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# THE STANDARD POLYNOMIAL AS AN IDENTITY ON SYMPLECTIC MATRICES

By
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A thesis submitted to the School of Graduate
Studies and Research of the University of Ottawa,
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at the
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# THE STANDARD POLYNOMIAL AS AN IDENTITY ON SYMPLECTIC MATRICES.

by Jay M.H. Adamsson

## Summary.

The *symplectic involution* $s$ is defined on $2n \times 2n$ matrices by

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}^s = \begin{pmatrix} D^t & -B^t \\ -C^t & A^t \end{pmatrix}$$

where $A$, $B$, $C$, and $D$ are $n \times n$ matrices, and $t$ is the standard transpose operation. This thesis investigates the value of the standard polynomial $S_k := \sum_{\sigma \in S_k} (-1)^\sigma x_{\sigma(1)} \cdots x_{\sigma(k)}$ evaluated over the ring of matrices which are symmetric with respect to the symplectic involution, denoted $H_n(F,s)$.

It was shown by Rowen that $S_{4n-2}$ is an identity for $H_n(F,s)$ for all $n$. He also shows that $S_{4n-3}$ is not an identity for $n = 1$ or $n = 2$. In this thesis the value of $S_{4n-3}$ for $n \geq 3$ is studied In particular, it is shown that $S_{4n-3}$ is not an identity for $n = 3$ and $n = 4$. Moreover, a reduction is obtained in the general case.

We choose a particular basis where each basis element can be represented on a graph of $2n$ points by a pair of labelled, directed edges. A pseudo-Eulerian path on this graph is defined as a path in which exactly one edge of each pair of edges is traversed exactly once. By counting the number of pseudo-Eulerian paths and assigning a value of -1 or +1 to each, the value of $S_k$ can be determined.

Our first result says that $S_{4n-3}$ is not an identity if $S_{4n-4}$ is not an identity. For the case n=3, the eight basis elements which will be used to show that $S_8$ is not an identity are:

$$\begin{array}{lll} x_1 = e_{12} + e_{54} & x_4 = e_{13} + e_{64} & x_7 = e_{11} + e_{44} \\ x_2 = e_{23} + e_{65} & x_5 = e_{32} + e_{56} & x_8 = e_{16} - e_{34} \\ x_3 = e_{31} + e_{46} & x_6 = e_{21} + e_{45} & \end{array}$$

Each pseudo-Eulerian path can be decomposed into three "cycles" and a "tail". By "inverting" some of the cycles, a new pseudo-Eulerian path can be obtained which has opposite sign. The total number of pseudo-Eulerian paths, which have to be counted to determine the value of $S_8$, can be reduced by pairing off paths which have been given opposite signs. Once the pairing off has been completed, there are thirty paths which are left, showing that the value of $S_8$ is non-zero.

In the case $n = 4$, the substitution used is:

$$\begin{array}{lll} x_1 = e_{12} + e_{65} & x_5 = e_{14} + e_{85} & x_9 = e_{13} + e_{75} \\ x_2 = e_{23} + e_{76} & x_6 = e_{43} + e_{78} & x_{10} = e_{31} + e_{57} \\ x_3 = e_{34} + e_{87} & x_7 = e_{32} + e_{67} & x_{11} = e_{11} + e_{55} \\ x_4 = e_{41} + e_{58} & x_8 = e_{21} + e_{56} & x_{12} = e_{18} - e_{45} \end{array}$$

The same method of pairing off paths is used in this instance to arrive at a total of 168 paths which are left unpaired. Thus, the value of $S_{12}$ is non-zero.

In the general case, the substitutions used are:

$$\begin{array}{ll} x_i = e_{i,i+1} + e_{i+1+n,i+n} & i = 1, \ldots, n-1 \\ x_n = e_{n,1} + e_{n+1,2n} & \\ x_{n+1} = e_{1,n} + e_{2n,n+1} & \\ x_i = e_{2n+2-i,2n+1-i} + e_{3n+1-i,3n+2-i} & i = n+2, \ldots, 2n \\ x_i = e_{1,2n+4-i} + e_{3n+4-i,n+1} & i = 2n+1, \ldots, 3n-3 \\ x_i = e_{3n+1-i,1} + e_{1+n,4n+1-i} & i = 3n-2, \ldots, 4n-6 \\ x_{4n-5} = e_{1,1} + e_{n+1,n+1} & \\ x_{4n-4} = e_{1,2n} - e_{n,n+1} & \end{array}$$

When the same procedure is applied, a reduction of the problem is obtained. However, the total number of all paths which contribute to this value is unknown.

## Introduction

In this thesis, we consider matrices over arbitrary fields. The *symplectic involution* $s$ is defined on $2n \times 2n$ matrices by

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}^s = \begin{pmatrix} D^t & -B^t \\ -C^t & A^t \end{pmatrix}$$

where $A$, $B$, $C$, and $D$ are $n \times n$ matrices, and $t$ is the standard transpose involution. A matrix is symmetric with respect to the symplectic involution if and only if it has the properties:

$$A = D^t$$
$$B = -B^t$$
$$C = -C^t.$$

The *standard polynomial* $S_k$ in k non-commuting variables is defined as

$$S_k = \sum_{\sigma \in S_k} (-1)^\sigma x_{\sigma(1)} \cdots x_{\sigma(k)},$$

where $S_k$ is the permutation group on $k$ elements. The Amitsur-Levitski Theorem [1] states that $S_{2n}$ is an identity for $M_n(F)$. Rowen [2] shows that $S_{4n-2}$ is an identity for the space of $2n \times 2n$ symplectically symmetric matrices $H_n(F,s)$ over a field $F$, and that this is minimal for $n = 1$ and 2. The goal of this thesis is to show the sharpness of this result for $H_3(F,s)$ and $H_4(F,s)$; i.e., that $S_{4n-3}$ is not an identity for $n = 3$ or $n = 4$. In addition, a partial result is given in the general case.

The space of $2n \times 2n$ symplectically symmetric matrices $H_n(F,s)$ can be given a basis of $2n^2 - n$ matrices:

$$\begin{array}{ll} e_{i,j} + e_{n+j,n+i} & 1 \le i,j \le n \\ e_{i,n+j} - e_{j,n+i} & 1 \le i < j \le n \\ e_{n+i,j} - e_{n+j,i} & 1 \le i < j \le n \end{array}$$

where $e_{ij}$ is the matrix with 1 in the $i,j$ position, and 0 elsewhere. Each of these basis elements can be represented on a graph of $2n$ points in the following way. We label the points from 1 to $2n$ and draw a directed edge with positive label from point $i$ to point $j$ to represent $e_{ij}$. Similarly, we introduce an edge from $i$ to $j$ with a negative label to represent $-e_{ij}$. In this manner, each of the basis elements in (1) is represented by a pair of directed and labelled edges on a graph.

Let $x_1, ..., x_m$ be a subset of the basis elements of $H_n(F,s)$ described above, and let $G_m$ be a graph of $2n$ points in which each of the basis elements $x_i$ is represented as described above. Then a *pseudo-Eulerian* path on $G_m$ is a path of length $m$ in which exactly one edge of the pair of edges associated with each $x_i$ is traversed exactly once. Let $\sigma$ be a permutation of $1, ..., m$. We use the following Reduction Principle, whose proof is straightforward.

**Reduction Principle.** *Let* $x_1, \dots, x_m$ *be a subset of the basis elements of* $H_n(F,s)$ *described above, and* $\sigma$ *be a permutation of* $1, \dots, m$. *Define* $A = x_{\sigma(1)} \cdots x_{\sigma(m)}$.
a. *All elements of* $A$ *are* $\pm 1$ *or 0,*
b. $A_{ij} = \pm 1$ *for some* $i, j \leq m$ *iff* $x_{\sigma(1)} \cdots x_{\sigma(m)}$ *is a pseudo-Eulerian path.*

In this paper, the value of $S_n(x_1, \dots, x_n)$ will be determined by counting the number of pseudo-Eulerian paths between two points, and by determining if each path contributes a positive or a negative value to the total sum.

## 1. Counting Pseudo-Eulerian Paths

Let $\tau$ be a permutation of 1 to $m$ defined by $\{i_1, \dots, i_m\}$ in which $\tau(j) = i_j, j = 1, \dots, m$. Assume the permutation $\tau$ defines a pseudo-Eulerian path from point $K$ to point $L$. Consider whether this path contributes a positive value of +1 or a negative value of $-1$ to the value of $M_{KL}$, where $M = S_m(x_1, \dots, x_m)$and $M_{KL}$ denotes the entry in the $K$th row and the $L$th column. To determine this value, we have to take two factors into account.

The first is the signature of $\tau$. By definition, the term $x_{i_1} \dots x_{i_m}$ in $S_m$ is preceded by $(-1)^\tau$. Thus, if $\tau$ is an odd permutation, the contribution of the pseudo-Eulerian path given by $\tau$ will be reversed.

Secondly, we have to consider the sign of the edge which is traversed. For many of the basis elements, one of the corresponding edges is positive, and the other negative. Each time the path crosses an edge which is negative, the contribution of the path will be reversed.

Consider the following examples. In $H_3(F,s)$, define

$$x_1 = e_{12} + e_{54},$$
$$x_2 = e_{23} + e_{65},$$
$$x_3 = e_{31} + e_{46},$$

which give the following diagram (where all edges are positive):

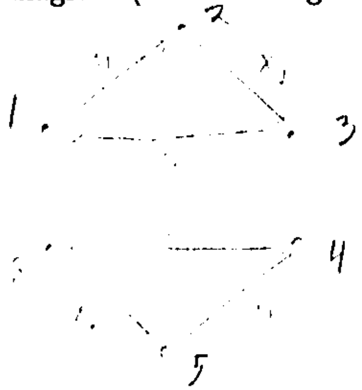


Figure 1

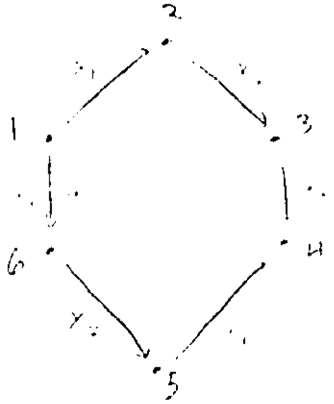
Let $M = S_3(x_1, x_2, x_3)$ and consider $M_{11}$. The only pseudo-Eulerian path from point 1 which returns to point 1 is $\tau = \{x_1, x_2, x_3\}$, i.e., $\tau = id$, the identity permutation on three elements. Thus, we conclude that $M_{11} = 1$. Now consider $M_{44}$. Once again, there is only one pseudo-Eulerian path from point 4 and back to point 4 again, namely the path given by $\tau = \{x_3, x_2, x_1\}$. This path is defined by the permutation $\tau$, which is an odd permutation of three elements. Thus, the contribution of $\tau$ to the value of $M_{44}$ is $-1$. Since this is the only path, the value of $M_{44}$ is $-1$.

For the next example, consider

$$\begin{aligned} x_1 &= e_{12} + e_{54}, \\ x_2 &= e_{23} + e_{65}, \\ x_3 &= e_{16} - e_{34}. \end{aligned}$$

These three elements of $H_3(F, s)$ give the following diagram, where all unlabelled edges are positive:

Figure 2



Again, let $M = S_3(x_1, x_2, x_3)$. There are two pseudo-Eulerian paths from point 1 to point 4. The first, $\{x_1, x_2, x_3\}$ is an even permutation, however, it crosses the leg of $x_3$ which is labelled negative, reversing its contribution from a positive value to a negative one. Hence, the contribution of the path $\{x_1, x_2, x_3\}$ to the value of $M_{14}$ is $-1$. The other path, $\{x_3, x_2, x_1\}$, is an odd permution which crosses the even leg of $x_3$. Thus, it too contributes the negative value of $-1$ to the overall value of $M_{14}$. The total value of $M_{14}$ is the sum of the contributions from all the corresponding pseudo-Eulerian paths, hence we determine that in this example, we have $M_{14} = -2$.

## 2. The Polynomial $S_{2n-3}$ in $H_n(F,s)$

We begin this section with the following

**Lemma 1.** *Let* $i, k \in N$ *with* $1 \leq i \leq k$.

$$\sum_{\substack{\sigma \in S_k \\ \sigma(i)=k}} (-1)^{\sigma} x_{\sigma(1)} \cdots x_{\sigma(k)}$$
$$= (-1)^{i+1} \sum_{\tau \in S_{k-1}} (-1)^{\tau} x_{\tau(1)} \cdots x_{\tau(i-1)} x_k x_{\tau(i)} \cdots x_{\tau(k-1)}$$

*Proof.* There is a correspondence between $\{\sigma : \sigma \in S_k, \sigma(i) = k\}$ and $\{\tau : \tau \in S_{k-1}\}$ given by:

$$\begin{aligned} &\tau(j) = \sigma(j) && 1 \leq j \leq i-1, \\ &\tau(j) = \sigma(j+1) && k \leq j \leq k-1. \end{aligned}$$

From this relation, we see that

$$x_{\sigma(1)} \cdots x_{\sigma(k)} = x_{\tau(1)} \cdots x_{\tau(i-1)} x_k x_{\tau(i)} \cdots x_{\tau(k-1)}$$

for corresponding $\sigma$ and $\tau$. This means that each monomial on the left hand side is repeated again on the right hand side. It remains to be shown that $(-1)^{\tau} = (-1)^{\sigma+k-i} = (-1)^{\sigma+1+i}$. When $i = k$, then obviously, for the corresponding $\tau$, we have that $(-1)^{\sigma} = (-1)^{\tau}$, and the result holds. Now proceed inductively. Let $\sigma \in S_k$ with $\sigma(i) = k$. Then there is a correspondance between $\{\sigma : \sigma \in S_k, \sigma(i) = k\}$ and $\{\sigma : \sigma \in S_k, \sigma(i-1) = k\}$ given by:

$$\begin{aligned} &\sigma'(j) = \sigma(j) && 1 \leq j \leq i-2, \\ &\sigma'(i-1) = k-1 = \sigma(i), && \\ &\sigma'(i) = \sigma(i-1), && \\ &\sigma'(j) = \sigma(j) && i+1 \leq i \leq k. \end{aligned}$$

The $\tau$ and $\tau'$ in $S_{k-1}$ which correspond to $\sigma$ and $\sigma'$ respectively are the same permutation. The only difference between $\sigma$ and $\sigma'$ are the values for $i-1$ and $i$, which are transposed. Thus, $(-1)^{\sigma} = -(-1)^{\sigma'}$. The lemma follows from this result.

**Theorem 1.** *Let* $x_1, \ldots, x_k \in H_3(F, s), I$*=the* $2n \times 2n$ *identity matrix. Then*

a. $S_k(x_1, \ldots, x_{k-1}, I) = 0$ *if* $k$ *is even*
b. $S_k(x_1, \ldots, x_k) = \sum_{i=1}^{k} (-1)^{k-i} S_{k-1}(x_1, \ldots, \hat{x}_i, \ldots, x_k)x_i$
c. $S_k(x_1, \ldots, x_{k-1}, I) = S_{k-1}(x_1, \ldots, x_{k-1})$ *when* $k$ *is odd*

*Proof.* **a.**

$$S_k(x_1, \ldots, x_k) = \sum_{\sigma \in S_k} (-1)^\sigma x_{\sigma(1)} \cdots x_{\sigma(k)}$$
$$= \sum_{i=1}^{k} \sum_{\substack{\sigma \in S_k \\ \sigma(i)=k}} (-1)^\sigma x_{\sigma(1)} \cdots x_{\sigma(k)}$$

Now let $x_k = I$:

$$\begin{aligned} S_k(x_1, \ldots, x_{k-1}, I) = & \sum_{\substack{\sigma \in S_k \\ \sigma(k)=k}} (-1)^\sigma x_{\sigma(1)} \cdots x_{\sigma(k-1)} I + \cdots \\ & + \sum_{\substack{\sigma \in S_k \\ \sigma(i)=k}} (-1)^\sigma x_{\sigma(1)} \cdots x_{\sigma(i-1)} I x_{\sigma(i+1)} \cdots x_{\sigma(k)} + \cdots \\ & + \sum_{\substack{\sigma \in S_k \\ \sigma(1)=k}} (-1)^\sigma I x_{\sigma(1)} \cdots x_{\sigma(k)} \end{aligned}$$
$$= \sum_{i=1}^{k} \sum_{\substack{\sigma \in S_k \\ \sigma(i)=k}} (-1)^\sigma x_{\sigma(1)} \cdots x_{\hat{\sigma}(i)} \cdots x_{\sigma(k)}$$
$$= \sum_{i=1}^{k} (-1)^{i+1} \sum_{\tau \in S_{k-1}} (-1)^\tau x_{\tau(1)} \cdots x_{\tau(k-1)}$$
$$= 0$$

Since $k$ is even, all the terms in this alternating sum cancel.

**b.** Collect all terms with $x_i$ at the end.

**c.** $S_k(x_1, \ldots, x_{k-1}, I) = \sum_{i=1}^{k} (-1)^{k-i} S_{k-1}(x_1, \ldots, \hat{x}_i, \ldots, x_k)x_i$

But $k-1$ is even, thus, by (a), $S_{k-1}(x_1, \ldots, \hat{x}_i, \ldots, x_k) = 0$ unless $i = k$. Thus.

$$S_k(x_1, \ldots, x_{k-1}, I) = S_{k-1}(x_1, \ldots, x_{k-1})$$

This result shows that, in order to prove that $S_{4n-3}$ is not an identity, it is only necessary to prove that $S_{4n-4}$ is not an identity. Therefore, this thesis will provide a set of $4n-4$ elements which will result in a non-zero result for $S_{4n-4}$, thereby establishing that $S_{4n-3}$ is not an identity.

It should be mentioned at this point that all the results which follow have been confirmed on computer by adding the values for all pseudo-Eulerian paths.

## 3. The Case $n = 3$

The cases where $n = 1$ and $n = 2$ have already been covered by Rowen. In the case where $n = 3$, the eight basis elements which will be used to show that $S_8$ is not an identity are:

$$
\begin{array}{lll}
x_1 = e_{12} + e_{54} & x_4 = e_{13} + e_{64} & x_7 = e_{11} + e_{44} \\
x_2 = e_{23} + e_{65} & x_5 = e_{32} + e_{56} & x_8 = e_{16} - e_{34} \\
x_3 = e_{31} + e_{46} & x_6 = e_{21} + e_{45} &
\end{array}
$$

which give the diagram

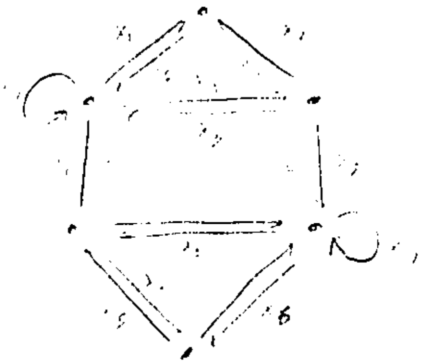


Figure 3

For ease of notation, if $\{x_{i_1}, \ldots, x_{i_j}\}$ is a pseudo-Eulerian path in this diagram, it will be denoted as $\{i_1, \ldots, i_j\}$.

There are a few observations to be made at this point about this set of substitutions. Whenever an integer is used to refer to a point on the diagram, it will be specified as such, in order to avoid confusion:

1. *There is a high degree of symmetry between the "top" and "bottom" halves of the diagram. If the edges labelled $x_8$ are ignored, there is a correspondence between points 1 and 4, between points 2 and 5, and between points 3 and 6. For this reason, the points 4, 5, and 6 will often be called points $1'$, $2'$, and $3'$, respectively.*

2. *Each edge which runs from point $A$ to point $B$ on the top half of the diagram has its corresponding edge going from point $B'$ to point $A'$ on the bottom.*

3. *A path $P$ from point $A$ to point $A'$ gives rise to another path $P^{-1}$, which also runs from point $A$ to point $A'$, but crosses the opposite leg of $x_8$. This*

*path is arrived at by simply taking the elements in the path P and considering them in reverse order. For example, $\{1,2,8\}$ is a path from point 1 to point $1'$ which crosses the negative leg of $x_8$. The reverse path $\{8,2,1\}$ is also a path from point 1 to point $1'$, but which crosses the positive leg of $x_8$.*

Let $M = S_8(x_1,\dots,x_8)$, and consider the value of $M_{16}$. The pseudo-Eulerian paths from point 1 to point 6 are too numerous to count directly, however, most can be eliminated by "pairing off".

Any pseudo-Eulerian path which goes from point 1 to point 6 can be subdivided into four parts. In order to define these parts, the following lemma is needed:

**Lemma 2.** *Suppose $P = \{i_1,\dots,i_8\}$ is a pseudo-Eulerian path. Let $X := \{j \in N \mid \text{the endpoint of } x_j \text{ is point } 1 \text{ or point } 4\}$. Then $\mid X \mid = 3$.*

*Proof.* There exists a $k$ for which $i_k = x_7$. The edge $x_7$ starts and ends at either point 1 or point 4, thus, $k \in X$. There are three other edges which begin at point 1 and two others which end at point 1. By observation 2 above, this gives five edges which end at either point 1 or point 4. Each time one of these edges ends at point 1 or 4, another must be used to leave that point. Hence, these edges occur in pairs, giving two edges which end at point 1 or 4. The final edge begins at point 1 at the beginning of the pseudo-Eulerian path P. Thus, there are three edges which end at either point 1 or point 4.

Arrange $X = \{i_{j_1}, i_{j_2}, i_{j_3}\}$ such that $j_1 < j_2 < j_3$. Then define

$$\begin{aligned} P_1 &:= \{i_1,\dots,i_{j_1}\} \\ P_2 &:= \{i_{j_1+1},\dots,i_{j_2}\} \\ P_3 &:= \{i_{j_2+1},\dots,i_{j_3}\} \\ Q &:= \{i_{j_3+1},\dots,i_8\} \end{aligned}$$

Each of the paths $P_i$ begin at either point 1 or 4, and end at point 1 or 4. These $P_i$'s will be termed *cycles*, since they begin and end at either point 1 or its symmetric counterpart point $1'$. The path Q, called the *tail*, begins at either point 1 or point 4 and ends at point 6.

**Example 1.** $P = \{1,2,3,8,4,7,6,5\}$ is a pseudo-Eulerian path. The edge $x_1$ goes from point 1 to point 2, $x_2$ goes from point 2 to point 3, and $x_3$ goes from point 3 to point 1. Thus, $P_1 = \{1,2,3\}$. Proceeding similarly, $P_2 = \{8,4\}$, $P_3 = \{7\}$, and $Q = \{6,5\}$.

**Example 2.** It may happen that all the $P_i$'s start and end at point 1. Consider $P = \{4,5,6,7,1,2,3,8\}$. Then $P_1 = \{4,5,6\}$, $P_2 = \{7\}$, $P_3 = \{1,2,3\}$, and $Q = \{8\}$.

Given any pseudo-Eulerian path $P$, define

$$\xi(P) = \begin{cases} -1 & \text{if } P \text{ contributes a negative value;} \\ +1 & \text{if } P \text{ contributes a positive value.} \end{cases}$$

Also define $\ell(P_i)$ to be the number of edges in the path $P_i$. In example 1, consider the cycle $P_2$. By observation 3, $P_2^{-1}$ is also a cycle, but one which runs over the opposite leg of $x_8$. Since $\ell(P_2) = 2$, an odd permutation is needed to convert $P_1P_2P_3Q$ into $P_1P_2^{-1}P_3Q$. But, since $P_2$ and $P_2^{-1}$ use different signed edges of $x_8$, an odd permutation combined with an oppositely signed leg give $\xi(P_1P_2P_3Q) = \xi(P_1P_2^{-1}P_3Q)$. To generalize:

**Lemma 3.** *Let $P = P_1P_2P_3Q$. Suppose $8 \in P_i$ Define*

$$P' = \begin{cases} P_1^{-1}P_2P_3Q, & \text{if } i = 1; \\ P_1P_2^{-1}P_3Q, & \text{if } i = 2; \\ P_1P_2P_3^{-1}Q, & \text{if } i = 3. \end{cases}$$

*Then*

$$\begin{array}{ll} \xi(P) = -\xi(P') & \text{if } \ell(P_i) \equiv 0, 1 mod 4 \\ \xi(P) = \xi(P') & \text{if } \ell(P_i) \equiv 2, 3 mod 4 \end{array}$$

*Proof.* Assume $\ell(P_i) \equiv 0, 1 mod 4$. Then an even number of transpositions are needed to convert $P_i$ into $P_i^{-1}$. However, $P_i$ and $P_i^{-1}$ cross oppositely signed legs of $x_8$. Thus, $P$ and $P'$ contribute opposite values.

Likewise, if $\ell(P_i) \equiv 2, 3 mod 4$, an odd number of transpositions are needed to convert $P_i$ into $P_i^{-1}$. Combining this with the oppositely signed leg of $x_8$ give that $P$ and $P'$ contribute the same value.

This relation can be generalized even further, by observing that the cycle which is reversed does not have to be a single $P_i$, but can be a string of $P_i$'s. For example, if $\ell(P_1P_2) \equiv 2$ or $3 mod 4$, then

$$\xi(P_1P_2P_3Q) = \xi((P_1P_2)^{-1}P_3Q) = \xi(P_2^{-1}P_1^{-1}P_3Q).$$

By using this result, we will be able to eliminate most of the pseudo-Eulerian paths, simply by pairing them off with another path of the opposite sign, arrived at by inverting one or more cycles.

We prescribe the following procedure for $P = P_1P_2P_3Q$:

**Step 1.** If $8 \in P_1$ and $\ell(P_1) \equiv 0, 1 mod 4$, define

$$\eta(P) := P_1^{-1}P_2P_3Q.$$

Then $\xi(P) \neq \xi(\eta(P))$ and we pair off $P$ and $\eta(P)$. For later use, we also calculate $\eta(\eta(P)) = \eta(P_1^{-1}P_2P_3Q) = P$.

**Step 2.** If $8 \in P_2$ and $\ell(P_2) \equiv 0, 1 mod 4$, define

$$\eta(P) := P_1 P_2^{-1} P_3 Q.$$

Then $\xi(P) \neq \xi(\eta(P))$ and we pair off $P$ and $\eta(P)$. For later use, we also calculate $\eta(\eta(P)) = \eta(P_1 P_2^{-1} P_3 Q) = P$.

**Step 3.** If $8 \in P_3$ and $\ell(P_3) \equiv 0, 1 mod 4$, define

$$\eta(P) := P_1 P_2 P_3^{-1} Q.$$

Then $\xi(P) \neq \xi(\eta(P))$ and we pair off $P$ and $\eta(P)$. For later use, we also calculate $\eta(\eta(P)) = \eta(P_1 P_2 P_3^{-1} Q) = P$.

**Step 4.** If $8 \in P_1 P_2 P_3$ and $\ell(P_1 P_2 P_3) \equiv 0, 1 mod 4$, then

$$\eta(P) = P_3^{-1} P_2^{-1} P_1^{-1} Q$$

and $\eta(\eta(P)) = P$. Now $\xi(P) \neq \xi(\eta(P))$ and we pair off $P$ and $\eta(P)$.

**Step 5.** If $8 \in P_1 P_2$ and $\ell(P_1 P_2) \equiv 0, 1 mod 4$, then

$$\eta(P) = P_2^{-1} P_1^{-1} P_3 Q$$

and $\eta(\eta(P)) = P$. Now $\xi(P) \neq \xi(\eta(P))$ and we pair off $P$ and $\eta(P)$.

**Step 6.** If $8 \in P_2 P_3$ and $\ell(P_2 P_3) \equiv 0, 1 mod 4$, then

$$\eta(P) = P_1 P_3^{-1} P_2^{-1} Q.$$

Now $\xi(P) \neq \xi(\eta(P))$ and we pair off $P$ and $\eta(P)$. In this case, if $8 \in P_3$ and $\ell(P_1 P_3) \equiv 0, 1 mod 4$, then by step 5 above, $\eta(\eta(P)) = \eta(P_1 P_3^{-1} P_2^{-1} Q) = P_3 P_1^{-1} P_2^{-1} Q$. Otherwise, $\eta(\eta(P)) = P$.

Otherwise, define $\eta(P) = P$. For later use, we record here that $\eta(\eta(P)) = P$ for all but step 6.

For example, if $P = \{1, 2, 3, 8, 4, 7, 6, 5\}$, then $P_1 = \{1, 2, 3\}$, $P_2 = \{8, 4\}$, $P_3 = \{7\}$, and $Q = \{6, 5\}$. Following the order given in the hierarchy, we first test $P_2$. Since $\ell(P_2) \equiv 2 mod 4$, we go on to the next one, which is $P_1 P_2 P_3$. Now, $\ell(P_1 P_2 P_3) = 6 \equiv 2 mod 4$. Continuing, we have $\ell(P_1 P_2) = 5 \equiv 1 mod 4$. Thus, it is determined that $\eta(P_1 P_2 P_3 Q) = (P_1 P_2)^{-1} P_3 Q$, with $\xi(P_1 P_2 P_3 Q) \neq \xi((P_1 P_2)^{-1} P_3 Q)$.

Using the $\eta$ function, we can pair off most of the pseudo-Eulerian paths which occur. All that remains is to count the ones that are not paired off. These paths are the ones that satisfy the following conditions 1 - 6:

1. $8 \notin P_1$ or$\ell(P_1) \equiv 2,3 mod 4$
2. $8 \notin P_2$ or $\ell(P_2) \equiv 2,3 mod 4$
3. $8 \notin P_3$ or $\ell(P_3) \equiv 2,3 mod 4$
4. $8 \notin P_1P_2P_3$ or $\ell(P_1P_2P_3) \equiv 2,3 mod 4$
5. $8 \notin P_1P_2$ or $\ell(P_1P_2) \equiv 2,3 mod 4$
6. $8 \notin P_2P_3$ or $\ell(P_2P_3) \equiv 2,3 mod 4$ or $\eta(\eta(P)) \neq P$.

We rewrite this as:

**Case 1.** $8 \in P_1$
$\ell(P_1) \equiv 2,3 mod 4$
$\ell(P_1P_2P_3) \equiv 2,3 mod 4$
$\ell(P_1P_2) \equiv 2,3 mod 4.$

**Case 2.** $8 \in P_2$
$\ell(P_2) \equiv 2,3 mod 4$
$\ell(P_1P_2P_3) \equiv 2,3 mod 4$
$\ell(P_1P_2) \equiv 2,3 mod 4$
$\ell(P_2P_3) \equiv 2,3 mod 4.$

**Case 3.** $8 \in P_3$
$\ell(P_3) \equiv 2,3 mod 4$
$\ell(P_1P_2P_3) \equiv 2,3 mod 4$
$\ell(P_2P_3) \equiv 2,3 mod 4.$

**Case 4.** $8 \in Q$.

**Case 5.** $8 \in P_1P_2P_3$
$\eta(\eta(P)) \neq P.$

**Case 1:** $8 \in P_1$
$\ell(P_1P_2P_3) \equiv 2,3 mod 4$
$\ell(P_1P_2) \equiv 2,3 mod 4$
$\ell(P_1) \equiv 2,3 mod 4$

**a.** $\ell(P_1) \equiv 2 mod 4$
$\Rightarrow$ $\ell(P_2P_3) \equiv 0,1 mod 4$ since $\ell(P_1P_2P_3) \equiv 2,3 mod 4$
$\ell(P_2) \equiv 0,1 mod 4$ since $\ell(P_1P_2) \equiv 2,3 mod 4$

$\ell(P_2) = 1 \Rightarrow \ell(P_3) \equiv 3,0 mod 4$ since $\ell(P_2P_3) \equiv 0,1 mod 4$

$\ell(P_3) = 1 \Rightarrow \ell(P_2) \equiv 0 mod 4 \Rightarrow \ell(P_2) = 4$

Therefore, there are three possibilities:

| **i.** | **ii.** | **iii.** |
|---|---|---|
| $\ell(P_1) = 2$ | $\ell(P_1) = 2$ | $\ell(P_1) = 2$ |
| $\ell(P_2) = 1$ | $\ell(P_2) = 1$ | $\ell(P_2) = 4$ |
| $\ell(P_3) = 3$ | $\ell(P_3) = 4$ | $\ell(P_3) = 1$ |
| $\ell(Q) = 2$ | $\ell(Q) = 1$ | $\ell(Q) = 1$ |

The actual lengths are known because of the limited number of terms available. The total length of $P_1P_2P_3Q$ is equal to eight. Thus, none of the cycles can be very long. Also, if a cycle has length congruent to zero mod four, its length cannot be zero since, by definition, all of the cycles have non-zero length.

**b.** $\ell(P_1) \equiv 3 mod 4$
$\ell(P_2P_3) \equiv 3, 0 mod 4$
$\ell(P_2) \equiv 3, 0 mod 4$

But, one of the cycles has to be the single loop $x_7$, which gives a cycle of length one. Thus, $7 \in P_2P_3$, and either $\ell(P_2) = 1$ or $\ell(P_3) = 1$. Since $\ell(P_2) \equiv 3, 0 mod 4$, by necessity, $\ell(P_3) = 1 \Rightarrow \ell(P_2) = 3$

Therefore, there is only one possibility:
i. $\ell(P_1) = 3$
$\ell(P_2) = 3$
$\ell(P_3) = 1$
$\ell(Q) = 1$

**Case 2:** $8 \in P_2$
$\ell(P_1P_2P_3) \equiv 2, 3 mod 4$
$\ell(P_1P_2) \equiv 2, 3 mod 4$
$\ell(P_2P_3) \equiv 2, 3 mod 4$
$\ell(P_2) \equiv 2, 3 mod 4$

**a.** $\ell(P_2) \equiv 2 mod 4$
$\Rightarrow$ $\ell(P_1P_3) \equiv 0, 1 mod 4$
$\ell(P_1) \equiv 0, 1 mod 4$
$\ell(P_3) \equiv 0, 1 mod 4$

$\ell(P_1) = 1 \Rightarrow \ell(P_3) \equiv 0 mod 4$

$\ell(P_3) = 1 \Rightarrow \ell(P_1) \equiv 0 mod 4$

Therefore, there are two possibilities:
i. $\ell(P_1) = 1$ ii. $\ell(P_1) = 4$
$\ell(P_2) = 2$ $\ell(P_2) = 2$
$\ell(P_3) = 4$ $\ell(P_3) = 1$
$\ell(Q) = 1$ $\ell(Q) = 1$

**b.** $\ell(P_2) \equiv 3 mod 4$
$\Rightarrow$ $\ell(P_1P_3) \equiv 3, 0 mod 4$
$\ell(P_1) \equiv 3, 0 mod 4$
$\ell(P_3) \equiv 3, 0 mod 4$

But, in this instance, there is no cycle whose length is equal to 1. Thus, $7 \notin P_1P_2P_3$. This is a contradiction, hence there are no possibilities.

**Case 3:** $8 \in P_3$

$\ell(P_1P_2P_3) \equiv 2, 3 mod 4$
$\ell(P_2P_3) \equiv 2, 3 mod 4$
$\ell(P_3) \equiv 2, 3 mod 4$

a. $\ell(P_3) \equiv 2 mod 4$
$\Rightarrow \quad \ell(P_1P_2) \equiv 0, 1 mod 4$
$\ell(P_2) \equiv 0, 1 mod 4$

$\ell(P_1) = 1 \Rightarrow \ell(P_2) \equiv 0 mod 4$

$\ell(P_2) = 1 \Rightarrow \ell(P_1) \equiv 3, 0 mod 4$

Therefore, there are three possibilities:

| i. | ii. | iii. |
|---|---|---|
| $\ell(P_1) = 1$ | $\ell(P_1) = 3$ | $\ell(P_1) = 4$ |
| $\ell(P_2) = 4$ | $\ell(P_2) = 1$ | $\ell(P_2) = 1$ |
| $\ell(P_3) = 2$ | $\ell(P_3) = 2$ | $\ell(P_3) = 2$ |
| $\ell(Q) = 1$ | $\ell(Q) = 2$ | $\ell(Q) = 1$ |

b. $\ell(P_3) \equiv 3 mod 4$
$\ell(P_1P_2) \equiv 3, 0 mod 4$
$\ell(P_2) \equiv 3, 0 mod 4$

But, $7 \in P_1P_2$, thus, either $\ell(P_1) = 1$ or $\ell(P_2) = 1$. Since $\ell(P_2) \equiv 3, 0 mod 4$, by necessity, $\ell(P_1) = 1 \Rightarrow \ell(P_2) = 3$

Therefore, there is only one possibility:
i. $\ell(P_1) = 1$
$\ell(P_2) = 3$
$\ell(P_3) = 3$
$\ell(Q) = 1$

Notice the similarity between **case 1** and **case 3**. These two cases give the same results, with only $P_1$ and $P_3$ interchanged. In fact, this is a general result which will be used later on.

At this point, the actual paths must be determined. As will be seen, the number of paths for each possibility remains small.

**Case 1.a.i** $\quad \ell(P_1) = 2$
$\ell(P_2) = 1$
$\ell(P_3) = 3$
$\ell(Q) = 2$

Starting with $Q$ and referring to the diagram, it is seen that the only path of length 2 from point 4 to point 6 is the path $\{6, 5\}$. Since $\ell(P_2) = 1$, we must have $P_2 = \{7\}$. Also, we know by assumption that $8 \in P_1$, and $\ell(P_1) = 2$, thus, either $P_1 = \{4, 8\}$ or $P_1 = \{8, 4\}$. This leaves $P_3$, which starts and ends at point 4 equal to $\{3,2,1\}$. Therefore, $P = \{4,8,7,3,2,1,6,5\}$ or $\{8,4,7,3,2,1,6,5\}$.

**Case 1.a.ii** $\ell(P_1) = 2$
$\ell(P_2) = 1$
$\ell(P_3) = 4$
$\ell(Q) = 1$

Once again starting with $Q$, since $\ell(Q) = 1$, the only possibility is that $Q = \{3\}$. Once again, $P_2 = \{7\}$ and $P_1 = \{4,8\}$ or $\{8,4\}$. That leaves $P_3 = \{1,2,5,6\}$ as the only possibility.

**Case 1.a.iii** $\ell(P_1) = 2$
$\ell(P_2) = 4$
$\ell(P_3) = 1$
$\ell(Q) = 1$

In this situation, the lengths of the cycles are the same as in the previous situation, only in a different order. Therefore, the actual cycles must be the same, but in a different order. Hence, the only possibilities are $P_1 = \{4,8\}$ or $\{8,4\}$, $P_2 = \{1,2,5,6\}$, $P_3 = \{7\}$, and $Q = \{3\}$.

**Case 1.b.i** $\ell(P_1) = 3$
$\ell(P_2) = 3$
$\ell(P_3) = 1$
$\ell(Q) = 1$

As before, with $\ell(Q) = 1$, the tail must be $Q = \{3\}$. The only cycles of length 3 which contain 8 are $\{1,2,8\}$ and $\{8,2,1\}$, so $P_1$ must be one of these. As before, $P_3 = \{7\}$ because it has length 1, leaving $P_2 = \{6,5,4\}$ as the only possibility

In total, when $8 \in P_1$, there are eight permutations which contribute to the total value of $M_{16}$. They are listed below, followed by each permutation specified as a product of disjoint cycles:

| | | |
|---|---|---|
| | $\{4,8,7,3,2,1,6,5\}$ | $(1,4,3,7,6)(2,8,5)$ |
| | $\{8,4,7,3,2,1,6,5\}$ | $(1,8,5,2,4,3,7,6,1)$ |
| | $\{4,8,7,1,2,5,6,3\}$ | $(1,4)(2,8,3,7,6,5)$ |
| (1) | $\{8,4,7,1,2,5,6,3\}$ | $(1,8,3,7,6,5,2,4)$ |
| | $\{1,2,5,6,7,4,8,3\}$ | $(1)(2)(3,5,7,8)(4,6)$ |
| | $\{1,2,5,6,7,8,4,3\}$ | $(1)(2)(3,5,7,4,6,8)$ |
| | $\{1,2,8,6,5,4,7,3\}$ | $(1)(2)(3,8)(4,6)(5)(7)$ |
| | $\{8,2,1,6,5,4,7,3\}$ | $(1,8,3)(2)(4,6)(5)(7)$ |

A quick check verifies that all these permutations give the same signed contribution to the value of $M_{16}$. In other words, $\xi(P)$ has the same value for all possible pseudo-Eulerian paths given above. Continuing in this manner with **Case 2** and **Case 3**, the other permutations are:

$$\begin{array}{ll}
\{7,4,8,1,2,5,6,3\} & (1,7,6,5,2,4)(3,8) \\
\{7,8,4,1,2,5,6,3\} & (1,7,6,5,2,8,3,4) \\
\{1,2,5,6,4,8,7,3\} & (1)(2)(3,5,4,6,8)(7) \\
\{1,2,5,6,8,4,7,3\} & (1)(2)(3,5,8)(4,6)(7)
\end{array}$$

$$\begin{array}{ll}
\{7,1,2,5,6,4,8,3\} & (1,7,8,3,2)(4,5,6) \\
\{7,1,2,5,6,8,4,3\} & (1,7,4,5,6,8,3,2) \\
\{1,2,3,7,4,8,6,5\} & (1)(2)(3)(4,7,6,8,5) \\
\{1,2,3,7,8,4,6,5\} & (1)(2)(3)(4,7,6)(5,8) \\
\{1,2,5,6,7,4,8,3\} & (1)(2)(3,5,7,8)(4,6) \\
\{1,2,5,6,7,8,4,3\} & (1)(2)(3,5,7,4,6,8) \\
\{7,4,5,6,1,2,8,3\} & (1,7,8,3,5)(2,4,6) \\
\{7,4,5,6,8,2,1,3\} & (1,7)(2,4,6)(3,5,8)
\end{array} \tag{2}$$

Thus, there are a total of 20 permutations arising from Cases 1 - 3, all of which contribute the same sign to $M_{16}$.

**Case 4.** The next group of permutations which need to be counted are those for which $\{8\} \in Q$. To count these permutations, another observation about $P_1P_2P_3$ has to be made at this point. At least one of $P_1P_2P_3$ has an odd length, since $\{7\} \in P_1P_2P_3$. Consider what happens when there are two terms of odd length.

**Lemma 4.** *Let $P = P_1P_2P_3Q$ and assume $8 \in Q$. Then:*

*a. All $P_i$'s go from point 1 to point 1,*

*b. If there are two cycles of odd length, denote by $P'$ the pseudo-Eulerian path obtained from $P$ by exchanging the two cycles of odd length. Then $\xi(P) \neq \xi(P')$.*

*Proof.* One needs an odd number of transpositions to exchange the two cycles. Hence, $\xi(P) \neq \xi(P')$ follows from the form of $S_m$.

In other words, exchanging the position of two paths of odd length, if both on the "top" portion of the diagram, will give two permutations with opposite contributions. Thus, all such paths cancel each other out, leaving only those paths which have positive length cycles, and the one cycle of length 1. Therefore, the two even length paths are of the form:

$\{1,6\}$
$\{1,2,5,6\}$
$\{4,3\}$
$\{4,5,2,3\}$

If $\{1,2,5,6\} \in P_1P_2P_3$, then we must have $\{4,3\} \in P_1P_2P_3$. If the $\{1,2,5,6\}$ cycle is replaced with $\{1,6\}$, and the $\{4,3\}$ with $\{4,5,2,3\}$, the result is two permutations which cancel each other out. Therefore, the only permutation of this type which does not cancel is where $\{1,6\} \in P_1P_2P_3$ and $\{4,3\} \in P_1P_2P_3$. This leaves six permutations:

$$\begin{array}{ll} \{7,1,6,4,3,8,2,5\} & (1,7,2)(3,6,8,5)(4) \\ \{7,4,3,1,6,8,2,5\} & (1,7,2,4)(3)(5,6,8) \\ \{1,6,7,4,3,8,2,5\} & (1)(2,6,8,5,3,7)(4) \\ \{4,3,7,1,6,8,2,5\} & (1,4)(2,3,7)(5,6,8) \\ \{1,6,4,3,7,8,2,5\} & (1)(2,6,8,5,7)(3,4) \\ \{4,3,1,6,7,8,2,5\} & (1,4,6,8,5,7,2,3) \end{array} \tag{3}$$

**Case 5.** The last set of permutations which have to be considered are those for which

$$\eta(\eta(P_1P_2P_3Q)) \neq P_1P_2P_3Q.$$

As seen previously, the only time this arises is in Step 6 above, which gives the following:

$\eta(P_1P_2P_3Q) = P_1P_3^{-1}P_2^{-1}Q$
$\eta(P_1P_3^{-1}P_2^{-1}Q) = P_3P_1^{-1}P_2^{-1}Q$
$\Rightarrow 8 \in P_3$

$\Rightarrow \quad \ell(P_1P_2P_3) \equiv 2,3 mod 4$
$\ell(P_2P_3) \equiv 0,1 mod 4$
$\ell(P_1P_3) \equiv 0,1 mod 4$
$\ell(P_3) \equiv 2,3 mod 4$

a. $\ell(P_3) \equiv 2 mod 4$
$\Rightarrow \quad \ell(P_1P_2) \equiv 0,1 mod 4$
$\ell(P_2) \equiv 2,3 mod 4$
$\ell(P_3) \equiv 2,3 mod 4$
Then $\{7\} \notin P_1P_2P_3$, so this possibility does not occur

b. $\ell(P_3) \equiv 3 mod 4$
$\Rightarrow \quad \ell(P_1P_2) \equiv 3,0 mod 4$
$\ell(P_2) \equiv 1,2 mod 4$
$\ell(P_1) \equiv 1,2 mod 4$

$\ell(P_1) = 1 \Rightarrow \ell(P_2) \equiv 2 mod 4$
$\ell(P_2) = 1 \Rightarrow \ell(P_1) \equiv 2 mod 4$

Thus, the lengths are:

$$\begin{array}{ll} \ell(P_1) = 1 & \ell(P_1) = 2 \\ \ell(P_2) = 2 & \ell(P_2) = 1 \\ \ell(P_3) = 3 & \ell(P_3) = 3 \\ \ell(Q) = 2 & \ell(Q) = 2 \end{array}$$

leaving the permutations

$$(4)\qquad \begin{array}{ll} \{7,4,3,1,2,8,6,5\} & (1,7,6,8,5,2,4)(3) \\ \{7,4,3,8,2,1,6,5\} & (1,7,6)(2,4,8,5)(3) \\ \{4,3,7,1,2,8,6,5\} & (1,4)(2,3,7,6,8,5) \\ \{4,3,7,8,2,1,6,5\} & (1,4,8,5,2,3,7,6) \end{array}$$

When the permutations (1) - (4) are taken together, it is easily confirmed that all permutations contribute the same sign to the value of $M_{16}$. Thus, the value of $M_{16}$ is $\pm 30$. Whether or not the value is positive or negative is unimportant. What is important is that the final result is non-zero. This establishes the theorem:

**Theorem 2.** *For $F$ a field whose charasteristic is not 2,3, or 5, then $S_9$ is not an identity for $H_3(F,s)$.*

## 4. The Case $n = 4$

For the next case, when $n = 4$, there are twelve basis elements which are used to show that $S_{4n-3}$ is not an identity. These are a generalization of the eight elements used in the previous case, for $n = 3$. These elements are:

$x_1 = e_{12} + e_{65}$
$x_2 = e_{23} + e_{76}$
$x_3 = e_{34} + e_{87}$
$x_4 = e_{41} + e_{58}$
$x_5 = e_{14} + e_{85}$
$x_6 = e_{43} + e_{78}$
$x_7 = e_{32} + e_{67}$
$x_8 = e_{21} + e_{56}$
$x_9 = e_{13} + e_{75}$
$x_{10} = e_{31} + e_{57}$
$x_{11} = e_{11} + e_{55}$
$x_{12} = e_{18} - e_{45}$

Figure 3a

To determine the value of $M_{18}$ in this case, the same methods are employed as for the previous case.

**Lemma 5.** *Suppose $P = \{i_1, \ldots, i_{12}\}$ is a pseudo-Eulerian path. Let $X := \{j \in N \mid$ the endpoint of $x_j$ is point $1$ or point $5\}$. Then $\mid X \mid = 4$.*

*Proof* Same as Lemma 2.

As in the previous case, define

$$\begin{aligned} P_1 &:= \{i_1, \ldots, i_{j_1}\} \\ P_2 &:= \{i_{j_1+1}, \ldots, i_{j_2}\} \\ P_3 &:= \{i_{j_2+1}, \ldots, i_{j_3}\} \\ P_4 &:= \{i_{j_3+1}, \ldots, i_{j_4}\} \\ Q &:= \{i_{j_4+1}, \ldots, i_{12}\} \end{aligned}$$

The observations which were made in the previous case can also be made here, with minor variations:

1. *There is a high degree of symmetry between the "top" and "bottom" halves of the diagram. If the edges labelled $x_{12}$ are ignored, there is a correspondance between points 1 and 5, between points 2 and 6, between points 3 and 7, and between points 4 and 8. For this reason, the points 5, 6, 7, and 8 will often be called points $1'$, $2'$, $3'$, and $4'$, respectively.*

2. *Each edge which runs from point $A$ to point $B$ on the top half of the diagram has its corresponding edge going from point $B'$ to point $A'$ on the bottom.*

3. *A path $P$ from point $A$ to point $A'$ gives rise to another path $P^{-1}$, which also runs from point $A$ to point $A'$, but crosses the opposite leg of $x_{12}$. This path is arrived at by simply taking the elements in the path $P$ and considering them in reverse. For example, $\{1,2,3,12\}$ is a path from point 1 to point 1' which crosses the negative leg of $x_{12}$. The reverse path $\{12,3,2,1\}$ is also a path from point 1 to point 1', but which crosses the positive leg of $x_{12}$.*

However, there are a couple of other facts which have to be taken into account. These results concern the number of odd-length cycles which are present in the path. It was already shown in the previous situation, that if there are two odd cycles, both on the top half of the diagram, then these two cancel out. The obvious question to ask at this time is what happens if two odd cycles are on opposite halves of the diagram. Call the edge which connects the top and bottom halves of the diagram, in this case the cycle containing $x_{12}$, the "crossing cycle".

**Lemma 6.** *After cancellation, there can only be at most one cycle of length congruent to $3 mod 4$, not including the crossing cycle. Furthermore, there is exactly one cycle of length congruent to $1 mod 4$.*

*Proof.* Let $P = P_1P_2P_3P_4Q$ be a pseudo-Eulerian path. Assume there are two non-crossing cycles of length congruent to $3 mod 4$, say $P_a$ and $P_b$. It was already shown that if $P_a$ and $P_b$ are on the same half, then reversing the position of $P_a$ and $P_b$ will give two paths of opposite contribution, thereby cancelling each other out. If $P_a$ and $P_b$ are on opposite halves, then observation 2 above shows that $P_a$ and $P_b$ can be exchanged by converting to $P_a^{-1}$ and $P_b^{-1}$ first. To convert $P_a$ to $P_a^{-1}$ requires an odd number of transpositions, as does converting $P_b$ to $P_b^{-1}$. Thus, an even number of transpositions are needed to convert both $P_a$ and $P_b$ to their inverses, and an odd number to exchange their positions. Thus, two permutations of opposite contribution are created, which cancel each other out. Similarily, if two cycles are of length congruent to $1 mod 4$, two permutations are created which cancel each other out.

Furthermore, exchanging the positions of two cycles of congruent lengths doesn't change the lengths of the groups of cycles under consideration. Thus, both of the permutations which cancel each other out will have the exact same pattern.

The case of $n = 4$ will be determined in the same manner as that of $n = 3$. Let $P = P_1P_2P_3P_4Q$:

**Step 1.** If $12 \in P_1$ and $\ell(P_1) \equiv 0, 1 mod 4$, define

$$\eta(P) := P_1^{-1}P_2P_3P_4Q.$$

Then $\xi(P) \neq \xi(\eta(P))$ and we pair off $P$ and $\eta(P)$. For later use, we also calculate $\eta(\eta(P)) = \eta(P_1^{-1}P_2P_3P_4Q) = P$.

**Step 2.** If $12 \in P_2$ and $\ell(P_2) \equiv 0, 1 mod 4$, define

$$\eta(P) := P_1P_2^{-1}P_3P_4Q.$$

Then $\xi(P) \neq \xi(\eta(P))$ and we pair off $P$ and $\eta(P)$. For later use, we also calculate $\eta(\eta(P)) = \eta(P_1P_2^{-1}P_3P_4Q) = P$.

**Step 3.** If $12 \in P_3$ and $\ell(P_3) \equiv 0, 1 mod 4$, define

$$\eta(P) := P_1P_2P_3^{-1}P_4Q.$$

Then $\xi(P) \neq \xi(\eta(P))$ and we pair off $P$ and $\eta(P)$. For later use, we also calculate $\eta(\eta(P)) = \eta(P_1P_2P_3^{-1}P_4Q) = P$.

**Step 4.** If $12 \in P_4$ and $\ell(P_4) \equiv 0, 1 mod 4$, define

$$\eta(P) := P_1P_2P_3^{-1}P_4^{-1}Q.$$

Then $\xi(P) \neq \xi(\eta(P))$ and we pair off $P$ and $\eta(P)$. For later use, we also calculate $\eta(\eta(P)) = \eta(P_1P_2P_3^{-1}P_4Q) = P$.

**Step 5.** If $12 \in P_1P_2P_3P_4$ and $\ell(P_1P_2P_3P_4) \equiv 0, 1 mod 4$, then

$$\eta(P) = P_4^{-1}P_3^{-1}P_2^{-1}P_1^{-1}Q$$

and $\eta(\eta(P)) = P$. Now $\xi(P) \neq \xi(\eta(P))$ and we pair off $P$ and $\eta(P)$.

**Step 6.** If $12 \in P_1P_2P_3$ and $\ell(P_1P_2P_3) \equiv 0, 1 mod 4$, then

$$\eta(P) = P_3^{-1}P_2^{-1}P_1^{-1}P_4Q$$

and $\eta(\eta(P)) = P$. Now $\xi(P) \neq \xi(\eta(P))$ and we pair off $P$ and $\eta(P)$.

**Step 7.** If $12 \in P_1P_2$ and $\ell(P_1P_2) \equiv 0, 1 mod 4$, then

$$\eta(P) = P_2^{-1}P_1^{-1}P_3P_4Q$$

and $\eta(\eta(P)) = P$. Now $\xi(P) \neq \xi(\eta(P))$ and we pair off $P$ and $\eta(P)$.

**Step 8.** If $12 \in P_2P_3P_4$ and $\ell(P_2P_3P_4) \equiv 0, 1 mod 4$, then

$$\eta(P) = P_1P_4^{-1}P_3^{-1}P_2^{-1}Q.$$

Now $\xi(P) \neq \xi(\eta(P))$ and $P$ and $\eta(P)$ pair off when $\eta(\eta(P)) = P$.

$$\eta(\eta(P)) = \begin{cases} P_3P_4P_1^{-1}P_2^{-1}Q & \text{if } \ell(P_1P_3P_4) \equiv 0, 1 mod 4 \\ P_4P_1^{-1}P_3^{-1}P_2^{-1}Q & \text{if } \ell(P_1P_3) \equiv 0, 1 mod 4 \\ P & \text{otherwise} \end{cases}$$

**Step 9.** If $12 \in P_2P_3$ and $\ell(P_2P_3) \equiv 0, 1 mod 4$, then

$$\eta(P) = P_1P_3^{-1}P_2^{-1}P_4Q.$$

Now $\xi(P) \neq \xi(\eta(P))$ and $P$ and $\eta(P)$ pair off when $\eta(\eta(P)) = P$.

$$\eta(\eta(P)) = \begin{cases} P_3P_1^{-1}P_2^{-1}P_4Q & \text{if } \ell(P_1P_3) \equiv 0, 1 mod 4 \\ P & \text{otherwise} \end{cases}$$

**Step 10.** If $12 \in P_3P_4$ and $\ell(P_3P_4) \equiv 0, 1 mod 4$, then

$$\eta(P) = P_1P_2P_4^{-1}P_3^{-1}Q.$$

Now $\xi(P) \neq \xi(\eta(P))$ and $P$ and $\eta(P)$ pair off when $\eta(\eta(P)) = P$.

$$\eta(\eta(P)) = \begin{cases} P_4P_2^{-1}P_1^{-1}P_3^{-1}Q & \text{if } \ell(P_1P_2P_4) \equiv 0, 1 mod 4 \\ P_1P_4P_2^{-1}P_3^{-1}Q & \text{if } \ell(P_2P_4) \equiv 0, 1 mod 4 \\ P & \text{otherwise} \end{cases}$$

Thus, to determine the unpaired paths, there are six cases to consider here:

**Case 1.** $12 \in P_1$
$\ell(P_1) \equiv 2, 3 mod 4$
$\ell(P_1P_2P_3P_4) \equiv 2, 3 mod 4$
$\ell(P_1P_2P_3) \equiv 2, 3 mod 4$
$\ell(P_1P_2) \equiv 2, 3 mod 4$

**Case 2.** $12 \in P_2$
$\ell(P_2) \equiv 2,3mod4$
$\ell(P_1P_2P_3P_4) \equiv 2,3mod4$
$\ell(P_1P_2P_3) \equiv 2,3mod4$
$\ell(P_1P_2) \equiv 2,3mod4$
$\ell(P_2P_3P_4) \equiv 2,3mod4$
$\ell(P_2P_3) \equiv 2,3mod4$

**Case 3.** $12 \in P_3$
$\ell(P_3) \equiv 2,3mod4$
$\ell(P_1P_2P_3P_4) \equiv 2,3mod4$
$\ell(P_1P_2P_3) \equiv 2,3mod4$
$\ell(P_2P_3P_4) \equiv 2,3mod4$
$\ell(P_2P_3) \equiv 2,3mod4$
$\ell(P_3P_4) \equiv 2,3mod4$

**Case 4.** $12 \in P_4$
$\ell(P_4) \equiv 2,3mod4$
$\ell(P_1P_2P_3P_4) \equiv 2,3mod4$
$\ell(P_2P_3P_4) \equiv 2,3mod4$
$\ell(P_3P_4) \equiv 2,3mod4$

**Case 5.** $12 \in Q$

**Case 6.** $12 \in P_1P_2P_3P_4$
$\eta(\eta(P)) \neq P$

**Case 1:** $12 \in P_1$
$\Rightarrow$ $\ell(P_1P_2P_3P_4) \equiv 2,3mod4$
$\ell(P_1P_2P_3) \equiv 2,3mod4$
$\ell(P_1P_2) \equiv 2,3mod4$
$\ell(P_1) \equiv 2,3mod4$

**a.** $\ell(P_1) \equiv 2mod4$
$\Rightarrow$ $\ell(P_2P_3P_4) \equiv 0,1mod4$
$\ell(P_2P_3) \equiv 0,1mod4$
$\ell(P_2) \equiv 0,1mod4$

$\ell(P_2) = 1$ $\Rightarrow \ell(P_3) \equiv 3,0mod4$
$\ell(P_3) \equiv 3mod4 \Rightarrow \ell(P_4) \equiv 0mod4$
$\ell(P_3) \equiv 0mod4 \Rightarrow \ell(P_4) \equiv 3,0mod4$

$\ell(P_3) = 1$ $\Rightarrow \ell(P_2) \equiv 0mod4$
$\Rightarrow \ell(P_4) \equiv 3,0mod4$

$\ell(P_4) = 1$ $\Rightarrow \ell(P_2) \equiv 0mod4$
$\Rightarrow \ell(P_3) \equiv 0mod4$

In the instances above, the fact that only one of $P_2$, $P_3$, and $P_4$ can have length congruent to 1 was used.

From the information outlined above, there are six possibilities:

$$\begin{array}{llllll}
\textbf{i.} & \ell(P_1)=2 & \textbf{ii.} & \ell(P_1)=2 & \textbf{iii.} & \ell(P_1)=2 \\
 & \ell(P_2)=1 & & \ell(P_2)=1 & & \ell(P_2)=1 \\
 & \ell(P_3)=3 & & \ell(P_3)=4 & & \ell(P_3)=4 \\
 & \ell(P_4)=4 & & \ell(P_4)=3 & & \ell(P_4)=4 \\
 & \ell(Q)=2 & & \ell(Q)=2 & & \ell(Q)=1
\end{array}$$

$$\begin{array}{llllll}
\textbf{iv.} & \ell(P_1)=2 & \textbf{v.} & \ell(P_1)=2 & \textbf{vi.} & \ell(P_1)=2 \\
 & \ell(P_2)=4 & & \ell(P_2)=4 & & \ell(P_2)=4 \\
 & \ell(P_3)=1 & & \ell(P_3)=1 & & \ell(P_3)=4 \\
 & \ell(P_4)=3 & & \ell(P_4)=4 & & \ell(P_4)=1 \\
 & \ell(Q)=2 & & \ell(Q)=1 & & \ell(Q)=1
\end{array}$$

**b.** $\ell(P_1) \equiv 3 mod 4$

$$\begin{array}{ll}
\Rightarrow & \ell(P_2P_3P_4) \equiv 3, 0 mod 4 \\
 & \ell(P_2P_3) \equiv 3, 0 mod 4 \\
 & \ell(P_2) \equiv 3, 0 mod 4
\end{array}$$

$$\begin{array}{ll}
\ell(P_3)=1 & \Rightarrow \ell(P_2) \equiv 3 mod 4 \\
 & \Rightarrow \ell(P_4) \equiv 0 mod 4 \\
\ell(P_4)=1 & \Rightarrow \ell(P_2P_3) \equiv 3 mod 4
\end{array}$$

Thus, there are three possibilities:

$$\begin{array}{llllll}
\textbf{i.} & \ell(P_1)=3 & \textbf{ii.} & \ell(P_1)=3 & \textbf{iii.} & \ell(P_1)=3 \\
 & \ell(P_2)=3 & & \ell(P_2)=3 & & \ell(P_2)=4 \\
 & \ell(P_3)=1 & & \ell(P_3)=4 & & \ell(P_3)=3 \\
 & \ell(P_4)=4 & & \ell(P_4)=1 & & \ell(P_4)=1 \\
 & \ell(Q)=1 & & \ell(Q)=1 & & \ell(Q)=1
\end{array}$$

**Case 2:** $12 \in P_2$

$$\begin{array}{ll}
\Rightarrow & \ell(P_1P_2P_3P_4) \equiv 2, 3 mod 4 \\
 & \ell(P_1P_2P_3) \equiv 2, 3 mod 4 \\
 & \ell(P_1P_2) \equiv 2, 3 mod 4 \\
 & \ell(P_2P_3P_4) \equiv 2, 3 mod 4 \\
 & \ell(P_2P_3) \equiv 2, 3 mod 4 \\
 & \ell(P_2) \equiv 2, 3 mod 4
\end{array}$$

**a.** $\ell(P_2) \equiv 2 mod 4$

$$\begin{array}{ll}
\Rightarrow & \ell(P_1P_3P_4) \equiv 0, 1 mod 4 \\
 & \ell(P_1P_3) \equiv 0, 1 mod 4 \\
 & \ell(P_1) \equiv 0, 1 mod 4 \\
 & \ell(P_3P_4) \equiv 0, 1 mod 4 \\
 & \ell(P_3) \equiv 0, 1 mod 4
\end{array}$$

$$\begin{aligned} \ell(P_1) = 1 \quad &\Rightarrow \ell(P_3) \equiv 0 mod 4 \\ &\Rightarrow \ell(P_4) \equiv 0, 1 mod 4 \end{aligned}$$

But $\ell(P_1P_3) \equiv 1 mod 4$ and $\ell(P_1P_3P_4) \equiv 1, 2 mod 4$
$\Rightarrow \ell(P_4) \equiv 0 mod 4$

$$\begin{aligned} \ell(P_3) = 1 \quad &\Rightarrow \ell(P_1) \equiv 0 mod 4 \\ &\Rightarrow \ell(P_4) \equiv 0 mod 4 \end{aligned}$$

$$\begin{aligned} \ell(P_4) = 1 \quad &\Rightarrow \ell(P_1) \equiv 0 mod 4 \\ &\Rightarrow \ell(P_3) \equiv 0 mod 4 \end{aligned}$$

This leaves three possibilities:

| i. | ii. | iii. |
|---|---|---|
| $\ell(P_1) = 1$ | $\ell(P_1) = 4$ | $\ell(P_1) = 4$ |
| $\ell(P_2) = 2$ | $\ell(P_2) = 2$ | $\ell(P_2) = 2$ |
| $\ell(P_3) = 4$ | $\ell(P_3) = 1$ | $\ell(P_3) = 4$ |
| $\ell(P_4) = 4$ | $\ell(P_4) = 4$ | $\ell(P_4) = 1$ |
| $\ell(Q) = 1$ | $\ell(Q) = 1$ | $\ell(Q) = 1$ |

b. $\ell(P_2) \equiv 3 mod 4$

$$\Rightarrow \quad \begin{aligned} &\ell(P_1P_3P_4) \equiv 3, 0 mod 4 \\ &\ell(P_1P_3) \equiv 3, 0 mod 4 \\ &\ell(P_1) \equiv 3, 0 mod 4 \\ &\ell(P_3P_4) \equiv 3, 0 mod 4 \\ &\ell(P_3) \equiv 3, 0 mod 4 \end{aligned}$$

$\Rightarrow \ell(P_4) = 1$ since there must be one cycle of length 1, and $P_4$ is the only possibility.
$\Rightarrow \ell(P_1P_3) \equiv 3 mod 4$

Therefore, there are two possibilities:

| i. | ii. |
|---|---|
| $\ell(P_1) = 3$ | $\ell(P_1) = 4$ |
| $\ell(P_2) = 3$ | $\ell(P_2) = 3$ |
| $\ell(P_3) = 4$ | $\ell(P_3) = 3$ |
| $\ell(P_4) = 1$ | $\ell(P_4) = 1$ |
| $\ell(Q) = 1$ | $\ell(Q) = 1$ |

For the next two cases, namely when $12 \in P_3$, and when $12 \in P_4$, the symmetry between these two cases and the previous two will be used. If the conditions are compared for the four cases, it will be observed that cases 1 and 4 have the same conditions, with only $P_1$ interchanged with $P_4$, and $P_2$ with $P_3$. The same symmetry will be seen with cases 2 and 3. Because of these similarities, the lengths of the cycles will remain the same, except for being interchanged as above. Then, the symmetry of the diagram will give the same paths, but in reverse order. In other words:

**Theorem 3.** *Let* $P = P_1P_2P_3P_4Q$ *be a pseudo-Eulerian path. Define* $P' := P_4^{-1}P_3^{-1}P_2^{-1}P_1^{-1}Q$. *Then:*

*a. If* $12 \in P_1$ *and* $P$ *is an unpaired path in Case 1 above, then* $P'$ *is an unpaired path in Case 4.*

*b. If* $12 \in P_2$ *and* $P$ *is an unpaired path in Case 2 above, then* $P'$ *is an unpaired path in Case 3.*

*Proof.* Observe that $P$ satisfies the conditions for an unpaired path in Case 1 (respectively Case 2) if and only if $P'$ satisfies the conditions for an unpaired path in Case 4 (respectively Case 3).

The actual pseudo-Eulerian paths are listed below, and are determined in the same manner as those when $n = 3$:

**Case 1.a.i** $\ell(P_1) = 2$
$\ell(P_2) = 1$
$\ell(P_3) = 3$
$\ell(P_4) = 4$
$\ell(Q) = 2$

This gives the following pseudo-Eulerian paths:

$$\begin{array}{lll}
 & \{5,12,11,4,3,9,8,7,2,1,10,6\} & (1,5,3,11,10)(2,12,6,9)(4)(7,8) \\
(5) & \{12,5,11,4,3,9,8,7,2,1,10,6\} & (1,12,6,9,2,5,3,11,10)(4)(7,8) \\
 & \{5,12,11,8,7,9,4,3,2,1,10,6\} & (1,5,7,4,8,3,11,10)(2,12,6,9) \\
 & \{12,5,11,8,7,9,4,3,2,1,10,6\} & (1,12,6,9,2,5,7,4,8,3,11,10)
\end{array}$$

**Case 1.a.iii** $\ell(P_1) = 2$
$\ell(P_2) = 1$
$\ell(P_3) = 4$
$\ell(P_4) = 4$
$\ell(Q) = 1$

$$\begin{array}{lll}
 & \{5,12,11,8,7,2,1,4,3,9,10,6\} & (1,5,7)(2,12,6)(3,11,10,9)(4,8) \\
(6) & \{12,5,11,8,7,2,1,4,3,9,10,6\} & (1,12,6,2,5,7)(3,11,10,9)(4,8) \\
 & \{5,12,11,4,3,2,1,8,7,9,10,6\} & (1,5,3,11,10,9,7)(2,12,6)(4)(8) \\
 & \{12,5,11,4,3,2,1,8,7,9,10,6\} & (1,12,6,2,5,3,11,10,9,7)(4)(8)
\end{array}$$

**Case 1.b.i** $\ell(P_1) = 3$
$\ell(P_2) = 3$
$\ell(P_3) = 4$
$\ell(P_4) = 1$
$\ell(Q) = 1$

$$\begin{array}{lll}
 & \{9,3,12,10,6,5,11,8,7,2,1,4\} & (1,9,7,11)(2,3,12,4,10)(5,6)(8) \\
(7) & \{12,3,9,10,6,5,11,8,7,2,1,4\} & (1,12,4,10,2,3,9,7,11)(5,6)(8) \\
 & \{9,3,12,10,2,1,11,8,7,6,5,4\} & (1,9,7,11,5,2,3,12,4,10,6)(8) \\
 & \{12,3,9,10,2,1,11,8,7,6,5,4\} & (1,12,4,10,6)(2,3,9,7,11,5)(8)
\end{array}$$

Notice that all possibilities have cycle lengths of:
a. One, three, and four, with a crossing cycle of length two,
b. One, four, and four, with a crossing cycle of length two, or
c. One, three, and four, with a crossing cycle of length three.
Each of these possibilities gives four permutations, a sample of each which is listed above. All other possibilities are obtained by rearranging the cycles listed above. All the paths contribute the same sign to the final value of $M_{18}$. In total, there are 36 permutations which arise from Case 1 and 20 from Case 2. Including cases 3 and 4, there are a total of 112 permutations.

**Case 5.** The next possibility to consider is when $12 \in Q$. For this instance, it is possible to proceed as in the previous case, ie. eliminating permutations one by one. However, there is another way to obtain the desired result, by first proving a theorem then applying it to the case in question.

**Theorem 4.** *Let* $\{x_1, x_2, \ldots, x_m\}$ *be a set of basis elements of the form* $e_{i,j} + e_{j+n,i+n}$ *where* $i, j \leq n$. *Then* $S_m(x_1, \ldots, x_m) = 0$ *if* $m \geq 2n$.

Proof. Let $\{x_1, x_2, \ldots, x_m\}$ be a set of basis elements of the form $e_{i,j} + e_{j+n,i+n}$ where $i, j \leq n$. The $x_i$ are matrices in $M_{2n}$, however, they can be considered as elements of $M_n \oplus M_n$ embedded in $M_{2n}$ as the top left corner and the bottom right corner. By the Amitsur-Levitsky Theorem, $S_m$ is an identity of $M_n$ for $m \geq 2n$. Therefore, $Sm(x_1, \ldots, x_M) = 0$.

In the case $n = 4$, there are four cycles, one of length one, and the other three of length at least two. Thus, $\ell(P_1P_2P_3P_4) \geq 7$. But the result above shows that if $\ell(P_1P_2P_3P_4) \geq 8$, then the standard polynomial evaluated using these terms will be equal to zero.
Consider what terms could be in $P_1P_2P_3P_4$. It is known that $12 \in Q$, and there are four edges in $P_1P_2P_3P_4$ which start at point 1. Thus, the first term in $Q$ must be $\{12\}$. What can follow 12? There are two possibilities, namely edges $\{5\}$ and $\{3\}$. But edge $\{5\}$ returns to point $1'$, thus it cannot be in $Q$. Similarly, any edge which returns to point $1'$ cannot be in $Q$, since it must have been used in $P_1P_2P_3P_4$. Therefore, there are only three possible tails: $Q = \{12, 3, 2, 7, 6\}$, $Q = \{12, 3, 6\}$, or simply $Q = \{12\}$. In the instance $Q = \{12, 3, 6\}$, or $Q = \{12\}$, then $\ell(P_1P_2P_3P_4) = 9$ and $\ell(P_1P_2P_3P_4) = 11$ respectively. By the theorem above, all these permutations will combine to zero. This leaves the instance where $\ell(P_1P_2P_3P_4) = 7$. Then, the only choices of cycles for $P_1, P_2, P_3, P_4$ are $\{11\}$, $\{1, 8\}$, $\{9, 10\}$, and $\{5, 4\}$. These possibilities give a total of 24 permutations which do not cancel. All of these permutations give the same result when the $\xi$ function is applied, and also give the same sign as those permutations discovered in Cases 1 - 4. Combined, there are now a total of 136 permutations of the same sign.

**Case 6.** The last group of permutations which has to be considered are those pseudo-Eulerian paths $P$ for which $\eta(\eta(P)) \neq P$. The first step once again is to determine what the lengths of the cycles must be in order for this situation to occur. There are three possible ways in which the situation being considered could arise. These possibilities are given in steps 8 - 10 earlier, and repeated below:

$\eta(P_1P_2P_3P_4Q) = P_1P_4^{-1}P_3^{-1}P_2^{-1}Q$
Then $\ell(P_1P_3P_4) \equiv 0, 1 mod 4 \Rightarrow \eta(P_1P_4^{-1}P_3^{-1}P_2^{-1}Q) = P_3P_4P_1^{-1}P_2^{-1}Q$
or $\ell(P_1P_4) \equiv 0, 1 mod 4 \Rightarrow \eta(P_1P_4^{-1}P_3^{-1}P_2^{-1}Q) = P_4P_1^{-1}P_3^{-1}P_2^{-1}Q$
$\eta(P_1P_2P_3P_4Q) = P_1P_3^{-1}P_2^{-1}P_4Q$
Then $\ell(P_1P_3) \equiv 0, 1 mod 4 \Rightarrow \eta(P_1P_3^{-1}P_2^{-1}P_4Q) = P_3P_1^{-1}P_2^{-1}P_4Q$
$\eta(P_1P_2P_3P_4Q) = P_1P_2P_4^{-1}P_3^{-1}Q$
Then $\ell(P_1P_2P_4) \equiv 0, 1 mod 4 \Rightarrow \eta(P_1P_2P_4^{-1}P_3^{-1}Q) = P_4P_2^{-1}P_1 - 1P_3^{-1}Q$
or $\ell(P_2P_4) \equiv 0, 1 mod 4 \Rightarrow \eta(P_1P_2P_4^{-1}P_3^{-1}Q) = P_1P_4P_2^{-1}P_3^{-1}Q$

The analysis of each of these patterns is carried out in the same manner as it was for the other patterns.

a. $\eta(P_1P_2P_3P_4Q) = P_1P_4^{-1}P_3^{-1}P_2^{-1}Q$
$\eta(P_1P_4^{-1}P_3^{-1}P_2^{-1}Q) = P_4P_1^{-1}P_3^{-1}P_2^{-1}Q$ with $\{12\} \in P_4$
$\Rightarrow$ $\ell(P_4) \equiv 2, 3 mod 4$
$\ell(P_1P_2P_3P_4) \equiv 2, 3 mod 4$
$\ell(P_2P_3P_4) \equiv 0, 1 mod 4$
$\ell(P_1P_3P_4) \equiv 2, 3 mod 4$
$\ell(P_1P_4) \equiv 0, 1 mod 4$

i. $\ell(P_4) \equiv 2 mod 4$
$\Rightarrow$ $\ell(P_1P_2P_3) \equiv 0, 1 mod 4$
$\ell(P_2P_3) \equiv 2, 3 mod 4$
$\ell P_1P_3) \equiv 0, 1 mod 4$
$\ell(P_1) \equiv 2, 3 mod 4$

$\Rightarrow$ either $\ell(P_2) = 1$ or $\ell(P_3) = 1$
$\Rightarrow$ $\ell(P_2P_3) \equiv 3 mod 4$
Since only one cycle can have length equivalent to 1
$\Rightarrow$ $\ell(P_1) \equiv 2 mod 4$
$\Rightarrow$ $\ell(P_3) \equiv 2, 3 mod 4$
$\Rightarrow$ $\ell(P_3) \equiv 2 mod 4$
$\ell(P_2) = 1$

Therefore, there is only one possibility:
$\ell(P_1) \equiv 2 mod 4$
$\ell(P_2) = 1$
$\ell(P_3) \equiv 2 mod 4$
$\ell(P_4) \equiv 2 mod 4$

**ii.** $\ell(P_4) \equiv 3 mod 4$
$\Rightarrow \quad \ell(P_1P_2P_3) \equiv 3, 0 mod 4$
$\ell(P_2P_3) \equiv 1, 2 mod 4$
$\ell(P_1P_2) \equiv 3, 0 mod 4$
$\ell(P_1) \equiv 1, 2 mod 4$

$\ell(P_1) = 1$
$\Rightarrow \quad \ell(P_2P_3) \equiv 2 mod 4$
$\ell(P_3) \equiv 2, 3 mod 4$
$\Rightarrow \quad \ell(P_2) \equiv 0, 3 mod 4$
$\Rightarrow \quad \ell(P_3) \equiv 2 mod 4$
$\ell(P_2) \equiv 0 mod 4$

$\ell(P_2) = 1$
$\Rightarrow \quad \ell(P_1) \equiv 2 mod 4$
$\Rightarrow \quad \ell(P_3) \equiv 2 mod 4$
$\Rightarrow \quad \ell(P_1P_2P_3) \equiv 1 mod 4 \qquad (\Rightarrow\Leftarrow)$

$\ell(P_3) = 1$
$\Rightarrow \quad \ell(P_1) \equiv 2 mod 4$
$\Rightarrow \quad \ell(P_2) \equiv 0 mod 4$

This gives two possibilities:

$\ell(P_1) = 1$
$\ell(P_2) = 4$
$\ell(P_3) = 2$
$\ell(P_4) = 3$
$\ell(Q) = 2$

or

$\ell(P_1) = 2$
$\ell(P_2) = 4$
$\ell(P_3) = 1$
$\ell(P_4) = 3$
$\ell(Q) = 2$

Continuing in this manner with the other possibilities:

b. $\eta(P_1P_2P_3P_4Q) = P_1P_4^{-1}P_3^{-1}P_2^{-1}Q$
$\eta(P_1P_4^{-1}P_3^{-1}P_2^{-1}Q) = P_3P_4P_1^{-1}P_2^{-1}Q$ with $\{12\} \in P_4$
$\Rightarrow \quad \ell(P_4) \equiv 2, 3 mod 4$
$\ell(P_1P_2P_3P_4) \equiv 2, 3 mod 4$
$\ell(P_2P_3P_4) \equiv 0, 1 mod 4$
$\ell(P_1P_3P_4) \equiv 0, 1 mod 4$

**i.** $\ell(P_4) \equiv 2 mod 4$
⇒ $\ell(P_1) \equiv 2 mod 4$
$\ell(P_2) \equiv 2 mod 4$
$\ell(P_3) = 1$
$\ell(P_4) \equiv 2 mod 4$

**ii.** $\ell(P_4) \equiv 3 mod 4$
⇒ $\ell(P_1) = 2$
$\ell(P_2) = 1$
$\ell(P_3) = 4$
$\ell(P_4) = 3$
$\ell(Q) = 2$

or

$\ell(P_1) = 1$
$\ell(P_2) = 2$
$\ell(P_3) = 4$
$\ell(P_4) = 3$
$\ell(Q) = 2$

c. $\eta(P_1 P_2 P_3 P_4 Q) = P_1 P_4^{-1} P_3^{-1} P_2^{-1} Q$
$\eta(P_1 P_4^{-1} P_3^{-1} P_2^{-1} Q) = P_3 P_4 P_1^{-1} P_2^{-1} Q$ with $\{12\} \in P_3$
⇒ $\ell(P_3) \equiv 2, 3 mod 4$
$\ell(P_1 P_2 P_3 P_4) \equiv 2, 3 mod 4$
$\ell(P_1 P_2 P_3) \equiv 2, 3 mod 4$
$\ell(P_2 P_3 P_4) \equiv 0, 1 mod 4$
$\ell(P_1 P_3 P_4) \equiv 0, 1 mod 4$

**i.** $\ell(P_3) \equiv 2 mod 4$
⇒ $\ell(P_1) \equiv 2 mod 4$
$\ell(P_2) \equiv 2 mod 4$
$\ell(P_3) \equiv 2 mod 4$
$\ell(P_4) = 1$

**ii.** $\ell(P_3) \equiv 3 mod 4$
⇒ $\ell(P_1) = 1$
$\ell(P_2) = 2$
$\ell(P_3) = 3$
$\ell(P_4) = 4$
$\ell(Q) = 2$

or

$\ell(P_1) = 2$
$\ell(P_2) = 1$
$\ell(P_3) = 3$
$\ell(P_4) = 4$
$\ell(Q) = 2$

d. $\eta(P_1P_2P_3P_4Q) = P_1P_3^{-1}P_2^{-1}P_4Q$

$\Rightarrow$ $\{12\} \in P_3$, and $\eta(P_1P_3^{-1}P_2^{-1}P_4Q) = P_3P_1^{-1}P_2^{-1}P_4Q$

In this instance, all of the possible configurations lead to a contradiction. Thus, there is no possible arrangement of lengths for which this possibility will occur.

e. $\eta(P_1P_2P_3P_4Q) = P_1P_2P_4^{-1}P_3^{-1}Q$

$\eta(P_1P_2P_4^{-1}P_3^{-1}Q) = P_4P_2^{-1}P_1^{-1}P_3Q$ with $\{12\} \in P_4$

$\Rightarrow$ $\ell(P_4) \equiv 2, 3 mod 4$
$\ell(P_1P_2P_3P_4) \equiv 2, 3 mod 4$
$\ell(P_2P_3P_4) \equiv 2, 3 mod 4$
$\ell(P_3P_4) \equiv 0, 1 mod 4$
$\ell(P_1P_2P_4) \equiv 0, 1 mod 4$

**i.** $\ell(P_4) \equiv 2 mod 4$
$\Rightarrow$ $\ell(P_1) = 1$
$\ell(P_2) \equiv 2 mod 4$
$\ell(P_3) \equiv 2 mod 4$
$\ell(P_4) \equiv 2 mod 4$

**ii.** $\ell(P_4) \equiv 3 mod 4$
$\Rightarrow$ $\ell(P_1) = 4$
$\ell(P_2) = 1$
$\ell(P_3) = 2$
$\ell(P_4) = 3$
$\ell(Q) = 2$

or

$\ell(P_1) = 4$
$\ell(P_2) = 2$
$\ell(P_3) = 1$
$\ell(P_4) = 3$
$\ell(Q) = 2$

f. $\eta(P_1P_2P_3P_4Q) = P_1P_2P_4^{-1}P_3^{-1}Q$

$\eta(P_1P_2P_4^{-1}P_3^{-1}Q) = P_1P_4P_2^{-1}P_3Q$ with $\{12\} \in P_4$

$\Rightarrow \quad \ell(P_4) \equiv 2, 3 mod 4$

$\ell(P_1P_2P_3P_4) \equiv 2, 3 mod 4$

$\ell(P_2P_3P_4) \equiv 2, 3 mod 4$

$\ell(P_3P_4) \equiv 0, 1 mod 4$

$\ell(P_1P_2P_4) \equiv 2, 3 mod 4$

$\ell(P_4) \equiv 0, 1 mod 4$

All possibilities once again lead to a contradiction in this configuration, so no new permutations are supplied here.

Once again, all of the permutations in this case are of two types. One type is when all but one of the cycles is of length congruent to two. The other is when the cycles are of lengths one, two, three, and four. The possible permutations for these lengths are given below:

**a.i.** This configuration will give twenty permutations. However, on analysis, all the permutations listed below will cancel with each other, except for the last four. Hence, this length configuration will contribute a total of four permutations.

$$\begin{array}{ll}
\{1,8,11,9,10,5,6,7,2,3,12,4\} & (1)(2,8,7,6,5,10,3,11,12,4,9) \\
\{1,8,11,9,10,12,3,2,7,6,5,4\} & (1)(2,8,2)(3,11,5,10,6,12,4,9,7) \\
\{9,10,11,1,8,5,6,7,2,3,12,4\} & (1,9,2,10,3,11,12,4)(5,8,7,6) \\
\{9,10,11,1,8,12,3,2,7,6,5,4\} & (1,9,7,3,11,5,8,2,10,6,12,4) \\
\{1,2,3,6,7,8,11,9,10,5,12,4\} & (1)(2)(3)(4,6,8,9,10,5,7,11,12) \\
\{1,2,3,6,7,8,11,9,1012,5,4\} & (1)(2)(3)(4,6,8,9,10,12)(5,7,11) \\
\{9,7,2,3,6,10,11,1,85,12,4\} & (1,9,8)(2,7,11,12,4,3)(5,6,10) \\
\{9,7,2,3,6,10,11,1,8,12,5,4\} & (1,9,8)(2,7,115,6,10,12,4,3) \\
\{9,3,6,7,2,10,11,1,8,5,12,4\} & (1,9,8)(2,3,6,10,5)(4,7,11,12) \\
\{9,3,6,7,2,10,11,1,8,12,5,4\} & (1,9,8)(2,3,610,12,4,7,11,5) \\
\{9,10,11,1,2,3,6,7,85,12,4\} & (1,9,8,7,6,3,11,12,4)(2,10,5) \\
\{9,10,11,1,2,3,6,7,8,12,5,4\} & (1,9,8,7,6,311,5,210,12,4) \\
\{1,8,11,9,7,2,3,6,10,5,12,4\} & (1)(2,8,6)(3,11,12,4,9,10,5,7) \\
\{1,8,11,9,7,2,3,6,10,12,5,4\} & (1)(2,8,6)(3,11,5,7)(4,9,10,12) \\
\{1,8,11,9,3,6,7,2,10,5,12,4\} & (1)(2,8)(3,11,12,4,9,10,5)(6)(7) \\
\{1,8,11,9,3,6,7,2,10,12,5,4\} & (1)(2,8)(3,11,5)(4,9,10,12,4)(6)(7) \\
\{1,8,11,9,10,5,12,4,3,2,7,6\} & (1)(2,8,4,9,3,11,7,12,6,5,10) \\
\{1,8,11,9,10,12,5,4,3,2,7,6\} & (1)(2,8,4,9,3,11,7,5,10)(6,12) \\
\{9,10,11,1,8,5,12,4,3,2,7,6\} & (1,9,3,11,7,12,6,5,8,4)(2,10) \\
\{9,10,11,1,8,12,5,4,3,2,7,6\} & (1,9,3,11,7,5,8,4)(2,10)(6,12)
\end{array} \tag{8}$$

**a.ii** This configuration will supply two permutations each time the pattern appears. Thus, there are four permutations which will appear for this configuration:

$$
\begin{array}{ll}
\{11,1,2,7,8,5,4,9,3,12,10,6\} & (1,11,10,12,6,5,8,9,3,2)(4,7) \\
\{11,1,2,7,8,5,4,12,3,9,10,6\} & (1,11,10,9,3,2)(4,7)(5,8,12,6) \\
\{5,4,1,2,7,8,11,9,3,12,10,6\} & (1,5,7,11,10,12,6,8,9,3)(2,4) \\
\{5,4,1,2,7,8,11,12,3,9,10,6\} & (1,5,7,11,10,9,3)(2,4)(6,8,12)
\end{array} \tag{9}
$$

There will also be a contribution of four permutations for b.i and ii, c.i and ii, and cases e.i and ii. Hence, a total of 32 permutations result from these situations. All these permutations contribute the same sign to the result as the previous permutations. Thus, the total number of unpaired permutations, and thus the value of $M_{18}$, is 168. Hence:

**Theorem 5.** *For $F$ a field whose characteristic is not 2, 3, or 7, then $S_{13}$ is not an identity for $H_4(F,s)$.*

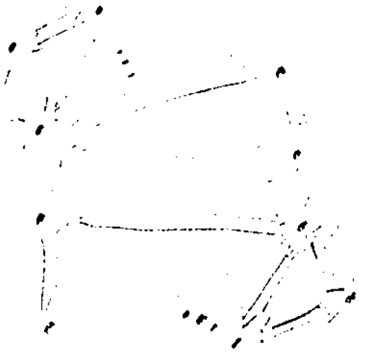
## 5. The General Case.

In the general case, the $4n-4$ substitutions which will be used are:

$$\begin{array}{ll}
x_i = e_{i,i+1} + e_{i+1+n,i+n} & i = 1,\dots,n-1 \\
x_n = e_{n,1} + e_{n+1,2n} & \\
x_{n+1} = e_{1,n} + e_{2n,n+1} & \\
x_i = e_{2n+2-i,2n+1-i} + e_{3n+1-i,3n+2-i} & i = n+2,\dots,2n \\
x_i = e_{1,2n+4-i} + e_{3n+4-i,n+1} & i = 2n+1,\dots,3n-3 \\
x_i = e_{3n+1-i,1} + e_{1+n,4n+1-i} & n = 3n-2,\dots,4n-6 \\
x_{4n-5} = e_{1,1} + e_{n+1,n+1} & \\
x_{4n-4} = e_{1,2n} - e_{n,n+1} &
\end{array}$$

In diagrammatic terms, these substitutions are seen to be a generalization of the case $n=4$ and $n=3$:

Figure 4



The following can be shown in the same way as Lemma 2:

**Lemma 7.** *Suppose* $P = \{i_1,\dots,i_{4n-4}\}$ *is a pseudo-Eulerian path. Let* $X := \{j \in N|$ *the endpoint of* $x_j$ *is point* $1$ *or point* $n+1\}$. *Then* $|X| = n$.

Define:

$$\begin{aligned}
P_1 &:= \{i_1,\dots,i_{j_1}\} \\
P_2 &:= \{i_{j_1+1},\dots,i_{j_2}\} \\
&\dots \\
P_n &:= \{i_{j_{n-1}+1},\dots,i_{j_n}\} \\
Q &:= \{i_{j_n+1},\dots,i_{4n-4}\}
\end{aligned}$$

As will be seen, the same methods as used previously can be applied to this case to determine the permutations which make a contribution to the value $M_{1,2n}$. In the first two steps, the entire list of permutations will be given. For the last case, a reduction is given, but a final result has not been obtained.

The cycle which contains the leg $x_{4n-4}$ is called the crossing cycle, and will be denoted as C. Also, the edge $x_{4n-3}$, which is the cycle of length one, will be denoted as $L$.

**Lemma 8.** After cancellation, there can be at most one cycle of length congruent to $3mod4$, not including the crossing cycle. Furthermore, there is exactly one cycle of length congruent to $1mod4$.

The steps in the general case are as follows:

**Step 1:** If $C \in P_1$ and $\ell(P_1) \equiv 0, 1mod4$, define

$$\eta(P) = P_1^{-1}P_2 \dots P_nQ$$

**Step 2:** If $C \in P_2$ and $\ell(P_2) \equiv 0, 1mod4$, define

$$\eta(P) = P_1P_2^{-1}P_3 \dots P_nQ$$

. . .

**Step n:** If $C \in P_n$ and $\ell(P_n) \equiv 0, 1mod4$, define

$$\eta(P) = P_1P_2 \dots P_{n-1}P_n^{-1}Q$$

**Step n+1:** If $C \in P_1 \dots P_n$ and $\ell(P_1 \dots P_n) \equiv 0, 1mod4$, define

$$\eta(P) = P_n^{-1} \dots P_1^{-1}Q$$

**Step n+2:** If $C \in P_1 \dots P_{n-1}$ and $\ell(P_1 \dots P_{n-1}) \equiv 0, 1mod4$, define

$$\eta(P) = P_{n-1}^{-1} \dots P_1^{-1}P_nQ$$

. . .

**Step 2n-1:** If $C \in P_1P_2$ and $\ell(P_1P_2) \equiv 0, 1mod4$, define

$$\eta(P) = P_2^{-1}P_1^{-1}P_3 \dots P_nQ$$

**Step 2n:** If $C \in P_2 \dots P_n$ and $\ell(P_2 \dots P_n) \equiv 0, 1mod4$, define

$$\eta(P) = P_1P_n^{-1} \dots P_2^{-1}Q$$

**Step 2n+1:** If $C \in P_2 \dots P_{n-1}$ and $\ell(P_2 \dots P_{n-1}) \equiv 0, 1mod4$, define

$$\eta(P) = P_1P_{n-1}^{-1} \dots P_2^{-1}P_nQ$$

. . .

**Step 3n-3:** If $C \in P_2P_3$ and $\ell(P_2P_3) \equiv 0, 1mod4$, define

$$\eta(P) = P_1P_3^{-1}P_2^{-1}P_4 \dots P_nQ$$

. . .

**Step $\frac{n(n+1)}{2}$:** If $C \in P_n - 1P_n$ and $\ell(P_n - 1P_n) \equiv 0, 1mod4$, define

$$\eta(P) = P_1P_2 \dots P_{n-2}P_n^{-1}P_{n-1}^{-1}Q$$

There are $n + 2$ cases to consider here. The first n cases are summarized as:

$\eta(P_1 P_2 \ldots P_n Q) = P_1 P_2 \ldots P_n Q$

$\Rightarrow \quad C \in P_a \qquad 1 \leq a \leq n$

$\ell(P_1 \ldots P_n) \equiv 2, 3 mod 4$

$\ell(P_1 \ldots P_{n-1}) \equiv 2, 3 mod 4$

$\ldots$

$\ell(P_1 \ldots P_a) \equiv 2, 3 mod 4$

$\ldots$

$\ell(P_a \ldots P_n) \equiv 2, 3 mod 4$

$\ldots$

$\ell(P_a) \equiv 2, 3 mod 4$

**a.** $\ell(P_a) \equiv 2 mod 4$

$L = P_b$ some $b \neq a$

$b < a \qquad \Rightarrow \ell(P_{b+1} \ldots P_a) \equiv 2 mod 4$

$\Rightarrow \ell(P_b \ldots P_a) \equiv 3 mod 4$

Suppose there exists $P_k, k \neq a$ with $\ell(P_k) \equiv 2 mod 4$

if $k > a$, then $\ell(P_1 \ldots P_{k-1}) \equiv 2, 3 mod 4$

$\Rightarrow \ell(P_1 \ldots P_k) \equiv 0, 1 mod 4 \qquad (\Rightarrow\Leftarrow)$

Likewise,if $k < a$, then $\ell(P_{k+1} \ldots P_n) \equiv 2, 3 mod 4$

$\Rightarrow \ell(P_k \ldots P_n) \equiv 0, 1 mod 4 \qquad (\Rightarrow\Leftarrow)$

It has already been established that there is only one term of length congruent to 1, and only one term other than C of length congruent to 3. Thus, all terms must be of length congruent to 0 mod 4 except for $P_a$, $P_b$, and possibly one other term of length congruent to 3 mod 4.

Suppose there exists $P_k$ with $\ell(P_k) \equiv 3 mod 4$ and $k \neq a$

if $k > a$, then $\ell(P_a \ldots P_k) \equiv 1 mod 4$

if $a > k > b$, then $\ell(P_k \ldots P_a) \equiv 1 mod 4$

Thus, it is necessary that $k < b$.

$b > a \qquad \Rightarrow \ell(P_a \ldots P_{b-1}) \equiv 2 mod 4$

$\Rightarrow \ell(P_a \ldots P_b) \equiv 3 mod 4$

As above, there are at most three cycles of length not congruent to 0 mod 4. Namely, they are $P_a$, $P_b$, and a cycle of length congruent to 3 mod 4. Once again, call this term $P_k$.

if $k < a$, then $\ell(P_k \ldots P_a) \equiv 1 mod 4$

if $a < k < b$, then $\ell(P_a \ldots P_k) \equiv 1 mod 4$

Thus, it is necessary that $k > b$.

Now consider the actual lengths of the cycles. There is one cycle of length one, one cycle of length congruent to 2 mod 4, and at least $n-3$ cycles of length congruent to 0 mod 4. The final cycle is of length congruent to either 3 or 0 mod 4. Each cycle of length congruent to 0 mod 4 is actually at least of total length 4. Therefore, the length of all the cycles together is at least $4n-6$. This leaves the length of $Q$ at most 2. Thus, all cycles are of length 4 or smaller.

**b.** $\ell(P_a) \equiv 3 mod 4$
$L = P_b$ some $b \neq a$

As above, the only terms which are not congruent to 0 mod 4 are $P_a$, $P_b$, and $P_k$.

if $b < a$ then $\ell(P_b \ldots P_a) \equiv 0 mod 4$ unless $b < k < a$,
in which case $\ell(P_b \ldots P_a) \equiv 3 mod 4$
Likewise, if $a < b$, then $a < k < b$

In this instance, there is one cycle of length 1, two of length congruent to 3 mod 4, and $n-3$ of length congruent to 0 mod 4. Thus, the cycles have a total length of at least $4n-5$. Therefore, each cycle is at most of length 4, and $\ell(Q) = 1$.

Now consider the possible cycles which can fit these patterns. First consider the pattern which has one cycle of length one, one of length three, $n-3$ cycles of length four, and a crossing cycle of length two. This leaves a length of two for $Q$, thus the value of $Q$ is $\{4n-6, n+2\}$, since this is the only length two path from point $n+1$ to point $2n$. Now consider the crossing cycle $P_a$. The only two possibilities for $P_a$ are $\{n+1, 4n-4\}$ and its inverse $\{4n-4, n+1\}$. It can be assumed that the remainder of the cycles are all on the top half of the diagram. If a cycle $P_i$ lies on the bottom half of the diagram, then $P_i^{-1}$ is a cycle on the top half of the diagram. Thus, all the possible cycles can be determined on the top half of the diagram, and afterwards inverted as necessary.

The three edges $\{n+1\}$, $\{n+2\}$ and $\{4n-6\}$ have been used, leaving $\{3n-3\}$, $\{n-1\}$, and $\{n\}$. There are two possibilities. First, two of these three edges can be used in a length four cycle, namely the cycle $\{3n-4, n-2, n-1, n\}$. This length four cycle is in the form of a large loop, and is shown below:

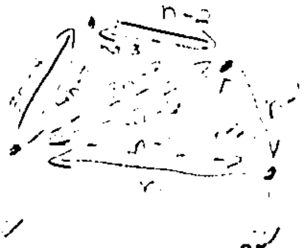


Figure 5

If this is done, there are now three other edges to consider, namely $\{3n-3\}$, $\{n+3\}$, and $\{4n-7\}$. As can be seen in the diagram, these are three edges in a triangle pattern, similar to that of the previous triple. The same choice arises in this situation. These three edges can either be made into the length three cycle, or they can be incorporated into a length four cycle. It is then seen that this pattern continues until the length three cycle is used.

Now consider what happens once the length three cycle is used. What remains unused are the points $2, 3, \ldots, m$ and all edges associated with them, all of which must be used in length four cycles. First, consider point $m$. There are two edges from point $m$, namely edges $\{m\}$ and $\{2n+1-m\}$ which must be used consecutively in a cycle. Since the cycle is of length four, the cycle incorporating these two edges must be $\{2n+3-m, m, 2n+1-m, 3n-5+m\}$ if $m \neq 2$ and $\{1, 2, 2n-1, 2n\}$ if $m = 2$. This pattern continues until all the edges are used. Thus, there are a total of $n-2$ possible length three cycles, each one of which gives a unique pattern of length four cycles.

Now consider the next pattern, with a crossing edge of length two, a cycle of length one, and $n-2$ cycles of length four. The tail $Q$ must therefore be of length one, and hence equal to $\{n\}$, and the crossing edge $P_a$ must be equal to $\{n+1, 4n-4\}$ or $\{4n-4, n+1\}$. Now a situation identical to that discussed above is present. The remaining cycles are of length four, and must be equal to $\{3n-3, n-1, n+2, 4n-6\}$, etc. Thus, the values of the cycles are uniquely determined here.

The third possibility is a crossing cycle of length three, one other cycle of length three, one of length one, and $n-3$ cycles of length four, leaving a tail of length one. Once again $Q$ and $P_a$ are known, $Q = \{n\}$ and $P_a = \{3n-3, n-1, 4n-4\}$ or $\{4n-4, n-1, 3n-3\}$. That leaves us with the following diagram:

Figure 6

This configuration is nearly identical to that seen in the first case above, and will yield the same result. Hence, there are $n-2$ locations to place the length three cycle, each of which uniquely determines the length four cycles.

Now it must be shown that all the possibilities outlined above actually contribute the same sign to the value of $M_{1,2n}$. Let $P = P_1P_2\dots P_nQ$ be a pseudo-Eulerian path. Let $P_a$ be the crossing cycle, $P_b$ the length one cycle, and $P_k$ a length three cycle. It has been shown that for case 1, either $k < b < a$ or $a < b < k$. Assume the former. The only odd length cycles are $P_k$ and $P_b$. If the order in which the length four cycles appear is changed, then the contribution of the permutation is unchanged. Thus, if $P_k = \{i_1, i_2, i_3\}$, then any pseudo-Eulerian path in which the length three cycle is $\{i_1, i_2, i_3\}$ will have the same contribution as P, so long as the length three cycle appears before the length one cycle.

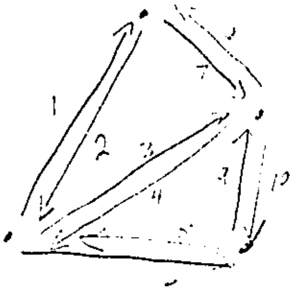
Now what about when the length three cycle appears after the length one cycle. Then the length one cycle must also appear after the crossing cycle. Take $P = P_1\dots P_a\dots P_b\dots P_k\dots P_nQ$. Then $\ell(P_1\dots P_n) \equiv 2, 3 mod 4$, thus, $\xi(P) = \xi(P_n^{-1}\dots P_1^{-1}Q)$. The inverted permutation is one of the permutations considered above. Thus, any permutation of this type which contains the three cycle $\{i_1, i_2, i_3\}$ on top or $\{i_3, i_2, i_1\}$ on bottom give the same contribution.

Now consider two permutations of this type in which the length three cycle is different. Compare two permutations which have different length three cycles.

$$P = P_1\dots P_nQ$$
$$P' = P_1'\dots P_n'.$$

Consider the diagram below. This diagram is a subsection of the entire diagram, with some of the edges renumbered for ease of illustration.

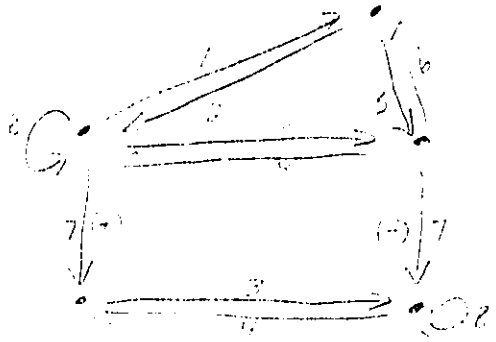
Figure 7



Assume the length three cycle $P_k$ is equal to $\{3, 9, 6\}$ in this diagram. All other cycles are uniquely determined. One of the other cycles is $P_j = \{1, 7, 8, 2\}$. Now look at the effect when $P$ is changed to $P'$, assuming the three cycle in $P'$ is $\{3, 8, 2\}$. In essence, this is looking at the effect of moving the length three cycle to the next position in the diagram. All of the cycles will remain the same, except for the cycle $P_j$, which will have to be changed to $\{1, 7, 9, 6\}$. The permutation which will change $\{1, 7, 8, 2, 3, 9, 6\}$ into $\{1, 7, 9, 6, 3, 8, 2\}$ is even. This shows that $\xi(P) = \xi(P')$ when the length three cycle in $P$ is a clockwise cycle. If the cycle goes counter-clockwise, then the permutation needed to change $P$ to $P'$ is the permutation needed to change $\{5, 10, 4, 1, 7, 8, 2\}$ into $\{1, 7, 4, 5, 10, 8, 2\}$. Once again, this is an even permutation, showing that $\xi(P) = \xi(P')$ for any two permutations of this type.

Now consider the next type of permutation, namely when there is a length two crossing cycle, a length one cycle, and $n - 2$ length four cycles. In this case, all of the cycles are uniquely determined, and all contribute the same value, since all but one of the cycles are of even length. To show that the contribution is the same as in the previous case, look at the sub-diagram below, again renumbered for convenience:

Figure 8



Let $P$ be a permutation of the first type considered above, where the length three cycle is $P_k = \{1, 5, 4\}$, $P_a = \{3, 7\}$, $P_b = \{8\}$ and $Q = \{2, 6\}$. If $P'$ is a permutation of the second type, then all of the cycles are the same, except for $P'_k = \{1, 5, 6, 2\}$, and $Q = \{4\}$. Thus, the permutation to convert $P$ into $P'$ is that needed to convert $\{1, 5, 4, 8, 3, 7, 2, 6\}$ into $\{1, 5, 6, 2, 8, 3, 7, 4\}$. This is an even permutation, showing that $\xi(P) = \xi(P')$.

For the permutations of type three, in which there is a crossing cycle $P_a$ of length three, another cycle $P_k$ of length three, and a cycle $P_b$ of length one, the situation arises again where one permutation of this type can be converted into another by either rearranging the length four permutations, or by shifting the length three cycle. Thus, all the cycles of this type contribute the same value. Now, using the previous figure once again, let $P$ be a type one permutation, and let $P'$ be a type three. Then, $P_a = \{3, 7\}$, $P_b = \{8\}$, $P_k = \{1, 5, 4\}$, and $Q = \{2, 6\}$. Also, $P'_a = \{1, 5, 7\}$, $P'_b = \{8\}$, $P'_k = \{3, 6, 2\}$, and $Q = \{4\}$. Thus, the conversion permutation in this case is that needed to convert $\{1, 5, 4, 8, 3, 7, 2, 6\}$ to $\{8, 3, 6, 2, 1, 5, 7, 4\}$. This is another even permutation, thus, $\xi(P) = \xi(P')$. It follows that all of the permutations for which $\eta(P) = P$ contribute the same value.

**Case n+1:** These are the permutations which have all cycles in the "top" half of the diagram, and $C \in Q$. Thus, there is one cycle of length 1, and

$n-1$ cycles of length at least 2. Analogous to the case when $n = 4$, all of these permutations will cancel out when the total length of all permutations is greater than or equal to $2n$. The only time when this doesn't happen is when the length of all cycles is equal to 2, except for the single loop $L$. This gives a total of $n!$ permutations, where the length two cycles are of the form:

$$\{1, 2n\}, \{2n+1, 3n-2\}, \{2n+2, 3n-1\}, \ldots, \{3n-3, 4n-6\}, \{n+1, n\}.$$

Once again, these permutations all contribute the same sign as in the previous set.

**Case n+2:** The final set of permutations are those for which $\eta(\eta(P)) \neq P$. The first result in this case is the following:

**Theorem 6.** *Let $P = P_1 \ldots P_n Q$. If $\eta(P) \neq P$, then*

$$\eta(P) = P_1 \ldots P_i P_n^{-1} \ldots P_{i+1}^{-1} Q$$

*or*

$$\eta(P) = P_j^{-1} \ldots P_1^{-1} P_{j+1} \ldots P_n Q.$$

Proof. Let $P_i \ldots P_j$ be the first set of cycles which are equivalent to 0 or 1 mod 4. Furthermore, assume $i \neq 1$ and $j \neq n$. Then:

$$\begin{array}{ll}
\textbf{a.} & \ell(P_i \ldots P_j) \equiv 0 mod 4 \\
\Rightarrow & \ell(P_{i-1} \ldots P_{j+1}) \equiv 2, 3 mod 4 \\
 & \ell(P_{i-1} \ldots P_j) \equiv 2, 3 mod 4 \\
 & \ell(P_i \ldots P_{j+1}) \equiv 2, 3 mod 4 \\
\Rightarrow & \ell(P_{i-1}) \equiv 2, 3 mod 4 \\
 & \ell(P_{j+1}) \equiv 2, 3 mod 4 \\
 & \ell(P_{i-1} P_{j+1}) \equiv 2, 3 mod 4 \\
\Rightarrow & \ell(P_{i-1}) \equiv \ell(P_{j+1}) \equiv 3 mod 4 \qquad (\Rightarrow\Leftarrow)
\end{array}$$

$$\begin{array}{ll}
\textbf{b.} & \ell(P_i \ldots P_j) \equiv 1 mod 4 \\
\Rightarrow & \ell(P_{i-1} \ldots P_{j+1}) \equiv 2, 3 mod 4 \\
 & \ell(P_{i-1} \ldots P_j) \equiv 2, 3 mod 4 \\
 & \ell(P_i \ldots P_{j+1}) \equiv 2, 3 mod 4 \\
\Rightarrow & \ell(P_{i-1}) \equiv 1, 2 mod 4 \\
 & \ell(P_{j+1}) \equiv 1, 2 mod 4 \\
 & \ell(P_{i-1} P_{j+1}) \equiv 1, 2 mod 4 \\
\Rightarrow & \ell(P_{i-1}) \equiv \ell(P_{j+1}) \equiv 1 mod 4 \qquad (\Rightarrow\Leftarrow)
\end{array}$$

Thus, either $i = 1$ or $j = n$.

What this theorem establishes is that the first group which has length congruent to 0 or 1 mod 4, and is therefore inverted, has to contain the final cycle $P_n$. When the function $\eta$ is applied to the resulting permutation, the first cycle which has length congruent to 0 or 1 mod 4 must contain the first cycle $P_1$. If the previous case, when $n = 4$ is reviewed, it will be seen that this property holds.

However, there is a problem with the analysis of this case. This problem lies in the fact that the lengths of some of the cycles cannot be determined in the manner outlined in the previous instances. As before, let $P = P_1 \dots P_n Q$ be a pseudo-Eulerian path. Then:

$$\eta(P_1 \dots P_n Q) = P_1 \dots P_i P_n^{-1} \dots P_{i+1}^{-1} Q$$
$$\eta(P_1 \dots P_i P_n^{-1} \dots P_{i+1}^{-1} Q) = P_{j+1} \dots P_n P_i^{-1} \dots P_1^{-1} P_j^{-1} \dots P_{i+1}^{-1} Q$$

From this the following relations can be derived:

$\ell(P_a) \equiv 2, 3 mod 4$
$\ell(P_1 \dots P_n) \equiv 2, 3 mod 4$
$\ell(P_1 \dots P_{n-1}) \equiv 2, 3 mod 4$
$\dots$
$\ell(P_1 \dots P_a) \equiv 2, 3 mod 4$
$\ell(P_2 \dots P_n) \equiv 2, 3 mod 4$
$\dots$
$\ell(P_i \dots P_n) \equiv 2, 3 mod 4$
$\dots$
$\ell(P_i \dots P_a) \equiv 2, 3 mod 4$
$\ell(P_{i+1} \dots P_n) \equiv 0, 1 mod 4$
$\ell(P_1 \dots P_i P_n^{-1} \dots P_{i+1}) \equiv 2, 3 mod 4$
$\dots$
$\ell(P_1 \dots P_i P_n^{-1} \dots P_j) \equiv 2, 3 mod 4$
$\ell(P_1 \dots P_i P_n^{-1} \dots P_{j+1}) \equiv 0, 1 mod 4$

The following can be drawn from this information:

As before, there is only one cycle of length congruent to 1 mod 4, and at most one non-crossing cycle of length congruent to 3 mod 4. Suppose these are $P_b$ and $P_k$ respectively. Then:

**a.** $b < i$
$\ell(P_b \dots P_a) \equiv 2, 3 mod 4$
$\ell(P_{b+1} \dots P_a) \equiv 2, 3 mod 4$
$\Rightarrow \quad \ell(P_{b+1} \dots P_c) \equiv 2$ for all $c \geq a$
$\quad \ell(P_b \dots P_c) \equiv 3$ for all $c \geq a$
$\Rightarrow \quad \ell(P_c) \equiv 0 mod 4$ for all $c > a$

Now $\ell(P_{i+1} \dots P_a) \equiv 0, 1 mod 4$
$\ell(P_i \dots P_a) \equiv 2, 3 mod 4$
$\ell(P_i) \not\equiv 0 mod 4$
$\ell(P_i) \not\equiv 1 mod 4$ because $b \neq i$
$\ell(P_i) \equiv 3 mod 4$
$\Rightarrow \quad \ell(P_i \dots P_a) \equiv 3 mod 4$
$\ell(P_{i-1} \dots P_a) \equiv 2, 3 mod 4$
$\Rightarrow \quad \ell(P_{i-1} \equiv 0 mod 4$
Likewise, $\ell(P_c) \equiv 0 mod 4$ for $b < c < i$
$\Rightarrow \quad \ell(P_{b+1} \dots P_a) \equiv 3 mod 4 \quad (\Rightarrow\Leftarrow)$
Therefore, $\ell(P_i) \equiv 2 mod 4$

Now consider the possible position of $P_k$.

$k < b$
$\ell(P_b \dots P_n) \equiv 3 mod 4$
$\Rightarrow \quad \ell(P_{k+1} \dots P_n) \equiv 3 mod 4$
$\ell(P_k \dots P_n) \equiv 2 mod 4$
$\ell(P_1 \dots P_n) \equiv 2 mod 4$
$\Rightarrow \quad \ell(P_c) \equiv 0 mod 4$ for all $c < b; c \neq k$
$\ell(P_c \dots P_n) \equiv 2 mod 4$ for all $b < c \leq i$
$\Rightarrow \quad \ell(P_c) \equiv 0 mod 4$ for all $b < c < i$
$\ell(P_1 \dots P_i P_n^{-1} \dots P_c^{-1}) \equiv 2 mod 4$ for all $i/leqc \leq c$
$\Rightarrow \quad \ell(P_c) \equiv 0 mod 4$ for all $i \leq c < j$
Also, $\ell(P_j) \equiv 2 mod 4$

$b < k < i$
$\Rightarrow \quad \ell(P_1 \dots P_n) \equiv 3 mod 4$
$\ell(P_c) \equiv 0 mod 4$ for all $c < i; c \neq b; c \neq k$
$\ell(P_c) \equiv 0 mod 4$ for all $i \leq c < j$

$i \leq k < j$
$\Rightarrow \quad \ell(P_c) \equiv 0 mod 4$ for all $c < i; c \neq b$
$\ell(P_c) \equiv 0 mod 4$ for all $i \leq c, j; c \neq k$

This will establish the length of most of the cycles, however, there are no constraints on the cycles from $P_{j+1}$ to $P_{a-1}$. The cycles within this range can have any length, providing that they do not contradict the rule concerning odd length cycles. In the previous cases, $n = 3$ and $n = 4$, this set of cycles was at most a single cycle, and the length of it could then be determined. However, for large $n$, this set will be a large number of cycles, with few restrictions on their lengths. For this reason, the number and sign of these permutations is unknown.

# REFERENCES

[1] Amitsur, A. S. and Levitzki, J., Minimal Identities for Algebras, Proc. Amer. Math. Soc. 1 (1950), 449-463.

[2] Rowen, L. H., A Simple Proof of Kostant's Theorem and an Analogue for the Symplectic Involution, Contemporary Mathematics 13 (1982), 207-215