VIBRATION
OF
LAMINATED ORTHOTROPIC
COMPOSITE PLATES AND SHELLS

A thesis submitted to
the School of Graduate Studies and Research
in partial fulfilment of the requirements for the degree of
Doctor of Philosophy in Mechanical Engineering

By

NONG LI

Department of Mechanical Engineering
University of Ottawa
Ottawa, Ontario, Canada

JUNE 1994

ISBN 0-612-00585-2

ABSTRACT

Almost all of the analytical solution techniques presented for composite plates and shells deal with either simply supported conditions or boundary conditions with at least a pair of opposite edges simply supported. In the present study, an alternative general approach, combining superposition and state space techniques is developed for the free vibration analysis of laminated orthotropic composite plates and shells having arbitrary boundary conditions.

This study concentrates on the antisymmetric angle-ply laminated plates and cross-ply laminated plates and shells. Three commonly adopted theories, i.e., classical theory, first-order shear deformation theory and third-order shear deformation theory, have been employed and compared with one another to investigate the influence of transverse shear deformation, structural aspect ratio, length-to-thickness ratio, degree of anisotropy and the number of layers on natural frequency.

Convergence tests have been carried out to guarantee the accuracy of the closed-form solutions. Wherever possible, numerical results generated by the present approach are compared with those reported in the published references.

Accurate non-dimensional fundamental frequencies are presented for laminated plates and shells with two adjacent edges, three edges and four edges clamped and other edges simply supported. Such analyses have not been reported in the literature previously. Also, vibration analysis of a cantilever angle-ply antisymmetric plate with a point support is conducted to demonstrate the applicability of the present technique.

It has been shown that the method works extremely well and excellent agreements are found between the present results and those generated by previous researchers. It has also been shown that more complicated boundary-value problems can be solved by this technique without any difficulty.

ACKNOWLEDGMENTS

The author would like to acknowledge the excellent assistance and guidance he has received from his supervisor, Dr. S. Mirza, who provided continuous encouragement, thoughtful suggestions and advice throughout this work.

I am thankful to the members of the examining committee, Dr. C. W. Bert, Dr. D. J. Gorman, Dr. J. Kirkhope and Dr. D. Redekop, for their valuable suggestions, which improved the quality of the thesis.

I wish to thank the Natural Science and Engineering Research Council of Canada for the financial support provided. Special thanks are expressed to Dr. D. J. Gorman for many helpful ideas before and during the study, and for providing an opportunity to join his research group prior to the program. Appreciation is also extended to the professors in the Department of Mechanical Engineering, University of Ottawa, for stimulating discussion in and outside the class rooms.

Finally, I would like to express my sincere gratitude to my wife, Erli Yin, for her love and thoughtful support. Without her, this work would not have been completed.

Contents

NOMENCLATURE

a, b	dimensions of plate or shell panel in x and y direction
A_1, A_2	Lamé parameters along ξ_1 and ξ_2 axes
A_{ij}, B_{ij}, D_{ij}	extensional, coupling and bending stiffnesses
c_1	$=4/h^2$
c_2	$=c_1/3$
[C]	coefficient matrix
d_{xy}	operator $=\partial^2/\partial x\partial y$
$(\ )_{,x}$	operator $=\partial(\ )/\partial x$
e_i $(i=1,\ldots,6)$	strain components
E_i	Young's modulus in i-direction
G_{ij}	shear modulus in i-j plane
h	thickness of laminate
$I_i,\bar{I}_i,\hat{I}_i$ $(i=1,\ldots,7)$	inertia coefficients
L	Lagrangian function
L_1, L_2, L_3	general Lamé parameters
L_{ij} $(i,j=1,\ldots,5)$	partial differential operators
k_1, k_2, k_6	middle surface curvatures
K_{ij}^2 $(i,j=4,5)$	transverse shear correction coefficients

k	number of terms used in the truncated displacement series
K	kinetic energy
K_m	$=m\pi/a$
K_n	$=n\pi/b$
m, n	series terms
M_1, M_2, M_6	moment resultants
$[M]$	eigenvalue matrix
$\|M\|$	determinant of the matrix $[M]$
N_1, N_2, N_6	in-plane force resultants
N	number of layers
$\{P\}$	column matrix of constants
Q_1, Q_2	shear force resultants
r_{ij}	eigenvectors associated with λ_i
R_1, R_2	principal radii of curvature
$[R]$	eigenvector matrix $=[r_{ij}]$
S_{ij} $(i,j=1,2,4,5,6)$	reduced stiffinesses
t	time
$Q_{ij}^{(k)}$ $(i,j=1,2,4,5,6)$	transformed reduced stiffinesses of the κth lamina
u, v, w	displacement components of middle surface
U	strain energy
U_1, U_2, U_3	displacement field
x, y, z	Cartesian surface coordinates

Z_i	reduced variables
ξ_1, ξ_2, ξ_3	orthogonal curvilinear coordinates
ϵ_i (i=1,2,4,5,6)	middle surface strains
ν_{ij}	Poisson's ratio for transverse strain in j-direction when stressed in the i-direction
θ	ply-angle with respect to x-axis of lamina
χ	rotation of transverse normal to middle surface in the x-z plane
φ	rotation of transverse normal to middle surface in the y-z plane
ω	circular frequency
ω^*	non-dimensional fundamental frequencies $=\omega a^2/h(\rho/E_2)^{1/2}$
ρ	density of laminate
λ_i	distinct eigenvalues of the matrix [C]
Ω	surface of laminate
$\{\delta\}$	column matrix consisting of the coefficients from displacement series

LIST OF TABLES

LIST OF FIGURES

Chapter 1

Introduction

1.1 Literature Survey

Laminated composite plates and shells are made up of two or more laminae (or layers) of uniform thickness bonded together to act as a unified structural element. Each lamina consists of fibre (eg. glass, boron, graphite) embedded in a matrix (or binder) material (eg. polymer, ceramics, metals). A lamina may be individually considered as being composed of an isotropic or orthotropic material so that its principal material directions are oriented to produce a structural element capable of resisting load in several directions.

Because of their high specific strength (failure stress/unit weight) and specific stiffness (stiffness/unit weight), composite plates and shells have been finding ever-increasing usage in many engineering applications, especially in aerospace industries (eg. aircraft and space station components). They are also used in the automobile industry and in the electronic industry as substrate materials of printed circuit boards. Thus analysis of characteristics of these types of materials is very important and of great interest to designers and researchers.

During the past decades, a wide variety of theories for static and dynamic analysis of composite plates and shells have been developed. Studying these theories shows that they can be classified into three general categories according to whether the effects of transverse shear deformation are considered or not. They are known as classical, first-order shear deformation and third-order shear deformation theories respectively.

Classical theory (CT)

This theory has been used more extensively due to its simplicity. It is based on the Love-Kirchhoff assumptions, known as the Love's first approximation theory [1], as follows: (i) the normals to the undeformed middle surface remain normal to the deformed middle surface and suffer no extension, (ii) the component of stress normal to the middle surface is small compared with other normal components of stress and can be neglected, (iii) displacements and strains are small compared to plate and shell thickness, (iv) the thickness of the plate and shell is small as compared with the least length of plate or the least radius of curvature of shell middle surface. It can be seen that the first assumption leads to neglecting transverse shear deformation. The second assumption requires the transverse normal stress to be equal to zero. The third and fourth assumptions imply that the theory is only suitable for thin plates and shells with small displacements.

The classical theory is essential in that it lays down the fundamentals for the higher order theories. Many researchers have made significant contributions in its development. Reissner and Stavsky [2-3] were probably the first to establish classical laminated anisotropic plate theory including bending-stretching coupling. It is known that the coupling between stretching and bending is the major difference between the macroscopic structural behaviour of laminated plate and homogenous plate. In reference [3], a special case of an antisymmetrical angle-ply plate ($B_{11}=B_{12}=B_{22}=B_{66}=0$) was considered and the coupling is caused by B_{16} and B_{26} terms.

The first application of Reissner-Stavsky theory to cylindrical bending of a long rectangular plate subjected to uniform normal pressure was considered by Stavsky[4] and by Dong et al.[5]. Later, Stavsky [6] extended the coupled bending-stretching theory to the vibration problem, but he did not give numerical results. The first published results of vibration analysis of laminated anisotropic plates were presented by Ashton and Waddoups [7] by using Rayleigh-Ritz method. Hikami [8] carried out a similar analysis and an experiment study for laminated orthotropic plates with combinations of simply supported and free edges. The vibration problem of clamped laminated plates was investigated by Bert et al.[9] and Ashton et al.[10], where experimental method and Rayleigh-Ritz method were both employed in each. Whitney and Leissa [11-12] obtained closed-form solutions for the natural frequencies of special layered orthotropic plates with simply supported boundary conditions.

The analysis including the bending-stretching effects for laminated shells was first presented by Ambartsumyan [13-14] as early as in 1953. However his theory dealt with what is now known as laminated orthotropic shells. It was Dong, Pister and Taylor [15] who presented an anisotropic laminated shell theory, which is based on the Kirchhoff-Love hypothesis and considered bending-stretching coupling. The theory is an extension of Reissner-Stavsky theory for laminated anisotropic plates to Donnell's shallow shell theory [16]. Dong [17] also obtained a general solution for natural frequencies of arbitrarily laminated cylindrical shells with various boundary conditions. A similar analysis was conducted by Bert et al., Stavsky et al. and Greenberg et al. [18-20] by using more accurate Love first-approximation shell theory and good agreement was achieved between them. Jones [21] investigated the problems of vibration and buckling of cross-ply laminated circular cylindrical shells by Donnell's shallow shell theory. Cheng and Ho [22] used Flügge's shell theory [23] to analyze buckling of laminated cylindrical shells. For axisymmetric vibration, Tasi [24] studied the effect of heterogeneity on cylindrical shells. For the dynamic analysis of cross-ply laminated circular cylindrical panels, Soldatos and Tzivanidis [25] and Soedel [26] obtained the natural frequencies for the panels by using Donnell-type theory. Bert and Kumar [27] presented a set of differential equations which, by use of tracers, could be reduced to the corresponding equations of Love's first approximation, Sanders' [28] and Donnell's theories, and analyzed the free vibration of moderately thick panels constructed of bimodulus material. Also, in reference [29] Soldatos obtained a closed-form solution for a cross-ply laminated circular cylindrical panel having simply supported edges, based on four different theories, namely, Donnell's, Love's, Sanders' and Flügge's theories.

First-order shear deformation theory (FSDT)

As mentioned above, since the classical theory assumes infinite rigidity in transverse shear and thus neglects transverse shear deformation, it is not suitable for laminated plates and shells in which material exhibits a finite rigidity in transverse shear. In addition, classical theory can be used only for thin plates and shells. Therefore attempts were made to refine the classical theory by incorporating transverse shear effects. For this purpose, the first assumption of the classical theory must be relaxed and replaced by the hypothesis that the normals to middle surface before deformation remain straight but not necessarily normal to the middle surface after deformation.

This type of theory including transverse shear deformation was presented by Stavsky [30] for laminated isotropic plates and Yang, Norris and Stavsky [31] for laminated anisotropic plates. Their work, known as YNS theory, may be thought as an extension of the Reissner-Mindlin shear deformation theory of elastic plates [32-34] to laminated plates. In YNS theory, the transverse shear stress is assumed to be constant over the entire cross section and governing equations are constructed by integration of the stress equations of motion over the plate thickness. As a result, a correction factor K is introduced to adjust the transverse shear stiffness. The first extensive application of YNS theory to various boundary-value problems was due to Whitney and Pagano [35], who considered cylindrical bending of antisymmetric angle-ply and cross-ply plate strips under sinusoidal load distribution and free vibration of antisymmetric angle-ply plate strips and discussed the value of K for composites. In contrast to this formulation is Whitney's theory [36], the essence of which is the result of a parabolic transverse shear stress variation over the plate thickness. Fortier and Rossettos [37] analyzed free vibration of thick rectangular plates of unsymmetric cross-ply construction while Sinha and Rath [38] considered both vibration and buckling for the same type of plates. Noor [39] treated buckling of thick rectangular plates of both symmetric and antisymmetric construction. Bert and Chen [40] provided a closed-form solution for the free vibration of finite-dimension angle-ply plates with simply supported edges, while Reddy [41] analyzed the same problem by finite element method. The finite element method was also used for investigations of laminated plates in references [42-45].

Khdeir [46-48] used the YNS theory to develop the Lévy solutions for bending, vibration and buckling of laminated antisymmetric angle-ply plate with two opposite edges simply supported and the rest subjected to a combination of clamped, simply supported and free boundary conditions. In references [49-50], Reddy, Khdeir and Librescu obtained similar solutions for laminated symmetric cross-ply plates. Li and Mirza [51-52] established analytical solutions for vibration and buckling of antisymmetric laminated angle-ply plates with two adjacent edges clamped and the others simply supported. Later, Li and Mirza [53] also studied the buckling problem of clamped sandwich plates using the reciprocal theorem method that was developed previously for the vibration and bending analysis of isotropic plates [54-60].

The effects of transverse shear deformation on the laminated orthotropic shells were considered by Dong and Tso [61]. In their theory, two correction factors K^2_{11}, K^2_{22} were introduced and determined from an analysis of plane waves in a plate with the same layered construction. At the same time, the effect of transverse shear deformation and transverse isotropy, as well as thermal expansion through the thickness of cylindrical shells were studied in reference [62-64]. Whitney and Sun [65-66] presented a refined shell theory specialized to anisotropic cylindrical shells, in which the displacements of the shell surface are expanded as linear functions of the thickness coordinate and the transverse displacement is expanded as a quadratic function of the thickness coordinate. The theory accounts for constant transverse shear stress over thickness and thus requires a correction to the transverse shear stiffness. Reddy [67] presented a generalization of Sanders' shell theory to laminated doubly-curved anisotropic shells, in which the transverse strains and rotations about the normal to the shell midplane were taken into account and exact solutions for bending and vibration of the cross-ply simply supported shell panels were developed.

Applications of those theories for various shell problems can be found in references [68-77]. Vibration and buckling problems of composite cylindrical shells were analyzed in references [68-73], while references [74-76] obtained closed-form solutions and reference [77] established exact solutions for bending and vibration of two-layer cross-ply cylindrical shells.

Third-order shear deformation theory (TSDT)

This theory is motivated by the fact that the first-order shear deformation theory does not satisfy the boundary conditions of zero thickness shear stresses on the inner and outer surfaces due to the assumption of a uniform distribution of transverse shear strains e_4 and e_5 across the thickness, although FSDT does model the lower, structurally important natural frequencies (including the fundamental frequency) very adequately. Besides, in Reissner-Mindlin type theories, the displacement field accounts for linear or higher-order variations of strains through the thickness. However, as indicated by the exact solutions [78-79], Mindlin type theories are unable to model adequately the behaviour of highly orthotropic composite structures. On the other hand, thick laminated plates and shells are basically three-dimensional problems and they are intractable as the number of layers becomes moderately large. To improve the accuracy of first-order theory and be able to treat the three-dimensional problems as two-dimensional ones, one has to develop a higher-order shear deformation theory.

Many higher-order theories [80-87] have been established for composite plates and shells since 1974. Lo et al.[82] presented a higher-order plate theory, which can provide an accurate prediction of the non-linear bending stress distribution according to Bert [88], but the displacement used in the theory does not satisfy the stress-free boundary conditions on the top and bottom surfaces of the plate. Cho et al.[89] and Reddy [90] introduced independently the "individual-layer" or "layerwise" based theories as distinguished from the conventional "packet" based theory. Levinson and Murthy [83-84] developed a third-order theory for the statics and dynamics of elastic plates and orthotropic laminated plates, however their theory is variationally inconsistent because both of them utilized directly the equilibrium equations and boundary conditions of the first-order shear deformation theory instead of establishing the equations of motion by means of the principle of virtual displacements. Later, Reddy [86] introduced a correction in these theories and obtained a set of modified governing equations and associated boundary conditions for laminated anisotropic plates by using the principle of virtual displacements. In reference [87], Reddy and Liu extended Reddy's previous work to laminated orthotropic shells.

In Reddy's third-order shear deformation theory for plates and shells, the same number of dependent variables as in the first-order shear deformation theory is used and the displacements of the midplane are expanded as cubic functions of the thickness coordinate, while the first-order shear deformation theory assumes that the displacements of the midplane vary as linear functions of the thickness coordinate. Furthermore, the same constant transverse deflection through the thickness is assumed as in the first-order shear deformation theory. As a result, a parabolic variation of transverse shear strains is realized and zero transverse shear stresses on the inner and outer surfaces are satisfied. Therefore, unlike the first-order shear deformation theory, no shear correction factors need to be introduced.

The static analysis of laminated composite plates using Reddy's third-order theory was first carried out in reference [86], where Navier-type solution was presented for symmetrically laminated cross-ply rectangular plates. Phan and Reddy [91] also used the finite element method to study a similar problem. Khdeir [92-93] obtained Lévy-type solutions for vibration and buckling of both symmetric and unsymmetric cross-ply plates with two opposite edges simply supported, while in references [94-95] forced vibration of cross-ply plates was analyzed.

Exact solutions of deflections and natural frequencies of simply supported cross-ply shells were obtained in reference [87] using Reddy's third-order shear deformation theory. Khdeir and Reddy [96] investigated the dynamic and static behaviour of laminated cross-ply shells. In references [97-98], Khdeir, Reddy and Frederick made use of the third-order, as well as first-order and classical theories to study the bending, vibration and buckling of cross-ply circular cylindrical shells. Soldatos [99] used Galerkin's method to analyze free vibrations of antisymmetric angle-ply laminated circular cylindrical panels, plates and cylinders.

Three types of the theories, i.e. classical, first-order shear deformation and third-order shear deformation theories are commonly used for analyzing laminated composite plates and shells. The classical theory has the simplest form but neglects transverse shear deformation. The first-order theory accounts for transverse shear deformation but does no satisfy the stress-free boundary conditions. The third-order theory is more comprehensive because it includes both transverse shear deformation and zero transverse shear stresses on the surfaces of structures. For further investigation, the readers are recommended to refer to the papers and books published by Bert [100-103], Reddy [104-105], Mirza [106] and Kapania [107].

1.2 Motivation and Objectives

By far, most of the solution techniques for composite plates and shells are numerical methods, such as finite element method [41-42,92], finite strip method [108], Galerkin's method [99] and Rayleigh-Ritz method [10]. It is obvious that the solutions generated by numerical methods are approximate in nature because the governing equations and the prescribed boundary conditions cannot be satisfied simultaneously. Thus the accuracy of numerical solutions is always questionable in the minds of design engineers. On the other hand, analytical solutions, which satisfy both governing equations and boundary conditions, are desirable to predict precisely the mechanical behaviours of composite structures.

In reviewing the published literature, we have noticed that closed-form solutions are not only very scarce but are available mostly for simply supported composite plates and shells, or those with at least a pair of simply supported edges on opposite sides. It is known that Navier-type solutions for simply supported boundary conditions are commonly employed for checking appropriate theories or other numerical methods, such as FEM, and there also exist a great number of plate and shell structures that have no two opposite edges simply supported. We, therefore, wish to explore an alternative technique to accurately analyze composite plates and shells having arbitrary boundary conditions.

The superposition technique is known as a powerful tool to deal with the boundary-value problem. It was used by Timoshenko [109] to treat bending problems of elastic plates. This method has been extensively applied by Gorman [110] for the free vibration analysis of thin elastic plates. In this method the governing differential equation is satisfied exactly throughout the domain of the plate and boundary conditions are satisfied to any desired degree of accuracy. Besides, unlike the Rayleigh-Ritz method, no shape functions are needed to be chosen for the analysis. So far, this technique has been utilized principally for the isotropic and orthotropic plates [110-112]. Extension of the superposition technique to the free vibration analysis of laminated orthotropic composite plates and shells is one of the goals of the work presented hereinafter.

The objectives of this doctoral research are as follows:

1. develop a new analytical approach combining the superposition and state-space techniques (CSST) for laminated orthotropic plates and shells having no two opposite edges simply supported.

2. establish closed-form solutions for free vibrations of antisymmetric angle-ply laminated plates and cross-ply laminated plates and shells in the framework of the classical, first-order shear deformation and third-order shear deformation theories and compare results among the three theories.

3. carry out convergence tests to demonstrate the accuracy of computed results obtained by the present approach and compare typical frequencies with those reported in the published references.

4. present accurate fundamental frequencies for laminated plates and shells with two adjacent edges, three edges and four edges clamped and the others simply supported, which have not been reported in the literature.

Chapter 2

Underlying Theories

In this chapter, the equations of motion of laminated plates and shells, based on the classical, first-order shear deformation and third-order shear deformation theories, are to be established for the free vibration analysis. Also, the boundary conditions associated with the equations of motion will be presented.

2.1 Classical Theory

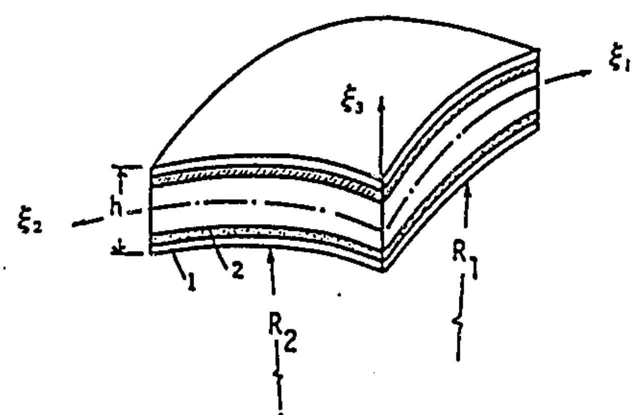

Figure 1 A laminated doubly curved element

Consider a doubly curved element made of N orthotropic laminae with uniform thickness shown in Figure 1. The orthogonal curvilinear coordinates ξ_1, ξ_2 and ξ_3 are chosen so that ξ_1 and ξ_2 coincide with the lines of curvature of mid-surface, $\xi_3=0$, and ξ_3 is a straight line perpendicular to the mid-surface.

According to the first two assumptions of the classical theory described in Chapter 1, we neglect the transverse shear deformation, that is, the shear strains e_3, e_4 and e_5 are assumed to be zero, and thus obtain the displacement field of the element,

$$
\begin{aligned}
U_1 &= (1+\frac{\xi_3}{R_1})u - \frac{\xi_3}{A_1} w_{,\xi_1} \\
U_2 &= (1+\frac{\xi_3}{R_2})v - \frac{\xi_3}{A_2} w_{,\xi_2} \\
U_3 &= w(\xi_1, \xi_2, t)
\end{aligned}
\tag{2.1.1}
$$

where t denotes time, u, v and w are the displacement components of the middle surface, R_1 and R_2 represent the radii of principal curvatures, and A_1 and A_2 are referred to as the Lamé parameters along ξ_1 and ξ_2 axes on the middle surface.

The strain-displacement relations for an orthogonal curvilinear coordinate system can be written as [113]:

$$e_1 = \frac{1}{L_1}u_{,\xi_1} + \frac{v}{L_1L_2}L_{1,\xi_2} + \frac{w}{L_1L_3}L_{1,\xi_3}$$

$$e_2 = \frac{1}{L_2}v_{,\xi_2} + \frac{u}{L_1L_2}L_{2,\xi_1} + \frac{w}{L_2L_3}L_{2,\xi_3}$$

$$e_3 = \frac{1}{L_3}w_{,\xi_3} + \frac{v}{L_2L_3}L_{3,\xi_2} + \frac{u}{L_1L_3}L_{3,\xi_1}$$

$$e_6 = \frac{L_2}{L_1}(\frac{v}{L_2})_{,\xi_1} + \frac{L_1}{L_2}(\frac{u}{L_1})_{,\xi_2} \qquad (2.1.2)$$

$$e_4 = \frac{L_1}{L_3}(\frac{u}{L_1})_{,\xi_3} + \frac{L_3}{L_1}(\frac{w}{L_3})_{,\xi_1}$$

$$e_5 = \frac{L_3}{L_2}(\frac{w}{L_3})_{,\xi_2} + \frac{L_2}{L_3}(\frac{v}{L_2})_{,\xi_3}$$

where L_i are the general Lamé parameters,

$$L_1 = A_1(1+\frac{\xi_3}{R_1}); \quad L_2 = A_2(1+\frac{\xi_3}{R_2}); \quad L_3 = 1;$$

$$A_i = |\bar{r}_{,\xi_i}| \quad (i=1,2) \qquad (2.1.3)$$

$$\xi_1 = \frac{x}{R_1}; \quad \xi_2 = \frac{y}{R_2}; \quad \xi_3 = z$$

here $\mathbf{r}$ denotes the position vector of a point on the middle surface, and x, y, and z are the Cartesian surface coordinates.

Substituting equation (2.1.1) into (2.1.2), using Donnell's approximation, i.e., neglecting the effects of u/R_1 and v/R_2 to the strains and recognizing that $z/R_1 << 1$ and $z/R_2 << 1$, one obtains,

$$e_1 = \epsilon_1 + zk_1; \quad e_2 = \epsilon_2 + zk_2; \quad e_6 = \epsilon_6 + zk_6 \tag{2.1.4}$$

It has long been known that of all the shell theories incorporating bending action, the Donnell theory is suitable for short and shallow shells. However, this study emphasizes developing a general method to solve equations brought forward by various theories rather than the theories themselves. For simplicity, Dennell's approximation is adopted.

In equation (2.1.4), the middle surface strains and curvatures are given by

$$\begin{aligned} &\epsilon_1 = u_{,x} + \frac{w}{R_1}; \quad \epsilon_2 = v_{,y} + \frac{w}{R_2}; \quad \epsilon_6 = u_{,y} + v_{,x}; \\ &k_1 = -w_{,xx}; \quad k_2 = -w_{,yy}; \quad k_6 = -2w_{,xy} \end{aligned} \tag{2.1.5}$$

The stress-strain relations for the κth orthotropic lamina can be expressed by the general Hooke's law as:

$$\begin{Bmatrix} \sigma_1^{(k)} \\ \sigma_2^{(k)} \\ \sigma_6^{(k)} \end{Bmatrix} = \begin{bmatrix} Q_{11}^{(k)} & Q_{12}^{(k)} & Q_{16}^{(k)} \\ & Q_{22}^{(k)} & Q_{26}^{(k)} \\ sym. & & Q_{66}^{(k)} \end{bmatrix} \begin{Bmatrix} e_1 \\ e_2 \\ e_6 \end{Bmatrix} \tag{2.1.6}$$

Here, $Q^{(k)}_{ij}$ are the transformed reduced stiffnesses. If the κth lamina has a principal direction of orthotropy oriented at an angle θ with respect to the x-axis of the lamina, we can write,

$$
\begin{aligned}
Q_{11}^{(k)} &= S_{11}\cos^4\theta + 2(S_{12}+2S_{66})\sin^2\theta\cos^2\theta + S_{22}\sin^4\theta \\
Q_{12}^{(k)} &= (S_{11}+S_{22}-4S_{66})\sin^2\theta\cos^2\theta + S_{12}(\sin^4\theta+\cos^4\theta) \\
Q_{22}^{(k)} &= S_{11}\sin^4\theta + 2(S_{12}+2S_{66})\sin^2\theta\cos^2\theta + S_{22}\cos^4\theta \\
Q_{16}^{(k)} &= (S_{11}-S_{12}-2S_{66})\sin\theta\cos^3\theta + (S_{12}-S_{22}+2S_{66})\sin^3\theta\cos\theta \\
Q_{26}^{(k)} &= (S_{11}-S_{12}-2S_{66})\sin^3\theta\cos\theta + (S_{12}-S_{22}+2S_{66})\sin\theta\cos^3\theta \\
Q_{66}^{(k)} &= (S_{11}+S_{22}-2S_{12}-2S_{66})\sin^2\theta\cos^2\theta + S_{66}(\sin^4\theta+\cos^4\theta)
\end{aligned}
\tag{2.1.7}
$$

where S_{ij} are the reduced stiffnesses,

$$
\begin{gathered}
S_{11}=\frac{E_1}{1-\nu_{12}\nu_{21}};\quad S_{12}=\frac{\nu_{12}E_2}{1-\nu_{12}\nu_{21}};\quad S_{22}=\frac{E_2}{1-\nu_{12}\nu_{21}}; \\
S_{66}=G_{12};\quad \frac{\nu_{12}}{E_1}=\frac{\nu_{21}}{E_2}
\end{gathered}
\tag{2.1.8}
$$

in which E_i is the Young'modulus in i-direction, ν_{ij} the Poisson's ratio for transverse strain in j-direction when stressed in the i-direction and G_{ij} the shear moduli in i-j planes.

The stress resultants are defined as:

$$
(N_i, M_i)=\sum_{k=1}^{N}\int_{h_{k-1}}^{h_k}\sigma_i^{(k)}(1,z)\,dz \qquad (i=1,2,6) \tag{2.1.9}
$$

Substituting equation (2.1.6) into (2.1.9) gives,

$$
\begin{aligned}
N_i &= A_{ij}\epsilon_j + B_{ij}k_j \\
M_i &= B_{ij}\epsilon_j + D_{ij}k_j
\end{aligned}
\tag{2.1.10}
$$

Here A_{ij}, B_{ij} and D_{ij} are the extensional, coupling and bending stiffnesses of the laminate,

$$(A_{ij}, B_{ij}, D_{ij}) = \sum_{k=1}^{N} \int_{h_{k-1}}^{h_k} Q_{ij}^{(k)}(1, z, z^2)\,dz \tag{2.1.11}$$

In order to obtained the equations of motion, the Hamilton's principle is used, which has an expression,

$$\int_{t_0}^{t_1} \delta L \; dt = 0 \tag{2.1.12}$$

where L is the Lagrangian function defined as

$$\begin{aligned} L &= U - K \\ &= \frac{1}{2}\sum_{k=1}^{N} \iint_{\Omega} \int_{h_{k-1}}^{h_k} \sigma_i^{(k)} e_i \, dxdydz - \frac{1}{2}\sum_{k=1}^{N} \iint_{\Omega} \int_{h_{k-1}}^{h_k} \rho\, \dot{U}_i \dot{U}_i \, dxdydz \end{aligned} \tag{2.1.13}$$

in which U and K are the strain energy and kinetic energy respectively, superposed dots denote differentiation with respect to time, and ρ is the density of the κth lamina. After substituting equations (2.1.1), (2.1.2) and (2.1.3) into equation (2.1.13) and taking the first variation of the Lagrangian function, one obtains

$$\begin{aligned} N_{1,x} + N_{6,y} &= I_1 \ddot{u} - I_2 \ddot{w}_{,x} \\ N_{6,x} + N_{2,y} &= I_1 \ddot{v} - I_2 \ddot{w}_{,y} \\ M_{1,xx} + 2M_{6,xy} + M_{2,yy} - \frac{N_1}{R_1} - \frac{N_2}{R_2} \\ &= I_1 \ddot{w} + I_2(\ddot{u}_{,x} + \ddot{v}_{,y}) - I_3(\ddot{w}_{,xx} + \ddot{w}_{,yy}) \end{aligned} \tag{2.1.14}$$

for the equations of motion, where I_i are the inertia coefficients,

$$I_i = \sum_{k=1}^{N} \int_{h_{k-1}}^{h_k} \rho z^{i-1} \, dz \qquad (i=1,2,3) \tag{2.1.15}$$

and the associated boundary conditions:

at x=constant,

$$\begin{aligned} N_1 &= N_1^* & \vee \quad & u = u^* \\ N_6 &= N_6^* & \vee \quad & v = v^* \\ M_1 &= M_1^* & \vee \quad & w_{,x} = w_{,x}^* \\ M_{1,x} + 2M_{6,y} &= M_{16}^* & \vee \quad & w = w^* \end{aligned} \tag{2.1.16}$$

at y=constant,

$$\begin{aligned} N_2 &= N_2^* & \vee \quad & v = v^* \\ N_6 &= N_6^* & \vee \quad & u = u^* \\ M_2 &= M_2^* & \vee \quad & w_{,y} = w_{,y}^* \\ M_{2,y} + 2M_{6,x} &= M_{26}^* & \vee \quad & w = w^* \end{aligned} \tag{2.1.17}$$

here the quantities with superscript stars are known.

It is noted that for the laminated composite plates and shells based on the classical theory, the coupling is due to the nature of the nonhomogeneity. Thus, in writing boundary conditions along an edge of a laminate, both in-plane and transverse effects must be considered simultaneously, i.e., four boundary conditions must be prescribed for each edge.

2.2 First-order Shear Deformation Theory

Consider the same element as shown in Figure 1. As stated before, since the first-order shear deformation theory assumes that the normals to the undeformed middle surface remain straight but not necessarily normal to the deformed surface, the displacement field can be described as:

$$
\begin{aligned}
U_1 &= u(x,y,t) + z\chi(x,y,t) \\
U_2 &= v(x,y,t) + z\varphi(x,y,t) \\
U_3 &= w(x,y,t)
\end{aligned}
\tag{2.2.1}
$$

where u, v and w are the displacements of the middle surface, and χ and φ denote the rotations of transverse normals to the middle surface in the x-z and y-z planes, respectively.

The strain-displacement relations are still determined by equation (2.1.2). Substituting equations (2.2.1) into (2.1.2) leads to

$$
\begin{aligned}
&e_1 = \epsilon_1 + z k_1; \quad e_2 = \epsilon_2 + z k_2; \\
&e_6 = \epsilon_6 + z k_6; \quad e_4 = \epsilon_4; \quad e_5 = \epsilon_5
\end{aligned}
\tag{2.2.2}
$$

Here the middle surface strains ϵ_i and curvatures κ_i are defined as

$$
\begin{aligned}
&\epsilon_1 = u_{,x} + \frac{w}{R_1}; \quad \epsilon_2 = v_{,y} + \frac{w}{R_2}; \quad \epsilon_6 = u_{,y} + v_{,x}; \quad \epsilon_4 = \varphi + w_{,y}; \\
&\epsilon_5 = \chi + w_{,x}; \quad k_1 = \chi_{,x}; \quad k_2 = \varphi_{,y}; \quad k_6 = \chi_{,y} + \varphi_{,x}
\end{aligned}
\tag{2.2.3}
$$

and the stress-strain relations can be written as

$$\begin{Bmatrix} \sigma_1^{(k)} \\ \sigma_2^{(k)} \\ \sigma_6^{(k)} \\ \sigma_4^{(k)} \\ \sigma_5^{(k)} \end{Bmatrix} = \begin{bmatrix} Q_{11}^{(k)} & Q_{12}^{(k)} & Q_{16}^{(k)} & 0 & 0 \\ & Q_{22}^{(k)} & Q_{26}^{(k)} & 0 & 0 \\ & & Q_{66}^{(k)} & 0 & 0 \\ & & & Q_{44}^{(k)} & 0 \\ sym. & & & & Q_{55}^{(k)} \end{bmatrix} \begin{Bmatrix} e_1 \\ e_2 \\ e_6 \\ e_4 \\ e_5 \end{Bmatrix} \tag{2.2.4}$$

in which $Q^{(k)}_{ij}$ are the same as those in equation (2.1.7) except for $Q^{(k)}_{44}$ and $Q^{(k)}_{55}$,

$$\begin{aligned} Q_{44}^{(k)} &= S_{44}\cos^2\theta + S_{45}\sin 2\theta + S_{55}\sin^2\theta \\ Q_{55}^{(k)} &= S_{55}\cos^2\theta - S_{45}\sin 2\theta + S_{44}\sin^2\theta \end{aligned} \tag{2.2.5}$$

Here, $S_{44}=G_{23}$ and $S_{55}=G_{13}$.

The stress resultants have the following expressions:

$$\begin{aligned} (N_i, M_i) &= \sum_{k=1}^{N} \int_{h_{k-1}}^{h_k} \sigma_i^{(k)}(1, z)\, dz \quad (i=1,\ 2,\ 6) \\ Q_i &= \sum_{k=1}^{N} \int_{h_{k-1}}^{h_k} \sigma_{6-i}^{(k)}\, dz \quad (i=1,\ 2) \end{aligned} \tag{2.2.6}$$

Substituting equation (2.2.4) into (2.2.6) gives,

$$\begin{aligned} N_i &= A_{ij}\epsilon_j + B_{ij}k_j; \quad M_i = B_{ij}\epsilon_j + D_{ij}k_j \qquad (i,\ j=1,\ 2,\ 6) \\ Q_1 &= A_{5j}\epsilon_j; \qquad Q_2 = A_{4j}\epsilon_j \qquad (j=4,\ 5) \end{aligned} \tag{2.2.7}$$

The stiffnesses of the laminate are

$$(A_{ij}, B_{ij}, D_{ij}) = \sum_{k=1}^{N} \int_{h_{k-1}}^{h_k} Q_{ij}^{(k)}(1, z, z^2)\ dz \qquad (i, j=1, 2, 6)$$

$$A_{ij} = \sum_{k=1}^{N} K_{ij}^2 \int_{h_{k-1}}^{h_k} Q_{ij}^{(k)}\ dz \qquad (i, j=4, 5) \tag{2.2.8}$$

where K^2_{ij} represent the transverse shear correction coefficients to adjust the transverse shear stiffness.

Following the procedure used for the classical theory, we substitute equations (2.2.1), (2.2.2) and (2.2.3) into the Hamilton's principle expressed by equation (2.1.13) (note: the index should take on the values, i=1, 2, 4, 5, 6 in this case.), integrate the expression by parts, and set the coefficients of δu, δv, δw, $\delta \chi$ and $\delta \varphi$ to zero separately. Thus we obtain,

$$\begin{aligned} N_{1,x} + N_{6,y} &= I_1 \ddot{u} + I_2 \ddot{\chi} \\ N_{6,x} + N_{2,y} &= I_1 \ddot{v} + I_2 \ddot{\varphi} \\ Q_{1,x} + Q_{2,y} - \frac{N_1}{R_1} - \frac{N_2}{R_2} &= I_1 \ddot{w} \\ M_{1,x} + M_{6,y} - Q_1 &= I_2 \ddot{u} + I_3 \ddot{\chi} \\ M_{6,x} + M_{2,y} - Q_2 &= I_2 \ddot{v} + I_3 \ddot{\varphi} \end{aligned} \tag{2.2.9}$$

for the equations of motion, where I_i are given by equation (2.1.15), and the boundary conditions:

at x=constant,

$$
\begin{array}{ccc}
N_1 - N_1^* & \vee & u - u^* \\
N_6 - N_6^* & \vee & v - v^* \\
M_1 - M_1^* & \vee & \chi - \chi^* \\
M_6 - M_6^* & \vee & \varphi - \varphi^* \\
Q_1 - Q_1^* & \vee & w - w^*
\end{array}
\qquad (2.2.10)
$$

at y=constant,

$$
\begin{array}{ccc}
N_2 - N_2^* & \vee & v - v^* \\
N_6 - N_6^* & \vee & u - u^* \\
M_2 - M_2^* & \vee & \varphi - \varphi^* \\
M_6 - M_6^* & \vee & \chi - \chi^* \\
Q_2 - Q_2^* & \vee & w - w^*
\end{array}
\qquad (2.2.11)
$$

where the quantities with superscript stars are known.

2.3 Third-order Shear Deformation Theory

In order to include the effect of transverse shear deformation and satisfy the boundary conditions on the top and bottom surfaces, the following displacement field is considered [87]:

$$U_1 = (1+\frac{z}{R_1})u + z\left[\chi - \frac{4}{3h^2}z^2(\chi + w_{,x})\right]$$

$$U_2 = (1+\frac{z}{R_2})v + z\left[\varphi - \frac{4}{3h^2}z^2(\varphi + w_{,y})\right] \qquad (2.3.1)$$

$$U_3 = w(x,y,t)$$

As described earlier for the displacement model of the first-order shear deformation theory, u, v and w again represent the displacement components of the middle surface, χ and φ are the rotations of the middle surface about the ζ_2 and ζ_1 axes respectively, R_1 and R_2 denote the radii of principal curvature, and h is the thickness of the laminate. It can be proved easily that the displacement field in equation (2.3.1) makes it possible to have the transverse shear strains, e_4 and e_5 equal to zero at the top and bottom surfaces. The unknown functions χ and φ are used to represent the action of the thickness shear strains on the middle surface.

The strains are obtained by using the strain-displacement relations (2.1.2) and the Donnell's approximation which neglects the effects of u/R_1 and v/R_2 as follows:

$$e_1 = \epsilon_1 + z(k_1 + z^2 k_1^1); \quad e_2 = \epsilon_2 + z(k_2 + z^2 k_2^1);$$

$$e_6 = \epsilon_6 + z(k_6 + z^2 k_6^1); \quad e_4 = \epsilon_4 + z^2 k_4^1; \quad e_5 = \epsilon_5 + z^2 k_5^1 \qquad (2.3.2)$$

here

$$\begin{gathered}
\epsilon_1 = u_{,x} + \frac{w}{R_1}; \quad \epsilon_2 = v_{,y} + \frac{w}{R_2}; \quad \epsilon_6 = u_{,y} + v_{,x}; \quad \epsilon_4 = \varphi + w_{,y}; \\
\epsilon_5 = \chi + w_{,x}; \quad k_1 = \chi_{,x}; \quad k_2 = \varphi_{,y}; \quad k_6 = \chi_{,y} + \varphi_{,x}; \\
k_1^1 = -c_2(\chi_{,x} + w_{,xx}); \quad k_2^1 = -c_2(\varphi_{,y} + w_{,yy}); \quad k_4^1 = -c_1(\varphi + w_{,y}); \\
k_5^1 = c_1(\chi + w_{,x}); \quad k_6^1 = -c_2(\varphi_{,x} + \chi_{,y} + 2w_{,xy})
\end{gathered} \tag{2.3.3}$$

in which $c_1 = 4/h^2$ and $c_2 = c_1/3$.

The stress-strain relations for the κth lamina are the same as in the first-order shear deformation theory. The equations of motion and the boundary conditions can be derived from the Hamilton's principle of equation (2.1.13), but the index has to take on the values, i=1, 2, 4, 5, 6. As a result, we obtain,

$$\begin{gathered}
N_{1,x} + N_{6,y} = \bar{I}_1 \ddot{u} + \bar{I}_2 \ddot{\chi} - \bar{I}_3 \ddot{w}_{,x} \\
N_{6,x} + N_{2,y} = \hat{I}_1 \ddot{v} + \hat{I}_2 \ddot{\varphi} - \hat{I}_3 \ddot{w}_{,y} \\
Q_{1,x} + Q_{2,y} - c_1(K_{1,x} + K_{2,y}) + c_2(P_{1,xx} + P_{2,yy} + 2P_{6,xy}) \\
-\frac{N_1}{R_1} - \frac{N_2}{R_2} = \bar{I}_3 \ddot{u}_{,x} + \hat{I}_3 \ddot{v}_{,y} + \bar{I}_5 \ddot{\chi}_{,x} + \hat{I}_5 \ddot{\varphi}_{,y} + I_1 \ddot{w} - c_2^2 I_7(\ddot{w}_{,xx} + \ddot{w}_{,yy}) \\
M_{1,x} + M_{6,y} - Q_1 + c_1 K_1 - c_2(P_{1,x} + P_{6,y}) = \bar{I}_2 \ddot{u} + \bar{I}_4 \ddot{\chi} - \bar{I}_5 \ddot{w}_{,x} \\
M_{6,x} + M_{2,y} - Q_2 + c_1 K_2 - c_2(P_{6,x} + P_{2,y}) = \hat{I}_2 \ddot{v} + \hat{I}_4 \ddot{\varphi} - \hat{I}_5 \ddot{w}_{,y}
\end{gathered} \tag{2.3.4}$$

for the equations of motion, and the boundary conditions:

at x=constant,

$$
\begin{array}{ccc}
N_1 - N_1^* & \vee & u - u^* \\
N_6 - N_6^* & \vee & v - v^* \\
M_1 - M_1^* & \vee & \chi - \chi^* \\
P_1 - P_1^* & \vee & w_{,x} - w_{,x}^* \\
M_6 - c_2 P_6 - M_6^* & \vee & \varphi - \varphi^* \\
Q_1 - c_1 K_1 + c_2(P_{1,x} + P_{6,y}) - Q_1^* & \vee & w - w^*
\end{array}
\qquad (2.3.5)
$$

at y=constant,

$$
\begin{array}{ccc}
N_2 - N_2^* & \vee & v - v^* \\
N_6 - N_6^* & \vee & u - u^* \\
M_2 - M_2^* & \vee & \varphi - \varphi^* \\
P_2 - P_2^* & \vee & w_{,y} - w_{,y}^* \\
M_6 - c_2 P_6 - M_6^* & \vee & \chi - \chi^* \\
Q_2 - c_1 K_2 + c_2(P_{2,y} + P_{6,x}) - Q_2^* & \vee & w - w^*
\end{array}
\qquad (2.3.6)
$$

where all the quantities with superscript star are assigned certain prescribed values and the stress resultants are given by

$$
\begin{aligned}
(N_i, M_i, P_i) &- \sum_{k-1}^{N} \int_{h_{k-1}}^{h_k} \sigma_i^{(k)} (1, z, z^3)\ dz \qquad (i-1, 2, 6) \\
(Q_i, K_i) &- \sum_{k-1}^{N} \int_{h_{k-1}}^{h_k} \sigma_{6-i}^{(k)} (1, z^2)\ dz \qquad (i-1, 2)
\end{aligned}
\qquad (2.3.7)
$$

The inertia coefficients are determined by

$$\bar{I}_1 = I_1 + \frac{2}{R_1} I_2; \quad \bar{I}_2 = I_2 + \frac{I_3}{R_1} - c_2 I_4 - \frac{c_2}{R_1} I_5;$$

$$\bar{I}_3 = c_2 I_4 + \frac{c_2}{R_1} I_5; \quad \bar{I}_4 = I_3 - 2c_2 I_5 + c_2^2 I_7; \quad \bar{I}_5 = c_2 I_5 - c_2^2 I_7; \tag{2.3.8}$$

$$I_i = \sum_{k=1}^{N} \int_{h_{k-1}}^{h_k} \rho z^{i-1}\, dz \qquad (i=1,\ 2,\ 3,\ 4,\ 5,\ 7)$$

The inertia coefficients $\hat{I}_i$ have the same expression as $\bar{I}_i$ except that R_1 is replaced by R_2.

The stress resultants can also be expressed in terms of the strain components by substituting equation (2.2.4) into (2.3.7), which results in,

$$\begin{aligned} N_i &= A_{ij}\epsilon_j + B_{ij}k_j + E_{ij}k_j^1 \\ M_i &= B_{ij}\epsilon_j + D_{ij}k_j + F_{ij}k_j^1 \qquad (i,\ j = 1,\ 2,\ 6) \\ P_i &= E_{ij}\epsilon_j + F_{ij}k_j + H_{ij}k_j^1 \\ Q_i &= A_{6-ij}\epsilon_j + D_{6-ij}k_j^1 \qquad (i=1,\ 2\ ;\ \ j=4,\ 5) \\ K_i &= D_{6-ij}\epsilon_j + F_{6-ij}k_j^1 \end{aligned} \tag{2.3.9}$$

where the laminate stiffnesses are

$$(A_{ij}, B_{ij}, D_{ij}, E_{ij}, F_{ij}, H_{ij}) = \sum_{k=1}^{N} \int_{h_{k-1}}^{h_k} Q_{ij}^{(k)} (1, z, z^2, z^3, z^4, z^6) \ dz$$

$$(i, j = 1, 2, 4, 5, 6) \qquad (2.3.10)$$

Chapter 3

Antisymmetric Angle-ply Laminated Plates

In this chapter, we develop an approach combining the superposition and state-space techniques (CSST) for the free vibration analysis of laminated plates without two opposite edges simply supported. Three types of theories, that is, classical, first-order and third-order theories, will be explored for the vibration problem of antisymmetric angle-ply laminated plates.

One of the most important characteristics of antisymmetric angle-ply laminated plates is that such structures exhibit coupling between bending and extension. As is well known, the elimination of the coupling not only makes the analysis easier than the laminated plates with coupling but also has no tendency to twist arising from thermally induced contractions that occur during cooling following the curing process. However, on the other hand coupling can play an essential role in many engineering applications in order to achieve design requirement. For instance, jet turbine fan blades are designed to pretwist by increasing the coupling. As a further example, if the shear stiffness of a laminate made of laminae with unidirectional fibres must be increased, one way to achieve this requirement is to position layers at some angle to the laminate axes. To stay within weight and cost requirement, an antisymmetric laminate may be the choice.

3.1 Governing Equations

The equations of motion for laminated plates and shells have been established in Chapter 2 using the three theories. These equations were expressed in terms of stress resultants. It is understood that there are usually three general approaches to deal with closed-form solutions for the boundary-value problems. They are known as displacement, stress and mixed displacement-stress methods. The displacement method is most commonly used in the analysis and will also be employed here. For this purpose, the equations of motion have to be expressed in terms of displacements.

Antisymmetric angle-ply laminated plates refer to plates having an even number of orthotropic laminae (or layers) with principal directions of orthotropy oriented at the angle of $+\theta$ and $-\theta$ with respect to the plate coordinate axes. Each lamina is assumed to be made from the same material. For such plates, the following plate stiffnesses and inertial coefficients are zero:

$$
\begin{gathered}
A_{i6}=D_{i6}=F_{i6}=H_{i6}=0 \quad (i=1,\ 2) \\
B_{ij}=E_{ij}=0 \quad (i,\ j=1,2) \\
A_{45}=B_{66}=D_{45}=E_{66}=F_{45}=H_{45}=I_2=\bar{I}_2=\bar{I}_3=0
\end{gathered}
\tag{3.1.1}
$$

Substituting equations (3.1.1) into (2.1.10), (2.2.7) and (2.3.9) respectively, then into the equations of motion expressed by equations (2.1.14), (2.2.9) and (2.3.4) and noting that $1/R_1$ and $1/R_2$ tend to be zero, we obtain the following equations for the classical theory,

$$\begin{bmatrix} L_{11} & L_{12} & L_{13} \\ & L_{22} & L_{23} \\ sym. & & L_{33} \end{bmatrix} \begin{Bmatrix} u \\ v \\ w \end{Bmatrix} = 0 \tag{3.1.2}$$

Where L_{ij} are the partial differential operators,

$$\begin{aligned} L_{11} &= A_{11}d_{xx} + A_{66}d_{yy} - I_1 d_{tt} \\ L_{12} &= (A_{12} + A_{66}) d_{xy} \\ L_{13} &= -3B_{16}d_{xxy} - B_{26}d_{yyy} \\ L_{22} &= A_{66}d_{xx} + A_{22}d_{yy} - I_1 d_{tt} \\ L_{23} &= -3B_{26}d_{xyy} - B_{16}d_{xxx} \\ L_{33} &= I_1 d_{tt} - I_3(d_{xxtt} + d_{yytt}) + D_{11}d_{xxxx} \\ &\quad + 2(D_{12} + 2D_{66}) d_{xxyy} + D_{22}d_{yyyy} \end{aligned} \tag{3.1.3}$$

in which $d_{xx} = \partial/\partial x \partial x$, etc.,

For the first-order shear deformation theory, the equations are,

$$\begin{bmatrix} L_{11} & L_{12} & L_{13} & L_{14} & L_{15} \\ & L_{22} & L_{23} & L_{24} & L_{25} \\ & & L_{33} & L_{34} & L_{35} \\ & & & L_{44} & L_{45} \\ sym. & & & & L_{55} \end{bmatrix} \begin{Bmatrix} u \\ v \\ w \\ \chi \\ \varphi \end{Bmatrix} = 0 \tag{3.1.4}$$

The operators L_{ij} used in the above equations are defined as,

$$
\begin{aligned}
&L_{11}=A_{11}d_{xx}+A_{66}d_{yy}-I_1d_{tt}\\
&L_{12}=(A_{12}+A_{66})d_{xy}\\
&L_{13}=0; \quad L_{14}=2B_{16}d_{xy}\\
&L_{15}=B_{16}d_{xx}+B_{26}d_{yy}\\
&L_{22}=(A_{22}d_{yy}+A_{66}d_{xx})-I_1d_{tt}\\
&L_{23}=0; \quad L_{24}=L_{15}; \quad L_{25}=2B_{26}d_{xy}\\
&L_{33}=I_1d_{tt}-A_{44}d_{yy}-A_{55}d_{xx}\\
&L_{34}=-A_{55}d_x; \quad L_{35}=-A_{44}d_y\\
&L_{44}=D_{11}d_{xx}+D_{66}d_{yy}-A_{55}-I_3d_{tt}\\
&L_{45}=(D_{12}+D_{66})d_{xy}\\
&L_{55}=D_{22}d_{yy}+D_{66}d_{xx}-A_{44}-I_3d_{tt}
\end{aligned}
\tag{3.1.5}
$$

For the third-order theory, the governing equations have the same form as in equation (3.1.4), however the differential elements are different,

$$
\begin{aligned}
&L_{11}=A_{11}d_{xx}+A_{66}d_{yy}-\bar{I}_1 d_{tt};\quad L_{12}=(A_{12}+A_{66})d_{xy}\\
&L_{13}=-c_2(3E_{16}d_{xxy}+E_{26}d_{yyy});\quad L_{14}=2(B_{16}-c_2E_{16})d_{xy}\\
&L_{15}=(B_{16}-c_2E_{16})d_{xx}+(B_{26}-c_2E_{26})d_{yy};\ L_{22}=A_{66}d_{xx}+A_{22}d_{yy}-\hat{I}_1 d_{tt}\\
&L_{23}=-c_2(3E_{26}d_{xyy}+E_{16}d_{xxx});\quad L_{24}=(B_{16}-c_2E_{16})d_{xx}+(B_{26}-c_2E_{26})d_{yy}\\
&L_{25}=2(B_{26}-c_2E_{26})d_{xy};\quad L_{33}=I_1d_{tt}-c_2^2I_7(d_{xxtt}+d_{yytt})+c_2^2[H_{11}d_{xxxx}\\
&\quad +2(H_{12}+2H_{66})d_{xxyy}+H_{22}d_{yyyy}]-[(A_{55}-2c_1D_{55}+c_1^2F_{55})d_{xx}+\\
&\quad (A_{44}-2c_1D_{44}+c_1^2F_{44})d_{yy}]\\
&L_{34}=\bar{I}_5d_{xtt}-(A_{55}-2c_1D_{55}+c_1^2F_{55})d_x-c_2(F_{11}-c_2H_{11})d_{xxx}\\
&\quad -c_2[(F_{12}-c_2H_{12})+2(F_{66}-c_2H_{66})]d_{xyy}\\
&L_{35}=\hat{I}_5d_{ytt}-(A_{44}-2c_1D_{44}+c_1^2F_{44})d_y-c_2(F_{22}-c_2H_{22})d_{yyy}\\
&\quad -c_2[(F_{12}-c_2H_{12})+2(F_{66}-c_2H_{66})]d_{xxy}\\
&L_{44}=(D_{11}-2F_{11}c_2+c_2^2H_{11})d_{xx}+(D_{66}-2F_{66}c_2+c_2^2H_{66})d_{yy}\\
&\quad -(A_{55}-2c_2D_{55}+c_1^2F_{55})-\bar{I}_4d_{tt}\\
&L_{45}=[D_{12}-2F_{12}c_2+D_{66}-F_6c_2-c_2(F_{12}-c_2H_{12})-\\
&\quad c_2(F_{66}-c_2H_{66})]d_{xy}\\
&L_{55}=(D_{22}-2F_{22}c_2+c_2^2H_{22})d_{yy}+(D_{66}-2F_{66}x2+c_2^2H_{66})d_{xx}\\
&\quad -(A_{44}-2c_2D_{44}+c_1^2F_{44})-\hat{I}_4d_{tt}
\end{aligned}
\tag{3.1.6}
$$

Until now we have completed the derivation of the governing equations for the antisymmetric angle-ply-laminated plates based on the three theories. The challenge that exists now is to solve the operator equations subjected to a set of prescribed boundary conditions, especially when the plates have no two opposite edges simply supported.

3.2 Superposition Technique

In order to demonstrate how to use the superposition technique for the laminated plates, we consider an antisymmetric angle-ply rectangular plate of dimensions a $\times$ b and with two adjacent edges clamped and others simply supported.

Using the classical theory, the boundary conditions of the plate can be described as follows:

at the boundary x=0,

$$u=v=w=w_{,x}=0 \tag{3.2.1}$$

at the boundary y=0,

$$u=v=w=w_{,y}=0 \tag{3.2.2}$$

at the boundary x=a,

$$u=w=M_1=N_6=0 \tag{3.2.3}$$

at the boundary y=b,

$$v=w=M_2=N_6=0 \tag{3.2.4}$$

To develop the exact solution for the plate, it is assumed that the general solution is constructed by the superposition of four components. We, therefore, write,

$$\begin{Bmatrix} u(x,y,t) \\ v(x,y,t) \\ w(x,y,t) \end{Bmatrix} = \sum_{k=1}^{4} \begin{Bmatrix} u^{k}(x,y,t) \\ v^{k}(x,y,t) \\ w^{k}(x,y,t) \end{Bmatrix} \tag{3.2.5}$$

In equation (3.2.5), each of the components has simply supported boundary conditions except that a distributed moment or an in-plane force is acting along one boundary. For instance, the first component is subjected to following boundary conditions:

at the boundaries y=0 and y=b,

$$v^{1} = w^{1} = M_{2}^{1} = N_{6}^{1} = 0 \tag{3.2.6}$$

at the boundary x=a,

$$u^{1} = w^{1} = M_{1}^{1} = N_{6}^{1} = 0 \tag{3.2.7}$$

at the boundary x=0,

$$u^{1} = w^{1} = N_{6}^{1} = 0; \quad M_{1}^{1} = M_{1}^{*} \tag{3.2.8}$$

Here, the quantity M^*_1 represents the amplitude of a distributed oscillating bending moment and may be expanded in a Fourier series as follows:

$$M_1^* = \sum_{n=1,2}^{\infty} E_n \sin K_n y \qquad (3.2.9)$$

where $K_n = n\pi/b$.

For the second component, its boundary conditions are identical to those of the first except that equation (3.2.8) should be replaced by

$$u^2 = w^2 = M_1^2 = 0; \quad N_6^2 = N_1^* \qquad (3.2.10)$$

in which N^*_1 is the amplitude of a distributed oscillating in-plane force and can be expanded in a series form, given by

$$N_1^* = \sum_{n=1,2}^{\infty} F_n \sin K_n y \qquad (3.2.11)$$

Following a similar procedure, the boundary conditions for the last two components can be written easily.

We now proceed to obtain the solution for the first component. To do this, the displacements of the middle surface may be expressed as products of unknown functions and known trigonometric and exponential functions, given as

$$\begin{Bmatrix} u^1(x,y,t) \\ v^1(x,y,t) \\ w^1(x,y,t) \end{Bmatrix} = \sum_{n=1}^{\infty} \begin{Bmatrix} U_n^1(x)\cos K_n y \\ V_n^1(x)\sin K_n y \\ W_n^1(x)\sin K_n y \end{Bmatrix} e^{i\omega t} \tag{3.2.12}$$

here ω is the circular frequency and $i=\surd{-1}$.

3.3 State-space Technique

The state-space technique is based on the principle of reducing the governing equations to a set of first-order differential equations while keeping the field equations in their exact form. With this method, it is convenient to obtain the solutions for higher-order differential equations.

Substituting equation (3.2.12) into governing equation (3.1.2), we obtain

$$\begin{aligned} U^1_{n,xx} &= C_1 U^1_n + C_2 V^1_{n,x} + C_3 W^1_n + C_4 W^1_{n,xx} \\ V^1_{n,xx} &= C_5 U^1_{n,x} + C_6 V^1_n + C_7 W^1_{n,x} + C_8 W^1_{n,xxx} \\ W^1_{n,xxxx} &= C_9 U^1_n + C_{10} V^1_{n,x} + C_{11} W^1_n + C_{12} W^1_{n,xx} \end{aligned} \tag{3.3.1}$$

where C_i are listed in the Appendix I.

The ordinary differential equations described in equations (3.3.1) can be reduced to a set of first-order differential equations by defining the variables,

$$Z_1 = U_n^1;\ Z_2 = U_{n,x}^1;\ Z_3 = V_n^1;\ Z_4 = V_{n,x}^1;$$
$$Z_5 = W_n^1;\ Z_6 = W_{n,x}^1;\ Z_7 = W_{n,xx}^1;\ Z_8 = W_{n,xxx}^1 \tag{3.3.2}$$

Equations (3.3.1) thus become

$$\{Z(x)\}_{,x} = [C]\{Z(x)\} \tag{3.3.3}$$

where $\{Z(x)\} = \{Z_1\ Z_2\ \dots\ Z_8\}^T$ and [C] is defined by

$$[C] = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ C_1 & 0 & 0 & C_2 & C_3 & 0 & C_4 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & C_5 & C_6 & 0 & 0 & C_7 & 0 & C_8 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ C_9 & 0 & 0 & C_{10} & C_{11} & 0 & C_{12} & 0 \end{bmatrix} \tag{3.3.4}$$

A general solution to equation (3.3.3) can be written as

$$\{Z(x)\} = \left[e^{[C]x}\right]\{P\} \tag{3.3.5}$$

where $\{P\} = \{p_1\ p_2\ \dots\ p_8\}^T$, a column matrix of constants which will be determined by equations (3.2.7) and (3.2.8), and $[e^{[C]x}]$ is expressed as:

$$[e^{[C]x}] = [R]\begin{bmatrix} e^{\lambda_1 x} & & & & \\ & e^{\lambda_2 x} & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & e^{\lambda_8 x} \end{bmatrix} \qquad (3.3.6)$$

Here the symbols λ_i represent the distinct eigenvalues of the matrix [C], and $[R]=[r_{ij}]$, the matrix of the corresponding eigenvectors.

Satisfying the boundary conditions described by equations (3.2.7) and (3.2.8) gives

$$[G]\{P\} = \{E\} \qquad (3.3.7)$$

here, $\{E\} = \{0\ 0\ \ldots\ E_n\}^T$ and [G] is a 8 × 8 matrix whose elements are all known. Thus from equation (3.3.7), we obtain

$$\{P\} = [G]^{-1}\{E\} \qquad (3.3.8)$$

We turn next to the solution for the second component. It is noted that the difference between the first and the second components is that there is an in-plane force rather than a bending moment acting along the edge x=a. Therefore the solution for the second component has the same form as that for the first except that {E} in equation (3.3.8) is replaced by

$$\{E\} = \{0\ 0\ \ldots\ F_n\ 0\ 0\}^T \tag{3.3.9}$$

Similarly, the solutions for the last components can be obtained by following the same procedures as those for the first. For example, the displacements of plate mid-surface may take the form,

$$\begin{Bmatrix} u^j(x,y,t) \\ v^j(x,y,t) \\ w^j(x,y,t) \end{Bmatrix} = \sum_{m=1}^{\infty} \begin{Bmatrix} U_m^j(y)\sin K_m x \\ V_m^j(y)\cos K_m x \\ W_m^j(y)\sin K_m x \end{Bmatrix} e^{i\omega t} \tag{3.3.10}$$

where $K_m = m\pi/a$, $j=3,4$ and the amplitudes of the distributed oscillating moment and in-plane force along the edge $y=0$ are expanded as

$$M_2^* = \sum_{m=1,2}^{\infty} G_m \sin K_m x \tag{3.3.11}$$

and

$$N_2^* = \sum_{m=1,2}^{\infty} H_m \sin K_m x \tag{3.3.12}$$

respectively.

3.4 Enforcement of Boundary Conditions

According to the superposition method the superimposed solution, that is, the one resulting from the summation of the solutions of the four components, must satisfy the prescribed boundary conditions of the original problem, which is described in equations (3.2.1)-(3.3.4). It is easy to verify that all of the boundary conditions have been satisfied automatically except for the following ones,

at the edge y=0,

$$u = w_{,y} = 0 \tag{3.4.1}$$

at the edge x=0,

$$v = w_{,x} = 0 \tag{3.4.2}$$

These equations indicate that the net slopes along edge y=0 and x=0 must be to zero, and the displacement in x direction along edge y=0 and the displacement in y direction along edge x=0 must be zero.

It is clear that we need to demonstrate the satisfaction of only one of these boundary conditions because of the similarity among them. We thus consider the first equation of equations (3.4.1). After substituting the solutions of the four components into this equation, we obtain

$$\sum_{n=1}^{\infty} U_n^1(x)+\sum_{n=1}^{\infty} U_n^2(x)+\sum_{m=1}^{\infty} U_m^3(0)\sin K_m x+\sum_{m=1}^{\infty} U_m^4(0)\sin K_m x=0 \tag{3.4.3}$$

Equation (3.4.3) can be rewritten by expanding the $U^1{}_n(x)$ and $U^2{}_n(x)$ in trigonometric series as follows:

$$\sum_{n=1}^{\infty} \bar{U}_n^1(E_n)+\sum_{n=1}^{\infty} \bar{U}_n^2(F_n)+U_m^3(G_m)+U_m^4(H_m)=0 \tag{3.4.4}$$

where

$$\begin{aligned} \bar{U}_n^1 &= \frac{2}{a}\int_0^a U_n^1(x)\sin K_m x \; dx \\ \bar{U}_n^2 &= \frac{2}{a}\int_0^a U_n^2(x)\sin K_m x \; dx \end{aligned} \tag{3.4.5}$$

In order to simplify equations (3.4.5), we may look at equations (3.3.5) and discuss them further.

Case I. If $\lambda_1, \lambda_2, \ldots, \lambda_8$ are all the real eigenvalues of the matrix [C], equation(3.3.5) can be written as

$$Z(x)=\left[\{r_1\}e^{\lambda_1 x} \quad \{r_2\}e^{\lambda_2 x} \quad \ldots. \quad \{r_8\}e^{\lambda_8 x}\right]\{P\} \tag{3.4.6}$$

where

$$\{r_k\}=\begin{Bmatrix} r_{k1} \\ r_{k2} \\ . \\ . \\ r_{k8} \end{Bmatrix} \quad ; \quad \{P\}=\begin{Bmatrix} p_1 \\ p_2 \\ . \\ . \\ p_8 \end{Bmatrix} \quad (k=1, \ldots, 8) \tag{3.4.7}$$

Case II. If λ_k and λ_{k+1} are a pair of conjugate eigenvalues of the matrix [C], equation (3.3.5) becomes

$$Z(x)=\left[\{r_1\}e^{\lambda_1 x}\ldots\{\bar{r}_k\}e^{Re(\lambda_k)x}\ \{\bar{r}_{k+1}\}e^{Re(\lambda_{k+1})x}\ldots\{r_8\}e^{\lambda_8 x}\right]\{\bar{P}\} \tag{3.4.8}$$

Here

$$\{\bar{r}_k\}=\begin{Bmatrix} \bar{r}_{1k} \\ \bar{r}_{2k} \\ . \\ . \\ \bar{r}_{8k} \end{Bmatrix}; \quad \{\bar{r}_{k+1}\}=\begin{Bmatrix} \bar{r}_{1k+1} \\ \bar{r}_{2k+1} \\ . \\ . \\ \bar{r}_{8k+1} \end{Bmatrix}; \quad \{\bar{P}\}=\begin{Bmatrix} p_1 \\ . \\ \bar{p}_k \\ \bar{p}_{k+1} \\ . \\ p_8 \end{Bmatrix} \tag{3.4.9}$$

with

$$\begin{gathered} \bar{r}_{jk}=Re(r_{jk})\cos Im(\lambda_k)x - Im(r_{jk})\sin Im(\lambda_k)x \\ \bar{r}_{jk+1}=Im(r_{jk})\cos Im(\lambda_k)x + Re(r_{jk})\sin Im(\lambda_k)x \\ \bar{p}_k=p_k+p_{k+1}; \quad \bar{p}_{k+1}=i(p_k-p_{k+1}); \quad i=\sqrt{-1} \\ (j=1,2,\ldots,8) \end{gathered} \tag{3.4.10}$$

Integrating equation (3.4.5) and making use of equations (3.4.6)-(3.4.10), we obtain

$$\bar{U}_n^j = \sum_{k=1}^{8} \bar{U}_{nk}^j \quad (j=1,\, 2) \tag{3.4.11}$$

where, if λ_k is a real eigenvalue,

$$\begin{gathered} \bar{U}_{nk}^j = \frac{2}{a}\, r_{1k}^j\, \theta_k^j\, p_k^j \\ \theta_k^j = \frac{K_m}{K_m^2+\lambda_k^2}(1-e^{\lambda_k a}\cos K_m a) \end{gathered} \tag{3.4.12}$$

and if λ_k and λ_{k+1} are a pair of conjugate eigenvalues,

$$\begin{gathered} \bar{U}_{nk}^j = \frac{1}{a}\left\{\bar{p}_k^j[\alpha_2^j(\theta_1+\theta_2) - \beta_2^j(\theta_3-\theta_4)] + \bar{p}_{k+1}^j[\beta_2^j(\theta_1+\theta_2) + \alpha_2^j(\theta_3-\theta_4)]\right\} \\ \theta_1 = \frac{\alpha_1 e^{\alpha_1 a}\cos K_m a \sin\beta_1 a + (K_m+\beta_1)(1-e^{\alpha_1 a}\cos K_m a \cos\beta_1 a)}{\alpha_1^2+(K_m+\beta_1)^2} \\ \theta_2 = \frac{-\alpha_1 e^{\alpha_1 a}\cos K_m a \sin\beta_1 a + (K_m-\beta_1)(1-e^{\alpha_1 a}\cos K_m a \cos\beta_1 a)}{\alpha_1^2+(K_m-\beta_1)^2} \\ \theta_3 = \frac{-\alpha_1(1-e^{\alpha_1 a}\cos K_m a \cos\beta_1 a) - (K_m-\beta_1)e^{\alpha_1 a}\cos K_m a \sin\beta_1 a}{\alpha_1^2+(K_m-\beta_1)^2} \\ \theta_4 = \frac{-\alpha_1(1-e^{\alpha_1 a}\cos K_m a \cos\beta_1 a) + (K_m+\beta_1)e^{\alpha_1 a}\cos K_m a \sin\beta_1 a}{\alpha_1^2+(K_m+\beta_1)^2} \\ \alpha_1 = Re(\lambda_k); \quad \beta_1 = Im(\lambda_k); \quad \alpha_2^j = Re(r_{1k}^j); \quad \beta_2^j = Im(r_{1k}^j) \end{gathered} \tag{3.4.13}$$

It is obvious that the quantities with superscript j in equations (3.4.11)-(3.4.13) refer to the jth component.

After the satisfaction of equations (3.4.1) and (3.4.2), an eigenvalue system of the homogeneous equations can be established as follows:

$$[M]\{\delta\} = 0 \tag{3.4.14}$$

Here [M] is a 4k × 4k eigenvalue matrix whose elements are determined by the solutions of the four components through enforcement of the boundary conditions, such as equation (3.4.4), and k is the number of terms used in the truncated series expansions of the displacements of middle surface, while $\{\delta\}$ is a 4k × 1 column matrix consisting of unknown constants E_n, F_n, etc. Since circular frequency ω is involved in the elements of the matrix [M], the natural frequencies can be evaluated by enforcing the determinant of the matrix [M] to vanish

$$|\boldsymbol{M}| = 0 \tag{3.4.15}$$

It should be pointed out that equation (3.4.15) can be used to generate not only fundamental frequency but also higher natural frequencies. In fact, any natural frequencies can be found from this equation by taking more terms in the displacement series. However, in the engineering sense, the fundamental frequency is often of greater interest than its higher natural frequencies because its forced response in many cases is the largest. Therefore only the fundamental frequency will be calculated for all types of the plates and shells considered.

3.5 Higher-Order Theory Cases

Following the same procedures described for the classical theory in the above sections, the solutions based on the first-order and third-order theories can also be generated by a combination of the superposition and state-space techniques for the antisymmetric angle-ply plate with two adjacent edges clamped and others simply supported. However since the higher-order theories account for more displacement components of the middle-surface than the classical theory, the problem will become more complicated.

The general solutions for the first-order and third-order theories are assumed to be

$$\begin{Bmatrix} u(x,y,t) \\ v(x,y,t) \\ w(x,y,t) \\ \chi(x,y,t) \\ \varphi(x,y,t) \end{Bmatrix} = \sum_{k=1}^{K^*} \begin{Bmatrix} u^k(x,y,t) \\ v^k(x,y,t) \\ w^k(x,y,t) \\ \chi^k(x,y,t) \\ \varphi^k(x,y,t) \end{Bmatrix} \tag{3.5.1}$$

where K^* takes the values, 4 and 6, indicating the first-order and third-order theory, respectively.

In equation (3.5.1), each of the components has such boundary conditions that all edges are simply supported except that one edge is subjected to a harmonic bending moment or an in-plane force of circular frequency ω and with a distributed amplitude which can be expanded in the series represented by equations (3.2.9) or (3.2.11).

Because the steps required in obtaining solutions to all the components are the same, only the first is discussed here.

The generalized displacement field for the first component is written as

$$\begin{Bmatrix} u^1(x,y,t) \\ v^1(x,y,t) \\ w^1(x,y,t) \\ \chi^1(x,y,t) \\ \varphi^1(x,y,t) \end{Bmatrix} = \sum_{n=1}^{\infty} \begin{Bmatrix} U_n^1(x)\cos K_n y \\ V_n^1(x)\sin K_n y \\ W_n^1(x)\sin K_n y \\ X_n^1(x)\sin K_n y \\ Y_n^1(x)\cos K_n y \end{Bmatrix} e^{i\omega t} \tag{3.5.2}$$

Substituting equation (3.5.2) into the governing equations (3.1.4), we obtain, for the first-order shear deformation theory (FSDT),

$$\begin{aligned} U^1_{n,xx} &= C_1 U^1_n + C_2 V^1_{n,x} + C_3 W^1_n + C_4 X^1_{n,x} + C_5 Y^1_n \\ V^1_{n,xx} &= C_6 U^1_{n,x} + C_7 V^1_n + C_8 W^1_{n,x} + C_9 X^1_n + C_{10} Y^1_{n,x} \\ W^1_{n,xx} &= C_{11} W^1_n + C_{12} X^1_{n,x} + C_{13} V^1_n \\ X^1_{n,xx} &= C_{14} U^1_{n,x} + C_{15} V^1_n + C_{16} W^1_{n,x} + C_{17} X^1_n + C_{18} Y^1_{n,x} \\ Y_{n,xx} &= C_{19} U^1_n + C_{20} V^1_{n,x} + C_{21} W^1_n + C_{22} X^1_{n,x} + C_{23} Y^1_n \end{aligned} \tag{3.5.3}$$

For the third-order shear deformation theory (TSDT), these equations take the form,

$$
\begin{aligned}
&U^1_{n,xx}=C_1U^1_n+C_2V^1_{n,x}+C_3W^1_n+C_4W^1_{n,xx}+C_5X^1_{n,x}+C_6Y^1_n\\
&V^1_{n,xx}=C_7U^1_{n,x}+C_8V^1_n+C_9W^1_{n,x}+C_{10}W^1_{n,xxx}+C_{11}X^1_n+C_{12}Y^1_n\\
&W^1_{n,xxxx}=C_{13}U^1_n+C_{14}V^1_{n,x}+C_{15}W^1_n+C_{16}W^1_{n,xx}+C_{17}X^1_{n,x}+C_{18}Y^1_n\\
&X^1_{n,xx}=C_{19}U^1_n+C_{20}V^1_n+C_{21}W^1_{n,x}+C_{22}W^1_{n,xxx}+C_{23}X^1_n+C_{24}Y^1_{n,x}\\
&Y^1_{n,xx}=C_{25}U^1_n+C_{26}V^1_{n,x}+C_{27}W^1_n+C_{28}W^1_{n,xx}+C_{29}X^1_{n,x}+C_{30}Y^1_n
\end{aligned}
\tag{3.5.4}
$$

Here C_i are listed in the Appendix I.

Following the state-space technique described in Section 3.3, we define the variables,

FSDT:

$$
\begin{gathered}
Z_1=U^1_n;\;\; Z_2=U^1_{n,x};\;\; Z_3=V^1_n;\;\; Z_4=V^1_{n,x};\;\; Z_5=W^1_n;\\
Z_6=W^1_{n,x};\;\; Z_7=X^1_n;\;\; Z_8=X^1_{n,x};\;\; Z_9=Y^1_n;\; Z_{10}=Y^1_{n,x}
\end{gathered}
\tag{3.5.5}
$$

TSDT:

$$
\begin{gathered}
Z_1=U^1_n;\; Z_2=U^1_{n,x};\; Z_3=V^1_n;\; Z_4=V^1_{n,x};\; Z_5=W^1_n;\; Z_6=W^1_{n,x}\\
Z_7=W^1_{n,xx};\; Z_8=W^1_{n,xxx};\; Z_9=X^1_n;\; Z_{10}=X^1_{n,x};\; Z_{11}=Y^1_n;\; Z_{12}=Y^1_{n,x}
\end{gathered}
\tag{3.5.6}
$$

Thus equations (3.5.3) and (3.5.4) are converted into a matrix form:

$$\{Z(x)\}_{,x}=[C]\{Z(x)\} \tag{3.5.7}$$

where,

FSDT:

$$[C]=\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ C_1 & 0 & 0 & C_2 & C_3 & 0 & 0 & C_4 & C_5 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & C_6 & C_7 & 0 & 0 & C_8 & C_9 & 0 & 0 & C_{10} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{11} & 0 & 0 & C_{12} & C_{13} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & C_{14} & C_{15} & 0 & 0 & C_{16} & C_{17} & 0 & 0 & C_{18} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ C_{19} & 0 & 0 & C_{20} & C_{21} & 0 & 0 & C_{22} & C_{23} & 0 \end{bmatrix}; \quad \{Z\}=\begin{Bmatrix} Z_1 \\ Z_2 \\ Z_3 \\ Z_4 \\ Z_5 \\ Z_6 \\ Z_7 \\ Z_8 \\ Z_9 \\ Z_{10} \end{Bmatrix} \tag{3.5.8}$$

TSDT:

$$[C]=\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ C_1 & 0 & 0 & C_2 & C_3 & 0 & C_4 & 0 & 0 & C_5 & C_6 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & C_7 & C_8 & 0 & 0 & C_9 & 0 & C_{10} & C_{11} & 0 & 0 & C_{12} \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ C_{13} & 0 & 0 & C_{14} & C_{15} & 0 & C_{16} & 0 & 0 & C_{17} & C_{18} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & C_{19} & C_{20} & 0 & 0 & C_{21} & 0 & C_{22} & C_{23} & 0 & 0 & C_{24} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ x25 & 0 & 0 & C_{26} & C_{27} & 0 & C_{28} & 0 & 0 & C_{29} & C_{30} & 0 \end{bmatrix}; \quad \{Z\}=\begin{Bmatrix} Z_1 \\ Z_2 \\ Z_3 \\ Z_4 \\ Z_5 \\ Z_6 \\ Z_7 \\ Z_8 \\ Z_9 \\ Z_{10} \\ Z_{11} \\ Z_{12} \end{Bmatrix} \tag{3.5.9}$$

A general solution to equation (3.5.7) is given by

$$\{Z(x)\} = \left[e^{[C]x}\right]\{P\} \tag{3.5.10}$$

where {P} is a constant column matrix associated with the boundary conditions and $[e^{[C]x}]$ is written as

$$\left[e^{[C]x}\right] = [R]\begin{bmatrix} e^{\lambda_1 x} & & & & \\ & e^{\lambda_2 x} & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & e^{\lambda_k x} \end{bmatrix} \tag{3.5.11}$$

here, k=10 for FSDT, k=12 for TSDT, λ_i denote the distinct eigenvalues of [C], and $\{R\}=\{r_{ij}\}$, represents the corresponding eigenvector matrix.

The constant matrix {P} in equation (3.5.10) is determined by the following boundary conditions:

at the edge x=0,

$$\begin{aligned} &FSDT: \quad u=w=\varphi=N_6=0; \quad M_1=M_1^* \\ &TSDT: \quad u=w=\varphi=P_1=N_6=0; \quad M_1=M_1^* \end{aligned} \tag{3.5.12}$$

at the edge x=a,

$$\textit{FSDT}: \qquad u = w = \varphi = N_6 = M_1 = 0$$
$$\textit{TSDT}: \qquad u = w = \varphi = P_1 = N_6 = M_1 = 0 \qquad (3.5.13)$$

After all the solutions of the components are obtained, the superposed solution has to satisfy the prescribed boundary conditions of the original plate, given by
at the boundary y=0,

$$\textit{FSDT}: \qquad u = \varphi = 0$$
$$\textit{TSDT}: \qquad u = \varphi = w_{,y} = 0 \qquad (3.5.14)$$

at the boundary x=0,

$$\textit{FSDT}: \qquad v = \chi = 0$$
$$\textit{TSDT}: \qquad v = \chi = w_{,x} = 0 \qquad (3.5.15)$$

Enforcement of equations (3.5.14) and (3.5.15) will lead to an eigenvalue system of the homogeneous equations,

$$[M]\{\delta\} = 0 \qquad (3.5.16)$$

Here, for FSDT, [M] is a 4k × 4k eigenvalue matrix, for TSDT, [M] has a dimension of 6k × 6k, and k is the number of terms used in the series, while {δ} is a column matrix consisting of the unknown constants E_n, etc.

For non-trivial solution of equation (3.5.16), the determinant of the coefficient matrix should be equal to zero, from which natural frequencies can be generated.

3.6 Plates with Three Edges Clamped

Having used the superposition and state-space techniques for the plate with two adjacent edges clamped, we now proceed to study antisymmetric angle-ply plates having three clamped and one simply supported boundaries. It is obvious that the steps required to obtain the exact solution of the plates are similar to those described before for the plate having two adjacent clamped boundaries. But more components need to be superimposed to construct the general solution.

The general solution is composed of the following components:

$$\begin{Bmatrix} u(x,y,t) \\ v(x,y,t) \\ w(x,y,t) \end{Bmatrix} = \sum_{k=1}^{6} \begin{Bmatrix} u^k(x,y,t) \\ v^k(x,y,t) \\ w^k(x,y,t) \end{Bmatrix} \tag{3.6.1}$$

for CT and,

$$\begin{Bmatrix} u(x,y,t) \\ v(x,y,t) \\ w(x,y,t) \\ \chi(x,y,t) \\ \varphi(x,y,t) \end{Bmatrix} = \sum_{k=1}^{K^*} \begin{Bmatrix} u^k(x,y,t) \\ v^k(x,y,t) \\ w^k(x,y,t) \\ \chi^k(x,y,t) \\ \varphi^k(x,y,t) \end{Bmatrix} \tag{3.6.2}$$

Here $K^*=6$, for FSDT and $K^*=9$, for TSDT.

It is seen that, corresponding to different plate theories there are different numbers of component solutions to be superposed. This is because higher-order theories need to satisfy more boundary conditions than for the classical theory. For instance, at each boundary there are four boundary conditions for the classical theory, however the number of boundary conditions increases to five and six for the first-order and third-order theories, respectively. Besides, among all the boundary conditions, some of them are satisfied automatically and others have to be enforced. As a result, more component solutions are required for the third-order shear deformation theory.

The next steps to develop solutions for each of the components are the same as before and thus need little explanation. The only thing which may be re-emphasized is the eigenvalue matrix [M] that is formed when the originally prescribed boundary conditions are enforced.

[M] is constructed by the coefficients of the homogenous equations expressed by, for example, equation (3.4.14) or (3.5.16). Its dimensions depend on how many components are used. In the cases of the classical and first-order theories, [M] is a $6k \times 6k$ matrix, while it becomes $9k \times 9k$ matrix when the third-order shear deformation theory is employed, where k is the number of the terms adopted in the series. Therefore a uniform form of the eigenvalue matrix can be written as:

$$[M] = [m_{ij}] \qquad (i, j = 1, 2, \ldots, K^{*} \times k) \tag{3.6.3}$$

where K^{*} represents total number of components used in the general solution.

3.7 Fully Clamped Plates

So far, because of the complexity of its boundary conditions, no closed-form solution has been established for the free vibration frequencies of antisymmetric angle-ply laminated plates with fully clamped boundaries. However this problem can be solved by the approach developed here, i.e., combining the superposition and state-space techniques (CSST) without difficulty.

The major difference between a fully clamped plate and the plates analyzed before lies in that one more boundary is required to be released by adding more components to the general solution of the former plate. Therefore for the classical theory, the general solution can still be expressed in equation (3.6.1), but the k should take on the values up to 8, while for the first-order and third-order shear deformation theory, the general solution is given by equation (3.6.2) and K^{*} is set equal to 8 and 12, respectively.

The procedures to obtain the solutions for the components as well as the original plate are exactly the same as for the plates with two adjacent or three edges clamped and the rest simply supported. We have presented detail descriptions of solution techniques for the latter plates.

Therefore it is unnecessary to repeat them here.

In writing the computational code for evaluating the fundamental frequencies, only one more subroutine will be added to include the effects of the additional clamped edge. This is easy to be realized by the standard algorithm coded during this study.

3.8 Cantilever Plates with Point Supports

In order to demonstrate that the approach developed in this study can be applied to solve more complicated problems, a cantilever plate with a point support along a free edge is considered in this section.

Suppose such a plate has one edge clamped along x=0, other three edges completely free and a point support at (a,y_i). In the case of classical theory, the general solution for the plate is constructed by nine components as follows:

$$\begin{Bmatrix} u(x,y,t) \\ v(x,y,t) \\ w(x,y,t) \end{Bmatrix} = \sum_{k=1}^{9} \begin{Bmatrix} u^k(x,y,t) \\ v^k(x,y,t) \\ w^k(x,y,t) \end{Bmatrix} \tag{3.8.1}$$

It is noted that, unlike the plates studied before, those components have to be imposed with so-called slip shear conditions in order to satisfy the boundary conditions of the original plate. The slip shear conditions refer to those with zero vertical edge reactions, zero in-plane forces, zero slope and zero in-plane displacements. For instance, the first component solution has the following boundary conditions,

along the edges y=0 and y=b,

$$u^1 = w^1_{,y} = N^1_2 = M^1_{2,y} + 2M^1_{6,x} = 0 \tag{3.8.2}$$

along the edge x=0,

$$u^1 = w^1 = N^1_6 = 0, \quad M^1_1 = M^*_1 \tag{3.8.3}$$

along the edge x=a,

$$v^1 = w^1_{,x} = N^1_1 = M^1_{2,x} + 2M^1_{6,y} = 0 \tag{3.8.4}$$

The boundary conditions of the second component are identical to those of the first component except that the amplitude of a distributed oscillating bending moment M_1^* is replaced by the amplitude of a corresponding in-plane force. Therefore the displacement solutions for the first two components can be written in a uniform expression,

$$\begin{Bmatrix} u^j(x,y,t) \\ v^j(x,y,t) \\ w^j(x,y,t) \end{Bmatrix} = \sum_{n=0,1}^{\infty} \begin{Bmatrix} U^j_n(x)\sin K_n y \\ V^j_n(x)\cos K_n y \\ W^j_n(x)\cos K_n y \end{Bmatrix} e^{i\omega t} \tag{3.8.5}$$

$$(j=1,\ 2)$$

Since the steps to obtain the solutions to the unknown functions $U^j_n(x)$, $V^j_n(x)$ and $W^j_n(x)$ in equation (3.8.5) are exactly the same as for the plates studied before, we do not present them and instead give more discussion of the remaining components.

The third and fourth components have the same slip shear conditions. However their driven edges are subjected to a harmonic rotation or displacement of circular frequency ω, which can be written as,

along the edges x=0,

$$u^j - w^j - N_6^j - M_1^j - 0 \qquad (j-3,\ 4) \tag{3.8.6}$$

along the edge x=a,

$$v^j - w^j_{,x} - N_1^j - M^j_{2,x} + 2M^j_{6,y} - 0 \qquad (j-3,\ 4) \tag{3.8.7}$$

along the edge y=b,

$$u^j - w^j_{,y} - N_2^j - M^j_{2,y} + 2M^j_{6,x} - 0 \qquad (j-3,\ 4) \tag{3.8.8}$$

along the edge y=0,

$$u^3 = N_2^3 = M_2^3 + 2M_6^3 = 0; \quad w_{,y}^3 = w_{,y}^{3*}$$
$$w_{,y}^4 = N_2^4 = M_2^3 + 2M_6^3 = 0; \quad u^4 = u^{4*} \tag{3.8.9}$$

where the quantities, $w_{,y}^{3*}$ and u^{4*} represent the distributed amplitudes of the harmonic rotation and displacement expressed by the series

$$w_{,y}^{3*} = \sum_{m=1,3}^{\infty} F_m \cos K_m x/2 \tag{3.8.10}$$

and

$$u^{4*} = \sum_{m=1,3}^{\infty} G_m \sin K_m x/2 \tag{3.8.11}$$

The general solutions for the third and fourth components are given by

$$\begin{Bmatrix} u^j(x,y,t) \\ v^j(x,y,t) \\ w^j(x,y,t) \end{Bmatrix} = \sum_{m=1,3}^{\infty} \begin{Bmatrix} U_n^j(y) \sin K_n x/2 \\ V_n^j(y) \cos K_n x/2 \\ W_n^j(y) \sin K_n x/2 \end{Bmatrix} e^{i\omega t} \tag{3.8.12}$$
$$(j = 3,\ 4)$$

The boundary conditions for the fifth to eighth components can be obtained following the same procedure as for the third and fourth components. This can be realized by changing the driven edge clockwise from the edge y=0 to the edge y=b while keeping the slip shear conditions for the remaining edges. For brevity, they are not presented here.

Next we consider the last component. Since there is a point support on the free edge of this component, we can describe its boundary conditions as follows:

at the boundaries y=0 and y=b,

$$u^9 = w^9_{,y} = N_2^9 = M^9_{2,y} + 2M^9_{6,x} = 0 \tag{3.8.13}$$

at the boundary x=0,

$$u^9 = w^9 = N_6^9 = M_1^9 = 0 \tag{3.8.14}$$

at the boundary x=a,

$$\begin{aligned} & v^9 = w^9_{,x} = N_1^9 = 0, \\ & M^9_{1,x} + 2M^9_{6,y} = V^* \end{aligned} \tag{3.8.15}$$

at the point (a, y_i),

$$w^9(a, y_i, t) = 0 \tag{3.8.16}$$

In equations (3.8.15) the quantity V^* denotes the amplitude of the vertical edge reaction which can be represented by a concentrated force P_i acting at the point (a, y_i). The concentrated force is expanded as a Dirac function in a series,

$$V^* = \sum_{n=0,1}^{\infty} P^* \cos K_n y_i \cos K_n y$$

$$P^* = \frac{2P_i}{\delta}, \quad \delta = \begin{cases} 2 & if \;\; n=0 \\ 1 & else \end{cases} \qquad (3.8,17)$$

It is appreciated that the solution for the ninth component has the same form as that of first one written in equations (3.8.5). It is also easy to verify that the boundary conditions given in equations (3.8.13) are satisfied identically by the displacement solution in equation (3.8.5).

After obtaining the solutions for all of the nine components, we can calculate the fundamental frequency by introducing the boundary conditions of the original problem and finding the non-trivial solutions of the eigenvalue equations. These boundary conditions require that the slope and the displacements must vanish along the edge x=0, the bending moments and vertical edge as well as the in-plane forces must vanish along other three edges, and the displacement at the supporting point must be zero.

Chapter 4

Cross-ply Laminated Plates

We now turn our attention to another type of laminated plates classified as cross-ply laminated plates. A combination of the superposition and state-space techniques (CSST) as established in Chapter 3 will be employed to analyze these plates. Depending on whether the coupling exists between the generalized displacements of the middle surface involved in the equations of motion, the free vibrations of symmetric and antisymmetric cross-ply laminated plates will be analyzed respectively. Also the three theories, that is, classical theory (CT), first-order shear deformation theory (FSDT) and third-order deformation theory (TSDT) will be used in the analysis.

4.1 Governing Equations

Cross-ply laminated plates indicate the plates with multiple specially orthotropic laminae (or layers) having their major principal material directions alternating at 0^0 and 90^0 with respect to the plate coordinate axes. For such plates, the following plate stiffnesses are zero:

$$A_{i6}=B_{i6}=D_{i6}=E_{i6}=F_{i6}=H_{i6}=0 \quad (i=1,\ 2)$$
$$A_{45}=D_{45}=F_{45}=0 \tag{4.1.1}$$

We substitute this equation into the strain-resultant relations described by equations (2.1.10), (2.2.7) and (2.3.9), then insert them in the equations of motion given by equations (2.1.14), (2.2.9) and (2.3.4), and set $1/R_1$ and $1/R_2$ equal to zero. As a result, the equations of motion are expressed in terms of the displacements as follows:

CT:

$$\begin{bmatrix} L_{11} & L_{12} & L_{13} \\ & L_{22} & L_{23} \\ sym. & & L_{33} \end{bmatrix} \begin{Bmatrix} u \\ v \\ w \end{Bmatrix} = 0 \tag{4.1.2}$$

where

$$\begin{aligned}
L_{11} &= A_{11}d_{xx}+A_{66}d_{yy}-I_1d_{tt} \\
L_{12} &= (A_{12}+A_{66})\,d_{xy} \\
L_{13} &= I_2d_{xtt}-B_{11}d_{xxx}-(B_{12}+2B_{66})\,d_{xyy} \\
L_{22} &= A_{66}d_{xx}+A_{22}d_{yy}-I_1d_{tt} \\
L_{23} &= I_2d_{ytt}-B_{22}d_{yy}-(B_{12}+2B_{66})\,d_{yxx} \\
L_{33} &= D_{11}d_{xxxx}+2(D_{12}+2D_{66})\,d_{xxyy}+ \\
&\qquad D_{22}d_{yyyy}-I_3\,(d_{xxtt}+d_{yytt})+I_1d_{tt}
\end{aligned} \tag{4.1.3}$$

FSDT and TSDT:

$$
\begin{bmatrix} L_{11} & L_{12} & L_{13} & L_{14} & L_{15} \\ & L_{22} & L_{23} & L_{24} & L_{25} \\ & & L_{33} & L_{34} & L_{35} \\ & & & L_{44} & L_{45} \\ sym. & & & & L_{55} \end{bmatrix} \begin{Bmatrix} u \\ v \\ w \\ \chi \\ \varphi \end{Bmatrix} = 0 \tag{4.1.4}
$$

It is noted that although FSDT and TSDT have the same form of operator equation described by equation (4.1.4), the differential elements are different. For FSDT, we can write

$$
\begin{aligned}
&L_{11}=A_{11}d_{tt}+A_{66}d_{yy}-I_1d_{tt} \\
&L_{12}=(A_{12}+A_{66})d_{xy};\ L_{13}=0 \\
&L_{14}=B_{11}d_{xx}+B_{66}d_{yy}-I_2d_{tt} \\
&L_{15}=(B_{12}+B_{66})d_{xy} \\
&L_{22}=A_{66}d_{xx}+A_{22}d_{yy}-I_1d_{tt} \\
&L_{23}=0;\quad L_{24}=L_{15} \\
&L_{25}=B_{66}d_{xx}+B_{22}d_{yy}-I_2d_{tt} \\
&L_{33}=I_1d_{tt}-A_{55}d_{xx}-A_{44}d_{yy} \\
&L_{34}=-A_{55}d_x;\quad L_{35}=-A_{44}d_y \\
&L_{44}=D_{11}d_{xx}+D_{66}d_{yy}-A_{55}-I_3d_{tt} \\
&L_{45}=(D_{12}+D_{66})d_{xy} \\
&L_{55}=D_{66}d_{xx}+D_{22}d_{yy}-A_{44}-I_3d_{tt}
\end{aligned} \tag{4.1.5}
$$

For TSDT, these elements become

$$L_{11}=A_{11}d_{xx}+A_{66}d_{yy}-\bar{I}_1d_{tt};\quad L_{12}=(A_{12}+A_{66})d_{xy};$$

$$L_{13}=\bar{I}_3d_{xtt}-c_2[E_{11}d_{xxx}+(E_{12}+2E_{66})d_{xyy}]$$

$$L_{14}=(B_{11}-c_2E_{11})d_{xx}+(B_{66}-c_2E_{66})d_{yy}-\bar{I}_2d_{tt};$$

$$L_{15}=[B_{12}+B_{66}-c_2(E_{12}+E_{66})]d_{xy}$$

$$L_{22}=A_{66}d_{xx}+a_{22}d_{yy}-\hat{I}_1d_{tt};\quad L_{23}=\hat{I}_3d_{ytt}-c_2[E_{22}d_{yyy}+(E_{12}+2E_{66})d_{xxy}]$$

$$L_{24}=(B_{66}-c_2E_{66}+B_{12}-E_{12}c_2)d_{xy};\quad L_{25}=(B_{22}-c_2E_{22})d_{yy}+(B_{66}-E_{66}c_2)d_{xx}-\hat{I}_2d_{tt}$$

$$L_{33}=[c_1(D_{55}-c_1F_{55})-(A_{55}-c_1D_{55}]d_{xx}+[c_1(D_{44}-c_1F_{44})-(A_{44}-c_1D_{44})]d_{yy}$$

$$+c_2^2[H_{11}d_{xxxx}+2(H_{12}+2H_{66})d_{xxyy}+H_{22}d_{yyyy}]+I_1d_{tt}-c_2^2(d_{xxtt}+d_{yytt})$$

$$L_{34}=[c_1(D_{55}-c_1F_{55})-(A_{55}-c_1D_{55})]d_x-c_2\{(F_{11}-c_2H_{11})d_{xxx}+[2F_{66}+F_{12}-$$

$$c_2(H_{12}+2H_{66}]d_{xxyy}\}+\bar{I}_5d_{xtt};$$

$$L_{35}=[c_1(D_{44}-c_1F_{44})-(A_{55}-c_1D_{44})]d_y-c_2(F_{22}-c_2H_{22})d_{yyy}-c_2[2F_{66}+F_{12}-$$

$$c_2(2H_{66}+H_{12})]d_{xxy}+\hat{I}_5d_{ytt}$$

$$L_{44}=c_1(D_{55}-c_1F_{55})-(A_{55}-c_1D_{55})+(D_{11}-2c_2F_{11}+c_2^2H_{11})d_{xx}+(D_{66}-2c_2F_{66}+$$

$$c_2^2H_{66})d_{yy}-\bar{I}_4d_{tt}$$

$$L_{45}=[D_{12}+D_{66}-2c_2(F_{12}+F_{66})+c_2^2(H_{12}+H_{66})]d_{xy}$$

$$L_{55}=c_1(D_{44}-c_1F_{44})-(A_{44}-c_1D_{44})+(D_{22}-2c_2F_{22}+c_2^2H_{22})d_{yy}+(D_{66}-2c_2F_{66}+$$

$$c_2^2H_{65})d_{xx}-\hat{I}_4d_{tt}$$

(4.1.6)

All the symbols used here are described in Chapter 2.

4.2 Symmetric Cross-ply Laminated Plates

When the plate has an odd number of cross-ply laminae and is symmetric with respect to the middle surface, the so-called symmetric cross-ply laminated plate is formed. In this case, further simplification can be realized, that is,

$$B_{ij} = E_{ij} = 0 \quad (i,\, j = 1,\, 2,\, 4,\, 5,\, 6) \tag{4.2.1}$$

Equation (4.2.1) shows that the coupling between bending and extension is eliminated. Furthermore, substituting equation (4.2.1) into the governing equations described by equations (4.1.1) and (4.1.2) leads to the uncoupling between displacements u, v and w, χ, φ. Thus the governing equations for symmetric cross-ply laminated plates become

CT:

$$Lw = 0 \tag{4.2.2}$$

where

$$L = D_{11}d_{xxxx} + 2(D_{12} + 2D_{66})d_{xxyy} + D_{22}d_{yyyy} + I_1 d_{tt} - I_3(d_{xxtt} + d_{yytt}) \tag{4.2.3}$$

FSDT and TSDT:

$$\begin{bmatrix} L_{11} & L_{12} & L_{13} \\ & L_{22} & L_{23} \\ sym. & & L_{33} \end{bmatrix} \begin{Bmatrix} w \\ \chi \\ \varphi \end{Bmatrix} = 0 \tag{4.2.4}$$

Here for FSDT, L_{ij} are given by

$$\begin{aligned} L_{11} &= I_1 d_{tt} - A_{55} d_{xx} - A_{44} d_{yy} \\ L_{12} &= -A_{55} d_{x} \\ L_{13} &= -A_{44} d_{y} \\ L_{22} &= D_{11} d_{xx} + D_{66} d_{yy} - A_{55} - I_3 d_{tt} \\ L_{23} &= (D_{12} + D_{66}) d_{xy} \\ L_{33} &= D_{66} d_{xx} + D_{22} d_{yy} - A_{44} - I_3 d_{tt} \end{aligned} \tag{4.2.5}$$

For TSDT, they become

$$L_{11}=[c_1(D_{55}-c_1F_{55})-(A_{55}-c_1D_{55})]d_{xx}+[c_1(D_{44}-c_1F_{44})-(A_{44}-c_1D_{44})]d_{yy}+$$
$$c_2^2[H_{11}d_{xxxx}+2(H_{12}+2H_{66})d_{xxyy}+4H_{22}d_{yyyy}]+I_1d_{tt}-c_2^2I_7(d_{xxtt}+d_{yytt})$$
$$L_{12}=[c_1(D_{55}-c_1F_{55})-(A_{55}-c_1D_{55})]d_x-c_2(F_{11}-c_2H_{11})d_{yyy}-c_2[F_{12}+2F_{66}-$$
$$c_2(H_{12}+2H_{66})]d_{xyy}+\bar{I}_5d_{xtt}$$
$$L_{13}=[c_1(D_{44}-c_1F_{44})-(A_{55}-c_1D_{44})]d_y-c_2(F_{22}-c_2H_{22})d_{yyy}-c_2[F_{12}+2F_{66}-$$
$$c_2(H_{12}+2H_{66})]d_{xxy}+\bar{I}_5d_{ytt} \tag{4.2.6}$$
$$L_{22}=c_1(D_{55}-c_1F_{55})-(A_{55}-c_1D_{55})+(D_{11}-2c_2F_{11}+c_2^2H_{11})d_{xx}+(D_{66}-2c_2F_{66}+$$
$$c_2^2H_{66})d_{yy}-\bar{I}_4d_{tt}$$
$$L_{23}=[D_{12}+D_{66}-2(F_{12}c_2+F_{66})+c_2^2(H_{12}+H_{66})]d_{xy}$$
$$L_{33}=c_1(D_{44}-c_1F_{44})-(A_{44}-c_1D_{44})+(D_{22}-2c_2F_{22}+c_2^2H_{22})d_{yy}+(D_{66}-2c_2F_{66}+$$
$$c_2^2H_{66})d_{xx}-\bar{I}_4d_{tt}$$

Now, let us consider a plate with two adjacent edges clamped and the rest simply supported. The general solution is composed of several components as described in Chapter 3, and we may therefore write,

CT:

$$w(x,y,t)=\sum_{k=1}^{2}w^k(x,y,t) \tag{4.2.7}$$

FSDT and TSDT:

$$\begin{Bmatrix} w(x,y,t) \\ \chi(x,y,t) \\ \varphi(x,y,t) \end{Bmatrix} = \sum_{k=1}^{K^*} \begin{Bmatrix} w^k(x,y,t) \\ \chi^k(x,y,t) \\ \varphi^k(x,y,t) \end{Bmatrix} \tag{4.2.8}$$

where $K^*=2$, for FSDT and $K^*=4$, for TSDT.

As discussed before, each of the components in equations (4.2.7) and (4.2.8) has simple support except that an oscillating bending moment or in-plane force is distributed along one edge. For the same reason as stated for the antisymmetric angle-ply plates, only the solution for the first component needs to be presented here.

Along the edge $x=0$ of the first component, there is a distributed harmonic bending moment of circular frequency ω and with a distributed amplitude given by

$$M_1^* = \sum_{n=1,2}^{\infty} E_n \sin K_n y \tag{4.2.9}$$

where $K_n = n\pi/b$.

The solution for the first component is taken in the form,

CT:

$$w^1(x,y,t) = \sum_{n=1}^{\infty} W_n^1(x) \sin K_n y e^{i\omega t} \tag{4.2.10}$$

FSDT and TSDT:

$$\begin{Bmatrix} w^1(x,y,t) \\ \chi^1(x,y,t) \\ \varphi^1(x,y,t) \end{Bmatrix} = \sum_{n=1}^{\infty} \begin{Bmatrix} W_n^1(x)\sin K_n y \\ \chi_n^1(x)\sin K_n y \\ \varphi_n^1(x)\cos K_n y \end{Bmatrix} e^{i\omega t} \tag{4.2.11}$$

Equation (4.2.11) is subject to the prescribed boundary conditions as follows,

at the boundary x=0,

$$\begin{aligned} &CT: && w=0, \quad M_1=M_1^* \\ &FSDT: && w=\varphi=0, \quad M_1=M_1^* \\ &TSDT: && w=\varphi=P_1=0, \quad M_1=M_1^* \end{aligned} \tag{4.2.12}$$

at the boundary x=a,

$$\begin{aligned} &CT: && w=M_1=0 \\ &FSDT: && w=\varphi=M_1=0 \\ &TSDT: && w=\varphi=P_1=M_1=0 \end{aligned} \tag{4.2.13}$$

where M^*_1 is given by equation (4.2.9), and other symbols are described in Chapter 2.

Substituting equations (4.2.10) and (4.2.11) into the governing equations given by equations (4.2.2) and (4.2.4) results in the ordinary differential equations, CT:

$$W^1_{n,xxxx} = C_1 w^1_n + C_2 W^1_{n,xx} \tag{4.2.14}$$

FSDT:

$$\begin{aligned} W^1_{n,xx} &= C_1 W^1_n + C_2 X^1_{n,x} + C_3 Y^1_n \\ X^1_{n,yy} &= C_4 W^1_{n,x} + C_5 X^1_n + C_6 Y^1_{n,x} \\ Y^1_{n,xx} &= C_7 W^1_n + C_8 X^1_{n,x} + C_9 Y^1_n \end{aligned} \tag{4.2.15}$$

TSDT:

$$\begin{aligned} X^1_{n,xx} &= C_1 W^1_{n,x} + C_2 W^1_{n,xxx} + C_3 X^1_n + C_4 Y^1_{n,x} \\ Y^1_{n,xx} &= C_5 W^1_n + C_6 W^1_{n,xx} + C_7 X^1_{n,x} + C_8 Y^1_n \\ W^1_{n,xxxx} &= C_9 W^1_n + C_{10} W^1_{n,xx} + C_{11} X^1_{n,x} + C_{12} Y^1_n \end{aligned} \tag{4.2.16}$$

where C_i are listed in the Appendix II.

Equations (4.2.14)-(4.2.16) may be written in a state-space form by introducing the variables,

CT:

$$Z_1 = W^1_n; \quad Z_2 = W^1_{n,x}; \quad Z_3 = W^1_{n,xx}; \quad Z_4 = W^1_{n,xxxx} \tag{4.2.17}$$

FSDT:

$$Z_1 = W_n^1;\quad Z_2 = W_{n,x}^1;\quad Z_3 = X_n^1$$
$$Z_4 = X_{n,x}^1;\quad Z_5 = Y_n^1;\quad Z_6 = Y_{n,x}^1 \tag{4.2.18}$$

TSDT:

$$Z_1 = X_n^1;\quad Z_2 = X_{n,x}^1;\quad Z_3 = Y_n^1;\quad Z_4 = Y_{n,x}^1$$
$$Z_5 = W_n^1;\quad Z_6 = W_{n,x}^1;\quad Z_7 = W_{n,xx}^1;\quad Z_8 = W_{n,xxx}^1 \tag{4.2.19}$$

Once we substitute equations (4.2.17)-(4.2.19) into equations (4.2.14)-(4.2.16), a unified matrix form can be written as

$$\{Z(x)\}_{,x} = [C]\{Z(x)\} \tag{4.2.20}$$

Depending on the specific theory used, the matrices [C] and {Z(x)} are given by equations (4.2.21), (4.2.22) or (4.2.23).

CT:

$$[C] = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ C_1 & 0 & C_2 & 0 \end{bmatrix};\quad \{Z\} = \begin{Bmatrix} Z_1 \\ Z_2 \\ Z_3 \\ Z_4 \end{Bmatrix} \tag{4.2.21}$$

FSDT:

$$[C] = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ C_1 & 0 & 0 & C_2 & C_3 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & C_4 & C_5 & 0 & 0 & C_6 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ C_7 & 0 & 0 & C_8 & C_9 & 0 \end{bmatrix}; \quad \{Z\} = \begin{Bmatrix} Z_1 \\ Z_2 \\ Z_3 \\ Z_4 \\ Z_5 \\ Z_6 \end{Bmatrix} \tag{4.2.22}$$

TSDT:

$$[C] = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ C_3 & 0 & 0 & C_4 & 0 & C_1 & 0 & C_2 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & C_7 & C_8 & 0 & C_5 & 0 & C_6 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & C_{11} & C_{12} & 0 & C_9 & 0 & C_{10} & 0 \end{bmatrix}; \quad \{Z\} = \begin{Bmatrix} Z_1 \\ Z_2 \\ Z_3 \\ Z_4 \\ Z_5 \\ Z_6 \\ Z_7 \\ Z_8 \end{Bmatrix} \tag{4.2.23}$$

A general solution of equation (4.2.20) can be written as

$$\{Z(x)\} = [R] \begin{bmatrix} e^{\lambda_1 x} & & & & \\ & e^{\lambda_2 x} & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & e^{\lambda_k x} \end{bmatrix} \{P\} \tag{4.2.24}$$

where k=4 for CT, k=6 for FSDT, and k=8 for TSDT. λ_i denote the distinct eigenvalues, [R] represents the eigenvector matrix of [C] and {P} is a constant column matrix dependent on the associated boundary conditions described by equations (4.2.12) and (4.2.13).

Following the same procedures, we can easily obtain solutions for the remaining components. Once the solutions for all the components are produced, the prescribed boundary conditions of the original problem must be satisfied, which are given by

at the boundary y=0,

$$\begin{aligned} &CT: && w_{,y} = 0 \\ &FSDT: && \varphi = 0 \\ &TSDT: && w_{,y} = \varphi = 0 \end{aligned} \tag{4.2.25}$$

at the boundary x=0,

$$\begin{aligned} &CT: && w_{,x} = 0 \\ &FSDT: && \chi = 0 \\ &TSDT: && w_{,x} = \chi = 0 \end{aligned} \tag{4.2.26}$$

Enforcement of equations (4.2.25) and (4.2.26) yields a homogeneous equations containing the circular frequency ω,

$$[M]\{\delta\} = 0 \tag{4.2.27}$$

For the non-trivial solution, the determinant of eigenvalue matrix should be zero, from which the natural frequencies are obtained by searching for ω.

4.3 Antisymmetric Cross-ply Laminated Plates

An antisymmetric cross-ply plate consists of an even number of orthotropic laminae (or layers) laid on each other with principal material directions alternating at 0^o and 90^o to the plate axes. As shown in equation (4.1.1), such plates have some zero stiffnesses, but they do possess coupling between bending and extension due to B_{11}, B_{12}, B_{22}, etc. The governing equations for the plates are described by equations (4.1.2) and (4.2.1), depending on the type of theory employed.

In order to simplify the analysis and demonstrate the approach developed in this research, we once again consider a plate having two adjacent boundaries clamped and others simply supported.

A general solution for the plate is superimposed by appropriate number of components and can be expressed as,

CT:

$$\begin{Bmatrix} u(x,y,t) \\ v(x,y,t) \\ w(x,y,t) \end{Bmatrix} = \sum_{k=1}^{4} \begin{Bmatrix} u^k(x,y,t) \\ v^k(x,y,t) \\ w^k(x,y,t) \end{Bmatrix} \tag{4.3.1}$$

FSDT and TSDT:

$$\begin{Bmatrix} u(x,y,t) \\ v(x,y,t) \\ w(x,y,t) \\ \chi(x,y,t) \\ \varphi(x,y,t) \end{Bmatrix} = \sum_{k=1}^{K^*} \begin{Bmatrix} u^k(x,y,t) \\ v^k(x,y,t) \\ w^k(x,y,t) \\ \chi^k(x,y,t) \\ \varphi^k(x,y,t) \end{Bmatrix} \tag{4.3.2}$$

where $K^*=4$, for FSDT, and $K^*=6$, for TSDT. Each of the components has simply supported boundary conditions but a harmonic bending moment or an in-plane force is imposed along one boundary. Taking the first component, we can write the distributed amplitude of the harmonic bending moment of circular frequency ω, along the edge $x=0$, in a series form,

$$M_1^* = \sum_{n=1,2}^{\infty} E_n \sin K_n y \tag{4.3.3}$$

here $K_n=n\pi/b$ and E_n are the constant coefficients of Fourier series.

A Lévy-type solution for this component is given by

CT:

$$\begin{Bmatrix} u^1(x,y,t) \\ v^1(x,y,t) \\ w^1(x,y,t) \end{Bmatrix} = \sum_{n=1}^{\infty} \begin{Bmatrix} U_n^1(x)\sin K_n y \\ V_n^1(x)\cos K_n y \\ W_n^1(x)\sin K_n y \end{Bmatrix} e^{i\omega t} \tag{4.3.4}$$

FSDT and TSDT:

$$\begin{Bmatrix} u^1(x,y,t) \\ v^1(x,y,t) \\ w^1(x,y,t) \\ \chi^1(x,y,t) \\ \varphi^1(x,y,t) \end{Bmatrix} = \sum_{n=1}^{\infty} \begin{Bmatrix} U_n^1(x)\sin K_n y \\ V_n^1(x)\cos K_n y \\ W_n^1(x)\sin K_n y \\ \chi_n^1(x)\sin K_n y \\ \varphi_n^1(x)\cos K_n y \end{Bmatrix} e^{i\omega t} \tag{4.3.5}$$

By substituting equations (4.3.4) and (4.3.5) into the governing equations (4.1.2) and (4.1.4) and following established procedures used in the early sections, we can write equations (4.3.6)-(4.3.8) for the three theories,

CT:

$$\begin{aligned} U^1_{n,xx} &= C_1 U^1_n + C_2 V^1_{n,x} + C_3 W^1_{n,x} + C_4 W^1_{n,xxx} \\ V^1_{n,xx} &= C_5 U^1_{n,x} + C_6 V^1_n + C_7 W^1_n + C_8 W^1_{n,xx} \\ W^1_{n,xx} &= C_9 U^1_{n,x} + C_{10} V^1_n + C_{11} W^1_n + C_{12} W^1_{n,xx} \end{aligned} \tag{4.3.6}$$

FSDT:

$$\begin{aligned} U^1_{n,xx} &= C_1 U^1_n + C_2 V^1_{n,x} + C_3 W^1_{n,x} + C_4 X^1_n + C_5 Y^1_{n,x} \\ V^1_{n,xx} &= C_6 U^1_{n,x} + C_7 V^1_n + C_8 W^1_n + C_9 X^1_{n,x} + C_{10} Y^1_n \\ W^1_{n,xx} &= C_{11} U^1_{n,x} + C_{12} V^1_n + C_{13} W^1_n + C_{14} X^1_{n,x} + C_{15} Y^1_n \\ X^1_{n,xx} &= C_{16} U^1_n + C_{17} V^1_{n,x} + C_{18} W^1_{n,x} + C_{19} X^1_n + C_{20} Y^1_{n,x} \\ Y^1_{n,xx} &= C_{21} U^1_{n,x} + C_{22} V^1_n + C_{23} W^1_n + C_{24} X^1_{n,x} + C_{25} Y^1_n \end{aligned} \tag{4.3.7}$$

TSDT:

$$
\begin{aligned}
U^1_{n,xx} &= C_1 U^1_n + C_2 V^1_{n,x} + C_3 W^1_{n,x} + C_4 W^1_{n,xxx} + C_5 X^1_n + C_6 Y^1_{n,x} \\
V^1_{n,xx} &= C_7 U^1_{n,x} + C_8 V^1_n + C_9 W^1_n + C_{10} W^1_{n,xx} + C_{11} X^1_{n,x} + C_{12} Y^1_n \\
W^1_{n,xx} &= C_{13} U^1_{n,x} + C_{14} V^1_n + C_{15} W^1_n + C_{16} W^1_{n,xx} + C_{17} X^1_{n,x} + C_{18} Y^1_n \\
X^1_{n,xx} &= C_{19} U^1_n + C_{20} V^1_{n,x} + C_{21} W^1_{n,x} + C_{22} W^1_{n,xxx} + C_{23} X^1_n + C_{24} Y^1_{n,x} \\
Y^1_{n,xx} &= C_{25} U^1_{n,x} + C_{26} V^1_n + C_{27} W^1_n + C_{28} W^1_{n,xx} + C_{29} X^1_{n,x} + C_{30} Y^1_n
\end{aligned}
\tag{4.3.8}
$$

The constants C_i used in the above equations are given in the Appendix II.

Using state-space technique and defining the variables Z_i for CT, FSDT and TSDT theories as follows,

CT:

$$
\begin{aligned}
&Z_1 = U^1_n;\quad Z_2 = U^1_{n,x};\quad Z_3 = V^1_n;\quad Z_4 = V^1_{n,x};\\
&Z_5 = W^1_n;\quad Z_6 = W^1_{n,x};\quad Z_7 = W^1_{n,xx};\quad Z_8 = W^1_{n,xxx}
\end{aligned}
\tag{4.3.9}
$$

FSDT:

$$
\begin{aligned}
&Z_1 = U^1_n;\quad Z_2 = U^1_{n,x};\quad Z_3 = V^1_n;\quad Z_4 = V^1_{n,x};\quad Z_5 = W^1_n;\\
&Z_6 = W^1_{n,x};\quad Z_7 = X^1_n;\quad Z_8 = X^1_{n,x};\quad Z_9 = Y^1_n;\quad Z_{10} = Y^1_{n,x}
\end{aligned}
\tag{4.3.10}
$$

TSDT:

$$Z_1 = U_n^1;\ Z_2 = U_{n,x}^1;\ Z_3 = V_n^1;\ Z_4 = V_{n,x}^1;\ Z_5 = W_n^1;\ Z_6 = W_{n,x}^1;$$
$$Z_7 = W_{n,xx}^1;\ Z_8 = W_{n,xxx}^1;\ Z_9 = X_n^1;\ Z_{10} = X_{n,x}^1;\ Z_{11} = Y_n^1;\ Z_{12} = Y_{n,x}^1 \qquad (4.3.11)$$

we can simplify equations (4.3.6)-(4.3.8) to the following form,

$$\{Z(x)\}_{,x} = [C]\{Z(x)\} \qquad (4.3.12)$$

Equations (4.3.13), (4.3.14) and (4.3.15) give the form of the matrix [C] and the vector {Z(x)} for the CT, FSDT and TSDT theories respectively.

$$[C] = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ C_1 & 0 & 0 & C_2 & 0 & C_3 & 0 & C_4 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & C_5 & C_6 & 0 & C_7 & 0 & C_8 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & C_9 & C_{10} & 0 & C_{11} & 0 & 12 & 0 \end{bmatrix}; \quad \{Z\} = \begin{Bmatrix} Z_1 \\ Z_2 \\ Z_3 \\ Z_4 \\ Z_5 \\ Z_6 \\ Z_7 \\ Z_8 \end{Bmatrix} \qquad (4.3.13)$$

$$[C]=\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ C_1 & 0 & 0 & C_2 & 0 & C_3 & C_4 & 0 & 0 & C_5 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & C_6 & C_7 & 0 & C_8 & 0 & 0 & C_9 & C_{10} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & C_{11} & C_{12} & 0 & C_{13} & 0 & 0 & C_{14} & C_{15} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ C_{16} & 0 & 0 & C_{17} & 0 & C_{18} & C_{19} & 0 & 0 & C_{20} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & C_{21} & C_{22} & 0 & C_{23} & 0 & 0 & C_{24} & C_{25} & 0 \end{bmatrix}; \quad \{Z\}=\begin{Bmatrix} Z_1 \\ Z_2 \\ Z_3 \\ Z_4 \\ Z_5 \\ Z_6 \\ Z_7 \\ Z_8 \\ Z_9 \\ Z_{10} \end{Bmatrix} \tag{4.3.14}$$

$$[C]=\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ C_1 & 0 & 0 & C_2 & 0 & C_3 & 0 & C_4 & C_5 & 0 & 0 & C_6 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & C_7 & C_8 & 0 & C_9 & 0 & C_{10} & 0 & 0 & C_{11} & C_{12} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & C_{13} & C_{14} & 0 & C_{15} & 0 & C_{16} & 0 & 0 & C_{17} & C_{18} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ C_{19} & 0 & 0 & C_{20} & 0 & C_{21} & 0 & C_{22} & C_{23} & 0 & 0 & C_{24} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & C_{25} & C_{26} & 0 & C_{27} & 0 & C_{28} & 0 & 0 & C_{29} & C_{30} & 0 \end{bmatrix}; \quad \{Z\}=\begin{Bmatrix} Z_1 \\ Z_2 \\ Z_3 \\ Z_4 \\ Z_5 \\ Z_6 \\ Z_7 \\ Z_8 \\ Z_9 \\ Z_{10} \\ Z_{11} \\ Z_{12} \end{Bmatrix} \tag{4.3.15}$$

A general solution of equation (4.3.12) is written in the form:

$$\{Z(x)\} = [e^{[C]x}]\{P\} \tag{4.3.16}$$

where {P} is a column matrix of unknown constants determined by the following boundary conditions,

at the edge x=0,

$$\begin{array}{ll} CT: & v=w=N_1=0; \quad M_1=M_1^* \\ FSDT: & v=w=\varphi=N_1=0; \quad M_1=M_1^* \\ TSDT: & v=w=\varphi=N_1=P_1=0; \quad M_1=M_1^* \end{array} \tag{4.3.17}$$

and along the edge x=a,

$$\begin{array}{ll} CT: & v=w=N_1=M_1=0 \\ FSDT: & v=w=\varphi=N_1=M_1=0 \\ TSDT: & v=w=\varphi=N_1=P_1=M_1=0 \end{array} \tag{4.3.18}$$

in which M^*_1 is given by equation (4.3.3). The other matrix $[e^{[Cx]}]$ involved in equation (4.3.16) is defined in equation (4.3.19),

$$[e^{[Cx]}] = [R]\begin{bmatrix} e^{\lambda_1 x} & & & & \\ & e^{\lambda_2 x} & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & e^{\lambda_k x} \end{bmatrix} \tag{4.3.19}$$

here k takes on the values of 8, 10 and 12, depending on CT, FSDT and TSDT respectively, λ_i represent the distinct eigenvalues and [R] is the matrix of the corresponding eigenvector.

The boundary conditions of the original plate require that

at the edge y=0,

$$\begin{aligned} &CT: && v = w_{,y} = 0 \\ &FSDT: && v = \varphi = 0 \\ &TSDT: && v = w_{,y} = \varphi = 0 \end{aligned} \tag{4.3.20}$$

and at the edge x=0,

$$\begin{aligned} &CT: && u = w_{,x} = 0 \\ &FSDT: && u = \chi = 0 \\ &TSDT: && u = w_{,x} = \chi = 0 \end{aligned} \tag{4.3.21}$$

Satisfying these boundary condition leads to an eigenvalue system of the homogeneous equation which can be expressed by equation (4.2.27). However, the dimensions of the eigenvalue matrix [M] vary depending on the theory used. If k terms are used in the displacement solutions of each components, the dimensions of the [M] matrix are 2k $\times$ 2k for CT, 4k $\times$ 4k for FSDT, and 6k $\times$ 6k for TSDT case. For the non-trivial solution, the determinant of [M] must be zero. Thus the natural frequencies (eigenvalues) can be evaluated by searching ω which makes the determinant vanish.

4.4 Plates with Three and Four Edges Clamped

In this section, the cross-ply laminated plates having three and four boundaries clamped are discussed. For the case of the plate with three edges clamped, the numerical results are generated by assuming the fourth edge to be simply supported. It should be noted that the remaining edge can be imposed with any kind of classical boundary conditions, such as free or simply supported, because the solution procedures are identical.

Care should be exercised in choosing the number of components in the superposition procedure, since it would vary with the type of the theory used as well as with the symmetric or antisymmetric construction. It should be pointed out that the general solution is still expressed by equations (4.3.1) and (4.3.2), but the k (or K^*) used to indicate the number of components is different. In the case of the classical theory, k in equation (4.3.1) takes on the values from 1 to 6 and 1 to 8 separately when three and four clamped edges are considered, while studying the first-order and third-order shear deformation theories, we set K^* equal to 6 and 9 in the case of three edges clamped, and 8 and 12 in equation (4.3.2) for the fully clamped case.

We do not need to repeat the description of the procedure used to obtain the component solutions because they are exactly the same as for the plates with two edges clamped. Following well-established steps, we can easily obtain the eigenvalue matrix from which the natural frequencies can be evaluated.

Chapter 5

Cross-ply Laminated Shells

In the previous chapters we have solved the free vibration problems of laminated plates without two opposite edges simply supported. It is well known that the laminated plates can be treated as special cases of laminated shells when the principal curvatures become zero. In order to enlarge the scope of this study we now develop the analysis for the cross-ply laminated shells. Depending on whether the transverse shear deformation is considered or not, vibration analysis is carried out based on the three types of theories. It will be seen that closed-form solutions for the problems are amenable by using the combination of superposition and state-space techniques (CSST).

5.1 Classical Theory

For a cross-ply laminated shell, because of the constitutive properties some shell stiffnesses vanish, which are shown by equations (4.1.1). Therefore using the classical theory, one can write the governing equations of the shell in tensor form as follows:

$$L_{ij}u_j = 0 \tag{5.1.1}$$

where u_j represent the displacements u, v and w as j takes on the values of 1, 2 and 3, and L_{ij} are the symmetric partial differential operators, given by

$$\begin{aligned}
&L_{11} = A_{11}d_{xx} + A_{66}d_{yy} - I_1 d_{tt} \\
&L_{12} = (A_{12} + A_{66})d_{xy} \\
&L_{13} = (A_{11}/R_1 + A_{12}/R_2)d_x - B_{11}d_{xxx} \\
&-(B_{12} + 2B_{66})d_{xyy} + I_2 d_{xtt} \\
&L_{22} = A_{66}d_{xx} + A_{22}d_{yy} - I_1 d_{tt} \\
&L_{23} = (A_{12}/R_1 + A_{22}/R_2)d_y - B_{22}d_{yyy} \\
&-(B_{12} + 2B_{66})d_{xxy} + I_2 d_{ytt} \\
&L_{33} = D_{11}d_{xxxx} + 2(D_{12} + 2D_{66})d_{xxyy} + D_{22}d_{yyyy} + \\
&(A_{11}/R_1^2 + 2A_{12}/R_1/R_2 + A_{22}/R_2^2) - 2(B_{11}/R_1 + B_{12}/R_2 \\
&\quad)d_x - 2(B_{12}/R_1 + R_{22}/R_2)d_{yy} - I_3(d_{xxtt} + d_{yytt}) + I_1 d_{tt}
\end{aligned} \tag{5.1.2}$$

In equations (5.1.2) R_i donate the radii of principal curvatures of the shell and the other symbols are defined in Chapter 2.

We consider, now, a shell of dimensions a × b with two adjacent edge clamped and other two simply supported. According to the CSST developed in this research, the solution for the shell is composed of 4 components. Each component of the shell is imposed with simply supported boundary conditions except for the fact that one edge is driven by a distributed harmonic bending moment or an in-plane force. For example, we impose the boundary conditions for the first component as follows:

along the edges x=0 and x=a,

$$v^1 - w^1 - N_1^1 - M_1^1 - 0 \tag{5.1.3}$$

along the edge y=0,

$$u^1 - w^1 - N_2^1 - M_2^1 - 0 \tag{5.1.4}$$

along the edge y=b,

$$u^1 - w^1 - N_2^1 - 0; \quad M_2^1 - M_2^* \tag{5.1.5}$$

The superscript 1, used in the above equations, refers to the first component and the M^*_2 represents the distributed amplitude of the harmonic moment which is expanded in a series form,

$$M_2^* = \sum_{m=1}^{\infty} E_m \sin K_m x \tag{5.1.6}$$

where $K_m = m\pi/a$.

Following the steps described for the first component, the boundary conditions for the remaining components can be easily established. However, for brevity, we do not present them and alternatively give full development of the first component.

The displacement solutions for the first component are assumed to be in Lévy-type form, given by

$$\begin{Bmatrix} u^1(x,y,t) \\ v^1(x,y,t) \\ w^1(x,y,t) \end{Bmatrix} = \sum_{n=1}^{\infty} \begin{Bmatrix} U_m^1(y) \cos K_m x \\ V_m^1(y) \sin K_m x \\ W_m^1(y) \sin Hxmx \end{Bmatrix} e^{i\omega t} \tag{5.1.7}$$

where, ω denotes the circular frequency of free vibration. It is obvious that the solution expressed by equation (5.1.7) satisfies identically the boundary conditions (5.1.3). The unknown functions $U'_m(y)$, $V'_m(y)$ and $W'_m(y)$ need to be determined by the governing equations and associated boundary conditions.

Substituting equation (5.1.7) into (5.1.1) yields,

$$U^1_{m,yy}=C_1U^1_m+C_2V^1_{m,y}+C_3W^1_m+C_4W^1_{m,yy}$$

$$V^1_{m,yy}=C_5U^1_{m,y}+C_6V^1_m+C_7W^1_{m,y}+C_8W^1_{m,yyy} \tag{5.1.8}$$

$$W^1_{m,yyyy}=C_9U^1_m+C_{10}V^1_{m,y}+C_{11}W^1_m+C_{12}W^1_{m,yy}$$

The complete expressions for the coefficients C_i are listed in the Appendix III.

Equations (5.1.8) can be converted to the state-space form, by writing

$$\{Z(y)\}_{,y}=[C]\{Z(y)\} \tag{5.1.9}$$

in which the vector $\{Z(y)\}=\{Z_1\ Z_2\ \ldots\ Z_8\}^T$, and its components replace the unknown functions and their derivatives,

$$Z_1=U^1_m;\quad Z_2=U^1_{m,y};\quad Z_3=V^1_m;\quad Z_4=V^1_{m,y};$$

$$Z_5=W^1_m;\quad Z_6=W^1_{m,y};\quad Z_7=W^1_{m,yy};\quad Z_8=W^1_{m,yyy} \tag{5.1.10}$$

The matrix [C] is given by

$$
[C]=\begin{bmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
C_1 & 0 & 0 & C_2 & C_3 & 0 & C_4 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & C_5 & C_6 & 0 & 0 & C_7 & 0 & C_8 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
C_9 & 0 & 0 & C_{10} & C_{11} & 0 & C_{12} & 0
\end{bmatrix}
\tag{5.1.11}
$$

A general solution of equation (5.1.9) is written as,

$$
\{Z(y)\}=[R]\begin{bmatrix}
e^{\lambda_1 y} & & & & \\
& e^{\lambda_2 y} & & & \\
& & \cdot & & \\
& & & \cdot & \\
& & & & e^{\lambda_8 x}
\end{bmatrix}\{P\}
\tag{5.1.12}
$$

Here λ_i denote the distinct eigenvalues of [C], [R] represents the matrix of the corresponding eigenvectors, and [P] is a constant column matrix determined by the boundary conditions (5.1.4) and (5.1.5).

Following the procedures established for the first component, we can easily obtain the solutions for the other components. With all the component solutions available, the next step is to develop the eigenvalue matrix for the natural frequencies. This can be realized by satisfying the boundary conditions of the original problem. These boundary conditions exist along x=0 and y=0 and are given by equations (5.1.13) and (5.1.14), along the boundary x=0,

$$u = w_{,x} = 0 \tag{5.1.13}$$

along the boundary y=0,

$$v = w_{,y} = 0 \tag{5.1.14}$$

It is evident that the other boundary conditions, such as, that the transverse displacement should be zero, are satisfied automatically. By substituting the superimposed solution into equations (5.1.13) and (5.1.14), and expanding some component solution in trigonometric series (as discussed in Chapter 3), we obtain a set of 4k simultaneous homogeneous algebraic equations containing unknown coefficients E_m, etc, where k is the number of terms used in each of the component solutions. At this stage of the solution, natural frequencies (eigenvalues) are evaluated by searching ω which causes the determinant of the coefficient matrix of the algebraic equations to vanish.

5.2 First-order Shear Deformation Theory

The governing equations based on the first-order shear deformation theory are given by equation (5.1.1); however, for this case, u_i represent the displacements u, v, w, χ and φ when i =1, 2,...,5 and the operators L_{ij} are defined by equations (5.2.1).

$$
\begin{aligned}
&L_{11}=A_{11}d_{xx}+A_{66}d_{yy}-I_1d_{tt}\\
&L_{12}=(A_{12}+A_{66})d_{xy}\\
&L_{13}=(A_{11}/R_1+A_{12}/R_2)d_x\\
&L_{14}=B_{11}d_{xx}+B_{66}d_{yy}-I_2d_{tt}\\
&L_{15}=(B_{12}+B_{66})d_{xy}\\
&L_{22}=A_{66}d_{xx}+A_{22}d_{yy}-I_1d_{tt}\\
&L_{23}=(A_{12}/R_1+A_{22}/R_2)d_y;\ L_{24}=L_{15}\\
&L_{25}=B_{66}d_{xx}+B_{22}d_{yy}-I_2d_{tt}\\
&L_{33}=A_{11}/R_1^2+2A_{12}/R_1/R_2+A_{22}/R_2^2\\
&\quad +I_1d_{tt}-A_{55}d_{xx}-A_{44}d_{yy}\\
&L_{34}=(B_{11}/R_1+B_{12}/R_2-A_{55})d_x\\
&L_{35}=(B_{12}/R_1+B_{22}/R_2-A_{44})d_y\\
&L_{44}=D_{11}d_{xx}+D_{66}d_{yy}-A_{55}-I_3d_{tt}\\
&L_{45}=(D_{12}+D_{66})d_{xy}\\
&L_{55}=D_{66}d_{xx}+D_{22}d_{yy}-A_{44}-I_3d_{tt}
\end{aligned}
\tag{5.2.1}
$$

For a laminated cross-ply shell having two adjacent edges clamped and the other two simply supported, it is assumed that its solution consists of four components, written as

$$\begin{Bmatrix} u(x,y,t) \\ v(x,y,t) \\ w(x,y,t) \\ \chi(x,y,t) \\ \varphi(x,y,t) \end{Bmatrix} = \sum_{k=1}^{4} \begin{Bmatrix} u^k(x,y,t) \\ v^k(x,y,t) \\ w^k(x,y,t) \\ \chi^k(x,y,t) \\ \varphi^k(x,y,t) \end{Bmatrix} \tag{5.2.2}$$

The boundary conditions for each of the components are the same as described before.

The general solution for the first component can be written as

$$\begin{Bmatrix} u^1(x,y,t) \\ v^1(x,y,t) \\ w^1(x,y,t) \\ \chi^1(x,y,t) \\ \varphi^1(x,y,t) \end{Bmatrix} = \sum_{m=1}^{\infty} \begin{Bmatrix} U_m^1(y) \cos K_m x \\ V_m^1(y) \sin K_m x \\ W_m^1(y) \sin K_m x \\ \chi_m^1(y) \cos K_m x \\ \varphi_m^1(y) \sin K_m x \end{Bmatrix} e^{i\omega t} \tag{5.2.3}$$

Substituting equation (5.2.3) into the governing equations results in five ordinary differential equations,

$$\begin{aligned}
U^1_{m,yy} &= C_1 U^1_m + C_2 V^1_{m,y} + C_3 W^1_m + C_4 X^1_m + C_5 Y^1_{m,y} \\
V^1_{m,yy} &= C_6 U^1_{m,y} + C_7 V^1_m + C_8 W^1_{m,y} + C_9 X^1_{m,y} + C_{10} Y^1_m \\
W^1_{m,yy} &= C_{11} U^1_m + C_{12} V^1_{m,y} + C_{13} W^1_m + C_{14} X^1_m + C_{15} Y^1_{m,y} \\
X^1_{m,yy} &= C_{16} U^1_m + C_{17} V^1_{m,y} + C_{18} W^1_m + C_{19} X^1_m + C_{20} Y^1_{m,y} \\
Y^1_{m,yy} &= C_{21} U^1_{m,y} + C_{22} V^1_m + C_{23} W^1_{m,y} + C_{24} X^1_{m,y} + C_{25} Y^1_m
\end{aligned} \tag{5.2.4}$$

Once again, the coefficients C_i used in equations (5.2.4) are listed in Appendix III.

Using the state-space technique and defining the variables Z_i as

$$Z_1 = U^1_m;\ Z_2 = U^1_{m,y};\ Z_3 = V^1_m;\ Z_4 = V^1_{m,y};\ Z_5 = W^1_m;$$
$$Z_6 = W^1_{m,y};\ Z_7 = X^1_m;\ Z_8 = X^1_{m,y};\ Z_9 = Y^1_m;\ Z_{10} = Y^1_{m,y} \tag{5.2.5}$$

equations (5.2.4) can be written in the symbolic form as,

$$\{Z(y)\}_{,y} = [C]\{Z(y)\} \tag{5.2.6}$$

The column vector $\{Z(y)\} = \{Z_1\ Z_2\ \ldots\ Z_{10}\}^T$, and the matrix [C] is given by

$$[C] = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ C_1 & 0 & 0 & C_2 & C_3 & 0 & C_4 & 0 & 0 & C_5 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & C_6 & C_7 & 0 & 0 & C_8 & 0 & C_9 & C_{10} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ C_{11} & 0 & 0 & C_{12} & C_{13} & 0 & C_{14} & 0 & 0 & C_{15} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ C_{16} & 0 & 0 & C_{17} & C_{18} & 0 & C_{19} & 0 & 0 & C_{20} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & C_{21} & C_{22} & 0 & 0 & C_{23} & 0 & C_{24} & C_{25} & 0 \end{bmatrix} \tag{5.2.7}$$

A general solution of equation (5.2.6) is taken in the following form:

$$\{Z(y)\} = [e^{[C]y}]\{P\} \tag{5.2.8}$$

where {P} is a column matrix of unknown constants determined by the boundary conditions given in equations (5.2.9) and (5.2.10).

At the edge y=0,

$$u^1 = w^1 = \chi^1 = N_2^1 = M_2^1 = 0 \qquad (5.2.9)$$

At the edge y=b,

$$u^1 = w^1 = \chi^1 = N_2^1 = 0; \quad M_2 = M_2^* \qquad (5.2.10)$$

In these equations M^*_2 represents the distributed amplitude of a harmonic bending moment of circular frequency ω, given by

$$M_2^* = \sum_{m=1}^{\infty} E_m \sin K_m x \qquad (5.2.11)$$

and $[e^{[C]y}]$ is written as

$$[e^{[C]y}] = [R] \begin{bmatrix} e^{\lambda_1 y} & & & & \\ & e^{\lambda_2 y} & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & e^{\lambda_{10} y} \end{bmatrix} \qquad (5.2.12)$$

Here, λ_i represent the distinct eigenvalues and [R] is the matrix of the corresponding eigenvectors.

Following the same procedures as described above for the first component, we can easily generate the solutions for the remaining cases. In order to obtain the fundamental frequency of the plate, the superimposed solution expressed by equation (5.2.2) should satisfy the following boundary conditions:

at the edge x=0,

$$u = \chi = 0 \tag{5.2.13}$$

at the edge y=0,

$$v = \varphi = 0 \tag{5.2.14}$$

Enforcement of equations (5.2.13) and (5.2.14) gives

$$[M]\{\delta\} = 0 \tag{5.2.15}$$

As pointed out earlier, if k terms are employed in the displacement solution for each of the components, [M] constitutes an eigenvalue matrix of dimensions 4k $\times$ 4k, while $\{\delta\}$ denotes a column matrix containing unknown Fourier coefficients, such as E_m, etc. Thus for non-trivial solution, the determinant of the coefficient matrix must be zero yielding a set of eigenvalues ω.

5.3 Third-order Shear Deformation Theory

We proceed to discuss the solution technique for shells based on the third-order shear deformation theory. In this case, the displacement components of the middle surface have the same form as for the first-order shear deformation theory, and the governing equations are expressed as

$$L_{ij}u_j = 0 \tag{5.3.1}$$

In the above equation, the parameters u_j denote the displacement components,

$$u_1 = u; \quad u_2 = v; \quad u_3 = w; \quad u_4 = \chi; \quad u_5 = \varphi \tag{5.3.2}$$

and the terms in the differential operator L_{ij} are given by equations (5.3.3),

$$L_{11}=A_{11}d_{xx}+A_{66}d_{yy}-\bar{I}_1 d_{tt};\quad L_{12}=(A_{12}+A_{66})d_{xy};$$

$$L_{13}=\bar{I}_3 d_{xtt}-c_2[E_{11}d_{xxx}+(E_{12}+2E_{66})d_{xyy}]+(A_{11}/R_1+A_{12}/R_2)d_x$$

$$L_{14}=(B_{11}-c_2E_{11})d_{xx}+(B_{66}-c_2E_{66})d_{yy}-\bar{I}_2 d_{tt};$$

$$L_{15}=[B_{12}+B_{66}-c_2(E_{12}+E_{66})]d_{xy};\qquad L_{22}=A_{66}d_{xx}+A_{22}d_{yy}-\hat{I}_1 d_{tt};$$

$$L_{23}=\hat{I}_3 d_{xtt}-c_2[E_{22}d_{yy}+(E_{12}+2E_{66})d_{xxy}]+(A_{12}/R_1+A_{22}/R_2)d_y$$

$$L_{24}=L_{Hx};\quad L_{25}=(B_{22}-c_2E_{22})d_{yy}+(B_{66}-E_{66}c_2)d_{xx}-\hat{I}_2 d_{tt}$$

$$\begin{aligned}L_{33}=&[c_1(D_{55}-c_1F_{55})-(A_{55}-c_1D_{55}]d_{xx}+[c_1(D_{44}-c_1F_{44})-\\&(A_{44}-c_1D_{44})]d_{yy}+c_2^2[H_{11}d_{xxxx}+2(H_{12}+2H_{66})d_{xxyy}+H_{22}d_{yyyy}]\\&-2c_2[(E_{11}/R_1+E_{12}/R_2)d_{xx}+(E_{12}/R_1+E_{22}/R_2)d_{yy}]+(A_{11}/R_1^2+\\&\qquad 2A_{12}/R_1/R_2+A_{22}/R_2^2)+I_1d_{tt}-c_2^2I_7(d_{xxtt}+d_{yytt})\end{aligned}\qquad (5.3.3)$$

$$\begin{aligned}L_{34}=&[c_1(D_{55}-c_1F_{55})-(A_{55}-c_1D_{55})]d_x-c_2\{(F_{11}-c_2H_{11})d_{xxx}+[2F_{66}+\\&F_{12}-c_2(H_{12}+2H_{66}]d_{xyy}\}+[B_{11}/R_1+B_{12}/R_2-c_2(E_{11}/R_1+E_{12}/R_2)]+\bar{I}_5d_{xtt};\end{aligned}$$

$$\begin{aligned}L_{35}=&[c_1(D_{44}-c_1F_{44})-(A_{55}-c_1D_{44})]d_y-c_2(F_{22}-c_2H_{22})d_{yyy}-c_2[2F_{66}+\\&F_{12}-c_2(2H_{66}+H_{12})]d_{xxy}+[B_{12}/R_1+B_{22}/R_2-c_2(E_{12}/R_1+E_{22}/R_2)]+\hat{I}_5d_{ytt}\end{aligned}$$

$$\begin{aligned}L_{44}=&c_1(D_{55}-c_1F_{55})-(A_{55}-c_1D_{55})+(D_{11}-2c_2F_{11}+c_2^2H_{11})d_{xx}+(D_{66}-\\&\qquad 2c_2F_{66}+c_2^2H_{66})d_{yy}-\bar{I}_4d_{tt}\end{aligned}$$

$$L_{45}=[D_{12}+D_{66}-2c_2(F_{12}+F_{66})+c_2^2(H_{12}+H_{66})]d_{xy}$$

$$\begin{aligned}L_{55}=&c_1(D_{44}-c_1F_{44})-(A_{44}-c_1D_{44})+(D_{22}-2c_2F_{22}+c_2^2H_{22})d_{yy}+(D_{66}-\\&\qquad 2c_2F_{66}+c_2^2H_{66})d_{xx}-\hat{I}_4d_{tt}\end{aligned}$$

To obtain a closed-form solution for a cross-ply laminated shell with two adjacent boundaries clamped and others simply supported, six components need to be superposed to construct the complete general solution. Each of these components has simple support except for a bending moment or an in-plane force acting along one edge. For the reasons stated earlier, only the first component is analyzed here.

The solution for the first component is assumed to be of Lévy-type expressed by equation (5.2.3) and the bending moment along the edge y=b is given by

$$M_2^* = \sum_{m=1}^{\infty} E_m \sin K_m x \tag{5.3.4}$$

Equation (5.3.1) can be rewritten by substituting the assumed displacement solution given by equation (5.2.3) as follows:

$$\begin{aligned}
U^1_{m,yy} &= C_1 U^1_m + C_2 V^1_{m,y} + C_3 W^1_m + C_4 W^1_{m,yy} + C_5 X^1_m + C_6 Y^1_{m,y} \\
V^1_{m,yy} &= C_7 U^1_{m,y} + C_8 V^1_m + C_9 W^1_{m,y} + C_{10} W^1_{m,yyy} + C_{11} X^1_{m,y} + C_{12} Y^1_m \\
W^1_{m,yy} &= C_{13} U^1_m + C_{14} V^1_m + C_{15} W^1_m + C_{16} W^1_{m,yy} + C_{17} X^1_m + C_{18} Y^1_{m,y} \\
X^1_{m,yy} &= C_{19} U^1_m + C_{20} V^1_{m,y} + C_{21} W^1_m + C_{22} W^1_{m,yy} + C_{23} X^1_m + C_{24} Y^1_{m,y} \\
Y^1_{m,yy} &= C_{25} U^1_{m,y} + C_{26} V^1_m + C_{27} W^1_{m,y} + C_{28} W^1_{m,yyy} + C_{29} X^1_{m,y} + C_{30} Y^1_m
\end{aligned} \tag{5.3.5}$$

The coefficients C_i are defined in Appendix III.

Equations (5.3.5) may be written in the state-space form by defining the following variables,

$$Z_1 = U_m^1;\; Z_2 = U_{m,y}^1;\; Z_3 = V_m^1;\; Z_4 = V_{m,y}^1;\; Z_5 = W_m^1;\; Z_6 = W_{m,y}^1;$$
$$Z_7 = W_{m,yy}^1;\; Z_8 = W_{m,yyy}^1;\; Z_9 = X_m^1;\; Z_{10} = X_{m,y}^1;\; Z_{11} = Y_m^1;\; Z_{12} = Y_{m,y}^1 \tag{5.3.6}$$

Therefore, we obtain

$$\{Z(y)\}_{,y} = [C]\{Z(y)\} \tag{5.3.7}$$

The matrices [C] and {Z(y)} are given by equations (5.3.8),

$$[C] = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ C_1 & 0 & 0 & C_2 & C_3 & 0 & C_4 & 0 & C_5 & 0 & 0 & C_6 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & C_7 & C_8 & 0 & 0 & C_9 & 0 & C_{10} & 0 & C_{11} & C_{12} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ C_{13} & 0 & 0 & C_{14} & C_{15} & 0 & C_{16} & 0 & C_{17} & 0 & 0 & C_{18} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ C_{19} & 0 & 0 & C_{20} & C_{21} & 0 & C_{22} & 0 & C_{23} & 0 & 0 & C_{24} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & C_{25} & C_{26} & 0 & 0 & C_{27} & 0 & C_{28} & 0 & C_{29} & C_{30} & 0 \end{bmatrix};\quad \{Z\} = \begin{Bmatrix} Z_1 \\ Z_2 \\ Z_3 \\ Z_4 \\ Z_5 \\ Z_6 \\ Z_7 \\ Z_8 \\ Z_9 \\ Z_{10} \\ Z_{11} \\ Z_{12} \end{Bmatrix} \tag{5.3.8}$$

A general solution of equation (5.3.7) can be written as:

$$\{Z(y)\} = [e^{[C]y}]\{P\} \tag{5.3.9}$$

where {P} is a column matrix of unknown constants determined by the associated boundary conditions and $[e^{[C]y}]$ is expressed as

$$[e^{[C]y}] - [R]\begin{bmatrix} e^{\lambda_1 y} & & & & \\ & e^{\lambda_2 y} & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & e^{\lambda_{12} y} \end{bmatrix} \tag{5.3.10}$$

The solutions for the rest of the components can also be established by the same procedures as for the first component. Once the solutions for all the components have been obtained, the boundary conditions of the original shell prescribed by equations (5.3.11) and (5.3.12) can be satisfied.

Along the boundary x=0,

$$u - w_{,x} - \chi - 0 \tag{5.3.11}$$

Along the boundary y=0,

$$v - w_{,y} - \varphi - 0 \tag{5.3.12}$$

Enforcement of equations (5.3.11) and (5.3.12) gives rise to an eigenvalue system of homogeneous equations

$$[M]\{\delta\}=0 \qquad (5.3.13)$$

The natural frequencies are determined by the condition of non-trivial $\{\delta\}$ in equation (5.3.13).

5.4 Shells with Three and Four Boundaries Clamped

As discussed in the previous chapters, CSST approach can easily treat the complicated boundary conditions by superposing more component solutions into the original problem. For the cross-ply laminated shells having more edges clamped, this method can be applied to obtain their analytical solutions without any difficulty.

Since the solution steps are identical to those for the laminated plates, it is decided to omit the detailed procedures and associated formulas for shells in this category. We have demonstrated in a step by step manner, how the solution for the vibration problems of laminated plates with three and four boundaries clamped using the CSST approach can be obtained.

It is important to note that for shells with three edges clamped and one edge simply supported, the number of the component solutions is 6, 6 and 9 when the CT, FSDT and TSDT are employed separately, while the number jumps to 8, 8 and 12 for the fully clamped shell if we treated the problem by CT, FSDT and TSDT theories.

Chapter 6

Computed Results and Discussion

Computations are carried out for antisymmetric laminated plates and cross-ply laminated plates and shells that have been analyzed in Chapters 3-5. Non-dimensional fundamental frequencies are evaluated for these structures with a wide range of structural parameters. In order to guarantee the accuracy of the closed-form solutions generated by the present technique, convergence tests are conducted and comparisons are made between the frequencies obtained by the CSST approach and those reported for the special cases in the published references.

6.1 Convergence Tests

Convergence tests have been performed typically for a square antisymmetric angle-ply plate with two edges clamped and the other two simply supported. Three types of theories, i.e., classical theory (CT), first-order shear deformation theory (FSDT) and third-order shear deformation theory (TSDT) are employed. Convergence results are tabulated in Table 6.1, which is obtained by increasing the number of terms in the displacement solution series and observing the variation of the non-dimensional fundamental frequency.

Table 6.1 Results of convergence tests

Number of terms	TSDT	FSDT	CT
2	20.3638	20.3789	29.3144
4	20.2690	20.3834	29.3420
6	20.3708	20.3853	29.3437
8	20.3715	20.3860	29.3438
10	20.3719	20.3863	29.3439

It has been found that solution converges so rapidly that there is no change at the third digit when 2 and 4 terms are employed. Furthermore, four-digit accuracy can be provided when more than 6 terms are taken into account. This conclusion is also true for all the plates and shells studied in the present work after the similar tests are conducted for each of these cases. Therefore it is decided to set the number of terms in the displacement solutions equal to 6 for all of the computed results presented here. This not only can guarantee the four-digit accuracy but also economize computation time.

6.2 Comparisons

First, we consider an isotropic plate whose closed-form solutions are available in Reference

[110]. It is well known that both isotropic and orthotropic plates and shells constitute the special cases of laminated plates and shells, since the former can be treated as a single layered laminate. To obtain numerical results, we utilize the program, generated during this research, for the symmetric cross-ply laminated plates. Computation has been performed by assuming the following material properties,

$$\frac{E_1}{E_2}=1, \qquad \nu_{12}=0.3$$

It is recalled that the shear moduli G_{12}, G_{13} and G_{23} are inter-dependent as given below:

$$G_{12}=\frac{E_1}{2(1+\nu_{12})}, \quad G_{13}=G_{23}=G_{12}$$

Table 6.2.1 shows the comparisons for a square plate having the three, CCSS, CCSC and CCCC boundary conditions, where the letter C is referred to **Clamped** and S denotes **Simply supported**. Thus CCSS represents clamped-clamped-simple-simple edge support counted counter-clockwise from the left side of the plate.

Table 6.2.1 Comparisons of non-dimensional fundamental frequencies of an isotropic plate. $\lambda=\omega a^2(\rho h/D)^{1/2}$

B. C.	Ref. [110]	CT	FSDT	TSDT
CCSS	27.05	27.054	25.283	25.308
CCSC	31.82	31.825	29.130	29.178
CCCC	35.982	35.984	32.524	32.593

It is seen that there is an excellent agreement between the results generated by the CSST technique based on the CT theory and those obtained by Gorman [110]. However, the frequencies obtained by the FSDT and the TSDT are always less than these produced from the CT theory even though there are only small differences in the results generated by the FSDT and the TSDT theories. This is due to the fact that the transverse shear deformation is included in FSDT and TSDT theories.

Next, consider a fully clamped orthotropic plate. In calculating its non-dimensional frequency, the same notations as in Reference [111] are used,

$$DHX = \frac{D_{12}+2D_{66}}{D_{11}}; \quad DHY = \frac{D_{12}+2D_{66}}{D_{22}}$$

Table 6.2.2 shows the comparison results between the data given by the present approach and those obtained in other publications. The results of Gorman were generated by superposition method [111]. Those of Marangoni et al. were obtained by Rayleigh-Ritz method [115]. Dickinson [116] developed his results by series solution method. Again a remarkable agreement is seen among all of the data.

Table 6.2.2 Comparisons of non-dimensional fundamental frequencies of a fully clamped orthotropic plate. $\lambda=\omega a^2(\rho h/D_{11})^{1/2}$

DHX	DHY	Ref.[111]	Ref.[115]	Ref.[116]	CT
2.0	2.0	39.70	39.71	39.71	39.696
2.0	1.0	45.64	45.64	45.64	45.636
2.0	0.5	55.56	55.56	55.56	55.566
1.0	0.5	42.40	42.40	42.40	42.395

Table 6.2.3 shows the comparison results for an antisymmetric angle-ply square plate with fully clamped edges, where the results of Whitney [117] were obtained by Fourier series method and those of Lin et al. were evaluated using an asymptotic method [118] developed by Bolotin [119-121]. Classical Kirchhoff assumptions and the C1 support condition in which all the displacement components along a clamped edges are zero were used. It is seen that there is an excellent agreement among all of the results.

Table 6.2.3 Comparisons of non-dimensional fundamental frequencies ω^* $(=\omega a^2/h(\rho/E_2)^{1/2})$ of an antisymmetric angle ply clamped square plate ($E_1/E_2=40$, $G_{12}/E_2=0.5$, $\nu_{12}=0.25$)

Number of layers	ply-angle (degree)	Ref.[117]	Ref.[118]	CT
2	5	37.582	36.880	36.801
	15	28.052	27.378	27.540
	25	24.940	24.317	24.584
	35	23.799	23.189	23.496
	45	23.488	22.838	23.198
4	5		40.267	40.446
	15		38.044	38.270
	25		37.025	37.338
	35		36.351	36.827
	45		36.086	36.660

6.3 Fundamental Frequencies

Fundamental frequencies of the plates and shells are computed for a wide range of structural parameters, namely, number of layers, plate aspect ratio, length-to-thickness ratio and degree of anisotropy under three sets of boundary conditions. For simplicity, all of the plates and shells studied here are assumed to be composed of laminae of equal-thickness, equal-density and equal-orthotropicity. In addition, in order to make the solution universal and observe the variation of the fundamental frequencies with structural properties, non-dimensional frequency ω^* is adopted, which is defined as,

$$\omega^* = \frac{\omega a^2}{h}\sqrt{\rho/E_2}$$

Unless otherwise indicated, the following material properties are considered in the computations.

$$\frac{E_1}{E_2} = 40, \quad \frac{G_{12}}{E_2} = \frac{G_{13}}{E_2} = 0.6, \quad \frac{G_{23}}{E_2} = 0.5, \quad \nu_{12} = 0.25$$

Furthermore, in the case of the first-order shear deformation theory, the transverse shear correction coefficient K^2_{ij} is taken as 5/6.

6.3.1 Antisymmetric Angle-ply Laminated Plates

Non-dimensional fundamental frequencies for these types of plates are tabulated in Tables A.1-A.7. Some of the typical plots which demonstrate explicitly the variation of the frequencies with respect to various plate parameters are shown in Figures A.1-A-11.

Let us first consider the results generated for the plates with two edge clamped and the others simply supported based on the first-order shear deformation theory. Figure A.1 shows the variation of the non-dimensional fundamental frequencies ω^* with plate aspect ratio and number of layers for the length-to-thickness ratio a/h=10 and the ply-angle $\theta=45^0$. It is seen that ω^* increase with the plate aspect ratio and number of layers. However, for a certain thickness of a plate, the number of layers has little effect on ω^* when the plate is composed of more than two layers. For evidence, when the number of layers increases from four to six the frequency increases only 2.7 % for a square plate with ply angle $\theta=45^0$ and a/h=10.

Figure A.2 demonstrates the effects of length-to-thickness ratio and plate aspect ratio on the frequencies ω* for a four-layer plate with $\theta=45^0$. It is clear that ω^* increase with plate aspect ratio a/b and length-to-thickness ratio a/h. The increase is seen to be rapid for smaller values of a/h, and then as the a/h increases, the rate of the increase of ω^* slows down and subsequently becomes nearly horizontal.

Figure A.3 indicates the variation of ω^* with plate aspect ratio and in-plane orthotropicity ratio for a four-layer plate, for the length-to-thickness ratio a/h=10 and the ply angle $\theta=45^0$. It can be found that ω^* increase with increasing a/b and E_1/E_2.

In Figure A.4, frequency variation as a function of the ply angle and the number of layers is shown. It is again shown that there is little significant increase of ω^* if more than two layers are used for a certain thickness. Moreover, ω^* are symmetric about $\theta=45^0$ and decrease with θ for two-layer plates and increase with θ for plates with more than two layers.

The frequency variation data written with respect to length-to-thickness ratio a/h for plates with CCSS, CCSC and CCCC boundary conditions and using the FSDT theories have been presented in Figure A.5. It is observed thrt ω^* increase with a/h ratio for all the three cases of boundary conditions considered.

Frequency comparisons for the three theories i.e., TSDT, FSDT and CT under CCSS, CCSC and CCCC boundary conditions have been presented in Figures A.6-A.8 for a four-layer plate. It is seen that the TSDT and the FSDT theories yield very close agreement in ω^* for all the values of E_1/E_2. The classical theory also yields almost the same results for the smaller values of E_1/E_2 (<5). The discrepancy among the results of the CT and TSDT or FSDT becomes larger with increasing E_1/E_2. For example, the CCSS square plate with a/h=10 and $\theta=45^0$ and E_1/E_2 relatively large (=40) produces a discrepancy of 43.9 % between the frequencies generated by the CT and the FSDT theories. The same plate shows a discrepancy of 44 % when the CT and the TSDT theories are used. However, the discrepancy becomes 9.8 % if we set $E_1/E_2=5$ for the CT and FSDT or TSTD theories. This implies that the CT theory can produce fairly accurate results for plates with lower orthotropicity ratio and thin thickness like elastic thin isotropic plates. However it can not predict correctly the frequencies for laminated composite plates that usually reveal higher orthotropicity ratio. The reason is that the CT theory neglects the transverse shear deformation.

6.3.2 Cross-ply Laminated Plates

Since there are symmetric and antisymmetric structures in this type of plates and they are treated by different governing equations, the non-dimensional fundamental frequencies have been generated individually.

Figures B.1-B.3 show the effects of different plate theories and length-to-thickness ratio a/h on the frequencies of symmetric cross-ply square plates with CCSS, CCSC and CCCC boundary conditions. It is noted that the CT theory generates solutions which are independent of a/h, while the TSDT and the FSDT theories produce results which show an increase with a/h. Furthermore, the TSDT and the FSDT predict results which are in close agreement with each other only for larger a/h. The effect of length-to-thickness ratio on ω^* indicates that the frequencies for thick plates are always lower than those obtained by classical theory based on the thin plate assumption. For thin plates, a/h has little effect on ω^*. This means that the three theories produce very close results for thin plates.

It is important for designers and researchers to know exactly what each theory is capable of predicting and what are the variations in results. It can be seen from Table B.1 that, in the case of a symmetric CCSS square plate ($0^0/90^0/0^0$), CT produces an error of about 112% for a/h=5 and 13.6% for a/h=15 as compared with the TSDT. When compared with the FSDT theories, the error becomes 120% and 14% respectively. For the same plate, in the case of a/h=2, the FDST produces a 12% lower frequency than the TDST, while in the case of a/h=15 the between the two theories is 0.27%. For the CCCC boundary, this discrepancy becomes 20% and 1.1% respectively.

The effects of boundary conditions and length-to-thickness ratio a/h on the plate frequencies obtained by the TSDT and the FSDT theories have been shown in Figures B.4 and B.5. The frequencies increase with the number of clamped edges, which can be explained by the physical property of the structures that increasing stiffness raises the natural frequencies.

Figures B.6 and B.7 have been plotted to emphasize the effects of the number of layers as well as the boundary conditions on the frequencies ω^* based on the FSDT and TSDT theories. Once again it is seen that the ω^* increase with the number of layers and clamped edges.

Similar observations can be made from the Figures B.8-B.14 for the antisymmetric cross-ply laminated plates. Since the behaviour is similar the detail of discussion for this case is not included here.

Non-dimensional fundamental frequencies are tabulated in Tables B.1-B.8 for both the symmetric and the antisymmetric cross-ply plates. In the case of the symmetric plates, the length-to-thickness ratio a/h vary between 2 and 15, while in the case of the antisymmetric plates, a/h is allowed to increase from 5 to 20.

6.3.3 Cross-ply Laminated Shells

Results have been generated for two types of shells: (i) cross-ply laminated cylindrical shell panels, and (ii) cross-ply laminated spherical shell panels. The number of layers is allowed to vary from 2 to 7, radius-to-side length ratio increases from 5 to 25 and side length-to-thickness ratio is taken as 15. As stated before, the shell panels have three, CCSS, CCSC and CCCC types of boundary conditions. Furthermore, the shells are composed of graphite-epoxy layers having the following material properties:

$$\frac{E_1}{E_2}=25, \quad \frac{G_{12}}{E_2}=\frac{G_{13}}{E_2}=0.5, \quad \frac{G_{23}}{E_2}=0.2, \quad \nu_{12}=0.25$$

Table C.1 shows the non-dimensional fundamental frequencies ω^* for a cylindrical shell panel based on the classical theory. The effect of the number of layers on the frequency variation of shell structures can be observed from the computations presented here. As the ratio R/a increases, i.e., as the shell panel becomes more shallow, the frequency decreases and as the number of layers increases, the frequency increases generally. However the variation of the frequency due to the radius-to-side length ratio is quite small when the R/a increases from 5 to 25.

Non-dimensional fundamental frequency data for spherical shell panels are tabulated in Table C.2. The frequency variation with respect to the number of layers and radius to side length is similar to that of the cylindrical shell panel. A comparison of the data presented in Table C.1 and in Table C.2 demonstrates that the fundamental frequency of a spherical shell panel is always a little larger than that of a cylindrical shell panel with the same R/a and a/h values and for the same number of layers.

For both the cylindrical shells and the spherical shells, it is interesting to note from Table C.1 and Table C.2, that there is a sharp increase in the frequency when the number of layers increases from 2 to 3, and after that the increase is rather small. This is because two-layer cross -ply shells exhibit much larger bending-stretching coupling behaviour than the shells having more layers. It is further noted that the shell panels with an odd number of layers have larger non-dimensional fundamental frequencies than the corresponding panels with an even number of layers. In the case of a cylindrical square panel with a/h=15 and R/a=5 as presented in Table 1, the frequency decreases by 8% when the number of layers varies from 3 to 4. The reason for this behaviour is the symmetry property of the structures. We may explain this behaviour by the fact that the antisymmetric cross-ply shell panels demonstrate more bending-stretching coupling action than the symmetric panels in the case of CCSS boundary conditions.

It is interesting to notice that the above conclusion that symmetric cross-ply shell panels have higher nondimensional fundamental frequencies than antisymmetric ones does not quit carry over to the shell panels with CCSC boundary conditions. This can be verified by Table C.3 and Table C.4. It is noted that the three-layer panels have higher frequencies than two-layer ones. However the panels having five or seven layers possess lower frequencies than those with four or six layers. This may be explained by the fact that boundary conditions have active effects on the frequencies. In other words, it is concluded that the fundamental frequencies for cross-ply shell panels depend on not only the number of layers but also their boundary conditions.

Figures C.1-C.3 show the effects of different theories as well as length-to-thickness ratio on the frequencies of a cylindrical two-layer shell panel with the three, CCSC, or CCSC or CCCC boundary conditions. It is once again apparent that the two shear deformation theories, TSDT and FSDT, generate very close fundamental frequencies for the laminated shell. However there is a discrepancy between TSDT and FSDT theories when very thick plates are considered. For example, in the case of an antisymmetric cross-ply cylindrical square CCSS shell panel ($0^0/90^0$) of R/a=5, the discrepancy is 6.8% for a/h=5 and 0.62% for a/h=20. It is also noted that this discrepancy becomes smaller in the case of a symmetric shell panel. For example, when the same shell panel but with symmetric layers ($0^0/90^0/0^0$) is considered, the discrepancy becomes 0.04% and 2.6% respectively. On the other hand, CT theory produces comparable but higher results than TSDT and FSDT when it is used for the shells with larger (>10) a/h ratios. For thick shell (a/h<10), the CT theory yields much larger errors for the frequencies and hence should be avoided.

Figures C.4 and C.5 present the variation of the frequencies ω^* with the boundary conditions and length-to-thickness ratio for a cylindrical two-layer shell panel using the TSDT and the FSDT theories. It is shown that ω^* increase with the number of clamped edges and a/h ratio.

Chapter 7

Summary

7.1 Conclusions

To the author's knowledge, this study has established, for the first time, a comprehensive analytical approach which combines superposition and state-space techniques for the vibration analysis of laminated orthotropic plates and shells based on the three types of existing theories. One of the significant advantages provided by this method lies in the fact that it has the capability of analyzing structures with any combinations of classical boundary conditions. Furthermore, the solutions obtained can satisfy the governing equations exactly throughout the domain of the structures and, the boundary conditions can be satisfied to any desired degree of accuracy by taking more terms in the displacement solutions.

Based on the results presented in the last chapter, the following conclusions can be drawn.

(1) The method (CSST) developed in this research works extremely well and only a small number of terms (six) are taken in the displacement solutions to provide accuracy to four significant digits for the fundamental frequencies of plates and shells studied here.

(2) Comparisons for some cases for which the results are available show that an excellent agreement exists between the data generated by this method and those given by other analytical and numerical methods.

(3) Comparisons of the classical theory (CT), first-order shear deformation theory (FSDT) and third-order shear deformation theory (TSDT) reveal that the CT theory over-estimates the fundamental frequencies for the laminated plates and shells, especially for structures with large thickness ($a/h > 10$). Hence it is recommended that the CT not be used for the laminated plates and shells having a thickness ratio a/h larger than 10. On the other hand, the FSDT and the TSDT can produce very comparable results in most of the cases. However for the very thick plates and shells there still exists a discrepancy (in some cases, more than 10%) in the frequencies predicted by the FSDT and TSDT. Besides, the FSDT is more simple in form, more steady in equation-solving and takes less computing time to obtain the solutions than the TSDT. In the case of thin structures, the three theories yield fairly identical results.

(4) Transverse shear deformation plays an essential role in the fundamental frequencies. It reduces the value of the vibration frequencies predicted by the classical theory. It is more pronounced in the composite structures due to the low transverse shear modulus relative to the in-plane Young's moduli. Therefore, transverse shear deformation can not be neglected in the static and dynamic analysis of composite plates and shells.

(5) The effect of the number of layers on the behaviours of the plates and shells shows that the fundamental frequencies increase with the number of layers. However for a certain thickness, this increase is more pronounced only when the number of layers increases from two to four. After that litter changes in the frequencies can be found.

(6) The effect of in-plane orthotropicity ratio on the non-dimensional fundamental frequencies of the structures reveals that the frequencies increase with increasing the ratio E_1/E_2. CT theory produces more discrepancy with the increase of E_1/E_2, while for smaller values of E_1/E_2 and small thickness, CT, TSDT and FSDT theories can obtain fairly close results.

(7) For the case of laminated cross-ply shells, spherical shell panels demonstrate somewhat higher non-dimensional frequencies than the cylindrical shell panels. Both the two types of shells exhibit much more bending-stretching coupling in the

two-layer shells than the shells composed of more than two layers.

7.2 Further Research

This work has considered the cantilever plates with a point support, and plates and shells having the three, CCSS, CCSC and CCCC types of boundary conditions and their solutions for the vibration problems. Further research can be extended to the following areas:

(1) Static and buckling analysis of laminated plates and shells. In fact, to a certain extent, the present method has been used for the buckling analysis of antisymmetric angle-ply laminated plates [52] by this author.

(2) Composite plates and shells with more complicated classical boundary conditions, such as free supports and elastic edge supports.

(3) Static and dynamic analysis of composite plates and shells subjected to in-plane forces or thermal stresses.

(4) Complete study of the fundamental frequencies for the cantilever plate with a point support, described in chapter 3 of this thesis, where only the analysis procedure was discussed.

(5) Analysis of the higher modes of vibration and their application in forced vibration analysis (modal analysis)

References

1. A. E. H. Love, "The small free vibrations and deformation of a thin elastic shell," Phil. Trans. Royal Society, Ser. A, Vol. 179, 491-546, 1888.

2. Y. Stavsky, "On the theory of heterogeneous anisotropic plates," Doctoral thesis, MIT, 1959.

3. E. Reissner and Y. Stavsky, "Bending and stretching of certain types of heterogeneous anisotropic elastic plates," Journal of Applied Mechanics, Vol. 28, 402-408, 1961.

4. Y. Stavsky, "Bending and stretching of laminated aeolotropic plates," Proceeding of the ASCE, Journal of the Engineering Mechanics Division, Vol. 87, No. EM6, 31-56, Dec. 1961.

5. S. B. Dong, R. B. Matthiesen, K. S. Pister and R. L. Tayor, "Analysis of structural laminates," AD-269541, ARL 76, Aeronautical Research Lab., Wright-Patterson Air Force Base, Ohio, Sept. 1961.

6. Y. Stavsky, "Thermoelastic vibration of heterogeneous membranes and inextensional plates," AIAA Journal, Vol. 1, No. 3, 722-723, 1963.

7. J. E. Ashton and M. E. Waddoups, "Analysis of anisotropic plates," Journal of Composite Materials, Vol. 3, 148-165, 1969.

8. Y. Hikami, "Transverse vibrations of laminated orthotropic plates," Report, Div. of Solid Mechanics, Structure and Mechanical Design, Case Western Reserve University, Cleveland, Ohio, April 1969.

9. C. W. Bert and B. L. Mayberry, "Free vibration of unsymmetrically laminated anisotropic plates with clamped edges," Journal of Composite Materials, Vol, 3., 282-293, 1969.

10. J. E. Ashton and J. D. Anderson, "The natural modes of vibration of boron-epoxy plates," Shock and Vibration Bulletin, Vol. 39, 81-91, 1969.

11. J. M. Whitney and A. W. Leissa, "Analysis of heterogeneous anisotropic plates," Journal of Applied Mechanics, Vol. 36, 261-266, 1969.

12. J. M. Whitney and A. W. Leissa, "Analysis of a simple supported laminated anisotropic rectangular plates," AIAA Journal, Vol. 8, No. 1, 28-33, 1970.

13. S. A. Ambartsumyan, "Calculation of laminated anisotropic shells," Izvestiia Akademiia Nauk Armenskoi SSR, Ser. Fiz. Mat. Est. Tekh., Vol. 6, No. 3, p.15, 1953.

14. S. A. Ambartsumyan, "Theory of anisotropic shells," Moscow, 1961, English translation, NASA-TT-F-118, 1964.

15. S. B. Dong, K. S. Pister and R. L. Taylor, "On the theory of laminated anisotropic shells and plates," Journal of the Aeronautic Science, Vol. 29, 969-974, 1962.

16 L. H. Donnell, "Stability of thin walled tubes in torsion," NACA Report 479, 1933.

17. S. B. Dong, "Free vibrations of laminated orthotropic cylindrical shells," Journal of Acoustical Society of America, Vol. 44, 1628-1635, 1968.

18. C. W. Bert, J. L. Baker and D. M. Egle, "Free vibrations of multilayered anisotropic cylindrical shells," Journal of Composite Materials, Vol. 3, 480-499, 1969.

19. Y. Stavsky and R. Loewy, "On vibrations of heterogeneous orthotropic cylindrical shells," Journal of Sound and Vibration, Vol. 15, 235-256, 1971.

22. J. B. Greenberg and Y. Stavsky, "Buckling and vibration of orthotropic composite cylindrical shells," Acta Mechanica, Vol. 36,15-29, 1980.

21. R. M. Jones, "Buckling and vibration of cross-ply laminated circular cylindrical shells," AIAA Journal, Vol. 13, 664-671, 1975.

22. S. Cheng and B. P. C. Ho, "Stability of heterogenous aeolotropic cylindrical shells under combined loading," AIAA Journal, Vol. 1, 892-898, 1963.

23. W. Flügge, "Stresses in shells," Springer, Berlin, 1960.

24. J. Tasi, "Effects of heterogeneity on the axisymmetric vibrations of cylindrical shells," Journal of Sound of Vibration, Vol. 14, 325-338, 1971.

25. K. Soldatos and G. J. Tzivanidis, "Buckling and vibration of cross-ply laminated circular cylindrical panels," Zeitschrift für angewandte Mathematik und Physik, Vol. 33, 230-340, 1982.

26. W. Soedel, "Simplified equations and solutions for the vibration of orthotropic cylindrical shells," Journal of Sound and Vibration, Vol. 87, 555-566, 1983.

27. C. W. Bert and M. Kumar, "Vibration of cylindrical shells of bimodulus composite materials," Journal of Sound and Vibration, Vol. 81, 107-121, 1982.

28. J. L. Sanders, Jr, "An improved first approximation theory for thin shells," NASA Report 24, 1959.

29. K. P. Soldatos, "A comparison of some shell theories used for the dynamic analysis of cross-ply laminated circular cylindric panels," Journal of Sound and Vibration, Vol. 97, No. 2, 305-319, 1984.

30. Y. Stavsky, "On the theory of symmetrically heterogeneous plates having the same thickness variation of the elastic moduli," In Topics in Applied Mechanics, E. Schwerin Memorial Volume (D. Abir, F. Ollendorff and M. Reiner, editors), 105-106, New York: Elsevier, 1965.

31. P. C. Yang, C. H. Norris and Y. Stavsky, "Elastic wave propagation in heterogeneous plates," International Journal of Solids and Structures, Vol. 2, 665-684, 1966.

32. E. Reissner, "On the theory of bending of elastic plates," Journal of Mathematics and Physics. Vol. 23, 184-191, 1944.

33. E. Reissner, "On transverse bending of plates, including the effect of transverse shear deformation," Journal of Applied Mechanics, Vol. 12, No. 55, A69-A77, 1945.

34. R. D. Mindlin, "Influence of rotatory inertia and shear on flexural motions of isotropic, elastic plates," Journal of applied Mechanics, Vol. 18, 31-38, 1951.

35. J. M. Whitney and N. J. Pagano, "Shear deformation in heterogeneous anisotropic plates," Journal of Applied Mechanics, Vol. 37, 1031-1036, 1970.

36. J. M. Whitney, "The effect of transverse shear deformation on the bending in heterogeneous plates," Journal of Composite Materials, Vol. 3, 534-547, 1969.

37. R. C. Fortier and J. N. Rossettos, "On the vibration of shear deformable curved anisotropic plates," Journal of Applied Mechanics, Vol. 40, 299-301, 1973.

38. P. K. Sinha and A. K. Rath, "Vibration and buckling of cross-ply laminated circular cylindrical panels," Aeronautical Quarterly, Vol. 26, 211-217, 1973.

39. A. K. Noor, "Stability of multilayered composite plates," Fibre Sci. Technol., Vol 8, 81-88, 1975.

40. C. W. Bert and T. L. Chen, "Effect of shear deformation on vibration of anisotropic angle-ply laminated rectangular plates," International Journal of Solids and Structures,

Vol. 14, 465-473, 1978.

41. J. N. Reddy, "Free vibration of antisymmetric angle-ply laminated plates including transverse shear deformation by the finite element method," Journal of Sound and Vibration, Vol. 66, 565-576, 1979.

42. C. W. Pryor Jr. and R. M. Barker, "A finite element analysis including transverse shear effects for applications to laminated plates," AIAA Journal, Vol. 9, 912-917, 1971.

43. A. K. Noor and M. D. Mathers, "Anisotropy and shear deformation in laminated composite plates,", AIAA Journal, Vol. 14, 282-285, 1976.

44. A. K. Noor and M. D. Mathers, "Finite element analysis of anisotropic plates," International Journal for Numerical Methods in Engineering, Vol. 11, 289-307, 1977.

45. E. Hinton, "A note on a finite element method for the free vibrations of laminated plates (short communication)," Earthquake Engineering and Structural Dynamics, Vol. 4, 511-514, 1976.

46. A. A. Khdeir, "An exact approach to the elastic state of stress of shear deformable antisymmetric angle-ply laminated plates," Composite Structures, Vol. 11, No. 4, 245-258, 1989.

47. A. A. Khdeir, "Free vibration of antisymmetric angle-ply laminated plates including various boundary conditions," Journal of Sound and Vibration, Vol. 122, No. 2, 377-388, 1988.

48. A. A. Khdeir, "Stability of antisymmetric angle-ply laminated plates," Journal of Engineering Mechanics, Vol. 115, No. 5, 952-962, 1988

49. J. N. Reddy, A. A. Khdeir and L. Librescu, "Lévy type solution for symmetrically laminated rectangular plates using first-order shear deformation theory," Journal of Applied Mechanics, Vol. 54, 740-742, 1987.

50. A. A. Khdeir, J. N. Reddy and L. Librescu, "Analytical solution of a refined shear deformation theory for rectangular composite plates," International Journal of Solids and Structures, Vol. 23, No. 10, 1447-1463, 1987.

51. N. Li and S. Mirza, "Free vibration analysis of antisymmetric angle-ply laminated composite plate," Proceedings of 14th Canadian Congress of Applied Mechanics," Vol.1, 139-140, May, 1993.

52. S. Mirza and N. Li, "Accurate buckling analysis of antisymmetrical laminated composite

plates," Proceedings of the Second Canadian International Composites Conference and Exhibition, 768-777, September,1993.

53. N. Li and S. Mirza, "Buckling analysis of clamped sandwich plates by the reciprocal theorem method," (to be published in the Journal of Computers and Structures).

54. N. Li, "The reciprocal theorem method for plate free vibration analysis-the completely free rectangular plate," Journal of Sound and Vibration, Vol. 158, No. 2, 307-316, 1992.

55. N. Li, "A method using the reciprocal theorem to solve the bending of the elastic semicircular plates," Applied Mathematics and Mechanics, Vol. 13, No. 12, 1127-1131, 1992.

56. N. Li and B. Fu, "Bending analysis of the circular plates with K internal point supports," Applied Mathematics and Mechanics, Vol. 12, No. 11, 634-642, 1991.

57. B. Fu and N. Li, "MRT of forced vibration for elastic thin rectangular plates (III)," Applied Mathematics and Mechanics, Vol. 12, No. 7, 663-680, 1991.

58. B. Fu and N. Li, "MRT of forced vibration for elastic thin rectangular plates (II)," Applied Mathematics and Mechanics, Vol. 11, No. 11, 1043-1054, 1990.

59. B. Fu and N. Li, "MRT of forced vibration for elastic thin rectangular plates (I)," Applied Mathematics and Mechanics, Vol. 10, No. 8, 727-749, 1989.

60. N. Li and B. Fu, "Application of the reciprocal theorem for calculating transverse displacement equation of thin elastic circular plates," Applied Mathematics and Mechanics, Vol. 9, No. 9, 891-898, 1988.

61. S. B. Dong and F. K. W. Tso, "On a laminated orthotropic shell theory including transverse shear deformation," Journal of Applied Mechanics, Vol. 39, 1091-1096, 1971.

62. S. T. Gulati and F. Essenberg, "Effects of anisotropy in axisymmetric cylindrical shells," Journal of Applied Mechanics, Vol. 34, 650-666, 1967.

63. J. A. Zukas and J. R. Vinson, "Laminated transversely isotropic cylindric shells," Journal of Applied Mechanics, Vol. 38, 400-407, 1971.

64. T. M. Hsu and J. T. S. Wang, "A theory of laminated cylindrical shells consisting of layers of orthotropic laminae," AIAA Journal, Vol. 8, 2141-2164, 1970.

65. J. M. Whitney and C. T. Sun, "A higher order theory for extensional motion of laminated anisotropic shells and plates," Journal of Sound and Vibration, Vol. 30, p. 85, 1973.

66. J. M. Whitney and C. T. Sun, "A refined theory for laminated anisotropic cylindrical shells," Journal of Applied Mechanics, Vol. 41, 471-476, 1974.

67. J. N. Reddy "Exact solutions for moderately thick laminated shells," Journal of Engineering Mechanics, Vol. 101, No. 5, 795-809, 1984.

68. R. B. Nelson, S. B. Dong and R. D. Kalra, "Vibration and waves in laminated orthotropic circular cylinders," Journal of Sound and Vibration, Vol. 18, 429-444, 1971.

69. L. G. Bradford and S. B. Dong, "Natural vibrations of orthotropic cylinders under initial stress," Journal of Sound and Vibration, Vol. 60, 157-175, 1978.

70. J. B. Greenberg and Y. Stavsky, "Buckling and vibration of orthotropic composite cylindrical shells," Acta Mechanica, Vol. 36, 15-29, 1980.

71. J. B. Greenberg and Y. Stavsky, "Vibrations of axially compressed laminated orthotropic cylindrical shells" Acta Mechanica, Vol 37, 13-18, 1980.

72. J. B. Greenberg and Y. Stavsky, "Vibrations of laminated filament wound cylindrical shells," AIAA Journal, Vol. 19, 1055-1062, 1981.

73. J. B. Greenberg and Y. Stavsky, "Buckling and Vibrations of Compressed, aeolotropic, composite cylindrical shells," Journal of Applied Mechanics, Vol. 49, 834-848, 1982.

74. C. W. Bert and V. S. Reddy, "Cylindrical shells of bimodulus material," Journal of Engineering Mechanics, Vol. 8, No. 5, 675-688, 1982.

75. C. W. Bert and M. Kumar, "Vibration of cylindrical shells of bimodulus composite materials," Journal of Sound and Vibration, Vol. 81, No. 1, 1982.

76. Y. S. Hsu, J. N. Reddy and C. W. Bert, "Thermoelasticity of circular cylindrical shells laminated of bimodulus composite material," Journal of Thermal Stresses, Vol. 4, 155-177, 1981.

77. M. E. Vanderpool and C. W. Bert, "Vibration of a materially monoclinic, thick-wall circular cylindrical shell," AIAA Journal, Vol. 19, No. 5, 634-641, 1981.

78. S. B. Dong and R. B. Nelson, "On natural vibration and waves in laminated orthotropic plates," Journal of Applied Mechanics, Vol. 39, 739-745, 1972.

79 S. V. Kulkarni and N. J. Pagano, "Dynamic characteristics of composite laminates," Journal of Sound and Vibration, Vol. 23, 127-143, 1972.

80. L. Librescu and R. Schmidt, "On the refined theory of multilayered anisotropic shells,"

Mechanika Polimerov, Part I, Vol. 6, 1974, Part II, Vol. 7, 1975.

81. R. B. Nelson and D. R. Lorch, "A refined theory for laminated orthotropic plates," Journal of Applied Mechanics, Vol. 41, 177-183, 1974.

82. K. H. Lo, R. M. Christensen and E. M. Wu, "A high-order theory of plate deformation, Part II, laminated plates," Journal of Applied Mechanics, Vol. 44, 669-676, 1977.

83. M. Levinson, "An accurate simple theory of the statics and dynamics of elastic plates," Mech. Res. Commun., Vol. 7, 343-350, 1980.

84. M. V. V. Murthy, "An improved transverse shear deformation theory for laminated anisotropic plates," NASA TP-1903, Nov. 1981.

85. A. Bhimaraddi and L. K. Stevens, "A high-order theory for free vibration of orthotropic, homogenous, and laminated rectangular plates," Journal of Applied Mechanics, Vol. 51, 195-198, 1984.

86. J. N. Reddy, "A simple higher-order theory for laminated composite plates," Journal of Applied Mechanics, Vol. 55, 745-752, 1984.

87. J. N. Reddy and C. Liu, "A higher-order shear deformation theory of laminated elastic shells," International Journal of Engineering Science, Vol. 23, No. 3, 319-330, 1985.

88. C. W. Bert, "A critical evaluation of new plate theories applied to laminated composites," Compos. Struct., Vol. 2, 329-347, 1984.

89. H. N. Cho, C. W. Bert and A. G. Striz, "New theory for bending of bimodular laminates," Society of Engineering Science, 24th Annual Meeting, Salt Lake City, UT, Sept. 20-23, 1987, Preprint ESP 24. 87034.

90. J. N. Reddy, "A generalization of two-dimensional theories of laminated composite plates," Communications in Applied Numerical Methods, Vol.3, 173-180, 1987.

91. N. D. Phan and J. N. Reddy, "Analysis of laminated composite plates using a high-order shear deformation theory," International Journal of Numerical Methods in Engineering, Vol. 21, 2201-2220, 1985.

92. A. A. Khdeir, "Free vibration and buckling of symmetric cross-ply laminated plates by an exact method," Journal of Sound and Vibration, Vol. 126, No. 3, 447-461, 1988.

93. A. A. Khdeir, "Free vibration and buckling of unsymmetric cross-ply laminated plates - using a refined theory," Journal of Sound and Vibration, Vol. 128, No. 3, 377-395, 1989.

94. A. A. Khdeir and J. N. Reddy, "On the forced motion of antisymmetric cross-ply laminated plates," International Journal of Mechanical Science, Vol. 31, No. 7, 499-510, 1989.

95. A. A. Khdeir and J. N. Reddy, "Exact solutions for the transient response of symmetric cross-ply laminates using a higher-order plate theory," Composites Science and Technology, Vol. 34, No. 3, 205-224, 1989.

96. A. A. Khdeir and J. N. Reddy, "Influence of edge conditions on the modal characteristics of cross-ply laminated shells," Computers and Structures, Vol. 34, No. 6, 817-826, 1990.

97. A. A. Khdeir, J. N. Reddy and D. Frederick, "A study of bending, vibration and buckling of cross-ply circular cylindrical shells with various shell theories," International Journal of Engineering Science, Vol. 27, No. 11, 1337-1351, 1989.

98. A. A. Khdeir, J. N. Reddy and D. Frederick, "On the transient response of cross-ply laminated circular cylindrical shells," International Journal of Impact Engineering, Vol. 9, No. 4, 475-484, 1990.

99. K. P. Soldatos, "Influence of thickness shear deformation on free vibrations of rectangular plates, cylindrical panels and cylinders of antisymmetric angle-ply construction," Journal of Sound and Vibration, Vol. 119, No. 1, 111-137, 1987.

100. C. W. Bert and P. H. Francis,"Composite material mechanics: Structural mechanics," AIAA Journal, Vol. 12, No. 9, 1173-1186, 1974.

101. C. W. Bert, "Recent research in composite and sandwich plate dynamics," Shock and Vibration Digest, Vol. 11, No. 10, 13-23, 1979.

102. C. W. Bert, "Vibration of composite structures." Recent advances in structural dynamics, Proceeding of International Conference, University of Southeampton, Vol. 2, 693-712, July, Southeampton, England, 1980.

103. C. W. Bert, "Advances in dynamics of composite structures," In Composite Structures 4, 1-17, ed. I. H. Marshall, Elsevier Applied Science, London and New York, 1987.

104. J. N. Reddy, "Finite element modelling of layered, anisotropic composite plates and shells: A review of recent research," Shock and Vibration Digest, Vol. 13, No. 12, 3-12, 1981.

105. J. N. Reddy, "Energy and variational methods in applied mechanics," John Wiley and Sons, New York, 1984.

106. S. Mirza, "Recent research in vibration of layered shells," ASME Journal of Pressure Vessel Technology, Vol 113, 321-325, 1991.

107. R. K. Kapania, "A review on the analysis of laminated shells," ASME Journal of Pressure Vessel Technology, Vol. 111, 88-96, 1989.

108. Y. K. Cheung, "Finite strip method in structural mechanics," Pergamom Press, 1976.

109. S. Timoshenko and S. Woinowsky-Krieger, "Theory of plates and shells," 2nd ed., McGraw-Hill, New York, 1959.

110. D. J. Gorman, "Free vibration analysis of rectangular plates," New York: Elsevier North Holland, 1982.

111. D. J. Gorman, "Accurate free vibration analysis of clamped orthotropic plates by the method of superposition," Journal of Sound and Vibration, Vol. 140, No. 3, 391-411, 1990.

112. N. Li, "Forced vibration analysis of the clamped orthotropic rectangular plates by the superposition method," Journal of Sound and Vibration, Vol. 158, No. 2, 307-316, 1992.

113. C. T. Wang, "Applied elasticity," New York: McGraw-Hill, 1953.

114. R. M. Jones, "Mechanics of composite materials," Washington D. C. : Scripta Book Company, 1975.

115. R. D. Marangoni, L. M. Cook and N. Basavanhally, "Upper and lower bounds to the natural frequencies of vibration of clamped rectangular orthotropic plates," International Journal of Solids and Structures, Vol. 14, 611-623, 1978.

116 S. M. Dickinson, "The flexural vibration of rectangular orthotropic plates," Journal of Applied Mechanics, Vol. 36, 101-106, 1969.

117 J. M. Whitney, "The effect of boundary conditions on the response of laminated composite," Journal of Composite Materials, Vol. 4, 192-203, 1970.

118 C.-C. Lin and W. W. King, "Free transverse vibration of rectangular unsymmetrically laminated plates," Journal of Sound and Vibration, Vol. 36, No. 1, 91-103, 1974.

119 V. V. Bolotin, "An asymptotic method for the study of the problem of eigenvalues for rectangular regions," in the Problems of Continuum Mechanics (Volume dedicated to N. I Muskhelishvili), 56-68, Philadelphia: Society for Industrial and Applied Mathematics, 1961.

120 V. V. Bolotin, "Dynamic edge effect in the elastic vibrations of plates," (in Russian) Inzhenernii Sbornik, Akademiia Nauk. SSSR 31, 3-14, 1961.

121 V. V. Bolotin, "The edge effect in the oscillations of elastic shells," (in Russian) Prikladnaya Matematikai Mekhanika, 24, 831-842, 1960.

122 N. Li and S. Mirza, "Vibration analysis of cross-ply laminated shells," Proceeding of SAMPE 1994: 15th International Conference & exhibition, European Chapter, France, 1994.

123 N. Li and D. J. Gorman, "Free vibration analysis of clamped rectangular plates with line support along the diagonals," Journal of Sound and Vibration, Vol. 173, No. 5, 591-598, 1994.

124 N. Li and D. J. Gorman, "Free vibration analysis of simply supported plates with internal line support along diagonals," Journal of Sound and Vibration, Vol. 165, No. 2, 361-368, 1993.

125 N. Li and D. J. Gorman, "Free vibration analysis of rectangular plates with free edges and line support along diagonals," AIAA Journal, Vol. 30, No. 9, 2351-2353, 1992.

APPENDIX I

COEFFICIENTS APPEARING IN EQUATIONS OF CHAPTER 3

C_i appearing in equations (3.3.1) are given as follows:

$$C_1=-g_1/g_2;\; C_2=-g_3/g_2;\; C_3=-g_4/g_2;\; C_4=-g_5/g_2;\; C_5=-g_6/g_8;$$

$$C_6=-g_7/g_8;\; C_7=-g_9/g_8;\; C_8=-g_{10}/g_8;\quad C_9=-[g_{11}+(g_{12}+g_{14}C_5)C_1]/\phi;$$

$$C_{10}=-[(g_{12}+g_{14}C_5)C_2+g_{13}+g_{14}C_6)/\phi;\; C_{11}=-[(g_{12}+g_{14}C_5)C_3+g_{15}]/\phi;$$

$$C_{12}=-[(g_{12}+g_{14}C_5)C_4+g_{14}C_7+g_{16}]/\phi;\quad \phi=g_{14}C_8+g_{17}$$

where

$$g_1=I_1\omega^2-A_{66}K_n^2;\; g_2=A_{11};\; g_3=(A_{12}+A_{66})K_n;\; g_4=B_{26}K_n^3;\; g_5=-3B_{16}K_n;$$

$$g_6=-g_3;\; g_7=I_1\omega^2-A_{22}K_n^2;\; g_8=A_{66};\; g_9=3B_{26}K_n^2;\; g_{10}=-B_{16};$$

$$g_{11}=-g_4;\; g_{12}=-g_5;\; g_{13}=g_9;\; g_{14}=-B_{16};\; g_{15}=D_{22}K_n^4-(I_1+I_3K_n^2)\omega^2;$$

$$g_{16}=I_1\omega^2-2(D_{12}+2D_{66})K_n^2;\; g_{17}=D_{11}$$

C_i in equation (3.5.3) are

$$C_1=[(D_{66}A_{66}-B_{16}B_{26}K_n^2)-I_1\omega^2 D_{66}]K_n/\phi_1;\; C_2=[2B_{26}B_{16}-D_{66}(A_{12}+A_{66})]K_n/\phi_1;$$

$$C_3=-B_{16}A_{44}K_n/\phi_1;\; C_4=(B_{16}D_{12}-B_{16}D_{66})K_n/\phi_1;$$

$$C_5=[(B_{22}D_{66}-D_{22}B_{16})K_n^2-A_{44}B_{16}+I_3\omega^2 B_{16}]/\phi_1;$$

$$C_6=[(A_{66}+A_{12})D_{11}-2B_{26}^2]K_n/\phi_1;\; C_7=[(A_{22}D_{11}-B_{12}B_{26})K_n^2-I_1\omega^2 B_{16}]/\phi_1;$$

$$C_8=-B_{16}A_{55}/\phi_1;\; C_9=[(D_{11}B_{26}-B_{16}D_{66})K_n^2-A_{55}B_{16}+I_2\omega^2 B_{16}]/\phi_1;$$

$$C_{10}=[2B_{26}D_{11}-B_{16}(D_{12}+D_{66})]K_n/\phi_1;\; C_{11}=(A_{44}K_n^2-I_1\omega^2)/A_{55};\; C_{12}=-1;$$

$$C_{13}=A_{44}K_n/A_{55};\; C_{14}=[B_{16}(A_{66}-A_{12})K_n]/\phi_2;\; C_{15}=[(B_{26}A_{66}-B_{16}A_{22})K_n^2+I_1\omega^2 B_{16}]/\phi_2;$$

$$C_{16}=A_{55}A_{66}/\phi_2;\; C_{17}=[(A_{66}D_{66}-B_{26}B_{16})K_n^2+A_{66}A_{55}-I_2\omega^2 A_{66}]/\phi_2;$$

$$C_{18}=[A_{66}(D_{12}+D_{66})-2B_{16}B_{26}]K_n/\phi_2;\; C_{19}=[(B_{26}A_{11}-B_{16})K_n^2+I_1\omega^2 B_{16}]/\phi_2;$$

$$C_{20}=[(A_{12}+A_{66})B_{16}-2B_{26}A_{11}]K_n/\phi_2;\; C_{21}=A_{44}A_{11}K_n/\phi_2;$$

$$C_{22}=[2B_{16}^2-A_{11}(D_{66}+D_{12})]K_n/\phi_2;\; C_{23}=[(A_{11}D_{22}-B_{26}B_{16})K_n^2+A_{11}A_{44}-I_3\omega^2 A_{11}]/\phi_2$$

C_i in equation (3.5.4) are

$$C_1=-(g_1g_{43}-g_{36}g_8)/\phi_1;\; C_2=-(g_3g_{43}-g_{38}g_8)/\phi_1;\; C_3=-(g_4g_{43}-g_{39}g_8)/\phi_1;$$

$$C_4=-(g_5g_{43}-g_{40}g_8)/\phi_1;\; C_5=-(g_6g_{43}-g_{41}g_8)/\phi_1;\; g_6=-(g_7g_{43}-g_{42}g_8)/\phi_1;$$

$$C_7=-(g_9g_{34}-g_{28}g_{15})/\phi_2;\; C_8=-(g_{10}g_{34}-g_{29}g_{15})/\phi_2;\; C_9=-(g_{12}g_{34}-g_{31}g_{15})/\phi_2;$$

$$C_{10}=-(g_{13}g_{34}-g_{32}g_{15})/\phi_2;\; C_{11}=-(g_{14}g_{34}-g_{33}g_{15})/\phi_2;\; C_{12}=-(g_{16}g_{34}-g_{35}g_{15})/\phi_2;$$

$$C_{13}=-(g_{17}+h_1C_1+h_6C_{25})/h_4;C_{14}=-(h_1C_2+h_6C_{26})/h_4;C_{15}=-(h_1C_3+g_{21}+h_6C_{27})/h_4;$$

$$C_{16}=-(h_1C_4+h_3+h_6C_{28})/h_4;C_{17}=-(h_1C_5+h_5+h_6C_{29})/h_4;C_{18}=-(h_1C_6+g_{26}+h_6C_{30})/h_4;$$

$$C_{19}=-(g_{28}+g_{30}C_7)/g_{34};\; C_{20}=-(g_{29}+g_{30}C_8)/g_{34};\; C_{21}=-(g_{30}C_9+g_{31})/g_{34};$$

$$C_{22}=-(g_{30}C_{10}+g_{32})/g_{34};\; C_{23}=-(g_{30}C_{11}+g_{33})/g_{34};\; C_{24}=-(g_{30}C_{12}+g_{35})/g_{34};$$

$$C_{25}=-(g_{37}C_1+g_{36})/g_{43};\; C_{26}=-(g_{37}C_2+g_{38})/g_{43};\; C_{27}=-(g_{37}c_3+g_{39})/g_{43};$$

$$C_{28}=-(g_{37}C_4+g_{40})/g_{43};\; C_{29}=-(g_{37}C_5+g_{41})/g_{43};\; C_{30}=(g_{37}C_6+g_{42})/g_{43}$$

where

$$g_1=\bar{I}_1\omega^2-A_{66}K_n^2;\ g_2=A_{11};\ g_3=(A_{12}+A_{66})K_n;\ g_4=c_2E_{26}K_n^3;\ g_5=-3c_2E_{16}K_n;$$

$$g_6=2(B_{16}-c_2E_{16})K_n;\ g_7=(c_2E_{26}-B_{26})K_n^2;\ g_8=B_{16}-c_2E_{16};\ g_9=-g_3;$$

$$g_{10}=\hat{I}_1\omega^2-A_{22}K_n^2;\ g_{11}=A_{66};\ g_{12}=3c_2E_{26}K_n^2;\ g_{13}=-c_2E_{16};\ g_{14}=c_2E_{26}-B_{26};$$

$$g_{15}=B_{16}-c_2E_{16};\ g_{16}=2(c_2E_{26}-B_{26})K_n;\ g_{17}=-g_4;\ g_{18}=-g_5;\ g_{19}=g_{12};$$

$$g_{20}=g_{13};\ g_{21}=c_2^2H_{22}K_n^4+(A_{44}-2c_1D_{44}+c_1^2F_{44})K_n^2-(I_1+c_2^2I_7K_n^2)\omega^2;$$

$$g_{22}=c_2^2I_7-2c_2^2(H_{12}+2H_{66})K_n^2-(A_{55}-2c_1D_{55}+c_1^2F_{44});\ g_{23}=c_2^2H_{11};$$

$$g_{24}=c_2[F_{12}-c_2H_{12}+2(F_{66}-c_2H_{66})]K_n^2-(A_{55}-2c_1D_{55}+c_1^2F_{55})-\bar{I}_5\omega^2;$$

$$g_{25}=c_2(F_{11}-c_2H_{11});\ g_{26}=(A_{44}-2c_1D_{44}+c_1^2F_{44})K_n-c_2(F_{22}-c_2H_{22})K_n^2+\hat{I}_5\omega^2K_n;$$

$$g_{27}=c_2[(F_{12}-c_2H_{12})+2(F_{66}-c_2H_{66})]K_n;\ g_{28}=-g_6;\ g_{29}=g_{15};\ g_{30}=g_{16};$$

$$g_{31}=g_{24};\ g_{32}=g_{25};\ g_{33}=\bar{I}_4\omega^2-(D_{66}-2F_{66}+c_2^2H_{66})K_n^2-(A_{55}-2c_2D_{55}+c_1^2F_{55});$$

$$g_{34}=D_{11}-2c_2F_{11}+c_2^2H_{11};\ g_{35}=c_2(F_{12}-c_2H_{12}+F_{66}-c_2H_{66})-(D_{12}-F_{12}c_2);$$

$$-(D_{66}-F_{66}c_2);\ g_{36}=g_7;\ g_{37}=g_8;\ g_{38}=-g_{16};\ g_{39}=-g_{26};\ g_{40}=-g_{27};\ g_{41}=-g_{25};$$

$$g_{42}=\hat{I}_4\omega^2-(D_{22}-2c_2F_{22}+c_2^2H_{22})K_n^2-(A_{44}-2c_2D_{44}+c_1^2F_{44});\ g_{43}=D_{66}-2F_{66}c_2+c_2^2H_{66}$$

APPENDIX II

COEFFICIENTS APPEARING IN EQUATIONS OF CHAPTER 4

C_i contained in equations (4.2.14) are

$$C_1=[(I_3K_n^2+I_1)\omega^2-D_{22}K_n^4]/D_{11};\ C_2=[I_3\omega^2-2(D_{12}+2D_{66})]/D_{11}$$

C_i in equations (4.2.15) are

$$C_1=-g_1/g_2;\ C_2=-g_2/g_2;\ C_3=-g_4/g_2;$$

$$C_4=-g_5/g_7;\ C_5=-g_6/g_7;\ C_6=-g_8/g_7;$$

$$C_7=-g_9/g_{12};\ C_8=-g_{10}/g_{12};\ C_9=-g_{11}/g_{12};$$

$$g_1=A_{44}K_n^2-I_1\omega^2;\ g_2=-A_{55};\ g_3=-A_{55};\ g_4=A_{44}K_n;$$

$$g_5=-A_{55};\ g_6=I_3\omega^2-D_{66}K_n^2-A_{55};\ g_7=D_{11};\ g_8=-(D_{12}+D_{66})k_n;$$

$$g_9=-A_{44};\ g_{10}=(D_{12}+D_{66})K_n;\ g_{11}=I_3\omega^2-D_{22}K_n^2-A_{44};\ g_{12}=D_{66}$$

C_i in equations (4.2.16) are

$$C_1=-g_1/g_4;\ C_2=g_2/g_4;\ C_3=g_3/g_4;\ C_4=g_5/g_4;$$

$$C_5=-g_6/g_{10};\ C_6=-g_7/g_{10};\ C_7=-g_8/g_{10};\ C_8=-g_9/g_{10};$$

$$C_9=-h_1/\phi;\ C_{10}=-h_2/\phi;\ C_{11}=-h_3/\phi;\ C_{12}=-h_4/\phi$$

where

$$g_1=c_1(D_{55}-c_1F_{55})-(A_{55}-c_1D_{55})+c_2[2F_{66}+F_{12}-c_2(2H_{66}+H_{12})]K_n^2-\bar{I}_5\omega^2;$$

$$g_2=-c_2(F_{11}-c_2H_{11});\ g_3=c_1(D_{55}-c_1F_{55})-(A_{55}-c_1D_{55})-K_n^2(D_{66}-2c_2F_{66}+c_2^2H_{66})$$

$$+\bar{I}_4\omega^2;\ g_4=D_{11}-2c_2F_{11}-c_2^2H_{11};\ g_5=-[D_{12}+D_{66}-2c_2(F_{12}+F_{66})+c_2^2(H_{12}+H_{16})]K_n;$$

$$g_6=[c_1(D_{44}-c_1F_{44})-(A_{44}-c_1D_{44})]K_n+c_2(F_{22}-c_2H_{22})K_n^3-\hat{I}_5K_n\omega^2;$$

$$g_7=-c_2[2F_{66}+F_{12}-c_2(2H_{66}+H_{12})]K_n;\ g_8=[D_{12}+D_{66}-2c_2(F_{12}+F_{66})+c_2^2(H_{12}+H_{66})]K_n;$$

$$g_9=c_1(D_{44}-c_1F_{44})-(A_{44}-c_1D_{44})-K_n^2(D_{22}-2c_2F_{22}+c_2^2H_{22})+\hat{I}_4\omega^2;$$

$$g_{10}=D_{66}-2c_2F_{66}+c_2^2H_{66};\ g_{11}=c_2^2H_{22}K_n^4-[c_1(D_{44}-c_1F_{44})-(A_{44}-c_1D_{44})]K_n^2$$

$$-(I_1+c_2^2K_n^2)\omega^2;\ g_{12}=c_1(D_{55}-c_1F_{55})-(A_{55}-c_1D_{55})-c_2^2(2H_{12}+4H_{66})K_n^2+c_2^2I_7\omega^2;$$

$$g_{13}=c_2^2H_{11};\ g_{14}=c_1(D_{55}-c_1F_{55})-(A_{55}-c_1D_{55})+c_2[2F_{66}+F_{12}-c_2(2H_{66}+H_{12})]K_n^2-\bar{I}_5\omega^2;$$

$$g_{15}=-c_2(F_{11}-c_2H_{11});\ g_{16}=[(A_{44}-c_1D_{44})-c_1(D_{44}-c_1F_{44})]K_n-c_2(F_{22}-c_2H_{22})K_n^3$$

$$+\hat{I}_5K_n\omega^2;\ g_{17}=c_2[2F_{66}+F_{12}-c_2(2H_{66}+H_{12})]K_n;\ h_1=g_{11}+g_{17}C_5+g_{15}C_4C_5;$$

$$h_2=g_{12}+g_{15}C_1+g_{27}C_6+g_{15}C_4C_6;\ h_3=g_{14}+g_{15}C_3+g_{17}C_7+g_{15}C_4C_7;$$

$$h_4=g_{16}+g_{17}C_8+g_{15}C_4C_8;\ \phi=g_{13}+g_{15}C_2$$

C_i in equations (4.3.6) are

$$C_1=-g_1/g_2;\ C_2=-g_3/g_2;\ C_3=g-4/g_2;\ C_4=-g_5/g_2;$$

$$C_5=-g_6/g_8;\ C_6=-g_7/g_8;\ C_7=-g_9/g_8;\ C_8=-g_{10}/g_8;$$

$$C_9=-(g_{11}+g_{12}C_1+g_{14}C_5+g_{12}C_2)/\phi;\ C_{10}=-(g_{13}+g_{14}C_6+g_{12}C_2C_6)/\phi;$$

$$C_{11}=-(g_{14}C_7+g_{15}+g_{12}C_2C_7)/\phi;\ C_{12}=-(g_{12}C_3+g_{14}C_8+g_{16}+g_{12}C_2C_8)/\phi$$

where

$$\phi = g_{12}C_4 + g_{17};\ g_1 = I_1\omega^2 - A_{66}K_n^2;\ g_2 = A_{11};\ g_3 = -(A_{12}+A_{66});$$

$$g_4 = (B_{12}+2B_{66})K_n^2 - I_2\omega^2;\ g_5 = -B_{11};\ g_6 = -g_3;\ g_7 = I_1\omega^2 - A_{22}K_n^2;$$

$$g_8 = A_{66};\ g_9 = B_{22}K_n^3 - I_2\omega^2 K_n;\ g_{10} = -(B_{12}+2B_{66});\ g_{11} = g_4$$

$$g_{12} = -B_{111};\ g_{13} = -g_9;\ g_{14} = -g_{10};\ g_{15} = D_{22}K_n^4 - (I_3K_n^2 + I_1)\omega^2;$$

$$g_{16} = -[2(D_{12}+2D_{66})K_n^2 - I_3\omega^2];\ g_{17} = D_{11};$$

C_i in equations (4.3.7) are

$$C_1 = (g_1g_{26} - g_{21}g_6)/\phi_1;\ C_2 = (g_3g_{26} - g_{23}g_6)/\phi_1;\ C_3 = (g_4g_{26} - g_{24}g_6)/\phi_1;$$

$$C_4 = (g_5g_{26} - g_{25}g_6)/\phi_1;\ C_5 = (g_7g_{26} - g_{27}g_6)/\phi_1;\ C_6 = (g_8g_{34} - g_{28}g_{14})/\phi_2;$$

$$C_7 = (g_9g_{34} - g_{29}g_{14})/\phi_2;\ C_8 = (g_{11}g_{34} - g_{31}g_{14})/\phi_2;\ C_9 = (g_{12}g_{34} - g_{32}g_4)/\phi_2;$$

$$C_{10} = (g_{13}g_{34} - g_{33}g_{14})/\phi_2;\ C_{11} = -g_{15}/g_{18};\ C_{12} = -g_{16}/g_{18};\ C_{13} = -g_{17}/g_{18};$$

$$C_{14} = -g_{19}/g_{18};\ C_{15} = -g_{20}/g_{18};\ C_{16} = -(g_{21}+g_{22}C_1)/g_{26};\ C_{17} = -(g_{23}+g_{22}C_2);$$

$$C_{18} = -(g_{24}+g_{22}C_3)/g_{26};\ C_{19} = -(g_{25}+g_{22}C_4)/g_{26};\ C_{20} = -(g_{27}+g_{22}C_5)/g_{26};$$

$$C_{21} = -(g_{28}+g_{30}C_6)/g_{34};\ C_{22} = -(g_{29}+g_{30}C_7)/g_{34};\ C_{23} = -(g_{31}+g_{30}C_8)/g_{34};$$

$$C_{24} = -(g_{32}+g_{30}C_9)/g_{34};\ C_{25} = -(g_{33}+g_{30}C_{10})/g_{34}$$

where

$$\phi_1=g_{22}g_6-g_2g_{26};\ \phi_2=g_{30}g_{14}-g_{10}g_{34};\ g_1=I_1\omega^2-A_{66}K_n^2;$$

$$g_2=A_{11};\ g_3=-(A_{12}+A_{66})K_n;\ g_4=0;\ g_5=I_2\omega^2-B_{66}K_n^2;$$

$$g_6=B_{11};\ g_7=-(B_{12}+B_{66})K_n;\ g_8=-g_3;\ g_9=I_1\omega^2-K_n^2A_{22};$$

$$g_{10}=A_{66};\ g_{11}=0;\ g_{12}=(B_{66}+B_{12})K_n;\ g_{13}=I_2\omega^2-B_{22}K_n^2;$$

$$g_{14}=B_{66};\ g_{15}=0;\ g_{16}=0;\ g_{17}=A_{44}K_n^2-I_1\omega^2;\ g_{18}=-A_{55};$$

$$g_{19}=-A_{55};\ g_{20}=A_{44};\ g_{21}=g_5;\ g_{22}=g_6;\ g_{23}=g_{12};\ g_{24}=g_{19};$$

$$g_{25}=I_3\omega^2-D_{66}K_n^2-A_{55};\ g_{26}=D_{11};g_{27}=-(D_{12}+D_{66})K_n;$$

$$g_{28}=-g_7;\ g_{29}=g_{13};\ g_{30}=g_{24};\ g_{31}=-g_{20};\ g_{32}=-g_{27};$$

$$g_{33}=I_3\omega^2-D_{22}K_n^2-A_{44};\ g_{34}=D_{66}$$

C_i in equations (4.3.8) are

$$C_1=(g_1g_{34}-g_7g_{28})/\phi_1;\ C_2=(g_3g_{34}-g_7g_{30})/\phi_1;\ C_3=(g_4g_{34}-g_7g_{31})/\phi_1;$$

$$C_4=(g_5g_{34}-g_7g_{32})/\phi_1;\ C_5=(g_6g_{34}-g_7g_{33})/\phi_1;\ C_6=(g_8g_{34}-g_7g_{35})/\phi_1;$$

$$C_7=(g_9g_{43}-g_{36}g_{16})/\phi_2;\ C_8=(g_{10}g_{43}-g_{37}g_{16})/\phi_2;\ C_9=(g_{12}g_{43}-g_{39}g_{16})/\phi_2;$$

$$C_{10}=(g_{13}g_{43}-g_{40}g_{16})/\phi_2;\ C_{11}=(g_{14}g_{43}-g_{41}g_{16})/\phi_2;\ C_{12}=(g_{15}g_{43}-g_{42}g_{16})/\phi_2;$$

$$C_{13}=-(h_4+h_2C_7+h_6C_{25})/h_4;\ C_{14}=-(g_{19}+h_2C_8+h_6C_{26})/h_4;$$

$$C_{15}=-(h_2C_9+g_{21}+h_6C_{27})/h_4;\ C_{16}=-(h_2C_{10}+h_3+h_6C_{28})/h_4;$$

$$C_{17}=-(h_2C_{11}+h_5+h_6C_{29})/h_4;\ C_{18}=-(h_2C_{12}+g_{26}+h_6C_{30})/h_4;$$

$$C_{19}=-(g_{28}+g_{29}C_1)/g_{34};\ C_{20}=-(g_{30}+g_{29}C_2)/g_{34};\ C_{21}=-(g_{31}+g_{29}C_3)/g_{34};$$

$$C_{22}=-(g_{32}+g_{29}C_4)/g_{34};\ C_{23}=-(g_{33}+g_{29}C_5)/g_{34};\ C_{24}=-(g_{35}+g_{29}C_6)/g_{34};$$

$$C_{25}=-(g_{36}+g_{38}C_7)/g_{43};\ C_{26}=-(g_{37}+g_{38}C_8)/g_{43};\ C_{27}=-(g_{39}+g_{38}C_9)/g_{43};$$

$$C_{28}=-(g_{40}+g_{38}C_{10})/g_{43};\ C_{29}=-(g_{41}+g_{38}C_{11})/g_{43};\ C_{30}=-(g_{43}+g_{38}C_{12})/g_{43}$$

where

$$\phi_1 = g_7 g_{29} - g_2 g_{34};\ \phi_2 = g_{38} g_{16} - g_{11} g_{43};\ g_1 = \bar{I}_1 \omega^2 - A_{66} K_n^2;$$

$$g_2 = A_{11};\ g_3 = -(A_{12}+A_{66})K_n;\ g_4 = c_2(E_{12}+2E_{66})K_n^2 - \bar{I}_3 \omega^2;$$

$$g_5 = -c_2 E_{11};\ g_6 = \bar{I}_2 \omega^2 - (B_{66} - c_2 E_{66})K_n;\ g_7 = B_{11} - c_2 E_{11};$$

$$g_8 = c_2(E_{12}+E_{66}) - (B_{12}+B_{66});\ g_9 = -g_3;\ g_{10} = \hat{I}_1 \omega^2 - A_{22} K_n^2;\ g_{11} = A_{66};$$

$$g_{12} = c_2 E_{22} K_n^3 - \hat{I}_3 \omega^2 K_n;\ g_{13} = -c_2(E_{12}+2E_{66});\ g_{14} = [B_{12}+B_{66} - c_2(E_{12}+E_{66})]K_n;$$

$$g_{15} = \hat{I}_2 \omega^2 - (B_{22} - c_2 E_{22})K_n^2;\ g_{16} = B_{66} - c_2 E_{66};\ g_{17} = c_2(E_{12}+2E_{66})K_n^2 - \bar{I}_3 \omega^2;$$

$$g_{18} = -c_2 E_{11};\ g_{19} = \hat{I}_3 K_n \omega^2 - c_2 E_{22} K_n^3;\ g_{20} = (E_{12}+2E_{66})c_2 K_n;$$

$$g_{21} = c_2^2 H_{22} K_n^4 + 2c_2(E_{12}/R_1 + E_{22}/R_2)K_n^2 - [c_1(D_{44} - c_1 F_{44}) - (A_{44} - c_1 D_{44})]K_n^2$$

$$-(I_1 + c_2^2 I_7 K_n^2)\omega^2;\ g_{22} = c_1(D_{55} - c_1 F_{55}) - (A_{55} - c_1 D_{55}) - 2c_2^2(H_{12}+2H_{66})K_n^2 +$$

$$c_2^2 I_7 \omega^2;\ g_{23} = c_2^2 H_{11};\ g_{24} = c_1(D_{55} - c_1 F_{55}) - (A_{55} - c_1 D_{55}) + c_2[2F_{66} + F_{12} -$$

$$c_2(2H_{66}+H_{12})]K_n^2 - \bar{I}_5 \omega^2;\ g_{25} = -c_2(F_{11} - c_2 H_{11});$$

$$g_{26} = [(A_{44} - c_1 D_{44}) - c_1(D_{44} - c_1 F_{44})]K_n - c_2(F_{22} - c_2 H_{22})K_n^3 + \hat{I}_5 K_n \omega^2;$$

$$g_{27} = c_2 K_n[2F_{66} + F_{12} - c_2(2H_{66}+H_{12})];\ g_{28} = g_6;\ g_{29} = g_7;\ g_{30} = -g_{14};\ g_{31} = g_{24};$$

$$g_{32} = g_{25};\ g_{33} = c_1(D_{55} - c_1 F_{55}) - (A_{55} - c_1 D_{55}) - K_n^2(D_{66} - 2c_2 F_{66} + c_2^2 H_{66}) + \bar{I}_4 \omega^2;$$

$$g_{34} = D_{11} - 2c_2 F_{11} + c_2^2 H_{11};\ g_{35} = -[D_{12}+D_{66} - 2c_2(F_{12}+F_{66}) + c_2^2(H_{12}+H_{66})]K_n;$$

$$g_{36} = -g_8;\ g_{37} = g_{15};\ g_{38} = g_{g_{16}};\ g_{39} = -g_{36};\ g_{40} = -g_{37};\ g_{41} = -g_{35};\ g_{42} = c_1(D_{44} -$$

$$c_1 F_{44}) - (A_{44} - c_1 D_{44}) - K_n^2(D_{22} - 2c_2 F_{22} + c_2^2 H_{22}) + \hat{I}_4 \omega^2;\ g_{43} = D_{66} - 2c_2 F_{66} + c_2^2 H_{66}$$

APPENDIX III

COEFFICIENTS APPEARING IN EQUATIONS OF CHAPTER 5

C_i in equations (5.1.8) are given by

$$C_1 = -g_1/g_2;\ C_2 = -g_3/g_2;\ C_3 = -g_4/g_2;\ C_4 = -g_5/g_2;$$

$$C_5 = -g_6/g_8;\ g_6 = -g_7/g_8;\ C_7 = -g_9/g_8;\ C_8 = -g_{10}/g_8;$$

$$C_9 = -(g_{11}+g_{12}C_1+g_{14}C_5C_1)/\phi;\ C_{10} = -(g_{12}C_2+g_{13}+g_{14}C_6+g_{14}C_5C_2)/\phi;$$

$$C_{11} = -(g_{12}C_3+g_{15}+g_{14}C_5C_3)/\phi;\ C_{12} = -(g_{12}C_4+g_{14}C_7+g_{16}+g_{14}C_{15}C_4)/\phi;$$

$$\phi = g_{14}C_8+g_{17};\ g_1 = I_1\omega^2-A_{11}K_m^2;\ g_2 = A_{66};\ g_3 = (A_{12}+A_{66})K_m;$$

$$g_4 = (A_{11}/R_1+A_{12}/R_2+B_{11}K_m^2-I_2\omega^2)K_m;\ g_5 = -(B_{12}+2B_{66})K_m;\ g_6 = -g_3;$$

$$g_7 = I_1\omega^2-A_{66}K_m^2;\ g_8 = A_{22};\ g_9 = [A_{12}/R_1+A_{22}/R_2+(B_{12}+2B_{66})]K_m^2-I_2\omega^2;$$

$$g_{10} = -B_{22};\ g_{11} = -g_4;\ g_{12} = -g_5;\ g_{13} = g_9;\ g_{14} = g_{10};$$

$$g_{15} = D_{11}K_m^4+A_{11}/R_1^2+2A_{12}/R_1/R_2+A_{22}/R_2^2+2(B_{11}/R_1+B_{12}/R_2)K_m^2-$$

$$(I_3K_m^2+I_1)\omega^2;\ g_{16} = I_3\omega^2-2[(D_{12}+2D_{66})K_m^2+B_{12}/R_1+B_{22}/R_2];\ g_{17} = D_{22}$$

C_i in equations (5.2.4) have the same form as those in equations (4.3.7) which are listed in the Appendix II. However, g_i are different,

$$g_1=I_1\omega^2-A_{11}K_m^2;\ g_2=A_{66};\ g_3=(A_{12}+A_{66})K_m;\ g_4=(A_{11}/R_1+A_{12}/R_2)K_m;$$

$$g_5=I_2\omega^2-B_{11}K_m^2;\ g_6=B_{66};\ g_7=(B_{12}+B_{66})K_m;\ g_8=-g_3;\ g_9=I_1\omega^2-A_{66}K_m^2;$$

$$g_{10}=A_{22};\ g_{11}=A_{12}/R_1+A_{22}/R_2;\ g_{12}=-g_7;\ g_{13}=I_2\omega^2-B_{66}K_m^2;\ g_{14}=B_{22};$$

$$g_{15}=-g_3;\ g_{16}=g_{11};\ g_{17}=A_{11}/R_1^2+2A_{12}/R_1/R_2+A_{22}/R_2^2+A_{55}K_m^2-I_1\omega^2;$$

$$g_{18}=-A_{44};\ g_{19}=A_{55}-B_{11}/R_1-B_{12}/R_2;\ g_{20}=B_{12}/R_1+B_{22}/R_2-A_{44};$$

$$g_{21}=g_5;\ g_{22}=g_6;\ g_{23}=-g_{12};\ g_{24}=-g_{19};\ g_{25}=I_3\omega^2-A_{55}-D_{11}K_m^2;$$

$$g_{26}=D_{66};\ g_{27}=(D_{12}+D_{66})K_m;\ g_{28}=-g_7;\ g_{29}=g_{13};\ g_{30}=g_{14};$$

$$g_{31}=g_{20};\ g_{32}=-g_{27};\ g_{33}=I_3\omega^2-D_{66}K_m^2-K_{44}^2A_{44};\ g_{34}=D_{22}$$

Similarly, C_i in equations (5.3.5) have the same expressions as given by equations (4.3.8), but g_i are defined as

$$g_1 = \bar{I}_1\omega^2 - A_{11}K_m^2;\ g_2 = A_{66};\ g_3 = (A_{12}+A_{66})K_m;$$

$$g_4 = [c_2E_{11}K_m^2 + (A_{11}/R_1 + A_{12}/R_2) - \bar{I}_3\omega^2]\omega^2;\ g_5 = -c_2K_m(E_{12}+2E_{66});$$

$$g_6 = \bar{I}_2\omega^2 - (B_{11}-c_2E_{11})K_m^2;\ g_7 = B_{66}-c_2E_{66};\ g_8 = [B_{12}+B_{66}-c_2(E_{12}+E_{66})]K_m;$$

$$g_9 = -g_3;\ g_{10} = \hat{I}_1\omega^2 - A_{66}K_m^2;\ g_{11} = A_{22};\ g_{12} = c_2(E_{12}+2E_{66})K_m^2 + A_{12}/R_1 + A_{22}/R_2$$

$$-\hat{I}_3\omega^2;\ g_{13} = -c_2E_{22};\ g_{14} = -(B_{66}-c_2E_{66}+B_{12}-E_{12}c_2)K_m;\ g_{15} = \hat{I}_2\omega^2 - (B_{66}$$

$$-c_2E_{66})K_m^2;\ g_{16} = B_{22}-c_2E_{22};\ g_{17} = -g_4;\ g_{18} = -g_5;\ g_{19} = g_{12};\ g_{20} = g_{13};$$

$$g_{21} = [A_{55}-2c_1D_{55}+c_1^2F_{55})]K_m^2 + c_2^2H_{11}K_m^4 + 2c_2(E_{11}/R_1+E_{12}/R_2)K_m^2$$

$$+A_{11}/R_1^2 + 2A_{12}/R_1/R_2 + A_{22}/R_2^2 - (I_1 + c_2^2I_7K_m^2)\omega^2;$$

$$g_{22} = (D_{44}-c_1F_{44})c_1 - (A_{44}-D_{44}c_1) - 2c_2^2(H_{12}+2H_{66})K_m^2 - 2c_2(E_{12}/R_1+E_{22}/R_2)$$

$$+c_2^2I_7\omega^2;\ g_{23} = c_2^2H_{22};\ g_{24} = (A_{55}-2c_1D_{55}+c_1^2F_{55})K_m - c_2(F_{11}-c_2H_{11})K_m^3 -$$

$$[(B_{12}-c_2E_{11})/R_1 - (B_{12}-c_2E_{12})/R_2]K_m + \bar{I}_5K_\omega^2;\ g_{25} = c_2[(2F_{66}+F_{12})$$

$$-c_2(H_{12}+2H_{66})]K_m;\quad g_{26} = c_1(D_{44}-c_1F_{44}) - (A_{44}-c_1D_{44}) + c_2[2F_{66}+F_{12}-$$

$$c_2(2H_{66}+H_{12})K_m^2 + (B_{12}-c_2E_{12})/R_1 + (B_{22}-c_2E_{22})/R_2 - \hat{I}_5\omega^2;$$

$$g_{27} = -c_2(F_{22}-c_2H_{22});\ g_{28} = g_6;\ g_{29} = g_7;\ g_{30} = -g_{14};g_{31} = -g_{24};\ g_{32} = -g_{25};$$

$$g_{33} = c_1(D_{55}-c_1F_{55}) - (A_{55}-D_{55}c_1) - (D_{11}-2c_2F_{11}+H_{11}c_2^2)K_m^2 + \bar{I}_4\omega^2;$$

$$g_{34} = D_{66} - 2F_{66}c_2 + c_2^2H_{66};\ g_{35} = [D_{12}+D_{66}-2c_2(F_{12}+F_{66})+c_2^2(H_{12}+H_{66})]K_m;$$

$$g_{36} = -g_8;\ g_{37} = g_{15};\ g_{38} = g_{16};\ g_{39} = g_{26};\ g_{40} = g_{27};\ g_{41} = -g_{35};\ g_{42} = c_1(D_{44}$$

$$-c_1F_{44}) - (A_{44}-c_1D_{44}) - (D_{66}-2c_2F_{66}+c_2^2H_{66})K_m^2 + \hat{I}_4\omega^2;\ g_{43} = D_{22} - 2c_2F_{22} - c_2^2H_{22}$$

Table A.1 Non-dimensional fundamental frequencies ω^* $(=\omega a^2/h(\rho/E_2)^{1/2})$ for an antisymmetric angle-ply CCSS plate based on FSDT as plate aspect ratio, ply-angle and number of layers vary. (a/h=10)

Number of layers	θ (degree)	a/b 1.0	1.5	2.0	2.5	3.0
2	15	15.55	19.00	24.11	30.70	38.41
	30	15.19	20.51	27.35	35.36	44.17
	45	15.33	22.76	31.58	41.20	51.24
	60	15.19	24.35	34.55	45.12	55.80
4	15	18.18	21.65	26.75	33.28	40.90
	30	19.70	25.68	33.07	41.51	47.90
	45	20.39	28.82	38.23	48.14	58.30
	60	19.70	29.80	40.37	51.07	61.85
6	15	18.51	21.99	27.10	33.63	41.25
	30	20.20	26.25	33.70	42.18	47.91
	45	20.94	29.43	38.87	48.77	58.93
	60	20.20	30.23	40.91	51.61	62.37
8	15	18.62	22.11	27.22	33.74	41.36
	30	20.36	26.44	33.91	42.40	47.91
	45	21.11	29.63	39.07	48.97	59.13
	60	20.36	30.91	41.08	51.77	62.46

Table A.2 Non-dimensional fundamental frequencies ω^* $(=\omega a^2/h(\rho/E_2)^{1/2})$ for an antisymmetric angle-ply CCSS plate based on FSDT as plate aspect ratio, ply-angle and length-to-thickness vary. (four layers)

a/h	θ (degree)	a/b 1.0	1.5	2.0	2.5	3.0
5	15	11.62	14.27	17.81	21.91	25.74
	30	12.69	16.22	20.35	26.51	30.00
	45	13.09	17.08	21.33	26.91	31.93
	60	12.69	17.60	22.67	27.85	33.10
15	15	21.89	25.79	31.61	39.31	48.65
	30	23.32	30.85	40.39	51.64	64.22
	45	24.02	35.41	48.79	63.35	78.56
	60	23.32	37.59	53.26	69.44	85.80
30	15	25.93	30.45	37.20	46.26	57.58
	30	27.08	36.64	49.13	64.37	82.21
	45	27.74	43.07	62.62	85.73	111.71
	60	27.08	47.56	73.04	101.87	132.76
50	15	27.17	31.93	39.04	48.62	60.66
	30	28.22	38.56	52.22	69.13	89.25
	45	28.89	45.68	67.82	95.06	126.94
	60	28.22	51.13	81.47	118.16	159.92

Table A.3 Non-dimesional fundamental frquencies ω^* $(=\omega a^2/h(\rho/E_2)^{1/2})$ for an antisymmetric angle-ply CCSS plate based on FSDT as plate aspect ratio, ply-angle and in-plane orthotropicity ratio vary. (four layers)

E_1/E_2	θ (degree)	a/b 1.0	1.5	2.0	2.5	3.0
5	15	11.06	15.07	20.90	28.12	36.31
	30	11.20	16.17	22.84	30.78	39.60
	45	11.28	17.47	25.37	34.42	44.09
	60	11.20	18.58	27.65	37.66	48.14
15	15	14.65	18.26	23.58	30.46	38.29
	30	15.27	20.69	27.67	35.81	44.74
	45	15.60	23.13	32.06	41.79	51.92
	60	15.27	24.50	34.79	45.48	54.96
25	15	16.51	19.99	25.18	31.83	39.57
	30	17.55	23.26	30.44	38.73	45.93
	45	18.06	26.14	35.40	45.28	55.46
	60	17.55	27.36	37.89	48.63	59.45
35	15	17.72	21.17	26.29	32.86	40.51
	30	19.09	24.99	32.33	40.72	47.38
	45	19.73	28.08	37.47	47.37	57.54
	60	19.09	29.14	39.70	50.43	60.41

Table A.4 Non-dimesional fundamental frquencies ω^* $(=\omega a^2/h(\rho/E_2)^{1/2})$ for antisymmetric angle-ply CCSC and CCCC plates based on FSDT as plate aspect ratio, ply-angle and number of layers vary. (a/h=10)

Number of layers	θ (degree)	CCSC		CCCC	
		a/b=1.0	a/b=1.5	a/b=1.0	a/b=1.5
2	15	16.16	21.05	18.60	22.74
	30	16.03	22.88	17.93	23.99
	45	16.64	25.77	17.90	26.46
	60	17.16	27.89	17.92	28.33
4	15	18.72	23.42	20.85	24.85
	30	20.44	27.80	21.98	28.58
	45	21.49	31.18	22.65	31.74
	60	21.22	32.21	21.98	32.62
6	15	19.04	23.73	21.11	25.10
	30	20.93	28.35	22.41	29.08
	45	22.00	31.69	23.14	32.24
	60	21.64	32.60	22.41	33.00
8	15	19.14	23.84	21.19	25.19
	30	21.09	28.52	22.54	29.24
	45	22.16	31.85	23.30	32.39
	60	21.77	32.57	22.54	33.13

Table A.5 Non-dimesional fundamental frquencies ω^* $(=\omega a^2/h(\rho/E_2)^{1/2})$ for antisymmetric angle-ply CCSC and CCCC plates based on FSDT as plate aspect ratio, ply-angle and length-to-thickness ratio vary. (four layers)

a/h	θ (degree)	CCSC a/b=1.0	CCSC a/b=1.5	CCCC a/b=1.0	CCCC a/b=1.5
5	15	12.04	15.43	12.50	15.67
	30	13.03	17.11	13.36	17.24
	45	13.36	17.05	13.74	17.19
	60	12.94	18.03	13.36	18.22
15	15	22.46	27.76	24.50	30.75
	30	24.31	33.75	27.18	35.42
	45	25.87	39.67	28.65	40.61
	60	26.27	42.70	27.18	43.21
30	15	26.54	32.62	33.99	38.78
	30	28.41	40.67	33.50	44.13
	45	30.71	50.71	33.40	52.32
	60	32.39	58.94	33.50	59.91
50	15	27.80	34.18	36.71	41.80
	30	29.71	43.03	35.70	47.44
	45	32.31	54.89	35.40	56.67
	60	24.49	65.91	35.70	66.65

Table A.6 Non-dimesional fundamental frquencies ω^* ($=\omega a^2/h(\rho/E_2)^{1/2}$) for antisymmetric angle-ply CCSC and CCCC plates based on FSDT as plate aspect ratio, ply-angle and in-plane orthotropicity ratio vary. (a/h=10, four layers)

	θ	CCSC		CCCC	
E_1/E_2	(degree)	a/b=1.0	a/b=1.5	a/b=1.0	a/b=1.5
5	15	11.93	17.59	14.15	19.07
	30	12.22	18.89	13.95	19.97
	45	12.64	20.73	13.90	21.45
	60	13.06	22.49	13.95	22.99
15	15	15.32	20.34	17.88	22.13
	30	16.12	23.06	18.03	24.18
	45	16.92	26.12	18.18	26.81
	60	17.25	28.02	18.03	28.46
25	15	17.10	21.94	19.52	23.59
	30	18.34	25.50	20.13	26.47
	45	19.29	28.85	20.53	29.49
	60	19.36	30.36	20.12	30.97
35	15	18.27	22.99	20.49	24.49
	30	19.85	27.15	21.47	27.99
	45	20.87	30.55	22.06	31.13
	60	20.70	31.72	21.47	32.14

Table A.7 Non-dimesional fundamental frquencies ω^* $(=\omega a^2/h(\rho/E_2)^{1/2})$ for antisymmetric angle-ply plates based on TSDT as plate aspect ratio, ply-angle and number of layers vary. (a/h=10)

Number of layers	θ (degree)	CCSS a/b 1.0	CCSS a/b 1.5	CCSC a/b 1.0	CCSC a/b 1.5	CCCC a/b 1.0	CCCC a/b 1.5
2	15	16.08	19.54	16.82	21.58	19.67	23.71
	30	15.68	21.22	16.56	23.75	18.79	25.13
	45	15.80	23.74	17.29	21.01	18.67	21.01
	60	15.68	25.69	18.00	21.84	18.79	21.84
4	15	18.24	21.82	18.83	23.60	21.19	25.30
	30	19.73	25.73	20.49	27.94	22.21	28.89
	45	20.37	28.91	21.56	26.21	22.79	26.21
	60	19.73	30.12	21.44	27.13	22.21	27.13
6	15	18.73	22.28	19.24	24.02	21.64	25.72
	30	20.36	26.48	21.12	28.68	22.83	29.63
	45	21.06	29.76	22.23	27.00	23.46	27.00
	60	20.36	30.86	22.05	27.95	23.83	27.95
8	15	18.98	22.44	19.65	24.18	22.13	25.89
	30	20.59	26.75	21.34	28.95	23.06	29.90
	45	21.30	30.07	22.47	27.30	23.71	27.28
	60	20.59	31.27	22.28	28.24	23.06	28.24

Table B.1 Non-dimensional fundamental frequencies ω^* $(=\omega a^2/h(\rho/E_2)^{1/2})$ for symmetric cross-ply plates. $(0^0/90^0/0^0)$

		CCSS		CCSC		CCCC	
		a/b		a/b		a/b	
a/h	theories	1.0	1.5	1.0	1.5	1.0	1.5
2	TDST	6.18	7.58	6.49	8.23	7.27	8.84
	FDST	5.43	6.95	5.72	7.38	5.78	7.42
	CT	24.94	32.90	30.02	36.86	41.82	46.22
5	TDST	11.75	14.66	12.52	16.25	13.60	17.09
	FDST	11.32	14.51	12.17	16.20	12.74	16.62
	CT	24.95	32.89	30.02	36.86	41.82	46.22
10	TDST	18.09	21.82	19.09	24.72	21.77	26.79
	FDST	17.87	21.78	18.93	24.90	21.22	26.62
	CT	24.95	32.89	30.02	36.86	41.82	46.22
15	TDST	21.96	25.80	22.97	29.25	27.53	32.88
	FDST	21.88	25.81	22.92	29.41	27.23	32.81
	CT	24.95	32.89	30.02	36.86	41.82	46.22

Table B.2 Non-dimensional fundamental frequencies ω^* $(=\omega a^2/h(\rho/E_2)^{1/2})$ for symmetric cross-ply plates $(0^0/90^0/90^0/0^0)$

a/h	theories	CCSS a/b 1.0	CCSS a/b 1.5	CCSC a/b 1.0	CCSC a/b 1.5	CCCC a/b 1.0	CCCC a/b 1.5
2	TDST	6.15	7.68	6.41	8.21	7.03	8.69
	FDST	5.62	7.27	5.82	7.52	5.89	7.57
	CT	29.07	36.86	31.25	44.90	41.82	52.84
5	TDST	12.27	15.73	13.08	17.11	14.05	17.85
	FDST	12.04	15.94	12.98	17.39	13.54	17.81
	CT	29.07	36.86	31.25	44.90	41.82	52.84
10	TDST	18.89	24.53	20.45	27.89	22.91	29.72
	FDST	18.75	24.90	20.48	28.70	22.59	30.22
	CT	29.07	36.86	31.25	44.90	41.82	52.84
15	TDST	22.64	29.26	24.51	34.19	28.71	37.27
	FDST	22.55	29.55	24.55	34.94	28.49	37.77
	CT	29.07	36.86	31.25	44.90	41.82	52.84

Table B.3 Non-dimensional fundamental frequencies ω^* $(=\omega a^2/h(\rho/E_2)^{1/2})$ for symmetric cross-ply plates $(0^0/90^0/0^0/90^0/0^0)$

		CCSS		CCSC		CCCC	
		a/b		a/b		a/b	
a/h	theories	1.0	1.5	1.0	1.5	1.0	1.5
2	TDST	6.21	7.74	6.43	8.22	6.98	8.65
	FDST	5.70	7.34	5.86	7.52	5.93	7.57
	CT	29.08	40.25	32.37	51.35	41.82	57.80
5	TDST	12.69	16.22	13.43	17.43	14.43	18.19
	FDST	12.43	16.44	13.29	17.67	13.88	18.11
	CT	29.08	40.25	32.37	51.35	41.82	57.80
10	TDST	19.65	26.01	21.36	29.24	23.89	31.12
	FDST	19.43	26.54	21.41	30.24	23.51	31.75
	CT	29.08	40.25	32.37	51.35	41.82	57.80
15	TDST	23.34	31.49	25.67	36.74	29.85	39.75
	FDST	23.16	31.96	25.72	37.86	29.51	40.50
	CT	29.08	40.25	32.37	51.35	41.82	57.80

Table B.4 Non-dimensional fundamental frequencies ω^* ($=\omega a^2/h(\rho/E_2)^{1/2}$) for symmetric cross-ply plates ($0^0/90^0/0^0/90^0/0^0/90^0/0^0/90^0/0^0$)

a/h	theories	CCSS		CCSC		CCCC	
		a/b		a/b		a/b	
		1.0	1.5	1.0	1.5	1.0	1.5
2	TDST	6.58	8.34	6.83	8.94	7.52	9.48
	FDST	5.74	7.41	5.87	7.54	5.95	7.60
	CT	29.08	44.92	34.00	59.86	41.83	65.56
5	TDST	13.18	17.20	13.96	18.47	15.06	19.30
	FDST	12.68	16.86	13.46	17.88	14.09	18.35
	CT	29.08	44.92	34.00	59.86	41.83	65.56
10	TDST	20.24	27.84	22.25	31.27	24.76	33.09
	FDST	19.97	27.97	22.11	31.47	24.18	32.94
	CT	29.08	44.92	34.00	59.86	41.83	65.56
15	TDST	23.84	34.11	26.82	34.00	30.72	42.69
	FDST	23.68	34.38	26.81	40.54	30.34	42.93
	CT	29.08	44.92	34.00	59.86	41.83	65.56

Table B.5 Non-dimensional fundamental frequencies ω^* $(=\omega a^2/h(\rho/E_2)^{1/2})$ for antisymmetric cross-ply square plates $(0^0/90^0)$

a/h	theories	CCSS		CCSC		CCCC	
		E_1/E_2		E_1/E_2		E_1/E_2	
		15	40	15	40	15	40
5	TDST	9.617	11.57	10.86	12.95	12.01	14.23
	FDST	9.289	10.74	10.35	11.75	11.35	12.71
	CT	12.07	16.55	15.17	19.59	18.29	22.61
10	TDST	11.64	14.82	13.68	17.35	15.46	19.56
	FDST	11.52	14.43	13.47	16.70	15.17	18.72
	CT	12.61	16.74	15.28	20.54	17.55	23.73
15	TDST	12.23	15.88	14.59	19.00	16.61	21.67
	FDST	12.17	15.69	14.48	18.64	16.46	21.18
	CT	12.72	16.89	15.41	20.73	17.70	23.95
20	TDST	12.47	16.35	14.96	19.74	17.10	22.63
	FDST	12.43	16.23	14.90	19.52	17.02	22.31
	CT	12.75	16.94	15.46	20.79	17.76	24.03

Table B.6 Non-dimensional fundamental frequencies ω^* ($=\omega a^2/h(\rho/E_2)^{1/2}$) for antisymmetric cross-ply square plates ($0^0/90^0/0^0/90^0$)

a/h	theories	CCSS		CCSC		CCCC	
		E_1/E_2		E_1/E_2		E_1/E_2	
		15	40	15	40	15	40
5	TDST	10.99	12.81	12.07	13.34	12.73	14.66
	FDST	10.89	12.50	11.87	13.26	12.81	13.98
	CT	17.02	25.88	20.98	31.15	23.89	38.74
10	TDST	14.72	19.30	17.01	21.69	19.05	23.85
	FDST	14.73	19.28	16.99	21.56	19.01	23.64
	CT	17.23	26.26	21.13	32.24	24.39	39.88
15	TDST	16.05	22.40	19.00	26.00	21.58	29.17
	FDST	16.03	22.45	19.02	26.01	21.59	29.15
	CT	17.34	26.45	21.26	32.73	24.56	38.06
20	TDST	16.59	23.95	19.92	28.37	22.77	32.19
	FDST	16.61	24.00	19.94	28.42	22.79	32.24
	CT	17.38	26.51	21.30	32.80	24.61	38.07

Table B.7 Non-dimensional fundamental frequencies ω^* $(=\omega a^2/h(\rho/E_2)^{1/2})$ for antisymmetric cross-ply square plates $(0^0/90^0/0^0/90^0/0^0/90^0)$

a/h	theories	CCSS		CCSC		CCCC	
		E_1/E_2		E_1/E_2		E_1/E_2	
		15	40	15	40	15	40
5	TDST	11.26	13.12	12.64	13.69	12.88	14.77
	FDST	11.08	12.66	12.04	13.39	12.96	14.40
	CT	17.74	26.23	21.72	33.80	24.54	39.80
10	TDST	15.21	19.97	17.55	22.37	19.63	24.44
	FDST	15.18	19.81	17.46	22.06	19.50	24.11
	CT	17.97	26.50	22.05	34.12	25.06	40.01
15	TDST	16.63	23.33	19.69	27.00	22.35	30.24
	FDST	16.62	23.29	19.67	26.88	22.31	30.05
	CT	18.07	27.87	22.17	34.50	25.63	40.13
20	TDST	17.23	25.04	20.69	29.59	23.64	33.54
	FDST	17.24	25.03	20.68	29.55	23.63	33.47
	CT	18.10	27.92	22.22	34.57	25.69	40.14

Table B.8 Non-dimensional fundamental frequencies ω^* ($=\omega a^2/h(\rho/E_2)^{1/2}$) for antisymmetric cross-ply square plates ($0^0/90^0 \ldots /90^0$, ten layers)

		CCSS		CCSC		CCCC	
		E_1/E_2		E_1/E_2		E_1/E_2	
a/h	theories	15	40	15	40	15	40
5	TDST	11.40	13.27	12.79	13.88	14.52	15.11
	FDST	11.17	12.73	12.02	13.44	13.03	14.63
	CT	18.17	27.47	22.15	34.22	25.35	39.88
10	TDST	15.46	20.31	17.82	22.73	19.92	25.24
	FDST	15.39	20.06	17.69	22.28	19.74	24.67
	CT	18.33	27.80	22.51	34.92	26.05	40.22
15	TDST	16.92	23.79	20.04	27.51	22.74	30.78
	FDST	16.90	23.69	19.99	27.29	22.66	30.47
	CT	18.43	28.57	22.63	35.38	26.17	40.32
20	TDST	17.56	25.58	21.07	30.20	24.08	34.21
	FDST	17.55	25.53	21.05	30.09	24.05	34.05
	CT	18.47	28.62	22.68	35.45	26.22	41.16

Table C.1 Non-dimensional fundamental frequencies ω^* $(=\omega a^2/h(\rho/E_2)^{1/2})$ for a cross-ply cylindrical CCSS square shell panel based on CT theory. (a/h=15)

No. of layers	R/a				
	5	10	15	20	25
2	14.654	14.453	14.417	14.406	14.401
3	23.342	23.249	23.232	23.226	23.223
4	21.477	21.377	21.360	21.355	21.354
5	23.347	23.257	23.236	23.229	23.226
6	22.528	22.435	22.418	22.414	22.412
7	23.348	23.254	23.236	23.230	23.227

Table C.2 Non-dimensional fundamental frequencies ω^* $(=\omega a^2/h(\rho/E_2)^{1/2})$ for a cross-ply spherical CCSS square shell panel based on CT theory. (a/h=15)

No. of layers	R/a				
	5	10	15	20	25
2	15.219	14.608	14.491	14.449	14.431
3	23.632	23.323	23.265	23.246	23.235
4	21.836	21.476	21.407	21.383	21.372
5	23.640	23.329	23.269	23.248	23.239
6	22.857	22.524	22.461	22.439	22.428
7	23.641	23.329	23.270	23.249	23.239

Table C.3 Non-dimensional fundamental frequencies ω^* ($=\omega a^2/h(\rho/E_2)^{1/2}$) for a cross-ply cylindrical CCSC square shell panel based on CT theory (a/h=15)

No. of layers	R/a				
	5	10	15	20	25
2	19.806	18.053	17.887	17.773	17.719
3	25.256	24.413	24.252	24.196	24.170
4	27.855	26.748	26.536	26.462	26.427
5	27.147	26.227	26.052	25.991	25.962
6	29.108	28.052	27.851	27.780	27.747
7	27.989	27.037	26.857	26.794	26.765

Table C.4 Non-dimensional fundamental frequencies ω^* $(=\omega a^2/h(\rho/E_2)^{1/2})$ for a cross-ply spherical CCSC square shell panel based on CT theory (a/h=15)

No. of layers	R/a				
	5	10	15	20	25
2	20.095	18.294	17.931	17.800	17.738
3	25.413	24.455	24.271	24.207	24.177
4	28.009	26.799	26.068	26.479	26.439
5	27.277	26.262	27.873	25.999	25.968
6	29.236	28.093	27.873	27.794	27.757
7	28.106	27.070	26.872	26.802	26.770

Table C.5 Non-dimensional fundamental frequencies ω^* ($=\omega a^2/h(\rho/E_2)^{1/2}$) for cross-ply cylindrical square shells ($0^0/90^0$)

a/h	theories	CCSS		CCSC		CCCC	
		R/a		R/a		R/a	
		5	20	5	20	5	20
5	TDST	9.463	9.468	10.97	10.56	11.91	11.53
	FDST	8.820	8.815	10.10	9.618	10.79	10.34
	CT	14.19	13.22	16.88	16.68	19.39	19.23
10	TDST	12.50	12.40	15.67	14.49	17.31	16.25
	FDST	12.20	12.09	15.22	13.26	16.70	15.57
	CT	14.43	14.26	18.46	17.53	21.01	20.21
15	TDST	13.72	13.43	18.39	16.11	20.31	18.27
	FDST	13.56	13.26	18.17	15.83	19.98	17.88
	CT	14.65	14.41	19.81	17.77	22.26	20.47
20	TDST	14.42	13.88	20.62	16.93	22.58	19.28
	FDST	14.33	13.79	20.49	16.76	22.40	19.04
	CT	14.98	14.47	21.40	17.95	22.58	20.64

Table C.6 Non-dimensional fundamental frequencies ω^* ($=\omega a^2/h(\rho/E_2)^{1/2}$) for cross-ply spherical square shells ($0^0/90^0$)

a/h	theories	CCSS R/a 5	CCSS R/a 20	CCSC R/a 5	CCSC R/a 20	CCCC R/a 5	CCCC R/a 20
5	TDST	9.559	9.480	11.01	10.56	12.32	11.56
	FDST	8.920	8.826	10.14	9.622	11.25	10.37
	CT	14.33	13.32	18.35	16.48	22.73	20.34
10	TDST	12.79	12.42	15.80	14.50	18.37	16.32
	FDST	12.49	12.11	15.36	13.99	17.82	15.65
	CT	14.59	14.29	18.61	17.55	25.79	20.26
15	TDST	14.31	13.47	18.67	16.13	22.27	18.41
	FDST	14.16	13.31	18.45	15.85	22.00	18.02
	CT	14.65	14.41	20.09	17.80	25.99	20.58
20	TDST	15.44	13.95	21.09	16.97	25.82	19.51
	FDST	15.43	13.85	20.97	16.79	25.77	19.27
	CT	15.46	14.54	21.88	17.99	26.41	20.84

Table C.7 Non-dimensional fundamental frequencies ω^* ($=\omega a^2/h(\rho/E_2)^{1/2}$) for cross-ply cylindrical square shells ($0^0/90^0/0^0$)

		CCSS		CCSC		CCCC	
		R/a		R/a		R/a	
a/h	theories	5	20	5	20	5	20
5	TDST	9.464	9.446	10.44	10.13	16.02	10.94
	FDST	9.460	9.444	10.38	10.06	15.33	10.58
	CT	22.80	22.28	22.55	22.53	32.34	32.44
10	TDST	14.55	14.46	16.20	15.44	18.09	17.42
	FDST	14.94	14.86	16.54	15.79	18.42	17.75
	CT	23.06	22.83	24.30	23.97	33.45	32.74
15	TDST	17.73	17.55	19.97	18.58	23.18	21.99
	FDST	18.19	18.00	20.97	19.01	23.76	22.61
	CT	23.34	23.23	25.26	24.19	34.40	33.37
20	TDST	19.63	19.35	22.13	20.39	27.02	25.19
	FDST	20.14	19.82	23.02	20.87	27.59	25.83
	CT	23.55	23.28	26.15	24.30	34.82	33.45

Table C.8 Non-dimensional fundamental frequencies ω^* ($=\omega a^2/h(\rho/E_2)^{1/2}$) for cross-ply spherical square shells ($0^0/90^0/0^0$)

a/h	theories	CCSS		CCSC		CCCC	
		R/a		R/a		R/a	
		5	20	5	20	5	20
5	TDST	9.532	9.450	10.46	10.14	16.40	10.94
	FDST	9.529	9.448	10.40	10.07	15.36	10.62
	CT	23.65	22.41	23.33	23.32	33.37	33.47
10	TDST	14.77	14.48	16.32	15.45	19.48	17.51
	FDST	15.15	14.87	16.65	15.80	19.83	17.84
	CT	23.28	23.09	24.95	24.07	34.17	33.51
15	TDST	18.23	17.57	20.23	18.60	25.57	22.16
	FDST	18.65	18.04	20.67	19.03	26.15	22.77
	CT	23.63	23.24	25.41	24.21	35.66	33.56
20	TDST	20.41	19.42	23.02	20.47	30.54	25.46
	FDST	20.84	19.86	23.46	20.90	31.12	26.08
	CT	24.12	23.32	26.49	24.32	37.48	33.63

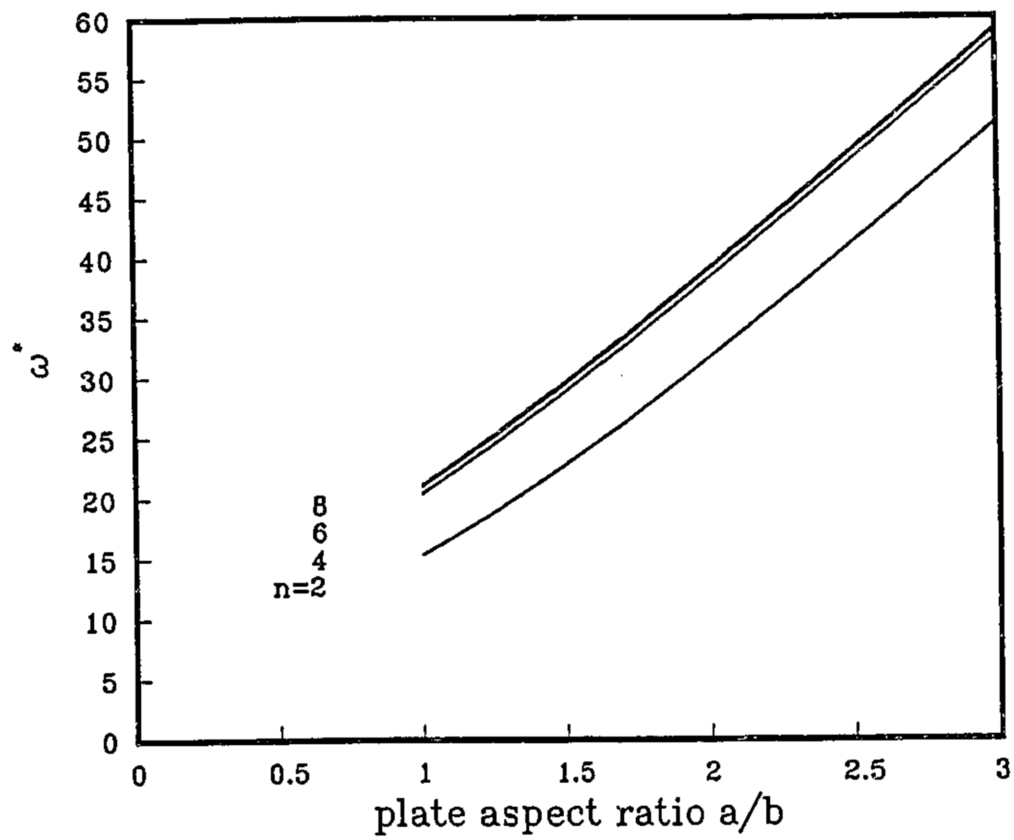

Figure A.1 Non-dimensional fundamental frequencies ω^* versus plate aspect ratio and number of layers of antisymmetric angle-ply CCSS plates.

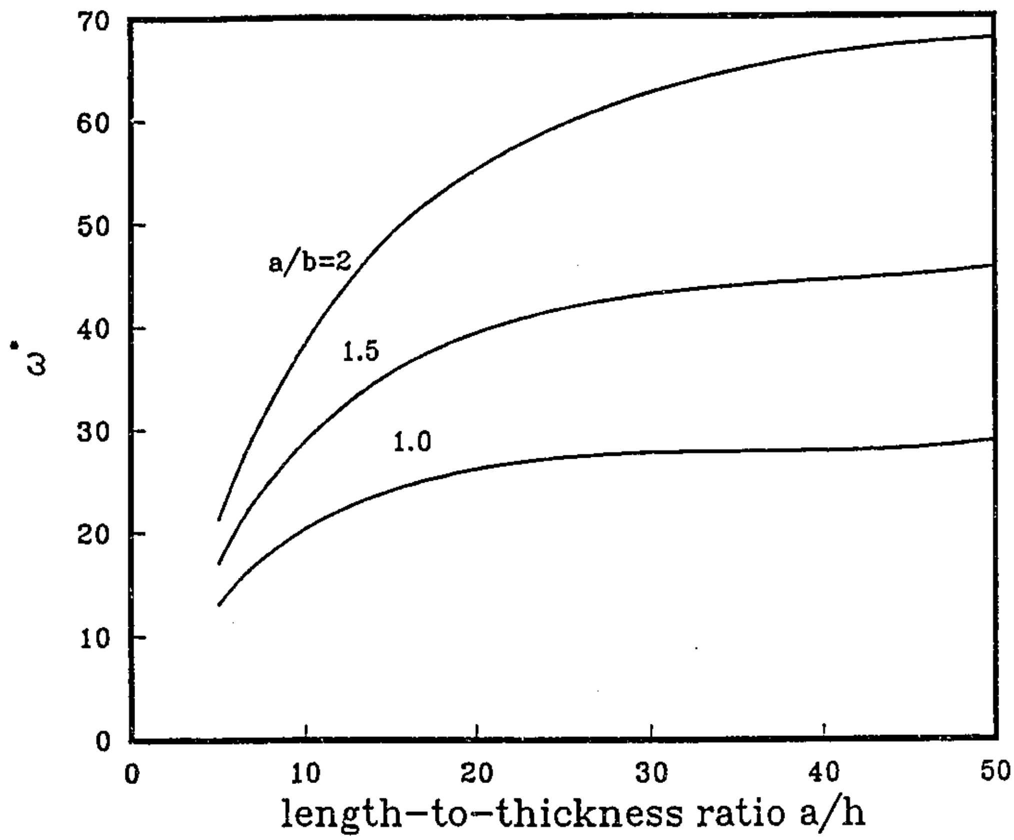

Figure A.2 Non-dimensional fundamental frequencies ω^* versus plate aspect ratio and length-to-thickness ratio of antisymmetric angle-ply CCSS plates.

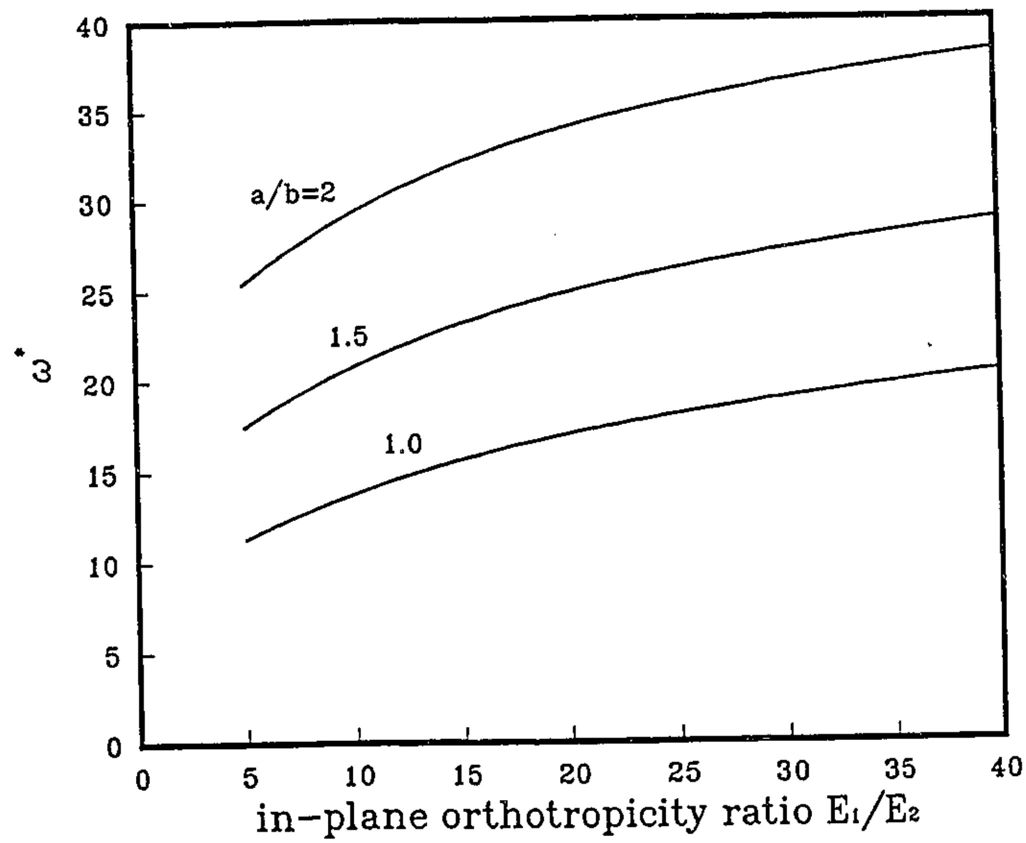

Figure A.3 Non-dimensional fundamental frequencies ω^* versus plate aspect ratio and in-plane orthotropicity ratio of antisymmetric angle-ply CCSS plates.

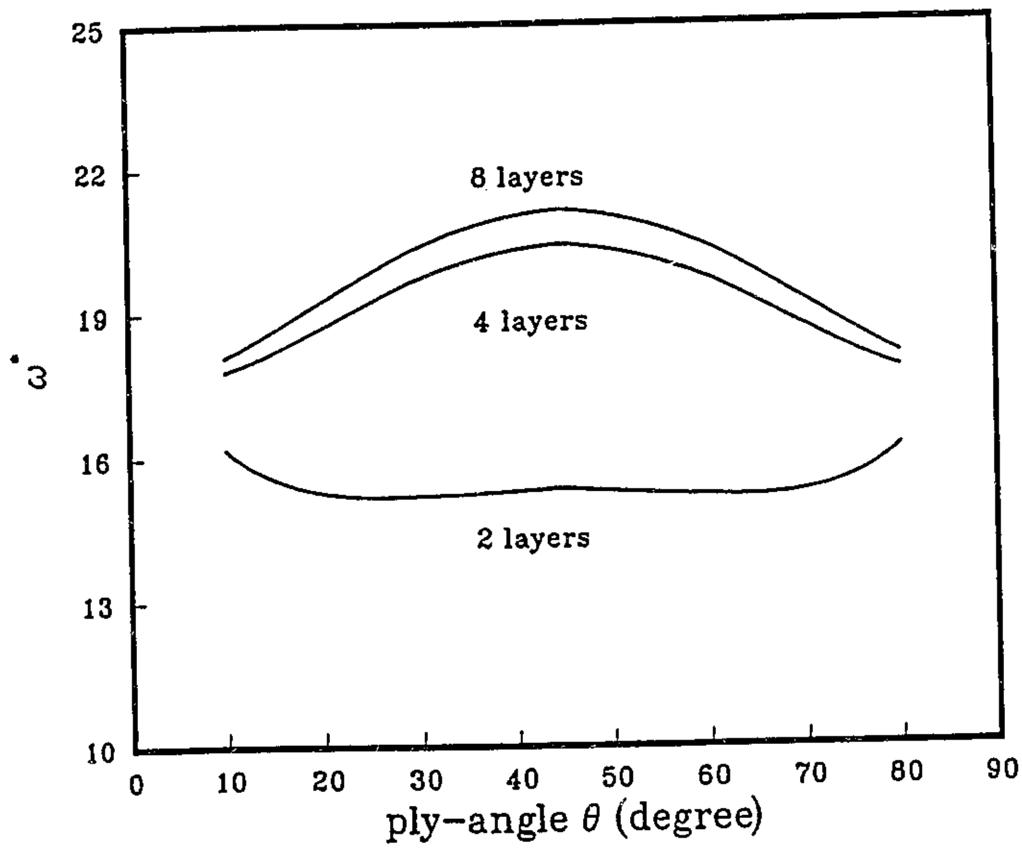

Figure A.4 Non-dimensional fundamental frequencies ω^* versus ply-angle and number of layers of antisymmetric angle-ply CCSS plates.

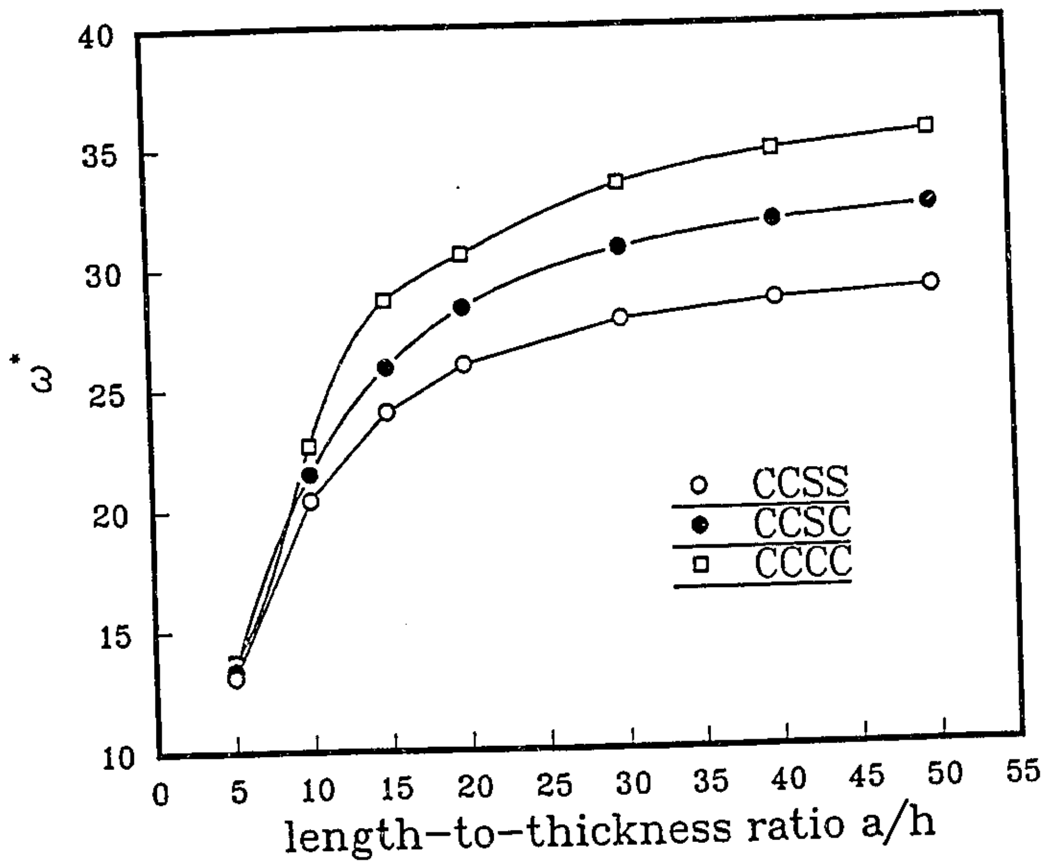

Figure A.5 Non-dimensional fundamental frequencies ω^* versus plate aspect ratio and boundary conditions of antisymmetric angle-ply plates based on FSDT theory.

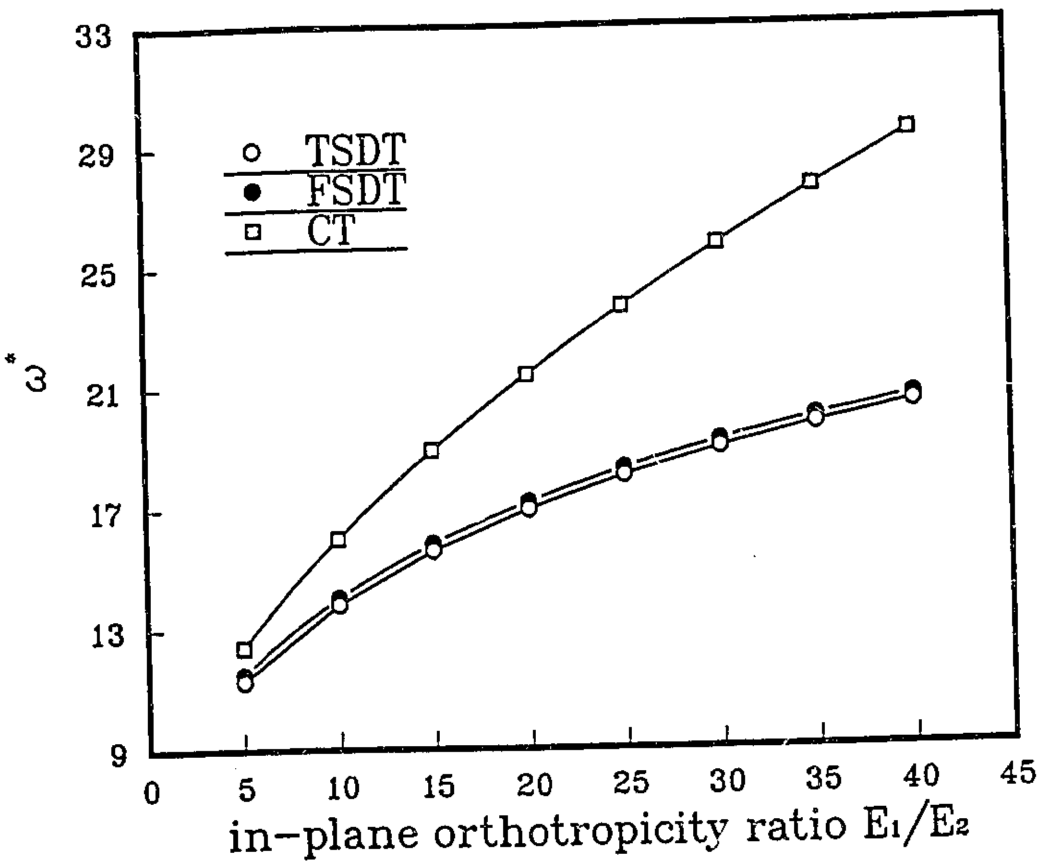

Figure A.6 Non-dimensional fundamental frequencies ω^* versus in-plane orthotropicity ratio of antisymmetric angle-ply CCSS plates based on TSDT, FSDT and CT theories.

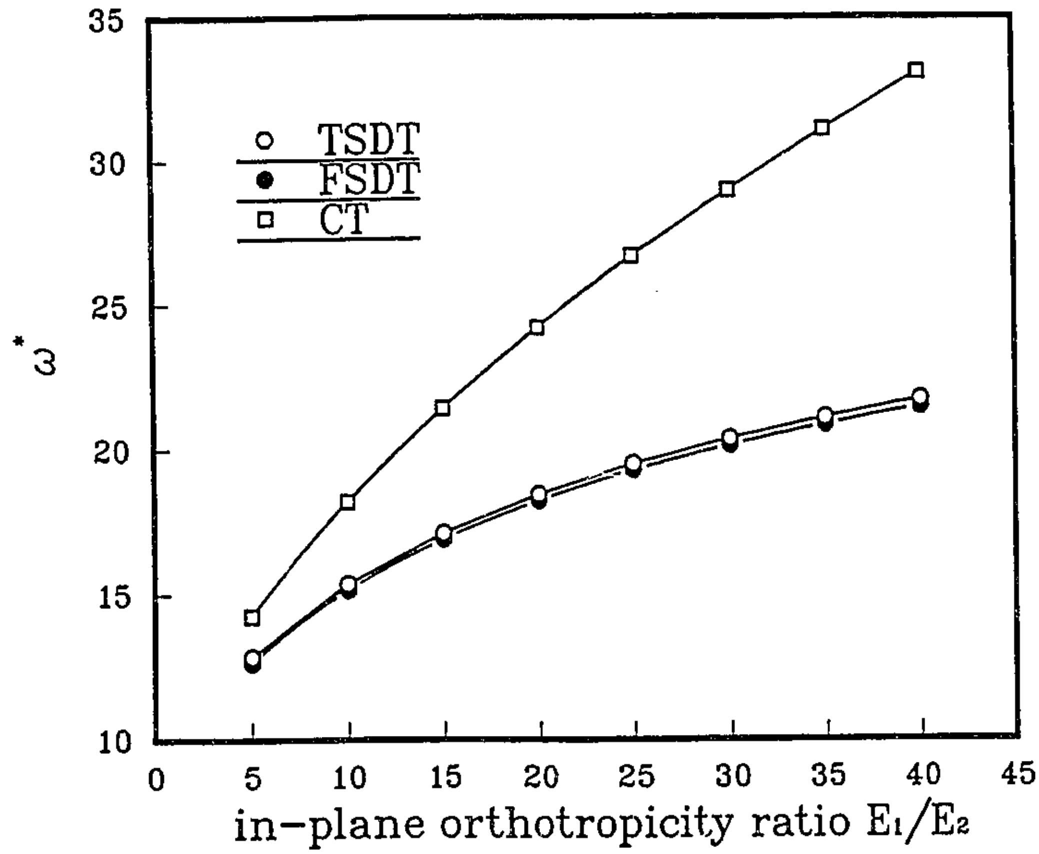

Figure A.7 Non-dimensional fundamental frequencies ω^* versus in-plane orthotropicity ratio of antisymmetric angle-ply CCSC plates based on TSDT, FSDT and CT theories.

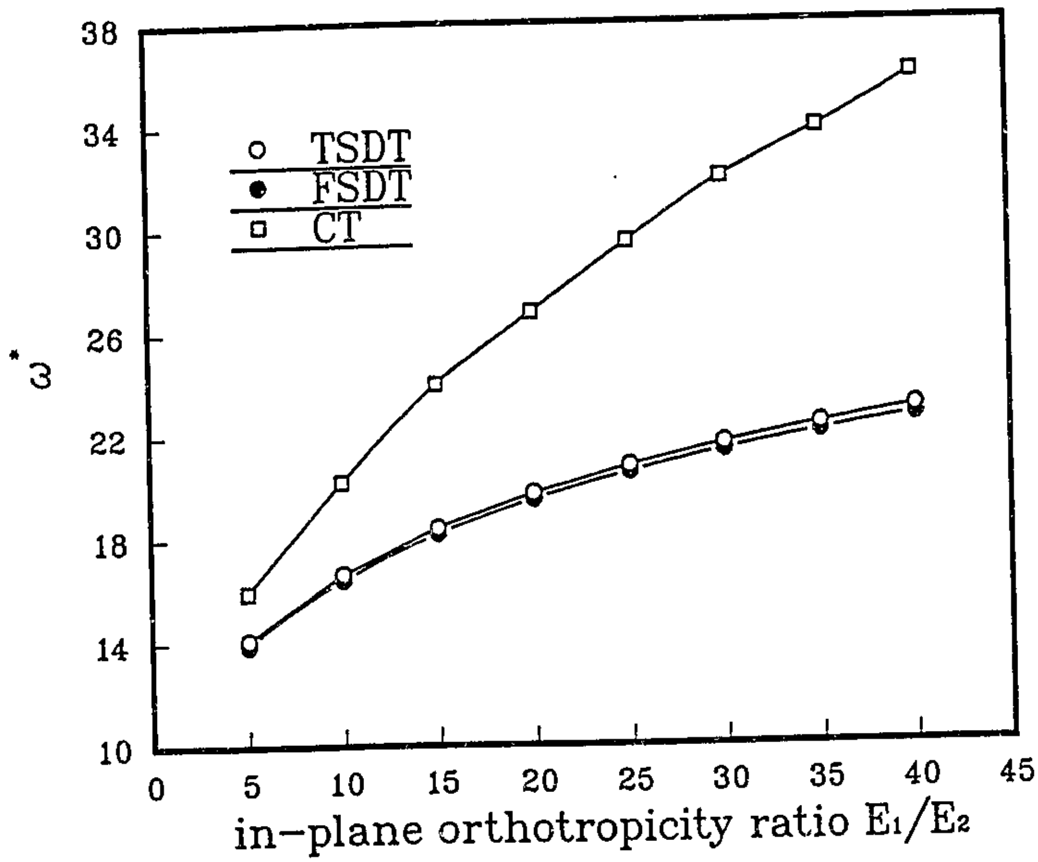

Figure A.8 Non-dimensional fundamental frequencies ω^* versus in-plane orthotropicity ratio of antisymmetric angle-ply CCCC plates based on TSDT, FSDT and CT theories.

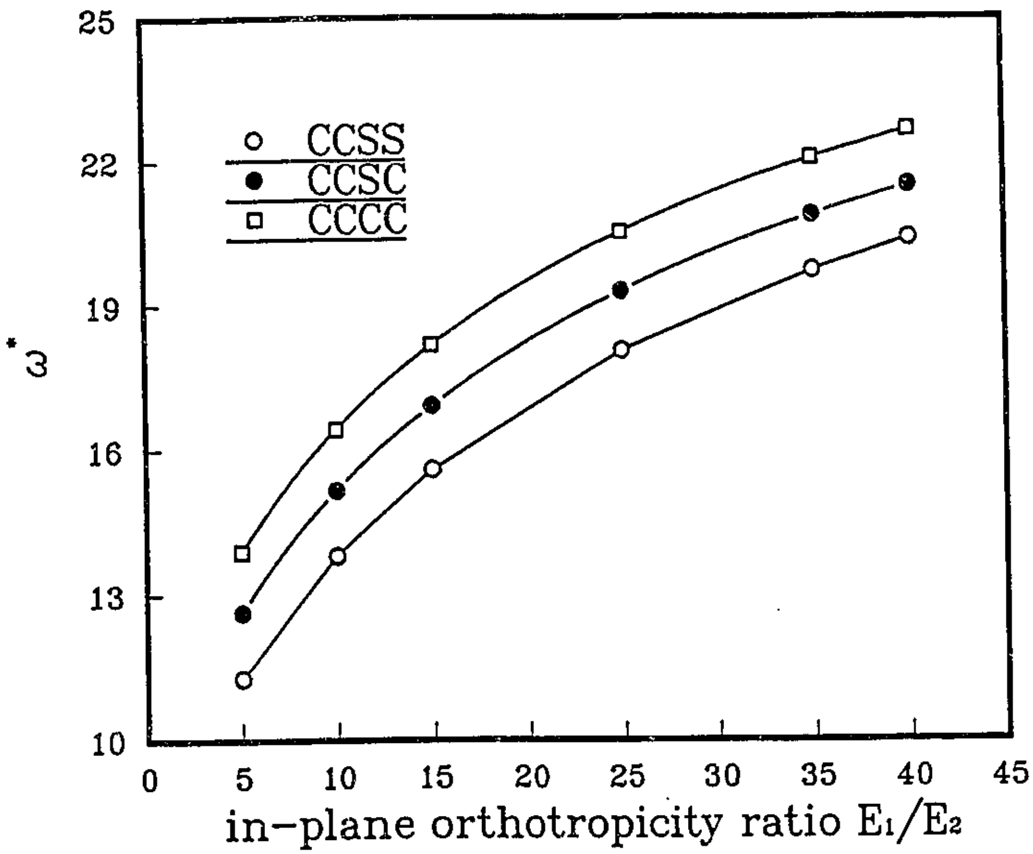

Figure A.9 Non-dimensional fundamental frequencies ω^* versus in-plane orthotropicity ratio and boundary conditions of antisymmetric angle-ply plates based on FSDT theory.

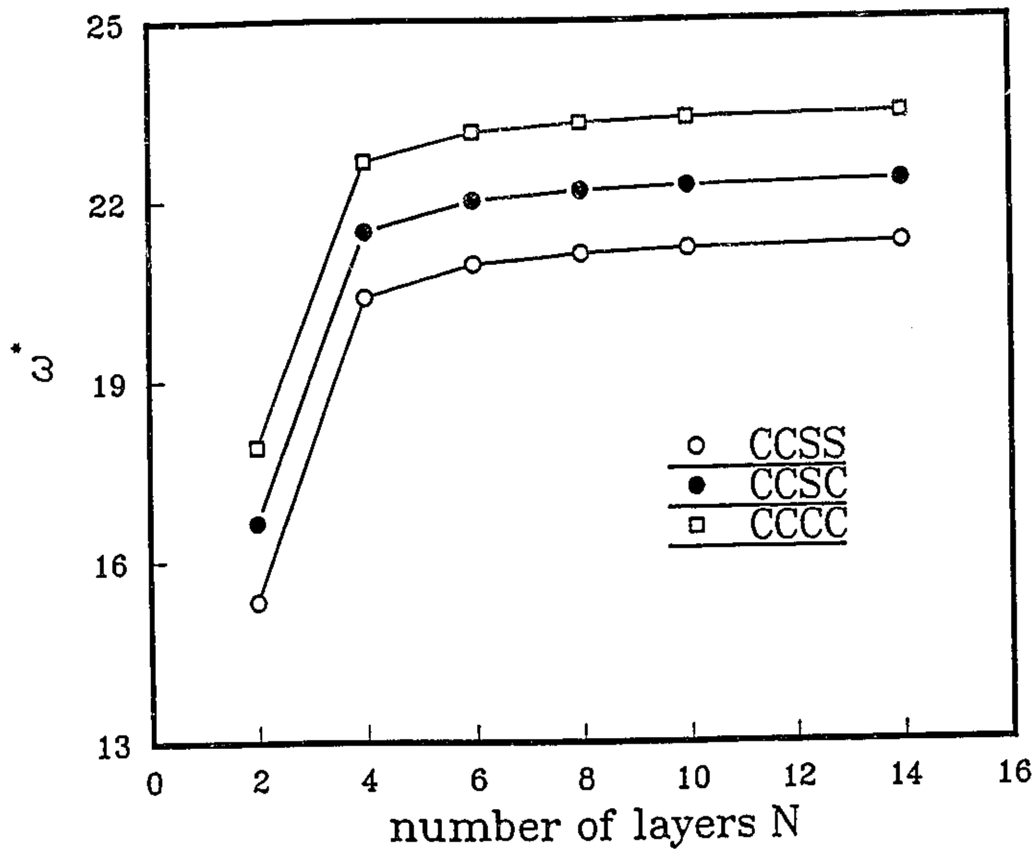

Figure A.10 Non-dimensional fundamental frequencies ω^* versus number of layers and boundary conditions of antisymmetric angle-ply plates based on FSDT theory.

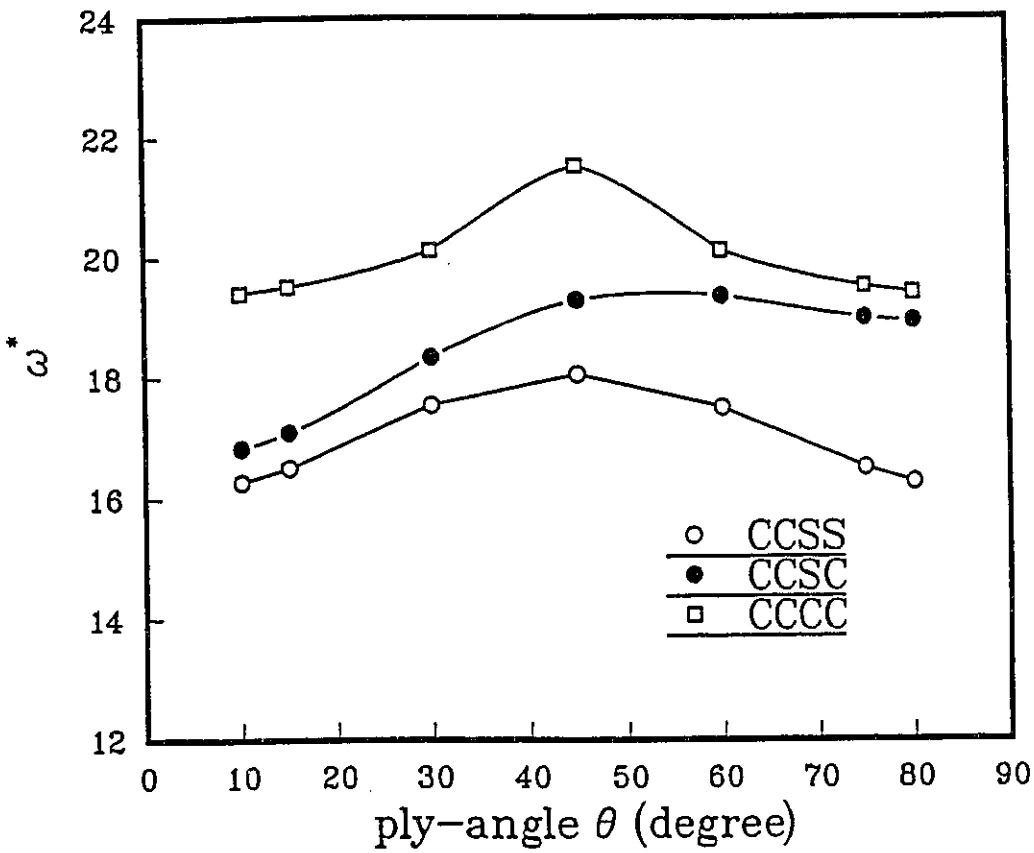

Figure A.11 Non-dimensional fundamental frequencies ω^* versus ply-angle and boundary conditions of antisymmetric angle-ply plates based on FSDT theory.

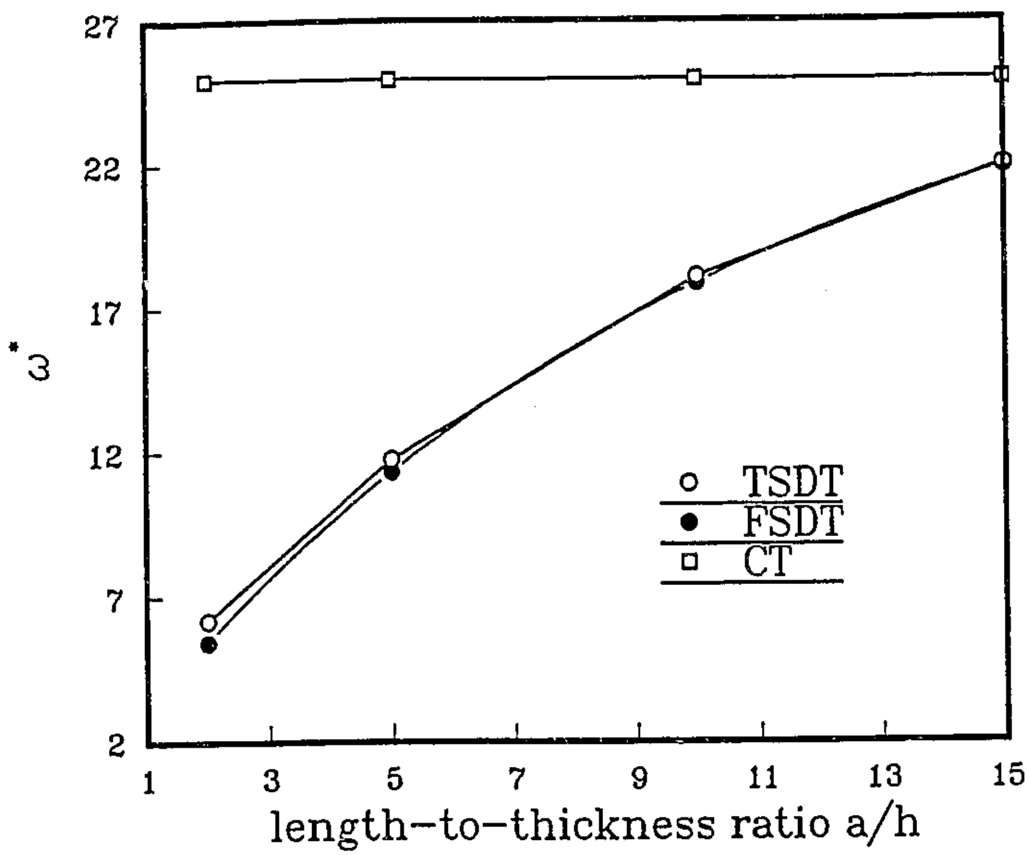

Figure B.1 Non-dimensional fundamental frequencies ω^* versus length-to-thickness ratio of symmetric cross-ply CCSS plates based on TSDT, FSDT and CT theories.

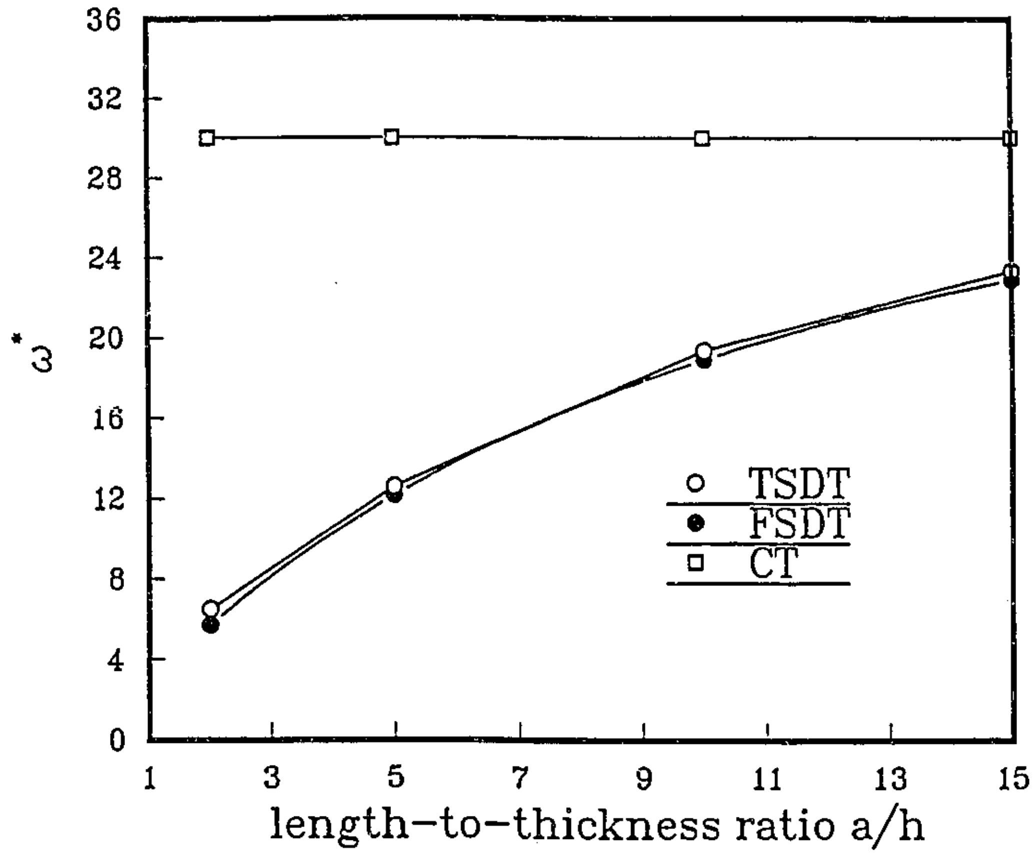

Figure B.2 Non-dimensional fundamental frequencies ω^* versus length-to-thickness ratio of symmetric cross-ply CCSC plates based on TSDT, FSDT and CT theories.

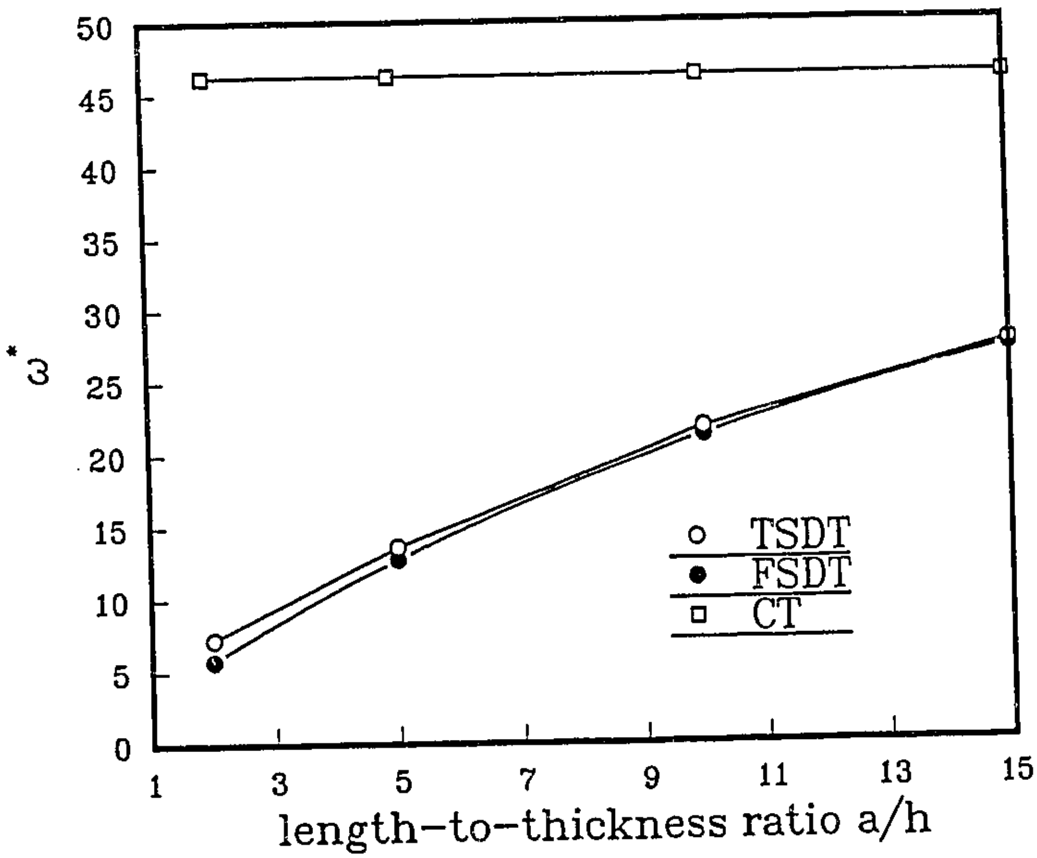

Figure B.3 Non-dimensional fundamental frequencies ω^* versus length-to-thickness ratio of symmetric cross-ply CCCC plates based on TSDT, FSDT and CT theories.

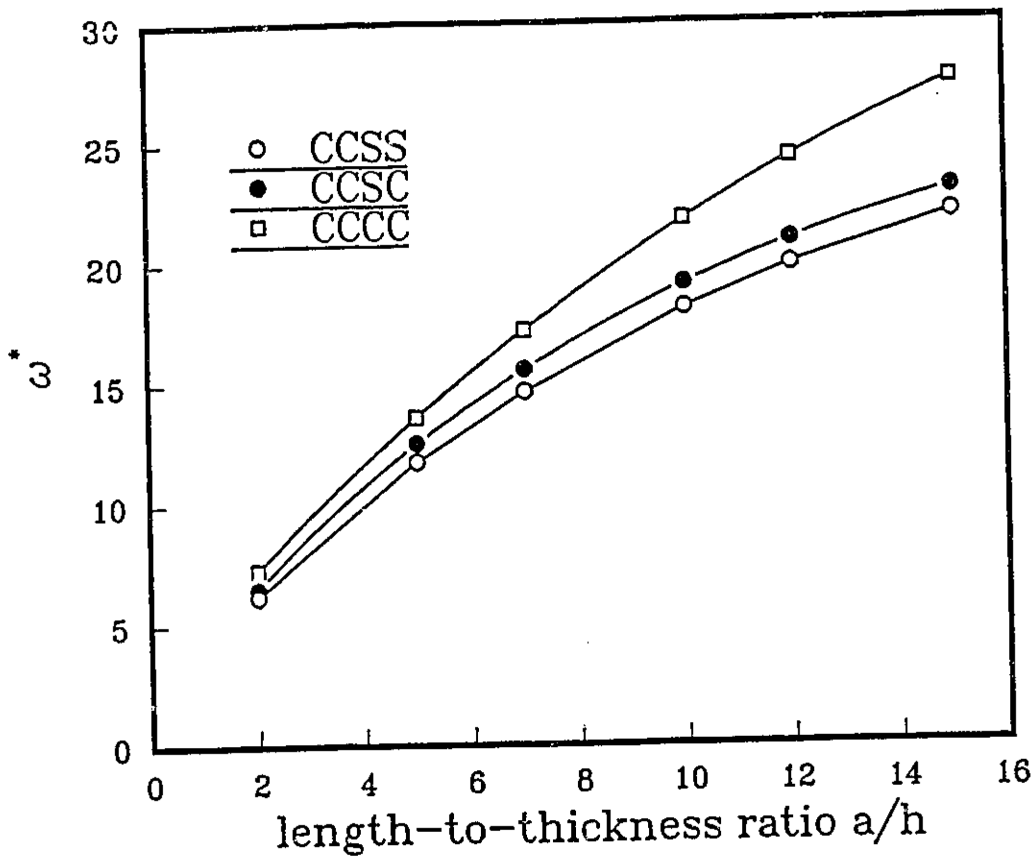

Figure B.4 Non-dimensional fundamental frequencies ω^* versus length-to-thickness ratio and boundary conditions of symmetric cross-ply plates based on TSDT theory.

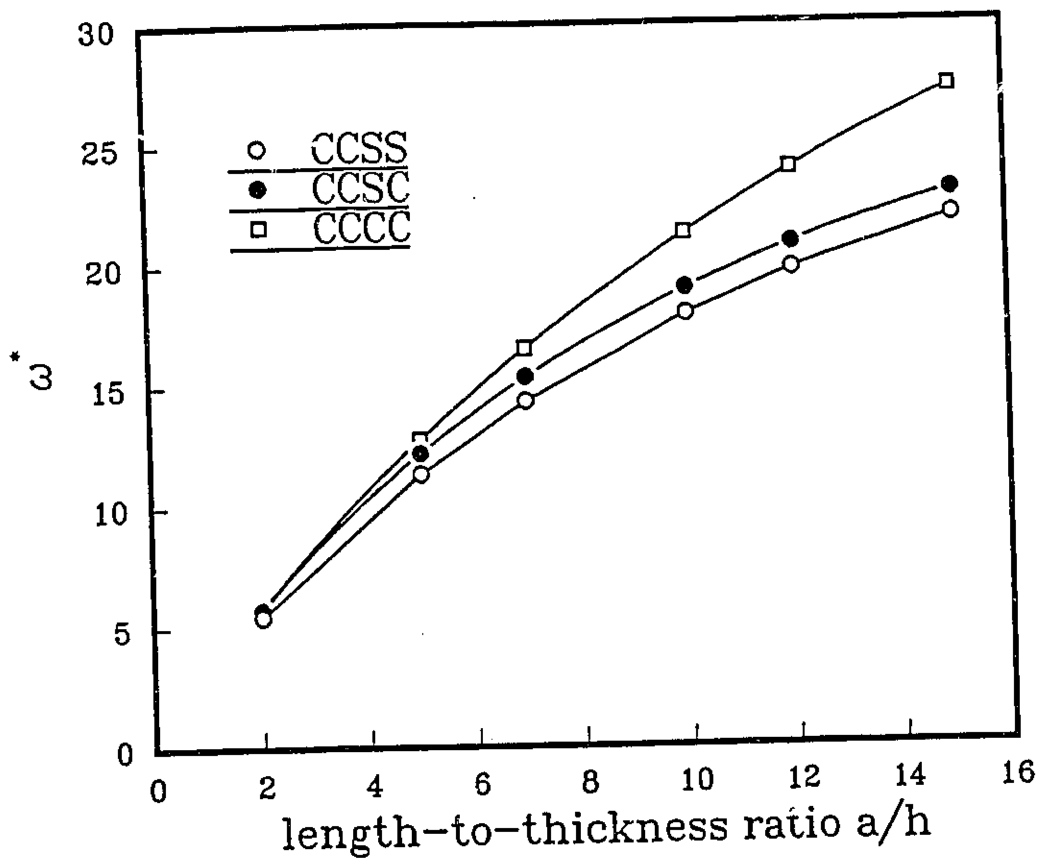

Figure B.5 Non-dimensional fundamental frequencies ω^* versus length-to-thickness ratio and boundary conditions of symmetric cross-ply plates based on FSDT theory.

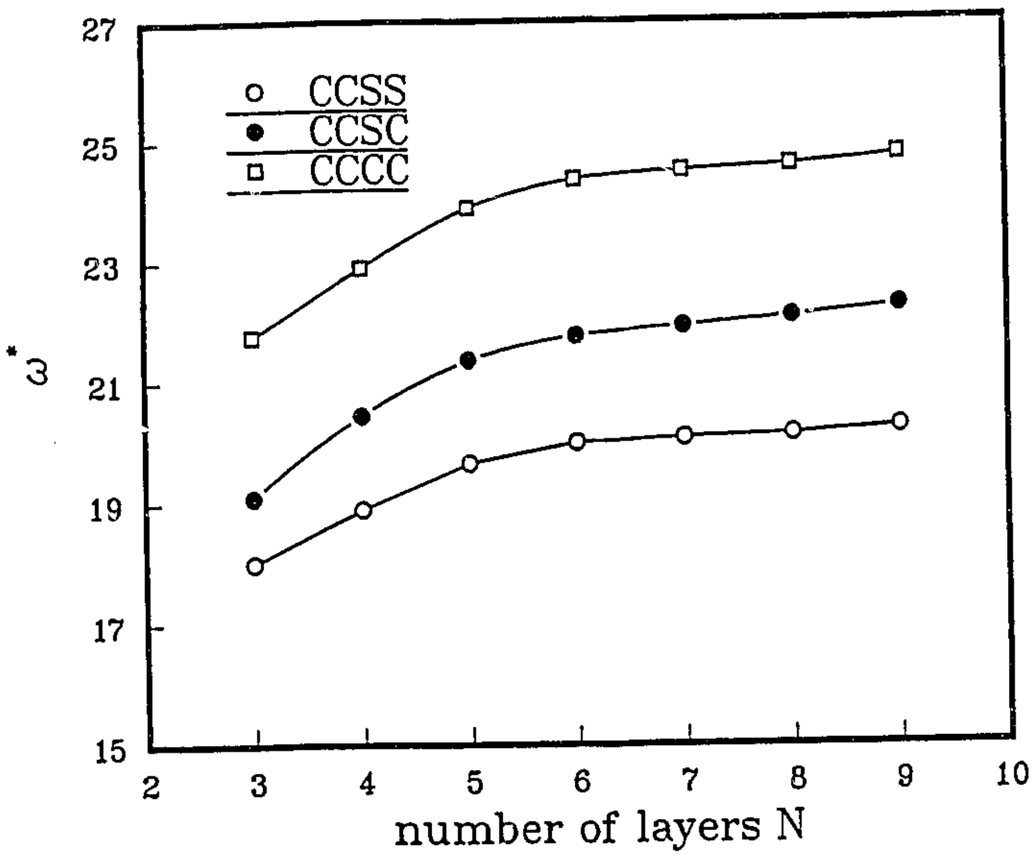

Figure B.6 Non-dimensional fundamental frequencies ω^* versus number of layers and boundary conditions of symmetric cross-ply plates based on TSDT theory.

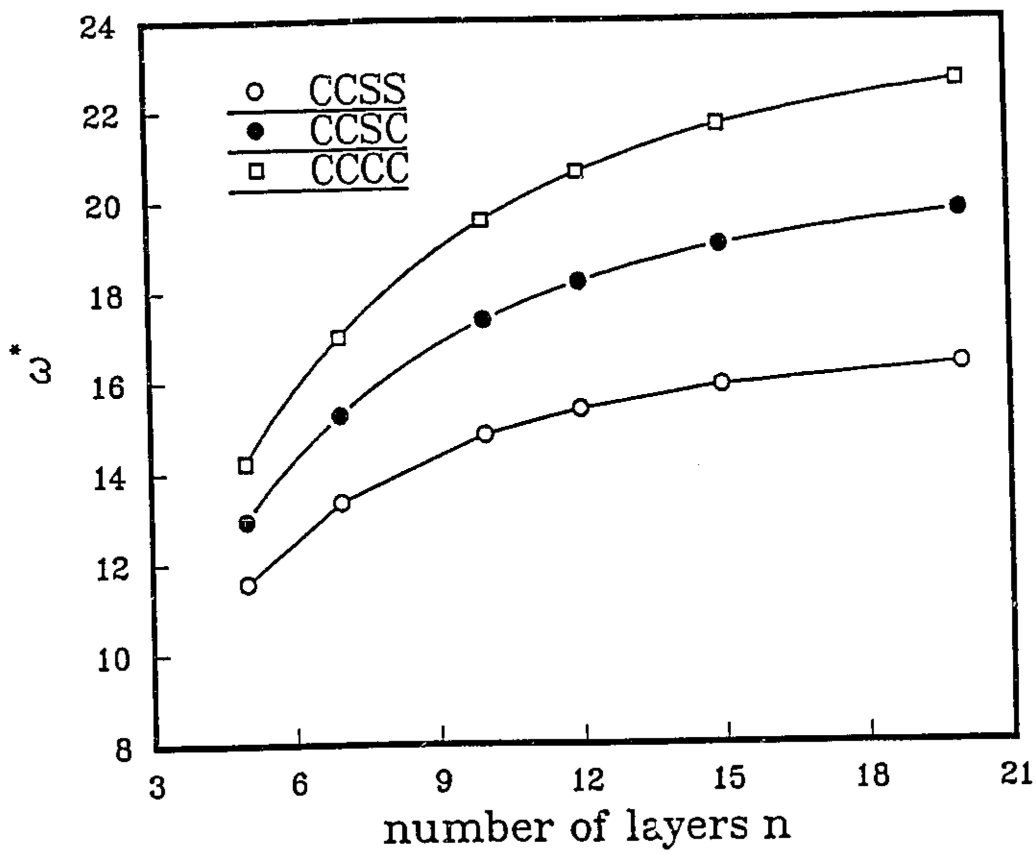

Figure B.7 Non-dimensional fundamental frequencies ω^* versus number of layers and boundary conditions of symmetric cross-ply plates based on FSDT theory.

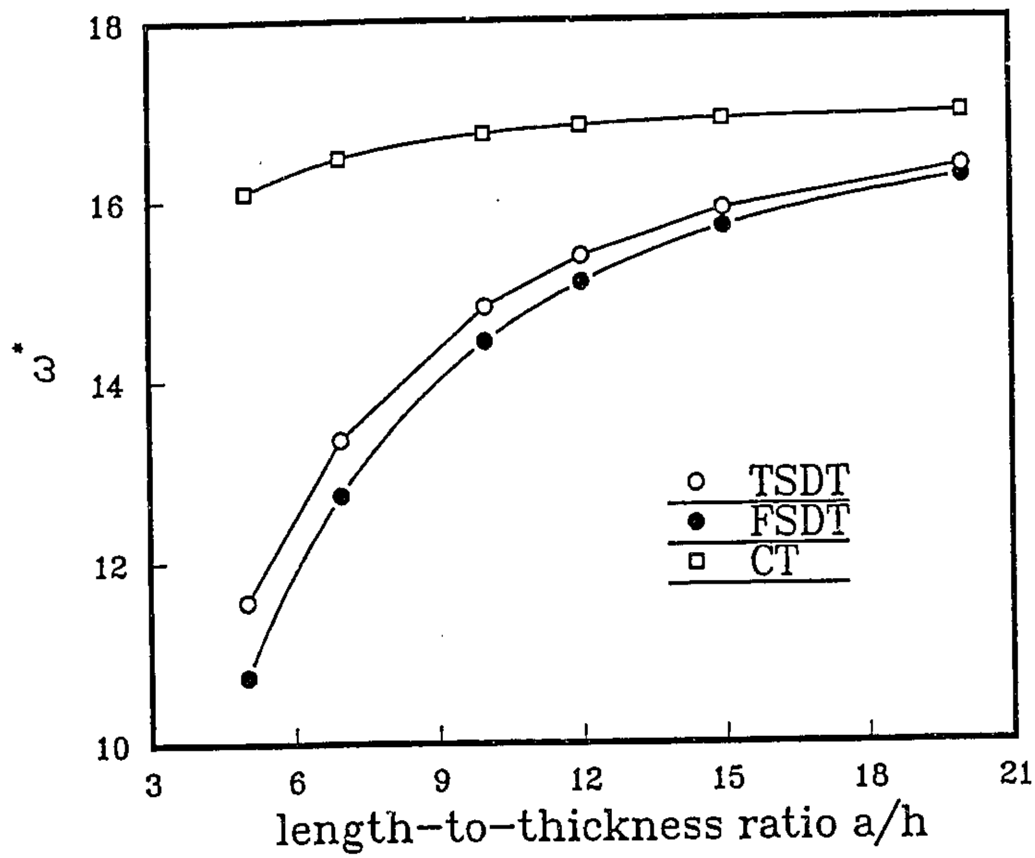

Figure B.8 Non-dimensional fundamental frequencies ω^* versus length-to-thickness ratio of antisymmetric cross-ply two-layer CCSS plates based on TSDT, FSDT and CT theories.

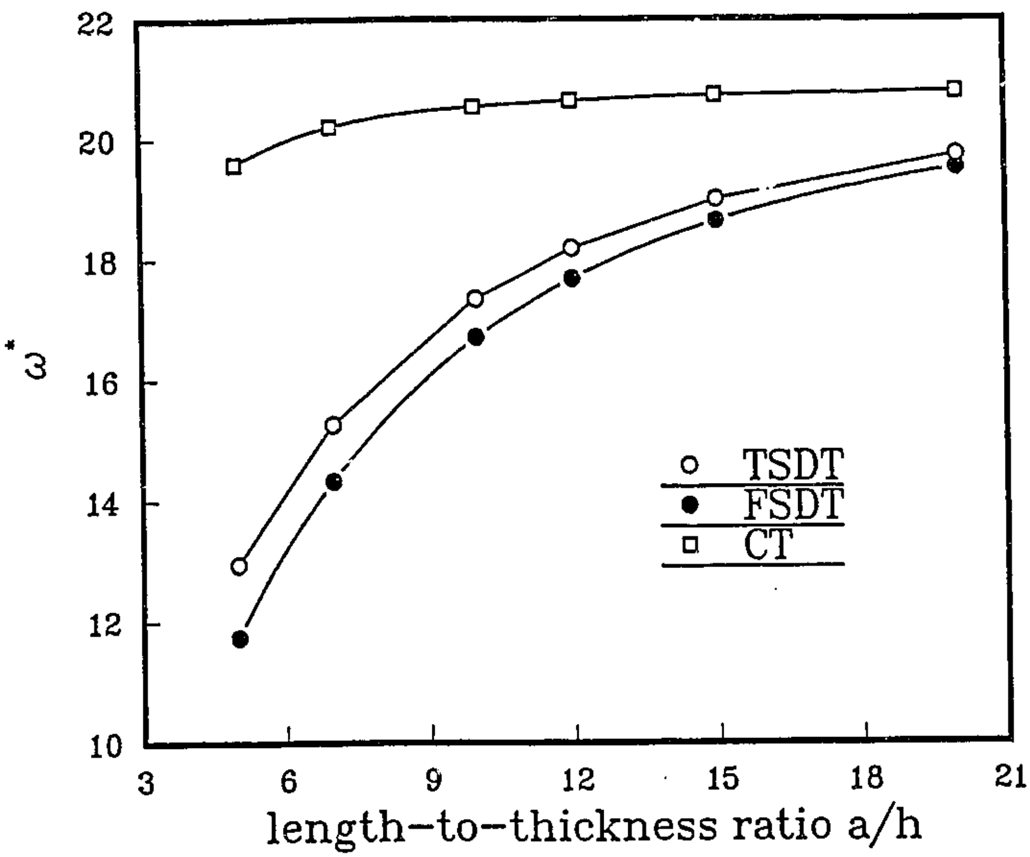

Figure B.9 Non-dimensional fundamental frequencies ω^* versus length-to-thickness ratio of antisymmetric cross-ply two-layer CCSC plates based on TSDT, FSDT and CT theories.

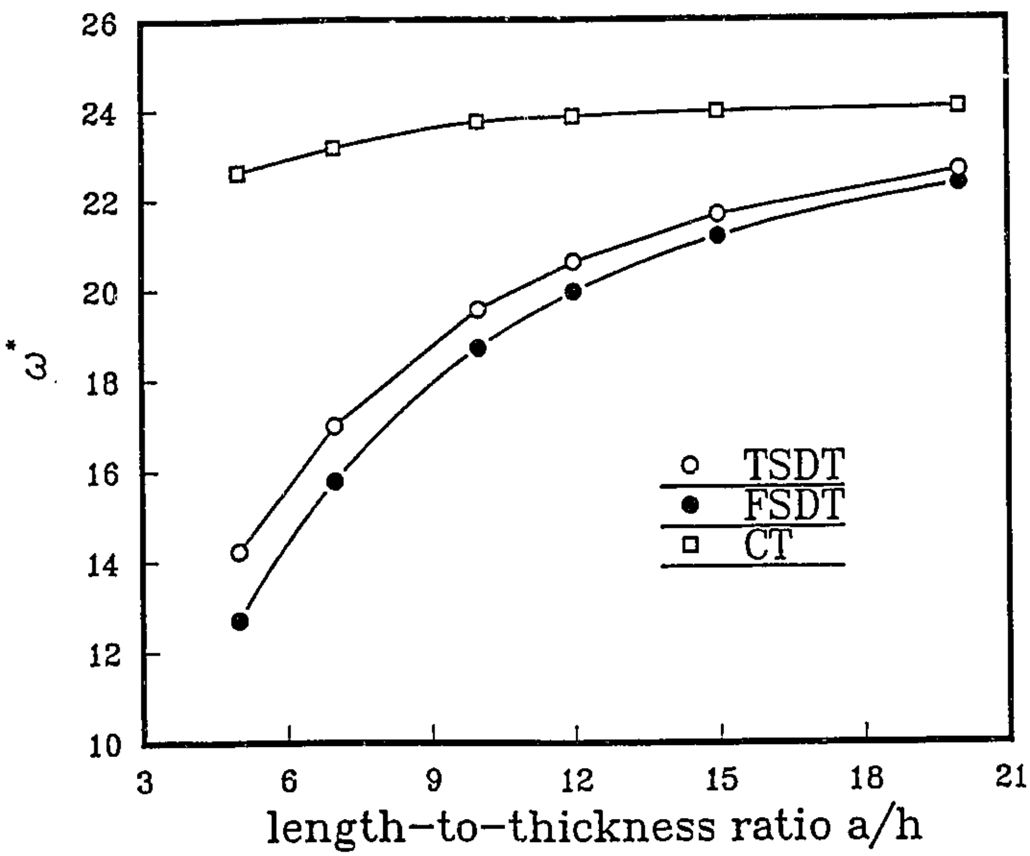

Figure B.10 Non-dimensional fundamental frequencies ω^* versus length-to-thickness ratio of antisymmetric cross-ply two-layer CCCC plates based on TSDT, FSDT and CT theories.

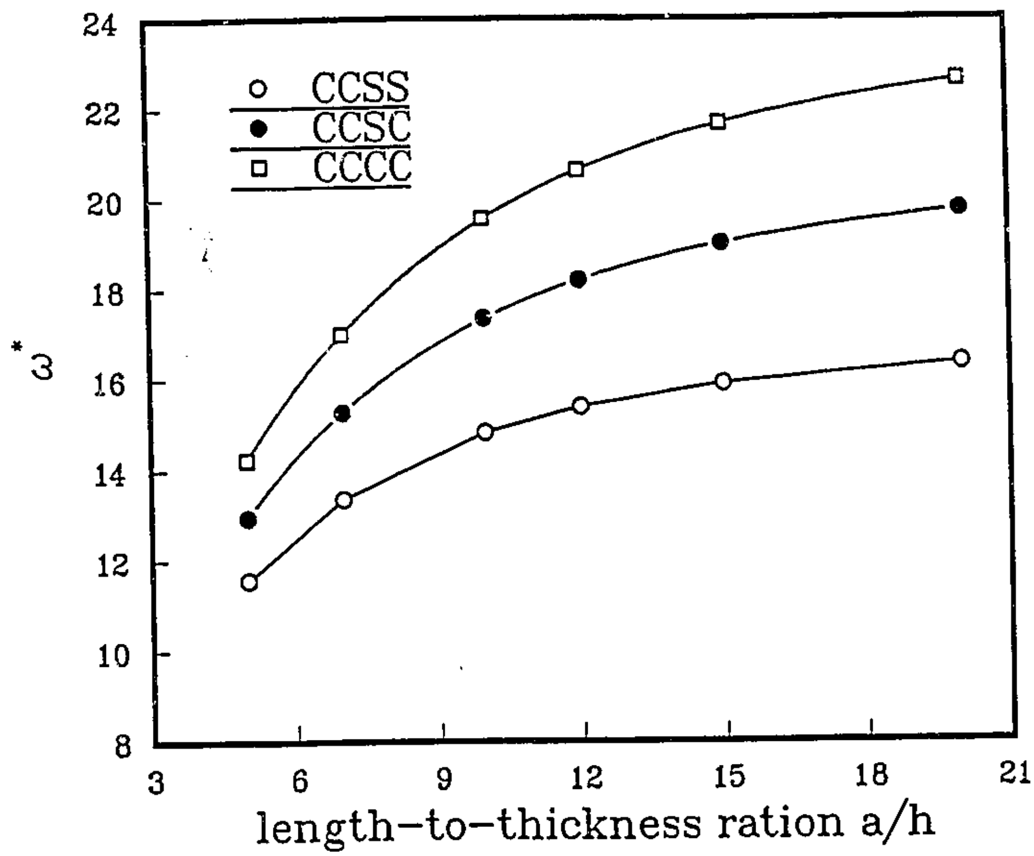

Figure B.11 Non-dimensional fundamental frequencies ω^* versus length-to-thickness ratio and boundary conditions of antisymmetric cross-ply two-layer plates based on TSDT theories.

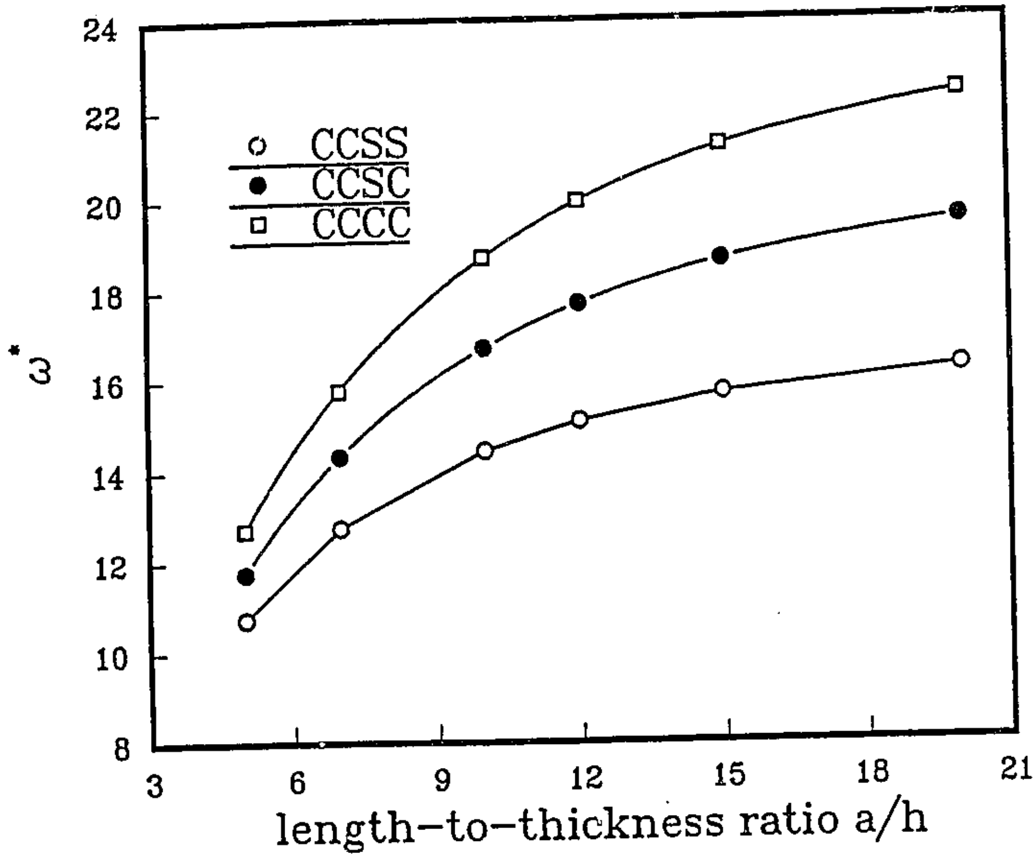

Figure B.12 Non-dimensional fundamental frequencies ω^* versus length-to-thickness ratio and boundary conditions of antisymmetric cross-ply two-layer plates based on FSDT theories.

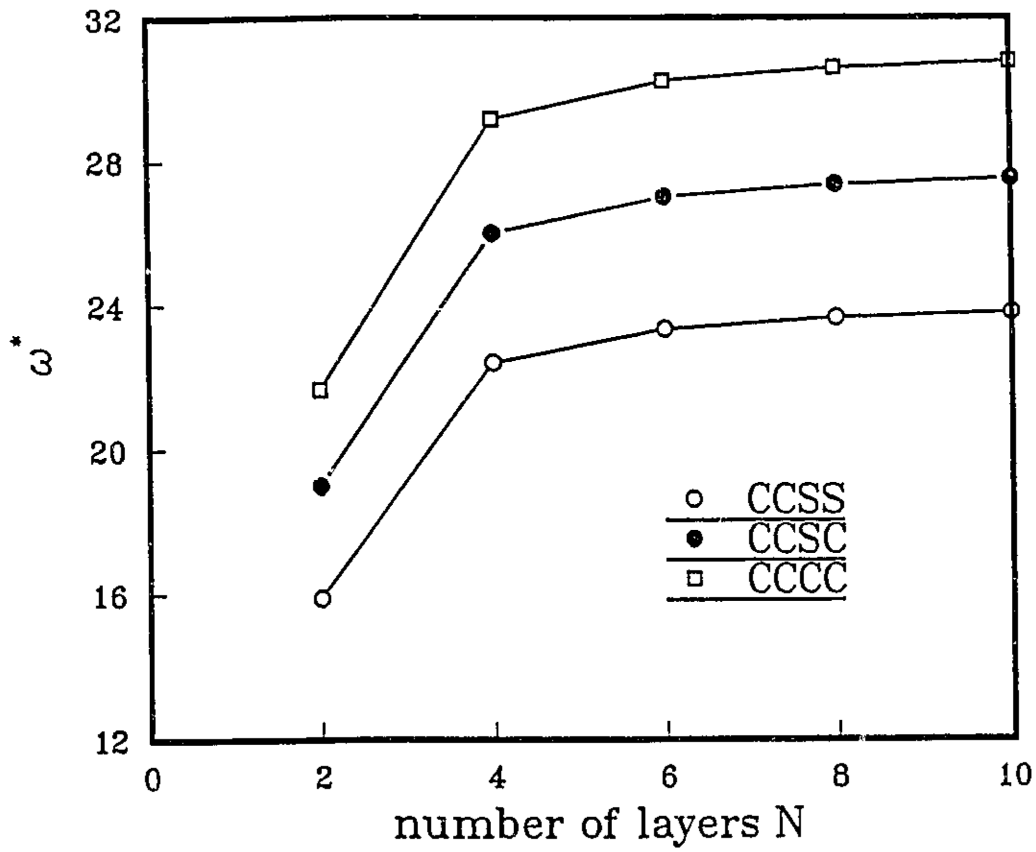

Figure B.13 Non-dimensional fundamental frequencies ω^* versus number of layers and boundary conditions of antisymmetric cross-ply two-layer plates based on TSDT theories.

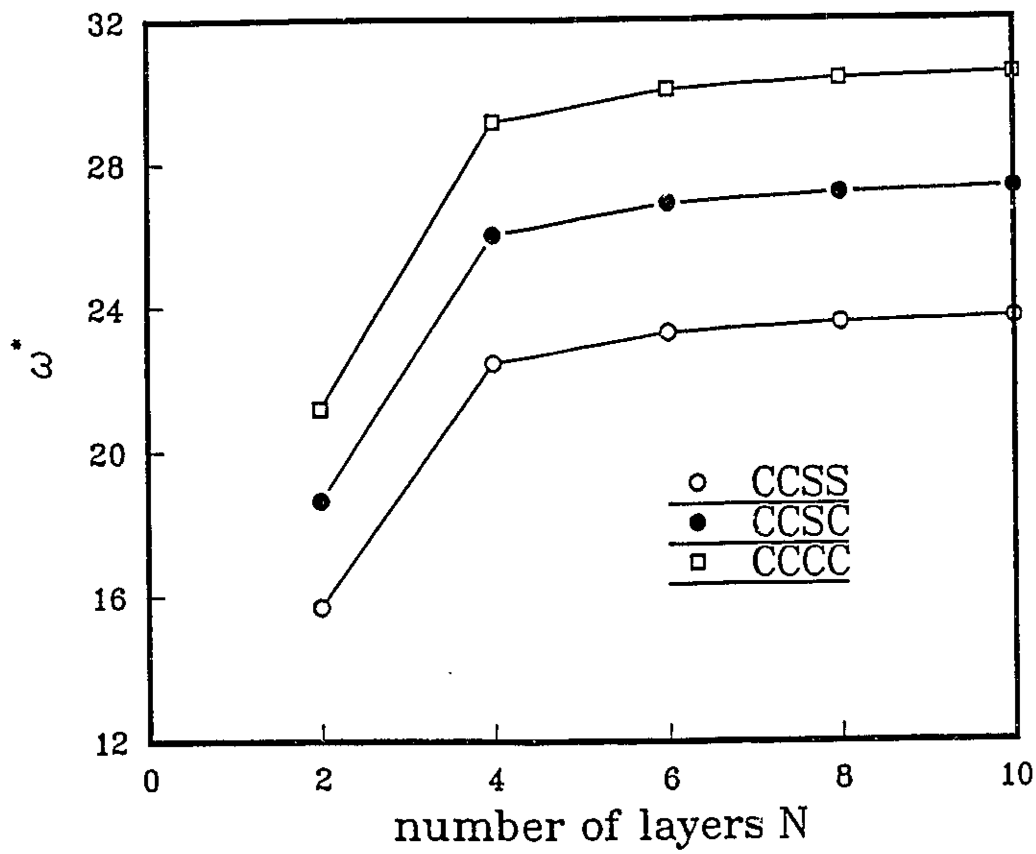

Figure B.14 Non-dimensional fundamental frequencies ω^* versus number of layers and boundary conditions of antisymmetric cross-ply two-layer plates based on FSDT theories.

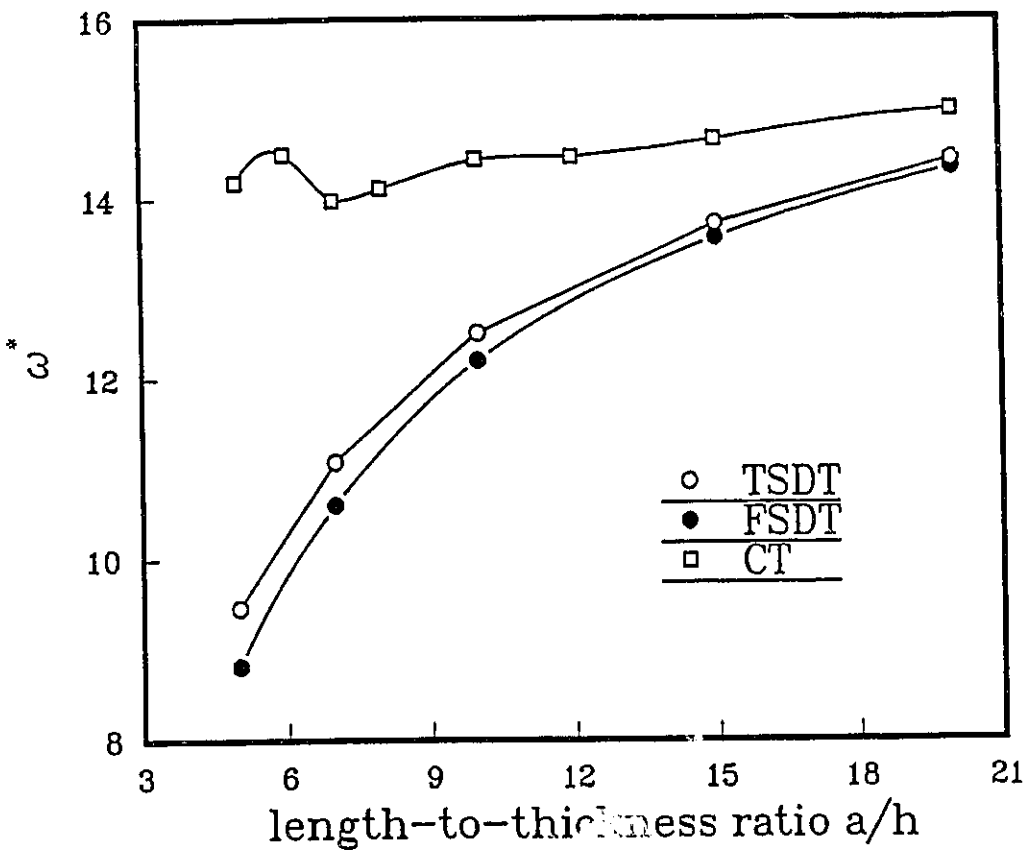

Figure C.1 Non-dimensional fundamental frequencies ω^* versus length-to-thickness ratio of cross-ply two-layer CCSS shells based on TSDT, FSDT and CT theories.

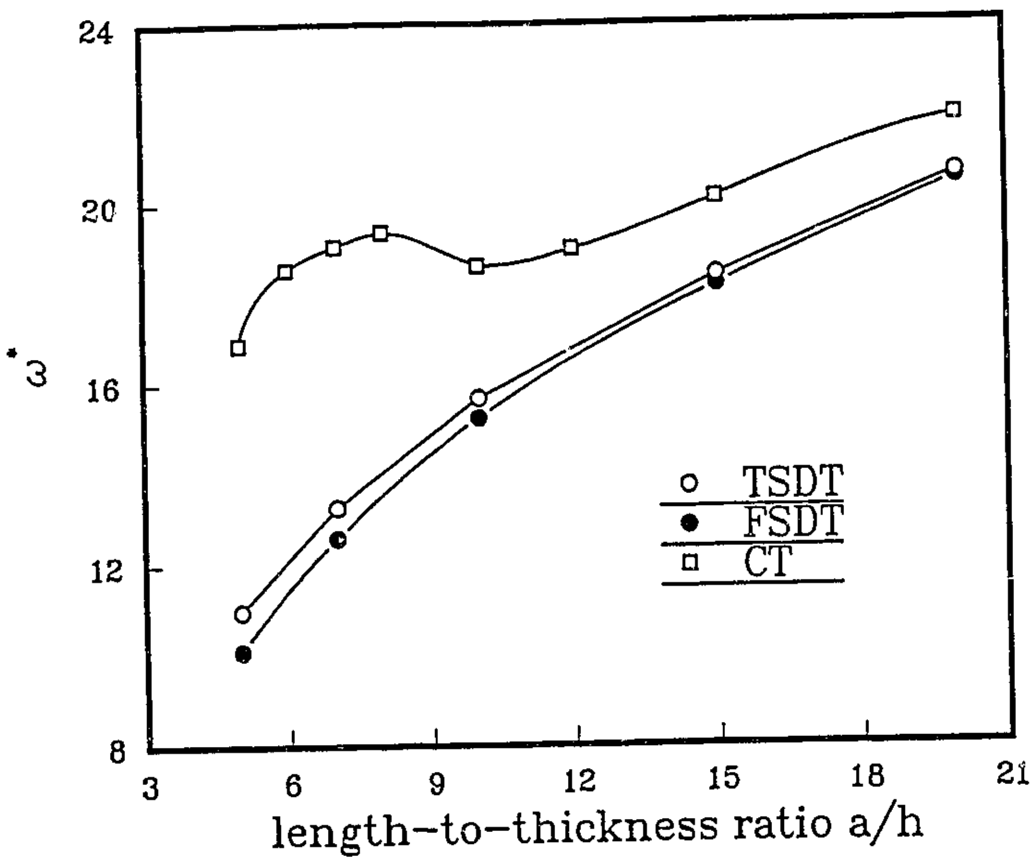

Figure C.2 Non-dimensional fundamental frequencies ω^* versus length-to-thickness ratio of cross-ply two-layer CCSC shells based on TSDT, FSDT and CT theories.

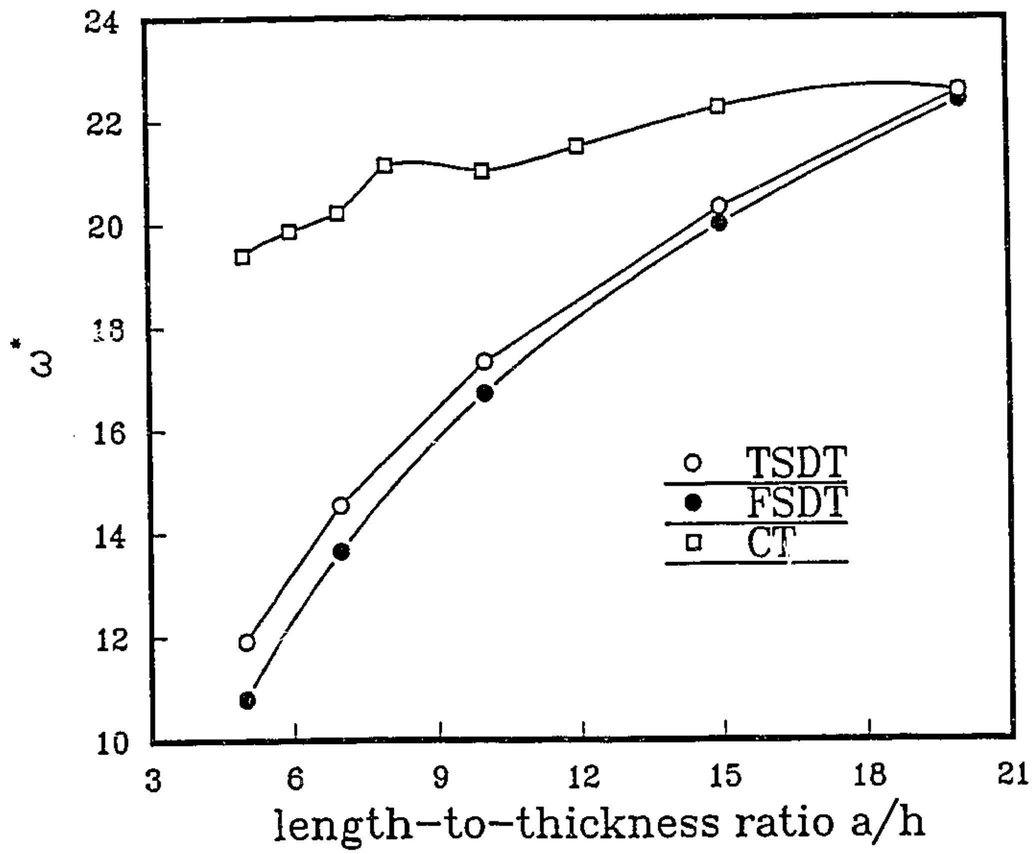

Figure C.3 Non-dimensional fundamental frequencies ω^* versus length-to-thickness ratio of cross-ply two-layer CCCC shells based on TSDT, FSDT and CT theories.

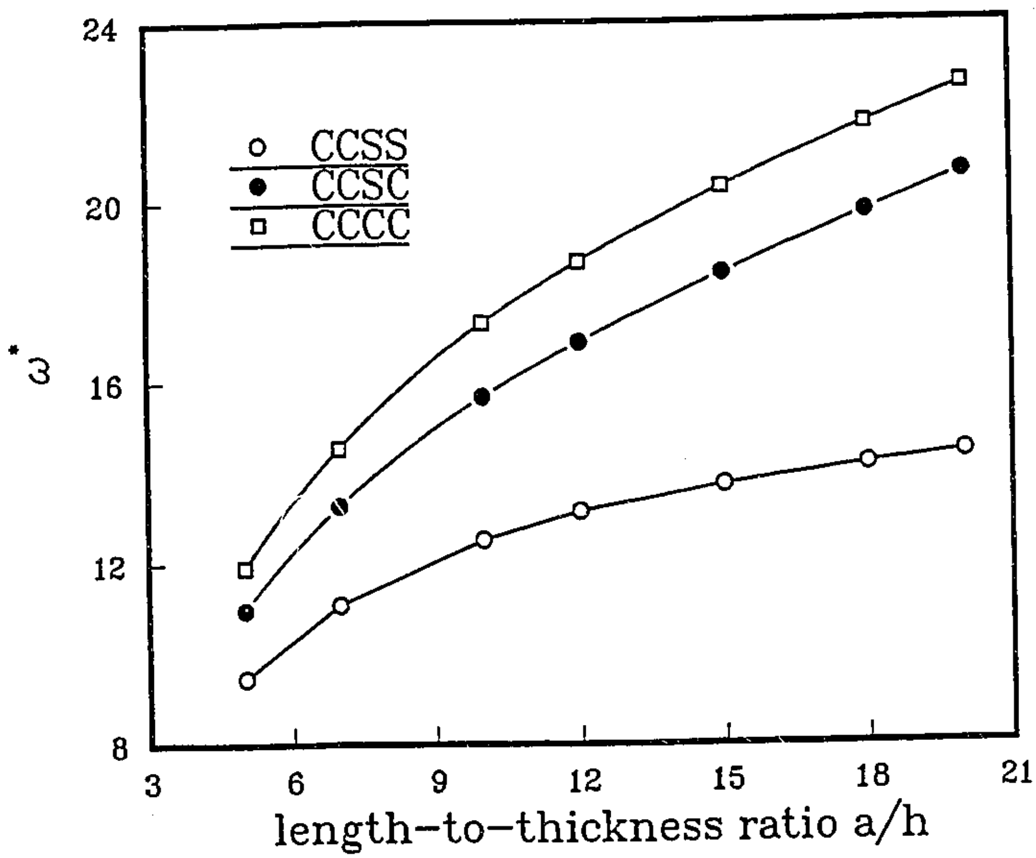

Figure C.4 Non-dimensional fundamental frequencies ω^* versus length-to-thickness ratio and boundary conditions of cross-ply two-layer shells based on TSDT theory.

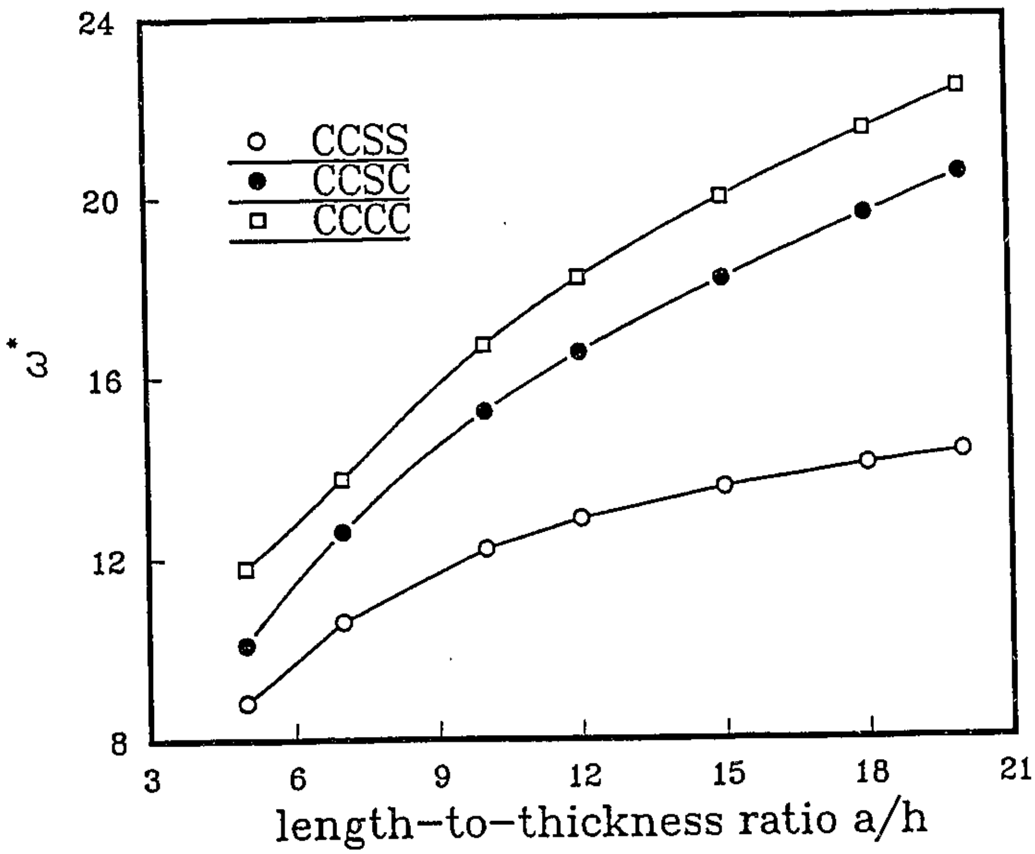

Figure C.5 Non-dimensional fundamental frequencies ω^* versus length-to-thickness ratio and boundary conditions of cross-ply two-layer shells based on FSDT theory.