

SPATIALLY INDUCED INDEPENDENCE AND CONCURRENCY WITHIN PRESHEAVES OF LABELLED TRANSITION SYSTEMS

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Abstract

In this thesis, we demonstrate how presheaves of labelled transition systems (LTS) acquire a very natural form of *spatially induced independence* on their actions when we allow a minimal amount of gluing on selected transitions within such systems. This gluing condition is characterized in the new model of *LTS-adapted presheaf*, and we also make use of the new model of *asynchronous labelled transition system with equivalence* (ALTSE) to characterize independence on actions. As such, our main result, the Theorem of Spatially Induced Independence, establishes functors from the categories of LTS-adapted presheaves to the categories of ALTSE-valued presheaves; it is a result that extends a proposition of Malcolm [SSTS] in the context of LTS-valued sheaves on complete Heyting algebras.

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Dedications

*I dedicate this work to my parents,
Ginette Fortier and Robert Garceau*

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Introduction

In this thesis we investigate how spatially distributed systems have an inherent potential to express independence in between internal processes and actions. We achieve this specifically in the case of presheaves of labelled transition systems (LTS) to represent such spatially distributed systems.

The study of distributed computing gained importance in the 1980s when it was realized that performing all manners of computation sequentially on a single supercomputer (mainframe) was not the most efficient way of solving certain problems admitting a divide and conquer strategy. In practice, we often encounter computational problems or tasks that can be decomposed into several independent subproblems or subtasks. We can then tackle such problems much more efficiently by delegating their subproblems to several independent machines performing simultaneously.

Initially, such a distribution of tasks was performed on a cluster of computers acting as a single coherent whole on a network. Today however, the notion of distributed system has a more general connotation, as distributed computing can occur within a single computer in which multiple software processes are engaged concurrently. In a general sense, a distributed system is any system that emerges from the interaction of several autonomous subsystems. These may refer to any kind of coupled system where separate components or objects (in the sense of an object-oriented system) are involved. Also, the word “concurrency” is a general term that is associated to any kind of processing or activity that arises within such distributed systems.

We may believe à priori that the theory of concurrency is only relevant in a context where multiple processor units are involved, but this is not the case. Even if all computations are performed sequentially, as soon as we have multiple threads or processes that interact and communicate with each other, there are definite distributed system problems such as resource sharing and deadlock that still have significance.

Another important aspect of distributed systems that still applies in the context of sequential systems is the notion of independence of actions. Basically, to informally explain a bit here, whenever actions are independent within a system, the order in which they are executed does not affect the final outcome of their execution. This has important applications in the context of model checking, and more specifically, when applying partial order reduction methods where we verify properties of a system that are invariant under permuting independent actions in a run (see Peled [POMC], [AOOA] and Lomuscio, Penczek, Qu [PMAS]). Our main theorem is essentially a statement about how such independence naturally arises in a context where actions are properly distributed via a presheaf on a complete Heyting algebra space.

The approach that we will take here in using presheaves on Heyting algebras to represent distributed systems has its roots in Goguen's Categorical General Systems Theory (see [CST] and [CFST]). In the paper *Sheaf Semantics for Concurrent Interacting Objects* [SSCO], Goguen proposes to model concurrent objects as set-valued sheaves, and then he represents a distributed system of concurrent objects as a categorical diagram of such sheaves. The morphisms in such a diagram are natural transformations between concurrent objects which, according to Goguen, represent the concept of *object inheritance*. Finally, he proposes that the categorical limit of such a diagram provides the behavior of the system as a whole. This theory is elegant from a mathematical standpoint as it advocates the use of sheaves to model both distributed systems and their components. In particular, Goguen's theory allows a bottom-up approach to distributed system specification, and the sheafification process automatically yields a global system behavior out of locally made specifications and connections on objects.

These ideas are carried forward by G. Malcolm in *Sheaves and Structures of Transition Systems* [SSTS], where concurrent objects are replaced with labelled transition systems (LTS)², and we get diagrams in the category of LTS, denoted $\mathbb{T}$, to represent distributed concurrent systems. However, the category $\mathbb{T}$ does not exactly coincide with a category of set-valued sheaves, and thus, we cannot think of every LTS morphism as representing object inheritance in the sense of Goguen³.

In general, LTS morphisms seem better suited for another kind of interpretation. The structure of a LTS morphism incorporates a partial map on actions (i.e. labels) of a LTS, that allows actions to vanish in the target LTS whenever the partial map is undefined. And, as Winskel and Nielsen pointed out in *Models for Concurrency* [MC], this partial map feature in question allows LTS morphisms to render the component-to-subcomponent relation within a distributed system. To be more precise, whenever we have a LTS morphism from a component A to a subcomponent B , we can allow actions in A that live outside of the boundary of B to vanish (i.e. to be undefined with respect to the partial map) when projecting to B . As such, LTS morphisms can give us a sense of where actions are located in space (as opposed to where they vanish). This allows LTS morphisms to play the role of spatial restrictions, i.e. they can describe how a system contains its subsystems from a spatial point of view.

In *Sheaves, Objects, and Distributed Systems* [SODS], Malcolm further demonstrates that for any distributed system that is represented as a diagram of (monoid) LTS, we can perform sheafification in such a way that the distributed system can be

²Malcolm uses monoid labelled transition systems (MLTS) in fact. But the category of MLTS has an adjunction with the category of LTS, and in this sense, the statements that he makes about MLTS extend naturally to LTS.

³It is true that, for a fixed labelling set L and a corresponding fiber category $\mathbb{T}_L$ generated by L in $\mathbb{T}$, Malcolm provides an adjunction between $\mathbb{T}_L$ and a category of set-valued sheaves. But in extending this adjunction to $\mathbb{T}$ as a whole, Malcolm loses the interpretation of LTS morphisms as natural transformations in the process. This means that only a subclass of LTS morphisms correspond with inheritance in this sense.

extended to a LTS-valued sheaf on a complete⁴ Heyting algebra (a bi-Heyting algebra if the diagram has a finite number of objects). More specifically, we can use the preorder generated by the underlying graph of the LTS diagram and look at downward closed sets with respect to this preorder to provide the elements of a Heyting algebra (meet and join operations are given by intersection and union on these downward closed sets). A Heyting algebra that is derived as such can be regarded as a space on which the concurrent system is distributed.

Within this thesis, we investigate concurrent systems that are spatially distributed over complete Heyting algebras in this manner. We believe that the proper way to achieve this in general is to use presheaves of the form $F : \mathcal{H}^{op} \rightarrow \mathcal{C}$; where $\mathcal{H}$ is a Heyting algebra that represents the space over which the system is distributed, and $\mathcal{C}$ is some chosen universe of process model.

Now, since LTS morphisms have the ability to represent spatial restrictions properly, we decided to focus our attention on the case of presheaves that take values in the category of LTS (fulfilling the role of $\mathcal{C}$ above). These are precisely presheaves of the form $\mathcal{T} : \mathcal{H}^{op} \rightarrow \mathbb{T}$ where $\mathcal{H}$ is a Heyting algebra. The theory of presheaves of LTS (i.e. $\mathbb{T}$ -valued presheaves on Heyting algebras) is vast and rich in potential. Our objective is to demonstrate how this is so by investigating one of its prominent features : spatially induced independence.

The basic idea behind such induced independence for presheaves of LTS is that when actions are carried out in spatially disjoint regions (or components), they cannot interfere with each other; as such, their transition relations commute in several ways. The framework under which this commutativity for independent actions is realized is based on extended forms of LTS, where an independence relation is incorporated into the structure. Two extended LTS models of this kind have been studied by Winskel and Nielsen in [MC], and they are *asynchronous labelled transition systems* (ALTS)

⁴We always work with complete Heyting algebras in this thesis; i.e. whenever we use the terms “Heyting algebra” or “Boolean algebra”, we actually mean that it is complete.

⁵ and *labelled transition systems with independence* (LTSI). To properly take into account the induced independence of actions within presheaves of LTS however, we will need a third model of this kind, that is *asynchronous labelled transition systems with equivalence* (ALTSE). These ALTSE are a new kind of model that we explore in this thesis, and they are a sort of combination of ALTS and LTSI models.

As far as we are aware, there are three cases where forms of SI-independence were implemented. One such case was with M. Mukund and M. Nielsen [CATS], where an interpretation of the Calculus of Communicating Systems (CCS) as an ALTS was given. This was achieved by adjusting the labelling system to keep track of the locations where actions occur. Also, Lomuscio, Penczek, and Qu [PMAS] have presented a model of multi-agent systems where an action is localized by assigning a set of agents to it; i.e. the set of agents that perform this action as a whole. In this system, actions are independent when their respective sets of agents are disjoint. Finally, G. Malcolm [SSTS] has presented a form of SI-independence that works in the case of sheaves of (monoid) labelled transition systems. This is closer to what we intend to develop in this thesis. Basically, given any $\mathbb{T}$ -valued presheaf $\mathcal{T} : \mathcal{H}^{op} \rightarrow \mathbb{T}$ and a region U in a Heyting algebra $\mathcal{H}$, we can construct a form of independence relation $\mathcal{I}(U)$ on the set of actions “local” to U , and this $\mathcal{I}(U)$ can be described informally as follows :

Principle of SI-independence : For any actions b, c in the set of actions that are local to U , we say that b is *spatially independent* of c (written $b \mathcal{I}(U) c$) if there exists a cover $\{V, W\}$ of U such that :

1. b vanishes in W and c vanishes in V , and
2. b is contained in V and c is contained in W .⁶

For $b \mathcal{I}(U) c$, this means precisely that the actions b and c each find themselves contained in a region where the other action does not interfere. If the relation $\mathcal{I}(U)$

⁵These originate from the work of Shields [CMach] and Bednarczyk [CAS].

⁶That an action is contained in a region relates to a formal concept presented in Definition 4.3.15.

given above effectively provides an independence relation that respects the axioms of an ALTS (or ALTSE, or LTSI), then we say that the presheaf $\mathcal{T}$ in question has *spatially induced independence* (SI-independence).

Malcolm essentially gave this independence relation $\mathcal{I}(U)$ in [SSTS] and proposed that if $\mathcal{T}$ is a sheaf, then $\mathcal{T}$ has SI-independence⁷. Yet, there are very natural forms of distributed systems that can be represented by $\mathbb{T}$ -valued presheaves that are not sheaves, and establishing SI-independence in these weaker contexts would be quite beneficial. Indeed, there are many distributed systems in computer science that can be modeled as presheaves over discrete spaces (such as Petri Nets), and imposing the condition of a sheaf in such contexts is usually too strong. On the other hand, SI-independence does not arise in all cases of presheaves of LTS, and this leads us to the following line of enquiry :

Is there a weaker condition than that of a sheaf, under which presheaves of LTS acquire SI-independence?

This is the question that we address in this thesis, and our answer is a positive one. We will embody this weaker condition for which SI-independence occurs in the form of *$\mathbb{T}$ -adapted presheaves* as presented in Chapter 5. These $\mathbb{T}$ -adapted presheaves are essentially presheaves of LTS where the properties of locality and gluing for the transition relations of actions hold in a restricted number of cases. Our main theorem for the thesis then follows with respect to these $\mathbb{T}$ -adapted presheaves. The Theorem of Spatially Induced Independence provides, for any Heyting algebra $\mathcal{H}$, a functor from the category of $\mathbb{T}$ -adapted presheaves on $\mathcal{H}$ to the category of ALTSE-valued presheaves on $\mathcal{H}$ ⁸ that exploits the independence relation $\mathcal{I}(U)$ as presented above.

⁷The terminology of SI-independence is new to this thesis however, and the statement of Malcolm is simply given without this terminology.

⁸By an ALTSE-valued presheaves on $\mathcal{H}$, we mean a presheaf on $\mathcal{H}$ that takes values in the category of ALTSE. The category of ALTSE is explored in Section 3.3.

The theory of SI-independence would definitely have important consequences in computer science. From a mathematical standpoint, SI-independence formalizes, in the broad context of category theory, the idea of commutativity of local actions or transformations that are carried out in disjoint regions in space. In computer science, the SI-independence that we develop in this thesis has a definite potential when it comes to applying partial order reduction methods within a distributed system.

Formally establishing and understanding SI-independence is somewhat laborious, as we need to expand on the theory of several categories along the way. Also, in this thesis, we decided to provide the material that we thought was necessary to facilitate further research in the theory of presheaves of LTS and SI-independence. These two aspects put together explain the considerable size of this thesis.

As for the structure of this thesis, we start in Chapter 1 by setting up some preliminaries necessary to expand on the theory of labelled transition systems. We provide some notation for the relational language of the category $\mathbf{Rel}$ of sets and binary relations in Section 1.1, as we will often deal with relational equations involving transition relations in LTS. In the section that follows (1.2), we elaborate on a category, $\mathbf{Set}_\varepsilon$, that is equivalent to the category of sets and partial maps. The $\mathbf{Set}_\varepsilon$ category is convenient when it comes to specifying LTS and their morphisms as we will see.

In Chapter 2, we set the ground for the thesis by laying down the notion of a labelled transition system. A good portion of the thesis is attributed to this chapter, as it is necessary to understand the LTS model properly if we are going to deal with presheaves of such structures later on. We investigate the category of labelled transition systems, $\mathbb{T}$, and some of its basic constructions in Section 2.1. In particular, we provide a new result that $\mathbb{T}$ is bicomplete in that section. Afterwards, we elaborate on an example of a product of registers in detail in Section 2.2, and we get a glimpse of the subject of presheaves of LTS in the process. In Section 2.3, we cover some theory relating to subobjects and we give a proof that $\mathbb{T}$ has no subobject classifier. Afterwards, we take a look at *monoidal labelled transition systems* (MLTS)

in Section 2.4. Doing so will make it easier to synchronize with the work of Malcolm and, also, it will facilitate certain forms of quotienting on sequences of actions that we will perform later on. We will exhibit an adjunction between $\mathbb{T}$ and the category of MLTS in this section as well. Sections 2.3 and 2.4 are in fact optional and do not affect the development of the core theory for this thesis; the reader may skip them in first reading if he is willing to accept the informal presentations mentioned in the text.

We then explore, in Chapter 3, LTS models with independence relations on their labels, i.e. asynchronous labelled transition systems (ALTS) and asynchronous labelled transition systems with equivalence (ALTSE). The ALTS model builds on top of the *concurrent alphabet* model, and we study the latter in Section 3.1 to begin with. Then, the ALTS model is covered in Section 3.2, and a functor is given from the category of ALTS to the category MLTS that exploits Mazurkiewicz’s *abstract trace equivalence* and *trace monoid* (from [TT]) in the process. In Section 3.3, we elaborate on ALTSE, and these are essentially the kind of structure that we want to recover within presheaves of LTS when applying the Theorem of Spatially Induced Independence.

The core subject matter of this thesis is initiated in Chapter 4 as we explore presheaves of LTS in a spatial sense. Our focus there is definitely on the theory that will help us find the minimal conditions that a presheaf of LTS requires in order to have SI-independence. In Section 4.1, we introduce presheaves of LTS and the idea of *action δ -gluing* (or *gluing of transitions for an action*). δ -gluing is a concept that we elaborated to allow a specific action a in a region U to decompose over a specific cover $\{V_j\}_{j \in J}$ of U , and this means that local transitions associated to the projections of a in each V_j can be glued together to provide global transitions for a . In Section 4.2, we investigate $\mathbb{T}$ -valued sheaves and how δ -gluing occurs indiscriminately in such sheaves. In Section 4.3, we describe how regions can contain effects and dependencies of actions (see Definitions 4.3.1 and 4.3.7), and how a regulated application of δ -gluing effectively yields regions that *contain actions* in the sense of containing both the effects and the dependencies of an action simultaneously. This will be instrumental in

defining the *well-contained actions* (WCA) axiom (Definition 4.3.19) to be imposed on presheaves of LTS. We will get $\mathbb{T}$ -adapted presheaves and SI-independence as such.

Finally, in Chapter 5, we will expand on the theory of $\mathbb{T}$ -adapted presheaves. There are various contexts in which SI-independence can be studied, and in Section 5.1, we start with the case of $\mathbb{T}$ -adapted presheaves on Heyting algebras in general. We emphasize the theory that will allow us to prove the Theorem of Spatially Induced Independence in the section that follows (Section 5.2). In Section 5.3, we will see that the theory established previously in Section 5.1 is somewhat simplified when we deal with $\mathbb{T}$ -adapted presheaves on Boolean algebras. However, we will develop an extended form of SI-independence in the process (see Definition 5.3.13 for $\mathcal{I}^+$), that takes more cases of spatially independent actions into account. The trade-off is that we do not get functors to render this extended form of SI-independence, as was achieved in the Theorem of Spatially Induced Independence of Section 5.2.

In the last chapter (6), we elaborate on *localized relational structures*, which are models (new to this thesis) that serve as intermediaries in constructing specific kinds of $\mathbb{T}$ -adapted presheaves (on discrete spaces). These are quite convenient because it is difficult in general to construct presheaves of LTS in an ad hoc fashion; we need to have efficient methods to provide examples of such presheaves. We describe the association of $\mathbb{T}$ -adapted presheaves to localized relational structures in Section 6.1, and we study two examples of LRS in the sections that follow : we look at Kings and Rooks on a Chessboard in Section 6.2 and Concurrent Unlimited Register Machines in Section 6.3.

Finally, in the appendix, we have added subsidiary proofs and a discussion (Appendix H) on a problem with the abstract trace equivalences that arise within adapted presheaves of LTS. It so happens that these equivalences do not match all the sequences of actions that should intuitively be matched. Through this discussion, we propose a solution that extends the abstract trace equivalence in question to provide more intuitive quotienting on sequences of actions within presheaves of LTS.

The next logical step to be taken with respect to our study would then be to link SI-independence to formal methods in computer science. That it has applications in partial order reduction methods, there is no doubt. But we can also look towards the work of Goguen [HAg], [HHTh], [HCoC] and Cirstea [SLAC], [CSHA], who have performed research in specification logics for object oriented systems. They basically use representation of systems that combine algebraic and coalgebraic features, with Goguen's Hidden Algebra mainly, and since there is a correspondence between LTS and coalgebras, it may be possible to apply the theory of SI-independence within such Hidden Algebras.

Chapter 1

Preliminaries

1.1 Notation for the relational language of Rel

When one specifies a labelled transition system, one deals with transition relations often, and it is convenient to use relational equations in the **Rel** category to provide such specifications.

Definition 1.1.1 [Category of sets and relations]. The category **Rel** uses sets as objects and binary relations as morphisms. More specifically, given sets A and B , we define a *relational morphism* $R : A \rightarrowtail B$ as a relation $R \subseteq A \times B$.

Two relations are equal whenever they are equal as sets (subsets of a cartesian product of sets). The composition of two relations $R : A \rightarrowtail B$ and $S : B \rightarrowtail C$ is given by :

$$S \circ R = \{ (x, z) \in A \times C \mid \exists y \in B, (x, y) \in R \text{ and } (y, z) \in S \}$$

The identity morphism on a set A is given by the diagonal relation on A , which is :

$$\Delta_A = \{ (x, x) \mid x \in A \}$$

Most of the time, these relations will reflect either transition relations or restrictions on some state space in this thesis. As such, whenever we have a pair

$(X, Y) \in R_m \circ \dots \circ R_1$, this means that there exists a path from state X to state Y that is provided by the successive firing of R_1 to R_m , i.e. there exists intermediary states X_i , with i ranging from 0 to m , such that $X_0 = X$ and $X_m = Y$ and $(X_{i-1}, X_i) \in R_i$ for all i from 1 to m . We represent such a case by using the following path notation :

$$X = X_0 \xrightarrow{R_1} X_1 \xrightarrow{R_2} X_2 \dots X_{m-1} \xrightarrow{R_m} X_m = Y^1$$

and we make the observation that this reverses the order in which the R_i are presented through relational composition.

The Rel category has boolean algebra structure on its hom-sets, i.e. $\mathbf{Hom}_{\mathbf{Rel}}(A, B) = \mathcal{P}(A \times B)$ for sets A and B , and powersets have boolean algebra structure when using set operations : inclusion $\subseteq$, intersection $\cap$, union $\cup$, complement $\overline{(-)}$, and $\emptyset$ and $A \times B$ for the minimal and maximal relations respectively.

The statement that $R_m \circ \dots \circ R_1 \subseteq S_n \circ \dots \circ S_1$ holds, is equivalent to saying that for every path $X = X_0 \xrightarrow{R_1} X_1 \xrightarrow{R_2} X_2 \dots X_{m-1} \xrightarrow{R_m} X_m = Y$ (with the R_i relations), there exists an alternate path $X = Y_0 \xrightarrow{S_1} Y_1 \xrightarrow{S_2} Y_2 \dots Y_{n-1} \xrightarrow{S_n} Y_n = Y$ with the same end points X to Y , that uses the S_i relations instead. Hence, to think in terms of paths is quite convenient, and probably the clearest way to think about relational composition and associated equations.

The composition of relations forms a congruence with respect to inclusion, i.e. for any pairs of relations $R_0, R : A \rightarrow B$ and $S_0, S : B \rightarrow C$ we have :

$$R_0 \subseteq R \text{ and } S_0 \subseteq S \Rightarrow S_0 \circ R_0 \subseteq S \circ R$$

¹For a given binary relation R , we use two different notations to represent the statement $(X, Y) \in R$ depending on the situation. There is the usual notation with $X R Y$, but we will also use $X \xrightarrow{R} Y$ to facilitate reading in certain parts of the text.

There is also an inverse map $(-)^{-1} : \text{Hom}_{\text{Rel}}(A, B) \rightarrow \text{Hom}_{\text{Rel}}(B, A)$ on relations, and it is defined as :

$$Y \stackrel{R^{-1}}{\rightarrow} X \Leftrightarrow X \stackrel{R}{\rightarrow} Y$$

for any $X \in A$ and $Y \in B$. This operation is an involution in the sense that $(R^{-1})^{-1} = R$ and $(S \circ R)^{-1} = R^{-1} \circ S^{-1}$ whenever R and S compose. Also, we have that $R \subseteq S$ implies $R^{-1} \subseteq S^{-1}$.

For a relation $R : A \rightarrowtail B$ and subsets $A_0 \subseteq A$ and $B_0 \subseteq B$, we write

$$R(A_0) = \{ Y \in B \mid \exists X \in A_0, (X, Y) \in R \}$$

to denote the *image* of A_0 under R . This usually represents a set of states reachable from those in A_0 through the one step firing of R . We can denote the *preimage* of B_0 under R by using the image of the inverse relation on B_0 , that is, $R^{-1}(B_0)$.

We say that R is *functional* if $x R y$ and $x R y'$ imply $y = y'$, and we say that R is *total* if $R^{-1}(B) = A$. When R is total and functional, we call R a function. We say that R is *injective* if R^{-1} is functional, and we say that R is *surjective* if R^{-1} is total. There are useful relational inclusions that are equivalent to these concepts, as depicted in the following table :

$R : A \rightarrowtail B$	
R is functional	$R \circ R^{-1} \subseteq \Delta_B$
R is total	$\Delta_A \subseteq R^{-1} \circ R$
R is injective	$R^{-1} \circ R = \Delta_A$
R is surjective	$\Delta_B = R \circ R^{-1}$

Finally, there are some distribution rules for composition over intersections and unions (whenever they are defined) we can work with :

$$\begin{aligned} (R \cap S) \circ P &\subseteq (R \circ P) \cap (S \circ P) \\ P \circ (R \cap S) &\subseteq (P \circ R) \cap (P \circ S) \\ (R \cup S) \circ P &= (R \circ P) \cup (S \circ P) \\ P \circ (R \cup S) &= (P \circ R) \cup (P \circ S) \end{aligned}$$

Finally, the inverse operation distributes over intersections and unions :

$$\begin{aligned} (R \cup S)^{-1} &= R^{-1} \cup S^{-1} \\ (R \cap S)^{-1} &= R^{-1} \cap S^{-1} \end{aligned}$$

There is an embedding from the category **Set** into the category **Rel**, that preserves the sets as objects, and that sends functions $f : A \rightarrow B$ to their associated graph $gr(f) = \{(x, f(x)) \mid x \in A\}$. Very often, we will deal with universal properties and constructions in **Set**, but we will work out equations in **Rel**. This is because in a presheaf of LTS, it is convenient to provide equations that combine spatial restrictions of state spaces (which are functions), with transition relations that operate on these state spaces (and which are, properly speaking, relations). Thus, we will abuse notation here and consider functions directly as relations through this embedding. For example, for any function $f, g : A \rightarrow B$, we will allow ourselves to write $f \cap g$, $f \cup g$, f^{-1} and composition in the relational language as if we were manipulating the associated graphs of these functions.

This provides the material needed to facilitate the treatment of relational equations. We also need some notation for labelling sets and labelling morphisms, which is the subject of the following section.

1.2 A representation of the category of sets and partial maps

We will work with a specific representation of the category of sets and partial maps, as presented by Winskel and Nielsen in [MC], that has a convenient format when it comes to describing the category of labelled transition systems. More specifically, this category will prove itself useful in describing the labelling set component of a labelled transition system, and the labelling morphisms that work as components in labelled transition system morphisms.

Definition 1.2.1 [Category of Labelling Sets (Winskel, Nielsen [MC])].

Let ε be a distinguished symbol. Consider the category $\mathbf{Set}_\varepsilon$, defined as follows :

- Objects : Sets that do not contain the distinguished symbol ε .
- Morphisms : A morphism $f : L \rightarrow_\varepsilon L'$ in $\mathbf{Set}_\varepsilon$ is a total set function $f : L \cup \{\varepsilon\} \rightarrow L' \cup \{\varepsilon\}$ such that $f(\varepsilon) = \varepsilon$.

Equality of morphisms is provided precisely when the corresponding total set functions are equal. Composition and identity morphisms are inherited from the category of sets and functions as well.

Remark 1.2.2 . In this section, for a given labelling set L , we will write L_ε to designate the set $L \cup \{\varepsilon\}$.

In the context in which we shall use the category $\mathbf{Set}_\varepsilon$ in this thesis, we will refer to the objects of $\mathbf{Set}_\varepsilon$ as *labelling sets*, and we will refer to morphisms in this category as *labelling morphisms*.

Clearly, the category $\mathbf{Set}_\varepsilon$ as defined is equivalent to the category of pointed sets, which is itself equivalent to the category of sets and partial maps. As such, $\mathbf{Set}_\varepsilon$ is equivalent to the Kleisli category of the $- + 1$ monad on $\mathbf{Set}$ (see Cockett and Lack [RC1], [RC2] and [RC3]). One should have no trouble translating the concepts on

which we shall develop using $\mathbf{Set}_\varepsilon$ to the more conventional representation of the category of sets and partial maps if one prefers to think in these terms. Basically, for a morphism $f : L \rightarrow_\varepsilon L'$, an equation of the form $f(a) = \varepsilon$ for $a \in L$ means that $f(a)$ is *undefined*. If $f(a) \neq \varepsilon$, then we say that $f(a)$ is *defined*.

There is an obvious embedding of $\mathbf{Set}_\varepsilon$ into $\mathbf{Rel}$, which preserves the sets as objects, and sends a morphism $f : L \rightarrow_\varepsilon L'$ to the graph of f , given by $gr(f) = \{(x, f(x)) \mid x \in L \text{ and } f(x) \neq \varepsilon\}$. This embedding generalizes the gr functor from $\mathbf{Set}$ to $\mathbf{Rel}$, as long as we consider sets that do not contain the distinguished ε symbol.

Once again, we will allow, for morphisms $f, g : L \rightarrow_\varepsilon L'$, a direct interpretation as relations through this embedding, and we will write things like $f \subseteq g$, or $f \cap g$, f^{-1} , etc., which of course, sometimes only makes sense in $\mathbf{Rel}$. Finally, we will say that a labelling morphism is partial (in the proper sense of the word), if it is functional but not total as a relation. (The proofs of the following two propositions are straightforward and we omit them in the final version of this thesis.)

Proposition 1.2.3 . *Consider any arrow $m : L \rightarrow_\varepsilon L'$ in $\mathbf{Set}_\varepsilon$. Then,*

$$m \text{ is a monomorphism in } \mathbf{Set}_\varepsilon \Leftrightarrow m : L_\varepsilon \rightarrow L'_\varepsilon \text{ is an injective map in } \mathbf{Set}$$

Proposition 1.2.4 . *A labelling morphism $\lambda : L \rightarrow_\varepsilon L'$ is an isomorphism if and only if it is a bijective map.*

And we will need some categorical constructions as well.

Definition 1.2.5 [Constructions in $\mathbf{Set}_\varepsilon$]. We provide the necessary constructions for a bicomplete category as follows :

1. **Zero object :** $\mathbf{Set}_\varepsilon$ has a zero object provided by the empty set $\emptyset$. This is because there is a single function from any set into the set $\emptyset_\varepsilon = \{\varepsilon\}$, and there is a single function out of the set $\emptyset_\varepsilon = \{\varepsilon\}$ that preserves ε into any other set containing ε .

2. **Indexed Products** : Given a J -indexed family of objects L_j in $\mathbf{Set}_\varepsilon$, we define the J -indexed product $\prod_{j \in J}^\varepsilon$ in $\mathbf{Set}_\varepsilon$ as :

$$\prod_{j \in J}^\varepsilon L_j := \prod_{j \in J} (L_j \cup \{\varepsilon\}) \setminus \{ a \in \prod_{j \in J} (L_j \cup \{\varepsilon\}) \mid \forall i \in J, \pi_i(a) = \varepsilon \}^2$$

where $\prod_{j \in J} (L_j \cup \{\varepsilon\})$ denotes the J -indexed cartesian product in $\mathbf{Set}$ with standard projection maps $\pi_i : \prod_{j \in J} (L_j \cup \{\varepsilon\}) \rightarrow L_i$ for each i in J .

For $\prod_{j \in J}^\varepsilon L_j$, we use the labelling projection maps $\pi_i^\varepsilon : \prod_{j \in J}^\varepsilon L_j \rightarrow_\varepsilon L_i$, for i in J , as :

$$\pi_i^\varepsilon(b) = \begin{cases} \pi_i(b) & \text{if } b \neq \varepsilon \\ \varepsilon & \text{otherwise} \end{cases}$$

for any $b \in (\prod_{j \in J}^\varepsilon L_j) \cup \{\varepsilon\}$.

3. **Indexed Coproducts** : Given a J -indexed family of objects L_j in $\mathbf{Set}_\varepsilon$, we define the J -indexed coproduct $\prod_{j \in J}^\varepsilon$ in $\mathbf{Set}_\varepsilon$ as :

$$\prod_{j \in J}^\varepsilon L_j := \prod_{j \in J} L_j$$

which is the J -indexed coproduct of the sets L_j in $\mathbf{Set}$ directly, using $\iota_i : L_i \rightarrow \prod_{j \in J} L_j$ as the canonical injection maps.

The labelling injection morphisms for $\prod_{j \in J}^\varepsilon L_j$ are provided by $\iota_i^\varepsilon : L_i \rightarrow_\varepsilon \prod_{j \in J}^\varepsilon L_j$, for i in J , as :

$$\iota_i^\varepsilon(b) = \begin{cases} \iota_i(b) & \text{if } b \neq \varepsilon \\ \varepsilon & \text{otherwise} \end{cases}$$

for any $b \in L_i \cup \{\varepsilon\}$.

Also, the ι_i^ε are monomorphisms for each i in J .

²We assume that $\varepsilon \notin \prod_{j \in J}^\varepsilon L_j$, i.e. ε does not label a function in $\prod_{j \in J}^\varepsilon L_j$. We will make similar assumptions of $\varepsilon \notin \prod_{j \in J}^\varepsilon L_j$, and so on for other sets that we will construct this way in $\mathbf{Set}_\varepsilon$.

4. **Equalizers** : Given a diagram $f, g : L \rightarrow_\varepsilon L'$ in $\mathbf{Set}_\varepsilon$, we define $E = \{ a \in L \mid f(a) = g(a) \} \subseteq L$ as an object in $\mathbf{Set}_\varepsilon$, and we use the inclusion map from E_ε into L_ε as the equalizer of f and g in $\mathbf{Set}_\varepsilon$.
5. **Coequalizers** Given a pair of morphisms $f, g : L \rightarrow_\varepsilon L'$ in $\mathbf{Set}_\varepsilon$. Take the coequalizer $q : L'_\varepsilon \rightarrow K$ of the maps $f, g : L_\varepsilon \rightarrow L'_\varepsilon$ in $\mathbf{Set}$ (we assume $\varepsilon \notin K$). Construct the set $Q = K \setminus \{q(\varepsilon)\}$. There is an obvious bijection from $\theta : K \rightarrow Q_\varepsilon$, which simply relabels $q(\varepsilon)$ to ε with

$$\theta(b) = \begin{cases} b & \text{if } b \neq q(\varepsilon) \\ \varepsilon & \text{otherwise} \end{cases}$$

for any $b \in K$.

We have that $\theta \circ q$ is a labelling morphism from L' to Q , and it is in fact, a coequalizer of f, g in $\mathbf{Set}_\varepsilon$.

Through the existence of such constructions, we can conclude that $\mathbf{Set}_\varepsilon$ has all limits and colimits.

Proposition 1.2.6 . $\mathbf{Set}_\varepsilon$ is bicomplete³

Proof. See Cockett and Lack [RC1], [RC2] and [RC3]. □

Finally, we need to provide a functor that creates freely generated monoids from labelling sets, and that expands labelling morphisms to monoid morphisms. This functor from $\mathbf{Set}_\varepsilon$ to $\mathbf{Mon}$ will be useful in expanding labelled transition systems to their monoid labelled format.

Definition 1.2.7 [**Functor from $\mathbf{Set}_\varepsilon$ to $\mathbf{Mon}$**].

Define a functor $(-)^* : \mathbf{Set}_\varepsilon \rightarrow \mathbf{Mon}$ as follows :

1. Given an object $L \in \mathbf{Set}_\varepsilon$, we define L^* as the freely generated monoid over L , and we write $\Lambda \in L^*$ for the empty word.

³This category is in fact bicomplete in a 2-categorical sense. We refer the reader to the work of Cockett and Lack, as indicted under the proposition.

2. Given a labelling morphism $\lambda : L_0 \rightarrow_\varepsilon L$, we define $\lambda^* : L_0^* \rightarrow L^*$ inductively as follows :

$$\lambda^*(\Lambda) = \Lambda$$

$$\lambda^*(wb) = \begin{cases} \lambda^*(w)\lambda(b) & \text{if } \lambda(b) \neq \varepsilon \\ \lambda^*(w) & \text{otherwise} \end{cases}$$

for any $w \in L_0^*$ and $b \in L_0$.

Remark 1.2.8 . It follows from this definition, that for any labelling morphism $\lambda : L_0 \rightarrow_\varepsilon L$ and any $a_1 \dots a_n \in L_0^*$ for some actions a_i in L_0 , that $\lambda^*(a_1 \dots a_n) = \lambda^*(a_1) \dots \lambda^*(a_n)$.

Proposition 1.2.9 . $(-)^* : \mathbf{Set}_\varepsilon \rightarrow \mathbf{Mon}$ is indeed a functor.

Proof. The proof is to be found in Appendix A. □

This clearly generalizes the adjunction from **Set** to **Mon** that sends a set to the freely generated monoid on that set (as long as we use sets that don't contain the ε symbol).⁴

Definition 1.2.10 [Functor from Mon to Set_ε]. We define a functor $G_{m\varepsilon} : \mathbf{Mon} \rightarrow \mathbf{Set}_\varepsilon$. Given monoids M and M' with neutral elements e and e' respectively (and ε not in M, M' , otherwise relabeled à priori), set $G_{m\varepsilon}(M) = (M \setminus \{e\})$ and for a monoid morphism $f : M \rightarrow M'$, set $G_{m\varepsilon}(f) : M \setminus \{e\} \rightarrow_\varepsilon M' \setminus \{e'\}$ in $\mathbf{Set}_\varepsilon$ as follows :

$$G_{m\varepsilon}(f)(\varepsilon) = \varepsilon$$

$$G_{m\varepsilon}(f)(b) = \begin{cases} f(b) & \text{if } f(b) \neq e' \\ \varepsilon & \text{otherwise} \end{cases}$$

⁴We also assume a version of **Mon** where none of the underlying sets of the monoids contain the distinguished symbol ε

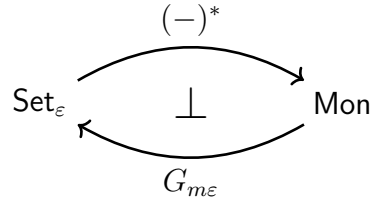
for any $b \in M \setminus \{e\}$.

Proposition 1.2.11 . $G_{m\varepsilon} : \mathbf{Mon} \rightarrow \mathbf{Set}_\varepsilon$ is indeed a functor.

Proof. The proof is to be found in Appendix A. □

And we can establish an adjunction between $(-)^*$ and $G_{m\varepsilon}$.

Proposition 1.2.12 . $(-)^*$ is the left adjoint of $G_{m\varepsilon}$.



Proof. The proof is to be found in Appendix A. □

We have now covered the necessary preliminaries and we can begin with the study of labelled transition systems.

Chapter 2

Labelled Transition Systems

A transition system is essentially a binary relation on a set of states that portrays how a system can evolve from one state to another. These transition systems are well-known models in computer science for the representation of processes and computational models, and they are commonly used to describe the operational semantics in calculi such as π -calculus or λ -calculus¹. The definition is as follows :

Definition 2.0.1 . A *transition system* is a pair $(S, \longrightarrow)$ where S is a set of states and $\longrightarrow \subseteq S \times S$ is a binary relation on S , called the *transition relation*. A pair $(X, Y) \in \longrightarrow$ is called a *transition*, and we write $X \longrightarrow Y$ for such a pair to mean that X can evolve to Y in the transition system.

We may recognize the definition of a simple directed graph here, where vertices are states and arrows are transitions in $\longrightarrow$. The choice of the term “transition system” as opposed to “simple directed graph” is a purely conceptual one. One other useful way of thinking about transition systems, for use in modal or temporal logic, is as relational structures, or Kripke frames : the worlds in question are the states and the accessibility relation is given by $\longrightarrow$.

¹For a good introduction to π -calculus, one can look into D. Sangiorgi and D. Walker’s book *The π -calculus : A Theory of Mobile Processes* [PiMob].

Now, transition systems are not always convenient to describe computational models in general because they are a bit too abstract. In practice, we want to be able to classify the activity within a transition system and make specifications about designated subsystems. A way to achieve this is by adding labels to transitions; as such, we obtain a *labelled transition system* as an indexed family of transition relations that perform together over a common universe of states. Labelled transition systems are classic models for representing computational systems. The objective of this chapter is to familiarize ourselves with their basic theory, so that we will be able to handle the categories of presheaves of labelled transition systems in Chapter 4 and 5. We do not undertake a full categorical treatment of labelled transition systems here. However, we certainly encourage the interested reader in developing the theory of the category of LTS in more detail, especially in matters relating to bifibration of the latter.

2.1 Category of Labelled Transition Systems

There is a variety of possible definitions for labelled transition systems in the literature. In this thesis, we use the definition from Winskel and Nielsen's *Models for Concurrency* [MC], with the exception that we do not integrate an initial state² for the system into the signature.

Definition 2.1.1 [Labelled Transition System].

A *labelled transition system* (LTS) is a structure $T = (S, L, \delta)$ where :

- S is a set of *states*;
- L is a set in $\mathbf{Set}_\varepsilon$ (i.e. a set that does not contain the distinguished ε symbol), and L 's elements are referred to as *labels* (or *actions*);
- $\delta \subseteq S \times L \times S$ is the *transition specifier*³ of the system.

²When such an initial state is incorporated, we get *pointed labelled transition systems*.

³This “ δ ” is usually called the transition relation. We gave it another name to facilitate the distinction with the transition relations associated to labels (Definition 2.1.5) and the underlying

A triple $(X, b, Y) \in \delta$ is called a *transition*, and we often write $X \xrightarrow{b}_T Y$ for such a triple to mean that the state X can evolve to the state Y through the action b in T .

Example 2.1.2 . We consider a labelled digraph as an example of a LTS as follows :

- States : $S = \{X, Y, Z, W\}$;
- Labels : $L = \{a, b, c\}$;
- Transitions : $\delta = \{(X, a, Y), (X, b, Z), (X, c, Z), (Y, c, Y), (Y, a, X), (Y, a, Z), (Z, c, W)\}$

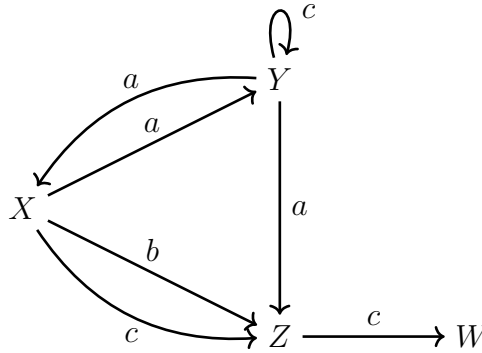
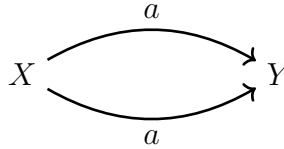


Figure 1: A labelled digraph example of a LTS

We remark that, for LTS in general, it is possible to use the same label for different transitions, as for (X, a, Y) , (Y, a, X) and (Y, a, Z) for example. However, it is not possible to have multiple transitions sharing the same source and target, i.e. we could not have something like :



We can more easily project how processes evolve inside such labelled transition systems if we accompany the latter with a notion of run, and the definition of such runs goes as follows :

transition relation of a LTS (Definition 2.1.8)

Definition 2.1.3 [Finite Run]. Given a labelled transition system $T = (S, L, \delta)$, a finite T -run is a connected sequence of transitions in T as follows :

$$X_0 \xrightarrow{a_1}_T X_1 \xrightarrow{a_2}_T X_2 \dots X_{n-1} \xrightarrow{a_n}_T X_n$$

i.e. where $(X_i, a_{i+1}, X_{i+1}) \in \delta$ for all i .

Thus, for the labelled digraph of Example 2.1.2, we get an example of a run as follows :

$$X \xrightarrow{a} Y \xrightarrow{c} Y \xrightarrow{a} X \xrightarrow{b} Z \xrightarrow{c} W$$

This run represents the following line of behavior : a process starts in a state X , then performs the action a to evolve to Y . It then performs the action c and stays in Y , and proceeds as depicted from one state to another in a sequential fashion. Upon reaching the state Z , the system has no choice but to evolve deterministically to W through the action c . Once in W , the system is in a deadlock state, since it can no longer perform actions to evolve out of that state according to our LTS specification.

Now, describing the transition specifier δ of a LTS directly isn't always convenient when specifying transition systems, and we should adopt other means of providing such specifications. We can think of a LTS as an amalgamation of transition systems (relational structures) that are indexed by the labelling set. To be more precise, we can create a correspondence between LTS and one object diagrams in **Rel**, and to formalize this properly, we need to define graphs and diagrams first :

Definition 2.1.4 [Graphs and Diagrams].

A *graph* is a tuple (V, E, dom, cod) where V is a set of vertices, E is a set of edges, $dom : E \rightarrow V$ is a map that assigns a source vertex to each edge, and $cod : E \rightarrow V$ is a map that assigns a target vertex to each edge.

A *graph morphism* from (V, E, dom, cod) to (V', E', dom', cod') is a pair of maps $(f : V \rightarrow V', g : E \rightarrow E')$ such that $dom' \circ g = f \circ dom$ and $cod' \circ g = f \circ cod$.

Given a graph J and a category $\mathcal{C}$, a *diagram of shape J* in $\mathcal{C}$ is a graph morphism from $D : J \rightarrow U(\mathcal{C})$ where $U(\mathcal{C})$ denotes the underlying graph of the category $\mathcal{C}$.

And the characterization of a LTS as a relational diagram goes as follows :

Definition 2.1.5 [Relational Diagram and Transition Relations].

Given a labelled transition system $T = (S, L, \delta)$, and using the graph

$$J = (\{\star\}, L, \text{dom} : L \rightarrow \{\star\}, \text{cod} : L \rightarrow \{\star\})$$

with one vertex $\star$, and labels L as arrows, we define the *relational diagram of T* as a diagram of shape J in $\mathbf{Rel}$, denoted $\xrightarrow{(-)}_T : J \rightarrow U(\mathbf{Rel})$, by setting :

On the vertex : $\xrightarrow{\star}_T := S$ as the set of states of the LTS, and

On the arrows : $\xrightarrow{b}_T := \{(X, Y) \in S^2 \mid (X, b, Y) \in \delta\}$ for any label $b \in L$.

For a label $b \in L$, we refer to $\xrightarrow{b}_T$ as the *transition relation associated to b* .

There is in fact a bijective correspondence between one object diagrams in $\mathbf{Rel}$ and labelled transition systems, i.e. we can recover a LTS by setting $S = \xrightarrow{\star}_T$, and defining L as the set of arrows at the source of the relational diagram. The transition specifier is then provided by $\delta = \{(X, b, Y) \in S \times L \times S \mid \exists b \in L, X \xrightarrow{b}_T Y\}$.

Personally, I find that the representation of a labelled transition system as a one object relational diagram is more natural than its representation as a tuple (S, L, δ) . This is simply because specifying LTS through a relational diagram and through relational equations is usually much more convenient than getting involved with the transition specifier δ , where no notion of composition is provided, and where one is forced to describe the set of triples $(X, a, Y) \in S \times L \times S$ in an ad hoc fashion. In fact, most of the specifications we provide throughout this thesis make use of the relational diagram of a LTS, and the following example demonstrates how this is done. In this example, we use the notation $\oplus$ and $\neg$ to symbolize the XOR and NOT operation on boolean values respectively, whereas $\wedge$ and $\vee$ will denote the AND and OR operations.

Example 2.1.6 [An n -bit Register with Operations]. Consider a register labelled transition system $\text{Reg} = (S, L, \delta)$ where

- $S = \{(x_n, \dots, x_1) \mid x_i \in \{0, 1\} \text{ for any } i \in \{1, \dots, n\}\}$; ⁴
- $L = \{\text{read}, \text{write}, \text{shift}, \text{clear}, \text{incr}\}$;
- δ is specified via Reg 's relational diagram $\xrightarrow{\text{Reg}}^{(-)}$, and

$\forall (x_n, \dots, x_1), (y_n, \dots, y_1) \in S :$

$$(1) (x_n, \dots, x_1) \xrightarrow{\text{clear}}_{\text{Reg}} (y_n, \dots, y_1) \Leftrightarrow \forall i, y_i = 0$$

$$(2) (x_n, \dots, x_1) \xrightarrow{\text{incr}}_{\text{Reg}} (y_n, \dots, y_1) \Leftrightarrow$$

$$\forall i \in \{1, \dots, n-1\}, y_{i+1} = (x_i \wedge \neg y_i) \oplus x_{i+1} \text{ and } y_1 = \neg x_1$$

$$(3) (x_n, \dots, x_1) \xrightarrow{\text{shift}}_{\text{Reg}} (y_n, \dots, y_1) \Leftrightarrow$$

$$\forall i \in \{1, \dots, n-1\}, y_{i+1} = x_i \text{ and } y_1 = 0$$

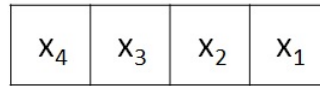
$$(4) (x_n, \dots, x_1) \xrightarrow{\text{read}}_{\text{Reg}} (y_n, \dots, y_1) \Leftrightarrow \forall i \in \{1, \dots, n\}, y_i = x_i$$

$$(5) (x_n, \dots, x_1) \xrightarrow{\text{write}}_{\text{Reg}} (y_n, \dots, y_1)$$

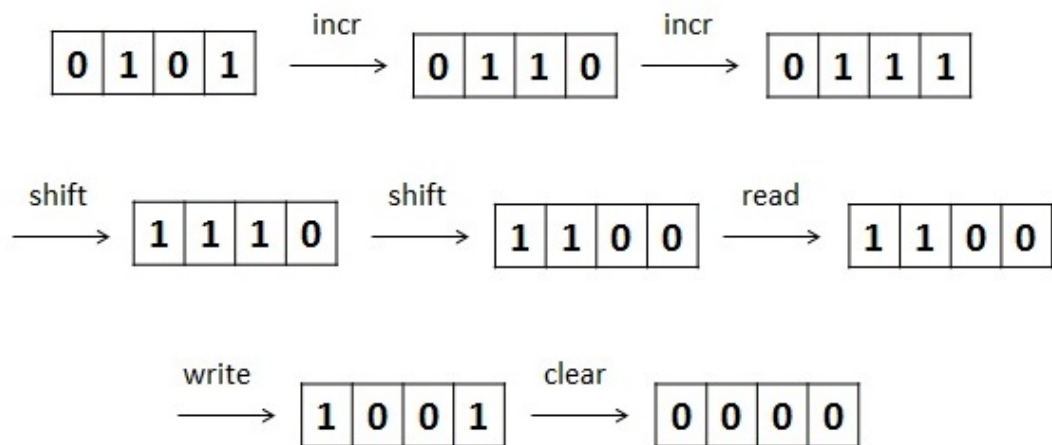
We can specify *read* and *write* operations more directly by setting $\xrightarrow{\text{read}}_{\text{Reg}} = \Delta_S$ and $\xrightarrow{\text{write}}_{\text{Reg}} = S \times S$.

⁴We write $(x_n, \dots, x_1)$ with x_1 as the most significant bit in the register.

Let $n = 4$. To represent the state $(x_4, x_3, x_2, x_1) \in S$, we use the following pictorial representation :



And we can visualize how the operations are performed on a 4-bit register through the following run :



Another example that we will encounter often is the following buffer :

Example 2.1.7 [Buffer (with two places)]. Consider a buffer labelled transition system $\text{Buff} = (S, L, \delta)$ where

- $S = \{(x, y) \mid x, y \in \{0, 1\}\};$
- $L = \{\text{input}, \text{send}\};$

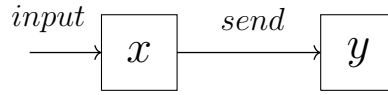


Figure 2: A representation of a buffer with two places, with an *input* action that substitutes the x value in the first place randomly, and with a *send* action that substitutes the value y in the second place with the value x in the first place.

- δ is specified via Buff 's relational diagram $\xrightarrow{(-)}_{\text{Buff}}$,

$\xrightarrow{\text{input}}_{\text{Buff}}$ is the smallest transition relation on S such that :

$$\forall x, x', y \in \{0, 1\}, \quad (x, y) \xrightarrow{\text{input}}_{\text{Buff}} (x', y)$$

Thus, any value can replace the first component x with the *input* action.

$\xrightarrow{\text{send}}_{\text{Buff}}$ is the smallest transition relation on S such that :

$$\forall x, y \in \{0, 1\}, \quad (x, y) \xrightarrow{\text{send}}_{\text{Buff}} (x, x)$$

And the value x is transferred to the second component by the *send* action.

We will encounter a few variants of this buffer later on when we look at spatial presheaves of LTS : a Buffer on a Discrete Space (Example 4.1.7), a Pullback Buffer Junction (Example 4.2.5), and a Discrete Space Buffer Junction (Example 5.3.12). The buffer examples are simple and illustrate the difference between a place of “dependency” and a place of “effect” for an action (these notions will be studied formally in Section 4.3). Mainly, the *send* action depends only on the value x in the first place

in order to perform, and it only has effects in the second place (where it changes the y value). The *input* action, on the other hand, does not depend on the value of any place to perform, and it simply has an effect in the first place as it changes the x value randomly.

The reason we will see so many examples of buffers, is to illustrate the consequences of dealing with Heyting algebra spaces (as provided in the Pullback Junction example mainly), in contrast to using Discrete spaces for the other examples, as far as dependencies and effects are involved.

Now, it will be useful to refer to the underlying transition system of a LTS when we talk about path shapes later on, and the definition is as follows :

Definition 2.1.8 [Underlying Transition System]. Given a LTS $T = (S, L, \delta)$ we define the *underlying (unlabelled) transition system of T* as the transition system $(S, \longrightarrow_T)$ where $\longrightarrow_T = \bigcup_{b \in L} (\overset{b}{\longrightarrow}_T)$. We often refer to $\longrightarrow_T$ as the *underlying transition relation of T* .

These underlying transition systems for LTS can come in handy if we wish to specify behavioral properties for the latter through temporal logics like LTL and CTL (because the latter require unlabeled transition systems for their semantics). Also, $\longrightarrow_T$ is quite useful when we want to describe certain patterns or shapes that the underlying transition system of a LTS should have. For example, we may want to work specifically with a class of LTS where the underlying transition systems are trees, sequential paths, posets, etc.⁵

We use $\longrightarrow_T^\bullet$ to talk about the transitive and reflexive closure of $\longrightarrow_T$. Also, we designate the set of states reachable in one step from a state X as $\longrightarrow_T(X)$, which is the image of X under $\longrightarrow_T$. We use $\longrightarrow_T^{-1}(X)$ to talk about states that can reach X in one step. Finally, we can also evoke the connected component of reachable states

⁵We recall that an unlabelled transition system is a simple directed graph (see Definition 2.0.1 and the discussion that follows), and thus, we can use graph theoretic vocabulary for these structures.

associated to X as $(\longrightarrow_T^\bullet \cup (\longrightarrow_T^{-1})^\bullet)(X)$.

We can address the formalization of LTS morphisms now. This requires that we introduce an *idle action*, designated by ε , to work in conjunction with labelling sets of LTS.

Definition 2.1.9 [Idle Action Extension of a LTS]. Given a labelled transition system $T = (S, L, \delta)$, we define the *extension of T with an idle action* as the structure $T_\varepsilon = (S, L_\varepsilon, \delta_\varepsilon)$ where $L_\varepsilon := L \sqcup \{\varepsilon\}$ and $\delta_\varepsilon := \delta \sqcup \{ (X, \varepsilon, X) \mid X \in S \}$.

Remark 2.1.10 . The extension in the previous definition is not a LTS because a LTS requires a labelling set that does not contain ε by definition.

And the labelling morphisms in $\mathbf{Set}_\varepsilon$ (as provided in Section 1.2) can now be integrated into the definition of morphisms for labelled transition systems, and this is achieved as follows :

Definition 2.1.11 [Labelled Transition System Morphism]. A *LTS morphism* from $T = (S, L, \delta)$ to $T' = (S', L', \delta')$ is a pair $(\sigma, \lambda) : T \rightarrow T'$ where $\sigma : S \rightarrow S'$ is a function and $\lambda : L \rightarrow_\varepsilon L'$ is a labelling morphism in $\mathbf{Set}_\varepsilon$, and the following holds :

$$\forall b \in L, \forall X, Y \in S, [(X, b, Y) \in \delta \Rightarrow (\sigma(X), \lambda(b), \sigma(Y)) \in \delta'_\varepsilon]^6$$

For a given LTS morphism (σ, λ) from $T = (S, L, \delta)$ to $T' = (S', L', \delta')$ and an action $b \in L$, we refer to $\lambda(b)$ as the *projection of b in T'* . When an action $b \in L$ projects to ε , we say that b *vanishes* in T'

Remark 2.1.12 . There is another way of specifying a LTS morphism through the relational diagrams of LTS. We must first extend the relational diagram $\xrightarrow{(-)}_T$ of a labelled transition system $T = (S, L, \delta)$ with the specification $\xrightarrow{\varepsilon}_T := \Delta_S$ for the idle action ε . Then, the condition that

$$\forall b \in L, \forall X, Y \in S, [(X, b, Y) \in \delta \Rightarrow (\sigma(X), \lambda(b), \sigma(Y)) \in \delta'_\varepsilon],$$

⁶We remark that it not necessary to use δ_ε in the antecedent of the implication because we will always have $(\sigma(X), \lambda(\varepsilon), \sigma(X)) \in \delta'_\varepsilon$ for any $(X, \varepsilon, X) \in \delta_\varepsilon$ since $\lambda(\varepsilon) = \varepsilon$.

in the definition of a LTS morphism, can be swapped with :

$$\forall b \in L, \forall X, Y \in S, [X \xrightarrow{b}_T Y \Rightarrow \sigma(X) \xrightarrow{\lambda(b)}_{T'} \sigma(Y)] \quad (1)$$

In fact, from now on, we shall assume that for the idle action ε , the specification $\xrightarrow{\varepsilon}_T := \Delta_S$ is always provided for the relational diagram $\xrightarrow{(-)}_T$ of any labelled transition system T .

Given a LTS morphism $(\sigma, \lambda) : T \rightarrow T'$ and an action b of T , if b vanishes in T' , i.e. $\lambda(b) = \varepsilon$, then for any transition $X \xrightarrow{b}_T Y$ in T , we get $\sigma(X) \xrightarrow{\varepsilon}_{T'} \sigma(Y)$ in T' , and this means $\sigma(X) = \sigma(Y)$ since $\xrightarrow{\varepsilon}_{T'} = \Delta_{S'}$. In other words, the projection of b in T' creates no observable change on states in T' if b vanishes in T' .

We will get a clearer sense of the role that ε plays in the context of spatial presheaves of LTS when we consider the example of Parallel Registers in the next section. For now, since we have LTS objects and morphisms at hand, we can initiate the categorical investigation of LTS.

Definition 2.1.13 [Category of Labelled Transition Systems]. The *category of labelled transition systems*, written $\mathbb{T}$, is defined through the following :

- Objects : Labelled transition systems (from Definition 2.1.1)
- Arrows : Labelled transition system morphisms (from Definition 2.1.11)

We let two morphisms (σ, λ) and (σ', λ') be equal iff $\sigma = \sigma'$ in Set and $\lambda = \lambda'$ in Set_ε . Composition of morphisms (σ, λ) and (σ', λ') is given by pointwise composition $(\sigma' \circ \sigma, \lambda' \circ \lambda)$ of functions, and the identity of a labelled transition system (S, L, δ) is given by the pair $(1_S, 1_L)$ where 1_S is the identity set function on S , and $1_L : L \rightarrow_\varepsilon L$ is the identity labelling morphism on L (which means it is the identity set function $1_{L_\varepsilon} : L_\varepsilon \rightarrow L_\varepsilon$).

It is rather straightforward to verify that the above provides a well-defined category, and we make a formal statement about that :

Proposition 2.1.14 . $\mathbb{T}$ is a well-defined category.

And we lay down some of the basic properties and constructions with respect to this category to end this section.

Proposition 2.1.15 . A LTS morphism $(\sigma, \lambda) : T \rightarrow T'$ is an isomorphism in $\mathbb{T}$ if and only if σ and λ are bijective maps.

Proof. (σ, λ) is an isomorphism if and only if it is an isomorphism componentwise. Also, σ is an isomorphism in **Set** if and only if it is a bijective map.

From Proposition 1.2.4, we get that λ is an isomorphism in $\mathbf{Set}_\varepsilon$ if and only if it is a bijective map. $\square$

There are three useful projection functors that we can define with respect to the components of LTS :

Definition 2.1.16 [LTS Component Functors].

1. The *states-projection functor* $p_s : \mathbb{T} \rightarrow \mathbf{Set}$ sends a LTS (S, L, δ) to the set of states S , and a LTS morphism (σ, λ) to the function σ .
2. The *labelling projection functor* $p_\ell : \mathbb{T} \rightarrow \mathbf{Set}_\varepsilon$ sends a LTS (S, L, δ) to the labelling set L , and a LTS morphism (σ, λ) to the labelling morphism λ in $\mathbf{Set}_\varepsilon$.
3. The *transitions-projection functor* $p_t : \mathbb{T} \rightarrow \mathbf{Set}$ sends a LTS (S, L, δ) to the set δ_ε , and a LTS morphism $(\sigma, \lambda) : (S, L, \delta) \rightarrow (S', L', \delta')$ to the function that maps (X, b, Y) in δ_ε to $(\sigma(X), \lambda(b), \sigma(Y))$ in δ'_ε .

With the category $\mathbb{T}$ at hand and the basic notation settled, we now introduce some of the categorical constructions that are possible in $\mathbb{T}$.⁷

⁷The way these constructions are presented follows a very set theoretical style, as opposed to a more categorical style. The advantage to using a more categorical approach is that it could facilitate proving certain properties for $\mathbb{T}$ such as bicompleteness (see Theorem 2.1.18). In particular, we suspect that the functor $p_s \times p_\ell : \mathbb{T} \rightarrow \mathbf{Set} \times \mathbf{Set}_\varepsilon$ forms a bifibration for $\mathbb{T}$, and proving this would certainly go a long way in developing the theory of $\mathbb{T}$.

Definition 2.1.17 [Constructions in $\mathbb{T}$]. We provide the necessary constructions for a bicomplete category as follows :

1. **Terminal object** : Combining the **Set** terminal object, that is a singleton set $\{\star\}$, and the zero object of $\mathbf{Set}_\varepsilon$, which is an empty set, we can form $1_{\mathbb{T}} = (\{\star\}, \emptyset, \emptyset)$, as a terminal object of $\mathbb{T}$. Indeed, for any labelled transition system $T = (S, L, \delta)$, there is precisely one LTS morphism $(\sigma, \lambda) : T \rightarrow 1_{\mathbb{T}}$ given by the unique map $\sigma : S \rightarrow \{\star\}$ and the unique labelling morphism $\lambda : L \rightarrow_\varepsilon \emptyset$.
2. **Initial object** : The empty set $\emptyset$ is an initial object in **Set** as much as it is in $\mathbf{Set}_\varepsilon$, and we get an initial object for $\mathbb{T}$ as $0_{\mathbb{T}} = (\emptyset, \emptyset, \emptyset)$. Considering any LTS $T = (S, L, \delta)$, there is precisely one LTS morphism $(\sigma, \lambda) : 0_{\mathbb{T}} \rightarrow T$ given by the unique map $\sigma : \emptyset \rightarrow S$ and the unique labelling morphism $\lambda : \emptyset \rightarrow_\varepsilon L$.
3. **Indexed Products** : Given a J -indexed family of labelled transition systems $T_j = (S_j, L_j, \delta_j)$, we define the J -indexed product as :

$$\prod_{j \in J} T_j := (S_J = \prod_{j \in J} S_j, L_J = \prod_{j \in J}^\varepsilon L_j, \delta_J)$$

where

- $\prod_{j \in J} S_j$ is the J -indexed product in **Set** with projection maps $\pi_i : \prod_{j \in J} S_j \rightarrow S_i$ for i in J , and
- $\prod_{j \in J}^\varepsilon L_j$ is the J -indexed product in $\mathbf{Set}_\varepsilon$ with labelling projection maps $\rho_i : \prod_{j \in J}^\varepsilon L_j \rightarrow_\varepsilon L_i$ for i in J , and
- $\delta_J := \{ (X, a, Y) \in S_J \times L_J \times S_J \mid \forall j \in J, \pi_j(X) \xrightarrow{\rho_j(a)}_{T_j} \pi_j(Y) \}$ ⁸.

The projections in $\mathbb{T}$ are given by the pairs $(\pi_i, \rho_i) : \prod_{j \in J} T_j \rightarrow T_i$.

4. **Indexed Coproducts** : Given a J -indexed family of labelled transition systems $T_j = (S_j, L_j, \delta_j)$, we define the J -indexed coproduct as :

$$\prod_{j \in J} T_j := (S_J = \prod_{j \in J} S_j, L_J = \prod_{j \in J}^\varepsilon L_j, \delta_J)$$

where

⁸This includes the case where $\rho_j(a) = \varepsilon$, in which case, we get $\pi_j(X) = \pi_j(Y)$ with $\xrightarrow{\varepsilon}_{T_j} = \Delta_{S_j}$.

- $\coprod_{j \in J} S_j$ is the J -indexed coproduct in $\mathbf{Set}$ with injection maps $\iota_i : S_i \rightarrow \coprod_{j \in J} S_j$ for each i in J , and
- $\coprod_{j \in J}^\varepsilon L_j$ is the J -indexed coproduct in $\mathbf{Set}_\varepsilon$ with injection labelling morphisms $\nu_i : L_i \rightarrow \coprod_{j \in J}^\varepsilon L_j$ for each i in J , and
- $\delta_J = \{ (X, a, Y) \in S_J \times L_J \times S_J \mid \exists i, \exists (X_i, a_i, Y_i) \in \delta_i, \iota_i(X_i) = X, \iota_i(Y_i) = Y, \nu_i(a_i) = a \}$.

The injections in $\mathbb{T}$ are given by the pairs $(\iota_i, \nu_i) : T_i \rightarrow \coprod_{j \in J} T_j$.

5. **Equalizers** : Given labelled transition systems $T = (S, L, \delta)$ and $T' = (S', L', \delta')$, and a pair of LTS morphisms $(\sigma_1, \lambda_1), (\sigma_2, \lambda_2) : T \rightarrow T'$, we can construct an equalizer of the latter by forming a pair with the equalizer of (σ_1, σ_2) in $\mathbf{Set}$ and the equalizer of (λ_1, λ_2) in $\mathbf{Set}_\varepsilon$. More precisely, we can form the subsets

$$S_e = \{X \in S \mid \sigma(X) = \sigma(Y)\} \subseteq S \quad \text{and} \quad L_e = \{a \in L \mid \lambda(a) = \lambda'(a)\} \subseteq L,$$

and use the subset of transitions $\delta_e = \delta \cap (S_e \times L_e \times S_e)$. We get that $T_e = (S_e, L_e, \delta_e)$ is a LTS and the inclusion maps σ_e from S_e to S and λ_e from $L \cup \{\varepsilon\}$ to $L \cup \{\varepsilon\}$ provide a pair $(\sigma_e, \lambda_e) : T_e \rightarrow T$ that is the equalizer for the pair of morphisms (σ_1, λ_1) and (σ_2, λ_2) in $\mathbb{T}$.

6. **Coequalizers** : Given labelled transition systems $T = (S, L, \delta)$ and $T' = (S', L', \delta')$, and a pair of LTS morphisms $(\sigma_1, \lambda_1), (\sigma_2, \lambda_2) : T \rightarrow T'$, we get a pair of labelling morphisms $\sigma_1, \sigma_2 : L' \rightarrow_\varepsilon L$, and we can take the labelling coequalizer (as in Definition 1.2.5) of the latter, say $\lambda_q : L \rightarrow_\varepsilon L_q$.

Let $\sim_q$ be the smallest equivalence relation on S such that :

- $\forall X \in S', \sigma_1(X) \sim_q \sigma_2(X)$, and
- $\forall b \in L, \forall X, Y \in S', [\text{ if } \lambda_q(b) = \varepsilon \text{ and } X \xrightarrow{b}_T Y, \text{ then } X \sim_q Y]$

We form the quotient set $S_q := S / \sim_q$ of S with respect to $\sim_q$ and we write $\sigma_q : S \rightarrow S_q$ for the corresponding quotient map. Finally, define

$$\delta_q := \{ (\sigma_q(X), \lambda_q(b), \sigma_q(Y)) \in S_q \times L_q \times S_q \mid (X, b, Y) \in \delta \}$$

We have that $T_q := (S_q, L_q, \delta_q)$ is a LTS and $(\sigma_q, \lambda_q) : T \rightarrow T_q$ is a LTS morphism that is the coequalizer of the pair of LTS morphisms (σ_1, λ_1) and (σ_2, λ_2) . Furthermore, $\theta \circ q$ is a surjective map.

With all indexed products and coproducts, equalizers and coequalizers, we can make the following statement.

Theorem 2.1.18 . *The category $\mathbb{T}$ is bicomplete.*

Proof. The proof is given in Appendix B. □

Remark 2.1.19 . We will see that $\mathbb{T}$ has no subobject classifier at the end of Section 2.3.

We will go through an example of a product of registers in detail in the next section. But before we proceed, we would like to make a few comments about variants that one can use with respect to the definition of a labelled transition system.

In our definition (the one given in Definition 2.1.1), for a given labelled transition system $T = (S, L, \delta)$ and an action $b \in L$, we allowed the possibility of an empty transition relation for b , i.e. $\xrightarrow{b}_T = \emptyset$. We will refer to an action b such that $\xrightarrow{b}_T = \emptyset$ as a *ghost action*. Certain variants of the definition of LTS reject ghost actions, i.e. impose the following rule :

$$\forall b \in L, \exists X, Y \in S, (X, b, Y) \in \delta$$

(and this is equivalent to $\forall b \in L, \xrightarrow{b}_T \neq \emptyset$).

We did not work with this rule because allowing ghost actions has several important consequences at the categorical level that we should be aware of. When we allow ghost actions, we basically get LTS objects in $\mathbb{T}$ of the form $(S = \emptyset, L, \delta = \emptyset)$ where $L \neq \emptyset$. This allows us to get a coreflection of Set_ε into $\mathbb{T}$ if we use an inclusion functor that sends a labelling set L to the labelled transition system $(\emptyset, L, \emptyset)$, and a labelling morphism $\lambda : L \rightarrow_\varepsilon L'$ to the LTS morphism $(0, \lambda) : (\emptyset, L, \emptyset) \rightarrow (\emptyset, L', \emptyset)$ where $0 : \emptyset \rightarrow \emptyset$ is the unique set function on the empty set. This means that we can

think of labelling sets as labelled transition systems directly in $\mathbb{T}$.

One important consequence of allowing ghost actions is Proposition 4.2.4, which states that if we have $\mathbb{T}$ -valued sheaf, then the sheaf property extends to the labelling component of the sheaf (gluing and locality of LTS extends to gluing and locality for labels properly). Another important consequence is that it facilitates dealing with subobjects. Indeed, when we single-handedly pick out transitions from a LTS, we always obtain a labelled transition subsystem (a sub-LTS) as long as we allow ghost actions (otherwise, we may get empty transition relations when removing transitions in a LTS).

Finally, integrating ghost actions facilitates an adjunction between the category of labelled transition systems and the category of monoid labelled transition systems (because the latter naturally have such ghost actions) as we will see in Section 2.4. But in sum, integrating ghost actions relaxes the category of LTS to a certain extent, and it is really a matter of choice whether one wants to have them lying around in a LTS structure or not.

There is another variant of the concept of LTS that has been studied by Winskel and Nielsen in [MC] where an initial state is integrated in the signature of the LTS, but we will simply use a different name to designate those as follows :

Definition 2.1.20 [Pointed Labelled Transition System]. A *pointed labelled transition system* is a structure $T = (X_0, S, L, \delta)$ where (S, L, δ) is a LTS and $X_0 \in S$ represents the *initial state* of the system.⁹

These are important if we want to establish a rapport with syntactic process representations as encountered in process algebras. Take for example, the case of certain process calculi such as CCS or π -calculus : At first, one syntactically constructs a

⁹Winskel and Nielsen use such pointed labelled transition systems in [MC], but they are simply called labelled transition systems in their work.

universe of processes via a generative grammar. Then, one equips the universe of processes thus generated with a transition relation to account for the operational semantics. The final construction is a transition system in which the states identify precisely the processes. Thus, pointing to a state in such transition systems corresponds precisely with designating a process in such process algebras. Winskel and Nielsen give an example of a process language *Proc* in [MC], that can be interpreted via categorical constructions on pointed labelled transition systems in particular.

It may be a good idea to mention here that Winskel and Cattani have elaborated a general representation of process models of this kind as set-valued presheaves (see [PresMC] and [ProfOB])¹⁰. The idea is that sections in these presheaves describe events (or traces, such as runs) and the inverse of the restriction maps in these presheaves describe the behavioural unfolding of such events with respect to some basic choice of path shape as a base category (runs use linear path shapes). This approach accounts for a more universal treatment of process modeling than labelled transition systems as it precipitates the subject into a more categorical framework.

We now consider an example of a product construction in $\mathbb{T}$, which basically reflects a form of parallel composition of register systems.

2.2 Example of Parallel Registers

In this section, we explore an example of a product of LTS and we take the opportunity to introduce the idea of SI-independence. The example to be considered is a set of parallel registers, and it is given as follows :

Example 2.2.1 [Parallel Registers]. Consider a family $\{\text{Reg}_i\}_{i=1}^n$ of 4-bit Registers with Operations as provided in Example 2.1.6, i.e. where each LTS register is given by $\text{Reg}_i = (S_i, L_i, \delta_i) = \text{Reg}$ as specified previously. Setting $A_n = \{1, 2, \dots, n\}$

¹⁰There is also [BisO] in which the theory of bisimulation as open maps has been developed by Winskel, Nielsen and Joyal with respect to these presheaf models.

for some $n \in \mathbb{N}$, we can take the A_n -indexed LTS product $\prod_{j \in A_n} \text{Reg}_j$ to represent n registers in parallel. We can think of a number j in A_n as providing the memory address for Reg_j in a computer's memory.

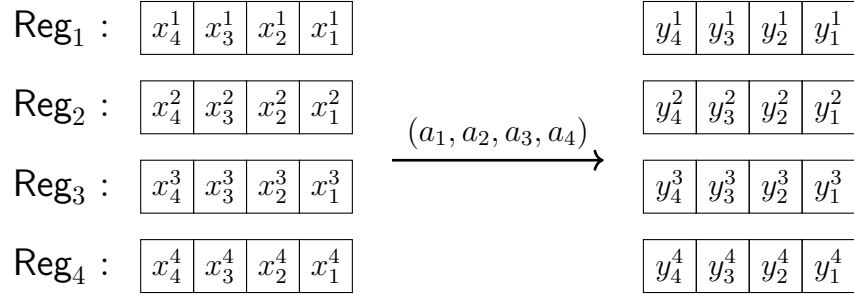
Considering the simple case where $n = 4$, we can represent the labelled transition system $T = \prod_{j \in A_4} \text{Reg}_j = (S, L, \delta)$ as follows :

1. $S = (\{0, 1\}^4)^4$ (or $S = \{0, 1\}^{16}$), and we use the variable $x_i^j \in \{0, 1\}$ to represent the bit value in the i^{th} position (from right to left) in the j^{th} register as in Figure 3.



Figure 3: Four Parallel Registers of 4 bits each.

2. $L = \{ (a_1, a_2, a_3, a_4) \in (L_{\text{Reg}} \cup \{\varepsilon\})^4 \mid (a_1, a_2, a_3, a_4) \neq (\varepsilon, \varepsilon, \varepsilon, \varepsilon) \}$ where $L_{\text{Reg}} = \{read, write, shift, clear, incr\}$ and where a_j represents an action in the j^{th} register Reg_j .
3. δ 's specification is given by the componentwise specification of the local transition relations on each register as in Figure 4. We get an example of a transition for the action $(\varepsilon, incr, \varepsilon, clear)$ in Figure 5.



if and only if

For all $j \in \{1, 2, 3, 4\}$,

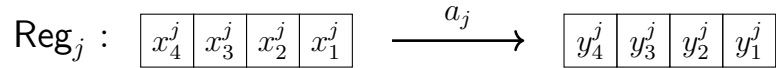


Figure 4: Specification of transitions for Parallel Registers

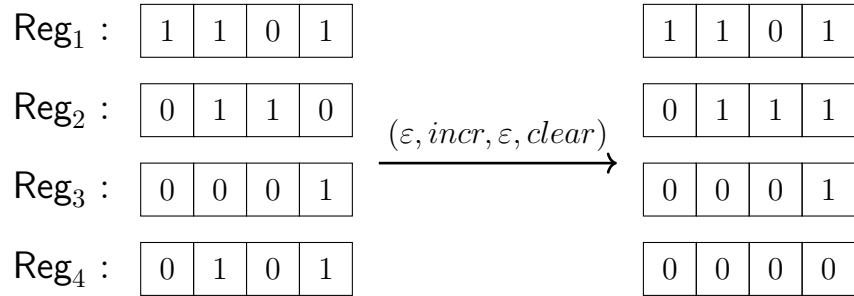


Figure 5: Example of transition for $(\varepsilon, incr, \varepsilon, clear)$ in the Parallel Registers

We can get a sense of how spatial presheaves of LTS and SI-independence work for these Parallel Registers. To achieve this, we must first consider the discrete space over $A_4 = \{1, 2, 3, 4\}$ that uses memory addresses as points. This yields a Boolean algebra space $\mathcal{B} = (\mathcal{P}(\{1, 2, 3, 4\}), \subseteq, \cap, \cup)$ where the elements of $\mathcal{P}(\{1, 2, 3, 4\})$ (referred to as regions) represent all possible groups of registers (by sets of corresponding memory addresses). Subset inclusion $\subseteq$ provides a partial order on $\mathcal{P}(\{1, 2, 3, 4\})$ that describes how different regions are included in one another, and set intersection $\cap$ and

set union $\cup$ provide the meet and join operations. The smallest region is $\emptyset$ and the largest region is $\{1, 2, 3, 4\}$.

We can think of $(\mathcal{P}(\{1, 2, 3, 4\}), \subseteq)$ as a category and we can set up a $\mathbb{T}$ -valued sheaf $\mathcal{T} : \mathcal{B}^{op} \rightarrow \mathbb{T}$ where $\mathcal{T}(U) = \prod_{j \in U} \mathbf{Reg}_j$ for each $U \subseteq \{1, 2, 3, 4\}$. For $V \subseteq U \subseteq \{1, 2, 3, 4\}$, the LTS restriction morphisms are given by the obvious LTS projections from $\mathcal{T}(U)$ to $\mathcal{T}(V)$ that arise for these indexed products of LTS. In such a case, we write $(\sigma_V^U, \lambda_V^U)$ to denote the LTS restriction morphism from $\mathcal{T}(U)$ to $\mathcal{T}(V)$, where σ_V^U is the states map component and λ_V^U is the labelling morphism component. For the global region $P = \{1, 2, 3, 4\}$, we have $\mathcal{T}(P) = T$, where T is the LTS that was just described.

Now, when we have regions $V \subseteq U$ and an action b in the labelling set of $\mathcal{T}(U)$ that projects to ε on the subregion V (i.e. with $\lambda_V^U(b) = \varepsilon$), we will say that b *vanishes* in V with respect to U . For example, consider the action $b = (incr, \varepsilon, \varepsilon, shift)$ that is part of the labelling set of $\mathcal{T}(P)$. We have that b projects to ε precisely on the subregions $\emptyset$, $\{2\}$, $\{3\}$, and $\{2, 3\}$ (see Figure 6), and we think of b as vanishing in these regions because b only performs an incrementation in $\mathbf{Reg}_1$ in region $\{1\}$, and a shift action in $\mathbf{Reg}_4$ in region $\{4\}$, but it does not do anything inside the regions that are subsets of $\{2, 3\}$ (where $\mathbf{Reg}_2$ and $\mathbf{Reg}_3$ reside).

An example of transition for the action $b = (incr, \varepsilon, \varepsilon, shift)$ in $\mathcal{T}(P)$ is given in Figure 7, and we clearly see there that b is not actually involved in the second and third register. Now, by contrast to saying that b vanishes in $\{2, 3\}$, we may say that b is “contained” in $\{1, 4\}$, which is the complement of $\{2, 3\}$ in P . By “contained”, we mean that all the information on which b depends in order to act in P , resides in the region $\{1, 4\}$, and also, all the changes that b creates on states essentially occur in $\{1, 4\}$.

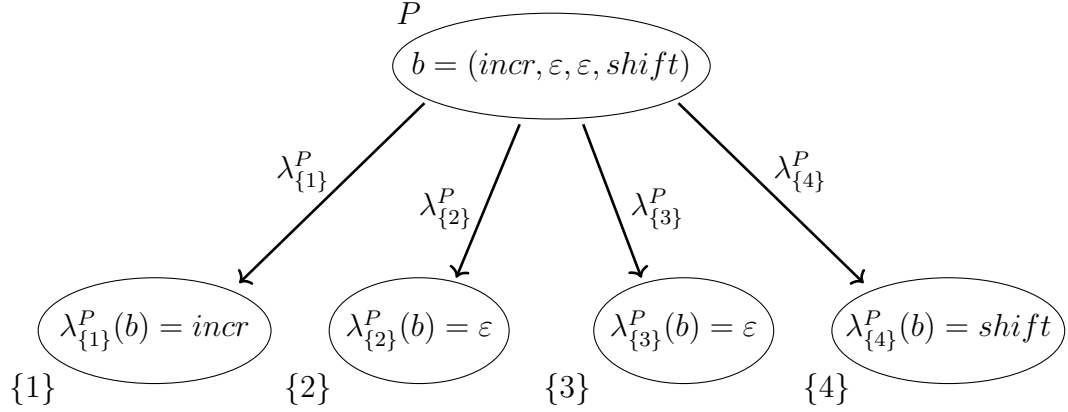


Figure 6: The projection of the global action b on each register individually; $\lambda_{\{j\}}^P$ denotes the labelling projection from the labelling set of $\mathcal{T}(P)$ to the labelling set of $\mathcal{T}(\{j\}) = \text{Reg}_j$, the j^{th} register.

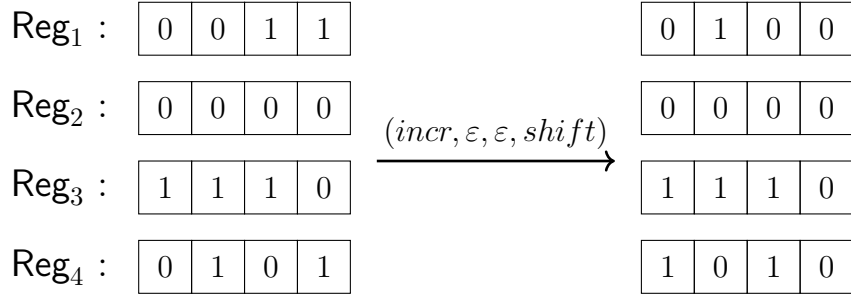


Figure 7: A transition for the action $(incr, \varepsilon, \varepsilon, shift)$

The notion of “containment” for actions is formalized in Definition 4.3.15 in Section 4.3 on Action Containment. When this notion will be formalized, the way SI-independence will be formulated is that :

For some region U , two actions b and c in the labelling set of $\mathcal{T}(U)$ are spatially independent in U if there exists a cover $\{V, W\}$ such that :

1. b vanishes in W and c vanishes in V , and
2. b is contained in V and c is contained in W .

Thus, the idea is that c does not interfere with the region V where b is contained, and vice-versa, b does not interfere with the region W where c is contained. Condition (1) above is simply : $\lambda_W^U(b) = \varepsilon$ and $\lambda_V^U(c) = \varepsilon$. Condition (2) is much more complicated to express and to impose however. Essentially, we will find a way to force condition (1) to imply condition (2) above. It turns out that this implication holds in any $\mathbb{T}$ -valued sheaf, and so it holds in the case of $\mathcal{T}$ for the Parallel Registers above because it is a sheaf that arises from a product (a limit) of registers.

For the case of the actions $b = (incr, \varepsilon, \varepsilon, shift)$ and $c = (\varepsilon, write, \varepsilon, \varepsilon)$ in the Parallel Registers for example, we have that both conditions (1) and (2) hold because condition (1) holds. Indeed, with $V = \{1, 3, 4\}$ and $W = \{2, 4\}$, we get a cover $\{V, W\}$ of $P = \{1, 2, 3, 4\}$ such that $\lambda_W^P(b) = \varepsilon$ and $\lambda_V^P(c) = \varepsilon$, i.e. b vanishes in W and c vanishes in V . We then get diamond-shaped diagrams like in Figure 8 that illustrate how b and c are independent of each other; i.e. they illustrate how we can change the order in which these b and c transitions fire without changing the final outcome (the state on the right in the diagram of Figure 8). Furthermore, in the diagram of this figure, we could add a middle arrow, labelled with $(incr, write, \varepsilon, shift)$, from the leftmost state to the the rightmost one. Indeed, when actions are independent, we may allow them to perform simultaneously. This typically occurs with sheaves such as $\mathcal{T}$ here.

In sum, to formally address SI-independence, we need to formally address action containment first, and to address the latter, we still have a lot of preparation to do. We thus pursue our investigation of LTS for a while until we arrive at an abstract representation of the properties of independence within a LTS by using asynchronous versions of the latter in Chapter 3. Then we will have what we need to evoke action containment and SI-independence properly.

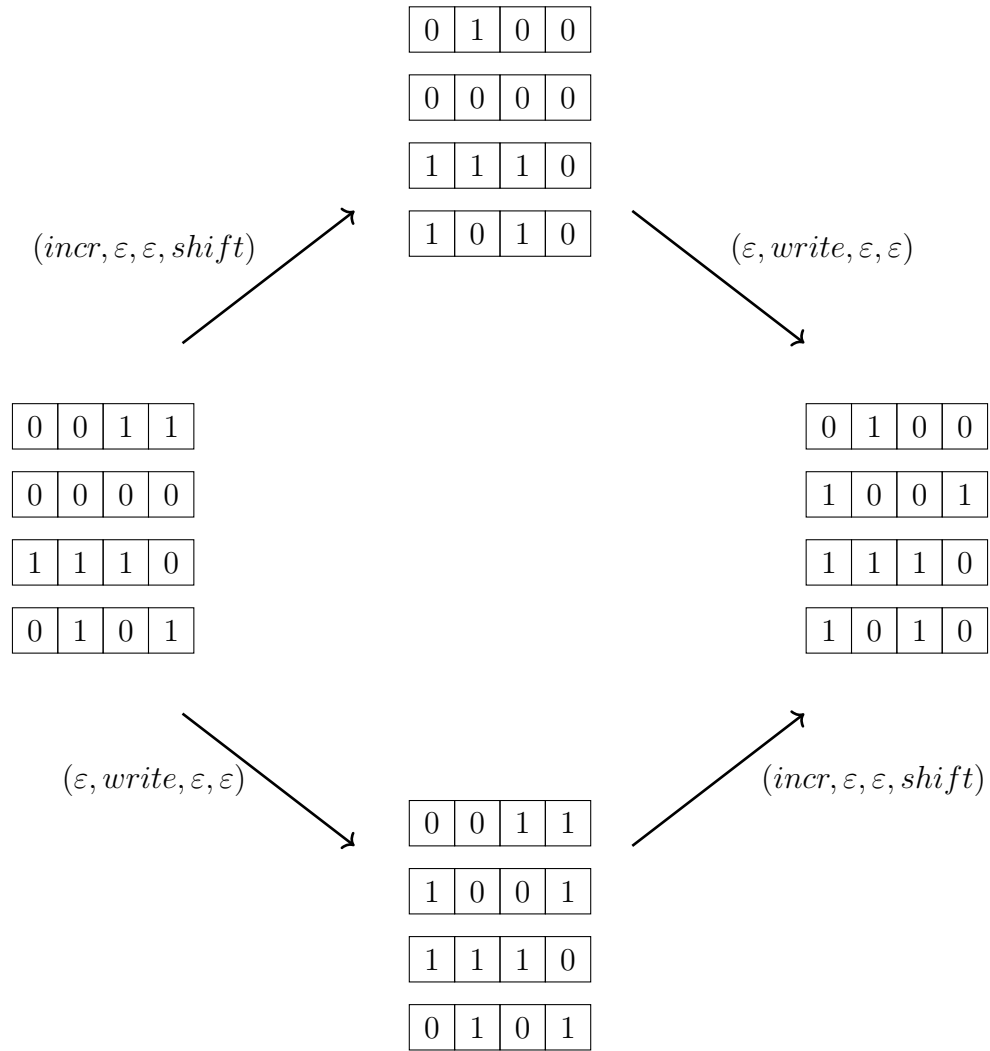


Figure 8: A diamond-shaped diagram depicting spatially independent actions.

2.3 Subobjects for $\mathbb{T}$

In this section, we provide a few results relating to subobjects in $\mathbb{T}$. As we indicated earlier in the introduction, this section is optional and subsequent theory in this thesis does not depend on the results presented here.

If we were to characterize as directly as possible the idea of a sub-LTS, it would probably go as follows :

Definition 2.3.1 [Sub-LTS]. Consider any $T = (S, L, \delta)$ in $\mathbb{T}$. A *sub-LTS* of T is a LTS $T' = (S', L', \delta')$ where $S' \subseteq S$, $L' \subseteq L$ and $\delta' \subseteq \delta$. This means precisely that a sub-LTS of $T = (S, L, \delta)$ is a tuple (S', L', δ') such that :

$$S' \subseteq S, \ L' \subseteq L, \text{ and } \delta' \subseteq \delta \cap (S' \times L' \times S') \quad ^{11}$$

This way of presenting sub-LTS simply means that you can remove transitions, states and labels, as much as you please, but whenever you remove a state, you must remove all transitions incident to that state, and whenever you remove a label, you must remove all transitions carrying that label. Implicitly, it is this fact that really helps us identify a sub-LTS of a LTS rapidly.

At this point, we have defined what the sub-LTS of an LTS are directly, but we have not yet proved that these sub-LTS indeed correspond with the subobjects of an LTS. The first step is to verify that if T' is a sub-LTS of T , then there is a monomorphism from T' into T , and we need a lemma for this.

Lemma 2.3.2 . *A morphism $(\sigma, \lambda) : T \rightarrow T'$ is a monomorphism if and only if σ and λ are injective maps.*

¹¹If we had chosen to reject ghost actions as part of the definition of LTS, it would've been necessary to introduce $\xrightarrow{a}_{T'} \neq \emptyset$ for every $a \in L'$ as an extra condition in this criteria.

Proof. Write $T = (S, L, \delta)$. Suppose (σ, λ) is a monomorphism. Consider any set functions $\sigma_1 : S_0 \rightarrow S$ and $\sigma_2 : S_0 \rightarrow S$ such that $\sigma \circ \sigma_1 = \sigma \circ \sigma_2$. Consider the LTS given by $T_0 = (S_0, L, \delta_0 = \emptyset)$. Then $(\sigma_1, 1_L), (\sigma_2, 1_L) : T_0 \rightarrow T$ are LTS morphisms because they trivially preserve an empty set of transitions. We get $(\sigma, \lambda) \circ (\sigma_1, 1_L) = (\sigma, \lambda) \circ (\sigma_2, 1_L)$ and so, $(\sigma_1, 1_L) = (\sigma_2, 1_L)$ by our assumption of (σ, λ) as a mono. Thus, $\sigma_1 = \sigma_2$, and this proves σ is a mono in Set , and thus, it is an injective map. Similarly, consider labelling morphisms $\lambda_1 : L_0 \rightarrow L$ and $\lambda_2 : L_0 \rightarrow L$ such that $\lambda \circ \lambda_1 = \lambda \circ \lambda_2$. Consider the LTS given by $T_0 = (S, L_0, \delta_0 = \emptyset)$. We get $(\sigma, \lambda) \circ (1_S, \lambda_1) = (\sigma, \lambda) \circ (1_S, \lambda_2)$ and so, $(1_S, \lambda_1) = (1_S, \lambda_2)$ by our assumption of (σ, λ) as a mono. Thus, $\lambda_1 = \lambda_2$, and this proves λ is a mono in Set_ε . By Proposition 1.2.3, λ is an injective map.

Conversely, suppose that σ and λ are injective maps. Then σ is clearly a mono, and λ is a mono by Proposition 1.2.3 once again. But then, consider any LTS morphisms $(\sigma_1, \lambda_1), (\sigma_2, \lambda_2) : T_0 \rightarrow T$ such that $(\sigma, \lambda) \circ (\sigma_1, \lambda_1) = (\sigma, \lambda) \circ (\sigma_2, \lambda_2)$. This means $\sigma \circ \sigma_1 = \sigma \circ \sigma_2$ and $\lambda \circ \lambda_1 = \lambda \circ \lambda_2$, and thus $\sigma_1 = \sigma_2$ and $\lambda_1 = \lambda_2$. Finally, we get $(\sigma_1, \lambda_1) = (\sigma_2, \lambda_2)$, and this establishes that (σ, λ) is a mono. $\square$

Remark 2.3.3 . Both of the constructed T_0 in the proof require that we allow ghost actions for LTS. We don't know if this is necessary to characterize monos.

We also need to characterize the image LTS associated to morphisms.

Definition 2.3.4 [Image of LTS Morphism]. Let $T_0 = (S_0, L_0, \delta_0)$ and $T = (S, L, \delta)$ be LTS, and let $(\sigma, \lambda) : T_0 \rightarrow T$ be a LTS morphism. Define the *image of* (σ, λ) as the labelled transition system $Im(\sigma, \lambda) = (S', L', \delta')$ where :

- $S' = \sigma(S_0)$ (and we have $S' \subseteq S$)
- $L' = \lambda(L_0) \setminus \{\varepsilon\}$ (and we have $L' \subseteq L$)
- $\delta' = \{ (\sigma(X), \lambda(b), \sigma(Y)) \in S' \times L' \times S' \mid (X, b, Y) \in \delta_0 \text{ and } \lambda(b) \neq \varepsilon \}$

With $S' \subseteq S$, $L' \subseteq L$ and $\delta' \subseteq \delta \cap (S' \times L' \times S')$, we have by Definition 2.3.1 that

$Im(\sigma, \lambda)$ is a sub-LTS of T .

And now we can state that the subobjects of an LTS are precisely its sub-LTS.

Proposition 2.3.5 . *Consider any labelled transition system $T_0 = (S_0, L_0, \delta_0)$ and $T = (S, L, \delta)$. There is a mono from T_0 into T in $\mathbb{T}$ if and only if T_0 is isomorphic to a sub-LTS of T .*

Proof. For the implication from right to left, suppose T_0 is isomorphic to a sub-LTS of T . Consider that sub-LTS in question and call it $T' = (S', L', \delta')$. It suffices to show that there is a monomorphism from T' into T to know via the isomorphism that there is a monomorphism from T_0 into T .

By definition, we have $S' \subseteq S$, $L' \subseteq L$ and $\delta' \subseteq \delta \cap (S' \times L' \times S')$. This means there is an inclusion map $\iota_S : S' \hookrightarrow S$, and there is an inclusion labelling morphism $\iota_L : L' \hookrightarrow_\epsilon L$. The pair (ι_S, ι_L) is in fact a LTS morphism from T' to T :

Consider any $(X, b, Y) \in \delta'$. We have $\delta' \subseteq \delta$, so $(X, b, Y) \in \delta$. But $(X, b, Y) \in S' \times L' \times S'$, so the inclusions are well-defined for each component and we get $\iota_S(X) = X$, $\iota_S(Y) = Y$, and $\iota_L(b) = b$. This means $(\iota_S(X), \iota_L(b), \iota_S(Y)) \in \delta$.

Finally, since ι_S and ι_L are injective maps, they are monos in their respective categories, and Lemma 2.3.2 states that (ι_S, ι_L) is a mono in $\mathbb{T}$.

For the implication from left to right, suppose that there is a monomorphism $(\sigma, \lambda) : T_0 \hookrightarrow T$ in $\mathbb{T}$. We know that $Im(\sigma, \lambda)$ is a sub-LTS of T , and it suffices to show that there is an isomorphism from T_0 to $Im(\sigma, \lambda)$. This isomorphism is precisely (σ, λ) (with a minor adjustment on the codomains of the components). Indeed, σ and λ are monos by Lemma 2.3.2, and they are thus injective maps by Proposition 1.2.3. But they are also surjective maps $\sigma(S_0)$ is the state space of $Im(\sigma, \lambda)$ and $\lambda_0(L_0) = \lambda(L_0) \setminus \{\epsilon\}$ is the labelling set of $Im(\sigma, \lambda)$. By Proposition 2.1.15, this means that

(σ, λ) is an isomorphism from T_0 to $Im(\sigma, \lambda)$. $\square$

There are two ways of deriving sub-LTS that will be most useful in the analysis of LTS, and these are labelling restrictions and (states) induced sub-LTS.

Definition 2.3.6 [Labelling Restriction]. Given a labelled transition system $T = (S, L, \delta)$ and $L' \subseteq L$, we define the *labelling restriction of T to L* as the sub-LTS $T \upharpoonright L' = (S, L', \delta')$ where $\delta' = \delta \cap (S \times L' \times S)$.

Definition 2.3.7 [Induced sub-LTS]. Given a LTS $T = (S, L, \delta)$ and $S' \subseteq S$, we define the *sub-LTS of T induced by S'* as the sub-LTS $T[S'] = (S', L, \delta')$ where $\delta' = \delta \cap (S' \times L \times S')$.

Remark 2.3.8 . We borrowed terminology from graph theory where the notion of induced subgraph is defined. The idea is practically the same if we think of states in a LTS as vertices in a graph.

These are clearly sub-LTS by Condition 2.3.1. The monomorphism from $T \upharpoonright L'$ to T is provided by $(1_S, \lambda)$ where 1_S is the identity on states and $\lambda : L' \rightarrow_\epsilon L$ is the inclusion map from L'_ϵ to L_ϵ . The monomorphism from $T[S']$ to T is provided by $(\sigma, 1_L)$ where 1_L is the labelling identity on L in Set_ϵ and $\sigma : S' \rightarrow S$ is the inclusion map from S' to S .

And we end this section with a proof that $\mathbb{T}$ has no subobject classifier. This is due to the fact that morphisms do not map transitions individually, but all of them together through labels. To be more precise, if we consider a label b in a system, then a morphism with such a system as its domain cannot distinguish between the subobjects that pick different subsets of transitions associated to b . We will see how this works in the proof that follows.

Proposition 2.3.9 . $\mathbb{T}$ has no subobject classifier.

Proof. We demonstrate that assuming the existence of a subobject classifier in $\mathbb{T}$ leads to a contradiction. Suppose there exists a subobject classifier Ω with a truth arrow $(\sigma_{true}, \lambda_{true}) : 1 = (\{*\}, \emptyset, \emptyset) \rightarrow \Omega$.

Consider the labelled transition system $T = (S, L, \delta)$ where $S = \{X, Y, Z\}$, $L = \{b\}$, and $\delta = \{(X, b, Y), (X, b, Z)\}$ and a sub-LTS, $T_0 = (S_0, L_0, \delta_0)$ where $S_0 = \{X, Y\}$, $L_0 = \{b\}$, $\delta = \{(X, b, Y)\}$, with the obvious monomorphism $(\iota_S, \iota_L) : T_0 \hookrightarrow T$ given by a pair of inclusion maps; that is, $\iota_S(X) = X$, $\iota_S(Y) = Y$ and $\iota_L(b) = b$.

Denote the unique arrows from T to 1 and from T_0 to 1 as (σ_T, λ_T) and $(\sigma_{T_0}, \lambda_{T_0})$ respectively. Then, there is a unique morphism $(\sigma_\chi, \lambda_\chi) : T \rightarrow \Omega$ such that the commutative diagram below is a pullback square.

$$\begin{array}{ccc}
 T_0 = & \begin{array}{c} X \xrightarrow{b} Y \end{array} & \xrightarrow{(\iota_S, \iota_L)} \begin{array}{c} X \xrightarrow{b} Y \\ \searrow b \\ Z \end{array} = T \\
 & \downarrow (\sigma_{T_0}, \lambda_{T_0}) & \downarrow (\sigma_\chi, \lambda_\chi) \\
 1 = & * & \xrightarrow{(\sigma_{true}, \lambda_{true})} \Omega
 \end{array}$$

Figure 9: Pullback diagram for T_0

Now, $(\sigma_\chi, \lambda_\chi) \circ (\iota_S, \iota_L) = (\sigma_{true}, \lambda_{true}) \circ (\sigma_{T_0}, \lambda_{T_0})$, which means that $\sigma_\chi(X) = (\sigma_\chi(\iota_S(X))) = \sigma_{true}(\sigma_{T_0}(X)) = \sigma_{true}(*)$; and similarly, $\sigma_\chi(Y) = \sigma_{true}(*)$.

Also, $\lambda_\chi(b) = (\lambda_\chi(\iota_L(b))) = \lambda_{true}(\lambda_{T_0}(b)) = \lambda_{true}(\varepsilon) = \varepsilon$, and we have $X \xrightarrow{b}_T Z$ and $(\sigma_\chi, \lambda_\chi)$ is a LTS morphism, so $\sigma_\chi(X) \xrightarrow{\lambda_\chi(b)=\varepsilon}_{\Omega} \sigma_\chi(Z)$, and this forces $\sigma_\chi(Z) = \sigma_\chi(X) = \sigma_{true}(*)$. But then, the identity morphism $(1_S, 1_L) : T \rightarrow T$ and $(\sigma_T, \lambda_T) : T \rightarrow 1$ also provide a commutative square as follows :

$$\begin{array}{ccc}
 T & \xrightarrow{(1_S, 1_L)} & T \\
 (\sigma_T, \lambda_T) \downarrow & & \downarrow (\sigma_\chi, \lambda_\chi) \\
 1 & \xrightarrow{(\sigma_{true}, \lambda_{true})} & \Omega
 \end{array}$$

Figure 10: Commutative square for T

The pullback structure for T_0 forces the existence of an arrow $(\sigma_u, \lambda_u) : T \rightarrow T_0$ such that $(\iota_S, \iota_L) \circ (\sigma_u, \lambda_u) = (1_S, 1_L)$, which means $\iota_S \circ \sigma_u = 1_S$. But $\sigma_u(Z) \in S_0 = \{X, Y\}$, and so $\iota_S(\sigma_u(Z)) \in \{X, Y\}$ since ι_S is an inclusion map. This means it cannot be equal to $Z = 1_S(Z)$, which contradicts $\sigma \circ \sigma_u = 1_S$. This proves there is no subobject classifier in $\mathbb{T}$. $\square$

2.4 Monoid Labelled Transition Systems

In this section, we introduce monoid labelled transition systems (MLTS). This is an optional section once again. One reason for which we study MLTS is that Malcolm uses them in [SSTS], and so we facilitate coordination with his work this way. The main reason, however, is that the theory of MLTS will help us in quotienting sequences of actions in freely generated MLTS to provide trace monoids (we will see this in detail when we study independence relations in LTS). Thus, we focus here on the theory that will serve towards that purpose. In particular, we provide an adjunction between the category of LTS and MLTS where the left adjoint associates a freely generated MLTS to any LTS.

As we generalize to MLTS, we automatically incorporate the idle action as the neutral element of the labelling monoid. The definition of monoid labelled transition system as given by Malcolm in [SSTS] is as follows :

Definition 2.4.1 [Monoid Labelled Transition System]. A *monoid labelled transition system* (or MLTS) is a structure $T = (S, M, \cdot, e, \delta)$ where :

1. $(M, \cdot, e)$ ¹² is a monoid, called the *labelling monoid* of T ;
2. (S, M, δ) is a LTS with set of states S , labels M , transition specifier δ ; ¹³
3. $\xrightarrow{e}_T = \Delta_S$
4. $\forall m, n \in M, \xrightarrow{m \cdot n}_T = \xrightarrow{n}_T \circ \xrightarrow{m}_T$

The idle transition is now incorporated as the neutral element e inside the monoid labelling set, and the mechanism by which actions vanish through labelling morphisms

¹²We will often write (S, M, δ) to designate a monoid labelled transition system $(S, M, \cdot, e, \delta)$ when the underlying monoid structure $(M, \cdot, e)$ is clear in the context.

¹³As we assume that (S, M, δ) is a LTS, we automatically get that M is in Set_ε and does not contain the distinguished symbol ε . This will facilitate interaction with the category of LTS.

by sending them to ε in $\mathbb{T}$, is now rendered by using monoid morphisms that send such actions to e .

We remark that the view of (S, M, δ) as a LTS allowed us to use the relational diagram notation $\xrightarrow{(-)}_T$ for this LTS as in Definition 2.1.5. Now, the labels have the additional structure of a monoid, so we can say more about such relational diagrams. If we look at $(M, \cdot, e)$ as a one object category where the arrows are the elements of M , then we can observe that the axioms (3) and (4) of a MLTS mean precisely that $\xrightarrow{(-)}_T$ is a one object *Rel*-valued presheaf that sends its single object to the set of states S in *Rel*. To be more precise, the third axiom says that the identity is preserved by $\xrightarrow{(-)}_T$, and the fourth axiom says that $\xrightarrow{(-)}_T$ preserves composition contravariantly, i.e. monoidal composition becomes relational composition with the order of composition reversed. We formalize this in a definition as follows :

Definition 2.4.2 [Relational Presheaf associated to MLTS]. Consider any monoid labelled transition system $T = (S, M, \cdot, e, \delta)$. We define the *relational presheaf associated to T* as the functor $\xrightarrow{(-)}_T: M^{op} \rightarrow Rel$ that is provided by the relational diagram defined on the graph $J = (\{\star\}, M)$ from Definition 2.1.5. This is in fact a contravariant functor on the one object category $(M, \cdot, e)$ as we have just explained.

Remark 2.4.3 . The map that sends a MLTS to its relational presheaf provides in fact a bijective correspondence between MLTS and one object *Rel*-valued presheaves. The rapport between LTS and relational presheaves has been studied by P. Sobociński in [RelPTS]. We will close this section with a few remarks about morphisms for such presheaves.

Example 2.4.4 [Dynamical Systems]. Given any dynamical system $\phi : \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}^n$, write $\phi(t, X)$ to represent the new position of a point X after an amount of time t has elapsed. We can derive a monoid labelled transition system T as follows :

- States : $S = \mathbb{R}^n$ (a point in space)
- Labelling monoid : $(\mathbb{R}, 0, +)$ (elapsed intervals of time)

- Transitions : $X \xrightarrow{t}_T Y \Leftrightarrow Y = \phi(t, X)$ for any $X, Y \in S$ and $t \in \mathbb{R}$

We now prove that this is a well-defined monoid labelled transition system. For any $X, Y \in S$, we get $X \xrightarrow{0}_T Y \Leftrightarrow Y = \phi(0, X) \Leftrightarrow X = Y$ by the properties of a dynamical system, and this means that $\xrightarrow{0}_T = \Delta_S$.

Also, consider any $t, s \in \mathbb{R}$. If $X \xrightarrow{t+s}_T Y$, then $Y = \phi(t + s, X)$, and setting $Z = \phi(t, X)$ (which means $X \xrightarrow{t}_T Z$), the properties of a dynamical system provide $\phi(t + s, X) = \phi(s, \phi(t, X)) = \phi(s, Z)$ which means $Z \xrightarrow{s}_T Y$. Thus, $(X, Y) \in (\xrightarrow{s}_T \circ \xrightarrow{t}_T)$. This proves that $\xrightarrow{t+s}_T \subseteq \xrightarrow{s}_T \circ \xrightarrow{t}_T$. If $(X, Y) \in (\xrightarrow{s}_T \circ \xrightarrow{t}_T)$, then there is $Z \in S$ such that $X \xrightarrow{t}_T Z \xrightarrow{s}_T Y$, and this means $Z = \phi(t, X)$, and $Y = \phi(s, Z) = \phi(s, \phi(t, X)) = \phi(t + s, X)$. Thus, $X \xrightarrow{t+s}_T Y$, and this proves $\xrightarrow{s}_T \circ \xrightarrow{t}_T \subseteq \xrightarrow{t+s}_T$. $\square$

We should now look into the matter of MLTS morphisms, and we take our definition from G. Malcolm's paper [SODS].

Definition 2.4.5 [MLTS morphism]. A MLTS morphism from $T = (S, M, \cdot, e, \delta)$ to $T' = (S', M', \cdot', e', \delta')$ is a pair of set maps $(\sigma : S \rightarrow S', \lambda : M \rightarrow M')$ such that :

- λ is a monoid morphism from $(M, \cdot, e)$ to $(M', \cdot', e')$;
- For any $X, Y \in S$ and $m \in M$, $X \xrightarrow{m}_T Y$ implies that $\sigma(X) \xrightarrow{\lambda(m)}_{T'} \sigma(Y)$.

Definition 2.4.6 [Category of MLTS]. We define the *category of monoid labelled transition systems*, $\mathbb{M}$, as the category that has MLTS for its objects (as in Definition 2.4.1) and MLTS morphisms for its arrows (as in Definition 2.4.5). Equality, composition, and identities are given by componentwise equality, composition and identities respectively of the components of a morphism regarded as set functions.

And it is straightforward to verify that this is a category.

Proposition 2.4.7 . $\mathbb{M}$ is a well-defined category

As we stated earlier, there exists an adjunction between $\mathbb{T}$ and $\mathbb{M}$. To elaborate this adjunction properly, we recall that an adjunction between $\mathbf{Set}_\varepsilon$ and $\mathbf{Mon}$ was provided in Proposition 1.2.12 :

$$\begin{array}{ccc} & (-)^* & \\ \text{Set}_\varepsilon & \xrightarrow{\quad} & \mathbf{Mon} \\ & \perp & \\ & G_{m\varepsilon} & \end{array}$$

where a labelling set L in Set_ε is sent to the freely generated monoid L^* in $\mathbf{Mon}$, and where the functor $G_{m\varepsilon}$ sends a monoid $(M, \cdot, e)$ to the underlying labelling set $M \setminus \{e\}$ in Set_ε . We will use the bijective correspondences $\theta_{L_0, M}$ on the hom-sets, as provided in the proof of Proposition 1.2.12 of the adjunction between Set_ε and $\mathbf{Mon}$, in order to build the proof of the adjunction between $\mathbb{T}$ and $\mathbb{M}$. This goes as follows :

Theorem 2.4.8 [Adjunction between $\mathbb{T}$ and $\mathbb{M}$]. *There is an adjunction between $\mathbb{T}$ and $\mathbb{M}$ with functors F_{tm} and G_{mt} defined as follows :*

$$\begin{array}{ccc} & F_{tm} & \\ \mathbb{T} & \xrightarrow{\quad} & \mathbb{M} \\ & \perp & \\ & G_{mt} & \end{array}$$

1. Define $F_{tm} : \mathbb{T} \rightarrow \mathbb{M}$ as the functor that sends a labelled transition system $T = (S, L, \delta)$ to

$$F_{tm}(T) = (S, M, \cdot, e, \tilde{\delta})$$

where the set of states is preserved, $M = L^*$, $e = \Lambda$ is the empty word of L^* and $\cdot$ is word concatenation in L^* . We define $\xrightarrow{\Lambda}_{F_{tm}(T)} := \Delta_S$, and for any $a_1 \dots a_n \in L^*$, $\xrightarrow{a_1 \dots a_n}_{F_{tm}(T)} := \xrightarrow{a_n}_T \circ \dots \circ \xrightarrow{a_1}_T$, and this determines the transition specifier of $F_{tm}(T)$ that we have designated as $\tilde{\delta}$.

Given a LTS morphism $(\sigma, \lambda) : T_0 \rightarrow T$, we define $F_{tm}(\sigma, \lambda) : F_{tm}(T_0) \rightarrow F_{tm}(T)$ as $F_{tm}(\sigma, \lambda) = (\sigma, \lambda^*)$ where λ^* is an extension of the labelling morphism λ that is provided by the functor $(-)^*$ from $\mathbf{Set}_\varepsilon$ to $\mathbf{Mon}$ in Definition 1.2.7.

2. Define $G_{mt} : \mathbb{M} \rightarrow \mathbb{T}$ as the functor that sends a monoid labelled transition system $(S, M, \cdot, e, \delta)$ to the labelled transition system (S, L, δ') where $L = G_{m\varepsilon}(M) = M \setminus \{e\}$ and $\delta' = \delta \setminus \{(X, e, X) \mid X \in S\}$ (thus G_{mt} simply forgets composition in the labelling monoid, and removes the neutral element so that the latter will not conflict with ε).

Given a MLTS morphism $(\sigma, \lambda) : T_0 \rightarrow T$, we define $G_{mt}(\sigma, \lambda) : G_{mt}(T_0) \rightarrow G_{mt}(T)$ as $G_{mt}(\sigma, \lambda) = (\sigma, G_{m\varepsilon}(\lambda))$ where $G_{m\varepsilon}$ is the functor from $\mathbf{Mon}$ to $\mathbf{Set}_\varepsilon$ in Definition 1.2.10.

We have that F_{tm} is left adjoint to G_{mt} .

Proof. The proof is provided in Appendix C. □

We should remark that these functors wouldn't work so easily if we had not allowed ghost actions in our systems. Indeed, for a labelled transition system T in general, $F_{tm}(T)$ will have a freely generated labelling monoid which will naturally admit words for which there are no corresponding transitions in $F_{tm}(T)$. If we are to apply G_{mt} afterwards, these inadmissible sequences would introduce ghost actions in $G_{mt}(F_{tm}(T))$. It seems possible to introduce a version of these functors where ghost actions are directly removed as we move from $\mathbb{M}$ to $\mathbb{T}$, but we decided to make things less complicated simply by allowing ghost actions for objects of $\mathbb{T}$.

Now, the following theorem was given by G. Malcolm in [SSTS] :

Theorem 2.4.9 [G. Malcolm]. $\mathbb{M}$ is complete

Before we close this section, we would like to make a few statements about the fact that we used the term “relational presheaf” in Definition 2.4.2 to qualify the

nature of $\xrightarrow{(-)}_T$ for a given monoid labelled transition system T . It so happens that relational presheaves are structures that are more general than *Rel*-valued presheaves. Relational presheaves are defined by K. Rosenthal in [ThQuan] (and are studied by P. Sobociński in [RelPTS] in parallel to MLTS) and their definition goes as follows :

Definition 2.4.10 [Relational Presheaf]. Given a category $\mathcal{C}$. A *relational presheaf* on $\mathcal{C}$ is a lax functor $\mathcal{R} : \mathcal{C}^{op} \rightarrow Rel$, i.e. $\mathcal{R}$ maps the objects A of $\mathcal{C}$ to sets $\mathcal{R}(A)$ in *Rel* and the arrows $f : A \rightarrow B$ in $\mathcal{C}$ to relations $\mathcal{R}(f) : \mathcal{R}(B) \rightarrow \mathcal{R}(A)$ in *Rel* such that :

1. $\mathcal{R}(1_A) \supseteq \Delta_{\mathcal{R}(A)}$ for all objects A in $\mathcal{C}$
2. $\mathcal{R}(g \circ f) \supseteq \mathcal{R}(f) \circ \mathcal{R}(g)$ for all arrows $A \xrightarrow{f} B \xrightarrow{g} C$ in $\mathcal{C}$

Any *Rel*-valued presheaf is also a relational presheaf because a *Rel*-valued presheaf satisfies stronger equations :

1. $\mathcal{R}(1_A) = \Delta_{\mathcal{R}(A)}$ for all objects A in $\mathcal{C}$
2. $\mathcal{R}(g \circ f) = \mathcal{R}(f) \circ \mathcal{R}(g)$ for all arrows $A \xrightarrow{f} B \xrightarrow{g} C$ in $\mathcal{C}$

In particular, since the relational presheaf $\xrightarrow{(-)}_T$ associated to a monoid labelled transition system T is a *Rel*-valued presheaf, then it is also a relational presheaf (and we decided to keep the name as such for Definition 2.4.2).

Two kinds of morphisms have been studied for relational presheaves :

Definition 2.4.11 . Fix a category $\mathcal{C}$ and two relational presheaves $\mathcal{R}, \mathcal{S} : \mathcal{C}^{op} \rightarrow Rel$.

1. An *rp-morphism* from $\mathcal{R}$ to $\mathcal{S}$ is an oplax¹⁴ natural transformation φ from $\mathcal{R}$ to $\mathcal{S}$ that consists of a family of **functions** (regarded as arrows in *Rel*) indexed by the objects of $\mathcal{C}$, say $\{\varphi_C\}_{C \in \mathcal{C}}$ where $\varphi_C : \mathcal{R}(C) \rightarrow \mathcal{S}(C)$, such that for any

¹⁴The use of the terms “oplax” and “lax” for natural transformations is not entirely consistent in the literature. We used the category theory website nLab’s formulation here: see <http://ncatlab.org/nlab/show/lax+natural+transformation>

morphism $f : B \rightarrow A$ in $\mathcal{C}$, we have $\varphi_B \circ \mathcal{R}(f) \subseteq \mathcal{S}(f) \circ \varphi_A$, i.e. the inclusion in the diagram of Figure 11 holds.

$$\begin{array}{ccc}
 \mathcal{R}(A) & \xrightarrow{\varphi_A} & \mathcal{S}(A) \\
 \mathcal{R}(f) \downarrow & \subsetneq & \downarrow \mathcal{S}(f) \\
 \mathcal{R}(B) & \xrightarrow{\varphi_B} & \mathcal{S}(B)
 \end{array}$$

Figure 11: An inclusion diagram for an oplax natural transformations in relational presheaves.

2. A *generalized rp-morphism* from $\mathcal{R}$ to $\mathcal{S}$ is an oplax natural transformation φ from $\mathcal{R}$ to $\mathcal{S}$ that consists of a family of **relations** indexed by the objects of $\mathcal{C}$, say $\{\varphi_C\}_{A \in \mathcal{C}}$ where $\varphi_C : \mathcal{R}(C) \rightarrow \mathcal{S}(C)$, such that for any morphism $f : B \rightarrow A$ in $\mathcal{C}$, we have $\varphi_B \circ \mathcal{R}(f) \subseteq \mathcal{S}(f) \circ \varphi_A$, i.e. the inclusion in the diagram of Figure 11 holds.

We can relate *rp-morphisms* to what we call *functional simulations*. In order to do this, we need the following definition :

Definition 2.4.12 [Simulation and Bisimulation]. Given labelled transition systems (or MLTS) $T = (S, L, \delta)$ and $T' = (S', L, \delta')$ with the same labelling set L , we define a *simulation* from T to T' as a relation $R \subseteq S \times S'$ on the set of states, such that for all $X, X' \in S$ and $Y \in S'$ and $b \in L$:

If $X \stackrel{R}{\sim} X'$ and $X \xrightarrow{b}_T Y$, then there exists $Y' \in S'$ such that $Y \stackrel{R}{\sim} Y'$ and $X' \xrightarrow{b}_{T'} Y'$.

Furthermore, we say that R is a *bisimulation* if its inverse relation R^{-1} is also a simulation (in which case, we say that T and T' are *bisimilar*). Finally, if R is a function, we say that it is a *functional simulation*.

Remark 2.4.13 . We say that T' *simulates* T if there is a simulation from T to T' .

We can provide an equivalent form to the concept of simulation in terms of the relational language. The proof of the following proposition is quite direct.

Proposition 2.4.14 [Simulations in the Relational Language]. *Let $T = (S, L, \delta)$ and $T' = (S', L, \delta')$ be labelled transition systems (or MLTS). A relation $R \subseteq S \times S'$ is a simulation from T to T' if and only if $\xrightarrow{b}_T \circ R^{-1} \subseteq R^{-1} \circ \xrightarrow{b}_{T'}$ for all $b \in L$, i.e. the obvious lax inclusion in the following diagram holds :*

$$\begin{array}{ccc}
 S & \xrightarrow{\quad b \quad}_T & S \\
 \uparrow R^{-1} & \lrcorner & \uparrow R^{-1} \\
 S' & \xrightarrow{\quad b \quad}_{T'} & S'
 \end{array}$$

Now, the difference between rp -morphisms and generalized rp -morphisms is that the former insist on components that are functions, whereas the latter allows any kind of relations for their components.

Given a one object category $(M, e, \cdot)$ and two monoid labelled transition systems $T = (S, M, e, \cdot, \delta)$ and $T' = (S', M, e, \cdot, \delta')$ with a common labelling monoid, we obtain that an rp -morphism (or generalized rp -morphism) from the relational presheaf $\xrightarrow{(-)}_T$ to $\xrightarrow{(-)}_{T'}$ would have a single component $R : S \rightarrowtail S'$ and the oplaxness condition would reflect as :

$$R \circ \xrightarrow{m}_T \subseteq \xrightarrow{m}_{T'} \circ R \quad (\text{for any } m \in M)$$

If this R is a function (i.e. an rp -morphism), then R is a functional simulation, and this can be established through the following proposition :

Proposition 2.4.15 . *For any relations $K : A \rightarrowtail A$, $K' : A' \rightarrowtail A'$, and $R : A \rightarrowtail A'$ such that R is a function,*

$$K \circ R^{-1} \subseteq R^{-1} \circ K' \Leftrightarrow R \circ K \subseteq K' \circ R$$

Proof. An easy proof. □

Corollary 2.4.16 . *Consider any relation $R : S \rightarrowtail S'$ in $\mathbf{Rel}$ and any monoid labelled transition systems $T = (S, M, e, \cdot, \delta)$ and $T' = (S', M, e, \cdot, \delta')$. Then the following are equivalent :*

- *R is the single component of an rp -morphism from $\xrightarrow{(-)}_T$ to $\xrightarrow{(-)}_{T'}$*
- *R is a functional simulation from T to T'*

Proof. By Proposition 2.4.14, R is a functional simulation from T to T' if and only if R is a function such that $\forall m \in M, \xrightarrow{m}_T \circ R^{-1} \subseteq R^{-1} \circ \xrightarrow{m}_{T'}$. By Proposition 2.4.15 above, we get that R is a functional simulation from T to T' if and only if R is a function such that $\forall m \in M, R \circ \xrightarrow{m}_T \subseteq \xrightarrow{m}_{T'} \circ R$, and this corresponds with the single component of an rp -morphism from $\xrightarrow{(-)}_T$ to $\xrightarrow{(-)}_{T'}$. □

Unfortunately, the equivalence in Proposition 2.4.15 does not extend to a rapport with simulations in general if R is not a function (i.e. if R is a generalized rp -morphism). Take for example, two labelled transition systems¹⁵ T and T' on a set of states $S = \{x, y, z\}$ and labelling set $L = \{m\}$. Set the transition relations as $\xrightarrow{m}_T = \{(x, z)\}$ and $\xrightarrow{m}_{T'} = \{(y, z)\}$, and consider $R = \{(x, y)\}$. As we can see in Figure 12, we get $R \circ \xrightarrow{m}_T \subseteq \xrightarrow{m}_{T'} \circ R$ because $\emptyset \subseteq \{(x, z)\}$. However $\xrightarrow{m}_T \circ R^{-1} = \{(y, z)\}$ is not included in $R^{-1} \circ \xrightarrow{m}_{T'} = \emptyset$. This means that R is not a simulation from T to T' .

¹⁵We should consider MLTS technically, but this is not a problem since the example extends naturally to freely generated MLTS.

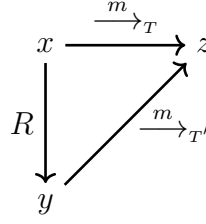


Figure 12: A counter-example to $R \circ \xrightarrow{m}_{\rightarrow_T} \subseteq \xrightarrow{m}_{\rightarrow_{T'}} \circ R \Rightarrow \xrightarrow{m}_{\rightarrow_T} \circ R^{-1} \subseteq R^{-1} \circ \xrightarrow{m}_{\rightarrow_{T'}}$.

What this means is that we got the wrong idea if we thought that generalized rp -morphisms expressed simulations in general. What they express in fact for MLTS is something quite unnatural :

$$\begin{aligned} R \circ \xrightarrow{m}_{\rightarrow_T} \subseteq \xrightarrow{m}_{\rightarrow_{T'}} \circ R &\Leftrightarrow (R \circ \xrightarrow{m}_{\rightarrow_T})^{-1} \subseteq (\xrightarrow{m}_{\rightarrow_{T'}} \circ R)^{-1} \\ &\Leftrightarrow \xrightarrow{m}_{\rightarrow_T}^{-1} \circ R^{-1} \subseteq R^{-1} \circ \xrightarrow{m}_{\rightarrow_{T'}}^{-1} \end{aligned}$$

which basically means that R is a simulation of the inverse of the transition relation of T to the inverse of the transition relation of T' ... the simulation is going backwards!

If ever we wanted to have a notion of “natural transformation” that accounts for simulations through relational equations, we would have no choice but to involve the inverse of the component relations somehow, and we would probably replace the condition $\varphi_B \circ \mathcal{R}(f) \subseteq \mathcal{S}(f) \circ \varphi_A$ with $\mathcal{R}(f) \circ \varphi_A^{-1} \subseteq \varphi_B^{-1} \circ \mathcal{S}(f)$ in the definition of the generalized rp -morphisms.

Chapter 3

Independence of Actions for Labelled Transition Systems

As we discussed in the introduction of this thesis, there are circumstances in a distributed system where processes (or actions) held in separate components or regions do not interfere with each other, i.e. are independent of each other. The objective of this chapter is to investigate a formalization of this kind of independence within the context of LTS. This will allow us to render *spatially induced independence* (SI-independence) for presheaves of LTS later on.

As far as the expression of such independence in LTS is concerned, there are two LTS models that have been studied by Winskel and Nielsen in [MC], where an independence relation is provided at the level of actions (or transitions), and they are the following :

- *Asynchronous labelled transition systems* (ALTS) are labelled transition systems equipped with an independence relation, given as a binary relation on the set of actions. It is then imposed that independent actions in this relation have associated transition relations that commute in various ways with respect to relational composition. These ALTS models originate (independently) from the work of Bednarczyk in [CAS] and Shields in [CMach].

- *Labelled transition systems with independence* (LTSI) are labelled transition systems equipped with an independence relation, given as a binary relation on the set of transitions (rather than the set of actions as in ALTS). Independent transitions have properties that essentially reflect the properties of transitions of independent actions within ALTS. An interesting aspect of LTSI is that an equivalence relation on transitions is derived from their independence relation, and this equivalence extends to runs in the underlying LTS of such LTSI. These LTSI are due to Winskel and Nielsen in [MC].

Presheaves of LTS have a natural potential to transform into presheaves of ALTS via a functor (if we add a few natural axioms to the former), and establishing the functor in question is the main result of this thesis. The form of independence that is generated via this functor is the one we refer to as SI-independence.

There are more properties exhibited by independent actions within such presheaves of LTS than those that are accounted for by ALTS, and we need to make a slight change to the usual definition of ALTS if we want to capture these properties properly. Thus, in Section 3.2, we provide a new definition of ALTS, where this minor change is incorporated, and we explore the basic theory of such ALTS there. In particular, we provide a functor from the category of ALTS to the category of MLTS that makes use of the Mazurkiewicz trace monoid (see Definition 3.1.5 ahead) that can be typically associated to an ALTS.

There are also equivalences on transitions that are naturally induced within presheaves of LTS. The elaboration of these equivalences is somewhat orthogonal to the elaboration of SI-independence and, unfortunately, we get that the equivalences in LTSI do not account for those in presheaves of LTS properly. Thus, to address the needs of presheaves of LTS at the level of these induced equivalences, we formulate a new concept, that of an *asynchronous labelled transition system with equivalence* (ALTSE). These ALTSE are investigated in Section 3.3, and they are essentially ALTS with an additional equivalence on transitions provided as a primitive notion (not derived from the independence relation) within their signature. The way in which these ALTSE

are formulated borrows ideas from the axiomatization of an LTSI, and this allows the equivalence on transitions to behave properly with respect to the independence relation of such systems. In particular, we will see how the equivalence on transitions extends to an equivalence on runs (as in LTSI), and we will close this chapter with a conjecture that the equivalence classe of a run is in bijective correspondance with the equivalence class of its underlying sequences of actions (see Conjecture 3.3.16).

Now, the concept of ALTS is built on top of *concurrent alphabets*, so we must explore those first in Section 3.1.

3.1 Concurrent Alphabets

The concept of an asynchronous labelled transition system directly builds on top of *concurrent alphabet* structures as presented by A. Mazurkiewicz in [TT]. Thus, we need to elaborate on the theory of the latter before we proceed to the analysis of ALTS. The definition of such concurrent alphabets is as follows :

Definition 3.1.1 [Concurrent Alphabets and Morphisms]. A *concurrent alphabet* is a pair (L, I) where L is a labelling set (the alphabet), which means that it is an object of Set_ε that does not contain the distinguished symbol ε , and $I \subseteq L \times L$ is a symmetric and irreflexive binary relation on the alphabet. Given $b, c \in L$, we say that b is *independent of* c whenever $b I c$.

A *concurrent alphabet morphism* from a concurrent alphabet (L, I) to another (L', I') is a labelling morphism $\lambda : L \rightarrow_\varepsilon L'$ in Set_ε that preserves the independence relation :

$$\forall b, c \in L, [b I c \text{ and } \lambda(b), \lambda(c) \neq \varepsilon] \Rightarrow \lambda(b) I' \lambda(c)$$

Setting equality, composition and identity for morphisms as given in Set_ε , we form the *category of concurrent alphabets*, written as $\mathbb{L}$.

Remark 3.1.2 . For a string $w \in L^*$ for some labelling set L , Mazurkiewicz uses the notation $Alph(w) = \{b \in L \mid b \text{ occurs has a letter in } w\}$ to designate the set of

actions that are part of the string w . For $b \in L$ and $w \in L^*$, he also uses the notation $b \ I \ w$ to represent the statement that $\forall c \in \text{Alph}(w), \ b \ I \ c$. We shall adopt these conventions here.

Remark 3.1.3 [Trace Languages]. We can turn concurrent alphabets into a form of abstract process model, where the notion of state is absent, by adding a set of *admissible sequences* (say $H \subseteq L^*$, respecting a few axioms) to the concurrent alphabet structures. These are referred to as *Mazurkiewicz trace languages* in [MC]. In the case where no notion of independence is provided, we get *Hoare trace languages*¹. Our intention however is not to make use of these abstract models, but simply to characterize trace monoids for which the concept of concurrent alphabet suffices.

We can extend an independence relation I to incorporate ε and facilitate the notion of concurrent alphabet morphism above. The extension in question is given as $I_\varepsilon \subseteq L_\varepsilon \times L_\varepsilon$ where $I_\varepsilon = I \sqcup (\{(b, \varepsilon) \mid b \in L_\varepsilon\} \cup \{(\varepsilon, b) \mid b \in L_\varepsilon\})$. Thus, for the morphism definition above, the condition that :

$$\forall b, c \in L, [b \ I \ c \text{ and } \lambda(b), \lambda(c) \neq \varepsilon] \Rightarrow \lambda(b) \ I' \lambda(c)$$

is now equivalent to :

$$\forall b, c \in L, b \ I \ c \Rightarrow \lambda(b) \ I'_\varepsilon \lambda(c) \text{ }^2$$

¹We can associate a Hoare Trace Language to any labelled transition system $T = (S, L, \delta)$ by setting $H = \{a_1 \dots a_n \in L^* \mid \xrightarrow{a_n}_T \circ \dots \circ \xrightarrow{a_1}_T \neq \emptyset\}$.

²We remark that it is not necessary to include I_ε in the hypothesis of the implication here because ε maps to ε , and the latter is always independent of every other action by definition. One thing we should observe, however, is that I_ε is not irreflexive (in contrast to I), and the only exception to irreflexivity is the pair $(\varepsilon, \varepsilon) \in I_\varepsilon$.

The category $\mathbb{L}$ as formulated in Definition 3.1.1 is described in [MC] by Winskel and Nielsen.³ However, it is not conceptually presented as a category of concurrent alphabets in that paper, and the formulation we took here simply aims to establish a connection with Mazurkiewicz's concept of concurrent alphabets. Indeed, our intention is to extend concurrent alphabets to trace monoids, and the way this is achieved actually builds up to a functor from $\mathbb{L}$ to **Mon** (Definition 3.1.12). This functor will then be extended to a functor from the category of ALTS to the category of MLTS⁴.

To construct the functor from $\mathbb{L}$ to **Mon**, we must first map concurrent alphabet objects to monoid objects. To achieve this, we need to quotient the freely generated monoid L^* , for a given concurrent alphabet (L, I) , by using a congruence relation on words ($\in L^*$) derived from the independence relation I . The way this equivalence works is that two words are equivalent if one can be obtained from the other by permuting adjacent actions that are independent. For example, if $b, c, d \in L$ and $c I b$ and $c I d$, then c commutes with b and d through the equivalence, and we get that the string bc is equivalent to the string cb , and the string $cbdc$ is equivalent to the string bdc in this case. We formalize the equivalence in question as follows :

Definition 3.1.4 [Abstract Trace Equivalence [TT]]. Let (L, I) be a concurrent alphabet and let L^* be the freely generated monoid over L . The *abstract trace equivalence* for I is the smallest congruence⁵ relation $\equiv_I$ on L^* such that :

$$\forall b, c \in L, b I c \Rightarrow bc \equiv_I cb$$

From this, we can derive the trace monoid of a concurrent alphabet by quotienting the words of L^* with the abstract trace equivalence.

³The category in question is given in Section 8.3.3 of [MC] and the notation $\mathbf{Set}_I$ is used to represent it.

⁴A similar kind of extension was achieved before when we extended the functor $(-)^* : \mathbf{Set}_\varepsilon \rightarrow \mathbf{Mon}$ to the functor $F_{tm} : \mathbb{T} \rightarrow \mathbb{M}$. Yet, the functor that we will obtain from the category of ALTS to the category of MLTS in this section is quite different from F_{tm} as we shall see.

⁵We would usually call the abstract trace equivalence, an abstract trace congruence. But the literature refers to it as a trace equivalence, and we adhere to this convention.

Definition 3.1.5 [Trace Monoid [TT]]. Let (L, I) be a concurrent alphabet and let L^* be the freely generated monoid over L . The *trace monoid* over (L, I) is the quotient monoid $M_{(L,I)} = L^* / \equiv_I$. The elements of this monoid are called the *abstract traces* of (L, I) . We write $[-]_I : L^* \rightarrow M_{(L,I)}$ to denote the quotient map in **Mon** whenever the labelling set L is clear in the context.

Remark 3.1.6 . The usual term to designate an “abstract trace” is simply that of a “trace”. We added the term “abstract” to distinguish abstract traces from runs in LTS (which can be thought of as traces). Similarly, the usual term for “abstract trace equivalence” is “trace equivalence” in Definition 3.1.4, and the reason we use the word “abstract” there is that we will derive an equivalence on runs (which are traces) later that we do not wish to confuse with the abstract trace equivalence.

Remark 3.1.7 . We remark that if $[a_1 \cdots a_n]_I = [a'_1 \cdots a'_m]_I$ in $M_{(L,I)}$, then $n = m$ and $a_{\sigma(i)} = a'_i$ for a permutation σ of $\{1, \dots, m\}$. That is, the abstract trace equivalence only commutes letters in the words; it does not alter the letters of a word. In particular, we get $\text{Alph}(a_1 \dots a_n) = \text{Alph}(a'_1 \dots a'_n)$.

Now, there are a few propositions and lemmas that we will need to establish the functor from $\mathbb{L}$ to **Mon** that we have in mind.

Proposition 3.1.8 . Consider any concurrent alphabet (L, I) . For all strings $w_0, w \in L^*$ and action $a \in L$, if $w_0 \equiv_I wa$, then there exists $u, v \in L^*$ such that $w_0 = uav$ and $a \notin \text{Alph}(v)$.

Proof. Suppose that $w_0 \equiv_I wa$ as in the above statement. By Remark 3.1.7, we have that $a \in \text{Alph}(w_0)$. But then, we can take the rightmost position in w_0 for which an a can be found. Let u be the substring of w_0 that precedes this occurrence of a in w_0 , and let v be the substring of w_0 that follows this occurrence of a in w_0 . We get $w_0 = uav$, and since this occurrence of a resides at the rightmost position for which there is an a , we have $a \notin \text{Alph}(v)$. $\square$

Proposition 3.1.9 [A. Mazurkiewicz, [TT]]. *Consider any concurrent alphabet (L, I) . For all strings $w, u, v \in L^*$ and symbol $a \notin \text{Alph}(v)$,*

$$uav \equiv_I wa \Rightarrow a I v.$$

Mazurkiewicz also makes the remark that given $w \equiv_I w'$ for some strings $w, w' \in L^*$, then for any action $a \in \text{Alph}(w) = \text{Alph}(w')$, we can cancel the rightmost appearance of a in w and w' , and the equivalence is preserved. We combine this idea with the above statement to yield the following :

Corollary 3.1.10 [A. Mazurkiewicz, [TT]]. *Consider any concurrent alphabet (L, I) . For all strings $w, u, v \in L^*$ and symbol $a \notin \text{Alph}(v)$,*

$$uav \equiv_I wa \Rightarrow [a I v \text{ and } uv \equiv_I w].$$

Combining Proposition 3.1.8 and Corollary 3.1.10, we get the following :

Corollary 3.1.11 . *Consider any concurrent alphabet (L, I) . For all strings $w_0, w \in L^*$ and action $a \in L$, if $w_0 \equiv_I wa$, then there exists $u, v \in L^*$ such that $w_0 = uav$ and $a I v$ and $uv \equiv_I w$.*

This last corollary is mainly what we will use to establish the functor from $\mathbb{L}$ to $\mathbf{Mon}$ that we talked about before the trace monoid definition. This goes as follows :

Definition 3.1.12 [Functor from $\mathbb{L}$ to $\mathbf{Mon}$]. We define a functor $F_{lm} : \mathbb{L} \rightarrow \mathbf{Mon}$ as follows :

- F_{lm} maps a concurrent alphabet object (L, I) in $\mathbb{L}$ to the trace monoid $F_{lm}(L, I) = M_{(L, I)}$ from Definition 3.1.5.

- Given a concurrent alphabet morphism $\lambda : (L_0, I_0) \rightarrow (L, I)$ in $\mathbb{L}$, we define $F_{lm}(\lambda) : M_{(L_0, I_0)} \rightarrow M_{(L, I)}$ as

$$F_{lm}(\lambda)([w]_{I_0}) = [\lambda^*(w)]_I^6$$

for any $[w]_{I_0} \in M_{(L_0, I_0)}$ where $w \in L_0^*$.

Proposition 3.1.13 . *F_{lm} is indeed a functor.*

Proof. We already know that $F_{lm}(L, I) = M_{(L, I)}$ is a well-defined monoid for any concurrent alphabet (L, I) . Consider any concurrent alphabet morphism $\lambda : (L_0, I_0) \rightarrow (L, I)$ in $\mathbb{L}$. We must verify that $F_{lm}(\lambda)$ is a well-defined monoid morphism from $M_{(L_0, I_0)}$ to $M_{(L, I)}$. We start by showing that it is a well-defined map, i.e. we show that :

$$\forall w, w' \in L_0^*, w \equiv_{I_0} w' \Rightarrow \lambda^*(w) \equiv_I \lambda^*(w')$$

We proceed by induction on the length of w (which is the same as the length of w' since $w \equiv_{I_0} w'$).

For the case $n = 0$, we only have $w = \Lambda = w'$, and thus $\lambda^*(w) = \lambda^*(w')$.

Fix some $n > 0$. Suppose that for any strings w, w' of size n , if $w \equiv_{I_0} w'$, then $\lambda^*(w) \equiv_I \lambda^*(w')$. Consider any strings $a_1 \dots a_{n+1}$ and $b_1 \dots b_{n+1}$ in L_0^* of size $n + 1$ such that $a_1 \dots a_{n+1} \equiv_{I_0} b_1 \dots b_{n+1}$. By Corollary 3.1.11, there is a $j \in \{1 \dots n + 1\}$ such that $a_{n+1} = b_j$ and $b_j \ I (b_{j+1} \dots b_{n+1})$ and $a_1 \dots a_n \equiv_{I_0} b_1 \dots b_{j-1} b_{j+1} \dots b_{n+1}$. By our induction hypothesis, we get $\lambda^*(a_1 \dots a_n) \equiv_I \lambda^*(b_1 \dots b_{j-1} b_{j+1} \dots b_{n+1})$. There are now two subcases depending on whether $\lambda(b_j) = \varepsilon$ or not.

⁶ $(-)^*$ is the functor from $\mathbf{Set}_\varepsilon$ to $\mathbf{Mon}$ from Definition 1.2.7; and so λ^* is a monoid morphism from L to L^* .

If $\lambda(b_j) = \varepsilon$, then the definition of λ^* gives $\lambda^*(a_{n+1}) = \lambda^*(b_j) = \Lambda$ and

$$\begin{aligned}
 \lambda^*(a_1 \dots a_{n+1}) &= \lambda^*(a_1 \dots a_n) \lambda^*(a_{n+1}) \\
 &= \lambda^*(a_1 \dots a_n) \\
 &\equiv_I \lambda^*(b_1 \dots b_{j-1} b_{j+1} \dots b_{n+1}) \\
 &= \lambda^*(b_1 \dots b_{j-1}) \lambda^*(b_{j+1} \dots b_{n+1}) \\
 &= \lambda^*(b_1 \dots b_{j-1}) \lambda^*(b_j) \lambda^*(b_{j+1} \dots b_{n+1}) \\
 &= \lambda^*(b_1 \dots b_{n+1})
 \end{aligned}$$

If $\lambda(b_j) \neq \varepsilon$, then for any $k > j$, we can show first that $\lambda^*(b_j) \lambda^*(b_k) \equiv_I \lambda^*(b_k) \lambda^*(b_j)$. Indeed, if $\lambda(b_k) = \varepsilon$, then $\lambda^*(b_j) \lambda^*(b_k) = \lambda^*(b_j) = \lambda^*(b_k) \lambda^*(b_j)$. And if $\lambda(b_k) \neq \varepsilon$, then the preservation of independence for λ gives us $\lambda(b_j) I \lambda(b_k)$, and in particular, $\lambda^*(b_j) \lambda^*(b_k) = \lambda(b_j) \lambda(b_k) \equiv_I \lambda(b_k) \lambda(b_j) = \lambda^*(b_k) \lambda^*(b_j)$, by definition of $\equiv_I$.

This means that :

$$\begin{aligned}
 \lambda^*(b_1) \dots \lambda^*(b_{j-1}) \lambda^*(b_{j+1}) \dots \lambda^*(b_{n+1}) \lambda^*(b_j) &\equiv_I \lambda^*(b_1) \dots \lambda^*(b_{j-1}) \lambda^*(b_j) \lambda^*(b_{j+1}) \dots \lambda^*(b_{n+1}) \\
 &= \lambda^*(b_1 \dots b_{n+1})
 \end{aligned}$$

Finally, we get

$$\begin{aligned}
 \lambda^*(a_1 \dots a_{n+1}) &= \lambda^*(a_1 \dots a_n) \lambda^*(a_{n+1}) \\
 &= \lambda^*(a_1 \dots a_n) \lambda^*(b_j) \\
 &\equiv_I \lambda^*(b_1 \dots b_{j-1} b_{j+1} \dots b_{n+1}) \lambda^*(b_j) \\
 &= \lambda^*(b_1) \dots \lambda^*(b_{j-1}) \lambda^*(b_{j+1}) \dots \lambda^*(b_{n+1}) \lambda^*(b_j) \\
 &\equiv_I \lambda^*(b_1 \dots b_{n+1})
 \end{aligned}$$

This proves that $F_{lm}(\lambda)$ is a well-defined map.

$F_{lm}(\lambda)$ is also a monoid morphism since for any $[w_1]_{I_0}, [w_2]_{I_0} \in M_{(L_0, I_0)}$, we have $F_{lm}(\lambda)([w_1]_{I_0} \cdot [w_2]_{I_0}) = F_{lm}(\lambda)([w_1 w_2]_{I_0}) = [\lambda^*(w_1 w_2)]_I = [\lambda^*(w_1) \lambda^*(w_2)]_I = [\lambda^*(w_1)]_I \cdot [\lambda^*(w_2)]_I = F_{lm}(\lambda)([w_1]_{I_0}) \cdot F_{lm}(\lambda)([w_2]_{I_0})$.

It remains to verify that F_{lm} preserves identities and composition. Consider the identity $1_{(L,I)} : (L, I) \rightarrow (L, I)$ on the concurrent alphabet (L, I) in $\mathbb{L}$. For any $[w]_I \in M_{(L,I)}$, we get $F_{lm}(1_{(L,I)})([w]_I) = [1_{(L,I)}^*(w)]_I = [w]_I$. Thus, $F_{lm}(1_{(L,I)})$ is the identity on $M_{(L,I)}$. Now, consider any concurrent alphabet morphisms $\lambda_1 : (L_0, I_0) \rightarrow (L_1, I_1)$ and $\lambda_2 : (L_1, I_1) \rightarrow (L_2, I_2)$ in $\mathbb{L}$. Then for any $[w]_{I_0} \in M_{(L_0,I_0)}$, we get $F_{lm}(\lambda_2 \circ \lambda_1)([w]_{I_0}) = [(\lambda_2 \circ \lambda_1)^*(w)]_{I_2} = [(\lambda_2^* \circ \lambda_1^*)(w)]_{I_2} = [(\lambda_2^*(\lambda_1^*(w)))]_{I_2} = F_{lm}(\lambda_2)([\lambda_1^*(w)]_{I_1}) = F_{lm}(\lambda_2)(F_{lm}(\lambda_1)([w]_{I_0})) = (F_{lm}(\lambda_2) \circ F_{lm}(\lambda_1))([w]_{I_0})$. This shows that $F_{lm}(\lambda_2 \circ \lambda_1) = F_{lm}(\lambda_2) \circ F_{lm}(\lambda_1)$. $\square$

We now have everything that we need with respect to the subject of concurrent alphabets to start our investigation of asynchronous labelled transition systems (ALTS).

3.2 Asynchronous Labelled Transition Systems

In this section, we explore *asynchronous labelled transition systems* (ALTS) and we provide a functor from the category of ALTS to the category of MLTS (Proposition 3.2.10). This functor will differ from the functor $F_{tm} : \mathbb{T} \rightarrow \mathbb{M}$ that we have encountered previously in Theorem 2.4.8. Indeed, with ALTS, we have a way to quotient certain sequences of actions through the use of their independence relation on actions, and this allows for a very natural extension to MLTS.

The following definition of ALTS is derived from Winskel and Nielsen in [MC], but we make some notable changes to it. First of all, we do not work with a pointed initial state. Secondly, we don't reject ghost actions. Thirdly, and most importantly, we integrate the new concept of *one-step amalgamation*, which is a property that occurs naturally within (adapted) presheaves of LTS. As such, the ALTS format that we present here is novel.

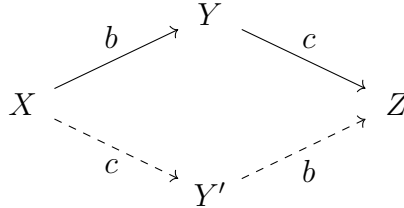
Also, given a labelled transition system T and action b in T , we will use the notation ${}_T\overleftarrow{b}$ to represent the relation $\xrightarrow{T}^{-1}_b$, i.e. $Y \xrightarrow{T}^{-1}_b X$ means that $X \xrightarrow{T}_b Y$ for states X and Y in T .

Definition 3.2.1 [Asynchronous Labelled Transition System].

An *asynchronous labelled transition system* is a structure $T = (S, L, \delta, I)$ where :

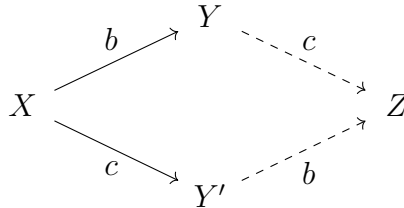
1. (S, L, δ) is a labelled transition system;
2. (L, I) is a concurrent alphabet;
3. **(Alternative Path) :**

For any $X, Y, Z \in S$ and $b, c \in L$, if $b I c$ and $X \xrightarrow{T}_b Y \xrightarrow{T}_c Z$, then there exists $Y' \in S$ such that $X \xrightarrow{T}_c Y' \xrightarrow{T}_b Z$.



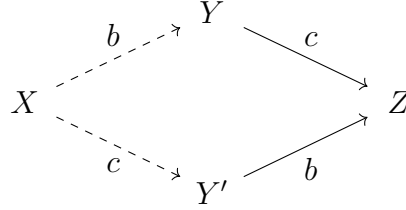
4. **(One-Step Co-amalgamation) :**

For any $X, Y, Y' \in S$ and $b, c \in L$, if $b I c$ and $Y \xrightarrow{T}^{-1}_b X \xrightarrow{T}_c Y'$, then there exists $Z \in S$ such that $Y \xrightarrow{T}_c Z \xrightarrow{T}^{-1}_b Y'$.



5. (One-Step Amalgamation) :

For any $Y, Y', Z \in S$ and $b, c \in L$, if $b I c$ and $Y \xrightarrow{c}_T Z \xleftarrow{b}_T Y'$, then there exists $X \in S$ such that $Y \xleftarrow{b}_T X \xrightarrow{c}_T Y'$.



The Alternative Path axiom can be formulated in simpler terms in the relational language as commutativity on the transition relations :

$$\forall(b, c) \in I, \quad \xrightarrow{c}_T \circ \xrightarrow{b}_T = \xrightarrow{b}_T \circ \xrightarrow{c}_T$$

The One-Step co-Amalgamation axiom is equivalent to the following :

$$\forall(b, c) \in I, \quad \xrightarrow{c}_T \circ \xleftarrow{b}_T \subseteq \xleftarrow{b}_T \circ \xrightarrow{c}_T$$

We might wonder as to what inclusion the other way around means, and it so happens that this is precisely equivalent to the statement of One-Step Amalgamation as :

$$\forall(b, c) \in I, \quad \xrightarrow{c}_T \circ \xleftarrow{b}_T \supseteq \xleftarrow{b}_T \circ \xrightarrow{c}_T$$

Thus, we could re-state the axioms of Amalgamation and co-Amalgamation as a single axiom, and write :

$$\forall(b, c) \in I, \quad \xrightarrow{c}_T \circ \xleftarrow{b}_T = \xleftarrow{b}_T \circ \xrightarrow{c}_T$$

As we explained, the axiom of One-Step Amalgamation is usually not a part of the definition of ALTS, and it is certainly interesting to see how it is a complement to the axiom of One-Step co-Amalgamation. However, the true reason why it was incorporated in the definition above will surface later when we see how such amalgamation arises in the context of presheaves of LTS.

Example 3.2.2 [Free Group Word Rewriting]. For the free group over a set X , and using the list of symbols $K = X \sqcup \{x^{-1} \mid x \in X\}$, we can formulate an ALTS, say T , as follows :

- States : K^* (with Λ the empty word)
- Labels : $L = X$
- Transitions : We define $\xrightarrow{x}_T$ to be the smallest transition relation (associated to $x \in L$) such that :

$$\forall x_1 \cdots x_k, y_1 \cdots y_n \in K^* \quad (\text{with } x_i, y_i \in K),$$

$$x_1 \cdots x_k x x^{-1} y_1 \cdots y_n \xrightarrow{x}_T x_1 \cdots x_k y_1 \cdots y_n$$

$$x_1 \cdots x_k x^{-1} x y_1 \cdots y_n \xrightarrow{x}_T x_1 \cdots x_k y_1 \cdots y_n$$

$$x^{-1} x \xrightarrow{x}_T \Lambda$$

$$x x^{-1} \xrightarrow{x}_T \Lambda$$

- Independence relation : $\forall x, y \in L, x I y \Leftrightarrow x \neq y$

Of course, when different symbols z and y in X are used, then the rewriting of one cannot interfere with the letters of the other, and we get diamond-shaped diagrams illustrating the Alternative Path axiom such as in Figure 13.

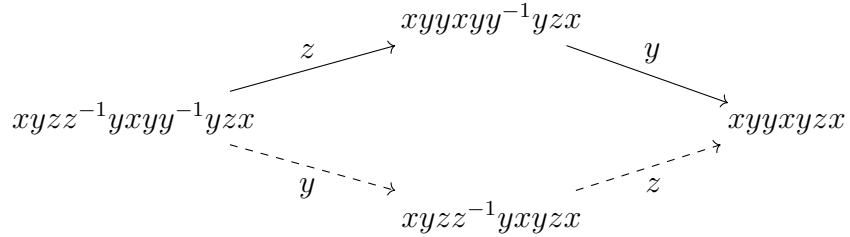


Figure 13: Example of alternative path for word rewriting.

On the other hand, when the same symbol is used, say y , we can have rewriting actions that overlap in a word. This leads to strange situations with transitions such

as $yy^{-1}y \xrightarrow{y}_T y$, where we cannot tell whether it was yy^{-1} that was eliminated, or if it was $y^{-1}y$. This is not exactly the type of independence that we will encounter with presheaves of LTS however, so we need not worry too much about such examples of ALTS.

Remark 3.2.3 . Through this example, we see that it is not enforced to have an action designating the simultaneous rewriting of y and z in the system; i.e. there is no middle arrow in the diagram of Figure 13. There is a way by which we can extend an ALTS in general with an operator “+” that accounts for the presence of such combined independent actions $a + b$ whenever $a \ I \ b$. However, this extension can be subsumed by the trace monoid if we let $a + b$ correspond with the equivalence class $[ab]_I = [ba]_I$, and so we simply stick to the trace monoid to account for such combined actions. We should note furthermore that it is not always desirable to have such combined actions $a + b$ for any independent actions a and b in an ALTS in general. For example, in a computer with a single processor unit, certain instructions that operate on the memory may be independent of each other, but that does not mean that we can perform two instructions at the same time.

Nevertheless, we will encounter “combined actions” of this format in presheaves of LTS sometimes, and we have a discussion about that in Appendix H at the end of the thesis.

We now extend LTS morphisms and concurrent alphabet morphisms to obtain ALTS morphisms. We present a definition of the latter that is derived from [MC], and it goes as follows :

Definition 3.2.4 [ALTS morphism]. An *ALTS morphism* from (S, L, δ, I) to (S', L', δ', I') is a LTS morphism $(\sigma : S \rightarrow S', \lambda : L \rightarrow_\varepsilon L')$ from (S, L, δ) to (S', L', δ') such that λ is a concurrent alphabet morphism from (L, I) to (L', I') .

Definition 3.2.5 [Category of ALTS]. We define the *category of asynchronous labelled transition systems*, denoted $\mathbb{A}$, that has for its objects ALTS from Definition 3.2.1, and that has for its arrows ALTS morphisms from Definition 3.2.4. Equality, composition and identities are inherited from $\mathbb{T}$.

We have that $\mathbb{L}$ is a coreflective subcategory of $\mathbb{A}$. The inclusion functor from $\mathbb{L}$ to $\mathbb{A}$ sends a concurrent alphabet (L, I) to the ALTS $(S = \emptyset, L, I, \delta = \emptyset)$, and a concurrent alphabet morphism λ to $(0, \lambda)$ where $0 : \emptyset \rightarrow \emptyset$ is the zero function. Again, this is because we allowed ghost actions that this is possible. There is a forgetful functor from $\mathbb{A}$ to $\mathbb{L}$ that drops the S and the δ components in (S, L, I, δ) for objects, and it sends an ALTS morphism (σ, λ) to the concurrent alphabet morphism λ . It can be easily verified that the latter is a right adjoint to the inclusion functor just described.

We also have that $\mathbb{T}$ is a coreflective subcategory of $\mathbb{A}$, and we formalize the adjoints as follows :

Definition 3.2.6 [Adjoints for $\mathbb{T}$ and $\mathbb{A}$].

1. Define the inclusion functor $F_{ta} : \mathbb{T} \rightarrow \mathbb{A}$ that sends a labelled transition system (S, L, δ) to $(S, L, \delta, I = \emptyset)$, and that leaves the morphisms unchanged.
2. Define the forgetful functor $G_{at} : \mathbb{A} \rightarrow \mathbb{T}$ that sends an asynchronous labelled transition system (S, L, δ, I) to its underlying labelled transition system (S, L, δ) , and that leaves the morphisms unchanged.

Proposition 3.2.7 [$\mathbb{T}$ is coreflective in $\mathbb{A}$]. F_{ta} is left adjoint to G_{at} .

Proof. This is a very straightforward proof. If (S, L, δ) is a LTS, then $F_{ta}(S, L, \delta) = (S, L, \delta, I = \emptyset)$ is clearly an ALTS : $I = \emptyset$ is irreflexive and symmetric, and all the ALTS axioms hold vacuously because they involve universal quantifiers over $I = \emptyset$. If $(\sigma, \lambda) : (S, L, \delta) \rightarrow (S', L', \delta')$ is a LTS morphism, then it is also an ALTS morphism from $(S, L, \delta, I = \emptyset) \rightarrow (S', L', \delta', I' = \emptyset)$ because the ALTS morphism requirement of preserving independence also holds vacuously. Identities and composition are clearly preserved with morphisms unchanged.

That G_{at} is a functor follows quite directly from the definition of ALTS.

For any labelled transition system $T_0 = (S_0, L_0, \delta_0)$ and any asynchronous labelled transition system $T = (S, L, \delta, I)$, we have that $Hom_{\mathbb{A}}((S_0, L_0, \delta_0, I_0 = \emptyset), T) = Hom_{\mathbb{T}}(T_0, (S, L, \delta))$. Indeed, any ALTS morphism from $(S_0, L_0, \delta_0, I_0 = \emptyset)$ to T is a LTS morphism from $T_0 = (S_0, L_0, \delta_0)$ to (S, L, δ) by definition. Also, for any LTS morphism $(\sigma, \lambda) : T_0 \rightarrow (S, L, \delta)$, we have that the ALTS morphism requirement of preserving independence holds vacuously with $I_0 = \emptyset$, i.e. (σ, λ) is a ALTS morphism from $(S_0, L_0, \delta_0, I_0 = \emptyset)$ to T . $\square$

Remark 3.2.8 . We do not know if G_{at} has a right adjoint or not. If such a right adjoint exists, say H , and if H leaves the morphisms unchanged, then $H(S, L, \delta) = (S, L, \delta, I)$ for a LTS (S, L, δ) is not simply provided by taking I as the largest independence relation for which (S, L, δ, I) is an ALTS (we have a counter-example to the bijective correspondences on the hom-sets in this case).

Now, it is possible to extend the concept of an ALTS to that of an asynchronous monoidal labelled transition system (and extend the notion of morphism as well), but we don't intend to use these asynchronous versions of MLTS; thus, we content ourselves with the construction of a functor from $\mathbb{A}$ to $\mathbb{M}$. To achieve this, we need the following lemma first :

Lemma 3.2.9 . *Given an asynchronous labelled transition system $T = (S, L, \delta, I)$, if $a_1 \dots a_n \equiv_I b_1 \dots b_n$ for some strings $a_1 \dots a_n$ and $b_1 \dots b_n$ in L^* ($a_i, b_i \in L$), then*

$$\xrightarrow{a_n}_T \circ \dots \circ \xrightarrow{a_1}_T = \xrightarrow{b_n}_T \circ \dots \circ \xrightarrow{b_1}_T$$

Proof. We remark that the empty word is only equivalent to itself, and there is nothing to prove in this case. We proceed by induction on $n \geq 1$ to show that for any $a_1 \dots a_n, b_1 \dots b_n$ in L^* , if $a_1 \dots a_n \equiv_I b_1 \dots b_n$, then $\xrightarrow{a_n}_T \circ \dots \circ \xrightarrow{a_1}_T = \xrightarrow{b_n}_T \circ \dots \circ \xrightarrow{b_1}_T$.

For the base case at $n = 1$, if $a_1 \equiv_I b_1$, then $a_1 = b_1$ and thus $\xrightarrow{a_1}_T = \xrightarrow{b_1}_T$.

Suppose the statement holds for some $n \geq 1$. If $a_1 \dots a_{n+1} \equiv_I b_1 \dots b_{n+1}$ for some strings $a_1 \dots a_{n+1}$ and $b_1 \dots b_{n+1}$ in L^* ($a_i, b_i \in L$), then Corollary 3.1.11 provides a $j \in \{1, \dots, n+1\}$ such that

$$a_{n+1} = b_j \text{ and } b_j I (b_{j+1} \dots b_{n+1}) \text{ and } a_1 \dots a_n \equiv_I b_1 \dots b_{j-1} b_{j+1} \dots b_{n+1}$$

By our induction hypothesis, we have :

$$\xrightarrow{a_n}_T \circ \dots \circ \xrightarrow{a_1}_T = \xrightarrow{b_{n+1}}_T \circ \dots \circ \xrightarrow{b_{j+1}}_T \circ \xrightarrow{b_{j-1}}_T \circ \dots \circ \xrightarrow{b_1}_T$$

Since $\xrightarrow{a_{n+1}}_T = \xrightarrow{b_j}_T$, we get :

$$\xrightarrow{a_{n+1}}_T \circ \xrightarrow{a_n}_T \circ \dots \circ \xrightarrow{a_1}_T = \xrightarrow{b_j}_T \circ \xrightarrow{b_{n+1}}_T \circ \dots \circ \xrightarrow{b_{j+1}}_T \circ \xrightarrow{b_{j-1}}_T \circ \dots \circ \xrightarrow{b_1}_T$$

By the Alternative Path axiom of an ALTS and $b_j I b_k$ for all $k \geq j+1$, we have that $\xrightarrow{b_j}_T$ commutes with all of the $\xrightarrow{b_{n+1}}_T, \dots, \xrightarrow{b_{j+1}}_T$. This finally provides

$$\xrightarrow{a_{n+1}}_T \circ \xrightarrow{a_n}_T \circ \dots \circ \xrightarrow{a_1}_T = \xrightarrow{b_{n+1}}_T \circ \dots \circ \xrightarrow{b_j}_T \circ \dots \circ \xrightarrow{b_1}_T$$

where the $\xrightarrow{b_j}_T$ has returned to its rightful place between $\xrightarrow{b_{j+1}}_T$ and $\xrightarrow{b_{j-1}}_T$. $\square$

It remains to be explicit about a functor from $\mathbb{A}$ to $\mathbb{M}$ that makes proper use of the trace monoid. We recall a previously defined functor $F_{tm} : \mathbb{T} \rightarrow \mathbb{M}$ that sends a LTS to its freely generated monoid LTS (see Definition 2.4.8). Also, we will need the forgetful functor $G_{at} : \mathbb{A} \rightarrow \mathbb{T}$ from Definition 3.2.6 that sends an ALTS to its underlying LTS. We can now define the functor from $\mathbb{A}$ to $\mathbb{M}$ in question.

Definition 3.2.10 [Functor from $\mathbb{A}$ to $\mathbb{M}$]. Define a functor $F_{am} : \mathbb{A} \rightarrow \mathbb{M}$ as follows :

- F_{am} sends an asynchronous labelled transition system $T = (S, L, I, \delta)$ to the monoid labelled transition system $F_{am}(T) = (S, M_{(L,I)}, \delta_{(L,I)})$, where the transition relation associated to $[a_1 \dots a_n]_I \in M_{(L,I)}$ ⁷ is given by :

$$\xrightarrow{[a_1 \dots a_n]_I}_{F_{am}(T)} = \xrightarrow{a_n}_T \circ \dots \circ \xrightarrow{a_1}_T$$

and where $\xrightarrow{[\Lambda]_I}_{F_{am}(T)} := \Delta_S$.

- F_{am} sends an ALTS morphism $(\sigma, \lambda) : (S, L, \delta, I) \rightarrow (S', L', \delta', I')$ to

$$F_{am}(\sigma, \lambda) = (\sigma, F_{lm}(\lambda))$$

where F_{lm} is the functor from $\mathbb{L}$ to $\mathbf{Mon}$ provided in Definition 3.1.12. This indeed provides a MLTS morphism from $(S, M_{(L,I)}, \delta_{(L,I)})$ to $(S', M_{(L',I')}, \delta_{(L',I')})$.

Remark 3.2.11 . For any $[w]_I \in M_{(L,I)}$ where $w \in L^*$, we get :

$$\xrightarrow{[w]_I}_{F_{am}(T)} = \xrightarrow{w}_{F_{tm}(G_{at}(T))}$$

Indeed, if $w = \Lambda$, then $\xrightarrow{[\Lambda]_I}_{F_{am}(T)} = \Delta_S = \xrightarrow{\Lambda}_{F_{tm}(G_{at}(T))}$. And if $w = a_1 \dots a_n$ for some $a_i \in L$, we get :

$$\xrightarrow{[a_1 \dots a_n]_I}_{F_{am}(T)} = \xrightarrow{a_n}_T \circ \dots \circ \xrightarrow{a_1}_T = \xrightarrow{a_n}_{G_{at}(T)} \circ \dots \circ \xrightarrow{a_1}_{G_{at}(T)} = \xrightarrow{a_1 \dots a_n}_{F_{tm}(G_{at}(T))}$$

by definition of F_{tm} .

Proposition 3.2.12 . F_{am} is indeed a functor from $\mathbb{A}$ to $\mathbb{M}$.

Proof. For an asynchronous labelled transition system $T = (S, L, I, \delta)$, we have that $F_{am}(T) = (S, M_{(L,I)}, \delta_{(L,I)})$ is a well-defined MLTS :

⁷We recall that $M_{(L,I)}$ is the trace monoid associated to (L, I) (see Definition 3.1.5).

$(S, M_{(L,I)}, \delta_{(L,I)})$ is a LTS with well-defined transition relations

$$\xrightarrow{[a_1 \dots a_n]_I}_{F_{am}(T)} := \xrightarrow{a_n}_T \circ \dots \circ \xrightarrow{a_1}_T$$

for any $[a_1 \dots a_n]_I \in M_{(L,I)}$, since this definition of $\xrightarrow{[a_1 \dots a_n]_I}_{F_{am}(T)}$ does not depend on the choice of representative; indeed, by Lemma 3.2.9, we have that $\xrightarrow{a_n}_T \circ \dots \circ \xrightarrow{a_1}_T = \xrightarrow{b_n}_T \circ \dots \circ \xrightarrow{b_1}_T$ whenever $a_1 \dots a_n \equiv_I b_1 \dots b_n$.

We have that the transition relation associated to the neutral element $[\Lambda]_I \in M_{(L,I)}$ is the diagonal relation on S as required. Also, given any $[a_1 \dots a_m]_I$ and $[b_1 \dots b_n]_I$ in $M_{(L,I)}$, we have :

$$\begin{aligned} \xrightarrow{[a_1 \dots a_m]_I \cdot [b_1 \dots b_n]_I}_{F_{am}(T)} &= \xrightarrow{[a_1 \dots a_m b_1 \dots b_n]_I}_{F_{am}(T)} \\ &= \xrightarrow{b_n}_T \circ \dots \circ \xrightarrow{b_1}_T \circ \xrightarrow{a_m}_T \circ \dots \circ \xrightarrow{a_1}_T \\ &= \xrightarrow{[b_1 \dots b_n]_I}_{F_{am}(T)} \circ \xrightarrow{[a_1 \dots a_m]_I}_{F_{am}(T)} \end{aligned}$$

This shows that $F_{am}(T)$ is a MLTS.

Consider any ALTS morphism $(\sigma, \lambda) : T \rightarrow T'$ with $T = (S, L, \delta, I)$ and $T' = (S', L', \delta', I')$. We have that (σ, λ) is a LTS morphism from $G_{at}(T) = (S, L, \delta)$ to $G_{at}(T') = (S', L', \delta')$ where G_{at} is the forgetful functor from $\mathbb{A}$ to $\mathbb{T}$ provided in Definition 3.2.6. By Theorem 2.4.8, we have that $F_{tm}(\sigma, \lambda) = (\sigma, \lambda^*)$ is a MLTS morphism from $F_{tm}(G_{at}(T))$ to $F_{tm}(G_{at}(T'))$ where F_{tm} is the freely generating functor from $\mathbb{T}$ to $\mathbb{M}$.

For $[a_1 \dots a_n] \in M_{(L,I)}$, if $X \xrightarrow{[a_1 \dots a_n]_I}_{F_{am}(T)} Y$ for some $X, Y \in S$, then we get $X \xrightarrow{a_1 \dots a_n}_{F_{tm}(G_{at}(T))} Y$ since $\xrightarrow{[a_1 \dots a_n]_I}_{F_{am}(T)} = \xrightarrow{a_1 \dots a_n}_{F_{tm}(G_{at}(T))}$ by Remark 3.2.11. But then, $(\sigma, \lambda^*) : F_{tm}(G_{at}(T)) \rightarrow F_{tm}(G_{at}(T'))$ is a MLTS morphism, and this means that the following transition holds :

$$\sigma(X) \xrightarrow{\lambda^*(a_1 \dots a_n)}_{F_{tm}(G_{at}(T'))} \sigma(Y)$$

By Remark 3.2.11 again, we get :

$$\xrightarrow{\lambda^*(a_1 \dots a_n)}_{F_{tm}(G_{at}(T'))} = \xrightarrow{[\lambda^*(a_1 \dots a_n)]_{I'}}_{F_{am}(T')} = \xrightarrow{F_{lm}(\lambda)([a_1 \dots a_n]_I)}_{F_{am}(T')}$$

Thus, $\sigma(X) \xrightarrow{F_{lm}(\lambda)([a_1 \dots a_n]_I)}_{F_{am}(T')} \sigma(Y)$, and this proves that $F_{am}(\sigma, \lambda) = (\sigma, F_{lm}(\lambda))$

is a MLTS morphism.

It remains to verify that the functor F_{am} preserves identities and composition. These essentially follow from the fact that F_{lm} is a functor (as established in Proposition 3.1.13). For the ALTS identity morphism $(1_S, 1_L) : (S, L, \delta, I) \rightarrow (S, L, \delta, I)$, we have that $F_{am}(1_S, 1_L) = (1_S, F_{lm}(1_L))$ is the MLTS identity morphism on $F_{am}(S, L, \delta, I)$ since $F_{lm}(1_L) = 1_{M(L, I)}$ is the monoid morphism identity on $M(L, I)$. Finally, for any ALTS morphisms $(\sigma_1, \lambda_1) : T_0 \rightarrow T_1$ and $(\sigma_2, \lambda_2) : T_1 \rightarrow T_2$, we get that $F_{am}((\sigma_2, \lambda_2) \circ (\sigma_1, \lambda_1)) = F_{am}(\sigma_2 \circ \sigma_1, \lambda_2 \circ \lambda_1) = (\sigma_2 \circ \sigma_1, F_{lm}(\lambda_2 \circ \lambda_1)) = (\sigma_2 \circ \sigma_1, F_{lm}(\lambda_2) \circ F_{lm}(\lambda_1)) = (\sigma_2, F_{lm}(\lambda_2)) \circ (\sigma_1, F_{lm}(\lambda_1)) = F_{am}(\sigma_2, \lambda_2) \circ F_{am}(\sigma_1, \lambda_1)$. $\square$

For a given asynchronous labelled transition system $T = (S, L, \delta, I)$, we should observe that the only effect that quotienting with $\equiv_I$ has on the freely generated monoid labelled transition system $F_{tm}(G_{at}(T))$ is to merge some parallel transitions in the transition specifier of $F_{tm}(G_{at}(T))$. To be more precise, given a transition (X, w, Y) in $F_{tm}(G_{at}(T))$ for a string w in L^* , and w being equivalent to a string w' with $\equiv_I$, then

$$\xrightarrow{w}_{F_{tm}(G_{at}(T))} = \xrightarrow{w'}_{F_{tm}(G_{at}(T))}$$

and so (X, w', Y) is also a transition in $F_{tm}(G_{at}(T))$. Hence, (X, w, Y) and (X, w', Y) are parallel transitions, and all such pairs of transitions (X, w, Y) and (X, w', Y) become identified by the quotienting.

Asynchronous labelled transition systems are rather elegant to work with in theory, but they have a few shortcomings with respect to the concurrency that naturally lives within presheaves of LTS. In particular, the LTS within such presheaves have an additional equivalence relation on runs (recall that these are traces of the form $: X_0 \xrightarrow{a_1} X_1 \xrightarrow{a_2} \dots \xrightarrow{a_n} X_n$). Such an equivalence allows a form of homotopy

amongst runs to surface, and it is through these that more interesting forms of traces arise. In order to capture such equivalences for ALTS, we need to explore ALTS with equivalence (ALTSE), which are the subject of the following section.

3.3 Asynchronous Labelled Transition Systems with Equivalence

As far as modeling concurrency goes in presheaves of LTS, we believe that *asynchronous labelled transition systems with equivalence* (ALTSE) is the way to go. These ALTS with equivalence are a new kind of elaboration provided in this thesis and it is a concept that borrows ideas from ALTS and labelled transition system with independence (LTSI). The concept of a LTSI was introduced by Winskel and Nielsen in [MC] and the approach singularly taken for these structures, as opposed to ALTS, is to generalize the independence relation to transitions rather than labels. Once this is achieved, one can derive a notion of equivalence on transitions by looking for patterns of transitions that form “independent squares”.

To be more precise, in a LTSI, we have that two distinct transitions (X, a, Y) and (X', a, Y') are equivalent precisely when :

1. There exist intermediary transitions (X, b, X') and (Y, b, Y') in which all of the following pairs are independent (see the left diagram in Figure 14) :

$$(X, a, Y) \text{ } I \text{ } (X, b, X') \text{ } I \text{ } (X', a, Y') \text{ } I \text{ } (Y, b, Y') \text{ } I \text{ } (X, a, Y)$$

2. Or symmetrically, if there exist intermediary transitions the other way around, i.e. there exists (X', b, X) and (Y', b, Y) in which all of the following pairs are independent (see the right diagram in Figure 14) :

$$(X, a, Y) \text{ } I \text{ } (X', b, X) \text{ } I \text{ } (X', a, Y') \text{ } I \text{ } (Y', b, Y) \text{ } I \text{ } (X, a, Y)$$

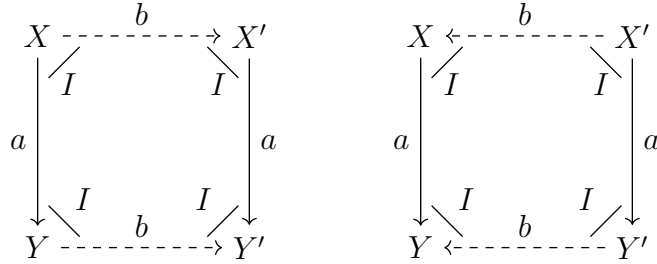


Figure 14: The equivalence on transitions involving a above requires intermediary independent transitions involving some other action b .

In a presheaf of LTS however, the naturally induced equivalence for transitions does not depend on the existence of intermediary transitions (such as the b -transitions in Figure 14), but stems from an equivalence on states given by the kernels of the restriction maps on the state spaces (see Definition 4.1.12). Therefore, LTSI do not have the format we require to account for the equivalence on transitions derived for presheaves of LTS.

Our alternative solution is to work with ALTSE instead, which use the equivalence on transitions as a primitive notion, rather than as a derivative of an independence relation on transitions. We should keep an eye on the fact that ALTSE have the same properties of alternative path, amalgamation and co-amalgamation as an ALTS does, but that each axiom involves the equivalence on transitions this time. The definition goes as follows :

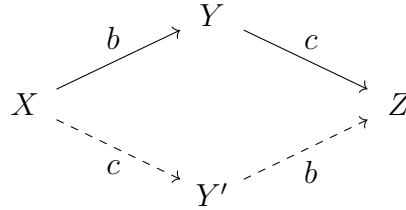
Definition 3.3.1 [Asynchronous Labelled Transition System with Equivalence]. An *asynchronous labelled transition system with equivalence* is a tuple $T = (S, L, \delta, I, \sim)$ where $I \subseteq L \times L$ is an irreflexive and symmetric binary relation on labels (i.e. the independence relation of an ALTS), and where $\sim \subseteq \delta \times \delta$ is an equivalence relation on transitions, and such that that the following hold for any $b, c \in L$:

1. $\forall X, Y, X', Y' \in S, \quad (X, b, Y) \sim (X', c, Y') \Rightarrow b = c$
2. $\forall X, Y, X', Y' \in S, \quad (X, b, Y) \sim (X', b, Y') \Rightarrow (X = X' \Leftrightarrow Y = Y')$

3. (Alternative Path)

For any states $X, Y, Z \in S$, if $b I c$ and $X \xrightarrow{b}_T Y \xrightarrow{c}_T Z$, then there exists $Y' \in S$ such that :

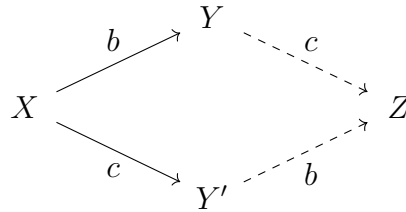
$$X \xrightarrow{c}_T Y' \xrightarrow{b}_T Z \text{ and } (X, b, Y) \sim (Y', b, Z) \text{ and } (Y, c, Z) \sim (X, c, Y')$$



4. (One-Step Co-amalgamation)

For any states $X, Y, Y' \in S$, if $b I c$ and $Y \xleftarrow{b}_T X \xrightarrow{c}_T Y'$ ⁸, then there exists $Z \in S$ such that :

$$Y \xrightarrow{c}_T Z \xleftarrow{b}_T Y' \text{ and } (X, b, Y) \sim (Y', b, Z) \text{ and } (Y, c, Z) \sim (X, c, Y')$$

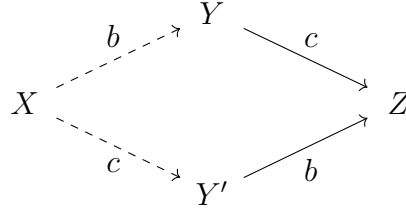


⁸Recall that we have defined $\xleftarrow{b}_T$ as $\xrightarrow{b^{-1}}_T$ earlier, just before Definition 3.2.1 of an ALTS.

5. (One-Step Amalgamation)

For any states $Y, Y', Z \in S$, if $b I c$ and $Y \xrightarrow{c}_T Z \xleftarrow{b}_T Y'$, then there exists $X \in S$ such that :

$$Y \xleftarrow{b}_T X \xrightarrow{c}_T Y' \text{ and } (X, b, Y) \sim (Y', b, Z) \text{ and } (Y, c, Z) \sim (X, c, Y').$$



Remark 3.3.2 . This definition contrasts with the one for LTSI by establishing an independence relation on the labelling set (as is typical of ALTS to do), as opposed to on the set of transitions δ . It is quite clear from the definition that if $(S, L, \delta, I, \sim)$ is an ALTSE, then (S, L, δ, I) is an ALTS.

The axioms above evoke the existence of alternative paths, amalgamations and co-amalgamations, and it turns out these are uniquely provided when the axioms apply.⁹

Proposition 3.3.3 [Uniqueness for Alternative Paths, Amalgamation and co-Amalgamation]. *Given an asynchronous labelled transition system with equivalence $T = (S, L, \delta, I, \sim)$, we have that for any $b, c \in L$:*

1. (Unique Alternative Path)

For any states $X, Y, Z \in S$, if $b I c$ and $X \xrightarrow{b}_T Y \xrightarrow{c}_T Z$, then there exists a unique $Y' \in S$ such that :

$$X \xrightarrow{c}_T Y' \xrightarrow{b}_T Z \text{ and } (X, b, Y) \sim (Y', b, Z) \text{ and } (Y, c, Z) \sim (X, c, Y')$$

⁹The uniqueness properties given by (1) and (2) in Proposition 3.3.3 also hold for LTSI.

2. (Unique One-Step co-Amalgamation)

For any states $X, Y, Y' \in S$, if $b I c$ and $Y \xrightarrow{T} \xleftarrow{b} X \xrightarrow{c} \xrightarrow{T} Y'$, then there exists a unique $Z \in S$ such that :

$$Y \xrightarrow{c} \xrightarrow{T} Z \xrightarrow{T} \xleftarrow{b} Y' \text{ and } (X, b, Y) \sim (Y', b, Z) \text{ and } (Y, c, Z) \sim (X, c, Y')$$

3. (Unique One-Step Amalgamation)

For any states $Y, Y', Z \in S$, if $b I c$ and $Y \xrightarrow{c} \xrightarrow{T} Z \xrightarrow{T} \xleftarrow{b} Y'$, then there exists a unique $X \in S$ such that :

$$Y \xrightarrow{T} \xleftarrow{b} X \xrightarrow{c} \xrightarrow{T} Y' \text{ and } (X, b, Y) \sim (Y', b, Z) \text{ and } (Y, c, Z) \sim (X, c, Y').$$

Proof. Suppose $b I c$. If $X \xrightarrow{b} \xrightarrow{T} Y \xrightarrow{c} \xrightarrow{T} Z$, and there are $Y', Y'' \in S$ such that

$$X \xrightarrow{c} \xrightarrow{T} Y' \xrightarrow{T} \xleftarrow{b} Z \text{ and } (X, b, Y) \sim (Y', b, Z) \text{ and } (Y, c, Z) \sim (X, c, Y')$$

and

$$X \xrightarrow{c} \xrightarrow{T} Y'' \xrightarrow{T} \xleftarrow{b} Z \text{ and } (X, b, Y) \sim (Y'', b, Z) \text{ and } (Y, c, Z) \sim (X, c, Y''),$$

then we have $(X, c, Y') \sim (X, c, Y'')$ from the transitivity of $\sim$ applied to $(X, c, Y') \sim (Y, c, Z) \sim (X, c, Y'')$. Finally, the second axiom for ALTSE forces $Y' = Y''$. The proofs of uniqueness for co-amalgamation and amalgamation are similar. □

When applying the axioms of Alternative Path, One-Step co-Amalgamation or One-Step Amalgamation, we basically generate a square pattern of transitions of a particular type. The definition of such a square pattern goes as follows :

Definition 3.3.4 [$I^\sim$ -square]. Given an asynchronous labelled transition system with equivalence $T = (S, L, \delta, I, \sim)$ and labels $b \neq c$ in L , we define a (b, c) -square in T to be a quadruple of transitions $\{(X, b, Y), (Y', b, Z), (Y, c, Z), (X, c, Y')\} \subseteq \delta$. We say this quadruple is an $I^\sim$ -square if $b I c$, and $(X, b, Y) \sim (Y', b, Z)$ and $(Y, c, Z) \sim (X, c, Y')$.

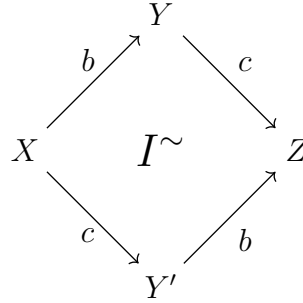


Figure 15: Representation of an $I^\sim$ -square.

And the following property is quite intuitive for such squares involving independent actions and equivalent transitions :

Proposition 3.3.5 . Consider any asynchronous labelled transition system with equivalence $T = (S, L, \delta, I, \sim)$ and any labels $b, c \in L$. If $b I c$ and $\{(X, b, Y), (Y', b, Z), (Y, c, Z), (X, c, Y')\}$ is a (b, c) -square in T , then :

$$(X, b, Y) \sim (Y', b, Z) \Leftrightarrow (Y, c, Z) \sim (X, c, Y')$$

i.e. if there is one equivalent pair of transitions in a (b, c) -square involving independent actions $b I c$, then we get an $I^\sim$ -square.

Proof. Suppose $(X, b, Y) \sim (Y', b, Z)$. The Alternative Path axiom provides a unique $Y'' \in S$ such that $X \xrightarrow{c}_T Y'' \xrightarrow{b}_T Z$ and $(X, b, Y) \sim (Y'', b, Z)$ and $(Y, c, Z) \sim (X, c, Y'')$. By transitivity of $\sim$, we get $(Y', b, Z) \sim (Y'', b, Z)$, and applying axiom (2) provides $Y' = Y''$. Thus, $(Y, c, Z) \sim (X, c, Y')$. A similar proof yields the converse statement. □

We now provide a definition for morphisms of ALTSE.

Definition 3.3.6 [ALTSE morphism]. Given two asynchronous labelled transition systems $T = (S, L, \delta, I, \sim_T)$ and $T' = (S', L', \delta', I', \sim_{T'})$, we define a *morphism of ALTSE* from T to T' as an ALTS morphism $(\sigma, \lambda) : (S, L, \delta, I) \rightarrow (S', L', \delta', I')$ that preserves the equivalence on transitions, that is :

$$(X, b, Y) \sim_T (W, b, Z) \text{ and } \lambda(b) \neq \varepsilon \Rightarrow (\sigma(X), \lambda(b), \sigma(Y)) \sim_{T'} (\sigma(W), \lambda(b), \sigma(Z))$$

for any transitions (X, b, Y) and (W, b, Z) in δ .

Definition 3.3.7 [Category of ALTSE]. We define the *category of asynchronous labelled transition systems with equivalence*, denoted $\mathbb{A}^\sim$, as the category where the objects are ALTSE (Definition 3.3.1) and the arrows are ALTSE morphisms (Definition 3.3.6), and the latter inherits equality, composition, and identities from $\mathbb{T}$.

It is straightforward to verify that $\mathbb{A}^\sim$ is a category. Also, there is a forgetful functor $G_a^\sim$ from $\mathbb{A}^\sim$ to $\mathbb{A}$, that simply drops the equivalence $\sim$ from the signature. If we compose this forgetful functor with $F_{am} : \mathbb{A} \rightarrow \mathbb{M}$ (provided in the previous section), we get an interpretation of ALTSE as MLTS (providing a trace monoid for labels). We provide a definition for this composition to facilitate references.

Definition 3.3.8 [Functor from $\mathbb{A}^\sim$ to $\mathbb{M}$]. Let $F_{am}^\sim : \mathbb{A}^\sim \rightarrow \mathbb{M}$ be the functor from $\mathbb{A}^\sim$ to $\mathbb{M}$ given by the composition of $G_a^\sim$ and F_{am} , that is $F_{am}^\sim = F_{am} \circ G_a^\sim$. This is represented in the following commutative diagram :

$$\begin{array}{ccc} \mathbb{A} & \xrightarrow{F_{am}} & \mathbb{M} \\ G_a^\sim \uparrow & \nearrow F_{am}^\sim & \\ \mathbb{A}^\sim & & \end{array}$$

Also, we recall that we have provided a forgetful functor $G_{at} : \mathbb{A} \rightarrow \mathbb{T}$ that sends

an ALTS to its underlying LTS, and we get the underlying LTS of an ALTSE by applying the functor $G_{at} \circ G_a^\sim$.

Whilst a functor from $\mathbb{A}$ to $\mathbb{A}^\sim$ might naturally exist, there is an easier way of working out an inclusion functor from $\mathbb{T}$ to $\mathbb{A}^\sim$ directly, and it is left adjoint to $G_{at} \circ G_a^\sim$.

Definition 3.3.9 [Inclusion of $\mathbb{T}$ into $\mathbb{A}^\sim$]. Define $F_{ta}^\sim : \mathbb{T} \rightarrow \mathbb{A}^\sim$ as the inclusion functor from $\mathbb{T}$ to $\mathbb{A}^\sim$ that sends a labelled transition system (S, L, δ) to $(S, L, \delta, I, \sim)$ where $I = \emptyset$ and $\sim = \Delta_\delta$ is the standard equality on the set of transitions δ . $F_{ta}^\sim$ leaves the morphisms unchanged.

Proposition 3.3.10 . $F_{ta}^\sim$ is a well-defined functor.

Proof. Consider any labelled transition system (S, L, δ) . We verify that $F_{ta}^\sim(S, L, \delta) = (S, L, \delta, I = \emptyset, \sim = \Delta_\delta)$ is an ALTSE. It is clear that Δ_δ is an equivalence relation. Also, the axioms (1) and (2) clearly hold since $(X, b, Y) \Delta_\delta (X', c, Y')$ implies that $X = X'$, $Y = Y'$, and $b = c$. The remaining axioms hold vacuously with $I = \emptyset$ (and the latter is irreflexive and symmetric).

For any LTS morphism (σ, λ) (which is $F_{ta}^\sim(\sigma, \lambda)$), independence is preserved vacuously. Also, if $(X, b, Y) \Delta_\delta (X', b, Y')$ and $\lambda(b) \neq \varepsilon$, then $X = X', Y = Y'$, so $\sigma(X) = \sigma(X')$, $\sigma(Y) = \sigma(Y')$, and this means $(\sigma(X), \lambda(b), \sigma(Y)) \Delta_\delta (\sigma(X'), \lambda(b), \sigma(Y'))$. Thus, we get that $F_{ta}^\sim(\sigma, \lambda)$ is an ALTSE morphism. Identities and composition are clearly preserved. $\square$

Proposition 3.3.11 $[\mathbb{T} \text{ is coreflective in } \mathbb{A}^\sim]$. $F_{ta}^\sim$ is left adjoint to $G_{at} \circ G_a^\sim$.

Proof. For any labelled transition system $T_0 = (S_0, L_0, \delta_0)$ and any asynchronous labelled transition system with equivalence $T = (S, L, \delta, I, \sim)$, we have :

$$\text{Hom}_{\mathbb{A}^\sim}((S_0, L_0, \delta_0, I_0 = \emptyset, \sim_{T_0} = \Delta_{\delta_0}), T) = \text{Hom}_{\mathbb{T}}(T_0, (S, L, \delta))$$

Indeed, any ALTSE morphism from $(S_0, L_0, \delta_0, I_0 = \emptyset, \sim_{T_0} = \Delta_{\delta_0})$ to T is a LTS morphism from $T_0 = (S_0, L_0, \delta_0)$ to (S, L, δ) by definition. Also, for any LTS morphism $(\sigma, \lambda) : T_0 \rightarrow (S, L, \delta)$, we have that the ALTSE morphism requirement of preserving independence holds vacuously with $I_0 = \emptyset$, and we have that the requirement of preserving $\sim_{T_0} = \Delta_{\delta_0}$ holds because $\sim$ is reflexive. Hence, we have that (σ, λ) is a ALTSE morphism from $(S_0, L_0, \delta_0, I_0 = \emptyset, \sim_{T_0} = \Delta_{\delta_0})$ to T . $\square$

We should now investigate the equivalence on runs that naturally arise within ALTSE. We recall the definition of a run here :

Definition 3.3.12 **[Finite Run]**. Given an asynchronous LTS with equivalence $T = (S, L, \delta, I, \sim)$, a finite T -run, say r , is a connected sequence of transitions in T as follows :

$$X_0 \xrightarrow{a_1}_T X_1 \xrightarrow{a_2}_T \dots \xrightarrow{a_n}_T X_n$$

i.e. where $(X_i, a_{i+1}, X_{i+1}) \in \delta$ for all i . We say that r is run on the sequence $a_1 \dots a_n$ in this case.

Remark 3.3.13 . Of course, we can replace the ALTSE with an ALTS or a LTS, and the definition is the same.

Remark 3.3.14 . We say that two runs $X_0 \xrightarrow{a_1}_T X_1 \xrightarrow{a_2}_T \dots \xrightarrow{a_n}_T X_n$ and $Y_0 \xrightarrow{b_1}_T Y_1 \xrightarrow{b_2}_T \dots \xrightarrow{b_n}_T Y_n$ are equal whenever $X_i = Y_i$ and $a_i = b_i$ for all i .

Now, for a given ALTS with equivalence T , we can establish an equivalence on T -runs through the use of $I^\sim$ -squares in T . This has been achieved before with LTSI by Winskel, Nielsen, and Joyal in [BisO], and we simply work out the idea for ALTSE here.

Definition 3.3.15 [Equivalence on ALTSE Runs]. Let $T = (S, L, \delta, I, \sim)$ be an ALTSE. We define $\approx_I$ to be the the smallest equivalence relation on T -runs such that for any natural numbers m, n such that $1 \leq m < n$, and any T -runs :

$$r_1 : X_0 \xrightarrow{a_1}_T X_1 \xrightarrow{a_2}_T \dots \xrightarrow{a_{m-1}}_T X_{m-1} \xrightarrow{a_m}_T X_m \xrightarrow{a_{m+1}}_T X_{m+1} \xrightarrow{a_{m+2}}_T \dots \xrightarrow{a_n}_T X_n$$

and

$$r_2 : X_0 \xrightarrow{a_1}_T X_1 \xrightarrow{a_2}_T \dots \xrightarrow{a_{m-1}}_T X_{m-1} \xrightarrow{a_{m+1}}_T X'_m \xrightarrow{a_m}_T X_{m+1} \xrightarrow{a_{m+2}}_T \dots \xrightarrow{a_n}_T X_n ,$$

if the set of transitions :

$$\{(X_{m-1}, a_m, X_m), (X_m, a_{m+1}, X_{m+1}), (X_{m-1}, a_{m+1}, X'_m), (X'_m, a_m, X_{m+1})\}$$

forms an $I^\sim$ -square, then $r_1 \approx_I r_2$.

We can see how these T -runs interact in the diagram of Figure 16 below, which depicts how one T -run can be obtained from the other by a local deformation where independent actions and equivalent transitions are involved.

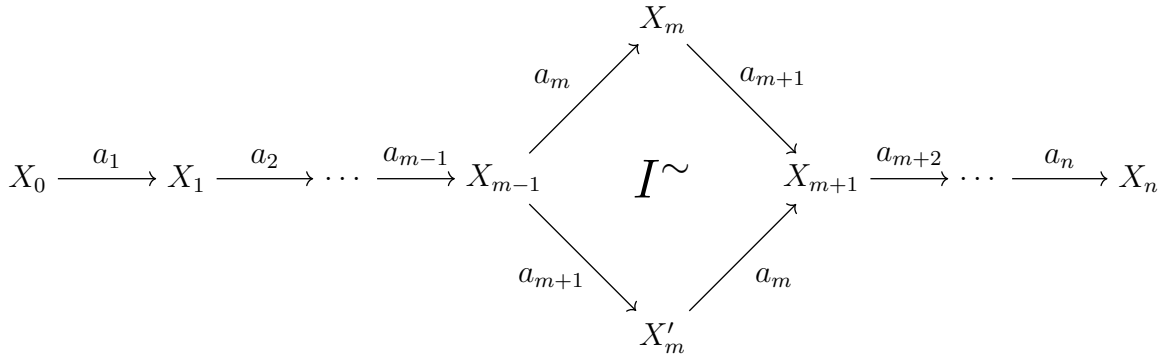


Figure 16: An equivalence on two runs that arises from a local deformation involving an $I^\sim$ -square.

There are important aspects of the theory of ALTSE on which we could expand here, but some proofs are still to be worked out. Nevertheless, we thought it would be interesting to present the following conjecture to give an idea of what we hope to address in the future.

Conjecture 3.3.16 . *Given an asynchronous labelled transition system with equivalence T with labelling set L , if*

$$X_0 \xrightarrow{a_1}_T X_1 \xrightarrow{a_2}_T \dots \xrightarrow{a_n}_T X_n$$

is a T -run, and if $b_1 \dots b_n$ is a word in L^ such that $a_1 \dots a_n \equiv_I b_1 \dots b_n$, then there exists a unique T -run $Y_0 \xrightarrow{b_1}_T Y_1 \xrightarrow{b_2}_T \dots \xrightarrow{b_n}_T Y_n$ such that :*

$$X_0 \xrightarrow{a_1}_T X_1 \xrightarrow{a_2}_T \dots \xrightarrow{a_n}_T X_n \approx_I Y_0 \xrightarrow{b_1}_T Y_1 \xrightarrow{b_2}_T \dots \xrightarrow{b_n}_T Y_n$$

This affirmation essentially support the idea that these equivalences on runs behave quite well. To be more precise, Conjecture 3.3.16 suggests that the equivalence class of a run $X_0 \xrightarrow{a_1}_T X_1 \xrightarrow{a_2}_T \dots \xrightarrow{a_n}_T X_n$ (with respect to $\approx_I$) has a bijective correspondence with the equivalence class $[a_1 \dots a_n]_I$ of the word $a_1 \dots a_n$ with respect to $\equiv_I$. This is a rather strong assertion that would be quite useful in partial order reduction methods because it tell us that it suffices to find all representatives of an equivalence class of $\equiv_I$ to find all representatives of an equivalence class of runs $\approx_I$. Furthermore, if we applied this result in the case of the equivalence on runs $\approx_I$ induced in a presheaf of LTS (as we will see later), we would get constructive ways of providing all the representatives of an equivalence class of runs.

We now have enough background on the theory of labelled transition systems and independence of actions to begin our study of presheaves of LTS, which is the main subject of this thesis.

Chapter 4

Presheaves of Labelled Transition Systems

In this chapter, we delve into the core subject matter, which is to understand presheaves of labelled transition systems. Mainly, we establish the conditions under which spatially induced independence (SI-independence) arises within such presheaves. We shall recall more precisely what this means in an instant. For now, we need some terminology.

We recall that, given any categories $\mathcal{H}$ and $\mathcal{D}$, a $\mathcal{D}$ -valued presheaf on $\mathcal{H}$ is a functor $F : \mathcal{H}^{op} \rightarrow \mathcal{D}$. We use a presheaf $F : \mathcal{H}^{op} \rightarrow \mathcal{D}$ in a spatial sense here, i.e. where the base category $\mathcal{H}$ represents a space : the objects of $\mathcal{H}$ represent *regions* and the arrows represent how regions are included in one another. This requires $\mathcal{H}$ to be at least a category associated to a poset (the partial order representing inclusion of regions), but for the chapters to follow, we will work under the minimal conditions of a complete Heyting algebra space, hence the letter $\mathcal{H}$ to represent such spaces. In fact, whenever we will say that $\mathcal{H}$ is a Heyting algebra, we will actually mean that $\mathcal{H}$ is a complete Heyting algebra within this thesis.¹

¹We will proceed in a similar fashion with Boolean algebras, i.e. we will use the term Boolean algebra to refer to a complete Boolean algebra.

Also, when $\mathcal{D} = \mathbf{Set}$ or $\mathbf{Set}_\varepsilon$, we will refer to the elements of $F(U)$, for some region U in $\mathcal{H}$, as the *sections of F over U* .

Now, for a given $\mathbb{T}$ -valued presheaf $\mathcal{T} : \mathcal{H}^{op} \rightarrow \mathbb{T}$ and a region U in a Heyting algebra $\mathcal{H}$, we can construct a form of independence relation $\mathcal{I}(U)$ on the set of actions local to U , and this $\mathcal{I}(U)$ can be described informally as follows :

For any actions b, c in the set of actions that are local to U , we set $b \mathcal{I}(U) c$ if and only if there is a proper cover $\{V, W\}$ of U such that :

1. b vanishes in W and c vanishes in V , and
2. b is contained in V and c is contained in W .

And with respect to these $\mathcal{I}(U)$ relations, we can formulate the Principle of SI-independence as we have stated it before in the introduction of this thesis :

Principle of SI-independence : Given a $\mathbb{T}$ -valued presheaf $\mathcal{T}$, we say that $\mathcal{T} = (\mathcal{S}, \mathcal{L}, \delta)$ has *spatially induced independence* if for any region U , we have that the relation $\mathcal{I}(U)$, as defined above, provides an independence relation on labels such that $\mathcal{T}(U) = (\mathcal{S}(U), \mathcal{L}(U), \delta(U), \mathcal{I}(U))$ is an ALTS. In which case, for $b \mathcal{I}(U) c$, we say that b is *spatially independent* of c with respect to U .

Thus, the objective of this chapter is to find efficient and minimal conditions under which a $\mathbb{T}$ -valued presheaf has SI-independence.

We also expressed in the introduction that $\mathbb{T}$ -valued sheaves do render SI-independence (this was proposed by Malcolm in [SSTS]). Yet, whilst sheaves of LTS are elegant in theory, we will see that they behave rather strangely when the LTS are distributed over a Boolean algebra space². The gluing of states is not the

²These Boolean algebras constitute an important case to consider because many distributed systems in computer science are constructed on top of discrete spaces (multi-agent systems for example); these provide Boolean algebra spaces whose structure is given by a powerset.

problem in question, but it is the gluing of labels and transitions that is sometimes counterintuitive. One problem arises in particular with respect to the idle action ε : if we take any action b in a region U , and if V is a subregion of U , then we can project b to V and systematically glue that projection back up with the idle action on the complement $U \setminus V$ to obtain an entirely different action in U (one which we usually don't intend to have). What this means is that the systematic presence of complements in the Boolean algebra yields an explosive amount of ε gluing, which allows one to systematically tear any action from a spatial context; and this leads to the creation of possibly many undesirable new actions. Proposition 4.2.10 formalizes the idea behind this problem, and we will see in Section 4.2 how it prevents the presheaves of transition systems associated to Petri Nets to become sheaves.

Now, we cannot simply reject gluing also, because it is this mechanism which is truly at the heart of rendering SI-independence. To remedy this situation, we propose a simple solution : we should force a minimal amount of gluing to occur for the transitions of actions, until such presheaves of LTS indeed acquire SI-independence. This controlled form of gluing is referred to as “ δ -gluing” for actions, and it is presented in the first section (4.1) of this chapter. In the section (4.2) that follows, we make a small detour into the world of $\mathbb{T}$ -valued sheaves, to see how δ -gluing of actions occurs there, and also to understand why a sheaf is too strong a condition to account for all interesting models of distributed systems. Finally, in Section 4.3, we explain how to apply this δ -gluing to force the dependencies of an action to be contained in a region³. This discussion on containment of dependencies is necessary to formulate the conditions for SI-independence. The idea is that an action b is spatially independent of another c if each of these actions do not interfere with the region in which the other has its dependencies.

³When a region V contains the dependencies of an action b , it means that the output of the action b only depends on information residing in V . We will formalize what this means in Definition 4.3.7.

Once we understand these ideal conditions under which SI-independence can work in a presheaf of LTS, we will simply define a new kind of presheaf, $\mathbb{T}$ -adapted presheaves, in which these conditions are enforced. In particular, these $\mathbb{T}$ -adapted presheaves will subsume the case of $\mathbb{T}$ -valued sheaves.

Before we begin however, we need some notation and terminology :

We represent a complete Heyting algebra through the following signature $(\mathcal{H}, \wedge, \vee, \leq, 1_{\mathcal{H}}, 0_{\mathcal{H}})$, where $\mathcal{H}$ is a set of regions, $\vee$ and $\wedge$ are the join and meet operations on regions respectively, $\leq$ is the partial order on regions that expresses for $V \leq U$ that V is a subregion of U . $1_{\mathcal{H}}$ is the global region of $\mathcal{H}$, with the property that $U \leq 1_{\mathcal{H}}$ for all regions U , and $0_{\mathcal{H}}$ is the empty region of $\mathcal{H}$, and it has the property that $0_{\mathcal{H}} \leq U$ for all regions U .

Fix a region U in $\mathcal{H}$. We say that a set of regions $\{V_i\}_{i \in J}$ in $\mathcal{H}$ is a *cover* of U if $U \leq \bigvee_{i \in J} V_i$, and we say that it is a *proper cover* of U if $U = \bigvee_{i \in J} V_i$. Given some regions $V, W \leq U$, we say that V *join-complements* W in U (or *j-complements* W in U) if $V \vee W = U$. Also, we say that V *complements* W in U if $V \vee W = U$ and $V \wedge W = 0_{\mathcal{H}}$.

Finally, given any categories $\mathcal{C}$ and $\mathcal{D}$, we will write $[\mathcal{C}, \mathcal{D}]$ to denote the category of functors from $\mathcal{C}$ to $\mathcal{D}$ that uses natural transformations between functors as morphisms.

4.1 $\mathbb{T}$ -valued Presheaves and δ -gluing

In this section, we lay down the notation and concepts that work for $\mathbb{T}$ -valued presheaves on complete Heyting algebras in general. But mainly, we want to arrive at the definition of δ -gluing for actions (Definition 4.1.17), which we also refer to as *gluing of transitions for actions* (δ is the symbol that relates to the set of transitions of a LTS). This is important because, when we enforce a δ -gluing condition in $\mathbb{T}$ -valued presheaves, we can express spatially induced independence.

Definition 4.1.1 [$\mathbb{T}$ -valued presheaves]. A $\mathbb{T}$ -valued presheaf on a complete Heyting algebra $\mathcal{H}$ is a functor $\mathcal{T} : \mathcal{H}^{op} \rightarrow \mathbb{T}$. The category of $\mathbb{T}$ -valued presheaves on $\mathcal{H}$ is the functor category $[\mathcal{H}^{op}, \mathbb{T}]$. Given a $\mathbb{T}$ -valued presheaf $\mathcal{T}$ and regions $V \leq U$ in $\mathcal{H}$, we refer to $\mathcal{T}(U)$ as the *transition system of $\mathcal{T}$ over U* , and we refer to

$$\mathcal{T}(V \leq U) := (res_V^U, \rho_V^U) : \mathcal{T}(U) \rightarrow \mathcal{T}(V)$$

as the *restriction morphism of $\mathcal{T}$ from U to V* (where res_V^U is the states map and ρ_V^U is the labelling morphism). We use the notation $\xrightarrow{(-)}_U$ to refer to the relational diagram associated to $\mathcal{T}(U)$, instead of writing $\xrightarrow{(-)}_{\mathcal{T}(U)}$ in the usual way (and we incorporate ε for this diagram as usual).

To get a better view of such a presheaf $\mathcal{T}$ of labelled transition systems, we can extend $\mathcal{T}$ to its presheaf components by using the functors $p_s : \mathbb{T} \rightarrow \mathbf{Set}$, $p_\ell : \mathbb{T} \rightarrow \mathbf{Set}_\varepsilon$ and $p_t : \mathbb{T} \rightarrow \mathbf{Set}$, as given previously in Definition 2.1.16, that project labelled transition systems to their components. We recall that, given a labelled transition system (S, L, δ) , we have that $p_s(S, L, \delta) = S$, $p_\ell(S, L, \delta) = L$, $p_t(S, L, \delta) = \delta$ project to states, labels and transitions respectively. Also, given a LTS morphism (σ, λ) , we have that $p_s(\sigma, \lambda) = \sigma$, $p_\ell(\sigma, \lambda) = \lambda$ and $p_t(\sigma, \lambda) = (\sigma, \lambda, \sigma)$ (λ is considered as a total map here, that includes mapping of the ε symbol).

Definition 4.1.2 . Let $\mathcal{T} : \mathcal{H}^{op} \rightarrow \mathbb{T}$ be a $\mathbb{T}$ -valued presheaf. We define the following presheaves with base $\mathcal{H}$ as follows :

- (1) $\mathcal{S} := p_s \circ \mathcal{T} : \mathcal{H}^{op} \rightarrow \mathbf{Set}$ is called the *state presheaf associated to $\mathcal{T}$* , which is a set-valued presheaf that provides states as sections, and for any regions $V \leq U$ in $\mathcal{H}$:
 - (a) $\mathcal{S}(U)$ is referred as the *state space of $\mathcal{T}$ over U* ;
 - (b) $\mathcal{S}(V \leq U)$ is written as $res_V^U : \mathcal{S}(U) \rightarrow \mathcal{S}(V)$ and is referred to as the *restriction of states from U to V* (other notation : $res_V^U(X) := X|_V$ when U is clear in the context and $X \in \mathcal{S}(U)$);
- (2) $\mathcal{L} := p_\ell \circ \mathcal{T} : \mathcal{H}^{op} \rightarrow \mathbf{Set}_\varepsilon$ is called the *labelling presheaf associated to $\mathcal{T}$* , which is a $\mathbf{Set}_\varepsilon$ -valued presheaf that provides actions (labels) as sections, and for any regions $V \leq U$ in $\mathcal{H}$:
 - (a) $\mathcal{L}(U)$ is called the *labelling set over U* (or *action set over U*);
 - (b) $\mathcal{L}(V \leq U)$ is written as $\rho_V^U : \mathcal{L}(U) \rightarrow \mathcal{L}(V)$ and is referred to as the *projection⁴ of labels from U to V* .
- (3) $\delta := p_t \circ \mathcal{T} : \mathcal{H}^{op} \rightarrow (\mathbf{Set} \times \mathbf{Set} \times \mathbf{Set})$ is called the *transition relation presheaf of $\mathcal{T}$* , and provides transitions as sections, and for any $V \leq U$ in $\mathcal{H}$:
 - (a) $\delta(U)$ is called the *transition specifier over U*
 - (b) $\delta(V \leq U) : \delta(U) \rightarrow \delta(V)$ is referred to as the *restriction of transitions from U to V* .

Thus, for any region U we can write $\mathcal{T}(U) = (\mathcal{S}(U), \mathcal{L}(U), \delta(U))$ for the transition system of $\mathcal{T}$ over U . We recall the definition of natural transformations to see how it is integrated in the case of $\mathbb{T}$ -valued presheaves.

⁴We use the term “projection” here instead of “restriction” because we do not want to confuse these ρ_V^U with the “labelling restrictions” from Definition 2.3.6

Definition 4.1.3 [Natural Transformations for $\mathbb{T}$ -valued Presheaves].

Let $\mathcal{T}$ and $\mathcal{T}'$ be two $\mathbb{T}$ -valued presheaves on $\mathcal{H}$. A *natural transformation* from $\mathcal{T}$ to $\mathcal{T}'$ is a family of LTS morphisms $\{(\sigma_U, \lambda_U) : \mathcal{T}(U) \rightarrow \mathcal{T}'(U)\}_{U \in \mathcal{H}}$ indexed by the regions of $\mathcal{H}$, such that for any regions $V \leq U$:

$$(\sigma_V, \lambda_V) \circ (res_V^U, \rho_V^U) = ((res_V^U)', (\rho_V^U)') \circ (\sigma_U, \lambda_U)$$

that is, the following diagram commutes :

$$\begin{array}{ccc} \mathcal{T}(U) & \xrightarrow{(\sigma_U, \lambda_U)} & \mathcal{T}'(U) \\ \downarrow (res_V^U, \rho_V^U) & & \downarrow ((res_V^U)', (\rho_V^U)') \\ \mathcal{T}(V) & \xrightarrow{(\sigma_V, \lambda_V)} & \mathcal{T}'(V) \end{array}$$

Now, since $\mathbb{T}$ is bicomplete, we know from Mac Lane's Categories for The Working Mathematician ([CWM], p.115-116) that the functor category $[\mathcal{H}^{op}, \mathbb{T}]$ is bicomplete.

Proposition 4.1.4 . *$[\mathcal{H}^{op}, \mathbb{T}]$ is bicomplete for any complete Heyting algebra $\mathcal{H}$. Furthermore, the limits and colimits in $[\mathcal{H}^{op}, \mathbb{T}]$ are given by componentwise limits and componentwise colimits respectively in $\mathbb{T}$.*

We recall some facts about natural transformations (given in [CWM] (p.91)) :

Given categories $\mathcal{C}$ and $\mathcal{D}$, and $\mathcal{C}$ small,

- (1) If $\mathcal{D}$ has all pullbacks, then a natural transformation in $[\mathcal{C}, \mathcal{D}]$ is a monomorphism iff it is componentwise a monomorphism in $\mathcal{D}$.
- (2) If $\mathcal{D}$ has all pushouts, then a natural transformation in $[\mathcal{C}, \mathcal{D}]$ is an epimorphism iff it is componentwise an epimorphism in $\mathcal{D}$.

Combining the above statements with Proposition 4.1.4, we get the following :

Proposition 4.1.5 . *Consider any complete Heyting algebra $\mathcal{H}$. Then the following are true :*

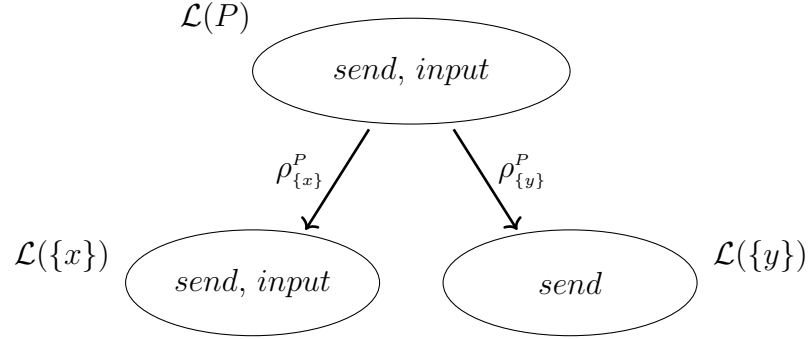
1. *A natural transformation in $[\mathcal{H}^{op}, \mathbb{T}]$ is a monomorphism iff it is componentwise a monomorphism in $\mathbb{T}$.*
2. *A natural transformation in $[\mathcal{H}^{op}, \mathbb{T}]$ is an epimorphism iff it is componentwise an epimorphism in $\mathbb{T}$.*

Now we are ready for some examples of $\mathbb{T}$ -valued presheaves, and we need a definition of discrete space first :

Definition 4.1.6 [Discrete Space]. Let P be a set. The *discrete space over P* is the pair $(P, \mathcal{P}(P))$ where the powerset $\mathcal{P}(P)$ is used as a discrete topology on P . We often refer to the elements of P as *places* in our applications, and we can derive a Boolean algebra space in the usual sense for such discrete spaces, by using the subsets of P as regions, and set inclusion $\subseteq$ as inclusion of regions. In fact, we will refer to this derived Boolean algebra space directly as the discrete space over P .

Example 4.1.7 [Buffer on a Discrete Space]. Consider a set of places $P = \{x, y\}$ and let $\mathcal{H}$ be the discrete space over P . Define a $\mathbb{T}$ -valued presheaf $\mathcal{T} = (\mathcal{S}, \mathcal{L}, \delta)$ on $\mathcal{H}$ as follows :

- State spaces :
 - $\mathcal{S}(P) = \{0, 1\} \times \{0, 1\}$,
 - $\mathcal{S}(\{y\}) = \mathcal{S}(\{x\}) = \{0, 1\}$, and
 - $\mathcal{S}(\emptyset) = \{*\}$ (a singleton set)
- Labelling sets :
 - $\mathcal{L}(P) = \mathcal{L}(\{x\}) = \{send, input\}$, and
 - $\mathcal{L}(\{y\}) = \{send\}$, and
 - $\mathcal{L}(\emptyset) = \emptyset$



- $res_{\{x\}}^P$ and $res_{\{y\}}^P$ are the left and right projections of $\mathcal{S}(P)$ respectively
- $\rho_{\{x\}}^P$ is the identity on $\{send, input\}$, and $\rho_{\{y\}}^P(send) = send$, $\rho_{\{y\}}^P(input) = \varepsilon$
- $\mathcal{T}(P)$ is precisely the buffer LTS from Example 2.1.7. We recall the specification of the transition relations here :

- $\xrightarrow{send}_P$ is the smallest transition relation on $\mathcal{S}(P)$ such that for all $m, n \in \{0, 1\}$:

$$(m, n) \xrightarrow{send}_P (m, m)$$

- $\xrightarrow{input}_P$ is the smallest transition relation on $\mathcal{S}(P)$ such that for all $m, m', n \in \{0, 1\}$:

$$(m, n) \xrightarrow{input}_P (m', n)$$

- $\mathcal{T}(\{x\})$'s transitions are specified by :

- $\xrightarrow{send}_{\{x\}} = \Delta_{\{0,1\}}$
- $\xrightarrow{input}_{\{x\}} = \{0, 1\}^2$, i.e. $m \xrightarrow{input}_{\{x\}} n$ for any $m, n \in \{0, 1\} = \mathcal{S}(\{x\})$

- $\mathcal{T}(\{y\})$ has only one action $send$ and its transition relation is $\xrightarrow{send}_{\{y\}} = \{0, 1\}^2$

All this information determines a $\mathbb{T}$ -valued presheaf on the discrete space over P . We can easily see that $(res_{\{x\}}^P, \rho_{\{x\}}^P)$ and $(res_{\{y\}}^P, \rho_{\{y\}}^P)$ are LTS morphisms. For example, given any transition $(m, n) \xrightarrow{send}_P (m, m)$ globally in P , we have that :

$$res_{\{y\}}^P(m, n) \xrightarrow{\rho_{\{y\}}^P(send)}_{\{y\}} res_{\{y\}}^P(m, m)$$

since $n \xrightarrow{send}_{\{y\}} m$ is a transition for *send* in $\{y\}$; indeed, y can expect any value to come from the outside with a *send* that has $\xrightarrow{send}_{\{y\}} = \{0, 1\}^2$.

We now consider Petri Nets and how we can associate presheaves of LTS to them.

Definition 4.1.8 [General Petri Nets]. A *General Petri Net*⁵ consists of a tuple (P, T, F, M_0, W) where :

1. P is a set of *places*;
2. T is a set of *transitions*⁶;
3. $F \subseteq (P \times T) \cup (T \times P)$ is a set of *arcs*;
4. $M_0 : P \rightarrow \mathbb{N}$ is a map that marks the initial state of the system (a *state* is a number of tokens assigned to each place);
5. $W : F \rightarrow \mathbb{N}^+$ is an assignment of a non-zero number to each arc, called the *weight* of that arc.

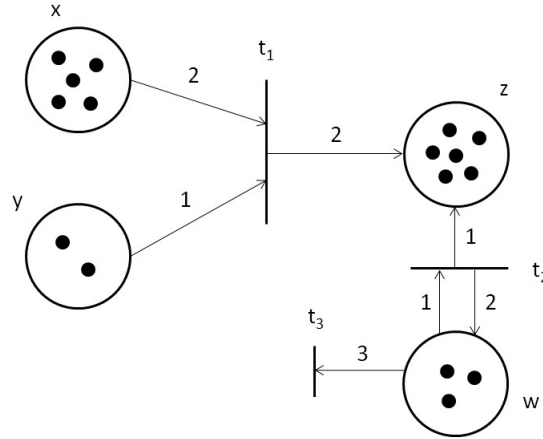
For $(x, t) \in F \cap (P \times T)$, $W(x, t)$ is the number of tokens you *remove* from the place x when firing the transition t . For $(t, y) \in F \cap (T \times P)$, $W(t, y)$ is the number of tokens you *add* to the place y when firing the transition t .

⁵This definition gives a “weighted” form of Petri Net, and is taken from nLab, although it differs in its notation. nLab’s notation uses maps $Pre : P \times T \rightarrow \mathbb{N}$ and $Post : T \times P \rightarrow \mathbb{N}$ to characterize arcs and weights simultaneously, i.e. $Pre(x, t) = 0$ means there is no arc from x to t and $Post(t, x) = 0$ means there is no arc from t to x . When these evaluate to a number other than zero, they characterize the presence of an arc and its weight simultaneously.

⁶The use of the term transition here is an unfortunate one, because these will actually represent labels in the associated LTS.

Example 4.1.9 . Consider the Petri Net specified as follows :

1. Places : $P = \{x, y, z, w\}$
2. Transitions : $T = \{t_1, t_2, t_3\}$
3. Arcs : $F = \{(x, t_1), (y, t_1), (t_1, z), (w, t_2), (t_2, w), (t_2, z), (w, t_3)\}$
4. Initial state : $M_0(x) = 5, M_0(y) = 2, M_0(z) = 0, M_0(w) = 2$
5. Weights : $W(x, t_1) = 2, W(y, t_1) = 1, W(t_1, z) = 2, W(w, t_2) = 1, W(t_2, w) = 2, W(t_2, z) = 1, W(w, t_3) = 3$



We can associate a labelled transition system to this Petri Net as follows :

Example 4.1.10 . For the Petri Net as provided in Example 4.1.9, we can associate a labelled transition system (globally) $T = (S, L, \delta)$ where :

1. $S = \mathbb{N}^{\{x, y, z, w\}}$, and we use tuple notation $(m_x, m_y, m_z, m_w) \in \mathbb{N}^4$ to represent $X \in \mathbb{N}^{\{x, y, z, w\}}$ such that $X(x) = m_x, X(y) = m_y, X(z) = m_z$ and $X(w) = m_w$.
2. $L = \{t_1, t_2, t_3\}$, which is the set of Petri Net transitions T .

3. δ is specified via the transition relations as follows :

- $\xrightarrow{t_1}_T$ is the smallest transition relation such that $\forall m_x, m_y, m_z, m_w \in \mathbb{N}$,
 $m_x \geq 2$ and $m_y \geq 1 \Rightarrow (m_x, m_y, m_z, m_w) \xrightarrow{t_1}_T (m_x - 2, m_y - 1, m_z + 2, m_w)$
- $\xrightarrow{t_2}_T$ is the smallest transition relation such that $\forall m_x, m_y, m_z, m_w \in \mathbb{N}$,
 $m_w \geq 1 \Rightarrow (m_x, m_y, m_z, m_w) \xrightarrow{t_2}_T (m_x, m_y, m_z + 1, m_w + 1)$
- $\xrightarrow{t_3}_T$ is the smallest transition relation such that $\forall m_x, m_y, m_z, m_w \in \mathbb{N}$,
 $m_w \geq 3 \Rightarrow (m_x, m_y, m_z, m_w) \xrightarrow{t_3}_T (m_x, m_y, m_z, m_w - 3)$

Now, the above Petri Net breaks down into sub-Petri Nets induced by subsets of P , and Figure 17 gives examples of such induced sub-Petri Nets.

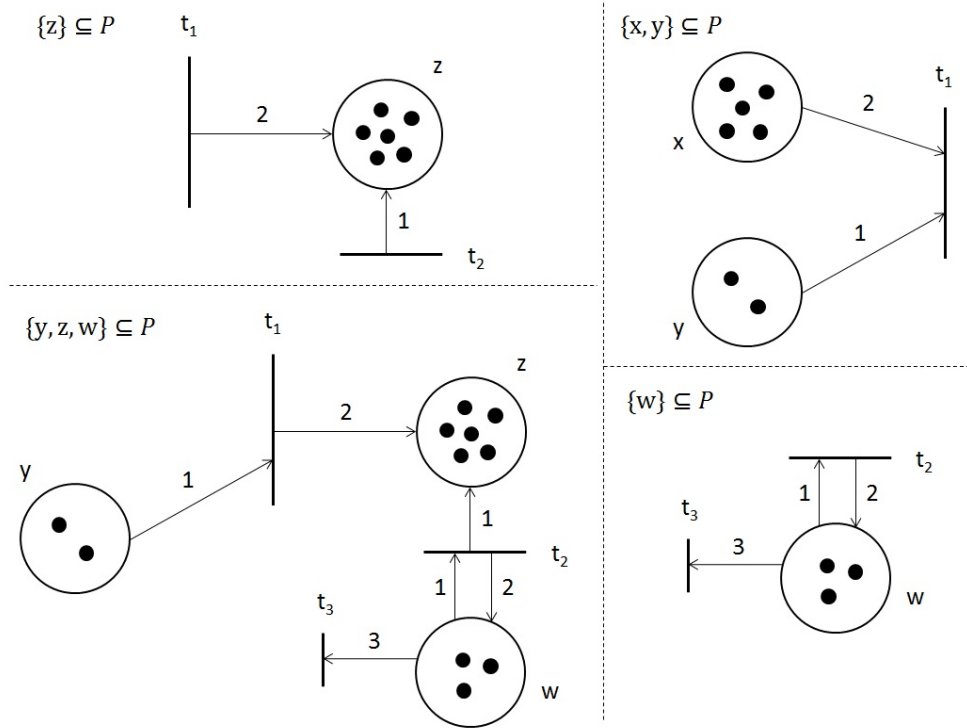


Figure 17: Examples of sub-Petri Nets induced by subsets of P (in association to Example 4.1.9).

Also, any Petri Net has a standard interpretation as a labelled transition system (and we saw an example above). Thus, we can decompose a Petri Net (with set of places P say) into sub-Petri Nets that decorate the discrete space over P , and we can associate a LTS to each such sub-Petri Net in the decomposition. This yields a natural representation of Petri Nets as $\mathbb{T}$ -valued presheaves, as follows :

Definition 4.1.11 [$\mathbb{T}$ -valued presheaf of a Petri Net]. Consider a General Petri Net (P, T, F, M_0, W) , and let $\mathcal{H}$ be the discrete space over the set of places P . We define a region for each transition, via the map $\psi : T \rightarrow \mathcal{P}(P)$, that sends a transition to the set of places it connects to through its arcs, i.e. we set $\psi(t) = \{x \in P \mid (x, t) \in F \text{ or } (t, x) \in F\}$. The $\mathbb{T}$ -valued presheaf associated to (P, T, F, M_0, W) , say $\mathcal{T}$, is given as follows :

Consider any region $U \in \mathcal{P}(P)$. We specify $\mathcal{T}(U)$ as :

- $\mathcal{S}(U) = \mathbb{N}^U$, i.e. the state space over U is the set of maps that assigns natural numbers (tokens) to the places in U (these are markings over U);
- $\mathcal{L}(U) = \{t \in T \mid \psi(t) \cap U \neq \emptyset\}$, i.e. the labelling set over U is the set of transitions that have an arc that connects to a place in U ;
- For a given $t \in \mathcal{L}(U)$, the transition relation associated to t (in U) is defined as follows :

For any states $X, Y \in \mathcal{S}(U)$, $X \xrightarrow{t}_U Y$ if and only if

- (1) $Y|_{U \setminus \psi(t)} = X|_{U \setminus \psi(t)}$ and,
- (2) $\forall x \in U \cap \psi(t)$, the following hold :
 - $(x, t) \in F \Rightarrow X(x) \geq W(x, t)$ (enough tokens in x to fire)
 - $(x, t) \in F$ and $(t, x) \notin F \Rightarrow Y(x) = X(x) - W(x, t)$
 - $(x, t) \notin F$ and $(t, x) \in F \Rightarrow Y(x) = X(x) + W(t, x)$
 - $(x, t) \in F$ and $(t, x) \in F \Rightarrow Y(x) = X(x) - W(x, t) + W(t, x)$

The first statement, $Y|_{U \setminus \psi(t)} = X|_{U \setminus \psi(t)}$, says that a transition has no effect on the places where it has no connection through its arcs in U . The statement that $\forall x \in U \cap \psi(t), (x, t) \in F \Rightarrow X(x) \geq W(x, t)$, verifies the firing condition for t , and we only verify these conditions for the arcs that connect to places in U . The last three conditions describe the three cases for modifying the marking of a place, depending on whether we have in-going or out-going arcs to that place in U .

This provides a well-defined labelled transition system $\mathcal{T}(U)$ for each region $U \subseteq P$. For given regions $V \subseteq U$, we define the state restriction $res_V^U : \mathcal{S}(U) \rightarrow \mathcal{S}(V)$ as $res_V^U(X) = X|_V$, which is the marking X of $\mathbb{N}^U$ restricted to V , and this is an element of the function space $\mathbb{N}^V$. It is clear that this $\mathcal{S}$ actually provides a set-valued sheaf of function spaces. Finally, we define the labelling projection $\rho_V^U : \mathcal{L}(U) \rightarrow \mathcal{L}(V)$ as

$$\rho_V^U(t) = \begin{cases} t & \text{if } V \cap \psi(t) \neq \emptyset \\ \varepsilon & \text{otherwise} \end{cases}$$

This yields a well-defined $\mathbb{T}$ -valued presheaf and we give part of the proof below. However, it so happens that the presheaf associated to a Petri Net this way can be formulated in terms of the presheaf of a *localized relational structure*, which is a structure we will investigate in Chapter 6. The proof that $\rho_U^U = 1_{\mathcal{L}(U)}$ and that $\rho_W^V \circ \rho_V^U = \rho_W^U$ for any regions $W \subseteq V \subseteq U$ is essentially addressed in the proof of Proposition 6.1.5 on labelling presheaves for LRS, but we repeat it here for convenience :

We have $\mathcal{L}(U) \subseteq T$ and ε is not an element of L , so ε is not an element of T (otherwise relabel this T à priori). This means that $\mathcal{L}(U)$ is an object of $\mathbf{Set}_\varepsilon$. Also, for any $t \in \mathcal{L}(U)$, we have that $U \cap \psi(t) \neq \emptyset$ by definition of $\mathcal{L}(U)$, and we get $\rho_U^U(t) = t$. Thus, $\rho_U^U = 1_{\mathcal{L}(U)}$ is the labelling identity on $\mathcal{L}(U)$ in $\mathbf{Set}_\varepsilon$.

Now, to see that $\mathcal{L}$ is a $\mathbf{Set}_\varepsilon$ -valued presheaf, it remains to verify that the labelling projections preserve composition i.e. for any regions $W \subseteq V \subseteq U$, we have $\rho_W^V \circ \rho_V^U = \rho_W^U$.

Consider any regions $W \subseteq V \subseteq U$, and any action $t \in \mathcal{L}(U)$.

- If $V \cap \psi(t) = \emptyset$, then $W \cap \psi(t) \subseteq V \cap \psi(t) = \emptyset$. In particular, this means $(\rho_W^V \circ \rho_V^U)(t) = \rho_W^V(\rho_V^U(t)) = \rho_W^V(\varepsilon) = \varepsilon = \rho_W^U(t)$.
- If $V \cap \psi(t) \neq \emptyset$ and $W \cap \psi(t) = \emptyset$, then

$$(\rho_W^V \circ \rho_V^U)(t) = \rho_W^V(\rho_V^U(t)) = \rho_W^V(t) = \varepsilon = \rho_W^U(t)$$

- If $V \cap \psi(t) \neq \emptyset$ and $W \cap \psi(t) \neq \emptyset$, then

$$(\rho_W^V \circ \rho_V^U)(t) = \rho_W^V(\rho_V^U(t)) = \rho_W^V(t) = t = \rho_W^U(t)$$

This shows $\rho_W^V \circ \rho_V^U = \rho_W^U$.

It follows that $(res_U^U, \rho_U^U) = (1_{\mathcal{S}(U)}, 1_{\mathcal{L}(U)})$ and that $(res_V^U, \rho_V^U) = (res_W^V, \rho_W^V) \circ (res_V^U, \rho_V^U)$ for any regions $W \subseteq V \subseteq U$ (we have already stated that $\mathcal{S}$ is a sheaf). It remains to verify that these (res_V^U, ρ_V^U) are actually LTS morphisms.

Proof that (res_V^U, ρ_V^U) is a LTS morphism. : Consider any $X \xrightarrow{t}_U Y$ in a region $U \in \mathcal{P}(P)$, and any subregion $V \subseteq U$. If $\rho_V^U(t) = \varepsilon$, then $V \cap \psi(t) = \emptyset$, which means that $V = V \setminus \psi(t) \subseteq U \setminus \psi(t)$. But then $X|_V = (X|_{U \setminus \psi(t)})|_V = (Y|_{U \setminus \psi(t)})|_V = Y|_V$ (we applied property (1) here for $\xrightarrow{t}_U$'s specification above), and this verifies the requirement for a LTS morphism that projects an action to ε .

If $\rho_V^U(t) \neq \varepsilon$, then we have $\rho_V^U(t) = t$ by definition of ρ_V^U . With $V \subseteq U$, we get $V \setminus \psi(t) \subseteq U \setminus \psi(t)$, so $(X|_V)|_{V \setminus \psi(t)} = X|_{V \setminus \psi(t)} = (X|_{U \setminus \psi(t)})|_{V \setminus \psi(t)} = (Y|_{U \setminus \psi(t)})|_{V \setminus \psi(t)} = Y|_{V \setminus \psi(t)} = (Y|_V)|_{V \setminus \psi(t)}$, and this verifies property (1) for $X|_V \xrightarrow{\rho_V^U(t)}_V Y|_V$ to hold.

We also have $V \cap \psi(t) \subseteq U \cap \psi(t)$, and it is rather straightforward to verify that all the universally quantified statements over $U \cap \psi(t)$ for t in (2) above, also hold over $V \cap \psi(t)$ for $\rho_V^U(t) = t$. To be more precise, we simply need to substitute the X and Y for $X|_V$ and $Y|_V$ in the equations provided in (2). Hence, we get that $X|_V \xrightarrow{\rho_V^U(t)}_V Y|_V$. $\square$

We will see how the above presheaf associated to a Petri Net matches the presheaf of the localized relational structure associated to such a Petri Net (this will be achieved in Example G.0.2).

Now, we can get an immediate sense of how SI-independence arises within these Petri Net presheaves (we consider the global region P here) :

Two actions (labels) $t, t' \in \mathcal{L}(P)$ are spatially independent if and only if

$$\psi(t) \cap \psi(t') = \emptyset$$

For Example 4.1.9, there is precisely one pair of labels at the global level for which the above condition holds, i.e. t_1 and t_3 are spatially independent because $\psi(t_1) = \{x, y, z\}$ and $\psi(t_3) = \{w\}$ are disjoint. To know that these actions are independent in the sense of an ALTS, we need to verify the alternative paths, one-step amalgamation and one-step co-amalgamation properties for t_1 and t_3 . In fact, we can transform $\mathcal{T}(P)$ into an ALTSE that has the SI-independence relation given by $I = \{(t_1, t_3), (t_3, t_1)\}$, and that has an equivalence on transitions provided in Definition 4.1.12 ahead. We simply give an intuitive sense of how this is true with an example of an $I^\sim$ -square depicted in Figure 18.

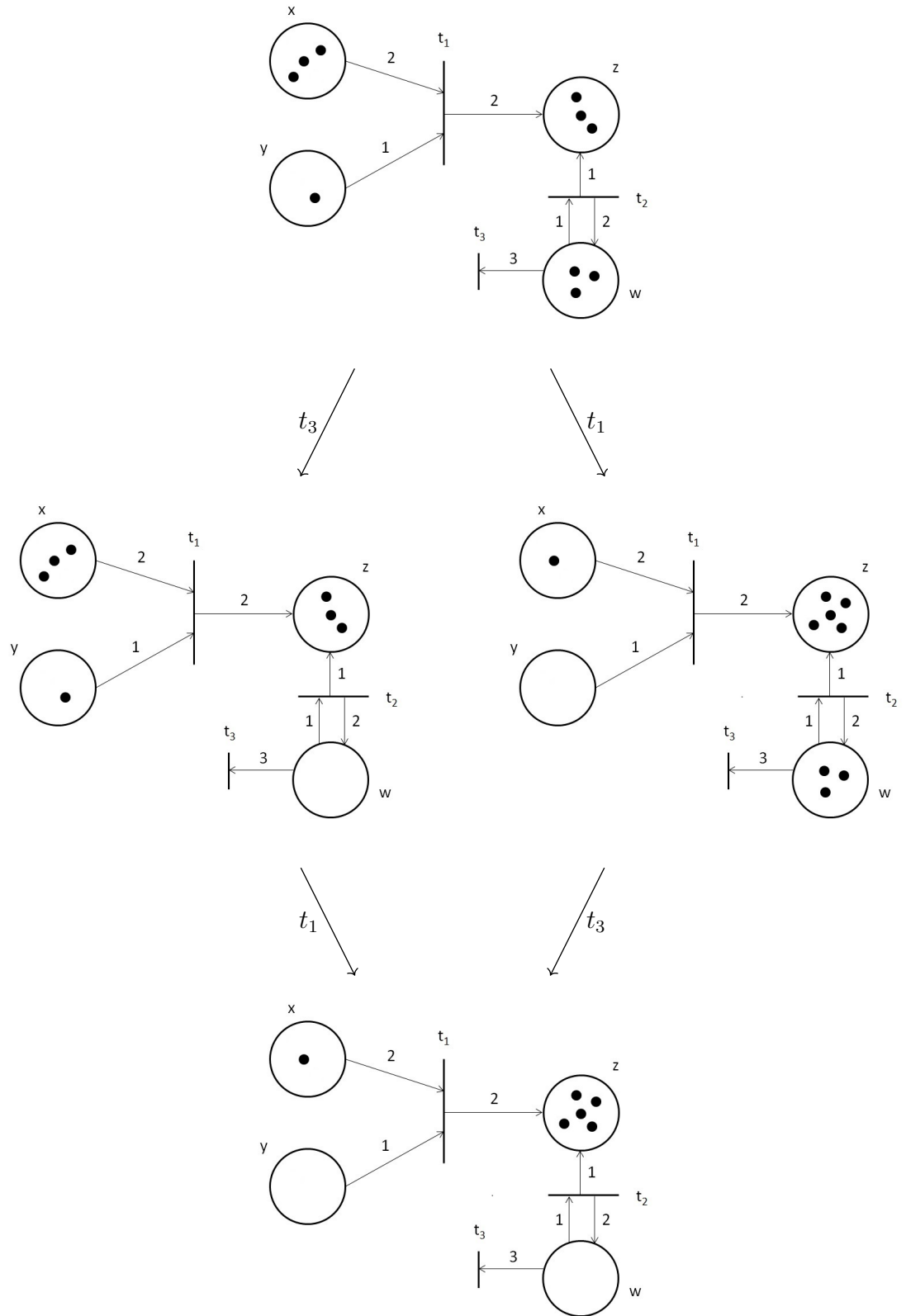
With the assumption that we can turn the Petri Net LTS $\mathcal{T}(P)$ into an ALTSE, we can make sense of equivalence classes for runs (as in Definition ??). Using the tuple notation $(m_x, m_y, m_z, m_w) \in \mathbb{N}^4$ to represent a state $X \in \mathbb{N}^{\{x, y, z, w\}}$ such that $X(x) = m_x$, $X(y) = m_y$, $X(z) = m_z$ and $X(w) = m_w$, consider the following run :

$$(3, 1, 3, 3) \xrightarrow{t_1}_T (1, 0, 5, 3) \xrightarrow{t_3}_T (1, 0, 5, 0)$$

There is only one other run equivalent to this one, and it is given by :

$$(3, 1, 3, 3) \xrightarrow{t_3}_T (3, 1, 3, 0) \xrightarrow{t_1}_T (1, 0, 5, 0)$$

These two runs are depicted in a single $I^\sim$ -square in Figure 18.


 Figure 18: Example of an $I^\sim$ -square

We should be more or less convinced that the I in question provides an ALTS for this system; this is basically because the whole region where t_1 is involved in its effects and dependencies (that is $\psi(t_1)$) is disjoint from the region where t_3 is involved (that is $\psi(t_3)$). This constitutes all that we have to say about Petri Net presheaves for now.

Going back to the theoretical development for $\mathbb{T}$ -valued presheaves, there is one important group of relations that we want in our arsenal with respect to such presheaves, and these are the equivalence relations provided by the mapping kernels of the state restriction maps res_V^U . These kernels will serve us in the definition of an equivalence on transitions that is key to transforming each labelled transition system $\mathcal{T}(U)$ of $\mathcal{T}$ into an ALTSE (i.e. into an object of $\mathbb{A}^\sim$).

Definition 4.1.12 [Equivalences on States]. Given a $\mathbb{T}$ -valued presheaf on a Heyting algebra $\mathcal{H}$ and any regions $V \leq U$ in $\mathcal{H}$, we write $\simeq_V^U$ to denote the equivalence relation on $\mathcal{S}(U)$ given by the mapping kernel of res_V^U , i.e. $\forall X, Y \in \mathcal{S}(U)$,

$$X \simeq_V^U Y \Leftrightarrow res_V^U(X) = res_V^U(Y)$$

Remark 4.1.13 . It can be easily verified that $\simeq_V^U = (res_V^U)^{-1} \circ res_V^U$.

A consequence of the presheaf structure of $\mathcal{T}$ is that for any regions $W \leq V \leq U$, if $res_V^U(X) = res_V^U(Y)$, then $res_W^U(X) = res_W^U(Y)$, and so we get the following :

Proposition 4.1.14 . *Given a $\mathbb{T}$ -valued presheaf on a Heyting algebra $\mathcal{H}$ and any regions $W, V, U \in \mathcal{H}$, we have :*

$$W \leq V \leq U \Rightarrow \simeq_V^U \subseteq \simeq_W^U$$

Now, for a $\mathbb{T}$ -valued presheaf and regions $V \leq U$, it is true that for any $X, Y \in \mathcal{S}(U)$ and $b \in \mathcal{L}(U)$, if $X \xrightarrow{b} U Y$, then $X|_V \xrightarrow{\rho_V^U(b)} V Y|_V$.

In the relational language, this last statement means that :

$$X \xrightarrow{res_V^U} X|_V \xrightarrow{\rho_V^U(b)} Y|_V \xrightarrow{(res_V^U)^{-1}} Y$$

and this is equivalent to saying that $(X, Y) \in ((res_V^U)^{-1} \circ \xrightarrow{\rho_V^U(b)} \circ res_V^U)$. We formalize this idea in a simple proposition.

Proposition 4.1.15 . *Given a $\mathbb{T}$ -valued presheaf on a Heyting algebra $\mathcal{H}$ and any regions $U \in \mathcal{H}$ and action $b \in \mathcal{L}(U)$, if $\{V_j\}_{j \in J}$ is a proper cover of U , then the following inclusion holds :*

$$\xrightarrow{b}_{\rightarrow U} \subseteq \bigcap_{j \in J} ((res_{V_j}^U)^{-1} \circ \xrightarrow{\rho_{V_j}^U(b)} \circ res_{V_j}^U)$$

In the case of a $\mathbb{T}$ -valued sheaf, the inclusion stated above works the other way around too, and we use the following definition to characterize such conditions where inclusion works both ways.

Definition 4.1.16 [Action Decomposition]. Given a proper cover $\{V_j\}_{j \in J}$ of a region U and an action $b \in \mathcal{L}(U)$, we say that b *decomposes over* $\{V_j\}_{j \in J}$ if :

$$\xrightarrow{b}_{\rightarrow U} = \bigcap_{j \in J} ((res_{V_j}^U)^{-1} \circ \xrightarrow{\rho_{V_j}^U(b)} \circ res_{V_j}^U)$$

The equation above is equivalent to the statement that for any states $X, Y \in \mathcal{S}(U)$,

$$X \xrightarrow{b}_{\rightarrow U} Y \Leftrightarrow X|_{V_j} \xrightarrow{\rho_{V_j}^U(b)} Y|_{V_j} \text{ for all } j \text{ in } J$$

We will show in Section 4.2 that these equations hold regardless of the choice of action and proper cover in $\mathbb{T}$ -valued sheaves.

We should stress, however, that the decomposition of an action is no substitute for gluing in general because it only works under the assumption of globally declared actions and states à priori. In particular, the idea of decomposition for an action b in U is not quite convincing because it does not actually tell us how to “glue” local

transitions $X_j \xrightarrow{\rho_{V_j}^U(b)}_{V_j} Y_j$ on a cover $\{V_j\}_{j \in J}$ of U to obtain a transition $X \xrightarrow{b}_U Y$ in U . This form of “gluing”, that associates an action’s transition relation to those of its projections on a cover, is what we refer to as δ -gluing for actions, a concept newly introduced in this thesis, and the definition is as follows :

Definition 4.1.17 [δ -gluing for Actions]. Consider a $\mathbb{T}$ -valued presheaf, a region U , any action $b \in \mathcal{L}(U)$ and a proper cover $\{V_j\}_{j \in J}$ of U . We say that b *glues its transitions over* $\{V_j\}_{j \in J}$ with respect to U (or b *has δ -gluing on* $\{V_j\}_{j \in J}$ with respect to U), if for any J -indexed set of transitions $\{ (X_j, \rho_{V_j}^U(b), Y_j) \in \delta(V_j) \}_{j \in J}$ such that $X_i|_{V_i \wedge V_j} = X_j|_{V_i \wedge V_j}$ and $Y_i|_{V_i \wedge V_j} = Y_j|_{V_i \wedge V_j}$ for all $i, j \in J$, there are unique $X, Y \in \mathcal{S}(U)$ such that :

$$X \xrightarrow{b}_U Y \text{ and } X|_{V_j} = X_j \text{ and } Y|_{V_j} = Y_j \text{ for all } j \in J$$

It turns out that if the states presheaf $\mathcal{S}$ is a sheaf, the uniqueness of the states X, Y in $\mathcal{S}(U)$, derived by gluing transitions in the above definition, follows naturally from the locality of $\mathcal{S}$. Better yet, if $\mathcal{S}$ is a sheaf, the notions of decomposition and δ -gluing for actions become equivalent notions.

Proposition 4.1.18 . Consider any $\mathbb{T}$ -valued presheaf and suppose that the states presheaf $\mathcal{S}$ is a sheaf. Consider any region U and any proper cover $\{V_j\}_{j \in J}$ of U , and any action $b \in \mathcal{L}(U)$. Then the following are equivalent :

1. The action b glues its transitions over $\{V_j\}_{j \in J}$
2. The action b decomposes over $\{V_j\}_{j \in J}$

Proof. [(1) $\Rightarrow$ (2)] Suppose (1) holds. From Proposition 4.1.15, it suffices to prove the following inclusion (from right to left) :

$$\xrightarrow{b}_U \supseteq \bigcap_{j \in J} ((res_{V_j}^U)^{-1} \circ \xrightarrow{\rho_{V_j}^U(b)}_{V_j} \circ res_{V_j}^U)$$

Thus, consider any $(X, Y) \in \bigcap_{j \in J} ((res_{V_j}^U)^{-1} \circ \xrightarrow{\rho_{V_j}^U(b)}_{V_j} \circ res_{V_j}^U)$.

Then for all $j \in J$, we have $X|_{V_j} \xrightarrow{\rho_{V_j}^U(b)}_{V_j} Y|_{V_j}$. By the consistency of restrictions on states, we get $(X|_{V_i})|_{V_i \wedge V_j} = X|_{V_i \wedge V_j} = (X|_{V_j})|_{V_i \wedge V_j}$, and similarly, $(Y|_{V_i})|_{V_i \wedge V_j} = Y|_{V_i \wedge V_j} = (Y|_{V_j})|_{V_i \wedge V_j}$ for any $i, j \in J$. Applying the fact that b glues its transitions over $\{V_j\}_{j \in J}$, we get $X', Y' \in \mathcal{S}(U)$ such that $X' \xrightarrow{b}_{\rightarrow U} Y'$ and $X'|_{V_j} = X|_{V_j}$ and $Y'|_{V_j} = Y|_{V_j}$ for all $j \in J$. By the locality property of $\mathcal{S}$, we get $X' = X$ and $Y' = Y$, and this proves the inclusion in question.

[(2) $\Rightarrow$ (1)] Suppose (2) holds. Consider any J -indexed set of transitions :

$$\{ (X_j, \rho_{V_j}^U(b), Y_j) \in \delta(V_j) \}_{j \in J}$$

such that $X_i|_{V_i \wedge V_j} = X_j|_{V_i \wedge V_j}$ and $Y_i|_{V_i \wedge V_j} = Y_j|_{V_i \wedge V_j}$ for every $i, j \in J$.

By the gluing property of $\mathcal{S}$ and the fact that the X_j agree on the overlaps of the cover $\{V_j\}_{j \in J}$ of U , we can perform gluing to yield an $X \in \mathcal{S}(U)$ such that $X|_{V_j} = X_j$ for all $j \in J$. Similarly, we can glue the Y_j to obtain $Y \in \mathcal{S}(U)$ such that $Y|_{V_j} = Y_j$ for all $j \in J$. But then, we have $X|_{V_j} \xrightarrow{\rho_{V_j}^U(b)}_{V_j} Y|_{V_j}$ for all j in J , and thus :

$$(X, Y) \in \bigcap_{j \in J} ((res_{V_j}^U)^{-1} \circ \xrightarrow{\rho_{V_j}^U(b)}_{V_j} \circ res_{V_j}^U) = \xrightarrow{b}_{\rightarrow U}$$

So, $X \xrightarrow{b}_{\rightarrow U} Y$ as desired. The uniqueness follows from the locality of $\mathcal{S}$ because the states X and Y are completely determined by the X_j and the Y_j respectively on the cover $\{V_j\}_{j \in J}$. $\square$

Thus, whilst action δ -gluing gives the intuition of how transition relations interact with local transition relations through gluing, decomposition gives the convenient equational format that we will manipulate in many cases.

Now, as we explained before, it is by allowing a small amount of δ -gluing for actions that we will be able to get SI-independence for $\mathbb{T}$ -valued presheaves. But, nevertheless, we should try to understand the case of sheaves first and see why we

cannot be content with them for the purpose of rendering SI-independence in general. Afterwards, in Section 4.3, we will explore how the notion of action δ -gluing can be used to express “containment of actions” (see Definition 4.3.15); a concept that is essential to the expression of SI-independence.

4.2 $\mathbb{T}$ -valued Sheaves

In this section we explore the consequences of dealing with sheaves of labelled transition systems over a complete Heyting algebra as a base space. Such $\mathbb{T}$ -valued sheaves have been explored in G. Malcolm’s work ([SSTS] and [SODS]), with the key difference that he worked out his results using monoid labelled transition systems instead of LTS (and so he worked with $\mathbb{M}$ -valued sheaves on Heyting algebras). In particular, Malcolm advocates the use of sheaves for specifying distributed processes, following the tradition of Goguen’s Systems Theory (see [CST] and [CFST]).

Our intention here, however, is to argue that these are too general to account for all interesting kinds of $\mathbb{T}$ -valued presheaves with SI-independence capabilities. To be more precise, $\mathbb{T}$ -valued sheaves seem to work well when using Heyting algebras that are not Boolean algebras as base spaces, but they sometimes yield unsatisfying results in the case of Boolean algebra for base spaces (and thus, this is inconvenient for the frequently encountered discrete spaces in computer science). In particular, they lead to an explosive amount of ε -gluing (see Proposition 4.2.10), and we will use this ε -gluing at the end of this section to show that Petri Net Presheaves are not sheaves in general.

Thus, first and foremost, we recall the definitions of sheaves and separated presheaves.

Definition 4.2.1 [Sheaf [SGLog]]. Given a complete category $\mathcal{C}$, a $\mathcal{C}$ -valued sheaf on a complete Heyting algebra $\mathcal{H}$ is a functor $\mathcal{F} : \mathcal{H}^{op} \rightarrow \mathcal{C}$ such that for $U = \bigvee_{j \in J} V_j$ in $\mathcal{H}$,

$$\mathcal{F}(U) \rightarrow \prod_{j \in J} \mathcal{F}(V_j) \rightrightarrows \prod_{i,j \in J} \mathcal{F}(V_i \wedge V_j)$$

is an equalizer diagram (where all the arrows arise from the obvious restrictions and the universal property of products and equalizers).

Definition 4.2.2 [Separated Presheaf [SGLog]]. A $\mathcal{C}$ -valued separated presheaf on a complete Heyting algebra $\mathcal{H}$ is a functor $\mathcal{F} : \mathcal{H}^{op} \rightarrow \mathcal{C}$ such that for $U = \bigvee_{j \in J} V_j$ in $\mathcal{H}$, the arrow $\mathcal{F}(U) \rightarrow \prod_{j \in J} \mathcal{F}(V_j)$ that arises from the obvious restrictions and the universal property on products, is a monomorphism. We refer to the mono property of these arrows as the *locality* property for $\mathcal{F}$.

To settle the definition in the context of $\mathbb{T}$, we get the following :

Definition 4.2.3 [$\mathbb{T}$ -valued sheaf]. A $\mathbb{T}$ -valued sheaf on a complete Heyting algebra $\mathcal{H}$ is a functor $\mathcal{T} : \mathcal{H}^{op} \rightarrow \mathbb{T}$ such that for $U = \bigvee_{j \in J} V_j$ in $\mathcal{H}$,

$$\mathcal{T}(U) \rightarrow \prod_{j \in J} \mathcal{T}(V_j) \rightrightarrows \prod_{i,j \in J} \mathcal{T}(V_i \wedge V_j)$$

is an equalizer diagram (where all the arrows arise from the obvious restrictions and the universal property of products and equalizers).

We can prove that if $\mathcal{T}$ is a sheaf, then so are its component presheaves. This is what the following proposition establishes.

Proposition 4.2.4 . *Suppose $\mathcal{T}$ is a $\mathbb{T}$ -valued sheaf on a complete Heyting algebra $\mathcal{H}$. Then the associated state presheaf $\mathcal{S}$ is a set-valued sheaf, and the associated labelling presheaf $\mathcal{L}$ is a $\mathbf{Set}_\varepsilon$ -valued sheaf.⁷*

Proof. The proof is provided in Appendix D. □

⁷Interestingly enough, the proof that $\mathcal{L}$ is a sheaf under the condition that $\mathcal{T}$ is a sheaf only follows if we allow ghost actions for labelled transition systems.

We now work out an example of pullback in $\mathbb{T}$ here, and these can always be associated to $\mathbb{T}$ -valued sheaves on an underlying Heyting algebra space of the shape $\mathcal{H} = \{0_{\mathcal{H}}, V_1 \wedge V_2, V_1, V_2, 1_{\mathcal{H}}\}$ as in Figure 19.

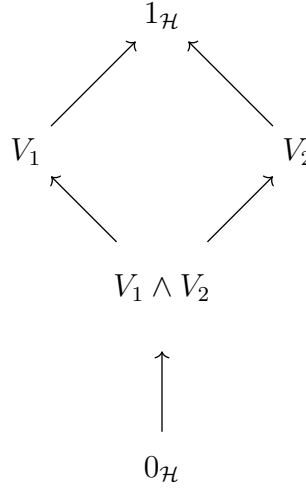


Figure 19: Diagram representing the Heyting algebra space that arises from pullbacks in $\mathbb{T}$ ⁸.

The $\mathbb{T}$ -valued sheaf example we consider is a junction of buffers as follows :

Example 4.2.5 [Pullback Buffer Junction].

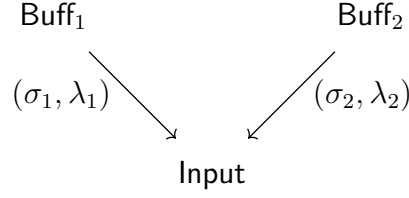
Consider a pair of buffers $\text{Buff}_1 = (S_1, L_1, \delta_1)$ and $\text{Buff}_2 = (S_2, L_2, \delta_2)$ as in Example 2.1.7. Write $S_i = \{(x^i, y^i) \in \{0, 1\}^2\}$ and $L_i = \{inp^i, send^i\}$.

We form a cospan diagram to connect the input region x^1 of Buff_1 with the input region x^2 of Buff_2 in a labelled transition system $\text{Input} = (S_I, L_I, \delta_I)$ (see Figure 20) with $S_I = \{0, 1\}$ and $L_I = \{inp\}$ and $\xrightarrow{inp}_C = \{0, 1\}^2$.

We set the LTS morphisms $(\sigma_1, \lambda_1) : \text{Buff}_1 \rightarrow \text{Input}$ and $(\sigma_2, \lambda_2) : \text{Buff}_2 \rightarrow \text{Input}$ as :

- $\sigma_1(x^1, y^1) = x^1$ for any $x^1, y^1 \in \{0, 1\}$ (i.e. proj. on the 1^{st} component)
- $\sigma_2(x^2, y^2) = x^2$ for any $x^2, y^2 \in \{0, 1\}$ (i.e. proj. on the 2^{nd} component)

⁸Not all arrows are shown in the diagram of this figure, i.e. a few region inclusions are omitted.

Figure 20: Connection of Buff_1 and Buff_2 on their input region.

- $\lambda_1(\text{send}^1) = \lambda_2(\text{send}^2) = \varepsilon$ and $\lambda_1(\text{inp}^1) = \lambda_2(\text{inp}^2) = \text{inp}$

Then, taking a limit (pullback) provides a LTS, say **Junction** (as in Figure 20), that describes the global behavior of the connected buffers taken as a single whole (this corresponds with Goguen's view that the limit of a system represents the global behavior of that system).

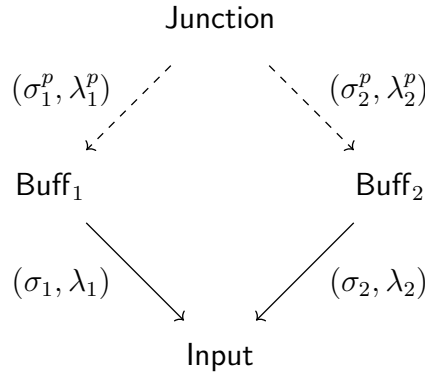


Figure 21: Pullback of diagram in Figure 20.

In fact, we get a sheaf on a Heyting algebra if we use as regions the downward closed sets (with respect to arrows) in the diagram of Figure 20. The downward closed sets are $\emptyset$, $\{\text{Input}\}$, $\{\text{Buff}_1, \text{Input}\}$, $\{\text{Buff}_2, \text{Input}\}$, $\{\text{Buff}_1, \text{Buff}_2, \text{Input}\}$. We decorate these regions with the labelled transition systems $1_{\mathbb{T}}$, **Input**, **Buff**₁, **Buff**₂, and **Junction** respectively. We use set inclusion on the downward closed sets to provide inclusion of regions. There is an obvious association of morphisms from the pullback

diagram of Figure 21 to the pairs of regions $V \subseteq U$ obtained as downward closed sets here.

The resulting **Junction** is a junction of buffers, where the input regions x^1 and x^2 are identified via σ_1 and σ_2 . Also, $\lambda_1(inp^1) = \lambda_2(inp^2) = inp$ forces the actions inp^1 and inp^2 to synchronize in the junction, i.e. they have no choice but to perform simultaneously. Finally, $\lambda_1(send^1) = \lambda_2(send^2) = \varepsilon$ allows $send^1$ and $send^2$ to perform independently of each other in the junction.

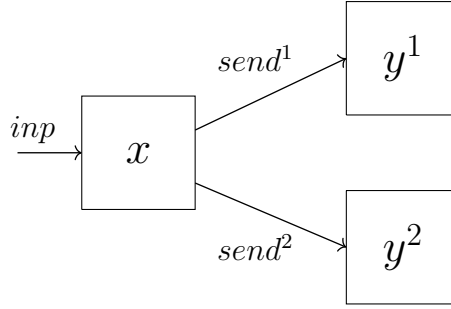


Figure 22: A visual representation of **Junction**, where x^1 from Buff_1 and x^2 of Buff_2 are identified as x .

To be more precise, $T = \mathbf{Junction}$ can be given as (S, L, δ) where $S = \{(x, y^1, y^2) \in \{0, 1\}^3\}$ and $L = \{(inp^1, inp^2), (send^1, \varepsilon), (\varepsilon, send^2), (send^1, send^2)\}$, and the transition relations for **Junction** can be specified as follows :

- $\xrightarrow{(inp^1, inp^2)}_T$ is the smallest transition relation on S such that :

$$\forall x, x', y^1, y^2 \in \{0, 1\}, \quad (x, y^1, y^2) \xrightarrow{(inp^1, inp^2)}_T (x', y^1, y^2)$$

- $\xrightarrow{(send^1, \varepsilon)}_T$ is the smallest transition relation on S such that :

$$\forall x, y^1, y^2 \in \{0, 1\}, \quad (x, y^1, y^2) \xrightarrow{(send^1, \varepsilon)}_T (x, x, y^2)$$

- $\xrightarrow{(\varepsilon, send^2)}_T$ is the smallest transition relation on S such that :

$$\forall x, y^1, y^2 \in \{0, 1\}, \quad (x, y^1, y^2) \xrightarrow{(\varepsilon, send^2)}_T (x, y^1, x)$$

- $\xrightarrow{(\text{send}^1, \text{send}^2)}_T$ is the smallest transition relation on S such that :

$$\forall x, y^1, y^2 \in \{0, 1\}, \quad (x, y^1, y^2) \xrightarrow{(\text{send}^1, \text{send}^2)}_T (x, x, x)$$

The LTS morphisms $(\sigma_1^p, \lambda_1^p) : \mathbf{Junction} \rightarrow \mathbf{Buff}_1$ and $(\sigma_2^p, \lambda_2^p) : \mathbf{Junction} \rightarrow \mathbf{Buff}_2$ are given as :

- $\sigma_1^p(x, y^1, y^2) = (x, y^1)$, and x gets matched with the x^1 component of $\mathbf{Buff}_1$.
- $\sigma_2^p(x, y^1, y^2) = (x, y^2)$, and x gets matched with the x^2 component of $\mathbf{Buff}_2$.

And λ_1^p and λ_2^p are projections on the first and second component respectively for the pairs provided in L , as follows :

- $\lambda_1^p(\text{inp}^1, \text{inp}^2) = \text{inp}^1$,
- $\lambda_1^p(\text{send}^1, \text{send}^2) = \lambda_1^p(\text{send}^1, \varepsilon) = \text{send}^1$,
- $\lambda_1^p(\varepsilon, \text{send}^2) = \varepsilon$,
- $\lambda_2^p(\text{inp}^1, \text{inp}^2) = \text{inp}^2$,
- $\lambda_2^p(\text{send}^1, \text{send}^2) = \lambda_2^p(\varepsilon, \text{send}^2) = \text{send}^2$,
- $\lambda_2^p(\text{send}^1, \varepsilon) = \varepsilon$.

We can visualize how spatial independence (in the sense of an ALTS) arises amongst certain actions in **Junction** through diamond-shaped patterns of alternative paths. With the SI-independence relations that we will provide later in Section 5.1, we get that $(\varepsilon, \text{send}^2)$ and $(\text{send}^1, \varepsilon)$ are independent in **Junction**. We get an example of an alternative path pattern in Figure 23.

Remark 4.2.6 . We get a middle arrow with action $(\text{send}^1, \text{send}^2)$ in Figure 23; it represents the simultaneous execution of send^1 and send^2 . These “middle arrows” (representing simultaneous executions of actions within a presheaf) can constitute a problem with respect to the trace monoid of an ALTS. Indeed, we would typically like to have the following equivalence on words $(\text{send}^1, \text{send}^2) \equiv (\varepsilon, \text{send}^2)(\text{send}^1, \varepsilon) \equiv$

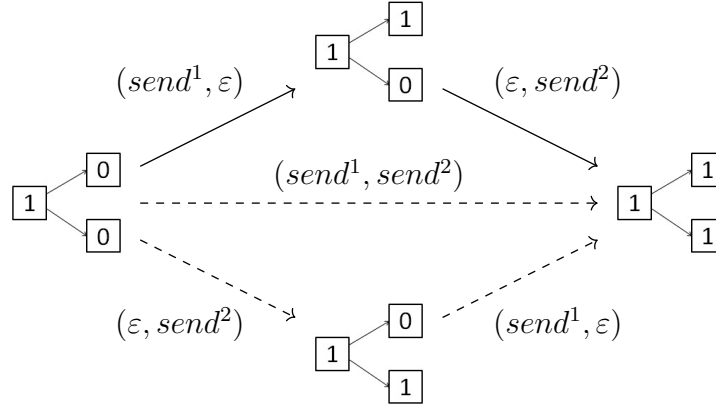


Figure 23: Example of an alternative path, plus a middle $(send^1, send^2)$ transition.

$(send^1, \varepsilon)(\varepsilon, send^2)$ in the **Junction** example, but the structure of an ALTS does not provide the existence of such middle arrows, and this means that the abstract trace equivalence of such ALTS does not provide us with adequate quotienting of sequences of actions in presheaves of LTS. We investigate this problem and propose a solution to it in Appendix H.

In Section 4.3, we will be interested in finding specific regions where actions are contained, and our idea is to look for j -complements⁹ of regions where actions project to ε . We describe informally here how this will be done through the above **Junction** example :

Write $V_1 = \{Input, Buff_1\}$ and $V_2 = \{Input, Buff_2\}$ to denote the regions of $Buff_1$ and $Buff_2$ respectively, and write $U = V_1 \cup V_2$. Consider the action $(send^1, \varepsilon)$ in the labelling set of the region U (this is the global region). We can get an idea of the subregion of U where this $(send^1, \varepsilon)$ action vanishes as a whole by taking the join of the regions where it projects to ε inside U . We obtain :

$$\bigvee \{ V \subseteq U \mid \rho_V^U(send^1, \varepsilon) = \varepsilon \} = V_2$$

Then, to get a sense of the subregion where this $(send^1, \varepsilon)$ is contained, we may,

⁹Recall that given regions $V \leq U$, a j -complement of V in U is a region $W \leq U$ such that $V \vee W = U$.

by contrast to the region V_2 where $(send^1, \varepsilon)$ vanishes, look for a j -complement of V_2 inside U . If we think of using the pseudo-complement of the Heyting algebra structure, we get the following :

$$\bigvee \{ V \subseteq U \mid V_2 \cap V = \emptyset \} = \emptyset$$

This empty region is clearly counter-intuitive if it is meant to characterize the region where $(send^1, \varepsilon)$ is “contained”. The latter clearly has effects and dependencies in V_1 , the region of Buff_1 . Indeed, $(send^1, \varepsilon)$ depends on the value of x in order to transfer it to y^1 , and it can create observable changes in states (effects) in y^1 when it fires. Thus, we should be careful in using the pseudo-complement of a Heyting algebra to characterize the region where an action is contained.

If we work in the context of a co-Heyting algebra¹⁰, there exists another kind of pseudo-complement (the logical subtraction, that is dual to $\Rightarrow$) that we can use, and with respect to V_2 in U , this new pseudo-complement provides the following :

$$\bigwedge \{ V \subseteq U \mid V_2 \cup V = U \} = V_1$$

i.e. instead of using the largest region that does not meet with V_2 , we can use the smallest region that forms a cover of the whole space with respect to V_2 . The V_1 region provided above does give the intuitive region where $(send^1, \varepsilon)$ is contained.

It might be good to know that in the case of a finite lattice, we always have a co-Heyting algebra (as well as a Heyting algebra, which yields a bi-Heyting algebra). This is because the infinite distributivity laws hold trivially. This is quite convenient, in particular, in the cases where we have finite diagrams of LTS to represent systems,

¹⁰A co-Heyting algebra, according to nLab, is a bounded distributive lattice L equipped with a binary subtraction operation $\setminus : L \times L \rightarrow L$ such that $x \setminus y \leq z$ iff $x \leq y \vee z$. These basically correspond with bounded lattices where the distributivity law holds for finite joins on infinite indexed families of meets :

$$U \vee \left(\bigwedge_{i \in I} V_i \right) = \bigwedge_{i \in I} (U \vee V_i)$$

because these will always extend to finite lattice spaces, and as such, we will have an efficient means of finding regions where actions are contained.

Now, before we proceed to our formal discussion about action containment in the next section, let us see how $\mathbb{T}$ -valued sheaves constitute a problem in the cases where the base space is a Boolean algebra. We need to consider a few propositions first.

Proposition 4.2.7 [Systematic δ -gluing in Sheaves]. *Given a $\mathbb{T}$ -valued sheaf on a Heyting algebra space $\mathcal{H}$, a region U in $\mathcal{H}$, and any action $b \in \mathcal{L}(U)$, we have that the action b glues its transitions over any proper cover of U .*

Proof. Suppose $\mathcal{T} = (\mathcal{S}, \mathcal{L}, \delta)$ is a $\mathbb{T}$ -valued sheaf on a Heyting algebra space $\mathcal{H}$. Consider any region U in $\mathcal{H}$ and action $b \in \mathcal{L}(U)$, and let $\{V_j\}_{j \in J}$ be a proper cover of U . Consider any J -indexed family of transitions $\{(X_j, \rho_{V_j}^U(b), Y_j) \in \delta(V_j)\}_{j \in J}$ such that $X_i|_{V_i \wedge V_j} = X_j|_{V_i \wedge V_j}$ and $Y_i|_{V_i \wedge V_j} = Y_j|_{V_i \wedge V_j}$ for all $i, j \in J$.

Now, consider the following labelled transition system :

$$T_0 = (S_0, L_0, \delta_0) = (\{X_0, Y_0\}, \{b_0\}, \{(X_0, b_0, Y_0)\})$$

with a single transition $X_0 \xrightarrow{b_0} Y_0$. For each $j \in J$, we set up a LTS morphism $(\sigma_j, \lambda_j) : T_0 \rightarrow \mathcal{T}(V_j)$ with $\sigma(X_0) = X_j$, $\sigma(Y_0) = Y_j$ and $\lambda_j(b_0) = \rho_{V_j}^U(b)$. These are clearly LTS morphisms. Also, for any $i, j \in J$, we have that :

$$(\rho_{V_j \cap V_i}^{V_j} \circ \lambda_j)(b_0) = \rho_{V_j \cap V_i}^{V_j}(\rho_{V_j}^U(b)) = \rho_{V_j \cap V_i}^U(b) = \rho_{V_j \cap V_i}^U(b) = \rho_{V_j \cap V_i}^{V_i}(\rho_{V_i}^U(b)) = (\rho_{V_j \cap V_i}^{V_i} \circ \lambda_i)(b_0)$$

and this means that $(\rho_{V_j \cap V_i}^{V_j} \circ \lambda_j) = (\rho_{V_j \cap V_i}^{V_i} \circ \lambda_i)$.

Also, $\sigma_j(X_0)|_{V_j \cap V_i} = X_j|_{V_j \cap V_i} = X_i|_{V_j \cap V_i} = \sigma_i(X_0)|_{V_j \cap V_i}$ and $\sigma_j(Y_0)|_{V_j \cap V_i} = Y_j|_{V_j \cap V_i} = Y_i|_{V_j \cap V_i} = \sigma_i(Y_0)|_{V_j \cap V_i}$, and this means $res_{V_i \cap V_j}^{V_j} \circ \sigma_j = res_{V_i \cap V_j}^{V_i} \circ \sigma_i$. Thus, for any $i, j \in J$, we have $(res_{V_i \cap V_j}^{V_j}, \rho_{V_i \cap V_j}^{V_j}) \circ (\sigma_j, \lambda_j) = (res_{V_i \cap V_j}^{V_j} \circ \sigma_j, \rho_{V_i \cap V_j}^{V_j} \circ \lambda_j) = (res_{V_i \cap V_j}^{V_i} \circ \sigma_i, \rho_{V_i \cap V_j}^{V_i} \circ \lambda_i) = (res_{V_i \cap V_j}^{V_i}, \rho_{V_i \cap V_j}^{V_i}) \circ (\sigma_i, \lambda_i)$.

We can apply the sheaf property of $\mathcal{T}$ to glue these (σ_j, λ_j) and get a LTS morphism $(\sigma, \lambda) : T_0 \rightarrow \mathcal{T}(U)$ such that $\sigma(X_0)|_{V_j} = \sigma_j(X_0)$ and $\sigma(Y_0)|_{V_j} = \sigma_j(Y_0)$ and $\rho_{V_j}^U(\lambda(b_0)) = \lambda_j(b_0)$ for all $j \in J$.

We have that $\rho_{V_j}^U(b) = \lambda_j(b_0) = \rho_{V_j}^U(\lambda(b_0))$ for all $j \in J$, and thus $b = \lambda(b_0)$ by locality of $\mathcal{L}$ (this applies Proposition 4.2.4 and Proposition 1.2.3). We get that $\lambda(b_0) \in \mathcal{L}(U)$ and thus $\lambda(b_0) \neq \varepsilon$. But then, we have that $\sigma(X_0) \xrightarrow{\lambda(b_0)}_U \sigma(Y_0)$, and this is $\sigma(X_0) \xrightarrow{b}_U \sigma(Y_0)$. Furthermore, with the transition $\sigma(X_0) \xrightarrow{b}_U \sigma(Y_0)$, we have $\sigma(X_0)|_{V_j} = \sigma_j(X_0) = X_j$ and $\sigma(Y_0)|_{V_j} = \sigma_j(Y_0) = Y_j$.

Finally, suppose there exists X and Y in $\mathcal{S}(U)$ such that $X \xrightarrow{a}_U Y$ and $X|_{V_j} = X_j$ and $Y|_{V_j} = Y_j$ for all j in J . We have that $\mathcal{S}$ has locality from Proposition 4.2.4, and this provides $X = \sigma(X_0)$ and $Y = \sigma(Y_0)$ since $\sigma(X_0)|_{V_j} = \sigma_j(X_0) = X_j$ and $\sigma(Y_0)|_{V_j} = \sigma_j(Y_0) = Y_j$ for all $j \in J$. $\square$

We saw that δ -gluing for actions is equivalent to decomposition of actions under the condition that $\mathcal{S}$ is a sheaf in Proposition 4.1.18, and we know from Proposition 4.2.4 that $\mathcal{S}$ is a sheaf if $\mathcal{T}$ is a sheaf. Thus, we get the following :

Corollary 4.2.8 . *Let $\mathcal{T} = (\mathcal{S}, \mathcal{L}, \delta)$ be a $\mathbb{T}$ -valued sheaf on a space $\mathcal{H}$. Then given any region U in $\mathcal{H}$ and action $b \in \mathcal{L}(U)$, and any proper cover $\{V_j\}_{j \in J}$ of U , we have that the action b decomposes over $\{V_j\}_{j \in J}$ (in the sense of Definition 4.1.16).*

What Proposition 4.2.7 (or Corollary 4.2.8) implies is that the structure of the space dictates precisely how actions can be cut and made into smaller independent local actions. Of course, the gluing requires an agreement on the overlaps, but in the case of covers that form partitions, as can be easily formed in discrete spaces, there is always agreement on the overlaps and this may constitute a problem.

Consider for example, the buffer over a discrete space as in Example 4.1.7. If the *send* action decomposes over the cover $\{\{x\}, \{y\}\}$ of $\{x, y\}$, we get to glue transitions like $0 \xrightarrow{\text{send}}_{\{y\}} 1$ and $0 \xrightarrow{\text{send}}_{\{x\}} 0$, and these yield $(0, 0) \xrightarrow{\text{send}}_{\{x, y\}} (0, 1)$ (the first

component of the pair corresponds with the value in $\{x\}$, and the second component corresponds with the value in $\{y\}$). This transition is inconsistent because the value of the first component, 0, was supposed to be sent by the action of *send* to the second component, but instead, we got a 1 value appearing there out of nowhere.

However, in the case of Petri Net presheaves, it so happens that actions sheaf over any cover.

Proposition 4.2.9 [Systematic δ -gluing in Petri Nets]. *Consider a Petri Net (P, T, F, M_0, W) and its associated $\mathbb{T}$ -valued presheaf $\mathcal{T}$ as in Definition 4.1.11. Then for any region U and any action $t \in \mathcal{L}(U)$, we have that the action t glues its transitions over any proper cover of U .*

Proof. Consider any region U and any action $t \in \mathcal{L}(U)$, and any proper cover $\{V_j\}_{j \in J}$ of U . Consider any J -indexed family of transitions $\{(X_j, \rho_{V_j}^U(t), Y_j) \in \delta(V_j)\}_{j \in J}$ such that $X_j|_{V_i \cap V_j} = X_i|_{V_i \cap V_j}$ and $Y_j|_{V_i \cap V_j} = Y_i|_{V_i \cap V_j}$ for all $i, j \in J$. We recall that $\mathcal{S}$ is a sheaf provided by $\mathcal{S}(V) = \mathbb{N}^V$ for any region V , and the restriction maps are function restrictions. Thus, there exists X and Y in $\mathcal{S}(U)$ such that $X|_{V_j} = X_j$ and $Y|_{V_j} = Y_j$ for all $j \in J$.

We prove that $X \xrightarrow{t}_U Y$. We must first show that $X|_{U \setminus \psi(t)} = Y|_{U \setminus \psi(t)}$, and since $\{V_j \setminus \psi(t)\}_{j \in J}$ is a cover of $U \setminus \psi(t)$, it suffices to show that $X|_{V_j \setminus \psi(t)} = Y|_{V_j \setminus \psi(t)}$ for any $j \in J$. Thus, consider any $j \in J$. If $\psi(t) \cap V_j = \emptyset$, then $\rho_{V_j}^U(t) = \varepsilon$ by definition. This means that $X|_{V_j} = Y|_{V_j}$ since $(res_{V_j}^U, \rho_{V_j}^U)$ is a LTS morphism. But then, $V_j \setminus \psi(t) = V_j$, and we get $X|_{V_j \setminus \psi(t)} = Y|_{V_j \setminus \psi(t)}$. Also, if $\psi(t) \cap V_j \neq \emptyset$, then $\rho_{V_j}^U(t) = t$ by definition, and since $X_j \xrightarrow{t}_{V_j} Y_j$, we have that $X_j|_{V_j \setminus \psi(t)} = Y_j|_{V_j \setminus \psi(t)}$ by definition of the transition relation $\xrightarrow{t}_{V_j}$ in this case. But then, $X|_{V_j \setminus \psi(t)} = (X|_{V_j})|_{V_j \setminus \psi(t)} = X_j|_{V_j \setminus \psi(t)} = Y_j|_{V_j \setminus \psi(t)} = (Y|_{V_j})|_{V_j \setminus \psi(t)} = Y|_{V_j \setminus \psi(t)}$.

Now, consider any $x \in U \cap \psi(t)$. We have that $x \in V_j \cap \psi(t) \neq \emptyset$ for some $j \in J$ since $\{V_j\}_{j \in J}$ covers U . In particular, $\psi(t) \cap V_j \neq \emptyset$, so $\rho_{V_j}^U(t) = t$. We remark that

$X(x) = X|_{V_j}(x) = X_j(x)$ and $Y(x) = Y|_{V_j}(x) = Y_j(x)$.

- Suppose that $(x, t) \in F$. Then $X_j \xrightarrow{t}_{V_j} Y_j$ implies that $X_j(x) \geq W(x, t)$, and this is $X(x) \geq W(x, t)$.
- Suppose that $(x, t) \in F$ and $(t, x) \notin F$. Then $X_j \xrightarrow{t}_{V_j} Y_j$ implies that $Y_j(x) \geq X_j(x) - W(x, t)$, and this is $Y(x) \geq X - W(x, t)$.
- Suppose that $(x, t) \notin F$ and $(t, x) \in F$. Then $X_j \xrightarrow{t}_{V_j} Y_j$ implies that $Y_j(x) \geq X_j(x) + W(t, x)$, and this is $Y(x) \geq X + W(t, x)$.
- Suppose that $(x, t) \in F$ and $(t, x) \in F$. Then $X_j \xrightarrow{t}_{V_j} Y_j$ implies that $Y_j(x) \geq X_j(x) - W(x, t) + W(t, x)$, and this is $Y(x) \geq X(x) - W(x, t) + W(t, x)$.

The conditions for $X \xrightarrow{t}_U Y$ are thus satisfied. Also, if there is any other $X' \xrightarrow{t}_U Y'$ such that $X'|_{V_j} = X_j$ and $Y'|_{V_j} = Y_j$ for all $j \in J$, then $X'|_{V_j} = X|_{V_j}$ and $Y'|_{V_j} = Y|_{V_j}$ for all $j \in J$. By locality of $\mathcal{S}$, we get $X' = X$ and $Y' = Y$. This proves that t has δ -gluing over $\{V_j\}_{j \in J}$. $\square$

But Petri Net presheaves are not $\mathbb{T}$ -valued sheaves in general actually (as we will see in Proposition 4.2.12), which means that the property of being a $\mathbb{T}$ -valued sheaf is certainly stronger than to simply say that every action glues its transitions over any cover as in Proposition 4.2.7. Indeed, this is because there is a lot more gluing that is happening in a $\mathbb{T}$ -valued sheaf in general. We illustrate, through the following proposition, how this additional gluing may become problematic with the ε action when using Boolean algebra spaces.

Proposition 4.2.10 [Gluing of ε on complements]. *Let $\mathcal{T} = (\mathcal{S}, \mathcal{L}, \delta)$ be a $\mathbb{T}$ -valued sheaf on a Heyting algebra space $\mathcal{H}$. Consider any region U in $\mathcal{H}$ and $\{V_1, V_2\}$ a proper cover of U such that $V_1 \wedge V_2 = 0_{\mathcal{H}}$. Then for any action $c \in \mathcal{L}(V_1)$, there exists an action $b \in \mathcal{L}(U)$ such that $\rho_{V_1}^U(b) = c$ and $\rho_{V_2}^U(b) = \varepsilon$. In particular, we get that :*

$$\xrightarrow{b}_U = ((\text{res}_{V_1}^U)^{-1} \circ \xrightarrow{c}_{V_1} \circ \text{res}_{V_1}^U) \cap \simeq_{V_2}^U$$

Proof. Consider a $\mathbb{T}$ -valued sheaf $\mathcal{T} = (\mathcal{S}, \mathcal{L}, \delta)$ on a Heyting algebra space $\mathcal{H}$, a region U in $\mathcal{H}$, a proper cover $\{V_1, V_2\}$ of U such that $V_1 \wedge V_2 = 0_{\mathcal{H}}$, and an action $c \in \mathcal{L}(V_1)$. Consider the labelled transition system $(S, L, \delta) = (\emptyset, \{d\}, \emptyset)$. Define two labelling morphisms $\lambda_1 : \{d\} \rightarrow_{\varepsilon} \mathcal{L}(V_1)$ and $\lambda_2 : \{d\} \rightarrow_{\varepsilon} \mathcal{L}(V_2)$ with $\lambda_1(d) = c$ and $\lambda_2(d) = \varepsilon$.

By Proposition 4.2.4, $\mathcal{L}$ is a sheaf, and since $\{V_1, V_2\}$ forms a proper cover of U and $V_1 \wedge V_2 = 0_{\mathcal{H}}$ (so there is no overlap), we obtain a unique labelling morphism $\lambda : \{d\} \rightarrow_{\varepsilon} \mathcal{L}(U)$ such that $\rho_{V_1}^U(\lambda(d)) = \lambda_1(d) = c$ and $\rho_{V_2}^U(\lambda(d)) = \lambda_2(d) = \varepsilon$. Setting $b = \lambda(d)$, we get that b decomposes over $\{V_1, V_2\}$ by Corollary 4.2.8. This means, by definition, that :

$$\xrightarrow{b}_U = ((res_{V_1}^U)^{-1} \circ \xrightarrow{\rho_{V_1}^U(b)}_{V_1} \circ res_{V_1}^U) \cap ((res_{V_2}^U)^{-1} \circ \xrightarrow{\rho_{V_2}^U(b)}_{V_2} \circ res_{V_2}^U)$$

But, we have that :

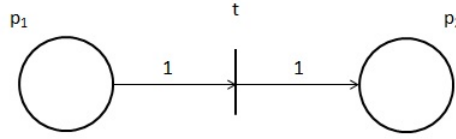
$$(res_{V_2}^U)^{-1} \circ \xrightarrow{\rho_{V_2}^U(b)}_{V_2} \circ res_{V_2}^U = (res_{V_2}^U)^{-1} \circ \xrightarrow{\varepsilon}_{V_2} \circ res_{V_2}^U = (res_{V_2}^U)^{-1} \circ res_{V_2}^U = \simeq_{V_2}^U$$

and $\rho_{V_1}^U(b) = c$, so we get the equation stated in the proposition. $\square$

What the above proposition says is that you can basically take any action in a subregion, and glue it with the idle action on a complement of that region (when such a complement exists). Consequently, we can apply such gluing with ε actions in profusion within $\mathbb{T}$ -valued sheaves on Boolean algebras, where complements of regions are systematically present. We can use the above proposition to demonstrate that the $\mathbb{T}$ -valued presheaves associated to Petri Nets as in Definition 4.1.11, are not sheaves in general, and we do so through the following counter-example :

Example 4.2.11 [Petri Net Counter-Example]. Define a Petri Net as follows :

1. Places : $P = \{p_1, p_2\}$
2. Transitions : $T = \{t\}$
3. Arcs : $F = \{(p_1, t), (t, p_2)\}$
4. Initial marking : $M_0 : P \rightarrow \mathbb{N}$ with $M_0(p_1) = M_0(p_2) = 0$
5. Weight : $W : F \rightarrow \mathbb{N}^+$ assigns $W(p_1, t) = W(t, p_2) = 1$

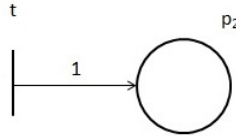


Proposition 4.2.12 . *The presheaf associated to the Petri Net from Example 4.2.11 is not a $\mathbb{T}$ -valued sheaf.*

Proof. Suppose it is a sheaf. We denote a global state $X : P \rightarrow \mathbb{N}$ with $X(p_1) = m$ and $X(p_2) = n$ as the pair (m, n) directly. At the global level, we get that $\xrightarrow{t}_P$ is the smallest transition relation such that for all $m, n \in \mathbb{N}$ with $m \geq 1$:

$$(m, n) \xrightarrow{t}_P (m - 1, n + 1)$$

The Petri Net in the region $\{p_2\}$ can be represented as :



and we get that $\xrightarrow{t}_{\{p_2\}}$ is the smallest transition relation such that for all $n \in \mathbb{N}$:

$$n \xrightarrow{t}_{\{p_2\}} n + 1$$

By Proposition 4.2.10, we can find an action b in $\mathcal{L}(P)$ such that

$$\xrightarrow{b}_P = ((res_{\{p_2\}}^U)^{-1} \circ \xrightarrow{t}_{\{p_2\}} \circ res_{\{p_2\}}^U) \cap \simeq_{\{p_1\}}^U$$

But this means that :

$$(m, n) \xrightarrow{b}_{\{p_1, p_2\}} (m', n') \Leftrightarrow m = m' \text{ and } n \xrightarrow{t}_{\{p_2\}} n' \Leftrightarrow m = m' \text{ and } n' = n + 1$$

for any $m, m', n, n' \in \mathbb{N}$. Clearly, this is not t 's effect in P as provided above (because m is not decremented here), i.e. there is no global action in this Petri Net that corresponds with b . Thus, this glued b action does not exist in $\mathcal{L}(P)$ and contradicts our choice of Petri Net. This means that this Petri Net's associated presheaf is not a sheaf. $\square$

Now, we briefly talked about finding the regions where actions are contained earlier on with the action $(send^1, \varepsilon)$ in the **Junction** example. We need to investigate this matter of action containment more thoroughly, and this is to be carried out in the following section.

4.3 Action Containment

The aim of this section is to provide an efficient mechanism by which the dependencies and effects of an action can be properly delineated within space. The final form of this mechanism is characterized by the axiom of *well-contained actions* (WCA) that we will impose on $\mathbb{T}$ -valued presheaves. To impose this axiom, we need a fair amount of action δ -gluing in a $\mathbb{T}$ -valued presheaf, one that involves the subregions where actions vanish. This process of well-containing actions is in fact essential to the proper rendering of spatially induced independence, and we shall get into the technical details of this elaboration here, until we arrive at the formulation of $\mathbb{T}$ -adapted presheaves at the end of this section.

We can get an intuitive understanding of how containment works for effects and dependencies individually as follows :

Imagine a class of students copying notes from the blackboard in their notebooks. The action $copy_A$ that a student A performs to copy the notes in question has its

effects contained in the notebook of student A (considered as a region). This is because $copy_A$ only creates changes in the state within A 's notebook, where a blank area of the notebook changes to a state where notes are written. On the other hand, the dependencies of $copy_A$ are contained in the blackboard (as a region) because the student only needs the information that is there in order to effectively copy the notes in question ($copy_A$ might also depend on A 's notebook, if A needs to find the next blank area where to write therein). In sum, $copy_A$'s effects and dependencies are contained in the join of the region of the blackboard and the region of A 's notebook.

Of course, we know that all students can copy the notes simultaneously, and the reason is that these copy actions are independent of each other in a spatial sense. Indeed, for two students A and B , $copy_A$ does not interfere with the blackboard nor with B 's notebook (i.e. it has no effects there), and the region “blackboard + B 's notebook” is precisely where $copy_B$ is contained as a whole. Vice-versa, $copy_B$ does not interfere with the the blackboard nor with A 's notebook, and the region “blackboard + A 's notebook” is precisely where $copy_A$ is contained as a whole. This addresses the requirement for SI-independence in between $copy_A$ and $copy_B$, in the sense of Definition 5.1.27 or Definition 5.3.13.

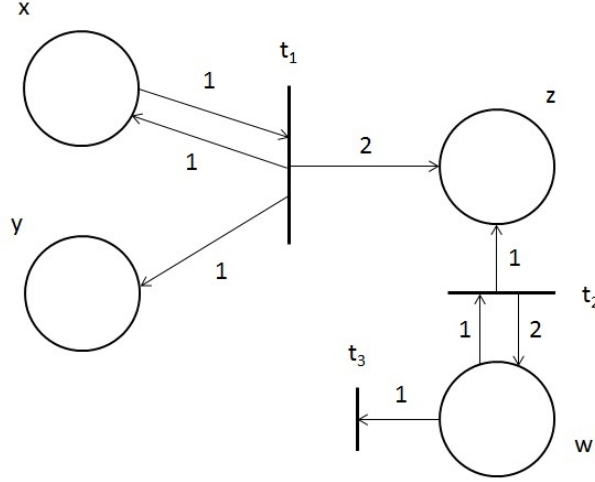
Now, we can begin formally addressing action containment. We first need to understand the different ways in which an action can interact with its projections on subregions, and we require a certain amount of terminology for that. In particular, we shall use the following definitions :

Definition 4.3.1 . Consider a $\mathbb{T}$ -valued presheaf on a Heyting algebra $\mathcal{H}$. Then, for any regions $V \leq U$ in $\mathcal{H}$ and any action $b \in \mathcal{L}(U)$:

- ε -region :** We say that V is an ε -region of b in U if $\rho_V^U(b) = \varepsilon$. We also say that b *vanishes in V* with respect to U in such a case.
- Transparent region :** We call V a *transparent region* of b in U if $\xrightarrow{b}_U \subseteq \simeq_V^U$. The action b is thought of as transparent in V in the sense that for any $X \xrightarrow{b}_U Y$, we have $X|_V = Y|_V$; i.e. the states in V are locally unmodified by the action of b in U . We also say in such circumstances that b has *no effects* in V .
- Effectful region :** We call V an *effectful region* of b in U if it is not a transparent region of b in U , i.e. if $\xrightarrow{b}_U \not\subseteq \simeq_V^U$. In such circumstances, we say that b *has an effect* in V , and this is because there exists a transition $X \xrightarrow{b}_U Y$ in U , such that $X|_V \neq Y|_V$. We refer to such transitions $X \xrightarrow{b}_U Y$ where $X|_V \neq Y|_V$ as *effects of b in V* (relative to U).
- Containment of effects :** We say that V *contains the effects of b* in U if there is a transparent region of b in U that is a j -complement of V in U . We shall also say in such circumstances that the *effects of b essentially occur* in V .

Remark 4.3.2 . For the above definitions, we shall omit the terms “with respect to U ” or “in U ” when U is clear in the context.

Example 4.3.3 . Consider the following Petri Net¹¹ :



We give examples for each type of region as provided in Definition 4.3.1, with respect to the actions t_1 and t_3 in $\mathcal{L}(P)$ where $P = \{x, y, z, w\}$ is the global region of the Petri Net presheaf.¹²

- The regions where t_1 vanishes with respect to P are : $\emptyset, \{w\}$
These are the regions to which t_1 doesn't connect with an arc. t_3 has a largest ε -region which is $\{w\}$.
- The transparent regions of t_1 in P are : $\emptyset, \{w\}, \{x\}, \{x, w\}$
We remark that these include ε -regions (see Proposition 4.3.4).
- The effectful regions of t_1 in P are given by $\mathcal{P}(P)$ minus the set of transparent regions. Here are a few examples :

$$\{y\}, \{z\}, \{y, z\}, \{y, x\}, \{z, x\}, \{y, z, x\}, \{y, w\}, \{y, z, x, w\} = P, \text{ etc.}$$

¹¹We consider this Petri Net regardless of a chosen marking (state) because we can analyze its associated presheaf regardless of such a marking.

¹²We recall that the definition of such presheaves was given earlier in Definition 4.1.11.

- The effects of t_1 essentially occur in (are contained in) the following regions :

$$\{y, z\}, \{y, z, x\}, \{y, z, w\}, \{y, z, x, w\}$$

It suffices here to look for regions that j -complement the largest transparent region $\{x, w\}$ of t_1 , and these are all the regions that contain $\{y, z\}$. Thus, if t_1 modifies the states in P when it fires, this is essentially because it modified the states in $\{y, z\}$.

- The regions where t_3 vanishes with respect to P are all the regions that don't contain the place w , because every other region connects to an arc of t_3 through w . t_3 has a largest ε -region which is $\{x, y, z\}$.
- The transparent regions of t_3 in P are the same as its ε -regions in P .
- The effectful regions of t_3 in P are given by $\mathcal{P}(P)$ minus the set of transparent regions, and these are the regions that contain the place w .
- The regions that contain the effects of t_3 are the same as the effectful regions of t_3 . Indeed, to contain the effects of t_3 , a region we must j -complement the largest transparent region of t_3 in P , which is $\{x, y, z\}$, and thus, they are precisely the regions that contain the place w .

The first thing that we should observe is that an action has no effects in a region where it vanishes, and we make that into a proposition.

Proposition 4.3.4 . *Consider any $\mathbb{T}$ -valued presheaf on a Heyting algebra $\mathcal{H}$. Then for any region $U \in \mathcal{H}$ and action $b \in \mathcal{L}(U)$, if b vanishes in V , then V is a transparent region of b in U ; that is, $\xrightarrow{b}_U \subseteq \simeq_V^U$.*

Proof. Suppose V is an ε -region of b in U . Then, $V \leq U$ and $\rho_V^U(b) = \varepsilon$. This means that for any $X \xrightarrow{b}_U Y$, we have $X|_V \xrightarrow{\varepsilon}_V Y|_V$, which means that $X|_V = Y|_V$ and thus $X \simeq_V^U Y$. This proves that $\xrightarrow{b}_U \subseteq \simeq_V^U$. $\square$

Thus, if W is a j -complement of an ε -region of b in U , then W is the j -complement of a transparent region, which also says the following :

Corollary 4.3.5 . *Consider any $\mathbb{T}$ -valued presheaf on a Heyting algebra $\mathcal{H}$, and any $U \in \mathcal{H}$ and $b \in \mathcal{L}(U)$. Then any j -complement of an ε -region of b contains the effects of b .*

The following proposition will also be useful, and it says that ε -regions behave properly with respect to labelling projections :

Proposition 4.3.6 [Stability of ε -regions under Projection]. *Given a $\mathbb{T}$ -valued presheaf, and any regions $V, K \leq U$ and $b \in \mathcal{L}(U)$, if b vanishes in K with respect to U , then $\rho_V^U(b)$ vanishes in $K \wedge V$ with respect to V .*

Proof. If K is an ε -region of b in U , then

$$\rho_{K \wedge V}^V(\rho_V^U(b)) = \rho_{K \wedge V}^U(b) = \rho_{K \wedge V}^K(\rho_K^U(b)) = \rho_{K \wedge V}^K(\varepsilon) = \varepsilon$$

□

Now, we have talked about containment of effects, but we can also talk about containment of dependencies. The definition for a region that contains the dependencies of an action is a bit more involved however, and it goes as follows :

Definition 4.3.7 [Containment of Dependencies]. Consider a $\mathbb{T}$ -valued presheaf, regions $V \leq U$, and an action $b \in \mathcal{L}(U)$. We say that b *depends only on* V (or V *contains the dependencies of* b), if there exists an ε -region K of b in U , and a j -complement W of K in U , and a relation $R : \mathcal{S}(V) \leftrightarrow \mathcal{S}(W)$ such that

$$R \circ res_V^U = res_W^U \circ \xrightarrow{b}_U$$

i.e. the following relational diagram commutes in *Rel* :

$$\begin{array}{ccc} \mathcal{S}(U) & \xrightarrow{\xrightarrow{b}_U} & \mathcal{S}(U) \\ \text{\scriptsize } res_V^U \downarrow & & \downarrow \text{\scriptsize } res_W^U \\ \mathcal{S}(V) & \xrightarrow{R} & \mathcal{S}(W) \end{array}$$

Remark 4.3.8 . When $b \in \mathcal{L}(U)$, we automatically have that U contains the dependencies of b .

And we can also formalize the idea that an action can be independent from a region as follows :

Definition 4.3.9 [Independence from a Region]. Consider any $\mathbb{T}$ -valued presheaf. Given regions $W \leq U$ and an action $b \in \mathcal{L}(U)$, we say that b is *independent of* W if there exists a region $V \leq U$ that contains the dependencies of b in U , and such that $V \wedge W = 0_{\mathcal{H}}$. If b is not independent of W , we say that it *depends on* W (this means that every region V that contains the dependencies of b must intersect non-trivially with W).

The requirement that W j -complements a region where b vanishes in U in Definition 4.3.7 means that W contains the effects of b (by Corollary 4.3.5), and the R expresses how the effects of b in W only depend on information residing within V . Personally, I like to conceive that the commutative diagram in Definition 4.3.7 expresses a form of “bisimulation” in between R and $\xrightarrow{b}_U$ (viewed as transition relations) through the restriction maps. What I mean by this informal notion of “bisimulation” is the following statement :

Proposition 4.3.10 . Consider any $\mathbb{T}$ -valued presheaf and suppose that the states presheaf $\mathcal{S}$ is a separated presheaf. Consider regions $V \leq U$ and an action $b \in \mathcal{L}(U)$. If there exists an ε -region K of b in U , and a region W that j -complements K in U , and a relation $R : \mathcal{S}(V) \rightarrow \mathcal{S}(W)$ such that $R \circ \text{res}_V^U = \text{res}_W^U \circ \xrightarrow{b}_U$ (so b depends only on V). Then the following are true :

1. For any states $X, Y \in \mathcal{S}(U)$ such that $X \stackrel{res_V^U}{\sim} X|_V$ and $X \xrightarrow{b}_U Y$, we have $X|_V \stackrel{R}{\sim} Y|_W$ and $Y \stackrel{res_W^U}{\sim} Y|_W$.

$$\begin{array}{ccc}
 X & \xrightarrow{b}_U & Y \\
 \downarrow res_V^U & & \downarrow res_W^U \\
 X|_V & \dashrightarrow^R & Y|_W
 \end{array}$$

2. For any states $X, Y' \in \mathcal{S}(U)$ such that $X|_V \stackrel{(res_V^U)^{-1}}{\sim} X$ and $X|_V \stackrel{R}{\sim} Y'$, there exists a unique $Y \in \mathcal{S}(U)$ such that $X \xrightarrow{b}_U Y$ and $Y' \stackrel{(res_W^U)^{-1}}{\sim} Y$. More precisely, this Y is the gluing of $Y' \in \mathcal{S}(W)$ and $X|_K \in \mathcal{S}(K)$.

$$\begin{array}{ccc}
 X & \dashrightarrow^b_U & Y \\
 \uparrow (res_V^U)^{-1} & & \uparrow (res_W^U)^{-1} \\
 X|_V & \xrightarrow{R} & Y'
 \end{array}$$

Proof. (1) Suppose that $X \stackrel{res_V^U}{\sim} X|_V$ and $X \xrightarrow{b}_U Y$ for some states $X, Y \in \mathcal{S}(U)$. Then $(X, Y|_W) \in (res_W^U \circ \xrightarrow{b}_U) = (R \circ res_V^U)$, and thus $X \stackrel{res_V^U}{\sim} X|_V \stackrel{R}{\sim} Y|_W$. Therefore, $X|_V \stackrel{R}{\sim} Y|_W$ and $Y \stackrel{res_W^U}{\sim} Y|_W$.

(2) Consider any $X|_V \stackrel{(res_V^U)^{-1}}{\sim} X$ and $X|_V \stackrel{R}{\sim} Y'$. Then $(X, Y') \in (R \circ res_{U,V}) = (res_W^U \circ \xrightarrow{b}_U)$, so there exists $Y \in \mathcal{S}(U)$ such that $X \xrightarrow{b}_U Y$ and $Y|_W = Y'$. Since K is an ε -region of b in U , we have $X|_K = Y|_K$. If any other Y_0 exists with the property that $X \xrightarrow{b}_U Y_0$ and $Y_0|_W = Y'$, then $X|_K = Y_0|_K$ also, and locality of $\mathcal{S}$ on the cover $\{W, K\}$ provides $Y = Y_0$. $\square$

Informally speaking, in the statement of Proposition 4.3.10, we can think of R and $\xrightarrow{b}_U$ as “simulating” each other’s behaviour through the restriction maps. Better yet,

they do so in a unique way at every step of the “simulation”. And since R specifies the transitions of b , and only requires information inside V (as declared in the proposition) to perform, it seems reasonable to think that b ’s performance also depends only on V . It would certainly help to consider some examples.

Example 4.3.11 [Dependencies for Buffer on Discrete Space]. Consider the Buffer on a Discrete Space from Example 4.1.7, and the actions $send, input \in \mathcal{L}(P)$ in the global region $P = \{x, y\}$. We recall that their transition relations are given as follows :

- $\xrightarrow{input}_P$ is the smallest transition relation on $\{0, 1\}^2$ such that :

$$(m, n) \xrightarrow{input}_P (m', n) \quad \text{for any } m, m', n \in \{0, 1\}$$

- $\xrightarrow{send}_P$ is the smallest transition relation on $\{0, 1\}^2$ such that :

$$(m, n) \xrightarrow{send}_P (m, m) \quad \text{for any } m, n \in \{0, 1\}$$

For the case of the *input* action, we have that $\{y\}$ is an ε -region of *input* in $P = \{x, y\}$, and $\{x\}$ is a j -complement of the latter. We show that *input* depends on no region (that is, *input* depends only on $\emptyset$). Consider the relation $R_{inp} : \mathcal{S}(\emptyset) \leftrightarrow \mathcal{S}(\{x\})$ with the singleton $\mathcal{S}(\emptyset) = \{*\}$ and $R_{inp} = \{(*, 0), (*, 1)\}$. It suffices to prove that $R_{inp} \circ res_{\emptyset}^P = res_{\{x\}}^P \circ \xrightarrow{input}_P$.

Thus, consider any $[(m, n), m'] \in (res_{\{x\}}^P \circ \xrightarrow{input}_P)$. Then we get :

$$(m, n) \xrightarrow{res_{\emptyset}^P} * \xrightarrow{R_{inp}} m'$$

Conversely, if $[(m, n), m'] \in (R_{inp} \circ res_{\emptyset}^P)$, we certainly have :

$$(m, n) \xrightarrow{input}_P (m', n) \xrightarrow{res_{\{x\}}^P} m'$$

Thus, *input* depends only on $\emptyset$.

We consider the case of *send* now, and show that it depends only on $\{x\}$. We have that *send* has no non-trivial ε -region (i.e. other than $\emptyset$), and thus we are forced to consider P for a j -complement of an ε -region of *send*. Consider the relation $R_s : \mathcal{S}(\{x\}) \rightarrow \mathcal{S}(P)$ with

$$R_s = \{ [m, (m, m)] \in \mathcal{S}(\{x\}) \times \mathcal{S}(P) \mid m \in \{0, 1\} \}$$

We get that if $[(m, n), (m', n')] \in (res_P^P \circ \xrightarrow{send}_P) = \xrightarrow{send}_P$, then $m = m' = n'$. We have $(m, n) \xrightarrow{res_{\{x\}}^P} m$ and $m \xrightarrow{R_s} (m, m) = (m', n')$. Thus, $[(m, n), (m', n')] \in (R_s \circ res_{\{x\}}^P)$. If $[(m, n), (m', n')] \in (R_s \circ res_{\{x\}}^P)$, then $(m, n) \xrightarrow{res_{\{x\}}^P} m \xrightarrow{R_s} (m', n')$. Thus, $m = m' = n'$, and we get $[(m, n), (m', n')] \in \xrightarrow{send}_P = (res_P^P \circ \xrightarrow{send}_P)$. This proves that $(res_P^P \circ \xrightarrow{send}_P) = (R_s \circ res_{\{x\}}^P)$, and it follows that *send* depends only on $\{x\}$.

Example 4.3.12 [Dependencies for Petri Nets]. Consider any Petri Net (P, T, F, M_0, W) , and recall the map $\psi : T \rightarrow P$ that sends a transition $t \in T$ to $\psi(t) = \{x \mid (x, t) \in F \text{ or } (t, x) \in F\}$ from the Petri Net Presheaf definition (Definition 4.1.11). I claim that t depends only on $\psi(t)$. In fact, it suffices to consider $R = \xrightarrow{t}_{\psi(t)} : \mathcal{S}(\psi(t)) \rightarrow \mathcal{S}(\psi(t))$ and we can prove that $R \circ res_{\psi(t)}^P = res_{\psi(t)}^P \circ \xrightarrow{t}_P$. This will be proven automatically when we make the localized relational structure representation of Petri Nets later in Section 6.1. Thus, we omit the proof until we arrive at this section.

Now, whilst j -complements of ε -regions contain effects as in Corollary 4.3.5, they do not contain dependencies of actions in $\mathbb{T}$ -valued presheaves in general, and this is unfortunate because an efficient rendering of SI-independence would rely on that. The following modified buffer on a discrete space gives such an example where a *send* action has a j -complement of an ε -region that does not contain the dependencies of *send*.

Example 4.3.13 [Modified Buffer on a Discrete Space (with ε -reading)].

We take the exact same $\mathbb{T}$ -valued presheaf as specified for the Buffer on a Discrete Space of Example 4.1.7 (the global region is given by $P = \{x, y\}$), but we use

$$\rho_{\{x\}}^P(send) = \varepsilon \quad \text{and} \quad \mathcal{L}(\{x\}) = \{input\}$$

instead of $\rho_{\{x\}}^P(send) = send$ and $\mathcal{L}(\{x\}) = \{input, send\}$. Thus, the only thing that is modified is that the global $send$ projects to ε in the region $\{x\}$, and $send$ does not exist anymore in the labelling set local to $\{x\}$. The rest of the specifications are the same. And we remind the reader that $\xrightarrow{send}_P$ is the smallest transition relation such that for all $m, n \in \{0, 1\}$:

$$(m, n) \xrightarrow{send}_P (m, m)$$

where the first component represents the value in the region $\{x\}$ (where $send$ takes its input) and the second component represents the value in the region $\{y\}$ (the region where a value is sent). Also, $\xrightarrow{send}_{\{y\}} = \{0, 1\}^2$, and this is all that we need to establish Proposition 4.3.14 that follows.

We see here that $\{y\}$ complements the ε -region $\{x\}$ of the global $send$ here, but $\{y\}$ clearly does not contain the dependencies of $send$ because the latter requires numerical information present in $\{x\}$ if it is going to be able to transfer it to $\{y\}$. We prove this formally to get better acquainted with containment of dependencies.

Proposition 4.3.14 . *In the above example, there exists an action, $send$, and a j -complement $\{y\}$ of an ε -region $\{x\}$ of $send$, in which the dependencies of $send$ are not contained. In particular, it does not hold that j -complements of ε -regions contain the dependencies of actions for $\mathbb{T}$ -valued presheaves in general.*

Proof. There are two possible regions that j -complement the ε -region $\{x\}$ of $send$ in P , and these are P and $\{y\}$.

It suffices to investigate the case of $\{y\}$, since if we can find a relation $R' : \mathcal{S}(\{y\}) \leftrightarrow \mathcal{S}(P)$ such that $R' \circ res_{\{y\}}^P = res_P^P \circ \xrightarrow{send}_U$ (as in Definition 4.3.7), then we will get a relation $R := res_{\{y\}}^P \circ R' : \mathcal{S}(\{y\}) \leftrightarrow \mathcal{S}(\{y\})$ with $R \circ res_{\{y\}}^P =$

$res_{\{y\}}^P \circ \xrightarrow{send}_U$. Thus, it suffices to show that there is no relation $R : \mathcal{S}(\{y\}) \leftrightarrow \mathcal{S}(\{y\})$ for which $R \circ res_{\{y\}}^P = res_{\{y\}}^P \circ \xrightarrow{send}_U$ holds to show that $\{y\}$ does not contain the dependencies of $send$.

Thus, suppose there exists a relation $R : \mathcal{S}(\{y\}) \leftrightarrow \mathcal{S}(\{y\})$ such that $R \circ res_{\{y\}}^P = res_{\{y\}}^P \circ \xrightarrow{send}_U$. Then we have $(m, n) \xrightarrow{send}_U (m, m) \xrightarrow{res_{\{y\}}^P} m$ for any $m, n \in \{0, 1\}$, and so $[(m, n), m] \in (R \circ res_{\{y\}}^P)$, and this yields $n \stackrel{R}{\sim} m$ for any $m, n \in \{0, 1\}$. Thus, $R = \{0, 1\}^2$.

Now, consider the global state $(1, 1)$. We have $1 \stackrel{R}{\sim} 0$, so $(1, 1) \xrightarrow{res_{\{y\}}^P} 1 \stackrel{R}{\sim} 0$. This means $[(1, 1), 0] \in (R \circ res_{\{y\}}^P)$, and we get $[(1, 1), 0] \in (res_{\{y\}}^P \circ \xrightarrow{send}_U)$. But we know that $\xrightarrow{send}_U$ is a functional relation and the only transition it provides out of the $(1, 1)$ state is $(1, 1) \xrightarrow{send}_U (1, 1)$. But then, $[(1, 1), 0] \in (res_{\{y\}}^P \circ \xrightarrow{send}_U)$ means that $(1, 1) \xrightarrow{send}_U (1, 1) \xrightarrow{res_{\{y\}}^P} 0$. This is a contradiction because $res_{\{y\}}^P(1, 1) = 1 \neq 0$.

Thus, it is not true that $send$ depends only on $\{y\}$. □

Basically, we proved that since R ignores the value in $\{x\}$, and since R has to provide any possible change of value in $\{y\}$, then it must perform in a completely non-deterministic fashion. Consequently, such an R cannot subsume the transition relation $\xrightarrow{send}_U$ properly.

Our intention now is to force j -complements of ε -regions to contain dependencies, but we must make a choice of a relation “ R ” (as in the definition of containment of dependencies) that works in general for any region. We simply use the transition relation for projected actions $\xrightarrow{\rho_V^U(b)}_V$ as follows :

Definition 4.3.15 [Containment of Action]. Consider any $\mathbb{T}$ -valued presheaf, and any regions $V \leq U$ and action $b \in \mathcal{L}(U)$. We say that b is *contained in* V (or that V *contains* b) if there exists an ε -region K of b that j -complements V such that :

$$\xrightarrow{\rho_V^U(b)} \circ res_V^U = res_V^U \circ \xrightarrow{b}$$

i.e. the following diagram commutes in Rel (for some region K) :

$$\begin{array}{ccc} \mathcal{S}(U) & \xrightarrow{\quad b \quad} & \mathcal{S}(U) \\ res_V^U \downarrow & & \downarrow res_W^U \\ \mathcal{S}(V) & \xrightarrow{\quad \rho_V^U(b) \quad} & \mathcal{S}(W) \end{array}$$

We say furthermore that V is the *proper region* of b in U if $V \wedge K = 0_{\mathcal{H}}$ ¹³

Remark 4.3.16 . The statement above only really adds inclusion from left to right $\xrightarrow{\rho_V^U(b)} \circ res_V^U \subseteq res_V^U \circ \xrightarrow{b}$ for $\mathbb{T}$ -valued presheaves in general because the right to left inclusion given by $\xrightarrow{\rho_V^U(b)} \circ res_V^U \supseteq res_V^U \circ \xrightarrow{b}$ already holds in such presheaves. Indeed, for any pair of states $(X, Y') \in (res_V^U \circ \xrightarrow{b})$, we get $X \xrightarrow{b} Y \xrightarrow{res_V^U} Y|_V = Y'$ for some $Y \in \mathcal{S}(U)$, and then $X|_V \xrightarrow{\rho_V^U(b)} Y|_V$ from the restriction on transitions, and thus $X \xrightarrow{res_V^U} X|_V \xrightarrow{\rho_V^U(b)} Y|_V = Y'$. So, $(X, Y') \in (\xrightarrow{\rho_V^U(b)} \circ res_V^U)$.

Thus, for such a V as in the above definition, we get that V simultaneously contains the effects of b (because it is a j -complement of an ε -region of b) and the dependencies of b (using $R = \xrightarrow{\rho_V^U(b)}$ as in Definition 4.3.7). This is why we adopted the term “ V contains b ” here, because V truly contains everything about the action b . It turns out that this notion of containment for actions is far more natural than it seems à priori; it is equivalent to action δ -gluing under the condition of the states presheaf $\mathcal{S}$ as a sheaf, as follows :

¹³We say “the” proper region because it would be unique as the complement of K in U . Furthermore, in such a case, we have that b is independent of K in the sense of Definition 4.3.9.

Proposition 4.3.17 [Equivalences for Action Containment]. *Consider any $\mathbb{T}$ -valued presheaf and suppose $\mathcal{S}$ is a sheaf. Consider any regions $V \leq U$ and action $b \in \mathcal{L}(U)$, and any ε -region K of b that j -complements V in U . Then the following statements are equivalent :*

1. $\frac{\rho_V^U(b)}{\rightarrow_V} \circ \text{res}_V^U = \text{res}_V^U \circ \frac{b}{\rightarrow_U}$ (i.e. V contains b)
2. For any $X' \xrightarrow{\rho_V^U(b)}_V Y'$ and $X'' \in \mathcal{S}(K)$ such that $X'|_{V \wedge K} = X''|_{V \wedge K}$, there exists unique $X, Y \in \mathcal{S}(U)$ such that $X \xrightarrow{b}_{\rightarrow_U} Y$ and $X|_V = X'$, $Y|_K = X|_K = X''$ and $Y|_V = Y'$.
3. The action b glues its transitions over $\{V, K\}$ (see Definition 4.1.17)
4. The action b decomposes over $\{V, K\}$ (see Definition 4.1.16)
5. $\frac{b}{\rightarrow_U} = ((\text{res}_V^U)^{-1} \circ \frac{\rho_V^U(b)}{\rightarrow_V} \circ \text{res}_V^U) \cap \simeq_K^U$

Proof. Proof to be found in Appendix E. □

Remark 4.3.18 . We may notice in the above proposition that the statement of containment of an action (in (1)) makes use of an equation that abstracts the ε -region “ K ”, as opposed to the other statements. Through this observation, we get that the existence of a single $\{V, K\}$ with one of the properties above suffices in establishing the property for any other proper cover $\{V, K'\}$, where the K is swapped by any other ε -region K' . For example, if $b \in \mathcal{L}(U)$ has δ -gluing over $\{V, K\}$ as in the above proposition, then for any ε -region K' such that $V \vee K' = U$, we have that b has δ -gluing over $\{V, K'\}$ as well.

Now that we have established properly the notion of containment of an action, we can state the axiom that will yield SI-independence if we enforce it on $\mathbb{T}$ -valued presheaves. This axiom is the following :

Definition 4.3.19 [WCA axiom]. We say that a $\mathbb{T}$ -valued presheaf has *well-contained actions* (WCA) if for any regions $V \leq U$ and any action $b \in \mathcal{L}(U)$:

If V is a j -complement of an ε -region of b in U , then V contains b .

The Modified Buffer as in Example 4.3.13 certainly does not enforce this as we saw. But the original Buffer on a Discrete Space from Example 4.1.7 has WCA (trivially in fact). Basically, the WCA axiom tries to make actions as independent as possible from the regions where they vanish. We say “as independent as possible” because in certain systems on Heyting algebra spaces such as the Pullback Junction of Buffers in Example 4.2.5, we cannot make the promise of finding a complement of just any ε -region. Indeed, within this example, we have a $send^1$ action in $\mathbf{Buff}_1$ (with region V_1) that actually depends on the input value to be found in the input region $(V_1 \wedge V_2)$ where it projects to ε in the intersection of the buffers (see Figure 24).

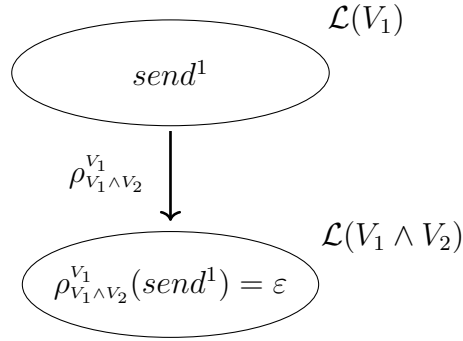


Figure 24: $send^1$ depends on $V_1 \wedge V_2$, where it projects to ε .

So far, we could say that the WCA axiom and the property of $\mathcal{S}$ as a sheaf seem like ideal conditions to work with in the context of $\mathbb{T}$ -valued presheaves. There happens to be a third condition that would be quite beneficial to our analysis. That condition is to allow the labelling presheaf $\mathcal{L}$ to be a separated presheaf. In particular, when $\mathcal{L}$ is a separated presheaf, we get largest ε -regions, and the presence of such regions will simplify the formulation of the WCA axiom as we will see. The following proposition initiates this idea that there exists a largest ε -region for any given action in such circumstances.

Proposition 4.3.20 . *Suppose we have a $\mathbb{T}$ -valued presheaf, where $\mathcal{L}$ is a separated presheaf, and consider any region U and $b \in \mathcal{L}(U)$. Then the property of being an ε -region of b in U is closed under arbitrary joins.*

Proof. Take any family $\{V_j\}_{j \in J}$ of ε -regions of b in U . Then $V_j \leq U$ for all j in J , so $\bigvee_{j \in J} V_j \leq U$. Write $W := \bigvee_{j \in J} V_j$.

With $W \leq U$, we get a well-defined projection $\rho_W^U(b) \in \mathcal{L}(U)$, and since $\rho_{V_j}^W(\rho_W^U(b)) = \rho_{V_j}^U(b) = \varepsilon$ for each j in J , we have a cover $\{V_j\}_{j \in J}$ of W in which $\rho_W^U(b) \in \mathcal{L}(W)$ projects to ε on each part. Yet, $\rho_{V_j}^W(\varepsilon) = \varepsilon$ for each j as well. Thus, by locality of $\mathcal{L}$, we get $\rho_W^U(b) = \varepsilon$ ¹⁴. This proves that W is an ε region of b in U . $\square$

Definition 4.3.21 . Given a $\mathbb{T}$ -valued presheaf on a Heyting algebra $\mathcal{H}$, where $\mathcal{L}$ is a separated presheaf, any region U and an action $b \in \mathcal{L}(U)$, we define the largest ε -region of b in U as :

$$\varepsilon_U(b) := \bigvee \{ V \in \mathcal{H} \mid V \leq U \text{ and } \rho_V^U(b) = \varepsilon \}$$

And from Proposition 4.3.20, it follows that :

Corollary 4.3.22 . *Given a $\mathbb{T}$ -valued presheaf where $\mathcal{L}$ is a separated presheaf, any region U and any action $b \in \mathcal{L}(U)$, we have that $\varepsilon_U(b)$ is indeed an ε -region of b in U . Also, for every region $V \leq U$:*

$$V \text{ is an } \varepsilon\text{-region of } b \text{ in } U \Leftrightarrow V \leq \varepsilon_U(b)$$

Corollary 4.3.23 . *Given a $\mathbb{T}$ -valued presheaf where $\mathcal{L}$ is a separated presheaf, we have that for any regions $V \leq U$ and action $b \in \mathcal{L}(U)$:*

$$V \text{ is a } j\text{-complement of an } \varepsilon\text{-region of } b \text{ in } U \Leftrightarrow V \text{ is a } j\text{-complement of } \varepsilon_U(b)$$

By applying the above corollary, we can reformulate the WCA axiom in such a context where $\mathcal{L}$ is a separated presheaf, as follows :

¹⁴This is mainly because the arrow $\mathcal{L}(W) \rightarrow \prod_{j \in J} \mathcal{L}(V_j)$ that arises from the obvious restrictions and the universal property on products, is a monomorphism by definition of $\mathcal{L}$ as a separated presheaf. From Proposition 1.2.3, we know that this means it is an injective map, and this is why $\rho_W^U(b)$ and ε in $\mathcal{L}(W)$ get matched.

Definition 4.3.24 [WCA axiom (with separated presheaf $\mathcal{L}$)]. Given a $\mathbb{T}$ -valued presheaf $\mathcal{T}$ whose labelling presheaf $\mathcal{L}$ is a separated presheaf, we say that $\mathcal{T}$ has *well-contained actions* (WCA) if for any regions $V \leq U$ and $b \in \mathcal{L}(U)$:

$$V \text{ is a } j\text{-complement of } \varepsilon_U(b) \text{ in } U \Rightarrow V \text{ contains } b$$

Combining Corollary 4.3.22 with Proposition 4.3.4, we get :

Corollary 4.3.25 . *Given a $\mathbb{T}$ -valued presheaf where $\mathcal{L}$ is a separated presheaf, a region U and an action $b \in \mathcal{L}(U)$, we have :*

$$\xrightarrow{b}_U \subseteq \simeq_{\varepsilon_U(b)}^U$$

Also, the largest ε -regions project properly on subregions.

Proposition 4.3.26 [ε_U - Stable under Projection]. *Given a $\mathbb{T}$ -valued presheaf where $\mathcal{L}$ is a separated presheaf, any region U and an action $b \in \mathcal{L}(U)$, we have $\varepsilon_W(\rho_W^U(b)) = \varepsilon_U(b) \wedge W$ for every region $W \leq U$.*

Proof. We have that $\varepsilon_U(b) \wedge W$ is an ε -region of $\rho_W^U(b)$ by Proposition 4.3.6, thus $\varepsilon_U(b) \wedge W \leq \varepsilon_W(\rho_W^U(b))$ by Corollary 4.3.22.

Setting $K = \varepsilon_W(\rho_W^U(b))$, we also have $K \leq W$ by definition, and

$$\rho_K^U(b) = \rho_K^W(\rho_W^U(b)) = \varepsilon$$

since $\varepsilon_W(\rho_W^U(b))$ is an ε -region of $\rho_W^U(b)$ in W . This means $\varepsilon_W(\rho_W^U(b))$ is an ε -region of b , and so $\varepsilon_W(\rho_W^U(b)) \leq \varepsilon_U(b)$. But then $\varepsilon_W(\rho_W^U(b)) \leq \varepsilon_U(b) \wedge W$, and we finally get $\varepsilon_W(\rho_W^U(b)) = \varepsilon_U(b) \wedge W$. $\square$

And we conclude this section by providing a simpler form to the equivalences stated in Proposition 4.3.17, in the context of $\mathcal{L}$ as a separated presheaf.

Proposition 4.3.27 . *Consider any $\mathbb{T}$ -valued presheaf where $\mathcal{L}$ is a separated presheaf and $\mathcal{S}$ is a sheaf. Then for any regions $V \leq U$ and any action $b \in \mathcal{L}(U)$, the following statements are equivalent :*

1. V contains b
2. b glues its transitions over $\{V, \varepsilon_U(b)\}$
3. b decomposes over $\{V, \varepsilon_U(b)\}$
4. $\xrightarrow{b}_U = ((res_V^U)^{-1} \circ \xrightarrow{\rho_V^U(b)}_V \circ res_V^U) \cap \simeq_{\varepsilon_U(b)}^U$

Proof. These equivalences follow directly by applying Proposition 4.3.17 using $\varepsilon_U(b)$ as the ε -region “K” stated in the latter. $\square$

In sum, we are now able to formulate the conditions under which we would like to work as far as the expression of spatially induced independence goes. These conditions are : $\mathcal{S}$ is a sheaf, $\mathcal{L}$ is a separated presheaf, and the WCA axiom. We embody the latter in the definition of $\mathbb{T}$ -adapted presheaves as follows :

Definition 4.3.28 [**$\mathbb{T}$ -adapted presheaf**]. A $\mathbb{T}$ -adapted presheaf on a complete Heyting algebra $\mathcal{H}$ is a $\mathbb{T}$ -valued presheaf $\mathcal{T} = (\mathcal{S}, \mathcal{L}, \delta)$ on $\mathcal{H}$, where

1. **(Sheaf of states)** : $\mathcal{S}$ is a set-valued sheaf
2. **(Separated presheaf of labels)** : $\mathcal{L}$ is a $\mathbf{Set}_\varepsilon$ -valued separated presheaf
3. **(Well-Contained Actions (WCA))** : For any regions $V \leq U$ and any action $b \in \mathcal{L}(U)$:

$$V \text{ is a } j\text{-complement of } \varepsilon_U(b) \text{ in } U \Rightarrow V \text{ contains } b$$

And this concludes our discussion on action containment.

Chapter 5

Spatially Induced Independence for $\mathbb{T}$ -Adapted Presheaves

In the previous chapter, we identified a δ -gluing condition (which relates to action containment) on actions with the hope that it would allow spatially induced independence (SI-independence) to manifest within presheaves of LTS. This condition was formalized in the Well-Contained Actions (WCA) axiom (see Definition 4.3.24), and this led us to the formulation of $\mathbb{T}$ -adapted presheaves¹.

We will now show that for each LTS decorating the regions of such an adapted presheaf, there is a spatially induced independence relation on labels (denoted $\mathcal{I}$) and a spatially induced equivalence on transitions that transform these LTS into ALTSE. More specifically, for any given complete Heyting algebra $\mathcal{H}$, we will construct an SI-independence functor embedding the category of $\mathbb{T}$ -adapted presheaves on $\mathcal{H}$ into the category of $\mathbb{A}^\sim$ -valued presheaves on $\mathcal{H}$ ². The construction of these SI-independence functors constitutes the main result of this thesis, which is the Theorem of Spatially Induced Independence (Theorem 5.2.2). Of course, what is important about these SI-independence functors is that they apply the principle of SI-independence to derive

¹ $\mathbb{T}$ is the category of labelled transition systems from Chapter 2.

²We recall that $\mathbb{A}^\sim$ is the category of asynchronous labelled transition systems with equivalence (ALTSE) from Section 3.3.

their independence relations. We remind the reader as to what this principle is and, to achieve this, we set up some binary relations first.

In a $\mathbb{T}$ -valued presheaf, given a region U , we define a binary relation $\mathcal{I}(U)$ ³ on $\mathcal{L}(U)$ such that for any two actions $b, c \in \mathcal{L}(U)$, we have $b \mathcal{I}(U) c$ if and only if there is a proper cover $\{V, W\}$ of U such that :

1. b vanishes in W and c vanishes in V , and
2. b is contained in V and c is contained in W .

And with respect to these relations, we can formulate the Principle of SI-independence.

Principle of SI-independence : Given a $\mathbb{T}$ -valued presheaf $\mathcal{T}$, we say that $\mathcal{T} = (\mathcal{S}, \mathcal{L}, \delta)$ has *spatially induced independence* if for any region U , we have that the relation $\mathcal{I}(U)$, as defined above, provides an independence relation on labels such that $\mathcal{T}(U) = (\mathcal{S}(U), \mathcal{L}(U), \delta(U), \mathcal{I}(U))$ is an ALTS. In which case, for $b \mathcal{I}(U) c$, we say that b is *spatially independent* of c with respect to U .

In the case of $\mathbb{T}$ -adapted presheaves, condition (1) above implies condition (2) (see Proposition 5.1.28), and this makes it much easier to systematically find patterns of SI-independence in between actions. Hence, the SI-independence relations that we will provide will simply impose condition (1). In such a case where (1) holds for some actions $b, c \in \mathcal{L}(U)$ and some proper cover $\{V, W\}$ of U within a $\mathbb{T}$ -adapted presheaf, we will get commutativity of the transition relations associated to these actions with $\xrightarrow{b}_U \circ \xrightarrow{c}_U = \xrightarrow{c}_U \circ \xrightarrow{b}_U$ (Alternative Paths) and $\xrightarrow{b}_U \circ {}_U\overleftarrow{c} = {}_U\overleftarrow{c} \circ \xrightarrow{b}_U$ ⁴ (One-step Amalgamation and One-step co-Amalgamation). We will in fact obtain much more because there are constructive and unique ways of obtaining equivalent runs when permuting SI-independent actions in these systems.

³There are variants for the definition of $\mathcal{I}(U)$, but the one given here constitutes a minimal case.

⁴We recall that ${}_U\overleftarrow{c} := \xrightarrow{c}_U^{-1}$.

The first section (5.1) of this chapter will develop some of the theory of $\mathbb{T}$ -adapted presheaves on Heyting algebras (we recall that we use the term “Heyting algebra” to mean “complete Heyting algebra”, and similarly, “Boolean algebra” means “complete Boolean algebra”), and will emphasize the theory that will allow us to prove the Theorem of Spatially Induced Independence. We reserve Section 5.2 to tackle the proof of the theorem in question. In the section that follows (i.e. Section 5.3), we demonstrate how the statements and results proved in Section 5.1 achieve a simpler format in the case where the base space of the $\mathbb{T}$ -adapted presheaf is a Boolean algebra. In particular, spatially induced independence (with $\mathcal{I}$) in between actions b and c acquires a very natural interpretation as spatial disjointness of the containing regions of b and c . The trade-off is that $\mathcal{I}$ loses some of its accuracy in the context of Boolean algebras, and we will provide an extended form of SI-independence, $\mathcal{I}^+$ (in Definition 5.3.13), that seems favorable in such Boolean algebra contexts (although both $\mathcal{I}$ and $\mathcal{I}^+$ only require $\mathbb{T}$ -adapted presheaves on Heyting algebras in order to be defined). However, this $\mathcal{I}^+$ does not yield SI-independence functors (as $\mathcal{I}$ does) from $\mathbb{T}$ -adapted presheaves to $A^\sim$ -valued presheaves.

5.1 $\mathbb{T}$ -adapted Presheaves on Heyting algebras

At the end of Section 4.3 in the previous chapter, we proposed an adapted form of $\mathbb{T}$ -valued presheaf, where a minimal amount of axioms were added, and this led us to define $\mathbb{T}$ -adapted presheaves. These adapted presheaves have just enough axioms so that the theorem characterizing SI-independence can be proved with respect to these structures. Hence, the object of this section is to understand some of the basic theory surrounding $\mathbb{T}$ -adapted presheaves on Heyting algebras, and to develop the lemmas that will serve us in proving the Theorem of Spatially Induced Independence in the next section.

Let us begin by recalling the definition of $\mathbb{T}$ -adapted presheaves as follows :

Definition 5.1.1 [**$\mathbb{T}$ -Adapted Presheaf**]. A $\mathbb{T}$ -adapted presheaf on a complete Heyting algebra $\mathcal{H}$ is a $\mathbb{T}$ -valued presheaf $\mathcal{T} = (\mathcal{S}, \mathcal{L}, \delta)$ on $\mathcal{H}$, where

1. **(Sheaf of states)** : $\mathcal{S}$ is a **Set**-valued sheaf.
2. **(Separated presheaf of labels)** : $\mathcal{L}$ is a $\mathbf{Set}_\varepsilon$ -valued separated presheaf.
3. **(Well-Contained Actions (WCA))** :

For any regions $V \leq U$ in $\mathcal{H}$ and any action $b \in \mathcal{L}(U)$:

If V is a j -complement of $\varepsilon_U(b)$ in U , then V contains b .⁵

We write $[\mathcal{H}^{op}, \mathbb{T}]_{adapt}$ for the category of $\mathbb{T}$ -adapted presheaves on $\mathcal{H}$, which is the full subcategory of $[\mathcal{H}^{op}, \mathbb{T}]$ where the objects are $\mathbb{T}$ -adapted presheaves.

As we emphasize the theory that allows us to derive SI-independence for $\mathbb{T}$ -adapted presheaves, we will have to neglect the study of the categories associated to such structures, i.e of $[\mathcal{H}^{op}, \mathbb{T}]_{adapt}$ for a Heyting algebra $\mathcal{H}$. However, we suspect that these are bicomplete, and we thus make the following conjecture :

Conjecture 5.1.2 . *For a given complete Heyting algebra $\mathcal{H}$, the category $[\mathcal{H}^{op}, \mathbb{T}]_{adapt}$ is bicomplete.*

Now, we remark that for some regions $V \leq U$ and action $b \in \mathcal{L}(U)$ in a $\mathbb{T}$ -adapted presheaf, if V contains b , then there is an ε -region of b that j -complements V in U (by definition), which means V is a j -complement of $\varepsilon_U(b)$ in U by Corollary 4.3.23. So, the following holds :

Proposition 5.1.3 . *For a $\mathbb{T}$ -adapted presheaf on a Heyting algebra, we have that for any regions $V \leq U$ and $b \in \mathcal{L}(U)$:*

$$V \text{ is a } j\text{-complement of } \varepsilon_U(b) \text{ in } U \Leftrightarrow V \text{ contains } b$$

⁵We recall that the expression “ V contains b (with respect to U)” here is formalized in Definition 4.3.15.

The equivalence provided in Proposition 5.1.3 facilitates the whole study of action containment when using $\mathbb{T}$ -adapted presheaves. Indeed, in such adapted presheaves, the statement “ $V \vee \varepsilon_U(b) = U$ ” becomes synonymous with “ V contains b (with respect to U)”, and the former statement is much easier to manipulate.

We establish now that action containment behaves properly in the cases where finite meets and projections are involved within $\mathbb{T}$ -adapted presheaves.

Proposition 5.1.4 [Action Containment - Closure under Finite Meets].

Consider any $\mathbb{T}$ -adapted presheaf on a Heyting algebra, and regions $V, V' \leq U$ and an action $b \in \mathcal{L}(U)$. If V and V' both contain b with respect to U , then $V \wedge V'$ contains b with respect to U .

Proof. Consider any such V and V' containing b as above. We get that $V \vee \varepsilon_U(b) = U$ and $V' \vee \varepsilon_U(b) = U$ (by Proposition 5.1.3), so

$$(V \wedge V') \vee \varepsilon_U(b) = (V \vee \varepsilon_U(b)) \wedge (V' \vee \varepsilon_U(b)) = U \wedge U = U$$

By the WCA axiom, we get that $V \wedge V'$ contains b . □

Proposition 5.1.5 [Stability of Action Containment under Projection].

Consider any $\mathbb{T}$ -adapted presheaf on a Heyting algebra, and regions $V, W \leq U$ and an action $b \in \mathcal{L}(U)$. If V contains b with respect to U , then $V \wedge W$ contains $\rho_W^U(b)$ with respect to W .

Proof. Suppose V contains b with respect to U and $W \leq U$. Then $V \vee \varepsilon_U(b) = U$, and so $(V \wedge W) \vee (\varepsilon_U(b) \wedge W) = W$. By Proposition 4.3.26, $\varepsilon_W(\rho_W^U(b)) = W \wedge \varepsilon_U(b)$, which means that $V \wedge W$ j -complements $\varepsilon_W(\rho_W^U(b))$, and thus $V \wedge W$ contains $\rho_W^U(b)$ in W by the WCA axiom. □

And there are also a few facts about transparent regions we will need later on in Section 5.3 :

Proposition 5.1.6 [Transparent Regions - Closure under arbitrary Meets].

Consider any $\mathbb{T}$ -valued presheaf where the states presheaf $\mathcal{S}$ is a sheaf⁶, and any region U and any action $b \in \mathcal{L}(U)$. Then the property of being a transparent region of b in U is closed under arbitrary joins.

Proof. Take any family $\{V_j\}_{j \in J}$ of transparent regions of b in U . Then $V_j \leq U$ for all j in J , so $\bigvee_{j \in J} V_j \leq U$. Write $W := \bigvee_{j \in J} V_j$.

Consider any transition $X \xrightarrow{b}_U Y$. We have $X|_{V_j} = Y|_{V_j}$ for all $j \in J$ since the V_j are transparent regions of b in U . Thus, $X|_W$ and $Y|_W$ are states that agree on the cover $\{V_j\}_{j \in J}$ of W since $(X|_W)|_{V_j} = X|_{V_j} = Y|_{V_j} = (Y|_W)|_{V_j}$. By the locality of $\mathcal{S}$, we get $X|_W = Y|_W$. This means b is transparent in W . $\square$

This means that we have largest transparent regions for actions :

Definition 5.1.7 . Given a $\mathbb{T}$ -adapted presheaf on a Heyting algebra $\mathcal{H}$, and any region U and $b \in \mathcal{L}(U)$, we define the largest transparent region of b in U as

$$\tau_U(b) := \bigvee \{ V \in \mathcal{H} \mid V \text{ is a transparent region of } b \text{ in } U \}$$

And from Proposition 5.1.6, it follows that :

Corollary 5.1.8 . For any $\mathbb{T}$ -adapted presheaf, region U , and action $b \in \mathcal{L}(U)$, $\tau_U(b)$ is indeed a transparent region of b in U , and for every transparent region V of b in U , we have $V \leq \tau_U(b)$.

Since we know that ε -regions are transparent regions (by Proposition 4.3.4), we can combine the above corollary to obtain the following :

Corollary 5.1.9 . For any $\mathbb{T}$ -adapted presheaf, region U , and action $b \in \mathcal{L}(U)$, we have $\varepsilon_U(b) \leq \tau_U(b)$.

⁶Thus, this proposition applies to $\mathbb{T}$ -adapted presheaves as well.

Corollary 5.1.10 . *For any $\mathbb{T}$ -adapted presheaf, region U , and action $b \in \mathcal{L}(U)$, we have :*

$$\xrightarrow{b}_U \subseteq \simeq_{\tau_U(b)}^U \subseteq \simeq_{\varepsilon_U(b)}^U$$

But these transparent regions are not as stable under projection as containing regions and ε -regions are. It is possible to have regions $V \leq U$ and an action $b \in \mathcal{L}(U)$ such that $\tau_V(\rho_V^U(b)) \neq \tau_U(b) \wedge V$. However, the following holds at least :

Proposition 5.1.11 . *Given a $\mathbb{T}$ -adapted presheaf, and any region U and action $b \in \mathcal{L}(U)$, we have $\tau_V(\rho_V^U(b)) \leq \tau_U(b) \wedge V$ for every region $V \leq U$.*

Proof. Of course, $\tau_V(\rho_V^U(b)) \leq V$. We show that $\tau_V(\rho_V^U(b))$ is a transparent region of b in U to get $\tau_V(\rho_V^U(b)) \leq \tau_U(b)$ by Corollary 5.1.8. Thus, consider any transition $X \xrightarrow{b}_U Y$. We have $X|_V \xrightarrow{\rho_V^U(b)}_V Y|_V$, and so, $(X|_V)|_{\tau_V(\rho_V^U(b))} = (Y|_V)|_{\tau_V(\rho_V^U(b))}$ since $\tau_V(\rho_V^U(b))$ is a transparent region of $\rho_V^U(b)$ in V by Corollary 5.1.8. But then, $X|_{\tau_V(\rho_V^U(b))} = (X|_V)|_{\tau_V(\rho_V^U(b))} = (Y|_V)|_{\tau_V(\rho_V^U(b))} = Y|_{\tau_V(\rho_V^U(b))}$. With states X and Y arbitrarily chosen for $X \xrightarrow{b}_U Y$ in $\mathcal{S}(U)$, we get that $\tau_V(\rho_V^U(b))$ is a transparent region of b in U , and thus, $\tau_V(\rho_V^U(b)) \leq \tau_U(b)$ by Corollary 5.1.8. $\square$

For a $\mathbb{T}$ -adapted presheaf, and region $V \leq U$ and action $b \in \mathcal{L}(U)$, the converse inclusion $\tau_V(\rho_V^U(b)) \geq \tau_U(b) \wedge V$ does not hold in general, and we can consider the following counter-example :

Example 5.1.12 . Consider a $\mathbb{T}$ -adapted presheaf $\mathcal{T} = (\mathcal{S}, \mathcal{L}, \delta)$ specified on the discrete space over $P = \{x, y\}$ as follows :

1. Let $\mathcal{S}(P) = \{0, 1\}^2$, $\mathcal{S}(\{x\}) = \mathcal{S}(\{y\}) = \{0, 1\}$, and $\mathcal{S}(\emptyset) = \{*\}$ (a singleton), $res_{\{x\}}^P$ is projection on the first component in $\mathcal{S}(\{x, y\}) = \{0, 1\}^2$ and $res_{\{y\}}^P$ is projection on the second component in the latter. This clearly gives $\mathcal{S}$ as a set-valued sheaf (it arises from a product of sets).
2. $\mathcal{L}(\{x, y\}) = \mathcal{L}(\{x\}) = \mathcal{L}(\{y\}) = \{b, c\}$, and $\mathcal{L}(\emptyset) = \emptyset$, and $\rho_{\{x\}}^P = \rho_{\{y\}}^P = \Delta_{\{b, c\}}$.
We can easily verify that the $\mathcal{L}$ presheaf that arises this way is separated.

3. The sets of transitions are depicted in Figure 25, and are given by :

$$\begin{aligned}\delta(P) &= \{ [(0, 0), b, (0, 0)], [(1, 1), c, (1, 1)] \} \\ \delta(\{x\}) &= \delta(\{y\}) = \{ (0, b, 0), (0, c, 1), (1, c, 1) \}\end{aligned}$$

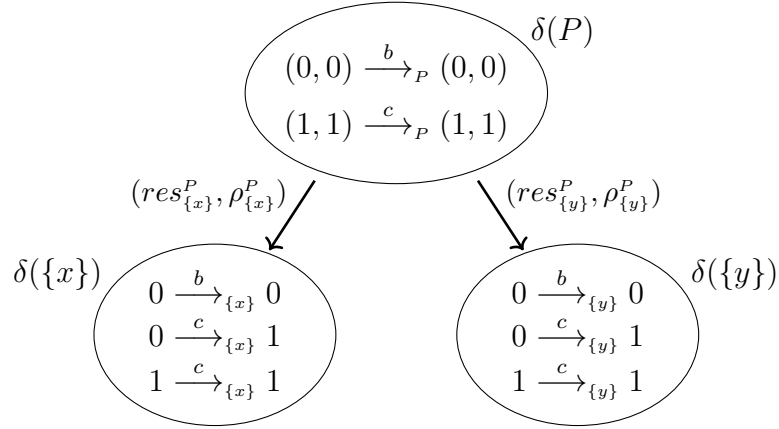


Figure 25: Specification of δ for Example 5.1.12.

We have that $\tau_P(c) = P$ and so $\tau_P(c) \cap \{x\} = \{x\}$. But $\tau_{\{x\}}(\rho_{\{x\}}^P(c)) = \tau_{\{x\}}(c) = \emptyset$ because $0 \xrightarrow{c}_{\{x\}} 1$ is an effect of c in $\{x\}$. Hence, it is not true that $\tau_{\{x\}}(\rho_{\{x\}}^P(c)) \supseteq \tau_P(c) \cap \{x\}$.

In a similar sense, we will get that natural transformations for $\mathbb{T}$ -adapted presheaves do not preserve transparent regions. However, we have that natural transformations preserve ε -regions and containing regions for $\mathbb{T}$ -adapted presheaves, and this will serve us in the proof of the Theorem of Spatially Induced Independence. The following lemma establishes the results in question.

Lemma 5.1.13 [Natural Transformations preserve ε -Regions and Containing Regions]. *Consider any two $\mathbb{T}$ -adapted presheaves $\mathcal{T}$ and $\mathcal{T}'$ on a Heyting algebra $\mathcal{H}$, and consider any natural transformation $\{(\sigma_W, \lambda_W)\}_{W \in \mathcal{H}} : \mathcal{T} \rightarrow \mathcal{T}'$ in $[\mathcal{H}^{op}, \mathbb{T}]_{adapt}$, which is a family of LTS morphisms $(\sigma_W, \lambda_W) : \mathcal{T}(W) \rightarrow \mathcal{T}'(W)$ indexed by regions W in $\mathcal{H}$. Then for any regions $V \leq U$ and any action $b \in \mathcal{L}(U)$ the following statements hold :*

1. If b vanishes in V with respect to U in $\mathcal{T}$, then $\lambda_U(b)$ vanishes in V with respect to U in $\mathcal{T}'$.
2. If b is contained in V with respect to U in $\mathcal{T}$, then $\lambda_U(b)$ is contained in V with respect to U in $\mathcal{T}'$.

Proof. The proof is given in Appendix F. □

Now, we expressed earlier on that $\mathbb{T}$ -adapted presheaves are weaker forms of $\mathbb{T}$ -valued sheaves, and we should check this assertion.

Proposition 5.1.14 . *A $\mathbb{T}$ -valued sheaf is also a $\mathbb{T}$ -adapted presheaf.*

Proof. Consider a $\mathbb{T}$ -valued sheaf $\mathcal{T} = (\mathcal{S}, \mathcal{L}, \delta)$. That $\mathcal{S}$ is a sheaf and $\mathcal{L}$ a separated presheaf is provided by Proposition 4.2.4. Now, consider any $V \leq U$ in $\mathcal{H}$ and $b \in \mathcal{L}(U)$ such that $V \vee \varepsilon_U(b) = U$. The sheaf structure of $\mathcal{T}$ automatically implies that b glues its transitions over any cover of U (by Proposition 4.2.7), so b glues its transitions over $\{V, \varepsilon_U(b)\}$ in particular. This is equivalent to the statement that V contains b by Proposition 4.3.27. □

This proposition establishes an important connection that allows our Theorem of Spatially Induced Independence to apply in the case of $\mathbb{T}$ -valued sheaves (since it will apply to $\mathbb{T}$ -adapted presheaves). Sheaves of labelled transition systems have been studied by Malcolm in [SSTS] in particular, and much of the work of Goguen [HAg] and Cirstea [CSHA] centers on establishing languages of formal specification that work for sheaves representing concurrent object oriented systems. Thus, with further research, it may be possible to introduce the subject of SI-independence for such concurrent object oriented systems, and this could serve as an additional tool in the arsenal of already established formal methods.

Now, we encountered two examples of $\mathbb{T}$ -valued sheaves before with the Pull-back Junction of Buffers from Section 4.2 and the Parallel Registers from Section 2.2. By Proposition 5.1.14 above, we get that these are examples of $\mathbb{T}$ -adapted presheaves.

For examples of $\mathbb{T}$ -adapted presheaves that are not sheaves, there is the Buffer on a Discrete Space from Example 4.1.7 and there are the Petri Net Presheaves⁷ from Example 4.1.11 (that is, these are generally not sheaves).

To construct $\mathbb{T}$ -adapted presheaves directly without using limits on diagrams is laborious however, and in the case of Boolean algebras, it is not always desirable to take limits. What we need is an efficient way of constructing examples on discrete spaces mainly, and we will be able to provide more examples of $\mathbb{T}$ -adapted presheaves when we reach Chapter 6 where we introduce localized relational structures.

For now, we focus on establishing the Theorem of Spatially Induced Independence, i.e. that $\mathbb{T}$ -adapted presheaves provide $\mathbb{A}^\sim$ -valued presheaves through spatially induced independence relations on actions and spatially induced equivalences on transitions⁸. We start by describing the spatially induced equivalence relations on transitions.

Definition 5.1.15 [SI-Equivalences on Transitions]. For a given $\mathbb{T}$ -adapted presheaf and a region U , we define the *spatially induced equivalence on transitions in U* (or SI-equivalence in U), denoted $\sim_U$, as the binary relation on $\delta(U)$ such that for all transitions $(X, b, Y), (X', c, Y') \in \delta(U)$, we have $(X, b, Y) \sim_U (X', c, Y')$ if and only if :

(1) $b = c$, and

(2) there exists a region V that contains b with respect to U such that ⁹ :

⁷We did not prove this yet. The proof is given in Example G.0.2.

⁸We recall that objects of $\mathbb{A}^\sim$ are LTS with an independence relation on their labels and an equivalence on their transitions that satisfy axioms of Alternative Path, One-Step Amalgamation and One-Step co-Amalgamation.

⁹We recall that the statement “ V contains b with respect U ” means that V j -complements an ε -region of b and :

$$\frac{\rho_V^U(b)}{\rightarrow_V} \circ res_V^U = res_V^U \circ \frac{b}{\rightarrow_U}$$

This is formalized in Definition 4.3.15, but the WCA axiom says that $V \vee \varepsilon_U(b) = U$ is a sufficient

$$X \simeq_V^U X' \text{ and } Y \simeq_V^U Y'$$

When we have $(X, b, Y) \sim_U (X', c, Y')$, we say that (X, b, Y) is *spatially equivalent* to (X', c, Y') in U .

Proposition 5.1.16 . *For any $\mathbb{T}$ -adapted presheaf, the SI-equivalence $\sim_U$ as defined above for a region U is an equivalence relation.*

Proof. The proof is provided in Appendix F. □

What these SI-equivalences express is that the true nature of a transition $X \xrightarrow{b}_U Y$ lies in a region where the action b is contained, i.e. if V is a region that contains b with respect to U , then $X|_V \xrightarrow{\rho_V^U(b)}_U Y|_V$ is the actual determinate of $X \xrightarrow{b}_U Y$, and all transitions in $\delta(U)$ that restrict to $X|_V \xrightarrow{\rho_V^U(b)}_U Y|_V$ are identified by the equivalence $\sim_U$.

We now prove that these $\sim_U$ respect the second axiom of an ALTSE.

Proposition 5.1.17 . *For any $\mathbb{T}$ -adapted presheaf and region U , and any transitions $(X, b, Y), (X', b, Y') \in \delta(U)$, if $(X, b, Y) \sim_U (X', b, Y')$, then*

$$X = X' \text{ if and only if } Y = Y'$$

Proof. Suppose $(X, b, Y) \sim_U (X', b, Y')$. By definition, we get a region V that contains b , such that $X \simeq_V^U X'$ and $Y \simeq_V^U Y'$. This means that $\{V, \varepsilon_U(b)\}$ is a proper cover of U . Now, if $X = X'$, then $Y|_{\varepsilon_U(b)} = X|_{\varepsilon_U(b)} = X'|_{\varepsilon_U(b)} = Y'|_{\varepsilon_U(b)}$. Also, with $Y'|_V = Y|_V$, we can apply the locality of $\mathcal{S}$ to get $Y = Y'$. Similarly, if $Y = Y'$, then $X|_{\varepsilon_U(b)} = Y|_{\varepsilon_U(b)} = Y'|_{\varepsilon_U(b)} = X'|_{\varepsilon_U(b)}$. Then, with $X'|_V = X|_V$, we can apply the locality of $\mathcal{S}$ to get $X = X'$. □

There are two lemmas that we will need to prove the Theorem of Spatially Induced Independence as far as these SI-equivalences are involved. They go as follows :

condition for V to contain b in a $\mathbb{T}$ -adapted presheaf.

Lemma 5.1.18 [LTS Restrictions preserve SI-Equivalences]. *Consider any $\mathbb{T}$ -adapted presheaf and regions $V \leq U$. Then for any transitions $(X, b, Y), (X', b, Y') \in \delta(U)$, if $(X, b, Y) \sim_U (X', b, Y')$ and $\rho_V^U(b) \neq \varepsilon$, then*

$$(X|_V, \rho_V^U(b), Y|_V) \sim_V (X'|_V, \rho_V^U(b), Y'|_V).$$

Proof. Consider any regions $V \leq U$ and any transitions $(X, b, Y), (X', b, Y') \in \delta(U)$. Suppose that $(X, b, Y) \sim_U (X', b, Y')$ and that $\rho_V^U(b) \neq \varepsilon$. Then there exists a region W that contains b with respect to U such that $X \simeq_W^U X'$ and $Y \simeq_W^U Y'$ (so $X|_W = X'|_W$ and $Y|_W = Y'|_W$). By Proposition 5.1.5, $V \wedge W$ contains $\rho_V^U(b)$ with respect to V . Furthermore, $(X|_V)|_{V \wedge W} = X|_{V \wedge W} = (X|_W)|_{V \wedge W} = (X'|_W)|_{V \wedge W} = X'|_{V \wedge W} = (X'|_V)|_{V \wedge W}$. Similarly, $(Y|_V)|_{V \wedge W} = (Y'|_V)|_{V \wedge W}$.

Also, we have $X|_V \xrightarrow{\rho_V^U(b)}_V Y|_V$ and $X'|_V \xrightarrow{\rho_V^U(b)}_V Y'|_V$ by the presheaf structure. With $V \wedge W$ containing $\rho_V^U(b)$ with respect to V , $X|_V \simeq_{V \wedge W}^U X'|_V$, and $Y|_V \simeq_{V \wedge W}^U Y'|_V$, we get $(X|_V, \rho_V^U(b), Y|_V) \sim_V (X'|_V, \rho_V^U(b), Y'|_V)$. $\square$

Lemma 5.1.19 [Natural Transformations preserve SI-Equivalences].

Consider any two $\mathbb{T}$ -adapted presheaves $\mathcal{T}$ (with SI-equivalence $\sim$) and $\mathcal{T}'$ (with SI-equivalence $\sim'$) on a Heyting algebra $\mathcal{H}$, and consider any natural transformation $\{(\sigma_W, \lambda_W)\}_{W \in \mathcal{H}} : \mathcal{T} \rightarrow \mathcal{T}'$ in $[\mathcal{H}^{op}, \mathbb{T}]_{adapt}$, which is a family of LTS morphisms $(\sigma_W, \lambda_W) : \mathcal{T}(W) \rightarrow \mathcal{T}'(W)$ indexed by regions W in $\mathcal{H}$.

Then for any regions $V \leq U$ and any transitions $(X, b, Y), (X', b, Y') \in \delta(U)$ such that $(X, b, Y) \sim_U (X', b, Y')$ and $\lambda_U(b) \neq \varepsilon$, we have that

$$(\sigma_U(X), \lambda_U(b), \sigma_U(Y)) \sim'_U (\sigma_U(X'), \lambda_U(b), \sigma_U(Y'))$$

Proof. The proof is given in Appendix F. $\square$

Now, we will prove that the SI-equivalences satisfy the axioms dictated for ALTSE (and we need to define a spatially induced independence relation for that first), but before we do so, we shall demonstrate that these SI-equivalences have a few interesting

properties of their own (that are not derivable simply from the axioms of an ALTSE). In particular, there are constructive ways of translating in between equivalent transitions. To be more precise, given any transition $X \xrightarrow{b}_U Y$, if we pick a local state X'' outside of the containing region of b (that matches X and Y where it intersects with them), then we can translate this $X \xrightarrow{b}_U Y$ to an equivalent transition $X' \xrightarrow{b}_U Y'$, which is constructed from the X , Y and X'' . This is achieved as follows :

Proposition 5.1.20 [Translation to Equivalent Transition]. *Consider any $\mathbb{T}$ -adapted presheaf and regions $V \leq U$ and an action $b \in \mathcal{L}(U)$ such that V contains b . Then, for any transition $X \xrightarrow{b}_U Y$ and local state $X'' \in \mathcal{S}(\varepsilon_U(b))$ such that $X|_{V \wedge \varepsilon_U(b)} = X''|_{V \wedge \varepsilon_U(b)}$, there exists unique X' and Y' in $\mathcal{S}(U)$ such that $X' \xrightarrow{b}_U Y'$ and $(X, b, Y) \sim_U (X', b, Y')$ and $Y'|_{\varepsilon_U(b)} = X'|_{\varepsilon_U(b)} = X''$. In particular, $X'|_V = X|_V$ and $Y'|_V = Y|_V$.*

Proof. Consider any regions $V \leq U$ and an action $b \in \mathcal{L}(U)$ such that V contains b . Consider any transition $X \xrightarrow{b}_U Y$ and $X'' \in \mathcal{S}(\varepsilon_U(b))$ such that $X|_{V \wedge \varepsilon_U(b)} = X''|_{V \wedge \varepsilon_U(b)}$. Since b glues its transitions over $\{V, \varepsilon_U(b)\}$, and since $X|_V \xrightarrow{\rho_V^U(b)}_V Y|_V$ and $X'' \in \mathcal{S}(\varepsilon_U(b))$ with $(X|_V)|_{V \wedge \varepsilon_U(b)} = X''|_{V \wedge \varepsilon_U(b)}$, the second statement in Proposition 4.3.17 (with $K := \varepsilon_U(b)$ in the proposition) yields unique X' and Y' such that $X' \xrightarrow{b}_U Y'$ and $X'|_V = X|_V$ and $Y'|_{\varepsilon_U(b)} = X'|_{\varepsilon_U(b)} = X''$, and $Y'|_V = Y|_V$ (remark : the X' and Y' here correspond with X and Y respectively in the statement of Proposition 4.3.17). But then, we have $X \simeq_V^U X'$ and $Y \simeq_V^U Y'$, so $(X, b, Y) \sim_U (X', b, Y')$. $\square$

Remark 5.1.21 . Given a $\mathbb{T}$ -valued presheaf, a region U and transitions $(X, b, Y), (X', b, Y') \in \delta(U)$, if $(X, b, Y) \sim_U (X', b, Y')$, then the X' and Y' are uniquely determined by X , Y and $Y'|_{\varepsilon_U(b)} = X'|_{\varepsilon_U(b)} \in \mathcal{S}(\varepsilon_U(b))$. This is because we get a region, say V , that contains b with respect to U and such that $X|_V = X'|_V$ and $Y|_V = Y'|_V$, and we can apply the locality of $\mathcal{S}$ on the cover $\{V, \varepsilon_U(b)\}$ for the equations $X|_V = X'|_V$ and $Y|_V = Y'|_V$ and $Y'|_{\varepsilon_U(b)} = X'|_{\varepsilon_U(b)}$.

For some regions $V \leq U$, the way this translation works (in the above proposition) is closely related to $\simeq_V^U$ acting as a bisimulation on all the actions that V contains

with respect to U . We will express that formally in Corollary 5.1.25, but we need a proposition and some definitions first.

Proposition 5.1.22 . *In a $\mathbb{T}$ -adapted presheaf, if V is a region that contains an action b with respect to a region U , we have :*

$$\xrightarrow{b}_U \circ \simeq_V^U = \simeq_V^U \circ \xrightarrow{b}_U$$

Proof. The proof is given in Appendix F. □

What this essentially says is that changes in states that occur outside of the region where an action is contained do not affect the behaviour of such an action. If we restrict the labelling set of $\mathcal{T}(U)$ to the set of labels that V contains with respect to U , we obtain a new system on which $\simeq_V^U$ acts as a bisimulation. In fact, $\simeq_V^U$ bisimulates both the forward and backward behavior of the transition relations in this restricted system (i.e. $\xrightarrow{b}_U$ and $\xrightarrow{b}^{-1}_U$ are bisimulated through $\simeq_V^U$), and this is formalized in Corollary 5.1.25 below.

Definition 5.1.23 . Given a $\mathbb{T}$ -adapted presheaf and regions $V \leq U$ in the latter, we define the *labelling set interior to V in U* as :

$$\text{int}_U(V) := \{ b \in \mathcal{L}(U) \mid b \text{ is contained in } V \text{ with respect to } U \}$$

Definition 5.1.24 . Given a labelled transition system $T = (S, L, \delta)$, we define the *inverse of T* (or *dual of T*) as the labelled transition system $T^{-1} = (S, L, \delta^{-1})$ where $\delta^{-1} = \{(Y, b, X) \mid (X, b, Y) \in \delta\}$. This is equivalent to saying that $\xrightarrow{b}_{T^{-1}} = \xrightarrow{b}^{-1}_T$ for any $b \in L$.

We recall that for a given labelled transition system $T = (S, L, \delta)$ and $L' \subseteq L$, we have denoted $T \upharpoonright L'$ as the restriction of T to the labelling set L' as in Definition 2.3.6. And now we get a corollary to Proposition 5.1.22 that expresses how $\simeq_V^U$ acts as a bisimulation¹⁰ on the actions contained in the region V .

¹⁰The definition of bisimulation is given in Definition 2.4.12.

Corollary 5.1.25 [$\simeq_V^U$ -Internal Bisimulation]. *Given a $\mathbb{T}$ -adapted presheaf and regions $V \leq U$, we have that $\simeq_V^U$ is a bisimulation on $\mathcal{T}(U) \upharpoonright \text{int}_U(V)$ ¹¹, and it is also a bisimulation on $(\mathcal{T}(U))^{-1} \upharpoonright \text{int}_U(V)$.*

Proof. Take any $b \in \text{int}_U(V)$. We have that b is contained in V by definition, and we get by Proposition 5.1.22 that $\xrightarrow{b}_U \circ \simeq_V^U = \simeq_V^U \circ \xrightarrow{b}_U$ (note that $\xrightarrow{b}_{\mathcal{T}(U) \upharpoonright \text{int}_U(V)} = \xrightarrow{b}_U$).

Using the fact that $\simeq_V^U = (\simeq_V^U)^{-1}$ by symmetry, and the fact that $[R = S \text{ implies } R^{-1} = S^{-1} \text{ for any relations } R \text{ and } S]$, we can derive the following equations from $\xrightarrow{b}_U \circ \simeq_V^U = \simeq_V^U \circ \xrightarrow{b}_U$:

$$\begin{aligned} \xrightarrow{b}_U \circ (\simeq_V^U)^{-1} &\subseteq (\simeq_V^U)^{-1} \circ \xrightarrow{b}_U \\ \xrightarrow{b}_U \circ ((\simeq_V^U)^{-1})^{-1} &\subseteq ((\simeq_V^U)^{-1})^{-1} \circ \xrightarrow{b}_U \\ \xrightarrow{b}_U^{-1} \circ (\simeq_V^U)^{-1} &\subseteq (\simeq_V^U)^{-1} \circ \xrightarrow{b}_U^{-1} \\ \xrightarrow{b}_U^{-1} \circ ((\simeq_V^U)^{-1})^{-1} &\subseteq ((\simeq_V^U)^{-1})^{-1} \circ \xrightarrow{b}_U^{-1} \end{aligned}$$

Since these equations hold for any $b \in \text{int}_U(V)$, we can apply Proposition 2.4.14, and the first equation means that $\simeq_V^U$ is a simulation on $\mathcal{T}(U) \upharpoonright \text{int}_U(V)$, whilst the second equation means that its inverse $(\simeq_V^U)^{-1}$ is a simulation on $\mathcal{T}(U) \upharpoonright \text{int}_U(V)$. This means that $\simeq_V^U$ is a bisimulation on $\mathcal{T}(U) \upharpoonright \text{int}_U(V)$. The third equation says that $\simeq_V^U$ is a simulation on $(\mathcal{T}(U))^{-1} \upharpoonright \text{int}_U(V)$, and the fourth equation says that its inverse $(\simeq_V^U)^{-1}$ is a simulation on $(\mathcal{T}(U))^{-1} \upharpoonright \text{int}_U(V)$. Therefore, $\simeq_V^U$ is a bisimulation on $(\mathcal{T}(U))^{-1} \upharpoonright \text{int}_U(V)$. $\square$

It turns out that when these simulations are enacted, they provide a unique constructive way of simulating a transition, and this yields an SI-equivalent transition in the process. This is just another formulation of Proposition 5.1.20 in truth, and it goes as follows :

¹¹When we say that a relation R is a bisimulation on a labelled transition system T , we mean that R is a bisimulation from T to itself.

Corollary 5.1.26 [Translation of Transitions (through $\simeq_V^U$)]. *In a $\mathbb{T}$ -adapted presheaf, if V is a region that contains an action b with respect to a region U , then we have :*

1. *For any $X \simeq_V^U X'$ and $X \xrightarrow{b}_U Y$, there exists a unique $Y' \in \mathcal{S}(U)$ such that $X' \xrightarrow{b}_U Y'$ and $(X, b, Y) \sim_U (X', b, Y')$. Moreover, this Y' is determined by the equations $Y' \simeq_{\varepsilon_U(b)}^U X'$ and $Y' \simeq_V^U Y$ (it is the gluing of $X'|_{\varepsilon_U(b)}$ and $Y|_V$).*

$$\begin{array}{ccc}
 X & \xrightarrow{b}_U & Y \\
 \simeq_V^U \downarrow & \wr & \downarrow \simeq_V^U \\
 X' & \xrightarrow{b}_U & Y'
 \end{array}$$

2. *For any $X \xrightarrow{b}_U Y$ and $Y \simeq_V^U Y'$, there exists a unique $X' \in \mathcal{S}(U)$ such that $X' \xrightarrow{b}_U Y'$ and $(X, b, Y) \sim_U (X', b, Y')$. Moreover, this X' is determined by the equations $X' \simeq_{\varepsilon_U(b)}^U Y'$ and $X' \simeq_V^U X$ (it is the gluing of $Y'|_{\varepsilon_U(b)}$ and $X|_V$).*

$$\begin{array}{ccc}
 X & \xrightarrow{b}_U & Y \\
 \downarrow \simeq_V^U & \wr & \downarrow \simeq_V^U \\
 X' & \xrightarrow{b}_U & Y'
 \end{array}$$

Proof. (1) Consider any $X' \simeq_V^U X$ and $X \xrightarrow{b}_U Y$. We get $X'|_{\varepsilon_U(b)} \in \mathcal{S}(\varepsilon_U(b))$ with $X|_{\varepsilon_U(b) \wedge V} = X'|_{\varepsilon_U(b) \wedge V} = (X'|_{\varepsilon_U(b)})|_{\varepsilon_U(b) \wedge V}$, and we can apply Proposition 5.1.20 to get unique states X_0 and Y_0 in $\mathcal{S}(U)$ such that $X_0 \xrightarrow{b}_U Y_0$ and $(X, b, Y) \sim_U (X_0, b, Y_0)$ and $X_0|_{\varepsilon_U(b)} = Y_0|_{\varepsilon_U(b)} = X'|_{\varepsilon_U(b)}$. In particular, Proposition 5.1.20 further states that $X_0|_V = X|_V$ and $Y_0|_V = Y|_V$. This means $X_0 = X'$ by locality of $\mathcal{S}$, and setting $Y' := Y_0$, we get $(X, b, Y) \sim_U (X', b, Y')$, and $Y' \simeq_{\varepsilon_U(b)}^U X'$ and $Y' \simeq_V^U Y$ also.

Suppose there exists another Y'' in $\mathcal{S}(U)$ such that $X \xrightarrow{b}_U Y''$ and $(X, b, Y) \sim_U (X', b, Y'')$. This means that $(X', b, Y'') \sim_U (X', b, Y')$, and by Proposition 5.1.17 we get $Y' = Y''$.

(2) Consider any $X \xrightarrow{b}_U Y$ and $Y \simeq_V^U Y'$. We get $Y'|_{\varepsilon_U(b)} \in \mathcal{S}(\varepsilon_U(b))$ with $X|_{\varepsilon_U(b) \wedge V} = Y|_{\varepsilon_U(b) \wedge V} = Y'|_{\varepsilon_U(b) \wedge V} = (Y'|_{\varepsilon_U(b)})|_{\varepsilon_U(b) \wedge V}$, and we can apply Proposition 5.1.20 to get unique states X_0 and Y_0 in $\mathcal{S}(U)$ such that $X_0 \xrightarrow{b}_U Y_0$ and $(X, b, Y) \sim_U (X_0, b, Y_0)$ and $X_0|_{\varepsilon_U(b)} = Y_0|_{\varepsilon_U(b)} = Y'|_{\varepsilon_U(b)}$. In particular, Proposition 5.1.20 further states that $X_0|_V = X|_V$ and $Y_0|_V = Y|_V$. This means $Y_0 = Y'$ by locality of $\mathcal{S}$, and setting $X' := X_0$, and we get $(X, b, Y) \sim_U (X', b, Y')$, and $X' \simeq_{\varepsilon_U(b)}^U Y'$ and $X' \simeq_V^U X$ also.

If there exists another X'' in $\mathcal{S}(U)$ such that $X'' \xrightarrow{b}_U Y$ and $(X, b, Y) \sim_U (X'', b, Y')$, then $(X'', b, Y') \sim_U (X', b, Y')$, and Proposition 5.1.17 gives $X' = X''$. $\square$

It remains to equip $\mathbb{T}$ -adapted presheaves with an independence relation, and then we will be able to verify that adapted presheaves yield $\mathbb{A}^\sim$ -valued presheaves.

Definition 5.1.27 [SI-Independence Relations]. Given a $\mathbb{T}$ -adapted presheaf and a region U , we define the *spatially induced independence relation on U* , denoted $\mathcal{I}(U)$, as the binary relation on $\mathcal{L}(U)$ such that for all $b, c \in \mathcal{L}(U)$, we have $b \mathcal{I}(U) c$ if and only if :

There exists a proper cover $\{V, W\}$ of U such that b vanishes in W and c vanishes in V .

We say that b is *spatially independent of c* in U when this condition holds.

We established a Principle of SI-independence earlier on (in the introduction of Chapter 5) that would require, furthermore, that V contains b and that W contains c in the above statement for $\mathcal{I}(U)$. But this is actually implied by the definition of

$\mathcal{I}(U)$ for $\mathbb{T}$ -adapted presheaves, and we prove this in the following proposition.

Proposition 5.1.28 . *Consider any $\mathbb{T}$ -adapted presheaf, a region U , and actions $b, c \in \mathcal{L}(U)$. Then for any proper cover $\{V, W\}$ of U such that b vanishes in W and c vanishes in V , we have that V contains b and W contains c with respect to U .*

Proof. Take any region U and any actions $b, c \in \mathcal{L}(U)$ in a $\mathbb{T}$ -adapted presheaf. Suppose that $\{V, W\}$ is a proper cover of U such that b vanishes in W and c vanishes in V . We get that $W \leq \varepsilon_U(b)$ and $V \leq \varepsilon_U(c)$. But then, $U = V \vee W \leq V \vee \varepsilon_U(b) \leq U$, and thus $V \vee \varepsilon_U(b) = U$. By the WCA axiom, we get that V contains b . We can easily verify that W contains c in a similar fashion. $\square$

There are other useful ways of looking at SI-independence relations, and the following proposition establishes some equivalences for the definition of these relations.

Proposition 5.1.29 [Equivalent Definitions for SI-Independence Relations]. *Given a $\mathbb{T}$ -adapted presheaf and a region U , and actions $b, c \in \mathcal{L}(U)$. Then the following statements are equivalent :*

1. $b \mathcal{I}(U) c$
2. $\varepsilon_U(b) \vee \varepsilon_U(c) = U$
3. b is contained in $\varepsilon_U(c)$ and c is contained in $\varepsilon_U(b)$

Proof. Consider any $\mathbb{T}$ -adapted presheaf, a region U , and actions $b, c \in \mathcal{L}(U)$.

[(1) $\Rightarrow$ (2)] Suppose that $b \mathcal{I}(U) c$. Then there is a proper cover $\{V, W\}$ of U such that b vanishes in W and c vanishes in V . This means that $W \leq \varepsilon_U(b)$ and $V \leq \varepsilon_U(c)$ respectively. But then, $U = W \vee V \leq \varepsilon_U(b) \vee \varepsilon_U(c) \leq U$, and thus $\varepsilon_U(b) \vee \varepsilon_U(c) = U$.

[(2) $\Rightarrow$ (1)] Suppose that $\varepsilon_U(b) \vee \varepsilon_U(c) = U$. Consider $V = \varepsilon_U(c)$ and $W = \varepsilon_U(b)$. Then $\{V, W\}$ is a proper cover of U such that b vanishes in W and c vanishes in V .

[(2) $\Leftrightarrow$ (3)] This is given by a direct application of Proposition 5.1.3.

□

We shall use these equivalent forms for $\mathcal{I}(U)$ often. We now verify that these $\mathcal{I}(U)$ are symmetric and irreflexive as this is required in the definition of an ALTS.

Proposition 5.1.30 . *For a given $\mathbb{T}$ -adapted presheaf, the SI-independence relations $\mathcal{I}(U)$ as provided in Definition 5.1.27 are symmetric and irreflexive.*

Proof. Consider any region U . It is clear that $\mathcal{I}(U)$ is symmetric from Definition 5.1.27. We do a proof by contradiction that for all $b \in \mathcal{L}(U)$, it is not true that $b \mathcal{I}(U) b$. Thus, suppose there exists $b \in \mathcal{L}(U)$ such that $b \mathcal{I}(U) b$. This means that $\varepsilon_U(b) = \varepsilon_U(b) \vee \varepsilon_U(b) = U$ by Proposition 5.1.29. But then, this means that $b = \rho_U^U(b) = \rho_{\varepsilon_U(b)}^U(b) = \varepsilon$, a contradiction, because $\varepsilon \notin \mathcal{L}(U)$. Therefore, it is not true that $b \mathcal{I}(U) b$, and this establishes that $\mathcal{I}(U)$ is irreflexive. □

Finally, we prove two lemmas that will be needed for the Theorem of Spatially Induced Independence with respect to the SI-independence relations.

Lemma 5.1.31 [LTS Restrictions preserve SI-Independence Relations].

Consider any $\mathbb{T}$ -adapted presheaf. For any regions $V \leq U$ and any actions $b, c \in \mathcal{L}(U)$ such that $b \mathcal{I}(U) c$ and $\rho_V^U(b), \rho_V^U(c) \neq \varepsilon$, we have that $\rho_V^U(b) \mathcal{I}(V) \rho_V^U(c)$.

Proof. Consider any regions $V \leq U$ and any actions $b, c \in \mathcal{L}(U)$ such that $b \mathcal{I}(U) c$ and $\rho_V^U(b), \rho_V^U(c) \neq \varepsilon$. With $b \mathcal{I}(U) c$ we get that $\{\varepsilon_U(b), \varepsilon_U(c)\}$ forms a proper cover of U by Proposition 5.1.29. This means that $\{V \wedge \varepsilon_U(b), V \wedge \varepsilon_U(c)\}$ forms a proper cover of V . Since $V \wedge \varepsilon_U(b) = \varepsilon_V(\rho_V^U(b))$ and $V \wedge \varepsilon_U(c) = \varepsilon_V(\rho_V^U(c))$ by Proposition 4.3.26, we get that $\{\varepsilon_V(\rho_V^U(b)), \varepsilon_V(\rho_V^U(c))\}$ forms a proper cover of V . This establishes $\rho_V^U(b) \mathcal{I}(V) \rho_V^U(c)$. □

The above will in fact establish that the LTS restriction morphisms within a $\mathbb{T}$ -adapted presheaf are ALTS morphisms within the next section. We need to show that the components of a natural transformation also have this potential to act as ALTS morphisms.

Lemma 5.1.32 [Nat. Transf. preserve SI-Independence Relations].

Consider any two $\mathbb{T}$ -adapted presheaves (on a Heyting algebra $\mathcal{H}$) $\mathcal{T} = (\mathcal{S}, \mathcal{L}, \delta)$ with SI-independence relation $\mathcal{I}$ and $\mathcal{T}' = (\mathcal{S}', \mathcal{L}', \delta')$ with SI-independence relation $\mathcal{I}'$. Consider any natural transformation $\{(\sigma_W, \lambda_W)\}_{W \in \mathcal{H}} : \mathcal{T} \rightarrow \mathcal{T}'$ in $[\mathcal{H}^{op}, \mathbb{T}]$, which is a family of LTS morphisms $(\sigma_W, \lambda_W) : \mathcal{T}(W) \rightarrow \mathcal{T}'(W)$ indexed by regions W in $\mathcal{H}$. Then for any actions $b, c \in \mathcal{L}(U)$, if $b \mathcal{I}(U) c$ and $\lambda_U(b), \lambda_U(c) \neq \varepsilon$, then $\lambda_U(b) \mathcal{I}'(U) \lambda_U(c)$.

Proof. The proof is given in Appendix F. □

We now have all the lemmas that we need to prove the Theorem of Spatially Induced Independence, and we proceed to this matter right away.

5.2 Theorem of Spatially Induced Independence

In this section, we prove that given a complete Heyting algebra $\mathcal{H}$, we can establish a functor from $[\mathcal{H}^{op}, \mathbb{T}]_{adapt}$ to $[\mathcal{H}^{op}, \mathbb{A}^\sim]$ that uses the SI-independence relations and SI-equivalences when it applies to a $\mathbb{T}$ -adapted presheaf. These functors are at the heart of what we have referred to as “spatially induced independence” for presheaves of LTS within this thesis, and they are defined as follows :

Definition 5.2.1 [SI-Independence Functors]. Given a complete Heyting algebra $\mathcal{H}$, we define a functor $K_{\mathcal{H}} : [\mathcal{H}^{op}, \mathbb{T}]_{adapt} \rightarrow [\mathcal{H}^{op}, \mathbb{A}^\sim]$ as follows :

1. A $\mathbb{T}$ -adapted presheaf $\mathcal{T} = (\mathcal{S}, \mathcal{L}, \delta)$ is sent to the $\mathbb{A}^\sim$ -valued presheaf $K_{\mathcal{H}}(\mathcal{T})$ given by $K_{\mathcal{H}}(\mathcal{T})(U) = (\mathcal{S}(U), \mathcal{L}(U), \delta(U), \mathcal{I}(U), \sim_U)$ for a region U in $\mathcal{H}$, and where $\mathcal{I}(U)$ is the independence relation for $K_{\mathcal{H}}(\mathcal{T})(U)$ and $\sim_U$ is its equivalence on transitions.
2. The natural transformations are given by the same families of LTS morphisms, i.e. given a natural transformation $\theta = \{(\sigma_U, \lambda_U) : \mathcal{T}(U) \rightarrow \mathcal{T}'(U)\}_{U \in \mathcal{H}}$ in $[\mathcal{H}^{op}, \mathbb{T}]_{adapt}$, we define $K_{\mathcal{H}}(\theta)$ as θ itself, and we write it as :

$$\{(\sigma_U, \lambda_U) : K_{\mathcal{H}}(\mathcal{T})(U) \rightarrow K_{\mathcal{H}}(\mathcal{T}')(U)\}_{U \in \mathcal{H}} : K_{\mathcal{H}}(\mathcal{T}) \rightarrow K_{\mathcal{H}}(\mathcal{T}') \text{ in } [\mathcal{H}^{op}, \mathbb{A}^\sim]$$

And now we arrive at the main theorem of this thesis. Through it, we should get the impression that the structures of ALTSE and adapted presheaves, as formalized in this thesis, constitute very natural mathematical objects of study, and that they authentically represent the concept of spatially induced independence.

Theorem 5.2.2 [Theorem of Spatially Induced Independence]. *For any complete Heyting algebra $\mathcal{H}$, we have that $K_{\mathcal{H}} : [\mathcal{H}^{op}, \mathbb{T}]_{adapt} \rightarrow [\mathcal{H}^{op}, \mathbb{A}^{\sim}]$, as given above in Definition 5.2.1, is indeed a functor.*

Proof. Consider any $\mathbb{T}$ -adapted presheaf $\mathcal{T} = (\mathcal{S}, \mathcal{L}, \delta)$ on a complete Heyting algebra $\mathcal{H}$. We prove that $K_{\mathcal{H}}(\mathcal{T})$ is an $\mathbb{A}^{\sim}$ -valued presheaf. Consider any region U . We show that $K_{\mathcal{H}}(\mathcal{T})(U) = (\mathcal{S}(U), \mathcal{L}(U), \delta(U), \mathcal{I}(U), \sim_U)$ is an ALTSE. We already know that $\sim_U$ is an equivalence relation on $\delta(U)$ (from Proposition 5.1.16) and that $\mathcal{I}(U)$ is an irreflexive and symmetric binary relation on $\mathcal{L}(U)$ from Proposition 5.1.30. And now, there are five axioms to verify now for an ALTSE, and we proceed in order.

The first axiom that $(X, b, Y) \sim_U (X', c, Y')$ implies $b = c$ for any transitions $(X, b, Y), (X', c, Y') \in \mathcal{L}(U)$ follows directly from the definition of $\sim_U$. The second axiom states that if $(X, b, Y) \sim_U (X', c, Y')$ for some transitions $(X, b, Y), (X', c, Y') \in \mathcal{L}(U)$, then $X = X'$ if and only if $Y = Y'$. This has been proven in Proposition 5.1.17 already. It remains to prove the axioms of alternative paths, one-step co-amalgamation and one-step amalgamation.

Suppose that $b \mathcal{I}(U) c$ for some actions $b, c \in \mathcal{L}(U)$. Then $\{\varepsilon_U(b), \varepsilon_U(c)\}$ covers U in $\mathcal{H}$ by Proposition 5.1.29. But then, $U = \varepsilon_U(b) \vee \varepsilon_U(c) \leq \tau_U(b) \vee \varepsilon_U(c) \leq U$, and thus $\tau_U(b) \vee \varepsilon_U(c) = U$, i.e $\tau_U(b)$ contains c . Similarly, we can show that $\tau_U(c)$ contains b .

Now, everything that follows the line on the next page, up to the end of the proof for One-Step Amalgamation (5), will only assume that $\tau_U(b)$ contains c and that $\tau_U(c)$ contains b , and this will actually serve in proving an extended result for

SI-independence in Theorem 5.3.17.

Since $\tau_U(b)$ contains c , we get $U = \tau_U(b) \vee \varepsilon_U(c) \leq \tau_U(b) \vee \tau_U(c) \leq U$, and so $\tau_U(b) \vee \tau_U(c) = U$, i.e. $\{\tau_U(b), \tau_U(c)\}$ is a proper cover of U .

(3 : Alternative Path)

Suppose $X \xrightarrow{b} \tau_U Y \xrightarrow{c} \tau_U Z$ for some $X, Y, Z \in \mathcal{S}(U)$. Then, $X \simeq_{\tau_U(b)}^U Y \simeq_{\tau_U(c)}^U Z$ since b is transparent in $\tau_U(b)$ and c is transparent on $\tau_U(c)$.

Since b is contained in $\tau_U(c)$, we can apply the second statement of translation of transitions in Corollary 5.1.26 to $X \xrightarrow{b} \tau_U Y$ and $Y \simeq_{\tau_U(c)}^U Z$, and we get $Y' \in \mathcal{S}(U)$ such that $Y' \xrightarrow{b} \tau_U Z$ and $(X, b, Y) \sim_U (Y', b, Z)$ and $Y' \simeq_{\varepsilon_U(b)}^U Z$ (it is also true that $Y' \simeq_{\tau_U(b)}^U Z$) and $Y' \simeq_{\tau_U(c)}^U X$. This is depicted in the left diagram of Figure 26.

Similarly, with c contained in $\tau_U(b)$, we can apply the first statement of translation of transitions in Corollary 5.1.26 to $Y \simeq_{\tau_U(b)}^U X$ and $Y \xrightarrow{c} \tau_U Z$, and we get $Y'' \in \mathcal{S}(U)$ such that $X \xrightarrow{c} \tau_U Y''$ and $(Y, c, Z) \sim_U (X, c, Y'')$, and also $Y'' \simeq_{\varepsilon_U(c)}^U X$ (it is also true that $Y'' \simeq_{\tau_U(c)}^U X$) and $Y'' \simeq_{\tau_U(b)}^U Z$. This is depicted in the right diagram of Figure 26.

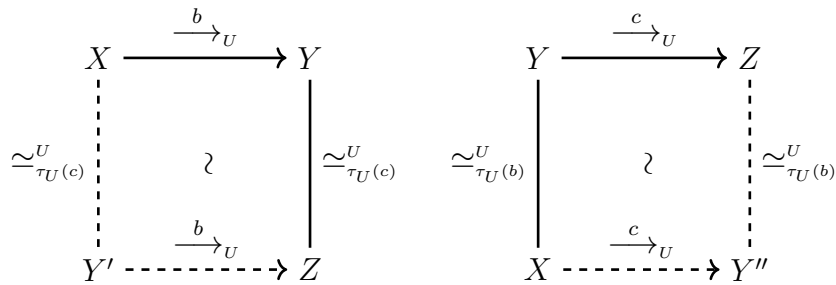


Figure 26: Translation for $X \xrightarrow{b} \tau_U Y$ with $Y \simeq_{\tau_U(c)}^U Z$ and for $Y \xrightarrow{b} \tau_U Z$ with $Y \simeq_{\tau_U(b)}^U X$.

But then, $Y'' \simeq_{\tau_U(c)}^U X \simeq_{\tau_U(c)}^U Y'$ and $Y'' \simeq_{\tau_U(b)}^U Z \simeq_{\tau_U(b)}^U Y'$, which means that

$Y''|_{\tau_U(c)} = Y'|_{\tau_U(c)}$ and $Y''|_{\tau_U(b)} = Y'|_{\tau_U(b)}$. Since $\{\tau_U(b), \tau_U(c)\}$ is a proper cover of U , we get $Y'' = Y'$ by locality of $\mathcal{S}$.

Hence, we have $X \xrightarrow{b} Y' \xrightarrow{a} Z$ and $(X, a, Y) \sim_U (Y', a, Z)$ and $(X, b, Y') \sim_U (Y, b, Z)$.

(4 : One-Step co-Amalgamation)

Suppose $Y \xleftarrow{b} X \xrightarrow{c} Y'$ for some $X, Y, Y' \in \mathcal{S}(U)$. Then, $Y \simeq_{\tau_U(b)}^U X \simeq_{\tau_U(c)}^U Y'$ since b is transparent on $\tau_U(b)$ and c is transparent on $\tau_U(c)$.

Since b is contained in $\tau_U(c)$, we can apply the first statement of translation of transitions in Corollary 5.1.26 to $X \simeq_{\tau_U(c)}^U Y'$ and $X \xrightarrow{b} Y$, and we get $Z \in \mathcal{S}(U)$ such that $Y' \xrightarrow{b} Z$ and $(X, b, Y) \sim_U (Y', b, Z)$, and also $Z \simeq_{\tau_U(b)}^U Y'$ (it is also true that $Z \simeq_{\tau_U(b)}^U Y$) and $Z \simeq_{\tau_U(c)}^U Y$. This is depicted in the left diagram of Figure 27.

Similarly, with c contained in $\tau_U(b)$, we can apply the first statement of translation of transitions in Corollary 5.1.26 to $X \simeq_{\tau_U(b)}^U Y$ and $X \xrightarrow{c} Y'$, and we get $Z' \in \mathcal{S}(U)$ such that $Y \xrightarrow{c} Z'$ and $(X, c, Y') \sim_U (Y, c, Z')$, and also $Z' \simeq_{\tau_U(c)}^U Y$ (it is also true that $Z' \simeq_{\tau_U(c)}^U Y$) and $Z' \simeq_{\tau_U(b)}^U Y'$. This is depicted in the right diagram of Figure 27.

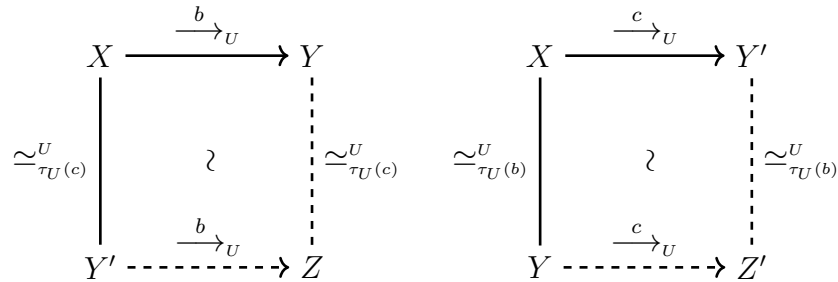


Figure 27: Translation for $X \xrightarrow{b} Y$ with $X \simeq_{\tau_U(c)}^U Y'$ and for $X \xrightarrow{c} Y'$ with $X \simeq_{\tau_U(b)}^U Y$.

But then, $Z \simeq_{\tau_U(c)}^U Y \simeq_{\tau_U(c)}^U Z'$ and $Z' \simeq_{\tau_U(b)}^U Y' \simeq_{\tau_U(b)}^U Z$, which means that

$Z|_{\tau_U(c)} = Z'|_{\tau_U(c)}$ and $Z|_{\tau_U(b)} = Z'|_{\tau_U(b)}$. Since $\{\tau_U(b), \tau_U(c)\}$ is a proper cover of U , we get $Z = Z'$ by locality of $\mathcal{S}$.

Hence, we have $Y \xrightarrow{c} \tau_U Z \xleftarrow{b} Y'$, and $(X, b, Y) \sim_U (Y', b, Z)$ and $(X, c, Y') \sim_U (Y, c, Z)$.

(5 : One-Step Amalgamation)

Suppose $Y \xrightarrow{c} \tau_U Z \xleftarrow{b} Y'$. Then, $Y \simeq_{\tau_U(c)}^U Z \simeq_{\tau_U(b)}^U Y'$ since b is transparent on $\tau_U(b)$ and c is transparent on $\tau_U(c)$.

Since b is contained in $\tau_U(c)$, we can apply the second statement of translation of transitions in Corollary 5.1.26 to $Y' \xrightarrow{b} \tau_U Z$ and $Z \simeq_{\tau_U(c)}^U Y$, and we get $X \in \mathcal{S}(U)$ such that $X \xrightarrow{b} \tau_U Y$ and $(Y', b, Z) \sim_U (X, b, Y)$, and also $X \simeq_{\tau_U(b)}^U Y$ (it is also true that $X \simeq_{\tau_U(b)}^U Y$) and $X \simeq_{\tau_U(c)}^U Y'$. This is depicted in the left diagram of Figure 28.

Similarly, with c contained in $\tau_U(b)$, we can apply the second statement of translation of transitions in Corollary 5.1.26 to $Y \xrightarrow{c} \tau_U Z$ and $Z \simeq_{\tau_U(b)}^U Y'$, and we get $X' \in \mathcal{S}(U)$ such that $X' \xrightarrow{c} \tau_U Y'$ and $(Y, c, Z) \sim_U (X', c, Y')$, and also $X' \simeq_{\tau_U(c)}^U Y'$ (it is also true that $X' \simeq_{\tau_U(c)}^U Y'$) and $X' \simeq_{\tau_U(b)}^U Y$. This is depicted in the right diagram of Figure 28.

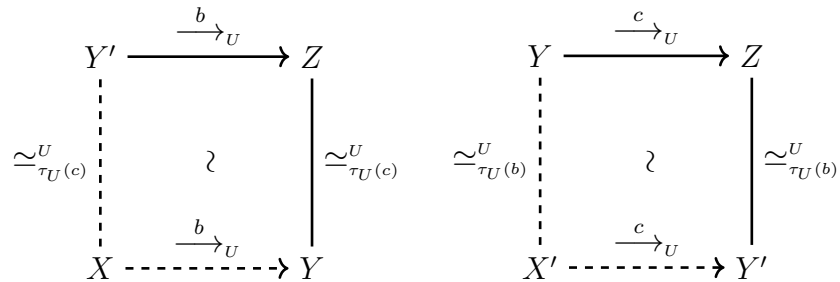


Figure 28: Translation for $Y' \xrightarrow{b} \tau_U Z$ with $Z \simeq_{\tau_U(c)}^U Y$ and for $Y \xrightarrow{c} \tau_U Z$ with $Z \simeq_{\tau_U(b)}^U Y'$.

But then, $X \simeq_{\tau_U(c)}^U Y' \simeq_{\tau_U(c)}^U X'$ and $X \simeq_{\tau_U(b)}^U Y \simeq_{\tau_U(b)}^U X'$, which means that $X|_{\tau_U(c)} = X'|_{\tau_U(c)}$ and $X|_{\tau_U(b)} = X'|_{\tau_U(b)}$. Since $\{\tau_U(b), \tau_U(c)\}$ is a proper cover of U , we get $X = X'$ by locality of $\mathcal{S}$. Hence, we have $X \xrightarrow{a}_U Y$ and $X \xrightarrow{b}_U Y'$, and $(X, a, Y) \sim_U (Y', a, Z)$ and $(X, b, Y') \sim_U (Y, b, Z)$.

This demonstrates that $K_{\mathcal{H}}(\mathcal{T})(U)$ is an ALTSE.

Furthermore, for any regions $U_0 \leq U$, the restriction morphisms $(res_{U_0}^U, \rho_{U_0}^U)$ of $\mathcal{T}$ provide ALTSE morphisms from $K_{\mathcal{H}}(\mathcal{T})(U)$ to $K_{\mathcal{H}}(\mathcal{T})(U_0)$. These are LTS morphisms from $K_{\mathcal{H}}(\mathcal{T})(U)$ to $K_{\mathcal{H}}(\mathcal{T})(U_0)$ by definition, and we proved that they preserve SI-independence with Lemma 5.1.31, and that they preserve SI-equivalences with Lemma 5.1.18. This establishes that they are ALTSE morphisms in fact.

We must verify that natural transformations are preserved by $K_{\mathcal{H}}$. Consider any $\mathbb{T}$ -adapted presheaves $\mathcal{T}$ and $\mathcal{T}'$ on $\mathcal{H}$. Consider any natural transformation $\{(\sigma_U, \lambda_U) : \mathcal{T}(U) \rightarrow \mathcal{T}'(U)\}_{U \in \mathcal{H}}$ in $[\mathcal{H}^{op}, \mathbb{T}]_{adapt}$. For a given region U , we have that (σ_U, λ_U) is a LTS morphism from $\mathcal{T}(U)$ to $\mathcal{T}'(U)$ by definition. We have proven in Lemma 5.1.32 that (σ_U, λ_U) preserves the SI-independence relation on U , and we have proven in Lemma 5.1.19 that it preserves the SI-equivalence on U . Therefore, (σ_U, λ_U) is an ALTSE morphism from $K_{\mathcal{H}}(\mathcal{T})$ to $K_{\mathcal{H}}(\mathcal{T}')$.

Finally, we must show that $K_{\mathcal{H}}$ preserves identities and composition. But this is clearly the case since we did not modify the components of a natural transformation with $K_{\mathcal{H}}$. Indeed, if $\{(1_{\mathcal{S}(U)}, 1_{\mathcal{L}(U)})\}_{U \in \mathcal{H}}$ is the identity on $\mathcal{T} = (\mathcal{S}, \mathcal{L}, \delta)$, then it certainly provides the identity for $K_{\mathcal{H}}(\mathcal{T}) = (\mathcal{S}, \mathcal{L}, \delta, \mathcal{I}, \sim)$. Furthermore, if $\theta = \{(\sigma_U, \lambda_U)\}_{U \in \mathcal{H}} : \mathcal{T} \rightarrow \mathcal{T}'$ and $\theta' = \{(\sigma'_U, \lambda'_U)\}_{U \in \mathcal{H}} : \mathcal{T}' \rightarrow \mathcal{T}''$ are natural transformations in $[\mathcal{H}^{op}, \mathbb{T}]_{adapt}$, then $K_{\mathcal{H}}(\mathcal{T})(\theta' \circ \theta) = \{(\sigma'_U \circ \sigma_U, \lambda'_U \circ \lambda_U)\}_{U \in \mathcal{H}} = \theta' \circ \theta = K_{\mathcal{H}}(\mathcal{T})(\theta') \circ K_{\mathcal{H}}(\mathcal{T})(\theta)$.

This proves that $K_{\mathcal{H}}$ is a functor from $[\mathcal{H}^{op}, \mathbb{T}]_{adapt}$ to $[\mathcal{H}^{op}, \mathbb{A}^{\sim}]$. □

For a given complete Heyting algebra $\mathcal{H}$, there is an inclusion functor from $[\mathcal{H}^{op}, \mathbb{T}]_{adapt}$ to $[\mathcal{H}^{op}, \mathbb{T}]$, say $\iota_{\mathcal{H}}$, and there is a forgetful functor from $[\mathcal{H}^{op}, \mathbb{A}^{\sim}]$ to $[\mathcal{H}^{op}, \mathbb{T}]$, say $G_{\mathcal{H}}$, and we get $G_{\mathcal{H}} \circ K_{\mathcal{H}} = \iota_{\mathcal{H}}$, i.e. the following diagram commutes :

$$\begin{array}{ccc}
 [\mathcal{H}^{op}, \mathbb{T}] & \xleftarrow{G_{\mathcal{H}}} & [\mathcal{H}^{op}, \mathbb{A}^{\sim}] \\
 \uparrow \iota_{\mathcal{H}} & & \nearrow K_{\mathcal{H}} \\
 [\mathcal{H}^{op}, \mathbb{T}]_{adapt} & &
 \end{array}$$

Finally, we can apply the functor $F_{am}^{\sim}$ from $\mathbb{A}^{\sim}$ to $\mathbb{M}$ (as given in Definition 3.3.8) to obtain a Mazurkiewicz trace monoid structure for an $\mathbb{A}^{\sim}$ -valued presheaf. This is undertaken in the following definition.

Definition 5.2.3 . For a given complete Heyting algebra $\mathcal{H}$, we define a functor $M_{\mathcal{H}} : [\mathcal{H}^{op}, \mathbb{A}^{\sim}] \rightarrow [\mathcal{H}^{op}, \mathbb{M}]$ that sends a $\mathbb{A}^{\sim}$ -valued presheaf $\mathcal{T}$ to the $\mathbb{M}$ -valued presheaf $M_{\mathcal{H}}(\mathcal{T}) = F_{am}^{\sim} \circ \mathcal{T}$.

For a natural transformation $\{(\sigma_U, \lambda_U)\}_{U \in \mathcal{H}} : \mathcal{T} \rightarrow \mathcal{T}'$ in $[\mathcal{H}^{op}, \mathbb{A}^{\sim}]$, we set $M_{\mathcal{H}}(\{(\sigma_U, \lambda_U)\}_{U \in \mathcal{H}}) = \{F_{am}^{\sim}(\sigma_U, \lambda_U)\}_{U \in \mathcal{H}}$, which is a natural transformation from $M_{\mathcal{H}}(\mathcal{T})$ to $M_{\mathcal{H}}(\mathcal{T}')$.

Proposition 5.2.4 . For a given complete Heyting algebra $\mathcal{H}$, $M_{\mathcal{H}}$ as specified in Definition 5.2.3 is indeed a functor from $[\mathcal{H}^{op}, \mathbb{A}^{\sim}]$ to $[\mathcal{H}^{op}, \mathbb{M}]$.

Proof. This simply follows from the fact that $F_{am}^{\sim}$ is a functor from $\mathbb{A}^{\sim}$ to $\mathbb{M}$ and the way $M_{\mathcal{H}}$ is set up. $\square$

Finally, for a given Heyting algebra $\mathcal{H}$, we get a functor $M_{\mathcal{H}} \circ K_{\mathcal{H}}$ from $[\mathcal{H}^{op}, \mathbb{T}]_{adapt}$ to $[\mathcal{H}^{op}, \mathbb{M}]$ that provides a trace monoid structure for $\mathbb{T}$ -adapted presheaves, and it uses the SI-independence relations to perform the quotienting on ALTS.

However, this quotienting does not seem optimal when it comes to presheaves of LTS. Indeed, there are actions that “glue” independent actions and that should be identified with the sequential firing of such independent actions. In Appendix H, we will make a conjecture that another kind of functor from $[\mathcal{H}^{op}, \mathbb{T}]_{adapt}$ to $[\mathcal{H}^{op}, \mathbb{M}]$ is possible, and it should perform the extra quotienting that we need.

Now, we have conceptualized spatially induced independence in the case of a Heyting algebra as a base space. In the case where a Boolean algebra is used as a base space, there are many statements provided in the previous section that are simplified a great deal through the presence of complements, and we explore how this is so in the following section.

5.3 $\mathbb{T}$ -adapted Presheaves on Boolean algebras

The dynamic of SI-independence is somewhat different in the case where $\mathbb{T}$ -adapted presheaves distribute LTS over Boolean algebra spaces (by contrast to Heyting algebra spaces in general). Our objective here is to demonstrate that the systematic presence of complements in Boolean algebra spaces provides a direct interpretation of the SI-independence relations $\mathcal{I}$ from Definition 5.1.27 as spatial disjointness of containing regions for actions. More importantly however, we will see that the presence of Boolean algebras forces every action to be independent of every region where they vanish, and this can actually make the $\mathcal{I}$ relations less efficient. Indeed, if we make a Boolean algebra version of the Pullback Junction of Buffers, we lose the possibility of saying that the $send^1$ and $send^2$ actions are spatially independent, because their containing regions will overlap on the input place of the junction. Thus, in this section based on Boolean algebras, we will provide an extended form of SI-independence relation, $\mathcal{I}^+$, that is a bit more sensible to such case scenarios and that distinguishes between containment of dependencies and containment of effects.¹²

¹²This $\mathcal{I}^+$ can be defined for $\mathbb{T}$ -adapted presheaves on Heyting algebras also, but we present it here because it seems to be more useful in the context of Boolean algebras.

Before we consider the case of Boolean algebra spaces however, let us consider the case of complete bi-Heyting algebra spaces first (a lattice that is both a Heyting algebra and a co-Heyting algebra). When we work with bi-Heyting algebra spaces, we allow the following distributivity law of finite joins on infinite indexed families of meets (which is given by the co-Heyting algebra structure in fact) :

$$U \vee \left(\bigwedge_{j \in J} V_j \right) = \bigwedge_{j \in J} (U \vee V_j)$$

This property allows us to make sense of a smallest containing region for actions.

Proposition 5.3.1 . *Consider any regions $V \leq U$. Then the property of being a region that j -complements V in U is closed under arbitrary meets.*

Proof. Consider any regions $V \leq U$ in a bi-Heyting algebra. Take any family $\{V_j\}_{j \in J}$ of regions that j -complement V in U . Write $W := \bigwedge_{j \in J} V_j$. We have $V_j \vee V = U$ for all j in J , and thus $W \vee V = \left(\bigwedge_{j \in J} V_j \right) \vee V = \bigwedge_{j \in J} (V_j \vee V) = \bigwedge_{j \in J} U = U$. This means that W is a j -complement of V in U . $\square$

Corollary 5.3.2 . *Suppose we have a $\mathbb{T}$ -adapted presheaf on a complete bi-Heyting algebra, and consider any region U and action $b \in \mathcal{L}(U)$. Then the property of being a region containing b with respect to U is closed under arbitrary meets.*

Proof. Take any family $\{V_j\}_{j \in J}$ of regions containing b with respect to U . Write $W := \bigwedge_{j \in J} V_j$. We have that $V_j \vee \varepsilon_U(b) = U$ for all j in J , and thus $W \vee \varepsilon_U(b) = U$ by Proposition 5.3.1. We can then apply the WCA axiom and get that W contains b with respect to U . $\square$

It follows that there exists a smallest containing region for any given action when we have bi-Heyting algebras.

Definition 5.3.3 [Smallest Containing Region]. Given a $\mathbb{T}$ -adapted presheaf on a complete bi-Heyting algebra $\mathcal{H}$, and any region U and action $b \in \mathcal{L}(U)$, we define the smallest containing region of b in U as

$$\psi_U(b) := \bigwedge \{ V \in \mathcal{H} \mid V \leq U \text{ and } V \vee \varepsilon_U(b) = U \}^{13}$$

And from Corollary 5.3.2, it follows that :

Corollary 5.3.4 . *In a $\mathbb{T}$ -adapted presheaf, if U is any region and $b \in \mathcal{L}(U)$, then $\psi_U(b)$ indeed contains b with respect to U , and for every region V that contains b with respect to U , we have $\psi_U(b) \leq V$.*

In other words, we get $\psi_U(b) \vee \varepsilon_U(b) = U$ for any region U and action $b \in \mathcal{L}(U)$ within a $\mathbb{T}$ -adapted presheaf on a bi-Heyting algebra. Unfortunately, these $\psi_U(b)$ regions are not as stable under projection as the $\varepsilon_U(b)$ regions, i.e. it is possible to have $\psi_V(\rho_V^U(b)) \neq \psi_U(b) \wedge V$.

Indeed, if we consider the Pullback Buffer Junction from Example 4.2.5 again (see Figure 29), we had an action $(send^1, \varepsilon) \in \mathcal{L}(V_1 \vee V_2)$ for which we actually identified the smallest containing region as $\psi_{V_1 \vee V_2}(send^1, \varepsilon) = V_1$. If we project this action to $V_1 \wedge V_2$, we get ε , so

$$\begin{aligned} \psi_{V_1 \wedge V_2}(\rho_{V_1 \wedge V_2}^{V_1 \vee V_2}(send^1, \varepsilon)) &= \psi_{V_1 \wedge V_2}(\varepsilon) = \bigwedge \{ V \leq (V_1 \wedge V_2) \mid V \vee \varepsilon_{V_1 \wedge V_2}(\varepsilon) = V_1 \wedge V_2 \} \\ &= \bigwedge \{ V \leq (V_1 \wedge V_2) \mid V \vee (V_1 \wedge V_2) = V_1 \wedge V_2 \} = 0_{\mathcal{H}} \\ &\neq V_1 \wedge V_2 = \psi_{V_1 \vee V_2}(send^1, \varepsilon) \wedge (V_1 \wedge V_2) \end{aligned}$$

¹³Notice that this $\psi_U(b)$ is definable in a weaker context where we have $\mathbb{T}$ -valued presheaf with $\mathcal{L}$ as a separated presheaf (as opposed to a $\mathbb{T}$ -adapted presheaf). In such contexts however, it does not provide a smallest containing region, i.e. it provides only a smallest j -complement to $\varepsilon_U(b)$. Also, for Proposition 5.3.8, we will be using this more general $\psi_U(b)$ in the context of a $\mathbb{T}$ -valued presheaf with $\mathcal{L}$ as a separated presheaf.

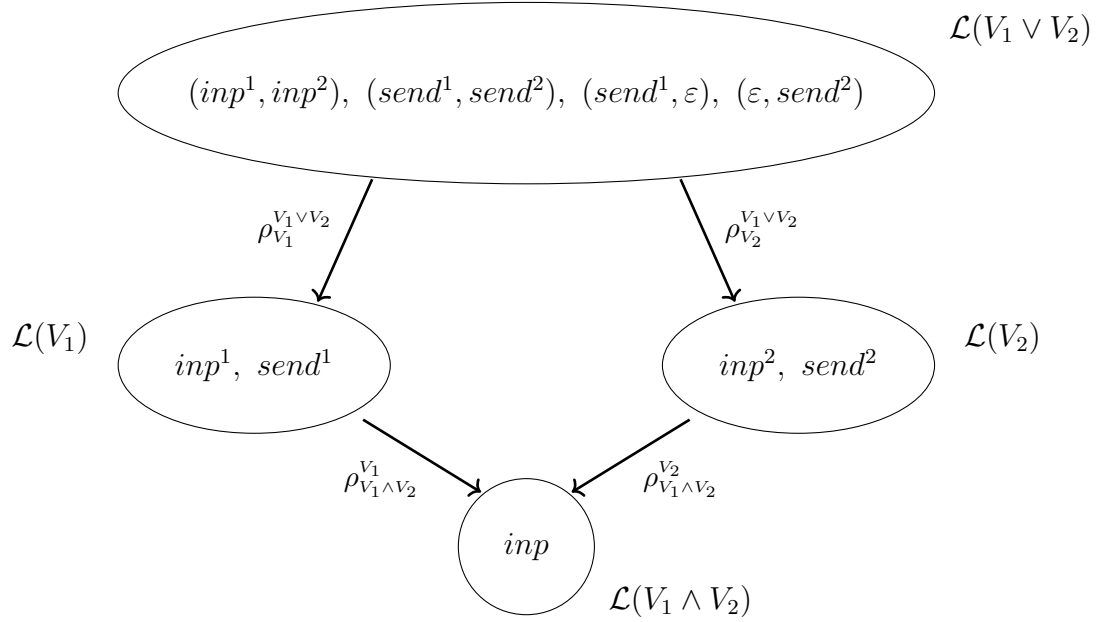


Figure 29: Specification of labelling sets for the Pullback Buffer Junction as provided in Example 4.2.5.

However, we do get the following :

Proposition 5.3.5 . *Given a $\mathbb{T}$ -adapted presheaf on a complete bi-Heyting algebra, and any regions $V \leq U$ and any action $b \in \mathcal{L}(U)$, we have :*

$$\psi_V(\rho_V^U(b)) \leq \psi_U(b) \wedge V$$

Proof. We have that ε_U -regions are stable under projection by Proposition 4.3.26, and so $\varepsilon_V(\rho_V^U(b)) = \varepsilon_U(b) \wedge V$. Thus, we get :

$$\begin{aligned} (\psi_U(b) \wedge V) \vee (\varepsilon_V(\rho_V^U(b))) &= (\psi_U(b) \wedge V) \vee (\varepsilon_U(b) \wedge V) \\ &= (\psi_U(b) \vee \varepsilon_U(b)) \wedge V \\ &= U \wedge V = V \end{aligned}$$

This means that $\psi_U(b) \wedge V$ contains $\rho_V^U(b)$ by the WCA axiom. By Corollary 5.3.4, we get $\psi_V(\rho_V^U(b)) \leq \psi_U(b) \wedge V$. $\square$

For this reason amongst many others, we decided to go ahead and develop the theory of $\mathbb{T}$ -adapted presheaves in an even stronger context, replacing the Heyting algebra base space not just with a bi-Heyting algebra, but with a Boolean algebra (fact : every Boolean algebra is a bi-Heyting algebra).

Thus, for the rest of this section, we fix $\mathcal{B}$ as a complete Boolean algebra. For any regions U and V in $\mathcal{B}$, we use the notation $\overline{V}$ to denote the complement of V in $\mathcal{B}$, and when $V \leq U$, we write $U \setminus V := U \wedge \overline{V}$ to denote the complement of V in U .

Proposition 5.3.6 . *Given a $\mathbb{T}$ -adapted presheaf on a complete Boolean algebra $\mathcal{B}$, and any region U and action $b \in \mathcal{L}(U)$, we have that $\psi_U(b) = U \setminus \varepsilon_U(b)$.*

Proof. We have that $(U \setminus \varepsilon_U(b)) \vee \varepsilon_U(b) = U$, which means that $U \setminus \varepsilon_U(b)$ contains b with respect to U by the WCA axiom. By Corollary 5.3.4, we get $\psi_U(b) \leq U \setminus \varepsilon_U(b)$.

Also, $U \setminus \varepsilon_U(b) = (\psi_U(b) \vee \varepsilon_U(b)) \setminus \varepsilon_U(b) = \psi_U(b) \setminus \varepsilon_U(b) = (U \wedge \psi_U(b)) \setminus \varepsilon_U(b) = (U \setminus \varepsilon_U(b)) \wedge \psi_U(b)$. This means $(U \setminus \varepsilon_U(b)) \leq \psi_U(b)$. $\square$

In other words, when it comes to $\mathbb{T}$ -adapted presheaves on Boolean algebras, we get that the smallest containing region $\psi_U(b)$ is the complement of $\varepsilon_U(b)$ in U for any region U and action $b \in \mathcal{L}(U)$. This is quite an appealing result which makes the case of Boolean algebras truly easier to deal with when talking about action containment. In Definition 4.3.15, we defined a proper region of b with respect to U as a region that contains U and that is disjoint from $\varepsilon_U(b)$. Thus, since $\psi_U(b) \wedge \varepsilon_U(b) = 0_{\mathcal{B}}$ in the context of a Boolean algebra space, we can refer to $\psi_U(b)$ as the *proper region* of b in U (it is unique of course). And we recall that by Definition 4.3.9, this means precisely that b is independent of $\varepsilon_U(b)$. Thus, every action is independent of the regions where they vanish. This contrasts with the case of $\mathbb{T}$ -adapted presheaves on Heyting algebras such as the Pullback Junction of Buffers from Example 4.2.5 where an action can depend on an ε -region (we saw that with the $(send^1, \varepsilon)$ action).

Now, in the case of Boolean algebra spaces, these proper regions for actions project well on subregions (as opposed to what Proposition 5.3.5 expresses in the case of bi-Heyting algebras) :

Proposition 5.3.7 [Stability of Proper Regions under Projection].

Given a $\mathbb{T}$ -adapted presheaf on a complete Boolean algebra, and any regions $V \leq U$ and $b \in \mathcal{L}(U)$, we have :

$$\psi_V(\rho_V^U(b)) = \psi_U(b) \wedge V$$

Proof. We have $\psi_V(\rho_V^U(b)) = V \setminus \varepsilon_V(\rho_V^U(b)) = V \wedge \overline{\varepsilon_V(\rho_V^U(b))} = V \wedge \overline{\varepsilon_U(b) \wedge V} = V \wedge \overline{\varepsilon_U(b)} \wedge \overline{V}$ since $\varepsilon_V(\rho_V^U(b)) = \varepsilon_U(b) \wedge V$ by Proposition 4.3.26, and then :

$$\begin{aligned} V \wedge \overline{(\varepsilon_U(b) \wedge V)} &= V \wedge (\overline{\varepsilon_U(b)} \vee \overline{V}) = (V \wedge \overline{\varepsilon_U(b)}) \vee (V \wedge \overline{V}) = V \wedge \overline{\varepsilon_U(b)} = \\ (U \wedge V) \wedge \overline{\varepsilon_U(b)} &= (U \wedge \overline{\varepsilon_U(b)}) \wedge V = (U \setminus \varepsilon_U(b)) \wedge V = \psi_U(b) \wedge V. \end{aligned} \quad \square$$

The presence of proper regions $\psi_U(b)$ facilitates the statement of the WCA axiom for $\mathbb{T}$ -adapted presheaves, and this is established in the following proposition.

Proposition 5.3.8 [WCA Equivalence for Boolean Algebras].

Given a $\mathbb{T}$ -valued presheaf on a complete Boolean algebra where $\mathcal{S}$ is a sheaf and $\mathcal{L}$ is a separated presheaf, we have that the following statements are equivalent :

1. (WCA axiom) *For any regions $V \leq U$ and any action $b \in \mathcal{L}(U)$, if $V \vee \varepsilon_U(b) = U$, then V contains b with respect to U .*
2. *For any region U and any action $b \in \mathcal{L}(U)$, we have that $\psi_U(b)$ contains b with respect to U .¹⁴*

¹⁴As remarked in the footnote provided with Definition 5.3.3, we do not have $\psi_U(b)$ as a smallest containing region with the assumptions of this proposition here. However, $\psi_U(b)$ as $U \setminus \varepsilon_U(b)$ is well-defined, and we make use of this.

Proof. If the WCA axiom holds, then consider any region U and any action $b \in \mathcal{L}(U)$. We automatically get that $\psi_U(b)$ contains b with respect to U because $\psi_U(b) \vee \varepsilon_U(b) = (U \setminus \varepsilon_U(b)) \vee \varepsilon_U(b) = U$.

Conversely, suppose that for any region U and any action $b \in \mathcal{L}(U)$, $\psi_U(b)$ contains b with respect to U . Consider any region V such that $V \vee \varepsilon_U(b) = U$. By Proposition 4.3.27, it suffices to prove that b decomposes over $\{V, \varepsilon_U(b)\}$, and this is precisely to say that the following equation holds :

$$\xrightarrow{b}_U = ((res_V^U)^{-1} \circ \xrightarrow{\rho_V^U(b)}_V \circ res_V^U) \cap \simeq_{\varepsilon_U(b)}^U$$

We already know that the inclusion from left to right holds in the above (this is simply from the presheaf structure (Proposition 4.1.15)). We verify that the inclusion from right to left also holds. Consider any $X, Y \in \mathcal{S}(U)$ such that $X|_V \xrightarrow{\rho_V^U(b)}_V Y|_V$ and $X|_{\varepsilon_U(b)} = Y|_{\varepsilon_U(b)}$ (this means $(X, Y) \in ((res_V^U)^{-1} \circ \xrightarrow{\rho_V^U(b)}_V \circ res_V^U) \cap \simeq_{\varepsilon_U(b)}^U$).

Since $V \vee \varepsilon_U(b) = U$, and since $\psi_U(b)$ is the complement of $\varepsilon_U(b)$ in U , we get that $\psi_U(b) \leq V$. This means $X|_{\psi_U(b)} = (X|_V)|_{\psi_U(b)} \xrightarrow{\rho_{\psi_U(b)}^U(b)}_{\psi_U(b)} (Y|_V)|_{\psi_U(b)} = Y|_{\psi_U(b)}$ and since

$$\xrightarrow{b}_U = ((res_{\psi_U(b)}^U)^{-1} \circ \xrightarrow{\rho_{\psi_U(b)}^U(b)}_{\psi_U(b)} \circ res_{\psi_U(b)}^U) \cap \simeq_{\varepsilon_U(b)}^U$$

is provided by our assumption that $\psi_U(b)$ contains b with respect to U (we applied Proposition 4.3.17 here), we get $X \xrightarrow{b}_U Y$. This shows the inclusion in question and we are done. $\square$

Through this equivalence, we can reformulate Definition 5.1.1 of a $\mathbb{T}$ -adapted presheaf in the context where we have a Boolean algebra as a base space, as follows :

Definition 5.3.9 [$\mathbb{T}$ -adapted Presheaf on a Boolean algebra].

A $\mathbb{T}$ -adapted presheaf on a complete Boolean algebra $\mathcal{B}$ is a $\mathbb{T}$ -valued presheaf $\mathcal{T} = (\mathcal{S}, \mathcal{L}, \delta)$ on $\mathcal{B}$, where :

1. **(Sheaf of states)** : $\mathcal{S}$ is a **Set**-valued sheaf
2. **(Separated presheaf of labels)** : $\mathcal{L}$ is a **Set _{ε}** -valued separated presheaf
3. **(Well-Contained Actions (WCA))** :

For any region U in $\mathcal{B}$ and $b \in \mathcal{L}(U)$, $\psi_U(b)$ contains b with respect to U .

The spatially induced equivalence on transitions is also simplified, as it suffices to deal with proper regions of actions now, and this goes as follows :

Proposition 5.3.10 . Consider any region U and action $b \in \mathcal{L}(U)$ in a $\mathbb{T}$ -adapted presheaf on a Boolean algebra. Then, for any transitions $(X, b, Y), (X', b, Y') \in \delta(U)$, we have that :

$$(X, b, Y) \sim_U (X', b, Y') \Leftrightarrow X \simeq_{\psi_U(b)}^U Y \text{ and } X' \simeq_{\psi_U(b)}^U Y'$$

Proof. Consider any transitions $(X, b, Y), (X', b, Y') \in \delta(U)$.

If $(X, b, Y) \sim_U (X', b, Y')$, then there exists a region V that contains b with respect to U such that $X \simeq_V^U Y$ and $X' \simeq_V^U Y'$. But then, $\psi_U(b) \leq V$ by Corollary 5.3.4, so $X \simeq_{\psi_U(b)}^U Y$ and $X' \simeq_{\psi_U(b)}^U Y'$.

Conversely, if $X \simeq_{\psi_U(b)}^U Y$ and $X' \simeq_{\psi_U(b)}^U Y'$, then we automatically obtain $(X, b, Y) \sim_U (X', b, Y')$ since $\psi_U(b)$ contains b with respect to U . $\square$

And we get a very intuitive interpretation of the spatially induced independence relation $\mathcal{I}$: that actions are spatially independent if and only if their proper regions are spatially disjoint. This goes as follows :

Proposition 5.3.11 [Standard SI-Independence in a Boolean Algebra].

Consider any region U and any action $b \in \mathcal{L}(U)$ in a $\mathbb{T}$ -adapted presheaf on a Boolean algebra $\mathcal{B}$. We have the following equivalence :

$$b \mathcal{I}(U) c \Leftrightarrow \psi_U(b) \wedge \psi_U(c) = 0_{\mathcal{B}}$$

for any $b, c \in \mathcal{L}(U)$, where $\mathcal{I}(U)$ is the independence relation given in Definition 5.1.27.

Proof. Taken any actions $b, c \in \mathcal{L}(U)$. Then,

$$\begin{aligned} & b \mathcal{I}(U) c \\ \Leftrightarrow & \varepsilon_U(b) \vee \varepsilon_U(c) = U \quad (\text{from Proposition 5.1.29}) \\ \Leftrightarrow & U \setminus (\varepsilon_U(b) \vee \varepsilon_U(c)) = 0_{\mathcal{B}} \\ \Leftrightarrow & (U \setminus \varepsilon_U(b)) \vee (U \setminus \varepsilon_U(c)) = 0_{\mathcal{B}} \\ \Leftrightarrow & \psi_U(b) \wedge \psi_U(c) = 0_{\mathcal{B}} \end{aligned}$$

□

Thus, the Theorem of Spatially Induced Independence applies for the case of a Boolean algebra precisely under the criterion of spatial disjointness of proper regions of actions for the SI-independence relations.

However, it so happens that, in the case of Boolean algebras, these SI-independence relations are not as efficient as they should be. Consider the following discrete space representation of the Buffer Junction (in comparison to the Pullback version of Example 4.2.5) :

Example 5.3.12 [Discrete Space Buffer Junction]. Consider a set of places $P = \{x, y, z\}$ and let $\mathcal{B}$ be the discrete space over P . Define a $\mathbb{T}$ -valued presheaf $\mathcal{T}$ on $\mathcal{B}$ as follows :

1. State spaces : $\mathcal{S}(U) = \{0, 1\}^U$ for any region $U \subseteq P$, and restriction maps are given by function restrictions.
2. Labelling sets : For any region $U \subseteq P$,
 - $send^1 \in \mathcal{L}(U) \Leftrightarrow \{x, y\} \cap U \neq \emptyset$
 - $send^2 \in \mathcal{L}(U) \Leftrightarrow \{x, z\} \cap U \neq \emptyset$
 - $inp \in \mathcal{L}(U) \Leftrightarrow \{x\} \cap U \neq \emptyset$
3. Labelling projections : For any regions $V \subseteq U \subseteq P$,

- If $\{x\} \cap U \neq \emptyset$, define $\rho_V^U(inp)$ as :

$$\rho_V^U(inp) = \begin{cases} inp & \text{if } \{x\} \cap V \neq \emptyset \\ \varepsilon & \text{otherwise} \end{cases}$$

- If $\{x, y\} \cap U \neq \emptyset$, define $\rho_V^U(send^1)$ as :

$$\rho_V^U(send^1) = \begin{cases} send^1 & \text{if } \{x, y\} \cap V \neq \emptyset \\ \varepsilon & \text{otherwise} \end{cases}$$

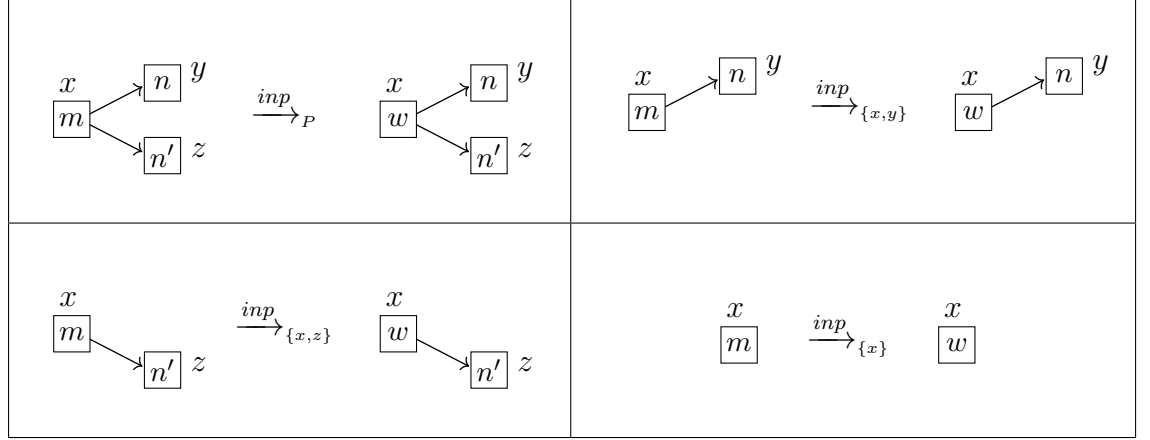
- If $\{x, z\} \cap U \neq \emptyset$, define $\rho_V^U(send^2)$ as :

$$\rho_V^U(send^2) = \begin{cases} send^2 & \text{if } \{x, z\} \cap V \neq \emptyset \\ \varepsilon & \text{otherwise} \end{cases}$$

4. The $\xrightarrow{send^1}_U$ transition relations are provided by the smallest transition relations such that for all $m, n, n', w \in \{0, 1\}$:

5. The $\xrightarrow{send^2}_U$ transition relations are provided by the smallest transition relations such that for all $m, n, n', w \in \{0, 1\}$:

6. The $\xrightarrow[\tau_U]{inp}$ transition relations are provided by the smallest transition relations such that for all $m, n, n', w \in \{0, 1\}$:



All this information determines a $\mathbb{T}$ -adapted presheaf on the discrete space over P , and it is not too complicated to show that the restrictions (res_V^U, ρ_V^U) provide LTS morphisms.

In the above example, we have that $send^1$ and $send^2$ are not independent in P in the sense of $\mathcal{I}(P)$, because $\psi_P(send^1) = \{x, y\}$ and $\psi_P(send^2) = \{x, z\}$, and these are not disjoint. This is unfortunate because these actions actually respect the equations of Alternative Paths $\xrightarrow[\tau_P]{send^1} \circ \xrightarrow[\tau_P]{send^2} = \xrightarrow[\tau_P]{send^2} \circ \xrightarrow[\tau_P]{send^1}$ and One-Step bi-Amalgamation $\xrightarrow[\tau_P]{send^1} \circ \xrightarrow[\tau_P]{send^2} = \xrightarrow[\tau_P]{send^2} \circ \xrightarrow[\tau_P]{send^1}$ in the sense of an ALTS.¹⁵

However, in the Pullback Buffer Junction, the global representation of these actions, $(send^1, \varepsilon)$ and $(\varepsilon, send^2)$, are actually independent, which means that the Heyting algebra representation gave us more information about independence of these actions. The reason why this is so is because we projected $(send^1, \varepsilon)$ and $(\varepsilon, send^2)$ to ε on a place where each of the actions actually had a dependence, and without

¹⁵The actions in question respect the conditions for independence in an ALTSE actually, using $\sim_U$ as an equivalence.

the more refined discrete space representation, the WCA axiom could not expose this projection to ε as “incorrect”.

The truth is that in our elaboration of action containment, we forced j -complements of $\varepsilon_U(b)$ as regions where dependencies and effects are contained simultaneously for an action. But generally speaking, regions that contain dependencies and regions that contain effects are entirely different kinds of regions. In the above Discrete Buffer Junction for example, $send^1$ depends precisely on $\{x\}$, and it has its effects precisely in $\{y\}$, and $\psi_P(send^1) = \{x, y\}$ has no choice but to consider the smallest region that contains both of these regions. Similarly, $send^2$ depends precisely on $\{x\}$, and it has its effects precisely in $\{z\}$, and $\psi_P(send^2) = \{x, z\}$ has to contain both of these regions to contain dependencies and effects at the same time. Thus, the fact that ψ_P does not discriminate between these different kinds of subregions prevents us from accessing a finer form of independence relation amongst actions.

However, there is a quick fix for this, and the idea for the above example is that even if $\psi_P(send^1) = \{x, y\}$ and $\psi_P(send^2) = \{x, z\}$ are not disjoint, it is true that $send^2$ has no effects in $\{x, y\}$, and it is true that $send^1$ has no effects in $\{x, z\}$, i.e. these actions should, in fact, be independent. As such, we can formulate an extended form of spatially induced independence with the help of transparent regions, and this goes as follows :

Definition 5.3.13 [Extended SI-Independence Relations]. Given a $\mathbb{T}$ -adapted presheaf on a complete Heyting algebra $\mathcal{H}$, and a region U in $\mathcal{H}$, we define the *extended spatially induced independence relation on U* , denoted $\mathcal{I}^+(U)$, as the binary relation on $\mathcal{L}(U)$ such that for all $b, c \in \mathcal{L}(U)$:

$$b \mathcal{I}^+(U) c \quad \Leftrightarrow \quad \tau_U(c) \text{ contains } b \text{ and } \tau_U(b) \text{ contains } c$$

We say that b is *spatially independent* of c with respect to U when the above condition holds.

Whenever actions are independent in the sense of $\mathcal{I}^+(U)$ for some region U , we get that their transparent regions form a proper cover of U . This is what the following proposition establishes.

Proposition 5.3.14 . *Given a $\mathbb{T}$ -adapted presheaf on a complete Heyting algebra $\mathcal{H}$, a region U in $\mathcal{H}$ and any actions $b, c \in \mathcal{L}(U)$, we have that :*

$$b \mathcal{I}^+(U) c \quad \Rightarrow \quad \tau_U(b) \vee \tau_U(c) = U$$

Proof. Suppose that $b \mathcal{I}^+(U) c$ as in the above statement. We get that $\tau_U(c)$ contains b in particular, and this means that $U = \tau_U(c) \vee \varepsilon_U(b) \leq \tau_U(c) \vee \tau_U(b) \leq U$. Thus, $\tau_U(c) \vee \tau_U(b) = U$. $\square$

There is an equivalent way of representing the extended SI-independence relations in the context of Boolean algebra spaces, and this goes as follows :

Proposition 5.3.15 . *Given a $\mathbb{T}$ -adapted presheaf on a complete Boolean algebra $\mathcal{B}$, a region U and any actions $b, c \in \mathcal{L}(U)$, we have :*

$$b \mathcal{I}^+(U) c \quad \Leftrightarrow \quad \psi_U(b) \leq \tau_U(c) \text{ and } \psi_U(c) \leq \tau_U(b)$$

Proof. Consider any region U and actions $b, c \in \mathcal{L}(U)$. If $b \mathcal{I}^+(U) c$, then we get that $\tau_U(c)$ contains b and $\tau_U(b)$ contains c . This means that $\psi_U(b) \leq \tau_U(c)$ and $\psi_U(c) \leq \tau_U(b)$ respectively by Corollary 5.3.4. On the other hand, if $\psi_U(b) \leq \tau_U(c)$ and $\psi_U(c) \leq \tau_U(b)$, then we get $U \setminus \tau_U(c) \leq U \setminus \psi_U(b) = \varepsilon_U(b)$ and $U \setminus \tau_U(b) \leq U \setminus \psi_U(c) = \varepsilon_U(c)$. But then, $U = \tau_U(c) \vee (U \setminus \tau_U(c)) \leq \tau_U(c) \vee \varepsilon_U(b) \leq U$ and $U = \tau_U(b) \vee (U \setminus \tau_U(b)) \leq \tau_U(b) \vee \varepsilon_U(c) \leq U$, from which we get $\tau_U(c) \vee \varepsilon_U(b) = U$ and $\tau_U(b) \vee \varepsilon_U(c) = U$. This means that $\tau_U(c)$ contains b and $\tau_U(b)$ contains c by the WCA axiom. $\square$

These extended SI-independence relations $\mathcal{I}^+(U)$ constitute finer forms of independence relations than the standard SI-independence relations $\mathcal{I}(U)$ as provided in Definition 5.1.27 :

Proposition 5.3.16 [Extended SI-Independence Relations are finer].

Given any $\mathbb{T}$ -adapted presheaf on a Heyting algebra, we have $\mathcal{I}(U) \subseteq \mathcal{I}^+(U)$ for any region U .

Proof. Consider any region U and any actions $b, c \in \mathcal{L}(U)$, and suppose that $b \mathcal{I}(U) c$. By Proposition 5.1.29, we get $U = \varepsilon_U(b) \vee \varepsilon_U(c)$. We have $\varepsilon_U(b) \leq \tau_U(b)$ from Corollary 5.1.9, and thus we get $U = \varepsilon_U(b) \vee \varepsilon_U(c) \leq \tau_U(b) \vee \varepsilon_U(c) \leq U$. This means $U = \tau_U(b) \vee \varepsilon_U(c)$, and thus $\tau_U(b)$ contains c by the WCA axiom. Similarly, we can show that b is contained in $\tau_U(c)$ with $\varepsilon_U(c) \leq \tau_U(c)$. Thus, $b \mathcal{I}^+(U) c$. $\square$

We had a difficulty earlier on with the Buffer Junction on a Discrete Space over $P = \{x, y, z\}$ from Example 5.3.12, where the $send^1$ and $send^2$ actions in $\mathcal{L}(P)$ were not SI-independent with the standard SI-independence relations given by $\mathcal{I}$. Fortunately, we have that they are SI-independent with the extended SI-independence relations : we have $\psi_P(send^1) = \{x, y\} = \tau_P(send^2)$ and $\psi_P(send^2) = \{x, z\} = \tau_P(send^1)$, and by Proposition 5.3.15, we get that $send^1 \mathcal{I}^+(P) send^2$.

Now, we need to establish that these extended SI-independence relations (together with the $\sim_U$) have the properties expected of them to provide ALTSE. However, it turns out that they do not exactly provide ALTSE in their actual form, but this is only up to a minor detail that the extended SI-independence relations are not irreflexive. We could always modify the definition of the $\mathcal{I}^+(U)$ and force the latter to be irreflexive by taking $\mathcal{I}^+(U) \setminus \Delta_{\mathcal{L}(U)}$ instead. With these $\mathcal{I}^+(U) \setminus \Delta_{\mathcal{L}(U)}$, we get ALTSE indeed, and this is established in the following theorem.

Theorem 5.3.17 [Extended Spatially Induced Independence].

Given any $\mathbb{T}$ -adapted presheaf $\mathcal{T} = (\mathcal{S}, \mathcal{L}, \delta)$ on a Heyting algebra and any region U , we have that $(\mathcal{S}(U), \mathcal{L}(U), \delta(U), \mathcal{I}^+(U) \setminus \Delta_{\mathcal{L}(U)}, \sim_U)$ is an ALTSE, where $\mathcal{I}^+(U) \setminus \Delta_{\mathcal{L}(U)}$ provides the independence relation of the ALTSE, and $\sim_U$ provides the equivalence on transitions.

Proof. Consider any $\mathbb{T}$ -adapted presheaf $\mathcal{T} = (\mathcal{S}, \mathcal{L}, \delta)$ on a Heyting algebra and any region U . The fact that $\sim_U$ is an equivalence has been established before in Proposition 5.1.16. That $\mathcal{I}^+(U) \setminus \Delta_{\mathcal{L}(U)}$ is irreflexive and symmetric is clear. Axiom (1) of an ALTSE follows from the definition of $\sim_U$, where $(X, b, Y) \sim_U (X', c, Y')$ certainly implies $b = c$ for any transitions $(X, b, Y), (X', c, Y') \in \delta(U)$. Axiom (2) of an ALTSE has been verified in Proposition 5.1.17.

Now, consider any actions $b, c \in \mathcal{L}(U)$, and suppose that $b \mathcal{I}^+(U) c$. The proofs for axioms (3), (4), and (5) are provided in the body of the proof of the Theorem of Spatially Induced Independence. Indeed, we have that $\tau_U(b)$ contains c and $\tau_U(c)$ contains b with respect to U , and these are the only assumptions that we need in order to follow the proofs for the axioms (3), (4), and (5) that are delimited in between two horizontal lines within the proof of Theorem 5.2.2.

This establishes that $(\mathcal{S}(U), \mathcal{L}(U), \delta(U), \mathcal{I}^+(U) \setminus \Delta_{\mathcal{L}(U)}, \sim_U)$ is an ALTSE. $\square$

The question remains now, why did we not simply focus on this extended SI-independence relation à priori for our first version of the Theorem of Spatially Induced Independence?

The answer is that we are far from having a functor from $[\mathcal{H}^{op}, \mathbb{T}]_{adapt}$ to $[\mathcal{H}^{op}, \mathbb{A}^\sim]$ like we had in the previous section with $\mathcal{I}$, i.e. if we use $\mathcal{I}^+$ instead. Indeed, if $(\mathcal{S}, \mathcal{L}, \delta)$ is a $\mathbb{T}$ -adapted presheaf, then it is not true that $(\mathcal{S}, \mathcal{L}, \delta, \mathcal{I}^+ \setminus \Delta, \sim)$ is a $\mathbb{A}^\sim$ -valued presheaf. The reason is that, whilst ε -regions and containing regions are stable under projections, it is seldom the case that transparent regions project properly in the presheaf structure. Indeed, we gave a counter-example to this with Example 5.1.12. This same example actually serves as a counter-example to the idea that extended SI-independence relations are preserved by LTS restriction morphisms within $\mathbb{T}$ -adapted presheaves in general. We explain how this is so here.

Example 5.3.18 . Consider the $\mathbb{T}$ -adapted presheaf on the discrete space over $P = \{x, y\}$ from Example 5.1.12. We had sets of transitions as depicted in Figure 30 for this system, and we recall that we had specified $\rho_{\{x\}}^P(b) = b$ and $\rho_{\{x\}}^P(c) = c$. We see in the figure that both the actions b and c are transparent in P , which means that $\tau_P(b) = \tau_P(c) = P$. But then, P clearly contains both b and c , so $b \mathcal{I}^+(P) c$.

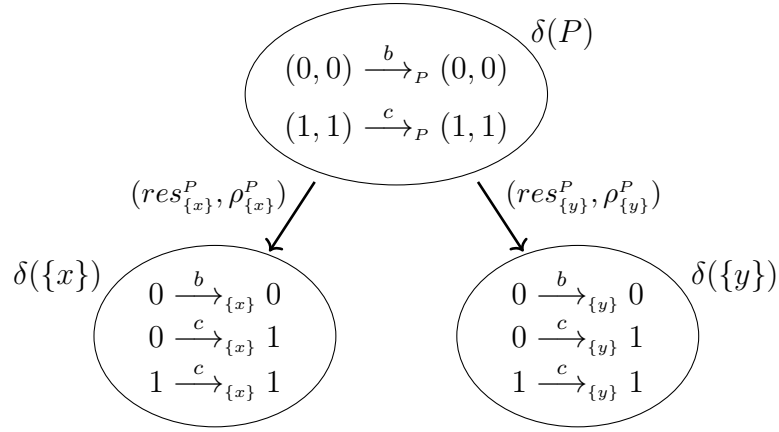


Figure 30: Specification of δ for Example 5.1.12.

We have $\tau_{\{x\}}(c) = \emptyset$ because $0 \xrightarrow{c}_{\{x\}} 1$ is an effect of c in the region $\{x\}$. Also, $b \neq \varepsilon$, so $\varepsilon_v = \varepsilon_v(b) = \emptyset$, because the empty set is the only subregion of $\{x\}$ apart from $\{x\}$. This means that $\tau_{\{x\}}(c) \vee \varepsilon_v(b) = \emptyset \neq \{x\}$, and this means that $\tau_{\{x\}}(c)$ does not contain b with respect to $\{x\}$. But this is the statement that $\tau_{\{x\}}(\rho_{\{x\}}^P(c))$ does not contain $\rho_{\{x\}}^P(b)$ with respect to $\{x\}$, and so it is not true that $\rho_{\{x\}}^P(b) \mathcal{I}^+(P) \rho_{\{x\}}^P(c)$. This shows that LTS restriction morphisms in $\mathbb{T}$ -adapted presheaves do not preserve the extended SI-independence relations in general.

Therefore, if one requires a maximal amount of SI-independent actions locally within a $\mathbb{T}$ -adapted presheaf, one can always use the extended SI-independence relations $\mathcal{I}^+(U)$. But if one requires a form of SI-independence structure that properly extends a $\mathbb{T}$ -adapted presheaf to a presheaf of MLTS with trace monoids, one might consider working with $\mathcal{I}$ instead.

We now proceed to an investigation of localized relational structures, and the latter will allow us to derive more concrete examples of $\mathbb{T}$ -adapted presheaves to work with.

Chapter 6

Localized Relational Structures

Now that the theory of SI-independence is laid out for $\mathbb{T}$ -adapted presheaves, we will need a methodical approach to efficiently construct a wide variety of such adapted presheaves to work with. When we have precise knowledge of the regions where the dependencies and the effects of actions are contained, we can make a direct specification of such actions in those regions. This is what we do with *localized relational structures* (LRS) over discrete spaces (these LRS are due to me) as we will see. These LRS are convenient to specify and serve as intermediaries in the elaboration of $\mathbb{T}$ -adapted presheaves. More precisely, we will associate one $\mathbb{T}$ -adapted presheaf to each localized relational structure. As such, we will study more examples of $\mathbb{T}$ -adapted presheaves and get a sense of how the theory of SI-independence can be applied. We will provide three examples of LRS here : Petri Net LRS, a Kings and Rooks LRS (to represent movements of chess pieces on a chessboard), and a Concurrent Unlimited Register Machine LRS.

6.1 Definition and LRS Presheaves

A localized relational structure (LRS) has a substructure, called a *localized label structure*, whose role is essentially to specify the labelling presheaf of a $\mathbb{T}$ -adapted presheaf. We investigate these substructures first, and then we will expand on LRS.

Definition 6.1.1 [Localized Label Structure]. A *localized label structure* (LLS) on a discrete space is a tuple $Q = (P, L, \psi^-, \psi^+)$ where :

1. P is a set of *places*. We refer to subsets of P as *regions* (so we will be using the discrete space associated to P);
2. L is a labelling set (an object of $\mathbf{Set}_\varepsilon$), that is, L is a set that does not contain the distinguished symbol ε . We will refer to the elements of L as *actions* (or as *labels*);
3. (P, L, ψ^-, ψ^+) is a directed multi-hypergraph in the following sense :
 - The set of vertices is given by the set of places P ;
 - The arrows are given by the labelling set L ;
 - $\psi^- : L \rightarrow \mathcal{P}(P)$, $\psi^+ : L \rightarrow \mathcal{P}(P)$ are the *incidence functions*, and for any action $b \in L$ we refer to $\psi^-(b)$ as the *input region of b* , and we refer to $\psi^+(b)$ as the *output region of b* .

In conjunction to such a localized label structure $Q = (P, L, \psi^-, \psi^+)$, we define a function $\psi : L \rightarrow \mathcal{P}(P)$ that sends a label $b \in L$ to the *proper region of b* , given by $\psi(b) = \psi^-(b) \cup \psi^+(b)$. And we will also use the following notation and terminology with respect to a region $U \subseteq P$ and $b \in L$:

- The *input region of b with respect to U* is defined as $\psi_U^-(b) := U \cap \psi^-(b)$.
- The *output region of b with respect to U* is defined as $\psi_U^+(b) := U \cap \psi^+(b)$.
- The *proper region of b with respect to U* is defined as $\psi_U(b) := U \cap \psi(b)$ ¹.

Finally, we will make use of the graph $G_Q = (\mathcal{P}(P), L, \psi^-, \psi^+)$ associated to Q , where vertices are regions, arrows are labels, and source and target maps are provided by ψ^- and ψ^+ respectively.

¹We will get these proper regions $\psi_U(b)$ to correspond with the ones defined previously in Section 5.3 when we get $\mathbb{T}$ -adapted presheaves later in this section.

Remark 6.1.2 . For a LLS as in the above definition, we have for any region U and any action $b \in L$ that $U \setminus \psi_U(b) = U \setminus \psi(b)$, $U \setminus \psi_U^+(b) = U \setminus \psi^+(b)$, and $U \setminus \psi_U^-(b) = U \setminus \psi^-(b)$. Also, for regions $V \subseteq U$, we have $\psi_V(b) \subseteq \psi_U(b) \subseteq \psi(b)$, $\psi_V^-(b) \subseteq \psi_U^-(b) \subseteq \psi^-(b)$, and $\psi_V^+(b) \subseteq \psi_U^+(b) \subseteq \psi^+(b)$.

For a given localized label structure $Q = (P, L, \psi^-, \psi^+)$, the intention behind the incidence functions is that for any action $b \in L$, the region $\psi^-(b)$ represents the region where the dependencies of b are contained as a whole, and the region $\psi^+(b)$ represents the region where the effects of b are contained. We can associate a labelling presheaf to a localized label structure, and this is realized as follows :

Definition 6.1.3 [Localized Label Structure Presheaf]. The *labelling presheaf* associated to a localized label structure $Q = (P, L, \psi^-, \psi^+)$, is the $\mathbf{Set}_\varepsilon$ -valued separated presheaf $\mathcal{L} : \mathcal{P}(P)^{op} \rightarrow \mathbf{Set}_\varepsilon$ ² as follows :

- $\mathcal{L}(U) = \{b \in L \mid \psi_U(b) \neq \emptyset\}$ for each region $U \subseteq P$, and we remark that $V \subseteq U \subseteq P$ implies that $\mathcal{L}(V) \subseteq \mathcal{L}(U)$;
- Given any regions $V \subseteq U$, we define the *labelling projection from U to V* , denoted $\rho_V^U := \mathcal{L}(V \subseteq U) : \mathcal{L}(U) \rightarrow_\varepsilon \mathcal{L}(V)$, as the relational inverse of the inclusion map from $\mathcal{L}(V)$ to $\mathcal{L}(U)$, and it is specified as follows :

$$\rho_V^U(b) = \begin{cases} b & \text{if } \psi_V(b) \neq \emptyset \\ \varepsilon & \text{otherwise} \end{cases}$$

for any $b \in \mathcal{L}(U)$.

Remark 6.1.4 . We did not impose $\psi(b) \neq \emptyset$ for an action b in a localized label structure, and a consequence is that we will not be able to physically localize an action b such that $\psi(b) = \emptyset$, i.e. $\psi(b) = \emptyset$ implies $b \notin \mathcal{L}(U)$ for any region U .

Proposition 6.1.5 . *The labelling presheaves associated to localized label structures as in Definition 6.1.3 are indeed $\mathbf{Set}_\varepsilon$ -valued separated presheaves.*

²We represent the discrete space associated to P as $\mathcal{P}(P)$, and it serves as the base space for $\mathcal{L}$.

Proof. Consider the labelling presheaf $\mathcal{L}$ of a localized label structure $Q = (P, L, \psi^-, \psi^+)$, and fix a region U . We have $\mathcal{L}(U) \subseteq L$ and ε is not an element of L , so ε is not an element of $\mathcal{L}(U)$. This means that $\mathcal{L}(U)$ is an object of $\mathbf{Set}_\varepsilon$. Also, we have for any $b \in \mathcal{L}(U)$ that $\psi_U(b) \neq \emptyset$ and we get $\rho_U^U(b) = b$. Thus, $\rho_U^U = 1_{\mathcal{L}(U)}$ is the labelling identity on $\mathcal{L}(U)$ in $\mathbf{Set}_\varepsilon$.

Now, to see that $\mathcal{L}$ is a $\mathbf{Set}_\varepsilon$ -valued presheaf, it remains to verify that the labelling projections preserve composition i.e. for any regions $W \subseteq V \subseteq U$, we have $\rho_W^V \circ \rho_V^U = \rho_W^U$.

Consider any regions $W \subseteq V \subseteq U$, and any action $b \in \mathcal{L}(U)$.

- If $\psi_V(b) = \emptyset$, then $\psi_W(b) = W \cap \psi(b) \subseteq V \cap \psi(b) = \psi_V(b) = \emptyset$, and we get $\psi_W(b) = \emptyset$ also. In particular, this means $(\rho_W^V \circ \rho_V^U)(b) = \rho_W^V(\rho_V^U(b)) = \rho_W^V(\varepsilon) = \varepsilon = \rho_W^U(b)$.
- If $\psi_V(b) \neq \emptyset$ and $\psi_W(b) = \emptyset$, then $(\rho_W^V \circ \rho_V^U)(b) = \rho_W^V(\rho_V^U(b)) = \rho_W^V(b) = \varepsilon = \rho_W^U(b)$.
- If $\psi_V(b) \neq \emptyset$ and $\psi_W(b) \neq \emptyset$, then $(\rho_W^V \circ \rho_V^U)(b) = \rho_W^V(\rho_V^U(b)) = \rho_W^V(b) = b = \rho_W^U(b)$.

This shows $\rho_W^V \circ \rho_V^U = \rho_W^U$.

We must now verify that $\mathcal{L}$ is a separated presheaf. Consider any proper cover $\{V_j\}_{j \in J}$ of a region U , and consider any two actions $b, c \in \mathcal{L}(U)$ such that $\rho_{V_j}^U(b) = \rho_{V_j}^U(c)$ for all $j \in J$.

If $\rho_{V_j}^U(b) = \varepsilon$ for all $j \in J$, then $V_j \cap \psi(b) = \psi_{V_j}(b) = \emptyset$ for all $j \in J$. This entails that $\psi_U(b) = U \cap \psi(b) = (\bigcup_{j \in J} V_j) \cap \psi(b) = \bigcup_{j \in J} (V_j \cap \psi(b)) = \emptyset$. This means that $b \notin \mathcal{L}(U)$, a contradiction.

Hence, there exists a j in J such that $\rho_{V_j}^U(b) = b = \rho_{V_j}^U(c)$, and this means $b = c$ by definition of $\rho_{V_j}^U$ applied to b . This proves that the labelling morphism $\mathcal{L}(U) \rightarrow \prod_{j \in J} \mathcal{L}(V_j)$ that arises from the product of the projections $\mathcal{L}(U) \rightarrow \mathcal{L}(V_j)$ is injective; in

particular, it is a monomorphism in $\mathbf{Set}_\varepsilon$ by Proposition 1.2.3. Thus, $\mathcal{L}$ is a separated presheaf. $\square$

And since the labelling projections ρ_V^U for the labelling presheaf of a LLS are relational inverses of inclusion maps $(\rho_V^U)^{-1}$, we get that the pairs $((\rho_V^U)^{-1}, \rho_V^U)$ form embedding-projection pairs in the following sense :

Proposition 6.1.6 [Embedding-Projection Pairs for LLS].

Given a labelling presheaf associated to a localized label signature, we have that for any regions $V \subseteq U$:

1. $\rho_V^U \circ (\rho_V^U)^{-1} = 1_{\mathcal{L}(V)}$
2. $(\rho_V^U)^{-1} \circ \rho_V^U \subseteq 1_{\mathcal{L}(U)}$ ³

Proof. An easy proof. $\square$

Independently of the formulation of $\mathbb{T}$ -adapted presheaves, we can start talking about ε -regions, containment of actions and spatially induced independence when we have a labelling presheaf of a localized label structure. Indeed, given such a $\mathbf{Set}_\varepsilon$ -valued separated presheaf $\mathcal{L}$ and a region U and an action $b \in \mathcal{L}(U)$, we can define an ε -region of b in U as a region $V \subseteq U$ such that $\rho_V^U(b) = \varepsilon$. The proof in Proposition 4.3.20, that establishes the existence of largest ε -regions, does not use the $\mathbb{T}$ -valued presheaf structure, and it works for any $\mathbf{Set}_\varepsilon$ -valued separated presheaf. As such, we get the following :

Proposition 6.1.7 [Largest ε -Region for LLS]. *Given a labelling presheaf $\mathcal{L}$ of a localized label structure, we have that the largest ε -region of an action b with respect to U is $\varepsilon_U(b) = U \setminus \psi_U(b)$.*

Proof. Consider the labelling presheaf $\mathcal{L}$ of a localized label structure, and fix a region U and an action $b \in \mathcal{L}(U)$. We have that $\psi_{U \setminus \psi_U(b)}(b) = (U \setminus \psi_U(b)) \cap \psi(b) =$

³We recall that this is inclusion for the relations that correspond to the graphs of these labelling morphisms.

$(U \setminus \psi(b)) \cap \psi(b) = \emptyset$, so $\rho_{U \setminus \psi_U(b)}^U(b) = \varepsilon$ by definition. Now, consider any region $V \subseteq U$ that is an ε -region of b in U . We have $\rho_V^U(b) = \varepsilon$, and so $V \cap \psi(b) = \emptyset$. But then, $V \subseteq \overline{\psi(b)}$, and we get $V \subseteq U \cap \overline{\psi(b)} = U \setminus \psi(b) = U \setminus \psi_U(b)$. This proves that $U \setminus \psi_U(b)$ is the largest ε -region of b with respect to U . $\square$

Of course, with $\psi_U(b) = U \setminus \varepsilon_U(b)$ as in the above proposition, we will be able to say $\psi_U(b)$ is the smallest containing region of b with respect to U , as long as we can elaborate a $\mathbb{T}$ -adapted presheaf for which the labelling presheaf is given by a localized label structure. In the same sense, we will be able to provide an SI-independence relation $\mathcal{I}$ that uses spatial disjointness of actions as a criterion. However, we give a definition for $\mathcal{I}$ right away because the latter's definition only relies on the structure of $\mathcal{L}$ once it is set in place. And there is another form of SI-independence, denoted $\mathcal{I}^\circ$, that we can use with respect to LLS also. These SI-independence relations are defined as follows :

Definition 6.1.8 [SI-Independence Relations for LLS]. Given a labelling presheaf $\mathcal{L}$ of a localized label structure and a region U , we define two binary relations $\mathcal{I}(U)$ and $\mathcal{I}^\circ(U)$ on $\mathcal{L}(U)$ as follows :

1. $b \mathcal{I}(U) c \Leftrightarrow \psi_U(b) \cap \psi_U(c) = \emptyset$
2. $b \mathcal{I}^\circ(U) c \Leftrightarrow (\psi_U^+(b) \cap \psi_U(c)) \cup (\psi_U^+(c) \cap \psi_U(b)) = \emptyset$

Given a labelling presheaf $\mathcal{L}$ for a localized label structure and a region U , it is not too difficult to see that $\mathcal{I}(U) \subseteq \mathcal{I}^\circ(U)$ if we recall that $\psi_U(b) = \psi_U^+(b) \cup \psi_U^-(b)$ for any action $b \in \mathcal{L}(U)$.

We now proceed to the definition of localized relational structures.

Definition 6.1.9 [Localized Relational Structure]. A *localized relational structure* (LRS) (on a discrete space) is a tuple $\Gamma = (P, L, \psi^-, \psi^+, \mathcal{S}, \langle - \rangle)$ where :

1. $Q = (P, L, \psi^-, \psi^+)$ is a localized label structure.
2. $\mathcal{S}$ is a set-valued sheaf on the discrete space over P called the state space sheaf of Γ .
3. $\langle - \rangle : G_Q \rightarrow Rel$ is a relational diagram of shape G_Q ⁴ in Rel such that $\langle U \rangle = \mathcal{S}(U)$ for each $U \subseteq P$. We refer to $\langle - \rangle$ as the *pre-relational diagram* of Γ , and it is called that way because it will serve as a basis on which the relational diagrams of a $\mathbb{T}$ -adapted presheaf can be specified.

Remark 6.1.10 . For a set-valued sheaf $\mathcal{S}$ on a discrete space over a set P , we get that for any region $U \subseteq P$, $\mathcal{S}(U)$ arises as a product of the state spaces associated to each point $p \in U$, i.e. $\mathcal{S}(U) \cong \prod_{p \in U} \mathcal{S}(\{p\})$. We can think of the elements of $\prod_{p \in U} \mathcal{S}(\{p\})$ as functions of the form $X : U \rightarrow \bigcup_{p \in U} \mathcal{S}(\{p\})$ where $X(p) \in \mathcal{S}(\{p\})$ for each $p \in U$. Thus, we will represent the elements of $\mathcal{S}(U)$ as functions of this form. The restriction map for regions $V \subseteq U$ in $\mathcal{S}$ are then simply given by applying a function restriction on a map $X \in \mathcal{S}(U)$, i.e. $res_V^U(X) = X|_V$ is the restriction of X to the set V .

Now, given a localized relational structure $\Gamma = (P, L, \psi^-, \psi^+, \mathcal{S}, \langle - \rangle)$, we will think of $\langle b \rangle : \mathcal{S}(\psi^-(b)) \rightarrow \mathcal{S}(\psi^+(b))$ as specifying b 's transition relation precisely where it is contained as a whole, with its dependencies in $\psi^-(b)$ and its effects in $\psi^+(b)$.

Before we associate $\mathbb{T}$ -adapted presheaves to localized relational structures, we start by describing the relational diagrams that are associated to the regions of such presheaves as follows :

⁴Recall that $G_Q = (\mathcal{P}(P), L, \psi^-, \psi^+)$ is the graph associated to Q from Definition 6.1.1.

Definition 6.1.11 [Relational Diagrams for Transition Relations in a LRS].

Given a localized relational structure $\Gamma = (P, L, \psi^-, \psi^+, \mathcal{S}, \langle - \rangle)$ and a region U in $\mathcal{P}(P)$, we define the *relational diagram associated to U in Γ* , denoted $\xrightarrow{(-)}_U$: $(\{\star\}, \mathcal{L}(U)) \rightarrow Rel$ ⁵, as follows :

- $\xrightarrow{\star}_U = \mathcal{S}(U)$
- For $b \in \mathcal{L}(U)$ and $X, Y \in \mathcal{S}(U)$, we let $X \xrightarrow{b}_U Y$ if and only if :
 - (1) $X|_{U \setminus \psi_U^+(b)} = Y|_{U \setminus \psi_U^+(b)}$
 - (2) $\exists (X', Y') \in \langle b \rangle$ such that $X|_{\psi_U^-(b)} = X'|_{\psi_U^-(b)}$ and $Y|_{\psi_U^+(b)} = Y'|_{\psi_U^+(b)}$

Condition (1) above says that, outside of the region where b has its effects in U , b must be transparent (indeed, the effects of b are contained in $\psi^+(b)$ as we will see). Condition (2) says that $\xrightarrow{b}_U$ operates precisely as $\langle b \rangle$ would, in the region where it intersects with the input and output regions of b in U . $\xrightarrow{b}_U$ is completely determined by $\langle b \rangle$ this way.

And we obtain a $\mathbb{T}$ -adapted presheaf for each LRS as follows :

Definition 6.1.12 [Presheaves for LRS]. Let $\Gamma = (P, L, \psi^-, \psi^+, \mathcal{S}, \langle - \rangle)$ be a localized relational structure. The *LRS presheaf* of Γ is a $\mathbb{T}$ -adapted presheaf given by the functor $\mathcal{T}_\Gamma = (\mathcal{S}_\Gamma, \mathcal{L}_\Gamma, \delta_\Gamma) : \mathcal{P}(P)^{op} \rightarrow \mathbb{T}$, where :

1. $\mathcal{S}_\Gamma$ is $\mathcal{S}$, which is the state space sheaf of Γ as given in Definition 6.1.9;
2. $\mathcal{L}_\Gamma$ is the labelling presheaf of the localized label structure (P, L, ψ^-, ψ^+) as provided in Definition 6.1.3;
3. For a region $U \subseteq P$, $\delta_\Gamma(U)$ is specified precisely by the relational diagram $\xrightarrow{(-)}_U$ of Γ in U as in Definition 6.1.11.

For any regions $V \subseteq U$, we get a well-defined labelled transition system $\mathcal{T}_\Gamma(U) = (\mathcal{S}_\Gamma(U), \mathcal{L}_\Gamma(U), \delta_\Gamma(U))$, and the LTS restriction morphism from $\mathcal{T}_\Gamma(U)$ to $\mathcal{T}_\Gamma(V)$ is

⁵ $(\{\star\}, \mathcal{L}(U))$ is a graph with a single vertex $\star$ and arrows $\mathcal{L}(U)$.

given by the pair (res_V^U, ρ_V^U) where res_V^U is the restriction map from $\mathcal{S}_\Gamma(U)$ to $\mathcal{S}_\Gamma(V)$ in $\mathcal{S}$, and ρ_V^U is the labelling projection from $\mathcal{L}_\Gamma(U)$ to $\mathcal{L}_\Gamma(V)$ in $\mathcal{L}$.

Theorem 6.1.13 . *The LRS presheaves (as in Definition 6.1.12) are indeed $\mathbb{T}$ -adapted presheaves.*

Proof. Consider any localized relational structure $\Gamma = (P, L, \psi^-, \psi^+, \mathcal{S}, \langle - \rangle)$. It is clear that $\mathcal{T}_\Gamma(U)$ is a well-defined LTS for any region U in $\mathcal{P}(P)$. Consider any regions $V \subseteq U$. We verify that the restriction morphism (res_V^U, ρ_V^U) is indeed a LTS morphisms. Consider any action $b \in \mathcal{L}(U)$ and any transition $X \xrightarrow{b}_U Y$ in $\mathcal{T}(U)$. Then $X|_{U \setminus \psi^+(b)} = Y|_{U \setminus \psi^+(b)}$, and we get an $(X', Y') \in \langle b \rangle$ such that $X|_{\psi_U^-(b)} = X'|_{\psi_U^-(b)}$ and $Y|_{\psi_U^+(b)} = Y'|_{\psi_U^+(b)}$. Also, since $b \in \mathcal{L}(U)$, we have that $\psi_U(b) \neq \emptyset$.

If $\rho_V^U(b) = \varepsilon$, then $\psi_V(b) = \emptyset$, and we must verify that $X|_V = Y|_V$. Since $X \xrightarrow{b}_U Y$, the first condition above provides $X|_{U \setminus \psi^+(b)} = Y|_{U \setminus \psi^+(b)}$. But then, $V \cap \psi(b) = \psi_V(b) = \emptyset$, and this means $V \subseteq U \setminus \psi_U(b)$. But $\psi_U^+(b) \subseteq \psi_U(b)$, and so $U \setminus \psi_U(b) \subseteq U \setminus \psi_U^+(b)$. Thus, with $V \subseteq U \setminus \psi_U^+(b)$, we get $X|_V = Y|_V$.

If $\rho_V^U(b) = b$, then we must verify that $X|_V \xrightarrow{b}_V Y|_V$. In this case, we get $V \setminus \psi^+(b) \subseteq U \setminus \psi^+(b)$, and since $X|_{U \setminus \psi^+(b)} = Y|_{U \setminus \psi^+(b)}$, we get $(X|_V)|_{V \setminus \psi^+(b)} = X|_{V \setminus \psi^+(b)} = Y|_{V \setminus \psi^+(b)} = (Y|_V)|_{V \setminus \psi^+(b)}$. This verifies the first condition for the transition $X|_V \xrightarrow{b}_V Y|_V$ to occur. We have $\psi_V^-(b) \subseteq \psi_U^-(b)$ and $\psi_V^+(b) \subseteq \psi_U^+(b)$, and with the (X', Y') in $\langle b \rangle$ as given above, we get that :

$$\begin{aligned} (X|_V)|_{\psi_V^-(b)} &= X|_{\psi_V^-(b)} = (X|_{\psi_U^-(b)})|_{\psi_V^-(b)} = (X'|_{\psi_U^-(b)})|_{\psi_V^-(b)} = X'|_{\psi_V^-(b)} \quad \text{and} \\ (Y|_V)|_{\psi_V^+(b)} &= Y|_{\psi_V^+(b)} = (Y|_{\psi_U^+(b)})|_{\psi_V^+(b)} = (Y'|_{\psi_U^+(b)})|_{\psi_V^+(b)} = Y'|_{\psi_V^+(b)} \end{aligned}$$

Thus, $X|_V \xrightarrow{b}_V Y|_V$ holds by definition, i.e. b has a local transition from the state $X|_V$ to the state $Y|_V$ in V .

We must now verify that $\mathcal{T}_\Gamma$ is a $\mathbb{T}$ -adapted presheaf. By definition, $\mathcal{S}_\Gamma$ is a sheaf. Also, from Proposition 6.1.5, we know that $\mathcal{L}_\Gamma$ is a separated presheaf. It remains to

check that the WCA axiom holds.

Consider any region U , and any action $b \in \mathcal{L}(U)$. Consider any region V that j -complements $\varepsilon_U(b)$ in U . We cannot have $\rho_V^U(b) = \varepsilon$ because, in this case, we get $V \cap \psi(b) = \emptyset$ and, with $\rho_{\varepsilon_U(b)}^U(b) = \varepsilon$, we get $\varepsilon_U(b) \cap \psi(b) = \emptyset$. This yields $\psi_U(b) = U \cap \psi(b) = (V \cup \varepsilon_U(b)) \cap \psi(b) = (V \cap \psi(b)) \cup (\varepsilon_U(b) \cap \psi(b)) = \emptyset$, and this contradicts $b \in \mathcal{L}(U)$. Thus, $\rho_V^U(b) = b$.

We need to show that V contains b with respect to U . By Proposition 4.3.27 and $\rho_V^U(b) = b$, it suffices to verify that :

$$\xrightarrow{b}_U = ((res_V^U)^{-1} \circ \xrightarrow{b}_V \circ res_V^U) \cap \simeq_{\varepsilon_U(b)}^U \quad (2)$$

We recall that the inclusion from left to right is already provided for the above equation, and we simply verify the inclusion from right to left.

Let X and Y be states in $\mathcal{S}(U)$ such that $X|_V \xrightarrow{b}_V Y|_V$ and $X \simeq_{\varepsilon_U(b)}^U Y$. By definition of $\xrightarrow{b}_V$, we get $(X|_V)|_{V \setminus \psi_V^+(b)} = (Y|_V)|_{V \setminus \psi_V^+(b)}$, and we get $(X', Y') \in \langle b \rangle$ such that $(X|_V)|_{\psi_V^-(b)} = X'|_{\psi_V^-(b)}$ and $(Y|_V)|_{\psi_V^+(b)} = Y'|_{\psi_V^+(b)}$. But then, we get $X|_{V \setminus \psi_V(b)} = Y|_{V \setminus \psi_V(b)}$, $X|_{\psi_V^-(b)} = X'|_{\psi_V^-(b)}$ and $Y|_{\psi_V^+(b)} = Y'|_{\psi_V^+(b)}$. We remark that

$$\varepsilon_U(b) \setminus \psi_U^+(b) = (U \setminus \psi_U(b)) \setminus \psi_U^+(b) = (U \setminus \psi_U(b)) = \varepsilon_U(b)$$

Then, we get :

$$\begin{aligned} U \setminus \psi_U^+(b) &= (V \cup \varepsilon_U(b)) \setminus \psi_U^+(b) = (V \setminus \psi_U^+(b)) \cup (\varepsilon_U(b) \setminus \psi_U^+(b)) \\ &= (V \setminus \psi_U^+(b)) \cup \varepsilon_U(b) = (V \setminus \psi_U^+(b)) \cup \varepsilon_U(b) \\ &= (V \setminus \psi^+(b)) \cup \varepsilon_U(b) = (V \setminus \psi_V^+(b)) \cup \varepsilon_U(b) \end{aligned}$$

Furthermore, we have :

$$(X|_{U \setminus \psi_U^+(b)})|_{V \setminus \psi_V^+(b)} = X|_{V \setminus \psi_V^+(b)} = Y|_{V \setminus \psi_V^+(b)} = (Y|_{U \setminus \psi_U^+(b)})|_{V \setminus \psi_V^+(b)}$$

and

$$(X|_{U \setminus \psi_U^+(b)})|_{\varepsilon_U(b)} = X|_{\varepsilon_U(b)} = Y|_{\varepsilon_U(b)} = (Y|_{U \setminus \psi_U^+(b)})|_{\varepsilon_U(b)}$$

Since $\{V \setminus \psi_V^+(b), \varepsilon_U(b)\}$ covers $U \setminus \psi_U^+(b)$, we get that $X|_{U \setminus \psi_U^+(b)} = Y|_{U \setminus \psi_U^+(b)}$ by locality of $\mathcal{S}$. This verifies the first condition for $X \xrightarrow{b}_U Y$ to hold.

For the second condition, we have $\psi_U^-(b) = U \cap \psi^-(b) = (V \cup \varepsilon_U(b)) \cap \psi^-(b) = (V \cap \psi^-(b)) \cup (\varepsilon_U(b) \cap \psi^-(b)) = \psi_V^-(b)$ because $\varepsilon_U(b) \cap \psi^-(b) = \emptyset$, and similarly, $\psi_U^+(b) = \psi_V^+(b)$. But then, we get $X|_{\psi_V^-(b)} = X'|_{\psi_V^-(b)}$ and $Y|_{\psi_V^+(b)} = Y'|_{\psi_V^+(b)}$ with $(X', Y') \in \langle b \rangle$. Thus, $X \xrightarrow{b}_U Y$ holds.

This proves the inclusion from right to left in Equation 2 above, and thus we get equality. This establishes that V contains b with respect to U . $\square$

And we should now verify that the ψ , ψ^- , and ψ^+ play their respective roles as far as containing dependencies and effects of an action goes in these LRS presheaves.

Proposition 6.1.14 . *Given a LRS presheaf and a region U , and an action $b \in \mathcal{L}(U)$, we have that the following hold :*

1. $\psi_U(b)$ as given in Definition 6.1.1 is indeed a proper region of b with respect to U in the sense of Definition 4.3.15.
2. $\psi_U^+(b)$ as given in Definition 6.1.1 is a region that contains the effects of b with respect to U in the sense of Definition 4.3.1.
3. $\psi_U^-(b)$ as given in Definition 6.1.1 is a region that contains the dependencies of b with respect to U in the sense of Definition 4.3.7.

Proof. The proof is provided in Appendix G $\square$

Proposition 6.1.15 . *Given a LRS presheaf, we have that $\mathcal{I}^\circ = \mathcal{I}^+$ if and only if*

For any region U and any action $b \in \mathcal{L}(U)$, we have $\tau_U(b) = U \setminus \psi_U^+(b)$.

Proof. Consider a LRS presheaf, a region U , and any actions $b, c \in \mathcal{L}(U)$:

$$\begin{aligned}
 b \mathcal{I}^\circ(U) c &\Leftrightarrow (\psi_U^+(b) \cap \psi_U(c)) \cup (\psi_U^+(c) \cap \psi_U(b)) = \emptyset \\
 &\Leftrightarrow \psi_U^+(b) \cap \psi_U(c) = \emptyset \text{ and } \psi_U^+(c) \cap \psi_U(b) = \emptyset \\
 &\Leftrightarrow \psi_U(c) \subseteq U \setminus \psi_U^+(b) = \tau_U(b) \text{ and } \psi_U(b) \subseteq U \setminus \psi_U^+(c) = \tau_U(c) \\
 &\Leftrightarrow \tau_U(b) \text{ contains } c \text{ and } \tau_U(c) \text{ contains } b
 \end{aligned}$$

□

Remark 6.1.16 . These cases where $\tau_U(b) \neq U \setminus \psi_U^+(b)$ for some region U and action $b \in \mathcal{L}(U)$ could technically be avoided, because $\psi_U^+(b)$ was designed precisely to contain the effects of b with respect to U . That is, we could probably impose simple conditions in the definition of a LRS to force $\tau_U(b) = U \setminus \psi_U^+(b)$ for any region U and action $b \in \mathcal{L}(U)$.

We end this section with the definition of LRS for Petri Nets.

Example 6.1.17 [LRS for Petri Nets]. Given a Petri Net (P, T, F, M_0, W) , we can define a localized relational structure $\Gamma = (P, L, \psi^-, \psi^+, \mathcal{S}, \langle - \rangle)$ where :

- The set of places P are matched in both structures;
- The labelling set L is the set of Petri Net transitions T (not to be confused with LTS transitions);
- $\psi^- = \psi^+ = \psi$ where $\psi : T \rightarrow \mathcal{P}(P)$ is a map that was previously defined for Petri Nets as $\psi(t) = \{ x \in P \mid \exists t \in T, (x, t) \in F \text{ or } (t, x) \in F \}$ for any $t \in L$ (this is the set of places to which t connects with an arc);
- $\mathcal{S}$ is given by the states sheaf of the Petri Net's associated presheaf as we saw in Definition 4.1.11 ($\mathcal{S}(U) = \mathbb{N}^U$ for each $U \subseteq P$, and the state restriction maps are function restrictions);
- For an action $t \in L$, and any $X', Y' \in \mathcal{S}(\psi(t))$ we set $(X', Y') \in \langle t \rangle$ if and only if $\forall x \in \psi(t)$, the following hold :

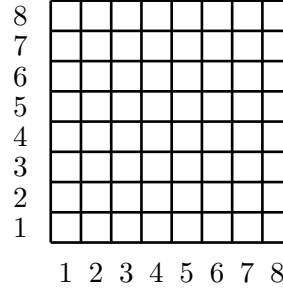
- (1) $(x, t) \in F \Rightarrow X'(x) \geq W(x, t)$
- (2) $(x, t) \in F$ and $(t, x) \notin F \Rightarrow Y'(x) = X'(x) - W(x, t)$
- (3) $(x, t) \notin F$ and $(t, x) \in F \Rightarrow Y'(x) = X'(x) + W(t, x)$
- (4) $(x, t) \in F$ and $(t, x) \in F \Rightarrow Y'(x) = X'(x) - W(x, t) + W(t, x)$

This provides a well-defined localized relational structure Γ . In Appendix G, we provide a proof that the presheaf associated to the Petri Net (P, T, F, M_0, W) as in Definition 4.1.11 corresponds with the $\mathbb{T}$ -adapted presheaf $\mathcal{T}_\Gamma = (\mathcal{S}_\Gamma, \mathcal{L}_\Gamma, \delta_\Gamma)$ of the localized relational structure Γ above.

We now investigate, in full detail, two examples of LRS and their associated presheaves. These examples will be a bit more visual, and we will see more clearly how we can separate the input and output regions to provide efficient $\mathcal{I}^\circ$ independence.

6.2 Kings and Rooks on a Chessboard

Consider an 8x8 chessboard (with only white squares) as follows :



We build a localized relational structure $Q = (P, L, \psi^-, \psi^+, \mathcal{S}, \langle - \rangle)$ as follows :

1. The set of places P is given by the set of tiles of the chessboard. We identify these tiles through their xy -coordinates, i.e. $P = \{1, 2, \dots, 8\}^2$. We write $\binom{x}{y} \in P$ with x representing the x -coordinate and y representing the y -coordinate of the tile, and we get, for example, that $\binom{5}{3} \in P$ represents the black tile on the board in Figure 31.

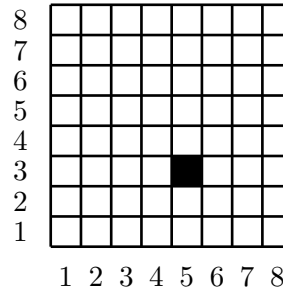


Figure 31: The black tile represents the place $\binom{5}{3} \in P$ on the chessboard.

2. The labelling set is given by pairs of distinct tiles that are horizontally aligned or vertically aligned, i.e. $L = \{[\binom{x}{y}, \binom{x'}{y}] \mid x, x', y \in \{1, \dots, 8\} \text{ and } x \neq x'\} \cup \{[\binom{x}{y}, \binom{x}{y'}] \mid x, y, y' \in \{1, \dots, 8\} \text{ and } y \neq y'\}$. These correspond precisely with vertical and horizontal arrows on the chessboard such as the following :

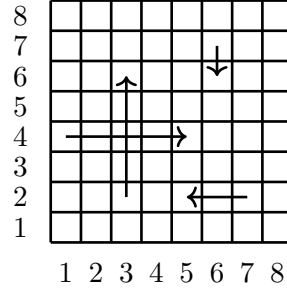


Figure 32: Arrows for the actions $[(\binom{1}{4}, \binom{5}{4})]$, $[(\binom{3}{2}, \binom{3}{6})]$, $[(\binom{6}{7}, \binom{6}{6})]$, and $[(\binom{7}{2}, \binom{5}{2})]$.

Thus, we refer to the actions in L as arrows, and the arrow $[(\binom{x}{y}, \binom{x'}{y'})]$ represents a movement that a king or a rook may effectuate, by moving from the tile $\binom{x}{y}$ to the tile $\binom{x'}{y'}$.

3. The states are provided by assigning rooks (designated r), kings (designated k) or no chess pieces (designated 0) to each tile, and we get $\mathcal{S}(U) = \{0, r, k\}^U$ where $U \subseteq P$. The restriction maps for these state spaces are provided by function restrictions.

We can consider an example of a global state $X \in \mathcal{S}(P)$ as follows :

$$X\left(\binom{x}{y}\right) = \begin{cases} r & \text{if } \binom{x}{y} \in \{(\binom{6}{7}, \binom{5}{2}), (\binom{3}{4})\} \\ k & \text{if } \binom{x}{y} \in \{(\binom{2}{7}, \binom{6}{5})\} \\ 0 & \text{otherwise} \end{cases}$$

for any tile $\binom{x}{y} \in P$. And this X can be represented on the chessboard as in Figure 33.

4. Input regions : To effectuate a motion given by $[(\binom{x}{y}, \binom{x'}{y'})]$, we must verify the nature of the chess piece to be displaced (i.e. we need to check the tile $\binom{x}{y}$), and we must make sure that the whole trajectory up to the tile where it lands (that is $\binom{x'}{y'}$) is clear. Whether $\binom{x'}{y'}$ has a chess piece or not does not matter; the chess piece that is moved from $\binom{x}{y}$ to $\binom{x'}{y'}$ will eat the chess piece that is there if there is any. This establishes the regions where the dependencies of an

8							
7		<i>k</i>				<i>r</i>	
6							
5					<i>k</i>		
4			<i>r</i>				
3							
2					<i>r</i>		
1							
	1	2	3	4	5	6	7

Figure 33: Representation of a state on the chessboard.

action are contained, and we declare ψ^- in terms of that.

For arrows of the form $[(\frac{x}{y}), (\frac{x'}{y})]$ where $x \neq x'$:

If $x \leq x'$, then $\psi^-([\frac{x}{y}), (\frac{x'}{y})] = \{(\frac{x_0}{y}) \in P \mid x \leq x_0 < x'\}$.

If $x \geq x'$, then $\psi^-([\frac{x}{y}), (\frac{x'}{y})] = \{(\frac{x_0}{y}) \in P \mid x \geq x_0 > x'\}$.

The above basically provide horizontal trajectories that includes everything from the tile $(\frac{x}{y})$ to the tile $(\frac{x'}{y})$, but that exclude $(\frac{x'}{y})$.

For arrows of the form $[(\frac{x}{y}), (\frac{x}{y'})]$ where $y \neq y'$:

If $y \leq y'$, then $\psi^-([\frac{x}{y}), (\frac{x}{y'})] = \{(\frac{x}{y_0}) \in P \mid y \leq y_0 < y'\}$.

If $y \geq y'$, then $\psi^-([\frac{x}{y}), (\frac{x}{y'})] = \{(\frac{x}{y_0}) \in P \mid y \geq y_0 > y'\}$.

These basically provide vertical trajectories that includes everything from the tile $(\frac{x}{y})$ to the tile $(\frac{x}{y'})$, but that exclude $(\frac{x}{y'})$.

5. Output regions : $\psi^+([\frac{x}{y}), (\frac{x'}{y'})] = \{(\frac{x}{y}), (\frac{x'}{y'})\}$, i.e. when a rook or a king moves from $(\frac{x}{y})$ to $(\frac{x'}{y'})$, an effect is created in the tile where the chess piece initiates its movement in $(\frac{x}{y})$ (the piece r or k is replaced with 0 there), and an effect

is created where the piece lands in $(\frac{x'}{y'})$ (the movement simply crushes the old value to be found in that tile with the one of the piece that is moved).

Thus, for example, we can consider the arrow $[(\frac{2}{5}), (\frac{6}{5})]$ as depicted on the left chessboard in Figure 34. The right chessboard in Figure 34 displays $\psi^-([(\frac{2}{5}), (\frac{6}{5})]) = \{(\frac{2}{5}), (\frac{3}{5}), (\frac{4}{5}), (\frac{5}{5})\}$ with minus signs on the corresponding tiles, and $\psi^+([(\frac{2}{5}), (\frac{6}{5})]) = \{(\frac{2}{5}), (\frac{6}{5})\}$ with plus signs on the corresponding tiles. We get that the proper region of $[(\frac{2}{5}), (\frac{6}{5})]$ is then given by $\psi([(\frac{2}{5}), (\frac{6}{5})]) = \psi^-([(\frac{2}{5}), (\frac{6}{5})]) \cup \psi^+([(\frac{2}{5}), (\frac{6}{5})]) = \{(\frac{2}{5}), (\frac{3}{5}), (\frac{4}{5}), (\frac{5}{5}), (\frac{6}{5})\}$, which is the area covered by the arrow as a whole.

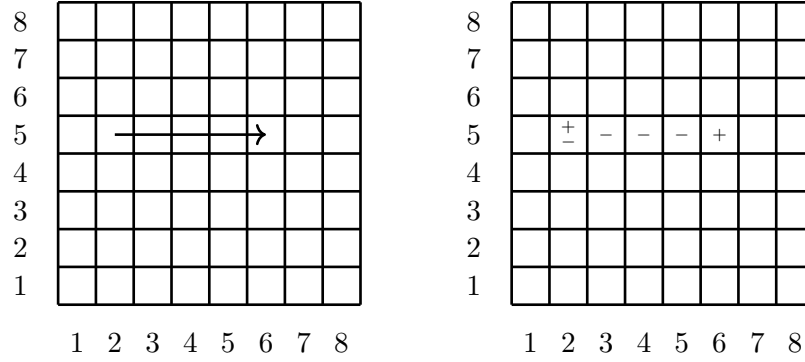


Figure 34: Input region (minus signs) and output region (plus signs) for $[(\frac{2}{5}), (\frac{6}{5})]$

6. Pre-Relational diagram : $\langle [(\frac{x}{y}), (\frac{x'}{y'})] \rangle$ for $[(\frac{x}{y}), (\frac{x'}{y'})] \in L$ is defined as follows :

For any $X \in \mathcal{S}(\psi^-([(\frac{x}{y}), (\frac{x'}{y'})]))$ and $Y \in \mathcal{S}(\psi^+([(\frac{x}{y}), (\frac{x'}{y'})]))$, we have :

$(X, Y) \in \langle [(\frac{x}{y}), (\frac{x'}{y'})] \rangle \Leftrightarrow$ the following hold :

- (a) $|x - x'| + |y - y'| > 1 \Rightarrow X(\frac{x}{y}) = r$
- (b) $|x - x'| + |y - y'| = 1 \Rightarrow X(\frac{x}{y}) \in \{r, k\}$
- (c) $X(\frac{x_0}{y_0}) = 0$ for all $(\frac{x_0}{y_0}) \in \psi^-([(\frac{x}{y}), (\frac{x'}{y'})]) \setminus \{(\frac{x}{y})\}$

$$(d) \ Y\binom{x}{y} = 0$$

$$(e) \ Y\binom{x'}{y'} = X\binom{x}{y}$$

The first equation says that if $\binom{x}{y}$ is more than one tile away from $\binom{x'}{y'}$, then only a rook can effectuate the movement corresponding to $[\binom{x}{y}, \binom{x'}{y'}]$. The second equation takes care of the other possible case, where the tiles $\binom{x}{y}$ and $\binom{x'}{y'}$ are adjacent, and it says that the action $[\binom{x}{y}, \binom{x'}{y'}]$ can be effectuated by a rook or a king in such circumstances. The third equation says that the tiles in between $\binom{x}{y}$ and $\binom{x'}{y'}$ must have no chess pieces on them to interfere if the movement is to be effectuated. The fourth equation says that we must remove the chess piece from the tile $\binom{x}{y}$ where the movement is initiated, and the fifth equation says that the chess piece that was on $\binom{x}{y}$ is moved to $\binom{x'}{y'}$ when $[\binom{x}{y}, \binom{x'}{y'}]$ fires.

All of the above provide a well-defined LRS. Setting the actions $a_1 = [\binom{2}{4}, \binom{7}{4}]$, $a_2 = [\binom{4}{2}, \binom{4}{7}]$, $a_3 = [\binom{5}{7}, \binom{4}{7}]$, and $a_4 = [\binom{6}{3}, \binom{6}{4}]$ (these are depicted as arrows in Figure 35), we can give an example of a global run (in P) on the chessboard as in Figure 36.

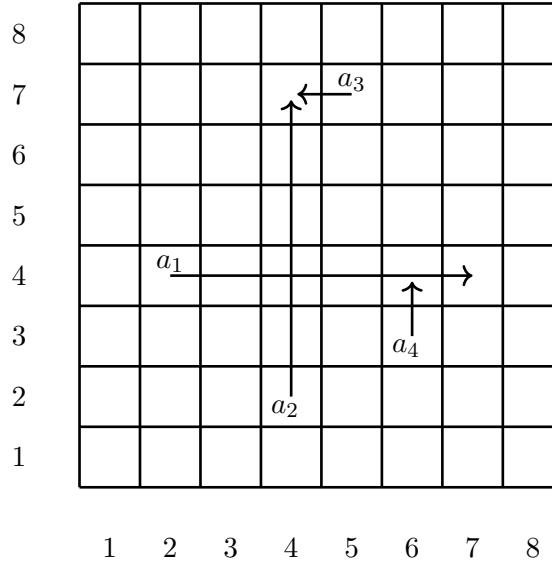


Figure 35: Representation of the actions $a_1 = [\binom{2}{4}, \binom{7}{4}]$, $a_2 = [\binom{4}{2}, \binom{4}{7}]$, $a_3 = [\binom{5}{7}, \binom{4}{7}]$, and $a_4 = [\binom{6}{3}, \binom{6}{4}]$ for $[\binom{2}{5}, \binom{6}{5}]$.

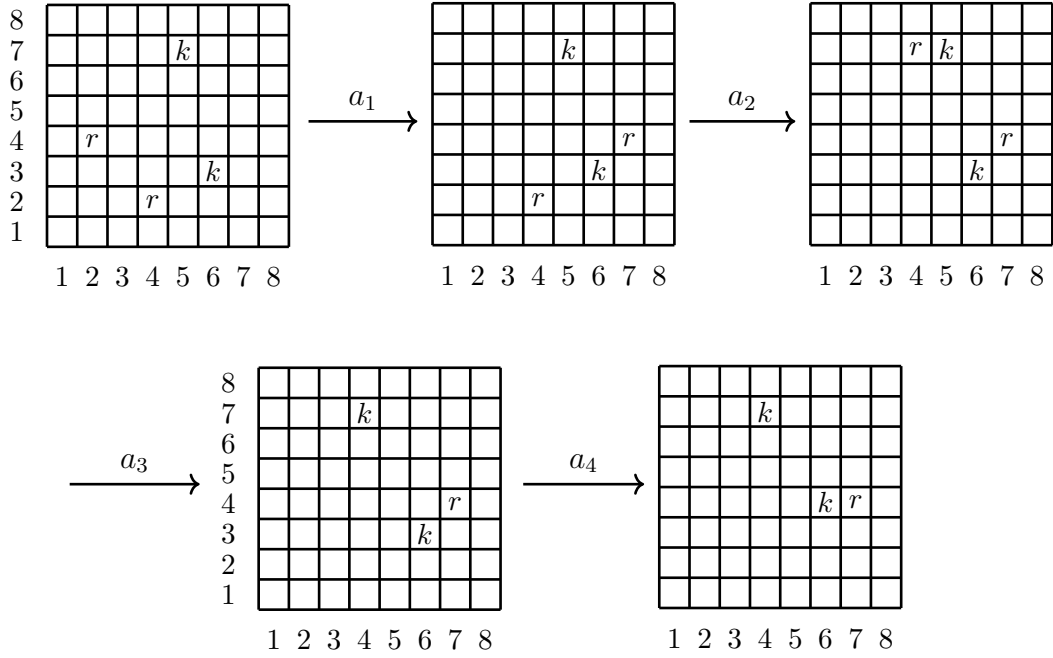


Figure 36: A global run on the chessboard.

With respect to the actions a_1, a_2, a_3, a_4 , we get that $\mathcal{I}$ gives the following pairs of independent actions :

- $a_1 \mathcal{I}(P) a_3$
- $a_2 \mathcal{I}(P) a_4$
- $a_3 \mathcal{I}(P) a_4$

and these basically correspond with pairs of actions whose arrows do not cross in Figure 35. For the run given in Figure 36, we can visualize all equivalent $\mathcal{T}(P)$ -runs with respect to the equivalence on runs that arises from $\mathcal{I}(P)$, and they are represented within a single diagram in Figure 37. We can see two $\mathcal{I}(P)$ -squares in the diagram in particular.

With respect to the actions a_1, a_2, a_3, a_4 and the SI-independence relations given

by $\mathcal{I}^\circ$ we get an extra pair of independent actions, which is $a_1 \mathcal{I}^\circ(P) a_2$. The reason we get independence here is that, whilst the arrows of a_1 and a_2 cross (so $\mathcal{I}$ did not perceive independence), we have that the tips of each of these arrows do not touch the proper region of the other arrow. This is precisely the condition for $a_1 \mathcal{I}^\circ(P) a_2$ to hold. We can derive the set of all equivalent $\mathcal{T}(P)$ -runs with respect to $\mathcal{I}^\circ(P)$ (depicted in Figure 38), and it is more informative than the previous set of equivalences (with $\mathcal{I}(P)$). This set of runs carries four $\mathcal{I}(P)^\sim$ -squares as we can see.

We have that $\mathcal{I}^\circ = \mathcal{I}^+$ in fact for this system. This is because the largest transparent region of an action excludes the tips of this action's arrow precisely, and these tips provide the output region of such an action. Thus, we get for any region U and any action $b \in \mathcal{L}(U)$:

$$\tau_U(b) = U \setminus \psi_U^+(b)$$

Thus, by Remark 6.1.15, we get that $\mathcal{I}^\circ = \mathcal{I}^+$. This means in particular, that the equivalent runs that are given in Figure 37 and Figure 38 demonstrate the contrast that exists between the use of $\mathcal{I}$ and the use of $\mathcal{I}^+$. Indeed, $\mathcal{I}^+$ is a finer relation and it is one that we can definitely put to good use in LRS in general.

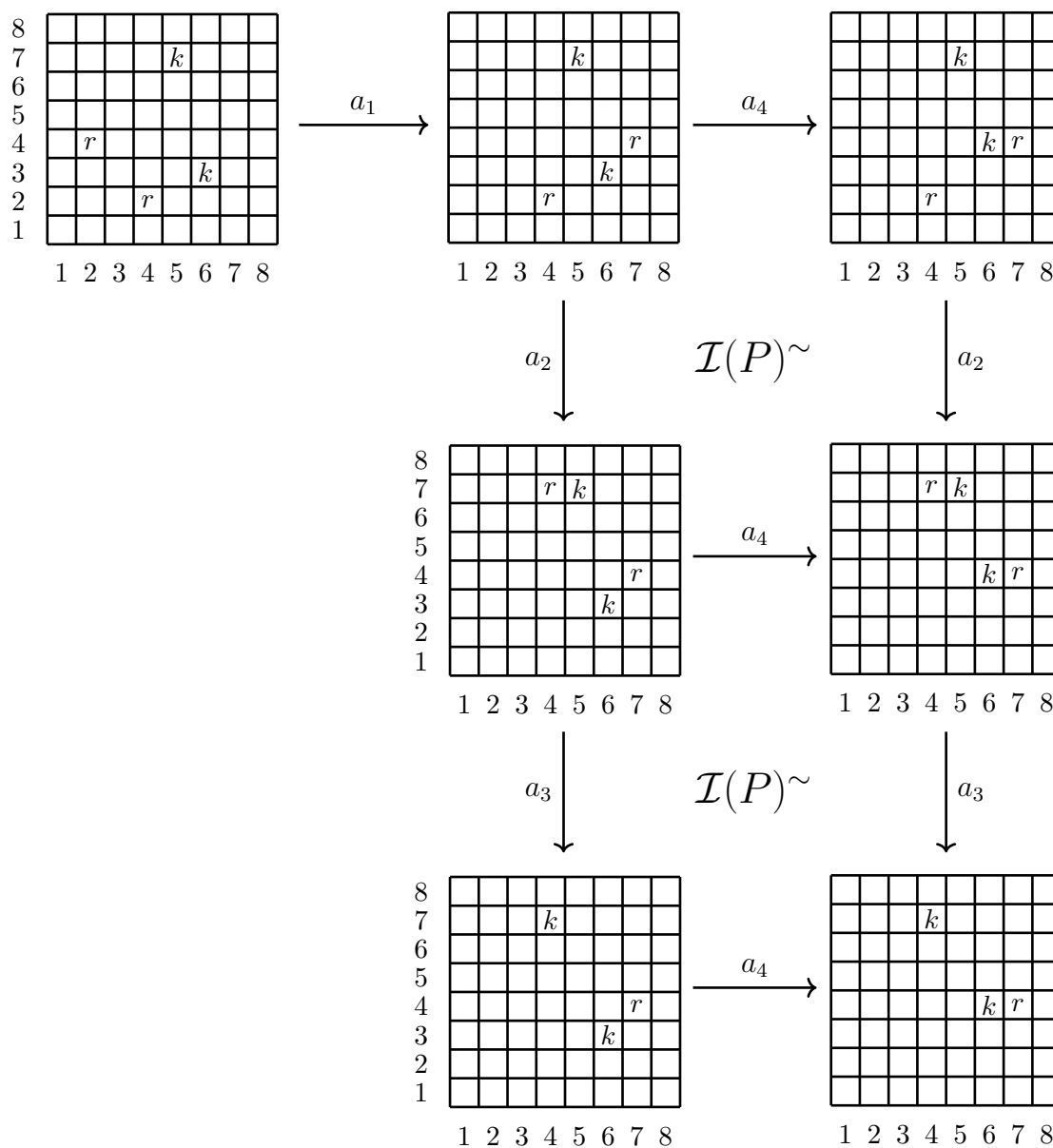


Figure 37: The set of equivalent runs (with $\mathcal{I}(P)$) that corresponds with the run from Figure 36.

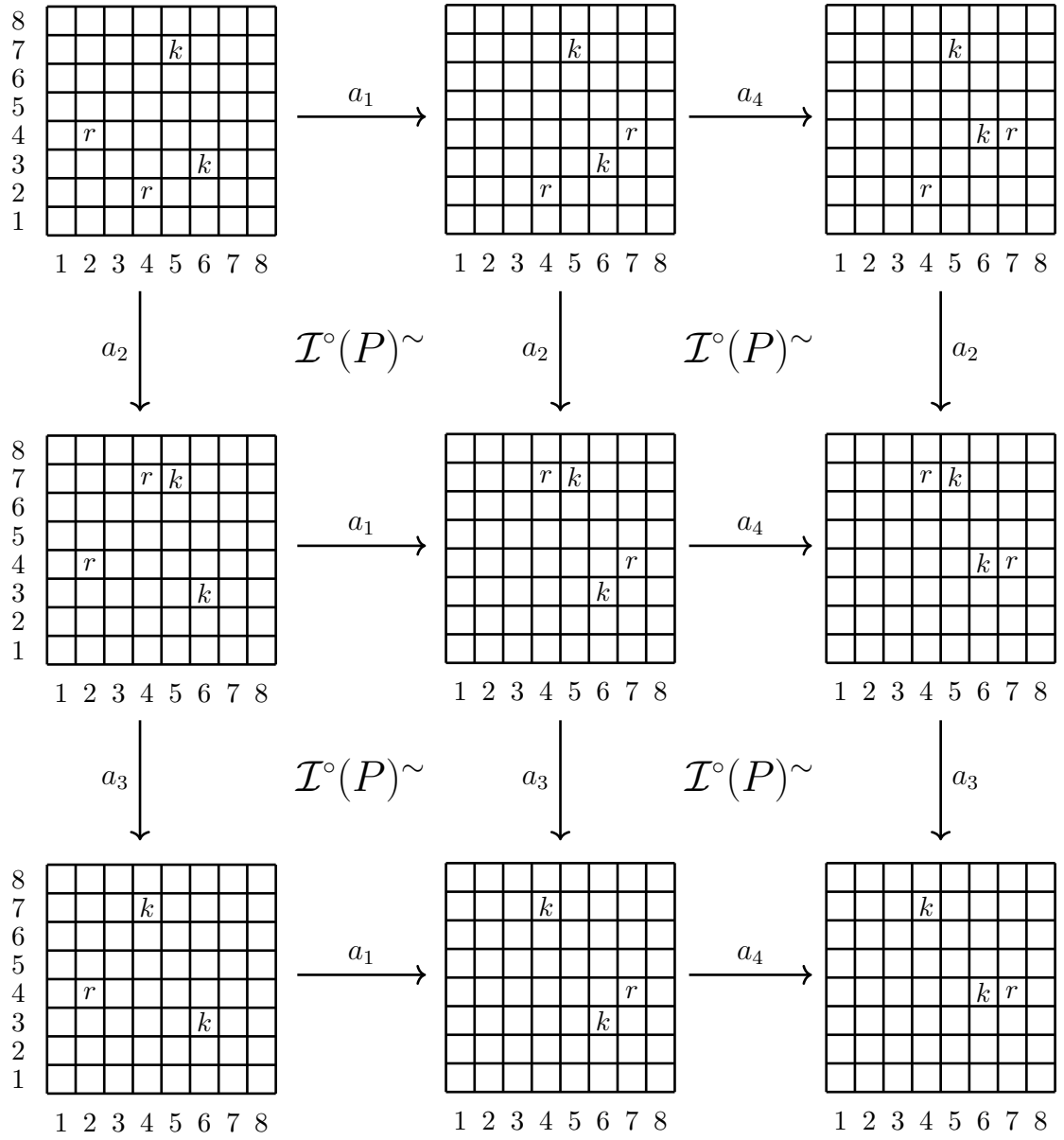


Figure 38: The set of equivalent runs (with $\mathcal{I}^o(P)$), that corresponds with the run from Figure 36.

This concludes our example for the Kings and Rooks on a Chessboard, and we investigate concurrent URM now.

6.3 Concurrent URM

Our next example illustrates how SI-independence can occur within a computer system where a block of memory is shared by multiple programs. We consider the simple case of systems realized by Unlimited Register Machines (URM), which are computational models introduced due to Lambek (1961) and Minsky (1961) independently⁶. We will borrow notation from N. J. Cutland [IRT] to represent the instruction sets here.

Definition 6.3.1 [Program for URM]. The set of *instructions for unlimited register machines* (URM) is given by $L = \{ T(i, j), S(i), Z(i), J(i, j, k) \mid i, j, k \in \mathbb{N} \}$, and these represent operations on a set of registers $\{r_i\}_{i \geq 0}$ that hold natural numbers. A *URM-program* is a finite list of instructions $K = (I_0, \dots, I_n)$, where $I_i \in L$ for each i . The instructions have the following effect on the system :

1. For $i, j \geq 0$, $T(i, j)$ is the *transfer instruction* that copies the value of the i^{th} register r_i into the j^{th} register r_j .
2. For $i \geq 0$, $Z(i)$ is the *zero instruction* that sets the value of the i^{th} register r_i to the value 0.
3. For $i \geq 0$, $S(i)$ is the *successor instruction* that increments the value of the i^{th} register r_i by one.
4. For $i, j, k \geq 0$, $J(i, j, k)$ is the *jump instruction*, that allows the program to jump to the k^{th} instruction I_k if the value of the i^{th} register matches the value of the j^{th} register. Otherwise, the program simply proceeds to the execution of the instruction that follows in the list.

We will get a more formal sense of how these operate with the LRS example that follows.

⁶The question of whom should be credited for the invention of register machines is discussed on Wikipedia : see https://en.wikipedia.org/wiki/Register_machine

The instructions given above will essentially designate actions that concurrent URM programs can execute on a shared set of registers. The execution of these instructions will perform as usual for a URM that has a single program.

Example 6.3.2 [Concurrent URM programs on Shared Memory]. Consider two URM-programs $K_1 = (I_0^1, \dots, I_n^1)$ and $K_2 = (I_0^2, \dots, I_m^2)$ where K_1 takes its instructions from :

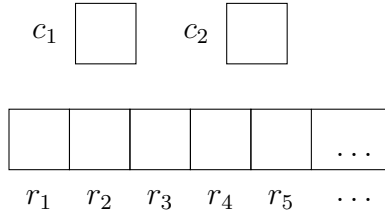
$$L_1 = \{ T_1(i, j), S_1(i), Z_1(i), J_1(i, j, k) \mid i, j, k \in \mathbb{N} \}$$

and where K_2 takes its instructions from :

$$L_2 = \{ T_2(i, r_j), S_2(i), Z_2(i), J_2(i, j, k) \mid i, j, k \in \mathbb{N} \}^7$$

We build a localized relational structure $\Gamma = (P, L, \psi^-, \psi^+, \mathcal{S}, \langle - \rangle)$ as follows :

1. Places : The set of places is given by $P = \{r_i\}_{i \geq 0} \cup \{c_1, c_2\}$ where the r_i are registers (taking values in $\mathbb{N}$) to be shared by the two concurrent programs K_1 and K_2 , and c_1 is the program counter of K_1 , and it holds the number of the position of the instruction that K_1 will perform next. c_2 has the same role as c_1 , but acts as the program counter of the program K_2 .



2. Sheaf of States : For any region $U \subseteq P$, we define $\mathcal{S}(U) = \mathbb{N}^U$, whose sections are functions assigning natural numbers to each of the registers and counters present in U .
3. Labelling Set : The labelling set is given by $L = L_1 \cup L_2$, which is any instruction that K_1 or K_2 can perform.

⁷ L_1 and L_2 correspond with the set of instructions in Definition 6.3.1, but we added a subscript of 1 or 2 to distinguish between the instructions that the first program (K_1) executes and the instructions that the second program (K_2) executes respectively.

4. Input regions : For every $i, j, k \in \mathbb{N}$, we have :

- $\psi^-(T_1(i, j)) = \{r_i, c_1\}$ and $\psi^-(T_2(i, j)) = \{r_i, c_2\}$
- $\psi^-(S_1(i)) = \{r_i, c_1\}$ and $\psi^-(S_2(i)) = \{r_i, c_2\}$
- $\psi^-(Z_1(i)) = \{c_1\}$ and $\psi^-(Z_2(i)) = \{c_2\}$
- $\psi^-(J_1(i, j, k)) = \{r_i, r_j, c_1\}$ and $\psi^-(J_2(i, j, k)) = \{r_i, r_j, c_2\}$

5. Output regions :

- $\psi^+(T_1(i, j)) = \{r_j, c_1\}$ and $\psi^+(T_2(i, j)) = \{r_j, c_2\}$
- $\psi^+(S_1(i)) = \{r_i, c_1\}$ and $\psi^+(S_2(i)) = \{r_i, c_2\}$
- $\psi^+(Z_1(i)) = \{r_i, c_1\}$ and $\psi^+(Z_2(i)) = \{r_i, c_2\}$
- $\psi^+(J_1(i, j, k)) = \{c_1\}$ and $\psi^+(J_2(i, j, k)) = \{c_2\}$

This is quite enlightening to see how containment of dependencies and containment of effects are set up, and how they vary from one action to the other. It should be clear at this point that these concepts of containment are very concrete, and that they can be applied to computer systems that go beyond those presented here. We will see through the equations of the pre-relational diagram that follows, that these specifications of input and output regions are correct.

6. Pre-Relational diagram :

For any $i, j, k \in \mathbb{N}$, we define the pre-relational diagrams associated to actions as follows :

- $\forall (X, Y) \in \mathcal{S}(\psi^-(T_1(i, j))) \times \mathcal{S}(\psi^+(T_1(i, j)))$, we set $(X, Y) \in \langle T_1(i, j) \rangle \Leftrightarrow$

$$I_{X(c_1)}^1 = T_1(i, j) \text{ and } Y(c_1) = X(c_1) + 1 \text{ and } Y(r_j) = X(r_i)$$

- $\forall (X, Y) \in \mathcal{S}(\psi^-(T_2(i, j))) \times \mathcal{S}(\psi^+(T_2(i, j)))$, we set $(X, Y) \in \langle T_2(i, j) \rangle \Leftrightarrow$

$$I_{X(c_2)}^2 = T_2(i, j) \text{ and } Y(c_2) = X(c_2) + 1 \text{ and } Y(r_j) = X(r_i)$$

- $\forall (X, Y) \in \mathcal{S}(\psi^-(S_1(i))) \times \mathcal{S}(\psi^+(S_1(i)))$, we set $(X, Y) \in \langle S_1(i) \rangle \Leftrightarrow$

$$I_{X(c_1)}^1 = S_1(i) \text{ and } Y(c_1) = X(c_1) + 1 \text{ and } Y(r_i) = X(r_i) + 1$$

- $\forall (X, Y) \in \mathcal{S}(\psi^-(S_2(i))) \times \mathcal{S}(\psi^+(S_2(i)))$, we set $(X, Y) \in \langle S_2(i) \rangle \Leftrightarrow$

$$I_{X(c_2)}^2 = S_2(i) \text{ and } Y(c_2) = X(c_2) + 1 \text{ and } Y(r_i) = X(r_i) + 1$$

- $\forall (X, Y) \in \mathcal{S}(\psi^-(Z_1(i))) \times \mathcal{S}(\psi^+(Z_1(i)))$, we set $(X, Y) \in \langle Z_1(i) \rangle \Leftrightarrow$

$$I_{X(c_1)}^1 = Z_1(i) \text{ and } Y(c_1) = X(c_1) + 1 \text{ and } Y(r_i) = 0$$

- $\forall (X, Y) \in \mathcal{S}(\psi^-(Z_2(i))) \times \mathcal{S}(\psi^+(Z_2(i)))$, we set $(X, Y) \in \langle Z_2(i) \rangle \Leftrightarrow$

$$I_{X(c_2)}^2 = Z_2(i) \text{ and } Y(c_2) = X(c_2) + 1 \text{ and } Y(r_i) = 0$$

- $\forall (X, Y) \in \mathcal{S}(\psi^-(J_1(i, j, k))) \times \mathcal{S}(\psi^+(J_1(i, j, k)))$, we set $(X, Y) \in \langle J_1(i, j, k) \rangle \Leftrightarrow$

$$I_{X(c_1)}^1 = J_1(i, j, k) \text{ and}$$

$$Y(c_1) = \begin{cases} k & \text{if } X(r_i) = X(r_j) \\ X(c_1) + 1 & \text{otherwise} \end{cases}$$

- $\forall (X, Y) \in \mathcal{S}(\psi^-(J_2(i, j, k))) \times \mathcal{S}(\psi^+(J_2(i, j, k)))$, we set $(X, Y) \in \langle J_2(i, j, k) \rangle \Leftrightarrow$

$$I_{X(c_2)}^2 = J_2(i, j, k) \text{ and}$$

$$Y(c_2) = \begin{cases} k & \text{if } X(r_i) = X(r_j) \\ X(c_2) + 1 & \text{otherwise} \end{cases}$$

All of the above provide a well-defined LRS and $\mathcal{I}^\circ = \mathcal{I}^+$ for this LRS. We also have that any two instructions carried out by the same program are never spatially independent because the program counter is always involved at the level of dependencies and effects. In between the two programs, we have the following examples of spatially independent actions :

- For any distinct $i, j, j' \in \mathbb{N}$, we have $T_1(i, j) \mathcal{I}^+(P) T_2(i, j')$
(we don't have $T_1(i, j) \mathcal{I}(P) T_2(i, j')$ however)
- For any $i, j \in \mathbb{N}$, we have $S_1(i) \mathcal{I}(P) Z_2(j)$ if and only if $i \neq j$, and also,
 $S_1(i) \mathcal{I}(P) S_2(j)$ if and only if $i \neq j$.
- For any $i, j, k, i', j', k' \in \mathbb{N}$, we have $J_1(i, j, k) \mathcal{I}^+(P) J_2(i', j', k')$ (we don't have
 $J_1(i, j, k) \mathcal{I}(P) J_2(i', j', k')$ however). These are $\mathcal{I}^+(P)$ independent because the
jump instructions only have effects in the individual counters of each program,
and they cannot interfere with each other as such.

Let us consider two concurrent programs as follows :

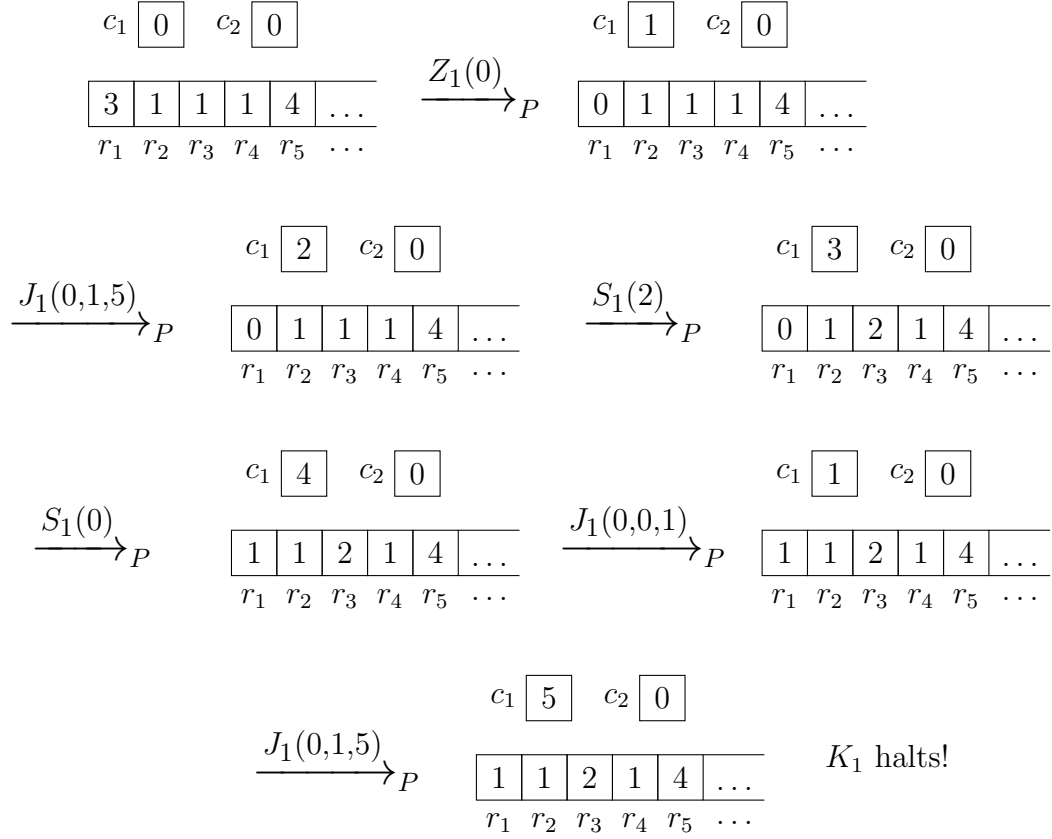
$$K_1 = (I_0^1, I_1^1, I_2^1, I_3^1, I_4^1) = (Z_1(0), J_1(0, 1, 5), S_1(2), S_1(0), J_1(0, 0, 1))$$

$$K_2 = (I_0^2, I_1^2, I_2^2, I_3^2, I_4^2) = (Z_2(4), J_2(2, 4, 5), S_2(3), S_2(4), J_2(4, 4, 1))$$
⁸

With instructions accessed by counter values ranging from 0 to 4, we remark that the instructions $I_1^1 = J_1(0, 1, 5)$ and $I_1^2 = J_2(0, 1, 5)$ halts each program individually because there are no instructions corresponding to a counter value of 5. Both programs perform addition of two numbers to be found in some registers. Individually speaking, the K_1 program performs addition of the values initially present in the registers r_1 and r_2 , and stores the result in r_2 . The register r_0 is kept as internal memory to the program K_1 , and it allows K_1 to keep track of incrementations performed with $S_1(2)$ in the register r_2 . Also, the program K_2 by itself performs addition of the values initially present in the registers r_2 and r_3 , and stores the result in r_3 . The register r_4 is kept as internal memory to the program K_2 , and it allows K_2 to keep track of incrementations performed with $S_2(3)$ in the register r_3 .

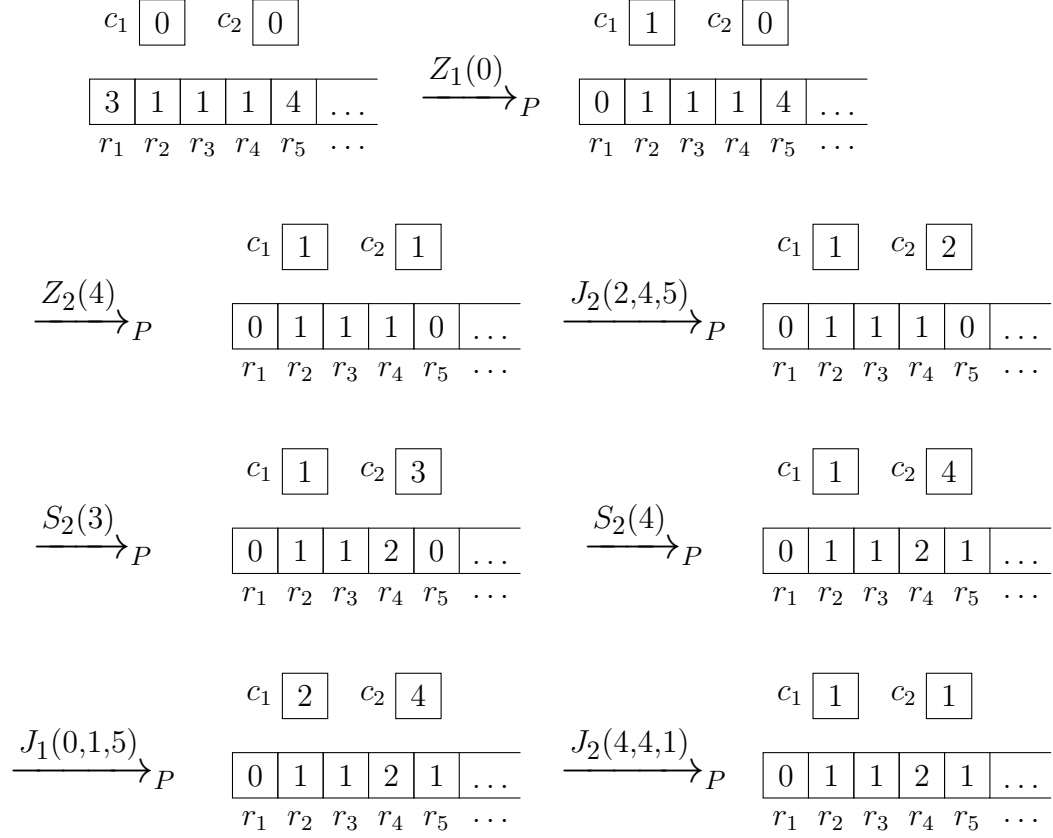
⁸Given $i, j, k \in \mathbb{N}$, it would probably be beneficial to redefine $\psi^-(J(i, j, k))$ to $\{c_1\}$ in the cases where $i = j$. Otherwise, our current definition would affect the dynamics of the spatially independent pairs for concurrent programs in a sort of non-constructive way. Indeed, we could redefine the program K_2 by replacing $J_1(4, 4, 1)$ with $J_1(0, 0, 1)$ equivalently, but this would prevent the $\mathcal{I}$ relations to be efficient in between the two programs.

For an example of a global run for K_1 , we have :



But when K_1 and K_2 perform concurrently, we have that the final outcome of both program halting, we have that r_2 contains the sum of the initial values in r_1 and r_2 (i.e. in the state when the program starts), whereas r_3 contains a value that ranges between the sum of the initial values in r_2 and r_3 , and the sum of the initial values in r_1 and r_2 and r_3 . In fact, r_3 will have the full sum of the initial values in r_1 and r_2 and r_3 if and only if K_1 runs faster than K_2 , i.e. if $J_2(2, 4, 5)$ is executed and halts program K_2 before r_2 contains the sum of the initial values in r_1 and r_2 .

For example, we have :



At the last state in the run above, if K_2 fires the action $J_2(2, 4, 5)$, then K_2 halts and the value in the register r_3 becomes 2, which is not the sum of the initial values of r_1 , r_2 , and r_3 . However, if the action $S_1(2)$ is performed, then when both programs halt, we will get that r_3 will contain the sum of the initial values of r_1 , r_2 , and r_3 , and r_2 will contain the sum of the initial values of r_1 and r_2 .

In other words, the final outcome depends on the order of activity. However, we may observe that this is only true for the computations of K_2 , i.e. K_1 gets the same task done (of computing the sum of the initial values in r_1 and r_2 , and stores the result in r_2) independently of K_2 's activity. This is quite interesting; it illustrates that there is a form of directionality for independence in between the programs K_1

and K_2 , and their actions. This is an aspect of SI-independence we did not cover so far.

Indeed, K_2 does not interfere with K_1 's task, but K_1 does interfere with K_2 , i.e. there is a one-sided interference. This, in fact, has an explanation in terms of regions that contain dependencies and regions that contain effects. To be more precise, the whole region where K_1 is contained in its dependencies can be represented as the union of the input regions of its instructions :

$$\begin{aligned}\psi^-(K_1) &:= \psi^-(Z_1(0)) \cup \psi^-(J_1(0, 1, 5)) \cup \psi^-(S_1(2)) \cup \psi^-(S_1(0)) \cup \psi^-(J_1(0, 0, 1)) \\ &= \{c_1, r_0, r_1, r_2\}\end{aligned}$$

and the whole region where K_1 is contained in its effects can be represented as the union of the output regions of its instructions :

$$\begin{aligned}\psi^+(K_1) &:= \psi^+(Z_1(0)) \cup \psi^+(J_1(0, 1, 5)) \cup \psi^+(S_1(2)) \cup \psi^+(S_1(0)) \cup \psi^+(J_1(0, 0, 1)) \\ &= \{c_1, r_0, r_2\}\end{aligned}$$

Whereas the region where K_2 is contained in its dependencies can be represented as :

$$\begin{aligned}\psi^-(K_2) &:= \psi^-(Z_2(4)) \cup \psi^-(J_2(2, 4, 5)) \cup \psi^-(S_2(3)) \cup \psi^-(S_1(4)) \cup \psi^-(J_2(4, 4, 1)) \\ &= \{c_2, r_2, r_3, r_4\}\end{aligned}$$

and the region where K_2 is contained in its effects can be represented as :

$$\begin{aligned}\psi^+(K_2) &:= \psi^+(Z_2(4)) \cup \psi^+(J_2(2, 4, 5)) \cup \psi^+(S_2(3)) \cup \psi^+(S_1(4)) \cup \psi^+(J_2(0, 0, 1)) \\ &= \{c_2, r_3, r_4\}\end{aligned}$$

And we see what happens now, as we have $\psi^+(K_2) \cap \psi^-(K_1) = \emptyset$, but $\psi^+(K_1) \cap \psi^-(K_2) = \{r_2\}$. This means that $\{r_2\}$ is the precise region where interference occurs in between the two programs, but better yet, it is only K_1 that interferes with K_2 in this region, not the other way around. What this illustrates here is that

the notion of independence in between actions can be set up in a unidirectional sense when using LRS; and this seems to actually extend the SI-independence of $\mathcal{I}^\circ$, which then becomes asymmetric. To be more precise, we can define a directed form of independence in a LRS as follows :

For a given region U , we define a binary relation $\mathcal{I}^\rightarrow(U)$ on $\mathcal{L}(U)$ as follows :

$$\text{For all actions } b, c \in \mathcal{L}(U), \quad b \mathcal{I}^\rightarrow(U) c \quad \Leftrightarrow \quad \psi_U^-(b) \cap \psi_U^+(c) = \emptyset$$

When $b \mathcal{I}^\rightarrow(U) c$, we can say that c *does not interfere with* b (so b is independent of the action c in a sense) with respect to U . We can say that c *can interfere with* b otherwise. With this definition, we get :

$$\mathcal{I}^\circ(U) = \mathcal{I}^\rightarrow(U) \cap (\mathcal{I}^\rightarrow(U))^{-1}$$

Thus, it might be interesting to study this form directed independence, and formalize what it means in terms of the transition relations for the actions it involves.

In the case of K_1 and K_2 , we can identify more precisely the culprits of interference. Indeed, we have that all actions in between these two systems are spatially independent (i.e. all their transition relations commute), except for one pair, which is $S_1(2)$ and $J_2(2, 4, 5)$. Indeed, K_1 interferes with K_2 precisely because $S_1(2)$ interferes with $J_2(2, 4, 5)$, i.e. $\psi^+(S_1(2)) \cap \psi^-(J_2(2, 4, 5)) = \{r_2\}$.

Thus, with LRS, we had the opportunity to get a much clearer vision of SI-independence in $\mathbb{T}$ -adapted presheaves through the examples given. In particular, we were able to emphasize the importance of discriminating between containment of dependencies and containment of effects, because they can provide substantially more spatially independent actions (with $\mathcal{I}^\circ$), then the notion of action containment by itself would (with $\mathcal{I}$). Furthermore, we saw that this discrimination in dependencies and effects led to a form of unidirectional independence in between actions. Indeed, this is uncharted subject matter in which we would definitely like the theory of SI-independence to expand.

Conclusion

In essence, the purpose of this thesis was to convince the reader that there is great interest in undertaking more research into the representation of distributed systems as presheaves. Through the sophisticated language of category theory, we saw how presheaves used in a spatial sense constitute an invaluable mathematical tool to represent distributed systems, and we certainly encourage their use in computer science. We basically had a glimpse of the insight that such presheaves could provide for distributed systems in the context of presheaves of labelled transition systems (LTS), as we were able to formulate spatially induced independence there.

The form of independence that we referred to as spatially induced independence within this thesis had been discovered by Malcolm [SSTS], in the context of $\mathbb{T}$ -valued sheaves on Heyting algebras (where $\mathbb{T}$ is the category of LTS). Malcolm essentially expressed that there were various forms of commutativity for the transition relations of spatially independent actions when we worked in the context of a sheaf. The question we answered within this thesis was whether it was possible to get similar results in a more general context of presheaves of LTS, i.e. replacing the condition of a $\mathbb{T}$ -valued sheaf with a weaker one. This was important because there are many cases of distributed systems that cannot be properly modelled by sheaves. As an answer to our question, we invented $\mathbb{T}$ -adapted presheaves (which include the case of $\mathbb{T}$ -valued sheaves), and we got the commutativity we needed amongst transition relations of spatially independent actions with them.

To be more precise, the realization of SI-independence within these $\mathbb{T}$ -adapted presheaves was given a concrete form with our main theorem, the Theorem of Spatially Induced Independence. In this theorem, we constructed, for any Heyting algebra $\mathcal{H}$, a functor from the category of $\mathbb{T}$ -adapted presheaves on $\mathcal{H}$ to the category of $\mathbb{A}^\sim$ -valued presheaves on $\mathcal{H}$ (where $\mathbb{A}^\sim$ is the category of asynchronous labelled transition systems with equivalence (ALTSE)).

In order to prove this theorem and understand its scope, we had to provide a lot of background and theory. Initially, we saw in Section 1.2 several results with the category of labelling morphisms $\mathbf{Set}_\varepsilon$ (that corresponds with the category of sets and partial maps) such as bicompleteness and an adjunction with $\mathbf{Mon}$. We then integrated these labelling morphisms into the structure of labelled transition system morphisms. We saw in particular that realizing spatially induced independence had something to do with incorporating these labelling morphisms (partial maps) as components into the structure of morphisms, as we were able to describe how actions or elements of a system vanish when we consider them outside of the regions where they are physically contained. Many parts of the theory simply relied on the use of the separated presheaf of partial maps that was given by $\mathcal{L}$ for labelling sets.

We did spend some time working with the category of labelled transition systems $\mathbb{T}$ in Chapter 2. Our intention there was to provide a stepping stone for further research into $\mathbb{T}$ -valued presheaves, and this requires a certain amount of understanding of $\mathbb{T}$. In particular, we saw the categorical constructions that made $\mathbb{T}$ into a bicomplete category. We then investigated subobjects and found a sub-LTS criterion to characterize the subobjects of a LTS. We also showed that $\mathbb{T}$ has no subobject classifier. Finally, we explored monoid labelled transition systems and their category $\mathbb{M}$, and we provided an adjunction between $\mathbb{T}$ and $\mathbb{M}$, where the left adjoint associates a freely generated monoid LTS for any given LTS.

Afterwards, we saw how to implement the notion of independence for a LTS as we explored asynchronous labelled transition systems (ALTS), and we saw a version

of such ALTS models where we added an equivalence on transitions (ALTSE). The ALTS in question allowed a form of quotienting on their underlying concurrent alphabet, and it was possible to provide a functor from $\mathbb{A}$ (the category of ALTS) to $\mathbb{M}$ that associated a Mazurkiewicz trace monoid as a labelling monoid to any given ALTS. Then, with our new concept of ALTSE, we laid out the foundations necessary to evoke spatially induced independence within presheaves of LTS. In particular, we provided a form of equivalence on runs in these ALTSE and we made two conjectures about how this equivalence behaved rather well within these systems.

We then tackled formally the subject of presheaves of LTS in Chapter 4. In this chapter, we investigated δ -gluing for actions and we saw how such gluing is systematically present with $\mathbb{T}$ -valued sheaves. We understood that imposing a certain amount of δ -gluing for actions, with respect to regions where actions vanish, was what made it possible to express containment of dependencies and effects for such actions. These concepts relating to action containment were at the heart of rendering SI-independence within $\mathbb{T}$ -valued presheaves. Thus, we proposed to impose an axiom of well-contained actions (WCA) on $\mathbb{T}$ -valued presheaves, that forced j -complements of ε -regions to contain actions. This gave us $\mathbb{T}$ -adapted presheaves (along with the states presheaf as a sheaf and the labelling presheaf as a separated presheaf) and SI-independence arose naturally within these presheaves.

The $\mathbb{T}$ -adapted presheaves in question were studied in Chapter 5. We emphasized the theory of SI-independence there, but we made a conjecture that $\mathbb{T}$ -adapted presheaves inherited the constructions of $\mathbb{T}$ -valued presheaves that would make them bicomplete (it would certainly be good thing to work out this result in the future). We saw two kinds of SI-independence relations on $\mathbb{T}$ -adapted presheaves on complete Heyting algebra, $\mathcal{I}$ and $\mathcal{I}^+$, where the former uses ε -regions of actions in order to be realized, whereas the latter uses transparent regions. For $\mathcal{I}$, and given a complete Heyting algebra $\mathcal{H}$, the Theorem of Spatially Induced Independence provided a functor from the category of $\mathbb{T}$ -adapted presheaves on $\mathcal{H}$ to the category of $\mathbb{A}^\sim$ -valued presheaves on $\mathcal{H}$. This was certainly an advantage of $\mathcal{I}$ over $\mathcal{I}^+$. However,

we showed that $\mathcal{I}^+$ provides finer SI-independence relations than $\mathcal{I}$, as the former distinguishes between containment of dependencies and containment of effects for actions. In particular, we saw that $\mathcal{I}^+$ was useful in the context of $\mathbb{T}$ -adapted presheaves on complete Boolean algebras. We also saw that for such $\mathbb{T}$ -adapted presheaves on complete Boolean algebras, we were able to give a convenient expression to the independence relations of $\mathcal{I}$ as spatial disjointness of the proper region of actions.

In Chapter 6 that followed, we investigated localized relational structures (LRS) and how we could associate $\mathbb{T}$ -adapted presheaves to the latter (in Section 6.1). We provided LRS for Petri Nets in particular, and we studied examples of $\mathbb{T}$ -adapted presheaves with a Kings and Rooks LRS (in Section 6.2) and a concurrent URM programs LRS (in Section 6.3). We were able to clearly see the difference between $\mathcal{I}$ and $\mathcal{I}^+$ at that point, and we were also able to evoke a form of unidirectional SI-independence $\mathcal{I}^\rightarrow$ in LRS systems through the URM example. The theory behind this unidirectional independence has not been worked on in this thesis and it would certainly be interesting to see how it develops in general for $\mathbb{T}$ -adapted presheaves.

There are definitely aspects of SI-independence on which we would like to know more and on which further research is possible. The most important would be to connect the subject of SI-independence with formal methods such as those presented for coalgebras (there are correspondences between coalgebras and LTS) in the work of Goguen [HAg], [HHTh], [HCoC] and Cirstea [SLAC], [CSHA]. For applications to partial order reduction methods, we would also need to connect the subject of SI-independence with the work of Peled [POMC], [AOOA] and Lomuscio, Penczek, Qu [PMAS]. The latter have essentially studied the effect of independence relations on LTS within the context of interpreted system logics, and its applications to partial order reductions methods.

We also mentioned in Section 2.1 that there were set-valued presheaf models of processes that were investigated by Winskel and Cattani in [ProfOB], and these could

act as a substitute for the representation of processes as pointed LTS⁹. Thus, it would definitely be interesting to investigate presheaves that take values into a category of such set-valued presheaves, as opposed to using presheaves of LTS like we did in this thesis.

Finally, presheaves of LTS certainly have an ability to represent dynamical systems in physics as we saw. Our hope however, is that more refined representations of such systems with these spatial presheaves of LTS would allow the subject of SI-independence to have an applications in physics. Whether an analogy can be made at that level or not is left as an open line of investigation.

⁹There is a form of adjunction, between LTS with functional simulations and set-valued presheaves, and it uses the Yoneda embedding.

Appendix A

Proofs for Section 1.2

Proposition A.0.1 . $(-)^* : \mathbf{Set}_\varepsilon \rightarrow \mathbf{Mon}$ is indeed a functor.

Proof. For any labelling set L in $\mathbf{Set}_\varepsilon$, it is clear that L^* is a monoid. Consider any labelling morphism $\lambda : L_0 \rightarrow_\varepsilon L$. We have that $\lambda^*(\Lambda) = \Lambda$ by definition, so the neutral element of L_0^* is mapped to the neutral element of L^* . Consider any $w_0, w_1 \in L_0^*$. If $w_0 = \Lambda$ or $w_1 = \Lambda$, then it is clear that $\lambda^*(w_0 w_1) = \lambda^*(w_0) \lambda^*(w_1)$. If $w_0 \neq \Lambda$ and $w_1 \neq \Lambda$, then we can write $w_0 = a_1 \dots a_n$ and $w_1 = b_1 \dots b_m$ for some $a_i, b_i \in L_0$. We get that $\lambda^*(w_0 w_1) = \lambda^*(a_1 \dots a_n b_1 \dots b_m) = \lambda^*(a_1) \dots \lambda^*(a_n) \lambda^*(b_1) \dots \lambda^*(b_m) = \lambda^*(a_1 \dots a_n) \lambda^*(b_1 \dots b_m) = \lambda^*(w_0) \lambda^*(w_1)$. This proves that λ^* is a monoid morphism from L_0^* to L^* .

Consider the identity $1_L : L \rightarrow_\varepsilon L$ on the labelling set L in $\mathbf{Set}_\varepsilon$. We have that for any $a_1 \dots a_n \in L^*$ where $a_i \in L$ for each i , $1_L(a_i) = a_i \neq \varepsilon$, and thus $1_L^*(a_i) = 1_L(a_i) = a_i$. But then, $1_L^*(a_1 \dots a_n) = 1_L^*(a_1) \dots 1_L^*(a_n) = 1_L(a_1) \dots 1_L(a_n) = a_1 \dots a_n$. Also, $1_L^*(\Lambda) = \Lambda$. Thus, 1_L^* is the identity on L^* in $\mathbf{Mon}$.

Finally, consider any labelling morphisms $\lambda_1 : L_0 \rightarrow_\varepsilon L_1$ and $\lambda_2 : L_1 \rightarrow_\varepsilon L_2$ in $\mathbf{Set}_\varepsilon$. We have that $(\lambda_2^* \circ \lambda_1^*)(\Lambda) = \lambda_2^*(\lambda_1^*(\Lambda)) = \lambda_2^*(\Lambda) = \Lambda = (\lambda_2 \circ \lambda_1)^*(\Lambda)$. For any $a_1 \dots a_n \in L_0^*$ where the a_i are in L_0 . Consider any $i \in \{1, \dots, n\}$.

- If $\lambda_1(a_i) = \varepsilon$, then $(\lambda_2 \circ \lambda_1)(a_i) = \lambda_2(\lambda_1(a_i)) = \varepsilon$, and we get $(\lambda_2 \circ \lambda_1)^*(a_i) =$

$$\Lambda = \lambda_2^*(\Lambda) = \lambda_2^*(\lambda_1^*(a_i)) = (\lambda_2^* \circ \lambda_1^*)(a_i).$$

- If $\lambda_1(a_i) \neq \varepsilon$ and $(\lambda_2 \circ \lambda_1)(a_i) = \lambda_2(\lambda_1(a_i)) = \varepsilon$, then $(\lambda_2 \circ \lambda_1)^*(a_i) = \Lambda = \lambda_2^*(\lambda_1(a_i)) = \lambda_2^*(\lambda_1^*(a_i)) = (\lambda_2^* \circ \lambda_1^*)(a_i)$.
- If $\lambda_1(a_i) \neq \varepsilon$ and $(\lambda_2 \circ \lambda_1)(a_i) = \lambda_2(\lambda_1(a_i)) \neq \varepsilon$, then $(\lambda_2 \circ \lambda_1)^*(a_i) = (\lambda_2 \circ \lambda_1)(a_i) = \lambda_2(\lambda_1(a_i)) = \lambda_2^*(\lambda_1(a_i)) = \lambda_2^*(\lambda_1^*(a_i)) = (\lambda_2^* \circ \lambda_1^*)(a_i)$.

But then, we have that

$$\begin{aligned} (\lambda_2 \circ \lambda_1)^*(a_1 \dots a_n) &= (\lambda_2 \circ \lambda_1)^*(a_1) \dots (\lambda_2 \circ \lambda_1)^*(a_n) \\ &= (\lambda_2^* \circ \lambda_1^*)(a_1) \dots (\lambda_2^* \circ \lambda_1^*)(a_n) \\ &= \lambda_2^*(\lambda_1^*(a_1)) \dots \lambda_2^*(\lambda_1^*(a_n)) \\ &= \lambda_2^*(\lambda_1^*(a_1) \dots \lambda_1^*(a_n)) \\ &= \lambda_2^*(\lambda_1^*(a_1 \dots a_n)) \\ &= (\lambda_2^* \circ \lambda_1^*)(a_1 \dots a_n) \end{aligned}$$

Thus, $(\lambda_2 \circ \lambda_1)^* = \lambda_2^* \circ \lambda_1^*$.

This proves that $(-)^*$ is a functor.

□

Proposition A.0.2 . $G_{m\varepsilon} : \mathbf{Mon} \rightarrow \mathbf{Set}_\varepsilon$ is indeed a functor.

Proof. For any monoid M , we have that $G_{m\varepsilon}(M) = M \setminus \{e\} \subseteq M$, which is assumed to not contain the distinguished symbol ε , so $G_{m\varepsilon}(M)$ is a labelling set in $\mathbf{Set}_\varepsilon$.

Consider any monoid morphism $f : M_0 \rightarrow M$ with e_0 and e as the neutral elements of M_0 and M respectively. We have that $G_{m\varepsilon}(f)(\varepsilon) = \varepsilon$ as required. Consider any $b \in M_0$. If $f(b) = e$, then $G_{m\varepsilon}(f)(b) = \varepsilon \in (M \setminus \{e\}) \cup \{\varepsilon\}$. If $f(b) \neq e$, then $G_{m\varepsilon}(f)(b) = f(b) \in (M \setminus \{e\}) \subseteq (M \setminus \{e\}) \cup \{\varepsilon\}$. Thus, $G_{m\varepsilon}(f)$ is a labelling morphism from $M_0 \setminus \{e_0\}$ to $M \setminus \{e\}$.

Consider the identity monoid morphism $1_M : M \rightarrow M$ in **Mon** with e as the neutral element of M . We get that $G_{m\varepsilon}(1_M)(\varepsilon) = \varepsilon$. Also, for any $b \in M \setminus \{e\}$, we have $(1_M)(b) = b \neq e$, and so $G_{m\varepsilon}(1_M)(b) = 1_M(b) = b$. This proves that $G_{m\varepsilon}(1_M)$ is the identity on the labelling set $M \setminus \{e\}$ in **Set** $_{\varepsilon}$.

Finally, consider any monoid morphisms $f_1 : M_0 \rightarrow M_1$ and $f_2 : M_1 \rightarrow M_2$ and let e_0, e_1 , and e_2 be the neutral elements of M_0, M_1 , and M_2 respectively. We have that $G_{m\varepsilon}(f_2 \circ f_1)(\varepsilon) = \varepsilon = G_{m\varepsilon}(f_2)(\varepsilon) = G_{m\varepsilon}(f_2)(G_{m\varepsilon}(f_1)(\varepsilon)) = (G_{m\varepsilon}(f_2) \circ G_{m\varepsilon}(f_1))(\varepsilon)$. Consider any $m \in M_0$.

- If $f_1(m) = e_1$, then $(f_2 \circ f_1)(m) = f_2(f_1(m)) = e_2$, and we get $G_{m\varepsilon}(f_2 \circ f_1)(m) = \varepsilon = G_{m\varepsilon}(f_2)(\varepsilon) = (G_{m\varepsilon}(f_2)(G_{m\varepsilon}(f_1)(m))) = (G_{m\varepsilon}(f_2) \circ G_{m\varepsilon}(f_1))(m)$.
- If $f_1(m) \neq e_1$ and $(f_2 \circ f_1)(m) = f_2(f_1(m)) = e_2$, then $G_{m\varepsilon}(f_2 \circ f_1)(m) = \varepsilon = G_{m\varepsilon}(f_2)(f_1(m)) = G_{m\varepsilon}(f_2)(G_{m\varepsilon}(f_1)(m)) = (G_{m\varepsilon}(f_2) \circ G_{m\varepsilon}(f_1))(m)$.
- If $f_1(m) \neq e_1$ and $(f_2 \circ f_1)(m) = f_2(f_1(m)) \neq e_2$, then $G_{m\varepsilon}(f_2 \circ f_1)(m) = \varepsilon = (f_2 \circ f_1)(m) = f_2(f_1(m)) = G_{m\varepsilon}(f_2)(f_1(m)) = G_{m\varepsilon}(f_2)(G_{m\varepsilon}(f_1)(m)) = (G_{m\varepsilon}(f_2) \circ G_{m\varepsilon}(f_1))(m)$.

Thus, $G_{m\varepsilon}(f_2 \circ f_1) = G_{m\varepsilon}(f_2) \circ G_{m\varepsilon}(f_1)$.

This proves that $G_{m\varepsilon}$ is a functor. □

Proposition A.0.3 . $(-)^*$ is the left adjoint of $G_{m\varepsilon}$.

$$\begin{array}{ccc}
 & (-)^* & \\
 \text{Set}_{\varepsilon} & \xrightarrow{\quad} & \text{Mon} \\
 & \perp & \\
 & G_{m\varepsilon} &
 \end{array}$$

Proof. We demonstrate that for any labelling set L_0 and any monoid $(M, \cdot, e)$, there is a bijective correspondence between $\text{Hom}_{\text{Mon}}(L_0^*, M)$ and $\text{Hom}_{\text{Set}_{\varepsilon}}(L_0, M \setminus \{e\})$. We

define a map $\theta_{L_0, M} : \mathbf{Hom}_{\mathbf{Mon}}(L_0^*, M) \rightarrow \mathbf{Hom}_{\mathbf{Set}_\varepsilon}(L_0, M \setminus \{e\})$ that sends a monoid morphism $\lambda : L_0^* \rightarrow M$ to the labelling morphism $\theta_{L_0, M}(\lambda) : L_0 \rightarrow_\varepsilon (M \setminus \{e\})$ defined by :

$$\begin{aligned} \theta_{L_0, M}(\lambda)(\varepsilon) &= \varepsilon \\ \theta_{L_0, M}(\lambda)(b) &= \begin{cases} \lambda(b) & \text{if } \lambda(b) \neq e \\ \varepsilon & \text{otherwise} \end{cases} \end{aligned}$$

for any $b \in L_0$.

We verify that $\theta_{L_0, M}(\lambda)$ is a labelling morphism in $\mathbf{Set}_\varepsilon$. We have $\theta_{L_0, M}(\lambda)(\varepsilon) = \varepsilon$ as required. Also, for any $b \in L_0$, if $\lambda(b) \neq e$, then $\theta_{L_0, M}(\lambda)(b) = \lambda(b) \in (M \setminus \{e\}) \cup \{\varepsilon\}$, and if $\lambda(b) = e$, then $\theta_{L_0, M}(\lambda)(b) = \varepsilon \in (M \setminus \{e\}) \cup \{\varepsilon\}$. This proves that $\theta_{L_0, M}(\lambda)$ is a labelling morphism from L_0 to $M \setminus \{e\}$.

Going the other way around, we can provide a map $\theta_{L_0, M}^{-1} : \mathbf{Hom}_{\mathbf{Set}_\varepsilon}(L_0, M \setminus \{e\}) \rightarrow \mathbf{Hom}_{\mathbf{Mon}}(L_0^*, M)$ that sends a labelling morphism $\lambda : L_0 \rightarrow_\varepsilon (M \setminus \{e\})$ to a monoid morphism $\theta_{L_0, M}^{-1}(\lambda) : L_0^* \rightarrow M$ defined inductively by :

$$\begin{aligned} \theta_{L_0, M}^{-1}(\lambda)(\Lambda) &= e \\ \theta_{L_0, M}^{-1}(\lambda)(wb) &= \begin{cases} \theta_{L_0, M}^{-1}(\lambda)(w) \cdot \lambda(b) & \text{if } \lambda(b) \neq \varepsilon \\ \theta_{L_0, M}^{-1}(\lambda)(w) & \text{otherwise} \end{cases} \end{aligned}$$

for any $w \in L_0^*$ and $b \in L_0$.

We verify that $\theta_{L_0, M}^{-1}(\lambda)$ is a monoid morphism from L_0^* to M . We have that the neutral element of L_0^* is mapped to the neutral element of M with $\theta_{L_0, M}^{-1}(\lambda)(\Lambda) = e$. We prove that for any $w_1, w_2 \in L_0^*$, we have $\theta_{L_0, M}^{-1}(\lambda)(w_1 w_2) = \theta_{L_0, M}^{-1}(\lambda)(w_1) \cdot \theta_{L_0, M}^{-1}(\lambda)(w_2)$, and we effectuate a proof by induction on the length of the word

$w_1 w_2 \in L_0^*$. If $w_1 w_2 = \Lambda$, then $w_1 = w_2 = \Lambda$, and we get $\theta_{L_0, M}^{-1}(\lambda)(w_1 w_2) = e = e \cdot e = \theta_{L_0, M}^{-1}(\lambda)(w_1) \cdot \theta_{L_0, M}^{-1}(\lambda)(w_2)$. If $w_2 \neq \Lambda$, then $w_2 = w'_2 b$ for some $w'_2 \in L_0^*$ and $b \in L_0$, and then there are two possibilities depending on whether $\lambda(b) = \varepsilon$ or not.

If $\lambda(b) = \varepsilon$, then :

$$\theta_{L_0, M}^{-1}(\lambda)(w_1 w_2) = \theta_{L_0, M}^{-1}(\lambda)(w_1 w'_2 b) = \theta_{L_0, M}^{-1}(\lambda)(w_1 w'_2) = \theta_{L_0, M}^{-1}(\lambda)(w_1) \cdot \theta_{L_0, M}^{-1}(\lambda)(w'_2)$$

by the induction hypothesis. But then,

$$\begin{aligned} \theta_{L_0, M}^{-1}(\lambda)(w_1) \cdot \theta_{L_0, M}^{-1}(\lambda)(w'_2) &= \theta_{L_0, M}^{-1}(\lambda)(w_1) \cdot \theta_{L_0, M}^{-1}(\lambda)(w'_2 b) \\ &= \theta_{L_0, M}^{-1}(\lambda)(w_1) \cdot \theta_{L_0, M}^{-1}(\lambda)(w_2) \end{aligned}$$

If $\lambda(b) \neq \varepsilon$, then :

$$\begin{aligned} \theta_{L_0, M}^{-1}(\lambda)(w_1 w_2) &= \theta_{L_0, M}^{-1}(\lambda)(w_1 w'_2 b) = \theta_{L_0, M}^{-1}(\lambda)(w_1 w'_2) \cdot \lambda(b) \\ &= \theta_{L_0, M}^{-1}(\lambda)(w_1) \cdot \theta_{L_0, M}^{-1}(\lambda)(w'_2) \cdot \lambda(b) \end{aligned}$$

by the induction hypothesis. But then,

$$\begin{aligned} \theta_{L_0, M}^{-1}(\lambda)(w_1) \cdot \theta_{L_0, M}^{-1}(\lambda)(w'_2) \cdot \lambda(b) &= \theta_{L_0, M}^{-1}(\lambda)(w_1) \cdot \theta_{L_0, M}^{-1}(\lambda)(w'_2 b) \\ &= \theta_{L_0, M}^{-1}(\lambda)(w_1) \cdot \theta_{L_0, M}^{-1}(\lambda)(w_2) \end{aligned}$$

The proof of $\theta_{L_0, M}^{-1}(\lambda)(w_1 w_2) = \theta_{L_0, M}^{-1}(\lambda)(w_1) \cdot \theta_{L_0, M}^{-1}(\lambda)(w_2)$ is similar in the case where $w_1 \neq \Lambda$. Thus, by induction on the length of a word $w_1 w_2 \in L_0^*$, we get that $\theta_{L_0, M}^{-1}(\lambda)(w_1 w_2) = \theta_{L_0, M}^{-1}(\lambda)(w_1) \cdot \theta_{L_0, M}^{-1}(\lambda)(w_2)$ for any $w_1, w_2 \in L_0^*$. This proves that $\theta_{L_0, M}^{-1}(\lambda)$ is a monoid morphism from L_0^* to M .

We now verify that $\theta_{L_0, M}^{-1}$ is the inverse of $\theta_{L_0, M}$, and we start by showing that it is a left inverse. Consider any monoid morphism $\lambda : L_0^* \rightarrow M$. We do a proof by induction on the length of a word $w \in L_0^*$, that $((\theta_{L_0, M}^{-1} \circ \theta_{L_0, M})(\lambda))(w) = \lambda(w)$.

For the base case, we have $w = \Lambda$, and

$$((\theta_{L_0, M}^{-1} \circ \theta_{L_0, M})(\lambda))(\Lambda) = \theta_{L_0, M}^{-1}(\theta_{L_0, M}(\lambda))(\Lambda) = e = \lambda(\Lambda)$$

For the inductive step, if $w = w_0b$ where $w_0 \in L_0^*$ and $b \in L_0$, then :

$$((\theta_{L_0,M}^{-1} \circ \theta_{L_0,M})(\lambda))(w) = ((\theta_{L_0,M}^{-1} \circ \theta_{L_0,M})(\lambda))(w_0b) = \theta_{L_0,M}^{-1}(\theta_{L_0,M}(\lambda))(w_0b)$$

- If $\lambda(b) = e$, then $\theta_{L_0,M}(\lambda)(b) = \varepsilon$, and then :

$$\theta_{L_0,M}^{-1}(\theta_{L_0,M}(\lambda))(w_0b) = \theta_{L_0,M}^{-1}(\theta_{L_0,M}(\lambda))(w_0) = \lambda(w_0)$$

by the induction hypothesis. But then, $\lambda(w_0) = \lambda(w_0) \cdot e = \lambda(w_0) \cdot \lambda(b) = \lambda(w_0b) = \lambda(w)$.

- If $\lambda(b) \neq e$, then $\theta_{L_0,M}(\lambda)(b) = \lambda(b) \neq \varepsilon$, and then :

$$\theta_{L_0,M}^{-1}(\theta_{L_0,M}(\lambda))(w_0b) = \theta_{L_0,M}^{-1}(\theta_{L_0,M}(\lambda))(w_0) \cdot \theta_{L_0,M}(\lambda)(b) = \lambda(w_0) \cdot \theta_{L_0,M}(\lambda)(b)$$

by the induction hypothesis. But then, $\lambda(w_0) \cdot \theta_{L_0,M}(\lambda)(b) = \lambda(w_0) \cdot \lambda(b) = \lambda(w_0b) = \lambda(w)$.

This proves that $((\theta_{L_0,M}^{-1} \circ \theta_{L_0,M})(\lambda))(w) = \lambda(w)$ in this case. And then, by induction, we get that $((\theta_{L_0,M}^{-1} \circ \theta_{L_0,M})(\lambda))(w) = \lambda(w)$ for any $w \in L_0^*$. This proves that $(\theta_{L_0,M}^{-1} \circ \theta_{L_0,M})(\lambda) = \lambda$ for any monoid morphism $\lambda : L_0^* \rightarrow M$. In particular, we get that $\theta_{L_0,M}^{-1}$ is the left inverse of $\theta_{L_0,M}$.

We prove that $\theta_{L_0,M}^{-1}$ is the right inverse of $\theta_{L_0,M}$. Consider any labelling morphism $\lambda : L_0 \rightarrow_{\varepsilon} (M \setminus \{e\})$ in Set_{ε} . We have

$$((\theta_{L_0,M} \circ \theta_{L_0,M}^{-1})(\lambda))(\varepsilon) = \theta_{L_0,M}(\theta_{L_0,M}^{-1}(\lambda))(\varepsilon) = \varepsilon = \lambda(\varepsilon)$$

Now, consider any $b \in L_0$. If $\lambda(b) = \varepsilon$, then $\theta_{L_0,M}^{-1}(\lambda)(b) = e$, and we have :

$$((\theta_{L_0,M} \circ \theta_{L_0,M}^{-1})(\lambda))(b) = \theta_{L_0,M}(\theta_{L_0,M}^{-1}(\lambda))(b) = \varepsilon = \lambda(b)$$

If $\lambda(b) \neq \varepsilon$, then $\theta_{L_0,M}^{-1}(\lambda)(b) = \lambda(b) \neq e$, and we have :

$$((\theta_{L_0,M} \circ \theta_{L_0,M}^{-1})(\lambda))(b) = \theta_{L_0,M}(\theta_{L_0,M}^{-1}(\lambda))(b) = \theta_{L_0,M}^{-1}(\lambda)(b) = \lambda(b)$$

This proves that $(\theta_{L_0, M} \circ \theta_{L_0, M}^{-1})(\lambda) = \lambda$ for any labelling morphism $\lambda : L_0 \rightarrow_{\varepsilon} (M \setminus \{e\})$. In particular, we get that $\theta_{L_0, M}^{-1}$ is the right inverse of $\theta_{L_0, M}$.

This proves $\theta_{L_0, M}^{-1}$ is the inverse of $\theta_{L_0, M}$ for any labelling set L_0 and monoid $(M, \cdot, e)$, and we get that $(-)^*$ is the left adjoint of $G_{m\varepsilon}$.

□

Appendix B

Proof that $\mathbb{T}$ is bicomplete

Theorem B.0.1 . $\mathbb{T}$ is bicomplete

Proof. We verify that Definition 2.1.17 has characterized the initial object, terminal object, products, coproducts, equalizers and coequalizer properly. The cases of the initial object and terminal object are clear. We verify the remaining constructions.

Indexed Products : Given a J -indexed family of LTS $T_j = (S_j, L_j, \delta_j)$, we define the J -indexed product as :

$$\prod_{j \in J} T_j := (S_J = \prod_{j \in J} S_j, L_J = \prod_{j \in J}^\varepsilon L_j, \delta_J)$$

where

- $\prod_{j \in J} S_j$ is the J -indexed product in **Set** with projection maps $\pi_i : \prod_{j \in J} S_j \rightarrow S_i$ for i in J , and
- $\prod_{j \in J}^\varepsilon L_j$ is the J -indexed product in **Set** _{ε} with labelling projection maps $\rho_i : \prod_{j \in J}^\varepsilon L_j \rightarrow_\varepsilon L_i$ for i in J , and
- $\delta_J := \{(X, a, Y) \in S_J \times L_J \times S_J \mid \forall j \in J, \pi_j(X) \xrightarrow{\rho_j(a)}_{T_j} \pi_j(Y)\}$ ¹.

¹This includes the case where $\rho_j(a) = \varepsilon$, in which case, we get $\pi_j(X) = \pi_j(Y)$ with $\xrightarrow{\varepsilon}_{T_j} = \Delta_{S_j}$.

The projections in $\mathbb{T}$ are given by the pairs $(\pi_i, \rho_i) : \prod_{j \in J} T_j \rightarrow T_i$.

We prove that this forms a product in $\mathbb{T}$. Write $T_J = \prod_{j \in J} T_j$.

Clearly, the (π_i, ρ_i) are morphisms. If we take any transition $X \xrightarrow{a}_{T_J} Y$ in T_J , then we get $\pi_i(X) \xrightarrow{\rho_i(a)}_{T_i} \pi_i(Y) \in \delta_i$ by definition of δ_J . This includes the case where $\rho_i(a) = \varepsilon$, in which case, we get $\pi_i(X) = \pi_i(Y)$ by definition of the extended relational diagram (see Equation 1).

Consider any J -indexed family of LTS morphisms $\{(\sigma_j, \lambda_j) : T \rightarrow T_j\}_{j \in J}$ with $T = (S, L, \delta)$. Then we have a J -indexed family of functions $\{\sigma_j : S \rightarrow S_j\}_{j \in J}$ and a J -indexed family of labelling morphisms $\{\lambda_j : L \rightarrow_{\varepsilon} L_j\}_{j \in J}$, which means that the universal property of the products $\prod_{j \in J} S_j$ and $\prod_{j \in J}^{\varepsilon} L_j$ in **Set** and **Set _{ε}** respectively give us :

- a unique function $\sigma : S \rightarrow \prod_{j \in J} S_j$ such that $\pi_i \circ \sigma = \sigma_i$ for each $i \in J$, and
- a unique labelling morphism $\lambda : L \rightarrow_{\varepsilon} \prod_{j \in J}^{\varepsilon} L_j$ such that $\rho_i \circ \lambda = \lambda_i$ for each $i \in J$

We verify that $(\sigma, \lambda) : T \rightarrow T_J$ is a LTS morphism. Consider any $X \xrightarrow{a}_T Y$ in T . Then, for any $i \in J$, we get $\sigma_i(X) \xrightarrow{\lambda_i(a)}_{T_i} \sigma_i(Y)$ in T_i since $(\sigma_i, \lambda_i) : T \rightarrow T_i$ is a LTS morphism (including $\lambda_i(a) = \varepsilon$). In which case, for any $i \in J$, we get $\pi_i(\sigma(X)) \xrightarrow{\rho_i(\lambda(a))}_{T_i} \pi_i(\sigma(Y))$ in T_i . But then, $\sigma(X) \xrightarrow{\lambda(a)}_{T_J} \sigma(Y)$ in T_J , by definition of δ_J .

If there is any other LTS morphism $(\sigma', \lambda') : T \rightarrow T_J$ such that $(\pi_i, \rho_i) \circ (\sigma', \lambda') = (\sigma_i, \lambda_i)$ for all $i \in J$, then we get $\pi_i \circ \sigma' = \sigma_i$ and $\rho_i \circ \lambda' = \lambda_i$ for all $i \in J$, and the uniqueness part of the universal property of $\prod_{j \in J} S_j$ and $\prod_{j \in J}^{\varepsilon} L_j$ yields that $\sigma' = \sigma$ and $\lambda' = \lambda$. This means $(\sigma', \lambda') = (\sigma, \lambda)$.

Indexed Coproducts : Given a J -indexed family of LTS $T_j = (S_j, L_j, \delta_j)$, we define

the J -indexed coproduct as :

$$\coprod_{j \in J} T_j := (S_J = \coprod_{j \in J} S_j, L_J = \coprod_{j \in J}^\varepsilon L_j, \delta_J)$$

where

- $\coprod_{j \in J} S_j$ is the J -indexed coproduct in **Set** with inclusion maps $\iota_i : S_i \rightarrow \coprod_{j \in J} S_j$ for each i in J , and
- $\coprod_{j \in J}^\varepsilon L_j$ is the J -indexed coproduct in **Set** $_\varepsilon$ with inclusion maps $\nu_i : L_i \rightarrow \coprod_{j \in J}^\varepsilon L_j$ for each i in J , and
- $\delta_J = \{(X, a, Y) \in S_J \times L_J \times S_J \mid \exists i, \exists (X_i, a_i, Y_i) \in \delta_i, \iota_i(X_i) = X, \iota_i(Y_i) = Y, \nu_i(a_i) = a\}$.

The injections in $\mathbb{T}$ are given by the pairs $(\iota_i, \nu_i) : T_i \rightarrow \coprod_{j \in J} T_j$.

We prove that this forms a coproduct in $\mathbb{T}$. Write $T_J = \coprod_{j \in J} T_j$.

First, we show that the (ι_i, ν_i) are LTS morphisms. Given $i \in J$, if we take any transition $X' \xrightarrow{a'}_{T_i} Y'$ in T_i with $a' \in L_i$, then $\nu_i(a') \neq \varepsilon$ because ν_i is an injective map (see Proof of Coproduct in Appendix A), and this means $\iota_i(X') \in S_J$, $\iota_i(Y') \in S_J$, and $\nu_i(a') \in L_J$. Thus, $\iota_i(X') \xrightarrow{\nu_i(a')}_{T_J} \iota_i(Y')$ in T_J by definition of δ_J .

Consider any J -indexed family of LTS morphisms $\{(\sigma_j, \lambda_j) : T_j \rightarrow T\}_{j \in J}$ with $T = (S, L, \delta)$. Then we have a J -indexed family of functions $\{\sigma_j : S_j \rightarrow S\}_{j \in J}$ and a J -indexed family of labelling morphisms $\{\lambda_j : L_j \rightarrow_\varepsilon L\}_{j \in J}$, which means that the universal property of the coproducts $\coprod_{j \in J} S_j$ and $\coprod_{j \in J}^\varepsilon L_j$ in **Set** and **Set** $_\varepsilon$ respectively give us :

- a unique function $\sigma : \coprod_{j \in J} S_j \rightarrow S$ such that $\sigma \circ \iota_i = \sigma_i$ for each $i \in J$, and
- a unique labelling morphism $\lambda : \coprod_{j \in J}^\varepsilon L_j \rightarrow_\varepsilon L$ such that $\lambda \circ \nu_i = \lambda_i$ for each $i \in J$

We verify that $(\sigma, \lambda) : T_J \rightarrow T$ is a LTS morphism. Consider any $X \xrightarrow{a}_{T_J} Y$ in T_J . Then, there exists $i \in J$ and a transition $X_i \xrightarrow{a_i}_{T_i} Y_i$ in T_i such that

$\iota_i(X_i) = X$, $\iota_i(Y_i) = Y$, and $\nu_i(a_i) = a$. We get that $\sigma_i(X_i) \xrightarrow{\lambda_i(a_i)}_T \sigma_i(Y_i)$ in T since $(\sigma_i, \lambda_i) : T_i \rightarrow T$ is a LTS morphism. But then, $\sigma(\iota_i(X_i)) \xrightarrow{\lambda(\nu_i(a_i))}_T \sigma(\iota_i(Y_i))$ in T , and this is $\sigma(X) \xrightarrow{\lambda(a)}_T \sigma(Y)$.

If there is any other LTS morphism $(\sigma', \lambda') : T \rightarrow T_J$ such that $(\sigma', \lambda') \circ (\iota_i, \nu_i) = (\sigma_i, \lambda_i)$ for all $i \in J$, then we get $\sigma' \circ \iota_i = \sigma_i$ and $\lambda' \circ \nu_i = \lambda_i$ for all $i \in J$, and the uniqueness part of the universal property of $\prod_{j \in J} S_j$ and $\prod_{j \in J}^\epsilon L_j$ yields that $\sigma' = \sigma$ and $\lambda' = \lambda$. This means $(\sigma', \lambda') = (\sigma, \lambda)$.

Equalizers : Given a pair of LTS morphisms $T = (S, L, \delta) \xrightleftharpoons[(\sigma_2, \lambda_2)]{(\sigma_1, \lambda_1)} T' = (S', L', \delta')$, we can form an equalizer by forming a pair with the equalizer of (σ_1, σ_2) in **Set** and the equalizer of (λ_1, λ_2) in **Set**_ε.

More precisely, consider the subset $S_e = \{X \in S \mid \sigma_1(X) = \sigma_2(X)\} \subseteq S$, the subset $L_e = \{a \in L \mid \lambda_1(a) = \lambda_2(a)\} \subseteq L$, and the subset of transitions $\delta_e = \delta \cap (S_e \times L_e \times S_e)$, and then take the inclusion map σ_e from S_e to S and the inclusion map λ_e from $L_e \cup \{\varepsilon\}$ to $L \cup \{\varepsilon\}$ (this is a labelling morphism). With the labelled transition system $T_e = (S_e, L_e, \delta_e)$, we show that $(\sigma_e, \lambda_e) : T_e \rightarrow T$ is the equalizer of the pair of LTS morphisms (σ_1, λ_1) and (σ_2, λ_2) .

We have that T_e is a sub-LTS of T (this is from Definition 2.3.1 and Equation 2.3.1), and we know that the inclusion maps give (σ_e, λ_e) as an LTS morphism from T_e to T , clearly.

Furthermore, for any $X \in S_e$, we get $\sigma_1(X) = \sigma_2(X)$, and this is $\sigma_1(\sigma_e(X)) = \sigma_2(\sigma_e(X))$ since $\sigma_e(X) = X$. In particular, we get $\sigma_1 \circ \sigma_e = \sigma_2 \circ \sigma_e$. Also, for any $a \in L_e$, we get $\lambda_1(a) = \lambda_2(a)$, and this is $\lambda_1(\lambda_e(a)) = \lambda_2(\lambda_e(a))$ since $\lambda_e(a) = a$. In particular, we get $\lambda_1 \circ \lambda_e = \lambda_2 \circ \lambda_e$. This shows $(\sigma_1, \lambda_1) \circ (\sigma_e, \lambda_e) = (\sigma_2, \lambda_2) \circ (\sigma_e, \lambda_e)$.

Now, consider any labelled transition system $T' = (S', L', \delta')$ and any LTS morphism $(\sigma, \lambda) : T' \rightarrow T$ such that $(\sigma_1, \lambda_1) \circ (\sigma, \lambda) = (\sigma_2, \lambda_2) \circ (\sigma, \lambda)$. Then, consider the LTS morphism $(\sigma_0, \lambda_0) : T' \rightarrow T_0$ given as $\sigma_0(X) = \sigma(X)$ for all $X \in S'$ and $\lambda_0(a) = \lambda(a)$ for all $a \in L'$. These component maps are well-defined since :

- For any X in S' , we get $\sigma_1(\sigma_0(X)) = \sigma_1(\sigma(X)) = \sigma_2(\sigma(X)) = \sigma_2(\sigma_0(X))$, which means $\sigma_0(X) \in S_e$, and
- for any a in L' , if $\lambda(a) \neq \varepsilon$, then we get $\lambda_1(\lambda_0(a)) = \lambda_1(\lambda(a)) = \lambda_2(\lambda(a)) = \lambda_2(\lambda_0(a))$ and this means $\lambda_0(a) \in L_e$. If $\lambda(a) = \varepsilon$, then $\lambda_0(a) = \varepsilon$, and this is in $L_e \cup \{\varepsilon\}$. So, this is a well-defined labelling morphism.

This (σ_0, λ_0) is a LTS morphism because for any $X \xrightarrow{a}_{T'} Y$ in T' , we get $\sigma(X) \xrightarrow{\lambda(a)}_T \sigma(Y)$, and this means $\sigma_0(X) \xrightarrow{\lambda_0(a)}_{T_e} \sigma_0(Y)$.

Finally, we have $(\sigma_e, \lambda_e) \circ (\sigma_0, \lambda_0) = (\sigma, \lambda)$ since for any $X \in S'$, we get $\sigma_e(\sigma_0(X)) = \sigma_e(\sigma(X)) = \sigma(X)$, and for any $a \in L'$, we get $\lambda_e(\lambda_0(a)) = \lambda_e(\lambda(a)) = \lambda(a)$. If there is any other LTS morphism $(\sigma'_0, \lambda'_0) : T' \rightarrow T_0$ such that $(\sigma_e, \lambda_e) \circ (\sigma'_0, \lambda'_0) = (\sigma, \lambda)$, then for any $X \in S'$, we get $\sigma_0(X) = \sigma_e(\sigma'_0(X)) = \sigma'_0(X)$. Also, we get for any $a \in L'$ that $\lambda_0(a) = \lambda_e(\lambda'_0(a)) = \lambda(a) = \lambda_e(\lambda'_0(a)) = \lambda'_0(a)$. In other words, $(\sigma_0, \lambda_0) = (\sigma'_0, \lambda'_0)$.

Coequalizers : Given a pair of LTS morphisms $T' = (S', L', \delta') \xrightarrow[(\sigma_2, \lambda_2)]{(\sigma_1, \lambda_1)} T = (S, L, \delta)$,

we get a pair of labelling morphisms $L' \xrightarrow[(\sigma_2, \lambda_2)]{(\sigma_1, \lambda_1)} L$, and we can take the labelling coequalizer (as in Definition 1.2.5) $\lambda_q : L \rightarrow_{\varepsilon} L_q$.

Let $\sim_q$ be the smallest equivalence relation on S such that :

$$\forall X \in S', \sigma_1(X) \sim_q \sigma_2(X), \text{ and}$$

$$\forall a \in L, \forall X, Y \in S', [\text{if } \lambda_q(a) = \varepsilon \text{ and } X \xrightarrow{a}_T Y, \text{ then } X \sim_q Y]$$

We form the quotient set $S_q := S / \sim_q$ of S with respect to $\sim_q$ and we write $\sigma_q : S \rightarrow S_q$ for the corresponding quotient map. Finally, define $\delta_q := \{(\sigma_q(X), \lambda_q(a), \sigma_q(Y)) \in S_q \times L_q \times S_q \mid (X, a, Y) \in \delta\}$.

We have that $T_q := (S_q, L_q, \delta_q)$ is a LTS and $(\sigma_q, \lambda_q) : T \rightarrow T_q$ is a LTS morphism that is the coequalizer of the pair of LTS morphisms (σ_1, λ_1) and (σ_2, λ_2) . We verify that this is the case.

First of all, we verify that $(\sigma_q, \lambda_q) : T \rightarrow T_q$ is a LTS morphism. Consider any $X \xrightarrow{a}_T Y$ in T . If $\lambda_q(a) \neq \varepsilon$, then $\lambda_q(a) \in L_q$ and this means $\sigma_q(X) \xrightarrow{\lambda_q(a)}_{T_q} \sigma_q(Y)$ by definition of δ_q . If $\lambda_q(a) = \varepsilon$, then $X \sim_q Y$, and this means $\sigma_q(X) = \sigma_q(Y)$.

We verify that $(\sigma_q, \lambda_q) \circ (\sigma_1, \lambda_1) = (\sigma_q, \lambda_q) \circ (\sigma_2, \lambda_2)$. We already have $\lambda_q \circ \lambda_1 = \lambda_q \circ \lambda_2$ since λ_q is the labelling coequalizer for the pair of labelling morphisms λ_1 and λ_2 . Also, for any $X \in S'$, we get $\sigma_1(X) \sim_q \sigma_2(X)$, and this means $\sigma_q(\sigma_1(X)) = \sigma_q(\sigma_2(X))$. Thus, $\sigma_q \circ \sigma_1 = \sigma_q \circ \sigma_2$. This proves $(\sigma_q, \lambda_q) \circ (\sigma_1, \lambda_1) = (\sigma_q, \lambda_q) \circ (\sigma_2, \lambda_2)$.

Consider any labelled transition system $T_0 = (S_0, L_0, \delta_0)$ and LTS morphism $(\sigma, \lambda) : T \rightarrow T_0$ such that $(\sigma, \lambda) \circ (\sigma_1, \lambda_1) = (\sigma, \lambda) \circ (\sigma_2, \lambda_2)$. Since λ_q is the coequalizer of λ_1 and λ_2 , and since $\lambda \circ \lambda_1 = \lambda \circ \lambda_2$, then there exists a unique labelling morphism $\lambda_u : L_q \rightarrow_{\varepsilon} L_0$ such that $\lambda_u \circ \lambda_q = \lambda$.

We build a set function $\sigma_u : S_q \rightarrow S_0$ by setting $\sigma_u(\sigma_q(X)) = \sigma(X)$ for any $X \in S$. We show that this is well-defined. Consider any $X, Y \in S$ such that $\sigma_q(X) = \sigma_q(Y)$. This means $X \sim_q Y$, and at the basis, we check the following two possible cases :

If there exists $Z \in S'$ such that $\sigma_1(Z) = X$ and $\sigma_2(Z) = Y$, then $\sigma(X) = \sigma(\sigma_1(Z)) = \sigma(\sigma_2(Z)) = \sigma(Y)$.

If $X \xrightarrow{a}_T Y$ and $\lambda_q(a) = \varepsilon$ for some $a \in L$, then $\lambda(a) = \lambda_u(\lambda_q(a)) = \lambda_u(\varepsilon) = \varepsilon$, and this means $\sigma(X) = \sigma(Y)$ since (σ, λ) is a LTS morphism.

This property that $\sigma(X) = \sigma(Y)$ extends naturally with symmetry and transitivity

of $\sim_q$, and we get thus, $\sigma(X) = \sigma(Y)$ for all $X, Y \in S$. This shows that σ_u is well-defined. We get $\sigma_u \circ \sigma_q = \sigma$ directly.

We verify that $(\sigma_u, \lambda_u) : T_q \rightarrow T_0$ is a LTS morphism. Consider any $X' \xrightarrow{b}_{T_q} Y'$ in T_q . Then there exists $X, Y \in S$ and $a \in L$ such that $\sigma_q(X) = X'$, $\sigma_q(Y) = Y'$, and $\lambda_q(a) = b$ (λ_q is a surjective map (see Proof of Coequalizer in Appendix A)) with $X \xrightarrow{a}_T Y$. We get that $\sigma(X) \xrightarrow{\lambda(a)}_{T_0} \sigma(Y)$ since (σ, λ) is a LTS morphism. In particular, we get that $\sigma_u(\sigma_q(X)) \xrightarrow{\lambda_u(\lambda_q(a))}_{T_0} \sigma_u(\sigma_q(Y))$, and this is $\sigma_u(X') \xrightarrow{\lambda_u(b)}_{T_0} \sigma_u(Y')$.

Finally, if there exists any other $(\sigma'_u, \lambda'_u) : T_q \rightarrow T_0$ such that $(\sigma'_u, \lambda'_u) \circ (\sigma_q, \lambda_q) = (\sigma, \lambda)$. Then $\lambda'_u \circ \lambda_q = \lambda$, and by uniqueness part of the universal property of λ_q as a coequalizer, we get $\lambda'_u = \lambda_u$. Also, for any $\sigma_q(X) \in S_q$, we get $\sigma_u(\sigma_q(X)) = \sigma(X) = \sigma'_u(\sigma_q(X))$, and this means $\sigma'_u = \sigma_u$. This shows that $(\sigma'_u, \lambda'_u) = (\sigma_u, \lambda_u)$.

□

Appendix C

Proof of adjunction between $\mathbb{T}$ and $\mathbb{M}$

Theorem C.0.1 . *There is an adjunction between $\mathbb{T}$ and $\mathbb{M}$:*

$$\begin{array}{ccc} & F_{tm} & \\ \mathbb{T} & \xrightarrow{\quad} & \mathbb{M} \\ & G_{mt} & \end{array} \quad \perp$$

where F_{tm} and G_{mt} are defined as follows :

1. Define $F_{tm} : \mathbb{T} \rightarrow \mathbb{M}$ as the functor that sends a labelled transition system $T = (S, L, \delta)$ to

$$F_{tm}(T) = (S, M, \cdot, e, \tilde{\delta})$$

where the set of states is preserved, $M = L^*$, $e = \Lambda$ is the empty word of L^* and $\cdot$ is word concatenation in L^* . We define $\xrightarrow{\Lambda}_{F_{tm}(T)} := \Delta_S$, and for any $a_1 \dots a_n \in L^*$, $\xrightarrow{a_1 \dots a_n}_{F_{tm}(T)} := \xrightarrow{a_n}_T \circ \dots \circ \xrightarrow{a_1}_T$, and this determines the transition specifier of $F_{tm}(T)$ that we have designated as $\tilde{\delta}$.

Given a LTS morphism $(\sigma, \lambda) : T_0 \rightarrow T$, we define $F_{tm}(\sigma, \lambda) : F_{tm}(T_0) \rightarrow F_{tm}(T)$ as $F_{tm}(\sigma, \lambda) = (\sigma, \lambda^*)$ where λ^* is an extension of the labelling morphism λ that

is provided by the functor $(-)^*$ from $\mathbf{Set}_\varepsilon$ to $\mathbf{Mon}$ in Definition 1.2.7.

2. Define $G_{mt} : \mathbb{M} \rightarrow \mathbb{T}$ as the functor that sends a monoid labelled transition system $(S, M, \cdot, e, \delta)$ to the labelled transition system (S, L, δ') where $L = M \setminus \{e\}$ and $\delta' = \delta \setminus \{(X, e, X) \mid X \in S\}$ (thus G_{mt} simply forgets composition in the labelling monoid, and removes the neutral element so that the latter will not conflict with ε).

Given a MLTS morphism $(\sigma, \lambda) : T_0 \rightarrow T$, we define $G_{mt}(\sigma, \lambda) : G_{mt}(T_0) \rightarrow G_{mt}(T)$ as $G_{mt}(\sigma, \lambda) = (\sigma, G_{m\varepsilon}(\lambda))$ where $G_{m\varepsilon}$ is provided as a functor from $\mathbf{Mon}$ to $\mathbf{Set}_\varepsilon$ in Definition 1.2.10.

We have that F_{tm} is left adjoint to G_{mt} .

Proof. F_{tm} is a functor :

Consider any labelled transition systems $T = (S, L, \delta)$. We have that $(L^*, \cdot, \Lambda)$ is a monoid, clearly. We have that $(S, L^*, \tilde{\delta})$ is a well-defined LTS with $\tilde{\delta} \subseteq S \times L^* \times S$ from the relational diagram specification. Axiom (3) of the MLTS definition is satisfied with $\xrightarrow{\Lambda}_{F_{tm}(T)} := \Delta_S$. Finally, consider any $m, n \in L^*$. If $m = a_1 \dots a_k$ and $n = b_1 \dots b_l$ for some $a_i, b_i \in L$, then $\xrightarrow{m \cdot n}_{F_{tm}(T)} = \xrightarrow{a_1 \dots a_k b_1 \dots b_l}_{F_{tm}(T)} = \xrightarrow{b_l}_{\rightarrow_T} \circ \dots \circ \xrightarrow{b_1}_{\rightarrow_T} \circ \xrightarrow{a_k}_{\rightarrow_T} \circ \dots \circ \xrightarrow{a_1}_{\rightarrow_T} = \xrightarrow{b_1 \dots b_l}_{F_{tm}(T)} \circ \xrightarrow{a_1 \dots a_k}_{F_{tm}(T)} = \xrightarrow{n}_{F_{tm}(T)} \circ \xrightarrow{m}_{F_{tm}(T)}$. If $n = \Lambda$, we get $\xrightarrow{m \cdot n}_{F_{tm}(T)} = \xrightarrow{m}_{F_{tm}(T)} = \xrightarrow{\Lambda}_{F_{tm}(T)} \circ \xrightarrow{m}_{F_{tm}(T)} = \xrightarrow{n}_{F_{tm}(T)} \circ \xrightarrow{m}_{F_{tm}(T)}$. Similarly, we get $\xrightarrow{m \cdot n}_{F_{tm}(T)} = \xrightarrow{n}_{F_{tm}(T)} \circ \xrightarrow{m}_{F_{tm}(T)}$ for $m = \Lambda$, and we get that Axiom (4) of the MLTS definition is verified. Thus, $F_{tm}(T)$ is a MLTS.

Consider any LTS morphism $(\sigma, \lambda) : T_0 \rightarrow T$ with $T = (S_0, L_0, \delta_0)$ and $T = (S, L, \delta)$. We get that $\lambda^* : L_0^* \rightarrow L^*$ is a monoid morphism from Proposition 1.2.9. Consider any $m \in L_0^*$ and any transition $X \xrightarrow{m}_{F_{tm}(T_0)} Y$ in $F_{tm}(T_0)$. If $m = \Lambda$, then $X = Y$, and we get $\sigma(X) = \sigma(Y)$. If $m \neq \Lambda$, then $m = a_1 \dots a_n$ for some $a_i \in L_0$. But then, $X \xrightarrow{a_1 \dots a_n}_{F_{tm}(T_0)} Y$, and we get $(X, Y) \in \xrightarrow{a_n}_{\rightarrow_{T_0}} \circ \dots \circ \xrightarrow{a_1}_{\rightarrow_{T_0}}$. This means that there exists $X_0, X_1, \dots, X_n \in S$ such that $X = X_0 \xrightarrow{a_1}_{\rightarrow_{T_0}} X_1 \xrightarrow{a_2}_{\rightarrow_{T_0}} \dots \xrightarrow{a_n}_{\rightarrow_{T_0}} X_n = Y$,

and we get $\sigma(X_i) \xrightarrow{\lambda(a_i)}_T \sigma(X_{i+1})$ for all i since $(\sigma, \lambda) : T_0 \rightarrow T$ is a LTS morphism. Consider any $i \in \{1, \dots, n\}$. If $\lambda(a_i) = \varepsilon$, then $\lambda^*(a_i) = \Lambda$ from the way λ^* is defined, and we get $\xrightarrow{\lambda^*(a_i)}_{F_{tm}(T)} = \xrightarrow{\Lambda}_{F_{tm}(T)} = \Delta_S = \xrightarrow{\varepsilon}_T = \xrightarrow{\lambda(a_i)}_T$. If $\lambda(a_i) \neq \varepsilon$, then $\lambda^*(a_i) = \lambda(a_i)$ by definition of λ^* , and we get $\xrightarrow{\lambda^*(a_i)}_{F_{tm}(T)} = \xrightarrow{\lambda(a_i)}_T$. Thus, $\xrightarrow{\lambda^*(a_i)}_{F_{tm}(T)} = \xrightarrow{\lambda(a_i)}_T$ for all $i \in \{1, \dots, n\}$. We derive that $\sigma(X_i) \xrightarrow{\lambda^*(a_i)}_{F_{tm}(T)} \sigma(X_{i+1})$ for all i , and we get that $(\sigma(X), \sigma(Y)) \in (\xrightarrow{\lambda^*(a_n)}_{F_{tm}(T)} \circ \dots \circ \xrightarrow{\lambda^*(a_1)}_{F_{tm}(T)})$ this way. This means that $(\sigma(X), \sigma(Y)) \in (\xrightarrow{\lambda^*(a_1) \dots \lambda^*(a_n)}_{F_{tm}(T)})$ (from the contravariance of $(-)$), and we get $\sigma(X) \xrightarrow{\lambda^*(a_1) \dots \lambda^*(a_n)}_{F_{tm}(T)} \sigma(Y)$ since λ^* is a monoid morphism. This proves that $F_{tm}(\sigma, \lambda) = (\sigma, \lambda^*)$ is a MLTS morphism from $F_{tm}(T_0)$ to $F_{tm}(T)$.

Given a labelled transition system $T = (S, L, \delta)$ and the LTS identity morphism $(1_S, 1_L) : T \rightarrow T$, we know from Proposition 1.2.9 that 1_{L^*} is the identity monoid morphism on L^* , and this means $F_{tm}(1_S, 1_L) = (1_S, 1_{L^*})$ is the MLTS identity morphism on $F_{tm}(T)$, and thus F_{tm} preserves identities.

Finally, given LTS morphisms $(\sigma_1, \lambda_1) : T_0 \rightarrow T_1$ and $(\sigma_2, \lambda_2) : T_1 \rightarrow T_2$, we have that $\lambda_2^* \circ \lambda_1^* = (\lambda_2 \circ \lambda_1)^*$ since $(-)^*$ is a functor, and get $F_{tm}(\sigma_2, \lambda_2) \circ F_{tm}(\sigma_1, \lambda_1) = (\sigma_2, \lambda_2^*) \circ (\sigma_1, \lambda_1^*) = (\sigma_2 \circ \sigma_1, \lambda_2^* \circ \lambda_1^*) = (\sigma_2 \circ \sigma_1, (\lambda_2 \circ \lambda_1)^*) = F_{tm}(\sigma_2 \circ \sigma_1, \lambda_2 \circ \lambda_1)$, and F_{tm} preserves composition.

This proves that F_{tm} is a functor.

G_{mt} is a functor :

Consider any monoid labelled transition system $(S, M, \cdot, e, \delta)$. Set $\delta' := \delta \setminus \{(X, e, X) \mid X \in S\}$. By definition, we have that (S, M, δ) is a LTS, and since $\delta' = \delta \setminus \{(X, e, X) \mid X \in S\} = \delta \cap (S \times M \setminus \{e\} \times S)$, we get that $G_{mt}(T) = (S, M \setminus \{e\}, \delta')$ is a sub-LTS of (S, M, δ) . In particular, it is a LTS.

Consider any MLTS morphism $(\sigma, \lambda) : T_0 \rightarrow T$ where $T_0 = (S_0, M_0, \cdot_0, e_0, \delta_0)$ and $T = (S, M, \cdot, e, \delta)$. We saw that $G_{m\varepsilon}(\lambda)$ is a labelling morphism from $M \setminus \{e\}$ to $M' \setminus \{e'\}$ in $\mathbf{Set}_\varepsilon$ with Proposition 1.2.11. Now, consider any $m \in M_0 \setminus \{e_0\}$ and any transition $X \xrightarrow{m}_{G_{mt}(T_0)} Y$. We have that $\xrightarrow{m}_{G_{mt}(T_0)} = \xrightarrow{m}_{T_0}$, and thus $X \xrightarrow{m}_{T_0} Y$. But then, we get that $\sigma(X) \xrightarrow{\lambda(m)}_T \sigma(Y)$ by definition of a MLTS morphism. If $G_{m\varepsilon}(\lambda)(m) = \varepsilon$, then $\lambda(a) = e$ by definition of $G_{m\varepsilon}$, and we get $\sigma(X) = \sigma(Y)$. If $G_{m\varepsilon}(\lambda)(m) \neq \varepsilon$, then $G_{m\varepsilon}(\lambda)(m) = \lambda(m) \in M \setminus \{e\}$, and we get $X \xrightarrow{G_{m\varepsilon}(\lambda)(m)}_{G_{mt}(T)} Y$. This proves that $G_{mt}(\sigma, \lambda) = (\sigma, G_{m\varepsilon}(\lambda))$ is a LTS morphism from $G_{mt}(T_0)$ to $G_{mt}(T)$.

Given a monoid labelled transition system $T = (S, M, \cdot, e, \delta)$ and the LTS identity morphism $(1_S, 1_M) : T \rightarrow T$, we know from Proposition 1.2.11 that $G_{m\varepsilon}(1_M)$ is the identity on the labelling set $M \setminus \{e\}$, and this means $G_{mt}(1_S, 1_M) = (1_S, G_{m\varepsilon}(1_M))$ is the LTS identity morphism on $G_{mt}(T)$, and G_{mt} preserves identities.

Finally, given MLTS morphisms $(\sigma_1, \lambda_1) : T_0 \rightarrow T_1$ and $(\sigma_2, \lambda_2) : T_1 \rightarrow T_2$, we have that $G_{m\varepsilon}(\lambda_2) \circ G_{m\varepsilon}(\lambda_1) = G_{m\varepsilon}(\lambda_2 \circ \lambda_1)$ since $G_{m\varepsilon}$ is a functor, and get $G_{mt}(\sigma_2, \lambda_2) \circ G_{mt}(\sigma_1, \lambda_1) = (\sigma_2, G_{m\varepsilon}(\lambda_2)) \circ (\sigma_1, G_{m\varepsilon}(\lambda_1)) = (\sigma_2 \circ \sigma_1, G_{m\varepsilon}(\lambda_2) \circ G_{m\varepsilon}(\lambda_1)) = (\sigma_2 \circ \sigma_1, G_{m\varepsilon}(\lambda_2 \circ \lambda_1)) = G_{mt}(\sigma_2 \circ \sigma_1, \lambda_2 \circ \lambda_1)$, and G_{mt} preserves composition.

This proves that G_{mt} is a functor.

We now demonstrate that for any labelled transition system $T_0 = (S_0, L_0, \delta_0)$ and any monoid labelled transition system $T = (S, M, \cdot, e, \delta)$, there is a bijective correspondence between $\mathbf{Hom}_{\mathbb{M}}(F_{tm}(T_0), T)$ and $\mathbf{Hom}_{\mathbb{T}}(T_0, G_{mt}(T))$, where $F_{tm}(T_0) = (S_0, L_0^*, \cdot, \Lambda, \tilde{\delta})$ and $G_{mt}(T) = (S, M \setminus \{e\}, \delta \setminus \{(X, e, X) \mid X \in S\})$. We define a map $\tilde{\theta}_{T_0, T} : \mathbf{Hom}_{\mathbb{M}}(F_{tm}(T_0), T) \rightarrow \mathbf{Hom}_{\mathbb{T}}(T_0, G_{mt}(T))$ that sends a MLTS morphism $(\sigma, \lambda) : F_{tm}(T_0) \rightarrow T$ to the LTS morphism $(\sigma, \theta_{L_0, M}(\lambda)) : T_0 \rightarrow G_{mt}(T)$ where $\theta_{L_0, M}(\lambda) : L_0 \rightarrow M \setminus \{e\}$ is given by a previously established bijection from $\theta_{L_0, M} : \mathbf{Hom}_{\mathbf{Mon}}(L_0^*, M) \rightarrow \mathbf{Hom}_{\mathbf{Set}_\varepsilon}(L_0, M \setminus \{e\})$ in Proposition 1.2.12. We will simply write $\theta := \theta_{L_0, M}$ and $\tilde{\theta} = \tilde{\theta}_{T_0, T}$ for convenience here.

We verify that $(\sigma, \theta(\lambda))$ is a LTS morphism from $T_0 = (S_0, L_0, \delta_0)$ to $G_{mt}(T) = (S, M \setminus \{e\}, \delta \setminus \{(X, e, X) \mid X \in S\})$. Consider any $a \in L_0$. If $\theta(\lambda)(a) = \varepsilon$, then $\lambda(a) = e$ by definition of θ , and we get $\xrightarrow{\theta(\lambda)(a)}_{G_{mt}(T)} = \Delta_S = \xrightarrow{\lambda(a)}_T$. If $\theta(\lambda)(a) \neq \varepsilon$, then $\theta(\lambda)(a) = \lambda(a)$ by definition of θ , and we get $\xrightarrow{\theta(\lambda)(a)}_{G_{mt}(T)} = \xrightarrow{\lambda(a)}_T$.

But then, for any transition $X \xrightarrow{a}_{T_0} Y$ in T_0 , we get that $X \xrightarrow{a}_{F_{tm}(T_0)} Y$ (since $\xrightarrow{a}_{F_{tm}(T_0)} = \xrightarrow{a}_{T_0}$), and then $\sigma(X) \xrightarrow{\lambda(a)}_T \sigma(Y)$ since (σ, λ) is a MLTS morphism. Thus, $\sigma(X) \xrightarrow{\theta(\lambda)(a)}_{G_{mt}(T)} \sigma(Y)$ (and this includes the case with $\theta(\lambda)(a) = \varepsilon$).

We can provide a map $\tilde{\theta}^{-1} : \mathbf{Hom}_{\mathbb{T}}(T_0, G_{mt}(T)) \rightarrow \mathbf{Hom}_{\mathbb{M}}(F_{tm}(T_0), T)$ that sends a LTS morphism $(\sigma, \lambda) : T_0 \rightarrow G_{mt}(T)$ to the LTS morphism $(\sigma, \theta^{-1}(\lambda)) : F_{tm}(T_0) \rightarrow T$ where $\theta^{-1}(\lambda) : L_0^* \rightarrow M$ is given by a previously established map $\theta^{-1} : \mathbf{Hom}_{\text{Set}_\varepsilon}(L_0, M \setminus \{e\}) \rightarrow \mathbf{Hom}_{\text{Mon}}(L_0^*, M)$ in Proposition 1.2.12 (it was shown in the proof of this proposition that θ^{-1} is the inverse of θ).

We verify that $(\sigma, \theta^{-1}(\lambda))$ is a MLTS morphism from $F_{tm}(T_0) = (S_0, L_0^*, \cdot, \Lambda, \tilde{\delta}_0)$ to $T = (S, M, \cdot, e, \delta)$. Consider any $a_1 \dots a_n \in L_0^*$, with each $a_i \in L_0$, and any transition $X \xrightarrow{a_1 \dots a_n}_{F_{tm}(T_0)} Y$ in $F_{tm}(T_0)$. We get that $(X, Y) \in \xrightarrow{a_n}_{T_0} \circ \dots \circ \xrightarrow{a_1}_{T_0}$, and this means that there exists $X_0, X_1, \dots, X_n \in S_0$ such that $X = X_0 \xrightarrow{a_1}_{T_0} X_1 \xrightarrow{a_2}_{T_0} \dots \xrightarrow{a_n}_{T_0} X_n = Y$. But then, $\sigma(X) = \sigma(X_0) \xrightarrow{\lambda(a_1)}_{G_{mt}(T)} \sigma(X_1) \xrightarrow{\lambda(a_2)}_{G_{mt}(T)} \dots \xrightarrow{\lambda(a_n)}_{G_{mt}(T)} \sigma(X_n) = \sigma(Y)$ since (σ, λ) is a LTS morphism from T_0 to $G_{mt}(T)$.

Consider any $i \in \{1, \dots, n\}$. If $\lambda(a_i) = \varepsilon$, then we have $\theta^{-1}(\lambda)(a_i) = e$ by definition of θ^{-1} , and we get $\xrightarrow{\lambda(a_i)}_{G_{mt}(T)} = \Delta_S = \xrightarrow{\theta^{-1}(\lambda)(a_i)}_T$. If $\lambda(a_i) \neq \varepsilon$, then we have $\theta^{-1}(\lambda)(a_i) = \lambda(a_i) \in M \setminus \{e\}$ by definition of θ^{-1} , and we get $\xrightarrow{\lambda(a_i)}_{G_{mt}(T)} = \xrightarrow{\theta^{-1}(\lambda)(a_i)}_T$.

This means that $\sigma(X) = \sigma(X_0) \xrightarrow{\theta^{-1}(\lambda)(a_1)}_T \sigma(X_1) \xrightarrow{\theta^{-1}(\lambda)(a_2)}_T \dots \xrightarrow{\theta^{-1}(\lambda)(a_n)}_T \sigma(X_n) = \sigma(Y)$, and we get :

$$(\sigma(X), \sigma(Y)) \in (\xrightarrow{\theta^{-1}(\lambda)(a_n)}_T \circ \dots \circ \xrightarrow{\theta^{-1}(\lambda)(a_1)}_T) = \xrightarrow{\theta^{-1}(\lambda)(a_1) \dots \theta^{-1}(\lambda)(a_n)}_T$$

It was established that $\theta^{-1}(\lambda)$ is a monoid morphism in Proposition 1.2.12. Therefore, $\sigma(X) \xrightarrow{\theta^{-1}(\lambda)(a_1 \cdots a_n)}_{\gamma_T} \sigma(Y)$. This proves that $(\sigma, \theta^{-1}(\lambda))$ is a MLTS morphism from $F_{tm}(T_0)$ to T .

Since θ and θ^{-1} are inverses of each other on the hom-sets $\mathbf{Hom}_{\mathbf{Set}_\varepsilon}(L_0, M \setminus \{e\})$ and $\mathbf{Hom}_{\mathbf{Mon}}(L_0^*, M)$, it is clear that the maps $\tilde{\theta}_{T_0, T}$ and $\tilde{\theta}_{T_0, T}^{-1}$ as defined above are inverses of each other. We get a bijective correspondence in between $\mathbf{Hom}_{\mathbb{M}}(F_{tm}(T_0), T)$ and $\mathbf{Hom}_{\mathbb{T}}(T_0, G_{mt}(T))$ thus, and this proves that F_{tm} is left adjoint to G_{mt} .

□

Appendix D

Proofs for Section 4.2

Proposition D.0.1 . *Suppose $\mathcal{T}$ is a $\mathbb{T}$ -valued sheaf on a complete Heyting algebra $\mathcal{H}$. Then the associated state presheaf $\mathcal{S}$ is a set-valued sheaf, and the associated labelling presheaf $\mathcal{L}$ is a $\mathbf{Set}_\varepsilon$ -valued sheaf.*

Proof. Suppose $\mathcal{T}$ is a $\mathbb{T}$ -valued sheaf on a complete Heyting algebra $\mathcal{H}$. Let U be region in $\mathcal{H}$ and suppose $\{V_j\}_{j \in J}$ is a proper cover of U .

($\mathcal{S}$ is a sheaf) : Consider any set S and any J -indexed family of functions $\{\sigma_j : S \rightarrow \mathcal{S}(V_j)\}_{j \in J}$, and suppose that $\text{res}_{V_j \wedge V_i}^{V_j}(\sigma_j) = \text{res}_{V_j \wedge V_i}^{V_i}(\sigma_i)$ for all i, j in J . Let $0_j : \emptyset \rightarrow_\varepsilon \mathcal{L}(V_j)$ and $0 : \emptyset \rightarrow_\varepsilon \mathcal{L}(U)$ be the unique labelling morphisms with their source in the initial object $\emptyset$ in $\mathbf{Set}_\varepsilon$. Consider the LTS $(S, L = \emptyset, \delta = \emptyset)$. We have that $(\sigma_j, 0_j) : (S, \emptyset, \emptyset) \rightarrow \mathcal{T}(V_j)$ are LTS morphisms (they do not need to preserve any transitions). In particular, $(\text{res}_{V_j \wedge V_i}^{V_j}, \rho_{V_j \wedge V_i}^{V_j})(\sigma_j, 0_j) = (\text{res}_{V_j \wedge V_i}^{V_j}(\sigma_j), \rho_{V_j \wedge V_i}^{V_j}(0_j)) = (\text{res}_{V_j \wedge V_i}^{V_i}(\sigma_i), \rho_{V_j \wedge V_i}^{V_i}(0_i))$ (remark that $\rho_{V_j \wedge V_i}^{V_j}(0_j) = \rho_{V_j \wedge V_i}^{V_i}(0_i)$ since they stem out of the initial object of $\mathbf{Set}_\varepsilon$). By the gluing property of $\mathcal{T}$, we get a LTS morphism $(\sigma, \lambda) : (S, \emptyset, \emptyset) \rightarrow \mathcal{T}(U)$ such that $(\text{res}_{V_j}^U, \rho_{V_j}^U)(\sigma, \lambda) = (\sigma_j, 0_j)$ for each j in J . Thus, $\text{res}_{V_j}^U(\sigma) = \sigma_j$ for each j in J and this means that $\sigma : S \rightarrow \mathcal{S}(U)$ is the gluing of the σ_j in question. This shows that $\mathcal{S}$ has the gluing property. Furthermore, if there is any other set function $\sigma' : S \rightarrow \mathcal{S}(U)$ such that $\text{res}_{V_j}^U(\sigma') = \sigma_j$ for each j in J , then consider the LTS morphisms $(\sigma', 0), (\sigma, 0) : (S, \emptyset, \emptyset) \rightarrow \mathcal{T}(U)$. We have

$(res_{V_j}^U, \rho_{V_j}^U)(\sigma', 0) = (\sigma_j, 0_j) = (res_{V_j}^U, \rho_{V_j}^U)(\sigma, 0)$, and by the locality of $\mathcal{T}$, we get that $(\sigma', 0) = (\sigma, 0)$. In particular, $\sigma = \sigma'$, and this proves that $\mathcal{S}$ has locality.

($\mathcal{L}$ is a sheaf) : Consider any labelling set L and any J -indexed family of labelling morphisms $\{\lambda_j : L \rightarrow_{\varepsilon} \mathcal{L}(V_j)\}_{j \in J}$, and suppose that $\rho_{V_j \wedge V_i}^{V_j}(\lambda_j) = \rho_{V_j \wedge V_i}^{V_i}(\lambda_i)$ for all i, j in J . Let $0_j : \emptyset \rightarrow \mathcal{S}(V_j)$ and $0 : \emptyset \rightarrow \mathcal{S}(U)$ be the unique set function with their source in the initial object $\emptyset$ in **Set**. Consider the LTS $(S = \emptyset, L, \delta = \emptyset)$, and this is only a LTS because we allowed ghost actions. We have that $(0_j, \lambda_j) : (\emptyset, L, \emptyset) \rightarrow \mathcal{T}(V_j)$ are LTS morphisms. In particular, $(res_{V_j \wedge V_i}^{V_j}, \rho_{V_j \wedge V_i}^{V_j})(0_j, \lambda_j) = (res_{V_j \wedge V_i}^{V_j}, \rho_{V_j \wedge V_i}^{V_j})(0_j, \lambda_j) = (res_{V_j \wedge V_i}^{V_i}, \rho_{V_j \wedge V_i}^{V_i})(0_i, \lambda_i)$. By the gluing property of $\mathcal{T}$, we get a LTS morphism $(\sigma, \lambda) : (\emptyset, L, \emptyset) \rightarrow \mathcal{T}(U)$ such that $(res_{V_j}^U, \rho_{V_j}^U)(\sigma, \lambda) = (0_j, \lambda_j)$ for each j in J . Thus, $\rho_{V_j}^U(\lambda) = \lambda_j$ for each j in J and this means that $\lambda : L \rightarrow_{\varepsilon} \mathcal{L}(U)$ is the gluing of the λ_j in question. This shows that $\mathcal{L}$ has the gluing property. Furthermore, if there is any other labelling morphism $\lambda' : L \rightarrow_{\varepsilon} \mathcal{L}(U)$ such that $\rho_{V_j}^U(\lambda') = \lambda_j$ for each j in J , then consider the LTS morphisms $(0, \lambda), (0, \lambda') : (\emptyset, L, \emptyset) \rightarrow \mathcal{T}(U)$. We have $(res_{V_j}^U, \rho_{V_j}^U)(0, \lambda) = (0_j, \lambda_j) = (res_{V_j}^U, \rho_{V_j}^U)(0, \lambda')$, and by the locality of $\mathcal{T}$, we get that $(0, \lambda) = (0, \lambda')$. In particular, $\lambda = \lambda'$, and this proves that $\mathcal{L}$ has locality. $\square$

Appendix E

Proofs for Section 4.3

Proposition E.0.1 [Equivalences for action containment]. *Consider any $\mathbb{T}$ -valued presheaf such that $\mathcal{S}$ is a sheaf, and consider any regions $V \leq U$ and action $b \in \mathcal{L}(U)$, and any ε -region K that j -complements V in U . Then the following statements are equivalent :*

1. $\frac{\rho_V^U(b)}{\rightarrow_V} \circ \text{res}_V^U = \text{res}_V^U \circ \xrightarrow{b}_U$ (i.e. V contains b)
2. For any $X' \xrightarrow{\rho_V^U(b)}_V Y'$ and $X'' \in \mathcal{S}(K)$ such that $X'|_{V \wedge K} = X''|_{V \wedge K}$, there exists unique $X, Y \in \mathcal{S}(U)$ such that $X \xrightarrow{b}_U Y$ and $X|_V = X'$, $Y|_K = X|_K = X''$ and $Y|_V = Y'$.
3. The action b glues its transitions over $\{V, K\}$ (see Definition 4.1.17)
4. The action b decomposes over $\{V, K\}$ (see Definition 4.1.16)
5. $\xrightarrow{b}_U = (\text{res}_V^{U-1} \circ \frac{\rho_V^U(b)}{\rightarrow_V} \circ \text{res}_V^U) \cap \simeq_K^U$

Proof. Consider any $\mathbb{T}$ -valued presheaf with $\mathcal{S}$ as a sheaf, and any regions $V \leq U$ and action $b \in \mathcal{L}(U)$, and any ε -region K that j -complements V in U .

[(1) $\Rightarrow$ (2)] Suppose (1) holds. Consider any transition $X' \xrightarrow{\rho_V^U(b)}_V Y'$ in V and state $X'' \in \mathcal{S}(K)$ such that $X'|_{V \wedge K} = X''|_{V \wedge K}$. Since X' and X'' agree as states on a proper cover $\{V, K\}$ of U , then the gluing property of $\mathcal{S}$ yields an $X \in \mathcal{S}(U)$ such that

$X|_V = X'$ and $X|_K = X''$. Since $X|_V \xrightarrow{\text{res}_V^U} X$ and $X|_V \xrightarrow{\rho_V^U(b)} Y'$, Proposition 4.3.10 yields a $Y \in \mathcal{S}(U)$ such that $X \xrightarrow{b} Y$ and $Y|_V = Y'$ and $Y|_K = X|_K = X''$. If there is any other $X_0, Y_0 \in \mathcal{S}(U)$ such that $X_0 \xrightarrow{b} Y_0$ and $X_0|_V = X'$, $Y_0|_K = X_0|_K = X''$ and $Y_0|_V = Y'$, then the locality of $\mathcal{S}$ applied to the cover $\{V, K\}$ of U yields $X_0 = X$ and $Y_0 = Y$.

[(2) $\Rightarrow$ (3)] Suppose (2) holds. Consider any $X' \xrightarrow{\rho_V^U(b)} Y'$ in V and $X'' \xrightarrow{\rho_K^U(b)} Y''$ in K with $X'|_{V \wedge K} = X''|_{V \wedge K}$ and $Y'|_{V \wedge K} = Y''|_{V \wedge K}$. By (2), we get unique $X, Y \in \mathcal{S}(U)$ such that $X \xrightarrow{a} Y$ and $X|_V = X'$, $Y|_K = X|_K = X''$ and $Y|_V = Y'$. Since $\rho_K^U(b) = \varepsilon$, we have $X'' = Y''$, and so $Y|_K = Y''$. This proves b glues its transitions over $\{V, K\}$.

[(3) $\Leftrightarrow$ (4)] It suffices to apply Proposition 4.1.18 to the case of $\{V, K\}$, i.e. b glues its transitions over $\{V, K\}$ iff b decomposes over $\{V, K\}$.

[(4) $\Rightarrow$ (5)] Suppose (4) holds. Then by definition :

$$\xrightarrow{b}_U \subseteq (\text{res}_V^{U-1} \circ \xrightarrow{\rho_V^U(b)}_V \circ \text{res}_V^U) \cap (\text{res}_K^{U-1} \circ \xrightarrow{\rho_K^U(b)}_K \circ \text{res}_K^U)$$

But $\text{res}_K^{U-1} \circ \xrightarrow{\rho_K^U(b)}_K \circ \text{res}_K^U = \text{res}_K^{U-1} \circ \xrightarrow{\varepsilon}_K \circ \text{res}_K^U = \text{res}_K^{U-1} \circ \Delta_{\mathcal{S}(K)} \circ \text{res}_K^U = \text{res}_K^{U-1} \circ \text{res}_K^U = \simeq_K^U$, and we are done.

[(5) $\Rightarrow$ (1)] Suppose (5) holds. The inclusion from right to left already holds from the presheaf structure and doesn't require the assumption of (5) (see the remark after Definition 4.3.15). For the inclusion from left to right : Consider any $(X, Y') \in (\xrightarrow{\rho_V^U(b)}_V \circ \text{res}_V^U)$. Then $X|_V \xrightarrow{\rho_V^U(b)} Y'$. Since b projects to ε in K , then $\rho_V^U(b)$ projects to ε in $V \wedge K$ (from Proposition 4.3.6). This means that $Y'|_{V \wedge K} = (X|_V)|_{V \wedge K} = X|_{V \wedge K} = (X|_K)|_{V \wedge K}$, and the agreement of $X|_K \in \mathcal{S}(K)$ and $Y' \in \mathcal{S}(V)$ on the cover $\{K, V\}$ of U allows them to glue to a unique $Y \in \mathcal{S}(U)$ such that $Y|_K = X|_K$ and $Y|_V = Y'$. But then, $X|_V \xrightarrow{\rho_V^U(b)} Y|_V$ and $X|_K = Y|_K$ means that $(X, Y) \in (\text{res}_{U,V}^{-1} \circ \xrightarrow{\rho_V^U(b)}_V \circ \text{res}_V^U) \cap \simeq_K^U = \xrightarrow{b}_U$. Thus, $(X, Y|_V) \in (\text{res}_V^U \circ \xrightarrow{b}_U)$, and since $Y' = Y|_V$, we obtain the inclusion in question. $\square$

Appendix F

Proofs for Section 5.1

Lemma F.0.1 [Natural Transformations preserve ε -Regions and Containing Regions]. *Consider any two $\mathbb{T}$ -adapted presheaves $\mathcal{T}$ and $\mathcal{T}'$ on a Heyting algebra $\mathcal{H}$, and consider any natural transformation $\{(\sigma_W, \lambda_W)\}_{W \in \mathcal{H}} : \mathcal{T} \rightarrow \mathcal{T}'$ in $[\mathcal{H}^{op}, \mathbb{T}]$, which is a family of LTS morphisms $(\sigma_W, \lambda_W) : \mathcal{T}(W) \rightarrow \mathcal{T}'(W)$ indexed by regions W in $\mathcal{H}$. Then for any regions $V \leq U$ and any action $b \in \mathcal{L}(U)$ the following statements hold :*

1. *If b vanishes in V with respect to U in $\mathcal{T}$, then $\lambda_U(b)$ vanishes in V with respect to U in $\mathcal{T}'$.*
2. *If b vanishes is contained in V with respect to U in $\mathcal{T}$, then $\lambda_U(b)$ is contained in V with respect to U in $\mathcal{T}'$.*

Proof. Consider any two $\mathbb{T}$ -adapted presheaves $\mathcal{T} = (\mathcal{S}, \mathcal{L}, \delta)$ and $\mathcal{T}' = (\mathcal{S}', \mathcal{L}', \delta')$ on a Heyting algebra $\mathcal{H}$, and any natural transformation $\{(\sigma_U, \lambda_U)\}_{U \in \mathcal{H}} : \mathcal{T} \rightarrow \mathcal{T}'$ in $[\mathcal{H}^{op}, \mathbb{T}]$. Consider any regions $V \leq U$, and write $(res_V^U, \rho_V^U) : \mathcal{T}(U) \rightarrow \mathcal{T}(V)$ and $((res_V^U)', (\rho_V^U)') : \mathcal{T}'(U) \rightarrow \mathcal{T}'(V)$ for the LTS restriction morphisms from U to V in $\mathcal{T}$ and $\mathcal{T}'$ respectively.

We can get an immediate sense of why this lemma is true because ε -regions can only get larger when applying LTS morphisms that preserve ε , and the commutative

diagrams provided by natural transformations assure that ε -regions are preserved also. Let us see how this is done.

[Proof for (1)] Consider any action $b \in \mathcal{L}(U)$ and suppose that b vanishes in V with respect to U in $\mathcal{T}$, i.e. $\rho_V^U(b) = \varepsilon$. By definition of a natural transformation in $[\mathcal{H}^{op}, \mathbb{T}]$, we have that the following diagram commutes in $\mathbb{T}$:

$$\begin{array}{ccc}
 \mathcal{T}(U) & \xrightarrow{(\sigma_U, \lambda_U)} & \mathcal{T}'(U) \\
 \downarrow (res_V^U, \rho_V^U) & & \downarrow ((res_V^U)', (\rho_V^U)') \\
 \mathcal{T}(V) & \xrightarrow{(\sigma_V, \lambda_V)} & \mathcal{T}'(V)
 \end{array}$$

For the labelling component morphisms, this implies that the following diagram commutes in $\mathbf{Set}_\varepsilon$:

$$\begin{array}{ccc}
 \mathcal{L}(U) & \xrightarrow{\lambda_U} & \mathcal{L}'(U) \\
 \downarrow \rho_V^U & & \downarrow (\rho_V^U)' \\
 \mathcal{L}(V) & \xrightarrow{\lambda_V} & \mathcal{L}'(V)
 \end{array}$$

and we get $(\rho_V^U)' \circ \lambda_U = \lambda_V \circ \rho_V^U$. But then, $(\rho_V^U)'(\lambda_U(b)) = \lambda_V(\rho_V^U(b)) = \lambda_V(\varepsilon) = \varepsilon$, and this means that V is an ε -region of $\lambda_U(b)$ in U with respect to $\mathcal{T}'$.

[Proof for (2)] Consider any action $b \in \mathcal{L}(U)$ and suppose that b is contained in V with respect to U in $\mathcal{T}$. This means that $V \vee \varepsilon_U(b) = U$. By (1) above, we have that b vanishes in $\varepsilon_U(b)$ with respect to U in $\mathcal{T}$, so $\lambda_U(b)$ vanishes in $\varepsilon_U(b)$ with respect to U in $\mathcal{T}'$. With $V \vee \varepsilon_U(b) = U$, we can apply the WCA axiom thus and we get that $\lambda_U(b)$ is contained in V with respect to U in $\mathcal{T}'$. $\square$

Proposition F.0.2 . *For any $\mathbb{T}$ -adapted presheaf, the SI-equivalence $\sim_U$ as defined above for a region U is an equivalence relation.*

Proof. Consider any $\mathbb{T}$ -adapted presheaf and a region U .

(Reflexive) : For any $(X, b, Y) \in \delta(U)$, we have $X \simeq_U^U X$ and $Y \simeq_U^U Y$, and since U contains b (i.e. $U \vee \varepsilon_U(a) = U$), we have that $(X, b, Y) \sim_U (X, b, Y)$.

(Symmetric) : Suppose that $(X, b, Y) \sim_U (X', b, Y')$ for some transitions $(X, b, Y), (X', b, Y') \in \delta(U)$. Then there exists a region V that contains b such that $X \simeq_V^U X'$ and $Y \simeq_V^U Y'$. But then $X' \simeq_V^U X$ and $Y' \simeq_V^U Y$, so $(X', b, Y') \sim_U (X, b, Y)$.

(Transitive) : Suppose that $(X, b, Y) \sim_U (X', b, Y') \sim_U (X'', b, Y'')$ for some transitions $(X, b, Y), (X', b, Y'), (X'', b, Y'') \in \delta(U)$. Then there exists regions V and V' that contain b and such that $X \simeq_V^U X'$ and $Y \simeq_V^U Y'$, and $X' \simeq_{V'}^U X''$ and $Y' \simeq_{V'}^U Y''$. We have that $V \wedge V'$ contains b by Proposition 5.1.4 and both $\simeq_V^U$ and $\simeq_{V'}^U$ are included in $\simeq_{V \wedge V'}^U$ (by Proposition 4.1.14). This implies that $X \simeq_{V \wedge V'}^U X'$ and $Y \simeq_{V \wedge V'}^U Y'$, and $X' \simeq_{V \wedge V'}^U X''$ and $Y' \simeq_{V \wedge V'}^U Y''$. The transitivity of $\simeq_{V \wedge V'}^U$ implies that $X \simeq_{V \wedge V'}^U X''$ and $Y \simeq_{V \wedge V'}^U Y''$, and this provides $(X, b, Y) \sim_U (X'', b, Y'')$. $\square$

Lemma F.0.3 [Natural Transformations preserve SI-Equivalences]. *Consider any two $\mathbb{T}$ -adapted presheaves $\mathcal{T}$ (with SI-equivalence $\sim$) and $\mathcal{T}'$ (with SI-equivalence $\sim'$) on a Heyting algebra $\mathcal{H}$, and consider any natural transformation $\{(\sigma_W, \lambda_W)\}_{W \in \mathcal{H}} : \mathcal{T} \rightarrow \mathcal{T}'$ in $[\mathcal{H}^{op}, \mathbb{T}]$, which is a family of LTS morphisms $(\sigma_W, \lambda_W) : \mathcal{T}(W) \rightarrow \mathcal{T}'(W)$ indexed by regions W in $\mathcal{H}$. Then for any regions $V \leq U$ and any transitions $(X, b, Y), (X', b, Y') \in \delta(U)$ such that $(X, b, Y) \sim_U (X', b, Y')$ and $\lambda_U(b) \neq \varepsilon$, we have that $(\sigma_U(X), \lambda_U(b), \sigma_U(Y)) \sim'_U (\sigma_U(X'), \lambda_U(b), \sigma_U(Y'))$.*

Proof. Consider any two $\mathbb{T}$ -adapted presheaves $\mathcal{T} = (\mathcal{S}, \mathcal{L}, \delta)$ and $\mathcal{T}' = (\mathcal{S}', \mathcal{L}', \delta')$ on a Heyting algebra $\mathcal{H}$, and any natural transformation $\{(\sigma_U, \lambda_U)\}_{U \in \mathcal{H}} : \mathcal{T} \rightarrow \mathcal{T}'$ in $[\mathcal{H}^{op}, \mathbb{T}]$. Consider any regions $V \leq U$, and write $(res_V^U, \rho_V^U) : \mathcal{T}(U) \rightarrow \mathcal{T}(V)$ and $((res_V^U)', (\rho_V^U)') : \mathcal{T}'(U) \rightarrow \mathcal{T}'(V)$ for the LTS restriction morphisms from U to V in $\mathcal{T}$ and $\mathcal{T}'$ respectively.

Consider any transitions $(X, b, Y), (X', b, Y') \in \delta(U)$ such that $(X, b, Y) \sim_U (X', b, Y')$ and $\lambda_U(b) \neq \varepsilon$. We verify that :

$$(\sigma_U(X), \lambda_U(b), \sigma_U(Y)) \sim'_U (\sigma_U(X'), \lambda_U(b), \sigma_U(Y'))$$

By definition of $\sim_U$, there exists a region V that contains b in U for $\mathcal{T}$, and such that $X|_V = X'|_V$ and $Y|_V = Y'|_V$. Also, we have that the following diagram commutes by definition of θ as a natural transformation :

$$\begin{array}{ccc} \mathcal{T}(U) & \xrightarrow{(\sigma_U, \lambda_U)} & \mathcal{T}'(U) \\ \downarrow (res_V^U, \rho_V^U) & & \downarrow ((res_V^U)', (\rho_V^U)') \\ \mathcal{T}(V) & \xrightarrow{(\sigma_V, \lambda_V)} & \mathcal{T}'(V) \end{array}$$

And for the states component maps, this implies that the following diagram commutes :

$$\begin{array}{ccc} \mathcal{S}(U) & \xrightarrow{\sigma_U} & \mathcal{S}'(U) \\ \downarrow res_V^U & & \downarrow (res_V^U)' \\ \mathcal{S}(V) & \xrightarrow{\sigma_V} & \mathcal{S}'(V) \end{array}$$

and we get $(res_V^U)' \circ \sigma_U = \sigma_V \circ res_V^U$.

Thus, we get $\sigma_U(X)|_V = \sigma_V(X|_V) = \sigma_V(X'|_V) = \sigma_U(X')|_V$ and $\sigma_U(Y)|_V = \sigma_V(Y|_V) = \sigma_V(Y'|_V) = \sigma_U(Y')|_V$. This means that $\sigma_U(X) \simeq_V^U \sigma_U(X')$ and $\sigma_U(Y) \simeq_V^U \sigma_U(Y')$. Now, since V contains b with respect to U in $\mathcal{T}$, we have by Lemma 5.1.13 that V contains $\lambda_U(b)$ with respect to U in $\mathcal{T}'$. This gives us :

$$(\sigma_U(X), \lambda_U(b), \sigma_U(Y)) \sim'_U (\sigma_U(X'), \lambda_U(b), \sigma_U(Y'))$$

as desired. $\square$

Proposition F.0.4 . *In a $\mathbb{T}$ -adapted presheaf, if V is a region that contains an action b in a region U , we have :*

$$\xrightarrow{b}_U \circ \simeq_V^U = \simeq_V^U \circ \xrightarrow{b}_U$$

Proof. Consider a $\mathbb{T}$ -adapted presheaf, regions $V \leq U$ and an action $b \in \mathcal{L}(U)$. Suppose that V is a region that contains an action b in a region U .

(For $\subseteq$) : Consider any $(X', Y) \in \xrightarrow{b}_U \circ \simeq_V^U$. Thus there exists $X \in \mathcal{S}(U)$ such that $X' \simeq_V^U X$ and $X \xrightarrow{b}_U Y$. This means $X'|_V = X|_V$ and $X|_{\varepsilon_U(b)} = Y|_{\varepsilon_U(b)}$. We get :

$$(X'|_{\varepsilon_U(b)})|_{\varepsilon_U(b) \wedge V} = X'|_{\varepsilon_U(b) \wedge V} = X|_{\varepsilon_U(b) \wedge V} = Y|_{\varepsilon_U(b) \wedge V} = (Y|_V)|_{\varepsilon_U(b) \wedge V}$$

And since $V \vee \varepsilon_U(b) = U$, we get a proper cover $\{V, \varepsilon_U(b)\}$ of U on which $X'|_{\varepsilon_U(b)} \in \mathcal{S}(\varepsilon_U(b))$ and $Y|_V \in \mathcal{S}(V)$ agree on the overlap. Thus, we can apply the gluing property of $\mathcal{S}$ to obtain $Y' \in \mathcal{S}(U)$ where $Y'|_{\varepsilon_U(b)} = X'|_{\varepsilon_U(b)}$ and $Y'|_V = Y|_V$ (this establishes $Y' \simeq_V^U Y$ in particular).

But then $X'|_V = X|_V \xrightarrow{\rho_V^U(b)}_V Y|_V = Y'|_V$ and $X'|_{\varepsilon_U(b)} = Y'|_{\varepsilon_U(b)}$, which means :

$$(X', Y') \in ((res_V^U)^{-1} \circ \xrightarrow{\rho_V^U(b)}_V res_V^U) \cap \simeq_{\varepsilon_U(b)}^U$$

Since V contains b , we know that $((res_V^U)^{-1} \circ \xrightarrow{\rho_V^U(b)}_V res_V^U) \cap \simeq_{\varepsilon_U(b)}^U = \xrightarrow{b}_U$ by Proposition 4.3.27, and this means $X' \xrightarrow{b}_U Y'$. In particular, $(X', Y) \in (\simeq_V^U \circ \xrightarrow{b}_U)$.

(For $\supseteq$) : Consider any $(X, Y') \in \simeq_V^U \circ \xrightarrow{b}_{\rightarrow U}$. Thus, there exists $Y \in \mathcal{S}(U)$ such that $X \xrightarrow{a}_{\rightarrow U} Y$ and $Y \simeq_V^U Y'$. This means $X|_{\varepsilon_U(b)} = Y|_{\varepsilon_U(b)}$ and $Y'|_V = Y|_V$. We get :

$$(Y'|_{\varepsilon_U(b)})|_{\varepsilon_U(b) \wedge V} = Y'|_{\varepsilon_U(b) \wedge V} = Y|_{\varepsilon_U(b) \wedge V} = X|_{\varepsilon_U(b) \wedge V} = (X|_V)|_{\varepsilon_U(b) \wedge V}$$

And since $V \vee \varepsilon_U(b) = U$, we get a proper cover $\{V, \varepsilon_U(b)\}$ of U on which $Y'|_{\varepsilon_U(b)} \in \mathcal{S}(\varepsilon_U(b))$ and $X|_V \in \mathcal{S}(V)$ agree on the overlap. Thus, we can apply the gluing property of $\mathcal{S}$ to obtain $X' \in \mathcal{S}(U)$ where $X'|_{\varepsilon_U(b)} = Y'|_{\varepsilon_U(b)}$ and $X'|_V = X|_V$ (this establishes $X' \simeq_V^U X$ in particular).

But then $X'|_V = X|_V \xrightarrow{\rho_V^U} Y|_V = Y'|_V$ and $X'|_{\varepsilon_U(b)} = Y'|_{\varepsilon_U(b)}$, which means that $X' \xrightarrow{b}_{\rightarrow U} Y'$ (see argument for $\subseteq$ above). In particular, $(X, Y') \in \xrightarrow{b}_{\rightarrow U} \circ \simeq_V^U$.

□

Lemma F.0.5 [Natural Transformations preserve SI-Independence Relations]. *Consider any two $\mathbb{T}$ -adapted presheaves $\mathcal{T} = (\mathcal{S}, \mathcal{L}, \delta)$ with SI-independence relation $\mathcal{I}$ and $\mathcal{T}' = (\mathcal{S}', \mathcal{L}', \delta')$ with SI-independence relation $\mathcal{I}'$ (both presheaves on a Heyting algebra $\mathcal{H}$). Consider any natural transformation $\{(\sigma_W, \lambda_W)\}_{W \in \mathcal{H}} : \mathcal{T} \rightarrow \mathcal{T}'$ in $[\mathcal{H}^{op}, \mathbb{T}]$, which is a family of LTS morphisms $(\sigma_W, \lambda_W) : \mathcal{T}(W) \rightarrow \mathcal{T}'(W)$ indexed by regions W in $\mathcal{H}$. Then for any actions $b, c \in \mathcal{L}(U)$, if $b \mathcal{I}(U) c$ and $\lambda_U(b), \lambda_U(c) \neq \varepsilon$, then $\lambda_U(b) \mathcal{I}'(U) \lambda_U(c)$.*

Proof. Consider any two $\mathbb{T}$ -adapted presheaf $\mathcal{T} = (\mathcal{S}, \mathcal{L}, \delta)$ and $\mathcal{T}' = (\mathcal{S}', \mathcal{L}', \delta')$ on a Heyting algebra $\mathcal{H}$, and any natural transformation $\{(\sigma_U, \lambda_U)\}_{U \in \mathcal{H}} : \mathcal{T} \rightarrow \mathcal{T}'$ in $[\mathcal{H}^{op}, \mathbb{T}]$.

Consider any regions $V \leq U$ and actions $b, c \in \mathcal{L}(U)$. Suppose that $b \mathcal{I}(U) c$ and suppose that $\lambda_U(b)$ and $\lambda_U(c)$ are not ε . Write $\varepsilon'_U(\lambda_U(b))$ and $\varepsilon'_U(\lambda_U(c))$ to characterize the largest ε -regions of $\lambda_U(b)$ and $\lambda_U(c)$ respectively (in U) with respect to $\mathcal{T}'$.

This means that $\{\varepsilon_U(b), \varepsilon_U(c)\}$ forms a proper cover of U by Proposition 5.1.29. Also, Proposition 5.1.13 establishes that $\varepsilon_U(b)$ is an ε -region of $\lambda_U(b)$ in U with respect to $\mathcal{T}'$. This means that $\varepsilon_U(b) \leq \varepsilon'_U(\lambda_U(b))$.

Similarly, we can show that $\varepsilon_U(c) \leq \varepsilon'_U(\lambda_U(c))$.

We then get :

$$U = \varepsilon_U(b) \vee \varepsilon_U(c) \leq \varepsilon'_U(\lambda_U(b)) \vee \varepsilon'_U(\lambda_U(c)) \leq U$$

and so $\varepsilon'_U(\lambda_U(b)) \vee \varepsilon'_U(\lambda_U(c)) = U$. Thus, $\{\varepsilon'_U(\lambda_U(b)), \varepsilon'_U(\lambda_U(c))\}$ is a proper cover of U , and by Proposition 5.1.29 again, we get that $\lambda_U(b) \mathcal{I}'(U) \lambda_U(c)$.

This proves that the SI-independence relation is preserved by λ_U . □

Appendix G

Proofs for Section 5.4

Proposition G.0.1 . *Given a LRS presheaf and a region U , and an action $b \in \mathcal{L}(U)$, we have that the following hold :*

1. $\psi_U(b)$ as given in Definition 6.1.1 is indeed a proper region of b with respect to U in the sense of Definition 4.3.15.
2. $\psi_U^+(b)$ as given in Definition 6.1.1 is a region that contains the effects of b with respect to U in the sense of Definition 4.3.1.
3. $\psi_U^-(b)$ as given in Definition 6.1.1 is a region that contains the dependencies of b with respect to U in the sense of Definition 4.3.7.

Proof. Consider a LRS presheaf, a region U , and an action $b \in \mathcal{L}(U)$.

(1) We have by Proposition 6.1.7 that $\varepsilon_U(b) = U \setminus \psi_U(b)$, and since we have a $\mathbb{T}$ -adapted presheaf, this means $\psi_U(b)$ is the proper region of b with respect to U .

(2) For any transition $X \xrightarrow{b}_U Y$, we have $X|_{U \setminus \psi_U^+(b)} = Y|_{U \setminus \psi_U^+(b)}$ by definition of $\xrightarrow{b}_U$. This means $U \setminus \psi_U^+(b)$ is a transparent region of b in U . But then, $(U \setminus \psi_U^+(b)) \vee \psi_U^+(b) = U$, so this means that $\psi_U^+(b)$ contains the effects of b .

(3) Let $V = \psi_U^-(b)$ and $W = \psi_U(b)$. We have $V \vee \psi_U^+(b) = W$, so $W \setminus \psi_U^+(b) \subseteq V$, and we have that $res_{W \setminus \psi_U^+(b)}^V$ is well-defined. Set the following relation :

$$R = ((res_{\psi_U^+(b)}^W)^{-1} \circ res_{\psi_U^+(b)}^{\psi^+(b)} \circ \langle b \rangle \circ (res_V^{\psi^-(b)})^{-1}) \cap ((res_{W \setminus \psi_U^+(b)}^W)^{-1} \circ res_{W \setminus \psi_U^+(b)}^V)$$

According to Definition 4.3.7, it suffices to verify that :

$$R \circ res_V^U = res_W^U \circ \xrightarrow{b}_U$$

We start by proving that $R \circ res_V^U \subseteq res_W^U \circ \xrightarrow{b}_U$. Consider any $(X, Y_0) \in (R \circ res_V^U)$. We get $X|_V \stackrel{R}{\sim} Y_0$, and thus there exists $(X', Y') \in \langle b \rangle$ such that :

$$X|_V \stackrel{(res_V^{\psi^-(b)})^{-1}}{\sim} X' \stackrel{\langle b \rangle}{\sim} Y' \stackrel{res_{\psi_U^+(b)}^{\psi^+(b)}}{\sim} Y'|_{\psi_U^+(b)} \stackrel{(res_{\psi_U^+(b)}^W)^{-1}}{\sim} Y_0 \quad (3)$$

and

$$X|_V \stackrel{res_{W \setminus \psi_U^+(b)}^V}{\sim} Y_0|_{W \setminus \psi_U^+(b)} \stackrel{res_{W \setminus \psi_U^+(b)}^W}{\sim}^{-1} Y_0 \quad (4)$$

Consider $Y \in \mathcal{S}(U)$ that is the gluing of Y_0 and $X|_{U \setminus W}$ on the cover $\{W, U \setminus W\}$ of U . This means in particular that $\{W \setminus \psi_U^+(b), U \setminus W\} = \{W \setminus \psi_U^+(b), (U \setminus W) \setminus \psi_U^+(b)\}$ is a proper cover of $U \setminus \psi_U^+(b)$. We get $X|_{U \setminus W} = Y|_{U \setminus W}$, and we also get $X|_{W \setminus \psi_U^+(b)} = (X|_V)|_{W \setminus \psi_U^+(b)} = Y_0|_{W \setminus \psi_U^+(b)}$ from Equation 4. This means that $X|_{U \setminus \psi_U^+(b)} = Y|_{U \setminus \psi_U^+(b)}$ by locality of $\mathcal{S}$.

Furthermore, from Equation 3, we derive $X'|_{\psi_U^-(b)} = (X|_V)|_{\psi_U^-(b)} = X|_{\psi_U^-(b)}$ and $Y'|_{\psi_U^+(b)} = Y_0|_{\psi_U^+(b)} = Y|_{\psi_U^+(b)}$. This means that $X \xrightarrow{b}_U Y$ by definition. But finally, we get $Y|_W = Y_0$, so $(X, Y_0) \in (res_W^U \circ \xrightarrow{b}_U)$. This proves that $R \circ res_V^U \subseteq res_W^U \circ \xrightarrow{b}_U$.

We now prove that $R \circ res_V^U \supseteq res_W^U \circ \xrightarrow{b}_U$. Consider any $(X, Y_0) \in (res_W^U \circ \xrightarrow{b}_U)$. We get a state $\in \mathcal{S}(U)$ such that $X \xrightarrow{b}_U Y$ and $Y|_W = Y_0$, and we need to show that $X|_V \stackrel{R}{\sim} Y_0$. By definition of $X \xrightarrow{b}_U Y$, we have $X|_{U \setminus \psi_U^+(b)} = Y|_{U \setminus \psi_U^+(b)}$ and there exists $(X', Y') \in \langle b \rangle$ such that $X'|_V = X'|_{\psi_U^-(b)} = X|_{\psi_U^-(b)} = X|_V$

and $Y'|_{\psi_U^+(b)} = Y|_{\psi_U^+(b)}$. With $\psi_U^+(b) \subseteq W$, we get $Y_0|_{\psi_U^+(b)} = (Y|_W)|_{\psi_U^+(b)} = Y|_{\psi_U^+(b)} = Y'|_{\psi_U^+(b)}$, and then the following holds :

$$X|_V \xrightarrow{(res_{\psi_U^-(b)}^-)^{-1}} X' \langle b \rangle Y' \xrightarrow{res_{\psi_U^+(b)}^+} Y'|_{\psi_U^+(b)} \xrightarrow{(res_{\psi_U^+(b)}^W)^{-1}} Y_0$$

Furthermore, with $W \setminus \psi_U^+(b) \subseteq U \setminus \psi_U^+(b)$, we get $X|_{W \setminus \psi_U^+(b)} = Y|_{W \setminus \psi_U^+(b)} = Y_0|_{W \setminus \psi_U^+(b)}$ (since $Y|_W = Y_0$), and this means that :

$$X|_V \xrightarrow{res_{W \setminus \psi_U^+(b)}^V} Y_0|_{W \setminus \psi_U^+(b)} \xrightarrow{(res_{W \setminus \psi_U^+(b)}^W)^{-1}} Y_0$$

By definition of R , we get $(X|_V, Y_0) \in R$. Thus, $(X, Y_0) \in (R \circ res_V^U)$, and this proves the inclusion $R \circ res_V^U \supseteq res_W^U \circ \xrightarrow{b}_U$. It follows that $R \circ res_V^U = res_W^U \circ \xrightarrow{b}_U$, and this proves that V contains the dependencies of b with respect to U . $\square$

Example G.0.2 [LRS for Petri Nets]. Given a Petri Net (P, T, F, M_0, W) , we can define a localized relational structure $\Gamma = (P, L, \psi^-, \psi^+, \mathcal{S}, \langle - \rangle)$ where :

- The set of places P are matched in both structures;
- The labelling set L is the set of Petri Net transitions T (not to be confused with LTS transitions);
- $\psi^- = \psi^+ = \psi$ where $\psi : T \rightarrow \mathcal{P}(P)$ is a map that was previously defined for Petri Nets as $\psi(t) = \{ x \in P \mid \exists t \in T, (x, t) \in F \text{ or } (t, x) \in F \}$ for any $t \in L$ (this is the set of places to which t connects with an arc);
- $\mathcal{S}$ is given by the states sheaf of the Petri Net's associated presheaf as we saw in Definition 4.1.11 ($\mathcal{S}(U) = \mathbb{N}^U$ for each $U \subseteq P$, and the state restriction maps are function restrictions);
- For an action $t \in L$, and any $X', Y' \in \mathcal{S}(\psi(t))$ we set $(X', Y') \in \langle t \rangle$ if and only if $\forall x \in \psi(t)$, the following hold :

$$(1) \quad (x, t) \in F \Rightarrow X'(x) \geq W(x, t)$$

$$(2) \quad (x, t) \in F \text{ and } (t, x) \notin F \Rightarrow Y'(x) = X'(x) - W(x, t)$$

$$(3) \quad (x, t) \notin F \text{ and } (t, x) \in F \Rightarrow Y'(x) = X'(x) + W(t, x)$$

$$(4) \quad (x, t) \in F \text{ and } (t, x) \in F \Rightarrow Y'(x) = X'(x) - W(x, t) + W(t, x)$$

This provides a well-defined localized relational structure Γ . We now provide a proof that the presheaf, say $\mathcal{T} = (\mathcal{S}, \mathcal{L}, \delta)$, associated to the Petri Net (P, T, F, M_0, W) as in Definition 4.1.11 corresponds with the $\mathbb{T}$ -adapted presheaf $\mathcal{T}_\Gamma = (\mathcal{S}_\Gamma, \mathcal{L}_\Gamma, \delta_\Gamma)$ of the localized relational structure Γ above.

Proof. We know that $\mathcal{S}_\Gamma = \mathcal{S}$ by definition. Also, $\mathcal{L}$ of the Petri Net's associated presheaf is defined precisely the same way as $\mathcal{L}_\Gamma$. Indeed, for any region U in $\mathcal{P}(P)$, we have $\mathcal{L}(U) = \{t \in T \mid U \cap \psi(t) \neq \emptyset\} = \{t \in L \mid \psi_U(t) \neq \emptyset\} = \mathcal{L}_\Gamma(U)$. The labelling projections are also defined in the same way with :

$$\rho_V^U(t) = \begin{cases} t & \text{if } \psi_V(t) \neq \emptyset \\ \varepsilon & \text{otherwise} \end{cases}$$

for any regions $V \subseteq U$ and action $t \in \mathcal{L}(U)$.

It suffices to prove now that $\delta(U) = \delta_\Gamma(U)$ for all regions U . Thus, consider any region U in $\mathcal{P}(P)$ and any action $t \in \mathcal{L}(U)$. We recall that $(X, t, Y) \in \delta(U)$ if and only if :

$$(i) \quad Y|_{U \setminus \psi(t)} = X|_{U \setminus \psi(t)} \text{ and,}$$

$$(ii) \quad \forall x \in U \cap \psi(t) (= \psi_U(t)), \text{ the following hold :}$$

- $(x, t) \in F \Rightarrow X(x) \geq W(x, t)$
- $(x, t) \in F \text{ and } (t, x) \notin F \Rightarrow Y(x) = X(x) - W(x, t)$
- $(x, t) \notin F \text{ and } (t, x) \in F \Rightarrow Y(x) = X(x) + W(t, x)$
- $(x, t) \in F \text{ and } (t, x) \in F \Rightarrow Y(x) = X(x) - W(x, t) + W(t, x)$

and that $(X, t, Y) \in \delta_\Gamma(U)$ if and only if :

$$(I) \quad Y|_{U \setminus \psi_U^+(t)} = X|_{U \setminus \psi_U^+(t)} \text{ and,}$$

(II) $\exists (X', Y') \in \langle t \rangle$ such that $X|_{\psi_U^-(t)} = X'|_{\psi_U^-(t)}$ and $Y|_{\psi_U^+(t)} = Y'|_{\psi_U^+(t)}$

We prove that $\delta(U) \subseteq \delta_\Gamma(U)$.

Consider any $(X, t, Y) \in \delta(U)$. Thus, (i) and (ii) hold above. With (i), we have that $X|_{U \setminus \psi^+(t)} = X|_{U \setminus \psi(t)} = Y|_{U \setminus \psi(t)} = Y|_{U \setminus \psi^+(t)}$, and this verifies condition (I) to get (X, t, Y) as a transition in $\delta_\Gamma(U)$. For condition (II), we need to find $(X', Y') \in \langle t \rangle$ such that $X'|_{\psi_U^-(t)} = X|_{\psi_U^-(t)}$ and $Y'|_{\psi_U^+(t)} = Y|_{\psi_U^+(t)}$.

Define $X', Y' \in \mathcal{S}(\psi(t))$ as follows :

$$X'|_{\psi_U(t)} = X|_{\psi_U(t)} \text{ and } Y'|_{\psi_U(t)} = Y|_{\psi_U(t)}$$

and for $x \in \psi(t) \setminus U$,

1. If $(x, t) \in F$ and $(t, x) \notin F$, set $X'(x) = W(x, t)$ and $Y'(x) = 0$ (this is equal to $X'(x) - W(x, t)$)
2. If $(x, t) \notin F$ and $(t, x) \in F$, set $X'(x) = 0$ and $Y'(x) = W(t, x)$ (this is equal to $X'(x) + W(t, x)$)
3. If $(x, t) \in F$ and $(t, x) \in F$, set $X'(x) = W(x, t)$ and $Y'(x) = W(t, x)$ (this is equal to $X'(x) - W(x, t) + W(t, x)$)

For these specifications, it is not too difficult to see that (1), (2), (3), and (4) hold for any $x \in \psi(t)$, i.e. to get $(X', Y') \in \langle t \rangle$. Indeed, if $x \in \psi_U(t)$, then the conditions hold for $X'|_{\psi_U(t)} = X|_{\psi_U(t)}$ and $Y'|_{\psi_U(t)} = Y|_{\psi_U(t)}$, by (ii) for X and Y . If $x \in \psi(t) \setminus U$ instead, then we get these conditions with the way X' and Y' are defined just above. Thus, $(X', Y') \in \langle t \rangle$, and we also have $X'|_{\psi_U^-(t)} = X'|_{\psi_U(t)} = X|_{\psi_U(t)} = X|_{\psi_U^-(t)}$ and $Y'|_{\psi_U^+(t)} = Y'|_{\psi_U(t)} = Y|_{\psi_U(t)} = Y|_{\psi_U^+(t)}$. This means (II) is satisfied, and we get $(X, t, Y) \in \delta_\Gamma(U)$.

Now, consider any $(X, t, Y) \in \delta_\Gamma(U)$. From (I), we get that $X|_{U \setminus \psi^+(t)} = Y|_{U \setminus \psi^+(t)}$, and this means $X|_{U \setminus \psi(t)} = Y|_{U \setminus \psi(t)}$, which fulfills condition (i) to get $(X, t, Y) \in \delta(U)$.

With (II), we also get $(X', Y') \in \langle t \rangle$ such that $X'|_{\psi_U^-(t)} = X|_{\psi_U^-(t)}$ and $Y'|_{\psi_U^+(t)} = Y|_{\psi_U^+(t)}$, and the conditions (1), (2), (3), and (4) given earlier hold for any $x \in \psi(t)$ with respect to X' and Y' . In particular, these conditions hold for any $x \in U \cap \psi(t) = \psi_U(t) \subseteq \psi(t)$ with respect to X' and Y' , and since $X|_{\psi_U(t)} = X|_{\psi_U^-(t)} = X'|_{\psi_U^-(t)} = X'|_{\psi_U(t)}$ and $Y|_{\psi_U(t)} = Y|_{\psi_U^+(t)} = Y'|_{\psi_U^+(t)} = Y'|_{\psi_U(t)}$, these conditions hold with respect to X and Y also. These are, in fact, the conditions given in (ii) and we get $(X, t, Y) \in \delta(U)$.

Thus, we have $\delta = \delta_\Gamma$, and it follows that $\mathcal{T} = \mathcal{T}_\Gamma$. □

Appendix H

Extension of the Abstract Trace Equivalences

We now propose a solution to an issue that arises with the spatially induced abstract trace equivalences within $\mathbb{T}$ -adapted presheaves. Mainly, in such presheaves, it is possible to have an action, say c , that represents the “gluing” of two independent actions b_1 and b_2 . The usual abstract trace equivalence of Mazurkiewicz (as in Definition 3.1.4) identifies the two words b_1b_2 and b_2b_1 , but it does not identify them with their combination c . We propose an extended equivalence that further identifies the words b_1b_2 and b_2b_1 with such a c (i.e a word with a single letter), when the latter exists. We end this appendix with a conjecture that quotienting with respect to this extended equivalence (a congruence on a freely generated monoid in fact) yields a form of trace monoid that may be more appropriate to the context of $\mathbb{T}$ -adapted presheaves than the usual trace monoid given by ALTS (with Definition 3.1.5 in Chapter 3).

From the Theorem of Spatially Induced Independence, we know that, for any $\mathbb{T}$ -adapted presheaf $(\mathcal{S}, \mathcal{L}, \delta)$ and a region U , we can associate an ALTSE given by $(\mathcal{S}(U), \mathcal{L}(U), \delta(U), \mathcal{I}(U), \sim_U)$. With respect to such an ALTSE, we obtain an abstract trace equivalence $\equiv_{\mathcal{I}(U)}$ from Definition 3.1.4 for ALTS, and an equivalence on runs from Definition 3.3.15. Now, recall the Parallel Registers from Example 2.2.1 and consider the set of transitions on a global region $A_4 = \{1, 2, 3, 4\}$ that is depicted in Figure 39.

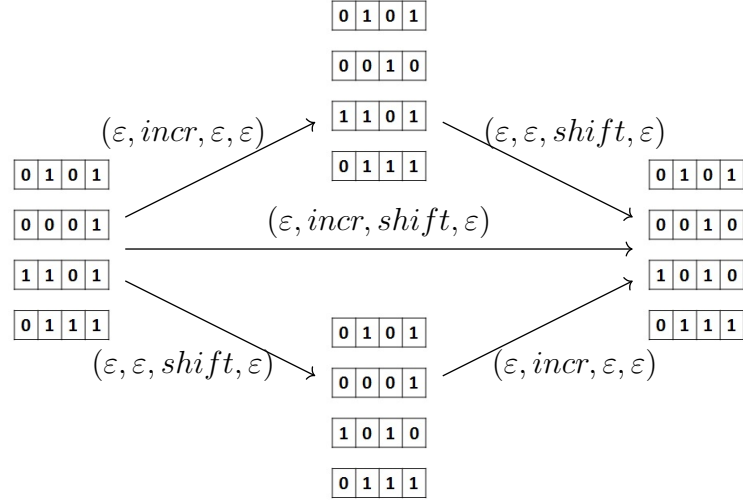


Figure 39: A set of transitions in PReg

Now, define a map $\sigma : \{0, 1, 2, 3\} \rightarrow \mathcal{S}(P)$ as in Figure 40.

$\sigma(0)$	$\sigma(1)$	$\sigma(2)$	$\sigma(3)$
$\begin{bmatrix} 0 & 1 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 0 & 1 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 0 & 1 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 0 & 1 & 0 & 1 \end{bmatrix}$
$\begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix}$
$\begin{bmatrix} 1 & 1 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 1 & 1 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 & 1 & 0 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 & 1 & 0 \end{bmatrix}$
$\begin{bmatrix} 0 & 1 & 1 & 1 \end{bmatrix}$	$\begin{bmatrix} 0 & 1 & 1 & 1 \end{bmatrix}$	$\begin{bmatrix} 0 & 1 & 1 & 1 \end{bmatrix}$	$\begin{bmatrix} 0 & 1 & 1 & 1 \end{bmatrix}$

Figure 40: The map $\sigma : \{0, 1, 2, 3\} \rightarrow \mathcal{S}(P)$

With respect to this set of transitions, we would like to say that the following runs are equivalent in the global region of PReg :

$$(1) \quad \sigma(0) \xrightarrow{(\varepsilon, incr, \varepsilon, \varepsilon)}_{\text{PReg}} \sigma(1) \xrightarrow{(\varepsilon, \varepsilon, shift, \varepsilon)}_{\text{PReg}} \sigma(2) ,$$

$$(2) \quad \sigma(0) \xrightarrow{(\varepsilon, \varepsilon, shift, \varepsilon)}_{\text{PReg}} \sigma(3) \xrightarrow{(\varepsilon, incr, \varepsilon, \varepsilon)}_{\text{PReg}} \sigma(2) , \text{ and}$$

$$(3) \quad \sigma(0) \xrightarrow{(\varepsilon, inc, shift, \varepsilon)}_{\text{PReg}} \sigma(2)$$

It is not the case that these are equivalent with the current equivalence on runs that arises from $(\mathcal{S}, \mathcal{L}, \delta, \mathcal{I}, \sim)$. The two runs given in (1) and (2) get matched, but not the third one. This is basically stemming from the fact that the sequences of actions $(\varepsilon, \varepsilon, shift, \varepsilon)(\varepsilon, incr, \varepsilon, \varepsilon) \equiv_{\mathcal{I}(U)} (\varepsilon, incr, \varepsilon, \varepsilon)(\varepsilon, \varepsilon, shift, \varepsilon)$ do not get matched with $(\varepsilon, inc, shift, \varepsilon)$ via $\equiv_{\mathcal{I}(U)}$.

Thus, we may get the impression that the current abstract trace equivalence and the equivalence on runs provide insufficient quotienting with respect to the quotienting potential that $\mathbb{T}$ -adapted presheaves offer. The object of this section is to propose a solution to this insufficient quotienting, by extending the abstract trace equivalence and the equivalence on runs in consideration of such “middle transitions” like $\sigma(0) \xrightarrow{(\varepsilon, inc, shift, \varepsilon)}_{\text{PReg}} \sigma(2)$ in Figure 39. We will work this out by using the standard SI-independence relations given by $\mathcal{I}$ here. The question of which results in this section hold for the case of $\mathcal{I}^+$ is an open problem (and it is certainly a bit more complicated to work out because transparent regions don’t behave as well as ε -regions).

Now, we need to understand how certain actions, like $(\varepsilon, inc, shift, \varepsilon)$ above, can arise as combinations of independent actions, like $(\varepsilon, \varepsilon, shift, \varepsilon)$ and $(\varepsilon, incr, \varepsilon, \varepsilon)$, and we propose the following concept to identify such combinations.

Definition H.0.1 [Combination of Independent Actions]. Consider a $\mathbb{T}$ -adapted presheaf on a Heyting algebra, a region U , and any actions b, c, d in $\mathcal{L}(U)$. We say that d is an $\mathcal{I}$ -combination of b and c in U if the following three conditions hold :

1. $b \mathcal{I}(U) c$,
2. $\rho_{\varepsilon_U(b)}^U(d) = \rho_{\varepsilon_U(b)}^U(c)$ and $\rho_{\varepsilon_U(c)}^U(d) = \rho_{\varepsilon_U(c)}^U(b)$,
3. d glues its local transitions over $\{\varepsilon_U(b), \varepsilon_U(c)\}$.

If d is an $\mathcal{I}$ -combination of two actions, then it is completely determined by the locality of $\mathcal{L}$ with respect to these actions (this means that if d_1 and d_2 are both an

$\mathcal{I}$ -combination of some actions b and c , then $d_1 = d_2$).

Now, the first thing we should verify is that the transition relation of an $\mathcal{I}$ -combination is given by the composition of the transition relations of the independent actions it combines. We need two lemmas to achieve this, and the first one goes as follows :

Lemma H.0.2 . *Consider a $\mathbb{T}$ -adapted presheaf on a complete Heyting algebra and any region U . Consider any actions $b, c \in \mathcal{L}(U)$ such that $b \mathcal{I}(U) c$. Then, for any $(X, Z) \in (\xrightarrow{c}_U \circ \xrightarrow{b}_U)$, there exists a unique $Y \in \mathcal{S}(U)$ such that $X \xrightarrow{b}_U Y \xrightarrow{c}_U Z$, and this Y is determined by the equations $Y \simeq_{\varepsilon_U(b)}^U X$ and $Y \simeq_{\varepsilon_U(c)}^U Z$ on the cover $\{\varepsilon_U(b), \varepsilon_U(c)\}$ of U .*

Proof. Consider a $\mathbb{T}$ -adapted presheaf on a complete Heyting algebra and any region U . Take $b, c \in \mathcal{L}(U)$ such that $b \mathcal{I}(U) c$, and consider any $(X, Z) \in (\xrightarrow{c}_U \circ \xrightarrow{b}_U)$. Then, by definition, there exists Y such that $X \xrightarrow{b}_U Y \xrightarrow{c}_U Z$. We have $X \simeq_{\varepsilon_U(b)}^U Y$ and $Y \simeq_{\varepsilon_U(c)}^U Z$ by the transparency of b and c respectively. If there is another $Y' \in \mathcal{S}(U)$ such that $X \xrightarrow{b}_U Y' \xrightarrow{c}_U Z$, then we also get $X \simeq_{\varepsilon_U(b)}^U Y'$ and $Y' \simeq_{\varepsilon_U(c)}^U Z$. Thus, $Y' \simeq_{\varepsilon_U(b)}^U Y$ and $Y' \simeq_{\varepsilon_U(c)}^U Y$, and by the locality of $\mathcal{S}$ on the cover $\{\varepsilon_U(b), \varepsilon_U(c)\}$ of U , we get $Y = Y'$. $\square$

And we also need to show that the largest ε -region of an $\mathcal{I}$ -combination is the meet of the largest ε -regions of the actions it combines.

Lemma H.0.3 . *Given a $\mathbb{T}$ -adapted presheaf on a complete Heyting algebra and a region U , if d is an $\mathcal{I}$ -combination of b and c in U , then $\varepsilon_U(b) \wedge \varepsilon_U(c) = \varepsilon_U(d)$.*

Proof. Consider a $\mathbb{T}$ -adapted presheaf on a complete Heyting algebra and a region U , and consider any action d that is an $\mathcal{I}$ -combination of some actions b and c in U . We have that :

$$\rho_{\varepsilon_U(b) \wedge \varepsilon_U(c)}^U(d) = \rho_{\varepsilon_U(b) \wedge \varepsilon_U(c)}^{\varepsilon_U(b)}(\rho_{\varepsilon_U(b)}^U(d)) = \rho_{\varepsilon_U(b) \wedge \varepsilon_U(c)}^{\varepsilon_U(b)}(\rho_{\varepsilon_U(b)}^U(c)) = \rho_{\varepsilon_U(b) \wedge \varepsilon_U(c)}^U(c) = \varepsilon$$

since c vanishes in $\varepsilon_U(b) \wedge \varepsilon_U(c)$. This means $\varepsilon_U(d) \leq \varepsilon_U(b) \wedge \varepsilon_U(c)$.

We also have that :

$$\rho_{\varepsilon_U(d) \wedge \varepsilon_U(b)}^{\varepsilon_U(d)}(\rho_{\varepsilon_U(d)}^U(b)) = \rho_{\varepsilon_U(d) \wedge \varepsilon_U(b)}^U(b) = \varepsilon$$

since b vanishes in $\varepsilon_U(d) \wedge \varepsilon_U(b)$.

Also, we have :

$$\begin{aligned} \rho_{\varepsilon_U(d) \wedge \varepsilon_U(c)}^{\varepsilon_U(d)}(\rho_{\varepsilon_U(d)}^U(b)) &= \rho_{\varepsilon_U(d) \wedge \varepsilon_U(c)}^U(b) = \rho_{\varepsilon_U(d) \wedge \varepsilon_U(c)}^{\varepsilon_U(c)}(\rho_{\varepsilon_U(c)}^U(b)) \\ &= \rho_{\varepsilon_U(d) \wedge \varepsilon_U(c)}^{\varepsilon_U(c)}(\rho_{\varepsilon_U(c)}^U(d)) = \rho_{\varepsilon_U(d) \wedge \varepsilon_U(c)}^U(d) = \varepsilon \end{aligned}$$

since d vanishes in $\varepsilon_U(d) \wedge \varepsilon_U(c)$. But $\{\varepsilon_U(d) \wedge \varepsilon_U(b), \varepsilon_U(d) \wedge \varepsilon_U(c)\}$ forms a cover of $\varepsilon_U(d)$, and by locality of $\mathcal{L}$, we get $\rho_{\varepsilon_U(d)}^U(b) = \varepsilon$. Thus, $\varepsilon_U(c) \leq \varepsilon_U(a)$. A similar proof shows that $\varepsilon_U(c) \leq \varepsilon_U(b)$, and we derive $\varepsilon_U(c) \leq \varepsilon_U(a) \wedge \varepsilon_U(b)$. Finally, we get $\varepsilon_U(d) = \varepsilon_U(b) \wedge \varepsilon_U(c)$. $\square$

These two lemmas can then be used to prove that any transition of an $\mathcal{I}$ -combination breaks down uniquely in a sequence of transitions for the actions it combines, and this goes as follows :

Proposition H.0.4 . *Consider a $\mathbb{T}$ -adapted presheaf on a complete Heyting algebra and a region U . If d is an $\mathcal{I}$ -combination of b and c in U , then for any $X, Z \in \mathcal{S}(U)$ such that $X \xrightarrow{d}_U Z$, there exists a unique Y such that $X \xrightarrow{b}_U Y \xrightarrow{c}_U Z$, and the latter is determined by the equations $Y \simeq_{\varepsilon_U(b)}^U X$ and $Y \simeq_{\varepsilon_U(c)}^U Z$ on the cover $\{\varepsilon_U(b), \varepsilon_U(c)\}$ of U .*

Proof. Suppose that d is an $\mathcal{I}$ -combination of b and c in U , and consider any $X, Z \in \mathcal{S}(U)$ such that $X \xrightarrow{d}_U Z$. We get $X|_{\varepsilon_U(c) \wedge \varepsilon_U(b)} = X|_{\varepsilon_U(d)} = Z|_{\varepsilon_U(d)} = Z|_{\varepsilon_U(c) \wedge \varepsilon_U(b)}$ by Lemma H.0.3 and, thus, we can take a state $Y \in \mathcal{S}(U)$ that is the gluing of $X|_{\varepsilon_U(b)}$ and $Z|_{\varepsilon_U(c)}$ on the cover $\{\varepsilon_U(b), \varepsilon_U(c)\}$ of U .

We get that :

$$X|_{\varepsilon_U(c)} \xrightarrow{\rho_{\varepsilon_U(c)}^U(d)}_{\varepsilon_U(c)} Z|_{\varepsilon_U(c)} = Y|_{\varepsilon_U(c)} \quad \text{and} \quad X|_{\varepsilon_U(b)} = Y|_{\varepsilon_U(b)}$$

But, $\rho_{\varepsilon_U(c)}^U(d) = \rho_{\varepsilon_U(c)}^U(b)$ and so $X|_{\varepsilon_U(c)} \xrightarrow{\rho_{\varepsilon_U(c)}^U(b)}_{\varepsilon_U(c)} Y|_{\varepsilon_U(c)}$. This means that $(X, Y) \in ((res_{\varepsilon_U(c)}^U)^{-1} \circ \frac{\rho_{\varepsilon_U(c)}^U(b)}{\xrightarrow{\varepsilon_U(c)}} \circ res_{\varepsilon_U(c)}^U) \cap \simeq_{\varepsilon_U(b)}^U = \xrightarrow{b}_U$ (the equality comes from the fact that $\varepsilon_U(c)$ contains b when $b \mathcal{I}(U)c$). This gives us $X \xrightarrow{b}_U Y$.

Also, we get that :

$$Y|_{\varepsilon_U(b)} = X|_{\varepsilon_U(b)} \xrightarrow{\rho_{\varepsilon_U(b)}^U(d)}_{\varepsilon_U(b)} Z|_{\varepsilon_U(b)} \quad \text{and} \quad Y|_{\varepsilon_U(c)} = Z|_{\varepsilon_U(c)}$$

But, $\rho_{\varepsilon_U(b)}^U(d) = \rho_{\varepsilon_U(b)}^U(c)$ and so $Y|_{\varepsilon_U(b)} \xrightarrow{\rho_{\varepsilon_U(b)}^U(c)}_{\varepsilon_U(b)} Z|_{\varepsilon_U(b)}$. This means that $(X, Y) \in ((res_{\varepsilon_U(b)}^U)^{-1} \circ \frac{\rho_{\varepsilon_U(b)}^U(c)}{\xrightarrow{\varepsilon_U(b)}} \circ res_{\varepsilon_U(b)}^U) \cap \simeq_{\varepsilon_U(c)}^U = \xrightarrow{c}_U$ (the equality comes from the fact that $\varepsilon_U(b)$ contains c when $b \mathcal{I}(U)c$). This gives us $X \xrightarrow{c}_U Y$.

Finally, the uniqueness of Y follows from Lemma H.0.2 since $b \mathcal{I}(U) c$. $\square$

Given an action d that is an $\mathcal{I}$ -combination of b and c , then Proposition H.0.4 makes the idea of decomposing the word d into the word bc quite convincing. This is because it tells us that we can split any c -transition in a unique and non-ambiguous way with b and c transitions fired in a sequence. This will allow us to extend the usual ALTSE equivalence on runs to take such splitting into account.

Now, whilst Proposition H.0.4 as in fact established $\xrightarrow{d}_U \subseteq \xrightarrow{c}_U \circ \xrightarrow{b}_U$, we should check that $\xrightarrow{d}_U \supseteq \xrightarrow{c}_U \circ \xrightarrow{b}_U$ also.

Corollary H.0.5 . *Consider a $\mathbb{T}$ -adapted presheaf on a complete Heyting algebra and a region U . If d is an $\mathcal{I}$ -combination of b and c in U , then $\xrightarrow{d}_U = \xrightarrow{c}_U \circ \xrightarrow{b}_U$.*

Proof. Consider an action d that is an $\mathcal{I}$ -combination of b and c in U . Proposition H.0.4 establishes that $\xrightarrow{d}_U \subseteq \xrightarrow{c}_U \circ \xrightarrow{b}_U$, and it remains to verify inclusion the other way around.

Thus, consider any $(X, Z) \in (\xrightarrow{c}_U \circ \xrightarrow{b}_U)$. By Lemma H.0.2, there is a unique $Y \in \mathcal{S}(U)$ such that $X \xrightarrow{b}_U Y \xrightarrow{c}_U Z$, and it is given by the gluing of $X|_{\varepsilon_U(b)}$ and

$Z|_{\varepsilon_U(c)}$ on the cover $\{\varepsilon_U(b), \varepsilon_U(c)\}$ of U (since $b \mathcal{I}(U) c$).

We get $X|_{\varepsilon_U(b)} = Y|_{\varepsilon_U(b)} \xrightarrow{\rho_{\varepsilon_U(b)}^U(c)}_{\varepsilon_U(b)} Z|_{\varepsilon_U(b)}$ and $X|_{\varepsilon_U(c)} \xrightarrow{\rho_{\varepsilon_U(c)}^U(b)}_{\varepsilon_U(c)} Y|_{\varepsilon_U(c)} = Z|_{\varepsilon_U(c)}$. Since $\rho_{\varepsilon_U(b)}^U(c) = \rho_{\varepsilon_U(b)}^U(d)$ and $\rho_{\varepsilon_U(c)}^U(b) = \rho_{\varepsilon_U(c)}^U(d)$, it follows that :

$$X|_{\varepsilon_U(b)} \xrightarrow{\rho_{\varepsilon_U(b)}^U(d)}_{\varepsilon_U(b)} Z|_{\varepsilon_U(b)} \quad \text{and} \quad X|_{\varepsilon_U(c)} \xrightarrow{\rho_{\varepsilon_U(c)}^U(d)}_{\varepsilon_U(c)} Z|_{\varepsilon_U(c)}$$

Since d glues its transitions over $\{\varepsilon_U(b), \varepsilon_U(c)\}$, we get that $X' \xrightarrow{d}_U Z'$ for some $X', Z' \in \mathcal{S}(U)$ such that $X'|_{\varepsilon_U(b)} = X|_{\varepsilon_U(b)}$ and $X'|_{\varepsilon_U(c)} = X|_{\varepsilon_U(c)}$, and $Z'|_{\varepsilon_U(b)} = Z|_{\varepsilon_U(b)}$ and $Z'|_{\varepsilon_U(c)} = Z|_{\varepsilon_U(c)}$. By the locality of $\mathcal{S}$, we get $X' = X$ and $Z' = Z$, i.e. $X \xrightarrow{d}_U Z$.

This proves that $\xrightarrow{c}_U \circ \xrightarrow{b}_U \subseteq \xrightarrow{d}_U$ and, in fact, $\xrightarrow{c}_U \circ \xrightarrow{b}_U = \xrightarrow{d}_U$. $\square$

The following shows that the SI-independence relations behave properly with respect to $\mathcal{I}$ -combinations.

Proposition H.0.6 . *Consider a $\mathbb{T}$ -adapted presheaf on a Heyting algebra and a region U . For any actions $b, c, d, e \in \mathcal{L}(U)$, if d is an $\mathcal{I}$ -combination of b and c in U , then :*

$$b \mathcal{I}(U) e \quad \text{and} \quad c \mathcal{I}(U) e \quad \Leftrightarrow \quad d \mathcal{I}(U) e$$

Proof. Suppose that d an $\mathcal{I}$ -combination of b and c in U . Recall that with Lemma H.0.3, we have $\varepsilon_U(d) = \varepsilon_U(b) \wedge \varepsilon_U(c)$. Consider any action $e \in \mathcal{L}(U)$.

If $b \mathcal{I}(U) e$ and $c \mathcal{I}(U) e$, then we get $\varepsilon_U(d) \vee \varepsilon_U(e) = (\varepsilon_U(b) \wedge \varepsilon_U(c)) \vee \varepsilon_U(e) = (\varepsilon_U(b) \vee \varepsilon_U(e)) \wedge (\varepsilon_U(c) \vee \varepsilon_U(e)) = U \vee U = U$. This proves that $d \mathcal{I}(U) e$.

If $d \mathcal{I}(U) e$, then $U = \varepsilon_U(d) \vee \varepsilon_U(e) = (\varepsilon_U(b) \wedge \varepsilon_U(c)) \vee \varepsilon_U(e) \leq \varepsilon_U(b) \vee \varepsilon_U(e) \leq U$. Thus, $\varepsilon_U(b) \vee \varepsilon_U(e) = U$. This proves that $b \mathcal{I}(U) e$. We can show that $c \mathcal{I}(U) e$ with a similar proof. $\square$

We can extend the abstract trace equivalence $\equiv_{\mathcal{I}(U)}$ to the following congruence :

Definition H.0.7 . Consider a $\mathbb{T}$ -adapted presheaf on a complete Heyting algebra and a region U . We define $\equiv_{\mathcal{L}(U)}$ as the smallest congruence relation on $\mathcal{L}(U)^*$ such that :

1. $\equiv_{\mathcal{I}(U)} \subseteq \equiv_{\mathcal{L}(U)}$, and
2. For any $b, c, d \in \mathcal{L}(U)$, if d is $\mathcal{I}$ -combination of b and c in U , then $c \equiv_{\mathcal{L}(U)} ab$.

Remark H.0.8 . We write $[w]_{\mathcal{L}(U)}$ to denote the equivalence class of $w \in \mathcal{L}(U)^*$ with respect to $\equiv_{\mathcal{L}(U)}$.

We can also extend the equivalence on runs [need to set this notation] $\approx_{\mathcal{I}(U)}$ for a region U , provided in Definition 3.3.15, and this goes as follows :

Definition H.0.9 . Consider a $\mathbb{T}$ -adapted presheaf $\mathcal{T} = (\mathcal{S}, \mathcal{L}, \delta)$ on a Heyting algebra $\mathcal{H}$ and a region U in $\mathcal{H}$. We define $\approx_{\mathcal{L}(U)}$ as the smallest equivalence relation on runs in $\mathcal{T}(U)$ such that :

1. $\approx_{\mathcal{I}(U)} \subseteq \approx_{\mathcal{L}(U)}$, and
2. Any T -runs :

$$X_0 \xrightarrow{a_1}_T X_1 \xrightarrow{a_2}_T \dots \xrightarrow{a_{m-1}}_T X_{m-1} \xrightarrow{a_m}_T X_m \xrightarrow{a_{m+1}}_T X_{m+1} \xrightarrow{a_{m+2}}_T \dots \xrightarrow{a_n}_T X_n$$

and

$$X_0 \xrightarrow{a_1}_T X_1 \xrightarrow{a_2}_T \dots \xrightarrow{a_{m-1}}_T X_{m-1} \xrightarrow{b}_T X_{m+1} \xrightarrow{a_{m+2}}_T \dots \xrightarrow{a_n}_T X_n$$

in $\mathcal{T}(U)$ are equivalent if b is an $\mathcal{I}$ -combination of a_m and a_{m+1} ¹.

We can see how these T -runs interact in the diagram of Figure 41 below, which depicts how one T -run can be obtained from the other by a local deformation where an $\mathcal{I}$ -combination is involved.

¹Remark that this forces the state X_m to be unique as such in the first run.

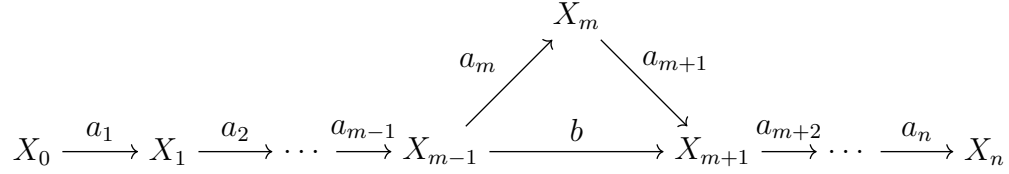


Figure 41

We can study behaviour of systems in terms of equivalence classes of runs and it would be nice to have a functor from $[\mathcal{H}^{op}, \mathbb{T}]_{adapt}$ to $[\mathcal{H}^{op}, \mathbb{M}]$ that would allow us to use a monoidal structure on labels like we did before with the Theorem of Spatially Induced Independence. We simply make a conjecture about this matter however, and it goes as follows :

Conjecture H.0.10 [Extended Spatial $\mathbb{M}$ -Functors]. *For any complete Heyting algebra $\mathcal{H}$, there is a functor F from $[\mathcal{H}^{op}, \mathbb{T}]_{adapt}$ to $[\mathcal{H}^{op}, \mathbb{M}]$ that sends a $\mathbb{T}$ -adapted presheaf $\mathcal{T} = (\mathcal{S}, \mathcal{L}, \delta)$ to $\mathcal{M} = (\mathcal{S}, \mathcal{L}^* / \equiv_{\mathcal{L}}, \delta_{\mathcal{M}})$, where for any region $U \in \mathcal{H}$, $(\mathcal{L}^* / \equiv_{\mathcal{L}})(U) = ((\mathcal{L}(U))^* / \equiv_{\mathcal{L}(U)})$ is given by the quotienting of the freely generated monoid $(\mathcal{L}(U))^*$ with the congruence $\equiv_{\mathcal{L}(U)}$, and $\delta_{\mathcal{M}}(U)$ is specified by :*

$$\frac{[a_1 \dots a_n]_{\mathcal{L}(U)}}{\rightarrow_{\mathcal{M}(U)}} = \frac{a_n}{\rightarrow_U} \circ \dots \circ \frac{a_1}{\rightarrow_U}$$

For any region $V \leq U$ in $\mathcal{H}$, given the restriction map (res_V^U, ρ_V^U) from U to V in $\mathcal{T}$, we define the restriction map from U to V in $\mathcal{M}$ by $(res_V^U, (\rho_V^U)_{\mathcal{M}})$ where :

$$(\rho_V^U)_{\mathcal{M}}([a_1 \dots a_n]_{\mathcal{L}(U)}) = [(\rho_V^U)^*(a_1 \dots a_n)]_{\mathcal{L}(V)}$$

Finally, a natural transformation $\{(\sigma_U, \lambda_U)\}_{U \in \mathcal{H}} : \mathcal{T}_0 \rightarrow \mathcal{T}_1$ in $[\mathcal{H}^{op}, \mathbb{T}]_{adapt}$ (i.e. (σ_U, λ_U) is a LTS morphism from $\mathcal{T}_0(U)$ to $\mathcal{T}_1(U)$) is sent to the natural transformation $\{(\sigma_U^{\mathcal{M}}, \lambda_U^{\mathcal{M}})\}_{U \in \mathcal{H}} : F(\mathcal{T}_0) \rightarrow F(\mathcal{T}_1)$ with $\sigma_U^{\mathcal{M}} = \sigma_U$ and $\lambda_U^{\mathcal{M}}$ is a monoid morphism from $(\mathcal{L}_0(U))^* / \equiv_{\mathcal{L}_0(U)}$ to $(\mathcal{L}_1(U))^* / \equiv_{\mathcal{L}_1(U)}$ given by :

$$\lambda_U^{\mathcal{M}}([a_1 \dots a_n]_{\mathcal{L}(U)}) = [(\lambda_U)^*(a_1 \dots a_n)]_{\mathcal{L}(V)}$$

To establish the above, one would probably use a combination of the proofs of the Theorem of Spatially Induced Independence and of Proposition 3.2.12 where the

functor F_{am} from $\mathbb{A}$ to $\mathbb{M}$ is established.

Finally, we claim that the runs

- (1) $\sigma(0) \xrightarrow{(\varepsilon, incr, \varepsilon, \varepsilon)}_{\text{PReg}} \sigma(1) \xrightarrow{(\varepsilon, \varepsilon, shift, \varepsilon)}_{\text{PReg}} \sigma(2)$,
- (2) $\sigma(0) \xrightarrow{(\varepsilon, \varepsilon, shift, \varepsilon)}_{\text{PReg}} \sigma(3) \xrightarrow{(\varepsilon, incr, \varepsilon, \varepsilon)}_{\text{PReg}} \sigma(2)$, and
- (3) $\sigma(0) \xrightarrow{(\varepsilon, inc, shift, \varepsilon)}_{\text{PReg}} \sigma(2)$,

as depicted in in Figure 39 at the beginning of this section, are equivalent runs in PReg with respect to $\approx_{\mathcal{L}(A_4)}$. In fact, with our concept of $\mathcal{I}$ -combination, we have that $d = (\varepsilon, inc, shift, \varepsilon)$ is indeed an $\mathcal{I}$ -combination of $b = (\varepsilon, \varepsilon, shift, \varepsilon)$ and $c = (\varepsilon, incr, \varepsilon, \varepsilon)$ in A_4 :

1. We have $b \mathcal{I}(A_4) c$, since $\varepsilon_{A_4}(b) = \{1, 2, 4\}$ and $\varepsilon_{A_4}(c) = \{1, 3, 4\}$, and we get that $\{\varepsilon_{A_4}(b), \varepsilon_{A_4}(c)\}$ forms a proper cover of $\{1, 2, 3, 4\}$.
2. We have $\rho_{\{1\}}^{A_4}(c) = \varepsilon = \rho_{\{1\}}^{A_4}(d)$, $\rho_{\{2\}}^{A_4}(c) = incr = \rho_{\{2\}}^{A_4}(d)$, and $\rho_{\{4\}}^{A_4}(c) = \varepsilon = \rho_{\{4\}}^{A_4}(d)$, so $\rho_{\varepsilon_U(b)}^{A_4}(c) = \rho_{\varepsilon_U(b)}^{A_4}(d)$ by locality of $\mathcal{L}$.

Also, $\rho_{\{1\}}^{A_4}(c) = \varepsilon = \rho_{\{1\}}^{A_4}(d)$, $\rho_{\{3\}}^{A_4}(c) = shift = \rho_{\{3\}}^{A_4}(d)$, and $\rho_{\{4\}}^{A_4}(c) = \varepsilon = \rho_{\{4\}}^{A_4}(d)$, so $\rho_{\varepsilon_U(c)}^{A_4}(c) = \rho_{\varepsilon_U(c)}^{A_4}(d)$ by locality of $\mathcal{L}$.

3. The fact that $(\varepsilon, incr, shift, \varepsilon)$ glues its transitions over $\{\varepsilon_{A_4}(b), \varepsilon_{A_4}(c)\}$ is provided by the fact that the presheaf in question for PReg is actually a sheaf, and we can simply apply Proposition 4.2.7 thus.

And this verifies that d is $\mathcal{I}$ -combination of b and c . This means furthermore that all of the runs :

$$(1) \quad \sigma(0) \xrightarrow{(\varepsilon, inc, \varepsilon, \varepsilon)}_{\text{PReg}} \sigma(1) \xrightarrow{(\varepsilon, \varepsilon, shift, \varepsilon)}_{\text{PReg}} \sigma(2) ,$$

$$(2) \quad \sigma(0) \xrightarrow{(\varepsilon, \varepsilon, shift, \varepsilon)}_{\text{PReg}} \sigma(3) \xrightarrow{(\varepsilon, inc, \varepsilon, \varepsilon)}_{\text{PReg}} \sigma(2) , \text{ and}$$

$$(3) \quad \sigma(0) \xrightarrow{(\varepsilon, inc, shift, \varepsilon)}_{\text{PReg}} \sigma(2)$$

are now equivalent with respect to $\approx_{\mathcal{L}(U)}$.

Thus, we are confident that these extended equivalences $\equiv_{\mathcal{L}(U)}$ and $\approx_{\mathcal{L}(U)}$ provide the right idea for quotienting with respect to these combinations of independent actions within $\mathbb{T}$ -adapted presheaves. Of course, there is still some work that needs to be done with respect to these extended equivalences. In particular, we would certainly require more examples that apply them, and we would like proofs for the stated conjectures. The case of $\mathcal{I}^+$ certainly needs to be addressed as well, and this is left as an open problem.

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