An Integer Polyhedron Related To The Design Of Survivable Communication Networks

by

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ABSTRACT

The linear programming cutting plane method has proven to be quite successful for solving certain "hard" combinatorial optimization problems, c.f. [1],[6],[12],[24],[26]. A great deal of this success is due to the use of problem specific cutting planes which define facets of the underlying integer polyhedra.

In this paper, we introduce a new class of valid inequalities for the polytope associated with the minimum cost 2-edge-connected subgraph problem, and give necessary and sufficient conditions for these inequalities to be facet inducing for this polytope. We believe it will be possible to use these inequalities efficiently in a cutting plane procedure for designing minimum cost survivable communication networks.

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Chapter 1

Introduction and Notation

1.1 Introduction

Communication networks have become more and more pervasive in today's world, thanks to advances in computer technology and in transmission technology. They range from Local Area Networks to cross-continent networks, and in many cases their role is vital.

There are two main issues in network design--economy and survivability. Economy refers to the construction cost. Survivability refers to the restoration of services in the event of catastrophic failures, such as the loss of a link or failure of a facility switch. The aim of network design is to minimize the construction cost while satisfying given survivability requirements. Star, tree and ring are typical topologies which have been used previously. The advent of optical fibre together with its nearly unlimited bandwidth has introduced some new characteristics to network design. A new topology for optical fiber networks is proposed which provides at least two diverse paths between any two offices. This leads naturally to the problem of designing certain two-connected networks.

In this paper, we study the networks that survive the loss of any single link and, for simplicity, we assume the construction cost to be the sum of each link cost (typically its length). This problem is known as the 2-*edge-connected spanning subgraph problem* (TECSP for short), and is of interest not only practically but theoretically as well. Mathematically, TECSP can be formulated as follows:

Given a graph $G=(V,E)$ and vector $c\in\mathbb{R}^E$ of edge costs, find a 2-edge-connected spanning subgraph having the minimum total edge cost.

As often is the case, we restrict ourselves to problems where G is the complete graph K_n, since any edge not existing in G can be included with a sufficiently large cost without affecting the optimal solution. For convenience, we will denote such a problem on n nodes by TECSP(n).

TECSP is closely related to the widely-studied (Symmetric) Travelling Salesman Problem (TSP for short) whose aim is to find a minimum cost Hamiltonian cycle in a given weighted complete graph. Like TSP, TECSP is also NP-complete as the problem of determining if a graph contains a Hamiltonian cycle can be reduced to TECSP (see [9]). There is no known efficient algorithm to solve these problems, and hence heuristic methods or branch-and-bound methods are often used to obtain supposedly good solutions. In this paper, we study TECSP(n) from an approach based on polyhedral theory and the linear programming cutting plane method, which always gives a lower bound on the cost of an optimal solution and sometimes even an optimal solution. This approach is complementary to heuristic and branch-and-bound methods as the lower bound shows how good the "supposedly good solution" is. First introduced by G.Dantzig, D.Fulkerson and S.Johnson [7], this approach has proven to be quite successful in solving a number of large TSP problems. In [10], a 120-city TSP problem is solved to optimality. In [6], 10 TSP problems ranging from 48 cities to 318 cities are solved to optimality. In [31], a 532-city problem is solved to optimality. A computation study on 74 TSP problems of size from 15 to 318 cities is conducted in [27], whose results also lend convincing support to the computation value of this method. In [29], 42 TSP problems of sizes from 48 to 2392 nodes are all solved to optimality in "reasonable" times.

In order to apply it, we must first associate with TECSP(n) a polytope Q^n, which is the convex hull of all 0-1 incidence vectors for the edge sets of 2-edge-connected subgraphs of Kn. Then we try to find its linear description, i.e., linear constraints. It should be pointed out that we cannot expect to find a complete description of Q^n for each n since the TECSP is NP-complete (see [21]). Next the linear programming cutting plane method is applied to optimize over this partial description of Q^n. Typically a commercial linear programming code is used to optimize the objective function cx over a small subset of the necessary constraints found for Q^n. If the optimal solution x obtained corresponds to a solution for TECSP(n), then it is also an optimal solution for TECSP(n). Otherwise it provides a lower bound on the value of an optimal solution of TECSP(n). We can stop here if we are satisfied with the lower bound obtained, or we can attempt to identify a constraint in the partial linear description found for Q^n which is violated by x and add this constraint to our current set of constraints and repeat the process again to get a better lower bound. Sometimes there is no such constraint in the partial description found, in which case we have to accept this lower bound as is. (For a more complete description of the linear programming cutting method, please see [26]).

Among the linear constraints for the polytope, some of them are necessary or "facet-inducing" while others are dominated, i.e., the former are more restrictive. A great deal of the success of the linear programming cutting plane method is due to the use of such necessary inequalities as cutting planes, however finding such necessary linear inequalities is in general difficult, as is identifying a constraint which is violated by x.

In [16], this approach proves to be successful in solving generalized 2-connected spanning subgraph problems of small sizes, from 28 to 39 nodes, where the underlying graph is not complete. In this paper, we only concern ourselves with the search for such facet-inducing inequalities, hoping our work will facilitate the use of the linear programming cutting plane approach for large problems. In Chapter 2, we describe the relationship between

TECSP and TSP, and introduce a new class of inequalities for TECSP. In Chapter 3, we show their validity for TECSP, and in Chapter 4 we show most of this new class of inequalities are indeed facet-inducing. In chapter 5, we address the equivalence problem for this new class of inequalities. We conclude in Chapter 6 by pointing out further problems.

We now introduce some basic concepts and terminology in graph theory and polyhedral theory.

1.2 General Notation

For any finite set X we let $|X|$ denote the cardinality of X. Given another set Y, we let $X\backslash Y$ denote the members of X which are not members of Y. Given sets A and B, we denote the set $\left\{(a,b)\,|\,a\in A \text{ and } b\in B\right\}$ by $A\times B$.

We let $\mathbb{R}$ denote the set of real numbers and let $\mathbb{R}^E$ denote the set of all real vectors indexed by E for any finite set E. If the members in E are ordered, i.e., $E=\{e_1,e_2,\ldots,e_n\}$, we sometimes use $\mathbb{R}^n$ instead of $\mathbb{R}^E$ and furthermore we let I_i denote the unit vector in $\mathbb{R}^n$ whose ith entry is one and is 0 elsewhere. For any $J\subseteq E$ and $x\in\mathbb{R}^E$, we let $x(J)$ denote $\left(\Sigma(x_i : i\in J)\right)$. We let $\vec{0}$ and $\vec{2}$ represent the vectors whose entries are all 0's and all 2's respectively.

For a finite set E, the *incidence vector* of $F\subseteq E$ is the vector $x\in\mathbb{R}^E$ defined by $x_e=\begin{cases}1 & \text{if } e\in F\\ 0 & \text{if } e\notin F\end{cases}$. We denote the incidence vector of F by x^F.

A set $X\subseteq\mathbb{R}^E$ is *linearly independent* if whenever $\sum_{x\in X}(\alpha_x x \mid x\in X)=\vec{0}$ for some $\alpha\in\mathbb{R}^X$, we have $\alpha=\vec{0}$. Otherwise, X is *linearly dependent*.

The *linear rank* of a set $S\subseteq\mathbb{R}^E$ is the cardinality of a largest linearly independent subset of S and is denoted by $r_l(S)$. The *linear rank* of a matrix A is the cardinality of a maximal independent subset of the columns of A.

1.3 Graph Theory

Some standard references can be found in [3]. For our purpose, a *graph* G is an ordered pair (V,E), where V is a finite set of members called *nodes* and E is a finite set of elements called *edges* such that each edge $e\in E$ corresponds to two distinct nodes in V, called *ends* of e. An edge $e\in E$ with ends w and v is sometimes denoted by (w,v), and we say edge e *joins* node w and node v. Two nodes u,v are said to be *adjacent* if $(u,v)\in E$, and if $e=(u,v)$, e is said to be *incident* with u and v. Here we will only consider graphs having no multiple edges, i.e., there is at most one edge joining any two nodes.

Given any graph G, we let V(G) and E(G) denote the node set and edge set of G respectively. For any $S \subseteq V(G)$, we use $\gamma(S)$ to denote the set of edges in E(G) with both ends in S, and we use $\delta(S)$ to denote the set of edges in E(G) with exactly one end in S and one end in $V(G) \setminus S$. For $V_1, V_2 \subseteq V(G)$, we denote by $[V_1:V_2]$ the set of edges in E(G) with one end in V_1 and the other end in V_2. The *degree* of a node $v \in V(G)$ is defined as the number of edges in G incident with node v, and is denoted by $\deg_G(v)$.

A graph H is called a *subgraph* of G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$ and every edge has the same ends in H as in G. If $V(H)=V(G)$, then H is called a *spanning* subgraph of G. Given $E' \subseteq E(G)$ and $V' \subseteq V(G)$, we denote by $G-E'-V'$ the subgraph of G whose node set is $V(G) \setminus V'$ and whose edge set is the set of edges in $E(G) \setminus E'$ with both ends in $V(G) \setminus V'$. In the case $E'=\{e\}$ and $V'=\{v\}$, we write $G-e-v$.

A graph G is called *complete* if every pair of nodes is joined by exactly one edge. The complete graph on n nodes is denoted by K_n. A *clique* of G is a subset W of the nodes such that any two nodes in W are adjacent in G.

A *path* P in a graph G is a finite non-null sequence $v_0 e_1 v_1 e_2 v_2 \ldots e_k v_k$ whose terms are alternately nodes and edges such that the nodes are distinct and, for $1 \le i \le k$, the ends of e_i are v_{i-1} and v_i. We say path P *joins* v_0 and v_k. The *length* of a path is $|E(P)|$, and a *Hamiltonian path* of G is a path of length $|V(G)|-1$.

A graph is *connected* if every two nodes in G are joined by a path. A *component* of graph G is any maximal connected subgraph of G. A graph G is called *2-edge-connected* if G-e is connected for any edge $e \in E(G)$. A graph G is called *2-node-connected* if G-v is connected for any node $v \in V(G)$.

A *cycle* C in a graph G is a connected subgraph of G such that every node in V(C) has degree 2 in C. The *length* of C is $|E(C)|$, and a *Hamiltonian cycle* of G is a cycle in G of length $|V(G)|$. Sometimes the 0-1 incidence vector of a Hamiltonian cycle is also referred to as a *tour*. A cycle is obviously 2-edge-connected.

A *forest* is a graph containing no cycles and a *tree* is a connected forest. A tree T has the property that $|V(T)|=|E(T)|+1$.

The *node-edge incidence matrix* of a graph G is a matrix $A \in \mathbb{R}^{V(G) \times E(G)}$ such that the entry of A indexed by node v and edge e has value 1 if v is an end of e, otherwise it has value 0.

1.4 Polyhedral Theory

Our discussion of polyhedral theory is brief and covers only what is essential for later sections. More detailed treatments of the subject can be found in [2],[25],[32].

For any $X \subseteq \mathbb{R}^E$ the *convex hull* of X, denoted by conv(X), is the set of all $y \subseteq \mathbb{R}^E$ such that y can be expressed as a convex combination of a finite subset of the members of X, i.e., $\text{conv}(X)=\left\{y\in\mathbb{R}^E \mid y=\Sigma(\lambda_x x \mid x\in\tilde{X}) \text{ for some finite set } \tilde{X}\subseteq X \text{ and some } \lambda\in\mathbb{R}^{\tilde{X}} \text{ such that } \Sigma(\lambda_x \mid x\in\tilde{X})=1,\ \lambda_x\geq 0 \text{ for each } x\in\tilde{X}\right\}$.

A *halfspace* is a set $H\subseteq\mathbb{R}^E$ of the form $\{x:ax\leq a_o\}$ and a *hyperplane* is a set $L\subseteq\mathbb{R}^E$ of the form $\{x:ax=a_o\}$ for some $a\in\mathbb{R}^E$ and $a_o\in\mathbb{R}$. An inequality $ax\leq a_o$ is *valid* for some $S\subseteq\mathbb{R}^E$ if S is contained in the halfspace defined by $ax\leq a_o$, i.e., $ay\leq a_o$ for any $y\in S$.

A *polyhedron* $P\subseteq\mathbb{R}^E$ is the intersection of finitely many halfspaces. Equivalently, P is the solution set of a finite system of linear equations and inequalities, and can be expressed in the form $P=\{x\in\mathbb{R}^E \mid Dx=d,\ Ax\leq b\}$. A *polytope* is a polyhedron which is bounded. A subset F of polyhedron P is a *face* of P if F is either the empty set or else the polyhedron obtained by taking the linear system which defines P and replacing some of the inequalities with the corresponding equations. F is called a *proper face* of P if $F\neq P$. A face $F\subseteq P$ is a *facet* of P if F is a proper non-empty maximal face of P.

If $ax\leq a_o$ is a valid inequality for P then we say the face $F=\{x\in P \mid ax=a_o\}$ is *induced* by $ax\leq a_o$. Furthermore, $ax\leq a_o$ is said to be *facet-inducing* if the face F induced by $ax\leq a_o$ is a facet of P. Any two valid inequalities $ax\leq a_o$ and $bx\leq b_o$ for P are called *equivalent* with respect to P if they induce the same face, i.e., $\{x\in P \mid ax=a_o\}=\{x\in P \mid bx=b_o\}$.

A finite set $X \subseteq \mathbb{R}^E$ is *affinely independent* if whenever $\sum_{x \in X}(\alpha_x x \mid x \in X) = \vec{0}$ and $\sum_{x \in X}(\alpha_x \mid x \in X) = 0$ for some $\alpha \in \mathbb{R}^X$, we also have $\alpha = \vec{0}$. The *affine rank* of a set $S \subseteq \mathbb{R}^E$ is the cardinality of a largest affinely independent subset of S and is denoted by $r_a(S)$. The *dimension* of a polyhedron $P \subseteq \mathbb{R}^E$ is defined as $r_a(P)-1$ and denoted by dim(P). We say P is of *full dimension* if $\dim(P)=|E|$ or equivalently if there does not exist some linear equation $ax=a_0$ satisfied by all $x \in P$. Given the linear system defining P, the set of constraints which are satisfied with equality by all $x \in P$ is called the *equality set* or *equation system* and is related to the dimension of P in the following way:

Theorem 1.1 Let $P \subseteq \mathbb{R}^E$ be a non-empty polyhedron and let $Ax=b$ represent the equality set for P. Then if the linear rank of A is ρ, we have $\dim(P)=|E|-\rho$.

Using the notion of equation system, we also have the following result on the equivalence of two inequalities:

Theorem 1.2 Given a polyhedron P with minimum equation system $Dx=d$, two facet-inducing inequalities $ax \le a_0$ and $bx \le b_0$ are equivalent for P if and only if $\begin{cases} b = \beta a + \lambda D \\ b_0 = \beta a_0 + \lambda d \end{cases}$ where $\beta > 0$ is a scalar and λ a vector.

A proper face of a non-empty polyhedron P is called a *vertex* if it has dimension 0 (i.e, if it consists of a single element). Often in combinatorial optimization, the description of a polyhedron P is given in terms of its vertices, precluding the use of linear programming techniques on the associated problem. Thus a major problem is to find a finite linear system which defines P, hopefully containing as few constraints as possible. To do this requires finding inequalities which induce facets of P. The following gives the relationship between facets and a minimal defining linear system for a polyhedron P.

Theorem 1.3 Let $P \subseteq \mathbb{R}^E$ be a polyhedron and suppose $P=\{x\in\mathbb{R}^E \mid Dx=d, Ax\leq b\}$. Then this is a minimal linear system sufficient to define P if and only if

i> the rows of D are linearly independent and $Dx=d$ forms the equality set, and

ii> each constraint of $Ax\leq b$ induces a distinct facet of P.

Finding a complete minimal linear system which defines a polyhedron P is often a difficult problem, but even a partial description can be useful for combinatorial optimization problems. Thus the problem of identifying facet-inducing inequalities for P is an important one. The following theorem provides two basic methods for proving an inequality is facet-inducing.

Theorem 1.4 Let F be a non-empty proper face of $P=\{x\in\mathbb{R}^E \mid Dx=d, Ax\leq b\}$, and let $Dx=d$ be an equality set of P and let the rows of D be indexed by a finite set V. Then the following statements are equivalent:

1> F is a facet of P;

2> $\dim(F)=\dim(P)-1$;

3> For any $a,b\in\mathbb{R}^E$ and $a_0,b_0\in\mathbb{R}$ satisfying $F=\{x\in P \mid ax=a_0\}$ $=\{x\in P \mid bx=b_0\}$ there exist $\lambda\in\mathbb{R}^V$ and positive $\beta\in\mathbb{R}$ such that $b=\beta a+\lambda D$ and $b_0=\beta a_0+\lambda d$.

Chapter 2

A new class of inequalities for TECSP

The polytope associated with TECSP(n) is $Q^n :=$ $\text{conv}\{x^F \in \mathbb{R}^E \mid (V,F) \text{ is a 2-edge-connected spanning subgraph of } K_n\}$, and the polytope associated with TSP(n) is $Q^n_T :=$

$$\text{conv}\{x^F \in \mathbb{R}^E \mid (V,F) \text{ is a Hamiltonian cycle of } K_n\}.$$

There has been extensive research on Q^n_T and its facets. Since any node in K_n must be incident with exactly two edges of a Hamiltonian cycle, any $x \in Q^n_T$ must satisfy

$$x(\delta(v)) = 2 \quad \text{for all} \quad v \in V.$$

These inequalities are called the *degree constraints*, and can also be written as $Ax = \vec{2}$ where A is the node-edge incidence matrix for K_n. Based on Theorem 1.1, we have $\dim(Q^n_T) \le \binom{n}{2} - n$ for $n \ge 3$ since $r_1(A) = n$. By actually constructing exactly $\binom{n}{2} - n + 1$ affinely independent tours, Grötschel and Padberg proved the following result in [17]:

Theorem 2.1 The dimension of Q^n_T is $\binom{n}{2} - n$ for $n \ge 3$.

This theorem implies that a facet-inducing inequality for Q^n_T has several equivalent forms, i.e., two seemingly distinct inequalities may actually induce the same facet for Q^n_T. It is worth noting that sometimes it is difficult to tell if two inequalities are equivalent. As a consequence of Theorem 1.1 and Theorem 2.1, we have the following:

Corollary 2.1 Let A be the node-edge incidence matrix for Kn. Then $Ax=\vec{2}$ forms a minimal equation system for Q_T^n when $n\geq 3$.

There are several well-known classes of facet-inducing inequalities for Q_T^n. Clearly for any $x\in Q_T^n$, we have

$$-x_e\leq 0 \text{ for all } e\in E.$$

These are called the *trivial inequalities* and are shown facet-inducing in [17]. Since no tour can contain a subtour, i.e., the 0-1 incidence vector of a cycle of length $w<n$, every $x\in Q_T^n$ must satisfy

$$x(\gamma(S))\leq|S|-1 \quad \text{for all} \quad S\subseteq V \quad \text{where} \quad 2\leq|S|\leq n-2.$$

These inequalities are proven facet-inducing in [18] and termed the *subtour elimination constraints*. Note that the subtour elimination constraints include the facet-inducing inequalities $x_e\leq 1$ for $e\in E$ by taking $|S|=2$. If we take any subtour elimination constraint $x(\gamma(S))\leq|S|-1$ and subtract half the sum of all degree constraints for $v\in S$ we obtain $-1/2\left(x(\delta(S))\right)\leq -1$. Thus the subtour elimination constraints can equivalently be described as

$$x(\delta(S))\geq 2 \text{ for all } S\subseteq V \text{ where } 2\leq|S|\leq n-2,$$

which is known as the *cut form* of these constraints. It is easy to see that the subtour elimination constraints for S and V\S are equivalent.

A class of facet-inducing inequalities for Q_T^n called *clique-tree inequalities* was introduced and proven facet-inducing by Grötschel and Pulleyblank in [20]. A *clique tree* is a connected graph C whose maximal cliques partition into two sets, the set of handles and the set of teeth, which satisfy the following properties:

<1> No two teeth intersect;

<2> No two handles intersect;

<3> Each tooth contains at least two and at most (n-2) nodes, and at least one node belonging to no handle.

<4> The number of teeth intersecting each handle is odd and at least three.

<5> If a tooth T and a handle H have a non-empty intersection, then $H \cap T$ is an articulation set of the clique tree ,i.e, $C \backslash H \cap T$ is not connected.

For any clique tree C we use H(C) and respectively T(C) to denote the set of handles respectively teeth of C. The corresponding clique tree inequality is

$$\sum(x(\gamma(H)):H \in H(C)) + \sum(x(\gamma(T)):T \in T(C))$$
$$\leq \sum(|H|:H \in H(C)) + \sum(|T|-t_T:T \in T(C)) - (|T(C)|+1)/2.$$

where for $T \in T(C)$, t_T denotes the number of $H \in H(C)$ which intersect T. The coefficients on the left hand side of the inequality are either 0, 1 or 2. Note that the subtour elimination constraints are a special case of the clique tree inequalities in which there is only one tooth and no handle.

A *comb* is a clique tree with one handle. First introduced by Chvátal [5], *comb inequalities* were generalized by Grötschel and Padberg in [17]. The comb inequalities are the clique tree inequalities having exactly one handle, and can be described as follows:

Let $K_n=(V,E)$ be the complete graph on n nodes, and let H, $T_1, T_2, \ldots, T_k \subset V$ satisfy

$$\begin{cases} |H \cap T_i| \geq 1 & 1 \leq i \leq k \\ |T_i \setminus H| \geq 1 & 1 \leq i \leq k \\ T_i \cap T_j = \emptyset & 1 \leq i \neq j \leq k \\ k \geq 3 \text{ odd} \end{cases}$$

then

$$x(\gamma(H)) + \sum_{i=1}^{k} x(\gamma(T_i)) \leq |H| + \sum_{i=1}^{k} (|T_i|-1) - \frac{k+1}{2}.$$

Note that the coefficients on the left hand side are still 0,1,2. Figure 2.1 shows a comb with the left hand side coefficients for each edge. Here edges with coefficient 0 are not shown for clarity.

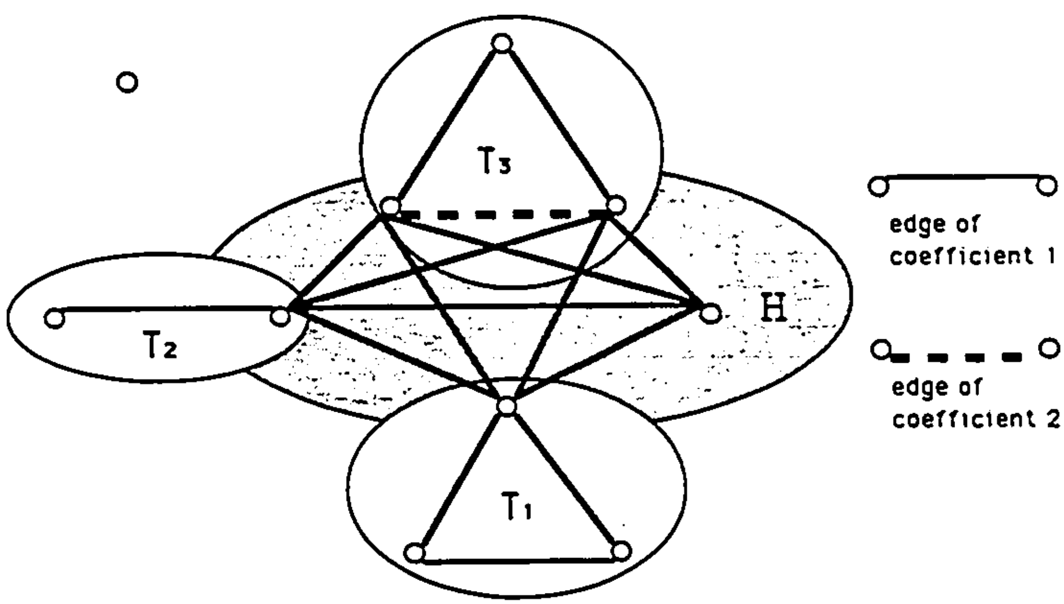

Figure 2.1 Edge coefficients of a comb inequality

In the case $|T_i|=2$ for all $1 \le i \le k$, the comb inequality simplifies to

$$x(\gamma(H)) + \sum_{i=1}^{k} x(\gamma(T_i)) \le |H| + \frac{k-1}{2}.$$

and is called a 2-*matching constraint* which was first introduced in [8].

As for the equivalence problem with respect to Q_T^n, we have the following:

Theorem 2.2 ([19]) Two combs C and C′ induce equivalent inequalities for Q_T^n if and only if $H(C')=V\backslash H(C)$ and $T(C')=T(C)$. Moreover, no comb inequality is equivalent to any trivial or subtour elimination inequality.

There has been some research done on the polytope Q^n. For any $x \in Q^n$, obviously we have $\begin{cases} -x_e \le 0 & \text{for any } e \in E(K_n) \\ x_e \le 1 & \text{for any } e \in E(K_n) \\ x(\delta(S)) \ge 2 & \text{for any } \emptyset \ne S \subset V(K_n) \end{cases}$. In a more general form, Q^n is studied in [13]. Q^n is also studied in [22] with the following results obtained.

Theorem 2.3 (Corollary 2.2, [22]) The dimension of Q^n is $\binom{n}{2}$ for $n \ge 4$.

By Theorem 2.3, the equivalence problem for Q^n is much simpler than for Q_T^n. Two facet-inducing inequalities for Q^n are equivalent only up to scaling.

Theorem 2.4 (Theorem 2.4, [22]) For $e \in E$, $x_e \geq 0$ is facet-inducing for Q^n if $n \geq 5$.

Theorem 2.5 (Theorem 2.5, [22]) For $e \in E$, $x_e \leq 1$ is facet-inducing for Q^n if $n \geq 4$.

Theorem 2.6 (Theorem 2.7, [22]) $x(\delta(S)) \geq 2$ for $S \subseteq V$ is facet-inducing for Q^n if $3 \leq |S| \leq |V|-3$.

Theorem 2.7 (Theorem 2.8, [22]) Two distinct cut constraints $x(\delta(S)) \geq 2$ and $x(\delta(S')) \geq 2$, defining facets for Q^n, are equivalent if and only if $S' = V \setminus S$.

Another class of inequalities called the *lifted 2-cover inequalities* are introduced in [14] for a more general polytope (of which Q^n is a special case), and these are shown to be facet-inducing under certain conditions. These inequalities can be described as follows: Let $H \subseteq V$ be a node set, let $T \subseteq \delta(H)$ be an edge set such that $|T| \geq 3$ and odd, and let $H_1, H_2, \ldots, H_p$, $p \geq 3$ be a partition of H into nonempty disjoint node sets such that no more than two edges in T intersect any H_i. Then the lifted 2-cover inequality is given by

$$x(E(H)) - \sum_{i=1}^{p} x(E(H_i)) + x(\delta(H)) - x(T) \geq p - (|T|-1)/2$$

It is noted in [14] that the *odd wheel inequalities* introduced in [22] for the 2-edge connected spanning subgraph polytope of general graphs is a special case of lifted 2-cover inequality.

The polytope Q^n is closely related to the polytope Q^n_T. It is easy to see from a graph theoretic point of view that a 2-edge-connected graph is a Hamiltonian cycle if and only if each node in the graph is of degree 2. So $Q^n_T=\{x\in Q^n | Ax=\vec{2}\}$ where A is the node-edge incidence matrix for K_n ,i.e., Q^n_T is a face of Q^n. Therefore "for every facet-inducing inequality for Q^n_T, there exists an equivalent form (with respect to Q^n_T) which is also facet-inducing for Q^n "(see [4]). Moreover this equivalent form is obtained by taking a positive scalar multiple of the inequality and adding a certain multiple of each degree constraint. More concretely, using Theorem 1.2 and Corollary 2.1, we have

Theorem 2.8 Two facet-inducing inequalities $ax\le a_0$ and $bx\le b_0$ are equivalent for Q^n_T if and only if $\begin{cases} b=\beta a+\lambda A \\ b_0=\beta a_0+\lambda\vec{2} \end{cases}$ where $\beta>0$ is a scalar and $\lambda\in R^V$ and A is the node-edge incidence matrix for K_n.

Based on the above, Boyd and Pulleyblank in [4] converted the 2-matching constraint to its equivalent form $ax\ge|\dot{V}|+(k+1)/2$ where $\dot{V}\subset V$ is the set of nodes not contained in H or any T_i for $1\le i\le k$,and

$$a_e=\begin{cases} 0 & \text{for } e\in\gamma(H) \text{ or } e\in\gamma(T_i) \text{ for } i=1,2,..,k \\ 1 & \text{elsewhere} \end{cases}$$

and proved it is facet-inducing for Q^n.

We now convert the comb inequalities into an equivalent form with respect to Q^n_T, and prove that they are valid for Q^n in this form and that in most cases they are indeed facet-inducing for Q^n.

We define a 2-*matching tooth* to be a tooth consisting of only two nodes, and divide the nodes of a comb into 6 different classes.

Class 1: those in the handle but not in any tooth;

Class 2: those in the handle as well as in a 2-matching tooth;

Class 3: those in a 2-matching tooth but not in the handle;

Class 4: those in a non-2-matching tooth but not in the handle;

Class 5: those in a non-2-matching tooth as well as in the handle;

Class 6: those not in the handle or in any tooth.

We denote the node set for each class by C_i for i=1,2,3,4,5,6. Figure 2.2 shows a comb with the different classes of nodes indicated.

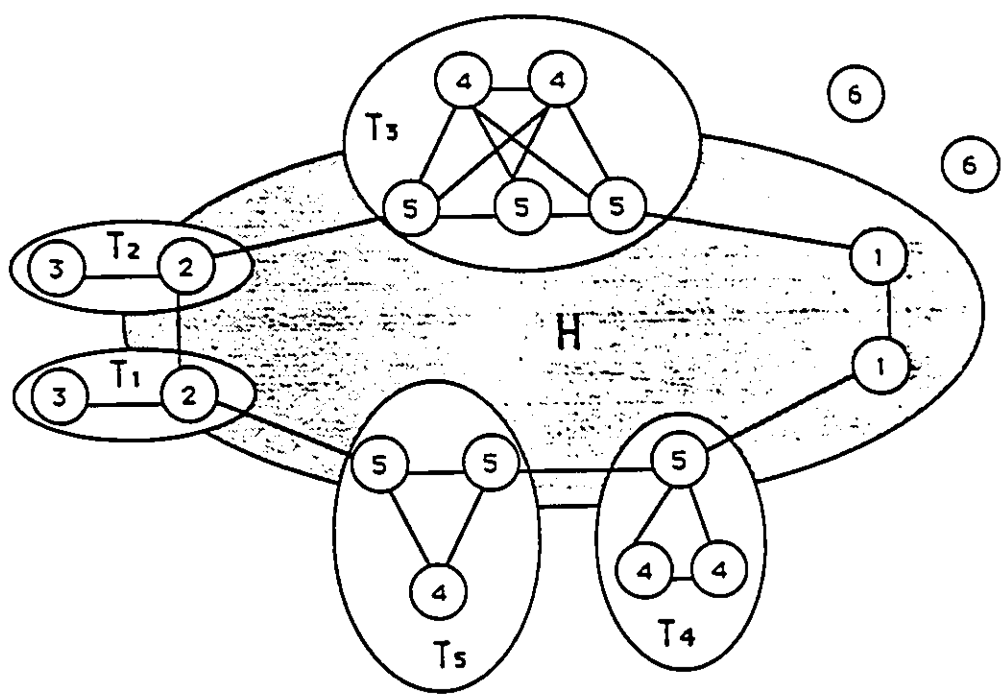

Figure 2.2 A comb with the different classes of nodes labelled

Let C be a comb with $|C_1|=s$, $|C_6|=r$ and p non-2-matching teeth and q 2-matching teeth where $p+q=k$. Suppose $T_1, T_2, \ldots, T_p$ are non-2-matching teeth and $T_{p+1}, T_{p+2}, \ldots, T_k$ are 2-matching teeth. To get the equivalent form of the comb inequality

$$x(\gamma(H)) + \sum_{i=1}^{k} x(\gamma(T_i)) \leq |H| + \sum_{i=1}^{k} (|T_i|-1) - \frac{k+1}{2},$$

we do the following:

1. negate the inequality;
2. multiply the two sides by 2;
3. add two times the degree constraint for the nodes in C_5 and one time the degree constraint for each other node.

Let $\underset{\circ}{T}_i = T_i \cap H$ and $\dot{T}_i = T_i \setminus H$, then the inequality becomes

$$bx \geq 2(|C_1|+|C_2|+|C_3|+|C_4|+2|C_5|+|C_6|) - 2(|H| + \sum_{i=1}^{k} (|T_i|-1) - \frac{k+1}{2})$$

$$= 2(|C_1|+|C_2|+|C_3|+|C_4|+2|C_5|+|C_6|) -$$

$$2(|C_1|+|C_2|+|C_5|+|C_4|+|C_5|-p+|C_2|- \frac{p+q+1}{2})$$

$$= 3p+2r+q+1, \quad \text{where}$$

$$b_e = \begin{cases} 0: & e \in \gamma(C_1 \cup C_2) \cup \gamma(\dot{T}_i) \cup \gamma(\underset{\circ}{T}_i) \cup \gamma(T_j) \text{ for } 1 \leq i \leq p, p+1 \leq j \leq k; \\ 1: & e \in [\underset{\circ}{T}_i : C_1 \cup C_2 \cup \dot{T}_i] \text{ for } 1 \leq i \leq p; \\ 3: & e \in [\underset{\circ}{T}_i : C_6 \cup \dot{T}_j] \text{ for } 1 \leq i \leq p, 1 \leq j \leq k \text{ and } i \neq j; \\ 2 & \text{elsewhere.} \end{cases}$$

We call this inequality the *complemented comb inequality*, and claim it is valid for Q^n. From now on, we reserve b as defined above and b_0 for 3p+2r+q+1 so we can represent this complemented comb inequality described above by $bx \geq b_0$. In case the comb inequality is a 2-matching inequality, we call the resulting inequality the *complemented 2-matching inequality*. The Figure 2.3 shows the comb, depicted in Figure 2.1, with the complemented comb inequality edge coefficients. Here edges with coefficient 2 are not shown for clarity.

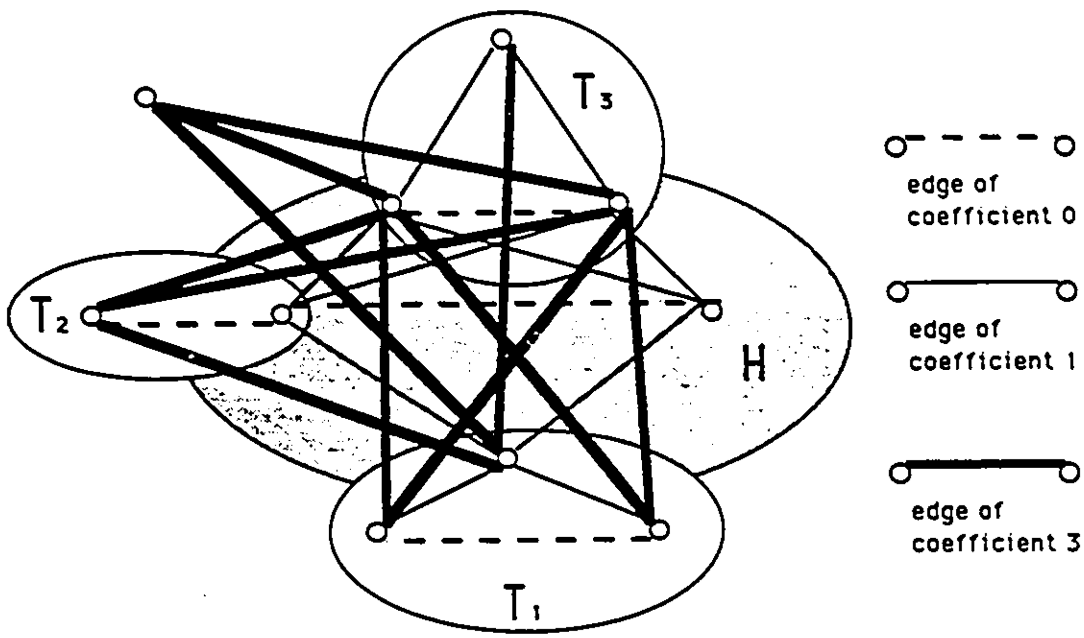

Figure 2.3 Edge coefficients of a complemented comb inequality

Chapter 3

Validity

In this chapter we give a proof of the validity of the complemented comb inequality.

Theorem 3.1 The complemented comb inequality $bx \geq b_0$ is valid for Q^n.

First we introduce a lemma to facilitate our proof of the validity of the complemented comb inequality.

Lemma 3.1 For any $x \in Q^n$, we have

<1> $x(\delta(v)) \geq 2$ for all $v \in C_6$;

<2> $x(\delta(\dot{T}_i)) - x(\gamma(T_i)) \geq 1$ for $p+1 \leq i \leq k$;

<3> $\frac{1}{2}\, x(\delta(\dot{T}_i)) \geq 1$ for $1 \leq i \leq p$;

<4> $\frac{1}{2}\, x(\delta(\underset{\circ}{T}_i)) \geq 1$ for $1 \leq i \leq p$;

<5> $\frac{1}{2}\, x(\delta(T_i)) \geq 1$ for $1 \leq i \leq p$;

<6> $x_e \geq 0$ for $e \in [C_6 : C_1 \cup C_2 \cup C_5]$;

<7> $x_e \geq 0$ for $e \in [C_3 : C_1 \cup C_2 \cup C_5] \setminus \bigcup_{i=p+1}^{k} \gamma(T_i)$;

<8> $x_e \geq 0$ for $e \in [\underset{..}{C_4} : C_1 \cup C_2 \cup C_5] \setminus \bigcup_{i=1}^{p} \gamma(T_i)$.

Furthermore, not all of the above inequalities can hold with equality simultaneously.

Proof:

For any $x \in Q^n$, we have $\begin{cases} x_e \geq 0 & \text{for any } e \in E(K_n) \\ x_e \leq 1 & \text{for any } e \in E(K_n) \\ x(\delta(S)) \geq 2 & \text{for any } \emptyset \neq S \subset V(K_n) \end{cases}$.

Therefore <1>,<3>,<4>,<5>,<6>,<7>,<8> are obvious. We obtain <2> by adding the two valid constraints $-x_e \geq -1$ and $x(\delta(\dot{T}_i)) \geq 2$ for $p+1 \leq i \leq k$ and $\{e\} = \gamma(T_i)$.

Consider any 2-edge-connected spanning subgraph of K_n with edge set F such that x^F satisfies all of the above inequalities with equality. Let $E' = \bigcup\left([C_6 \cup \dot{T}_i : C_6 \cup \dot{T}_j] : 1 \leq i \neq j \leq k\right)$, and let $E^* = F \cap E'$. We claim that $|\delta(v) \cap E^*| = 2$ for any $v \in C_6$ and that $|\delta(\dot{T}_i) \cap E^*| = 1$ for $1 \leq i \leq k$.

For any $v \in C_6$, we have

$$x^F(\delta(v)) = x^F(\delta(v) \cap E') + x^F[C_6 : C_1 \cup C_2 \cup C_5].$$

Since $x^F(\delta(v)) = 2$ by <1> and $x^F_e = 0$ for all $e \in [C_6 : C_1 \cup C_2 \cup C_5]$ by <6>, it follows that $x^F(\delta(v) \cap E') = 2$. Thus $|\delta(v) \cap E^*| = 2$ as required.

For T_i, $p+1 \leq i \leq k$, we have

$$x^F(\delta(\dot{T}_i)) = x^F(\delta(\dot{T}_i) \cap E') + x^F(\gamma(T_i)) + x^F([\dot{T}_i : C_1 \cup C_2 \cup C_5] \setminus \gamma(T_i)).$$

By <7>, $x^F_e = 0$ for all $e \in [\dot{T}_i : C_1 \cup C_2 \cup C_5] \setminus \gamma(T_i)$, and thus $x^F(\delta(\dot{T}_i) \cap E') = x^F(\delta(\dot{T}_i)) - x^F(\gamma(T_i)) = 1$ by <2>. If follows that $|\delta(\dot{T}_i) \cap E^*| = 1$ as required.

For T_i, $1 \leq i \leq p$, we have

$$x^F(\delta(\dot{T}_i)) = x^F(\delta(\dot{T}_i) \cap E') + x^F[\dot{T}_i : \underset{\circ}{T}_i] + x^F([\dot{T}_i : C_1 \cup C_2 \cup C_5] \setminus \gamma(T_i)).$$

By <8>, $x^F_e = 0$ for all $e \in [\dot{T}_i : C_1 \cup C_2 \cup C_5] \setminus \gamma(T_i)$. Also, by <3>,<4>, and <5> we have $x^F(\delta(\dot{T}_i)) = x^F(\delta(\underset{\circ}{T}_i)) = x^F(\delta(T_i)) = 2$, which implies that $x^F[\dot{T}_i : \underset{\circ}{T}_i] = 1$. Thus $x^F(\delta(\dot{T}_i) \cap E') = x^F(\delta(\dot{T}_i)) - 1 = 2 - 1 = 1$ by <3>. It follows that $|\delta(\dot{T}_i) \cap E^*| = 1$ as required.

Since $2|E^*| = \sum(|\delta(v)\cap E^*|:v\in C_6) + \sum(|\delta(T_i)\cap E^*|:1\le i\le k)$, it now follows that $2|E^*|=2|C_6|+k$. But k is odd and thus $|E^*|=|C_6|+k/2$ is not an integer, which leads to a contradiction. ∎

Proof of Theorem 3.1:

Let $\hat{b}x\ge\hat{b}_o$ be the valid inequality for Q^n obtained by adding all the inequalities in <1>,<2>,...,<8>. Since not all of these inequalities can hold with equality simultaneously by Lemma 3.1, we have $\hat{b}x>\hat{b}_o$ for all $x\in Q^n$. Furthermore, since $\hat{b}$, $\hat{b}_o$ and all the vertices of Q^n are integer, it follows that

$$\hat{b}x\ge\hat{b}_o+1 \text{ for all } x\in Q^n.$$

Clearly $\hat{b}_o+1=2|C_6|+3p+q+1=b_o$. We now complete the proof by showing $b=\hat{b}$.

Choose v_i, $w_i\in C_i$ for $1\le i\le 6$ such that $w_i\ne v_i$ and the two nodes of each pair (v_2,v_3), (w_2,w_3), (v_4,v_5), (w_4,w_5) are in the same tooth. (Please see Figure 3.1 in which T_1 and T_2 are the non-2-matching teeth.) Then the coefficients for edges (v_i,w_j) for $1\le i,j\le 6$ can be tabulated as follows:

	w_1	w_2	w_3	w_4	w_5	w_6
v_1	0	0	2	2	1	2
v_2	0	0	2	2	1	2
v_3	2	2	2	2	3	2
v_4	2	2	2	2	3	2
v_5	1	1	3	3	2	3
v_6	2	2	2	2	3	2

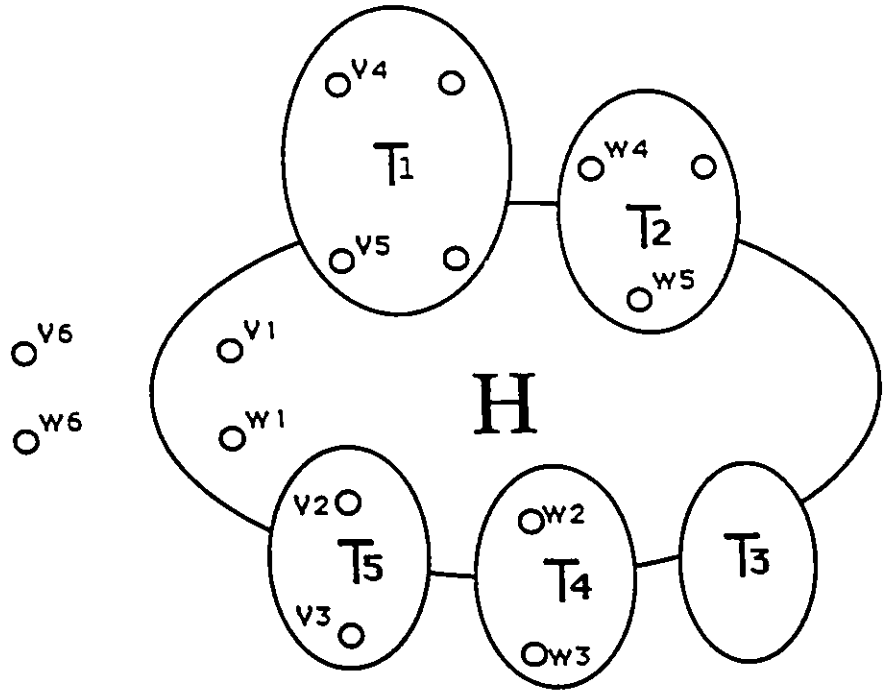

Figure 3.1

It can be verified that $b_e=\hat{b}_e$ for edges $e=(v_i,w_j)$ for $1\le i,j\le 6$. It can also be verified that $b_e=0=\hat{b}_e$ for edges $e\in(v_2,v_3)\cup\gamma(T_i)\cup\gamma(\mathring{T}_i)$, $1\le i\le p$ and that $b_e=1=\hat{b}_e$ for edge $e=(v_4,v_5)$. Therefore $b=\hat{b}$ since these edges cover all the possibilities. ∎

The following theorem is obvious using Lemma 3.1 and Theorem 3.1. We will use it later to prove particular Hamiltonian cycles satisfy $bx=b_o$.

Theorem 3.2 For any $x\in Q^n$, $bx=b_o$ if and only if all inequalities in <1>,<2>,...,<8> hold with equality except for exactly one, which is violated by 1.

Chapter 4

Facet-inducing

In this chapter, we show the following theorem.

Theorem 4.1 The complemented comb inequality $bx \geq b_0$ is facet-inducing for Q^n if and only if $r=0$ or $(q+s) \geq 1$ where s, q, r are the number of nodes in C_1, C_2, C_6 in comb C respectively.

We don't want to prove Theorem 4.1 from scratch, typically using Theorem 1.4. Instead, we want to exploit the fact that $\dim(Q^n_T) = \binom{n}{2} - n$ and that Q^n_T is a face of Q^n to simplify our proof, using methods described in [4].

Given a valid inequality $ax \leq a_0$ for a polyhedron F, we define $F_a^= = \{x \in F \mid ax = a_0\}$. Given a non-empty face F of a full dimensional polytope $P \subseteq \mathbb{R}^E$, we define a set $D \subseteq \mathbb{R}^E$ to be an *independent direction set* for $F_a^=$ if

<1> For every $d \in D$, there exists $\hat{x}^d \in F_a^=$ such that $x^d := \hat{x}^d + d \in P$;

<2> For every $d \in D$, $ad=0$;

<3> For some minimal equation system $A^F x = b^F$ for F, $\{A^F d : d \in D\}$ are linearly independent.

Theorem 4.2 (Corollary 2.2 in [4]) Let $ax \le a_0$ be an inequality which is valid for a full dimensional polyhedron $P \subseteq \mathbb{R}^E$ and facet-inducing for a non-empty face F of P. Let $A^F x = b^F$ be a minimal equation system for F. If there exists an independent direction set for $F_a^=$ of size $r_1(A^F)$, then $ax \le a_0$ is also facet-inducing for P.

Since Q_T^n is a non-empty face of Q^n, $bx \ge b_0$ is valid for Q^n, and $Ax = \vec{2}$ is a minimal equation system for Q_T^n by Corollary 2.1, we obtain the following corollary from Theorem 4.2.

Corollary 4.2 The complemented comb inequality $bx \ge b_0$ is facet-inducing for Q^n if there is $D = \{d_1, d_2, \ldots, d_n\} \subseteq \mathbb{R}^E$ such that for each $1 \le i \le n$:

<P1> there exists $\bar{x}_i \in Q_T^n \cap \{x \mid bx = b_0\}$ such that $x_i := \bar{x}_i + d_i \in Q^n$;

<P2> $b \cdot d_i = 0$;

<P3> $Ad_1, Ad_2, \ldots, Ad_n$ are linearly independent where A is the node-edge incidence matrix for the complete graph K_n.

We now set out to find such an independent direction set D. First we describe two specific Hamiltonian cycles and introduce a lemma that will be used later.

Suppose comb C has k teeth. We pair T_{2i} and T_{2i+1} for $1 \le i \le (k-1)/2$ and construct a Hamiltonian tour H_1 as shown in Figure 4.1.

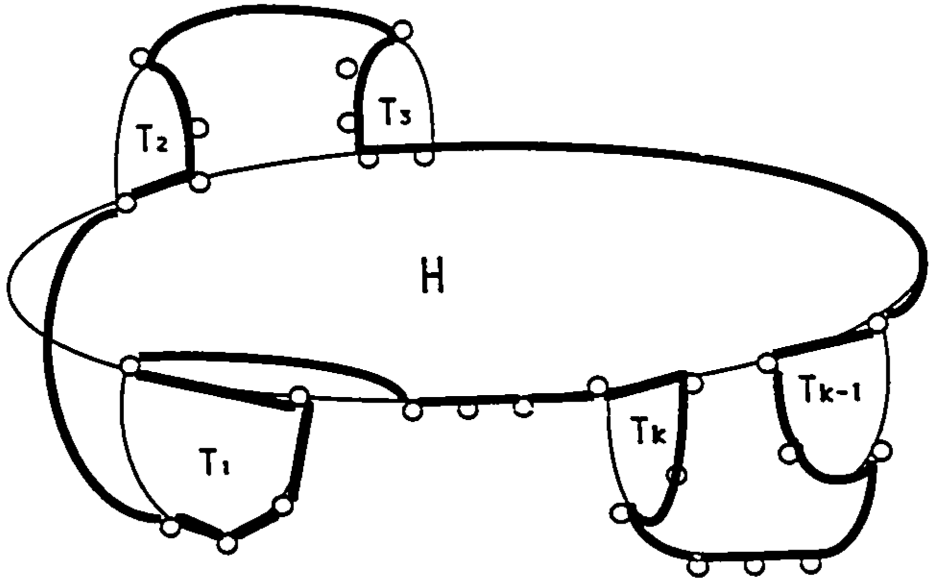

Figure 4.1 Hamiltonian cycle H_1 satisfying $bx=b_0$

For each pair T_{2i} and T_{2i+1}, H_1 first traverses each node in $\underset{\circ}{T}_{2i}$ then each node in $\dot{T}_{2i}$, then goes to $\dot{T}_{2i+1}$ traversing each node in $\dot{T}_{2i+1}$ then each node in $\underset{\circ}{T}_{2i+1}$ before going to the next pair. In the case $i=(k-1)/2$, H_1 traverses each node in C_6 before going to $\dot{T}_k$. After traversing each node in $\underset{\circ}{T}_k$, H_1 goes to C_1 traversing each node of C' before traversing each node in T_1. In T_1, H_1 first traverses each node in $\underset{\circ}{T}_1$ then each node in $\dot{T}_1$ before returning to the node in $\underset{\circ}{T}_2$ that H_1 starts with.

Suppose C has k teeth with at least one 2-matching tooth. Without loss of generality we assume that T_1 is a 2-matching tooth. Similar to the construction of H_1, we pair T_{2i} and T_{2i+1} for $1 \leq i \leq (k-1)/2$ and construct a Hamiltonian tour H_2 as shown in Figure 4.2.

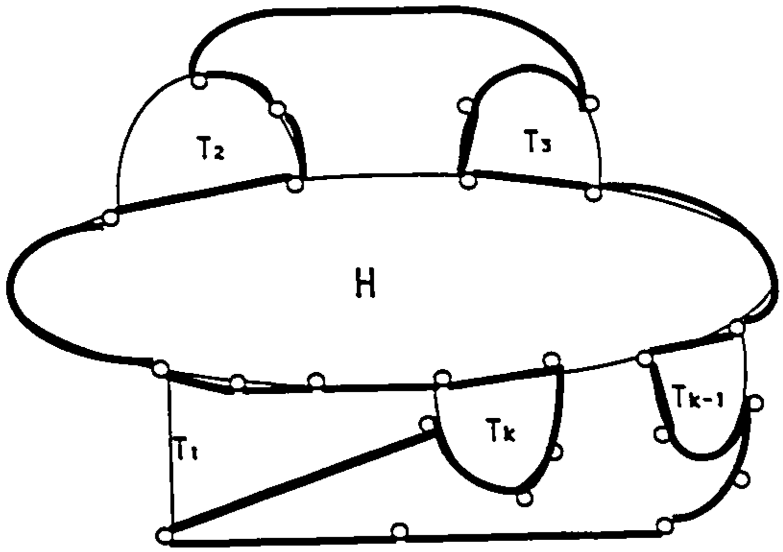

Figure 4.2 Hamiltonian cycle H_2 satisfying $bx=b_0$

H_2 is the same as H_1, except for in teeth T_1, C_1 and T_k. After visiting each node in C_6, H_2 traverses the node w in $\dot{T}_1$ then each node in $\dot{T}_k$ and then each node in $\underset{\circ}{T}_k$. H_2 then traverses each node in C_1 and then the node v in $\underset{\circ}{T}_1$ before returning to the node in $\underset{\circ}{T}_2$ that H_2 starts with.

Lemma 4.1 <1> Given a comb C, Hamiltonian cycle H_1 satisfies $bx=b_0$; <2> Given a comb C with at least one 2-matching tooth, Hamiltonian cycle H_2 satisfies $bx=b_0$.

Proof:

<1> H_1 satisfies $bx=b_0$ because the incidence vector x^H of H_1 satisfies all the inequalities in <1>,<2>,...,<8> in Lemma 3.1 with equality except for $x_e \geq 0$ where $e=(v,u)$ and u is the first node in $\underset{\circ}{T}_2$ that H_1 starts with and v is the last node in $\dot{T}_1$ that H_1 visits. We have $x_e^H=1$. Thus by Theorem 3.2, it follows that $bx^H=b_0$.

<2> H_2 satisfies $bx=b_0$ because the incidence vector x^H of H_2 satisfies all the inequalities in <1>,<2>,...,<8> in Lemma 3.1 with equality except for $x(\delta(\dot{T}_1))-x(\gamma(T_1))\geq 1$. We have $x^H(\gamma(T_1))=0$ and $x^H(\delta(\dot{T}_1))=2$, i.e., $x^H(\delta(\dot{T}_1))-x^H(\gamma(T_1))=2$. Thus by Theorem 3.2 we have $bx^H=b_0$. ∎

Given the comb C, let G be the weighted complete graph with $V(G)=V(K_n)$ and weight distribution b. Let G_0 be the spanning subgraph with $E(G_0)=\{e\in E(G)\mid \text{weight of } e \text{ is } 0\}$. G_0 consists of several components. To be specific, all nodes in C_1,C_2,C_3 are in the same component with nodes in C_1, C_2 forming a clique; of each non-2-matching tooth T_i, nodes in $\dot{T}_i$ form a component and so do those in $\underset{\circ}{T}_i$; each node in C_6 forms a component by itself. Altogether G_0 has $2p+r+1$ components if $|C_1|+|C_2|\geq 1$ and $2p+r$ components if $|C_1|=|C_2|=0$.

For the graph G with the weight distribution b as defined above, we say v *induces* a *tight triangle* if there exist $u,w\in V(G)$ and a Hamiltonian tour H satisfying $bx=b_0$ such that

<1> $\text{weight}((v,u))+\text{weight}((v,w))=\text{weight}((w,u))$;

<2> edges (v,w) and (v,u) are not in $E(H)$ but edge (u,w) is.

Theorem 4.3 In the graph G, node v induces a tight triangle if and only if $v\in C_1\cup C_2\cup C_4\cup C_5$.

Proof:

We construct a Hamiltonian cycle H_3 as shown below in Figure 4.3. H_3 is a special case of H_1 and thus satisfies $bx=b_0$ by Lemma 4.1. Let w, v_4 be the first and last nodes in T_2 that H_3 visits and v_5,u be the first and last nodes in T_1 that H_3 visits.

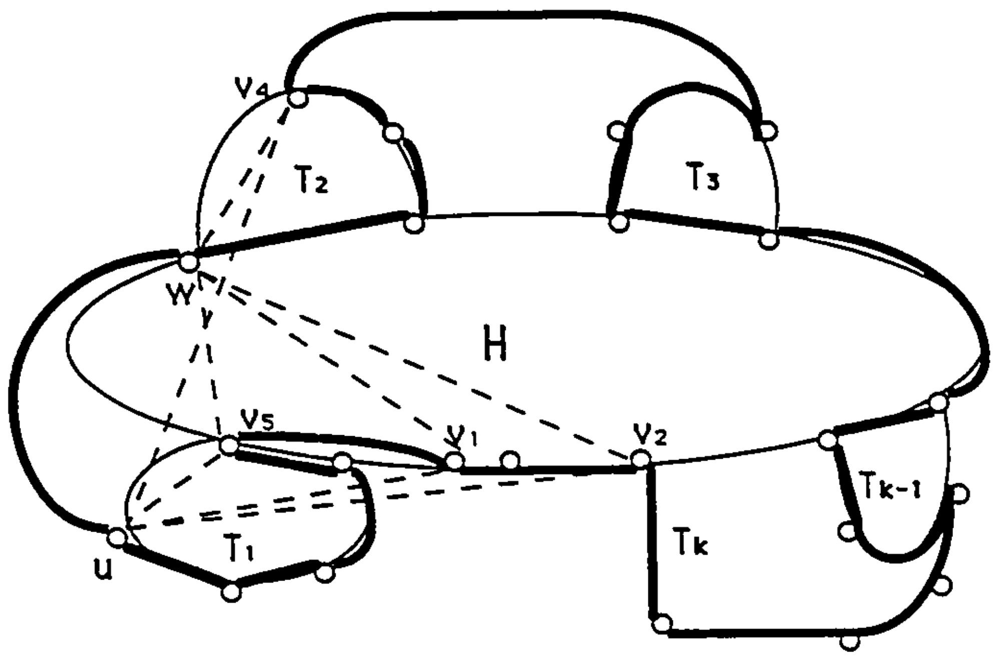

Figure 4.3 Hamiltonian cycle H_3 satisfying $bx=b_0$

If $C_i \neq \emptyset$ (i=1,2), then let v_i be any node in C_i for i=1,2. Without loss of generality, we assume $v_2 \in T_k$ (see Figure 4.3). Then node v_i induces a tight triangle (v_i,u,w) with respect to weight distribution b and Hamiltonian cycle H_3 for i=1,2 because

if T_2 is a 2-matching tooth, then

$$weight((v_i,w))=0$$

$$weight((v_i,u))=2$$

$$weight((u,w))=2, \quad \text{and}$$

if T_2 is a non-2-matching tooth, then

$$weight((v_1,w))=1$$

$$weight((v_1,u))=2$$

$$weight((u,w))=3.$$

If $C_4\neq\emptyset$, then without loss of generality we assume that $v_4\in T_2$ is any node in C_4, i.e., T_2 is a non-2-matching tooth (see Figure 4.3). It is easy to verify that v_4 induces a tight triangle (v_4,w,u) with respect to Hamiltonian cycle H_3, because

$$weight((v_4,w))=1$$

$$weight((v_4,u))=2$$

$$weight((u,w))=3.$$

If $C_5\neq\emptyset$, then without loss of generality we assume that $v_5\in T_1$ is any node in C_5, i.e., T_1 is a non-2-matching tooth (see Figure 4.3). It is easy to verify that v_5 induces a tight triangle (v_5,w,u) with respect to and Hamiltonian cycle H_3, because

if T_2 is a 2-matching tooth, then

$$weight((v_5,w))=1$$

$$weight((v_5,u))=1$$

$$weight((u,w))=2, \quad \text{and}$$

if T_2 is a non-2-matching tooth, then

$$weight((v_5,w))=2$$

$$weight((v_5,u))=1$$

$$weight((u,w))=3.$$

On the other hand, any edge incident with a node in C_6 has weight 2 or 3, but all edges in G have weight 0,1,2,3. So no node in C_6 induces a tight triangle. We now deal with nodes in C_3.

Let v be a node in C_3. Without loss of generality, let T_1 be a 2-matching tooth and $v \in T_1$ and $w \in \dot{\underset{\circ}{T}}_1$. The edges incident with v have weight 2 or 3 except for edge (v,w) which has weight 0. So if v induces a tight triangle then the weights of the three edges have to be (0,2,2) or (0,3,3). In either case, edge (w,v) has to be present. But no edge incident with w has weight 3, so case (0,3,3) is impossible. Suppose case (0,2,2) is possible, i.e., there is a node u and Hamiltonian cycle H such that edges (v,w), $(v,u) \notin E(H)$ but edge $(w,u) \in E(H)$ and weight((v,u))=weight((w,u))=2. So $u \in C_3 \cup C_4 \cup C_6$. Letting x^H be the edge incidence vector for H, then we have $x^H_e = 1$ where e=(w,u) and $x^H(\delta(\dot{T}_1)) - x^H(\gamma(T_1)) \geq 2$ because $x^H(\gamma(T_1)) = 0$ as (v,w) is not in H. By Theorem 3.2, we have that H does not satisfy $bx = b_0$ as at least two inequalities (<2> and one of <6>,<7>,<8>) are strict. So no node in C_3 induces a tight triangle. ∎

To facilitate our proof of Theorem 4.1, we now introduce some lemmas.

Lemma 4.2 Let H be a Hamiltonian tour in K_n satisfying $bx=b_0$. Then H+e is a 2-edge-connected spanning subgraph satisfying $bx=b_0$ for any $e \in E(G_0)$ not in H, and so is H+(v,w)+(v,u)−(w,u) for each node v that induces a tight triangle (v,w,u) with respect to H.

The above lemma follows directly from the fact that $e \in E(G_0)$ has weight 0 in G and the definition of a tight triangle.

Lemma 4.3 Let A be the node-edge incidence matrix for the complete graph K_n, and let $e=(v_i,v_j)$ be an edge of K_n. Then $Ax^e = I_i + I_j$ where x^e is the 0-1 edge incidence vector for $\{e\}$ and I_i and I_j are the unit vectors.

The above lemma is obvious as Ax^e is the column of A indexed by e.

Lemma 4.4 Let A be the node-edge incidence matrix for the complete graph K_m and let $e_1, e_2, \ldots, e_{m-1}$ be edges that induce a spanning tree in K_m. Then $Ad_1, Ad_2, \ldots, Ad_{m-1}$ and I_i are linearly independent for any $1 \le i \le m$ where $d_1, d_2, \ldots, d_{m-1}$ are the edge incidence vectors for edge sets $\{e_1\}, \{e_2\}, \ldots, \{e_{m-1}\}$.

Proof:

Suppose there exist $\lambda_1, \lambda_2, \ldots, \lambda_m$ such that

$$\lambda_1 Ad_1 + \lambda_2 Ad_2 + \ldots + \lambda_{m-1} Ad_{m-1} + \lambda_m I_i = 0.$$

As a tree, F has at least two leaves, one of which is v_j such that $j \neq i$. Without loss of generality, we assume e_1 is incident with this leaf v_j. Based on Lemma 4.3, Ad_1 has 1 in its jth entry while $Ad_2, Ad_3, .., Ad_{m-1}$ and I_i have 0 in the jth entry. So $\lambda_1 = 0$. Moreover graph $F-\{e_1\}-\{v\}$ is also a tree, and has a leaf which is not v_i. By the same argument, we can prove $\lambda_2 = \ldots = \lambda_{m-1} = 0$. Thus $\lambda_m = 0$, i.e., Ad_1, $Ad_2, \ldots, Ad_{m-1}$ and I_i are linearly independent for any $1 \le i \le m$. ∎

Proof of Theorem 4.1:

Based on Theorem 4.3, any component of G_0 containing no node in C_6 has at least one node that induces a tight triangle. Let M be an arbitrary component of G_0 containing m nodes but no node in C_6. We choose the edges $\{e_1, e_2, \ldots, e_{m-1}\}$ that induce a spanning tree of M and choose a node $v_i \in M$ that induces a tight triangle (v_i, w, u). For convenience, we denote $(v_i,w)+(v_i,u)-(w,u)$ by e_m, and call $d_m = d_{m1} + d_{m2} - d_{m3}$ the incidence vector for e_m where d_{m1}, d_{m2}, d_{m3} are the incidence vectors for (v_i,w), (v_i,u) and (w,u) respectively. Let $D_M = \{d_j | d_j$ is the incidence vector for e_j for $1 \le j \le m\}$. We claim D_M is an independent direction set.

<P1> Recall that $bx \ge b_0$ is equivalent to a comb constraint for Q_T^n and thus, by Theorem 2.2, does not induce the facet $x_e = 1$ of Q_T^n for each $e \in E$. Hence for each e_j $(1 \le j \le m-1)$, there exists a Hamiltonian tour H such that $bx^H = b_0$, and H does not contain e_j. Furthermore, by the definition of a tight triangle, there exists a Hamiltonian cycle H such that node v_i induces a tight triangle (v_i, u, w). So there exists a Hamiltonian tour H such that $H + e_j$ $(1 \le j \le m)$ is a 2-edge-connected spanning subgraph, i.e., $(x^H + d_j) \in Q^n$;

<P2> for $1 \le j \le m-1$, we have $b \cdot d_j = 0$ because the weight of e_j is 0 according to the weight distribution b, and we have $b \cdot d_m = 0$ because v_i induces a tight triangle.

<P3> Ad_j is the column of A indexed by e_j for $1 \le j \le m-1$ and Ad_m is $2 \cdot I_i$ based on Lemma 4.3. So $Ad_1, Ad_2, \ldots, Ad_m$ are linearly independent based on Lemma 4.4.

If $r=0$, we claim $D=\cup D_M$ for all components M of G_0 is an independent direction set containing n vectors. First each component contributes as many vectors as the nodes in the component, so D contains n members. Second, for any two edge incidence vectors d_i and d_j from different components, Ad_i and Ad_j do not have any non-zero entries in common positions. Since the edge incidence vectors from a common component satisfy <P3>, it thus follows that the vectors in D also satisfy <P3>. Therefore $bx\geq b_0$ is facet-inducing for Q^n by Corollary 4.2.

Suppose $r\geq 1$ and $q\geq 1$. Given the comb C with k teeth, suppose T_1 is a 2-matching tooth where $\dot{T}_1=\{w\}$ and $\underset{\circ}{T}_1=\{v_j\}$. Let v_i be a node in C_6 and u be a node in T_k. We construct a Hamiltonian tour H_4 as shown below in Figure 4.4 such that v_i is the last node in C_6 and u is the first node in T_k that H_4 visits. By Lemma 4.1, H_4 satisfies $bx=b_0$ as it is a special case of H_2.

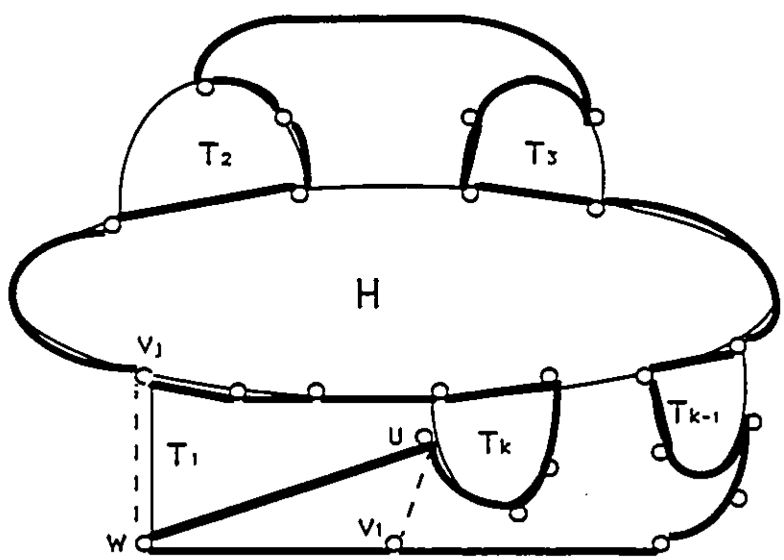

Figure 4.4 Hamiltonian cycle H_4

Clearly $H+(w,v_j)+(v_i,u)-(w,u)$ is a 2-edge-connected subgraph. Let d_i be the incidence vector for $(w,v_j)+(v_i,u)-(w,u)$. Then $Ad_i=I_i+I_j$. The node v_i forms a component itself in G_o. To conform to our previous notation, let $D_M=\{d_i\}$ for each component containing an node v_i in C_6. We claim $D=\cup D_M$ for all components M of G_o is an independent direction set containing n vectors. First each component contributes as many vectors as the nodes in that component, so D contains n members. Second, for each component M containing only a node v_i in C_6, d_i satisfies <P1> as $H+(w,v_j)+(v_i,u)-(w,u)$ is a 2-edge-connected subgraph and d_i also satisfies <P2> because $b \cdot d_i=$ weight(w,v_j)+weight(v_i,u)-weight(w,u) $=0$. All vectors in D satisfy <P3> because for each node v_i in C_6, Ad_i has 1 in its ith entry while for any other vector $d \in D$, Ad has 0 in its ith entry. Thus based on the result of the previous case, D is an independent direction set consisting of n vectors, i.e., $bx \geq b_o$ is facet-inducing for Q^n when $r \geq 1$ and $q \geq 1$ by Corollary 4.2.

Suppose $r \geq 1$ and $q=0$ but $s \geq 1$. Given the comb C with k=p non-2-matching teeth, suppose $v_j \in C_1$. Let v_i be any node in C_6 and let u be a node in $\dot{T}_1$ and w be the node in $\underset{\circ}{T}_2$ in the Hamiltonian cycle H_5 as shown below in Figure 4.5 such that w is the first node in T_2 and u, v_i are the last nodes in T_1 and C_6 that H_5 visits. H_5 is a special case of H_1 and thus satisfies $bx=b_o$ by Lemma 4.1.

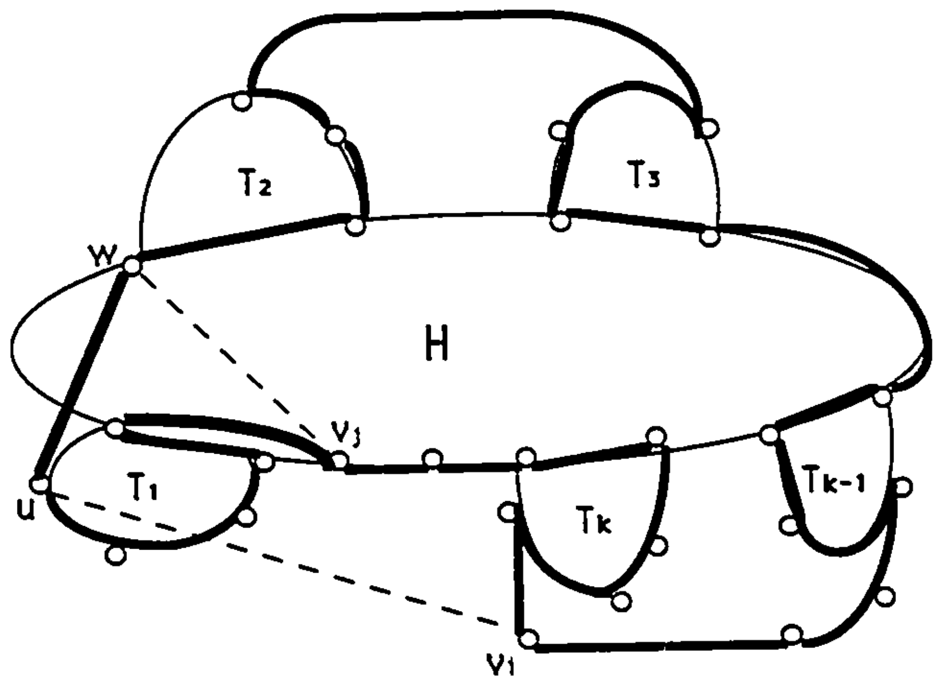

Figure 4.5 Hamiltonian cycle H_5

Clearly $H+(w,v_j)+(v_i,u)-(w,u)$ is a 2-edge-connected subgraph. Let d_i be the incidence vector for $(w,v_j)+(v_i,u)-(w,u)$ and $D_M=\{d_i\}$ for each component M containing a node v_i in C_6. Similar to the case above, we can prove $\cup D_M$ for all components M of G_0 is an independent direction set containing n vectors. , i.e., $bx \geq b_0$ is facet-inducing for Q^n by Corollary 4.2.

Suppose $r \geq 1$ and $(q+s)=0$. We show $bx \geq b_0$ is not facet-inducing for Q^n by showing $x^H(\delta(v_i))=2$ for all $v_i \in C_6$ and 2-edge-connected subgraphs H. Since $bx \geq b_0$ and $x^H(\delta(v_i))=2$ are not linearly independent by Theorem 1.4 and Theorem 2.2, the affine rank of $\left\{x \in Q^n : bx=b_0\right\} \leq \binom{n}{2}-2$ by Theorem 1.1, i.e., $bx \geq b_0$ can not be facet-inducing.

Suppose $x^H(\delta(v_i))\geq 3$ for some v_i in C_6. If $x^H(\delta(v_i))\geq 4$, then $bx^H\geq b_0+1$ by Theorem 3.2. So suppose $x^H(\delta(v_i))=3$. Then by Theorem 3.2, all other inequalities in <1>,<2>,...,<8> must be satisfied with equality. Therefore $x^H(\delta(v))=2$ for all $v\in C_6$ and $v\neq v_i$, i.e., all nodes in C_6 except v_i have degree 2 in H. Furthermore each tooth T_i for $1\leq i\leq k$ has $x^H(\delta(T_i))=2$, i.e., the number of odd degree nodes in H in each tooth is even. Overall we have an odd number of odd degree nodes in H, which leads to a contradiction. So $bx\geq b_0$ is not facet-inducing for Q^n when $r\geq 1$ and $(q+s)=0$. ∎

Chapter 5

Equivalence

We have been trying to convert facet-inducing inequalities for Q_T^n into an equivalent form such that they are also facet-inducing or at least valid for Q^n. As some of the inequalities are equivalent for Q_T^n, one interesting and also nagging problem is the equivalence issue with respect to Q^n.

Theorem 5.1 Let F be a face of polyhedron P. Suppose $ax \le a_o$ and $\hat{a}x \le \hat{a}_o$ are both valid for F and P. If $ax \le a_o$ and $\hat{a}x \le \hat{a}_o$ are not equivalent with respect to F then they are not equivalent with respect to P.

Proof:

We prove the converse. Suppose they are equivalent with respect to P and $F=\{x\in P \mid Ax=d\}$. Then $\{x\in P \mid ax=a_o\}=\{x\in P \mid \hat{a}x=\hat{a}_o\}$. So $\{x\in F \mid ax=a_o\}=\{x\in P \mid ax=a_o\}\cap\{x \mid Ax=d\}=\{x\in P \mid \hat{a}x=\hat{a}_o\}\cap\{x \mid Ax=d\}=\{x\in F \mid \hat{a}x=\hat{a}_o\}$. So they are equivalent with respect to F. ∎

Corollary 5.1 Two inequalities, both valid for Q_T^n and Q^n, are not equivalent with respect to Q^n if they are not equivalent with respect to Q_T^n.

The complemented comb inequalities are not equivalent, with respect to Q^n, to other known facet-inducing inequalities for Q^n as stated in the following theorem.

Theorem 5.2 The complemented comb inequality $bx \geq b_0$ is not equivalent to <1> $x_e \leq 1$ for $e \in E$ or

<2> $x_e \geq 0$ for $e \in E$ or

<3> $x(\delta(S)) \geq 2$ for $3 \leq |S| \leq |V|-3$.

Furthermore, a complemented comb inequality is equivalent to a lifted 2-cover inequality if and only if it corresponds to a complemented 2-matching inequality.

Proof:

Parts <1>,<2>,<3> follow directly from Theorem 2.2 and Corollary 5.1.

In the case $|H_i|=1$ for $i=1,2,\ldots,p$ and the edges in T are disjoint, a lifted 2-cover inequality is identical to the complemented 2-matching constraint with handle $V(K_n)\backslash H$ and teeth corresponding to the end nodes for each edge in T. Furthermore, since the lifted 2-node cover inequalities have left-hand side coefficients 0,1, they cannot be equivalent to any other complemented comb constraint since a complemented comb constraint has left-hand side coefficients 0,1,2 and 3 whenever the corresponding comb has at least one non-2-matching tooth. ∎

Note that Theorem 5.2 shows that the complemented comb inequalities that are not 2-matching constraints define a new class of facet-inducing inequalities for Q^n.

As to the equivalence problem regarding complemented comb inequalities, we have the following:

Theorem 5.3 The facet-inducing complemented comb inequalities $bx \geq b_0$ and $\hat{b}x \geq \hat{b}_0$ for Q^n associated with combs C and C′ respectively are equivalent if and only if $H(C') = V \setminus H(C)$ and $T(C') = T(C)$ and $s = q = r = 0$ where s, q, r are the number of nodes in C_1, C_2, C_6 in comb C respectively.

Proof:

"⟸" Since Q^n is of full dimension, we only have to prove that they are simply positive multiples of each other.

If $s = q = r = 0$, then $b_0 = \hat{b}_0 = 3p+1$. We now prove $b = \hat{b}$. Let u, v be any two nodes. We run all the possibilities for u and v.

1> If $u, v \in \dot{T}_i$ for some $\dot{T}_i$ in C then $u, v \in \underset{\circ}{T}_i$ in C′. We have $b_e = \hat{b}_e = 0$ where $e = (u, v)$.

2> If $u, v \in \underset{\circ}{T}_i$ for some $\underset{\circ}{T}_i$ in C then $u, v \in \dot{T}_i$ in C′. We have $b_e = \hat{b}_e = 0$ where $e = (u, v)$.

3> If $u \in \dot{T}_i$ and $v \in \underset{\circ}{T}_j$ for some $\dot{T}_i$ and $\underset{\circ}{T}_j$ in C where $i \neq j$, then $u \in \underset{\circ}{T}_i$ and $v \in \dot{T}_j$ in C′. We have $b_e = \hat{b}_e = 3$.

4> If $u \in \dot{T}_i$ and $v \in \dot{T}_j$ for some $\dot{T}_i$ and $\dot{T}_j$ in C where $i \neq j$, then $u \in \underset{\circ}{T}_i$ and $v \in \underset{\circ}{T}_j$ in C′. We have $b_e = \hat{b}_e = 2$.

5> If $u \in \underset{\circ}{T}_i$ and $v \in \underset{\circ}{T}_j$ for some $\underset{\circ}{T}_i$ and $v \in \underset{\circ}{T}_j$ where $i \neq j$, then $u \in \dot{T}_i$ and $v \in \dot{T}_j$ in C′. We have $b_e = \hat{b}_e = 2$.

Therefore we have $b = \hat{b}$ and $b_0 = \hat{b}_0$, i.e., these two inequalities are equivalent.

"$\Longrightarrow$" If the complemented comb inequalities $bx \ge b_0$ and $\hat{b}x \ge \hat{b}_0$ are equivalent for Q^n, then they are also equivalent for Q^n_T by Corollary 5.1. Therefore comb C and comb C' satisfy $H(C')=V \setminus H(C)$ and $T(C')=T(C)$ by Theorem 2.2.

We now prove that these two inequalities are not equivalent if $s \ge 1$ or $q \ge 1$ or $r \ge 1$. First if $p=0$, then $q \ge 3$. Let u,v be any two distinct nodes in C_2 of C, then $b_e=0$ where $e=\{u,v\}$. Nodes u,v are in C_3 of C', thus $\hat{b}_e=2$. But $b_0=3p+2r+q+1>0$. So $bx \ge b_0$ and $\hat{b}x \ge \hat{b}_0$ are not equivalent for Q^n.

We now deal with the cases in which $p \ge 1$. Suppose T_1 is a non-2-matching tooth and $u \in \underset{\circ}{T_1}$ and $v \in \dot{T}_1$ in C. Then $u \in \dot{T}_1$ and $v \in \underset{\circ}{T_1}$ in C'. Thus $b_e=\hat{b}_e=1$ where $e=(u,v)$.

Suppose $q \ge 1$. Let T_2 be a 2-matching tooth and $w \in \underset{\circ}{T_2}$. Then $b_e=1$ and $\hat{b}_e=2$ where $e=(w,u)$.

Suppose $r \ge 1$. Let w be a node in C_6 of C. Then $w \in C_1$ in C'. Thus $b_e=3$ and $\hat{b}_e=2$ for $e=(w,u)$.

Suppose $s \ge 1$. Let w be a node in C_1 of C. Then $w \in C_6$ in C'. Thus $b_e=1$ and $\hat{b}_e=2$ for $e=(w,u)$.

Therefore b and $\hat{b}$ are not multiples of each other , i.e., those two inequalities are not equivalent if $s \ge 1$ or $q \ge 1$ or $r \ge 1$. Thus if the two inequalities are equivalent then $s=q=r=0$. ∎

Note that Theorem 5.3 shows that two complemented comb inequalities are not necessarily equivalent for Q^n even when the corresponding original comb inequalities are equivalent for Q^n_T.

Chapter 6

Conclusion

As previously mentioned, the linear programming cutting plane method has proven to be quite successful in solving large TSP problems. Typically, people start with the equation set $Ax=\vec{2}$ and inequality set $0\le x\le 1$ as the initial set of constraints, then apply linear programming techniques and identify and add in constraints as needed. In many cases subtour elimination and comb constraints alone were sufficient for solving large problems to optimality. Considering the close relationship between Q^n and Q^n_T and the effectiveness of subtour elimination and comb constraints for Q^n_T, we thus expect good results in solving TECSP using this method.

In order to use the linear programming cutting plane method for TECSP(n), we need to be able to either find a constraint known for Q^n which is violated by an intermediate solution or prove such constraint does not exist in the set of constraints known for Q^n. Termed the *separation problem,* this problem is hard in general and can be formally stated as follows:

Problem 1. Given a point $x\in\mathbb{R}^E$ and a family Γ of inequalities in $\mathbb{R}^E$, identify one or more inequalities in Γ that is violated by x or prove that no such inequality exists in Γ.

As for Q_T^n, Problem 1 can be solved in $O(n^4)$ for the subtour elimination inequalities using the Gomory-Hu algorithm [11] for the determination of a minimum weighted cut in an undirected graph; based on the same algorithm, Problem 1 can also be solved in $O(n^4)$ for the 2-matching inequalities (see [28]). Generally for 2-matching inequalities, heuristics are used instead of this algorithm due to time and memory considerations (see [29]). At present, no polynomial algorithm is known for Problem 1 for general comb and clique tree inequalities. In [30], heuristics for comb inequalities and special clique trees (called basic) are proposed. We hope that some of these algorithms and heuristics can be easily adapted for the complemented comb inequalities for Q^n.

As to the aspect of facets of Q^n, we have the following problems.

Problem 2. When $r \geq 1$ and $q = s = 0$, the complemented comb inequality $bx \geq b_0$ is not facet-inducing for Q^n. What is the conversion procedures for comb inequalities in this case such that the resulting inequality is facet-inducing for Q^n? Are there other conversion procedures for comb inequalities in general ?

Problem 3. What is the conversion procedure for clique tree inequalities such that the resulting form is facet-inducing for Q^n?

A problem closely related to TECSP is the minimum cost 2-node-connected spanning subgraph problem whose aim is to find a minimum cost 2-node-connected spanning subgraph in a given complete weighted graph. Let $Q_V^n=\{x^E \mid (V,E)$ is 2-node-connected spanning subgraph of $K_n\}$. It is easy to see from a graph theoretic point of view that any 2-node-connected graph is also a 2-edge-connected, and thus $Q_V^n \subseteq Q^n$. Therefore any valid inequality for Q^n is also valid for Q_V^n. In particular, the complemented comb inequalities are valid for Q_V^n. Furthermore as a special class of complemented comb inequalities, the complemented 2-matching inequalities are also facet-inducing for Q_V^n (see [4]). However, not all complemented comb inequalities are facet-inducing for Q_V^n. As an example, Figure 6.1 shows a comb whose associated complemented comb inequality $bx \geq 6$ is not facet-inducing for Q_V^n. The vector b is shown in Figure 6.1 by the edge coefficients for reader's convenience. Note that edges e for which $b_e=2$ are not shown.

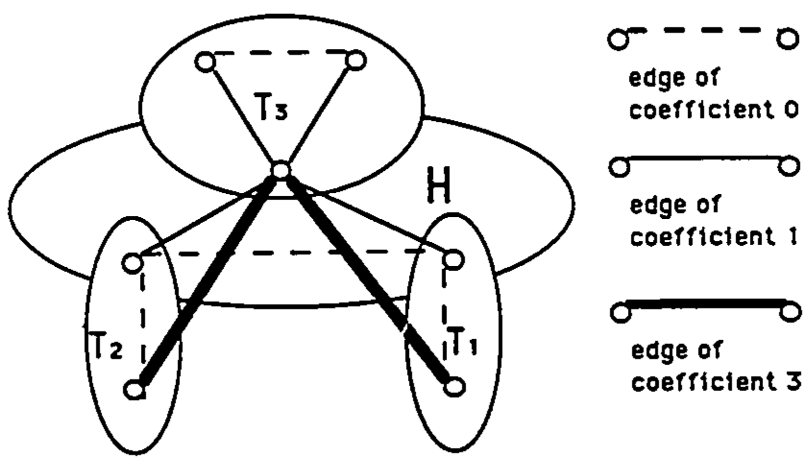

Figure 6.1

This naturally leads to the following problem:

Problem 6. What are the necessary and sufficient conditions for a complemented comb inequality to be facet-inducing for Q_V^n ?

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