

Vector-valued automorphic forms and vector bundles

by

Hicham Saber

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Department of Mathematics and Statistics
Faculty of Science
University of Ottawa

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Abstract

In this thesis we prove the existence of vector-valued automorphic forms for an arbitrary Fuchsian group and an arbitrary finite dimensional complex representation of this group. For small enough values of the weight as well as for large enough values, we provide explicit formulas for the spaces of these vector-valued automorphic forms (holomorphic and cuspidal).

To achieve these results, we realize vector-valued automorphic forms as global sections of a certain family of holomorphic vector bundles on a certain Riemann surface associated to the Fuchsian group. The dimension formulas are then provided by the Riemann-Roch theorem.

In the cases of 1 and 2-dimensional representations, we give some applications to the theories of generalized automorphic forms and equivariant functions.

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Introduction

This thesis deals with the topic of vector-valued automorphic forms. These are considered as a natural extension of classical automorphic forms and are defined as follows. Let $\mathrm{SL}(2, \mathbb{R})$ be the group of real 2×2 matrices with determinant one, and $\mathrm{PSL}(2, \mathbb{R}) = \mathrm{SL}(2, \mathbb{R})/\{\pm I\}$. The group $\mathrm{SL}(2, \mathbb{R})$ acts on the Poincaré half-plane

$$\mathbb{H} = \{z \in \mathbb{C} : \mathrm{Im} z > 0\}$$

by Möbius transformations, *i.e.*,

$$\gamma z = \frac{az + b}{cz + d}, \quad z \in \mathbb{H}, \quad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbb{R}).$$

If Γ is a discrete subgroup of $\mathrm{SL}(2, \mathbb{R})$, and R is an n -dimensional complex representation of Γ , that is a homomorphism

$$R : \Gamma \longrightarrow \mathrm{GL}(n, \mathbb{C}),$$

then an unrestricted vector-valued automorphic form for Γ of multiplier R and integral weight k is a meromorphic map $F : \mathbb{H} \longrightarrow \mathbb{C}^n$ satisfying

$$F(\gamma \cdot z) = J_\gamma^k(z) R(\gamma) F(z), \quad z \in \mathbb{H}, \quad \gamma \in \Gamma,$$

where $J_\gamma(z) = cz + d$ if $\gamma = \begin{pmatrix} * & * \\ c & d \end{pmatrix}$. When F is meromorphic (resp. holomorphic) at the cusps of Γ , then we say that F is a vector-valued automorphic form (resp. holomorphic vector-valued automorphic form) for Γ of multiplier

R and weight k . This definition can be extended to real weights k using the notion of multiplier system, see [23]. In this thesis we content ourselves to the case of integral weights.

The theory of vector-valued automorphic forms has been around for a long time, first, as a generalization of the classical theory of scalar automorphic forms, then as natural objects appearing in mathematics and physics. For instance, Selberg suggested vector-valued forms as a tool to study modular forms for finite index subgroups of the modular group [49] and they also appear as Jacobi forms in the work of M. Eichler and D. Zagier [10], and in [57]. In the last decade, there has been a growing interest in the study of vector-valued forms, and several important results have been obtained by M. Knopp, G. Mason and other authors [4, 17, 30, 31, 35, 36, 37].

In order to prove the existence of vector-valued automorphic forms or to compute the dimensions of their vector spaces, people used a wide range of tools coming from geometry, spectral theory, differential equations and classical complex analysis. For instance, J. Fischer [14], D. A. Hejhal [22, 23], and W. Roelcke [47, 48] used the Selberg trace formula to compute the dimensions of the spaces of holomorphic vector-valued automorphic forms, but they imposed the tight condition of the unitarity of the representation R , or more generally the multiplier system. This condition was also imposed by R. Borcherds [6, 7] for the metaplectic group of a general Fuchsian group of the first kind. In the special case of the modular group $SL(2, \mathbb{Z})$, G. Mason [37] employed linear differential equations to construct vector-valued automorphic forms of negative weight. In the paper [17], T. Gannon used the Riemann-Hilbert problem to prove the existence of vector-valued automorphic forms for $SL(2, \mathbb{Z})$, with a possible generalization to the case of Γ being a Fuchsian group of genus zero as announced there. Classical methods such as Poincaré series have been used by M. Knopp, and G. Mason [30] to solve the existence problem, but always restricted to the special case of $\Gamma = SL(2, \mathbb{Z})$. A different approach based on the Riemann-Roch theorem has been used by C. Meneses [39], but only when the weight is two and the representation is

unitary. E. Freitag [15] also used the Riemann-Roch theorem in the case when Γ has a finite index subgroup Γ_0 such that the image of the elements of Γ_0 by the multiplier system are simultaneously diagonalizable. In other words, up to a change of basis, the components of a vector-valued automorphic form for Γ are simply classical automorphic forms for Γ_0 . A sketch of a proof has been given in the case of unitary multiplier system.

These different restrictions on the representation R or the group Γ have the disadvantage of excluding many important cases which are highly relevant to the theory of automorphic forms. For example, weight two vector-valued automorphic forms for the symmetric power representations [34], which are not unitary, play an essential role in the theory of periods of modular forms, see Shimura's work in [56]. Also, in [42] it is shown that the theory of quasi-modular forms [38] can be covered by vector-valued modular forms with respect to the symmetric tensor representations. In the case of 2-dimensional representations, one misses the theory of equivariant functions developed by A. Sebbar and others, see [11, 12, 51, 54]. Even in the 1-dimensional case, these restrictions are strict enough to exclude the emerging theory of generalized modular forms, see [28, 25, 26, 29, 33, 44, 45].

What we propose in this thesis is to prove the existence of vector-valued automorphic forms in full generality, *i.e.*, without any restriction on the group Γ or the representation. Also, to show that given a discrete subgroup Γ of $\mathrm{SL}(2, \mathbb{R})$, an n -dimensional complex representation R of Γ , and an integer $k \in \mathbb{Z}$, the dimensions $d_{\Gamma, R, k}$, $s_{\Gamma, R, k}$ of the spaces $M_k(\Gamma, R)$, $S_k(\Gamma, R)$ of holomorphic, cusp vector-valued automorphic forms for Γ of multiplier R and weight k are infinite when Γ is Fuchsian group of the second kind, and they are finite when Γ is Fuchsian group of the First kind. In this last case, we provide two rational numbers $k_{\Gamma, R}^+$, and $k_{\Gamma, R}^-$ depending on Γ and R such that:

- (A) For $k < k_{\Gamma}^-$, we have $d_{\Gamma, R, k} = 0$, and $s_{\Gamma, R, k} = 0$.
- (B) For $k > k_{\Gamma}^+$, we have explicit formulas for $d_{\Gamma, R, k}$ and $s_{\Gamma, R, k}$ expressed in

terms of k , n , and some invariants of Γ and R .

Our method consists of associating to the triplet (Γ, R, k) a holomorphic vector bundle $\mathcal{E}_{\Gamma, R, k}$ on a certain, well known, Riemann surface X_Γ . Then we prove that the global sections of $\mathcal{E}_{\Gamma, R, k}$, when lifted to \mathbb{H} , correspond to vector-valued automorphic forms for Γ of multiplier R and weight k . In case X_Γ is noncompact, we show that $d_{\Gamma, R, k}$ and $s_{\Gamma, R, k}$ are both infinite. When X_Γ is compact, then we use the Riemann-Roch theorem to give explicit formulas for $d_{\Gamma, R, k}$ and $s_{\Gamma, R, k}$ as explained in (A) and (B). The existence of vector-valued automorphic forms will be proved by showing that $d_{\Gamma, R, k}$ and $s_{\Gamma, R, k}$, when they are finite, are asymptotically equivalent to k , that is

$$\frac{d_{\Gamma, R, k}}{k} \longrightarrow \frac{n m_\Gamma}{2}, \text{ as } k \rightarrow \infty,$$

and

$$\frac{s_{\Gamma, R, k}}{k} \longrightarrow \frac{n m_\Gamma}{2}, \text{ as } k \rightarrow \infty,$$

where m_Γ is some positive constant depending on Γ , see Remark 3.1.5.

It should be pointed that in a joint work with A. Sebbar [50], we proved the existence of vector-valued automorphic forms in the case of subgroups of $\mathrm{PSL}(2, \mathbb{R})$ rather than subgroups of $\mathrm{SL}(2, \mathbb{R})$. This was achieved as follows: we used the solution to the Riemann-Hilbert problem for non-compact Riemann surfaces [13] to attach to each pair (Γ, R) of a discrete subgroup Γ of $\mathrm{PSL}(2, \mathbb{R})$ and an n -dimensional complex representation R of Γ , a holomorphic vector bundle $\mathcal{E}_{\Gamma, R}$ on X_Γ in such a way that the global sections of $\mathcal{E}_{\Gamma, R}$ correspond to vector-valued automorphic forms for Γ of multiplier R and weight 0. Then by the Kodaira vanishing theorem for compact Riemann surfaces [19], in case X_Γ is compact, or by the fact that X_Γ is a Stein variety [21] in case it is not compact, we proved the existence of n linearly independent vector-valued automorphic forms for Γ of multiplier R and a certain integral weight k .

The 2-dimensional case is of special interest since it has a connection to ρ -equivariant functions. More precisely, given a discrete subgroup Γ of

$\mathrm{SL}(2, \mathbb{R})$ and a representation $\rho : \Gamma \rightarrow \mathrm{GL}(2, \mathbb{C})$, a meromorphic function h on \mathbb{H} is called a ρ -equivariant function with respect to Γ if

$$h(\gamma z) = \rho(\gamma)h(z) \text{ for all } \gamma \in \Gamma, z \in \mathbb{H}, \quad (0.0.1)$$

where the action on both sides is by linear fractional transformations. When ρ is the defining representation of Γ , that is $\rho(\gamma) = \gamma$ for all γ in Γ , h is simply an equivariant function, see (4.2.1).

ρ -equivariant functions and 2-dimensional vector-valued automorphic forms are connected in the following way: if $F = (f_1, f_2)^t$ is an unrestricted vector-valued automorphic form for Γ of multiplier ρ and weight k , then one can check that if f_2 is nonzero, then the function $h_F = f_1/f_2$ is a ρ -equivariant function for Γ . The main result of our joint work [52] with A. Sebbar is that the converse is also true: for any ρ -equivariant function there exists an unrestricted vector-valued automorphic form $F = (f_1, f_2)^t$ for Γ of multiplier R and some weight $k \in \mathbb{Z}$ such that:

$$h = f_1/f_2.$$

In this thesis we enhance this result by proving that for any 2-dimensional representation ρ of Γ , there exists a ρ -equivariant function of the form

$$h_F = f_1/f_2,$$

where $F = (f_1, f_2)^t$ is a holomorphic vector-valued automorphic form for Γ of multiplier R and a certain nonnegative weight k .

The 1-dimensional case corresponds to the theory of generalized automorphic forms. Due to their similarity with classical automorphic forms and their richness, the generalized automorphic forms are receiving an increasing interest, specially after the work of M. Knopp and G. Mason in [28], see [25, 26, 29, 33, 44, 45]. A meromorphic function f on \mathbb{H} is called a generalized automorphic form of weight k for a discrete subgroup Γ of $\mathrm{SL}(2, \mathbb{Z})$ if for all $\gamma \in \Gamma$ and $z \in \mathbb{H}$, we have

$$f(\gamma \cdot z) = \mu(\gamma) J_\gamma^k(z) f(z)$$

where $\mu : \Gamma \rightarrow \mathbb{C}$ is a character. In addition, f should be meromorphic at the cusps. The fundamental difference between generalized automorphic forms and classical automorphic forms is that the character μ need not be unitary. The existence theorem of vector-valued automorphic forms in dimension 1 guarantees the existence of generalized automorphic forms for an arbitrary character of an arbitrary discrete group Γ . Our main contribution to this theory will be the computation of the dimensions of the spaces of holomorphic and cusp generalized automorphic forms in the sense of (A) and (B).

This thesis is organized as follows: in chapter one we shall review some classical results from the theories of holomorphic vector bundles on Riemann surfaces, Fuchsian groups, and automorphic forms.

In chapter two we realize vector-valued automorphic forms as global sections of holomorphic vector bundles. More precisely, given a Fuchsian subgroup Γ of $\mathrm{SL}(2, \mathbb{R})$, a finite-dimensional representation R of Γ , and an integer k such that the pair (R, k) is simple, see Definition 2.1.2, we construct a holomorphic vector bundle $\mathcal{E}_{\Gamma, R, k}$ over the Riemann surface $X_{\Gamma} = \Gamma \backslash \mathbb{H}^*$ associated to Γ . The spaces $M_k(\Gamma, R)$, $S_k(\Gamma, R)$ of holomorphic, cusp vector-valued automorphic forms for Γ of multiplier R and weight k will be respectively isomorphic to $H^0(X_{\Gamma}, \mathcal{O}(\mathcal{E}_{\Gamma, R, -k}))$ and $H^0(X_{\Gamma}, \mathcal{O}(-\mathcal{D}_{\mathfrak{S}, R, -k} + \mathcal{E}_{\Gamma, R, -k}))$, where $\mathcal{D}_{\mathfrak{S}, R, -k}$ is a certain holomorphic line bundle depending on the cuspidal points of X_{Γ} . As a result, we deduce that the dimensions $d_{\Gamma, R, k}$, $s_{\Gamma, R, k}$ of $M_k(\Gamma, R)$, $S_k(\Gamma, R)$ are finite if and only if Γ is Fuchsian group of the first kind, see Theorem 2.5.9.

In chapter three we introduce the notion of the holomorphic degree of a holomorphic vector bundle, which consists of associating to each holomorphic vector bundle \mathcal{E} , defined on a compact Riemann surface, an integer $d(\mathcal{E})$ in a such a way that:

$$d(\mathcal{E}) < 0 \implies H^0(X, \mathcal{O}(\mathcal{E})) \text{ is trivial.}$$

We will give some basic properties of the holomorphic degree and some useful

bounds of $d(\mathcal{E})$. Applying the Riemann-Roch theorem to the holomorphic vector bundles $\mathcal{E}_{\Gamma,R,-k}$ and $-\mathcal{D}_{\mathfrak{S},R,-k} + \mathcal{E}_{\Gamma,R,-k}$, we prove the results mentioned in (A) and (B).

The last chapter provides applications of our results in the special cases of 1 and 2-dimensional representations. In particular, we compute the dimensions of the spaces of holomorphic and cusp generalized automorphic forms. Also, we show the existence of ρ -equivariant functions, and prove their parametrization by 2-dimensional unrestricted vector-valued automorphic forms of multiplier ρ . This parametrization relies on the existence of global solutions to a certain second degree differential equation, as well as on the fact that the Schwarz derivative of a ρ -equivariant function is a weight 4 unrestricted automorphic form for Γ . We end the chapter by constructing examples of ρ -equivariant functions when ρ is the monodromy representation of second degree ordinary differential equations.

Chapter 1

Preliminaries

The aim of this chapter is to give a review of some basic facts from the theories of holomorphic vector bundles on Riemann surfaces, Fuchsian groups, and automorphic forms.

1.1 Vector Bundles

The content of this section is entirely based on the references:

[2]: Chapter 1: §1.5, §1.8.

[13]: Chapter 1: §29, §30, and Chapter 2: §16.

[19] Chapter 0: §0.5, Chapter 1: §1.1, and Chapter 1: §2.1 .

[20] Chapter C.

[?] Chapter 6: §6.5.

X will be a Riemann surface, that is, a one-dimensional complex manifold, and n a positive integer.

1.1.1 Basic properties

Definition 1.1.1. Let X be a Riemann surface, \mathcal{E} be a complex manifold, and $p : \mathcal{E} \rightarrow X$ be a holomorphic map. Suppose that each fiber $\mathcal{E}_x := p^{-1}(\{x\})$ has the structure of an n -dimensional vector space over \mathbb{C} . Then $p : \mathcal{E} \rightarrow X$, or simply \mathcal{E} , is called a *holomorphic vector bundle* of rank n on X if every point $x \in X$ has an open neighborhood U such that :

1. There exists a biholomorphic map

$$\phi_U : \mathcal{E}_U := p^{-1}(U) \rightarrow U \times \mathbb{C}^n$$

taking \mathcal{E}_x to $\{x\} \times \mathbb{C}^n$ for every $x \in U$.

2. For every $x \in U$, the map ϕ_U is a linear isomorphism from \mathcal{E}_x to $\{x\} \times \mathbb{C}^n \cong \mathbb{C}^n$.

The map $\phi_U : \mathcal{E}_U \rightarrow U \times \mathbb{C}^n$ is called a *trivialization* of \mathcal{E} over U . A vector bundle of rank 1 is called a *line bundle*.

Definition 1.1.2. Let \mathcal{E} and \mathcal{F} be two holomorphic vector bundles on X of rank n and m respectively. A holomorphic map $\Psi : \mathcal{E} \rightarrow \mathcal{F}$ is called a homomorphism (resp. isomorphism) if for all $x \in X$

1. $\Psi(\mathcal{E}_x) \subseteq \mathcal{F}_x$
2. $\Psi_x := \Psi|_{\mathcal{E}_x} : \mathcal{E}_x \rightarrow \mathcal{F}_x$ is a linear homomorphism (resp. isomorphism).

\mathcal{E} and \mathcal{F} are called *isomorphic* if there exists an isomorphism between them. In particular, \mathcal{E} is called a *trivial* holomorphic vector bundles on X if it is isomorphic to the trivial holomorphic vector bundle $X \times \mathbb{C}^n$.

Note that for any pair of trivializations ϕ_U and ϕ_V we have a holomorphic map

$$g_{UV} : U \cap V \rightarrow \text{GL}(n, \mathbb{C}),$$

such that

$$\phi_{UV} := \phi_U \circ \phi_V^{-1} : (U \cap V) \times \mathbb{C}^n \longrightarrow (U \cap V) \times \mathbb{C}^n$$

is given by

$$\phi_{UV}(x, t) = (x, g_{UV}(x)t) \quad \text{for every } x \in (U \cap V) \times \mathbb{C}^n.$$

The map g_{UV} is called a *transition function* for \mathcal{E} relative to the trivializations ϕ_U and ϕ_V . Also, one has the relation

$$g_{UV} g_{VW} = g_{UW} \quad \text{on } U \cap V \cap W. \quad (1.1.1)$$

Definition 1.1.3. Let U be an open subset of X . Let $\mathcal{O}(U)$ (resp. $\mathcal{M}(U)$) denotes the group of holomorphic (resp. meromorphic) functions on U , and $\text{GL}(n, \mathcal{O}(U))$ (resp. $\text{GL}(n, \mathcal{M}(U))$) the group of all $n \times n$ invertible matrices with coefficients in $\mathcal{O}(U)$ (resp. $\mathcal{M}(U)$). Together with the natural restrictions when $V \subseteq U$, they define sheaves of groups \mathcal{O} , \mathcal{M} , $\text{GL}(n, \mathcal{O})$, and $\text{GL}(n, \mathcal{M})$ on X ($\text{GL}(n, \mathcal{O})$ and $\text{GL}(n, \mathcal{M})$ are not abelian for $n \geq 2$).

If $\mathcal{U} = (U_i)_I$ is an open cover of X , we define $Z^1(\mathcal{U}, \text{GL}(n, \mathcal{O}))$ to be the set of all 1-cocycles with values in $\text{GL}(n, \mathcal{O})$ with respect to \mathcal{U} , *i.e.*, all families $(g_{UV})_{\mathcal{U}}$ with

$$g_{UV} \in \text{GL}(n, \mathcal{O}(U \cap V))$$

and verifying (1.1.1).

Remark 1.1.1. Note that for $n \geq 2$ the set $Z^1(\mathcal{U}, \text{GL}(n, \mathcal{O}))$ is not a group with respect to component-wise multiplication.

As seen above, any holomorphic vector bundle gives rise to one a 1-cocycle with respect to an open cover \mathcal{U} of X by means of the transition functions. Conversely, we have:

Theorem 1.1.2. *Let $\mathcal{U} = (U_i)_I$ be an open cover of X , and $(g_{UV})_{\mathcal{U}}$ be an element of $Z^1(\mathcal{U}, \text{GL}(n, \mathcal{O}))$. Then up to isomorphism, there exists a unique holomorphic vector bundle \mathcal{E} on X of rank n having transition functions $(g_{UV})_{\mathcal{U}}$.*

Also, we have

Theorem 1.1.3. *Let $\mathcal{U} = (U_i)_I$ be an open cover of X , and $(g_{UV})_{\mathcal{U}}, (h_{UV})_{\mathcal{U}}$ be elements of $Z^1(\mathcal{U}, GL(n, \mathcal{O}))$. Suppose that there exist elements $f_U \in GL(n, \mathcal{O}(U))$, $U \in \mathcal{U}$, such that*

$$h_{UV} = f_U g_{UV} f_V^{-1} \quad \text{on } U \cap V,$$

then $(g_{UV})_{\mathcal{U}}$ and $(h_{UV})_{\mathcal{U}}$ represent the same holomorphic vector bundle.

In principle, all operations on vector spaces induce operations on vector bundles. For example, if \mathcal{E} and \mathcal{F} are holomorphic vector bundles over X of rank n and m having transition functions $(g_{UV})_{\mathcal{U}}$ and $(h_{UV})_{\mathcal{U}}$ one can define:

1. The dual bundle \mathcal{E}^* , given by the transition functions

$$g_{UV}^*(x) = (g_{UV}^{-1})^T(x).$$

2. The direct sum $\mathcal{E} \oplus \mathcal{F}$, given by the transition functions

$$g_{UV}(x) \oplus h_{UV}(x) \in GL(\mathbb{C}^n \oplus \mathbb{C}^m).$$

3. The tensor product $\mathcal{E} \otimes \mathcal{F}$, given by the transition functions

$$g_{UV}(x) \otimes h_{UV}(x) \in GL(\mathbb{C}^n \otimes \mathbb{C}^m).$$

4. The exterior product $\bigwedge^k \mathcal{E}$, given by the transition functions

$$\bigwedge^k g_{UV}(x) \in GL(\bigwedge^k \mathbb{C}^n).$$

In particular, $\bigwedge^n \mathcal{E}$ is a line bundle given by the transition functions

$$\det(g_{UV})(x) \in GL(1, \mathbb{C}) = \mathbb{C}^*,$$

also called the *determinant bundle* of \mathcal{E} .

We now come to the notion of a section over a holomorphic vector bundle.

Definition 1.1.4. Let $p : \mathcal{E} \rightarrow X$ be a holomorphic vector bundle of rank n , and U be an open subset of X . A meromorphic section σ of \mathcal{E} over U is a meromorphic map $\sigma : U \rightarrow \mathcal{E}$ such that $p \circ \sigma = id_U$, i.e., $\sigma(x) \in \mathcal{E}_x$ for all $x \in U \setminus P_\sigma$, where P_σ is the set of poles of σ in U . If $U = X$, then σ is called a global meromorphic section of \mathcal{E} .

If σ has no poles, it is called a holomorphic section.

If ϕ_U is a trivialization of \mathcal{E} over U , then we can associate to each holomorphic section σ of \mathcal{E} over U a unique meromorphic map $f_U : U \rightarrow \mathbb{C}^n$ such that

$$\phi_U(\sigma(x)) = (x, f_U(x)), \quad \text{for all } x \in U \setminus P_\sigma.$$

If ϕ_V is a trivialization of \mathcal{E} over V , then we have

$$f_U = g_{UV} f_V \quad \text{on } (U \cap V) \setminus P_\sigma,$$

where g_{UV} is the transition function of \mathcal{E} relative to ϕ_U and ϕ_V . Thus we have

Proposition 1.1.4. *In terms of trivializations $\mathcal{E}_U : U \rightarrow U \times \mathbb{C}^n$, a meromorphic section σ of \mathcal{E} over $\bigcup_{U \in \mathcal{U}} U$, corresponds to a collection $(f_U)_U$ of meromorphic maps $f_U : U \rightarrow \mathbb{C}^n$ such that*

$$f_U = g_{UV} f_V \quad \text{on } (U \cap V) \setminus P_\sigma,$$

where the g_{UV} , $U, V \in \mathcal{U}$, are transition functions of \mathcal{E} relative to $(\phi_U)_U$.

□

1.1.2 Divisors

In this subsection, we define the notion of a divisor of a section of a holomorphic vector bundle and give its basic properties.

Let U be an open subset of \mathbb{C} , and $F : U \rightarrow \mathbb{C}^n$ be a meromorphic map. Then for any w in \mathbb{H} , there exists an open disc D_w centered at w , and a unique integer n_w such that

$$F(z) = (z - w)^{n_w} G_w(z), \quad z \in D_w,$$

where G_w is a holomorphic map on D_w satisfying $G_w(w) \neq 0$. The integer n_w is called the order of F at w and is denoted by $\text{ord}_w(F)$.

Remark 1.1.5. If $\Psi: D_w \rightarrow \text{GL}(n, \mathbb{C})$ is a holomorphic map, Then

$$\text{ord}_w(F) = \text{ord}_w(\Psi F).$$

Suppose that \mathcal{U} is an open cover of X . Let \mathcal{E} be a holomorphic vector bundle over X having transition functions $(g_{UV})_{\mathcal{U}}$, σ be a meromorphic section of \mathcal{E} over X , and $(f_U)_{\mathcal{U}}$ be its corresponding collection of meromorphic maps. Let $P \in U$ and take an open neighborhood U_P of P in U such that under a coordinate chart φ for X , we have $\varphi(U_P) = \mathbb{D}$ and $\varphi(P) = 0$, where \mathbb{D} is the unit disc. The map $F_{U,P} = f_U \circ \varphi^{-1}$ is then meromorphic on \mathbb{D} , and we define the order $\nu_P(\sigma)$ of σ at P by

$$\nu_P(\sigma) = \text{ord}_0(F_{U,P}). \quad (1.1.2)$$

According to [58], the integer $\nu_P(\sigma)$ is invariant under coordinate changes. Hence, it is enough to show that $\nu_P(\sigma)$ is independent of U . Indeed, if P lies in $U \cap V$, then we have

$$f_U = g_{UV} f_V \quad \text{on } (U \cap V) \setminus P_\sigma.$$

Since $g_{UV} \in \text{GL}(n, \mathcal{O}(U \cap V))$, by the above remark, we have

$$\text{ord}_P(f_U) = \text{ord}_P(f_V).$$

The divisor of σ is defined by the formal sum

$$\text{div}(\sigma) = \sum_{P \in X} \nu_P(\sigma) P. \quad (1.1.3)$$

Similarly, the divisor of a meromorphic function f on X is given by

$$\operatorname{div}(f) = \sum_{P \in X} \nu_P(f)P, \quad (1.1.4)$$

where $\nu_P(f)$ is the order of f at P . More generally

Definition 1.1.5. A divisor D on X is a locally finite formal sum

$$D = \sum_{P \in X} d_P P, \quad d_P \in \mathbb{Z},$$

where locally finite means that for any point P in X , there exists a neighborhood U of P such that $d_P \neq 0$ for only finitely many P in U . If U is an open subset of X , we set

$$D|_U = \sum_{P \in U} d_P P.$$

$\operatorname{Div}(X)$ will be the abelian group of all divisors on X . For $D, D' \in \operatorname{Div}(X)$, set $D \geq D'$ if $d_P \geq d'_P$ for all $P \in X$. A divisor D is called effective if $D \geq 0$.

The support $\operatorname{Supp}(D)$ of D will be the closure in X of the set

$$\{P \in X \mid d_P \neq 0\}.$$

Remark 1.1.6. When X is compact, a locally finite sum is simply a finite sum. Hence $\operatorname{Supp}(D)$ is a finite set.

Now, we focus on the relationship between divisors and line bundles. Let D be a divisor on X . By the local finiteness assumption on D , one can find an open cover \mathcal{U} of X and functions $f_U \in \mathcal{M}(U)$, $U \in \mathcal{U}$, such that

$$\operatorname{div}(f_U) = D \quad \text{on } U.$$

Since f_U and f_V have the same divisor $D|_{U \cap V}$ on $U \cap V$, we see that

$$g_{UV} := f_U/f_V \in \mathcal{O}^*(U \cap V) = \operatorname{GL}(1, \mathcal{O}(U \cap V)).$$

It is clear that $(g_{UV})_{\mathcal{U}}$ is an element of $Z^1(\mathcal{U}, \text{GL}(n, \mathcal{O}))$. Hence it defines a line bundle \mathcal{D} on X , and by construction, $\sigma_D = (f_U)_{\mathcal{U}}$ is a global meromorphic section of \mathcal{D} such that

$$\text{div}(\sigma_D) = D.$$

If the set of line bundles constructed in the above way is denoted by \mathcal{A}_D , then it can be shown that any two line bundles in \mathcal{A}_D are naturally isomorphic. We use this fact to identify all elements of \mathcal{A}_D to one line bundle that will be called the *associated line bundle* to the divisor D , and will be denoted by $|D|$.

Suppose that \mathcal{L} is a holomorphic line bundle given by transition functions $(g_{UV})_{\mathcal{U}}$. If \mathcal{L} has a nonzero global meromorphic section $\sigma = (f_U)_{\mathcal{U}}$, then by Proposition 1.1.4 we have

$$g_{UV} = f_U/f_V,$$

that is $\mathcal{L} = |\text{div}(\sigma)|$. From this we deduce that a line bundle is associated to a divisor (resp. an effective divisor) if and only if it has a nonzero global meromorphic (resp. holomorphic) section.

Theorem 1.1.7. *Any holomorphic line bundle \mathcal{L} on X has a nontrivial global meromorphic section. In other words it is of the form $\mathcal{L} = |D|$ for some divisor D on X .*

Recall that when X is compact, the support of any divisor on X is a finite set.

Definition 1.1.6. The degree of a divisor $D = \sum_{P \in X} d_P P$, on a compact Riemann surface X is defined by

$$\text{deg}(D) = \sum_{P \in X} d_P.$$

We have the following

Proposition 1.1.8. *Suppose that X is compact. If f is a meromorphic function on X , then*

$$\text{deg}(\text{div}(f)) = 0.$$

The degree of a holomorphic line bundle over a compact Riemann surface X is defined as follows. Let \mathcal{U} be an open cover of X , \mathcal{L} a holomorphic line bundle over X having transition functions $(g_{UV})_{\mathcal{U}}$, σ a meromorphic section of \mathcal{L} over X , and $(f_U)_{\mathcal{U}}$ its corresponding collection of meromorphic maps. Such σ always exists according to Theorem 1.1.7. If $\sigma_1 = (h_U)_{\mathcal{U}}$ is another meromorphic section of \mathcal{L} , then from the relations

$$f_U = g_{UV}f_V, \quad h_U = g_{UV}h_V \quad \text{on } U \cap V$$

we see that

$$h_U/f_U = h_V/f_V, \quad \text{on } U \cap V.$$

Hence $f := \sigma_1/\sigma$ is a global meromorphic function on X , and we have

$$\text{div}(\sigma_1) = \text{div}(\sigma) + \text{div}(f).$$

Therefore

$$\deg(\text{div}(\sigma_1)) = \deg(\text{div}(\sigma)) + \deg(\text{div}(f)) = \deg(\text{div}(\sigma)),$$

since $\deg(\text{div}(f)) = 0$ according to Theorem 1.1.8. Thus, the divisors of global meromorphic sections have the same degree.

Definition 1.1.7. Let X be a compact Riemann surface, and let \mathcal{E} be a holomorphic vector bundle over X , and \mathcal{L} be a holomorphic line bundle over X .

1. The degree of \mathcal{L} is defined by

$$\deg(\mathcal{L}) = \deg(\text{div}(\sigma)),$$

where σ is any global meromorphic section of \mathcal{L} .

2. The degree of \mathcal{E} is defined by

$$\deg(\mathcal{E}) = \deg(\det(\mathcal{E})),$$

where $\det(\mathcal{E})$ is the determinant bundle of \mathcal{E} .

The degree has the following useful property.

Proposition 1.1.9. *Let X be a compact Riemann surface, and let \mathcal{E} and \mathcal{F} be holomorphic vector bundles over X , and \mathcal{L} be a holomorphic line bundle over X . We have*

$$\deg(\mathcal{E} \oplus \mathcal{F}) = \deg(\mathcal{E}) + \deg(\mathcal{F}),$$

and

$$\deg(\mathcal{L} \otimes \mathcal{E}) = n \deg(\mathcal{L}) + \deg(\mathcal{E}),$$

where n is the rank of \mathcal{E} .

1.1.3 The Riemann-Roch theorem.

The goal of this subsection is to state the Riemann-Roch theorem for holomorphic vector bundles over compact Riemann surfaces.

Let \mathcal{E} be a holomorphic vector bundle over X of rank n , and D a divisor on X , σ a meromorphic section of \mathcal{E} over X , and $(f_U)_{\mathcal{U}}$ its corresponding collection of meromorphic maps. For any open subset U of X set $\mathcal{O}(\mathcal{E})(U)$ (resp. $\mathcal{M}(\mathcal{E})(U)$) to be the $\mathcal{O}(U)$ -module of holomorphic (resp. meromorphic) sections of \mathcal{E} over U . Together with the natural restrictions when $V \subseteq U$, they define sheaves $\mathcal{O}(\mathcal{E})$ and $\mathcal{M}(\mathcal{E})$ of \mathcal{O} -modules on X . The vector space $H^0(X, \mathcal{O}(\mathcal{E}))$ (resp. $H^0(X, \mathcal{M}(\mathcal{E}))$) of global sections of $\mathcal{O}(\mathcal{E})$ (resp. $\mathcal{M}(\mathcal{E})$) is exactly the vector space of global holomorphic (resp. meromorphic) sections on X .

Remark 1.1.10. The dimension of the \mathbb{C} -vector space $H^0(X, \mathcal{O}(\mathcal{E}))$ will be denoted by $h^0(\mathcal{E})$ when it is finite.

Example 1.1.11. If (U, z_U) , $U \in \mathcal{U}$, is a covering of X by coordinate neighborhoods, then on $U \cap V$ the function

$$g_{UV} = dz_U/dz_V$$

lies in $\mathrm{GL}(1, \mathcal{O}(U \cap V)) = \mathcal{O}^*(U \cap V)$, and it is clear that

$$(g_{UV})_{\mathcal{U}} \in Z^1(\mathcal{U}, \mathrm{GL}(1, \mathcal{O})).$$

The attached holomorphic line bundle is called the *canonical line bundle* of X , and will be denoted by \mathcal{K}_X .

The sheaf $\mathcal{O}(\mathcal{K}_X)$ is isomorphic to the sheaf Ω_X of holomorphic 1-forms on X , the latter is defined by taking $\Omega(U)$ to be the $\mathcal{O}(U)$ -module of holomorphic 1-forms on U together with the natural restrictions. Thus the sections of \mathcal{K}_X over U can be identified with holomorphic 1-forms on U . We will adopt this identification for the rest of this thesis.

Similarly, if for an open subset U of X we set

$$\mathcal{O}_D(\mathcal{E})(U) = \{\sigma \in \mathcal{M}(\mathcal{E})(U) \mid \mathrm{div}(\sigma) \geq -D|_U\},$$

then $\mathcal{O}_D(\mathcal{E})(U)$ is an $\mathcal{O}(U)$ -module, and together with the natural restrictions, we get a sheaf $\mathcal{O}_D(\mathcal{E})$ of \mathcal{O} -modules on X . By definition, we have

$$H^0(X, \mathcal{O}_D(\mathcal{E})) = \{\sigma \in H^0(X, \mathcal{M}(\mathcal{E})) \mid \mathrm{div}(\sigma) \geq -D\}.$$

Moreover, if $|D|$ is the associated line bundle to the divisor D , and σ is a section over $|D|$ such that $\mathrm{div}(\sigma) = D$, then the multiplication by σ gives an isomorphism of sheaves

$$\mathcal{O}_D(\mathcal{E}) \longrightarrow \mathcal{O}(|D| \otimes \mathcal{E}) : f \mapsto \sigma f.$$

This induces the isomorphism

$$H^0(X, \mathcal{O}_D(\mathcal{E})) \longrightarrow H^0(X, \mathcal{O}(|D| \otimes \mathcal{E})) : f \mapsto \sigma f, \quad (1.1.5)$$

which we use to identify $H^0(X, \mathcal{O}_D(\mathcal{E}))$ with $H^0(X, \mathcal{O}(|D| \otimes \mathcal{E}))$. We have the following fundamental theorem:

Theorem 1.1.12. *Suppose that X is compact, and let \mathcal{E} be a holomorphic vector bundle over X of rank n . Then $H^0(X, \mathcal{O}(\mathcal{E}))$ is a finite dimensional \mathbb{C} -vector space.*

Definition 1.1.8. The genus g of a compact Riemann surface X is defined to be the dimension of the space of holomorphic 1-forms on X , that is $g = h^0(\mathcal{K}_X)$.

Finally, the Riemann-Roch theorem for holomorphic vector bundles on a compact Riemann surfaces reads as follows:

Theorem 1.1.13. *Suppose that X is compact and having genus g . If \mathcal{E} is a holomorphic vector bundle over X of rank n , then*

$$h^0(\mathcal{E}) - h^0(\mathcal{K}_X \otimes \mathcal{E}^*) = \deg(\mathcal{E}) - n(g - 1).$$

Remark 1.1.14. To ease the notations, in the sequel, the tensor product $\mathcal{L} \otimes \mathcal{E}$ of a vector bundle \mathcal{E} and a line bundle \mathcal{L} over X will be denoted by $\mathcal{L} + \mathcal{E}$.

1.2 Fuchsian groups and their Riemann surfaces.

All the content of this section is taken from the references:

[5]: Chapter 4.

[24]: Chapter 2.

[46] Chapter 1: §1.4 .

[55] Chapter 1.

We give the classification of Möbius transformations, and some standard results on Fuchsian groups and their Riemann surfaces.

1.2.1 Classification of Möbius transformations

Let $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ and a, b, c, d be complex numbers with $ad - bc \neq 0$, then the map $g : \widehat{\mathbb{C}} \rightarrow \widehat{\mathbb{C}}$ defined by

$$g(z) = \frac{az + b}{cz + d}$$

is called a *Möbius transformation*. The set \mathbb{M} of all Möbius transformations equipped with the composition of maps is a group. Moreover, the map given by $A \mapsto g_A$, where

$$g_A(z) = \frac{az + b}{cz + d}, \quad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

is a group homomorphism $\phi : \text{GL}(2, \mathbb{C}) \rightarrow \mathbb{M}$, and

$$\ker \phi = \left\{ \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}, a \neq 0 \right\}.$$

In general, we shall be more concerned with the restriction of ϕ to $\text{SL}(2, \mathbb{C})$. The kernel of this restriction is

$$\ker \phi \cap \text{SL}(2, \mathbb{C}) = \{-I, I\}.$$

Hence \mathbb{M} is isomorphic to $\text{PSL}(2, \mathbb{C}) = \text{SL}(2, \mathbb{C})/\{-I, I\}$.

Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{GL}(2, \mathbb{C})$ and $\text{tr}(A) = a + d$, then the function

$$\frac{\text{tr}^2(A)}{\det(A)}$$

is invariant under the transformation $A \rightarrow \lambda A$, $\lambda \neq 0$, and so it induces a function on \mathbb{M} , namely

$$\text{Tr}(g) = \frac{\text{tr}^2(A)}{\det(A)},$$

where A is any matrix which projects on g . Notice that $\text{Tr}(g)$ is invariant under conjugation.

We now proceed to classify Möbius transformations. First, we introduce some normalized Möbius transformations. For each $k \neq 0$ in \mathbb{C} , we define m_k by

$$m_k(z) = kz \text{ if } k \neq 1,$$

and

$$m_1(z) = z + 1.$$

Notice that for all $k \neq 0$, we have

$$\text{Tr}(m_k) = k + \frac{1}{k} + 2.$$

If $g \neq I$ is any Möbius transformation, we denote by $\alpha \in \widehat{\mathbb{C}}$ its fixed point if it has a unique one, and by α and β in $\widehat{\mathbb{C}}$, $\alpha \neq \beta$, if it has two. Now let h be any Möbius transformation such that

$$h(\alpha) = \infty, \quad h(\beta) = 0, \quad h(g(\beta)) = 1 \text{ if } g(\beta) \neq \beta,$$

then

$$hgh^{-1}(\infty) = \infty, \quad hgh^{-1}(0) = \begin{cases} 0 & \text{if } g(\beta) = \beta \\ 1 & \text{if } g(\beta) \neq \beta. \end{cases}$$

If g fixes α and β , then hgh^{-1} fixes 0 and ∞ and thus $hgh^{-1} = m_k$ for some $k \neq 1$. If g fixes α only, then hgh^{-1} fixes ∞ only and $hgh^{-1}(0) = 1$ and thus $hgh^{-1} = m_1$. Therefore, any nonidentity Möbius transformation is conjugate to one of the standard form m_k .

Definition 1.2.1. Let $g \neq I$ be any Möbius transformation. We say that

1. g is *parabolic* if and only if g is conjugate to m_1 (equivalently g has a unique fixed point in $\widehat{\mathbb{C}}$);
2. g is *loxodromic* if and only if g is conjugate to m_k for some k satisfying $|k| \neq 1$ (g has exactly two fixed points in $\widehat{\mathbb{C}}$);
3. g is *elliptic* if and only if g is conjugate to m_k for some k satisfying $|k| = 1$ (g has exactly two fixed points in $\widehat{\mathbb{C}}$).

It is convenient to subdivide the loxodromic class by reference to invariant discs rather than fixed points.

Definition 1.2.2. Let g be a loxodromic transformation. We say that g is *hyperbolic* if $g(D) = D$ for some open disc or half-plane D in $\widehat{\mathbb{C}}$. Otherwise g is said to be *strictly loxodromic*.

Now, using the fact that $\text{Tr}(g)$ is invariant under conjugation, we can give a complete classification of all Möbius transformations. Indeed, we have the following results.

Theorem 1.2.1. *Let f and g be two Möbius transformations, neither of which is the identity. Then $\text{Tr}(f) = \text{Tr}(g)$ if and only if f and g are conjugate.*

Theorem 1.2.2. *Let $g \neq I$ be any Möbius transformation. Then*

1. *g is parabolic if and only if $\text{Tr}(g) = 4$;*
2. *g is elliptic if and only if $\text{Tr}(g) \in [0, 4)$;*
3. *g is hyperbolic if and only if $\text{Tr}(g) \in (4, +\infty)$;*
4. *g is strictly loxodromic if and only if $\text{Tr}(g) \notin [0, +\infty)$.*

1.2.2 Fuchsian groups

Let $\mathbb{H} = \{z \in \mathbb{C}, \text{Im}(z) > 0\}$ be the Poincaré upper half-plane, $\text{SL}(2, \mathbb{R})$ be the group of real 2×2 matrices with determinant one, $\widehat{\mathbb{R}} := \mathbb{R} \cup \{\infty\}$, and $\widehat{\mathbb{H}} := \mathbb{H} \cup \widehat{\mathbb{R}}$. If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}(2, \mathbb{R})$ and $g_A(z) = \frac{az + b}{cz + d}$ is the associated Möbius transformation, then g_A maps $\widehat{\mathbb{H}}$ into $\widehat{\mathbb{H}}$. Indeed, for z in \mathbb{H}

$$\text{Im}(g_A(z)) = \frac{\text{Im}(z)}{|cz + d|^2}. \quad (1.2.1)$$

Also, it is clear that g_A maps $\widehat{\mathbb{R}}$ into $\widehat{\mathbb{R}}$.

Let $\mathbb{M}_{\mathbb{R}}$ denote the group of real Möbius transformations. As we have seen, the map $\phi : \mathrm{SL}(2, \mathbb{R}) \rightarrow \mathbb{M}_{\mathbb{R}}$ given by $A \mapsto g_A$ is a surjective group homomorphism, and $\ker \phi = \{\pm 1\}$ (1 here stands for I_2 , this will be used for the rest of the thesis). Thus $\mathbb{M}_{\mathbb{R}}$ can be identified with

$$\mathrm{PSL}(2, \mathbb{R}) = \mathrm{SL}(2, \mathbb{R}) / \{\pm 1\}.$$

Recall that an automorphisms of \mathbb{H} is a biholomorphic bijection from \mathbb{H} to \mathbb{H} .

Theorem 1.2.3. *The group of automorphisms of \mathbb{H} is $\mathbb{M}_{\mathbb{R}} = \mathrm{PSL}(2, \mathbb{R})$.*

The group $\mathrm{PSL}(2, \mathbb{C})$ inherits the topology of $\mathrm{SL}(2, \mathbb{C})$, which is a topological space with respect to the standard norm of \mathbb{C}^4 . A subgroup Γ of $\mathrm{PSL}(2, \mathbb{C})$ is called *discrete* if the subspace topology on Γ is the discrete topology. Thus Γ is discrete if and only if: For any sequence T_n in Γ , $T_n \rightarrow T \in \mathrm{GL}(2, \mathbb{C})$ implies $T_n = T$ for all sufficiently large n . Giving $\mathrm{PSL}(2, \mathbb{R}) \subseteq \mathrm{PSL}(2, \mathbb{C})$ the subspace topology of $\mathrm{PSL}(2, \mathbb{C})$, we have

Definition 1.2.3. A discrete subgroup of $\mathrm{PSL}(2, \mathbb{R})$ is called a *Fuchsian* group.

Example 1.2.4. The *modular group* $\mathrm{PSL}(2, \mathbb{Z}) = \mathrm{SL}(2, \mathbb{Z}) / \{\pm 1\}$ and its subgroups are Fuchsian groups.

For γ in $\mathrm{PSL}(2, \mathbb{R})$, let $\mathrm{Fix}(\gamma)$ be the set of fixed points of γ in $\widehat{\mathbb{H}}$. By Theorem 1.2.2, γ cannot be strictly loxodromic, and one can show that

1. If γ is elliptic, then $\mathrm{Fix}(\gamma) = \{z_\gamma\}$, for some z_γ in \mathbb{H} .
2. If γ is parabolic, then $\mathrm{Fix}(\gamma) = \{s_\gamma\}$, for some s_γ in $\widehat{\mathbb{R}}$.
3. If γ is hyperbolic, then $\mathrm{Fix}(\gamma) = \{a_\gamma, b_\gamma\}$, for some a_γ, b_γ in $\widehat{\mathbb{R}}$.

Definition 1.2.4. Let Γ be a Fuchsian subgroup of $\mathrm{PSL}(2, \mathbb{R})$, and z be a point of $\widehat{\mathbb{H}}$. Then:

1. The point z is called cuspidal or a cusp (resp. elliptic, hyperbolic) if it is the fixed point of a parabolic (resp. elliptic, hyperbolic) element of Γ .
2. The set of all elliptic (resp. cuspidal) points of Γ will be denoted by E_Γ (resp. C_Γ).
3. The stabilizer Γ_z of z in Γ is the subgroup defined by

$$\Gamma_z = \{\gamma \in \Gamma \mid \gamma.z = z\}.$$

We have

Proposition 1.2.5. *Let Γ be a Fuchsian subgroup of $PSL(2, \mathbb{R})$, and z be a point of $\widehat{\mathbb{H}}$. Then*

1. *For any $\gamma \in PSL(2, \mathbb{R})$, we have*

$$(\gamma\Gamma\gamma^{-1})_{\gamma.z} = \gamma(\Gamma_z)\gamma^{-1}.$$

In particular, if $\gamma \in \Gamma$, then $\Gamma_{\gamma.z} = \gamma(\Gamma_z)\gamma^{-1}$.

2. Γ_z is cyclic.
3. If $z \in \mathbb{H}$, then Γ_z is finite and $\Gamma_z = \{\pm 1\}$ if z is not elliptic.

Definition 1.2.5. Let z be a point of \mathbb{H} . The cardinality of Γ_z will be called the order of z , and will be denoted by n_z . In particular, $n_z = 1$ if z is not elliptic.

Remark 1.2.6. Since the conjugate of an elliptic (resp. parabolic) element of $PSL(2, \mathbb{C})$ is also elliptic (resp. parabolic), we see that Γ acts on E_Γ and C_Γ .

1.2.3 The Riemann surface of a Fuchsian group.

In this subsection, Γ will be a Fuchsian subgroup of $\mathrm{PSL}(2, \mathbb{R})$, and

$$\mathbb{H}^* := \mathbb{H} \cup C_\Gamma.$$

Since Γ acts on C_Γ , it then acts on \mathbb{H}^* , and so we can form the quotient $X = X_\Gamma := \Gamma \backslash \mathbb{H}^*$. Our aim here is to give X a structure of a Riemann surface.

First we define a topology on \mathbb{H}^* as follows. \mathbb{H} is given its usual topology. For a cusp $s \neq \infty$, we take as a fundamental system of neighborhoods all sets of the form:

$$\{s\} \cup \{ \text{the interior of a circle in } \mathbb{H} \text{ tangent to the real axis at } s \}$$

If ∞ is cusp, then a fundamental system of neighborhoods of ∞ are the sets:

$$\{\infty\} \cup \{z \in \mathbb{H} \mid \Im(z) > c\},$$

for all positive numbers c . Endowed with this topology, \mathbb{H}^* becomes a Hausdorff space, and Γ acts on \mathbb{H}^* by homeomorphisms.

We give X a structure of a Riemann surface in the following way. Let ϕ denotes the natural projection map of \mathbb{H}^* onto $X = \Gamma \backslash \mathbb{H}^*$, p be an arbitrary point of \mathbb{H}^* , and $P = \phi(p)$. Then

Lemma 1.2.7. *There exists an open neighborhood U of p in \mathbb{H}^* such that*

$$\Gamma_p = \{\gamma \in \Gamma \mid (\gamma.U) \cap U \neq \emptyset\}.$$

Hence we have a natural injection of $\Gamma_p \backslash U \rightarrow X$, and $\Gamma_p \backslash U$ is an open neighborhood of P in X . If P is neither an elliptic point nor a cusp, then Γ_p contains only the identity, so that the map $U \rightarrow \Gamma_p \backslash U$ is a homeomorphism. We take the pair $(\Gamma_p \backslash U, \phi^{-1})$ as a member of the local charts defining the complex structure on X .

In case p is elliptic, setting

$$\alpha(z) = \frac{z - p}{z - \bar{p}}, \quad z \in \mathbb{H},$$

then $\alpha(\mathbb{H})$ is the unit disc \mathbb{D} , $\alpha(p) = 0$, and $\alpha\Gamma_p\alpha^{-1}$ is the group

$$\langle \sigma'_p : w \mapsto \zeta_p w \rangle,$$

where ζ_p is a primitive n_p -th root of unity, n_p is the order of p . Then we define a map $\psi : \Gamma_p \backslash U \rightarrow \mathbb{C}$ by

$$\psi(\phi(z)) = \alpha(z)^{n_p}.$$

We see that ψ is a homeomorphism onto an open subset of \mathbb{C} . Thus we include $(\Gamma_p \backslash U, \psi)$ in our local charts.

Finally, if p is a cusp of Γ , and $\alpha \in \mathrm{SL}(2, \mathbb{R})$ is such that $\alpha \cdot s = \infty$, then we have

$$\alpha\Gamma_p\alpha^{-1} = \langle t_h : z \mapsto z + h \rangle,$$

h being a positive number. We define a homeomorphism ψ of $\Gamma_p \backslash U$ onto an open subset of \mathbb{C} by

$$\psi(\phi(z)) = \exp(2\pi i \alpha(z)/h),$$

and include $(\Gamma_p \backslash U, \psi)$ in the local charts.

The atlas formed by these local charts gives X the structure of a Riemann surface.

Definition 1.2.6. Keeping the above notations, we have :

1. A point P in X will be called elliptic, if it corresponds to an elliptic point of Γ . The function $t = w^{n_p}$ will be called the local parameter at P (or at p).
2. A point P in X will be called a cusp, if it corresponds to a cusp of Γ . The function $q_h := \exp(2\pi iz/h)$ will be called the local parameter at P (or at p).

An important class of Fuchsian groups consists of Fuchsian groups of the first kind; they are defined as follows:

Definition 1.2.7. Γ is called a Fuchsian group of the first kind, if $X_\Gamma = \Gamma \backslash \mathbb{H}^*$ is compact. A subgroup Γ_1 of $\mathrm{SL}(2, \mathbb{R})$ is a Fuchsian group of the first kind, if its associated subgroup $\overline{\Gamma}_1 := \{\pm 1\} \backslash \Gamma_1 \cdot \{\pm 1\}$ of $\mathrm{PSL}(2, \mathbb{R})$ is of the first kind.

As we have seen above, Γ acts on the sets E_Γ (resp. C_Γ) of elliptic (resp. cuspidal) points of Γ . Set

$$\mathfrak{E}_\Gamma := \Gamma \backslash E_\Gamma, \quad \mathfrak{S}_\Gamma := \Gamma \backslash C_\Gamma. \quad (1.2.2)$$

Proposition 1.2.8. *If Γ is of the first kind, then \mathfrak{E}_Γ and \mathfrak{S}_Γ are both finite sets. Their cardinality will be respectively denoted by e_Γ and c_Γ .*

Also, we have following useful result:

Proposition 1.2.9. *Suppose that Γ is of the first kind, then any finite index subgroup of Γ is also a Fuchsian group of the first kind.*

Suppose that $p \in \mathbb{H}$ is an elliptic fixed point of Γ . By Proposition 1.2.5, the orders of $\gamma.p$ and p are equal for all $\gamma \in \Gamma$. Therefore, the order of an elliptic point can be lifted to X .

Definition 1.2.8. If $P \in X$ is not a cuspidal point, then we define the order of P to be the order of any point in \mathbb{H} corresponding to P .

1.2.4 Congruence subgroups of $\mathrm{PSL}(2, \mathbb{Z})$

The modular group $\mathrm{PSL}(2, \mathbb{Z})$ and its finite index subgroups form a very rich category of Fuchsian groups of the first kind. Here we list some well known examples.

Let

$$T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, P = ST = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}.$$

Then S is elliptic of order 2 with fixed point i , and P is elliptic of order 3 and fixes $\rho = e^{2\pi i/3}$. Moreover $\mathrm{PSL}(2, \mathbb{Z})$ is generated by S and T , or equivalently, by S and P .

An important class of subgroups of $\mathrm{PSL}(2, \mathbb{Z})$ are the so-called *congruence subgroups*. A subgroup Γ of $\mathrm{PSL}(2, \mathbb{Z})$ is called a congruence subgroup if it contains some principal congruence subgroup $\Gamma(n)$ for some positive integer n , where $\Gamma(n)$ is defined by

$$\Gamma(n) = \{\gamma \in \mathrm{SL}(2, \mathbb{Z}), \gamma \equiv \pm 1 \pmod{n}\} / \{\pm 1\}.$$

The smallest such n is called the *level* of Γ .

One easily shows that $\Gamma(n)$ is a normal subgroup of $\mathrm{PSL}(2, \mathbb{Z})$ of finite index and that the conjugates in $\mathrm{PSL}(2, \mathbb{Z})$ of a congruence subgroup are also congruence subgroups.

Example 1.2.10. Important examples of congruence subgroups of $\mathrm{PSL}(2, \mathbb{Z})$ are:

$$\Gamma_1(n) = \{\gamma \in \mathrm{SL}(2, \mathbb{Z}), \gamma \equiv \pm \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \pmod{n}\} / \{\pm 1\},$$

$$\Gamma_0(n) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbb{Z}), c \equiv 0 \pmod{n} \right\} / \{\pm 1\},$$

$$\Gamma^0(n) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbb{Z}), b \equiv 0 \pmod{n} \right\} / \{\pm 1\}.$$

It is clear that $\Gamma(n) \subseteq \Gamma_1(n) \subseteq \Gamma_0(n)$ and that $\Gamma_0(n)$, $\Gamma_1(n)$, $\Gamma^0(n)$ are of level n . Note that $\Gamma_0(n)$ is conjugate to $\Gamma^0(n)$ by $\begin{pmatrix} n & 0 \\ 0 & 1 \end{pmatrix}$

1.3 Automorphic forms.

In this section we give the definition of automorphic forms and provide some well known examples.

1.3.1 Definition of automorphic forms.

In this subsection we follow The second chapter of [55].

Let $k \in \mathbb{Z}$, $\gamma \in \mathrm{GL}^+(2, \mathbb{R})$, Γ be a Fuchsian, and f be a meromorphic function on \mathbb{H} , we set

$$f|_k \gamma = (\det \gamma)^{\frac{k}{2}} J_\gamma^{-k} f \circ \gamma, \quad (1.3.1)$$

where $J_\gamma(z) = cz + d$ if $\gamma = \begin{pmatrix} * & * \\ c & d \end{pmatrix}$. We have for all $\alpha, \gamma \in \Gamma$,

$$J_{\alpha\gamma}(z) = J_\alpha(\gamma z) J_\gamma(z), \quad z \in \mathbb{H}. \quad (1.3.2)$$

Notice that for k odd, $J_{-\gamma}^k = -J_\gamma^k$, hence $f|_k(-\gamma) = -f|_k \gamma$. If k is even, then $f|_k(-\gamma) = f|_k \gamma$.

Definition 1.3.1. Let Γ be a Fuchsian subgroup of $\mathrm{SL}(2, \mathbb{R})$, and k be an integer. A meromorphic function f on \mathbb{H} is called an unrestricted automorphic form of weight k for Γ , if it satisfies

$$(i) \quad f|_k \gamma = f \text{ for all } \gamma \in \Gamma.$$

If in addition

$$(ii) \quad f \text{ is meromorphic at every cusp of } \Gamma,$$

then f is called an automorphic form of weight k for Γ . When $k = 0$, f is called an automorphic function.

By a cusp of Γ we mean a cusp of its associated subgroup $\bar{\Gamma} := \{\pm 1\} \backslash \Gamma \cdot \{\pm 1\}$ of $\mathrm{PSL}(2, \mathbb{R})$. The last condition in the definition means the following. If Γ has no cusp, then this condition is empty. In case Γ has a cusp s , then the stabilizer of s in $\bar{\Gamma}$ is equal to $\bar{\Gamma}_s$ for some subgroup $\Gamma_s \subseteq \Gamma$. Moreover, if $\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbb{R})$ is such that $\alpha \cdot s = \infty$, then we have

$$\alpha \bar{\Gamma}_s \alpha^{-1} \{\pm 1\} = \left\langle \pm \begin{pmatrix} 1 & h \\ 0 & 1 \end{pmatrix} \right\rangle,$$

with h being a positive real number referred to as the cusp width at s . From (i) we deduce that $f_s := f|_k(\alpha^{-1})$ is invariant under the group $\alpha \Gamma_s \alpha^{-1}$.

Case I: k is even. Here f_s is invariant under the translation

$$t_h : z \longmapsto z + h,$$

hence $f_s = G_s(q_h)$ for some meromorphic function G_s in the punctured disc $: 0 < |q_h| < r$, where r is a positive number, and $q_h := \exp(2\pi iz/h)$. We say that f is meromorphic at the cusp s if G_s is meromorphic at $q_h = 0$, this means that f , when it is nonzero, has a Laurent expansion

$$\sum_{n \geq n_s} a_n q_h^n, \quad 0 < |q_h| < r,$$

where n_s is the order $\mathrm{ord}_s(G_s)$ of G_s at 0. We define the order of f at s by $\mathrm{ord}_s(f) = \mathrm{ord}_0(G_s)$.

Case II: k is odd. If Γ contains -1 , then condition (i) implies that $f = -f$. Hence there is no nontrivial automorphic form of weight k for Γ . Therefore we may assume that $-1 \notin \Gamma$. Then $\alpha \Gamma_s \alpha^{-1}$ is generated by $\begin{pmatrix} 1 & h \\ 0 & 1 \end{pmatrix}$, or by $-\begin{pmatrix} 1 & h \\ 0 & 1 \end{pmatrix}$. We say that the cusp s is *regular* or *irregular*, accordingly. If s is regular, then the meromorphy at s is treated in the same way as in **Case I**. If s is irregular, then $f_s(z + h) = -f_s$. Letting

$$g_s(z) = \exp(-\pi iz/h) f_s, \quad z \in \mathbb{H},$$

then g_s is invariant under the translation $t_h : z \longmapsto z + h$, hence $g_s = G_s(q_h)$ for some meromorphic function G_s in the punctured disc $: 0 < |q_h| < r$,

where r is a positive number, and $q_h := \exp(2\pi iz/h)$. We say that f is meromorphic at the cusp s if G_s is meromorphic at $q_h = 0$. We define the order of f at s by $\text{ord}_s(f) = \text{ord}_0(G_s) + 1/2$.

f is said to be holomorphic (resp. cuspidal) at s if $\text{ord}_s(f) \geq 0$ (resp. $\text{ord}_s(f) > 0$). An unrestricted automorphic form which is meromorphic at any point of \mathbb{H}^* , is called a *meromorphic automorphic form*. An automorphic form which is holomorphic at any point of \mathbb{H}^* , is called a *holomorphic automorphic form*. A holomorphic automorphic form which is cuspidal at any cusp is called a *cusp automorphic form*, or simply a *cusp form*. The vector space of automorphic forms of weight k for Γ will be denoted by $A_k(\Gamma)$. The vector spaces of unrestricted, holomorphic, cusp automorphic forms of weight k for Γ will be respectively denoted by $G_k(\Gamma)$, $M_k(\Gamma)$, and $S_k(\Gamma)$.

If Γ is Fuchsian group of the first kind, then $M_k(\Gamma)$ and $S_k(\Gamma)$ have finite dimensions. Indeed, we have

Theorem 1.3.1. *Let Γ be a Fuchsian group of the first kind, and k be an even integer. Let g be the genus of $X_\Gamma = \Gamma \backslash \mathbb{H}^*$, e and c be the number of elliptic and cuspidal points of X_Γ respectively. If n_1, \dots, n_e are the orders of the elliptic points, then*

$$\dim M_k(\Gamma) = \begin{cases} (k-1)(g-1) + (k/2)c + \sum_{i=1}^e [k(n_i - 1/2n_i)] & (k > 2) \\ g + c - 1 & (k = 2, c > 2) \\ g & (k = 2, c = 0) \\ 1 & (k = 0) \\ 0 & (k < 0), \end{cases}$$

and

$$\dim S_k(\Gamma) = \begin{cases} (k-1)(g-1) + (k/2-1)c + \sum_{i=1}^e [k(n_i - 1/2n_i)] & (k > 2) \\ g & (k = 2) \\ 1 & (k = 0, m = 0) \\ 1 & (k = 0, m > 0) \\ 0 & (k < 0), \end{cases}$$

where $[\cdot]$ denotes the integral part of a real number.

1.3.2 Examples of modular forms

The main reference for this subsection is the first chapter of [38]. When working with the modular group $\mathrm{PSL}(2, \mathbb{Z})$ and its subgroups, the word automorphic is replaced by modular.

We start our examples by the well known Eisenstein series. They are defined for every even integer $k \geq 2$ and $z \in \mathbb{H}$ by

$$G_k(z) = \sum_n \sum_m ' \frac{1}{(nz + m)^k},$$

where the symbol $\sum '$ means that the summation is over the pairs (m, n) different from $(0, 0)$. Notice that this series is not absolutely convergent for $k = 2$. If we normalize the Eisenstein series by letting

$$E_k = 2\zeta(k)G_k,$$

where ζ is the Riemann Zeta function, then we get

$$E_k(z) = 1 - \frac{2k}{B_k} \sum_{n=1}^{\infty} \sigma_{k-1}(n)q^n, \quad q = e^{2\pi iz}.$$

Here B_k is the k -th *Bernoulli* number and $\sigma_{k-1}(n) = \sum_{d|n} d^{k-1}$. The most familiar Eisenstein series are:

$$E_2(z) = 1 - 24 \sum_{n=1}^{\infty} \sigma_1(n)q^n,$$

$$E_4(z) = 1 + 240 \sum_{n=1}^{\infty} \sigma_3(n)q^n,$$

$$E_6(z) = 1 - 504 \sum_{n=1}^{\infty} \sigma_5(n)q^n.$$

For $k \geq 4$, the series E_k are holomorphic modular forms of weight k . The Eisenstein series E_2 is holomorphic on \mathbb{H} and at the cusps, but it is not a modular form as it does not satisfy the modularity condition. The Eisenstein series E_2 is an example of a *quasimodular form* and plays an important role in the theory of vector valued modular forms, see [37]. Moreover, E_2 satisfies

$$E_2(z) = \frac{1}{2\pi i} \frac{\Delta'(z)}{\Delta(z)}, \quad (1.3.3)$$

where Δ is the weight 12 cusp form for $\mathrm{PSL}(2, \mathbb{Z})$ given by

$$\Delta(z) = q \prod_{n \geq 1} (1 - q^n)^{24}.$$

The Eisenstein series satisfy the Ramanujan relations

$$\frac{6}{\pi i} E_2' = E_2^2 - E_4, \quad (1.3.4)$$

$$\frac{3}{2\pi i} E_4' = E_4 E_2 - E_6, \quad (1.3.5)$$

$$\frac{1}{\pi i} E_6' = E_6 E_2 - E_4^2. \quad (1.3.6)$$

The Dedekind j -function given by

$$j(z) = \frac{E_4^3 - E_6^2}{\Delta}$$

is a modular function and it generates the function field of modular functions for $\mathrm{PSL}(2, \mathbb{Z})$.

Chapter 2

Vector-valued automorphic forms and vector bundles

The aim of this chapter is to realize vector-valued automorphic forms as global sections of holomorphic vector bundles. More precisely, given a Fuchsian subgroup Γ of $\mathrm{SL}(2, \mathbb{R})$, a finite-dimensional representation R of Γ , and an integer k such that the pair (R, k) is simple, see Definition 2.1.2, we construct a holomorphic vector bundle $\mathcal{E}_{\Gamma, R, k}$ over the Riemann surface $X_{\Gamma} = \Gamma \backslash \mathbb{H}^*$ associated to Γ . The global meromorphic sections of $\mathcal{E}_{\Gamma, R, k}$ will correspond to vector-valued automorphic forms of weight k and multiplier R . More interestingly, the spaces $M_k(\Gamma, R)$, $S_k(\Gamma, R)$ of holomorphic, cusp vector-valued automorphic forms of weight k and multiplier R , will be respectively isomorphic to $H^0(X_{\Gamma}, \mathcal{O}(\mathcal{E}_{\Gamma, R, -k}))$ and $H^0(X_{\Gamma}, \mathcal{O}(-\mathcal{D}_{\mathfrak{S}, R, -k} + \mathcal{E}_{\Gamma, R, -k}))$, where $\mathcal{D}_{\mathfrak{S}, R, -k}$ is a certain holomorphic line bundle depending on the cuspidal points of X_{Γ} . As an immediate consequence, we deduce that $M_k(\Gamma, R)$ and $S_k(\Gamma, R)$ are finite-dimensional \mathbb{C} -vector spaces if and only if Γ is Fuchsian group of the first kind, see Theorem 2.5.9.

2.1 Notations

For the remainder of this thesis, n is a positive integer, Γ is a Fuchsian subgroup of $\mathrm{SL}(2, \mathbb{R})$, R is a representation of Γ in $\mathrm{GL}(n, \mathbb{C})$. We will denote the matrix $R(\gamma)$, $\gamma \in \Gamma$, by R_γ . R^* will be the adjoint representation to R , that is

$$R_\gamma^* = (R_{\gamma^{-1}})^T,$$

where $(\cdot)^T$ is the transpose operator. $\bar{\Gamma}$ will be the image of Γ in $\mathrm{PSL}(2, \mathbb{R})$, that is $\bar{\Gamma} = \{\pm 1\} \backslash \Gamma \cdot \{\pm 1\}$. E will be the set of elliptic fixed points of $\bar{\Gamma}$ and C the set of its cusps in $\mathbb{R} \cup \{\infty\}$. We define

$$\begin{aligned} \mathbb{H}^* &:= \mathbb{H} \cup C, & \mathfrak{S} &:= \bar{\Gamma} \backslash C, & \mathfrak{E} &:= \bar{\Gamma} \backslash E, \\ X = X(\bar{\Gamma}) &:= \bar{\Gamma} \backslash \mathbb{H}^*, & X' = X'(\bar{\Gamma}) &:= \bar{\Gamma} \backslash (\mathbb{H} - E), \\ Y' = Y'(\bar{\Gamma}) &:= \mathbb{H} - E. \end{aligned}$$

We have an unbranched covering map

$$\pi : Y' \longrightarrow X',$$

and since Γ acts properly and discontinuously on Y' , then the group of covering transformations is

$$\mathrm{Deck}(Y'/X') = \bar{\Gamma}.$$

Moreover, this is a Galois covering in the sense that for all y_1 and y_2 in Y' with $\pi(y_1) = \pi(y_2)$, there exists $\sigma \in \bar{\Gamma}$ such $\sigma(y_1) = y_2$.

Also, when Γ does not contain -1 , we can identify Γ with $\bar{\Gamma}$ via the canonical map, and so Γ will be viewed as a group of matrices as well as a group of Möbius transformations.

If k is an integer, then we define the slash operator $|_k$ on meromorphic maps $F : \mathbb{H} \longrightarrow \mathbb{C}^m$, m being an integer, by

$$F|_k \gamma = J_\gamma^{-k} F \circ \gamma, \quad \gamma \in \mathrm{GL}^+(2, \mathbb{R}), \quad (2.1.1)$$

where $J_\gamma(z) = cz + d$ if $\gamma = \begin{pmatrix} * & * \\ c & d \end{pmatrix}$.

Definition 2.1.1. Let k be an integer. An unrestricted vector-valued automorphic form for Γ of multiplier R and weight k is a meromorphic map $F : \mathbb{H} \rightarrow \mathbb{C}^n$ satisfying

$$F|_k \gamma = R_\gamma F.$$

The \mathbb{C} -vector space of these maps will be denoted by $G_k(\Gamma, R)$.

If $R = R_1 \oplus R_2$ is the direct sum of two representations R_1 and R_2 , then it is clear that $G_k(\Gamma, R) = G_k(\Gamma, R_1) \oplus G_k(\Gamma, R_2)$, and when Γ contains -1 , then after [3], the representation R will be called even (resp. odd) if $R(-1) = I_n$ (resp. $R(-1) = -I_n$). Since -1 commutes with Γ , then modulo a conjugation in $GL(n, \mathbb{C})$, any representation R may be decomposed into a direct sum $R = R_+ \oplus R_-$ of even and odd representations. It follows from Definition 2.1.1 that for an even (resp. odd) representation R there are no nontrivial unrestricted vector-valued automorphic forms of odd (resp. even) weight. Therefore

$$G_k(\Gamma, R) = \begin{cases} G_k(\Gamma, R_+) & \text{if } k \text{ is even,} \\ G_k(\Gamma, R_-) & \text{if } k \text{ is odd.} \end{cases}$$

This shows that it suffices to treat the case when both R and k are even, or when they are both odd.

To avoid trivial cases, we adopt the following definition:

Definition 2.1.2. The pair (R, k) , k being an integer, is called simple if

1. Γ does not contain -1 , or
2. Γ contains -1 , R and k have the same parity.

Remark 2.1.1. In the sequel, we only consider the integers k such that (R, k) is a simple pair. In particular, R is either even or odd when $-1 \in \Gamma$.

Definition 2.1.3. The integer $\varepsilon = \varepsilon(R)$ defined by

$$\varepsilon = \begin{cases} 0 & \text{if } R \text{ is even, or } -1 \notin \Gamma \\ 1 & \text{if } R \text{ is odd} \end{cases}$$

is called the parity index of R .

Notice that ε is the smallest nonnegative integer k such that (R, k) is a simple pair.

- Remark 2.1.2.**
1. In this work, almost all the constructed objects will depend on at least three parameters, namely : Γ , R , and the weight k . When the context is clear, we will only keep the relevant indices.
 2. If the context is clear, an object defined on Y' which is invariant under the covering transformations, will be considered as an object defined on X' .

2.2 The family $\mathcal{E}_{\Gamma, R, k}$ of vector bundles

The goal of this section is to associate to each triplet (Γ, R, k) , with (R, k) being a simple pair, a holomorphic vector bundle $\mathcal{E}_{\Gamma, R, k}$ on X . This will be achieved by constructing a 1-cocycle in $Z^1(\mathcal{U}, \text{GL}(n, \mathcal{O}_X))$, where \mathcal{U} is an open cover of X . Then $\mathcal{E}_{\Gamma, R, k}$ will be its associated holomorphic vector bundle.

2.2.1 The unbranched case.

Let (R, k) be a simple pair, we construct a matrix-valued automorphic form of weight k and multiplier R on the unbranched covering Y' of X' . More precisely, we have

Theorem 2.2.1. *There exists a holomorphic map*

$$\Psi_k = \Psi_{R,k} : Y' \longrightarrow GL(n, \mathbb{C})$$

such that for all $\sigma \in \Gamma$, we have

$$\Psi_k|_{\sigma} = \Psi_k R_{\sigma^{-1}}.$$

Proof. Since $\pi : Y' \longrightarrow X'$ is an unbranched Galois covering, there exists an open covering $\mathcal{U} = (U_i)_{i \in I}$ of X' and homeomorphisms

$$\phi_i = (\pi, \eta_i) : \pi^{-1}(U_i) \longrightarrow U_i \times \bar{\Gamma}$$

which are compatible with the action of $\bar{\Gamma}$ in the sense that $\phi_i(y) = (x, \sigma)$ implies $\phi_i(\tau y) = (x, \tau\sigma)$. In other words, the mapping $\eta_i : \pi^{-1}(U_i) \longrightarrow \bar{\Gamma}$ satisfies $\eta_i(\tau y) = \tau\eta_i(y)$ for all $y \in \pi^{-1}(U_i)$ and $\tau \in \bar{\Gamma}$.

On $Y_i = \pi^{-1}(U_i)$ we define $\Psi_{k,i} : Y_i \longrightarrow GL(n, \mathbb{C})$ by

$$\Psi_{k,i}(y) = J_{\eta_i(y)^{-1}}^{-k}(y) R_{\eta_i(y)^{-1}}, \quad y \in Y_i,$$

which is well defined since the expression

$$J_{\sigma}^{-k} R_{\sigma}$$

is well defined on $\bar{\Gamma}$ for a simple pair (R, k) . Moreover, $\Psi_{k,i}$ is holomorphic since it is locally constant, and so $\Psi_{k,i} \in GL(n, \mathcal{O}(Y_i))$. Now, if $y \in Y_i$ and $\sigma \in \bar{\Gamma}$, then

$$\begin{aligned} \Psi_{k,i}(\sigma y) &= J_{\eta_i(\sigma y)^{-1}}^{-k}(\sigma y) R_{\eta_i(\sigma y)^{-1}} \\ &= J_{\eta_i(y)^{-1}\sigma^{-1}}^{-k}(\sigma y) R_{\eta_i(y)^{-1}\sigma^{-1}} \\ &= J_{\sigma^{-1}}^{-k}(\sigma y) J_{\eta_i(y)^{-1}}^{-k}(y) R_{\eta_i(y)^{-1}} R_{\sigma^{-1}} \\ &= J_{\sigma}^k(y) \Psi_{k,i}(y) R_{\sigma^{-1}}. \end{aligned}$$

Hence, for $\sigma \in \Gamma$,

$$\Psi_{k,i}|_k \sigma = \Psi_{k,i} R_{\sigma^{-1}} \quad \text{on } Y_i.$$

Therefore, $\Psi_{k,i}$ has the required property on Y_i . If we set

$$F_{k,ij} = F_{R,k,ij} = \Psi_{k,i} \Psi_{k,j}^{-1} \in \text{GL}(n, \mathcal{O}(Y_i \cap Y_j)),$$

then for all $y \in Y_i \cap Y_j$ we have $F_{k,ij} \sigma = F_{k,ij}$, that is, the $F_{k,ij}$ is invariant under covering transformations and hence may be considered as an element of $\text{GL}(n, \mathcal{O}(U_i \cap U_j))$. Thus, we have a cocycle

$$(F_{k,ij}) \in Z^1(\mathcal{U}, \text{GL}(n, \mathcal{O})).$$

As X' is a noncompact Riemann surface, there exist elements

$$F_{k,i} = F_{R,k,i} \in \text{GL}(n, \mathcal{O}(U_i))$$

such that

$$F_{k,ij} = F_{k,i} F_{k,j}^{-1} \quad \text{on } U_i \cap U_j, \quad (2.2.1)$$

see [13]. We now look at the $F_{k,i}$ as elements of $\text{GL}(n, \mathcal{O}(Y_i))$ that are invariant under covering transformations and set

$$\tilde{\Psi}_i = F_{k,i}^{-1} \Psi_{k,i} \in \text{GL}(n, \mathcal{O}(Y_i)).$$

Then, for every $\sigma \in \Gamma$, we have

$$\tilde{\Psi}_{k,i}|_k \sigma = (F_{k,i} \sigma)^{-1} \Psi_{k,i}|_k \sigma = F_i^{-1} \Psi_{k,i} R_{\sigma^{-1}} = \tilde{\Psi}_i R_{\sigma^{-1}}.$$

Moreover, on $Y_i \cap Y_j$ we have

$$(\tilde{\Psi}_{k,i})^{-1} \tilde{\Psi}_{k,j} = \Psi_{k,i}^{-1} F_{k,i} F_{k,j}^{-1} \Psi_{k,j} = \Psi_{k,i}^{-1} F_{k,ij} \Psi_{k,j} = \Psi_{k,i}^{-1} \Psi_{k,i} \Psi_{k,j}^{-1} \Psi_{k,j} = 1.$$

Thus, the $\tilde{\Psi}_{k,i}$'s define a global function $\Psi_k \in \text{GL}(n, \mathcal{O}(Y'))$ with

$$\Psi_k|_k \sigma = \Psi_k R_{\sigma^{-1}}, \quad \sigma \in \Gamma.$$

□

We see that the $\Psi_{R,k}$ constructed above depends on the choice of the elements $F_{R,k,i} \in \mathrm{GL}(n, \mathcal{O}(U_i))$, $i \in I$, let us call it $\Psi_{k,F}$. Suppose that $G_{R,k,i} \in \mathrm{GL}(n, \mathcal{O}(U_i))$, $i \in I$, is another choice, then by (2.2.1), we have

$$F_{k,i}F_{k,j}^{-1} = F_{k,ij} = G_{k,i}G_{k,j}^{-1} \text{ on } U_i \cap U_j.$$

Hence, the element $A_{F,G} \in \mathrm{GL}(n, \mathcal{O}(X'))$ given by

$$A_{F,G}|_{U_i} = G_{k,i}^{-1}F_{k,i} \text{ on } U_i$$

is well-defined and we

$$\Psi_{k,G} = A_{F,G}\Psi_{k,F}. \quad (2.2.2)$$

Now, we take an arbitrary choice of the elements $F_{R,k,i} \in \mathrm{GL}(n, \mathcal{O}(U_i))$, $i \in I$, and we fix it for the rest of the thesis. Its corresponding map will be our $\Psi_{R,k}$.

Remark 2.2.2. From the proof of Theorem 2.2.1, we see that

$$F_{R,k,i}^*(F_{R,k,j}^*)^{-1} = F_{R^*,-k,i}F_{R^*,-k,j}^{-1} \text{ on } U_i \cap U_j,$$

and hence, as in the above discussion, we have

$$(\Psi_{R,k})^* = \Delta_{R,k}\Psi_{R^*,-k},$$

for some $\Delta_{R,k} \in \mathrm{GL}(n, \mathcal{O}(X'))$.

2.2.2 The elliptic case.

Now, let e be an elliptic fixed point of $\bar{\Gamma}$, then the stabilizer of e in $\bar{\Gamma}$ is equal to $\bar{\Gamma}_e$ for some subgroup $\Gamma_e \subseteq \Gamma$. Moreover, $\bar{\Gamma}_e = \langle \sigma_e \rangle$ with $\sigma_e = \bar{\gamma}_e$, for some γ_e in Γ_e . If \bar{e} denotes the complex conjugate of e , then we have a character μ_e on Γ_e defined by

$$\mu_e(\gamma) = J_\gamma(\bar{e}), \quad \gamma \in \Gamma_e, \quad (2.2.3)$$

such that $\mu_e(-1) = -1$, when $-1 \in \Gamma$. Also, the holomorphic function on \mathbb{H} given by

$$f_e(z) = \frac{1}{z - \bar{e}}, \quad z \in \mathbb{H},$$

verifies

$$f_e|_1\gamma = \mu_e(\gamma)f_e, \quad \gamma \in \Gamma_e.$$

Lemma 2.2.3. *There exists a meromorphic map $\Psi_{k,e} = \Psi_{R,k,e}$ in \mathbb{H} , having a pole only at e , such that for all $\gamma \in \Gamma_e$, we have*

$$\Psi_{k,e}|_k\gamma = \Psi_{k,e}R_{\gamma^{-1}}.$$

Proof. If

$$\alpha(z) = \frac{z - e}{z - \bar{e}}, \quad z \in \mathbb{H},$$

then $\alpha(\mathbb{H})$ is the unit disc \mathbb{D} , $\alpha(e) = 0$, and $\alpha\bar{\Gamma}_e\alpha^{-1}$ is the group

$$\langle \sigma'_e : w \mapsto \zeta_e w \rangle$$

where ζ_e is a primitive n_e -th root of unity, n_e being the order of e .

Let $\tilde{R} = \tilde{R}_{k,e} := \mu_e^{-k}R$, then \tilde{R} is a well defined representation of $\bar{\Gamma}_e$. Since $\tilde{R}_{\gamma_e}^{n_e} = 1$, then \tilde{R}_{γ_e} is diagonalizable and for some $A_{k,e} = A_{R,k,e} \in \text{GL}(n, \mathbb{C})$ we have

$$A_{k,e}\tilde{R}_{\sigma_e}A_{k,e}^{-1} = \text{diag}(\lambda_1, \dots, \lambda_n), \quad \lambda_j^{n_e} = 1.$$

Write $\lambda_j = \zeta_e^{m_{k,j}}$, $0 \leq m_{k,j} \leq n_e - 1$, and for $w \in \mathbb{D}^*$ set

$$\Phi_{k,e}(w) = \Phi_{R,k,e}(w) := \text{diag}(w^{-m_{k,1}}, \dots, w^{-m_{k,n}}).$$

We have

$$\begin{aligned} \Phi_{k,e}(\sigma'_e w) &= \Phi_{k,e}(\zeta_e w) \\ &= \text{diag}(\zeta_e^{-m_{k,1}} w^{-m_{k,1}}, \dots, \zeta_e^{-m_{k,n}} w^{-m_{k,n}}) \\ &= \text{diag}(\lambda_1^{-1} w^{-m_{k,1}}, \dots, \lambda_n^{-1} w^{-m_{k,1}}) \\ &= \Phi_{k,e}(w) \text{diag}(\lambda_1, \dots, \lambda_n) \\ &= \Phi_{k,e}(w) A_{k,e} \tilde{R}_{\sigma_e^{-1}} A_{k,e}^{-1}. \end{aligned}$$

Therefore,

$$\Phi_{k,e}(\sigma'_e)A_{k,e} = \Phi_{k,e}A_{k,e}\tilde{R}_{\sigma^{-1}},$$

If we set

$$\tilde{\Phi}_{k,e} = (\Phi_{k,e}\alpha)A_{k,e},$$

then for all $\sigma \in \overline{\Gamma}_e$,

$$\tilde{\Phi}_{k,e}(\sigma) = \tilde{\Phi}_{k,e}\tilde{R}_{\sigma^{-1}}.$$

Now, let

$$\Psi_{k,e} = f_e^k \tilde{\Phi}_{k,e},$$

then for all $\gamma \in \Gamma_e$,

$$\Psi_{k,e}|_k \gamma = \mu_e^k(\gamma) f_e^k \tilde{\Phi}_{k,e} \tilde{R}_{(\gamma)^{-1}} = \mu_e^k(\gamma) \tilde{\Psi}_{k,e} \tilde{R}_{(\gamma)^{-1}}.$$

But for $\gamma \in \Gamma_e$ we have

$$\tilde{R}_{(\gamma)^{-1}} = \mu_e^{-k}(\gamma^{-1}) R_{\gamma^{-1}}.$$

Hence for all $\gamma \in \Gamma_e$, we have

$$\Psi_{k,e}|_k \gamma = \Psi_{k,e} R_{\gamma^{-1}}$$

as desired. □

Notice that the construction of $\Psi_{R,k,e}$ depends on the choice of the $A_{R,k,e}$. We fix a choice for the matrix $A_{R,k,e}$ once for all, and its corresponding map will be our $\Psi_{R,k,e}$.

Remark 2.2.4. The relationship between $(\Psi_{R,k,e})$ and $\Psi_{R^*,-k,e}$ is given by the following: Since

$$A_{R,k,e} \left(\mu_e^{-k}(\gamma_e) R_{\gamma_e} \right) A_{R,k,e}^{-1} = \text{diag}(\zeta_e^{m_{R,k,1}}, \dots, \zeta_e^{m_{R,k,n}}),$$

where $(m_{R,k,j})_j$ is a sequence of integers in $[0, n_e)$, then

$$A_{R,k,e}^* \left(\mu_e^k(\gamma_e) R_{\gamma_e}^* \right) (A_{R,k,e}^*)^{-1} = \text{diag}(\zeta_e^{n_e - m_{R,k,1}}, \dots, \zeta_e^{n_e - m_{R,k,n}}).$$

But

$$\text{diag}(\zeta_e^{n_e - m_{R,k,1}}, \dots, \zeta_e^{n_e - m_{R,k,n}}) = A_{R^*, -k, e}(\mu_e^k(\gamma_e) R_{\gamma_e}^*) A_{R^*, -k, e}^{-1}.$$

This implies that the matrix

$$B_{R,k,e} = A_{R,k,e}^* A_{R^*, -k, e}^{-1}$$

commutes with

$$\text{diag}(\zeta_e^{n_e - m_{R,k,1}}, \dots, \zeta_e^{n_e - m_{R,k,n}}),$$

hence with $\Phi_{R^*, -k, e}$, and we have

$$(\Psi_{R,k,e})^* = (\alpha(z))^{n_e} B_{R,k,e} \Psi_{R^*, -k, e}.$$

Definition 2.2.1. With the above notations, we define the k -order of R at e to be

$$\nu_{k,e}(R) = -(1/n_e) \sum_{j=1}^n m_{R,k,j},$$

where the integers $0 \leq m_{R,k,j} \leq n_e - 1$ are such that

$$A_{R,k,e}(\mu_e^{-k}(\gamma_e) R_{\gamma_e}) A_{R,k,e}^{-1} = \text{diag}(\zeta_e^{m_{R,k,1}}, \dots, \zeta_e^{m_{R,k,n}}).$$

Since it only depends on the class of e modulo Γ , we define the k -order of R at an elliptic point $P \in X$ to be

$$\nu_{k,P}(R) = \nu_{k,e}(R),$$

with $e \in \mathbb{H}$ is an elliptic point corresponding to P .

2.2.3 The cuspidal case.

Similarly, If s is a cusp of $\bar{\Gamma}$, then the stabilizer of s in $\bar{\Gamma}$ is equal to $\bar{\Gamma}_s$ for some subgroup $\Gamma_s \subseteq \Gamma$. Also, $\bar{\Gamma}_s = \langle \bar{\gamma}_s \rangle$ with γ_s in Γ_s , and we have a character μ_s on Γ_s defined by

$$\mu_s(\gamma) = J_\gamma(s), \quad \gamma \in \Gamma_s, \quad (2.2.4)$$

such that $\mu_s(-1) = -1$, when $-1 \in \Gamma$. Notice that for $\gamma = \begin{pmatrix} a_\gamma & b_\gamma \\ c_\gamma & d_\gamma \end{pmatrix} \in \Gamma_s$, a simple computation shows that

$$s = (a_\gamma - d_\gamma)/2c_\gamma.$$

This implies that

$$\mu_s(\gamma) = \text{Trace}(\gamma)/2, \quad \gamma \in \Gamma_s, \quad (2.2.5)$$

which is also valid for $s = \infty$. Since the trace of γ is ± 2 , we see that

$$\mu_s(\gamma) = \pm 1 \quad \gamma \in \Gamma_s.$$

Moreover, the holomorphic function on \mathbb{H} defined by

$$f_s(z) = \begin{cases} \frac{1}{z-s}, & z \in \mathbb{H}, \quad \text{if } s \neq \infty \\ 1 & \text{if } s = \infty, \end{cases}$$

verifies

$$f_s|_1\gamma = \mu_s(\gamma)f_s, \quad \gamma \in \Gamma_s.$$

If $\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}(2, \mathbb{R})$ is such that $\alpha \cdot s = \infty$, then, using the notations of [55], we have

$$\alpha \bar{\Gamma}_s \alpha^{-1} \cdot \{\pm 1\} = \langle t_h : z \mapsto z + h \rangle,$$

with h being a positive real number referred to as the cusp width of s . Also, from $s = \alpha^{-1} \cdot \infty = -d/c$, we have $f_s = c_\alpha / J_\alpha$ for some nonzero constant $c_\alpha \in \mathbb{C}$.

Lemma 2.2.5. [50] *There exists a holomorphic map $\Psi_{k,s} = \Psi_{R,k,s}$ in \mathbb{H} , such that for all $\gamma \in \Gamma_s$, we have*

$$\Psi_{k,s}|_k\gamma = \Psi_{k,s}R_{\gamma^{-1}}.$$

Proof. Let $\tilde{R} = \tilde{R}_{k,s} := \mu_s^{-k} R$, then \tilde{R} is a well-defined representation of $\overline{\Gamma}_s$. Take $B_{k,s} = B_{R,k,s} \in \mathcal{M}(n, \mathbb{C})$ such that

$$\tilde{R}_{\overline{\gamma}_s} = \exp(2\pi i B_{k,s}).$$

If we set

$$\Phi_{k,s}(z) = \exp\left(-2\pi i \frac{z}{h} B_{k,s}\right), \quad z \in \mathbb{H},$$

we have

$$\begin{aligned} \Phi_{k,s}(t_h z) &= \exp\left(-2\pi i \frac{z+h}{h} B_{k,s}\right) \\ &= \exp\left(-2\pi i \frac{z}{h} B_{k,s} - 2\pi i B_{k,s}\right) \\ &= \tilde{R}_{(\overline{\gamma}_s)^{-1}} \Phi_{k,s}(z) \\ &= \Phi_{k,s}(z) \tilde{R}_{(\overline{\gamma}_s)^{-1}}. \end{aligned}$$

Hence if

$$\Psi_{k,s} = f_s^k \Phi_{k,s} \alpha = c_\alpha^k \Phi_{k,s}|_k \alpha,$$

then $\Psi_{k,s}$ has the required property. □

Remark 2.2.6. Note that $\Psi_{k,s}$ depends on the choice of $B_{k,s}$. In the sequel we take $B_{k,s}$ such that

$$0 \leq \Re(\lambda) < 1$$

for every eigenvalue λ of $B_{k,s}$, which is always possible, see [18]. With this choice, one can check that [18, 16]

$$B_{R^*, -k, s} = I_n - (B_{R, k, s})^t.$$

Therefore

$$(\Phi_{R, k, s})^* = q_h(\alpha) \Phi_{R^*, -k, s},$$

where $q_h(\alpha(z)) = \exp(2\pi i \alpha(z)/h)$, $z \in \mathbb{C}$.

Definition 2.2.2. Let $B_{k,s} = B_{R,k,s} \in \mathcal{M}(n, \mathbb{C})$ such that

$$\mu_s^{-k}(\gamma_s)R_{\gamma_s} = \exp(2\pi i B_{k,s}),$$

and for all eigenvalues λ of $B_{k,s}$

$$0 \leq \Re(\lambda) < 1.$$

Then the number

$$\nu_{k,R}(s) = -\text{Trace}(B_{k,s})$$

is called the k -order of R at s . Since it only depends on the class of s modulo Γ , we define the k -order of R at a cuspidal point $P \in X$ to be

$$\nu_{k,P}(R) = \nu_{k,s}(R),$$

where $s \in \mathbb{H}^*$ is a cusp corresponding to P .

Definition 2.2.3. If for a point $P \in X \setminus (\mathfrak{E} \cup \mathfrak{S})$ we set $\nu_{k,P}(R) = 0$, then the k -divisor of R is defined by

$$D_{R,k} = \sum_{P \in X} \nu_{k,P}(R) P.$$

2.2.4 Construction of the 1-cocycle.

We now come to the main construction in this section. Write

$$\mathfrak{E} \cup \mathfrak{S} = \{\bar{a}_i\}_{i \geq 1}$$

for the discrete closed set of classes of cusps and elliptic fixed points in X . In particular, they correspond to inequivalent points a_i in \mathbb{H}^* . For each \bar{a}_i we choose a neighborhood in X in the following way:

If \bar{a}_i is a cusp, let U_i be a neighborhood of \bar{a}_i in X given by

$$U_i = (\overline{\Gamma_{a_i}} \setminus D_{a_i}) \cup \{\bar{a}_i\},$$

where D_{a_i} is a horocycle in \mathbb{H} tangent at a_i [55] (if $a_i = \infty$, this horocycle is a half-plane). Here $\overline{\Gamma_{a_i}}$ is infinite cyclic.

If \bar{a}_i is an elliptic fixed point, then U_i is given by

$$U_i = \overline{\Gamma_{a_i}} \setminus D_{a_i},$$

where D_{a_i} is an open disc in \mathbb{H} centered at a_i . Here $\overline{\Gamma_{a_i}}$ is a finite cyclic group.

Further, these neighborhoods can be taken such that for $\bar{a}_i \neq \bar{a}_j$, we have $U_i \cap U_j = \emptyset$. Also, for each \bar{a}_i choose a chart z for X such that $z(U_i) = \mathbb{D}$ and $z(a_i) = 0$ where \mathbb{D} is the unit disc.

Set $U_0 = X'$, $V_0 = \mathbb{H} - E = \pi^{-1}(X') = Y'$, and $V_i = \pi^{-1}(U_i - \{\bar{a}_i\})$, $i \geq 1$. Then $\mathcal{U} = (U_i)_{i \geq 0}$ is an open covering of X , and we have

$$V_i = \bigsqcup_{\bar{\gamma} \in \overline{\Gamma_{a_i}}} Z_\gamma,$$

where γ is a representative of $\bar{\gamma}$ in $\overline{\Gamma_{a_i}}$, and Z_γ is a connected component of V_i . If $\Psi_{k,0} = \Psi_{R,k}$ is the map constructed in Theorem 2.2.1, then we have the following.

Proposition 2.2.7. [50] *If \bar{a}_i is a cusp, then there exists a holomorphic map*

$$\Psi_{k,i} : V_i \longrightarrow GL(n, \mathbb{C})$$

such that $\Psi_{k,i} \Psi_{k,0}^{-1}$ is invariant under $\text{Deck}(Z_\gamma/U_i - \{a_i\})$ for all connected components Z_γ of V_i .

Proof. We have

$$V_i = \overline{\Gamma} D_{a_i} = \bigsqcup_{\bar{\gamma} \in \overline{\Gamma_{a_i}}} \gamma D_{a_i} = \bigsqcup_{\bar{\gamma} \in \overline{\Gamma_{a_i}}} D_{\gamma a_i}.$$

Here we have a disjoint union of connected components $Z_\gamma = \gamma D_{a_i}$ of V_i and γ is a chosen representative of $\bar{\gamma}$ in $\overline{\Gamma_{a_i}}$. Then

$$Z_\gamma \longrightarrow U_i - \{\bar{a}_i\}$$

is a universal covering with the covering transformations given by the fundamental group

$$\pi_1(U_i - \{\bar{a}_i\}) = \overline{\Gamma_{\gamma a_i}}.$$

By Theorem 2.2.1 and Lemma 2.2.5, the maps $\Psi_{k,\gamma a_i}$ and $\Psi_{k,0}$ on Z_γ have the same automorphic behaviour with respect to $\Gamma_{\gamma a_i}$, and so $\Psi_{k,\gamma a_i} \Psi_{k,0}^{-1}$ is invariant under $\Gamma_{\gamma a_i}$, hence under $\overline{\Gamma_{\gamma a_i}}$. The map $\Psi_{k,i}$ defined on V_i by

$$\Psi_{k,i}|_{Z_\gamma} = \Psi_{k,\gamma a_i}$$

is then a holomorphic map on V_i such that $\Psi_{k,i} \Psi_{k,0}^{-1}$ is invariant under $\text{Deck}(Z_\gamma/U_i - \{a_i\})$ for all connected components Z_γ of V_i .

□

Proposition 2.2.8. [50] *If \bar{a}_i is an elliptic fixed point, then there exists a holomorphic map*

$$\Psi_{k,i} : V_i \longrightarrow GL(n, \mathbb{C})$$

such that $\Psi_{k,i} \Psi_{k,0}^{-1}$ is invariant under $\text{Deck}(Z_\gamma/U_i - \{a_i\})$ for all connected components Z_γ of V_i . Moreover, $\Psi_{k,i}$ is meromorphic at each point of $\pi^{-1}\{\bar{a}_i\}$.

Proof. If \bar{a}_i is the class of an elliptic fixed point a_i in \mathbb{H} , then

$$U_i - \{\bar{a}_i\} = \overline{\Gamma_{a_i}} \backslash D_{a_i}^*,$$

where $D_{a_i}^* = D_{a_i} - \{a_i\}$ is the punctured disc. Here, the stabilizer $\overline{\Gamma_{a_i}}$ of cyclic of finite order k_i . Furthermore, we have

$$V_i = \bigsqcup_{\bar{\gamma} \in \overline{\Gamma_{a_i}}} \gamma D_{a_i}^* = \bigsqcup_{\bar{\gamma} \in \overline{\Gamma_{a_i}}} Z_\gamma,$$

where the $Z_\gamma = \gamma D_{a_i}^*$ are the connected components of V_i . In the meantime, the group of covering transformations of the unbranched covering $Z_\gamma \longrightarrow D_{a_i}^*$ is given by the fundamental group

$$\pi_1(U_i - \{\bar{a}_i\}) = \overline{\Gamma_{\gamma a_i}}.$$

Indeed, it is isomorphic to the covering

$$\begin{aligned} D^* &\longrightarrow D^* \\ z &\longmapsto z^{k_i} \end{aligned}$$

for which the covering transformations are given by the group $\langle \sigma_i : z \mapsto \zeta_i z \rangle$ where ζ_i is a primitive k_i -th root of unity, k_i is the order of a_i .

By Theorem 2.2.1 and Lemma 2.2.3, the maps $\Psi_{k,\gamma a_i}$ and $\Psi_{k,0}$ on Z_γ have the same automorphic behavior with respect to $\Gamma_{\gamma a_i}$, and so $\Psi_{k,\gamma a_i} \Psi_{k,0}^{-1}$ is invariant under $\Gamma_{\gamma a_i}$, hence under $\overline{\Gamma_{\gamma a_i}}$. The map $\Psi_{k,i}$ defined on V_i by

$$\Psi_{k,i}|_{Z_\gamma} = \Psi_{k,\gamma a_i}$$

is then a holomorphic map on $Z_\gamma = \gamma D_{a_i}^*$, and meromorphic at $\gamma a_i \in \gamma D_{a_i}$. Hence, $\Psi_{k,i}$ is holomorphic in V_i and meromorphic at each point of $\pi^{-1}\{\bar{a}_i\}$. Also, $\Psi_{k,i} \Psi_{k,0}^{-1}$ is invariant under $\text{Deck}(Z_\gamma/U_i - \{a_i\})$ for all connected components of Z_γ of V_i . \square

Now define the 1-cocycle

$$F_{k,ij} = \Psi_{k,i} \Psi_{k,j}^{-1} \in \text{GL}(n, \mathcal{O}(V_i \cap V_j)) \text{ if } i \neq j \text{ and } F_{k,ii} = \text{id}. \quad (2.2.6)$$

Theorem 2.2.9. [50] *The 1-cocycle $(F_{k,ij})$ defines a 1-cocycle $(\tilde{F}_{k,ij})$ in $Z^1(\mathcal{U}, \text{GL}(n, \mathcal{O}))$ such that $F_{k,ij} = \pi^* \tilde{F}_{k,ij}$.*

Proof. We need to prove that the $F_{k,ij}$'s defined in (2.2.6) can be considered as elements of $\text{GL}(n, \mathcal{O}(U_i \cap U_j))$. By construction, for $i \neq 0$ and $j \neq 0$ we have

$$U_i \cap U_j = V_i \cap V_j = \emptyset, \quad U_0 \cap U_i = U_i - \{a_i\}, \quad V_0 \cap V_i = V_i.$$

Thus we only need to prove that $\Psi_{k,0} \Psi_{k,i}^{-1} \in \text{GL}(n, \mathcal{O}(V_i))$ defines an element of $\text{GL}(n, \mathcal{O}(U_i - \{a_i\}))$, that is to show that $\Psi_{k,0} \Psi_{k,i}^{-1}$ is invariant under the action of $\text{Deck}(Z_\gamma/U_i - \{a_i\})$ which is a direct consequence of

Proposition 2.2.7 and Proposition 2.2.8. Therefore, there exists an element $(\widetilde{F}_{k,ij}) \in Z^1(\mathcal{U}, \mathrm{GL}(n, \mathcal{O}))$ such that $F_{k,ij} = \pi^* \widetilde{F}_{k,ij}$.

□

According to Theorem 1.1.2, a 1-cocycle in $Z^1(\mathcal{U}, \mathrm{GL}(n, \mathcal{O}))$ gives rise to a holomorphic vector bundle on X .

Definition 2.2.4. We define $\mathcal{E}_{\Gamma, R, k}$ to be the holomorphic vector bundle $p : \mathcal{E}_{\Gamma, R, k} \rightarrow X$ of rank n whose transition functions are the $(\widetilde{F}_{k,ij})$. If the context is clear $\mathcal{E}_{\Gamma, R, k}$ will be denoted by \mathcal{E}_k .

If R is the trivial character χ_0 sending all elements of Γ to 1, then $\mathcal{E}_{\Gamma, \chi_0, k}$ is a line bundle which will be denoted by $\mathcal{L}_{\chi_0, k} = \mathcal{L}_k$. Since χ_0 is an even representation, then \mathcal{L}_k is defined only for even k 's when $-1 \in \Gamma$, and for any k when $-1 \notin \Gamma$.

2.3 Behavior at the cusps

In this section we shall make explicit the behavior of unrestricted vector-valued automorphic forms at cusps.

Let s be a cusp of $\overline{\Gamma}$. As we have seen in §2.2.3, the stabilizer of s in $\overline{\Gamma}$ is equal to $\overline{\Gamma}_s$ for some subgroup $\Gamma_s \subseteq \Gamma$. Also, $\overline{\Gamma}_s = \langle \overline{\gamma}_s \rangle$ with γ_s in Γ_s . Moreover, if $\alpha \in \mathrm{SL}(2, \mathbb{R})$ is such that $\alpha \cdot s = \infty$, then we have

$$\alpha \overline{\Gamma}_s \alpha^{-1} = \langle t_h : z \mapsto z + h \rangle,$$

h being the cusp width of s .

Let F be an unrestricted vector-valued automorphic form for Γ of multiplier R and weight k , $F_s := \Psi_{-k, s} F$, and $\widetilde{F}_s := F_s \alpha^{-1}$. By Lemma 2.2.5, F_s is invariant under Γ_s , hence under $\overline{\Gamma}_s$. Therefore \widetilde{F}_s is invariant under $\alpha \overline{\Gamma}_s \alpha^{-1}$, and so

$$\widetilde{F}_s t_h = \widetilde{F}_s.$$

Hence $\widetilde{F}_s(z) = G_s(q_h)$ for some meromorphic map

$$G_s : \{q_h; 0 < |q_h| < r\} \longrightarrow \mathbb{C}^n,$$

where r is a positive number, and $q_h := \exp(2\pi iz/h)$. We say that F is meromorphic at the cusp s if G_s is meromorphic at $q_h = 0$, this means that G_s , when it is nonzero, has a Laurent expansion

$$\sum_{n \geq n_s} a_n q_h^n, \quad 0 < |q_h| < r,$$

where n_s is the order $\text{ord}_s(G_s)$ of G_s at 0. We define the order of F at s by

$$\text{ord}_s(F) := \text{ord}_0(G_s).$$

F is said to be holomorphic at s if $\text{ord}_s(F) \geq 0$. An unrestricted vector-valued automorphic form which is meromorphic (resp. holomorphic) at every point of \mathbb{H}^* , is called a *vector-valued automorphic form* (resp. *holomorphic vector-valued automorphic form*). The vector space of vector-valued automorphic forms for Γ of multiplier R and weight k will be denoted by $A_k(\Gamma, R)$, and its subspace of holomorphic vector-valued automorphic forms will be denoted by $M_k(\Gamma, R)$.

2.4 The divisor of a vector-valued automorphic form

The goal of this section is to associate to each vector-valued automorphic form F a divisor $\text{div}(F)$ defined on the Riemann surface X .

Suppose that F is a vector-valued automorphic form for Γ of multiplier R and weight k . Then by Remark 1.1.5, we have $\text{ord}_{\gamma w}(F) = \text{ord}_w(F)$ for any $w \in \mathbb{H}^*$, $\gamma \in \Gamma$. This means that the order of F can be lifted to X .

If $P \in X$ corresponds to a point $w \in \mathbb{H}$, then set $\nu_P(F) = \text{ord}_w(F)/n_w$, where n_w is the order of the point w , see Definition 1.2.5. If $P \in X$ corresponds to a cusp s , then set $\nu_P(F) = \text{ord}_s(F)$. We define the divisor of F on

X to be the formal sum

$$\operatorname{div}(F) = \sum_{P \in X} \nu_P(F) P. \quad (2.4.1)$$

From this we see that F is holomorphic in \mathbb{H}^* exactly when $\operatorname{div}(F) \geq 0$.

Proposition 2.4.1. *we have*

$$M_k(\Gamma, R) = \{ F \in A_k(\Gamma, R) \mid \operatorname{div}(F) \geq 0 \}.$$

□

As usual, one can define a cusp vector-valued automorphic form of multiplier R and weight k , to be an element F of $M_k(\Gamma, R)$ vanishing at all the cusps of Γ . More precisely, suppose that s is cusp, then using notations of Lemma 2.2.5, and Definition 2.2.2, we have

$$\mu_s^k(\gamma_s) R_{\gamma_s} = \exp(2\pi i B_{R,-k,s}),$$

such that the eigenvalues $\lambda_{-k,s,j}$ of $B_{R,k,s}$ satisfy the condition

$$0 \leq \Re(\lambda_{-k,s,j}) < 1, \quad 0 \leq j \leq n.$$

In terms of the local parameter q_h at s , we have

$$F|_k \alpha^{-1} = c_\alpha^k \exp\left(2\pi i \frac{z}{h} B_{R,-k,s}\right) G_s(q_h). \quad (2.4.2)$$

By the Jordan decomposition theorem, we can write $B_{R,-k,s}$ in the form

$$B_{R,-k,s} = D_{R,-k,s} + N_{R,-k,s},$$

where $D_{R,-k,s}$ is diagonalizable matrix, $N_{R,-k,s}$ is nilpotent matrix, and

$$D_{R,-k,s} N_{R,-k,s} = N_{R,-k,s} D_{R,-k,s}.$$

Therefore

$$\exp\left(2\pi i \frac{z}{h} B_{R,-k,s}\right) = \exp\left(2\pi i \frac{z}{h} N_{R,-k,s}\right) \exp\left(2\pi i \frac{z}{h} D_{R,-k,s}\right).$$

Let n_R be the smallest integer such that $N_{R,-k,s}^{n_R} = 0$. From the definition of the exponential of a matrix, we see that the entries of the matrix

$$\exp\left(2\pi i \frac{z}{h} N_{R,-k,s}\right)$$

are polynomials in z of degree less than n_R .

Since $D_{R,-k,s}$ is diagonalizable matrix, we have

$$D_{R,-k,s} = A_{R,-k,s} \operatorname{diag}(\lambda_{-k,s,1}, \dots, \lambda_{-k,s,n_R}) A_{R,-k,s}^{-1},$$

for some $A = A_{R,k,e} \in \operatorname{GL}(n, \mathbb{C})$. By (2.4.2), we have

$$A^{-1} F|_k \alpha^{-1} =$$

$$c_\alpha^k \left(A^{-1} \exp\left(2\pi i \frac{z}{h} N_{R,-k,s}\right) A \right) \exp\left(2\pi i \frac{z}{h} \operatorname{diag}(\lambda_{-k,s,1}, \dots, \lambda_{-k,s,n_R})\right) A^{-1} G_s(q_h).$$

Set

$$F_s = A^{-1} F|_k \alpha^{-1}, \quad H_s = A^{-1} G_s(q_h), \quad Q(z) = c_\alpha^k A^{-1} \exp\left(2\pi i \frac{z}{h} N_{R,-k,s}\right) A.$$

Then it is clear that $\operatorname{ord}_0(H_s) = \operatorname{ord}_0(G)$, and that the entries of $Q(z)$ are polynomials in z . Also, we have

$$\begin{aligned} & \exp\left(2\pi i \frac{z}{h} \operatorname{diag}(\lambda_{-k,s,1}, \dots, \lambda_{-k,s,n_R})\right) = \\ & \operatorname{diag}\left(\exp\left(2\pi i \lambda_{-k,s,1} \frac{z}{h}\right), \dots, \exp\left(2\pi i \lambda_{-k,s,n_R} \frac{z}{h}\right)\right) := \\ & \operatorname{diag}\left(q_h^{\lambda_{-k,s,1}}, \dots, q_h^{\lambda_{-k,s,n_R}}\right) \end{aligned}$$

With these notations, we get

$$F_s(z) = Q(z) \operatorname{diag}\left(q_h^{\lambda_{-k,s,1}}, \dots, q_h^{\lambda_{-k,s,n_R}}\right) H_s(q_h).$$

We say that F vanishes at s if the right hand side of this equation vanishes when $z \rightarrow i\infty$ (or $q_h \rightarrow 0$).

We want to describe this condition in terms of the order of F at s , *i.e.*, in terms of $\text{ord}_s(F) = \text{ord}_0(G_s) = \text{ord}_0(H_s)$. Let

$$\delta_{R,-k,s} = \min\{\Re(\lambda_{-k,s,j}), 0 \leq j \leq n\}.$$

Suppose that $\delta_{R,-k,s} > 0$. Since the entries of $Q(z)$ are polynomials in z , it suffices to ask that H_s is holomorphic in q_h ($\text{ord}_0(H_s) \geq 0$) in order to get the vanishing of F at s . If $\delta_{R,-k,s} = 0$, we need that $\text{ord}_s(F) = \text{ord}_0(H_s) \geq 1$. This leads to the following definition.

Definition 2.4.1. Let $B_{R,k,s} = B_{k,s} \in \mathcal{M}(n, \mathbb{C})$ such that

$$\mu_s^{-k}(\gamma_s)R_{\gamma_s} = \exp(2\pi i B_{k,s}),$$

and

$$0 \leq \Re(\lambda_{k,s}) < 1,$$

for all eigenvalues $\lambda_{k,s}$ of $B_{k,s}$. Let

$$\delta_{R,k,s} = \min\{\Re(\lambda_{k,s}), \lambda_{k,s} \text{ is an eigenvalue of } B_{k,s}\}.$$

Then

1. The number

$$\rho_{k,s}(R) = \begin{cases} 0 & \text{if } \delta_{R,k,s} > 0 \\ 1 & \text{if } \delta_{R,k,s} = 0 \end{cases}$$

is called the k -cuspidal order of R at s . Since it only depends on the class of s modulo Γ , we define the k -cuspidal order of R at a cuspidal point $P \in X$ to be

$$\rho_{k,P}(R) = \rho_{k,s}(R),$$

where $s \in \mathbb{H}^*$ is a cusp corresponding to P .

2. The k -cuspidal Divisor of R is defined by

$$D_{\mathfrak{S},R,k} = \sum_{P \in \mathfrak{S}} \rho_{k,P}(R)P.$$

The associated line bundle $|D_{\mathfrak{S},R,k}|$ to the divisor $D_{\mathfrak{S},R,k}$ will be denoted by $\mathcal{D}_{\mathfrak{S},R,k}$.

We see that an element F in $A_k(\Gamma, R)$ vanishes at a cuspidal point s of Γ if and only if $\nu_P(F) \geq \rho_{-k,s}(R)$, in this case we say that F is *cuspidal* at s . An element F in $M_k(\Gamma, R)$ is called a *cuspidal vector-valued automorphic form* of multiplier R and weight k , if it is cuspidal at all cusps of Γ . The vector space of cuspidal vector-valued automorphic forms for Γ of multiplier R and weight k will be denoted by $S_k(\Gamma, R)$. We summarize the above discussion in the following result

Proposition 2.4.2. *In terms of divisors, $S_k(\Gamma, R)$ can be written as*

$$S_k(\Gamma, R) = \{ F \in M_k(\Gamma, R) \mid \text{div}(F) \geq D_{\mathfrak{S}, R, -k} \}.$$

□

Now, suppose that χ is a character on Γ and that $f \in A_l(\Gamma, \chi)$. If $F \in A_k(\Gamma, R)$, then $fF \in A_{(k+l)}(\Gamma, \chi R)$. In the following, we shall express the divisor of fF in terms of $\text{div}(f)$ and $\text{div}(F)$. Set $\tilde{R} = \chi R$. If $P \in X$ corresponds to a point of \mathbb{H} , then it is clear that

$$\nu_P(fF) = \nu_P(f) + \nu_P(F).$$

If $P \in X$ corresponds to a cusp s , the above formula is no longer valid. Indeed, using notations of Lemma 2.2.5, and Definition 2.2.2, we have

$$\mu_s^k(\gamma_s)R_{\gamma_s} = \exp(2\pi i B_{R, -k, s}),$$

with eigenvalues $\lambda_{k,s}$ of $B_{R, k, s}$ verifying $0 \leq \Re(\lambda_{k,s}) < 1$. Similarly

$$\mu_s^l(\gamma_s)\chi_{\gamma_s} = \exp(2\pi i b_{\chi, -l, s}),$$

with $0 \leq \Re(b_{\chi, l, s}) < 1$. This implies that

$$\mu_s^{(k+l)}(\gamma_s)\tilde{R}_{\gamma_s} = \exp(2\pi i (b_{\chi, -l, s}I_n + B_{R, -k, s})).$$

Since the eigenvalues of $b_{\chi, l, s}I_n + B_{R, k, s}$ are the eigenvalues of $B_{R, k, s}$ plus the scalar $b_{\chi, l, s}$, we see that

$$B_{\tilde{R}, -(k+l), s} = (b_{\chi, -l, s}I_n + B_{R, -k, s}) + \text{diag}(-m_1, \dots, -m_n),$$

where $m_j \in \{0, 1\}$, $1 \leq j \leq n$.

Using notations of Lemma 2.2.5, then in terms of the local parameter q_h at s , we have

$$\begin{aligned} c_\alpha^{-k} \exp(-2\pi i \frac{z}{h} B_{R,-k,s}) F|_k \alpha^{-1} &= G_s(q_h), \\ c_\alpha^{-l} \exp(-2\pi i \frac{z}{h} b_{\chi,-l,s}) f|_l(\alpha^{-1}) &= g_s(q_h), \\ c_\alpha^{-(k+l)} \exp(-2\pi i \frac{z}{h} B_{\tilde{R},-(k+l),s}) (fF)|_{k+l}(\alpha^{-1}) &= \tilde{G}_s(q_h), \end{aligned}$$

and by definition, we have

$$\nu_P(F) = \text{ord}_0(G_s), \quad \nu_P(f) = \text{ord}_0(g_s), \quad \nu_P(fF) = \text{ord}_0(\tilde{G}_s).$$

From this we see that

$$c_\alpha^{-(k+l)} \exp\left(-2\pi i \frac{z}{h} (b_{\chi,-l,s} I_n + B_{R,-k,s})\right) (fF)|_{k+l}(\alpha^{-1}) = g_s(q_h) G_s(q_h),$$

and so

$$\exp\left(-2\pi i \frac{z}{h} (b_{\chi,-l,s} I_n + B_{R,-k,s} - B_{\tilde{R},-(k+l),s})\right) \tilde{G}_s(q_h) = g_s(q_h) G_s(q_h).$$

Hence

$$\text{diag}(q_h^{m_1}, \dots, q_h^{m_n}) \tilde{G}_s(q_h) = g_s(q_h) G_s(q_h).$$

Let $g_{s,j}$ (resp. $\tilde{g}_{s,j}$), $1 \leq j \leq n$, be the components of G_s (resp. \tilde{G}_s). Then for some $1 \leq j_0, j_1 \leq n$, we have

$$\begin{aligned} m_{j_0} + \text{ord}_0(\tilde{G}_s) &= \text{ord}_0(g_s) + \text{ord}_0(g_{s,j_0}), \\ m_{j_1} + \text{ord}_0(\tilde{g}_{s,j_1}) &= \text{ord}_0(g_s) + \text{ord}_0(G_s). \end{aligned}$$

From the definition of the order, we deduce that

$$\begin{aligned} \text{ord}_0(g_s) + \text{ord}_0(G_s) &\leq m_{j_0} + \text{ord}_0(\tilde{G}_s) \\ m_{j_1} + \text{ord}_0(\tilde{G}_s) &\leq \text{ord}_0(g_s) + \text{ord}_0(G_s), \end{aligned}$$

which yield

$$m_{j_1} + \leq (\text{ord}_0(g_s) + \text{ord}_0(G_s) - \text{ord}_0(\tilde{G}_s)) \leq m_{j_0},$$

Hence

$$0 \leq \nu_P(f) + \nu_P(F) - \nu_P(fF) \leq 1.$$

Thus, we have proved

Theorem 2.4.3. *Let χ be a character, and f be an element of $A_l(\Gamma, \chi)$. If $F \in A_k(\Gamma, R)$, then $fF \in A_{(k+l)}(\Gamma, \chi R)$, and we have*

$$\text{div}(fF) = \text{div}(f) + \text{div}(F) - D_{f,F},$$

where

$$D_{f,F} = \sum_{Q \in \mathfrak{S}} a_{Q,f,F} Q$$

is a divisor having support in \mathfrak{S} with coefficients $a_{Q,f,F} \in \{0, 1\}$.

2.5 $M_k(\Gamma, R)$ and $S_k(\Gamma, R)$ as global sections of vector bundles

We now come to the main construction of this chapter. We establish the correspondence between vector-valued automorphic forms of weight k and multiplier k for Γ , and global sections of the the holomorphic vector bundles $\mathcal{E}_k = \mathcal{E}_{\Gamma, R, k}$ on the Riemann surface X .

In this section we keep the notations of §2.2.4 used to construct the vector bundle \mathcal{E}_k . Recall that the transition functions of \mathcal{E}_k are the elements of the 1-cocycle $(\tilde{F}_{k,ij}) \in Z^1(\mathcal{U}, \text{GL}(n, \mathcal{O}))$, and that $F_{k,ij} = \pi^* \tilde{F}_{k,ij}$, with

$$F_{k,ij} = \Psi_{k,i} \Psi_{k,j}^{-1} \in \text{GL}(n, \mathcal{O}(V_i \cap V_j)) \text{ if } i \neq j \text{ and } F_{k,ii} = \text{id},$$

where $\Psi_{k,i}$ is the map constructed in Proposition 2.2.7 and Proposition 2.2.8. Now, suppose that G is a collection $(G_i)_{V_i}$ of meromorphic functions G_i on

V_i satisfying

$$G_i = F_{k,ij}G_j \quad \text{on} \quad V_i \cap V_j. \quad (2.5.1)$$

If G is invariant under Γ , that is

$$G_i\gamma = G_i, \quad \text{for all } \gamma \in \Gamma, \quad (2.5.2)$$

then there exists a collection $\tilde{G} = (\tilde{G}_i)_{U_i}$ of meromorphic functions \tilde{G}_i on U_i such that

$$G_i = \pi^*\tilde{G}_i \quad \text{on} \quad V_i.$$

Note that, by construction, V_i is Γ -invariant. Since $F_{k,ij} = \pi^*\tilde{F}_{k,ij}$, then, by (2.5.1), we have

$$\tilde{G}_i = \tilde{F}_{k,ij}\tilde{G}_j \quad \text{on} \quad U_i \cap U_j.$$

Hence by Proposition 1.1.4, we deduce that $\tilde{G} = (\tilde{G}_i)_{U_i}$ is a meromorphic section of \mathcal{E}_k .

Conversely, if $\tilde{G} = (\tilde{G}_i)_{U_i}$ is a meromorphic section of \mathcal{E}_k , then

$$G = \pi^*\tilde{G} = (\pi^*\tilde{G}_i)_{V_i}$$

clearly satisfies (2.5.1) and (2.5.2). Thus we have proved the following

Proposition 2.5.1. *There is a correspondence between the global meromorphic sections of \mathcal{E}_k , and the collections $(G_i)_{V_i}$ of meromorphic functions G_i on V_i satisfying*

$$G_i = F_{k,ij}G_j \quad \text{on} \quad V_i \cap V_j,$$

and

$$G_i\gamma = G_i, \quad \text{for all } \gamma \in \Gamma.$$

□

Theorem 2.5.2. *We have a linear isomorphism between $A_{-k}(\Gamma, R)$ and $H^0(X, \mathcal{M}(\mathcal{E}_k))$ given by*

$$F \mapsto \Theta_F = (\Psi_{k,i}F)_{V_i},$$

with inverse

$$\Theta \mapsto F_\Theta = (\Psi_{k,i}^{-1} \pi^* \Theta_i)_{V_i},$$

where $\Theta = (\Theta_i)_{U_i}$.

Proof. By Proposition 2.5.1, the only nontrivial part is to prove that F_Θ is globally defined on \mathbb{H} . For this, it suffices to show that

$$\Psi_{k,i}^{-1} \pi^* \Theta_i = \Psi_{k,j}^{-1} \pi^* \Theta_j \quad \text{on} \quad V_i \cap V_j.$$

Since $\pi^* \tilde{F}_{k,ij} = \Psi_{k,i} \Psi_{k,j}^{-1}$ on $V_i \cap V_j$, we have

$$\begin{aligned} \Psi_{k,i}^{-1} \pi^* \Theta_i &= \Psi_{k,i}^{-1} \pi^* (\tilde{F}_{k,ij} \Theta_j) \\ &= \Psi_{k,i}^{-1} \pi^* (\tilde{F}_{k,ij}) \pi^* (\Theta_j) \\ &= \Psi_{k,i}^{-1} F_{k,ij} \pi^* \Theta_j \\ &= \Psi_{k,i}^{-1} \Psi_{k,i} \Psi_{k,j}^{-1} \pi^* \Theta_j \\ &= \Psi_{k,j}^{-1} \pi^* \Theta_j. \end{aligned}$$

□

The next step is to find the relationship between the divisor of an element F of $A_{-k}(\Gamma, R)$ and its corresponding element Θ_F of $H^0(X, \mathcal{M}(\mathcal{E}_k))$, see (1.1.3) and (2.4.1). First, suppose that $P \in X'$ corresponds to a point $z_0 \in V_0$, then in a local chart Θ_F is just $\Psi_0 F$. Since Ψ_0 lies in $\text{GL}(n, \mathcal{O}(V_0))$, then by Remark 1.1.5, we have $\nu_P(\Theta_F) = \nu_P(F)$.

In case P corresponds to an elliptic point e of order n_e , then using the notations in the proof of Lemma 2.2.3 and letting $\alpha(z) = w$, the local parameter at P is $t = w^{n_e}$. Letting $\Theta_F = G(t)$, we have in terms of w

$$\begin{aligned} G(w^{n_e}) &= \Psi_e(\alpha^{-1} w) A_{k,e} F(\alpha^{-1} w) = \\ &f_e^k(\alpha^{-1} w) \text{diag}(w^{-m_1}, \dots, w^{-m_n}) A_{k,e} F(\alpha^{-1} w), \end{aligned}$$

which implies that

$$n_e \nu_P(\Theta_F) = \text{ord}_0(\widetilde{F}_e), \quad (2.5.3)$$

where \widetilde{F}_e is defined near $w = 0$ by

$$\widetilde{F}_e(w) = \text{diag}(w^{-m_1}, \dots, w^{-m_n}) A_{k,e} F(\alpha^{-1}w). \quad (2.5.4)$$

If $F_e = (A_{k,e}F)\alpha^{-1}$, then it is clear that $\text{ord}_0(F_e) = \text{ord}_e(F)$. Moreover, if n_i denotes the order of the i^{th} component of F_e , $1 \leq i \leq n$, then we have

$$\text{ord}_0(\widetilde{F}_e) = n_{i_0} - m_{i_0},$$

for some index i_0 , $1 \leq i_0 \leq n$. Since $\text{ord}_e(F) = \text{ord}_0(F_e) = n_j$ for some j , then we have

$$\text{ord}_0(\widetilde{F}_e) \leq n_j - m_j = \text{ord}_e(F) - m_j.$$

In other words,

$$m_j \leq \text{ord}_e(F) - \text{ord}_0(\widetilde{F}_e).$$

Let $m_e = \max\{m_i, 1 \leq i \leq n\}$,

$$F^*(w) := \text{diag}(w^{m_e - m_1}, \dots, w^{m_e - m_n})(w^{-\text{ord}_0(F_e)} F_e(w)),$$

and

$$\widehat{F}_e(w) := w^{-\text{ord}_0(\widetilde{F}_e)} \widetilde{F}_e(w).$$

Then F^* is holomorphic in w , and by (2.5.4), we have

$$\widehat{F}_e(w) = w^{(\text{ord}_0(F_e) - \text{ord}_0(\widetilde{F}_e) - m_e)} F^*(w).$$

If $\text{ord}_0(F_e) - \text{ord}_0(\widetilde{F}_e) - m_e > 0$, then $\widehat{F}_e(0) = 0$ which contradicts the definition of the order of \widetilde{F}_e at 0, and so $\text{ord}_0(F_e) - \text{ord}_0(\widetilde{F}_e) \leq m_e$. Hence

$$m_j \leq \text{ord}_e(F) - \text{ord}_0(\widetilde{F}_e) \leq m_e. \quad (2.5.5)$$

We are led to the following definition.

Definition 2.5.1. The elliptic error $\tau_e(F)$ of F at the elliptic point $e \in \mathbb{H}$ is defined by

$$\tau_e(F) = \frac{\text{ord}_e(F) - \text{ord}_0(\widetilde{F}_e)}{n_e}.$$

Notice that by (2.5.5), we have

$$0 \leq \tau_e(F) \leq 1 - 1/n_e.$$

Since $\tau_e(F)$ depends only on the class of e modulo Γ , then:

1. If $P \in X$ corresponds to an elliptic point $e \in \mathbb{H}$, then we define

$$\tau_P(F) = \tau_e(F).$$

2. If $\Theta \in H^0(X, \mathcal{M}(\mathcal{E}_k))$, then we define the elliptic error of Θ at an elliptic point P of X by

$$\tau(\Theta) = \tau(F_\Theta),$$

where F_Θ is the element of $A_{-k}(\Gamma, R)$ constructed in Theorem 2.5.2.

3. We set

$$\tau(F) = \tau(\Theta_F) = \sum_{P \in \mathfrak{E}} \tau_P(F)P = \sum_{P \in \mathfrak{E}} \tau_P(\Theta_F)P.$$

Using the fact that $n_e \nu_P(F) = \text{ord}_e(F)$ and (2.5.3), we have

$$\nu_P(\Theta_F) = \nu_P(F) - \tau_P(F). \quad (2.5.6)$$

Finally, suppose that P corresponds to a cusp s . Since F has weight $-k$, then from §2.3, we see that the behavior of F at s is by definition the behavior of Θ_F at P . Therefore

$$\nu_P(\Theta_F) = \nu_P(F). \quad (2.5.7)$$

Combining (2.5.6) and (2.5.7), we get the following.

Theorem 2.5.3. *Let F be an element of $A_{-k}(\Gamma, R)$, and Θ_F be its corresponding element in $H^0(X, \mathcal{M}(\mathcal{E}_k))$. We have*

$$\text{div}(\Theta_F) = \text{div}(F) - \tau(F).$$

At this point we need to give an explicit formula for the line bundle $\mathcal{L}_k = \mathcal{E}_{\Gamma, \chi_0, k}$ for even integers k , where χ_0 is the trivial character sending all elements of Γ to 1, see Definition 2.2.4.

It is well-known that to each automorphic form f of even weight k is associated a $(k/2)$ -form $\eta = f(dz)^{k/2}$ on X , see [55].

Proposition 2.5.4. [55] *Let f be a nonzero automorphic form for Γ of even weight k , and $\eta = f(dz)^{k/2}$ be the associated $(k/2)$ -form on X . Then*

$$\operatorname{div}(f) = \operatorname{div}(\eta) + (k/2) \left(\sum_{P \in \mathfrak{E}} (1 - 1/n_e) P + \sum_{Q \in \mathfrak{S}} Q \right).$$

Definition 2.5.2. As in [55], we define the integral part of a rational divisor

$$D = \sum_{P \in X} a_P P, \quad a_P \in \mathbb{Q},$$

by

$$[D] = \sum_{P \in X} [a_P] P,$$

where $[x]$ is the integral part of $x \in \mathbb{R}$.

We have the following useful lemma.

Lemma 2.5.5. [55] *Let $A = \sum_{P \in X} a_P P$, $a_P \in \mathbb{Q}$, be a rational divisor on X . Then for an integral divisor D on X , i.e., having coefficients in \mathbb{Z} , we have*

$$D \geq -A \iff D \geq -[A].$$

Remark 2.5.6. Notice that for Θ in $H^0(X, \mathcal{M}(\mathcal{E}_k))$ and F_Θ the corresponding element of $A_{-k}(\Gamma, R)$, see Theorem 2.5.3, we have

$$\operatorname{div}(\Theta) = [\operatorname{div}(F_\Theta)].$$

Recall that the line bundle associated to a divisor D on X is denoted by $|D|$, see §1.1.2.

Proposition 2.5.7. *Let \mathcal{K}_X be the canonical bundle of X ,*

$$D_{\mathfrak{S}} = \sum_{P \in \mathfrak{S}} P, \quad D'_{\mathfrak{E}} = \sum_{P \in \mathfrak{E}} (1 - 1/n_P)P,$$

and $\mathcal{D}_{\mathfrak{S}} = |D_{\mathfrak{S}}|$. Then for k even, we have

$$\mathcal{L}_{-k} = (k/2)\mathcal{K}_X + (k/2)\mathcal{D}_{\mathfrak{S}} + |[(k/2)D'_{\mathfrak{E}}]|.$$

In particular

$$\mathcal{L}_{-2} = \mathcal{K}_X + \mathcal{D}_{\mathfrak{S}}.$$

Proof. Let Θ be an element of $H^0(X, \mathcal{M}(\mathcal{L}_{-k}))$, and F_{Θ} be the corresponding element in $A_k(\Gamma, R)$. By Proposition 2.5.7 and Remark 2.5.6, we have

$$\operatorname{div}(\Theta) = [\operatorname{div}(F_{\Theta})] = [\operatorname{div}(\eta_{F_{\Theta}}) + (k/2) \left(\sum_{P \in \mathfrak{E}} (1 - 1/n_e)P + \sum_{Q \in \mathfrak{S}} Q \right)],$$

that is

$$\operatorname{div}(\Theta) = \operatorname{div}(\eta_{F_{\Theta}}) + (k/2)D_{\mathfrak{S}} + [(k/2)D'_{\mathfrak{E}}].$$

Hence

$$|\operatorname{div}(\Theta)| = |\operatorname{div}(\eta_{F_{\Theta}})| + (k/2)\mathcal{D}_{\mathfrak{S}} + |[(k/2)D'_{\mathfrak{E}}]|.$$

Since $\mathcal{L}_{-k} = |\operatorname{div}(\Theta)|$ and $(k/2)\mathcal{K}_X = |\operatorname{div}(\eta_{F_{\Theta}})|$, we have

$$\mathcal{L}_{-k} = (k/2)\mathcal{K}_X + (k/2)\mathcal{D}_{\mathfrak{S}} + |[(k/2)D'_{\mathfrak{E}}]|,$$

as desired. When $k = 2$, the divisor $(k/2)[D'_{\mathfrak{E}}] = [D'_{\mathfrak{E}}]$ is equal to the zero divisor, and hence $\mathcal{L}_{-2} = \mathcal{K}_X + \mathcal{D}_{\mathfrak{S}}$. □

We now come to the main theorem of this chapter. Recall that the $(-k)$ -cuspidal divisor of R is given by $D_{\mathfrak{S}, R, -k} = \sum_{P \in \mathfrak{S}} \rho_{-k, P}(R)P$, see Definition 2.4.1.

Theorem 2.5.8. *If $\mathcal{D}_{\mathfrak{S},R,-k} = |D_{\mathfrak{S},R,-k}|$ is the line bundle associated to $D_{\mathfrak{S},R,-k}$, then we have*

$$M_k(\Gamma, R) \cong H^0(X, \mathcal{O}(\mathcal{E}_{-k})),$$

and

$$S_k(\Gamma, R) \cong H^0(X, \mathcal{O}(-\mathcal{D}_{\mathfrak{S},R,-k} + \mathcal{E}_{-k})) \cong \{ \Theta \in H^0(X, \mathcal{O}(\mathcal{E}_{-k})) \mid \text{div}(\Theta) \geq D_{\mathfrak{S},R,-k} \}.$$

Proof. Let Θ be an element of $H^0(X, \mathcal{M}(\mathcal{E}_{-k}))$, and F_Θ be its corresponding element in $A_k(\Gamma, R)$. By Theorem 2.5.3, we have

$$\text{div}(\Theta) = \text{div}(F_\Theta) - \tau(\Theta).$$

According to Definition 2.5.1, the coefficients of $\tau(\Theta)$ are in $[0, 1)$, hence $[\tau(\Theta)] = 0$. By Lemma 2.5.5, we have

$$\text{div}(F_\Theta) \geq 0 \iff \text{div}(\Theta) \geq -\tau(\Theta) \iff \text{div}(\Theta) \geq -[\tau(\Theta)] = 0.$$

Similarly, since $D_{\mathfrak{S},R,-k}$ is an integral divisor, we have

$$\begin{aligned} \text{div}(F_\Theta) \geq D_{\mathfrak{S},R,-k} &\iff \text{div}(\Theta) - D_{\mathfrak{S},R,-k} \geq -\tau(\Theta) \iff \\ \text{div}(\Theta) - D_{\mathfrak{S},R,-k} \geq -[\tau(\Theta)] = 0 &\iff \text{div}(\Theta) \geq D_{\mathfrak{S},R,-k}. \end{aligned}$$

Now, the result is a straightforward consequence of (1.1.5), Theorem 2.5.2, Proposition 2.4.1, and Proposition 2.4.2. \square

As a first consequence we have

Theorem 2.5.9. *Let Γ be a Fuchsian group. Then for any simple pair (R, k) , $k \in \mathbb{Z}$, the dimensions of $M_k(\Gamma, R)$ and $S_k(\Gamma, R)$ are finite if and only if Γ is of the first kind.*

Proof. If Γ is of the first kind, then X is a compact Riemann surface. From Theorem 1.1.12 and Theorem 2.5.8 we deduce that the dimensions of $M_k(\Gamma, R)$ and $S_k(\Gamma, R)$ are finite.

If Γ is not of the first kind, then X is a noncompact Riemann surface. This implies that any holomorphic vector bundle \mathcal{V} on a X is trivial, see [13]. Therefore $H^0(X, \mathcal{O}(\mathcal{V}))$ is isomorphic to $\mathcal{O}(\mathcal{V})^n$, where n is the rank of \mathcal{V} . But the dimension of $\mathcal{O}(\mathcal{V})$ is infinite since any effective divisor is the divisor of a holomorphic function on X , see [13]. Hence $H^0(X, \mathcal{O}(\mathcal{V}))$ is an infinite-dimensional vector space. By the Theorem 2.5.8, $M_k(\Gamma, R)$ and $S_k(\Gamma, R)$ can be realized as global holomorphic sections of some holomorphic vector bundles on X , hence they have infinite dimensions. \square

From now on Γ will be a Fuchsian group of the first kind, and so X will be a compact Riemann surface. We set

1. g_X for the genus of X .
2. $c_X = |\mathfrak{S}|$ for the number of cuspidal points of X , which is finite since Γ is of the first kind.
3. $e_X = |\mathfrak{E}|$ for the number of elliptic points of X , which is finite since Γ is of the first kind.
4. $d_{\Gamma, R, k} = d_k$ and $s_{\Gamma, R, k} = s_k$ for the dimension of $M_k(\Gamma, R)$ and $S_k(\Gamma, R)$ respectively.

Recall that the parity index of R is denoted by $\varepsilon = \varepsilon(R)$, see Definition 2.1.3.

Theorem 2.5.10. *Let $D_{\mathfrak{E}} = \sum_{P \in \mathfrak{E}} P$, and $\mathcal{D}_{\mathfrak{E}} = |D_{\mathfrak{E}}|$ be its associated holomorphic line bundle. Then for any line bundle \mathcal{L} on X , we have*

$$h^0(-\mathcal{D}_{\mathfrak{E}} + \mathcal{L} + \mathcal{L}_{-(k-\varepsilon)} + \mathcal{E}_{-\varepsilon}) \leq h^0(\mathcal{L} + \mathcal{E}_{-k}) \leq h^0(\mathcal{D}_{\mathfrak{E}} + \mathcal{L} + \mathcal{L}_{-(k-\varepsilon)} + \mathcal{E}_{-\varepsilon}).$$

Proof. Let L be a divisor on X such that $\mathcal{L} = |L|$. From the definition of the parity index, we see that the space $A_{k-\varepsilon}(\Gamma)$ of automorphic forms of weight $k - \varepsilon$ for Γ is nontrivial. We fix a choice of a nonzero element $f \in A_{k-\varepsilon}(\Gamma)$, and we let α be its corresponding element of $H^0(X, \mathcal{M}(\mathcal{L}_{-(k-\varepsilon)}))$. Then by (1.1.5), $H^0(X, \mathcal{O}(\mathcal{L} + \mathcal{E}_{-k}))$ is isomorphic to

$$\{\Theta \in H^0(X, \mathcal{M}(\mathcal{E}_{-k})) \mid \operatorname{div}(\Theta) \geq -L\},$$

which is, by Theorem 2.5.3 and Theorem 2.5.2, isomorphic to

$$\{F \in A_k(\Gamma, R) \mid \operatorname{div}(F) - \tau(F) \geq -L\}.$$

Since the multiplication by f gives an isomorphism

$$A_\varepsilon(\Gamma, R) \longrightarrow A_k(\Gamma, R) : G \mapsto fG,$$

we have

$$\begin{aligned} \{F \in A_k(\Gamma, R) \mid \operatorname{div}(F) - \tau(F) \geq -L\} &\cong \\ \{G \in A_\varepsilon(\Gamma, R) \mid \operatorname{div}(fG) - \tau(fG) \geq -L\}. \end{aligned}$$

On the other hand, using Theorem 2.4.3, we have

$$\operatorname{div}(fG) = \operatorname{div}(f) + \operatorname{div}(G) - D_{f,G},$$

where

$$D_{f,G} = \sum_{P \in \mathfrak{S}} a_{P,f,G} P$$

is a divisor having support in \mathfrak{S} with coefficients $a_{P,f,G} \in \{0, 1\}$. Hence

$$\begin{aligned} \{G \in A_\varepsilon(\Gamma, R) \mid \operatorname{div}(fG) - \tau(fG) \geq -L\} &= \\ \{G \in A_\varepsilon(\Gamma, R) \mid \operatorname{div}(f) + \operatorname{div}(G) - \tau(fG) - D_{f,G} \geq -L\}, \end{aligned}$$

which is, by Theorem 2.5.3 and Theorem 2.5.2, isomorphic to

$$\{\Theta \in H^0(X, \mathcal{M}(\mathcal{E}_{-\varepsilon})) \mid \operatorname{div}(\alpha) + \tau(f) + \operatorname{div}(\Theta) + \tau(G_\Theta) - \tau(fG_\Theta) - D_{f,G_\Theta} \geq -L\}$$

which is in turn equal to

$$\{\Theta \in H^0(X, \mathcal{M}(\mathcal{E}_{-\varepsilon})) \mid \operatorname{div}(\alpha) + \operatorname{div}(\Theta) + L - D_{f, G_\Theta} \geq -(\tau(f) + \tau(\Theta) - \tau(fG_\Theta))\}.$$

By Lemma 2.5.5, the latter space is equal to

$$\{\Theta \in H^0(X, \mathcal{M}(\mathcal{E}_{-\varepsilon})) \mid \operatorname{div}(\alpha) + \operatorname{div}(\Theta) + L - D_{f, G_\Theta} \geq -[\tau(f) + \tau(G_\Theta) - \tau(fG_\Theta)]\}$$

which is equal to

$$\begin{aligned} \{\Theta \in H^0(X, \mathcal{M}(\mathcal{E}_{-\varepsilon})) \mid \operatorname{div}(\alpha) + \operatorname{div}(\Theta) + L \geq D_{f, G_\Theta} - [\tau(f) + \tau(G_\Theta) - \tau(fG_\Theta)]\} \\ =: V_{k, L}. \end{aligned}$$

According to Definition 2.5.1, the divisors $\tau(f)$, $\tau(G_\Theta)$, and $\tau(fG_\Theta)$ have coefficients in $[0, 1)$. Hence, the divisor $\tau(f) + \tau(G_\Theta) - \tau(fG_\Theta)$ has coefficients in $(-1, 2)$. If Θ_1 is the element of $H^0(X, \mathcal{M}(\mathcal{E}_{-k}))$ corresponding to the element fG_Θ of $A_k(R, \Gamma)$, then from the relation $\operatorname{div}(fG_\Theta) = \operatorname{div}(f) + \operatorname{div}(G_\Theta) - D_{f, G_\Theta}$, and Theorem 2.5.3 applied to G_Θ , we have

$$\operatorname{div}(\Theta_1) + \tau(fG_\Theta) = \operatorname{div}(\alpha) + \tau(f) + \operatorname{div}(\Theta) + \tau(G_\Theta) - D_{f, G_\Theta},$$

which is equivalent to

$$\operatorname{div}(\Theta_1) - \operatorname{div}(\alpha) - \operatorname{div}(\Theta) + D_{f, G_\Theta} = \tau(f) + \tau(G_\Theta) - \tau(fG_\Theta).$$

Therefore the coefficients of $\tau(f) + \tau(G_\Theta) - \tau(fG_\Theta)$ are all integers, and so lie in $\{0, 1\}$. Hence

$$0 \leq [\tau(\alpha) + \tau(\Theta) - \tau(fG_\Theta)] \leq \sum_{P \in \mathfrak{E}} P = D_{\mathfrak{E}}.$$

By Theorem 2.4.3, the divisor D_{f, G_Θ} has support in \mathfrak{S} with coefficients in $\{0, 1\}$, and so $D_{f, G_\Theta} \leq D_{\mathfrak{S}}$. Therefore, we have

$$\{\Theta \in H^0(X, \mathcal{M}(\mathcal{E}_{-\varepsilon})) \mid \operatorname{div}(\alpha) + \operatorname{div}(\Theta) + L \geq D_{\mathfrak{S}}\} \subseteq V_{k, L},$$

and

$$V_{k,L} \subseteq \{ \Theta \in H^0(X, \mathcal{M}(\mathcal{E}_{-\varepsilon})) \mid \operatorname{div}(\alpha) + \operatorname{div}(\Theta) + L \geq -\mathcal{D}_{\mathfrak{E}} \}.$$

But (1.1.5) implies that

$$\begin{aligned} & \{ \Theta \in H^0(X, \mathcal{M}(\mathcal{E}_{-\varepsilon})) \mid \operatorname{div}(\alpha) + \operatorname{div}(\Theta) + L \geq D_{\mathfrak{E}} \} \cong \\ & H^0(X, \mathcal{O}(-\mathcal{D}_{\mathfrak{E}} + \mathcal{L} + \mathcal{L}_{-(k-\varepsilon)} + \mathcal{E}_{-\varepsilon})), \end{aligned}$$

and

$$\begin{aligned} & \{ \Theta \in H^0(X, \mathcal{M}(\mathcal{E}_{-\varepsilon})) \mid \operatorname{div}(\alpha) + \operatorname{div}(\Theta) + L \geq -\mathcal{D}_{\mathfrak{E}} \} \cong \\ & H^0(X, \mathcal{O}(\mathcal{D}_{\mathfrak{E}} + \mathcal{L} + \mathcal{L}_{-(k-\varepsilon)} + \mathcal{E}_{-\varepsilon})). \end{aligned}$$

By Theorem 2.5.8, we conclude that

$$h^0(-\mathcal{D}_{\mathfrak{E}} + \mathcal{L} + \mathcal{L}_{-(k-\varepsilon)} + \mathcal{E}_{-\varepsilon}) \leq h^0(\mathcal{L} + \mathcal{E}_{-k}) \leq h^0(\mathcal{D}_{\mathfrak{E}} + \mathcal{L} + \mathcal{L}_{-(k-\varepsilon)} + \mathcal{E}_{-\varepsilon}).$$

□

Taking $\mathcal{L} = -\mathcal{D}_{\mathfrak{E},R,-k}$ or the trivial line bundle, then by Theorem 2.5.8, we get the following.

Corollary 2.5.11. *We have*

$$h^0(-\mathcal{D}_{\mathfrak{E}} + \mathcal{L}_{-(k-\varepsilon)} + \mathcal{E}_{-\varepsilon}) \leq d_k \leq h^0(\mathcal{D}_{\mathfrak{E}} + \mathcal{L}_{-(k-\varepsilon)} + \mathcal{E}_{-\varepsilon}).$$

$$h^0(-\mathcal{D}_{\mathfrak{E}} - \mathcal{D}_{\mathfrak{E},R,-k} + \mathcal{L}_{-(k-\varepsilon)} + \mathcal{E}_{-\varepsilon}) \leq s_k \leq h^0(-\mathcal{D}_{\mathfrak{E},R,-k} + \mathcal{D}_{\mathfrak{E}} + \mathcal{L}_{-(k-\varepsilon)} + \mathcal{E}_{-\varepsilon}).$$

□

The above two results will be of fundamental use in the next chapter when computing the dimensions of $M_k(\Gamma, R)$ and $S_k(\Gamma, R)$.

Chapter 3

The dimension Formula

3.1 The degree of $\mathcal{E}_{\Gamma,R,k}$

Our main tool to compute the dimensions of $M_k(\Gamma, R)$ and $S_k(\Gamma, R)$ is the Riemann-Roch theorem applied to the holomorphic vector bundles $\mathcal{E}_{-k} = \mathcal{E}_{\Gamma,R,-k}$ and $-\mathcal{D}_{\mathfrak{S},R,-k} + \mathcal{E}_{-k}$, where $\mathcal{D}_{\mathfrak{S},R,-k}$ is the holomorphic line bundle associated to the $(-k)$ -cuspidal divisor of R , see Definition 2.4.1. Therefore we need to compute the degree of \mathcal{E}_{-k} .

Let $\Theta_1, \dots, \Theta_n$ be global meromorphic sections of the holomorphic vector bundle \mathcal{E}_{-k} such that

$$\Theta = \Theta_1 \wedge \dots \wedge \Theta_n \tag{3.1.1}$$

is a nonzero section of the holomorphic line bundle $\bigwedge^n \mathcal{E}_{-k}$ (the determinant bundle of \mathcal{E}_{-k}). These always exist according to [50]. By Theorem 2.5.2, each Θ_i corresponds to an element F_i of $A_k(\Gamma, R)$. Hence $F = F_1 \wedge \dots \wedge F_n$ is an element of $A_{nk}(\Gamma, \det(R))$.

Recall that for an integer l , the space of automorphic forms of weight l and trivial character for Γ is denoted by $A_l(\Gamma)$. Set $\chi = \det(R)$, and let g be a fixed element of $A_{2nk}(\Gamma)$. Then $F_1 = F^2/g$ lies in $A_0(\Gamma, \chi^2)$, and so $f = F'_1/F_1$ is a weight 2 automorphic form for Γ . Write $\eta = f dz$ for its

corresponding 1–form on X . To find the degree of \mathcal{E}_{-k} , we follow the same strategy used in [43]. First, we express the residues of η in terms of the orders of Θ and g , then use the fact that the sum of residues of η is 0.

Lemma 3.1.1. *Let f be a weight 2 automorphic form, and η be its associated 1–form on X . Suppose that $P \in X$ corresponds to $x \in \mathbb{H}^*$, then*

1. *If $x \in \mathbb{H}$, then $\text{Res}_P(\eta) = (1/n_P)\text{Res}_x(f)$, where n_P is the order of P .*
2. *If x is a cusp s , and*

$$\sum_{n \geq n_s} a_n(f) q_h^n$$

is the expansion of f at s , q_h being the local parameter at s , and h the cusp width at s , then we have

$$(2\pi i/h)\text{Res}_P(\eta) = a_0(f).$$

Proof. Suppose that $x \in \mathbb{H}$. If

$$\alpha(z) = \frac{z - x}{z - \bar{x}}, \quad z \in \mathbb{H},$$

then $\alpha(\mathbb{H})$ is the unit disc \mathbb{D} , $\alpha(e) = 0$. With $\alpha(z) = w$, the local parameter at P is $t = w^{n_x}$, where $n_x = n_P$ is the order of P . Set $\beta := \alpha^{-1}$. By definition, $\text{Res}_x(f)$ is the residue of the 1–form $f(z)dz$ on \mathbb{H} . Writing

$$\beta^*(f(z)dz) = h(w)dw$$

and using the fact that the residue of a differential form is invariant under changes of local parameter, we deduce that

$$\text{Res}_x(f) = \text{Res}_0(h). \tag{3.1.2}$$

if $\eta = h_1(t)dt$, then according to [55], we have

$$h(w) = n_x w^{n_x - 1} h_1(w^{n_x}). \tag{3.1.3}$$

Suppose that the t -expansion of h_1 near $t = 0$ is given by

$$h_1(t) = \sum_{n \geq n_0} c_n t^n,$$

where $n_0 = \text{ord}_0(h_1)$, then

$$h_1(w^{n_x}) = \sum_{n \geq n_0} c_n w^{n n_x}.$$

Using (3.1.3), we find

$$\text{Res}_0(h) = n_x \text{Res}_0(h_1) = n_x \text{Res}_P(\eta),$$

and from (3.1.2), we conclude that

$$\text{Res}_P(\eta) = (1/n_P) \text{Res}_x(f).$$

As for 2., if x is a cusp $s \in \mathbb{H}^*$ with cusp width h , then if we take $\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}(2, \mathbb{R})$ such that $\alpha \cdot s = \infty$, the local parameter at P is given by $q_h = \exp(2\pi iz/h)$. Suppose that $\eta = F(q_h) dq_h$, then according to [55], we have

$$(2\pi/h) F(q_h) = q_h^{-1} \sum_{n \geq n_s} a_n(f) q_h^n. \quad (3.1.4)$$

Thus

$$(2\pi/h) \text{Res}_0(F) = a_0(f).$$

Since $\text{Res}_0(F) = \text{Res}_P(\eta)$, we conclude that

$$(2\pi i/h) \text{Res}_P(\eta) = a_0(f).$$

□

For non-cuspidal points of X , we have the following

Lemma 3.1.2. *If $P \in X'$, then*

$$\text{Res}_P(\eta) = \begin{cases} 2\nu_P(\Theta) - 2\nu_{-k,P}(R) - \nu_P(g) & \text{if } P \text{ is an elliptic point} \\ 2\nu_P(\Theta) - \nu_P(g) & \text{otherwise.} \end{cases}$$

Proof. Let $x \in \mathbb{H}$ be above P . Then

$$\text{Res}_P(\eta) = \text{Res}_x(f) = \nu_x(F_1) = 2\nu_x(F) - \nu_x(g) = 2\nu_P(F) - \nu_P(g),$$

and so it suffices to express $\nu_P(F)$ in terms of $\nu_P(\Theta)$. Suppose that x is an elliptic fixed point e of Γ . Using the notations in the proof of Lemma 2.2.3 and letting $\alpha(z) = w$, the local parameter at P is $t = w^{n_e}$, where $n_e = n_P$ is the order of P . If $\Theta_i = G_i(t)$, then by the correspondence in Theorem 2.5.2, we have in terms of w

$$\begin{aligned} G_i(w^{n_e}) &= \Psi_{-k,e}(\alpha^{-1}w) A_{-k,e} F_i(\alpha^{-1}w) \\ &= f_e^{-k}(\alpha^{-1}w) \text{diag}(w^{-m-k,1}, \dots, w^{-m-k,n}) A_{-k,e} F_i(\alpha^{-1}w). \end{aligned}$$

If $\Theta = G(t)$, then $G = G_1 \wedge \dots \wedge G_n$ and

$$G(w^{n_e}) = f_e^{-nk}(\alpha^{-1}w) w^{-(\sum_{j=1}^n m_{-k,j})} \det(A_{-k,e}) F(\alpha^{-1}w).$$

Hence

$$\nu_P(\Theta) = \nu_P(F) - (1/n_e) \sum_{j=1}^n m_{-k,j} = \nu_P(F) + \nu_{-k,P}(R).$$

In case x is not an elliptic point, then in a local chart Θ_i is given by $\Psi_{-k,0}F_i$, and so $\Theta = \det(\Psi_{-k,0})F$. Since $\Psi_{-k,0} \in \text{GL}(n, \mathcal{O}(Y'))$, we have $\nu_p(\Theta) = \nu_p(F)$.

□

As for the cusps, we have:

Lemma 3.1.3. *Suppose that $P \in X$ corresponds to a cusp $s \in \mathbb{H}^*$, then*

$$\text{Res}_P(\eta) = 2\nu_P(\Theta) - 2\nu_{-k,P}(R) - \nu_P(g).$$

Proof. Let $\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}(2, \mathbb{R})$ such that $\alpha \cdot s = \infty$, then the local parameter at P is given by $q_h = \exp(2\pi iz/h)$, h being the cusp width at

s . Suppose that $\Theta_i = G_i(q_h)$ and $\Theta = G(t)$, then using notations from the proof of Lemma 2.2.5, we have

$$G_i(q_h) = f_s^{-k}(\alpha^{-1}z) \exp\left(-2\pi i \frac{z}{h} B_{-k,s}\right) F_i(\alpha^{-1}z).$$

This, together with the fact that for a square matrix A ,

$$\det(\exp(A)) = \exp(\text{trace}(A)),$$

gives

$$G(q_h) = f_s^{-nk}(\alpha^{-1}z) \exp\left(-2\pi i \frac{z}{h} \text{trace}(B_{-k,s})\right) F(\alpha^{-1}z) = .$$

$$G(q_h) = f_s^{-nk}(\alpha^{-1}z) q_h^{\nu_{-k,P}(R)} F(\alpha^{-1}z),$$

As $-\text{trace}(B_{-k,s})$ is $\nu_{-k,P}(R)$.

Since $s = \alpha^{-1} \cdot \infty = -d/c$, we have $f_s(\alpha^{-1}z) = c_\alpha J_{\alpha^{-1}}(z)$ for some nonzero constant $c_\alpha \in \mathbb{C}$. Hence

$$G(q_h) = c_\alpha q_h^{\nu_{-k,P}(R)} F|_{nk}(\alpha^{-1}).$$

Applying the logarithmic derivative on both sides we get

$$\frac{2\pi i q_h}{h} \frac{(\partial_{q_h} G)(q_h)}{G(q_h)} = \frac{2\pi i \nu_{-k,P}(R)}{h} + \frac{(F|_{nk}(\alpha^{-1}))'}{F|_{nk}(\alpha^{-1})},$$

and since $f = 2F'/F - g'/g$, we get

$$f|_2(\alpha^{-1}) = 2 \frac{(F|_{nk}(\alpha^{-1}))'}{F|_{nk}(\alpha^{-1})} - \frac{(g|_{2nk}(\alpha^{-1}))'}{g|_{2nk}(\alpha^{-1})}.$$

Therefore

$$f|_2(\alpha^{-1}) = \frac{4\pi i q_h}{h} \frac{(\partial_{q_h} G)(q_h)}{G(q_h)} - \frac{4\pi i \nu_{-k,P}(R)}{h} - \frac{(g|_{nk}(\alpha^{-1}))'}{g|_{nk}(\alpha^{-1})}.$$

Comparing the constant terms of the q_h -expansions and using Lemma 3.1.1, we have

$$(2\pi i/h)\text{Res}_P(\eta) = (4\pi i/h)\nu_P(\Theta) - (4\pi i/h)\nu_{-k,P}(R) - (2\pi i/h)\nu_P(g)$$

and hence the formula. □

Combining the preceding two lemmas with the fact that

$$\sum_{p \in X} \text{Res}_p(\eta) = 0,$$

as X is compact, we get

$$\text{div}(\Theta) = D_{R,-k} + (1/2) \text{div}(g),$$

where $D_{R,-k}$ is the $(-k)$ -divisor of R , see Definition 2.2.3. If η_g is the (nk) -form associated to g , then by Proposition 2.5.4 we have

$$\text{div}(\Theta) = D_{R,-k} + (1/2) \text{div}(\eta_g) + (nk/2)(D'_{\mathfrak{E}} + D_{\mathfrak{S}}).$$

Hence

$$\begin{aligned} \deg(\mathcal{E}_{-k}) &= \deg(\text{div}(\Theta)) = \\ &= \deg(D_{R,-k}) + nk(g_X - 1) + (nk/2)(\deg(D'_{\mathfrak{E}}) + \deg(D_{\mathfrak{S}})), \end{aligned}$$

where g_X is the genus of the compact Riemann surface X . We summarize the above in the following:

Theorem 3.1.4. *Let $c_X = |\mathfrak{S}|$ be the number of cuspidal points of X , g_X be the genus of X . Then the degree of \mathcal{E}_{-k} is given by*

$$\deg(\mathcal{E}_{-k}) = \frac{nk}{2} \left(2g_X - 2 + c_X + \sum_{P \in \mathfrak{E}} (1 - 1/n_P) \right) + \deg(D_{R,-k}).$$

□

Remark 3.1.5. The number m_X defined by

$$m_X = 2g_X - 2 + c_X + \sum_{P \in \mathfrak{E}} (1 - 1/n_P)$$

is positive, see [55], and $2\pi m_X$ is called the hyperbolic area of X . Also, it can be shown that

$$m_X \geq 1/42,$$

see [55].

According to Definition 2.2.3 the k -divisor of R is given by

$$D_{R,k} = \sum_{P \in X} \nu_{k,P}(R) P.$$

Set

$$\widehat{D}_{R,k} = \sum_{P \in X} \Re(\nu_{k,P}(R)) P. \quad (3.1.5)$$

The above theorem implies that the degree of $D_{R,k}$ is a rational number. Hence

$$\deg(D_{R,k}) = \deg(\widehat{D}_{R,k}). \quad (3.1.6)$$

But from Definition 2.2.1, and Definition 2.2.2, we see that

$$-n < \Re(\nu_{k,P}(R)) \leq 0,$$

for all $P \in \mathfrak{E} \cup \mathfrak{S}$, and since the support of $D_{R,k}$ is by definition in $\mathfrak{E} \cup \mathfrak{S}$, we conclude that

$$-n (c_X + e_X) < \deg(\widehat{D}_{R,k}) \leq 0.$$

Now, using (3.1.6), we find

Proposition 3.1.6. *Let $D_{R,k}$ be the k -divisor of R , we have*

$$-n (c_X + e_X) < \deg(D_{R,k}) \leq 0.$$

□

3.2 The holomorphic degree of a vector bundle.

Recall that the dimensions of $M_k(\Gamma, R)$ and $S_k(\Gamma, R)$ are respectively denoted by

$d_k = d_{\Gamma, R, k}$ and $s_k = s_{\Gamma, R, k}$. Applying the Riemann-Roch theorem to the holomorphic vector bundles \mathcal{E}_{-k} and $-\mathcal{D}_{\mathfrak{S}, R, -k} + \mathcal{E}_{-k}$ yields

$$\begin{aligned} h^0(\mathcal{E}_{-k}) - h^0(\mathcal{K}_X + \mathcal{E}_{-k}^*) &= \deg(\mathcal{E}_{-k}) - n(g_X - 1), \\ h^0(-\mathcal{D}_{\mathfrak{S}, R, -k} + \mathcal{E}_{-k}) - h^0(\mathcal{K}_X + \mathcal{D}_{\mathfrak{S}, R, -k} + \mathcal{E}_{-k}^*) &= \\ \deg(-\mathcal{D}_{\mathfrak{S}, R, -k} + \mathcal{E}_{-k}) - n(g_X - 1), \end{aligned}$$

where \mathcal{E}_{-k}^* is the dual holomorphic vector bundle of \mathcal{E}_{-k} . Combining this with Theorem 2.5.8 gives:

$$\begin{aligned} d_k &= \deg(\mathcal{E}_{-k}) - n(g_X - 1) + h^0(\mathcal{K}_X + \mathcal{E}_{-k}^*), \\ s_k &= \deg(-\mathcal{D}_{\mathfrak{S}, R, -k} + \mathcal{E}_{-k}) - n(g - 1) + h^0(\mathcal{K}_X + \mathcal{D}_{\mathfrak{S}, R, -k} + \mathcal{E}_{-k}^*). \end{aligned}$$

Therefore we need to compute $h^0(\mathcal{K}_X + \mathcal{E}_{-k}^*)$ and $h^0(\mathcal{K}_X + \mathcal{D}_{\mathfrak{S}, R, -k} + \mathcal{E}_{-k}^*)$.

The key point in computing d_k and s_k in the case of automorphic forms (*i.e.*, R is the trivial character χ_0) relies on the vanishing of these two dimensions, which is in fact equivalent to the non-existence of holomorphic automorphic forms of negative weight. In higher dimensions, this is no longer valid since for some representations vector-valued automorphic forms of negative weight do exist, see (4.2.7). To handle this problem we shall introduce the notion of the *holomorphic degree of a vector bundle*, which associates to each holomorphic vector bundle \mathcal{E} , defined over a compact Riemann surface, an integer $d(\mathcal{E})$ in a such way that:

$$d(\mathcal{E}) < 0 \implies H^0(X, \mathcal{O}(\mathcal{E})) \text{ is trivial.}$$

We will give some basic properties of the holomorphic degree, and some useful bounds of $d(\mathcal{E})$ that will serve in the next section.

For this section X will be an arbitrary compact Riemann surface of genus g .

Proposition 3.2.1. *Let \mathcal{E} be a vector bundle over a compact Riemann surface X . There exists a constant $C_{\mathcal{E}} \in \mathbb{Z}$ such that*

$$\deg(\operatorname{div}(\Theta)) \leq C_{\mathcal{E}} \quad \text{for all } \Theta \in H^0(X, \mathcal{M}(\mathcal{E})).$$

Proof. If no such constant exists, then we will have a sequence

$$(\Theta_n)_{n \in \mathbb{N}} \in H^0(X, \mathcal{M}(\mathcal{E}))$$

such that $\deg(\operatorname{div}(\Theta_n)) > n$. Let $D_n = \operatorname{div}(\Theta_n)$, and $\mathcal{D}_n = |D_n|$ be its associated holomorphic line bundle. By the Riemann-Roch theorem, we have

$$h^0(\mathcal{D}_n) = \deg(\mathcal{D}_n) + h^0(\mathcal{K}_X - \mathcal{D}_n) - (g - 1).$$

If we take $n > \deg(\mathcal{K}_X) = 2(g - 1)$, then $h^0(\mathcal{K}_X - \mathcal{D}_n) = 0$, and so

$$h^0(\mathcal{D}_n) = \deg(\mathcal{D}_n) - (g - 1).$$

Hence

$$h^0(\mathcal{D}_n) > n - (g - 1). \tag{3.2.1}$$

Let $L(D_n) = \{f \in \mathcal{M}(X) \mid \operatorname{div}(f) + D_n \geq 0\}$. Then the injective homomorphism

$$L(D_n) \longrightarrow H^0(X, \mathcal{M}(\mathcal{E})) : f \mapsto f\Theta_n$$

has its image in $H^0(X, \mathcal{O}(\mathcal{E}))$. Indeed, if $f \in L(D_n)$, then

$$\operatorname{div}(f\Theta_n) = \operatorname{div}(f) + \operatorname{div}(\Theta_n) = \operatorname{div}(f) + D_n \geq 0.$$

Hence $L(D_n)$ imbeds in $H^0(X, \mathcal{O}(\mathcal{E}))$. Now, using the standard fact that $L(D_n)$ is isomorphic to $H^0(X, \mathcal{O}(\mathcal{D}_n))$, we deduce that $H^0(X, \mathcal{O}(\mathcal{D}_n))$ imbeds in $H^0(X, \mathcal{O}(\mathcal{E}))$. Therefore $h^0(\mathcal{E}) \geq h^0(\mathcal{D}_n)$. If $n > 2(g - 1)$, then according to (3.2.1), we have

$$h^0(\mathcal{E}) > n - (g - 1),$$

Thus $h^0(\mathcal{E}) = \infty$, which contradicts Theorem 1.1.12. \square

As a consequence, $\max\{\deg(\operatorname{div}(\Theta)); \Theta \in H^0(X, \mathcal{M}(\mathcal{E}))\}$ is finite.

Definition 3.2.1. Let \mathcal{E} be a holomorphic vector bundle over a compact Riemann surface X , then the holomorphic degree $d(\mathcal{E})$ of \mathcal{E} is defined by

$$d(\mathcal{E}) = \max\{\deg(\operatorname{div}(\Theta)) : \Theta \in H^0(X, \mathcal{M}(\mathcal{E}))\}.$$

Here are some basic properties of the holomorphic degree:

Proposition 3.2.2. *Let X be a compact Riemann surface, \mathcal{E} a holomorphic vector bundle over X and \mathcal{L} a holomorphic line bundle over X . Then we have*

1. $d(\mathcal{L}) = \deg(\mathcal{L})$.
2. $d(\mathcal{L} + \mathcal{E}) = d(\mathcal{L}) + d(\mathcal{E})$.
3. If $d(\mathcal{E}) < 0$, then $h^0(\mathcal{E}) = 0$.

Proof. By definition of $d(\mathcal{L})$, there exists an element $\alpha \in H^0(X, \mathcal{M}(\mathcal{L}))$ such that

$$d(\mathcal{L}) = \deg(\operatorname{div}(\alpha)) = \deg(\mathcal{L})$$

as \mathcal{L} is a line bundle, which proves 1.

To prove 2., take $\alpha \in H^0(X, \mathcal{M}(\mathcal{L}))$ and $\Theta \in H^0(X, \mathcal{M}(\mathcal{E}))$ such that

$$\deg(\operatorname{div}(\alpha)) = d(\mathcal{L}), \quad \deg(\operatorname{div}(\Theta)) = d(\mathcal{E}),$$

then $\alpha\Theta \in H^0(X, \mathcal{M}(\mathcal{L} + \mathcal{E}))$ and we have

$$\deg(\operatorname{div}(\alpha\Theta)) = \deg(\operatorname{div}(\alpha)) + \deg(\operatorname{div}(\Theta)) = d(\mathcal{L}) + d(\mathcal{E}).$$

Let $\Theta_1 \in H^0(X, \mathcal{M}(\mathcal{L} + \mathcal{E}))$. Then $\Theta_1/\alpha \in H^0(X, \mathcal{M}(\mathcal{E}))$ and we have

$$\deg(\operatorname{div}(\Theta_1)) = \deg(\operatorname{div}(\alpha)) + \deg(\operatorname{div}(\Theta_1/\alpha)) \leq$$

$$\deg(\operatorname{div}(\alpha)) + \deg(\operatorname{div}(\Theta)) = d(\mathcal{L}) + d(\mathcal{E}).$$

Therefore

$$d(\mathcal{L} + \mathcal{E}) \leq d(\mathcal{L}) + d(\mathcal{E}) = \deg(\operatorname{div}(\alpha\Theta)) \leq d(\mathcal{L} + \mathcal{E}).$$

Hence $d(\mathcal{L} + \mathcal{E}) = d(\mathcal{L}) + d(\mathcal{E})$.

As for 3., suppose that $d(\mathcal{E}) < 0$. If there exists a nonzero element $\Theta \in H^0(X, \mathcal{O}(\mathcal{E}))$, then we will have

$$0 \leq \deg(\operatorname{div}(\Theta)) \leq d(\mathcal{E}),$$

which is impossible. Hence $h^0(\mathcal{E}) = 0$. □

We have the following upper bound of $d(\mathcal{E})$:

Theorem 3.2.3. *Let \mathcal{E} be a holomorphic vector bundle over X of holomorphic degree $d(\mathcal{E})$. We have*

$$d(\mathcal{E}) \leq h^0(\mathcal{E}) + g - 1.$$

Proof. Take an element Θ of $H^0(X, \mathcal{M}(\mathcal{E}))$ such that $\deg(\operatorname{div}(\Theta)) = d(\mathcal{E})$, and set $D = \operatorname{div}(\Theta)$ and $\mathcal{D} = |D|$. As in the proof of Proposition 3.2.1, we have an embedding of $H^0(X, \mathcal{O}(\mathcal{D}))$ in $H^0(X, \mathcal{O}(\mathcal{E}))$. Therefore

$$h^0(\mathcal{D}) \leq h^0(\mathcal{E}).$$

By the Riemann-Roch theorem, we have

$$h^0(\mathcal{D}) = \deg(\mathcal{D}) + h^0(\mathcal{K}_X - \mathcal{D}) - (g - 1),$$

which implies that

$$\deg(\mathcal{D}) - (g - 1) \leq h^0(\mathcal{D}).$$

Since $\deg(\mathcal{D}) = \deg(D) = d(\mathcal{E})$, we have

$$d(\mathcal{E}) - (g - 1) \leq h^0(\mathcal{D}) \leq h^0(\mathcal{E}),$$

hence

$$d(\mathcal{E}) \leq h^0(\mathcal{E}) + g - 1.$$

□

Now we will give a lower bound for $d(\mathcal{E})$. Applying the Riemann-Roch theorem to the holomorphic vector bundle \mathcal{E} , we get

$$h^0(\mathcal{E}) = h^0(\mathcal{K}_X + \mathcal{E}^*) + \deg(\mathcal{E}) - n(g - 1),$$

where n is the rank of \mathcal{E} . Thus

$$\deg(\mathcal{E}) - n(g - 1) \leq h^0(\mathcal{E}). \quad (3.2.2)$$

Let P be any point of X , and $\mathcal{L} = |P|$ be the line bundle associated to the divisor P . If \mathcal{E}_1 is defined by

$$\mathcal{E}_1 = -(d(\mathcal{E}) + 1)\mathcal{L} + \mathcal{E},$$

then we have

$$d(\mathcal{E}_1) = -(d(\mathcal{E}) + 1)d(\mathcal{L}) + d(\mathcal{E}) = -(d(\mathcal{E}) + 1) + d(\mathcal{E}) = -1.$$

Hence, by Proposition 3.2.2, we have $h^0(\mathcal{E}_1) = 0$. By (3.2.2) applied to \mathcal{E}_1 , we have

$$\deg(\mathcal{E}_1) - n(g - 1) \leq h^0(\mathcal{E}_1) = 0.$$

Using Proposition 1.1.9, we get

$$-n(d(\mathcal{E}) + 1) \deg(\mathcal{L}) + \deg(\mathcal{E}) - n(g - 1) \leq 0,$$

and since $\deg(\mathcal{L}) = 1$, we conclude that

$$(1/n) \deg(\mathcal{E}) - g \leq d(\mathcal{E}).$$

Thus we have proved:

Theorem 3.2.4. *Let \mathcal{E} be a rank n holomorphic vector bundle over X of holomorphic degree $d(\mathcal{E})$. We have*

$$\frac{\deg(\mathcal{E})}{n} - g \leq d(\mathcal{E}).$$

□

3.3 The dimension of $M_k(\Gamma, R)$ and $S_k(\Gamma, R)$

We now come to one of the main results of this thesis. The goal is to give formulas for the dimensions of $M_k(\Gamma, R)$ and $S_k(\Gamma, R)$. We will show that there are some constants $k_{R,X}^+$ and $k_{R,X}^-$ such that:

1. For $k < k_{R,X}^-$, $M_k(\Gamma, R)$ will be trivial.
2. For $k > k_{R,X}^+$, we will have an explicit formula for the dimensions of $M_k(\Gamma, R)$ and $S_k(\Gamma, R)$ expressed in terms of some invariants of R and X .

Recall that

$$D_{\mathfrak{E}} = \sum_{P \in \mathfrak{E}} P, \quad D_{\mathfrak{S}} = \sum_{P \in \mathfrak{S}} P,$$

$D_{R,-k}$ is the $(-k)$ -divisor of R , and that $D_{\mathfrak{S},R,-k}$ is the $(-k)$ -cuspidal divisor of R , see Definition 2.2.3 and Definition 2.4.1. Their associated line bundles are respectively denoted by $\mathcal{D}_{\mathfrak{E}}$, $\mathcal{D}_{\mathfrak{S}}$, $\mathcal{D}_{R,-k}$, and $\mathcal{D}_{\mathfrak{S},R,-k}$. Also, recall that

$$m_X = 2g_X - 2 + c_X + \sum_{P \in \mathfrak{E}} (1 - 1/n_P),$$

and that the parity index of R is denoted by $\varepsilon = \varepsilon(R)$ according to Definition 2.1.3.

Applying the Riemann-Roch theorem to the holomorphic vector bundles \mathcal{E}_{-k} and $-\mathcal{D}_{\mathfrak{S},R,-k} + \mathcal{E}_{-k}$, we get:

$$h^0(\mathcal{E}_{-k}) - h^0(\mathcal{K}_X + \mathcal{E}_{-k}^*) = \deg(\mathcal{E}_{-k}) - n(g_X - 1),$$

$$h^0(-\mathcal{D}_{\mathfrak{S},R,-k} + \mathcal{E}_{-k}) - h^0(\mathcal{K}_X + \mathcal{D}_{\mathfrak{S},R,-k} + \mathcal{E}_{-k}^*) = \deg(-\mathcal{D}_{\mathfrak{S},R,-k} + \mathcal{E}_{-k}) - n(g_X - 1).$$

Combining this with Theorem 2.5.8 gives:

$$d_k = \deg(\mathcal{E}_{-k}) - n(g_X - 1) + h^0(\mathcal{K}_X + \mathcal{E}_{-k}^*),$$

$$s_k = \deg(-\mathcal{D}_{\mathfrak{S},R,-k} + \mathcal{E}_{-k}) - n(g_X - 1) + h^0(\mathcal{K}_X + \mathcal{D}_{\mathfrak{S},R,-k} + \mathcal{E}_{-k}^*),$$

where \mathcal{E}_{-k}^* is the dual holomorphic vector bundle of \mathcal{E}_{-k} . We have the following formula for \mathcal{E}_{-k}^* .

Lemma 3.3.1. *If R^* is the adjoint representation of R , then*

$$\mathcal{E}_{R,-k}^* = \mathcal{D}_{\mathfrak{e}} + \mathcal{D}_{\mathfrak{S}} + \mathcal{E}_{R^*,k}.$$

Proof. This is a direct consequence of the Remarks (2.2.2, 2.2.4, 2.2.6), and Theorem 1.1.3. The only nontrivial part is the contribution of the line bundles $\mathcal{D}_{\mathfrak{e}} = |D_{\mathfrak{e}}|$ and $\mathcal{D}_{\mathfrak{S}} = |D_{\mathfrak{S}}|$. Using notations of Remark 2.2.4 and Remark 2.2.6, this can be justified by the fact that the local parameter at an elliptic (resp. cuspidal) point $P \in X$ corresponding to $e \in \mathbb{H}$ (resp. $s \in \mathbb{H}^*$) is $t = w^{ne}$ (resp. $\exp(2\pi iz/h)$), and that the section defining $D_{\mathfrak{e}}$ (resp. $D_{\mathfrak{S}}$) is given near P by the function t (resp. q_h), see [19]. \square

As a consequence we have:

Proposition 3.3.2. *We have*

$$d_k = \deg(\mathcal{E}_{-k}) - n(g_X - 1) + h^0(\mathcal{K}_X + \mathcal{D}_{\mathfrak{e}} + \mathcal{D}_{\mathfrak{S}} + \mathcal{E}_{R^*,k}),$$

and

$$s_k = \deg(-\mathcal{D}_{\mathfrak{S},R,-k} + \mathcal{E}_{-k}) - n(g_X - 1) + h^0(\mathcal{K}_X + \mathcal{D}_{\mathfrak{e}} + \mathcal{D}_{\mathfrak{S}} + \mathcal{D}_{\mathfrak{S},R,-k} + \mathcal{E}_{R^*,k}).$$

\square

We will show that for k large enough, the numbers $h^0(\mathcal{K}_X + \mathcal{D}_{\mathfrak{e}} + \mathcal{D}_{\mathfrak{S}} + \mathcal{E}_{R^*,k})$ and $h^0(\mathcal{K}_X + \mathcal{D}_{\mathfrak{e}} + \mathcal{D}_{\mathfrak{S}} + \mathcal{D}_{\mathfrak{S},R,-k} + \mathcal{E}_{R^*,k})$ vanish, thus providing exact formulas for the dimensions d_k and s_k . We only treat the first case, the second one follows in a similar way. Using Theorem 2.5.10, we have

$$h^0(\mathcal{K}_X + \mathcal{D}_{\mathfrak{e}} + \mathcal{D}_{\mathfrak{S}} + \mathcal{E}_{R^*,k}) \leq h^0(\mathcal{K}_X + 2\mathcal{D}_{\mathfrak{e}} + \mathcal{D}_{\mathfrak{S}} + \mathcal{L}_{(k+\varepsilon)} + \mathcal{E}_{R^*,-\varepsilon}).$$

Hence, by Proposition 3.2.2, it suffices to show that

$$d(\mathcal{K}_X + 2\mathcal{D}_{\mathfrak{e}} + \mathcal{D}_{\mathfrak{S}} + \mathcal{L}_{(k+\varepsilon)} + \mathcal{E}_{R^*,-\varepsilon}) < 0,$$

which is equivalent to

$$d(\mathcal{K}_X) + 2d(\mathcal{D}_{\mathfrak{E}}) + d(\mathcal{D}_{\mathfrak{S}}) + d(\mathcal{L}_{(k+\varepsilon)}) + d(\mathcal{E}_{R^*, -\varepsilon}) < 0,$$

that is

$$(2g_X - 2 + 2e_X + c_X) + d(\mathcal{L}_{(k+\varepsilon)}) + d(\mathcal{E}_{R^*, -\varepsilon}) < 0. \quad (3.3.1)$$

By Theorem 3.1.4, the degree of $\mathcal{L}_{(k+\varepsilon)}$ is given by

$$d(\mathcal{L}_{(k+\varepsilon)}) = -\frac{(k+\varepsilon)m_X}{2} + \deg(D_{\chi_0, k+\varepsilon}),$$

where χ_0 is the trivial character sending all elements of Γ to 1, see Definition 2.2.4. By definition, the support of the divisor

$$D_{\chi_0, k+\varepsilon} = \sum_{P \in X} \nu_{k+\varepsilon, P}(\chi_0) P$$

lies in $\mathfrak{E} \cup \mathfrak{S}$, with coefficients in $(-1, 0]$. Therefore

$$\deg(D_{\chi_0, k+\varepsilon}) \leq 0.$$

Hence, to get (3.3.1), it suffices to have

$$d(\mathcal{E}_{R^*, -\varepsilon}) + (2g_X - 2 + c_X + 2e_X) - \frac{(k+\varepsilon)m_X}{2} < 0,$$

or equivalently,

$$\frac{2}{m_X} (d(\mathcal{E}_{R^*, -\varepsilon}) + 2g_X - 2 + c_X + 2e_X) - \varepsilon < k.$$

Now, using Theorem 3.1.4 and Proposition 3.3.2, we get

Theorem 3.3.3. *Let Γ be a Fuchsian group of the first kind, and X be the associated compact Riemann surface. If*

$$k_{X,R}^+ = \frac{2}{m_X} (d(\mathcal{E}_{R^*, -\varepsilon}) + 2g_X - 2 + c_X + 2e_X) - \varepsilon,$$

then

1. For $k > k_{R,X}^+$, the dimension of $M_k(\Gamma, R)$ is given by

$$d_k = \frac{nk m_X}{2} - n(g_X - 1) + \deg(D_{R,-k}).$$

2. For $k > k_{X,R}^+ + (2/m_X) \deg(D_{\mathfrak{S},R,-k})$, the dimension of $S_k(\Gamma, R)$ is given by

$$s_k = d_k - n \deg(D_{\mathfrak{S},R,-k}).$$

□

If we relax the lower bound of k , we get the following:

Corollary 3.3.4. For

$$k \geq 84 (h^0(\mathcal{E}_{R^*,-\varepsilon}) + g_X + c_X + 2e_X),$$

we have

1. The dimension of $M_k(\Gamma, R)$ is given by

$$d_k = \frac{nk m_X}{2} - n(g_X - 1) + \deg(D_{R,-k}).$$

2. The dimension of $S_k(\Gamma, R)$ is given by

$$s_k = d_k - n \deg(D_{\mathfrak{S},R,-k}).$$

Proof. By Remark 3.1.5, we have

$$(2/m_X) \leq (2)42 = 84.$$

Also, by Theorem 3.2.3, we have

$$d(\mathcal{E}_{R^*,-\varepsilon}) \leq h^0(\mathcal{E}_{R^*,-\varepsilon}) + (g_X - 1).$$

By definition, $D_{\mathfrak{S},R,-k} \leq D_{\mathfrak{S}}$, and so $\deg(D_{\mathfrak{S},R,-k}) \leq \deg(D_{\mathfrak{S}}) = c_X$. Combining these inequalities with the fact that $(2g_X - 2 + c_X)/m_X \leq 1$, and $\varepsilon \in \{0, 1\}$, we have

$$k_{X,R}^+ \leq k_{X,R}^+ + (2/m_X) \deg(D_{\mathfrak{S},R,-k}) \leq k_{X,R}^+ + (2/m_X)c_X =$$

$$\begin{aligned}
& (2/m_X) (d(\mathcal{E}_{R^*, -\varepsilon}) + 2g_X - 2 + 2c_X + 2e_X) - \varepsilon \leq \\
& (2/m_X) (h^0(\mathcal{E}_{R^*, -\varepsilon}) + (g_X - 1) + c_X + 2e_X) + 2/(nm_X)(2g_X - 2 + c_X) \leq \\
& 84 (h^0(\mathcal{E}_{R^*, -\varepsilon}) + (g_X - 1) + c_X + 2e_X) + (2/n) \leq \\
& 84 (h^0(\mathcal{E}_{R^*, -\varepsilon}) + g_X + c_X + 2e_X).
\end{aligned}$$

We conclude using Theorem 3.3.3. \square

Now, recall from Proposition 3.1.6 that

$$-n (c_X + e_X) < \deg(D_{R,k}) \leq 0,$$

so that $|\deg(D_{R,k})| < n (c_X + e_X)$. Since $0 \leq \deg(D_{\mathfrak{G}, R, -k}) \leq c_X$, then by Corollary 4.1.3, we have

Corollary 3.3.5. *Let Γ be a Fuchsian group of the first kind. The dimensions d_k of $M_k(\Gamma, R)$, and s_k of $S_k(\Gamma, R)$ are asymptotically equivalent to k . More precisely, we have*

$$\frac{d_k}{k} \longrightarrow \frac{n m_X}{2}, \text{ as } k \rightarrow \infty,$$

and

$$\frac{s_k}{k} \longrightarrow \frac{n m_X}{2}, \text{ as } k \rightarrow \infty.$$

Combining this result with Theorem 2.5.9, we have

Theorem 3.3.6. *Holomorphic, cusp vector-valued automorphic forms exist for any Fuchsian group Γ and any n -dimensional representation R of Γ .*

Using the upper bounds of d_k and s_k from Corollary 2.5.11, we have

Theorem 3.3.7. *Let Γ be a Fuchsian group of the first kind, X be the associated compact Riemann surface. If*

$$k_{X,R}^- = \frac{-2}{m_X} (d(\mathcal{E}_{R, -\varepsilon}) + e_X) - \varepsilon,$$

then

1. For $k < k_{R,X}^-$, the dimension d_k of $M_k(\Gamma, R)$ is zero.
2. For $k < k_{X,R}^- + (2/m_X) \deg(D_{\mathfrak{S},R,-k})$, the dimension s_k of $S_k(\Gamma, R)$ is zero.

Proof. By Corollary 2.5.11, we have

$$d_k \leq h^0(\mathcal{D}_{\mathfrak{E}} + \mathcal{L}_{-(k-\varepsilon)} + \mathcal{E}_{-\varepsilon}),$$

$$s_k \leq h^0(-\mathcal{D}_{\mathfrak{S},R,-k} + \mathcal{D}_{\mathfrak{E}} + \mathcal{L}_{-(k-\varepsilon)} + \mathcal{E}_{-\varepsilon}).$$

Hence it suffices to show that

$$d(\mathcal{D}_{\mathfrak{E}} + \mathcal{L}_{-(k-\varepsilon)} + \mathcal{E}_{-\varepsilon}) < 0, \quad (3.3.2)$$

$$d(-\mathcal{D}_{\mathfrak{S},R,-k} + \mathcal{D}_{\mathfrak{E}} + \mathcal{L}_{-(k-\varepsilon)} + \mathcal{E}_{-\varepsilon}) < 0. \quad (3.3.3)$$

We only treat the first case. As in the proof of the Theorem 3.3.3, we have

$$\begin{aligned} d(\mathcal{D}_{\mathfrak{E}} + \mathcal{L}_{-(k-\varepsilon)} + \mathcal{E}_{R,-\varepsilon}) &= \\ d(\mathcal{D}_{\mathfrak{E}}) + d(\mathcal{L}_{-(k-\varepsilon)}) + d(\mathcal{E}_{R,-\varepsilon}) &\leq \\ e_X + \frac{(k+\varepsilon)m_X}{2} + d(\mathcal{E}_{-\varepsilon}). \end{aligned}$$

To get (3.3.2), it is enough to have

$$e_X + \frac{(k+\varepsilon)m_X}{2} + d(\mathcal{E}_{-\varepsilon}) < 0,$$

that is

$$k < \frac{-2}{m_X} (d(\mathcal{E}_{R,-\varepsilon}) + e_X) - \varepsilon = k_{X,R}^-.$$

□

As in Corollary 4.1.3, we have the following simpler upper bound for k :

Corollary 3.3.8. *If*

$$k < -84 (h^0(\mathcal{E}_{R,-\varepsilon}) + (g_X - 1) + e_X) - 1,$$

then $M_k(\Gamma, R)$ and $S_k(\Gamma, R)$ are trivial.

□

Remark 3.3.9. Notice that for values of k in the interval $[k_{X,R}^-, k_{X,R}^+]$ we have:

$$d_k \geq \deg(\mathcal{E}_{-k}) - n(g_X - 1),$$

and

$$s_k \geq \deg(-\mathcal{D}_{\mathfrak{S},R,-k} + \mathcal{E}_{-k}) - n(g_X - 1).$$

This is a direct consequence of Proposition 3.3.2.

3.4 Finite image representations

Due to its richness and simplicity, the case of finite image representations has been treated by many authors, see [3, 15]. Our aim in this section is to provide bounds for the weight k simpler than $k_{X,R}^+$ and $k_{X,R}^-$. For example, we will see that $k_{X,R}^-$ can be replaced by 0. For the sake of clarity and to avoid technical details, we restrict ourselves to the case of even representations, that is $\varepsilon = \varepsilon(R) = 0$.

Let

$$\Gamma_R := \ker(R), \quad \overline{\Gamma}_R := \{\pm 1\} \backslash \Gamma_R \cdot \{\pm 1\}, \quad X_R := \overline{\Gamma}_R \backslash \mathbb{H}^*.$$

Since the image of R is finite, Γ_R has finite index in Γ . Therefore, Γ_R is a Fuchsian group of the first kind, and X_R is a compact Riemann surface, see Proposition 1.2.9. Suppose that $F \in A_k(\Gamma, R)$, by definition we have

$$F|_k \gamma = R_\gamma F, \quad \gamma \in \Gamma,$$

and so for $\gamma \in \Gamma_R$ we get

$$F|_k \gamma = F.$$

Hence all components of F are weight k automorphic forms for Γ_R , that is $F \in (A_k(\Gamma_R))^n$. As Γ_R has finite index in Γ , Γ and Γ_R have the same cusps, and the cuspidal behavior of F is the same with respect to Γ and Γ_R . Therefore we have the following two embeddings

$$M_k(\Gamma, R) \longrightarrow (M_k(\Gamma_R))^n : F \longrightarrow F, \quad (3.4.1)$$

$$S_k(\Gamma, R) \longrightarrow (S_k(\Gamma_R))^n : F \longrightarrow F. \quad (3.4.2)$$

Since $M_k(\Gamma_R)$ is trivial for $k < 0$, we deduce that $M_k(\Gamma, R)$ and $S_k(\Gamma, R)$ are both trivial for $k < 0$.

The next step is to simplify the bound

$$k_{X,R}^+ = \frac{2}{m_X} (d(\mathcal{E}_{R^*, -\varepsilon}) + 2g_X - 2 + c_X + 2e_X) - \varepsilon$$

given by Theorem 3.3.3 (here $\varepsilon = 0$). Suppose that $d(\mathcal{E}_{R^*, -\varepsilon}) - (g_X - 1) > 0$, and take $\Theta_1 \in H^0(X, \mathcal{M}(\mathcal{E}_{R^*, 0}))$ such that $\deg(\operatorname{div}(\Theta_1)) = d(\mathcal{E}_{R^*, 0})$. Set $D = \operatorname{div}(\Theta_1)$ and $\mathcal{D} = |D|$ its associated holomorphic line bundle. By the Riemann-Roch theorem, we have

$$h^0(\mathcal{D}) = \deg(\mathcal{D}) + h^0(\mathcal{K}_X - \mathcal{D}) - (g_X - 1) \geq d(\mathcal{E}_{R^*, -\varepsilon}) - (g_X - 1) > 0.$$

Hence $H^0(X, \mathcal{O}(\mathcal{D}))$ is nontrivial. Let δ be an element of $H^0(X, \mathcal{M}(\mathcal{D}))$ such that $\operatorname{div}(\delta) = D$, and take $\sigma \in H^0(X, \mathcal{O}(\mathcal{D}))$. Then $f = \sigma/\delta$ is a meromorphic function on X satisfying

$$\operatorname{div}(f) + D = \operatorname{div}(\sigma) \geq 0.$$

Therefore,

$$\operatorname{div}(f \Theta_1) = \operatorname{div}(f) + D \geq 0,$$

and hence $\Theta = f \Theta_1 \in H^0(X, \mathcal{O}(\mathcal{E}_{R^*, 0}))$. Let F be the element of $M_0(\Gamma, R^*)$ corresponding to Θ . Since the elements of $M_0(\Gamma_R)$ correspond to holomorphic functions on the compact Riemann surface X_R , we have $M_0(\Gamma_{R^*}) \subseteq \mathbb{C}^n$. By the embedding of $M_0(\Gamma, R^*)$ in $M_0(\Gamma_R^*)$, see (3.4.1), we deduce that Θ is a nonzero vector in \mathbb{C}^n . Therefore, $\operatorname{div}(F)$ is the trivial divisor. Since $[\operatorname{div}(F)] = \operatorname{div}(\Theta)$ and $\deg(\operatorname{div}(f)) = 0$, we have

$$0 = \deg(\operatorname{div}(\Theta)) = \deg(\operatorname{div}(\Theta_1)) = \deg(D) = d(\mathcal{E}_{R^*, 0}).$$

We conclude that if $d(\mathcal{E}_{R^*, -\varepsilon}) - (g_X - 1) > 0$, then $d(\mathcal{E}_{R^*, 0}) = 0$, and so $g_X = 0$. Hence, in all cases

$$d(\mathcal{E}_{R^*, 0}) \leq g_X.$$

Using the fact that $m_X \geq 2g_X - 2 + c_X$, we see that

$$k_{X,R}^+ \leq 2 + \frac{2}{m_X} (g_X + 2e_X).$$

Now, by Theorem 3.3.3 we conclude that

Theorem 3.4.1. *Let Γ be a Fuchsian group of the first kind, X be the associated compact Riemann surface, and R be an even finite image representation. If*

$$f_{X,R} = 2 + \frac{2}{m_X} (g_X + 2e_X),$$

then

1. For $k > f_{R,X}$, the dimension of $M_k(\Gamma, R)$ is given by

$$d_k = \frac{nk m_X}{2} - n(g_X - 1) + \deg(D_{R,-k}).$$

2. For $k > f_{X,R} + (2/m_X) \deg(D_{\mathfrak{S},R,-k})$, the dimension of $S_k(\Gamma, R)$ is given by

$$s_k = d_k - n \deg(D_{\mathfrak{S},R,-k}).$$

3. For $k < 0$, $M_k(\Gamma, R)$ and $S_k(\Gamma, R)$ are both trivial.

□

As we have seen in the discussion preceding Theorem 3.4.1, $M_0(\Gamma, R)$ is a subset of \mathbb{C}^n . Combining this with the definition of vector valued automorphic forms of weight 0 and multiplier R , and the fact that the elements of $S_0(\Gamma, R) \subseteq M_0(\Gamma, R) \subseteq \mathbb{C}^n$ vanish at the cusps, we have

Corollary 3.4.2. *Let Γ be a Fuchsian group of the first kind, X be the associated compact Riemann surface, and R be an even finite image representation. Then $S_0(\Gamma, R)$ is trivial, and we have*

$$M_0(\Gamma, R) = \{v \in \mathbb{C}^n \mid R_\gamma(v) = v \text{ for all } \gamma \in \Gamma\}.$$

Chapter 4

Applications

4.1 Generalized automorphic forms

Since the work of M. Knopp and G. Mason in [28], the theory of generalized automorphic forms has been receiving an increasing interest, see [25, 26, 29, 32, 33, 44, 45]. We propose here to compute the dimensions of the vector spaces of generalized automorphic forms. In this section R will be a 1–dimensional representation (a character) and will be denoted by χ . The holomorphic vector bundle $\mathcal{E}_{\Gamma, \chi, k}$ is then a line bundle, and will be denoted by $\mathcal{I}_{\Gamma, \chi, k} = \mathcal{I}_k$.

The generalized automorphic forms look like classical modular forms with a multiplier system except for the fact that the character χ need not be unitary. They can be defined as 1–dimensional vector valued automorphic forms. More precisely,

Definition 4.1.1. Let (χ, k) , $k \in \mathbb{Z}$, be a simple pair. Then

1. An element of $A_k(\Gamma, \chi)$ is called a generalized automorphic form for Γ of multiplier χ and weight k .
2. An element of $G_k(\Gamma, \chi)$ (resp. $M_k(\Gamma, \chi)$, $S_k(\Gamma, \chi)$) is called an unre-

stricted (resp. holomorphic, cusp) generalized automorphic form for Γ of multiplier χ and weight k .

Recall that the dimensions of $M_k(\Gamma, \chi)$ and $S_k(\Gamma, \chi)$ are denoted by $d_{\Gamma, \chi, k} = d_k$ and $s_{\Gamma, \chi, k} = s_k$ respectively. Also, according to Definition 2.4.1, $\mathcal{D}_{\mathfrak{S}, \chi, -k}$ denotes the holomorphic line bundle associated to the k -cuspidal divisor of χ

$$D_{\mathfrak{S}, \chi, k} = \sum_{P \in \mathfrak{S}} \rho_{k, P}(\chi) P.$$

To compute d_k and s_k , we follow the same steps as in §3.3. Indeed, applying the Riemann-Roch theorem to \mathcal{I}_{-k} and $-\mathcal{D}_{\mathfrak{S}, \chi, -k} + \mathcal{I}_{-k}$ yields

$$h^0(\mathcal{I}_{-k}) - h^0(\mathcal{K}_X + \mathcal{I}_{-k}^*) = \deg(\mathcal{I}_{-k}) - (g_X - 1),$$

$$h^0(-\mathcal{D}_{\mathfrak{S}, \chi, -k} + \mathcal{I}_{-k}) - h^0(\mathcal{K}_X + \mathcal{D}_{\mathfrak{S}, \chi, -k} + \mathcal{I}_{-k}^*) = \deg(-\mathcal{D}_{\mathfrak{S}, \chi, -k} + \mathcal{I}_{-k}) - (g_X - 1),$$

where \mathcal{I}_{-k}^* is the dual holomorphic line bundle of \mathcal{E}_{-k} . Combining this with Theorem 2.5.8, and the fact that $\mathcal{I}_{-k}^* = -\mathcal{I}_{-k}$ gives:

$$d_k = \deg(\mathcal{I}_{-k}) - (g - 1) + h^0(\mathcal{K}_X - \mathcal{I}_{-k}),$$

$$s_k = \deg(-\mathcal{D}_{\mathfrak{S}, R, -k} + \mathcal{I}_{-k}) - (g - 1) + h^0(\mathcal{K}_X + \mathcal{D}_{\mathfrak{S}, R, -k} - \mathcal{I}_{-k}).$$

We will show that for k large enough, the numbers $h^0(\mathcal{K}_X - \mathcal{I}_{-k})$ and $h^0(\mathcal{K}_X + \mathcal{D}_{\mathfrak{S}, \chi, -k} - \mathcal{I}_{-k})$ vanish. Thus we find an exact formula for the dimensions d_k and s_k . As usual, we only treat the first case. By Proposition 3.2.2, $h^0(\mathcal{K}_X - \mathcal{I}_{-k})$ is zero if

$$d(\mathcal{K}_X - \mathcal{I}_{-k}) = d(\mathcal{K}_X) - d(\mathcal{I}_{-k}) < 0,$$

that is

$$2(g_X - 1) - d(\mathcal{I}_{-k}) < 0.$$

By Theorem 3.1.4, the degree of \mathcal{I}_{-k} is given by

$$d(\mathcal{I}_{-k}) = (k/2)m_X + \deg(D_{\chi, -k}).$$

Hence, $d(\mathcal{K}_X - \mathcal{I}_{-k}) < 0$ if and only if

$$2(g_X - 1) - (k/2)m_X - \deg(D_{X,-k}) < 0.$$

By definition, the support of the divisor

$$D_{X,-k} = \sum_{P \in X} \nu_{-k,P}(\chi) P$$

lies in $\mathfrak{E} \cup \mathfrak{S}$, with coefficients in $(-1, 0]$, and so

$$-\deg(D_{X,-k}) \leq c_X + e_X.$$

Therefore, to get $d(\mathcal{K}_X - \mathcal{I}_{-k}) < 0$ it suffices to have

$$2(g_X - 1) - (k/2)m_X + c_X + e_X < 0,$$

in other words,

$$(2/m_X)(2(g_X - 1) + c_X + e_X) < k.$$

But

$$(2/m_X)(2(g_X - 1) + c_X + e_X) = m_X + \sum_{P \in \mathfrak{E}} \frac{1}{n_P}.$$

Thus we conclude that $h^0(\mathcal{K}_X - \mathcal{I}_{-k}) = 0$ if

$$2 + \frac{2}{m_X} \sum_{P \in \mathfrak{E}} \frac{1}{n_P} < k.$$

Using Theorem 3.1.4 and Proposition 3.3.2, we have

Theorem 4.1.1. *Let Γ be a Fuchsian group of the first kind, X be its compact Riemann surface, and let*

$$m_X = 2g_X - 2 + c_X + \sum_{P \in \mathfrak{E}} (1 - 1/n_P),$$

$$l_{X,X}^+ = 2 + \frac{2}{m_X} \sum_{P \in \mathfrak{E}} \frac{1}{n_P}.$$

Then:

1. For $k > l_{X,X}^+$, the dimension of $M_k(\Gamma, \chi)$ is given by

$$d_k = (k/2) m_X - (g_X - 1) + \deg(D_{X,-k}).$$

2. For $k > l_{X,X}^+ + (2/m_X) \deg(D_{\mathfrak{E},X,-k})$, the dimension of $S_k(\Gamma, \chi)$ is

$$s_k = d_k - \deg(D_{\mathfrak{E},X,-k}).$$

We have the following simplification for the lower bound of k .

Corollary 4.1.2. *Suppose that $g_X \geq 1$. Then*

1. For $k > 4$, the dimension of $M_k(\Gamma, \chi)$ is

$$d_k = (k/2) m_X - (g_X - 1) + \deg(D_{X,-k}).$$

2. For $k > 6$, the dimension of $S_k(\Gamma, \chi)$ is

$$s_k = d_k - \deg(D_{\mathfrak{E},X,-k}).$$

Proof. First we show that m_X is greater than c_X and $\sum_{P \in \mathfrak{E}} (1/n_P)$. Indeed, if $g_X \geq 1$, then $2(g_X - 1) \geq 0$, and hence all the terms in the expression of m_X are nonnegative. This implies that $m_X \geq c_X$. If Γ has an elliptic point P , then by definition $n_P \geq 2$. Therefore

$$\sum_{P \in \mathfrak{E}} \frac{1}{n_P} \leq \sum_{P \in \mathfrak{E}} (1 - 1/n_P) \leq m_X.$$

If Γ is torsion-free, this inequality is obviously satisfied, since $\sum_{P \in \mathfrak{E}} (1/n_P)$ is interpreted as 0. Now, we have

$$l_{X,X}^+ = 2 + (2/m_X) \sum_{P \in \mathfrak{E}} (1/n_P) \leq 2 + 2 = 4.$$

Also, since $D_{\mathfrak{E},R,-k} \leq D_{\mathfrak{E}}$, we have $\deg(D_{\mathfrak{E},R,-k}) \leq \deg(D_{\mathfrak{E}}) = c_X$, and so

$$l_{X,X}^+ + (2/m_X) \deg(D_{\mathfrak{E},X,-k}) \leq 4 + (2/m_X) c_X \leq 4 + 2 = 6.$$

The formulas follow from Theorem 4.1.1. □

When g_X is zero, we have the following.

Corollary 4.1.3. *Suppose that $g_X = 0$. If $c_X \geq 2$, then*

1. *For $k > 4$, the dimension of $M_k(\Gamma, \chi)$ is*

$$d_k = (k/2)m_X + \deg(D_{\chi, -k}) + 1.$$

2. *For $k > 4 + \max\{6, 4/m_X\}$, the dimension of $S_k(\Gamma, \chi)$ is*

$$s_k = d_k - \deg(D_{\mathfrak{S}, \chi, -k}).$$

Moreover, we have $\max\{6, 4/m_X\} \leq 168$.

Proof. Suppose that $g_X = 0$. If $c_X \geq 2$, then all the terms in the expression of m_X are nonnegative. As in the preceding proof, we conclude that $l_{\chi, X}^+ \leq 4$. Since $\deg(D_{\mathfrak{S}, R, -k}) \leq c_X$, we have

$$l_{\chi, X}^+ + (2/m_X) \deg(D_{\mathfrak{S}, \chi, -k}) \leq 4 + (2/m_X) c_X.$$

Suppose that $c_X > 2$. Since $m_X \geq (c_X - 2)$, we have

$$(2/m_X) c_X \leq 2c_X/(c_X - 2) = 2 + 4/(c_X - 2) \leq 6.$$

Hence

$$l_{\chi, X}^+ + (2/m_X) \deg(D_{\mathfrak{S}, \chi, -k}) \leq 4 + \max\{6, 4/m_X\}.$$

The result follows from Theorem 4.1.1.

By Remark 3.1.5, we have

$$(4/m_X) \leq (4)42 = 168.$$

Hence $\max\{6, 4/m_X\} \leq 168$. □

Remark 4.1.4. In the remaining case, that is $g_X = 0$ and $c_X < 2$, one can show that Γ is generated by elliptic elements, and so the image of χ is finite. Therefore, our generalized automorphic forms are classical automorphic forms.

Rewriting Theorem 3.3.7 in the 1-dimensional case gives the following:??

Theorem 4.1.5. *Let Γ be a Fuchsian group of the first kind, X be its compact Riemann surface, and*

$$m_X = 2g_X - 2 + c_X + \sum_{P \in \mathfrak{E}} (1 - 1/n_P).$$

Then:

1. *For $k < 0$, the dimension d_k of $M_k(\Gamma, \chi)$ is zero.*
2. *For $k < (2/m_X) \deg(D_{\mathfrak{S}, \chi, -k})$, the dimension s_k of $S_k(\Gamma, \chi)$ is zero.*

Corollary 4.1.6. *If $k < 0$, then $M_k(\Gamma, \chi)$ and $S_k(\Gamma, \chi)$ are trivial.*

4.2 ρ -equivariant functions

Throughout this section, Γ will be a Fuchsian subgroup of $\mathrm{SL}(2, \mathbb{R})$, and $\rho : \Gamma \rightarrow \mathrm{GL}(2, \mathbb{C})$ will be a 2-dimensional complex representation of Γ .

Definition 4.2.1. A meromorphic function h on \mathbb{H} is called a ρ -equivariant function with respect to Γ if

$$h(\gamma \cdot z) = \rho(\gamma) \cdot h(z) \quad \text{for all } z \in \mathbb{H}, \gamma \in \Gamma,$$

where the action on both sides is by Möbius transformations. The set of ρ -equivariant functions for Γ will be denoted by $E_\rho(\Gamma)$.

In the case ρ is the defining representation of Γ , that is $\rho(\gamma) = \gamma$ for all $\gamma \in \Gamma$, then elements of $E_\rho(\Gamma)$ are simply called *equivariant functions*. These were studied extensively in [11, 12, 51, 54] and have various connections to modular forms, quasi-modular forms, and elliptic functions. In particular, one shows that the set of equivariant functions for a Fuchsian group Γ without the trivial one $h_0(z) = z$ has a vector space structure isomorphic to the space

of weight 2 unrestricted automorphic forms for Γ . Nontrivial examples are constructed from automorphic forms. Indeed, if f is a weight k automorphic form for Γ , then

$$h_f(z) = z + k \frac{f(z)}{f'(z)}$$

is equivariant for Γ . These are referred to as the *rational equivariant functions* [12].

The first main result is that ρ -equivariant functions are parameterized by 2-dimensional unrestricted vector-valued automorphic form of multiplier ρ . More precisely, if $F = (f_1, f_2)^t$ is an unrestricted vector-valued automorphic form of multiplier ρ and an arbitrary weight such that f_2 is nonzero, then $h_F(z) = f_1(z)/f_2(z)$ is a ρ -equivariant function. We will show that, in fact, every ρ -equivariant function arises in this way. To achieve this parametrization, we use the fact that the Schwarz derivative of a ρ -equivariant function is a weight 4 unrestricted automorphic form for Γ , as well as the existence of global solutions to a certain second degree differential equation. The second main result of this section is that the ρ -equivariant functions always exist. This will follow from Theorem 2.5.9 and Theorem 3.3.5.

We end this section by constructing examples of ρ -equivariant functions, specially when ρ is the monodromy representation of second degree ordinary differential equations.

4.2.1 Differential equations

Let D be a domain in \mathbb{C} and let f be a meromorphic function on D . Its Schwarz derivative, $S(f)$, is defined by

$$S(f) = \left(\frac{f''}{f'} \right)' - \frac{1}{2} \left(\frac{f''}{f'} \right)^2.$$

This is an important tool in projective geometry and differential equations. The main properties that will be useful to us are summarized as follows (see [40] for more details):

Proposition 4.2.1. *We have*

1. *If y_1 and y_2 are two linearly independent solutions to a differential equation $y'' + Qy = 0$ where Q is a meromorphic function on D , then $S(y_1/y_2) = 2Q$.*
2. *If f and g are two meromorphic functions on D , then $S(f) = S(g)$ if and only if $f = \frac{ag + b}{cg + d}$ for some $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(2, \mathbb{C})$.*
3. *$S(f \circ \gamma)(z) = (cz + d)^4 S(f)$ provided $\gamma \cdot z \in D$, where $\gamma = \begin{pmatrix} * & * \\ c & d \end{pmatrix}$.*

In particular, we have

Proposition 4.2.2. *If f is a ρ -equivariant for Γ , then $S(f)$ is an unrestricted automorphic form of weight 4 for Γ .*

Now, consider the second order ordinary differential equation (ODE)

$$x'' + Px' + Qx = 0,$$

where P and Q are holomorphic functions on D . This ODE has two linearly independent holomorphic solutions on D if D is simply connected. For a fixed $z_0 \in D$, set

$$y(z) = x(z) \exp\left(\int_{z_0}^z \frac{1}{2}P(w)dw\right).$$

The above ODE reduces to an ODE in normal form

$$y'' + gy = 0, \tag{4.2.1}$$

with

$$g = Q - \frac{1}{2}P' - \frac{1}{4}P^2.$$

When the domain D is not simply connected, we may not expect to find global solutions to (4.2.1) on D . However, under some conditions on g , global solutions do exist as it is illustrated in the following theorem which will be crucial for the rest of this section.

Theorem 4.2.3. *Let D be a domain in \mathbb{C} . Suppose h is a nonconstant meromorphic function on D such that $S(h)$ is holomorphic in D , and let $g = \frac{1}{2}S(h)$. Then the differential equation*

$$y'' + gy = 0$$

has two linearly independent holomorphic solutions in D .

Proof. Let $\{U_i, i \in I\}$ be a covering of D by open discs with $\dim V(U_i) = 2$ for all $i \in I$ where $V(U_i)$ denotes the space of holomorphic solutions to $y'' + gy = 0$ on U_i . Choose L_i and K_i to form a basis for $V(U_i)$. Using property (1) of Proposition 4.2.1, we have $S(K_i/L_i) = 2g = S(h)$ on U_i . Now, using property (2) of Proposition 4.2.1, we have $K_i/L_i = \alpha_i \cdot h$ for $\alpha_i \in \text{GL}(2, \mathbb{C})$. On the other hand, on each connected component W of $U_i \cap U_j$, we have

$$(K_i, L_i)^t = \alpha_W (K_j, L_j)^t, \quad \alpha_W \in \text{GL}(2, \mathbb{C}),$$

since each of (K_i, L_i) and (K_j, L_j) is a basis of $V(W)$. Hence, on W we have

$$\frac{K_i}{L_i} = \alpha_W \cdot \frac{K_j}{L_j},$$

and therefore

$$\alpha_i h = \alpha_W \alpha_j h.$$

It follows that

$$\alpha_i \alpha_j^{-1} = \alpha_W$$

as h is meromorphic and nonconstant and thus it takes more than three distinct values on the domain D . Therefore, α_W does not depend on W . Moreover, on $U_i \cap U_j$ we have

$$\alpha_i^{-1} (K_i, L_i)^t = \alpha_j^{-1} (K_j, L_j)^t. \quad (4.2.2)$$

If we define f_1 and f_2 on U_i by

$$(f_1, f_2)^t = \alpha_i^{-1} (K_i, L_i)^t,$$

then using (4.2.2), we see that f_1 and f_2 are well defined all over D and they are two linearly independent solutions to $y'' + gy = 0$ on all of D as they are linearly independent over U_i . \square

4.2.2 The correspondence

The aim of this subsection is to prove the existence of ρ -equivariant functions as well as their parametrization by unrestricted vector-valued automorphic forms of multiplier ρ .

We start with the following proposition.

Proposition 4.2.4. *Let $F(z) = (f_1(z), f_2(z))^t$ be a nonzero unrestricted vector-valued automorphic form for ρ of a certain weight. If f_2 is nonzero, then $h_F := f_1/f_2$ is a ρ -equivariant function for Γ .*

Proof. Suppose that F is of weight k , $k \in \mathbb{Z}$, and that $\rho(\gamma) = \begin{pmatrix} a_\gamma & b_\gamma \\ c_\gamma & d_\gamma \end{pmatrix}$, $\gamma \in \Gamma$. Since $F|_k \gamma = \rho(\gamma) F$, we have

$$f_1(\gamma) = J_\gamma^k (a_\gamma f_1 + b_\gamma f_2), \quad (4.2.3)$$

and

$$f_2(\gamma) = J_\gamma^k (c_\gamma f_1 + d_\gamma f_2). \quad (4.2.4)$$

This implies that

$$\frac{f_1(\gamma)}{f_2(\gamma)} = \frac{a_\gamma f_1 + b_\gamma f_2}{c_\gamma f_1 + d_\gamma f_2},$$

that is

$$h_F(\gamma) = \frac{a_\gamma h_F + b_\gamma}{c_\gamma h_F + d_\gamma} = \rho(\gamma) \cdot h_F. \quad \square$$

The slash operator has the following useful property known as Bol's identity.

Proposition 4.2.5. [27] *Let r be a nonnegative integer, $F(z)$ a complex function and $\gamma \in SL(2, \mathbb{C})$, then*

$$(F|_{-r}\gamma)^{(r+1)}(z) = F^{(r+1)}|_{r+2}\gamma(z).$$

□

As a consequence, we have

Corollary 4.2.6. *Let r be a nonnegative integer, g an unrestricted automorphic form of weight $2(r+1)$ for Γ , and D a domain in \mathbb{H} that is stable under the action of Γ . Denote by $V_r(D)$ the solution space on D to the differential equation*

$$f^{(r+1)} + gf = 0.$$

Suppose that $f \in V_r(D)$. Then for all $\gamma \in \Gamma$, we have $f|_{-r}\gamma \in V_r(D)$.

Proof. Suppose that $f \in V_r(D)$. Using Bol's identity and the fact that $g|_{2(r+1)} = g$, we have

$$\begin{aligned} (f|_{-r}\gamma)^{(r+1)} + g(f|_{-r}\gamma) &= f^{(r+1)}|_{(r+2)}\gamma + g(f|_{-r}\gamma) = \\ J_\gamma^{-(r+2)} \left(f^{(r+1)}(\gamma) + (g|_{2(r+1)}\gamma) f(\gamma) \right) &= (f^{(r+1)} + gf)|_{(r+2)}\gamma = (0)|_{(r+2)}\gamma = 0. \end{aligned}$$

□

Corollary 4.2.7. *The operator $|_{-r}$ provides a representation ρ_r of Γ in $GL(V_r(D))$. Moreover, if f_1, f_2, \dots, f_{r+1} form a basis of V_r (if the basis exists), then*

$$F = (f_1, f_2, \dots, f_{r+1})^t$$

is an unrestricted vector valued automorphic form of multiplier ρ_r and weight $-r$ for Γ .

Recall from Definition 2.1.1 that the space of unrestricted vector-valued automorphic forms for Γ of multiplier ρ and weight $k \in \mathbb{Z}$, is denoted by $G_k(\Gamma, \rho)$. Our first main result of this section is the following.

Theorem 4.2.8. *The map*

$$G_{-1}(\Gamma, \rho) \longrightarrow E_\rho(\Gamma)$$

$$F \mapsto h_F$$

is surjective.

Proof. Suppose that h is a ρ -equivariant function for Γ . According Proposition 4.2.2, its Schwarz derivative $S(h)$ is an unrestricted automorphic form of weight 4 for Γ . Let $g = \frac{1}{2}S(h)$ and D the complement in \mathbb{H} of the set of poles of g . Then D is a domain that is stable under Γ since g is an unrestricted automorphic form for Γ .

Using the same notation as in the previous section, we have, for $r = 1$, $S(f_1/f_2) = S(h)$ where $\{f_1, f_2\}$ are two linearly independent solutions in $V(D)$ provided by Theorem 4.2.3. Hence, by Proposition 4.2.1

$$\frac{f_1}{f_2} = \alpha \cdot h, \quad \alpha \in \text{GL}(2, \mathbb{C}).$$

Also, using Corollary 4.2.7 with $r = 1$, we deduce that $F_1 = (f_1, f_2)^t$ is a vector-valued unrestricted automorphic form for Γ of multiplier ρ_1 and weight -1 . Therefore,

$$\alpha^{-1}\rho_1\alpha = \rho.$$

Hence $F = \alpha^{-1}F_1$ lies in $G_{-1}(\Gamma, \rho)$, and $h_F = h$ on D . Since g has only double poles, then by looking at the form of the solutions near a singular point, and using the fact that f_1 and f_2 are holomorphic and thus single-valued, we see that f_1 and f_2 can be extended to meromorphic functions on \mathbb{H} . \square

Remark 4.2.9. If there exists an unrestricted automorphic form f of weight $k + 1$ for Γ , then $(f_1, f_2) \longrightarrow (ff_1, ff_2)$ yields an isomorphism between $G_{-1}(\Gamma, \rho)$ and $G_k(\Gamma, \rho)$, and therefore, the above surjection in the theorem extends to $G_k(\Gamma, \rho)$ whenever it is nontrivial.

We now prove the existence of ρ -equivariant functions. Recall that a representation is called decomposable if it is the direct sum of two representations. If ρ is decomposable, then $\rho = \chi_1 \oplus \chi_2$ for some two characters χ_1 and χ_2 on Γ . If k is a large enough integer, then by Theorem 2.5.9 and Theorem 3.3.5, $M_k(\Gamma, \chi_2)$ is nontrivial. Let f_2 be a nonzero element of $M_k(\Gamma, \rho_2)$, then for any $f_1 \in M_k(\Gamma, \rho_1)$, we have $F(z) = (f_1(z), f_2(z))^t \in M_k(\Gamma, \rho)$. Hence, by Proposition 4.2.4, $h_F = f_1/f_2$ is a ρ -equivariant function for Γ .

Suppose that ρ is indecomposable. As usual, modulo a conjugation in $\mathrm{GL}(2, \mathbb{C})$, the representation ρ is either even, odd, or a direct sum $\rho = \rho_+ \oplus \rho_-$ of an even and an odd representations. Since ρ is indecomposable, the latter case is excluded. Hence, by Theorem 2.5.9 and Theorem 3.3.5, there exists $k_0 \in \mathbb{N}$ such that for all $k \geq k_0$, $M_k(\Gamma, \rho)$ is nontrivial.

We now show that for a certain integer $k \geq k_0$, there exists an element $F(z) = (f_1(z), f_2(z))^t \in M_k(\Gamma, \rho)$ such that f_2 is nonzero. Suppose the converse is true, then for $k \geq k_0$, any nontrivial element $F \in M_k(\Gamma, \rho)$ has the form $F = (f_1, 0)^t$ for some nonzero holomorphic function f_1 on \mathbb{H} . According to (4.2.3), we have for all $\gamma \in \Gamma$

$$f_1|_k \gamma = a_\gamma f_1, \quad (4.2.5)$$

where $\rho(\gamma) = \begin{pmatrix} a_\gamma & b_\gamma \\ c_\gamma & d_\gamma \end{pmatrix}$. Using the fact that $|_k$ is a linear operator, we deduce that the map

$$\alpha : \Gamma \longrightarrow \mathbb{C}^* \quad \gamma \mapsto a_\gamma$$

is character on Γ . Thus $f_1 \in G_k(\Gamma, \alpha)$. Also, from (4.2.4), we see that $c_\gamma = 0$ for all $\gamma \in \Gamma$, and so F and f_1 have the same cuspidal behavior with respect to Γ . Since $F \in M_k(\Gamma, \rho)$, we conclude that $f_1 \in M_k(\Gamma, \alpha)$, and hence we have an embedding of $M_k(\Gamma, \rho)$ into $M_k(\Gamma, \alpha)$ given by $F = (f_1, 0)^t \longrightarrow f_1$. Consequently, for all $k \geq k_0$, we have

$$d_{\Gamma, \rho, k} \leq d_{\Gamma, \alpha, k}, \quad (4.2.6)$$

where $d_{\Gamma, \rho, k}$ and $d_{\Gamma, \alpha, k}$ respectively denote the dimensions of $M_k(\Gamma, \chi)$ and

$M_k(\Gamma, \alpha)$. But from Theorem 3.3.5, we know that

$$\frac{d_{\Gamma, \rho, k}}{k} \longrightarrow m_X, \text{ as } k \rightarrow \infty,$$

and

$$\frac{d_{\Gamma, \alpha, k}}{k} \longrightarrow \frac{m_X}{2}, \text{ as } k \rightarrow \infty.$$

where by see Remark 3.1.5

$$m_X = 2g_X - 2 + c_X + \sum_{P \in \mathfrak{E}} (1 - 1/n_P) \geq 1/42$$

Hence for k large enough, we have

$$\frac{d_{\Gamma, \rho, k}}{k} > \frac{d_{\Gamma, \alpha, k}}{k},$$

that is

$$d_{\Gamma, \rho, k} > d_{\Gamma, \alpha, k},$$

which contradicts (4.2.6). Therefore, for a certain integer $k \geq k_0$, there exists an element $F(z) = (f_1(z), f_2(z))^t \in M_k(\Gamma, \rho)$ such that f_2 is nonzero. Then by Proposition 4.2.4, $h_F = f_1/f_2$ is a ρ -equivariant function for Γ .

Thus we have proved the following existence theorem for ρ -equivariant functions.

Theorem 4.2.10. *There exists a ρ -equivariant function $h_F = f_1/f_2$, where $F(z) = (f_1(z), f_2(z))^t \in M_k(\Gamma, \rho)$ for a certain nonnegative integer k . \square*

4.2.3 Examples

In this subsection, we shall illustrate our parametrization by constructing examples of unrestricted vector valued automorphic forms, and their corresponding ρ -equivariant functions.

Recall that the Γ -equivariant functions are simply the ρ_0 -equivariant functions for Γ , with $\rho_0(\gamma) = \gamma$ for all $\gamma \in \Gamma$. One can easily see that

$$F_0(z) = \begin{pmatrix} z \\ 1 \end{pmatrix}, \quad z \in \mathbb{H}$$

is an unrestricted vector-valued automorphic form for Γ of multiplier ρ_0 and weight -1 , and

$$h_{F_0}(z) = z, \quad z \in \mathbb{H}.$$

As we mentioned before, if we take a weight k unrestricted automorphic form f for Γ , then

$$h_f(z) = z + k \frac{f(z)}{f'(z)}, \quad z \in \mathbb{H}$$

is an equivariant function for Γ . This corresponds to the $(k-1)$ unrestricted vector-valued automorphic form for Γ of multiplier ρ_0 given by

$$F(z) = \begin{pmatrix} k f + z f' \\ f' \end{pmatrix}, \quad z \in \mathbb{H}. \quad (4.2.7)$$

Our next example is the monodromy representation of a differential equation. Indeed, let U be a domain in \mathbb{C} such that $\mathbb{C} \setminus U$ contains at least two points. The universal covering of U is then \mathbb{H} as it cannot be $\mathbb{P}_1(\mathbb{C})$ because U is noncompact and it cannot be \mathbb{C} because of Picard's theorem.

Let $\pi : \mathbb{H} \rightarrow U$ be the covering map. We consider the differential equation on U

$$y'' + P y' + Q y = 0, \quad (4.2.8)$$

where P and Q are two holomorphic functions on U . This differential equation has a lift to \mathbb{H}

$$y'' + \pi^* P y' + \pi^* Q y = 0. \quad (4.2.9)$$

Let V be the solution space to (4.2.9) which is a 2-dimensional vector space since \mathbb{H} is simply connected. Let γ be a covering transformation in $\text{Deck}(\mathbb{H}/U)$ which is isomorphic to the fundamental group $\pi_1(U)$ and let $f \in V$. Then $\gamma^* f = f \circ \gamma^{-1}$ is also a solution in V . This defines the monodromy representation of $\pi_1(U)$:

$$\rho : \pi_1(U) \rightarrow \text{GL}(V).$$

If f_1 and f_2 are two linearly independent solutions in V , we set $F = (f_1, f_2)^t$. Then we have

$$F \circ \gamma = \rho(\gamma) F,$$

that is $F \in G_0(\pi_1(U), \rho)$. Therefore, the quotient f_1/f_2 is a ρ -equivariant function on \mathbb{H} for the group $\pi_1(U)$, which is a torsion-free Fuchsian group.

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