The extended Shapley value for organizations

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We generalize the Shapley value to organizations where each worker may choose a different level of contribution to the production of the output. We uniquely characterize the new value thanks to three axioms.

1 Introduction

In *Theory of games and economic behavior* (1944), John von Neumann and Oskar Morgenstern laid the foundations for what is now considered the modern theory of games. Since their seminal contribution, the fundamental concept of fair distribution in an economy has been a focus of sustained interest. Understanding what fair distribution entails provides insight into such issues as how workers should be remunerated. A very important solution proposed to this problem was that of Lloyd Shapley (1953) who introduced a function to numerically evaluate the "value" to each player from playing a particular game. This value is known as the Shapley value.

The Shapley value assumes that workers can exert only two levels of effort denoted by 0 and 1. While this assumption permits the study of the distribution of profits in a simple environment, it provides little guidance with respect to how firms should allocate the joint surplus resulting from varying levels of labor input per individual worker. In the real world, however, firms employ part-time and/or full-time workers. In this paper, we are interested in the distribution of joint surplus within an organization in which there are varying levels of labor input. Should the quantity of hours worked affect a worker's compensation? While the analysis of the problem of fair distribution often highlights the potential intermix of economic, political and social factors, our approach is to stress the role of an individual worker in the firm.

We consider a simple model. The framework developed here models an economy with n workers facing a finite list of alternative options, and in which there are h levels of labor input. The relationship that exists between a worker and the firm is depicted in characteristic function form, which can be interpreted as the production function. The production function represents available technology and labor is an essential input to production. For simplicity, fixed costs of production are assumed to be zero and the price of output is normalized to 1. The government does not enforce minimum wage standards. Surplus from the production process is fully distributed among the participants. We have not specified the subsistence-level of workers: we assume workers have alternative means to cover subsistence should they choose not to enter the firm. Each worker can choose his optimal level of h subject to a variety of factors such as inheritance, preference for leisure and health among others. We extend the Shapley value to this environment and uniquely characterize the new value using a set of three axioms: symmetry, efficiency and strong monotonicity.

The structure of this paper is organized as follows. In Section 2 we conduct a literature review outlining the different interpretations and extensions of the classical Shapley value. Section 3 introduces basic definitions. Section 4 is devoted to the generalization of our extended Shapley value for Organizations. Section 5 provides a simple numerical application. In Sections 6 and 7 we provide three axioms that uniquely characterize our extended Shapley value for organizations, and Section 8 concludes our study.

2 Literature Review

The modern theory of games assumes that the players of a game can evaluate every "prospect" that might arise as a result of a play. The question of how to numerically evaluate every "prospect" is therefore of critical importance. Lloyd Shapley (1953) proposed an important solution to this problem. He summarized the opportunities facing each player in a game by a single number, which represents the "value" of playing the game. To identify this "value", Shapley makes the following assumptions. Players agree to play a game and form a coalition. Starting from one player, the coalition adds an additional player one at a time until everyone has been admitted. The order of entry of players into the game is random and each arrangement is equally probable. He next introduces a characteristic function form v and defines the "value" of playing the game for player i as $\phi_i(v)$. $\phi_i(v)$ known as the classical Shapley value, is a function that maps each player i from the universe of players to a real number. It is an allocation procedure that distributes the surplus generated by the grand coalition.

The classical Shapley value satisfies four basic axioms, which are symmetry, efficiency, null player and additivity. Young (1985) introduced a new axiom named strong monotonicity, to replace the basic null player and additivity axioms of the classical Shapley value. The principle of monotonicity is particularly important in applications where the underlying game is constantly changing, for example, in cost allocation problems. It states that if a worker's contribution to all coalitions increases or stays the same then the worker's payoff should not decrease. He proves that the Shapley value is a symmetric and efficient allocation procedure that is strongly monotonic.

The classical Shapley value has been applied to several environments. For example, Shapley and Shubik (1954) were the first to apply the Shapley value to the class of simple games, which are natural models of voting rules. They study the division of power among voters using the then existing rules of the

Security Council and the United States Congress. They use a characteristic function form v to assign a value to a coalition S as such - S is winning if $v(\mathrm{S}) = 1$ and losing if $v(\mathrm{S}) = 0$. The Shapley-Shubik index for any player i is equal to the proportion of random orders in which he is a "pivotal" player, that is, his vote transforms a losing coalition into a winning coalition. It measures the number of times the vote of a player can alter the state of affairs upon his entry into the game.

The classical Shapley value assumes that workers can exert only two levels of effort, "work" and "not work" denoted by 1 and 0 respectively. While it addresses the question of how to distribute the surplus of joint work within an organization, it can only be applied to a simple environment where workers face only two levels of effort. In real world, however, workers and firms have the option to choose from a larger set of effort levels. In practice, a worker's compensation should be based on the effort level he/she brings to the firm. The question of how to distribute the joint surplus resulting from varying levels of labor input remained unanswered.

Hsiao and Raghavan (1993) extend the classical Shapley value to address this gap in the existing literature. They allow for workers to have a finite list of actions and assign weights to actions, to develop a ranking system on the set of actions. They define the set of actions $\mathrm{A} = \{0, 1, ..., m\}$ and weight function $w : \mathrm{A} \rightarrow \mathbb{R}$ such that $w(0) \leq w(1).... \leq w(m)$. The value of the game is defined in matrix form. The resulting allocation procedure is an extended weighted Shapley value that satisfies four basic axioms.

Freixas and Zwicker (2003) use the notion of abstention in weighted voting games to generalize the concept of allowing for j levels of approval in the input and k levels of approval in the output for (j, k) weighted games. Freixas (2005) extends the Shapley-Shubik index to this environment. He introduces the concept of probability to the Shapley-Shubik index by assigning each j-partition an equal probability of $\frac{1}{j}$ and estimates the probability of a player p becoming i-critical; i=1,2,..,$k-1$. Lastly, he axiomatizes the raw Shapley-Shubik index for (j,2) simple games.

Pongou, Tchantcho and Tedjeugang (2015) propose a simple model of trial-based tournaments and derive relative measure of performance to rank workers for each job. Although, their framework is similar to that of Hsiao and Raghavan (1993) and Freixas and Zwicker (2003), they offer a very different interpretation. They then compare the ranking rule generated by their tournament model with the generalized Shapley value from Freixas (2005) (see Pongou, Tchantcho and Tedjeugang (2014) for a comparison of this ranking rule with the Banzhaf index). They show that the Shapley value is only a weakly increasing function of tournament rank and therefore does not always fairly reward productivity. However, in the class of linear and top-down biased organizations that include the class of von Neumann-Morgenstern organizations, a worker's pay and tournament rank are ordinally equivalent. The results of this study have practical implications for the design of organizations that seek to promote fairness in pay.

The framework developed in this paper is similar to that of Hsiao and Raghavan (1993), Freixas and Zwicker (2003), and Pongou, Tchantcho and Tedjeugang (2015). We introduce the finite set of actions H with cardinality $|\mathrm{H}| = h \geq 3$ and allow for h levels of labor input in the production process. In contrast to the previous studies, the set H enjoys a natural ordering and eliminates the need to qualitatively order tasks by assigning each task a specific weight. We make no assumptions on the importance of individual tasks, but rather use varying levels of hours contributed to the production process to determine the distribution of joint surplus within an organization. The set of axioms we use to characterize our extended Shapley value is different from that used in Hsiao and Raghavan (1993) and Freixas (2005). In particular our characterization is similar to that of Young(1985) but in a marginal framework.

3 Preliminaries

In this section we introduce preliminary definitions.

• The firm is defined by (N,H,f) where H $= \{-1\} \cup \{0, 1, 2, ..., h\}$ is a finite set of actions with cardinality $|\mathrm{H}| = h \geq 3$, f is the production function representing available technology and N$= \{1, ..., n\}$ is a non-empty set of workers in an economy. We assume all workers start with an initial labor contribution of $h = -1$, which means that they have not entered the firm. A worker with $h = 0$ has entered the firm but is not working. A worker with $h \geq 1$ is considered to be working.

• S $\subset$ N is a coalition of workers; S $= \{1, ..., s\}$.

The total surplus generated by a coalition of workers is fully distributed among them.

• Z is the set of all possible labor allocations for all workers in S.
Z $= \{\bar{Z}^a, \bar{Z}^b, ..., \bar{Z}^j\}$ where $\bar{Z}^a = \{\bar{z_1^a}, \bar{z_2^a}, ..., \bar{z_i^a}\}$; $i = 1, 2, ..., n$; $j \in \mathbb{N}$ and $\bar{z_i^a} \geq 0$. $\bar{z_i^a}$ is the ith worker's hours associated with the labor allocation $\bar{Z}^a$

Definition 1: f is a non-negative real valued function defined on the set Z f: Z $\to \mathbb{R}$, such that $f(\bar{Z}) = 0$ when $\bar{Z} = (0, 0, ..., 0)$ and when $\bar{Z} = (-1, -1, ..., -1)$.

The function $f(\bar{Z})$ specifies the output that a labor allocation $\bar{Z}$ of workers is able to produce under the conditions of the model. For simplicity, we assume the price of output to be equal to 1. Therefore, $f(\bar{Z})$ also specifies the value of output created by labor allocation $\bar{Z}$.

Definition 2: An allocation procedure is a function Φ that to every production function f on N associates an allocation $\Phi(f(\bar{Z})) = \{\phi_1, \phi_2, .., \phi_i\}$, such that $\sum_i \phi_i = f(\bar{Z})$, $i = 1, 2, ..., n$.

ϕ_i is the production surplus which accrues to worker i in labor allocation $\bar{Z}$.

4 Generalization of the extended Shapley Value for Organizations with h levels of labor input

In this section, we generalize the extended Shapley Value for Organizations with h ordered alternatives in the input level.

In contrast to the classical Shapley Value, we distinguish between workers who do not enter the firm and workers who enter the firm but contribute nothing, i.e. their entry into the firm leaves total product unchanged. We implement this distinction to allow for firms to hire part-time and seasonal workers. In practice, the hours assigned to part-time workers fluctuate with the level of production undertaken by the firm. In slow periods, these workers may not work but are still considered as employees of the firm. They enter the firm but do not contribute to the production process. This situation is contrasted with workers who are not employees of the firm, that is, they do not enter the firm in their lifetime.

We begin by assuming all workers start with an initial labor contribution of $h = -1$. Therefore, we set $\bar{Z}_0 = (-1, -1, -1, -1, ..., -1)$ as the initial labor allocation in our model. At this stage, the workers have not entered the firm.

We define $|Z| \in \mathbb{R}$ to be the total number of workers who have entered the firm in labor allocation Z. This includes part-time and seasonal workers who enter the firm but do not contribute to the production process.

We next introduce the steps involved in accurately estimating the value of $|Z|$ in our model. Given a 1 x n column vector Z, we define the function q: $\mathbb{R}^{\mathrm{N}} \to \mathbb{R}^{\mathrm{N}}$ such that:

$$q(Z) = \begin{pmatrix} q_1(z_1) \\ q_2(z_2) \\ q_3(z_3) \\ . \\ . \\ q_n(z_n) \end{pmatrix} \text{ where } q_i(z_i) = \begin{cases} 1 \text{ if } z_i \geq 0;\ i = 1, 2, ..., n \\ 0 \text{ otherwise} \end{cases}$$

The function q maps a labor allocation Z to a vector of 0's and 1's. The value 0 is assigned to workers who do not enter the firm, i.e. z_i = -1. The value 1 is assigned to workers who enter the firm irrespective of whether they contribute to the production process, i.e. $z_i \geq 0$.

We next define the function p: $\mathbb{R}^{\mathrm{N}} \to \mathbb{R}$ such that:
$p(q(Z)) = q_1 + q_2 + ... + q_n \in \mathbb{R}$;

Lastly, we set $p(q(Z)) = |Z|$

The function p sums the entries of its pre-image $q(Z)$. The vector $q(Z)$ defined above consists only of 0's and 1's, where the value 1 signifies a worker's entry into the firm. Therefore, $p(q(Z))$ is non-negative and counts the number of workers who enter the firm in labor allocation Z.

We next analyze the effects of assuming an initial labor allocation different from 0, i.e. $\bar{Z}_0 = (-1, -1, ..., -1)$.

Let $\bar{Z} = \{\ \bar{z_1}, \bar{z_2}, ..., \bar{z_i}..\}$ be a predetermined labor allocation with $\bar{z_i} \geq 0$ for $i \in$ N. If worker i enters the firm, he contributes the hours specified in $\bar{Z}$ and $z_i = \bar{z_i}$. If $z_i \neq \bar{z_i}$ then worker i has not entered the firm and $z_i = -1$. Therefore, $z_i \leq \bar{z_i}$.

The classical Shapley value assumes an initial labor allocation of $\bar{Z} = (0, 0, ..., 0)$. In this model, we assume an initial labor allocation of $\bar{Z}_0 = (-1, -1, ..., -1)$. To account for this difference, we introduce a function r in our extended Shapley value as follows.

Let r: $\mathbb{R} \to \mathbb{R}$ such that:

$$r(x) = \begin{cases} -1 & \text{if} \quad x \geq 0 \\ \ \ 0 & \quad \text{otherwise} \end{cases}$$

The function r maps its pre-image to the value -1, if its pre-image is greater than or equal to zero.
We discuss two cases below.

1. If $\bar{z_i} \geq 0$ then $r(\bar{z_i}) = -1$.
2. If $\bar{z_i} \ngeq 0$ then $r(\bar{z_i}) = 0$.

We next show that inclusion of the function r in our extended Shapley value offsets changes in marginal product, caused primarily by our choice of the initial labor allocation, that is $\bar{Z}_0 = (-1, -1, ..., -1)$.

Let $f^i(Z)$ be the marginal product of worker i. It is the addition to total product from worker i providing z_i hours.

We define $e_i = \begin{pmatrix} 0 \\ 0 \\ 1 \\ \cdot \\ \cdot \\ 0 \end{pmatrix}$ where $\begin{cases} \text{all entries are equal to 0} \\ \text{except the ith entry which is equal to 1} \end{cases}$

Then $f^i(Z) = f(z + [z_i + 1 + r(z_i)]e_i) - f(z)$;

We analyze the marginal product of worker i for three cases.

1. For $z_i = -1$:
   $f(z + e_i[z_i + 1 + r(z_i)]) - f(z) = f(z + e_i[-1 + 1 + 0]) - f(z) = 0$

   If worker i does not enter the firm, his marginal product is 0.

2. For $z_i = \bar{z}_i = 0$:
   $f(z + e_i[z_i + 1 + r(z_i)]) - f(z) = f(z + e_i[0 + 1 - 1]) - f(z) = 0$

   Similar to the classical Shapley value, if worker i enters the firm but does not contribute to the production process, his marginal product is 0.

3. For $z_i = \bar{z}_i > 0$:
   $f(z + e_i[z_i + 1 + r(z_i)]) - f(z) = f(z + e_i[\bar{z}_i + 1 - 1]) - f(z)$
   $f(z + e_i[z_i + 1 + r(z_i)]) - f(z) = f(z + e_i[\bar{z}_i]) - f(z) > 0$

   The expression for the marginal product obtained above is identical to that in the classical Shapley value when $\bar{z}_i > 0$.

Therefore, we conclude that the inclusion of the function r in our extended Shapley value ensures consistency between our results and those obtained from the classical Shapley value.

We next generalize the extended Shapley Value for Organizations with h ordered alternatives in the input level, given an initial labor allocation of $\bar{Z}_0 = (-1, -1, ..., -1)$.

For any predetermined labor allocation $\bar{Z}$, the extended Shapley Value for Organizations is given by:

$$\phi_i(f(\bar{Z})) = \sum_{\substack{Z \rhd \bar{Z} \\ z_i=-1}} \frac{(|Z|)!(n-|Z|-1)!}{n!}[f(z+e_i[z_i+1+r(z_i)]) - f(z)] \quad (1)$$

$$Z \rhd \bar{Z} \quad \text{if} \quad \begin{cases} z_i = \bar{z}_i & \text{if} \quad z_i > -1 \\ z_i = -1 & \text{otherwise} \end{cases}$$

$|Z| \in \mathbb{R}$
$i = 1, 2, ..., n$

To understand the intuition behind this value, we assume that workers can enter the firm in $n!$ ways. When the ith worker enters the firm, the previous workers have entered in $(|Z|)!$ ways. The remaining workers can enter in $(n - |Z| - 1)!$ ways. The value of $|Z|$ can be estimated using the steps outlined at the beginning of this section.

5 Numerical Example

In this section, we provide a simple numerical example that illustrates the distribution of joint surplus within an organization in which there are varying levels of labor input. We show that a worker's compensation is determined by the quantity of hours he contributes to the production process.

We first make following simplifying assumptions. We consider an economy consisting of 2 workers and an organization that allows for $h \leq 2$ alternatives in the input level. Therefore, N=$\{1, 2\}$ and H = $\{-1\} \cup \{0, 1, 2\}$.
Workers begin with an initial labor allocation of $h = -1$. We assume both workers are equally skilled and have equal marginal products for each hour worked. Labor is in an essential input to production and it experiences constant returns to scale. The assumption of constant returns to scale is for mere simplicity. The same method can be applied when labor experiences decreasing returns to scale. For the firm, the first hour employed has a marginal product of 70 for the first man worked. Following which, every additional hour worked has a constant marginal product of 10 per worker.

The set of all possible labor allocations $\bar{Z}$ is defined as such:
Z = $\{(0, 0), (0, 1), (1, 0), (1, 1), (0, 2), (2, 0), (1, 2), (2, 1), (2, 2)\}$

The function $f(\bar{Z})$ specifies the output that a labor allocation $\bar{Z}$ of workers is able to produce under the conditions of the model. For simplicity, we assume the price of output to be equal to 1. Therefore, $f(\bar{Z})$ also specifies the value of output created by labor allocation $\bar{Z}$.

The table below shows the value of output created per labor allocation. For example, consider labor allocation $\bar{Z}$ = (1,1), the first hour employed has a marginal product of 70 and every additional hour, in this case 1 hour, has a marginal product of 10. Therefore, total product is 80. Similarly for labor allocation $\bar{Z}$ = (2,2), the first hour employed has a marginal product of 70 and every additional hour, in this case 3 hours, has a marginal product of 10. Therefore, total product is 100.

$\bar{Z}$	$f(\bar{Z})$
(0,0)	0
(0,1)	70
(1,0)	70
(1,1)	80
(0,2)	80
(2,0)	80
(1,2)	90
(2,1)	90
(2,2)	100

We next apply the extended Shapley value for organizations given by equation (1), to determine the surplus that accrues to each of the two workers. The mathematics of this exercise is shown in the appendix section.

For labor allocation $\bar{Z}$, let $\phi_1(f(\bar{Z}))$ be the surplus that accrues to worker 1 and $\phi_2(f(\bar{Z}))$ be the surplus that accrues to worker 2. The table below shows the distribution of total output among worker 1 and worker 2 per labor allocation.

$\bar{Z}$	$f(\bar{Z})$	ϕ_1	ϕ_2
(0,0)	0	0	0
(0,1)	70	0	70
(1,0)	70	70	0
(1,1)	80	40	40
(0,2)	80	0	80

$\bar{Z}$	$f(\bar{Z})$	ϕ_1	ϕ_2
(2,0)	80	80	0
(1,2)	90	40	50
(2,1)	90	50	40
(2,2)	100	50	50

Consider labor allocation $\bar{Z}$ = (1,1):
Total product is 80 and $\phi_1 = \phi_2 = 40$.
This result implies that if the hours contributed by worker 1 and worker 2 are equal, then their corresponding compensation will also be equal.

Similarly, for labor allocation $\bar{Z}$ = (1,2):
Total product is 90, $\phi_1 = 40$ and $\phi_2 = 50$.
Since worker 2 provides an additional hour of labor compared to worker 1, his compensation is higher by 10, which is also the marginal product of each additional hour worked.

The results of our numerical exercise satisfy the following properties:

1. $(f(\bar{Z})) = \phi_1(f(\bar{Z}))) + \phi_2(f(\bar{Z}))$;
   Total value of output created is fully distributed.
2. If $\bar{Z} = (0,0)$, then $\phi_1(f(\bar{Z})) = 0$ and $\phi_2(f(\bar{Z}))$ =0.
   A worker who enters the firm but does not contribute to the production process, receives a payoff of 0.

We next characterize our extended Shapley value for organizations using three axioms.

6 Axioms

Axiom 1: Symmetry

An allocation procedure ϕ_i is *symmetric* if for any two workers i and j and any production function f

1. $z_i = z_j = -1$
2. $\bar{z}_i = \bar{z}_j$ where $\bar{z}_i, \bar{z}_j \geq 0$
3. $f^i(\bar{Z}) = f^j(\bar{Z})$

Implies that $\phi_i(f(\bar{Z}) = \phi_j(f(\bar{Z})$; $\qquad \forall \bar{Z} \in$ Z, $i, j \in$ N, $i \neq j$

Axiom 2: Efficiency

An allocation procedure ϕ_i is *efficient* for any production function f if

$$\sum_i \phi_i(f(\bar{Z})) = f(\bar{Z}); \qquad \forall \bar{Z} \in \mathrm{Z};\ i \in \mathrm{N}$$

Axiom 3: Strong Monotonicity An allocation procedure ϕ_i satisfies *strong monotonicity* if for any two production functions f and g,

$$f^i(\bar{Z}) \geq g^i(\bar{Z}) \text{ implies that } \phi_i(f(\bar{Z})) \geq \phi_i(g(\bar{Z})); \qquad \forall \bar{Z} \in \mathrm{Z};\ i \in \mathrm{N}$$

7 Theorem

An allocation procedure is efficient, symmetric and strongly monotonic if and only if it is the extended Shapley value for organizations defined by equation (1).

Proof

The structure of this proof is as follows. We first show that the extended Shapley value for organizations, $\phi_i(f(\bar{Z}))$, is *symmetric, efficient* and *strongly monotonic*. We next prove that an allocation procedure that is *symmetric, efficient* and *strongly monotonic* is the extended Shapley value for organizations.

Let us begin by proving that $\phi_i(f(\bar{Z}))$ is *symmetric*.

For any two workers i and $j \in \mathrm{N}$, let $z_i = z_j = -1$ and $\bar{z}_i = \bar{z}_j$.

From equation (1), the extended Shapley Value for workers i and j is given by:

$$\phi_i(f(\bar{Z})) = \sum_{\substack{Z \rhd \bar{Z} \\ z_i=-1}} \frac{(|Z|)!(n-|Z|-1)!}{n!}[f(z+e_i[z_i+1+r(z_i)]) - f(z)]$$

$$\phi_j(f(\bar{Z})) = \sum_{\substack{Z \rhd \bar{Z} \\ z_j=-1}} \frac{(|Z|)!(n-|Z|-1)!}{n!}[f(z+e_j[z_j+1+r(z_j)]) - f(z)]$$

The marginal contribution of worker i to total product is given by:
$f^i(\bar{Z}) = f(z+e_i[z_i+1+r(z_i)]) - f(z)$

Similarly, the marginal contribution of worker j to total product is given by:
$f^j(\bar{Z}) = f(z+e_j[z_j+1+r(z_j)]) - f(z)$

Case 1

If $z_i = z_j = -1$ then substituting $z_i = -1$ in $f^i(\bar{Z})$, we get

$f^i(\bar{Z}) = f(z + e_i[-1 + 1 + 0]) - f(z)$

$\Rightarrow f^i(\bar{Z}) = f(z) - f(z) = 0$

Similarly, substituting $z_j = -1$ in $f^j(\bar{Z})$, we get

$f^j(\bar{Z}) = f(z + e_j[-1 + 1 + 0]) - f(z)$

$\Rightarrow f^j(\bar{Z}) = f(z) - f(z) = 0$

Therefore, we can conclude that $\phi_i(f(\bar{Z})) = \phi_j(f(\bar{Z}))$ when $z_i = z_j = -1$.

Case 2

If $\bar{z_i} = \bar{z_j} \geq 0$,

$f^i(\bar{Z}) = f(z + e_i[\bar{z_i} + 1 - 1]) - f(z) = f(z + e_i[\bar{z_i}]) - f(z)$

$\Rightarrow f^i(\bar{Z}) = f(z + e_i[\bar{z_i}]) - f(z)$

Since $\bar{z_i} = \bar{z_j} \Rightarrow f(z + e_i[\bar{z_i}]) - f(z) = f(z + e_i[\bar{z_j}]) - f(z)$

Worker i receives equal payoffs from contributing $\bar{z_i}$ or $\bar{z_j}$ hours. Therefore, we can conclude that $\phi_i(f(\bar{Z})) = \phi_j(f(\bar{Z}))$ when $\bar{z_i} = \bar{z_j}$. This result establishes symmetry.

We next show that $\phi_i(f(\bar{Z}))$ is *efficient*.

For a given labor allocation $\bar{Z} \in$ Z, assume every worker contributes his predetermined hours as shown in $\bar{Z}$, where $\bar{z_i} \geq 0$ for $i \in$ N.

Assume every worker enters the firm as follows: worker 1 enters first, then worker 2, then 3, continuing in the same order till the last worker enters in the nth position.

Let $\bar{Z}_0$=(-1,-1,-1,-1,....,-1) be the initial state of all workers in the set N.

The marginal contribution of worker 1 is given by:

$$\phi_1(f(\bar{Z})) = [f(\bar{Z}_0 + e_1[\bar{z_1} + 1 - 1]) - f(z)] = f(\underbrace{\bar{Z}_0 + e_1[\bar{z_1}]}_{w_1}) - f(\bar{Z}_0)$$

The marginal contribution of worker 2 is given by:

$$\phi_2(f(\bar{Z})) = [f(w_1 + e_2[\bar{z_2} + 1 - 1]) - f(w_1)] = \mathrm{f}(\underbrace{w_1 + e_2[\bar{z_2}]}_{w_2}) - f(w_1)$$

The marginal contribution of worker 3 is given by:

$\phi_3(f(\bar{Z})) = [f(w_2 + e_3[\bar{z_3} + 1 - 1]) - f(w_2)] = f(\underbrace{w_2 + e_3[\bar{z_3}]}_{w_3}) - f(w_2)$

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Similarly, the marginal contribution of worker n is given by:

$\phi_n(f(\bar{Z})) = [f(w_{n-1} + e_n[\bar{z_n} + 1 - 1]) - f(w_{n-1})] = f(\bar{Z}) - f(w_{n-1})$

Taking the summation on both sides of the above n equations, we have

$\sum_i \phi_i(f(\bar{Z})) = f(\bar{Z}) - f(\bar{Z}_0)$.

As $f(\bar{Z}_0) = 0$ by assumption, $\sum_i \phi_i(f(\bar{Z})) = f(\bar{Z})$ for i = 1,2,...,n.
We can repeat the process for any order of entry for workers in the firm, and obtain the same outcome. This result establishes efficiency. It implies that the total surplus generated by a coalition of workers is fully distributed among them.

We next show that $\phi_i(f(\bar{Z}))$ is *strongly monotonic.*

Let f and g be any two production functions on the set N of workers. For a given labor allocation $\bar{Z} \in$ Z, assume every worker contributes his predetermined hours as shown in $\bar{Z}$, where $\bar{z_i} \geq 0$ for $i \in$ N.

Let $\bar{Z}_0$=(-1,-1,-1,-1,....,-1) be the initial state of all workers in the set N.

From equation (1), we have:

$$\phi_i(f(\bar{Z})) = \sum_{\substack{Z \rhd \bar{Z} \\ z_i = -1}} \frac{(|Z|)!(n - |Z| - 1)!}{n!}[f(z + e_i[z_i + 1 + r(z_i)]) - f(z)]$$

$$\phi_i(g(\bar{Z})) = \sum_{\substack{Z \rhd \bar{Z} \\ z_i = -1}} \frac{(|Z|)!(n - |Z| - 1)!}{n!}[g(z + e_i[z_i + 1 + r(z_i)]) - g(z)]$$

The marginal contribution of worker i in each firm is given by:

$f^i(\bar{Z})) = [f(z + e_i[\bar{z_i} + 1 - 1]) - f(z)] = f(z + e_i[\bar{z_i}]) - f(z)$

$g^i(\bar{Z})) = [g(z + e_i[\bar{z_i} + 1 - 1]) - g(z)] = g(z + e_i[\bar{z_i}]) - g(z)$

We can re-write the extended Shapley value for organizations as shown below.

$$\phi_i(f(\bar{Z})) = \sum_{\substack{Z \rhd \bar{Z} \\ z_i=-1}} \frac{(|Z|)!(n-|Z|-1)!}{n!}[f^i(\bar{Z})]$$

$$\phi_i(g(\bar{Z})) = \sum_{\substack{Z \rhd \bar{Z} \\ z_i=-1}} \frac{(|Z|)!(n-|Z|-1)!}{n!}[g^i(\bar{Z})]$$

It is clear that if $f^i(\bar{Z}) \geq g^i(\bar{Z})$ for all $\bar{Z} \in \mathrm{Z}$ then $\phi_i(f(\bar{Z})) \geq \phi_i(g(\bar{Z}))$. This result establishes strong monotonicity. It implies that if a cooperative game changes so that some worker's contribution to all coalitions increases or stays the same then the corresponding worker's allocation should increase or stay the same.
We conclude the first part of our proof, that the extended Shapley value for organizations is *symmetric, efficient* and *strongly monotonic.* □

We next prove that an allocation procedure that is *symmetric, efficient* and *strongly monotonic* is the extended Shapley value for organizations.

Strong Monotonicity means that for any two production functions f and g,

$$f^i(\bar{Z}) = g^i(\bar{Z}) \text{ for all } \bar{Z} \in \mathrm{Z} \text{ implies that } \phi_i(f(\bar{Z})) = \phi_i(g(\bar{Z})).$$

Consider a symmetric production function g which is identically zero for all $\bar{Z} \in \mathrm{Z}$, so that $g^i(\bar{Z}) = 0$ for all $i \in \mathrm{N}$.
By *Symmetry,* $\phi_i(g(\bar{Z})) = \phi_j(g(\bar{Z}))$ for all workers $i \neq j$.
By *Efficiency,* $\sum_i \phi_i(g(\bar{Z})) = 0$ and therefore, $\phi_i(g(\bar{Z})) = 0$ for all $i \in \mathrm{N}$.

By (2) for any production function f and any worker $i \in \mathrm{N}$,
If $f^i(\bar{Z}) = g^i(\bar{Z})$ for all $\bar{Z} \in \mathrm{Z}$ then $f^i(\bar{Z}) = 0$, which in turn implies that $\phi_i(f(\bar{Z})) = 0$ as $\phi_i(g(\bar{Z})) = 0$

Consider a non-empty subset R of N, where the N is the set of workers. Z is the set of all possible labor allocations for all workers in R. According to Shapley, every game can be expressed as a sum of primitive games. Using this fact, we can express the production function f as follows:

$$f = \sum_{\emptyset \neq \mathrm{R} \subseteq \mathrm{N}} c_{\mathrm{R}} f_{\mathrm{R}} \tag{2}$$

We extend this concept to an environment where workers can choose to provide h levels of labor input. For simplicity, we choose a predetermined labor allocation $\bar{Z}$ for all workers in R, where $\bar{z_i} = \bar{z_j} \geq 0$; $\forall i, j \in$ R and $i \neq j$. Then,

$$c_{\mathrm{R}} f_{\mathrm{R}} = \begin{cases} c_{\mathrm{R}} f_{\mathrm{R}} & \text{if} \quad \mathrm{R} \subseteq \mathrm{N} \\ 0 & \text{if} \quad \mathrm{R} \not\subseteq \mathrm{N} \end{cases}$$

$$f_{\mathrm{R}}(\bar{Z}) = \begin{cases} f_{\mathrm{R}}(\bar{Z}) & \text{if} \quad \mathrm{R} \subseteq \mathrm{N} \\ 0 & \text{if} \quad \mathrm{R} \not\subseteq \mathrm{N} \end{cases}$$

Assuming constant marginal product $\forall i \neq j$ and by *symmetry* and *efficiency*, the Shapley value is

$$\phi_i(f(\bar{Z})) = \sum_{\emptyset \neq \mathrm{R} \subseteq \mathrm{N}} c_{\mathrm{R}} f_{\mathrm{R}}(\bar{Z}) = \sum_{\mathrm{R}: i \in \mathrm{R}} \frac{c_{\mathrm{R}} f_{\mathrm{R}}(\bar{Z})}{|\mathrm{R}|} \tag{3}$$

$|\mathrm{R}| \in \mathbb{R}$ is the total number of workers who have entered the firm in labor allocation $\bar{Z}$. Using the functions p and q defined in the Preliminaries section, we can obtain the value for $|\mathrm{R}|$.

We define an Index I of f as the minimum number of non-zero terms in some expression for f of the form (3). The theorem is proved by the induction on I.

Case 1
If I $= 0$, worker i is a null worker and receives a payoff of 0.
$\phi_i(f(\bar{Z})) = 0$ for all $i \in$ N.

Case 2
If I $= 1$, $f = c_{\mathrm{R}} f_{\mathrm{R}}$ for some R $\subseteq$ N.
For $i \notin$ R, $f^i(\bar{Z}) = 0$; $\forall \bar(Z) \in$ Z. Therefore, $\phi_i(f(\bar{Z})) = 0$.

For all $i, j \in$ R where $\bar{z_i} = \bar{z_j} \geq 0$ and the initial labor allocation is given by $z_i = z_j = -1$, *By symmetry*, $\phi_i(f(\bar{Z})) = \phi_j(f(\bar{Z}))$.

By *efficiency*, we have $\sum_i \phi_i(f(\bar{Z})) = f(\bar{Z}) = c_{\mathrm{R}} f_{\mathrm{R}}(\bar{Z})$.

$$\phi_i(f(\bar{Z})) = \frac{c_{\mathrm{R}} f_{\mathrm{R}}(\bar{Z})}{|\mathrm{R}|} \text{ for all } i \in R$$

We conclude that, $\phi_i(f(\bar{Z}))$ is the extended Shapley value when the Index of I is 0 or 1.

Case 3

Let $\phi_i(f(\bar{Z}))$ be the extended Shapley value when the index of f is at most I and let f have an of index of I+1, then

$$f = \sum_{k=1}^{I+1} c_{\mathrm{R}_k} f_{\mathrm{R}_k}(\bar{Z}), \qquad \forall c_{\mathrm{R}_k} \neq 0 \tag{4}$$

$$\text{Let } R = \bigcap_{k=1}^{I+1} \mathrm{R}_k \tag{5}$$

Assume $i \notin$ R, we define a production function g as follows:

$$g = \sum_{k:i\in \mathrm{R}_k} c_{\mathrm{R}_k} f_{\mathrm{R}_k}(\bar{Z}) \tag{6}$$

The index of g is at most I and $g^i(\bar{Z}) = f^i(\bar{Z})$ for all $\bar{Z} \in$ Z.
By induction and strong monotonicity we conclude,

$$\phi_i(f(\bar{Z})) = \phi_i(g(\bar{Z})) = \sum_{k:i\in \mathrm{R}_k} \frac{c_{\mathrm{R}_k} f_{\mathrm{R}_k}(\bar{Z})}{|\mathrm{R}|}$$

Therefore, $\phi_i(f(\bar{Z}))$ is the extended Shapley value when $i \notin$ R.

Now, let $i \in$ R. By *symmetry* $\phi_i(f(\bar{Z}))$ is a constant c for all members of R. Similarly, the Shapley value is a constant c' for all members of R as $\bar{z}_i = \bar{z}_j$ for all $i, j \in$ R and $i \neq j$. By *efficiency* both allocations sum to $f(\bar{Z})$ and are also equal for all i not in R. Therefore, c = c' and $\phi_i(f(\bar{Z}))$ is also the extended Shapley value when $i \in$ R.
We conclude the second part of our proof that an allocation procedure that is *symmetric, efficient* and *strongly monotonic* is the extended Shapley value for organizations. □

Therefore, we have proved that an allocation procedure is efficient, symmetric and strongly monotonic if and only if it is the extended Shapley value for organizations defined by equation (1).

8 Conclusion

The classical Shapley value assumes that workers can exert only two levels of effort: "work" and "not work" denoted by 1 and 0, respectively. In the real world, however, firms employ part-time and/or full-time workers with varying levels of labor input per individual worker. In practice, a worker's compensation should be based on the effort level he/she brings to the firm.

Many studies have addressed this gap in the existing literature by allowing for workers to face a finite list of alternative options. In contrast to previous studies, we make no assumptions on the importance of tasks but rather use varying levels of hours contributed to production process to study the distribution of joint surplus within an organization. We also distinguish between workers who do not enter the firm and workers who enter the firm but contribute nothing. This distinction allows for firms to hire part-time and seasonal workers as required.

We extend the classical Shapley value to organizations allowing for varying levels of hours per individual worker and axiomatize the new value using three axioms: symmetry, efficiency and strong monotonicity.

9 References

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10 Appendix

For $\bar{Z}$ **= (0,0)**

$$\phi_1(f(\bar{Z})) = \underbrace{\frac{(0)!(2-1)!}{2!}[f((0,0)+e_1[0-1+1])-f(0,0)]}_{\textbf{worker 1 enters 1st}} + \underbrace{\frac{(1)!(2-1-1)!}{2!}[f((0,0)+e_1[0-1+1])-f(0,0)]}_{\textbf{worker 1 enters 2nd}}$$

$$\phi_1(f(\bar{Z})) = \tfrac{(0)!(2-1)!}{2!}(0-0) + \tfrac{(1)!(2-1-1)!}{2!}(0-0) = 0$$

$$\phi_2(f(\bar{Z})) = \underbrace{\frac{(0)!(2-1)!}{2!}[f((0,0)+e_2[0-1+1])-f(0,0)]}_{\textbf{worker 2 enters 1st}} + \underbrace{\frac{(1)!(2-1-1)!}{2!}[f((0,0)+e_2[0-1+1])-f(0,0)]}_{\textbf{worker 2 enters 2nd}}$$

$$\phi_2(f(\bar{Z})) = \tfrac{(0)!(2-1)!}{2!}(0-0) + \tfrac{(1)!(2-1-1)!}{2!}(0-0) = 0$$

For $\bar{Z}$ **= (0,1)**

$$\phi_1(f(\bar{Z})) = \underbrace{\frac{(0)!(2-1)!}{2!}[f((0,0)+e_1[0+1-1])-f(0,0)]}_{\textbf{worker 1 enters 1st}} + \underbrace{\frac{(1)!(2-1-1)!}{2!}[f((0,1)+e_1[0+1-1])-f(0,1)]}_{\textbf{worker 1 enters 2nd}}$$

$$\phi_1(f(\bar{Z})) = \tfrac{(0)!(2-1)!}{2!}(0-0) + \tfrac{(1)!(2-1-1)!}{2!}(70-70) = 0$$

$$\phi_2(f(\bar{Z})) = \underbrace{\frac{(0)!(2-1)!}{2!}[f((0,0)+e_2[1+1-1])-f(0,0)]}_{\textbf{worker 2 enters 1st}} + \underbrace{\frac{(1)!(2-1-1)!}{2!}[f((0,0)+e_2[1+1-1])-f(0,0)]}_{\textbf{worker 2 enters 2nd}}$$

$$\phi_2(f(\bar{Z})) = \tfrac{(0)!(2-1)!}{2!}(70-0) + \tfrac{(1)!(2-1-1)!}{2!}(70-0) = 70$$

For $\bar{Z}$ **= (1,0)**

$$\phi_1(f(\bar{Z})) = \underbrace{\frac{(0)!(2-1)!}{2!}[f((0,0)+e_1[1+1-1])-f(0,0)]}_{\textbf{worker 1 enters 1st}} + \underbrace{\frac{(1)!(2-1-1)!}{2!}[f((0,0)+e_1[1+1-1])-f(0,0)]}_{\textbf{worker 1 enters 2nd}}$$

$$\phi_1(f(\bar{Z})) = \tfrac{(0)!(2-1)!}{2!}(70-0) + \tfrac{(1)!(2-1-1)!}{2!}(70-0) = 70$$

$$\phi_2(f(\bar{Z})) = \underbrace{\frac{(0)!(2-1)!}{2!}[f((0,0)+e_2[0+1-1])-f(0,0)]}_{\textbf{worker 2 enters 1st}} + \underbrace{\frac{(1)!(2-1-1)!}{2!}[f((1,0)+e_2[0+1-1])-f(1,0)]}_{\textbf{worker 2 enters 2nd}}$$

$\phi_2(f(\bar{Z})) = \frac{(0)!(2-1)!}{2!}(0-0) + \frac{(1)!(2-1-1)!}{2!}(70-70) = 0$

For $\bar{Z}$ $\mathbf{= (1,1)}$

$$\phi_1(f(\bar{Z})) = \underbrace{\frac{(0)!(2-1)!}{2!}[f((0,0)+e_1[1+1-1])-f(0,0)]}_{\textbf{worker 1 enters 1st}} + \underbrace{\frac{(1)!(2-1-1)!}{2!}[f((0,1)+e_1[1+1-1])-f(0,1)]}_{\textbf{worker 1 enters 2nd}}$$

$\phi_1(f(\bar{Z})) = \frac{(0)!(2-1)!}{2!}(70-0) + \frac{(1)!(2-1-1)!}{2!}(80-70) = 40$

$$\phi_2(f(\bar{Z})) = \underbrace{\frac{(0)!(2-1)!}{2!}[f((0,0)+e_2[1+1-1])-f(0,0)]}_{\textbf{worker 2 enters 1st}} + \underbrace{\frac{(1)!(2-1-1)!}{2!}[f((1,0)+e_2[1+1-1])-f(1,0)]}_{\textbf{worker 2 enters 2nd}}$$

$\phi_2(f(\bar{Z})) = \frac{(0)!(2-1)!}{2!}(70-0) + \frac{(1)!(2-1-1)!}{2!}(80-10) = 40$

For $\bar{Z}$ $\mathbf{= (0,2)}$

$$\phi_1(f(\bar{Z})) = \underbrace{\frac{(0)!(2-1)!}{2!}[f((0,0)+e_1[0+1-1])-f(0,0)]}_{\textbf{worker 1 enters 1st}} + \underbrace{\frac{(1)!(2-1-1)!}{2!}[f((0,2)+e_1[0+1-1])-f(0,2)]}_{\textbf{worker 1 enters 2nd}}$$

$\phi_1(f(\bar{Z})) = \frac{(0)!(2-1)!}{2!}(0-0) + \frac{(1)!(2-1-1)!}{2!}(80-80) = 0$

$$\phi_2(f(\bar{Z})) = \underbrace{\frac{(0)!(2-1)!}{2!}[f((0,0)+e_2[2+1-1])-f(0,0)]}_{\textbf{worker 2 enters 1st}} + \underbrace{\frac{(1)!(2-1-1)!}{2!}[f((0,0)+e_2[2+1-1])-f(0,0)]}_{\textbf{worker 2 enters 2nd}}$$

$\phi_2(f(\bar{Z})) = \frac{(0)!(2-1)!}{2!}(80-0) + \frac{(1)!(2-1-1)!}{2!}(80-0) = 80$

For $\bar{Z}$ $\mathbf{= (2,0)}$

$$\phi_1(f(\bar{Z})) = \underbrace{\frac{(0)!(2-1)!}{2!}[f((0,0)+e_1[2+1-1])-f(0,0)]}_{\textbf{worker 1 enters 1st}} + \underbrace{\frac{(1)!(2-1-1)!}{2!}[f((0,0)+e_1[2+1-1])-f(0,0)]}_{\textbf{worker 1 enters 2nd}}$$

$\phi_1(f(\bar{Z})) = \frac{(0)!(2-1)!}{2!}(80-0) + \frac{(1)!(2-1-1)!}{2!}(80-0) = 80$

$$\phi_2(f(\bar{Z})) = \underbrace{\frac{(0)!(2-1)!}{2!}[f((0,0)+e_2[0+1-1])-f(0,0)]}_{\textbf{worker 2 enters 1st}} + \underbrace{\frac{(1)!(2-1-1)!}{2!}[f((2,0)+e_2[0+1-1])-f(2,0)]}_{\textbf{worker 2 enters 2nd}}$$

$\phi_2(f(\bar{Z})) = \frac{(0)!(2-1)!}{2!}(0-0) + \frac{(1)!(2-1-1)!}{2!}(80-80) = 0$

For $\bar{Z}$ $\mathbf{= (1,2)}$

$$\phi_1(f(\bar{Z})) = \underbrace{\frac{(0)!(2-1)!}{2!}[f((0,0)+e_1[1+1-1])-f(0,0)]}_{\textbf{worker 1 enters 1st}} + \underbrace{\frac{(1)!(2-1-1)!}{2!}[f((0,2)+e_1[1+1-1])-f(0,2)]}_{\textbf{worker 1 enters 2nd}}$$

$\phi_1(f(\bar{Z})) = \frac{(0)!(2-1)!}{2!}(70-0) + \frac{(1)!(2-1-1)!}{2!}(90-80) = 40$

$$\phi_2(f(\bar{Z})) = \underbrace{\frac{(0)!(2-1)!}{2!}[f((0,0)+e_2[2+1-1]) - f(0,0)]}_{\textbf{worker 2 enters 1st}} + \underbrace{\frac{(1)!(2-1-1)!}{2!}[f((1,0)+e_2[2+1-1]) - f(1,0)]}_{\textbf{worker 2 enters 2nd}}$$

$$\phi_2(f(\bar{Z})) = \tfrac{(0)!(2-1)!}{2!}(80-0) + \tfrac{(1)!(2-1-1)!}{2!}(90-70) = 50$$

For $\bar{Z}$ **= (2,1)**

$$\phi_1(f(\bar{Z})) = \underbrace{\frac{(0)!(2-1)!}{2!}[f((0,0)+e_1[2+1-1]) - f(0,0)]}_{\textbf{worker 1 enters 1st}} + \underbrace{\frac{(1)!(2-1-1)!}{2!}[f((0,1)+e_1[2+1-1]) - f(0,1)]}_{\textbf{worker 1 enters 2nd}}$$

$$\phi_1(f(\bar{Z})) = \tfrac{(0)!(2-1)!}{2!}(80-0) + \tfrac{(1)!(2-1-1)!}{2!}(90-70) = 50$$

$$\phi_2(f(\bar{Z})) = \underbrace{\frac{(0)!(2-1)!}{2!}[f((0,0)+e_2[1+1-1]) - f(0,0)]}_{\textbf{worker 2 enters 1st}} + \underbrace{\frac{(1)!(2-1-1)!}{2!}[f((2,0)+e_2[1+1-1]) - f(2,0)]}_{\textbf{worker 2 enters 2nd}}$$

$$\phi_2(f(\bar{Z})) = \tfrac{(0)!(2-1)!}{2!}(70-0) + \tfrac{(1)!(2-1-1)!}{2!}(90-80) = 40$$

For $\bar{Z}$ **= (2,2)**

$$\phi_1(f(\bar{Z})) = \underbrace{\frac{(0)!(2-1)!}{2!}[f((0,0)+e_1[2+1-1]) - f(0,0)]}_{\textbf{worker 1 enters 1st}} + \underbrace{\frac{(1)!(2-1-1)!}{2!}[f((0,2)+e_1[2+1-1]) - f(0,2)]}_{\textbf{worker 1 enters 2nd}}$$

$$\phi_1(f(\bar{Z})) = \tfrac{(0)!(2-1)!}{2!}(80-0) + \tfrac{(1)!(2-1-1)!}{2!}(100-80) = 50$$

$$\phi_2(f(\bar{Z})) = \underbrace{\frac{(0)!(2-1)!}{2!}[f((0,0)+e_2[2+1-1]) - f(0,0)]}_{\textbf{worker 2 enters 1st}} + \underbrace{\frac{(1)!(2-1-1)!}{2!}[f((2,0)+e_2[2+1-1]) - f(2,0)]}_{\textbf{worker 2 enters 2nd}}$$

$$\phi_2(f(\bar{Z})) = \tfrac{(0)!(2-1)!}{2!}(80-0) + \tfrac{(1)!(2-1-1)!}{2!}(100-80) = 50$$