.2.3 are all unitary.

Example 3.3.12.
We have seen that $\mathbf{Hilb}_{fd}$ is a compact closed category that is dagger symmetric monoidal. We will check equation (30). Let H be a Hilbert space with orthonormal

basis $\{ e_i \mid i = 1, \ldots, n \}$. It suffices to show that $\epsilon^{\dagger}_{H \otimes H^*}(1) = \sum_{i=1}^{n} e_i \otimes e_i{}^*$. Let $\sum c_{jk} e_j \otimes e_k{}^* \in H \otimes H^*$. Then

$$\langle \epsilon_{H \otimes H^*}(\Sigma c_{jk} e_j \otimes e_k{}^*), 1 \rangle = \epsilon_{H \otimes H^*}(\Sigma c_{jk} e_j \otimes e_k{}^*) = \Sigma c_{jj}.$$

But,

$$\langle \Sigma c_{jk} e_j \otimes e_k{}^*, \sum_{i=1}^{n} e_i \otimes e_i{}^* \rangle = \Sigma c_{jk} \sum_{i=1}^{n} \langle e_j, e_i \rangle \cdot \langle e_k{}^*, e_i{}^* \rangle = \Sigma c_{jj}.$$

So we must have that $\epsilon^{\dagger}_{H \otimes H^*}(1) = \sum_{i=1}^{n} e_i \otimes e_i{}^*$, as required. Hence $\mathbf{Hilb}_{fd}$ is dagger compact closed.

Example 3.3.13.
Rel is also dagger compact closed. Recall that for a set X we defined

$$\eta_X = \{ (*, (x, x)) \mid x \in X \} \quad \text{and} \quad \epsilon_X = \{ ((x, x), *) \mid x \in X \}.$$

So clearly $\sigma \circ \epsilon^{\dagger}_X = \epsilon^{\dagger}_X = \eta_X$, i.e., equation (30) is satisfied.

The following proposition is due to Selinger.

Proposition 3.3.14. *Let $\mathcal{C}$ be a dagger compact closed category and let $f : A \to B$ in $\mathcal{C}$. Then $f^{*\dagger} = f^{\dagger *} : A^* \to B^*$.*

Proof. Recall from Proposition 3.2.2 that

$$f^* \stackrel{\text{def}}{=} \rho_{A^*} \circ (id_{A^*} \otimes \epsilon_B) \circ \alpha_{A^*,B,B^*} \circ ((id_{A^*} \otimes f) \otimes id_{B^*}) \circ (\eta_A \otimes id_{B^*}) \circ \lambda^{-1}_{B^*}.$$

Then

$$f^{\dagger *} = \rho_{B^*} \circ (id_{B^*} \otimes \epsilon_A) \circ \alpha_{B^*,A,A^*} \circ ((id_{B^*} \otimes f^{\dagger}) \otimes id_{A^*}) \circ (\eta_B \otimes id_{A^*}) \circ \lambda^{-1}_{A^*}$$

and

$$\begin{aligned} f^{*\dagger} &= (\lambda^{-1}_{B^*})^{\dagger} \circ (\eta_A \otimes id_{B^*})^{\dagger} \circ ((id_{A^*} \otimes f) \otimes id_{B^*})^{\dagger} \circ \alpha^{\dagger}_{A^*,B,B^*} \circ (id_{A^*} \otimes \epsilon_B)^{\dagger} \circ \rho^{\dagger}_{A^*} \\ &= \lambda_{B^*} \circ (\eta^{\dagger}_A \otimes id_{B^*}) \circ ((id_{A^*} \otimes f^{\dagger}) \otimes id_{B^*}) \circ \alpha^{-1}_{A^*,B,B^*} \circ (id_{A^*} \otimes \epsilon^{\dagger}_B) \circ \rho^{-1}_{A^*}. \end{aligned}$$

To see that $f^{*\dagger} = f^{\dagger *}$ consider the following commutative diagram.

$$
\begin{tikzcd}
A^* \arrow[r, "\lambda^{-1}"] \arrow[ddd, "\rho^{-1}"'] \arrow[dr, phantom, "(13)"] & I \otimes A^* \arrow[rr, "\eta_B \otimes A^*"] \arrow[ddr, "\epsilon_B^\dagger \otimes A^*"] \arrow[dr, phantom, "(30)"] & & (B^* \otimes B) \otimes A^* \arrow[rrr, "(B^* \otimes f^\dagger) \otimes A^*"] \arrow[dr, "\alpha"] \arrow[drr, phantom, "\alpha \text{ nat.}"] & & & (B^* \otimes A) \otimes A^* \arrow[ddd, "\alpha"] \\
& & & & B^* \otimes (B \otimes A^*) \arrow[drr, "B^* \otimes (f^\dagger \otimes A^*)"] \arrow[dddl, "B^* \otimes \sigma"'] \arrow[dd, phantom, "\sigma \text{ nat. and } \otimes \text{ functor}"] & & \\
& & (B \otimes B^*) \otimes A^* \arrow[uur, "\sigma \otimes A^*"] & & & & \\
A^* \otimes I \arrow[uuur, "\sigma"'] \arrow[dd, "A^* \otimes \epsilon_B^\dagger"'] \arrow[r, phantom, "\sigma \text{ nat.}"] & & & & & & B^* \otimes (A \otimes A^*) \arrow[ddll, "B^* \otimes \sigma"'] \arrow[dd, "B^* \otimes \epsilon_A"] \arrow[dl, phantom, "(30)"] \\
& & (14) & B^* \otimes (A^* \otimes B) \arrow[dr, "B^* \otimes (A^* \otimes f^\dagger)"] & & & \\
A^* \otimes (B \otimes B^*) \arrow[uuurr, "\sigma"] \arrow[d, "\alpha^{-1}"'] & & \sigma \text{ nat.} & & B^* \otimes (A^* \otimes A) \arrow[rr, "B^* \otimes \eta_A^\dagger"'] & \sigma \text{ nat.} & B^* \otimes I \arrow[d, "\rho"] \arrow[dl, phantom, "(13)"] \\
(A^* \otimes B) \otimes B^* \arrow[rrr, "(A^* \otimes f^\dagger) \otimes B^*"'] \arrow[uurrr, "\sigma"] & & & (A^* \otimes A) \otimes B^* \arrow[ur, "\sigma"] \arrow[rr, "\eta_A^\dagger \otimes B^*"'] & & I \otimes B^* \arrow[ur, "\sigma"] \arrow[r, "\lambda"'] & B^*
\end{tikzcd}
$$

□

Definition 3.3.15. Let $\mathcal{C}$ be a dagger compact closed category. Then $(-)_* \stackrel{\text{def}}{=} (-)^{*\dagger} = (-)^{\dagger *}$ defines a covariant functor on $\mathcal{C}$. Moreover, as $(-)^\dagger$ and $(-)^*$ are strong, so is $(-)_*$.

There are some important equations that hold in a dagger compact closed category. Later, we will extend these to the nuclear ideal setting. These equations appear (modulo equation (30)) in Kelly and Laplaza's paper on coherence for compact closed categories [20].

$$
\begin{tikzcd}
I \arrow[r, "\eta_{A\otimes B}"] \arrow[d, "\lambda^{-1} = \rho^{-1}"'] & B^* \otimes A^* \otimes A \otimes B \\
I \otimes I \arrow[r, "\eta_A \otimes \eta_B"'] & A^* \otimes A \otimes B^* \otimes B \arrow[u, "\sigma \otimes B"']
\end{tikzcd}
\tag{31}
$$

$$
\begin{tikzcd}
A \otimes B \otimes B^* \otimes A^* \arrow[r, "\epsilon_{A\otimes B}"] \arrow[d, "A \otimes \sigma"'] & I \\
A \otimes A^* \otimes B^* \otimes B \arrow[r, "\epsilon_A \otimes \epsilon_B"'] & I \otimes I \arrow[u, "\lambda = \rho"']
\end{tikzcd}
\tag{32}
$$

$$
\begin{tikzcd}
I \arrow[rr, "\eta_I"] \arrow[dr, "\lambda^{-1} = \rho^{-1}"'] & & I^* \otimes I \\
& I \otimes I \arrow[ur, "\cong"'] &
\end{tikzcd}
\tag{33}
$$

$$
\begin{tikzcd}
& {} \arrow[dr, "\epsilon^\dagger_{A^*}"] & \\
I \arrow[ur, dash] \arrow[rr, "\eta_A"] \arrow[d, "\cong"'] & & A^* \otimes A \\
I^* \arrow[rr, "\eta^{*\dagger}_A"'] & & (A^* \otimes A)^* \arrow[u, "\cong"']
\end{tikzcd}
\tag{34}
$$

Chapter 4

Complete Positivity

In this chapter, we provide a detailed review of Selinger's article on completely positive maps in the setting of dagger compact closed categories [26].

4.1 Positive Maps and Positive Matrices

We begin with Selinger's notion of a positive morphism in a general dagger category.

Definition 4.1.1. A morphism $f : A \longrightarrow A$ in a dagger category $\mathcal{C}$ is ***positive*** if there is an object B and a morphism $g : A \longrightarrow B$ in $\mathcal{C}$ such that $f = g^\dagger \circ g$.

A positive morphism $f : A \longrightarrow A$ may have several decompositions of the form $f = g^\dagger \circ g$. Uniqueness is not required.

We now characterize the positive maps in the familiar categories $\mathbf{Hilb}_{fd}$, **Hilb**, **Rel**, and **LFR**.

Example 4.1.2.
Let $T : H \longrightarrow H$ be a positive morphism in $\mathbf{Hilb}_{fd}$ or **Hilb**. Then we may write $T = S^\dagger \circ S$, for some $S : H \longrightarrow K$. Thus

$$T^\dagger = (S^\dagger \circ S)^\dagger = S^\dagger \circ S^{\dagger\dagger} = S^\dagger \circ S = T,$$

i.e., T is self-adjoint. Moreover,

$$\langle Tx, x\rangle = \langle S^\dagger \circ Sx, x\rangle = \langle Sx, Sx\rangle \geq 0.$$

So T is a positive operator as defined in Section 2.1. Conversely, if T is a self-adjoint positive operator, then T factors uniquely as $S^\dagger \circ S$ with S a positive operator. Hence the positive maps in $\mathbf{Hilb}_{fd}$ and $\mathbf{Hilb}$ are precisely the positive operators.

Example 4.1.3.
Let $R : A \to A$ be a positive morphism in **Rel**. Write $R = S^\dagger \circ S$ with $S : A \to B$. Then

$$\begin{aligned} R &= S^\dagger \circ S \\ &= \{\, (a, a') \mid \exists\, b \in B \text{ with } (a,b) \in S \text{ and } (b, a') \in S^\dagger \,\} \\ &= \{\, (a, a') \mid \exists\, b \in B \text{ with } (a,b) \in S \text{ and } (a', b) \in S \,\}. \end{aligned}$$

So R is symmetric and *partially reflexive* in the sense that if $(a, a') \in R$ then $(a, a) \in R$. Conversely, suppose $R : A \to A$ is a symmetric partially reflexive relation. Then we can view R as a set and define a relation $S : A \to R$ by:

$$S = \{\, (a, (a, a')) \mid (a, a') \in R \,\} \cup \{\, (a', (a, a')) \mid (a, a') \in R \,\}.$$

It is clear that $R \subseteq S^\dagger \circ S$. Suppose $(a, a') \in S^\dagger \circ S$. Then $\exists\, (x, y) \in R$ with $(a, (x, y)) \in S$ and $((x, y), a') \in S^\dagger$. There are four possibilities:

$$\begin{cases} a = x = a'; \\ a = x,\ a' = y; \\ a = y,\ a' = x; \\ a = y = a'. \end{cases}$$

In the first case, we have $(a, (x, y)) = (a, (a, y)) \in S$ so that $(a, y) \in R$. But R is partially reflexive, so $(a, a) = (a, a') \in R$. In the second case we have $(a, (x, y)) = (a, (a, a')) \in S$ thus $(a, a') \in R$. Similarly, in the third case we get $(a', a) \in R$. As R is reflexive, $(a, a') \in R$. The last case is similar to the first but uses the symmetry of R. Hence $R = S^\dagger \circ S$, so R is positive. Therefore the positive maps in **Rel** are the symmetric partially reflexive relations.

Example 4.1.4.
If $R : A \to A$ is locally finite, then so is the relation S defined above. Hence the positive maps in **LFR** are the locally finite relations which are symmetric and partially reflexive.

Recall from Section 2.3 that all maps of the form $T : H \to H$ in $\mathbf{Hilb}_{fd}$ are trace class. Selinger generalizes this notion to any dagger compact closed category as follows.

Definition 4.1.5. Let $f : A \to A$ in a compact closed category $\mathcal{C}$. Its ***trace*** $trf : I \to I$ is defined by

$$\begin{tikzcd} I \arrow[r, "\eta_A"] \arrow[rrrd, "trf"'] & A^* \otimes A \arrow[r, "\sigma"] & A \otimes A^* \arrow[r, "f \otimes id_{A^*}"] & A \otimes A^* \arrow[d, "\epsilon_A"] \\ & & & I \end{tikzcd}$$

Lemma 4.1.6. *In a dagger compact closed category, we have the following properties.*
(a) If $f : A \to A$ *is positive, then* $h^\dagger \circ f \circ h : C \to C$ *is positive, for any* $h : C \to A$.
(b) For each object A, id_A *is positive.*
(c) If $f : A \to A$ *and* $h : C \to C$ *are positive, then* $f \otimes h : A \otimes C \to A \otimes C$ *is positive.*
(d) If $f : A \to A$ *is positive, then* $f^\dagger = f$.
(e) If $f : A \to A$ *is positive, then* f^* *is positive.*
(f) If $f : A \to A$ *is positive, then* $trf : I \to I$ *is positive.*
(g) If $f, h : A \to A$ *are positive, then* $tr(h \circ f) : I \to I$ *is positive.*

Proof.
(a): If $f = g^\dagger \circ g$ then

$$\begin{aligned} h^\dagger \circ f \circ h &= h^\dagger \circ (g^\dagger \circ g) \circ h \\ &= (h^\dagger \circ g^\dagger) \circ (g \circ h) \\ &= (g \circ h)^\dagger \circ (g \circ h) \qquad \text{(by (23))}. \end{aligned}$$

(b): For each object A, we have

$$\begin{aligned} id_A &= id_A \circ id_A \\ &= id_A^{\dagger} \circ id_A && \text{(by (22)).} \end{aligned}$$

(c): If $f = g^{\dagger} \circ g$ and $h = k^{\dagger} \circ k$, then

$$\begin{aligned} f \otimes h &= (g^{\dagger} \circ g) \otimes (k^{\dagger} \circ k) \\ &= (g^{\dagger} \otimes k^{\dagger}) \circ (g \otimes k) \\ &= (g \otimes k)^{\dagger} \circ (g \otimes k) && \text{(by (25)).} \end{aligned}$$

(d): If $f = g^{\dagger} \circ g$ then

$$\begin{aligned} f^{\dagger} &= (g^{\dagger} \circ g)^{\dagger} \\ &= g^{\dagger} \circ g^{\dagger\dagger} && \text{(by (23))} \\ &= g^{\dagger} \circ g && \text{(by (24)).} \end{aligned}$$

(e): If $f = g^{\dagger} \circ g$ then

$$\begin{aligned} f^{*} &= (g^{\dagger} \circ g)^{*} \\ &= g^{*} \circ g^{\dagger *} && \text{(by Prop. 3.2.2)} \\ &= g^{*} \circ g^{*\dagger} && \text{(by Prop. 3.3.14)} \\ &= h^{\dagger} \circ h && \text{(where } h = g^{*\dagger}\text{).} \end{aligned}$$

(f): If $f = g^{\dagger} \circ g$ then

$$\begin{aligned} trf &= tr(g^{\dagger} \circ g) \\ &= \epsilon_A \circ ((g^{\dagger} \circ g) \otimes id_{A^*}) \circ \sigma \circ \eta_A \\ &= \epsilon_A \circ ((g^{\dagger} \otimes id_{A^*}) \circ (g \otimes id_{A^*})) \circ \sigma \circ \eta_A \\ &= (\epsilon_A \circ (g^{\dagger} \otimes id_{A^*})) \circ ((g \otimes id_{A^*}) \circ \sigma \circ \eta_A) \\ &= (\epsilon_A \circ (g^{\dagger} \otimes id_{A^*})) \circ ((g \otimes id_{A^*}) \circ \epsilon_A^{\dagger}) && \text{(by (30))} \\ &= (\epsilon_A \circ (g^{\dagger} \otimes id_{A^*})) \circ (\epsilon_A \circ (g \otimes id_{A^*})^{\dagger})^{\dagger} && \text{(by (23))} \\ &= (\epsilon_A \circ (g^{\dagger} \otimes id_{A^*})) \circ (\epsilon_A \circ (g^{\dagger} \otimes id_{A^*}^{\dagger}))^{\dagger} && \text{(by (25))} \\ &= (\epsilon_A \circ (g^{\dagger} \otimes id_{A^*})) \circ (\epsilon_A \circ (g^{\dagger} \otimes id_{A^*}))^{\dagger} && \text{(by (22)).} \end{aligned}$$

(g): If $f, h : A \longrightarrow A$ are positive with $f = g^\dagger \circ g$ then

$$\begin{aligned} tr(h \circ f) &= tr(h \circ (g^\dagger \circ g)) \\ &= tr((h \circ g^\dagger) \circ g) \\ &= tr(g \circ h \circ g^\dagger) \qquad\qquad (tr(j \circ k) = tr(k \circ j)), \end{aligned}$$

which is positive by (a) and (f). $\square$

Remark 4.1.7. In **Rel** the map $\sigma_{A,A} = \{((a,a'),(a',a)) \mid a, a' \in A\}$ satisfies $\sigma = \sigma^\dagger$. However, $\sigma_{A,A}$ does not satisfy the partial reflexivity condition and thus is not positive. Hence the converse of (d) fails.

Recall that in the category $\mathbf{Hilb}_{fd}$ we have $H \multimap (-) = H^* \otimes (-)$. So we have the bijective correspondence

$$Hom(H, K) \cong Hom(\mathbb{C} \otimes H, K) \cong Hom(\mathbb{C}, H^* \otimes K) \cong H^* \otimes K.$$

Via this correspondence we can view a morphism $f : H \longrightarrow K$ as an element of $H^* \otimes K$ called its *matrix.* If a map $f : H \longrightarrow H$ is positive, the corresponding element in $H^* \otimes H$ is called a *positive matrix.* The following definition generalizes these concepts to arbitrary dagger compact closed categories.

Definition 4.1.8. In a dagger compact closed category, the ***name*** or ***matrix*** of a morphism $f : A \longrightarrow B$ is the map $\ulcorner f \urcorner : I \longrightarrow A^* \otimes B$ defined by:

$$\begin{tikzcd} I \arrow[r, "\eta_A"] \arrow[dr, "\ulcorner f \urcorner"'] & A^* \otimes A \arrow[d, "id_{A^*} \otimes f"] \\ & A^* \otimes B \end{tikzcd} \tag{35}$$

A ***positive matrix*** is a morphism $\ulcorner f \urcorner : I \longrightarrow A^* \otimes A$ that is the name of a positive map $f : A \longrightarrow A$.

In particular, η_A is a positive matrix for any $A \in \mathcal{C}$ since $\eta_A = \ulcorner id_A \urcorner$.

In a compact closed category the maps η and ϵ are dinatural transformations [5, 13, 16]. This means that the following diagrams commute for any map $f : A \to B$:

$$\begin{tikzcd}
& A^* \otimes A \arrow[dr, "A^* \otimes f"] & \\
I \arrow[ur, "\eta_A"] \arrow[dr, "\eta_B"'] & & A^* \otimes B \\
& B^* \otimes B \arrow[ur, "f^* \otimes B"'] &
\end{tikzcd} \tag{36}$$

$$\begin{tikzcd}
& A^* \otimes A \arrow[dr, "\epsilon_A"] & \\
B^* \otimes A \arrow[ur, "f^* \otimes A"] \arrow[dr, "B^* \otimes f"'] & & I \\
& B^* \otimes B \arrow[ur, "\epsilon_B"'] &
\end{tikzcd} \tag{37}$$

These equations can be found in [20]. The first equation gives us an equivalent characterization of positive matrices in a dagger compact closed category.

Remark 4.1.9. A positive matrix

$$I \xrightarrow{\eta_A} A^* \otimes A \xrightarrow{A^* \otimes h} A^* \otimes B \xrightarrow{A^* \otimes h^\dagger} A^* \otimes A$$

is the same thing as a map

$$I \xrightarrow{\eta_B} B^* \otimes B \xrightarrow{h^* \otimes h^\dagger} A^* \otimes A$$

as the following diagram shows:

$$\begin{tikzcd}
I \arrow[r, "\eta_A"] \arrow[d, "\eta_B"'] \arrow[dr, phantom, "(36)"] & A^* \otimes A \arrow[r, "A^* \otimes h"] \arrow[dr, phantom, "\otimes \text{ func.}"] & A^* \otimes B \arrow[d, "A^* \otimes h^\dagger"] \\
B^* \otimes B \arrow[urr, "h^* \otimes B"] \arrow[rr, "h^* \otimes h^\dagger"'] & & A^* \otimes A
\end{tikzcd}$$

4.2 Completely Positive Maps

There is a special class of morphisms in $\mathbf{Hilb}_{fd}$ called completely positive operators.

Definition 4.2.1. A morphism $T : H^* \otimes H \to K^* \otimes K$ in $\mathbf{Hilb}_{fd}$ is a ***completely positive map*** (CPM) if T maps positive matrices to positive matrices. Additionally, this positivity property must be retained when T is tensored with arbitrary identity maps.

Selinger extends this idea to general dagger compact closed categories as follows.

Definition 4.2.2. Let $f : A^* \otimes A \to B^* \otimes B$ in a dagger compact closed category. The map f is ***completely positive*** if the following map is positive:

$$\begin{tikzcd}
B \otimes A^* \arrow[r, "\rho^{-1}"] & B \otimes A^* \otimes I \arrow[r, "B \otimes A^* \otimes \epsilon_A^\dagger"] & B \otimes A^* \otimes A \otimes A^* \arrow[d, "B \otimes f \otimes A^*"] \\
 & & B \otimes B^* \otimes B \otimes A^* \arrow[d, "\epsilon_B \otimes B \otimes A^*"] \\
 & & I \otimes B \otimes A^* \arrow[d, "\lambda"] \\
 & & B \otimes A^* .
\end{tikzcd} \tag{38}$$

Completely positive maps can be characterized in several equivalent ways.

Proposition 4.2.3. *Let $f : A^* \otimes A \to B^* \otimes B$. The following are equivalent:*
(a) $f : A^* \otimes A \to B^* \otimes B$ is completely positive.
(b) There exists a positive map $g : B \otimes A^* \to B \otimes A^*$ such that

$$\begin{tikzcd}
A^* \otimes A \arrow[r, "\lambda^{-1}"] \arrow[ddddrr, "f"'] & I \otimes A^* \otimes A \arrow[r, "\eta_B \otimes A^* \otimes A"] & B^* \otimes B \otimes A^* \otimes A \arrow[d, "B^* \otimes g \otimes A"] \\
 & & B^* \otimes B \otimes A^* \otimes A \arrow[d, "B^* \otimes B \otimes \eta_A^\dagger"] \\
 & & B^* \otimes B \otimes I \arrow[d, "\rho"] \\
 & & B^* \otimes B
\end{tikzcd}$$

(c) There exists an object C and a morphism $h : A \to C \otimes B$ such that

$$\begin{tikzcd}
A^* \otimes A \arrow[r, "h_* \otimes h"] \arrow[drr, "f"'] & B^* \otimes C^* \otimes C \otimes B \arrow[r, "B^* \otimes \eta_C^\dagger \otimes B"] & B^* \otimes I \otimes B \arrow[d, "B^* \otimes \lambda"] \\
 & & B^* \otimes B
\end{tikzcd}$$

(d) There exists an object C and a morphism $k : C \otimes A \to B$ such that

$$\begin{tikzcd}
A^* \otimes A \arrow[r, "\rho^{-1} \otimes A"] \arrow[drr, "f"'] & A^* \otimes I \otimes A \arrow[r, "A^* \otimes \eta_C \otimes A"] & A^* \otimes C^* \otimes C \otimes A \arrow[d, "k_* \otimes k"] \\
 & & B^* \otimes B
\end{tikzcd}$$

Proof.
[a $\Longrightarrow$ b] Suppose $f : A^* \otimes A \to B^* \otimes B$ is completely positive. Then we take $g : B \otimes A^* \to B \otimes A^*$ to be the composite in (38), which is positive by assumption. We will omit tensor symbols in the following diagram to reduce the size of the objects involved. We have:

$$
\begin{tikzcd}
A^* A \arrow[rr, "\lambda^{-1}"] \arrow[drr, "\lambda^{-1}"'] \arrow[dddddddddddd, "id"'] & & I\, A^* A \arrow[rr, "\eta_B\, id"] \arrow[drrr, "id\, \rho^{-1} A"] \arrow[dr, phantom, "\text{coh.}"] & & B^* B\, A^* A \arrow[rrr, "B^* \rho^{-1} A"] \arrow[dr, phantom, "\otimes\text{ func.}"] & & & B^* B\, A^* I\, A \arrow[dd, "id\, \epsilon_A^\dagger\, A"] \\
& & A^* I\, A \arrow[rrr, "\lambda^{-1} id"] \arrow[dd, "A^* \epsilon_A^\dagger\, A"] \arrow[dr, phantom, "\otimes\text{ func.}"] & & & I\, A^* I\, A \arrow[urr, "\eta_B\, id"] \arrow[dd, "id\, \epsilon_A^\dagger\, A"] \arrow[r, phantom, "\otimes\text{ func.}"] & & \\
& & & & & & & B^* B\, A^* A\, A^* A \arrow[dd, "id\, f\, id"] \\
& & A^* A\, A^* A \arrow[rrr, "\lambda^{-1} id"] \arrow[ddr, "f\, id"] \arrow[dddddddd, "id\, \eta_A^\dagger"] \arrow[drr, phantom, "\lambda\text{ nat.}"] & & & I\, A^* A\, A^* A \arrow[urr, "\eta_B\, id"'] \arrow[dd, "I\, f\, id"] \arrow[dr, phantom, "\otimes\text{ func.}"] & & \\
& & & & & & & B^* B\, B^* B\, A^* A \arrow[ddlll, "B^* \epsilon_B\, id"] \arrow[dddl, "id\, \eta_A^\dagger"] \arrow[dd, "B^* \epsilon_B\, id"] \\
& & & B^* B\, A^* A \arrow[rr, "\lambda^{-1} id"] \arrow[dddd, "id"] \arrow[dr, phantom, "(20)"] & & I\, B^* B\, A^* A \arrow[urr, "\eta_B\, id"] & & \\
& (19) & & & B^* I\, B\, A^* A \arrow[dddl, "\rho\, id"'] \arrow[dddrr, "id\, \eta_A^\dagger"] \arrow[dr, phantom, "\otimes\text{ func.}"] & & \otimes\text{ func.} & B^* I\, B\, A^* A \arrow[dddl, "id\, \eta_A^\dagger"'] \arrow[dd, "B^* \lambda\, A"] \\
& & & & & & B^* B\, B^* B\, I \arrow[dd, "B^* \epsilon_B\, id"'] & \\
& & & & \otimes\text{ func.} & & & B^* B\, A^* A \arrow[dd, "id\, \eta_A^\dagger"] \arrow[l, phantom, "\lambda\text{ nat.}"] \\
& & & B^* B\, A^* A \arrow[ddrrr, "id\, \eta_A^\dagger"] \arrow[d, phantom, "\otimes\text{ func.}"] & & & B^* I\, B\, I \arrow[dr, "B^* \lambda\, I"] \arrow[dd, "\rho\, id"] & \\
& & & \, & & & & B^* B\, I \arrow[dd, "\rho"] \arrow[l, phantom, "\text{coh.}"] \\
& & A^* A\, I \arrow[rrrr, "f\, I"] \arrow[dll, "A^* \rho"] \arrow[drr, phantom, "\rho\text{ nat.}"] & & & & B^* B\, I \arrow[ur, "id"'] & \\
A^* A \arrow[rrrrrrr, "f"'] & & & & & & & B^* B
\end{tikzcd}
$$

Hence (b) holds.

[b $\Longrightarrow$ c] Suppose (b) holds and let $j : B \otimes A^* \to D$ be such that the positive map g factors as $j^\dagger \circ j$. By assumption f is the composite

$$\begin{tikzcd}
A^* \otimes A \arrow[r, "\lambda^{-1}"] \arrow[ddddrr, "f"'] & I \otimes A^* \otimes A \arrow[r, "\eta_B \otimes A^* \otimes A"] & B^* \otimes B \otimes A^* \otimes A \arrow[d, "B^* \otimes j \otimes A"] \\
 & & B^* \otimes D \otimes A \arrow[d, "B^* \otimes j^\dagger \otimes A"] \\
 & & B^* \otimes B \otimes A^* \otimes A \arrow[d, "B^* \otimes B \otimes \eta_A^\dagger"] \\
 & & B^* \otimes B \otimes I \arrow[d, "\rho"] \\
 & & B^* \otimes B
\end{tikzcd}$$

We define $h : A \to D^* \otimes B$ to be the map

$$A \xrightarrow{\rho^{-1}} A \otimes I \xrightarrow{A \otimes \eta_B} A \otimes B^* \otimes B \xrightarrow{j_* \otimes B} D^* \otimes B$$

and we claim that (c) holds, i.e, the following diagram commutes.

$$\begin{tikzcd}
A^* \otimes A \arrow[r, "h_* \otimes h"] \arrow[drr, "f"'] & B^* \otimes D \otimes D^* \otimes B \arrow[r, "B^* \otimes \eta_{D^*}^\dagger \otimes B"] & B^* \otimes I \otimes B \arrow[d, "B^* \otimes \lambda"] \\
 & & B^* \otimes B
\end{tikzcd}$$

We show this in two steps. First,

$$
\begin{tikzcd}
A^* \otimes A \arrow[rrrr, "f"] \arrow[ddd, "h_* \otimes h"'] \arrow[ddr, "\rho^{-1}"] & & & & B^* \otimes B \arrow[dll, "\rho^{-1}"'] \arrow[ddl, "B^* \otimes \rho^{-1}"'] \arrow[ddddd, "id"] \\
& \rho \text{ nat.} & B^* \otimes B \otimes I \arrow[dr, "id"'] & \text{coh.} & \\
& A^* \otimes A \otimes I \arrow[ur, "f \otimes I"] \arrow[dd, "id \otimes \eta_B"] & & B^* \otimes B \otimes I \arrow[dd, "id \otimes \eta_B"] & \\
B^* \otimes D \otimes D^* \otimes B \arrow[dd, "B^* \otimes \eta^\dagger_{D^*} \otimes B"'] \arrow[dr, phantom, "(*)"] & & \otimes \text{ func.} & & \\
& A^* \otimes A \otimes B^* \otimes B \arrow[rr, "f \otimes id"] & & B^* \otimes B \otimes B^* \otimes B \arrow[dlll, "B^* \otimes \epsilon_B \otimes B"] \arrow[dr, phantom, "(19)"] & \\
B^* \otimes I \otimes B \arrow[rrrr, "B^* \otimes \lambda"'] & & & & B^* \otimes B
\end{tikzcd}
$$

It remains to establish the equation $(*)$. For this we replace f and h. This gives the following commutative diagram.

$$
\begin{tikzcd}[column sep=small]
A^* A \arrow[rr, "\rho^{-1}"] \arrow[dd, "\rho^* \rho^{-1}"'] \arrow[dr, "\lambda^{-1} \rho^{-1}"'] & & A^* A I \arrow[rr, "id \eta_B"] \arrow[d, "\lambda^{-1} id"] \arrow[dl, phantom, "\text{coh.}"] & & A^* A B^* B \arrow[rr, "\lambda^{-1} id"] \arrow[d, phantom, "\otimes \text{ func.}"] & & I A^* A B^* B \arrow[dd, "\eta_B id"] \\
& \arrow[d, phantom, "(16)"] & I A^* A I \arrow[urrrr, "id \eta_B"'] \arrow[dd, "\eta_B id"] & & \arrow[d, phantom, "(=)"] & & \\
I^* A^* A I \arrow[urr, "\cong"'] \arrow[drr, "\eta_B^{*\dagger} id"'] \arrow[ddd, "\eta_B^{*\dagger} id \eta_B"'] & (34) & & & {} & & B^* B A^* A B^* B \arrow[dd, "B^* j id"] \arrow[dddlll, "B^* j j_* B"'] \\
& \otimes \text{ func.} & B^* B A A^* I \arrow[ddll, "id \eta_B"] & & & & \\
& & & (=) & & & B^* D A B^* B \arrow[dd, "B^* j^\dagger id"] \\
B^* B A^* A B^* B \arrow[uuurrrrrr, "id"] \arrow[rrr, "B^* j j_* B"] \arrow[ddddddd, "B^* j j_* B"'] & (=) & & B^* D D^* B \arrow[dl, "B^* \epsilon_D B"] & \text{Rem. 4.1.9} & & \\
& & B^* I B \arrow[dddddr, "B^* \lambda^{-1} B"] & (32) & & & B^* B A^* A B^* B \arrow[llll, "B^* \epsilon_{B \otimes A^*} B"'] \arrow[dll, "id \sigma B"'] \arrow[dd, "id \eta_A^\dagger id"] \arrow[dd, bend right=60, "id \epsilon_{A^*} id"'] \\
& & & & B^* B B^* A^* A B \arrow[dd, "id \epsilon_{A^*} B"'] & (34) & \\
& (34) & & & \sigma \text{ nat.} & & B^* B I B^* B \arrow[dll, "id \sigma B"'] \arrow[dd, "\rho id"] \\
& & (=) & & B^* B B^* I B \arrow[drr, "B^* \lambda B"] \arrow[ddl, "B^* \epsilon_B B"'] & \text{coh.} & \\
& & & & & \lambda \text{ nat.} & B^* B B^* B \arrow[dd, "B^* \epsilon_B id"] \\
& & & B^* I I B \arrow[drrr, "B^* \lambda B"] & & & \\
B^* D D^* B \arrow[rrrrrr, "B^* \eta^\dagger_{D^*} B"'] \arrow[uuuuuurr, bend left=20, "B^* \epsilon_D B"] \arrow[uuuuuurr, bend right=20, "B^* \eta^\dagger_{D^*} B"'] & & & & & & B^* I B
\end{tikzcd}
$$

Hence (c) holds.

[c $\Longrightarrow$ d] Suppose (c) holds, i.e., there is an object C and a morphism $h : A \to C \otimes B$ such that

$$\begin{tikzcd}
A^* \otimes A \arrow[r, "h_* \otimes h"] \arrow[drr, "f"'] & B^* \otimes C^* \otimes C \otimes B \arrow[r, "B^* \otimes \eta_C^\dagger \otimes B"] & B^* \otimes I \otimes B \arrow[d, "B^* \otimes \lambda"] \\
& & B^* \otimes B
\end{tikzcd}$$

We use the object C^* and define $k : C^* \otimes A \to B$ to be the map

$$C^* \otimes A \xrightarrow{C^* \otimes h} C^* \otimes C \otimes B \xrightarrow{\eta_C^\dagger \otimes B} I \otimes B \xrightarrow{\lambda} B.$$

We must show that the following diagram commutes.

$$\begin{tikzcd}
B^*\,A \arrow[rr, "\rho^{-1}\,A"] \arrow[dd, "h_*\,h"'] \arrow[dr, phantom, "\rho\text{ nat.}"] & & A^*\,I\,A \arrow[rrr, "A^*\,\eta_{C^*}\,A"] \arrow[dr, "h_*\,I\,h"] \arrow[rr, phantom, "\otimes\text{ func.}"] & & & A^*\,C\,C^*\,A \arrow[dddd, "h_*\,C\,C^*\,h"] \\
& & & B^*\,C^*\,I\,C\,B \arrow[dddrr, "id\,\eta_{C^*}\,id"] & & \\
B^*\,C^*\,C\,B \arrow[urrr, "\rho^{-1}\,id"] \arrow[dddr, "B^*\,\rho^{-1}\,B"] \arrow[dddddd, "B^*\,\eta_C^\dagger\,B"'] & & & & & \\
& & B^*\,I\,C^*\,C\,B \arrow[ull, "B^*\,\lambda\,id"'] \arrow[ddrr, "\cong"] \arrow[ur, phantom, "(19)"] \arrow[dl, phantom, "\text{coh.}"] & & & \\
& & & & & B^*\,C^*\,C\,C^*\,C\,B \arrow[ulll, "B^*\,\epsilon_{C^*}\,id"'] \arrow[dl, "B^*\,\eta_C^*\,id"'] \arrow[dd, "B^*\,\eta_C^*\,\eta_C^\dagger\,B"] \arrow[ll, phantom, "(34)"] \arrow[dl, phantom, "\otimes\text{ func.}", shift left=6] \\
& B^*\,C^*\,C\,I\,B \arrow[uur, "B^*\,\sigma\,B"] \arrow[dd, "B^*\otimes\eta_C^\dagger\,id"] \arrow[rrr, phantom, "\sigma\text{ nat.}", shift right=6] \arrow[dl, phantom, "\otimes\text{ func.}"] & & & B^*\,I^*\,C^*\,C\,B \arrow[dr, "id\,\eta_C^\dagger\,B"] & \\
& & & & & B^*\,I^*\,I\,B \arrow[dd, "\lambda_*\,\lambda"] \\
& B^*\,I\,I\,B \arrow[rrr, "\cong"] \arrow[drrrr, "\rho\,\lambda"] \arrow[dr, phantom, "\text{coh.}"] & & & B^*\,I\,I^*\,B \arrow[ur, "B^*\,\sigma\,B"] \arrow[dr, phantom, "(16)"] & \\
B^*\,I\,B \arrow[ur, "B^*\,\rho^{-1}\,B"] \arrow[rrrrr, "B^*\,\lambda"'] & & & & & B^*\,B
\end{tikzcd}$$

Hence (d) holds.

[d $\Longrightarrow$ a] Suppose (d) holds, i.e., there is an object C and a morphism $k : C \otimes A \to B$ such that

$$\begin{tikzcd}
A^* \otimes A \arrow[r, "\rho^{-1} \otimes A"] \arrow[drr, "f"'] & A^* \otimes I \otimes A \arrow[r, "A^* \otimes \eta_C \otimes A"] & A^* \otimes C^* \otimes C \otimes A \arrow[d, "k_* \otimes k"] \\
 & & B^* \otimes B
\end{tikzcd}$$

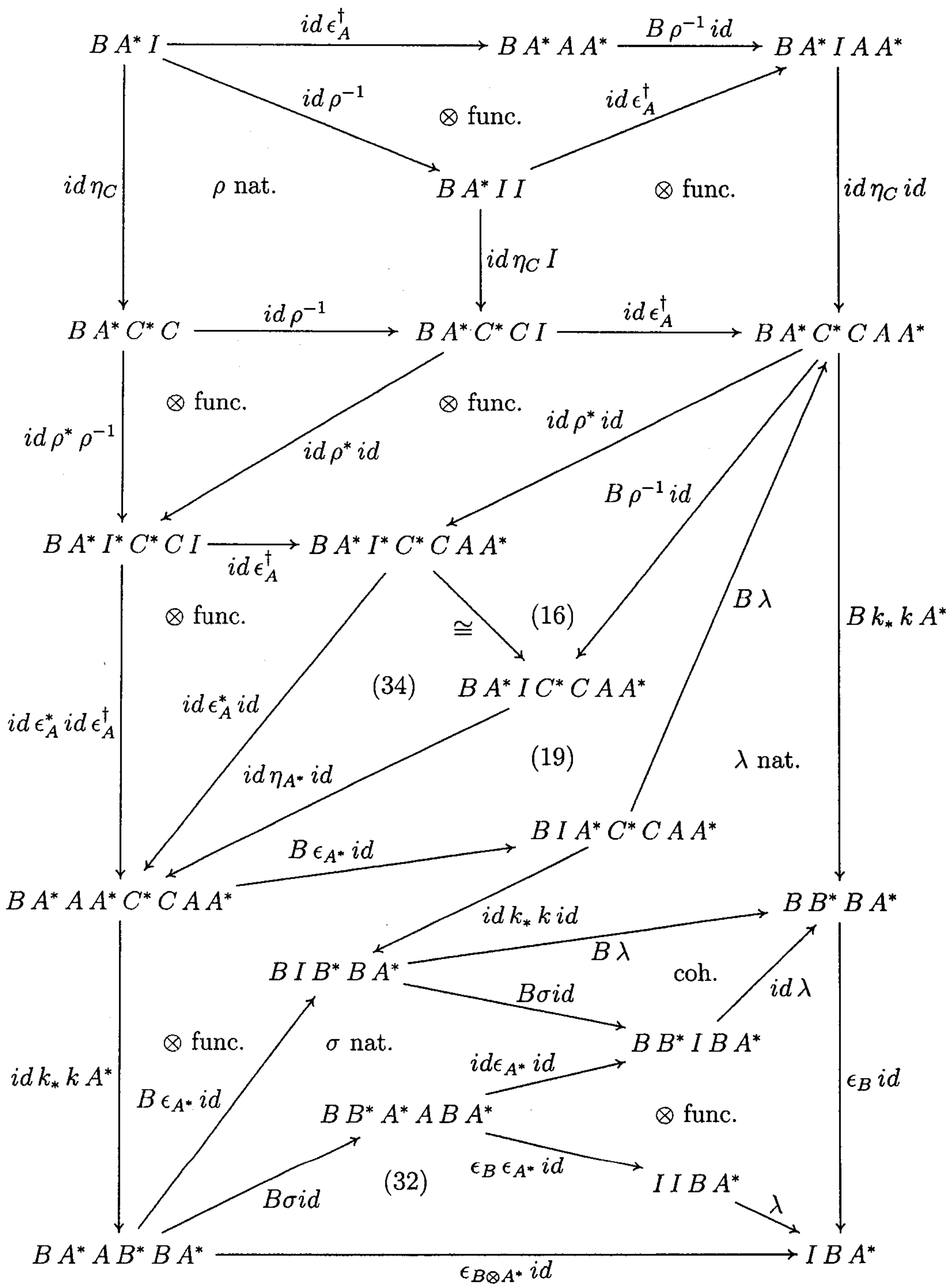
We must show that f is completely positive. We define p to be the map

$$B \otimes A^* \xrightarrow{k^\dagger \otimes A^*} C \otimes A \otimes A^* \xrightarrow{C \otimes \epsilon_A} C \otimes I \xrightarrow{\rho} C$$

and we claim that the composite (38) is $p^\dagger \circ p$. This is shown by the following diagram.

$$\begin{tikzcd}
B\,A^* \arrow[rr, "\rho^{-1}"] \arrow[dddd, "id"'] & & B\,A^*\,I \arrow[rr, "id\,\epsilon_A^\dagger"] \arrow[d, "id\,\eta_{B\otimes A^*}"'] \arrow[dr, "id\,\eta_C"] & & B\,A^*\,A\,A^* \arrow[dd, "B\,f\,A^*"] \\
& & B\,A^*\,A\,B^*\,B\,A^* \arrow[l, phantom, "(19)"] \arrow[dl, "\epsilon_{B\otimes A^*}\,id"'] \arrow[ddd, "id\,p"] \arrow[dr, phantom, "\text{Rem. 4.1.9}"] & B\,A^*\,C^*\,C \arrow[ur, phantom, "(*)"] \arrow[dd, "id\,p^*\,p^\dagger"] & \\
& I\,B\,A^* \arrow[ddl, "\lambda"] \arrow[ddd, "I\,p"] \arrow[dr, phantom, "\otimes\text{ func.}"] & & & B\,B^*\,B\,A^* \arrow[ddd, "\epsilon_B\,id"] \\
& & & B\,A^*\,A\,B^*\,B\,A^* \arrow[ddr, "\epsilon_{B\otimes A^*}\,id"'] & \\
B\,A^* \arrow[dd, "p"'] \arrow[r, phantom, "\lambda\text{ nat.}"] & & B\,A^*\,A\,B^*\,C \arrow[uur, "id\,p^\dagger"] \arrow[dl, "\epsilon_{B\otimes A^*}\,id"] \arrow[r, phantom, "\otimes\text{ func.}"] & & \\
& I\,C \arrow[rrr, "I\,p^\dagger"] \arrow[dl, "\lambda"] \arrow[drr, phantom, "\lambda\text{ nat.}"] & & & I\,B\,A^* \arrow[d, "\lambda"] \\
C \arrow[rrrr, "p^\dagger"'] & & & & B\,A^*
\end{tikzcd}$$

It remains to establish the equation $(*)$. To do this we replace p and f. We have the following commutative diagram.



$B\,A^*\,I$
$id\,\epsilon_A^\dagger$
$B\,A^*\,A\,A^*$
$B\,\rho^{-1}\,id$
$B\,A^*\,I\,A\,A^*$
$id\,\rho^{-1}$
$\otimes$ func.
$id\,\epsilon_A^\dagger$
$id\,\eta_C$
ρ nat.
$B\,A^*\,I\,I$
$\otimes$ func.
$id\,\eta_C\,id$
$id\,\eta_C\,I$
$B\,A^*\,C^*\,C$
$id\,\rho^{-1}$
$B\,A^*\,C^*\,C\,I$
$id\,\epsilon_A^\dagger$
$B\,A^*\,C^*\,C\,A\,A^*$
$\otimes$ func.
$\otimes$ func.
$id\,\rho^*\,id$
$id\,\rho^*\,\rho^{-1}$
$id\,\rho^*\,id$
$B\,\rho^{-1}\,id$
$B\,A^*\,I^*\,C^*\,C\,I$
$B\,A^*\,I^*\,C^*\,C\,A\,A^*$
$id\,\epsilon_A^\dagger$
$\otimes$ func.
$\cong$
(16)
$B\,\lambda$
$B\,k_*\,k\,A^*$
$id\,\epsilon_A^*\,id$
(34)
$B\,A^*\,I\,C^*\,C\,A\,A^*$
$id\,\epsilon_A^*\,id\,\epsilon_A^\dagger$
$id\,\eta_{A^*}\,id$
(19)
λ nat.
$B\,I\,A^*\,C^*\,C\,A\,A^*$
$B\,\epsilon_{A^*}\,id$
$B\,A^*\,A\,A^*\,C^*\,C\,A\,A^*$
$id\,k_*\,k\,id$
$B\,B^*\,B\,A^*$
$B\,I\,B^*\,B\,A^*$
$B\,\lambda$
coh.
$B\sigma id$
$id\,\lambda$
$\otimes$ func.
σ nat.
$B\,B^*\,I\,B\,A^*$
$id\epsilon_{A^*}\,id$
$id\,k_*\,k\,A^*$
$B\,\epsilon_{A^*}\,id$
$\epsilon_B\,id$
$B\,B^*\,A^*\,A\,B\,A^*$
$\otimes$ func.
$\epsilon_B\,\epsilon_{A^*}\,id$
(32)
$I\,I\,B\,A^*$
$B\sigma id$
λ
$B\,A^*\,A\,B^*\,B\,A^*$
$\epsilon_{B\otimes A^*}\,id$
$I\,B\,A^*$

Therefore f is completely positive. □

We should mention that Selinger defines a graphical language for dagger compact closed categories in [26]. In this language, proofs such as the one above are greatly simplified.

Example 4.2.4.
CPMs in $\mathbf{Hilb}_{fd}$ are precisely those operators discussed at the start of the section.

Example 4.2.5.
Let $R : A \times A \to B \times B$ be a completely positive map in **Rel**. Then by Proposition 4.2.3 (d), there is a set C and a relation $K : C \times A \to B$ such that $R = (K_* \times K) \circ (id \times \eta_C \times id) \circ (\rho^{-1} \times id)$. Unravelling this, we find that

$$((a_1, a_2), (b_1, b_2)) \in R \iff \exists\, c \in C \text{ with } \begin{cases} ((c, a_1), b_1) \in K \text{ and} \\ ((c, a_2), b_2) \in K. \end{cases}$$

It follows from this that R satisfies the following properties:

(1) $((a_1, a_2), (b_1, b_2)) \in R \Longrightarrow ((a_2, a_1), (b_2, b_1)) \in R$, i.e., $R = \sigma R \sigma$;

(2) $((a_1, a_2), (b_1, b_2)) \in R \Longrightarrow ((a_1, a_1), (b_1, b_1)) \in R$.

Conversely, let $R : A \times A \to B \times B$ and suppose that R satisfies conditions (1) and (2). We claim that R is completely positive. Viewing R as a set, we take $D = R$ and define a map $J : D \times A \to B$ as follows:

$$\begin{aligned} J = \{\, ((((a_1, a_2), (b_1, b_2)), a_1), b_1) \mid ((a_1, a_2), (b_1, b_2)) \in R \,\} \\ \cup \{\, ((((a_1, a_2), (b_1, b_2)), a_2), b_2) \mid ((a_1, a_2), (b_1, b_2)) \in R \,\}. \end{aligned}$$

We show that $R = (J_* \times J) \circ (id \times \eta_D \times id) \circ (\rho^{-1} \times id)$, i.e., that

$$((a_1, a_2), (b_1, b_2)) \in R \iff \exists\, d \in D \text{ with } \begin{cases} ((d, a_1), b_1) \in J \text{ and} \\ ((d, a_2), b_2) \in J. \end{cases}$$

If $((a_1, a_2), (b_1, b_2)) \in R$, then $((a_1, a_2), (b_1, b_2)) = d \in D$ satisfies $((d, a_1), b_1) \in J$ and $((d, a_2), b_2) \in J$. On the other hand, suppose $\exists\, d \in D$ with $((d, a_1), b_1) \in J$ and

$((d, a_2), b_2) \in J$. We have $d = ((a_3, a_4), (b_3, b_4)) \in R$ with

$$\begin{cases} ((((a_3, a_4), (b_3, b_4)), a_1), b_1) \in J \text{ and} \\ ((((a_3, a_4), (b_3, b_4)), a_2), b_2) \in J. \end{cases}$$

There are four possibilities:

$$\begin{cases} a_3 = a_1, b_3 = b_1, a_3 = a_2, b_3 = b_2 \text{ or} \\ a_3 = a_1, b_3 = b_1, a_4 = a_2, b_4 = b_2 \text{ or} \\ a_4 = a_1, b_4 = b_1, a_3 = a_2, b_3 = b_2 \text{ or} \\ a_4 = a_1, b_4 = b_1, a_4 = a_2, b_4 = b_2. \end{cases}$$

In the first case we have:

$$((a_3, a_4), (b_3, b_4)) \in R \stackrel{(2)}{\Longrightarrow} ((a_3, a_3), (b_3, b_3)) = ((a_1, a_2), (b_1, b_2)) \in R.$$

In the second case:

$$((a_3, a_4), (b_3, b_4)) = ((a_1, a_2), (b_1, b_2)) \in R.$$

In the third case we have:

$$((a_3, a_4), (b_3, b_4)) = ((a_2, a_1), (b_2, b_1)) \in R \stackrel{(1)}{\Longrightarrow} ((a_1, a_2), (b_1, b_2)) \in R.$$

In the last case we have:

$$\begin{aligned} ((a_3, a_4), (b_3, b_4)) \in R &\stackrel{(1)}{\Longrightarrow} ((a_4, a_3), (b_4, b_3)) \in R \\ &\stackrel{(2)}{\Longrightarrow} ((a_4, a_4), (b_4, b_4)) = ((a_1, a_2), (b_1, b_2)) \in R. \end{aligned}$$

Thus completely positive maps in **Rel** are precisely those maps satisfying conditions (1) and (2).

Note that there are operators which are positive in the sense that they map positive matrices to positive matrices, yet which fail to be completely positive. Consider the following example in **Rel**.

Example 4.2.6.
Let $A = \{ a_1, a_2 \}$ and $B = \{ b_1, b_2 \}$. Then there are exactly four positive maps on A: $P_1 = \emptyset$, $P_2 = \{ (a_1, a_1) \}$, $P_3 = \{ (a_2, a_2) \}$, and $P_4 = A \times A$. We define $R : A \times A \to B \times B$ by

$$\begin{aligned} R = \{ & ((a_1, a_1), (b_1, b_2)), \\ & ((a_1, a_1), (b_2, b_1)), \\ & ((a_1, a_1), (b_1, b_1)), \\ & ((a_1, a_1), (b_2, b_2)), \\ & ((a_1, a_2), (b_2, b_2)) \}. \end{aligned}$$

Since $R \neq \sigma R \sigma$, R is not completely positive. However, R maps the positive subsets P_i of $A \times A$ to positive subsets of $B \times B$:

$$\begin{aligned} R|_{P_1} &= \emptyset; \\ R|_{P_2} &= B \times B; \\ R|_{P_3} &= \emptyset; \\ R|_{P_2} &= B \times B. \end{aligned}$$

Proposition 4.2.7. *If $f : A^* \otimes A \to B^* \otimes B$ is completely positive, then $f = f_*$.*

Proof. We use Proposition 4.2.3 (d) to write f as the composite

$$\begin{tikzcd} A^* \otimes A \arrow[r, "\rho^{-1} \otimes A"] & A^* \otimes I \otimes A \arrow[r, "A^* \otimes \eta_C \otimes A"] & A^* \otimes C^* \otimes C \otimes A \arrow[d, "k_* \otimes k"] \\ & & B^* \otimes B. \end{tikzcd}$$

Then the following diagram shows that $f = f_*$.

$$
\begin{tikzcd}
A^* \otimes A \arrow[rrr, "\rho^{-1} \otimes A"] \arrow[dr, "A^* \otimes \lambda^{-1}"] \arrow[ddd, "A^* \otimes \rho^*"'] \arrow[drr, phantom, "\text{coh.}"] & & & A^* \otimes I \otimes A \arrow[dd, "A^* \otimes \eta_C \otimes A"] \\
& A^* \otimes I \otimes A \arrow[urr, "id"] \arrow[ddl, "\cong"] \arrow[dd, phantom, "(=)"] & & \\
& & (34) & A^* \otimes C^* \otimes C \otimes A \arrow[d, "k_* \otimes k"] \arrow[dl, phantom, "(=)"] \\
A^* \otimes I^* \otimes A \arrow[uuurrr, "\cong"'] \arrow[r, "A^* \otimes \eta_C^{*\dagger} \otimes A"'] \arrow[uu, phantom, "(16)"] & A^* \otimes C^* \otimes C \otimes A \arrow[urr, "id"] \arrow[rr, "k^{*\dagger} \otimes k"'] & & B^* \otimes B
\end{tikzcd}
$$

□

The converse is not true. For example, in **Rel** we have $\sigma^{*\dagger}_{A,A} = \sigma_{A,A}$ for all A. However, the composite (38) reduces to $\sigma_{A,A}$ which we know is not positive. So $\sigma_{A,A}$ is not completely positive.

Proposition 4.2.8. *In a dagger compact closed category, a positive matrix $g : I \to A^* \otimes A$ is the same thing as a completely positive map $f : I^* \otimes I \to A^* \otimes A$, modulo the isomorphism $I^* \otimes I \cong I$.*

Proof. Let $g : I \to A^* \otimes A$ be a positive matrix. Then $g = (A^* \otimes h) \circ \eta_A$, for some positive map $h : A \to A$. We show that the map

$$(I^* \otimes I) \xrightarrow{\cong} I \xrightarrow{g} A^* \otimes A$$

is completely positive using Proposition 4.2.3 (b). We have a map $h \otimes I^* : A \otimes I^* \to A \otimes I^*$, which is positive by Lemma 4.1.6 (b) and (c). Moreover the following diagram commutes.

$$
\begin{tikzcd}
I^* \otimes I \arrow[r, "\lambda^{-1}"] \arrow[dd, "\cong"'] \arrow[dr, phantom, "\lambda\text{ nat.}"] & I \otimes I^* \otimes I \arrow[rr, "\eta_A \otimes id"] \arrow[d, "\cong"] \arrow[r, phantom, "\otimes\text{ func.}"] & & A^* \otimes A \otimes I^* \otimes I \arrow[rr, "A^* \otimes h \otimes id"] \arrow[dl, "\cong"'] \arrow[dddr, "id \otimes \eta_I^\dagger"] \arrow[d, phantom, "(33)"] \arrow[drr, phantom, "\otimes\text{ func.}"] & & A^* \otimes A \otimes I^* \otimes I \arrow[dd, "id \otimes \eta_I^\dagger"] \\
& I \otimes I \otimes I \arrow[r, "\eta_A \otimes id"] \arrow[dd, "I \otimes \lambda"] & A^* \otimes A \otimes I \otimes I \arrow[ddrr, "id \otimes \lambda"'] \arrow[d, phantom, "\lambda\text{ nat.}"] & & & \\
I \otimes I \arrow[ur, "\lambda^{-1}"] \arrow[ddd, "\lambda"'] \arrow[r, phantom, "\text{coh.}"] & & {} & & & A^* \otimes A \otimes I \arrow[ddd, "\rho"] \\
& I \otimes I \arrow[rrr, "\eta_A \otimes I"] \arrow[ddl, "\rho"] \arrow[dr, phantom, "\rho\text{ nat.}"] & & & A^* \otimes A \otimes I \arrow[ur, "A^* \otimes h \otimes I"] \arrow[dl, "\rho"] \arrow[dr, phantom, "\rho\text{ nat.}"] & \\
& & & A^* \otimes A \arrow[drr, "A^* \otimes h"] \arrow[d, phantom, "(=)"] & & \\
I \arrow[urrr, "\eta_A"] \arrow[rrrrr, "g"'] & & & {} & & A^* \otimes A
\end{tikzcd}
$$

Conversely, let $f : I^* \otimes I \to A^* \otimes A$ be completely positive. Then by Proposition

4.2.3 (b) there exists a positive map $k : A \otimes I^* \to A \otimes I^*$ such that

$$\begin{tikzcd}
I^* \otimes I \arrow[r, "\lambda^{-1}"] \arrow[ddddrr, "f"'] & I \otimes I^* \otimes I \arrow[r, "\eta_A \otimes I^* \otimes I"] & A^* \otimes A \otimes I^* \otimes I \arrow[d, "A^* \otimes k \otimes I"] \\
& & A^* \otimes A \otimes I^* \otimes I \arrow[d, "A^* \otimes A \otimes \eta_I^\dagger"] \\
& & A^* \otimes A \otimes I \arrow[d, "\rho"] \\
& & A^* \otimes A
\end{tikzcd}$$

We claim that the map

$$I \xrightarrow{\cong} (I^* \otimes I) \xrightarrow{f} A^* \otimes A$$

is a positive matrix. So we need a positive map $j : A \to A$ such that $(A^* \otimes j) \circ \eta_A$ gives the above map. We define j as follows:

$$\begin{tikzcd}
A \arrow[r, "\rho^{-1}"] \arrow[ddrrrr, "j"'] & A \otimes I \arrow[r, "A \otimes \eta_I"] & A \otimes I^* \otimes I \arrow[r, "k \otimes I"] & A \otimes I^* \otimes I \arrow[d, "A \otimes \eta_I^\dagger"] \\
& & & A \otimes I \arrow[d, "\rho"] \\
& & & A
\end{tikzcd}$$

The map j is positive by Lemma 4.1.6. Moreover, the following diagram commutes.

$$
\begin{tikzcd}
I \arrow[rr, "\lambda^{-1}=\rho^{-1}"] \arrow[rrr, bend right, "\eta_I"'] \arrow[dddr, "\lambda^{-1}=\rho^{-1}"] \arrow[dddddd, "\eta_A"'] & & I\otimes I \arrow[r, "\cong"] \arrow[d, phantom, "(33)"] & I^*\otimes I \arrow[rrr, "\lambda^{-1}"] & & & I\otimes I^*\otimes I \arrow[dd, "\eta_A\otimes id"] \\
 & & \arrow[d, phantom, "\lambda\text{ nat.}"] & & & & \\
 & & & & & & A^*\otimes A\otimes I^*\otimes I \arrow[d, "A^*\otimes k\otimes I"] \\
\arrow[d, phantom, "\rho\text{ nat.}"] & I\otimes I \arrow[uuurrrrr, "I\otimes\eta_I"] \arrow[dr, "\eta_A\otimes I"] & & \arrow[r, phantom, "\otimes\text{ func.}"] & & & A^*\otimes A\otimes I^*\otimes I \arrow[dd, "id\otimes\eta_I^\dagger"] \\
 & & A^*\otimes A\otimes I \arrow[r, "id"] & A^*\otimes A\otimes I \arrow[uurrr, "id\otimes\eta_I"] & & & \\
 & & \arrow[r, phantom, "\text{coh.}"] & & \arrow[r, phantom, "(=)"] & & A^*\otimes A\otimes I \arrow[d, "id\otimes\rho"] \\
A^*\otimes A \arrow[uurr, "\rho^{-1}"] \arrow[uurrr, "A^*\otimes\rho^{-1}"'] \arrow[rrrrrr, "A^*\otimes j"'] & & & & & & A^*\otimes A
\end{tikzcd}
$$

□

In a dagger compact closed category, CPMs enjoy the following categorical properties.

Lemma 4.2.9.

(a) The identity map on $A^*\otimes A$ is completely positive.

(b) If $f : A^*\otimes A \to B^*\otimes B$ and $g : B^*\otimes B \to C^*\otimes C$ are completely positive, then so is $g\circ f : A^*\otimes A \to C^*\otimes C$.

(c) If $f : A^*\otimes A \to B^*\otimes B$ and $g : C^*\otimes C \to D^*\otimes D$ are completely positive, then

so is
$C^* \otimes A^* \otimes A \otimes C \xrightarrow{\cong} A^* \otimes A \otimes C^* \otimes C \xrightarrow{f \otimes g} B^* \otimes B \otimes D^* \otimes D \xrightarrow{\cong} D^* \otimes B^* \otimes B \otimes D$.
(d) If $f : A \to B$ is any morphism, then $f_* \otimes f : A^* \otimes A \to B^* \otimes B$ is completely positive.

Proof.
(a) For $f = id_{A^* \otimes A}$, the composite (38) reduces to the positive map $\epsilon_A^\dagger \circ \epsilon_A$:

$$\begin{tikzcd}
A \otimes A^* \arrow[r, "\rho^{-1}"] \arrow[dd, "\epsilon_A"'] \arrow[dr, phantom, "\rho \text{ nat.}"] & A \otimes A^* \otimes I \arrow[r, "A \otimes A^* \otimes \epsilon_A^\dagger"] \arrow[d, "\epsilon_A \otimes I"] \arrow[dr, phantom, "\otimes \text{ func.}"] & A \otimes A^* \otimes A \otimes A^* \arrow[d, "\epsilon_A \otimes A \otimes A^*"] \\
& I \otimes I \arrow[r, "I \otimes \epsilon_A^\dagger"'] \arrow[dr, phantom, "\lambda \text{ nat.}"] & I \otimes A \otimes A^* \arrow[d, "\lambda"] \\
I \arrow[ur, "\rho^{-1} = \lambda^{-1}"] \arrow[rr, "\epsilon_A^\dagger"'] & & A^* \otimes A
\end{tikzcd}$$

(b) Let $f : A^* \otimes A \to B^* \otimes B$ and $g : B^* \otimes B \to C^* \otimes C$ be completely positive, and denote the positive composites from (38) as j and k:

$$\begin{tikzcd}
B \otimes A^* \arrow[r, "\rho^{-1}"] \arrow[ddddrr, "j"'] & B \otimes A^* \otimes I \arrow[r, "B \otimes A^* \otimes \epsilon_A^\dagger"] & B \otimes A^* \otimes A \otimes A^* \arrow[d, "B \otimes f \otimes A^*"] \\
& & B \otimes B^* \otimes B \otimes A^* \arrow[d, "\epsilon_B \otimes B \otimes A^*"] \\
& & I \otimes B \otimes A^* \arrow[d, "\lambda"] \\
& & \\
& & B \otimes A^*
\end{tikzcd}$$

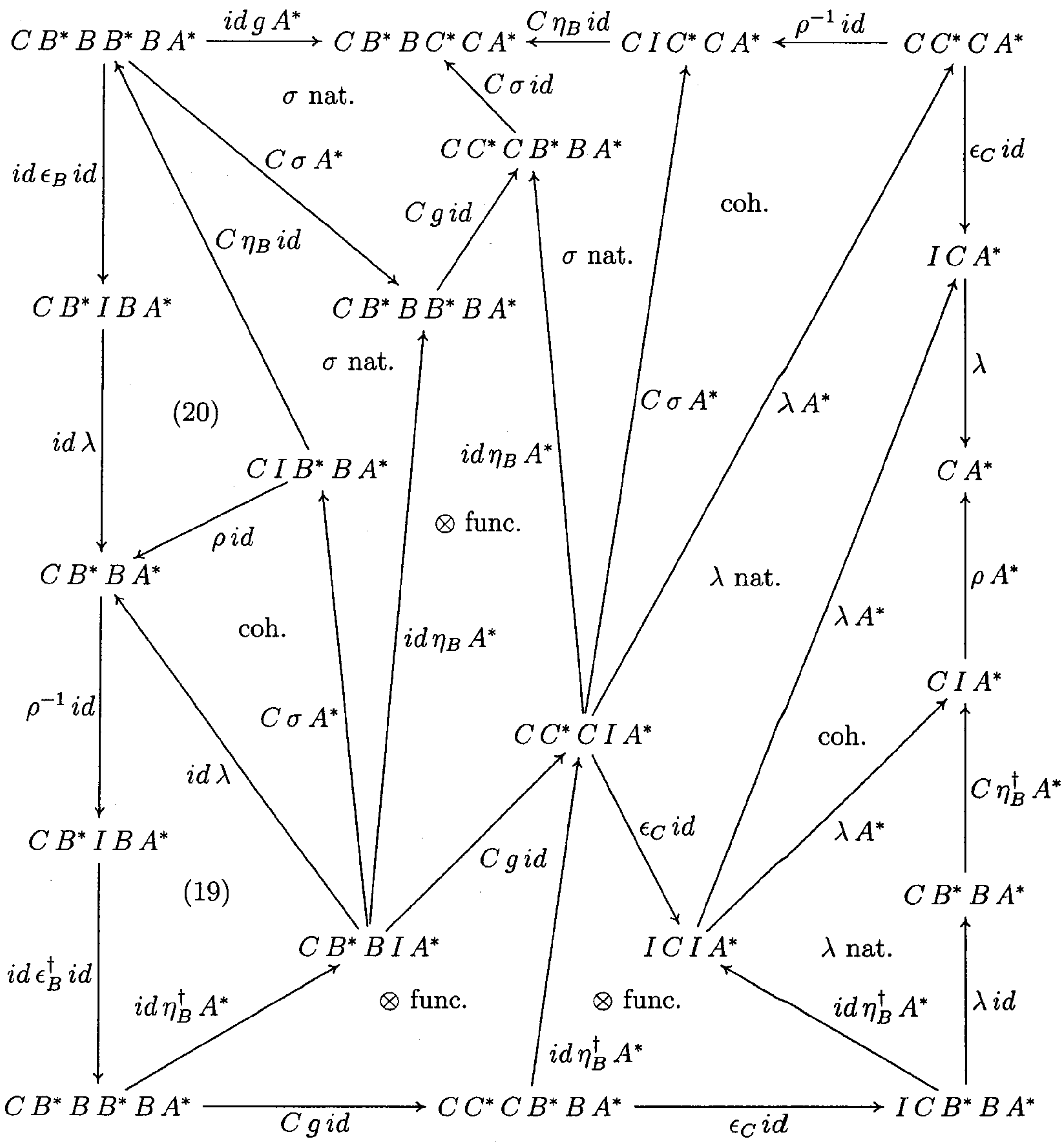
$$\begin{tikzcd}
C \otimes B^* \arrow[r, "\rho^{-1}"] \arrow[rrddd, "k"'] & C \otimes B^* \otimes I \arrow[r, "C \otimes B^* \otimes \epsilon_B^\dagger"] & C \otimes B^* \otimes B \otimes B^* \arrow[d, "C \otimes g \otimes B^*"] \\
& & C \otimes C^* \otimes C \otimes B^* \arrow[d, "\epsilon_C \otimes C \otimes B^*"] \\
& & I \otimes C \otimes B^* \arrow[d, "\lambda"] \\
& & C \otimes B^*
\end{tikzcd}$$

We show that (38) applied to gf gives the map

$$\begin{tikzcd}
C \otimes A^* \arrow[r, "\rho^{-1} \otimes A^*"] & C \otimes I \otimes A^* \arrow[r, "C \otimes \eta_B \otimes A^*"] & C \otimes B^* \otimes B \otimes A^* \arrow[d, "j \otimes k"] \\
& & C \otimes B^* \otimes B\, A^* \arrow[d, "C \otimes \eta_B^\dagger \otimes A^*"] \\
& & C \otimes I \otimes A^* \arrow[d, "\rho \otimes A^*"] \\
& & C \otimes A^*
\end{tikzcd}$$

which is positive by Lemma 4.1.6, parts (a) and (c). The diagram we need is too large to present here so we break it up into smaller pieces.

$$
\begin{tikzcd}
C\,A^* \arrow[rr, "\rho^{-1}"] \arrow[d, "\rho^{-1}\,A^*"'] \arrow[dr, phantom, "\text{coh.}"] & & C\,A^*\,I \arrow[rr, "id\,\epsilon_A^\dagger"] \arrow[dl, "\rho^{-1}\,id"'] \arrow[dr, phantom, "\rho\text{ nat.}"] & & C\,A^*\,A\,A^* \arrow[dl, "\rho^{-1}\,id"'] \arrow[ddd, "C\,f\,A^*"] \\
C\,I\,A^* \arrow[r, "id\,\rho^{-1}"] \arrow[d, "C\,\eta_B\,A^*"'] \arrow[dr, phantom, "\rho\text{ nat.}"] & C\,I\,A^*\,I \arrow[rr, "id\,\epsilon_A^\dagger"] \arrow[dddl, "C\,\eta_B\,id"] \arrow[dr, phantom, "\otimes\text{ func.}"] & & C\,I\,A^*\,A\,A^* \arrow[ddddlll, "C\,\eta_B\,id"] \arrow[dddd, "id\,f\,A^*"] \arrow[dr, phantom, "\rho\text{ nat.}"] & \\
C\,B^*\,B\,A^* \arrow[dd, "id\,\rho^{-1}"'] & & & & \\
 & & & & C\,B^*\,B\,A^* \arrow[ddl, "\rho^{-1}\,id"'] \arrow[ddd, "C\,g\,A^*"] \\
C\,B^*\,B\,A^*\,I \arrow[d, "id\,\epsilon_A^\dagger"'] & \arrow[dr, phantom, "\otimes\text{ func.}"] & & & \\
C\,B^*\,B\,A^*\,A\,A^* \arrow[d, "id\,f\,A^*"'] & & \arrow[d, phantom, "\otimes\text{ func.}"] & C\,I\,B^*\,B\,A^* \arrow[dlll, "C\,\eta_B\,id"'] \arrow[d, "id\,g\,A^*"] \arrow[dr, phantom, "\rho\text{ nat.}"] & \\
C\,B^*\,B\,B^*\,B\,A^* \arrow[r, "id\,g\,A^*"'] & C\,B^*\,B\,C^*\,C\,A^* & & C\,I\,C^*\,C\,A^* \arrow[ll, "C\,\eta_B\,id"] & C\,C^*\,C\,A^* \arrow[l, "\rho^{-1}\,id"]
\end{tikzcd}
$$



Placing the first diagram on top of the second gives the desired equation.

(c) Let $f : A^* \otimes A \to B^* \otimes B$ and $g : C^* \otimes C \to D^* \otimes D$ be completely positive, and denote the positive composites from (38) as follows:

$$\begin{tikzcd}
B\otimes A^* \arrow[r, "\rho^{-1}"] \arrow[ddddrr, "j"'] & B\otimes A^*\otimes I \arrow[r, "B\otimes A^*\otimes \epsilon_A^\dagger"] & B\otimes A^*\otimes A\otimes A^* \arrow[d, "B\otimes f\otimes A^*"] \\
& & B\otimes B^*\otimes B\otimes A^* \arrow[d, "\epsilon_B\otimes B\otimes A^*"] \\
& & I\otimes B\otimes A^* \arrow[d, "\lambda"] \\
& & B\otimes A^*
\end{tikzcd}$$

$$\begin{tikzcd}
D\otimes C^* \arrow[r, "\rho^{-1}"] \arrow[ddddrr, "k"'] & D\otimes C^*\otimes I \arrow[r, "D\otimes C^*\otimes \epsilon_C^\dagger"] & D\otimes C^*\otimes C\otimes C^* \arrow[d, "D\otimes g\otimes C^*"] \\
& & D\otimes D^*\otimes D\otimes C^* \arrow[d, "\epsilon_D\otimes D\otimes C^*"] \\
& & I\otimes D\otimes C^* \arrow[d, "\lambda"] \\
& & D\otimes C^*
\end{tikzcd}$$

We show that (38) applied to

$$C^*\otimes A^*\otimes A\otimes C \xrightarrow{\cong} A^*\otimes A\otimes C^*\otimes C \xrightarrow{f\otimes g} B^*\otimes B\otimes D^*\otimes D \xrightarrow{\cong} D^*\otimes B^*\otimes B\otimes D$$

gives the map

$$\begin{tikzcd}
B\otimes D\otimes C^*\otimes A^* \arrow[r, "B\otimes\sigma"] & B\otimes A^*\otimes D\otimes C^* \arrow[d, "j\otimes k"] \\
& B\otimes A^*\otimes D\otimes C^* \arrow[d, "B\otimes\sigma^{-1}"] \\
& B\otimes D\otimes C^*\otimes A^*
\end{tikzcd}$$

which is positive by Lemma 4.1.6, parts (a) and (c). This shown as follows.

$$
\begin{tikzcd}[column sep=small]
B\,D\,C^*\,A^* \arrow[rr, "\rho^{-1}"] \arrow[d, "B\,\sigma"'] \arrow[dr, phantom, "\text{coh.}"] & & B\,D\,C^*\,A^*\,I \arrow[rr, "id\,\epsilon^\dagger_{A\otimes C}"] \arrow[dl, "id\,\lambda^{-1}"'] \arrow[d, phantom, "(32)"] & & B\,D\,C^*\,A^*\,A\,C\,C^*\,A^* \arrow[dd, "id\,\sigma"] \\
 & B\,D\,C^*\,A^*\,I\,I \arrow[rr, "id\,\epsilon^\dagger_A\,\epsilon^\dagger_C"] \arrow[dd, "id\,\sigma\,I"] \arrow[ddr, phantom, "\sigma\text{ nat.}"] & & B\,D\,C^*\,A^*\,A\,A^*\,C\,C^* \arrow[ur, "id\,\sigma"] \arrow[dd, "id\,\sigma\,id"'] \arrow[dr, phantom, "\text{coh.}"] & \\
B\,A^*\,D\,C^* \arrow[dd, "\rho^{-1}\,\rho^{-1}"'] & & & & B\,D\,A^*\,A\,C^*\,C\,C^*\,A^* \arrow[dd, "id\,f\,g\,id"] \\
 & B\,D\,A^*\,I\,C^*\,I \arrow[rr, "id\,\epsilon^\dagger_A\,C^*\,\epsilon^\dagger_C"'] \arrow[d, phantom, "\sigma\text{ nat.}"] & & B\,D\,A^*\,A\,A^*\,C^*\,C\,C^* \arrow[ur, "id\,\sigma"] \arrow[ddl, "id\,f\,A^*\,g\,C^*"] \arrow[d, phantom, "\sigma\text{ nat.}"] & \\
B\,A^*\,I\,D\,C^*\,I \arrow[ur, "B\,\sigma\,I"] \arrow[dd, "id\,\epsilon A^\dagger\,id\,\epsilon^\dagger_C"'] & {} & & {} & B\,D\,B^*\,B\,D^*\,D\,C^*\,A^* \arrow[dd, "id\,\sigma\,id"] \\
 & & B\,D\,B^*\,B\,A^*\,D^*\,D\,C^* \arrow[urr, "id\,\sigma"'] \arrow[dl, phantom, "\sigma\text{ nat.}"] \arrow[d, phantom, "\text{coh.}"] & & \\
B\,A^*\,A\,A^*\,D\,C^*\,C\,C^* \arrow[uurrr, "B\,\sigma\,id"] \arrow[ddd, "B\,f\,id\,g\,C^*"'] & {} & {} & & B\,D\,D^*\,B^*\,B\,D\,C^*\,A^* \arrow[ddd, "\epsilon_{B\otimes D}\,id"] \\
 & & & B\,D\,D^*\,B^*\,B\,A^*\,D\,C^* \arrow[ur, "id\,\sigma"'] \arrow[ddd, "\epsilon_{B\otimes D}\,id"] \arrow[r, phantom, "\otimes\text{ func.}"] & {} \\
 & B\,B^*\,D\,D^*\,B\,A^*\,D\,C^* \arrow[urr, "B\,\sigma\,id"] \arrow[ddd, "\epsilon_B\,\epsilon_D\,id"] \arrow[dr, phantom, "(32)"] & & & \\
B\,B^*\,B\,A^*\,D\,D^*\,D\,C^* \arrow[uuurr, bend left, "B\,\sigma\,id"] \arrow[ur, "id\,\sigma\,id"'] \arrow[ddd, "\epsilon_B\,id\,\epsilon_D\,id"'] \arrow[ddr, phantom, "\sigma\text{ nat.}"] & & {} & & I\,B\,D\,C^*\,A^* \arrow[ddd, "\lambda"] \\
 & & & I\,B\,A^*\,D\,C^* \arrow[ur, "id\,\sigma"'] \arrow[dr, phantom, "\text{coh.}"] & \\
 & I\,I\,B\,A^*\,D\,C^* \arrow[urr, "\lambda\,id"] & & & {} \\
I\,B\,A^*\,I\,D\,C^* \arrow[rr, "\lambda\,\lambda"'] \arrow[ur, "I\,\sigma\,id"] & & B\,A^*\,D\,C^* \arrow[rr, "B\,\sigma^{-1}"'] & & B\,D\,C^*\,A^*
\end{tikzcd}
$$

(d) Let $f : A \to B$. We show that $f_* \otimes f$ is completely positive using Proposition

4.2.3 (d). We take $C = I$ and $k : I \otimes A \to B$ to be the composite $f \circ \lambda_A$. Then we have

$$\begin{tikzcd}
A^* \otimes A \arrow[r, "\rho^{-1} \otimes A"] \arrow[rrrdd, "A^* \otimes A"] \arrow[rrrddd, "f_* \otimes f"'] \arrow[rr, phantom, shift right=6ex, "\text{coh.}"] & A^* \otimes I \otimes A \arrow[rr, "A^* \otimes \eta_I \otimes A"] \arrow[dr, "A^* \otimes \lambda^{-1} \otimes A"'] \arrow[rr, phantom, shift right=6ex, "(33)"] & & A^* \otimes I^* \otimes I \otimes A \arrow[dd, "\lambda_* \otimes \lambda"] \\
& & A^* \otimes I \otimes I \otimes A \arrow[ur, "\cong"] \arrow[dr, "\rho \otimes \lambda"] \arrow[ur, phantom, shift right=4ex, "(16)"] & \\
& & \arrow[r, phantom, "(=)"] & A^* \otimes A \arrow[d, "f^* \otimes f"] \\
& & & B^* \otimes B
\end{tikzcd}$$

as required. $\square$

Completely positive maps preserve positivity in the following sense.

Lemma 4.2.10. *If $f : A^* \otimes A \to B^* \otimes B$ is completely positive and $g : I \to A^* \otimes A$ is a positive matrix, then $f \circ g : I \to B^* \otimes B$ is a positive matrix.*

Proof. Let $f : A^* \otimes A \to B^* \otimes B$ be completely positive and let $g : I \to A^* \otimes A$ be a positive matrix. We use Proposition 4.2.3 (d) to write

$$\begin{tikzcd}
A^* \otimes A \arrow[r, "\rho^{-1} \otimes A"] \arrow[drr, "f"'] & A^* \otimes I \otimes A \arrow[r, "A^* \otimes \eta_C \otimes A"] & A^* \otimes C^* \otimes C \otimes A \arrow[d, "k_* \otimes k"] \\
& & B^* \otimes B
\end{tikzcd}$$

and we write g as the composite

$$I \xrightarrow{\eta_A} A^* \otimes A \xrightarrow{A^* \otimes h} A^* \otimes D \xrightarrow{A^* \otimes h^\dagger} A^* \otimes A.$$

Next we define a map $m : B \to C \otimes D$ as follows:

$$B \xrightarrow{k^\dagger} C \otimes A \xrightarrow{C \otimes h} C \otimes D.$$

We claim that fg is the positive matrix

$$I \xrightarrow{\eta_B} B^* \otimes B \xrightarrow{B^* \otimes m} B^* \otimes C \otimes D \xrightarrow{B^* \otimes m^\dagger} B^* \otimes B.$$

We use Remark 4.1.9 to get the following:

$$\begin{tikzcd}
I \arrow[rrrr, "\eta_B"] \arrow[drrr, "\eta_{C \otimes D}"] \arrow[ddr, "\eta_A"'] \arrow[dddrrr, "\eta_D"] \arrow[ddddd, "g"'] & & & & B^* \otimes B \arrow[ddd, "B^* \otimes m"] \\
& & & D^* \otimes C^* \otimes C \otimes D \arrow[ddddr, "m^* \otimes m^\dagger"'] & \\
& A^* \otimes A \arrow[dd, "A^* \otimes h"] & & \arrow[r, phantom, "\text{Rem. 4.1.9}"] & \\
\arrow[r, phantom, "(=)"] & \arrow[rr, phantom, "\text{Rem. 4.1.9}"] & & D^* \otimes D \arrow[ddlll, "h^* \otimes h^\dagger"] & B^* \otimes C \otimes D \arrow[dd, "B^* \otimes m^\dagger"] \\
& A^* \otimes D \arrow[dl, "A^* \otimes h^\dagger"] & \arrow[rr, phantom, "(*)"] & & \\
A^* \otimes A \arrow[rrrr, "f"'] & & & & B^* \otimes B
\end{tikzcd}$$

It remains to establish the equation $(*)$, which is straightforward once we replace f and m:

$$
\begin{tikzcd}
I \arrow[rrrr, "\eta_{C\otimes D}"] \arrow[ddr, "\lambda^{-1}"] \arrow[ddd, "\eta_D"'] \arrow[from=1-1, to=2-4, phantom, "(31)"] & & & & D^*\otimes C^*\otimes C\otimes D \arrow[dddd, "h^*\otimes id\otimes h^\dagger"] \arrow[dddddddl, "h^*\otimes id\otimes h^\dagger"'] \\
& & & C^*\otimes C\otimes D^*\otimes D \arrow[ur, "\sigma\otimes D"] & \\
& I\otimes I \arrow[urr, "\eta_C\otimes\eta_D"] \arrow[dr, "I\otimes\eta_D"] & & & \\
D^*\otimes D \arrow[rr, "D^*\otimes\lambda^{-1}"'] \arrow[dd, "h^*\otimes h^\dagger"'] \arrow[dddrr, "\rho^{-1}\otimes D"'] \arrow[from=4-1, to=3-2, phantom, "\lambda\text{ nat.}"] & & I\otimes D^*\otimes D \arrow[uur, "\eta_C\otimes id"'] \arrow[ddd, "\sigma\otimes D"] \arrow[from=4-3, to=2-3, phantom, "\otimes\text{ func.}"] \arrow[from=4-3, to=1-5, phantom, "\sigma\text{ nat.}"] & & \\
\arrow[from=5-1, to=4-3, phantom, "\text{coh.}"] & & & & A^*\otimes C^*\otimes C\otimes A \arrow[ddd, "k_*\otimes k"] \\
A^*\otimes A \arrow[dd, "\rho^{-1}\otimes A"'] \arrow[from=6-1, to=7-2, phantom, "\rho\text{ nat.}"] & & & \arrow[r, phantom, "(=)"] & \\
& & D^*\otimes I\otimes D \arrow[uuuuuurr, "D^*\otimes\eta_C\otimes D"] \arrow[dll, "h^*\otimes I\otimes h^\dagger"] \arrow[from=7-3, to=8-4, phantom, "\otimes\text{ func.}"] & & \\
A^*\otimes I\otimes A \arrow[rrr, "A^*\otimes\eta_C\otimes A"'] & & & A^*\otimes C^*\otimes C\otimes A \arrow[r, "k_*\otimes k"'] & B^*\otimes B
\end{tikzcd}
$$

□

Lemma 4.2.11. *Let $f : A^*\otimes A \to B^*\otimes B$ in a dagger compact closed category $\mathcal{C}$. Then f is completely positive if and only if $id_{C^*}\otimes f\otimes id_C$ preserves positive matrices for all objects $C\in\mathcal{C}$, i.e, for all positive matrices $M : I \to C^*\otimes A^*\otimes A\otimes C$, the map*

$$I \xrightarrow{M} C^* \otimes A^* \otimes A \otimes C \xrightarrow{id_{C^*} \otimes f \otimes id_C} C^* \otimes B^* \otimes B \otimes C$$

is a positive matrix.

Proof.

If f is completely positive then by Lemma 4.2.9 (c), so is $id_{C^*} \otimes f \otimes id_C$. By Lemma 4.2.10, $id_{C^*} \otimes f \otimes id_C$ preserves positive matrices.

Conversely, suppose that $(id_{C^*} \otimes f \otimes id_C) \circ M$ is a positive matrix for all objects C and for all positive matrices $M : I \to C^* \otimes A^* \otimes A \otimes C$. This holds in particular for $C = A^*$ and the positive matrix $M = \ulcorner \epsilon_A^\dagger \circ \epsilon_A \urcorner$. Thus $(id_A \otimes f \otimes id_{A^*}) \circ M = \ulcorner g \urcorner$, for some positive map $g : B \otimes A^* \to B \otimes A^*$, so we have the following commutative diagram:

$$\begin{tikzcd}
I \arrow[r, "\eta_{A\otimes A^*}"] \arrow[dd, "\eta_{B\otimes A^*}"'] & A \otimes A^* \otimes A \otimes A^* \arrow[r, "id_{A\otimes A^*} \otimes \epsilon_A"] & A \otimes A^* \otimes I \arrow[d, "id_{A\otimes A^*} \otimes \epsilon_A^\dagger"] \\
 & & A \otimes A^* \otimes A \otimes A^* \arrow[d, "id_A \otimes f \otimes id_{A^*}"] \\
A \otimes B^* \otimes B \otimes A^* \arrow[rr, "id_{A\otimes B^*} \otimes g"'] & & A \otimes B^* \otimes B \otimes A^*
\end{tikzcd}$$

From this it follows that

$$\begin{tikzcd}
A^* \otimes A \arrow[r, "\lambda^{-1}"] \arrow[ddddrr, "f"'] & I \otimes A^* \otimes A \arrow[r, "\eta_B \otimes A^* \otimes A"] & B^* \otimes B \otimes A^* \otimes A \arrow[d, "B^* \otimes g \otimes A"] \\
 & & B^* \otimes B \otimes A^* \otimes A \arrow[d, "B^* \otimes B \otimes \eta_A^\dagger"] \\
 & & B^* \otimes B \otimes I \arrow[d, "\rho"] \\
 & & B^* \otimes B
\end{tikzcd}$$

commutes. Hence f is completely positive by Proposition 4.2.3 (b). $\square$

4.3 Selinger's CPM Construction

We now state Selinger's CPM construction for dagger compact categories.

Definition 4.3.1. **(CPM Construction)** Let $\mathcal{C}$ be a dagger compact closed category. We define a new category $\mathbf{CPM}(\mathcal{C})$ whose objects are the same as the objects of $\mathcal{C}$. A morphism $f : A \to B$ in $\mathbf{CPM}(\mathcal{C})$ is a completely positive map $f : A^* \otimes A \to B^* \otimes B$ in $\mathcal{C}$. Composition of morphisms is as in $\mathcal{C}$.

By Lemma 4.2.9 (a) and (b), $\mathbf{CPM}(\mathcal{C})$ is indeed a category. Moreover, Lemma 4.2.9 (d) yields a functor $F : \mathcal{C} \to \mathbf{CPM}(\mathcal{C})$ defined by:

$$F(A) = A \quad \text{and} \quad F(f) = f_* \otimes f.$$

The following theorem is due to Selinger [26].

Theorem 4.3.2. *Let $\mathcal{C}$ be a dagger compact closed category. Then $\mathbf{CPM}(\mathcal{C})$ is again dagger compact closed and the functor $F : \mathcal{C} \to \mathbf{CPM}(\mathcal{C})$ preserves the dagger compact closed structure.*

Proof. $\mathbf{CPM}(\mathcal{C})$ inherits its tensor product on objects from $\mathcal{C}$. On morphisms it is given by Lemma 4.2.9 (c). Each object A in $\mathbf{CPM}(\mathcal{C})$ has a dual A^* from $\mathcal{C}$. The natural isomorphisms α, λ, ρ, and σ as well as the maps η and ϵ are given by the images of the respective maps in $\mathcal{C}$ under F. If $f : A^* \otimes A \to B^* \otimes B$ is a morphism in $\mathbf{CPM}(\mathcal{C})$, $f^\dagger$ in $\mathbf{CPM}(\mathcal{C})$ is given by $f^\dagger : B^* \otimes B \to A^* \otimes A$ in $\mathcal{C}$. There are several equations to verify. See [26] for a sample calculation. □

Example 4.3.3.
The category $\mathbf{CPM}(\mathbf{Hilb}_{fd})$ has finite-dimensional Hilbert spaces as objects, and completely positive operators as morphisms. $\mathbf{CPM}(\mathbf{Hilb}_{fd})$ is the full subcategory of simple objects in the category $\mathbf{W}$ of [27].

Example 4.3.4.
The category $\mathbf{CPM}(\mathbf{Rel})$ has sets as objects. A morphism from A to B is a relation $R : A \times A \to B \times B$ satisfying $R = \sigma R \sigma$ and $((a_1, a_2), (b_1, b_2)) \in R \Rightarrow ((a_1, a_1), (b_1, b_1)) \in R$.

Chapter 5

Nuclearity

Our goal is to extend Selinger's CPM construction to include the category **Hilb**. We have seen that **Hilb** is a dagger symmetric monoidal category but fails to be compact closed. Thus Selinger's CPM construction is not applicable. However, this category does share a great deal of structure with its finite-dimensional counterpart, making it an ideal setting for the axiomatization of quantum mechanics. For this reason, we seek something more general than Selinger's dagger compact closed categories.

When passing from the category of finite-dimensional Hilbert spaces to the category of all Hilbert spaces, we require an appropriate categorical framework. To capture the structure of **Hilb**, we use the notions of tensored †-categories and nuclear ideals.

5.1 Tensored †-Categories

Definition 5.1.1. Let $\mathcal{C}$ be a dagger category. Then $\mathcal{C}$ is ***tensored*** if it is symmetric monoidal, the isomorphisms α, λ, ρ and σ are all unitary, $(f \otimes g)^\dagger = f^\dagger \otimes g^\dagger$, and there is a covariant *conjugate functor* $\overline{(-)} : \mathcal{C} \to \mathcal{C}$ which commutes with the dagger functor and has natural isomorphisms:

- $\overline{\overline{A}} \cong A$ (generally taken to be equality);
- $\tau_{A,B} : \overline{A} \otimes \overline{B} \xrightarrow{\cong} \overline{B \otimes A}$;
- $\iota : I \xrightarrow{\cong} \overline{I}$;

which are unitary and satisfy the symmetric monoidal functor equations:

$$
\begin{tikzcd}
 & \overline{A}\otimes(\overline{B}\otimes\overline{C}) \arrow[r, "\overline{A}\otimes\tau_{B,C}"] & \overline{A}\otimes\overline{(C\otimes B)} \arrow[dr, "\tau_{A,C\otimes B}"] & \\
(\overline{A}\otimes\overline{B})\otimes\overline{C} \arrow[ur, "\alpha_{\overline{A},\overline{B},\overline{C}}"] \arrow[dr, "\tau_{A,B}\otimes\overline{C}"'] & & & \overline{(C\otimes B)\otimes A} \\
 & \overline{(B\otimes A)}\otimes\overline{C} \arrow[r, "\tau_{B\otimes A,C}"'] & \overline{C\otimes(B\otimes A)} \arrow[ur, "\overline{\alpha^{-1}_{C,B,A}}"'] &
\end{tikzcd}
\tag{39}
$$

$$
\begin{tikzcd}
\overline{A}\otimes I \arrow[r, "\rho_{\overline{A}}"] \arrow[d, "\overline{A}\otimes\iota"'] & \overline{A} \\
\overline{A}\otimes\overline{I} \arrow[r, "\tau_{A,I}"'] & \overline{I\otimes A} \arrow[u, "\overline{\lambda_A}"']
\end{tikzcd}
\tag{40}
$$

$$
\begin{tikzcd}
I\otimes\overline{A} \arrow[r, "\lambda_{\overline{A}}"] \arrow[d, "\iota\otimes\overline{A}"'] & \overline{A} \\
\overline{I}\otimes\overline{A} \arrow[r, "\tau_{I,A}"'] & \overline{A\otimes I} \arrow[u, "\overline{\rho_A}"']
\end{tikzcd}
\tag{41}
$$

$$
\begin{tikzcd}
\overline{A}\otimes\overline{B} \arrow[r, "\sigma_{\overline{A},\overline{B}}"] \arrow[d, "\tau_{A,B}"'] & \overline{B}\otimes\overline{A} \arrow[d, "\tau_{B,A}"] \\
\overline{B\otimes A} \arrow[r, "\overline{\sigma_{B,A}}"'] & \overline{A\otimes B}
\end{tikzcd}
\tag{42}
$$

dagger compact closed category	tensored †-category
$(-)^\dagger$	$(-)^\dagger$
$(-)^*$	$\overline{(-)^\dagger} = \overline{(-)}^\dagger$
$(-)_*$	$\overline{(-)}$

Table 1: Dagger compact closed structures expressed as tensored dagger structures.

Additionally, given $f : I \to I$ we require:

$$\begin{tikzcd} I \arrow[r, "f^\dagger"] \arrow[d, "\iota"'] & I \arrow[d, "\iota"] \\ \overline{I} \arrow[r, "\overline{f}"'] & \overline{I} \end{tikzcd} \tag{43}$$

Our definition of a tensored dagger category is a modified version of a tensored $*$-category, as defined by Abramsky et. al. in [2]. To reflect this, we will refer to such a category as a tensored †-category from here on. In the original formulation, the conjugate functor is strong and satisfies the usual symmetric monoidal equations. In particular τ is an isomorphism $\overline{A} \otimes \overline{B} \overset{\cong}{\to} \overline{A \otimes B}$. However, in order to efficiently generalize Selinger's construction we use τ to denote the natural isomorphism $\overline{A} \otimes \overline{B} \overset{\cong}{\to} \overline{B \otimes A}$ instead. The first four equations in our definition are the monoidal equations modulo this twist.

We now provide examples of these categories. First of all, any dagger compact closed category $\mathcal{C}$ is a tensored †-category. In this case we have the covariant functor $\overline{(-)} \overset{\text{def}}{=} (-)_* : \mathcal{C} \to \mathcal{C}$. Table 1 illustrates the correspondence in notation for dagger compact closed categories and tensored †-categories.

What about **LFR** and **Hilb**? We have seen that these categories share a dagger symmetric monoidal structure with **Rel** and $\mathbf{Hilb}_{fd}$ respectively, but not the full dagger compact closed structure. Nonetheless, it turns out that both are tensored

$\dagger$-categories.

Example 5.1.2.
In **LFR** we define $\overline{(-)} \stackrel{\text{def}}{=} id$. Then the necessary isomorphisms are all identities, and clearly id satisfies the symmetric monoidal functor equations. Lastly, we verify equation (43). Since $\iota = id$, it suffices to show that $R = R^\dagger$ for any $R : \{*\} \to \{*\}$. But there are only two such maps: $R = \emptyset$ and $R = \{(*, *)\}$, both of which satisfy $R = R^\dagger$. Hence (43) holds and **LFR** is a tensored $\dagger$-category.

Example 5.1.3.
In the case of **Hilb**, the situation is quite different. Here, the conjugate functor $\overline{(-)}$ acts by conjugation. Recall that for a Hilbert space $H = (H, +, \cdot, \langle -, - \rangle)$, its conjugate $\overline{H} = (H, +, \bar{\cdot}, \langle -, - \rangle^-)$ is the Hilbert space with the same underlying set, addition as in H, but with scalar multiplication and inner product given by:

$$\alpha \bar{\cdot} x = \overline{\alpha} x \quad \text{and} \quad \langle x, y \rangle^- = \overline{\langle x, y \rangle} = \langle y, x \rangle.$$

For a morphism $T : H \to K$, applying the conjugate functor yields the morphism $\overline{T} : \overline{H} \to \overline{K}$ which is defined to be T viewed as a map $\overline{H} \to \overline{K}$. Together with its dagger symmetric monoidal structure, this makes **Hilb** a tensored $\dagger$-category. The first isomorphism is the identity, the second is the symmetry map σ, and $\mathbb{C} \cong \overline{\mathbb{C}}$ via conjugation. The monoidal functor equations follow from the fact that conjugation preserves the action of the morphisms. For the last equation, we let $T_y : \mathbb{C} \to \mathbb{C}$ denote multiplication by y. Then $T_y^\dagger = T_{\overline{y}} : \mathbb{C} \to \mathbb{C}$ and $\overline{T_y} = T_{\overline{y}} : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ (multiplication in $\overline{\mathbb{C}}$ by $\overline{y}$). We have

$$\iota \circ {T_y}^\dagger(x) = \iota(\overline{y}x) = \overline{\overline{y}x} = y\overline{x} = \overline{y} \bar{\cdot} \overline{x} = T_{\overline{y}}(\iota(x)) = \overline{T_y} \circ \iota(x).$$

Therefore (43) holds and **Hilb** is a tensored $\dagger$-category.

5.2 Nuclear Ideals

Recall that in a dagger compact closed category $\mathcal{C}$, we have natural bijections

$$Hom(A \otimes B, C) \cong Hom(B, A^* \otimes C),$$

$\forall A, B, C \in \mathcal{C}$. These yields natural bijections

$$Hom(A, B) \cong Hom(A \otimes I, B) \cong Hom(I, A^* \otimes B) = Hom(I, \overline{A} \otimes B).$$

In the tensored †-category **Hilb**, Theorem 2.2.13 gave us the restricted condition

$$HSO(H, K) \cong Hom(I, \overline{H} \otimes K).$$

This is the idea behind the notion of a nuclear ideal which is due to Abramsky, Blute and Panangaden [2]. In a sense, a nuclear ideal structure can measure the extent to which a tensored †-category is compact closed.

Definition 5.2.1. Let $\mathcal{C}$ be a tensored †-category. A ***nuclear ideal*** for $\mathcal{C}$ consists of the following structure:

- For all objects $A, B \in \mathcal{C}$, a subset $\mathcal{N}(A, B) \subseteq Hom(A, B)$. We denote the union of these subsets as $\mathcal{N}(\mathcal{C})$ or $\mathcal{N}$. We refer to the elements of $\mathcal{N}$ as ***nuclear maps***. The class $\mathcal{N}$ must be closed under composition with arbitrary morphisms in $\mathcal{C}$, and closed under $\otimes$, $(-)^\dagger$, and $\overline{(-)}$.
- A bijection $\theta : \mathcal{N}(A, B) \to Hom(I, \overline{A} \otimes B)$. If $f : A \to B$ is a nuclear morphism, note that we can use the bijection θ and the †-functor to construct morphisms of the form:

  1. $\theta(f) : I \to \overline{A} \otimes B$
  2. $\theta(f)^\dagger : \overline{A} \otimes B \to I$
  3. $\theta(f^\dagger) : I \to \overline{B} \otimes A$
  4. $\theta(f^\dagger)^\dagger : \overline{B} \otimes A \to I$.

We refer to these morphisms as the ***transposes*** of f. The bijection θ must also satisfy the following properties:

1. **Preservation of tensored †-structure**

(a) If $f : A \to B$ and $g : C \to D$ are nuclear, then $\theta(f \otimes g) = \theta(f) \otimes \theta(g)$. More precisely, the following diagram commutes:

$$\begin{tikzcd}
I \arrow[rr, "\theta(f \otimes g)"] \arrow[d, "\lambda_I^{-1}"'] & & \overline{A \otimes C} \otimes B \otimes D \\
I \otimes I \arrow[r, "\theta(f) \otimes \theta(g)"'] & \overline{A} \otimes B \otimes \overline{C} \otimes D \arrow[r, "\sigma_{\overline{A} \otimes B, \overline{C}} \otimes D"'] & \overline{C} \otimes \overline{A} \otimes B \otimes D \arrow[u, "\tau_{C,A} \otimes B \otimes D"']
\end{tikzcd} \tag{44}$$

Furthermore, the transposes of a nuclear morphism of the form $f : I \to A$ are given by the composites below.

$$\begin{tikzcd}
I \arrow[r, "\theta(f)"] \arrow[d, "f"'] & \overline{I} \otimes A \\
A \arrow[r, "\lambda_A^{-1}"'] & I \otimes A \arrow[u, "\iota \otimes A"']
\end{tikzcd} \tag{45}$$

$$\begin{tikzcd}
\overline{I} \otimes A \arrow[r, "\theta(f)^\dagger"] \arrow[d, "\iota^{-1} \otimes A"'] & I \\
I \otimes A \arrow[r, "\lambda_A"'] & A \arrow[u, "f^\dagger"']
\end{tikzcd} \tag{46}$$

$$\begin{tikzcd}
I \arrow[r, "\theta(f^\dagger)"] \arrow[d, "\iota"'] & \overline{A} \otimes I \\
\overline{I} \arrow[r, "\overline{f}"'] & \overline{A} \arrow[u, "\rho_{\overline{A}}^{-1}"']
\end{tikzcd} \tag{47}$$

$$\begin{tikzcd}
\overline{A} \otimes I \arrow[r, "\theta(f^\dagger)^\dagger"] \arrow[d, "\rho_{\overline{A}}"'] & I \\
\overline{A} \arrow[r, "\overline{f^\dagger}"'] & \overline{I} \arrow[u, "\iota^{-1}"']
\end{tikzcd} \tag{48}$$

(b) If $f : A \to B$ is nuclear, then $\theta(\overline{f}) = \theta(f^\dagger) = \overline{\theta(f)}$. Again, more precisely, we have:

$$
\begin{tikzcd}
 & A \otimes \overline{B} \arrow[dr, "\sigma_{A,\overline{B}}"] & \\
I \arrow[ur, "\theta(\overline{f})"] \arrow[rr, "\theta(f^\dagger)"] \arrow[d, "\iota"'] & & \overline{B} \otimes A \\
\overline{I} \arrow[rr, "\overline{\theta(f)}"'] & & \overline{\overline{A} \otimes B} \arrow[u, "\tau^{-1}_{B,\overline{A}}"']
\end{tikzcd}
\tag{49}
$$

2. **Naturality** For any $f : A \to C$ and $g : B \to D$ in $\mathcal{C}$, the following diagram commutes:

$$
\begin{tikzcd}
\mathcal{N}(A, B) \arrow[r, "\theta"] \arrow[d, "\mathcal{N}(f^\dagger, g)"'] & Hom(I, \overline{A} \otimes B) \arrow[d, "Hom(I, \overline{f} \otimes g)"] \\
\mathcal{N}(C, D) \arrow[r, "\theta"'] & Hom(I, \overline{C} \otimes D)
\end{tikzcd}
\tag{50}
$$

For $k \in \mathcal{N}(A, B)$, we have

$$(\overline{f} \otimes g) \circ \theta(k) = \theta(g \circ k \circ f^\dagger).$$

Note that the function $\mathcal{N}(f^\dagger, g)$ is well defined since $\mathcal{N}$ is closed under composition with arbitrary maps.

3. **Compactness** Let $f : A \to B$ and $g : B \to C$ be nuclear. Then the following commutes:

$$
\begin{tikzcd}
A \arrow[r, "\rho_A^{-1}"] \arrow[rrdddd, "gf"'] & A \otimes I \arrow[r, "A \otimes \theta(g)"] & A \otimes \overline{B} \otimes C \arrow[d, "\sigma_{A,\overline{B}} \otimes C"] \\
 & & \overline{B} \otimes A \otimes C \arrow[d, "\theta(f^\dagger)^\dagger \otimes C"] \\
 & & I \otimes C \arrow[d, "\lambda_C"] \\
 & & C
\end{tikzcd}
\tag{51}
$$

Remark 5.2.2. In the compactness condition, if A is nuclear and $f = g = id_A$, (51) reduces to the compact closed equation (19). The second compact closed equation (20) follows from this definition as well:

$$
\begin{tikzcd}
\overline{A} \arrow[rr, "\lambda^{-1}"] \arrow[dr, "\rho^{-1}"'] \arrow[ddddddd, "id"'] & & I \otimes \overline{A} \arrow[rrr, "\theta(id_A) \otimes A"] \arrow[drr, "\theta(id_{\overline{A}}) \otimes \overline{A}"'] & & & \overline{A} \otimes A \otimes \overline{A} \arrow[dd, "\overline{A} \otimes \sigma"] \arrow[ddddd, bend right, "\overline{A} \otimes \theta(id_{\overline{A}})^{\dagger}"'] \\
& \overline{A} \otimes I \arrow[ur, "\sigma"] \arrow[ddr, "\overline{A} \otimes \theta(id_{\overline{A}})"'] & (13) & (49) & A \otimes \overline{A} \otimes \overline{A} \arrow[ur, "\sigma \otimes \overline{A}"'] & \\
& & \sigma \text{ nat.} & (14) & & \overline{A} \otimes \overline{A} \otimes A \arrow[ddd, "\overline{A} \otimes \theta(id_A)^{\dagger}"] \\
& & \overline{A} \otimes A \otimes \overline{A} \arrow[uurr, "\sigma"] \arrow[dr, "\sigma \otimes \overline{A}"'] & & (49) & \\
& & & A \otimes \overline{A} \otimes \overline{A} \arrow[uuuurr, "\sigma"] \arrow[ddr, "\theta(id_{\overline{A}})^{\dagger} \otimes \overline{A}"'] & & \\
& (51) & & \sigma \text{ nat.} & & \overline{A} \otimes I \arrow[dd, "\rho"] \\
& & & & I \otimes \overline{A} \arrow[ur, "\sigma"] \arrow[dr, "\lambda"'] & (13) \\
\overline{A} \arrow[rrrrr, "id"'] & & & & & \overline{A}
\end{tikzcd}
\tag{52}
$$

Definition 5.2.3. Let $\mathcal{N}$ be a nuclear ideal for $\mathcal{C}$. An object A of $\mathcal{C}$ is ***$\mathcal{N}$-nuclear*** or ***nuclear*** if $\mathcal{N}(A, -) = Hom(A, -)$. This is equivalent to id_A being nuclear by the ideal property.

Remark 5.2.4. In a nuclear ideal $\mathcal{N}(\mathcal{C})$, the tensor unit I is nuclear since we have

$$\mathcal{N}(I, I) \xrightarrow{\theta} Hom(I, \overline{I} \otimes I) \xrightarrow{\cong} Hom(I, I).$$

A nuclear ideal for a tensored $\dagger$-category generalizes the notion of a dagger compact closed category. In particular, each dagger compact closed category $\mathcal{C}$ is endowed with the nuclear ideal $\mathcal{N}$ of all maps, i.e., for all objects A and B in $\mathcal{C}$, $\mathcal{N}(A, B) =$

$Hom(A, B)$. This is equivalent to all objects being nuclear. The adjunction $A \otimes (-) \vdash A^* \otimes (-)$ yields a natural bijection $\theta : \mathcal{N}(A, B) \to Hom(I, A^* \otimes B)$ as we discussed above.

We now consider the converse. The following appears as Theorem 5.9 in [2].

Proposition 5.2.5. *Let $\mathcal{N}(\mathcal{C})$ be a nuclear ideal in which all objects are nuclear. Then $\mathcal{C}$ is a compact closed category.*

For each object $A \in \mathcal{C}$, the unit and counit maps are given by

$$\eta_A = \theta(id_A) \qquad \text{and} \qquad \epsilon_A = \theta(id_A)^\dagger \circ \sigma.$$

Therefore the dagger compact closed equation (30) holds by construction. This means that $\mathcal{C}$ is in fact a dagger compact closed category. In summary, a dagger compact closed category $\mathcal{C}$ is the same thing as a tensored †-category $\mathcal{C}$ equipped with a nuclear ideal $\mathcal{N}(\mathcal{C})$ in which all objects are nuclear.

The categories **LFR** and **Hilb** have (proper) nuclear ideals.

Example 5.2.6.
Finite relations form a nuclear ideal $\mathcal{N}$ for **LFR**. Let $S : B \to C$ be finite, and let $R : A \to B$, $T : C \to D$ be arbitrary maps in **LFR**. Then

$$\begin{aligned} SR &= \{\, (a, c) \mid \exists b \in B \text{ with } (a, b) \in R, (b, c) \in S \,\} \\ &= \bigcup_{(b,c) \in S} \{\, (a, c) \mid a \in R_b \,\} \\ &= \bigcup_{(b,c) \in S} R_b \times \{\, c \,\}, \end{aligned}$$

which is a finite union of finite sets, and hence finite. Similarly, the composite TS is the finite relation

$$TS = \bigcup_{(b,c) \in S} \{\, b \,\} \times T_c.$$

Thus $\mathcal{N}$ is closed under composition with morphisms in **LFR**. Additionally, if S and S' are finite relations then

$$|S \otimes S'| = |S| \cdot |S'|,$$

and

$$|S^{\dagger}| = |S| = |\overline{S}|.$$

So $\mathcal{N}$ is closed under $\otimes$, $(-)^{\dagger}$, and $\overline{(-)}$. We define $\theta : \mathcal{N}(A, B) \to Hom(I, A \times B)$ by $\theta(S) = I \times S$. This is clearly a well-defined bijection. Several simple calculations show that θ preserves the tensored †-structure. For naturality, let $S : A \to B$ be finite, and let $R : A \to C$ and $T : B \to D$ be arbitrary. Then

$$\begin{aligned}
(*, (c, d)) \in (R \times T) \circ \theta(S) & \\
&\iff \exists\, (a, b) \text{ with } (*, (a, b)) \in \theta(S) \text{ and } ((a, b), (c, d)) \in R \times T \\
&\iff \exists\, (a, b) \in S \text{ with } (a, c) \in R \text{ and } (b, d) \in T \\
&\iff \exists\, (a, b) \in S \text{ with } (c, a) \in R^{\dagger} \text{ and } (b, d) \in T \\
&\iff (c, d) \in T \circ S \circ R^{\dagger} \\
&\iff (*, (c, d)) \in \theta(T \circ S \circ R^{\dagger}).
\end{aligned}$$

Hence equation (50) holds. A similar calculation verifies the compactness condition.

Example 5.2.7.

Let $\mathcal{N}$ denote the class of Hilbert-Schmidt operators in **Hilb**, then $\mathcal{N}$ is a nuclear ideal for **Hilb**. Fix Hilbert spaces H, K, M, N with respective orthonormal bases B_H, B_K, B_M and B_N. We saw in Remark 2.2.11 that $\mathcal{N}$ is a two-sided ideal in **Hilb**. Let $S : H \to K$ and $S' : M \to N$ be nuclear. We have

$$\begin{aligned}
\sum_{b_h \in B_H} \sum_{b_m \in B_M} \|(S \otimes S')(b_h \otimes b_m)\|^2 &= \sum_{b_h \in B_H} \sum_{b_m \in B_M} \|(Sb_h) \otimes (S'b_m)\|^2 \\
&= \sum_{b_h \in B_H} \sum_{b_m \in B_M} \langle (Sb_h) \otimes (S'b_m), (Sb_h) \otimes (S'b_m) \rangle \\
&= \sum_{b_h \in B_H} \sum_{b_m \in B_M} \langle (Sb_h), (Sb_h) \rangle \langle (S'b_m), (S'b_m) \rangle \\
&= \sum_{b_h \in B_H} \sum_{b_m \in B_M} \|Sb_h\|^2 \|S'b_m\|^2 \\
&= \Big(\sum_{b_h \in B_H} \|Sb_h\|^2 \Big) \Big(\sum_{b_m \in B_M} \|\|S'b_m\|^2 \Big) < \infty,
\end{aligned}$$

$$\sum_{b_k\in B_K}\sum_{b_h\in B_H}|\langle S^\dagger b_k, b_h\rangle|^2=\sum_{b_k\in B_K}\sum_{b_h\in B_H}|\langle b_k, Sb_h\rangle|^2=\sum_{b_h\in B_H}\sum_{b_k\in B_K}|\langle Sb_h, b_k\rangle|^2<\infty,$$

and

$$\sum_{b_h\in B_H}\sum_{b_k\in B_K}|\langle \overline{S}b_h, b_k\rangle^-|^2=\sum_{b_h\in B_H}\sum_{b_k\in B_K}|\langle \overline{S}b_h, b_k\rangle|^2=\sum_{b_h\in B_H}\sum_{b_k\in B_K}|\langle Sb_h, b_k\rangle|^2<\infty.$$

Hence $\mathcal{N}$ is closed under $\otimes$, $(-)^\dagger$, and $\overline{(-)}$. Recall the bijection $\theta : HSO(H,K) \to Hom(I, \overline{H}\otimes K)$ from Remark 2.2.14. We saw that θ was an isomorphism of vector spaces. By this linearity, it suffices to check the nuclear ideal equations on bases. For the space $HSO(H,K)$ we had the orthonormal basis $\{\,T_{b_h,b_k} \mid b_h \in B_H, b_k \in B_K\,\}$, where

$$T_{x,y}(u) \stackrel{\text{def}}{=} \langle u, x\rangle y,$$

for $u \in H$. Specifically, we had $\theta(T_{b_h,b_k}) = [1 \mapsto b_h \otimes b_k]$. Let $T_{b_h,b_k} : H \to K$ and $T_{b_m,b_n} : M \to N$ be nuclear. Note that

$$\begin{aligned}(T_{b_h,b_k}\otimes T_{b_m,b_n})(u\otimes v) &= \langle u, b_h\rangle b_k \otimes \langle v, b_m\rangle b_n\\ &= (\langle u, b_h\rangle\langle v, b_m\rangle) b_k\otimes b_n\\ &= (\langle u\otimes v, b_h\otimes b_m\rangle) b_k \otimes b_n\\ &= T_{b_h\otimes b_m, b_k\otimes b_n}(u\otimes v),\end{aligned}$$

i.e., $(T_{b_h,b_k}\otimes T_{b_m,b_n}) = T_{b_h\otimes b_m, b_k\otimes b_n}$. Therefore, $\theta(T_{b_h,b_k}\otimes T_{b_m,b_n}) = [1 \mapsto b_h\otimes b_m\otimes b_k\otimes b_n]$. On the other hand,

$$\begin{aligned}\tau\circ(\sigma\otimes id)\circ(\theta(T_{b_h,b_k})\otimes\theta(T_{b_m,b_n}))\circ\rho^{-1} = [1 &\mapsto 1\otimes 1\\ &\mapsto b_h\otimes b_k\otimes b_m\otimes b_n\\ &\mapsto b_m\otimes b_h\otimes b_k\otimes b_n\\ &\mapsto b_h\otimes b_m\otimes b_k\otimes b_n].\end{aligned}$$

Thus equation (44) holds. It is clear that the transposes of a nuclear map of the form $T : \mathbb{C} \to H$ satisfy the required equations. Next we compute $T^\dagger_{b_h,b_k}$. For $u \in H$ and $v \in K$ we have

$$\langle u, T_{b_k,b_h}v\rangle = \langle u, \langle v, b_k\rangle b_h\rangle = \langle b_k, v\rangle\langle u, b_h\rangle = \langle\langle u, b_h\rangle b_k, v\rangle = \langle T_{b_h,b_k u,v}\rangle,$$

so $T^{\dagger}_{b_h,b_k} = T_{b_k,b_h}$. Equation (49) follows from this fact. For naturality, we let $R : H \longrightarrow M$ and $S : K \longrightarrow N$ be arbitrary maps in **Hilb**. Then

$$ST_{b_h,b_k}R^{\dagger}(u) = S(\langle R^{\dagger}u, b_h\rangle b_k) = \langle R^{\dagger}u, b_h\rangle Sb_k = \langle u, Rb_h\rangle Sb_k = T_{Rb_h,Sb_k}u,$$

and

$$\theta(ST_{b_h,b_k}R^{\dagger}) = [1 \mapsto Rb_h \otimes Sb_k] = (\overline{R} \otimes S) \circ \theta(T_{b_h,b_k}).$$

This is the required naturality equation. The compactness equation follows similarly from the linearity of θ and the adjoint structure of **Hilb**.

In **LFR** an object is nuclear if and only if it is finite. Similarly, an object in **Hilb** is nuclear if and only if it is finite-dimensional.

Lemma 5.2.8. *In a nuclear ideal $\mathcal{N}$, the isomorphism ι satisfies $\overline{\iota} = \iota^{-1}$.*

Proof.

$$\begin{tikzcd}
\overline{I} \arrow[rrrr, "\overline{\iota}"] \arrow[drr, "\overline{\theta(id_I)}"] \arrow[ddd, "\overline{I}"'] & & & & I \arrow[ddd, "I"] \\
& & \overline{I}\otimes I \arrow[urr, "\overline{\rho}"'] \arrow[dr, "\iota^{-1}\otimes I"] \arrow[u, phantom, "(47)"] \arrow[rr, phantom, "(41)"] & & \\
& I \arrow[ur, "\theta(id_I)"] \arrow[ul, phantom, "(49)"] \arrow[dr, phantom, "(47)"] & & I\otimes I \arrow[uuur, "\lambda"'] \arrow[dr, "\rho"] \arrow[r, phantom, "(9)"] \arrow[dl, phantom, "\rho\text{ nat.}"] & \\
\overline{I} \arrow[rrrr, "\iota^{-1}"'] \arrow[ur, "\iota^{-1}"] \arrow[uurr, "\rho^{-1}"'] & & & & I
\end{tikzcd}$$

□

Since the isomorphism ι is unitary, we have the following corollary.

Corollary 5.2.9. *In a nuclear ideal, $\overline{\iota} = \iota^{-1} = \iota^{\dagger}$.*

Chapter 6

CPMs in a Nuclear Ideal

In this chapter, we generalize Selinger's CPM construction to the setting of nuclear ideals. This represents the main contribution of our thesis.

6.1 Positive Maps, Positive Matrices, and Trace Class Maps in a Nuclear Ideal

Positivity in a tensored $\dagger$-category $\mathcal{C}$ can be defined independently of any nuclear ideal structure on $\mathcal{C}$. We adopt Selinger's notion of positive maps.

Definition 6.1.1. A morphism $f : A \longrightarrow A$ in a tensored $\dagger$-category $\mathcal{C}$ is ***positive*** if there exists an object B and a morphism $g : A \longrightarrow B$ in $\mathcal{C}$ such that $f = g^\dagger g$.

If we restrict the definition and require the object B to be nuclear, we can no longer guarantee that identity maps are positive. This justifies our definition of positivity. However, we will be interested in the special case when the object B is indeed nuclear.

Definition 6.1.2. Let $\mathcal{N}(\mathcal{C})$ be a nuclear ideal. A morphism $f : A \longrightarrow A$ in $\mathcal{C}$ is ***$\mathcal{N}$-nuclear-positive*** or ***nuclear-positive*** if there exists a nuclear object B and a morphism $g : A \longrightarrow B$ in $\mathcal{C}$ such that $f = g^\dagger g$. In particular this implies that f is itself a nuclear morphism.

We have already determined the positive maps in **Hilb** and **LFR** in Examples 4.1.2 and 4.1.4. We now calculate the nuclear-positive maps in **Hilb** and **LFR**, relative to the usual nuclear ideals.

Example 6.1.3.
Let $R : A \to A$ be a nuclear-positive map in **LFR**, relative to the nuclear ideal of finite relations. Then R is nuclear and positive, i.e., R is finite, symmetric and partially reflexive. We claim that these are sufficient conditions for nuclear-positivity in **LFR**. Let $R : A \to A$ be a finite, symmetric partially reflexive relation. We take $B = R$ viewed as a (finite) set and define $S : A \to B$ by

$$S = \{\, (a, (a, a')) \mid (a, a') \in R \,\} \cup \{\, (a', (a, a')) \mid (a, a') \in R \,\}.$$

Then S is finite and hence locally finite and $R = S^\dagger S$ by the same argument we used in Example 4.1.3. Therefore the nuclear-positive maps in **LFR** are exactly the nuclear, positive maps.

One might ask if the nuclear-positive maps coincide with the nuclear, positive maps in general. The following example shows that this is not the case.

Example 6.1.4.
Let $T : H \to H$ be a nuclear-positive map in **Hilb**. Then there is a finite-dimensional Hilbert space K and a morphism $S : H \to K$ such that $T = S^\dagger S$. It follows that T has finite-rank in the sense that the range of T is finite. Note that being a finite-rank operator is stronger than being a Hilbert-Schmidt operator. Altogether then T is a positive, finite-rank (Hilbert-Schmidt) operator. Conversely, let $T : H \to H$ be a morphism in **Hilb** satisfying these conditions. We show that T is nuclear-positive. First, since T is positive, T has a positive square root $S : H \to H$ by Theorem 2.1.9. If we let $R(L)$ denote the range of a map L, then it is clear that $R(T) \subseteq R(S)$. In fact we claim that $R(S) = R(T)$. As constructed, the map S is the limit of a sequence $\{S_n\}$ of maps with $R(S_n) \subseteq R(T)$ for each n. Moreover, as $R(T)$ is a finite-dimensional subspace of H, $R(T)$ is closed. Thus $R(S) \subseteq R(T)$. Now let $P : H = R(S) \oplus R(S)^\perp \to R(S)$ denote the usual orthogonal projection. Clearly $P^\dagger$ is the inclusion map, and $T = S^\dagger S = S^\dagger P^\dagger P S = (PS)^\dagger PS$. Hence T is nuclear-positive. Therefore the nuclear-positive maps in **Hilb** are the positive finite-rank operators.

This means that a nuclear, positive map $T : H \to H$ in **Hilb** need not be nuclear-positive. This is because not all Hilbert-Schmidt operators are finite-dimensional. In fact each Hilbert-Schmidt operator $T : H \to H$ is the limit of a sequence of finite-dimensional operators on H (See Exercise 2.8.37 [18]).

Recall from Section 2.3 that a map $T : H \to H$ in **Hilb** is trace class if and only if it can be written as the composite of two Hilbert-Schmidt operators. We extend the notion of trace class maps to arbitrary nuclear ideals.

Definition 6.1.5. Let $\mathcal{N}(\mathcal{C})$ be a nuclear ideal and let $f : A \to A$ in $\mathcal{C}$. If there exists an object B and nuclear morphisms $g : A \to B$ and $h : B \to A$ such that $f = hg$, then f is called ***trace class*** and its ***trace*** $trf : I \to I$ is defined by

$$\begin{tikzcd}
I \arrow[r, "\theta(g)"] \arrow[drrr, "trf"'] & \overline{A} \otimes B \arrow[r, "\tau_{A,\overline{B}}"] & \overline{\overline{B}} \otimes A \arrow[r, "\overline{\theta(h)}^{\dagger}"] & \overline{I} \arrow[d, "\iota^{-1}"] \\
 & & & I
\end{tikzcd}$$

One must verify that the trace is independent of the decomposition. In both **Hilb** and **LFR** this is not an issue.

It follows easily from this definition that trace class maps $f : A \to A$ are closed under composition with arbitrary maps in $Hom_{\mathcal{C}}(A, A)$. Also, if A is nuclear then $f = id_A f = f id_A$ is trace class for all $f \in Hom_{\mathcal{C}}(A, A)$. In this case we get that $trf = \theta(id_A)^{\dagger} \circ (id_{\overline{A}} \otimes f) \circ \theta(id_A)$, which coincides with Selinger's trace for dagger compact closed categories.

We now adapt Lemma 4.1.6 to this situation.

Lemma 6.1.6. *Let $\mathcal{N}(\mathcal{C})$ be a nuclear ideal. We have the following properties.*
(a) If $f : A \to A$ is positive (nuclear-positive), then $h^{\dagger} \circ f \circ h : C \to C$ is positive (nuclear-positive), for any $h : C \to A$.
(b) For each object A, id_A is positive. Moreover, if A is nuclear then id_A is nuclear-positive.

(c) If $f : A \to A$ and $h : C \to C$ are positive (nuclear-positive), then $f \otimes h : A \otimes C \to A \otimes C$ is positive (nuclear-positive).
(d) If $f : A \to A$ is positive, then $f^\dagger = f$.
(e) If $f : A \to A$ is positive (nuclear-positive), then $\overline{f}$ is positive (nuclear-positive).
(f) If $f : A \to A$ is nuclear-positive, then $tr f : I \to I$ is positive.
(g) If $f, h : A \to A$ are nuclear-positive, then $tr(h \circ f) : I \to I$ is positive.

Proof. Most of the proof is identical to that of Lemma 4.1.6. There are modifications in parts (e), (f), and (g).
(e): If $f = g^\dagger \circ g$ then

$$\overline{f} = \overline{g^\dagger \circ g} = \overline{g^\dagger} \circ \overline{g} = \overline{g}^\dagger \circ \overline{g}.$$

(f): If $f : A \to A$ is nuclear-positive, then there is a nuclear object B and a map $g : A \to B$ such that $f = g^\dagger g$. In particular both g and $g^\dagger$ are nuclear so f is trace class. We have

$$\begin{aligned} tr f &= tr(g^\dagger \circ g) \\ &= \iota^{-1} \circ \overline{\theta(g^\dagger)}^\dagger \circ \tau \circ \theta(g) \\ &= \iota^{-1} \circ (\tau \circ \theta(g^{\dagger\dagger}) \circ \iota^{-1})^\dagger \circ \tau \circ \theta(g) \qquad \text{(by (49))} \\ &= \iota^{-1} \circ \iota \circ \theta(g)^\dagger \circ \tau^{-1} \circ \tau \circ \theta(g) \\ &= \theta(g)^\dagger \circ \theta(g). \end{aligned}$$

(g): Let $f, h : A \to A$ be nuclear-positive. Write $f = p^\dagger p$, where $p : A \to B$ and B is nuclear. Then

$$tr(hf) = tr(h(p^\dagger p)) = tr((hp^\dagger)p).$$

We claim that $tr(jk) = tr(kj)$ for all pairs of nuclear maps $j : X \to Y$ and $k : Y \to X$. Assuming this, we get $tr(hf) = tr(php^\dagger)$, which is positive be parts (a) and (f). We now establish the claimed trace equation as follows.

$$\begin{tikzcd}
I \arrow[rr, "\theta(k)"] \arrow[ddd, "\theta(j)"'] \arrow[dr, "\iota"] \arrow[ddddrrrr, "(\theta(j)^\dagger \circ \theta(k^\dagger))^\dagger" description] & & \overline{Y} \otimes X \arrow[rr, "\tau"] \arrow[d, phantom, "(49)"] & & \overline{\overline{X} \otimes Y} \arrow[ddd, "\overline{\theta(j)}^\dagger"] \\
& & \overline{I} \arrow[urr, "\overline{\theta(k^\dagger)}"] \arrow[ddrr, "\overline{\theta(j)^\dagger \circ \theta(k^\dagger)}"'] \arrow[rr, phantom, "\overline{(-)}\text{ func.}"] & & \\
& \dagger\text{ func.} & & & \\
\overline{X} \otimes Y \arrow[d, "\tau"'] \arrow[drrrr, "\theta(k^\dagger)^\dagger" description] \arrow[dr, phantom, "(49)"] & & & (43) & \overline{I} \arrow[d, "\iota^{-1}"] \\
\overline{\overline{Y} \otimes X} \arrow[rr, "\overline{\theta(k)}^\dagger"'] & & \overline{I} \arrow[rr, "\iota^{-1}"'] & & I
\end{tikzcd}$$

□

Definition 6.1.7. Let $\mathcal{N}(\mathcal{C})$ be a nuclear ideal and let $f : A \longrightarrow B$ be nuclear. The ***name*** or ***matrix*** of f is the map $\ulcorner f \urcorner \stackrel{\text{def}}{=} \theta(f) : I \longrightarrow \overline{A} \otimes B$. In the case that A is a nuclear object, we recover the usual matrix:

$$\begin{tikzcd}
I \arrow[r, "\theta(id_A)"] \arrow[dr, "\ulcorner f \urcorner"'] & \overline{A} \otimes A \arrow[d, "id_{\overline{A}} \otimes f"] \\
& \overline{A} \otimes B
\end{tikzcd}$$

A ***positive matrix*** is a morphism $\ulcorner f \urcorner : I \longrightarrow \overline{A} \otimes A$ that is the name of a positive map $f : A \longrightarrow A$. Similarly, a ***nuclear-positive matrix*** is a morphism $\ulcorner f \urcorner : I \longrightarrow \overline{A} \otimes A$ that is the name of a nuclear-positive map $f : A \longrightarrow A$.

6.2 Completely Positive Maps in a Nuclear Ideal

We now define the notion of a completely positive map in a tensored †-category equipped with a nuclear ideal.

Definition 6.2.1. Let $\mathcal{N}(\mathcal{C})$ be a nuclear ideal. A morphism $f : \overline{A} \otimes A \to \overline{B} \otimes B$ is ***completely positive*** if there exists a nuclear object C and a morphism $k : C \otimes A \to B$ such that the following diagram commutes:

$$\begin{tikzcd}
\overline{A} \otimes A \arrow[r, "\rho^{-1} \otimes A"] \arrow[drr, "f"'] & \overline{A} \otimes I \otimes A \arrow[r, "\overline{A} \otimes \theta(id_C) \otimes A"] & \overline{A} \otimes \overline{C} \otimes C \otimes A \arrow[d, "\overline{k} \otimes k"] \\
 & & \overline{B} \otimes B
\end{tikzcd} \tag{53}$$

Again there is an equivalent characterization of complete positivity.

Proposition 6.2.2. *The following are equivalent:*
(a) $f : \overline{A} \otimes A \to \overline{B} \otimes B$ *is completely positive.*
(b) There exists a nuclear object C *and a morphism* $h : A \to C \otimes B$ *such that*

$$\begin{tikzcd}
\overline{A} \otimes A \arrow[r, "\overline{h} \otimes h"] \arrow[drr, "f"'] & \overline{B} \otimes \overline{C} \otimes C \otimes B \arrow[r, "\overline{B} \otimes \theta(id_C)^{\dagger} \otimes B"] & \overline{B} \otimes I \otimes B \arrow[d, "\overline{B} \otimes \lambda"] \\
 & & \overline{B} \otimes B
\end{tikzcd}$$

Proof. Let $f : \overline{A} \otimes A \to \overline{B} \otimes B$ be completely positive. Then there is a nuclear object C and a morphism $k : C \otimes A \to B$ with

$$\begin{tikzcd}
\overline{A} \otimes A \arrow[r, "\rho^{-1} \otimes A"] \arrow[drr, "f"'] & \overline{A} \otimes I \otimes A \arrow[r, "\overline{A} \otimes \theta(id_C) \otimes A"] & \overline{A} \otimes \overline{C} \otimes C \otimes A \arrow[d, "\overline{k} \otimes k"] \\
 & & \overline{B} \otimes B
\end{tikzcd}$$

Since C is nuclear, id_C is nuclear and so is $\overline{id_C} = id_{\overline{C}}$. Hence $\overline{C}$ is nuclear. We define $h : A \to \overline{C} \otimes B$ to be the map

$$A \xrightarrow{\lambda^{-1}} I \otimes A \xrightarrow{\theta(id_C) \otimes A} \overline{C} \otimes C \otimes A \xrightarrow{\overline{C} \otimes k} \overline{C} \otimes B.$$

We use θ_C to denote $\theta(id_C)$ in the following commutative diagram and in subsequent diagrams.

$$
\begin{tikzcd}
\overline{A}\,A \arrow[rr, "\overline{\lambda^{-1}}\,\lambda^{-1}"] \arrow[ddr, "\overline{A}\,\lambda^{-1}"] \arrow[ddddd, "\rho^{-1}\,A"'] \arrow[dddr, phantom, "\text{coh.}"] & & \overline{A}\,\overline{I}\,I\,A \arrow[rr, "\overline{A}\,\overline{\theta_C}\,\theta_C\,A"] & & \overline{A}\,\overline{C}\,C\,\overline{C}\,C\,A \arrow[ddd, "\overline{k}\,C\,\overline{C}\,k"] \arrow[dddddddl, "id\,\theta^{\dagger}_{\overline{C}}\,id"] \\
 & & \overline{A}\,\overline{I}\,\overline{C}\,C\otimes A \arrow[urr, "\overline{A}\,\overline{\theta_C}\,id"] \arrow[ddr, phantom, "(49)"] & & \\
 & \overline{A}\,I\,A \arrow[dddl, "id"] \arrow[dd, phantom, "\otimes\ \text{func.}"] & \arrow[dl, phantom, "(40)"] & & \\
 & & & \overline{A}\,C\,\overline{C}\,\overline{C}\,C\,A \arrow[uuur, "\overline{A}\,\sigma\,id"] & \overline{B}\,C\,\overline{C}\,B \arrow[ddd, "\overline{B}\,\theta^{\dagger}_{\overline{C}}\,B"] \\
 & & \overline{A}\,I\,\overline{C}\,C\,A \arrow[uuu, "\overline{A}\,\iota\,id"'] \arrow[ur, "\overline{A}\,\theta_{\overline{C}}\,id"'] \arrow[ddddll, "\rho\,id"'] \arrow[ddddll, "\overline{A}\,\lambda\,id"] \arrow[dd, phantom, "\text{coh.}"] \arrow[ddr, phantom, "(51)"] & & \arrow[dd, phantom, "\otimes\ \text{func.}"] \\
\overline{A}\,I\,A \arrow[ddd, "\overline{A}\,\theta_C\,A"'] & & & & \\
 & & & & \overline{B}\,I\,B \arrow[dd, "\overline{B}\,\lambda"] \\
 & & & \overline{A}\,\overline{C}\,I\,C\,A \arrow[ur, "\overline{k}\,I\,k"] \arrow[dlll, "\overline{A}\,\rho\,id"'] \arrow[dlll, bend left, "id\,\lambda"] \arrow[dl, phantom, "\text{coh.}"] \arrow[dr, phantom, "\lambda\ \text{nat.}"] & \\
\overline{A}\,\overline{C}\,C\otimes A \arrow[rrrr, "\overline{k}\,k"'] \arrow[uuuuuuurr, "\overline{\lambda^{-1}}\,id"] & & & & \overline{B}\,B
\end{tikzcd}
$$

Hence (b) holds.

For the converse, we assume that there exists a nuclear object C and a morphism $h : A \to C \otimes B$ such that

$$\begin{tikzcd}
\overline{A} \otimes A \arrow[r, "\overline{h} \otimes h"] \arrow[drr, "f"'] & \overline{B} \otimes \overline{C} \otimes C \otimes B \arrow[r, "\overline{B} \otimes \theta(id_C)^\dagger \otimes B"] & \overline{B} \otimes I \otimes B \arrow[d, "\overline{B} \otimes \lambda"] \\
& & \overline{B} \otimes B
\end{tikzcd}$$

We define $k : \overline{C} \otimes A \to B$ to be the map

$$\overline{C} \otimes A \xrightarrow{\overline{C} \otimes h} \overline{C} \otimes C \otimes B \xrightarrow{\theta(id_C)^\dagger \otimes B} I \otimes B \xrightarrow{\lambda} B.$$

Now consider the following commutative diagram.

$$
\begin{tikzcd}
\overline{A}A \arrow[rr, "\rho^{-1}A"] \arrow[dd, "\overline{h}h"'] \arrow[dr, phantom, "\rho\text{ nat.}"] & & \overline{A}IA \arrow[rrr, "\overline{A}\theta_{\overline{C}}A"] \arrow[dr, "\overline{h}Ih"] \arrow[drr, phantom, "\otimes\text{ func.}"] & & & \overline{A}C\overline{C}A \arrow[dddd, "\overline{A}C\overline{C}h"] \\
& & & \overline{B}\overline{C}ICB \arrow[dddrr, "id\,\theta_{\overline{C}}\,id"] \arrow[dl, phantom, "(51)"] & & \\
\overline{B}\overline{C}CB \arrow[urrr, "\rho^{-1}id"] \arrow[dddr, "\overline{B}\rho^{-1}B"] \arrow[dddddd, "\overline{B}\theta^{\dagger}_{C}B"'] & & & & & \\
& & \overline{B}I\overline{C}CB \arrow[ull, "\overline{B}\lambda\,id"'] \arrow[ddrr, "\overline{B}\iota\,id"] & & \overline{B}C\overline{C}\overline{C}CB \arrow[ll, "\overline{B}\theta^{\dagger}_{\overline{C}}id"'] \arrow[ddl, phantom, "(49)"] & \\
& & & & & \overline{B}\overline{C}C\overline{C}CB \arrow[ul, "\overline{B}\sigma\,id"] \arrow[dl, "\overline{B}\theta^{\dagger}_{C}id"'] \arrow[dd, "\overline{B}\overline{\theta^{\dagger}_{C}}\theta^{\dagger}_{C}B"] \arrow[ddl, phantom, "\otimes\text{ func.}"] \\
& \overline{B}\overline{C}CIB \arrow[uur, "\overline{B}\sigma B"'] \arrow[dd, "\overline{B}\theta^{\dagger}_{C}id"] \arrow[uu, phantom, "\text{coh.}"] \arrow[dl, phantom, "\otimes\text{ func.}"] & & & \overline{B}\overline{I}\overline{C}CB \arrow[dr, "id\,\theta^{\dagger}_{C}B"'] \arrow[dll, phantom, "\sigma\text{ nat.}"] & \\
& & & & & \overline{B}\overline{I}IB \arrow[dd, "\overline{\lambda}\lambda"] \\
& \overline{B}IIB \arrow[rrr, "id\,\iota B"] \arrow[drrrr, "\rho\lambda"] \arrow[dr, phantom, "\text{coh.}"] & & & \overline{B}I\overline{I}B \arrow[ur, "\overline{B}\sigma B"] \arrow[dr, phantom, "(40)"] & \\
\overline{B}IB \arrow[ur, "\overline{B}\rho^{-1}B"'] \arrow[rrrrr, "\overline{B}\lambda"'] & & & & & \overline{B}B
\end{tikzcd}
$$

This shows that f is completely positive. $\square$

In the following examples we calculate the completely positive maps in **LFR** and **Hilb**, with respect to the nuclear ideals exhibited in Examples 5.2.6 and 5.2.7.

Example 6.2.3.
Let $R : A \times A \to B \times B$ be a completely positive map in **LFR**. Then there is a finite set C and a locally finite relation $K : C \times A \to B$ such that

$$((a_1, a_2), (b_1, b_2)) \in R \iff \exists\, c \in C \text{ with } \begin{cases} ((c, a_1), b_1) \in K \text{ and} \\ ((c, a_2), b_2) \in K. \end{cases}$$

Hence

$$R = \bigcup_{c \in C} \{\, ((a_1, a_2), (b_1, b_2)) \mid ((c, a_1), b_1), ((c, a_2), b_2) \in K \,\}.$$

For each $c \in C$, we define

$$X_c = \{\, (a, b) \mid ((c, a), b) \in K \,\},$$

which we view as a relation from A to B. Then each X_c is locally finite (since K is locally finite), and

$$R = \bigcup_{c \in C} X_c \otimes X_c = \bigcup_{c \in C} X_c \times X_c.$$

Conversely, suppose the locally finite relation $R : A \times A \to B \times B$ is of the form

$$R = \bigcup_{i=1}^{n} S_i \times S_i,$$

with $S_i : A \to B$ locally finite for each i. We show that R is completely positive. First let $C = \{\, S_i \times S_i \mid i = 1, \ldots, n \,\}$ viewed as a (finite) set. Then define $K : C \times A \to B$ to be the relation,

$$K = \bigcup_{i=1}^{n} \{\, ((S_i \times S_i, a), b) \mid (a, b) \in S_i \,\}.$$

It follows from the local finiteness of the S_i that K is locally finite. Moreover,

$$\begin{aligned} ((a_1, a_2), (b_1, b_2)) \in R &\iff \exists\, i \text{ with } ((a_1, a_2), (b_1, b_2)) \in S_i \times S_i \\ &\iff \exists\, i \text{ with } (a_1, b_1), (a_2, b_2) \in S_i \\ &\iff ((S_i \times S_i, a_1), b_1), ((S_i \times S_i, a_2), b_2) \in K. \end{aligned}$$

Hence R is completely positive.

Example 6.2.4.

Let $T : \overline{H} \times H \to \overline{K} \times K$ be a completely positive map in **Hilb**. Then there is a finite-dimensional Hilbert space M and a bounded linear map $S : M \otimes H \to K$ such that

$$\begin{tikzcd}
\overline{H} \otimes H \arrow[r, "\rho^{-1} \otimes H"] \arrow[drr, "T"'] & \overline{H} \otimes \mathbb{C} \otimes H \arrow[r, "\overline{H} \otimes \theta(id_M) \otimes H"] & \overline{H} \otimes \overline{M} \otimes M \otimes H \arrow[d, "\overline{S} \otimes S"] \\
 & & \overline{K} \otimes K
\end{tikzcd}$$

Let $\{ e_i \mid i \in I \}$ and $\{ f_1, \dots, f_n \}$ be orthonormal bases for H and M respectively. Then $\{ e_i \otimes e_j \mid i, j \in I \}$ is an orthonormal basis for $\overline{H} \otimes H$. Also, $id_M = \Sigma_{k=1}^{n} T_{f_k, f_k}$, where the maps T_{f_k, f_l} are the ones defined in Theorem 2.2.13. It follows by Remark 2.2.14 that $\theta(id_M)(1) = \Sigma_{k=1}^{n} f_k \otimes f_k$. So we have

$$T(e_i \otimes e_j) = \sum_{k=1}^{n} \overline{S}(e_i \otimes f_k) \otimes S(f_k \otimes e_j).$$

Now if we let S_k denote the restriction of S to the Hilbert space $span(f_k) \otimes H$ for $k = 1, \dots, n$, then each S_k can be viewed as a map $\hat{S}_k$ from H into K via the isomorphism λ. Moreover, for all basis elements $e_i \otimes e_j$ in $\overline{H} \otimes H$ we have

$$\begin{aligned}
\sum_{k=1}^{n} (\overline{\hat{S}_k} \otimes \hat{S}_k)(e_i \otimes e_j) &= \sum_{k=1}^{n} \overline{\hat{S}_k}(e_i) \otimes \hat{S}_k(e_j) \\
&= \sum_{k=1}^{n} \overline{S_k}(e_i \otimes f_k) \otimes S_k(f_k \otimes e_j) \\
&= T(e_i \otimes e_j).
\end{aligned}$$

Thus T is the finite sum

$$T = \sum_{k=1}^{n} \overline{\hat{S}_k} \otimes \hat{S}_k$$

with each $\hat{S}_k : H \to K$ a morphism in **Hilb**. For the converse, let $T : \overline{H} \times H \to \overline{K} \times K$. Suppose that there are bounded linear maps $R_1, \dots, R_n : H \to K$ such that $T = \sum_{k=1}^{n} \overline{R_k} \otimes R_k$. To show that T is completely positive we take M to be the finite-dimensional Hilbert space $\mathbb{C}^n$ with standard orthonormal basis $\{ f_1, \ldots, f_n \}$. We define $S : M \otimes H \to K$ to be the map determined by

$$f_k \otimes e_i \mapsto R_k(e_i).$$

Since each R_k is bounded, so is S. For all basis elements $e_i \otimes e_j$ in $\overline{H} \otimes H$ we have

$$\begin{aligned}
&(\overline{S} \otimes S) \circ (\overline{H} \otimes \theta(id_M) \otimes H) \circ (\rho^{-1} \otimes H)(e_i \otimes e_j) \\
&\qquad = (\overline{S} \otimes S) \circ (\overline{H} \otimes \theta(id_M) \otimes H)(e_i \otimes 1 \otimes e_j) \\
&\qquad = (\overline{S} \otimes S)(e_i \otimes (\Sigma_{k=1}^n f_k \otimes f_k) \otimes e_j) \\
&\qquad = \Sigma_{k=1}^n \overline{S}(e_i \otimes f_k) \otimes S(f_k \otimes e_j) \\
&\qquad = \Sigma_{k=1}^n \overline{R_k}(e_i) \otimes R_k(e_j) \\
&\qquad = T(e_i \otimes e_j).
\end{aligned}$$

Hence T is completely positive.

As in the case of dagger compact closed categories, we have the following proposition.

Proposition 6.2.5. *If $f : \overline{A} \otimes A \to \overline{B} \otimes B$ is completely positive, then $f = \overline{f}$.*

Proof.

$$\begin{tikzcd}
\overline{A}\otimes A \arrow[rrrr, "\rho^{-1}\otimes A"] \arrow[drr, "\overline{A}\otimes\lambda^{-1}"'] \arrow[ddd, "\overline{A}\otimes\overline{\rho^{-1}}"'] & & \arrow[d, phantom, "\text{coh.}"] & & \overline{A}\otimes I\otimes A \arrow[dd, "\overline{A}\otimes\theta(id_C)\otimes A"] \\
& \arrow[d, phantom, "(41)"] & \overline{A}\otimes I\otimes A \arrow[urr, "id"'] \arrow[ddll, "\overline{A}\otimes\iota\otimes A"'] & & \\
& \arrow[r, phantom, "(=)"] & & \arrow[r, phantom, "(49)"] & \overline{A}\otimes\overline{C}\otimes C\otimes A \arrow[d, "\overline{k}\otimes k"] \\
\overline{A}\otimes\overline{I}\otimes A \arrow[uuurrrr, "\overline{A}\otimes\iota^{-1}\otimes A"'] \arrow[rr, "\overline{A}\otimes\overline{\theta(id_C)}\otimes A"'] & & \overline{A}\otimes\overline{C}\otimes C\otimes A \arrow[urr, "id"] \arrow[rr, "\overline{k}\otimes k"'] & \arrow[u, phantom, "(=)"] & \overline{B}\otimes B
\end{tikzcd}$$

□

We will require the next lemma in the proof of our main theorem.

Lemma 6.2.6. *If $f:\overline{A}\otimes A\to\overline{B}\otimes B$ is completely positive, then so is $\sigma f\sigma$.*

Proof. Let f be completely positive with

$$\begin{tikzcd}
\overline{A}\otimes A \arrow[r, "\rho^{-1}\otimes A"] \arrow[drr, "f"'] & \overline{A}\otimes I\otimes A \arrow[r, "\overline{A}\otimes\theta(id_C)\otimes A"] & \overline{A}\otimes\overline{C}\otimes C\otimes A \arrow[d, "\overline{k}\otimes k"] \\
& & \overline{B}\otimes B
\end{tikzcd}$$

Since C is nuclear, so is $\overline{C}$. We claim that $\sigma f \sigma$ is the completely positive map

$$\begin{tikzcd}
A\otimes\overline{A} \arrow[r, "\rho^{-1}\otimes\overline{A}"] \arrow[drr, "\sigma f\sigma"'] & A\otimes I\otimes\overline{A} \arrow[r, "A\otimes\theta(id_{\overline{C}})\otimes\overline{A}"] & A\otimes C\otimes\overline{C}\otimes\overline{A} \arrow[d, "k\sigma\otimes\overline{k\sigma}"] \\
& & B\otimes\overline{B}
\end{tikzcd}$$

The following diagram establishes this claim:

$$\begin{tikzcd}
A\,\overline{A} \arrow[rr, "\sigma"] \arrow[dd, "\rho^{-1}\,\overline{A}"'] & & \overline{A}\,A \arrow[rr, "\rho^{-1}\,A"] \arrow[d, phantom, "\text{coh.}"] & & \overline{A}\,I\,A \arrow[dd, "\overline{A}\,\theta(idC)\,A"] \\
& & I\,\overline{A}\,A \arrow[urr, "\sigma\,A"] \arrow[dr, "\theta(id_C)\,id"] \arrow[rr, phantom, "\sigma\text{ nat.}"] & & \\
A\,I\,\overline{A} \arrow[urr, "\sigma"] \arrow[dr, "A\,\theta(id_C)\,\overline{A}"] \arrow[dd, "A\,\theta(id_{\overline{C}})\,\overline{A}"'] \arrow[rr, phantom, "\sigma\text{ nat.}"] & & & \overline{C}\,C\,\overline{A}\,A \arrow[r, "\sigma\,A"] & \overline{A}\,\overline{C}\,C\,A \arrow[dd, "\overline{k}\,k"] \\
\arrow[r, phantom, "(49)"] & A\,\overline{C}\,C\,\overline{A} \arrow[urr, "\sigma"] \arrow[dl, "A\,\sigma\,\overline{A}"] & & & \\
A\,C\,\overline{C}\,\overline{A} \arrow[d, "\sigma\,\overline{\sigma}"'] \arrow[d, bend left=60, "\sigma\,\sigma"] \arrow[rr, phantom, "\text{coh.}"] \arrow[d, phantom, shift left=4ex, "(42)"] & & & & \overline{B}\,B \arrow[d, "\sigma"] \arrow[dl, phantom, "\sigma\text{ nat.}"] \\
C\,A\,\overline{A}\,\overline{C} \arrow[rrrr, "k\,\overline{k}"'] \arrow[uuurrrr, bend left, "\sigma"] & & & & B\,\overline{B}
\end{tikzcd}$$

This completes the proof. $\square$

Remark 6.2.7. In a nuclear ideal $\mathcal{N}$, the maps $\theta(id_I)$ and $\theta(id_I)^\dagger$ are inverse.

Applying (45) and (46) to the map $f = id_I$ gives this result:

$$
\begin{tikzcd}
I \arrow[rr, "\theta(id_I)"] \arrow[dr, "\lambda^{-1}"'] & & \overline{I} \otimes I \\
& I \otimes I \arrow[ur, "\iota \otimes I"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
\overline{I} \otimes I \arrow[rr, "\theta(id_I)^\dagger"] \arrow[dr, "\iota^{-1} \otimes I"'] & & I \\
& I \otimes I \arrow[ur, "\lambda"'] &
\end{tikzcd}
$$

Proposition 6.2.8. *Let A be a nuclear object in a nuclear ideal $\mathcal{N}$ and let $f : \overline{I} \otimes I \to \overline{A} \otimes A$ be completely positive. Then*

$$I \xrightarrow{\cong} \overline{I} \otimes I \xrightarrow{f} \overline{A} \otimes A$$

is a positive matrix.

Proof. Let A be nuclear and let $f : \overline{I} \otimes I \to \overline{A} \otimes A$ be completely positive. Then there is a nuclear object C and a map $k : C \otimes I \to A$ such that

$$
\begin{tikzcd}
\overline{I} \otimes I \arrow[r, "\rho^{-1} \otimes I"] \arrow[drr, "f"'] & \overline{I} \otimes I \otimes I \arrow[r, "\overline{I} \otimes \theta(id_C) \otimes I"] & \overline{I} \otimes \overline{C} \otimes C \otimes I \arrow[d, "\overline{k} \otimes k"] \\
& & \overline{A} \otimes A
\end{tikzcd}
$$

We claim that

$$I \xrightarrow{\cong} \overline{I} \otimes I \xrightarrow{f} \overline{A} \otimes A$$

is the positive matrix

$$I \xrightarrow{\theta(id_A)} \overline{A} \otimes A \xrightarrow{id_{\overline{A}} \otimes k^\dagger} \overline{A} \otimes C \otimes I \xrightarrow{id_{\overline{A}} \otimes k} \overline{A} \otimes A.$$

To see this, consider the following diagram:

$$
\begin{tikzcd}
I \arrow[rr, "\theta(id_I)"] \arrow[dr, "\lambda^{-1}"'] \arrow[dddd, "\theta(id_A)"'] \arrow[rrrrdd, bend right, "\theta(id_{C\otimes I})"] \arrow[rrd, phantom, "\lambda\text{ nat.}"] & & \overline{I}\otimes I \arrow[rr, "\rho^{-1}\otimes I"] \arrow[dr, "\lambda^{-1}"'] \arrow[rrd, phantom, "\text{coh.}"] & & \overline{I}\otimes I\otimes I \arrow[dd, "\overline{I}\otimes\theta(id_C)\otimes I"] \\
& I\otimes I \arrow[rr, "I\otimes\theta(id_I)"] \arrow[dr, "\theta(id_C)\otimes\theta(id_I)"'] \arrow[rr, phantom, shift right=4ex, "\otimes\text{ func.}"] & & I\otimes\overline{I}\otimes I \arrow[ur, "\sigma\otimes I"'] \arrow[dl, "\theta(id_C)\otimes id"] \arrow[r, phantom, "\sigma\text{ nat.}"] & \\
\arrow[r, phantom, "(44)"] & & \overline{C}\otimes C\otimes\overline{I}\otimes I \arrow[rr, "\sigma\otimes I"] & & \overline{I}\otimes\overline{C}\otimes C\otimes I \arrow[dd, "\overline{k}\otimes k"] \arrow[ddll, "\overline{k}\otimes id"'] \arrow[dd, phantom, shift right=8ex, "\otimes\text{ func.}"] \\
\arrow[r, phantom, "(50)"] & & & & \\
\overline{A}\otimes A \arrow[rr, "\overline{A}\otimes k^{\dagger}"'] & & \overline{A}\otimes C\otimes I \arrow[rr, "\overline{A}\otimes k"'] & & \overline{A}\otimes A
\end{tikzcd}
$$

□

Corollary 6.2.9. *A completely positive map* $f : \overline{I}\otimes I \to \overline{I}\otimes I$ *satisfies* $f^{\dagger} = f$.

Proof. If $f : \overline{I}\otimes I \to \overline{I}\otimes I$ is completely positive, then $f\circ\theta(id_I) : I \to \overline{I}\otimes I$ is a positive matrix by Proposition 6.2.8. This means that

$$f\circ\theta(id_I) = (\overline{I}\otimes h)\circ\theta(id_I)$$

for some positive map $h : I \to I$. But by Remark 6.2.7, $\theta(id_I)$ is invertible so we have

$$f = \overline{I}\otimes h.$$

Recall that the positive map h satisfies $h^{\dagger} = h$ by Lemma 4.1.6 (d). Then we have

$$f^{\dagger} = (\overline{I}\otimes h)^{\dagger} = \overline{I}\otimes h^{\dagger} = \overline{I}\otimes h = f.$$

□

Unlike in the setting of dagger compact closed categories, the converse of Proposition 6.2.8 does not hold in general. With an additional assumption we get a partial converse.

Proposition 6.2.10. *Let A be a nuclear object and let $g : I \longrightarrow \overline{A} \otimes A$ be a nuclear-positive matrix in a nuclear ideal $\mathcal{N}$. Then*

$$\overline{I} \otimes I \xrightarrow{\cong} I \xrightarrow{g} \overline{A} \otimes A$$

is completely positive.

Proof. Let $g : I \longrightarrow \overline{A} \otimes A$ be a nuclear-positive matrix with A a nuclear object. Then there is a nuclear object B and a morphism $j : A \longrightarrow B$ such that

$$\begin{tikzcd}
I \arrow[rr, "\theta(id_A)"] \arrow[rrdd, "g"'] & & \overline{A} \otimes A \arrow[d, "\overline{A} \otimes j"] \\
& & \overline{A} \otimes B \arrow[d, "\overline{A} \otimes j^{\dagger}"] \\
& & \overline{A} \otimes A
\end{tikzcd}$$

We claim that

$$\overline{I} \otimes I \xrightarrow{\cong} I \xrightarrow{g} \overline{A} \otimes A$$

is the completely positive map

$$\begin{tikzcd}
\overline{I} \otimes I \arrow[r, "\rho^{-1} \otimes I"] & \overline{I} \otimes I \otimes I \arrow[r, "\overline{I} \otimes \theta(id_B) \otimes I"] & \overline{I} \otimes \overline{B} \otimes B \otimes I \arrow[d, "\overline{\rho} \otimes \rho"] \\
& & \overline{A} \otimes A \arrow[d, "\overline{j^{\dagger}} \otimes j^{\dagger}"] \\
& & \overline{B} \otimes B
\end{tikzcd}$$

The following commutative diagram establishes this claim:

$$
\begin{tikzcd}
\overline{I}\otimes I \arrow[rr, "\rho^{-1}\otimes I"] \arrow[dd, "\rho"'] \arrow[dr, phantom, "\text{coh.}"] & & \overline{I}\otimes I\otimes I \arrow[rr, "\overline{I}\otimes\theta(id_B)\otimes I"] \arrow[dl, "\overline{I}\otimes\rho"'] \arrow[dr, phantom, "\rho\text{ nat.}"] & & \overline{I}\otimes\overline{B}\otimes B\otimes I \arrow[dl, "id\otimes\rho"'] \arrow[dd, "\overline{\rho}\otimes\rho"] \\
& \overline{I}\otimes I \arrow[rr, "\overline{I}\otimes\theta(id_B)"] \arrow[dl, "\rho"'] \arrow[dd, "\iota^{-1}\otimes I"'] \arrow[dr, phantom, "\otimes\text{ func.}"] & & \overline{I}\otimes\overline{B}\otimes B \arrow[dl, "\iota^{-1}\otimes id"'] \arrow[dd, "\overline{\rho}\otimes B"] \arrow[dr, phantom, "\otimes\text{ func.}"] & \\
\overline{I} \arrow[dd, "\iota^{-1}"'] \arrow[dr, phantom, "\rho\text{ nat.}"] & & I\otimes\overline{B}\otimes B \arrow[dr, "\lambda\otimes B"] \arrow[r, phantom, "(41)"] \arrow[d, phantom, "\lambda\text{ nat.}"] & {} & \overline{B}\otimes B \arrow[ddd, "\overline{j^\dagger}\otimes j^\dagger"] \\
& I\otimes I \arrow[ur, "I\otimes\theta(id_B)"] \arrow[dl, "\lambda=\rho"'] & {} & \overline{B}\otimes B \arrow[ur, "id"'] \arrow[ddl, "j^\dagger\otimes B"'] \arrow[dr, phantom, "\otimes\text{ func.}"] & \\
I \arrow[urrr, "\theta(id_B)"'] \arrow[d, "\theta(id_A)"'] \arrow[drr, phantom, "(50)"] & & & & {} \\
\overline{A}\otimes A \arrow[rr, "\overline{A}\otimes j"'] & & \overline{A}\otimes B \arrow[rr, "\overline{A}\otimes j^\dagger"'] & & \overline{A}\otimes A
\end{tikzcd}
$$

□

Remark 6.2.11. In Proposition 6.2.8 we actually constructed a nuclear-positive matrix. Hence, if A is a nuclear object in a nuclear ideal $\mathcal{N}$, a completely positive map $f : \overline{I}\otimes I \to \overline{A}\otimes A$ is the same thing as a nuclear-positive matrix $g : I \to \overline{A}\otimes A$, modulo the isomorphism $\overline{I}\otimes I \cong I$.

Lemma 4.2.9 generalizes to our notion of completely positive maps. We have added part (e) as we will need this condition as well.

Lemma 6.2.12.

(a) The identity map on $\overline{A} \otimes A$ is completely positive.

(b) If $f : \overline{A} \otimes A \to \overline{B} \otimes B$ and $g : \overline{B} \otimes B \to \overline{C} \otimes C$ are completely positive, then so is $g \circ f : \overline{A} \otimes A \to \overline{C} \otimes C$.

(c) If $f : \overline{A} \otimes A \to \overline{B} \otimes B$ and $g : \overline{C} \otimes C \to \overline{D} \otimes D$ are completely positive, then so is $\overline{C} \otimes \overline{A} \otimes A \otimes C \xrightarrow{\cong} \overline{A} \otimes A \otimes \overline{C} \otimes C \xrightarrow{f \otimes g} \overline{B} \otimes B \otimes \overline{D} \otimes D \xrightarrow{\cong} \overline{D} \otimes \overline{B} \otimes B \otimes D$.

(d) If $f : A \to B$ is any morphism, then $\overline{f} \otimes f : \overline{A} \otimes A \to \overline{B} \otimes B$ is completely positive.

(e) If $f : \overline{A} \otimes A \to \overline{B} \otimes B$ is completely positive, then so is $f^{\dagger}$.

Proof.

(a) We take C to be the nuclear object I and the map $k = \lambda : I \otimes A \to A$. Then $id_{\overline{A} \otimes A}$ can be written as follows:

$$
\begin{tikzcd}
\overline{A} \otimes A \arrow[rrr, "\rho^{-1} \otimes A"] \arrow[dddd, "id"'] & & & \overline{A} \otimes I \otimes A \arrow[dll, "id"] \arrow[dd, "\overline{A} \otimes \theta(id_I) \otimes A"] \\
& \overline{A} \otimes I \otimes A \arrow[dd, "\overline{A} \otimes \lambda^{-1} \otimes A"'] \arrow[dl, phantom, "\text{coh.}"] & & \\
& & (45) & \overline{A} \otimes \overline{I} \otimes I \otimes A \arrow[dd, "\overline{\lambda} \otimes \lambda"] \\
& \overline{A} \otimes I \otimes I \otimes A \arrow[urr, "\overline{A} \otimes \iota \otimes id"] \arrow[drr, "\rho \otimes \lambda"'] & (40) & \\
\overline{A} \otimes A \arrow[rrr, "id"'] & & & \overline{A} \otimes A
\end{tikzcd}
$$

Hence $id_{\overline{A} \otimes A}$ is completely positive.

(b) Let $f : \overline{A} \otimes A \to \overline{B} \otimes B$ and $g : \overline{B} \otimes B \to \overline{C} \otimes C$ be completely positive. So we have nuclear objects D and E and maps $k : D \otimes A \to B$ and $j : E \otimes B \to C$ such that the following commute:

$$\begin{tikzcd}
\overline{A} \otimes A \arrow[r, "\rho^{-1} \otimes A"] \arrow[drr, "f"'] & \overline{A} \otimes I \otimes A \arrow[r, "\overline{A} \otimes \theta(id_D) \otimes A"] & \overline{A} \otimes \overline{D} \otimes D \otimes A \arrow[d, "\overline{k} \otimes k"] \\
& & \overline{B} \otimes B
\end{tikzcd}$$

$$\begin{tikzcd}
\overline{B} \otimes B \arrow[r, "\rho^{-1} \otimes B"] \arrow[drr, "g"'] & \overline{B} \otimes I \otimes B \arrow[r, "\overline{B} \otimes \theta(id_E) \otimes B"] & \overline{B} \otimes \overline{E} \otimes E \otimes B \arrow[d, "\overline{j} \otimes j"] \\
& & \overline{C} \otimes C
\end{tikzcd}$$

Since both D and E are nuclear, so is $E \otimes D$. We define m to be the composite $j \circ (E \otimes k) : E \otimes D \otimes A \to C$, and we claim that gf is the composite:

$$\begin{tikzcd}
\overline{A} \otimes A \arrow[r, "\rho^{-1} \otimes A"] \arrow[drr, "gf"'] & \overline{A} \otimes I \otimes A \arrow[r, "\overline{A} \otimes \theta(id_{E \otimes D}) \otimes A"] & \overline{A} \otimes \overline{D} \otimes \overline{E} \otimes E \otimes D \otimes A \arrow[d, "\overline{m} \otimes m"] \\
& & \overline{C} \otimes C
\end{tikzcd}$$

Replacing f, g and m in the above shows this:

$$
\begin{tikzcd}
\overline{A}\,A \arrow[rr, "\rho^{-1}A"] \arrow[dd, "\rho^{-1}A"'] \arrow[dr, phantom, "(=)"] & & \overline{A}\,I\,A \arrow[rr, "\overline{A}\,\theta_{E\otimes D}\,A"] \arrow[dl, "\overline{A}\,\lambda^{-1}\,A"'] \arrow[dr, phantom, "(44)"] & & \overline{A}\,\overline{D}\,\overline{E}\,E\,D\,A \arrow[dddd, "\overline{k}\,id\,k"] \arrow[ddddddll, "\overline{k}\,id\,k"] \\
& \overline{A}\,I\,I\,A \arrow[rr, "\overline{A}\,\theta_E\,\theta_D\,A"] \arrow[ddd, "id\,\theta_D\,A"'] \arrow[dr, phantom, "\otimes\ \text{func.}"] & & \overline{A}\,\overline{E}\,E\,\overline{D}\,D\,A \arrow[ur, "\overline{A}\,\sigma\,id"] & \\
\overline{A}\,I\,A \arrow[ur, "\overline{A}\,\lambda^{-1}\,A"] \arrow[dd, "\overline{A}\,\theta_D\,A"'] \arrow[dr, phantom, "\lambda\ \text{nat.}"] & & & & \\
& \overline{A}\,I\,\overline{D}\,D\,A \arrow[uurr, "\overline{A}\,\theta_E\,id"] \arrow[dd, "\overline{A}\,\sigma\,id"'] \arrow[dr, phantom, "\sigma\ \text{nat.}"] & & & \\
\overline{A}\,\overline{D}\,D\,A \arrow[ur, "\overline{A}\,\lambda^{-1}\,A"] \arrow[d, "\overline{k}\,k"'] \arrow[dr, "\rho^{-1}\,id"'] \arrow[r, phantom, "\text{coh.}"] & & & & \overline{B}\,\overline{E}\,E\,B \arrow[dd, "\overline{j}\,j"] \arrow[dl, phantom, "(=)"] \\
\overline{B}\,B \arrow[d, "\rho^{-1}B"'] \arrow[r, phantom, "\rho\ \text{nat.}"] & \overline{A}\,\overline{D}\,I\,D\,A \arrow[uuuuurrr, "id\,\theta_E\,id"] \arrow[dl, "\overline{k}\,I\,k"] \arrow[dr, phantom, "\otimes\ \text{func.}"] & & & \\
\overline{B}\,I\,B \arrow[rr, "\overline{B}\,\theta_E\,B"'] & & \overline{B}\,\overline{E}\,E\,B \arrow[rr, "\overline{j}\,j"'] & & \overline{C}\,C
\end{tikzcd}
$$

Hence gf is completely positive.

(c) Let $f : \overline{A} \otimes A \to \overline{B} \otimes B$ and $g : \overline{C} \otimes C \to \overline{D} \otimes D$ be completely positive. So we have nuclear objects E and F and maps $k : E \otimes A \to B$ and $j : F \otimes C \to D$ such that the following commute:

$$
\begin{tikzcd}
\overline{A} \otimes A \arrow[r, "\rho^{-1} \otimes A"] \arrow[rrd, "f"'] & \overline{A} \otimes I \otimes A \arrow[r, "\overline{A} \otimes \theta(id_E) \otimes A"] & \overline{A} \otimes \overline{E} \otimes E \otimes A \arrow[d, "\overline{k} \otimes k"] \\
& & \overline{B} \otimes B
\end{tikzcd}
$$

$$
\begin{tikzcd}
\overline{C} \otimes C \arrow[r, "\rho^{-1} \otimes C"] \arrow[rrd, "g"'] & \overline{C} \otimes I \otimes C \arrow[r, "\overline{C} \otimes \theta(id_F) \otimes C"] & \overline{C} \otimes \overline{F} \otimes F \otimes C \arrow[d, "\overline{j} \otimes j"] \\
& & \overline{D} \otimes D
\end{tikzcd}
$$

Since both E and F are nuclear, so is $F \otimes E$. We define $n : F \otimes E \otimes A \otimes C \to B \otimes D$ to be the composite

$$F \otimes E \otimes A \otimes C \xrightarrow{F \otimes k \otimes C} F \otimes B \otimes C \xrightarrow{\sigma \otimes C} B \otimes F \otimes C \xrightarrow{B \otimes j} B \otimes D.$$

We claim that the following commutes:

$$
\begin{tikzcd}
\overline{C}\,\overline{A}\,A\,C \arrow[r, "\rho^{-1} \otimes id"] \arrow[d, "\sigma \otimes C"'] & \overline{C}\,\overline{A}\,I\,A\,C \arrow[r, "id \otimes \theta(id_{F \otimes E}) \otimes id"] & \overline{C}\,\overline{A}\,\overline{E}\,\overline{F}\,F\,E\,A\,C \arrow[d, "\overline{n} \otimes n"] \\
\overline{A}\,A\,\overline{C}\,C \arrow[r, "f \otimes g"'] & \overline{B}\,B\,\overline{D}\,D \arrow[r, "\sigma \otimes D"'] & \overline{D}\,\overline{B}\,B\,D
\end{tikzcd}
$$

We establish this by replacing f, g, and n to get:

$$
\begin{tikzcd}
\overline{C}\,\overline{A}AC \arrow[rr, "\rho^{-1}id"] \arrow[ddd, "\sigma C"'] \arrow[dr, phantom, "\text{coh.}"] & & \overline{C}\,\overline{A}IAC \arrow[rr, "id\,\theta_{F\otimes E}\,id"] \arrow[dl, "id\,\lambda^{-1}\,id"] \arrow[dr, phantom, "(44)"] & & \overline{C}\,\overline{A}\,\overline{E}\,\overline{F}FEAC \arrow[ddd, "\overline{C}\,\overline{k}\,id\,k\,C"] \\
& \overline{C}\,\overline{A}IIAC \arrow[dr, "id\,\theta_F\,\theta_E\,id"] & & & \\
& & \overline{C}\,\overline{A}\,\overline{F}F\overline{E}EAC \arrow[uurr, "id\,\sigma\,id"] & \sigma\text{ nat.} & \\
\overline{A}A\overline{C}C \arrow[ddd, "\rho^{-1}A\rho^{-1}C"'] & & \sigma\text{ nat.} & & \overline{C}\,\overline{B}\,\overline{F}FBC \arrow[ddd, "\overline{C}\sigma\sigma C"] \\
& \overline{C}I\overline{A}IAC \arrow[uuu, "\overline{C}\sigma\,id"] \arrow[dddr, "\overline{C}\theta_F\overline{A}\theta_E\,id"] & \text{coh.} & & \\
& & & \overline{C}\,\overline{F}F\overline{B}BC \arrow[uur, "\overline{C}\sigma\,id"] & \text{coh.} \\
\overline{A}IA\overline{C}IC \arrow[uur, "\sigma C"] \arrow[dddd, "\overline{A}\theta_E\,id\,\theta_F\,C"'] & & & & \overline{C}\,\overline{F}\,\overline{B}BFC \arrow[dddd, "\overline{j}\,id\,j"] \\
& \sigma\text{ nat.} & \overline{C}\,\overline{F}F\overline{A}\,\overline{E}EAC \arrow[uuuuu, "\overline{C}\sigma\,id"] \arrow[uuuuuuurr, "\overline{C}\sigma\,id"] \arrow[uur, "id\,\overline{k}\,k\,C"] & & \\
& & \sigma\text{ nat.} & \overline{B}B\overline{C}\,\overline{F}FC \arrow[uuu, "\sigma C"] \arrow[uur, "\sigma\,id"] \arrow[ddl, "id\,\overline{j}\,j"] & \\
& & \otimes\text{ func.} & \sigma\text{ nat.} & \\
\overline{A}\,\overline{E}EA\overline{C}\,\overline{F}FC \arrow[uuurr, "\sigma C"] \arrow[uurrr, "\overline{k}\,k\,id"] \arrow[rr, "\overline{k}\,k\,\overline{j}\,j"'] & & \overline{B}B\overline{D}D \arrow[rr, "\sigma D"'] & & \overline{D}\,\overline{B}BD
\end{tikzcd}
$$

(d) Let $f : A \to B$. We take $C = I$ and $k : I \otimes A \to B$ to be the composite $f \circ \lambda_A$.

Then we have

$$
\begin{tikzcd}
\overline{A}\otimes A \arrow[r, "\rho^{-1}\otimes A"] \arrow[rrrdd, "id"] \arrow[rrrddd, "\overline{f}\otimes f"'] & \overline{A}\otimes I\otimes A \arrow[rr, "\overline{A}\otimes\theta_I\otimes A"] \arrow[dr, "\overline{A}\otimes\lambda^{-1}\otimes A"'] \arrow[d, phantom, "\text{coh.}"] & \arrow[d, phantom, "(45)"] & \overline{A}\otimes\overline{I}\otimes I\otimes A \arrow[dd, "\overline{\lambda}\otimes\lambda"] \\
 & & \overline{A}\otimes I\otimes I\otimes A \arrow[ur, "\overline{A}\,\iota\, id"] \arrow[dr, "\rho\otimes\lambda"] \arrow[r, phantom, "(40)"] & \\
 & & \arrow[d, phantom, "(=)"] & \overline{A}\otimes A \arrow[d, "\overline{f}\otimes f"] \\
 & & & \overline{B}\otimes B
\end{tikzcd}
$$

as required.

(e) Let $f : A \to B$ be a completely positive map of the form

$$
\begin{tikzcd}
\overline{A}\otimes A \arrow[r, "\rho^{-1}\otimes A"] \arrow[rrd, "f"'] & \overline{A}\otimes I\otimes A \arrow[r, "\overline{A}\otimes\theta(id_C)\otimes A"] & \overline{A}\otimes\overline{C}\otimes C\otimes A \arrow[d, "\overline{k}\otimes k"] \\
 & & \overline{B}\otimes B
\end{tikzcd}
$$

Then $f^\dagger$ is the map

$$\begin{tikzcd}
\overline{B}\otimes B \arrow[r, "\overline{k}^\dagger \otimes k^\dagger"] \arrow[drr, "f^\dagger"'] & \overline{A}\otimes\overline{C}\otimes C\otimes A \arrow[r, "\overline{A}\otimes\theta(id_C)^\dagger\otimes A"] & \overline{A}\otimes I\otimes A \arrow[d, "(\rho^{-1})^\dagger\otimes A"] \\
& & \overline{A}\otimes A
\end{tikzcd}$$

which we can rewrite as

$$\begin{tikzcd}
\overline{B}\otimes B \arrow[r, "\overline{k^\dagger} \otimes k^\dagger"] \arrow[drr, "f^\dagger"'] & \overline{A}\otimes\overline{C}\otimes C\otimes A \arrow[r, "\overline{A}\otimes\theta(id_C)^\dagger\otimes A"] & \overline{A}\otimes I\otimes A \arrow[d, "\overline{A}\otimes\lambda"] \\
& & \overline{A}\otimes A
\end{tikzcd}$$

Hence $f^\dagger$ is completely positive by Proposition 6.2.2. □

Lemma 6.2.13. *If $f : \overline{A}\otimes A \to \overline{B}\otimes B$ is completely positive with A and B nuclear objects and $g : I \to \overline{A}\otimes A$ is a nuclear-positive matrix, then $fg : I \to \overline{B}\otimes B$ is a nuclear-positive matrix.*

Proof. Let A and B be nuclear objects. Suppose $f : \overline{A}\otimes A \to \overline{B}\otimes B$ is completely positive and $g : I \to \overline{A}\otimes A$ is a nuclear-positive matrix. Then by Proposition 6.2.10,

$$\overline{I}\otimes I \xrightarrow{\cong} I \xrightarrow{g} \overline{A}\otimes A$$

is completely positive. Hence

$$\overline{I}\otimes I \xrightarrow{\cong} I \xrightarrow{g} \overline{A}\otimes A \xrightarrow{f} \overline{B}\otimes B$$

is also completely positive by Lemma 6.2.12 (b). Then by Proposition 6.2.8,

$$I \xrightarrow{g} \overline{A}\otimes A \xrightarrow{f} \overline{B}\otimes B$$

is a positive matrix. In fact, fg is a nuclear-positive matrix by Remark 6.2.11. □

Lemma 6.2.14. *Let $\mathcal{N}$ be a nuclear ideal and let $f : \overline{A} \otimes A \to \overline{B} \otimes B$ with A and B nuclear. Then f is completely positive if and only if $id_{\overline{C}} \otimes f \otimes id_C$ preserves nuclear-positive matrices for all nuclear objects $C \in \mathcal{C}$, i.e, for all nuclear-positive matrices $M : I \to \overline{C} \otimes \overline{A} \otimes A \otimes C$, the map*

$$I \xrightarrow{M} \overline{C} \otimes \overline{A} \otimes A \otimes C \xrightarrow{id_{\overline{C}} \otimes f \otimes id_C} \overline{C} \otimes \overline{B} \otimes B \otimes C$$

is a nuclear-positive matrix.

The proof is similar to that of Lemma 4.2.11.

6.3 A CPM Construction for Nuclear Ideals

At this point we have all the tools necessary to extend Selinger's CPM construction to the setting of nuclear ideals.

Definition 6.3.1. (**CPM Construction for Nuclear Ideals**) Let $\mathcal{N}$ be a nuclear ideal for a tensored †-category $\mathcal{C}$. We define a new category $\mathbf{CPM}_{\mathcal{N}}(\mathcal{C})$ whose objects are the same as the objects of $\mathcal{C}$. A morphism $f : A \to B$ in $\mathbf{CPM}_{\mathcal{N}}(\mathcal{C})$ is a completely positive map $f : \overline{A} \otimes A \to \overline{B} \otimes B$ in $\mathcal{C}$. Composition of morphisms is as in $\mathcal{C}$.

Lemma 6.2.12 (a) guarantees that $\mathbf{CPM}_{\mathcal{N}}(\mathcal{C})$ has all identity maps, and part (b) ensures that composition in $\mathbf{CPM}_{\mathcal{N}}(\mathcal{C})$ is well-defined. Moreover, $\mathbf{CPM}_{\mathcal{N}}(\mathcal{C})$ inherits the identity and associativity laws from $\mathcal{C}$. Hence $\mathbf{CPM}_{\mathcal{N}}(\mathcal{C})$ is a well-defined category. Furthermore, Lemma 6.2.12 (d) yields a functor $F : \mathcal{C} \to \mathbf{CPM}_{\mathcal{N}}(\mathcal{C})$ defined by:

$$F(A) = A \quad \text{and} \quad F(f) = \overline{f} \otimes f.$$

Functoriality is easy to check; namely,

$$F(id_A) = \overline{id_A} \otimes id_A = id_{\overline{A}} \otimes id_A = id_{\overline{A} \otimes A}$$

for all $A \in \mathbf{CPM}_{\mathcal{N}}(\mathcal{C})$, and for morphisms $f : A \to B$ and $g : B \to C$ in $\mathbf{CPM}_{\mathcal{N}}(\mathcal{C})$ we have

$$F(gf) = \overline{gf} \otimes gf = \overline{g}\overline{f} \otimes gf = (\overline{g} \otimes g) \circ (\overline{f} \otimes f) = F(g) \circ F(f).$$

Theorem 6.3.2. *Let $\mathcal{N}$ be a nuclear ideal for a tensored $\dagger$-category $\mathcal{C}$. Then $\mathbf{CPM}_{\mathcal{N}}(\mathcal{C})$ is also a tensored $\dagger$-category and the functor $F : \mathcal{C} \to \mathbf{CPM}_{\mathcal{N}}(\mathcal{C})$ preserves the tensored dagger structure.*

Proof.

Part I: $\mathbf{CPM}_{\mathcal{N}}(\mathcal{C})$ is a tensored $\dagger$-category.

The category $\mathbf{CPM}_{\mathcal{N}}(\mathcal{C})$ inherits a tensor product on objects from $\mathcal{C}$; on morphisms it is given by Lemma 6.2.12 (c). We check that this defines a functor $\otimes$: $\mathbf{CPM}_{\mathcal{N}}(\mathcal{C}) \times \mathbf{CPM}_{\mathcal{N}}(\mathcal{C}) \to \mathbf{CPM}_{\mathcal{N}}(\mathcal{C})$. First, for objects A and B in $\mathbf{CPM}_{\mathcal{N}}(\mathcal{C})$ we need $id_A \otimes id_B = id_{A\otimes B}$ in $\mathbf{CPM}_{\mathcal{N}}(\mathcal{C})$. This means that in $\mathcal{C}$ we need

$$\begin{tikzcd}
\overline{B}\otimes\overline{A}\otimes A\otimes B \arrow[r, "\sigma\otimes id_B"] \arrow[drr, "id_{\overline{B}}\otimes id_{\overline{A}}\otimes id_A\otimes id_B"'] & \overline{A}\otimes A\otimes\overline{B}\otimes B \arrow[r, "id_{\overline{A}}\otimes id_A\otimes id_{\overline{B}}\otimes id_B"] & \overline{A}\otimes A\otimes\overline{B}\otimes B \arrow[d, "\sigma\otimes id_B"] \\
 & & \overline{B}\otimes\overline{A}\otimes A\otimes B
\end{tikzcd}$$

which holds by naturality of σ in $\mathcal{C}$. Secondly, we need to show that $gf \otimes kh = (g\otimes k)\circ(f\otimes h)$ for all maps $f : A \to B$, $g : B \to C$, $h : D \to E$, and $k : E \to F$ in

$\mathbf{CPM}_{\mathcal{N}}(\mathcal{C})$. Thus in $\mathcal{C}$ we need the following

$$\begin{tikzcd}
\overline{D}\otimes\overline{A}\otimes A\otimes D \arrow[rrr, "\sigma\otimes D"] \arrow[d, "\sigma\otimes D"'] \arrow[dr, phantom, "(=)"] & & & \overline{A}\otimes A\otimes\overline{D}\otimes D \arrow[dd, "gf\otimes kh"] \\
\overline{A}\otimes A\otimes\overline{D}\otimes D \arrow[urrr, "id"'] \arrow[d, "f\otimes h"'] \arrow[ddddrr, "gf\otimes kh"] & & (=) & \\
\overline{B}\otimes B\otimes\overline{E}\otimes E \arrow[d, "\sigma\otimes E"'] \arrow[dd, bend left, "id"] \arrow[d, phantom, shift left=8, "(=)"] & & & \overline{C}\otimes C\otimes\overline{F}\otimes F \arrow[dd, "\sigma\otimes F"] \\
\overline{E}\otimes\overline{B}\otimes B\otimes E \arrow[d, "\sigma\otimes E"'] & \otimes\text{ func. on }\mathcal{C} & & (=) \\
\overline{B}\otimes B\otimes\overline{E}\otimes E \arrow[rr, "g\otimes k"'] & & \overline{C}\otimes C\otimes\overline{F}\otimes F \arrow[uur, "id"] \arrow[r, "\sigma\otimes F"'] & \overline{F}\otimes\overline{C}\otimes C\otimes F
\end{tikzcd}$$

Hence $\otimes$ is indeed a functor.

The tensor unit I from $\mathcal{C}$ is also the tensor unit for $\mathbf{CPM}_{\mathcal{N}}(\mathcal{C})$. The isomorphisms α, λ, ρ, and σ are given by the images under F of the respective maps in $\mathcal{C}$. For each of these, naturality follows directly from naturality in $\mathcal{C}$ and the tensored dagger equations (39) - (42). For example, for naturality of σ in $\mathbf{CPM}_{\mathcal{N}}(\mathcal{C})$ we need

$$\begin{tikzcd}
A\otimes B \arrow[r, "\sigma"] \arrow[d, "f\otimes g"'] & B\otimes A \arrow[d, "g\otimes f"] \\
B'\otimes A' \arrow[r, "\sigma"'] & A'\otimes B'
\end{tikzcd}$$

for all $f : A \to A'$ and $g : B \to B'$. This means that in $\mathcal{C}$ we need the following

$$\begin{tikzcd}
\overline{B}\otimes\overline{A}\otimes A\otimes B \arrow[rr, "\overline{\sigma}\otimes\sigma"] \arrow[dr] \arrow[dd, "\sigma\otimes B"'] & & \overline{A}\otimes\overline{B}\otimes B\otimes A \arrow[rr, "\sigma\otimes A"] \arrow[d, phantom, "(42)"] & & \overline{B}\otimes B\otimes\overline{A}\otimes A \arrow[dd, "g\otimes f"] \\
\text{coh.} & {} \arrow[ur, "\sigma\otimes\sigma"'] & {} & & \\
\overline{A}\otimes A\otimes\overline{B}\otimes B \arrow[uurrrr, "\sigma"] \arrow[dd, "f\otimes g"'] & & \sigma \text{ nat. in } \mathcal{C} & & \overline{B'}\otimes B'\otimes\overline{A'}\otimes A' \arrow[dd, "\sigma\otimes A'"] \\
& & & {} \arrow[dr] \arrow[d, phantom, "(42)"] & \text{coh.} \\
\overline{A'}\otimes A'\otimes\overline{B'}\otimes B' \arrow[rr, "\sigma\otimes B'"'] \arrow[uurrrr, "\sigma"] & & \overline{B'}\otimes\overline{A'}\otimes A'\otimes B' \arrow[rr, "\overline{\sigma}\otimes\sigma"'] \arrow[ur, "\sigma\otimes\sigma"] & {} & \overline{A'}\otimes\overline{B'}\otimes B'\otimes A'
\end{tikzcd}$$

Additionally, the five symmetric monoidal category equations for $\mathbf{CPM}_{\mathcal{N}}(\mathcal{C})$ follow from these equations in $\mathcal{C}$ combined with the equations (39) - (42). Therefore $\mathbf{CPM}_{\mathcal{N}}(\mathcal{C})$ is symmetric monoidal.

The category $\mathbf{CPM}_{\mathcal{N}}(\mathcal{C})$ inherits the dagger functor from $\mathcal{C}$. Lemma 6.2.12 (e) ensures that $(-)^\dagger : \mathbf{CPM}_{\mathcal{N}}(\mathcal{C}) \to \mathbf{CPM}_{\mathcal{N}}(\mathcal{C})$ is well-defined on maps. As well, we require the isomorphisms α, λ, ρ, and σ to be unitary. This follows easily from the fact that each is unitary in $\mathcal{C}$ in conjunction with the functoriality of $\overline{(-)}$ on $\mathcal{C}$. Consider λ for example. In $\mathcal{C}$ we have

$$(\overline{\lambda}^\dagger\otimes\lambda^\dagger)\circ(\overline{\lambda}\otimes\lambda) = (\overline{\lambda}^\dagger\circ\overline{\lambda})\otimes(\lambda^\dagger\circ\lambda) = (\overline{\lambda^\dagger}\circ\overline{\lambda})\otimes(\lambda^\dagger\circ\lambda) = (\overline{\lambda^\dagger\circ\lambda})\otimes(\lambda^\dagger\circ\lambda) = \overline{id}\otimes id = id,$$

and similarly

$$(\overline{\lambda}\otimes\lambda)\circ(\overline{\lambda}^\dagger\otimes\lambda^\dagger) = id.$$

Therefore we have the required equations $\lambda^\dagger\circ\lambda = id$ and $\lambda\circ\lambda^\dagger = id$ in $\mathbf{CPM}_{\mathcal{N}}(\mathcal{C})$.

The requirement that $(f \otimes g)^\dagger = f^\dagger \otimes g^\dagger$ for all maps f and g in $\mathbf{CPM}_{\mathcal{N}}(\mathcal{C})$ is a consequence of the same equation in $\mathcal{C}$.

Finally, we need a covariant conjugate functor $\overline{(-)}$ on $\mathbf{CPM}_{\mathcal{N}}(\mathcal{C})$. For $A \in \mathbf{CPM}_{\mathcal{N}}(\mathcal{C})$ we define $\overline{A}$ as in $\mathcal{C}$. For a morphism $f : A \to B$ in $\mathbf{CPM}_{\mathcal{N}}(\mathcal{C})$, we need a morphism $\overline{f} : \overline{A} \to \overline{B}$ in $\mathbf{CPM}_{\mathcal{N}}(\mathcal{C})$, i.e., a completely positive map $\overline{f} : A \otimes \overline{A} \to B \otimes \overline{B}$ in $\mathcal{C}$. By Lemma 6.2.6 we may take $\overline{f} = \sigma f \sigma$, where $f : \overline{A} \otimes A \to \overline{B} \otimes B$ is the completely positive map in $\mathcal{C}$. Note that by Proposition 6.2.5, $\overline{f} = \sigma f \sigma = \sigma \overline{f} \sigma$.

For functoriality we need $\overline{id_A} = id_{\overline{A}}$, for all $A \in \mathbf{CPM}_{\mathcal{N}}(\mathcal{C})$, and $\overline{gf} = \overline{g}\overline{f}$, for all maps $f : A \to B$ and $g : B \to C$ in $\mathbf{CPM}_{\mathcal{N}}(\mathcal{C})$. This is an easy computation:

$$\overline{id_A} = \sigma(id_{\overline{A}} \otimes id_A)\sigma = id_A \otimes id_{\overline{A}} = id_{\overline{A}}\ ;$$

$$\overline{gf} = \sigma g f \sigma = \sigma g \sigma \sigma f \sigma = \overline{g}\overline{f}.$$

Next we show that the conjugate functor commutes with the dagger functor:

$$\overline{f}^\dagger = (\sigma f \sigma)^\dagger = \sigma^\dagger f^\dagger \sigma^\dagger = \sigma f^\dagger \sigma = \overline{f^\dagger}.$$

The necessary natural isomorphisms

- $\overline{\overline{A}} \cong A$
- $\tau : \overline{A} \otimes \overline{B} \xrightarrow{\cong} \overline{B \otimes A}$
- $\iota : I \xrightarrow{\cong} \overline{I}$

are given by the images under F of the respective isomorphisms in $\mathcal{C}$.

It remains to verify the tensored $\dagger$-category equations (39) - (43). The first four equations hold easily since they hold in $\mathcal{C}$. We verify the nontrivial equation (43). Let $f : I \to I$ in $\mathbf{CPM}_{\mathcal{N}}(\mathcal{C})$. We must show that the following diagram commutes:

$$\begin{tikzcd}
I \arrow[r, "f^\dagger"] \arrow[d, "\iota"'] & I \arrow[d, "\iota"] \\
\overline{I} \arrow[r, "\overline{f}"'] & \overline{I}
\end{tikzcd}$$

Thus we need the following diagram in $\mathcal{C}$:

$$\begin{tikzcd}
\overline{I}\otimes I \arrow[rrr, "f^\dagger"] \arrow[d, "\overline{\iota}\otimes\iota"'] & & & \overline{I}\otimes I \arrow[d, "\overline{\iota}\otimes\iota"] \\
I\otimes\overline{I} \arrow[r, "\sigma"'] & \overline{I}\otimes I \arrow[r, "f"'] & \overline{I}\otimes I \arrow[r, "\sigma"'] & I\otimes\overline{I}
\end{tikzcd}$$

Since $f : \overline{I}\otimes I \to \overline{I}\otimes I$ is completely positive, we have $f^\dagger = f$ by Corollary 6.2.9. So it suffices to prove that $\sigma\circ(\overline{\iota}\otimes\iota) = id$. This is shown below

$$\begin{tikzcd}
\overline{I}\otimes I \arrow[rr, "\overline{\iota}\otimes\iota"] \arrow[rr, phantom, "\otimes\text{ func.}" below, shift right=6ex] \arrow[dr, "\overline{\iota}\otimes I"'] \arrow[ddd, "id"'] \arrow[ddd, phantom, "(=)" right, shift left=6ex] & & I\otimes\overline{I} \arrow[ddd, "\sigma"] \arrow[ddd, phantom, "\sigma\text{ nat.}" left, shift right=6ex] \\
& I\otimes I \arrow[ur, "I\otimes\iota"'] \arrow[d, "id=\sigma"] & \\
& I\otimes I \arrow[dr, "\iota\otimes I"] & \\
\overline{I}\otimes I \arrow[ur, "\overline{\iota}\otimes I"] \arrow[rr, "id"'] \arrow[rr, phantom, "\text{Lemma 5.2.8}" above, shift left=6ex] & & \overline{I}\otimes I
\end{tikzcd}$$

Hence $\mathbf{CPM}_{\mathcal{N}}(\mathcal{C})$ is a tensored $\dagger$-category.

Part II: F preserves the tensored dagger structure.

The functor $F : \mathcal{C} \to \mathbf{CPM}_{\mathcal{N}}(\mathcal{C})$ is the identity on objects. This means that F is strict in the following sense:

- $F(A\otimes B) = F(A)\otimes F(B)$;
- $F(I) = I$;
- $F(A^\dagger) = F(A)^\dagger$;
- $F(\overline{A}) = \overline{F(A)}$.

Moreover, the structural isomorphisms in $\mathbf{CPM}_{\mathcal{N}}(\mathcal{C})$ are given by the images under F of the respective maps in $\mathcal{C}$ by construction. That is to say, $\tau = F(\tau)$ and $\iota = F(\iota)$. $\square$

We can apply our CPM construction to the tensored †-categories **LFR** and **Hilb** equipped with the usual nuclear ideals.

Example 6.3.3.
The category $\mathbf{CPM}_{\mathcal{N}}(\mathbf{LFR})$ has sets as objects and completely positive locally finite relations as morphisms. We saw in Example 6.2.3 that these morphisms can be viewed as finite unions of certain locally finite relations.

Example 6.3.4.
In the category $\mathbf{CPM}_{\mathcal{N}}(\mathbf{Hilb})$, the objects are Hilbert spaces and the morphisms are completely positive maps, as characterized in Example 6.2.4.

These are two new examples of tensored †-categories which should prove to be interesting to study in the future.

Chapter 7

Conclusions

The theory of completely positive maps in a nuclear ideal would benefit if accompanied by a graphical language. In [26], Selinger expresses his theory graphically. He proves that his language is sound and complete with respect to dagger compact closed categories. In doing so, he deals with coherence issues once and for all, and consequently his proofs are greatly simplified. I intend to develop a similar language appropriate for my theory. Selinger's graphical language should provide a suitable starting point.

It would be interesting to extend the Abramsky-Coecke formalism for abstract quantum mechanics [3] to the infinite-dimensional setting using nuclear ideals. In the original formulation, completely positive maps played an important role. It is my expectation that my theory of CPMs will play a similar role. Such work could lead to further models of Selinger's language for quantum programming [27]. Furthermore, the setting of nuclear ideals would give infinite-dimensional Hilbert spaces a more substantial part in these theories of quantum programming.

Although the identity map on an infinite-dimensional Hilbert space is not nuclear, there is an evident sense in which it is approximated by nuclear maps. Indeed it is approximated by finite-dimensional projections. This idea can be formalized using *shape theory* [10]. This was carried out for general nuclear ideals in [6]. I would like to develop a shape-theoretic procedure of this nature for the CPM setting.

Nuclear ideals have also had applications to other areas of mathematics such as in the field of vertex algebras [9]. It would be interesting to see if the theory of CPMs that I have developed has any applications there [7].

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