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FREE VIBRATION ANALYSIS OF NON-RECTANGULAR
QUADRILATERAL PLATES

by
H.T. SALIBA

Thesis presented to the School of Graduate Studies
as partial fulfillment of the requirements
for the degree of Ph.D. in
Mechanical Engineering

UNIVERSITY OF OTTAWA
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Henri T. Saliba, Ottawa, Canada, 1986.

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ABSTRACT

Through an extension of Gorman's superposition techniques, a detailed investigation of the title problem is undertaken. In Part A, a critical assesment of the available literature dealing with this problem is provided. The classical theory of thin plate motion is then briefly discussed in Chapter 1 along with the method of superposition as it relates to dynamic problems of rectangular plates. In Part B, the problem of the non-rectangular quadrilateral plates is then discussed in Chapter 2 where it is shown how any polygonal plate can be divided into two sets of rectangular and right angle triangular regions. Also in Chapter 2, the basic building blocks used throughout this investigation are studied. Consequently, solutions for the triangular and rectangular elements mentioned above are given. It is then shown how these solutions are combined to arrive at a solution to the problem at hand. It is in Part C that this technique is applied, in Chapter 3 to the solution of the simply supported symmetrical trapezoidal plate, and in Chapter 4 to the fully clamped plate, resulting in a highly accurate analytical solution for these important engineering problems, and therefore establishing a systematic approach to the solution of a very large family of non-rectangular quadrilateral plates. Numerical results along with some modal shapes are also provided in Chapter 5, and compared with previously available data where good agreement is shown. Excellent agreement with experimental results, in the case of the fully clamped plate is reported. Finally, a brief but comprehensive discussion on the validity and accuracy of the technique is provided in Part D. Some discussion on the effect of rotatory inertia and shear on the flexural motion of plates is briefly given along with few words on internal damping effects.

LIST OF SYMBOLS

a	base length of the triangular part of plate(x direction).
a'	inclined side length of the quadrilateral plate(x' direction).
a_r	side length of the rectangular part of the plate divided by two(x direction).
b	height of the triangular part of the plate(y direction).
b'	lateral dimension in the y' direction of plates (e) and (f) of triangular element basic building blocks.
b_1	dimension of the smallest of the two parallel sides of the trapezoidal plate.
b_2	dimension of the longer of the two parallel sides of the trapezoidal plate.
D	$= Eh^3/[12(1 - \nu^2)]$, plate flexural rigidity.
E	Young's modulus of the plate material.
h	plate thickness
W	amplitude of the plate lateral displacement.
x, y	plate spatial coordinates.
x', y'	plates (e) and (f) spatial coordinates.
λ^2	$= \omega a^2 \sqrt{\rho/D}$, plate free vibration eigenvalue.
ρ	mass of plate per unit area.
ω	circular frequency of plate vibration.
ϕ_1	$= b/a$, triangular part aspect ratio.
ϕ_r	$= b/a_r$, rectangular part aspect ratio.

ν Poisson's ratio for plate material; ($\nu = 0.3$).
 ξ $= x/a$, dimensionless plate spatial coordinate.
 ξ' $= x'/a'$, dimensionless plate spatial coordinate.
 η $= y/b$, dimensionless plate spatial coordinate.
 η' $= y'/b'$, dimensionless plate spatial coordinate.
 β plate solution parameter, as defined in text.
 γ plate solution parameter, as defined in text.

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PART A

INTRODUCTION

INTRODUCTION

Thin plates are straight, plane surface structures whose thickness is small compared to their other dimensions. They could have simply supported, fixed, and free boundary conditions, including elastic supports and restraints, or, in some cases point supports. Modern engineering demands that engineers have good knowledge of the vibration behavior of structures beyond the usual beam and rod vibration examples. Vibrating plate structures are not only encountered by civil and aeronautical engineers, but also by industrial, chemical, mechanical, and nuclear engineers. The two-dimensional structural action of plates results in lighter structures offering numerous application and economic advantages, which have contributed considerably to the wide use of thin plates in virtually all fields of engineering. Although thin shell structures offer the above mentioned advantages to an even greater extent, it must be noted that the greater savings in material achieved by the three-dimensional load-carrying capacity of shells is often offset by higher fabrication costs. Furthermore, while there are numerous structural members requiring plane surfaces which bar the use of curved surface structures, many structures, such as ships and large containers, among others, require enclosure that can be easily accomplished by plates without any additional covering resulting in further saving in both material and labor.

In many design problems, specifications merely ensuring that plates will withstand applied static loads prove to be inadequate. It is for this reason that

analysis of the free vibration of plates has been and still is gaining momentum. A study of the literature reveals that highly accurate analytical solutions can be found for many rectangular thin plate free vibration problems [1,2,3,4]. However, because no exact solutions to the governing differential equation expressed in skew coordinates are known to exist in variables separable form, no exact solutions exist for straight-edged quadrilateral thin flat plates which do not have rectangular shapes, such as trapezoidal plates and parallelogram plates.

For parallelogram and trapezoidal plate problems, it is found that most of the analyses presented so far have been reviewed by Leissa [1,5,6,7], and proved to be approximate in nature in that they either do not satisfy exactly the differential equation, the prescribed boundary conditions, or both. A brief but comprehensive critical assessment of the relevant available literature will follow. Analysis of this type of plate has traditionally presented the analyst with serious difficulties. Even the simplest case when all edges are simply supported requires an intricate solution, unlike the case of the rectangular plate. In the present analysis, such difficulties are obviated, and a highly accurate analytical type solution is provided for this particular engineering problem. Through exploitation of the method of superposition, and by extending the techniques developed by Gorman [2] for the free vibration analyses of rectangular plates to non-rectangular plates, a solution for the above problem is obtained by superimposing a number of rectangular plate solutions that can readily be obtained by classical methods [8,9,10]. Unlike numerical and other approximate solutions, this analytical solution satisfies exactly the differential equation everywhere. It also satisfies all boundary conditions to any desired degree of accuracy. This method of investigation will be demonstrated by presenting the full analysis of the free vibration problem of symmetrical trapezoidal plates.

RELEVANT LITERATURE REVIEW

Trapezoidal Plates

An exhaustive literature study by the author revealed that most available relevant analyses up to the year 1980 have been reviewed by Leissa. It appears that the earliest significant work on the problem of the free vibration analysis of the simply supported trapezoidal plate was conducted by Klein[11] in 1955. He solved the case of a simply supported symmetrical trapezoidal plate by using the collocation method, where the assumed shape function satisfied the differential equation at three points along the axis of symmetry. The boundary conditions were satisfied as follows. The condition of zero displacement was satisfied on all edges, while the condition of zero moment was satisfied at two points on the parallel sides, and some point along the other two edges. This led to a third order characteristic determinant for the frequencies. Numerical results obtained were presented in graphical form. However, the accuracy of these results is poor, as expected, since both the boundary conditions and the differential equation were violated.

A better approach to the same problem was adopted by Reipert[12] where he formulated a solution in terms of functions which satisfy the differential equation and the parallel edges boundary conditions exactly. The satisfaction of the boundary conditions along the remaining two edges led to a characteristic determinant for the frequencies. The method uses a simple functional series that may be considered as an extension of the well known Lévy's method. The method can be applied to trapezoidal plates whose parallel edges are simply supported, with arbitrary boundary conditions on the other pair of edges. Numerical results will be shown in later chapter for purposes of comparison.

The problem of the cantilevered trapezoidal plate in the special case when only half of the symmetrical trapezoid is considered was discussed by

Nagaraja[13] in 1961. He used the Rayleigh-Ritz method with deflection functions which are products of characteristic beam functions. Considerable algebraic complications were encountered, and numerical integration had to be used.

Taking advantage of a known relation, ie. the eigenvalues of a simply supported polygonal plate are the squares of the eigenvalues of a polygonal membrane of the same shape with fixed edges and their eigenfunctions are identical, in 1971, Chopra and Durvasula investigated the natural frequencies and mode shapes of simply supported symmetrical[14], and unsymmetrical[15] trapezoidal plates by solving the simpler problem of the corresponding membranes. The method of solution is approximate and quite standard. The deflection surface is expressed in terms of a truncated double Fourier sine series in the transformed coordinates and the Galerkin method is applied. Numerical results were presented in the form of tables and graphs. Some of these results will be shown later for purposes of comparison. Although the method resulted in acceptable numerical accuracy, it is approximate in nature and is restricted to simple support conditions. Also, some errors in the presentation of results were revealed by this author and will be discussed later in forthcoming chapters.

In 1973, a finite element approach was discussed by Orris and Petyt[16] to obtain the natural frequencies and nodal patterns for both, simply supported and clamped plates of rectangular, trapezoidal, and triangular configurations. The obtained numerical results were compared with those found by other authors, in particular by Chopra and Durvasula.

The most recent theoretical study of the free vibration of trapezoidal plates was that of Nagaya[17]. In this paper, Nagaya deals with the flexural vibration of a thin elastic solid plate with straight line boundaries. The classical plate theory and the classical solution for vibration of a solid circular plate expressed in terms of Bessel and modified Bessel functions are used in the analysis. Boundary conditions of the plate are considered for clamped, simply supported and free edges. The cartesian coordinate x ; is taken to be

normal to the i^{th} boundary. By a transformation of coordinates, boundary conditions can be expressed in terms of the polar coordinates r and θ . To satisfy these conditions Fourier expansion of the expression of r in terms of θ along the boundary line are performed. The Fourier coefficients are obtained by the addition of those for the separately considered boundaries. The boundary conditions for all-simply supported and all-clamped plates symmetric about an axis are explicitly given in Fourier series. Thus the frequency equation for symmetric and anti-symmetric modes of vibration of the plate can be obtained. Numerical results for various modes were presented for trapezoidal, and rhombic plates with all-clamped and all-simply supported edges. Good agreement is noted between Nagaya's results and those obtained in the present work. It must be noted however, that algebraic complications were often encountered by Nagaya, and as a result, numerical integrations were performed.

Most recently, Maruyama, Ichinomiya, and Narita have experimentally investigated the free transverse vibration of symmetrical and unsymmetrical clamped trapezoidal plates[18]. In their investigation the authors applied the real time technique of time averaged holographic interferometry to determine the natural frequencies and mode shapes for the transverse vibrations of clamped trapezoidal plates. The experimental natural frequencies were expressed in terms of a dimensionless frequency parameter and shown graphically as a function of the ratio of the lengths of the two parallel sides. The experimental results were compared with the available analytical ones, and satisfactory agreement was reported. This experimental study proved to be very useful in the verification of the validity of the present investigation.

To the author's knowledge, no other published literature exists on the subject of free vibration analysis of thin trapezoidal plates.

Parallelogram Plates

The situation is similar in the case of skew plates where some solutions have been obtained by approximate methods for a few of the many possible combinations of boundary conditions[1]. Most of these solutions are mainly concerned with calculating the fundamental frequency. In 1975, using polynomials and the Rayleigh-Ritz method, Hasegawa[19] calculated the fundamental frequency of clamped skew plates. Hamada[20,21] used the Lagrangian multiplier method with trigonometric functions to obtain a lower bound for the fundamental frequency of clamped rhombic plates. The boundary collocation method was used by Conway and Farnham[22]. In dealing with this problem, they considered clamped and simply supported skew plates employing solutions of the differential equation in polar coordinates. The problem of the simply supported skew plate was considered by Tsydzik. Tsydzik solved this problem by using the perturbation method (see Reference 1). Seth gave an exact solution for a particular simply supported parallelogram plate[23]. Particular emphasis exists in the literature for the case of the cantilevered parallelogram because of its importance as an aerodynamic lifting or stabilizing surface. References dealing with this particular problem are numerous and the reader is referred to Leissa's work[1] for a comprehensive list of references, some of which are experimental studies. Monforton[24] used the finite element method for the solution of the free vibration problem of clamped skew plates.

More recently, an approximate but more comprehensive study of the natural vibration characteristics of skew plates which are either clamped or simply supported on all four edges was conducted by Durvasula[25,26]. The vibration problem of skew plates with different edge conditions involving simple and clamped supports was considered by Nair and Durvasula[27] using the variational method of Ritz. Natural frequencies and modes of vibration were obtained for different combinations of side ratios and skew angles. In a different paper, these same two authors used the partition method to analyze a similar problem. The partition method has been referred to by various names such as the subdomain method and the method of inversion. In fact, it is a

modified form of the collocation method. This method consists of expressing the solution of the governing differential equation in appropriately chosen trial functions satisfying the boundary conditions. The given domain is to be partitioned into subdomains, and over each, the differential equation is to be integrated. The error in the differential equation is to be set to zero. In Reference [28], the authors apply this method to vibration problems of rectangular and skew plates with clamped boundary conditions. A comparison of numerical results with earlier investigations based on other methods, such as Galerkin and finite element methods, showed good agreement. The Rayleigh-Ritz method with B-spline functions as coordinate functions was discussed by Mizusawa, Kajita, and Naruoka[29] in dealing with the problem of free vibration of skew plates. In Reference [30], the authors use a method which they developed to find upper and lower bounds for the lowest frequencies of vibration of fixed rhombic plates. The method combines two types of inequalities connecting estimates for the eigenvalues with known integrals or trial functions. The method does not require the trial functions to satisfy any boundary conditions, yet is like the Rayleigh-Ritz method computationally in that a set of linear combinations of trial functions is optimized, leading to a matrix eigenvalue problem. Unlike the Rayleigh-Ritz method, both upper and lower bounds on the eigenvalues result. The authors limit their numerical calculations to the lowest frequencies. Reference [31] presents an approximate method for determining the natural frequencies of vibration of simply supported isotropic and orthotropic skew plates. The approach is based on a reduction method which uses as its basis the natural frequencies of a clamped skewed membrane. The paper demonstrates the approach numerically by treating the cases of a uniformly stretched skew membrane, a simply supported isotropic skew plate, a simply supported specially orthotropic skew plate and a simply supported generally orthotropic skew plate.

Most recently, Srinivasan and Babu[32] presented a numerical method for the analysis of cantilever quadrilateral plates of general shape. The plate is divided into a mesh. Using quadrilateral coordinates and the integral equation of beams, the expressions for the strain energy and the kinetic energy

are developed in a discrete-point format. Governing equations for the free vibration are obtained in a matrix form by minimizing the Lagrangian of the system. The problem is then solved for frequencies and shapes.

This concludes the available relevant literature on the subject of free vibration of skew plates. Unlike the case of trapezoidal plates, it is seen here that a considerable number of studies on the free vibration of skew plates have been carried out. However, similar to the trapezoidal plates' case, these studies use a variety of approximate methods such as the Rayleigh-Ritz method, the Galerkin method, the partition method, the finite element method and other approximate and numerical methods. In both cases comparison of these studies indicates that the solutions became less accurate with increasing skew angles. Also, the proper selection of displacement functions is essential to obtain accuracy and stable convergence of results. In the following chapters, the numerous advantages of the superposition method used in the present analysis will be shown by analyzing the case of symmetrical trapezoidal plates. A procedure will also be outlined for the analysis of skew plates as well as trapezoidal plates which do not have any symmetry.

Chapter 1

THE UNDERLYING THEORY

The objective of this analysis is to outline and discuss the superposition techniques leading to analytically exact solutions for the free vibration problems of non-rectangular quadrilateral plates, and to develop and provide such solution for the free vibration problem of symmetrical trapezoidal plates and unsymmetrical trapezoidal plates that have one right angle. However, it seems appropriate to start with a brief discussion on the underlying theory of the free vibration of plates in general and the method of superposition as introduced by Gorman in particular. Although the author appreciates the importance of providing the reader with the general theory and procedure that are followed throughout this work, and form the basis of the final results, it is left up to the reader to consult Gorman's work for any detailed derivations or explanation.

1.1 The Differential Equation

The correct differential equation governing the pure bending of plates subjected to lateral static loading is presented with detailed derivation by Timoshenko[33], and is written as;

$$\frac{\partial^4 W(x, y)}{\partial x^4} + \frac{2\partial^4 W(x, y)}{\partial x^2 \partial y^2} + \frac{\partial^4 W(x, y)}{\partial y^4} = \frac{q(x, y)}{D}, \quad (1.1)$$

where $q(x, y)$ is the applied static loading.

The governing differential equation for the free vibration of rectangular plates is obtained by replacing the lateral force $q(x, y)$ of Equation (1.1) by the inertial force and by introducing the time variable parameter t .

$$\frac{\partial^4 W(x, y, t)}{\partial x^4} + \frac{2\partial^4 W(x, y, t)}{\partial x^2 \partial y^2} + \frac{\partial^4 W(x, y, t)}{\partial y^4} + \frac{\rho}{D} \frac{\partial^2 W(x, y, t)}{\partial t^2} = 0, \quad (1.2)$$

by replacing $W(x, y, t)$ by its equivalent $W(x, y)T(t)$ (separable variables), it can be shown that $T(t) = A \sin(\omega t + \alpha)$. And Equation (1.2) can be written as;

$$\frac{\partial^4 W(x, y)}{\partial x^4} + \frac{2\partial^4 W(x, y)}{\partial x^2 \partial y^2} + \frac{\partial^4 W(x, y)}{\partial y^4} - \frac{\omega^2 \rho W(x, y)}{D} = 0, \quad (1.3)$$

in its dimensionless form Equation (1.3) can be written as;

$$\frac{\partial^4 W(\xi, \eta)}{\partial \xi^4} + \frac{2\partial^4 W(\xi, \eta)}{\phi^2 \partial \xi^2 \partial \eta^2} + \frac{\partial^4 W(\xi, \eta)}{\phi^4 \partial \eta^4} - \lambda^4 W(\xi, \eta) = 0, \quad (1.4)$$

or,

$$\frac{\partial^4 W(\xi, \eta)}{\partial \eta^4} + \frac{2\phi^2 \partial^4 W(\xi, \eta)}{\partial \eta^2 \partial \xi^2} + \frac{\phi^4 \partial^4 W(\xi, \eta)}{\partial \xi^4} - \phi^4 \lambda^4 W(\xi, \eta) = 0. \quad (1.5)$$

1.2 The Lévy Type Solution

In 1820, Navier presented a paper to the French Academy of Sciences on the solution of bending of simply supported rectangular plates by double trigonometric series. This solution is sometimes called the forced solution. At the turn of the century, in 1899, a solution by single Fourier series was introduced by Lévy[34]. This powerful method obtains the solution of Equation (1.5) in the form;

$$\lim_{k \rightarrow \infty} W(\xi, \eta) = \sum_{m=1}^k Y_m(\eta) \sin(m\pi\xi). \quad (1.6)$$

By substituting this solution in Equation (1.5) and rearranging we get;

$$\frac{d^4 Y_m(\eta)}{d\eta^4} - 2\phi^2 (m\pi)^2 \frac{d^2 Y_m(\eta)}{d\eta^2} + \phi^4 [(m\pi)^4 - \lambda^4] Y_m(\eta) = 0. \quad (1.7)$$

The solution of Equation (1.7) depends on whether $\lambda^2 - (m\pi)^2$ is negative or positive. If $\lambda^2 > (m\pi)^2$ then;

$$Y_m(\eta) = A_m \cosh \beta_m \eta + B_m \sinh \beta_m \eta + C_m \sin \gamma_m \eta + D_m \cos \gamma_m \eta, \quad (1.8)$$

and if $\lambda^2 < (m\pi)^2$ then;

$$Y_m(\eta) = A_m \cosh \beta_m \eta + B_m \sinh \beta_m \eta + C_m \sinh \gamma_m \eta + D_m \cosh \gamma_m \eta, \quad (1.9)$$

where,

$$\begin{aligned} \beta_m &= \phi \sqrt{\lambda^2 + (m\pi)^2}, \\ \gamma_m &= \phi \sqrt{\lambda^2 - (m\pi)^2} \quad \text{or} \quad \gamma_m = \phi \sqrt{(m\pi)^2 - \lambda^2}, \end{aligned} \quad (1.10)$$

whichever is real. And where A_m, B_m, C_m , and D_m are constants to be determined by means of prescribed boundary conditions. Unless stated otherwise, the coordinate system used will always be that of Figure 1.1 throughout this work.

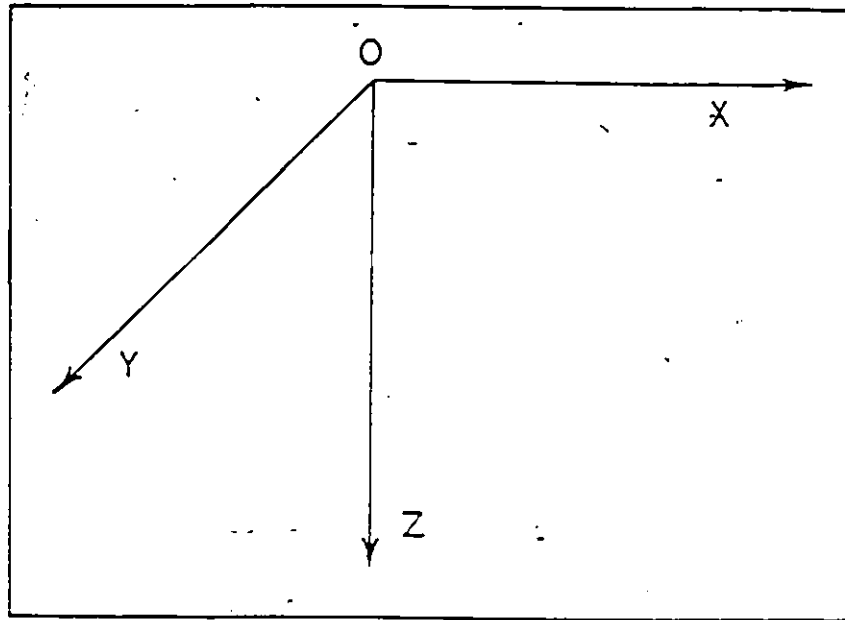


Figure 1.1- Conventional rectangular coordinates system.

1.3 Limitations of Lévy solution

Although, Lévy's method which uses a single trigonometric series is more general than Navier's solution, it does not have an entirely general character since in its original form it can only be applied to rectangular plate vibration problems where at least two opposite edges have simple supports. However, it has been shown that by making use of the superposition method[2,3,4], this Lévy-type solution is readily employed to solve not only rectangular plate vibration problems of all possible combinations of classical boundary conditions, but also to analyze numerous rectangular plates with non-classical type boundary conditions. Furthermore, it is the objective of the present analysis to extend the use of this combination of Lévy-type solution and superposition techniques to analyze free vibration problems of non-rectangular plates. Next, a word about the classical boundary conditions and their mathematical formulations.

1.4 Classical Boundary Conditions

Whenever classical boundary conditions are mentioned, our attention is directed toward the three types of boundary conditions that have been studied thoroughly in the classical literature. They are the clamped, the simply supported, and the free edge conditions. Although we are dealing with the problem of free vibration, the formulation of these boundary conditions is identical to that found in Timoshenko's work[33] on static plate analysis. This formulation is presented here in both conventional and dimensionless coordinate systems. Reference is made here to Figures 1.1 and 1.2.

Clamped Edges:

Conventional Coordinates;

$$W(x, y) = \frac{\partial W(x, y)}{\partial x} = 0. \quad (1.11)$$

Dimensionless Coordinates;

$$W(\xi, \eta) = \frac{\partial W(\xi, \eta)}{\partial \xi} = 0. \quad (1.12)$$

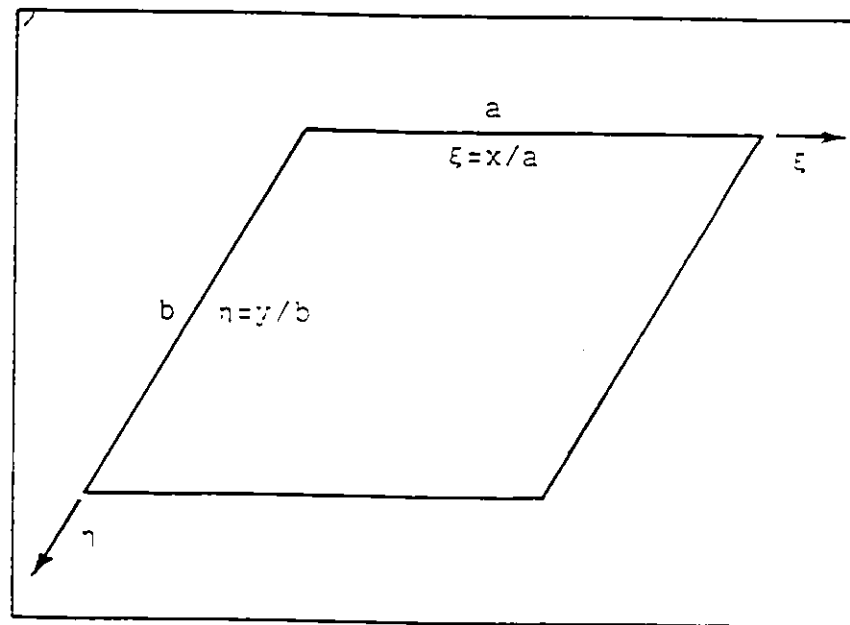


Figure 1.2- Dimensionless rectangular coordinate system.

Supported Edges:

Conventional Coordinates;

$$W(x, y) = \frac{\partial^2 W(x, y)}{\partial x^2} = 0. \quad (1.13)$$

Dimensionless Coordinates;

$$W(\xi, \eta) = \frac{\partial^2 W(\xi, \eta)}{\partial \xi^2} = 0. \quad (1.14)$$

Free Edges:

Conventional Coordinates;

$$\frac{\partial^2 W(x, y)}{\partial x^2} + \nu \frac{\partial^2 W(x, y)}{\partial y^2} = 0, \quad |_{x=a} \quad (1.15 - a)$$

$$\frac{\partial^3 W(x, y)}{\partial x^3} + \nu \frac{\partial^3 W(x, y)}{\partial x \partial y^2} = 0, \quad |_{x=a} \quad (1.15 - b)$$

$$\frac{\partial^2 W(x, y)}{\partial y^2} + \nu \frac{\partial^2 W(x, y)}{\partial x^2} = 0, \quad |_{y=b} \quad (1.15 - c)$$

$$\frac{\partial^3 W(x, y)}{\partial y^3} + \nu \frac{\partial^3 W(x, y)}{\partial y \partial x^2} = 0, \quad |_{y=b} \quad (1.15 - d)$$

Dimensionless Coordinates;

$$\frac{\partial^2 W(\xi, \eta)}{\partial \xi^2} + \frac{\nu}{\phi^2} \frac{\partial^2 W(\xi, \eta)}{\partial \eta^2} = 0, \quad |_{\xi=1} \quad (1.16 - a)$$

$$\frac{\partial^3 W(\xi, \eta)}{\partial \xi^3} + \frac{\nu^*}{\phi^2} \frac{\partial^3 W(\xi, \eta)}{\partial \xi \partial \eta^2} = 0, \quad |_{\xi=1} \quad (1.16 - b)$$

$$\frac{\partial^2 W(\xi, \eta)}{\partial \eta^2} + \nu \phi^2 \frac{\partial^2 W(\xi, \eta)}{\partial \xi^2} = 0, \quad |_{\eta=1} \quad (1.16 - c)$$

$$\frac{\partial^3 W(\xi, \eta)}{\partial \eta^3} + \nu^* \phi^2 \frac{\partial^3 W(\xi, \eta)}{\partial \eta \partial \xi^2} = 0, \quad |_{\eta=1} \quad (1.16 - d)$$

where the displacement and coordinate x are divided by plate dimension a , while the coordinate y is divided by dimension b . And where $\phi = b/a$.

As we progress further ahead, it will become apparent that mathematical formulations for bending moments and vertical edge reactions are also required, as presented in the next paragraph.

1.5 Edge Bending Moments and Vertical Reactions

The mathematical expressions for bending moments and vertical edge reactions, distributed along the edges of rectangular plate, have been developed in the conventional coordinates by Timoshenko[33]. These expressions are as follows.

Distributed Bending Moments:

Conventional Coordinates;

$$M_x = -D \left[\frac{\partial^2 W(x,y)}{\partial x^2} + \nu \frac{\partial^2 W(x,y)}{\partial y^2} \right], \quad (1.17 - a)$$

$$M_y = -D \left[\frac{\partial^2 W(x,y)}{\partial y^2} + \nu \frac{\partial^2 W(x,y)}{\partial x^2} \right], \quad (1.17 - b)$$

Dimensionless Coordinates;

$$\frac{M_\xi a}{D} = - \left[\frac{\partial^2 W(\xi, \eta)}{\partial \xi^2} + \frac{\nu}{\phi^2} \frac{\partial^2 W(\xi, \eta)}{\partial \eta^2} \right], \quad (1.18 - a)$$

$$\frac{M_\eta \phi b}{D} = - \left[\frac{\partial^2 W(\xi, \eta)}{\partial \eta^2} + \nu \phi^2 \frac{\partial^2 W(\xi, \eta)}{\partial \xi^2} \right], \quad (1.18 - b)$$

where reference is made to Figure 1.3.

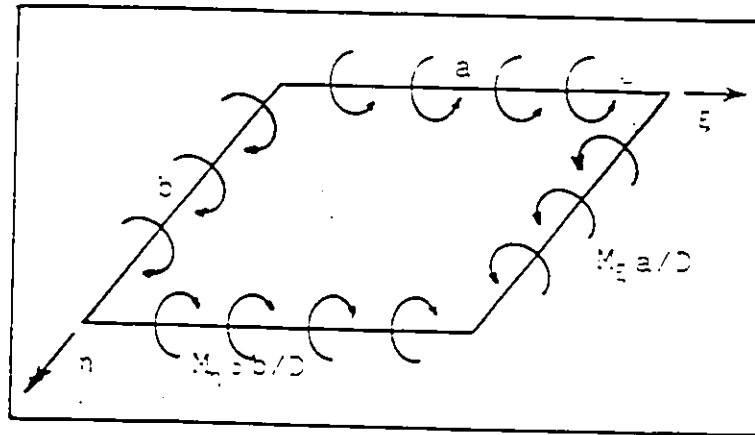


Figure 1.3- Rectangular plate edge bending moments

Vertical Edge Reactions:

Conventional coordinates;

$$V_x = -D \left[\frac{\partial^3 W(x, y)}{\partial x^3} + \nu \frac{\partial^3 W(x, y)}{\partial x \partial y^2} \right], \quad (1.19 - a)$$

$$V_y = -D \left[\frac{\partial^3 W(x, y)}{\partial y^3} + \nu \frac{\partial^3 W(x, y)}{\partial y \partial x^2} \right], \quad (1.19 - b)$$

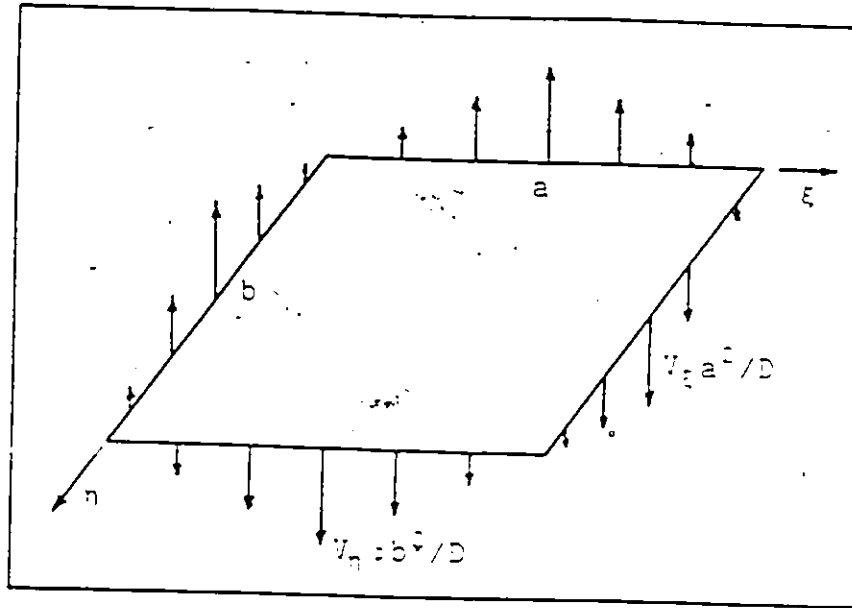


Figure 1.4- Vertical edge reactions on a rectangular plate.

Dimensionless Coordinates;

$$\frac{V_\xi a^2}{D} = - \left[\frac{\partial^3 W(\xi, \eta)}{\partial \xi^3} + \frac{\nu}{\phi^2} \frac{\partial^3 W(\xi, \eta)}{\partial \xi \partial \eta^2} \right], \quad (1.20 - a)$$

$$\frac{V_n \phi b^2}{D} = - \left[\frac{\partial^3 W(\xi, \eta)}{\partial \eta^3} + \nu^* \phi^2 \frac{\partial^3 W(\xi, \eta)}{\partial \eta \partial \xi^2} \right], \quad (1.20 - b)$$

where $\nu^* = 2 - \nu$, and where reference is made to Figure 1.4.

The vertical concentrated force that will act at each corner of a rectangular plate, as explained by Timoshenko[33], is represented by the following expressions.

Corner Vertical Reaction:

Conventional Coordinates;

$$R = 2D(1 - \nu) \frac{\partial^2 W(x, y)}{\partial x \partial y} \quad (1.21)$$

Dimensionless Coordinates;

$$R = \frac{2D}{b^3}(1 - \nu) \frac{\partial^2 W(\xi, \eta)}{\partial \xi \partial \eta} \quad (1.22)$$

It will become clear in future chapters that general expressions for the bending moments and vertical reactions along oblique lines in the plate are also needed. These expressions were also discussed with detailed derivations by Timoshenko[33], and are given here for convenience.

$$M_n = M_x \cos^2 \alpha_i + M_y \sin^2 \alpha_i - 2M_{xy} \sin \alpha_i \cos \alpha_i, \quad (1.23)$$

in which α_i is the angle between the normal n and the x axis. In its dimensionless form, Equation (1.23) becomes;

$$\frac{M_n b^2}{aD} = - \left[\theta_1 \phi^2 \frac{\partial^2 W}{\partial \xi^2} + \theta_2 \frac{\partial^2 W}{\partial \eta^2} + \theta_3 \phi \frac{\partial^2 W}{\partial \xi \partial \eta} \right], \quad (1.24)$$

where,

$$\theta_1 = \cos^2 \alpha_i + \nu \sin^2 \alpha_i,$$

$$\theta_2 = \sin^2 \alpha_i + \nu \cos^2 \alpha_i,$$

$$\theta_3 = (1 - \nu) \sin 2\alpha_i.$$

The expression for vertical edge reactions is;

$$V = -D \left[v_1 \frac{\partial^3 W}{\partial x^3} + v_2 \frac{\partial^3 W}{\partial y^3} + v_3 \frac{\partial^3 W}{\partial x \partial y^2} + v_4 \frac{\partial^3 W}{\partial y \partial x^2} \right]. \quad (1.25)$$

In its dimensionless form, Equation (1.25) is written as;

$$\frac{V_n a^2}{D} = - \left[V_1 \frac{\partial^3 W}{\partial \xi^3} + V_2 \frac{\partial^3 W}{\partial \eta^3} + V_3 \frac{\partial^3 W}{\partial \xi^2 \partial \eta} + V_4 \frac{\partial^3 W}{\partial \xi \partial \eta^2} \right], \quad (1.26)$$

where,

$$V_1 = \cos \alpha + (1 - \nu) \frac{\sin \alpha \sin 2\alpha}{2},$$

$$V_2 = \left[\sin \alpha + (1 - \nu) \frac{\cos \alpha \sin 2\alpha}{2} \right] / \phi^3,$$

$$V_3 = \left[\sin \alpha - (1 - \nu) \left(\sin \alpha \cos 2\alpha + \frac{\cos \alpha \sin 2\alpha}{2} \right) \right] / \phi,$$

$$V_4 = \left[\cos \alpha + (1 - \nu) \left(\cos \alpha \cos 2\alpha - \frac{\sin \alpha \sin 2\alpha}{2} \right) \right] / \phi^2.$$

So far, we have looked at all the basic equations that will be used throughout this work. A word about the superposition method as it applies to rectangular plate free vibration problems will follow.

1.6 Method of Superposition

It has been mentioned that Lévy type solutions are easily found, and well established for the family of rectangular plates with at least two opposite edges simply supported. A second family of rectangular plates is that of plates for which no two opposite edges are simply supported. Most of these plates were analyzed by Gorman using the method of superposition. In this method, two or more appropriate plate problems whose Lévy-type solutions can be obtained are superimposed, and the constants appearing in their boundary condition formulation are adjusted in such a way so that their combination provides boundary conditions similar to those specified in the original problem. The plate problems that are superimposed are often called building blocks.

1.7 Discussion and Conclusion

We have looked so far at the rectangular plate free vibration underlying theory, and have also stated all the basic equations for the exact analytical solution of any rectangular plate free vibration problem for which the following assumptions apply.

- Plate thickness is small compared to its lateral dimensions.
- For higher vibration modes, plate thickness is small compared to the distance between nodal lines.
- Lateral displacement W is small compared to the thickness of the plate.
- Negligible rotatory inertia effects.
- No significant in-plane forces.

Fortunately, most plate vibration problems satisfy the above assumptions well enough for most practical purposes.

PART B

NON-RECTANGULAR QUADRILATERAL PLATES

Chapter 2

NONRECTANGULAR QUADRILATERAL PLATES BASIC BUILDING BLOCKS

2.1 Geometry of Non-Rectangular Quadrilateral Plates

Any non-rectangular quadrilateral plate can be divided into triangular and rectangular subareas. Consider for instance the skew plate shown in Figure 2.1. It is seen that this plate can easily be divided into three parts, a rectangle and two right angle triangles, as shown in Figure 2.2. In the particular case where $\alpha = \alpha'$ a parallelogram is obtained. However, if $\alpha \neq \alpha'$, a skew plate of general shape is obtained.

As a second example, consider the trapezoidal plate of Figure 2.3. This plate can also be divided into a rectangular and two right angle triangular elements as shown in Figure 2.4. In the particular case where $\alpha = \alpha'$, the plate becomes symmetrical, and one need only analyze half of the plate as will be seen in Part C of this work. If $\alpha \neq \alpha'$, an unsymmetrical trapezoid results.

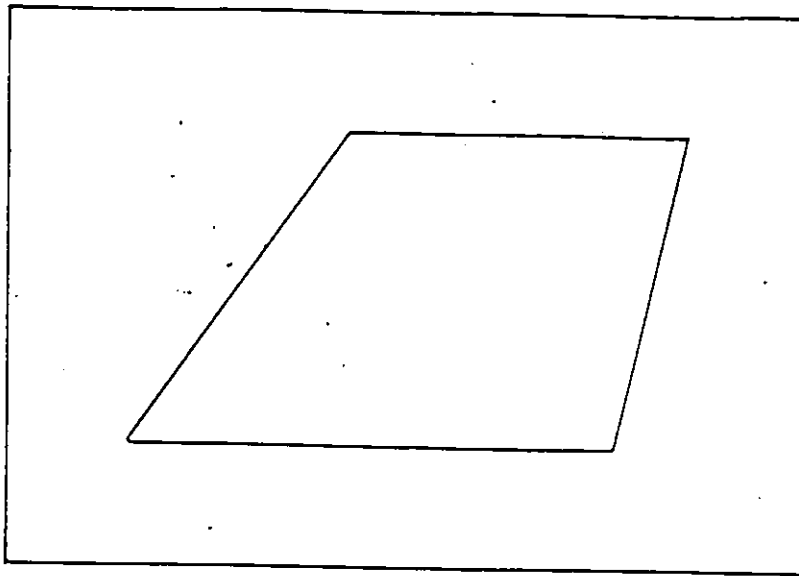


Figure 2.1- Quadrilateral plate of non-rectangular shape.

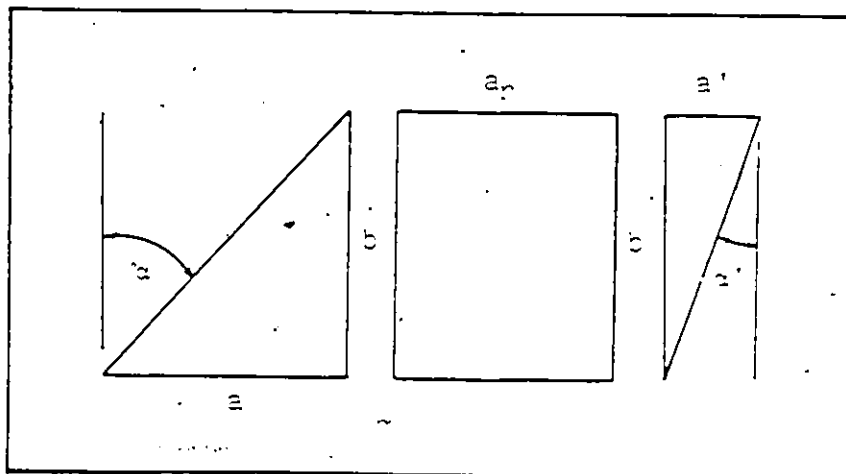


Figure 2.2- Division of a non-rectangular quadrilateral plate.

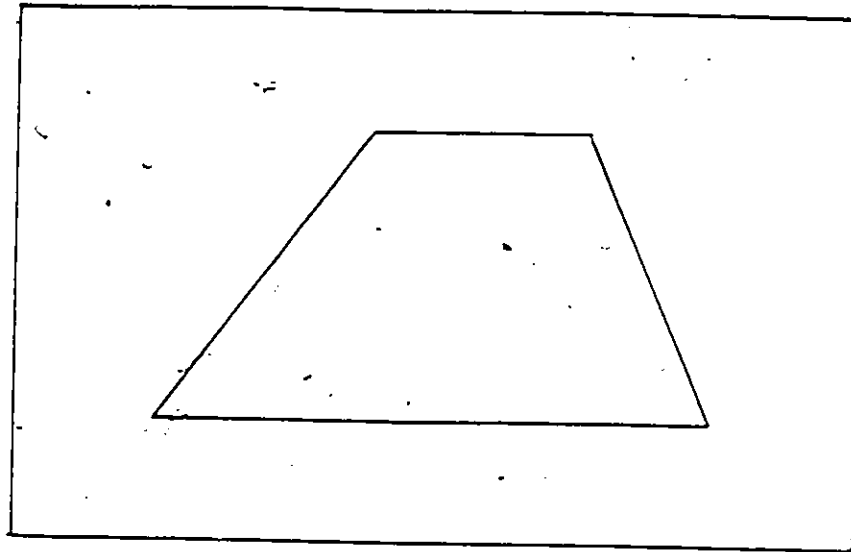


Figure 2.3- Quadrilateral plate of trapezoidal shape.

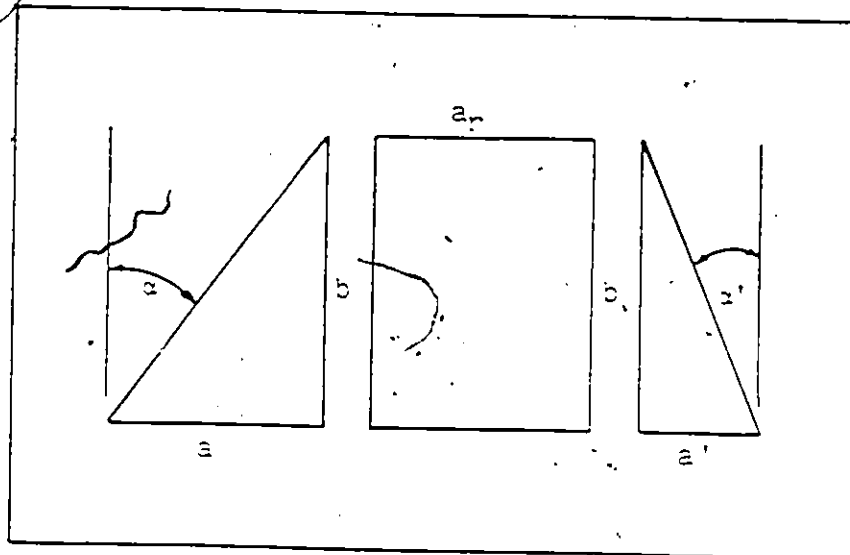


Figure 2.4- Division of a trapezoidal plate.

Other quadrilateral non-rectangular plates of irregular shapes can also be divided into rectangular and right angle triangular elements. It is therefore reasonable to say that a solution to the free vibration problem of any non-rectangular plate can be arrived at if one has solutions for the right angle triangular and rectangular elements, since these solutions can be joined together by enforcing the continuity conditions across the inter-segment lines. Therefore, it is logical to start the present analysis by introducing the basic building blocks that will be used in the solution of both, the right angle triangle and the rectangle discussed above.

2.2 Solution of The Triangular Element

The first step in the superposition techniques is to select the proper set of basic building blocks. Several possible choices are always available. The solution might look somewhat different for different choices of proper building blocks, but the final numerical results will always converge to the exact eigenvalues as was discussed in Reference [35]. Advantages and disadvantages of each individual choice may not be always obvious. However, careful consideration of the minimum number of building blocks required is always a criterion. The ease of solution, the amount of derivation required and the possibility of direct representation of the boundary conditions must always be taken into consideration. However, it must be mentioned here that although the selection of proper building blocks in general will have some effect on the degree of ease and possibly the amount of computation involved in order to arrive at the final results, if the set of building blocks selected properly represents the boundary conditions of the original plate, numerical results will always converge to the exact eigenvalues rapidly and accurately. After careful consideration of all the influencing factors, it was decided that the problem at hand is best represented by the set of building blocks discussed in this chapter. As will be seen, two parallel solutions, one using building blocks with sine functions and the other using building blocks with cosine functions, are discussed and developed simultaneously. It is felt that the extra work and

effort required to develop two alternative solutions will be compensated for by strengthening the confidence in the final results. If the same results are obtained by two independently derived solutions, one may be reasonably sure of the validity, correctness, and convergence of the method.

2.2-1 Basic Building Blocks(Sine Function)

Consider the set of building blocks shown in Figure 2.5. Each plate has simple support along three of its edges. The fourth edge has either a forbidden lateral motion and a prescribed harmonic moment of radian frequency, ω , or a zero moment and a forced harmonic displacement of radian frequency, ω , as it is clearly indicated by the relevant edge of each of the building blocks. Considering the first building block or plate (a), the forced harmonic moment along the edge $\eta = 1$ is expressed as;

$$\left. \frac{M_n b^2}{aD} \right|_{\eta=1} = \sum_{m=1}^{\infty} E_{1m} \sin(m\pi\xi). \quad (2.1)$$

The Lévy type solution for this building block is;

$$W_1(\xi, \eta) = \sum_{m=1}^{k^*} E_{1m} \theta_{11m} (\sinh \beta_{1m} \eta + C_{11m} \sin \gamma_{1m} \eta) \sin(m\pi\xi) + \sum_{m=k^*+1}^{\infty} E_{1m} \theta_{12m} (\sinh \beta_{1m} \eta + C_{12m} \sinh \gamma_{1m} \eta) \sin(m\pi\xi), \quad (2.2)$$

where,

$$\begin{aligned} \beta_{1m} &= \phi_1 \sqrt{\lambda^2 + (m\pi)^2}, & \phi_1 &= b/a, \\ \gamma_{1m} &= \phi_1 \sqrt{\lambda^2 - (m\pi)^2} \quad \text{or} \quad \phi_1 \sqrt{(m\pi)^2 - \lambda^2}, \end{aligned}$$

whichever is real. Also;

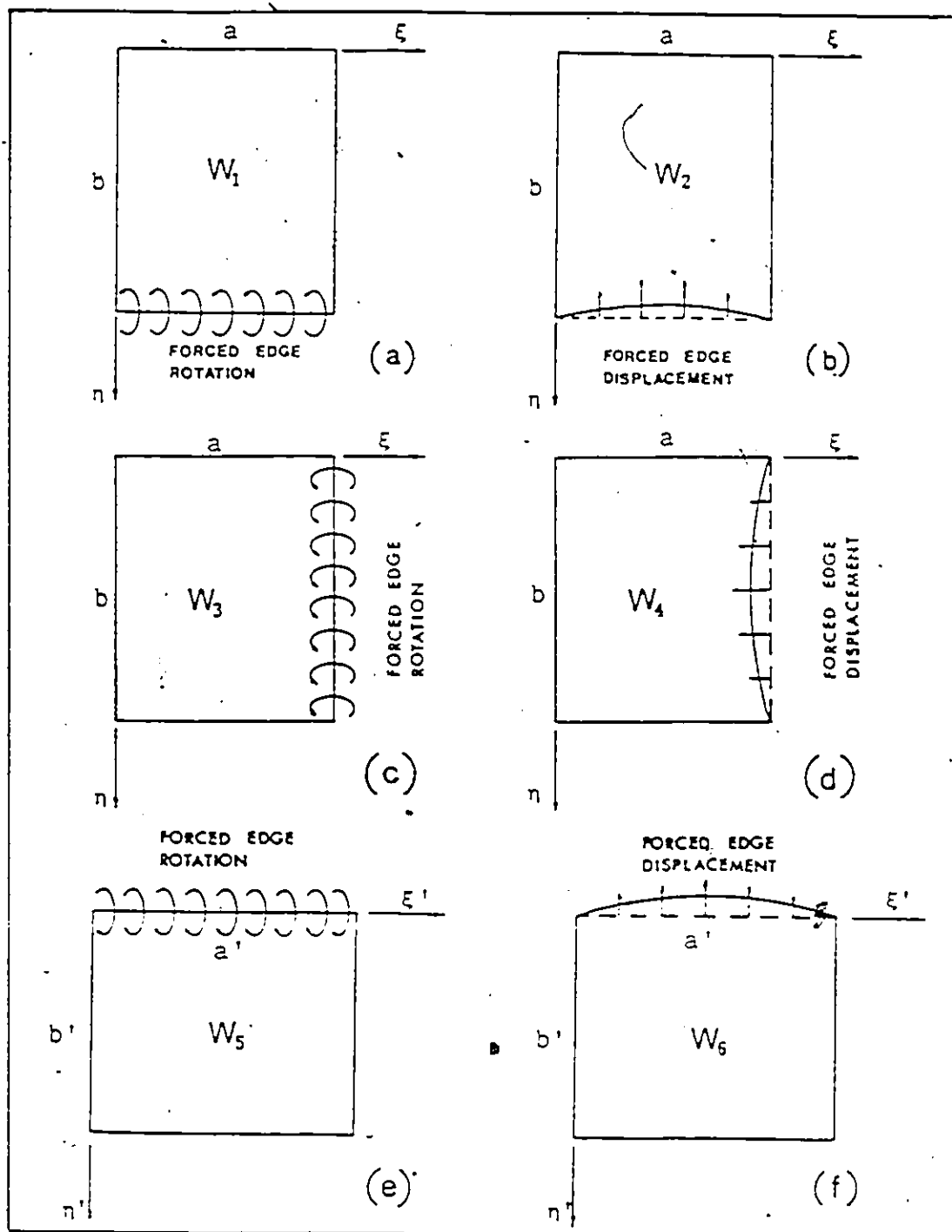


Figure 2.5- First set of basic building blocks used in the solution of the triangular element.

$$\begin{aligned}
C_{11m} &= -\sinh\beta_{1m}/\sin\gamma_{1m}, \\
C_{12m} &= -\sinh\beta_{1m}/\sinh\gamma_{1m}, \\
\theta_{11m} &= -1/(\beta_{1m}^2 \sinh\beta_{1m} - C_{11m}\gamma_{1m}^2 \sin\gamma_{1m}), \\
\theta_{12m} &= -1/(\beta_{1m}^2 \sinh\beta_{1m} + C_{12m}\gamma_{1m}^2 \sinh\gamma_{1m}).
\end{aligned}$$

The second building block or Plate (b) has a harmonic lateral motion along the edge $\eta = 1$ given by;

$$W_2(\xi, \eta) \Big|_{\eta=1} = \sum_{m=1}^{\infty} V_{2m} \sin(m\pi\xi). \quad (2.3)$$

The Lévy type solution for this building block is;

$$\begin{aligned}
W_2(\xi, \eta) &= \sum_{m=1}^{k^*} V_{2m} \theta_{21m} (\sinh\beta_{1m}\eta + C_{21m} \sin\gamma_{1m}\eta) \sin(m\pi\xi) + \\
&\quad \sum_{m=k^*+1}^{\infty} V_{2m} \theta_{22m} (\sinh\beta_{1m}\eta + C_{22m} \sinh\gamma_{1m}\eta) \sin(m\pi\xi), \quad (2.4)
\end{aligned}$$

where,

$$\begin{aligned}
C_{21m} &= [\beta_{1m}^2 - \nu\phi_1^2(m\pi)^2] \sinh\beta_{1m} / [\gamma_{1m}^2 + \nu\phi_1^2(m\pi)^2] \sin\gamma_{1m}, \\
C_{22m} &= -[\beta_{1m}^2 - \nu\phi_1^2(m\pi)^2] \sinh\beta_{1m} / [\gamma_{1m}^2 - \nu\phi_1^2(m\pi)^2] \sinh\gamma_{1m}, \\
\theta_{21m} &= 1/(\sinh\beta_{1m} + C_{21m} \sin\gamma_{1m}), \\
\theta_{22m} &= 1/(\sinh\beta_{1m} + C_{22m} \sinh\gamma_{1m}).
\end{aligned}$$

For plates (c) and (d) it is seen that their solutions are easily obtained from Equations (2.2) and (2.4) by interchanging the variables ξ and η . It is

necessary also to recognize that the aspect ratio ϕ_1 must be replaced by its inverse, and the quantity λ^2 by $\phi_1^2 \lambda^2$.

$$W_3(\xi, \eta) = \sum_{m=1}^{k^*} E_{3m} \theta_{31m} (\sinh \beta_{3m} \xi + C_{31m} \sin \gamma_{3m} \xi) \sin(m\pi\eta) + \sum_{m=k^*+1}^{\infty} E_{3m} \theta_{32m} (\sinh \beta_{3m} \xi + C_{32m} \sinh \gamma_{3m} \xi) \sin(m\pi\eta), \quad (2.5)$$

where,

$$\begin{aligned} \beta_3 &= \phi_3 \sqrt{\phi_1^2 \lambda^2 + (m\pi)^2}, & \phi_3 &= a/b = 1/\phi_1, \\ \gamma_{1m} &= \phi_3 \sqrt{\phi_1^2 \lambda^2 - (m\pi)^2} & \text{or} & \quad \phi_3 \sqrt{(m\pi)^2 - \phi_1^2 \lambda^2}, \end{aligned}$$

whichever is real, and

$$\begin{aligned} C_{31m} &= -\sinh \beta_{3m} / \sin \gamma_{3m}, \\ C_{32m} &= -\sinh \beta_{3m} / \sinh \gamma_{3m}, \\ \theta_{31m} &= -1/(\beta_{3m}^2 \sinh \beta_{3m} - C_{31m} \gamma_{3m}^2 \sin \gamma_{3m}), \\ \theta_{32m} &= -1/(\beta_{3m}^2 \sinh \beta_{3m} + C_{32m} \gamma_{3m}^2 \sinh \gamma_{3m}). \end{aligned}$$

The forced harmonic moment along the edge $\xi = 1$ is given by;

$$\left. \frac{M_n a^2}{bD} \right|_{\xi=1} = \sum_{m=1}^{\infty} E_{3m} \sin(m\pi\eta), \quad (2.6)$$

and

$$W_4(\xi, \eta) = \sum_{m=1}^{k^*} V_{4m} \theta_{41m} (\sinh \beta_{3m} \xi + C_{41m} \sin \gamma_{3m} \xi) \sin(m\pi\eta) + \sum_{m=k^*+1}^{\infty} V_{4m} \theta_{42m} (\sinh \beta_{3m} \xi + C_{42m} \sinh \gamma_{3m} \xi) \sin(m\pi\eta), \quad (2.7)$$

where,

$$\begin{aligned} C_{41m} &= [\beta_{3m}^2 - \nu\phi_3^2(m\pi)^2] \sinh\beta_{3m} / [\gamma_{3m}^2 + \nu\phi_3^2(m\pi)^2] \sin\gamma_{3m}, \\ C_{42m} &= -[\beta_{3m}^2 - \nu\phi_3^2(m\pi)^2] \sinh\beta_{3m} / [\gamma_{3m}^2 - \nu\phi_3^2(m\pi)^2] \sinh\gamma_{3m}, \\ \theta_{41m} &= 1 / (\sinh\beta_{3m} + C_{41m} \sin\gamma_{3m}), \\ \theta_{42m} &= 1 / (\sinh\beta_{3m} + C_{42m} \sinh\gamma_{3m}). \end{aligned}$$

The forced harmonic lateral motion along the edge $\xi = 1$ is given by;

$$W_4(\xi, \eta) \Big|_{\xi=1} = \sum_{m=1}^{\infty} V_{4m} \sin(m\pi\eta). \quad (2.8)$$

Examining Plates (e) and (f), it is seen that they differ from Plates (a) and (b) only in that the prescribed harmonic moments and displacements are enforced along the edge $\eta' = 0$. Therefore, their solutions are also obtained from Equations (2.2) and (2.4) simply by replacing η by $1 - \eta'$ and ξ by ξ' . Note that Plates (e) and (f) have the dimensions b' and a' and their aspect ratio is different from that of the other four plates. Considering Plate (e), the harmonic moment along the edge $\eta' = 0$ is expressed as;

$$\frac{M_n b'^2}{a' D} \Big|_{\eta'=0} = \sum_{m=1}^{\infty} E_{5m} \sin(m\pi\xi'). \quad (2.9)$$

The Lévy type solution for this building block is;

$$\begin{aligned} W_5(\xi', \eta') &= \sum_{m=1}^{k^*} E_{5m} \theta_{51m} [\sinh\beta_{5m}(1 - \eta') + C_{51m} \sin\gamma_{5m}(1 - \eta')] \sin(m\pi\xi') \\ &+ \sum_{m=k^*+1}^{\infty} E_{5m} \theta_{52m} [\sinh\beta_{5m}(1 - \eta') + \\ &C_{52m} \sinh\gamma_{5m}(1 - \eta')] \sin(m\pi\xi'), \end{aligned} \quad (2.10)$$

where,

$$\begin{aligned}\beta_{5m} &= \phi_5 \sqrt{\lambda'^2 + (m\pi)^2}, \\ \gamma_{5m} &= \phi_5 \sqrt{\lambda'^2 - (m\pi)^2} \quad \text{or} \quad \phi_5 \sqrt{(m\pi)^2 - \lambda'^2},\end{aligned}$$

whichever is real, and

$$\begin{aligned}C_{51m} &= -\sinh\beta_{5m}/\sin\gamma_{5m}, \\ C_{52m} &= -\sinh\beta_{5m}/\sinh\gamma_{5m}, \\ \theta_{51m} &= -1/(\beta_{5m}^2 \sinh\beta_{5m} - C_{51m} \gamma_{5m}^2 \sin\gamma_{5m}), \\ \theta_{52m} &= -1/(\beta_{5m}^2 \sinh\beta_{5m} + C_{52m} \gamma_{5m}^2 \sinh\gamma_{5m}),\end{aligned}$$

also, $\phi_5 = b'/a' = \cos\alpha/\phi_1$, $\lambda' = \lambda/\sin^2\alpha$, and α is the skew angle as will be seen later in this chapter.

The harmonic lateral motion along the edge $\eta' = 0$ of Plate (f), is given by;

$$W_6(\xi', 0) = \sum_{m=1}^{\infty} V_{6m} \sin(m\pi\xi'). \quad (2.11)$$

The Lévy type solution of this building block is;

$$\begin{aligned}W_6(\xi', \eta') &= \sum_{m=1}^{k^*} V_{6m} \theta_{61m} [\sinh\beta_{5m}(1 - \eta') + C_{61m} \sin\gamma_{5m}(1 - \eta')] \sin(m\pi\xi') \\ &+ \sum_{m=k^*+1}^{\infty} V_{6m} \theta_{62m} [\sinh\beta_{5m}(1 - \eta') + \\ &C_{62m} \sinh\gamma_{5m}(1 - \eta')] \sin(m\pi\xi'),\end{aligned} \quad (2.12)$$

where,

$$\begin{aligned} C_{61m} &= [\beta_{5m}^2 - \nu\phi_5^2(m\pi)^2] \sinh\beta_{5m} / [\gamma_{5m}^2 + \nu\phi_5^2(m\pi)^2] \sin\gamma_{5m}, \\ C_{62m} &= -[\beta_{5m}^2 - \nu\phi_5^2(m\pi)^2] \sinh\beta_{5m} / [\gamma_{5m}^2 - \nu\phi_5^2(m\pi)^2] \sinh\gamma_{5m}, \\ \theta_{61m} &= 1 / (\sinh\beta_{5m} + C_{61m} \sin\gamma_{5m}), \\ \theta_{62m} &= 1 / (\sinh\beta_{5m} + C_{62m} \sinh\gamma_{5m}). \end{aligned}$$

We now have dynamic response solutions for all plate problems of Figure 2.5. An alternative set of building blocks is obtained simply by replacing the simple support conditions along two opposite edges of each of the above building blocks by slip shear conditions (zero slope and zero vertical edge reaction), as shown in Figure 2.6.

2.2-2 Alternative Building Blocks (Cosine Function)

Consider now the set of building blocks shown in Figure 2.6. Each plate has slip shear conditions along two opposite edges. The third edge has simple support conditions. The fourth edge has either a forbidden lateral motion and a prescribed harmonic moment of radian frequency ω , or a zero moment and a forced harmonic displacement of radian frequency ω as it is clearly indicated by the relevant edge of each of the building blocks. Considering the first building block or Plate (a'), the forced harmonic moment along the edge $\eta = 1$ is expressed as;

$$\left. \frac{M_n b^2}{aD} \right|_{\eta=1} = \sum_{m=0}^{\infty} E_{1m} \cos(m\pi\xi). \quad (2.13)$$

The Lévy type solution for this building block is;

$$\begin{aligned} W_1(\xi, \eta) &= \sum_{m=0}^{k^*} E_{1m} \theta_{11m} (\sinh\beta_{1m}\eta + C_{11m} \sinh\gamma_{1m}\eta) \cos(m\pi\xi) + \\ &\quad \sum_{m=k^*+1}^{\infty} E_{1m} \theta_{12m} (\sinh\beta_{1m}\eta + C_{12m} \sinh\gamma_{1m}\eta) \cos(m\pi\xi), \quad (2.14) \end{aligned}$$

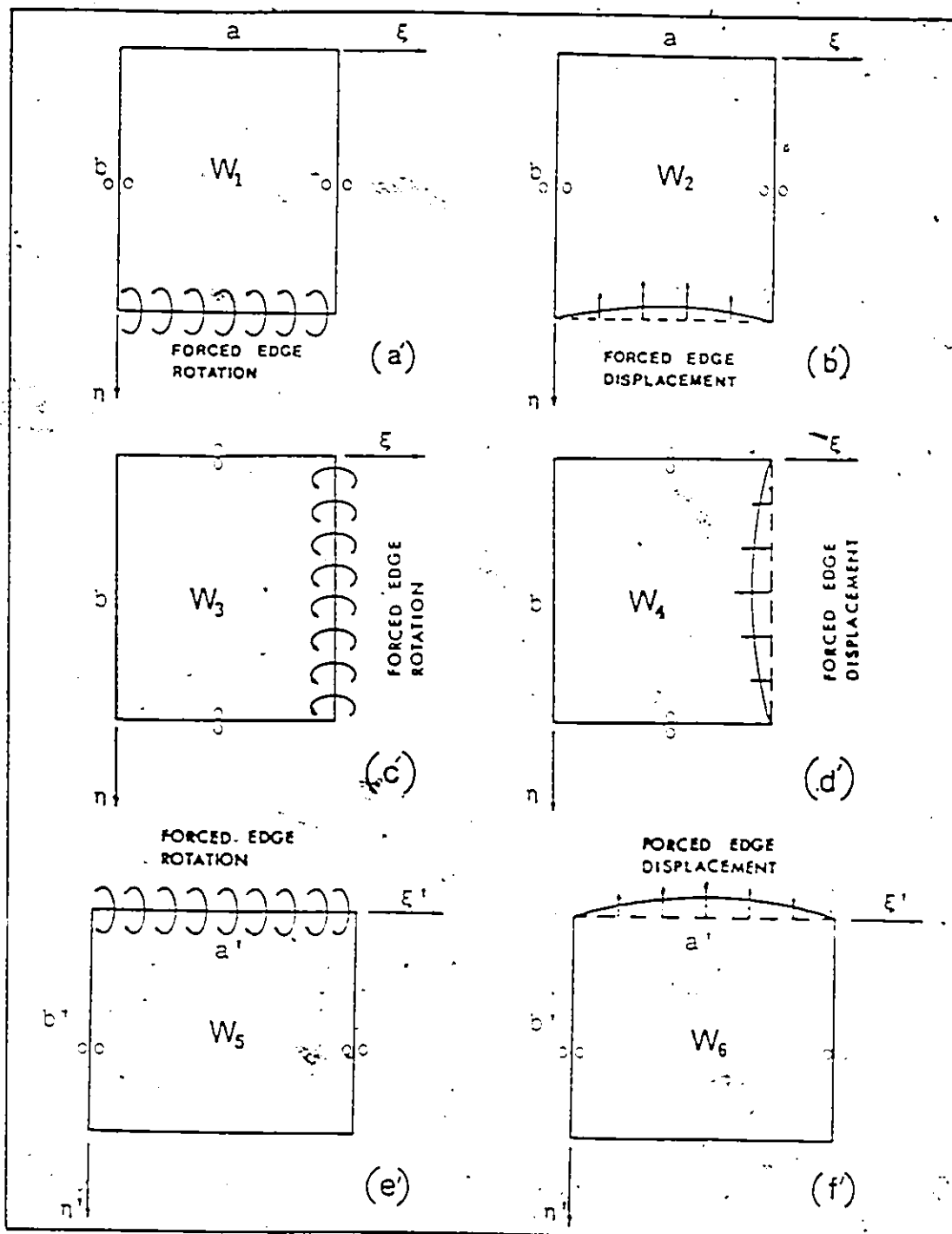


Figure 2.6- Alternative set of basic building blocks used in the solution of the triangular element.

The second building block or Plate (b') has a harmonic lateral motion along the edge $\eta = 1$ given by;

$$W_2(\xi, \eta) \Big|_{\eta=1} = \sum_{m=0}^{\infty} V_{2m} \cos(m\pi\xi). \quad (2.15)$$

The Lévy type solution for this building block is;

$$W_2(\xi, \eta) = \sum_{m=0}^{k^*} V_{2m} \theta_{21m} (\sinh \beta_{1m} \eta + C_{21m} \sinh \gamma_{1m} \eta) \cos(m\pi\xi) + \sum_{m=k^*+1}^{\infty} V_{2m} \theta_{22m} (\sinh \beta_{1m} \eta + C_{22m} \sinh \gamma_{1m} \eta) \cos(m\pi\xi), \quad (2.16)$$

For Plates (c') and (d') it is seen that their solutions are easily obtained from Equations (2.14) and (2.16) by interchanging the variables ξ and η . The aspect ratio ϕ_1 must also be replaced by its inverse and the quantity λ^2 must be replaced by $\phi_1^2 \lambda^2$. Considering Plate (c'), the harmonic moment along the edge $\xi = 1$ is given by;

$$\frac{M_n a^2}{bD} \Big|_{\xi=1} = \sum_{m=0}^{\infty} E_{3m} \cos(m\pi\eta). \quad (2.17)$$

The corresponding Lévy type solution is;

$$W_3(\xi, \eta) = \sum_{m=0}^{k^*} E_{3m} \theta_{31m} (\sinh \beta_{3m} \xi + C_{31m} \sinh \gamma_{3m} \xi) \cos(m\pi\eta) + \sum_{m=k^*+1}^{\infty} E_{3m} \theta_{32m} (\sinh \beta_{3m} \xi + C_{32m} \sinh \gamma_{3m} \xi) \cos(m\pi\eta). \quad (2.18)$$

Plate (d') has a harmonic lateral motion along the edge $\eta = 1$ given by;

$$W_4(\xi, \eta) \Big|_{\xi=1} = \sum_{m=0}^{\infty} V_{4m} \cos(m\pi\eta), \quad (2.19)$$

and its Lévy type solution is;

$$W_4(\xi, \eta) = \sum_{m=0}^{k^*} V_{4m} \theta_{41m} (\sinh \beta_{3m} \xi + C_{41m} \sinh \gamma_{3m} \xi) \cos(m\pi\eta) + \sum_{m=k^*}^{\infty} V_{4m} \theta_{42m} (\sinh \beta_{3m} \xi + C_{42m} \sinh \gamma_{3m} \xi) \cos(m\pi\eta). \quad (2.20)$$

For Plates (e') and (f'), solutions are obtained from Equations (2.14) and (2.16) simply by replacing η by $1 - \eta'$ and ξ by ξ' . Note that Plates (e') and (f') have the dimensions b' and a' , and that their aspect ratio is different from that of the other four plates. As for Plate (e'), it has a prescribed harmonic moment along the edge $\eta' = 0$ given by;

$$\frac{M_n b'^2}{a' D} \Big|_{\eta'=0} = \sum_{m=0}^{\infty} E_{5m} \cos(m\pi\xi'), \quad (2.21)$$

and a Lévy type solution given by;

$$W_5(\xi', \eta') = \sum_{m=0}^{k^*} E_{5m} \theta_{51m} [\sinh \beta_{5m} (1 - \eta') + C_{51m} \sinh \gamma_{5m} (1 - \eta')] \cos(m\pi\xi') + \sum_{m=k^*+1}^{\infty} E_{5m} \theta_{52m} [\sinh \beta_{5m} (1 - \eta') + C_{52m} \sinh \gamma_{5m} (1 - \eta')] \cos(m\pi\xi'). \quad (2.22)$$

The last building block in Figure 2.6, or Plate (f'), has a harmonic lateral displacement along the edge $\eta' = 0$ prescribed as;

$$W_6(\xi', \eta') \Big|_{\eta'=0} = \sum_{m=0}^{\infty} V_{6m} \cos(m\pi \xi'). \quad (2.23)$$

The Lévy type solution for this building block is;

$$W_6(\xi', \eta') = \sum_{m=0}^{k^*} V_{6m} \theta_{61m} [\sinh \beta_{5m}(1 - \eta') + C_{61m} \sinh \gamma_{5m}(1 - \eta')] \cos(m\pi \xi') \\ + \sum_{m=k^*+1}^{\infty} V_{6m} \theta_{62m} [\sinh \beta_{5m}(1 - \eta') + C_{62m} \sinh \gamma_{5m}(1 - \eta')] \cos(m\pi \xi') \quad (2.24)$$

Variables appearing in all of the above equations are as defined in the previous section.

2.2-3 Superposition of Building Blocks

One thus has dynamic response solutions available for all plate problems of Figures 2.5 and 2.6. These plate solutions act as the basic building blocks for obtaining solutions for the triangular plate element free vibration problem. The first four building blocks are superimposed immediately on top of each other. Plates (e) and (f) are superimposed on top of this assembly with their base $\eta' = 0$ lying along the diagonal of the first assembly and with their outer edge $\eta' = 1$ passing through its outer corner as shown in Figure 2.7. Considering now the right angle triangular plate segment enclosed by the diagonal of the first four building blocks of Figure 2.7 and their edges $\xi = 1$ and $\eta = 1$, we note that one distributed forced edge moment and one distributed forced edge displacement appears along each edge of this region. Adjustment of the coefficients appearing in the expression of these edge moments and displacements will allow us to prescribe either simple support or clamped boundary conditions or a combination of both as will become clear in future chapters.

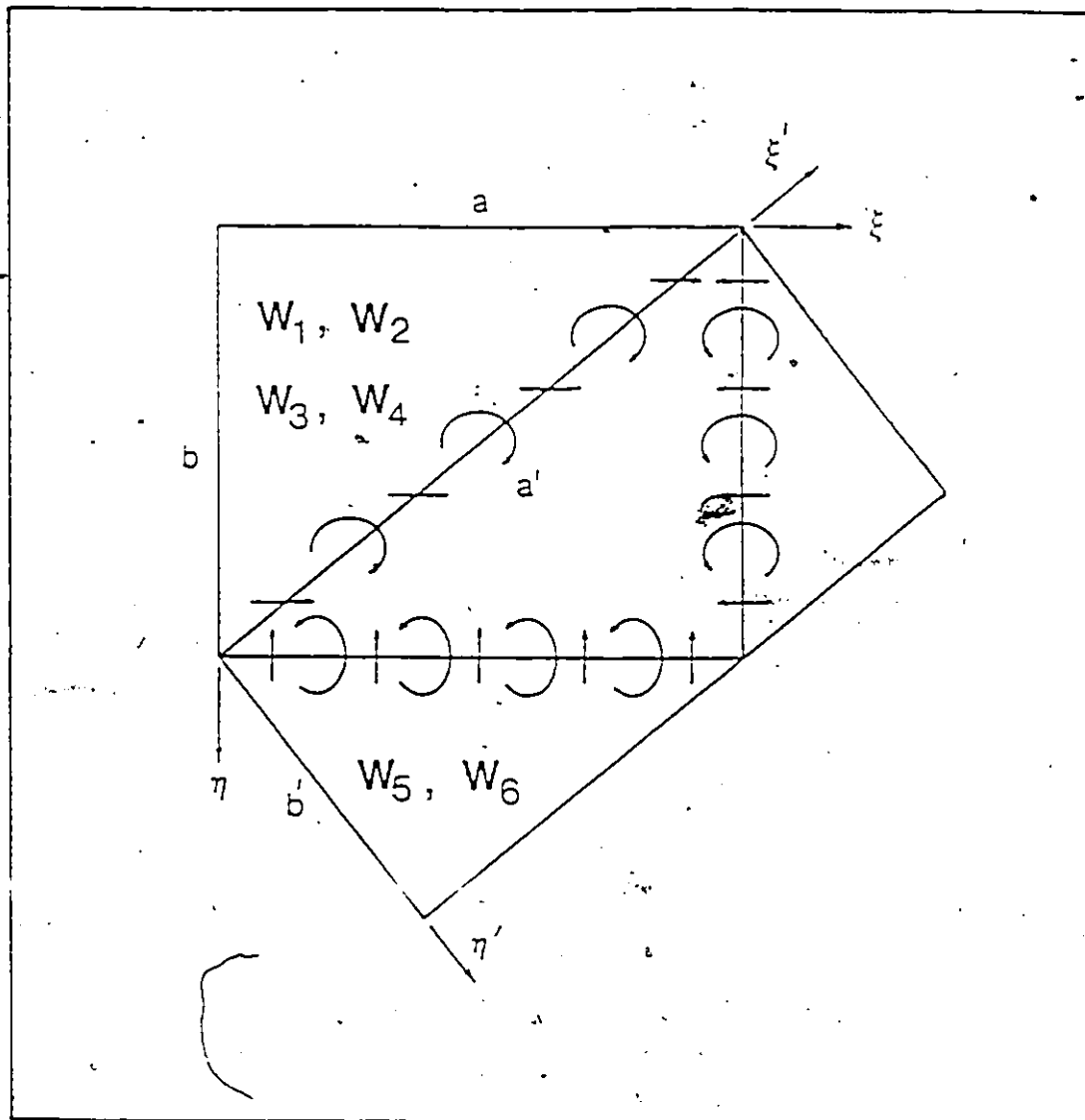


Figure 2.7- Superposition of building blocks used in the solution of the triangular plate element.

2.3 Solution of the Rectangular Element

The basic building blocks needed in the solution of the rectangular segment of Figures 2.2 and 2.4 will depend on the particular application and the given boundary conditions. Superposition techniques pertinent to the solution of free vibration problems of rectangular plates are very well established and has been extensively discussed in References [2,3,4,35]. However, solutions for the basic building blocks used in the application part of this dissertation are provided in the following sections.

2.3-1 Solution of The Basic Building Blocks

Attention is now given to the set of building blocks shown in Figure 2.8. These building blocks will be used individually or in combination depending upon the particular application. The first building block or Plate (a) has simple supports on two opposite edges, slip shear on the third edge while the fourth edge has no specified boundary conditions. It is along this fourth edge that a triangular element may be joined to this building block by enforcing continuity across this boundary. In view of these boundary conditions, the Lévy type solution is;

$$W_{rs}(\xi, \eta) = \sum_{m=1}^{k^*} [A_m \cosh \beta_{rm}(1-\xi) + D_m \cos \gamma_{rm}(1-\xi)] \sin(m\pi\eta) \\ + \sum_{m=k^*+1}^{\infty} [A_m \cosh \beta_{rm}(1-\xi) + D_m \cosh \gamma_{rm}(1-\xi)] \sin(m\pi\eta), \quad (2.25)$$

where,

$$\beta_{rm} = \frac{1}{\phi_r} \sqrt{\phi_r^2 \lambda_r^2 + (m\pi)^2}, \\ \gamma_{rm} = \frac{1}{\phi_r} \sqrt{\phi_r^2 \lambda_r^2 - (m\pi)^2} \quad \text{or} \quad \frac{1}{\phi_r} \sqrt{(m\pi)^2 - \phi_r^2 \lambda_r^2},$$

whichever is real, and

$$\phi_r = \frac{b}{a_r}, \quad \lambda_r^2 = \frac{\phi_1^2}{\phi_r^2} \lambda^2, \quad \frac{a_r}{a} = \frac{\phi_1}{\phi_r},$$

and where A_m and D_m are to be obtained by enforcing the continuity conditions across the edge $\xi = 0$. Plate (b) has slip shear conditions along the edges $\xi = 0$ and $\xi = 1$, simple support conditions along the edge $\eta = 0$, a forbidden lateral movement and a prescribed harmonic rotation along its fourth edge $\eta = 1$ given by;

$$\left. \frac{\partial W_{1rs}}{\partial \eta} \right|_{\eta=1} = \sum_{m=0}^{\infty} E_{1rm} \cos(m\pi\xi) \quad (2.26)$$

The Lévy type solution obtained by enforcing these boundary conditions is;

$$\begin{aligned} W_{1rs}(\xi, \eta) = & \sum_{m=0}^{k^*} E_{1rm} \theta_{11rm} [\sinh \beta_{1rm} \eta + C_{11rm} \sin \gamma_{1rm} \eta] \cos(m\pi\xi) \\ & + \sum_{m=k^*+1}^{\infty} E_{1rm} \theta_{12rm} [\sinh \beta_{1rm} \eta + C_{12rm} \sinh \gamma_{1rm} \eta] \cos(m\pi\xi), \end{aligned} \quad (2.27)$$

where,

$$\begin{aligned} \beta_{1rm} &= \phi_r \sqrt{\lambda_r^2 + (m\pi)^2}, \\ \gamma_{1rm} &= \phi_r \sqrt{\lambda_r^2 - (m\pi)^2}, \quad \text{or} \quad \phi_r \sqrt{(m\pi)^2 - \lambda_r^2}, \end{aligned}$$

whichever is real, and

$$\begin{aligned} C_{11rm} &= -\sinh \beta_{1rm} / \sin \gamma_{1rm}, \\ C_{12rm} &= -\sinh \beta_{1rm} / \sinh \gamma_{1rm}, \\ \theta_{11rm} &= 1 / (\beta_{1rm} \cosh \beta_{1rm} + C_{11rm} \gamma_{1rm} \cos \gamma_{1rm}), \\ \theta_{12rm} &= 1 / (\beta_{1rm} \cosh \beta_{1rm} + C_{12rm} \gamma_{1rm} \cosh \gamma_{1rm}). \end{aligned}$$

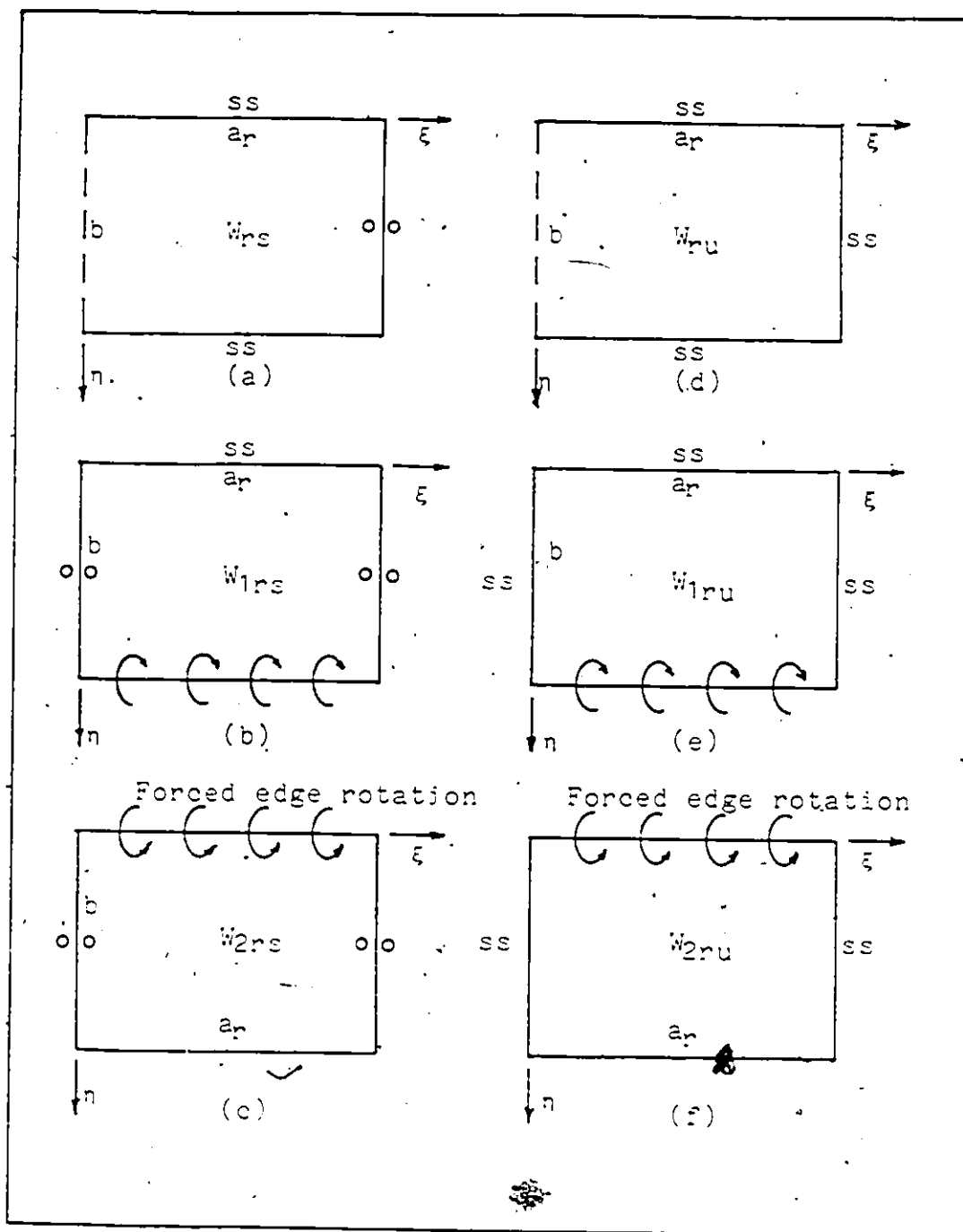


Figure 2.8- Basic building blocks used in the solution of the rectangular plate element.

The third building block or Plate (c) has the same boundary conditions as Plate (b), with the exception of the forbidden displacement and the prescribed harmonic rotation that are now along the edge $\eta = 0$, and the edge $\eta = 1$ has the simple support conditions. The harmonic rotation is given the following expression;

$$\left. \frac{\partial W_{2rs}}{\partial \eta} \right|_{\eta=0} = \sum_{m=0}^{\infty} -E_{2rm} \cos(m\pi\xi), \quad (2.28)$$

and the Lévy type solution is;

$$\begin{aligned} W_{2rs}(\xi, \eta) = & \sum_{m=0}^{k^*} E_{2rm} \theta_{11rm} [\sinh \beta_{1rm}(1-\eta) + C_{11rm} \sin \gamma_{1rm}(1-\eta)] \cos(m\pi\xi) \\ & + \sum_{m=k^*+1}^{\infty} E_{2rm} \theta_{12rm} [\sinh \beta_{1rm}(1-\eta) + \\ & C_{12rm} \sinh \gamma_{1rm}(1-\eta)] \cos(m\pi\xi). \end{aligned} \quad (2.29)$$

Considering now the remaining three building blocks, or Plates (d), (e) and (f), it is seen that they differ from Plates (a), (b) and (c) respectively in that the slip shear conditions are replaced by simple support conditions. In view of these changes, The Lévy type solutions are found to be

$$\begin{aligned} W_{ru}(\xi, \eta) = & \sum_{m=1}^{k^*} [A_m \sinh \beta_{rm}(1-\xi) + D_m \sin \gamma_{rm}(1-\xi)] \sin(m\pi\eta) \\ & + \sum_{m=k^*+1}^{\infty} [A_m \sinh \beta_{rm}(1-\xi) + D_m \sinh \gamma_{rm}(1-\xi)] \sin(m\pi\eta), \end{aligned} \quad (2.30)$$

$$\begin{aligned}
W_{1ru}(\xi, \eta) = & \sum_{m=1}^{k^*} E_{1rm} \theta_{11rum} [\sinh \beta_{1rm} \eta + C_{11rum} \sin \gamma_{1rm} \eta] \sin(m\pi \xi) \\
& + \sum_{m=k^*+1}^{\infty} E_{1rm} \theta_{12rum} [\sinh \beta_{1rm} \eta + C_{12rum} \sinh \gamma_{1rm} \eta] \sin(m\pi \xi), \quad (2.31)
\end{aligned}$$

$$\begin{aligned}
W_{2ru}(\xi, \eta) = & \sum_{m=1}^{k^*} E_{2rm} \theta_{11rum} [\sinh \beta_{1rm} (1 - \eta) + C_{11rum} \sin \gamma_{1rm} (1 - \eta)] \sin(m\pi \xi) \\
& + \sum_{m=k^*+1}^{\infty} E_{2rm} \theta_{12rum} [\sinh \beta_{1rm} (1 - \eta) + \\
& C_{12rum} \sinh \gamma_{1rm} (1 - \eta)] \sin(m\pi \xi). \quad (2.32)
\end{aligned}$$

where,

$$\begin{aligned}
C_{11rum} &= -\sinh \beta_{1rm} / \sin \gamma_{1rm}, \\
C_{12rum} &= -\sinh \beta_{1rm} / \sinh \gamma_{1rm}, \\
\theta_{11rum} &= 1 / (\beta_{1rm} \cosh \beta_{1rm} - C_{11rum} \gamma_{1rm} \cos \gamma_{1rm}), \\
\theta_{12rum} &= 1 / (\beta_{1rm} \cosh \beta_{1rm} + C_{12rum} \gamma_{1rm} \cosh \gamma_{1rm}).
\end{aligned}$$

The prescribed harmonic rotations are given by;

$$\begin{aligned}
\left. \frac{\partial W_{1ru}}{\partial \eta} \right|_{\eta=1} &= \sum_{m=1}^{\infty} E_{1rm} \sin(m\pi \xi), \\
\left. \frac{\partial W_{2ru}}{\partial \eta} \right|_{\eta=0} &= \sum_{m=1}^{\infty} -E_{2rm} \sin(m\pi \xi).
\end{aligned}$$

2.3-2 Superposition of Building Blocks

We now have available dynamic response solutions for all plate problems of Figure 2.8. Plate (a) used alone will represent a rectangular element with simple support conditions and an axis of symmetry along the edge $\xi = 1$ where slip shear conditions are prescribed. Should the need arise to join another triangular element to this rectangular element along this edge, one may do so by not prescribing any edge boundary condition, so that enforcement of continuity conditions becomes possible. By superimposing Plates (a), (b) and (c) on top of each other, we obtain a rectangular element with zero displacement and a forced harmonic rotation along the outer edges, $\eta = 0$ and $\eta = 1$. By properly adjusting the coefficients appearing in the expressions of these rotations, clamped edge conditions can be satisfied. For simple support conditions along the edge $\xi = 1$, or for anti-symmetric free vibration modes of the rectangular element discussed above, Plate (d) is used alone for simple support conditions along the two outer edges, and is super imposed above Plates (e) and (f) for clamped edge conditions. ▽

We now have solutions for both plate elements. The next step is to discuss the technique of joining these elements together.

2.4 Joining of Triangular and Rectangular Elements

In order to join the two elements together, they are put back to back as shown in Figure 2.9. The following continuity conditions are then enforced across the inter-segment line;

- 1.- Continuity of displacement.
- 2.- Continuity of slope.
- 3.- Continuity of moment.
- 4.- Zero net vertical edge reaction.

This procedure is best understood by illustrative examples as will be demonstrated in the application part of this dissertation.

We now have all the basic derivations, techniques, and building block solutions necessary to move to the next chapter where the problem of the free vibration of trapezoidal plates is discussed in detail.

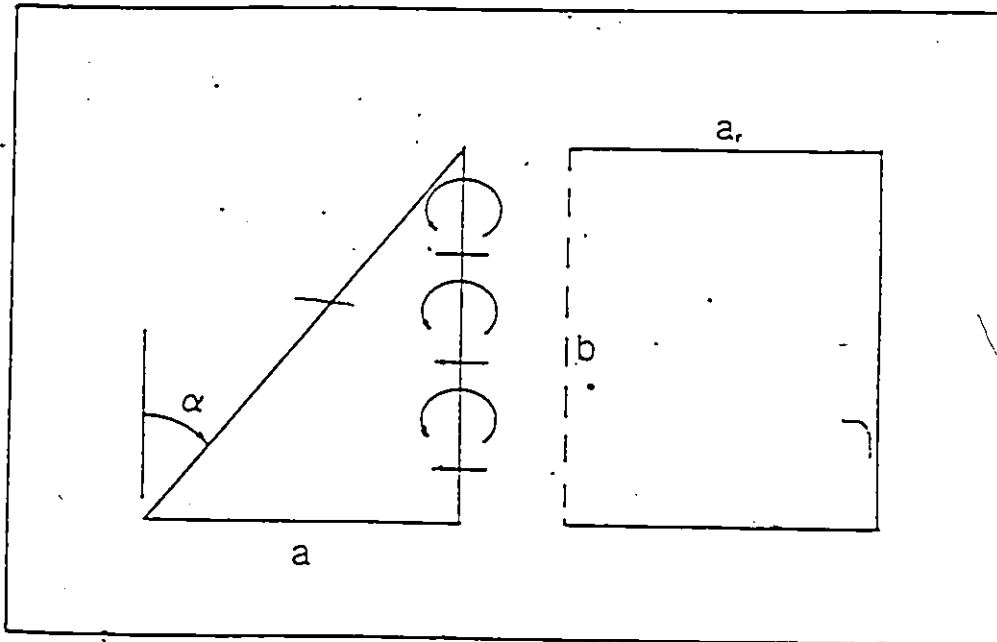


Figure 2.9- Joining of the two plate elements.

PART C

APPLICATION

Chapter 3

SIMPLY SUPPORTED SYMMETRICAL TRAPEZOIDAL PLATES

3.1 Basic Building Blocks and their Superposition

The plate under consideration is shown in Figure 3.1. Physical considerations dictate that all possible free vibration modes must be symmetric or anti-symmetric with respect to the central axis. Dealing with each of these two possible families of modes separately will help greatly simplify the analysis. The basic building blocks used in this solution are shown in Figures 3.2 and 3.3. Solutions for these building blocks are available from the previous chapter. The building blocks of Figure 3.2 are superimposed in the manner discussed in Section 2.2-3, then joined with plate (a) of Figure 3.3 to obtain a solution to the symmetric mode free vibration problem of the trapezoidal plate under study. If the triangular part is joined to Plate (b) of this figure, a solution to the anti-symmetric mode will result.

3.2 Solution by Enforcement of Boundary Conditions

With the appropriate building blocks properly in place, one need only constrain the coefficients appearing in these solutions in such a way that the net bending moment and the net displacement vanish along each of the outer

edges of the triangular region of the plate of Figure 3.1, creating four sets of homogeneous constraint equations D'_1 , M'_1 , D'_3 , and M'_3 as represented schematically in the three term expansion coefficient matrix of Figure 3.4. Four continuity conditions are then enforced along the inter-segment of the rectangular and triangular regions. The set of equations D'_2 then arises from the condition of continuity of displacement, M'_2 arises from the condition of continuity of moments, S'_2 arises from the condition of continuity of slope across the inter-segment line, and finally the set of equations R'_2 arises from the condition of continuity of vertical edge reaction. Then, by requiring the determinant of this coefficient matrix to vanish, eigenvalues are obtained.

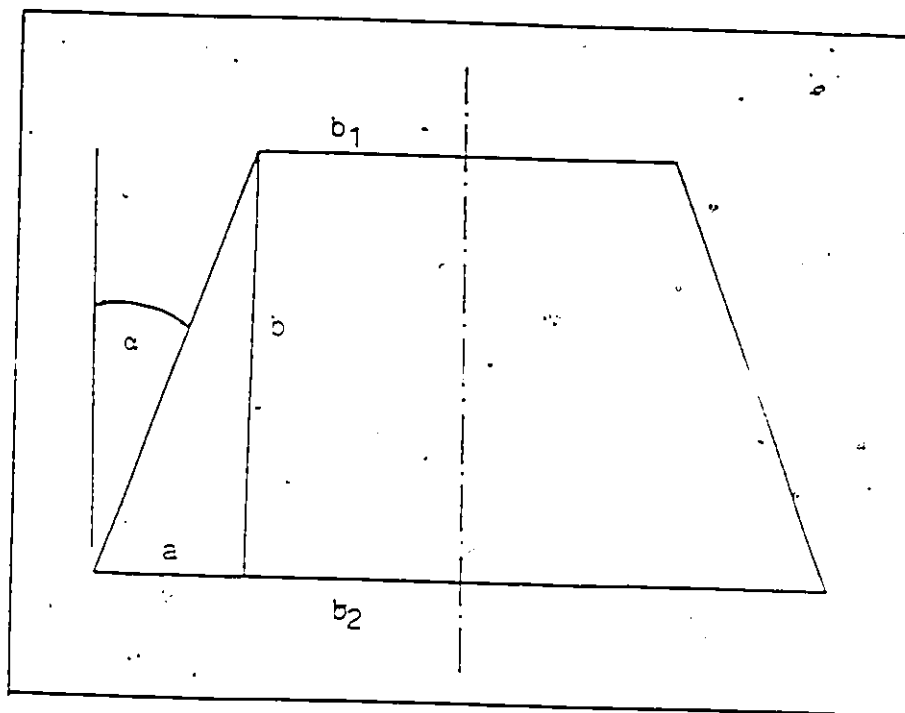


Figure 3.1- Simply supported symmetrical trapezoidal plate.

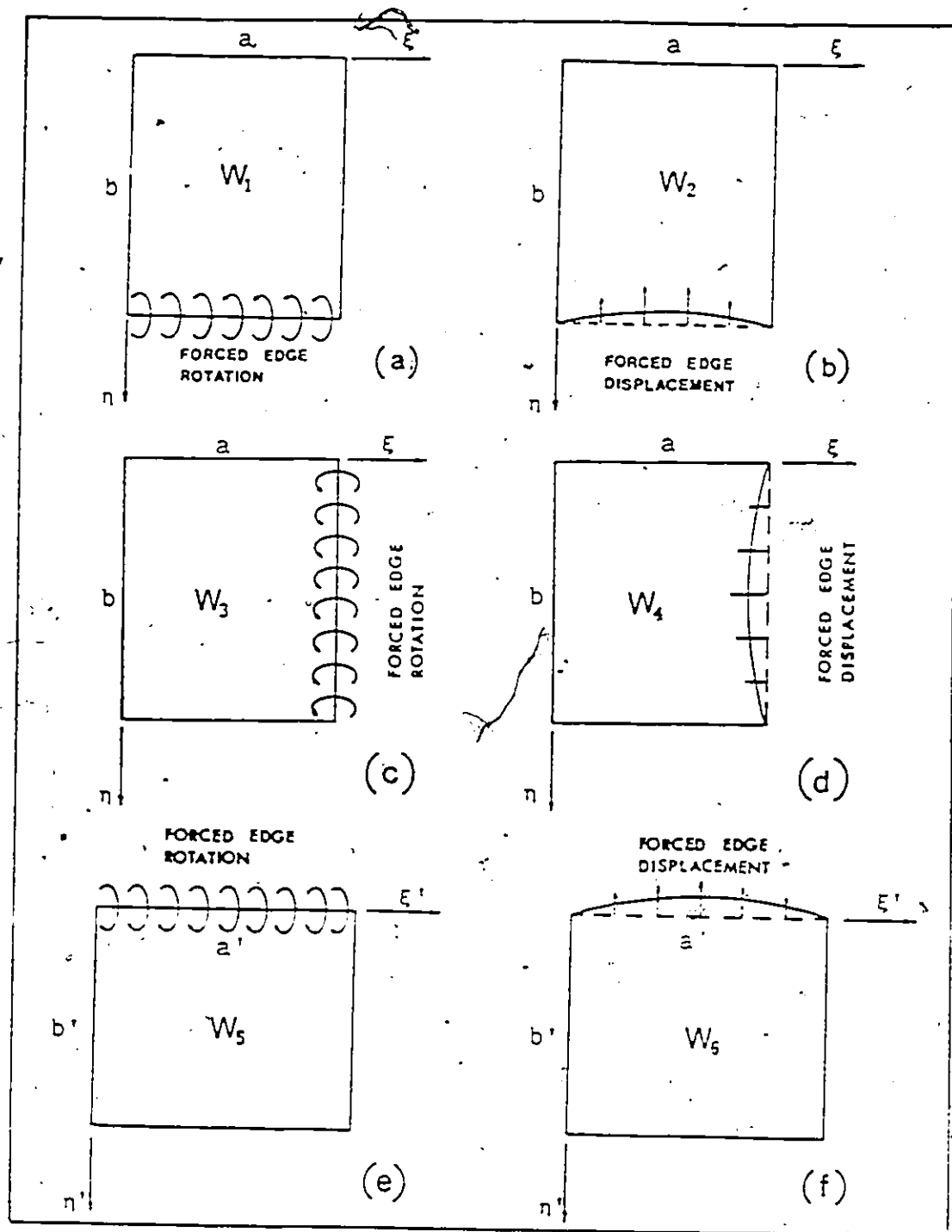


Figure 3.2- First set of basic building blocks used in the analysis of the simply supported symmetrical trapezoidal plate of Figure 3.1.

3.3 Satisfaction of Displacement Requirements

When superimposed, each of the basic building blocks has its own contributions expressed in the form of a trigonometric series. In order to be able to add relevant contributions of different building blocks together, it is necessary to express these contributions in the same type of Fourier series as will be seen in the following sections.

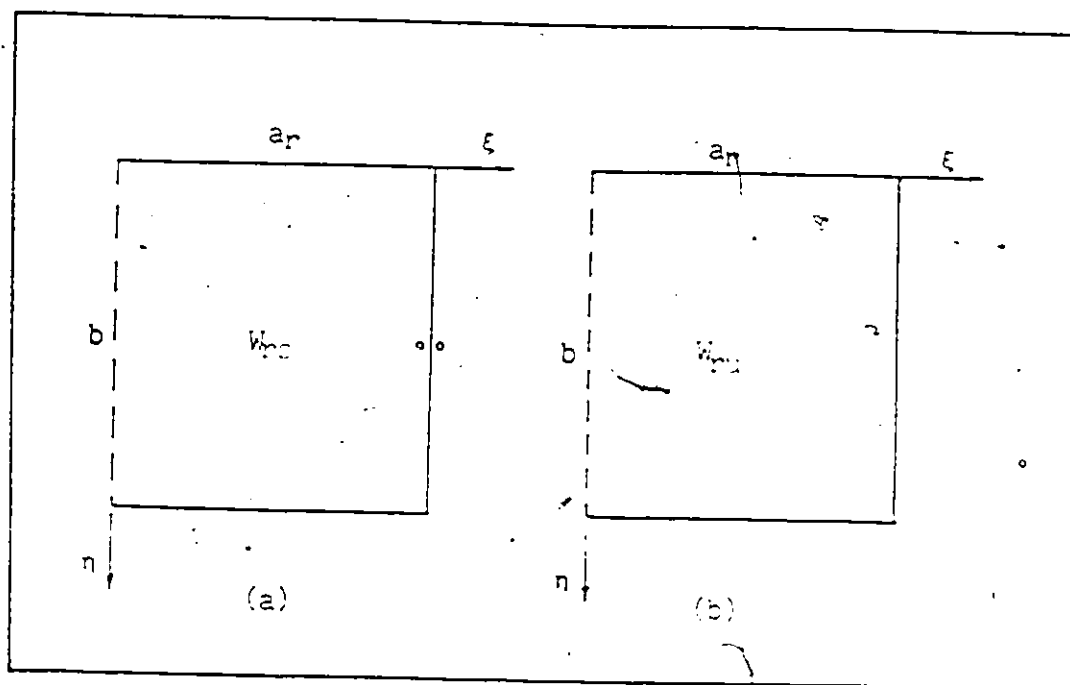


Figure 3.3- Second set of building blocks used in the analysis of the simply supported symmetrical trapezoidal plate of Figure 3.1.

	E_{1m}	V_{2m}	E_{3m}	V_{4m}	E_{5m}	V_{6m}	A_m	D_m
D'_1		-			- - -	- - -		
D'_2				-	- - -	- - -	-	-
D'_3	- - -	- - -	- - -	- - -		-		
M'_1	-				- - -	- - -		
M'_2			-		- - -	- - -	-	-
M'_3	- - -	- - -	- - -	- - -	-			
S'_2	- - -	- - -	-	-	- - -	- - -	-	-
VE R'_2	- - -	- - -	-	-	- - -	- - -	-	-

Figure 3.4- Schematic representation of a three term expansion coefficient matrix.

$$W_2(\xi, 1) = \sum_{m=1}^{\infty} V_{2m} \sin(m\pi \xi).$$



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$$\begin{aligned}
W_5(\xi, 1) = & \sum_{m=1}^{k^*} E_{5m} \theta_{51m} [\sinh \beta_{5m} (1 - \xi) + C_{51m} \sinh \gamma_{5m} (1 - \xi)] \sin(m\pi F_2 \xi) \\
& + \sum_{m=k^*+1}^{\infty} E_{5m} \theta_{52m} [\sinh \beta_{5m} (1 - \xi) + \\
& C_{52m} \sinh \gamma_{5m} (1 - \xi)] \sin(m\pi F_2 \xi). \quad (3.2)
\end{aligned}$$

In its present form, $W_5(\xi)$ can not be added to $W_2(\xi, 1)$ from Equation (3.1). In order to accomplish this task, $W_5(\xi)$ is expanded in a Fourier sine series of the form;

$$F(\xi) = \sum_{n=1}^{\infty} A_n \sin(n\pi \xi), \quad (3.3)$$

where A_n is given as;

$$A_n = 2 \int_0^1 F(\xi) \sin(n\pi \xi) d\xi. \quad (3.4)$$

Let,

$$W_5(\xi) = \sum_{m=1}^{k^*} E_{5m} \theta_{51m} [I + C_{51m} II] + \sum_{m=k^*+1}^{\infty} E_{5m} \theta_{52m} [I + C_{52m} III],$$

then,

$$I = 2 \int_0^1 \sinh \beta_{5m} (1 - \xi) \sin(m\pi F_2 \xi) \sin(n\pi \xi) d\xi.$$

Taking advantage of the following trigonometric identity,

$$\sin(m\pi F_2 \xi) \sin(n\pi \xi) = \frac{1}{2} \cos(m\pi F_2 - n\pi) \xi - \frac{1}{2} \cos(m\pi F_2 + n\pi) \xi,$$

I, can be expressed as the sum of A_1 and A_2 where,

$$A_1 = \int_0^1 \sinh \beta_{5m} (1 - \xi) \cos(m\pi F_2 - n\pi) \xi d\xi,$$

$$A_2 = - \int_0^1 \sinh \beta_{5m} (1 - \xi) \cos(m\pi F_2 + n\pi) \xi d\xi.$$

Appendix A-I provides a list of the type of integrals used throughout this analysis. Performing the above integrations to find,

$$A_1 = \beta_{5m} [\cosh \beta_{5m} - \cos(m\pi F_2 - n\pi)] / [\beta_{5m}^2 + (m\pi F_2 - n\pi)^2],$$

$$A_2 = \beta_{5m} [\cos(m\pi F_2 + n\pi) - \cosh \beta_{5m}] / [\beta_{5m}^2 + (m\pi F_2 + n\pi)^2].$$

The expression for II is;

$$II = 2 \int_0^1 \sin \gamma_{5m} (1 - \xi) \sin(m\pi F_2 \xi) \sin(n\pi \xi) d\xi.$$

Taking advantage of the same trigonometric identity as above, II , can be written as the sum of A_3 and A_4 where,

$$A_3 = \int_0^1 \sin \gamma_{5m} (1 - \xi) \cos(m\pi F_2 - n\pi) \xi d\xi,$$

$$A_4 = - \int_0^1 \sin \gamma_{5m} (1 - \xi) \cos(m\pi F_2 + n\pi) \xi d\xi.$$

performing the above integration, the following results are found;

$$\text{if } \gamma_{5m}^2 \neq (m\pi F_2 - n\pi)^2,$$

$$A_3 = \gamma_{5m} [\cos \gamma_{5m} - \cos(m\pi F_2 - n\pi)] / [(m\pi F_2 - n\pi)^2 - \gamma_{5m}^2],$$

$$\text{but if } \gamma_{5m}^2 = (m\pi F_2 - n\pi)^2, \text{ then } A_3 = \sin \gamma_{5m} / 2.$$

$$\text{If } \gamma_{5m}^2 \neq (m\pi F_2 + n\pi)^2,$$

$$A_4 = \gamma_{5m} [\cos(m\pi F_2 + n\pi) - \cos \gamma_{5m}] / [(m\pi F_2 + n\pi)^2 - \gamma_{5m}^2],$$

$$\text{or if } \gamma_{5m}^2 = (m\pi F_2 + n\pi)^2, \text{ then } A_4 = -\sin \gamma_{5m} / 2.$$

The expression for III is;

$$III = 2 \int_0^1 \sinh \gamma_{5m} (1 - \xi) \sin(m\pi F_2 \xi) \sin(n\pi \xi) d\xi.$$

Expressing this integral as the sum of A_5 and A_6 , it is easily seen that the expression of A_5 is found from that of A_1 by replacing β_{5m} by γ_{5m} . Similarly, the expression of A_6 is also found from that of A_2 by replacing β_{5m} by γ_{5m} .

The contribution of $W_6(\xi', \eta')$ to the displacement along this edge is found from equation (2.12) by replacing η' by ξ , and ξ' by $F_2 \xi$.

$$\begin{aligned} W_6(\xi, 1) = & \sum_{m=1}^{k^*} V_{6m} \theta_{61m} [\sinh \beta_{5m} (1 - \xi) + C_{61m} \sin \gamma_{5m} (1 - \xi)] \sin(m\pi F_2 \xi) \\ & + \sum_{m=k^*+1}^{\infty} V_{6m} \theta_{62m} [\sinh \beta_{5m} (1 - \xi) + \\ & C_{62m} \sinh \gamma_{5m} (1 - \xi)] \sin(m\pi F_2 \xi), \end{aligned} \quad (3.5)$$

which after performing the Fourier expansion can be written as;

$$W_6(\xi, 1) = \sum_{m=1}^{k^*} V_{6m} \theta_{61m} [I + C_{61m} II] + \sum_{m=k^*+1}^{\infty} V_{6m} \theta_{62m} [I + C_{62m} III],$$

where, $I = A_1 + A_2$, $II = A_3 + A_4$, and $III = A_5 + A_6$.

We now have the contributions of all relevant building blocks to the displacement along the edge $\eta = 1$ expressed in the same type of Fourier series. Therefore, for zero displacement, one could impose the constraint that the sum of the coefficients before each of the Fourier trigonometric functions must equal zero. Or for this edge,

$$D'_1 = A_{i,j} E_{1m} + A_{i,j+k} V_{2m} + A_{i,j+2k} E_{3m} + A_{i,j+3k} V_{4m} + \\ A_{i,j+4k} E_{5m} + A_{i,j+5k} V_{6m} + A_{i,j+6k} A_m + A_{i,j+7k} D_m = 0,$$

where,

$$A_{i,j} = A_{i,j+2k} = A_{i,j+3k} = A_{i,j+4k} = A_{i,j+7k} = 0.$$

$$A_{i,i+k} = 1.$$

$$A_{i,j+4k} = \theta_{51m} [A_1 + A_2 + C_{51m} (A_3 + A_4)],$$

$$A_{i,j+5k} = \theta_{61m} [A_1 + A_2 + C_{61m} (A_3 + A_4)],$$

$$\text{or if } \lambda'^2 < (m\pi)^2,$$

$$A_{i,j+4k} = \theta_{52m} [A_1 + A_2 + C_{52m} (A_5 + A_6)],$$

$$A_{i,j+5k} = \theta_{62m} [A_1 + A_2 + C_{62m} (A_5 + A_6)].$$

This procedure is repeated for the contributions of all relevant building blocks to the various edge boundary conditions, as well as the continuity conditions across the inter-segment line, of the plate under study. In order to avoid repetition as much as possible, results will be listed and only discussed when necessary. It must also be stressed here that symbols used are uniquely defined throughout the analysis unless clearly stated otherwise.

3.3-2 Displacement along the Edge $\xi = 1$

The first three building blocks or W_1, W_2 and W_3 have no contribution to the displacement along this edge. W_4 has a forced harmonic displacement given by Equation (2.8) as;

$$W_4(1, \eta) = \sum_{m=1}^{\infty} V_{4m} \sin(m\pi\eta). \quad (3.6)$$

It can be shown that along the edge $\xi = 1$, $\eta' = \eta$ and $\xi' = 1 - F_1\eta$, where $F_1 = \sin^2\alpha$.

The contribution of W_5 to the displacement along this edge is obtained from Equation (2.10) by replacing η' by η and ξ' by $1 - F_1\eta$. The resulting expression is;

$$\begin{aligned} W_5(1, \eta) = & \sum_{m=1}^{k^*} E_{5m} \theta_{51m} [\sinh \beta_{5m} (1 - \eta) + \\ & C_{51m} \sin \gamma_{5m} (1 - \eta)] \sin [m\pi (1 - F_1\eta)] \\ & + \sum_{m=k^*+1}^{\infty} E_{5m} \theta_{52m} [\sinh \beta_{5m} (1 - \eta) + \\ & C_{52m} \sinh \gamma_{5m} (1 - \eta)] \sin [m\pi (1 - F_1\eta)]. \end{aligned}$$

Expanding in a Fourier sine series of the type shown by Equation (3.6), W_5 is expressed as follows;

$$W_5(\eta) = \sum_{m=1}^{k^*} E_{5m} \theta_{51m} [I + C_{51m} II] + \sum_{m=k^*+1}^{\infty} E_{5m} \theta_{52m} [I + C_{52m} III],$$

where, $I = B_1 + B_2$, $II = B_3 + B_4$, $III = B_5 + B_6$, and where,

$$B_1 = \frac{\beta_{5m} [\cosh \beta_{5m} \cos(m\pi) - \cos(m - n - mF_1)\pi]}{[\beta_{5m}^2 + (m\pi F_1 + n\pi)^2]},$$

$$B_2 = \frac{\beta_{5m} [\cos(m - n + mF_1)\pi - \cosh \beta_{5m} \cos(m\pi)]}{[\beta_{5m}^2 + (m\pi F_1 - n\pi)^2]},$$

$$B_3 = \frac{\gamma_{5m} [\cos(\gamma_{5m} - m\pi) - \cos(mF_1 - m + n)\pi]}{[(m\pi F_1 + n\pi)^2 - \gamma_{5m}^2]},$$

or if $\gamma_{5m}^2 = (m\pi F_1 + n\pi)^2$, then, $B_3 = \sin(\gamma_{5m} + m\pi)/2$,

$$B_4 = \frac{\gamma_{5m} [\cos(mF_1 - m - n)\pi - \cos(\gamma_{5m} - m\pi)]}{[(m\pi F_1 - n\pi)^2 - \gamma_{5m}^2]},$$

or if $\gamma_{5m}^2 = (m\pi F_1 - n\pi)^2$, then, $B_4 = \sin(m\pi - \gamma_{5m})/2$.

B_5 and B_6 are found from the expressions of B_1 and B_2 respectively, simply by replacing β_{5m} by γ_{5m} .

The contribution of W_6 to the displacement along the edge $\xi = 1$ is obtained from Equation (2.12) by replacing η' by η and ξ' by $1 - F_1\eta$. The resulting equation is;

$$\begin{aligned}
W_6(1, \eta) = & \sum_{m=1}^{k^*} V_{6m} \theta_{61m} [\sinh \beta_{5m} (1 - \eta) + \\
& C_{61m} \sin \gamma_{5m} (1 - \eta)] \sin [m\pi (1 - F_1 \eta)] \\
& + \sum_{m=k^*+1}^{\infty} V_{6m} \theta_{62m} [\sinh \beta_{5m} (1 - \eta) + \\
& C_{62m} \sin \gamma_{5m} (1 - \eta)] \sin [m\pi (1 - F_1 \eta)].
\end{aligned}$$

In its expanded form, W_6 becomes;

$$W_6(1, \eta) = \sum_{m=1}^{k^*} V_{6m} \theta_{61m} [I + C_{61m} II] + \sum_{m=k^*+1}^{\infty} V_{62m} [I + C_{62m} III],$$

where again $I = B_1 + B_2$, $II = B_3 + B_4$ and $III = B_5 + B_6$.

This concludes the contribution of the rectangular element to the displacement along this edge. The relevant elements of the coefficient matrix are the following;

$$A_{i+k,j} = A_{i+k,j+k} = A_{i+k,j+2k} = 0,$$

$$A_{i+k,i+3k} = 1.,$$

$$A_{i+k,j+4k} = \theta_{51m} [B_1 + B_2 + C_{51m} (B_3 + B_4)],$$

$$A_{i+k,j+5k} = \theta_{61m} [B_1 + B_2 + C_{61m} (B_3 + B_4)],$$

or if $\lambda'^2 < (m\pi)^2$,

$$A_{i+k,j+4k} = \theta_{52m} [B_1 + B_2 + C_{52m} (B_5 + B_6)],$$

$$A_{i+k,j+5k} = \theta_{62m} [B_1 + B_2 + C_{62m} (B_5 + B_6)].$$

In order to satisfy the condition of displacement continuity along this common segment, the contribution of the rectangular element is considered. For symmetric modes this contribution is found from Equation (2.25) by replacing ξ by zero. Since the two displacements, that of the rectangular and triangular elements, must be the same, their difference is equated to zero. Therefore, for symmetric modes;

$$A_{i+k,i+6k} = -\cosh\beta_{rm},$$

$$A_{i+k,i+7k} = -\cos\gamma_{rm},$$

$$\text{or if } \phi_r^2 \lambda_r^2 < (m\pi)^2,$$

$$A_{i+k,i+7k} = -\cosh\gamma_{rm}.$$

And for anti-symmetric modes,

$$A_{i+k,i+6k} = -\sinh\beta_{rm},$$

$$A_{i+k,i+k} = -\sin\gamma_{rm},$$

$$\text{or if } \phi_r^2 \lambda_r^2 < (m\pi)^2,$$

$$A_{i+k,i+7k} = -\sinh\gamma_{rm}.$$

Next, we look at the displacement along the third edge or $\eta' = 0$.

3.3-3 Displacement Along the Edge $\eta' = 0$

It can be shown that along this edge, $\xi' = \xi$ and $\eta = 1 - \xi$. The only building block that has no contribution to the displacement here is W_5 . W_6 provides the forced harmonic displacement the expression of which is obtained from Equation (2.11) by replacing ξ' by ξ .

$$W_6(\xi, 0) = \sum_{m=1}^{\infty} V_{6m} \sin(m\pi\xi). \quad (3.7)$$

Starting now by the evaluation of the contribution of the first building block to the displacement, we find that by replacing η by $1 - \xi$ in Equation (2.2) we obtain;

$$W_1(\xi) = \sum_{m=1}^{k^*} E_{1m} \theta_{11m} [\sinh \beta_{1m}(1 - \xi) + C_{11m} \sin \gamma_{1m}(1 - \xi)] \sin(m\pi\xi) \\ + \sum_{m=k^*+1}^{\infty} E_{1m} \theta_{12m} [\sinh \beta_{1m}(1 - \xi) + C_{12m} \sinh \gamma_{1m}(1 - \xi)] \sin(m\pi\xi),$$

which after expansion assumes the form;

$$W_1(\xi) = \sum_{m=1}^{k^*} E_{1m} \theta_{11m} [I + C_{11m} II] + \sum_{m=k^*+1}^{\infty} E_{1m} \theta_{12m} [I + C_{12m} II],$$

where, $I = C_1 + C_2$, $II = C_3 + C_4$, and $III = C_5 + C_6$ and where,

$$C_1 = \beta_{1m} [\cosh \beta_{1m} - \cos(m\pi - n\pi)] / [\beta_{1m}^2 + (m\pi - n\pi)^2],$$

$$C_2 = \beta_{1m} [\cos(m\pi + n\pi) - \cosh \beta_{1m}] / [\beta_{1m}^2 + (m\pi + n\pi)^2],$$

if $\gamma_{1m}^2 = (m\pi - n\pi)^2$, then,

$$C_3 = \gamma_{1m}[\cos\gamma_{1m} - \cos(m\pi - n\pi)]/[(m\pi - n\pi)^2 - \gamma_{1m}^2],$$

or if $\gamma_{1m}^2 = (m\pi - n\pi)^2$, $C_3 = \sin\gamma_{1m}/2$, and if $\gamma_{1m}^2 \neq (m\pi - n\pi)^2$, then,

$$C_4 = \gamma_{1m}[\cos(m\pi + n\pi) - \cos\gamma_{1m}]/[(m\pi + n\pi)^2 - \gamma_{1m}^2],$$

or if $\gamma_{1m}^2 = (m\pi + n\pi)^2$, $C_4 = -\sin\gamma_{1m}/2$.

C_5 and C_6 are found simply by replacing β_{1m} by γ_{1m} in the expressions of C_1 and C_2 respectively.

The contribution of W_2 to the displacement along this edge is found from Equation (2.4), which after replacing η by $1 - \eta$ and performing the proper Fourier expansion assumes the following form;

$$W_2(\xi) = \sum_{m=1}^{k^*} V_{2m}\theta_{21m}[I + C_{21m}II] + \sum_{m=k^*+1}^{\infty} V_{2m}\theta_{22m}[I + C_{22m}III],$$

where, $I = C_1 + C_2$, $II = C_3 + C_4$, and $III = C_5 + C_6$.

Contribution of W_3 is now considered by replacing η by $1 - \xi$ in Equation (2.5). The resulting expression is then expanded in a sine Fourier series similar to that of Equation (3.7) to obtain;

$$W_3(\xi) = \sum_{m=1}^{k^*} E_{3m}\theta_{31m}[I + C_{31m}II] + \sum_{m=k^*+1}^{\infty} E_{3m}\theta_{32m}[I + C_{32m}III],$$

where, $I = D_1 + D_2$, $II = D_3 + D_4$, and $III = D_5 + D_6$, and where,

$$D_1 = \beta_{3m} [\cosh \beta_{3m} \cos(n\pi) - \cos(m\pi)] / [\beta_{3m}^2 + (m\pi + n\pi)^2],$$

$$D_2 = \beta_{3m} [\cos(m\pi) - \cosh \beta_{3m} \cos(n\pi)] / [\beta_{3m}^2 + (m\pi - n\pi)^2],$$

if $\gamma_{3m}^2 \neq (m\pi + n\pi)^2$, then,

$$D_3 = \gamma_{3m} [\cos(m\pi) - \cos(\gamma_{3m} + n\pi)] / [\gamma_{3m}^2 - (m\pi + n\pi)^2],$$

or if $\gamma_{3m}^2 = (m\pi + n\pi)^2$,

$$D_3 = [\cos(m\pi) - \cos(2\gamma_{3m} + m\pi)] / 4\gamma_{3m},$$

and if $\gamma_{3m}^2 \neq (m\pi - n\pi)^2$,

$$D_4 = \gamma_{3m} [\cos(\gamma_{3m} - n\pi) - \cos(m\pi)] / [\gamma_{3m}^2 - (m\pi - n\pi)^2],$$

or if $\gamma_{3m}^2 = (m\pi - n\pi)^2$,

$$D_4 = [\cos(2\gamma_{3m} + m\pi) - \cos(m\pi)] / 4\gamma_{3m}.$$

D_5 and D_6 are now found by replacing β_{3m} by γ_{3m} in the expressions of D_1 and D_2 respectively.

The contribution of W_4 to displacement along this edge is now obtained by considering Equation (2.7). In its expanded form this contribution is written as:

$$W_4(\xi) = \sum_{m=1}^{k^*} V_{4m} \theta_{41m} [I + C_{41m} II] + \sum_{m=k^*+1}^{\infty} V_{4m} \theta_{42m} [I + C_{42m} III],$$

where, $I = D_1 + D_2$, $II = D_3 + D_4$, and $III = D_5 + D_6$.

This concludes our discussion on the contributions to displacement along this edge. The corresponding coefficients are;

$$A_{i+2k,j} = \theta_{11m} [C_1 + C_2 + C_{11m} (C_3 + C_4)],$$

$$A_{i+2k,j+k} = \theta_{21m} [C_1 + C_2 + C_{21m} (C_3 + C_4)],$$

or if $\lambda^2 < (m\pi)^2$,

$$A_{i+2k,j} = \theta_{12m} [C_1 + C_2 + C_{12m} (C_5 + C_6)],$$

$$A_{i+2k,j+k} = \theta_{22m} [C_1 + C_2 + C_{22m} (C_5 + C_6)],$$

and

$$A_{i+2k,j+2k} = \theta_{31m} [D_1 + D_2 + C_{31m} (D_3 + D_4)],$$

$$A_{i+2k,j+3k} = \theta_{41m} [D_1 + D_2 + C_{41m} (D_3 + D_4)],$$

or if $\phi_1^2 \lambda^2 < (m\pi)^2$,

$$A_{i+2k,j+2k} = \theta_{32m} [D_1 + D_2 + C_{32m} (D_5 + D_6)],$$

$$A_{i+2k,j+3k} = \theta_{42m} [D_1 + D_2 + C_{42m} (D_5 + D_6)].$$

We also have,

$$A_{i+2k,j+4k} = A_{i+2k,j+5k} = A_{i+2k,j+6k} = 0,$$

and

$$A_{i+2k,i+5k} = 1.$$

By now, the reader must be questioning the validity of this procedure since W_1 and W_2 were divided by the side length a , while W_3 and W_4 were divided by the side length b . Also W_5 and W_6 were divided by a' and W_{rs} and W_{ru} were divided by b' . Therefore, simply adding these quantities together will not represent the actual displacement.

$$\frac{W_1 + W_2}{a} + \frac{W_3 + W_4}{b} + \frac{W_5 + W_6}{a'} \neq W.$$

But, if a correction factor was introduced, the above inequality can be written as;

$$\frac{W_1 + W_2}{a} + \frac{b}{a} \frac{W_3 + W_4}{b} + \frac{a'}{a} \frac{W_5 + W_6}{a'} = \frac{W}{a}.$$

However, these correction factors did not have to be considered thus far simply because they were automatically incorporated within the values of E_{3m} , V_{4m} , E_{5m} , V_{6m} , A_m , and D_m . Therefore, care must be exercised when dealing with the remaining boundary conditions.

3.4 Satisfaction of Bending Moment Requirements

The general expression of bending moment was discussed in Chapter 1, and given by Equation (1.24).

3.4-1 Bending Moment Along the Edge $\eta = 1$

Along this edge, the general moment equation may be written as follows:

$$\frac{M_n b^2}{aD} = - \left[\theta_{11} \phi^2 \frac{\partial^2 W}{\partial \xi^2} + \theta_{12} \frac{\partial^2 W}{\partial \eta^2} + \theta_{13} \phi \frac{\partial^2 W}{\partial \xi \partial \eta} \right], \quad (3.8)$$

where,

$$\theta_{11} = \cos^2 \alpha_1 + \nu \sin^2 \alpha_1,$$

$$\theta_{12} = \sin^2 \alpha_1 + \nu \cos^2 \alpha_1,$$

$$\theta_{13} = (1 - \nu) \sin 2\alpha_1.$$

For the first building block or $W_1(\xi, \eta)$, $\alpha_1 = 90^\circ$, $\theta_{11} = \nu$, $\theta_{12} = 1$, and $\theta_{13} = 0$, and where,

$$\frac{M_n b^2}{aD} = - \left[\nu \phi_1^2 \frac{\partial^2 W_1}{\partial \xi^2} + \frac{\partial^2 W_1}{\partial \eta^2} \right],$$

which is the same as Equation (1.18). However, along this edge, W_1 has a forced harmonic moment given by;

$$\left. \frac{M_n b^2}{aD} \right|_{\eta=1} = \sum_{m=1}^{\infty} E_{1m} \sin(m\pi\xi). \quad (3.9)$$

The second, third and fourth building blocks have simple support conditions along this edge, and hence, have no contribution to the bending moment.

For W_5 and W_6 , it is seen that the general expression for the bending moment may be written as;

$$\frac{M_n b'^2}{a'D} = - \left[\theta_{11} \phi_5^2 \frac{\partial^2 W}{\partial \xi'^2} + \theta_{12} \frac{\partial^2 W}{\partial \eta'^2} + \theta_{13} \phi_5 \frac{\partial^2 W}{\partial \xi' \partial \eta'} \right],$$

where, $\alpha_1 = 180^\circ - \alpha$. Also, it can be shown that if the above expression is multiplied by $1/F_2$, we obtain the proper dimensionless moment $M_n b^2/aD$.

$$\frac{b'}{a} \frac{M_n b'^2}{a'D} \frac{1}{F_2} = \frac{M_n b^2}{aD},$$

where the factor b'/a is incorporated into E_{5m} and V_{6m} as was discussed in Section 3.3-3. For each term of W_5 we have

$$\left. \frac{\partial^2 W_5}{\partial \xi'^2} \right|_{\eta=1} = -E_{5m} \theta_{51m} (m\pi)^2 [\sinh \beta_{5m} (1 - \xi) + C_{51m} \sin \gamma_{5m} (1 - \xi)] \sin(m\pi F_2 \xi),$$

$$\left. \frac{\partial^2 W_5}{\partial \eta'^2} \right|_{\eta=1} = E_{5m} \theta_{51m} [\beta_{5m}^2 \sinh \beta_{5m} (1 - \xi) - C_{51m} \gamma_{5m}^2 \sin \gamma_{5m} (1 - \xi)] \sin(m\pi F_2 \xi),$$

$$\left. \frac{\partial^2 W_5}{\partial \xi' \partial \eta'} \right|_{\eta=1} = -E_{5m} \theta_{51m} (m\pi) [\beta_{5m} \cosh \beta_{5m} (1 - \xi) + C_{51m} \gamma_{5m} \cos \gamma_{5m} (1 - \xi)] \cos(m\pi F_2 \xi),$$

or if $\lambda'^2 < (m\pi)^2$, then,

$$\left. \frac{\partial^2 W_5}{\partial \xi'^2} \right|_{\eta=1} = E_{5m} \theta_{52m} (m\pi)^2 [\sinh \beta_{5m} (1 - \xi) + C_{52m} \sinh \gamma_{5m} (1 - \xi)] \sin(m\pi F_2 \xi),$$

$$\left. \frac{\partial^2 W_5}{\partial \eta'^2} \right|_{\eta=1} = E_{5m} \theta_{52m} [\beta_{5m}^2 \sinh \beta_{5m} (1 - \xi) + C_{52m} \gamma_{5m}^2 \sinh \gamma_{5m} (1 - \xi)] \sin(m\pi F_2 \xi),$$

and

$$\left. \frac{\partial^2 W_5}{\partial \xi' \partial \eta'} \right|_{\eta=1} = -E_{5m} \theta_{52m} (m\pi) [\beta_{5m} \cosh \beta_{5m} (1 - \xi) + C_{52m} \gamma_{5m} \cosh \gamma_{5m} (1 - \xi)] \cos(m\pi F_2 \xi).$$

Performing the proper Fourier expansion, let,

$$I = 2 \int_0^1 \cosh \beta_{5m} (1 - \xi) \cos(m\pi F_2 \xi) \sin(n\pi \xi) d\xi,$$

then, $I = E_1 + E_2$, where,

$$E_1 = \int_0^1 \cosh \beta_{5m} (1 - \xi) \sin(m\pi F_2 + n\pi) \xi d\xi,$$

$$E_1 = (m\pi F_2 + n\pi) [\cosh \beta_{5m} - \cos(m\pi F_2 + n\pi)] / [\beta_{5m}^2 + (m\pi F_2 + n\pi)^2],$$

and

$$E_2 = - \int_0^1 \cosh \beta_{5m} (1 - \xi) \sin(m\pi F_2 - n\pi) \xi d\xi,$$

$$E_2 = (m\pi F_2 - n\pi) [\cos(m\pi F_2 - n\pi) - \cosh \beta_{5m}] / [\beta_{5m}^2 + (m\pi F_2 - n\pi)^2].$$

Now let;

$$II = 2 \int_0^1 \cos \gamma_{5m} (1 - \xi) \cos(m\pi F_2 \xi) \sin(n\pi \xi) d\xi,$$

then, $II = E_3 + E_4$, where,

$$E_3 = \int_0^1 \cos \gamma_{5m} (1 - \xi) \sin(m\pi F_2 + n\pi) \xi d\xi,$$

$$E_3 = (m\pi F_2 + n\pi) [\cos \gamma_{5m} - \cos(m\pi F_2 + n\pi)] / [(m\pi F_2 + n\pi)^2 - \gamma_{5m}^2],$$

if $\gamma_{5m}^2 = (m\pi F_2 + n\pi)^2$, then, $E_3 = \sin \gamma_{5m} / 2$.

And

$$E_4 = - \int_0^1 \cos \gamma_{5m} (1 - \xi) \sin(m\pi F_2 - n\pi) \xi d\xi,$$

$$E_4 = (m\pi F_2 - \gamma_{5m}) [\cos(m\pi F_2 - n\pi) - \cos \gamma_{5m}] / [(m\pi F_2 - n\pi)^2 - \gamma_{5m}^2],$$

if $\gamma_{5m} = (m\pi F_2 - n\pi)$, then, $E_4' = -\sin \gamma_{5m} / 2$,

and if $\gamma_{5m} = -(m\pi F_2 - n\pi)$, then $E_4 = \sin \gamma_{5m} / 2$.

Finally, let,

$$III = 2 \int_0^1 \cosh \gamma_{5m} (1 - \xi) \cos(m\pi F_2 \xi) \sin(n\pi \xi) d\xi,$$

then, $III = E_5 + E_6$, where E_5 and E_6 are obtained from E_1 and E_2 respectively by replacing β_{5m} by γ_{5m} . If we now let,

$$A_{1n} = -\theta_{51m} (m\pi)^2 [A_1 + A_2 + C_{51m} (A_3 + A_4)],$$

$$A_{2n} = \theta_{51m} [\beta_{5m}^2 (A_1 + A_2) - C_{51m} \gamma_{5m}^2 (A_3 + A_4)],$$

$$A_{3n} = -\theta_{51m} (m\pi) [\beta_{5m} (E_1 + E_2) + C_{51m} \gamma_{5m} (E_3 + E_4)],$$

or if $\lambda'^2 < (m\pi)^2$, then let,

$$A_{1n} = -\theta_{52m}(m\pi)^2[A_1 + A_2 + C_{52m}(A_5 + A_6)],$$

$$A_{2n} = \theta_{52m}[\beta_{5m}^2(A_1 + A_2) - C_{52m}\gamma_{5m}^2(A_5 + A_6)],$$

$$A_{3n} = -\theta_{52m}(m\pi)[\beta_{5m}(E_1 + E_2) + C_{52m}\gamma_{5m}(E_5 + E_6)].$$

The coefficients of E_{5m} in the expression for bending moment along this edge are given by the following equation;

$$A_{i+3k,j+4k} = -[\theta_{11}\phi_5^2 A_{1n} + \theta_{12}A_{2n} + \theta_{13}\phi_5 A_{3n}]/F_2.$$

The coefficients of V_{6m} are now found from the above equation simply by replacing θ_{51m} , C_{51m} , θ_{52m} , and C_{52m} by θ_{61m} , C_{61m} , θ_{62m} , and C_{62m} respectively in the expressions of A_{1n} , A_{2n} , and A_{3n} . The rectangular element has no contribution along this edge.

3.4-2 Bending Moment Along the Inter-Segment $\xi = 1$

Along this inter-segment line, the bending moment generated by the rectangular element is equated to the bending moment generated by the triangular element. The first two building blocks W_1 and W_2 , as well as the fourth building block W_4 have no contribution to bending moments along this common edge. Therefore,

$$A_{i+4k,j} = A_{i+4k,j+k} = A_{i+4k,j+3k} = 0.$$

The third building block W_3 has a forced harmonic moment given by Equation (2.6) as;

$$\left. \frac{M_n a^2}{bD} \right|_{\xi=1} = \sum_{m=1}^{\infty} E_{3m} \sin(m\pi\eta). \quad (3.10)$$

In order to obtain the quantity $M_n b'^2 / aD$, Equation (3.10) is multiplied by a correction factor which can be shown to be ϕ_1^2 . Therefore,

$$A_{i+4k, j+2k} = \phi_1^2.$$

For W_5 and W_6 , the bending moment along this line is obtained using the general equation of bending moments about oblique lines in the plane of a rectangular plate given by Equation (1.24) and rewritten here for convenient reference as;

$$\frac{M_n b'^2}{a'D} = - \left[\theta_{21} \phi_5^2 \frac{\partial^2 W}{\partial \xi'^2} + \theta_{22} \frac{\partial^2 W}{\partial \eta'^2} + \theta_{23} \phi_5 \frac{\partial^2 W}{\partial \xi' \partial \eta'} \right],$$

where,

$$\theta_{21} = \cos^2 \alpha_2 + \nu \sin^2 \alpha_2,$$

$$\theta_{22} = \sin^2 \alpha_2 + \nu \cos^2 \alpha_2,$$

$$\theta_{23} = (1 - \nu) \sin 2\alpha_2,$$

and where, $\alpha_2 = 90^\circ - \alpha$. Performing the necessary differentiation the followings are obtained;

$$\left. \frac{\partial^2 W_5}{\partial \xi'^2} \right|_{\xi=1} = -E_{5m} \theta_{51m} (m\pi)^2 [\sinh \beta_{5m} (1 - \eta) + C_{51m} \sin \gamma_{5m} (1 - \eta)] \sin [m\pi (1 - F_1 \eta)],$$

$$\left. \frac{\partial^2 W_5}{\partial \eta'^2} \right|_{\xi=1} = E_{5m} \theta_{51m} [\beta_{5m}^2 \sinh \beta_{5m} (1 - \eta) - C_{51m} \gamma_{5m}^2 \sin \gamma_{5m} (1 - \eta)] \sin [m\pi (1 - F_1 \eta)],$$

$$\left. \frac{\partial^2 W_5}{\partial \xi' \partial \eta'} \right|_{\xi=1} = -E_{5m} \theta_{51m} (m\pi) [\beta_{5m} \cosh \beta_{5m} (1 - \eta) + C_{51m} \gamma_{5m} \cos \gamma_{5m} (1 - \eta)] \cos[m\pi(1 - F_1 \eta)],$$

or if $\lambda'^2 < (m\pi)^2$, then the following expressions result,

$$\left. \frac{\partial^2 W_5}{\partial \xi'^2} \right|_{\xi=1} = -E_{5m} \theta_{52m} (m\pi)^2 [\sinh \beta_{5m} (1 - \eta) + C_{52m} \sinh \gamma_{5m} (1 - \eta)] \sin[m\pi(1 - F_1 \eta)],$$

$$\left. \frac{\partial^2 W_5}{\partial \eta'^2} \right|_{\xi=1} = E_{5m} \theta_{52m} [\beta_{5m}^2 \sinh \beta_{5m} (1 - \eta) - C_{52m} \gamma_{5m}^2 \sinh \gamma_{5m} (1 - \eta)] \sin[m\pi(1 - F_1 \eta)],$$

$$\left. \frac{\partial^2 W_5}{\partial \xi' \partial \eta'} \right|_{\xi=1} = -E_{5m} \theta_{52m} (m\pi) [\beta_{5m} \cosh \beta_{5m} (1 - \eta) + C_{52m} \gamma_{5m} \cosh \gamma_{5m} (1 - \eta)] \cos[m\pi(1 - F_1 \eta)].$$

consideration of the above expressions reveals that Fourier sine expansions for the first two sets of derivatives are readily available from previous sections. Expanding the mixed derivative expression, we let,

$$I = 2 \int_0^1 \cosh \beta_{5m} (1 - \eta) \cos[m\pi(1 - F_1 \eta)] \sin(n\pi \eta) d\eta,$$

then, $I = G_1 + G_2$, where,

$$G_1 = \int_0^1 \cosh \beta_{5m} (1 - \eta) \sin[m\pi - (m\pi F_1 - n\pi)\eta] d\eta,$$

$$G_1 = \frac{(m\pi F_1 - n\pi) [\cos(m\pi - m\pi F_1 + n\pi) - \cosh \beta_{5m} \cos(m\pi)]}{[\beta_{5m}^2 + (m\pi F_1 - n\pi)^2]},$$

$$G_2 = - \int_0^1 \cosh \beta_{5m} (1 - \eta) \sin[m\pi - (m\pi F_1 + n\pi)\eta] d\eta,$$

$$G_2 = \frac{(m\pi F_1 + n\pi) [\cosh \beta_{5m} \cos(m\pi) - \cos(m\pi - m\pi F_1 - n\pi)]}{[\beta_{5m}^2 + (m\pi F_1 + n\pi)^2]}.$$

Now, let;

$$II = 2 \int_0^1 \cos \gamma_{5m} (1 - \eta) \cos[m\pi(1 - F_1\eta)] \sin(n\pi\eta) d\eta,$$

then, $II = G_3 + G_4$, where,

$$G_3 = \int_0^1 \cos \gamma_{5m} (1 - \eta) \sin[m\pi - (m\pi F_1 - n\pi)\eta] d\eta,$$

$$G_3 = \frac{(m\pi F_1 - n\pi) [\cos(m\pi - m\pi F_1 + n\pi) - \cos(m\pi + \gamma_{5m})]}{[(m\pi F_1 - n\pi)^2 - \gamma_{5m}^2]},$$

or if $\gamma_{5m} = (m\pi F_1 - n\pi)$, then $G_3 = \sin(m\pi - \gamma_{5m})/2$, and if $\gamma_{5m} = (n\pi - m\pi F_1)$, and if $\gamma_{5m} = (n\pi - m\pi F_1)$, then, $G_3 = \sin(m\pi + \gamma_{5m})/2$, also,

$$G_4 = - \int_0^1 \cos \gamma_{5m} (1 - \eta) \sin[m\pi - (m\pi + n\pi)\eta] d\eta,$$

$$G_4 = \frac{(m\pi F_1 + n\pi) [\cos(m\pi + \gamma_{5m}) - \cos(m\pi - m\pi F_1 - n\pi)]}{[(m\pi F_1 + n\pi)^2 - \gamma_{5m}^2]},$$

or if $\gamma_{5m} = (m\pi F_1 + n\pi)$, then, $G_4 = \sin(\gamma_{5m} - m\pi)/2$.

Now let $III = G_5 + G_6$, where G_5 and G_6 are obtained from The expressions of G_1 and G_2 respectively by replacing β_{5m} by γ_{5m} .

Bending moment coefficients of E_{5m} are then evaluated and written as shown below. The correction factor discussed earlier can be shown to be $1/F_2$ for this particular situation.

$$A_{i+4k, j+4k} = -[\theta_{21}\phi_5^2 A_{1n} + \theta_{22}A_{2n} + \theta_{23}\phi_5 A_{3n}]/F_2,$$

where,

$$A_{1n} = -\theta_{51m}(m\pi)^2[B_1 + B_2 + C_{51m}(B_3 + B_4)],$$

$$A_{2n} = \theta_{51m}[\beta_{5m}^2(B_1 + B_2) - C_{51m}\gamma_{5m}^2(B_3 + B_4)],$$

$$A_{3n} = -\theta_{51m}(m\pi)[\beta_{5m}(G_1 + G_2) + C_{51m}\gamma_{5m}(G_3 + G_4)],$$

or if $\lambda'^2 < (m\pi)^2$,

$$A_{1n} = -\theta_{52m}(m\pi)^2[B_1 + B_2 + C_{52m}(B_5 + B_6)],$$

$$A_{2n} = \theta_{52m}[\beta_{5m}^2(B_1 + B_2) + C_{52m}\gamma_{5m}^2(B_5 + B_6)],$$

$$A_{3n} = -\theta_{52m}(m\pi)[\beta_{5m}(G_1 + G_2) + C_{52m}\gamma_{5m}(G_5 + G_6)].$$

The coefficients of V_{6m} are found to have the same expression as those of E_{5m} , where A_{1n} , A_{2n} , and A_{3n} are obtained from the above expressions by replacing θ_{51m} , θ_{52m} , C_{51m} , and C_{52m} by θ_{61m} , θ_{62m} , C_{61m} , and C_{62m} respectively.

These contributions must now be equated to the contributions of the rectangular element for which the bending moment expression reduces to;

$$\frac{M_n b}{D} = - \left[\phi_r^2 \frac{\partial^2 W}{\partial \xi^2} + \nu \frac{\partial^2 W}{\partial \eta^2} \right]$$

This is equivalent to $M_n b^2 / aD$ since a factor of b/a has already been built into A_m and D_m . Considering first the symmetric modes;

$$\left. \frac{\partial^2 W_{rs}}{\partial \xi^2} \right|_{\xi=0} = [A_m \beta_{rm}^2 \cosh \beta_{rm} - D_m \gamma_{rm}^2 \cos \gamma_{rm}] \sin(m\pi\eta),$$

$$\left. \frac{\partial^2 W_{rs}}{\partial \eta^2} \right|_{\xi=0} = -(m\pi)^2 [A_m \cosh \beta_{rm} + D_m \cos \gamma_{rm}] \sin(m\pi\eta),$$

or if $\phi_r^2 \lambda_r^2 < (m\pi)^2$,

$$\left. \frac{\partial^2 W_{rs}}{\partial \xi^2} \right|_{\xi=0} = [A_m \beta_{rm}^2 \cosh \beta_{rm} + D_m \gamma_{rm}^2 \cosh \gamma_{rm}] \sin(m\pi\eta),$$

$$\left. \frac{\partial^2 W_{rs}}{\partial \eta^2} \right|_{\xi=0} = -(m\pi)^2 [A_m \cosh \beta_{rm} + D_m \cosh \gamma_{rm}] \sin(m\pi\eta).$$

And therefore,

$$A_{i+4k, i+6k} = \phi_r^2 \beta_{rm}^2 \cosh \beta_{rm} - \nu (m\pi)^2 \cosh \beta_{rm},$$

$$A_{i+4k, i+7k} = -\phi_r^2 \gamma_{rm}^2 \cos \gamma_{rm} - \nu (m\pi)^2 \cos \gamma_{rm},$$

or if, $\phi_r^2 \lambda_r^2 < (m\pi)^2$, then,

$$A_{i+4k, i+7k} = \phi_r^2 \gamma_{rm}^2 \cosh \gamma_{rm} - \nu (m\pi)^2 \cosh \gamma_{rm}.$$

Note that the signs of the above coefficients have been changed so that they can be added to the relevant coefficients from the triangular element and their sum equated to zero. In the case of the anti-symmetric modes these coefficients become,

$$A_{i+4k, i+6k} = \phi_r^2 \beta_{rm}^2 \sinh \beta_{rm} - \nu (m\pi)^2 \sinh \beta_{rm},$$

$$A_{i+4k, i+7k} = -\phi_r^2 \gamma_{rm}^2 \sinh \gamma_{rm} - \nu (m\pi)^2 \sinh \gamma_{rm},$$

$$\text{or if } \phi_r^2 \lambda_r^2 < (m\pi)^2,$$

$$A_{i+4k, i+7k} = \phi_r^2 \gamma_{rm}^2 \sinh \gamma_{rm} + \nu (m\pi)^2 \sinh \gamma_{rm},$$

Consideration is next given to the third edge where $\eta' = 0$.

3.4-3 Bending Moment Along the Edge $\eta' = 0$

For the first four building blocks, the expression of bending moment at any point along this edge may be written as;

$$\frac{M_n b^2}{aD} = - \left[\theta_{31} \phi_1^2 \frac{\partial^2 W}{\partial \xi^2} + \theta_{32} \partial^2 W \partial \eta^2 + \theta_{33} \phi_1 \frac{\partial^2 W}{\partial \xi \partial \eta} \right],$$

where,

$$\theta_{31} = \cos^2 \alpha_3 + \nu \sin^2 \alpha_3,$$

$$\theta_{32} = \sin^2 \alpha_3 + \nu \cos^2 \alpha_3,$$

$$\theta_{33} = (1 - \nu) \sin 2\alpha_3,$$

and where $\alpha_3 = \alpha$.

Considering now the first building block W_1 , and performing the necessary derivations, the followings are obtained;

$$\left. \frac{\partial^2 W_1}{\partial \xi^2} \right|_{\eta'=0} = -E_{1m} \theta_{11m} (m\pi)^2 [\sinh \beta_{1m} (1 - \xi) + C_{11m} \sin \gamma_{1m} (1 - \xi) \sin(m\pi \xi)].$$

$$\left. \frac{\partial^2 W_1}{\partial \eta^2} \right|_{\eta'=0} = E_{1m} \theta_{11m} [\beta_{1m}^2 \sinh \beta_{1m} (1 - \xi) + C_{11m} \gamma_{1m}^2 \sin \gamma_{1m} (1 - \xi)] \sin(m\pi \xi),$$

$$\left. \frac{\partial^2 W_1}{\partial \xi \partial \eta} \right|_{\eta'=0} = E_{1m} \theta_{11m} (m\pi) [\beta_{1m} \cosh \beta_{1m} (1 - \xi) + C_{11m} \gamma_{1m} \cos \gamma_{1m} (1 - \xi)] \cos(m\pi \xi),$$

or if $\lambda^2 < (m\pi)^2$, then,

$$\left. \frac{\partial^2 W_1}{\partial \xi^2} \right|_{\eta'=0} = -E_{1m} \theta_{12m} (m\pi)^2 [\sinh \beta_{1m} (1 - \xi) + C_{12m} \sinh \gamma_{1m} (1 - \xi)] \sin(m\pi \xi),$$

$$\left. \frac{\partial^2 W_1}{\partial \eta^2} \right|_{\eta'=0} = E_{1m} \theta_{12m} [\beta_{1m}^2 \sinh \beta_{1m} (1 - \xi) + C_{12m} \gamma_{1m}^2 \sinh \gamma_{1m} (1 - \xi)] \sin(m\pi \xi),$$

$$\left. \frac{\partial^2 W_1}{\partial \xi \partial \eta} \right|_{\eta'=0} = E_{1m} \theta_{12m} (m\pi) [\beta_{1m} \cosh \beta_{1m} (1 - \xi) + C_{12m} \gamma_{1m} \cosh \gamma_{1m} (1 - \xi)] \cos(m\pi \xi).$$

It is seen here that only the mixed derivative expression needs to be expanded in a Fourier sine series, for Fourier expansions are readily available for the first two derivatives. Let;

$$I = 2 \int_0^1 \cosh \beta_{1m} (1 - \xi) \cos(m\pi \xi) \sin(n\pi \xi) d\xi,$$

then, $I = H_1 + H_2$, where,

$$H_1 = \int_0^1 \cosh \beta_{1m} (1 - \xi) \sin(m\pi + n\pi) \xi d\xi,$$

$$H_1 = \frac{(m\pi + n\pi) [\cosh \beta_{1m} - \cos(m\pi + n\pi)]}{[\beta_{1m}^2 + (m\pi + n\pi)^2]},$$

and

$$H_2 = - \int_0^1 \cosh \beta_{1m} (1 - \xi) \sin(m\pi - n\pi) \xi d\xi,$$

$$H_2 = \frac{(m\pi - n\pi) [\cos(m\pi - n\pi) - \cosh \beta_{1m}]}{[\beta_{1m}^2 + (m\pi - n\pi)^2]}.$$

Now let;

$$II = 2 \int_0^1 \cos \gamma_{1m} (1 - \xi) \cos(m\pi \xi) \sin(n\pi \xi) d\xi,$$

then, $II = H_3 + H_4$, where,

$$H_3 = \int_0^1 \cos \gamma_{1m} (1 - \xi) \sin(m\pi + n\pi) \xi d\xi,$$

$$H_3 = \frac{(m\pi + n\pi) [\cos \gamma_{1m} - \cos(m\pi + n\pi)]}{[(m\pi + n\pi)^2 - \gamma_{1m}^2]},$$

or if $\gamma_{1m}^2 = (m\pi + n\pi)^2$, then, $H_3 = \sin \gamma_{1m} / 2$, and

$$H_4 = - \int_0^1 \cos \gamma_{1m} (1 - \xi) \sin(m\pi - n\pi) \xi d\xi,$$

$$H_4 = \frac{(m\pi - n\pi)[\cos(m\pi - n\pi) - \cos \gamma_{1m}]}{[(m\pi - n\pi)^2 - \gamma_{1m}^2]},$$

or if $\gamma_{1m} = (m\pi - n\pi)$, then, $H_4 = -\sin \gamma_{1m}/2$, and if $\gamma_{1m} = (n\pi - m\pi)$, $H_4 = \sin \gamma_{1m}/2$. The last term is;

$$III = 2 \int_0^1 \cosh \gamma_{1m} (1 - \xi) \cos(m\pi \xi) \sin(n\pi \xi) d\xi,$$

or, $III = H_5 + H_6$, where H_5 and H_6 are obtained from the expressions of H_1 and H_2 respectively by replacing β_{1m} by γ_{1m} . Therefore, the bending moment coefficients of E_{1m} along this edge are;

$$A_{i+sk,j} = -[\theta_{31}\phi_1^2 A_{1n} + \theta_{32}A_{2n} + \theta_{33}\phi_1 A_{3n}],$$

where,

$$A_{1n} = -\theta_{11m}(m\pi)^2[C_1 + C_2 + C_{11m}(C_3 + C_4)],$$

$$A_{2n} = \theta_{11m}[\beta_{1m}^2(C_1 + C_2) - C_{11m}\gamma_{1m}^2(C_3 + C_4)],$$

$$A_{3n} = \theta_{11m}(m\pi)[\beta_{1m}(H_1 + H_2) + C_{11m}\gamma_{1m}(H_3 + H_4)],$$

and if $\lambda^2 < (m\pi)^2$, then,

$$A_{1n} = -\theta_{12m}(m\pi)^2[C_1 + C_2 + C_{12m}(C_5 + C_6)],$$

$$A_{2n} = \theta_{12m}[\beta_{1m}^2(C_1 + C_2) + C_{12m}\gamma_{1m}^2(C_5 + C_6)],$$

$$A_{3n} = \theta_{12m}(m\pi)[\beta_{1m}(H_1 + H_2) + C_{12m}\gamma_{1m}(H_5 + H_6)].$$

The bending moment coefficients of V_{2m} along this edge are found to have the same expression as those of E_{1m} , where A_{1n} , A_{2n} , and A_{3n} are obtained

by replacing θ_{11m} , C_{11m} , θ_{12m} , and C_{12m} by θ_{21m} , C_{21m} , θ_{22m} , and C_{22m} respectively in the above equations.

Performing the necessary partial differentiation on W_3 , it is seen here also that appropriate Fourier expansion are readily available from previous sections for the second partial derivative of W_3 with respect to ξ as well as with respect to η .

$$\left. \frac{\partial^2 W_3}{\partial \xi^2} \right|_{\eta'=0} = E_{3m} \theta_{31m} [\beta_{3m}^2 \sinh \beta_{3m} \xi - C_{31m} \gamma_{3m}^2 \sin \gamma_{3m} \xi] \sin[m\pi(1 - \xi)],$$

$$\left. \frac{\partial^2 W_3}{\partial \eta^2} \right|_{\eta'=0} = -E_{3m} \theta_{31m} (m\pi)^2 [\sinh \beta_{3m} \xi + C_{31m} \sin \gamma_{3m} \xi] \sin[m\pi(1 - \xi)],$$

$$\left. \frac{\partial^2 W_3}{\partial \xi \partial \eta} \right|_{\eta'=0} = E_{3m} \theta_{31m} (m\pi) [\beta_{3m} \cosh \beta_{3m} \xi + C_{31m} \gamma_{3m} \cos \gamma_{3m} \xi] \cos[m\pi(1 - \xi)].$$

And if $\phi_1^2 \lambda^2 < (m\pi)^2$, then,

$$\left. \frac{\partial^2 W_3}{\partial \xi^2} \right|_{\eta'=0} = E_{3m} \theta_{32m} [\beta_{3m}^2 \sinh \beta_{3m} \xi + C_{32m} \gamma_{3m}^2 \sinh \gamma_{3m} \xi] \sin[m\pi(1 - \xi)],$$

$$\left. \frac{\partial^2 W_3}{\partial \eta^2} \right|_{\eta'=0} = -E_{3m} \theta_{32m} (m\pi)^2 [\sinh \beta_{3m} \xi + C_{32m} \sinh \gamma_{3m} \xi] \sin[m\pi(1 - \xi)],$$

$$\left. \frac{\partial^2 W_3}{\partial \xi \partial \eta} \right|_{\eta'=0} = E_{3m} \theta_{32m} (m\pi) [\beta_{3m} \cosh \beta_{3m} \xi + C_{32m} \gamma_{3m} \cosh \gamma_{3m} \xi] \cos[m\pi(1 - \xi)].$$

Expanding the mixed derivative in a Fourier sine series, let;

$$I = 2 \int_0^1 \cosh \beta_{3m} \xi \cos[m\pi(1 - \xi)] \sin(n\pi \xi) d\xi,$$

then, $I = P_1 + P_2$, where,

$$P_1 = \int_0^1 \cosh \beta_{3m} \xi \sin[m\pi - (m\pi - n\pi)\xi] d\xi,$$

$$P_1 = \frac{(m\pi - n\pi) [\cosh \beta_{3m} \cos(n\pi) - \cos(m\pi)]}{[\beta_{3m}^2 + (m\pi - n\pi)^2]},$$

and

$$P_2 = - \int_0^1 \cosh \beta_{3m} \xi \sin[m\pi - (m\pi + n\pi)\xi] d\xi,$$

$$P_2 = \frac{(m\pi + n\pi) [\cos(m\pi) - \cosh \beta_{3m} \cos(n\pi)]}{[\beta_{3m}^2 + (m\pi + n\pi)^2]}.$$

Now, let;

$$II = 2 \int_0^1 \cos \gamma_{3m} \xi \cos[m\pi(1-\xi)] \sin(n\pi\xi) d\xi,$$

then $II = P_3 + P_4$, where,

$$P_3 = \int_0^1 \cos \gamma_{3m} \xi \sin[m\pi - (m\pi - n\pi)\xi] d\xi,$$

$$P_3 = \frac{(m\pi - n\pi)[\cos(n\pi - \gamma_{3m}) - \cos(m\pi)]}{[(m\pi - n\pi)^2 - \gamma_{3m}^2]}.$$

But if $\gamma_{3m} = (m\pi - n\pi)$, then,

$$P_3 = [\cos(m\pi - 2\gamma_{3m}) - \cos(m\pi)]/4\gamma_{3m},$$

or if $\gamma_{3m} = (n\pi - m\pi)$,

$$P_3 = [\cos(m\pi) - \cos(m\pi + 2\gamma_{3m})]/4\gamma_{3m}.$$

And

$$P_4 = - \int_0^1 \cos \gamma_{3m} \xi \sin[m\pi - (m\pi + n\pi)\xi] d\xi,$$

$$P_4 = \frac{(m\pi + n\pi)[\cos m\pi - \cos(n\pi + \gamma_{3m})]}{[(m\pi + n\pi)^2 - \gamma_{3m}^2]},$$

or if $\gamma_{3m} = (m\pi + n\pi)$, then,

$$P_4 = [\cos(m\pi) - \cos(m\pi - 2\gamma_{3m})]/4\gamma_{3m}.$$

The remaining term is now written as;

$$III = 2 \int_0^1 \cosh \gamma_{3m} \xi \cos[m\pi(1-\xi)] \sin(n\pi\xi) d\xi,$$

where III can be expressed as the sum of P_5 and P_6 , and where P_5 and P_6 are obtained from the expressions of P_1 and P_2 respectively by replacing β_{3m} by γ_{3m} . The coefficients of E_{3m} in the expression of bending moment along this edge are then;

$$A_{i+5k, j+2k} = -[\theta_{31} \phi_1^2 A_{1n} + \theta_{32} A_{2n} + \theta_{33} \phi_1 A_{3n}],$$

where,

$$A_{1n} = \theta_{31m} [\beta_{3m}^2 (D_1 + D_2) - C_{31m} \gamma_{3m}^2 (D_3 + D_4)],$$

$$A_{2n} = -\theta_{31m} (m\pi)^2 [D_1 + D_2 + C_{31m} (D_3 + D_4)],$$

$$A_{3n} = \theta_{31m} (m\pi) [\beta_{3m} (P_1 + P_2) + C_{31m} \gamma_{3m} (P_3 + P_4)],$$

or if $\phi_1^2 \lambda^2 < (m\pi)^2$, then,

$$A_{1n} = \theta_{32m} [\beta_{3m}^2 (D_1 + D_2) + C_{32m} \gamma_{3m}^2 (D_5 + D_6)],$$

$$A_{2n} = -\theta_{32m} (m\pi)^2 [D_1 + D_2 + C_{32m} (D_5 + D_6)],$$

$$A_{3n} = \theta_{32m} (m\pi) [\beta_{3m} (P_1 + P_2) + C_{32m} \gamma_{3m} (P_5 + P_6)].$$

The bending moment coefficients of V_{4m} along this edge are found to have the same expression as those of E_{3m} , where A_{1n} , A_{2n} , and A_{3n} are obtained from the above equations by replacing θ_{31m} , C_{31m} , θ_{32m} , and C_{32m} by θ_{41m} , C_{41m} , θ_{42m} , and C_{42m} respectively.

The fifth building block or W_5 has a prescribed harmonic bending moment along this edge as given by Equation (2.9) and written here for convenient reference.

$$\left. \frac{M_n b'^2}{a' D} \right|_{\eta'=0} = \sum_{m=1}^{\infty} E_{5m} \sin(m\pi\xi).$$

The correction factor is found to be $1/F_2$, and hence, the coefficients of E_{5m} are;

$$A_{i+5k, i+4k} = 1/F_2.$$

The remaining building blocks have no contribution towards the net bending moment along this edge. Therefore,

$$A_{i+5k, j+5k} = A_{i+5k, j+6k} = A_{i+5k, j+7k} = 0.$$

It remains to satisfy the conditions of continuity of slope and continuity of vertical edge reactions across the inter-segment line. The condition of continuity of slope is discussed next.

3.5 Satisfaction of The Condition of Continuity of Slope

The slope is obtained by considering the first partial derivative of the shape function with respect to the appropriate variable. In general, the slope in the direction of the normal n is expressed as;

$$\frac{\partial W}{\partial n} = \frac{\partial W}{\partial y} \frac{dy}{dn} + \frac{\partial W}{\partial x} \frac{dx}{dn}.$$

Reference is made to Figure 3.6 where it can be seen that $dx/dn = \cos\alpha$ and $dy/dn = \sin\alpha$. Therefore,

$$\frac{\partial W}{\partial n} = \frac{\partial W}{\partial x} \cos\alpha + \frac{\partial W}{\partial y} \sin\alpha.$$

For the first four building blocks $\alpha = 0$ since the common edge is normal to the x or ξ axis. Hence, the slope is simply the first derivative with respect to ξ . For the first building block W_1 we have;

$$\left. \frac{\partial W_1}{\partial \xi} \right|_{\xi=1} = E_{1m} \theta_{11m}(m\pi) [\sinh \beta_{1m} \eta + C_{11m} \sinh \gamma_{1m} \eta] \cos(m\pi),$$

or if $\lambda^2 < (m\pi)^2$, then,

$$\left. \frac{\partial W_1}{\partial \xi} \right|_{\xi=1} = E_{1m} \theta_{12m}(m\pi) \cos(m\pi) [\sinh \beta_{1m} \eta + C_{12m} \sinh \gamma_{1m} \eta].$$

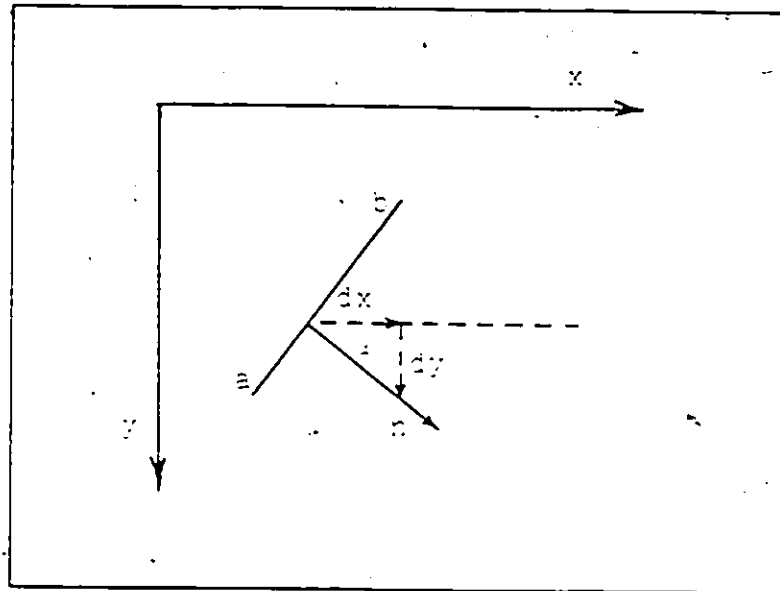


Figure 3.6- boundary at an angle.

performing the necessary Fourier sine series transformations to find the coefficients of E_{1m} for normal slope across this common inter-segment line to be;

$$A_{i+6k,j} = \theta_{11m}(m\pi)\cos(m\pi)[D_{s1} + C_{11m}D_{s2}],$$

$$\text{or if } \lambda^2 < (m\pi)^2,$$

$$A_{i+6k,j} = \theta_{12m}(m\pi)\cos(m\pi)[D_{s1} + C_{12m}D_{s3}],$$

where,

$$\begin{aligned} D_{s1} &= -\frac{2n\pi\cos(n\pi)\sinh\beta_{1m}}{[\beta_{1m}^2 + (n\pi)^2]}, \\ D_{s2} &= \frac{2n\pi\cos(n\pi)\sin\gamma_{1m}}{[\gamma_{1m}^2 - (n\pi)^2]}, \\ D_{s3} &= -\frac{2n\pi\cos(n\pi)\sinh\gamma_{1m}}{[\gamma_{1m}^2 + (n\pi)^2]}. \end{aligned}$$

Similarly, the coefficients of V_{2m} are found to be;

$$A_{i+6k,j+k} = \theta_{21m}(m\pi)\cos(m\pi)[D_{s1} + C_{21m}D_{s2}],$$

$$\text{or if } \lambda^2 < (m\pi)^2,$$

$$A_{i+6k,j+k} = \theta_{22m}(m\pi)\cos(m\pi)[D_{s1} + C_{22m}D_{s3}].$$

Considering now the third building block W_3 , the following derivatives are obtained;

$$\begin{aligned} \left. \frac{\partial W_3}{\partial \xi} \right|_{\xi=1} &= E_{3m}\theta_{31m}[\beta_{3m}\cosh\beta_{3m} + \\ &\quad C_{31m}\gamma_{3m}\cos\gamma_{3m}]\sin(m\pi\eta), \end{aligned}$$

$$\text{or if } \phi_1^2\lambda^2 < (m\pi)^2,$$

$$\left. \frac{\partial W_3}{\partial \xi} \right|_{\xi=1} = E_{3m} \theta_{32m} [\beta_{3m} \cosh \beta_{3m} + C_{32m} \gamma_{3m} \cosh \gamma_{3m}] \sin(m\pi\eta).$$

It is seen here that the above expressions have the form of a proper Fourier sine series, and therefore, no expansion is necessary. Consequently, the coefficients of E_{3m} in relation to the slope are found to be;

$$A_{i+6k, i+2k} = \theta_{31m} [\beta_{3m} \cosh \beta_{3m} + C_{31m} \gamma_{3m} \cos \gamma_{3m}],$$

$$\text{or if } \phi_1 \lambda^2 < (m\pi)^2;$$

$$A_{i+6k, i+2k} = \theta_{32m} [\beta_{3m} \cosh \beta_{3m} + C_{32m} \gamma_{3m} \cosh \gamma_{3m}].$$

Similarly, the coefficients of V_{4m} are found to be;

$$A_{i+6k, i+3k} = \theta_{41m} [\beta_{3m} \cosh \beta_{3m} + C_{41m} \gamma_{3m} \cos \gamma_{3m}],$$

$$\text{or if } \phi_1 \lambda^2 < (m\pi)^2,$$

$$A_{i+6k, i+3k} = \theta_{42m} [\beta_{3m} \cosh \beta_{3m} + C_{42m} \gamma_{3m} \cosh \gamma_{3m}].$$

The contributions of W_5 and W_6 are now considered. We start by expanding the proper derivatives in appropriate Fourier sine series to find;

$$A_{i+6k, j+4k} = F_{21} A_{1n} + F_{22} A_{2n},$$

$$\text{where, } F_{21} = \cos \alpha_2 \sin \alpha, \quad F_{22} = \sin \alpha_2 / \cos \alpha, \quad \text{and where,}$$

$$A_{1n} = \theta_{51m} (m\pi) [E_{s1} + E_{s2} + C_{51m} (E_{s3} + E_{s4})],$$

$$A_{2n} = -\theta_{51m} [\beta_{5m} (F_{s1} + F_{s2}) + C_{51m} \gamma_{5m} (F_{s3} + F_{s4})],$$

$$\text{or if } \lambda'^2 < (m\pi)^2,$$

$$A_{1n} = \theta_{52m}(m\pi)[E_{s1} + E_{s2} + C_{52m}(E_{s5} + E_{s6})],$$

$$A_{2n} = -\theta_{52m}[\beta_{5m}(F_{s1} + F_{s2}) + C_{52m}\gamma_{5m}(F_{s5} + F_{s6})].$$

Also,

$$A_{i+6k, j+5k} = F_{21}A_{1n} + F_{22}A_{2n},$$

where A_{1n} and A_{2n} are obtained from the above expressions by replacing θ_{51m} , C_{51m} , θ_{52m} , and C_{52m} by θ_{61m} , C_{61m} , θ_{62m} , and C_{62m} respectively. Furthermore, the values of E_{s1} through E_{s6} are found from the coefficients of the Fourier expansions to be;

$$E_{s1} = \frac{[\beta_{5m}\sin(m\pi F_1 - m\pi - n\pi) - (m\pi F_1 - n\pi)\sinh\beta_{5m}\cos(m\pi)]}{[\beta_{5m}^2 + (m\pi F_1 - n\pi)^2]},$$

$$E_{s2} = \frac{[(m\pi F_1 + n\pi)\sinh\beta_{5m}\cos(m\pi) - \beta_{5m}\sin(m\pi F_1 - m\pi + n\pi)]}{[\beta_{5m}^2 + (m\pi F_1 + n\pi)^2]},$$

$$E_{s3} = \frac{[\cos(m\pi - \pi/2 - m\pi F_1 + n\pi) - \cos(\gamma_{5m} + m\pi - \pi/2)]}{-2[(m\pi F_1 - n\pi) + \gamma_{5m}]} + \frac{[\cos(\gamma_{5m} - m\pi + \pi/2) - \cos(m\pi F_1 - n\pi - m\pi + \pi/2)]}{2[(m\pi F_1 - n\pi) - \gamma_{5m}]},$$

if $\gamma_{5m} = (m\pi F_1 - n\pi)$,

$$E_{s3} = \frac{1}{2}\sin(\gamma_{5m} - m\pi + \pi/2) + \frac{1}{2\gamma_{5m}}\cos(m\pi - \gamma_{5m} - \pi/2),$$

or if $\gamma_{5m} = (n\pi - m\pi F_1)$,

$$E_{s3} = \frac{1}{2} \sin(\gamma_{5m} + m\pi - \pi/2) + \frac{1}{2\gamma_{5m}} \cos(\pi/2 - \gamma_{5m} - m\pi),$$

also,

$$E_{s4} = \frac{[\cos(\gamma_{5m} + m\pi - \pi/2) - \cos(m\pi - m\pi F_1 - n\pi - \pi/2)]}{2[(m\pi F_1 + n\pi) + \gamma_{5m}]} + \frac{[\cos(m\pi F_1 - m\pi + n\pi + \pi/2) - \cos(\gamma_{5m} - m\pi + \pi/2)]}{[(m\pi F_1 + n\pi) - \gamma_{5m}]},$$

or if $\gamma_{5m} = (m\pi F_1 + n\pi)$,

$$E_{s4} = \frac{1}{2} \sin(m\pi - \gamma_{5m} - \pi/2) + \frac{1}{2\gamma_{5m}} \cos(m\pi + \gamma_{5m} - \pi/2).$$

E_{s5} and E_{s6} are found from the expressions of E_{s1} and E_{s2} respectively by replacing β_{5m} by γ_{5m} .

Similarly, it is found that,

$$F_{s1} = \frac{\beta_{5m} \sinh \beta_{5m} \cos(m\pi) - (m\pi F_1 + n\pi) \sin(m\pi - m\pi F_1 - n\pi)}{\beta_{5m}^2 + (m\pi F_1 + n\pi)^2},$$

$$F_{s2} = \frac{(m\pi F_1 - n\pi) \sin(m\pi - m\pi F_1 + n\pi) - \beta_{5m} \sinh \beta_{5m} \cos(m\pi)}{\beta_{5m}^2 + (m\pi F_1 + n\pi)^2},$$

$$F_{s3} = \frac{\sin(m\pi F_1 + n\pi - m\pi) - \sin(\gamma_{5m} - m\pi)}{2[(m\pi F_1 + n\pi) - \gamma_{5m}]} + \frac{\sin(\gamma_{5m} + m\pi) - \sin(m\pi - m\pi F_1 - n\pi)}{2[(m\pi F_1 + n\pi) + \gamma_{5m}]},$$

or if $\gamma_{5m} = (m\pi F_1 + n\pi)$, then,

$$F_{s3} = \frac{1}{2} \cos(\gamma_{5m} - m\pi) + \frac{1}{2\gamma_{5m}} \sin(m\pi + \gamma_{5m}),$$

and

$$F_{s4} = \frac{\sin(\gamma_{5m} - m\pi) - \sin(m\pi F_1 - n\pi - m\pi)}{2[(m\pi F_1 - n\pi) - \gamma_{5m}]} + \frac{\sin(m\pi - m\pi F_1 + n\pi) - \sin(\gamma_{5m} + m\pi)}{2[(m\pi F_1 - n\pi) + \gamma_{5m}]},$$

or if $\gamma_{5m}^2 = (m\pi F_1 - n\pi)^2$,

$$F_{s4} = \frac{1}{2\gamma_{5m}} \sin(m\pi - \gamma_{5m}) - \frac{1}{2} \cos(\gamma_{5m} - m\pi).$$

Here also, F_{s5} and F_{s6} are obtained by replacing β_{5m} by γ_{5m} in the expressions of F_{s1} and F_{s2} respectively.

Adding up all six contributions results in the net value of the slope taken normal to the inter-segment line. If this slope is to be continuous across this line, it must be equated to $\partial W_{rs}/\partial n$ in the case of symmetrical modes and to $\partial W_{ru}/\partial n$ for anti-symmetric modes. It can be shown that,

$$\frac{\partial W_{rs}}{\partial n} = \frac{\partial W_{rs}}{\partial \xi} \frac{\phi_r}{\phi_1}.$$

The same is true for $\frac{\partial W_{rs}}{\partial \xi}$. Evaluating the above derivatives to have;

$$\left. \frac{\partial W_{rs}}{\partial \xi} \right|_{\xi=0} = -(A_m \beta_{rm} \sinh \beta_{rm} - D_m \gamma_{rm} \sin \gamma_{rm}) \sin(m\pi\eta),$$

$$\left. \frac{\partial W_{ru}}{\partial \xi} \right|_{\xi=0} = -(A_m \beta_{rm} \cosh \beta_{rm} + D_m \gamma_{rm} \cos \gamma_{rm}) \sin(m\pi\eta),$$

or if $\phi_r^2 \lambda_r^2 < (m\pi)^2$, then,

$$\left. \frac{\partial W_{rs}}{\partial \xi} \right|_{\xi=0} = -(A_m \beta_{rm} \sinh \beta_{rm} + D_m \gamma_{rm} \sinh \gamma_{rm}) \sin(m\pi\eta),$$

$$\left. \frac{\partial W_{ru}}{\partial \xi} \right|_{\xi=0} = -(A_m \beta_{rm} \cosh \beta_{rm} + D_m \gamma_{rm} \cosh \gamma_{rm}) \sin(m\pi\eta).$$

Changing sign, premultiplying by the proper correction factor, then adding to the net contribution of the rectangular element and equating to zero, the coefficients of A_m and D_m are found. For the case of symmetric modes,

$$A_{i+6k, i+6k} = \beta_{rm} \sinh \beta_{rm} \frac{\phi_r}{\phi_1},$$

$$A_{i+6k, i+7k} = -\gamma_{rm} \sin \gamma_{rm} \frac{\phi_r}{\phi_1}.$$

And for anti-symmetric modes,

$$A_{i+6k, i+6k} = \beta_{rm} \cosh \beta_{rm} \frac{\phi_r}{\phi_1},$$

$$A_{i+6k, i+7k} = \gamma_{rm} \cos \gamma_{rm} \frac{\phi_r}{\phi_1},$$

or if $\phi_r^2 \lambda_r^2 < (m\pi)^2$, then for symmetric modes,

$$A_{i+6k,i+7k} = \gamma_{rm} \sinh \gamma_{rm} \frac{\phi_r}{\phi_1},$$

and for anti-symmetric modes,

$$A_{i+6k,i+7k} = \gamma_{rm} \cosh \gamma_{rm} \frac{\phi_r}{\phi_1}.$$

Next, consideration is given to the condition of continuity of vertical edge reaction along the inter-segment line.

3.6 Satisfaction of the Edge Reaction Continuity Condition

In general, the vertical edge reaction is given by equation 1.26. However, along the inter-segment line, for the contributions of W_1 through W_4 , the normal direction coincide with the x or ξ axis, and therefore, Equation 1.26 reduces to;

$$\frac{V_\xi a^2}{D} = - \left[\frac{\partial^3 W}{\partial \xi^3} + \frac{\nu^*}{\phi_1^2} \frac{\partial^3 W}{\partial \xi \partial \eta^2} \right].$$

Performing the required derivations for W_1 to find;

$$\left. \frac{\partial^3 W_1}{\partial \xi^3} \right|_{\xi=1} = -E_{1m} \theta_{11m} (m\pi)^3 [\sinh \beta_{1m} \eta + C_{11m} \sin \gamma_{1m} \eta] \cos(m\pi),$$

$$\left. \frac{\partial^3 W_1}{\partial \xi \partial \eta^2} \right|_{\xi=1} = E_{1m} \theta_{11m} (m\pi) [\beta_{1m}^2 \sinh \beta_{1m} \eta - C_{11m} \gamma_{1m}^2 \sin \gamma_{1m} \eta] \cos(m\pi),$$

or if $\lambda^2 < (m\pi)^2$,

$$\left. \frac{\partial^3 W_1}{\partial \xi^3} \right|_{\xi=1} = -E_{1m} \theta_{12m} (m\pi)^3 [\sinh \beta_{1m} \eta + C_{12m} \sinh \gamma_{1m} \eta] \cos(m\pi),$$

$$\left. \frac{\partial^3 W_1}{\partial \xi \partial \eta^2} \right|_{\xi=1} = E_{1m} \theta_{12m} (m\pi) [\beta_{1m}^2 \sinh \beta_{1m} \eta - C_{12m} \gamma_{1m}^2 \sinh \gamma_{1m} \eta] \cos(m\pi).$$

The coefficients of E_{1m} in the expression of vertical edge reaction become;

$$A_{i+7k,j} = -(A_{1n} + \frac{\nu}{\phi_1^2} A_{2n}),$$

where,

$$\begin{aligned} A_{1n} &= -\theta_{11m} (m\pi)^3 \cos(m\pi) (D_{s1} + C_{11m} D_{s2}), \\ A_{2n} &= \theta_{11m} (m\pi) \cos(m\pi) (\beta_{1m}^2 D_{s1} - C_{11m} \gamma_{1m}^2 D_{s2}), \end{aligned}$$

and if $\lambda^2 < (m\pi)^2$,

$$\begin{aligned} A_{1n} &= -\theta_{12m} (m\pi)^3 \cos(m\pi) (D_{s1} + C_{12m} D_{s3}), \\ A_{2n} &= \theta_{12m} (m\pi) \cos(m\pi) (\beta_{1m}^2 D_{s1} + C_{12m} \gamma_{1m}^2 D_{s3}). \end{aligned}$$

The coefficients of V_{2m} are now evaluated, and found to have the same expression as those of E_{1m} , where A_{1n} and A_{2n} are also found from the above equations by replacing θ_{11m} , C_{11m} , θ_{12m} , and C_{12m} by θ_{21m} , C_{21m} , θ_{22m} , and C_{22m} respectively. Considering now the contributions of W_3 we find;

$$\left. \frac{\partial^3 W_3}{\partial \xi^3} \right|_{\xi=1} = E_{3m} \theta_{31m} [\beta_{3m}^3 \cosh \beta_{3m} - C_{31m} \gamma_{3m}^3 \cos \gamma_{3m}] \sin(m\pi \eta),$$

$$\left. \frac{\partial^3 W_3}{\partial \xi \partial \eta^2} \right|_{\xi=1} = -E_{3m} \theta_{31m} (m\pi)^2 [\beta_{3m} \cosh \beta_{3m} + C_{31m} \gamma_{3m} \cos \gamma_{3m}] \sin(m\pi\eta),$$

or if $\phi_1^2 \lambda^2 < (m\pi)^2$,

$$\left. \frac{\partial^3 W_3}{\partial \xi^3} \right|_{\xi=1} = E_{3m} \theta_{32m} [\beta_{3m}^3 \cosh \beta_{3m} + C_{32m} \gamma_{3m}^3 \cosh \gamma_{3m}] \sin(m\pi\eta),$$

$$\left. \frac{\partial^3 W_3}{\partial \xi \partial \eta^2} \right|_{\xi=1} = -E_{3m} \theta_{32m} (m\pi)^2 [\beta_{3m} \cosh \beta_{3m} + C_{32m} \gamma_{3m} \cosh \gamma_{3m}] \sin(m\pi\eta).$$

And the coefficients of V_{3m} in the expression of vertical edge reaction are;

$$A_{i+7k, i+2k} = -(A_{1n} + \frac{\nu^*}{\phi_1^2} A_{2n}),$$

where,

$$A_{1n} = \theta_{31m} (\beta_{3m}^3 \cosh \beta_{3m} - C_{31m} \gamma_{3m}^3 \cos \gamma_{3m}),$$

$$A_{2n} = -\theta_{31m} (m\pi)^2 (\beta_{3m} \cosh \beta_{3m} + C_{31m} \gamma_{3m} \cos \gamma_{3m}),$$

or if $\phi_1^2 \lambda^2 < (m\pi)^2$, then,

$$A_{1n} = \theta_{32m} (\beta_{3m}^3 \cosh \beta_{3m} + C_{32m} \gamma_{3m}^3 \cosh \gamma_{3m}),$$

$$A_{2n} = -\theta_{32m} (m\pi)^2 (\beta_{3m} \cosh \beta_{3m} + C_{32m} \gamma_{3m} \cosh \gamma_{3m}).$$

Consideration of the contribution of W_4 shows that the coefficients of V_{4m} have the same expression as those of E_{3m} , where A_{1n} and A_{2n} are obtained

by replacing θ_{31m} , C_{31m} , θ_{32m} , and C_{32m} in the above equations by θ_{41m} , C_{41m} , θ_{42m} , and C_{42m} respectively.

Attention is now given to the contributions of W_5 and W_6 . Starting by performing the necessary derivations the followings are obtained,

$$\left. \frac{\partial^3 W_5}{\partial \xi'^3} \right|_{\xi=1} = -E_{5m} \theta_{51m} (m\pi)^3 [\sinh \beta_{5m} (1 - \eta) + C_{51m} \sin \gamma_{5m} (1 - \eta)] \cos[m\pi(1 - F_1 \eta)],$$

$$\left. \frac{\partial^3 W_5}{\partial \eta'^3} \right|_{\xi=1} = -E_{5m} \theta_{51m} [\beta_{5m}^3 \cosh \beta_{5m} (1 - \eta) - C_{51m} \gamma_{5m}^3 \cos \gamma_{5m} (1 - \eta)] \sin[m\pi(1 - F_1 \eta)],$$

$$\left. \frac{\partial^3 W_5}{\partial \xi'^2 \partial \eta'} \right|_{\xi=1} = E_{5m} \theta_{51m} (m\pi)^2 [\beta_{5m} \cosh \beta_{5m} (1 - \eta) + C_{51m} \gamma_{5m} \cos \gamma_{5m} (1 - \eta)] \sin[m\pi(1 - F_1 \eta)],$$

$$\left. \frac{\partial^3 W_5}{\partial \xi' \partial \eta'^2} \right|_{\xi=1} = E_{5m} \theta_{51m} (m\pi) [\beta_{5m}^2 \sinh \beta_{5m} (1 - \eta) - C_{51m} \gamma_{5m}^2 \sin \gamma_{5m} (1 - \eta)] \cos[m\pi(1 - F_1 \eta)],$$

and if $\lambda'^2 < (m\pi)^2$,

$$\left. \frac{\partial^3 W_5}{\partial \xi'^3} \right|_{\xi=1} = -E_{5m} \theta_{52m} (m\pi)^3 [\sinh \beta_{5m} (1 - \eta) + C_{52m} \sinh \gamma_{5m} (1 - \eta)] \cos[m\pi(1 - F_1 \eta)],$$

$$\left. \frac{\partial^3 W_5}{\partial \eta'^3} \right|_{\xi=1} = -E_{5m} \theta_{52m} [\beta_{5m}^3 \cosh \beta_{5m} (1 - \eta) + C_{52m} \gamma_{5m}^3 \cosh \gamma_{5m} (1 - \eta)] \sin[m\pi(1 - F_1 \eta)],$$

$$\left. \frac{\partial^3 W_5}{\partial \xi'^2 \partial \eta'} \right|_{\xi=1} = E_{5m} \theta_{52m} (m\pi)^2 [\beta_{5m} \cosh \beta_{5m} (1 - \eta) + C_{52m} \gamma_{5m} \cosh \gamma_{5m} (1 - \eta)] \sin[m\pi(1 - F_1 \eta)],$$

$$\left. \frac{\partial^3 W_5}{\partial \xi' \partial \eta'^2} \right|_{\xi=1} = E_{5m} \theta_{52m} (m\pi) [\beta_{5m}^2 \sinh \beta_{5m} (1 - \eta) + C_{52m} \gamma_{5m}^2 \sinh \gamma_{5m} (1 - \eta)] \cos[m\pi(1 - F_1 \eta)].$$

Appropriate Fourier series expansions are readily available for all of the above derivatives from previous sections. Substitution into Equation 1.26 leads to the following expression for the coefficients of E_{5m} ,

$$A_{i+7k, j+4k} = -(V_1 A_{1n} + V_2 A_{2n} + V_3 A_{3n} + V_4 A_{4n}) \sin^3 \alpha,$$

where,

$$\begin{aligned} A_{1n} &= -\theta_{51m} (m\pi)^3 [E_{s1} + E_{s2} + C_{51m} (E_{s3} + E_{s4})], \\ A_{2n} &= -\theta_{51m} [\beta_{5m}^3 (F_{s1} + F_{s2}) - C_{51m} \gamma_{5m}^3 (F_{s3} + F_{s4})], \\ A_{3n} &= \theta_{51m} (m\pi)^2 [\beta_{5m} (F_{s1} + F_{s2}) + C_{51m} \gamma_{5m} (F_{s3} + F_{s4})], \\ A_{4n} &= \theta_{51m} (m\pi) [\beta_{5m}^2 (E_{s1} + E_{s2}) - C_{51m} \gamma_{5m}^2 (E_{s3} + E_{s4})], \end{aligned}$$

or if $\lambda'^2 < (m\pi)^2$, then,

$$\begin{aligned}
A_{1n} &= -\theta_{52m}(m\pi)^3[E_{s1} + E_{s2} + C_{52m}(E_{s5} + E_{s6})], \\
A_{2n} &= -\theta_{52m}[\beta_{5m}^3(F_{s1} + F_{s2}) + C_{52m}\gamma_{5m}^3(F_{s5} + F_{s6})], \\
A_{3n} &= \theta_{52m}(m\pi)^2[\beta_{5m}(F_{s1} + F_{s2} + C_{52m}\gamma_{5m}(F_{s5} + F_{s6}))], \\
A_{4n} &= \theta_{52m}(m\pi)[\beta_{5m}^2(E_{s1} + E_{s2}) + C_{52m}\gamma_{5m}^2(E_{s5} + E_{s6})].
\end{aligned}$$

Here also, the coefficients of V_{6m} are found to have the same expression as those of E_{5m} where, A_{1n} through A_{4n} are found by replacing θ_{51m} , C_{51m} , θ_{52m} , and C_{52m} by θ_{61m} , C_{61m} , θ_{62m} , and C_{62m} respectively in the above Equations.

Turning to the contribution of the rectangular element, Equation 1.26 reduces to the following;

$$\frac{V_{\xi}a^2}{D} = - \left[\frac{\partial^3 W}{\partial \xi^3} + \frac{\nu}{\phi_r^2} \frac{\partial^2 W}{\partial \xi \partial \eta^2} \right] \frac{\phi_r^3}{\phi_1^3}.$$

Evaluating the necessary derivatives to obtain;

$$\left. \frac{\partial^3 W_{rs}}{\partial \xi^3} \right|_{\xi=0} = -(A_m \beta_{rm}^3 \sinh \beta_{rm} + D_m \gamma_{rm}^3 \sin \gamma_{rm}) \sin(m\pi\eta),$$

$$\left. \frac{\partial^3 W_{rs}}{\partial \xi \partial \eta^2} \right|_{\xi=0} = (m\pi)^2 (A_m \beta_{rm} \sinh \beta_{rm} - D_m \gamma_{rm} \sin \gamma_{rm}) \sin(m\pi\eta),$$

$$\left. \frac{\partial^3 W_{ru}}{\partial \xi^3} \right|_{\xi=0} = -(A_m \beta_{rm}^3 \cosh \beta_{rm} - D_m \gamma_{rm}^3 \cos \gamma_{rm}) \sin(m\pi\eta),$$

$$\left. \frac{\partial^3 W_{ru}}{\partial \xi \partial \eta^2} \right|_{\xi=0} = (m\pi)^2 (A_m \beta_{rm} \cosh \beta_{rm} + D_m \gamma_{rm} \cos \gamma_{rm}) \sin(m\pi\eta),$$

and if $\lambda_r^2 < (m\pi)^2$, then,

$$\left. \frac{\partial^3 W_{re}}{\partial \xi^3} \right|_{\xi=0} = -(A_m \beta_{rm}^3 \sinh \beta_{rm} + D_m \gamma_{rm}^3 \sinh \gamma_{rm}) \sin(m\pi\eta),$$

$$\left. \frac{\partial^3 W_{rs}}{\partial \xi \partial \eta^2} \right|_{\xi=0} = (m\pi)^2 (A_m \beta_{rm} \sinh \beta_{rm} + D_m \gamma_{rm} \sinh \gamma_{rm}) \sin(m\pi\eta),$$

$$\left. \frac{\partial^3 W_{ru}}{\partial \xi^3} \right|_{\xi=0} = -(A_m \beta_{rm}^3 \cosh \beta_{rm} + D_m \gamma_{rm}^3 \cosh \gamma_{rm}) \sin(m\pi\eta),$$

$$\left. \frac{\partial^3 W_{ru}}{\partial \xi \partial \eta^2} \right|_{\xi=0} = (m\pi)^2 (A_m \beta_{rm} \cosh \beta_{rm} + D_m \gamma_{rm} \cosh \gamma_{rm}) \sin(m\pi\eta).$$

It is clearly seen here that the above derivatives have the appropriate form of a Fourier sine series and can readily be used to evaluate the coefficients of A_m and D_m which are found to be as follows;

$$A_{i+7k, i+6k} = (A_{1n} + \frac{\nu_r}{\phi_r^2} A_{2n}) \frac{\phi_r^3}{\phi_1^3},$$

and

$$A_{i+7k, i+7k} = (A_{3n} + \frac{\nu}{\phi_r^2} A_{4n}) \frac{\phi_r^3}{\phi_1^3},$$

where for symmetric modes,

$$A_{1n} = -\beta_{rm}^3 \sinh \beta_{rm},$$

$$A_{2n} = (m\pi)^2 \beta_{rm} \sinh \beta_{rm},$$

$$A_{3n} = -\gamma_{rm}^3 \sin \gamma_{rm},$$

$$A_{4n} = -(m\pi)^2 \gamma_{rm} \sin \gamma_{rm},$$

and for anti-symmetric modes,

$$A_{1n} = -\beta_{rm}^3 \cosh \beta_{rm},$$

$$A_{2n} = (m\pi)^2 \beta_{rm} \cosh \beta_{rm},$$

$$A_{3n} = \gamma_{rm}^3 \cos \gamma_{rm},$$

$$A_{4n} = (m\pi)^2 \gamma_{rm} \cos \gamma_{rm}.$$

However, if $\lambda_r^2 < (m\pi)^2$, then for symmetric modes,

$$A_{3n} = -\gamma_{rm}^3 \sinh \gamma_{rm},$$

$$A_{4n} = (m\pi)^2 \gamma_{rm} \sinh \gamma_{rm},$$

and for anti-symmetric modes,

$$A_{3n} = -\gamma_{rm}^3 \cosh \gamma_{rm},$$

$$A_{4n} = (m\pi)^2 \gamma_{rm} \cosh \gamma_{rm}.$$

3.7 Generation of Eigenvalues and Mode Shapes

All the elements of the coefficient matrix of Figure 3.4 are now available as a function of the eigenvalue λ^2 . The problem is now an eigenvalue problem. A trial value of λ^2 is now assumed, and the determinant of the coefficient matrix is evaluated. The assumed eigenvalue is then incremented, and the corresponding value of the determinant computed. This procedure is repeated until an eigenvalue that causes the determinant to vanish is found. However, not all eigenvalues found in this manner result in vibratory motion. Some of them may represent trivial solutions, in which case the resulting shape would have zero displacement. These eigenvalues will be discussed further in Part D of this thesis. Once a genuine eigenvalue is determined, then one of the unknowns in the shape function is set to unity, say $E_{1m} = 1$ for $m = 1$, and the set of equations is then solved for the remaining unknowns. Once E_{1m} through D_m are found, the shape is then easily generated. Program I in Appendix A-II is designed to perform these tasks. This computer program, as well as the computed eigenvalues and mode shapes for this problem will be discussed in details in Part D of this dissertation.

3.8 Alternative Solution (Cosine Function)

A set of alternative building blocks using cosine function solutions was discussed in Section 2.2-2, and is shown here in Figure 3.7 for convenient reference. An alternative solution to the free vibration problem of the simply supported symmetrical trapezoidal plate is arrived at by using this set of alternative building blocks. This solution is sometimes referred to as the cosine function solution, and was developed not to duplicate the work that has already been done and discussed, but to demonstrate the validity and convergence of the superposition method in solving this type of dynamic problems as will be discussed in more detailed manner in Part D of this manuscript. However, since the same procedure and techniques used in the previous solution applies here, and in order not to be repetitive, details of this solution are not being given here, but are shown in Appendix A-III for completeness.

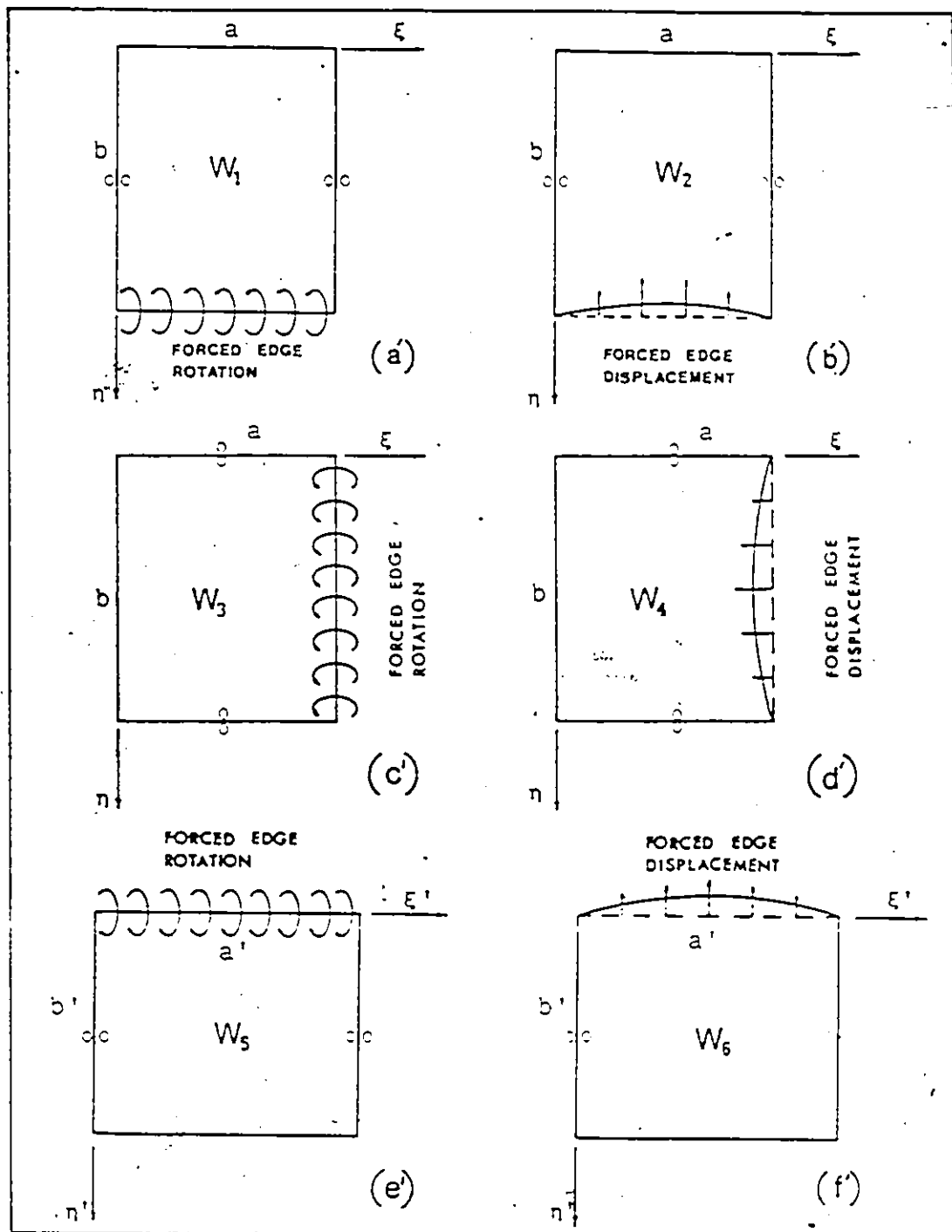


Figure 3.7- Alternative first set of basic building blocks used in the analysis of the simply supported symmetrical trapezoidal plate of Figure 3.1.

Chapter 4

FULLY CLAMPED SYMMETRICAL TRAPEZOIDAL PLATES

General

Steps and techniques that constitute the basis of this work were outlined in Chapter 2. As the first application to these techniques, the simply supported trapezoidal plate was discussed in Chapter 3 where the reader was exposed, first hand, to the implementation of these techniques by clearly explaining in details the steps involved. As a second application, we now consider the case of the fully clamped symmetrical trapezoidal plate.

4.1 Basic Building Blocks and their Superposition

Reference is made here to Figure 4.1 where as discussed in Section 3.1, due to symmetry about the central axis, all possible free vibration modes must be symmetric or anti-symmetric with respect to this axis. Each of these two possible families of modes will be dealt with separately. The basic building blocks used in this solution are shown in Figures 4.2 and 4.3. It is clear from Figure 4.2 that building blocks used in the solution of the triangular element of the simply supported case are also used here for the fully clamped case.

Therefore, the solution of the triangular element is readily available from

Chapter 3. However, instead of adjusting the coefficient appearing in this solution to have zero net moment along the edges $\eta = 1$ and $\eta' = 0$, they are adjusted for zero rotation along these edges, the requirement of zero displacement remaining unchanged. On the other hand, two new building blocks, (b) and (c) for symmetric modes, and (e) and (f) for anti-symmetric modes, are required on top of those of Figure 3.3 for the rectangular element to replace the simple support conditions along the relevant edges by clamped edge conditions. Solutions for these building blocks were discussed in Chapter 2.

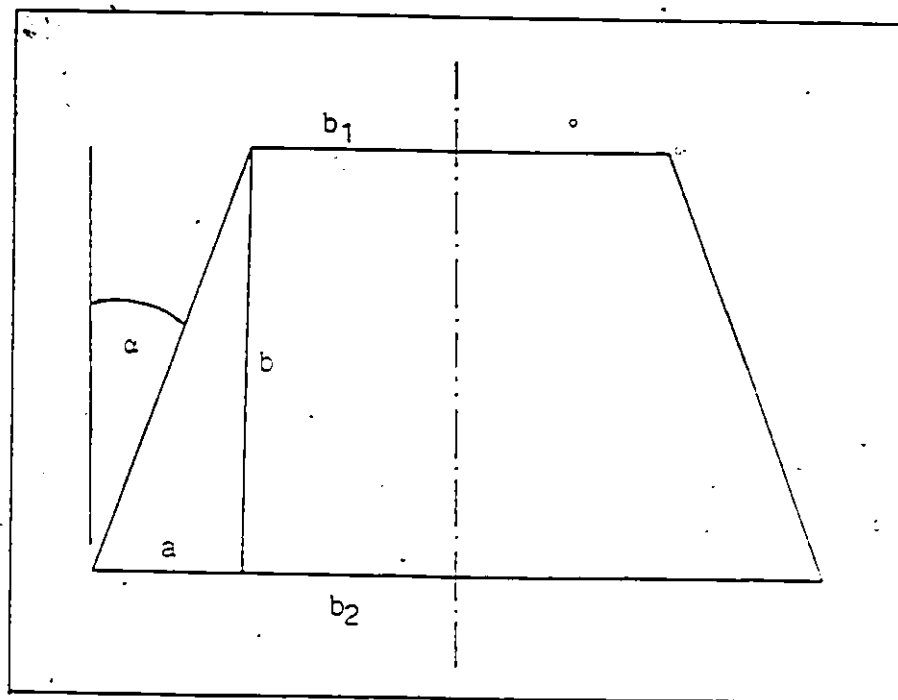


Figure 4.1- Fully clamped symmetrical trapezoidal plate.

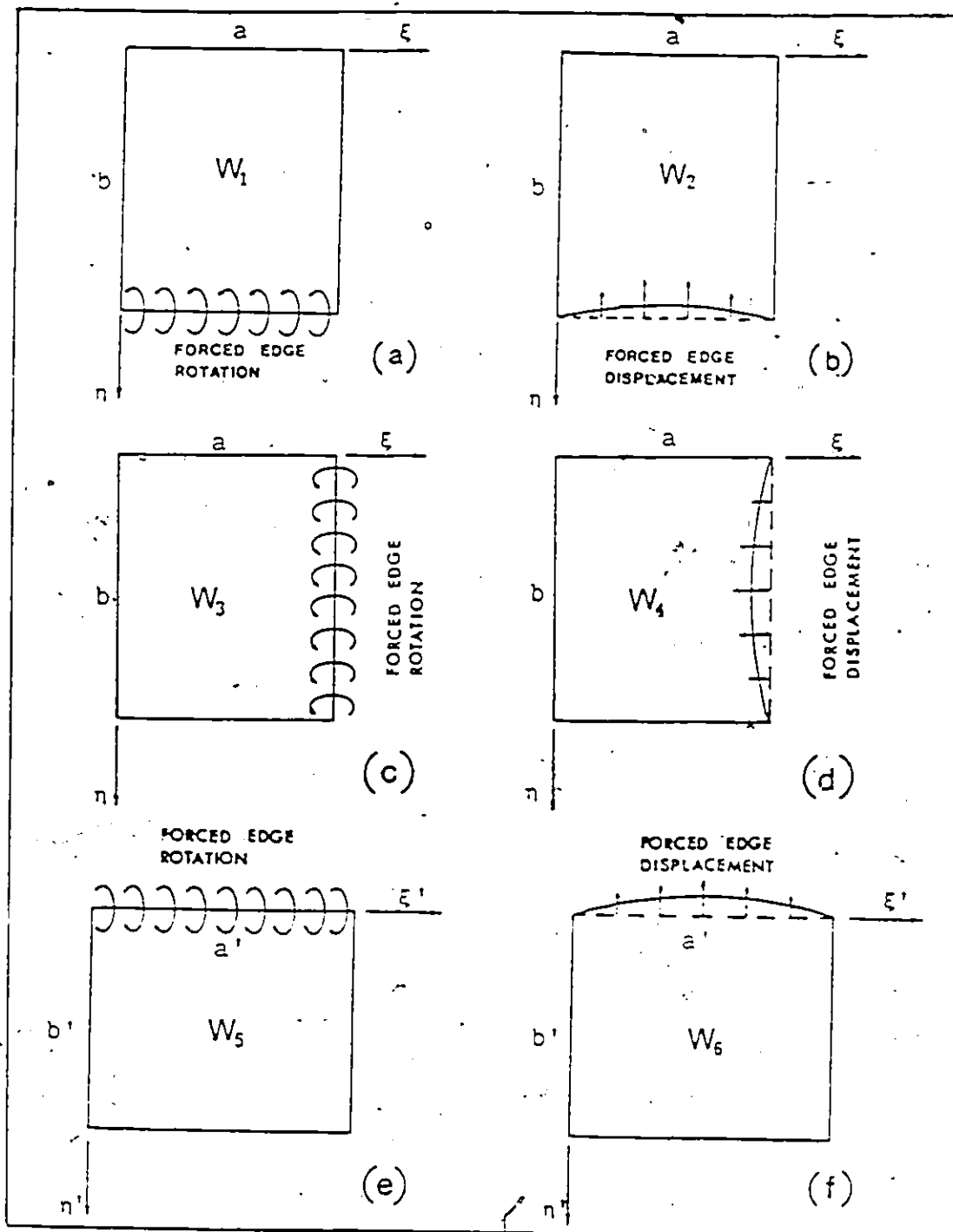


Figure 4.2- First set of basic building blocks used in the analysis of the fully clamped symmetrical trapezoidal plate of Figure 4.1.

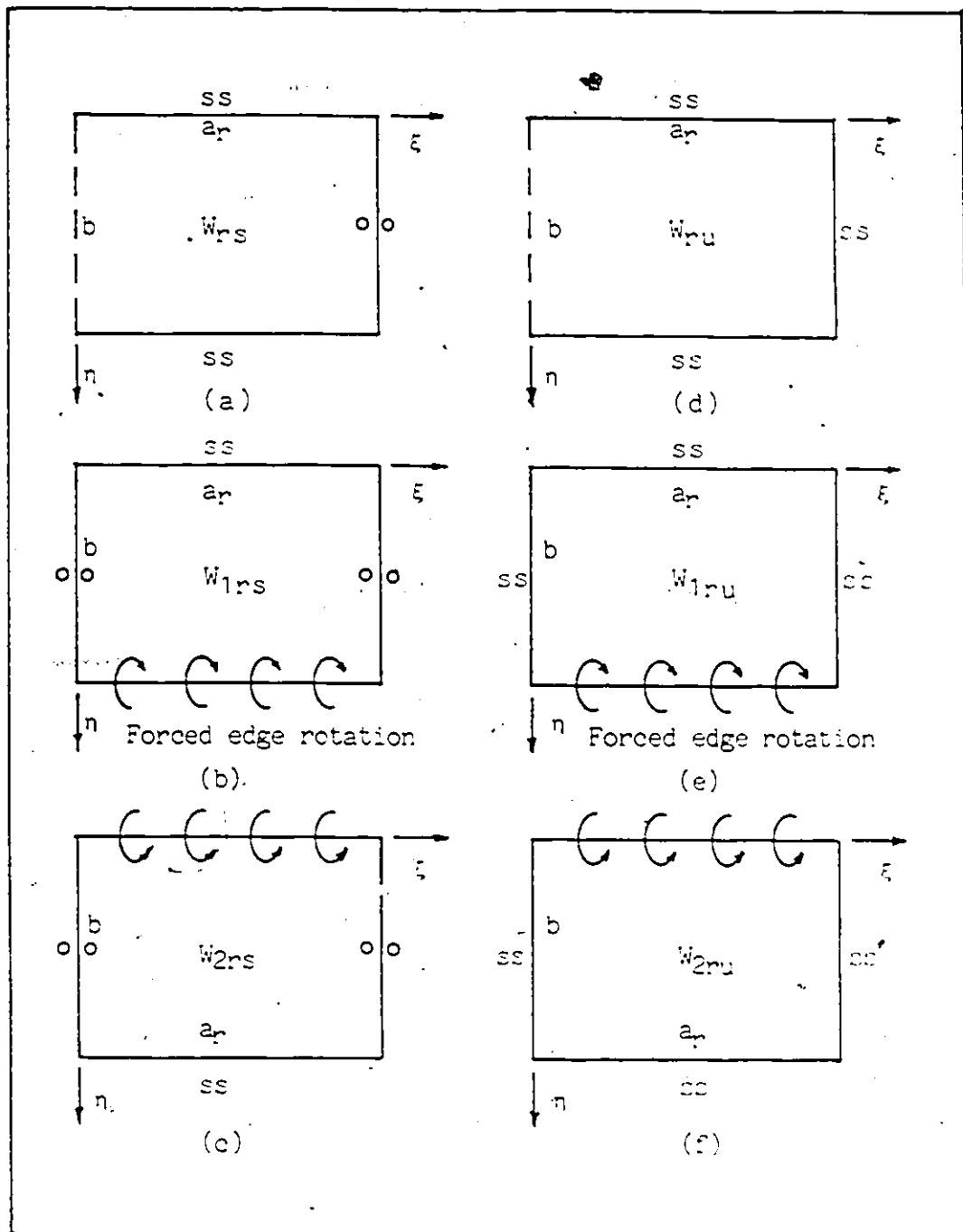


Figure 4.3- Second set of basic building blocks used in the analysis of the fully clamped symmetrical trapezoidal plate of Figure 4.1.

4.2 Solution by Enforcement of Boundary Conditions

Having placed all appropriate building blocks properly in place, we only need to constrain the coefficients appearing in their solutions in such a way that the net rotation and the net displacement vanish along each of the outer edges of both, triangular and rectangular, elements of Figure 4.1, creating six sets of homogeneous constraint equations, D'_1 , S'_1 , D'_3 , S'_3 , S'_{r1} , and S'_{r2} as represented schematically in the three term expansion coefficient matrix of Figure 4.4. The four continuity conditions are then enforced along the inter-segment of the rectangular and triangular elements as discussed in Section 3.2, creating the remaining four sets of homogeneous constraint equations D'_2 , M'_2 , S'_2 , and R'_2 . Eigenvalues are then obtained by requiring the determinant of this coefficient matrix to vanish.

A simple comparison of Figures 3.4 and 4.4 reveals that the coefficient matrix of Figure 4.4 may easily be obtained from that of Figure 3.4 simply by replacing the two sets of homogeneous equations M'_1 and M'_3 , requiring the net bending moment to vanish along the two outer edges of the triangular element, by the two sets of homogeneous equations S'_1 and S'_3 ; setting the rotation along these edges to zero. We must also add the contributions of the two extra building blocks discussed above to the continuity equations along the inter-element line, and to the rotation along the outer edges of the rectangular element. Therefore, we only need to discuss the above mentioned changes. The other coefficients remain unchanged, and are available from the previous chapter.

4.3 Satisfaction of Displacement Requirements

The only modification to be made here, in order to get from the coefficient matrix of Figure 3.4 to the coefficient matrix at hand, is the addition of the contributions of the two new building blocks, W_{1rs} , and W_{2rs} for symmetric modes, and W_{1ru} , and W_{2ru} for anti-symmetric modes. For symmetric modes, these contributions are found to be;

	E_{1m}	V_{2m}	E_{3m}	V_{4m}	E_{5m}	V_{6m}	A_m	D_m	E_{1rm}	E_{2rm}
D'_1										
D'_2										
D'_3										
S'_1										
M'_2										
S'_3										
S'_2										
VR'_2										
S'_{r1}										
S'_{r2}										

Figure 4.4- Schematic representation of a three term expansion coefficient matrix, symmetric modes.

	E_{1m}	V_{2m}	E_{3m}	V_{4m}	E_{5m}	V_{6m}	A_m	D_m	E_{1rm}	E_{2rm}
D_1'										
D_2'										
D_3'										
S_1'										
M_2'										
S_3'										
S_2'										
R_2'										
S_{-1}'										
S_{-2}'										

Figure 4.5- Schematic representation of a three term expansion coefficient matrix, anti-symmetric modes.

$$A_{i+k,j+8k} = -\theta_{11rsm}(B_{1rs} + C_{11rsm}B_{2rs}),$$

$$A_{i+k,j+9k} = -\theta_{11rsm}(C_{1rs} + C_{11rsm}C_{2rs}),$$

or if $\lambda_r^2 < (m\pi)^2$, then,

$$A_{i+k,j+8k} = -\theta_{12rsm}(B_{1rs} + C_{12rsm}B_{3rs}),$$

$$A_{i+k,j+9k} = -\theta_{12rsm}(C_{1rs} + C_{12rsm}C_{3rs}),$$

where,

$$B_{1rs} = -\frac{2n\pi\cos(n\pi)\sinh\beta_{1rm}}{\beta_{1rm}^2 + (n\pi)^2},$$

$$B_{2rs} = \frac{2n\pi\cos(n\pi)\sin\gamma_{1rm}}{\gamma_{1rm}^2 - (n\pi)^2},$$

$$B_{3rs} = -\frac{2n\pi\cos(n\pi)\sinh\gamma_{1rm}}{\gamma_{1rm}^2 + (n\pi)^2},$$

or if $\gamma_{1rm} = n\pi$, then $B_{2rs} = 1$, and

$$C_{1rs} = \frac{2n\pi\sinh\beta_{1rm}}{\beta_{1rm}^2 + (n\pi)^2},$$

$$C_{2rs} = \frac{2n\pi\sin\gamma_{1rm}}{(n\pi)^2 - \gamma_{1rm}^2},$$

$$C_{3rs} = \frac{2n\pi\sinh\gamma_{1rm}}{(n\pi)^2 + \gamma_{1rm}^2},$$

or if $\gamma_{1rm} = n\pi$, then,

$$C_{2rs} = (\sin\gamma_{1rm} - \gamma_{1rm}\cos\gamma_{1rm})/\gamma_{1rm}.$$

For anti-symmetric modes we have;

$$A_{i+k,j+8k} = A_{i+k,j+9k} = 0.$$

4.4 Satisfaction of Bending Moment Requirements

Since the plate is fully clamped, the only restriction on bending moments in the boundary conditions is the continuity requirement along the inter-segment line where the triangular and rectangular elements are joined together. Contributions to the bending moment about this inter-segment are found from the previous chapter with the exception of the contributions of the two added building blocks as discussed earlier. For symmetric modes, these contributions are found to be;

$$A_{i+4k, j+8k} = (A_{1n} + \frac{\nu}{\phi_r^2} A_{2n}) \phi_r^2,$$

where,

$$\begin{aligned} A_{1n} &= -\theta_{11rsm} (m\pi)^2 (B_{1rs} + C_{11rsm} B_{2rs}), \\ A_{2n} &= \theta_{11rsm} (\beta_{1rm}^2 B_{1rs} - C_{11rsm} \gamma_{1rm}^2 B_{2rs}), \end{aligned}$$

or if $\lambda_r^2 < (m\pi)^2$, then,

$$\begin{aligned} A_{1n} &= -\theta_{12rsm} (m\pi)^2 (B_{1rs} + C_{12rsm} B_{3rs}), \\ A_{2n} &= \theta_{12rsm} (\beta_{1rm}^2 B_{1rs} + C_{12rsm} \gamma_{1rm}^2 B_{3rs}). \end{aligned}$$

And

$$A_{i+4k, j+9k} = (A_{1n} + \frac{\nu}{\phi_r^2} A_{2n}) \phi_r^2,$$

where, A_{1n} and A_{2n} are obtained from the above expression by replacing B_{1rs} , B_{2rs} and B_{3rs} by C_{1rs} , C_{2rs} and C_{3rs} respectively. For anti-symmetric modes, the two new building blocks have no contribution to the bending moment along the inter-segment line, and therefore,

$$A_{i+4k, j+8k} = A_{i+4k, j+9k} = 0.$$

4.5 Satisfaction of Rotation Requirements

Here, the major deviation of the coefficient matrix under consideration from that of Figure 3.4 is in the replacement of the two sets of homogeneous equations M'_1 and M'_3 by the S'_1 and S'_3 requiring the rotation to vanish along the two outer edges of the triangular element instead of the bending moment.

4.5-1 Rotation Along the Edge $\eta = 1$

It was shown in Section 3.5 that the general equation representing the rotation about any line in the plane of a plate is given by;

$$\frac{\partial W}{\partial n} = \frac{\partial W}{\partial x} \cos \alpha + \frac{\partial W}{\partial y} \sin \alpha.$$

Consideration of the contributions of the first two building blocks W_1 and W_2 to the rotation about this edge leads to the following coefficients,

$$A_{i+3k,i} = \theta_{11m}(\beta_{1m} \cosh \beta_{1m} + C_{11m} \gamma_{1m} \cos \gamma_{1m}),$$

$$A_{i+3k,i+k} = \theta_{21m}(\beta_{1m} \cosh \beta_{1m} + C_{21m} \gamma_{1m} \cos \gamma_{1m}),$$

or if $\lambda^2 < (m\pi)^2$, then,

$$A_{i+3k,i} = \theta_{12m}(\beta_{1m} \cosh \beta_{1m} + C_{12m} \gamma_{1m} \cosh \gamma_{1m}),$$

$$A_{i+3k,i+k} = \theta_{22m}(\beta_{1m} \cosh \beta_{1m} + C_{22m} \gamma_{1m} \cosh \gamma_{1m}).$$

Contributions of W_3 and W_4 to rotation about the edge $\eta = 1$ provide the following coefficients,

$$A_{i+3k,j+2k} = m\pi \theta_{31m} \cos(m\pi)(A_{s1} + C_{31m} A_{s2}),$$

$$A_{i+3k,j+3k} = m\pi \theta_{41m} \cos(m\pi)(A_{s1} + C_{41m} A_{s2}),$$

or if $\phi_1^2 \lambda^2 < (m\pi)^2$, then,

$$A_{i+3k,j+2k} = m\pi\theta_{32m}\cos(m\pi)(A_{s1} + C_{32m}A_{s3}),$$

$$A_{i+3k,j+3k} = m\pi\theta_{42m}\cos(m\pi)(A_{s1} + C_{42m}A_{s3}),$$

where,

$$A_{s1} = -2n\pi\cos(n\pi)\sinh\beta_{3m}/[\beta_{3m}^2 + (n\pi)^2],$$

$$A_{s2} = 2n\pi\cos(n\pi)\sin\gamma_{3m}/[\gamma_{3m}^2 - (n\pi)^2],$$

$$A_{s3} = -2n\pi\cos(n\pi)\sinh\gamma_{3m}/[\gamma_{3m}^2 + (n\pi)^2].$$

Consider now the contributions of the two building blocks W_5 and W_6 . Evaluation of these contributions leads to the following coefficients,

$$A_{i+3k,j+4k} = F_{11}A_{1n} + F_{12}A_{2n},$$

where,

$$F_{11} = \cos\alpha\cos\alpha_1,$$

$$F_{12} = \sin\alpha_1/\sin\alpha,$$

and where,

$$A_{1n} = \theta_{51m}(m\pi)[B_{s1} + B_{s2} + C_{51m}(B_{s3} + B_{s4})],$$

$$A_{2n} = -\theta_{51m}[\beta_{5m}(C_{s1} + C_{s2}) + C_{51m}\gamma_{5m}(C_{s3} + C_{s4})],$$

or if $\lambda'^2 < (m\pi)^2$, then,

$$A_{1n} = \theta_{52m}(m\pi)[B_{s1} + B_{s2} + C_{52m}(B_{s5} + B_{s6})],$$

$$A_{2n} = -\theta_{52m}[\beta_{5m}(C_{s1} + C_{s2}) + C_{52m}\gamma_{5m}(C_{s5} + C_{s6})].$$

Also,

$$A_{i+3k,j+5k} = F_{11}A_{1n} + F_{12}A_{2n},$$

where, A_{1n} and A_{2n} are obtained from the above expressions by replacing θ_{51m} , C_{51m} , θ_{52m} and C_{52m} by θ_{61m} , C_{61m} , θ_{62m} and C_{62m} respectively. And where,

$$B_{s1} = \frac{(m\pi F_2 + n\pi) \sinh \beta_{5m} - \beta_{5m} \sin(m\pi F_2 + n\pi)}{\beta_{5m}^2 + (m\pi F_2 + n\pi)^2},$$

$$B_{s2} = \frac{\beta_{5m} \sin(m\pi F_2 - n\pi) - (m\pi F_2 - n\pi) \sinh \beta_{5m}}{\beta_{5m}^2 + (m\pi F_2 - n\pi)^2},$$

$$B_{s3} = \frac{\gamma_{5m} [\sin \gamma_{5m} - \sin(m\pi F_2 + n\pi)]}{(m\pi F_2 + n\pi)^2 - \gamma_{5m}^2},$$

$$B_{s4} = \frac{\gamma_{5m} [\sin(m\pi F_2 - n\pi) - \sin \gamma_{5m}]}{(m\pi F_2 - n\pi)^2 - \gamma_{5m}^2},$$

but if $\gamma_{5m} = (m\pi F_2 + n\pi)$, then, then,

$$B_{s3} = (\sin \gamma_{5m} - \gamma_{5m} \cos \gamma_{5m}) / 2\gamma_{5m},$$

or if $\gamma_{5m} = (m\pi F_2 - n\pi)$ then,

$$B_{s4} = (\gamma_{5m} \cos \gamma_{5m} - \sin \gamma_{5m}) / 2\gamma_{5m},$$

and if $\gamma_{5m} = (n\pi - m\pi F_2)$,

$$B_{s4} = (\sin \gamma_{5m} - \gamma_{5m} \cos \gamma_{5m}) / 2\gamma_{5m},$$

B_{s5} and B_{s6} are obtained from the expressions of B_{s1} and B_{s2} respectively by replacing β_{5m} by γ_{5m} . And where,

$$\begin{aligned}
C_{s1} &= \frac{\beta_{5m} \sinh \beta_{5m} + (m\pi F_2 - n\pi) \sin(m\pi F_2 - n\pi)}{\beta_{5m}^2 + (m\pi F_2 - n\pi)^2}, \\
C_{s2} &= -\frac{\beta_{5m} \sinh \beta_{5m} + (m\pi F_2 + n\pi) \sin(m\pi F_2 + n\pi)}{\beta_{5m}^2 + (m\pi F_2 + n\pi)^2}, \\
C_{s3} &= \frac{(m\pi F_2 - n\pi) \sin(m\pi F_2 - n\pi) - \gamma_{5m} \sin \gamma_{5m}}{(m\pi F_2 - n\pi)^2 - \gamma_{5m}^2}, \\
C_{s4} &= \frac{\gamma_{5m} \sin \gamma_{5m} - (m\pi F_2 + n\pi) \sin(m\pi F_2 + n\pi)}{(m\pi F_2 + n\pi)^2 - \gamma_{5m}^2},
\end{aligned}$$

or if $\gamma_{5m}^2 = (m\pi F_2 - n\pi)^2$, then,

$$C_{s3} = (\gamma_{5m} \cos \gamma_{5m} - \sin \gamma_{5m}) / 2\gamma_{5m},$$

and if $\gamma_{5m} = (m\pi F_2 + n\pi)$,

$$C_{s4} = (\sin \gamma_{5m} - \gamma_{5m} \cos \gamma_{5m}) / 2\gamma_{5m}.$$

C_{s5} and C_{s6} are obtained from the above expressions of C_{s1} and C_{s2} respectively by replacing β_{5m} by γ_{5m} .

4.5-2 Rotation Along the Inter-Segment Line $\xi = 1$

The coefficients found in the set of equations S'_2 of Figure 3.4 remain unchanged except for the addition of the coefficients relating to E_{1rm} and E_{2rm} of the added building blocks. In the case of symmetric modes, these building blocks have no contribution to the slope across the inter-segment line, and

therefore,

$$A_{i+6k,j+8k} = A_{i+6k,j+9k} = 0.$$

In the case of anti-symmetric modes,

$$A_{i+6k,j+8k} = \theta_{11rum}(m\pi)(B_{1rs} + C_{11rum}B_{2rs})\phi_r/\phi_1,$$

$$A_{i+6k,j+9k} = \theta_{11rum}(m\pi)(C_{1rs} + C_{11rum}C_{2rs})\phi_r/\phi_1,$$

or if $\lambda_r^2 < (m\pi)^2$, then,

$$A_{i+6k,j+8k} = \theta_{12rum}(m\pi)(B_{1rs} + C_{12rum}B_{3rs})\phi_r/\phi_1,$$

$$A_{i+6k,j+9k} = \theta_{12rum}(m\pi)(C_{1rs} + C_{12rum}C_{3rs})\phi_r/\phi_1,$$

where, $m = 1, 2, \dots$

4.5-3 Rotation Along the Edge $\eta' = 0$

The set of homogeneous algebraic equations S_3 is now obtained by considering the various relevant contributions to rotations along this edge. Starting with the contributions of the first two building blocks of Figure 4.2 to get,

$$A_{i+5k,j} = F_{31}A_{1n} + F_{32}A_{2n},$$

where,

$$F_{31} = \cos\alpha\cos\alpha_3,$$

$$F_{32} = \sin\alpha\sin\alpha_3,$$

$$A_{1n} = \theta_{11m}(m\pi)[G_{s1} + G_{s2} + C_{11m}(G_{s3} + G_{s4})],$$

$$A_{2n} = \theta_{11m}[\beta_{1m}(H_{s1} + H_{s2}) + C_{11m}\gamma_{1m}(H_{s3} + H_{s4})],$$

or if $\lambda^2 < (m\pi)^2$, then,

$$\begin{aligned} A_{1n} &= \theta_{12m}(m\pi)[G_{s1} + G_{s2} + C_{12m}(G_{s5} + G_{s6})], \\ A_{2n} &= \theta_{12m}[\beta_{1m}(H_{s1} + H_{s2}) + C_{12m}\gamma_{1m}(H_{s5} + H_{s6})]. \end{aligned}$$

Also,

$$A_{i+5k,j+k} = F_{31}A_{1n} + F_{32}A_{2n},$$

where, A_{1n} and A_{2n} are obtained from the above expressions by replacing θ_{11m} , C_{11m} , θ_{12m} and C_{12m} by θ_{21m} , C_{21m} , θ_{22m} and C_{22m} respectively. And where,

$$G_{s1} = (m\pi + n\pi)\sinh\beta_{1m}/[\beta_{1m}^2 + (m\pi + n\pi)^2],$$

$$G_{s2} = (n\pi - m\pi)\sinh\beta_{1m}/[\beta_{1m}^2 + (m\pi - n\pi)^2],$$

$$G_{s3} = (m\pi + n\pi)\sin\gamma_{1m}/[(m\pi + n\pi)^2 - \gamma_{1m}^2],$$

$$G_{s4} = (n\pi - m\pi)\sin\gamma_{1m}/[(m\pi - n\pi)^2 - \gamma_{1m}^2],$$

but if $\gamma_{1m} = (m\pi + n\pi)$, then,

$$G_{s3} = (\sin\gamma_{1m} - \gamma_{1m}\cos\gamma_{1m})/2\gamma_{1m},$$

or if $\gamma_{1m} = (m\pi - n\pi)$, then,

$$G_{s4} = (\gamma_{1m}\cos\gamma_{1m} - \sin\gamma_{1m})/2\gamma_{1m},$$

and if $\gamma_{1m} = (n\pi - m\pi)$,

$$G_{s4} = (\sin\gamma_{1m} - \gamma_{1m}\cos\gamma_{1m})/2\gamma_{1m}.$$

G_{s5} and G_{s6} are found from the expressions of G_{s1} and G_{s2} respectively by replacing β_{1m} by γ_{1m} . Also,

$$H_{s1} = \beta_{1m}\sinh\beta_{1m}/[\beta_{1m}^2 + (m\pi - n\pi)^2],$$

$$H_{s2} = -\beta_{1m}\sinh\beta_{1m}/[\beta_{1m}^2 + (m\pi + n\pi)^2],$$

$$H_{s3} = -\gamma_{1m}\sin\gamma_{1m}/[(m\pi - n\pi)^2 - \gamma_{1m}^2],$$

$$H_{s4} = \gamma_{1m}\sin\gamma_{1m}/[(m\pi + n\pi)^2 - \gamma_{1m}^2],$$

or if $\gamma_{1m}^2 = (m\pi - n\pi)^2$, then,

$$H_{s3} = (\gamma_{1m}\cos\gamma_{1m} - \sin\gamma_{1m})/2\gamma_{1m},$$

and if $\gamma_{1m}^2 = (m\pi + n\pi)^2$, then,

$$H_{s4} = (\sin\gamma_{1m} - \gamma_{1m}\cos\gamma_{1m})/2\gamma_{1m}.$$

H_{s5} and H_{s6} are now obtained from the expressions of H_{s1} and H_{s2} respectively by replacing β_{1m} by γ_{1m} .

Turning now to the contributions of W_3 and W_4 , the coefficients of E_{2m} are found to be,

$$A_{i+5k, j+2k} = F_{31}A_{1n} + F_{32}A_{2n},$$

where,

$$A_{1n} = \theta_{31m}[\beta_{3m}(P_{s1} + P_{s2}) + C_{31m}\gamma_{3m}(P_{s3} + P_{s4})],$$

$$A_{2n} = \theta_{31m}(m\pi)[Q_{s1} + Q_{s2} + C_{31m}(Q_{s3} + Q_{s4})],$$

or if $\phi_1^2 \lambda^2 < (m\pi)^2$, then,

$$A_{1n} = \theta_{32m}[\beta_{3m}(P_{s1} + P_{s2}) + C_{32m}\gamma_{3m}(P_{s5} + P_{s6})],$$

$$A_{2n} = \theta_{32m}(m\pi)[Q_{s1} + Q_{s2} + C_{32m}(Q_{s5} + Q_{s6})].$$

Also, the coefficients of V_{4m} are,

$$A_{i+5k, j+3k} = F_{31}A_{1n} + F_{32}A_{2n},$$

where, A_{1n} and A_{2n} are obtained by replacing θ_{31m} , C_{31m} , θ_{32m} and C_{32m} in the above expressions, by θ_{41m} , C_{41m} , θ_{42m} and C_{42m} respectively. And where,

$$P_{s1} = \beta_{3m} \sinh \beta_{3m} \cos(n\pi) / [\beta_{3m}^2 + (m\pi + n\pi)^2],$$

$$Q_{s1} = (m\pi - n\pi) \sinh \beta_{3m} \cos(n\pi) / [\beta_{3m}^2 + (m\pi - n\pi)^2],$$

$$P_{s2} = -\beta_{3m} \sinh \beta_{3m} \cos(n\pi) / [\beta_{3m}^2 + (m\pi - n\pi)^2],$$

$$Q_{s2} = -(m\pi + n\pi) \sinh \beta_{3m} \cos(n\pi) / [\beta_{3m}^2 + (m\pi + n\pi)^2],$$

$$P_{s3} = \gamma_{3m} \sin(\gamma_{3m} + n\pi) / [\gamma_{3m}^2 - (m\pi + n\pi)^2],$$

$$Q_{s3} = (m\pi - n\pi) \sin(\gamma_{3m} + n\pi) / [(\gamma_{3m}^2 - (m\pi - n\pi)^2)],$$

$$P_{s4} = -\gamma_{3m} \sin(\gamma_{3m} + n\pi) / [\gamma_{3m}^2 - (m\pi - n\pi)^2],$$

$$Q_{s4} = -(m\pi + n\pi) \sin(\gamma_{3m} - n\pi) / [\gamma_{3m}^2 - (m\pi + n\pi)^2],$$

but if $\gamma_{3m} = (m\pi + n\pi)$, then,

$$P_{s3} = [2\gamma_{3m} \cos(m\pi) + \sin(2\gamma_{3m} - m\pi)] / 4\gamma_{3m},$$

$$Q_{s4} = [2\gamma_{3m} \cos(m\pi) - \sin(2\gamma_{3m} - m\pi)] / 4\gamma_{3m},$$

or if $\gamma_{3m}^2 = (m\pi - n\pi)^2$,

$$P_{s4} = -[2\gamma_{3m}\cos(m\pi) + \sin(2\gamma_{3m} + m\pi)]/4\gamma_{3m},$$

or if $\gamma_{3m} = (m\pi - n\pi)$, then,

$$Q_{s3} = [\sin(2\gamma_{3m} - m\pi) - 2\gamma_{3m}\cos(m\pi)]/4\gamma_{3m},$$

and if $\gamma_{3m} = (n\pi - m\pi)$,

$$Q_{s3} = [2\gamma_{3m}\cos(m\pi) - \sin(2\gamma_{3m} - m\pi)]/4\gamma_{3m}.$$

P_{s5} , P_{s6} , Q_{s5} and Q_{s6} are obtained by replacing β_{3m} by γ_{3m} in the expressions of P_{s1} , P_{s2} , Q_{s1} and Q_{s2} respectively.

The only relevant contributions remaining for consideration are those of W_5 and W_6 which lead to the following coefficients,

$$A_{i+5k, i+4k} = \theta_{51m}(\beta_{5m}\cosh\beta_{5m} + C_{51m}\gamma_{5m}\cos\gamma_{5m}),$$

$$A_{i+5k, i+5k} = \theta_{61m}(\beta_{5m}\cosh\beta_{5m} + C_{61m}\gamma_{5m}\cos\gamma_{5m}),$$

or if $\lambda'^2 < (m\pi)^2$, then,

$$A_{i+5k, i+4k} = \theta_{52m}(\beta_{5m}\cosh\beta_{5m} + C_{52m}\gamma_{5m}\cosh\gamma_{5m}),$$

$$A_{i+5k, i+5k} = \theta_{62m}(\beta_{5m}\cosh\beta_{5m} + C_{62m}\gamma_{5m}\cosh\gamma_{5m}).$$

4.5-4 Rotation Along the Edge $\eta = 0$

The contributing building blocks here are those of Figure 4.3, where Blocks (a), (b) and (c) are used for symmetric modes, and Blocks (d), (e) and (f) are used for anti-symmetric modes. In the case of symmetric modes, consideration of W_{rs} leads to the following coefficients for A_m and D_m ,

$$A_{i+9k, j+6k} = m\pi A_{1rss},$$

$$A_{i+9k, j+7k} = m\pi A_{2rss},$$

or if $\phi_r^2 \lambda_r^2 < (m\pi)^2$, then,

$$A_{i+9k, j+7k} = m\pi A_{3rss},$$

where,

$$A_{1rss} = \delta \beta_{rm} \sinh \beta_{rm} / [\beta_{rm}^2 + (n\pi)^2],$$

$$A_{2rss} = \delta \gamma_{rm} \sin \gamma_{rm} / [\gamma_{rm}^2 - (n\pi)^2],$$

$$A_{3rss} = \delta \gamma_{rm} \sinh \gamma_{rm} / [\gamma_{rm}^2 + (n\pi)^2],$$

or if $\gamma_{rm} = n\pi$, then,

$$A_{2rss} = \delta (\gamma_{rm} \cos \gamma_{rm} + \sin \gamma_{rm}) / 2\gamma_{rm}.$$

Also, $n = 0, 1, 2, \dots$, and

$$\delta = \begin{cases} 1 & \text{if } n = 0, \\ 2 & \text{if } n \neq 0. \end{cases}$$

Turning now to the contributions of W_{1rs} and W_{2rs} , the following coefficients are observed,

$$A_{i+9k,i+8k} = \theta_{11rsm}(\beta_{1rm} + C_{11rsm}\gamma_{1rm}),$$

$$A_{i+9k,i+9k} = -1,$$

or if $\lambda_r^2 < (m\pi)^2$, then,

$$A_{i+9k,i+8k} = \theta_{12rsm}(\beta_{1rm} + C_{12rsm}\gamma_{1rm}).$$

In the case of anti-symmetric modes, the contributions of W_{ru} lead to the following coefficients,

$$A_{i+9k,j+6k} = m\pi A_{1rus},$$

$$A_{i+9k,j+7k} = m\pi A_{2rus},$$

or if $\phi_r^2 \lambda_r^2 < (m\pi)^2$, then,

$$A_{i+9k,j+7k} = m\pi A_{3rus},$$

where,

$$A_{1rus} = 2n\pi \sinh \beta_{rm} / [\beta_{rm}^2 + (n\pi)^2],$$

$$A_{2rus} = 2n\pi \sin \gamma_{rm} / [(n\pi)^2 - \gamma_{rm}^2],$$

$$A_{3rus} = 2\widetilde{n}\pi \sinh \gamma_{rm} / [\gamma_{rm}^2 + (n\pi)^2],$$

or if $\gamma_{rm} = n\pi$, then,

$$A_{2rus} = (\sin \gamma_{rm} - \gamma_{rm} \cos \gamma_{rm}) / \gamma_{rm}.$$

Consideration of the contributions of the last two building blocks to the slope across this edge results in the following coefficients,

$$A_{i+9k,i+8k} = \theta_{11rum}(\beta_{1rm} + C_{11rum}\gamma_{1rm}),$$

$$A_{i+9k,i+9k} = -1,$$

or if $\lambda_r^2 < (m\pi)^2$, then,

$$A_{i+9k,i+8k} = \theta_{12rum}(\beta_{1rm} + C_{12rum}\gamma_{1rm}).$$

4.5-5 Rotation Along the Edge $\eta = 1$

Starting with the symmetric modes, from the contributions of W_{rs} to the rotation along this edge, we obtain,

$$A_{i+8k,j+6k} = m\pi \cos(m\pi) A_{1rss},$$

$$A_{i+8k,j+7k} = m\pi \cos(m\pi) A_{2rss},$$

or if $\phi_r^2 \lambda_r^2 < (m\pi)^2$, then,

$$A_{i+8k,j+7k} = m\pi \cos(m\pi) A_{3rss}.$$

And from the contributions of W_{1rs} and W_{2rs} one obtains,

$$A_{i+8k,i+8k} = 1,$$

$$A_{i+8k,i+9k} = -\theta_{11rsm}(\beta_{1rm} + C_{11rsm}\gamma_{1rm}),$$

or if $\lambda_r^2 < (m\pi)^2$, then,

$$A_{i+8k,i+9k} = -\theta_{12rsm}(\beta_{1rm} + C_{12rsm}\gamma_{1rm}).$$

In the case of anti-symmetric modes, consideration of the relevant building blocks and their contributions leads to the following coefficients,

$$A_{i+8k,j+6k} = m\pi \cos(m\pi) A_{1rus},$$

$$A_{i+8k,j+7k} = m\pi \cos(m\pi) A_{2rus},$$

or if $\phi_r^2 \lambda_r^2 < (m\pi)^2$, then,

$$A_{i+8k, j+7k} = m\pi \cos(m\pi) A_{3rus}.$$

And

$$A_{i+8k, i+8k} = 1,$$

$$A_{i+8k, i+9k} = -\theta_{11rum}(\beta_{1rm} + C_{11rum}\gamma_{1rm}),$$

or if $\lambda_r^2 < (m\pi)^2$, then,

$$A_{i+8k, i+9k} = -\theta_{12rum}(\beta_{1rm} + C_{12rum}\gamma_{1rm}).$$

We have thus far accounted for all the boundary conditions as well as the continuity conditions across the inter-element line with the exception of the continuity of vertical edge reaction which will be considered next.

4.6 Satisfaction of the Continuity of Vertical Edge Reaction

The only change to the relevant coefficients of Figure 3.4 in order to obtain the correct coefficients of Figure 4.4 is the addition of the coefficients of E_{1rm} and E_{2rm} . In the case of symmetric modes, no change is required since the two building Blocks (b) and (c) of Figure 4.3 have no contribution to the net vertical edge reaction along the inter-segment line. Considering the anti-symmetric modes we obtain,

$$A_{i+7k, j+8k} = (A_{1n} + \frac{\nu_r}{\phi_r^2} A_{2n}) \phi_r^3 / \phi_1^3,$$

where,

$$A_{1n} = -\theta_{11rum}(m\pi)^3(B_{1rs} + C_{11rum}B_{2rs}),$$

$$A_{2n} = \theta_{11rum}(m\pi)(\beta_{1rm}^2 B_{1rs} - C_{11rum}\gamma_{1rm}^2 B_{2rs}),$$

or if $\lambda_r^2 < (m\pi)^2$, then,

$$A_{1n} = -\theta_{12rum}(m\pi)^3(B_{1rs} + C_{12rum}B_{3rs}),$$

$$A_{2n} = \theta_{12rum}(m\pi)(\beta_{1rm}^2 B_{1rs} + C_{12rum}\gamma_{1rm}^2 B_{3rs}).$$

And

$$A_{i+7k,j+9k} = (A_{1n} + \frac{\nu_r}{\phi_r^2} A_{2n})\phi_r^3/\phi_1^3,$$

where, A_{1n} and A_{2n} are obtained from the above expressions by replacing B_{1rs} , B_{2rs} and B_{3rs} by C_{1rs} , C_{2rs} and C_{3rs} respectively.

The coefficient matrix of Figure 4.4 is now completed, and eigenvalues for the free vibration problem of the fully clamped symmetrical trapezoidal plate are found by requiring the determinant of this matrix to vanish. An alternative solution to this problem using cosine function expansions has also been worked out by modifying the coefficient matrix of the alternative solution discussed in Chapter 3 for the simply supported case. However, in the interest of not being repetitive, this solution will not be discussed here except to say that it led to the exact same results as the above solution.

Chapter 5

PRESENTATION OF RESULTS

In previous chapters, the superposition techniques in solving dynamic problems were discussed. Solutions for both, the simply supported and fully clamped symmetrical trapezoidal plates were obtained. Although a procedure to obtain numerical results has been discussed, no numerical values were given. The present chapter deals with the computational aspect of the problem, providing the necessary information for the use of the specially designed computer programs which are listed in appendix A-II. From Figure 5.1, it is clear that in view of the geometrical configuration of the symmetrical trapezoid, a given trapezoid is completely defined if the angle α is given along with one of its bases b_1 or b_2 . Since there are an infinite number of possible combinations of these parameters, it is physically impossible to include here numerical results for each and every one of these plates. Instead, listings of computer programs are provided in the appendices making it possible for the reader to generate eigenvalues for any particular plate dimensions one may require. Furthermore, numerical results are provided here for a wide range of plate aspect ratios for both simply supported and fully clamped plates.

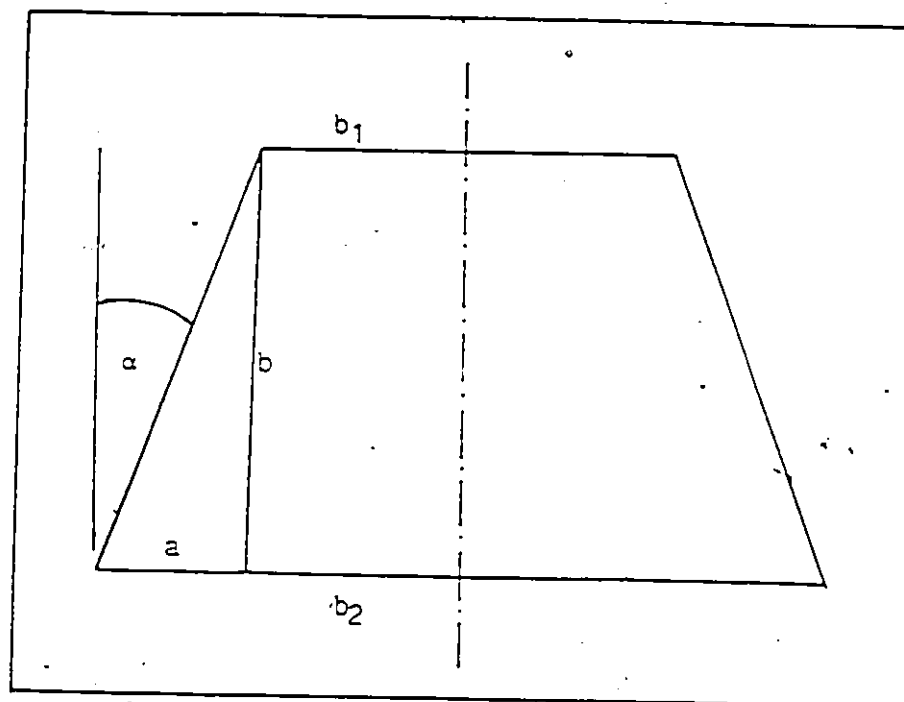


Figure 5.1- Symmetrical trapezoidal plate.

5.1 Simply Supported Plates

An analytically exact solution for the free vibration problem of simply supported symmetrical trapezoidal plates was developed in Chapter 3. Expressions in the form of trigonometric series for the various elements of the coefficient matrix were also given reducing the problem to a simple eigenvalue search. As mentioned earlier, a trapezoid is uniquely defined if α and one of the two bases b_1 or b_2 are given. The value of α may be expressed as the ratio of b to a which has been given the symbol ϕ_1 , and the value of either b_1 or b_2 may be defined by providing the value of ϕ_r which represents the ratio of b to b_1 , or in terms of Chapter 3 symbols, $\phi_r = b/a_r$ where, $a_r = b_1$ and $b_2 = a + a_r$. Once ϕ_1 and ϕ_r are defined, the only unknown in the expressions of the various elements is λ^2 . The value of λ^2 that causes the determinant of the coefficient matrix to vanish is an eigenvalue. The procedure followed in all of the computer programs listed in the appendices is to

give λ^2 a predetermined numerical value that is judged to be smaller than the eigenvalue corresponding to the first fundamental mode. The determinant of the coefficient matrix is then evaluated and recorded. The numerical value of λ^2 is then incremented by a carefully chosen increment which must be small enough so that no eigenvalue is missed, and the corresponding determinant recorded. This procedure continues until a systematically decreasing positive determinant, or an increasing negative determinant, changes sign indicating a range of twice the value of the present increment where an eigenvalue may be located. with the lower and upper limits of this range used as limits for λ^2 , and with smaller and smaller increments, this procedure is repeated until an eigenvalue is located. Values of λ^2 obtained in this manner may not necessarily represent a genuine eigenvalue that represents a real physical vibratory motion of the plate. This kind of eigenvalues, if one may call them eigenvalues, will be discussed later in the following chapter. At the present time, it suffices to keep in mind that they exist, and that care must be taken not to mistake them for genuine eigenvalues.

Once an eigenvalue is located, one of the unknowns present in the final shape function is given a numerical value of unity, and thus reducing the n homogeneous algebraic equations obtained when satisfying the various boundary conditions, into $n - 1$ non-homogeneous and therefore, solvable equations which we then solve for the remaining $n - 1$ unknowns, which are in fact the adjustable parameters of the various forcing functions used in the development of the solution. These parameters are then assigned their values in the shape function to obtain the necessary shape data which are used to plot the various mode shapes as will be seen later in this section.

Focussing our attention on Program I or (TSLS) of Appendix A-II, it is seen that by making use of explanatory comments, this program is made both, easy to use and simple to modify if necessary. The first part of the program is concerned with the identification of the various parameters. POI is used to represent Poisson's ratio ν and is assigned the value 0.3. This value may be changed to reflect the correct value of a given material. QLIM is a limit

which if reached the program will switch from the original set of equations to a simplified set in order to avoid numerical instabilities as will be explained in detail in the next chapter. KS is the number of points for which shape data are required. KS may be given any integer. The limitation here is the availability of storage memory in the computer being used. K is the number of terms used in the Fourier expansion. The value of K determines the size of the coefficient matrix. For this particular application, K may be given any integer between 7 and 16 depending on the accuracy required in the evaluation of the eigenvalue as will be seen in the following chapter. At this stage, the user must provide the numerical values of ϕ_1 (PHI1) and ϕ_r (PHIR) that will uniquely define the dimensions of the plate under consideration. He must also supply numerical values for ALMDS which is the starting value or lower limit of λ^2 , and for DLIM and DEL. Where, DLIM is the value of λ^2 at which the computation must stop and DEL is the increment by which the value of λ^2 is incremented at the end of each iteration. If DEL is given a numerical value of zero, this will trigger the program to switch from the eigenvalue search to the solution and generation of mode shape data. The numerical value assigned to λ^2 is taken as being an exact eigenvalue here, and not as a trial value. This program is capable of handling both, the symmetric and anti-symmetric vibration modes. MODE must be assigned a numerical value of zero if symmetric modes are under study. Any other positive numerical value would signal the program to switch to anti-symmetric modes. A second program based on the cosine alternative solution was also written. However, this program is not listed here since both programs gave identical results. The program is now ready for execution, and the results obtained are as shown in the following tables and figures. Starting with Figure 5.2, it is seen that the information stored are clear and self contained. Symbols are as illustrated in the small diagram shown in the upper left corner of the figure. Although there is a considerable number of eigenvalues stored in these curves, this is not intended to be the purpose of this or any other figure. The primary objective here is to show the variation of the eigenvalue λ^2 with the change of the plate aspect ratios. It is seen how the eigenvalue λ^2 , and consequently the circular

frequency ω increases as the ratio b/b_2 increases. However, this increase in eigenvalue with the increase of b/b_2 is more pronounced for higher values of α . For better accuracy, these eigenvalues are also shown in tabulated form in table 5.1. Figure 5.3 shows the curve of λ^2 as a function of b/b_2 for $\alpha = 45^\circ$. Figure 5.4 shows the eigenvalue λ^2 as a function of the ratio b/b_2 for $\alpha = 15^\circ$. The first three symmetric modes are shown in Figure 5.4-(a), while Figure 5.4-(b) shows the first three anti-symmetric modes. Corresponding values are also given in tabulated form for better accuracy in Table 5.2. Comparison of results was also made in Table 5.3. More eigenvalues for a wide range of aspect ratios ϕ_1 and ϕ_r are provided in Table 5.4. Some examples of computer generated mode shapes for a plate with $\phi_1 = \phi_r = 2.0$ are shown in Figure 5.5.

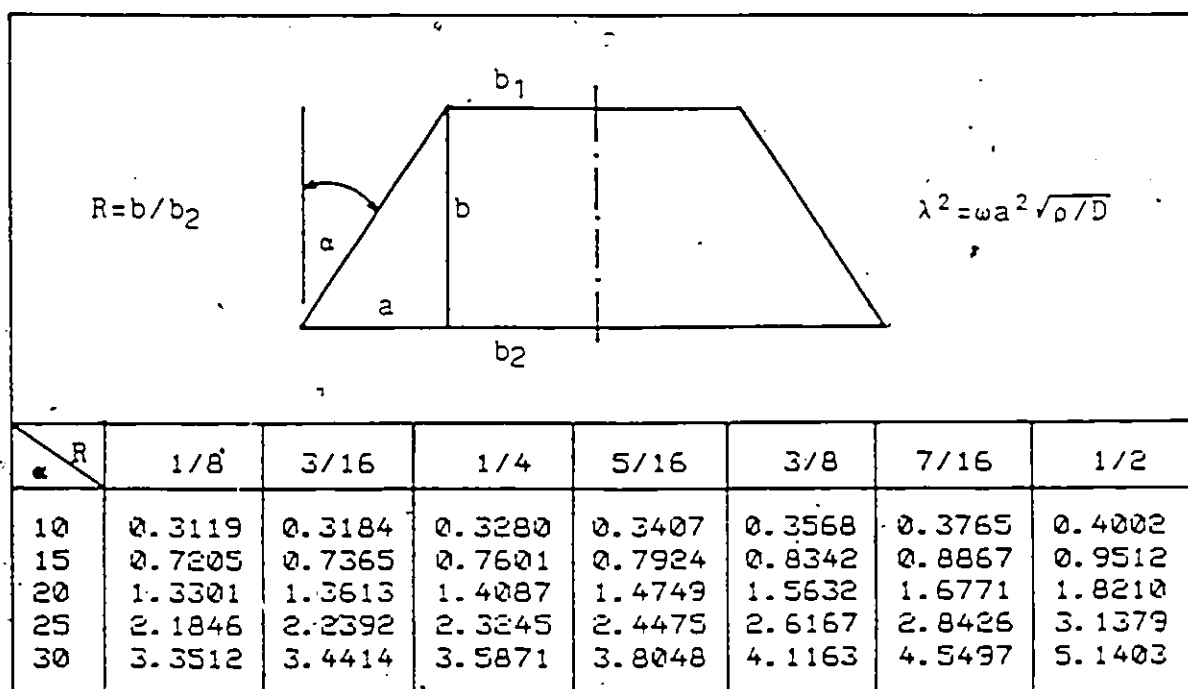


Table 5.1- First free vibration eigenvalues of simply supported symmetrical trapezoidal plates of various combinations of plate aspect ratio and α as indicated.

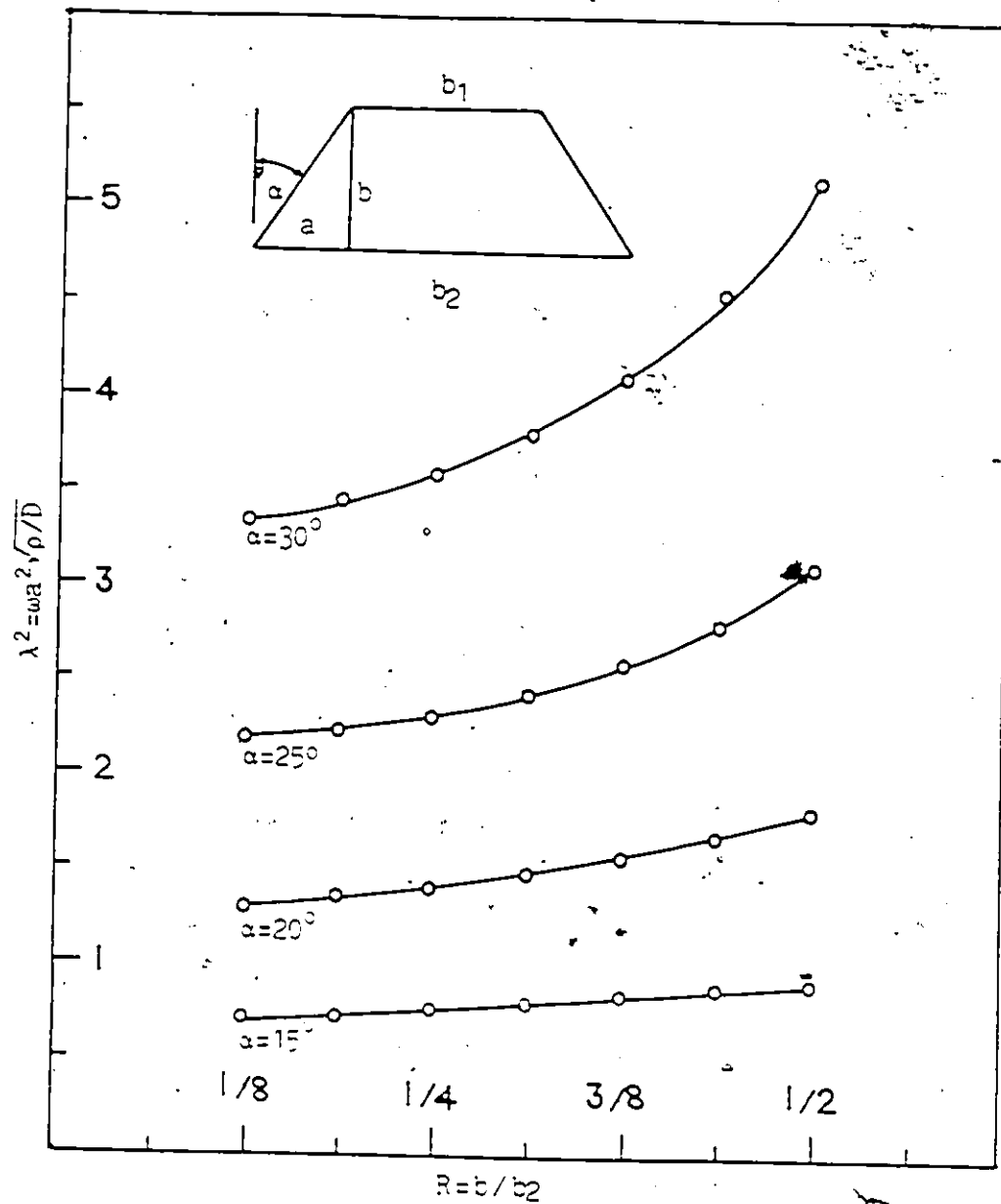


Figure 5.2- First mode free vibration eigenvalues of a simply supported symmetrical trapezoidal plate as a function of plate aspect ratio for different values of α as indicated.

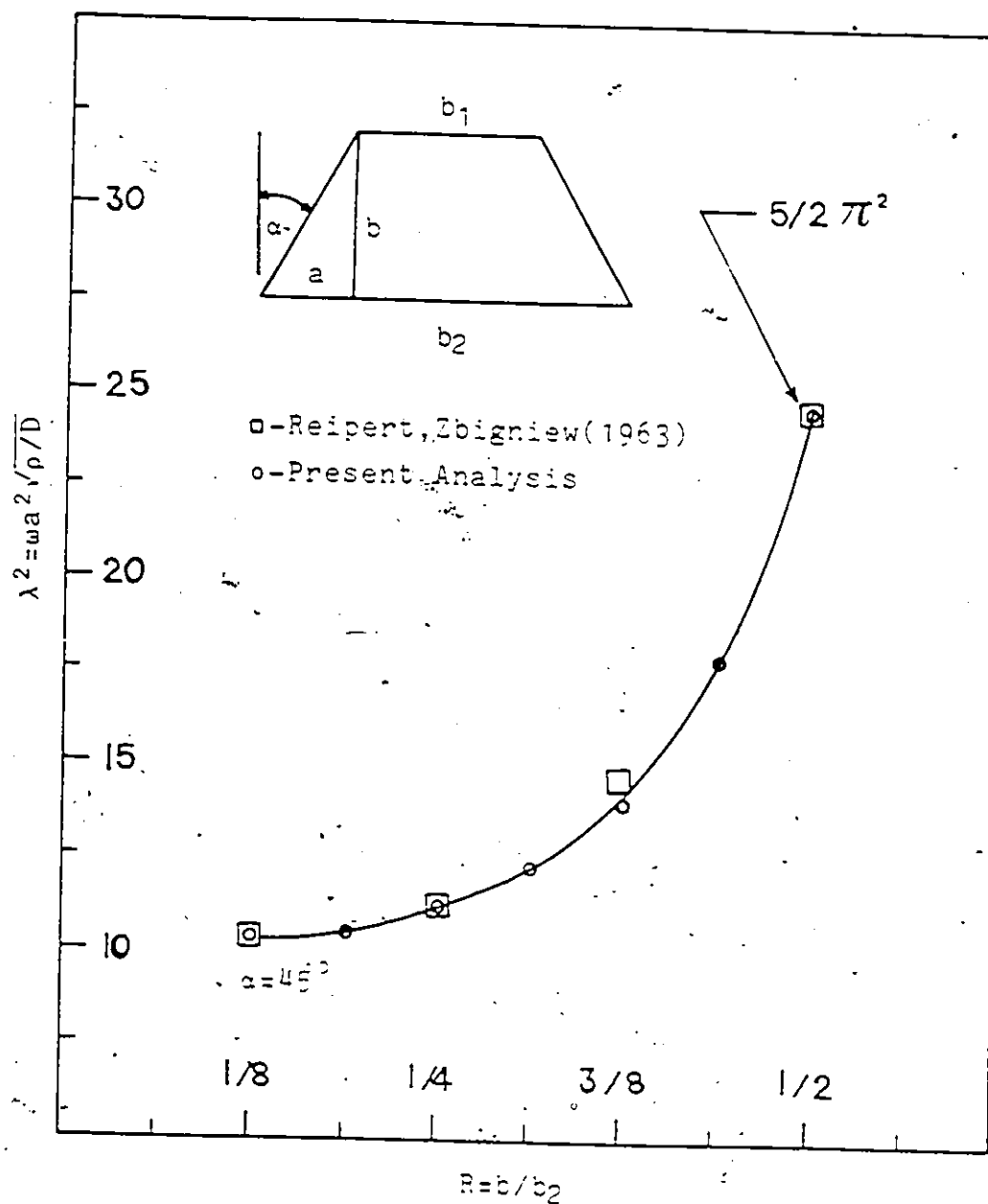


Figure 5.3- First mode free vibration eigenvalues of a simply supported symmetrical trapezoidal plate as a function of plate aspect ratio for $\alpha = 45^\circ$.

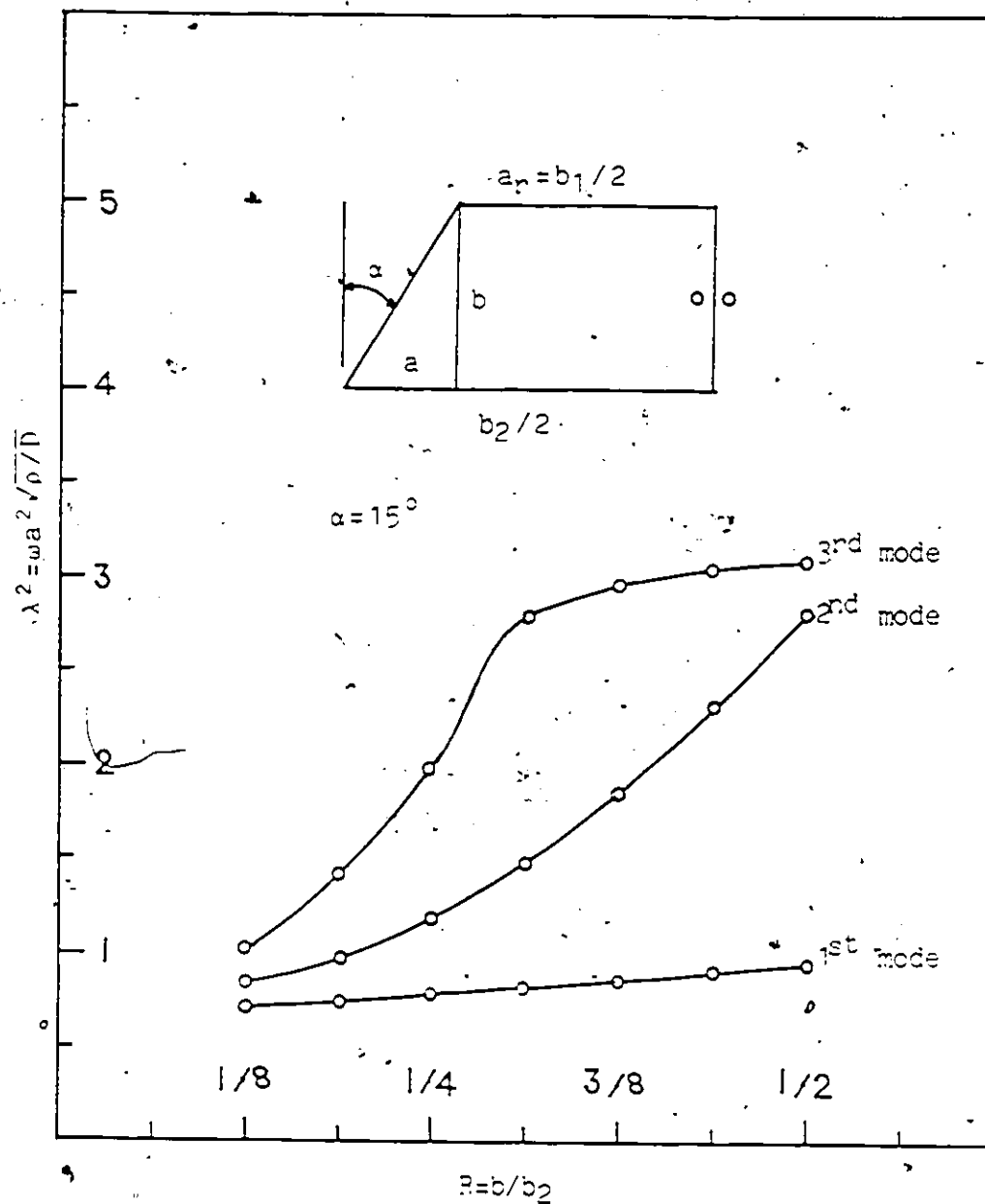


Figure 5.4-(a) First three symmetric mode free vibration eigenvalues of a simply supported symmetrical trapezoidal plate as a function of plate aspect ratio for $\alpha = 15^\circ$.

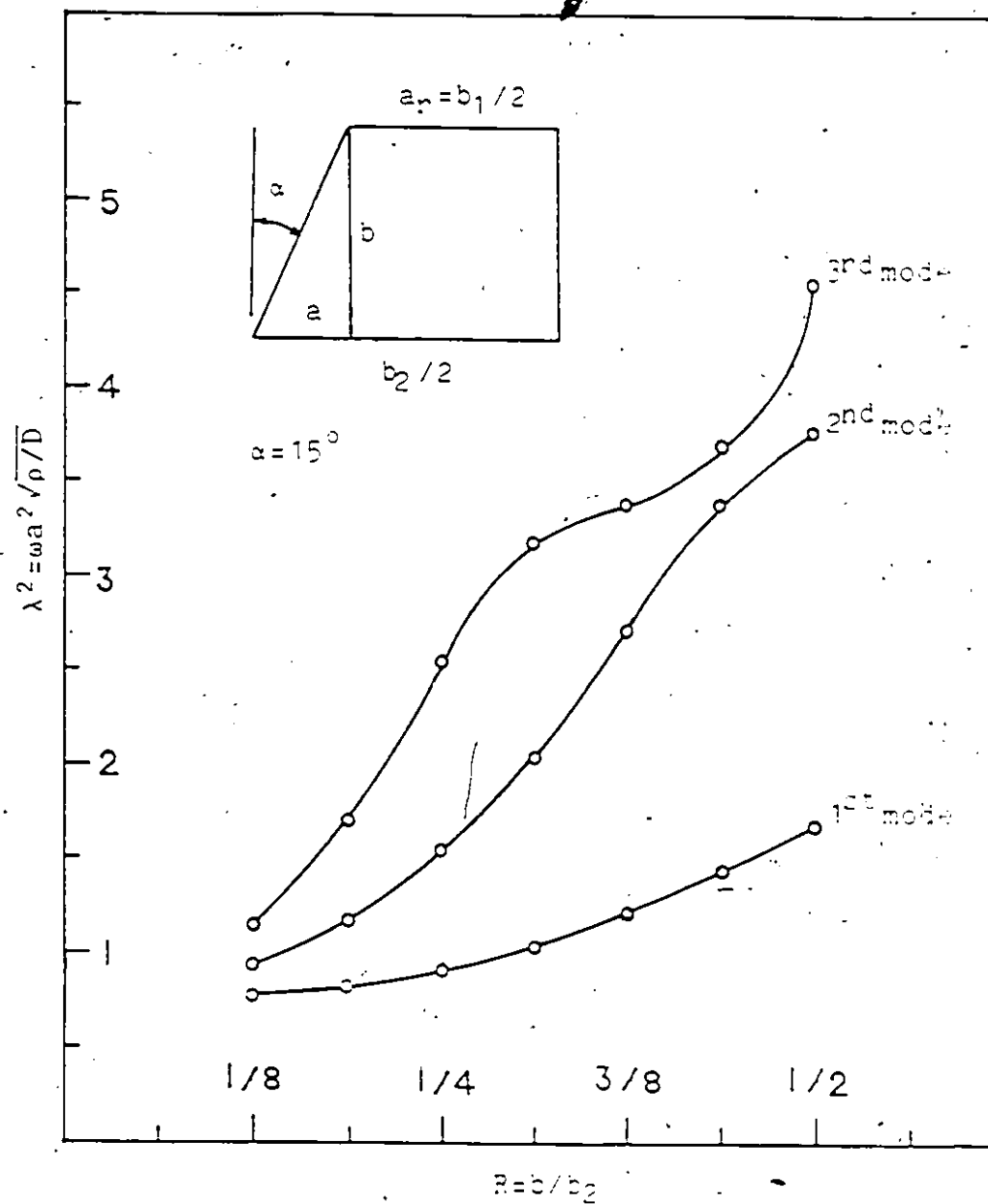


Figure 5.4-(b) First three anti-symmetric mode free vibration eigenvalues of a simply supported symmetrical trapezoidal plate as a function of plate aspect ratio for $\alpha = 15^\circ$.

$R = b/b_2$ $\alpha = 15^\circ$ $\lambda^2 = \omega a^2 \sqrt{\rho/D}$							
mode \ R	1/8	3/16	1/4	5/16	3/8	7/16	1/2
1	0.7205	0.7365	0.7601	0.7924	0.8342	0.8867	0.9512
2	0.8159	0.9594	1.1719	1.4605	1.8328	2.2944	2.8344
3	1.0067	1.4046	1.9912	2.7742	2.9603	3.0181	3.1078
Symmetric + Anti-symmetric							
1	0.7563	0.8201	0.9147	1.0434	1.2102	1.4189	1.6737
2	0.8994	1.1543	1.5311	2.0411	2.6903	3.4008	3.7711
3	1.1377	1.7099	2.5490	3.1672	3.3500	3.6703	4.5410

Table 5.2- First three symmetric and anti-symmetric mode free vibration eigenvalues of a simply supported symmetrical trapezoidal plate with $\alpha = 15^\circ$ and aspect ratio as indicated.

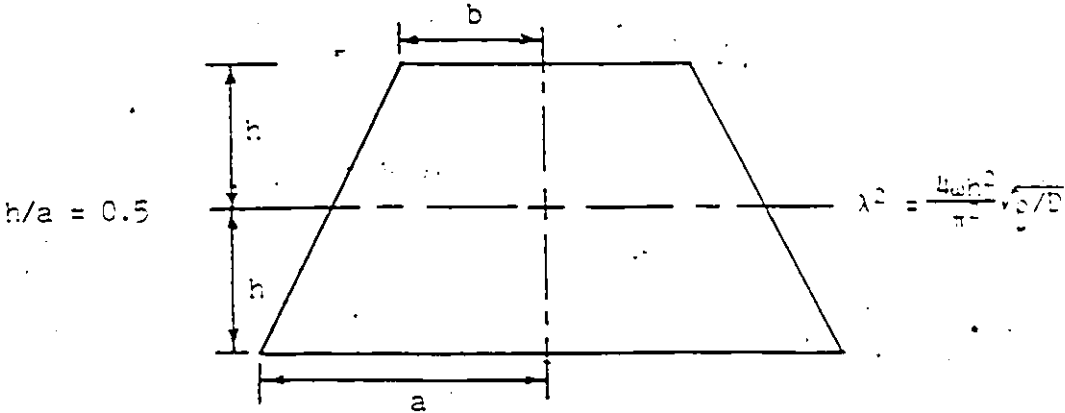
						
b/a mode	1.0	0.8	0.6	0.4	0.2	0.0
1	1.25 (1.25)	1.3131 (1.3135)	1.4154 (1.4177)	1.5887 (1.6003)	1.9435 (1.9378)	2.500 (2.502)
2	3.25 (3.25)	3.7880 (3.790)	4.2892 (4.295)	4.8268 (4.544)	5.0499 (5.091)	6.500 (6.515)
3	4.25 (4.25)	4.3272 (4.328)	4.7421 (4.744)	5.7749 (5.786)	7.0133 (7.027)	8.500 (8.53)
4	6.25 (6.25)	6.7885 (6.80)	7.6789 (7.70)	8.9350 (8.99)	9.9093 (10.00)	12.50 (12.60)

Table 5.3- First four symmetric mode free vibration eigenvalues of a simply supported trapezoidal plate as shown above. Parenthesis indicate eigenvalues as reported by Chopra and Durvasula(1971).

$\phi_1 = b/a$ $\phi_r = b/a_r$ $\lambda^2 = \omega a^2 \sqrt{\rho/D}$					
$\phi_1 \backslash \phi_r$	mode	0.8	1.2	1.6	2.0
0.4	1	15.854	7.068	3.982	2.551
	2	19.477	8.778	4.995	3.221
	3	26.551	12.257	7.005	4.556
0.8	1	16.940	7.541	4.279	2.756
	2	27.496	13.085	7.607	5.041
	3	46.573	23.133	14.000	9.416
1.2	1	17.911	8.122	4.664	3.031
	2	35.633	17.963	10.960	7.488
	3	61.680	28.775	16.288	10.474
1.6	1	18.937	8.728	5.088	3.344
	2	43.028	22.478	14.132	9.765
	3	66.097	29.926	17.010	11.064
2.0	1	19.910	9.315	5.424	3.675
	2	48.651	26.051	16.233	10.855
	3	68.076	31.495	18.748	12.870

Table 5.4-(a) First three symmetric mode free vibration eigenvalues of the simply supported trapezoidal plate shown above.

$\phi_1 = b/a$ $\phi_r = b/a_r$ $\lambda^2 = \omega a^2 \sqrt{\rho/D}$					
$\phi_1 \rightarrow$ $\phi_r \leftarrow$	mode	0.8	1.2	1.6	2.0
0.4	1	17.279	7.711	4.362	2.803
	2	22.588	10.235	5.878	3.806
	3	31.302	14.552	8.290	5.468
0.8	1	20.887	9.577	5.546	3.618
	2	36.008	17.651	10.538	6.923
	3	57.215	28.739	16.914	10.985
1.2	1	24.947	12.071	7.046	4.701
	2	48.972	25.401	15.724	10.777
	3	72.689	33.667	19.378	12.568
1.6	1	28.800	14.372	8.913	5.901
	2	58.170	30.989	19.299	12.990
	3	81.772	38.662	23.380	16.016
2.0	1	32.184	16.594	10.394	7.351
	2	64.370	34.980	21.666	14.639
	3	88.931	44.130	28.053	19.956

Table 5.4-(b) First three anti-symmetric mode free vibration eigenvalues of the simply supported trapezoidal plate shown above.

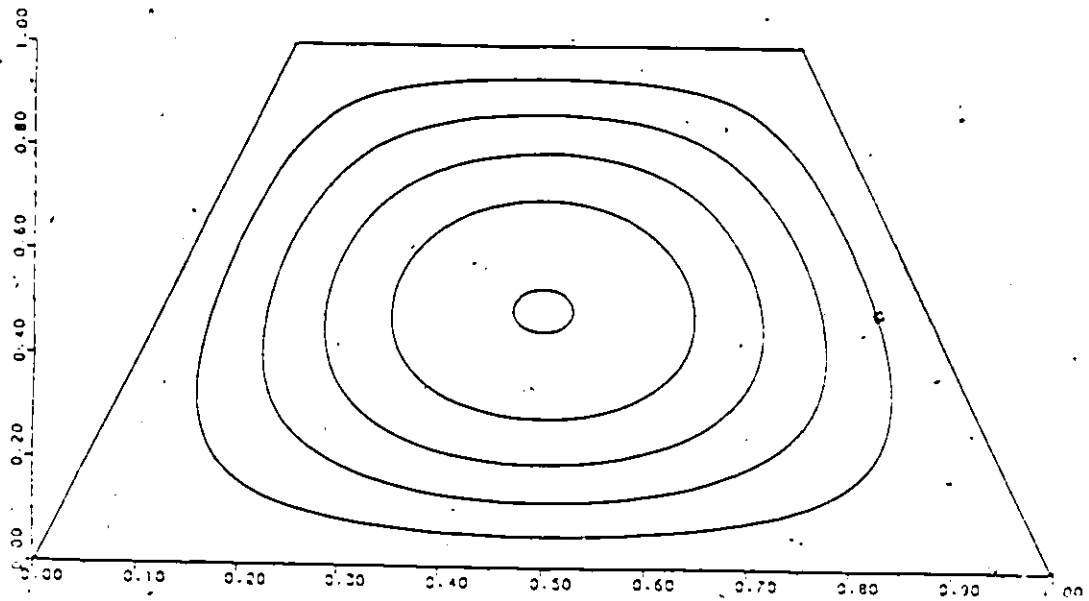


Figure 5.5-(a) First symmetric mode shape for the simply supported symmetrical trapezoidal plate. $\phi_1 = 2.0$, $\phi_r = 2.0$. - - -, nodal line; ---, downward deflection; —, upward deflection.

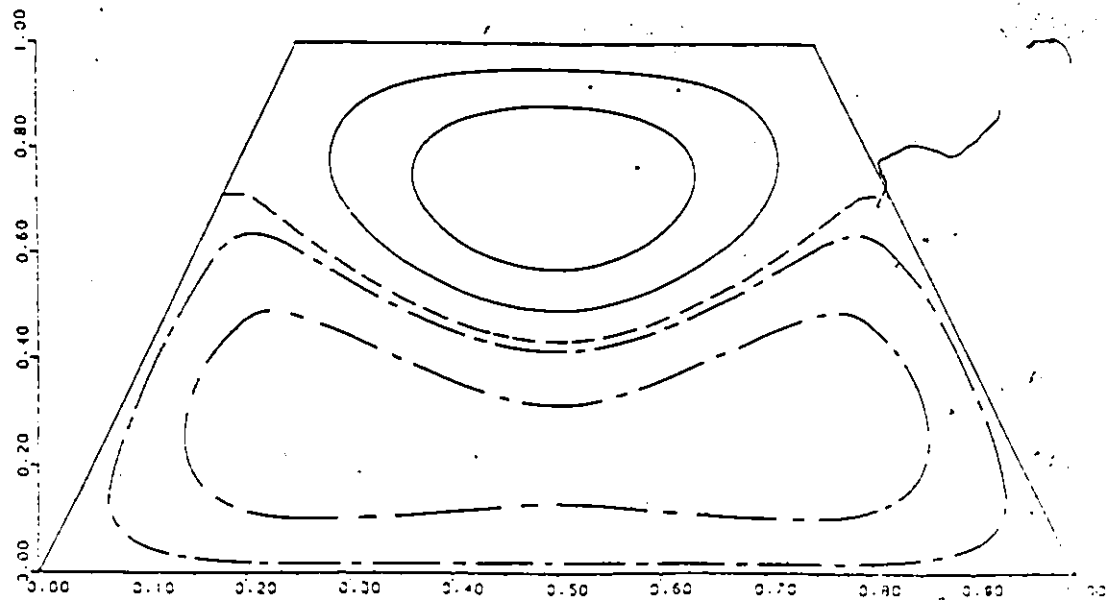


Figure 5.5-(b). Second symmetric mode shape for the simply supported symmetrical trapezoidal plate. $\phi_1 = 2.0$, $\phi_r = 2.0$. - - -, nodal line; - - -, downward deflection; —, upward deflection.

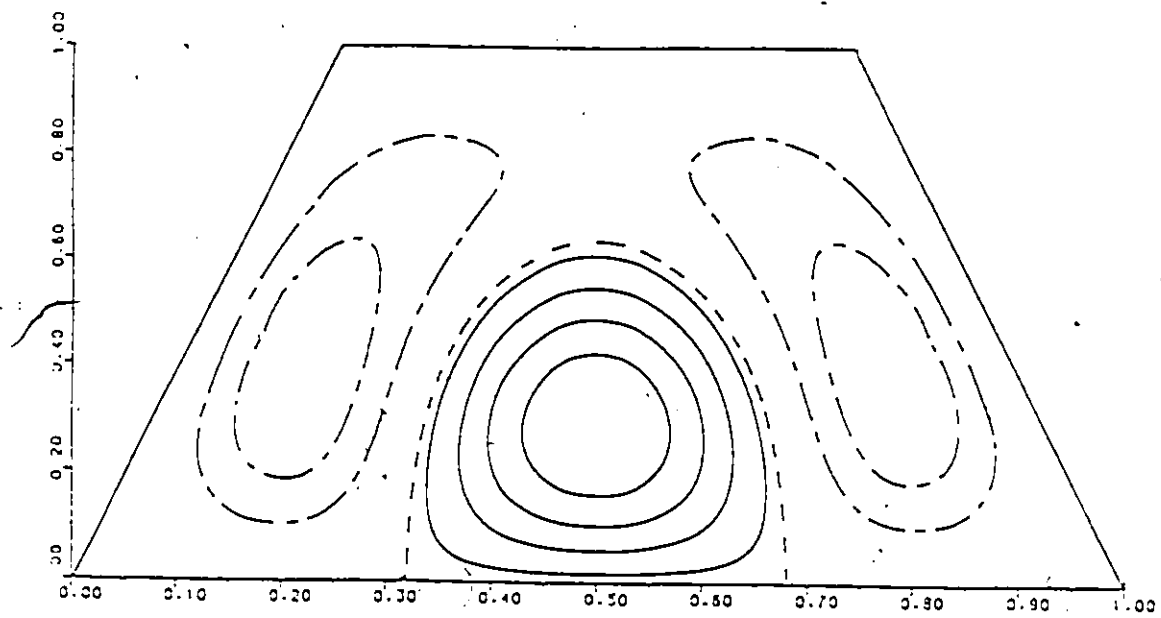


Figure 5.5-(c) Third symmetric mode shape for the simply supported symmetrical trapezoidal plate. $\phi_1 = 2.0$, $\phi_r = 2.0$. - - -, nodal line; -.-., downward deflection; —, upward deflection.

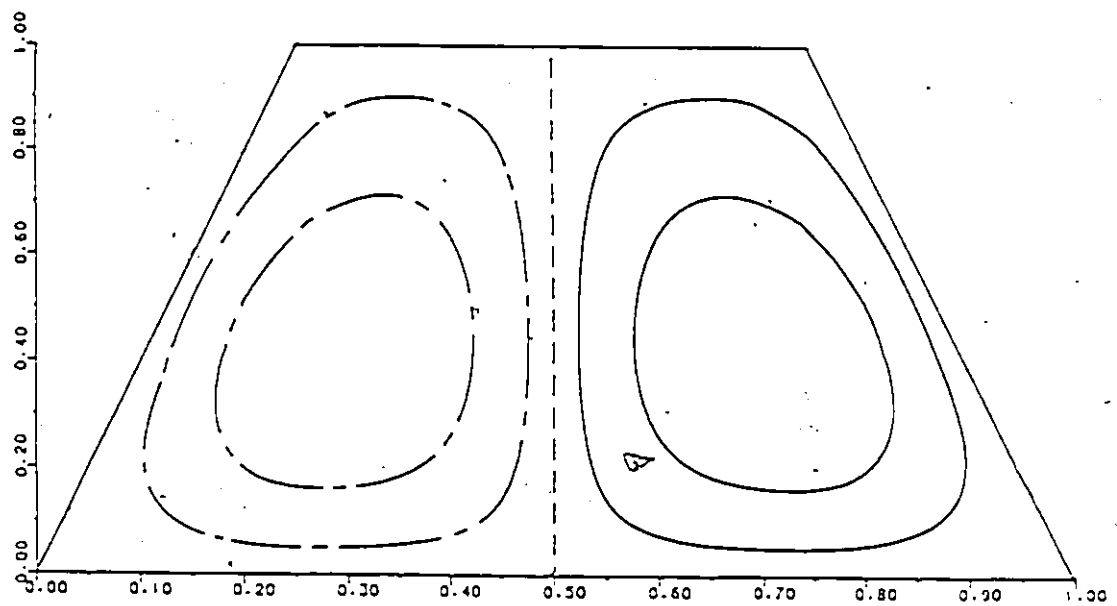


Figure 5.5-(d) First anti-symmetric mode shape for the simply supported symmetrical trapezoidal plate. $\phi_1 = 2.0$, $\phi_r = 2.0$. - - -, nodal line; - - -, downward deflection; —, upward deflection.

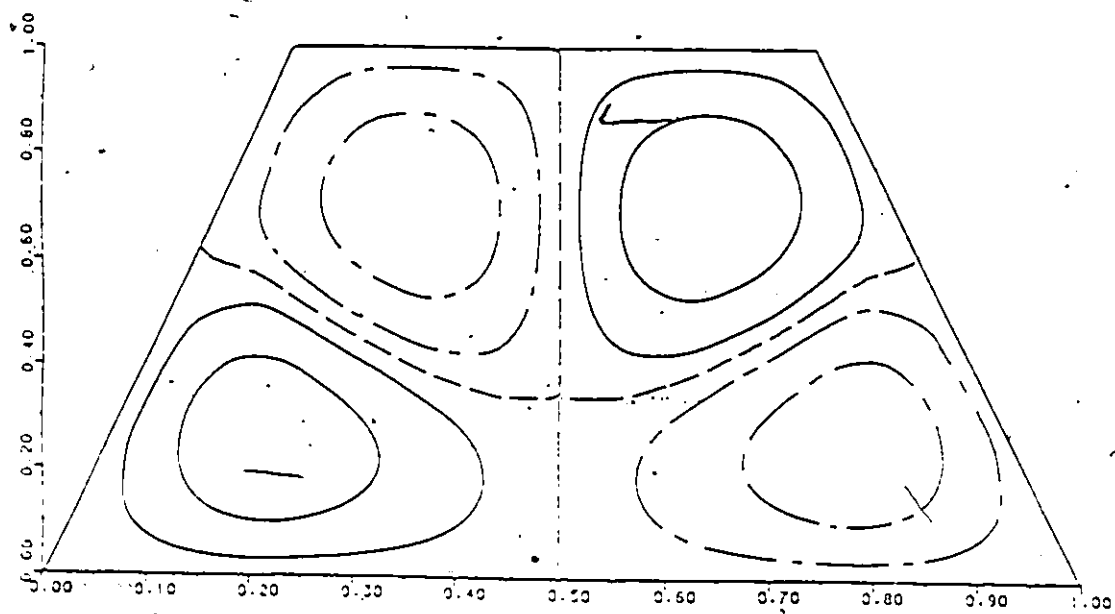


Figure 5.5-(e) Second anti-symmetric mode shape for the simply supported symmetrical trapezoidal plate. $\phi_1 = 2.0$, $\phi_r = 2.0$. - - -, nodal line; ---, downward deflection; —, upward deflection.

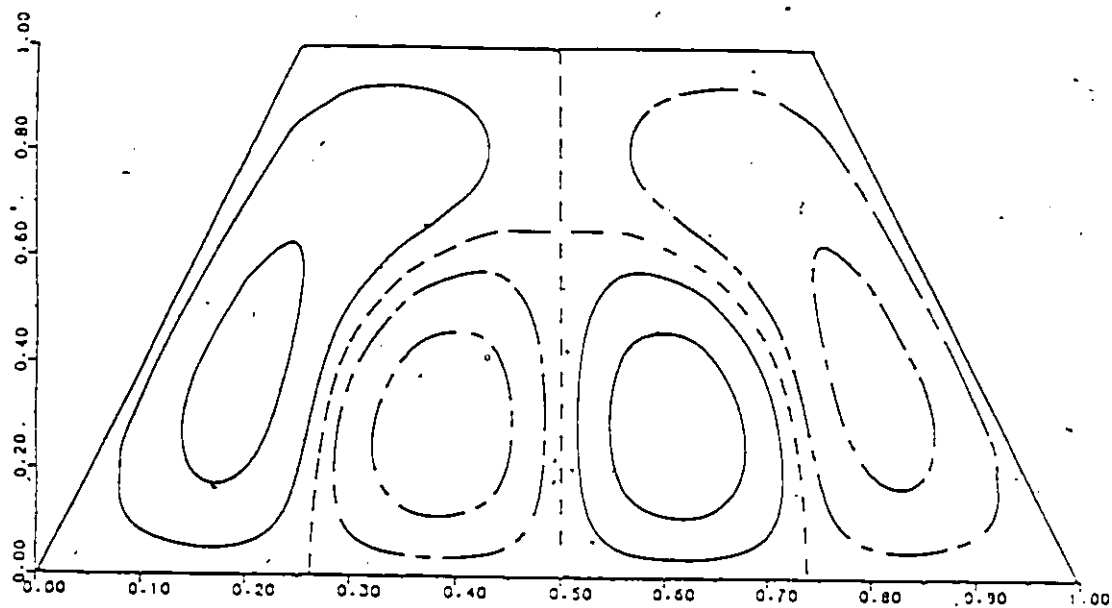


Figure 5.5-(f) Third anti-symmetric mode shape for the simply supported symmetrical trapezoidal plate. $\phi_1 = 2.0$, $\phi_r = 2.0$. - - -, nodal line; ---, downward deflection; —, upward deflection.

5.2 Fully Clamped Plates

The solution of the fully clamped symmetrical trapezoidal plate was discussed in full depth in the previous chapter. Similar to the simply supported case, no numerical results were given. It is in this section that the use of Program II or (TSLC) of Appendix A-II is to be explained. This program is in principal similar to the previous program (TSLS). In fact, Program TSLC is obtained from Program TSLS simply by implementing the proper changes in the formulation of the problem as was discussed in Chapter 4. From the user point of view, the two programs are similar in every aspect. The same parameters have to be specified, and the same computational procedure is followed. Program TSLC is also capable of switching from the eigenvalue search mode to shape data generation mode if DEL is assigned a zero value. Assigning zero value to MODE will instruct the computer to generate data relevant to symmetric vibration modes, while any other positive value would switch the program to generate data relevant to anti-symmetric modes. Symmetric and anti-symmetric data were generated and stored both graphically and in tabulated form. Figure 5.6 shows first mode eigenvalues as a function of plate aspect ratio for α equals to 15, 20, 25 and 30 degrees. Figure 5.7 shows the same for the first three symmetric and anti-symmetric modes for $\alpha = 15^\circ$. In plotting these graphs, data from Tables 5.5 and 5.6 were used. Comparison of results with experimental eigenvalues was possible here as shown in Table 5.7. More eigenvalues for a wide range of plate aspect ratios are also provided in Table 5.8 for both symmetric and anti-symmetric modes. The presentation of results is concluded by Figure 5.8 where an example of computer generated mode shapes for a clamped symmetrical trapezoidal plate with aspect ratios $\phi_1 = \phi_r = 2.0$ is shown for both, symmetric and anti-symmetric modes.

The next and final chapter will be devoted to the discussion of results particularly where comparison is made with previously available data. The assumptions listed at the end of Chapter 1 will also be discussed in some detail in order to study their effects on the final results.

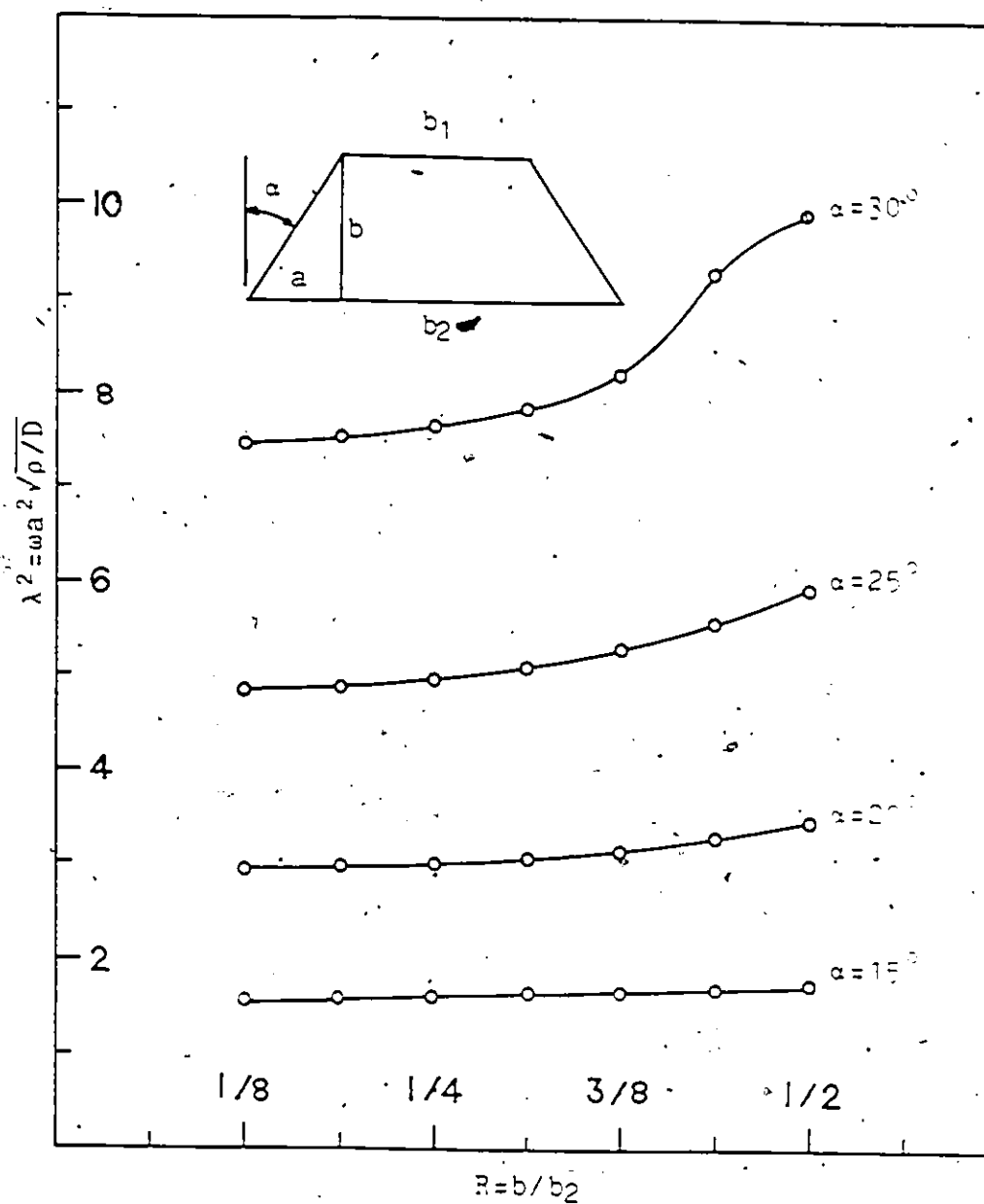


Figure 5.6- First mode free vibration eigenvalues, for fully clamped symmetrical trapezoidal plates, as a function of plate aspect ratio, for different values of α .

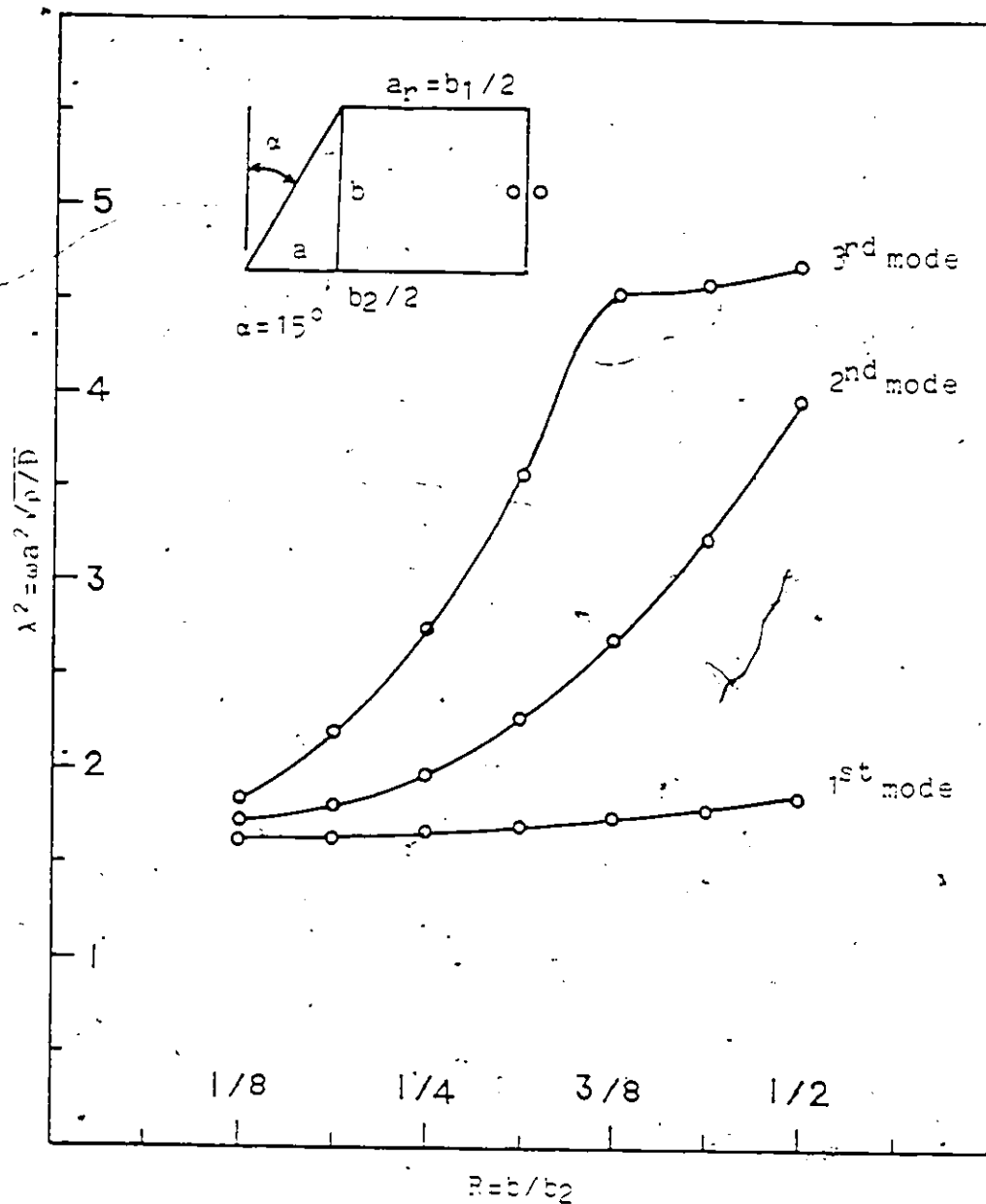


Figure 5.7-(a) First three symmetric mode free vibration eigenvalues, for fully clamped symmetrical trapezoidal plates, as a function of plate aspect ratio, for $\alpha = 15^\circ$.

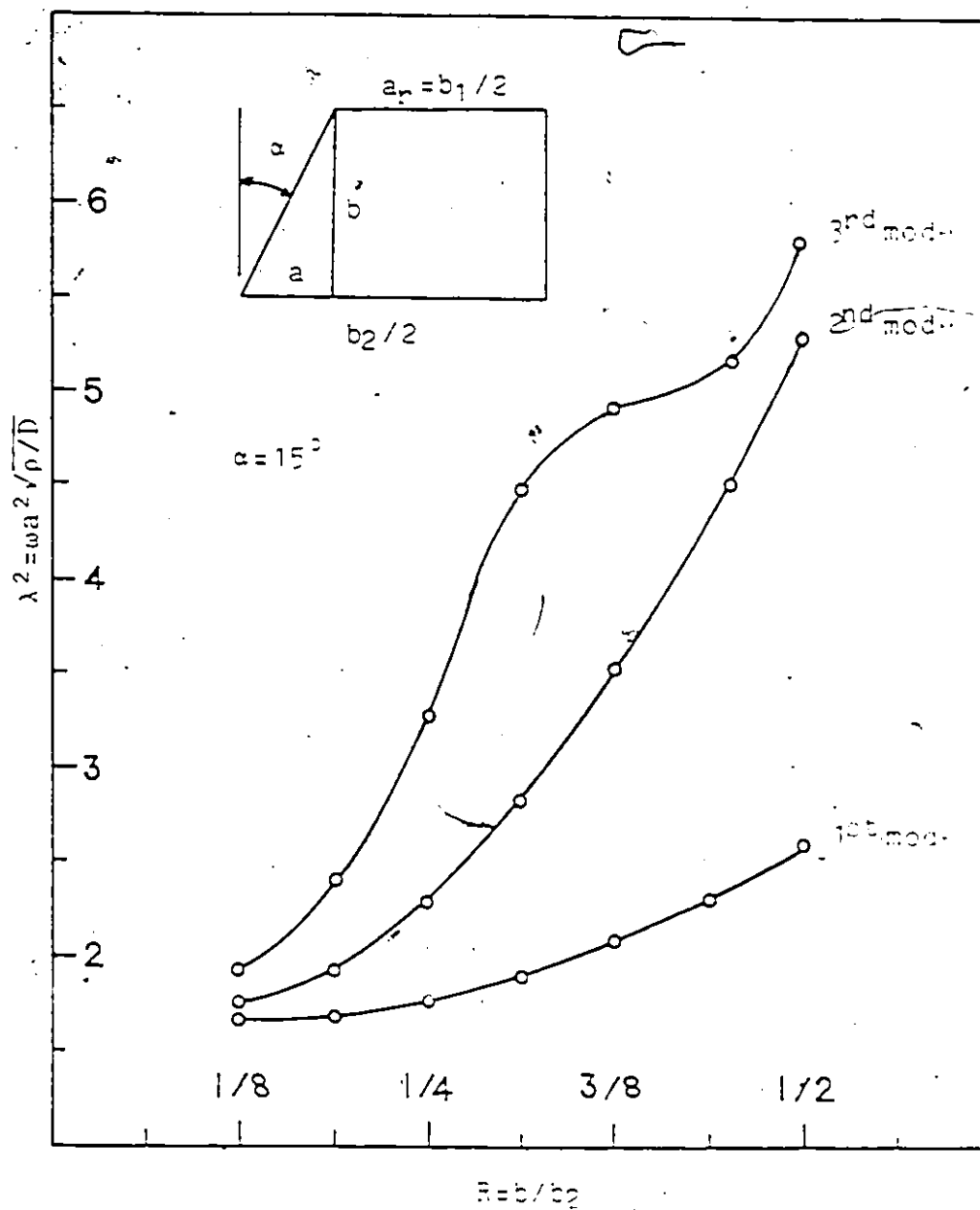
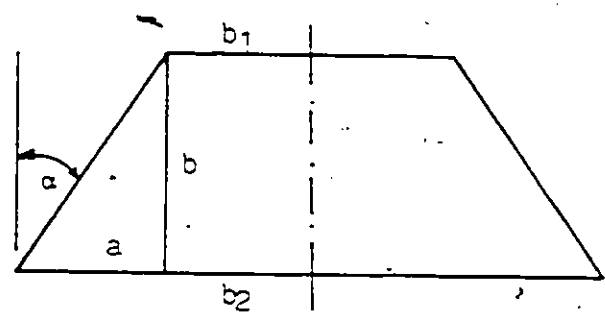


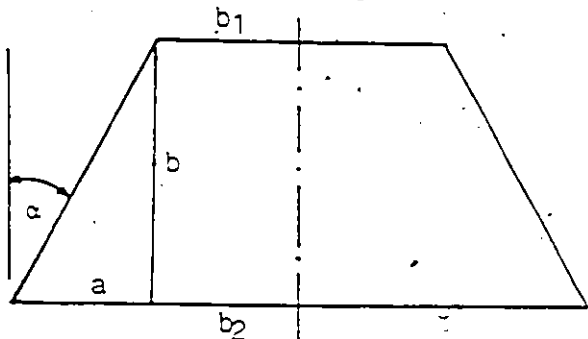
Figure 5.7-(b) First three anti-symmetric mode free vibration eigenvalues, for fully clamped symmetrical trapezoidal plates, as a function of plate aspect ratio, for $\alpha = 15^\circ$.



$\alpha \backslash R$	1/8	3/16	1/4	5/16	3/8	7/16	1/2
10	0.6987	0.7032	0.7104	0.7210	0.7360	0.7565	0.7838
15	1.6137	1.6246	1.6427	1.6702	1.7102	1.7668	1.8447
20	2.9779	2.9995	3.0362	3.0940	3.1815	3.3101	3.4947
25	4.8887	4.9270	4.9944	5.1052	5.2809	5.5523	5.9616
30	7.4959	7.5601	7.6782	7.8830	8.2265	8.7921	9.6965

Table 5.5- First free vibration eigenvalues of fully clamped symmetrical trapezoidal plates of various combinations of plate aspect ratio and α as indicated.

$R = b/b_2$
 $\alpha = 15^\circ$



$\lambda^2 = \omega a^2 \sqrt{\rho/D}$

$\begin{matrix} R \rightarrow \\ \text{mode} \end{matrix}$	1/8	3/16	1/4	5/16	3/8	7/16	1/2
1	1.6137	1.6246	1.6427	1.6702	1.7102	1.7668	1.8447
2	1.6744	1.7824	1.9677	2.2556	2.6681	3.2213	3.9920
3	1.8061	2.1415	2.7179	3.5750	4.5337	4.5975	4.6778
Symmetric + + Anti-symmetric							
1	1.6361	1.6816	1.7582	1.8770	2.0507	2.2923	2.6136
2	1.7307	1.9338	2.2853	2.8222	3.5670	4.5174	5.3204
3	1.9039	2.4103	3.2665	4.4928	4.8917	5.1453	5.6434

Table 5.6- First three symmetric and anti-symmetric mode free vibration eigenvalues of fully clamped symmetrical trapezoidal plates with $\alpha = 15^\circ$, and aspect ratio as indicated.

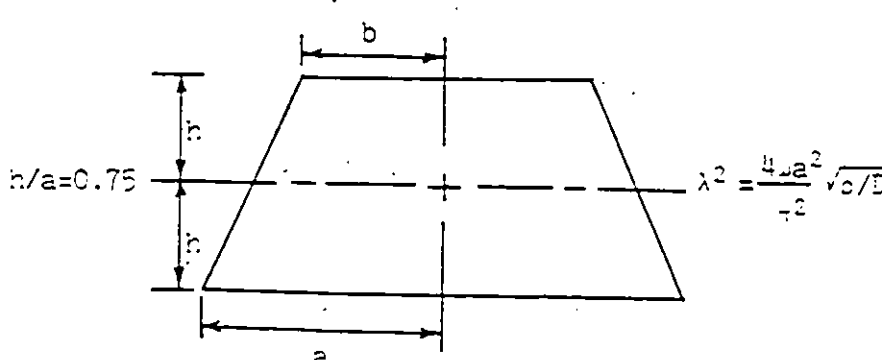
				
$\begin{matrix} b/a \\ m+n \end{matrix}$	0.2	0.4	0.6	0.8
1 1	3.048 (3.03)	2.720 (2.65)	2.497 (2.46)	2.358 (2.32)
1 2	4.184 (4.22)	3.782 (3.78)	3.613 (3.61)	3.533 (3.53)
3 1	5.276 (5.367)	4.927 (5.039)	4.565 (4.591)	4.153 (4.22)
1 3	5.466 (5.561)	5.032 (5.060)	4.850 (4.883)	4.790 (4.810)
Symmetric + Anti-symmetric				
2 1	4.251 (4.29)	3.857 (3.88)	3.482 (3.50)	3.172 (3.18)
2 2	5.448 (5.513)	4.826 (4.860)	4.357 (4.372)	4.084 (4.10)
2 3	6.464 (6.585)	5.788 (5.831)	5.388 (5.405)	5.204 (5.237)

Table 5.7- Comparison of experimental eigenvalues of clamped symmetrical trapezoidal plates with the analytical ones. Parenthesis indicate experimental eigenvalues as reported by Maruyama, Ichinomiya and Narita[17](1983).

$\phi_l = b/a$ $\phi_r = b/a_r$ $\lambda^2 = \omega a^2 \sqrt{\rho/D}$					
$\phi_l \rightarrow$ $\phi_r \downarrow$	mode	0.8	1.2	1.6	2.0
0.4	1	35.257	15.679	8.823	5.648
	2	37.798	16.882	9.530	6.115
	3	43.416	19.578	11.123	7.170
0.8	1	36.021	16.073	9.067	5.816
	2	45.487	20.892	12.036	7.843
	3	65.333	31.327	18.551	12.322
1.2	1	37.157	16.675	9.462	6.058
	2	55.569	26.690	15.887	10.617
	3	86.862	43.403	24.831	15.956
1.6	1	38.393	17.431	9.984	6.488
	2	65.363	32.817	20.195	13.229
	3	99.758	48.093	25.291	18.461

Table 5.8-(a) First three symmetric mode free vibration eigenvalues of the fully clamped trapezoidal plate shown above.

$\phi_1 = b/a$ $\phi_r = b/a_r$ $\lambda^2 = \omega a^2 \sqrt{\rho/D}$					
$\phi_r \backslash \phi_1$	mode	0.8	1.2	1.6	2.0
0.4	1	36.171	16.115	9.079	5.817
	2	40.167	18.020	10.202	6.561
	3	47.532	21.573	12.307	7.957
0.8	1	39.427	17.790	10.121	6.534
	2	54.243	25.456	14.864	9.783
	3	77.862	38.127	22.860	15.300
1.2	1	43.952	20.341	11.804	7.741
	2	70.625	35.125	21.389	14.515
	3	102.827	48.229	27.522	17.791
1.6	1	48.952	23.368	13.907	9.307
	2	83.971	43.509	26.983	18.289
	3	115.420	67.464	31.300	21.013

Table 5.8-(b) First three anti-symmetric mode free vibration eigenvalues of the fully clamped trapezoidal plate shown above.

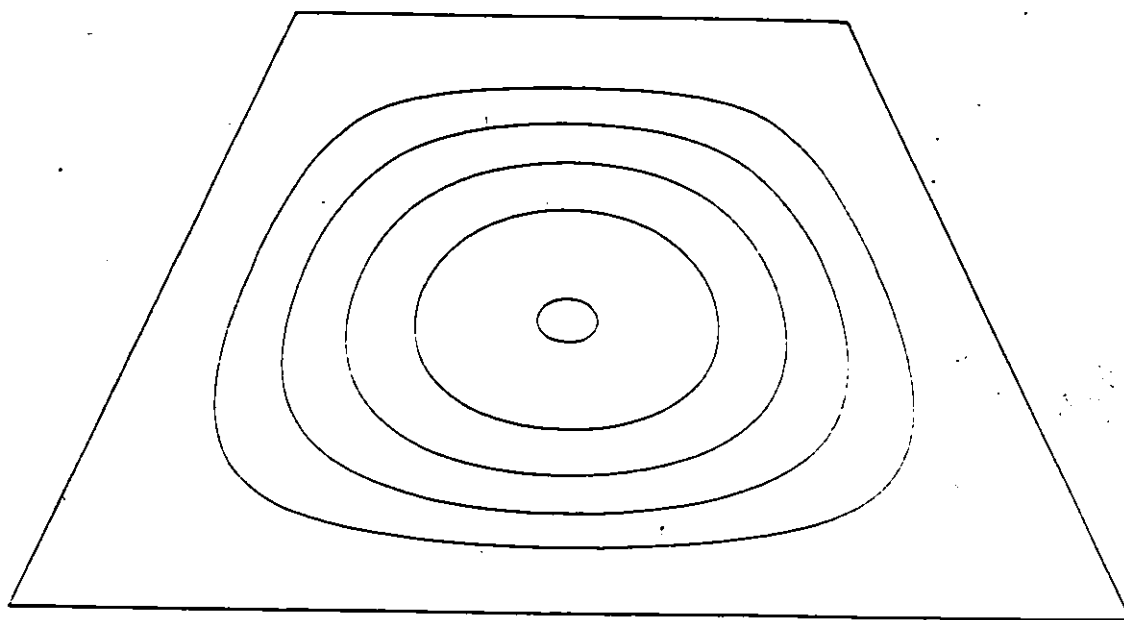


Figure 5.8-(a) First symmetric mode shape for the fully clamped symmetrical trapezoidal plate. $\phi_1 = 2.0$, $\phi_r = 2.0$. - - -, nodal line; - · -, downward deflection; —, upward deflection.

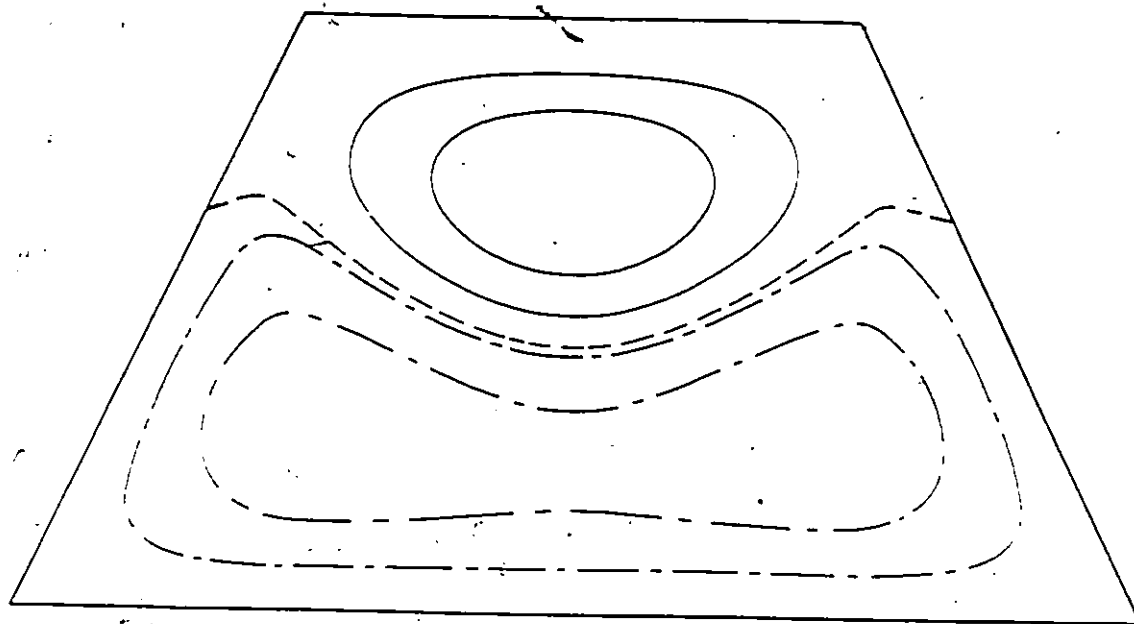


Figure 5.8-(b) Second symmetric mode shape for the fully clamped symmetrical trapezoidal plate. $\phi_1 = 2.0$, $\phi_r = 2.0$. - - -, nodal line; -.-, downward deflection; —, upward deflection.

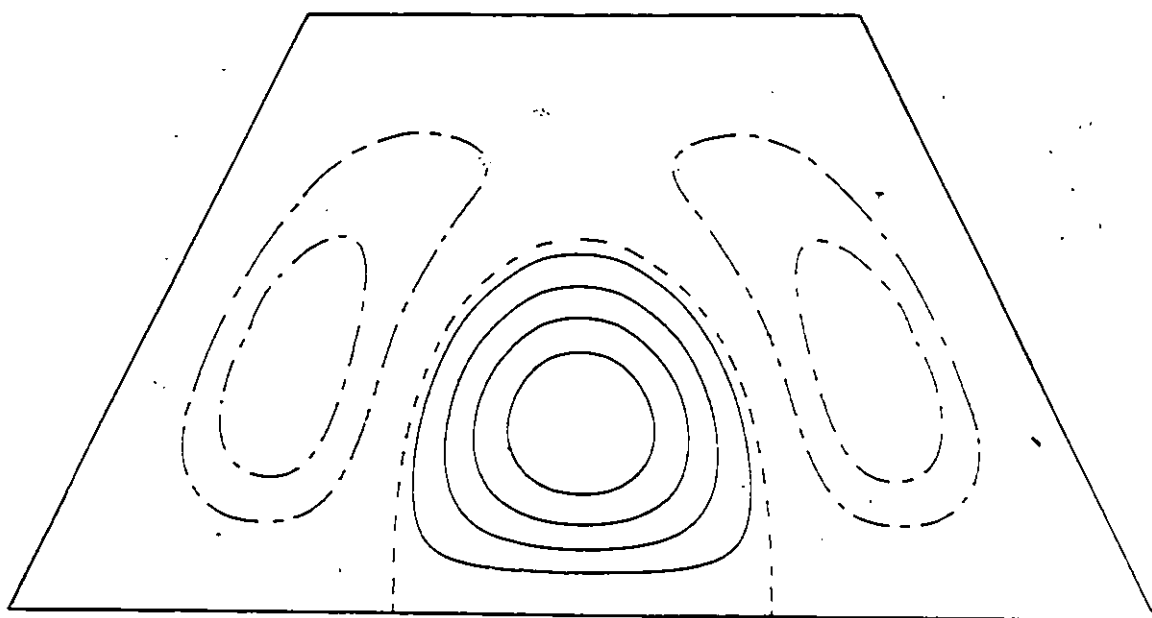


Figure 5.8-(c) Third symmetric mode shape for the fully clamped symmetrical trapezoidal plate. $\phi_1 = 2.0$, $\phi_r = 2.0$. - - -, nodal line; - · -, downward deflection; —, upward deflection.

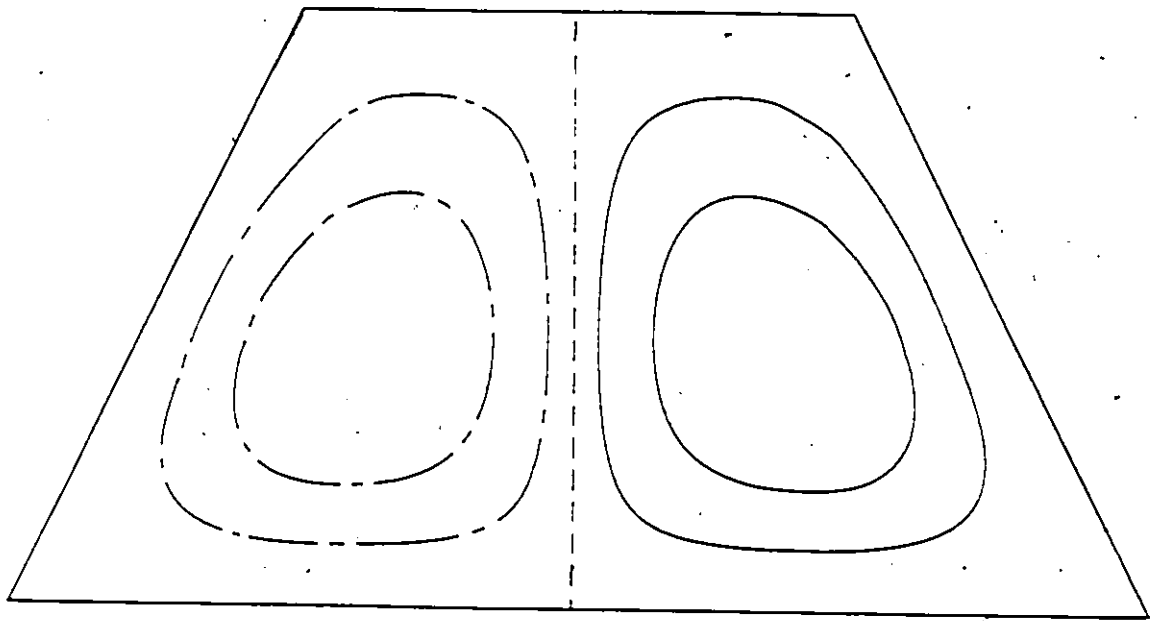


Figure 5.8-(d) First anti-symmetric mode shape for the fully clamped symmetrical trapezoidal plate. $\phi_1 = 2.0$, $\phi_r = 2.0$. - - -, nodal line: - . - , downward deflection; — , upward deflection.

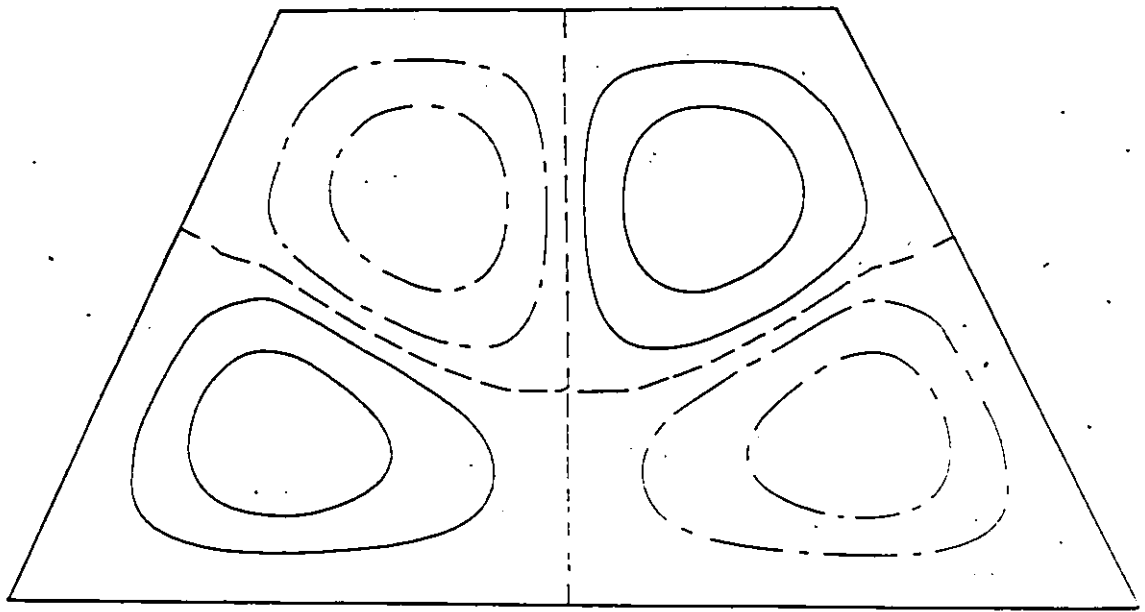


Figure 5.8-(e) Second anti-symmetric mode shape for the fully clamped symmetrical trapezoidal plate. $\phi_1 = 2.0$, $\phi_r = 2.0$. - - -, nodal line; ---, downward deflection; —, upward deflection.

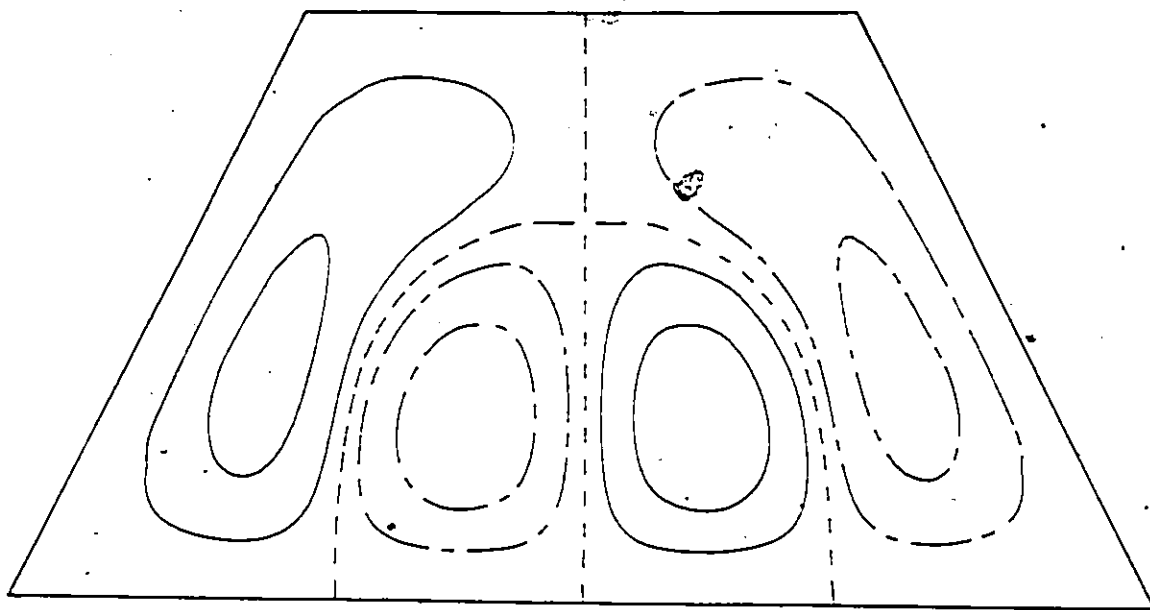


Figure 5.8-(f) Third anti-symmetric mode shape for the fully clamped symmetrical trapezoidal plate. $\phi_1 = 2.0$, $\phi_r = 2.0$. - - -, nodal line; - · -, downward deflection; —, upward deflection.

PART D

DISCUSSION

Chapter 6

DISCUSSION

Numerical results pertaining to the free vibration problem of both, the simply supported and the fully clamped symmetrical trapezoidal plates, were obtained and shown in the previous chapter. Comparison tables were also provided in each case. The purpose of the present chapter is to provide a brief but comprehensive discussion on the theory, method and techniques used throughout this analysis. Based on the numerical results of Chapter 5, some observations were made, and will be discussed here along with the influence of rotatory inertia and shear on the motion of the plate, and their effects on the frequencies.

6.1 Discussion of Results

Some of the results obtained were presented graphically for demonstration purposes. In Figure 5.2 and 5.6, the variation of the first mode eigenvalues with the change in α and the aspect ratio b/b_2 is illustrated for both the simply supported and the fully clamped plates respectively. In the case of the simply supported plate, Figure 5.3 provides a comparison of results obtained

from the present analysis with those of Reipert and Zbigniew[12], for the case of $\alpha = 45^\circ$. A good general agreement is observed. It is interesting to note the convergence to the exact eigenvalue of the right angle isosceles triangle as R approaches $1/2$. For more accurate comparison of results, attention is focussed on Table 5.3. In this table, numerical comparison is made for the symmetric vibration mode eigenvalues for the simply supported trapezoidal plate shown, where $h/a = 0.5$ and b/a ranges from 0.0 (triangle) to 1.0 (rectangle). Eigenvalues in parenthesis are those found in Reference [14]. Although the method utilized by the authors of this reference is an approximate method, good agreement is found. The accuracy of the present analytically exact method is demonstrated in the limiting case of $b/a = 0.0$ (triangle) where eigenvalues obtained by the present analysis coincide exactly with the known exact eigenvalues for this geometry[36]. However, for the benefit of the reader, it must be mentioned here that the definition of the eigenvalue λ^2 as given by the authors of Reference [14] does not agree with their tabulated numerical results. A simple look at Figures 7 through 10 of this reference shows that λ^2 is increasing as the ratio h/a increases. This should not be the case since as defined, eigenvalues should decrease as the ratio h/a increases. This fact becomes evident in the case of the simply supported rectangular plate where the exact expression for λ^2 is given by;

$$\lambda^2 = m^2 + (na/h)^2,$$

where,

$$\lambda^2 = (4\omega a^2 \pi^2) \sqrt{\rho/D},$$

and where a and h are as shown in Table 5.3. It is this author's opinion that the numerical results of Reference [14] are based on a definition for λ^2 similar to that given in this table.

Satisfactory general agreement was also noted when comparison of results was made with those of Reference [17]. The author of this reference utilizes

a combination of Bessel function solutions for the governing differential equation and Fourier series expansions for the boundary conditions which, in most cases, led to complicated algebraic equations and integrals that could only be solved numerically. The advantages of the present superposition method are evident from the simplicity of the Lévy-type solutions involved. The convergence of the present method is illustrated by Figure 6.1 where, a typical convergence curve of eigenvalues λ^2 as a function of the number of terms in the Fourier expansion k , for one of the most demanding cases (sharp angle), is shown. It is clear from Figure 6.1 that for the most demanding cases, 16 term expansion is sufficient. However, in most cases, 8 term expansions proved to be sufficient for three and in some cases four decimal points accuracy.

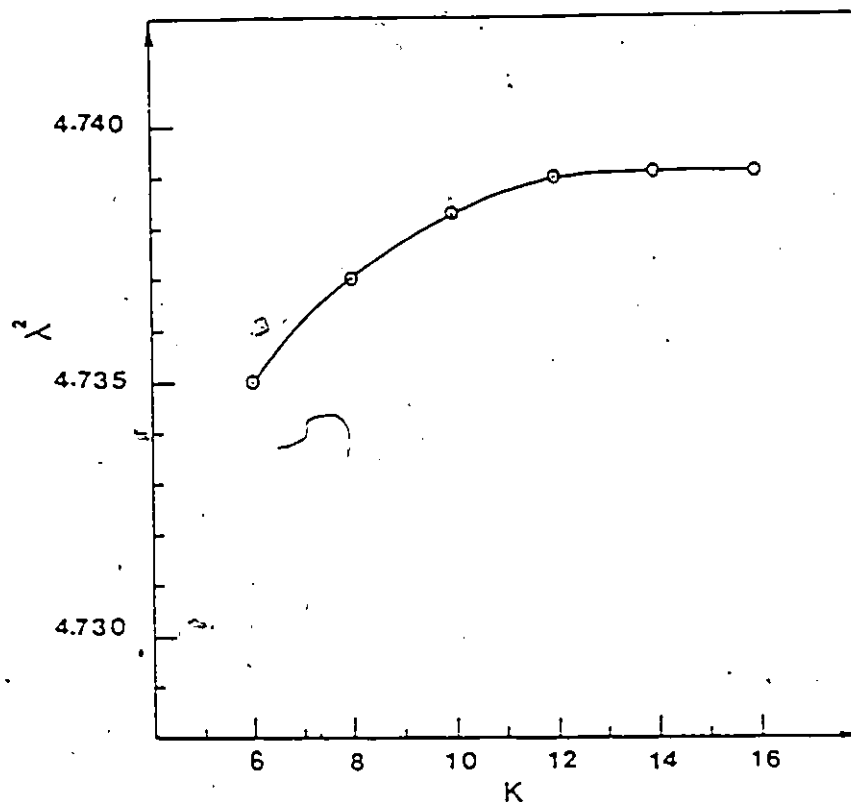


Figure 6.1- Typical convergence test curve.

In the case of fully clamped plates, comparison of results with experimental data was made possible by Table 5.7 for both symmetric and anti-symmetric modes. m and n in this table give the numbers of anti-nodes in the horizontal and vertical directions respectively. Eigenvalues in parenthesis are those found in Reference [18]. In their investigation, the authors of this reference applied the real time technique of time averaged holographic interferometry to determine the natural frequencies and the corresponding mode shapes for the transverse vibrations of clamped trapezoidal plates. A simple look at Table 5.7 reveals the excellent agreement of the presently generated theoretical eigenvalues and the experimental results of Reference [18]. This close agreement shows not only the validity and correctness of both the classical thin plate theory and the superposition techniques used in evaluating the eigenvalues listed in Chapter 5, but also the fast convergence of the present analysis to the exact eigenvalues.

Turning now to Figures 5.4-(a), 5.4-(b), 5.7-(a) and 5.7-(b), it is seen how the eigenvalue λ^2 and consequently the circular frequency ω increases with the increase of $R = b/b_2$ for both the simply supported and the fully clamped plate. In these figures, the eigenvalues are divided into symmetric and anti-symmetric corresponding to vibration modes that are symmetric or anti-symmetric with respect to the central axis of symmetry of the plate. The interesting point to note here is the sudden change in the curves of the second and third mode eigenvalues as the aspect ratio R of the plate increases. This phenomenon was also noted by the authors of Reference [14] where they refer to it as the frequency crossing. This can be explained physically. As the plate aspect ratio $R = b/b_2$ increases, the height of the plate increases with respect to its other dimensions. At some point, the plate becomes too short in the direction of the ξ axis to accommodate the increasing number of anti-nodes in the horizontal direction, causing a vibratory motion with anti-nodes in the vertical direction and consequently, a sudden change in the eigenvalues curve is observed. The second interesting point to note is that in some cases, depending on the size of the skew angle and the mode of vibration, it was noticed that this crossover from horizontal to vertical anti-nodes or visa versa

✓

would often happen through a series of mixed anti-nodes which cannot be categorized as either, but in reality are some what in between as can be seen in Figures 5.5 and 5.8. (these figures show an example of computer generated and plotted symmetric and anti-symmetric modal shapes for the simply supported and the fully clamped symmetrical trapezoidal plates with $\phi_1 = \phi_r = 2.0$). Another note of interest is the one on one resemblance in the modal shapes of Figure 5.5 representing the simply supported plate and those of Figure 5.8 representing the fully clamped plate. However, it must be noted that the distance from the boundary lines to the first contour line is greater in the modal shapes of Figure 5.8 than in those of Figure 5.5, reflecting the infinite rigidity of the boundary of the clamped plate and therefore the zero slope condition. Although the two cases are similar, the corresponding eigenvalues and consequently the circular frequencies in the case of the clamped plate are higher, also due to the rigidity of the boundary and its resistance to rotation in the latter case.

The validity, correctness, and convergence of the superposition method explored in this investigation have been demonstrated through the comparison of numerical results with previously published analytical data. Some of these data were exact eigenvalues corresponding to limiting cases such as triangles or rectangles. Some were obtained through approximate methods such as the Galerkin method. The development of an alternative solution based on the alternative building blocks of Figure 2.6 provided a powerful proof of the legitimacy of the present analysis as both solutions converged to the same eigenvalues at all times. Furthermore, through the excellent agreement of the presently generated analytical values with the experimental ones of Reference [18], it is clear that both the classical thin plate theory and the superposition techniques are applicable for the range of the numerical results of Chapter 5. At this point, the reader must be asking, what if some of the assumptions stated at the end of Chapter 1 were violated? This question will be answered in the next section.

6.2 Effects of Rotatory Inertia and Shear

In view of the five assumptions stated at the end of Chapter 1, the classical two-dimensional theory of flexural motions of thin elastic plates, leading to Lagrange's equation was used in conjunction with the superposition techniques to arrive at the results shown in Chapter 5. According to this theory, the wave velocity of straight crested waves is inversely proportional to the wave length. This was confirmed by the exact solution, by Rayleigh[37], Lamb[38] and Timoshenko[39], of the general equations of the linear theory of elasticity, for the case of straight crested flexural waves, as long as the waves are long in comparison with the thickness of the plate. However, the classical plate theory cannot be expected to give good results for sharp transients or for the frequencies of modes of vibration of high order because, as the wave length diminishes, the velocity in the three-dimensional theory has its upper limit the velocity of rayleigh surface waves. In the case of bars, a similar situation exists in regard to the classical one-dimensional theory leading to Bernoulli-Euler equation of motion of elastic bars. However, in the latter case, Rayleigh[40] introduced the effect of rotatory inertia, resulting in a finite upper limit for the velocity, but its magnitude remained too large. Later, Timoshenko[41,42] included the effect of transverse shear deformation, and therefore, obtained a one-dimensional theory of flexural motions of bars which gives satisfactory results for short waves and high modes of vibration.

In the case of plates, the effect of transverse shear deformation on the bending of elastic plates was discussed by Reissner[43,44]. In 1951, Mindlin[45] discussed the influence of rotatory inertia and shear on flexural motions of isotropic elastic plates. At various stages of his analysis, Mindlin directed attention to the very close similarities between his theory and Reissner's theory[43,44] of flexural equilibrium of plates. Furthermore, Mindlin demonstrated how a two-dimensional theory of flexural motions of isotropic elastic plates is deduced from the three-dimensional equations of elasticity. His theory included the effects of rotatory inertia and shear in the same manner as

Timoshenko's one-dimensional theory of bars, leading to the following equation of motion for rectangular homogeneous plates;

$$\left(\nabla^2 - \frac{2\rho(1+\nu)}{EhK} \frac{\partial^2}{\partial t^2} \right) \left(D \nabla^2 - \frac{\rho h^2}{12} \frac{\partial^2}{\partial t^2} \right) W + \rho \frac{\partial^2 W}{\partial t^2} = 0, \quad (6.1)$$

where ∇^2 is the Laplacian operator, and where K is the equivalent of Timoshenko's shear coefficient. This coefficient is variously chosen as 2/3 and 8/9 by Timoshenko. A formula for this coefficient, involving Poisson's ratio, was suggested by the author of Reference [45], where by its use, the limiting velocity for very short waves was made identical with the velocity of Rayleigh surface waves, and velocities intermediate between very long and very short waves were brought into close agreement with the three dimensional theory as illustrated by Figure 6.2 reproduced from the same reference. In this figure, c is the wave velocity of a wave length λ which is given in the form of the transcendental equation [37,38,39];

$$\frac{4c_s^2 \sqrt{(c_s^2 - \psi c^2)(c_s^2 - c^2)}}{(2c_s^2 - c^2)^2} = \frac{\tanh \frac{\pi h}{\lambda c_s} \sqrt{c_s^2 - \psi c^2}}{\tanh \frac{\pi h}{\lambda c_s} \sqrt{c_s^2 - c^2}}, \quad 0 < \frac{c}{c_s} < 1,$$

where c_s is the velocity of shear waves and is given by $c_s = \sqrt{Gh/\rho}$, and $\psi = (1 - 2\nu)/2(1 - \nu)$. The expression for the constant K is;

$$4\sqrt{(1 - \psi K^2)(1 - K^2)} = (2 - K^2)^2, \quad 0 < K < 1. \quad (6.2)$$

We now have available to us the differential equations governing the vibrating motion of the plate obtained from both, classical and advanced vibration theories. In the advanced theory, the natural frequencies can be expressed

as the frequencies obtained from the classical theory multiplied by a correction factor N .

$$\omega_a = \omega_c N.$$

The problem is now reduced to finding the correction factors N which can be expressed as;

$$N = \omega_a / \omega_c.$$

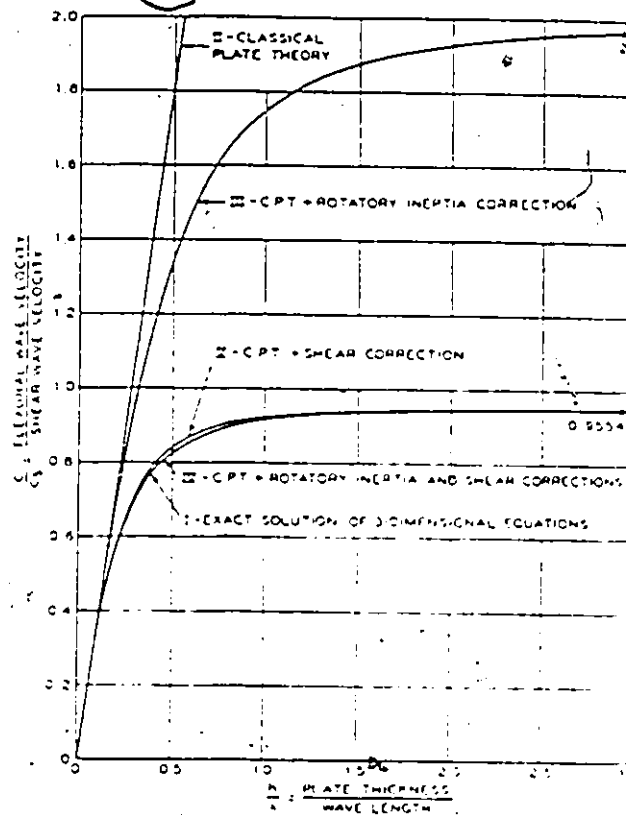


Figure 6.2- Shear and rotatory inertia correction[45].

And therefore, to obtain these factors it is necessary to solve the differential equations obtained from both, classical and advanced vibration theories. However, it must be remembered that in most cases, in view of the complexity of shape and boundary conditions of interest, solutions to these differential equations are often out of reach. It is therefore clear that in general, no exact correction factor can be found for the case where it is needed. However, approximate correction factors can always be found. Since it is believed that these factors will be of more value to the experimenter than none at all, they will be considered in the rest of this section.

A vibrating plate at higher modes exhibit a net work of node lines dividing it into smaller plates. The node lines are, in effect, simple supports. Therefore, at higher modes, any plate will act as it consists mainly of smaller simply supported plates. One might then assume that the correction factors corresponding to simply supported rectangular plate at higher modes will be applicable to other plates in an approximate way.

A frequency equation for the simply supported rectangular plate is obtained by substituting the following displacement function in Equation (6.1);

$$W = \sin(m\pi\xi)\sin(n\pi\eta)[C_1\sin(\omega_a t) + C_2\cos(\omega_a t)].$$

The resulting frequency equation is;

$$D[(m\pi/a)^2 + (n\pi/b)^2] - [(\rho h^2/12) + 2D\rho(1+\nu)/EhK]\{[(m\pi/a)^2 + (n\pi/b)^2]^2 - \rho\}\omega_a^2 + [2\rho^2(1+\nu)h^2/12EK]\omega_a^4 = 0.$$

In the classical theory we have;

$$\omega_c = \sqrt{D/\rho}[(m\pi/a)^2 + (n\pi/b)^2].$$

From the above two equations we find;

$$N^2 = \left(\frac{\omega_a}{\omega_c} \right)^2 = \left\{ \left[\left(\frac{\rho h^2}{12} + 2D\rho \frac{(1+\nu)}{EhK} \right) \left(\left(\frac{m\pi}{a} \right)^2 + \left(\frac{n\pi}{b} \right)^2 \right) - \rho \right] \right. \\ \left. - \left[\left(\frac{\rho h^2}{12} + 2D\rho \frac{(1+\nu)}{EhK} \right) \left(\left(\frac{m\pi}{a} \right)^2 + \left(\frac{n\pi}{b} \right)^2 \right) - \rho \right]^2 \right. \\ \left. - 2\rho^2 Dh^2 \frac{(1+\nu)}{3EK} \left(\left(\frac{m\pi}{a} \right)^2 + \left(\frac{n\pi}{b} \right)^2 \right)^2 \right\} / \\ \left\{ D\rho h^2 \frac{(1+\nu)}{3EK} \left(\left(\frac{m\pi}{a} \right)^2 + \left(\frac{n\pi}{b} \right)^2 \right)^2 \right\}.$$

This equation is the same as Equation (23) of Reference [46]. In applying the above correction factors to the trapezoidal plate case, it is suggested to use $a = (b_1 + b_2)/2$. It is to be expected that the approximation will be better with increasing mode number. The accuracy of this approximation can not be estimated since no exact solutions are known to exist. It is therefore advisable not to rely on this approximation without experimental verifications except to have a broad idea of the frequency range of the vibrating system. One must also keep in mind that a vibrating trapezoidal plate, or any non-rectangular quadrilateral plate, at higher modes most often exhibit a net work of curvilinear node lines rather than straight node lines.

6.3 Fine Points in Generating Eigenvalues

The Three Types of Values Encountered

It has been shown, in the previous chapter, how eigenvalues are generated with the help of the two programs of Appendix A-II. It was also mentioned that care must be exercised in selecting the appropriate ones for some of them may not be genuine. There are three types of these values. the first type consists of those numbers that give the illusion of converging toward an eigenvalue, but do not result in a vanishing coefficient matrix. These values can not be

called eigenvalues, and are easily detected since they always show discontinuity in the eigenvalue curve. The second type consists of what is referred to as rejection modes eigenvalues[2]. Although they result in a vanishing coefficient matrix, these values do not result in a real physical vibratory motion of the plate under consideration. It is found that in each study there exists certain combinations of building block shapes which correspond to zero net plate displacement. Satisfaction of the specified boundary conditions is observed. Net displacement of the total assembly is zero because the non-zero displacement of one group of building blocks is equal and opposite to the contribution of the remaining blocks. A non-trivial solution will exist for the coefficients appearing in the forcing functions even though no free vibration can occur. Associated eigenvalues are false and must be rejected. Fortunately, the presence of this type of false eigenvalues is virtually eliminated by the fact that in any analysis, at least two independent solutions (triangular and rectangular element solutions) are joined together side by side. In order for a false eigenvalue to exist, all the solutions of the involved elements must satisfy the above described conditions simultaneously. Nevertheless, this type of eigenvalues do exist in some rare cases and must be dismissed. Their detection is easily achieved since they result in zero net displacement. Finally, the third type of values is that of genuine eigenvalues that not only result in vanishing coefficient matrix and hence a non-trivial solution for the coefficients of the forcing functions, but also in a real vibratory motion of the plate under consideration.

Numerical Instabilities

Given the nature of the equations involved in this analysis, and due to the presence of hyperbolic functions, there was, at first, some concern on the part of the author relating to possible numerical instabilities and their effects on the method used in this study. However, as the work progressed, it became clear that any possibility of numerical instabilities could be eliminated by simplifying the equations involved. After careful manipulation of all the

equations; the only term judged to be a potential cause of any numerical instability was found to be in the form of $\sinh(x)/\cosh(x)$, or $\cosh(x)/\sinh(x)$, which can be replaced by unity for large enough values of x . The simplified equations and the critical values of x , at which the programs of Appendix A-II switch from the original equations to the simplified ones, are checked by evaluating the numerical values of both at the point of crossover to make sure the exact same results were obtained in each case. Programs TSLS and TCLS are each equipped with a set of simplified equations to deal with any possibility of numerical instability that might arise.

6.4 Internal Damping Effects

In the preceding analysis it was assumed that the vibrating plate possesses no damping. While this mathematically convenient assumption closely approximate the actual dynamic behavior of the plate, there is always some damping effect in any real situation. It is due to this damping that the free vibration dies out when, after initial excitation, the plate is left alone. External damping forces produced by the surrounding media, such as water, air, and soil can be predicted, controlled, and kept to an insignificant level. However, what we are concerned with here is the internal damping forces which are attributed to the hysteresis of elastic material; i.e., the work required to produce certain deformation is larger than the energy required in the reloading process. Consequently, there is a loss of energy which can have a pronounced effect on the vibration only when the deformations are in the plastic range. Since, strictly speaking, these internal forces are stress-dependent, their exact consideration in the analysis is cumbersome. However, their influence on the free and steady-state vibrations is small. In the case of a single degree of freedom system, for instance, a strong damping force capable of reducing the amplitude of free vibration to half within one period will have only a negligible effect of $\pm 0.6\%$ on the first circular frequency.

CONCLUSION

The present analysis is devoted to the dynamic analysis of non-rectangular quadrilateral plates. The classical solution of the differential equation of motion is used in conjunction with Gorman's superposition techniques to develop a technique leading to an analytically exact solution to the title problem. This technique is then applied to solve the free vibration problem of the simply supported trapezoidal plates of Chapter 3. In Chapter 4, the problem of the fully clamped plate is solved. Numerical results for the above two cases are given in Chapter 5, where comparison of eigenvalues with both analytical and experimental results is made. Unlike numerical and other approximate solutions, the present analytical method satisfies exactly the differential equation throughout the plate. It also satisfies all boundary conditions to any desired degree of accuracy. It is assumed that the effect of rotatory inertia and shear is, in most cases, of negligible order of magnitude. This is true only if the assumptions stated at the end of chapter one are observed. To provide, however, a generally usable tool for cases when the rotatory inertia and shear effects must be taken into consideration, these effects are discussed in Chapter 6, providing an approximate correction factor for the higher mode frequencies. However, the reader must keep in mind that this correction factor is approximate, and that no estimate could be given as to its level of reliability. Use of the two programs of Appendix A-II is also explained.

The mathematical technique described in this analysis is not limited in application to problems with axis of symmetry. Its application could be extended to any polygonal plate with arbitrary boundary conditions by prudently choosing a sufficient number, of triangular and rectangular elements to properly represent the geometry of the plate under study, and of building blocks to satisfy its boundary conditions. The only requirement is that these boundary conditions and the differential equation do not introduce any nonlinearities. In fact, the solution of the free vibration problem of the simply supported unsymmetrical trapezoidal plate, obtained by dividing the simply supported symmetrical trapezoidal plate into two along its axis of symmetry, is the same as that of the anti-symmetric modes of the latter plate. The anti-symmetric mode solution of the fully clamped plate is also the solution of this unsymmetrical trapezoidal plate with simple support along the common edge of the two right angles and clamped along the other three. However, the reader must be made aware of the fact that a large number of right angle triangles and rectangles may be needed in order to represent some of the more complicated quadrilateral non-rectangular shapes. Consequently, at some point the analyst must be prepared to fully analyze the situation in order to decide if the present analysis is feasible in view of the increasing number of building blocks required for the solution of such plate, and therefore the large size coefficient matrix involved.

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APPENDICES

APPENDIX A-I

COMMONLY USED INTEGRALS

Beginning in Chapter 3 we have often found it necessary to expand analytical functions in Fourier-type series. Fortunately, only a very limited number of products of these functions need to be integrated. In this appendix, a list of such integrals is provided. Although very brief, it is sufficient for all the integration required in this analysis. Integrations are always over the interval from 0 to 1.

$$\begin{aligned}\int_0^1 \cos(Ax)\cos(Bx)dx &= \frac{1}{2} \left[\frac{\sin(A-B)}{A-B} + \frac{\sin(A+B)}{A+B} \right], \\ \int_0^1 \sin(Ax)\sin(Bx)dx &= \frac{1}{2} \left[\frac{\sin(A-B)}{A-B} - \frac{\sin(A+B)}{A+B} \right], \\ \int_0^1 \sin(Ax)\cos(Bx)dx &= -\frac{1}{2} \left[\frac{\cos(A-B)}{A-B} + \frac{\cos(A+B)}{A+B} - \frac{2A}{A^2-B^2} \right],\end{aligned}$$

in the above three integrals $A^2 \neq B^2$.

$$\begin{aligned}\int_0^1 \sin(Ax) \cosh(Bx) dx &= \frac{A(1 - \cos A \cosh B) + B \sin A \sinh B}{A^2 + B^2}, \\ \int_0^1 \cos(Ax) \sinh(Bx) dx &= \frac{B \cos A \cosh B + A \sin A \sinh B - B}{A^2 + B^2}, \\ \int_0^1 \cos(Ax) \cosh(Bx) dx &= \frac{B \cos A \sinh B + A \sin A \cosh B}{A^2 + B^2}.\end{aligned}$$

in the next three integrals $A^2 \neq C^2$,

$$\begin{aligned}\int_0^1 \cos(Ax + B) \cos(Cx + D) dx &= \frac{\sin(A - C + B - D) - \sin(B - D)}{2(A - C)} + \\ &\quad \frac{\sin(A + B + C + D) - \sin(B + D)}{2(A + C)},\end{aligned}$$

$$\begin{aligned}\int_0^1 \sin(Ax + B) \sin(Cx + D) &= \frac{\sin(A + B - C - D) - \sin(B - D)}{2(A - C)} - \\ &\quad \frac{\sin(A + B + C + D) - \sin(B + D)}{2(A + B)},\end{aligned}$$

$$\begin{aligned}\int_0^1 \sin(Ax + B) \cos(Cx + D) dx &= \frac{\cos(B - D) - \cos(A - C + B - D)}{2(A - C)} + \\ &\quad \frac{\cos(B + D) - \cos(A + B + C + D)}{2(A + C)}.\end{aligned}$$

$$\begin{aligned}\int_0^1 \sinh(Ax + B) \cos(Cx + D) dx &= \\ &\quad A \frac{\cosh(A + B) \cos(C + D) - \cosh B \cos D}{A^2 + C^2} + \\ &\quad C \frac{\sinh(A + B) \sin(C + D) - \sinh B \sin D}{A^2 + C^2}.\end{aligned}$$

$$\int_0^1 \cosh(Ax + B) \sin(Cx + D) dx =$$

$$A \frac{\sinh(A + B) \sin(C + D) - \sinh B \sin D}{A^2 + C^2} +$$

$$C \frac{\cosh B \cos D - \cosh(A + B) \cos(C + D)}{A^2 + C^2}$$

$$\int_0^1 \sinh(Ax + B) \sin(Cx + D) dx =$$

$$A \frac{\cosh(A + B) \sin(C + D) - \cosh B \sin D}{A^2 + C^2} +$$

$$C \frac{\sinh B \cos D - \sinh(A + B) \cos(C + D)}{A^2 + C^2}$$

$$\int_0^1 \cosh(Ax + B) \cos(Cx + D) dx =$$

$$A \frac{\sinh(A + B) \cos(C + D) - \sinh B \cos D}{A^2 + C^2} +$$

$$C \frac{\cosh(A + B) \sin(C + D) - \cosh B \sin D}{A^2 + C^2}$$

Program I, TSLS

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93 FORMAT(' ',F9.5,2X,F8.4,2X,F7.5,2X,F7.5,2X,F7.4,3X,I2,/)
94 FORMAT(' ', 'KKX.KKKXX KXX.KXXK K.KXXKX K.KXXKX KX.KXXK KX')
MODE=0.
RATIO= 1
EDGE=-3
C=1.
PI=4.*DATAN(C)
PHI1=1./PHI1
AL=DATAN(PHI1)
PHI3=1./PHI1
PHI5=DCOS(AL)*DCOS(AL)/PHI1
PHI1S=PHI1*PHI1
PHI3S=PHI3*PHI3
PHI5S=PHI5*PHI5
PHI5C=PHI5S*PHI5
PHIR=PHIR*PHIR
F1=DCOS(AL)*DCOS(AL)
F2=DSIN(AL)*DSIN(AL)
AL1=PI-AL
AL2=(PI/2.)-AL
AL3=PI+AL
TD11=DCOS(AL1)*DCOS(AL1)-POI*DSIN(AL1)*DSIN(AL1)
TD12=DSIN(AL1)*DSIN(AL1)+POI*DCOS(AL1)*DCOS(AL1)
TD13=(1-POI)*DSIN(2.*AL1)
TD21=DCOS(AL2)*DCOS(AL2)-POI*DSIN(AL2)*DSIN(AL2)
TD22=DSIN(AL2)*DSIN(AL2)-POI*DCOS(AL2)*DCOS(AL2)
TD23=(1-POI)*DSIN(2.*AL2)
TD31=DCOS(AL3)*DCOS(AL3)-POI*DSIN(AL3)*DSIN(AL3)
TD32=DSIN(AL3)*DSIN(AL3)-POI*DCOS(AL3)*DCOS(AL3)
TD33=(1-POI)*DSIN(2.*AL3)
V1=DCOS(AL2)+(1-POI)*DSIN(AL2)*DSIN(2.*AL2)
V2=(DSIN(AL2)+(1-POI)*DCOS(AL2)*DSIN(2.*AL2))/PHI5C
V3=(DSIN(AL2)-(1-POI)*DCOS(AL2)*DCOS(2.*AL2)-DCOS(AL2)*DSIN(2.*AL2))/PHI5
V4=(DCOS(AL2)+(1-POI)*(DCOS(AL2)*DCOS(2.*AL2)-DSIN(AL2)*DSIN(2.*AL2)))/PHI5S
F21=DCOS(AL2)*DSIN(AL)
F22=DSIN(AL2)/DCOS(AL)
PRINT1,PHI1,K,AL
1 FORMAT('1', 'PHI1 =',F8.4,10X,'K =',I5,10X,'ALPHA =',F10.3,10X)
*
* Initialize the matrix A
*
2 I=8*K
DO 3 M=1,I
DO 3 N=1,I
3 A(M,N)=0.0

```

*
* Start the computation of the Coefficient matrix elements
*

```

DO 4 I=1,K
  A(I,I+K)      = 1.
  A(I+K,I+3*K)  = 1.
  A(I+2*K,I+3*K) = 1.
  A(I+3*K,I)    = 1.
  A(I+4*K,I+3*K) = 1.*PHI1S
  A(I+5*K,I+4*K) = 1./F2
4 CONTINUE
DO 100 I=1,K
DO 100 J=1,K
  ALMDPS=ALMDS/F2
  ALMDRS=ALMDS*PHI1S/PHIRS
  EMP=J*PI
  ENP=I*PI
  EMPS=EMP*EMP
  ENPS=ENP*ENP
  B1MS=PHI1S*(ALMDS+EMPS)
  B1M=DSQRT(B1MS)
  TEST1M=ALMDS-EMPS
  G1MS=PHI1S*TEST1M
  IF (G1MS.LT.0.) G1MS=-G1MS
  G1M=DSQRT(G1MS)
  B3MS=PHI3S*(PHI1S*ALMDS+EMPS)
  B3M=DSQRT(B3MS)
  TEST3M=PHI1S*ALMDS-EMPS
  G3MS=PHI3S*TEST3M
  IF (G3MS.LT.0.) G3MS=-G3MS
  G3M=DSQRT(G3MS)
  B5MS=PHI5S*(ALMDPS-EMPS)
  B5M=DSQRT(B5MS)
  TEST5M=ALMDPS-EMPS
  G5MS=PHI5S*TEST5M
  IF (G5MS.LT.0.) G5MS=-G5MS
  G5M=DSQRT(G5MS)
  BRMS=(PHIRS*ALMDRS-EMPS)/PHIRS
  BRM=DSQRT(BRMS)
  TESTRM=PHIRS*ALMDRS-EMPS
  GRMS=TESTRM/PHIRS
  IF (GRMS.LT.0.) GRMS=-GRMS
  GRM=DSQRT(GRMS)
  X1=EMP*F2-ENP
  X1S=X1*X1
  X2=EMP*F2-ENP
  X2S=X2*X2

```

```

X3=EMP*F1+ENP
X3S=X3*X3
X4=EMP*F1-ENP
X4S=X4*X4
X5=EMP-ENP
X5S=X5*X5
X6=EMP+ENP
X6S=X6*X6
IF (TEST1M.LT.0.) GO TO 20G
C11M=-DSINH(B1M)/DSIN(G1M)
TD11M=-1./ (B1MS*DSINH(B1M)-C11M*G1MS*DSIN(G1M))
C21M=(B1MS-POI*PHI1S*EMPS)*DSINH(B1M)/ ((G1MS-POI*PHI1S*EMPS)*
IDSIN(G1M))
TD21M=1./ (DSINH(B1M)-C21M*DSIN(G1M))
C1=B1M*(DCOSH(B1M)-DCOS(X5))/(B1MS-X5S)
C2=B1M*(DCOS(X6)-DCOSH(B1M))/(B1MS-X6S)
TEST=G1MS-X5S
IF (TEST.EQ.0.) GO TO 19
C3=((DCOS(X5)-DCOS(G1M))/(X5-G1M))-((DCOS(G1M)-DCOS(X5))/(X5-G1M))
C3=C3/2.
GO TO 21
19 C3=DSIN(G1M)/2.
21 TEST=G1MS-X6S
IF (TEST.EQ.0.) GO TO 22
C4=((DCOS(G1M)-DCOS(X6))/(X6-G1M))-((DCOS(X6)-DCOS(G1M))/(X6-G1M))
C4=C4/2.
GO TO 23
22 C4=-DSIN(G1M)/2.
23 H1=X6*(DCOSH(B1M)-DCOS(X6))/(B1MS-X6S)
H1=X5*(DCOS(X5)-DCOSH(B1M))/(B1MS-X5S)
TEST=G1MS-X6S
IF (TEST.EQ.0.) GO TO 37
H3=((DCOS(G1M)-DCOS(X6))/(X6-G1M))-((DCOS(G1M)-DCOS(X6))/(X6-G1M))
H3=H3/2.
GO TO 38
37 H3=DSIN(G1M)/2.
38 TEST=G1MS-X5S
IF (TEST.EQ.0.) GO TO 39
H4=((DCOS(X5)-DCOS(G1M))/(X5-G1M))-((DCOS(X5)-DCOS(G1M))/(X5-G1M))
H4=H4/2.
GO TO 41
39 H4=-DSIN(G1M)/2.
IF (X5.LT.0.) H4=-H4
41 DS1=-2.*ENP*DCOS(ENP)*DSINH(B1M)/(B1MS-ENPS)
DS2=2.*ENP*DCOS(ENP)*DSIN(G1M)/(G1MS-ENPS)
DS3=-2.*ENP*DCOS(ENP)*DSINH(G1M)/(G1MS-ENPS)
GO TO 210

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200 IF (B1MS.GT.QLIM) GO TO 205
C12M=-DSINH(B1M)/DSINH(G1M)
TD12M=-1./((B1MS*DSINH(B1M)+C12M*G1MS*DSINH(G1M))
C22M=-(B1MS-POI*PHI1S*EMPS)*DSINH(B1M)/((G1MS-POI*PHI1S*EMPS)*
DSINH(G1M))
TD22M=1./((DSINH(B1M)+C22M*DSINH(G1M))
C1=B1M*(DCOSH(B1M)-DCOS(X5))/(B1MS+X5S)
C2=B1M*(DCOS(X6)-DCOSH(B1M))/(B1MS+X6S)
C5=G1M*(DCOSH(G1M)-DCOS(X5))/(G1MS+X5S)
C6=G1M*(DCOS(X6)-DCOSH(G1M))/(G1MS+X6S)
H1=X6*(DCOSH(B1M)-DCOS(X6))/(B1MS+X6S)
H2=X5*(DCOS(X5)-DCOSH(B1M))/(B1MS+X5S)
H5=X6*(DCOSH(G1M)-DCOS(X6))/(G1MS+X6S)
H6=X5*(DCOS(X5)-DCOSH(G1M))/(G1MS+X5S)
DS1=-2.*ENP*DCOS(ENP)*DSINH(B1M)/(B1MS+ENPS)
DS2=2.*ENP*DCOS(ENP)*DSIN(G1M)/(G1MS-ENPS)
DS3=-2.*ENP*DCOS(ENP)*DSINH(G1M)/(G1MS+ENPS)
GO TO 210
205 C12M=-1
TD12M=-1./((B1MS+C12M*G1MS)
C22M=-(B1MS-POI*PHI1S*EMPS)/(G1MS-POI*PHI1S*EMPS)
TD22M=1./((1.+C22M)
C1=B1M/(B1MS+X5S)
C2=-B1M/(B1MS+X6S)
C5=G1M/(G1MS+X5S)
C6=-G1M/(G1MS+X6S)
H1=X6/(B1MS+X6S)
H2=-X5/(B1MS+X5S)
H5=X6/(G1MS+X6S)
H6=-X5/(G1MS+X5S)
DS1=-2.*ENP*DCOS(ENP)/(B1MS-ENPS)
DS2=2.*ENP*DCOS(ENP)*DSIN(G1M)/(G1MS-ENPS)
DS3=-2.*ENP*DCOS(ENP)/(G1MS-ENPS)
210 IF (TEST3M.LT.0.) GO TO 220
C31M=-DSINH(B3M)/DSIN(G3M)
TD31M=-1./((B3MS*DSINH(B3M)-C31M*G3MS*DSIN(G3M))
C41M=(B3MS-POI*PHI3S*EMPS)*DSINH(B3M)/((G3MS-POI*PHI3S*EMPS)*
DSIN(G3M))
TD41M=1./((DSINH(B3M)+C41M*DSIN(G3M))
D1=B3M*(DCOSH(B3M)*DCOS(ENP)-DCOS(ENP))/(B3MS+X6S)
D2=B3M*(DCOS(ENP)-DCOSH(B3M)*DCOS(ENP))/(B3MS+X5S)
TEST=G3MS-X6S
IF (TEST.EQ.0.) GO TO 24
D3=((DCOS(ENP)-DCOS(G3M+ENP))/(G3M+X6))+((DCOS(ENP)-DCOS(G3M-
1ENP))/(G3M-X6))
D3=D3/2.
GO TO 25

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24 D3=(DCOS(EMP)-DCOS(2.*G3M+EMP))/(4.*G3M)
25 TEST=G3MS-X3S
   IF (TEST.EQ.0) GO TO 26
   D4=((DCOS(G3M-EMP)-DCOS(EMP))/(G3M-X3)) + ((DCOS(G3M+EMP)-DCOS(
EMP))/(G3M+X3))
   D4=D4/2.
   GO TO 27
26 D=(DCOS(2.*G3M+EMP)-DCOS(EMP))/(4.*G3M)
27 P1=X3*(DCOSH(B3M)*DCOS(EMP)-DCOS(EMP))/(B3MS-X3S)
   P2=X6*(DCOS(EMP)-DCOSH(B3M)*DCOS(EMP))/(B3MS-X6S)
   TEST=G3MS-X3S
   IF (TEST.EQ.0.) GO TO 42
   P3=((DCOS(EMP-G3M)-DCOS(EMP))/(G3M-X3)) + ((DCOS(EMP)-DCOS(EMP-G3M))
1/(G3M-X3))
   P3=P3/2.
   GO TO 43
42 P3=(DCOS(EMP-2.*G3M)-DCOS(EMP))/(4.*G3M)
   IF (X3.LT.0.) P3=-P3
43 TEST=G3MS-X6S
   IF (TEST.EQ.0.) GO TO 44
   P4=((DCOS(EMP)-DCOS(EMP+G3M))/(G3M+X6)) + ((DCOS(G3M-EMP)-DCOS(EMP))
1/(G3M-X6))
   P4=P4/2.
   GO TO 45
44 P4=(DCOS(EMP)-DCOS(EMP-2.*G3M))/(4.*G3M)
45 CONTINUE
   GO TO 230
220 IF (B3MS.GT.QLIM) GO TO 225
   C32M=-DSINH(B3M)/DSINH(G3M)
   TD32M=-1./(B3MS*DSINH(B3M)-C32M*G3MS*DSINH(G3M))
   C42M=-(B3MS-POI*PHI3S*EMPS)*DSINH(B3M)/((G3MS-POI*PHI3S*EMPS)*
DSINH(G3M))
   TD42M=1./(DSINH(B3M)-C42M*DSINH(G3M))
   D1=B3M*(DCOSH(B3M)*DCOS(EMP)-DCOS(EMP))/(B3MS-X6S)
   D2=B3M*(DCOS(EMP)-DCOSH(B3M)*DCOS(EMP))/(B3MS-X3S)
   D3=G3M*(DCOSH(G3M)*DCOS(EMP)-DCOS(EMP))/(G3MS-X6S)
   D4=G3M*(DCOS(EMP)-DCOSH(G3M)*DCOS(EMP))/(G3MS-X3S)
   P1=X3*(DCOSH(B3M)*DCOS(EMP)-DCOS(EMP))/(B3MS-X3S)
   P2=X6*(DCOS(EMP)-DCOSH(B3M)*DCOS(EMP))/(B3MS-X6S)
   P3=X3*(DCOSH(G3M)*DCOS(EMP)-DCOS(EMP))/(G3MS-X3S)
   P4=X6*(DCOS(EMP)-DCOSH(G3M)*DCOS(EMP))/(G3MS-X6S)
   GO TO 230
225 C32M=-1.
   TD32M=-1./(B3MS+C32M*G3MS)
   C42M=-(B3MS-POI*PHI3S*EMPS)/(G3MS-POI*PHI3S*EMPS)
   TD42M=1./(1+C42M)
   D1=B3M*DCOS(EMP)/(B3MS-X6S)

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D2=-B3M*DCOS(ENP)/(B3MS+X3S)
D5=G3M*DCOS(ENP)/(G3MS+X6S)
D6=-G3M*DCOS(ENP)/(G3MS+X3S)
P1=X5*DCOS(ENP)/(B3MS+X3S)
P2=-X6*DCOS(ENP)/(B3MS+X6S)
P5=X5*DCOS(ENP)/(G3MS+X3S)
P6=-X6*DCOS(ENP)/(G3MS+X6S)
230 IF (TEST5M.LT.0.) GO TO 240
C51M=-DSINH(B5M)/DSIN(G5M)
TD51M=-1./((B3MS*DSINH(B5M)-C51M*G5MS*DSIN(G5M))
C61M=(B5MS-POI*PHI5S*EMPS)*DSINH(B5M)/((G5MS+POI*PHI5S*EMPS)*
DSIN(G5M))
TD61M=1./((DSINH(B5M)+C61M*DSIN(G5M))
A1=B5M*(DCOSH(B5M)-DCOS(X1))/(B5MS+X1S)
A2=B5M*(DCOS(X2)-DCOSH(B5M))/(B5MS+X2S)
TEST=G5MS-X1S
IF (TEST.EQ.0.) GO TO 11
A3=((DCOS(X1)-DCOS(G5M))/(X1+G5M))+((DCOS(G5M)-DCOS(X1))/(X1-G5M))
A3=A3/2.
GO TO 12
11 A3=DSIN(G5M)/2.
12 TEST=G5MS-X2S
IF (TEST.EQ.0.) GO TO 13
A4=((DCOS(G5M)-DCOS(X2))/(X2+G5M))+((DCOS(X2)-DCOS(G5M))/(X2-G5M))
A4=A4/2.
GO TO 14
13 A4=-DSIN(G5M)/2.
14 B1=B5M*(DCOSH(B5M)*DCOS(EMP)-DCOS(EMP-X3))/(B5MS+X3S)
B2=B5M*(DCOS(EMP-X4)-DCOSH(B5M)*DCOS(EMP))/(B5MS-X4S)
TEST=G5MS-X3S
IF (TEST.EQ.0.) GO TO 15
B3=((DCOS(G5M-EMP)-DCOS(X3-EMP))/(X3-G5M))-((DCOS(EMP-X3)-DCOS(
IG5M+EMP))/(X3+G5M))
B3=B3/2.
GO TO 16
15 B3=DSIN(G5M-EMP)/2.
16 TEST=G5MS-X4S
IF (TEST.EQ.0.) GO TO 17
B4=((DCOS(X4-EMP)-DCOS(G5M-EMP))/(X4-G5M))-((DCOS(G5M+EMP)-DCOS(
EMP-X4))/(X4+G5M))
B4=B4/2.
GO TO 18
17 B4=DSIN(EMP-G5M)/2.
18 E1=X2*(DCOSH(B5M)-DCOS(X2))/(B5MS+X2S)
E2=X1*(DCOS(X1)-DCOSH(B5M))/(B5MS+X1S)
TEST=G5MS-X2S
IF (TEST.EQ.0.) GO TO 28

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E3=((DCOS(G5M)-DCOS(X2))/(X2-G5M))+((DCOS(G5M)-DCOS(X2))/(X2-G5M))
E3=E3/2.
GO TO 29
28 E3=DSIN(G5M)/2.
29 TEST=G5MS-X1S
IF (TEST.EQ.0.) GO TO 31
E4=((DCOS(X1)-DCOS(G5M))/(X1-G5M))+((DCOS(X1)-DCOS(G5M))/(X1-G5M))
E4=E4/2.
GO TO 32
31 E4=-DSIN(G5M)/2.
IF (X1.LT.0.) E4=-E4
32 G1=X4*(DCOS(EMP-X4)-DCOSH(B5M)*DCOS(EMP))-(B5MS-X4S)
G2=X3*(DCOSH(B5M)*DCOS(EMP)-DCOS(EMP-X3))-(B5MS-X3S)
TEST=G5MS-X4S
IF (TEST.EQ.0.) GO TO 33
G3=((DCOS(EMP-G5M)-DCOS(EMP-X4))/(G5M-X4))+((DCOS(EMP-X4)-DCOS(
1EMP-G5M))/(G5M+X4))
G3=G3/2.
GO TO 34
33 G3=DSIN(EMP-G5M)/2.
IF (X4.LT.0.) G3=-G3
34 TEST=G5MS-X3S
IF (TEST.EQ.0.) GO TO 35
G4=((DCOS(EMP-X3)-DCOS(EMP-G5M))/(G5M-X3))+((DCOS(EMP-G5M)-DCOS(
1EMP-X3))/(G5M+X3))
G4=G4/2.
GO TO 36
35 G4=DSIN(G5M-EMP)/2.
36 ES1=(B5M*DSIN(X4-EMP)-X4*DSINH(B5M)*DCOS(EMP))-(B5MS-X4S)
ES2=(X3*DSINH(B5M)*DCOS(EMP)-B5M*DSIN(X3-EMP))-(B5MS-X3S)
TEST=G5MS-X4S
IF (TEST.LT.0.) TEST=-TEST
IF (TEST.EQ.0.) GO TO 46
ES3=((DCOS(EMP-(PI/2.-X4)-DCOS(G5M-EMP-(PI/2.-X4))-(X4-G5M))-
1(DCOS(G5M-(PI/2.-EMP)-DCOS(X4-(PI/2.-EMP))-(X4-G5M))
ES3=ES3/2.
GO TO 46
46 IF (X4.LT.0.) GO TO 47
ES3=DSIN(G5M-(PI/2.-EMP))/2.-((DCOS(EMP-G5M-(PI/2.-X4)-DCOS
1(EMP-G5M-(PI/2.-X4)))/(4.*G5M))
GO TO 48
47 ES3=DSIN(G5M-EMP-(PI/2.))/2.-((DCOS(PI/2.-G5M-EMP)-DCOS
1(G5M-(PI/2.-EMP)))/(4.*G5M))
48 TEST=G5MS-X3S
IF (TEST.LT.0.) TEST=-TEST
IF (TEST.EQ.0.) GO TO 49
ES4=((DCOS(G5M-EMP-(PI/2.-X3)-DCOS(EMP-(PI/2.-X3))-(X3-G5M))-

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1((DCOS(X3+(PI/2.))-EMP)-DCOS(G5M+(PI/2.))-EMP))/(X3-G5M))
ES4=ES4/2.
GO TO 51
49 ES4=(DSIN(EMP-G5M-(PI/2.))/2.)+(DCOS(EMP-G5M-(PI/2.))-DCOS
1(EMP-G5M-(PI/2.)))/(4.*G5M))
51 FS1=(B5M*DSINH(B5M)*DCOS(EMP)-X3*DSIN(EMP-X3))/(B5MS-X3S)
FS2=(X4*DSIN(EMP-X4)-B5M*DSINH(B5M)*DCOS(EMP))/(B5MS-X4S)
TEST=G5MS-X3S
IF (TEST.LT.0.) TEST=-TEST
IF (TEST.EQ.0.) GO TO 52
FS3=((DSIN(X3-EMP)-DSIN(G5M-EMP))/(X3-G5M))+
1((DSIN(G5M+EMP)-DSIN(EMP-X3))/(X3-G5M))
FS3=FS3/2.
GO TO 53
52 FS3=(DCOS(G5M-EMP)/2.)+(DSIN(EMP-G5M)/(2.*G5M))
53 TEST=G5MS-X4S
IF (TEST.LT.0.) TEST=-TEST
IF (TEST.EQ.0.) GO TO 54
FS4=((DSIN(G5M-EMP)-DSIN(X4-EMP))/(X4-G5M))-
1((DSIN(EMP-X4)-DSIN(G5M+EMP))/(X4-G5M))
FS4=FS4/2.
GO TO 55
54 FS4=(DSIN(EMP-G5M)/(2.*G5M))-(DCOS(G5M-EMP)/2.)
55 CONTINUE
GO TO 250
240 IF (B5MS.GT.QLIM) GO TO 245
C52M=-DSINH(B5M)/DSINH(G5M)
TD52M=-1./(B5MS*DSINH(B5M)+C52M*G5MS*DSINH(G5M))
C62M=-(B5MS-POI*PHI5S*EMPS)*DSINH(B5M)/((G5MS-POI*PHI5S*EMPS*
1DSINH(G5M))
TD62M=1./(DSINH(B5M)+C62M*DSINH(G5M))
A1=B5M*(DCOSH(B5M)-DCOS(X1))/(B5MS-X1S)
A2=B5M*(DCOS(X2)-DCOSH(B5M))/(B5MS-X2S)
A5=G5M*(DCOSH(G5M)-DCOS(X1))/(G5MS-X1S)
A6=G5M*(DCOS(X2)-DCOSH(G5M))/(G5MS-X2S)
B1=B5M*(DCOSH(B5M)*DCOS(EMP)-DCOS(EMP-X3))/(B5MS-X3S)
B2=B5M*(DCOS(EMP-X4)-DCOSH(B5M)*DCOS(EMP))/(B5MS-X4S)
B5=G5M*(DCOSH(G5M)*DCOS(EMP)-DCOS(EMP-X3))/(G5MS-X3S)
B6=G5M*(DCOS(EMP-X4)-DCOSH(G5M)*DCOS(EMP))/(G5MS-X4S)
E1=X2*(DCOSH(B5M)-DCOS(X2))/(B5MS-X2S)
E2=X1*(DCOS(X1)-DCOSH(B5M))/(B5MS-X1S)
E5=X2*(DCOSH(G5M)-DCOS(X2))/(G5MS-X2S)
E6=X1*(DCOS(X1)-DCOSH(G5M))/(G5MS-X1S)
G1=X4*(DCOS(EMP-X4)-DCOSH(B5M)*DCOS(EMP))/(B5MS-X4S)
G2=X3*(DCOSH(B5M)*DCOS(EMP)-DCOS(EMP-X3))/(B5MS-X3S)
G5=X4*(DCOS(EMP-X4)-DCOSH(G5M)*DCOS(EMP))/(G5MS-X4S)
G6=X3*(DCOSH(G5M)*DCOS(EMP)-DCOS(EMP-X3))/(G5MS-X3S)

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ES1=(B5M*DSIN(X4-EMP)-X4*DSINH(B5M)*DCOS(EMP))/(B5MS-X4S)
ES2=(X3*DSINH(B5M)*DCOS(EMP)-B5M*DSIN(X3-EMP))/(B5MS-X3S)
ES3=(G5M*DSIN(X4-EMP)-X4*DSINH(G5M)*DCOS(EMP))/(G5MS-X4S)
ES6=(X3*DSINH(G5M)*DCOS(EMP)-G5M*DSIN(X3-EMP))/(G5MS-X3S)
FS1=(B5M*DSINH(B5M)*DCOS(EMP)-X3*DSIN(EMP-X3))/(B5MS-X3S)
FS2=(X4*DSIN(EMP-X4)-B5M*DSINH(B5M)*DCOS(EMP))/(B5MS-X4S)
FS3=(G5M*DSINH(G5M)*DCOS(EMP)-X3*DSIN(EMP-X3))/(G5MS-X3S)
FS6=(X4*DSIN(EMP-X4)-G5M*DSINH(G5M)*DCOS(EMP))/(G5MS-X4S)
GO TO 250
245 C52M=-1.
TD52M=-1./(B5MS+C52M*G5MS)
C62M=-(B5MS-POI*PHI3S*EMPS)/(G5MS-POI*PHI3S*EMPS)
TD62M=1./(1.+C62M)
A1=B5M/(B5MS+X1S)
A2=-B5M/(B5MS+X2S)
A3=G5M/(G5MS+X1S)
A6=-G5M/(G5MS+X2S)
B1=B5M*DCOS(EMP)/(B5MS-X3S)
B2=-B5M*DCOS(EMP)/(B5MS-X4S)
B3=G5M*DCOS(EMP)/(G5MS+X3S)
B6=-G5M*DCOS(EMP)/(G5MS+X4S)
E1=X2/(B5MS+X2S)
E2=-X1/(B5MS+X1S)
E3=X2/(G5MS+X2S)
E6=-X1/(G5MS+X1S)
G1=-X4*DCOS(EMP)/(B5MS+X4S)
G2=X3*DCOS(EMP)/(B5MS+X3S)
G3=-X4*DCOS(EMP)/(G5MS+X4S)
G6=X3*DCOS(EMP)/(G5MS+X3S)
ES1=-X4*DCOS(EMP)/(B5MS-X4S)
ES2=X3*DCOS(EMP)/(B5MS+X3S)
ES3=-X4*DCOS(EMP)/(G5MS-X4S)
ES6=X3*DCOS(EMP)/(G5MS+X3S)
FS1=B5M*DCOS(EMP)/(B5MS-X3S)
FS2=-B5M*DCOS(EMP)/(B5MS-X4S)
FS3=G5M*DCOS(EMP)/(G5MS+X3S)
FS6=-G5M*DCOS(EMP)/(G5MS-X4S)
250 CONTINUE

```

```

*
* This step will compute elements with ALMDS > EMPS
*

```

```

IF (TESTIN.LT.0.) GO TO 61
A(I-2*N,J) = TD11M*(C1-C2-C11M*(C3-C4))
A(I-2*N,J+K) = TD21M*(C1-C2+C21M*(C3-C4))
A1N = -TD11M*EMPS*(C1+C2+C11M*(C3-C4))
A2N = TD11M*(B1MS*(C1+C2)-C11M*G1MS*(C3+C4))
A3N = TD11M*EMP*(B1M*(H1+H2)+C11M*G1M*(H3+H4))

```

```

A(I+5*K,J) = -(TD31*PHI1S*A1N+TD32*A2N+TD33*PHI1*A3N)
A1N = -TD21M*EMPS*(C1+C2+C21M*(C3+C4))
A2N = TD21M*(B1MS*(C1+C2)-C21M*G1MS*(C3+C4))
A3N = TD21M*EMP*(B1M*(H1+H2)+C21M*G1M*(H3+H4))
A(I+5*K,J-K) = -(TD31*PHI1S*A1N+TD32*A2N+TD33*PHI1*A3N)
A(I+6*K,J) = TD11M*EMP*DCOS(EMP)*(DS1+C11M*DS2)
A(I+6*K,J-K) = TD21M*EMP*DCOS(EMP)*(DS1+C21M*DS2)
A1N = -TD11M*EMPS*EMP*DCOS(EMP)*(DS1+C11M*DS2)
A2N = TD11M*EMP*DCOS(EMP)*(B1MS*DS1-C11M*G1MS*DS2)
A(I+7*K,J) = -(A1N+POIS*PHI3S*A2N)
A1N = -TD21M*EMPS*EMP*DCOS(EMP)*(DS1+C21M*DS2)
A2N = TD21M*EMP*DCOS(EMP)*(B1MS*DS1-C21M*G1MS*DS2)
A(I+7*K,J+K) = -(A1N+POIS*PHI3S*A2N)
GO TO 62

```

*
*
*

This step will generate elements with ALMDS < EMPS

```

61 A(I+2*K,J) = TD12M*(C1+C2+C12M*(C5+C6))
A(I+2*K,J-K) = TD22M*(C1+C2+C22M*(C5+C6))
A1N = -TD12M*EMPS*(C1+C2+C12M*(C5+C6))
A2N = TD12M*(B1MS*(C1+C2)+C12M*G1MS*(C5+C6))
A3N = TD12M*EMP*(B1M*(H1+H2)+C12M*G1M*(H5+H6))
A(I+5*K,J) = -(TD31*PHI1S*A1N+TD32*A2N+TD33*PHI1*A3N)
A1N = -TD22M*EMPS*(C1+C2+C22M*(C5+C6))
A2N = TD22M*(B1MS*(C1+C2)+C22M*G1MS*(C5+C6))
A3N = TD22M*EMP*(B1M*(H1+H2)+C22M*G1M*(H5+H6))
A(I+5*K,J-K) = -(TD31*PHI1S*A1N+TD32*A2N+TD33*PHI1*A3N)
A(I+6*K,J) = TD12M*EMP*DCOS(EMP)*(DS1+C12M*DS3)
A(I+6*K,J-K) = TD22M*EMP*DCOS(EMP)*(DS1+C22M*DS3)
A1N = -TD12M*EMPS*EMP*DCOS(EMP)*(DS1+C12M*DS3)
A2N = TD12M*EMP*DCOS(EMP)*(B1MS*DS1-C12M*G1MS*DS3)
A(I+7*K,J) = -(A1N+POIS*PHI3S*A2N)
A1N = -TD22M*EMPS*EMP*DCOS(EMP)*(DS1+C22M*DS3)
A2N = TD22M*EMP*DCOS(EMP)*(B1MS*DS1-C22M*G1MS*DS3)
A(I+7*K,J-K) = -(A1N+POIS*PHI3S*A2N)

```

*
*
*

This step will generate elements with PHI1S*ALMDS > EMPS

```

62 IF (TEST3M.LT.0.) GO TO 63
A(I+2*K,J-2*K) = TD31M*(D1+D2-C31M*(D3-D4))
A(I+2*K,J-3*K) = TD41M*(D1+D2-C41M*(D3-D4))
A1N = TD31M*(B3MS*(D1+D2)-C31M*G3MS*(D3-D4))
A2N = -TD31M*EMPS*(D1+D2+C31M*(D3-D4))
A3N = TD31M*EMP*(B3M*(P1+P2)+C31M*G3M*(P3-P4))
A(I+5*K,J-2*K) = -(TD31*PHI1S*A1N+TD32*A2N+TD33*PHI1*A3N)
A1N = TD41M*(B3MS*(D1+D2)-C41M*G3MS*(D3-D4))
A2N = -TD41M*EMPS*(D1+D2+C41M*(D3-D4))

```

```

A3N      = TD41M*EMP*(B3M*(P1+P2)+C41M*G3M*(P3+P4))
A(I+5*K,J+3*K) = -(TD31*PHI1S*A1N+TD32*A2N-TD33*PHI1*A3N)
A(J+6*K,J+2*K) = TD31M*(B3M*DCOSH(B3M)+C31M*G3M*DCOS(G3M))
A(J+6*K,J+3*K) = TD41M*(B3M*DCOSH(B3M)+C41M*G3M*DCOS(G3M))
A1N      = TD31M*(B3MS*B3M*DCOSH(B3M)-C31M*G3MS*G3M*DCOS(G3M))
1
A2N      = -TD31M*EMPS*(B3M*DCOSH(B3M)-C31M*G3M*DCOS(G3M))
A(J+7*K,J+2*K) = -(A1N+POIS*PHI3S*A2N)
A1N      = TD41M*(B3MS*B3M*DCOSH(B3M)-C41M*G3MS*G3M*DCOS(G3M))
1
A2N      = -TD41M*EMPS*(B3M*DCOSH(B3M)-C41M*G3M*DCOS(G3M))
A(J+7*K,J+3*K) = -(A1N+POIS*PHI3S*A2N)
GO TO 64

```

```

*
*   This step will generate elements with PHI1S*ALMDS < EMPS
*

```

```

63 A(I+2*K,J+2*K) = TD32M*(D1-D2-C32M*(D5-D6))
A(I+2*K,J+3*K) = TD42M*(D1-D2-C42M*(D5+D6))
A1N      = TD32M*(B3MS*(D1-D2)+C32M*G3MS*(D5-D6))
A2N      = -TD32M*EMPS*(D1-D2+C32M*(D5+D6))
A3N      = TD32M*EMP*(B3M*(P1+P2)+C32M*G3M*(P5+P6))
A(I+5*K,J+2*K) = -(TD31*PHI1S*A1N+TD32*A2N-TD33*PHI1*A3N)
A1N      = TD42M*(B3MS*(D1+D2)+C42M*G3MS*(D5+D6))
A2N      = -TD42M*EMPS*(D1+D2+C42M*(D5+D6))
A3N      = TD42M*EMP*(B3M*(P1+P2)+C42M*G3M*(P5+P6))
A(I+5*K,J+3*K) = -(TD31*PHI1S*A1N+TD32*A2N-TD33*PHI1*A3N)
IF (B3MS.GT.QLIM) GO TO 255
A(J+6*K,J+2*K) = TD32M*(B3M*DCOSH(B3M)-C32M*G3M*DCOSH(G3M))
A(J+6*K,J+3*K) = TD42M*(B3M*DCOSH(B3M)-C42M*G3M*DCOSH(G3M))
A1N      = TD32M*(B3MS*B3M*DCOSH(B3M)-C32M*G3MS*G3M*DCOSH(G3M))
1
A2N      = -TD32M*EMPS*(B3M*DCOSH(B3M)-C32M*G3M*DCOSH(G3M))
A(J+7*K,J+2*K) = -(A1N+POIS*PHI3S*A2N)
A1N      = TD42M*(B3MS*B3M*DCOSH(B3M)-C42M*G3MS*G3M*DCOSH(G3M))
1
A2N      = -TD42M*EMPS*(B3M*DCOSH(B3M)-C42M*G3M*DCOSH(G3M))
A(J+7*K,J+3*K) = -(A1N+POIS*PHI3S*A2N)
GO TO 64
255 A(J+6*K,J+2*K) = TD32M*(B3M-C32M*G3M)
A(J+6*K,J+3*K) = TD42M*(B3M-C42M*G3M)
A1N      = TD32M*(B3MS*B3M-C32M*G3MS*G3M)
A2N      = -TD32M*EMPS*(B3M+C32M*G3M)
A(J+7*K,J+2*K) = -(A1N+POIS*PHI3S*A2N)
A1N      = TD42M*(B3MS*B3M-C42M*G3MS*G3M)
A2N      = -TD42M*EMPS*(B3M+C42M*G3M)
A(J+7*K,J+3*K) = -(A1N+POIS*PHI3S*A2N)

```

THIS STEP WILL GENERATE ELEMENTS WITH ALMDPS. > EMPS

```

64      IF (TEST5M.LT.0.) GO TO 65
        A(I,J-4*K) = TD51M*(A1+A2+C51M*(A3+A4))
        A(I,J-5*K) = TD61M*(A1+A2+C61M*(A3+A4))
        A(I+K,J-4*K) = TD51M*(B1+B2+C51M*(B3+B4))
        A(I+K,J-5*K) = TD61M*(B1+B2+C61M*(B3+B4))
        A1N = -TD51M*EMPS*(A1-A2+C51M*(A3+A4))
        A2N = TD51M*(B5MS*(A1-A2)-C51M*G5MS*(A3+A4))
        A3N = -TD51M*EMP*(B5M*(E1+E2)+C51M*G5M*(E3+E4))
        A(I+3*K,J-4*K) = -(TD11*PHI5S*A1N+TD12*A2N+TD13*PHI5*A3N)/F2
        A1N = -TD61M*EMPS*(A1-A2+C61M*(A3+A4))
        A2N = TD61M*(B5MS*(A1+A2)-C61M*G5MS*(A3+A4))
        A3N = -TD61M*EMP*(B5M*(E1+E2)+C61M*G5M*(E3+E4))
        A(I+3*K,J-5*K) = -(TD11*PHI5S*A1N+TD12*A2N+TD13*PHI5*A3N)/F2
        A1N = -TD51M*EMPS*(B1-B2+C51M*(B3+B4))
        A2N = TD51M*(B5MS*(B1+B2)-C51M*G5MS*(B3+B4))
        A3N = -TD51M*EMP*(B5M*(G1+G2)+C51M*G5M*(G3+G4))
        A(I+4*K,J-4*K) = -(TD21*PHI5S*A1N+TD22*A2N+TD23*PHI5*A3N)/F2
        A1N = -TD61M*EMPS*(B1+B2+C61M*(B3+B4))
        A2N = TD61M*(B5MS*(B1+B2)-C61M*G5MS*(B3+B4))
        A3N = -TD61M*EMP*(B5M*(G1+G2)+C61M*G5M*(G3+G4))
        A(I+4*K,J-5*K) = -(TD21*PHI5S*A1N+TD22*A2N+TD23*PHI5*A3N)/F2
        A1N = TD51M*EMP*(ES1+ES2+C51M*(ES3-ES4))
        A2N = -TD51M*(B5M*(FS1+FS2)+C51M*G5M*(FS3+FS4))
        A(I+6*K,J-4*K) = F21*A1N+F22*A2N
        A1N = TD61M*EMP*(ES1+ES2+C61M*(ES3-ES4))
        A2N = -TD61M*(B5M*(FS1+FS2)+C61M*G5M*(FS3+FS4))
        A(I+6*K,J-5*K) = F21*A1N+F22*A2N
        A1N = -TD51M*EMPS*EMP*(ES1-ES2+C51M*(ES3-ES4))
        A2N = -TD51M*(B5MS*B5M*(FS1+FS2)-C51M*G5MS*G5M*(FS3+
        FS4))
        A3N = TD51M*EMPS*(B5M*(FS1+FS2)+C51M*G5M*(FS3+FS4))
        A4N = TD51M*EMP*(B5MS*(ES1+ES2)-C51M*G5MS*(ES3-ES4))
        A(I+7*K,J-4*K) = -(V1*A1N+V2*A2N+V3*A3N+V4*A4N)*F2*DSIN(AL)
        A1N = -TD61M*EMPS*EMP*(ES1+ES2+C61M*(ES3-ES4))
        A2N = -TD61M*(B5MS*B5M*(FS1+FS2)-C61M*G5MS*G5M*(FS3+
        FS4))
        A3N = TD61M*EMPS*(B5M*(FS1+FS2)+C61M*G5M*(FS3+FS4))
        A4N = TD61M*EMP*(B5MS*(ES1+ES2)-C61M*G5MS*(ES3-ES4))
        A(I+7*K,J-5*K) = -(V1*A1N+V2*A2N+V3*A3N+V4*A4N)*F2*DSIN(AL)
        GO TO 66

```

THIS STEP WILL GENERATE ELEMENTS WITH ALMDPS < EMPS

```

65      A(I,J-4*K) = TD52M*(A1+A2+C52M*(A5+A6))
        A(I,J-5*K) = TD62M*(A1+A2+C62M*(A5+A6))

```

```

A(I+K,J+4*K) = TD52M*(B1+B2+C52M*(B5+B6))
A(I+K,J+5*K) = TD62M*(B1+B2+C62M*(B5+B6))
A1N          = -TD52M*EMPS*(A1+A2+C52M*(A5-A6))
A2N          = TD52M*(B5MS*(A1-A2)+C52M*G5MS*(A5-A6))
A3N          = -TD52M*EMP*(B5M*(E1+E2)+C52M*G5M*(E5+E6))
A(I+3*K,J+4*K) = -(TD11*PHI5S*A1N+TD12*A2N-TD13*PHI5*A3N)/F2
A1N          = -TD62M*EMPS*(A1+A2+C62M*(A5-A6))
A2N          = TD62M*(B5MS*(A1-A2)+C62M*G5MS*(A5-A6))
A3N          = -TD62M*EMP*(B5M*(E1+E2)+C62M*G5M*(E5+E6))
A(I+3*K,J+5*K) = -(TD11*PHI5S*A1N+TD12*A2N-TD13*PHI5*A3N)/F2
A1N          = -TD52M*EMPS*(B1+B2+C52M*(B5+B6))
A2N          = TD52M*(B5MS*(B1+B2)+C52M*G5MS*(B5+B6))
A3N          = -TD52M*EMP*(B5M*(G1+G2)+C52M*G5M*(G5+G6))
A(I+4*K,J+4*K) = -(TD21*PHI5S*A1N+TD22*A2N-TD23*PHI5*A3N)/F2
A1N          = -TD62M*EMPS*(B1+B2+C62M*(B5+B6))
A2N          = TD62M*(B5MS*(B1+B2)+C62M*G5MS*(B5+B6))
A3N          = -TD62M*EMP*(B5M*(G1+G2)+C62M*G5M*(G5+G6))
A(I+4*K,J+5*K) = -(TD21*PHI5S*A1N+TD22*A2N-TD23*PHI5*A3N)/F2
A1N          = TD52M*EMP*(ES1+ES2+C52M*(ES5+ES6))
A2N          = -TD52M*(B5M*(FS1+FS2)+C52M*G5M*(FS5+FS6))
A(I+6*K,J+4*K) = F21*A1N+F22*A2N
A1N          = TD62M*EMP*(ES1-ES2+C62M*(ES5+ES6))
A2N          = -TD62M*(B5M*(FS1+FS2)+C62M*G5M*(FS5+FS6))
A(I+6*K,J+5*K) = F21*A1N+F22*A2N
A1N          = -TD52M*EMPS*EMP*(ES1+ES2+C52M*(ES5+ES6))
A2N          = -TD52M*(B5MS*B5M*(FS1+FS2)+C52M*G5MS*G5M*(FS5+
1          FS6))
A3N          = TD52M*EMPS*(B5M*(FS1+FS2)+C52M*G5M*(FS5+FS6))
A4N          = TD52M*EMP*(B5MS*(ES1+ES2)+C52M*G5MS*(ES5+ES6))
A(I+7*K,J+4*K) = -(V1*A1N+V2*A2N+V3*A3N+V4*A4N)/F2*DSIN(AL)
A1N          = -TD62M*EMPS*EMP*(ES1+ES2+C62M*(ES5+ES6))
A2N          = -TD62M*(B5MS*B5M*(FS1+FS2)+C62M*G5MS*G5M*(FS5+
1          FS6))
A3N          = TD62M*EMPS*(B5M*(FS1+FS2)+C62M*G5M*(FS5+FS6))
A4N          = TD62M*EMP*(B5MS*(ES1+ES2)+C62M*G5MS*(ES5+ES6))
A(I+7*K,J+5*K) = -(V1*A1N+V2*A2N+V3*A3N+V4*A4N)/F2*DSIN(AL)
66 IF (TESTRM.LT.0.) GO TO 67
   IF (MODE.GT.0.) GO TO 68
*
*   This step will generate elements with PHIRS*ALMDSR > EMPS
*
A(J+K,J+6*K) = -1
A(J+K,J+7*K) = -DCOS(GRM)/DCOSH(GRM)
A1N          = BRMS
A2N          = -EMPS
A(J+4*K,J+6*K) = (PHIRS*A1N+POI*A2N)
A1N          = -GRMS*DCOS(GRM)/DCOSH(GRM)

```

```

A2N = -EMPS*DCOS(GRM)/DCOSH(GRM)
A(J+4*K,J+7*K) = (PHIRS*A1N+POI*A2N)
A(J+6*K,J+6*K) = BRM*DSINH(BRM)*PHIR/(PHI1*DCOSH(BRM))
A(J+6*K,J+7*K) = -GRM*DSIN(GRM)*PHIR/(PHI1*DCOSH(GRM))
A1N = -BRMS*BRM*DSINH(BRM)/DCOSH(BRM)
A2N = EMPS*BRM*DSINH(BRM)/DCOSH(BRM)
A(J+7*K,J+6*K) = (A1N+(POIS*A2N*PHIRS))*PHIRS*PHIR/(PHI1S*PHI1)
A1N = -GRMS*GRM*DSIN(GRM)/DCOSH(GRM)
A2N = -EMPS*GRM*DSIN(GRM)/DCOSH(GRM)
A(J+7*K,J+7*K) = (A1N+(POIS*A2N*PHIRS))*PHIRS*PHIR/(PHI1S*PHI1)
GO TO 100
68 A(J+K,J+6*K) = -1
A(J+K,J+7*K) = -DSIN(GRM)/DSINH(GRM)
A1N = BRMS
A2N = -EMPS
A(J+4*K,J+6*K) = (PHIRS*A1N+POI*A2N)
A1N = -GRMS*DSIN(GRM)/DSINH(GRM)
A2N = -EMPS*DSIN(GRM)/DSINH(GRM)
A(J+4*K,J+7*K) = (PHIRS*A1N+POI*A2N)
A(J+6*K,J+6*K) = BRM*DCOSH(BRM)*PHIR/(PHI1*DSINH(BRM))
A(J+6*K,J+7*K) = GRM*DCOS(GRM)*PHIR/(PHI1*DSINH(GRM))
A1N = -BRMS*BRM*DCOSH(BRM)/DSINH(BRM)
A2N = EMPS*BRM*DCOSH(BRM)/DSINH(BRM)
A(J+7*K,J+6*K) = (A1N+(POIS*A2N*PHIRS))*PHIRS*PHIR/(PHI1S*PHI1)
A1N = GRMS*GRM*DCOS(GRM)/DSINH(GRM)
A2N = EMPS*GRM*DCOS(GRM)/DSINH(GRM)
A(J+7*K,J+7*K) = (A1N+(POIS*A2N*PHIRS))*PHIRS*PHIR/(PHI1S*PHI1)
GO TO 100

```

*
*
*

This step will generate elements with PHIRS*ALMDRS < EMPS

```

67 IF (BRMS.GT.OLIM) GO TO 260
IF (MODE.GT.0.) GO TO 69
A(J-K,J+6*K) = -1
A(J-K,J+7*K) = -1
A1N = BRMS
A2N = -EMPS
A(J+4*K,J+6*K) = (PHIRS*A1N+POI*A2N)
A1N = GRMS
A2N = -EMPS
A(J+4*K,J+7*K) = (PHIRS*A1N+POI*A2N)
A(J+6*K,J+6*K) = BRM*DSINH(BRM)*PHIR/(PHI1*DCOSH(BRM))
A(J+6*K,J+7*K) = GRM*DSINH(GRM)*PHIR/(PHI1*DCOSH(GRM))
A1N = -BRMS*BRM*DSINH(BRM)/DCOSH(BRM)
A2N = EMPS*BRM*DSINH(BRM)/DCOSH(BRM)
A(J+7*K,J+6*K) = (A1N+(POIS*A2N*PHIRS))*PHIRS*PHIR/(PHI1S*PHI1)
A1N = -GRMS*GRM*DSINH(GRM)/DCOSH(GRM)

```

```

A2N      = EMPS*GRM*DSINH(GRM)/DCOSH(GRM)
A(J+7*K,J+7*K) = (A1N+(POIS*A2N/PHIRS))*PHIRS*PHIR / (PHI1S*PHI1)
GO TO 100.
69 A(J+K,J+6*K) = -1
A(J+K,J+7*K) = -1
A1N      = BRMS
A2N      = -EMPS
A(J+4*K,J+6*K) = (PHIRS*A1N+POI*A2N)
A1N      = GRMS
A2N      = -EMPS
A(J+4*K,J+7*K) = (PHIRS*A1N+POI*A2N)
A(J+6*K,J+6*K) = BRM*DCOSH(BRM)*PHIR/(PHI1*DSINH(BRM))
A(J+6*K,J+7*K) = GRM*DCOSH(GRM)*PHIR/(PHI1*DSINH(GRM))
A1N      = -BRMS*BRM*DCOSH(BRM)/DSINH(BRM)
A2N      = EMPS*BRM*DCOSH(BRM)/DSINH(BRM)
A(J+7*K,J+6*K) = (A1N+(POIS*A2N/PHIRS))*PHIRS*PHIR / (PHI1S*PHI1)
A1N      = -GRMS*GRM*DCOSH(GRM)/DSINH(GRM)
A2N      = EMPS*GRM*DCOSH(GRM)/DSINH(GRM)
A(J+7*K,J+7*K) = (A1N+(POIS*A2N/PHIRS))*PHIRS*PHIR / (PHI1S*PHI1)
GO TO 100

*
*   If MODE > 0., the program will switch to anti-symmetric modes
*
260 IF (MODE.GT.0.) GO TO 70
A(J+K,J+6*K) = -1
A(J+K,J+7*K) = -1
A1N      = BRMS
A2N      = -EMPS
A(J+4*K,J+6*K) = (PHIRS*A1N+POI*A2N)
A1N      = GRMS
A2N      = -EMPS
A(J+4*K,J+7*K) = (PHIRS*A1N+POI*A2N)
A(J+6*K,J+6*K) = BRM*PHIR/PHI1
A(J+6*K,J+7*K) = GRM*PHIR/PHI1
A1N      = -BRMS*BRM
A2N      = EMPS*BRM
A(J+7*K,J+6*K) = (A1N+(POIS*A2N/PHIRS))*PHIRS*PHIR / (PHI1S*PHI1)
A1N      = -GRMS*GRM
A2N      = EMPS*GRM
A(J+7*K,J+7*K) = (A1N+(POIS*A2N/PHIRS))*PHIRS*PHIR / (PHI1S*PHI1)
GO TO 100
70 A(J+K,J+6*K) = -1
A(J+K,J+7*K) = -1
A1N      = BRMS
A2N      = -EMPS
A(J+4*K,J+6*K) = (PHIRS*A1N+POI*A2N)
A1N      = GRMS

```

```

A2N      = -EMPS
A(J+4*K,J+7*K) = (PHIRS*A1N+POI*A2N)
A(J+6*K,J+6*K) = BRM*PHIR/PHI1
A(J+6*K,J+7*K) = GRM*PHIR/PHI1
A1N      = -BRMS*BRM
A2N      = EMPS*BRM
A(J+7*K,J+6*K) = (A1N+(POIS*A2N/PHIRS))*PHIRS*PHIR / (PHI1S*PHI1)
A1N      = -GRMS*GRM
A2N      = EMPS*GRM
A(J+7*K,J+7*K) = (A1N+(POIS*A2N/PHIRS))*PHIRS*PHIR / (PHI1S*PHI1)
100 CONTINUE
I=8*K

```

```

*
*   If DEL=0., the program will switch to shape data generation
*

```

```

IF (DEL.EQ.0.) GO TO 103
CALL DETERM (A,I,DET)
WRITE(6,101) ALMDS,DET
101 FORMAT(' ',5X,'ALMDS =',F10.6,5X,'DET =',D15.8,)
IF (ALMDS.LT.DLIM) GO TO 102
IF (ALMDS.GT.0.001) GO TO 90
GO TO 400

```

```

102 ALMDS=ALMDS+DEL
GO TO 2

```

```

103 CALL DETSOL (A,I,X)
X(8*K)=-1
DO 104 I=1,K
E1M(I)=X(I)
V2M(I)=X(I+K)
E3M(I)=X(I+2*K)
V4M(I)=X(I+3*K)
E5M(I)=X(I+4*K)
V6M(I)=X(I+5*K)
AM(I)=X(I+6*K)
ITEST=8*K

```

```

IF (I.EQ.ITEST) GO TO 104
DM(I)=X(I+7*K)

```

```

104 CONTINUE
DM(K)=-1.
L=2*K+1

```

```

DO 105 I=1,KS1
DO 105 J=1,L

```

```

105 W(I,J)=0.

```

```

*
*   THIS PORTION OF THE PROGRAM WILL GENERATE SHAPE DATA
*

```

```

ALMDPS=ALMDS/F2

```

```

ALMDRS=ALMDS*PHI1S/PHIRS
ETA=0.
DO 309 I=1,KS1
  PSI=1.
  DO 308 J=1,I
    PSIP=PSI*DSIN(AL)*DSIN(AL)-(1.-ETA/DCOS(AL))*DCOS(AL)
    ETAP=ETA-1-PSI
    W11=0.
    W22=0.
    W33=0.
    W44=0.
    W55=0.
    W66=0.
  DO 307 M=1,K
    EMP=M*PI
    EMPS=EMP*EMP
    B1MS=PHI1S*(ALMDS+EMPS)
    B1M=DSQRT(B1MS)
    TEST1M=ALMDS-EMPS
    G1MS=PHI1S*TEST1M
    IF (G1MS.LT.0.) G1MS=-G1MS
    G1M=DSQRT(G1MS)
    B3MS=PHI3S*(PHI1S*ALMDS+EMPS)
    B3M=DSQRT(B3MS)
    TEST3M=PHI1S*ALMDS-EMPS
    G3MS=PHI3S*TEST3M
    IF (G3MS.LT.0.) G3MS=-G3MS
    G3M=DSQRT(G3MS)
    B5MS=PHI5S*(ALMDPS+EMPS)
    B5M=DSQRT(B5MS)
    TEST5M=ALMDPS-EMPS
    G5MS=PHI5S*TEST5M
    IF (G5MS.LT.0.) G5MS=-G5MS
    G5M=DSQRT(G5MS)
    IF (TEST1M.LT.0.) GO TO 270
    C11M=-DSINH(B1M),DSIN(G1M)
    TD11M=-1./((B1MS*DSINH(B1M)-C11M*G1MS*DSIN(G1M))
    C21M=(B1MS-POI*PHI1S*EMPS)*DSINH(B1M)/((G1MS-POI*PHI1S*EMPS)*
    DSIN(G1M))
    TD21M=1.-DSINH(B1M)+C21M*DSIN(G1M)
    KW1=B1M(M)*TD11M*(DSINH(B1M*ETA)+C11M*DSIN(G1M*ETA))*DSIN(EMP*PSI)
    KW2=V2M(M)*TD21M*(DSINH(B1M*ETA)+C21M*DSIN(G1M*ETA))*DSIN(EMP*PSI)
    GO TO 274
270 IF (B1MS.GT.QLIM) GO TO 271
    C12M=-DSINH(B1M)/DSIN(G1M)
    TD12M=-1./((B1MS*DSINH(B1M)-C12M*G1MS*DSIN(G1M))
    C22M=-((B1MS-POI*PHI1S*EMPS)*DSINH(B1M)/((G1MS-POI*PHI1S*EMPS)*

```

```

1DSINH(G1M))
TD22M=1./ (DSINH(B1M)+C22M*DSINH(G1M))
KW1=E1M(M)*TD12M*(DSINH(B1M*ETA)+C12M*DSINH(G1M*ETA))*DSIN(EMP*
1PSI)
KW2=V2M(M)*TD22M*(DSINH(B1M*ETA)+C22M*DSINH(G1M*ETA))*DSIN(EMP*
1PSI)
GO TO 274
271 C12M=-1.
TD12M=-1./ (B1MS+C12M*G1MS)
C22M=-(B1MS-POI*PHI1S*EMPS)/(G1MS-POI*PHI1S*EMPS)
TD22M=1./ (1+C22M)
B=1.
TEST=B1M-B1M*ETA
IF (TEST.GT.60.) B=0.
A1M=DEXP((B1M*ETA-B1M)*B)*B
A2M=DEXP((G1M*ETA-G1M)*B)*B
KW1=E1M(M)*TD12M*(A1M-C12M*A2M)*DSIN(EMP*PSI)
KW2=V2M(M)*TD22M*(A1M-C22M*A2M)*DSIN(EMP*PSI)
IF (ETA.EQ.0.) KW1=0.
IF (PSI.EQ.0.) KW1=0.
IF (ETA.EQ.0.) KW2=0.
IF (PSI.EQ.0.) KW2=0.
274 IF (TEST3M.LT.0.) GO TO 275
C31M=-DSINH(B3M)/DSIN(G3M)
TD31M=-1./ (B3MS*DSINH(B3M)-C31M*G3MS*DSIN(G3M))
C41M=(B3MS-POI*PHI3S*EMPS)*DSINH(B3M)/((G3MS-POI*PHI3S*EMPS)*
1DSINH(G3M))
TD41M=1./ (DSINH(B3M)+C41M*DSIN(G3M))
KW3=E3M(M)*TD31M*(DSINH(B3M*PSI)-C31M*DSIN(G3M*PSI))*DSIN(EMP*ETA)
KW4=V4M(M)*TD41M*(DSINH(B3M*PSI)+C41M*DSIN(G3M*PSI))*DSIN(EMP*ETA)
GO TO 279
275 IF (B3MS.GT.QLIM) GO TO 276
C32M=-DSINH(B3M)/DSIN(G3M)
TD32M=-1./ (B3MS*DSINH(B3M)+C32M*G3MS*DSIN(G3M))
C42M=-(B3MS-POI*PHI3S*EMPS)*DSINH(B3M)/((G3MS-POI*PHI3S*EMPS)*
1DSINH(G3M))
TD42M=1./ (DSINH(B3M)-C42M*DSIN(G3M))
KW3=E3M(M)*TD32M*(DSINH(B3M*PSI)+C32M*DSIN(G3M*PSI))*DSIN(EMP*
1ETA)
KW4=V4M(M)*TD42M*(DSINH(B3M*PSI)+C42M*DSIN(G3M*PSI))*DSIN(EMP*
1ETA)
GO TO 279
276 C32M=-1.
TD32M=-1./ (B3MS+C32M*G3MS)
C42M=-(B3MS-POI*PHI3S*EMPS)/(G3MS-POI*PHI3S*EMPS)
TD42M=1./ (1+C42M)
B=1.

```

```

TEST=B3M-B3M*PSI
IF (TEST.GT.60.) B=0.
A1M=DEXP((B3M*PSI-B3M)*B)*B
A2M=DEXP((G3M*PSI-G3M)*B)*B
KW3=E3M(M)*TD32M*(A1M+C32M*A2M)*DSIN(EMP*ETA)
KW4=V4M(M)*TD42M*(A1M+C42M*A2M)*DSIN(EMP*ETA)
IF (ETA.EQ.0.) KW3=0.
IF (PSI.EQ.0.) KW3=0.
IF (ETA.EQ.0.) KW4=0.
IF (PSI.EQ.0.) KW4=0.
279 XX1=B5M-B5M*ETAP
XX2=G5M-G5M*ETAP
IF (TEST5M.LT.0.) GO TO 280
C51M=-DSINH(B5M)/DSINH(G5M)
TD51M=-1./ (B5MS*DSINH(B5M)-C51M*G5MS*DSINH(G5M))
C61M=(B5MS-POI*PHI5S*EMPS)*DSINH(B5M)/ ((G5MS-POI*PHI5S*EMPS)*
1DSINH(G5M))
TD61M=1./ (DSINH(B5M)+C61M*DSINH(G5M))
KW5=E5M(M)*TD51M*(DSINH(XX1)-C51M*DSINH(XX2))*DSIN(EMP*PSIP)
KW6=V6M(M)*TD61M*(DSINH(XX1)+C61M*DSINH(XX2))*DSIN(EMP*PSIP)
GO TO 284
280 IF (B5MS.GT.QLIM) GO TO 281
C52M=-DSINH(B5M)/DSINH(G5M)
TD52M=-1./ (B5MS*DSINH(B5M)-C52M*G5MS*DSINH(G5M))
C62M=- (B5MS-POI*PHI5S*EMPS)*DSINH(B5M)/ ((G5MS-POI*PHI5S*EMPS)*
1DSINH(G5M))
TD62M=1./ (DSINH(B5M)+C62M*DSINH(G5M))
KW5=E5M(M)*TD52M*(DSINH(XX1)+C52M*DSINH(XX2))*DSIN(EMP*PSIP)
KW6=V6M(M)*TD62M*(DSINH(XX1)-C62M*DSINH(XX2))*DSIN(EMP*PSIP)
GO TO 284
281 C52M=-1.
TD52M=-1./ (B5MS+C52M*G5MS)
C62M=- (B5MS-POI*PHI5S*EMPS) / (G5MS-POI*PHI5S*EMPS)
TD62M=1./ (1.+C62M)
B=1.
TEST=B5M*ETAP
IF (TEST.GT.60.) B=0.
A1M=DEXP(-B5M*ETAP*B)*B
A2M=DEXP(-G5M*ETAP*B)*B
KW5=E5M(M)*TD52M*(A1M+C52M*A2M)*DSIN(EMP*PSIP)
KW6=V6M(M)*TD62M*(A1M+C62M*A2M)*DSIN(EMP*PSIP)
IF (PSIP.EQ.0.) KW5=0.
IF (PSIP.EQ.0.) KW6=0.
IF (ETAP.EQ.1.) KW5=0.
IF (ETAP.EQ.1.) KW6=0.
IF (ETAP.EQ.0.) KW5=0.
IF (ETAP.EQ.0.) KW6=V6M(M)*DSIN(EMP*PSIP)

```

```

284 W11=W11+XW1
    W22=W22+XW2
    W33=W33+XW3
    W44=W44+XW4
    W55=W55+XW5
    W66=W66+XW6
287 CONTINUE
    W(I,J+KS)=W11+W22+W33+W44+W55+W66
    PSI=PSI-1./DFLOAT(KS)
288 CONTINUE
    ETA=ETA+1./DFLOAT(KS)
289 CONTINUE
    ETA=0.
    DO 315 I=1,KS1
    PSI=0.
    DO 314 J=1,KS1
    W11=0.
    DO 313 M=1,K
    EMP=M*PI
    EMPS=EMP*EMP
    BRMS=(PHIRS*ALMDRS+EMPS)/PHIRS
    BRM=DSQRT(BRMS)
    TESTRM=PHIRS*ALMDRS-EMPS
    GRMS=TESTRM/PHIRS
    IF (GRMS.LT.0.) GRMS=-GRMS
    GRM=DSQRT(GRMS)
    XX1=BRM-BRM*PSI
    XX2=GRM-GRM*PSI
    IF (TESTRM.LT.0.) GO TO 285
    IF (MODE.EQ.1.) GO TO 321
    NW1=((AM(M)*DCOSH(XX1)/DCOSH(BRM))-(DM(M)*DCOSH(XX2)/DCOSH(GRM)))
    1*DSIN(EMP*ETA)
    GO TO 312
321 NW1=((AM(M)*DSINH(XX1)/DSINH(BRM))-(DM(M)*DSIN(XX2)/DSINH(GRM)))
    1*DSIN(EMP*ETA)
    GO TO 312
285 IF (MODE.EQ.1.) GO TO 322
    IF (BRMS.GT.QLIM) GO TO 286
    NW1=((AM(M)*DCOSH(XX1)/DCOSH(BRM))-(DM(M)*DCOSH(XX2)/DCOSH(GRM)))
    1*DSIN(EMP*ETA)
    GO TO 312
286 B=1.
    TEST=BRM*PSI
    IF (TEST.GT.60.) B=0.
    A1M=DEXP(-BRM*PSI*B)*B
    A2M=DEXP(-GRM*PSI*B)*B
    NW1=(AM(M)*A1M+DM(M)*A2M)*DSIN(EMP*ETA)

```

```

      GO TO 312
322 IF (BRMS.GT.QLIM) GO TO 287
      KW1=((AM(M)*DSINH(XX1)/DSINH(BRM))+(DM(M)*DSINH(XX2)/DSINH(GRM)))
      1*DSIN(EMP*ETA)
      GO TO 312
287 B=1.
      TEST=BRM*PSI
      IF (TEST.GT.60.) B=0.
      AIM=DEXP(-BRM*PSI*B)*B
      ADM=DEXP(-GRM*PSI*B)*B
      KW1=(AM(M)*AIM+DM(M)*ADM)*DSIN(EMP*ETA)
      IF (PSI.EQ.1.) KW1=0.
312 W11=W11-KW1
313 CONTINUE
      JREV=KS+1-J
      W(I,JREV)=W11
      PSI=PSI+1./DFLOAT(KS)
314 CONTINUE
      ETA=ETA+1./DFLOAT(KS)
315 CONTINUE
*
*   IF RATIO < 0. the displacement ratio of plate to boundary
*   will not be computed.
*
      IF (RATIO.LT.0) GO TO 318
      SUM=0.
      SUM1=0.
      DO 316 I=1,21
      KK=KS-I
      DO 316 J=1,KK
      IF (I.NE.1) GO TO 421
      SUM1=SUM1+DABS(W(I,J))
421 IF (I.NE.21) GO TO 422
      SUM1=SUM1-DABS(W(I,J))
422 CONTINUE
      SUM=SUM-DABS(W(I,J))
316 CONTINUE
      DO 317 I=2,20
      J=KS-I
      SUM1=SUM1-DABS(W(I,J))
      IF (MODE.EQ.0) GO TO 317
      SUM1=SUM1+DABS(W(I,1))
317 CONTINUE
      SUM=SUM-SUM1
      QK1=81
      IF (MODE.GT.0) QK1=100
      QK2=570

```

```

        IF (MODE.GT.0) QK2=351.
        RAT=SUM*QK1 / (SUM1*QK2)
        WRITE(6,319) RAT
319 FORMAT('1'.10X,'RATIO = '.F13.5, '/')
318 CALL NODE (W)
400 STOP
      END
      SUBROUTINE DETERM (A,N,DET)
      IMPLICIT REAL*8(A-H,C-Z)
      DIMENSION A(130,130)
      SIGN=1.
      LAST=N-1
*
*      Start overall loop for (N-1) pivots
*
      DO 100 I=1,LAST
*
*      Find the largest remaining term in Ith. column for pivot
*
      BIG=0.
      DO 30 K=I,N
      TERM=DABS(A(K,I))
      IF (TERM-BIG)50,50,30
30  BIG=TERM
      L=K
50  CONTINUE
*
*      Check whether a non-zero term has been found
*
      IF (BIG)60,60,30
*
*      Lth. row has the biggest term----is I=L
*
80  IF (I-L)90,120,90
*
*      I is not equal to L,switch rows I and L
90  SIGN=-SIGN
      DO 100 J=1,N
      TEMP=A(I,J)
      A(I,J)=A(L,J)
100  A(L,J)=TEMP
*
*      Now start pivotal reduction
*
120  PIVOT=A(I,I)
      NEXTI=I+1
*

```

```

*   For each of the rows after the Ith.
DO 200 J=NEXTR,N
*
*   Multiplying constant for the Jth. row is
CONST=A(I,I)/PIVOT
*
*   Now reduce each term of the Jth. row
DO 200 K=I,N
200 A(J,K)=A(J,K)-CONST*A(I,K)
*
*   End of pivotal reduction---now compute determinant
DET=SIGN
DO 300 I=1,N
300 DET=DET*A(I,I)*10.
GO TO 61
60 DET=0.
61 RETURN
END
SUBROUTINE DETSOL (A,N,X)
IMPLICIT REAL*8(A-H,O-Z)
DIMENSION A(130,130),X(130)
SIGN=1.
M=N-1
LAST=M-1
*
*   Start overall loop for (N-1) pivots
DO 200 I=1,LAST
*
*   Find the largest remaining term in Ith. column for pivot
BIG=0.
DO 50 K=I,M
TERM=DABS(A(K,I))
IF (TERM-BIG)50,50,30
30 BIG=TERM
L=K
50 CONTINUE
*
*   Check whether a non-zero term has been found
IF (BIG)60,60,60
*
*   Ith. row has the biggest term-----is I=L

```

```

*
80  IF (I-L)90,120,90
*
*      I is not equal to L, switch rows I and L
*
90  DO 100 J=1,N
    TEMP=A(I,J)
    A(I,J)=A(L,J)
100  A(L,J)=TEMP
*
*      Now start pivotal reduction
*
120  PIVOT=A(I,I)
    NENTR=I+1
*
*      For each of the rows after the Ith.
*
DO 200 J=NENTR,M
*
*      Multiplying constant for the Jth. row is;
*
CONST=A(J,I)/PIVOT
*
*      Now reduce each term of the Jth. row
*
DO 200 K=I,N
200  A(J,K)=A(J,K)-CONST*A(I,K)
*
*      End of pivotal reduction-- perform back substitution
*
M=M-1
DO 500 I=1,M
*
*      IREV is the backward index, going from BACK to 1
*
IREV=M-1-I
*
*      Get Y(IREV) in preparation
*
Y=A(IREV,N)
IF (IREV-M) 400,500,400
*
*      Not working on last row, I is 2 or greater
*
400  DO 450 J=2,I
*
*      Work backward for X(N),X(N-1)-----substituting previously

```

```

*      found values
*
      K=N+1-J
450  Y=Y-A(IREV,K)*X(K)
*
*      Finally, compute X(IREV)
*
500  X(IREV)=Y/A(IREV,IREV)
60  RETURN
END
SUBROUTINE NODE(W)
  IMPLICIT REAL*8(A-H,O-Z)
  DIMENSION W(21,41)
  CHARACTER*1 SIGN(21,41),PLUS,MINUS,ZERO,STAR
  PLUS = '+'
  ZERO = '0'
  MINUS = '-'
  STAR = '*'
  N=21
  K=2*N-1
  DO 6 I=1,N
    DO 6 J=1,K
      6 SIGN(I,J)=' '
    DO 1 I=1,N
      II=I-N-1
      DO 1 J=1,II
        IF (W(I,J).LE.0) GO TO 11
        SIGN(I,J)=PLUS
      11 IF (W(I,J).GE.0) GO TO 12
        SIGN(I,J)=MINUS
      12 IF (W(I,J).NE.0) GO TO 1
        SIGN(I,J)=ZERO
      1 CONTINUE
      DO 2 I=1,N
        II=I-N-1
        SIGN(I,II)=STAR
        SIGN(II,I)=STAR
      2 CONTINUE
      DO 3 I=1,K
        3 SIGN(N,I)=STAR
      DO 7 I=1,N
        7 SIGN(1,I)=STAR
      DO 4 I=1,N
        4 WRITE(6,5) (SIGN(I,J),J=1,K)
      5 FORMAT(' ',41A1)
      RETURN
    END

```

Program II, TCLS

```

*
*
*          Program II
*
*          TCLS
*
*
*
*
*      IMPLICIT REAL*8(A-H,O-Z)
*      DIMENSION A(160,160),W(21,41),E1M(16),V2M(16),E3M(16),V4M(16),
*      E5M(16),V6M(16),AM(16),DM(16),E1RM(16),E2RM(16),X(160)
*
*      Define problem parameters
*
*      DO 1000 I=1,160
*1000 N(I)=0.
*      POI=0.3
*      POIS=2-POI
*      QLIM=3600.
*      KS=20
*      KS1=KS-1
*90 WRITE(6,91)
*91 FORMAT('  ALMDS      DLIM      DEL      PHI1      PHIR      K')
*      WRITE(6,94)
*      READ(5,92) ALMDS,DLIM,DEL,PHI1,PHIR,K
*92 FORMAT(F9.5,2X,F8.4,2X,F7.5,2X,F7.5,2X,F7.4,3X,I2)
*
*      If ALMDS < 0., program will terminate.
*
*      IF( ALMDS.LT.0.001)GO TO 400
*      WRITE(6,93) ALMDS,DLIM,DEL,PHI1,PHIR,K
*93 FORMAT('  ',F9.5,2X,F8.4,2X,F7.5,2X,F7.5,2X,F7.4,3X,I2,/)
*94 FORMAT('  ',XXX.XXXXXX XXX.XXXXX XXX.XXXXX XXX.XXXXX XXX.XXXXX XXX')
*      MODE=1.
*      RATIO= 1
*      EDGE=-3
*      C=1.
*      PI=4.*DATAN(C)
*      PHIN=1./PHI1

```

```

AL=DATAN(PHIN)
PHI3=1./PHI1
PHI5=DCOS(AL)*DCOS(AL)/PHI1
PHI1S=PHI1*PHI1
PHI3S=PHI3*PHI3
PHI5S=PHI5*PHI5
PHI5C=PHI5S*PHI5
PHIR=PHIR*PHIR
F1=DCOS(AL)*DCOS(AL)
F2=DSIN(AL)*DSIN(AL)
AL1=PI-AL
AL2=(PI/2.)-AL
AL3=PI+AL
TD11=DCOS(AL1)*DCOS(AL1)-POI*DSIN(AL1)*DSIN(AL1)
TD12=DSIN(AL1)*DSIN(AL1)+POI*DCOS(AL1)*DCOS(AL1)
TD13=(1-POI)*DSIN(2.*AL1)
TD21=DCOS(AL2)*DCOS(AL2)-POI*DSIN(AL2)*DSIN(AL2)
TD22=DSIN(AL2)*DSIN(AL2)+POI*DCOS(AL2)*DCOS(AL2)
TD23=(1-POI)*DSIN(2.*AL2)
TD31=DCOS(AL3)*DCOS(AL3)-POI*DSIN(AL3)*DSIN(AL3)
TD32=DSIN(AL3)*DSIN(AL3)+POI*DCOS(AL3)*DCOS(AL3)
TD33=(1-POI)*DSIN(2.*AL3)
V1=DCOS(AL2)+(1-POI)*DSIN(AL2)*DSIN(2.*AL2)/2.
V2=(DSIN(AL2)+(1-POI)*DCOS(AL2)*DSIN(2.*AL2)/2.)/PHI5C
V3=(DSIN(AL2)-(1-POI)*(DSIN(AL2)*DCOS(2.*AL2)-DCOS(AL2)*DSIN(2.*
IAL2)/2.))/PHI5
V4=(DCOS(AL2)+(1-POI)*(DCOS(AL2)*DCOS(2.*AL2)-DSIN(AL2)*DSIN(2.*
IAL2)/2.))/PHI5S
F11=DCOS(AL)*DCOS(AL1)
F12=DSIN(AL1)/DSIN(AL)
F21=DCOS(AL2)*DSIN(AL)
F22=DSIN(AL2)/DCOS(AL)
F31=DCOS(AL)*DCOS(AL3)
F32=DSIN(AL)*DSIN(AL3)
PRINT1,PHI1,K,AL
1 FORMAT('1', 'PHI1 =', F6.1, 10X, 'K =', I3, 10X, 'ALPHA =', F10.5,
*
*      Initialize the matrix A
*
2 I=10*K
DO 3 M=1,I
DO 3 N=1,I
3 A(M,N)=0.0
*
*      Start the computation of coefficient matrix elements
*
DO 4 I=1,k

```

```

A(I,I+K)      = 1.
A(I+K,I+3*K) = 1.
A(I+2*K,I+3*K) = 1.
A(I+4*K,I+2*K) = 1.*PHI1S
A(I+8*K,I+8*K) = 1.
A(I+9*K,I+9*K) = -1.
CONTINUE
DO 100 I=1,K
DO 100 J=1,K
ALMDPS=ALMDS/F2
ALMDRS=ALMDS*PHI1S/PHIRS
EMP=J*PI
ENP=I*PI
EMPS=EMP*EMP
ENPS=ENP*ENP
EMPC=(J-1)*PI
ENPC=(I-1)*PI
EMPCS=EMPC*EMPC
ENPCS=ENPC*ENPC
N=I-1
DI=2.
IF (N.EQ.0) DI=1.
B1MS=PHI1S*(ALMDS+EMPS)
B1M=DSQRT(B1MS)
TEST1M=ALMDS-EMPS
G1MS=PHI1S*TEST1M
IF (G1MS.LT.0.) G1MS=-G1MS
G1M=DSQRT(G1MS)
B3MS=PHI3S*(PHI1S*ALMDS+EMPS)
B3M=DSQRT(B3MS)
TEST3M=PHI1S*ALMDS-EMPS
G3MS=PHI3S*TEST3M
IF (G3MS.LT.0.) G3MS=-G3MS
G3M=DSQRT(G3MS)
B5MS=PHI5S*(ALMDPS-EMPS)
B5M=DSQRT(B5MS)
TEST5M=ALMDPS-EMPS
G5MS=PHI5S*TEST5M
IF (G5MS.LT.0.) G5MS=-G5MS
G5M=DSQRT(G5MS)
BRMS=(PHIRS*ALMDRS-EMPS)/PHIRS
BRM=DSQRT(BRMS)
TESTRM=PHIRS*ALMDRS-EMPS
GRMS=TESTRM/PHIRS
IF (GRMS.LT.0.) GRMS=-GRMS
GRM=DSQRT(GRMS)
ENPKS=ENPCS

```

```

EMPXS=EMPCS
IF (MODE.GT.0) EMPXS=EMPS
IF (MODE.GT.0) ENPXS=ENPS
B1RMS=PHIRS*(ALNDRS-EMPXS)
B1RM=DSQRT(B1RMS)
TEST1R=ALNDRS-EMPXS
G1RMS=PHIRS*TEST1R
IF (TEST1R.LT.0.) G1RMS=-G1RMS
G1RM=DSQRT(G1RMS)
ENPX=DSQRT(ENPKS)
EMPX=DSQRT(EMPXS)
X1=EMP*F2-ENP
X1S=X1*X1
X2=EMP*F2-ENP
X2S=X2*X2
X3=EMP*F1-ENP
X3S=X3*X3
X4=EMP*F1-ENP
X4S=X4*X4
X5=EMP-ENP
X5S=X5*X5
X6=EMP-ENP
X6S=X6*X6
IF (TEST1M.LT.0.) GO TO 200
C11M=-DSINH(B1M)/DSIN(G1M)
TD11M=-1./((B1MS*DSINH(B1M)-C11M*G1MS*DSIN(G1M)))
C21M=(B1MS-POI*PHI1S*EMPS)*DSINH(B1M)/((G1MS-POI*PHI1S*EMPS)*
DSIN(G1M))
TD21M=1./((DSINH(B1M)+C21M*DSIN(G1M))
C1=B1M*(DCOSH(B1M)-DCOS(X5))/(B1MS-X5S)
C2=B1M*(DCOS(X6)-DCOSH(B1M))/(B1MS-X6S)
TEST=G1MS-X5S
IF (TEST.EQ.0.) GO TO 19
C3=(DCOS(X5)-DCOS(G1M))/(X5-G1M)-((DCOS(G1M)-DCOS(X5))/(X5-G1M))
C3=C3/2.
GO TO 21
19 C3=DSIN(G1M)/2.
21 TEST=G1MS-X6S
IF (TEST.EQ.0.) GO TO 22
C4=(DCOS(G1M)-DCOS(X6))/(X6-G1M)-((DCOS(X6)-DCOS(G1M))/(X6-G1M))
C4=C4/2.
GO TO 23
22 C4=-DSIN(G1M)/2.
23 H1=X6*(DCOSH(B1M)-DCOS(X6))/(B1MS-X6S)
H2=X5*(DCOS(X5)-DCOSH(B1M))/(B1MS-X5S)
TEST=G1MS-X6S
IF (TEST.EQ.0.) GO TO 37

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```

H3=((DCOS(G1M)-DCOS(X6))/(X6+G1M))+((DCOS(G1M)-DCOS(X6))/(X6-G1M))
H3=H3/2.
GO TO 38
37 H3=DSIN(G1M)/2.
38 TEST=G1M-X5
IF (TEST.EQ.0.) GO TO 39
H4=((DCOS(X5)-DCOS(G1M))/(X5+G1M))+((DCOS(X5)-DCOS(G1M))/(X5-G1M))
H4=H4/2.
GO TO 41
39 H4=-DSIN(G1M)/2.
IF (X5.LT.0.) H4=-H4
41 DS1=-2.*ENP*DCOS(ENP)*DSINH(B1M)/(B1M-ENPS)
DS2=-2.*ENP*DCOS(ENP)*DSIN(G1M)/(G1M-ENPS)
DS3=-2.*ENP*DCOS(ENP)*DSINH(G1M)/(G1M+ENPS)
GS1=(X6*DSINH(B1M)-B1M*DSIN(X6))/(B1M+X6S)
GS2=(B1M*DSIN(X5)-X5*DSINH(B1M))/(B1M+X5S)
TEST=G1M-X6.
IF (TEST.EQ.0.) GO TO 410
GS3=((DCOS(G1M-(PI/2.))-DCOS(X6-(PI/2.)))/(2.*(X6-G1M)))
1+((DCOS((PI/2.)-X6)-DCOS(G1M+(PI/2.)))/(2.*(X6+G1M)))
GO TO 411
410 GS3=(DSIN(G1M-(PI/2.))/2.)-((DCOS(G1M-(PI/2.))-DCOS((PI/2.)-G1M))
1/(4.*G1M))
411 TEST=G1M-X5S
IF (TEST.EQ.0.) GO TO 412
GS4=((DCOS(X5-(PI/2.))-DCOS(G1M-(PI/2.)))/(2.*(X5-G1M)))
1+((DCOS(G1M+(PI/2.))-DCOS((PI/2.)-X5))/(2.*(X5+G1M)))
GO TO 414
412 IF (X5.LT.0.) GO TO 413
GS4=((DCOS(G1M-(PI/2.))-DCOS((PI/2.)-G1M))/(2.*G1M))-(DSIN(G1M-
1(PI/2.))/2.)
GO TO 414
413 GS1=((DCOS(G1M-(PI/2.))-DCOS(G1M-(PI/2.)))/(2.*G1M))-(DSIN(G1M-
1(PI/2.))/2.)
414 HS1=(B1M*DSINH(B1M)-X5*DSIN(X5))/(B1M-X5S)
HS2=-(B1M*DSINH(B1M)+X6*DSIN(X6))/(B1M-X6S)
TEST=G1M-X5S
IF (TEST.EQ.0.) GO TO 415
HS3=((DSIN(G1M)-DSIN(X5))/(2.*(X5-G1M)))-((DSIN(X5)-DSIN(G1M))
1(2.*(X5+G1M)))
GO TO 416
415 HS3=(DCOS(G1M)/2.)-(DSIN(G1M)/(2.*G1M))
416 TEST=G1M-X6
IF (TEST.EQ.0.) GO TO 417
HS4=((DSIN(G1M)-DSIN(X6))/(2.*(X6-G1M)))-((DSIN(G1M)+DSIN(X6))
1(2.*(X6+G1M)))
GO TO 210

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417 HS4=(DSIN(G1M)/(2.*G1M))-(DCOS(G1M)/2.)
GO TO 210
200 IF (B1MS.GT.QLIM) GO TO 205
C12M=-DSINH(B1M)/DSINH(G1M)
TD12M=-1./(B1MS*DSINH(B1M)-C12M*G1MS*DSINH(G1M))
C22M=-(B1MS-POI*PHI1S*EMPS)*DSINH(B1M)/(G1MS-POI*PHI1S*EMPS*
1DSINH(G1M))
TD22M=1./(DSINH(B1M)+C22M*DSINH(G1M))
C1=B1M*(DCOSH(B1M)-DCOS(X5))/(B1MS+X5S)
C2=B1M*(DCOS(X6)-DCOSH(B1M))/(B1MS+X6S)
C5=G1M*(DCCSH(G1M)-DCOS(X5))/(G1MS+X5S)
C6=G1M*(DCOS(X6)-DCOSH(G1M))/(G1MS+X6S)
H1=X6*(DCOSH(B1M)-DCOS(X6))/(B1MS+X6S)
H2=X5*(DCOS(X5)-DCOSH(B1M))/(B1MS+X5S)
H5=X6*(DCOSH(G1M)-DCOS(X6))/(G1MS+X6S)
H6=X5*(DCOS(X5)-DCOSH(G1M))/(G1MS+X5S)
DS1=-2.*ENP*DCOS(ENP)*DSINH(B1M)/(B1MS-ENPS)
DS2=2.*ENP*DCOS(ENP)*DSIN(G1M)/(G1MS-ENPS)
DS3=-2.*ENP*DCOS(ENP)*DSINH(G1M)/(G1MS-ENPS)
GS1=(X6*DSINH(B1M)-B1M*DSIN(X6))/(B1MS+X6S)
GS2=(B1M*DSIN(X5)-X5*DSINH(B1M))/(B1MS+X5S)
GS5=(X6*DSINH(G1M)-G1M*DSIN(X6))/(G1MS+X6S)
GS6=(G1M*DSIN(X5)-X5*DSINH(G1M))/(G1MS+X5S)*
HS1=(B1M*DSINH(B1M)+X5*DSIN(X5))/(B1MS+X5S)
HS2=-(B1M*DSINH(B1M)+X6*DSIN(X6))/(B1MS+X6S)
HS5=(G1M*DSINH(G1M)+X5*DSIN(X5))/(G1MS+X5S)
HS6=-(G1M*DSINH(G1M)+X6*DSIN(X6))/(G1MS+X6S)
GO TO 210
205 C12M=-1
TD12M=-1./(B1MS+C12M*G1MS)
C22M=-(B1MS-POI*PHI1S*EMPS)/(G1MS-POI*PHI1S*EMPS)
TD22M=1./(1.-C22M)
C1=B1M/(B1MS+X5S)
C2=-B1M/(B1MS+X6S)
C5=G1M/(G1MS+X5S)
C6=-G1M/(G1MS+X6S)
H1=X6/(B1MS+X6S)
H2=-X5/(B1MS+X5S)
H5=X6/(G1MS+X6S)
H6=-X5/(G1MS+X5S)
DS1=-2.*ENP*DCOS(ENP)/(B1MS-ENPS)
DS2=2.*ENP*DCOS(ENP)*DSIN(G1M)/(G1MS-ENPS)
DS3=-2.*ENP*DCOS(ENP)/(G1MS-ENPS)
GS1=X6/(B1MS+X6S)
GS2=-X5/(B1MS+X5S)
GS5=X6/(G1MS+X6S)
GS6=-X5/(G1MS+X5S)

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HS1=B1M/(B1MS+X5S)
HS2=-B1M/(B1MS-X6S)
HS3=G1M/(G1MS+X5S)
HS6=-G1M/(G1MS-X6S)
210 IF (TEST3M.LT.0.) GO TO 220
C31M=-DSINH(B3M)/DSIN(G3M)
TD31M=-1./((B3MS*DSINH(B3M)-C31M*G3MS*DSIN(G3M)))
C41M=(B3MS-POI*PHI3S*ENPS)*DSINH(B3M)/((G3MS-POI*PHI3S*ENPS)*
1DSIN(G3M))
TD41M=1./((DSINH(B3M)+C41M*DSIN(G3M)))
D1=B3M*(DCOSH(B3M)*DCOS(ENP)-DCOS(EMP))/(B3MS+X6S)
D2=B3M*(DCOS(EMP)-DCOSH(B3M)*DCOS(ENP))/(B3MS+X5S)
TEST=G3MS-X6S
IF (TEST.EQ.0.) GO TO 24
D3=((DCOS(EMP)-DCOS(G3M+ENP))/(G3M+X6S))+((DCOS(EMP)-DCOS(G3M-
1ENP))/(G3M-X6S))
D3=D3/2.
GO TO 25
24 D3=(DCOS(EMP)-DCOS(2.*G3M+EMP))/(4.*G3M)
25 TEST=G3MS-X5S
IF (TEST.EQ.0.) GO TO 26
D4=((DCOS(G3M-ENP)-DCOS(EMP))/(G3M+X5S))+((DCOS(G3M+ENP)-DCOS(
1EMP))/(G3M-X5S))
D4=D4/2.
GO TO 27
26 D4=(DCOS(2.*G3M+EMP)-DCOS(EMP))/(4.*G3M)
27 P1=X5*(DCOSH(B3M)*DCOS(ENP)-DCOS(EMP))/(B3MS+X5S)
P2=X6*(DCOS(EMP)-DCOSH(B3M)*DCOS(ENP))/(B3MS+X6S)
TEST=G3MS-X5S
IF (TEST.EQ.0.) GO TO 42
P3=((DCOS(ENP-G3M)-DCOS(EMP))/(G3M+X5S))+((DCOS(EMP)-DCOS(ENP-G3M))
1/(G3M-X5S))
P3=P3/2.
GO TO 43
42 P3=(DCOS(EMP-2.*G3M)-DCOS(EMP))/(4.*G3M)
IF (X5.LT.0.) P3=-P3
43 TEST=G3MS-X6S
IF (TEST.EQ.0.) GO TO 44
P4=((DCOS(EMP)-DCOS(ENP+G3M))/(G3M+X6S))+((DCOS(G3M-ENP)-DCOS(EMP))
1/(G3M-X6S))
P4=P4/2.
GO TO 45
44 P4=(DCOS(EMP)-DCOS(EMP-2.*G3M))/(4.*G3M)
45 AS1=-2.*ENP*DCOS(ENP)*DSINH(B3M)/(B3MS+ENPS)
AS2=2.*ENP*DCOS(ENP)*DSIN(G3M)/(G3MS-ENPS)
PS1=B3M*DSINH(B3M)*DCOS(ENP)/(B3MS+X6S)
PS2=-B3M*DSINH(B3M)*DCOS(ENP)/(B3MS+X5S)

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TEST=G3M-X6
IF (TEST.EQ.0.) GO TO 419
PS3=(DSIN(G3M-ENP)/(2.*(G3M+X6)))+(DSIN(G3M-ENP)/(2.*(G3M-X6)))
GO TO 420
-19 PS3=(DCOS(ENP)/2.)+(DSIN(2.*G3M-ENP)/(4.*G3M))
-20 TEST=G3MS-X5S
IF (TEST.EQ.0.) GO TO 421
PS4=-(DSIN(G3M-ENP)/(2.*(G3M+X5)))+(DSIN(G3M-ENP)/(2.*(G3M-X5)))
GO TO 422
-21 PS4=-(DCOS(ENP)/2.)-(DSIN(2.*G3M-ENP)/(4.*G3M))
-22 QS1=X5*DSINH(B3M)*DCOS(ENP)/(B3MS-X5S)
QS2=-X6*DSINH(B3M)*DCOS(ENP)/(B3MS-X6S)
TEST=G3MS-X5S
IF (TEST.EQ.0.) GO TO 423
QS3=(DCOS(G3M-ENP-(PI/2.))/(2.*(X5-G3M)))-(DCOS(G3M-ENP+(PI/2.))
1/(2.*(X5-G3M)))
GO TO 425
-23 IF (X5.LT.0.) GO TO 424
QS3=(DSIN(ENP-(PI/2.))/2.)-(DCOS(2.*G3M-ENP-(PI/2.))/(4.*G3M))
GO TO 425
-24 QS3=(DSIN((PI/2.)-ENP)/2.)-(DCOS(2.*G3M-ENP-(PI/2.))/(4.*G3M))
-25 TEST=G3M-X6
IF (TEST.EQ.0.) GO TO 426
QS4=(DCOS(G3M-ENP-(PI/2.))/(2.*(G3M-X6)))-(DCOS(G3M-ENP+(PI/2.))
1/(2.*(G3M+X6)))
GO TO 230
-26 QS4=(DSIN((PI/2.)-ENP)/2.)-(DCOS(2.*G3M-ENP-(PI/2.))/(4.*G3M))
GO TO 230
-27 IF (B3MS.GT.QLIM) GO TO 225
CB2M=-DSINH(B3M)*DSINH(G3M)
TD32M=-1./(B3MS*DSINH(B3M)-CB2M*G3MS*DSINH(G3M))
C42M=-(B3MS-POI*PHI3S*EMPS)*DSINH(B3M)/(G3MS-POI*PHI3S*EMPS*
DSINH(G3M))
TD42M=1./(DSINH(B3M)-C42M*DSINH(G3M))
D1=B3M*(DCOSH(B3M)*DCOS(ENP)-DCOS(ENP))/(B3MS-X6S)
D2=B3M*(DCOS(ENP)-DCOSH(B3M)*DCOS(ENP))/(B3MS-X5S)
D3=G3M*(DCOSH(G3M)*DCOS(ENP)-DCOS(ENP))/(G3MS-X6S)
D4=G3M*(DCOS(ENP)-DCOSH(G3M)*DCOS(ENP))/(G3MS-X5S)
D5=X5*(DCOSH(B3M)*DCOS(ENP)-DCOS(ENP))/(B3MS-X5S)
D6=X6*(DCOS(ENP)-DCOSH(B3M)*DCOS(ENP))/(B3MS-X6S)
D7=X5*(DCOSH(G3M)*DCOS(ENP)-DCOS(ENP))/(G3MS-X5S)
D8=X6*(DCOS(ENP)-DCOSH(G3M)*DCOS(ENP))/(G3MS-X6S)
AS1=-2.*ENP*DCOS(ENP)*DSINH(B3M)/(B3MS-ENPS)
AS2=-2.*ENP*DCOS(ENP)*DSINH(G3M)/(G3MS-ENPS)
PS1=B3M*DSINH(B3M)*DCOS(ENP)/(B3MS-X6S)
PS2=-B3M*DSINH(B3M)*DCOS(ENP)/(B3MS-X5S)
PS3=G3M*DSINH(G3M)*DCOS(ENP)/(G3MS-X6S)

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PS6=-G3M*DSINH(G3M)*DCOS(ENP)/(G3MS+X5S)
QS1=X5*DSINH(B3M)*DCOS(ENP)/(B3MS+X5S)
QS2=-X6*DSINH(B3M)*DCOS(ENP)/(B3MS+X6S)
QS3=X5*DSINH(G3M)*DCOS(ENP)/(G3MS+X5S)
QS6=-X6*DSINH(G3M)*DCOS(ENP)/(G3MS+X6S)
GO TO 230
225 C32M=-1.
TD32M=-1./(B3MS+C32M*G3MS)
C42M=-(B3MS-POI*PHI3S*EMPS)/(G3MS-POI*PHI3S*EMPS)
TD42M=1./(1-C42M)
D1=B3M*DCOS(ENP)/(B3MS+X6S)
D2=-B3M*DCOS(ENP)/(B3MS+X5S)
D5=G3M*DCOS(ENP)/(G3MS+X6S)
D6=-G3M*DCOS(ENP)/(G3MS+X5S)
P1=X5*DCOS(ENP)/(B3MS+X5S)
P2=-X6*DCOS(ENP)/(B3MS+X6S)
P5=X5*DCOS(ENP)/(G3MS+X5S)
P6=-X6*DCOS(ENP)/(G3MS+X6S)
AS1=-2.*ENP*DCOS(ENP)/(B3MS-ENPS)
AS3=-2.*ENP*DCOS(ENP)/(G3MS-ENPS)
PS1=B3M*DCOS(ENP)/(B3MS+X6S)
PS2=-B3M*DCOS(ENP)/(B3MS+X5S)
PS5=G3M*DCOS(ENP)/(G3MS+X6S)
PS6=-G3M*DCOS(ENP)/(G3MS+X5S)
QS1=X5*DCOS(ENP)/(B3MS+X5S)
QS2=-X6*DCOS(ENP)/(B3MS+X6S)
QS3=X5*DCOS(ENP)/(G3MS+X5S)
QS6=-X6*DCOS(ENP)/(G3MS+X6S)
230 IF (TEST3M.LT.0.) GO TO 240
C51M=-DSINH(B5M)/DSIN(G5M)
TD51M=-1.(B5MS*DSINH(B5M)-C51M*G5MS*DSIN(G5M))
C61M=(B5MS-POI*PHI5S*EMPS)*DSINH(B5M)/(G5MS-POI*PHI5S*EMPS)*
DSIN(G5M)
TD61M=1.(DSINH(B5M)+C61M*DSIN(G5M))
A1=B5M*(DCOSH(B5M)-DCOS(X1))/(B5MS+X1S)
A2=B5M*(DCOS(X2)-DCOSH(B5M))/(B5MS+X2S)
TEST=G5MS-X1S
IF (TEST.EQ.0.) GO TO 11
A3=((DCOS(X1)-DCOS(G5M))/(X1-G5M))-((DCOS(G5M)-DCOS(X1))/(X1-G5M))
A3=A3/2.
GO TO 12
11 A3=DSIN(G5M)/2.
12 TEST=G5MS-X2S
IF (TEST.EQ.0.) GO TO 13
A4=((DCOS(G5M)-DCOS(X2))/(X2-G5M))-((DCOS(X2)-DCOS(G5M))/(X2-G5M))
A4=A4/2.
GO TO 14

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13 A4=-DSIN(G5M)/2.
14 B1=B5M*(DCOSH(B5M)*DCOS(EMP)-DCOS(EMP-X3))/(B5MS+X3S)
   B2=B5M*(DCOS(EMP-X4)-DCOSH(B5M)*DCOS(EMP))/(B5MS+X4S)
   TEST=G5MS-X3S
   IF (TEST.EQ.0.) GO TO 15
   B3=((DCOS(G5M-EMP)-DCOS(X3-EMP))/(X3-G5M))+((DCOS(EMP-X3)-DCOS(
15 G5M+EMP))/(X3+G5M))
   B3=B3/2.
   GO TO 16
15 B3=DSIN(G5M-EMP)/2.
16 TEST=G5MS-X4S
   IF (TEST.EQ.0.) GO TO 17
   B4=((DCOS(X4-EMP)-DCOS(G5M-EMP))/(X4-G5M))+((DCOS(G5M-EMP)-DCOS(
17 EMP-X4))/(X4+G5M))
   B4=B4/2.
   GO TO 18
17 B4=DSIN(EMP-G5M)/2.
18 E1=X2*(DCOSH(B5M)-DCOS(X2))/(B5MS+X2S)
   E2=X1*(DCOS(X1)-DCOSH(B5M))/(B5MS-X1S)
   TEST=G5MS-X2S
   IF (TEST.EQ.0.) GO TO 28
   E3=((DCOS(G5M)-DCOS(X2))/(X2+G5M))+((DCOS(G5M)-DCOS(X2))/(X2-G5M))
   E3=E3/2.
   GO TO 29
28 E3=DSIN(G5M)/2.
29 TEST=G5MS-X1S
   IF (TEST.EQ.0.) GO TO 31
   E4=((DCOS(X1)-DCOS(G5M))/(X1-G5M))+((DCOS(X1)-DCOS(G5M))/(X1+G5M))
   E4=E4/2.
   GO TO 32
31 E4=-DSIN(G5M)/2.
   IF (X1.LT.0.) E4=-E4
32 G1=X4*(DCOS(EMP-X4)-DCOSH(B5M)*DCOS(EMP))/(B5MS+X4S)
   G2=X3*(DCOSH(B5M)*DCOS(EMP)-DCOS(EMP-X3))/(B5MS+X3S)
   TEST=G5MS-X4S
   IF (TEST.EQ.0.) GO TO 33
   G3=((DCOS(EMP-G5M)-DCOS(EMP-X4))/(G5M-X4))+((DCOS(EMP-X4)-DCOS(
33 EMP+G5M))/(G5M+X4))
   G3=G3/2.
   GO TO 34
33 G3=DSIN(EMP-G5M)/2.
   IF (X4.LT.0.) G3=-G3
34 TEST=G5MS-X3S
   IF (TEST.EQ.0.) GO TO 35
   G4=((DCOS(EMP-X3)-DCOS(EMP-G5M))/(G5M-X3))+((DCOS(EMP-G5M)-DCOS(
35 EMP+X3))/(G5M+X3))
   G4=G4/2.

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GO TO 36
35 G4=DSIN(G5M-EMP)/2.
36 BS1=(X2*DSINH(B5M)-B5M*DSIN(X2))/(B5MS+X2S)
   BS2=(B5M*DSIN(X1)-X1*DSINH(B5M))/(B5MS+X1S)
   TEST=G5M-X2
   IF (TEST.EQ.0.) GO TO 401
   BS3=((DCOS(G5M-(PI/2.))-DCOS(X2-(PI/2.)))/(2.*(X2-G5M)))
   1/((DCOS((PI/2.)-X2)-DCOS(G5M+(PI/2.)))/(2.*(X2+G5M)))
   GO TO 402
401 BS3=DSIN(G5M-(PI/2.))/2.-(DCOS(G5M+(PI/2.))-DCOS((PI/2.)-G5M))
   1/(4.*G5M)
402 TEST=G5MS-X1S
   IF (TEST.EQ.0.) GO TO 403
   BS4=((DCOS(X1-(PI/2.))-DCOS(G5M-(PI/2.)))/(2.*(X1-G5M)))
   1/((DCOS(G5M-(PI/2.))-DCOS((PI/2.)-X1))/(2.*(X1+G5M)))
   GO TO 405
403 IF (X1.GT.0.) GO TO 404
   BS4=((DCOS(G5M-(PI/2.))-DCOS(G5M+(PI/2.)))/(4.*G5M))
   1-(DSIN(G5M+(PI/2.))/2.)
   GO TO 405
404 BS4=((DCOS(G5M+(PI/2.))-DCOS((PI/2.)-G5M))/(4.*G5M))
   1-(DSIN(G5M-(PI/2.))/2.)
405 CS1=(B5M*DSINH(B5M)+X1*DSIN(X1))/(B5MS+X1S)
   CS2=-(B5M*DSINH(B5M)+(X2*DSIN(X2)))/(B5MS+X2S)
   TEST=G5MS-X1S
   IF (TEST.EQ.0.) GO TO 406
   CS3=((DSIN(G5M)+DSIN(X1))/(2.*(X1+G5M)))-((DSIN(X1)-DSIN(G5M))
   1/(2.*(X1-G5M)))
   GO TO 407
406 CS3=(DCOS(G5M)/2.)-(DSIN(G5M)/(2.*G5M))
407 TEST=G5M-X2
   IF (TEST.EQ.0.) GO TO 408
   CS4=((DSIN(G5M)-DSIN(X2))/(2.*(X2-G5M)))-((DSIN(G5M)+DSIN(X2))
   1/(2.*(X2+G5M)))
   GO TO 409
408 CS4=(DSIN(G5M)/(2.*G5M))-(DCOS(G5M)/2.)
409 ES1=(B5M*DSIN(X4-EMP)-X4*DSINH(B5M)*DCOS(EMP))/(B5MS+X4S)
   ES2=(X3*DSINH(B5M)*DCOS(EMP)-B5M*DSIN(X3-EMP))/(B5MS+X3S)
   TEST=G5MS-X4S.
   IF (TEST.LT.0.) TEST=-TEST
   IF (TEST.EQ.0.) GO TO 46
   ES3=((DCOS(EMP-(PI/2.))-X4)-DCOS(G5M+EMP-(PI/2.)))/(X4+G5M))+
   1/((DCOS(G5M+(PI/2.)-EMP)-DCOS(X4+(PI/2.)-EMP))/(X4-G5M))
   ES3=ES3/2.
   GO TO 48
46 IF (X4.LT.0.) GO TO 47
   ES3=(DSIN(G5M+(PI/2.)-EMP)/2.)+((DCOS(EMP-G5M-(PI/2.))-DCOS

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1(EMP+G5M-(PI/2.)))/(4.*G5M))
GO TO 48
47 ES3=(DSIN(G5M-EMP-(PI/2.))/2.)-(DCOS((PI/2.-G5M-EMP)-DCOS
1(G5M+(PI/2.-EMP)))/(4.*G5M))
TEST=G5MS-X3S
IF (TEST.LT.0.) TEST=-TEST
IF (TEST.EQ.0.) GO TO 49
ES4=((DCOS(G5M-EMP-(PI/2.))-DCOS(EMP-(PI/2.-X3)))/(X3-G5M))-
1(DCOS(X3-(PI/2.-EMP)-DCOS(G5M-(PI/2.-EMP)))/(X2-G5M))
ES4=ES4/2.
GO TO 51
49 ES4=(DSIN(EMP-G5M+(PI/2.))/2.)+(DCOS(EMP-G5M-(PI/2.))-DCOS
1(EMP-G5M-(PI/2.)))/(4.*G5M))
51 FS1=(B5M*DSINH(B5M)*DCOS(EMP)-X3*DSIN(EMP-X3))/(B5MS-X3S)
FS2=(X4*DSIN(EMP-X4)-B5M*DSINH(B5M)*DCOS(EMP))/(B5MS-X4S)
TEST=G5MS-X3S
IF (TEST.LT.0.) TEST=-TEST
IF (TEST.EQ.0.) GO TO 52
FS3=((DSIN(X3-EMP)-DSIN(G5M-EMP))/(X3-G5M))-
1((DSIN(G5M-EMP)-DSIN(EMP-X3))/(X3-G5M))
FS3=FS3/2.
GO TO 53
52 FS3=(DCOS(G5M-EMP)/2.)+(DSIN(EMP-G5M)/(2.*G5M))
53 TEST=G5MS-X4S
IF (TEST.LT.0.) TEST=-TEST
IF (TEST.EQ.0.) GO TO 54
FS4=((DSIN(G5M-EMP)-DSIN(X4-EMP))/(X4-G5M))-
1((DSIN(EMP-X4)-DSIN(G5M-EMP))/(X4-G5M))
FS4=FS4/2.
GO TO 55
54 FS4=(DSIN(EMP-G5M)/(2.*G5M))-(DCOS(G5M-EMP)/2.)
55 CONTINUE
GO TO 150
240 IF (B5MS.GT.Q1IM) GO TO 245
C62M=-DSINH(B5M)*DSINH(G5M)
TD62M=-1./(B5MS*DSINH(B5M)+C62M*G5MS*DSINH(G5M))
C62M=-(B5MS*POI*PHI5S*EMPS)*DSINH(B5M)/(G5MS*POI*PHI5S*EMPS*
1DSINH(G5M))
TD62M=1.-(DSINH(B5M)-C62M*DSINH(G5M))
A1=B5M*(DCOSH(B5M)-DCOS(X1))/(B5MS-X1S)
A2=B5M*(DCOS(X2)-DCOSH(B5M))/(B5MS-X2S)
A5=G5M*(DCOSH(G5M)-DCOS(X1))/(G5MS-X1S)
A6=G5M*(DCOS(X2)-DCOSH(G5M))/(G5MS-X2S)
B1=B5M*(DCOSH(B5M)*DCOS(EMP)-DCOS(EMP-X3))/(B5MS-X3S)
B2=B5M*(DCOS(EMP-X4)-DCOSH(B5M)*DCOS(EMP))/(B5MS-X4S)
B5=G5M*(DCOSH(G5M)*DCOS(EMP)-DCOS(EMP-X3))/(G5MS-X3S)
B6=G5M*(DCOS(EMP-X4)-DCOSH(G5M)*DCOS(EMP))/(G5MS-X4S)

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E1=X2*(DCOSH(B5M)-DCOS(X2))/(B5MS+X2S)
E2=X1*(DCOS(X1)-DCOSH(B5M))/(B5MS+X1S)
E5=X2*(DCOSH(G5M)-DCOS(X2))/(G5MS+X2S)
E6=X1*(DCOS(X1)-DCOSH(G5M))/(G5MS+X1S)
G1=X4*(DCOS(EMP-X4)-DCOSH(B5M)*DCOS(EMP))/(B5MS+X4S)
G2=X3*(DCOSH(B5M)*DCOS(EMP)-DCOS(EMP-X3))/(B5MS+X3S)
G5=X4*(DCOS(EMP-X4)-DCOSH(G5M)*DCOS(EMP))/(G5MS+X4S)
G6=X3*(DCOSH(G5M)*DCOS(EMP)-DCOS(EMP-X3))/(G5MS+X3S)
BS1=(X2*DSINH(B5M)-B5M*DSIN(X2))/(B5MS+X2S)
BS2=(B5M*DSIN(X1)-X1*DSINH(B5M))/(B5MS+X1S)
BS5=(X2*DSINH(G5M)-G5M*DSIN(X2))/(G5MS+X2S)
BS6=(G5M*DSIN(X1)-X1*DSINH(G5M))/(G5MS+X1S)
CS1=(B5M*DSINH(B5M)+X1*DSIN(X1))/(B5MS+X1S)
CS2=(B5M*DSINH(B5M)+X2*DSIN(X2))/(B5MS+X2S)
CS5=(G5M*DSINH(G5M)+X1*DSIN(X1))/(G5MS+X1S)
CS6=(G5M*DSINH(G5M)+X2*DSIN(X2))/(G5MS+X2S)
ES1=(B5M*DSIN(X4-EMP)-X4*DSINH(B5M)*DCOS(EMP))/(B5MS+X4S)
ES2=(X3*DSINH(B5M)*DCOS(EMP)-B5M*DSIN(X3-EMP))/(B5MS+X3S)
ES5=(G5M*DSIN(X4-EMP)-X4*DSINH(G5M)*DCOS(EMP))/(G5MS+X4S)
ES6=(X3*DSINH(G5M)*DCOS(EMP)-G5M*DSIN(X3-EMP))/(G5MS+X3S)
FS1=(B5M*DSINH(B5M)*DCOS(EMP)-X3*DSIN(EMP-X3))/(B5MS+X3S)
FS2=(X4*DSIN(EMP-X4)-B5M*DSINH(B5M)*DCOS(EMP))/(B5MS+X4S)
FS5=(G5M*DSINH(G5M)*DCOS(EMP)-X3*DSIN(EMP-X3))/(G5MS+X3S)
FS6=(X4*DSIN(EMP-X4)-G5M*DSINH(G5M)*DCOS(EMP))/(G5MS+X4S)
GO TO 250

```

245 CS2M=-1.

```

TD52M=-1./(B5MS+CS2M*G5MS)
C62M=-(B5MS-POI*PHI5S*EMPS)/(G5MS-POI*PHI5S*EMPS)
TD62M=1./(1.+C62M)
A1=B5M/(B5MS+X1S)
A2=-B5M/(B5MS+X2S)
A5=G5M/(G5MS+X1S)
A6=-G5M/(G5MS+X2S)
B1=B5M*DCOS(EMP)/(B5MS+X3S)
B2=-B5M*DCOS(EMP)/(B5MS+X4S)
B5=G5M*DCOS(EMP)/(G5MS+X3S)
B6=-G5M*DCOS(EMP)/(G5MS+X4S)
E1=X2/(B5MS+X2S)
E2=-X1/(B5MS+X1S)
E5=X2/(G5MS+X2S)
E6=-X1/(G5MS+X1S)
G1=-X4*DCOS(EMP)/(B5MS+X4S)
G2=X3*DCOS(EMP)/(B5MS+X3S)
G5=-X4*DCOS(EMP)/(G5MS+X4S)
G6=X3*DCOS(EMP)/(G5MS+X3S)
BS1=X2/(B5MS+X2S)
BS2=-X1/(B5MS+X1S)

```

```

BS5=X2/(G5MS+X2S)
BS6=-X1/(G5MS+X1S)
CS1=B5M/(B5MS+X1S)
CS2=-B5M/(B5MS+X2S)
CS3=G5M/(G5MS+X1S)
CS6=-G5M/(G5MS+X2S)
ES1=-X1*DCOS(EMP)/(B5MS+X4S)
ES2=X3*DCOS(EMP)/(B5MS+X3S)
ES5=-X4*DCOS(EMP)/(G5MS+X4S)
ES6=X3*DCOS(EMP)/(G5MS+X3S)
FS1=B5M*DCOS(EMP)/(B5MS+X3S)
FS2=-B5M*DCOS(EMP)/(B5MS+X4S)
FS3=G5M*DCOS(EMP)/(G5MS+X3S)
FS6=-G5M*DCOS(EMP)/(G5MS+X4S)
250 CONTINUE
IF (TEST1R.LT.0.) GO TO 251
C11RM=-DSINH(B1RM)/DSIN(G1RM)
TD11RM=1./(B1RM*DCOSH(B1RM)+C11RM*G1RM*DCOS(G1RM))
441 B1RS=-2.*ENP*DCOS(ENP)*DSINH(B1RM)/(B1RMS-ENPS)
B2RS=2.*ENP*DCOS(ENP)*DSIN(G1RM)/(G1RMS-ENPS)
IF (G1RM.EQ.ENP) B2RS=1.
C1RS=2.*ENP*DSINH(B1RM)/(ENPS+B1RMS)
IF (G1RM.EQ.ENP) GO TO 442
C2RS=2.*ENP*DSIN(G1RM)/(ENPS-G1RMS)
GO TO 253
442 C1RS=DSIN(G1RM-(PI/2.))-DCOS(G1RM-(PI/2.))*G1RM
GO TO 253
251 IF (B1RMS.GT.QLIM) GO TO 252
C12RM=-DSINH(B1RM)/DSINH(G1RM)
TD12RM=1./(B1RM*DCOSH(B1RM)+C12RM*G1RM*DCOSH(G1RM))
444 B1RS=-2.*ENP*DCOS(ENP)*DSINH(B1RM)/(B1RMS-ENPS)
B3RS=-2.*ENP*DCOS(ENP)*DSINH(G1RM)/(G1RMS-ENPS)
C1RS=2.*ENP*DSINH(B1RM)/(ENPS-B1RMS)
C3RS=2.*ENP*DSINH(G1RM)/(ENPS-G1RMS)
GO TO 253
252 C12RM=-1.
TD12RM=1./(B1RM+C12RM*G1RM)
B1RS=-2.*ENP*DCOS(ENP)/(B1RMS-ENPS)
B3RS=-2.*ENP*DCOS(ENP)/(G1RMS-ENPS)
C1RS=2.*ENP/(ENPS+B1RMS)
C3RS=2.*ENP/(ENPS-G1RMS)
253 CONTINUE

```

* This step will compute elements with ALMDS = ENPS
 *

```

IF (TEST1M.LT.0.) GO TO 61
A(I-2*K,J) = TD11M*(C1-C2-C11M*(C3+C4))

```

```

A(I+2*K,J+K) = TD21M*(C1+C2+C21M*(C3+C4))
A(J+3*K,J)   = TD11M*(B1M*DCOSH(B1M)+C11M*G1M*DCOS(G1M))
A(J-3*K,J+K) = TD21M*(B1M*DCOSH(B1M)+C21M*G1M*DCOS(G1M))
A1N           = TD11M*EMP*(GS1-GS2+C11M*(GS3+GS4))
A2N           = TD11M*(B1M*(HS1+HS2)+C11M*G1M*(HS3+HS4))
A(I-5*K,J)    = F31*A1N-F32*A2N
A1N           = TD21M*EMP*(GS1-GS2+C21M*(GS3+GS4))
A2N           = TD21M*(B1M*(HS1+HS2)+C21M*G1M*(HS3+HS4))
A(I-5*K,J+K)  = F31*A1N+F32*A2N
A(I-6*K,J)    = TD11M*EMP*DCOS(EMP)*(DS1+C11M*DS2)
A(I-6*K,J+K)  = TD21M*EMP*DCOS(EMP)*(DS1+C21M*DS2)
A1N           = -TD11M*EMPS*EMP*DCOS(EMP)*(DS1+C11M*DS2)
A2N           = TD11M*EMP*DCOS(EMP)*(B1M*DS1-C11M*G1M*DS2)
A(I-7*K,J)    = -(A1N-POIS*PHI3S*A2N)
A1N           = -TD21M*EMPS*EMP*DCOS(EMP)*(DS1+C21M*DS2)
A2N           = TD21M*EMP*DCOS(EMP)*(B1M*DS1-C21M*G1M*DS2)
A(I-7*K,J+K)  = -(A1N-POIS*PHI3S*A2N)
GO TO 62

```

```

*
* This step will generate elements with ALMDS < EMPS
*

```

```

61 A(I+2*K,J)   = TD12M*(C1-C2+C12M*(C5+C6))
A(I-2*K,J+K)   = TD22M*(C1-C2+C22M*(C5+C6))
A1N            = TD12M*EMP*(GS1-GS2+C12M*(GS5+GS6))
A2N            = TD12M*(B1M*(HS1+HS2)+C12M*G1M*(HS5+HS6))
A(I-5*K,J)     = F31*A1N-F32*A2N
A1N            = TD22M*EMP*(GS1-GS2+C22M*(GS5+GS6))
A2N            = TD22M*(B1M*(HS1+HS2)+C22M*G1M*(HS5+HS6))
A(I-5*K,J+K)   = F31*A1N+F32*A2N
A(I-6*K,J)     = TD12M*EMP*DCOS(EMP)*(DS1-C12M*DS3)
A(I-6*K,J+K)   = TD22M*EMP*DCOS(EMP)*(DS1-C22M*DS3)
A1N            = -TD12M*EMPS*EMP*DCOS(EMP)*(DS1+C12M*DS3)
A2N            = TD12M*EMP*DCOS(EMP)*(B1M*DS1-C12M*G1M*DS3)
A(I-7*K,J)     = -(A1N-POIS*PHI3S*A2N)
A1N            = -TD22M*EMPS*EMP*DCOS(EMP)*(DS1+C22M*DS3)
A2N            = TD22M*EMP*DCOS(EMP)*(B1M*DS1+C22M*G1M*DS3)
A(I-7*K,J+K)   = -(A1N-POIS*PHI3S*A2N)
IF (B1M.GT.QLIN) GO TO 256
A(J-3*K,J)     = TD12M*(B1M*DCOSH(B1M)-C12M*G1M*DCOSH(G1M))
A(J-3*K,J+K)   = TD22M*(B1M*DCOSH(B1M)-C22M*G1M*DCOSH(G1M))
GO TO 62
256 A(J-3*K,J)  = TD12M*(B1M-C12M*G1M)
A(J-3*K,J+K)   = TD22M*(B1M-C22M*G1M)

```

```

*
* This step will generate elements with PHI3S*ALMDS > EMPS
*

```

```

62 IF (TEST3M.LT.0.) GO TO 63

```

```

A(I+2*K,J+2*K) = TD31M*(D1+D2+C31M*(D3+D4))
A(I+2*K,J+3*K) = TD41M*(D1+D2+C41M*(D3+D4))
A(I+3*K,J+2*K) = EMP*TD31M*DCOS(EMP)*(AS1+C31M*AS2)
A(I+3*K,J+3*K) = EMP*TD41M*DCOS(EMP)*(AS1+C41M*AS2)
A1N
A2N
A(I+5*K,J+2*K) = F31*A1N-F32*A2N
A1N
A2N
A(I+5*K,J+3*K) = F31*A1N-F32*A2N
A(J+6*K,J+2*K) = TD31M*(B3M*DCOSH(B3M)+C31M*G3M*DCOS(G3M))
A(J+6*K,J+3*K) = TD41M*(B3M*DCOSH(B3M)+C41M*G3M*DCOS(G3M))
A1N
1
A2N
A(J+7*K,J+2*K) = -(A1N+POIS*PHI3S*A2N)
A1N
1
A2N
A(J+7*K,J+3*K) = -(A1N+POIS*PHI3S*A2N)
GO TO 64

```

*
*
*

This step will generate elements with PHI3S*ALMDS < EMPS

```

63 A(I+2*K,J+2*K) = TD32M*(D1+D2+C32M*(D5+D6))
A(I+2*K,J+3*K) = TD42M*(D1+D2+C42M*(D5+D6))
A(I+3*K,J+2*K) = EMP*TD32M*DCOS(EMP)*(AS1+C32M*AS3)
A(I+3*K,J+3*K) = EMP*TD42M*DCOS(EMP)*(AS1+C42M*AS3)
A1N
A2N
A(I+5*K,J+2*K) = F31*A1N-F32*A2N
A1N
A2N
A(I+5*K,J+3*K) = F31*A1N-F32*A2N
IF (B3MS.GT.QLIM) GO TO 255
A(J+6*K,J+2*K) = TD32M*(B3M*DCOSH(B3M)+C32M*G3M*DCOSH(G3M))
A(J+6*K,J+3*K) = TD42M*(B3M*DCOSH(B3M)+C42M*G3M*DCOSH(G3M))
A1N
1
A2N
A(J+7*K,J+2*K) = -(A1N+POIS*PHI3S*A2N)
A1N
1
A2N
A(J+7*K,J+3*K) = -(A1N+POIS*PHI3S*A2N)
GO TO 64
255 A(J+6*K,J+2*K) = TD32M*(B3M-C32M*G3M)

```

```

A(J-6*K,J+3*K) = TD42M*(B3M+C42M*G3M)
A1N              = TD32M*(B3MS*B3M+C32M*G3MS*G3M)
A2N              = -TD32M*EMPS*(B3M+C32M*G3M)
A(J-7*K,J+2*K) = -(A1N+POIS*PHI3S*A2N)
A1N              = TD42M*(B3MS*B3M+C42M*G3MS*G3M)
A2N              = -TD42M*EMPS*(B3M+C42M*G3M)
A(J-7*K,J+3*K) = -(A1N+POIS*PHI3S*A2N)

```

*
* This step will generate elements with ALWDPS > EMPS
*

```

04 IF (TEST5M.LT.0.) GO TO 65
A(I,J+4*K) = TD51M*(A1+A2-C51M*(A3-A4))
A(I,J+5*K) = TD61M*(A1+A2+C61M*(A3-A4))
A(I-K,J+4*K) = TD51M*(B1+B2-C51M*(B3-B4))
A(I-K,J+5*K) = TD61M*(B1+B2+C61M*(B3-B4))
A1N         = TD51M*EMP*(B51-B52+C51M*(B53-B54))
A2N         = -TD51M*(B5M*(C51+C52)-C51M*G5M*(C53+C54))
A(I-3*K,J+4*K) = F11*A1N+F12*A2N
A1N         = TD61M*EMP*(B51-B52+C61M*(B53-B54))
A2N         = -TD61M*(B5M*(C51+C52)-C61M*G5M*(C53+C54))
A(I-3*K,J+5*K) = F11*A1N+F12*A2N
A1N         = -TD51M*EMPS*(B1+B2+C51M*(B3-B4))
A2N         = TD51M*(B5MS*(B1+B2)-C51M*G5MS*(B3+B4))
A3N         = -TD51M*EMP*(B5M*(G1+G2)+C51M*G5M*(G3+G4))
A(I-4*K,J+4*K) = -(TD21*PHI5S*A1N-TD22*A2N+TD23*PHI5*A3N)/F2
A1N         = -TD61M*EMPS*(B1+B2+C61M*(B3-B4))
A2N         = TD61M*(B5MS*(B1+B2)-C61M*G5MS*(B3+B4))
A3N         = -TD61M*EMP*(B5M*(G1+G2)+C61M*G5M*(G3+G4))
A(I-4*K,J+5*K) = -(TD21*PHI5S*A1N-TD22*A2N+TD23*PHI5*A3N)/F2
A(J-5*K,J-4*K) = TD51M*(B5M*DCOSH(B5M)-C51M*G5M*DCOS(G5M))
A(J-5*K,J-5*K) = TD61M*(B5M*DCOSH(B5M)+C61M*G5M*DCOS(G5M))
A1N         = TD51M*EMP*(ES1-ES2+C51M*(ES3-ES4))
A2N         = -TD51M*(B5M*(FS1+FS2)-C51M*G5M*(FS3+FS4))
A(I-6*K,J+4*K) = F21*A1N+F22*A2N
A1N         = TD61M*EMP*(ES1-ES2+C61M*(ES3-ES4))
A2N         = -TD61M*(B5M*(FS1+FS2)-C61M*G5M*(FS3+FS4))
A(I-6*K,J+5*K) = F21*A1N+F22*A2N
A1N         = -TD51M*EMPS*EMP*(ES1-ES2-C51M*(ES3-ES4))
A2N         = -TD51M*(B5MS*B5M*(FS1+FS2)-C51M*G5MS*G5M*(FS3+FS4))
A3N         = TD51M*EMPS*(B5M*(FS1+FS2)+C51M*G5M*(FS3+FS4))
A4N         = TD51M*EMP*(B5MS*(ES1-ES2)-C51M*G5MS*(ES3-ES4))
A(I-7*K,J-4*K) = -(V1*A1N+V2*A2N+V3*A3N+V4*A4N)*F2*DSIN(AL)
A1N         = -TD61M*EMPS*EMP*(ES1-ES2+C61M*(ES3+ES4))
A2N         = -TD61M*(B5MS*B5M*(FS1+FS2)-C61M*G5MS*G5M*(FS3+FS4))
A3N         = TD61M*EMPS*(B5M*(FS1+FS2)+C61M*G5M*(FS3+FS4))

```

```

A4N      = TD61M*EMP*(B5MS*(ES1+ES2)-C61M*G5MS*(ES3+ES4))
A(I+7*K,J+5*K) = -(V1*A1N+V2*A2N+V3*A3N+V4*A4N)*F2*DSIN(AL)
GO TO 66

*
* This step will generate elements with ALMDPS < EMPS
*
65 A(I,J+4*K) = TD52M*(A1-A2-C52M*(A3+A6))
A(I,J+5*K) = TD62M*(A1-A2-C62M*(A3+A6))
A(I+K,J+4*K) = TD52M*(B1-B2-C52M*(B3+B6))
A(I+K,J+5*K) = TD62M*(B1-B2-C62M*(B3+B6))
A1N      = TD52M*EMP*(BS1-BS2+C52M*(BS3+BS6))
A2N      = -TD52M*(B5M*(CS1-CS2)-C52M*G5M*(CS3+CS6))
A(I+3*K,J+4*K) = F11*A1N+F12*A2N
A1N      = TD62M*EMP*(BS1-BS2+C62M*(BS3+BS6))
A2N      = -TD62M*(B5M*(CS1+CS2)+C62M*G5M*(CS3+CS6))
A(I+3*K,J+5*K) = F11*A1N+F12*A2N
A1N      = -TD52M*EMPS*(B1-B2-C52M*(B3+B6))
A2N      = TD52M*(B5MS*(B1+B2)-C52M*G5MS*(B3+B6))
A3N      = -TD52M*EMP*(B5M*(G1-G2)+C52M*G5M*(G3+G6))
A(I+4*K,J+4*K) = -(TD21*PHI5S*A1N-TD22*A2N-TD23*PHI5*A3N)/F2
A1N      = -TD62M*EMPS*(B1+B2+C62M*(B3+B6))
A2N      = TD62M*(B5MS*(B1+B2)+C62M*G5MS*(B3+B6))
A3N      = -TD62M*EMP*(B5M*(G1+G2)+C62M*G5M*(G3+G6))
A(I+4*K,J+5*K) = -(TD21*PHI5S*A1N-TD22*A2N-TD23*PHI5*A3N)/F2
A1N      = TD52M*EMP*(ES1+ES2+C52M*(ES3+ES6))
A2N      = -TD52M*(B5M*(FS1+FS2)+C52M*G5M*(FS3+FS6))
A(I+6*K,J+4*K) = F21*A1N+F22*A2N
A1N      = TD62M*EMP*(ES1+ES2+C62M*(ES3+ES6))
A2N      = -TD62M*(B5M*(FS1+FS2)+C62M*G5M*(FS3+FS6))
A(I+6*K,J+5*K) = F21*A1N+F22*A2N
A1N      = -TD52M*EMPS*EMP*(ES1+ES2-C52M*(ES3+ES6))
A2N      = -TD52M*(B5MS*B5M*(FS1+FS2)-C52M*G5MS*G5M*(FS3+FS6))
A3N      = TD52M*EMPS*(B5M*(FS1+FS2)-C52M*G5M*(FS3+FS6))
A4N      = TD52M*EMP*(B5MS*(ES1+ES2)+C52M*G5MS*(ES3+ES6))
A(I+7*K,J+4*K) = -(V1*A1N+V2*A2N+V3*A3N+V4*A4N)*F2*DSIN(AL)
A1N      = -TD62M*EMPS*EMP*(ES1+ES2-C62M*(ES3+ES6))
A2N      = -TD62M*(B5MS*B5M*(FS1+FS2)+C62M*G5MS*G5M*(FS3+FS6))
A3N      = TD62M*EMPS*(B5M*(FS1+FS2)+C62M*G5M*(FS3+FS6))
A4N      = TD62M*EMP*EMPS*(ES1+ES2-C62M*G5MS*(ES3+ES6))
A(I+7*K,J+5*K) = -(V1*A1N+V2*A2N+V3*A3N+V4*A4N)*F2*DSIN(AL)
IF (B5MS.GT.QLIM) GO TO 257
A(J+5*K,J+4*K) = TD52M*(B5M*DCOSH(B5M)+C52M*G5M*DCOSH(G5M))
A(J+5*K,J+5*K) = TD62M*(B5M*DCOSH(B5M)+C62M*G5M*DCOSH(G5M))
GO TO 66
257 A(J+5*K,J+4*K) = TD52M*(B5M+C52M*G5M)

```

```

A(J+5*K,J+5*K) = TD62M*(B5M+C62M*G5M)
66 IF (TESTRM.LT.0.) GO TO 67
IF (MODE.GT.0.) GO TO 68

```

```

*
*
*

```

This step will generate elements with PHIRS*ALMDS > EMPS

```

A(J+K,J+6*K) = -1
A(J+K,J+7*K) = -DCOS(GRM)/DCOSH(GRM)
A1N = BRMS
A2N = -EMPS
A(J+4*K,J+6*K) = (PHIRS*A1N+POI*A2N)
A1N = -GRMS*DCOS(GRM)/DCOSH(GRM)
A2N = -EMPS*DCOS(GRM)/DCOSH(GRM)
A(J+4*K,J+7*K) = (PHIRS*A1N+POI*A2N)
A(J+6*K,J+6*K) = BRM*DSINH(BRM)*PHIR/(PHI1*DCOSH(BRM))
A(J+6*K,J+7*K) = -GRM*DSIN(GRM)*PHIR/(PHI1*DCOSH(GRM))
A1N = -BRMS*BRM*DSINH(BRM)/DCOSH(BRM)
A2N = EMPS*BRM*DSINH(BRM)/DCOSH(BRM)
A(J+7*K,J+6*K) = (A1N+(POIS*A2N/PHIRS))*PHIRS*PHIR/(PHI1S*PHI1)
A1N = -GRMS*GRM*DSIN(GRM)/DCOSH(GRM)
A2N = -EMPS*GRM*DSIN(GRM)/DCOSH(GRM)
A(J+7*K,J+7*K) = (A1N+(POIS*A2N/PHIRS))*PHIRS*PHIR/(PHI1S*PHI1)
A1RS = DI*BRM*DSINH(BRM)/(BRMS+ENPCS)
TEST = GRM-ENPC
IF (TEST.EQ.0.) GO TO 430
A2RS = DI*GRM*DSIN(GRM)/(GRMS-ENPCS)
GO TO 431
-30 A2RS = (DI*DCOS(GRM)/2.)+(DI*DSIN(GRM)/(2.*GRM))
-31 A(I+8*K,J+6*K) = EMP*DCOS(EMP*A1RS/DCOSH(BRM))
A(I+8*K,J+7*K) = EMP*DCOS(EMP*A2RS/DCOSH(GRM))
A(I+9*K,J+6*K) = EMP*A1RS/DCOSH(BRM)
A(I+9*K,J+7*K) = EMP*A2RS/DCOSH(GRM)
GO TO 254
66 A(J+K,J+6*K) = -1
A(J+K,J+7*K) = -DSIN(GRM)*DSINH(GRM)
A1N = BRMS
A2N = -EMPS
A(J+4*K,J+6*K) = (PHIRS*A1N+POI*A2N)
A1N = -GRMS*DSIN(GRM)*DSINH(GRM)
A2N = -EMPS*DSIN(GRM)*DSINH(GRM)
A(J+4*K,J+7*K) = (PHIRS*A1N+POI*A2N)
A(J+6*K,J+6*K) = BRM*DCOSH(BRM)*PHIR/(PHI1*DSINH(BRM))
A(J+6*K,J+7*K) = GRM*DCOS(GRM)*PHIR/(PHI1*DSINH(GRM))
A1N = -BRMS*BRM*DCOSH(BRM)/DSINH(BRM)
A2N = EMPS*BRM*DCOSH(BRM)/DSINH(BRM)
A(J+7*K,J+6*K) = (A1N+(POIS*A2N/PHIRS))*PHIRS*PHIR/(PHI1S*PHI1)
A1N = GRMS*GRM*DCOS(GRM)*DSINH(GRM)

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A2N      = EMPS*GRM*DCOS(GRM)/DSINH(GRM)
A(J-7*K,J+7*K) = (A1N-(POIS*A2N/PHIRS))*PHIRS*PHIR / (PHI1S*PHI1)
A1RS     = 2.*ENP*DSINH(BRM)/(BRMS+ENPS)
TEST=GRM-ENP
IF TEST.EQ.0.) GO TO 432
A2RS     = 2.*ENP*DSIN(GRM)/(ENPS-GRMS)
GO TO 433
432 A2RS   = DSIN(GRM-PI/2.))-(DCOS(GRM-(PI/2.)),GRM)
-33 A(I-6*K,J+6*K) = EMP*DCOS(EMP)*A1RS/DSINH(BRM)
A(I-6*K,J+7*K) = EMP*DCOS(EMP)*A2RS/DSINH(GRM)
A(I-9*K,J+6*K) = EMP*A1RS/DSINH(BRM)
A(I-9*K,J+7*K) = EMP*A2RS/DSINH(GRM)
GO TO 254
*
* This step will generate elements for which PHIRS*ALMDRS < EMPS
*
67 IF BRMS.GT.QLIM) GO TO 260
IF MODE.GT.0.) GO TO 69
A(J-K,J+6*K) = -1
A(J-K,J+7*K) = -1
A1N          = BRMS
A2N          = -EMPS
A(J-4*K,J+6*K) = (PHIRS*A1N+POI*A2N)
A1N          = GRMS
A2N          = -EMPS
A(J-4*K,J+7*K) = (PHIRS*A1N+POI*A2N)
A(J-6*K,J+6*K) = BRM*DSINH(BRM)*PHIR / (PHI1*DCOSH(BRM))
A(J-6*K,J+7*K) = GRM*DSINH(GRM)*PHIR / (PHI1*DCOSH(GRM))
A1N          = -BRMS*BRM*DSINH(BRM)/DCOSH(BRM)
A2N          = EMPS*BRM*DSINH(BRM)/DCOSH(BRM)
A(J-7*K,J+6*K) = (A1N-(POIS*A2N/PHIRS))*PHIRS*PHIR / (PHI1S*PHI1)
A1N          = -GRMS*GRM*DSINH(GRM)/DCOSH(GRM)
A2N          = EMPS*GRM*DSINH(GRM)/DCOSH(GRM)
A(J-7*K,J+7*K) = (A1N-(POIS*A2N/PHIRS))*PHIRS*PHIR / (PHI1S*PHI1)
A1RS         = DI*BRM*DSINH(BRM)/(BRMS+ENPCS)
A2RS         = DI*GRM*DSINH(GRM)/(GRMS+ENPCS)
A(I-8*K,J+6*K) = EMP*DCOS(EMP)*A1RS/DCOSH(BRM)
A(I-8*K,J+7*K) = EMP*DCOS(EMP)*A2RS/DCOSH(GRM)
A(I-9*K,J+6*K) = EMP*A1RS/DCOSH(BRM)
A(I-9*K,J+7*K) = EMP*A2RS/DCOSH(GRM)
GO TO 254
69 A(J-K,J+6*K) = -1
A(J-K,J+7*K) = -1
A1N          = BRMS
A2N          = -EMPS
A(J-4*K,J+6*K) = (PHIRS*A1N+POI*A2N)
A1N          = GRMS

```

```

A2N      = -EMPS
A(J+4*K,J+7*K) = (PHIRS*A1N+POI*A2N)
A(J+6*K,J+6*K) = BRM*DCOSH(BRM)*PHIR/(PHI1*DSINH(BRM))
A(J+6*K,J+7*K) = GRM*DCOSH(GRM)*PHIR/(PHI1*DSINH(GRM))
A1N      = -BRMS*BRM*DCOSH(BRM)/DSINH(BRM)
A2N      = EMPS*BRM*DCOSH(BRM)/DSINH(BRM)
A(J+7*K,J+6*K) = (A1N+(POIS*A2N/PHIRS))*PHIRS*PHIR/(PHI1S*PHI1)
A1N      = -GRMS*GRM*DCOSH(GRM)/DSINH(GRM)
A2N      = EMPS*GRM*DCOSH(GRM)/DSINH(GRM)
A(J+7*K,J+7*K) = (A1N+(POIS*A2N/PHIRS))*PHIRS*PHIR/(PHI1S*PHI1)
A1RS     = 2.*ENP*DSINH(BRM)/(BRMS+ENPS)
A3RS     = 2.*ENP*DSINH(GRM)/(GRMS+ENPS)
A(I+8*K,J+6*K) = EMP*DCOS(EMP)*A1RS/DSINH(BRM)
A(I+8*K,J+7*K) = EMP*DCOS(EMP)*A3RS/DSINH(GRM)
A(I+9*K,J+6*K) = EMP*A1RS/DSINH(BRM)
A(I+9*K,J+7*K) = EMP*A3RS/DSINH(GRM)
GO TO 254

```

* If MODE > 0., program will switch to anti-symmetric modes

260 IF (MODE.GT.0.) GO TO 70

```

A(J+K,J+6*K) = -1
A(J+K,J+7*K) = -1
A1N          = BRMS
A2N          = -EMPS
A(J+4*K,J+6*K) = (PHIRS*A1N+POI*A2N)
A1N          = GRMS
A2N          = -EMPS
A(J+4*K,J+7*K) = (PHIRS*A1N+POI*A2N)
A(J+6*K,J+6*K) = BRM*PHIR/PHI1
A(J+6*K,J+7*K) = GRM*PHIR/PHI1
A1N          = -BRMS*BRM
A2N          = EMPS*BRM
A(J+7*K,J+6*K) = (A1N+(POIS*A2N/PHIRS))*PHIRS*PHIR/(PHI1S*PHI1)
A1N          = -GRMS*GRM
A2N          = EMPS*GRM
A(J+7*K,J+7*K) = (A1N+(POIS*A2N/PHIRS))*PHIRS*PHIR/(PHI1S*PHI1)
A1RS         = DI*BRM/(BRMS+ENPCS)
A3RS         = DI*GRM/(GRMS+ENPCS)
A(I+8*K,J+6*K) = EMP*DCOS(EMP)*A1RS
A(I+8*K,J+7*K) = EMP*DCOS(EMP)*A3RS
A(I+9*K,J+6*K) = EMP*A1RS
A(I+9*K,J+7*K) = EMP*A3RS
GO TO 254

```

```

70 A(J+K,J+6*K) = -1
A(J+K,J+7*K) = -1
A1N          = BRMS

```

```

A2N = -EMPS
A(J-4*K,J-5*K) = (PHIRS*A1N-POI*A2N)
A1N = GRMS
A2N = -EMPS
A(J-4*K,J-7*K) = (PHIRS*A1N-POI*A2N)
A(J-5*K,J-6*K) = BRM*PHIR,PHI1
A(J-6*K,J-7*K) = GRM*PHIR,PHI1
A1N = -BRMS*BRM
A2N = EMPS*BRM
A(J-7*K,J-6*K) = (A1N-(POIS*A2N*PHIRS)*PHIRS*PHIR*(PHI1*PHI1))
A1N = -GRMS*GRM
A2N = EMPS*GRM
A(J-7*K,J-7*K) = (A1N-(POIS*A2N*PHIRS)*PHIRS*PHIR*(PHI1*PHI1))
A1RS = 2*ENP/(BRMS+ENPS)
A2RS = 2*ENP/(GRMS+ENPS)
A(I-8*K,J-6*K) = EMP*OCOS(EMP)*A1RS
A(I-8*K,J-7*K) = EMP*OCOS(EMP)*A2RS
A(I-9*K,J-6*K) = EMP*A1RS
A(I-9*K,J-7*K) = EMP*A2RS
254 CONTINUE
IF (TESTIR.LT.0.) GO TO 446
*
* This step will generate elements for which A2RS > EMPS
*
IF (MODE.GT.0) GO TO 445
A(I-K,J-8*K) = -TD11RM*(B1RS-C11RM*B2RS)
A1N = -TD11RM*EMPCS*(B1RS-C11RM*B2RS)
A2N = TD11RM*(B1RMS*B1RS-C11RM*G1RMS*B2RS)
A(I-4*K,J-8*K) = (A1N*PHIRS-A2N*POI)
A(J-9*K,J-8*K) = TD11RM*(B1RM-C11RM*G1RM)
A(I-K,J-9*K) = -TD11RM*(C1RS-C11RM*G1RS)
A1N = -TD11RM*EMPCS*(C1RS-C11RM*G1RS)
A2N = TD11RM*(B1RMS*C1RS-C11RM*G1RMS*G1RS)
A(I-4*K,J-9*K) = (A1N*PHIRS-A2N*POI)
A(J-8*K,J-9*K) = -TD11RM*(B1RM-C11RM*G1RM)
GO TO 100
445 A(I-6*K,J-8*K) = -TD11RM*EMP*(B1RS-C11RM*B2RS)*PHIR,PHI1
A1N = -TD11RM*EMPS*EMP*(B1RS-C11RM*B2RS)
A2N = TD11RM*EMP*(B1RMS*B1RS-C11RM*G1RMS*B2RS)
A(I-7*K,J-8*K) = (A1N*PHIRS-POIS*A2N*PHIR*(PHI1*PHI1))
A(J-9*K,J-8*K) = TD11RM*(B1RM-C11RM*G1RM)
A(I-6*K,J-9*K) = -TD11RM*EMP*(C1RS-C11RM*G1RS)*PHIR,PHI1
A1N = -TD11RM*EMPS*EMP*(C1RS-C11RM*G1RS)
A2N = TD11RM*EMP*(B1RMS*C1RS-C11RM*G1RMS*G1RS)
A(I-7*K,J-9*K) = (A1N*PHIRS-POIS*A2N*PHIR*(PHI1*PHI1))
A(J-8*K,J-9*K) = -TD11RM*(B1RM-C11RM*G1RM)
GO TO 100

```

```

*
*   This step will generate terms for which ALMDS < EMPS
*
446 IF (MODE.GT.0) GO TO 447
  A(I+K,J+8*K) = -TD12RM*(B1RS+C12RM*B3RS)
  A1N          = -TD12RM*EMPCS*(B1RS+C12RM*B3RS)
  A2N          = TD12RM*(B1RMS*B1RS-C12RM*G1RMS*B3RS)
  A(I+4*K,J+8*K) = (A1N*PHIRS+POI*A2N)
  A(I+K,J+9*K) = -TD12RM*(C1RS+C12RM*C3RS)
  A1N          = -TD12RM*EMPCS*(C1RS+C12RM*C3RS)
  A2N          = TD12RM*(B1RMS*C1RS-C12RM*G1RMS*C3RS)
  A(I+4*K,J+9*K) = (A1N*PHIRS+POI*A2N)
  GO TO 448
447 A(I+6*K,J+8*K) = -TD12RM*EMP*(B1RS+C12RM*B3RS)*PHIR/PHI1
  A1N              = -TD12RM*EMPS*EMP*(B1RS+C12RM*B3RS)
  A2N              = TD12RM*EMP*(B1RMS*B1RS+C12RM*G1RMS*B3RS)
  A(I+7*K,J+8*K) = (A1N*PHIRS+POIS*A2N)*PHIR/(PHI1S*PHI1)
  A(I+6*K,J+9*K) = -TD12RM*EMP*(C1RS-C12RM*C3RS)*PHIR/PHI1
  A1N              = -TD12RM*EMPS*EMP*(C1RS+C12RM*C3RS)
  A2N              = TD12RM*EMP*(B1RMS*C1RS+C12RM*G1RMS*C3RS)
  A(I+7*K,J+9*K) = (A1N*PHIRS+POIS*A2N)*PHIR/(PHI1S*PHI1)
448 IF (B1RMS.GT.QLIM) GO TO 449
  A(J+9*K,J+8*K) = TD12RM*(B1RM+C12RM*G1RM)
  A(J+8*K,J+9*K) = -TD12RM*(B1RM-C12RM*G1RM)
  GO TO 100
449 A(J+9*K,J+8*K) = 0.0
  A(J+8*K,J+9*K) = 0.0
100 CONTINUE
  I=10*K
*
*   If DEL=0., program will switch to shape data generation mode.
*
  IF (DEL.EQ.0.) GO TO 103
  CALL DETERM (A,I,DET)
  WRITE(6,101) ALMDS,DET
101 FORMAT(' ',3X,'ALMDS =',F10.6,3X,'DET =',D15.6)
  IF (ALMDS.LT.DLIM) GO TO 102
  IF (ALMDS.GT.0.001) GO TO 90
  GO TO 400
102 ALMDS=ALMDS+DEL
  GO TO 2
103 CALL DETSOL (A,I,X)
  X(10*K)=-1
  DO 104 I=1,K
  E1M(I)=X(I)
  V2M(I)=X(I+K)
  E3M(I)=X(I+2*K)

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```

V4M(I)=X(I+3*K)
E5M(I)=X(I+4*K)
V6M(I)=X(I+5*K)
AM(I)=X(I+6*K)
DM(I)=X(I+7*K)
E1RM(I)=X(I+8*K)
E2RM(I)=X(I+9*K)
104 CONTINUE
L=2*KS+1
DO 105 I=1,KS1
DO 105 J=1,L
105 W(I,J)=0.

```

```

*
* This portion of the program will generate shape data.
*

```

```

ALMDPS=ALMDS/F2
ALMDRS=ALMDS*PHI1S/PHIRS
ETA=0.
DO 309 I=1,KS1
PSI=1.
DO 308 J=1,I
PSIP=PSI*DSIN(AL)*DSIN(AL)+(1.-ETA)*DCOS(AL)*DSIN(AL)
ETAP=ETA-1+PSI
W11=0.
W22=0.
W33=0.
W44=0.
W55=0.
W66=0.
DO 307 M=1,N
EMP=M*PI
EMPS=EMP*EMP
B1MS=PHI1S*(ALMDS-EMPS)
B1M=DSQRT(B1MS)
TEST1M=ALMDS-EMPS
G1MS=PHI1S*TEST1M
IF (G1MS.LT.0.) G1MS=-G1MS
G1M=DSQRT(G1MS)
B3MS=PHI3S*(PHI1S*ALMDS-EMPS)
B3M=DSQRT(B3MS)
TEST3M=PHI1S*ALMDS-EMPS
G3MS=PHI3S*TEST3M
IF (G3MS.LT.0.) G3MS=-G3MS
G3M=DSQRT(G3MS)
B5MS=PHI5S*(ALMDPS+EMPS)
B5M=DSQRT(B5MS)
TEST5M=ALMDPS-EMPS

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G5MS=PHI5S*TEST5M
IF (G5MS.LT.0.) G5MS=-G5MS
G5M=DSQRT(G5MS)

*
*
*
      HERE

IF (TEST1M.LT.0.) GO TO 270
C11M=-DSINH(B1M)/DSIN(G1M)
TD11M=-1./((B1MS*DSINH(B1M)-C11M*G1MS*DSIN(G1M)))
C21M=(B1MS-POI*PHI1S*EMPS)*DSINH(B1M)/((G1MS-POI*PHI1S*EMPS)*
DSIN(G1M))
TD21M=1./((DSINH(B1M)+C21M*DSIN(G1M)))
XW1=E1M(M)*TD11M*(DSINH(B1M*ETA)+C11M*DSIN(G1M*ETA))*DSIN(EMP*PSI)
XW2=V2M(M)*TD21M*(DSINH(B1M*ETA)+C21M*DSIN(G1M*ETA))*DSIN(EMP*PSI)
GO TO 274
270 IF (B1MS.GT.QLIM) GO TO 271
C12M=-DSINH(B1M)/DSINH(G1M)
TD12M=-1./((B1MS*DSINH(B1M)+C12M*G1MS*DSINH(G1M)))
C22M=-((B1MS-POI*PHI1S*EMPS)*DSINH(B1M)/((G1MS-POI*PHI1S*EMPS)*
DSINH(G1M)))
TD22M=1./((DSINH(B1M)+C22M*DSINH(G1M)))
XW1=E1M(M)*TD12M*(DSINH(B1M*ETA)+C12M*DSINH(G1M*ETA))*DSIN(EMP*
PSI)
XW2=V2M(M)*TD22M*(DSINH(B1M*ETA)+C22M*DSINH(G1M*ETA))*DSIN(EMP*
PSI)
GO TO 274
271 C12M=-1.
TD12M=-1./((B1MS-C12M*G1MS)
C22M=-((B1MS-POI*PHI1S*EMPS)/(G1MS-POI*PHI1S*EMPS)
TD22M=1./((1-C22M)
B=1.
TEST=B1M-B1M*ETA
IF (TEST.GT.60.) B=0.
A1M=DEXP((B1M*ETA-B1M)*B)*B
A2M=DEXP((G1M*ETA-G1M)*B)*B
XW1=E1M(M)*TD12M*(A1M-C12M*A2M)*DSIN(EMP*PSI)
XW2=V2M(M)*TD22M*(A1M-C22M*A2M)*DSIN(EMP*PSI)
IF (ETA.EQ.0.) XW1=0.
IF (PSI.EQ.0.) XW1=0.
IF (ETA.EQ.0.) XW2=0.
IF (PSI.EQ.0.) XW2=0.
274 IF (TEST3M.LT.0.) GO TO 275
C31M=-DSINH(B3M)/DSIN(G3M)
TD31M=-1./((B3MS*DSINH(B3M)-C31M*G3MS*DSIN(G3M)))
C41M=(B3MS-POI*PHI3S*EMPS)*DSINH(B3M)/((G3MS-POI*PHI3S*EMPS)*
DSIN(G3M))
TD41M=1./((DSINH(B3M)+C41M*DSIN(G3M)))

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NW3=E3M(M)*TD31M*(DSINH(B3M*PSI)-C31M*DSIN(G3M*PSI))*DSIN(EMP*ETA)
NW4=V4M(M)*TD41M*(DSINH(B3M*PSI)-C41M*DSIN(G3M*PSI))*DSIN(EMP*ETA)
GO TO 279
275 IF(B3MS.GT.QLIM) GO TO 276
C32M=-DSINH(B3M)/DSINH(G3M)
TD32M=-1./((B3MS*DSINH(B3M)-C32M*G3MS*DSINH(G3M))
C42M=-(B3MS-POI*PHI3S*EMPS)*DSINH(B3M)/(G3MS-POI*PHI3S*EMPS*
DSINH(G3M))
TD42M=1./((DSINH(B3M)-C42M*DSINH(G3M))
NW3=E3M(M)*TD32M*(DSINH(B3M*PSI)-C32M*DSINH(G3M*PSI))*DSIN(EMP*
ETA)
NW4=V4M(M)*TD42M*(DSINH(B3M*PSI)-C42M*DSINH(G3M*PSI))*DSIN(EMP*
ETA)
GO TO 279
276 C32M=-1.
TD32M=-1./((B3MS-C32M*G3MS)
C42M=-(B3MS-POI*PHI3S*EMPS)/(G3MS-POI*PHI3S*EMPS)
TD42M=1./((1.-C42M)
B=1.
TEST=B3M-B3M*PSI
IF(TEST.GT.60.) B=0.
A1M=DENP((B3M*PSI-B3M)*B)*B
A2M=DENP((G3M*PSI-G3M)*B)*B
NW3=E3M(M)*TD32M*(A1M-C32M*A2M)*DSIN(EMP*ETA)
NW4=V4M(M)*TD42M*(A1M-C42M*A2M)*DSIN(EMP*ETA)
IF(ETA.EQ.0.) NW3=0.
IF(PSI.EQ.0.) NW3=0.
IF(ETA.EQ.0.) NW4=0.
IF(PSI.EQ.0.) NW4=0.
279 NW1=B3M-B3M*ETAP
NW2=G3M-G3M*ETAP
IF (TEST3M.LT.0.) GO TO 280
C51M=-DSINH(B5M)/DSINH(G5M)
TD51M=-1./((B5MS*DSINH(B5M)-C51M*G5MS*DSINH(G5M))
C61M=-(B5MS-POI*PHI5S*EMPS)*DSINH(B5M)/(G5MS-POI*PHI5S*EMPS*
DSINH(G5M))
TD61M=1./((DSINH(B5M)-C61M*DSINH(G5M))
NW3=E5M(M)*TD51M*(DSINH(NW1)-C51M*DSINH(NW2))*DSIN(EMP*PSI)
NW4=V6M(M)*TD61M*(DSINH(NW1)-C61M*DSINH(NW2))*DSIN(EMP*PSI)
GO TO 281
280 IF(B5MS.GT.QLIM) GO TO 281
C52M=-DSINH(B5M)/DSINH(G5M)
TD52M=-1./((B5MS*DSINH(B5M)-C52M*G5MS*DSINH(G5M))
C62M=-(B5MS-POI*PHI5S*EMPS)*DSINH(B5M)/(G5MS-POI*PHI5S*EMPS*
DSINH(G5M))
TD62M=1./((DSINH(B5M)-C62M*DSINH(G5M))
NW3=E5M(M)*TD52M*(DSINH(NW1)-C52M*DSINH(NW2))*DSIN(EMP*PSI)

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```

XW6=V6M(N)*TD62M*(DSINH(XN1)+C62M*DSINH(XN2)+DSIN(EMP*PSIP)
GO TO 284
281 C52M=-1.
TD52M=-1./(B5MS-C52M*G5MS)
C62M=-1.(B5MS-POI*PHISS*EMPS)/(G5MS-POI*PHISS*EMPS)
TD62M=1./(1.-C62M)
B=1.
TEST=B5M*ETAP
IF(TEST.GT.60.) B=0.
A1M=DEXP(-B5M*ETAP*B)*B
A2M=DEXP(-G5M*ETAP*B)*B
XW5=E5M(M)*TD52M*(A1M-C52M*A2M)*DSIN(EMP*PSIP)
XW6=V6M(N)*TD62M*(A1M-C62M*A2M)*DSIN(EMP*PSIP)
IF(PSIP.EQ.0.) XW5=0.
IF(PSIP.EQ.3.) XW6=0.
IF(ETAP.EQ.1.) XW5=0.
IF(ETAP.EQ.1.) XW6=0.
IF(ETAP.EQ.0.) XW5=0.
IF(ETAP.EQ.0.) XW6=V6M(N)*DSIN(EMP*PSIP)
284 W11=W11+XW1
W22=W22+XW2
W33=W33+XW3
W44=W44+XW4
W55=W55+XW5
W66=W66+XW6
307 CONTINUE
W(I,J-KS)=W11+W22+W33+W44+W55+W66
PSI=PSI-1./DFLOAT(KS)
308 CONTINUE
ETA=ETA-1./DFLOAT(KS)
309 CONTINUE
ETA=0.
DO 313 I=1,KS:
PSI=0.
DO 314 J=1,KS:
W11=0.
DO 315 M=1,K
NC=M-1
EMP=M*PI
EMPS=EMP*EMP
EMPC=NC*PI
IF (MODE.GT.0) EMPC=EMP.
EMPCS=EMPC*EMPC
BRMS=(PHIRS*ALNDRS-EMPS)/PHIRS
BRM=DSQRT(BRMS)
BIRMS=PHIRS*(ALNDRS-EMPCS)
BIRM=DSQRT(BIRMS)

```

```

TESTRM=PHIRS*ALMDRS-EMPS
GRMS=TESTRM/PHIRS
IF (GRMS.LT.0.) GRMS=-GRMS
GRM=DSQRT(GRMS)
TESTIR=ALMDRS-EMPCS
GIRMS=PHIRS*TESTIR
IF (GIRMS.LT.0.) GIRMS=-GIRMS
GIRM=DSQRT(GIRMS)
XX1=BRM-BRM*PSI
XX2=GRM-GRM*PSI
IF (TESTRM.LT.0.) GO TO 285
IF (MODE.GT.0) GO TO 321
NW1=((AM(M)*DCOSH(XX1)*DCOSH(BRM))-(DM(M)*DCOSH(XX2)*DCOSH(GRM))
1*DSIN(EMP*ETA)
GO TO 312
321 NW1=((AM(M)*DSINH(XX1)*DSINH(BRM))-(DM(M)*DSINH(XX2)*DSINH(GRM))
1*DSIN(EMP*ETA)
GO TO 312
285 IF (MODE.GT.0) GO TO 322
IF (BRMS.GT.QLIM) GO TO 286
NW1=((AM(M)*DCOSH(XX1)*DCOSH(BRM))-(DM(M)*DCOSH(XX2)*DCOSH(GRM))
1*DSIN(EMP*ETA)
GO TO 312
286 B=1.
TEST=BRM*PSI
TESTS=TEST*TEST
IF (TESTS.GT.QLIM) B=0.
A1M=DEXP(-BRM*PSI*B)*B
A2M=DEXP(-GRM*PSI*B)*B
* IF (GRMS.LT.QLIM) A2M=DCOSH(GRM*(1-PSI))*DCOSH(GRM)
NW1=(AM(M)*A1M-DM(M)*A2M)*DSIN(EMP*ETA)
GO TO 312
322 IF (BRMS.GT.QLIM) GO TO 287
NW1=((AM(M)*DSINH(XX1)*DSINH(BRM))-(DM(M)*DSINH(XX2)*DSINH(GRM))
1*DSIN(EMP*ETA)
GO TO 312
287 B=1.
TEST=BRM*PSI
TESTS=TEST*TEST
IF (TESTS.GT.QLIM) B=0.
A1M=DEXP(-BRM*PSI*B)*B
A2M=DEXP(-GRM*PSI*B)*B
* IF (GRMS.LT.QLIM) A2M=DSINH(GRM*(1-PSI))*DSINH(GRM)
NW1=(AM(M)*A1M-DM(M)*A2M)*DSIN(EMP*ETA)
IF (PSI.EQ.1.) NW1=0.
312 CONTINUE
IF (TESTIR.LT.0.) GO TO 283

```

```

C11RM=-DSINH(B1RM)/DSINH(G1RM)
TD11RM=1./(B1RM*DCOSH(B1RM)+C11RM*G1RM*DCOS(G1RM))
IF (MODE.GT.0) GO TO 452
W1R=E1RM(M)*TD11RM*(DSINH(B1RM*ETA)+C11RM*DSINH(G1RM*ETA))*
1DCOS(EMP*PSI)
W2R=E2RM(M)*TD11RM*(DSINH(B1RM*(1.-ETA))+C11RM*DSINH(G1RM*(1.-ETA)
1))*DCOS(EMP*PSI)
GO TO 459
452 W1R=E1RM(M)*TD11RM*(DSINH(B1RM*ETA)+C11RM*DSINH(G1RM*ETA))*
1DSIN(EMP*PSI)
W2R=E2RM(M)*TD11RM*(DSINH(B1RM*(1.-ETA))+C11RM*DSINH(G1RM*(1.-ETA)
1))*DSIN(EMP*PSI)
GO TO 459
453 IF (B1RM.GT.QLIM) GO TO 455
C12RM=-DSINH(B1RM)/DSINH(G1RM)
TD12RM=1./(B1RM*DCOSH(B1RM)+C12RM*G1RM*DCOSH(G1RM))
IF (MODE.GT.0) GO TO 454
W1R=E1RM(M)*TD12RM*(DSINH(B1RM*ETA)+C12RM*DSINH(G1RM*ETA))*
1DCOS(EMP*PSI)
W2R=E2RM(M)*TD12RM*(DSINH(B1RM*(1.-ETA))+C12RM*DSINH(G1RM*(1.-ETA)
1))*DCOS(EMP*PSI)
GO TO 459
454 W1R=E1RM(M)*TD12RM*(DSINH(B1RM*ETA)+C12RM*DSINH(G1RM*ETA))*
1DSIN(EMP*PSI)
W2R=E2RM(M)*TD12RM*(DSINH(B1RM*(1.-ETA))+C12RM*DSINH(G1RM*(1.-ETA)
1))*DSIN(EMP*PSI)
GO TO 459
455 C12RM=-1.
TD12RM=1./(B1RM-C12RM*G1RM)
TEST=B1RM*(1.-ETA)
TESTS=TEST*TEST
B=1.
IF (TESTS.GT.QLIM) B=0.
A2N=DEXP(-TEST*B)*B
TEST=B1RM*ETA
TESTS=TEST*TEST
B=1.
IF (TESTS.GT.QLIM) B=0.
A2N=DEXP(-TEST*B)*B
TEST=G1RM*(1.-ETA)
TESTS=TEST*TEST
B=1.
IF (TESTS.GT.QLIM) B=0.
A2N=DEXP(-TEST*B)*B
TEST=G1RM*ETA
TESTS=TEST*TEST
B=1.

```

```

IF (TESTS.GT.QLIM) B=0.
A4N=DEXP(-TEST*B)*B
IF (MODE.GT.0.) GO TO 458
W1R=E1RM(M)*TD12RM*(A1N-C12RM*A2N)*DCOS(EMP*PSI)
W2R=E2RM(M)*TD12RM*(A3N-C12RM*A4N)*DCOS(EMP*PSI)
GO TO 459
458 W1R=E1RM(M)*TD12RM*(A1N-C12RM*A2N)*DSIN(EMP*PSI)
W2R=E2RM(M)*TD12RM*(A3N-C12RM*A4N)*DSIN(EMP*PSI)
459 W11=W11+KW1-W1R-W2R
313 CONTINUE
JREV=KS1+1-J
W(I,JREV)=W11
PSI=PSI+1./DFLOAT(KS)
314 CONTINUE
ETA=ETA+1./DFLOAT(KS)
315 CONTINUE
*
* If RATIO < 0., the displacement plate to boundary ratio will
* not be computed.
IF (RATIO.LT.0) GO TO 318
SUM=0.
SUM1=0.
DO 316 I=1,21
KK=KS-I
DO 316 J=1,KK
IF (I.NE.1) GO TO 621
SUM1=SUM1-DABS(W(I,J))
621 IF (I.NE.21) GO TO 622
SUM1=SUM1-DABS(W(I,J))
622 SUM=SUM-DABS(W(I,J))
316 CONTINUE
DO 317 I=2,20
J=KS-I
SUM1=SUM1-DABS(W(I,J))
IF (MODE.EQ.0) GO TO 317
SUM1=SUM1-DABS(W(I,1))
317 CONTINUE
SUM=SUM-SUM1
QK1=81
IF (MODE.GT.0) QK1=100
QK2=570
IF (MODE.GT.0) QK2=551
RAT=SUM*(QK1/(SUM1+QK2))
WRITE(6,319) RAT
319 FORMAT('1',10X,'RATIO = ',F18.5,////)
CONTINUE
318 CALL NODE (W)

```

```

400 STOP
END
SUBROUTINE DETERM (A,N,DET)
IMPLICIT REAL*8(A-H,O-Z)
DIMENSION A(160,160)
SIGN=1.
LAST=N-1
DO 200 I=1, LAST
BIG=0.
DO 50 K=I, N
TERM=DABS(A(K,I))
IF (TERM-BIG)50,50,30
30 BIG=TERM
L=K
50 CONTINUE
IF (BIG)80,60,80
80 IF (I-L)90,120,90
90 SIGN=-SIGN
DO 100 J=I, N
TEMP=A(I,J)
A(I,J)=A(L,J)
100 A(L,J)=TEMP
120 PIVOT=A(I,I)
NEXTR=I+1
DO 200 J=NEXTR, N
CONST=A(J,I)/PIVOT
DO 300 K=I, N
200 A(J,K)=A(J,K)-CONST*A(I,K)
DET=SIGN
DO 300 I=1, N
300 DET=DET*A(I,I)*10.
GO TO 61
60 DET=0.
61 RETURN
END
SUBROUTINE DETSOL (A,N,M)
IMPLICIT REAL*8(A-H,O-Z)
DIMENSION A(160,160),X(160)
SIGN=1.
M=M-1
LAST=M-1
DO 200 I=1, LAST
BIG=0.
DO 50 K=I, M
TERM=DABS(A(K,I))
IF (TERM-BIG)50,50,30
30 BIG=TERM

```

```

      L=K
50  CONTINUE
      IF (BIG)80,60,80
80  IF (I-L)90,120,90
90  DO 100 J=1,N
      TEMP=A(I,J)
      A(I,J)=A(L,J)
100  A(L,J)=TEMP
120  PIVOT=A(I,I)
      NEXTR=I+1
      DO 200 J=NEXTR,M
      CONST=A(J,I)/PIVOT
      DO 200 K=I,N
200  A(J,K)=A(J,K)-CONST*A(I,K)
      M=N-1
      DO 500 I=1,M
      IREV=M-1-I
      Y=A(IREV,N)
      IF (IREV-M) 400,500,-80
400  DO 450 J=2,I
      K=N+1-J
450  Y=Y-A(IREV,K)*X(K)
500  X(IREV)=Y/A(IREV,IREV)
60  RETURN
      END
      SUBROUTINE NODE(W)
      IMPLICIT REAL*8(A-H,O-Z)
      DIMENSION W(21,41)
      CHARACTER*1 SIGN(21,41),PLUS,MINUS,ZERO,STAR
      PLUS = '+'
      ZERO = '0'
      MINUS = '-'
      STAR = '*'
      N=21
      K=2*N-1
      DO 6 I=1,N
      DO 6 J=1,K
6  SIGN(I,J)=' '
      DO 1 I=1,N
      II=I-N-1
      DO 1 J=1,II
      IF (W(I,J).LE.0) GO TO 11
      SIGN(I,J)=PLUS
11  IF (W(I,J).GE.0) GO TO 12
      SIGN(I,J)=MINUS
12  IF (W(I,J).NE.0) GO TO 1
      SIGN(I,J)=ZERO

```

```

1 CONTINUE
  DO 2 I=1,N
    II=I+N-1
    SIGN(I,1)=STAR
    SIGN(I,II)=STAR
2 CONTINUE
  DO 3 I=1,K
3 SIGN(N,I)=STAR
  DO 7 I=1,N
7 SIGN(1,I)=STAR
  DO 4 I=1,N
4 WRITE(6,5) (SIGN(I,J),J=1,K)
5 FORMAT(' ',41A1)
  RETURN
  END

```

APPENDIX A-III

COSINE FUNCTION ALTERNATIVE SOLUTION

Displacement Along the Edge $\eta = 1$

Considering plate (a') of Figure 3.7 it is seen that this plate has a forbidden lateral motion and hence no contribution toward displacements along the edge $\eta = 1$. Plate (b') on the other hand has a forced edge displacement that can be adjusted to balance the net contribution of the remaining plates, and is given by;

$$W_2(\xi, \eta) = \sum_{m=0,1}^{\infty} V_{2m} \cos(m\pi\xi).$$

The contribution of Plate (c') is;

$$\begin{aligned} W_3(\xi, 1) = & \sum_{m=0,1}^{k^*} E_{3m} \theta_{31m} [\sinh \beta_{3m} \xi + C_{31m} \sin \gamma_{3m} \xi] \cos(m\pi) \\ & + \sum_{m=k^*+1}^{\infty} E_{3m} \theta_{32m} [\sinh \beta_{3m} \xi + C_{32m} \sinh \gamma_{3m} \xi] \cos(m\pi). \end{aligned}$$

performing the necessary Fourier transformation to find;

$$A_{i,j+2k} = \theta_{32m}[A_1 + C_{32m}A_3]\cos(m\pi),$$

$$A_{i,j+3k} = \theta_{41m}[A_1 + C_{41m}A_2]\cos(m\pi),$$

or if $\phi_1^2 \lambda^2 < (m\pi)^2$ then,

$$A_{i,j+2k} = \theta_{32m}[A_1 + C_{32m}A_3]\cos(m\pi),$$

$$A_{i,j+3k} = \theta_{42m}[A_1 + C_{42m}A_3]\cos(m\pi),$$

where the second of these equation is the contribution of Plate (d'), and where,

$$A_1 = \frac{\delta\beta_{3m}}{\beta_{3m}^2 + (n\pi)^2} [\cosh\beta_{3m}\cos(n\pi) - 1],$$

$$A_2 = \frac{\delta\gamma_{3m}}{\gamma_{3m}^2 - (n\pi)^2} [1 - \cos(\gamma_{3m} + n\pi)],$$

or if $\gamma_{3m} = n\pi$ then,

$$A_{2n} = \delta(1 - \cos 2\gamma_{3m})/4\gamma_{3m},$$

and A_3 is obtained from the expression of A_1 by replacing β_{3m} by γ_{3m} .

Considering now the contributions of Plates (e') and (f'), to find;

$$A_{i,j+4k} = \theta_{51m}[B_1 + B_2 + C_{51m}(B_3 + B_4)],$$

$$A_{i,j+5k} = \theta_{61m}[B_1 + B_2 + C_{61m}(B_3 + B_4)],$$

or if $\lambda'^2 < (m\pi)^2$ then,

$$A_{i,j+4k} = \theta_{52m}[B_1 + B_2 + C_{52m}(B_5 + B_6)],$$

$$A_{i,j+5k} = \theta_{62m} [B_1 + B_2 + C_{62m}(B_5 + B_6)],$$

where,

$$\begin{aligned} B_1 &= \frac{\delta\beta_{5m}[\cosh\beta_{5m} - \cos(m\pi F_2 - n\pi)]}{2[\beta_{5m}^2 + (m\pi F_2 - n\pi)^2]}, \\ B_2 &= \frac{\delta\beta_{5m}[\cosh\beta_{5m} - \cos(m\pi F_2 + n\pi)]}{2[\beta_{5m}^2 + (m\pi F_2 + n\pi)^2]}, \\ B_3 &= \frac{\delta\gamma_{5m}[\cos\gamma_{5m} - \cos(m\pi F_2 - n\pi)]}{2[(m\pi F_2 - n\pi)^2 - \gamma_{5m}^2]}, \\ B_4 &= \frac{\delta\gamma_{5m}[\cos\gamma_{5m} - \cos(m\pi F_2 + n\pi)]}{2[(m\pi F_2 + n\pi)^2 - \gamma_{5m}^2]}, \end{aligned}$$

or if $\gamma_{5m}^2 = (m\pi F_2 - n\pi)^2$ then, $B_3 = \delta \sin \gamma_{5m} / 4$, and if $\gamma_{5m}^2 = (m\pi F_2 + n\pi)^2$ then, $B_4 = \delta \sin \gamma_{5m} / 4$. Also, B_5 and B_6 are obtained from the expressions of B_1 and B_2 respectively, by replacing β_{5m} by γ_{5m} .

Displacement Along the Edge $\xi = 1$

From the contributions of Plates (a') and (b') to displacement along this edge, it is found that;

$$A_{i+k,j} = \theta_{11m} [C_1 + C_{11m} C_2] \cos(m\pi),$$

$$A_{i+k,j+k} = \theta_{21m} [C_1 + C_{21m} C_2] \cos(m\pi),$$

or if $\lambda^2 < (m\pi)^2$ then,

$$A_{i+k,j} = \theta_{12m} [C_1 + C_{12m} C_3] \cos(m\pi),$$

$$A_{i+k,j+k} = \theta_{22m} [C_1 + C_{22m} C_3] \cos(m\pi),$$

where,

$$C_1 = \frac{\delta\beta_{1m}[\cosh\beta_{1m}\cos(n\pi) - 1]}{\beta_{1m}^2 + (n\pi)^2},$$

$$C_2 = \frac{\delta\gamma_{1m}[1 - \cos(\gamma_{1m} + n\pi)]}{\gamma_{1m}^2 - (n\pi)^2},$$

or if $\gamma_{1m} = n\pi$ then, $C_2 = \delta(1 - \cos 2\gamma_{1m})/4\gamma_{1m}$, and C_3 is obtained from the expression of C_1 by replacing β_{1m} by γ_{1m} .

Plate (c') has no contribution while Plate (d') has a forced displacement given by;

$$W_4(1, \eta) = \sum_{m=0,1}^{\infty} V_{4m} \cos(m\pi\eta).$$

Considering now the two plates (e') and (f') to find;

$$A_{i+k, j+4k} = \theta_{51m} [D_1 + D_2 + C_{51m} (D_3 + D_4)],$$

$$A_{i+k, j+5k} = \theta_{61m} [D_1 + D_2 + C_{61m} (D_3 + D_4)],$$

or if $\lambda'^2 < (m\pi)^2$ then,

$$A_{i+k, j+4k} = \theta_{52m} [D_1 + D_2 + C_{52m} (D_5 + D_6)],$$

$$A_{i+k, j+5k} = \theta_{62m} [D_1 + D_2 + C_{62m} (D_5 + D_6)],$$

where,

$$D_1 = \delta\beta_{5m} \frac{\cosh\beta_{5m} \cos(m\pi) - \cos[m\pi - (m\pi F_1 + n\pi)]}{2[\beta_{5m}^2 + (m\pi F_1 + n\pi)^2]},$$

$$D_2 = \delta\beta_{5m} \frac{\cosh\beta_{5m} \cos(m\pi) - \cos[m\pi - (m\pi F_1 - n\pi)]}{2[\beta_{5m}^2 + (m\pi F_1 - n\pi)^2]},$$

$$D_3 = \delta\gamma_{5m} \frac{\cos(\gamma_{5m} - m\pi) - \cos[(m\pi F_1 + n\pi) - m\pi]}{2[(m\pi F_1 + n\pi)^2 - \gamma_{5m}^2]},$$

$$D_4 = \delta\gamma_{5m} \frac{\cos(\gamma_{5m} - m\pi) - \cos[(m\pi F_1 - n\pi) - m\pi]}{2[(m\pi F_1 - n\pi)^2 - \gamma_{5m}^2]},$$

or if $\gamma_{5m}^2 = (m\pi F_1 + n\pi)^2$ then, $D_3 = \delta \sin(\gamma_{5m} + m\pi)/4$, and if $\gamma_{5m}^2 = (m\pi F_1 - n\pi)^2$ then, $D_4 = \delta \sin(\gamma_{5m} - m\pi)/4$.

In order to enforce the displacement continuity requirement along this intersegment line, the net displacement of the rectangular element along this line is evaluated and expanded in an appropriate Fourier series. The following expressions result;

$$A_{i+k,j+6k} = -\cosh\beta_{rm}A_{nr}, \quad (\text{symmetric})$$

$$A_{i+k,j+6k} = -\sinh\beta_{rm}A_{nr}, \quad (\text{anti-symmetric})$$

$$A_{i+k,j+7k} = -\cos\gamma_{rm}A_{nr}, \quad (\text{symmetric})$$

$$A_{i+k,j+7k} = -\sin\gamma_{rm}A_{nr}, \quad (\text{anti-symmetric})$$

or if $\phi_r^2\lambda_r^2 < (m\pi)^2$ then,

$$A_{i+k,j+7k} = -\cosh\gamma_{rm}A_{nr}, \quad (\text{symmetric})$$

$$A_{i+k,j+7k} = -\sinh\gamma_{rm}A_{nr}, \quad (\text{anti-symmetric})$$

where $A_{nr} = 0$ if $m = n$, and if $m \neq n$ then,

$$A_{nr} = \delta m\pi[1 - \cos(m\pi + n\pi)]/[(m\pi)^2 - (n\pi)^2].$$

Displacement Along the Edge $\eta' = 0$

From the contributions of Plats (a') and (b') toward displacements along this edge it is found that;

$$A_{i+2k,j} = \theta_{11m}[E_1 + E_2 + C_{11m}(E_3 + E_4)],$$

$$A_{i+2k,j+k} = \theta_{21m}[E_1 + E_2 + C_{21m}(E_3 + E_4)],$$

or if $\lambda^2 < (m\pi)^2$ then,

$$A_{i+2k,j} = \theta_{12m}[E_1 + E_2 + C_{12m}(E_5 + E_6)],$$

$$A_{i+2k,j+k} = \theta_{22m}[E_1 + E_2 + C_{22m}(E_5 + E_6)],$$

where;

$$E_1 = \delta\beta_{1m} \frac{\cosh\beta_{1m} - \cos(m\pi - n\pi)}{2[\beta_{1m}^2 + (m\pi - n\pi)^2]},$$

$$E_2 = \delta\beta_{1m} \frac{\cosh\beta_{1m} - \cos(m\pi + n\pi)}{2[\beta_{1m}^2 + (m\pi + n\pi)^2]},$$

$$E_3 = \delta\gamma_{1m} \frac{\cos\gamma_{1m} - \cos(m\pi - n\pi)}{2[(m\pi - n\pi)^2 - \gamma_{1m}^2]},$$

$$E_4 = \delta\gamma_{1m} \frac{\cos\gamma_{1m} - \cos(m\pi + n\pi)}{2[(m\pi + n\pi)^2 - \gamma_{1m}^2]},$$

or if $\gamma_{1m}^2 = (m\pi - n\pi)^2$ then, $E_3 = \delta\sin\gamma_{1m}/4$, and if $\gamma_{1m}^2 = (m\pi + n\pi)^2$ then, $E_4 = \delta\sin\gamma_{1m}/4$. E_5 and E_6 are also found by replacing β_{1m} by γ_{1m} in the expressions of E_1 and E_2 respectively.

Turning now to the contributions of Plates (c') and (d') to find;

$$A_{i+2k,j+2k} = \theta_{31m}[G_1 + G_2 + C_{31m}(G_3 + G_4)],$$

$$A_{i+2k,j+3k} = \theta_{41m}[G_1 + G_2 + C_{41m}(G_3 + G_4)],$$

or if $\phi_1^2\lambda^2 < (m\pi)^2$ then,

$$A_{i+2k,j+2k} = \theta_{32m}[G_1 + G_2 + C_{32m}(G_5 + G_6)],$$

$$A_{i+2k,j+3k} = \theta_{42m}[G_1 + G_2 + C_{42m}(G_5 + G_6)],$$

where,

$$G_1 = \delta\beta_{3m} \frac{\cosh\beta_{3m}\cos(n\pi) - \cos(m\pi)}{2[\beta_{3m}^2 + (m\pi + n\pi)^2]},$$

$$G_2 = \delta\beta_{3m} \frac{\cosh\beta_{3m}\cos(n\pi) - \cos(m\pi)}{2[\beta_{3m}^2 + (m\pi - n\pi)^2]},$$

$$G_3 = \delta\gamma_{3m} \frac{\cos(\gamma_{3m} + n\pi) - \cos(m\pi)}{2[(m\pi + n\pi)^2 - \gamma_{3m}^2]},$$

$$G_4 = \delta\gamma_{3m} \frac{\cos(\gamma_{3m} + n\pi) - \cos(m\pi)}{2[(m\pi - n\pi)^2 - \gamma_{3m}^2]},$$

or if $\gamma_{3m}^2 = (m\pi + n\pi)^2$ then,

$$G_3 = \delta \frac{\cos(m\pi) - \cos(2\gamma_{3m} + m\pi)}{8\gamma_{3m}},$$

and if $\gamma_{3m}^2 = (m\pi - n\pi)^2$ then,

$$G_4 = \delta \frac{\cos(m\pi) - \cos(2\gamma_{3m} + m\pi)}{8\gamma_{3m}}.$$

G_5 and G_6 are obtained from the expressions of G_1 and G_2 respectively by replacing β_{3m} by γ_{3m} .

Plate (e') has forbidden lateral motion along this edge, and Plate (f') has a forced lateral motion given by;

$$W_6(\xi, 0) = \sum_{m=0,1}^{\infty} V_{6m}\cos(m\pi\xi).$$

Bending Moment Along the Edge $\eta = 1$

Relevant equations were discussed and given in details in Section 3.4-1. Following steps outlined in this section, the contribution of the first plate is found to be;

$$\left. \frac{M_n b^2}{aD} \right|_{\eta=1} = \sum_{m=0,1}^{\infty} E_{1m} \cos(m\pi\xi),$$

and Plate (b') has no contribution to bending since it has forbidden bending moment along this edge. Contributions of Plates (c') and (d') lead to the following;

$$A_{i+3k,j+2k} = -[\nu\phi_1^2 A_{1n} + A_{2n}],$$

where,

$$A_{1n} = \theta_{31m}[\beta_{3m}^2 A_1 - C_{31m} \gamma_{3m}^2 A_2] \cos(m\pi),$$

$$A_{2n} = -\theta_{31m}(m\pi)^2 [A_1 + C_{31m} A_2] \cos(m\pi),$$

or if $\phi_1^2 \lambda^2 < (m\pi)^2$, then,

$$A_{1n} = \theta_{32m}[\beta_{3m}^2 A_1 + C_{32m} \gamma_{3m}^2 A_3] \cos(m\pi),$$

$$A_{2n} = -\theta_{32m}(m\pi)^2 [A_1 + C_{32m} A_3] \cos(m\pi),$$

and

$$A_{i+3k,j+3k} = -[\nu\phi_1^2 A_{1n} + A_{2n}],$$

where,

$$A_{1n} = \theta_{41m}[\beta_{3m}^2 A_1 - C_{41m} \gamma_{3m}^2 A_2] \cos(m\pi),$$

$$A_{2n} = -\theta_{41m}(m\pi)^2 [A_1 + C_{41m} A_2] \cos(m\pi),$$

or if $\phi_1^2 \lambda^2 < (m\pi)^2$ then,

$$A_{1n} = \theta_{42m} [\beta_{3m}^2 A_1 + C_{42m} \gamma_{3m}^2 A_3] \cos(m\pi),$$

$$A_{2n} = -\theta_{42m} (m\pi)^2 [A_1 + C_{42m} A_3] \cos(m\pi).$$

The contributions of the last two plates result in the following coefficients;

$$A_{i+3k, j+4k} = -[\theta_{11} \phi_5^2 A_{1n} + \theta_{12} A_{2n} + \theta_{13} \phi_5 A_{3n}] / F_2,$$

where,

$$A_{1n} = -\theta_{51m} (m\pi)^2 [B_1 + B_2 + C_{51m} (B_3 + B_4)],$$

$$A_{2n} = \theta_{51m} [\beta_{5m}^2 (B_1 + B_2) - C_{51m} \gamma_{5m}^2 (B_3 + B_4)],$$

$$A_{3n} = \theta_{51m} (m\pi) [\beta_{5m} (H_1 + H_2) + C_{51m} \gamma_{5m} (H_3 + H_4)],$$

or if $\lambda'^2 < (m\pi)^2$ then,

$$A_{1n} = -\theta_{52m} (m\pi)^2 [B_1 + B_2 + C_{52m} (B_5 + B_6)],$$

$$A_{2n} = \theta_{52m} [\beta_{5m}^2 (B_1 + B_2) + C_{52m} \gamma_{5m}^2 (B_5 + B_6)],$$

$$A_{3n} = \theta_{52m} (m\pi) [\beta_{5m} (H_1 + H_2) + C_{52m} \gamma_{5m} (H_5 + H_6)],$$

and where,

$$H_1 = \delta(n\pi + m\pi F_2) \frac{\cosh \beta_{5m} - \cos(n\pi + m\pi F_2)}{2[\beta_{5m}^2 + (m\pi F_2 + n\pi)^2]},$$

$$H_2 = \delta(n\pi - m\pi F_2) \frac{\cos(n\pi - m\pi F_2) - \cosh \beta_{5m}}{2[\beta_{5m}^2 + (n\pi - m\pi F_2)^2]},$$

$$H_3 = \delta(n\pi + m\pi F_2) \frac{\cos \gamma_{5m} - \cos(n\pi + m\pi F_2)}{2[(n\pi + m\pi F_2)^2 - \gamma_{5m}^2]},$$

$$H_4 = \delta(n\pi - m\pi F_2) \frac{\cos \gamma_{5m} - \cos(n\pi - m\pi F_2)}{2[(n\pi - m\pi F_2)^2 - \gamma_{5m}^2]},$$

but if $\gamma_{5m} = (n\pi + m\pi F_2)$ then, $H_3 = \delta \sin \gamma_{5m} / 4$, and if $\gamma_{5m} = (n\pi - m\pi F_2)$ then, $H_4 = -\delta \sin \gamma_{5m} / 4$, or if $\gamma_{5m} = (m\pi F_2 - n\pi)$ then, $H_4 = \delta \sin \gamma_{5m} / 4$. H_5 and H_6 are found by replacing β_{5m} by γ_{5m} in the expressions of H_1 and H_2 respectively. And the coefficients of V_{6m} are found from those of E_{5m} simply by replacing θ_{51m} , C_{51m} , θ_{52m} and C_{52m} by θ_{61m} , C_{61m} , θ_{62m} and C_{62m} respectively.

Bending Moment Along the Edge $\xi = 1$

In Section 3.4-2, we have discussed in detail the reasoning and procedure followed to derive the various coefficients relevant to this inter-segment or common edge. The same procedure is followed here to arrive at the following contributions and coefficients.

Consideration of the contributions of the first and second building blocks of Figure 3.7 leads to the coefficients of E_{1m} and V_{2m} along this edge, which are found to be;

$$A_{i+4k,j} = -[\phi_1^2 A_{1n} + \nu A_{2n}],$$

where,

$$A_{1n} = -\theta_{11m}(m\pi)^2 [C_1 + C_{11m}C_2] \cos(m\pi),$$

$$A_{2n} = \theta_{11m}[\beta_{1m}^2 C_1 - C_{11m}\gamma_{1m}^2 C_2] \cos(m\pi),$$

or if $\lambda^2 < (m\pi)^2$ then,

$$A_{1n} = -\theta_{12m}(m\pi)^2 [C_1 + C_{12m}C_3] \cos(m\pi),$$

$$A_{2n} = \theta_{12m}[\beta_{1m}^2 C_1 - C_{12m}\gamma_{1m}^2 C_3] \cos(m\pi).$$

Coefficients of V_{2m} are obtained from the above expressions by replacing θ_{11m} , C_{11m} , θ_{12m} and C_{12m} by θ_{21m} , C_{21m} , θ_{22m} and C_{22m} respectively.

Block c' has the forced edge moment along this edge which is given by;

$$\left. \frac{M_n a^2}{bD} \right|_{\xi=1} = \sum_{m=0,1}^{\infty} E_{3m} \cos(m\pi\eta),$$

and Plate (d') has a forbidden moment and hence no contribution toward bending moments along the edge under consideration.

Turning to the contributions of the remaining building blocks, Plates (e') and (f'), it is seen that;

$$A_{i+4k, j+4k} = -[\theta_{21}\phi_5^2 A_{1n} + \theta_{22}A_{2n} + \theta_{23}\phi_5 A_{3n}]/F_2,$$

where,

$$\begin{aligned} A_{1n} &= -\theta_{51m}(m\pi)^2[D_1 + D_2 + C_{51m}(D_3 + D_4)], \\ A_{2n} &= \theta_{51m}[\beta_{5m}^2(D_1 + D_2) - C_{51m}\gamma_{5m}^2(D_3 + D_4)], \\ A_{3n} &= \theta_{51m}(m\pi)[\beta_{5m}(P_1 + P_2) + C_{51m}\gamma_{5m}(P_3 + P_4)], \end{aligned}$$

or if $\lambda'^2 < (m\pi)^2$ then,

$$\begin{aligned} A_{1n} &= -\theta_{52m}(m\pi)^2[D_1 + D_2 + C_{52m}(D_5 + D_6)], \\ A_{2n} &= \theta_{52m}[\beta_{5m}^2(D_1 + D_2) + C_{52m}\gamma_{5m}^2(D_5 + D_6)], \\ A_{3n} &= \theta_{52m}(m\pi)[\beta_{5m}(P_1 + P_2) + C_{52m}\gamma_{5m}(P_5 + P_6)], \end{aligned}$$

and where,

$$\begin{aligned} P_1 &= \delta(m\pi F_1 - n\pi) \frac{\cos[m\pi - (m\pi F_1 - n\pi)] - \cosh\beta_{5m}\cos(m\pi)}{2[\beta_{5m}^2 + (m\pi F_1 - n\pi)^2]}, \\ P_2 &= \delta(m\pi F_1 + n\pi) \frac{\cos[m\pi - (m\pi F_1 + n\pi)] - \cosh\beta_{5m}\cos(m\pi)}{2[\beta_{5m}^2 + (m\pi F_1 + n\pi)^2]}, \\ P_3 &= \delta(m\pi F_1 - n\pi) \frac{\cos[m\pi - (m\pi F_1 - n\pi)] - \cos(m\pi + \gamma_{5m})}{2[(m\pi F_1 - n\pi)^2 - \gamma_{5m}^2]}, \\ P_4 &= \delta(m\pi F_1 + n\pi) \frac{\cos[m\pi - (m\pi F_1 + n\pi)] - \cos(m\pi - \gamma_{5m})}{2[(m\pi F_1 + n\pi)^2 - \gamma_{5m}^2]}, \end{aligned}$$

but if $\gamma_{5m} = m\pi F_1 - n\pi$ then, $P_3 = \delta \sin(m\pi - \gamma_{5m})/4$, or if $\gamma_{5m} = n\pi - m\pi F_1$ then, $P_3 = \delta(m\pi + \gamma_{5m})/4$, and if $\gamma_{5m} = (m\pi F_1 + n\pi)$ then, $P_4 = \delta \sin(m\pi - \gamma_{5m})/4$. The coefficients of V_{6m} are now found from those of E_{5m} simply by replacing θ_{51m} , C_{51m} , θ_{52m} and C_{52m} by θ_{61m} , C_{61m} , θ_{62m} and C_{62m} respectively.

Considering now the rectangular element We find;

$$A_{i+4k,j+6k} = [\phi_r^2 A_{1n} + \nu A_{2n}] \phi_r^2,$$

where for symmetric modes,

$$A_{1n} = \beta_{rm}^2 \cosh \beta_{rm} A_{nr},$$

$$A_{2n} = -(m\pi)^2 \cosh \beta_{rm} A_{nr},$$

and for anti-symmetric modes,

$$A_{1n} = \beta_{rm}^2 \sinh \beta_{rm} A_{nr},$$

$$A_{2n} = -(m\pi)^2 \sinh \beta_{rm} A_{nr}.$$

And for the coefficients of D_m we find;

$$A_{i+4k,j+7k} = [\phi_r^2 A_{1n} + \nu A_{2n}] \phi_r^2,$$

where for symmetric modes;

$$A_{1n} = -\gamma_{rm}^2 \cos \gamma_{rm} A_{nr},$$

$$A_{2n} = -(m\pi)^2 \cos \gamma_{rm} A_{nr},$$

and for anti-symmetric modes;

$$A_{1n} = -\gamma_{rm}^2 \sin \gamma_{rm} A_{nr},$$

$$A_{2n} = -(m\pi)^2 \sin \gamma_{rm} A_{nr},$$

or if $\phi_r^2 \lambda_r^2 < (m\pi)^2$ then for symmetric modes;

$$A_{1n} = -\gamma_{rm}^2 \cosh \gamma_{rm} A_{nr},$$

$$A_{2n} = -(m\pi)^2 \cosh \gamma_{rm} A_{nr},$$

and for anti-symmetric modes;

$$A_{1n} = -\gamma_{rm}^2 \sinh \gamma_{rm} A_{nr},$$

$$A_{2n} = -(m\pi)^2 \sinh \gamma_{rm} A_{nr},$$

Bending Moment Along the Edge $\eta' = 0$

Following steps discussed in Section 3.4-3 We find;

$$A_{i+5k,j} = -[\theta_{31}\phi_1^2 A_{1n} + \theta_{32}A_{2n} + \theta_{33}\phi_1 A_{3n}],$$

where,

$$A_{1n} = -\theta_{11m}(m\pi)^2[E_1 + E_2 + C_{11m}(E_3 + E_4)],$$

$$A_{2n} = \theta_{11m}[\beta_{1m}^2(E_1 + E_2) - C_{11m}\gamma_{1m}^2(E_3 + E_4)],$$

$$A_{3n} = -\theta_{11m}(m\pi)[\beta_{1m}(Q_1 + Q_2) + C_{11m}\gamma_{1m}(Q_3 + Q_4)],$$

or if $\lambda^2 < (m\pi)^2$ then,

$$A_{1n} = -\theta_{12m}(m\pi)^2[E_1 + E_2 + C_{12m}(E_5 + E_6)],$$

$$A_{2n} = \theta_{12m}[\beta_{1m}^2(E_1 + E_2) + C_{12m}\gamma_{1m}^2(E_5 + E_6)],$$

$$A_{3n} = -\theta_{12m}(m\pi)[\beta_{1m}(Q_1 + Q_2) + C_{12m}\gamma_{1m}(Q_5 + Q_6)].$$

Also,

$$A_{i+5k,j+k} = -[\theta_{31}\phi_1^2 A_{1n} + \theta_{32}A_{2n} + \theta_{33}\phi_1 A_{3n}],$$

where A_{1n} , A_{2n} and A_{3n} are obtained by replacing θ_{11m} , C_{11m} , θ_{12m} and C_{12m} in the above expressions by θ_{21m} , C_{21m} , θ_{22m} and C_{22m} respectively. E_1 through E_6 are as previously defined, and

$$Q_1 = \delta(m\pi + n\pi) \frac{\cosh\beta_{1m} - \cos(m\pi + n\pi)}{2[\beta_{1m}^2 + (m\pi + n\pi)^2]},$$

$$Q_2 = \delta(m\pi - n\pi) \frac{\cosh\beta_{1m} - \cos(m\pi - n\pi)}{2[\beta_{1m}^2 + (m\pi - n\pi)^2]},$$

$$Q_3 = \delta(m\pi + n\pi) \frac{\cos\gamma_{1m} - \cos(m\pi + n\pi)}{2[(m\pi + n\pi)^2 - \gamma_{1m}^2]},$$

$$Q_4 = \delta(m\pi - n\pi) \frac{\cos\gamma_{1m} - \cos(m\pi - n\pi)}{2[(m\pi - n\pi)^2 - \gamma_{1m}^2]},$$

or if $\gamma_{1m} = (m\pi - n\pi)$ then, $Q_4 = \delta \sin \gamma_{1m}/4$, and if $\gamma_{1m} = (n\pi - m\pi)$ then, $Q_4 = -\delta \sin \gamma_{1m}/4$. Q_5 and Q_6 are found by replacing β_{1m} by γ_{1m} in the expressions of Q_1 and Q_2 respectively. For contributions of W_3 and W_4 We find;

$$A_{i+5k, j+2k} = [-\theta_{31}\phi_1^2 A_{1n} + \theta_{32}A_{2n} + \theta_{33}\phi_1 A_{3n}],$$

where,

$$\begin{aligned} A_{1n} &= \theta_{31m}[\beta_{3m}^2(G_1 + G_2) - C_{31m}\gamma_{3m}^2(G_3 + G_4)], \\ A_{2n} &= -\theta_{31m}(m\pi)^2[G_1 + G_2 + C_{31m}(G_3 + G_4)], \\ A_{3n} &= -\theta_{31m}(m\pi)[\beta_{3m}(R_1 + R_2) + C_{31m}\gamma_{3m}(R_3 + R_4)], \end{aligned}$$

or if $\phi_1^2 \lambda^2 < (m\pi)^2$ then,

$$\begin{aligned} A_{1n} &= \theta_{32m}[\beta_{3m}^2(G_1 + G_2) + C_{32m}\gamma_{3m}^2(G_5 + G_6)], \\ A_{2n} &= -\theta_{32m}(m\pi)^2[G_1 + G_2 + C_{32m}(G_5 + G_6)], \\ A_{3n} &= -\theta_{32m}(m\pi)[\beta_{3m}(R_1 + R_2) + C_{32m}\gamma_{3m}(R_3 + R_4)]. \end{aligned}$$

Also,

$$A_{i+5k, j+3k} = -[\theta_{31}\phi_1^2 A_{1n} + \theta_{32}A_{2n} + \theta_{33}\phi_1 A_{3n}],$$

where A_{1n} , A_{2n} and A_{3n} are obtained by replacing θ_{31m} , C_{31m} , θ_{32m} and C_{32m} by θ_{41m} , C_{41m} , θ_{42m} and C_{42m} respectively in the above expressions, and where,

$$\begin{aligned} R_1 &= \delta(m\pi - n\pi) \frac{\cosh \beta_{3m} \cos(n\pi) - \cos(m\pi)}{2[\beta_{3m}^2 + (m\pi - n\pi)^2]}, \\ R_2 &= \delta(m\pi + n\pi) \frac{\cosh \beta_{3m} \cos(n\pi) - \cos(m\pi)}{2[\beta_{3m}^2 + (m\pi + n\pi)^2]}, \\ R_3 &= \delta(m\pi - n\pi) \frac{\cos(n\pi - \gamma_{3m}) - \cos(m\pi)}{2[(m\pi - n\pi)^2 - \gamma_{3m}^2]}, \\ R_4 &= \delta(m\pi + n\pi) \frac{\cos(n\pi + \gamma_{3m}) - \cos(m\pi)}{2[(m\pi + n\pi)^2 - \gamma_{3m}^2]}, \end{aligned}$$

or if $\gamma_{3m} = (m\pi - n\pi)$ then,

$$R_3 = \delta[\cos(m\pi - 2\gamma_{3m}) - \cos(m\pi)]/8\gamma_{3m},$$

but if $\gamma_{3m} = (n\pi - m\pi)$,

$$R_3 = \delta[\cos(m\pi) - \cos(m\pi + 2\gamma_{3m})]/8\gamma_{3m},$$

and if $\gamma_{3m} = (m\pi + n\pi)$ then,

$$R_4 = \delta[\cos(m\pi - 2\gamma_{3m}) - \cos(m\pi)]/8\gamma_{3m}.$$

R_5 and R_6 are obtained by replacing β_{3m} by γ_{3m} in the expressions of R_1 and R_2 respectively.

The fifth building block, Plate (e'), has no contribution to the bending moment here, and Plate (f') has the forced harmonic moment given by;

$$\left. \frac{M_n b'^2}{a'D} \right|_{\eta'=0} = \sum_{m=0,1}^{\infty} E_{5m} \cos(m\pi \xi).$$

Slope of Normal Lines to the Edge $\xi = 1$

Along this edge, W_1 and W_2 have slip shear conditions, and therefore, no contribution to slopes along this edge. Contributions of W_3 and W_4 lead to the following;

$$A_{i+6k, i+2k} = \theta_{31m} [\beta_{3m} \cosh \beta_{3m} + C_{31m} \gamma_{3m} \cos \gamma_{3m}],$$

$$A_{i+6k, i+2k} = \theta_{41m} [\beta_{3m} \cosh \beta_{3m} + C_{41m} \gamma_{3m} \cos \gamma_{3m}],$$

or if $\phi_1^2 \lambda^2 < (m\pi)^2$ then,

$$A_{i+6k, i+2k} = \theta_{32m} [\beta_{3m} \cosh \beta_{3m} + C_{32m} \gamma_{3m} \cosh \gamma_{3m}],$$

$$A_{i+6k, i+2k} = \theta_{42m} [\beta_{3m} \cosh \beta_{3m} + C_{42m} \gamma_{3m} \cosh \gamma_{3m}].$$

Turning now to the contributions of W_5 and W_6 We obtain the following coefficients;

$$A_{i+6k, j+4k} = F_{21} A_{1n} + F_{22} A_{2n},$$

where,

$$A_{1n} = -\theta_{51m} (m\pi) [C_{s1} + C_{s2} + C_{51m} (C_{s3} + C_{s4})],$$

$$A_{2n} = -\theta_{51m} [\beta_{5m} (D_{s1} + D_{s2}) + C_{51m} \gamma_{5m} (D_{s3} + D_{s4})],$$

or if $\lambda'^2 < (m\pi)^2$ then,

$$A_{1n} = -\theta_{52m} (m\pi) [C_{s1} + C_{s2} + C_{52m} (C_{s5} + C_{s6})],$$

$$A_{2n} = -\theta_{52m} [\beta_{5m} (D_{s1} + D_{s2}) + C_{52m} \gamma_{5m} (D_{s5} + D_{s6})].$$

And

$$A_{i+6k, j+5k} = F_{21} A_{1n} + F_{22} A_{2n},$$

where, A_{1n} and A_{2n} are obtained by replacing θ_{51m} , C_{51m} , θ_{52m} and C_{52m} in the above expressions by θ_{61m} , C_{61m} , θ_{62m} and C_{62m} respectively. And where,

$$F_{21} = \cos\alpha_2 \sin\alpha,$$

$$F_{22} = \sin\alpha_2 / \cos\alpha,$$

$$C_{s1} = \delta \frac{\beta_{5m} \sin(m\pi F_1 - n\pi - m\pi) - (m\pi F_1 - n\pi) \sinh\beta_{5m} \cos(m\pi)}{2[\beta_{5m}^2 + (m\pi F_1 - n\pi)^2]},$$

$$C_{s2} = \delta \frac{\beta_{5m} \sin(m\pi F_1 + n\pi - m\pi) - (m\pi F_1 + n\pi) \sinh\beta_{5m} \cos(m\pi)}{2[\beta_{5m}^2 + (m\pi F_1 + n\pi)^2]},$$

$$C_{s3} = \delta \frac{\gamma_{5m} \sin(m\pi F_1 - m\pi - n\pi) - (m\pi F_1 - n\pi) \sin(\gamma_{5m} - m\pi)}{2[(m\pi F_1 - n\pi)^2 - \gamma_{5m}^2]},$$

$$C_{s4} = \delta \frac{\gamma_{5m} \sin(m\pi F_1 + n\pi - m\pi) - (m\pi F_1 + n\pi) \sin(\gamma_{5m} - m\pi)}{2[(m\pi F_1 + n\pi)^2 - \gamma_{5m}^2]},$$

but if $\gamma_{5m} = (m\pi F_1 - n\pi)$ then,

$$C_{s3} = \delta \frac{\gamma_{5m} \cos(\gamma_{5m} - m\pi) - \sin(\gamma_{5m} - m\pi)}{4\gamma_{5m}},$$

or if $\gamma_{5m} = (n\pi - m\pi F_1)$ then,

$$C_{s3} = \delta \frac{\sin(\gamma_{5m} - m\pi) - \gamma_{5m} \cos(\gamma_{5m} - m\pi)}{4\gamma_{5m}},$$

and if $\gamma_{5m} = (m\pi F_1 + n\pi)$ then,

$$C_{s4} = \delta \frac{\gamma_{5m} \cos(\gamma_{5m} - m\pi) - \sin(\gamma_{5m} - m\pi)}{4\gamma_{5m}}.$$

C_{s5} and C_{s6} are obtained from the expressions of C_{s1} and C_{s2} respectively by replacing β_{5m} by γ_{5m} . Also,

$$D_{s1} = \delta \frac{\beta_{5m} \sinh \beta_{5m} \cos(m\pi) + (m\pi F_1 + n\pi) \sin(m\pi F_1 + n\pi - m\pi)}{[\beta_{5m}^2 + (m\pi F_1 + n\pi)^2]},$$

$$D_{s2} = \delta \frac{\beta_{5m} \sinh \beta_{5m} \cos(m\pi) + (m\pi F_1 - n\pi) \sin(m\pi F_1 - n\pi - m\pi)}{[\beta_{5m}^2 + (m\pi F_1 - n\pi)^2]},$$

$$D_{s3} = \delta \frac{(m\pi F_1 + n\pi) \sin(m\pi F_1 + n\pi - m\pi) - \gamma_{5m} \sin(\gamma_{5m} - m\pi)}{2[(m\pi F_1 + n\pi)^2 - \gamma_{5m}^2]},$$

$$D_{s4} = \delta \frac{(m\pi F_1 - n\pi) \sin(m\pi F_1 - n\pi - m\pi) - \gamma_{5m} \sin(\gamma_{5m} - m\pi)}{2[(m\pi F_1 - n\pi)^2 - \gamma_{5m}^2]},$$

but if $\gamma_{5m} = (m\pi F_1 + n\pi)$ then,

$$D_{s3} = \delta \frac{\gamma_{5m} \cos(\gamma_{5m} + m\pi) + \sin(\gamma_{5m} + m\pi)}{4\gamma_{5m}},$$

or if $\gamma_{5m} < 2 = (m\pi F_1 - n\pi)^2$ then,

$$D_{s4} = \delta \frac{\gamma_{5m} \cos(\gamma_{5m} + m\pi) + \sin(\gamma_{5m} + m\pi)}{4\gamma_{5m}}.$$

Now replacing β_{5m} by γ_{5m} in the above expressions of D_{s1} and D_{s2} , We obtain D_{s5} and D_{s6} respectively.

Concentrating now on the rectangular element, for symmetric modes, We obtain;

$$A_{i+6k,j+6k} = \beta_{rm} \sinh \beta_{rm} A_{nr} \phi_r / \phi_1,$$

$$A_{i+6k,j+7k} = -\gamma_{rm} \sin \gamma_{rm} A_{nr} \phi_r / \phi_1,$$

or if $\phi_r^2 \lambda_r^2 < (m\pi)^2$ then,

$$A_{i+6k,j+7k} = \gamma_{rm} \sinh \gamma_{rm} A_{nr} \phi_r / \phi_1,$$

and for anti-symmetric modes We have;

$$A_{i+6k,j+6k} = \beta_{rm} \cosh \beta_{rm} A_{nr} \phi_r / \phi_1,$$

$$A_{i+6k,j+7k} = -\gamma_{rm} \cos \gamma_{rm} A_{nr} \phi_r / \phi_1,$$

or if $\phi_r^2 \lambda_r^2 < (m\pi)^2$ then,

$$A_{i+6k,j+7k} = \gamma_{rm} \cosh \gamma_{rm} A_{nr} \phi_r / \phi_1,$$

where A_{nr} is as defined earlier.

Vertical Edge Reaction Along $\xi = 1$

The first two building blocks have slip shear conditions along this edge, and therefore, have no contribution toward the vertical edge reaction here. Consideration of the contributions of Plates (c') and (d') leads to the following;

$$A(i, +7k, i + 2k) = -[A_{1n} + \frac{\nu}{\phi_1^2} A_{2n}],$$

where,

$$A_{1n} = \theta_{31m} [\beta_{3m}^3 \cosh \beta_{3m} - C_{31m} \gamma_{3m}^3 \cos \gamma_{3m}],$$

$$A_{2n} = -\theta_{31m} (m\pi)^2 [\beta_{3m} \cosh \beta_{3m} + C_{31m} \gamma_{3m} \cos \gamma_{3m}],$$

or if $\phi_1^2 \lambda^2 < (m\pi)^2$ /quad then,

$$A_{1n} = \theta_{32m} [\beta_{3m}^3 \cosh \beta_{3m} + C_{32m} \gamma_{3m}^3 \cosh \gamma_{3m}],$$

$$A_{2n} = -\theta_{32m} (m\pi)^2 [\beta_{3m} \cosh \beta_{3m} + C_{32m} \gamma_{3m} \cosh \gamma_{3m}].$$

Also,

$$A_{i+7k,i+3k} = -[A_{1n} + \frac{\nu}{\phi_1^2} A_{2n}],$$

where A_{1n} and A_{2n} are obtained from the above expressions by replacing θ_{31m} , C_{31m} , θ_{32m} and C_{32m} by θ_{41m} , C_{41m} , θ_{42m} and C_{42m} respectively.

Turning now to the contributions of W_5 and W_6 We find;

$$A_{i+7k,j+4k} = -[V_1 A_{1n} + V_2 A_{2n} + V_3 A_{3n} + V_4 A_{4n}] \sin^3 \alpha,$$

where V_1, V_2, V_3 and V_4 are given by Equation (1.26), and where,

$$\begin{aligned} A_{1n} &= \theta_{51m}(m\pi)^3 [C_{s1} + C_{s2} + C_{51m}(C_{s3} + C_{s4})], \\ A_{2n} &= -\theta_{51m}[\beta_{5m}^3(D_{s1} + D_{s2}) - C_{51m}\gamma_{5m}^3(D_{s3} + D_{s4})], \\ A_{3n} &= \theta_{51m}(m\pi)^2 [\beta_{5m}(D_{s1} + D_{s2}) + C_{51m}\gamma_{5m}(D_{s3} + D_{s4})], \\ A_{4n} &= -\theta_{51m}(m\pi)[\beta_{5m}^2(C_{s1} + C_{s2}) - C_{51m}\gamma_{5m}^2(C_{s3} + C_{s4})], \end{aligned}$$

or if $\lambda'^2 < (m\pi)^2$ then,

$$\begin{aligned} A_{1n} &= \theta_{52m}(m\pi)^3 [C_{s1} + C_{s2} + C_{52m}(C_{s5} + C_{s6})], \\ A_{2n} &= -\theta_{52m}[\beta_{5m}^3(D_{s1} + D_{s2}) + C_{52m}\gamma_{5m}^3(D_{s5} + D_{s6})], \\ A_{3n} &= \theta_{52m}(m\pi)^2 [\beta_{5m}(D_{s1} + D_{s2}) + C_{52m}\gamma_{5m}(D_{s5} + D_{s6})], \\ A_{4n} &= -\theta_{52m}(m\pi)[\beta_{5m}^2(C_{s1} + C_{s2}) + C_{52m}\gamma_{5m}^2(C_{s5} + C_{s6})]. \end{aligned}$$

And

$$A_{i+7k,j+57} = -[V_1 A_{1n} + V_2 A_{2n} + V_3 A_{3n} + V_4 A_{4n}] \sin^3 \alpha,$$

where A_{1n} through A_{4n} are obtained from the above expressions by replacing $\theta_{51m}, C_{51m}, \theta_{52m}$ and C_{52m} by $\theta_{61m}, C_{61m}, \theta_{62m}$ and C_{62m} respectively.

Focusing attention on the contributions of the rectangular element We have;

$$A_{i+7k,j+6k} = [A_{1n} + \frac{\nu_r}{\phi_r^2} A_{2n}] \phi_r^3 / \phi_1^3,$$

where for symmetric modes;

$$\begin{aligned} A_{1n} &= -\beta_{rm}^3 \sinh \beta_{rm} A_{nr}, \\ A_{2n} &= (m\pi)^2 \beta_{rm} \sinh \beta_{rm} A_{nr}, \end{aligned}$$

and for anti-symmetric modes;

$$A_{1n} = -\beta_{rm}^3 \cosh \beta_{rm} A_{nr},$$

$$A_{2n} = (m\pi)^2 \beta_{rm} \cosh \beta_{rm} A_{nr}.$$

Also,

$$A_i + 7k, j + 7k = [A_{1n} + \frac{\nu}{\phi_r^2} A_{2n}] \phi_r^3 / \phi_1^3,$$

where for symmetric modes;

$$A_{1n} = -\gamma_{rm}^3 \sin \gamma_{rm} A_{nr},$$

$$A_{2n} = -(m\pi)^2 \gamma_{rm} \sin \gamma_{rm} A_{nr},$$

or if $\phi_r^2 \lambda_r^2 < (m\pi)^2$ then,

$$A_{1n} = -\gamma_{rm}^3 \sinh \gamma_{rm} A_{nr},$$

$$A_{2n} = (m\pi)^2 \gamma_{rm} \sinh \gamma_{rm} A_{nr},$$

and for anti-symmetric modes;

$$A_{1n} = -\gamma_{rm}^3 \cos \gamma_{rm} A_{nr},$$

$$A_{2n} = -(m\pi)^2 \gamma_{rm} \cos \gamma_{rm} A_{nr},$$

or if $\phi_r^2 \lambda_r^2 \leq (m\pi)^2$ then,

$$A_{1n} = -\gamma_{rm}^3 \cosh \gamma_{rm} A_{nr},$$

$$A_{2n} = (m\pi)^2 \gamma_{rm} \cosh \gamma_{rm} A_{nr}.$$