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Equivariant Forms

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Equivariant Forms

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Abstract

The goal of this thesis is to develop the theory of the so-called equivariant forms. Precisely, we study and classify all meromorphic functions of the extended upper-half plane $\mathfrak{H}^*$ that commute with the action of a finite index subgroup of $\mathrm{SL}_2(\mathbb{Z})$ on $\mathfrak{H}^*$. It is shown that they are intimately connected to modular forms, differential forms and quasimodular forms, and hence inherit their structures. A close connection with different geometric objets such as differential forms and sections of line bundles is also established. Finally, to show more the richness of such objects, some applications to the critical points of modular forms are given.

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Dedication

To my Parents and family.

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Introduction

The main objects under study in thesis are the so-called *equivariant forms.* These are meromorphic functions on $\mathfrak{H}$, the upper-half of the complex plane $\mathbb{C}$, which commute with the action of a discrete subgroup Γ of $\mathrm{SL}_2(\mathbb{R})$, the group of 2 by 2 matrices with real entries and determinant 1. More precisely, an element $\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$ acts on $\mathfrak{H}$ (and on $\mathbb{C}$) by

$$\alpha \cdot z = \frac{az+b}{cz+d}\,, \quad z \in \mathfrak{H}\,.$$

A meromorphic function h on $\mathfrak{H}$ is an *equivariant form* for Γ if it satisfies

$$h(\alpha \cdot z) = \alpha \cdot h(z)\,,\ z \in \mathfrak{H}\,,\ \alpha \in \Gamma\,,$$

in addition to some conditions at the cusps of Γ.

These functions appeared first in the work by Brady [Bra] and Heins [Hei] as quotients of pseudo-periods of the Weierstrass elliptic ζ functions. In fact, these authors were concerned with these functions in the framework of the theory of elliptic functions. Not a long ago, these type of functions appeared in the work by Abdellah Sebbar and Ahmed Sebbar in [AAS1] in the framework of the theory of modular forms and where the terminology of *equivariant forms* was first initiated. In the paper *loc. cit.*, the authors studied the converse of a problem that appears in an earlier paper [McSe1] by John McKay and Abdellah Sebbar in which a close connection was established between the geometry of certain genus zero discrete subgroups of $\mathrm{SL}_2(\mathbb{R})$ and the analytic properties of automorphic forms obtained by applying the Schwarz derivative to the generator of the function field of the Riemann surface corresponding to the group Γ, known as a Hauptmodul. More precisely, the *Schwarz derivative* of a meromorphic function f on a domain in $\mathbb{C}$ is given by

$$\{f, z\} = 2\left(\frac{f''}{f'}\right)' - \left(\frac{f''}{f'}\right)^2 .$$

If f is an automorphic function for a genus zero discrete group Γ, then $\{f, z\}$ is a weight 4 meromorphic modular form on Γ. More interestingly, the group of automorphy for $\{f, z\}$ has more symmetry than Γ, and in the case of a Hauptmodul f, it is the normalizer of Γ inside $\mathrm{SL}_2(\mathbb{R})$. The converse problem asks if f is a meromorphic function of $\mathfrak{H}$ such that its Schwarz derivative $F(z) = \{f, z\}$ is a weight 4 automorphic form on a discrete group Γ, what is the size of G_f, the invariance group of f? Using the properties of the Schwarz derivative and the modularity of F, this is equivalent to say that, for $\alpha \in \Gamma$,

$$f(\alpha \cdot z) = \Phi_\alpha \cdot f(z) ,$$

for some matrix Φ_α in $\mathrm{GL}_2(\mathbb{C})$. Therefore, we have a group homomorphism

$$\begin{array}{rccl} \Phi : & \Gamma & \longrightarrow & \mathrm{GL}_2(\mathbb{C}) \\ & \alpha & \longmapsto & \Phi_\alpha . \end{array}$$

The invariance group G_f of f is then the kernel of Φ, and the problem is reduced to the study of its size. Interestingly, this kernel may be trivial and this occurs in particular when Φ is the identity, that is, the natural injection of Γ into $\mathrm{GL}_2(\mathbb{C})$. Equivalently, one asks whether there are meromorphic functions on $\mathfrak{H}$ such that

$$f(\alpha \cdot z) = \alpha \cdot f(z), \text{ for all } z \in \mathfrak{H} \text{ and } \alpha \in \Gamma ,$$

and a solution to this problem is then an equivariant form.

In the work [AAS1], the authors establish a wide class of solutions known as the *rational equivariant forms* (see Subsection 3.2.1) for the modular group $\mathrm{SL}_2(\mathbb{Z})$ that are parameterized by modular forms yielding various interesting applications to modular differential equations and the analysis of the critical points of modular forms for $SL_2(\mathbb{Z})$. One of the ingredients in this investigation relies on a rationality criteria of the residues at poles of certain meromorphic functions attached to the equivariant forms. More precisely, if this criteria holds for an equivariant form h, then there exists a weight k modular form f for $SL_2(\mathbb{Z})$ such that

$$h(z) = z + k\frac{f(z)}{f'(z)}.$$

In this thesis, we undertake the task to pursue this study for all the finite index subgroups of $SL_2(\mathbb{Z})$. In particular, we provide a complete classification of all the equivariant forms that include the rational forms in [AAS1] as well as the irrational ones. Our investigation involves the theory of modular forms as well as the theory of quasimodular forms initiated by Zagier and Kaneko in [KaZa]. Furthermore, our classification endows the set $\mathcal{E}(\Gamma)$ of equivariant forms on Γ with the structure of an infinite dimensional vector space when we remove the trivial equivariant form $h_t(z) = z$. Moreover, the zero element is represented by the *fundamental* equivariant form

$$h_0(z) = z + 12\frac{\Delta}{\Delta'} = z + \frac{6}{i\pi E_2(z)},$$

where Δ is the modular discriminant and E_2 is the weight 2 Eisenstein series. A finite dimensional subspace of this latter is the class of equivariant forms that do not have fixed points on $\mathfrak{H}$ and some condition at cusps, which we call *equivariant forms without fixed points* (see Definition 3.2.4) and to which h_0 belong.

In the elliptic point of view, for a lattice $L = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$ with $\tau = \omega_1/\omega_2 \in \mathfrak{H}$, the Weierstrass ζ-function is defined by $\zeta' = -\wp$ where $\wp$ is the Weierstrass elliptic $\wp$-function. If η_1 and η_2 are the pseudo-periods of ζ, then the fundamental equivariant form is given by

$$h_0 = \omega_1\eta_2.$$

Surprisingly, there is a geometric interpretation to the equivariant forms for a modular subgroup Γ. Indeed, we show that the space $\mathcal{E}(\Gamma)$ is isomorphic to the space of meromorphic differential 1-forms over the Riemann surface X_Γ obtained by compactifying the quotient $\Gamma \setminus \mathfrak{H}$. In other words, the elements of $\mathcal{E}(\Gamma)$ are identified with the sections of the cotangent bundle (canonical line bundle) to the genus g Riemann surface X_Γ. The zero section is then given by the fundamental equivariant form h_0. In fact, as the modular discriminant Δ does not vanish on $\mathfrak{H}$, one notices that $h_0(z)$ does not have a fixed point on $\mathfrak{H}$. It turns out the absence of fixed points provides a special class of equivariant forms. For instance, the equivariant forms without fixed points with an additional condition at cusps correspond to the holomorphic sections of the above-mentioned canonical line bundle. In particular, if the group Γ has genus 0, then the only such equivariant form without fixed points correspond to the zero section and thus should be equal to the fundamental example h_0.

In the modular forms setting, the above differential forms define in a canonical way weight 2 meromorphic modular forms for Γ. The correspondence between an equivariant form h for Γ in $\mathcal{E}(\Gamma)$ and a weight 2 modular form f for Γ is given by

$$h(z) = z + \frac{1}{\tilde{E}_2 + f},$$

where $\tilde{E}_2 = (i\pi/6)E_2$. Again, the equivariant forms without fixed points correspond to the holomorphic weight 2 cusp forms for Γ. In the genus zero case, it is known that there are no nonzero cusp forms of weight 2, which makes h_0 as the unique equivariant form without fixed points. Moreover the trivial equivariant form $h(z) = z$ is obtained if one considers $f = \infty$ which allows us to look at $h(z) = z$ as the point at infinity of the space $\mathcal{E}(\Gamma)$.

Meanwhile, there are other ways to associate modular forms to equivariant forms which deserve a deep understanding. Indeed, we show that the cross-ratio of any four distinct equivariant forms h_i, $1 \leq i \leq 4$, for Γ

$$f = \frac{(h_1 - h_3)(h_2 - h_4)}{(h_2 - h_3)(h_1 - h_4)}$$

is a modular function. At the same time, the Schwarz derivative of an equivariant form is actually a modular form of weight four. This is not surprising as the Schwarz derivative is simply the infinitesimal counterpart of the cross-ratio.

Besides the importance of equivariant forms as intriguing objects by themselves and which have an elliptic, a modular and a geometric interpretation, there are various fields in which they can find applications. The more immediate area of application is the study of critical points of classical modular forms by means of the corresponding equivariant forms. We show for instance that the equivariance of the fundamental example h_0 implies immediately that there are infinitely many nonequivalent zeros to the Eisenstein series E_2. In other words, the discriminant Δ has infinitely many nonequivalent critical points.

Some of the developments that we do not consider in this thesis include the theory of equivariant K-theory, the higher dimensional counterpart attached to the Hilbert modular forms and the real-analytic counterpart attached to the Maass wave forms among others.

We now proceed to the description of the various chapters in this thesis.

In the first chapter, we recall some aspects of the theory of modular forms for discrete subgroups of $\mathrm{SL}_2(\mathbb{R})$. We discuss their connection with differential forms on the compact Riemann surface $\Gamma \setminus \mathfrak{H}^*$. We also introduce the notion of quasimodular forms on Γ following Kaneko and Zagier [KaZa]. A generalization, that will be used in Chapter 5, of properties of quasimodular forms is discussed at the end of this chapter.

The second chapter deals with equivariant forms on the modular group $\mathrm{SL}_2(\mathbb{Z})$. We provide in this part some properties of the Schwarz derivative which is at the origin of these functions. Then we move on to discuss the main subject of this chapter. We give the precise definition of an equivariant form for $\mathrm{SL}_2(\mathbb{Z})$ as in [AAS1] with an emphasis on the so-called rational equivariant forms. We then provide two types of generalizations. First, we consider the case of modular form with a multiplier system, and then move on to exhibit a larger class that include the rational equivariant forms as well as the irrational ones. In the last section, we exhibit an important role that the fixed points play in characterizing the equivariant forms.

In Chapter 3, we generalize the results of the previous chapter to any subgroup of the modular group. The main tool is the notion of the slash operator on equivariant forms. We then study the conditions under which an equivariant form h is rational. This is based on the rationality of the residues of the poles in $\mathfrak{H}$ and at cusps of the function $\widehat{h}(z) = (h(z) - z)^{-1}$. In the last section, we show that the fundamental equivariant form h_0 is unique in the sense that it is the only one without fixed points provided a geometric condition on Γ is met.

The aim of Chapter 4 is to give the complete classification of equivariant forms. Furthermore, we show that they can be endowed with an algebraic structure using two approaches involving the quasimodular forms on one side and the the modular forms on the other. In particular, we prove that for an equivariant form h on a modular subgroup Γ, we have

$$h(z) = z + \frac{1}{\widetilde{E}_2 + f} \ ,$$

where $\widetilde{E}_2 = (\pi\imath/6)E_2$ with E_2 being the Eisenstein series of weight 2, and f is a weight 2 modular form for Γ.

This new form of equivariant forms has various consequences. We obtain, for instance, one-to-one correspondences between non-trivial equivariant forms and

- the set of weight 2 depth 1 quasimodular forms on Γ that transforms like $\widetilde{E}_2$; i.e. the set $\{\widetilde{E}_2 + f |\ f \in \mathfrak{M}_2^m(\Gamma)\}$.
- the space of weight 2 meromorphic modular forms on Γ.
- the space of degree one differentials $\mathrm{Dif}(X_\Gamma)$ on $X_\Gamma = \Gamma \setminus \mathfrak{H}^*$. Hence a justification of the name "*equivariant forms*".

Other correspondences are between the set of equivariant forms without fixed points for Γ and the space of weight 2 holomorphic modular forms on Γ. These correspondences show in particular that the set $\mathcal{E}(\Gamma)\setminus\{z\}$ inherits a structure of a vector space. Moreover, we can view the trivial equivariant form $h(z) = z$ as the point at infinity.

In the rest of this chapter we establish a link to some classical tools from projective differential geometry, namely the cross-ratio and the Schwarz derivative which is a projective connection. The cross-ratio is invariant under Möbius transformations as well as the Schwarz derivative, which is the infinitesimal counterpart of the cross-ratio. In fact, the cross-ratio of four distinct equivariant forms is a modular form while the Scwarz derivative of an equivariant form is a weight 4 modular form. To illustrate this we give the Dedekind j-function and the Klein elliptic λ-function as cross-ratios of certain equivariant forms. The rest of this chapter is dedicated to viewing the equivariant forms as sections of a line bundle, using a direct connection as well as a quasimodular and a modular points of view.

These connections with various algebraic-geometric objects show the richness of equivariant forms, which leads to many applications. In the last chapter, we give an application of equivariant forms to the critical points of modular forms. More precisely, the fundamental equivariant form $h_0 = z + 12\Delta\,/\,\Delta' = z + 1/\,\widetilde{E}_2$ is used to prove that the Eisenstein series E_2 has infinitely many zeros in the half-strip $\mathfrak{S} = \{\tau \in \mathfrak{H},\ -\frac{1}{2} < \mathrm{Re}(\tau) \leq \frac{1}{2}\}$. Consequently, the modular discriminant Δ has infinitely many critical points in the half-strip $\mathfrak{S}$. We also present a detailed account of the distribution of these critical points.

Chapter 1

Modular forms and quasimodular forms

In this chapter we give an exposition of the theory of discrete groups and the theory of modular forms that will be useful in the remaining chapters. This will consist of a summary of the main properties of modular forms for discrete subgroups of $\mathrm{SL}_2(\mathbb{R})$ and their connection with other classical objects such as differential forms. We will also introduce the theory of quasimodular forms, a generalization of modular forms by [KaZa]. We end this chapter by giving a generalization of the definition of a quasimodular form.

1.1 Discrete subgroups of $\mathbf{SL}_2(\mathbb{R})$

In this section we briefly describe some of the properties of Fuchsian subgroups of the first kind of $\mathrm{SL}_2(\mathbb{R})$, the group of uni-modular matrices with real entries, where by Fuchsian subgroup of the first kind we mean a discrete subgroup such that the corresponding Riemann surface $\Gamma \setminus \mathfrak{H}^*$, $\mathfrak{H} = \{z \in \mathbb{C} | \, \Im(z) > 0\}$, obtained by joining the set of inequivalent cusps to the quotient $\Gamma \setminus \mathfrak{H}$ is compact. These groups acts on the upper-half plane $\mathfrak{H}$ by linear rational fractions. They are classified by their trace as follows.

If α is not the identity element, then it has a single fixed point on $\mathbb{R} \cup \{\infty\}$ if and only if $|Tr(\alpha)| = 2$, in which case α is called parabolic and the fixed point is a cusp. Also, α has a fixed point in $\mathfrak{H}$ if and only if $|Tr(\alpha)| < 2$, in which case α is called an elliptic element and the fixed point is in $\mathfrak{H}$. The case $|Tr(\alpha)| > 2$ is equivalent to having two fixed points on the real line, and A is called hyperbolic.

A key property of a Fuchsian subgroup of the first kind Γ is that the set of inequivalent cusps (respectively, elliptic points) is finite [Shi]. An example of a Fuchsian subgroup not of the first kind is $SL_2(\mathbb{R})$. Moreover, there is a class of subgroups that shares with Γ the same set of cusps, namely the class of subgroups that are commensurable with Γ. A group G is called commensurable with Γ if $G \cap \Gamma$ has finite index in both G and Γ. Note that, in this case, G is also a Fuchsian subgroup of the first kind [Shi].

The following representation will be used to establish the modularity of certain modular forms in Chapter 3. Let Γ be a subgroup of $SL_2(\mathbb{Z})$ of genus g, meaning that the corresponding compact Riemann surface $\Gamma \setminus \mathfrak{H}^*$ is of genus g. Let ν_∞ be the number of inequivalent cusps and r be the number of inequivalent elliptic points. Let $m_1, ..., m_r$ be the orders of the stabilizer of all conjugacy classes of elliptic points. Then we say that Γ has signature $(g;\ m_1, ...,\ m_r;\ \nu_\infty)$. The algebraic structure of the group can be determined by its signature. In fact, the group has a presentation:

generators :

$$A_1,\ B_2, ...,\ A_g,\ B_g;\ E_1, ...,\ E_r;\ P_1, ...,\ P_{\nu_\infty}$$

relations :

$$E_1^{m_1} = ... = E_r^{m_r} = \prod_{i=1}^{\nu_\infty} P_i \prod_{i=1}^{r} E_i \prod_{i=1}^{g} A_i B_i A_i^{-1} B_i^{-1}.$$

The generators P_i are parabolic, the E_i are elliptic, and the A_i and B_i are hyperbolic.

The modular group $SL_2(\mathbb{Z})$, the group of uni-modular matrices with integer entries, is an example of a Fuchsian subgroup of the first kind and for which (and its finite index subgroups) the set of cusps is $\mathbb{Q} \cup \{\infty\}$ that are equivalent to ∞. It is a reference in the theory of modular forms and other topics because of the many arithmetic and geometric properties it has. The modular group is generated by the elliptic and parabolic matrices $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ or, equivalently, by the elliptic elements S and $P := ST = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$. Note that as the genus of $\mathrm{SL}_2(\mathbb{Z}) \setminus \mathfrak{H} \cup \{\infty\}$ is zero there are no hyperbolic generators. The following result gives us some information on the automorphisms of $\mathrm{SL}_2(\mathbb{Z})$ that will be used in Chapter 3.

Theorem 1.1.1 ([Ran]) *Let ϕ be an automorphism of $SL_2(\mathbb{Z})$. Then ϕ is determined uniquely by its action on the generators S and P, and we must have, for some $L \in SL_2(\mathbb{Z})$, $\phi(S) = L^{-1}S^tL$, $\phi(P) = L^{-1}P^uL$, where $t = \pm 1$, $u = \pm 1$.*

One can deduce from this theorem that the trace is conserved by the automorphisms of $\mathrm{SL}_2(\mathbb{Z})$, and as a corollary, we have

Corollary 1.1.2 ([Ran]) *Any two elements of $SL_2(\mathbb{Z}) \setminus \{\pm 1_2\}$ having the same trace are conjugate to each other by an element of $SL_2(\mathbb{Z})$.*

A second important example of a Fuchsian subgroup of the first kind is the so-called congruence subgroups of $\mathrm{SL}_2(\mathbb{Z})$. Recall that a subgroup Γ of $\mathrm{SL}_2(\mathbb{Z})$ is called a congruence subgroup if it contains some principal congruence subgroup $\Gamma(N)$, for some positive integer N, which defined as follows.

Let N be a positive integer, then

$$\Gamma(N) = \left\{ \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a \equiv d \equiv 1 \mod (N),\ b \equiv c \equiv 0 \mod (N) \right\}.$$

Since $\Gamma(N)$ is of finite index in $\mathrm{SL}_2(\mathbb{Z})$, it therefore follows that any congruence subgroup of $\mathrm{SL}_2(\mathbb{Z})$ is of finite index in $\mathrm{SL}_2(\mathbb{Z})$. It is worth to mention that $\Gamma(N)$ is normal in $SL_2(\mathbb{Z})$. In fact, we have the following exact sequence

$$0 \longrightarrow \Gamma(N) \longrightarrow SL_2(\mathbb{Z}) \longrightarrow SL_2(\mathbb{Z}/N\mathbb{Z}) \longrightarrow 0$$

where $SL_2(\mathbb{Z}/N\mathbb{Z})$ is the group of matrices of determinant 1 with entries in $\mathbb{Z}/N\mathbb{Z}$. Note that $\Gamma(1) = SL_2(\mathbb{Z})$. We would like also to notice that if Γ is a congruence subgroup then so is any of its conjugates.

Example 1.1.1 Some of the most important congruence subgroups are

$$\Gamma_1(N) = \left\{ \alpha \in SL_2(\mathbb{Z}) |\ \alpha \equiv \pm \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \mod (N) \right\},$$

$$\Gamma^0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) |\ b \equiv 0 \mod (N) \right\}$$

which are obviously congruence subgroups of level N, meaning the least integer N for which $\Gamma(N) \subset \Gamma_1(N)$. An other important example of congruence subgroups, is the subgroup

$$\Gamma_0(N) = \left\{ \pm \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) |\ c \equiv 0 \mod (N) \right\},$$

which is conjugate to the subgroup $\Gamma^0(N)$ by the matrix $\begin{pmatrix} N & 0 \\ 0 & 1 \end{pmatrix}$.

Note that the set of cusps of these subgroups is represented by $\mathbb{Q} \cup \{\infty\}$ (which is that of $SL_2(\mathbb{Z})$), as they are commensurable with $SL_2(\mathbb{Z})$. In fact, this set represents the set of cusps for any group commensurable with $SL_2(\mathbb{Z})$.

We end this subsection by discussing some characteristics of the cusp width for a congruence subgroup.

Given Γ a congruence subgroup and $s \in \mathbb{P}^1(\mathbb{Q})$ a cusp, let Γ_s (respectively G_s) denotes the isotropy group of s inside Γ (respectively inside $SL_2(\mathbb{Z})$). Then Γ_s is a subgroup of G_s of finite index, say m. The index m is called the cusp width of Γ at s. It is also referred to as the smallest positive integer m such that $T^m \in \gamma \Gamma_s \gamma^{-1}$, where $\gamma \in SL_2(\mathbb{Z})$ with $\gamma \cdot s = \infty$.

For a congruence subgroup Γ of the modular group of level N, Larcher [Lar] proved that if d is the least cusp width in Γ, it is possible to conjugate by an element of the modular group so that the cusp width at ∞ of the resulting congruence subgroup is d and that the cusp width at 0 becomes N. Note that since $T^N \in \Gamma$, we have $d \leq N$. More generally, we have

Proposition 1.1.3 (**[Se1]**) *Modular conjugation only permutes the cusp widths for a congruence subgroup.*

1.2 Modular forms

The content of the section can be found in most books on modular forms. We refer for instance to [Mil], [Ran] and [Shi].

1.2.1 The definition of a modular form

We first define the slash operator on meromorphic functions on the upper half plane $\mathfrak{H}$ via the action of $\mathrm{GL}_2(\mathbb{C})$, the group of two-by-two invertible matrices with complex entries.

Let $\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{C})$, and $z \in \mathfrak{H}$. Let k be an even positive integer, and denote by $j_\alpha(z) = cz + d$; a definition that will be used throughout this work. Then α acts on a meromorphic function f as follows

$$f(z)|_k[\alpha] \quad = \quad \det(\alpha)^{k/2} j_\alpha(z)^{-k} f(\alpha \cdot z) \tag{1.2.1}$$

We will see that $j : \mathrm{SL}_2(\mathbb{R}) \times \mathfrak{H} \to \mathbb{C}^*$ defines what is called an automorphic factor (see Section §1.2.2) and therefore satisfies

$$j_{\alpha\beta}(z) = j_\alpha(\beta \cdot z) j_\beta(z)$$

for all $\alpha,\ \beta \in \mathrm{PSL}_2(\mathbb{R})$.

We notice that, since $-\alpha \cdot z = \alpha \cdot z$, when k is odd $j_{-\alpha}(z)^k = -j_\alpha(z)^k$, so that $f(z)|_k[-\alpha] = -f(z)|_k[\alpha]$.

Definition 1.2.1 *Let k be a positive integer. A function f on $\mathfrak{H}$ is called a meromorphic modular form or simply a modular form of weight k for a discrete subgroup Γ of $SL_2(\mathbb{R})$ if*

1. *f is meromorphic on $\mathfrak{H}$,*
2. *for all $\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$ and $z \in \mathfrak{H}$, we have $f(z)|_k[\alpha] = f(z)$,*
3. *f is meromorphic at the cusps.*

The last condition means the following. First, this condition is ignored if Γ has no cusps. Otherwise, let s be a cusp of Γ. Let $\gamma \in \mathrm{SL}_2(\mathbb{R})$ such that $\gamma \cdot s = \infty$. Then, the function $f(z)|_k[\gamma^{-1}]$ is invariant under $\gamma\Gamma_s\gamma^{-1} =< T^{l_s} >$, l_s being the cusp width at s and Γ_s the isotropy group of s inside Γ. Hence, it has a Fourier expansion in the local parameter at infinity $q_s := e^{2\pi i z/l_s}$ if k is even and $q_s = e^{\pi i z/l_s}$ if k is odd. Then the meromorphy condition translates into saying that in the Fourier series

$$f(z)|_k[\gamma^{-1}] = \sum_{n=n_s}^{\infty} a_n^s q_s^n$$

the integer n_s is finite. If $n_s \geq 0$ for all cusps s and if f is holomorphic on $\mathfrak{H}$ then f is called a holomorphic modular form. A holomorphic modular form is called a cusp form if it vanishes at all cusps; in other words $n_s > 0$ for all cusps s.

For $k = 0$ the modular form is called a modular function. We would also like to notice that a holomorphic modular form of weight 0 is constant.

Let us denote respectively by $\mathfrak{M}_k^m(\Gamma)$, $\mathfrak{M}_k(\Gamma)$, $\mathfrak{S}_k(\Gamma)$ the spaces of meromorphic modular forms, holomorphic modular forms and cusp forms on Γ.

1.2.2 Modular forms with multiplier systems

The aim of this section is to give a more general example of modular forms. This will be needed is subsequent chapters. More precisely, we will generalize the definition of a modular form to what is called a *modular form with a multiplier system.* We follow the treatment of [Ran].

First, we introduce the notion of an automorphic factor.

An automorphic factor μ of weight $k \in \mathbb{R}$ for Γ, a Fuchsian subgroup of $\mathrm{SL}_2(\mathbb{R})$, is a map $\mu : \ \Gamma \times \mathfrak{H} \longrightarrow \mathbb{C}^\times$ with the properties

- For all $\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$, and $z \in \mathfrak{H}$, $|\mu_\alpha(z)| = |cz+d|^k$.
- For all $\alpha,\ \gamma \in \Gamma$, and $z \in \mathfrak{H}$, $\mu_{\alpha\gamma}(z) = \mu_\alpha(\gamma \cdot z)\mu_\gamma(z)$.
- For all $z \in \mathfrak{H}$, $\mu_{-1_2}(z) = \mu_{1_2}(z)$, 1_2 being the identity matrix.

If $-1_2 \notin \Gamma$, the last condition can be used to extend the automorphic factor μ to a function on $\overline{\Gamma} \times \mathfrak{H}$, where $\overline{\Gamma} = \Gamma\{\pm 1_2\}/\{\pm 1_2\}$ is the homogeneous subgroup corresponding to Γ. The function on $\Gamma \times \mathfrak{H}$ given by $(\alpha,\ z) \mapsto j_\alpha(z)^k$ for $k \in \mathbb{R}$, $\alpha \in \Gamma$ and $z \in \mathfrak{H}$, is an example of an automorphic factor of weight k for Γ. Since j and μ are holomorphic and do not vanish on $\mathfrak{H}$, it follows from Liouville's theorem and the fact that j is an automorphic factor that $\mu_\alpha(z) = \nu(\alpha) j_\alpha(z)^k$ for all $\alpha \in \Gamma$ and $z \in \mathfrak{H}$, where $\nu(\alpha)$ depends only on α and $|\nu(\alpha)| = 1$. The factor $\nu(\alpha)$ is called a multiplier, and the function ν defined on Γ is called a *multiplier system* of weight k for Γ. Note that, from the value of μ at $(1_2,\ z)$ and the definition of ν, we have $\nu(1_2) = 1$.

Definition 1.2.2 *Let μ be an automorphic factor of weight $k \in \mathbb{R}$ for Γ and ν the associated multiplier system. A function $f : \ \mathfrak{H} \longrightarrow \mathbb{C}$ is called an unrestricted modular form for Γ of weight k, with automorphic factor μ (or, equivalently, with multiplier system ν) if it satisfies*

i. f is meromorphic on $\mathfrak{H}$.

ii. $f(z)|_k[\alpha] := \nu(\alpha)^{-1} j_\alpha(z)^{-k} f(\alpha \cdot z) = f(z)$, *for all* $\alpha \in \Gamma$ *and* $z \in \mathfrak{H}$. *We choose to use the same notation for the slash operator* $f(z)|_k[\alpha]$ *given in* (1.2.1). *The context will be clarified if the two cases occur.*

Fix f as in the definition. Let s be a cusp, and let $\gamma \in \mathrm{SL}_2(\mathbb{R})$ such that $\gamma \cdot s = \infty$. Then, as $T^{l_s} \in \gamma\Gamma_s\gamma^{-1}$, l_s being the cusp width at s, and as a multiplier system does not need to be constant on Γ, we have $T^{l_s} = \gamma\alpha\gamma^{-1}$ for some $\alpha \in \Gamma_s$ and so

$$f(z)|_k[\alpha\gamma^{-1}] = f(z)|_k[T^{l_s}\gamma^{-1}] = f(z+l_s)|_k[\gamma^{-1}] = \nu(T^{l_s}) f(z)|_k[\gamma^{-1}] \ .$$

Set $\tilde{f}_{\gamma^{-1}}(z) := \nu(T^{l_s})^{-1} f(z)|_k[\gamma^{-1}]$. Then, we have $\tilde{f}_{\gamma^{-1}}(T^{l_s} \cdot z) = \tilde{f}_{\gamma^{-1}}(z)$, and hence it has a Fourier expansion in the local parameter $q_s = e^{2\pi i z/l_s}$, possibly with infinitely many negative powers of q_s.

We notice that this is independent of the choice of γ, and that two equivalent cusps modulo Γ have the same Fourier expansion in q_s.

Differentiating both sides of $f(\alpha \cdot z) = \nu(\alpha) j_\alpha(z)^k f(z)$ and using $\frac{d}{dz}\alpha \cdot z = (j_\alpha(z))^{-2}$, we get

$$f'(\alpha \cdot z) = \nu(\alpha) j_\alpha(z)^{k+2} f'(z) + ck\nu(\alpha) j_\alpha(z)^{k+1} f(z) \ . \tag{1.2.2}$$

Definition 1.2.3 *An unrestricted modular form for* Γ *of weight* $k \in \mathbb{R}$ *and multiplier system* ν *is called a modular form and a multiplier system* ν *if the function* $\tilde{f}_{\gamma^{-1}}$ *is meromorphic at infinity. In other words, if the Fourier series of* $\tilde{f}_{\gamma^{-1}}$ *has only a finite number of negative indices).*

A modular form with a multiplier system ν *is called a holomorphic modular form with multiplier system* ν *if* f *is holomorphic on* $\mathfrak{H}$ *and* $\tilde{f}_{\gamma^{-1}}$ *is holomorphic at infinity. If, in addition,* $\tilde{f}_{\gamma^{-1}}$ *vanishes at infinity it is called cusp form with multiplier system* ν. *We denote the spaces of weight* k *meromorphic modular forms, holomorphic modular forms and cusp forms with multiplier system* ν *respectively by* $\mathfrak{M}_k^m(\Gamma,\nu)$, $\mathfrak{M}_k(\Gamma,\nu)$ *and* $\mathfrak{S}_k(\Gamma,\nu)$. *Clearly* $\mathfrak{S}_k(\Gamma,\nu) \subset \mathfrak{M}_k(\Gamma,\nu) \subset \mathfrak{M}_k^m(\Gamma,\nu)$.

As in this definition, we will skip the reference to the group, the weight and the multiplier system when the context is clear.

We would like to notice that if $k \in \mathbb{Z}$ then the multiplier system is a character on Γ. We shall give an example of multiplier systems of general type as the existence of these objects is not always obvious. The following shows how to construct multiplier systems on $\mathrm{SL}_2(\mathbb{Z})$.

Consider an automorphic factor μ of weight $k \in \mathbb{R}$ with associated multiplier system ν. Set $\nu(T) = w$. It is shown in [Ran] that

$$\nu(S) = w^{-3} \quad \text{and} \quad w = \chi(T)e^{\pi \imath k/6} \quad , \quad \text{where } S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

and χ is a character on $\mathrm{SL}_2(\mathbb{Z})$ with the following values

- $\chi(-1_2) = 1$.
- $\chi(T) = \xi_6$, where ξ_6 is a sixth root of unity.

It is sufficient to define this multiplier system only for S and T as these two matrices generate the modular group $\mathrm{SL}_2(\mathbb{Z})$. We notice that for ν such that $\nu(\alpha) = 1$ for all $\alpha \in \mathrm{SL}_2(\mathbb{Z})$, the associated automorphic factor coincides with that given in Section §1.2.

Remark 1.2.1 There is a more general notion of modular forms with multiplier systems ν not necessarily of unitary modulus called *generalized modular forms* [KnMa]. As the definition of a generalized modular form is similar to what is presented in this subsection, we thought that it is convenient to just refer to [KnMa] for a detailed exposition.

Finally, we define the order of the function $\tilde{f}_\gamma$ at infinity to be the smallest integer n_s such that the corresponding Fourier coefficient is not zero. More precisely, if $n_s < 0$, then we say that $\tilde{f}_\gamma$ has a pole of order $-n_s$ at infinity, and if $n_s > 0$ then $\tilde{f}_\gamma$ is said to have a zero of order n_s at infinity. In the case $n_s = 0$, the function $\tilde{f}_\gamma$

does not vanish at infinity nor has a pole. We will use this definition for any function with a Fourier expansion in a local parameter.

1.2.3 Examples of modular forms

The following typical examples of modular forms will be used extensively in the coming chapters.

The Eisenstein series are defined for every even integer $k \geq 2$ and $z \in \mathfrak{H}$ by

$$G_k(z) = \frac{1}{2} \sum_{m,\ n}{}' \frac{1}{(mz+n)^k} \ ,$$

where the symbol $\sum'$ means that we are summing over the pairs $(m,\ n) \neq (0,\ 0)$. Note that this series is not absolutely convergent for $k = 2$.

We are interested in the following representation of Eisenstein series

$$\begin{aligned} E_k(z) &= 1 - \tfrac{2k}{B_k} \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n \\ &= 1 - \tfrac{2k}{B_k} \sum_{n=1}^{\infty} \frac{n^{k-1} q^n}{1-q^n}, \qquad q = e^{2\pi \imath z}. \end{aligned} \tag{1.2.3}$$

Here B_k is the k-th Bernoulli number and $\sigma_k(n) = \sum_{d|n} d^k$. This is connected to the first definition by [Ser, Chapter 4]

$$G_k(z) = 2\zeta(k) E_k(z) \ .$$

The most familiar Eisenstein series are:

$$E_2(z) = 1 - 24 \sum_{n=1}^{\infty} \sigma_1(n) q^n, \tag{1.2.4}$$

$$E_4(z) = 1 + 240 \sum_{n=1}^{\infty} \sigma_3(n) q^n, \tag{1.2.5}$$

$$E_6(z) = 1 - 504 \sum_{n=1}^{\infty} \sigma_5(n) q^n. \tag{1.2.6}$$

The series E_4 and E_6 are respectively holomorphic modular forms of weight 4 and 6. However, although the Eisenstein series E_2 is holomorphic on $\mathfrak{H}$ and at the cusps, it is not a modular form as it does not satisfy the modularity condition. The Eisenstein series E_2 is an example of a *quasimodular form* (see Section §1.4) and plays an important role in the construction of equivariant forms as will be seen in Chapter 4. We will also give in Chapter 5 some properties of this series that appear in [ElSe1].

Another example is the Dedekind eta function. It is a weight $\frac{1}{2}$ modular form on $\mathrm{SL}_2(\mathbb{Z})$ with a non-trivial multiplier system and is given by the infinite product

$$\eta(z) = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n), \; q = e^{2\pi i z} .$$

Its multiplier system takes the values $e^{\pi i/12}$ and $\sqrt{-i}$ at the generators T and S of $\mathrm{SL}_2(\mathbb{Z})$, respectively. This leads us to the modular discriminant, which is the unique (normalized at infinity) weight 12 cusp form for $\mathrm{SL}_2(\mathbb{Z})$. It is defined by

$$\Delta(z) = \eta(z)^{24} = q \prod_{n=1}^{\infty} (1 - q^n)^{24} .$$

Another important example, that we will encounter in Chapter 4, is the modular invariant j-function. This modular function is defined by

$$j(z) = \frac{E_4(z)^3 - E_6(z)^2}{1728\Delta} .$$

It is a Hauptmodul for $\mathrm{SL}_2(\mathbb{Z})$, in the sense that it generates the function field of the Riemann surface $\mathrm{SL}_2(\mathbb{Z}) \setminus \mathfrak{H}^*$.

We now provide an example of modular forms on congruence subgroups with non-trivial multiplier systems.

The Jacobi theta functions have been intensively studied because of their many important applications. They are defined by

$$\begin{aligned}\vartheta_2(z) &= \sum_{n=-\infty}^{\infty} q^{(n+1/2)^2} ,\\ \vartheta_3(z) &= \sum_{n=-\infty}^{\infty} q^{n^2} ,\\ \vartheta_4(z) &= \sum_{n=-\infty}^{\infty} (-1)^n q^{n^2} ,\end{aligned}$$

In [Ran], it is shown that The functions ϑ_2, ϑ_3 and ϑ_4 are holomorphic modular forms of weight $\frac{1}{2}$ for the conjugate congruence subgroups $\Gamma_0(2)$, $(ST)^{-1}\Gamma_0(2)(ST)$ and $\Gamma^0(2) = (ST)^{-2}\Gamma_0(2)(ST)^2$, respectively. Their associated multiplier systems are u, v and w respectively and are defined by

$$\begin{array}{lclclcl} u(-1_2) &=& v(-1_2) &=& w(-1_2) &=& -i, \\ v(T^2) &=& u((ST)T^2(ST)^{-l}) &=& w((ST)^2T^2(ST)^{-2}) &=& 1, \\ v(S) &=& u((ST)S(ST)^{-l}) &=& w((ST)^2S(ST)^{-2}) &=& e^{-\pi\imath/4} . \end{array}$$

Moreover, these modular forms do not vanish on $\mathfrak{H}$ and satisfy the Jacobi identity

$$\vartheta_2^4 + \vartheta_4^4 = \vartheta_3^4 .$$

Rankin also provides numerous interesting equations connecting theta functions to other modular forms sometimes on larger groups. For instance we have

$$\Delta = \left(2^{-1}\vartheta_2\vartheta_3\vartheta_4\right)^8 ,$$

and

$$E_4(z) = \frac{1}{2}\left(\vartheta_2^8 + \vartheta_3^8 + \vartheta_4^8\right) .$$

Another important fact on theta functions is that they have an eta-products

$$\vartheta_2(z) = 2\frac{\eta(4z)^2}{\eta(2z)}$$

$$\vartheta_3(z) = \frac{\eta(2z)^5}{\eta(z)^2\eta(4z)^2}$$

$$\vartheta_4(z) = \frac{\eta(z)^2}{\eta(2z)},$$

where $\eta = \Delta^{1/24}$ is the Dedekind eta function.

1.3 Differential forms and the Riemann-Roch theorem

In the theory of Riemann surfaces, differential forms play a central role. Together with the Riemann-Roch theorem they constitute a fundamental tool for computing dimensions of certain vector spaces. For instance, one can use them to compute how many functions there are with a prescribed number of poles and zeros up to a scalar multiple. In this section, we shall state the Riemann-Roch theorem and recall some facts about differential forms and their connection with modular forms that will be used to compute the dimension of the space of modular forms on a Fuchsian subgroup Γ of the first kind of $SL_2(\mathbb{R})$. We refer to [Mil], [Shi] and the references therein for more details on this subjects.

1.3.1 The Riemann-Roch theorem

To state the Riemann-Roch theorem we need first to recall some facts on the notion of a divisor on a compact Riemann surface.

Let $\mathfrak{R}$ be a compact Riemann surface and denote by $\mathcal{K}$ the field of meromorphic functions on $\mathfrak{R}$. We identify $\mathbb{C}$ with the subfield of $\mathcal{K}$ consisting of all constant functions. Then $\mathcal{K}$ is an algebraic function field, meaning that $\mathcal{K}$ is a finite algebraic extension of $\mathbb{C}(f)$ for $f \in \mathcal{K} \setminus \mathbb{C}$.

Denote by $\mathcal{D}$ the set of all formal finite sums $\sum_l c_l P_l,\ c_l \in \mathbb{Z},\ P_l \in \mathfrak{R}$. Then $\mathcal{D}$ is a free $\mathbb{Z}-$module. We call any of its elements a divisor of $\mathfrak{R}$ or of $\mathcal{K}$. To a divisor $D = \sum_{P\in\mathfrak{R}} c_P P$ we associate the following data:

$$\begin{aligned} \deg D &= \sum c_P, \quad \text{the degree of } D; \\ v_P(D) &= \quad c_P, \quad \text{the order of } D \text{ at } P. \end{aligned}$$

A divisor D is called an effective divisor, a condition denoted by $D \geq 0$, if $v_P(D) \geq 0$ for all $P \in \mathfrak{R}$. If E is another divisor then $D \geq E$ if the divisor $D - E \geq 0$.

Next, we consider the valuation ring $S_P := \{f \in \mathcal{K} |\ f(P) \neq \infty\}$, which has $\mathcal{K}$ as its quotient field. For a local parameter t at P and a nonzero function $f \in S_P$, we have

$$f(z) = \sum_{v=v_P}^{\infty} a_v t(z)^v,$$

where z is the local variable at P. Then, the order at P is $\mathrm{ord}_P(f) = v_p$, and f has a zero of order v_P at P, a pole of order $-v_P$ at P, or neither a pole nor a zero at P according to whether $v_P > 0$, $v_P < 0$ or $v_P = 0$.

The divisor of $f \in \mathcal{K}^\times$ is defined by

$$\mathrm{div}(f) = \sum_{P\in\mathfrak{R}} \mathrm{ord}_P(f) P \ .$$

This sum is finite as $\mathfrak{R}$ is compact. Thus, the map $f \mapsto \mathrm{div}(f)$ is a homomorphism of $\mathcal{K}^\times$ to $\mathcal{D}$, so that

$$\mathrm{div}(fg) = \mathrm{div}(f) + \mathrm{div}(g), \qquad \mathrm{div}(f^{-1}) = -\mathrm{div}(f) \ .$$

The image of this homomorphism is called the *group of principal divisors*. Let (f) denote the divisor of $f \in \mathcal{K}^\times$. For a divisor D, we put

$$L(D) := \{f \in \mathcal{K} |\ f = 0 \text{ or } \mathrm{div}(f) + D \geq 0\} \ .$$

Then $L(D)$ is a vector space over $\mathbb{C}$ of finite dimension. Denote its dimension by $l(D)$.

Two divisors are said to be linearly equivalent if their difference is principal. The map $D \mapsto \deg(D)$ is a homomorphism $\mathcal{D} \mapsto \mathbb{Z}$ whose kernel contains the principal divisors. Note that for any two linearly equivalent divisors D_1, D_2, $\deg(D_1) = \deg(D_2)$ and $l(D_1) = l(D_2)$, and we can talk about equivalence classes of linearly equivalent divisors which we call *divisor classes.* The quotient group is known as the Picard group of the Riemann surface $\mathfrak{R}$, denoted $\mathrm{Pic}(\mathfrak{R})$.

We can construct a one dimensional vector space $\mathrm{Dif}(\mathfrak{R})$ over $\mathcal{K}$ together with an additive map $d: \mathcal{K} \mapsto \mathrm{Dif}(\mathfrak{R})$ satisfying

$$d(fg) = gdf + fdg, \qquad df = 0 \Leftrightarrow f \in \mathbb{C}^\times .$$

Elements of $\mathrm{Dif}(\mathfrak{R})$ are called (meromorphic) differential forms (of degree 1) on $\mathfrak{R}$. If $f \in \mathcal{K} \setminus \mathbb{C}$, we have $\mathrm{Dif}(\mathfrak{R}) = \mathcal{K} \cdot df$. This means that for any $w \in \mathrm{Dif}(\mathfrak{R})$ there exists $h \in \mathcal{K}$ such that $w = h \cdot df$, and then $h = w/df$. In particular, dg/df is a well-defined element of $\mathcal{K}$ for every $g \in \mathcal{K}$. We define the divisor of w as follows. For $P \in \mathfrak{R}$, choose $t \in \mathcal{K}$ such that $v_P(t) = 1$, and put $v_P(w) := v_P(w/dt)$. This is independent of the choice of t. The divisor of w is then

$$\mathrm{div}(w) := \sum_{P \in \mathfrak{R}} v_P(w) P .$$

Then, we have $\mathrm{div}(fw) = \mathrm{div}(f) + \mathrm{div}(w)$. Thus the divisor $\mathrm{div}(w)$ for all $w \in \mathrm{Dif}(\mathfrak{R})^\times$ forms a divisor class, called the *canonical class* of $\mathfrak{R}$ or of $\mathcal{K}$. We say that $w \in \mathrm{Dif}(\mathfrak{R})$ is *holomorphic* or of the *first kind* if $w = 0$ or $\mathrm{div}(w) \geq 0$.

Theorem 1.3.1 (Riemann-Roch) *Let $\mathfrak{R}$ be a compact Riemann surface of genus g. Let $w \in \mathrm{Dif}(\mathfrak{R})^\times$. Then for any divisor D, $l(D)$ is finite and*

$$l(D) = \deg(D) + 1 - g + l(\mathrm{div}(w) - D).$$

Proposition 1.3.2 *For any* $w \in \mathrm{Dif}(\mathfrak{R})^\times$,

$$\deg(\mathrm{div}(w)) = 2g - 2\ .$$

Consequently,

Corollary 1.3.3
- *For the zero divisor we have* $L(0) = \mathbb{C}$, *so that* $l(0) = 1$.
- *For* $w \in \mathrm{Dif}(\mathfrak{R})^\times$, $l(\mathrm{div}(w)) = g$.
- *For a divisor* D,

$$\textit{if } \deg(D) < 0 \textit{ then } l(D) = 0,$$

$$\textit{if } \deg(D) > 2g - 2 \textit{ then } \quad l(D) = \deg(D) - g + 1\ .$$

1.3.2 Differential forms from modular forms

We restrict our attention in this subsection to the case of the Riemann surfaces $\Gamma\backslash\mathfrak{H}^*$, where Γ is a Fuchsian group of $\mathrm{SL}_2(\mathbb{R})$ of the first kind. We will establish here the connection between differential forms on $\mathfrak{R}$ and modular forms on Γ.

A meromorphic function on $\mathfrak{R} = \Gamma \backslash \mathfrak{H}^*$ gives rise to a modular function on Γ, and vice et versa. This is induced by the projection map $\pi : \mathfrak{H}^* \longrightarrow \mathfrak{R}$. We thus identify $\mathcal{K}$ with the field of modular functions on Γ. For k an even integer, the space of weight k meromorphic modular forms $\mathfrak{M}_k^m(\Gamma)$ is a one-dimensional vector space over $\mathcal{K}$. Moreover, given a non-constant modular function f on Γ, so that f' is a weight 2 (meromorphic) modular form on Γ and hence $f'^m \neq 0$ is a weight $2m$ modular form for any $m \in \mathbb{Z}$. On the other hand, as the weight is additive, we have for f and g, modular forms of the same weight, the function f/g is a weight 0 (meromorphic) modular form.

For $f \in \mathfrak{M}_2^m(\Gamma)$, we can view $f(z)dz$ as a differential form on $\mathfrak{R}$. Indeed, let $\psi \in \mathcal{K} \setminus \mathbb{C}$. Since $\psi' = d\psi/dz \in \mathfrak{M}_2^m(\Gamma)$, so that $f/\psi' \in \mathfrak{M}_0^m(\Gamma) = \mathcal{K}$. Then we put $f(z)dz = (f/\psi')d\psi$. This is independent of the choice of ψ. Conversely, if $w \in \mathrm{Dif}(\mathfrak{R})$, then $f := w/d\psi \in \mathcal{K}$, i.e. $f\psi' \in \mathfrak{M}_2^m(\Gamma)$ and $w = (f\psi')dz$. Thus the map $f \mapsto fdz$ is an isomorphism of $\mathfrak{M}_2^m(\Gamma)$ onto $\mathrm{Dif}(\mathfrak{R})$.

One can now construct an associative graded algebra

$$\mathfrak{T} := \bigoplus_{n=-\infty}^{\infty} \mathrm{Dif}^n(\mathfrak{R})$$

with the following conditions:

a) $\mathrm{Dif}^0(\mathfrak{R}) = \mathcal{K},\ \mathrm{Dif}^1(\mathfrak{R}) = \mathrm{Dif}(\mathfrak{R})$;

b) For each n, $\mathrm{Dif}^n(\mathfrak{R})$ is a one dimensional vector space over $\mathcal{K}$;

c) For $w_1 \in \mathrm{Dif}^m(\mathfrak{R}),\ w_2 \in \mathrm{Dif}^n(\mathfrak{R})$, the product w_1w_2 is defined and belongs to $\mathrm{Dif}^{m+n}(\mathfrak{R})$ and $w_1w_2 \neq 0$ if $w_1 \neq 0$ and $w_2 \neq 0$.

Similarly to the case of $k = 2$, the space $\mathrm{Dif}^k(\mathfrak{R})$ is isomorphic to $\mathfrak{M}_{2k}^m(\Gamma)$ via $f \mapsto f(dz)^k$. Furthermore, we have

$$\deg(\mathrm{div}(w)) = k(2g-2), \text{ for all } w \in \mathrm{Dif}^k(\mathfrak{R})^\times .$$

1.3.3 Dimension formula

The above subsections will now be used to compute the dimension of the spaces of holomorphic modular forms of a certain weight on a Fuchsian group Γ of the first kind. We will give the formula only for even positive weight, as these are the one of interest to us. For more details on dimensions formulae see [Shi].

The divisor of a weight k, k even, modular form f is defined as follows. Let $P \in \mathfrak{R}$. If P corresponds to a point z_0 of $\mathfrak{H}$, take a holomorphic isomorphism λ of $\mathfrak{H}$ onto the unit disc such that $\lambda(z_0) = 0$. If the order of the isotropy group of z_0 is e, then f has an expansion at z_0 in the local parameter $t = \lambda(z)^e$, say

$$f(z) = \sum_{n \geq v_{z_0}(f)}^{\infty} a_n t^n .$$

We set $v_P(f) = v_{z_0}(f)$. If P corresponds to a cusp s, then $f(z)|_k[\gamma^{-1}]$ has an expansion in the local parameter $q_s := \exp(2\pi i z/l_s)$ as in Section §1.2, where $\gamma \in \mathrm{SL}_2(\mathbb{R})$ such that $\gamma \cdot s = \infty$ and l_s is the cusp width at s. Then we define $v_P(f) = n_s$.
The divisor of f is then defined by

$$\mathrm{div}(f) = \sum_{P \in \mathfrak{R}} v_P(f) P \ .$$

The spaces $\mathfrak{M}_k(\Gamma)$ and $\mathfrak{S}_k(\Gamma)$ correspond then respectively to

$$\{f \in \mathfrak{M}_k^m(\Gamma) | \ \mathrm{div}(f) \geq 0\}$$

and

$$\{f \in \mathfrak{M}_k^m(\Gamma) | \ \mathrm{div}(f) \geq \sum_{j=1}^{r} Q_j\} \ ,$$

where $Q_1, \cdots, Q_r$ are the points of $\mathfrak{R}$ corresponding to the set of inequivalent cusps of Γ.

Proposition 1.3.4 *([Shi]) Let $E_1, \cdots, E_l$ be the points of $\mathfrak{R}$ corresponding to the elliptic fixed points of Γ, of order respectively $e_1, \cdots, e_l$, and let $Q_1, \cdots, Q_r$ be as above. Let $f \in \mathfrak{M}_k(\Gamma)^\times$, k even, let $w = f(z)(dz)^{k/2}$. Then,*

$$\mathrm{div}(f) = \mathrm{div}(w) + (k/2)\left(\sum_{i=1}^{l}(1 - e_i^{-1})P_i + \sum_{j=1}^{r} Q_j \ ,\right)$$

$$\deg(\mathrm{div}(f)) = (k/2)\left((2g-2) + \sum_{i=1}^{l}(1 - e_i^{-1}) + r\right) \ .$$

Taking into account that for $w \in \mathrm{Dif}(\mathfrak{R})^\times$, $\deg(\mathrm{div}(w)) = 2g - 2$, we have

Corollary 1.3.5 *The space $\mathfrak{S}_2(\Gamma)$ is isomorphic to the space of differential forms of the first kind on $\mathfrak{R}$, through the map $f \mapsto f dz$. In particular,*

$$\dim \mathfrak{S}_2(\Gamma) = g \ .$$

This corollary implies in particular that $\mathfrak{S}_2(\Gamma)$ is trivial if $g(\Gamma) = 0$.

Let $f_0 \in \mathfrak{M}_k^m(\Gamma)^\times$, and $D = \operatorname{div}(f_0)$. Since every element $f \in \mathfrak{M}_k^m(\Gamma)^\times$ can be written in the form $f = f_0 h$ with $h \in \mathfrak{M}_0^m(\Gamma)$, $\operatorname{div}(f) \geq 0$ if and only of $\operatorname{div}(h) \geq -D$. Therefore,

$$\dim \mathfrak{M}_k(\Gamma) = \dim\{h \in \mathcal{K} |\ \operatorname{div}(h) \geq -D\}$$

and

$$\dim \mathfrak{S}_k(\Gamma) = \dim \left\{ h \in \mathcal{K} |\ \operatorname{div}(h) \geq -D + \sum_{j=1}^{r} Q_j \right\} .$$

By applying the Riemann-Roch Theorem to the divisors $-D$ and $-D + \sum_{j=1}^{r} Q_j$, one gets

Theorem 1.3.6 *([Shi]) The notation being the same as above, we have,*

$$\dim \mathfrak{M}_k(\Gamma) = \begin{cases} (k-1)(g-1) + rk/2 + \sum_{i=1}^{l} \left[\frac{k(e_i - 1)}{2e_i} \right] & k > 2 , \\ g + r - 1 & k = 2,\ r > 0 , \\ g & k = 2,\ r = 0 , \\ 1 & k = 0 , \\ 0 & k < 0 . \end{cases}$$

and

$$\dim \mathfrak{S}_k(\Gamma) = \begin{cases} (k-1)(g-1) + r\left(\frac{k}{2} - 1\right) + \sum_{i=1}^{l} \left[\frac{k(e_i - 1)}{2e_i} \right] & k > 2 , \\ g & k = 2 , \\ 1 & k = 0,\ r = 0 , \\ 0 & k = 0,\ m > 0 , \\ 0 & k < 0 . \end{cases}$$

where $[k(e_i - 1)/2e_i]$ denotes the integer part of the rational $k(e_i - 1)/2e_i$.

1.4 Quasimodular forms

Quasimodular forms are a generalization of modular forms introduced by M. Kaneko and D. Zagier [KaZa]. Unlike modular forms, quasimodular forms are nearly modular under the action of Γ. It has been proved that they do have many applications in various fields. For instance, they have been used to derive identities involving convolution of the divisor power sigma functions [Ro], and in the theory of Painlevé equations where they were used to describe solutions to the so-called Chazy equation [Zag]. We notice that the Eisenstein series E_2 plays an important role in quasimodular forms for $\mathrm{SL}_2(\mathbb{Z})$ and its subgroups.

Quasimodular forms arise from the almost holomorphic modular forms in the following sense. An almost holomorphic modular forms on Γ is a function in $\mathfrak{H}$ which transforms like a modular form but, instead of being holomorphic, is a polynomial in $1/y$ (with $y = \Im(z)$) with holomorphic coefficients. The motivating example being the non-holomorphic Eisenstein series $E_2^* := E_2(z) - \dfrac{3}{\pi y}$ which is modular of weight 2. More precisely, an almost holomorphic modular form of weight k and depth p on Γ is a function of the form

$$F(z) = \sum_{r=0}^{p} f_r(z)(-4\pi y)^{-r},$$

where each f_r is a holomorphic function of moderate growth, and transforms like a weight k modular form. We denote by $\widehat{M}_k^{(\leq p)} = \widehat{M}_k^{(\leq p)}(\Gamma)$ the space of such forms and by $\widehat{M}_* = \bigoplus_k \widehat{M}_k$, $\widehat{M}_k = \bigcup_p \widehat{M}_k^{(\leq p)}$ the graded ring of all almost holomorphic modular forms and the ring of all almost holomorphic modular forms of weight k and depth $\leq p$, where Γ is omitted from the notations. For the basic example $E_2^* \in \widehat{M}_k^{\leq p}(\mathrm{SL}_2(\mathbb{Z}))$, we have $f_0 = E_2$, the Eisenstein series, and $f_1 = 12$.

The space $\widetilde{M}_k^{(\leq p)} = \widetilde{M}_k^{(\leq p)}(\Gamma)$ of quasimodular forms of weight k and depth $\leq p$ on Γ is then defined as the space of constant terms $f_0(z)$ of $F(z)$ as F runs over $\widehat{M}_k^{(\leq p)}$. Note that the almost holomorphic modular form F is uniquely determined by its constant term f_0. Hence the ring of quasimodular forms (respectively of weight k and depth $\leq p$) on Γ, denoted by $\widetilde{M}_* = \bigoplus_k \widetilde{M}_k$ (respectively $\widetilde{M}_k = \bigcup_p \widetilde{M}_k^{(\leq p)}$) is canonically isomorphic to the ring $\widehat{M}_*$ (respectively to $\widehat{M}_k$). Another direct definition of quasimodular forms is as follows. A quasimodular form of weight k and depth

$\leq p$ on Γ is a holomorphic function f in $\mathfrak{H}$ such that, for fixed $z \in \mathfrak{H}$ and variable $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$, the function $f(z)|_k[\alpha] := \jmath_\alpha(z)^{-k} f(\gamma \cdot z)$ is a polynomial of degree $\leq p$ in $\frac{c}{\jmath_\alpha(z)}$ with the same coefficients as above. We have the following proposition, in which Γ is a non-cocompact discrete subgroup of $\mathrm{SL}_2(\mathbb{R})$ and $\phi \in \widetilde{M}_2(\Gamma)$ is a quasimodular form of weight 2 on Γ which is not modular, e.g., $\phi = E_2$ if Γ is a subgroup of $\mathrm{SL}_2(\mathbb{Z})$. In fact, we have the following

Proposition 1.4.1 *For $\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$, we have*

$$E_2(\alpha \cdot z) \;=\; \jmath_\alpha(z)^2 E_2(z) + \frac{6c}{\pi \imath} \jmath_\alpha(z) \;. \tag{1.4.1}$$

This proposition can be proved in different ways, for instance one can use the fact that E_2 is the logarithmic derivative of the modular discriminant

$$\Delta := \frac{1}{1728}(E_4^3 - E_6^2)$$

which is a cusp form of weight 12; the derivation being $\dfrac{1}{2\pi\imath}\dfrac{d}{dz}$.

The following result summarizes most of the properties of quasimodular forms

Theorem 1.4.2 ([KaZa]) *(i) The space of quasimodular forms on Γ is closed under differentiation. More precisely, we have*

$$D\left(\widetilde{M}_k^{(\leq p)}\right) \subset \widetilde{M}_{k+2}^{(\leq p+1)}$$

for all $k,\ p \geq 0$. Here D denotes the differential operator $\dfrac{1}{2\pi\imath}\dfrac{d}{dz}$.

(ii) Every quasimodular form on Γ is a polynomial in ϕ with modular coefficients. More precisely, we have

$$\widetilde{M}_k^{(\leq p)}(\Gamma) = \bigoplus_{r=0}^{p} \mathfrak{M}_{k-2r}(\Gamma) \cdot \phi^r$$

for all $k,\ p \geq 0$, where $\mathfrak{M}_\jmath(\Gamma)$ denote the space of weight $\jmath$ modular forms on Γ.

(iii) *Every quasimodular form on* Γ *can be written uniquely as a linear combination of derivatives of modular forms and of* ϕ. *More precisely, for all* $k,\ p \geq 0$ *we have*

$$\widetilde{M}_k^{(\leq p)}(\Gamma) = \begin{cases} \bigoplus_{r=0}^{p} D^r(\mathfrak{M}_{k-2r}(\Gamma)) & \text{if } p < k/2, \\ \bigoplus_{r=0}^{k/2-1} D^r(\mathfrak{M}_{k-2r}(\Gamma)) \oplus \mathbb{C} \cdot D^{k/2-1}\phi & \text{if } p \geq k/2. \end{cases}$$

In the case $\Gamma = \mathrm{SL}_2(\mathbb{Z})$, with $\phi = E_2$, the proposition implies the property that the graded ring $\mathbb{C}[E_2,\ E_4,\ E_6]$ is closed under differentiation; provides another proof to the Ramanujan differential system in Chapter 5, Section §5.1.1; and produces another basic example of a quasimodular form, namely, the derivative of a modular form.

We end this chapter by generalizing the definition of a quasimodular form. The above theorem implies that any holomorphic quasimodular form for Γ is holomorphic function that transforms this way

$$f(z)|_k[\alpha] = \sum_{r=0}^{p} f_r(z)\phi(z)^r \left(\frac{c}{cz+d}\right)^r, \text{ for all } \alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

under the action of Γ, where f_r is a holomorphic modular function on Γ of weight $k-2r$ and ϕ is a weight 2 depth 1 holomorphic quasimodular form on Γ. If we ignore the growth condition, then we can define a meromorphic quasimodular form on a discrete subgroup of $\mathrm{SL}_2(\mathbb{R})$ as follows

Definition 1.4.3 *A meromorphic function* f *of* $\mathfrak{H}$ *is called a meromorphic quasimodular form on* Γ *of weight* k *and depth* p *if*

$$f(z)|_k[\alpha] = \sum_{r=0}^{p} f_r(z)\phi(z)^r \left(\frac{c}{cz+d}\right)^r, \text{ for all } \alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

where f_r *is a meromorphic modular function on* Γ *of weight* $k-2r$ *and* ϕ *is a weight 2 depth 1 holomorphic quasimodular form on* Γ.

Thus defined, a similar statement to that in Theorem 1.4.2 holds with $\mathfrak{M}_j(\Gamma)$ replaced by $\mathfrak{M}_j^m(\Gamma)$. The quasimodular ϕ in that proposition, however, may be chosen to be holomorphic. We will use this generalization in Chapter 5 to list all *equivariant forms*.

Chapter 2

Equivariant forms for the modular group

The notion of equivariant forms originates from the study of certain Schwarz differential equation involving modular forms by [AAS1]. The goal was to study the modularity of meromorphic solutions f such that the Schwarz derivative of f, denoted $\{f, z\}$, is a weight 4 modular form for a discrete subgroup G of $\mathrm{SL}_2(\mathbb{R})$. This notion also appears in the context of elliptic functions in [Bra, Hei]. Indeed, for a lattice $L = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$ with $z = \omega_2/\omega_1 \in \mathfrak{H}$, the Weierstrass ζ-function is defined by $\zeta' = -\wp$ where $\wp$ is the Weierstrass elliptic $\wp$-function. If η_1 and η_2 are the pseudo-periods of ζ, then the fundamental equivariant form is given by

$$h_0 = \omega_1 \eta_2$$

depends only on z and satisfy $h_0(\alpha \cdot z) = \alpha \cdot h_0(z)$, $\alpha \in \mathrm{SL}_2(\mathbb{Z})$. We will not pursue our investigation in this direction, we will however study equivariant forms in connection with modular forms.

In this chapter, we begin by giving some preliminaries about the origin of these special functions. We then provide the formal definition of an equivariant form for the modular group $SL_2(\mathbb{Z})$. We also prove some results about these equivariant forms and give further generalizations.

2.1 The Schwarz derivative

The Schwarz derivative is a differential operator that appears in many contexts of mathematics. In particular, it is a useful toll in the theory of analytic functions and in the theory of automorphic forms, see [Ford]. It also plays an important role in [AAS2], where it leads to the notion of *equivariant forms* and is used to solve certain differential equations involving modular forms.

For a meromorphic function on a domain (open and connected) of $\mathbb{C}$, the Schwarz derivative, denoted $\{f, z\}$, is defined by

$$\begin{aligned} \{f, z\} &= \left(\frac{f''}{f'}\right)' - \frac{1}{2}\left(\frac{f''}{f'}\right)^2 \\ &= \frac{2f'f''' - 3f''^2}{2f'^2} \,. \end{aligned} \tag{2.1.1}$$

It satisfies the following rules.

Proposition 2.1.1 *([McSe1]) We have*

- **Chain rule:** *If w is a function of z then*

$$\{f, z\} = (dw/dz)^2\{f, w\} + \{w, z\} \,. \tag{2.1.2}$$

- *If f is a linear fractional transform of z, then $\{f, z\} = 0$.*

- **Inversion formula:** *If $w'(z_0) \neq 0$ for some point z_0, then in a neighborhood of z_0,*

$$\{z, w\} = -(dz/dw)^2\{w, z\} \,. \tag{2.1.3}$$

This proposition can be verified using the definition of the Schwarz derivative, and suggests the question of finding meromorphic functions $w(z)$ for which $\{w, z\} = Q(z)$, for some given meromorphic function Q on some domain. This problem is equivalent to solving the differential equation

$$y''(z) + \frac{1}{2}Q(z)y(z) = 0 , \tag{2.1.4}$$

for which one can expect to obtain two linearly independent solutions $y_1(z)$ and $y_2(z)$. The quotient of these $w(z) = \frac{y_2(z)}{y_1(z)}$ is a solution to $\{w, z\} = Q(z)$. Conversely, if w is a locally univalent function such that $\{w, z\} = Q(z)$, then $y_1 = \frac{w}{\sqrt{w'}}$, $y_2 = \frac{1}{\sqrt{w'}}$ are two linearly independent solutions to (2.1.4). This yields, as a corollary,

Proposition 2.1.2 *We have*

- $\{f, z\} = 0$ *if and only if f is a linear fractional transform of z.*
- $\{w_1, z\} = \{w_2, z\}$ *if and only if each function is a linear fraction of the other.*

Moreover, we have

Corollary 2.1.3 *Let* $\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathbb{C})$. *Then*

$$\{w, z\} = \frac{\det \alpha}{(cz + d)^4}\{w, \alpha \cdot z\} \tag{2.1.5}$$

Lastly, given a meromorphic function f, then, from the definition of the Schwarz derivative, one deduces the following

Proposition 2.1.4 ([McSe1]) *The Schwarz derivative $\{f, z\}$ of f has a double pole at the critical points of f and is holomorphic elsewhere including at simple poles of f.*

2.2 The origin of equivariant forms in the context of Schwarz differential equations

In studying solutions to certain Schwarz equations [AAS2], the authors show how the notion of equivariant forms did appear as exceptional solutions. More precisely, they were answering the following question: Given a meromorphic function f on $\mathfrak{H}$ such that $\{f, z\} =: F(z)$ is a weight 4 modular form for a certain group G_F, what can be said about the invariance group G_f of f? In other words, is f modular for any discrete group ? In fact, this problem is the converse to another question which was first raised and solved in [McSe1], namely given a modular function f for a subgroup G_f of $\mathrm{SL}_2(\mathbb{R})$ then $F(z) = \{f, z\}$ is a weight 4 modular form for a group G_F containing G_f. In particular, if G_f is a genus 0 discrete subgroup of $\mathrm{SL}_2(\mathbb{R})$ and f is a Hauptmodul, that is to say that f generates the function field over this Riemann surface corresponding to G_f and having a simple pole at infinity, then G_F is nothing but the normalizer of G_f in $\mathrm{SL}_2(\mathbb{R})$. In conclusion,

Proposition 2.2.1 ([McSe1]) *We have*

i. *If f is a modular function for G_f then $F(z)$ is a weight 4 modular form for G_f.*

ii. *If G_f is of genus 0 and f is a Hauptmodul for G_f, then $F(z)$ is weight 4 holomorphic modular form for the normalizer of G_f inside $SL_2(\mathbb{R})$.*

To make this statement concrete, we give the example (see [McSe1]) of the principal congruence subgroup $\Gamma(2)$ of $\mathrm{SL}_2(\mathbb{Z})$. The group $\Gamma(2)$ is of genus 0, has no elliptic elements, and in this case, the *lambda-elliptic function* λ due to Klein is a Hauptmodul for $\Gamma(2)$. It is given by

$$\lambda(z) = \frac{\vartheta_2^4}{\vartheta_3^4} = 16q - 128q^2 + 704q^3 - 3072q^4 + 11488q^5 + \dots\ ,\quad q = e^{\pi i z}.$$

Then, this proposition implies that $\{\lambda, z\}$ is a weight 4 holomorphic modular form for $\mathrm{SL}_2(\mathbb{Z})$, the normalizer of $\Gamma(2)$ inside $\mathrm{SL}_2(\mathbb{R})$. However, the space of weight

4 modular forms for $SL_2(\mathbb{Z})$ is one dimensional and is generated by the Eisenstein series E_4 (1 2 5) Therefore, $\{\lambda, z\}$ is a constant multiple of E_4 Finally, a comparison of the first Fourier coefficients shows that

$$\{\lambda, z\} = \pi^2 E_4$$

More generally, it is shown in [AAS2] that for all values of n for which $\Gamma(n)$ is of genus 0, notably $2 \leq n \leq 5$, if f_n is a Hauptmodul for $\Gamma(n)$, then

$$\{f_n, z\} = \frac{4\pi^2}{n^2} E_4$$

In particular, this says that f_n for the above values of n is a solution to the Schwarz equation

$$\{f, z\} = \frac{4\pi^2}{n^2} E_4 \tag{2 2 1}$$

For the value $n = 1$, $\Gamma(1) = SL_2(\mathbb{Z})$, and in this case the modular j function,

$$j = \frac{E_4^3}{\Delta}$$

is a Hauptmodul for $SL_2(\mathbb{Z})$ and is no longer a solution to (2 2 1) Therefore, it is natural to ask for a solution of (2 2 1) in the case $n = 1$

Theorem 2.2.2 ([AAS2]) *The function on $\mathfrak{H}$ defined by*

$$h_4(z) = z + 4\frac{E_4}{E_4'} \tag{2 2 2}$$

is a solution to the Schwarz equation (2 2 1)

The proof of this theorem is based on the following differential system of Ramanujan [Ramj] satisfied by the Eisenstein series E_2, E_4 and E_6

$$\frac{1}{2\pi i}\frac{dE_2}{dz} = \frac{1}{12}(E_2^2 - E_4), \tag{2 2 3}$$

$$\frac{1}{2\pi i}\frac{dE_4}{dz} = \frac{1}{3}(E_2 E_4 - E_6), \tag{2 2 4}$$

$$\frac{1}{2\pi i}\frac{dE_6}{dz} = \frac{1}{2}(E_2E_6 - E_4^2). \qquad (2.2.5)$$

The function in Theorem 2.2.2 turned out to be very special and obeys a certain transformation equation under the action of the modular group, namely,

Theorem 2.2.3 ([AAS2]) *The function h_4 in Theorem 2.2.2 satisfies*

$$f(\alpha \cdot z) = \alpha \cdot f(z)$$

for all $\alpha \in SL_2(\mathbb{Z})$ and $z \in \mathfrak{H}$.

This theorem can be proved using the facts that $\mathrm{SL}_2(\mathbb{Z})$ in generated by the matrices S and T, given at the beginning, and that E_4 is weight 4 modular form for $\mathrm{SL}_2(\mathbb{Z})$.

The function h_4 in Theorem 2.2.2 is the first example of an *equivariant form.*

We now return to the details of the first question that we asked about the size of G_f if we only know that of G_F, the invariance group of $F(z) = \{f, z\}$. In fact, this question may posed in an equivalent way. By definition F satisfies the modular equation

$$\{f(\alpha \cdot z), \alpha \cdot z\} = F(\alpha \cdot z) = (cz+d)^4 F(z) = (cz+d)^4\{f(z), z\}\ ,$$

for all $\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G_F,\ z \in \mathfrak{H}$. However, as seen in Section §2.1,

$$\{f(\alpha \cdot z), \alpha \cdot z\} = (cz+d)^4\{f(\alpha \cdot z), z\}.$$

Therefore, by Proposition 2.1.2, $f(\alpha \cdot z)$ is a linear fraction of $f(z)$

$$f(\alpha \cdot z) = \Phi_\alpha \cdot f(z)\ ,$$

for some (invertible) matrix Φ_α in $\mathrm{GL}_2(\mathbb{C})$. Thus, we have a group homomorphism

$$\begin{array}{rccl} \Phi\ : & G_F & \longrightarrow & \mathrm{GL}_2(\mathbb{C}) \\ & \alpha & \longmapsto & \Phi_\alpha\ . \end{array}$$

The invariance group G_f of f is then the kernel of Φ. The notion of *equivariant forms* arises from the question of when is $\ker\Phi$ trivial. For instance, it is the case if Φ is the natural injection of G_F into $\mathrm{GL}_2(\mathbb{C})$; that is to say, when

$$f(\alpha\cdot z)=\alpha\cdot f(z),\ \text{ for all } z\in\mathfrak{H} \text{ and } \alpha\in G_F\ .$$

This is the type of functions we are studying in this work.

2.3 Equivariant forms for the modular group $\mathrm{SL}_2(\mathbb{Z})$

The aim of this section is to provide the first family of equivariant forms involving holomorphic modular forms.

2.3.1 Equivariant forms for $\mathrm{SL}_2(\mathbb{Z})$

As seen in the previous section, the function h_4 in Theorem 2.2.2 commutes with the action of $\mathrm{SL}_2(\mathbb{Z})$. In this subsection we give the precise definition of a general type functions that are called *equivariant forms*. The terminology will be justified in Chapter 5.

Definition 2.3.1 *A meromorphic function h on $\mathfrak{H}$ is called an equivariant form for $SL_2(\mathbb{Z})$ if it satisfies*

1. *$h(\alpha\cdot z)=\alpha\cdot h(z)$, for all $\alpha\in SL_2(\mathbb{Z})$, and*
2. *the function $h(z)-z$ is meromorphic at infinity.*

A clarification to the second condition in this definition is as follows. Using condition 1. with $\alpha=T$, we see that $h(z+1)=h(z)+1$ so that the function $h(z)-z$ is 1-periodic, and hence has a Fourier expansion in the local parameter $q=e^{2\pi iz}$. The meromorphy at infinity is then equivalent to saying that in this Fourier expansion the number of negative powers of q is finite.

We notice that the function $h_t(z) = z$ is a trivial example for such definition.

The function h_4 in Theorem 2.2.3 is the first example of such functions. In fact, we only need to check the second condition as the first one follows from Theorem 2.2.3. However, it is straightforward from the expression of E_4 that it is meromorphic on $\mathfrak{H}$; the meromorphy at infinity follows from the $q-$expansion of E_4', which is of the form $cq(1+O(q))$, $c = 480\pi i$. We now provide a more general example inspired from the above one.

Theorem 2.3.2 ([AAS1]) *Let f be a meromorphic modular form of weight k (k even) for $SL_2(\mathbb{Z})$. Then the function*

$$h_f(z) = z + k\frac{f(z)}{f'(z)}$$

is an equivariant form for $SL_2(\mathbb{Z})$.

We shall give the proof of this theorem as it provides a generalization of the first example of an equivariant form.

Proof: To verify that h_f satisfies the first condition of the definition it suffices to verify that for S and T, as $SL_2(\mathbb{Z})$ is generated by these two matrices. Now, from the invariance of f under translation, it is immediate that $h_f(z+1) = h_f(z) + 1$. Also, from the transformation formula of f under S, namely

$$f\left(\frac{-1}{z}\right) = z^k f(z) \ ,$$

one gets

$$f'\left(\frac{-1}{z}\right) = kz^{k+1}f(z) + z^{k+2}f'(z) \ .$$

Hence,

$$\begin{aligned} h_f\left(\frac{-1}{z}\right) &= \frac{-1}{z} + k\frac{f(z)}{kzf(z) + z^2 f'(z)} \\ &= \frac{-zf'(z)}{kzf(z) + z^2 f'(z)} \ . \end{aligned}$$

On the other hand, one has easily shows that $\dfrac{-1}{h_f(z)}$ is equal to the above, whence $h_f(\alpha\ z) = \alpha\ h_f(z)$ for all $\alpha \in \mathrm{SL}_2(\mathbb{Z})$ The second condition follows from the fact that f has a q-expansion of the form $q^{n_0}(a_{n_0} + O(q))$, for some finite index n_0 ∎

Before we provide other examples involving Eisenstein series, we give some key properties these functions

Proposition 2.3.3 ([AAS1]) *The modular forms cf, c a non-zero constant, and f^n, $n \in \mathbb{Z}^\times$, give rise to the same equivariant form h_f as f*

This proposition follows from the fact that the logarithmic derivative of f is the same for scalar multiples cf As for f^n, its weight is kn and its logarithmic derivative is nf'/f, so that $knf/nf' = kf/f'$ Note that the restriction to integer powers of f is made merely to avoid dealing with branch points

Since these objects are not familiar, we give another example that involves the Eisenstein series E_6

Example 2.3.1 Let E_6 be the weight 6 Eisenstein series given by (1 2 6) Then the function

$$h_6(z) = z + 6\frac{E_6}{E_6'} \tag{2 3 1}$$

is an equivariant form for $\mathrm{SL}_2(\mathbb{Z})$ This also follows from Theorem 2 3 2, however, we would like to make use of the Ramanujan identity (2 2 5) and the transformation formula (1 4 1) for E_2 to prove this fact We will prove this for the generators S and T of $\mathrm{SL}_2(\mathbb{Z})$

The function can be also written as

$$h_6(z) = z + 6\frac{E_6}{\pi i(E_2E_6 - E_4^2)} = z + \frac{1}{(\pi i/6)E_2 - E_4^2/E_6}$$

From this expression, it is easy to see that $h_6(z+1) = h_6(z) + 1$. The verification of

$$h_6\left(\frac{-1}{z}\right) = \frac{-1}{h_6(z)}$$

is as follows. The left hand side is equal to

$$h_6\left(\frac{-1}{z}\right) = \frac{-(E_2 - E_4^2/E_6)}{z(E_2 - E_4^2/E_6) + 6/(\pi i)},$$

which can be verified to be equal to the right hand side

$$\frac{-1}{h_6(z)} = \frac{-(E_2 - E_4^2/E_6)}{z(E_2 - E_4^2/E_6) + 6/(\pi i)}.$$

Thus h_6 is an equivariant form for $SL_2(\mathbb{Z})$. The meromorphicity at infinity follows from the invariance under T and the Fourier expansion of E_6 and its derivative.

2.3.2 A first generalization

All the results of the above section can now be generalized to the case of modular forms with multiplier systems.

Theorem 2.3.4 *Let $f \in \mathfrak{M}_k^m(SL_2(\mathbb{Z}), \nu)$, where $k \in \mathbb{R}$ and ν is a multiplier system associated to an automorphic factor on $SL_2(\mathbb{Z})$ of weight k. Then the function*

$$h(z) := z + \frac{kf(z)}{f'(z)}$$

is an equivariant form for $SL_2(\mathbb{Z})$.

Proof: Let $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$. Then, we have

$$\begin{aligned} h(\alpha \cdot z) &= \frac{az+b}{cz+d} + \frac{k(cz+d)^k \nu(\alpha) f}{ck(cz+d)^{k+1}\nu(\alpha) f + (cz+d)^{k+2}\nu(\alpha) f'} \\ &= \frac{(az+b)(ckf + (cz+d)f') + kf}{(cz+d)(ckf + (cz+d)f')} \\ &= \frac{akf + (az+b)f'}{ckf + (cz+d)f'}, \end{aligned}$$

where we have used the modular identity $ad - bc = 1$ to simplify the numerator. In the meantime, we have

$$\alpha \cdot h(z) = \frac{ah(z) + b}{ch(z) + d} = \frac{(az + b)f'(z) + akf(z)}{(cz + d)f'(z) + ckf} .$$

The meromorphy of $h(z) - z$ at infinity follows similarly as

$$h(z) - z = \frac{kf(z)}{f'(z)} = \frac{k\tilde{f}(z)}{\tilde{f}'(z)} ,$$

and by noticing that the factor $\nu(T)$ in $\tilde{f} = f(T \cdot z)$ is independent of z, so that $f'(z+1) = \nu(T)\tilde{f}'(z)$. ∎

Remark 2.3.1 So far, all the preceding examples involve only holomorphic modular forms. Actually, the purpose of this restriction was to give the first few examples and properties of these very special functions. We will shows in the next section that we can construct a more general class of equivariant forms.

2.3.3 A second generalization

A fundamental question would be to ask whether, all equivariant forms arise this way in the sense that they all arise from reciprocal of logarithmic derivatives of modular forms. It turns out that this type of equivariant forms accounts only for a small class.

Example 2.3.2 Let h be the function defined by

$$h(z) = z + \frac{4E_4}{E_4' + E_6} .$$

Then h is an equivariant form for $SL_2(\mathbb{Z})$. To verify that we only check the first condition of the definition as the second follows similarly as in the above examples.

Here again we use the fact that $\mathrm{SL}_2(\mathbb{Z})$ is generated by S and T. The invariance of the Eisenstein series E_4 and E_6 and of E_4' under T implies that

$$h(z+1) = h(z) + 1 \ .$$

For $h(-1/z)$, we have

$$\begin{aligned} h\left(\frac{-1}{z}\right) &= \frac{-1}{z} + \frac{4z^4 E_4}{(4z^5 E_4 + z^6 E_4' + z^6 E_6} \\ &= \frac{-(E_4' + E_6)}{z(E_4' + E_6) + 4E_4} \ . \end{aligned}$$

On the other hand,

$$\begin{aligned} \frac{-1}{h(z)} &= \frac{-1}{z + 4E_4/\left(E_4' + E_6\right)} \\ &= \frac{-(E_4' + E_6)}{z(E_4' + E_6) + 4E_4} \ . \end{aligned}$$

This proves our assertion.

Remark 2.3.2 The equivariant form in the above example can not be represented as in Theorem 2.3.2. In fact, it is the first example of a larger class of equivariant form as it will be seen later. The verification of this fact could be done by using the fact that $E_4' + E_6$ is a quasimodular form or using Ramanujan's differential system. Indeed, using the Ramanujan identity for E_4' and the expansions of E_4, E_6 and E_4' about the zero ρ, the cubic root of unity, of E_4 one finds that the residue at this point of $(E_4' + E_6)/E_4$ has residue $1 + 2\pi\imath/3$, which is not a rational.

More generally, we have

Theorem 2.3.5 *Let f and g be (meromorphic) modular forms for $SL_2(\mathbb{Z})$ of respective weights k and $k+2$, where $k \in \mathbb{R}$. Then, the function*

$$h(z) = z + \frac{kf(z)}{f'(z) + g(z)}$$

is an equivariant form for $SL_2(\mathbb{Z})$.

Proof: The meromorphy of $h(z) - z$ at ∞ follows from the fact that f and g are meromorphic on $\mathfrak{H}$ and at infinity and from their Fourier expansion at ∞. The first condition can be verified as follows. Let $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$. Then, we have

$$\begin{aligned} h(\alpha \cdot z) &= \frac{az+b}{cz+d} + \frac{k(cz+d)^k f}{ck(cz+d)^{k+1} f + (cz+d)^{k+2}(f'+g)} \\ &= \frac{(az+b)(ckf + (cz+d)(f'+g)) + kf}{(cz+d)(ckf + (cz+d)(f'+g))} \\ &= \frac{akf + (az+b)(f'+g)}{ckf + (cz+d)(f'+g)} , \end{aligned}$$

In the meantime, we have

$$\begin{aligned} \alpha \cdot h(z) &= \frac{ah(z)+b}{ch(z)+d} \\ &= \frac{(az+b)(f'+g) + akf}{(cz+d)(f'+g) + ckf} . \end{aligned}$$

This concludes the proof. ∎

Remark 2.3.3 Theorem 2.3.5 can be generalized only to modular forms with constant multiplier systems for the equivariance property would force both f and g to have the same multiplier system. However, using the definition of a multiplier system mentioned in Chapter 1, it can be shown that a multiplier system ν for both weights k and $k+2$ ($k \in \mathbb{R}$) is constant.

2.4 Equivariant forms without fixed points

In this subsection we consider a subset of the set of equivariant forms having very interesting properties. Let h be an equivariant form, and let $\alpha \in \mathrm{SL}_2(\mathbb{Z})$ be an elliptic element with fixed point $z_0 \in \mathfrak{H}$. Then, we have

$$h(\alpha \cdot z_0) = h(z_0) = \alpha \cdot h(z_0) .$$

Therefore $h(z_0)$ is also a fixed point of α, and thus

$$h(z_0) = z_0 \text{ or } \overline{z}_0$$

as an elliptic element fixes exactly two complex conjugate complex numbers.

Definition 2.4.1 *An equivariant form h is said to be* equivariant form without fixed points *if*

- *there is no $z \in \mathfrak{H}$ such that $h(z) = z$;*
- *infinity is not a fixed point of h in the sense that*

$$\lim_{z \mapsto \infty} (h(z) - z) \neq 0 \ .$$

From the above discussion, we have

Proposition 2.4.2 ([AAS1]) *Let h be an equivariant form without fixed points for $SL_2(\mathbb{Z})$. Then, for any elliptic element $\alpha \in SL_2(\mathbb{Z})$ with fixed point $z_0 \in \mathfrak{H}$, we have*

$$h(z_0) = \overline{z}_0 \ .$$

Example 2.4.1 Let Δ be the modular discriminant. Then, we know that

$$\frac{1}{2\pi i} \frac{\Delta'}{\Delta} = E_2 \ ,$$

where E_2 is the Eisenstein series in (1.2.4). Hence, the function

$$h_0 := z + 12\frac{\Delta}{\Delta'} = z + \frac{6}{\pi i E_2}$$

is an equivariant form for $\mathrm{SL}_2(\mathbb{Z})$ a consequence of the Theorem 2.3.2. As the Eisenstein series is holomorphic on $\mathfrak{H}$ and at infinity, it follows that the function $h_0(z) - z$ never vanishes on $\mathfrak{H}^*$. Hence, h_0 is an equivariant form without fixed points for

$\mathrm{SL}_2(\mathbb{Z})$. Furthermore, as $\imath$ and $\rho := (-1+\imath\sqrt{3})/2$ are representatives of the elliptic points of $\mathrm{SL}_2(\mathbb{Z})$, and by Proposition 2.4.2 we have the special values of h_0

$$h_0(\imath) = -\imath, \qquad h_0(\rho) = \overline{\rho}\,.$$

The equivariant forms h_4 in (2.2.2) and h_6 in (2.3.1) have fixed points as $h_4(\rho) = \rho$ and $h_6(\imath) = \imath$. We notice that using the Ramanujan differential (2.2.4) and (2.2.5) we have that $E_4'(\rho) = -2\pi\imath E_6(\rho)/3$ and $E_6'(\imath) = -\pi\imath E_4(\imath)^2$. As E_4 and E_6 vanish respectively only at ρ and $\imath$, the fact that $E_4'(\rho) \neq 0$ and $E_6'(\imath) \neq 0$ follows; i.e. h_4 and h_6 have a fixed point.

This example and its generalization play an important role for the algebraic structure of the set equivariant forms as will be seen in Chapter 4. We will also show therein that the two classes of equivariant forms, the one that arises from logarithmic derivatives of modular forms and that of equivariant forms without fixed points, form subspaces of the *space of equivariant forms.*

We end this section by giving an example of an equivariant form with fixed points at cusps only.

Example 2.4.2 Consider the following function

$$h(z) := z + \frac{12\Delta(z)}{\Delta'(z) + E_4^2(z)E_6(z)}\,,$$

where E_4, E_6 are respectively the Eisenstein series of weight 4 and 6 and Δ is the modular discriminant. Then, by Theorem 2.3.5, h is an equivariant form for $\mathrm{SL}_2(\mathbb{Z})$. Also, h has no fixed points on $\mathfrak{H}$ as the denominator in $h(z) - z$ is holomorphic on $\mathfrak{H}$ and Δ does not vanish on $\mathfrak{H}$, yet, at infinity

$$\lim_{z\mapsto\infty}(h(z) - z) = 0$$

as $\Delta' + E_4^2E_6 = 1 + O(q)$ and hence $h(z) - z = 12\Delta(1 + O(q))$ vanishes at infinity since Δ does. This proves our assertion.

Chapter 3

Rational equivariant forms

In this chapter we investigate the connection between equivariant forms and modular forms. More precisely, we study conditions under which an equivariant form h arises from a modular form f via the formula $h(z) = z + k\dfrac{f(z)}{f'(z)}$. Such equivariant form will be called *rational equivariant form*; a term that will be justified below.

3.1 Equivariant forms for modular subgroups

The aim of this section is to generalize the definition of an equivariant form for $\mathrm{SL}_2(\mathbb{Z})$ to any subgroup of finite index of $\mathrm{SL}_2(\mathbb{Z})$. In fact such definition is easily generalized to any discrete subgroup $\mathrm{SL}_2(\mathbb{R})$. However, for the sake of clarity and in order to be able to provide concrete examples, we restrict ourselves to subgroups of $\mathrm{SL}_2(\mathbb{Z})$.

3.1.1 The slash operator on equivariant forms

In this section we define the notions of *unrestricted equivariant functions* and *equivariant forms* for a subgroup of $\mathrm{SL}_2(\mathbb{Z})$, and also introduce the notion of the slash operator on unrestricted equivariant functions.

Definition 3.1.1 *A function $h : \mathfrak{H} \to \mathbb{C}$ is called an* unrestricted equivariant function *for a finite index subgroup* Γ *of* $SL_2(\mathbb{Z})$ *if it commutes with the action of* Γ *on* $\mathfrak{H}$*. In other words, if it satisfies*

$$h(\alpha \cdot z) = \alpha \cdot h(z)$$

for all $\alpha \in \Gamma$ *and* $z \in \mathfrak{H}$*.*

In this definition, by "unrestricted" we mean that we do not have to worry about the type of singularities that h may have, especially at cusps.

We now define the slash operator for unrestricted equivariant functions. There is an action of $\mathrm{SL}_2(\mathbb{R})$ on the space of meromorphic functions on $\mathfrak{H}$ given as follows. For f meromorphic on $\mathfrak{H}$ and $\gamma = \begin{pmatrix} p & q \\ r & s \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$

$$f||_2[\gamma](z) = j_\gamma(z)^{-2} f(z) - r j_\gamma(z)^{-1} \ .$$

Indeed, for elements of $\mathrm{SL}_2(\mathbb{Z})$, $\beta = \begin{pmatrix} p & q \\ r & s \end{pmatrix}$, $\gamma = \begin{pmatrix} p & q \\ r & s \end{pmatrix}$, we have on one hand

$$f||_2[\beta\gamma](z) = j_{\beta\gamma}(z)^{-2} f(\beta\gamma \cdot z) - (cp + dr) j_{\beta\gamma}(z)^{-1} \ .$$

On the other hand,

$$\begin{aligned} f||_2[\beta]||_2[\gamma](z) &= j_\gamma(z)^{-2} f||_2[\beta](\gamma \cdot z) - r j_\gamma(z)^{-1} \\ &= j_\gamma(z)^{-2} \left(j_\beta(\gamma \cdot z)^{-2} f(\beta\gamma \cdot z) - c j_\beta(\gamma \cdot z)^{-1} \right) - r j_\gamma(z)^{-1} \\ &= j_{\beta\gamma}(z)^{-2} f(\beta\gamma \cdot z) - c j_\gamma(z)^{-2} j_\beta(\gamma \cdot z)^{-1} - r j_\gamma(z)^{-1} \ . \end{aligned}$$

To conclude, an easy computation shows that

$$c j_\gamma(z)^{-2} j_\beta(\gamma \cdot z)^{-1} + r j_\gamma(z)^{-1} = (cp + dr) j_{\beta\gamma}(z)^{-1} \ .$$

The slash operator for the equivariant functions is defined through the above action on the function $\widehat{h}(z) = (h(z) - z)^{-1}$, where h is an equivariant function.

Using this definition we have

Proposition 3.1.2 *Let* h *be a function on* $\mathfrak{H}$*. Then* $h(z)$ *is an equivariant function for* Γ *if and only if* $\widehat{h}||_2[\gamma](z) = \widehat{h}(z)$ *for all* $\gamma \in \Gamma$*. Moreover, if* $-1_2 \in \Gamma$*, then* $\widehat{h}||_2[-\gamma](z) = \widehat{h}||_2[\gamma](z)$*.*

Proof: The first part follows from the definition of the slash operator. Indeed, for $\gamma = \begin{pmatrix} p & q \\ r & s \end{pmatrix} \in \Gamma$ we have

$$h(\gamma \cdot z) = \gamma \cdot h(z) \Leftrightarrow \widehat{h}(\gamma \cdot z) = \jmath_\gamma(z)\jmath_\gamma(h(z))\widehat{h}(z) \Leftrightarrow$$

$$\jmath_\gamma(z)^{-2}\widehat{h}(\gamma \cdot z) = \frac{\jmath_\gamma(h(z))}{\jmath_\gamma(z)}\widehat{h}(z) .$$

However, $\jmath_\gamma(h(z)) = c(h(z) - z) + \jmath_\gamma(z)$ so that $\frac{\jmath_\gamma(h(z))}{\jmath_\gamma(z)}\widehat{h}(z) = \jmath_\gamma(z)^{-2}\widehat{h}(z) + r\jmath_\gamma(z)$. Thus, h is equivariant if and only if $\widehat{h}||_2[\gamma](z) = \widehat{h}(z)$. The second part is a consequence of $\jmath_{-\gamma}(w) = -\jmath_\gamma(w)$ for $w \in \mathbb{C}$. ∎

The second part of this proposition simply says that we may assume that Γ contains -1_2. We will see in Chapter 4 that this proposition has other consequences.

3.1.2 Equivariant forms: The general definition

We now give the precise definition of an equivariant form for a subgroup Γ of $\mathrm{SL}_2(\mathbb{Z})$ containing -1_2; a condition that we assume for the rest of this work.

Definition 3.1.3 *An unrestricted equivariant function $h : \mathfrak{H} \to \mathbb{C}$ for Γ is called an equivariant form for Γ if it is meromorphic on $\mathfrak{H}$ and at the cusps of Γ (in the sense defined below).*

First, we disregard the condition of the meromorphy at cusps if Γ has no cusps. Otherwise, let s be a cusp of Γ. Then in this definition, the meromorphy condition at the cusp s means the following. Choose $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$ such that $\gamma \cdot s = \infty$. Then the isotropy group of s, $\Gamma_s = \{\alpha \in \Gamma | \ \alpha \cdot s = s\}$, is conjugate by γ to the cyclic group generated by T^{l_s}, for some positive integer $l_s \in \mathbb{Z}$ which is known as the cusp width of Γ at the cusp s. This, therefore, implies that the function $\widehat{h}||_2[\gamma^{-1}](z)$ is invariant under $\gamma\Gamma_s\gamma^{-1} = \langle T^{l_s} \rangle$, which is equivalent to saying that it is l_s-periodic.

Hence, it has a Fourier expansion in the local parameter $q_s = \exp(2\pi i z/l_s)$ of the form

$$\widehat{h}||_2[\gamma^{-1}](z) = \sum_{m \geq m_s} a_m q_l^m.$$

We say that h is meromorphic at s if the integer m_s is finite.

Remark 3.1.1
- The above condition is independent of the choice of γ.
- If the above condition is satisfied at a cusp s, then it is satisfied at any equivalent cusp to s under Γ.

In the following we show how to obtain an equivariant form for a conjugate group.

Proposition 3.1.4 *Let Γ_1, Γ_2 be two subgroups of $SL_2(\mathbb{Z})$. Suppose that Γ_1 and Γ_2 are conjugate subgroups of $SL_2(\mathbb{Z})$, that is there is an element $\alpha \in SL_2(\mathbb{Z})$ such that $\Gamma_1 = \alpha\Gamma_2\alpha^{-1}$. Then if h_1 is an equivariant form for Γ_1, the function*

$$h_2(z) = \alpha^{-1} \circ h_1 \circ \alpha(z)$$

is an equivariant form for Γ_2.

Proof: The equivariance of h_2 follows from the fact that every $\gamma_2 \in \Gamma_2$ has the form $\gamma_2 = \alpha^{-1}\gamma_1\alpha$ for some $\gamma_1 \in \Gamma_1$. Thus, h_2 is an equivariant function for Γ_2. To conclude it suffices to show that h_2 is meromorphic at cusps. For this we will show that

$$\widehat{h}_2(z) = \widehat{h}_1||_2[\alpha](z) \ ,$$

with $\widehat{h}_i(z) = (h_i(z) - z)^{-1}$, $i = 1, 2$.

We have for $\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$$\widehat{h}_2(z) = \frac{1}{\alpha^{-1}h_1(\alpha \cdot z) - z} = \frac{j_{\alpha^{-1}}(h_1(\alpha \cdot z))}{dh_1(\alpha \cdot z) + czh_1(\alpha \cdot z) - az - b}$$

$$= \frac{-ch_1(\alpha \cdot z) + c\alpha \cdot z - c\alpha \cdot z + a}{j_\alpha(z)(h_1(\alpha \cdot z) - \alpha \cdot z)}$$

$$= \frac{-c}{j_\gamma(z)} + \frac{\widehat{h}_1(\alpha \cdot z)}{j_\alpha(z)^2} = \widehat{h}_1||_2[\alpha](z) \ ,$$

since $-c\alpha \cdot z + a = j_\alpha(z)^{-1}$.

The meromorphy of h_2 at a cusp s of Γ_2 follows from that of h_1 at the cusp $\alpha \cdot s$ since h_1 is an equivariant form so that $\widehat{h}_2(z) = \widehat{h}_1||_2[\alpha](z)$ is meromorphic at infinity. ∎

The identity $\widehat{h}_2(z) = \widehat{h}_1||_2[\alpha](z)$ is interesting as it shows that the behavior of h_1 at cusps is exactly that of h_2 at infinity, and constitutes a property of the slash operator on equivariant forms.

3.2 Equivariant forms and modular forms

In this section we study equivariant forms in connection with modular forms leading to what we call rational equivariant forms. We also focus on the particular case when the equivariant forms do not have fixed points.

3.2.1 Rational equivariant forms

The first examples of equivariant forms for $\mathrm{SL}_2(\mathbb{Z})$ we have encountered were those arising from reciprocal of logarithmic derivatives of modular forms on the modular group. They have the following particularity. For a weight $k \in \mathbb{Z}^+$ modular form, let $h_f(z) = z + kf(z)/f'(z)$ be the associated equivariant form. Then, one easily sees that any zero of $\widehat{h}_f^{-1} = h_f(z) - z$, which is also a zero of f, is a simple zero and that the residue of $\widehat{h}_f$ at this zero is a rational number which is given by n/k, where n is the multiplicity and k is the weight. We will show that if this property holds for an equivariant form h for a subgroup Γ in addition to a rationality consideration at the cusps, then $h = h_f$ for a certain modular form f for Γ. This also includes the case of modular forms with a multiplier system. Indeed, if f has a multiplier system then f^m

would have a trivial multiplier system for some $m \in \mathbb{R}$, which is due to the finiteness of the number of generators of the subgroup Γ. On the other hand, Proposition 2.3.3 implies that f and f^m produce the same equivariant form.

Definition 3.2.1 *An equivariant form h is called a rational equivariant form if it arises as $h = h_f$, for some modular form of the general type.*

Theorem 3.2.2 *Let $f \in \mathfrak{M}_k^m(\Gamma, \nu)$. Then the function h_f given by*

$$h_f(z) = z + \frac{kf(z)}{f'(z)}$$

is an equivariant form on Γ.

Proof: To prove that h_f is an equivariant form, we only need to look at the behavior at cusps.

Let s be a cusp of Γ. Choose $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$ such that $\gamma \cdot s = \infty$ and, so, the isotropy group of s is conjugate by γ to the group generated by T^{l_s}, for some $l_s \in \mathbb{Z}$. Then we have

$$\begin{aligned}
\widehat{h}_f||_2[\gamma^{-1}] &= j_{\gamma^{-1}}(z)^{-2}\widehat{h}_f(\gamma^{-1} \cdot z) + cj_{\gamma^{-1}}(z)^{-1} \\
&= \frac{j_{\gamma^{-1}}(z)^{-2} f'(\gamma^{-1} \cdot z)}{kf(\gamma^{-1} \cdot z)} + cj_{\gamma^{-1}}(z)^{-1} \\
&= \frac{j_{\gamma^{-1}}(z)^{-2} f'(\gamma^{-1} \cdot z) + ckj_{\gamma^{-1}}(z)^{-1} f(\gamma^{-1} \cdot z)}{kf(\gamma^{-1} \cdot z)} \\
&= \frac{j_{\gamma^{-1}}(z)^{-k-2} f'(\gamma^{-1} \cdot z) + ckj_{\gamma^{-1}}(z)^{-k-1} f(\gamma^{-1} \cdot z)}{kf(\gamma^{-1} \cdot z)} \\
&= \frac{(f(z)|_k[\gamma^{-1}])'}{kf(z)|_k[\gamma^{-1}]} \qquad \text{(by (1.2.2))} \\
&= \frac{(\tilde{f}_{\gamma^{-1}}(z))'}{k\tilde{f}_{\gamma^{-1}}(z)},
\end{aligned}$$

where

$$\tilde{f}_{\gamma^{-1}}(z) = \nu\left(T^{l_s}\right)^{-1} f(z)|_k[\gamma^{-1}]$$

and $f(z)|_k[\gamma^{-1}]$ denotes the action of the slash operator on the weight k modular form f (see Section §1.2.2). Hence, as the function $\tilde{f}_{\gamma^{-1}}(z)$ is invariant under T^{l_s}, so that it has a Fourier expansion

$$\tilde{f}_{\gamma^{-1}}(z) = \sum_{n=n_s}^{\infty} a_n^s q_s^n$$

in the local parameter $q_s := e^{2\pi i z/l_s}$, l_s being the cusp width at s, with n_s finite as f is meromorphic at cusps, the function $\widehat{h}_f||_2[\gamma^{-1}](z)$ has a Fourier expansion in the parameter q_s with only a finite number of negative indices. Thus h_f is meromorphic on $\mathfrak{H}$ and at the cusps, that is to say that h_f is an equivariant form for Γ. Note that two conjugate subgroups of $\mathrm{SL}_2(\mathbb{Z})$ share the same set of cusps. ∎

The function has the particularity that $\widehat{h}_f$ has simple poles on $\mathfrak{H}$ and at cusps. Also, one conclude that the behavior at cusps of h_f is given by that of f.

In the following proposition we show how conjugation conserves rational equivariant forms.

Proposition 3.2.3 *Let Γ_1, Γ_2 be conjugate subgroups of $SL_2(\mathbb{Z})$, say $\Gamma_1 = \alpha\Gamma_2\alpha^{-1}$ for $\alpha \in SL_2(\mathbb{Z})$. Let $f \in \mathfrak{M}_k^m(\Gamma, \nu)$, and denote by $h_f(z)$ the equivariant form $z + kf(z)/f'(z)$. Then the function*

$$\alpha^{-1}h_f(\alpha\cdot z) = z + \frac{k\tilde{f}_\alpha(z)}{(\tilde{f}_\alpha(z))'}$$

is an equivariant form for Γ_2, where

$$\tilde{f}_\alpha(z) = \nu\left(T^{l_s}\right)^{-1} f(z)|_k[\alpha]$$

is as in Section §1.2.2.

Proof: The fact that $\alpha^{-1}h_f\alpha$ is an equivariant form for Γ_2 follows from Proposition 3.1.4. The expression of $g_f := \alpha^{-1}h_f\alpha$ is a consequence of $\widehat{g}_f(z) = \widehat{h}_f||_2[\alpha](z)$

with $\widehat{g}_f(z) = (g_f(z) - z)^{-1}$, $\widehat{h}_f(z) = (h_f(z) - z)^{-1}$ and $\widehat{h}_f||_2[\alpha](z) = \dfrac{(\tilde{f}_\alpha(z))'}{k\tilde{f}_\alpha(z)}$. ∎

Recall the theta functions from Section §1.2.3, which are modular forms with non-trivial multiplier systems. Then using their eta-products and the fact that the Eisenstein series E_2 is the logarithmic derivative of the η function, we have

$$\left.\begin{aligned} f_2(z) &:= z + \frac{\vartheta_2(z)}{2\vartheta_2'(z)} = z + \frac{1}{8\pi i(4E_2(4z) - E_2(2z))}, \\ f_3(z) &:= z + \frac{\vartheta_3(z)}{2\vartheta_3'(z)} = z + \frac{24}{2\pi i(5E_2(2z) - E_2(z) - 4E_2(4z))}, \\ f_4(z) &:= z + \frac{\vartheta_4(z)}{2\vartheta_4'(z)} = z + \frac{1}{4\pi i(E_2(z) - E_2(2z))}. \end{aligned}\right\} \tag{3.2.1}$$

These functions are rational equivariant forms for the conjugate subgroups of $\Gamma_0(2)$ (see Section §1.2.3).

Another important example is the case of modular forms with character. Let $N \geq 2$ be an integer and let χ be character modulo N. Let f be a weight $k \in \mathbb{Z}$, $k \geq 2$, (holomorphic) modular form on the principal congruence subgroup $\Gamma(N)$ with character χ. These type of functions play an important role in the study of spaces of modular forms [Ko]. For instance, the structure of the space of holomorphic modular forms of integer weight $k \geq 2$ and constant multiplier system on $\Gamma(N)$ decomposes as a direct sum of the spaces of holomorphic modular forms of the same weight and Dirichlet character modulo N on $\Gamma(N)$, i.e.,

$$\mathfrak{M}_k(\Gamma(N),\ 1) = \oplus_\chi \mathfrak{M}_k(\Gamma(N),\ \chi).$$

Recall that a Dirichlet character modulo N is a group homomorphism $\chi :\ (\mathbb{Z}/N\mathbb{Z})^\times \longrightarrow \mathbb{C}^\times$. Note that there is a finite number of such characters.

The following result is a generalization of Theorem 2.3.5 that goes beyond the rational equivariant forms.

Theorem 3.2.4 *Let f and g be modular forms for Γ of respective weight k and $k+2$, where k is a positive integer $k \in \mathbb{R}$. Then, the function*

$$h(z) = z + \frac{kf(z)}{f'(z) + g(z)}$$

is an equivariant form for Γ.

The proof of this result follows a similar pattern to that of Theorem 2.3.5 for the equivariance and that of Theorem 3.2.2 for the analysis at cusps, though the situation is simpler in this case as the multiplier system is constant.

3.2.2 Equivariant forms without fixed points

The first example of an equivariant form without fixed points we dealt with was the (rational) equivariant form involving the modular discriminant (which is a cusp form), namely

$$h_0(z) = z + \frac{12\Delta(z)}{\Delta'(z)}.$$

Recall that the modular discriminant is given by

$$\Delta(z) = \eta^{24}(z) = q\prod_{n=1}^{\infty}(1-q^n)^{24},\ q = e^{2\pi i z},$$

where η is the Dedekind eta function. As mentioned earlier the logarithmic derivative of the modular discriminant is the Eisenstein series E_2

$$\frac{1}{2\pi i}\frac{\Delta'(z)}{\Delta(z)} = E_2(z).$$

The equivariant form h_0 has the property that $\widehat{h}_0(z) = (h_0(z) - z)^{-1}$ does not vanish on $\mathfrak{H}$ and is holomorphic at infinity. We look at h_0 as an equivariant form for any subgroup Γ of $\mathrm{SL}_2(\mathbb{Z})$. In fact, if h is an equivariant form $\mathrm{SL}_2(\mathbb{Z})$ then it is also for any of its subgroups, and the behavior of h at all cusps is exactly that at infinity as infinity represents all cusps of $\mathrm{SL}_2(\mathbb{Z})$.

Definition 3.2.5 *Let h be an equivariant form on Γ. Then h is said to be an equivariant form without fixed points if the following two conditions are met*

1. *h has no fixed points in $\mathfrak{H}$;*
2. *h has no fixed points at cusps. In other words, for every cusp s, if $\alpha \in SL_2(\mathbb{Z})$ is such that $\alpha \cdot \infty = s$, then $\lim_{z \mapsto \infty} \widehat{h}||_2[\alpha](z)$ is finite.*

As a first example of this type of function, the equivariant form without fixed points h_0 is still an equivariant form without fixed points on any subgroup of the modular group. We will give a large class of equivariant form without fixed points in the next chapter. We will also show that for genus zero subgroups, h_0 is the only equivariant form without fixed points.

Now, recall the examples of the rational equivariant forms in (3.2.1). Then from their expressions, it is easily seen that they do not have fixed points on $\mathfrak{H}$ and at cusps, that is, they are equivariant form without fixed points. To see that, we will treat the case of ϑ_2 as the other two cases are dealt similarly. The fact that f_2 has no fixed points on $\mathfrak{H}$ follows from the fact that $4E_2(4z) - E_2(z)$ is holomorphic on $\mathfrak{H}$. For the cusps, we know that $\Gamma_0(2)$ has exactly two inequivalent cusps represented by 0 and ∞. At infinity, with $\widehat{f_2}(z) = (f_2(z) - z)^{-1}$, we have

$$\lim_{z \mapsto \infty} \widehat{h}_2(z) = \lim_{z \mapsto \infty} 8\pi i(4E_2(4z) - E_2(2z)) = 24\pi i \ .$$

For the cusp at 0, we know that $S \cdot 0 = \infty$, where $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. Hence, using the transformation formula for E_2 (see (1.4.1) in Chapter 1), we have

$$\begin{aligned} \widehat{f_2}||_2[S](z) &= z^{-2} 8\pi i(4E_2(-4/z) - E_2(-2/z)) - z^{-1} \\ &= z^{-2} 8\pi i \left(\frac{z^2}{4} E_2(z/4) - \frac{z^2}{4} E_2(z/2) - \frac{3z}{\pi i} \right) - z^{-1} \\ &= 8\pi i \left(\frac{1}{4} E_2(z/4) - \frac{1}{4} E_2(z/2) - \frac{3}{\pi i z} \right) - z^{-1} \ . \end{aligned}$$

Thus,

$$\lim_{z \mapsto \infty} \widehat{f_2}||_2[S](z) = 0,$$

as

$$E_2(z) = 1 + O(q)$$

and so tends to 1 at infinity. Therefore f_2 is an equivariant form without fixed points.

The verification of the other two cases, whose equivariance subgroups are conjugate to that of f_2, could be deduced from the following. This proposition simply says that conjugation preserves the class of equivariant forms without fixed points.

Proposition 3.2.6 *The notation being the same as in Proposition 3.1.4, if h_1 is an equivariant form without fixed points for Γ_1 then so is h_2 for Γ_2.*

Proof: We only need to verify the fixed points as it follows from Proposition 3.1.4. On $\mathfrak{H}$, if z is a fixed point of h_2 then $\alpha \cdot z$ is a fixed point of h_1. At cusps, the identity $\widehat{h_2}(z) = \widehat{h_1}||_2[\alpha](z)$ shows that if s is a cusp fixed by h_2 then $\alpha \cdot s$ is a cusp fixed by h_1. ∎

3.3 Rational equivariant forms: The converse

The aim of this section is to investigate the conditions under which an equivariant form is a rational equivariant form. More precisely, we study the optimal analytic conditions that allow an equivariant form to arise from a modular form.

3.3.1 The converse

The purpose of this subsection is to give the necessary and sufficient conditions for a equivariant form to be a rational equivariant form. For the sake of a simple exposition, we restrict ourselves to the class of congruence subgroups of $SL_2(\mathbb{Z})$. We will use the group representation of a Fuchsian group of the first kind mentioned in Section §1.1.

Let $h(z) = h_f(z) = z + kf(z)/f'(z)$ be a rational equivariant form, where f is a weight k modular form, then the poles of $\widehat{h}(z) = f'(z)/kf(z)$ are located at the zeros of f. Moreover, any such pole z_0 is simple and the residues of $\widehat{h}$ at z_0 is given by the rational number n/k where n is the multiplicity of f at z_0. As for the cusps one can show that h_f satisfies $\lim_{z \mapsto \infty} \widehat{h}||_2[\gamma](z) = a_\gamma \in 2\pi\imath\mathbb{Q}$ for all $\gamma \in \mathrm{SL}_2(\mathbb{Z})$, indeed, using the Fourier expansion of $(f(z)|_k[\gamma]$, one sees that $\widehat{h}||_2[\gamma](z) = \dfrac{(f(z)|_k[\gamma])'}{kf(z)|_k[\gamma]}$ is given by $2\pi n_0\imath/k$ where n_0 is the order of $f(z)|_k[\gamma]$ at ∞.

Lemma 3.3.1 *Let Γ be a congruence subgroup of $SL_2(\mathbb{Z})$ and let h be an equivariant form for Γ. Then in the closure of a fundamental domain of Γ the function $\widehat{h}(z) = (h(z) - z)^{-1}$ has only a finite number of poles.*

Proof: We will show that the function $\widehat{h}||_2[\gamma^{-1}](z)$ does not vanish in a neighborhood of infinity for all cusps s that are vertices of the chosen fundamental domain, where $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ such that $\gamma \cdot s = \infty$. Since h is an equivariant form, we know that $\widehat{h}(z)$ is meromorphic on $\mathfrak{H}$ and has a Fourier expansion at infinity. Hence, there is $A > 0$ such that for every $z \in \mathfrak{H}$ with $\Im(z) > A$ we have $\widehat{h}(z) \neq 0$, that is to say that $h(z) \neq z$. If a vertex of the fundamental domain of Γ is at a cusp s and that this cusp is a pole of $\widehat{h}(z)$, then again the function $\widehat{h}||_2[\gamma^{-1}](z)$ does not vanish in a neighborhood of infinity in $\mathfrak{H}$. Therefore, all the poles of $\widehat{h}(z)$ at vertices of this fundamental domain can be isolated. Note that at these latter the number of the poles is finite. In the rest of the fundamental domain, the meromorphic function $\widehat{h}(z)$ has a finite number of poles and the lemma follows. ∎

The notation being as in the above lemma, we have

Lemma 3.3.2 *Suppose that the poles in $\mathfrak{H}$ of $\widehat{h}(z)$ are simple and have rational residues, then these residues have bounded denominators.*

Proof: We will show that any equivalent points have the same residue, and then use the fact that in the closure of a fundamental domain of Γ the meromorphic function $\widehat{h}(z)$ has only a finite number of poles by the previous lemma, to deduce that if the residues are rational numbers then their denominators are bounded. Let z_0 be a pole of $\widehat{h}(z)$. Then, as z_0 is a simple pole of $\widehat{h}(z)$, and in particular $h(z_0) = z_0$, we have

$$\begin{aligned}\mathrm{Res}_{z_0}(\widehat{h}(z)) &= \lim_{z\mapsto z_0} \frac{z - z_0}{h(z) - z} \\ &= \lim_{z\mapsto z_0} \frac{1}{\frac{h(z) - z}{z - z_0}} \\ &= \lim_{z\mapsto z_0} \frac{1}{\frac{h(z) - h(z_0)}{z - z_0} - 1} \\ &= \frac{1}{h'(z_0) - 1} .\end{aligned}$$

To conclude the proof it suffices to prove that $h'(z_0) = h'(\alpha \cdot z_0)$, for every $\alpha \in \Gamma$. Since $h(\alpha \cdot z) = \alpha \cdot h(z)$ and by differentiating both sides of this equality, and using $\frac{d}{dz}\alpha \cdot z = (\jmath_\alpha(z))^{-2}$, we get

$$\begin{aligned}\frac{d}{dz}h(\alpha \cdot z) &= \frac{1}{\jmath_\alpha(z)^2}h'(\alpha \cdot z) \\ &= \frac{h'(z)}{\jmath_\alpha(h(z))^2} .\end{aligned}$$

Substituting z with z_0 yields the lemma. ∎

We will also need the following lemma.

Lemma 3.3.3 *Let $\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$ be a parabolic element. Then the fixed point of α is $s = (a - d)/2c$. Furthermore, $\jmath_\alpha(s) = \pm 1$.*

Proof: A simple computation shows that α fixes $s = (a - d)/2c$, the unique fixed point of the parabolic element α. The second part follows from the definition of $j_\alpha(z)$ and that the trace of a parabolic element is ± 2 for the parabolic elements of $\mathrm{SL}_2(\mathbb{Z})$. ∎

Theorem 3.3.4 *Let Γ be a congruence subgroup. Suppose that h is an equivariant form for Γ satisfying the following conditions:*

1. *the poles in $\mathfrak{H}$ of the function $\widehat{h}(z)$ are simple with rational residues.*
2. *for every cusp s,*
$$\lim_{z \mapsto \infty} \widehat{h}||_2[\gamma^{-1}](z) = a_s$$
exists and satisfies $a_s \in 2\pi i\mathbb{Q}$, where $\gamma \in SL_2(\mathbb{Z})$ is such that $\gamma \cdot s = \infty$.

Then there is a weight k, $k \in \mathbb{N}$, modular form f on Γ such that
$$h(z) = z + \frac{kf(z)}{f'(z)}.$$

Proof: Assume that the conditions *1.* and *2.* hold. Let k is a positive integer chosen to be a multiple of the denominator of the residue of $\widehat{h}(z)$ at any (simple) pole, which is possible according to Lemma 3.3.2 and such that $ka_s \in 2\pi i\mathbb{Z}$ for any cusp s which is possible since we have finitely many nonequivalent cusps. Let $z_0 \in \mathfrak{H}$, not being a pole of $\widehat{h}(z)$. Define a function f by
$$f(z) = \exp\left(\int_{z_0}^{z} k\widehat{h}(u)du\right).$$
This function is well defined as the integral is independent of the path of integration. Indeed, let Σ_1, Σ_2 be two paths joining z_0 and z that do not containing any pole of $\widehat{h}(z)$. Then
$$\int_{\Sigma_1 \cup \Sigma_2} k\widehat{h}(u)du = 2\pi k i \sum \mathrm{Res}(\widehat{\mathrm{h}}(\mathrm{z}),\ \mathrm{z}) \ \in \ 2\pi \mathrm{i}\mathbb{Z},$$

where the sum is over the poles of $\widehat{h}(z)$ interior to $\Sigma_1 \cup \Sigma_2$ which we orient positively. Thus,

$$\int_{\Sigma_1} k\widehat{h}(u)du = \int_{\Sigma_2} k\widehat{h}(u)du + 2\pi mi$$

for some $m \in \mathbb{Z}$, and so f is well-defined. We extend f to a meromorphic function on the set S of poles of $\widehat{h}(z)$ in the following way. Let r (an integer) be the residue of $k\widehat{h}(z)$ at a pole z_1. If $r > 0$ we define $f(z_1) = 0$ to make f holomorphic at z_1 and the order of f at z_1 is r. If $r < 0$ then z_1 is a pole of f of order $-r$. Thus f is a well-defined meromorphic function of $\mathfrak{H}$ satisfying

$$h(z) = z + \frac{kf(z)}{f'(z)}.$$

We may assume that k is even.

We now proceed to verify the modularity of f. Let $\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$. Then, we have $f(\alpha \cdot z) = g_\alpha(z) f(z)$, where

$$g_\alpha(z) = \exp\left(\int_z^{\alpha\, z} k\widehat{h}(u)du\right).$$

Taking the logarithmic derivative of g_α yields

$$\frac{g'_\alpha(z)}{g_\alpha(z)} = \frac{d}{dz}(\alpha \cdot z)k\widehat{h}(\alpha \cdot z) - k\widehat{h}(z) = \frac{kc}{j_\alpha(z)},$$

since $\frac{d}{dz}\alpha \cdot z = (j_\alpha(z))^{-2}$, and

$$\frac{d}{dz}(\alpha \cdot z)\widehat{h}(\alpha \cdot z) = j_\alpha(z)^{-2}\left(j_\alpha(z)^2\widehat{h}(z) + cj_\alpha(z)\right)$$

as h is an equivariant form. Hence, $g_\alpha(z) = D_\alpha(cz + d)^k$, for some $D_\alpha \in \mathbb{C}^\times$. In fact, this defines a group character $D: \Gamma \longrightarrow \mathbb{C}^{\times}$. Indeed, let $\alpha,\ \beta \in \Gamma$. We have

$$\begin{aligned} g_{\alpha\beta}(z) &= D_{\alpha\beta}(\jmath_{\alpha\beta}(z))^k \\ &= \exp\left(\int_z^{\alpha\beta\, z} k\widehat{h}(u)du\right) \\ &= \exp\left(\int_{\beta\, z}^{\alpha\,(\beta\, z)} k\widehat{h}(u)du\right)\exp\left(\int_z^{\beta\, z} k\widehat{h}(u)du\right) \\ &= g_\alpha(\beta\cdot z)g_\beta(z)\ , \end{aligned}$$

which implies that $D_{\alpha\beta}\jmath_{\alpha\beta}(z)^k = D_\alpha \jmath_\alpha(\beta\cdot z)^k D_\beta \jmath_\beta(z)^k$. In the meantime, $\jmath_{\alpha\beta}(z) = \jmath_\alpha(\beta\cdot z)\jmath_\beta(z)$ since $\jmath$ is an automorphic factor. Therefore,

$$D_{\alpha\beta} = D_\alpha D_\beta.$$

This proves, in particular, that f is a weight k modular form on Γ with multiplier system D.

Next, we use the conditions of the theorem to investigate the nature of the character D. We will do so for α being one of the elliptic, parabolic and hyperbolic generators of Γ which are in finite number.

• α **is elliptic:** First note that $D_{1_2} = 1$ and since α has a finite order, say m_α, we have $D_\alpha^{m_\alpha} = 1$, that is, D_α is an m_α−th root of unity.

• α **is parabolic:** Let s be the cusp fixed by α. Choose $\gamma = \begin{pmatrix} p & q \\ r & s \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$ so that $\gamma\cdot s = \infty$. Making the substitution $u = \gamma^{-1}\cdot w$ in the expression of $g_\alpha(z)$, we get

$$\begin{aligned} g_\alpha(z) &= \exp\left(\int_{\gamma z}^{\gamma\alpha z} j_{\gamma^{-1}}(w)^{-2}\widehat{h}(\gamma^{-1}\cdot w)kdw\right) \\ &= \exp\left(\int_{\gamma z}^{\gamma\alpha z} \left(-r j_{\gamma^{-1}}(w)^{-1} + \widehat{h}||_2[\gamma^{-1}](w)\right) kdw\right) \\ &= j_{\gamma^{-1}}(\gamma\alpha\cdot z)^k j_{\gamma^{-1}}(\gamma\cdot z)^{-k} \exp\left(\int_{\gamma\cdot z}^{\gamma\alpha z} \widehat{h}||_2[\gamma^{-1}](w)kdw\right) \\ &= j_\gamma(\alpha\cdot z)^{-k} j_\gamma(z)^k \exp\left(\int_{\gamma z}^{T^{p_s}\gamma z}\left(a_s + \sum_{n\geq 1} a_n \exp(2\pi\imath nw/l_s)\right) kdw\right) , \end{aligned}$$

since $j_{\gamma^{-1}}(\gamma\cdot u) = j_\gamma(u)$ for all $u\in\mathbb{C}$ and by assumption

$$\widehat{h}||_2[\gamma^{-1}](z) = a_s + \sum_{n\geq 1} a_n q_s^n,\ q_s = e^{2\pi\imath z/l_s} .$$

As for the bound $T^{p_s}\gamma\cdot z$ integration, since α belongs to the isotropy group Γ_s of s it follows that $\gamma\alpha = T^{p_s}\gamma$ for some p_s divisible by the cusp width l_s at the cusp s. Hence,

$$g_\alpha(z) = j_\gamma(\alpha\cdot z)^{-k} j_\gamma(z)^k \exp\left(ka_s p_s\right) = j_\gamma(\alpha\cdot z)^{-k} j_\gamma(z)^k ,$$

as $ka_s \in 2\pi i\mathbb{Z}$ and that the function $z\mapsto \exp(2\pi iz/l_s)$ is periodic with period l_s. Notice that the difference of the bounds of integration $T^{p_s}\gamma\cdot z$ and $\gamma\cdot z$ equals to p_s which is divisible by l_s. Thus $D_\alpha = (j_\alpha(z)j_\gamma(\alpha\cdot z))^{-k} j_\gamma(z)^k$.

Taking the limit as z tends to s so that αz tends to s and by Lemma 3.3.3 $j_\alpha(z)^k$ tends to 1, we get that $D_\alpha = 1$.

- α **is hyperbolic:** The case of a hyperbolic matrix reduces to the parabolic case. More precisely, since Γ is a congruence subgroup, say of level N, we will use the fact that a power of this element can be written as a product of parabolic matrices. Indeed, if $\alpha\in\Gamma$ is hyperbolic, then there exists $n_\alpha\in\mathbb{Z}$ such that $\alpha^{n_\alpha}\in\Gamma(N)$ as $\Gamma(N)$ has finite index in Γ. Put $\alpha_1 := \alpha^{n_\alpha} = \begin{pmatrix} 1+aN & bN \\ cN & 1+dN\end{pmatrix}$, so that

$$\alpha_1^2 = \begin{pmatrix} 1+a^2N^2+2aN+bcN^2 & \star \\ \star & 1+d^2N^2+2dN+bcN^2\end{pmatrix} .$$

Meanwhile, if we let $T_N = \begin{pmatrix} 1 & N \\ 0 & 1 \end{pmatrix}$, $S_N = \begin{pmatrix} 1 & 0 \\ N & 1 \end{pmatrix} \in \Gamma(N)$, we have

$$\beta := T_N^{(a^2+2bc+d^2)+2(a+d)/N} S_N = \begin{pmatrix} 1 + N^2(a^2 + d^2 + 2bc) + 2N(a+d) & \star \\ \star & 1 \end{pmatrix}.$$

Note that, since $\det(\alpha_1) = 1$, we have $a + d = N(bc - ad)$, that is $N|(a + d)$ and so $\beta \in \Gamma(N)$. Now, since the elements α_1^2 and β have the same trace, there exists, by Corollary 1.1.2, $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ such that $\alpha_1^2 = \gamma\beta\gamma^{-1}$. Hence, α^{2n_α} is the product of parabolic elements belonging to $\Gamma(N) \subset \Gamma$. Therefore, as D is a group character, $D_{\alpha^{2n_\alpha}} = (D_\alpha)^{2n_\alpha} = 1$ using the parabolic case.

In conclusion, we have established that f is weight k modular form with character D such that D_α is an m_α−th (respectively n_α-th) root of unity if α is an elliptic (respectively hyperbolic) generator and $D_\alpha = 1$ for parabolic generators. Let m be the least common multiple of the exponents m_α and n_α. Then f^m is a modular form of weight km (with trivial multiplier system). However, by Proposition 2.3.3 (in Chapter 2), f and f^m give rise to the same equivariant form. Therefore, without loss of generality, we may assume $D_\alpha = 1$ for all $\alpha \in \Gamma$ to make f a modular form for Γ.

To conclude we only need to check the meromorphy at the cusps. Let s be cusp and choose as before $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ such that $\gamma \cdot s = \infty$. Then we have

$$
\begin{aligned}
f(z)|_k[\gamma^{-1}] &= \jmath_{\gamma^{-1}}(z)^{-k} f(\gamma^{-1}\cdot z) \\
&= \jmath_{\gamma^{-1}}(z)^{-k} \exp\left(\int_{z_0}^{\gamma^{-1} z} k\widehat{h}(u)du \right) \\
&= \jmath_{\gamma^{-1}}(z)^{-k} \exp\left(\int_{\gamma z_0}^{z} \jmath_{\gamma^{-1}}(w)^{-2}\widehat{h}(\gamma^{-1}\cdot w)kdw \right) \\
&= \jmath_{\gamma^{-1}}(z)^{-k} \exp\left(\int_{\gamma z_0}^{z} \left(\widehat{h}||_2[\gamma^{-1}](w) - r\jmath_{\gamma^{-1}}(z)^{-1}\right) kdw \right) \\
&= \jmath_{\gamma^{-1}}(\gamma\cdot z_0)^{-k} \exp\left(\int_{\gamma z_0}^{z} \left(a_s + \sum_{n\geq 1} a_n \exp(2\pi\imath wn/l_s) \right) kdw \right) \\
&= \jmath_{\gamma}(z_0)^{k} \exp(ka_s z)\cdot \\
&\qquad \exp\left(ka_s\alpha\cdot z_0 + \sum_{n\geq 1} a_n(\exp(2\pi\imath nz/l_s) - \exp(2\pi\imath n\gamma\cdot z_0/l_s)) \right) \\
&= \exp(ka_s z)\cdot \text{ holomorphic term at infinity },
\end{aligned}
$$

since z and $\alpha\cdot z_0$ belong to $\mathfrak{H}$ so that the series $\sum_{n\geq 1} a_n(\exp(2\pi\imath nz/l_s) - \exp(2\pi\imath\gamma\cdot z_0/l_s))$ converges normally. Therefore, $f(z)|_k[\gamma^{-1}] = q_s^{n_s}\cdot$ *holomorphic term at infinity*, since $ka_s = 2\pi\imath kn_s \in 2\pi\imath\mathbb{Z}$. Thus f is a meromorphic modular form of weight k on Γ. ∎

Remark 3.3.1

- A proof of the particular case $\Gamma = \mathrm{SL}_2(\mathbb{Z})$ appears in [AAS1], where the situation is much simpler as the set of inequivalent cusps is represented by infinity. Also, the set of generators the genus zero group $\mathrm{SL}_2(\mathbb{Z})$ consists of the parabolic element T and the elliptic element S only, whereas for a subgroup we may have all three sorts of transformations.
- The conditions of this theorem are optimal since there are equivariant forms

which do not satisfy either condition *1.* or condition *2.* For instance, the equivariant form $h(z) = z + \dfrac{4E_4}{E_4' + E_6}$ from Example 2.3.2 has a residue $\mathrm{Res}_\rho \widehat{h} = 1 - 3/2\pi i$ at ρ which is not rational and thus does not satisfy condition *1.* Also the equivariant form

$$h(z) \;=\; z \,+\, 12\,\frac{\Delta}{\Delta' + E_{14}} \;=\; z + 12q + \mathrm{o}(q) \;\text{as}\;\; q \to \infty$$

does not satisfy the second condition of the theorem as $\widehat{h}$ has a pole at ∞.

3.3.2 The effect of the Schwarz derivative

The following result was implicit when we motivated the notion of equivariant forms in Chapter 2. It yields another close connection between equivariant forms and modular forms.

Proposition 3.3.5 *Let h be an equivariant form for a modular subgroup Γ. Then the Schwarz derivative $\{h, z\}$ of h is a modular form of weight 4 for Γ.*

Proof: This follows by direct computation using the equivariance of h and Proposition 2.1.4. ∎

This effect of the Schwarz derivative becomes more visible if the equivariant form is rational. This is very closely related to what is known as the Cohen-Rankin bracket.

Let f and g be two modular forms of weights k and l, respectively. Then, for an integer $n \geq 0$, the n−th Cohen-Rankin bracket is defined by

$$[f,g]_n = \sum_{r+s=n} \binom{k+n-1}{s}\binom{l+n-1}{r} D^r f D^s g \;,\; r,\; s \geq 0$$

where $D^j = \dfrac{d^j}{dz^j}$. For instance

$$[f,g]_0 = fg \;, \quad [f,g]_1 = kfg' - lf'g$$

and

$$[f,g]_2 = \frac{k(k+1)}{2} f g'' - (k+1)(l+1) f'g' + \frac{l(l+1)}{2} f''g \ .$$

We have,

Proposition 3.3.6 *([Zag]) For $f \in \mathfrak{M}_k^m(\Gamma)$ and $g \in \mathfrak{M}_l^m(\Gamma)$ and for every $n \geq 0$, the function $[f,g]_n$ belongs to $\mathfrak{M}_{k+l+2n}^m(\Gamma)$.*

This proposition implies in particular that for $f = g$, we have

$$[f,f]_2 = -(k+1)((k+1)f'^2 - kff'')$$

is a weight $2k+4$ modular form on Γ.

Let now f be a weight k modular form on a modular subgroup Γ and let h_f be the corresponding rational equivariant form. From Proposition 2.1.4 the poles of $\{h_f, z\}$ are located at the critical points of h_f. However, a simple computation shows that

$$h_f'(z) = \frac{(k+1)f'^2 - kff''}{f'^2} = \frac{-[f,f]_2}{(k+1)f'^2} \ .$$

Hence, we have

Proposition 3.3.7 *Let f be a modular form of weight k on Γ. Then $f'^2 h_f'$ is a weight $2k+4$ modular form on Γ. Moreover, the poles of $\{h_f, z\}$ are located at the zeros of the second Cohen-Rankin bracket $[f,f]_2$ of f.*

To illustrate this proposition we consider the case of the Eisenstein series E_4. The second Cohen-Rankin bracket of E_4 is a weight $2 \times 4 + 4 = 12$ holomorphic modular form for $\mathrm{SL}_2(\mathbb{Z})$. Therefore, it is a linear combination of the weight 12 Eisenstein series E_{12} and the modular discriminant Δ. Hence,

$$h_4'(z) = \frac{5E_4'(z)^2 - 4E_4(z)E_4''(z)}{E_4'(z)^2} \ ,$$

where h_4 is the rational equivariant form associated with E_4. An investigation of the first few Fourier coefficients shows that

$$[E_4, E_4]_2 = 768\pi^2\Delta ,$$

so that

$$h_4'(z) = \frac{3840\pi^2\Delta}{E_4'^2} .$$

Therefore, h_4 has no critical points in $\mathfrak{H}$ and thus the Schwarz derivative of h_4 is holomorphic on $\mathfrak{H}$. We will treat in the next section (Proposition 3.4.1) the example of the fundamental equivariant form h_0 which correspond to $f = \Delta$ which, however, do have critical points in $\mathfrak{H}$.

3.4 The genus zero condition

The purpose of this section is to show that if the subgroup Γ is of genus zero, then the equivariant forms without fixed points are equal to h_0, where

$$h_0(z) = z + \frac{6}{\pi \imath E_2(z)} = z + \frac{12\Delta(z)}{\Delta'(z)} .$$

3.4.1 Some properties of h_0

In the following we summarize some properties of h_0 which is the fundamental example of equivariant forms form without fixed points.

Proposition 3.4.1 *[AAS1] We have*

i. If z_0 is a fixed elliptic point of $SL_2(\mathbb{Z})$ in $\mathfrak{H}$, then $h_0(z_0) = \overline{z}_0$.

ii. The poles of h_0 are located at the zeros of E_2 and are all simple.

iii. The critical points of h_0 are located at the zeros of the Eisenstein series E_4.

Proof: (Sketch)

For *i.*, this follows from Proposition 2.4.2, as h_0 is an equivariant form without fixed points on $\mathrm{SL}_2(\mathbb{Z})$. The proof of *iii.* of this proposition relies on Ramanujan's differential relation

$$\frac{1}{2\pi i}\frac{dE_2}{dz} = \frac{1}{12}(E_2^2 - E_4)\ .$$

Indeed, a direct computation shows that

$$h_0'(z) = 1 + \frac{6E_2'(z)}{\pi i E_2^2(z)} = \frac{E_4(z)}{E_2^2(z)}\ .$$

Hence h_0' vanishes exactly at the zeros of E_4. For *ii.*, it is clear form the expression of h_0 that its poles are the zeros of E_2. Again using Ramanujan's equations, if a zero of E_2 is not simple then it is a zero of E_4 and thus it lies on the $\mathrm{SL}_2(\mathbb{Z})$-orbit of the zero $\rho := e^{2\pi i/3}$ of E_4. However, by *i.*, h_0 does not have a pole at the elliptic fixed points. Therefore, the poles of h_0 are simple. ∎

Remark 3.4.1 The number of poles of h_0 or, equivalently, the number of zeros of E_2 is infinite. See Chapter 5 for more details on properties of the zeros of E_2.

As a consequence of this proposition, one obtains the following particular values of the Eisenstein series E_2 at the elliptic points i and the cubic root of unity ρ. We have

$$E_2(i) = \frac{3}{\pi}, \qquad E_2(\rho) = \frac{2\sqrt{3}}{\pi}\ .$$

Another consequence is the following equation connecting the Schwarz derivative of h_0, Δ and E_4.

Proposition 3.4.2 *[AAS1] We have*

$$\{h_0, z\} = -2^6 3^2 \pi^2 \frac{\Delta}{E_4^2}\ .$$

This proposition can be established either by a direct computation using Ramanujan's differential system or as follows. Recall from Proposition 2.1.2 in Chapter 2 that $\{h, z\}$ has a double pole at the critical points of h and that if h is an equivariant form for $\mathrm{SL}_2(\mathbb{Z})$ then $\{h, z\}$ is a weight 4 meromorphic modular form on $\mathrm{SL}_2(\mathbb{Z})$. Hence, the function $E_4^2\{h, z\}$ is weight 12 holomorphic modular form on $\mathrm{SL}_2(\mathbb{Z})$ and, therefore, is a linear combination of E_4^3 and Δ which constitute a basis for the space of weight 12 modular forms on $\mathrm{SL}_2(\mathbb{Z})$ which is of dimension 2. An investigation of the first few Fourier coefficients yields the result.

3.4.2 The converse

In this subsection, we show that the fundamental example h_0 is unique for the genus zero congruence groups in the sense that it has no fixed points in addition to a condition at the cusps. Let us first introduce the Serre operator acting on a weight k modular form f by

$$\delta_k := \frac{12}{2\pi i}\frac{d}{dz} - kE_2,$$

and carrying it to a modular form of weight $k+2$ and also carrying cusp forms to cusp forms. Recall that the space $\mathfrak{S}_2(\Gamma)$ of weight 2 cusp forms on Γ has dimension g, the genus of Γ (see Corollary 1.3.5). In particular, this space is trivial when the group Γ is of genus 0.

Theorem 3.4.3 *Let h be an equivariant form without fixed points on a congruence subgroup Γ of $SL_2(\mathbb{Z})$ of genus 0. Suppose that, for every cusp s, if $\gamma \in SL_2(\mathbb{Z})$ is such that $\gamma \cdot s = \infty$, we have*

$$\lim_{z \mapsto \infty} \widehat{h}||_2[\gamma^{-1}](z) = a_s$$

is finite and satisfies $(a_s/6) \in \pi i\mathbb{Z}^+$. Then

$$h(z) = h_0(z) = z + \frac{6}{\pi i E_2(z)}.$$

Proof: It is clear from the conditions of the theorem that

$$f(z) = \exp\left(\int_{z_0}^{z} \frac{12dw}{h(w) - w}\right)$$

for z_0 fixed in $\mathfrak{H}$ defines a non-vanishing holomorphic function of $\mathfrak{H}$. We will prove that f is a non-vanishing weight 12 cusp form on Γ, then use the operator δ_{12} to conclude that the quotient $\frac{\delta_{12} f}{f}$ is a weight 2 cusp form on Γ, hence, is trivial as Γ is of genus 0.

We only need to prove the modularity for the elliptic and parabolic generators as the genus zero condition implies that Γ can be generated by these types of elements only.

Let $\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$. We have $f(\alpha \cdot z) = g_\alpha(z) f(z)$, where

$$g_\alpha(z) = \exp\left(\int_{z}^{\alpha \cdot z} 12\widehat{h}(u)du\right).$$

Taking the logarithmic derivative of g_α yields

$$\frac{g'_\alpha(z)}{g_\alpha(z)} = \frac{12c}{cz + d},$$

since $\frac{d}{dz}\alpha \cdot z = (j_\alpha(z))^{-2}$, and h is equivariant. Hence, $g_\alpha(z) = D_\alpha(cz + d)^{12}$, for some $D_\alpha \in \mathbb{C}^\times$ defining a character of Γ which we now analyze for the elliptic and parabolic generators as follows:

- α **is elliptic:** Since α has a finite order, say m_α, $D_\alpha^{m_\alpha} = 1$, that is D_α is an m_α-th root of unity.

• α **is parabolic:** Let s be the cusp fixed by α. Choose $\gamma = \begin{pmatrix} p & q \\ r & s \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$ so that $\gamma \cdot s = \infty$. Making the substitution $u = \gamma^{-1} \cdot w$ in the expression of $g_\alpha(z)$ yields

$$\begin{aligned} g_\alpha(z) &= \exp\left(\int_{\gamma\, z}^{\gamma\alpha\, z} \jmath_{\gamma^{-1}}(w)^{-2}\widehat{h}(\gamma^{-1}\cdot w)12dw\right) \\ &= \jmath_{\gamma^{-1}}(\gamma\alpha\cdot z)^{12}\jmath_{\gamma^{-1}}(\gamma\cdot z)^{-12}\exp\left(\int_{\gamma\, z}^{\gamma\alpha\, z} \widehat{h}||_2[\gamma^{-1}](w)12dw\right) \\ &= \jmath_\gamma(\alpha\cdot z)^{-12}\jmath_\gamma(z)^{12}\exp\left(\int_{\gamma\, z}^{T^{p_s}\gamma\, z}\left(a_s + \sum_{n\geq 1} a_n \exp(2\pi\imath n w/l_s)\right)12dw\right)\ , \end{aligned}$$

since $\jmath_{\gamma^{-1}}(\gamma\cdot u) = \jmath_\gamma(u)$ for all $u \in \mathbb{C}$ and by assumption

$$\widehat{h}||_2[\gamma^{-1}](z) = a_s + \sum_{n\geq 1} a_n q_s^n,\ q_s = e^{2\pi\imath z/l_s} \text{ with } (a_s/6) \in \pi\imath\mathbb{Z}^+.$$

As for the bound $T^{p_s}\gamma\cdot z$ integration, since α belongs to the isotropy group Γ_s of s it follows that $\gamma\alpha = T^{p_s}\gamma$ for some p_s divisible by the cusp width l_s at the cusp s. Hence,

$$g_\alpha(z) \ = \ \jmath_\gamma(\alpha\cdot z)^{-12}\jmath_\gamma(z)^{12}\exp\left(12a_s p_s\right) = \jmath_\gamma(\alpha\cdot z)^{-12}\jmath_\gamma(z)^{12}\ ,$$

as $a_s \in 2\pi\imath\mathbb{Z}$ and that the function $z \mapsto \exp(2\pi\imath z/l_s)$ is periodic with period l_s. Notice that the difference of the bounds of integration $T^{p_s}\gamma\cdot z$ and $\gamma\cdot z$ equals to p_s which is divisible by l_s. Thus $D_\alpha = (\jmath_\alpha(z)\jmath_\gamma(\alpha\cdot z))^{-12}\jmath_\gamma(z)^{12}$. Taking the limit as z tends to s so that αz tends to s and by Lemma 3.3.3 $\jmath_\alpha(z)^{12}$ tends to 1, we get that $D_\alpha = 1$.

Thus f is a non-vanishing weight 12 modular form on Γ with a character D that is trivial on parabolic elements.

For the holomorphy at the cusps, let s be a cusp and let, as above, $\gamma = \begin{pmatrix} p & q \\ r & s \end{pmatrix} \in$ $\mathrm{SL}_2(\mathbb{Z})$ such that $\gamma \cdot s = \infty$ Then, we have

$$\begin{aligned}
f(z)|_{12}[\gamma^{-1}] &= \jmath_{\gamma^{-1}}(z)^{-12} f(\gamma^{-1}\cdot z) \\
&= \jmath_{\gamma^{-1}}(z)^{-12} \exp\left(\int_{\gamma z_0}^{z} \jmath_{\gamma^{-1}}(w)^{-2}\widehat{h}(\gamma^{-1}\cdot w)12dw\right) \\
&= \jmath_{\gamma^{-1}}(z)^{-12} \exp\left(\int_{\gamma z_0}^{z} \left(\widehat{h}||_2[\gamma^{-1}](w) - r\jmath_{\gamma^{-1}}(z)^{-1}\right) 12dw\right) \\
&= \jmath_{\gamma^{-1}}(\gamma\cdot z_0)^{-12} \exp\left(\int_{\gamma z_0}^{z} \left(a_s + \sum_{n\geq 1} a_n \exp(2\pi \imath w n/l_s)\right) 12dw\right) \\
&= \jmath_{\gamma}(z_0)^{12} \exp(12 a_s z)\cdot \\
&\qquad \exp\left(12 a_s \alpha\cdot z_0 + 12\sum_{n\geq 1} a_n(\exp(2\pi\imath n z/l_s) - \exp(2\pi\imath n\gamma\cdot z_0/l_s))\right) \\
&= \exp(12 a_s z)\cdot \text{ holomorphic term at infinity },
\end{aligned}$$

Thus, $f(z)|_{12}[\gamma^{-1}] = q^m \cdot$ *holomorphic term at infinity*, $q = e^{2\pi\imath z}$, is holomorphic and vanishes at infinity as $12a_s = 2\pi\imath m$, $m \geq 1$; that is to say that f is a weight 12 cusp form for Γ with character D.

Let us assume for now that this character is also trivial on elliptic elements. Since $m \geq 1$, $\dfrac{f}{\Delta}$ is a weight 0 holomorphic modular form on Γ, since Δ does not vanish on $\mathfrak{H}$ and has a simple zero at infinity. Hence f/Δ is constant, i.e., $f = c\Delta$, for some non-zero constant c, which proves the assertion of the theorem. Also, comparing the $q-$expansions of f and Δ we get that $m = 1$. So far, this does not imply the uniqueness of h_0, as the space of weight 12 cusp forms on Γ is not necessarily one-dimensional and therefore one can take different quotients of f by different cusp forms.

We now proceed to prove the uniqueness of h_0 for the genus 0 group Γ. If f is another cusp form non-vanishing on $\mathfrak{H}$ and satisfying the same conditions, $\delta_{12} f$ is a weight 14 cusp form on Γ. We divide this cusp form by f to get

$$\begin{aligned}\frac{\delta_{12}f}{f} &= \frac{1}{2\pi i}\frac{f'}{f} - E_2 \\ &= \frac{1}{2\pi i}\frac{f'}{f} - \frac{1}{2\pi i}\frac{\Delta'}{\Delta},\end{aligned}$$

as E_2 is the logarithmic derivative of the modular discriminant Δ. The function $\delta_{12}f/f$ is actually holomorphic on $\mathfrak{H}$ and at cusps. Indeed, we have

$$\frac{\delta_{12}f}{f} = \frac{12}{2\pi i}\left(\frac{1}{h(z)-z} - \frac{1}{h_0(z)-z}\right)$$

is a sum of holomorphic function on $\mathfrak{H}$ and at the cusps since they both have no fixed points on $\mathfrak{H}^*$, that is, $\frac{\delta_{12}f}{f}$ is weight 2 modular form on Γ. Moreover, if we denote by $l(z) := \frac{\delta_{12}f}{f}$ and if s and γ are as before

$$\begin{aligned} l(z)|_2[\gamma^{-1}] &= \frac{1}{2\pi i j_{\gamma^{-1}}(z)^2}\left(\widehat{h}(\gamma^{-1}\cdot z) - \widehat{h}_0(\gamma^{-1}\cdot z)\right) \\ &= \frac{1}{2\pi i}\left(\widehat{h}||_2[\gamma^{-1}](z) - \widehat{h}_0||_2[\gamma^{-1}](z)\right),\end{aligned}$$

with $\widehat{h}_0(z)$ as usual denotes $(h_0(z)-z)^{-1}$. However, $\widehat{h}||_2[\gamma^{-1}](z) = 2\pi i + O(q_s)$ and $\widehat{h}_0||_2[\gamma^{-1}](z) = 12\tilde{E}_2(z) = 2\pi i + O(q)$. Hence $l(z)$ is a weight 2 cusp form on Γ which is of genus 0. According to the discussion preceding the theorem, we have $l(z) = 0$. Therefore

$$h(z) = h_0(z) = z + \frac{6}{\pi i E_2}.$$

If $D_\alpha \neq 1$ for the elliptic generators, then $D_\alpha^{m_\alpha} = 1$, m_α being the order of α and we let $t(z) = f(z)^n$ which now has a trivial character, n being the least common multiple of the orders of the elliptic generators. Both f and f^n give rise to the same equivariant form (Proposition 2.3.3 in Chapter 2). The proof then follows exactly the same steps with Δ replaced by Δ^n and δ_{12} by δ_{12n}. ∎

We end this section by commenting on the conditions of the above theorem.

The condition on the genus of the group is mandatory for the uniqueness of the equivariant form without fixed points h_0. As for the behavior of the equivariant form without fixed points at cusps, this is a minimal condition for an equivariant form without fixed points to arise from a cusp form; i.e., it is a rational equivariant form for this corresponding cusp form. We will prove in the next chapter this result using a different argument.

Chapter 4

The complete classification of equivariant forms and their geometry

In this chapter we characterize and classify all the equivariant forms for a subgroup of the modular group using the theory of modular forms and quasimodular forms. As a consequence, the set of nontrivial equivariant forms will be endowed with a vector space structure and the trivial equivariant form $h(z) = z$ will be looked at as "the point at infinity". We will also interpret the equivariant forms in a geometric setting involving meromorphic differential 1-forms and sections of a line bundle over a compact Riemann surface or equivalently over an algebraic curve. It should be mentioned that the group under consideration can be any discrete subgroup of $\mathrm{SL}_2(\mathbb{R})$, but to simplify the exposition, we restrict ourselves to finite index subgroups of $\mathrm{SL}_2(\mathbb{Z})$.

4.1 Equivariant forms and quasimodular forms

In this section we establish a close connection between equivariant forms and quasimodular forms. This connection will enable us to recover many of the results of the previous chapters as well as to establish new results.

4.1.1 From quasimodular forms to equivariant forms

Our main concern will be the construction of equivariant forms from a certain class of quasimodular forms on Γ.

Let first ϕ be a weight 2 and depth 1 holomorphic quasimodular forms on a finite index subgroup Γ of $\mathrm{SL}_2(\mathbb{Z})$ satisfying

$$\phi(z)|_2[\alpha] := (cz+d)^{-2}\phi(\alpha\cdot z) = \phi(z) + \frac{c}{cz+d} \tag{4.1.1}$$

for all $\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$; in other words, $\phi||_2[\alpha](z) = \phi(z)$. Then we have

Theorem 4.1.1 *The function*

$$h_\phi(z) = z + \frac{1}{\phi(z)}$$

is an equivariant form without fixed points for Γ.

Proof: Let $\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$. Then we have

$$\begin{aligned} h_\phi(\alpha\cdot z) &= \frac{az+b}{cz+d} + \frac{1}{j_\alpha(z)^2\phi(z) + cj_\alpha(z)} \\ &= \frac{(az+b)j_\alpha(z)\phi(z) + acz + bc + 1}{j_\alpha(z)^2\phi(z) + cj_\alpha(z)} \\ &= \frac{(az+b)\phi(z) + a}{j_\alpha(z)\phi(z) + c}\,, \end{aligned}$$

where we have used the modular identity $ad - bc = 1$. Meanwhile, a direct computation shows that

$$\begin{aligned} \alpha \cdot h_\phi(z) &= \frac{ah(z)+b}{ch(z)+d} \\ &= \frac{(az+b)\phi(z)+a}{j_\alpha(z)\phi(z)+c} , \end{aligned}$$

which proves the equivariance property of h.

Now, as ϕ is holomorphic on $\mathfrak{H}$, the function $\widehat{h}_\phi(z)$ does not vanish on $\mathfrak{H}$, that is to say that h_ϕ has no fixed points in $\mathfrak{H}$. At cusps, from the above analysis we see that none of the cusps is a fixed point of $h_\phi(z)$, since $\widehat{h}_\phi||_2[\gamma^{-1}](z)$ is holomorphic at infinity. ∎

We would like to notice that the equivariance property follows also from Proposition 3.1.2.

This theorem can be easily generalized to the following. We will omit the proof as it is proved along the same steps.

Theorem 4.1.2 *Let ϕ be as in (4.1.1), and let f be a meromorphic weight 2 modular form on Γ. Then, the function*

$$h(z) := z + \frac{1}{\phi(z)+f}$$

is an equivariant form for Γ. Moreover, if f is holomorphic then h is an equivariant form without fixed points on Γ.

These theorems provide a class of equivariant forms constructed from certain quasimodular forms. We will show in the following subsection that, indeed, there is a bijection between a subset of equivariant forms and this class of quasimodular forms.

4.1.2 From equivariant forms to quasimodular forms

The aim of this subsection is to give the general form of an equivariant form using the theory of quasimodular forms.

Let h be a non-trivial equivariant form on Γ; i.e. $h(z) \neq z$. Then the function $\widehat{h}(z) = \dfrac{1}{h(z) - z}$ transforms as follows under the action of Γ. Let $\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$ and $z \in \mathfrak{H}$. We have

$$\begin{aligned} \widehat{h}(\alpha \cdot z) &= \frac{1}{h(\alpha \cdot z) - \alpha \cdot z} \\ &= \frac{1}{\alpha \cdot h(z) - \alpha \cdot z} \\ &= \frac{1}{\dfrac{ah(z) + b}{ch(z) + d} - \dfrac{az + b}{cz + d}} \\ &= \frac{(ch(z) + d)(cz + d)}{h(z) - z} \\ &= c(cz + d) + \frac{(cz + d)^2}{h(z) - z} \\ &= c(cz + d) + (cz + d)^2 \widehat{h}(z). \end{aligned}$$

Using this, we have

Theorem 4.1.3 *Let h be an equivariant form on Γ. Then there exists a weight 2 meromorphic modular form $f_{h,\phi}$ on Γ such that*

$$h(z) = z + \frac{1}{\phi + f_{h,\phi}}, \tag{4.1.2}$$

where ϕ is a fixed weight 2 and depth 1 holomorphic quasimodular on Γ that transforms as in (4.1.1).

Proof: Denote as before by $\widehat{h}(z)$ the function $\dfrac{1}{h(z) - z}$. Then, since $h(z)$ is meromorphic on $\mathfrak{H}$, $g(z)$ is also a meromorphic on $\mathfrak{H}$. At a cusp s, we let $\gamma =$

$\begin{pmatrix} t & u \\ v & w \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$ such that $\gamma \cdot z = \infty$. Then we have

$$\begin{aligned} \widehat{h}(z)\|_2[\gamma^{-1}] &= \jmath_{\gamma^{-1}}(z)^{-2}\widehat{h}(\gamma^{-1} \cdot z) + \frac{-v}{\jmath_{\gamma^{-1}}(z)} \\ &= \widehat{h}\|_2[\gamma^{-1}](z) \ , \end{aligned}$$

which is meromorphic at infinity.

Next, from Theorem 1 4.2 and its generalization Section §1.4, there exists $f_0 = f_{0,h,\phi} \in \mathfrak{M}_0^m(\Gamma)$ and $f_{h,\phi} \in \mathfrak{M}_2^m(\Gamma)$ such that $\widehat{h} = f_0\phi + f_{h,\phi}$. Hence

$$h(z) = z + \frac{1}{f_0\phi + f_h}$$

It remains to prove that $f_0 = 1$ Indeed, let $\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$ We have

$$h(\alpha \cdot z) = \frac{(az+b)\jmath_\alpha(z)(f_0\phi + f_{h,\phi}) + (az+b)cf_0 + 1}{\jmath_\alpha(z)^2(f_0\phi + f_{h,\phi}) + c\jmath_\alpha(z)f_0}$$

and

$$\alpha \cdot h(z) = \frac{(az+b)(f_0\phi + f_{h,\phi}) + a}{\jmath_\alpha(z)(f_0\phi + f_{h,\phi}) + c}.$$

Since h is equivariant, equating the above relations yields $f_0 = 1$ as desired. ∎

This theorem has the following immediate consequences.

Corollary 4.1.4 *There is a bijection between the set of non-trivial (meaning not equal to z) equivariant forms and the set of weight 2 depth 1 quasimodular forms on* Γ *that transforms like* ϕ*; i.e. the set* $\{\phi + f \mid f \in \mathfrak{M}_2^m(\Gamma)\}$.

Furthermore, if fix a weight 2 and depth 1 quasimodular form for Γ, say $\phi = \tilde{E}_2 = (\imath\pi/6)E_2$, we have

Corollary 4.1.5 *The map* $h \mapsto f_{h,\phi}$ *define a bijection from the set of non-trivial equivariant forms and the space of weight 2 meromorphic modular forms on* Γ

Suppose now that $h(z)$ is an equivariant form without fixed points, that is, $\widehat{h}(z)$ never vanishes. Then $\widehat{h}(z)$ is a holomorphic quasimodular form of weight 2 and depth 1. The holomorphy at cusps is also straightforward as the function $\widehat{h}||_2[\gamma](z)$ is holomorphic for all $\gamma \in \Gamma$.

Corollary 4.1.6 *The set of equivariant forms without fixed points for* Γ *and the space of weight 2 holomorphic modular forms on* Γ *are in one-to-one correspondence.*

In the next section, we will retrieve the above two corollaries using the theory of modular forms.

Recall now that the dimension of the space of holomorphic weight 2 modular forms on Γ is equal to $g+r-1$ and that of the space of cusp forms is g, where g is the genus of Γ and r is the number of inequivalent cusps under Γ. In particular, if $g = 0$, then these dimensions reduce respectively to $r - 1$ and 0. For instance, for $\mathrm{SL}_2(\mathbb{Z})$ the space of weight 2 holomorphic modular forms on $\mathrm{SL}_2(\mathbb{Z})$ is zero dimensional since all cusps are equivalent to infinity modulo $\mathrm{SL}_2(\mathbb{Z})$. Therefore, this corollary provides another proof of Theorem 3.4.3 for $\Gamma = \mathrm{SL}_2(\mathbb{Z})$.

4.2 Equivariant forms, modular forms and differential forms

In this section, we establish many of the results of the previous section by establishing a direct correspondence between the set of non-trivial equivariant forms and the space of modular forms of weight 2. We will explain why this is more effective as compared to the point of view of quasimodular forms from the previous section.

4.2.1 Equivariant forms as modular forms

In the following proposition we give the relation between equivariant forms and modular forms.

Proposition 4.2.1 *Let h_1, h_2 be two nontrivial equivariant forms on Γ; i.e. not equal to $h(z) = z$. Then the function*

$$f_{1,2} := \widehat{h}_1(z) - \widehat{h}_2(z)$$

is a weight 2 meromorphic modular form on Γ.

Proof: First, as each of the equivariant forms is meromorphic on $\mathfrak{H}$ and at cusps, it follows that $f_{1,2}$ is also meromorphic on $\mathfrak{H}$ and at cusps. Therefore, it remains to prove that $f_{1,2}$ satisfies the modularity condition.

Let $\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$. Using the transformation formula for an equivariant form, we have

$$\begin{aligned} f_{1,2}(\alpha \cdot z) &= \frac{1}{\alpha \cdot h_1(z) - \alpha \cdot z} - \frac{1}{\alpha \cdot h_2(z) - \cdot z} \\ &= \left(c(cz+d) + \frac{(cz+d)^2}{h_1(z) - z} \right) - \left(c(cz+d) + \frac{(cz+d)^2}{h_2(z) - z} \right) \\ &= (cz+d)^2 \left(\widehat{h}_1(z) - \widehat{h}_2(z) \right). \end{aligned}$$

Thus $f_{1,2}$ is a meromorphic modular form on Γ of weight 2. ∎

Now, if we fix h_2 in Proposition 4.2.1, then as h_1 runs over the space of nontrivial equivariant forms, $f_{1,2}$ runs over the space of weight 2 modular forms and vice et versa. Hence the following correspondence.

Theorem 4.2.2 *Let ϕ be as in (4.1.1). Then any non-trivial equivariant form h on Γ has the form*

$$h(z) = z + \frac{1}{\phi + f_{\phi,h}},$$

for some weight 2 modular form f on Γ. Consequently, the set of non-trivial equivariant forms is in a one-to-one correspondence with the space of weight 2 meromorphic modular forms on Γ.

Proof: Fix $h_2 = h_\phi$ with $h_1 = h$ as in Proposition 4.2.1 so that $f_{1,2} = f_{\phi,h}$. The map $h \mapsto f_{\phi,h}$ gives the correspondence. ∎

As a direct consequence, we have

Corollary 4.2.3 *The set of equivariant forms without fixed points is in bijection with the space of weight 2 holomorphic modular forms.*

Once again, the map $h \mapsto f_{\phi,h}$ gives the correspondences between the set of non-trivial equivariant forms and the space of weight 2 meromorphic modular forms on Γ and the set of equivariant forms without fixed points and the space of weight 2 holomorphic modular forms on Γ.

Remark 4.2.1 In all results presented in either this section or in Section §5.1.3 the choice of the quasimodular form ϕ is not canonical in the following sense. If f is a weight 2 holomorphic modular form on Γ, then the function $\phi_1 = \phi + f$ is again a weight 2 depth 1 quasimodular form on Γ that behaves similar to ϕ. This simply says that interchanging ϕ and ϕ_1 only permutes the elements in our correspondences, and hence does not affect these results.

We shall introduce some notations that we adopt for the rest of this work. For a modular subgroup Γ, we will use $\mathcal{E}(\Gamma)$ for the set of all equivariant forms for Γ, We also denote by $\mathcal{E}^w(\Gamma)$ for the set of equivariant forms without fixed points and $\mathcal{E}_0^w(\Gamma)$ for the set of equivariant forms without fixed points such that the function $f_{h,\phi}$ of Theorem 4.1.3 is a cusp form. Furthermore, if we formally set $f = \infty$ in the relation

$$h(z) = z + \frac{1}{\phi + f}$$

then we get $h(z) = z$. We thus look at the trivial equivariant form $h(z) = z$ as the point at infinity of the space $\mathcal{E}(\Gamma)$. We denote the set of equivariant forms, including the trivial form $h(z) = z$ by $\mathcal{E}^*(\Gamma)$. It would be interesting to consider $\mathcal{E}^*(\Gamma)$ as a projectivization of the space $\mathcal{E}(\Gamma)$.

4.2.2 The connection with differential forms

Differential forms play a central role in differential geometry and particularly in the theory of Riemann surfaces and algebraic curves. In this subsection we establish the connection between equivariant forms and differential forms on the Riemann surface $\mathfrak{H}^*$. To an equivariant form h on a finite index subgroup Γ of $\mathrm{SL}_2(\mathbb{Z})$ we associate the meromorphic degree 1 differential $w = (\widehat{h}(z))^{-1}dz$, where $\widehat{h}(z)$ denotes as usual $h(z) - z$. Then, since h is an equivariant form, we get a degree one differential satisfying

$$\alpha^* w = w + \frac{c}{j_\alpha(z)} dz \ , \ \alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma \ ,$$

where $\alpha^* w = (\widehat{h}(\alpha \cdot z))^{-1} d(\alpha \cdot z)$ This is equivalent to say that

$$\alpha^* w - w = \frac{c}{j_\alpha(z)} dz \qquad (4.2.1)$$

for all $\alpha \in \Gamma$ as $\dfrac{d}{dz} \alpha \cdot z = j_\alpha(z)^{-2}$. Conversely, suppose we are given a degree 1 meromorphic differential w on $\mathfrak{H}^*$ such that the above condition holds for all $\alpha \in \Gamma$. Write w as $w = f(z)dz$, for some meromorphic function $f : \mathfrak{H}^* \longrightarrow \mathbb{C}$. Then, we have

$$\alpha^* w - w = j_\alpha(z)^{-2} f(\alpha \cdot z) dz = f(z)|_2[\alpha] dz - f(z) dz = \frac{c}{j_\alpha(z)} dz \ .$$

Hence, $f(z)|_2[\alpha] = f(z) + \dfrac{c}{j_\alpha(z)}$, that is to say that f is a weight 2 depth 1 meromorphic quasimodular form. We then let $h(z) = z + 1/f(z)$, which is an equivariant form by Theorem 4.1.1. Hence, the correspondence between equivariant forms and degree 1 meromorphic differentials on $\mathfrak{H}^*$ satisfying (4.2.1).

Let now w_1, w_2 be two such differentials. Then the degree 1 meromorphic differential $w = w_1 - w_2$ is invariant under the action of Γ. Therefore, there is a weight 2 (meromorphic) modular form $f_{1,2}$ on Γ such that $w = f_{1,2}(z)dz$. Hence, fixing $w_2 = (\widehat{h}_0(z))^{-1}dz$, $h_0(z) = z + 1/\widetilde{E}_2$, we get a one-to-one correspondence between the space of degree 1 meromorphic differentials on $\mathfrak{H}^*$ invariant under the action of Γ, which we identified with the space of degree 1 meromorphic differentials on $\Gamma \setminus \mathfrak{H}^* = X_\Gamma$, and the set of degree 1 meromorphic differentials on $\mathfrak{H}^*$ satisfying (4.2.1).

Theorem 4.2.4 *There is a one-to-one correspondence between the space* $\mathrm{Dif}(X_\Gamma)$ *and the space* $\mathcal{E}^*(\Gamma)$ *of non-trivial equivariant forms .*

One could also use the correspondence in Corollary 4.1.6 to connect differential forms and equivariant forms. Recall, from Section §1.3.2, that the space $\mathrm{Dif}^{k/2}(X_\Gamma)$ of degree $k/2$, $k \in 2\mathbb{N}$, meromorphic differential forms on the compact Riemann surface $X_\Gamma = \Gamma \setminus \mathfrak{H}^*$ is isomorphic to the space $\mathfrak{M}_k^m(\Gamma)$ of meromorphic modular forms via $f \mapsto f(dz)^{k/2}$.

Also, from Corollary 1.3.5 and Corollary 4.2.3, we have

Corollary 4.2.5 *The space of degree 1 holomorphic differential is in a one-to-one correspondence with* $\mathcal{E}_0^w(\Gamma)$.

Let us fix $h = h_0$, the fundamental equivariant form. If Γ has genus zero, then $\mathcal{E}_0^w(\Gamma)$ is reduced to the trivial equivariant form h_0, which in accordance with the uniqueness of h_0 as the only equivariant form for Γ without fixed point. In this case, the correspondence of the above corollary is canonical.

4.2.3 Operations on the set of equivariant forms

In this subsection, we make explicit the vector space operations on $\mathcal{E}(\Gamma)$ that are inherited from the correspondences in the above section. We shall start with the addition. Let ϕ be a fixed weight 2 depth 1 holomorphic quasimodular forms on Γ satisfying (4.1.1). For instance we may take $\phi = \tilde{E}_2 = (i\pi/6)E_2$. If h_1 and h_2 are two elements of $\mathcal{E}(\Gamma)$, it is not difficult to see that the sum $h_1 \oplus h_2$ of h_1 and h_2 is given by the relation

$$\widehat{h_1 \oplus h_2} = \widehat{h}_1 + \widehat{h}_2 - \phi \,. \tag{4.2.2}$$

Recall that we have set $\widehat{h}(z) = (h(z) - z)^{-1}$. The *zero element* is then given by $h_\phi(z) = z + \dfrac{1}{\phi}$. As for the opposite h^- of h, it satisfies

$$\widehat{h^-} = 2\phi - \widehat{h}.$$

To complete the description of the vector space structure on $\mathcal{E}(\Gamma)$, we give the scalar multiplication. Let $c \in \mathbb{C}$, and let h be an element of $\mathcal{E}(\Gamma)$. Then, the scalar multiple $c \odot h$ of h satisfies

$$\widehat{c \odot h} = c\widehat{h} + (1-c)\phi.$$

Remark 4.2.2 This structure is not canonical as one may choose a different quasimodular form than ϕ. The new structure is still the one transferred from that on $\mathfrak{M}_2^m(\Gamma)$.

Therefore, the sets $\mathcal{E}^w(\Gamma)$ and $\mathcal{E}_0^w(\Gamma)$ are subspaces of $\mathcal{E}^\times(\Gamma)$. Note that, by Theorem 4.1.1, these subspaces clearly contain the zero element h_ϕ. Using Corollary 4.2.3 and the dimension formula Theorem 1.3.6, we deduce

Corollary 4.2.6 *The subspaces $\mathcal{E}^w(\Gamma)$ and $\mathcal{E}_0^w(\Gamma)$ are finite dimensional subspaces of $\mathcal{E}^*(\Gamma)$ with dimension respectively $g+r-1$ and g. Here g is the genus of Γ and r the number of inequivalent cusps modulo Γ.*

Remark 4.2.3 If f is a nonzero weight 2 modular form for Γ, then $1/f$ becomes a modular form of weight -2. If one considers the space of equivariant form $\widetilde{\mathcal{E}}(\Gamma) = \mathcal{E}(\Gamma) \setminus \{h_\phi\}$ for a fixed weight 2 and depth 1 quasimodular form ϕ, then this space is in bijection with the space of modular forms of weight -2, which is again infinite dimensional. Now the zero element becomes the trivial equivariant form $h(z) = z$ which allows us to define an almost canonical vector space structure. The operations depend linearly on the ϕ. The equivariant form h_ϕ becomes then the point at infinity. It can be shown that the subset of rational equivariant forms from Section §3.2.1 without h_ϕ if it is rational is a linear subspace of $\widetilde{\mathcal{E}}(\Gamma)$ of infinite dimension.

4.3 The cross-ratio of equivariant forms and modular functions

The cross-ratio plays an important role in projective differential geometry [Ov]. It is defined for four points z_1, z_2, z_3, z_4 of the projective line $\mathbb{P}^1(\mathbb{C})$ by

$$\frac{(z_1 - z_3)(z_2 - z_4)}{(z_2 - z_3)(z_1 - z_4)} .$$

A well-known property of the cross-ratio is that it is invariant under Möbius transformations. Hence, it can be looked at as a geometric invariant of the projective line. Moreover, the Schwarz derivative is the infinitesimal part of the cross-ratio. In connection with equivariant forms, the Schwarz derivative maps equivariant forms to weight 4 modular forms. We will consider in this section the effect of this invariant on the projective space $\mathcal{E}(\Gamma)$; a phenomenon that worths to be investigated.

Proposition 4.3.1 *Let h_1, h_2, h_3, h_4 be four equivariant forms on Γ, a finite index subgroup of $SL_2(\mathbb{Z})$. Define a function f as the cross-ratio of these four elements*

$$f = (h_1, h_2; h_3, h_4) = \frac{(h_1 - h_3)(h_2 - h_4)}{(h_2 - h_3)(h_1 - h_4)} . \qquad (4.3.1)$$

Then, if $h_2 \neq h_3$ and $h_1 \neq h_4$, the function f is a modular function on Γ.

Proof: First, as the h_l's, $1 \leq l \leq 4$, are equivariant forms for Γ and as the cross ratio is invariant under any Möbius transformations, the function f is invariant under Γ . The meromorphy property on $\mathfrak{H}$ and at cusps follows from that of these four equivariant forms. ∎

Remark 4.3.1 We would like to notice that [Bra] has obtained the same parametrization, which was motivated by the work of Heins [Hei] on the theory of elliptic functions. To illustrate this let $L = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$ be a lattice with $\tau = \omega_2/\omega_1 \in \mathfrak{H}$ and f a pseudo-periodic function with pseudo-period $H(\omega|L)$. Then $H(\tau|L)/H(1|L)$ commutes with the action of the modular group. For example, for the Weierstrass ζ-function which is defined by $\zeta' = -\wp$, where $\wp$ is the Weierstrass elliptic $\wp$-function. If η_1 and η_2 are the pseudo-periods of ζ, then the fundamental equivariant form is given by $h_0 = \omega_1\eta_2$. However, our motivation is essentially geometric and is inspired by the intimate connection between the cross-ratio and the Schwarz derivative in projective geometry [Ov], where the Schwarz derivative appears as a measure of the effect of diffeomorphisms of the projective line affects this cross-ratio [Ov]. It actually appears a quadratic differential, meaning, if f is such a diffeomorphism then

$$(f(z_1), f(z_2); f(z_3), f(z_4)) = (z_1, z_2; z_3, z_4) - 2\epsilon^2\{f, z\}(z_1) + O(\epsilon^3) ,$$

where ϵ depends on the distance between the points z_i, $1 \leq i \leq 4$. In our case the Schwarz derivative plays the role of mapping equivariant forms to weight 4 modular forms. A investigation that we intend to pursue in future work.

Now, the symmetric group S_4 acts on the cross-ratio h by permuting $h_1,\ h_2,\ h_3,\ h_4$, and, hence, produces the following symmetric relations

$$(h_1, h_2; h_4, h_3) = \frac{1}{f} , \qquad (h_1, h_3; h_2, h_4) = 1 - f ,$$

$$(h_1, h_3; h_4, h_2) = \frac{1}{1-f} , \qquad (h_1, h_4; h_3, h_2) = \frac{f}{f-1} ,$$

$$(h_1, h_4; h_2, h_3) = \frac{f-1}{f} .$$

The modular function f is invariant under all other permutations, which one can easily check that they form the Klein four-group ({ identity, (1,2)(3,4), (1,3)(2,4), (1,4)(2,3) }). In fact, one can consider only the action of the symmetric group S_3 as shown above by fixing one equivariant form and permuting the others. We would like to notice that the transformations

$$z \mapsto \frac{z}{z-1},\ z \mapsto 1-z,\ z \mapsto \frac{1}{z},\ z \mapsto \frac{1}{1-z},\ \text{and } z \mapsto \frac{z-1}{z},$$

together with $z \mapsto z$ form a group that is isomorphic to S_3.

To illustrate this phenomenon we will show how to obtain certain classical modular functions, which are in fact Hauptmoduln, of certain genus zero groups.

It is shown in [Ran] that the Klein modular function $\lambda = \frac{\vartheta_2^4}{\vartheta_3^4}$, which is a Hauptmodul for the the genus 0 principal congruence subgroup $\Gamma(2)$ which is of index 6 in $SL_2(\mathbb{Z})$, transforms under representatives of conjugacy classes of the quotient by the modular group $SL_2(\mathbb{Z})/\Gamma(2)$ and produces the same relation. This was shown using only transformations of the theta function ϑ_2 and ϑ_3. These relations are

$$\begin{aligned} \lambda(z/(z+1)) &= \frac{1}{\lambda}, & \lambda(-1/z) &= 1-\lambda, \\ \lambda(-1/(z+1)) &= \frac{1}{1-\lambda}, & \lambda(z+1) &= \frac{\lambda}{\lambda-1}, \\ \lambda(-(z+1)/z) &= \frac{\lambda-1}{\lambda}. \end{aligned}$$

We provide a proof of these relations which is a consequence of the above action on cross-ratio of equivariant forms. Recall the equivariant forms

$$\begin{aligned} f_2(z) &= z + \frac{\vartheta_2(z)}{2\vartheta_2'(z)}, \\ f_3(z) &= z + \frac{\vartheta_3(z)}{2\vartheta_3'(z)}, \\ f_4(z) &= z + \frac{\vartheta_4(z)}{2\vartheta_4'(z)}, \end{aligned}$$

which are equivariant forms without fixed points on level 2 congruence subgroup, and, in particular, for $\Gamma(2)$.

Proposition 4.3.2 *The cross-ratio* $(z,\ f_4;\ f_2,\ f_3)$ *equals to the Klein modular function* λ.

Proof: First, we would like to notice that all the functions in the cross-ratio $(z,\ f_4;\ f_2,\ f_3)$ are equivariant for congruence subgroup containing the principal congruence subgroup $\Gamma(2)$, so in particular for $\Gamma(2)$. Also we know from the above that this cross-ratio is a modular function on $\Gamma(2)$, which is of genus 0. Hence, it is a rational function of λ, which is a Hauptmodul of the genus zero principal congruence subgroup $\Gamma(2)$. To conclude it suffices to show that this cross-ratio has a simple zero at ∞ (in the local parameter $t = e^{\pi i z}$) since the function λ does. However, this follows from the Fourier expansions of the theta functions. Precisely, since the theta series have Fourier expansions of the form

$$\vartheta_2(z) = 2t^{1/4}\sum_{n=1}^{\infty} t^{n(n-1)},\ \vartheta_3(z) = 1 + 2\sum_{n=1}^{\infty} t^{n^2},$$

$$\vartheta_4(z) = 1 + 2\sum_{n=1}^{\infty} t^{n^2},\ t = e^{\pi i z}\ ,$$

we see easily that

$$(z,\ f_4;\ f_2,\ f_3) = \frac{\vartheta_2(\vartheta_3'\vartheta_4 - \vartheta_3\vartheta_4')}{\vartheta_3(\vartheta_2'\vartheta_4 - \vartheta_2\vartheta_4')} = 16t(1 + O(t))\ .$$

An investigation of the Fourier coefficient of t gives the result. ∎

Remark 4.3.2 A consequence of the action of S_3 on the cross-ratio λ is that it satisfies the above transformations. This action translates to λ by noting that the quotient group $\mathrm{SL}_2(\mathbb{Z})/\Gamma(2)$ is isomorphic to S_3.

This proposition could be proved using the following identities involving these theta series and λ [Ran], however the computation is tedious.

$$\begin{aligned}\frac{\lambda'}{\lambda} &= \pi i\vartheta_4^4\\ \frac{\lambda'}{\lambda(1-\lambda)} &= \pi i\vartheta_3^4\\ \frac{\lambda'}{1-\lambda} &= \pi i\vartheta_2^4\ .\end{aligned}$$

A similar result holds for the modular invariant j, which is a Hauptmodul for the modular group $\mathrm{SL}_2(\mathbb{Z})$. Let h_k denote the rational equivariant form corresponding to the weight k, an even positive integer, Eisenstein series E_k on the modular group defined earlier in Chapter 1.

Proposition 4.3.3 *We have*

$$1728(z,\ h_6;\ h_4,\ h_0) = j\ ,$$

where $h_0 = z + 6/\pi i E_2$.

Proof: Here again, one uses the Fourier expansion of the Eisenstein series E_2, E_4 and E_6 to show that the cross-ratio has a simple pole at infinity (in the local parameter $q = e^{2\pi iz}$). Indeed, since the Eisenstein series have expansion $E_k(z) = 1 + c_k\sum_{n=1}^{\infty}\sigma_{k-1}(n)q^n$ in the local parameter $q = e^{2\pi iz}$ with $c_2 = -24,\ c_4 = 240,\ c_6 = -504$, it follows that $3E_4'E_6 - 2E_4E_6' = 3456\pi iq(1 + O(q))$ and $\pi iE_2E_6 - E_6' = \pi i(1 + O(q))$, so that

$$(z,\ h_6;\ h_4,\ h_0) = \frac{2E_4(\pi iE_2E_6 - E_6')}{3E_4'E_6 - 2E_4E_6'} = \frac{1}{1728q}(1 + O(q)).$$

Hence it is a linear fraction of the $j-$function as it is the Hauptmodul for the modular group. An investigation of the coefficient of $1/q$ gives now the result. ∎

4.4 Equivariant forms and sections of line bundles

The goal of this section is to present a correspondence between equivariant forms and sections of a certain line bundles. In order to achieve this goal and make this section self-contained we recall the definitions and some properties of these objects. We refer, for instance to [Gun] and [Mil] for more details on the content of this section.

4.4.1 Equivariant forms as sections of line bundles

In this part we present the connection between equivariant forms and sections of certain line bundle on a quotient of $\mathfrak{H} \times \mathbb{C}$.

Let us first recall the definition of the objects we will be dealing with in the rest of this chapter.

Definition 4.4.1 *Let X be a (compact) Riemann surface. A vector bundle L on X of rank $n \geq 1$ is a map of complex manifolds*

$$\pi : \; L \longrightarrow \; X$$

such that for some open covering $\bigcup_i U_i$ of X, $\pi^{-1}(U_i) \cong U_i \times \mathbb{C}^n$. If $n = 1$, this is called a line bundle.

We now define the notion of a section of a line bundle.

Definition 4.4.2 *Let L, $p : \; L \longrightarrow \; X$, be a line bundle on a Riemann surface X, and let U be an open subset of X. A* holomorphic section of L over U *is a holomorphic map $s : \; U \longrightarrow \; L$ such that $p \circ s = id_U$. The group of all sections of L over U is denoted by $S(U, \; L)$.*
Note that if $L = U \times \mathbb{C}$, the trivial bundle, then $S(U, \; L)$ can be identified with the set of holomorphic functions on U.

Similarly, one can define a meromorphic section of L over U being a holomorphic section s over $U \setminus V$ such that V is a discrete subset of U and s has a pole at every point of V.

Unless specified, by a section we mean a meromorphic section.

We now represent equivariant forms as sections of line bundles.

Define an action, which is the action of the slash operator, of Γ on $\mathfrak{H} \times \mathbb{C}$ as follows $\gamma * (z,\ t) := (\gamma \cdot z,\ j_\gamma(z)^2 t + c j_\gamma(z))$, where $\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$. Denote by $\mathcal{Q}$ the quotient by Γ of $\mathfrak{H} \times \mathbb{C}$. Then the first projection $\mathfrak{H} \times \mathbb{C} \longrightarrow \mathfrak{H}$ induces a surjective map

$$p:\ \mathcal{Q} \longrightarrow Y_\Gamma,\ Y_\Gamma = \Gamma \setminus \mathfrak{H}$$

such that locally $p^{-1}([z]) \cong \{[z]\} \times \mathbb{C}$, where $[z]$ denotes the class of $z \in \mathfrak{H}$ modulo Γ. Thus $\mathcal{Q}$ has a structure of a line bundle over Y_Γ. Denote by $L(Y_\Gamma, \mathcal{Q})$ the space of sections of the line bundle $\mathcal{Q}$ over Y_Γ. Let $\pi :\ \mathfrak{H} \longrightarrow Y_\Gamma$ denote the canonical map. Then for an open neighborhood U of $z \in \mathfrak{H}$ a section of $\mathcal{Q}$ over the open $\pi(U)$ neighborhood of $[z]$ (π is an open map) is a map $s :\ \pi(U) \longrightarrow p^{-1}(\pi(U)) \subseteq \mathcal{Q}$ meromorphic on $\pi(U)$ such that $s \circ p = id_{\pi(U)}$. In particular, this says that we can view $s([z]) = s(\pi(z)) = \{(z, f(z))\}$, where $\{(z,t)\}$ denotes the class of (z,t) in $\mathcal{Q}$ and $f :\ U \longrightarrow \mathbb{C}$ is a meromorphic function. Now, as the orbit of $(z,t) \in \mathfrak{H} \times \mathbb{C}$ and $\alpha * (z,t)$ are the same for all $\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$, we get

$$s(\pi(z)) = \{(z, f(z))\} = s(\pi(\alpha \cdot z)) = \{(\alpha \cdot z, f(\alpha \cdot z))\}$$

$$= \{\alpha^{-1} * (\alpha \cdot z, f(\alpha \cdot z))\} = \{(z, j_{\alpha^{-1}}(z)^2 f(\alpha \cdot z) - c j_{\alpha^{-1}}(z))\}\ .$$

Hence, $f(z) = f(\alpha \cdot z) = j_{\alpha^{-1}}(z)^2 f(\alpha \cdot z) - c j_{\alpha^{-1}}(z)$. However, $j_\alpha(z)$ satisfies the 1-cocycle relation and in particular we have $j_\alpha(z).j_{\alpha^{-1}}(z) = 1$. This means that f is weight 2 depth 1 quasimodular form that behaves the same way as $\widetilde{E}_2$, so that by Corollary 4.1.4 we have $h_f(z) = z + 1/f$ is an equivariant form. In conclusion,

Theorem 4.4.3 *The space $L(Y_\Gamma, \mathcal{Q})$ of sections of the line bundle $\mathcal{Q}$ over Y_Γ is in a one-to-one correspondence with the set of meromorphic unrestricted equivariant functions.*

The line bundle $\mathcal{Q}$ over Y_Γ could be extended in the same way to a line bundle $\mathcal{Q}^*$ over $X_\Gamma = \Gamma \setminus \mathfrak{H}^*$, and hence the correspondence.

4.4.2 Equivariant forms as sections of line bundles: A modular forms point of view

To represent equivariant forms as sections of a line bundle, we show how modular forms arise as sections of a line bundle on $\Gamma \setminus \mathfrak{H}^*$, and use Theorem 4.2.2.

In the following we give a close connection between automorphic factors and isomorphisms of line bundles on $\Gamma \setminus \mathfrak{H}$ [Mil].

Recall that an automorphic factor on Γ is a holomorphic function

$$\mu : \ \Gamma \times \mathfrak{H} \longrightarrow \mathbb{C}^\times$$

satisfying the so-called 1-cocycle relation

$$\mu_{\alpha\beta}(z) = \mu_\alpha(\beta \cdot z)\mu_\beta(z)$$

for all $\alpha,\ \beta \in \Gamma$ and $z \in \mathfrak{H}$.

Let μ be an automorphic factor for Γ. Then the projection map $p : \ \Gamma \setminus \mathfrak{H} \times \mathbb{C} \longrightarrow \Gamma \setminus \mathfrak{H}$ defines a line bundle on $\Gamma \setminus \mathfrak{H}$, where an action of Γ on $\mathfrak{H} \times \mathbb{C}$ is given by $\gamma * (z,\ t) := (\gamma \cdot z,\ \mu_\gamma(z)t)$, where $\gamma \in \Gamma,\ z \in \mathfrak{H}$ and $t \in \mathbb{C}$.

Conversely, if $\psi :\ L \longrightarrow\ \Gamma \setminus \mathfrak{H}$ is a line bundle and $\pi :\ \mathfrak{H} \longrightarrow\ \Gamma \setminus \mathfrak{H}$ is the canonical map, then $\pi^*(L) := \{(h,\ l) \in \mathfrak{H} \times L |\ \psi(l) = \pi(h)\}$ is a line bundle on $\mathfrak{H}$, and Γ act on π^* trough its action on $\mathfrak{H}$. Moreover, if $i :\ \mathfrak{H} \times \mathbb{C} \longrightarrow\ \pi^*(L)$ is a given isomorphism, then the action of Γ on $\pi^*(L)$ can be transferred to an action on $\mathfrak{H} \times \mathbb{C}$ defined by $\gamma * (z,\ t) := (\gamma \cdot z,\ \mu_\gamma(z)t)$, which is a multiplication by a non-zero scalar

$\mu_\gamma(z)$. Since under this action the orbits of $(z,\ t)$ and $\gamma * (z,\ t)$ coincide, we deduce that $\mu_\gamma(z)$ is an automorphic factor. In conclusion, this shows

Proposition 4.4.4 ([Mil]) *There is a one-to-one correspondence between the set of pairs $(L,\ \imath)$ and the set of automorphic factors, where L is a line bundle on $\Gamma \setminus \mathfrak{H}$ and $\imath$ is an isomorphism $\mathfrak{H} \times \mathbb{C} \overset{\imath}{\cong} \pi^*(L)$.*

The following result is also fundamental.

Proposition 4.4.5 *([Gun]) A line bundle L on a Riemann surface X is trivial, meaning that it is isomorphic to $X \times \mathbb{C}$, if and only if it has a no-where vanishing section*

This proposition also implies that every line bundle on $\mathfrak{H}$ is trivial, and hence isomorphic to $\mathfrak{H} \times \mathbb{C}$. Consequently, we have a classification of the line bundles on $\Gamma \setminus \mathfrak{H}$.

We return to the case of line bundles on $\Gamma \setminus \mathfrak{H}$ and show how modular forms arise as sections of the line bundle $\Gamma \setminus \mathfrak{H} \times \mathbb{C}$.

Let L be a line bundle on $\Gamma \setminus \mathfrak{H}$. Then the group of sections $S(\Gamma \setminus \mathfrak{H}, L)$ on $\Gamma \setminus \mathfrak{H}$ is

$$\{F \in S(\mathfrak{H}, \pi^*(L)) |\ F \text{ commutes with the action of } \Gamma \text{ on } \pi^*(L)\}\ .$$

Therefore, if F is a section of $\mathfrak{H} \times \mathbb{C}$ over $\mathfrak{H}$, then F commutes with the action of Γ on $\mathfrak{H} \times \mathbb{C}$ defined above. This section can be given by

$$\begin{array}{rccc} F: & \mathfrak{H} & \longrightarrow & \mathfrak{H} \times \mathbb{C} \\ & z & \longmapsto & (z,\ f(z)), \end{array}$$

where $f : \mathfrak{H} \longrightarrow \mathbb{C}$ is meromorphic. The commutativity of F with the action of Γ implies that

$$f(\gamma \cdot z) = \mu_\gamma(z) f(z).$$

In particular, if $\mu_\gamma(z) = (cz + d)^2$, then f is a weight 2 meromorphic modular form on Γ.

The line bundle $\Gamma \setminus \mathfrak{H} \times \mathbb{C}$ can be extended to a line bundle on the quotient of the extended half-plane $\mathfrak{H}^*$ (obtained by adding the set of inequivalent cusps).

Theorem 4.4.6 ([Mil]) *There is a one-to-one correspondence between sections of the line bundle $\Gamma \setminus \mathfrak{H}^* \times \mathbb{C}$ over $\Gamma \setminus \mathfrak{H}^*$ (for the automorphic factor $\mu_\gamma(z) = (cz+d)^2$) and weight 2 modular forms for Γ.*

Now, we have seen in Theorem 4.2.2 that every equivariant form has the form $h(z) = z + 1/(\widetilde{E}_2(z) + f_h(z))$ for some weight 2 modular form. Therefore, the map $F_h(z) = (z, f_h(z)) \mapsto h$, where F_h is a section of the line bundle $\mathfrak{H} \times \mathbb{C}$ over $\mathfrak{H}$, yields

Corollary 4.4.7 *The space of non-trivial equivariant forms $\mathcal{E}^*(\Gamma)$ is in a one-to-one correspondence with meromorphic sections of the line bundle $\Gamma \setminus \mathfrak{H}^* \times \mathbb{C}$ over $\Gamma \setminus \mathfrak{H}^*$.*

4.4.3 Equivariant forms as sections of a line bundle: A quasi-modular forms point of view

We use in this subsection the results of [Lee] to obtain a representation of equivariant forms as sections a line bundle of a similar type.

Let f be a quasimodular form on Γ. Then, by definition, it satisfies the transformation

$$f(z)|_k[\gamma] = (cz+d)^{-k} f(\gamma \cdot z) = \sum_{i=0}^{p} f_i(z) \frac{c^i}{(cz+d))^i} ,$$

where each f_i is a meromorphic function of $\mathfrak{H}$. The quasimodular polynomial is defined by

$$F_k^p(z,\ X) := \sum_{i=0}^{p} f_i(z) X^i.$$

Denote by $\mathfrak{Q}_k^p(\Gamma)$ the space of quasimodular polynomials of weight k and degree at most p. From this definition, it follows that there is a one-to-one correspondence between quasimodular forms and quasimodular polynomials.

We now describe an action of Γ on $\mathfrak{H} \times \mathbb{C}_p[X]$, where $\mathbb{C}_p[X]$ denotes the space of polynomials of degree at most p and complex coefficients. Let $\mathbb{F}$ denote the field of meromorphic functions on $\mathfrak{H}$, and let $\mathbb{F}_p[X]$ be the complex vector space of polynomials in X and degree at most p, $p \geq 0$. Define an action of $\mathrm{SL}_2(\mathbb{R})$ on this vector space by

$$F(z, X)\|_l := j_\gamma(z)^{-l} F\left(\gamma \cdot z, j_\gamma(z)^2 (X - \Re(\gamma, z))\right) , \tag{4.4.1}$$

where $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R})$, $F(z, X) \in \mathbb{F}_k[X]$, $l \in \mathbb{N}$, $\Re(\gamma, z) = c/j_\gamma(z)$. Then this particularly implies that quasimodular polynomials of degree at most p and weight l are invariant under this action of Γ. Fix a nonnegative integer p. Then given a polynomial in $\mathbb{C}_p[X]$ of the form

$$F(X) = \sum_{j=0}^{p} c_j X^j$$

and $l \leq p$, we define a left action of $\mathrm{SL}_2(\mathbb{R})$ on $\mathfrak{H} \times \mathbb{C}_p[X]$ by

$$\gamma \bullet_p^l (z, F(X)) = \left(\gamma \cdot z, \sum_{r=0}^{p} \sum_{j=r}^{p} c_j \Phi_r^{l,j}(\gamma, z) X^r\right) , \tag{4.4.2}$$

where $\Phi_r^{l,j}$ $\mathrm{SL}_2(\mathbb{R}) \times \mathfrak{H} \longrightarrow \mathbb{C}$ is given by

$$\Phi_r^{l,j}(\gamma, z) = \binom{j}{r} j_\gamma(z)^{l-2r} \Re(\gamma, z)^{j-r} .$$

We now show how this action is used to define a vector bundle structure on $\Gamma \backslash \mathfrak{H}$.

Let ν_l^k denote the quotient of $\mathfrak{H} \times \mathbb{C}_k[X]$ under this action. Then the projection map $p : \mathfrak{H} \times \mathbb{C}_k[X] \longrightarrow \mathfrak{H}$ induces a surjective map $\sigma : \nu_l^k \longrightarrow \Gamma \backslash \mathfrak{H}$ such that locally we have $\sigma^{-1}(z) \cong \mathbb{C}_k[X]$ for each $z \in \Gamma \backslash \mathfrak{H}$. This implies that ν_l^k has a complex vector bundle structure over $\Gamma \backslash \mathfrak{H}$ whose fiber is the $(k+1)$-dimensional complex vector space $\mathbb{C}_k[X]$.

In conclusion, this shows

Theorem 4.4.8 ([Lee]) *The space of quasimodular forms of weight k and depth $\leq p$ is canonically isomorphic to the space of holomorphic sections of the vector bundle ν_l^p over $\Gamma \backslash \mathfrak{H}$.*

Remarks 4.4.1

- If $p = 0$, in which case the space of quasimodular forms is exactly that of weight 2 modular forms on Γ, then this theorem reduces to the correspondence obtained earlier in the above subsection.
- Lee proved this theorem only for the case of holomorphic quasimodular forms. His proof could, however, be easily generalized to include meromorphic quasimodular forms since the holomorphy was used only in the definition of a quasimodular form. Also, his proof is true is we extend the vector bundle ν_l^p to a vector bundle on the extended quotient $\Gamma \setminus \mathfrak{H}^*$.

A consequence, we have

Corollary 4.4.9 *The space $\mathcal{E}(\Gamma)$, which is in a one-to-one correspondence with the set of quasimodular polynomials of weight 2 and degree exactly 1 and leading coefficient 1, can be represented as sections of the line bundle $\Gamma \setminus \mathfrak{H}^* \times \mathbb{C}$ over $\Gamma \setminus \mathfrak{H}^*$.*

In this corollary, we used the fact that the set of quasimodular polynomials of weight 2 and degree exactly 1 is isomorphic to $\mathcal{E}^*(\Gamma)$. Recall, that any element $h \in \mathcal{E}(\Gamma)$ satisfies

$$((\widehat{h}(z))^{-1})|_2[\alpha] = (\widehat{h}(z))^{-1} + \frac{c}{j_\alpha(z)} ,$$

for $\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$ and $\widehat{h}(z) = h(z) - z$, hence is a weight 2 depth 1 quasimodular form producing a quasimodular polynomial of weight 2 and degree 1.

Remark 4.4.1 The action defined in the first subsection is different from that of [Lee] in which case the action leave invariant the quasimodular polynomials. Moreover, it gives a direct representation of equivariant forms.

Chapter 5

Applications

The previous chapter shows that equivariant forms are connected with different algebraic-geometric objects that arise often in different contexts of mathematics. In Chapter 3 we proved that some of the relevant information of equivariant forms could be read from their residues. These developments prove that these special type functions are indeed very simple to deal with and could be used in different settings.

We give in this chapter some of their first few applications.

5.1 Zeros of the Eisenstein series E_2

The goal of this section is obtain information on the zeros of the Eisenstein series E_2 using the theory of equivariant forms. Precisely, we will make use of the fundamental equivariant form to obtain such information.

5.1.1 Motivation

Recall that the Eisenstein series are defined for every even integer $k \geq 2$ and $\tau \in \mathfrak{H}$ by

$$\begin{aligned} E_k(\tau) &= 1 - \tfrac{2k}{B_k} \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n \\ &= 1 - \tfrac{2k}{B_k} \sum_{n=1}^{\infty} \frac{n^{k-1} q^n}{1-q^n}, \qquad q = e^{2\pi i \tau}, \end{aligned} \tag{5.1.1}$$

where B_k is the k-th Bernoulli number and $\sigma_k(n) = \sum_{d|n} d^k$.

As was shown earlier these series play an important role in the theory of modular forms and quasi-modular forms and have been the topic of extensive investigation for a long time from various points of view. For instance, from the analytic point of view, the study of the zeros of $E_k(z)$, $k \geq 4$, has been carried out by several authors. In 1963, K. Wohlfahrt proved in [Woh] that the zeros of E_k , $4 \leq k \leq 26$, are simple and lie in the arc of the unit circle $\{z = e^{i\theta} : \pi/2 \leq \theta \leq 3\pi/2\}$ in the fundamental domain $\mathfrak{F} = \{\tau \in \mathfrak{H},\ |\tau| \geq 1 \text{ and } |\mathrm{Re}(\tau)| \leq 1/2\}$ of the modular group $\mathrm{SL}_2(\mathbb{Z})$. He also conjectured that this holds for all $k \geq 4$. In 1970, F.K.C. Rankin and H.P.F. Swinnerton-Dyer [Rf-Sd] proved Wohlfahrt's conjecture. In 1982, R.A. Rankin [Ran1] generalized their result to a certain class of Poincaré series. However, nothing has been proven for the Eisenstein series E_2 which is important in many fields. For example, it is shown in [Zag] that the Eisenstein series E_2 is a key master in the theory of Painlevé equations as it is used to describe all solutions of the so-called Chazy equation

$$f''' - ff'' + \frac{3}{2} f'^2 = 0 \, .$$

In fact, even whether it has a finite or an infinite number of zeros has not been known.

In the next subsection, we use properties of equivariant forms to prove that there are infinitely many non-equivalent zeros of E_2 in $\mathfrak{H}$. In fact, since E_2 is not exactly a modular form but rather a quasimodular form, we will see that two zeros τ_0 and

τ_1 of E_2 are $\mathrm{SL}_2(\mathbb{Z})$-equivalent, that is $\tau_1 = \gamma \cdot \tau_0$ for $\gamma \in \mathrm{SL}_2(\mathbb{Z})$, if and only if $\tau_1 = \tau_0 + n$ for an integer n. Thus, we restrict our investigation to the half-strip $\mathfrak{S} = \{\tau \in \mathfrak{H},\ -\frac{1}{2} < \mathrm{Re}(\tau) \leq \frac{1}{2}\}$ in which we prove that there are infinitely many zeros for E_2. Moreover, these zeros present an interesting distribution in $\mathfrak{S}$. More precisely, the fundamental domain $\mathfrak{F}$ and infinitely many of its conjugates within $\mathfrak{S}$ contain no zero of E_2, while there are infinitely many conjugates of $\mathfrak{F}$ which contain zeros of E_2.

5.1.2 Zeros of the Eisenstein series E_2

In this subsection we prove, using the fundamental equivariant form h_0, that the series E_2 has infinitely many zeros; a fact that has not been known before. Recall that, by Proposition 3.4.1, the poles of h_0 are located at the zeros of E_2 and conversely.

We will first establish some results needed for the proof of the main result. Also, to make this subsection self-contained, we recall the main properties of E_2 to be used. Set as earlier $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $S_n = \begin{pmatrix} 1 & 0 \\ n & 1 \end{pmatrix}$ for positive integers n.

We have seen that the Eisenstein series E_2 transforms under the action of the modular group as follows (see [Ran]). For $\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$, we have

$$E_2(\alpha \cdot \tau) = (c\tau + d)^2 E_2(\tau) + \frac{6c}{\pi i}(c\tau + d), \qquad (5.1.2)$$

where $\alpha \cdot \tau = \dfrac{a\tau + b}{c\tau + d}$.

It also appears in Ramanujan's differential relations

$$\frac{1}{2\pi i}\frac{dE_2}{d\tau} = \frac{1}{12}(E_2^2 - E_4)\ , \qquad (5.1.3)$$

$$\frac{1}{2\pi i}\frac{dE_4}{d\tau} = \frac{1}{3}(E_2 E_4 - E_6)\ , \qquad (5.1.4)$$

$$\frac{1}{2\pi i}\frac{dE_6}{d\tau} = \frac{1}{2}(E_2 E_6 - E_4^2)\ . \qquad (5.1.5)$$

Thus the graded ring $\mathbb{C}[E_2,\ E_4,\ E_6]$ is closed under the differential operator $\frac{d}{d\tau}$. We have seen in the first chapter that the space of all modular forms on $\mathrm{SL}_2(\mathbb{Z})$ is exactly the graded ring $\mathbb{C}[E_4,\ E_6]$.

We shall at this stage give some special values of E_2 at i and at the cubic root of unity $\rho = \frac{-1+i\sqrt{3}}{2}$:

$$E_2(i) = \frac{3}{\pi} \tag{5.1.6}$$

$$E_2(\rho) = \frac{2\sqrt{3}}{\pi}\ . \tag{5.1.7}$$

This follows from the transformation formula for E_2 together with the appropriate transformations that fix i and ρ.

Proposition 5.1.1 *The Eisenstein series E_2 has a unique zero τ_0 on the imaginary axis and a zero τ_1 on the axis $Re(z) = \frac{1}{2}$.*

Proof: It is clear that for $\tau = iy$, the series $E_2(\tau)$ is real and increasing on $(0,\ \infty)$. Meanwhile, $\lim_{y\to 0} E_2(iy) = -\infty$ and $\lim_{y\to\infty} E_2(iy) = 1$. It follows that E_2 has a unique zero, say τ_0 on the purely imaginary axis.

Similarly, $E_2(\tau)$ is real for $\tau = 1/2\ +\ iy,\ y > 0$. Furthermore, $\lim_{y\mapsto 0} E_2\left(\frac{1}{2}+iy\right) = -\infty$. Indeed, for $\alpha = S_2^{-1} = \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}$ we have

$$E_2\left(\frac{1}{2}+iy\right) = -\frac{1}{4y^2}\left(E_2\left(-\frac{1}{2}+\frac{i}{4y}\right) - \frac{24y}{\pi}\right). \tag{5.1.8}$$

This gives the desired limit since $E_2\left(-\frac{1}{2}+\frac{i}{4y}\right)$ tends to 1 as y tends to 0. Combining this with the fact that $E_2(\rho) = E_2(\rho+1) = \frac{2\sqrt{3}}{\pi}$ yields the existence of a zero τ_1 of real part $1/2$ and whose imaginary part is less than $\sqrt{3}/2$. Here again we used the transformation formula in (5.1.2) with $\alpha = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. The uniqueness of τ_0 follows from the strict monotony of E_2 on the imaginary axis. ∎

As for the location of these two zeros, and taking into account the special value of E_2 at $\imath$ and ρ given respectively by (5.1.6) and (5.1.7), we have

Proposition 5.1.2 *The zeros τ_0 and τ_1 are contained respectively in the fundamental domains $S\mathfrak{F}$, $S_2\mathfrak{F}$.*

It is worth mentioning that these two zeros appear in [FaSe] where they are studied as equilibrium points of Green's functions. In fact, $\tau_0 = 0.5235217000...\imath$ and $\tau_1 = 0.5 + 0.1309384864...\imath$.

Unlike the case of modular forms, the set of zeros of E_2 is not invariant under every conjugation by elements of $\mathrm{SL}_2(\mathbb{Z})$. In fact we have

Proposition 5.1.3 *Two zeros of E_2 are equivalent if and only if one is a translate of the other by an integer*

Proof: Suppose that z_1, z_2 are any two zeros of E_2 in the half plane $\mathfrak{H}$ that are equivalent modulo $\mathrm{SL}_2(\mathbb{Z})$. Say, $z_1 = \alpha \cdot z_2$, $\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Then z_2 and $z_1 = \alpha \cdot z_2$ are both poles of $h_0 = z + 6/\pi\imath E_2(z)$, the fundamental equivariant form. Since h_0 is equivariant, and in order to have $\alpha \cdot h(z)$ infinite at z_2, we must have $c = 0$ and thus $ad = 1$. Therefore $a = d = \pm 1$ and α acts as a translation. ∎

As a consequence we have

Corollary 5.1.4 *No two distinct zeros of E_2 in the half-strip $\mathfrak{S}$ are equivalent modulo the modular group $SL_2(\mathbb{Z})$.*

Lemma 5.1.5 *The fundamental equivariant form carries $SL_2(\mathbb{Z})$-conjugates of any zero z_0 of E_2 onto $\mathbb{Q} \cup \{\infty\}$. Also, suppose that $h_0(z_1) = -d/c$, for some $z_1 \in \mathfrak{H}$ and $d/c \in \mathbb{Q}$ in a reduced form. Then there is $(a,\ b) \in \mathbb{Z}^2$ such that $\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$ and $E_2(\alpha \cdot z_1) = 0$.*

Proof: If z_0 is a zero of E_2 and $\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$, $c \neq 0$, then the fundamental equivariant form h_0 maps $\alpha \cdot z_0$ to the rational a/c. Moreover, let z_1 be as in the statement of this lemma, and choose a, $b \in \mathbb{Z}$ such that $\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$. Then, since h_0 is an equivariant form for $\mathrm{SL}_2(\mathbb{Z})$, we see that $\alpha \cdot h_0(z_1) = h_0(\alpha \cdot z_1) = \infty$; that is to say that $\alpha \cdot z_1$ is a pole of h_0. The lemma now follows from the above discussion. ∎

We will now use this property of h_0 to prove the main results of this subsection.

Theorem 5.1.6 *The Eisenstein series E_2 has infinitely many zeros in the half-strip $\mathfrak{S} = \{\tau \in \mathfrak{H},\ -\frac{1}{2} < Re(\tau) \leq \frac{1}{2}\}$.*

Proof: Let τ_0 be the unique zero of E_2 on the imaginary axis. Let $\alpha = \begin{pmatrix} t & u \\ v & w \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$ with $tv \neq 0$.

Then, by Lemma 5.1.5, we know that the fundamental equivariant form h_0 carries $\alpha \cdot \tau_0$ onto $r_0 = t/v$, and thus it maps any open neighborhood D_0 of $\alpha \cdot \tau_0$, which we choose in the interior of the fundamental domain $\alpha S\mathfrak{F}$ and on which it is holomorphic, onto an open neighborhood U_0 of r_0. Let $r_1 = b_1/a_1$ be a reduced fraction in $\mathbb{Q} \cap U_0 \setminus \{r_0\}$. Then there exists $z_1 \in D_0 \setminus \{\alpha \cdot \tau_0\}$ such that $h_0(z_1) = b_1/a_1$. Therefore, by Lemma 5.1.5, we can choose c_1, $d_1 \in \mathbb{Z}$ such that $b_1 d_1 - a_1 c_1 = 1$. Then $\gamma_1 := \begin{pmatrix} d_1 & -c_1 \\ -a_1 & b_1 \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$ and $E_2(\tau_1) = 0$, where we have set $\tau_1 = \gamma_1 \cdot z_1$. Moreover, τ_1 is not equivalent to τ_0 modulo $\mathrm{SL}_2(\mathbb{Z})$, otherwise we would have according to Proposition 5.1.3 that $\tau_0 := T^n \gamma_1 \cdot z_1$ for some $n \in \mathbb{Z}$. Since $z_1 \in \alpha S\mathfrak{F}$, write $z_1 = \alpha \cdot z_1'$ for some $z_1' \in S\mathfrak{F}$. We have $\tau_0 = T^n \gamma_1 \alpha \cdot z_1'$ with τ_0 and z_1' being in the fundamental domain $S\mathfrak{F}$. Therefore, $T^n \gamma_1 \alpha = 1$, and hence $\tau_0 = z_1'$, and $\alpha \cdot \tau_0 = z_1$; a contradiction since we have chosen $z_1 \in D_0 \setminus \{\alpha \tau_0\}$. Thus τ_1 is a zero of E_2 that is not equivalent to τ_0.

It remains to show that two distinct rational numbers lead to two distinct zeros of E_2. Let $r_2 = b_2/a_2$ be a rational number in $U_0 \setminus \{r_0,\ r_1\}$. In the same way we construct a zero of E_2, $\tau_2 = \gamma_2 \cdot z_2$, that is not equivalent to τ_0 modulo $\mathrm{SL}_2(\mathbb{Z})$, with $z_2 \in \alpha S\mathfrak{F}$. Then τ_2 is not equivalent to τ_1 modulo $\mathrm{SL}_2(\mathbb{Z})$. Indeed if $\tau_1 = T^m \cdot \tau_2$ for some $m \in \mathbb{Z}$, then $\gamma_1\alpha \cdot z_1' = T^m\gamma_2\alpha \cdot z_2'$ with z_1' and z_2' being in the same fundamental domain $S\mathfrak{F}$. It follows that $\gamma_1\alpha = T^m\gamma_2\alpha$, and consequently $r_1 = r_2$. This contradicts our choice of r_2. Hence, τ_2 is another zero of E_2 that is not equivalent to neither τ_0 nor τ_1. Finally, since the open set U_0 contains infinitely many rational numbers, we deduce that E_2 has infinitely many zeros in the half strip $\mathfrak{S}$. ∎

Again using the equivariance of h_0 for $\mathrm{SL}_2(\mathbb{Z})$, we have the immediate consequence

Corollary 5.1.7 *The function h_0 has infinitely many zeros in $\mathfrak{H}$ that are not equivalent modulo $SL_2(\mathbb{Z})$.*

Proof: Let z_1 be any zero of E_2 in the half-strip $\mathfrak{S}$. Then, composing by the element $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$, we get $h_0(S \cdot z_1) = 0$, so that $S \cdot z_1$ is a zero of h_0. Further, as all the zeros of E_2 in $\mathfrak{S}$ are not equivalent modulo $\mathrm{SL}_2(\mathbb{Z})$, then so are their images by S. ∎

Another important corollary of this theorem is the following.

Corollary 5.1.8 *The discriminant Δ has infinitely many critical points.*

Recall that E_2 is the logarithmic derivative of the discriminant Δ, and that the modular discriminant does not vanish on $\mathfrak{H}$.

5.1.3 Distribution of the zeros of E_2

In this section, we will show that there are infinitely many fundamental regions within the half strip $\mathfrak{S}$ that contain zeros of E_2 using again the equivariant form h_0.

Theorem 5.1.9 *There is a positive integer c_0 such that for all integers $c \geq c_0$, there is a fundamental domain with a vertex at $1/c$ containing a zero of E_2.*

Proof: Let τ_0 denote again the unique zero of E_2 on the imaginary axis, and let $\alpha = \begin{pmatrix} t & u \\ v & w \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$, so that $tv \neq 0$. As in the proof of Theorem 5.1.6 the function h_0 maps any neighborhood of $\alpha \cdot \tau_0$ onto a neighborhood of t/v. In particular, h_0 maps a neighborhood D_0 of $S_1\tau_0$, chosen to be in the interior of $S_1S\mathfrak{F}$, onto a neighborhood U_0 of 1 (recall that $S_1 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$). There exists a positive integer c_0 such that for all $c \geq c_0$, $1 + 1/c \in U_0$. For each $c \geq c_0$, let $z_c \in D_0$ be such that $h_0(z_c) = 1 + 1/c$. Therefore, if $\gamma_c = \begin{pmatrix} -1 & 1 \\ -1-c & c \end{pmatrix}$, then, as in the proof of Theorem 5.1.6, ${\gamma_c}^{-1} \cdot z_c$ is a zero E_2 belonging to ${\gamma_c}^{-1}S_1S\mathfrak{F}$. If we set $S_c = {\gamma_c}^{-1}S_1S = \begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix}$ for $c \geq c_0$, then we have constructed a zero of E_2 in the fundamental domain $S_c\mathfrak{F}$ which has a vertex at the cusp $1/c$. ∎

Remark 5.1.1

- Thanks to Proposition 5.1.3, the above theorem can be extended to include the cusps 0 and 1/2.
- An immediate consequence of this theorem is again the infiniteness of the number of zeros of the Eisenstein series E_2. Furthermore, it follows from Corollary 5.1.4 that all these zeros are inequivalent modulo $\mathrm{SL}_2(\mathbb{Z})$ as all these fundamental domains are contained in the half-strip $\mathfrak{S}$.

5.2 More on the zeros of E_2

In this section we give more properties of the zeros of the Eisenstein series E_2. More precisely we study the multiplicity and focus on the fundamental domains that contain no zeros of E_2.

5.2.1 Multiplicity of the zeros of E_2

For the multiplicity of the zeros of E_2, using the Ramanujan differential relation (5.1.3) for E_2, we have

Theorem 5.2.1 *The zeros of the Eisenstein series E_2 are all simple.*

Proof: Let z_0 be a zero of E_2. By (5.1.3), we have

$$\frac{1}{2\pi i}\frac{dE_2(z_0)}{d\tau} = \frac{1}{12}(E_2(z_0)^2 - E_4(z_0)) = \frac{-1}{12}E_4(z_0).$$

Therefore, to prove that this zero is simple, it suffices to show that $E_4(z_0) \neq 0$. It is known that E_4 has all its zeros at $\rho = \dfrac{-1+i\sqrt{3}}{2}$ and its conjugates modulo $\mathrm{SL}_2(\mathbb{Z})$ (see for instance [Ran]). Thus, it is enough to show that $E_2(\alpha \cdot \rho) \neq 0$ for all $\alpha \in \mathrm{SL}_2(\mathbb{Z})$. Using (5.1.2) and (5.1.7), we have for $\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$:

$$E_2(\alpha \cdot \rho) = (c\rho + d)^2 \frac{2\sqrt{3}}{\pi} + \frac{6c}{\pi i}(c\rho + d) = \frac{2\sqrt{3}}{\pi}(c^2 - cd + d^2)\ ,$$

which does not vanish unless $c = d = 0$ which is not the case since $ad - bc = 1$. This shows that E_2 does not vanish on the orbit of ρ and consequently E_4 and E_2 have no common zeros. ∎

5.2.2 More on the distribution of the zeros of E_2

In this subsection we investigate fundamental domains that contain no zeros of E_2.

We start by studying the existence of a zero of E_2 in the fundamental domain $\mathfrak{F}$ of the modular group $\mathrm{SL}_2(\mathbb{Z})$.

Proposition 5.2.2 *The Eisenstein series E_2 has no zeros in the fundamental domain $\mathfrak{F}$ of $SL_2(\mathbb{Z})$.*

Proof: Let $\tau_0 = iy_0$ be the unique zero of E_2 on the imaginary axis. Using the transformation formula (1.4.1) for E_2, we have

$$0 < E_2(-1/\tau_0) = \frac{6}{\pi} y_0 < 1.$$

This follows from the fact that $\mathrm{Im}(\tau_0) < 1$ (since $\tau_0 \in S\mathfrak{F}$) and thus $\mathrm{Im}(-1/\tau_0) > \mathrm{Im}(\tau_0)$, and the fact that E_2 is strictly increasing on the imaginary axis with the value 0 at τ_0 and the value 1 at $i\infty$. Therefore,

$$y_0 < \frac{\pi}{6}. \tag{5.2.1}$$

If $\tau = x + iy \in \mathfrak{F}$ is a zero of E_2, then $\mathrm{Im}(\tau) > \sqrt{3}/2 > \pi/6 > y_0$ and therefore

$$\frac{1}{24}|1 - E_2(\tau)| = \left| \sum_{n=1}^{\infty} \sigma_1(n) e^{2\pi i n \tau} \right| \leq \sum_{n=1}^{\infty} \sigma_1(n) e^{-2\pi n y} < \sum_{n=1}^{\infty} \sigma_1(n) e^{-2\pi n y_0}.$$

The latter sum is simply $1/24(1 - E_2(\tau_0)) = 1/24$. Therefore

$$\frac{1}{24}|1 - E_2(\tau)| < \frac{1}{24}.$$

Hence $E_2(\tau)$ cannot be 0 if $\tau \in \mathfrak{F}$. ∎

In the above proof, we have used the inequality $\sqrt{3}/2 > \pi/6$ which is obvious numerically, but is a consequence of a simpler inequality such as $\pi < 4$. In what follows we will rely on another inequality which is also numerically obvious:

$$e^{-\pi\sqrt{3}} < \frac{1}{200}. \tag{5.2.2}$$

It simply says that $0.00433 < 0.005$.

We will now investigate more fundamental domains that do not contain any zeros of E_2. For a fixed integer $c \geq 2$ we set again $S_c = \begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix}$ and $S_{b,d}(c) = \begin{pmatrix} 1 & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$, $b,\ d \in \mathbb{Z}$ and $\delta_b = \begin{pmatrix} 0 & -1 \\ 1 & b \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$, $b \in \mathbb{Z}$. The fundamental domain $S_{b,d}(c)\mathfrak{F}$ has a vertex at the cusp $1/c$ as does $S_c\mathfrak{F}$. Also $\delta_b\mathfrak{F}$ has a vertex at the cusp 0 as does $S\mathfrak{F}$.

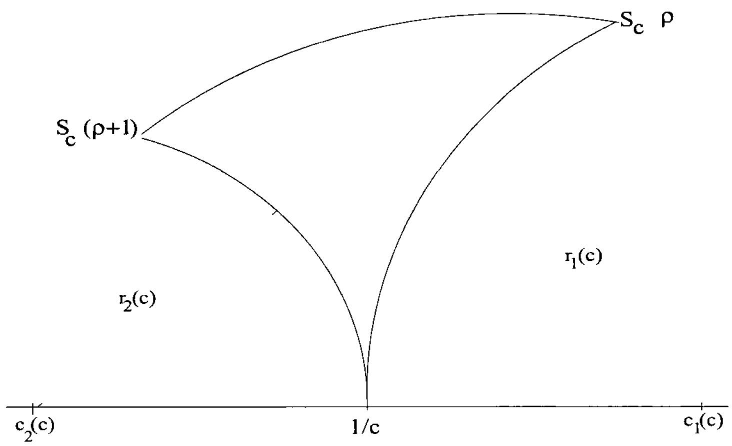

Figure 5 1 Fundamental domain at the cusp $1/c$

Let us examine more closely the fundamental domain $S_c\mathfrak{F}$ Its vertices are

$$\frac{1}{c}, \quad S_c \quad \rho = \frac{c - \frac{1}{2} + i\frac{\sqrt{3}}{2}}{c^2 - c + 1}, \quad S_c \quad (\rho + 1) = \frac{c + \frac{1}{2} + i\frac{\sqrt{3}}{2}}{c^2 + c + 1}$$

It is clear that $\operatorname{Im}(S_c \quad \rho) > \operatorname{Im} S_c \quad (\rho + 1)$ and $\operatorname{Re} S_c \quad \rho > 1/c > \operatorname{Re} S_c \quad (\rho + 1)$ Thus we have the following situation for the fundamental region $S_c\mathfrak{F}$ (see Figure 1)

The edge joining $1/c$ and $S_c \quad \rho$ is an arc of the circle $\mathcal{C}_1(c)$ centered at $c_1(c) = (c-1)/c(c-2)$ and having radius $r_1(c) = 1/c(c-2)$, while the edge joining $1/c$ and $S_c \quad (\rho + 1)$ is an arc of the circle $\mathcal{C}_2(c)$ centered at $c_2(c) = (c+1)/c(c+2)$ with radius $r_2(c) = 1/c(c+2)$ In particular, any other fundamental domain having the cusp $1/c$ as a vertex is either within the circle $\mathcal{C}_1(c)$ or within the circle $\mathcal{C}_2(c)$

The case $c = 2$ needs to be clarified as the radius $r_1(2)$ is infinite and in this case the arc joining $1/2$ and $S_2 \quad \rho$ is the vertical segment $\left[1/2, \ 1/2 + i\sqrt{3}/6\right]$ (see Figure 2) Moreover, as we are restricting the study to the half-strip $\mathfrak{S}$, we only consider those fundamental domains with vertex at the cusp $1/2$ that lie under the arc of the circle $\mathcal{C}_2(2)$ It has center at $c_2(2) = 3/10$ and radius $r_2(2) = 1/10$

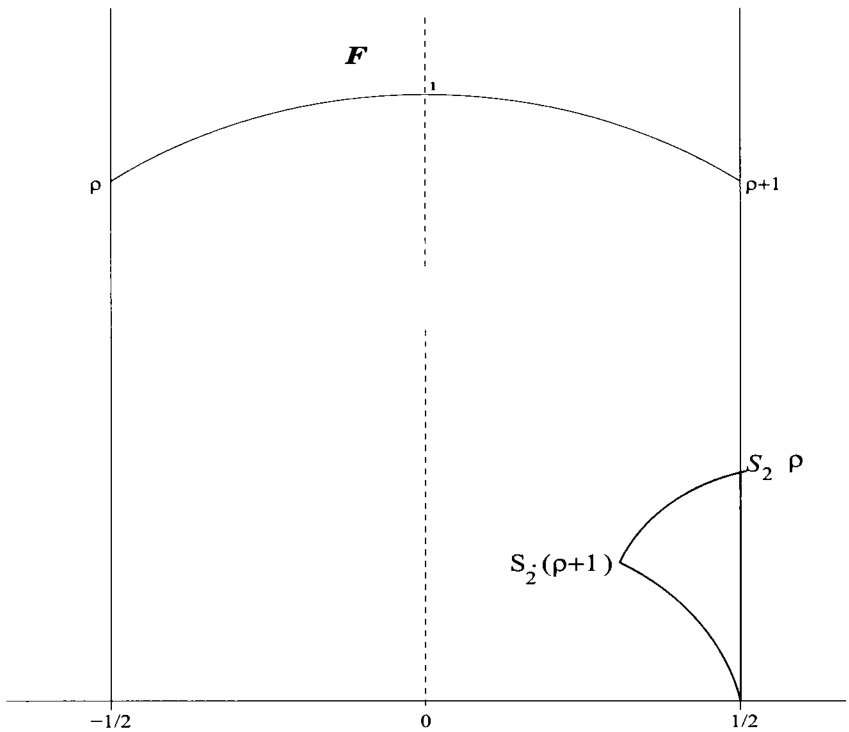

Figure 5.2: Fundamental domain at the cusp 1/2

Lemma 5.2.3 *If we set*

$$M = \frac{1}{24}\left(1 - E_2\left(\frac{\imath\sqrt{3}}{2}\right)\right),$$

then we have

$$24^2\left(M^2 + \frac{M}{\pi}\right) < 1 \qquad (5\ 2\ 3)$$

Proof: Set $q = exp(-\pi\sqrt{3})$ We have

$$0 < M = \sum_{n\geq 1}\sigma_1(n)\,q^n = \sum_{n\geq 1}\frac{nq^n}{1-q^n} \leq \frac{1}{1-q}\sum_{n\geq 1} nq^n = \frac{q}{(1-q)^3}$$

Hence, using (5 2 2), we have

$$M \leq \frac{40000}{7880599}$$

Therefore,

$$24^2\left(M^2 + \frac{M}{\pi}\right) < 24^2\left(M^2 + \frac{M}{3}\right) \leq \frac{61444600320000}{62103840598801} < 1$$

∎

In the following, we will prove that the only fundamental domains having a vertex at the cusp $1/c$ who might contain a zero of E_2 are the $\beta_c\mathfrak{F}$, and the only fundamental domain having a vertex at the cusp 0 that might contain a zero is $S\mathfrak{F}$

Theorem 5.2.4 *If $b \neq 0$, then E_2 has no zeros in $S_{b\,d}(c)\mathfrak{F}$ or in $\delta_b\mathfrak{F}$*

Proof: Suppose first that $c \geq 3$, and suppose there is a zero z_0 of E_2 in the fundamental domain $S_{b\,d}(c)\mathfrak{F}$ where $S_{b\,d}(c) = \begin{pmatrix} 1 & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$ If $b \neq 0$, then, according to the discussion preceding the above lemma, the fundamental domain $S_{b\,d}(c)\mathfrak{F}$ is either within the circle $\mathcal{C}_1(c)$ or $\mathcal{C}_2(c)$ We will show that in fact z_0 is outside the circles $\mathcal{C}_1(c)$ and $\mathcal{C}_2(c)$, which is a contradiction

We have

$$E_2(S_{b\,d}(c)^{-1}\ \ z_0) = \frac{-6c}{\pi\imath}(-cz_0 + 1),$$

so that

$$\begin{aligned}\sum_{n=1}^{\infty} \sigma_1(n) e^{2\pi n i S_{b\, d}(c)^{-1}\ z_0} &= \frac{1}{24} + \frac{c}{4\pi i}(-cz_0 + 1) \\ &= -\frac{c^2}{4\pi i}\left(z_0 - \left(\frac{1}{c} + \frac{\pi i}{6c^2}\right)\right), \qquad (5\ 2\ 4)\end{aligned}$$

Since $S_{b\, d}(c)^{-1}\ \ z_0 \in \mathfrak{F}$, we have

$$\operatorname{Im}\left(S_{b\, d}(c)^{-1}\ \ z_0\right) \geq \frac{\sqrt{3}}{2} \qquad (5\ 2\ 5)$$

Hence

$$\left|\sum_{n=1}^{\infty} \sigma_1(n) e^{2\pi n i S_{b\, d}^{-1}\ z_0}\right| \leq \sum_{n=1}^{\infty} \sigma_1(n) e^{-n\pi\sqrt{3}} = M \qquad (5\ 2\ 6)$$

Therefore

$$\left|z_0 - \left(\frac{1}{c} + \frac{\pi i}{6c^2}\right)\right| \leq M\frac{4\pi}{c^2}, \qquad (5\ 2\ 7)$$

that is z_0 belongs to the disk $\mathcal{D}_0(c)$ of center $c_0(c) = \dfrac{1}{c} + \dfrac{\pi i}{6c^2}$ and radius $r_0(c) = M\dfrac{4\pi}{c^2}$ We will now show that the disk $\mathcal{D}_0(c)$ lies outside the circles $\mathcal{C}_1(c)$ and $\mathcal{C}_2(c)$ by showing respectively that $|c_0(c) - c_1(c)| > r_1(c) + r_0(c)$ and that $|c_0(c) - c_2(c)| > r_2(c) + r_0(c)$ Because the cusp $1/c$ and $c_0(c)$ are on the same vertical axis, we have

$$|c_1(c) - c_0(c)|^2 = r_1(c)^2 + \left(\frac{\pi}{6c^2}\right)^2$$

Thus in order to prove that $|c_0(c) - c_1(c)| > r_0(c) + r_1(c)$ we only need to prove that

$$r_0(c)^2 + 2r_0(c)r_1(c) < \left(\frac{\pi}{6c^2}\right)^2$$

In other words,

$$2\pi M^2 + \frac{Mc}{c-2} < \frac{\pi}{288}$$

In the meantime, for $c \geq 4$, we have $c/(c-2) = 1 + 2/(c-2) \leq 2$ Thus it is enough to prove that $2\pi M^2 + 2M < \pi/288$, which is a consequence of Lemma 5 2 3

Similarly, we prove hat $|c_2 - c_0| > r_2 + r_0$. Indeed, as above, it is enough to show that

$$2\pi M^2 + \frac{Mc}{c+2} < \frac{\pi}{288},$$

which is a consequence of Lemma 5.2.3 since $c/(c+2) < 1$. Notice that $|c_2 - c_0| > r_2 + r_0$ is also valid for the cases $c = 2$ and $c = 3$. This proves the theorem for $c \geq 4$ and also for $c = 2$ since the circle $\mathcal{C}_1(c)$ is the vertical line $\operatorname{Re} z = 1/2$ and thus we only need to estimate the distance $|c_2 - c_0|$.

The case $c = 3$ involves different estimates since we cannot apply Lemma 5.2.3 for the above choice of M. As we noticed above z_0 is outside the circle $\mathcal{C}_2(3)$, and we only need to show that it is outside $\mathcal{C}_1(3)$. On the other hand, the fundamental domain $S_{-1,-2}(3)\mathfrak{F}$ is adjacent (on the right) to $S_3\mathfrak{F}$, see figure 3, and the disc $\mathcal{D}_0(3)$ is outside the circle $\mathcal{C}_3$ which joins the vertices $1/3$ and $S_{-1,-2}(3) \cdot \rho$. Indeed, this circle is centered at $8/21$ and has radius $1/21$. Moreover

$$|c_0(3) - 8/21| = \frac{\sqrt{324 + 49\pi^2}}{378} \approx 0.07518,$$

and

$$r_0(3) + \frac{1}{21} = \frac{4\pi M}{9} + \frac{1}{21} < \frac{4\pi}{9 \cdot 200} + \frac{1}{21} \approx 0.0546.$$

It follows that the only possible values of (b, d) for which $S_{b,d}\mathfrak{F}$ might contain a zero are $(b, d) = (-1, -2)$ leading to $S_{-1,-2}(3)\mathfrak{F}$ and $(b, d) = (0, 1)$ leading to $S(3)\mathfrak{F}$. We now show that $z_0 \notin S_{-1,-2}(3)\mathfrak{F}$ by exhibiting a smaller disc $\mathcal{D}(3)$ containing z_0 and lying outside the circle $\mathcal{C}_1(3)$ as the disc $\mathcal{D}_0(3)$ does not necessarily meet this condition. The transformation $S_{-1,-2}$ maps $\mathcal{D}_0(3)$ onto a disc $\mathcal{D}_0'(3)$ centered at

$$c_0'(3) = S_{-1,-2}(3)^{-1} \cdot c_0(3) \ = \frac{6i}{\pi} + \frac{2}{3}$$

and with radius $r_0'(3)$ that can easily be shown to satisfy $r_0'(3) < 0.26$. Therefore, we obtain a more precise lower bound to $\operatorname{Im} S_{-1,-2}(3) \cdot z_0$ as compared to (5.2.5):

$$\operatorname{Im} S_{-1,-2}(3) \cdot z_0 > \frac{6}{\pi} - 0.26.$$

We now replace M in Lemma 5.2.3 by

$$M' = \frac{1}{24}\left(1 - E_2\left(i(6/\pi - 0.26)\right)\right),$$

and obtain

$$2\pi M'^2 + 3M' < \pi/288.$$

Hence, as in the general case, we conclude that

$$\left| z_0 - \left(\frac{1}{3} + \frac{i\pi}{54}\right)\right| \leq M'\frac{4\pi}{9},$$

and therefore, the disc $D(3) = \mathcal{D}(1/3 + i\pi/54,\ 4\pi M'/9)$ is outside the circle $\mathcal{C}_1(3)$. It follows that there is no zero of E_2 in $S_{-1,-2}(3)\mathfrak{F}$ and thus in any $S_{b,d}(3)\mathfrak{F}$ for $b \neq 0$.

Finally, for the case of the cusp at 0, if z_0 is a zero of E_2 in $\delta_b\mathfrak{F}$, then z_0 is contained inside the circle centered at $\frac{\pi i}{6}$ and having radius $4M\pi$ which is clearly contained in $S\mathfrak{F}$. Therefore $b = 0$ since, otherwise, $\delta_b\mathfrak{F}$ and $S\mathfrak{F}$ would be disjoint. ∎

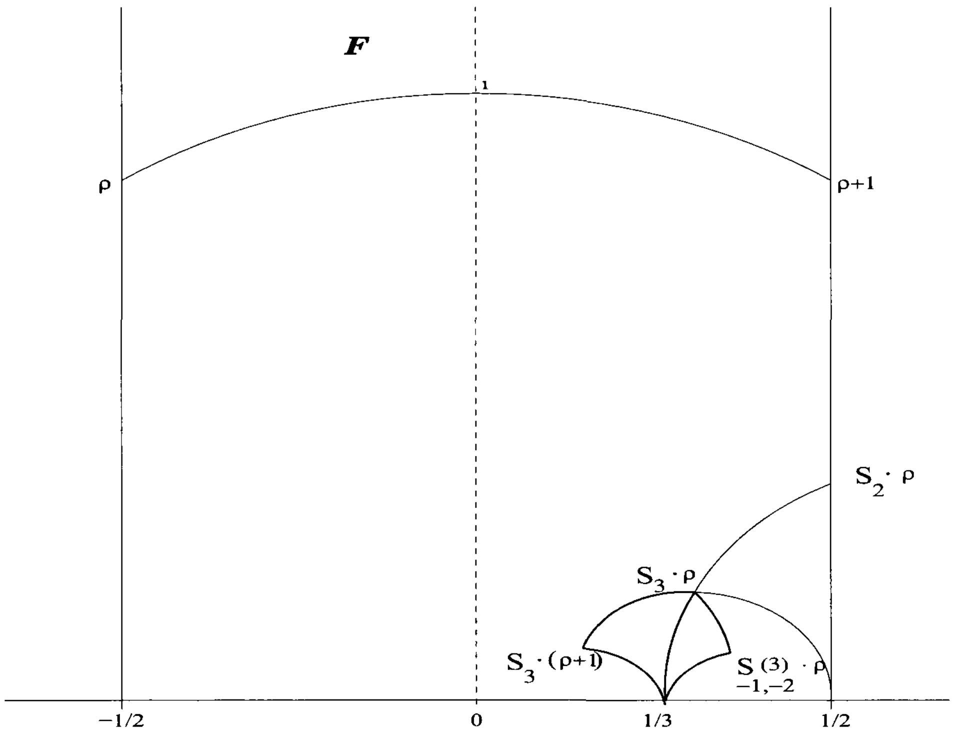

Figure 5.3: Fundamental domains at the cusp 1/3

5.3 Epilogue

We have seen that the equivariant forms present a rich mathematical structure and are closely related to various topics such as elliptic functions, modular forms, quasi-modular forms, differential forms and sections of line bundles and therefore have a real potential for applications and development. It is important to note that in various other fields of mathematics, there are similar equivariant objects which play an important role and which can be connected to the equivariant forms presented in this work. Here are few examples:

- In equivariant K-theory, there is a similar notion, namely, the notion of equivariant differential forms. They originate form the action of a compact Lie group G on a smooth manifold M and appear as global sections of the algebra over the sheaf of rings of invariant functions $\mathcal{C}_G^\infty$ for the action of G on the underlying manifold to G by conjugation, (see [BlGe] for more details).

- In projective differential geometry, there is the notion of concomitants that are maps between projective spaces commuting with the action of a group. We have seen that the equivariant forms mix very well with two fundamental tools from projective differential geometry, namely the cross-ratio and the Schwarz derivative. See [Per], p 45.

- In both functional analysis and representation theory [KnaSt], there is the notion of intertwining operators which are the object of extensive research in both fields. The fact that the upper half-plane $\mathfrak{H}$ is a homogeneous space, as it is the quotient of the real Lie group $\mathrm{SL}_2(\mathbb{R})$ with a compact subgroup of rotations, brings the equivariant forms as intertwining operators much closer to representation theory.

- In theoretical physics, in studying the so-called Nahm equations which resemble Ramanujan differential relations and Yang-Baxter equations, one introduces

[Sut] the notion of Jarvis functions which are equivariant under the action of a subgroup of $GL_2^+(\mathbb{Q})$ and for which our equivariant forms present concrete examples in some special contexts.

It is our hope that the theory of equivariant forms can play an important role in many of these topics, providing concrete examples and solutions to different problems in mathematics, in the same way the theory of modular forms has influenced several fields of mathematics.

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