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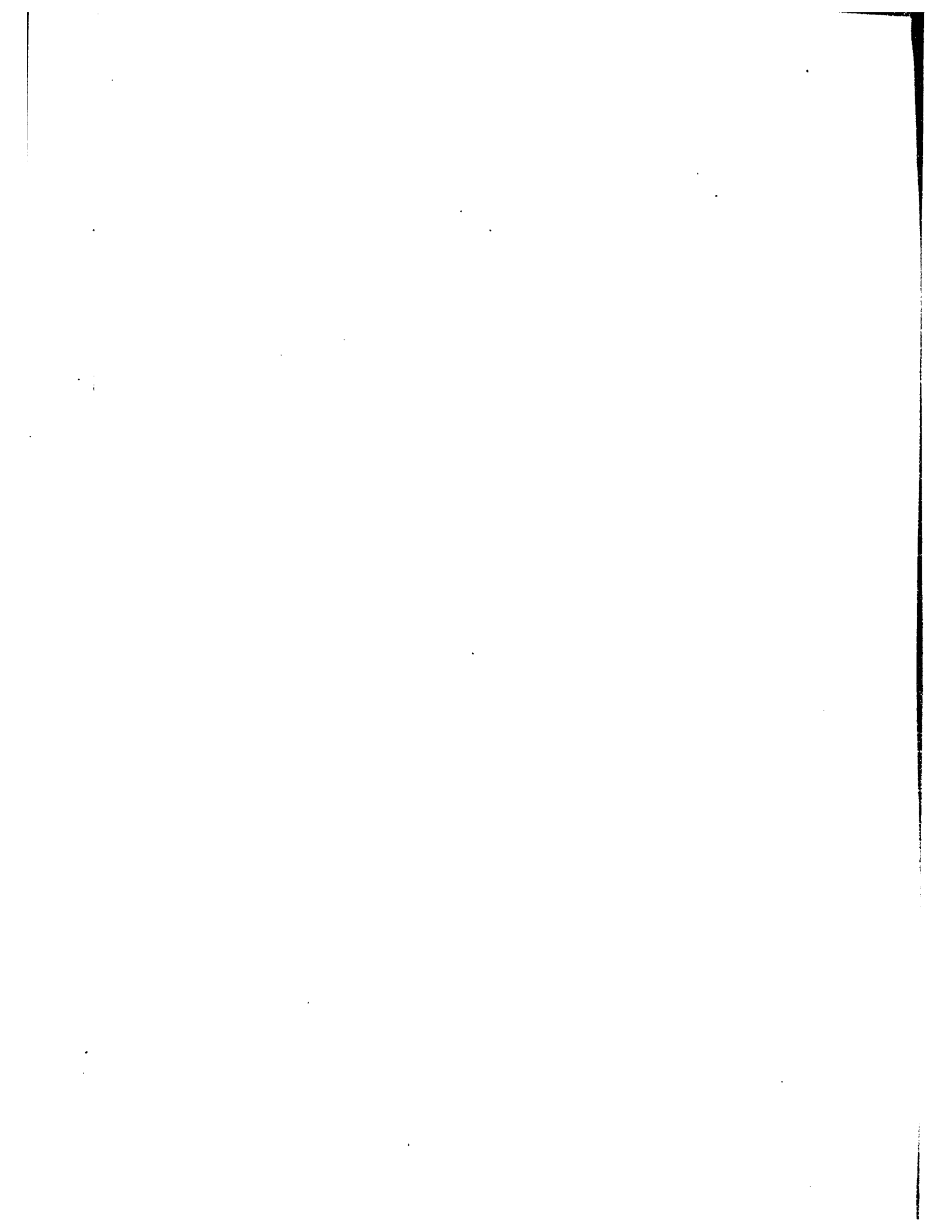
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ON THE BOTT PERIODICITY

A thesis submitted

by

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to

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of the University of Ottawa

in partial fulfillment of the requirements

for the degree of

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in the subject of

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ABSTRACT

In this thesis we first introduce K-theory. Then we present the formulation of the Bott periodicity of the homotopy groups of the infinite unitary group in the framework of K-theory. Finally we prove the equivalence of this K-theory formulation with the original one given by Bott.

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INTRODUCTION

Let X be a topological space with base point x_0 , and $\pi_n(X, x_0) = [S^n, X]_*$ be the set of based homotopy classes of base point preserving maps of the n -sphere $(S^n, *)$ into (X, x_0) . Then, for $n \geq 1$, $\pi_n(X, x_0)$ admits a group structure and is called the n -th homotopy group of X . For $n > 1$, $\pi_n(X, x_0)$ is abelian.

Let F denote either the real field R or the complex field C , and let $U_F(k)$ denote the Lie group of endomorphisms of F^k leaving the standard inner product invariant. $U_F(k)$ is isomorphic to the subgroup of $U_F(k+1)$ consisting of the elements u for which $u(e_{k+1}) = e_{k+1}$ where e_1, \dots, e_{k+1} is the standard basis of F^{k+1} . Therefore, $U_F(k)$ is naturally imbedded in $U_F(k+1)$ and we define $U_F = \bigcup_{k \geq 1} U_F(k)$ with the weak topology. We denote $U_F = 0$ for $F = R$ and $U_F = U$ for $F = C$.

The original periodicity theorems of R. Bott [2] show the existence of isomorphisms $\pi_k U \cong \pi_{k+2} U$, $\pi_k 0 \cong \pi_{k+8} 0$, $k = 0, 1, 2, \dots$. Later on Atiyah and Hirzebruch [1] reformulated these results in the framework of K -theory. Our purpose is to give the K -theory formulation of the periodicity theorem in the complex case and prove in some detail the equivalence

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of these two formulations.

In Chapter I, we give the main definitions and constructions concerning bundles.

In Chapter II, two operations on vector spaces, namely the direct sum and the tensor product are extended to vector bundles. With these two operations $\text{Vect}X$, the set of isomorphism classes of vector bundles over X , becomes a semi-ring. We define then KX as the ring completion of $\text{Vect}X$. We define also $\hat{K}X$, a certain ideal of KX , and give another description of the functors K and \hat{K} .

In Chapter III, we formulate the periodicity in terms of K -theory, i.e. the \hat{K} -cup product $\hat{K}X \otimes \hat{K}S^2 \rightarrow \hat{K}(X \wedge S^2)$ is a natural isomorphism. Furthermore, using the fact that the functor \hat{K} is representable by B_U , the classifying space of the group U , we show that this cup product implies the existence of a weak homotopy equivalence $\gamma : B_U \rightarrow \Omega^2 B_U$, which induces the isomorphisms of Bott.

In Chapter IV, the Puppe sequence and half-exact functors are introduced. As examples, we see that \hat{K} and $\hat{K}(- \wedge S^2)$ are half-exact.

In Chapter V, we describe the weak homotopy equivalence γ and show that γ induces a group isomorphism $\hat{K}X \rightarrow \hat{K}(X \wedge S^2)$. On the other hand, the \hat{K} -cup product $\hat{K}X \otimes \hat{K}S^2 \rightarrow \hat{K}(X \wedge S^2)$

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is induced by a map α . Finally, we deform γ into α . Thus the original theorem of Bott implies that the \tilde{K} -cup product is an isomorphism. Together with the result of Chapter III, this shows the equivalence of the two formulations of the Bott periodicity.

CHAPTER I

GENERALITIES ON BUNDLES

1.1. A bundle is a triple (E, p, B) where E and B are topological spaces and $p : E \rightarrow B$ is a map. The space B is called the base space, E is called the total space, and the map p is called the projection. For each $b \in B$, the space $p^{-1}(b)$ is called the fibre over $b \in B$. A space F is a (typical) fibre of a bundle (E, p, B) provided every fibre $p^{-1}(b)$ is homeomorphic to F .

A bundle (E', p', B') is a sub-bundle of (E, p, B) provided E' is a subspace of E , B' is a subspace of B , and $p' = p|_{E'}$.

1.2. Example: The product bundle over B with fibre F is $(B \times F, p, B)$ where p is the first projection.

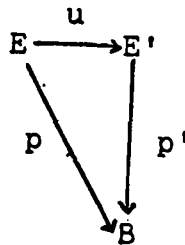
1.3. Let (E, p, B) and (E', p', B') be two bundles. A bundle morphism $(u, f) : (E, p, B) \rightarrow (E', p', B')$ is a pair of maps $u : E \rightarrow E'$, $f : B \rightarrow B'$ such that the following diagram

$$\begin{array}{ccc} E & \xrightarrow{u} & E' \\ p \downarrow & & \downarrow p' \\ B & \xrightarrow{f} & B' \end{array}$$

commutes, i.e. $p'u = fp$.

Let (E, p, B) and (E', p', B') be two bundles over B . A

B-morphism $u : (E, p, B) \rightarrow (E', p', B)$ is a map $u : E \rightarrow E'$ such that the following diagram



commutes, i.e. $p = p'u$.

A B-morphism $u : (E, p, B) \rightarrow (E', p', B)$ is called a B-isomorphism if there exists a B-morphism $v : (E', p', B) \rightarrow (E, p, B)$ such that $uv = I_{E'}$, $vu = I_E$.

A bundle (E, p, B) is trivial with fibre F if (E, p, B) is B-isomorphic to the product bundle $(B \times F, p, B)$.

1.4. Let $\xi = (E, p, B)$ be a bundle and $f : B_1 \rightarrow B$ be a map. The induced bundle of ξ under f , denoted by $f^*(\xi)$ has base space B_1 , total space $E_1 \subseteq B_1 \times E$ consisting of all $(b_1, x) \in B_1 \times E$ such that $f(b_1) = p(x)$, and projection $p_1 : (b_1, x) \mapsto b_1$.

Let \mathbb{C} be the field of complex numbers. A k -dimensional complex vector bundle is a bundle (E, p, B) such that each fibre $p^{-1}(b)$ has a structure of k -dimensional vector space over \mathbb{C} . Moreover, each $b \in B$ has an open neighborhood U and an U -isomorphism $h : U \times \mathbb{C}^k \rightarrow p^{-1}(U)$ such that the restriction $b \times \mathbb{C}^k \rightarrow p^{-1}(b)$ is a vector space isomorphism.

Let $\xi = (E, p, B)$ and $\xi' = (E', p', B')$ be two complex vector bundles. A morphism of vector bundles $(u, f) : \xi \rightarrow \xi'$ is a morphism of the underlying bundles such that the restriction $u : p^{-1}(b) \rightarrow p'^{-1}(b)$ is linear for each $b \in B$. In particular, if $B = B'$, $f = I_B$, then $(u, I_B) : \xi \rightarrow \xi'$ is called a B-morphism of vector bundles and it is usually denoted by $u : \xi \rightarrow \xi'$. Moreover, if there exists a B-morphism of vector bundles $v : \xi' \rightarrow \xi$ such that $uv = I_{E'}$, $vu = I_E$ then u is a B-isomorphism of vector bundles.

1.5. For a topological group G , a (right) G-space is a space X together with a continuous (right) action $X \times G \rightarrow X$.

A G-space X is effective if $xg = x$ implies $g = 1$. Let X be an effective G-space and $X^* \subseteq X \times X$ consisting of (x, xg) , $x \in X$, $g \in G$. Define $\tau : X^* \rightarrow G$ by the equation

$$x\tau(x, x') = x'.$$

τ is well-defined for if $g_1, g_2 \in G$ such that $xg_1 = xg_2 = x'$, then, $xg_1 g_2^{-1} = (xg_2)g_2^{-1} = x$, $g_1 g_2^{-1} = 1$, $g_1 = g_2$. The function τ is called the translation function.

A G-space X is called principal if X is an effective G-space with a continuous translation function $\tau : X^* \rightarrow G$.

Let X/G be the space of orbits and $\pi : X \rightarrow X/G$ the natural projection. Then $\alpha(X) = (X, \pi, X/G)$ is a bundle.

A bundle (X, p, B) is called a G-bundle if (X, p, B) and $\alpha(X)$ are isomorphic for some G-space structure on X by an isomorphism $(1, f) : \alpha(X) \rightarrow (X, p, B)$ where $f : X/G \rightarrow B$ is a homeomorphism. A principal G-bundle is a G-bundle (X, p, B) , where X is a principal G-space.

Let B be a topological space. An open covering $\{U_\lambda\}_{\lambda \in \Lambda}$ of B is numerable if there is a locally finite partition of unity $u_i : B \rightarrow [0, 1]$, $i \in S$ such that $\overline{u_i^{-1}(0,1]}$ refines $\{U_\lambda\}$. [5, p. 169].

A principal G-bundle ξ over B is numerable if there is a numerable cover $\{U_\lambda\}_{\lambda \in \Lambda}$ of B such that $\xi|_{U_\lambda}$ is trivial for each $\lambda \in \Lambda$.

Let ξ be a numerable principal G-bundle over B , and $f : B' \rightarrow B$ be a map, then $f^*(\xi)$ is a numerable principal G-bundle over B' . [9, p. 48]. If $f_t : B' \rightarrow B$ is a homotopy, then $f_0^*(\xi)$ and $f_1^*(\xi)$ are isomorphic numerable principal G-bundles. [9, p. 51].

For each space B , let $k_G(B)$ be the set of isomorphism classes of numerable principal G-bundles over B . Let $[\xi]$ denote the class containing ξ . For a homotopy class $[f] : X \rightarrow Y$, define $k_G([f]) : k_G Y \rightarrow k_G X$ by $[\xi] \mapsto [f^*(\xi)]$. The class $[f^*(\xi)]$ is independent of ξ in $[\xi]$ and f in $[f]$. [9, p. 43]. Consequently, $k_G([f])$ is well-defined. k_G is

is a contravariant functor from the category of spaces and homotopy classes of maps to the category of sets. We denote $[A, B]$ the set of homotopy classes of maps $A \rightarrow B$.

Let $\omega = (E_0, p_0, B_0)$ be a fixed numerable principal G -bundle. For each space X , define $\phi_\omega : [-, B_0] \rightarrow k_G(-)$ by $\phi_\omega(X)([u]) = [u^*(\omega)]$, $[u] \in [X, B_0]$. ϕ_ω is a natural transformation of contravariant functors: $[-, B_0] \rightarrow k_G(-)$. [9, p. 52].

A principal G -bundle $\omega = (E_0, p_0, B_0)$ is universal provided ω is numerable and $\phi_\omega : [-, B_0] \rightarrow k_G(-)$ is a natural equivalence. The space B_0 is called a classifying space of G .

A bundle ω is called n -universal or universal for dimension $\leq n$ provided $\phi_\omega(X)$ is a bijection for each CW-complex X with $\dim. X \leq n$.

Milnor has shown that to every topological group G , there exists a universal G -bundle and consequently every topological group G has a classifying space. [9, p. 53].

1.6. Example: The Stiefel manifold of (orthonormal) k -frames in \mathbb{C}^n , denoted $V_k(\mathbb{C}^n)$ is the subspace of $(\mathbb{C}^n)^k$, consisting of the k -tuples (v_1, \dots, v_k) , $v_i \in \mathbb{C}^n$, $i=1, \dots, k$, such that $\langle v_i, v_j \rangle = \delta_{ij}$. With each k -frame (v_1, \dots, v_k) there is associated the k -dimensional subspace $\langle v_1, \dots, v_k \rangle$ spanned by v_1, \dots, v_k . Moreover, each k -dimensional subspace

of \mathbb{C}^n is of the form $\langle v_1, \dots, v_k \rangle$.

The Grassman manifold of k -dimensional subspaces of \mathbb{C}^n , denoted by $G_k(\mathbb{C}^n)$, is the set of k -dimensional subspaces of \mathbb{C}^n with the quotient topology defined by the function:

$$p : (v_1, \dots, v_k) \longmapsto \langle v_1, \dots, v_k \rangle \text{ of } V_k(\mathbb{C}^n)$$
onto $G_k(\mathbb{C}^n)$.

Let $GL(n, \mathbb{C})$ denote the set of $n \times n$ matrices with determinant different from 0. $GL(n, \mathbb{C})$ is an open subset of \mathbb{C}^{n^2} and therefore a manifold and a Lie group. Define the unitary group $U(n)$ by $U(n) = \{u \in GL(n, \mathbb{C}) : \langle u(x), u(y) \rangle = \langle x, y \rangle, \forall x, y \in \mathbb{C}^n\}$. $U(n)$ is a closed subgroup of $GL(n, \mathbb{C})$ and therefore, $U(n)$ is a Lie group. [8, p. 84].

$U(n)$ acts transitively on $V_k(\mathbb{C}^n)$ and on $G_k(\mathbb{C}^n)$. Therefore, $V_k(\mathbb{C}^n)$ is homeomorphic to $U(n)/U(n-k)$ and $G_k(\mathbb{C}^n)$ is homeomorphic to $U(n)/U(k) \times U(n-k)$. Hence, $U(n)/U(n-k)$, $U(n)/U(k) \times U(n-k)$ both admit a structure of analytic manifold. [7, p. 92 - 93]. Therefore, by identifying $V_k(\mathbb{C}^n)$ with $U(n)/U(n-k)$ and $G_k(\mathbb{C}^n)$ with $U(n)/U(k) \times U(n-k)$, $V_k(\mathbb{C}^n)$ and $G_k(\mathbb{C}^n)$ are analytic manifolds of dimension $k(2n-k)$ and $2k(n-k)$ respectively.

One shows that the bundle $(V_k(\mathbb{C}^n), p, G_k(\mathbb{C}^n))$ is a principal $U(k)$ -bundle, [9, p. 79], which is universal in $\dim. \leq 2(n-k)$. [9, p. 83].

CHAPTER II

K - THEORY

The class of all finite-dimensional complex vector bundles over a space B and B -morphisms of vector bundles forms a category, denoted by VB_B . [9, p. 25]. If B is a one point space, denoted by 0 , then, VB_0 can be viewed as the category of complex vector spaces.

2.1. Let $VB_0(p, q)$ denote the product category consisting of p copies of VB_0 and q copies of VB_0^* , the dual category of VB_0 . Let $F : VB_0(p, q) \rightarrow VB_0$ be a functor of $(p + q)$ variables: $\underbrace{VB_0 \times \dots \times VB_0}_{p+q} \rightarrow VB_0$ which is covariant in each of the first p -variables and contravariant in each of the last q -variables. [11, p. 58]. F is a continuous functor if for any complex vector spaces $V_1, \dots, V_{p+q}, W_1, \dots, W_{p+q}$, any space Z , and any family of maps $\{u_i : Z \rightarrow L(V_i, W_i)\}$, the function $Z \rightarrow L(F(V_1, \dots, V_p, W_{p+1}, \dots, W_{p+q}), F(W_1, \dots, W_p, V_{p+1}, \dots, V_{p+q}))$ defined by $z \mapsto F(u_1(z), \dots, u_{p+q}(z))$ is continuous.

2.2. Examples:

(1) \oplus is a functor: $VB_0(2, 0) \rightarrow VB_0$ assigning to $(V_1, V_2) \in VB_0(2, 0)$, the vector space $V_1 \oplus V_2$ and to a

pair of linear maps $f_1 : V_1 \rightarrow W_1$, $f_2 : V_2 \rightarrow W_2$, the map $f_1 \oplus f_2 : V_1 \oplus V_2 \rightarrow W_1 \oplus W_2$ defined by $(f_1 \oplus f_2)(v_1, v_2) = (f_1(v_1), f_2(v_2))$. $v_1 \in V_1, v_2 \in V_2$.

For any two maps $u_1 : Z \rightarrow L(V_1, W_1)$, $u_2 : Z \rightarrow L(V_2, W_2)$

consider the function ϕ defined by the composition

$$\begin{array}{ccc} Z & \xrightarrow{(u_1, u_2)} & L(V_1, W_1) \times L(V_2, W_2) \\ & \searrow \phi & \downarrow \oplus \\ & & L(V_1 \oplus V_2, W_1 \oplus W_2) \end{array}$$

where \oplus is defined as $\oplus (u_1(z), u_2(z)) = u_1(z) \oplus u_2(z)$.

The function \oplus is linear for if $\psi_1 : V_1 \rightarrow W_1$, $\psi_1' : V_1 \rightarrow W_1$, $\psi_2 : V_2 \rightarrow W_2$, $\psi_2' : V_2 \rightarrow W_2$, $a, b \in \mathbb{C}$ then,

$$\begin{aligned} \oplus (a(\psi_1, \psi_2) + b(\psi_1', \psi_2')) &= \oplus (a\psi_1 + b\psi_1', a\psi_2 + b\psi_2') \\ &= (a\psi_1 + b\psi_1') \oplus (a\psi_2 + b\psi_2') \end{aligned}$$

$$\begin{aligned} \text{and } (a\psi_1 + b\psi_1') \oplus (a\psi_2 + b\psi_2')(v_1, v_2) &= (a\psi_1(v_1) + b\psi_1'(v_1), a\psi_2(v_2) + b\psi_2'(v_2)) \\ &= (a\psi_1(v_1), a\psi_2(v_2)) + (b\psi_1'(v_1), b\psi_2'(v_2)) \\ &= a(\psi_1(v_1), \psi_2(v_2)) + b(\psi_1'(v_1), \psi_2'(v_2)) = a(\psi_1 \oplus \psi_2)(v_1, v_2) \\ &\quad + b(\psi_1' \oplus \psi_2')(v_1, v_2) = (a(\psi_1 \oplus \psi_2) + b(\psi_1' \oplus \psi_2'))(v_1, v_2). \end{aligned}$$

Being linear \oplus is continuous and hence ϕ is continuous:

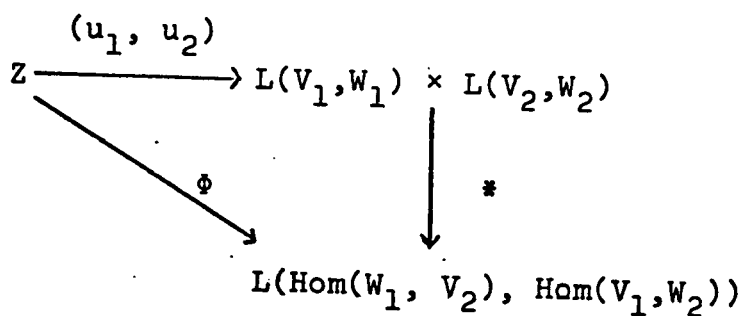
\oplus is a continuous functor.

(ii) \otimes is a functor : $VB_0(2, 0) \rightarrow VB_0$ assigning to $(V_1, V_2) \in VB_0(2, 0)$, the vector space $V_1 \otimes_{\mathbb{C}} V_2$, which is the tensor product over \mathbb{C} of two vector spaces, and to $f_1 : V_1 \rightarrow W_1, f_2 : V_2 \rightarrow W_2$, the linear map $f_1 \otimes f_2 : V_1 \otimes V_2 \rightarrow W_1 \otimes W_2$ defined by $(f_1 \otimes f_2)(v_1, v_2) = f_1(v_1) \otimes f_2(v_2)$. For any two maps $u_1 : Z \rightarrow L(V_1, W_1), u_2 : Z \rightarrow L(V_2, W_2)$, consider the function ϕ defined by the composition:

$$\begin{array}{ccc}
 Z & \xrightarrow{(u_1, u_2)} & L(V_1, W_1) \times L(V_2, W_2) \\
 & \searrow \phi & \downarrow \otimes \\
 & & L(V_1 \otimes V_2, W_1 \otimes W_2)
 \end{array}$$

where $\otimes : L(V_1, W_1) \times L(V_2, W_2) \rightarrow L(V_1 \otimes V_2, W_1 \otimes W_2)$ is defined as $\otimes(f, g) = f \otimes g$. For $a_1, a_2 \in \mathbb{C}, \phi_1, \phi_2 \in L(V_1, W_1), \psi \in L(V_2, W_2)$, $\otimes(a_1\phi_1 + a_2\phi_2, \psi) = (a_1\phi_1 + a_2\phi_2) \otimes \psi = a_1(\phi_1 \otimes \psi) + a_2(\phi_2 \otimes \psi)$. Similarly, $\otimes(\phi, b_1\psi_1 + b_2\psi_2) = b_1(\phi \otimes \psi_1) + b_2(\phi \otimes \psi_2)$. So that \otimes is bilinear, and so is continuous, since the vector spaces are finite dimensional. ϕ is continuous, and so \otimes is a continuous functor.

(iii) To vector spaces V_1, V_2 , define $\text{Hom}(V_1, V_2)$ as the space of all linear maps from V_1 to V_2 . $\text{Hom}(V_1, V_2) \in \text{VB}_0$. Let $f_1 : V_1 \rightarrow W_1, f_2 : V_2 \rightarrow W_2$ be linear maps, then $\text{Hom}(f_1, f_2) \in L(\text{Hom}(W_1, V_2), \text{Hom}(V_1, W_2))$ is defined as $\text{Hom}(f_1, f_2)(f) = f_2 \circ f \circ f_1$ for each $f \in \text{Hom}(W_1, V_2)$. Consequently, Hom is a functor: $\text{VB}_0 \times \text{VB}_0 \rightarrow \text{VB}_0$ which is contravariant in the first variable and covariant in the second variable. Let Z be any space and $u_1 : Z \rightarrow L(V_1, W_1), u_2 : Z \rightarrow L(V_2, W_2)$, the composite function ϕ defined in the following diagram is continuous if $*$ is continuous.



where $*(u_1(z), u_2(z))(f) = u_2(z) \circ f \circ u_1(z) \forall f \in \text{Hom}(W_1, V_2)$.

$*$ is bilinear: for each $f \in \text{Hom}(W_1, V_2), a_1, a_2 \in \mathbb{C}, \phi_1,$

$\phi_2 \in L(V_1, W_1) \psi \in L(V_2, W_2), *((a_1\phi_1 + a_2\phi_2), \psi)(f)$

$$= \psi \circ f \circ (a_1\phi_1 + a_2\phi_2) = a_1\psi \circ f \circ \phi_1 + a_2\psi \circ f \circ \phi_2$$

$$= \{a_1[*(\phi_1, \psi)] + a_2[*(\phi_2, \psi)]\}(f). \text{ Therefore } *(a_1\phi_1 + a_2\phi_2, \psi)$$

$$= a_1[*(\phi_1, \psi)] + a_2[*(\phi_2, \psi)]. \text{ Similarly, } *(\phi, a_1\psi_1 + a_2\psi_2)$$

$$= a_1[*(\phi, \psi_1)] + a_2[*(\phi, \psi_2)]. \text{ } * \text{ is bilinear and therefore}$$

continuous: Hom is a continuous functor.

2.3. Some Results About Continuous Functors: These results allow the extension to vector bundles of several operations defined for vector spaces, like the direct sum, the tensor product and so on.

Theorem 1: For each continuous functor $F : VB_0(p, q) \rightarrow VB_0$ there exists a family of functors $F_B : VB_B(p, q) \rightarrow VB_B$, one for each space B such that $F_{B_1}(f^*(\xi_1), \dots, f^*(\xi_{p+q}))$ and $f^*F_B(\xi_1, \dots, \xi_{p+q})$ are B_1 -isomorphic bundles for each map $f : B_1 \rightarrow B$. Moreover $F = F_0$. [9, p. 65]

Theorem 2: Let $F, G : VB_0(p, q) \rightarrow VB_0$ be two continuous functors and $\phi : F \rightarrow G$ be a morphism of functors. Then for each space B , there exists a morphism $\phi_B : F_B \rightarrow G_B$ of functors for which $\phi_B(\xi_1, \dots, \xi_{p+q}) : F_B(\xi_1, \dots, \xi_{p+q}) \rightarrow G_B(\xi_1, \dots, \xi_{p+q})$ restricted to the fibre over $z \in B$ is simply $\phi(\xi_{1,z}, \dots, \xi_{p+q,z}) : F(\xi_{1,z}, \dots, \xi_{p+q,z}) \rightarrow G(\xi_{1,z}, \dots, \xi_{p+q,z})$ and ϕ_B is unique with this property. [9, p. 66]

Theorem 3: Let $F, G, H : VB_0(p, q) \rightarrow VB_0$ be continuous functors and $\phi : F \rightarrow G, \psi : G \rightarrow H$ be morphisms of functors, then

- (i) $(\psi\phi)_B = \psi_B\phi_B$
- (ii) If $\phi : F \rightarrow G$ is an isomorphism, then $\phi_B : F_B \rightarrow G_B$ is an isomorphism. [9, p. 66]

2.4. We have a continuous functor $\oplus : VB_0(2,0) \rightarrow VB_0$.

By Theorem 1, for each space B, it defines a functor

$\oplus : VB_B(2,0) \rightarrow VB_B$, by assigning to (ξ, η) , the bundle $\xi \oplus \eta$. The tensor product in VB_0 is a continuous functor

$\otimes : VB_0(2, 0) \rightarrow VB_0$. It therefore defines a functor

$\otimes : VB_B(2, 0) \rightarrow VB_B$, by assigning to (ξ, η) , the bundle $\xi \otimes \eta$.

Hence we have two operations \oplus , the Whitney sum, and \otimes , the tensor product in VB_B .

By Theorems 2 and 3, the usual properties of the direct sum and the tensor product of vector spaces prolong to the Whitney sum and tensor product of vector bundles.

The usual properties in VB_0 are the following:

$$V_1 \oplus V_2 \cong V_2 \oplus V_1$$

$$(V_1 \oplus V_2) \oplus V_3 \cong V_1 \oplus (V_2 \oplus V_3)$$

$$V_1 \otimes V_2 \cong V_2 \otimes V_1$$

$$(V_1 \otimes V_2) \otimes V_3 \cong V_1 \otimes (V_2 \otimes V_3)$$

$$V_1 \otimes (V_2 \oplus V_3) \cong (V_1 \otimes V_2) \oplus (V_1 \otimes V_3).$$

Therefore, in VB_B , we have the following:

$$\xi \oplus \eta \cong \eta \oplus \xi$$

$$(\xi \oplus \eta) \oplus \zeta \cong \xi \oplus (\eta \oplus \zeta)$$

$$\xi \otimes \eta \cong \eta \otimes \xi$$

$$(\xi \otimes \eta) \otimes \zeta \cong \xi \otimes (\eta \otimes \zeta)$$

$$\xi \otimes (\eta \oplus \zeta) \cong (\xi \otimes \eta) \oplus (\xi \otimes \zeta).$$

The trivial line bundle is the product bundle

$\theta^1 = (B \times C, p, B)$. The canonical isomorphism $V \otimes C \cong V$ determines a natural equivalence of functors $F : V \rightarrow V \otimes C$ and $G : V \rightarrow V$. By Theorem 2, the extension of this natural equivalence determines an isomorphism $\xi \otimes \theta^1 \cong \xi$. Consider the bundle $0 = (B, I_B, B)$. The vector space consisting of one point o only has the properties: $V \oplus o \cong V$, $V \otimes o \cong o$. These two properties are extended to: $\xi \oplus 0 \cong \xi$, $\xi \otimes 0 \cong 0$ in VB_B .

Alternating Descriptions of Whitney Sum " \oplus ".

Let $\xi, \eta \in VB_B$, there are two other ways of defining the Whitney sum, namely:

(i) Let $\xi = (E_1, p_1, B)$, $\eta = (E_2, p_2, B)$.

Define $\xi \oplus_1 \eta = (E, p, B)$ by

$$E = \{(e_1, e_2) \in E_1 \times E_2 : p_1(e_1) = p_2(e_2)\}$$

and $p : E \rightarrow B$ is defined as: $p(e_1, e_2) = p_1(e_1) = p_2(e_2)$.

(ii) Consider a vector bundle $\zeta = (E_1 \times E_2, p_1 \times p_2, B \times B)$ and the diagonal map $\Delta : B \rightarrow B \times B$ defined by $\Delta(b) = (b, b)$.

Define $\xi \oplus_2 \eta = \Delta^*(\zeta)$.

We are going to prove that these two definitions are equivalent to the one we have described above up to isomorphism, namely,

$$\xi \oplus_1 \eta \cong \xi \oplus_2 \eta \cong \xi \oplus \eta.$$

(a) To show $\xi \oplus_1 \eta \cong \xi \oplus_2 \eta$.

Proof: $\xi \oplus_2 \eta = \Delta^*(E_1 \times E_2, p_1 \times p_2, B \times B)$. Each

$x \in E(\xi \oplus_2 \eta)$, $x = (b, e_1, e_2)$ with

$\Delta(b) = (b, b) = (p_1 \times p_2)(e_1, e_2) = (p_1(e_1), p_2(e_2))$. There-

fore $x = (b, e_1, e_2)$ with $p_1(e_1) = p_2(e_2) = b$. Define

$u : E(\xi \oplus_2 \eta) \rightarrow E(\xi \oplus_1 \eta)$ by $u(b, e_1, e_2) = (e_1, e_2)$.

The map $v : E(\xi \oplus_1 \eta) \rightarrow E(\xi \oplus_2 \eta)$ defined by $v(e_1, e_2)$

$= (p(e_1, e_2), e_1, e_2)$ is the inverse of u . Therefore, u is

an isomorphism and so $\xi \oplus_1 \eta \cong \xi \oplus_2 \eta$.

(b) To show $\xi \oplus_1 \eta \cong \xi \oplus \eta$.

Proof: Let ξ be a k -dimensional complex vector bundle, and

η be a m -dimensional complex vector bundle, both over B . Let

$\{(U_i, h_i)\}$, $\{(U_i, k_i)\}$ be local charts for ξ and η respectively.

Let $\{g_{i,j}\}$ be the transition functions of ξ , and $\{f_{i,j}\}$ be the

transition functions of η , $i, j \in A$. Then $g_{i,j}$, $f_{i,j}$ are maps

$g_{i,j} : U_i \cap U_j \rightarrow L(\mathbb{C}^k, \mathbb{C}^k)$, $f_{i,j} : U_i \cap U_j \rightarrow L(\mathbb{C}^m, \mathbb{C}^m)$. [9, p. 61]

By the continuous functor $\oplus : VB_0(2, 0) \rightarrow VB_0$, we get transition functions $h_{i,j} : U_i \cap U_j \rightarrow L(\mathbb{C}^k, \mathbb{C}^k) \oplus L(\mathbb{C}^m, \mathbb{C}^m) \subseteq L(\mathbb{C}^{k+m}, \mathbb{C}^{k+m})$ defined by $h_{i,j}(b) = g_{i,j}(b) \oplus f_{i,j}(b)$. $\xi \oplus \eta$ is defined to be the vector bundle with transition functions $\{h_{i,j}\}$, $i, j \in A$. [9, p. 65]. Consider the sum space (coproduct) $Z = \coprod_i \{U_i \times (L(\mathbb{C}^k, \mathbb{C}^k) \oplus L(\mathbb{C}^m, \mathbb{C}^m))\}$. Define an equivalence relation in Z by $(b, s, i) \sim (b', s', j)$ iff $b = b'$ and $s' = h_{j,i}(b) \circ s$ (\circ is the composition in $L(\mathbb{C}^{k+m}, \mathbb{C}^{k+m})$). Let E be the quotient space and let us denote the equivalence classes by $\langle b, s, i \rangle$. Define $p : E \rightarrow B$ by $p \langle b, s, i \rangle = b$, then, the bundle with $\{h_{i,j}\}$ as transition functions is: $\xi \oplus \eta = (E, p, B)$. [9, p. 63]. Consider ξ as the vector bundle constructed from its transition functions $\{g_{i,j}\}$, then $\xi = (E_1, p_1, B)$ is such that $E_1 = \{ \langle b, s_1, j \rangle : s_1 \in L(\mathbb{C}^k, \mathbb{C}^k) \}$, $p_1 : \langle b, s_1, j \rangle \mapsto b$ and similarly, $E_2 = \{ \langle b, s_2, j \rangle : s_2 \in L(\mathbb{C}^m, \mathbb{C}^m) \}$, $p_2 : \langle b, s_2, j \rangle \mapsto b$. Define $u : \xi \oplus \eta \rightarrow \xi \oplus \eta$ by assigning

$$\langle b, s, i \rangle \mapsto (\langle b, s_1, i \rangle, \langle b, s_2, i \rangle)$$

where $s = s_1 \oplus s_2 \in L(\mathbb{C}^k, \mathbb{C}^k) \oplus L(\mathbb{C}^m, \mathbb{C}^m)$. u is well-defined

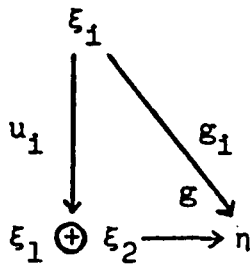
$$\begin{aligned} &\text{as } (b, s, i) \sim (b, s', j), s' = h_{j,i}(b) \circ s. \text{ Then } s' = s'_1 \oplus s'_2 \\ &= (g_{j,i}(b) \oplus f_{j,i}(b)) \circ (s_1 \oplus s_2) = g_{j,i}(b) \circ s_1 \oplus f_{j,i}(b) \circ s_2 \end{aligned}$$

$s'_1 = g_{j,1}(b) \circ s_1, s'_2 = f_{j,1}(b) \circ s_2 \Rightarrow (b, s'_1, j) \sim (b, s_1, i),$
 $(b, s'_2, j) \sim (b, s_2, i).$ Now define $v : \xi \oplus_1 \eta \rightarrow \xi \oplus \eta$
 by $v(\langle b, s_1, j \rangle, \langle b, s_2, j \rangle) = \langle b, s_1 \oplus s_2, j \rangle$, then
 $uv = \text{identity}, vu = \text{identity}, u$ is an isomorphism and
 $\xi \oplus_3 \eta \cong \xi \oplus_1 \eta.$

Remarks:

(i). The Whitney sum " \oplus " is the coproduct in VB_B .

Let us take $\oplus_1 = \oplus$, and let ξ_1, ξ_2 be any two vector
 bundles in VB_B . Define a morphism: $u_1 : \xi_1 \rightarrow \xi_1 \oplus \xi_2$ by
 $u_1(x) = (x, 0)$ where 0 denotes the 0-vector in the fibre of
 ξ_2 over $p_1(x)$. i.e. $u_1(x) = (x, s_2 p_1(x))$ with $s_2 = 0$ -cross
 section of ξ_2 , [9, p. 11]. u_1 is continuous. And define a
 morphism $u_2 : \xi_2 \rightarrow \xi_1 \oplus \xi_2$ in a similar way by $u_2(x) = (0, x)$.
 Let $\{g_i : \xi_i \rightarrow \eta\}, i = 1, 2$, be B-morphisms. Define $g : \xi_1 \oplus \xi_2 \rightarrow \eta$
 by $g(x_1, x_2) = g_1(x_1) + g_2(x_2)$



This is well-defined as $p_\eta(g_1(x_1)) = p_{\xi_1}(x_1) = p_{\xi_2}(x_2)$
 $= p_\eta(g_2(x_2))$, so $g_1(x_1), g_2(x_2)$ are in the same fibre in η

and $gu_1(x) = g(x,0) = g_1(x) + g_2(0) = g_1(x)$. i.e.

$gu_1 = g_1$, similarly $gu_2 = g_2$. Finally, it is clear that g is unique with the property $gu_i = g_i$, $i = 1, 2$.

(ii) In the category VB_B , the product of two vector bundles ξ_1 and ξ_2 coincides with the coproduct of ξ_1 and ξ_2 i.e., coincides with $\xi_1 \oplus \xi_2$. Define $p_i : \xi_1 \oplus \xi_2 \rightarrow \xi_i$ $i = 1, 2$ by

$$p_1(x_1, x_2) = x_1, \quad p_2(x_1, x_2) = x_2.$$

p_i is a B-morphism: $\xi_1 \oplus \xi_2 \rightarrow \xi_i$. Let $u_1 : \eta \rightarrow \xi_1$, $u_2 : \eta \rightarrow \xi_2$ be arbitrary B-morphisms. Define $u : \eta \rightarrow \xi_1 \oplus \xi_2$ by $u(x) = (u_1(x), u_2(x))$ as $p_{\xi_1} u_1(x) = p_\eta(x)$, $p_{\xi_2} u_2(x) = p_\eta(x)$, so $(u_1(x), u_2(x)) \in E(\xi_1 \oplus \xi_2)$ and u is a well-defined B-morphism. Moreover, $p_i u(x) = p_i(u_1(x), u_2(x)) = u_i(x)$, $i = 1, 2$; and u is unique with the property $p_i u = u_i$, $i = 1, 2$.

2.5. Let $\text{Vect}X$ be the set of isomorphism classes of complex vector bundles over X . Define \oplus and \otimes in $\text{Vect}X$ by $[\xi] \oplus [\eta] = [\xi \oplus \eta]$ and $[\xi] \otimes [\eta] = [\xi \otimes \eta]$. In VB_0 , if V_1 is isomorphic to V_2 , W_1 is isomorphic to W_2 , then $V_1 \oplus W_1 \cong V_2 \oplus W_2$, $V_1 \otimes W_1 \cong V_2 \otimes W_2$. These properties are extended to VB_X : If $\xi \cong \xi'$, $\eta \cong \eta'$, then

$\xi \oplus \eta \cong \xi' \oplus \eta'$, $\xi \otimes \eta \cong \xi' \otimes \eta'$. Hence, the operations \oplus and \otimes in $\text{Vect}X$ are well-defined, and $\text{Vect}X$ admits a natural commutative semi-ring structure, where a semi-ring is required to satisfy all axioms of a ring except the existence of an additive inverse. The 0 of $\text{Vect}X$ is the class of the 0-bundle (X, I_X, X) and the 1 is the class of θ^1 (see p. 13).

Let S, S' be two semi-rings. A function $f : S \rightarrow S'$ is a semi-ring morphism if $f(a + b) = f(a) + f(b)$ and $f(ab) = f(a)f(b)$. For instance, let $\xi \in \text{VB}_X$, and ξ_b be the fibre of ξ over $b \in X$. ξ_b admits a complex vector space structure with dimension denoted by $\dim \xi_b$. $\dim \xi_b$ is independent of $b \in X$ and also independent of the choice of ξ in the class $[\xi]$. Define the rank as a function $\text{rk} : \text{Vect}X \rightarrow \mathbb{Z}$ by $\text{rk}[\xi] = \dim \xi_b$. Let V_1, V_2 be two vector spaces over \mathbb{C} . $\dim(V_1 \otimes V_2) = (\dim V_1)(\dim V_2)$, $\dim(V_1 \oplus V_2) = \dim V_1 + \dim V_2$, so that $\text{rk}([\xi] \oplus [\eta]) = \text{rk}([\xi \oplus \eta]) = \dim(\xi \oplus \eta)_b = \dim(\xi_b \oplus \eta_b) = \dim \xi_b + \dim \eta_b = \text{rk}[\xi] + \text{rk}[\eta]$, and $\text{rk}([\xi] \otimes [\eta]) = \text{rk}[\xi \otimes \eta] = \dim(\xi \otimes \eta)_b = \dim(\xi_b \otimes \eta_b) = (\dim \xi_b)(\dim \eta_b) = (\text{rk}[\xi])(\text{rk}[\eta])$. Hence, the function $\text{rk} : \text{Vect}X \rightarrow \mathbb{Z}$ is a semi-ring morphism.

2.6. The ring completion of a semi-ring S is a pair (S^*, θ) , where S^* is a ring and $\theta : S \rightarrow S^*$ is a semi-ring morphism such that if $f : S \rightarrow R$ is any semi-ring morphism into

a ring R , there exists a unique ring morphism $g : S^* \rightarrow R$ such that the diagram

$$\begin{array}{ccc} S & & \\ \theta \downarrow & \searrow f & \\ S^* & \xrightarrow{g} & R \end{array}$$

commutes, i.e. $f = g\theta$.

Theorem: For any semi-ring S , the ring completion (S^*, θ) exists and is unique up to isomorphism. S^* consists of equivalence classes $\langle a, b \rangle$ of pairs $(a, b) \in S \times S$ where $(a, b) \sim (a', b')$ iff there exists $c \in S$ such that $a + b' + c = a' + b + c$. We shall use the simpler notation $\langle a, b \rangle = a - b$. The function $\theta : S \rightarrow S^*$ is defined by $\theta(a) = \langle a, 0 \rangle$. [9, p. 103]

2.7. In Section 2.5, we proved that $\text{Vect}X$ admits a commutative semi-ring structure for any topological space X . We define KX to be the ring completion of $\text{Vect}X$.

Let X and Y be topological spaces and f be a map from Y to X . Define $\text{Vect}(f) : \text{Vect}X \rightarrow \text{Vect}Y$ by $[\xi] \mapsto [f^*(\xi)]$.

The operation \oplus is a functor: $\text{VB}_X(2, 0) \rightarrow \text{VB}_X$, by 2.3,

Theorem 1, $f^*(\xi \oplus \eta) \cong f^*(\xi) \oplus f^*(\eta)$. The operation \otimes

is a functor: $\text{VB}_X(2, 0) \rightarrow \text{VB}_X$, by 2.3 Theorem 1, again,

$f^*(\xi \otimes \eta) \cong f^*(\xi) \otimes f^*(\eta)$. Hence,

$$\begin{aligned} \text{Vect}(f)([\xi] \oplus [\eta]) &= \text{Vect}(f)([\xi \oplus \eta]) = [f^*(\xi \oplus \eta)] \\ &= [f^*(\xi) \oplus f^*(\eta)] = [f^*(\xi)] \oplus [f^*(\eta)] \\ &= (\text{Vect}(f)[\xi]) \oplus (\text{Vect}(f)[\eta]) \end{aligned}$$

Similarly,

$$\text{Vect}(f)([\xi] \otimes [n]) = (\text{Vect}(f)[\xi]) \otimes (\text{Vect}(f)[n]).$$

Therefore, $\text{Vect}(f)$ is a semi-ring morphism. Moreover,

$$\text{Vect}(I_X) = I_{\text{Vect}X}$$

and if $f : Y \rightarrow X$, $g : Z \rightarrow Y$ are maps, then,

$$\begin{aligned} [\text{Vect}(f \circ g)]([\xi]) &= [(f \circ g)^*(\xi)] = [g^* \circ f^*(\xi)] \quad [9, \text{p. 19}] \\ &= \text{Vect}(g) \cdot \text{Vect}(f)[\xi]. \end{aligned}$$

Thus, Vect is a contravariant functor from the category of topological spaces to the category of semi-rings.

Consider the following diagram:

$$\begin{array}{ccc} \text{Vect}X & \xrightarrow{\theta} & KX \\ \downarrow & \text{Vect}(f) & \downarrow Kf \\ \text{Vect}Y & \xrightarrow{\theta} & KY \end{array}$$

where θ is the canonical imbedding of the semi-ring into its ring completion. Since $\theta \circ \text{Vect}(f)$ is a semi-ring morphism into a ring KY , it induces uniquely a ring morphism $Kf : KX \rightarrow KY$, and K is a contravariant functor from the category of topological spaces to the category of rings.

Since $\text{rk} : \text{Vect}X \rightarrow Z$ is a semi-ring morphism, it induces uniquely a ring morphism $\phi : KX \rightarrow Z$ such that the diagram commutes. For simplicity, we also denote ϕ by rk .

$$\begin{array}{ccc} \text{Vect}X & & \\ \downarrow \theta & \searrow \text{rk} & \\ KX & \xrightarrow{\phi} & Z \end{array}$$

Define \tilde{K} as the reduced K functor by the relation $\tilde{K} = \text{Ker}(\text{rk} : K \rightarrow \mathbf{Z})$ which assigns to each space X , the ideal $\tilde{K}X = \text{Ker}(\text{rk} : KX \rightarrow \mathbf{Z})$ and to each map $f : Y \rightarrow X$, the restriction of Kf to $\tilde{K}X$. Since $\text{rk}[\xi] = \text{rk}[f^*(\xi)]$, [9, p. 26], it follows that $Kf|_{\tilde{K}X} : \tilde{K}X \rightarrow \tilde{K}Y$. Hence \tilde{K} is a contravariant functor from the category of topological spaces to the category of rings.

Define a ring morphism $\epsilon : \mathbf{Z} \rightarrow KX$ by

$$\epsilon(n) = \begin{cases} [\theta^n] & n \geq 0 \\ -[\theta^{-n}] & n \leq 0 \end{cases}$$

where $\theta^n = (X \times \mathbb{C}^n, p, X)$ and p is the first projection. Then, $\text{rk} \circ \epsilon = I_{\mathbf{Z}}$ and the following exact sequence of abelian groups

$$0 \rightarrow \tilde{K}X \xrightarrow{i} KX \xrightarrow[\epsilon]{\text{rk}} \mathbf{Z} \rightarrow 0$$

splits (where i is the inclusion). So $KX \cong \tilde{K}X \oplus \mathbf{Z}$.

2.8. Another Description of \tilde{K} .

Two vector bundles ξ and η over a space X are s-equivalent, denoted $\xi \sim \eta$, if there exists integers $q \geq 0, n \geq 0$ such that $\xi \oplus \theta^n \cong \eta \oplus \theta^q$. s-equivalence is an equivalence relation, and isomorphic vector bundles are s-equivalent. Consequently, s-equivalence can be regarded as an equivalence relation in $\text{Vect}X$ by defining $[\xi] \sim [\eta]$ if $\xi \sim \eta$ for any $\xi \in [\xi]$ and $\eta \in [\eta]$.

Let X be a finite CW-complex, and $\xi \in VB_X$, then there exists $\eta \in VB_X$, such that $\xi \oplus \eta \cong \theta^m$, for some integer $m \geq 0$. [9, p. 31]

Theorem: Let X be a finite CW-complex. Then the function $\alpha : VectX \rightarrow \hat{K}X$ defined by $\alpha([\xi]) = [\xi] - rk[\xi]$ is a surjection and $\alpha([\xi]) = \alpha([\eta])$ iff $[\xi] \sim [\eta]$. To form $[\xi] - rk[\xi]$, we imbed by $\epsilon : Z \hookrightarrow KX$. Thus, for the category of finite CW-complex, the elements of $\hat{K}X$ can be identified with s -equivalence classes in $VectX$. [9, p. 105]

2.9. Another Description of K : Let \underline{C} be a small category with short exact sequences. Let \underline{FC} be the free-abelian group generated by the set of objects of \underline{C} , and D be the subgroup of \underline{FC} generated by elements of the form $A - A' - A''$ whenever $0 \rightarrow A' \rightarrow A \rightarrow A'' \rightarrow 0$ is exact. Define the Grothendieck group \underline{GC} of \underline{C} to be the quotient group $\underline{GC} = \underline{FC}/D$.

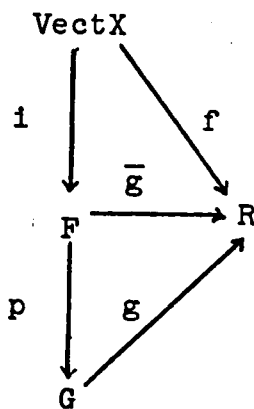
For each space X , consider the semi-ring $VectX$. $VectX$ can be regarded as a category with isomorphism classes of complex vector bundles over X as objects and the corresponding classes of X -morphisms as morphisms. Let F be the free-abelian group generated by $VectX$, and D be the subgroup generated by elements of the form: $[\xi] - [\xi'] - [\xi'']$ whenever $0 \rightarrow \xi' \xrightarrow{u} \xi \xrightarrow{v} \xi'' \rightarrow 0$ is exact. [9, p. 35]. Since every exact sequence of vector bundles splits, [9, p.37], D is generated by elements of the

form $[\xi' \oplus \xi''] - [\xi'] - [\xi'']$ for all $[\xi'], [\xi''] \in \text{Vect}X$.
The corresponding Grothendieck group is $G = F/D$.

Define a multiplication in F in the following way: for any $\alpha = \sum_i n_i [\xi_i]$, $\beta = \sum_i m_i [\eta_i] \in F$, where $[\xi_i], [\eta_i] \in \text{Vect}X$, $m_i, n_i \in \mathbb{Z}$ and for all except a finite number of $n_i, m_i = 0$, define $\alpha \times \beta = \sum_{i,j} n_i m_j [\xi_i \otimes \eta_j]$. Since $\xi_i \otimes \eta_j \cong \eta_j \otimes \xi_i$ and $(\xi_i \otimes \eta_j) \otimes \zeta_k \cong \xi_i \otimes (\eta_j \otimes \zeta_k)$ (by 2.4), this multiplication is commutative and associative. One checks easily that it is also distributive with respect to the addition. Hence F is a commutative ring. Moreover, for a generator $[\xi_1 \oplus \xi_2] - [\xi_1] - [\xi_2] \in D$ and $\alpha = \sum_i m_i [\eta_i] \in F$, $\alpha \times \{([\xi_1 \oplus \xi_2] - [\xi_1] - [\xi_2])\} = \sum_i m_i [(\eta_i \otimes \xi_1) \oplus (\eta_i \otimes \xi_2)] - \sum_i m_i [\eta_i \otimes \xi_1] - \sum_i m_i [\eta_i \otimes \xi_2]$, i.e. $\alpha \times \{([\xi_1 \oplus \xi_2] - [\xi_1] - [\xi_2])\} \in D$, D is an ideal and $G = F/D$ is a commutative ring.

Let $f : \text{Vect}X \rightarrow R$ be any semi-ring morphism into a ring R . By the universal property of the free abelian group, there exists a unique homomorphism $\bar{g} : F \rightarrow R$ such that $f = \bar{g} \circ i$, where $i : \text{Vect}X \rightarrow F$ is the canonical injection. Let $\alpha = \sum_i n_i [\xi_i]$, $\beta = \sum_i m_i [\eta_i]$ in F , $\alpha \times \beta = \sum_{i,j} n_i m_j [\xi_i \otimes \eta_j]$, $\bar{g}(\alpha \times \beta) = \sum_{i,j} n_i m_j \bar{g}[\xi_i \otimes \eta_j] = \sum_{i,j} n_i m_j f[\xi_i \otimes \eta_j] = \sum_{i,j} n_i m_j f[\xi_i] \cdot f[\eta_j] = \sum_{i,j} n_i m_j \bar{g}[\xi_i] \cdot \bar{g}[\eta_j] = \bar{g}(\alpha) \bar{g}(\beta)$.

Hence \bar{g} is a ring homomorphism. And finally $\bar{g}[\xi \oplus \xi']$
 $= f[\xi \oplus \xi'] = f([\xi] \oplus [\xi']) = f[\xi] + f[\xi'] = \bar{g}[\xi] + \bar{g}[\xi']$,
and so $\bar{g}([\xi \oplus \xi'] - [\xi] - [\xi']) = 0$, therefore $\bar{g}|D = 0$,
and there exists a unique ring homomorphism $g : G \rightarrow R$ such
that the following diagram commutes,



where p is the projection: $F \rightarrow G = F/D$. Let $\theta = p \circ i$, θ
is a semi-ring morphism and (G, θ) has the universal property
of the ring completion of the semi-ring $\text{Vect}X$. By uniqueness,
 $G \cong KX$.

CHAPTER III

THE PERIODICITY IN K-THEORY

All spaces considered in this chapter are finite CW-complexes.

3.1. By the tensor product of vector bundles, we define a ring structure on KX and $\tilde{K}X$ is an ideal in KX , which is the kernel of $\text{rk} : KX \rightarrow \mathbb{Z}$.

Let $\pi_X : X \times Y \rightarrow X$ and $\pi_Y : X \times Y \rightarrow Y$ denote the two projections. K , being contravariant, π_X and π_Y induce the homomorphisms

$$\pi_X^* : KX \rightarrow K(X \times Y), \quad \pi_Y^* : KY \rightarrow K(X \times Y).$$

Define a function $\phi : KX \times KY \rightarrow K(X \times Y)$ by

$\phi : (a, b) \longmapsto \pi_X^*(a) \cdot \pi_Y^*(b)$ where \cdot denotes the multiplication in $K(X \times Y)$. The function $\phi : (a, b) \longmapsto \pi_X^*(a) \cdot \pi_Y^*(b)$ is bilinear and so induces a unique abelian group homomorphism $\alpha : KX \otimes KY \rightarrow K(X \times Y)$ such that the diagram

$$\begin{array}{ccc} KX \times KY & \xrightarrow{\phi} & K(X \times Y) \\ \downarrow p & \nearrow \alpha & \\ KX \otimes KY & & \end{array}$$

commutes. Therefore, α is defined by $\alpha(a \otimes b) = \pi_X^*(a) \cdot \pi_Y^*(b)$.

It is called the external K-cup product.

If $\text{rk}(a) = \text{rk}(b) = 0$, then $\text{rk}(\pi_X^*(a)) = \text{rk}(\pi_Y^*(b)) = 0$.

Hence the restriction of the external K -cup product to $\hat{K}X \otimes \hat{K}Y$ maps into $\hat{K}(X \times Y)$. It is the external \hat{K} -cup product $\tilde{\alpha} : \hat{K}X \otimes \hat{K}Y \rightarrow \hat{K}(X \times Y)$.

Proposition 1: Let $(X, x_0), (Y, y_0)$ be two pointed spaces, $X \vee Y = (X \times y_0) \cup (x_0 \times Y)$, and the reduced product $X \wedge Y$ defined by $X \wedge Y = X \times Y / X \vee Y$. Let $\pi : X \times Y \rightarrow X \wedge Y = X \times Y / X \vee Y$ be the canonical projection and $r : X \vee Y \rightarrow X \times Y$ be the canonical injection. Then the induced sequence of abelian groups $0 \rightarrow \hat{K}(X \wedge Y) \xrightarrow{\pi^*} \hat{K}(X \times Y) \xrightarrow{r^*} \hat{K}(X \vee Y) \rightarrow 0$ is exact.

Proposition 2: Let $\tilde{\alpha} : \hat{K}X \otimes \hat{K}Y \rightarrow \hat{K}(X \times Y)$ be the external \hat{K} -cup product and r^* is the homomorphism induced by $r : X \vee Y \rightarrow X \times Y$, then $r^* \tilde{\alpha} = 0$. [9 p. 117]

From these two results follows the existence of a homomorphism $\beta : \hat{K}X \otimes \hat{K}Y \rightarrow \hat{K}(X \wedge Y)$ called the \hat{K} -cup product (or reduced tensor product) making the following diagram commutative.

$$\begin{array}{ccccccc}
 0 & \rightarrow & \hat{K}(X \wedge Y) & \xrightarrow{\pi^*} & \hat{K}(X \times Y) & \xrightarrow{r^*} & \hat{K}(X \vee Y) \rightarrow 0 \\
 & & \swarrow \beta & & \uparrow \tilde{\alpha} & & \nearrow 0 \\
 & & & & \hat{K}X \otimes \hat{K}Y & &
 \end{array}$$

We describe now the \tilde{K} -cup product in terms of elements.

Consider $\tilde{K}X$ as the set of all s-equivalence classes in $\text{Vect}X$.

That is, we identify an element $[\xi] - \text{rk}[\xi] \in \tilde{K}X$ with a s-class $[\bar{\xi}]$ containing $[\xi]$ in $\text{Vect}X$. Let ξ, η be vector bundles over X and Y respectively. If $\text{rk}[\xi] = m, \text{rk}[\eta] = n$, then the external cup product of $[\bar{\xi}]$ and $[\bar{\eta}]$ is represented by:

$$\begin{aligned} \tilde{\alpha}([\bar{\xi}] \otimes [\bar{\eta}]) &= \tilde{\alpha}([\xi] - [\theta^m]) \otimes ([\eta] - [\theta^n]) \\ &= \{\pi_X^*[\xi] - [\theta^m]\} \cdot \{\pi_Y^*[\eta] - [\theta^n]\} \text{ (as } \pi_X^*[\theta^m] = [\theta^m]) \\ &= \pi_X^*[\xi] \otimes \pi_Y^*[\eta] - \pi_X^*[\xi] \otimes [\theta^n] - [\theta^m] \otimes \pi_Y^*[\eta] + [\theta^m] \otimes [\theta^n] \end{aligned}$$

Let $[\xi]$ represent $[\bar{\xi}]$ and $[\eta]$ represent $[\bar{\eta}]$, then the external

cup product of $[\xi]$ and $[\eta]$ is: $\pi_X^*[\xi] \otimes \pi_Y^*[\eta] + \pi_X^*(-[\xi]) \otimes [\theta^n]$
 $+ [\theta^m] \otimes \pi_Y^*(-[\eta])$ as $[\theta^m] \otimes [\theta^n] \in 0\text{-class}$. Now, denote by

$\xi \otimes \eta \in \text{VB}_{X \times Y}$ the product $\xi \otimes \eta = \pi_X^*(\xi) \otimes \pi_Y^*(\eta)$. Using

this notation, the external cup product becomes $[\xi] \otimes [\eta]$

$+ (-[\xi]) \otimes [\theta^n] + [\theta^m] \otimes (-[\eta])$ and is represented by $(\xi \otimes \eta)$

$\oplus (\xi^- \otimes \theta^n) \oplus (\theta^m \otimes \eta^-)$ where $\xi^- \in -[\xi]$ and $\eta^- \in -[\eta]$

[10, III p. 19] Define $\xi \# \eta = (\xi \otimes \eta) \oplus (\xi^- \otimes \theta^n) \oplus (\theta^m \otimes \eta^-)$,

a representative of $\tilde{\alpha}([\bar{\xi}] \otimes [\bar{\eta}])$. If $\xi \sim \xi', \eta \sim \eta'$, then

$\xi' \# \eta' \sim \xi \# \eta$. Hence the s-class $[\bar{\xi} \# \bar{\eta}]$ depends only on

$[\bar{\xi}]$ and $[\bar{\eta}]$. Define $[\bar{\xi}] \# [\bar{\eta}] = [\bar{\xi} \# \bar{\eta}]$. This is well-defined

and moreover, $r^*(\xi \# \eta)$, i.e. the restriction of $\xi \# \eta$ to

$X \vee Y$, belongs to the 0 s-class. Therefore, the exact sequence

$$\begin{array}{ccccccc}
 0 & \rightarrow & \tilde{K}(X \wedge Y) & \xrightarrow{\pi^*} & \tilde{K}(X \times Y) & \xrightarrow{r^*} & \tilde{K}(X \vee Y) \rightarrow 0 \\
 & & \swarrow \beta & & \uparrow \# & & \searrow 0 \\
 & & \tilde{K}X & \otimes & \tilde{K}Y & &
 \end{array}$$

implies that $[\bar{\xi}] \# [\bar{\eta}] \in I_m \pi^*$ and β is precisely the homomorphism induced by $\#$. Thus, we have a description of the \tilde{K} -cup product β in terms of elements and we shall denote simply $\beta([\bar{\xi}] \otimes [\bar{\eta}])$ by $[\bar{\xi}] \# [\bar{\eta}]$. [3]

3.2. The periodicity theorem of Bott can be expressed in K-theory as follows:

The external K-cup product $\alpha : KX \otimes KS^2 \rightarrow K(X \times S^2)$ is a natural isomorphism for any space X. [9, p. 128]

The sequence of spaces: $X \vee S^2 \rightarrow X \times S^2 \rightarrow X \wedge S^2$ induces the following split exact sequence

$$0 \rightarrow K(X \wedge S^2) \rightarrow \tilde{K}(X \times S^2) \rightarrow \tilde{K}(X \vee S^2) \rightarrow 0.$$

Moreover, the natural inclusions $X \rightarrow X \vee Y$, $Y \rightarrow X \vee Y$ induce an isomorphism $\tilde{K}(X \vee Y) \rightarrow \tilde{K}X \oplus \tilde{K}Y$. [9, p. 116]. Consider the following diagram:

$$\begin{array}{l}
 KX \otimes KS^2 = (\tilde{K}X \oplus Z) \otimes (\tilde{K}S^2 \oplus Z) \\
 \quad \quad \quad = \tilde{K}X \otimes \tilde{K}S^2 \oplus \tilde{K}S^2 \oplus \tilde{K}X \oplus Z \\
 \downarrow \alpha \\
 K(X \times S^2) = \tilde{K}(X \times S^2) \oplus Z = \tilde{K}(X \wedge S^2) \oplus \tilde{K}(X \vee S^2) \oplus Z \\
 \quad \quad \quad = \tilde{K}(X \wedge S^2) \oplus \tilde{K}S^2 \oplus \tilde{K}X \oplus Z.
 \end{array}$$

The α morphism restricted to the last three groups is the identity, and restricted to the first one, can be viewed as the \mathbb{K} -cup product; hence, the periodicity theorem can also be formulated as follows:

The \mathbb{K} -cup product $\beta : \mathbb{K}X \otimes \mathbb{K}S^2 \rightarrow \mathbb{K}(X \wedge S^2)$ is a natural isomorphism for any space X .

3.3. We consider again the Grassman manifold $G_n(\mathbb{C}^{2n})$ of n -dimensional subspace of \mathbb{C}^{2n} (p. 6). The natural imbedding of \mathbb{C}^{2n} into \mathbb{C}^{2n+1} induces an inclusion map $i : G_n(\mathbb{C}^{2n}) \rightarrow G_n(\mathbb{C}^{2n+1})$. Define an inclusion $j : G_n(\mathbb{C}^{2n+1}) \rightarrow G_{n+1}(\mathbb{C}^{2n+2})$ by $j(V) = V \oplus W$ for each $V \in G_n(\mathbb{C}^{2n+1})$, where W is the subspace of \mathbb{C}^{2n+2} generated by $\{e_{2n+2}\}$. The composition $j \circ i$ defines an inclusion $G_n(\mathbb{C}^{2n}) \rightarrow G_{n+1}(\mathbb{C}^{2n+2})$. Thus, we have the inclusions: $G_1(\mathbb{C}^2) \subset G_2(\mathbb{C}^4) \subset \dots \subset G_n(\mathbb{C}^{2n}) \subset G_{n+1}(\mathbb{C}^{2n+2}) \subset \dots$, and we define $B_u = \bigcup_{n \geq 1} G_n(\mathbb{C}^{2n})$ with the weak topology. We have the following representation theorem:

There is an isomorphism (i.e. natural equivalence) of contravariant functors defined on the homotopy category of finite CW-complexes $\theta : [-, B_u] \rightarrow \mathbb{K}(-)$. [9, p. 107]. θ is defined by $\theta_X[g] = g^*(\gamma_n^{2n}) - n$ where $g : X \rightarrow G_n(\mathbb{C}^{2n})$ and

γ_n^{2n} is a subbundle of the product bundle $(G_n(\mathbb{C}^{2n}) \times_{\mathbb{C}^{2n}, p, G_n(\mathbb{C}^{2n})})$ with the total space consisting of $(V, x) \in G_n(\mathbb{C}^{2n}) \times \mathbb{C}^{2n}$ such that $x \in V$.

From this isomorphism, follows the periodicity of the homotopy groups of U or more precisely, the existence of a weak homotopy equivalence $B_U \rightarrow \Omega^2 B_U$. Since $\tilde{K}S^2 = \mathbb{Z}$ [9, p.109], then the periodicity theorem of 3.2 can be formulated: there is a natural isomorphism $\tilde{K}X \rightarrow \tilde{K}(X \wedge S^2)$ and by the representation theorem above, a natural isomorphism

$$[X, B_U] \rightarrow [X \wedge S^2, B_U] = [X, \Omega^2 B_U]$$

which is therefore induced by a map $B_U \rightarrow \Omega^2 B_U$. Now setting $X = S^1$, and since $\pi_i B_U = \pi_{i-1} U$, for all i [9, p. 108], we get

$$\pi_{i-1} U = \pi_i B_U \rightarrow \pi_i \Omega^2 B_U = \pi_{i+2} B_U = \pi_{i+1} U,$$

for all i , i.e. the above map $B_U \rightarrow \Omega^2 B_U$ is a weak homotopy equivalence and the homotopy groups of U are periodic with period 2.

CHAPTER IV

THE PUPPE SEQUENCE AND HALF-EXACT FUNCTORS

The reference for this chapter is [4]

4.1. Let $f : (X, x_0) \rightarrow (Y, y_0)$ be a base point preserving map of pointed topological spaces. Define the mapping cone Cf of f by

$$Cf = (X \times I) \cup Y / \{(x, 0) = f(x), (x, 1) = * = (x_0, t), x \in X, t \in I\},$$

where I is the unit interval and $*$ is defined as the base point of Cf . The space Y is a subspace of Cf with its base point $y_0 = f(x_0)$ coinciding with the base point of Cf . Let $f' : Y \rightarrow Cf$ be the natural inclusion map, then f' is base point preserving. Hence we obtain the mapping cone Cf' of f' and the base point preserving inclusion map $f'' : Cf \rightarrow Cf'$. By iterating this procedure, we obtain an infinite sequence called the Puppe sequence:

$$X \xrightarrow{f} Y \xrightarrow{f'} Cf \xrightarrow{f''} Cf' \xrightarrow{f'''} Cf'' \rightarrow Cf''' \rightarrow \dots$$

For a space X with base point x_0 , define the suspension $\Sigma X = X \times I / \{(x_0, t) = (x, 0) = (x, 1) = *, x \in X, t \in I\}$ where $*$ is defined as the base point of ΣX . It is clear that $\Sigma X = Cf/Y$. For a map $f : (X, x_0) \rightarrow (Y, y_0)$, define $\Sigma f : \Sigma X \rightarrow \Sigma Y$ by $\Sigma f \langle x, t \rangle = \langle f(x), t \rangle$. Σf is a base point preserving map. Let $\pi : Cf \rightarrow \Sigma X$ be the natural projection. We obtain again an infinite sequence

$$X \xrightarrow{f} Y \xrightarrow{f'} Cf \xrightarrow{\pi} \Sigma X \xrightarrow{\Sigma f} \Sigma Y \xrightarrow{\Sigma f'} \Sigma Cf \xrightarrow{\Sigma \pi} \Sigma^2 X \rightarrow \dots$$

which is also called the Puppe sequence. These two above sequences are related by the following diagram which is homotopy commutative.

$$\begin{array}{ccccccccccc}
 X & \xrightarrow{f} & Y & \xrightarrow{f'} & Cf & \xrightarrow{f''} & Cf' & \xrightarrow{f'''} & Cf'' & \xrightarrow{f^{(4)}} & Cf'''' & \rightarrow \dots \\
 \downarrow \text{id.} & & \downarrow \text{id.} & & \downarrow \text{id.} & & \downarrow = & & \downarrow = & & \downarrow = & \\
 X & \xrightarrow{f} & Y & \xrightarrow{f'} & Cf & \xrightarrow{\pi} & \Sigma X & \xrightarrow{\Sigma f} & \Sigma Y & \xrightarrow{\Sigma f'} & \Sigma Cf & \rightarrow \dots
 \end{array}$$

where "=" stands for a homotopy equivalence.

4.2. Let \underline{C} be the category of pointed finite CW-complexes with homotopy classes of base point preserving maps as morphisms. Let Gr be the category of groups. A contravariant functor $t : \underline{C} \rightarrow Gr$ is half-exact if for all $f : X \rightarrow Y$, the sequence $t(Cf) \rightarrow t(Y) \rightarrow t(X)$ is exact.

By 2.7, \tilde{K} is a contravariant functor from the category of topological spaces to the category of groups. Let $f \simeq g : X \rightarrow Y$ be base point preserving. Then $\tilde{K}(f) : \tilde{K}Y \rightarrow \tilde{K}X$ is defined as

$$\begin{aligned}
 (\tilde{K}f)([\xi] - [n]) &= [f^*(\xi)] - [f^*(n)] = [g^*(\xi)] - [g^*(n)] \\
 &= (\tilde{K}g)([\xi] - [n]).
 \end{aligned}$$

This shows that $\tilde{K}f = \tilde{K}g$ and so \tilde{K} may be considered as a contravariant functor $\underline{C} \rightarrow Gr$. Let $X, Y \in \underline{C}$ and $f : X \rightarrow Y$ be in \underline{C} , then $\tilde{K}(Cf) \rightarrow \tilde{K}Y \rightarrow \tilde{K}X$ is exact. Hence \tilde{K} is a half-exact functor.

[9, p. 114]. Consider the contravariant functor

$\hat{K}(- \wedge S^2) : \underline{C} \rightarrow \text{Gr}$ which assigns to each $X \in \underline{C}$, a group $\hat{K}(X \wedge S^2)$ (cf. 3.1 for the definition of the reduced product) and to each $f : Y \rightarrow X$, the group homomorphism

$\hat{K}(f \wedge I_{S^2}) : \hat{K}(X \wedge S^2) \rightarrow \hat{K}(Y \wedge S^2)$ where $f \wedge I_{S^2}$ is defined

as follows: Consider the map $f \times I_{S^2} : Y \times S^2 \rightarrow X \times S^2$. The restriction $f \times I_{S^2}|_{Y \vee S^2} : Y \vee S^2 \rightarrow X \vee S^2$ and hence induces a map $Y \wedge S^2 = Y \times S^2 / Y \vee S^2 \rightarrow X \wedge S^2 = X \times S^2 / X \vee S^2$ denoted by $f \wedge I_{S^2}$. To show that $\hat{K}(- \wedge S^2)$ is half-exact, consider

the following diagram for each $f : X \rightarrow Y$ in \underline{C} :

$$\begin{array}{cccccccccccccccc}
 X & \xrightarrow{f} & Y & \xrightarrow{f'} & Cf & \xrightarrow{f''} & Cf' & \xrightarrow{f'''} & Cf'' & \xrightarrow{f^{(4)}} & Cf''' & \xrightarrow{f^{(5)}} & Cf^{(4)} & \xrightarrow{f^{(6)}} & Cf^{(5)} & \xrightarrow{\quad} & Cf^{(6)} \\
 \downarrow \text{id.} & & \downarrow \text{id.} & & \downarrow \text{id.} & & \downarrow \text{id.} & & \downarrow \text{id.} & & \downarrow \text{id.} & & \downarrow \text{id.} & & \downarrow \text{id.} & & \downarrow \text{id.} \\
 X & \xrightarrow{f} & Y & \xrightarrow{f'} & Cf & \xrightarrow{\pi} & \Sigma X & \xrightarrow{\Sigma f} & \Sigma Y & \xrightarrow{\Sigma f'} & \Sigma Cf & \xrightarrow{\Sigma \pi} & \Sigma^2 X & \xrightarrow{\Sigma^2 f} & \Sigma^2 Y & \xrightarrow{\quad} & \Sigma^2 Cf
 \end{array}$$

Since $Cf \wedge S^2 = \Sigma^2 Cf = Cf^{(6)}$, $Y \wedge S^2 = \Sigma^2 Y = Cf^{(5)}$,

$X \wedge S^2 = \Sigma^2 X = Cf^{(4)}$, and $f^{(6)} : Cf^{(4)} \rightarrow Cf^{(5)}$, the sequence

$\hat{K}(Cf^{(6)}) \rightarrow \hat{K}(Cf^{(5)}) \rightarrow \hat{K}(Cf^{(4)})$ is exact. Hence $\hat{K}(Cf \wedge S^2)$

$\rightarrow \hat{K}(Y \wedge S^2) \rightarrow \hat{K}(X \wedge S^2)$ is exact and $\hat{K}(- \wedge S^2)$ is a half-exact functor.

If t is a half-exact functor, then t transforms the Puppe sequence in an exact sequence, since in the Puppe sequence,

any space is the mapping cone of the preceding map.

Theorem: Let $\phi : t \rightarrow t'$ be a natural transformation of half-exact functors. Assume that $\phi(S^i) : t(S^i) \rightarrow t'(S^i)$ is an isomorphism for all $i \leq n$. Then $\phi(X)$ is an isomorphism for every finite CW-complex of $\dim X \leq n$.

Proof: (taken from [4]) Every finite CW-complex can be obtained by successively attaching wedges of cells, starting from a finite set of points. Let $\alpha(X)$ denote the number of such attachments to obtain X . We prove the theorem by induction on $\alpha(X)$. Thus, assume the theorem to be proved for X' and $X = X' \cup_f (V_\lambda e_\lambda^{j+1})$, with $\dim X' < n$, $j + 1 \leq n$ (by e^k , we denote a k -cell) $X = Cf$. Applying t and t' to the Puppe sequence defined by f , i.e.

$$V_\lambda(S_\lambda^j) \xrightarrow{f} X' \xrightarrow{f'} X \xrightarrow{\pi} V_\lambda(S_\lambda^{j+1}) \xrightarrow{\Sigma f} \Sigma X' \rightarrow$$

and using the naturality of $\phi : t \rightarrow t'$, we obtain a commutative diagram:

$$\begin{array}{ccccccccc} t(V_\lambda(S_\lambda^j)) & \leftarrow & t(X') & \leftarrow & t(X) & \leftarrow & t(V_\lambda(S_\lambda^{j+1})) & \leftarrow & t(\Sigma X') \\ \downarrow \phi_1 & & \downarrow \phi_2 & & \downarrow \phi_3 & & \downarrow \phi_4 & & \downarrow \phi_5 \\ t'(V_\lambda(S_\lambda^j)) & \leftarrow & t'(X') & \leftarrow & t'(X) & \leftarrow & t'(V_\lambda(S_\lambda^{j+1})) & \leftarrow & t'(\Sigma X') \end{array}$$

Since $t(A \vee B) = tA \times tB$ and using the assumption, ϕ_1 and ϕ_4 are isomorphisms. By the inductive hypothesis, ϕ_2 is an isomorphism. The space X' can be considered as the mapping

cone of some map $g : V_{\lambda'}(S_{\lambda'}^{j-1}) \rightarrow X''$, so that

$\Sigma X' = \Sigma(Cg) = C\Sigma g$, thus, $\alpha(\Sigma X') \leq \alpha(X')$. The induction hypothesis implies that ϕ_5 is also an isomorphism. Therefore, $\phi_3 : t(X) \rightarrow t'(X)$ is an isomorphism by the 5-lemma.

CHAPTER V

THE EQUIVALENCE BETWEEN THE TWO FORMULATIONS

In 3.3, we proved that if the \tilde{K} -cup product

$$\tilde{K}X \otimes \tilde{K}S^2 \rightarrow \tilde{K}(X \wedge S^2)$$

is an isomorphism, then there is a map

$$B_U \rightarrow \Omega^2 B_U$$

which is a weak homotopy equivalence and consequently, the homotopy groups of the infinite unitary group U are periodic with period 2. We shall now prove the converse. This can be done for the complex case by algebraic methods and this was the original proof by Atiyah [6]. We shall follow the proof given later on by Bott [3], which has the advantage of being applicable to the real and quaternionic cases. The idea is the following. Bott originally proved that there is a weak homotopy equivalence

$$\gamma : B_U \rightarrow \Omega^2 B_U$$

and γ induces an isomorphism

$$\tilde{K}X \rightarrow \tilde{K}(X \wedge S^2)$$

for every finite CW-complex X . On the other hand, the \tilde{K} -cup product

$$\tilde{K}X \otimes \tilde{K}S^2 \rightarrow \tilde{K}(X \wedge S^2)$$

is induced by a map α , which is homotopic to $\gamma^* : B_U \wedge S^2 \rightarrow B_U$.

This proves that the isomorphism induced by γ is actually the \hat{K} -cup product

$$\hat{K}X \otimes \hat{K}S^2 \rightarrow \hat{K}(X \wedge S^2).$$

We give now the details of the proof.

5.1. We construct first the map

$$\gamma : B_U \rightarrow \Omega^2 B_U.$$

To simplify the notation we write $G_n^m = G_n(\mathbb{C}^m)$ for the Grassman manifold of n -planes of \mathbb{C}^m (cf. p. 6) and we use the same notation for a map $X \times Y \rightarrow Z$ and its associate

$X \rightarrow Z^Y$. We define a map

$$\lambda_n : U(n) \rightarrow \Omega G_n^{2n}$$

in the following way. We identify the elements of G_n^{2n} with the n -dimensional subspaces of the complex vector space $\mathbb{C}^n \otimes_{\mathbb{R}} V_2^0$, where V_2^0 is a real vector space of dimension 2 with basis $\{e_1, e_2\}$. For $a \in U(n)$, define

$$\lambda_n(a, \phi) = \begin{cases} [x \otimes_{\mathbb{R}} e_1 \cos \frac{\phi}{2} + a(x) \otimes_{\mathbb{R}} e_2 \sin \frac{\phi}{2}], & 0 \leq \phi \leq \pi \\ [x \otimes_{\mathbb{R}} (e_1 \cos \frac{\phi}{2} + e_2 \sin \frac{\phi}{2})], & \pi \leq \phi \leq 2\pi \end{cases}$$

where $[\quad]$ means "subspace of \mathbb{C}^{2n} generated by" for all $x \in \mathbb{C}^n$. Therefore, λ_n is a map from $U(n)$ into the space of loops of G_n^{2n} at $\mathbb{C}^n \otimes_{\mathbb{R}} e_1$. Actually the map λ_n for $\pi \leq \phi \leq 2\pi$

is useless and we shall omit it in the future computations. We define now another map

$$f_n : G_n^{2n} \rightarrow \Omega U(2n)$$

in the following way. For $A \in G_n^{2n}$ and $0 \leq \theta \leq 2\pi$, $f_n(A, \theta)$ is the unitary transformation:

$$f_n(A, \theta)(x) = \begin{cases} xe^{i\theta}, & x \in A, & 0 \leq \theta \leq \pi \\ xe^{-i\theta}, & x \in A^\perp, & 0 \leq \theta \leq \pi \\ xe^{i\theta}, & x \in A_0, & \pi \leq \theta \leq 2\pi \\ xe^{-i\theta}, & x \in A_0^\perp, & \pi \leq \theta \leq 2\pi \end{cases}$$

where A^\perp is the orthogonal complement of A with respect to the usual metric of \mathbb{C}^{2n} and A_0 is the base point of G_n^{2n} .

The elements of G_{2n}^{4n} are identified with the $2n$ -dimensional subspaces of $\mathbb{C}^{2n} \otimes_{\mathbb{C}} V_2$, where V_2 is the complex vector space with basis $\{e_1, e_2\}$. Finally, as the classifying space

$$B_U = \bigcup_{n \geq 1} G_n^{2n} \text{ (cf. p. 29) we can define}$$

$$\gamma : B_U \rightarrow \Omega^2 B_U$$

by the composite $\Omega \lambda_{2n} \circ f_n$.

γ does not preserve the base points, since λ_{2n} does not. But B_U is simply connected and therefore γ is homotopic to a

base point preserving map which we denote also by γ .

γ induces

$$\gamma_* : [S^1, B_U] \rightarrow [S^1, \Omega^2 B_U] = [S^{1+2}, B_U]$$

But $\pi_1 B_U \cong \pi_{1-1} U$ (cf. p. 30) and therefore, $\gamma_* : \pi_{1-1} U \rightarrow \pi_{1+1} U$ is an isomorphism for every i by the Bott periodicity theorem, i.e. γ is a weak homotopy equivalence.

By 4.2 and 3.3, $\hat{K}(-)$ and $\hat{K}(- \wedge S^2)$ are half-exact functors on the category of finite CW-complexes and $\hat{K}(-)$ is represented by $[-, B_U]$. Hence, $[-, B_U]$ and $[- \wedge S^2, B_U]$ are half-exact. γ induces a natural transformation:

$[-, B_U] \rightarrow [-, \Omega^2 B_U] = [- \wedge S^2, B_U]$ which is an isomorphism for the sphere S^1 . Thus, by the theorem proved in 4.2,

$$\gamma_X : [X, B_U] \rightarrow [X \wedge S^2, B_U]$$

is an isomorphism for every finite CW-complex.

5.2. Consider the elements of $\hat{K}X$ as s-equivalence classes in $\text{Vect}X$ and imbed $\hat{K}(X \wedge Y)$ into $\hat{K}(X \times Y)$ by π . (cf. p. 26) The \hat{K} -cup product of an m -dimensional vector bundle ξ over X and an n -dimensional vector bundle η over Y is, as we have seen in 3.1,

$$\xi \# \eta = \xi \otimes \eta + \xi^- \otimes \theta^n + \theta^m \otimes \eta^-,$$

where we write $+$ instead of \oplus to simplify. Let

γ_n^{2n} and γ_1^2 be the vector bundles described on p. 30.

$$\overline{[\gamma_n^{2n} \# \gamma_1^2]} \in \mathcal{K}(G_n^{2n} \wedge G_1^2) \xrightarrow{\pi^*} \mathcal{K}(G_n^{2n} \times G_1^2) = [G_n^{2n} \times G_1^2, B_U].$$

$\overline{[\gamma_n^{2n} \# \gamma_1^2]}$ is represented by a map

$$\alpha : G_n^{2n} \times G_1^2 \rightarrow B_U = \bigcup_{i \geq 1} G_1^{2i}$$

Since $G_n^{2n} \times G_1^2$ is compact, there exists an integer k such that the image of α is contained in G_{n+k}^{2n+2k} . Let A_0 be the base point of G_n^{2n} , L_0 be the base point of G_1^2 , and A^\perp, L^\perp be the orthogonal complements of $A \in G_n^{2n}$ and $L \in G_1^2$ respectively. Using the relation

$$\xi \# \eta = \xi \otimes \eta + \xi^- \otimes \theta^n + \theta^m \otimes \eta^-,$$

α can be described as:

$$\alpha(A, L) = (A \otimes_{\mathbb{C}} L + A^\perp \otimes_{\mathbb{C}} L_0) + (A_0 \otimes_{\mathbb{C}} L^\perp + A_0^\perp \otimes_{\mathbb{C}} L_0)$$

where $A_0^\perp \otimes_{\mathbb{C}} L_0$ represents a trivial bundle added for reasons of symmetry. Since $A \otimes_{\mathbb{C}} L, A^\perp \otimes_{\mathbb{C}} L_0$ are n -dimensional subspaces in \mathbb{C}^{4n} , $A \otimes_{\mathbb{C}} L + A^\perp \otimes_{\mathbb{C}} L_0$ is a $2n$ -dimensional subspace in \mathbb{C}^{8n} . Similarly, $A_0 \otimes_{\mathbb{C}} L^\perp + A_0^\perp \otimes_{\mathbb{C}} L_0$ is a $2n$ -dimensional subspace in \mathbb{C}^{8n} . Note that they are in the same \mathbb{C}^{8n} , so that $\alpha(A, L)$ is a $4n$ -dimensional subspace in \mathbb{C}^{8n} .

Thus α is a map: $G_n^{2n} \times G_1^2 \rightarrow G_{4n}^{8n}$, which induces the \mathbb{K} -cup product

$$\mathbb{K}X \otimes \mathbb{K}S^2 \rightarrow \mathbb{K}(X \wedge S^2).$$

5.3. We now show that the map α is homotopic to the map associated to γ , i.e. the composite (that we still call γ)

$$G_n^{2n} \times [0, 2\pi] \times [0, \pi] \xrightarrow{f_n \times 1} U(2n) \times [0, \pi] \xrightarrow{\lambda_{2n}} G_{2n}^{4n}$$

Hence

$$\begin{aligned} \gamma(A, \theta, \phi) &= \lambda_{2n}(f_n(A, \theta), \phi) \\ &= [x \otimes_{\mathbb{R}} e_1 \cos \frac{\phi}{2} + f_n(A, \theta)(x) \otimes_{\mathbb{R}} e_2 \sin \frac{\phi}{2}] \end{aligned}$$

and γ sends the base point $(A_0, 0, 0)$ to $e^{2n} \otimes e_1$, base point of G_{2n}^{4n} . Let $I_2 = [0, 2\pi] \times [0, \pi]$ and define

$$\rho : I_2 \rightarrow G_1^2$$

by $\rho(\theta, \phi) = [e_1 \cos \frac{\phi}{2} + e_2 e^{i\theta} \sin \frac{\phi}{2}]$. In I_2 , identify (θ, ϕ) with $(\theta + 2\pi, \phi)$, $(\theta, 0)$ with a point and (θ, π) with another point for all θ . We obtain a quotient space \bar{I}_2 which is homeomorphic to G_1^2 , i.e. S^2 . Define $G_1^2 \rightarrow G_1^2$ by $L \mapsto \bar{L}$ where \bar{L} is the space obtained by conjugating each element of L . All the straight lines $L = \bar{L}$ constitute a circle $S^1 \subset G_1^2$ which divides G_1^2 into D^+ and D^- , where $D^+ = \rho(I_2 | \theta \leq \pi)$ and $D^- = \rho(I_2 | \theta \geq \pi)$.

For the continuous function γ defined above, we have:

$$\begin{aligned} \gamma(A, \theta + 2\pi, \phi) &= \gamma(A, \theta, \phi) \text{ and } \gamma(A, \theta, 0) = [x \otimes_{\mathbb{R}} e_1] \\ &= \mathbb{C}^n \otimes_{\mathbb{R}} e_1 \end{aligned}$$

$$\gamma(A, \theta, \pi) = [f_n(A, \theta)(x) \otimes_{\mathbb{R}} e_2] = \mathbb{C}^n \otimes_{\mathbb{R}} e_2.$$

Consequently γ induces a map from the quotient

$$\gamma : G_n^{2n} \times \bar{I}_2 \rightarrow G_{2n}^{4n}$$

which, by using the identification $\rho : \bar{I}_2 \rightarrow G_1^2$, is actually a map

$$\gamma : G_n^{2n} \otimes G_1^2 \rightarrow G_{2n}^{4n}.$$

Thus, for $A \in G_n^{2n}$, $L \in G_1^2$, we identify L with (θ, ϕ) . If $L \in D^+$, we have $\theta \leq \pi$ and

$$\begin{aligned} \gamma(A, L) &= [x \otimes_{\mathbb{R}} e_1 \cos \frac{\phi}{2} + xe^{i\theta} \otimes_{\mathbb{R}} e_2 \sin \frac{\phi}{2}]_{x \in A} \\ &\quad + [x \otimes_{\mathbb{R}} e_1 \cos \frac{\phi}{2} + xe^{-i\theta} \otimes_{\mathbb{R}} e_2 \sin \frac{\phi}{2}]_{x \in A^1}. \end{aligned}$$

By the isomorphisms

$$\begin{aligned} \mathbb{C}^{2n} \otimes_{\mathbb{C}} V_2 &\cong \mathbb{C}^{2n} \otimes_{\mathbb{C}} (\mathbb{C} \otimes_{\mathbb{R}} V_2^0) \cong (\mathbb{C}^{2n} \otimes_{\mathbb{C}} \mathbb{C}) \otimes_{\mathbb{R}} V_2^0 \\ &\cong \mathbb{C}^{2n} \otimes_{\mathbb{R}} V_2^0 \end{aligned}$$

we have

$$\begin{aligned} x \otimes_{\mathbb{C}} e^{i\theta} (\sin \frac{\phi}{2}) e_2 &\mapsto x \otimes_{\mathbb{C}} (e^{i\theta} \otimes_{\mathbb{R}} e_2 \sin \frac{\phi}{2}) \mapsto xe^{i\theta} \otimes_{\mathbb{R}} e_2 \sin \frac{\phi}{2} \\ x \otimes_{\mathbb{C}} e_1 \cos \frac{\phi}{2} &\mapsto x \otimes_{\mathbb{C}} (e^{i0} \otimes_{\mathbb{R}} e_1 \cos \frac{\phi}{2}) \mapsto x \otimes_{\mathbb{R}} e_1 \cos \frac{\phi}{2} \end{aligned}$$

Thus,

$$\begin{aligned}
 & [x \otimes_{\mathbb{R}} e_1 \cos \frac{\phi}{2} + x e^{i\theta} \otimes_{\mathbb{R}} e_2 \sin \frac{\phi}{2}]_{x \in A} \\
 &= [x \otimes_{\mathbb{C}} e_1 \cos \frac{\phi}{2} + x \otimes_{\mathbb{C}} e^{i\theta} e_2 \sin \frac{\phi}{2}]_{x \in A} \\
 &= [x \otimes_{\mathbb{C}} (e_1 \cos \frac{\phi}{2} + e^{i\theta} e_2 \sin \frac{\phi}{2})]_{x \in A} = A \otimes_{\mathbb{C}} L.
 \end{aligned}$$

Similarly,

$$[x \otimes_{\mathbb{R}} e_1 \cos \frac{\phi}{2} + x e^{-i\theta} \otimes_{\mathbb{R}} e_2 \sin \frac{\phi}{2}]_{x \in A^{\dagger}} = A^{\dagger} \otimes_{\mathbb{C}} \bar{L}.$$

Thus, for $L \in D^+$, $\gamma(A, L) = (A \otimes_{\mathbb{C}} L) + (A \otimes_{\mathbb{C}} \bar{L})$.

Similarly, for $L \in D^-$, $\gamma(A, L) = (A_0 \otimes_{\mathbb{C}} L) + (A_0^{\dagger} \otimes_{\mathbb{C}} \bar{L})$.

Since

$$\gamma : G_n^{2n} \times G_1^2 \rightarrow G_{2n}^{4n}$$

and

$$\alpha : G_n^{2n} \times G_1^2 \rightarrow G_{4n}^{8n},$$

we imbed G_{2n}^{4n} into G_{4n}^{8n} so that γ and α are mapping into the same space. The imbedding can be described as

$$W \mapsto W + A_0 \otimes_{\mathbb{C}} V_2$$

for every $W \in G_{2n}^{4n}$. Hence γ is now described as:

$$\gamma(A, L) = \begin{cases} (A \otimes_{\mathbb{C}} L + A \otimes_{\mathbb{C}} \bar{L}) + A_0 \otimes_{\mathbb{C}} V_2; & L \in D^+ \\ (A_0 \otimes_{\mathbb{C}} L + A_0^{\dagger} \otimes_{\mathbb{C}} \bar{L}) + A_0 \otimes_{\mathbb{C}} V_2; & L \in D^-. \end{cases}$$

Now we deform γ into α in two steps:

(1) Consider

$$A_0 \otimes_{\mathbb{C}} V_2 = A_0 \otimes_{\mathbb{C}} (L + L^{\perp}) = A_0 \otimes_{\mathbb{C}} L + A_0 \otimes_{\mathbb{C}} L^{\perp}.$$

Since $L \cap L^{\perp} = \{0\}$, we deform the mapping

$$(A, L) \rightarrow A_0 \otimes_{\mathbb{C}} L + A_0 \otimes_{\mathbb{C}} L^{\perp}$$

by rotating A_0 in the first summand of $\frac{\pi}{2}$ and so obtain:

$$(A, L) \longmapsto A_0^\perp \otimes_{\mathbb{C}} L + A_0 \otimes_{\mathbb{C}} L^\perp.$$

By (i), γ is deformed into a continuous mapping:

$$(A, L) \longmapsto \begin{cases} (A \otimes_{\mathbb{C}} L + A^\perp \otimes_{\mathbb{C}} \bar{L}) + (A_0^\perp \otimes_{\mathbb{C}} L + A_0 \otimes_{\mathbb{C}} L^\perp); & L \in D^+ \\ (A_0 \otimes_{\mathbb{C}} L + A_0^\perp \otimes_{\mathbb{C}} \bar{L}) + (A^\perp \otimes_{\mathbb{C}} L + A \otimes_{\mathbb{C}} L^\perp); & L \in D^- \end{cases}$$

(ii) In the second step, the above mapping is unchanged for $L \in D^+$, while for $L \in D^-$, we deform:

$$(A, L) \longmapsto A_0^\perp \otimes_{\mathbb{C}} \bar{L} + A_0 \otimes_{\mathbb{C}} L$$

into

$$(A, L) \longmapsto A_0^\perp \otimes_{\mathbb{C}} L + A_0 \otimes_{\mathbb{C}} \bar{L}$$

by rotating L onto \bar{L} , and this homotopy is relative to $G_n^{2n} \times \partial D^-$, since $L = \bar{L}$ on ∂D^- . Hence for $(A, L) \in G_n^{2n} \times D^-$, the above mapping is deformed into:

$$\begin{aligned} (A, L) &\longmapsto (A_0 \otimes_{\mathbb{C}} L + A_0^\perp \otimes_{\mathbb{C}} L) + (A_0^\perp \otimes_{\mathbb{C}} \bar{L} + A_0 \otimes_{\mathbb{C}} L^\perp) \\ &= (A_0 + A_0^\perp) \otimes_{\mathbb{C}} L + (A_0^\perp \otimes_{\mathbb{C}} \bar{L} + A_0 \otimes_{\mathbb{C}} L^\perp) \\ &= \mathbb{C}^{2n} \otimes_{\mathbb{C}} L + (A_0^\perp \otimes_{\mathbb{C}} \bar{L} + A_0 \otimes_{\mathbb{C}} L^\perp) \\ &= (A + A^\perp) \otimes_{\mathbb{C}} L + (A_0^\perp \otimes_{\mathbb{C}} \bar{L} + A_0 \otimes_{\mathbb{C}} L^\perp) \\ &= (A \otimes_{\mathbb{C}} L + A^\perp \otimes_{\mathbb{C}} L) + (A_0^\perp \otimes_{\mathbb{C}} \bar{L} + A_0 \otimes_{\mathbb{C}} L^\perp) \end{aligned}$$

Define

$$\psi : G_1^2 \rightarrow G_1^2$$

by

$$\psi(L) = \begin{cases} \bar{L} & L \in D^+ \\ L & L \in D^- \end{cases}$$

For $L \in D^+$, $L = [e_1 \cos \frac{\phi}{2} + e_2 e^{i\theta} \sin \frac{\phi}{2}]$, $\theta \leq \pi$

$$\bar{L} = [e_1 \cos \frac{\phi}{2} + e_2 e^{-i\theta} \sin \frac{\phi}{2}] = [e_1 \cos \frac{\phi}{2} + e_2 e^{i(2\pi-\theta)} \sin \frac{\phi}{2}],$$

$2\pi - \theta \geq \pi$. Therefore, $\bar{L} \in D^-$ and ψ maps G_1^2 onto D^- , the lower hemisphere.

In the second step, γ is deformed into a map

$$(A, L) \mapsto \begin{cases} (A \otimes_{\mathbb{C}} L + A^+ \otimes_{\mathbb{C}} \bar{L}) + (A_0^+ \otimes_{\mathbb{C}} L + A_0 \otimes_{\mathbb{C}} L^+); & L \in D^+ \\ (A \otimes_{\mathbb{C}} L + A^+ \otimes_{\mathbb{C}} L) + (A_0^+ \otimes_{\mathbb{C}} \bar{L} + A_0 \otimes_{\mathbb{C}} L^+); & L \in D^- \end{cases}$$

Using ψ , γ can be written in the form:

$$(A, L) \mapsto (A \otimes_{\mathbb{C}} L + A^+ \otimes_{\mathbb{C}} \psi(L)) + (A_0^+ \otimes_{\mathbb{C}} \psi(\bar{L}) + A_0 \otimes_{\mathbb{C}} L^+).$$

Since $G_1^2 = S^2$, ψ is a map $S^2 \rightarrow S^2$ which is not surjective.

Therefore, it is homotopic to a constant map $L \mapsto L_0$ for every

$L \in G_1^2$. Hence γ is finally deformed into a map

$$(A, L) \mapsto (A \otimes_{\mathbb{C}} L + A^+ \otimes_{\mathbb{C}} L_0) + (A_0^+ \otimes_{\mathbb{C}} L_0 + A_0 \otimes_{\mathbb{C}} L^+)$$

which is precisely α .

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