# Societal Loss in R2: The Welfare Effects of the Majority Decision Rule in the Spatial Voting Model

Charles Bouwer Supervisor: Nguyen Van Quyen

University of Ottawa

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**Abstract:** The Euclidean distance metric is applied to two dimensions to determine the loss to society of different majority voting rules. The social objective function is the policy which minimizes the distance from voters' preference points. The sample is small due to computational feasibility. The results of this spatial voting model indicate that an average loss of 4% is reasonable and decreases when the data is highly polarized.

# Introduction:

This paper seeks to answer the question: What effect does the majority voting rule have on the social optimization of elections? To this end a straightforward spatial voting model is used "to introduce the philosophical conceptions of utilitarianism" [Edge1925] with a, to my knowledge, unique approach. The socially optimal policy is computed using the Euclidean distance metric as a utility function. The social objective function is the policy which minimizes the distance from all voters' preference points. Then every subset of the population has an optimal point computed to simulate the amounts of voters needed under different proportional rules and/or presence of parties. Finally, a loss function is created by comparing these values to the complete set optimum.

The paper is divided into seven sections:

I. Literature Review
II. Framework
III. Model
IV. Output
V. Results
VI. References
VII. Annex

# I. Literature Review:

The concept of a social welfare function, the notation for preferences, the completeness of sets of alternatives, the conditions for non-negativity, non-imposition, non-dictatorial, and the independence of irrelevant alternatives can be attributed to Kenneth Arrow and are still relevant today[Arrow1950]. Around the same time the foundation for Game Theory was laid by John Nash who developed the payoff function which maps strategies to real numbers and proved every finite game has an equilibrium point[Nash1951] using Brouwer Fixed Point Theorem. Complimentary to Nash's work was that of Oskar Morgenstern and John Von Neumann whose presentation of set theoretic approaches to game theory and solutions to n player cooperative games form the basis of many assumptions around present day models. Especially their consideration of information flows.

Anthony Downs defined governments influence on the economy as "that agency in the division of labor which has the power to coerce all other agents in society" [Downs1957]. With less emphasis on the enabling and regulation of markets, the redistribution of wealth and the pursuit of social order political, parties were defined as seeking solely to benefit from holding office and individuals as rational actors with single peaked preferences. This leads to a similar behaviour as firms and is the analogous counter point of the social welfare function, that out of many self-maximizing agents an equilibrium can be established which is socially optimal. This restriction on preferences was continued until in order to facilitate the original voting models. [Black1958, Bord1984] It is however possible to include single peaked preferences by definition of the metric without explicitly assuming them. The voter's ideal point or bliss point in the proximity model yields such single peaked preferences.

Presently there is a working definition of the structure and application of the spatial model [Scho2007] and much work being done around polarization [Patt2017] given the result of the 2016 United Stated Presidential. Early theoretical economic models of representative democracy [Besley1997] have been expanded to include realistic elements such as agents who change from voter to candidate based on past dissatisfaction [Laver2007]. The extension into higher dimensions has led to the study of mixed equilibrium in some [Dugg2005] and the robustness of the spatial model in others[Hen2001]. When comparing information asymmetries [Kess2005] and information cascades [Ellis2010] under direct and representative democracy the set of alternatives among which the policy is to be chosen has proven to be critical. With the rise in popularity of spatial models an increase in alternatives such as cross entropy models [Cab2014] has been observed.

Building on the theoretical corpus much empirical work has been done testing spatial voting models. Voter heterogeneity [Hen2001] and predictive power [Jeon2007] in the uncovered set of preferences are possible as more data becomes available which give a more complete profile of voters. Some factors related to identity (race, language, religion or culture) have been found to polarize voters to parties with which they identify[Anso2016]. All these papers owe much to recent increases in computing power. Just as Economics borrows from Mathematics, Sociology, Psychology, Finance, and other disciplines it has benefited from an entire stream of Computer Science which applies the theoretical complexity of computing to game theory [Dask2008, Chen2009, and Babi2016].

## II. Framework:

The geometric form of Nash's proof [Nash1951] used to describe equilibrium fits particularly well with social optimization. The method of economic analysis in this model was inspired by the former and "Goodness of Fit in Optimizing Models" [Varian1990]. When evaluating outcomes, it is "primarily important that one finds the right object to maximize" [Pete2016]. The social value function in this paper is composed of the Euclidean distance metric. It measures the departure of a policy point from an individual's preferred policy. This also provides insight into where a policy should be positioned to win certain voters. This interaction defines points of interests. While the implementation of the distance between two points is simple finding the points of interest is not as straightforward and the non-linear functions are difficult to solve. The solution is found to be the spatial median or L-1 median. Much work has been done to improve the L-1 median algorithm [Vardi1999], algorithms for the spatial median [Kent2010], or a novel algorithm for estimating geometric median [Cohen2016]. It is similar to the K-means algorithm [MacQ1979] and the two have been compared for: clustering [Kark2005], principal component analysis [Bruce2014], and sparsity [Kash2008]. The geometric medians provide the principal components for the set of candidates' policy points. Although they are theoretically able to set any point in this model the metric defines points which are socially optimal and at the same time optimal strategies for winning a certain subset of voters. This game has its foundation in the one-dimensional equilibrium presented by Harold Hotelling [Hote1929] in which landlords compete on a line. The metric affects the outcomes as the distance model implies that the voter's utility function increases towards their ideal point, peaks, and then drops off in all directions. Using direction alone would imply that the voter utility function is unbounded. In practice a finite number of candidates dictates the function peaks at the most extreme candidate in the direction of the voter [Merr1993].

## III. Model:

The model analyzes the effects of the majority voting decision rule on a given population within the framework of the spatial voting model discussed more broadly above. The Euclidean distance metric lends itself intuitively to economics as the desire for redistribution of someone with high income should differ from someone with low income and can be rationalized on a variety of topics. For two points in $R^2$:$(x_1, y_1), (x_2, y_2), distance = [(x_2 - x_1)^2 + (y_2 - y_1)^2]^{1/2}$

This is easily extended to any dimensionality allowing for analysis of many types of functional relationships including smooth polynomial functions. The Euclidean distance results in single peaked preferences which are maximized when policy points match individuals bliss points [Merr1993] and gives less weight to outliers than the mean. It also stipulates that individuals have a policy bliss point which will maximize their utility.

An optimal candidate position set is constructed by estimating the spatial median of every subset of voters' ideal points. These represent the optimal strategy for winning any subset of the populations vote as well as the socially optimal point for that many individuals. While there are algorithms for efficiently estimating the spatial median [Kent2010], and directed graph search of Nash equilibrium, [Dask2008] [Chen2009] [Babi2016] as mentioned above, it is found to exponentially hard for n player binary action games. So, this n player game with k actions is at least exponentially hard. This is one reason that the simulated population will remain small.

These medians, except for the full subset, are sub optimal allocations for society as ideally the entire population would be considered. They provide insight into the pure strategies of any number of politicians from 0 to the size of the population [Nash1949]. This method also allows for an easy extension of this paper modelling of the effects of abstention because once a completely connected graph is created it is possible to remove nodes(individuals) and observe the implications. The choice of R2 was motivated by the measure of fit, that 80-95% of variation in roll call voting in the U.S can be explained in 2 dimensions [Pool2001], against the computational requirements of higher dimensions. It also allows for other, possibly higher dimensional, connections to describe the flow of information in the model.

This model assumes that there is full information so there is no exchange or cases of asymmetries. This assumption also decreases the complexity of the game(graph) to be analyzed. The entire paper could be dedicated to the consideration of the topological spread of ideas, [Saenz2015] or the scaling of connections on a graph, decaying with $k^{-a}$ $2<a<3$, networks occur with strong evidence in less than 4% of cases [Broido2018]. These are outside the scope of the paper.

This model is static but could be extended to repeated elections to test for convergence. There is an elegant simplex proof for how in a convex set of dimension m, a proportion of voters greater than $(m/(m + 1))n$ provides a stable majority [Green1979]. However, under full information it does not seem possible for any number of candidates greater than 1 to capture and hold such a majority for more than 1 period. Other theories which could be added to make the model more reflective of the real world have been found to result in stability. While they are not included in this model some examples of possible extensions and how they could be implemented will follow the working model presented next in this paper.

An explanation of the essential Mathematica code, some verification code is omitted, will be presented here and the code and random seeds required to reproduce the results are in *Annex*. A complete output run is over 500 pages due to the combinatorial nature of the problem and will be omitted.

SeedRandom[“”] Allows for the data to be reproduced. See *Annex* for list of seeds.

Define a population of voters:

$$n = 10$$

Chosen due to exponential time required for computation, as per the binomial coefficient there are $\sum_{k=0}^{n}\binom{n}{k} = 2^{n}$ subsets of a set of n elements. In future implementation of one of the algorithms outlined above or use of significantly better processing power would allow larger simulation.

Define a function which will generate the simulated preferences:

$$F = NormalDistribution[5,6]$$

This instance uses a normal distribution with a mean= 5 and standard deviation=6. Several symmetric and asymmetric distributions are used with varying parameters to test the outcomes of 12 different datasets. Further discussion will follow in the *Results* section.

Create the matrix of individuals ideal points:

$$preferences = RandomVariate[F, \{n, 2\}]$$

Several different functions such as $RandomPoint$ are implemented depending on whether the data was drawn from a distribution or a region.

The Euclidean distance metric is defined:

$$dtotal[f\_] := Total[Sqrt[Total[(Transpose[f] - \{x, y\}){}^\wedge 2]]]$$

Which weights all individuals’ preferences equally and is therefore a good aggregator for a social value function.

The least distance metric is applied to the full matrix of preferences which results in spatial median which minimizes the total distance from every ideal point:

$$minpref\ = NMinimize[dtotal[preferences], \{x, y\}]$$

This presents a global non-linear unconstrained optimization problem which is the heart of the model. The non-trivial case where all points do not have the same coordinates is not easily described explicitly. Many gradient based methods sequential quadratic programming method, the augmented Lagrangian method, and the (nonlinear) interior point method, have requirements for constraints and differentiability. Solving the first and second order conditions for this problem is difficult. For these reasons Mathematic presents several better options for global search algorithms: $Random\ Search, Simulated\ Annealing, Differential\ Evolution, Nelder\ Mead$. While all the algorithms were tested they yielded similar computation times and identical results given the specified accuracy. For this reason, the $Nminimize$ function was called with automatically selected method. Mathematica chose $Nelder\ Mead$ which, similar to the above proof [Green1979], compares points on a simplex to determine the best path[Neld1965].

The solution to the minimization problem is recorded:

$$prefopt = \{x/.minpref[[2]], y/.minpref[[2]]\}$$

The total distance to all ideal points is recorded: $prefdist = minpref[[1]]$

Next this process is applied to all possible combinations of voters.

Create a subset for each possible combination of elements (bliss points):

$$prefsub\ =\ Subsets[preferences, \{1, n\}]$$

Map the distance metric over every subset:

$$prefsubdist = Map[dtotal, Subsets[preferences, \{1, n\}]]$$

Now calculate the policy points which minimize the total sum of distance from each individual in each subset:

$$prefsubopt\ =\ \{x, y\}\ /.NMinimize[\#, \{x, y\}][[2]]\ \&\ /@\ prefsubdist$$

This represents the optimal policy point within each subset. Depending on which fraction of the vote is required to win these are the points which will be used to calculate if the rule creates a loss for society as a whole. The distance from each of these subset optimums to individuals bliss points:

$$loss = DistanceMatrix[prefsubopt, preferences]$$

The loss from each of these subset optimal points to the original preferences is calculated:

$$lossopt = Total[loss, \{2\}]$$

Finally, to make the cases comparable the loss function is defined as:

$$losspercent = 1 - lossopt/prefdist$$

Or one minus the ratio of the socially optimal loss to the subset optimal loss. This means that as the total distance in the voting space approaches infinity the loss approaches 0. Following the *Extensions*, the *Results* of this simulation for all the simulated data points is presented.

Extensions:

Adding a random element specific to each candidate in voter utility functions [Lin1999]. Which results in non-convergent equilibria being only locally stable and disappear as voter uncertainty increases [Tove2009]. A random disturbance ε drawn from a probability distribution could represent uncertainty. Applied to the bliss point to yield an actual and perceive optimal utility or uncertainty about voters[Calv1985]. It could also be applied to the policy points to model uncertainty.

A lower dimensional linear map which spans the possible candidates' platforms and create equilibrium in less than [Green1979] worst case[Cres2010]. Examining the one dimensional span of the space.

Adding a cost of changing policies [Tove2009]. A penalty in the form of distance, relative distance, random disturbance or simple accounting could be added to the policy point.

Adding a small distance which represents the ability of politicians to finagle and are uncontestable within the radius[Wuffle1989] [Brau2002]. Effectively turning the policy points into regions(disks). A parameter ϕ could describe this region.

Adding voter beliefs [Ogden2017]. Parties then gain more from appealing to voters with more precise beliefs. This is similar to bounded rationality or voter uncertainty [Tove2009] and could be implemented with an error term or a function for calculating perceived policy points based on distance, connections or characteristics of the voter/candidate.

Adding different metrics directional, proximity, discounted proximity which define types of voters and politicians[Coll2010][Merr1993]. This could be done with different welfare functions for certain regions.

Giving some candidates a larger voter base or other advantage such as bias under uncertainty [Xeft2012]. Advantage on the non-policy dimension allows the favored candidate to propose policies that deviate from the vote-maximizing point and towards his own ideal point[Pere2009]. Biased clustering of a given number of ideal points around some policy point or within a policy region.

Allowing for adaptive platforms and better modelling of the electoral process, in which candidates exchanging information with voters and modifying platforms. [Stad1998]. Including directed edges on the space which represent the flow of information.

Abstention and lower voter turnout[Stad2000]. Reducing the number of voting nodes.

By introducing politicians who are interested not only in winning but also the policy result of the election [Calv1985]. This form of motivation could be implemented by restricting certain parties' policy points. Adding boundaries or regions which represent certain policies which they will adhere to.

## IV. Output:

The model was run on data from 8 different distributions with varying parameters to produce 12 datasets of 10 samples of 10 individuals each with 1023 subsets (empty set removed). To highlight the different effects of distributions the four following will be presented in *Results* and the rest available for reproduction in *Annex*:

A: $Normal\ Distribution\ \mu = 5, \sigma = 2\ \ Average\ k = 6\ loss = \ -4.525\%$

B: $Mixed\ Normal\ \mu_1 = 5\ , \mu_2 = 20, \sigma_1 = \sigma_2 = 2\ \ Average\ k = 6\ loss =$ $-5.849\%$

C: $Beta\ Distribution\ \alpha = \beta = 0.5\ Average\ k = 6\ loss = \ -5.52\%$

D:

$Elipses\ with\ centroids: (5{,}5), (500{,}5) and\ semiaxes\ lengths: r_{x1}, r_{y1} = rx_2, r_{y2} = (4{,}2)\ Average\ k = 6\ loss = \ -0.072\%$

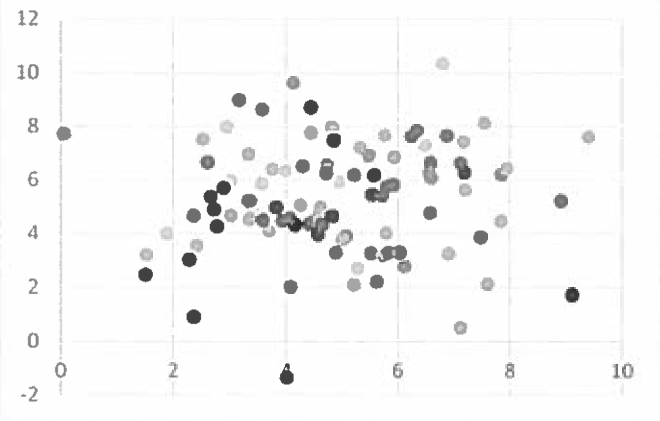


Figure A1: Preferences

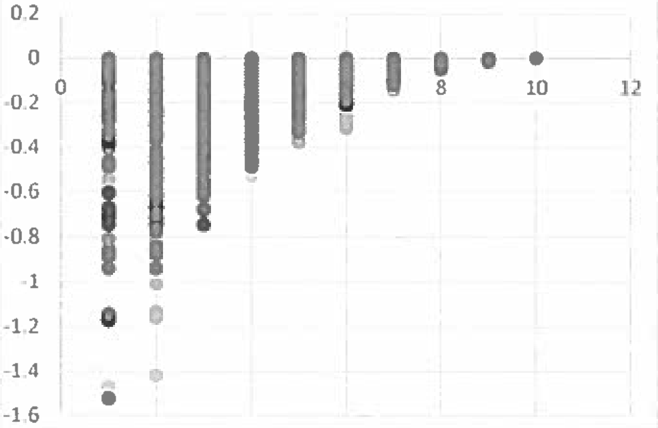


Figure A2: Loss

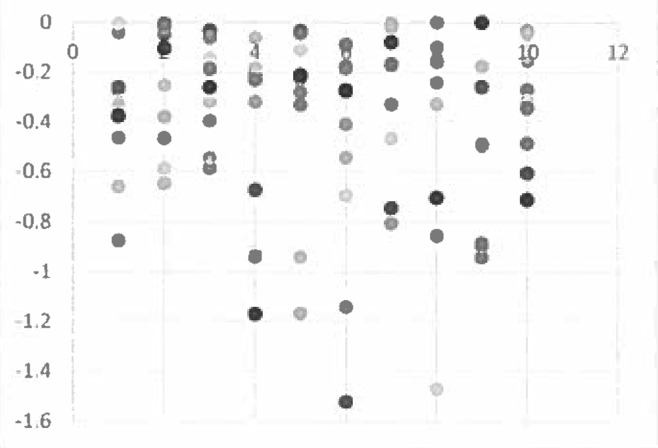


Figure A3: k=1 Subset Loss

Figure B1: Preferences

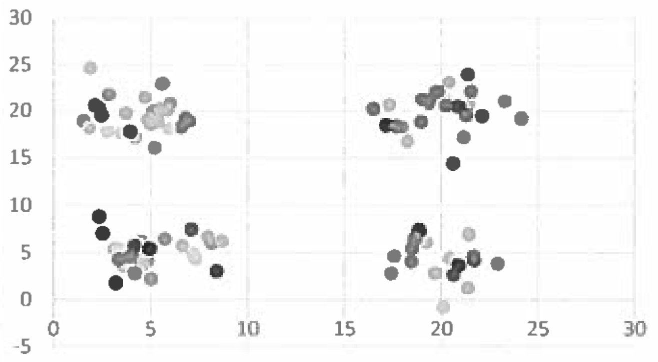


Figure B2: Loss

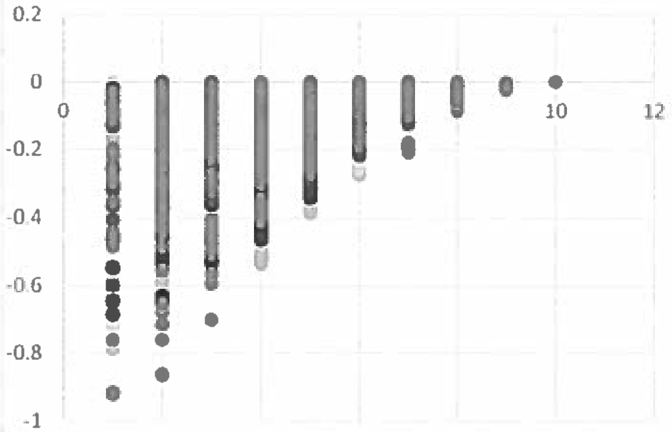


Figure B3: k=1 Subset Loss

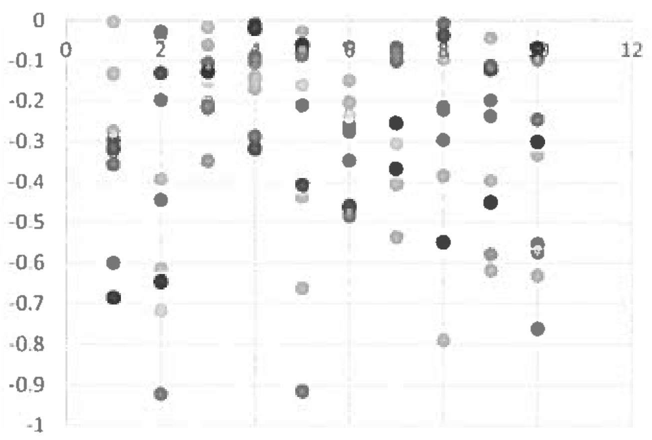


Figure C1: Preferences

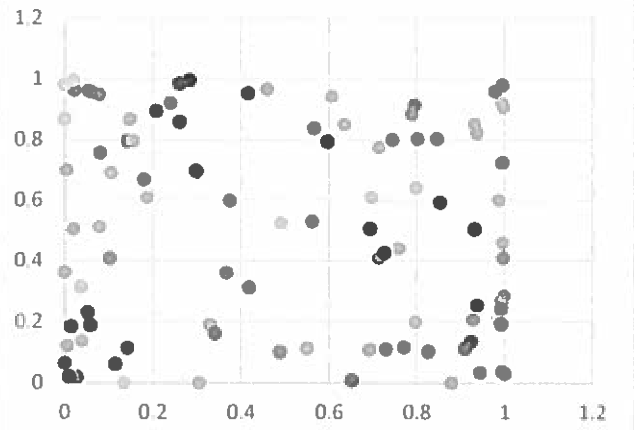


Figure C2: Loss

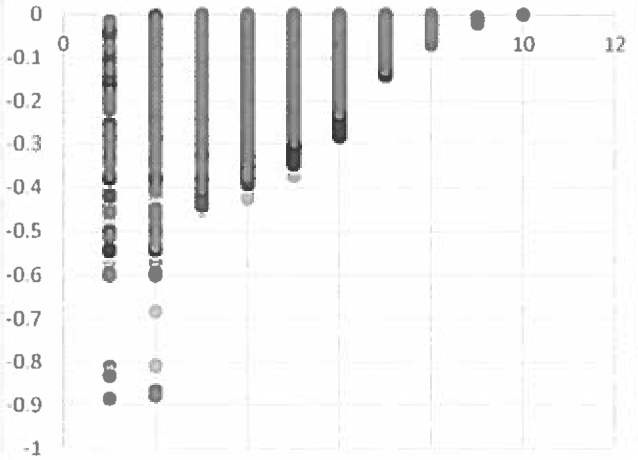


Figure C3: k=1 Subset Loss

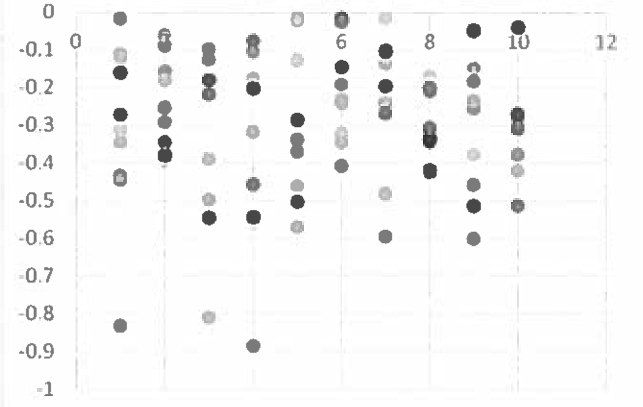


Figure D1: Preferences

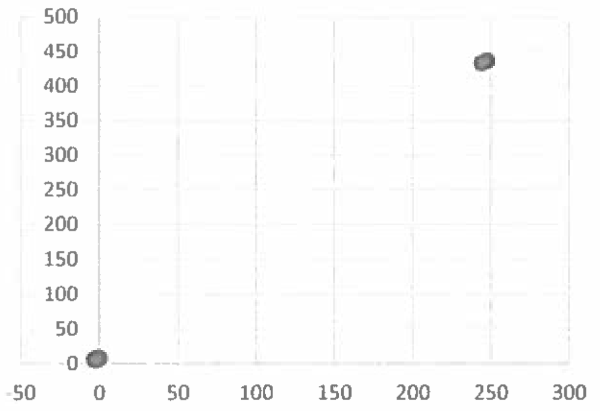


Figure D2: Loss

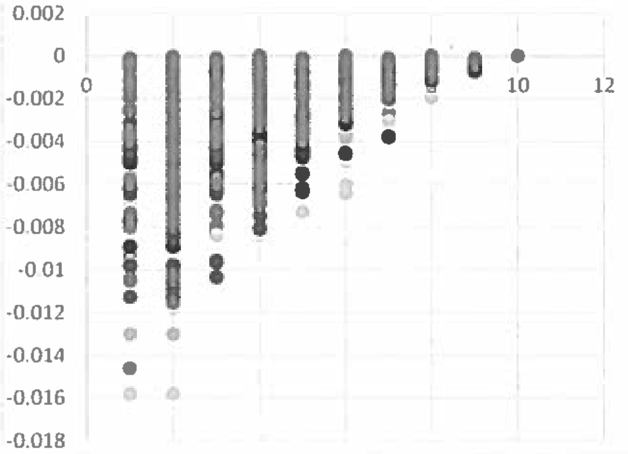
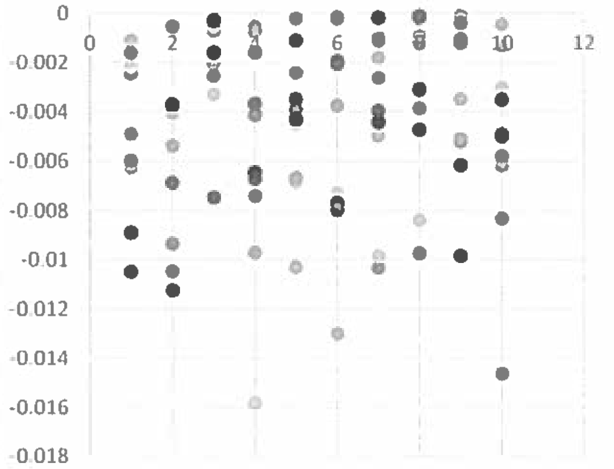


Figure D3: k=1 Subset Loss



## V. Results:

The main results of the experiment are as follows:

- The socially optimal number of parties as defined by minimizing the Euclidean distance to all voters' ideal points occurs when there is one candidate who places his policy point at the spatial median of all the population's ideal points. The first is the most intuitive result and has been obtained under other models[Tove2009][Lin1999], but to my knowledge has never been tested with a similar model. *Figures: A2, B2, C2, D2.*

- The average loss of a 50% majority rule (6/10 points) is decreasing in standard deviation and polarization. As the total distance in the voting space increases to infinity the loss will go to zero. This has interesting implications for society as ideas diverge society does not gain by including the opposite viewpoint. A further analysis of every ideal point would show the losers in each scenario and could inform a dynamic game. This may also have important implications for voters who find themselves in a near empty core and are willing to move from either polarized group. *Figures: A1, A3, B1, B3, D1, D3.*

- Some suboptimal points may be preferred to others by equal numbers of voters but have different results for society. The k=5 subsets optimal points all have different loss values. This calls in to question:

    I. The robustness of the spatial voting model with different data distributions. *Output A, B, C, D.*

    II. The previous results that: In a majority proportional to 0< b < 1: An alternative x is socially at least as good as y if and only if the number who prefer x to y (x>y) is equal to b/1-b times the number who prefer y to x (y>x) [Slu1979]. *Output A, B, C, D.*

III. The previous result: that if we impose Arrow's conditions of collective rationality, IIA, and the Pareto principle on the social welfare function, then it must be dictatorial[Bord1984]. *Output D.*

- In both symmetric and asymmetric distributions which are not distant the median voter theorem is recovered as there is a representative in the k=1 subset whose optimal point presents a near 0 loss to society. *Figures A3, B3, C3.*

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## VII. Annex:

Code:

Wolfram Mathematica 11.2

```
(* Clear the environment*)
ClearAll["Global'*"];
(* Define the population*)
n = 10
(* Set the seed*)
SeedRandom[]
(* Define the distribution*)
F = []
(* Create the matrix of preferences*)
preferences = RandomVariate[F, {n, 2}]
(* Define the distance metric*)
dtotal[f_] := Total[Sqrt[Total[(Transpose[f] - {x, y})^2]]]
(* Minimize for the whole population*)
minpref = NMinimize[dtotal[preferences], {x, y}]
(* Record the optimal point and total distance*)
prefopt = {x /. minpref[[2]], y /. minpref[[2]]}
prefdist = minpref[[1]]
(* Verify*)
SpatialMedian[preferences]
(* Define the subsets empty set omitted*)
prefsub = Subsets[preferences, {1, n}]
(* Map the function over the subsets*)
prefsubdist = Map[dtotal, Subsets[preferences, {1, n}]]
(* Find the optimal point for each subset*)
prefsubopt = {x, y} /. NMinimize[#, {x, y}][[2]] & /@ prefsubdist
(* Create the distance matrix to the optimal point of each
subset*)
loss = DistanceMatrix[prefsubopt, preferences]
(*Record the distance to the preferences*)
```

```
lossopt = Total[loss, {2}]
(* Define the loss function*)
losspercent = 1 - lossopt/prefdist
(* Export results *)
lossperxl = Table[losspercent, 1]
Export[".xlsx", lossperxl, "XLSX"]
Export[".xlsx", preferences, "XLSX"]
```

Seeds and functions:

```
1052[0-9] F=NormalDistribution[5, 2]
1054[0-9] F=NormalDistribution[5, 4]
1056[0-9] F=NormalDistribution[5, 6]
“quad1051[0-9]” F=ProbabilityDistribution[(x - 5)^2 + 1, {x, 0,
10}]
“beta55[0-9]” F = BetaDistribution[2, 5]
“mix5202[0-9]” F = MixtureDistribution[{0.5, 0.5},
{NormalDistribution[5, 2], NormalDistribution[20, 2]}]
“exp2[0-9]” F = ExponentialDistribution[2]
“chisq3[0-9]” F = ChiSquareDistribution[3]
"elip51042[0-9]” replace F with:
reg1 = TransformedRegion[Disk[{5, 5}, {4, 2}],
RotationTransform[Pi/3]]
reg2 = TransformedRegion[Disk[{10, 5}, {4, 2}],
 RotationTransform[Pi/3]]
pref1 = RandomPoint[reg1, n/2]
pref2 = RandomPoint[reg2, n/2]
preferences = Join[pref1, pref2]
"elip51042[0-9]” replace F with:
reg1 = TransformedRegion[Disk[{5, 5}, {4, 2}],
RotationTransform[Pi/3]]
reg2 = TransformedRegion[Disk[{10, 5}, {4, 2}],
 RotationTransform[Pi/3]]
pref1 = RandomPoint[reg1, n/2]
```

```
pref2 = RandomPoint[reg2, n/2]

preferences = Join[pref1, pref2]
```

"elip550042[0-9]” replace F with:

```
reg1 = TransformedRegion[Disk[{5, 5}, {4, 2}],

RotationTransform[Pi/3]]

reg2 = TransformedRegion[Disk[{500, 5}, {4, 2}],

 RotationTransform[Pi/3]]

pref1 = RandomPoint[reg1, n/2]

pref2 = RandomPoint[reg2, n/2]

preferences = Join[pref1, pref2]
```

“sine1010[0-9]"F = ProbabilityDistribution[Sin[x] + 1, {x, 0, 10}]