Honeymoon Oberwolfach Problem

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Introduction

The Honeymoon Oberwolfach Problem is a scheduling problem stated as follows:

'Given n newlywed couples and round tables of sizes $2m$, where $2n = 2\ell m$ for some integer ℓ, is there a way the couples could be seated over a span of several nights so that the spouses are always side-by-side, but every person sits next to every person other than their spouse exactly once?'[1].

We denote this problem as HOP$(2n; 2m)$.

The following results were proved in [1].

- For n odd: HOP$(2n; 2m)$ has a solution whenever $m|n$.
- For m even, $m \geq 8$: HOP$(2n; 2m)$ has a solution whenever $m|n$.
- For m odd, $m \geq 5$, $n \neq 4m$: HOP$(2n; 2m)$ has a solution whenever $m|n$ and HOP$(4m; 2m)$ has a solution.

The primary goal for this project is to find solutions for the following two important cases:

- HOP$(12; 12)$, which by [1] generates a solution for HOP$(12\ell; 12)$ for all $\ell \geq 3$, and
- HOP$(12; 6)$, which by [1] generates a solution for HOP$(12\ell; 6)$ for all $\ell \geq 4$.

The secondary goal is to solve the remaining exceptional cases HOP$(24; 12)$, HOP$(24; 6)$, and HOP$(36; 6)$ thereby complete the solutions to HOP$(2n; 2m)$ for $m = 6$ and $m = 3$.

Methodology and Results

Although seemingly general, this question can be modelled and solved using graph theory. An example of how we model and solve this problem can be seen in Figures 1 and 2 below.

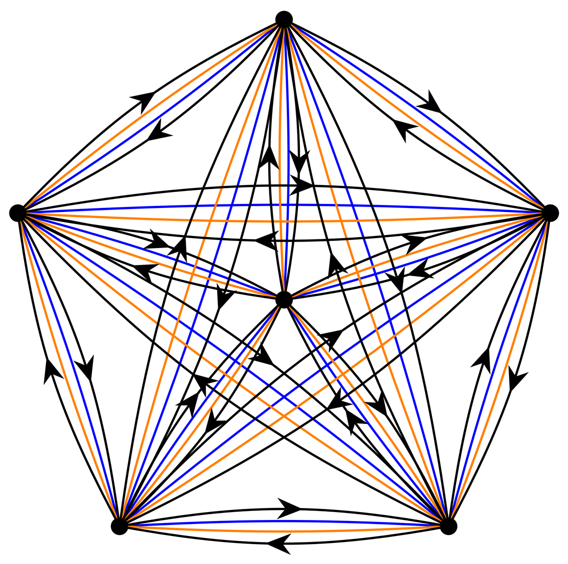

Figure 1: A graph representing the required seating of 6 couples.

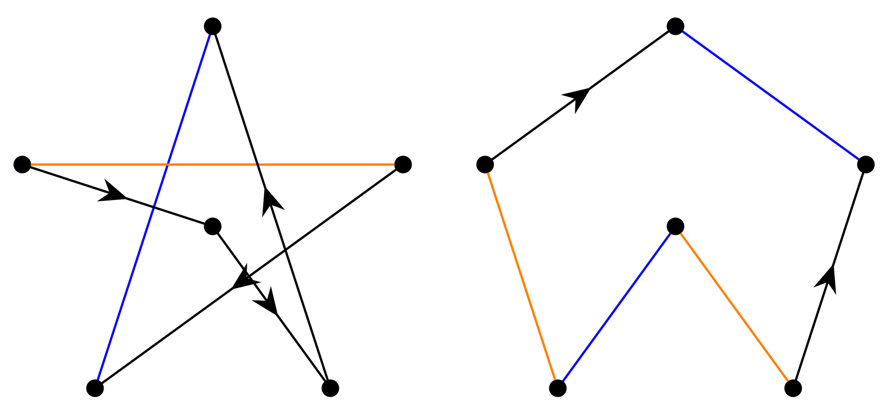

Figure 2: Solution to HOP$(12, 12)$

Denote the graph in Figure 1 by G. Notice in G:

- the vertices represent the couples
- an edge between two vertices, X and Y, represents:
  - the woman in X sitting next to the woman in Y if the edge is orange.
  - the man in X sitting next to the man in Y if the edge is blue.
  - the man in X sitting next to the woman in Y if the edge is directed from X to Y.
  - the man in Y sitting next to the woman in X if the edge is directed from Y to X.

Thus the edges between two vertices represent all possible side-by-side sitting arrangements between the 4 people in 2 couples. Each of these edges must be used exactly once.

Methodology and Results

To see what a solution in G looks like, consider the following solution to HOP$(12; 12)$, Figure 3. In each couple, (X, x), the uppercase letter represents the man and the lower case letter the woman. After some time, the reader should be able to verify that this is, in fact, a solution.

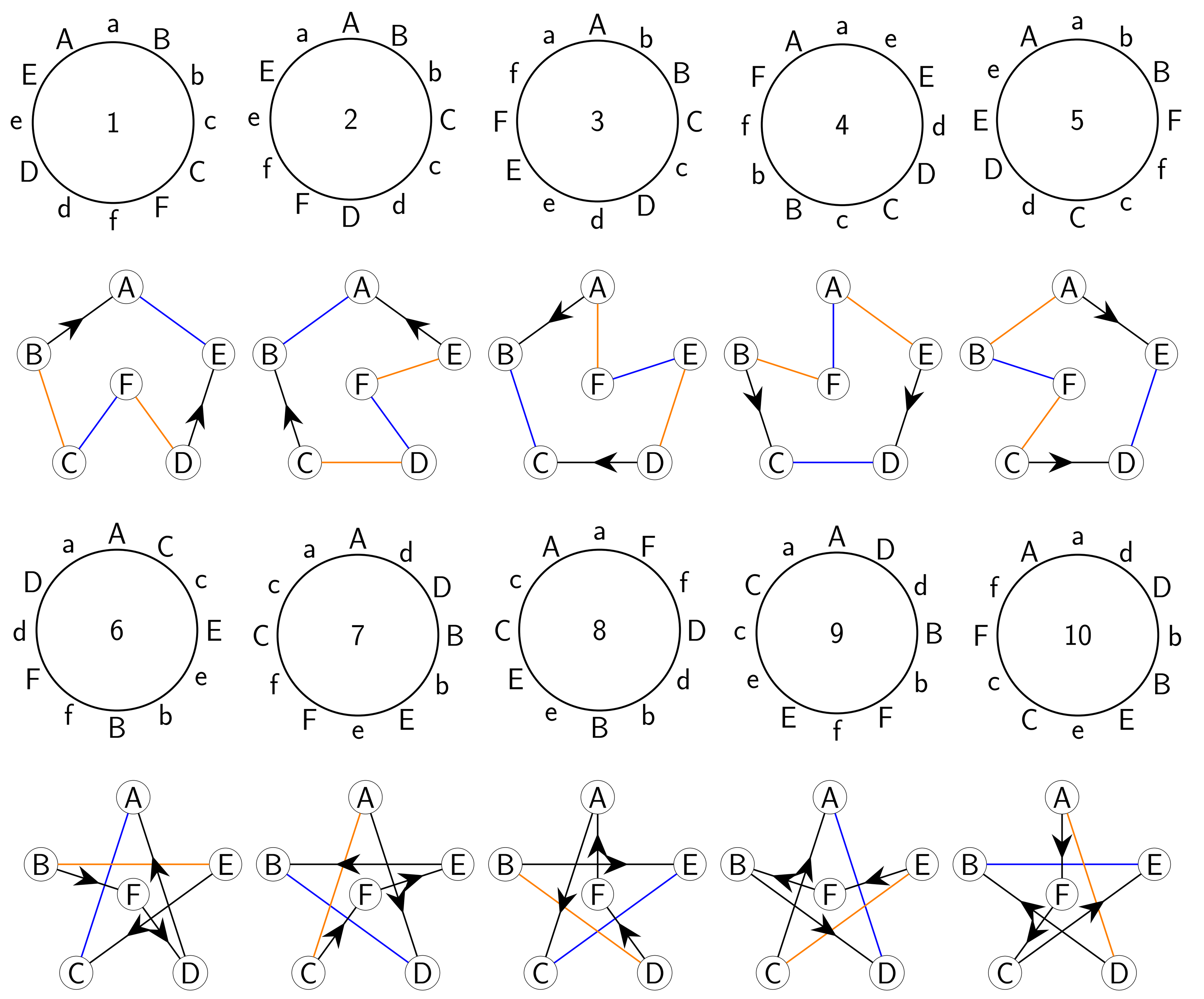

Figure 3: Solution to HOP$(12, 12)$ and the corresponding graphs

Observe that each of these table arrangements can be represented by a cycle in G. Furthermore, these 10 cycles satisfy the following important properties:

- each edge of G lies in exactly one of these cycles (this set of cycles forms a *decomposition* of G),
- each cycle contains all 6 vertices, and
- in each cycle:
  - in every run of blue and orange edges, the colours alternate, and
  - in every run of black edges, all edges have the same direction pointing away from the orange edge at one end and towards the blue edge on the other end.

Knowing this, it should not be too hard to convince yourself that any cycle decomposition of G satisfying these properties represents a solution to HOP$(12; 12)$.

Furthermore, our solution in Figure 3 has the following symmetry: cycles $2 - 5$ are obtained from cycle 1 by rotating around the fixed vertex F, and similarly cycles $7 - 10$ are obtained from cycle 6. Cycles 1 and 5, shown again in Figure 2, are called *starter cycles* of our solution.

Figures 4-6 give the two starter cycles for our solutions to HOP$(12; 6)$, HOP$(24; 12)$, and HOP$(24; 6)$, respectively.

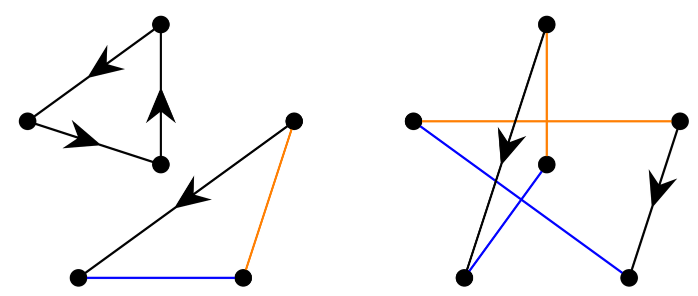

Figure 4: Solution to HOP$(12, 6)$

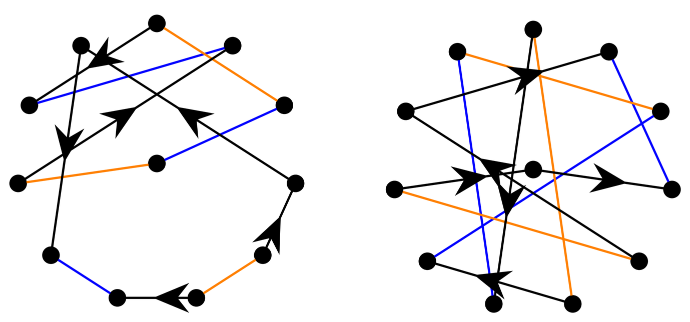

Figure 5: Solution to HOP$(24, 12)$

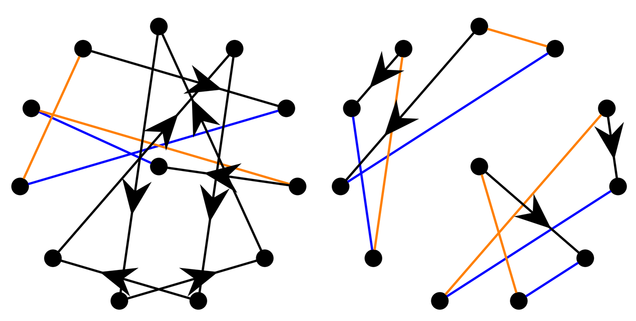

Figure 6: Solution to HOP$(24, 6)$

Conclusion

Our method successfully supplied solutions to several cases of the HOP, especially the important HOP$(12; 6)$ and HOP$(12; 12)$.

Despite this, many cases of the Honeymoon Oberwolfach Problem remain open. In particular, there is our last desired case HOP$(36, 6)$ whose solution would give us a complete solution for tables of size 6. It would require finding, for the graph in Figure 7, a cycle decomposition that satisfies properties similar to those mentioned earlier. More generally, it remains to solve HOP$(4m; 2m)$ and HOP$(8m; 2m)$ for all odd $m \geq 5$.

The Honeymoon Oberwolfach Problem has also been defined for tables of unequal sizes. While some solutions are already known [1], in general, the problem remains wide open.

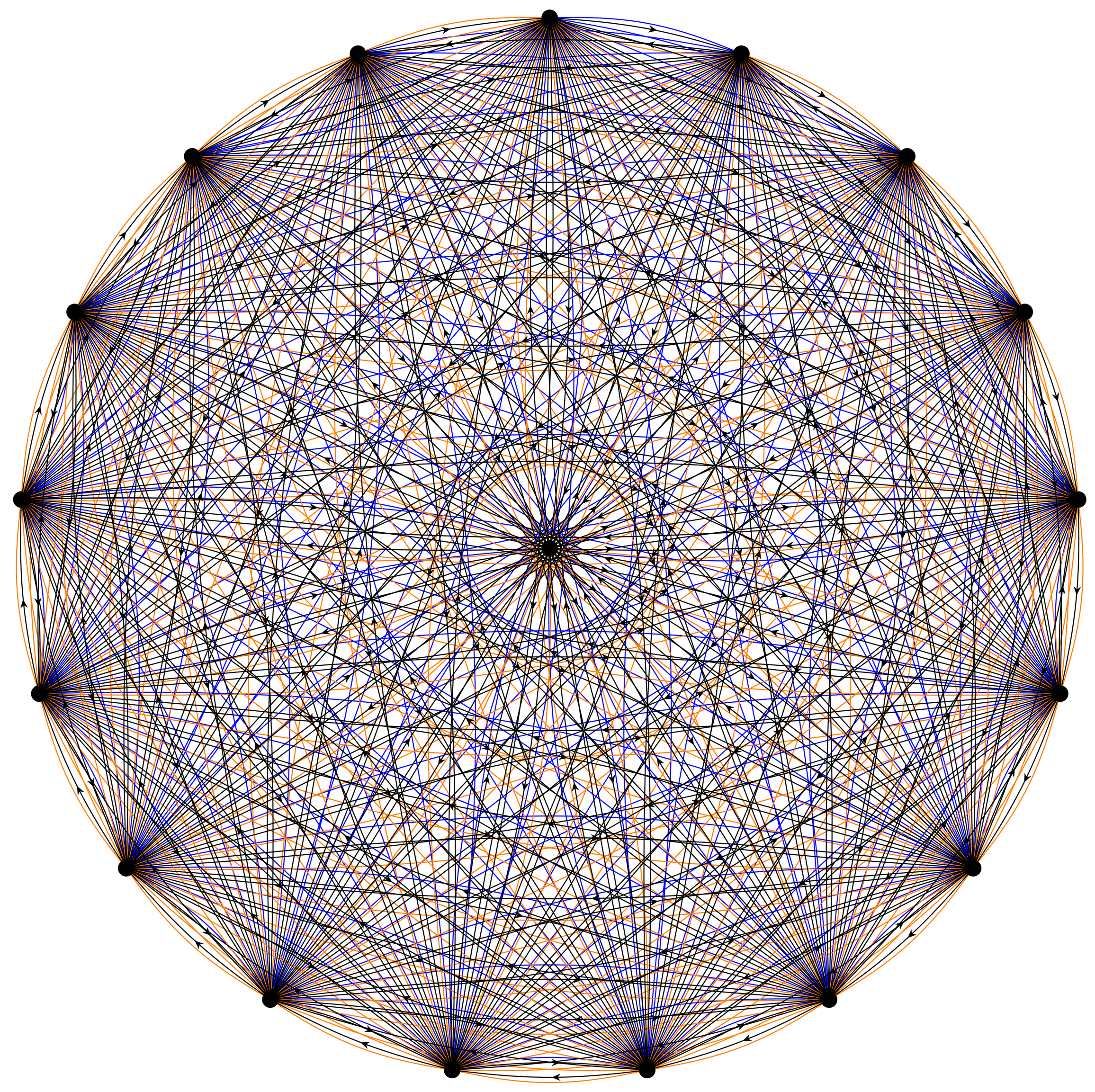

Figure 7: Model of HOP$(36; 6)$

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This poster was made entirely from scratch (figures and all) using the program LaTeX.

References

[1] Andrea Burgess, Nevena Francetić, Dene Lepine, and Mateja Šajna.
On the Honeymoon Oberwalfach Problem.
Manuscript in preparation. 2017.

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