

**THINGS KIDS THINK WITH: THE ROLE OF THE PHYSICAL PROPERTIES OF
MATHEMATICAL TOOLS IN CHILDREN'S LEARNING IN THE CONTEXT OF
ADDITION OF FRACTIONS**

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Abstract

This research was designed to examine the role of the physical properties of the mathematical tools, in children's learning in the context of adding two fractions. My two research questions were: (1) How does the feedback from the mathematical tools play a mediating role between the physical actions of the child with respect to the mathematical affordances of the tools and the child's *thinking* about and *learning* and *knowing* of solving addition of fractions problems? And (2) What role is played by mathematical tools in the emergence of a Zone of Proximal Development during the child's solving of addition of fractions problems? To address these questions, I interviewed 13 grade 7 students in Ottawa, Ontario, in groups of two and in three rounds of 30-minute interviews per group. The results showed that the physical properties of the tools play a role in how children perceived the mathematical affordances of the tool, attached mathematical meaning to the tools, created mathematical artefacts and solved the addition of fractions problems. Moreover, the findings show that in children's interactions with mathematical tools, at times, the Zone of Proximal Development (ZPD) emerged, with the guidance provided by the tools. I conclude that children's interaction with the tools provided them the possibility of learning newer forms of reflections, expressions and actions in relation to adding two fractions. This learning was a result of a complex and intertwined relationship between the immediate physical properties and affordances of the tool, the traces of the thoughts of the designer of the tools, as well as the children's previous knowing of fractions. With this study, I extend the Vygotskian notion of the more knowledgeable other within the ZPD to include not only agents (children and adults) but also tools.

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Chapter 1

Introduction

A few years ago, I thought a grade 7 mathematics class with 14 very bright dyslexic children. In that class, I kept thinking about how I could teach mathematics if I was to use written mathematical symbols as little as possible. That year, a lot of my time was spent in the basement cutting wood and nailing pieces to make ‘things’ that might help my students. For me, the most challenging aspect was to come up with things that might help me teach the operations of fractions. The fraction board – which is used as one of the tools in this study – was one of the things I made, to show my children the steps involved in adding two fractions, as they coloured the strips and moved the pieces around (for further explanation refer to page 20). My interest in why and how these ‘things’ helped (or did not help at all) the children in better knowing mathematics led me to do a small-scale research project and to write a report¹ about it. In that research project, I aimed to see how 5 fifth graders thought about the addition of fractions as they interacted with the Cuisenaire rods, fraction circles, and my newly designed fraction board.

An interesting outcome of this study was what I referred to as the notion of ‘feedback’ from the tools. Martin & Schwartz’s (2005) framework – the theoretical framework I used in my interim report – defined stable environments as systems, which based on their design, offer clear feedback and strong constraints on what counts as a correct interpretation; systems such as an airplane’s control panel. They further noted that feedback is an essential part of a stable environment. In my study, I looked at children’s interaction with three different environments, two of which were commercially available (Cuisenaire rods and fraction circles) and the third of which I made in my basement (the fraction board). I showed how the feedback integrated within fraction circles and the fraction board helped students to adapt the environment, self-correct, and interpret the new arrangement of the environment. For example, sizes and colours were the feedback from the fraction circles. This integrated feedback made the environments stable. However, in the particular activity done with the Cuisenaire rods, nothing in the rods alone could provide the students with any feedback to

¹ This report is called an interim report at the University of Ottawa.

adapt the environment, self correct and reinterpret the mathematical concept. The lack of immediate designed feedback within this environment led the children to use different physical properties of the tools and/or my comments and questions as possible feedback. When this small study was finished, although I noticed that feedback was essential in every system and physical properties of the tools matter in the ways in which children interact with them to solve task, I was left with questions such as the following:

- What can count as feedback? And what are different types of feedback from the tools?
- How do some of the properties of the tools assist the children in the process of problem solving, and some others do not?
- How do children ‘see’ these properties in relation to the task of adding two fractions?
- Why do some children ‘see’ the feedback whereas other children do not?
- What is the role of feedback from the tools in children’s learning of fractions?

These questions led to the research presented in this dissertation.

Context

Within the field of mathematics education, there is a strong impetus to incorporate different mathematical tools into the teaching and learning of mathematics in general and into the teaching and learning of fractions in particular (Abrahamson et al., 2011; Abrahamson, 2009; Bartolini Bussi, 2011, 1996; Bartolini Bussi & Mariotti, 2008; Friedman, 1978; Prince & Felder, 2006; Sowell, 1989). Nevertheless, the concerns that have been articulated with regard to the role of the physical properties of mathematical tools and their constraints and affordances for the learning/knowing of fractions remain largely unanswered.

I define *mathematical tool* as any *tool-like object* for which its mathematical *affordances* are *perceived* by a child who is using it to solve a mathematical task. Examples include a piece of paper, an apple, fraction circles, or Java-Bars. I use Engeström’s (2009) conceptualisation of *object* as any focus of attention. I emphasise here that an object becomes a mathematical object if the person(s) perceive(s) its mathematical *affordances*. For example, a paper clip can become a mathematical object if we can perceive the ‘measuring’ affordances that are provided by the paper clips for non-standard measurement. I elaborate on Vygotsky’s view of signs and tools, as well as the role of perception and the concept of affordance in more detail in chapter 3.

To examine the role that is played by the physical properties of mathematical tools in the learning of mathematics, I selected the mathematical concept of fractions. The reason for my focus on the fractions is that many research studies, the Ontario Mathematics curriculum, and a number of reform documents acknowledge fractions as one of the most challenging concepts to teach and learn in elementary-school mathematics (CLIPS, 2011; Kieren, 1992, 1993; Lamon, 2002; Steffe & Olive, 2010). Lamon (2007) noted that fractions are one of the topics in elementary-school mathematics that are among ‘the most difficult to teach, the most mathematically complex, the most cognitively challenging, and the most essential to success in higher mathematics and science’ (p. 23). Many different research studies propose that one way to assist children in learning fractions is to employ different mathematical tools in the classroom (Ball, 1992, 1993, 1995; Behr et al., 1983; Cramer & Henry, 2002; Lamon, 2002; Lamon, 2007; Misquitta, 2011; Pape & Tochoshanov, 2001; Pirie & Kieren, 1994; Steffe, 2004, 2003, 2002). For example, Martin & Schwartz suggested that in order to solve a fraction problem, a child needs to use her/his understanding of whole numbers and to develop a new interpretation of the fractions (2005). They then further argued that reinterpretation is complex and is particularly difficult through thinking alone (Martin, 2009; Martin et al., 2012). Hence, they recommended that it is crucial for students to have opportunities to use different mathematical tools while learning fractions.

The Ontario mathematics curriculum and other supportive pedagogical documents (i.e., reform documents) also promote the use of mathematical tools in teaching and learning fractions. They suggest that mathematical tools assist both children and teachers. Mathematical tools help students by deepening and extending their knowing and learning of mathematical concepts. They also assist teachers by providing them with useful insights into students’ thinking and consequently provide support to help enhance their thinking ‘by analysing students’ concrete representations of mathematical concepts and listening carefully to their reasoning’ (OME, 2005, p.27). To help students’ learning, the current Grades 1 – 8 Ontario Mathematics Curriculum (OME, 2005) presents progressive messages on incorporating different mathematical tools to help children in learning fractions. Table 1.1 shows some examples of specific curriculum expectations for Grades 2-6 prompting the use of a variety of tools and concrete materials to advance the teaching and learning of fraction:

Number Sense	Specific Expectations
Grade 2	○ determine, through investigation using concrete materials, the relationship between the number of fractional parts of a whole and the size of the fractional parts ○ regroup <u>fractional</u> parts into wholes, <u>using concrete materials</u> ○ compare <u>fractions using concrete materials</u>, without using standard fractional notation
Grade 4	○ represent <u>fractions using concrete materials</u>, words, and standard fractional notation, and explain the meaning of the denominator as the number of the fractional parts of a whole or a set, and the numerator as the number of fractional parts being considered; ○ demonstrate and explain the relationship between equivalent <u>fractions</u>, <u>using concrete materials</u> ○ count forward by halves, thirds, fourths, and tenths to beyond one whole, <u>using concrete materials</u> and number lines
Grade 5	○ represent, compare, and order <u>fractional</u> amounts with like denominators, including proper and improper fractions and mixed numbers, <u>using a variety of tools</u> (e.g., fraction circles, Cuisenaire rods, number lines) ○ demonstrate and explain the concept of <u>equivalent fractions</u>, <u>using concrete materials</u>
Grade 6	○ represent, compare, and order <u>fractional amounts</u> with unlike denominators, including proper and improper fractions and mixed numbers, <u>using a variety of tools</u> (e.g., fraction circles, Cuisenaire rods, drawings, number lines, calculators)
<i>Table 1.1.</i> Promotion of the use of mathematical tools for the learning of fractions in the Ontario Mathematics Curriculum (underling added).	

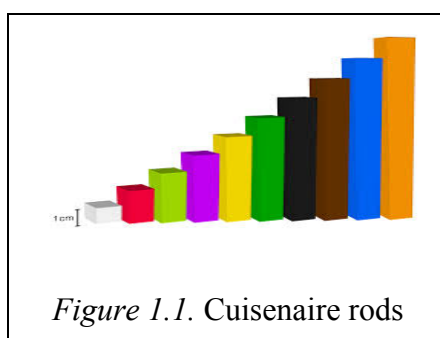
Furthermore, numerous mathematical reform documents within the province of Ontario promote the use of mathematical tools to facilitate the teaching and learning of mathematics. For example, the report of the Curriculum Implementation in Intermediate Mathematics (CIIM) Research Project (Suurtamm & Graves, 2007) states that ‘Engaging with manipulative materials (mathematical tools) allows teachers and students alike the

opportunities to behave and think mathematically as they, jointly become aware of patterns, make conjectures, compare results, and provide explanations (p. 89). Similarly another reform document from the Ontario Ministry of Education, named ‘Guide to Effective Instruction Mathematics’ devotes its fifth volume on Number Sense and Numeration Grade 4 to 6 to the fractions. This volume presents fractions as one of the most challenging concepts of mathematics in the junior grades. It progressively recommends the use of mathematical tools as an instructional strategy for teachers to help students create a better understanding of the concept. Some examples of these suggestions include:

- counting fraction pieces to beyond one whole using concrete materials and number lines (e.g., use fraction circles to count fourths: One fourth, two fourths, three fourths, four fourths, five fourths, six fourths. . .)
- provide experiences of comparing and ordering fractions using concrete and pictorial representations of fractions;
- determine equivalent fractions using concrete and pictorial models (Number Sense and Numeration, Grades 4 to 6 – Volume 5, p. 23).

What Is the Issue?

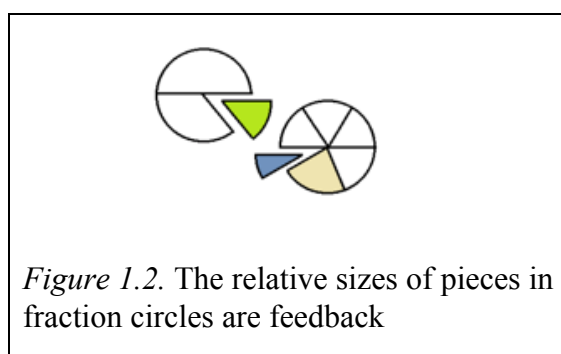
The incorporation of mathematical tools into the teaching and learning of fractions is recommended by both research and different ministry documents, which conceptualise tools as being useful. Yet, these tools also have limitations. The usefulness (or not) of a mathematical tool depends not only on its physical properties and affordances of the tools, but also on the child’s perceptions while interacting with them. A tool that is useful for one child to solve a particular task may not be useful for another child for the same task or for the same child for a different task (Pimm, 2002). Similarly, a tool that is useful for working on one task may not be useful for working on another (Clements & McMillen, 1996; Cramer & Wyberg, 2009). For example, it is one thing for a child to show $\frac{1}{2}$ or $\frac{1}{3}$ using Cuisenaire rods (see Figure 1.1) but it is quite another to use the rods to add $\frac{1}{2}$ and $\frac{1}{3}$.



The literature in mathematics education includes such unanswered questions as ‘What is it about pattern blocks that did not support students’ thinking on fraction order tasks?’ (Cramer & Wyberg, 2009, p. 14) and ‘Is it the sensory characters [of the manipulative] that make mathematical tools helpful?’ (Clements & McMillen, 1996, p. 270). These questions are posed to identify the strengths and limitations of the tools used in learning fractions. However, they are not addressed in terms of the role that is played by physical properties or in terms of the feedback from the mathematical tools in solving a mathematical task. These questions, the questions which arose from my interim report, along with my own experience as a teacher and a learner of mathematics, led me to look more closely at the interrelationships between the physical properties of the mathematical tools used to teach the addition of fractions and children’s knowing of and learning about the addition of fractions.

The Specifics of My Study

This study is a continuation of my interim report within which I looked at the role of feedback from mathematical tools used in children’s problem solving. I describe feedback as the physical properties of a mathematical tool and/or the interrelationships among its various components that make the mathematical affordances of the tool more apparent. For example, in fraction circles, I consider the relationships among the sizes of the pieces to be a form of feedback (see Figure 1.2).



My purpose in this study was to investigate how children use the physical properties of the mathematical tools to think about and/or solve addition of fractions problems and the role of feedback from the tools in this process. I investigated these interrelationships with respect to the Zone of Proximal Development (ZPD). The ZPD is described by Vygotsky as ‘the distance between the actual developmental level (independent problem solving) and the level of potential development (problem solving under adult guidance or in collaboration

with more capable peers)’ (Vygotsky, 1978, p. 69). Following Vygotsky’s view, in the field of mathematics education the more knowledgeable others are usually conceptualised as agents such as teachers, adults and peers. Furthermore, the communication and interactions that take place within the ZPD have usually been referred to as sign-mediated and inter-subjective (Lerman & Meira, 2001; Roth & Radford, 2010), meaning that within these studies the notion of ZPD is looked at between children and adults or between children themselves.

To look at the interrelationships between the feedback from the mathematical tools, what children did and said, and the mathematical task I examined:

- the children’s perceptions of the mathematical tools;
- the children’s interactions with the tools, which lead them to work on or to solve the mathematical task at hand; and
- the children’s talk about the mathematical tools and their actions, which might indicate the emergence of the ZPD.

The two research questions I addressed were:

1. How does the feedback from the mathematical tools play a mediating role between the physical actions of the child with respect to the mathematical affordances of the tools and the child’s *thinking* about and *learning* and *knowing* of solving addition of fractions problems?
2. What role is played by mathematical tools in the emergence of a Zone of Proximal Development during the child’s solving of addition of fractions problems?

To address these research questions, I collected data by interviewing 13 children from two grade 7 classes. The interviews were conducted in teams of two (and one team of three) and in three parts. In each part of the interviews, conducted over three different days, children were given different types of tools to work with in order to solve two addition of fractions problems. The transcribed data was then analysed to answer the two research questions.

Outline of the Dissertation

The dissertation is divided into 8 chapters. In Chapter 2 (Literature Review) I look at the studies that investigated the effectiveness of mathematical tools to solve mathematical tasks as well as the challenges that are associated with children’s use. This section is followed by how the benefit (or not) of the tools is conceptualised in relation to the learning

of fractions. I then look more closely at the physical properties of mathematical tools through the notion of feedback. Next, to set the basis for my theoretical choices, I review how mathematical tools have been conceptualised by two different perspectives: radical constructivism and Vygotskian theory. I conclude Chapter 2 with an explanation of the influence of the reviewed literature on the design of my study. In Chapter 3 (Theoretical Framework) I discuss the Vygotskian concepts of tools and signs, the notions of the zone of proximal development, plus Gibson's theory of affordances. These ideas provide the theoretical underpinning of my research, within which I designed the study and analysed the data. In Chapter 4 (Methodology), I explain how these theoretical ideas created a base for the methodological aspects of the study (i.e., the design, data collection, analysis) as well as the rationale for the choices I made. There are two chapters presenting the findings. Chapter 5 details the findings in relation to the creation and the importance of artefacts and the role of feedback in creating artefacts. Chapter 6 details the findings in relation to the emergence of the zone of proximal development and the role of the more knowledgeable other(s). In Chapter 7 (Discussion), I first provide a general synthesis of the study and its findings in relation to the process of gradual perceptual change, the notion of the creation of the artefacts, and the tool-mediated Zone of Proximal Development. I also point to two theory-related issues that need further exploration. In Chapter 8 (Conclusion), I address the research questions, in the context of the broader literature. I explain the implications and limitations of this study. I conclude by discussing the contributions of this study to the field of mathematics education.

Throughout this writing, I frequently include transcripts and/or brief examples of children's interaction with different mathematical tools either from the fractions literature or from my own data. I did this for three reasons: first, to better illustrate the discussion; second, to show how I saw children's actual work in relation the theoretical framework; and third, to show how I looked at my data and, later, justified the choices that I made in analysing the data. Explanation of the data and the transcripts can be found in Chapter 4.

Before proceeding to the next chapter, there are two clarifying notes that I need to make. Firstly, although this study was investigating the role of mathematical tools in the learning of mathematics, to narrow down the scope, I chose the mathematical *concept* of the addition of fractions. By the word concept, I refer to generalised mathematical meanings that are accepted within the historically contingent cultural practice of mathematics; what Lerman (2014) referred to as the subject of mathematics with its precise "strong grammar" (p. 15).

Examples of different mathematical concepts are listed in the Ontario curriculum.

The addition of fractions is introduced to the children in relation to the concept of fractions as a whole. I believe every child learns every fraction concept differently and at a different time in (or out of) his/her schooling. I will, therefore, provide a brief overview of the development of fractions, from when school age children are introduced to the concept of fractions to when they are introduced to the concept of the addition of fractions. I will then provide some discussion of curriculum structures relating to the fractional knowing upon which the knowing of the addition of fractions is built.

Secondly, in this dissertation, I make reference to several kinds of mathematical tools. I therefore provide a brief description of each tool.

Note 1. Learning trajectory and a developmental perspective of fractions

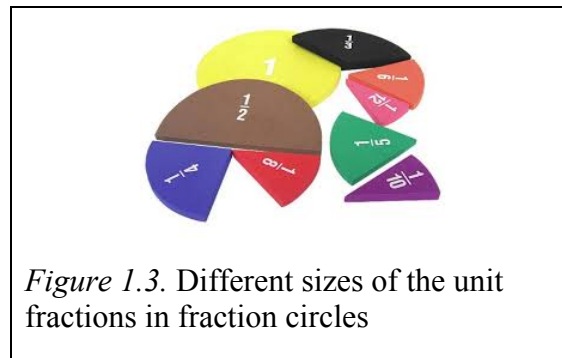
The concept of fractions and its teaching and learning has been examined for several decades and by many researchers from different developmental perspectives. For the limits of this study, I use Confrey et al.'s (2012) longitudinal study in which they provided a learning trajectory and a developmental perspective for many mathematical concepts taught in elementary and high schools, including fractions.

A developmental perspective. In their trajectory Confrey et al. (2012) structured the learning of fractions into three components: (a) working with unit and non-unit fractions; (b) equivalent fractions; and (c) operations with fractions on which, for the purpose of this study, I will only focus on the addition of fractions.

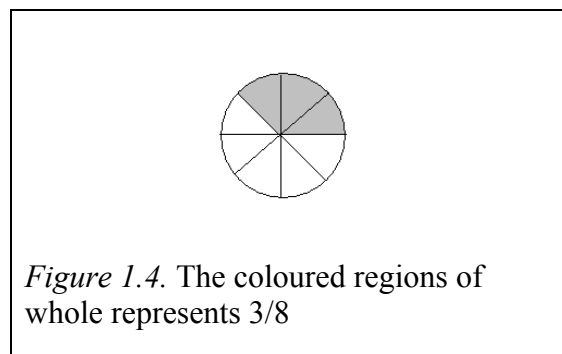
(a) Working with unit and non-unit fractions. According to Confrey et al. (2012), learning fractions starts with learning equi-partitioning. A fraction is described relative to the whole to which it refers. This whole is called a unit. The unit refers to 'the point of reference for a fraction', and is usually assumed to be 'one' (Steffe & Olive, 2010). For example, the fraction $\frac{2}{3}$ is considered to be two thirds of one whole unit. In order to learn equi-partitioning, young students equally partition collections of items and single wholes; for example, students begin to understand ' $\frac{1}{2}$ ' as *half of* the whole collection or as *one half* of a single whole – that is, as one piece out of two pieces shared fairly between two people.

Later, for the children to learn that having more equal parts in a whole makes each share smaller, children need to work with unit fractions. A unit fraction refers to the size of each piece (share) into which the whole is portioned (Steffe & Olive, 2010). A unit fraction is a fraction with a numerator of one. Working with tools such as fraction circles or fraction

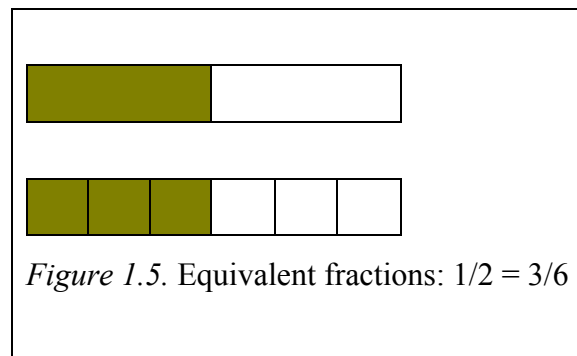
strips helps students to see that the strips for $1/2$, $1/3$, $1/4$, $1/5$ and so on are all sized differently (see figure 1.3).



A critical next step to knowing the unit fraction is to model and name non-unit fractions. To begin, students name regions of wholes as a/b , where a is smaller than b (Confrey et al., 2012). In this step, students learn that the size of a/b is equal to the sizes of the unit fraction $1/b$ repeated a times. For example, the fractional amount of $3/8$ means that where the whole is equi-partitioned into 8 parts, three parts of $1/8$ pieces are coloured (see figure 1.4)



(b) Equivalent fractions. After learning about the fractional amount of a/b , students learn about equivalent fractions: first by splitting a region of fraction models into smaller equi-partitions and/or by composing smaller regions into larger units (Flores et al., 2006; Kamii & M., 1999). For example, they can create three equal parts on one half of a circle and can reason that they have created three ‘sixths’ because the same action could be applied to the other half, thus creating an identical result (in this case $3/6 = 1/2$) (See figure 1.5).



In general, if a/b is a fraction number and if c is a whole number (other than zero), then $a/b = (ac)/(bc)$. The two fractions a/b and $(ac)/(bc)$ are referred to as equivalent fractions (Steffe & Olive, 2010). At this stage students learn to use different representations for a/b and ca/cb to reason about and justify the equivalence for different values of c (Confrey et al., 2012). After learning the concept of equivalent fractions, students are ready to learn to perform operations with fractions, starting with addition and subtraction.

(c) Operations with fractions (addition of fractions). To learn about the addition of fractions, students can start by decomposing a fraction into its added parts; for example, $3/5 = 1/5 + 1/5 + 1/5$ (Steffe & Olive, 2010). Confrey et al. (2012) noted that, before introducing the concept of adding fractions, students should learn about the structure of a fraction a/b . In other words, students need to learn that a/b can be decomposed into a parts of size $1/b$ th. This helps them to learn that fractions can be joined or added only if the fractions are taken as parts of the same whole (Confrey et al., 2012). Decomposing a/b into a parts each of size $1/b$ can be used as a basis for transitioning to adding fractions with common denominators – that is, $a/b + c/b = (a+c)/b$. By adding fractions with common denominators, students learn that the denominator describes the size of the part, and that the numerator describes the number of those parts (Siegler et al., 2010).

In order to add fractions with un-like denominators, Confrey et al. (2012) further explain that students should be provided with examples to show that, in adding fractions with un-like denominators, simply adding the numerators to each other and the denominators to each other does not work. For example, students can see that $1/2 + 2/4$ does not equal $3/6$ because $3/6$ is equal to $1/2$, and that $1/2 + 2/4$ is more than $1/2$. Then students learn to add fractions with un-like denominators and with smaller denominators (like, for example, $1/2$, $1/3$ and $1/4$), which helps them gain a better understanding of the fact that the denominator shows the size of the part and that the numerator shows the number of those parts (Clarke et

al., 2008). The students' previous learning of the concept of equivalent fractions and of the fact that in order to add two fractions the denominators should be the same size, provides a basis for them to find a common denominator based on equivalent fractions. At this stage students learn that, when solving problems in which fractions are referring to the same whole but with un-like denominators, they must find a common denominator in order to solve the problem (Siegler et al., 2010).

Ontario mathematics curriculum structure. The introduction of the addition of fractions is logically situated within the concept of fractions as a whole. Since this study was conducted in the province of Ontario, I use the current Ontario mathematics curriculum for grades 1–8 (Ontario Ministry of Education, 2005) to situate the concept of the addition of fractions within the broader picture of fractions.

Fractions are introduced to children in grade one. From grade one to grade three, the interaction of children with any fraction-related concept is through the use of mathematical tools and does not involve the use of numbers presented in standard fractional notation.

In grade 1, the only encounter of the children with fractions is to investigate equal-sized parts of a whole, using names such as halves, fourths or quarters.

In grade 2, children learn the interrelationship between the number of fractional parts of a whole and the size of the fractional parts. For example, the curriculum suggests that children in grade two will learn that a paper plate divided into fourths has larger parts than a paper plate divided into eighths. In the same year, the children will learn to regroup fractional parts into wholes, using concrete materials. For example, they could use a whole strip of fraction strips and one strip of $\frac{1}{4}$ to make the fraction of $\frac{4}{5}$.

In grade 3, children learn to divide whole objects and sets of objects into equal parts, and to identify the parts using fractional names (e.g., one half; three thirds; two fourths or two quarters).

In grade 4, children learn to use standard fractional notation, and to explain the meaning of the denominator and the numerator – as the number of fractional parts of a whole or a set and as the number of fractional parts respectively. Comparing and ordering fractions are introduced to the children in this grade so that they will know, for example, that $\frac{4}{5}$ is bigger than $\frac{3}{5}$. Children will also learn the relationship between equivalent fractions (e.g., $\frac{1}{2} = \frac{3}{6}$), using concrete materials (e.g., fraction circles, fraction strips, pattern blocks) and drawings.

The simple form of addition of fractions using concrete materials and number lines is

introduced in grade 4 as well. Children will learn to count forward by halves, thirds, fourths and tenths (e.g., to use fraction circles to count fourths: ‘one fourth, two fourths, three fourths,...’).

In grade 5, using a variety of tools (e.g., fraction circles, Cuisenaire rods, number lines) and using standard fractional notation, children learn to compare and order fractional amounts with like denominators, including proper and improper fractions and mixed fractions.

In grade 6, working with fractions with un-like denominators is introduced to students. Children in this grade learn to compare and order fractional amounts with unlike denominators, including proper and improper fractions and mixed fractions.

In grade 7, children are introduced to the addition of fractions with simple like and un-like denominators. They are to use a variety of tools (e.g., fraction circles, Cuisenaire rods, drawings, calculators) as well as algorithms.

In grade 8, children finally learn to represent the multiplication and division of fractions using a variety of tools and strategies.

Common models of representing fractions. There are three common representations used to model fractions: area models, set models, and linear models. These models help students develop a sense of fractional quantity. The following is a brief description of each model and suggestions of different types of tools to be used for each type.

- **Area Models:** In an area model, one shape represents the whole. The whole is divided into fractional parts. Although the fractional parts are equal in area, they are not necessarily the same size and shape. A variety of materials can serve as area models. Fraction circles, pattern blocks, and square tiles are tools that represent this model.
- **Set Models:** In a set model, a collection of objects represents the whole amount. Subsets of the whole make up the fractional parts. Students can use set models to solve problems. A variety of materials can serve as set models. Real objects such as pencil crayons, counter and square tiles are among tools that represent this model.
- **Linear Models:** In a linear model, a length is identified as the whole unit and is divided into fractional parts. Line-segment drawings and a variety of tools can be used as linear models. Interlocking cubes, Cuisenaire rods, and paper strips are among tools that represent this model. Furthermore, I designed the fraction board (see note 2) on the basis of the linear model to represent the addition of fractions.

Note 2. List of Mathematical Tools

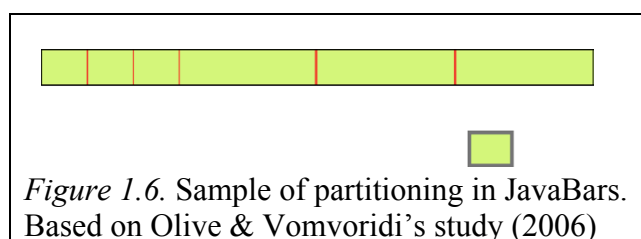
In the following, I provide a list of mathematical tools that I refer to in this study and are noted in the literature. I also provide a brief description for each.

1. Fraction Circles

Fraction circles are sets of transparent circles of various colours. Each circle is broken into equal fractional parts and uses the same-sized whole. The circles included are one whole circle as well as circles divided into halves, thirds, quarters, fifths, sixths, eighths, tenths, and twelfths.

2. JavaBars (Biddlecomb & Olive, 2000)

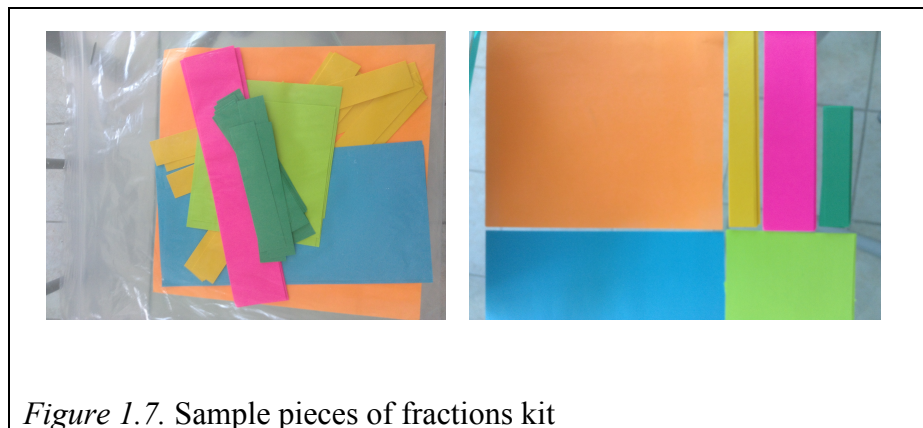
JavaBars is a computer program in which students can draw rectangles (bars) of various dimensions (see Figure 1.6).



The program was designed to foster students' construction of key operations such as partitioning, iterating, and dis-embedding in producing bars that are fractional amounts of other bars. In the program, any drawn bar can be set as the UNIT BAR, the unit by which other bars made from that bar can be measured. For example, a student can COPY the unit bar and then, using the PARTS menu, partition that bar into some number of equal parts (e.g., they can dial to 10 and mark the bar into 10 equal parts, see Figure 3). Students can partition a partition (e.g., partition each of the 10 parts into, say, three equal parts), CLEAR a partition (erase all marks on the bar), BREAK a bar into its parts, or PULL OUT a part from a bar. When students use PULL OUT, the part simultaneously remains in the bar and is dis-embedded from the bar. To make longer bars, students can JOIN parts or bars together, or use REPEAT to iterate a part or bar some number of times. Students can also FILL parts or bars in different colours.

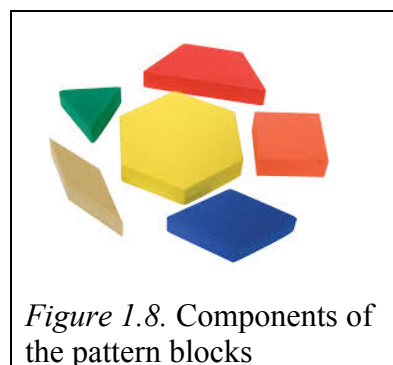
3. The fractions kit (Pirie & Kieren, 1994)

Fractions kits were designed by Tom Kieren. A fractions kit contains rectangles based on a common standard sheet as a unit, representing halves, thirds, fourths, sixths, eighths, twelfths, and twenty-fourths. The following Figure (1.7) is a sample of the fractions kit that was prepared by Tom Kieren for a presentation in a conference. This set contains coloured-paper pieces of different sized fractions: 1 orange piece of one unit, two blue halves, six pink $\frac{1}{6}$ s, 12 yellow $\frac{1}{12}$ s and 24 green $\frac{1}{24}$ s.



4. Pattern blocks

Pattern blocks (see figure 1.8) are pieces of different geometrical shapes. They allow children to see how shapes can be decomposed into other shapes. The standard pattern blocks contain: Equilateral triangle (Green), Regular rhombus (Blue), Trapezoid (Red) and Hexagon (Yellow)

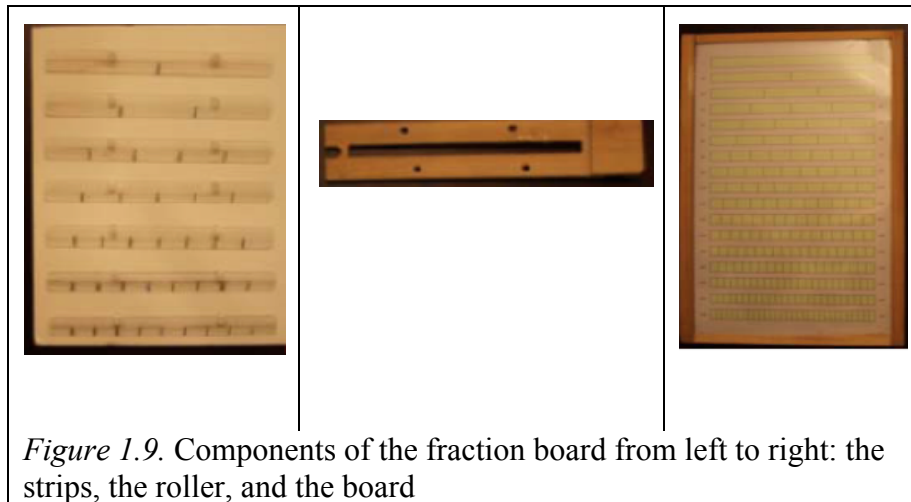


5. The fraction board (Abtahi, 2013)

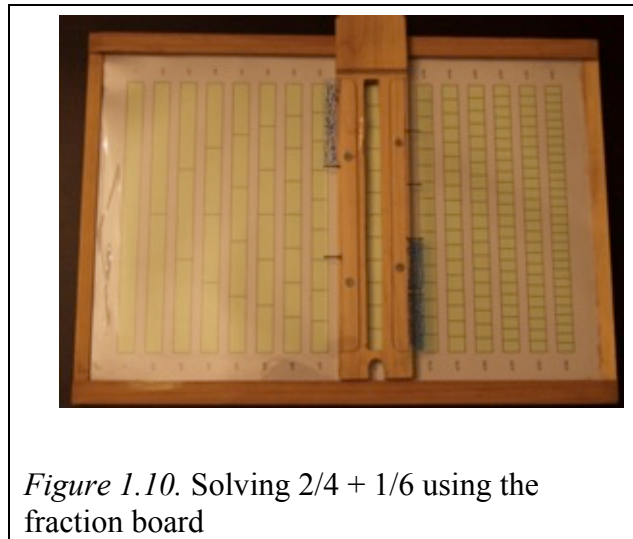
The fraction board has three parts:

- The board: A wooden board, which represents fraction strips from one whole to $\frac{x}{30}$;

- Fraction strips: Transparent plastic Fraction Strips to represent the denominators of the fractions. The numerators will be coloured by students on these Fraction Strips;
- The roller: A mobile wooden structure that can be loaded by fraction strips. It fits on the board and rolls up and down (see Figure 1.9).



The board uses the Lowest Common Multiple (LCM) strategy to solve the addition of fractions. When given a question ($\frac{2}{4} + \frac{1}{6}$), students pick two fraction strips that represent parts equal to the denominator ($\frac{1}{4}$ and $\frac{1}{6}$). Then they colour the nominators on the fraction strips (two parts on $\frac{1}{4}$ strip and one part on $\frac{1}{6}$ strip) and they load the fraction strips on the roller and fit the roller on the board. They move the roller down to find a target strip on the board. The target strip is the one where its vertical lines match with all the vertical lines of both fraction strips (in this case $\frac{1}{12}$). Then, the student will count how many little pieces on the target strip will match the coloured parts of the first fraction strip and will colour the target strip accordingly (colour 6 pieces on $\frac{1}{12}$). They do the same thing for the second Fraction Strip ($\frac{2}{12}$). Now they can see how $\frac{2}{4} + \frac{1}{6} = \frac{8}{12}$ (see Figure 1.10).



6. Cuisenaire Rods

Cuisenaire rods were devised in the 1920s not by Georges Cuisenaire, but by his wife², Mrs Cuisenaire, who was a Belgian educator and a violin player. Cuisenaire rods were designed to work on the number relations, including fractional numbers. The invention remained almost unknown outside the village of Thuin for about 23 years, until in 1953 an Egyptian-born, British mathematician and education specialist Caleb Gattegno went to visit Georges Cuisenaire and to observe lessons in his class. Gattegno named the mathematics devices ‘Cuisenaire rods’ and began popularizing them since he believed the rods allowed students to expand on their mathematical abilities. With Gattegno’s help, the use of the rods for both mathematics and language teaching was developed and popularised in many countries around the world.

Cuisenaire Rods (see Figure 1.1) are a collection of rectangular rods, with different colours and sizes. The smallest rod is 1 cm long and the longest rod is 10 cm long. The sizes and the corresponding colours of the rods are as follows:

White rod: 1 cm

Red rod: 2 cm

Light green: 3 cm

Purple: 4 cm

Yellow: 5 cm

² I could not find Mrs. Cuisenaire’s first name.

Dark green: 6 cm

Black: 7cm

Brown: 8 cm

Blue: 9 cm

Orange: 10 cm

If organised from the smallest to the longest rod, the rods grow in 1cm increments. Moreover, the sizes of the rods are interrelated; that is, the orange rod of 10 is twice as big as the yellow rod of 5, or it takes 3 white rods of one to cover a lime green rod of three (see Figure 1.11).



Figure 1.11. The relative sizes of rods in Cuisenaire Rods

7. Fraction strips

Fraction strips were developed in the nineteen-sixties by Dr. Albert B. Bennett. To protest the widespread ‘lecture approach’ to teaching, in 1968, Professor Bennett organised a mathematics lab at the University of New Hampshire and started writing activities for teachers. Fraction strips, among other mathematical tools, are devised as a result of the activities in the mathematics lab. A complete set of fraction strips contain pieces to illustrate one whole, halves, thirds, fourths, fifths, sixths, tenths, and twelfths. These bars represent the part-to-whole linear model (see Figure 1.12).

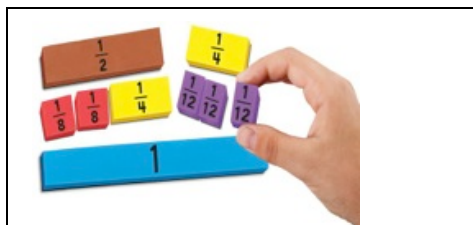


Figure 1.12. The relative sizes of strips in Fraction Strips

Chapter 2

Literature Review –

Benefits and Challenges of the Mathematical Tools

In this chapter, I provide a review of literature related to key aspects of my study. I begin the chapter by considering past studies that investigated the uses of mathematical tools in classrooms. First, I discuss different conceptualisations of the effectiveness of the mathematical tools in learning and teaching of mathematics. I then discuss the challenges associated with the use of mathematical tools, by critically examining a widely used conceptualisation of mathematical tools as ‘concrete objects’. I then discuss the benefits and challenges of the use of mathematical tools in the learning of fractions, which leads me to pay closer attention to the physical properties and specific designs of the mathematical tools used to teach and learn fractions. I do this by discussing the notion of feedback in more detail as I situate the notion of feedback within the broader literature, in order to create a base for positioning the role of the physical properties of tools in learning of fractions. Finally, to lead to my theoretical framework (explained in Chapter 3), I conclude this chapter by examining the conceptualisations of mathematical tools with a focus on two theoretical perspectives: Vygotskian and (radical) constructivist, paying special attention to the notion of ‘others’.

Conceptualisation of the Effectiveness of Mathematical Tools

For decades, researchers in the field of mathematics education have been keen to demonstrate the effectiveness of mathematical tools for learning and teaching (Abrahamson et al., 2011, Abrahamson, 2009; Ball 1993; Bartolini Bussi, 2011, 1996; Pape & Tchoshanov, 2001; Pirie & Kieren 1994; Steffe, 2004). These tools, in general, were looked at as objects that can be handled in a sensory manner to promote mathematical thinking and/or problem solving (Bartolini Bussi, 2007; Steffe, 2004; Swan & Marshall, 2010;). Examples of mathematical tools that can be used in a mathematics classroom are blocks, dice, an abacus, paper, pencils, a blackboard and selected information-technology tools (Bartolini Bussi & Mariotti, 2008). The effectiveness of mathematical tools has been conceptualised from a variety of angles, including their role in fostering children’s mathematical learning; in teaching mathematical concepts; in reconstruction of mathematical tasks; and in promoting communication and social interactions.

Researchers have shown that mathematical tools are useful for children to construct,

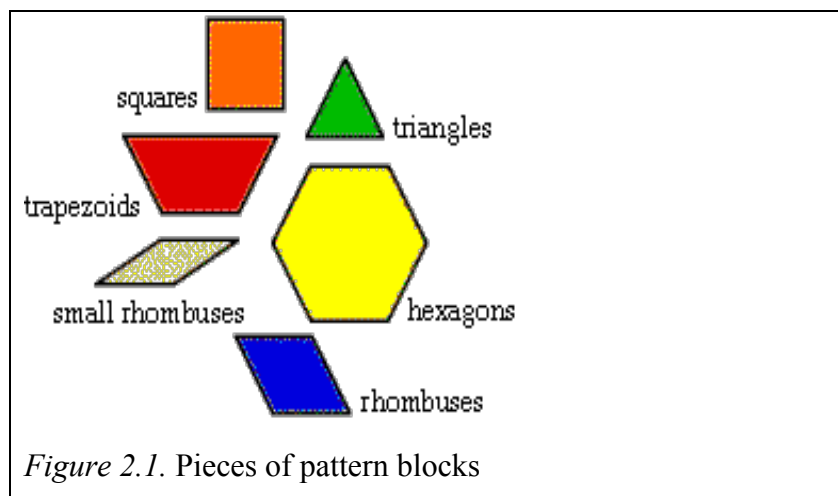
expand and communicate their understanding of different mathematical concepts (Cobb, 2002; Lamon 2002; Moyer, 2002). Von Glasersfeld (1995) defined understanding as comprehension through actions and thoughts in a given situation to the extent that one can solve problems posed by the situation with regard to the why and the how. Suurtamm & Graves (2007) also noted that mathematical tools help students perceive relationships between different mathematical concepts, successfully make connections between the concrete and the abstract, and communicate their reasoning more effectively. Moreover, students, while working with mathematical tools and by making their own models of mathematical ideas, achieve a great means of building and explicating their mathematical thinking (Pimm, 2002; Wright, 2014).

Different studies have illustrated the effectiveness of mathematical tools for teaching mathematics. These studies concluded that the use of mathematical tools in mathematics instruction gives students hands-on experience in building mathematical ideas (Ball, 1992), and promotes conversations that focus on mathematical understanding rather than on observable calculations and methods (Cobb et al., 2001). Moreover, teachers can gain insight into children's knowing of mathematics by observing their actions on mathematical tools (NCTM, 2000).

Behr et al., (1983) conceptualised the effectiveness of mathematical tools in terms of their role in the re-construction of a mathematical task. They stated that 'properly conceived and sequenced materials can provide for continual reconstruction of the problem condition and concurrently can permit a dynamic interaction between the problem solver and problem conditions' (p. 19). Norman (1993), in his book entitled *Things that Make Us Smart*, also noted that mathematical tools 'don't make us smarter [...] they change the task' (p. 78). For example, if a child is to add $\frac{1}{2}$ and $\frac{1}{3}$, then fraction circles can possibly change the task from finding the common denominator by writing mathematical symbols to finding pieces that physically fit both on the $\frac{1}{2}$ and $\frac{1}{3}$.

The view that I subscribe to conceptualises the effectiveness of mathematical tools in terms of promoting the communication and social interaction associated with their use in classrooms. Research that views mathematical inquiry as a network of social practices strongly endorses the incorporation of mathematical tools into the teaching and learning of mathematics (Bartolini Bussi & Mariotti, 2008; van Oers, 2010). From this view, I consider mathematical tools as 'tools' when the focus is on their customary structures and physical properties (such as sizes and colours in fraction strips, or the number of beads in an abacus's

wire). At this stage the mathematical tools can be used to work on a mathematical problem; they can equally be used to build a castle or a paper airplane, or to be sat on as a chair. Mathematical tools as ‘tools’ become effective in mathematical problem solving only when they are tied to signs such as mathematical language, and mathematical symbols. For example, a child in Cramer and Wyberg’s (2009) study used pattern blocks to model $\frac{3}{4}$. To explain what he did, he said: ‘I used the yellow hexagon as the whole and four of the brown trapezoids would make one whole and so I put three to make $\frac{3}{4}$ ’ (p. 12) (See figure 2.1).



This talk emerged from the activity with the mathematical tool and referred to both of its parts (i.e., yellow hexagon, brown trapezoids) and to the action that the child accomplished with the tool: ‘[...] so I put three to make $\frac{3}{4}$ ’. These utterances probably never would have emerged if the child were not using the pattern blocks.

According to Bartolini Bussi & Mariotti (2008), the relationship between mathematical tools and signs can be examined within the context of the social use of the materials in working on or solving a mathematical task. For example, a child may write his/her findings in a symbolic form (e.g., $\frac{1}{2} + \frac{1}{4} = \frac{3}{4}$), after having used fraction circles to solve a fractions problem. The social use of mathematical tools also can be seen when children are asked to carry out a task in pairs or alone. Working in pairs provokes communication (verbal and written signs) (Bartolini Bussi & Mariotti, 2008). But I believe that working alone with mathematical tools illustrates the social use of the tools, considering the social origin the tools (i.e., interaction with the thoughts and ideas of its creator/designer).

Conceptualisation of the Challenges Associated with the Use of Mathematical Tools

Incorporating mathematical tools in mathematics teaching and learning is not always effective, especially when the focus is on the useful-ness of the tools as being *concrete materials*. For years, and following Piaget, mathematical tools were conceptualised as concrete objects and as useful building blocks in the formation of more abstract and hypothetical thinking (Wilensky, 1991). Piaget (1964) noted ‘Once the operations [working with concrete objects] have been attained, this experience is no longer needed and the coordination of actions can take place by themselves in the form of construction for abstract, structures’ (p. 180). Dienes (1971), in building on Piaget’s perspective of stages, conceptualised mathematical tools in terms of the role that they play in the ‘abstraction process’ (p. 337), which is a process that ‘proceeds from the concrete to the final stage of wielding a mathematical formal system’ (p. 337). He referred to ‘concrete’ as the ‘immediate contact’ of the child with objects and events (Dienes, 1971, p. 335). Behr et al., (1983) in the Rational Number Project – the longest running multi-university research project in the history of mathematics education – conceptualised the role of mathematical tools in terms of ‘facilitating the acquisition and use of rational-number concepts as the child’s understanding moves from concrete to abstract’ (p. 12).

But how concrete is ‘concrete’? The main arguments of several studies (including McNeil & Uttal, 2009; Pimm, 2002; Pirie & Kieren, 1989; Wilensky, 1991) were formulated around the fact that it is not actually the mathematical tools as ‘concrete’ objects that help the child in mathematical problem solving. Rather, it is the interaction of the child with the tools and the child’s thinking about his/her actions that make the mathematical tools useful. Pimm (2002) argued that mathematical tools do not contain or generate mathematics; instead, children do this, with their minds. Wilensky (1991) questioned the ‘concreteness’ of concrete materials. He conceptualised concreteness as ‘a property which measures the degree of our relatedness to the object (the richness of our representations, interactions, connections with it)’ (p. 36). For example, for a child who has never seen or used an abacus, an abacus is as abstract as the numbers that it is trying to represent.

McNeil & Fyfe (2012) also noted it is not the concrete-ness of mathematical tools that makes them useful, on the contrary, children often have difficulty grasping the relationship between mathematical tools and the mathematical concepts that they are intended to represent, and that their comprehension of mathematical tools depends strongly on instruction. Their dual-representation hypothesis provided a theoretical account of these

difficulties. They explained that the principle of the dual-representation hypothesis is that any concrete symbol (e.g., mathematical tools) ‘can be thought of in two different ways: (a) as an object in its own right and (b) as a representation of something else’ (p. 43). For example, if the relationship between the number of beads in an abacus’s wire and the notion of addition is not clear to a child, then he/she needs to learn not only the addition, but also the functionality of the abacus as a system and its relationship with the mathematical concept; in other words, he/she needs to learn two separate systems and the relationship between them. From this perspective, mathematical tools are materials with which children interact, interpret the results of their own actions and solve particular tasks (Martin & Schwartz, 2005; McNeil & Alibali, 2005; McNeil & Fyfe, 2012; McNeil & Jarvin, 2007; Tall, 2011).

I also follow a similar view, namely, I view mathematical tools as any tool that children interact with to think about mathematics, to think about their actions, and to solve a mathematical task. In such a case, I view an orange as a mathematical tool if a child halves the orange and gave $\frac{1}{3}$ of the half to someone else and says: you got $\frac{1}{6}$ of the whole orange.

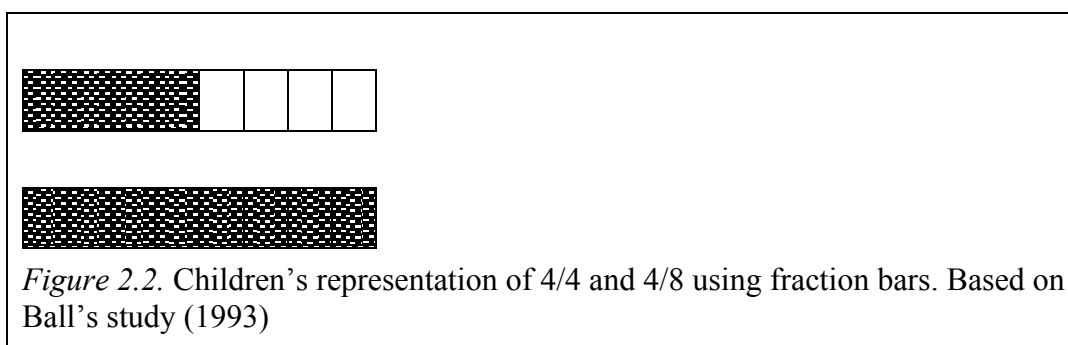
Considering the above-mentioned effectiveness and challenges associated with the use of mathematical tools in learning of mathematics, I now focus on how mathematical tools have been incorporated into the teaching and learning of fractions.

Mathematical Tools in the Learning (or not) of Fractions

The incorporation of mathematical tools into the learning of fractions has its own benefits and challenges. At times, the use of mathematical tools positively impacts students’ learning of fractions (Cramer & Henry, 2002; DeCastro, 2008; Misquitta, 2011). For example, data collected by Martin & Schwartz (2005) shows that using different mathematical tools had a significant role in children’s learning of fractions (Martin & Schwartz, 2005). This data particularly shows that ‘for the same children in the same session on the same class of problems, physically manipulating the pieces had strong benefits compared to drawing on pictures of the pieces’ (p. 576). Another example would be multiple studies done by Kieren over two decades showing how using mathematical tools such as fractions kits or Pizza fractions kits can help students to learn fractions. In one longitudinal study, Pirie and Kieren showed a case of a third grader (Teresa) who learnt to add two fractions using the fractions kit (Pirie & Kieren, 1989). Pirie and Kieren noted that Teresa began the task of adding two fractions not knowing what to do. She said ‘I don’t know’ and ‘I

think you just add the tops and the bottom' (Pirie & Kieren, 1989, p. 163). She was then given the fractions kit to work on a series of tasks: 'Using her kit she noticed that one fourth, three eights, and two sixteenths together exactly cover three fourths' (Pirie & Kieren, 1989, p. 163). Later, Teresa could 'add' $1/3 + 1/6 + 6/12$ using the kit. After a while, she was able to say: 'You can do $2/3 + 5/6$ because twelfths fit on both' (Pirie & Kieren, 1989, p. 167). Later, when asked 'What is $1/2 + 3/4 + 2/5 + 7/10$?' Teresa, without using the kit, said: 'Twentieths will fit on all of them. Two times ten makes twenty, so one times ten or ten twentieths. Four times five makes twenty so three times five is fifteen twentieths...' (Pirie & Kieren, 1989, p. 169). The authors noted that Teresa went beyond the concrete concept of addition, making statements like: 'Addition is easy. You can make up the right kind of fractions just by multiplying the denominators and then just get the right numerators by multiplying by the right amounts' (Pirie & Kieren, 1989, p. 169).

Although mathematical tools have an important place in the learning of fractions, the extent to which they are useful (or not) depends on how the child interacts with the tools and on what task. The following example, extracted from Ball's (1993) study, shows how the physical properties of the mathematical tools (i.e., fraction bars, in the case of her study) limited the communications that might have emerged if the tool was not used. In this example (see Figure 2.2), the students were comparing $4/4$ with $4/8$ using the fraction bars. Ball stated that the use of fraction bars 'force them [the children] to the right answer', as they did not 'have to consider directly the role of the common unit, for it is implicit within the material' (1993, p. 162).



She then explains that if the children make their own models, they may have to struggle with some essential concepts in fractions such as *unit*, as a nine-year-old did when he drew his model (see Figure 2.3):

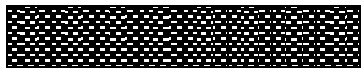
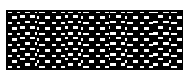


Figure 2.3. Children's drawing of $4/4$ and $4/8$. Based on Ball's study (1993)

This drawing made it seem as if $4/4$ might be equal to $4/8$. Ball noted that:

He and his classmates struggled with the question of whether the rectangles had to be the same size in order to compare two fractions. One classmate asserted that they did, because otherwise, your drawing would convince you of something that wasn't true. Another student, however, argued that it didn't really matter how big you made the rectangles because you could see that $4/4$ took up the entire rectangle, while $4/8$ took up only half of it (p. 162).

Ball concluded that these valuable discussions would probably have never come up if the students were to use only the fraction bars.

Similar examples can be found in Cramer and Wyberg's (2009) study. They investigated the effectiveness of different mathematical tools on students' understanding of fractions. They identified strengths and limitations of different tools used. For example they noted pattern blocks had limited value in helping children better learn the part-whole model. They showed that pattern blocks also had limited value in children's understanding of adding and subtracting fractions. Moreover, they showed that, for example, a paper fraction chart based on a paper-folding model supported the children in ordering fractions with same numerators but was less useful in assisting the children on estimation tasks. At the end of their study they posed questions such as 'What is it about pattern blocks that did not support students' thinking on fraction order tasks?' (Cramer & Wyberg, 2009, p. 14).

In my study, I tackled questions such as the one posed by Cramer and Wyberg and looked at what is (or is not) in the tools (in terms of their physical properties) that might support some activities but not others. I looked at these properties of the tools, using the notion of feedback. In the following section, I pay closer attention to what I call 'feedback' from the mathematical tools.

What do I Mean by Feedback From the Mathematical Tools?

As mentioned earlier in chapter 1, I describe feedback from a mathematical tool as the physical properties of a mathematical tool and the interrelationships among its physical components that make the mathematical affordances of the tool more apparent. I use Gibson's analogy of affordance, as whatever it is about the tool that contributes to the sort of activity that happens (Gibson, 1977; Gibson & Gibson, 1955). Consequently, the notion of feedback I am referring to is those properties of the tools that make mathematical affordances of the tool more apparent, for example, the sizes, the colours, and the interrelationship between the sizes of the rods in a set of Cuisenaire Rods. A crucial point to make explicit is that feedback from the mathematical tools is useful only according to how it is seen by the child who is using them and in relation to the task that they are used for. That is, it is ultimately the task at hand that differentiates between useful and non-useful affordances of the tools. For example, the flatness and hardness of the surface of a table are called its affordances if I want to put my plate on the table. The same properties (flatness and hardness) are also called the table's affordance if I want to stand on the table to fix a light bulb. Yet, the flatness and hardness of the table are not called its affordances if I hide under the table while playing hide and seek.

Although the notion of feedback is one that I have introduced in order to have a closer look at the properties of the tools, in the following section, I will situate the concept of 'feedback' within the already existing mathematics education literature. I highlight different types of feedback in different types of tools.

Situating 'Feedback' in the Literature and Some Examples

Bartolini Bussi and Mariotti (2008) contended that 'mathematical tools are designed according to a particular goal; therefore they embed a specific mathematical knowledge' (p. 126). Considering my view of *mathematical tools* – as any *tool-like object* that its mathematical *affordances* are *perceived* by the child who is using it to solve a mathematical task – I argue that *some* mathematical tools are designed according to a particular mathematical goal and some are not. An example of a mathematical tool that is not designed according to a mathematical goal is a piece of scotch tape: one can measure 24 cm pieces of the scotch tape, mark and cut 8 cm of the tape, and calculate who got what fraction of the scotch tape (i.e., one person had $\frac{1}{3}$ and the other had $\frac{2}{3}$ of the piece of tape).

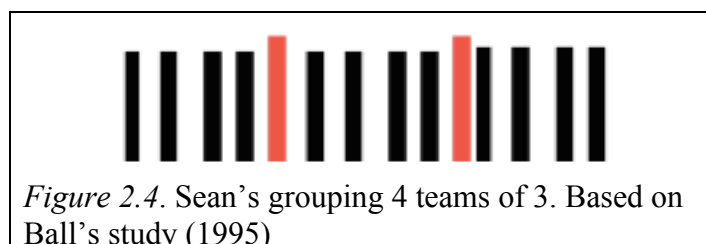
Based on how mathematical tools are designed, some tools have the mathematical

perception and knowing of their designer(s) incorporated into the elements of their design – that is within the physical properties of the tools. I refer to these elements of design as possible feedback from the tools. And the feedback becomes useful only if it is perceived by the child who is using the tool to solve a mathematical task.

Some examples of tools in the fractions literature that have the fractional knowing of their designer(s) incorporated into the elements of their design include: pattern blocks and a fraction chart (Cramer & Wyberg, 2009), the pizza fractions kit (Kieren & Davis, 1996), the fractions kit (Pirie & Kieren, 1994), non-transparent plastic fraction circles (Olive & Vomvoridi, 2006), and computer micro-worlds (i.e., TIMA sticks, fraction bars, JavaBars and sticks) (Hackenberg & Tillema, 2009; Orrill, 2003; Olive & Steffe, 2002; Steffe, 2004). A common element of all these tools is the specific mathematical design of their physical properties.

Some examples of the tools that are not designed in relation to fractions but were used in the literature on fractions are as follows: plastic chips (Cramer & Wyberg, 2009), paper for folding activities (Empson & Turner, 2006), a marker and a white board (Ball, 1995), a pencil and paper, cubes (Empson, 1999) and discrete quantities (i.e., bags of spices) (Olive & Vomvoridi, 2006). A common element of these tools is that none is designed to provide immediate feedback to the child about how to think about fractions. Examples within the literature show that children constructed their own feedback as they interacted with these tools. In the following, I provide two examples from the literature to better clarify what I mean by ‘feedback’ from the mathematical tools and the construction of the feedback and how they might be useful or not.

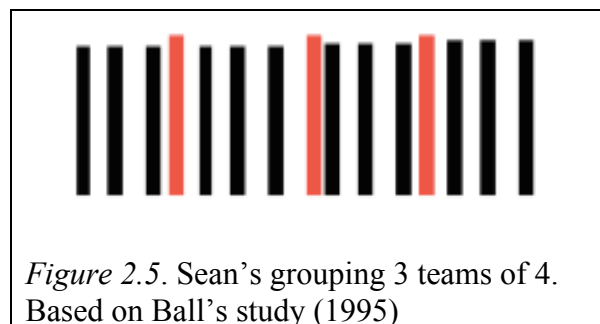
Example 2.1 – Sean’s constructed feedback. The following example is drawn from Ball’s (1995) study. Sean was asked to use paper and pencil to solve the following task: ‘ $\frac{3}{4}$ of the crayons in Mrs. Rundquist’s box of a dozen crayons are broken. How many unbroken crayons are there?’ Sean modified the mathematical tool of a white board to constructing his own feedback; ‘He drew 12 sticks to represent the 12 crayons, and marked off groups of four crayons’ (p. 356) (see Figure 2.4).



Sean modified the tool of paper to construct an artefact. He then used the feedback from the created artefact. The feedback was the way in which he had the groups drawn on the page. He explained:

Well, I um counted these and I got, I went 1, 2, 3, 4 and I put a line down. So it's... then I went 1, 2, 3, 4 and I put another line down and I add them up and it's 8, and I put another line 1, 2, 3, 4. And that was 12... (p. 356)

Sean then used the constructed feedback to 'change his mind'. Ball stated: 'Sean looking at his own drawing immediately changed his mind' (p. 356). He again used his constructed feedback (i.e., the grouping of the lines) to explain: 'A quarter wouldn't be that ... Because um, because that's a third. There is only three groups. There's supposed to be four groups' (p. 356). Sean re-modified the artefact by drawing 'new lines to mark off four groups of three crayons' (p. 356) (see Figure 2.5).



Subsequently, he used his newly constructed feedback to say:

Because it is three fourths, that is what I said, it is three fourths so three crayons is a fourth, so three and (pointing at each group as he spoke) that's a fourth, that is a fourth and that's a fourth, so that's three fourths (p. 357)

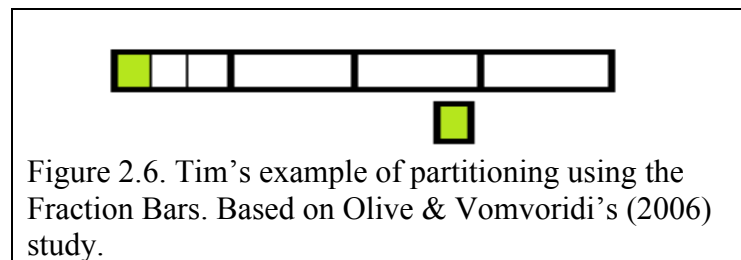
In this interaction Sean used the constructed feedback from his own drawing, that is the way in which the black lines were grouped, to solve the fraction problems.

Example 2.2 – Tim's (mis)use of feedback by design. Feedback from the mathematical tools is not always useful for every child. The feedback is only useful to the extent that it is seen by the child who is using the tool to solve a mathematical task. The following example is drawn from Olive & Vomvoridi's (2006) study. This example shows how the feedback from the tool was not useful for the child (Tim) to solve the fraction task. In this example, Tim used the micro-world of JavaBars to set up the following problem:

He used the computer actions to partition an on-screen unit bar into four equal parts.

He then used the computer actions to partition the first part on the left into three equal parts. The resulting bar consisted of six unequal parts that we would regard as 3 one-twelfths and 3 one-fourths (p. 25).

Tim was asked what fraction ‘the small piece below the original bar’ represents (p. 25) (see Figure 2.6).



Feedback in this case is the interrelationships between the sizes of each part. The feedback is constructed from the ways in which JavaBars are designed. Olive & Vomvoridi noted that:

Tim's concept of a unit fraction was based primarily on the number of parts that were present (regardless of size) and did not take into consideration the part-to-whole relation of one part to a referent whole (p. 25).

For Tim the feedback from the JavaBars was not useful, hence he concluded that the little piece showed the fraction of $1/6$. The following piece of transcript extracted from this study shows Tim's interaction:

- Interviewer: O.K. So any idea what fraction this is of the unit bar [pointing to the small $1/12$ -piece that Tim pulled out of the bar]?
- Tim: One seventh? Is it one seventh?
- Interviewer: Do you think one seventh? Why do you think one seventh?
- Tim: Cause there's seven in all and we pulled one out.
- Interviewer: Show me the seven that you have.
- Tim: [Pointing to each of the parts in his partitioned bar] one, two, three, four, five, six. Oh six! One sixth (p. 25).

Tim's not-so-clear knowing of fractions led him to not be able to see any usefulness in the feedback, as he said the little piece underneath the bar represented 'One sixth'. Although one might argue that the feedback from the JavaBars was designed in the tool, Tim's not-so-clear knowing of fractions prevented him from seeing the interrelation between the sizes of different parts in the bar, that is that the sizes of the pieces are not equal so they

can not be expressed in a fraction from.

Situating My View of Tools and Other-ness in the Literature and Some Examples

So far in this chapter, I have explored the role of mathematical tools from a variety of angles: as tools in their own right as well as their role in fostering children's mathematical thinking, in teaching mathematical concepts and in promoting the communication and social interactions associated with their use in classrooms. Although, over decades, researchers were eager to demonstrate the effectiveness of mathematical tools, their conceptualisation of these tools were influenced by and were recursively related to the underlying theoretical frameworks, and more specifically, the influence of the work of neo-Piagetian (radical) constructivists and Vygotsky-influenced socio-cultural theories.

In general, Vygotskians saw mathematical tools as materials which children use to negotiate a *shared meaning* within the class community (Bartolini Bussi & Mariotti, 2008; van Oers, 1996), putting emphasis on the notion of 'shared meaning' through the social use of mathematical tools as children interact with the tools to work on a task, in order to talk about the tools, the task or to think about them. Furthermore, they conceptualised the use of mathematical tools in terms of "participation" in a socially organised activity. Whereas, constructivists viewed mathematical tools as materials to be "manipulated" to construct individual knowing. In particular, my reading of studies guided by the (radical) constructivist view raised a concern. What was strikingly missing from close to 54 constructivist studies that I reviewed (such as Abrahamson et al., 2011; Hackenberg & Tillema, 2009; Hackenberg, 2010; Steffe & Kieren, 1994) was that they considered children's interaction with the tools as 'independent work', without considering the social origin of the (mathematical) tools. By social origin, I refer to the designer/creator's thought that is manifested in the design of the tools that later becomes part of the child's social interaction with them. For example, Steffe (2004), in one of his studies, analysed Jason's interaction with the tool to explain that 'he could *independently* engage in the operations that were necessary to produce such explanations or actions, and beyond that he could *independently* produce a unit fraction that was commensurate with, say, three fifteenths. I consider such *independent* productions as necessary in order to judge that a child has constructed a commensurate fractional schema' (p. 159). Steffe viewed Jason's interaction with the mathematical tools as 'independent'; hence his argument that the child is constructing knowledge 'independently' aligned with the radical constructivists' view of knowledge construction. However, Steffe's statement –

referring to the ‘independent’ engagement of the child with the tools may be revisited, taking into consideration the social origin of the material. In consequence, the unanswered question will be: How is the child ‘independently’ constructing knowledge if he/she is involved in a socially originated and socially organised interaction with the mathematical tools?

I argue that radical constructivists’ view of ‘independent interaction of children with the tools stems from the ways in which they view ‘others’ and the role of others in children’s learning. In the following sections, I briefly look at how these ‘others’ are conceptualised from a Vygotskian perspective, from constructivists’ view and finally my own view of others.

Before moving on, I shall emphasise that I see a clear distinction between constructivism – of any kind, including social constructivism – and Vygotsky’s view of learning. My main concern resides on the term “construction”, which I believe by its nature, entails an individualistic view to learning which Vygotsky’s view of ‘others’ - for example in the more knowledgeable other - is in contrast with.

Bruner (1996), in a talk he gave at a conference marking the centenary of the birth of both Vygotsky and Piaget in Geneva, clarifies that constructivism draws on an individualistic philosophy which is quite distinct from that drawn on by Vygotsky. He explains that in Piaget’s view, the world cannot be known directly. Our learning and knowing of the world is constructed with the mediation of logical operations of assimilation and accommodation. Our knowing is then a *construction*: a construction that its viability can be tested through interactions with others. While Vygotsky, like Piaget, argued for mediated form of learning, he did not think of knowing and learning in terms of an expression of logical operations (Bruner, 1996). Learning in Vygotsky’s view expresses itself in interaction with others. That is why the notion of the ZPD is so central to Vygotsky’s theory. Bruner (1996) views the ZPD as a zone in which culture gets internalised by the mediation of others.

Therefore, I believe that Vygotsky was not a (social) constructivist as to him our knowing is not individually constructed. On the contrary he claimed that learning is only possible in our interactions with others. Now, to make my point about the role of ‘others’ in children’s learning explicit, I briefly compare and contrast the conceptualisation of ‘others’ from Vygotsky’s, radical constructivists’ and social constructivists’ views. Then, I explain my own view about ‘other’-ness.

Different conceptualisations of the role of others. In my view, what was fundamentally different in the Vygotskian view with those of the constructivists was their perspective of the role of ‘others’ in humans learning in general and in children’s learning in

particular. In the following few paragraphs I will briefly explain this difference.

Vygotskian's conceptualisations of 'Others'. Interpreters of Vygotsky have made it clear that others play a major role in children's learning. Rogoff (1990) recognized this significance by stating that children's cognitive development is understood only as taking place with social support in interaction with others, and in the presence of 'sociohistorically developed tools that mediate intellectual activity' (p. 35). Van Oers (1996) argued that:

One of the basic tenets of the Vygotskian approach to education is the assumption that individual learning is dependent on the social interaction. However it should be clear from the outset that this is not merely a statement of correlation between individual learning and social context [with others]. This thesis should be interpreted in its strongest possible form, proposing that qualities of thinking are actually generated by the organizational features of the social interaction (p. 2).

Wertsch referred to Vygotsky's theory to conclude that rather than deriving explanations of psychological activity from the individual's character and then adding the secondary social influences, scholars should focus on 'the social unit of activity and regard individual higher cognitive processing as derived from that' (Wertsch, 1985, p. 35). Lerman (1989, 1993, 1994, 1996, 2006), using Vygotsky's notion of social interaction, describes that what things signify is learned by us as we grow with others into our cultures (1996) and with the help of others, in the strongest form as mentioned by Van Oers.

Radical constructivists' conceptualization of 'Others'. The principles of radical constructivism are formed based on Piaget's perspective, but are carried further, in several steps, by von Glasersfeld (1994, 1990, 1989). Neo-Piagetian radical constructivists at no point in their examination of the process of learning discussed the contribution of 'others.' The radical constructivists' view states that even though the individual shares and negotiates information from others as part of the learning environment, ultimately we learn individually (Thomas, 1994; Steffe & Thompson, 2000; Steffe & Kieren 1994; Steffe & Tzur, 1994). Radical constructivists consider the role of others only alongside other sensorimotor materials in the learning of an individual. Thomas proposed that 'to know a piece of world is to have a viable notion of that piece of the world, viability can be achieved through comparison of one's knowledge with others' (2003, p. 43). As is made clear in Thomas's proposal, to radical constructivists, others play a role, but not necessarily in the process of learning, but in the process of checking the viability of the end product, which is the newly formed concept or

activity. These ‘others’ play an important role in the stabilization and solidification of our experiential reality by providing us with second-order viability. Von Glasersfeld (1995) defined this viability as a state of affairs where the conceptual structures of others are thought to have at least some of the knowledge that we ourselves have found to be viable in our dealings with experiences.

Social constructivists’ conceptualization of ‘Others’. One attempt to address the issue of the downplayed role of social interaction (the role of others) was suggested by constructivist scholars including Confery (1991), Cobb (1994a, 1994b, 1989), Ernest (1992, 1994, 2010), Davis & Sumara (2003, 2002) and Cobb et al., (1992). The attempt of these scholars was to incorporate social interaction, *as viewed by Vygotsky*, into an overall constructivist position. Cobb (1994b) explained his work as having been influenced to some extent by Vygotsky’s analysis of the role that Others play in learning. Cobb (1994) then justified his claim by saying: ‘Like Piaget, Vygotsky views learners as active organizers of their experiences but, in contrast, he emphasizes the social and cultural dimensions of development [...] Social interaction therefore constitutes a crucial source of opportunities to learn mathematics in that the process of constructing mathematical knowledge involves cognitive conflict, reflection, and active cognitive reorganization. As such, mathematical learning is, from our perspective, an interactive as well as constructive activity’ (p. 127).

The amalgamation of social interactions – *as viewed by Vygotsky* – into the constructivists’ perspective created a wide range of reactions within the mathematics-education community. The reaction covered the spectrum from necessary (Sfard, 1998), to productive and complementary (Confery, Cobb, Ernest) and to highly incoherent (Lerman). The incoherence that Lerman described is a reaction to the social constructivists’ attempt to place a much greater emphasis on the social context. He argued (1989) that ‘as long as individual is at the heart of the process [the construction of knowledge] any social interaction is itself interpreted individually’ (p. 44). To Lerman this is in disagreement with Vygotsky’s theory. Vygotskians do not assume strong emphasis or less emphasis on social interactions: to them learning is not possible without the presence of others.

As shown with above-mentioned argument, from the perspective of Vygotsky and his followers, a child's learning is impossible without his or her interaction with others, whereas, for constructivists, social interaction is useful in child development because it provides a base for viable adaptation. Constructivists’ view of “social interaction” makes it possible to have greater emphasis (i.e., social constructivism), or less emphasis (i.e., radical constructivism)

on social interactions. However, a Vygotskian view involves more than just an “emphasis” on social interaction. From this perspective, learning is a *product* of social interaction.

My conceptualization of ‘Others’. In my opinion, which stems from Vygotsky’s view, learning is impossible without the presence of mediating agents and (more knowledgeable) others. I do not see others only useful enough to provide me with a second order viability to my thoughts and actions. On the contrary, I merely learn as I interact with others – read others, listen to others, and talk to others. That is, by no means, I am at the heart of my process of learning. I, as a person, am an amalgam and extension of all the ‘others’ that I have interacted with during my history of existence as a being. I always have the other-ness of others present in my thoughts and in my actions (even if I am unconscious about it).

In summary, I view tools as socially rooted and originated and I recognise children’s interactions with tools as socially generated and organised experience. Moreover, I view others as non-separate-able parts of the learning process. My perspective with regards to the role of the other in learning as well as the role of tools in interactions of children with tools is more compatible with the Vygotskian perspective and not the constructivists (neither radical, nor social).

Summary

In this chapter, I examined different conceptualisations of mathematical tools in terms of their benefits and challenges. Mathematical tools are conceptualised as being effective for learners. Not only, are tools thought to help children expand and communicate their mathematical knowing, they also are thought to help them see the relationship between mathematical concepts. Mathematical tools are also useful because they tend to reconstruct the mathematical tasks into different forms and types of tasks. Another benefit of mathematical tools is the way in which they promote communication and social interactions in classrooms.

Mathematical tools, however, are not always useful and effective. One challenge associated with the use of mathematical tools is that children can have difficulties seeing the mathematical concept that the tools are intended to represent. This makes the mathematical tools useful for some children (ones who see the mathematics in the tools) and not so useful for others. To address this challenge, tools can be thought of in two distinct ways (McNeil & Uttal, 2009): as an object in its own right; and as a representation of something else (i.e., mathematical concepts).

All these views played an important role in how I conceived and designed my study. In particular, the ways in which the challenges of mathematical tools were looked at and were addressed led me to view mathematical tools as *any tool-like thing* that children interact with to think about mathematics, to think about their actions, and to solve a mathematical task. As a result, as part of my design of the activities for the interviews, I used different things such as ribbons, scotch tapes, and cardboard to investigate how children interact with these tools to think about and or to solve addition of fractions problems. Furthermore, the findings of these studies helped me apprehend that a tool only becomes a mathematical tool if a child sees their mathematical affordances, in relation to solving a mathematical task.

Although these studies conceptualise the effectiveness (or not) of the mathematical tools, they do not entirely look at the role that is played by the physical properties and design of the tools (i.e., their feedback) in thinking about mathematics or in solving mathematical problems. This points to a need to see what is (or is not) in the tools – in terms of their physical properties – that might support some mathematical activities but not others and for some children and not others.

In the next chapter, I explain the theoretical framework of this study, in which I employed Gibson's theory of affordances and the socio-cultural view of Vygotsky to look at the role of the physical properties of tools in children's learning during work on addition of fractions problems.

Chapter 3

Theoretical Framework – Artefacts and Their Possible Role in the Emergence of the ZPD

In this chapter, I explain how Vygotsky's perspective underpinned the theoretical base of my study.

From a Vygotskian perspective, I consider the child's interactions with mathematical tools to be *dialectic* and *mediated*. They are *mediated* in the sense that instead of acting directly, in unmediated ways in a social and physical world, our actions are indirect and mediated by *tools* and *signs* (Cole, 1990; Moll, 1992; Vygotsky, 1978; Wertsch, 1984; Wertsch, 1985). The child's interactions are *dialectic* in the sense that the child's external actions modify the mathematical tool, and the newly modified tool modifies his/her mathematical thinking. By modifying, I mean making changes: such as, by moving fraction circles pieces into a new arrangement.

For example, in Olive & Vomvoridi's (2006) study, Tim modified the arrangements of the pieces of fraction circles as he 'attempted to make a circle using the green sectors' (p. 26). He then found that one sector was missing, and thus he had an incomplete disk with five green sectors (see Figure 3.1).

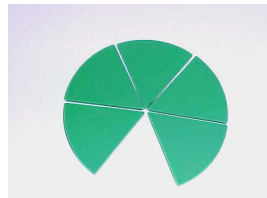


Figure 3.1. Fraction circles of 1/6ths with a missing piece

When asked what fraction of the circle was missing, his mathematical thinking was modified as he created the following artefact sign: 'One fifth, just right there, but we're missing one' (p. 26). Such as the case for Tim, in the children's interaction with the mathematical tools, changes to mathematical thinking is not always *useful*. By useful I mean acceptable in a local culture of the mathematics subject.

In this study, I investigated whether and how feedback from mathematical tools could provide guidance for the useful modification of the tools and of the mathematical thinking and learning of children. My two research questions are:

1. How does the feedback from the mathematical tools play a mediating role between the physical actions of the child with respect to the mathematical affordances of the tools and the child's thinking about and learning and knowing of the addition of fractions?
2. What role is played by mathematical tools in the emergence of a Zone of Proximal Development during the child's solving of addition of fractions problems?

In order to systematically analyse the role of feedback in the dialectic process of the children's interaction with tools, I looked at: tools as physical 'things'; the dialectic process of children's modifications to the mathematical tool and the modifications to their mathematical thinking and ways of problem solving; and children's learning during their work on the addition of fractions problems. In order to explore tools as physical 'things', I used Gibson's view of affordances. To investigate the dialectic modifications to the tools and to the children's thinking as they interact with the tools, I used a Vygotskian perspective on the tie between tools and signs as well as Vygotsky's perspective on perception. To investigate children's learning I focused on the emergence of the Zone of Proximal Development

In the following sections, I make the rationale for my theoretical selections explicit. I first explain how I see affordances of the tools as physical things. Then, I discuss the relationships among the tools, the signs and the mathematical tasks. After, I discuss Vygotsky's notion of the object/meaning ratio, to see how children perceive different meanings (mathematical or otherwise) in their interactions with the tools. I conclude by explaining how all the elements (i.e., affordances, signs and tools, perceptions, and meanings) fit within the ZPD.

Tools as 'Things' and Their Affordances

Concerned with how the environment supports thinking and action, Gibson (1977) contended that 'in any interaction involving an agent with some other system, conditions that enable that interaction include some properties of the agent along with some properties of the other system' (p. 68). For my study, the 'system' was the mathematical tool and the agents were the children. Therefore, using Gibson's (1977) analogy, my reference to the term *affordance* would be to whatever it was about the mathematical tool that contributed to the kind of interaction that happened. In consequence, I referred to *perception* as whatever it was

about the child's thinking/knowing (of the tools and the addition of fractions) that contributed to the kind of interaction that happened.

In order to conduct a careful analysis of a child's interaction with the mathematical tools, it was crucial for me to consider where to locate the reference for the term 'affordance'. For example, is the affordance that fraction circles provide for showing half of a unit (or $1/2$) a property of the fraction circles, a property of the child interacting with them or a property of something else? Gibson (1977) argued that an affordance is the specific property of the tools that interacts with the perception of the child in such a way that an activity can be supported. The added dimension connecting the properties of the tool (i.e., affordances) to the properties of the child (i.e., perception) was the 'activity' – that is, what the child was doing with the tool. Thus, the child's perceived affordances of the mathematical tools could not be separated from the mathematical task at hand. For example, the physical properties of fraction circles may provide affordances for a child to build a bridge. If the task at hand is to build a bridge then the perceived affordances are useful. Yet, if the task is to add $1/3$ and $1/2$, then the perceived affordances (i.e., making a bridge) are not useful. That is, the useful (or not useful) affordances of a mathematical tool depend not only on the child's perception but also on the task.

The above argument also implies that interacting with an environment that provides an affordance for some activity does not entail that the activity will happen; the occurrence of the activity is intertwined with the perception of the person who is interacting with the tool and the task (i.e., what he/she is doing with the tool). To look at children's perceptions of the affordances of the tools, I start by explaining how children attach signs to tools as they interact with the tools to solve a mathematical task.

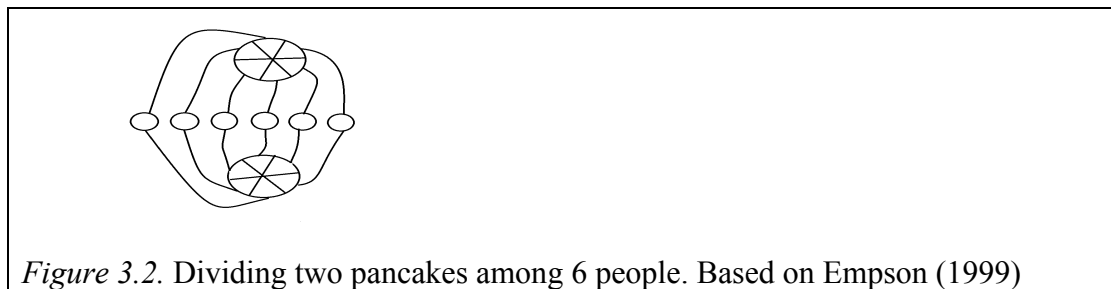
Tools, Signs and Artefacts

Two important assumptions by Vygotsky provide the theoretical foundation of my study: 1) a child's development and learning depend on the presence of the mediating agents of tools and signs; and 2) the tie between tools and signs creates artefacts.

The notion of tool I refer to is based on Marx's view of working tools whereby that man uses the physical and mechanical properties of objects to reach his goals (Marx & Engels, 1865). The Marxian view of tool was then extended by Vygotsky as a means of external activity (i.e., labour) with which humans influence the environment. Hammers, nails and chairs are examples of tools. Signs, on the other hand, are means of internal activity that

affect humans internally. Languages, various systems for counting, and algebraic symbol systems are examples of signs. For the purpose of this study, I consider both x^2 and an abacus as mathematical objects, though an abacus is a mathematical tool and x^2 is a mathematical sign.

Vygotsky (1978) believed that an essential difference between signs and tools are the ways in which they orient human behaviour. A tool's function is *externally* oriented: 'It is a means by which human external activity is aimed at mastering nature' (p. 55). A sign, on the other hand, is *internally* oriented: 'It is a means of internal activity aimed at mastering oneself' (p. 55). In order to better explain the difference between signs and tools, I use the drawing extracted from Empson's (1999) study and set out below (see Figure 3.2).



A child has used a tool (i.e., a pencil) to manipulate an object in the environment (paper, in this case). This action is externally oriented. However, the drawing is to produce an internalization of a mathematical concept to show dividing two pancakes between 6 people. This action is internally oriented. The pencil is a tool and the drawing is a sign.

From this perspective, mathematical tools are considered 'tools' when the focus is on their physical properties. At this point, the realization of the affordances of the tools openly depends on the child's perception. Children can perceive the affordances of the tools as a doorstopper, as being useful to build a paper airplane, or as being suitable to be sat on as a chair; they can equally be perceived as being useful for working on a mathematical problem. For example, the affordances provided by the sizes and shapes of the Cuisenaire rods might be perceived as appropriate to build a train, or to find a piece that is $\frac{1}{3}$ of the dark green piece (i.e., the red piece). Another example could be a pencil; where its writing affordance can be perceived to draw the sun or equally its length affordance can be perceived to measure the side of a table.

It is in conjunction with the child's solving of a mathematical problem that the

mathematical tool acquires a new meaning and form. The mathematical tools acquire a new form because it is in solving a mathematical task that the child needs to perceive mathematical affordances provided by the tools. These affordances become apparent to the child by the help of more knowledgeable others. The mathematical tools acquire new meaning because it is in solving a mathematical task that mathematical tools are tied to the signs – especially in talking about the task, drawing and using symbols (Bartolini Bussi & Mariotti, 2008; Bartolini Bussi, 2011; Arzarello, 2010).

Focusing on how mathematical tools acquire a new meaning, I introduce Vygotsky's notion of artefact (1978). An artefact is created when tools are tied to the signs. Hence, it is in solving/working on a task that mathematical tools become mathematical artefacts. I use the example of a drawing to show how, in the process of solving a mathematical task, the tie between signs and tools creates an artefact. I borrowed the drawing in Figure 7 from Empson's (1999) study in which Cameron divided two pancakes among six people. Cameron used mathematical tools (paper and pencil) and signs (lines and circles) to create a figure. It was the task (i.e., sharing pancakes) that gave this drawing a new form. This drawing does not merely represent lines and circles; it represents two pancakes that are being divided among six people. Moreover, it was the task that gave new meaning to the tie between the tools (paper and pen) and the signs (circles and lines). The tie between the sign and the tools created an artefact (see Figure 3.2) that Cameron used to create the following artefact sign: 'I divided it into six. And I gave them each a sixth from that and then I had the other one to do the same thing with, so I did the same thing with the other one' (Empson, 1999, p. 315). This example shows how Cameron created an artefact by attaching mathematical signs to his use of tools.

The creation of a tie between the tools and the signs is an extremely complex process, which is highly intertwined with how children perceive mathematical affordance of the tools in relation to the task. In this study, the process by which the child ties signs and tools is of particular interest to me because I hypothesise that it is through this process that the ZPD emerges. In the following, I look more closely into the interrelationships among the tools as physical things, the perceived affordances of the tool and the task. Assuming the tasks in this study are adding two fractions, I shall look at how children perceive affordances of the tools. Vygotsky's notions of the Object/Meaning ratio offers a systematic approach to looking at the gradual yet complex process of change in the child's perception as she/he interacts with a mathematical tool to make meanings for the tool as well as for the mathematical concept.

Object/Meaning Ratio

A special feature of human perception is the perception of real objects (Vygotsky, 1978). This involves the perception not only of colours and shapes but also of meaning; we do not see a round object with two hands we see a 'clock'. The attachment of meaning to an object is a process that develops through the use of signs in interactions with the object. To better explain this process I use Vygotsky's object/meaning ratio. I start with a short account of a child's development in Vygotskian theory because it helps to clarify my discussion.

Vygotsky argued that at first the perception of a human being could be expressed figuratively as a ratio in which the numerator is the object and the meaning is denominator – object/meaning. This ratio shows that for a young child, the object is dominant and the meaning of the object is subordinate. At this stage the physical properties of things play an important role in a child's interaction with them. For instance, a stick can be a horse but a postcard cannot be a horse. It is only later, when the child can make use of signs and symbols in her/his interaction with the objects, that the meaning becomes the central point and objects are moved from being dominant to being subordinate, thus giving rise to the meaning/object ratio. At this stage, Vygotsky noted that, for example, to show the location of a horse on a map, a child could put a box of matches down and say, 'This is a horse'. The perception of the child can now be expressed as meaning/object. This figure of perception in which the meaning dominates is the result of tying signs to the tools; the box of matches is a symbol (sign) to represent the horse.

Vygotsky's object/meaning view has a two-fold theoretical implication for the study of children's perceptions as they interact with mathematical tools to solve a task. First, it provides a base for analysing the gradual changes of the child's perception of the affordances of the tools as the child interacts with the tools to work on a mathematical activity. Second, it provides a base for analysing the gradual changes of the child's perception of the interrelationship between the tools and the mathematical meaning they are intending to represent.

Provided that the child is interacting with a mathematical tool to solve a mathematical task, at the initial stage of the child's encounter with the mathematical tool, the child's perception can be presented by the object/meaning. This figure of perception applies to how the child perceives the affordances of the tool (i.e., the meaning of the tools as an object in relation to the task at hand) as well as how the child perceives the mathematics represented by the tool (i.e., the mathematical meaning). At this stage, the tool is dominant and its

meaning(s) – as an object or its mathematical meaning – is subordinate. Hence, this is the stage of which the physical properties of the mathematical tool play an important role in the child's interaction with them, both to perceive the affordances provided by the tool and to perceive the mathematics represented by the tool. For example, in fraction circles, the relationship between the sizes as well as the colours of the pieces plays an important role in how the child perceives the affordances of the fraction circles and the fractions.

In order to invert this ratio, that is, in order for a tool to be used as a symbol (a sign) for a mathematical concept, the child needs to increasingly tie signs to their use of the tool. Children do this by talking about what they do, talking about the tasks, drawing, and using mathematical symbols. It is in the gradual process of inverting the object/meaning ratio to a meaning/object ratio that children grasp the interrelationship between the affordances of mathematical tools and the meaning of the mathematical concepts that they are intended to represent.

I argue that it is in the process of inverting object/meaning to meaning/object that mathematical tools *gradually* become mathematical artefacts as they are tied to signs. I emphasise the word 'gradually' because, as I mentioned above, it is often difficult for children to grasp the relationship between mathematical tools and the mathematical concepts that they are intended to represent (McNeil & Uttal, 2009; Norman, 1993; Rabardel & Samurcay, 2001). I also argue that it is when object/meaning is inverted to meaning/object that the child can use any object (i.e., apple) as a mathematical tool.

Through an example, I make my own view of object/meaning → meaning/object more explicit. Again, I use the case of Teresa in Pirie and Kieren's (1989) study, mentioned in chapter 2, where she began the task of adding two fractions, not knowing what to do. She then was given fractions kits and 'Using her kit she noticed that one fourth, three eighths, and two sixteenths together exactly cover three fourths' (Pirie & Kieren, 1989, p. 163). At this stage, Teresa's perception can be expressed figuratively by object/meaning, where the physical properties of the pieces of the fractions kit were dominant in Teresa's interaction with the kit and the meaning of the addition of fractions was subordinate. Later, Teresa could 'add' $\frac{1}{3} + \frac{1}{6} + \frac{6}{12}$ using the kit. After a while, she was able to tie signs to her interaction with fractions kit: 'You can do $\frac{2}{3} + \frac{5}{6}$ because twelfths fit on both' (Pirie & Kieren, 1989, p. 167). This figure of perception can be expressed as meaning/object when the meaning of the addition of fraction dominates and the properties of the tools are subordinate. Later, when asked 'What is $\frac{1}{2} + \frac{3}{4} + \frac{2}{5} + \frac{7}{10}$?' Teresa, without using the kit, said: 'Twentieths will

fit on all of them. Two times ten makes twenty, so one times ten or ten twentieths. Four times five makes twenty so three times five is fifteen twentieths.’ (Pirie & Kieren, 1989, p. 169).

This example shows how the object/meaning perception of Teresa (where she used the fractions kit to solve the task) was inverted to become meaning/object (where she used signs and symbols to solve the task). In this process, Teresa increasingly used signs, while interacting with the fractions kits.

Teresa’s case is by no means an isolated one in the literature. I noted similar tool/sign interactions in many other studies (e.g., Mack, 2001; Olive & Steffe, 2002). What is not specifically addressed in any of these studies is the role of feedback from the mathematical tools in the child’s interaction with the mathematical tool (tying the tools to the signs) to solve a mathematical problem. This is the main focus of my study.

The Emergence of the Zone of Proximal Development

My Vygotskian view of learning assumes that children learn through sign mediation in activity; ‘*learning-leading-development* is a consequence of *sign mediation*’ (Vygotsky, 1978, p. 200). The second half of this statement (i.e., sign mediation) is made explicit in my analysis of the object/meaning ratio in which I explained that the child, to different degrees, uses signs in her interaction with the mathematical tools. Now, I emphasise the first half of the above quotation: ‘*learning-leading-development*’.

Vygotsky (1978) argued that ‘properly organized learning results in development’ (p. 68). To clarify this statement, I use Teresa’s case once again (see above). Pirie and Kieren’s (1989) study showed Teresa’s developmental process – from the point when she started using the fractions kit to solve the addition-of-fractions problems to the point when she could perform the task without the kit. Teresa’s development was a result of an organised series of learning. Teresa learnt how the sizes of the pieces in the fractions kit were related to one another. She learnt that different pieces could be put together to create a new piece (a new fraction). She learnt that $\frac{1}{12}$ fits on both $\frac{2}{3}$ and $\frac{5}{6}$. From a Vygotskian perspective, Teresa’s development can be viewed as learning-leading-development. In this study, I was interested in seeing the learning that happens as children interact with the mathematical tools. I investigated this interaction within the ZPD. Lerman and Meira (2001) stated that ‘the ZPD is not something that pre-exists; it is not carried around, like a box, by the child’ (p. 203). The ZPD emerges as a field for ‘interaction and communication where learning leads development’ (p. 204). How then does one recognize the emergence of the ZPD?

When a child participates in interactions with mathematical tools to solve a mathematical task, there is a systematic relationship between what he/she does, what he/she says and the knowing that takes place (the knowing of mathematics or the knowing of the tools). Roth and Radford (2010) conceptualised *knowing* as ‘the possibilities that become available to the participants for thinking, reflecting, arguing, and acting in a certain historically contingent cultural practice’ (p. 301). For example, in the case of my study, the ‘certain historically contingent cultural practice’ is the methods of adding two fractions and consequently, children would know this cultural practice if they can reflect, argue and act accordingly. But opportunities for thinking, reflecting and acting do not just happen on their own. Knowing happens through *learning* – what Radford (2013) referred to as a social and sign-mediated process of becoming acquainted with historical and cultural forms of expression, action and reflection. Hence, one possible way of looking at children’s learning is to look at the ways in which children express, reflect and act change as they participate in an interaction (among themselves, with the teachers, or with the tools).

How does learning happen? For Lerman (2014), the zone of proximal development is ‘the mechanism through which learning happens’ (p. 22). Vygotsky (1978) proposes that a fundamental feature of learning is that it creates the zone of proximal development because ‘learning awakens a variety of internal developmental processes that are able to operate only when the child is interacting with people in his environment’ (p. 67).

To look at the children’s participation in interactions with the tools and their consequent learning within the zone of proximal development, I use and extend the conceptualisations of Wertsch (1993) and of Roth and Radford (2010). Roth and Radford (2010) saw the ZPD as ‘the emergence of a new form of collective consciousness, something that cannot be achieved if we act in solitary fashion’ (p. 306). Wertsch’s (1993) interpretation of the ZPD pointed to the possibility of learning through the process of collaboration among individuals who deliberately interact to accomplish a goal. In both perspectives, the ‘more knowledgeable other’ is not viewed in terms of its institutional and/or societal position(s) (e.g., teachers versus students, or parents versus children). Roth and Radford’s (2010) notion of the participant ‘in the know’ – the one whom Wertsch (1993) called ‘the source of authority’ – claims that *the more knowledgeable other* arises through collaborative interaction of the participants in which the role of being the more knowledgeable other alternates among them.

I illustrate these discussions with an example from Empson’s (1999) study. I realize,

however, that the following example is only a snapshot of Jonathan's social experience. Sally, Jonathan and Ms. K were discussing the relationship between two eighths and a quarter:

Jonathan: But two eighths make a quarter.

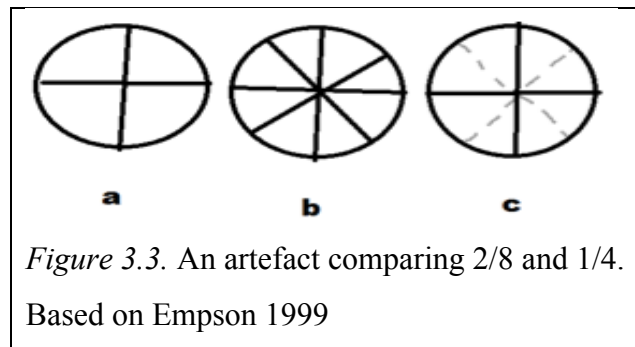
Ms. K: Why? [to Jonathan] Why do you think two eighths make a quarter?

Jonathan: Because eighths are pretty small, and quarters are kind of small, so eighths, if you put them together, you would get a quarter.

Sally: See, look. Here's a fourth (drawing Figure 3.3a)

Sally: (drawing Figure 3.3b) Here's an eighth. And you want to make a fourth, right?

All you have to do is erase four [erases the four lines that halve each fourth]. All you have to do is erase the four lines, to get to fourths, eighths can turn into fourths (see Figure 3.3c).' (p. 321)



As Empson noted:

After Sally's explanation, Ms. Kolan demonstrated equivalence of [two eighths to one fourth], by cutting up giant paper cut outs [...]. At each step, she prompted children to tell her what she was doing or asked a child to remind the group of the conjectures they were checking (p. 321).

Jonathan's initial explanation of the relationship between $2/8$ and $1/4$ might show his level of potential development. Based on Vygotsky's theory, guidance, such as that provided by Sally and Ms. Kolan, could help Jonathan, over time, to do the following: participate in a socially shared experiences, learn the concept and, consequently, solve similar problems independently (i.e., the level of actual development). The above examples show that, from Vygotsky's perspective, learning is impossible without the presence of more knowledgeable

others.

In the course of children's interaction with the tool, tool-mediated learning sometimes occurs. For mathematical learning to be tool-mediated, a child needs to perceive the mathematical affordances of the tools with respect to his/her mathematical knowing and with respect to the task at hand. Within the ZPD, this learning happens under the guidance of the more knowledgeable other(s). In this research, I looked at the mathematical tools as being the more knowledgeable other(s). That is, in looking at children's interaction with the mathematical tools my unit of analysis would be: *the child* (with perception of the tools and his/her knowing of fractions) + the physical properties of *the tool* + *the task* of solving an addition of fractions problem, as I see them highly intertwined.

Summary

In this chapter, I organised a theoretical framework with which to look at the complex interrelationships between children's perceptions, affordances of tools, and the possible learning that happens as children interact with tools to solve a mathematical task. To look at children's perceptions, I used Vygotsky's notion of the object/meaning ratio. To look at the affordances of tools, I used Gibson's theory of affordances. I used the notion of artefact to explain how children attached signs to their use of tools. Finally to look at learning, I used Vygotsky's concept of the ZPD. In the next chapter, I explain how my theoretical framework created a solid base for the design of my study.

Chapter 4

Methodology

In this chapter, I provide detailed information about the methodology of my study. To give a general context of my methodological choices, I begin by giving an overview of the whys and hows of the data collection method and the manner in which recruitment took place, and then describe the study's participants, as well as the procedures by which I collected data. I also position myself in the study. I then provide a detailed description of the design of the instruments used in this study and address the data analysis methods. I conclude this chapter by highlighting the rationale for some of the choices I made to analyse my data.

Research Design

This study involved interviewing 13 grade 7 children. The children were interviewed in teams of two and each team participated in three rounds of interviews. In each round of interviews, I used different mathematical tools and different mathematical tasks. In the first round of interviews, the children interacted with Cuisenaire rods to work on fractions. This round was designed to help the children play with Cuisenaire rods and to get better acquainted with different properties of the tools in relation to unit and unit fractions. The second and third rounds of interviews were designed so that the children could interact with specific types of tools, to think about and solve addition of fraction problems. In the second round of interviews, the children used tools that have some fraction related properties already designed into them. These tools include Cuisenaire rods, fraction strips and fraction board. In the third round of interviews, the children used tools that did not have any fraction related properties incorporated in their design, such as, papers, adhesive and masking tapes and ribbons. Data collected from the second and the third interviews was then analysed to answer both research questions.

The Process of Recruiting the Participants

The research for this study occurred during the 2014-2015 academic school year. It took place in the secondary section of a private school in the Ottawa area, in which children were mostly from a particular ethnic background. In late October 2014, I contacted the school's principal to ask whether it would be possible for me to conduct my research in the school. Through an email, I provided the principal with an overview of the aim of the study as well as details on what would happen during interviews and after the interviews (see the

information letter, Appendix). I met the principal and the grade seven mathematics teacher in early November. In this meeting, we decided that I would observe the grade 7 classes (7A and 7B) to build relationships with the children before commencing the study. I visited the two grade seven classes 2-3 times per week, for a period of 2 months, prior to starting the interviews.

In my first visit I explained to the children that I was a student at the University of Ottawa doing a study on how children use ‘things’ to think about and work on mathematics problems. We brainstormed and talked about what might be some possible ‘things’; the list included ruler, pen, pencil, manipulatives (suggested by the teacher), dice, etc. I then told them that I would be attending their mathematics class for the next little while. In my subsequent visits, in the mathematics lesson periods – where the teacher would teach a lesson – I mostly observed the classes. In mathematics working periods – where students worked on worksheets or on exercises from the textbook – I walked around and answered questions or worked with the children who wanted to work with me. Each visit lasted about 2 hours.

During the time that I visited the school, the teacher started and finished teaching of fractions; including equivalent fractions, adding fractions with common denominators, and adding fractions with denominators not the same. The children were taught the addition of fractions as the following process:

1. multiply the denominators to get the common denominator
2. do whatever you have done to the bottom, to the top
3. add the two fractions.

The children were asked to verbally repeat these steps, any time they were asked to go to the board to solve an addition of fractions problem. The children were then tested on how to add two fractions. All the children in both classes passed their addition of fractions test using the method mentioned above process. I did not observe the children in these two classes using any mathematical tools in their mathematics classes, more specifically in their fraction lessons.

While visiting, I also ran a 50-minute mathematics workshop for each class, in which the children used mathematical tools to solve mathematical tasks. The tasks in this workshop were not related to of fractions. For example, in one station students were asked to use plastic solids to find the relationship between the volume of prisms and the volume of the pyramids.

Participants

As mentioned earlier in chapter 1, in the recent version of the Ontario Mathematics Curriculum for Grades 1 to 8 (2005) the addition of fractions is introduced in grade 7. The curriculum in grade 7 states that by the end of Grade 7, students should be able to:

add and subtract fractions with simple like and unlike denominators, using a variety of tools (e.g., fraction circles, Cuisenaire rods, drawings, calculators) and algorithm (p. 100)

Since the addition of fractions is introduced in grade 7, I looked for children in grade 7.

For all of the children in both grade-seven classes, an invitation of participation and a permission slip were sent home, for both the parents and the children to sign, in early January 2015. I had 13 forms signed and returned by January 14, 2015. All 13 children, who wished to participate and consented, participated in the study. I grouped the children in teams of two (and one team of three) based on the order by which the consent forms were piled up. I started the interviews on January 16th, 2015 and finished the interviews on January 22nd, 2015. Table 4.1 shows how the teams were formed:

Team 1	Z and A
Team 2	A and M
Team 3	A, S and J
Team 4	K and H
Team 5	Y and S
Team 6	F and S
Table 4.1. The teams	

Table 4.2 shows the schedule for the interviews:

Teams/Date	Jan 16 th	Jan 19 th	Jan 20 th	Jan 21	Jan 22
Team 1	Interview 1	Interview 2	Interview 3		
Team 2	Interview 1	Interview 2	Interview 3		
Team 3	Interview 1	Interview 2		Interview 3	
Team 4	Interview 1	Interview 2		Interview 3	
Team 5	Interview 1	Interview 2	Interview 3		Interview 3
Team 6	Interview 1				Interview 2 and 3
Table 4.2. The schedule for the three rounds of interviews for the six teams					

Whys and Hows of Data Collection

All data for this study were collected from 18 task-based interviews. Task-based interviews are interviews in which a child or group of children talk as they work on a

mathematical task or set of tasks (Maher & Sigley, 2014). In the following paragraphs, I describe the rationale for selecting task-based interviews as my mode of data collection and why I found the method useful within the purpose of my study. Then, I provide a detailed account of how I collected my data.

Why Task-Based Interviews?

In terms of gathering data, Brown and Dowling (2010) identified five categories of interest: 1) what people know; 2) what people do; 3) what people think or feel; 4) how people think; and 5) how people learn and construct meaning. They argued that an *interview* is a method that is concerned with the last two interests – that is, how people think and how they construct meaning (Brown & Dowling, 2010). In this study, my main aim was to see how the use of mathematical tools plays a role in how children think about and construct meaning for solving addition of fractions problems. In particular, my goal was to see what children did and said as they interacted with the tools. Hence, I decided to use interviews to have some insight into how children think about and learn/construct meaning in solving addition of fractions problems.

My rationale for choosing task-based interviews was that they provide ways in which children can talk as they work on a set of tasks (Maher & Sigley, 2014). The method of task-based interview was particularly useful for three reasons:

- It gave me insight into children's thinking, learning and problem solving
- It gave me the possibility of designing tasks specific to my study
- It gave me the flexibility to modify my questions and/or the task, during the interviews based on my judgment

Insight into children's thinking and learning. To explore children's learning during work on addition of fractions problems, I used Radford's (2013) conceptualisation of learning as a social and sign-mediated process of becoming acquainted with historical and cultural forms of expression, action, and reflection. I focused on how the children's interaction with the tools provided them with new possibilities for reflecting, arguing and acting; possibilities that were made available to them as they gradually became acquainted with the affordances of and feedback from the tools. I did so by thinking about what the children said as they interacted with the tools.

The following interaction is an example of how I viewed the new possibilities for reflecting, arguing and acting as they became available to the children only by their

interaction with the mathematical tools. In this interaction, children were using fraction strips to add $\frac{1}{2} + \frac{2}{5}$ (see Figure 4.1).

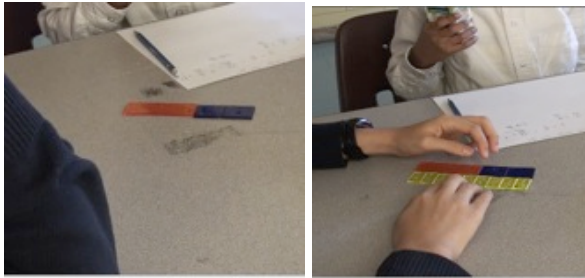


Figure 4.1. Forms of action and expression related to the process of adding two fractions

S: So how much of this would equal to? It equals to one tenth *[picks up a 1/10 strip and puts it underneath the 1/2 strip]*.

Yasmine: Why did you use 1/10? *[pointing to a 1/10 strip]* [...] Why didn't you use this? *[takes a 1/12 piece and puts it on the desk]* Would that work?

S: *[pointing to the 1/5 strip]* We found the least common denominator... The common denominator for both of these is 10.

Later they used the pieces of 1/12 and lined up underneath the pieces of 1/2 and 1/5 and they realised that 1/12 did not cover the 1/5 pieces (see Figure 4.2):

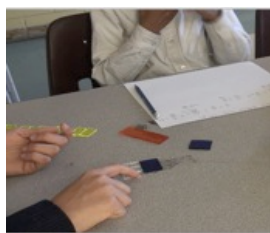


Figure 4.2. Forms of action and expression due to interaction with the tools

S: For one fifth it does not work... You still have that section that is left *[pointing to the uncovered space from the end of 1/12 to the end of 1/5, on the desk]*.

S: It is like half a piece that you cannot fill... There is like *[pointing again to the left-over piece on the desk]* like that little strip *[slides her finger on the table to show the little strip]*.

In this interaction, the children first decided to choose the pieces of 1/10. When asked

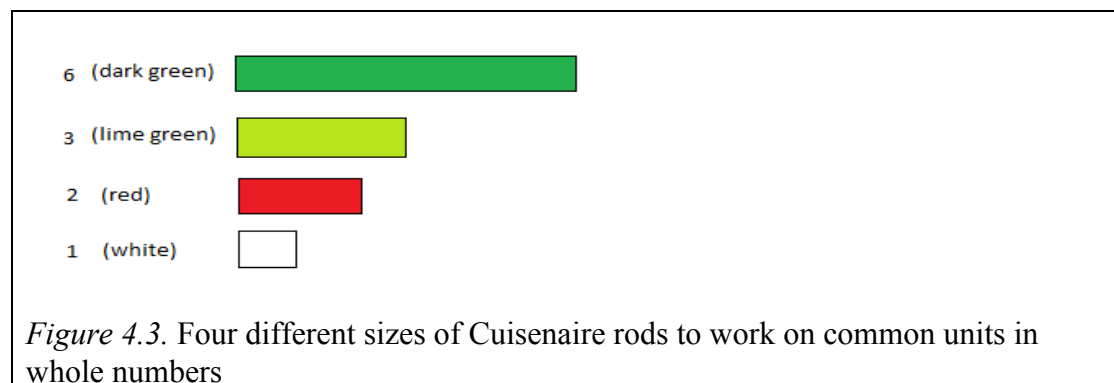
for their reason why they chose the $\frac{1}{10}$ pieces, their argument was that ‘The common denominator for both of these is 10’. Later in the interaction, and prompted by my questions, they used the physical properties of the tools to act, reflect and argue differently, as they said ‘For one fifth it does not work... You still have that section that is left [*pointing to the uncovered space from the end of $\frac{1}{12}$ to the end of $\frac{1}{5}$, on the desk*] or ‘It is like half a piece that you cannot fill... There is like [*pointing again to the left-over piece on the desk*] like that little strip [*slides her finger on the table to show the little strip*].

This example shows how the process of the task-based interviews provided the children with an opportunity to talk about the task of adding two fractions. By examining what they said, then, I could gain insights into the new possibilities for reflecting, arguing and acting that became available to the children as they interacted with the mathematical tools.

Careful construction of tasks. Task-based interviews are designed so that children can interact not only with the interviewer and a small group but also with a task environment that is carefully designed for the purposes of the interview. Hence, a carefully constructed task is a key component of the task-based interview in mathematics education (Maher & Sigley, 2014). In this study, I designed a series of tasks so that in the process of children’s interaction with different types of tools, I could see how they thought about and/or learnt adding two fractions.

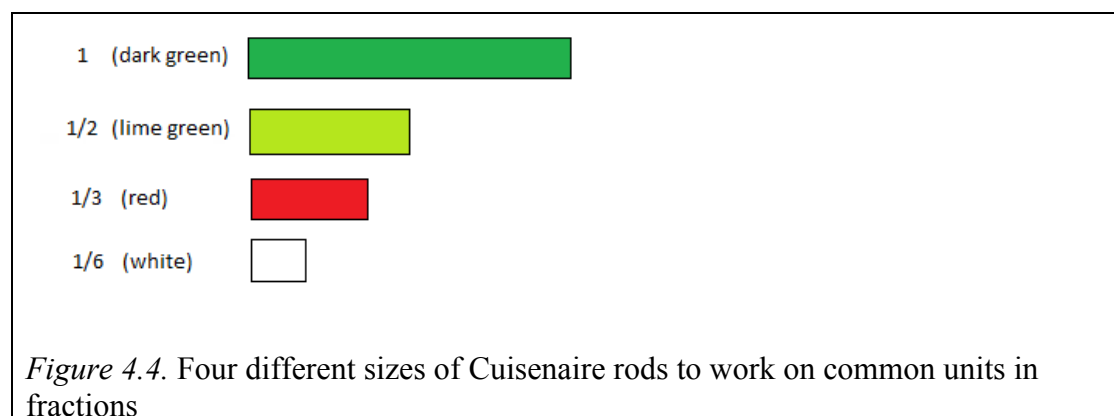
The first interview. In the first round of interviews, children worked on a preliminary task. I designed the preliminary task to emphasise the role of the unit and the unit fraction using the mathematical tool of Cuisenaire Rods. This activity was designed to lead the children to work with mathematical tool and think about the unit fractions. The task had two parts:

Part 1- Use your 4 colours Cuisenaire rods (dark green, lime green, red, and white) and scissors to size and cut ribbons that are 2 rods long (see Figure 4.3).



If both of you follow the same steps, would all your ribbons look the same? Why? Does the unit of ‘two-rod long’ show the differences? What better unit can you think of?

The objective of this activity is to re-emphasise the importance of a common unit in addition with two whole numbers. The second part of this interview was designed to connect the notion of common units in whole numbers to the importance of common units in fractions. In this activity, and as part of the task, I used the same 4 colours of rods – dark green, lime green, red, and white. However, this time I called the dark green a ‘unit’ or ‘one’. The goal of this activity is now to add (or put together) the red rod and lime green rod, considering they were representing fractional numbers (see Figure 4.4).



Due to the multiple-stepped nature of this task, I presented these tasks verbally. I presented the second part when the first part finished.

The second interview. In the second round of interviews children interacted with selected mathematical tools to solve two addition of fractions tasks. The mathematical tools available to the children in the second round of interviews were: Cuisenaire Rods, fraction strips, and the fraction board. For this study, I only selected tools that represent fractions based on the linear model (see page 16). These tools were also selected so that their feedback

– embedded into the properties of their designs – would make the *fractions*-related affordances of the mathematical tools more apparent. I emphasise *fractions-related* feedback because other mathematical tools, such as a ruler, are also designed with the mathematical perception of their designer incorporated into them – in the case of the ruler, its 1 cm incrementing lines on its edge. However, the feedback from a ruler is not designed and might not easily be perceived to make the *fractional* affordances of the ruler more apparent; instead, the feedback is designed to make the measuring affordances of the ruler more apparent. For example, the feedback from the Cuisenaire Rods includes their colours as well as the interrelationship between their sizes. That is, if the orange rod is picked as one whole, then a yellow rod would represent a half (size-wise 2 yellow rods fit on one orange rod) and a red rod would represent $\frac{1}{5}$ (size-wise five red rods fit on one orange rod). The feedback from fraction strips includes both the numerical fractions written on each piece, the colours of strips, as well as the interrelationship between their sizes. For example, two $\frac{1}{2}$ strips fit on one whole strip.

In the second round of interviews students were asked to solve two addition of fractions problems, in as many different ways as they could, using fraction strips, Cuisenaire Rods, and/or the fraction board. They, however, were not asked to use all three tools. This task was designed to look at children's perceptions of the feedback from the mathematical tools as they used different tools to solve different tasks. All the tools were set on the table so that they could choose any tool they wanted. There was no introduction from me as to which tool is what. However, I addressed, as short as possible, the children's questions through out the interviews, if they asked about the functionality of the fraction board.

For the second interview I designed the following task:

Using these tools:

In how many ways can you solve $\frac{1}{2} + \frac{2}{5}$?

In how many ways can you solve $\frac{1}{6} + \frac{2}{5}$?

This task was given to the children on a piece of paper. I read the task to them as I was pointing to the tools that were organized in three different containers and were put on the desk.

Table 4.3 gives a general overview of which tools were used by the children to work on what tasks:

Teams/ Task		Cuisenaire Rods	Fraction strips	fraction board
Team 1	$1/2 + 2/5$		X	
	$1/6 + 2/5$	X		X
Team 2	$1/2 + 2/5$		X	X
	$1/6 + 2/5$			X
Team 3	$1/2 + 2/5$		X	
	$1/6 + 2/5$			X
Team 4	$1/2 + 2/5$	X		
	$1/6 + 2/5$	X		X
Team 5	$1/2 + 2/5$	X		
	$1/6 + 2/5$			X
Team 6	$1/2 + 2/5$		X	
	$1/6 + 2/5$	X		X
Table 4.3. The tools used by each team in the second round of interviews				

As shown in table 4.3, to solve $1/6 + 2/5$, all teams used the fraction board. In addition to the fraction board, one team used the Cuisenaire Rods. To solve $1/2 + 2/5$ two teams used the Cuisenaire rods and three teams used fraction strips.

The third interview. The third round of interviews was designed so that the children used mathematical tools for which the feedback was not designed to make the fractions related affordances more apparent. The purpose of these rounds was for the children to create their own artefacts and construct their own feedback, which in turn, would help them to solve the addition of fractions tasks. The mathematical tools used in the third round of interviews were: pencils, markers, grid papers, transparent papers, green and blue pieces of Poster board, blank A4 print paper, scotch tape, red masking tape, glue, ribbons, 3 different sizes of rulers (1 m, 50 cm, 15 cm), and pairs of scissors. In the third round of interviews, children were asked to work with different types of tools to solve two additions of fractions problem.

For the third interview I designed the following task:

Billy is given these tools to solve two addition of fractions problems. In how many different ways can you help Billy to solve the task?

$$1/3 + 1/4$$

$$2/7 + 1/8$$

This task was also given to the children on a piece of paper. Again, I read the task to them as I was pointing to the tools that were on the desk.

Much like the second round of interviews the children all took different approaches to problem solving. But, in contrast to the second round of interviews, the children did not use the already designed fractional elements of the physical properties of the tools embedded in the tools to think about or to solve the task, because there were none. Instead children, in varying degrees, used the guidance provided by the feedback from the tools or the guidance provided by their own knowing of the addition of fractions to perceive different affordances of the tools and consequently constructed a variety of different artefacts. They then used the feedback from their constructed artefacts to think about and/or to solve the task.

Flexibility of modifying questions and tasks. The method of task-based interviews was useful because it offered me a particular kind of flexibility, both in terms of the types of questions I asked in the interview and also allowing me to modify the tasks depending on my judgment (Maher & Sigley, 2014). Within this protocol, I was able to ask questions, which would help me gain further information about children's mathematical thinking and learning. I asked questions such as the following:

Yasmine: if you have this long big thingy as a unit... would you be able to tell me ... Could you tell me what would be two over seven? in that whole long thing... *[pointing to the 56cm ribbon]*

Yasmine: oh...why do we need to have the same number at the bottom?

Yasmine: does it matter in fractions if the spaces are equal?

Yasmine: why did you choose 30?

I also asked questions to gain insight into how or why children used certain affordances of the tools and not others or to clarify why (or why not) children modified the tools in a certain way, or chose one tool over another. The kinds of questions that I asked were meant to further explore the role of feedback in the children's modifications of the mathematical tools. These questions included:

Yasmine: why did you decide to cut the tape? A: in case we want to move them

Yasmine: oh okay... so 56 says how many little spaces are between each line there? *[pointing to the ribbon]*

Yasmine: if this is two fifths *[pointing to the purple Cuisenaire Rod]* what would be a one fifth?

Yasmine: I was wondering how then you can make sure that this [*shows the $\frac{2}{5}$ with two fingers*] fits in that [*points to the $\frac{1}{30}$ line*]

Moreover, although I originally designed these tasks to explore the particular ways in which children would think about the tasks of adding two fractions in relation to the affordances of and the feedback from the mathematical tools, based on what children did and said, I modified one of the tasks after the first interview. The modification was as follows:

The initial task was:

Billy is given these tools to solve two additions of fractions problems. In how many different ways can you help Billy to solve the task?

The modified task was:

Billy is given these tools to solve an addition of fractions problems. Billy does not know the concept of the common denominator. In how many different ways can you help Billy to solve the task?

My rationale for this modification to the task was that in almost all the second round of interviews and the interview with the first group of the third rounds of interviews, the children, knowing a mathematical procedure for adding two fractions used the notion of common denominator, started solving the task by working on the common denominator, as opposed to using the tools to see how and why they need to find the common denominator. Adding this layer of constraint made the rest of the groups either find a way to explain the common denominator to Billy, or to find a different way to solve the task.

The Process of Collecting Data

I conducted a total of 18 task-based interviews over the course of 5 working days. There were six different groups of children. There were three rounds of interviews categorised, based on the type of task and types of tools the children were using. Each group of children did all three rounds of interviews. Each team did only one interview per day. Interviews took between 30 to 45 minutes per session, depending on the children's pace. All children, however, solved all the tasks. I did the first round of interviews with all the groups, then moved to the second round of interviews with all the groups and then did the all of the third round interviews.

For the children to feel comfortable, I needed to tell them, in words and in actions,

that I was only interested in their thinking. That is, it was okay to make mistakes; what was more important than a wrong answer was the thinking that underlies it (Ginsburg 1997). Therefore, at the beginning of each interview session, I explicitly mentioned to the children that I am only interested in how they think as they work with the tools, and not in the final answer.

Use of the video camera. To record the children's interactions with the tools, I used one video camera during all interview sessions. I used a video camera because I needed to know how children used the feedback provided by the tool to modify the tools – for example by cutting, moving, pasting – to think about and or to solve the tasks. The camera was zoomed in to capture all of the students' work, but not their faces. One photo camera was used to capture particular students' modifications to the mathematical tools. All of the student work generated during the two sessions was collected.

From my explanation of the study in class, the children knew that the interviews would be video recorded. They also knew that I would be capturing only the work area and their hands and not their faces. Before starting the first interviews of the first rounds, for each group of children, I showed them my camera and let them play with it for a while. They turned the camera on and off and took some short videos of the walls and the desks. I flipped over the screen of the camera so that they could see what they had been recording. After 5-10 minutes of playing with the camera, I told the children that I was going to put the camera on a tri-pod to video record their hands and the space that they were working in. We, the children and I, worked together to find the best location for the camera and the best zoom that captured everybody's hands (excluding mine). Again, I flipped the screen of the camera towards the children so they could see their hands on the screen. We spent some time seeing what we could see on the screen and doing silly things with our hands and the tools on the desk. Then I started the interviews. I did not notice the children being concerned with the existence of the camera during the interviews. Nevertheless, the camera was there.

My role in the process of data collection. I, as a researcher, a designer and a teacher was present in all aspects of this study, including in the process of data collection.

Firstly, the type of tasks that I have designed at times made the children uncomfortable. In all the tasks, I asked the children to solve the problems in different ways using different tools. As mentioned earlier, these groups of children had not been using mathematical tools to solve fractions problems. Using the tools to solve the task, at times, was not that comfortable. The following are some examples of this discomfort:

- S1: I do not know... I have never done it with these...I have always done it in my brain
- S2: [...] So two out of five. *[she takes two whites rods and puts them underneath the yellow rods]*... I am not sure how to do this.
- H: eh... because I don't really know yet how we would do it.

Secondly, in all the children's interactions with the fraction board, I was an extension of the fraction board, by explaining how it worked. For example:

- Yasmine: This is called fraction board. And these are the strips *[picking up one strip]*... so these are called fraction strips and they are divided...
- S: to three
- A: three parts
- Yasmine: and if we want to say it in fraction form?
- S: one third

I did not however explain the roller and the relation between the strips, the chart, and the roller.

I was part of the process of data collection, with the questions that I asked during the interviews. Since the aim of this study was to see how children use mathematical tools to solve the tasks, at times, I asked questions with which I tried to lead them to use the tools and to think about or solve the task – so that children's problem-solving activity was mediated by the feedback from the tools rather than the students mathematical knowing. For example:

- S: What are we suppose to do right now? then we have to create one that has the same denominator
- Y: least common denominator?
- Yasmine: Okay... how do we do that?
- Y: you multiply the two denominators
- S: yah...so it will be five times two and two times five and then the top times two
- Yasmine: Okay... how can we do it with this? *[pointing to the board]*

This is an example that shows that my question 'how can we do it with this' led the children to use the tool to solve the task, rather than using their already known method of adding two fractions 'you multiply the two denominators'.

Although asking these questions would have made me the more knowledgeable other, I asked them intentionally and on certain occasions. At times, the children, in *multiple attempts*, could not perceive the useful affordance of the tools and as a result used their knowing the addition of fractions to solve the task. I asked the question either when the children decided that the problem was solved (using the previously known process of the additions of fractions) or they could not modify the tools any further to create a more useful feedback to help them think about or solve the task. I viewed my role as another form of feedback.

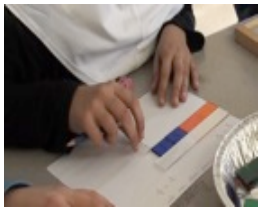
Data Sets

I transcribed the video recordings of all three rounds of interviews into a single data set. However, for the first round of interviews, I only transcribed what the children said, whereas for the second and third rounds of interviews, I added much more detail, explained below. The reason for not fully transcribing the first round of interviews was that my research question focused on the role of feedback in solving the addition of fractions problems. The task for the first round of interviews was designed to re-emphasize the importance of unit and for the children to use some mathematical tools to work on fractions problems.

For each transcription file of the second and third interviews, I created a three-column table, showing time/screen capture/what was said and done. I transcribed what was said, using a combination of speech recognition and audio manipulation software. Each video recording was played by VLC video processing software to change the speed of the speech. I used a combination of transcribing by listening and typing, and transcribing by the use of MS Word speech recognition program, in which I re-spoke what was said in the interview into the software to be converted to text. I found the method of word to text conversion at times challenging, so I switched back and forth between typing and talking. I also transcribed all the modifications to the tools; modifications included: pointing to the pieces, moving the pieces, marking/writing/drawing on the paper, cutting, and measuring. To distinguish between what they said and what they did, I made all modifications to the tool (what they did) [*in a bracket and italic*]. Screen shots of the children's modifications to the tools were included in the transcript to capture what the children were doing. I also noted the time that the each screen shot was taken. To be anonymous, I used the first letter of each child's name. For the children with similar first letter of name, I used the first two letters of their name. Table 4.4 is an example of a few rows of a data file:

02:19		S: <i>[tapes the strip of scotch tape on a paper]</i> okay now we can see it <i>[picks up the ruler]</i>
02:27		S: how long this is? <i>[measures the strip of tape]</i>
02:32		J: wait wait wait wait <i>[aligns the ruler with the tape A helps]</i>
02:46		J: <i>[marks the tape every 3cm]</i> S: like that <i>[every three cm draws a line on the tape]</i>
Table 4.4. Illustration of the layout of the transcripts in a data file		

In only one interview, the three children decided to first work separately on different tasks. For that interview, in each selected time, I included two or three screen shots, one for each child, to show what he/she was doing at that time. Table 4.5 is an example of a few rows of a data file chosen from the above-mentioned interview:

01:39				Y to A: These are called fraction strips.
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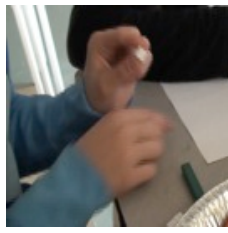
01:47				Y to A: So I have different kinds. This is divided into three pieces
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Table 4.5. Illustration of the layout of the transcripts for the interview in which the children worked seperately

I then compared each transcription text against the video recording and edited and repeated the process until I believed that the written text was only what was said in the video. The average length of the transcript data for each interview was about 21 pages. The 12 conducted interviews (6 second rounds and 6 third rounds) created 254 pages of transcript data and more than 1,220 screen shots images.

Data Analysis

My initial analysis started as I was transcribing the data. As I was typing and listening I paid attention to instances that children were able to use the tools to solve the tasks or example of where the tools did not particularly help them to solve the tasks. I made notes about my observations on when the tools were useful and when they were not.

When the transcribing was finished, I re-read all the files multiple times and noticed that there were other nuances which made the categorisation of useful/not useful tools too broad. Hence I created 5 files and organised all the activities in to the followings sub-categories:

1. Tools messing up the activities
2. Tool works – they used their mathematics knowing to use the tool
3. Tool did not work – they used their mathematics knowing to use the tool
4. Tool works – they used the tools to do the mathematics
5. Tool did not work – they used the tools to do the mathematics

While organising the activities in the above-mentioned categories, I noticed that activities, as they unfold, were much more complex than being put into five groups. For example there were activities in which children initiated the use of a tool based on their previous knowing of fractions but then as the interaction progressed they used the properties

of the tools to solve the task. I did not know where to put these activities.

So, instead of categorising the data, I decided to look at the process of artefacting: how children tied signs to the tools to attach newer meaning to the tools. Vygotsky's notion of the object/meaning ratio helped me to systematically look at data and focus on the times that children were using the properties of the tools to solve the task (object/meaning) and also on times that children attached new meaning to the tools. For example, when children used the properties of the Cuisenaire rods to represent 0.5. I examined the inversion of object/meaning ratio to meaning/object ration for all the activities. I specifically focused on the role of the physical properties of the tools in this inversion. This focus helped me to answer my first research question.

Furthermore, my theoretical framework says that learning is becoming acquainted with new forms of expressions and actions and it emerges within a ZPD. Hence, to think about my second research question, I looked at if and how children's ways of expression, reflection and action in relation to the task of adding two fractions changed as they interacted with the tools to add the two factions. While I looked at learning within the ZPD, I constantly looked for who/what was the more knowledgeable other and how the role of the more knowledgeable other alternated in the activities.

To be systematic in looking at the role of the physical properties of and feedback from the mathematical tools as well as the learning (changes in expression and actions) within the ZPD, I re-categorised the transcripts for each data file into two parts: what children said; and what they did. I added a fourth column to the already existing 3-column table of each data file and moved 'what children have done', that is the parts that *[were written in bracket and in italic]* to the new column, Table 4.6 is an example:

Time	Screen shot	<i>What they did</i>	What they said
02:19		<i>S: [tapes the tape on a paper]</i>	
			okay now we can see it
		<i>[picks up the ruler]</i>	

02:27			S: how long this is?
		<i>[measures the tape]</i>	
02:32			J: wait wait wait wait
		<i>[aligns the ruler with the tape. A helps]</i>	
<i>Table 4.6. Illustration of the layout of the transcripts ready for analysis – columned into what they did and what they said</i>			

In-depth interrogation of these two categories and of the interrelationships between them was the starting point of my subsequent readings of the transcripts in the next stage of my analysis.

Trustworthiness

In this section, I address the trustworthiness of this study. Shenton (2004) suggested that to ensure credibility of a research, the researcher needs to adopt well-established methods. In terms of the process of data collection, I have explicated the rationale for my choice of task-based interviews in a think-aloud format. I have also explained my role in this study, including the selection of the tasks and the tools as well as my role in all sessions of interviews. My methods of data analysis are derived from those that have been successfully used in previous comparable projects. A Vygotskian view of learning has been used in the field of mathematics education for decades.

Merriam & Tisdell (2015) believe that external validity is to explain the extent to which the findings of one study can be applied to other situations. Since the findings of this project are specific to a very small number of participants and specific tools, it is not feasible to declare that the findings and conclusions are applicable to other situations and populations. However, using the same theoretical framework and method of data analysis (analysis of what children said and did with focus on affordances of the tools and emergence of ZPD) may include other tools, with or without design-related feedback as well as other participants

with their varied knowing of addition of fractions.

To ensure, as much as possible, that this study's findings emerged from the data and not my own predispositions, extracts of transcripts and screen shots of the videos are presented with the findings so that my interpretations can be evaluated by other researchers.

Presentation of the Findings

The findings of this study are presented in two different chapters. In Chapter 5, I will discuss the findings as it related to children's attachment of meaning to the use of tools and the consequent creation of the artefacts. In chapter 6, I discuss the emergence of the ZPD and will show how children learnt with the guidance provided by the tools and their own knowing of fractions. In both chapters, I frequently provide excerpts from the transcripts – with photos of what children did – to further support my arguments.

The Choices I Have Made and My Rationale for Them

In order to confine my focus to the aim of this dissertation, I had to make some choices in selecting and analysing data. I made these choices deliberately and rationally. The following are these choices and my rationale for them:

- Over the course of the analysis, I was aware of the fact that the children were interacting with the mathematical tools in the presence of me and of other children. Therefore the participants in these interactions were not always just the children and the tools. Nevertheless, my focus in the analysis was on the moments when the children were thinking, reflecting and acting interactively with the mathematical tools. These interactions might have been triggered by questions/suggestions from others.
- In my first few readings of the transcripts, I noted that children used the feedback from the mathematical tools not only to solve the addition of fraction tasks, but also to construct the two fractions that they needed to add. Since in this study, my focus was on the concept of the addition of fractions, I focused my attention exclusively on the guidance provided by the mathematical tools in solving the addition of the two fractions and not necessarily on the construction of the two fractions.
- Since I confined my focus of analysis to children's interactions as they attempt to solve the addition of fractions problems, I did not include the data that was gathered in the first round of interviews in my analysis. The task in the first round

of interviews was designed for the children to think about fractions and unit fractions as they interacted with the tools. I am also convinced that the amount of data gathered from children's interactions in the second and third rounds of interviews gave me a diverse enough range of interactions to systematically analyse the data, solidify the findings, and answer my research questions.

Chapter 5

Feedback and Its Importance

In this chapter, I present the findings on the importance of the feedback from the mathematical tools in the process of solving addition of fractions problems. Discussion of these findings will be the basis for addressing my first research question:

How does the feedback from the mathematical tools play a mediating role between the physical actions of the child with respect to the mathematical affordances of the tools and the child's thinking about and learning and knowing of the addition of fractions?

To systematically analyse the role of feedback from the mathematical tools in adding two fractions, I start by discussing the findings on how children perceived the feedback from mathematical tools and/or their own constructed feedback, on the basis of the perceived affordances of the tools. Then, I illustrate how children attached mathematical meanings to the tools as they used the feedback from the tools to solve the tasks of adding two fractions. I do so by examining children's gradual perceptual change within the inversion of object/meaning to meaning/object ratio. Then, I discuss the significance of the creation of artefacts in the process of problem solving – by inversion of the object/meaning to meaning/object ratio. To examine the crucial role of the artefacts in problem solving, I illustrate how children created the artefacts, how they used the feedback from the artefacts in different steps of thinking about and solving the addition of fraction tasks, and how the feedback from the artefacts mediated between children's modification to the tools and to their own thinking/knowing. I start with a general overview of main findings.

Overview of Main Findings

In general, the analysis of data showed the following two findings in relation to the role of feedback from the tools in children's interaction with the tools to solve addition of fraction tasks:

1. Children's interactions with the tools were mediated by the feedback from the different artefacts that they created throughout their interactions with the tools. As mentioned in chapter 3, one of the theoretical foundations of this study was that instead of acting directly, in unmediated ways in a social and physical world, all our actions are indirect and mediated (Wertsch, 1993) and that from a Vygotskian perspective, these actions are mediated

by tools and signs (Vygotsky, 1978). With this finding, I propose to extend the notion of mediation to show that it was neither the tools nor the signs that *independently* mediated the children's interactions with the tools. Rather, it was the children's created artefacts that played a mediating role between the children's modifications to the tools as well as the modifications to their mathematical thinking.

2. In the children's interactions with the tools and in all problem-solving stages, both attaching mathematical meaning(s) to the tools and the consequential process of creating an artefact were a gradual and complex process, which were closely related to: (a) The ways in which the children perceived the mathematical affordances of the tools – through the feedback provided by the tools – to create and to use the artefacts; (b) The children's mathematical knowing of fractions in general and of the addition of fractions in particular; and (c) The task of adding two fractions.

In the rest of this chapter, I present these findings in more depth, supported by detailed analysis of the children's interactions with the tools.

Tools and the Feedback From the Tools

In this study, I introduced the notion of 'feedback from the mathematical tools' to analyse the role that the physical properties of tools play in children's interaction with the tools and in their resulting thinking about and knowing/learning during work on addition of fractions problems. I described the term 'feedback' as those specific physical properties of a mathematical tool and/or the interrelationships among its components that make the mathematical *affordances* of the tool more apparent. To examine children's use of feedback from the mathematical tools in the process of thinking about or solving addition of fractions problems, I first need to discuss where and how I applied the term 'affordances', as perceived by the children, in their interactions with the mathematical tools.

Noted in chapter 3, for children to problem solve with the mediation of the mathematical tools, it is necessary for them to perceive the useful mathematical affordances of the tools. Following Gibson's (1977) view, I used the term *affordances* to refer to all that is related to the mathematical tools and their physical properties that contributes to the types of interaction that happen – in the case of this study, working on the addition of fractions problems. To refer to the physical properties of the tools, I used a more neutral description, independent of the child who was using them. By physical properties of the tools, I mean properties such as rigidity, non-rigidity, stickiness, edges, shapes, sizes and the relationships

among them.

In general, I noted that although it was the physical properties of the tools that provided affordances for doing some things but not others, it was up to the children's perceptions and mathematical knowing to discern the useful affordances of the tools, in relation to adding the two fractions. This means that I called the properties of the tools their affordances *only* if the children perceived them in relation to the task of adding two fractions and from my own standpoint 'feedback(s)' are those properties of the tools that make perceiving the mathematical-related affordances of the tools easier.

In the second interviews, there was feedback already designed in the physical properties of these tools. In the third round of interviews this was not the case: the feedback was not designed in the tools. In other words, the tools used in the second round of interviews had the fractional perception of their designer(s) embedded in their design, whereas the tools used in the third round of interviews did not have the same feedback embedded in their design. One finding of my study is that the differences between the types of feedback offered in each round of interviews impacted the children's interactions with the tools:

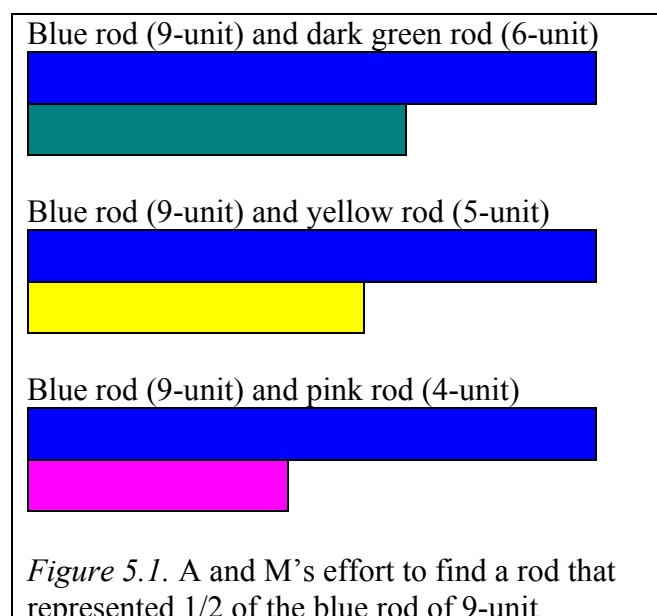
- In the second round of interviews, due to the feedback provided by the interrelationship between the components of the tools, the children were able to perceive the fractional affordances of the tools more easily. Hence, most of the children were able to solve the addition of fractions without substantial difficulty. Here I would also argue that the constraints provided by the specific design of these tools made some other possible actions with the tools unachievable (such as cutting them, for example). Therefore, it was what the tools could afford and what the tools could not afford (i.e., their constraints) that gave the children less choice in terms of what they could do with them, thus making the tools easier to work with.
- In the third round of interviews, since there was no fractional feedback embedded in the tools provided, the children could not use them to think about or work on the task. They had to construct the feedback by perceiving different physical properties and affordances of the tool. By constructing feedback, I mean for the children to modify the tools by measuring, cutting, marking or drawing so that they constructed an artefact that would help them think about or solve the addition of fractions. An important point to raise here is that the limited constraints (more afford-abilities) of the tools made it more difficult for the children to initiate the task. Hence, in almost all of the third interviews, the children created artefacts that would function in the linear model of representing fractions – in a

manner similar to the functionality of the tools that were used in second rounds of interviews– using the measuring affordance of the ruler and different affordance tools of masking tape, scotch tape, papers and ribbons.

- Although the tools in the second round of interviews had the feedback designed in them, it did not mean that all children were able to perceive the feedback and use the tools to think about or to solve the tasks. There were circumstances in the second rounds of interviews in which children could not perceive any feedback from the tools. Hence, they solved the task by mostly writing mathematical symbols and by using the previous steps of adding two fractions.

I now use three examples – one for each finding just mentioned above – to present how I referred to and used the terms ‘affordances’ and ‘feedback’. I also discuss how children either used (or could not use) the feedback designed in the tools or how they constructed feedback to be able to think about or to solve the task of adding two fractions.

Example 5.1 – Useful embedded feedback. The first example shows how A and M – in multiple attempts – tried to present $\frac{1}{2}$ and $\frac{2}{5}$, using the Cuisenaire rods. They first chose the blue rod of 9 to be their whole one unit. Then they perceived different coloured rods – with different sizes – to see which one could afford being the fractional amount of $\frac{1}{2}$ with respect to the length of the blue rod of 9 as a whole. They first perceived the size of the green rod, followed by those of the yellow rod and then of the pink rod, to possibly afford being half of the blue rod (see Figure 5.1).



The feedback from the rods assisted the children to solve the problem, in the sense that they could compare the sizes and see that none of the rods picked (dark green, yellow and pink) would represent $\frac{1}{2}$ of the blue rod of 9. That is, when put underneath the rod of 9, they were all either smaller or bigger than a half of 9.

Later, they changed their unit (one whole). This time they perceived the orange rod of 10 as their unit (one whole) and used the feedback from the rods to perceive which other rod could afford being its $\frac{1}{2}$ and which could afford being $\frac{2}{5}$. The following excerpts from the transcript show this interaction:

02:45		<i>A: [picks up an orange rod of ten]</i>	This could be one	1
		<i>Both: [puts two dark green underneath it]</i>		2
03:01		<i>M: [puts two yellow underneath it]</i>		3
			A: okay M: yeah okay that is good A: yeah M: so we need one of these A: and then two fifths	4
		<i>A: [puts two lime green underneath the orange]</i>		5
			No... that is three	6
03:17		<i>M: [pick up the red rod]</i>		7
			M: the red	8

03:28		<i>Both:[Line up the reds underneath the orange rod]</i>		9
			A: Okay, these are fifths, how many of those? M: two A: two fifths	10
03:36		<i>[puts rod red rods next to the yellow rod]</i>		11
			A: so this like this	12
Team 2 – Second Interview – Jan, 19, 2015				

In this interaction, A and M perceived the orange rod (10-unit) to afford presenting the one whole unit as they said: ‘This could be one’ (turn 1). Then they perceived the sizes of the dark green rod (4-unit), then of the yellow rod (5-unit), to possibly afford being half of the orange rod (turns 2-3). The feedback from the rods helped them to select the yellow rod (5-unit) to represent the $\frac{1}{2}$ of the orange rod of 10 (turn 4). The next interaction was for A and M to perceive and find a rod that was $\frac{1}{5}$ of the orange rod of 10 (turn 4). They first tried the affordances of the lime green rods (turn 5) and stated: ‘No... that is three’ (turn 6). Then they perceived the sizes of red rod (2-unit) as they stated ‘the red’ (turn 7-8). They used the feedback from the rods to perceive how the red rods of 2 afforded being $\frac{1}{5}$ of the orange rod (10-unit). They lined up five red rods (2-unit) underneath the orange rod (10-unit) and it fit (turn 9). They then said:

A: Okay, these are fifths, how many of those?

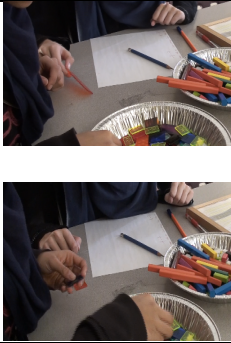
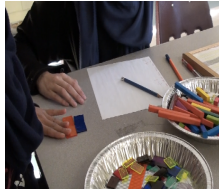
M: two

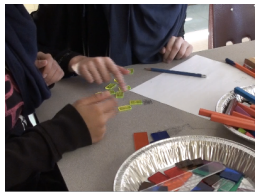
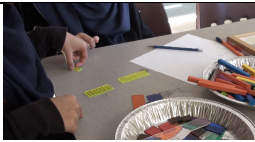
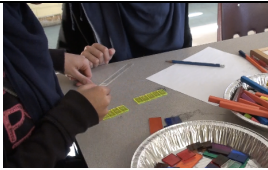
A: two fifths (turn 10)

Then they lined up two red rods – presenting the fractional amount of $\frac{2}{5}$ – and one yellow rod – presenting the fractional amount of $\frac{1}{2}$ – (see image in turn 12) and stated ‘so this like this’ (turn 12). Here, A and M perceived the useful affordances of four rods – an orange (10-unit), a yellow (5-unit) and two red rods (2-unit) in relation to the task of presenting $\frac{1}{2}$ and $\frac{2}{5}$. They used the feedback from the rods to find a rod that represented $\frac{1}{2}$ of a rod of the orange rod of ten, which is the yellow rod of 5 units and a rod that represented $\frac{1}{5}$ of a rod of the orange rod of ten, which is a red rod of 2-unit. Hence, they

presented $1/2 + 2/5$.

Example 5.2 – Not so useful embedded feedback. This example shows children's interaction with the tools to solve $1/2 + 2/5$, in which the feedback from the tools, although designed in the properties of the tools, was not perceived by the children. Therefore, the feedback was not useful in helping the children add the two fractions. The following excerpts from the transcript show this interaction:

01:02			F: what is the question? S: one half	1
		<i>F: [picks up the $\frac{1}{2}$ strip puts it on the table]</i>		2
			S: and then F: and two fifths	3
01:03		<i>S and F: [both reaching to the fraction strips container]</i>		4
01:09			S: so what did you get?	5
		<i>[looks at the $\frac{1}{2}$ strip, flips it over to see the number on it]</i>		6
			half	7
		<i>F: [looks and picks up 2 $1/5$ strips]</i>		8
01:17		<i>F: [puts the strip in two row]</i>	We need to add them	9
01:18 — 01:28			[Pause 10 sec]	10
01:33			S: we can do... two times F: five S: five	11

		<i>F: [reaches to the container and moves the pieces]</i>		12
			S: equals ten and then one times... lets do it...	13
		<i>S: [writes on the paper] F: [drops a pieces and looks at the paper]</i>		14
01:52			S: so... so you have to do five out of ten plus four out of ten	15
01:53		<i>S: [picks up the 1/10 pieces] F and S: [pick up all the 1/10 pieces]</i>		16
02:12			S: how many do we have? Two three four five ... ten	17
02:33		<i>F: [moves her pieces to line them up with S's. They did not connect the pieces]</i>		18
02:35		<i>F: [picks up the whole strip]</i>		19
			S: so it equals nine out of ten	20
Team 5 – Second Interview – Jan, 19, 2015				

In this interaction with fraction strips, S and F were able to use the feedback from the strips to locate $\frac{1}{2}$ and $\frac{2}{5}$ (turns 1-4). The feedback from the strips in this interaction was the written numbers on the strips. Because when S asked F ‘so what did you get?’ (turn 5), F first ‘[looks at the $\frac{1}{2}$ strip, flips it over to see the number on it]’ (turn 6) and then she said ‘half’ (turn 7). Although the feedback was useful for the children to present $\frac{1}{2}$ and $\frac{2}{5}$, it was not useful enough to help them in adding the two fractions as they said: ‘we need to add

them' (turn 9) and then they paused (turn 10). The way in which the two sets of strips of $\frac{1}{2}$ and $\frac{2}{5}$ were arranged, one above the other (turn 9), did not provide S and F with useful feedback to assist them in the process of problem solving. They could not perceive the addition-of-fractions affordances of the pieces. Hence, they thought about the problem through the previously known process of finding the common denominator as they said:

S: we can do... two times

F: five

S: five ... equals ten and then one times ... lets do it...

S: *[writes on the paper $\frac{5}{10} + \frac{4}{10}$]*

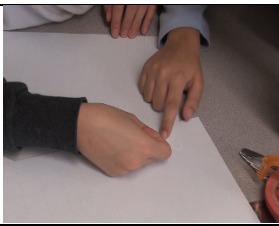
F: *[puts the paper down and looks at the paper]*

S: so ... so you have to do five out of ten plus four out of ten (turns 12-17)

In this interaction, the children did not use the physical properties of the tools to find the common denominator. They instead used their knowing of the addition of fractions as S stated 'we can do... two times... five... [...] equals ten' and to say 'so... so you have to do five out of ten plus four out of ten'. Then they used the feedback from the strips to find the strips of $\frac{1}{10}$ and to line them up into parts, as S said: 'so it equals nine out of ten' (turn 20). In this interaction, even though both S and F knew the process of adding two fractions, they still found it difficult to use the feedback from the fraction strips to add $\frac{1}{2} + \frac{2}{5}$.

Example 5.3 – Constructed feedback. In this interaction, M and A were asked to use available tools (papers, tapes, rulers, scissors, etc.) to add $\frac{1}{8}$ and $\frac{2}{7}$. M and A started their interactions by looking at the tools on the table:

00:39			M: Okay what do we have here? Tape and ribbon, scissors A: Tracing paper and paper M: I think this is useful like what is this?	1
		<i>[picking up a graph paper]</i>		2
			what is each	3
		<i>[counts with her finger]</i>		4
			one two three five six seve...	5

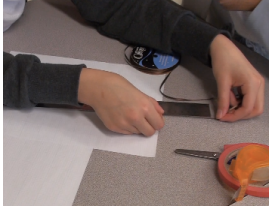
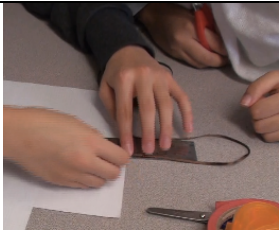
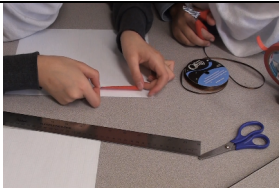
01:07			M: [<i>starts over, F helps</i>] is it eight by eight? A: yeah eight M: eight by eight so sixty four in each one	6
01:10 - 01:42			[30 sec pause]	7
01:43			A: Seven over eight, how can we show... M: we can cut pieces of ribbon with the ruler	8
		[<i>picks up ribbon and the ruler</i>]		9
Team 2 – Third Interview – Jan, 20, 2015				

In this interaction M and A started to see what types of tools they have available to themselves, M asked: ‘Okay what do we have here? Tape and ribbon, scissors’ and A added ‘Tracing paper and papers’ (turn 1). They soon perceived the grids (8x8 partitioned squares) on the grid paper as possible useful affordances for adding the two fractions, M said: ‘I think this is useful like what is this? [*picking up a grid paper*] what is each [*counts with her finger*] one two three five six seven... (turns 2-5).

At this point, M had perceived the partitioned affordances of the grid paper to be the useful mathematical affordances in relation to the task of adding $\frac{1}{8}$ and $\frac{2}{7}$. Therefore, for M, the grids were feedback from the paper. However, A was not quite clear of how the grids on the grid paper could be perceived as the useful mathematical affordance to provide feedback on presenting seven over eight, she said: ‘Seven over eight, how can we show?’ (turn 8). With the not-so-clear perception of the mathematical affordances of the grid, M perceived the cut-ability affordance of the ribbon and stated: ‘we can cut pieces of ribbon with the ruler’ as she [*picks up ribbon and the ruler*]’ (turns 8-9).

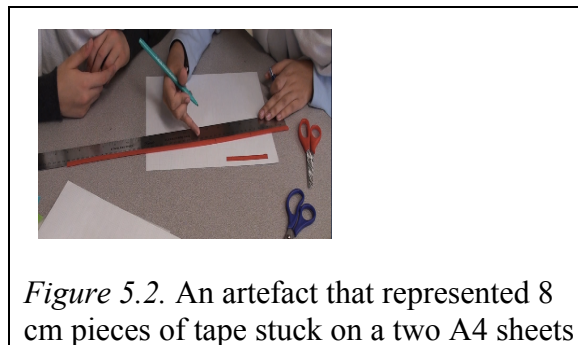
In these interactions M and A tried to use the physical properties of the tools (e.g., papers, rulers and ribbons) to perceive their possible affordances in relation to the task of adding two fractions. However, there was no feedback from any of these tools to make the mathematical affordances of these tools more apparent. The lack of feedback from the ribbons led A and M to construct ‘*something*’ that would help them perceive the

mathematical affordances of the ribbons easily apparent. That is, they had to construct the feedback themselves. The following interactions show what unfolds afterwards:

02:12			A: How long? M: we can do it in 56 because that is the common denominator between these two we can do it in sevenths 56 so... like each one is eight centemetre	1
		<i>[measures one eight cm ribbon]</i>		2
02:29		<i>M: [tries to hold the ribbon on the ruler]</i>		3
			M: should we use a tape so that it doesn't move around? So we can stick it on the paper and it does not move	4
		<i>[moves the ribbon to the side]</i>		5
			A: yeah sure	6
		<i>[gets the tape]</i>		7
			M: no I meant like cut the pieces of tape that are	8
02:57		<i>M: [puts the tape on the ruler – 8cm picks up the tape, sticks it on a piece of paper]</i>		9
Team 5 – Third Interview – Jan, 22, 2015				

A's perception was still not-so-clear about the mathematical affordances of the ribbons, she asked how long each piece of ribbon should be, she said 'How long?' (turn 1). Knowing the concept of the addition of fractions, M explained that they could measure 56.

She then rationalised her suggestion by stating: ‘because that is the common denominator between these two we can do it in sevenths 56 so’ (turn 1), as she tried to cut the ribbon. Yet the wobbly-ness of the ribbon was perceived as not being able to afford steady holding of the ruler and easy cutting. Hence, M perceives the sticky-ness affordances of the tape as more useful: she said: ‘should we use a tape so that it doesn’t move around? So we can stick it on the paper and it does not move’. They then cut an 8 cm piece of tape and stuck it on an A4 piece of paper, and created an artefact (see Figure 5.2).



They still were figuring out the affordances of different tools to help them create a feedback which in turn will help them to perceive the mathematical affordance of the tools. They perceived that the length of an A4 paper was not long enough for a long piece of tape of 56 cm. But a 2-A4-paper long sheet would afford holding 56 cm tape as they said:

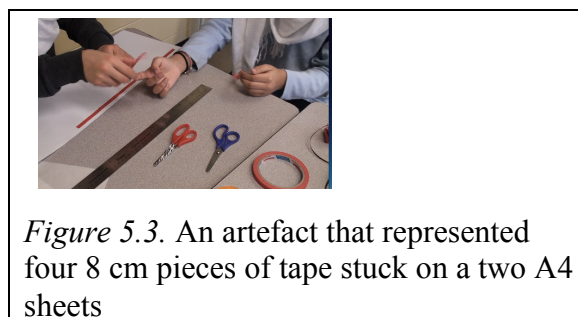
M: *[picks up the tape, sticks it on a piece of paper]*

A: that is not long enough for all of them

M: we can just line up two beside each other *[points to another paper]*

A: yeah

They created the following artefact (see Figure 5.3):



The children then measured a long piece of tape, marked it every 8 cm, cut each piece and stuck it on the A4 papers and underneath the first pieces of masking tape they wrote $\frac{1}{7}$.

Through these interactions, I showed that M and A used the physical properties of papers, rulers and tapes to perceive their affordances and to construct feedback that would help them to represent the fractional amount of $1/7$.

Discussion of example 5.1 - 5.3. With regard to the three examples set out above, I indicated my underlying considerations/assumptions about ‘feedback’ from and ‘affordances’ of mathematical tools – both the feedback(s) that were incorporated into the designs of the tools or children’s constructed feedback. I illustrated that, in the dialectic process of the children’s interaction with the mathematical tools, if it was perceived by the children, the feedback from the tools played a role between the physical properties of the tools and the children’s thinking about and/or solving the addition of fractions.

Moreover, I noted that the degree of usefulness of the feedback was related to the children’s knowing of the addition of fractions. So not only did the children for whom the knowing of the addition of fractions was not so clear have difficulties using the feedback from the tools to perceive their mathematical affordances (Pimm, 2002), but also some children for whom the knowing of the addition of fractions was clear had some of these difficulties as well.

Now the questions are: how do children use the feedback in the process of problem solving? What is the process within which children *perceived* not ANY affordances of the tools but THE mathematical affordances, in relation to the task of adding two fractions? I will address these questions through examining the children’s *perceptual* change as they interacted with the tools to solve the tasks.

Gradual Perceptual Change

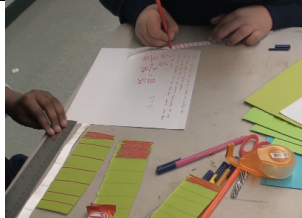
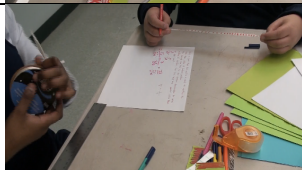
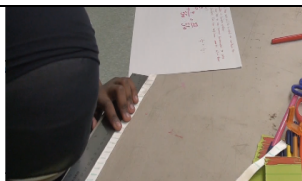
As mentioned in my theoretical framework, I utilised Vygotsky’s view of ‘perception of real objects’ (Vygotsky, 1978) to systematically look at and make sense of what I called ‘children’s perception’, as they worked with mathematical tools to think about and/or work on the tasks. As I mentioned in chapter 3, Vygotsky elaborated upon the discussion of ‘children’s perception’ through what he referred to as the object/meaning ratio.

In my study, children constantly interacted with the mathematical tools to add two fractions. Following Vygotsky’s notion of the object/meaning ratio, I noted that, during the initial stages of a child’s encounter with a mathematical tool, his/her perception of the tool could be presented using the ratio of object/meaning. This figure of perception represents how the children perceived the physical properties and affordances of the tools – that is, the

meaning of the tools as objects, respective to the tasks at hand. At this stage, the tools and their physical properties are dominant and their meaning(s) – as an object or as the mathematical meanings – is subordinate. That is, a tool can be represented by object/meaning.

$$\text{Tool} = \frac{\text{Object}}{\text{meaning}}$$

Hence, this is the stage when the physical properties of the mathematical tools played an important role in the children's interaction with them, both to perceive the affordances provided by the tools and to perceive their mathematical meaning(s). For example, in the following interaction H and K perceived the physical properties and affordances of a ruler and ribbons to present $2/7 + 1/8$. The following excerpt from the transcript shows this interaction:

10:52		<i>K; [picks up the ruler] H: [marking the other half ribbon]</i>		1
11:00			K: probably... or I can use this...	2
		<i>[picks up the ribbon roll to cut a new piece]</i>		3
			I can... I think what I can do is to make a long one that's 56	4
			K: so I am doing each about a centimetre... so basically what I'm doing, I am just making the lines of the fraction out of 56	5
		<i>K: [tries to make the other end of the ribbon straight]</i>		6
			K: And yeah I am done... so what we can do... that can probably shade 23	7
Team 4 – Third Interview – Jan, 21, 2015				

In this interaction, the children first perceived the physical properties and affordances

of the ribbon and a ruler: that is the measure of 1cm increments on the ruler and the cut-ability of the ribbon. These affordances of the tools played an important role in the children's interaction with the tools. In this interaction, the objects (ruler and ribbons) and their physical properties were dominant and the meaning of $2/7 + 1/8$ was subordinate. At this stage, the figurative perception of the children can be represented by the object/meaning ratio. That is,

$$\frac{\text{Object}}{\text{meaning}} = \frac{\text{Object-measures on the ruler and ribbons}}{\text{meaning-mathematical meaning of } \frac{1}{8} + \frac{2}{7}}$$

In order to invert this ratio – that is, in order for the tools to be used as a symbol (a sign) for a mathematical concept – the children needed to increasingly tie the signs to their use of the tool. The analysis showed that the children at times were able to use the physical properties of the tools and their own knowing of fractions to attach mathematical meanings to the tools. Hence the inversion of the ratio of object/meaning to meaning/object, where the mathematical meaning was dominant and the physical properties of the tools were subordinate. For example, in the interaction set out below F and S tried to perceive the physical properties of the Cuisenaire rods in relation to the task of making 0.5. The interaction, however, reached a tension point, as they found it challenging to perceive decimals with their rods. As S stated: 'It's going in decimal and here is no decimal there [*pointing to the container*]'.

The following excerpt from the transcript shows how S and F dealt with the tension of creating a decimal number using the Cuisenaire rods and perceived the 0.5-ness affordance of the tools.

11:39			S: if this is one	1
		[<i>showing the white</i>]		2
11:40			S: if we make this one	3
		[<i>showing the red rod</i>]		4
11:42			S: this would be	5
		[<i>showing the white</i>]		6
			S: a point five F: a point five... yah F: this is point five	7

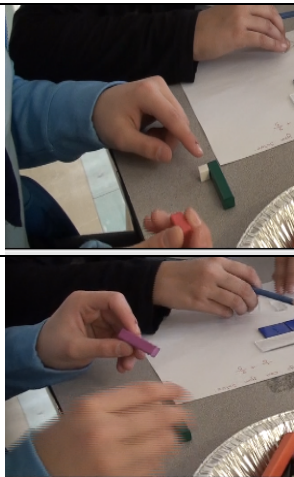
Team 6 – Second Interview – Jan 19 th 2015

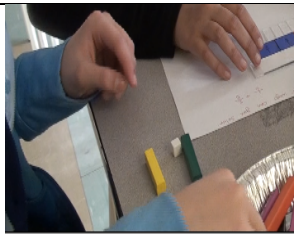
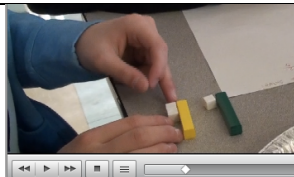
This interaction shows how S and F perceived the physical properties of the rods and attached a new mathematical meaning to the rods. They perceived the red rod (2-units) to represent 1 ‘if we make this one’ then the white rod (1-unit) could be perceived as 0.5. Showing the white rod they said: ‘this is point five’. The children perceived new affordances of the rods and attached the meaning of 0.5 to the pieces of rod. At this time, for S and F, the pieces of red rod (2-unit) did not mean just a piece of plastic, for example suitable to build a train with; instead it meant 0.5. This was where the mathematical meaning dominated and the physical properties became subordinate, thus giving rise to the meaning/object ratio.

$$\frac{\text{meaning}}{\text{object}} = \frac{0.5}{\text{relative length}}$$

Furthermore, the data analysis showed that it was during the gradual process of inverting the object/meaning ratio to a meaning/object ratio that children thought about the interrelationship among the affordances of mathematical tools and the meaning of the mathematical concepts in relation to the task at hand. Here I re-emphasise the fact that the mathematical meanings were not presented by the tools but rather by the children in their interaction with the tools, as they perceived the mathematical meaning in relation to the task.

In the following, I provide an example that shows how S’s and A’s perception of the objects (i.e., the mathematical tools) and its gradual change from object/meaning to meaning/object. The following sequence shows how S interacted with different rods in relation to the task of creating 2/5 after she said: ‘Now I need to figure out what I could use for two over five’.

	<i>[She then takes the red rod]</i> <i>[She puts the red rod back]</i>		1
		S: I think I want one that worth five.	2
	<i>[She takes the pink rod of four]</i>		3
		S: I think this one is four so I need to find the block	

		that is five	
	<i>[She takes the yellow rod and exchanges it with the pink]</i>		4
		S: I think this is five. So yes compared to that one, this one is five. So two out of five.	5
	<i>[She takes two whites and puts them underneath the yellow]</i>		6
Team 3 – Second Interview – Jan, 19, 2015			

This interaction shows how S first perceived the affordances of the tool – that is the meaning of the tool as an object in relation to the task at hand – to present $\frac{2}{5}$ with the Cuisenaire rods. S used the physical properties of different rods (i.e., the interrelationships among their sizes) to perceive the affordances of the rods. She took a red rod and then a pink rod in the process of representing $\frac{2}{5}$. At this stage, the tool is dominant and its meaning – its mathematical meaning – is subordinate, thus giving rise to the object/meaning ratio. She later perceived the affordances of the yellow rod as being useful to afford her to represent $\frac{2}{5}$ as she said: ‘I think this is five. So yes compared to that one, this one is five. So two out of five’. The figure of perception at this time can be presented by the ratio of meaning/object, in which the mathematical meaning dominates as the result of tying signs to the tools, as S ‘*takes two whites and puts them underneath the yellow*’. At this point S attached a new meaning to the pieces of the rods: she perceived $\frac{2}{5}$ -ness in the way in which two white rods of one were placed underneath a yellow rod of five.

The scenarios described above are only a few examples of other similar scenarios in which children attached new mathematical meanings to the tools, as their object/meaning ratio perception gradually changed to that of the meaning/object. That is, in all the interviews and with all the tools, at some point, children were able to perceive the mathematical affordances of the tools (object/meaning), using the feedback from the tools. They were able to use the feedback from the tools to attach signs to their use of tools and hence to attach mathematical meaning to the tools. Therefore, they could invert the ratio of object/meaning to one of meaning/object to think about or solve the addition of fractions tasks.

Attaching mathematical meanings to the tools does not necessarily mean that children were able to add the two fractions and solve the tasks. Solving the task was intertwined with the mathematical knowing of the children as well as the ways in which they attached mathematical meaning to the tools.

To examine this intertwined relationship between the mathematical knowing of the children and attaching meaning to the tools, I propose to further expand the notion of the perceptual change – from object/meaning to meaning/object – to include the creation of artefacts. In other words, when children perceive a new mathematical meaning being afforded by the tool, the tool is not just a tool anymore, but a mathematical artefact.

In the next section, I discuss the process of creating an artefact, as an extension to the discussion of perceptual change from the object/meaning ratio to the meaning/object ratio.

The Artefacts and Their Mediating role

In the above section, I explained how children attached mathematical meaning to the tools. Yet, in order to solve an addition of fraction task, children need to attach multiple meanings to the same (or different) tools to solve the tasks. For example, to add $\frac{2}{7} + \frac{1}{8}$, they are required to attach 1/7-ness to the tools (of ribbons, for example), then 1/8-ness, then they needed to possibly attach the meaning equivalent to the concept of the common denominator to the tool, and solve the task. How would they do that?

To address this question, I present the findings about the creation of *artefacts*. That is, I discuss that it was by creating mathematical artefacts that children used the tools – with mathematical meaning(s) attached to them – in the process of problem solving. In accordance with the views of Vygotsky, mathematical artefacts are created as the children tie signs to their use of tools, while interacting with the tools to solve mathematical tasks.

In the rest of this section, I show my findings related to the creation of the artefacts in more depth, supported by detailed analysis of the children's interactions with the tools. I start by discussing the types of artefacts that they created and the role of the artefacts in helping the children to think about and solve the tasks. Next, I state the reasons why the children created the artefacts and I highlight the factors that influenced the construction of these artefacts.

Reasons for creating the artefacts and factors affecting these creations. In their interactions with the mathematical tools, children attached signs to the tools, for example, by talking about what they do, by talking about the various parts of the tools, or by using

mathematical symbols. These signs are attached to the tools according to not only the ways in which children perceive the physical properties of the tools in relation to the tasks, but also to their mathematical knowing of fractions. That is, the children used their mathematical knowing(s) and the feedback from the tools to perceive (new) affordances of the tools and to *create* and *re-create* artefacts and then used the artefacts – and the feedback that they provided – to solve fractions problems.

In relation to the task of adding the two fractions, the children, in most of the interactions, *created* artefacts for two reasons. The first reason for the creation of artefacts was that the artefacts would in turn assist them in thinking about different steps of adding two fractions. The second reason was that they created artefacts to assist them in ‘re-presenting’ different steps of adding two fractions with the tools. By re-presenting I mean, for example, when children used their knowing of the addition of fractions first to add the two fractions and then use the tools to illustrate – or re-present – the final answer to the question; this is in contrast to what I call ‘using the physical properties of the tools to solve the task’.

Moreover, the children *re-created* new artefacts in order to help them to think about the next step in the problem-solving process or in order to solve the task. In the process of problem solving, the children re-created artefacts because they could not perceive any useful affordances provided by the previously created one(s). By useful affordances, I mean ones that could have been perceived by the children in assisting them in the problem-solving process. However, this does not mean that all of the children, regardless of their knowing of fractions and/or their perceptions of the affordances of the tools, were able to use the guidance provided by the tools to create artefacts and/or to solve the various tasks.

Types of artefacts created by the children. With regard to the task at hand, the children mostly created the artefacts by following the previously learnt steps of adding two fractions: (1) represent the two fractions to be added; (2) find the common denominator; (3) find the equivalent fractions with the common denominator being the new denominator for both of them; and (4) add the two fractions together. There was an exception, as one team of children did not use these steps to solve the task. Later, I explain this team’s interactions in detail.

Set out below are some general examples of the artefacts that the children created with respect to each of the four steps mentioned above (1 to 4). These examples are taken from the second and third rounds of interviews.

Table 5.1 shows some artefacts that represented the unit fractions – that is, artefacts

that showed the number of equal parts in one whole (e.g., $1/7$, $1/8$ or $1/5$).

Representing $1/5$ and $1/2$ using fraction strips, by using the feedback (interrelationships between their sizes) designed into the strips.	
$1/5$ and $1/6$ using Cuisenaire Rods. Putting a unit of one underneath the yellow rod of 5 to represent $1/5$. And a unit of one underneath green rod of 6 to represent $1/6$.	
$1/5$ and $1/2$ using Cuisenaire Rods. Using the interrelationship between the sizes. Red rod of 2 represented $1/5$ in relation to the orange rod of 10. Yellow rod of 5 represented $1/2$ in relation to the orange rod of 10.	
$1/7$ and $1/8$ using ruler and red masking tape. They divided a masking tape of 56 cm into 8 pieces to represent $1/7$ (the first row) and divided a masking tape of 56 cm into 7 pieces to represent $1/8$ (the second row).	
Table 5.1. Artefacts that represented the unit fractions	

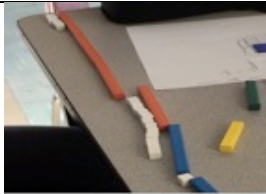
Table 5.2 shows some artefacts that represented the two fractions to be added – that is, an artefact that showed $2/7$ or $2/5$.

$2/5$ and $1/2$ using fraction strips. Using the interrelationship between the sizes of the strips	
$2/7$ and $1/8$ using transparent paper. Using the measuring affordance of a ruler to divide a strip of 24 cm by 7. They then coloured two parts. They did similar for $1/8$.	

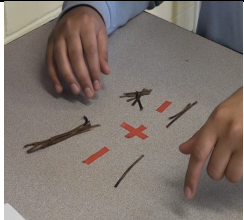
$2/7$ and $1/8$ using poster board. They used the cut-ability affordance of a poster board to cut a strip into two parts. Then they drew 7 lines on one and 6 lines on the other to show $1/7$ and $1/8$. They then coloured two parts on $1/7$ strips and one part on $1/8$.	
$2/7$ and $1/8$ using scotch tape and ruler. They used measuring affordance of a ruler and sticky-ness and shiny-ness affordances of scotch tape to create a line of 24 cm. they then marked every 7 cm on the tape and every 8 cm on top of the tape. Then they coloured 2 lines on $1/7$ part of the tape.	
Table 5.2. Artefacts that represented the two fractions to be added	

Table 5.3 shows shows some artefacts that represented the equivalent fractions with the new denominator being the common denominator.

$16/56$ and $7/56$ using paper and ruler. They used the measuring affordance of a ruler and cut-ability of a poster board to make two strips of 56cm. They marked the 56 lines and coloured 16 pieces on one and 7 pieces on the other to show $1/8$ and $2/7$	
$16/56$ and $7/56$ using the ruler. They used the measuring affordance of a ruler to put lines on the ruler to show $16/56$ and $7/56$.	
$12/30$ and $5/30$ using fraction board. They used the interrelationship between the sizes of the strips and the sizes of the chart to colour $12/30$ and $5/30$.	

* $5/12$ and $6/12$ using fraction strips. They used the interrelationship between the sizes of the strips to show $5/12$ and $6/12$. They meant to use $1/10$ ths.	
$12/30$ and $5/30$ using rods. They used the interrelationship between the sizes of the rods to create two lines each 30 units. Underneath the first one they put 5 units of one to show $5/30$. Underneath the second one they put 12-units of one to show $12/30$.	
Table 5.3. Artefacts that represented the equivalent fractions	

Finally, Table 5.4 shows are shows some artefacts that represented the adding together of the two fractions.

$1/2 + 2/5$ using fraction strips	
$1/8 + 2/7$ using ribbons and masking tape. Children used the cut-ability affordances of masking tape and ribbons to represent $2/7 + 1/8$	
$1/8 + 2/7$ using ribbons and red making tape	
$1/8 + 2/7$ using the ruler and a line	
Table 5.4. Artefacts that represented the equivalent fractions	

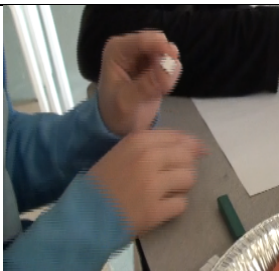
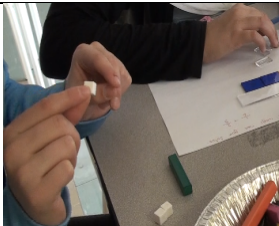
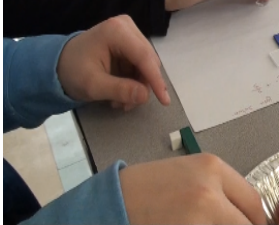
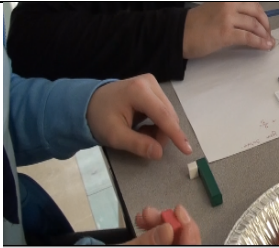
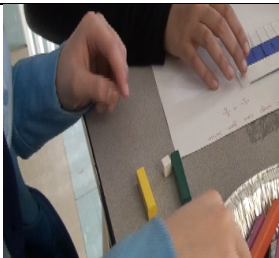
The children, working in relation to the task of adding two fractions, created all of the artefacts in these interactions to help think about and solve the task. Particularly, my analysis of all of the children's interactions with the tools confirmed that the process of creating an artefact, in all problem-solving stages, was gradual and complex and was closely related to:

- The ways in which the children perceived the mathematical affordances of the tools – through the feedback provided by the tools – to create and to use the artefacts;
- The children's mathematical knowing of fractions in general and of the addition of fractions in particular; and
- The task of adding two fractions.

More specifically, children whose knowing of fractions was not sufficiently developed had difficulties solving problems using the feedback from the tools. I refer to this knowing as a 'not-so-clear knowing'. Moreover, if the children's perceptions of the affordances of the tools were not clear enough, then the children once again had difficulties solving problems under the guidance provided by the tools. I refer to this perception as a 'not-so-clear perception'.

In the discussion set out below, I examine two interactions of the children with the tools in order to thoroughly illustrate the gradual and complex process of creating the artefacts and in order to highlight the multifaceted interrelationship between the creation of an artefact, children's mathematical knowing and the task.

Example 5.4 – Creation of artefact – $\frac{1}{6}$ and $\frac{2}{5}$. The series of events set out in this example show S's interaction with the Cuisenaire Rods as she attempted to use the physical properties and the feedback from the rods to solve $\frac{1}{6} + \frac{2}{5}$. In her participation in the interaction with the Cuisenaire Rods, S used the feedback from the rods as well as her knowing of the addition of fractions to think about and solve the task. She started the interaction by creating an artefact that represented the two fractions of $\frac{1}{5}$ and $\frac{1}{6}$ using the rods. Yet, at the beginning of this interaction, she was not able to perceive any feedback from her created artefact that would guide her in the process of adding the two fractions. In other words, the artefact that she created was not useful to her. The following excerpt from the transcript shows this interaction:

02:03			Y to S: Would you like to tell me what you are doing? S: Sure. So this is one for me	1
		<i>[Holding a one unit Rod]</i>		2
			One out six plus two over five	3
02:11			S: So this is one out of six for me	4
02:16		.	S: So now I am going to keep it over here	5
		<i>[Puts the rod of 1 underneath the dark green rod of 6]</i>		6
02:17			S: Now I need to figure out what I could use for two over five.	7
		<i>[She then takes the red rod] [She puts the red rod back]</i>		8
02:23			S: I think I want one that worth five.	9
		<i>[She takes the pink rod of four]</i>		10
			S: I think this one is four so I need to find the block that is five	
02:41		<i>[She takes the yellow rod and exchanges it with the pink]</i>		11
			S: I think this is five. So yes compared to that one, this one is five. So two out of five.	12
02:47		<i>[She takes two whites and puts them underneath the yellow]</i>		13

			S: Now I can try... [3 second pause] I am not sure how to do this.	14
Team 3 – Second Interview – Jan, 19, 2015				

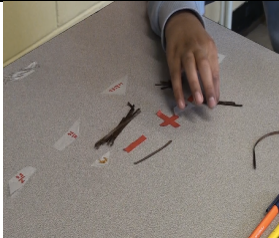
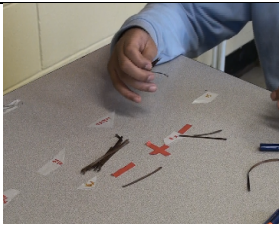
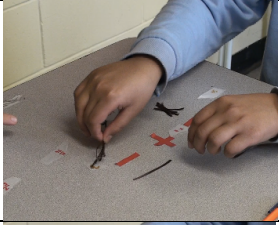
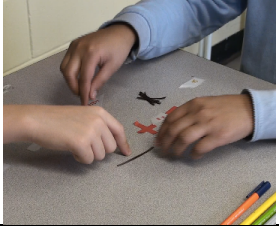
In the above extract, S participated in an interaction with Cuisenaire Rods. In this interaction, she used the feedback from the Cuisenaire Rods – that is, the interrelationship between the sizes of the rods – to create an artefact. The reason for the creation of this artefact was to help her to represent the fractions of $\frac{1}{6}$ and $\frac{2}{5}$ with the rods. She first explained that the white unit of one represented ‘one’ for her: ‘So this is one for me [*holding a one unit Rod*]’ (turns 1-2). She then created a new artefact by putting the ‘*the rod of 1 underneath the dark green rod of 6*’. She afterwards, using the feedback provided from the sizes of the rods – that the white unit of one being, size-wise, $\frac{1}{6}$ of a dark green unit of six – perceived this new artefact to represent the fraction of $\frac{1}{6}$: ‘So this is one out of six for me’ (turns 4,6). Now S attached a new meaning to the rods, as she perceived $\frac{1}{6}$ -ness in the way in which the green rod of size was placed above the white rod of one.

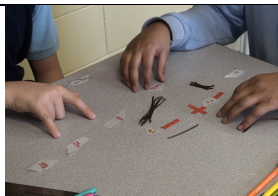
Perceiving the affordances of other rods, S attempted to create an artefact that, as she explained, she ‘could use for two over five’ (turn 7). She tried a red rod of two and ‘*[She put the red rod back]*’ (turn 8). To her the red rod of two could not afford her to create an artefact that represented $\frac{2}{5}$, as she said ‘I think I want one that worth five’ (turn 9). She then took a pink rod of four. Using the feedback from the rods, she perceived that the rod of four could not afford making $\frac{2}{5}$: ‘I think this one is four so I need to find the block that is five’ (turn 10). She then picked a yellow rod of five. She used the feedback from the interrelation between the size of the pink rod of four and the yellow rod of five and stated: ‘I think this is five. So yes compared to that one, this one is five. So two out of five’ (turn 12). Through this statement S clearly demonstrated that she had used the feedback from the Cuisenaire Rods to create $\frac{1}{5}$, by comparing the size of the pink rod with the size of the yellow rod.

The interaction, however, reached a tension point as S, pausing for three seconds, stated: ‘I am not sure how to do this’ (turn 14). S could not perceive any feedback from the created artefacts (see the image in turn 13) to guide her in the process of adding the two fractions of $\frac{1}{5}$ and $\frac{1}{6}$. She could not attach any meaning to her artefacts that might help her in the process of adding the two fractions. Therefore, at this stage her created artefact was not useful. However, S continues working with the rods to later use this created artefact to add

the two $\frac{2}{5}$ and $\frac{1}{6}$. I discuss the rest of this interaction, in chapter 6.

Example 5.5 – Creation of artefact – $\frac{1}{8} + \frac{2}{7}$. The second interaction shows how K and A used the cut-ability affordances of the ribbons and masking tape to create an artefact that represented $\frac{1}{8} + \frac{2}{7}$. In the participation of K and A with the ribbons and tapes, there was no feedback to guide them in the process of problem solving. K and A started adding signs to the tools they used based on their previous knowing of fractions. The following excerpt from the transcript shows how they explained what they did and the artefact they created:

09:48			A: We like visualised it. We took two of these	1
		<i>[pointing to the two pieces of ribbon on top of the masking tape]</i>		2
			A: Like we took some ribbons	3
10:01			A: And then for 7 pieces we cut	4
		<i>[picks up the ribbon from the table]</i>		5
			A: Put seven for the denominator and two more for the numerator	6
10:06			A: and then we put a plus sign	7
		<i>[pointing to the + sign]</i>		8
			A: Then we cut... This is eight	9
		<i>[pointing to the second denominator]</i>		10
10:15			K: and then we put one because it is the numerator	12
		<i>[pointing to the numerator of the second fraction]</i>		13

			K: and then we found the least common denominator. So you have to do the opposite. So you have to times it so that they can be the same number. So seven times eight equals 56 and eight time seven equals 56.	14
10:34			K: So here you have to put the 56. Then you have to multiply what the top is.	15
		<i>[pointing to the pieces of paper with the numbers of 56 written on the denominator and 16 and 7 on the numrators]</i>		16
Team 6 – Third Interview – Jan 22 nd , 2015				

In the above extract, K and A participated in an interaction with ribbons and masking tape. Since there was no feedback designed in these tools, the children had to perceive different affordances of the tools to create an artefact, whose feedback in turn, would help them to think about or to add $1/8 + 2/7$. They perceived the cut-ability affordance of the ribbon to create an artefact that would help them ‘visualised it [the two fractions]’ (turn 1). By attaching signs to the use of tools, they explained that to represent the fraction of $1/7$, they cut 2 pieces of ribbon to represent the numerator; they used masking tape to present the fractional line, and then another 7 ribbons to represent the denominator. They repeated the same steps to present the fraction of $2/8$. They created an artefact whose feedback did not help them in adding two fractions. Later, they explained that they ‘found the least common denominator’ (turn 13), by multiplying 7 and 8, they said: ‘seven times eight equals 56’, which they wrote down on a piece of paper. They then wrote $7/56$ and $16/56$ on two small pieces of papers and explained that the answer is $23/56$. The artefact that A and K created here did not help them to add two fractions using the physical properties of the tools. It only helped them to re-present the two fractions with the tools and to solve the task using their previously learnt steps of adding two fractions.

In the above examples, I explained how children created artefacts as they attached signs to the tools to solve the tasks. In the process of adding two fractions children created multiple artefacts, both useful ones and ones that were not quite useful. A few important

points to raise here are as follows:

- Not all of the problem-solving activities – within both rounds of interviews – included representations from all of these categories of artefacts (i.e., the 4 steps of adding fractions a to d).
- Not all of the created artefacts in each category were similar. In other words, children using the same tool and attempting to perform the same task created different artefacts depending on how they perceived the affordances of the tool and depending on their mathematical knowing. For example, in one interview the children used pieces of fractions strips to create the artefact illustrated below (see Figure 5.4) and to solve $1/2 + 2/5$. They argued that, since there was one $1/10$ -piece missing to get to the whole, it follows that $1/2 + 2/5$ is $9/10$.

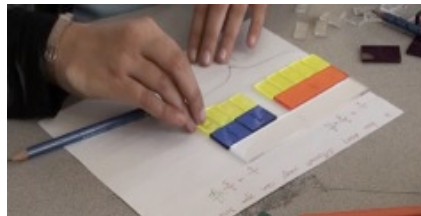


Figure 5.4. An artefact that represented $1/2 + 2/5$

Another group of children, using the same tools of fraction strips and attempting to perform the same task of $1/2 + 2/5$, created the artefact illustrated in Figure 5.5. This team argued that, since nine pieces of $1/10$ fit underneath two pieces of $1/5$ and a piece of $1/2$, it follows that $1/2 + 2/5$ is $9/10$.



Figure 5.5. An artefact that represented $1/2 + 2/5$

- The artefacts created by the children were not at all times useful. Artefacts were useful or limited in their usefulness based on how the children, with their knowing of the addition of fractions, perceived the feedback received from the artefact. This was

the case even for the third round of interviews in which the children had to construct the feedback from the artefacts. For example, the artefact illustrated in Figure 5.6. was created by the children to solve $1/8 + 2/7$. It shows two rows of red masking tape partitioned into seven and eight parts (rows 1 and 2 from the top), each of which was 56 cm in length. The third row had a length equal to two $1/7$ pieces and one $1/8$ piece.

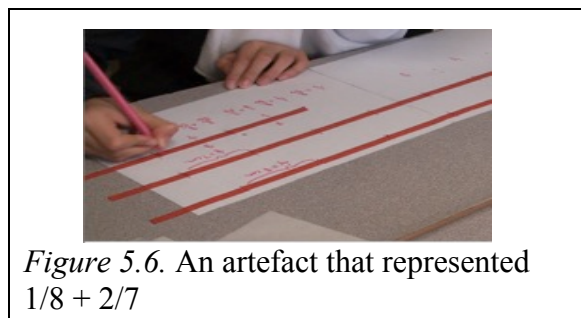


Figure 5.6. An artefact that represented $1/8 + 2/7$

After the creation of this artefact, the children were not able to perceive the affordances of and the feedback from the artefact to guide them to add the two fractions (i.e., to find the length of the third row) and to act accordingly and thus solve the task. This artefact, despite being created by the children, was not useful. But what was it (or was not) related to this artefact that was not useful? I look into this question by re-examining the notion of feedback from the artefacts.

Artefacts and the Feedback From the Artefacts

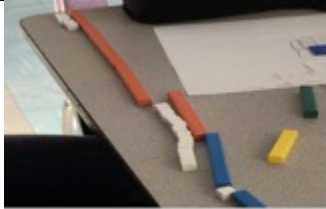
Earlier in this chapter, I explained how in both rounds of interviews (the second round and the third round), as well as across all types of tools, the children used or constructed the feedback from the tools to perceive their mathematical affordances and then used the perceived affordances as a guide to solve the mathematical task at hand. Mainly, in the second round of interviews, the children used the feedback embedded in the design of the tool to initiate the problem-solving activity. Whereas, in all of the activities in the third round of interviews, the children had to construct the feedback to modify the affordances that the tool could provide in terms of solving the mathematical task of adding two fractions. In this section, I take the analysis a bit deeper to discuss how the children used the feedback from the tools to create artefacts and/or modify their created artefacts. In turn, they used the modified artefacts to think about or solve the task.

Previously, I noted that in each problem solving activity, the children created multiple artefacts. The feedback from each of these artefacts then guided the children in thinking

about or solving the task. A finding of this study is that in both rounds of interviews (the second round and the third round) as well as across all types of tools, the children attempted to create new artefacts because either they did not perceive the feedback provided by the existing artefact to be useful, or they did not perceive the feedback provided by the existing artefact to provide sufficient guidance to solve the task.

The sequence of images set out in the next three examples show how the artefact changed through each problem-solving activity, with each artefact providing newer feedback guiding the children to think about or solve the task.

Example 5.6 – Change of artefacts using Cuisenaire rods.

		
<i>Figure 5.6.1.</i> Artefact to show the two denominators of 30. $x/30 + x/30$	<i>Figure 5.6.2.</i> Artefact to show $5/30 + 12/30$	<i>Figure 5.6.3.</i> Artefact to show $17/30$

In this sequence a child made different artefacts using the same feedback (the interrelation between the sizes of the pieces of the Cuisenaire rods) to add $1/6 + 2/5$. The newly created artefacts in each turn, made the newer affordances of the artefact more apparent, hence constructing newer feedback. First, the child used the relationships among the sizes of the Cuisenaire rods to make two rows of thirty (Figure 5.6.1). Then they made the same properties of the rods to create an artefact. The artefact now is not merely the pieces of rods: it is the ways in which the white rods of one unit were arranged underneath the orange rods of 10 units (and the combination of rods of blue (9-unit) and white (1-unit) to create more 10-unit rods). The feedback in this artefact now made the affordances of the rods in creating two fractions of $5/30$ and $12/30$ more apparent (Figure 5.6.2). Similarly the ways, in which 17 white rods of 1 unit are put underneath a row of 30, created an artefact with the feedback to make $17/30$ -making affordances of the rods more apparent (Figure 5.6.3).

That is, although in all the steps of the above sequence the child used the same tools of Cuisenaire rods, but the types of feedback from each created artefact in different steps

made certain affordances of the rods more apparent, such as the $5/30$ -making affordances of the rods or the $17/30$ affordances of the rods, for that matter. Consequently, in the steps of the creation of each artefact, children used the feedback from the artefact to think about the next step or to solve the task: $5/30 + 12/30 = 17/30$

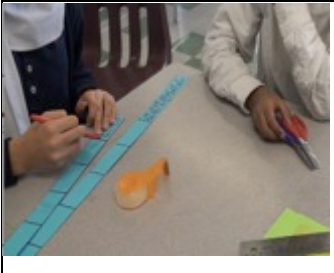
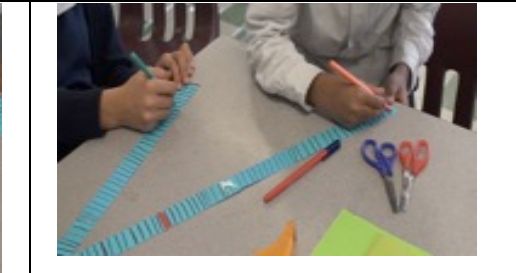
Example 5.7 – Change of artefacts using fraction strips.

	
<i>Figure 5.7.1. Artefact showing $1/2 + 2/5$</i>	<i>Figure 5.7.2. Artefact showing $1/2 + 2/5 = 9/10$</i>

Similarly, in this interaction, a child made different artefacts using the feedback for the pieces of fraction strips – interrelation between their sizes – to solve $1/2 + 2/5$. However, each created artefact provided a different newer feedback in the process of problem solving. The child created the first artefact by putting the orange piece of $1/2$ next to two dark blue pieces of $1/5$ (Figure 5.7.1). These three pieces put in a row are not like any other pieces of the fractions strips scattered around the table. They are specifically put in a row so that the feedback from this newly created artefact makes the $1/2 + 2/5$ – making affordances of the strip more apparent. Later the child created a newer artefact (Figure 5.7.2), which had 9 pieces of 10ths underneath the previously made artefact of $1/2 + 2/5$. This newly created artefact, which included the combination of the pieces arranged in two rows, provided newer feedback. It showed that one piece of $1/2$ and two pieces of $1/5$ put in a row had the same length as 9 pieces of $1/10$ put in a row. Children used the feedback from the last artefact (Figure 5.8.2) to solve the task:

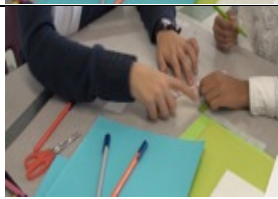
$$1/2 + 2/5 = 9/10$$

Example 5.8 – Change of artefacts using poster board and marker.

		
<i>Figure 5.8.1.</i> Artefact showing $2/7$ and $1/8$	<i>Figure 5.8.2.</i> Artefact showing $x/56$	<i>Figure 5.8.3.</i> Artefacts showing $7/56$ and $16/56$

In sequence 3, the children used the measuring affordances of ruler and the cut-ability affordances of the poster board to solve $2/7 + 1/8$. First, they created an artefact, which included two strips, each 56cm, and partitioned them into 7 and 8 parts. There was no immediate feedback from the tools, or the created artefact to help them think about or solve $2/7 + 1/8$. Then the children constructed feedback by colouring two parts on the $1/7$ strip and one part on the $1/8$ strip (Figure 5.8.1). Now they created an artefact from which the feedback makes the $2/7$ and $1/8$ – making affordances of the poster board more apparent. Yet, the feedback from this artefact was not useful enough to lead the children to think about the task of adding $2/7$ and $1/8$. Therefore, they created a newer artefact using the measuring affordances of the ruler. That is, they used the strips of poster board and partitioned them into 56 parts (Figure 5.8.2). They then used the created artefact to colour 7 parts of one and 16 parts of the other to solve the task. At the end, they used the feedback from the final artefact – that is two strips, partitioned into 56 parts, in one strip there were 7 parts coloured and in the second strip there were 16 parts coloured. So they solved the task: $16/56 + 7/56 = 23/56$

An important issue to point to here is that the process of constructing the feedback was as complex as the process of creating the artefact. This was because it was the ways in which the children perceived the mathematical affordances of the developing artefact-to-be that led the children to modify the existing artefact in order to create a newer one. For example, in an interaction of two children with pieces of transparent paper, the children made two strips of 17 cm each. They divided one strip into seven equal parts and the second strip into eight equal parts. They then coloured the first part on the eighth strip to show $1/8$. The following interaction shows how the children tried to make an artefact that represented $2/7$ and that at the same time could afford guiding them to solve $1/8 + 2/7$:

10:32			S: okay now colour these two...	1
		<i>[shows the second and third parts of the 1/7]</i>		2
			S: because we need to	3
		<i>...[points to the first coloured part on the 1/8 strip]</i>		4
10:33			S: ... skip one	5
		<i>[shows the first part on the 1/7 strip]</i>		6
			cause to add...these two	7
		<i>[shows the second and third parts of the 1/7]</i>		8
Team 5 – Third Interview – Jan, 20, 2015				

S explained to Y that he needed to colour '*[the second and third parts of the 1/7]*' (turn 2). Her reason for colouring the second and third parts of the 1/7 strip was related to how the strips of 1/8 were coloured on the first part. She said because we need to *[points to the first coloured part on the 1/8 strip]*, further stating 'skip one' (turn 5). She provided an explanation for skipping a part: 'casue to add...these two' (turn 7). S's suggestion to skip a part was due to her perception of the mathematical affordances of the artefact-to-be. She wanted to use the feedback from the newer artefact to add 1/8 and 2/7.

The above example is representative of 5 similar occasions, in which the children either disagreed about how to construct the feedback or they had to explain to each other their rationale for constructing certain feedback.

In all cases of the construction of all the feedback – both ones that children agreed on and ones that they did not – the goal was always to create a newer artefact whose affordances would better guide them to think about the problem or to solve it.

Summary

In this chapter, I have employed Gibson's concept of affordances and Vygotsky's notion of the object/meaning ratio and artefact to analyse children's interactions with two different types of mathematical tools, both ones with certain mathematical elements designed into them (e.g., fraction strips and fraction board) and ones without (paper and masking tape).

My analysis of both rounds of interviews (the second and third rounds) and my

analysis across all types of tools show that the children participated in dialectic and mediated interactions with the tools while they attempted to solve different addition of fractions problems. More specifically, they used the feedback from the tools to perceive the affordances of the tools and then they used the perceived affordances to solve the mathematical task.

Through the examination of children's perceptual change as well as their creation of artefacts, I showed how the children's perceptions, of the tools and of the mathematical meanings, were highly intertwined with the ways in which children build the meaning(s) of the tool, of their own actions, and of the mathematical task at hand. I also discussed that the process of creating an artefact as children tie signs to the use of tools is a *gradual* one because it is difficult for children to grasp the relationship between mathematical tools and the mathematical concepts that they are intending to represent (McNeil & Uttal, 2009; Norman, 1993). Finally, I provided examples from the data to demonstrate how children used the feedback from the tools or constructed feedback to create an artefact and how the feedback from the artefact mediated between children's interaction with the tools and their thinking about or solving the addition of fractions tasks. Attachment of mathematical meaning to the tools suggests a system of relationships among the tasks, the children's perception of the physical properties of the tools, and the mathematical *learning* that is happening as the children use the tools. In the next chapter, I examine the possible learning(s) that happened as children interacted with the tools to solve the tasks.

Chapter 6

Findings on the ZPD and the More Knowledgeable Other(s)

In this chapter, I will show the findings regarding the emergence (or not) of the zone of proximal development, as well as regarding the role of the more knowledgeable other, as the children interacted with the tools to solve the addition of fractions tasks, across both rounds of interviews (i.e., the second round and the third round) and for all tool types.

Discussion of these findings will be the basis for addressing my second research question:

What role is played by mathematical tools in the emergence of a Zone of Proximal Development during the child's solving of addition of fractions problems?

Showing the results of the emergence of ZPD as children interacted with the tools, I begin by closely looking at the role of 'the more knowledgeable other(s)'. In my examination of the more knowledgeable other(s) in children's interaction with the tools, I pay special attention to who/what was (were) the more knowledgeable other(s) and how the role of the more knowledgeable other(s) alternated between the children and the tools. I then examine how the ZPD(s) emerged and how, in the emergence of the ZPD, the role played by the physical properties of the tools (and their affordances) became apparent. To do this, I look at the emergence of the ZPD, by focusing on the possibilities that became available to the children for newer forms of expression, action and reflection in relation to adding two fractions, as they interacted with the tools.

In the process of closely examining the notions of the more knowledgeable other and the ZPD, I bore in mind the conceptualisation of Goos, Galbraith and Renshaw (2002), of Graven and Lerman (2014), and of Roth and Radford (2010). All these researchers conceptualised the ZPD as bi-directional and the role of the more knowledgeable other as alternating between the participants. I extend the alternating view of the more knowledgeable other to include scenarios within which I viewed the role of the more knowledgeable other not as alternating but rather as co-constructed. I start with a general summary of main findings.

Summary of Main Findings

In general, the analysis of data showed the following two findings in relation to ZPD(s) and 'more knowledgeable other(s)':

- In the children's interaction with the tool, tool-mediated learning sometimes

occurred. By tool-mediated learning, I mean the possibilities that became available to the children as they interacted with the tools and as they perceived the affordances of the tools to become acquainted with newer forms of expression, action and reflection within a certain historically contingent cultural practice (Radford, 2013); in the case of the interactions in my study the socially contingent practice was to use the common denominator to add two fractions. This finding is an extension to the Vygotskian view of ZPD, in which ZPD(s) are sign-mediated and emerge inter-subjectively (i.e., between people) (Bartolini Bussi, 2007; Goos, 2014; Lerman and Meira, 2001; Roth and Radford, 2010; Roth, 2014; Steele, 2001).

- Within the ZPD, at times, the learning happened under the guidance of the more knowledgeable other(s). My analysis shows that in the children's interactions with the tools, children used the feedback from the tools to think about or to solve the addition of fractions task, which made the tools the more knowledgeable others. This finding is also an extension to the already existing view of Vygotsky's theory in which the more knowledgeable others are conceptualised as people such as adults or peers.

In the rest of this chapter, I present these findings in more depth, supported by detailed examples of the children's interactions with the tools.

The More Knowledgeable Others

Vygotsky, in his description of ZPD, explicitly referred to 'problem solving under adult guidance or in collaboration with more capable peers' (Vygotsky, 1978, p. 69), referring to the more knowledgeable other(s) as an 'adult' or 'more capable peers'. Following Vygotsky, other researchers also looked at the 'more knowledgeable others' as being people (Bartolini Bussi, 2007; Goos et al., 2002; Goos, 2014; Graven and Lerman, 2014; Lerman and Meira, 2001; Roth and Radford, 2010; Roth, 2014; Steele, 2001; Tudge, 1992; Valsiner, 1984).

In this study, I am expanding the inter-subjective notion of ZPD to include the guidance provided by the tools. I build on Roth and Radford's (2010) conceptualisation of 'participation' – in which the zone of proximal development is viewed as an interactional achievement that allows all participants to become teachers and learners (p. 305) to carefully describe the more knowledgeable others. By the more knowledgeable others I mean *the*

participant(s) in an interaction whose knowing or feedback are used in different stages of the process of the problem-solving activity to think about the problem and/or to solve it. In children's interaction with the tools, I consider the 'participants' to be the children and the tools and I see the role of the more knowledgeable other as alternating between the children and the tools. Now, how do I see the children or the tools as more knowledgeable others? And how do I see the role of the more knowledgeable other as alternating?

Children as More Knowledgeable Others

Children who participated in this study had previous knowing about fractions in general, and about the addition of fractions, in particular. Consequently, throughout their interactions with the tools, they used their knowing to interact with the tools and/or to solve the tasks. To examine children's knowing and employing Roth and Radford's conceptualisation, by 'children's knowing' of the addition of fractions I mean the ways in which the children were able to express, reflect and act with regards to the task of adding two fractions and in relation to the previously known process of addition of fractions, namely finding the common denominator, finding the equivalent fractions, and then adding the two fractions together.

My analysis showed that in order to think about the steps of adding two fractions and/or to solve the tasks, at times, children made statements that referred to their previously known process of the addition of fractions. These statements were uttered either with or without any explicit reference to the physical properties of the tools with which they were interacting. In such circumstances, I refer to the children as the participant(s) whose knowing was used in a particular stage of the process of the problem-solving activity to think about the problem and/or to solve it. That is, the children became the more knowledgeable others. The following three examples show how I refer to children as the more knowledgeable others.

Example 6.1 – J as the more knowledgeable other – no references to the tool.

J: so you calculate that and whatever you do to the denominator ... this is the denominator [*shows seven on the sheet of paper*] you have to do to the numerator... so it's two times eight and one times seven

Example 6.2 – K's as the more knowledgeable other – no references to the tool.

K: so what we can do... we can find a denominator... that's... A denominator... that like... that they are both... that is multiple of seven and eight?

The above utterances are evidence of children's knowing of the common denominator. What I refer to as 'knowing' is the way in which they were able to express and act. Their knowing of the addition of fractions made it possible for them to think, reflect and act in accordance with the socially contingent practice of using the common denominator. Hence, I refer to the children as the more knowledgeable others, as opposed to, for example, the physical properties of the tools.

Example 6.3 – Y and S as the more knowledgeable others – references to the tool.

In the next interaction, the children were working with the fractions board to add $\frac{2}{5}$ and $\frac{1}{2}$.

02:38-03:10			S: What are we suppose to do right now?	1
			S: then we have to create one that has the same denominator	2
			Y: least common denominator?	3
			Yasmine: Okay... how do we do that?	4
			Y: you multiply the two denominators	5
			S: yah...so it will be five times two and two times five and then the top times two	6

I chose this example to show what I mean by utterances that were made with or without references to the tools, by comparing and contrasting turn 2 with turns 5 and 6. In turn 2, to explain what they did when adding the two fractions, S stated: 'then we have *to create one* that has the same denominator'. Perceiving the physical properties of the fraction board, she suggested that in order to add the two fractions they needed 'to create one [strip] that has the same denominator'. Even though in this utterance S referred to the physical properties of the tool, she referred to it in relation to her previous mathematical knowing of the addition of fractions, and she said that they needed to create a strip that 'has the same denominator'. Hence, S was the more knowledgeable other. Later, in turns 5 and 6, the

children said:

Y: you multiply the two denominators

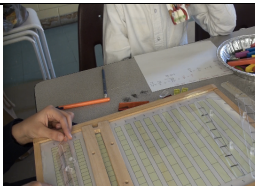
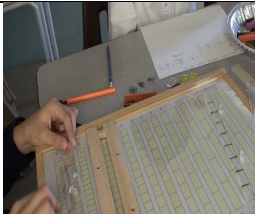
S: yah... so it will be five times two and two times five and then the top times two

These utterances were made as Y and S used their mathematical knowing of the addition of fractions with no particular reference to the physical properties of the fraction board, again making Y and S the more knowledgeable others.

Tools as More Knowledgeable Others

In analysing the data, I noticed that to think about and/or to solve the addition of fraction tasks, the children made statements that referred to the physical properties of and the feedback from the tools. At time, in such interactions, newer possibilities for thinking, reflecting, arguing, and acting became available to the children. These new possibilities for expression and action were not related to their previously known concept of addition of fractions. On the contrary, they only emerged as the children used the feedback from the tools to think about or solve the tasks. In such circumstances, I refer to the tools as the participant(s) whose feedback was used in a stage of the process of the problem-solving activity to think about the problem and/or to solve it. That is, the tools become the more knowledgeable other. The next example illustrates this event.

Example 6.4 – fraction board as the as the more knowledgeable other. In this interaction Y and S used fraction board to add $1/6 + 2/5$.

19:54			S: cause we have two	1
		<i>[picks up the 1/5 strip and puts it in front of the 1/30 line of the chart on the board]</i>		2
19:55			S: this one is double and each on it six...	3
20:02			S: here we have one two three four five	4
		<i>[counting the pieces on the 1/30 line with the 1/6 strip]</i>		5

			And that would end up to be seventeen.	6
Team 5 – Second Interview – Jan19th, 2015				

In this interaction, S used the feedback from the fraction board to add $2/5 + 1/6$. She took the strip of $1/5$ and put right above the $1/30$ line in the chart of the board and explained that ‘S: cause we have two this one is double and each on it six’ (turns 1-3). The feedback from the tools in this interaction was the way in which two parts on the strip of $1/5$ covered 12 parts on the $1/30$ line on the chart. She then stated that the one part on the strip of $1/6$ s would cover 5 on the $1/30$ line on the chart, she said: ‘here we have one two three four five’ as she counted ‘*[the pieces on the 1/30 line with the 1/6 strip]*’. She then stated that ‘And that would end up to be seventeen’ (turn 6).

Interacting with the fraction strip and using the feedback from it provided S with newer possibilities for thinking, reflecting, and acting with regards to adding two fractions. That is, for S, $2/5 + 1/6$ equalled $17/30$ not because she followed the already known process of adding two fractions, namely:

$$\begin{aligned}
 &2/5 + 1/6 \\
 &= 12/30 + 5/30 \\
 &= 17/30
 \end{aligned}$$

In fact, $2/5 + 1/6 = 17/30$ because two parts on the strip of $1/5$ would cover 12 parts on the $1/30$ line of that chart on the board. Similarly, 1 part on the strip of $1/6$ would cover 5 parts on the $1/30$ line of that chart on the board, and as she said: ‘And that would end up to be seventeen’ parts on the line of $1/30$, so to make $2/5 + 1/6$ equal $17/30$. These new possibilities for expression and action only emerged as S used the feedback from the fraction board to think about and solve the tasks. The tool is a participant whose feedback was used in different stages of the process of the problem-solving activity to solve $2/5 + 1/6$. That is the tools became the more knowledgeable other.

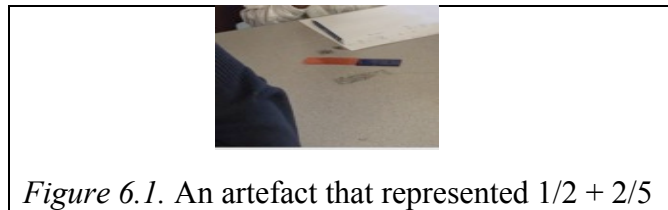
Alternation of the Role of More Knowledgeable Other(s)

To examine the alternation of the role of the more knowledgeable other I use Wertsch’s (1993) and Roth and Radford’s (2010) interpretations. In both perspectives, the ‘more knowledgeable other’ is not viewed in terms of its institutional and/or societal position(s) (e.g., teachers versus students, or parents versus children). Roth and Radford’s (2010) notion of the participant ‘in the know’ – the one whom Wertsch (1993) called ‘the

source of authority’ – claims that *the more knowledgeable other* arises through collaborative interaction of the participants in which the role of being the more knowledgeable other alternates among them.

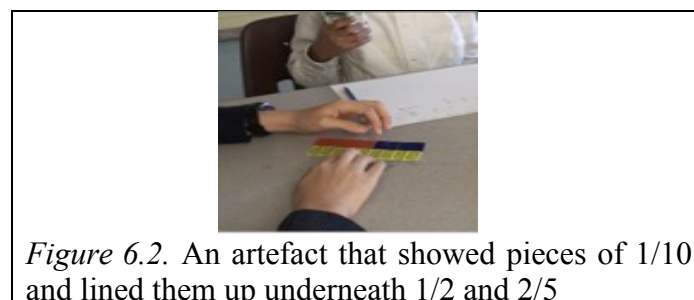
In example 6.5, I show how I view the alternation of the role of the more knowledgeable other, as feedback and knowing from different participants were used in different stages of the problem solving activity. In the following section, I have picked three examples (6.5, 6.6, and 6.7) from the same problem-solving activity, to emphasise this alternation.

Example 6.5 – Alternation – S as more knowledgeable other. To solve $1/2 + 2/5$, S and A put the pieces of $1/2$ and $1/5$ s in a row on the desk (see Figure 6.1).



The participants in this interaction were S and A, as well as the pieces of fraction strips. To add $1/2$ and $2/5$, S, with her mathematical knowing of the addition of fractions [*reaching into the fraction-strips container*] said: ‘So how much of this would equal to? It equals to one tenth’. In this stage of the problem-solving process, it was S’s knowing that was used to think about or solve the problem. She knew, from her previously learning of the addition of fractions that $1/10$ is the type of piece that should be chosen. In fact, this is what they selected. At this particular stage of the interaction, S was the more knowledgeable other.

Example 6.6: – Alternation – Fraction strips as more knowledgeable other. Continuing with the same interaction mentioned in example 6.5, A and S collected all of the available pieces of $1/10$ and lined them up underneath $1/2$ and $2/5$ (Figure 6.2).



Then they used the feedback from the strips (i.e., the interrelationship between the sizes of the fraction strips) to solve the task as they counted and said: ‘one, two, three, four, five... nine... nine... now this is nine out of ten’. In this interaction the participants were S, A and the pieces of fraction strips. At this stage of the problem-solving process, as the children used the feedback provided by the physical properties of the fraction strips to solve the task, they saw that nine pieces of the $\frac{1}{10}$ strip would fit underneath one piece of half ($\frac{1}{2}$) and two pieces of $\frac{1}{5}$. Hence, *at this particular stage*, the fraction strips were the more knowledgeable other, in the sense that the interrelationship between the sizes of the strips guided the children in solving the problem, with nine pieces of $\frac{1}{10}$ fitting underneath $\frac{1}{2}$ and two pieces of $\frac{1}{5}$.

Example 6.7 – Alternation – Yasmine as more knowledgeable other. Later, in the same activity, I asked the children why they used the pieces of $\frac{1}{10}$ and not, for example, the pieces of $\frac{1}{12}$. I said: ‘and you said why not one over twelve? [*holding a the $\frac{1}{12}$ strip*]’. In response to my question, S said ‘we could try it but I do not think it would work’. She then moved away the $\frac{1}{10}$ line, and started collecting the $\frac{1}{12}$ s. In this interaction, the participants were S, A, the pieces of fraction strips, and I. In this stage of the problem-solving process, it was my question that helped the children in thinking about a novel way of solving the problem. Hence, at this particular stage of the interaction, I was the more knowledgeable other.

In this section, I have explained how children’s knowing and the feedback from the tools guided the children through the process of problem solving, making them the more knowledgeable others. I specifically focused on children’s knowing(s) because such a focus in turn made it possible for me to examine the learning that emerged (or did not emerge) from their interactions with the tools. In the next section, I will examine children’s possible learning during their work on addition of fractions problems, through the emergence of the ZPD.

Learning Within the ZPD

As mentioned before, I utilised Roth and Radford’s (2010) conceptualisation of *knowing* as ‘the possibilities that become available to the participants for thinking, reflecting, arguing, and acting in a certain historically contingent cultural practice’ (p. 301). Yet the

possibilities of thinking, reflecting and acting do not just happen on their own. Knowing happens through *learning* – what Radford (2013) referred to as a social and sign-mediated process of becoming acquainted with historical, cultural forms of expression, action and reflection. For Vygotskians, learning happens through the ZPD (Lerman, 2014; Vygotsky, 1978; Radford, 2013).

To examine the children's participation in the interactions with the tools and their consequent learning within the zone of proximal development, I used the conceptualisation of Wertsch (1993) and of Roth and Radford (2010). Roth and Radford (2010) looked at the ZPD as 'the emergence of a new form of collective consciousness, something that cannot be achieved if we act in solitary fashion' (p. 306). Wertsch's (1984) interpretation of the ZPD pointed to the possibility of learning in collaboration among individuals, who deliberately interact to accomplish a goal.

In my study, a common element running through all of the problem-solving activities, in both rounds of interviews and among all of the types of tool, was the fact that the children participated in interactions with the tools to solve a mathematical task in which the participants were both the children and the tools. I should mention that, over the course of the analysis, I was aware of the fact that the children were interacting with the mathematical tools in the presence of other children and me. Therefore, the participants in these interactions were not always just the children and the tools. Nevertheless, my emphasis in this study was on the moments when the children, individually or in a team, were thinking, reflecting and acting interactively with a focus on the physical properties of the mathematical tools. These interactions might have been triggered by questions/suggestions from others (including me).

My analysis shows that, in the children's interaction with the tool, tool-mediated learning sometimes occurred. By tool-mediated learning, I mean that the children become acquainted with newer forms of expression, action and reflection in reference to the physical properties and affordances of the tools. Detailed analysis of all problem-solving activities shows that the children frequently drew from the feedback provided by the physical properties of the tools to think about and/or solve the addition-of-fractions problems. Therefore, over the course of the children's interaction with the mathematical tools, a ZPD at times emerged under the guidance provided by the feedback from the tools. My focus here was on careful examination of the role of the feedback (the physical properties of the tools) and of the guidance provided by them in the emergence of the ZPD.

In the rest of this chapter, I show these findings in more depth, supported by detailed examples of the children's interaction with the tools. I present five examples and within each example, I present:

- a. how I viewed the emergence of the tool-mediated ZPD, by examining the possibilities that became available to the children for newer forms of expression and action, in relation to their interaction with the tools.
- b. the alternation of the role of the more knowledgeable. I will follow this discussion with a closer look at who/what was the more knowledgeable other(s), if anything/anybody at all.

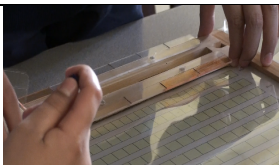
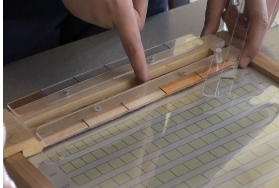
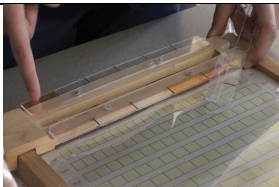
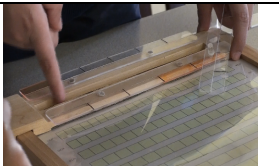
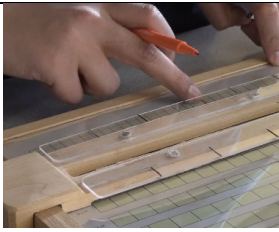
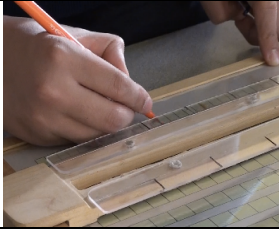
Example 6.8 – A, H and fraction board. In this interaction, A and H used the fraction board to solve $2/5 + 1/6$. For the first five minutes of their interactions with the fraction board, A and H tried to perceive its affordances by asking questions or by interacting with its different parts.

The alternation of the role of the more knowledgeable other. In this part of interaction, I assisted the children in this process of perceiving the affordances of the tools by asking questions or by making comments. For example:

- A: so what are these for? This one... *[points to the chart]*
 Yasmine these are all divided into different pieces
 A: I do not get what this thing is for then *[picks up the roller]*
 Yasmine oh we did not get what is this good for? Maybe we can... cause this really goes in there *[picks up the 1/6 strip and puts it on the roller]*

In the above interactions, at times I was the more knowledgeable other because my comments and actions helped the children in the process of problem solving. Comments included 'these are all divided into different piece' and 'Maybe we can... cause this really goes in there'. Actions included picking up '*[the 1/6 strip and [putting] it on the roller]*'.

The following excerpt from the transcript shows how H and A then solved the problem (i.e., $1/6 + 2/5$) using the fraction board.

14:33		<i>H: [stands up and touches the two sides of the roller]</i>		1
			H: but... how do you	2
14:54			H: oh... I see... so you see... we are trying to	3
		<i>[slides his hand underneath the coloured parts of the 1/5]</i>		4
15:00		<i>H: [move his finger into the centre of the roller]</i>	H: ya... you see... now I get it... H: so you see... and then count it	5
			H: and then this one too	6
		<i>[points to the 1/6 part]</i>		7
			H: this would be five	8
		<i>[pointing to the centre of the roller on the 1/6 part]</i>		9
		<i>[counting the pieces on the board through the roller's centre]</i>	H: and this one would be one two three four... six seven eight nine ten eleven twelve	10
17:06		<i>[starts from the end of the 12 spot mark and counts 5 parts forward]</i>	H: and then you count one two three four five	11
17:09		<i>H: [he then marks the chart on the 17 part]</i>		12
Team 2 – Second Interview – Jan, 19, 2015				

In this interaction the participants are the fraction board and its components, and H. H

was able to perceive a few affordances of the fraction board, like for instance the role of the roller and the relationship between the roller and the strips. He still was not very sure of how to use the properties of the fraction board to solve the task of $2/5 + 1/6$ as he asked ‘but... how do you?’ (turn 2), and almost immediately thereafter, he said ‘oh... oh... I get it now...’. Then, sliding ‘his hand underneath the coloured parts of the $1/5$ ’ he said: ‘oh... I see... so you see... we are trying to so you see... and then count it [*move his finger into the centre of the roller*]’ (turns 3-5) (see the image in turn 5).

In this interaction, H used the feedback from the fraction board as a basis for saying ‘so you see ... and then count it’. The feedback in this case was the way in which the marked lines were designed on the chart on the board to have a relationship with the parts of $1/5$ on the strips, as he used his finger to show how far he needed to count on the chart lines (see the image in turn 6-7).

Then he again used the feedback from the fraction board to point to the coloured part of the $1/6$ strip and said: ‘and then this one too’ (turn 6). He, using the feedback from the fraction board, said ‘this would be five’ (turn 8) while ‘[*pointing to the centre of the roller on the $1/6$ part*]’ (turn 9) (see the image in turn 8).

He then counted the two coloured parts of $1/5$ on the chart and said: ‘and this one would be one two three four... six seven eight nine ten eleven twelve’. At this stage of interaction, the fraction board was the more knowledgeable other, because H was using the feedback provided by the fraction board to add the two fractions. The feedback was the way in which the lines on the chart were designed in relation to the lines on the $1/5$ and $1/6$ strip.

At the end, H deleted all of the marks on two strips and coloured 17 sections on the $1/30$ line of the chart, to yield the answer of $17/30$. Here again, H used the physical properties of the board to add $2/5$ and $1/6$. At this stage of the interaction the more knowledgeable other was the fraction board, as the feedback provided by these properties helped H in the process of problem solving in terms of counting the matching parts on the chart board.

In this interaction, I did not notice any alternation of the role of the more knowledgeable other, because as the interaction unfolded, H used the feedback from the tools to think about that task, and not necessarily his own previous knowing of the addition of fractions.

The emergence of tool-mediated ZPD. In this particular part of the interaction the participants were H and the pieces of the fraction board, where H solved $1/6 + 2/5$ using the pieces of the fraction board. In this interaction, the fraction board made it possible for H to

express, reflect and act in a newer way, in relation to his already-known way of expressing the steps of adding two fractions. In other words, in this interaction, $1/6 + 2/5$ was equal to $17/30$ – not because 30 is the common denominator and not because he needed to do whatever he did to the denominator to the numerators ($5/30 + 12/30$), but instead because two coloured parts in the strip of $1/5$ (i.e., $2/5$) covered 12 parts in the $1/30$ line of the chart, because one coloured part of the $1/6$ strip matched five parts of the $1/30$ line of the chart, and because the act of putting together all of the coloured parts in the line of $1/30$ yielded $12 + 5 = 17$ parts on $1/30$ line which equals to $17/30$.

This new way of reflecting and expressing the addition of two fractions emerged as H participated in an interaction with the fraction board. Moreover, it was the interrelationship among the sizes of the parts of the strips and the lines on the chart that guided H through new forms of reflecting, talking and acting. Therefore, I propose that H's learning of the new affordances of the tool and of the newer way of talking about adding two fractions developed through the emergence of the ZPD, with guidance provided by the physical properties of the fraction board.

Example 6.9 – S, Y and fraction strips. The following series of events show S and Y's interaction with the fraction strips as they attempted to use the feedback from the fraction strips to solve $1/2 + 2/5$. In their interaction with the fraction strips, S and Y used the feedback from the strips as well as their knowing of the addition of fractions to add $1/2$ and $2/5$. They used the feedback from the fraction strips to create an artefact that represented the two fractions of $1/2$ and $2/5$. The children initially used their own mathematical knowing of the addition of fractions to find the common denominator ($1/10$) and to solve the task. Yet, later on, prompted by my question, they used the feedback from the fraction strips to explain how and why they used $1/10$ as the common denominator and they proceeded to re-solve the task.

The difference between this the interaction and the interaction in example 6.8 is that in this interaction, children could not immediately perceive any useful feedback from the tools in relation to adding the two fractions. Hence, the ZPD did not emerge as they initially interacted with the tools. The newer forms of expression and action became available to them only after I prompted the children to compare and contrast the properties of different strips of the fraction strips.

The alternation of the role of the more knowledgeable other. The excerpts of the

transcripts set out below show S and Y's interaction with the fraction strips as they attempted to solve $1/2 + 2/5$.

05:41			S: so a half and two fifths...lets try half and two fifths again	1
		<i>[puts the half down and gets a 1/5 strip, flips over to see the number]</i>		2
			I like the two fifths...put the other fifth here	3
		<i>[she puts the second 1/5 strips next to the other one]</i>		4
Team 5 – Second Interview – Jan, 19, 2015				

In turns 1 to 4, S used the feedback from the tool (i.e., the written numbers on the strips) to perceive the mathematical affordances provided by the fraction strips and to modify the arrangement of the pieces. She put '*[the half down and got a 1/5 strip]*' (turns 1-2). Asking for another strip of $1/5$, she said 'I like the two fifths...put the other fifth here' and 'she puts the second $1/5$ strips next to the other one'. She thus created an artefact that represented $1/2 + 2/5$ (see the image in turns 1-2). S created this artefact by the using the feedback from the strips. The feedback at this stage is the relative sizes of $1/2$ and $1/5$ strips. At this stage of interaction the more knowledgeable others are the fraction strips, as the feedback from them was used to create the new artefact.

S, however, did not immediately perceive any useful feedback from the newly arranged fraction strips to assist her in adding the two fractions. She used her knowing of the addition of fractions and stated that strips of $1/10$ would be useful. The following excerpt from the transcript shows this interaction.

05:52			S: so how much of this would equal to?	5
		<i>[reaching into the fraction-strips container]</i>		
06:09			S: it equals to one tenth	6
		<i>[picks a 1/10 strips and puts it underneath the 1/2]</i>		7
			one tenth	8
		<i>[picks another 1/10 and puts</i>		9

		<i>it next to the other]</i>		
06:17		<i>Y and S: [starts lining up the 1/10 strips underneath 1/2 and 2/5]</i>		10
Team 5 – Second Interview – Jan, 19, 2015				

In relation to the task of adding $1/2$ and $2/5$, the feedback from the strips was not useful enough to guide the children through the process of problem solving. Instead of using the feedback from the tool, S used her mathematical knowing of the addition of fractions to perceive that $1/10$ was the piece that could afford what was needed: ‘So how much of this would equal to? *[reaching into the fraction-strips container]* It equals to one tenth’ (turn 5-6), she said. At this stage, S was the more knowledgeable other because it was her own knowing of the common denominator which assisted her in choosing the pieces of $1/10$ and not necessarily the physical properties of the fraction strips.

Then they picked up the pieces of $1/10$ and lined them up underneath the pieces of $1/2$ and $2/5$. Thus, they created an artefact that represented the equivalent fractions to $1/2$ (i.e., $5/10$) and $2/5$ (i.e., $4/10$). Lastly, S and Y used the feedback provided by their newly created artefact – one that showed the equivalent fractions (i.e., pieces of $1/10$ s lined up underneath the pieces of $1/2$, $1/5$ and $1/5$) – to add the two fractions. The following excerpt of transcript shows this interaction.

06:33		<i>[starts recounting using one finger]</i>		11
			S: one two three four five... nine... nine... now this is nine out of ten	12
Team 5 – Second Interview – Jan, 19, 2015				

Even though the perceived usefulness of $1/10$ was related to the mathematical knowing of S, S and Y perceived the feedback from the fraction strips – that is, the interrelationships among the sizes of the strips – and used it to solve the problem as they started *[lining up the 1/10 strips underneath 1/2 and 2/5]* (turn 10) and said: ‘one, two, three,

four, five... nine... nine... now this is nine out of ten' (turn 11). At this stage of the activity, it was the physical properties of the fraction strips that assisted the children in the process of problem solving. The children, using the relationships among the sizes of the fraction strips, perceived how nine pieces of the $\frac{1}{10}$ strip fit underneath a half piece and the two-fifths pieces. At this stage of the problem-solving process, the fraction strips were the more knowledgeable other.

Later, I asked the children why they used the $\frac{1}{10}$ pieces instead of, for example, the $\frac{1}{12}$ pieces. The following extracts from the transcript show my question and the children's response:

06:47			Yasmine: okay... why did you use 1/10?	1
		<i>[pointing to a 1/10 strip]</i>		2
			Yasmine: Why didn't you use this	3
		<i>[takes a 1/12 piece and puts it on the desk]</i>		4
			Yasmine: Would that work? You think?	5
06:53			S: we found the least common denominator... the common denominator for both of these is ten Yasmine: Okay... Y: the least common denominator if you add five times two is ten S: So	6
07:02			Y: so we added all of these to get the least common denominator... this is already read... we just need to get the least common denominator	7
		<i>[sliding his finger on the air over the 1/10 line]</i>		8
Team 5 – Second Interview – Jan, 19, 2015				

When asked why they used the $\frac{1}{10}$ pieces instead of, for example, the $\frac{1}{12}$ pieces (turns 1-5), they made statements that were related to their mathematical knowing of the

addition of fractions and not necessarily to the affordances of the strips per se. S said, ‘we found the least common denominator... the common denominator for both of these is ten’ (turn 6), and Y said, ‘the least common denominator if you add five times two is ten’ (turn 7). How do I see the emergence of the ZPD?

The emergence (or not) of tool-mediated ZPD. In this particular part of the interaction the participants were Y, S and the fraction strips. In some stages of problem solving, children used their knowing of fractions to think about the steps of adding two fractions and in other stages they used the feedback from the fractions strips. They used their knowing of the addition of fractions to express why they chose the pieces of $1/10$: ‘10 is the least common denominator’. Whereas they used the physical properties of the strips add $1/2 + 2/5$ as they counted how many $1/10$ would fit underneath: ‘one two three four five... nine... nine... now this is nine out of ten’.

Although, the ZPD did not emerge in children’s interaction with the fractions strips in terms of finding the common denominator, I argue that a ZPD emerged when children were adding $1/2$ and $2/5$ using the physical properties of the tools. The reason I argue for the emergence of the ZPD in this stage is that the way in which the children expressed, acted, and reflected in relation to the concept of the addition of fractions changed. This newer form of reflection and action was made possible for the children as they interacted with the fraction strips. That is, for the children, $1/2 + 2/5$ was not $9/10$ because of the following steps of adding the fraction:

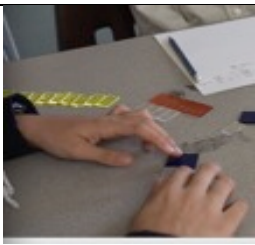
$$\begin{aligned} &1/2 + 2/5 \\ &= 5/10 + 4/10 \\ &= 9/10 \end{aligned}$$

Instead, $1/2 + 2/5$ was $9/10$ because 9 strips of $1/10$ fit underneath a strip of $1/2$ and two strips of $1/5$.

Later on, prompted by my questions, Y and S decided to use the mathematical affordances provided by the strips (rather than to rely entirely on their knowing of mathematics) to examine why the $1/12$ pieces do not work. At this stage, I was the more knowledgeable other because it was through my question that the children thought about a newer way of solving the problem and of expressing and reflecting on why the $1/10$ strip was the best choice.

07:22		<i>[holding to the 1/12 stip]</i>	Yasmine: and you said why not one over twelve?	1
07:29			S: we could try it but I do not think it would work	2
		<i>[moves away the 1/10 line, starts collecting the 1/12s]</i>		3
			Y: I do not think it would work S: me neither...	4
Team 5 – Second Interview – Jan, 19, 2015				

This interaction was initiated by my question about the reason why the pieces of $\frac{1}{12}$ were not the most appropriate choice. The children reluctantly decided to use the pieces of $\frac{1}{12}$ to see if they worked. As S said, ‘I do not think it would work’ and, as Y said, ‘me neither’ (turn 4). Then they modified the strips by moving away the $\frac{1}{10}$ pieces, by collecting the $\frac{1}{12}$ s, and by lining up the $\frac{1}{12}$ underneath the orange $\frac{1}{2}$ piece and the blue $\frac{1}{5}$ pieces, and thus they perceived that the $\frac{1}{12}$ pieces did not fit within the $\frac{1}{5}$ s. The following excerpt from the transcript shows how the rest of the interaction unfolded, as they tried to see why 12 is not the common denominator between 2 and 5:

07:31			S: for one fifth it does not work... you still have that section that is left	1
		<i>[Pointing to the uncovered space from the end of 1/12 to the end of 1/5, on the desk]</i>		2
			Y: they are like two pieces extra	3
			S: no... it is like half a piece that you can not fill... there is like	4
		<i>[pointing again to the left over piece on the desk]</i>		5
08:16			S: like that little strip	6
		<i>[slides her finger on the table to show the little strip]</i>		7

08:28			S: so that is why it equals one over ten	8
		<i>[pointing to the 1/10 row]</i>		9
Team 5 – Second Interview – Jan, 19, 2015				

In this interaction, S lined up the $\frac{1}{12}$ pieces underneath the blue $\frac{1}{5}$ pieces and created a new artefact to see if the pieces of $\frac{1}{12}$ would work in solving $\frac{1}{2} + \frac{2}{5}$. It is evident that the feedback from the new artefact dialectically modified their perception of the mathematical affordances of the strips, as she said that ‘For one fifth it does not work... you still have that section that is left’ (turn 1). The feedback in this situation was the relationships among the sizes of the pieces – in this case the pieces of $\frac{1}{12}$ and $\frac{1}{5}$. In other words, the way in which the pieces of $\frac{1}{12}$ did not fully fit underneath the $\frac{1}{5}$ pieces resulted in the creation of a small leftover element.

In this part of Y and S’s interaction with the fraction strips, new possibilities for expression, reflection and action became available to S and Y. At this stage, the children’s statements in regard to the un-fit-ability of the pieces of $\frac{1}{12}$ were related to the affordances of the strips and not to their mathematical knowing per se. Such utterances included:

For one fifth it does not work... you still have that section that is left (turn 1);

it is like half a piece that you cannot fill... there is like... like that little strip (turn 4);

and

S: like that little strip (turn 6).

In this participation, Y and S solved the problem of why ‘12’ was not the common denominator between 2 and 5 using the fraction strips. At this stage, the children used the feedback from the tool to solve the task. This participation made it possible for Y and S to become acquainted with a different form of expression, action and reflection as they perceived different affordances of the fraction strips. Y and S learnt new affordances of the tools and the fact that the $\frac{1}{12}$ pieces did not help them in adding $\frac{1}{2}$ and $\frac{2}{5}$ – not only because 12 is not the common denominator, but also because, if lined up properly, the $\frac{1}{12}$ pieces do not fit on the $\frac{1}{5}$ pieces. This knowing made new forms of expression and reflection possible for the children. Examples included statements related to the perceived feedback from the tools, such as ‘that section is left’, ‘half a piece that you cannot fill’ and

‘like that little strip’, instead of statements related to the knowing of mathematics like ‘12 is not the common denominator’ and ‘12 is not 10’.

This new approach to reflecting on and expressing opinions about the addition of two fractions emerged as the children participated in an interaction with the fraction strips. Moreover, it was the relationships among the sizes of the fraction strips that guided them through new forms of reflecting, talking and acting. Therefore I propose that the children’s learning of the new affordances of the tool and of the novel way of talking about the common denominator occurred through the emergence of the ZPD, with guidance provided by the physical properties of the fraction strips.

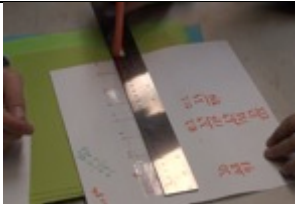
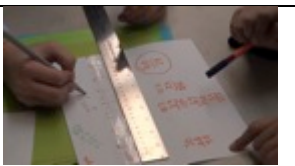
Example 6.10 – J, S, ruler and clear scotch tape. In this interaction, J and S used the affordances of the ruler and the transparent scotch tape to add $\frac{2}{7}$ and $\frac{1}{8}$. The difference between this interaction and the two others (mentioned immediately above) is that throughout the entire interaction each of the two children perceived the affordances of their created artefact differently from the other. For the first child (S), the created artefact was not useful enough to add two fractions, whereas for the second child (J) it was. So, the second child was using the feedback provided by their created artefact to add the two fractions (making the tool the more knowledgeable other) and at the same time was explaining the rationale for her actions to the second child (making her the more knowledgeable other to S at the same time). In this interaction, at times, I found it challenging to pinpoint the alternations between the more knowledgeable others.

In this interaction, children made an artefact, which was a line of scotch tape, 24 cm long. The children’s reason for selecting 24 cm was that they could divide 24 by 8, as indicated when they said ‘3, 3, 3, each right?’ They then subdivided the tape, first into eight pieces (each part being 3 cm long) and then into seven pieces (each part being 3.4 cm long). To show the $\frac{2}{7}$ the children coloured two parts of the seventh row. Here the interaction reached a point of tension because the children were not sure where to colour the $\frac{1}{8}$ piece in order to ensure that the feedback from their constructed artefact (one that showed $\frac{2}{7} + \frac{1}{8}$) would help them to solve the task (see the image in turn 1, below)

The alternation of the role of the more knowledgeable other. S’s not-so-clear perception of the feedback from the sizes of each piece made her doubt the usefulness of what J had perceived. Yet after three seconds of pause, she asked J ‘what did you just do?’

The following excerpt from the transcript shows the interaction of S and J as J

explained to S how she used the feedback from the sizes of each part. It is in the following interaction that I saw, J and the artefact being the more knowledgeable others, at the same time.

07:02			S: I think... it might not work this way... I don't	1
		<i>[3 sec pause]</i>		2
07:07			S: what did you just do? J: I added these two together	3
		<i>[pointing to the coloured parts of the tape with the back of the marker]</i>	S: oh I think oh	4
			J: nine point eight S: nine point eight... what is it in fractions that is what we need to know out of ten... or what?	5
07:16			J: wait.. this (pointing to the tape) is out of twenty four so J: nine point eight but you cannot have decimals in fractions S: You can you can J: oh okay	6
07:48		<i>S: [write down six point eight on the paper]</i>		7
			S: oh isn't that J: yes that is exactly what I have S: so I guess it is nine point eight out of twenty four	8
Team 3 – Third Interview – Jan, 21, 2015				

S, not quite perceiving the usefulness of the feedback provided by the sizes of the sections, stated: 'I think... it might not work this way... I don't' (turn 1). After a three second pause, she asked J 'what did you just do?'(turn 3). Then J used the feedback from the artefact

to explain ‘I added these two together’ (turn 3), while she pointed to ‘*[the coloured parts of the tape with the back of the marker]*’ (turn 4).

J *[writing at the bottom of the paper]* added the sizes of one part of $\frac{1}{8}$ (3 cm) and two parts of $\frac{1}{7}$ ($3.4 + 3.4$). She said: ‘nine point eight’ (turn 5). At this point the more knowledgeable other(s) were both J and the created artefact. J was the more knowledgeable other because of her knowing of fractions and her perception of the affordances of the artefact – that is that she had coloured one part of 3 cm and two parts of 3.4 cm and she said ‘I added these two together [...] nine point eight’. The artefact comprised the more knowledgeable other because the ways in which the parts had been drawn on the tape and the interrelationships among the sizes of each part in relation to the whole helped J to solve the task.

S still did not perceive the affordances of the artefact in terms of their sizes in relation to the task of adding two fractions. She therefore asked about the fractional number, as she said: ‘nine point eight ... what is it in fractions? That is, what we need to know out of ten... or what?’ J, being the more knowledgeable other, used her knowing of fractions as well as the feedback from the artefact – that is, the relationship between the size of the entire line (24 cm) and its coloured parts – and said: ‘wait... this *[pointing to the tape]* is out of twenty four so ...’ (turn 6), hence giving the result of 9.8 out of 24.

At this point, the more knowledgeable other(s) were again both J and the physical properties of the created artefact. J was the more knowledgeable other because of her knowing of fractions and her perception of the affordances of the artefact. The artefact was the more knowledgeable other because, even though they had been created by the children, the ways in which the parts had been drawn on the tape and the interrelationships among the sizes of each part in relation to the whole helped the children to solve the task. This interaction was guided by both J and the created artefact. At this stage of the interaction, I found it to be particularly challenging to pinpoint the alternation in the role of the more knowledgeable other. In turn 6, however, J’s not-so-clear knowing of fractions led her to say ‘but you cannot have decimals in fractions’. This time, S, as the more knowledgeable other, affirmed that it was in fact possible: ‘S: You can you can’. S, in the end still not fully convinced, stated: ‘so I guess it is nine point eight out of twenty four’.

The emergence (or not) of a tool-mediated ZPD. In this interaction, the participants were S, J and the artefact that they had constructed using such tools as tape, a marker and a piece of cardboard. J and S used the physical properties of the tools to create an

artefact to help them solve $2/7 + 1/8$. The interaction of J and S with the created artefact indicated the existence of a complex relationship between the mathematical knowing of the children on the one hand and their perceptions (and the changes in their perceptions) of the mathematical affordances of the created artefact on the other hand. These knowing(s) and perception(s) were mediated by the children's participation in the interactions with the tools with regard to the goal of the interaction – that is, to solve $2/7 + 1/8$. I argue that a newer way of reflecting on and expressing the addition of two fractions emerged as the children participated in a collective interaction with the artefact. Hence

$$\begin{aligned} 2/7 + 1/8 \\ = 9.8/24 \end{aligned}$$

It was the interrelationships among the sizes of the parts within the artefact that guided them through this newer form of reflecting, talking and acting. Even though it was not entirely clear for S, J perceived the relationship between the sizes of the entire tape (24 cm) and the sizes of the coloured pieces (3 cm + 3.4 cm + 3.4 cm) as constituting feedback from the created artefact that was useful for solving the task. In this case $2/7 + 1/8$ was not $23/56$, because of the children's previously known steps of adding two fractions; that is:

$$\begin{aligned} 1/8 + 2/7 \\ = 7/56 + 16/56 \\ = 23/56 \end{aligned}$$

Instead $1/8 + 2/7 = 9.8/24 (= 23/56)$, because the children have coloured 1 part of 3 cm and two parts of 3.4 cm from an entire tape which was 24 cm. Therefore:

$$\begin{aligned} 1/8 &= 3 \text{ cm} \\ 2/7 &= 3.4 \text{ cm} + 3.4 \text{ cm} = 6.8 \text{ cm} \end{aligned}$$

The whole tape was 24 cm. Hence:

$$\begin{aligned} 1/8 + 2/7 &= (3 + 3.4 + 3.4) / 24 \\ &= 9.8/24 \end{aligned}$$

This new approach to reflecting and expressing the addition of two fractions emerged as the children participated in an interaction with their created artefact. Moreover, it was the relationships among the sizes of the coloured parts and the whole that provided the children with the possibilities of the newer forms of reflecting, talking and acting. Therefore, I propose that the children's learning of the new affordances of the tool occurred through the emergence of a ZPD, with guidance provided at times by J, or by S, or by the created artefact.

Example 6.11 – A, M, ruler, paper and masking tape. In the interaction in this example, M and A used the affordances of the ruler (the 1 cm measuring marks thereon) and the affordances of the tape (its stickiness) to create two lines of 56 cm each, one divided into seven pieces and the other divided into eight pieces (Figure 6.3).

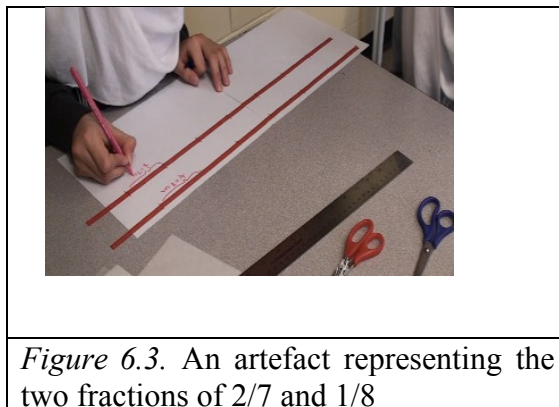


Figure 6.3. An artefact representing the two fractions of $\frac{2}{7}$ and $\frac{1}{8}$

Then they attempted to add the two fractions of $\frac{2}{7}$ and $\frac{1}{8}$, as they said:

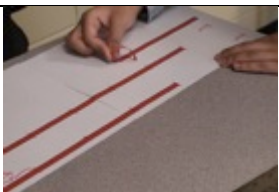
M: so that is the tape now we have to do the addition thing... do we do we solve it right?

Yasmine: yes

A: Two over seven plus one over eight

In the paragraphs below, I discuss how the children used the affordances of their constructed artefact – that is, the two rows of masking tape – to work on the task. Initially, they used one of the affordances of their newly constructed artefact (i.e., the movability of the pieces in each row) to create a newer artefact. They took two pieces out of the $\frac{1}{7}$ row (the row that was divided by seven) and they took one piece out of the $\frac{1}{8}$ row; they then proceeded to stick them on the A4 paper, one after the other to create a new row of tapes. The following excerpt from the transcript shows this interaction:

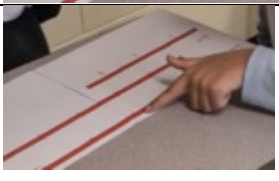
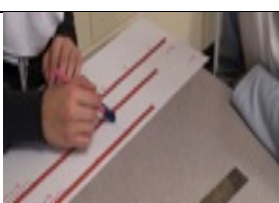
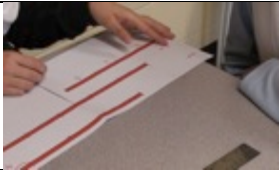
12:33			A: so it is this	1
		<i>[shows 2 parts on the 1/7 strip]</i>		2
12:38		M: <i>[takes out the pieces]</i>		3

13:04		<i>A: [gets one 1/7 piece and sticks it on the bottom of the page]</i>		4
13:16		<i>M: [gets another 1/7 piece and sticks along side of the first one]</i>		5
			A: this is two over seven	6
		<i>A: [pulls out one 1/8 part]</i>		7
			A: and this is one over eight	8
		<i>A: [sticks the piece to the side of two 1/7s]</i>		9
13:40		<i>M: [extends the cut marks of the tape on the paper, writes 1/7 under the first two parts and 1/8 under the third]</i>	M: this is one over seven A: uhum	10
Team 2 – third Interview – Jan, 20, 2015				11

They created an artefact using the pieces of masking tape. They perceived this artefact (three pieces of tape) to represent two $\frac{1}{7}$ pieces and one $\frac{1}{8}$ piece as they moved the pieces into a new row, they stated: ‘this is two over seven’ and ‘this is one over eight’.

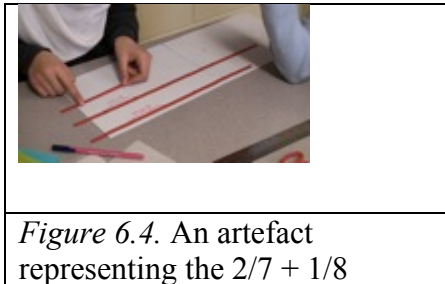
They then attempted to use the feedback from the entire artefact – the three rows of tape – to solve $\frac{2}{7} + \frac{1}{8}$. This feedback was comprised of both the sizes of the rows and the interrelationships among the positions of the tapes on the paper – that is, from where they started to where they ended. Participant A used her finger to draw an imaginary line from the end of the third row – the row representing $\frac{2}{7} + \frac{1}{8}$ – to the eighth row with a pause on the seventh row. The following excerpt shows this interaction:

13:45		<i>A: [draws a line with her finger, from where the 1/8 part finishes, to the 1/8 strip]</i>		1
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		<i>A: [her finger lands only close to a marked part... she moves her finger on the mark]</i>		2
13:48			A: this is almost equal to this	3
		<i>[showing the 2/8 line on the first row]</i>		4
		<i>M: [traces the line with the tip of the marker]</i>		5
			M: we are not sure if it is... it might be... and it might be not be... oh no actually it should be... it should be equal to both, because we're out of the common denominator...	6
14:07		<i>M: [points to the beginning of the third row]</i>	but we needed to start this at the same place	7
Team 2 – third Interview – Jan, 20, 2015				

Even though in this interaction the children attempted to use the feedback from the created artefact, the feedback from the created three-row artefact was not useful enough for the children and did not provide them with the guidance needed to solve the task. In an attempt to solve $2/7 + 1/8$, A traced from the spot where the tapes on the third row finished to the first row and stated that 'it is almost equal to this' (turn 1-2). To show that her finger had landed on the line, she marked this spot using two pieces of one eighth ($2/8$) on the first row of eighths. In order to re-check the claim, M also used the feedback from the three-row artefact (i.e., the position of the rows of tape in relation to one another) and traced an imaginary line from the end of the third row to the first row, whereupon she stated: 'it might be... and it might be not be...' (turn 6). She used her knowing of mathematics, however, to state that the end of the third row should line up with one of the lines on the first and second rows. As she said: 'actually it should be... it should be equal to both, because we're out of the common denominator' (turn 6). She then concluded that the way in which the artefact had

been created did not provide useful feedback to help them to solve the task. M argued that they needed to align the beginnings of the three rows, so that they would start from the same place: ‘we needed to start this at the same place’ (turn 7). Hence, they took the three pieces of the third row (two $\frac{1}{7}$ s and one $\frac{1}{8}$) from the left end and moved them to the right end (see Figure 6.4).



Nevertheless, the newly created artefact still did not provide them with useful feedback to solve the task. The following statements show how they could not perceive any properties of the created artefact as a useful form of feedback:

M: hmmm... still doesn't fit... does not fit

A: maybe it just won't

M: maybe not because

[pause 5 sec]

M: so maybe yeah it doesn't have to

[pause 17 seconds]

M: I kindda don't know where to go

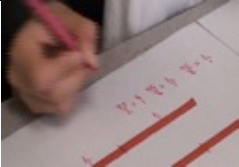
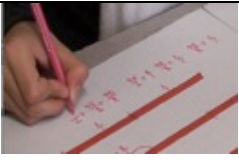
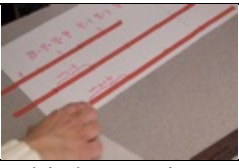
[pause 18 seconds]

M: I do not know

In the interaction mentioned above, the children created a newer artefact. This time, they positioned the pieces so that all three rows were aligned at one end (the right-hand end). The re-positioning of the pieces created a new piece of feedback, in the sense that the end of the third row ($\frac{2}{7} + \frac{1}{8}$) now was aligned with different places in the first ($\frac{1}{8}$ ths) and second ($\frac{1}{7}$ ths) rows. However, this feedback still was not useful enough and did not help them to add $\frac{2}{7}$ and $\frac{1}{8}$. Here the activity reached a tension point. They could not perceive the feedback from the artefact in such a way that would assist them in solving the problem.

Therefore, after some long pauses (together comprising around 40 seconds), they concluded that they did not know and that they ‘kindda don’t know where to go’.

After another three seconds pause, they used their mathematical knowing of the common denominator to conclude that ‘we can convert them to the common denominator’ and then add $2/7$ and $1/8$. The following excerpt shows this interaction:

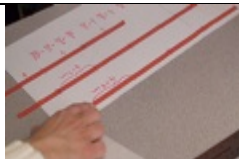
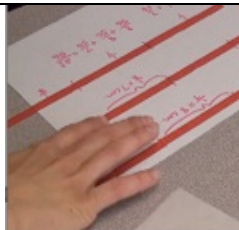
	<i>[Pause 3 sec]</i>	M: okay then we can... since we showed him the common denominator thing... we can convert them to the common denominator A: okay M: so we can do over fifty six... one over seven, quickly so	1
	<i>[starts writing on the paper: $1/7 = 8/56$ $1/7 = 8/56$ $1/8 = 7/56$]</i>		2
		seven over fifty six	3
	M: <i>[writes $8/56 + 8/56 + 7/56$]</i>		4
		M: that is 16... twenty one twenty two A: 8 plus 8 is 16 oh yeah what am I saying?	5
	M: <i>[writes $23/56$]</i>		6
Team 2 – third Interview – Jan, 20, 2015			

In this interaction, the children used the previously known process of the common denominator as they said ‘we can convert them to the common denominator’ (turn 1). Although the children used their previously learnt concept of the addition of fractions to think about ‘converting to the common denominator’, they later used the feedback from their created artefact to add the two fractions. In this interaction, they perceived the feedback from the artefact – that is, the fractional amount of each piece of the red tape in the third row – to

write $1/7$, $1/7$ and $1/8$. Then, they used their mathematical knowing of fractions to write $1/7 = 8/56$, $1/7 = 8/56$ and $1/8 = 7/56$; that is, one mathematical sentence for each piece of the red masking tape in the third row. They then wrote $8/56 + 8/56 + 7/56 = 23/56$. My rationale for arguing that they still used feedback from the artefact to write the above-stated mathematical sentences is that, after finding the common denominator, they did not write $16/56 + 7/56$ but instead they used the feedback provided by the layout of the third row to write $8/56 + 8/56 + 7/56$.

As I mentioned earlier, as part of my presence during the interviews, I asked questions so that the children's problem-solving activities would be mediated by the feedback from the tools rather than by their mathematical knowing. Hence, I continued the above-described interaction by asking various questions in order to lead A and M to think about a newer way of problem solving – hence making me the more knowledgeable other at this stage of the activity.

After M and A had solved the task, I pointed to the feedback provided by the length of each piece in the third row, and asked: 'so how big are these?' [*Pointing to the $1/7$ and $1/8$ pieces on the first and second rows*]. At this stage of the interaction I was the more knowledgeable other, as I was making the mathematical affordance of the artefact more apparent. My guidance at this stage made it possible for the children to perceive new affordances and feedback from the artefact that they had created. The following excerpt shows the interaction that was led by my questions:

17:15			Yasmine: okay good... so how big are these?	1
		[points to the $1/7$ piece on the $1/7$ row]		2
			M: eight cm long Yasmine: eight cm long	3
17:24			Yasmine: and how big are	4
		[points to the $1/8$ piece on the tape]		5
			A: those are seven	6

			M: oh that is true we didn't think of that A: think of what? M: So that it could be like because this is 7 cm and this is 8 cm and this is 8 cm	7
		<i>[pointing to each part of the third line]</i>		8
			we could have just added them like that... which is just the same thing that we did here but we did it with the calculations and that's really with the manipulatives	9
17:51		<i>M: [writes on top of each part of the third line $8 + 8 + 7$]</i>		10
			equal twenty three M: out of 56 because that is	11
		<i>[shows the length of $1/8$ strip]</i>		12
			fifty six	13
Team 2 – third Interview – Jan, 20, 2015				

In the above interaction, I was the more knowledgeable other, pointing to the newer affordances of the children's created artefacts. My questions about the sizes of each piece in the first and second rows led M to perceive new affordances of the artefact that they had created. In regard to the interrelationships among the sizes of the pieces (turns 1-6), M said 'oh that is true we didn't think of that'. However, A did not perceive the new feedback from the artefact and stated: 'think of what?' M, becoming the more knowledgeable other, explained the interrelationships among the sizes in the third row: 'So that it could be like because this is 7 cm and this is 8 cm and this is 8 cm' (turn 7). She then explained how she could have used the feedback from the sizes of the tapes to solve the task and how that would have been 'really with the manipulatives', as opposed to 'with the calculations'. She said: 'we could have just added them like that... which is just the same thing that we did here but we did it with the calculations and that's really with the manipulatives' (turn 9).

She then wrote down $8 + 8 + 7$ and said 'equal twenty three'. Then Y, still not so clear, asked 'and how do you know it is 23 over of what?' On the other hand, using the same artefact, M perceived the feedback – the interrelationship between the sizes of the pieces –

and stated that the 23 is ‘out of 56’; she justified herself by pointing to the length of the first row and saying ‘out of 56 because that is fifty six [*shows the length of 1/8 strip*]’ (turn 12-13).

At this stage of the interaction, M solved $1/8 + 2/7$ using the artefact that the two children had made – that is, using pieces of masking tape. Guided by my question, the physical properties of the created artefact provided M with an opportunity to express, reflect and act in a new way, in relation to her already-known way of expressing the steps for adding two fractions. In other words, in this interaction, $1/8 + 2/7$ was equal to $23/56$, but not because of the following steps:

$$\begin{aligned} &1/8 + 2/7 \\ &= 7/56 + 16/56 \\ &= 23/56 \end{aligned}$$

Instead, $2/7 + 1/8$ was found to equal $23/56$ because of the sizes of the pieces of tape in the third row – the row that represented two pieces of $1/7$ and one piece of $1/8$; in other words, each piece of $1/7$ equalled 8 cm and each piece of $1/8$ equalled 7 cm:

1/7	1/7	1/8
8 cm	8 cm	7 cm

$$\begin{aligned} &= 1/7 + 1/7 + 1/8 \\ &= 8 \text{ cm} + 8 \text{ cm} + 7 \text{ cm} \\ &= 23 \text{ cm} \end{aligned}$$

It was 23 out of 56 because the original size of the tape was 56 cm.

This new way of reflecting on and expressing the addition of two fractions emerged as M participated in an interaction with the strips of masking tape. Moreover, it was the sizes of each part of the tape that guided M through new ways of reflecting, talking and acting. Therefore I propose that M’s learning of the new affordances of the tool and of the newer way of talking about adding two fractions occurred through the emergence of the ZPD, with guidance provided by the physical properties of the created artefact. I mention once again that, even though I initiated the newer way of thinking about the addition of fractions through the questions that I asked, M was able to reflect and act in a new way not only to solve the task, but also to explain it to A.

Example 6.12 – A, H, fraction strips. In this interaction, A and H used the affordances of the fraction strips to add $1/2$ and $2/5$. The difference between this example and

all of the other examples mentioned above is the fact that, as the activities unfolded, even though I was able to argue that the ZPD emerged in the children's interaction with the fraction strips as they became better acquainted with newer forms of adding two fractions, it was rather difficult for me to point to who/what was the more knowledgeable other.

In their participation with the fraction strips, A and H used the feedback from the fraction strips as well as their knowing of the general concept of fractions (but not of the addition of fractions) to solve $1/2 + 2/5$.

Before an attempt was made to solve the task, A and H looked in the container containing fraction strips to find pieces of $1/2$ and $1/5$. They had some difficulty, but in the end they found the pieces and put them on the desk. A and H used the feedback from the fraction strips to create an artefact that represented the fractions of $1/2$ and $2/5$ (see Figure 7.3). At this stage of interaction, the feedback from the strips (that is the interrelationships among the sizes) guided A and H to think about the task. So I refer to the tool as the more knowledgeable other.



Figure 6.5. An artefact created by A and H representing $1/2$ and $2/5$

Then H, using his knowing of the general concept of fractions, stated that they need the whole unit of 'one'. He said: 'wait wait wait... we need the... one...oh ...here [*he then organises the $1/2$ and two $1/5$ pieces underneath the 1 whole*]' . By organising the $1/2$ and $2/5$ strips underneath the strip for a whole unit he created a newer artefact (see Figure 7.4). In this stage of the interaction A's knowing of the fraction was used to think about the task, so I become the more knowledgeable other.



Figure 6.6. An artefact created by A and H representing the relationship between the strip of one whole and $1/2$ and $2/5$

The following excerpts from the transcript show how the interaction of H and A unfolded as they attempted to add $1/2$ and $2/5$:

02:13			A: so that is... H: aa? We A: we need to see what piece is missing	1
02:16		<i>[They both go to the fraction strips container] H: [picks a piece of 1/10] A: [picks a piece of 1/12]</i>		2
02:18			H: wait... no... no...	3
		<i>A: [picks up the 1/12 pieces]</i>		4
			A: so then one ten	5
02:23		<i>H: [puts the 1/10 pieces into the left over place at the end of 1/2 and 2/5s]</i>		6
			H: so that means they are nine tenths	7
02:30			A: so we got nine tenth basically	8
		<i>H: [moves the 1/2 and 1/5 pieces away pointing to the one]</i>		9

		<i>whole piece]</i>		
			H: One minus one tenth is nine tenth A: I think we got nine ten	10

In this interaction, the participants were H, A and the pieces of the fraction strips. A and H used the the feedback from the created artefact – that is, the way in which the piece of $\frac{1}{2}$ and two pieces of $\frac{1}{5}$ were set up in relation to the strip for a whole unit – to say: ‘we need to see what piece is missing’ (turn 1). They did not use their previous knowing of the addition of fractions. In other words, they did not look for either the common denominator or the equivalent fractions. Instead, they used the feedback from their created artefact as well as their general knowing of fractions to work on the task. At this stage of the interaction, I find it rather difficult to say who/what was the more knowledgeable other. Was it A who was looking for the missing piece or was it the feedback from the way in which the strips were organised?

Next, the children used the feedback provided by the sizes of $\frac{1}{10}$ and $\frac{1}{12}$ to try and then find out which of them would fit within the missing piece (turn 2). At this stage, I refer to the tool as the more knowledgeable, as it was the feedback from the strips that help the children to perceive the affordances provided by $\frac{1}{10}$ strip, as they perceived that the piece of one tenth could afford filling up the missing section, as A said, ‘so then one ten’ (turn 5). They lined up one piece of $\frac{1}{10}$ at the end of the second $\frac{1}{5}$ piece, thus creating a newer artefact (see image in turn 6). The feedback from this created artefact, as well as H’s knowing of the general concept of fractions led H to state, ‘so that means they are nine tenths’. At this stage of the interaction, it was neither the knowing of H nor the feedback from the tool, independently, that provided the children the guidance to solve the task. The feedback in this situation was the interrelationship between the sizes and the positioning of the two rows of strips (i.e., the white strip of one whole unit compared to the sizes of blue $\frac{1}{5}$ + blue $\frac{1}{5}$ + orange $\frac{1}{2}$ + yellow $\frac{1}{10}$).

When they were asked how they did it, their responses were related to the affordances of the strips and not to their mathematical knowing of the addition of fractions per se. The following excerpt from the transcript shows this explanation:

			Yasmine: okay... how did you do that?	1
			A: because what we did... we got one half	2
02:48			A: then it was two fifth. We know one fifth plus one fifth is two fifth	3
		<i>[puts the 1/5 next to the 1/2]</i>		4
03:01			A: And we measured them with the whole number...	5
		<i>[pointing to the whole one piece]</i>		6
			the whole one... and the we had a little spot missing	7
03:11			H: and then... A: we had a little spot missing...	8
		<i>[picks up the 1/10 and puts in the spot]</i>		9
03:16			A: we fit that... we fit... the only thing that fit was the one tenth	10
03:21			A: and we knew... we knew	11
		<i>[picks up the 1/10 piece]</i>		12
			and then we minused it and we got nine tenths	13

In this interaction, A and H explained how they used the feedback from the created artefact to add $1/2$ and $2/5$.

Within their approach in their explanation, H and A did not refer to the steps of adding fractions (e.g., finding the common denominator and equivalent fractions); instead, they explained a newer approach, using the feedback from their created artefact. They said that they took a half unit and two one-fifth strips – explaining that, if they put together two

strips of one fifth, they would get two fifths:

A: because what we did... we got one half. then it was two fifth. We know one fifth plus one fifth is two fifth [*puts the 1/5 next to the 1/2*] (turns 2-4)

Next, they explained how they used the whole unit of one to add $1/2 + 2/5$. A explained that they measured $1/2$ and $2/5$ against the whole unit of one: ‘we measured them with the whole number...’ (turn 5). In turn 6, A explained what he perceived as the useful feedback from the artefact, which assisted them in the process of adding the two fractions. As A said:

A: the whole one... and the we had a little spot missing

H: and then...

A: we had a little spot missing... (turns 7-8)

Again, in turn 9 they explained that the feedback that they perceived to be useful in terms of helping them to add the two fractions. According to A, ‘we fit that...we fit... the only thing that fit was the one tenth.’ The interrelationship between the way in which the $1/2$ piece and the two $1/5$ pieces were organised in relation to the one whole constructed a feedback for the children; they perceived that they could find the missing pieces and find the strip that fits there, which was one tenth. Then, using their mathematical knowing, they stated that, in order to find the combined size of $1/2$ and $2/5$, they needed to subtract $1/10$ from the whole unit of one. Hence the answer would be $9/10$:

A: and we knew... we knew... and then we minused it and we got nine tenths (turns 11, 13)

This participation made it possible for H and A to become acquainted with a different form of expression, action and reflection, as they perceived different affordances of the fraction strips. This dialectical knowing made new forms of expression and reflection possible for the children. Put another way:

$$\begin{aligned} &1/2 + 2/5 \\ &= 1 - 1/10 \\ &= 9/10 \end{aligned}$$

This form of expression was different from the one with which they previously had been made familiar. Put another way:

$$\begin{aligned} &1/2 + 2/5 \\ &= 5/10 + 4/10 \end{aligned}$$

$$= 9/10$$

This new form of reflecting and expressing about the addition of two fractions emerged as the children participated in a collective interaction with the fraction strips. An important issue to raise here is that the feedback from the tool was not provided by the tools in and of themselves; instead, the guidance was provided from what children '*made*', using the tools – that is, the artefact. The strips of one whole, $1/2$ and two $1/5$ s scattered on the table would have not assisted the children to use the approach that they actually used (i.e., 'to minus it and to get nine tenths'). Instead, it was the complex combination of the mathematics that they already knew, the perceived affordances of the fraction strips in relation to their knowing, and the properties of the tools (sizes of the pieces) that assisted the children in creating the artefact and in solving the task. I will further examine this point in Chapter 7.

Summary

In this chapter, I have showed how, in the children's interactions with the tools, the zone(s) of proximal development emerged (not necessarily inter-subjectively) and how the children either used the feedback from the tools or their previous knowing of fractions to solve the tasks, making the children and the tools the more knowledgeable others. More specifically, I argue for the emergence of a ZPD, only if the children's interaction with the tools made it possible for the children to become acquainted with different (and newer) forms of expression, action and reflection on the concepts pertaining to the addition of fractions, as the children perceived different affordances of the tools. Furthermore, I noted that the emergence of the ZPD was related to the children's perception (or not-so-clear perception) of the affordances of the tools as well as to their knowing (or not-so-clear knowing) of fractions.

The general issue that I bring up here is whether the zone of proximal development emerged as the children interacted with the mathematical tools. Based on the evidence that I have outlined in this section, I believe that it did in fact emerge. My claim about the mediating role played by the affordances of the tools, as exemplified in the cases mentioned above, suggests a possible extension of the common interpretation of Vygotsky's ZPD. Specifically, it suggests the inclusion of the resources provided by the physical properties and of the affordances of the mathematical tools as 'the more knowledgeable other' as, at various times, it was under the guidance provided by the physical properties of the tools that the children solved the problems.

All this raises the question of how I see a *tool* as a more knowledgeable other. My

view draws from an essential property of the tools – that is, that tools (including mathematical tools) are culturally and historically based (Wertsch & Rupert, 1993; Wertsch 1984, 1985) and that their respective designs are socially originated by humans. In emphasising the social origin of the design, I do not mean that each tool was designed all at once through an act of an isolated individual. I am referring to the historical evolution of the tools as they have mediated our actions over time. Thus, I argue that the physical properties of the mathematical tools carry within them the social elements of their design and development. The more knowledgeable otherness of the tools is based on and is originated from the collection of the perceptions of many individuals who have historically designed, used and modified the tools and as Maheux (2015) puts it: the children are being exposed to a way of being in the world that has developed over time and within a specific context.

Chapter 7

Discussion

In the two previous chapters (chapters 5 and 6), I presented the findings of my study, supported by detailed analysis of the children's interactions with the mathematical tools. In this chapter, I provide a general synthesis of the findings as a whole. In comparison with chapters 5 and 6 where I presented the findings separately – in terms of the properties of the mathematical tools and the creation of the artefact on the one hand (chapter 5) and the possible emergence of tool-mediated ZPD on the other hand (chapter 6) – in this chapter I look at all of the findings collectively.

I first provide a general synthesis of the findings. Next, I discuss three main findings that arose in my study, compare my findings to those of other researchers and highlight the contributions of my study. I end this chapter by discussing two issues that emerged from the findings that made me think in greater detail namely: the alternation of the more knowledgeable others; and the process of perceiving.

General Synthesis

Tools and signs are inseparable parts of the teaching and learning of mathematics. From a Vygotskian perspective, in this research study, I investigated a sign/tool-mediated view of learning. I discussed such questions as: how tools are perceived, how signs are tied to the use of tools and, consequently, how tools are being used in the learning that happens during children's work on addition of fractions problems. I looked at Vygotsky's perspectives on the perception of the tools and learning through the lens of Gibson's view of affordances. My focus in this study was on the participation of the children in interactions with mathematical tools. My main reason for this focus was to re-emphasise the tool-mediated learning/problem solving that happens in mathematics classrooms. Consequently, from a Vygotskian view, I studied the tool-mediated learning within the emergence of the ZPD.

My analysis of both rounds of interviews (the second and third rounds) and my analysis across all types of tools show that the children participated in dialectic and mediated interactions with the tools while they attempted to solve different addition of fractions problems. In all these interactions the participants were the children *and* the tools. The children used the resources provided by the physical properties of the mathematical tools to solve the problems. More specifically, they used the feedback from the tools to perceive the

affordances of the tools and then they used the perceived affordances as a guide to solve the mathematical task.

Within this participation, there was a complex system of relationships among what the children did with the tools, what they said and the learning/knowing that took place. To look closely at each of these components (the doing, the talking and the learning), I first examined the tools as objects in their own right, exploring their physical properties and then their affordances. I then focused on the children and, in particular, on their perceptions with regard to the affordances of the tools and with regard to their learning and knowing. I also used and proposed to extend the Vygotskian notion of the ZPD to examine the learning that happened while the children interacted with the tools. This learning includes the learning about mathematics and the learning about the tools and their mathematical affordances.

I emphasise that for all of the children, one historical and cultural form of expression and action pertaining to the addition of fractions was available prior to their interaction with the tools. They were able to solve addition of fractions problems using the process of finding the lowest common denominator. Nevertheless, their interaction with the tools provided them with new possibilities for reflecting, arguing and acting. This knowing became available to them as they gradually became acquainted with the possibilities offered by the affordance of the tools, through the feedback from the tools. For example, at the end of an interaction with the fraction strips to add $\frac{1}{2}$ and $\frac{2}{5}$, the children found that the reason why the $\frac{1}{12}$ strip could not be picked as the right choice was not only that it was not the lowest common denominator, between 5 and 2, but also that, if lined up properly, the pieces of $\frac{1}{12}$ positioned underneath the pieces of $\frac{1}{2} + \frac{1}{5} + \frac{1}{5}$ would not end up being equal in length. Statements like ‘that section is left’, ‘half a piece that you cannot fill’ and ‘like that little strip’ indicate how the children used the feedback from the strips to perceive different affordances of the tools. These new forms of talking, acting and reflecting were made possible as the ZPD emerged through the guidance provided by the physical properties of the fraction strips and through the feedback received from the same, thus making the tools the more knowledgeable other.

In the next sections, I discuss in detail three issues that arose from the findings: first, gradual perceptual change; second, the creation of artefacts; and, third, tool-mediated ZPD. I discuss my views with respect to my findings and the findings of others.

First Finding – Gradual Perceptual Change

In chapter 5, I discussed how children used the physical properties of the tools, as objects, to add two fractions. I showed that it was in the complex and gradual process of perceiving the affordances of the tools that the children attached mathematical meaning to their use of these tools. Meaning was attached to each tool as children's perceptual change occurred in the process of inverting the object/meaning ratio into a meaning/object ratio as the children perceived the affordances of the mathematical tools in relation to the various tasks. My findings in that chapter supports and extends the work and findings of other scholars.

Firstly, I discuss how mathematical meanings were not represented by the tool; rather, different children perceived different mathematical meanings while using the same tool to work on the same task. Although this finding is similar to the findings of other scholars such as Norman (1993), McNeil and Uttal (2009), and Pimm (2002), I further provide a rational basis on which to examine how children attach (different) mathematical meanings to tools. With respect to the theoretical notion of the inversion of the object/meaning ratio into the meaning/object ratio, my findings show how different children perceived the affordances of the tools differently through the use of feedback from the tools and how, based on their perception of the affordances, they attached different mathematical meanings to the tools in relation to the task of adding two fractions. So, there is nothing inherent about the tools to represent the mathematical concept of fractions. On the contrary, it is up to the children and their perception of the tools and knowing of fractions to perceive useful mathematical affordances of the tools to work the addition of fractions problems. The affordances of the tools in relation to the task of adding two fractions include, for example, the interrelationships among the sizes of the Cuisenaire rods, the 1 cm incremental lines on the ruler, the cut-ability of the ribbon and the stickiness of the masking tape.

Secondly, I show that the ways in which the children used the tools varied significantly from interaction to interaction. This finding aligns with the studies of scholars like Cramer and Wyberg (2009) and Clements and McMillen (1996) in which they discuss the strengths and weaknesses of different mathematical tools by asking various questions such as 'What is it about pattern blocks that did not support students' thinking on fraction order tasks?' (Cramer & Wyberg, 2009, p. 14). My study provides a base for addressing such questions in order to explain the fact that children's interactions with the tools were strongly related to the ways in which the children perceived the mathematical affordances of the tools

– through the feedback provided by the tools and the children’s mathematical knowing of fractions in general, and through the addition of fractions in particular. It was not solely the tools that did or did not support the children’s thinking, but rather it was the combination of the children’s perception of the affordances of the tools, their knowing of fractions and the task at hand that supported their thinking about and solving of the addition of fractions problems.

Thirdly, my finding in terms of children’s gradual perceptual changes extends findings of other scholars such as Norman (1993), McNeil & Uttal (2009) and Rabardel & Samurçay (2001). These scholars showed that while interacting with mathematical tools, children often have difficulty grasping the relationship between the tools and the mathematical meaning that they are intending to represent (Norman, 1993; McNeil & Uttal, 2009; Rabardel & Samurçay, 2001). In order to investigate the difficulty of the children as they interacted with mathematical tools, for example, McNeil and Uttal (2009) provided a theoretical account to explain that any mathematical tool ‘can be thought of in two different ways: (a) as an object in its own right and (b) as a representation of something else’ (p. 43). My study extends these findings by illustrating in detail why and how children had difficulties adding fractions with the tools: the processes of perceiving affordances of the tools and attaching meaning to the tools were complex and gradual, and highly related to the knowing of the child and properties of the tools. Hence, children had difficulties grasping the relationship between the tools and the mathematical meaning either because they could not perceive affordances of the tools, or they could not attach mathematical meaning to the tools, or even if they could attach mathematical meanings to the tools and create an artefact, the feedback from the artefact was not perceived as useful.

Second Finding – Creation of Artefacts

In chapter 5, I also discussed how children used the physical properties of the tools to create artefacts. For example, they picked up and moved pieces of fraction strips in accordance with their respective sizes, they cut and pasted the ribbons, and they used the ruler to draw lines. They created artefacts as they tied mathematical symbols and meanings to the tools by, for example, talking about the tools or their actions. For instance, they made statements like ‘this is one half plus two fifths [*pointing to strips of fraction strips put in a row*]’, ‘the whole is 24 [*pointing to a tape on the green Poster board*]’ and the ‘ $\frac{1}{7}$ part is 8 cm [*showing a pieces of red masking tape*]’. Thus, the tools are no longer just tools. They are

artefacts. The children then used the created artefacts – and the feedback from them – to solve or think about the fractions problems.

I showed that the creation of artefacts to solve mathematical tasks is a complex action that is highly intertwined with the ways in which children build up the meaning of the tool, the meaning of their own actions and the meaning of the mathematical task. The process of creating an artefact as children tie signs to the use of tools is also a *gradual* process because, as is mentioned above, it is difficult for children to grasp the relationship between the mathematical tools and the mathematical concepts that they are intended to represent (Norman, 1993; McNeil & Uttal, 2009). Some important points to mention are:

- Some of the children, regardless of their knowing of fractions and/or their perception of the affordances of the tools, were unable to use the feedback provided by the artefact to think about or to solve the tasks. This implies that, if the children's knowing of the mathematical concepts was not so clear and/or was not necessarily useful, then the children experienced difficulties attaching signs to the tools and creating artefacts.
- Some of the created artefacts were not useful. In other words, at times the children created an artefact, but were unable to perceive any useful feedback from it to solve the task. For example, in cutting up pieces of masking tape the children made an artefact with three rows. In the first row there were eight pieces of $\frac{1}{8}$ and, in the second row, there were seven pieces of $\frac{1}{7}$. In the third row, they put two pieces of $\frac{1}{7}$ and one piece of $\frac{1}{8}$ to show $\frac{2}{7} + \frac{1}{8}$. The created artefact did not help the children to think about or to add $\frac{1}{8}$ and $\frac{2}{7}$.

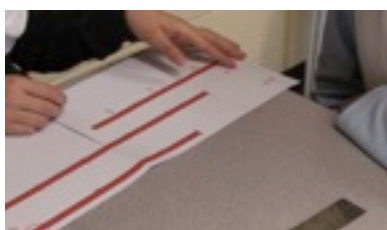


Figure 7.2. An artefact that represents $\frac{2}{7} + \frac{1}{8}$

The finding of my study in relation to the creation of artefact supports and extends the work of scholars such as Bartolini Bussi and Mariotti (2008) and van Oers (2010). According to Bartolini Bussi and Mariotti (2008), the concept of artefact is very general – it includes, objects produced by human beings over the course of history, such as sounds and gestures,

utensils and implements, oral and written forms of natural language, texts and books, and musical instruments. Examples of artefacts that are used more specifically in mathematics classrooms include books, paper, pencils, the blackboard, manipulatives and information-technology tools (Bartolini Bussi and Mariotti, 2008; van Oers, 2010). Nevertheless, following Vygotsky's notion of artefact, I argue that an artefact is not a 'thing' – an object independent of the perception of the child who is using it and independent of what the child is using it for. Rather, artefacts are created as children tie signs to their interactions with the tools in order to work with the tools in the process of working on or solving a particular task. In mathematics classrooms, children tie signs to mathematical tools by, for example, talking about what they do, talking about the mathematical concept or writing mathematical symbols as they interact with the tools. These signs are, however, attached to the tools according to the perceived *physical properties* of the tools and of the tasks. For example, if I use a ruler to measure a line, then the signs (e.g., the language or numbers) that I attach to the use of the tool are very different from the signs that I would tie to the same tool if I were to use it to stop a wobbly table. In these two instances, beginning with two different goals – measuring a line and stopping a wobbly table – I created two different artefacts using one tool (i.e., the ruler). What I argue here is that the artefact-ness of an artefact depends on how the child perceives the physical properties and affordances of the tools in relation to the goal of the activity. In other words, a ruler, as it sits on a shelf, is a tool and only becomes an artefact when it is used by a child in relation to an activity (mathematical or otherwise).

Third Finding – The Tool-Mediated Zone of Proximal Development

The general issue that I brought up in chapter 6 was whether the ZPD emerged as children interacted with mathematical tools. From a Vygotskian perspective, the ZPD is a zone within which learning happens with the helps of others. So, to analyse the emergence of a ZPD, I looked at children's learning. My conceptualisation of learning stems from what Radford (2013) referred to as a social and sign-mediated process of becoming acquainted with historical, cultural forms of expression, action and reflection. I noted the emergence of the ZPD where children became acquainted with newer forms of expression, action and reflection *only* with reference to the physical properties of the tools and with the guidance provided by the feedback from the tools. That is, I made a distinction between when the children used their own knowing of the addition of fraction and when they used the physical properties of the tools to think about or solve the task. For example, at times, the children

made statements in relation to their previously known concept of fractions or of the addition of fractions to think about the problem-solving process; statements included ‘we can convert them to the common denominator’ or ‘we can find a denominator [...] that is multiple of seven and eight?’ At other times, they used the physical properties of the tools to think about or to solve the task. They made statements like ‘For one fifth it does not work... you still have that section that is left’ and ‘it is like half a piece that you cannot fill... there is like... like that little strip’. I studied the emergence of the ZPD only when children made utterances in relation to the physical properties of the tools and not their previous knowing of the addition of fractions.

My findings with relation to the emergence of the ZPD and the more knowledgeable other are in line with the conceptualisation of other scholars such as Goos, Galbraith & Renshaw (2002), Graven & Lerman (2014), and Roth & Radford (2010). They all pointed out that ZPD provides the possibility of collaboration, where the more knowledgeable other arose from the collaborative interaction of the participants and where this role alternates between the participants (i.e., child-child interaction or child-teacher interaction). Yet, my claim about the mediating role played by the feedback from the tools suggests a possible extension to our common interpretation of Vygotsky’s ZPD as being inter-subjective. Specifically, I suggested looking at the participations not only among the children, but also between the tools and the children. Furthermore, I suggested the inclusion of the recourses provided by the physical properties of the mathematical tools. I argued that, at times, tools provided guidance in the process of adding two fractions, making tools ‘the more knowledgeable other’ (Abtahi, 2014, 2015, 2016). My rationale for viewing tools as potential more knowledgeable others stems from my view of tools. Following Wertsch (1994) and del Río, & Alvarez (1995), I see tools as being socially designed, created and developed, over time and in a particular historical context. A compass, for example, within the specific elements of its design – its handle, legs and hinge, and the way in which they fit together as a whole – reflects the perceptions of many individuals who, over time, have used, modified and reused this tool. The more knowledgeable other-ness of a compass carries within it traces of its social-cultural design and development. Therefore, stated in general terms, I assume that the ZPD is a sign-/tool-mediated zone of guided actions and talk, within which children become acquainted with newer historical and cultural forms of expression, argument, action, with the traces of the history and culture being embedded in an agent or in a tool.

With this discussion as a background, there are two theoretical issues that I would like

to discuss: the alternation of the more knowledgeable others and the process of perceiving.

First Theoretical Concern – The Alternation of the More Knowledgeable Others

I designed this study to examine if and how within an emerged ZPD, tools can be more knowledgeable others, as much as people can. My findings show that tools can be more knowledgeable others as, at times, children use the guidance from the tools to learn new ways of adding two fractions. However, a claim about which I would like to raise a concern is the necessity of the guidance within the ZPD being provided by a more knowledgeable ‘other’. This is the claim that, in order for learning to happen in an interaction, guidance from an ‘other’ is required, with this other being a tool or an agent. As mentioned throughout my dissertation, one can trace this claim to several sources such as Vygotsky (1978), Lerman (2014), Roth and Radford (2010), Goos (2014) and Wertsch and Rupert (1993). Conversely, I believe that it is worth considering the possibility of interactions within which learning happens not necessarily with one being the ‘other’ that is more knowledgeable.

In our day-to-day interactions, there are times when it is not difficult to identify the more knowledgeable other and the learner(s). For example, if I ask a person the meaning of the French word ‘sagesse’, the person who gives me the answer is the more knowledgeable other and I am the learner. Yet, there are times in some interactions when it is not easy to determine who is more knowledgeable, who is learning, and whether and how this role might be alternating. And there are times in an interaction when an attempt to identify THE more knowledgeable other could be an oversimplification of the complex interrelationship that exists between the participants. I propose that, at times, more knowledgeable otherness is mutually co-constructed by the participants as the interaction unfolds.

My assertion about the more knowledgeable-ness of the tools provides a basis for examining and questioning the possibility and usefulness of pointing to an ‘other’ that, in certain interactions, is more knowledgeable. A point I wish to pursue here is the possibility of the emergence of the ZPD(s) and the resulting learning without one participant necessarily being ‘the’ more knowledgeable other.

As highlighted in chapter 6, at times, I had difficulty pointing to the alternation of the role of the more knowledgeable other because the children’s knowing of fractions, their perceptions of the affordances of the artefacts, the physical properties of the artefact and the task at hand were so intricately intertwined that any characterisation of the interaction in a bi-directional fashion of alternation that purported to reduce the complexity of this multifaceted

interaction to a mere back-and-forth movement appeared to be an oversimplification.

Example 6.12, in chapter 6 is a nice example. In this example, I highlighted the emergence of the ZPD(s) and learning, as new forms of reflecting, acting and expressing in relation to adding two fractions became available to the children (A and H) as they participated in a collective interaction with the fraction strips. Yet, there was a point of discomfort for me as I tried to examine what/who was the more knowledgeable other in A and H's interaction with the strips, as well as to distinguish how and when the role alternated.

As I explained in chapter 6, it was neither the specific design of the tool nor the children's knowing of fractions that independently guided the children through the process of problem solving. Instead, the process unfolded firstly through the ways in which the children perceived the affordances of the fraction strips and used their knowing of fractions to create artefacts, and secondly through the ways in which the children perceived the new affordances of the artefacts that they had created. The multiple created artefacts carried within them the mathematical knowing of the children who had created them, and the ways in which the physical properties of the tool had been used to create the same. So in the process of solving the addition of fraction problem, the guidance was not provided by the children or by the tool; but rather it was complexly provided by the interrelationship between the affordances of the strips as well as by the knowing and actions of children themselves. This complexity leads me to question the necessity of the existence of an 'other' which is more knowledgeable, in some interactions. Hence there can be interactions (as was the case in example 6.12) within which learning happens and the more knowledgeable-ness is co-constructed as the interactions unfold, without one 'other' being responsible for it.

Second Theoretical Concern – Thoughts on Perceiving and Learning

In this study, I have employed Gibson (1977)'s concept of affordances and Vygotsky's notion of the object/meaning ratio to analyse the children's interactions with different mathematical tools. In particular, I showed the gradual changes in the children's perceptions of the affordances of the mathematical tools and of the mathematical meaning of the tools over the course of their interactions with them. The findings throughout all of the interactions showed how the children's perceptions, of the tools and of the mathematical meanings, were highly intertwined with the mathematical task and with their mathematical knowing.

Furthermore, within the emergence of the ZPD, I showed the newer possibilities of

expressing, reflecting and acting that became available to the children as they interacted with the tools to add two fractions. Using Roth and Radford's (2010) analogy of learning (i.e., a social and sign-mediated process of becoming acquainted with historical and cultural forms of expression, action and reflection), I characterised the children's acquaintance with these newer forms of reflection, expression and action as '*learning*'.

Whilst reading many of the children's interactions with the tools in both interviews, I saw that the 'children's gradual changes of perception' also provided them with the newer forms of expression, reflection and action. What made me somewhat confused was the relationship between the children's gradual perceptual change and learning, if both provided the children with newer forms of expression, reflection and action. I start by elaborating my point of confusion.

As shown in detail in chapter 5, the children, throughout all of their interactions, made statements illustrating the attachment of mathematical meaning to their use of tools (i.e., showing their perceptual change from the object/meaning ratio to the meaning/object ratio). For example, in an interaction of the children with pieces of ribbon, a child pointed to a white Cuisenaire rod and said 'this is zero point five' whilst, at another time, another child picking up a piece of ribbon said 'this is one over five'. It is important to acknowledge that there is nothing inherently ' $1/5$ ' about a piece of ribbon, nor anything inherently ' 0.5 ' about a rod. Rather, it was through a process – doing things with the ribbons or rods and thinking about fractions – that the children, at one point, perceived some ' 0.5 -ness' in the rod or some ' $1/5$ -ness' in the ribbon. So what is this process? As I explain in detail in chapter 5, at the point at which a child first perceived 0.5 in a rod, that piece of rod ceased to be a mere piece of plastic suitable for train-making, but henceforth was ' 0.5 '. Seeing a 0.5 -ness affordance of the rod accordingly changed what the children did with the rods (their actions) and what they said (their expressions and reflections).

These new forms of expression and action were made possible as the children interacted with the rods. Thus, following Roth and Radford's (2010) conceptualisation of learning, I could refer to what happened here as '*learning*'. Yet, this is exactly what I was not comfortable with. Because I could not figure out what was being learnt there.

Instead I decided to take a closer look at the process of perceptual change from the object/meaning ratio to the meaning/object ratio when the object becomes subordinate and the meaning becomes dominant (i.e., seeing the clock as a clock rather than as a round thing with two handles, or seeing the rod as 0.5 rather than as a piece of plastic).

In chapter 5, I discussed two processes. First, for the children to perceive the affordances of the artefacts, they used the feedback from the artefacts as well as their knowing of mathematics in relation to the task of adding two fractions. Second, in order to attach a mathematical meaning to their use of the artefacts, the children went through a gradual and complex process within which they inverted the object/meaning to meaning/object ratio.

What I would like to propose here is that, in the ‘process of perceiving’, the children also become acquainted with new forms of expression, reflection and action. Using Vygotsky’s concept of the object/meaning ratio, I refer to the process of perceiving as one within which a child starts by seeing the physical properties of the artefact (as an object of its own), gradually perceives their affordances in relation to the task and then attaches a mathematical meaning to their use of artefacts. Hence, much like learning, the ‘process of perceiving’ is also a complex and gradual process within which newer forms of expression, reflection and action become available to the children.

Summary

In this chapter, I presented a discussion on the findings of this study with regards to the role of the physical properties of and the feedback from the tools. I explained how the feedback from the tools was used by the children to perceive the mathematical affordances of the tools, to attach mathematical meaning to the tools and to solve the addition of fractions task. Furthermore, I showed how the children’s expression and reflections and actions with regards to the concept of the addition of fractions changed as they used the guidance provided by the feedback from the tools – built in the design of the tools or constructed by the children themselves. I referred to the emergence of the newer forms of expression and actions as learning within the Zone of Proximal Development. This learning was, at times, guided by the feedback from the tools making them the more knowledgeable other.

I further raised two points. First, I argued that not in every learning interaction is someone explicitly a more knowledgeable ‘other’. In some human interaction (both with tools and with others), there are times that the more knowledgeable other-ness is constructed as the interaction unfolds; as such, reducing the alternation of the role of the more knowledgeable other to a binary of back and forth is simplistic. Second, I argued that the process of perceiving is as complex and gradual as is the process of learning. And, in both processes (perceiving and learning), the ways in which children express, act and reflect in

relation to the task at hand, gradually changes.

I now use the arguments provided in this chapter to explicitly address my two research questions.

Chapter 8

Conclusion

In this chapter, I begin by addressing my research questions. At the same time I link my findings to the theoretical framework and the extant body of literature. Then I discuss the implications of this study. I finish by stating the contributions of this study to the literature.

Addressing the Research Questions

In chapters 5 and 6 (findings) and in chapter 7 (discussion), I implicitly address the research questions in considerable depth. More specifically, in chapter 5, I lay out a basis for addressing the first research question and, in chapter 6, I lay out a basis for addressing the second research question. In this chapter, I explicitly address the research questions based on the evidence that has already been presented in the three previous chapters. I used data gathered from two sets of interviews of 13 grade 7 students to address the research questions, which are as follows:

1. How does the feedback from the mathematical tools play a mediating role between the physical actions of the child with respect to the mathematical affordances of the tools on the one hand and the child's *thinking* about and *learning* and *knowing* of the addition of fractions on the other hand?
2. What role is played by mathematical tools in the emergence of a Zone of Proximal Development during the child's solving of addition of fractions problems?

Research question 1. This research question focuses on the role that the physical properties of the tools played in the children's interaction with the tools in terms of thinking about and solving the tasks of adding two fractions. Earlier, I define feedback as the physical properties of a mathematical tool and/or the interrelationships among its components that make the tool's mathematical affordances more apparent. Throughout their interactions with the mathematical tools, the children constantly used the feedback from the tools. In a sense, the feedback was a mediator between the structure and physical properties of the mathematical tools and the children's thinking about and learning of the addition of fractions. These feedback included those that had been designed into the inherent properties of the tools (i.e., the Cuisenaire rods, the fraction strips and the fraction board) and those that had not been designed into the inherent properties of the tools (e.g., masking tape, ribbons, paper and

the ruler) and were constructed by children.

More specifically, the children used the feedback from the tools to attach mathematical signs and meanings to the tools and hence to created artefacts. In turn, they used the modified artefacts to think about or solve the task. The process of attaching mathematical signs to the use of tools – within children’s gradual perceptual changes – was a gradual and complex process that was related to the ways in which the children used the feedback from the tools to perceive the affordances of the tools, as well as to their knowing of fractions and to the task of adding two fractions. This gradual and complex process explains the difficulties to which Norman (1993), McNeil and Uttal (2009), and Pimm (2002) referred – in particular, the fact that it is harder for children to grasp the relationship between mathematical tools and the mathematical concepts that they are intending to represent. Nonetheless, my findings contrast with what McNeil and Uttal (2009) and Norman (1993) referred to as ‘the mathematical concepts that they [tools] are intending to represent’. I believe that tools, by themselves, are not the representatives of anything. They only represent ‘something’ (mathematical or otherwise) to a child if the child actually perceives their affordances in relation to the task (mathematical or otherwise) and is able to attach signs to tools, in accordance with the task.

Overall, I show that, in the gradual process of artefact creation, feedback played a role in terms of how the children perceived the physical properties of the tools, how they attached signs to their use of tools, and how the newer forms of learning the addition of fractions happened as the children created artefacts with the goal of solving the addition of fraction problems. I further explain the ‘*learning*’ that happened by addressing the second research question.

Research question 2. Research question 2 focused on the emergence of the zone of proximal development in the interactions of the children with the mathematical tools. In the view of Lerman (2014), learning happens through the ZPD. In order to answer my second research question, I focused on a specific conceptualisation of learning – as a social and sign-mediated process of becoming acquainted with historical and cultural forms of expression, action and reflection (Radford, 2013). The examination of what the children said in their interactions with the tools led me to notice changes in the ways in which the children reflected and expressed themselves in relation to the process of adding two fractions. Moreover, a careful examination of what the children did with the tools led me to see how these actions and hence arguments changed as the children gradually used the feedback from

the tools to create artefacts and then used the created artefacts to solve various tasks. I related these changes in expression, reflection, actions and arguments to the newer form of learning that happened while the children were using the resources associated with the physical properties of the tools (the feedback) to think about and/or to solve the addition of fractions. My Vygotskian view of learning (i.e., that learning happens within the ZPD), therefore presupposes the emergence of ZPD(s). Thus, based on the evidence that I laid out immediately above and in the three previous chapters, I claim that the Zone(s) of Proximal Development(s) emerge as children interact with the tools to solve a task. Since, in their interactions with the tools, the children at times used the resources provided by the physical properties of the tools, I propose that, at times, the physical properties of the tools actually become the more knowledgeable other.

Implications for Practice

My study has been built on numerous other studies that have sought to analyse the usefulness of the use of the tools in mathematics classrooms. The findings of my study extend these findings by explicitly explaining the gradual and complex nature of the process that occurs when children try to perceive mathematical affordances provided by tools, by attaching signs and meanings to their use of these tools. Hence, mathematical tools are not useful merely because they are designed with mathematical meanings built into them or because their mathematical meanings are perceived by the teacher. Mathematical tools, including any tool whose mathematical affordances are perceived by the child (such as a ball or an apple), become useful to a child *only* if the child perceives their affordances in relation to a mathematical task. For example, Cuisenaire rods sitting on a cupboard do not represent $12/30$, $5/30$ or the addition of the two fractions ($12/30 + 5/30 = 17/30$), for that matter. They do this *only* if the child perceives the rods affordances in relation to the task of adding $2/5$ and $1/6$.

The result of such a proposal would lead to a re-examination of how mathematical tools might be useful in the process of learning the addition of fractions, especially given the fact that perceiving the affordances of the tools and attaching meaning to them is as gradual and complex of a process as it is the learning of the mathematical concepts, itself. Thus, in a mathematics classroom it is important to spend time ensuring that gradual perceptual changes are indeed taking place among the children as much as it is important to spend time ensuring that the learning of the mathematical concepts with the use of tools is taking place.

Limitations of This Study

The participants in this study did not have prior experiences with using designed mathematical tools to think about or solve fraction problems. Since the Ontario Mathematics Curriculum encourages the use of mathematical tools, many children in the Ontario have past experiences with the use of mathematical tools in their learning of fractions. In that sense, this study might not be a representative of the majority of school children in Ottawa. However, the theoretical findings of this study in relation to the role of feedback in children's interactions with the tools and in the emergence of the ZPD can be considered in other situations in which children may or may not have prior experiences interacting with the mathematical tools.

Moreover, the participants in this study have been acquainted with one form of expression and action in terms of adding two fractions. They already knew the procedure of finding the common denominator to add two fractions. The children's interactions with the mathematical tools would have been different if the children had not had prior experience with finding the common denominator, as in nine out of 12 interviews the children referred to or used the concept of the common denominator to think about or add fractions using the tools. However, based on the findings of my study, any child's interaction with the tools, with or without prior knowing of addition of fractions, would be different as each child has his/her knowing of fractions and his/her own perception of the tools and of its affordances in relation to adding two fractions.

In the second round of the interviews of this study, to be systematic, I deliberately selected tools that represent fractions only based on the linear model (i.e., Cuisenaire rods, fraction strips and the fraction board) and not the set or the area model. Inadvertently, however, in the third rounds of the interviews, I provided the children with tools that mostly could represent the linear model of fractions; tools such as rulers, masking tapes, scotch tape, paper, and ribbons. Hence, in almost all the third interviews children created artefacts that would function in the linear model of representing fractions, similar to the functionality of the fraction strips, for example. So this study does not provide a general view for all different fractional modelling and mainly focuses on the linear model. Had I provided the children with other forms of tools (counting chips or fraction circles), I could have studied how children would use different modelling of fractions in adding two fractions.

Contributions of This Study

My study extends the currently used theoretical underpinning of studies on the usefulness of mathematical tools in two ways:

- By combining Gibson's notion of affordances and Vygotsky's view of perception; and
- By extending the Zone of Proximal Development to include the tool-mediated emergence of ZPD.

Within the field of mathematics education, mathematical tools and their effects in the teaching and learning of mathematical concepts have been studied under different theoretical frameworks within the socio-cultural view such as (e.g., Mariana Bartolini Bussi and Michael Roth), within the constructive view (e.g., Paul Cobb, Leslie Steffe and Amy Hackenberg), within the enactivist view (e.g., David Reid and Jean-François Maheux) and so on. In this section, I only discuss the theoretical implications of this study for the theoretical discussion within the socio-cultural view. My focus in this study was on how children, with their own perceptions, utilise the physical properties of tools to construct meaning for the mathematical concept, by solving a mathematical task. Therefore, I constructed a framework by using Gibson's notion of affordances to look at the physical properties of mathematical tools and their affordances in relation to the task. In order to look into the children's perceptions and changes in perception as they interacted with the tools (i.e., the interrelationship between the tools and mathematical meanings), I used the Vygotsky's view of perception within the *object/meaning* ratio.

What this framework offers to the field of mathematics education is that it makes it possible to examine the usefulness of the tools in a more coherent way:

- The elements that I draw from Gibson's affordances provide a lens through which to examine the affordances of the tools – not as something that is inherent within the tool, but rather as something that is dynamically changing based on the perception of the child who is using the tool and equally based on the task at hand.
- The elements that I draw from Vygotsky's perception provide a lens through which to examine the complex and gradual perceptual changes of the children, though *only* with respect to the mathematical affordances of the tools, let alone with respect to the use of the tools to solve a mathematical task.
- My proposal regarding the combination of the two theories would create a lens through which to examine the usefulness of the tools. That is, looking at the physical properties of

the tools and their particular design separated from the child who is using it and the task that is used for, would not give a rounded picture neither of the usefulness of the tool, nor of the learning of the child.

While addressing my second research question, I explicitly explained how I viewed the emergence of tool-mediated ZPD as the children interacted with the tools to solve a task. This is an extension of the existing body of literature in mathematics education, which views the interactions within the ZPD as sign-mediated and inter-subjective. Examples include the views of such scholars as Bartolini Bussi (2007), Goos (2014), Lerman and Meira (2001), Radford (2013), Roth (2014), Roth and Radford (2010) and Steele (2001), among many others. My study extends the interactions within the ZPD to be sign- and/or tool-mediated and extends the notion of the ‘more knowledgeable other’ to include the resources provided by the physical properties of the tools that are used by children in mathematics classrooms.

Future Work

The following are some possible methodological and theoretical extensions to further develop this work:

Methodological extension.

- To conduct a similar study, but with younger participants who have not learnt the addition of fractions yet. Such study will make it possible to examine different formations of historically contingent cultural practices of adding two fractions, which may or may not include the notion of finding the common denominator.
- To conduct a study with tools selected from different indigenous backgrounds from inside and outside Canada, to see how cultural tools possibly effect the formation of the mathematical concepts, for example the addition of fractions. Such study would aim to explicitly elaborate “*whose* historically contingent culture” are we talking about, when we talk about the practices of mathematics.

Theoretical extensions.

- To further look at tools and their affordances from a socio-cultural perspective and not necessarily from the Gibson’s theory of affordances. Looking at Vygotskian take on the affordances, may lead to a more coherent view of tool/sign mediation in the examination of children’s interactions with mathematical tools.
- To focus on the un-addressed theoretical issue I raised in chapter 7, in terms of what the ‘other’ is in the notion of the more knowledgeable other. In particular, expanding

on my main concern about the bi-directional view of ZPD, which I believe brings down the complexity of some of our interactions and communications to a simplistic binary of back and forth.

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Appendix

Students and Parents Information Letters and Consent Forms

October 2014

Dear student,

Research Project: Things Kids Think With: The role of physical properties of the mathematical tools in children's leaning of the addition of fractions

I am Yasmine Abtahi. I work and study at the University of Ottawa. The objective of this research is to see how educational materials can help children better understand math, especially the addition of fractions. I would like to do a research project with students in your class. This research will help schools and teachers in the future. I am doing this study under the direction of my supervisor Dr. Richard Barwell.

What will happen in the research?

All students in your class, who wish to participate, will be included in this research. Students who have signed and returned this form will be grouped in teams of two. Each team will work on 6 different math activities with me. You will use educational materials to work on the math activities.

During this time I will:

- Ask you to leave the class for a duration of 45 minutes at times to work on the mathematics activities. There will be a maximum of 4 interviews.
- Ask you to think-aloud as they do the activities
- Make notes about what happens.
- Record what happens with a video recorder. I will only video record your work area, and not the faces.
- Photocopy all their written work.
- Write a report about this research project.

What will happen to the recordings and notes?

I will take all the information back to the university. It will be kept for 10 years after the project has finished. I will only share the recordings with authorised researchers (i.e., my thesis supervisor and my three committee members). We will make sure that no-one else knows who is in the project. We will write reports about the research project using the information. When we have finished the information it will be destroyed.

What happens next?

Thank you for reading this information. If you agree to take part in this research you will need to fill out a consent form. This form is where you give permission for me to do the project with you. Please note that taking part in this study is voluntary. Even if you agree now, you can change your mind at any time.

Thank you for reading this letter.

Yasmine Abtahi

Consent form: Please circle 'yes' or 'no' to show if you agree or not then sign your name at the end.

Leaving the class: I agree to I leave my class to do two math activities with Yasmine Abtahi in a separate room within the school	YES	NO
Observations: I agree that Yasmine Abtahi will observe my work, ask me some questions and make notes.		
Video Recordings: I agree that Yasmine Abtahi can video record what happens in my activities; only the working area, and not my face.		
Photocopies of my work: I agree Yasmine Abtahi may make photocopies of my work.		
Reports about the research: I agree that the researchers will write reports about the research project.		

I understand what will happen in the research project.

I understand that the recordings will only be shared with authorised (i.e., my thesis supervisor and my three committee members) and that no-one else will know who is in the project.

I know that taking part is voluntary and that I can change my mind at any time and that if I change my mind, it will not make any difference to my grades or report cards or anything like that. If you decide to withdraw from the study, the data collected from you will be destroyed.

I am aware that there are two copies of this information and consent form, one of which I can keep.

I know that if I don't like anything that happens in the research project, I should tell my teacher about it.

If you have concerns regarding the ethical conduct of the study, Please contact the Protocol Officer for research ethics, information available at the back of this letter.

Your Name: _____

Your signature: _____

Date: _____

Researcher's signature: _____

Date: _____

Signature of a witness (if you did not fill in this form yourself):

Date: _____

You can contact me, Yasmine Abtahi or the Protocol Officer for research ethics in the following ways:

Write to: Yasmine Abtahi
University of Ottawa
Faculty of Education
145 J-J Lussier
Ottawa, Ontario, K1N 6N5

Email: yabtahi@uottawa.ca

Telephone: (613) 562-5387

Write to: Protocol Officer of the Social Sciences and
Humanities Research Ethics Board
University of Ottawa
Tabaret Hall
550 Cumberland St, room 154
Ottawa, Ontario, K1N 6N5

Email: ethics@uottawa.ca

October 2014

Dear Parent/ Guardian

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- Record what happens with a video recorder. I will only video record your child's work area, and not the faces.
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- Write a report about this research project.

What will happen to the recordings and notes?

I will take all the information back to the university. It will be kept for 10 years after the project has finished. I will only share the recordings with authorised researchers (i.e., my thesis supervisor and my three committee members). We will make sure that no-one else knows who is in the project. We will write reports about the research project using the information. When we have finished the information it will be destroyed.

What happens next?

Thank you for reading this information. If you agree that your child can take part in this research you will need to fill out a consent form and return it to the school. This form is where you give permission for me to do the project with your child. Please note that taking part in this study is voluntary. Even if you agree now, you can change your mind at any time.

If you would like to know more about the project, you can talk to your child's teacher, or contact me, or my supervisor, or the university – details are shown on the back of this letter.

Thank you for reading this letter.

Yasmine Abtahi
University of Ottawa

Contact details for the Research Project: Things Kids Think With: The role of physical properties of the mathematical tools in children's learning of the addition of fractions

You can contact me, Yasmine Abtahi, in the following ways:

Write to: Yasmine Abtahi
University of Ottawa
Faculty of Education
145 J-J Lussier
Ottawa, Ontario, K1N 6N5

Email: yabtahi@uottawa.ca

You can contact my supervisor, Dr Richard Barwell, in the following ways:

Telephone: 613-562-5800, extension 6797.

Write to: Richard Barwell
University of Ottawa
Faculty of Education
145 J-J Lussier
Ottawa, Ontario, K1N 6N5

Email: richard.barwell@uottawa.ca

If you have any questions about your child's rights as a participant or to talk with someone not connected with the project, you can use the following contact information:

Telephone: (613) 562-5387

Write to: Protocol Officer of the Social Sciences and
Humanities Research Ethics Board
University of Ottawa
Tabaret Hall
550 Cumberland St, room 154
Ottawa, Ontario, K1N 6N5

Email: ethics@uottawa.ca

Consent form: Please circle 'yes' or 'no' to show if you agree for your child to participate in followings then sign your name at the end.

Leaving the class: I agree for my child to leave his/her class to do two math activities with Yasmine Abtahi in a separate location within the school	YES	NO
Observations: I agree that Yasmine Abtahi will observe my child's work, ask him/her some questions and make notes.		
Video Recordings: I agree that Yasmine Abtahi can video record what happens in the activities; only the working area, and not their faces.		
Photocopies of my work: I agree Yasmine Abtahi may make photocopies of my child's work.		
Reports about the research: I agree that the researchers will write reports about the research project.		

I understand what will happen in the research project.

I understand that the recordings will only be shared with authorised researchers (i.e., my thesis supervisor and my three committee members) and that no-one else will know who is in the project.

I know that my child's participation is voluntary and that he/she or I can change mind at any time and that if I or my child change my mind, it will not make any difference to my child's grades or report cards or anything like that.

If your child decides to withdraw from the study, the data collected from him/her will be destroyed.

I am aware that there are two copies of this information and consent form, one of which I can keep.

I know that if I am not happy about anything that happens in the research project, I can contact the university at the address given on the back of the letter.

Name of child: _____

Signature of parent: _____

Date: _____

Signature of parent: _____

Date: _____

Researcher's signature: _____

Date: _____

Signature of a witness (if you did not fill in this form yourself)

Date: _____

October 2014

Dear colleague,

Research Project: Things Kids Think With: The role of physical properties of the mathematical tools in children's leaning of the addition of fractions

I am Yasmine Abtahi. I work and study at the University of Ottawa. The objective of this research is to see how educational materials can help children better understand math, especially the addition of fractions. I would like to do a research project with students in your class. This research will help schools and teachers in the future. I am doing this study under the direction of my supervisor Dr. Richard Barwell

What will happen in the research?

All students in your class, who wish to participate, will be included in this research. Students who have signed and returned this form will be grouped in teams of two. Each team will work on 6 different math activities with me. They use educational materials to work on the math activities. During this time I will:

- Ask students who are being interview to leave the class for the duration of 45 minutes at a time to work on the mathematics activities.
- There will be a maximum of 4 interviews.
- Ask students to think-aloud as they do the activities
- Make notes about what happens.
- Record what happens with a video recorder. I will only video record their work area.
- Photocopy all their written work.
- Write a report about this research project

What will happen to the recordings and notes?

I will take all the information back to the university. It will be kept for 10 years after the project has finished. I will only share the recordings with authorised researchers (i.e., my thesis supervisor and my three committee members). We will make sure that no-one else knows who is in the project. We will write reports about the research project using the information. When we have finished the information it will be destroyed.

Please note that taking part is voluntary. Even if your students agree to participate in this study now, they can change their mind at any time.

If you would like to know more about the project, you can contact me at any time, or contact the university – details are shown on the back of this letter.

Thank you for reading this information.

Yasmine Abtahi

University of Ottawa

Contact details for the Research Project: Things Kids Think With: The role of physical properties of the mathematical tools in children's leaning of the addition of fractions

You can contact me, Yasmine Abtahi, in the following ways:

Telephone: 613-314-8620.

Write to: Yasmine Abtahi
University of Ottawa
Faculty of Education
145 J-J Lussier
Ottawa, Ontario, K1N 6N5

Email: yabtahi@uottawa.ca

You can contact my supervisor, Dr Richard Barwell, in the following ways:

Telephone: 613-562-5800, extension 6797.

Write to: Richard Barwell
University of Ottawa
Faculty of Education
145 J-J Lussier
Ottawa, Ontario, K1N 6N5

Email: richard.barwell@uottawa.ca

If you have any questions about your child's rights as a participant or to talk with someone not connected with the project, you can use the following contact information:

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Write to: Protocol Officer of the Social Sciences and
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University of Ottawa
Tabaret Hall
550 Cumberland St, room 154
Ottawa, Ontario, K1N 6N5

Email: ethics@uottawa.ca

Approval of the Study by the Schools Principal

Wa alaykum as Salaam wa Rahmatullah Sr. Yasmine,

Yes, they have approved it. The next step would be to come in and meet with me and the respective teachers to arrange to introduce the study to the students and to secure the research participant signatures.

JAK,

Dr.Mohammed Saleem,
School Principal

Abraar School: Islamic Education for the Next Generation

www.abraarschool.com

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Elementary Campus: 70 Fieldrow Street, Ottawa, ON K2G 2Y7 | ph.613-226-1396 | fx. 613-226-2745

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On Wed, Oct 22, 2014 at 3:06 PM, Yasmine Abtahi

----- Forwarded message -----

From: **Yasmine Abtahi** <Yasmine.Abtahi@uottawa.ca>

Date: Wed, Sep 24, 2014 at 9:15 AM

Subject: Research study in Mathematcis

To: 'smunawar@abraarschool.com' <smunawar@abraarschool.com>

Salaam alaikum,

My name is Yasmine Abtahi. I am a PhD student at the University of Ottawa. I am doing research on how educational material can help children better understand math, specially the addition of fractions. I would like to if your school would be interested to participate in this study. This research will help schools and teachers in the future. I am doing this study under

the direction of my supervisor Dr. Richard Barwell.

Following please find the details of the project:

-

What will happen in the research?

All 7 students in your school, who wish to participate, will be included in this research. Students who have signed and returned this form will be grouped in teams of two. Each team will work on two different math activities with me. During this time I will:

- Ask students who are being interviewed to leave the class for the duration of 45 minutes to work on some mathematics activities.
- There will be maximum of 4 interviews per pair.
- Interview students and ask them to think-aloud as they do the activities
- Make notes about what happens.
- Record what happens with a video recorder. I will only video tape their work area. No face will be included in the recording.
- Photocopy all their written work.
- Write a report about this research project.

What will happen to the recordings and notes?

I will take all the information back to the university. It will be kept for 10 years after the project has finished. I will only share the recordings with authorised researchers. We will make sure that no-one else knows who is in the project. We will write reports about the research project using the information. If you would like to know more about the project, you can contact me at any time, or contact the university – details are shown on the back of this letter.

Thank you for reading this information.

Yasmine Abtahi

University of Ottawa

Tasks and Samples of Interview Questions

Tasks used in the interviews:

I designed the tasks 1-A, 1-B and 2-A.

Task 2-B is borrowed from the literature.

Important notes about Task 2-B:

- 1- It is borrowed from a Susan Pirie and Thomas Kieren (1994) study.
- 2- Fraction Kits is a kit designed by Professor Tom Kieren. It contains rectangles, based on a common standard sheet as a unit, representing halves, thirds, fourths, sixths, eighths, twelfths, and twenty-fourths.
- 3- I received 3 sets of Fraction Kits from Professor Kieren, for the purpose of this study.

i

Tasks for the first interview session:

Task 1-A

The objective of this activity is to re-emphasise the importance of a common unit in the concept of addition. The goal is for students to see that they cannot add (put together) pieces of different sizes to measure an object, without taking their sizes into account.

Material:

- | | |
|---|---|
| · 4 colours <i>Cuisenaire</i> rods – dark green, lime green, red, and white | |
| · ribbons - 1 metre per group | |
| · tape | 6 (dark green)  |
| · paper | 3 (lime green)  |
| · ruler | 2 (red)  |
| · pencil | 1 (white)  |
| · scissors | |

Part 1- Use your 4 colours *Cuisenaire* rods (dark green, lime green, red, and white) and scissors to size and cut ribbons that are 2 rods long. If both of you follow the same steps, would all your ribbons look the same? Why? Does the unit of ‘two-rod long’ show the differences? What better unit can you think of?

Examples of prompting questions are:

- Are all the ribbons the same?
- Why do you think they are different?
- What can we do to fix this?

Part 2- this part is designed to connect part A to the concept of the addition of fractions. In this activity, I used the same 4 colours of rods – dark green, lime green, red, and white. However, this time I called the dark green a ‘unit’ or ‘one’. The goal of this activity is now to add (or put together) the red rod and lime rod considering they were representing fractional numbers.



The objective is to use the outcome of the first activity to conclude that a ‘common unit’ (that is a rod that fits on both lime green and red rods) is needed to add the rods accurately.

Task 1-B

Use the tools that you have to solve $\frac{1}{3} + \frac{1}{4}$

(Tools are ruler, paper, pen, markers, scissors, and transparent sheets)

Examples of probing questions are:

- Which one of these do you want to use?
- Why do you think that would help you?

Tasks for the second interview session:

Task 2-A

Use your tools to solve

a) $\frac{1}{3} + \frac{1}{4}$

b) $\frac{2}{5} + \frac{1}{6}$

(Tools are the fraction board, fraction strips, and the fraction kit and Cuisenaire Rods)

Examples of prompting questions are:

- What do you think these strips show?
- How can I show $\frac{2}{5}$, using these?
- How do the strips fit on the roller, if you look at these two ‘bumps’ at the back of the strips?

Task 2-B

Part 1- Using your kit, notice that one fourth, three eighths, and two sixteenths together exactly cover three fourths or that taken together one fourth, three eighths, and two sixteenths are equal in amount to three fourths. We can write $\frac{1}{4} + \frac{3}{8} + \frac{2}{16} = \frac{3}{4}$. Use your kit to find as many quantities or combinations of quantities that make exactly three fourths as you can. Draw diagrams of your findings. Write fractional number sentences like the one above.

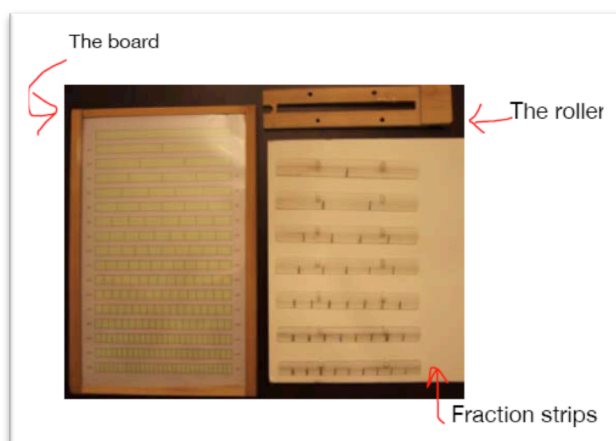
Part 2- Using halves, thirds, sixths, twelfths and twenty fourths, make two thirds in as many ways as you can.

Tasks for the third interview session:

Use these tools to add $\frac{1}{3} + \frac{2}{5}$

Tools are paper, ruler, dot paper, pencil, and ruler

A picture of the fraction board:

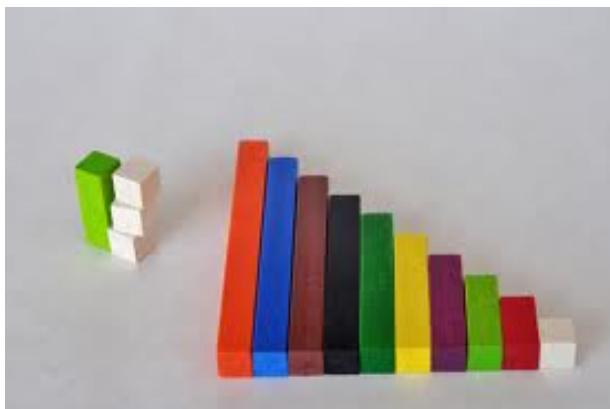


2- Fraction strips

Fraction strips are rectangular pieces of plastic strips to represent different parts of the same whole. They can be moved manipulated to see how various parts can be added together to make the whole or compare different fractional amounts for equivalency.



3- Cuisenaire Rods are a collection of rods in 10 color-related sizes. When arranged in order of length in a pattern commonly called a 'staircase' each rod differs from the next by one centimeter. The rods' related lengths provide a model, allowing you to assign a value to one rod and then assign values to the other rods by using the relationships among the rods.

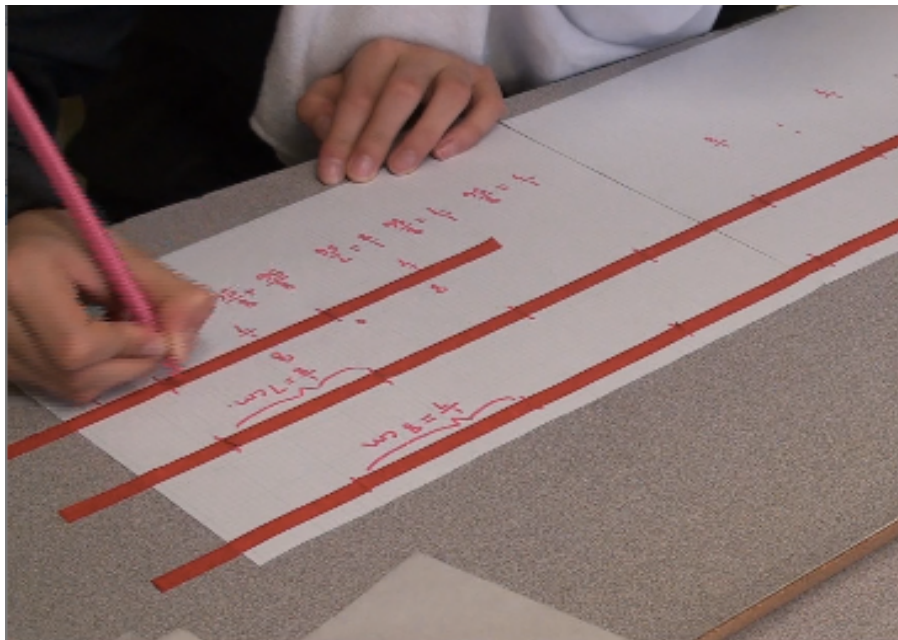


Samples of Students' Created Artefacts

Third interview

M, A

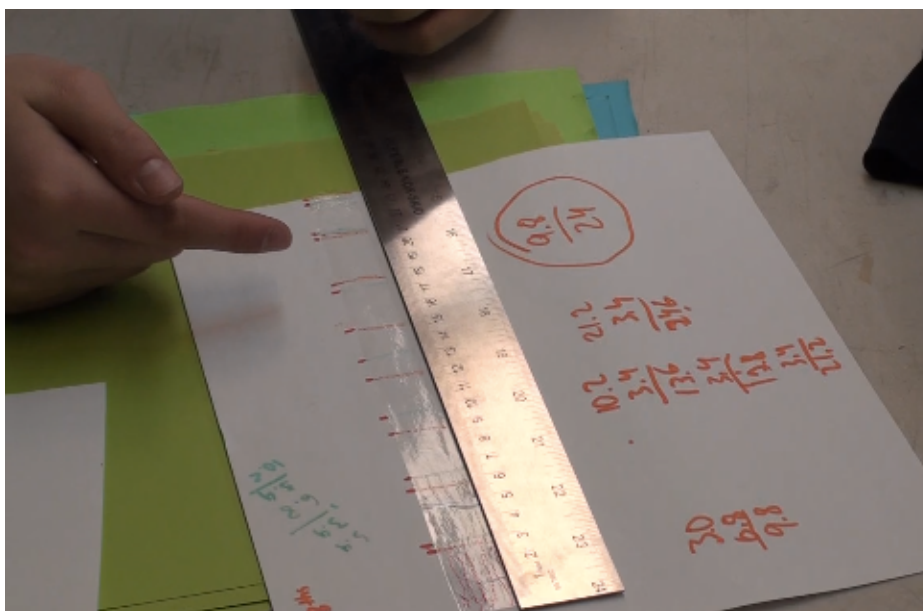
23 January 2015



Third interview

J, S and A

21 January 2015



Third interview

J, S and A

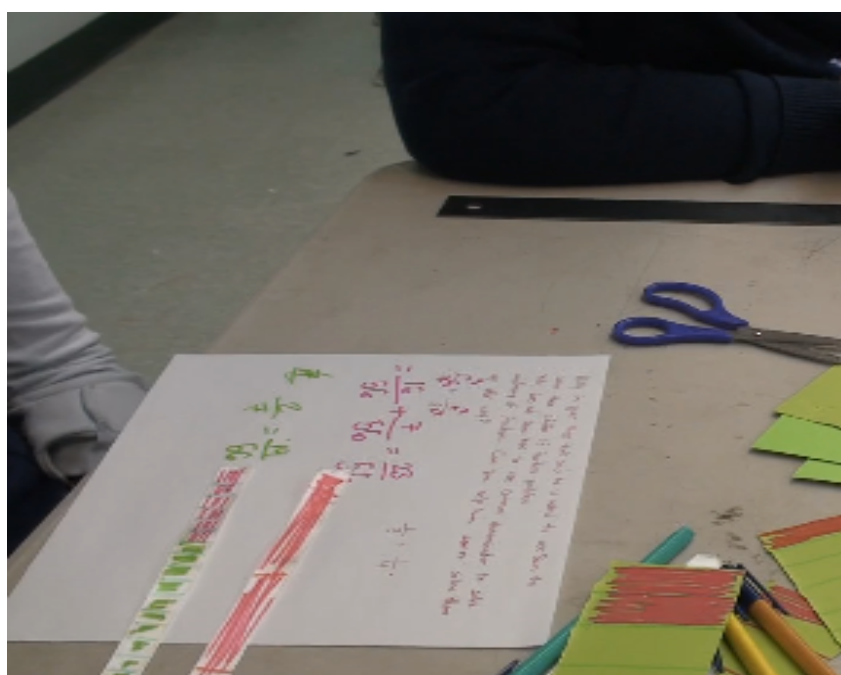
22 January 2015



Third interview

K and H

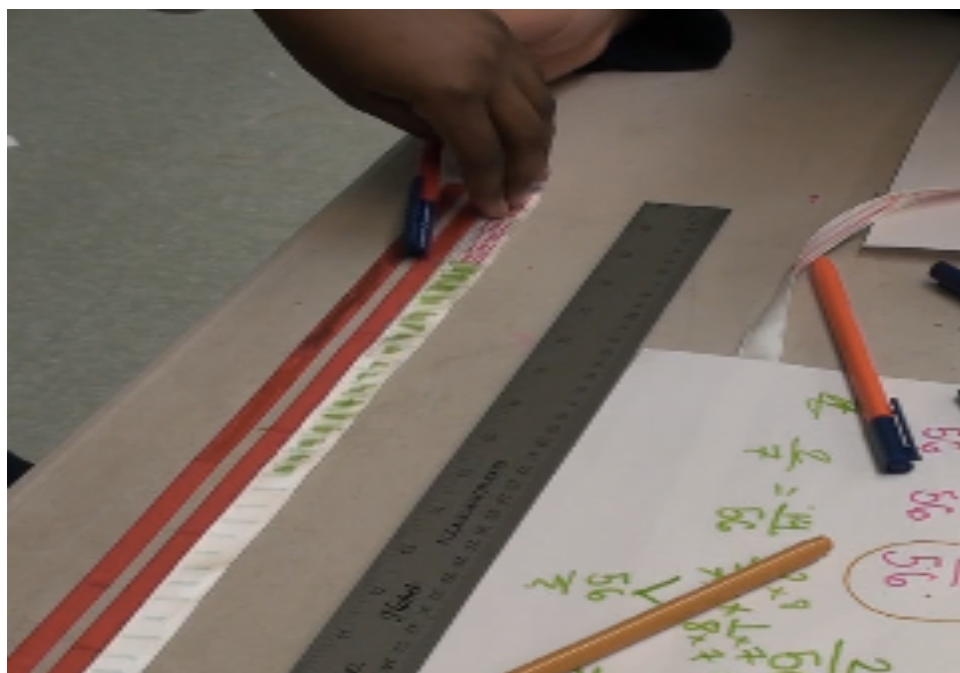
22 January 2015



Third interview

K and H

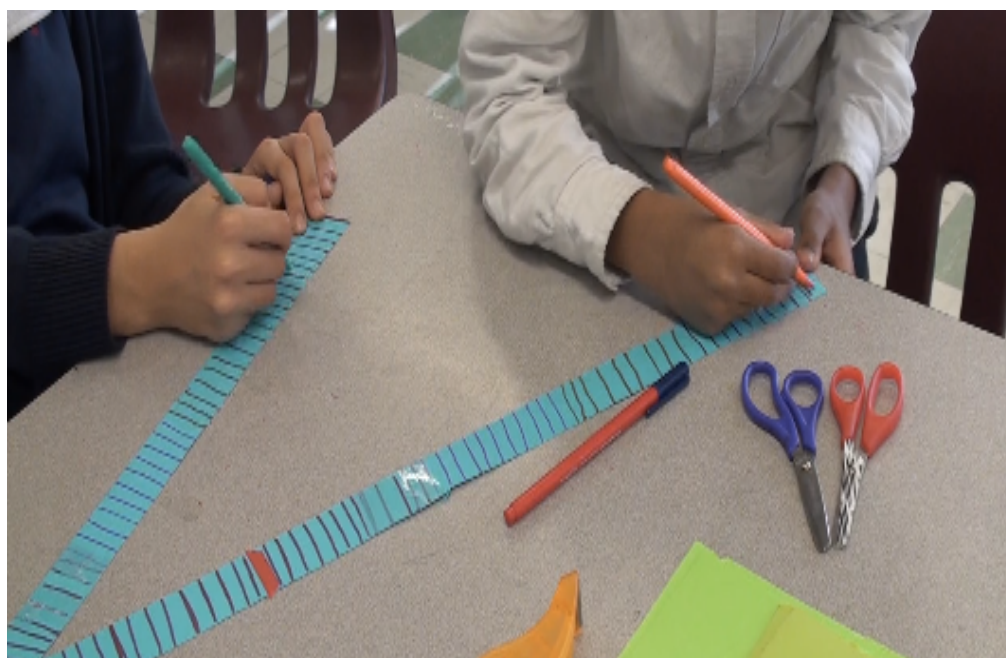
21 January 2015



Third interview

Y and S

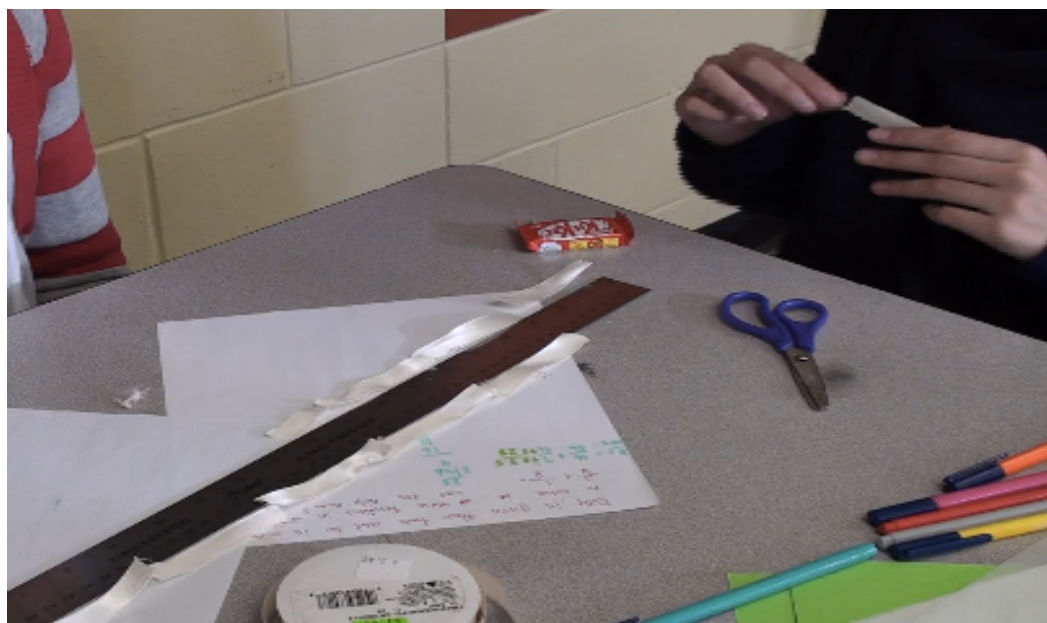
22 January 2015



Third interview

S and F

22 Jan 2015



Third interview

S and F

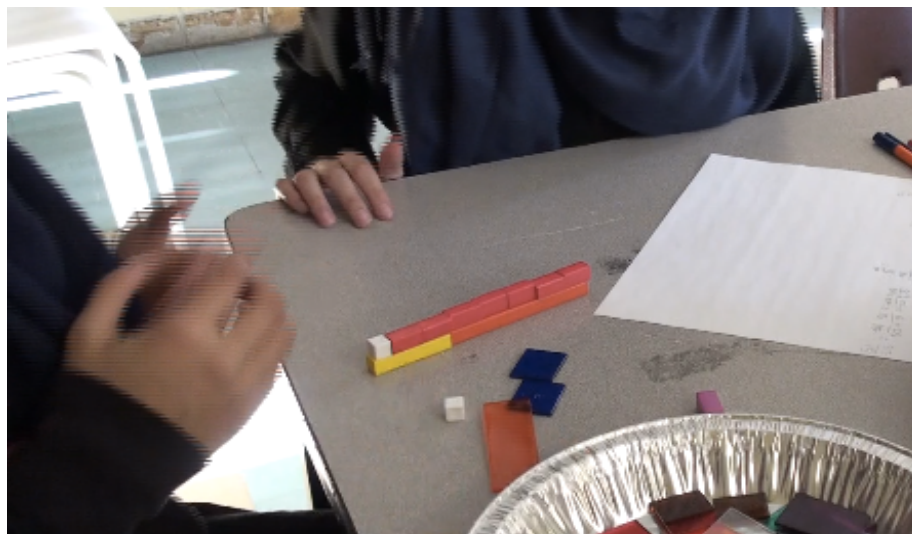
22 Jan 2015



Second Interview

S and F

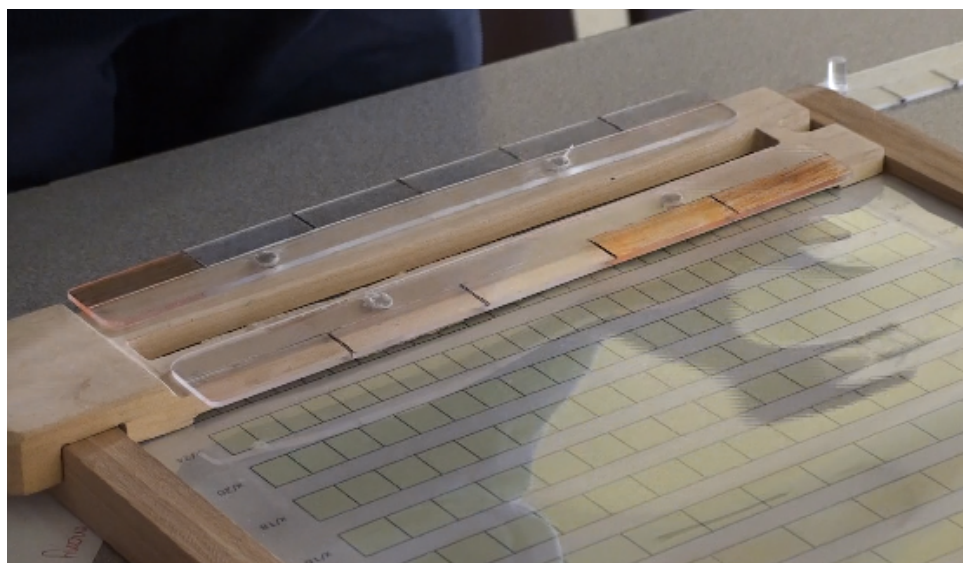
21 Jan 2015



Second interview

A and H

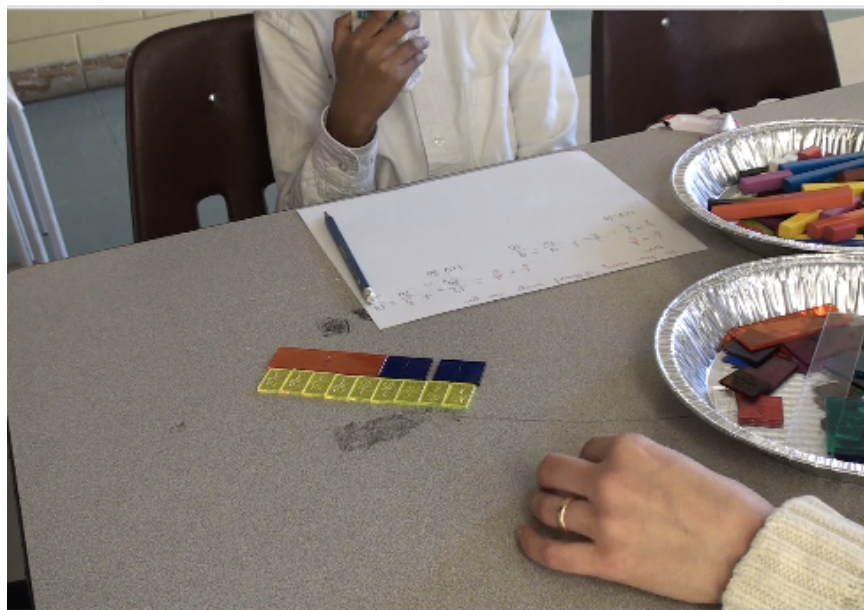
20 Jan 2015



Second interview

Y and S

19 Jan 2015



Second interview

Y and S

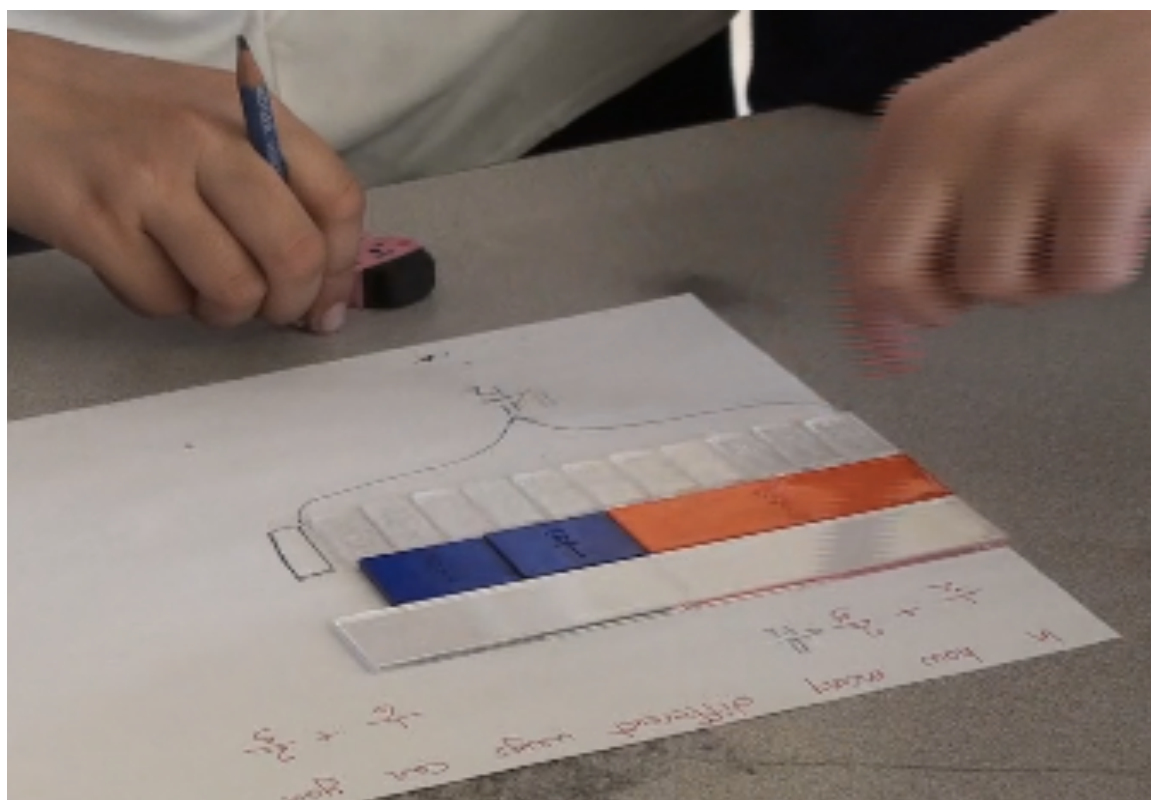
21 Jan 2015



Second interview

J, A, S

19 Jan 2015



Second interview

J, A, S

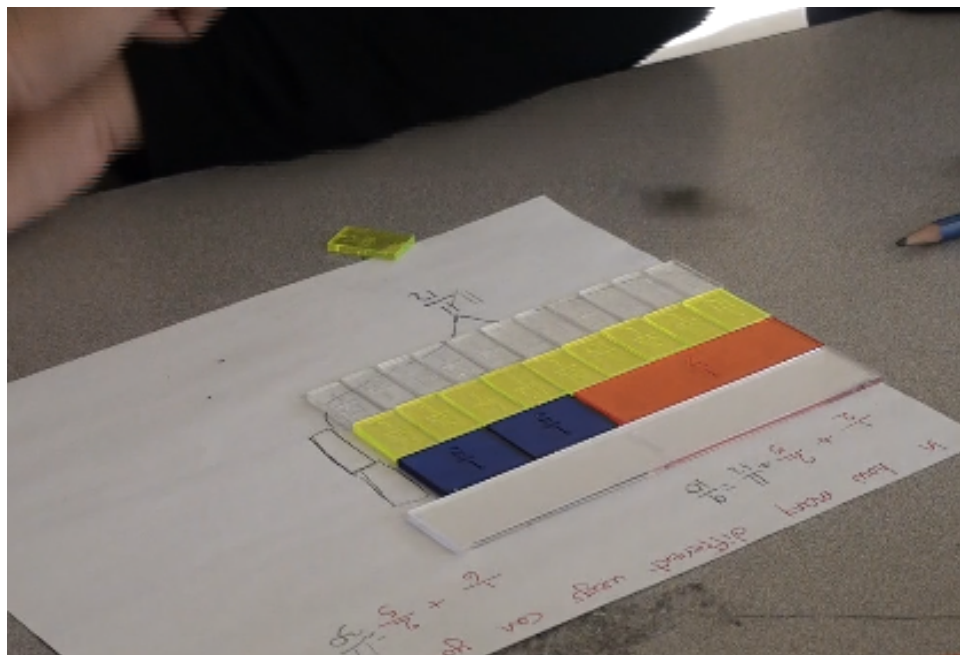
19 Jan 2015



Second interview

J, A, S

19 Jan 2015



Third interview

K and Z

23 January 2015

