

BUCKLING ANALYSIS OF STEEL PLATES REINFORCED WITH GFRP

by

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Abstract

Glass Fiber Reinforced Polymers (GFRP) plates have recently received attention as a viable option for reinforcing existing steel members. Possible application involve web strengthening of existing plate girders and reinforcing corroded flanges and/or webs where a steel plate under combination of shear and/or normal stresses can be governed by their buckling strength. Using GFRP as a retrofit material is attractive from several respects such as easy application, achieving high additional strength with low additional weight, and corrosion resistance. Since the elastic properties of the GFRP, adhesive, and steel are orders of magnitudes apart, reliable predictions of the buckling strength of such systems necessitates careful 3D modelling involving significant modelling and computational effort. Within this context, the present study develops a simplified buckling theory for steel plates symmetrically reinforced with GFRP plates and subjected to in-plane biaxial normal stresses and shear. The theory idealizes the steel and GFRP as Kirchhoff plates while accounting for the transverse shear deformations within the adhesive layers.

A variational formulation is first developed based on the principle of stationary potential energy. The validity of the variational formulation is then assessed through systematic comparisons with results based 3D finite element models for a variety of buckling problems. The variational principle thus validated, is then used to develop a finite element formulation. The new element features four nodes with five degrees of freedom per node. Results based on the finite element are compared to results based on 3D modelling to assess its validity. The element is then used to investigate the effect of GFRP thickness, adhesive thickness, and adhesive shear modulus on the critical pressure of composite systems for practical retrofitting problems.

It is shown that GFRP thickness is particularly effective in increasing the capacity of the composite system, while the effect of the adhesive layer shear modulus low to moderate. Conversely, an increase in the adhesive thickness is found to correspond to a decrease in buckling capacity of the composite system.

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List of Symbols

a, b	: Plate dimensions along x, y direction
a_i, b_i	: Coefficients used in approximate buckling displacement functions in Chapter 3
$\bar{A}_i$	: Coefficient used in second variation of total potential energy in terms of mechanical and physical properties of the plates and adhesive
$A_1 - A_4$	: Four arbitrary points in going from Configuration 1 to Configuration 4
b_f	: Width of the flange of the section introduced in Example 1 in Chapter 5
D	: Bending rigidity of the plate
D_m	: Membrane rigidity of the plate
E_s	: Modulus of elasticity of steel
E_g	: Modulus of elasticity of GFRP
$f_i(x, y)$	: Polynomials used in approximated displacement functions in Chapter 3
G_a	: Shear modulus of adhesive
h_i	: Plate thickness
H_i	: Shape functions used to interpolate buckling transverse displacement fields and rotations about x and y
L_i	: Bilinear shape functions used in Chapter 4
L_x, L_y	: Half length of the shell element in x direction
O_i	: Origins of local coordinate system
P_0	: Buckling pressure of the un-corroded single plate introduced in Example 1 in Chapter 5
P_0	: Buckling pressure of the corroded single plate introduced in Example 1 in Chapter 5
P_{xx}	: Normal pressure along x direction acting on the mid-surface of the plate
P_{yy}	: Normal pressure along y direction acting on the mid-surface of the plate
P_{xy}	: Shear traction acting on the mid-surface of the plate

t_a	: Thickness of adhesive layer in Chapter 5
t_g	: Thickness of GFRP plate in Chapter 5
t_s	: Thickness of un-corroded steel plate in Chapter 5
t_{sr}	: Thickness of corroded steel plate in Chapter 5
u_{ib}, v_{ib}	: In-plane buckling displacement fields along x and y direction respectively
$u_p(x, y)$	: In-plane pre-buckling displacement field along x direction
u^*, v^*, w^*	: Total displacement fields along x , y and z respectively
U_p	: Pre-buckling internal strain energy stored in the plate
U^*	: Total internal strain energy in going from Configuration 1 to Configuration 4
$u_b(x, y)$	: Approximate in-plane displacement functions in x direction in Chapter 3
u_b^j, v_b^j	: Buckling in-plane displacements of node i used in Chapter 4 ($j = 1, 2, 3, 4$)
$\bar{u}_{bn}, \bar{v}_{bn}$	: In plane buckling displacement field in x and y direction for the GFRP plates
$v_p(x, y)$	: In-plane pre-buckling displacement field along y direction
V_p	: Pre-buckling load potential energy lost by the external loads
V^*	: Total load potential energy in going from Configuration 1 to Configuration 4
$\tilde{v}_b(x, y)$	: Approximate in-plane displacement functions in y direction in Chapter 3
w_b	: Transverse buckling displacement field
$w_b(x, y)$	: Approximate in-plane displacement functions in z direction in Chapter 3
w_b^j	: Transverse buckling displacement of node i used in Chapter 4 ($j = 1, 2, 3, 4$)
$\bar{w}_{bn}$	: Transverse buckling displacement field used for all layers
x, y, z	: Variables; or axes of Cartesian coordinate
α, β	: Constants used in in-plane pre-buckling displacement functions
$\gamma_{xy,p}$	: Pre-buckling shear strain component within xy plane
$\gamma_{xz,p}$	: Pre-buckling shear strain component within xz plane

$\gamma_{yz,p}$	: Pre-buckling shear strain component within yz plane
$\gamma_{xy,bl}$	: Buckling-linear shear strain component within xy plane
$\gamma_{xy,bn}$	: Buckling-nonlinear shear strain component within xy plane
$\gamma_{xz,bl}$	: Buckling-linear shear strain component within xz plane
$\gamma_{xz,bn}$	: Buckling-nonlinear shear strain component within xz plane
$\gamma_{yz,bl}$	: Buckling-linear shear strain component within yz plane
$\gamma_{yz,bn}$	: Buckling-nonlinear shear strain component within yz plane
γ_{xy}^*	: Total shear strain component within xy plane
γ_{xz}^*	: Total shear strain component within xz plane`
γ_{yz}^*	: Total shear strain component within yz plane
$\epsilon_{x,p}, \epsilon_{y,p}$	: Pre-buckling normal strain components within the plate along x and y direction
$\epsilon_{x,bl}, \epsilon_{y,bl}$	: Buckling-linear normal strain components within the plate along x and y direction
$\epsilon_{x,bn}, \epsilon_{y,bn}$	: Buckling-nonlinear normal strain components within the plate along x and y direction
$\epsilon_x^*, \epsilon_y^*$	: Total normal strain components along x and y direction
ζ	: $(x - y)/\sqrt{2}$
η	: y/L_y
λ	: Load factor (Eigen Value)
ν_s	: Poisson ration of steel
ξ	: x/L_x
Π_p	: Total potential energy of pre-buckling analysis
Π^*	: Total potential energy in going from Configuration 1 to Configuration 4

σ_x	: Normal stress along x direction
σ_y	: Normal stress along y direction
τ_{xy}	: In-plane shear stress
τ_{xz}, τ_{yz}	: Transverse shear stresses
$\left[\overline{C}_i\right]_{4 \times 4}$	: Matrix of sixteen integration constants used in Eq. 4.10
$\left[\overline{D}_1\right]_{12 \times 12}$	: Matrix of sixteen integration constants used in Eq. 4.10
$[K]_{20 \times 20}$	: Stiffness matrix of eigenvalue equation used in Eq. 4.11
$[K_g]_{20 \times 20}$	: Geometric stiffness matrix of eigenvalue equation used in Eq. 4.11

1 Introduction

1.1 Introduction and motivation

Carbon Fiber Reinforced Polymers (CFRP) and Glass Fiber Reinforced Polymers (GFRP) plates have recently received attention as viable options for reinforcing existing steel members. Possible applications involve web strengthening of existing plate girders and reinforcing corroded flanges and/or webs where a steel plate under combination of shear and/or normal stresses can be governed by their buckling strength. Using FRPs as a retrofit material is attractive from several respects such as easy application, achieving high additional strength with low additional weight, and corrosion resistance. The main differences between GFRP and CFRP plates are in their elasticity modulus and their practical thicknesses. CFRP have a high modulus of elasticity (ranging from 112 GPa to 450 GPa) and are practically used in thin sheets (1.3-12 mm), while GFRP have a lower modulus of elasticity (ranging from 11.46 GPa to 87 GPa) and are typically used in thicker layers (1.5-19 mm). Reliable and effective predictions of the resistance of resulting steel-FRP composite system is important for their effective design of such systems. Towards this, the present study aims at developing effective and accurate numerical solutions for the analysis of composite systems consisting of structural plates reinforced with GFRP plates through adhesive layers.

1.2 Literature review

The present section provides a survey of the literature relevant to the proposed study. It consists of two main sub-Sections. Sub-Section 1.2.1 presents studies aimed at characterizing the mechanical properties of commonly used GFRP, CFRP, and adhesive materials, while Sub-Section 1.2.2 provides an overview of analytical and computational studies on plate buckling including single plates, multi-layer plates, and sandwich plates as well as studies on steel members reinforced with GFRP.

1.2.1 Mechanical Properties

Owing to their superior mechanical properties, abundancy, light-weight, low density and high toughness, fiber-reinforced polymers (FRP) have increasingly been used as retrofitting materials for existing structural members. FRPs fall into two common categories; Carbon fiber-reinforced polymers (CFRP) and glass fiber-reinforced polymers (GFRP). FRP is bonded to structural members using an adhesives material. The properties of FRP and the adhesives greatly influence the strength and behaviour of the resulting composite system. Thus, Section 1.2.1 aims at reviewing the mechanical properties of CFRP (Sub-Section 1.2.1.1), GFRP (Sub-Section 1.2.1.2), and adhesives (Sub-Section 1.2.1.3).

1.2.1.1 Properties of CFRP

Several studies have investigated the structural and material properties of CFRP. In the present section, a description for some of these studies is provided, followed by a summary of CFRP mechanical properties as listed in various studies.

Miller et al. (2001) conducted a study on the application of composite steel members used in bridge engineering. A CFRP plate with a modulus of elasticity of 112 GPa and 5.25 mm thickness was bonded to a steel girder with a high-strength epoxy Araldite AV8113/ HV8113. Four full-scale bridge girders were tested to evaluate the effectiveness of the system. The results showed an increase of 10-37 % in the elastic stiffness and 17-25 % in the ultimate capacity compared to the unreinforced case.

Tavakkolizadeh and Saadatmanesh (2003) presented a study on the behaviour of steel-concrete composite girders strengthened with CFRP sheets under static loading. The thickness of the CFRP sheets was kept constant while the number of layers was varied from one to five. The test results showed increase of ultimate-load carrying capacity of the composite structure compared to those obtained from the tests on unreinforced case.

Linghoff et al. (2010) conducted a study on the behaviour of steel beams strengthened with three configurations of carbon-fiber-reinforced-polymer laminates by studying the failure modes and interfacial shear stresses in the bond line between steel and CFRP sheets. The steel beams were tested in four-point bending. The results showed that using CFRP laminates bonded to the tension flange of a steel beam can increase the bending strength by up to 20 %.

Potyrala and Rius (2011) reported a study on FRP composites properties and use of these materials in civil engineering applications. The authors listed a number of properties of FRP (CFRP and GFRP) composites such as density, modulus, Poisson's ratio and tensile strength. The authors also presented simplified formulas to evaluate properties of the composites based on the properties of their components.

Aguilera and Fam (2013) conducted an experimental study on strengthening rectangular hollow steel section T-joints against web buckling induced by transverse compression. A 2 mm thick high-modulus CFRP pultruded plate with unidirectional fibers was investigated. The high modulus of the CFRP was observed to limit the ultimate strain and consequently, CFRP could not accommodate the bending deformation. The authors also tested hollow steel sections reinforced with GFRP plates (much thicker than CFRP plates with a lower modulus) and concluded that GFRP reinforced specimens were more efficient than CFRP reinforced specimens.

Table 1.1- Mechanical properties of CFRP

(1)	(2)	(3)	(4)	(5)	(6)
Reference	Material type	Young Modulus (GPa)	maximum Strength (MPa)	Poisson's Ratio	Source
Miller et al. (2001)	CFRP	112	930.0	0.37	Manufacturer Specifications
Tavakkolizade M. and Saadatmanesh H. (2003)	CFRP	144	2137	0.34	Experiment
D. Linghoff et al. (2010)	CFRP 1	200	3300	Not mentioned	Manufacturer Specifications
	CFRP 2	330	1500		
	CFRP 3	165	3100		
Potyrala and Rius (2011)	CFRP(HS) ^(*)	230	Not mentioned	Not mentioned	Experiment
	CFRP(HM) ^(**)	370			
Aguilera and Fam (2013)	CFRP	450	1200	Not mentioned	Manufacturer Specifications

^(*) HS: High strength CFRP, ^(**) HM: High modulus CFRP

Table 1.1 provides a summary of CFRP properties as characterized by different researchers. Listed are the Material types (Column 2), Young modulus (Column 3), maximum strength (Column 4),

Poisson's ratio (Column 5), and the source of information (Column 6). A large variability is observed in the results. The Young modulus ranged from 112 GPa to 450 GPa and the maximum strength ranged from 930 MPa to 3300MPa.

1.2.1.2 Properties of GFRP

A few studies have investigated the structural and material properties of GFRP. In the present section, a description of each study is provided followed by a comparative summary of GFRP mechanical properties as listed in various studies.

El Damatty and Abushagur (2003) conducted an experimental investigation to evaluate the shear and peel stiffness of steel members bonded to GFRP plates through an adhesive. The experiment involved shear lap testing on $110\text{ mm} \times 100\text{ mm} \times 19\text{ mm}$ GFRP sheets bonded to hollow steel sections. The in-plane behaviour of the composite system was evaluated by using displacement measurements while the out of plane behaviour was evaluated using strain measurement at the outer face of FRP sheets.

Harries and El-Tawil (2011) conducted a study on Steel-GFRP composite systems. Their study focused on (1) strengthening steel members with GFRPs, (2) stabilizing buckling-critical steel elements with GFRPs, (3) fatigue and fracture of Steel-GFRP composite and (4) bonding properties of adhesives.

Shokrieh and Omid (2011) conducted a study on strain rate effects on the transverse tensile and compressive properties of Glass/Epoxy composite materials. Using strain rate control The authors conducted experimental tensile and compressive tests on the composite systems.

Mounier et al. (2012) conducted an extensive experimental study on the material properties of GFRP based on Resonant Ultrasound Spectroscopy laser method (RUS) to characterize their longitudinal and transverse Young moduli. Results showed negligible directional difference and thus concluded that GFRP can reasonably be assumed as an isotropic material. The study also characterized thermal properties of GFRP. It was shown that thermal effects significantly influence their bond properties when embedded in a composite system.

Table 1.2 provides a summary of GFRP properties as characterized by different researchers. Listed are the Material types (Column 2), Young modulus (Column 3), maximum strength (Column 4), Poisson's ratio (Column 5), and the source of information (Column 6). A large variability is

observed in the results. Young modulus ranged from 11.46 GPa to 87 GPa and the tensile strength ranged from 2.56 MPa to 896 MPa.

Table 1.2- Mechanical properties of GFRP

(1)	(2)	(3)	(4)	(5)	(6)
Reference	Material type	Young Modulus (GPa)	maximum Strength (MPa)	Poisson's Ratio	Source
El. Damatty and M.Abushagur (2003)	GFRP (EXTREN500) (under tension)	17.21	206.8	0.33	Manufacturer Specifications
Harries and El-Tawil (2011)	GFRP (under tension)	42	896	Not mentioned	Not mentioned
Mahmood M Shokrieh et al. (2011)	GFRP (under tension)	11.46	68.66	0.3	Experiment
	GFRP (under compression)	13.86	141.8		
Potyrala and Rius (2011)	GFRP (E- Glass) (under tension)	72	1.720	Not mentioned	Experiment
	GFRP (S- Glass) (under tension)	87	2.530		
D. Mounier et al. (2012)	GFRP (under tension)	75 ± 5	Not mentioned	0.250 ± 0.05	Experiment

1.2.1.3 Properties of Adhesive

Adhesive materials used in bonding steel to FRPs exhibit large variability. Thus, many researchers conducted experimental investigations on the mechanical properties and behaviour of adhesive used in composite systems. Table 1.3 provides a summary of adhesive properties as characterized by different researchers. Listed are the Material types (Column 2), Young modulus (Column 3), Poisson's ratio (Column 4), and the source of information (Column 5). The Young modulus is observed to range from 107 MPa to 7000 MPa.

Table 1.3- Mechanical properties of adhesive

(1)	(2)	(3)	(4)	(5)
Reference	type	Young Modulus (MPa)	Poisson's Ratio	Source
Miller et al. (2001)	AV8113/HV8113	107	Not mentioned	Manufacturer
	Plexus MA555	Not mentioned		
El. Damatty and M.Abushagur (2003)	MA 420	Not mentioned	Not mentioned	Manufacturer
Schnerch et al. (2004)	SS620	410	Not mentioned	Manufacturer
Harries and El-Tawil (2011)	High modulus	4500	Not mentioned	Not mentioned
	Low modulus	400		
Linghoff et al. (2010)	Epoxy 1	7000	Not mentioned	Manufacturer
	Epoxy 2	4500		

1.2.2 Analytical and Computational Studies for plate bending/buckling

1.2.2.1 General

This section is subdivided into four parts. Sub-Section 1.2.2.2 provides an overview of the studies on buckling of homogeneous plates. In the present document, the term “multilayer plates” is used to denote plates consisting of multiple layers with mechanical properties of comparable magnitudes. In the present study, the term “sandwich plates” denotes systems with a soft and thick core bonded to two thin face sheets with a significantly stronger material. Sub-Section 1.2.2.3 provides a review of studies on multilayer plates. Studies relevant to the sandwich plates are reviewed in Sub-Sections 1.2.2.4 and studies on the steel members bonded to GFRP plates are reviewed in Sub-Section 1.2.2.5.

1.2.2.2 Studies on homogeneous plates

The Kirchhoff-Love plate theory, also known as classical plate theory, was developed based on extending Euler-Bernoulli beam kinematic assumptions to thin plates. The assumptions forming the basis of the Kirchhoff-Love plate theory (e.g., Timoshenko (1959)) are: (1) straight lines normal to the mid-surface remain normal after deformation, (2) the thickness of the plate does not change during deformation, and (3) straight lines normal to the mid-surface remain straight after deformation. The Kirchhoff hypotheses were modified by Reissner (1945) and Mindlin (1951) to develop a First-order Shear deformable theory (FSDT) which is more suitable for thick plates. Under FSDT, the in-plane displacements are also assumed to have a linear distribution across the thickness and the normal to the mid-surface is assumed to remain straight after deformation while the difference between FSDT and the Kirchhoff-Love plate bending theory is that a fiber initially normal to the mid-surface does not remain perpendicular to the mid-surface. In FSDT theories, a shear correction coefficient is needed in order to account for the variation of shear stresses and strains throughout the plate thickness. In order to overcome the limitation of FSDTs, higher order shear deformable theories (HSDT) have been developed in recent years.

Hildebrand et al. (1949) were the first to develop a plate and shell theory based on higher order shear deformable. Other higher order shear deformable theories include the work of Reddy (1984), which has been extended to the analysis of the composite plates. In HSDT, the in-plane displacements are assumed to have a polynomial distribution across the plate thickness. The HSDT theories will be discussed in the following sections.

1.2.2.3 Studies on multilayer plates

This section provides the studies on multilayer plates based on two main methods: (1) the Equivalent Single layer theory (ESL) and (2) the Layer-wise Laminate theory (LWT). Each method can be applied either with HSDT or FSDT. The main deficiency of ESL theories is that since the transverse displacements across the multilayer plate thickness are assumed continuous, the resulting strains are also continuous, while the transverse shear stresses become discontinuous at the layer interfaces given the different material properties of the layers. Unlike the ESL theories, the Layer-wise theories (LWT) assume separate displacement distribution across the thickness of each layer and displacement continuity across layers interfaces is enforced using constraint

equations. Figure 1.1 depicts the in-plane displacement distribution across the thickness of a multilayer plate based on ESL and LWT theories.

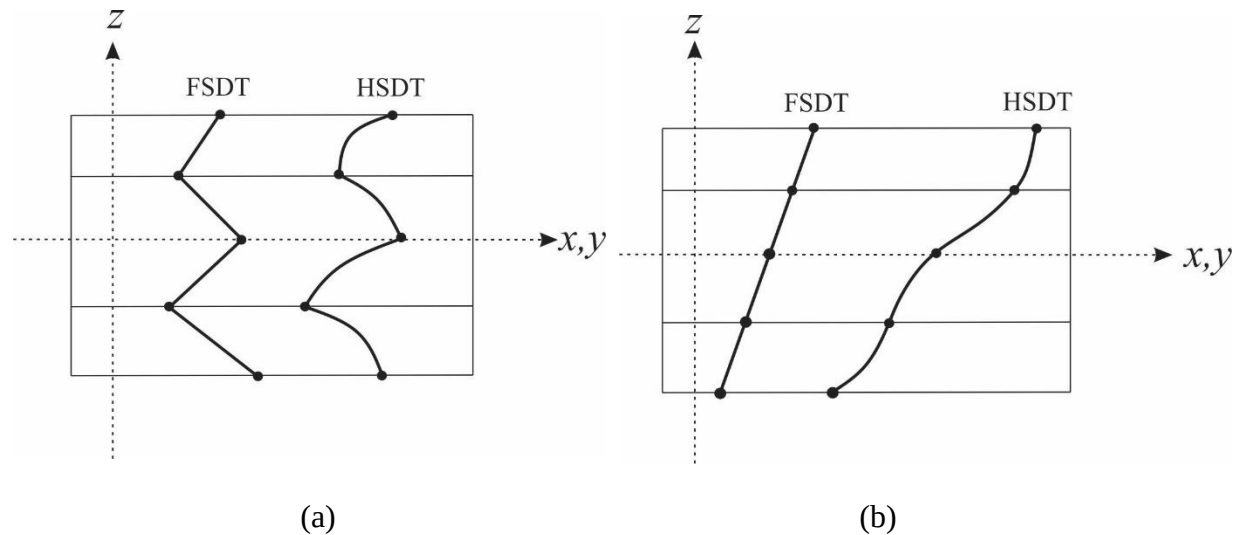


Figure 1.1- In-plane displacement distribution across the thickness based on (a) LWT theory, (b) ESL theory

1.2.2.3.1 Studies based on ESL theories

Reddy (1984) developed a simple HSDT for analyzing laminated multilayer plates. The theory postulated a parabolic distribution of the transverse shear strains through the thickness of the multilayer plate.

Di Sciuva (1987) developed a FSDT for moderately thick multilayer shells and plates. The in-plane displacement fields were assumed to be linear while the transverse displacements were assumed constant across the thickness, thus reducing the problem from 3D to 2D. The displacement distributions satisfied the continuity condition at the interfaces.

Vallabhan et al. (1993) developed a model for the nonlinear geometric analysis of laminated glass units consisting of two thin glass plates bonded together by a thin polyvinyl butyral (PVB) core material. The authors adopted variational calculus to minimize the potential energy and obtained the governing equilibrium equations and associated boundary conditions. The equations were then numerically solved using the finite difference method. Lateral displacements and strains at the top and bottom of the glass units were compared to those based on experiments

Liew et al. (1994) derived closed-form analytical solutions for the buckling of simply supported rectangular symmetric multilayer plates and based on the Navier's method. Unlike conventional

solutions, their formulation considered the effects of in-plane pre-buckling deformations. Transverse shear effects were incorporated through a first order shear deformable theory (FSDT). The governing differential equations were obtained by evoking the stationary conditions of the total potential energy.

Nguyen-Xuan et al. (2013) developed an efficient formulation based on a fifth order shear deformable theory (FiSDT) for static, dynamic and buckling analysis of rectangular and circular multilayer plates. The solution is based on an isogeometric finite element approach which offers the possibility of integrating finite element analysis into conventional CAD design tools. Under this approach, accurate results were attained with relatively coarse meshing.

1.2.2.3.2 Studies based on Layer-Wise-Theories (LWT)

Di Scuiava (1986) developed a zigzag layer-wise theory in which the in-plane displacement fields are expressed in terms of those of the reference plane, and the transverse displacement is assumed constant throughout the plate thickness. The results obtained from their theory were then compared to those based on the theory of elasticity, the classical Kirchhoff laminated plate theory, and the Mindlin shear deformable theory.

Reddy (1987) proposed a layer-wise HSDT nonlinear plate bending theory based on the von Karman strain expressions for thick multilayer plates and evaluate their bending behaviour. The theory is based on the assumptions of (1) zero traction boundary conditions on the top and bottom faces of the plate, and (2) in-plane displacements with cubic distribution throughout the plate thickness. Closed-form solutions for simply supported plates were presented and the results were then compared to those based on 3D elasticity theory to show the accuracy of the developed formulation.

Cho and Paramerter (1992) developed a zigzag plate bending theory combining both Reddy's HSDT and the zigzag layer-wise theory developed by Di Scuiava (1986). The in-plane displacement fields were based on a cubic distribution in the transverse direction resulting in the same number of displacement fields as FSDTs. The displacements satisfied the transverse shear stress continuity condition at the interfaces and zero-shear at the outer faces. Thus, no shear correction factor was needed in their theory. Because the number of variables was reduced, the study offered an efficient solution compared to other theories.

1.2.2.4 Studies on sandwich plates

Benson and Maters (1967) formulated the governing equations and associated boundary conditions for sandwich plates with orthotropic core (honeycomb) by considering both the transverse shear and normal strains in all layers. Using variational principles, the stationary conditions of the total potential energy were evoked under assumptions that: (1) the shear stresses across the core are assumed to be independent of the coordinate along the plate thickness, (2) thin-plate theory is used to characterize the behaviour of the faces and (3) The face materials are assumed to be isotropic, and (3) both faces were assumed to have equal stiffnesses. Closed form solutions were then presented for specific boundary conditions. Also, the Rayleigh-Ritz method was used to provide approximate solutions for more general boundary conditions.

Heder (1991) conducted an analytical solution for the buckling loads of sandwich panels with various boundary conditions. Their results were shown to be in good agreement with those based on the finite element analysis. The assumptions made were: (1) the face and core materials are isotropic, (2) the faces have equal thicknesses, (3) the faces are thin compared to the core, (4) all deflections are small, and (5) the transverse normal stresses are negligible. Their analytical solutions were observed to be higher than those based on the finite element analysis by about 15%. The author reasoned that the thicker solid elements used in the finite element analysis were associated with lower shear stiffness and thus lead to lower buckling load predictions than those based on the analytical solution developed by the authors.

Frostig and Baruch (1993) conducted an analytical study on the buckling behaviour of sandwich beams consisting of two composite laminates or metallic faces and a thick core made of foam or low-strength honeycomb. Their analytical solution is applicable to general boundary conditions but, for simplicity, a closed form solution based on HSDT was developed for simply supported beams with identical skins.

In a subsequent study, Frostig (1998) developed a closed form solution for plate buckling of sandwich panels with flexible thick cores. The analysis was based on a higher-order theory which captures the compressibility of the core layer. Numerical results were presented for the case of simply supported panels. Given the flexibility of the core, for some structural configurations, local buckling of the skins was observed to be more critical than global buckling of the sandwich system.

Sokolinsky and Frostig (1999) developed a closed form solution based on a linearized buckling analysis of high-order shear deformable theory for sandwich plates. Their solution is based on assumptions that (1) the sandwich panel response remains in the linear elastic range, (2) the face sheets undergo moderate deformations, (3) the thick core undergoes small deformations, (4) the height of the core is allowed to change under loading and (5) the interfaces provide full bonding between core and the face sheets. The authors investigated the effect of the boundary conditions on the buckling capacity of sandwich plates.

Dawe and Yuan (2001) developed a finite strip solution for predicting the buckling stresses of rectangular sandwich plates. In their study, the core was represented by three dimensional solids and the faceplates were considered as composite laminates. Two idealizations were considered for the faceplates; a shear deformable solution and a classical thin plate solution. The in-plane displacement fields of the core were assumed to vary quadratically across the thickness and the transverse displacements of the core were assumed to vary linearly across the height. The displacement fields of the finite strip were presented by 12 generalized displacements for shear deformable faceplates and 8 generalized displacements for the classical non-shear deformable solution.

Kant and Swaminathan (2002) developed a static analytical formulation based on high order shear deformable theory. A closed form solution was presented using Navier's technique. The theoretical model accounted for the effects of transverse shear deformation, transverse normal stress/strain and a nonlinear variation of in-plane displacements with respect to the thickness coordinate. The results based on this study and the other studies based on FSDT and HSDT were then compared to those based on 3D FEA in Abaqus and were shown to be in better agreement with Abaqus.

Khalili et al. (2013) developed an improved HSDT for the buckling analysis of sandwich plates with soft orthotropic cores. The study was based on the assumptions that: (1) transverse shear stresses vanish at the top and bottom faces of the composite system, (2) transverse shear stresses at the interfaces are continuous, (3) the core is softer and thicker than the faces, (4) a third-order shear deformable theory is adopted for the face sheets, and (5) quadratic and cubic functions are assumed for the transverse and in-plane displacements of the soft core. The nonlinear von-Karman strain-displacement relations were adopted. An analytical solution for the buckling analysis of simply supported sandwich plates under various in-plane compressive loads was developed using

Navier's solution. The results were then compared to those based on high-order equivalent single layer theory (ESL). The overall buckling loads based on the present study were observed to be less than those based on the ESL.

1.2.2.5 Studies on steel members bonded to GFRP

El. Damatty and Abushagur (2003) developed an analytical solution to characterize the in-plane and out of plane behaviour of shear lap tests for GFRP sheets bonded to hollow structural steel sections. The GFRP sheets were treated as plates on elastic foundation while the steel section was considered infinitely rigid compared to the GFRP and adhesive. The model neglected the interaction between the shear and peel behaviours, and omitted the visco-elastic behaviour of the adhesive, but provided insight on the behaviour of adhesive bonding GFRP to steel sections.

Accord and Earls (2006) conducted an analytical study to investigate the effect of GFRP strips on controlling the local buckling of steel beam flanges during plastic hinging. The study employed nonlinear finite element modeling within the ADINA software. The steel I-section was modeled using the four-node shell finite element MITC4 which captures transverse shearing effects within the element. Fully integrated eight-node 3D continuum finite elements were used to model the volumes of the GFRP and the interface material. The load-deflection relations obtained have shown that the bare steel model experienced severe local buckling at the compression flange, in contrast with GFRP reinforced members, which were stabilized against local buckling and forced into different failure modes.

Siddique and El Damatty (2012) assessed the enhancement in the buckling capacity of steel plates bonded to glass fiber reinforced polymer (GFRP) sheets with adhesive interlayer based on a finite element modelling. The numerical model was based on a 13-node consistent degenerated triangular sub-parametric shell element developed by Koziey and Mirza (1997). This element is free from shear-locking. To model the adhesive layer, two dimensional continuous springs were adopted to represent the shear interaction between both plates. Thus, the composite system was modelled using 26-nodes. The buckling capacity of the steel plates reinforced with GFRP plates was then determined for different ratios of plate slenderness.

In another study, Siddique and El Damatty (2013) adopted their model to study the effect of using GFRP plates for enhancing local buckling behaviour of wide flange steel beams. The results

showed a significant increase in load carrying capacity and deflection at failure of the strengthened members in comparison with those related to unreinforced members.

In a subsequent study, Siddique et al. (2013) conducted a numerical investigation to assess the effectiveness of using GFRP plates to improve the over-strength and ductility factors of moment resisting steel frames. The GFRP plates were bonded to the flanges of steel beams to strengthen them against local buckling and were modeled using a set of a continuous linear spring system. In their model they assumed that: (1) all inelastic deformations were assumed to be concentrated into a hinge of zero length located at each end of the girder, (2) the point of contra-flexure in the beam was located at girder mid-span. Based on assumptions (1) and (2), they considered half of the cantilever beam in their model. The finite element model in Siddique and El Damatty (2012) was adopted to determine the moment-rotation relationship of steel beams retrofitted with GFRP plates. The results were verified against the experimental results reported in El. Damatty and Abushagur (2003). Using the shell element model, the steel beams with and without the GFRP were incrementally loaded until failure by one of the following modes: (1) elastic buckling of the system associated with local buckling of the flanges of the beam, (2) inelastic buckling of the system, (3) full plastification of the steel, (4) shear failure of the adhesive, (5) peeling failure of the adhesive or (6) GFRP rupture. The results showed that for frames with slender beams, both the lateral load-carrying capacity and deflection at failure were enhanced by adding GFRP plates.

Pham and Mohareb (2014)a developed finite element formulations for analysis of wide-flange steel beams reinforced by a GFRP plate. The solution is based on extended version of assumptions adopted in Vlasov and Gjelsvik beam theory, i.e., (1) the section contours of the beam and GFRP plate were assumed to remain un-deformed in their own plane, (2) The beam and GFRP plate were assumed to follow the classical Kirchhoff assumption (i.e., a normal to the middle surface is assumed to remain normal throughout deformation), (3) The beam and GFRP plate were assumed to be perfectly bonded to the adhesive at the interfaces, (4) The adhesive was assumed to act as a weak material in comparison with steel and GFRP, (5) The thickness of the adhesive layer was assumed to remain constant throughout deformation, and (6) The steel material, GFRP, an adhesive were assumed to be linearly elastic isotropic. Two developments were presented in this study, i.e., a shear-deformable theory (SDE) and non-shear deformable theory (NSDE). Based on a SDE, a 10-degree-of freedom model was developed for the longitudinal-transverse analysis and another 16-degree-of-freedom model for lateral torsional response. For the NSDE, an 8-degree-of

freedom model was developed for the longitudinal-transverse analysis and another 12-degree-of-freedom model for lateral torsional response.

In a subsequent study, Pham and Mohareb (2014)b developed a shear deformable thin-walled beam theory for the analysis of steel beams reinforced with a GFRP to one of the flanges based on the assumptions made in Pham and Mohareb (2014)a. The governing equilibrium equations and boundary conditions were formulated based on the principle of stationary total potential energy and were solved using the finite difference technique. Comparisons with 3D FEA results based on the C3D8R element in Abaqus showed the validity of the solution.

In another study, Pham and Mohareb (2015)a developed a general closed form solution for the non-shear deformable theory for the analysis of the steel beams reinforced with GFRP plate. All the assumptions made in this study were similar to the previous works. The study also provided a comparison with 3D FEA solutions and discussed the limitations of non-shear deformable solutions in the predictions of lateral-torsional response.

An another 3D FEA solution, Pham and Mohareb (2015)b investigated the stress state in adhesive materials bonding wide flange steel beams to the GFRP plate and subjected to transverse bending. The authors presented contour plots for stress field based on 3D FEA. A parametric study was then performed to investigate the effect of the shear modulus of adhesive, adhesive layer thickness and GFRP plate thickness on the maximum von-Mises stress. In a companion study, Pham and Mohareb (2015)c extended the work to wide flange steel beam to the GFRP and subjected to twist. The study investigated the effects of the adhesive mechanical properties on the maximum von Mises stresses in adhesive materials. Table 1.4 provides a comparative review of the studies on multilayer plates (Section 1.2.2.3), sandwich plates (Section 1.2.2.4) and steel members reinforced with FRP plates (Section 1.2.2.5).

Table 1.4- Summary of studies reviewed in Sections 1.2.2.3, 1.2.2.4 and 1.2.2.5

Studies on Multilayer Plates																	
Source	Type of analysis		Methodology		Contribution of strains in Energy formulation				Solution Type								
	Buckling	Bending	ESL	LWT	Normal in-plane strain	Normal transverse strain	Shear in-plane strain	Shear transverse strain	Analytical solution	Finite element solution							
Reddy (1984)	-	✓	✓	-	✓	-	✓	✓	✓	-							
Di Scuiua (1987)	-	✓	✓	-	✓	✓	✓	✓	✓	-							
Vallabhan et al. (1993)	-	✓	✓	-	✓	-	✓	✓	✓	-							
Liew et al. (1994)	✓	-	✓	-	✓	-	✓	✓	✓	-							
Nguyen-Xuan et al. (2013)	✓	-	✓	-	✓	-	✓	✓	✓	✓							
Di Scuiua (1986)	✓	-	-	✓	✓	✓	✓	✓	✓	-							
Reddy (1987)	-	✓	-	✓	✓	-	✓	✓	✓	-							
Cho and Paramerter (1992)	-	✓	-	✓	✓	-	✓	✓	✓	-							
Studies on Sandwich Plates																	
Source	Type of Analysis		Contribution of strains in Energy formulation								Solution types						
	Buckling	Bending	Face sheets				Core				Analytical Solution	Finite Element	Finite Difference	Finite Strip	3D FEA in software	Experimental	
			Normal in-plane strain	Normal transverse strain	Shear in-plane strain	Shear transverse strain	Normal in-plane strain	Normal transverse strain	Shear in-plane strain	Shear transverse strain							
Benson and Maters (1967)	-	✓	✓	-	✓	-	-	✓	-	✓	✓	-	-	-	-	-	
Heder (1991)	✓	-	✓	-	✓	-	-	-	-	✓	✓	-	-	-	-	-	
Frostig and Baruch (1993)	✓	-	✓	-	-	-	-	✓	✓	✓	✓	-	-	-	-	-	
Frostig (1998)	✓	-	✓	-	✓	-	-	✓	-	✓	✓	-	-	-	-	-	
Sokolinsky and Frostig (1999)	✓	-	✓	✓	✓	✓	✓	✓	✓	✓	-	-	✓	-	-	-	
Dawe and Yuan (2001)	✓	-	✓	-	-	✓	-	✓	✓	✓	✓	-	-	✓	-	-	
Kant and Swaminathan (2002)	✓	-	✓	✓	✓	✓	✓	✓	✓	✓	✓	-	-	-	-	-	
Manshadi et al. (2011)	✓	-	✓	-	✓	✓	✓	✓	✓	✓	✓	-	-	-	-	-	
Studies on steel members reinforced with GFRP plates																	
El. Damatty and Abushagur (2003)		-	✓	✓	✓	✓	✓	-	-	✓	✓	✓	-	-	-	-	✓
Accord and Earls (2006)		✓	-	-	-	-	-	-	-	-	-	-	-	-	✓	-	-
Siddique and El Damatty (2012)		✓	-	✓	✓	✓	✓	-	-	✓	✓	✓	✓	-	-	-	✓
Siddique and El Damatty (2013)		✓	-	✓	✓	✓	✓	-	-	✓	✓	✓	✓	-	-	-	✓
Siddique et al. (2014)		✓	-	✓	✓	✓	✓	-	-	✓	✓	-	-	-	-	✓	-
Pham and Mohareb (2014)a)	SDT ^(*)	-	✓	✓	-	✓	-	-	-	✓	✓	-	✓	-	-	-	-
	NSDT ^(**)	-	✓	✓	-	-	-	-	-	✓	✓	-	✓	-	-	-	-
Pham and Mohareb (2014)b)	SDT	-	✓	✓	-	✓	-	-	-	✓	✓	✓	-	-	-	-	-
Pham and Mohareb (2015)a)	NSDT	-	✓	✓	-	-	-	-	-	✓	✓	✓	-	-	-	✓	-
Pham and Mohareb (2015)b)		-	✓	-	-	-	-	-	-	-	-	-	-	-	-	✓	-
Pham and Mohareb (2015)c)		-	✓	-	-	-	-	-	-	-	-	-	-	-	-	✓	-
Present Study		✓	-	✓	-	✓	-	-	-	✓	✓	✓	✓	-	-	✓	-

(*) SDT: Shear Deformable Theory, (**) NSDT: Non-shear Deformable Theory

1.3 Scope of the present study

Table 1.4 has provided a summary of studies on multilayer, sandwich plates and steel members reinforced with GFRP plates. Under the proposed theory, a new variational principle and finite element formulation are developed for analysis of steel plates reinforced with GFRP plates. The specifics of the present study are: (1) GFRP and steel plate are assumed to follow the thin-plate theory assumptions, (2) adhesive is assumed to act as a weak elastic material with a small modulus of elasticity compared to that of the adjoining materials. Consequently, the in-plane normal stresses in the adhesive layer are considered negligible compared to the normal stresses of plates, and (3) The 3D problem is transformed into 2D problem resulting into a computationally efficient solution.

1.4 Objectives

The present study aims at:

- (1) Developing a buckling variational principle for composite system includes steel plates strengthened with two GFRP plates through an adhesive layer
- (2) Verifying the validity of the variational principle through comparisons to 3D FEA results
- (3) Developing a finite element formulation and conducting verifications by solving practical examples
- (4) Using the finite element formulation developed, to conduct parametric studies for practical engineering problems
- (5) Compiling the results of parametric runs to provide design recommendations for engineering steel plates reinforced with GFRP.

1.5 Layout of the thesis

The present introductory chapter is followed by five chapters. Chapter 2 develops a variational principle for buckling analysis of single plates and composite plates including a steel plate reinforced with another identical steel plate through an adhesive layer and a steel plate reinforced with two GFRP plates through two adhesive layers. Chapter 3 provides a critical assessment of the correctness of the variational statements developed in Chapter 2 through comparison of results against 3D solutions. Chapter 4 presents a finite element formulation for buckling analysis of a steel plate reinforced with two GFRP plates based on the variational principle developed in

Chapter 2 and validated in Chapter 3. Chapter 5 provides first a verification study for the finite element solution developed in Chapter 4 and then provides a parametric study aimed at investigating the effect of key geometric and material parameters on the critical loads of the composite system. A summary, conclusions, and recommendations based on the study are then provided in Chapter 6.

2 Variational Formulation

2.1 Introduction

This chapter aims to develop a potential energy formulation for the buckling analysis of single and composite plates. The kinematics of the classical plate buckling, are first reviewed and the energy formulation for a single plate is presented in Section 2.3. The formulation is then extended in Section 2.4 for the case of two plates bonded together through an adhesive layer. In Section 2.5, the energy formulation is further extended for the buckling analysis of a steel plate reinforced with two GFRP plates through two adhesive layers.

2.2 General Assumptions

For a single plate, two steel plates bonded together through an adhesive layer and a steel plate reinforced with Two GFRP plates through adhesive layers, the solutions developed are based on the following two assumptions.

- (1) All materials (steel, GFRP and adhesive) are considered to be linearly elastic isotropic, and
- (2) All displacement variables used in derivations are related to the mid-surfaces of the layers.

Additional assumptions specific to each of the three problems investigated are provided under separate sections (2.3.1, 2.4.2, and 2.5.2)

2.3 Single Plate

2.3.1 Assumptions

The formulation is developed based on the assumptions of Kirchhoff-Love plate theory, namely:

- (1) a straight line normal to the mid-surface remains normal to the mid-surface after deformation,
- (2) straight lines normal to the mid-surface remain straight after deformation, (3) the thickness of the plate does not change during deformation, and (4) only the in-plane normal stresses and in-plane shear stresses are assumed to contribute to the internal strain energy while the contribution of other stress components are negligible.

2.3.2 Statement of the problem, coordinates and sign conventions

A single plate with dimensions $a \times b$ and thickness h (Figure 2.1) is subjected to normal pressure P_{xx} and P_{yy} or shear traction P_{xy} , all acting within the mid-surface of the plate. As shown in Figure 2.1, the right-handed coordinate system x, y and z with origin O is used for the plate. It is required to find the critical loading combination $\lambda(P_{xx}, P_{yy}, P_{xy})$ at which the system would buckle out of its own plane.

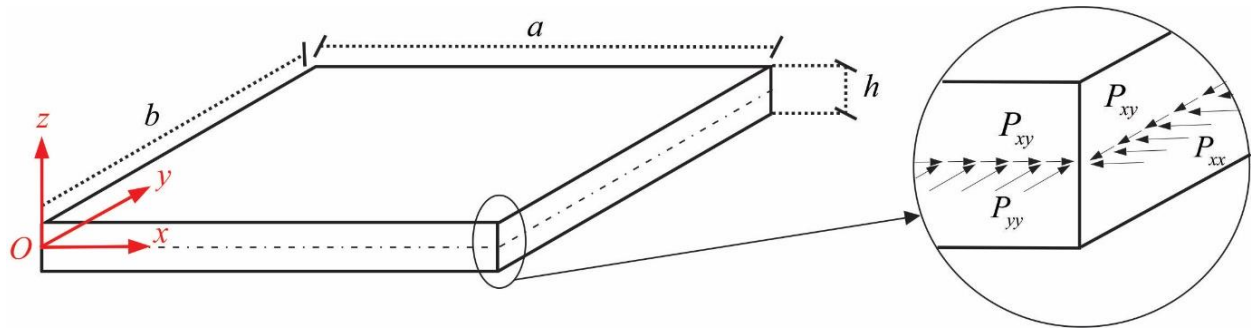


Figure 2.1- A single plate $a \times b \times h$ under loading combination (P_{xx}, P_{yy}, P_{xy})

2.3.3 Kinematics

As shown in Figure 2.2(1,2), the plate is assumed to deform from Configuration 1 to Configuration 2 under the applied reference loads by undergoing in-plane pre-buckling displacements $u_p(x, y)$ and $v_p(x, y)$. The load is then assumed to increase to λP_{xx} , λP_{yy} and λP_{xy} so that the plate reaches Configuration 3 which is onset of buckling state. The associated pre-buckling displacements are assumed to linearly increase to λu_p and λv_p , where λ is a constant to be determined. At Configuration 3, the plate has a tendency to buckle (i.e., going from Configuration 3 to configuration 4 under no increase in loads). As the plate buckles, its mid-surface undergoes out-of-plane buckling displacement $w_b(x, y)$ to reach Configuration 4 (Figure 2.2 (4)). Figure 2.2 shows the locations A_1, A_2, A_3 and A_4 for an arbitrary point in going from Configuration 1 to Configuration 4. At Configuration 4 the total displacement fields are obtained by summing the in-plane displacement fields at the onset of buckling to those during buckling, yielding

$$u^* = \lambda u_p, \quad v^* = \lambda v_p, \quad w^* = w_b \quad (2.1)$$

where superscripts $*$ denote displacements in the final buckling state (i.e., in going from Configuration 1 to Configuration 4) while all subscripts p denote pre-buckling displacements (i.e., in going from Configuration 1 to Configuration 2) and subscripts b denote buckling displacements (i.e., in going from Configuration 3 to Configuration 4). The above notation convention is also extended to other fields such as strains and stresses.

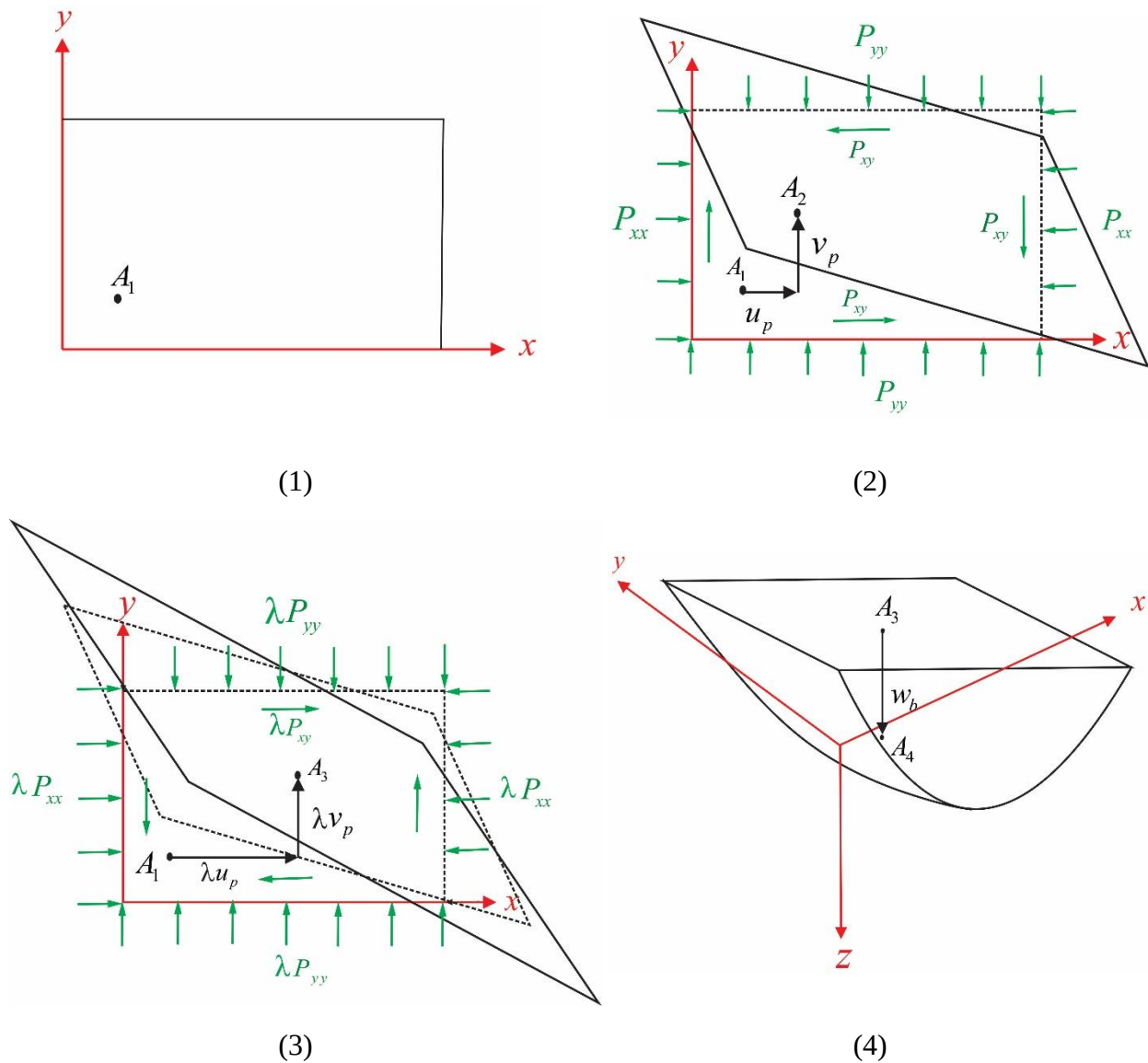


Figure 2.2- Initial configurations of the mid-surface of the plate, (1) Plan view of configuration 1, (2) Plan view of Configuration 2, (3) Plan view of Configuration 3, (4) Configuration 4

2.3.4 Strains in Terms of Displacements

Table 2.1 provides the buckling and pre-buckling strain–displacement relationships including linear and non-linear components. As a convention, all primes denote the differentiation of the argument function with respect to x and all dots denote differentiation with respect to y .

Table 2.1-Strain-Displacement Relationships of single plate theory

Pre-buckling strains		Buckling strains	
Linear	Nonlinear	Linear	Nonlinear
$\varepsilon_{x,p} = u'_p$	neglected	$\varepsilon_{x,bl} = -zw''_b$	$\varepsilon_{x,bn} = \frac{1}{2}(w'_b)^2$
$\varepsilon_{y,p} = \dot{v}_p$	neglected	$\varepsilon_{y,bl} = -z\ddot{w}_b$	$\varepsilon_{y,bn} = \frac{1}{2}(\dot{w}_b)^2$
$\gamma_{xy,p} = \dot{u}_p + v'_p$	neglected	$\gamma_{xy,bl} = -2z\dot{w}'_b$	$\gamma_{xy,bn} = w'_b\dot{w}_b$

2.3.5 Strain Decomposition

From Eq. (2.1) by substituting into the displacement fields gained in Table 2.1, the total strains ε_x^* , ε_y^* and γ_{xy}^* can be decomposed as

$$\varepsilon_x^* = \lambda\varepsilon_{x,p} + \varepsilon_{x,bl} + \varepsilon_{x,bn}, \quad \varepsilon_y^* = \lambda\varepsilon_{y,p} + \varepsilon_{y,bl} + \varepsilon_{y,bn}, \quad \gamma_{xy}^* = \lambda\gamma_{xy,p} + \gamma_{xy,bl} + \gamma_{xy,bn} \quad (2.2)$$

where all the components of Eq. (2.2) have been defined in Table 2.1.

2.3.6 Stress-Strain Relationships

The generalized Hooke's law gives the relationship between stresses and strains as

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \frac{E_s}{1-\nu_s^2} \begin{bmatrix} 1 & \nu_s & 0 \\ \nu_s & 1 & 0 \\ 0 & 0 & \frac{1-\nu_s}{2} \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} \quad (2.3)$$

where E_s is Young's Modulus and ν_s is Poisson's Ratio.

2.3.7 Pre-Buckling Analysis

Prior to performing a buckling analysis, the configuration of the plate under general reference loads (going from Configuration 1 to Configuration 2) needs to be determined from equilibrium considerations. The associated formulation is determined from pre-buckling analysis. The total potential energy related to pre-buckling state Π_p is sum of the internal strain energy stored in the plate U_p and the total potential energy lost by the external loads V_p

$$\Pi_p = U_p + V_p \quad (2.4)$$

The internal strain energy stored in the plate is

$$U_p = \frac{1}{2} \iiint_v (\sigma_{x,p} \varepsilon_{x,p} + \sigma_{y,p} \varepsilon_{y,p} + \tau_{xy,p} \gamma_{xy,p}) dx dy dz \quad (2.5)$$

The second contribution of the total potential energy is

$$V_p = - \int_0^a \int_0^b h \left[P_x u'_p + P_y \dot{v}_p + P_{xy} (\dot{u}_p + v'_p) \right] dx dy \quad (2.6)$$

From Eqs.(2.3), (2.5) and(2.6), by substituting into Eq.(2.4) one obtains

$$\begin{aligned} \Pi_p = & \frac{1}{2} \iiint_v \left(\frac{E_s}{1-\nu_s^2} (\varepsilon_{xp})^2 + \frac{E_s}{1-\nu_s^2} (\varepsilon_{yp})^2 + \frac{2\nu_s E_s}{1-\nu_s^2} \varepsilon_{yp} \varepsilon_{xp} + \frac{E_s}{2(1+\nu_s)} (\gamma_{xyp})^2 \right) dx dy dz \\ & - \int_0^a \int_0^b h \left[P_x u'_p + P_y \dot{v}_p + P_{xy} (\dot{u}_p + v'_p) \right] dx dy \end{aligned} \quad (2.7)$$

From the strain-displacement relation provided in Table 2.1, by substituting into Eq.(2.7), and taking first variation of the total potential energy with respect to the pre-buckling displacement field, one obtains

$$\begin{aligned} \overline{\Pi}_p = & \iint_A \left\{ \frac{E_s h}{1-\nu_s^2} \left[u'_p \overline{u'_p} + \dot{v}_p \overline{\dot{v}_p} + \nu \dot{v}_p \overline{u'_p} + \nu u'_p \overline{\dot{v}_p} \right] \right\} dx dy \\ & + \iint_A \frac{E_s h}{4(1+\nu_s)} \left(2\dot{u}_p \overline{\dot{u}_p} + 2\nu'_p \overline{\dot{v}_p} + 2\overline{\dot{u}_p} \nu'_p + 2\dot{u}_p \overline{\dot{v}_p} \right) dx dy \\ & - \int_0^a \int_0^b h \left[P_x \overline{u'_p} + P_y \overline{\dot{v}_p} + P_{xy} (\overline{\dot{u}_p} + \overline{\dot{v}_p}) \right] dx dy \end{aligned} \quad (2.8)$$

where $\overline{(\quad)}$ denotes the first variation of the argument function (or functional) with respect to the pre-buckling displacement fields. By performing integration by parts, one obtains the equilibrium field equations as

$$\begin{aligned} u_p'' + \nu \dot{v}_p' + \frac{1}{2}(1-\nu)(\ddot{u}_p + \dot{v}_p') &= 0 \\ \ddot{v}_p + \nu \dot{u}_p' + \frac{1}{2}(1-\nu)(v_p'' + \dot{u}_p') &= 0 \end{aligned} \quad (2.9)$$

and the boundary conditions

$$\begin{aligned} \frac{E_s h}{1-\nu_s^2} \left[(1+\nu) u_p' + \frac{1}{2}(1-\nu)(\dot{u}_p + v_p') \right] - P_x - hP_{xy} &= 0 \\ \frac{E_s h}{1-\nu_s^2} \left[(1+\nu) \dot{v}_p + \frac{1}{2}(1-\nu)(\dot{u}_p + v_p') \right] - P_y - hP_{xy} &= 0 \end{aligned} \quad (2.10)$$

The solution of Eq.(2.10) provides the pre-buckling displacements in terms of external loads. It can be shown that Eq. (2.9) reverts to the well know classical equilibrium equations, (e.g. Brush and Almroth (1975)).

2.3.8 Buckling analysis

2.3.8.1 Total potential energy

The total potential energy Π^* in going from Configuration 1 to Configuration 4 is given by

$$\begin{aligned} \Pi^* = U^* + V^* &= \frac{1}{2} \iiint_v (\sigma_x^* \varepsilon_x^* + \sigma_y^* \varepsilon_y^* + \tau_{xy}^* \gamma_{xy}^*) dx dy dz \\ &- \int_0^a \int_0^b h \left[P_x u_p' + P_y \dot{v}_p + P_{xy} (\dot{u}_p + v_p') \right] dx dy \end{aligned} \quad (2.11)$$

where U^* is the total internal strain energy stored in the plate in going from Configuration 1 to Configuration 4 and V^* is the total potential energy lost by the applied loads.

2.3.8.2 Condition of neutral stability

The neutral stability state (Configuration 3 in Figure 2.2) is obtained by setting to zero the variation of second variation of total potential energy given in Eq.(2.11). From the stress-strain relations in Eq.(2.3), and the strain-displacement relations in Table 2.1, by substituting into Eq. (2.11),

performing the integrations with respect to z , the second variation of the total potential energy

$\overline{\overline{\Pi^*}}$ with respect to buckling displacement fields is found to take the form

$$\begin{aligned} \frac{1}{2} \overline{\overline{\Pi^*}} = & \iint_A \left[\frac{E_s h^3}{24(1-\nu_s^2)} (\overline{\overline{w_b''^2}} + \overline{\overline{w_b'^2}}) + \frac{\nu_s E_s h^3}{12(1-\nu_s^2)} \overline{\overline{w_b'' w_b'}} + \frac{E_s h^3}{12(1+\nu_s)} \overline{\overline{w_b'^2}} \right] dx dy \\ & + \frac{1}{2} \lambda \left\{ \iint_A \left[\frac{E_s h}{1-\nu^2} (u_p' \overline{\overline{w_b'^2}} + \dot{v}_p \overline{\overline{w_b'^2}}) + \frac{\nu E_s h}{1-\nu_s^2} (\dot{v}_p \overline{\overline{w_b'^2}} + u_p' \overline{\overline{w_b'^2}}) + \frac{E_s h}{1+\nu_s} (\dot{u}_p + v_p') \overline{\overline{w_b' w_b'}} \right] dx dy \right\} \end{aligned} \quad (2.12)$$

where $\overline{(\quad)}$ denotes the first variation of the argument function with respect to the buckling displacements and $\overline{\overline{(\quad)}}$ denotes the second variation. All the terms multiplied by magnifier λ , are destabilizing terms and the remaining are stabilizing terms. The condition of neutral stability is obtained by setting the variation of Eq.(2.12) with respect to the buckling displacements to zero, thus yielding

$$\begin{aligned} \delta \frac{1}{2} \overline{\overline{\Pi^*}} = & \iint_A \left\{ \frac{E_s h^3}{12(1-\nu_s^2)} \left[(\overline{\overline{w_b''}} + \nu \overline{\overline{w_b'}}) \delta \overline{\overline{w_b''}} + (\overline{\overline{w_b'}} + \nu \overline{\overline{w_b''}}) \delta \overline{\overline{w_b'}} + 2(1-\nu) \overline{\overline{w_b'}} \delta \overline{\overline{w_b'}} \right] \right\} dx dy \\ & + \lambda \frac{E_s h}{1-\nu_s^2} \left\{ \iint_A \left[(u_p' + \nu \dot{v}_p) \overline{\overline{w_b'}} + \frac{1}{2}(1-\nu)(\dot{u}_p + v_p') \overline{\overline{w_b}} \right] \delta \overline{\overline{w_b'}} dx dy \right\} \\ & + \lambda \frac{E_s h}{1-\nu_s^2} \left\{ \iint_A \left[(\dot{v}_p + \nu u_p') \overline{\overline{w_b}} + \frac{1}{2}(1-\nu)(\dot{u}_p + v_p') \overline{\overline{w_b}} \right] \delta \overline{\overline{w_b}} dx dy \right\} = 0 \end{aligned} \quad (2.13)$$

By performing integration by parts on Eq.(2.13), and grouping the coefficients of similar arbitrary functions, one recovers the condition of neutral stability

$$D(\overline{\overline{w_b''''}} + \overline{\overline{w_b''}} + 2\overline{\overline{w_b''}}) - \lambda D_m \left[(u_p' + \nu \dot{v}_p) \overline{\overline{w_b''}} + (\dot{v}_p + \nu u_p') \overline{\overline{w_b}} + (1-\nu)(\dot{u}_p + v_p') \overline{\overline{w_b'}} \right] = 0 \quad (2.14)$$

where $D = E_s h^3 / 12(1-\nu_s^2)$ is the bending rigidity of the plate and $D_m = E_s h / (1-\nu_s^2)$ is its membrane rigidity. The associated boundary conditions are also recovered and take the form

$$\left\langle \left\{ -D \left[\overline{\overline{w_b''''}} + \nu \overline{\overline{w_b''}} + 2(1-\nu) \overline{\overline{w_b''}} \right] + \lambda D_m (u_p' + \nu \dot{v}_p) \left[\overline{\overline{w_b''}} + 0.5(1-\nu)(\dot{u}_p + v_p') \overline{\overline{w_b'}} \right] \right\} \delta \overline{\overline{w_b}} \right\rangle_{x=a} = 0 \quad (2.15)$$

$$\left\langle \left\{ -D \left[\bar{\ddot{w}}_b + \nu \bar{\dot{w}}_b'' + 2(1-\nu) \bar{\dot{w}}_b'' \right] + \lambda D_m \left(u_p' + \nu \dot{v}_p \right) \left[\bar{w}_b' + 0.5(1-\nu) \left(\dot{u}_p + v_p' \right) \bar{\dot{w}}_b \right] \right\} \delta \bar{w}_b \right\rangle_{y=b} = 0 \quad (2.16)$$

$$\left\{ D \left[\bar{w}_b'' + \nu \bar{\ddot{w}}_b + 2(1-\nu) \bar{\dot{w}}_b'' \right] \delta \bar{w}_b' \right\}_{x=a} = 0 \quad (2.17)$$

$$\left\{ D \left[\bar{\ddot{w}}_b + \nu \bar{w}_b'' + 2(1-\nu) \bar{\dot{w}}_b'' \right] \delta \bar{\dot{w}}_b \right\}_{y=b} = 0 \quad (2.18)$$

$$\left[\bar{\dot{w}}_b' \delta \bar{w}_b \right]_{(x=a, y=b)} = 0 \quad (2.19)$$

$$\left[\bar{\dot{w}}_b' \delta \bar{w}_b \right]_{(x=a, y=0)} = 0 \quad (2.20)$$

$$\left[\bar{\dot{w}}_b' \delta \bar{w}_b \right]_{(x=0, y=b)} = 0 \quad (2.21)$$

$$\left[\bar{\dot{w}}_b' \delta \bar{w}_b \right]_{(x=0, y=0)} = 0 \quad (2.22)$$

Equation 2.14 is the well-known plate buckling equation (e.g, Timohsenko and Gere (1961))

2.4 Two plates bonded through a soft layer

2.4.1 Statement of the problem, coordinates and sign conventions

Two plates with dimensions $a \times b$ and equal thicknesses $h_1 = h_2$ (Figure 2.3) are bonded through a thin soft layer with thickness $h_3 \ll h_1 = h_2$. Generally speaking, all variables and dimensions related to the top plate, bottom plate and soft layer are denoted by Subscripts 1, 2 or 3 respectively. The composite system is assumed to be loaded through normal pressure P_{xx} and P_{yy} or shear traction P_{xy} . It is required to find the critical loading combination $\lambda(P_{xx}, P_{yy}, P_{xy})$ at which the composite system would buckle out of its own plane.

The right-handed local coordinate system x, y and z_1 with origin O_1 , x, y and z_2 with origin O_2 and x, y and z_3 with origin O_3 are used for the top plate, bottom plate and adhesive layer respectively (Figure 2.3). Axes x and y are oriented within the plate mid-surface while the z -axis is normal to the plane.

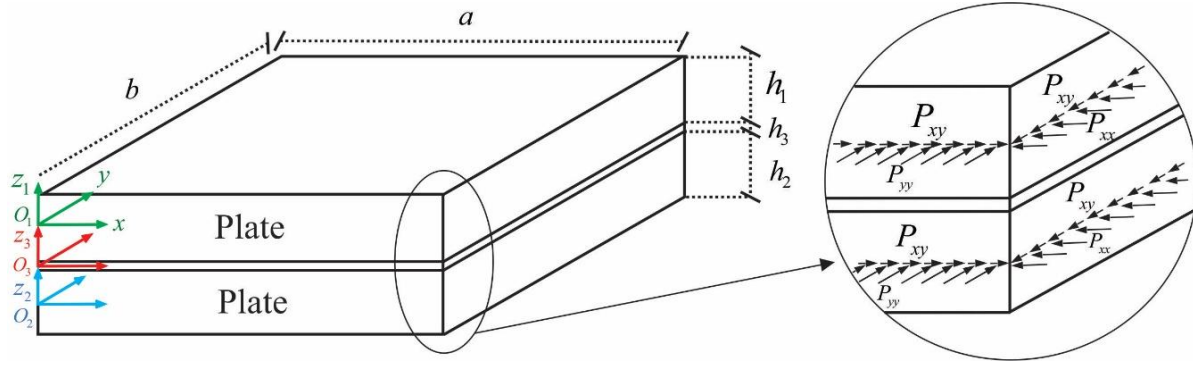


Figure 2.3- Two plates bonded through a soft layer under loading combination (P_{xx}, P_{yy}, P_{xy})

2.4.2 Assumptions

The following assumptions are made

- (1) The soft layer is assumed to be thin compared to both plates,
- (2) The soft layer is assumed to act as a weak elastic material with a small modulus of elasticity compared to that of the plate. Consequently, the in-plane normal stresses in the soft layer are considered negligible compared to the normal stresses in the plates.
- (3) Within the plates, only the in-plane normal and shear stresses are assumed to contribute to the internal strain energy while the other stress components are negligible.
- (4) The in-plane buckling displacements of a point on mid-surfaces of the plates are assumed to be equal and opposite. Consequently, the number of independent variables reduces to three independent fields u_{1b} , v_{1b} and w_b .
- (5) The transverse displacements throughout the thickness of all three layers are assumed equal, i.e., the system can be considered incompressible in the transverse direction, and
- (6) The in-plane buckling displacements of the soft layer are assumed to be a function of the in-plane displacements of top and bottom plates and have a linear distribution across the thickness.

2.4.3 Kinematics

All three plates are assumed to deform from Configuration 1 to Configuration 2 (Figure 2.4(1,2)) under the applied reference loads by undergoing in-plane pre-buckling displacements $u_p(x, y)$ and $v_p(x, y)$. The applied load is then assumed to increase to λP_{xx} , λP_{yy} and λP_{xy} so that the

composite system reaches Configuration 3 (Figure 2.4 (3)) which is onset of buckling state. The associated pre-buckling displacements are assumed to linearly increase to λu_p and λv_p , where λ is a constant to be determined. At Configuration 3, the plate has a tendency to buckle (i.e., to go from Configuration 3 to Configuration 4 without any increase in loads). As the plate buckles, its mid-surface undergoes out-of-plane buckling displacement $w_b(x, y)$ to reach Configuration 4 (Figure 2.4 (4)). At Configuration 4 the total displacement fields are obtained by summing the in-plane displacement fields at the onset of buckling to those during buckling, yielding

$$\begin{aligned} (u^*)_i &= (\lambda u_p + u_b)_i, & (v^*)_i &= (\lambda v_p + v_b)_i \\ (w^*)_i &= (w_b)_i \end{aligned} \quad i = 1, 2, 3 \quad (2.23)$$

where superscripts * denote displacements in the final buckling state (i.e., in going from Configuration 1 to Configuration 4).

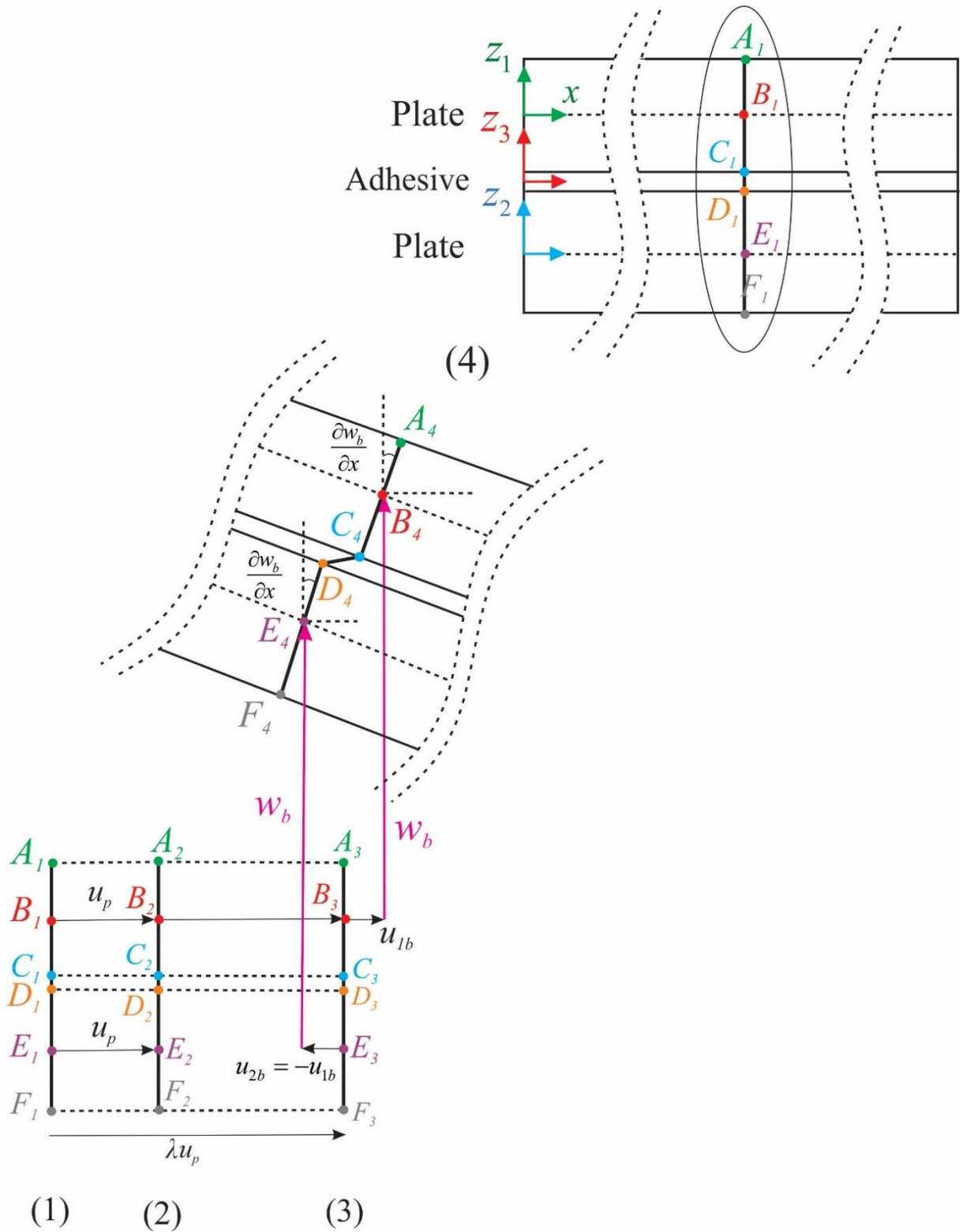


Figure 2.4- An arbitrary vertical section of an arbitrary plane of the composite system in the xz plane and associated Initial configurations in going from undeformed configuration to buckled configuration, (1) Configuration 1, (2) Configuration 2, (3) Configuration 3, (4) Configuration 4

2.4.4 In-plane displacement fields within the soft layer

Point C_1 (Figure 2.4) is located at the bottom surface of the top steel plate and the top of the soft layer. Also, point D_1 (Figure 2.4) is located at the top of the bottom plate and at the bottom of the soft layer. The corresponding in plane displacements $u_{C_1}^*$, $u_{D_1}^*$, $v_{C_1}^*$, $v_{D_1}^*$ are expressed in terms of the total displacements of mid-surface of the top and bottom plates as

$$\begin{aligned} u_{C_1}^* &= u_1^* - \frac{h_1}{2} w'_b = (\lambda u_{1p} + u_{1b}) - \frac{h_1}{2} w'_b \\ u_{D_1}^* &= u_2^* + \frac{h_1}{2} w'_b = (\lambda u_{2p} + u_{2b}) + \frac{h_1}{2} w'_b \\ v_{C_1}^* &= v_1^* - \frac{h_1}{2} \dot{w}_b = (\lambda v_{1p} + v_{1b}) - \frac{h_1}{2} \dot{w}_b \\ v_{D_1}^* &= v_2^* + \frac{h_1}{2} \dot{w}_b = (\lambda v_{2p} + v_{2b}) + \frac{h_1}{2} \dot{w}_b \end{aligned} \quad (2.24)$$

Assuming linear interpolation for the displacement fields within the soft layer thickness, one obtains the in-plane displacement within the soft layer as a function of depth z_3 as

$$\begin{aligned} u_3^*(z) &= \left(\frac{1}{2} + \frac{z_3}{h_3} \right) u_{C_1}^* + \left(\frac{1}{2} - \frac{z_3}{h_3} \right) u_{D_1}^* \\ v_3^*(z) &= \left(\frac{1}{2} + \frac{z_3}{h_3} \right) v_{C_1}^* + \left(\frac{1}{2} - \frac{z_3}{h_3} \right) v_{D_1}^* \end{aligned} \quad (2.25)$$

From Eq.(2.24), by substituting into Eq.(2.25), one obtains total in-plane displacements of soft layer as

$$\begin{aligned} u_3^*(z) &= \lambda u_p + \frac{z_3}{h_3} (2u_{1b} - h_1 w'_b) \\ v_3^*(z) &= \lambda v_p + \frac{z_3}{h_3} (2v_{1b} - h_1 \dot{w}_b) \end{aligned} \quad (2.26)$$

2.4.5 Strains in Terms of Displacements

2.4.5.1 Plates

Table 2.2 provides the buckling and pre-buckling strain–displacement relationships including linear and non-linear components.

Table 2.2- Strain-Displacement relationship within the plates

Pre-buckling strains		Buckling strains	
Linear	Nonlinear	Linear	Nonlinear
$\varepsilon_{x,p} = u'_p$	neglected	$(\varepsilon_{x,bl})_i = u'_{ib} - z_i w''_b$	$(\varepsilon_{x,bn})_i = \frac{1}{2}(w'_b)^2$
$\varepsilon_{y,p} = \dot{v}_p$	neglected	$(\varepsilon_{y,bl})_i = \dot{v}_{ib} - z_i \ddot{w}_b$	$(\varepsilon_{y,bn})_i = \frac{1}{2}(\dot{w}_b)^2$
$\gamma_{xy,p} = \dot{u}_p + v'_p$	neglected	$(\gamma_{xy,bl})_i = \dot{u}_{ib} + v'_{ib} - 2z_i \dot{w}'_b$	$(\gamma_{xy,bn})_i = w'_b \dot{w}_b$

Note: $i=1$ and $i=2$ denote top and bottom plates respectively.

2.4.5.2 Soft layer

Table 2.3 provides the strain-displacement relationship of soft layer. As a convention $()_{z_3}$ denote derivative of the displacement field with respect to z_3 .

Table 2.3- Strain-Displacement relationship within soft layer

Pre-buckling strains		Buckling strains	
Linear	Nonlinear	Linear	Nonlinear
$\gamma_{xy,p} = \dot{u}_p + v'_p$	neglected	$(\gamma_{xy,bl})_3 = \dot{u}_{3b} + v'_{3b} - 2z_3 \dot{w}'_b$	$(\gamma_{xy,bn})_3 = w'_b \dot{w}_b$
$\gamma_{xz,p}$	neglected	$(\gamma_{xz,bl})_3 = u_{z_3,3b} + w'_b - 2z_3 w'_{z_3,b}$	$(\gamma_{xz,bn})_3 = w'_b w_{z_3,b}$
$\gamma_{yz,p}$	neglected	$(\gamma_{yz,bl})_3 = v_{z_3,3b} + \dot{w}_b - 2z_3 \dot{w}_{z_3,b}$	$(\gamma_{yz,bn})_3 = \dot{w}_b w_{z_3,b}$

2.4.6 Stress-Strain Relationships

2.4.6.1 Plates

The generalized Hooke's law gives the relationship between stress and strains associated with plates as

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix}_i = \frac{E_s}{1-\nu_s^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu_s}{2} \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix}_i \quad i=1,2 \quad (2.27)$$

where E_s is Young's Modulus and ν_s is Poisson's Ratio of plate.

2.4.6.2 Soft Layer

The generalized Hooke's law gives the relationships between stress and strains associated with soft layer as

$$\begin{Bmatrix} \tau_{xy} \\ \tau_{xz} \\ \tau_{yz} \end{Bmatrix}_3 = G_a \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} \gamma_{xy} \\ \gamma_{xz} \\ \gamma_{yz} \end{Bmatrix}_3 \quad (2.28)$$

where G_a is shear modulus of soft.

2.4.7 Strain decomposition

2.4.7.1 Plates

From Eq.(2.23), by substituting into the displacement fields of Table 2.2, the total buckling strains $(\varepsilon_x^*)_i$, $(\varepsilon_y^*)_i$, and $(\gamma_{xy}^*)_i$ can be decomposed as

$$\begin{aligned} (\varepsilon_x^*)_i &= (\lambda \varepsilon_{x,p} + \varepsilon_{x,bl} + \varepsilon_{x,bn})_i \\ (\varepsilon_y^*)_i &= (\lambda \varepsilon_{y,p} + \varepsilon_{y,bl} + \varepsilon_{y,bn})_i \\ (\gamma_{xy}^*)_i &= (\lambda \gamma_{xy,p} + \gamma_{xy,bl} + \gamma_{xy,bn})_i \end{aligned} \quad i=1,2 \quad (2.29)$$

2.4.7.2 Soft Layer

From Eq.(2.23), by substituting into the displacement fields of Table 2.3, the total buckling strains $(\gamma_{xy}^*)_3$, $(\gamma_{xz}^*)_3$, and $(\gamma_{yz}^*)_3$ can be decomposed as

$$\begin{aligned} (\gamma_{xy}^*)_3 &= (\lambda \gamma_{xy,p} + \gamma_{xy,bl} + \gamma_{xy,bn})_3 \\ (\gamma_{xz}^*)_3 &= (\lambda \gamma_{xz,p} + \gamma_{xz,bl} + \gamma_{xz,bn})_3 \\ (\gamma_{yz}^*)_3 &= (\lambda \gamma_{yz,p} + \gamma_{yz,bl} + \gamma_{yz,bn})_3 \end{aligned} \quad (2.30)$$

2.4.8 Energy Formulation

The total potential energy of the composite system Π_t^* in going from Configuration 1 to Configuration 4 is given by

$$\Pi_i^* = \sum_{i=1}^3 \Pi_i^* \quad (2.31)$$

where $i = 1$ to $i = 3$ denote contributions of the total potential energy associated with top plate, bottom plate and soft layer respectively.

2.4.8.1 Contribution of Plates

The total potential energy of plates is sum of internal strain energies stored within the plates U_i^* and the total potential energies lost by applied loads V_i^* , i.e.,

$$\begin{aligned} \sum_{i=1}^2 \Pi_i^* &= \sum_{i=1}^2 (U_i^* + V_i^*) \\ &= \sum_{i=1}^2 \left[\frac{1}{2} \iiint_V [(\sigma_x^*)_i (\varepsilon_x^*)_i + (\sigma_y^*)_i (\varepsilon_y^*)_i + (\tau_{xy}^*)_i (\gamma_{xy}^*)_i] dx dy dz_i \right] \\ &\quad - \sum_{i=1}^2 \left\{ \int_0^a \int_0^b h_i [P_x u'_p + P_y \dot{v}_p + P_{xy} (\dot{u}_p + v'_p)] dx dy \right\} \end{aligned} \quad (2.32)$$

From Eqs.(2.27) and (2.29), by substituting into Eq.(2.32), one obtains

$$\begin{aligned} \sum_{i=1}^2 \Pi_i^* &= \sum_{i=1}^2 \frac{1}{2} \left[\iiint_V \frac{E_s}{1-\nu_s^2} (\lambda \varepsilon_{x,p} + \varepsilon_{x,b})_i^2 dx dy dz_i + \iiint_V \frac{E_s}{1-\nu_s^2} (\lambda \varepsilon_{y,p} + \varepsilon_{y,b})_i^2 dx dy dz_i \right] \\ &\quad \sum_{i=1}^2 \frac{1}{2} \left[\iiint_V \frac{2\nu_s E_s}{1-\nu_s^2} (\lambda \varepsilon_{x,p} + \varepsilon_{x,b})_i (\lambda \varepsilon_{y,p} + \varepsilon_{y,b})_i dx dy dz_i + \iiint_V \frac{E_s}{2(1+\nu_s)} (\lambda \gamma_{xy,p} + \gamma_{xy,b})_i^2 dx dy dz_i \right] \\ &\quad - \sum_{i=1}^2 \left\{ \int_0^a \int_0^b h_i [P_x u'_p + P_y \dot{v}_p + P_{xy} (\dot{u}_p + v'_p)] dx dy \right\} \end{aligned} \quad (2.33)$$

2.4.8.2 Contribution of Soft Layer

The total potential energy of soft layer is sum of internal strain energies stored within the soft layer U_i^* and the load potential energies lost by external applied loads V_i^* , i.e.,

$$\begin{aligned}
\Pi_{i=3}^* &= U_{i=3}^* + V_{i=3}^* \\
&= \frac{1}{2} \iiint_V \left[(\tau_{xy}^*)_3 (\gamma_{xy}^*)_3 + (\tau_{xz}^*)_3 (\gamma_{xz}^*)_3 + (\tau_{yz}^*)_3 (\gamma_{yz}^*)_3 \right] dx dy dz_3 \\
&\quad - \int_0^a \int_0^b h_3 \left[P_x u'_p + P_y \dot{v}_p + P_{xy} (\dot{u}_p + v'_p) \right] dx dy
\end{aligned} \tag{2.34}$$

From Eqs (2.28) and (2.30), by substituting into Eq. (2.34), one obtains

$$\begin{aligned}
\Pi_{i=3}^* &= \iiint_V \frac{1}{2} G_a \left[(\lambda \gamma_{xy,p} + \gamma_{xy,b})_{i=3}^2 + (\lambda \gamma_{xz,p} + \gamma_{xz,b})_{i=3}^2 + (\lambda \gamma_{yz,p} + \gamma_{yz,b})_{i=3}^2 \right] dx dy dz_3 \\
&\quad - \int_0^a \int_0^b h_3 \left[P_x u'_p + P_y \dot{v}_p + P_{xy} (\dot{u}_p + v'_p) \right] dx dy
\end{aligned} \tag{2.35}$$

2.4.8.3 Total Potential Energy of the composite system

From Eqs. (2.33) and (2.35), by substituting into Eq. (2.31), the total potential energy of the composite system take the form

$$\begin{aligned}
\Pi_t^* &= \sum_{i=1}^2 \frac{1}{2} \left[\iiint_V \frac{E_s}{1-\nu_s^2} (\lambda \varepsilon_{x,p} + \varepsilon_{x,b})_i^2 dx dy dz_i + \iiint_V \frac{E_s}{1-\nu_s^2} (\lambda \varepsilon_{y,p} + \varepsilon_{y,b})_i^2 dx dy dz_i \right] \\
&\quad + \sum_{i=1}^2 \frac{1}{2} \iiint_V \frac{2\nu_s E_s}{1-\nu_s^2} (\lambda \varepsilon_{x,p} + \varepsilon_{x,b})_i (\lambda \varepsilon_{y,p} + \varepsilon_{y,b})_i dx dy dz_i + \sum_{i=1}^2 \frac{1}{2} \iiint_V \frac{E_s}{2(1+\nu_s)} (\lambda \gamma_{xy,p} + \gamma_{xy,b})_i^2 dx dy dz_i \\
&\quad + \iiint_V \frac{1}{2} G_a \left[(\lambda \gamma_{xy,p} + \gamma_{xy,b})_3^2 + (\lambda \gamma_{xz,p} + \gamma_{xz,b})_3^2 + (\lambda \gamma_{yz,p} + \gamma_{yz,b})_3^2 \right] dx dy dz_3 \\
&\quad - \sum_{i=1}^3 \left\{ \int_0^a \int_0^b h_i \left[P_x u'_p + P_y \dot{v}_p + P_{xy} (\dot{u}_p + v'_p) \right] dx dy \right\}
\end{aligned} \tag{2.36}$$

2.4.9 Condition of Neutral Stability of the composite system

From the strain-displacement relations in Eq. (2.36), Table 2.2, and Table 2.3, by substituting into Eq. (2.36), taking the second variation of the total potential energy expression with respect to the buckling displacement fields, and performing the integral with respect to z along the thickness

(Appendix A.1), the second variation of the total potential energy $\overline{\overline{\Pi_t^*}}$ is found to take the form

$$\begin{aligned} \overline{\Pi}^* = \iint \{ & \bar{A}_1 \bar{u}_{1b}^2 + \bar{A}_2 \bar{u}_{1b}'^2 + \bar{A}_3 \bar{u}_{1b}^2 + \bar{A}_1 \bar{v}_{1b}^2 + \bar{A}_2 \bar{v}_{1b}'^2 + \bar{A}_3 \bar{v}_{1b}'^2 + \bar{A}_4 \bar{w}_b'^2 + \bar{A}_4 \bar{w}_b^2 \\ & + \bar{A}_5 \bar{w}_b'^2 + \bar{A}_5 \bar{w}_b^2 + \bar{A}_6 \bar{w}_b'^2 + \bar{A}_7 \bar{w}_b'' \bar{w}_b + \bar{A}_8 \bar{u}_{1b} \bar{w}_b' + \bar{A}_8 \bar{v}_{1b} \bar{w}_b + \bar{A}_9 \bar{u}_{1b} \bar{v}_{1b} + \bar{A}_{10} \bar{u}_{1b} \bar{v}_{1b}' \\ & + \bar{A}_{11} \bar{u}_{1b} \bar{w}_b' + \bar{A}_{11} \bar{v}_{1b} \bar{w}_b' + \lambda (\bar{A}_{12} \bar{w}_b'^2 + \bar{A}_{13} \bar{w}_b^2 + \bar{A}_{14} \bar{w}_b \bar{w}_b') \} dx dy \end{aligned} \quad (2.37)$$

where the stabilizing coefficients ($\bar{A}_1$ to $\bar{A}_{11}$) and the destabilizing coefficients ($\bar{A}_{12}$ to $\bar{A}_{14}$) are summarized in Table 2.4. The neutral stability state (Configuration 3 in Figure 2.4) is obtained by setting to zero the variation of second variation of total potential energy in Eq. (2.37).

Table 2.4- Stabilizing and destabilizing Coefficients of Eq.(2.37)

$\bar{A}_1$	$4G_a/h_3$	$\bar{A}_8$	$4G_a(1-h_1/h_3)$
$\bar{A}_2$	$2E_s h_1/(1-\nu_s^2)$	$\bar{A}_9$	$4\nu E_s h_1/(1-\nu_s^2)$
$\bar{A}_3$	$G_a h_3/3 + E_s h_1/(1+\nu_s)$	$\bar{A}_{10}$	$2G_a h_3/3 + 2E_s h_1/(1+\nu_s)$
$\bar{A}_4$	$G_a(h_1^2/h_3 + h_3 - 2h_1)$	$\bar{A}_{11}$	$-G_a[(2h_1 h_3 + 2h_3^2)/3]$
$\bar{A}_5$	$E_s h_1^3/6(1-\nu_s^2)$	$\bar{A}_{12}$	$[2E_s h_1/(1-\nu_s^2)](u_p' + \nu \dot{v}_p)$
$\bar{A}_6$	$E_s h_1^3/3(1+\nu_s) + G_a[(h_3^3 + h_1^2 h_3 + 2h_3^2 h_1)/3]$	$\bar{A}_{13}$	$[2E_s h_1/(1-\nu^2)](\dot{v}_p + \nu u_p')$
$\bar{A}_7$	$\nu E_s h_1^3/3(1-\nu_s^2)$	$\bar{A}_{14}$	$2(\dot{u}_p + \nu_p') [G_a h_3 + E_s h_1/(1+\nu)]$

2.5 Steel plate reinforced with two GFRP plates

2.5.1 Statement of the problem, coordinates and sign conventions

A steel plate of dimensions $a \times b$ and thickness h_3 (Figure 2.5) is reinforced with two GFRP plates with thicknesses $h_1 = h_5$ through adhesive layers with thicknesses $h_2 = h_4 \ll h_1, h_3, h_5$. As a matter of convention, all variables and dimensions related to the top GFRP plate, top adhesive layer, steel plate, bottom adhesive layer and bottom GFRP plate are denoted by Subscripts 1 to 5, respectively. The right-handed local coordinate systems x, y and z_1 with origin O_1 (Figure 2.5), x, y and z_2 with origin O_2 and x, y and z_3 with origin O_3 are defined for the top and bottom GFRP plates and the steel plate respectively. The composite system is assumed to be loaded through normal pressure P_{xx} and P_{yy} and shear traction P_{xy} (Figure 2.5). It is required to find the

critical loading combination $\lambda(P_{xx}, P_{yy}, P_{xy})$ at which the composite system would buckle out of its own plane.

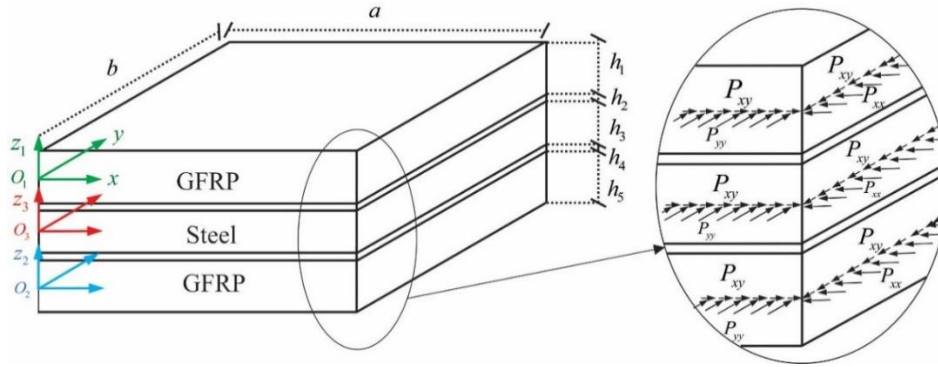


Figure 2.5- Steel plate reinforced with two GFRP plates subjected to normal and shear pressures

2.5.2 Assumptions

The following assumptions are made

- (1) The adhesive layers are assumed to be thin compared to the steel and GFRP plates.
- (2) Within the steel and GFRP plates, only the in-plane normal and shear stresses are assumed to contribute in the internal strain energy while the other stress components are negligible.
- (3) Any point within the steel plate mid-surface is assumed undergo a transverse buckling displacement w_b . The mid-surfaces of the GFRP plates are assumed to undergo the same transverse buckling displacement w_b (i.e., the sandwich system is assumed incompressible in the transverse direction) while undergoing in-plane displacement u_{ib}, v_{ib} ($i=1$ for the top plate and $i=5$ for the bottom plate).
- (4) The in-plane buckling displacements u_{1b}, v_{1b} at the mid-surface of the top GFRP plate, are assumed to be equal and opposite to those of the bottom GFRP plate u_{5b}, v_{5b} .
- (5) Perfect bonding is assumed between all interfaces, and
- (6) The in-plane buckling displacements within the adhesive layers are assumed to have a linear distribution across the layer thickness.

2.5.3 Kinematics

As shown in Figure 2.6 (1) and (2), the composite system deforms from Configuration 1 to Configuration 2 under the applied reference loads by undergoing in-plane pre-buckling displacements $u_p(x, y)$ and $v_p(x, y)$. The applied load (Figure 2.6(3)), is then assumed to increase to λP_{xx} , λP_{yy} and λP_{xy} so that the composite system reaches Configuration 3 at the onset of buckling state. The associated pre-buckling displacements are assumed to linearly increase to λu_p and λv_p , where λ is a constant to be determined. At Configuration 3, the plate has a tendency to buckle (i.e., going from Configuration 3 to Configuration 4 without any increase in loads). As the plate buckles, its mid-surface undergoes out-of-plane buckling displacement $w_b(x, y)$ to reach Configuration 4 (Figure 2.6 (4)). At Configuration 4, the in-plane mid-surface displacements for the top GFRP, steel, and bottom GFRP plates are obtained by summing the in-plane displacement fields at the onset of buckling to those during buckling, yielding

$$\begin{aligned} (u^*)_1 &= (\lambda u_p + u_b)_1, & (v^*)_1 &= (\lambda v_p + v_b)_1 \\ (u^*)_3 &= (\lambda u_p)_3, & (v^*)_3 &= (\lambda v_p)_3 \\ (u^*)_5 &= (\lambda u_p + u_b)_5, & (v^*)_5 &= (\lambda v_p + v_b)_5 \end{aligned} \quad (2.38)$$

where superscripts * denote displacements in the final buckling state (i.e., in going from Configuration 1 to Configuration 4).

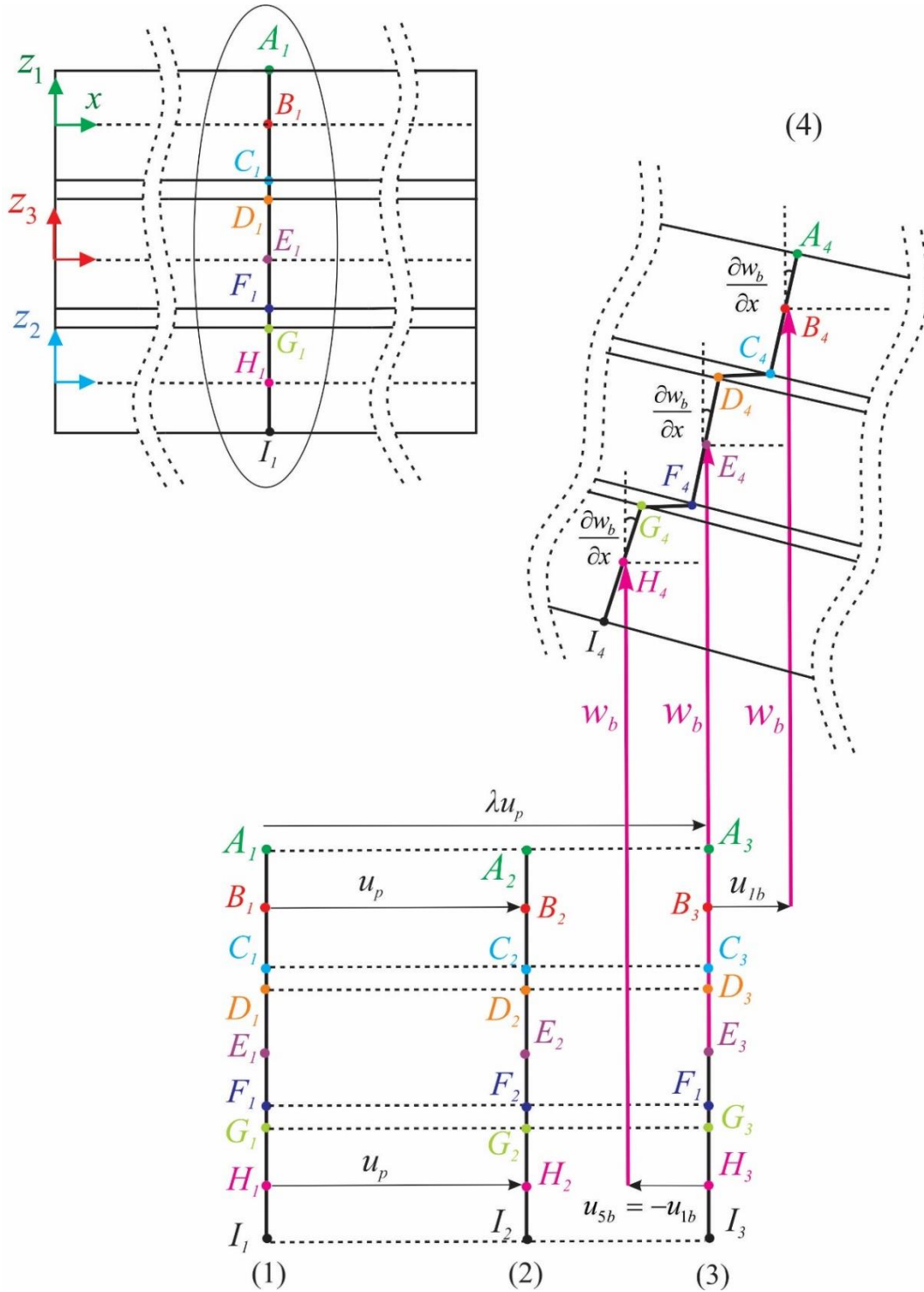


Figure 2.6 An arbitrary vertical section of an arbitrary plane of the composite system in the xz plane and associated Initial configurations in going from undeformed configuration to buckled configuration, (1) Configuration 1, (2) Configuration 2, (3) Configuration 3, (4) Configuration 4

2.5.4 In-plane displacements within adhesive layers

2.5.4.1 Top adhesive layer

As shown in Figure 2.6, point C_1 is located at the bottom surface of the top GFRP and the top of the top adhesive layer. Also, point D_1 is at the top of the steel plate and at the bottom of the top adhesive layer. The total in plane displacements (in going from Configuration 1 to Configuration 4) of the these two points are $u_{C_1}^*$, $u_{D_1}^*$, $v_{C_1}^*$ and $v_{D_1}^*$. These displacements are expressed in terms of the total displacements of the middle surface of the top GFRP and plates as

$$\begin{aligned} u_{C_1}^* &= u_1^* - \frac{h_1}{2} w'_b = (\lambda u_p + u_{1b}) - \frac{h_1}{2} w'_b \\ u_{D_1}^* &= u_3^* + \frac{h_1}{2} w'_b = (\lambda u_p) + \frac{h_3}{2} w'_b \\ v_{C_1}^* &= v_1^* - \frac{h_g}{2} \dot{w}_b = (\lambda u_p + u_{1b}) - \frac{h_1}{2} \dot{w}_b \\ v_{D_1}^* &= v_3^* + \frac{h_1}{2} \dot{w}_b = (\lambda u_p) + \frac{h_3}{2} \dot{w}_b \end{aligned} \quad (2.39)$$

Assuming linear interpolation for the displacement fields within the top adhesive layer thickness, one obtains the in-plane displacement within the adhesive layer as a function of depth z as

$$\begin{aligned} u_2^*(z_3) &= \lambda u_p + \left[\left(\frac{-h_3}{2h_2} u_{1,b} \right) + \left(\frac{h_1 h_3}{4h_2} + \frac{h_3^2}{4h_2} + \frac{h_3}{2} \right) w'_b + z_3 \left(\frac{2u_{1,b} - h_1 w'_b - h_3 w'_b}{2h_2} \right) \right] \\ v_2^*(z_3) &= \lambda v_p + \left[\left(\frac{-h_3}{2h_2} v_{1,b} \right) + \left(\frac{h_1 h_3}{4h_2} + \frac{h_3^2}{4h_2} + \frac{h_3}{2} \right) \dot{w}_b + z_3 \left(\frac{2v_{1,b} - h_1 \dot{w}_b - h_3 \dot{w}_b}{2h_2} \right) \right] \end{aligned} \quad (2.40)$$

2.5.4.2 Bottom adhesive layer

As shown in Figure 2.6, point F_1 is located at the bottom surface of the steel plate and the top of the bottom adhesive layer. Also, point G_1 is at the top of the bottom GFRP and at the bottom of the bottom adhesive layer. The total in plane displacements of the these two points are $u_{F_1}^*$, $u_{G_1}^*$, $v_{F_1}^*$ and $v_{G_1}^*$. These displacements are expressed in terms of the total displacements of the middle surface of the bottom GFRP and plates as

$$\begin{aligned}
u_{F_1}^* &= u_3^* + \frac{h_3}{2} w'_b = (\lambda u_p) + \frac{h_3}{2} w'_b \\
u_{G_1}^* &= u_5^* - \frac{h_5}{2} w'_b = (\lambda u_p + u_{5b}) - \frac{h_5}{2} w'_b \\
v_{F_1}^* &= v_3^* + \frac{h_3}{2} \dot{w}_b = (\lambda v_p) + \frac{h_3}{2} \dot{w}_b \\
v_{G_1}^* &= v_5^* - \frac{h_5}{2} \dot{w}_b = (\lambda v_p + v_{5b}) - \frac{h_5}{2} \dot{w}_b
\end{aligned} \tag{2.41}$$

Assuming linear interpolation for the displacement fields within the top adhesive layer thickness, one obtains the in-plane displacement within the adhesive layer as a function of depth z as

$$\begin{aligned}
u_4^*(z_3) &= \lambda u_p + \left[\left(\frac{-h_3}{2h_4} u_{5,b} \right) + \left(\frac{h_5 h_3}{4h_4} + \frac{h_3^2}{4h_4} + \frac{h_3}{2} \right) w'_b + z_3 \left(\frac{-2u_{5,b} + h_5 w'_b + h_3 w'_b}{2h_4} \right) \right] \\
v_4^*(z_3) &= \lambda v_p + \left[\left(\frac{-h_5}{2h_4} v_{5,b} \right) + \left(\frac{h_5 h_3}{4h_4} + \frac{h_3^2}{4h_4} + \frac{h_3}{2} \right) \dot{w}_b + z_3 \left(\frac{-2v_{5,b} + h_5 \dot{w}_b + h_3 \dot{w}_b}{2h_4} \right) \right]
\end{aligned} \tag{2.42}$$

2.5.5 Strains in Terms of Displacements

2.5.5.1 GFRP Plates

Table 2.5 provides the buckling and pre-buckling strain-displacement relationships including linear and non-linear components.

Table 2.5- Strain-Displacement relations within GFRP plates

Pre-buckling strains		Buckling strains	
Linear	Nonlinear	Linear	Nonlinear
$\varepsilon_{x,p} = u'_p$	neglected	$(\varepsilon_{x,bl})_i = u'_{ib} - z_{1,2} w''_b$	$(\varepsilon_{x,bn})_i = \frac{1}{2} (w'_b)^2$
$\varepsilon_{y,p} = \dot{v}_p$	neglected	$(\varepsilon_{y,bl})_i = \dot{v}_{ib} - z_{1,2} \ddot{w}_b$	$(\varepsilon_{y,bn})_i = \frac{1}{2} (\dot{w}_b)^2$
$\gamma_{xy,p} = \dot{u}_p + v'_p$	neglected	$(\gamma_{xy,bl})_i = \dot{u}_{ib} + v'_{ib} - 2z_{1,2} \dot{w}'_b$	$(\gamma_{xy,bn})_i = w'_b \dot{w}_b$

Note: $i = 1$ and $i = 5$ denote top and bottom GFRP plates respectively.

2.5.5.2 Steel plate

Table 2.6 provides the buckling and pre-buckling strain–displacement relationships including linear and non-linear components.

Table 2.6 -Strain-Displacement relations within steel plate

Pre-buckling strains		Buckling strains	
Linear	Nonlinear	Linear	Nonlinear
$\varepsilon_{x,p} = u'_p$	neglected	$(\varepsilon_{x,bl})_3 = -z_3 w''_b$	$(\varepsilon_{x,bn})_3 = \frac{1}{2}(w'_b)^2$
$\varepsilon_{y,p} = \dot{v}_p$	neglected	$(\varepsilon_{y,bl})_3 = -z_3 \ddot{w}_b$	$(\varepsilon_{y,bn})_3 = \frac{1}{2}(\dot{w}_b)^2$
$\gamma_{xy,p} = \dot{u}_p + v'_p$	neglected	$(\gamma_{xy,bl})_3 = -2z_3 \dot{w}'_b$	$(\gamma_{xy,bn})_3 = w'_b \dot{w}_b$

2.5.5.3 Adhesive layer

Table 2.7 provides the strain-displacement relationship of adhesive layers. As a convention $()_{z_3}$ denote derivative of the displacement field with respect to z_3 .

Table 2.7 -Strain-displacement relations associated with adhesive layer

Pre-buckling strains		Buckling strains	
Linear	Nonlinear	Linear	Nonlinear
$\gamma_{xy,p} = \dot{u}_p + v'_p$	neglected	$(\gamma_{xy,bl})_i = \dot{u}_{ib} + v'_{ib} - 2z_3 \dot{w}'_b$	$(\gamma_{xy,bn})_i = w'_b \dot{w}_b$
$\gamma_{xz,p}$	neglected	$(\gamma_{xz,bl})_i = u_{z_3,ib} + w'_b - 2z_3 w'_{z_3,b}$	$(\gamma_{xz,bn})_i = w'_b w_{z_3,b}$
$\gamma_{yz,p}$	neglected	$(\gamma_{yz,bl})_i = v_{z_3,ib} + \dot{w}_b - 2z_3 \dot{w}_{z_3,b}$	$(\gamma_{yz,bn})_i = \dot{w}_b w_{z_3,b}$

Note: $i=2$ and $i=4$ denote top and bottom adhesive layers respectively.

2.5.6 Stress-Strain Relationships

2.5.6.1 GFRPs and Steel Plate

The generalized Hooke's law gives the relationship between stresses and strains associated with steel and GFRP plates as

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix}_i = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix}_i \quad i=1,3,5 \quad (2.43)$$

where E_g and ν_g will be Young's Modulus and Poisson's ratio of GFRP and E_s and ν_s will be those related to steel.

2.5.6.2 Adhesive Layers

The generalized Hooke's law gives the relationships between stress and strains associated with adhesive layer as

$$\begin{Bmatrix} \tau_{xy} \\ \tau_{xz} \\ \tau_{yz} \end{Bmatrix}_i = G_a \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} \gamma_{xy} \\ \gamma_{xz} \\ \gamma_{yz} \end{Bmatrix}_i \quad i=2,4 \quad (2.44)$$

where G_a is shear modulus of adhesive.

2.5.7 Strain decomposition

2.5.7.1 GFRPs and Plates

From Eq. (2.38), by substituting into the displacement fields of Table 2.5 and

Table 2.6 the total buckling strains $(\varepsilon_x^*)_i$, $(\varepsilon_y^*)_i$, and $(\gamma_{xy}^*)_i$ can be decomposed as

$$\begin{aligned} (\varepsilon_x^*)_i &= (\lambda \varepsilon_{x,p} + \varepsilon_{x,bl} + \varepsilon_{x,bn})_i \\ (\varepsilon_y^*)_i &= (\lambda \varepsilon_{y,p} + \varepsilon_{y,bl} + \varepsilon_{y,bn})_i \\ (\gamma_{xy}^*)_i &= (\lambda \gamma_{xy,p} + \gamma_{xy,bl} + \gamma_{xy,bn})_i \end{aligned} \quad i=1,3,5 \quad (2.45)$$

2.5.7.2 Adhesive Layer

The total buckling strains $(\gamma_{xy}^*)_i$, $(\gamma_{xz}^*)_i$, and $(\gamma_{yz}^*)_i$ in the adhesive layers can be decomposed as

$$\begin{aligned}
(\gamma_{xy}^*)_i &= (\lambda\gamma_{xy,p} + \gamma_{xy,bl} + \gamma_{xy,bn})_i \\
(\gamma_{xz}^*)_i &= (\lambda\gamma_{xz,p} + \gamma_{xz,bl} + \gamma_{xz,bn})_i \\
(\gamma_{yz}^*)_i &= (\lambda\gamma_{yz,p} + \gamma_{yz,bl} + \gamma_{yz,bn})_i
\end{aligned} \quad i = 2, 4 \quad (2.46)$$

2.5.8 Energy Formulation

The total potential energy of the composite system Π_t^* in going from Configuration 1 to Configuration 4 is given by

$$\Pi_t^* = \sum_{i=1}^5 \Pi_i^* \quad (2.47)$$

where $i = 1$ to $i = 5$ denote contributions of the total potential energy associated with all five layers.

2.5.8.1 Contribution of GFRP Plates

The total potential energy of GFRP plates is sum of the total potential energy of top plate (Π_1^*) and bottom GFRP plate (Π_5^*), i.e.,

$$\begin{aligned}
\Pi_1^* + \Pi_5^* &= (U_i^* + V_i^*)_{i=1} + (U_i^* + V_i^*)_{i=5} \\
&= \left\{ \frac{1}{2} \iiint_v [(\sigma_x^*)_i (\varepsilon_x^*)_i + (\sigma_y^*)_i (\varepsilon_y^*)_i + (\tau_{xy}^*)_i (\gamma_{xy}^*)_i] dx dy dz_1 \right. \\
&\quad \left. - \left[\int_0^a \int_0^b h_i [P_x u'_p + P_y \dot{v}_p + P_{xy} (\dot{u}_p + v'_p)] dx dy \right] \right\}_{i=1} \\
&\quad + \left\{ \frac{1}{2} \iiint_v [(\sigma_x^*)_i (\varepsilon_x^*)_i + (\sigma_y^*)_i (\varepsilon_y^*)_i + (\tau_{xy}^*)_i (\gamma_{xy}^*)_i] dx dy dz_2 \right. \\
&\quad \left. - \left[\int_0^a \int_0^b h_i [P_x u'_p + P_y \dot{v}_p + P_{xy} (\dot{u}_p + v'_p)] dx dy \right] \right\}_{i=5}
\end{aligned} \quad (2.48)$$

From Eqs. (2.43) and (2.45), by substituting into Eq.(2.48), one obtains

$$\begin{aligned}
\Pi_1^* + \Pi_5^* = & \left\{ \frac{1}{2} \left[\iiint_V \frac{E_g}{1-\nu_g^2} (\lambda \varepsilon_{x,p} + \varepsilon_{x,b})_i^2 dx dy dz_1 + \iiint_V \frac{E_g}{1-\nu_g^2} (\lambda \varepsilon_{y,p} + \varepsilon_{y,b})_i^2 dx dy dz_1 \right. \right. \\
& + \iiint_V \frac{2\nu_g E_g}{1-\nu_g^2} (\lambda \varepsilon_{x,p} + \varepsilon_{x,b})_i (\lambda \varepsilon_{y,p} + \varepsilon_{y,b})_i dx dy dz_1 \\
& + \left. \iiint_V \frac{E_g}{2(1+\nu_g)} (\lambda \gamma_{xy,p} + \gamma_{xy,b})_i^2 dx dy dz_1 \right] \\
& - \left. \left[\int_0^a \int_0^b h_i \left[P_x u'_p + P_y \dot{v}_p + P_{xy} (\dot{u}_p + v'_p) \right] dx dy \right] \right\}_{i=1} \\
& + \left\{ \frac{1}{2} \left[\iiint_V \frac{E_g}{1-\nu_g^2} (\lambda \varepsilon_{x,p} + \varepsilon_{x,b})_i^2 dx dy dz_2 + \iiint_V \frac{E_g}{1-\nu_g^2} (\lambda \varepsilon_{y,p} + \varepsilon_{y,b})_i^2 dx dy dz_2 \right. \right. \\
& + \iiint_V \frac{2\nu_g E_g}{1-\nu_g^2} (\lambda \varepsilon_{x,p} + \varepsilon_{x,b})_i (\lambda \varepsilon_{y,p} + \varepsilon_{y,b})_i dx dy dz_2 \\
& + \left. \iiint_V \frac{E_g}{2(1+\nu_g)} (\lambda \gamma_{xy,p} + \gamma_{xy,b})_i^2 dx dy dz_2 \right] \\
& - \left. \left[\int_0^a \int_0^b h_i \left[P_x u'_p + P_y \dot{v}_p + P_{xy} (\dot{u}_p + v'_p) \right] dx dy \right] \right\}_{i=5}
\end{aligned} \tag{2.49}$$

2.5.8.2 Contribution of steel plate

The total potential energy of steel plate (Π_3^*) is sum of internal strain energy stored within steel plate (U_3^*) and potential energy lost by the reference loads (V_3^*), i.e.,

$$\begin{aligned}
\Pi_3^* = U_3^* + V_3^* = & \left\{ \frac{1}{2} \iiint_V \left[(\sigma_x^*)_i (\varepsilon_x^*)_i + (\sigma_y^*)_i (\varepsilon_y^*)_i + (\tau_{xy}^*)_i (\gamma_{xy}^*)_i \right] dx dy dz_i \right. \\
& - \left. \left[\int_0^a \int_0^b h_i \left[P_x u'_p + P_y \dot{v}_p + P_{xy} (\dot{u}_p + v'_p) \right] dx dy \right] \right\}_{i=3}
\end{aligned} \tag{2.50}$$

From Eqs. (2.43) and (2.45), by substituting into Eq.(2.50), one obtains

$$\begin{aligned}
\Pi_3^* = & \left\{ \frac{1}{2} \left[\iiint_V \frac{E_s}{1-\nu_s^2} (\lambda \varepsilon_{x,p} + \varepsilon_{x,b})_i^2 dx dy dz_i + \iiint_V \frac{E_s}{1-\nu_s^2} (\lambda \varepsilon_{y,p} + \varepsilon_{y,b})_i^2 dx dy dz_i \right. \right. \\
& + \iiint_V \frac{2\nu_s E_s}{1-\nu_s^2} (\lambda \varepsilon_{x,p} + \varepsilon_{x,b})_i (\lambda \varepsilon_{y,p} + \varepsilon_{y,b})_i dx dy dz_i + \iiint_V \frac{E_s}{2(1+\nu_s)} (\lambda \gamma_{xy,p} + \gamma_{xy,b})_i^2 dx dy dz_i \left. \right] \\
& \left. - \int_0^a \int_0^b h_i \left[P_x u'_p + P_y \dot{v}_p + P_{xy} (\dot{u}_p + v'_p) \right] dx dy \right\}_{i=3}
\end{aligned} \tag{2.51}$$

2.5.8.3 Contribution of adhesive layers

The total potential energy of adhesive layers is sum of the total potential energy of top adhesive layer (Π_2^*) and bottom adhesive layer (Π_4^*), i.e.,

$$\begin{aligned}
\sum_{i=2,4} \Pi_i^* &= \sum_{i=2,4} (U_i^* + V_i^*) \\
&= \sum_{i=2,4} \left\{ \frac{1}{2} \iiint_V \left[(\tau_{xy}^*)_i (\gamma_{xy}^*)_i + (\tau_{xz}^*)_i (\gamma_{xz}^*)_i + (\tau_{yz}^*)_i (\gamma_{yz}^*)_i \right] dx dy dz_3 \right. \\
&\quad \left. - \int_0^a \int_0^b h_i \left[P_x u'_p + P_y \dot{v}_p + P_{xy} (\dot{u}_p + v'_p) \right] dx dy \right\}
\end{aligned} \tag{2.52}$$

From Eq. (2.46), by substituting into Eq. (2.52), one obtains

$$\begin{aligned}
\sum_{i=2,4} \Pi_i^* &= \sum_{i=2,4} \left\{ \iiint_V \frac{1}{2} G_a \left[(\lambda \gamma_{xy,p} + \gamma_{xy,b})_i^2 + (\lambda \gamma_{xz,p} + \gamma_{xz,b})_i^2 + (\lambda \gamma_{yz,p} + \gamma_{yz,b})_i^2 \right] dx dy dz_3 \right. \\
&\quad \left. - \int_0^a \int_0^b h_i \left[P_x u'_p + P_y \dot{v}_p + P_{xy} (\dot{u}_p + v'_p) \right] dx dy \right\}
\end{aligned} \tag{2.53}$$

2.5.8.4 Total Potential Energy of the composite system

The total potential energy of the composite system, Π_t^* , is sum of the total potential energy of all layers obtained in Eqs.(2.48), (2.50) and (2.52), i.e.,

$$\Pi_t^* = \sum_{i=1}^5 \Pi_i^* \tag{2.54}$$

2.5.9 Condition of Neutral Stability of the composite system

By taking second variation of the total potential energy Π_t^* with respect to the buckling displacement fields, From the strain-displacement relations and stress-strain relations (i.e., Eqs. (2.43), (2.44), (2.45) and (2.46) and from Table 2.5,

Table 2.6 and Table 2.7), by substituting into Eq. (2.54) and performing integration across the thickness (Appendix A.2), one obtains

$$\begin{aligned} \overline{\Pi}_t^* = \iint \bigg\{ & \bar{A}_1 \bar{u}_{1b}^2 + \bar{A}_2 \bar{u}_{1b}'^2 + \bar{A}_3 \bar{u}_{1b}^2 + \bar{A}_1 \bar{v}_{1b}^2 + \bar{A}_2 \bar{v}_{1b}'^2 + \bar{A}_3 \bar{v}_{1b}'^2 \\ & + \bar{A}_4 \bar{w}_b'^2 + \bar{A}_4 \bar{w}_b^2 + \bar{A}_5 \bar{w}_b'^2 + \bar{A}_5 \bar{w}_b^2 + (\bar{A}_6 + \bar{A}_7 + \bar{A}_8) \bar{w}_b'^2 \\ & + \bar{A}_9 \bar{w}_b'' \bar{w}_b + \bar{A}_{10} \bar{u}_{1b} \bar{w}_b' + \bar{A}_{10} \bar{v}_{1b} \bar{w}_b' + \bar{A}_{11} \bar{u}_{1b}' \bar{v}_{1b} + \bar{A}_{12} \bar{u}_{1b} \bar{v}_{1b}' \\ & + (\bar{A}_{13} + \bar{A}_{14}) \bar{u}_{1b} \bar{w}_b' + (\bar{A}_{13} + \bar{A}_{14}) \bar{v}_{1b} \bar{w}_b' \\ & + \lambda (\bar{A}_{15} \bar{w}_b'^2 + \bar{A}_{16} \bar{w}_b^2 + \bar{A}_{17} \bar{w}_b \bar{w}_b') \bigg\} dx dy \end{aligned} \quad (2.55)$$

where the coefficients related to stabilizing terms ($\bar{A}_1$ to $\bar{A}_{14}$) and the coefficients related to destabilizing terms ($\bar{A}_{15}$ to $\bar{A}_{17}$) are summarized in Table 2.8 . The neutral stability state (Configuration 3 in Figure 2.6) is obtained by setting to zero the variation of second variation of total potential energy in Eq. (2.55).

Table 2.8- Coefficients of Eq.(2.55)

$\bar{A}_1$	$\frac{2G_a}{h_2}$
$\bar{A}_2$	$\frac{2E_g h_1}{(1-\nu_g^2)}$
$\bar{A}_3$	$2G_a \left\{ \frac{h_3^2}{4h_2} + \frac{1}{3h_2^2} \left[\left(\frac{h_3}{2} + h_2 \right)^3 - \left(\frac{h_3}{2} \right)^3 \right] - \frac{h_3}{2h_2^2} \left[\left(\frac{h_3}{2} + h_2 \right)^2 - \left(\frac{h_3}{2} \right)^2 \right] \right\} + \frac{E_g h_1}{(1+\nu_g)}$
$\bar{A}_4$	$2G_a \left[\frac{(h_1 + h_3)^2}{4h_2} + h_2 - h_3 - h_1 \right]$
$\bar{A}_5$	$\frac{E_g h_1^3}{6(1-\nu_g^2)} + \frac{E_s h_3^3}{12(1-\nu_s^2)}$

$\bar{A}_6$	$2G_a \left\{ h_2 \left(\frac{h_1 h_3}{2h_2} + \frac{h_3^2}{2h_2} + h_3 \right)^2 + \frac{1}{3} \left[\left(\frac{h_3}{2} + h_2 \right)^3 - \left(\frac{h_3}{2} \right)^3 \right] \left(\frac{h_1 + h_3}{h_2} + 2 \right)^2 \right\}$
$\bar{A}_7$	$2G_a \left[\left(\frac{h_3}{2} \right)^2 - \left(\frac{h_3}{2} + h_2 \right)^2 \right] \left(\frac{h_1 h_3}{2h_2} + \frac{h_3^2}{2h_2} + h_3 \right) \left(\frac{h_1 + h_3}{h_2} + 2 \right)$
$\bar{A}_8$	$\frac{E_g h_1^3}{3(1+\nu_g)} + \frac{E_s h_3^3}{6(1+\nu_s)}$
$\bar{A}_9$	$\frac{\nu_g E_g h_1^3}{3(1+\nu_g)} + \frac{\nu_s E_s h_3^3}{6(1+\nu_s)}$
$\bar{A}_{10}$	$2G_a \left(2 - \frac{h_1 + h_3}{h_2} \right)$
$\bar{A}_{11}$	$\frac{4\nu_g E_g h_1}{1 - \nu_g^2}$
$\bar{A}_{12}$	$2G_a \left\{ \frac{2}{3h_2^2} \left[\left(\frac{h_3}{2} + h_2 \right)^3 - \left(\frac{h_3}{2} \right)^3 \right] + \frac{h_3^2}{2h_2} - \frac{h_3}{h_2^2} \left[\left(\frac{h_3}{2} + h_2 \right)^2 - \left(\frac{h_3}{2} \right)^2 \right] \right\} + \frac{2E_g h_1}{1+\nu_g}$
$\bar{A}_{13}$	$2G_a \left\{ \frac{-2}{3} \left[\left(\frac{h_3}{2} + h_2 \right)^3 - \left(\frac{h_3}{2} \right)^3 \right] \left(\frac{h_1 + h_3}{h_2^2} + \frac{2}{h_2} \right) - h_3 \left(\frac{h_1 h_3}{2h_2} + \frac{h_3^2}{2h_2} + h_3 \right) \right\}$
$\bar{A}_{14}$	$2G_a \left\{ \frac{h_3}{2h_2} \left(\frac{h_1 + h_3}{h_2} + 2 \right) \left[\left(\frac{h_3}{2} + h_2 \right)^2 - \left(\frac{h_3}{2} \right)^2 \right] + \left[\left(\frac{h_3}{2} + h_2 \right)^2 - \left(\frac{h_3}{2} \right)^2 \right] \left(\frac{h_1 h_3}{2h_2^2} + \frac{h_3^2}{2h_2^2} + \frac{h_3}{h_2} \right) \right\}$
$\bar{A}_{15}$	$\frac{2E_g h_1 u'_p}{1 - \nu_g^2} + \frac{2\nu E_g h_1 \dot{v}_p}{1 - \nu_g^2} + \frac{E_s h_3 u'_p}{1 - \nu_s^2} + \frac{\nu E_s h_3 \dot{v}_p}{1 - \nu_s^2}$
$\bar{A}_{16}$	$\frac{2E_g h_1 \dot{v}_p}{1 - \nu_g^2} + \frac{2\nu_g E_g h_1 u'_p}{1 - \nu_g^2} + \frac{E_s h_3 \dot{v}_p}{1 - \nu_s^2} + \frac{\nu_s E_s h_3 u'_p}{1 - \nu_s^2}$
$\bar{A}_{17}$	$(\dot{u}_p + \nu'_p) \left(4G_a h_2 + \frac{2E_g h_1}{1+\nu_g} + \frac{E_s h_3}{1+\nu_s} \right)$

3 Verification of the variational formulation

3.1 General

This chapter provides an assessment of the validity of the total potential energy formulation developed in Chapter 2 for a single plate and two plates bonded together through a soft layer. Five problems are considered in this Chapter. These are (1) a single plate under pure shear, (2) a single plate under biaxial normal pressure, (3) two plates bonded through a soft layer under pure shear, (4) two plates bonded through a soft layer under biaxial normal pressure, and (5) cylindrical bending problem.

3.2 Outline of Methodology

Towards the goal outlined in Section 3.1, the following steps are followed for each problem:

- (1) A 3D FEA model is developed under Abaqus and the critical load combinations are determined along with the buckled mode shapes.
- (2) Through regression analysis, the buckled mode shapes obtained from the 3D FEA in Abaqus are approximated using mathematical functions.
- (3) The mathematical distribution of the buckled shapes developed through regression in (b) are then used in conjunction with the energy based neutral stability functional developed in Sections 2.3.8.2 and 2.4.9 to transform the functional into a relatively simple Eigen-value analysis.
- (4) The Eigen-value problems formulated in Step (3) are then solved and an energy based critical load combination is determined and compared to that obtained under Abaqus in Step (1). Close agreement between the critical load combinations as predicted by Steps (1) and (4) is indicative of the correctness and validity of the variational formulations developed under Sections 2.3.8.2 and 2.4.9.

3.3 Example 1- Single plate under pure shear traction

3.3.1 Statement of the problem

A single plate with dimensions $a \times a \times h = 4m \times 4m \times 0.1m$ (Figure 3.1) is subjected to a reference pure shear traction $P_{xy} = 1 \text{ KPa}$. Modulus of elasticity is $E_s = 200 \text{ GPa}$ and Poisson's ratio is $\nu_s = 0.3$. All edges of the plate are assumed to be fully restrained relative to rotation. It is required to find the critical pressure λP_{xy} at which the plate would buckle out of its plane.

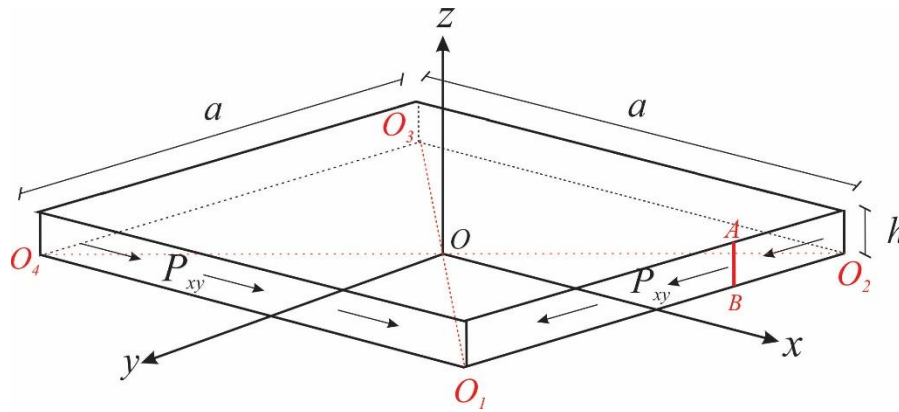


Figure 3.1- Single plate $a \times a \times h$ under reference pure shear traction P_{xy}

3.3.2 Details of the Abaqus Model

3.3.2.1 Finite Element

The plate in this problem (and the subsequent ones) is modeled by using the C3D8 solid element in Abaqus. Element C3D8 is a 3D continuum element with 8 nodes and three translational degrees of freedom per node, i.e., the element has a total of 24 degrees of freedom. Two elements are taken across the plate thickness.

3.3.2.2 Boundary conditions

For a generic edge line of the plate such as line AB (Figure 3.1), all the nodes are constrained to move equally along the x and y directions to restrain the edges against rotations about x and y , i.e.,

$$u_A = u_B \quad , \quad v_A = v_B \quad (3.1)$$

These conditions are applied for all the nodes through the thickness of the plate.

To apply the symmetry conditions about diagonals O_1O_3 and O_2O_4 (Figure 3.1), the following constraints are enforced on nodes O_1 , O_2 , O_3 and O_4

$$u_{O_1} = v_{O_1} \quad , \quad u_{O_3} = v_{O_3} \quad , \quad u_{O_2} = -u_{O_4} \quad , \quad v_{O_2} = -v_{O_4} \quad (3.2)$$

These constraints are also enforced on all the nodes through the plate thickness. All constraints were enforced by using the *EQUATION feature in ABAQUS which allows the user to enter any set of linear equations relating various degrees of freedom within the structure.

3.3.2.3 Analysis procedures

The present study is concerned with the linear buckling analysis of plates. This type of analysis is supported in Abaqus and is evoked by using the *BUCKLE procedure. Under this type of procedure, the user enters a set of reference loads. Given the reference loads, the program conducts a linear elastic static pre-buckling analysis to determine the pre-buckling displacements, strains, and stresses, followed by an Eigen-value analysis to determine the critical load combination, by determining the load multiplier λ applied to the reference loads at which the plate would have a tendency to buckle.

In the *BUCKLE analysis procedure in Abaqus, the user is unable to directly access the pre-buckling u_p, v_p under the user input reference loads based on the pre-buckling analysis. Thus, in the present problem (and subsequent ones), a linearly elastic static analysis was first conducted using the *STATIC procedure in Abaqus under loads identical to those input to the *BUCKLE analysis and the pre-buckling analysis displacement fields were extracted, since these pre-buckling displacement fields will form the basis of the energy based solution to be assessed in this study (step (4) in Section 3.2).

3.3.2.4 Mesh Study

Two elements were taken across the plate thickness. Along the plate sides, meshes based on the 30, 50, 70, 100, 200 and 400-element were investigated. As observed in Table 3.1, the solution based on the 400×400 -element is in agreement with the 200×200 -element solution within four significant digits. Thus, convergence is deemed to be achieved for 200×200 -element mesh.

Table 3.1- Mesh study for 3D FEA in Abaqus for a single plate under pure shear traction

Number of subdivision per side	Critical load (MPa)
30	2369
50	1925
70	1805
100	1743
200	1699
400	1699

3.3.3 Developing approximate functions

Under the input reference loads, the nodal displacements based on the static analysis are extracted and mathematically described through simple functions (Section 3.3.3.1, Eq.(3.3)). Also, the nodal displacements based on the first buckling mode as predicted by the 3D FEA in Abaqus are extracted (Section 3.3.3.2, Eq.(3.4)). The displacements are then visualized to provide insight on proposing suitable approximate displacement function with only a few degrees of freedom. The unknown degrees of freedom are then determined by minimizing the summation of square of the differences between the Abaqus buckling displacements and the proposed displacement function. The details are provided in the following sections:

3.3.3.1 Pre-buckling in plane displacement functions

The pre-buckling displacements under reference load $P_{xy} = 1 \text{ KPa}$ were obtained from the static analysis of the plate in 3D FEA in Abaqus and were found to be exactly described by the functions

$$\begin{aligned} u_p &= -\alpha(y + a/2) \\ v_p &= -\alpha(x + a/2) \end{aligned} \quad (3.3)$$

where, for the present problem, $\alpha = 0.65 \times 10^{-8}$, $a = 4 \text{ m}$.

3.3.3.2 Buckling transverse displacement function

The nodal transverse displacements w_b based on the first buckling mode, as predicted by the FEA solution, are extracted (Figure 3.2 (a)). The displacements are then visualized to provide insight

on proposing suitable approximating displacement function w_b with a few unknown degrees of freedom which satisfy all the boundary conditions applied on the plate:

$$w_b(x, y) = A_1 f_1(x, y) f_2(x, y) \quad (3.4)$$

where

$$\begin{aligned} f_1(x, y) &= (x - a/2)^2 (x + a/2)^2 (y - a/2)^2 (y + a/2)^2 \\ f_2(x, y) &= (10^{-3}) \times \sum_{i=0}^{11} b_i \zeta^i \end{aligned} \quad (3.5)a,b$$

and

$$\zeta = (x - y)/\sqrt{2} \quad (3.6)$$

with $a = 4 \text{ m}$. Function $f_1(x, y)$ ensures that $w_b(x, y)$ meets the essential boundary conditions:

$$w_{b,(x=-a/2)} = 0, \quad w_{b,(x=a/2)} = 0, \quad w_{b,(y=-a/2)} = 0, \quad w_{b,(y=a/2)} = 0$$

and coefficients b_i are determined by minimizing the summation of square of the differences between the Abaqus buckling displacements and those based on proposed by the approximation given in Eq.(3.5) b, i.e.,

$$\Delta w = \sqrt{\frac{\sum_{j=1}^m \sum_{i=1}^n \left[\frac{w_{b,(ij)} - w_{b(ij)}}{w_{b,(ij)}} \right]^2}{n \times m}} = \min \quad (3.7)$$

where m and n are number of nodes along each edge (201×201 -nodes in present problem). The coefficients b_i as obtained from the minimization of Eq. (3.7) are provided in Table 3.2.

Table 3.2- The coefficients b_i of approximate transverse displacement function w_b , Example 1

b_0	b_1	b_2	b_3	b_4	b_5	b_6	b_7	b_8	b_9	b_{10}
995.1	2.211	-1353	1.321	614.4	0	-128.0	0	13.14	0	0

From the coefficients in Table 3.2, by substituting into Eq.(3.4), one obtains the approximate transverse displacement function at the mid-surface of the plate. A visual comparison is provided in Figure 3.2 (b).

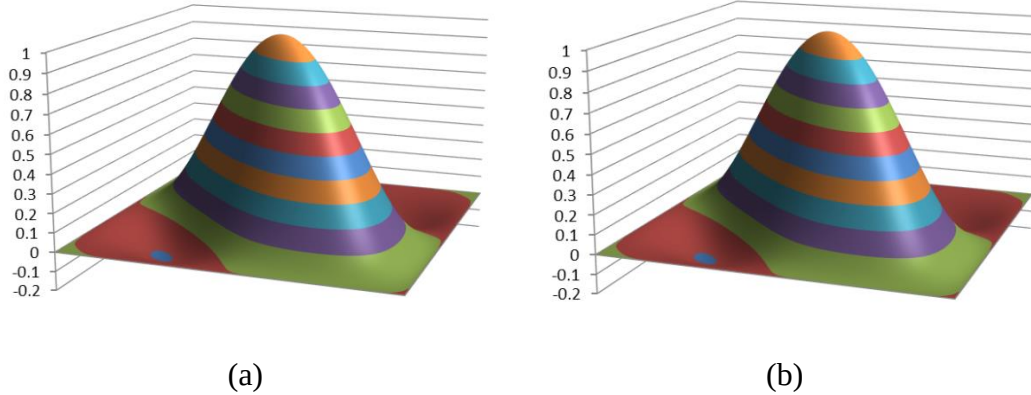


Figure 3.2- Transverse displacement functions based on (a) Abaqus (w_b) - (b) approximation (w_b)

3.3.4 Comparison of Results

From the approximate displacement function obtained in Eq.(3.4), by substituting into the Eigen-value problem in Section 2.3.8, a prediction of the critical load is obtained. The buckling load as predicted by the single degree of freedom solution and that based on the Abaqus 3D model are provided in Table 3.3. It is observed that the energy solution slightly overestimates the buckling pressure by about 2.5%, i.e., is in excellent agreement with the buckling prediction based on the 3D FEA. This is a natural outcome of the fact that the approximation in Eq. (3.4) provides an accurate representation of the buckling mode shapes. Also, it is indicative of the fact that the variational expression developed for the single plate problem provides an excellent approximation of the buckling capacity of the single plate considered in the present problem.

Table 3.3- Comparison of critical load between 3D FEA and Energy solution for Example 1

3D FEA (MPa)	Energy formulation (MPa)	Percentage Difference
$P_{xy,cr} = 1699$	$P_{xy,cr} = 1743$	2.52 %

3.4 Example 2- Single plate under biaxial normal pressure

3.4.1 Statement of the problem

A simply supported single plate with dimensions $a \times b \times h = 6m \times 4m \times 0.1m$ (Figure 3.3) is subjected to normal pressures $P_{xx} = 1 KPa$ and $P_{yy} = 1 KPa$. Modulus of elasticity is $E_s = 200 GPa$ and Poisson's ratio is $\nu_s = 0.3$. The mid-surface of the plate all around the edges are simply supported relative to the transverse displacement. It is required to find the critical loading combination $\lambda(P_{xx}, P_{yy})$ at which the plate would buckle out of its own plane.

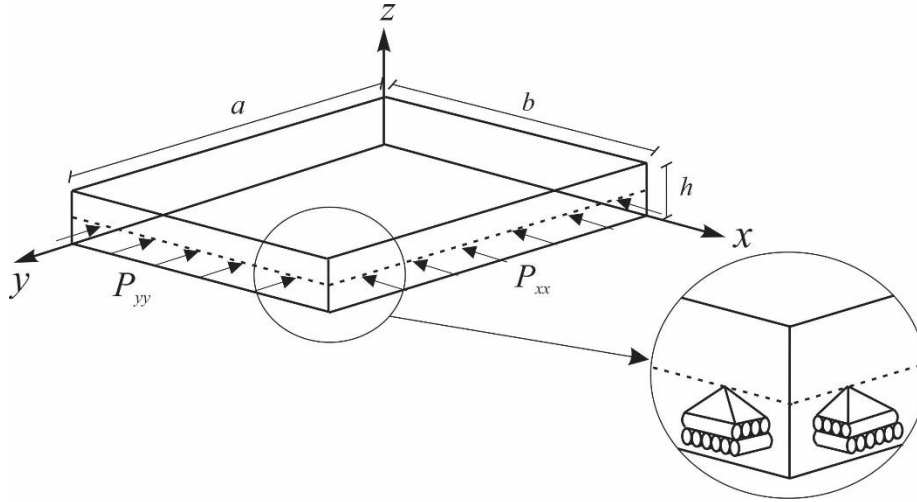


Figure 3.3 - Single plate $a \times b \times h$ under biaxial normal pressure combination (P_{xx}, P_{yy})

3.4.2 Details of the Abaqus 3D Model

3.4.2.1 Boundary conditions

For the edge of an arbitrary surface $O_1O_6O_5O_{10}$ normal to the y axis (Figure 3.4), the following constraints are applied

$$u_{O_1} = -u_{O_6}, u_{O_2} = -u_{O_7}, u_{O_3} = -u_{O_8}, u_{O_4} = -u_{O_9}, u_{O_5} = -u_{O_{10}} \quad (3.8)$$

Also for the edge of an arbitrary surface $M_1M_6M_5M_{10}$ normal to the x axis (Figure 3.4), the following constraints are applied

$$v_{M_1} = -v_{M_6}, \quad v_{M_2} = -v_{M_7}, \quad v_{M_3} = -v_{M_8}, \quad v_{M_4} = -v_{M_9}, \quad v_{M_5} = -v_{M_{10}} \quad (3.9)$$

The above constraints are applied for all the edges around the plate. Also, all the nodes of the mid-surface of the plate are restrained in the vertical movement to enforce the simply supported condition. All the conditions are enforced by using the *EQUATION feature in Abaqus which allows the user to enter any set of linear equations relating various degrees of freedom within the structure.

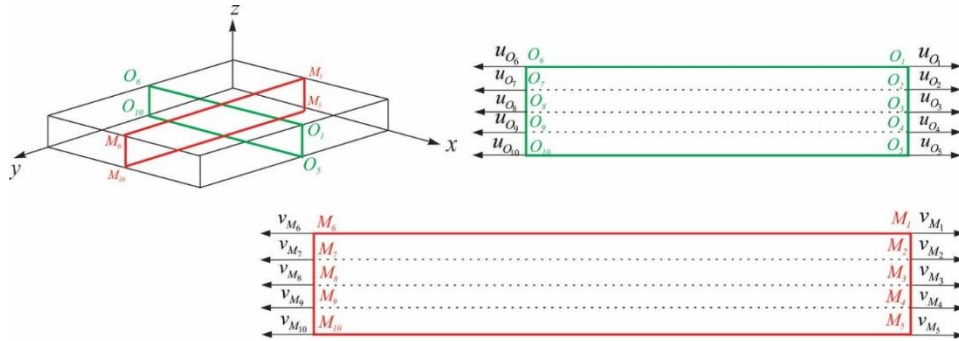


Figure 3.4- Single plate $a \times a \times h$ and initial in-plane displacements of two arbitrary sections

3.4.3 Mesh study

Table 3.4 provides the results of a mesh sensitivity study for the 3D FEA in Abaqus. Four elements are taken across the plate thickness. Along the plate sides, meshes based on the 20×30 , 40×60 , 80×120 , 160×240 , 300×450 and 400×600 -element are investigated. The solution based on the 400×600 -element mesh is observed to be in agreement with that based on the 160×240 -element solution, within four significant digits. Thus, convergence is deemed to be achieved for the 160×240 element mesh.

Table 3.4- Mesh study for 3D FEA in Abaqus for single plate under biaxial normal pressure

Number of subdivisions per side	Critical load (MPa)
20×30	283.1
40×60	186.8
80×120	162.5
160×240	156.4
300×450	155.9
400×600	155.9

3.4.4 Developing approximate displacement functions

Under the input reference loads, the nodal displacements based on the static analysis are extracted and mathematically described through simple functions (Section 3.4.4.1). Also, the nodal displacements based on the first buckling mode as predicted by the FEA solution are extracted (Section 3.4.4.2).

3.4.4.1 Pre-buckling in plane displacement functions

The pre-buckling displacements under reference biaxial pressure $P_{xx} = 1 \text{ KPa}$ and $P_{yy} = 1 \text{ KPa}$ were obtained from a static analysis of the plate in 3D FEA in Abaqus and were found to be exactly described by the functions

$$u_p = -\alpha(a/2 - x), \quad v_p = -\beta(b/2 - y) \quad (3.10)$$

For the present problem $a = 6 \text{ m}$, $b = 4 \text{ m}$, $\alpha = \beta = 0.351 \times 10^{-8}$.

3.4.4.2 Buckling transverse displacement function

The nodal transverse buckling displacement w_b as predicted by the FEA solution, are extracted (Figure 3.5 (a)) for the first buckling mode. The displacements are then visualized to provide insight on proposing suitable approximating displacement function w_b with a single degree of freedom which satisfies all the boundary conditions. The following function was deemed to accurately replicate the deformation pattern.

$$w_b = A_2 \sin(\pi x/a) \sin(\pi y/b) \quad (3.11)$$

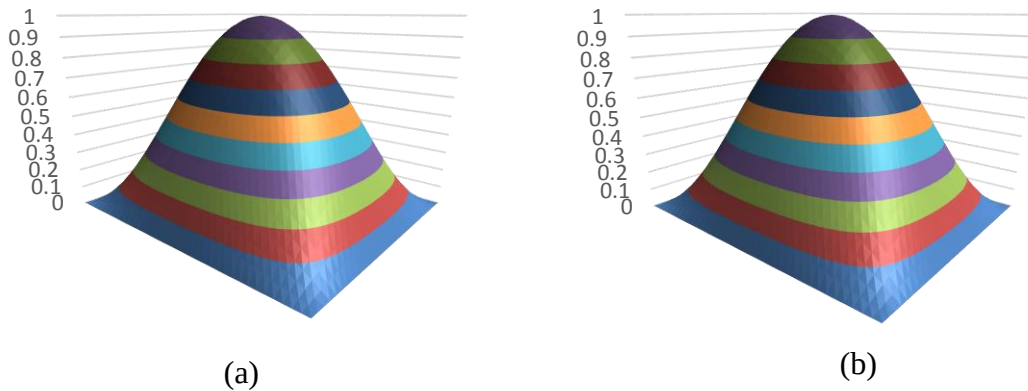


Figure 3.5- Transverse displacement function based on (a) w_b Abaqus (w_b), (b) approximation (w_b)

3.4.5 Comparison of results

The critical load is determined from the displacement function w_b proposed in Eq. (3.11), by substituting into the variational expression developed in Chapter 2 (Section 2.3.8), the buckling load as predicted by the single degree of freedom energy solution and that based on the Abaqus 3D model are provided in Table 3.5. The present energy-based solution is found to overpredict the critical load by about 4%, again suggesting the accuracy of the energy expression developed in Section 2.3.8.

Table 3.5- Comparison of critical load between 3D FEA and Energy solution for example 2

3D FEA (MPa)	Energy formulation (MPa)	Percentage Difference
$P_{xx,cr} = P_{yy,cr} = 156.4$	$P_{xx,cr} = P_{yy,cr} = 163$	4.01 %

3.5 Example 3- Two plates bonded with a soft layer under pure shear

3.5.1 Statement of the problem

Two plates with dimensions $a \times a \times h = 4m \times 4m \times 0.1m$ (Figure 3.6) are bonded together through a 25mm thick soft layer. The composite system is subjected to pure shear traction. Modulus of elasticity and Poisson's ratio of steel are $E_s = 200GPa$, $\nu_s = 0.3$, and those of soft layer are $E_a = 0.02GPa$ and $\nu_a = 0.3$. All edges of the plate are fully fixed relative to rotations and transverse displacements. It is required to find the critical shear traction λP_{xy} at which the composite system would buckle.

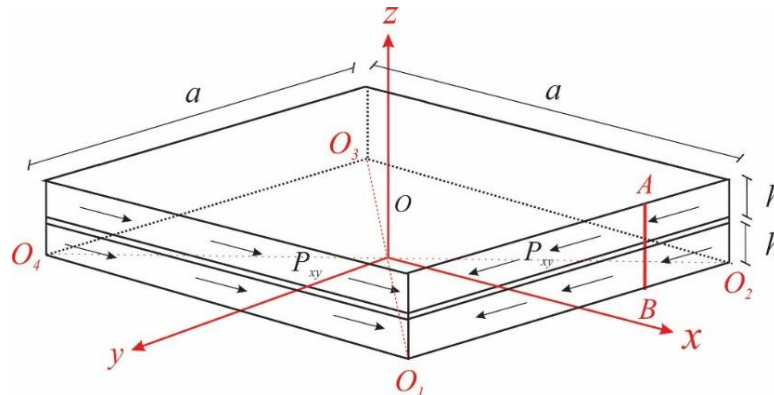


Figure 3.6- Two plates bonded through an adhesive layer under pure shear traction P_{xy}

3.5.2 Details of the Abaqus 3D Model

3.5.2.1 Boundary conditions

For a generic edge line of the plate such as line AB (Figure 3.6), all the nodes are constrained to move equally along the x and y directions to restrain the edges against rotations about x and y . Also, because all in-plane displacements at the mid-surface of the top and bottom plates must be equal and opposite (assumption made in Chapter 2, Section 2.4.2), the only means to achieve this requirement while meeting the requirement of equal displacements is to set all mid-surface displacements at the edges to zero. To apply symmetry conditions about diagonals O_1O_3 and O_2O_4 (Figure 3.6), the following constraints are enforced to the nodes O_1, O_2, O_3 and O_4 and all the nodes through thickness of the composite system.

$$\begin{aligned} u_{O_1} &= v_{O_1} \quad , \quad u_{O_3} = v_{O_3} \\ u_{O_2} &= -u_{O_4} \quad , \quad v_{O_2} = -v_{O_4} \end{aligned} \quad (3.12)$$

where all the constraints are enforced by using the *EQUATION keyword in Abaqus.

3.5.3 Mesh study

Table 3.6 provides the results of a mesh sensitivity study for the 3D FEA in ABAQUS. Four elements are taken across each plate thickness and one element is taken across the adhesive layer. Along the plate sides, meshes based on the 20, 40, 80, 100, 200 and 300-element are investigated. The solution based on the 200×200 -element mesh is observed to be in agreement with that based on the 300×300 -element solution, within three significant digits. Thus, convergence is deemed to be achieved for the 200×200 -element mesh.

Table 3.6- Mesh study for 3D FEA in Abaqus for Example 3

Number of subdivisions per side	Critical load (MPa)
20	3176
40	2000
80	1709
100	1674
200	1644
300	1641

3.5.4 Developing approximate functions

Under the input reference loads, the nodal displacements based on the static analysis are extracted and mathematically described through simple functions (Section 3.5.4.1). Also, the nodal displacements based on the first buckling mode as predicted by the FEA solution are extracted (Section 3.5.4.2 and 3.5.4.3). The displacements are then visualized to provide insight on proposing suitable approximating displacement functions with only a few unknown degrees of freedom. The unknown degrees of freedom are then determined by minimizing the summation of square of the differences between the Abaqus buckling displacements and the proposed displacement function.

3.5.4.1 Pre-buckling displacement functions

The pre-buckling displacements under reference shear traction $P_{xy} = 1 \text{ KPa}$ were obtained from the static analysis of the plate in 3D FEA in Abaqus and were found to be exactly described by the functions

$$u_p = -\alpha(y + a/2) \quad , \quad v_p = -\alpha(x + a/2) \quad (3.13)$$

where, for the present problem, one has $\alpha = 0.65 \times 10^{-8}$, $a = 4 \text{ m}$.

3.5.4.2 Buckling in-plane displacement functions

The nodal in-plane displacements based on the first buckling mode u_b and v_b as predicted by the FEA solution are extracted (Figure 3.7 (a) and (c)). The displacements are then visualized to provide insight on proposing suitable approximating displacement function u_b and $\tilde{v}_b$ with several unknown degrees of freedom which satisfy all the boundary conditions of the plate.

$$u_b = A_3 f_4(x, y) \quad , \quad \tilde{v}_b = A_4 f_5(x, y) \quad (3.14)\text{a,b}$$

where $f_4(x, y)$ and $f_5(x, y)$ are complete polynomials chosen among Pascal triangle terms in order to satisfy the boundary conditions. The functions take the forms

$$\begin{aligned}
f_4(x, y) = 10^{-4} \times & \left(a_1x + a_2y + a_3x^3 + a_4x^2y + a_5xy^2 + a_6y^3 + a_7x^5 + a_8x^4y + a_9x^3y^2 + a_{10}x^2y^3 \right. \\
& + a_{11}xy^4 + a_{12}y^5 + a_{13}x^7 + a_{14}x^6y + a_{15}x^5y^2 + a_{16}x^4y^3 + a_{17}x^3y^4 + a_{18}x^2y^5 \\
& + a_{19}xy^6 + a_{20}y^7 + a_{21}x^9 + a_{22}x^8y + a_{23}x^7y^2 + a_{24}x^6y^3 + a_{25}x^5y^4 + a_{26}x^4y^5 \\
& + a_{27}x^3y^6 + a_{28}x^2y^7 + a_{29}xy^8 + a_{30}y^9 + a_{31}x^{11} + a_{32}x^{10}y + a_{33}x^9y^2 + a_{34}x^8y^3 \\
& \left. + a_{35}x^7y^4 + a_{36}x^6y^5 + a_{37}x^5y^6 + a_{38}x^4y^7 + a_{39}x^3y^8 + a_{40}x^2y^9 + a_{41}xy^{10} + a_{42}y^{11} \right)
\end{aligned}
\tag{3.15}$$

and

$$\begin{aligned}
f_5(x, y) = 10^{-4} \times & \left(b_1x + b_2y + b_3x^3 + b_4x^2y + b_5xy^2 + b_6y^3 + b_7x^5 + b_8x^4y + b_9x^3y^2 + b_{10}x^2y^3 \right. \\
& + b_{11}xy^4 + b_{12}y^5 + b_{13}x^7 + b_{14}x^6y + b_{15}x^5y^2 + b_{16}x^4y^3 + b_{17}x^3y^4 + b_{18}x^2y^5 \\
& + b_{19}xy^6 + b_{20}y^7 + b_{21}x^9 + b_{22}x^8y + b_{23}x^7y^2 + b_{24}x^6y^3 + b_{25}x^5y^4 + b_{26}x^4y^5 \\
& + b_{27}x^3y^6 + b_{28}x^2y^7 + b_{29}xy^8 + b_{30}y^9 + b_{31}x^{11} + b_{32}x^{10}y + b_{33}x^9y^2 + b_{34}x^8y^3 \\
& \left. + b_{35}x^7y^4 + b_{36}x^6y^5 + b_{37}x^5y^6 + b_{38}x^4y^7 + b_{39}x^3y^8 + b_{40}x^2y^9 + b_{41}xy^{10} + b_{42}y^{11} \right)
\end{aligned}
\tag{3.16}$$

where $a_1 - a_{42}$ and $b_1 - b_{42}$ are the coefficients to determined by minimizing the summation of squares of the differences between the Abaqus buckling displacements and the proposed displacement function; i.e.,

$$\Delta u = \sqrt{\frac{\sum_{j=1}^m \sum_{i=1}^n \left[\frac{u_{b,(ij)} - u_{b(i,j)}}{u_{b,(ij)}} \right]^2}{n \times m}} = \min, \quad \Delta v = \sqrt{\frac{\sum_{j=1}^m \sum_{i=1}^n \left[\frac{v_{b,(ij)} - \tilde{v}_{b(i,j)}}{v_{b,(ij)}} \right]^2}{n \times m}} = \min \tag{3.17} \text{ a,b}$$

where m and n are number of nodes at each edges (201×201 -nodes). The coefficients $a_1 - a_{42}$ and $b_1 - b_{42}$ as obtained from the minimization process in Eq. 3.17a,b are provided in Table 3.7.

A visual comparison of the displacement fields based on the Abaqus 3D model and that based on Eq. 3.14 is provided in Figure 3.7 (a -d). It is clear that the proposed functions successfully replicate the distribution of in-plane buckling displacements.

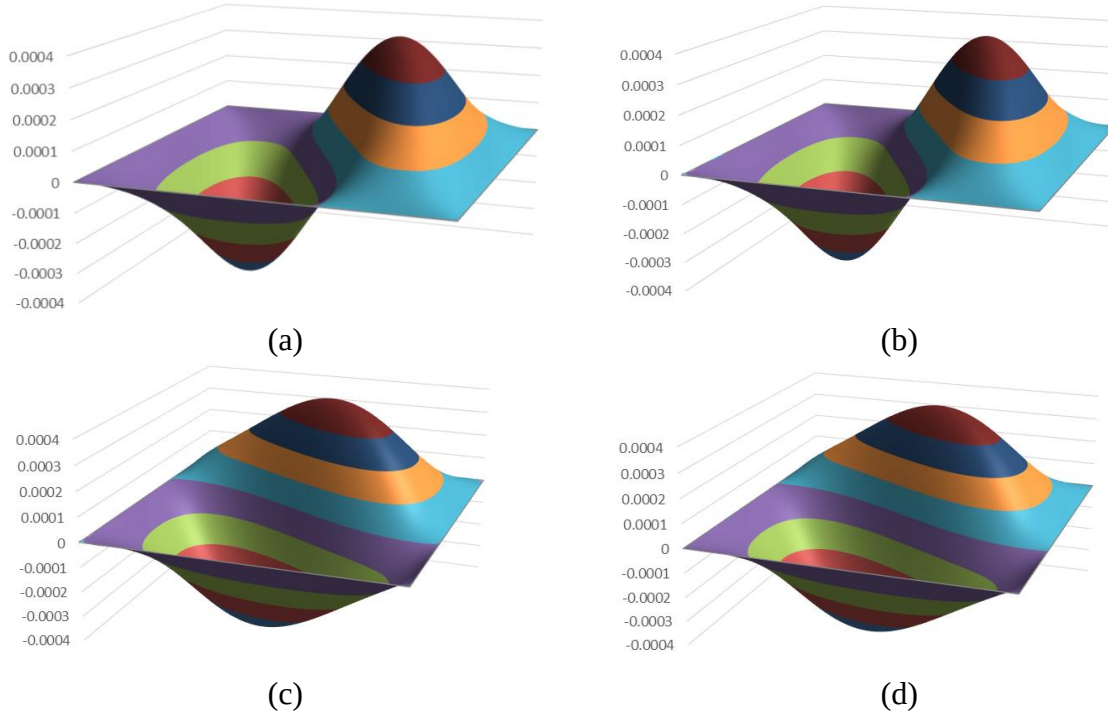


Figure 3.7- In-plane buckling displacement functions based on, (a) Abaqus (u_b), (b) approximation (u_b), (c) Abaqus (v_b), (d) approximation ($\tilde{v}_b$)

Table 3.7- Coefficients $a_1 - a_{42}$ and $b_1 - b_{42}$ defined in Eqs. (3.15) and (3.16) obtained from regression analysis

a_1	6.671	a_{15}	-0.476	a_{29}	0.008	b_1	3.621	b_{15}	-0.633	b_{29}	0.000
a_2	3.624	a_{16}	-0.743	a_{30}	0.000	b_2	6.672	b_{16}	-0.744	b_{30}	0.000
a_3	-3.433	a_{17}	-0.633	a_{31}	0.000	b_3	-1.723	b_{17}	-0.480	b_{31}	0.000
a_4	-4.532	a_{18}	-0.399	a_{32}	0.000	b_4	-3.972	b_{18}	-0.181	b_{32}	0.000
a_5	-3.967	a_{19}	-0.154	a_{33}	0.000	b_5	-4.528	b_{19}	-0.048	b_{33}	-0.002
a_6	-1.728	a_{20}	-0.026	a_{34}	0.001	b_6	0.321	b_{20}	0.000	b_{34}	-0.006
a_7	0.621	a_{21}	0.000	a_{35}	-0.006	b_7	1.072	b_{21}	0.008	b_{35}	-0.007
a_8	1.554	a_{22}	0.007	a_{36}	-0.008	b_8	2.255	b_{22}	0.029	b_{36}	-0.008
a_9	2.388	a_{23}	0.034	a_{37}	-0.007	b_9	2.391	b_{23}	0.069	b_{37}	-0.006
a_{10}	2.250	a_{24}	0.075	a_{38}	-0.006	b_{10}	1.551	b_{24}	0.142	b_{38}	-0.001
a_{11}	1.073	a_{25}	0.115	a_{39}	-0.002	b_{11}	0.623	b_{25}	0.113	b_{39}	0.000
a_{12}	0.298	a_{26}	0.112	a_{40}	0.000	b_{12}	-0.026	b_{26}	0.075	b_{40}	0.000
a_{13}	-0.048	a_{27}	0.069	a_{41}	0.000	b_{13}	-0.154	b_{27}	0.035	b_{41}	0.000
a_{14}	-0.184	a_{28}	0.029	a_{42}	0.000	b_{14}	-0.385	b_{28}	0.007	b_{42}	0.000

3.5.4.3 Buckling transverse displacement function

The nodal transverse buckling displacements based on the first buckling mode w_b as predicted by the FEA solution are extracted (Figure 3.8 (a)). The displacements are visualized to provide insight on proposing suitable approximating displacement function w_b with only few unknown degrees of freedom which satisfies the boundary conditions of the plate, i.e.,

$$w_b = A_5 f_6(x, y) f_7(x, y) \quad (3.18)$$

in which

$$f_6(x, y) = (x - a/2)^2 (x + a/2)^2 (y - a/2)^2 (y + a/2)^2 \quad (3.19)$$

Function $f_6(x, y)$ ensures that $w_b(x, y)$ meets the essential boundary conditions

$$w_{b,(x=-a/2)} = 0, \quad w_{b,(x=a/2)} = 0, \quad w_{b,(y=-a/2)} = 0, \quad w_{b,(y=a/2)} = 0$$

Function $f_7(x, y)$ is a complete polynomial chosen among Pascal triangle terms in order to satisfy the boundary conditions. It is chosen to take the form

$$f_6(x, y) = 10^{-3} \times (c_0 + c_1 x^2 + c_2 xy + c_3 y^2 + c_4 x^4 + c_5 x^3 y + c_6 x^2 y^2 + c_7 xy^3 + c_8 y^4) \quad (3.20)$$

in which $C_0 - C_8$ are the coefficients to be determined by minimizing the summation of square of the differences between the Abaqus buckling displacements and the proposed displacement function, i.e.,

$$\Delta w = \sqrt{\frac{\sum_{j=1}^m \sum_{i=1}^n \left[\frac{w_{b,(ij)} - w_{b,(ij)}}{w_{b,(ij)}} \right]^2}{n \times m}} = \min \quad (3.21)$$

where m and n are number of nodes at each edges (201×201-nodes). The results of coefficients $C_0 - C_8$ are provided in Table 3.8.

Table 3.8- Coefficients $c_0 - c_8$ defined in Eqs. (3.20) as determined through regression analysis

c_0	c_1	c_2	c_3	c_4	c_5	c_6	c_7	c_8
3.912	-1.322	-4.345	-1.322	0.07511	0.2741	1.111	0.2741	0.07511

From the coefficients provided in Table 3.8, by substituting into Eq.(3.18), one obtains the approximate transverse displacement functions at the mid-surface of the plate. A visual comparison is provided in Figure 3.8 (a) and (b).

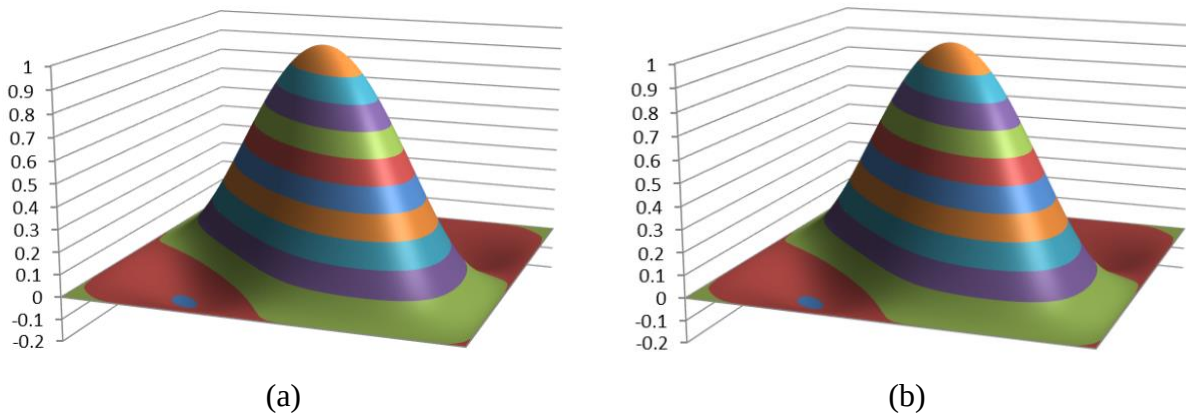


Figure 3.8- Transverse displacement function based on (a) w_b Abaqus (w_b) - (b) approximation (w_b)

3.5.5 Comparison of results

From the displacement functions u_b , $\tilde{v}_b$ and w_b proposed in (3.14) a,b and (3.18), by substituting into the variational expression developed in Chapter 2 (Section 2.4.9) and setting the variation of second variation of total potential energy to zero, the critical load is determined.

The buckling load as predicted by the energy solution and that based on the Abaqus 3D model are provided in Table 3.9. It is observed that the energy solution overestimate the buckling pressure of the system with excellent agreement. This is an outcome of the fact that the approximation provided in Eqs. (3.14) a,b and (3.18) provides an accurate representation of the buckling mode shapes.

Table 3.9- Comparison of critical shear traction obtained based on FEA in Abaqus and energy formulation for Example 3

3D FEA (MPa)	Energy formulation (MPa)	Percentage Difference
$P_{xy,cr} = 1644$	$P_{xy,cr} = 1671$	1.61 %

3.6 Example 4- Two plates bonded through a soft layer under biaxial normal pressure

3.6.1 Statement of the problem

Two simply supported plates with dimensions $a \times a \times h = 6m \times 4m \times 0.1m$ (Figure 3.9) are bonded through a 25 mm thick soft layer. The composite system is subjected to normal pressures $P_{xx} = 1 \text{ KPa}$, $P_{yy} = 1 \text{ KPa}$. Modulus of elasticity and Poisson's ratio of plates are $E_s = 200 \text{ GPa}$, $\nu_s = 0.3$, and those of soft layer are $E_a = 0.02 \text{ GPa}$ and $\nu_a = 0.3$. The mid-surface of the plates all around the edges are simply supported relative to the transverse displacement. It is required to find the critical loading $\lambda(P_{xx}, P_{yy})$ at which the composite system would buckle out of its own plate.

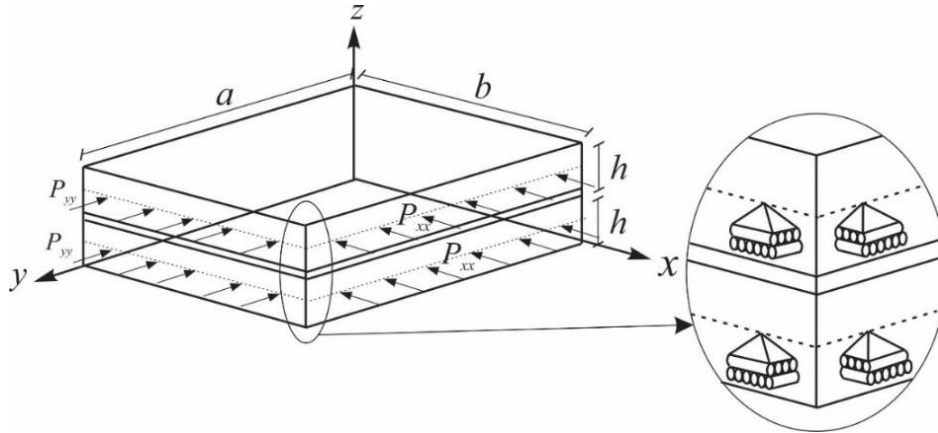


Figure 3.9- Two simply supported plates bonded through a soft layer under biaxial normal pressure

3.6.2 Details of the Abaqus 3D Model

3.6.2.1 Boundary conditions

For the edge of an arbitrary surface $O_1O_6O_5O_{10}$ normal to the y axis (Figure 3.10), the following constraints are applied

$$u_{O_1} = -u_{O_6}, u_{O_2} = -u_{O_7}, u_{O_3} = -u_{O_8}, u_{O_4} = -u_{O_9}, u_{O_5} = -u_{O_{10}} \quad (3.22)$$

Also for the edge of an arbitrary surface $M_1M_6M_5M_{10}$ normal to the x axis (Figure 3.10), the following constraints are applied

$$v_{M_1} = -v_{M_6}, \quad v_{M_2} = -v_{M_7}, \quad v_{M_3} = -v_{M_8}, \quad v_{M_4} = -v_{M_9}, \quad v_{M_5} = -v_{M_{10}} \quad (3.23)$$

The above constraints are applied for all the edges around the plate. Also, all the nodes of the mid-surface of the plate are restrained in the vertical movement to enforce the simply supported condition. All the conditions are enforced by using the *EQUATION feature in Abaqus which allows the user to enter any set of linear equations relating various degrees of freedom within the structure.

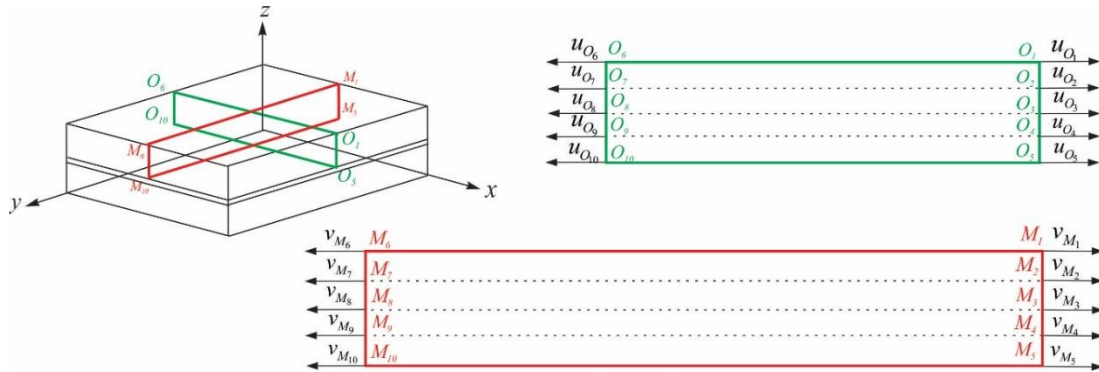


Figure 3.10- Two plates bonded through a soft layer and in-plane displacements of two arbitrary surfaces

3.6.3 Mesh study

Table 3.10 provides the results of a mesh sensitivity study for the 3D FEA in Abaqus. Four elements are taken across each plate thickness and one element is taken across the adhesive layer. Along the plate sides, meshes based on 60×90 , 160×240 , 200×300 and 250×375 -elements are investigated. The solution based on 250×375 -element mesh is observed to be in agreement with that based on the 160×240 -element solution, within four significant digits. Thus, convergence is deemed to be achieved for 160×240 -element mesh.

Table 3.10- Mesh study for 3D FEA for Example 4

Number of subdivisions	Critical load (MPa)
60×90	189.0
160×240	178.9
200×300	178.7
250×375	178.6

3.6.4 Developing approximate functions

Under the input reference loads, the nodal displacements based on the static analysis are extracted and mathematically described through simple functions (Section 3.6.4.1). Also, the nodal displacements based on the first buckling mode as predicted by the FEA solution are extracted (Sections 3.6.4.2 and 3.6.4.3). The displacements are then visualized to provide insight on proposing suitable approximating displacement functions with a few unknown degrees of freedom. The unknown degrees of freedom are then determined by minimizing the summation of square of the differences between the Abaqus buckling displacements and the proposed displacement function.

3.6.4.1 Pre-buckling in plane displacement functions

The pre-buckling displacements under reference load $P_{xx} = 1 \text{ KPa}$ and $P_{yy} = 1 \text{ KPa}$ were obtained from the static analysis of the plate in 3D FEA in Abaqus and were found to be exactly described by the functions

$$u_p = -\alpha(a/2 - x) \quad , \quad v_p = -\beta(b/2 - y) \quad (3.24)$$

For the present problem ($a = 6 \text{ m}$, $b = 4 \text{ m}$), one has $\alpha = \beta = 0.351 \times 10^{-8}$.

3.6.4.2 Buckling in-plane displacement functions

The nodal in-plane displacements based on the first buckling mode u_b and v_b as predicted by the FEA solution are extracted (Figure 3.11(a) and (c)). The displacements are visualized to provide insight on proposing suitable approximating displacement function u_b and $\tilde{v}_b$ with a few unknown degrees of freedom which satisfies all the boundary conditions applied on the plate.

$$u_b = A_6 f_8(x, y) \quad , \quad \tilde{v}_b = A_7 f_9(x, y) \quad (3.25)_{a,b}$$

where

$$f_8(x, y) = \cos(\pi x/a) [a_0 + a_1 y + a_2 y^2 + a_3 y^3 + a_4 y^4] \quad (3.26)$$

$$f_9(x, y) = \cos(\pi y/b) [b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4] \quad (3.27)$$

Coefficients $a_0 - a_4$ and $b_0 - b_4$ are determined by minimizing the summation of squares of the differences between the Abaqus buckling displacements and the proposed displacement function, i.e.,

$$\Delta u = \sqrt{\frac{\sum_{j=1}^m \sum_{i=1}^n \left[\frac{u_{b,(ij)} - u_{b(i,j)}}{u_{b,(ij)}} \right]^2}{n \times m}} = \min, \quad \Delta v = \sqrt{\frac{\sum_{j=1}^m \sum_{i=1}^n \left[\frac{v_{b,(ij)} - \tilde{v}_{b(i,j)}}{v_{b,(ij)}} \right]^2}{n \times m}} = \min \quad (3.28)a,b$$

where m and n are number of nodes at each edges (161×241 -nodes). The coefficients obtained from regression analysis are provided in Table 3.11 and

Table 3.12. By substituting into Eq.(3.25) a,b, one obtains the approximate in-plane displacement functions at the mid-surface of the plate. A visual comparison between in-plane displacement distributions based on the 3D FEA solution and the approximation obtained by regression is provided in Figure 3.11 (a-d).

Table 3.11- Coefficients $10^5 \times (a_0 - a_4)$ defined in Eq.(3.26)

a_0	a_1	a_2	a_3	a_4
-145.1	29.00	49.13	21.04	2.005

Table 3.12- Coefficients $10^5 \times (b_0 - b_4)$ used in Eq.(3.27)

b_0	b_1	b_2	b_3	b_4
-59.23	0	-40.42	13.10	-1.110

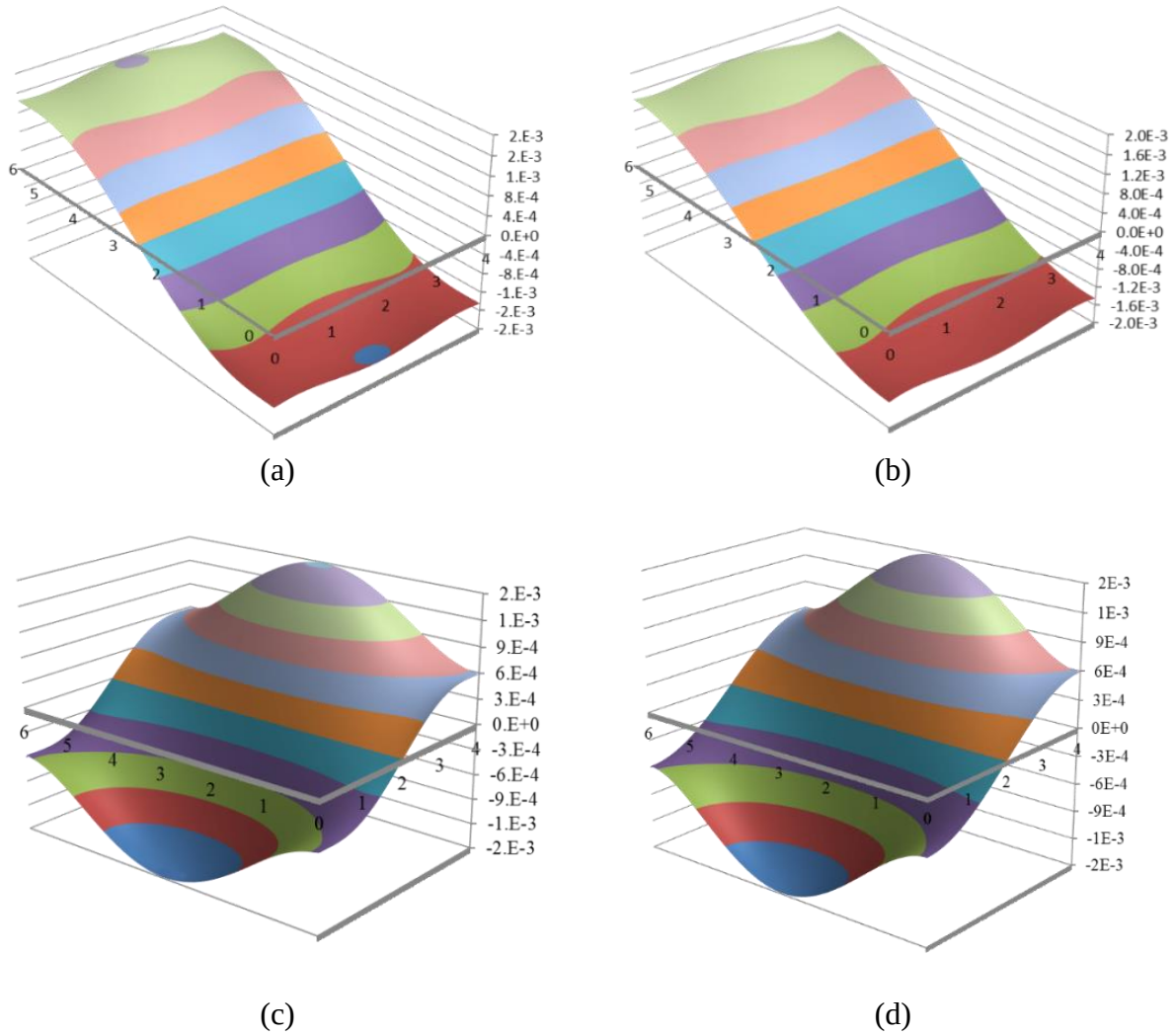


Figure 3.11- In-plane buckling displacement functions based on, (a) Abaqus (u_b), (b) approximation (u_b), (c) Abaqus (v_b), (d) approximation ($\tilde{v}_b$)

3.6.4.3 Buckling transverse displacement function

The nodal transverse buckling displacements w_b based on the first buckling mode as predicted by the FEA solution are extracted (Figure 3.12 (a)). The displacements are then visualized to provide insight on proposing suitable approximating displacement function w_b with single degree of freedom which satisfies all the boundary conditions applied on the plate. The following function was deemed appropriate.

$$w_b = A_b \sin(\pi x/a) \sin(\pi y/b) \quad (3.29)$$

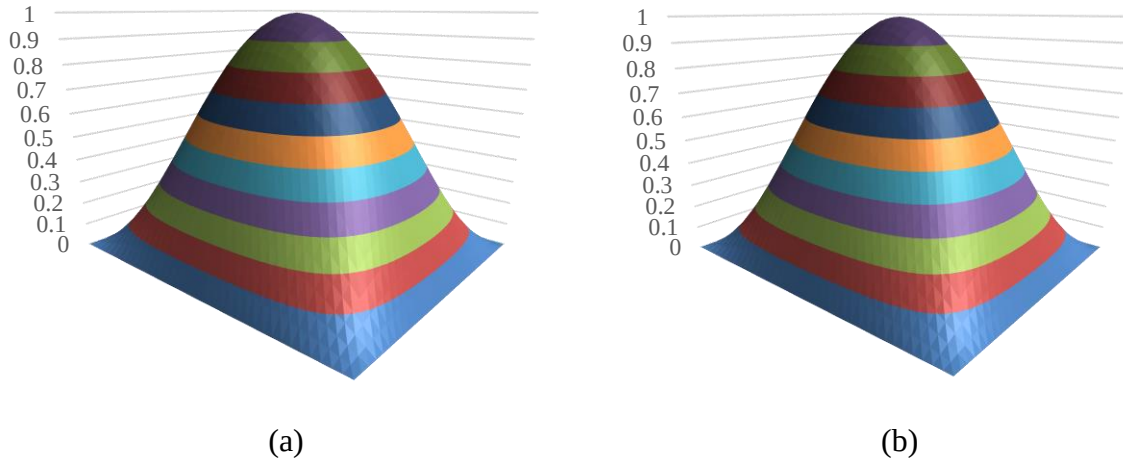


Figure 3.12- Transverse displacement function based on (a) w_b Abaqus (w_b) - (b) approximation (w_b)

3.6.5 Comparison of results

From the displacement functions u_b , $\tilde{v}_b$ and w_b proposed in Eqs. (3.25) a,b and (3.29), by substituting into the variational expression developed in Chapter 2 (Section 2.4.9) and setting the variation of second variation of total potential energy to zero, the critical load is determined. The buckling load as predicted by the energy solution and that based on the Abaqus 3D model are provided in Table 3.13. It is observed that the energy solution overestimate the buckling pressure of the system only by 1.6%, i.e., excellent agreement is obtained. The proximity of buckling load prediction suggests the validity of the variational expression developed for the problem.

Table 3.13- Comparison of critical load obtained from 3D FEA and Energy solution, Example 4

3D FEA (MPa)	Energy formulation (MPa)	Percentage Difference
$P_{xy,cr} = 178.9$	$P_{xy,cr} = 181.8$	1.6 %

3.7 Example 5- Two-plate cylindrical bending problem

3.7.1 Statement of the problem

Two plates with dimensions $a \times b \times h = 4m \times 0.5m \times 0.2m$ (Figure 3.13) are bonded together through a soft layer with thickness $0.1m$. The composite system is subjected to normal pressure

$P_{yy} = 1 \text{ KPa}$. Modulus of elasticity and Poisson's ratio of plates are $E_s = 200 \text{ GPa}$, $\nu_s = 0.3$, and those of soft layer are $E_a = 0.02 \text{ GPa}$ and $\nu_a = 0.3$, respectively. The edges $O_1O_4O_5O_8$ and $O_2O_3O_6O_7$ are simply supported against movement along the z direction. It is required to find the critical loading λP_{yy} at which the composite system would buckle out of its own plate.

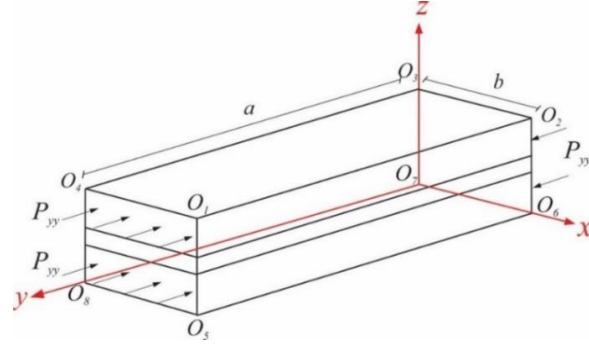


Figure 3.13- Two plates bonded together through a soft layer under uniaxial normal pressure (cylindrical bending)

3.7.2 Details of the Abaqus 3D Model

3.7.2.1 Boundary conditions

For the edge of an arbitrary surface $M_1M_6M_5M_{10}$ normal to the x axis (Figure 3.14) the following constrains are applied

$$v_{M_1} = -v_{M_6}, v_{M_2} = -v_{M_7}, v_{M_3} = -v_{M_8}, v_{M_4} = -v_{M_9}, v_{M_5} = -v_{M_{10}} \quad (3.30)$$

The above constrains are applied for all the edges around the plate. All the movements along x direction are restrained to satisfy cylindrical bending condition.

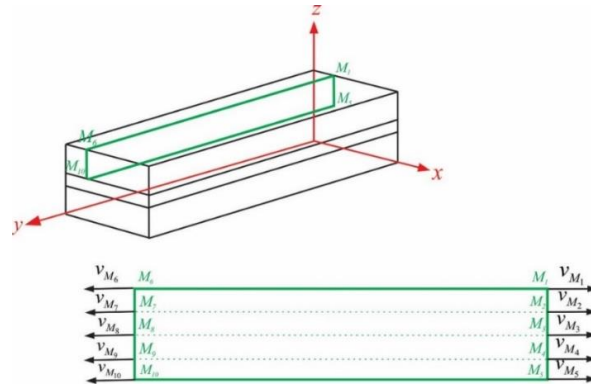


Figure 3.14- Two plates bonded together through a soft layer and in-plane displacements of an arbitrary surface

3.7.3 Mesh study

Table 3.14 provides the results of a mesh sensitivity study for the 3D FEA in Abaqus. Four elements are taken across each plate thickness and one element is taken across the adhesive layer. Along the plate sides, meshes based on 24×100 , 24×200 , 24×300 , 24×400 -element and 24×600 are investigated. The solution based on 24×400 -element mesh is observed to be in agreement with that based on the 24×600 -element solution, within four significant digits. Thus, convergence is deemed to be achieved for the 24×400 -element mesh.

Table 3.14- Mesh study for 3D FEA, Example 5

Number of subdivisions	Critical load (MPa)
24×100	451.0
24×200	445.9
24×300	445.0
24×400	444.7
24×600	444.6

3.7.4 Developing approximate functions

Under the input reference loads, the nodal displacements based on the static analysis are extracted and mathematically described through simple functions (Section 3.7.4.1). Also, the nodal displacements based on the first buckling mode as predicted by the FEA solution are extracted (Sections 3.7.4.2 and 3.7.4.3). The displacements are then visualized to provide insight on proposing suitable approximating displacement functions with a few unknown degrees of freedom. The unknown degrees of freedom are then determined by minimizing the summation of square of the differences between the Abaqus buckling displacements and the proposed displacement function.

3.7.4.1 Pre-buckling in plane displacement functions

The pre-buckling displacements under reference load $P_{yy} = 1 \text{ KPa}$ were obtained from the static analysis of the plate in 3D FEA in Abaqus and were found to be exactly described by the functions

$$v_p = -\alpha(a/2 - y) \quad (3.31)$$

in which $\alpha = 0.46 \times 10^{-8}$, $a = 4\text{ m}$.

3.7.4.2 Buckling in-plane displacement functions

The nodal in-plane displacement based on the first buckling mode v_b as predicted by the FEA solution is extracted (Figure 3.15 (a)). The displacements are then visualized to provide insight on proposing suitable approximating displacement function $\tilde{v}_b$ with single degree of freedom which satisfies all the boundary conditions applied on the plate.

$$\tilde{v}_b = A_9 f_{11}(x, y) \quad (3.32)$$

Where

$$f_{11}(x, y) = \cos\left(\frac{\pi y}{a}\right) \quad (3.33)$$

where for the present problem $a = 4\text{ m}$.

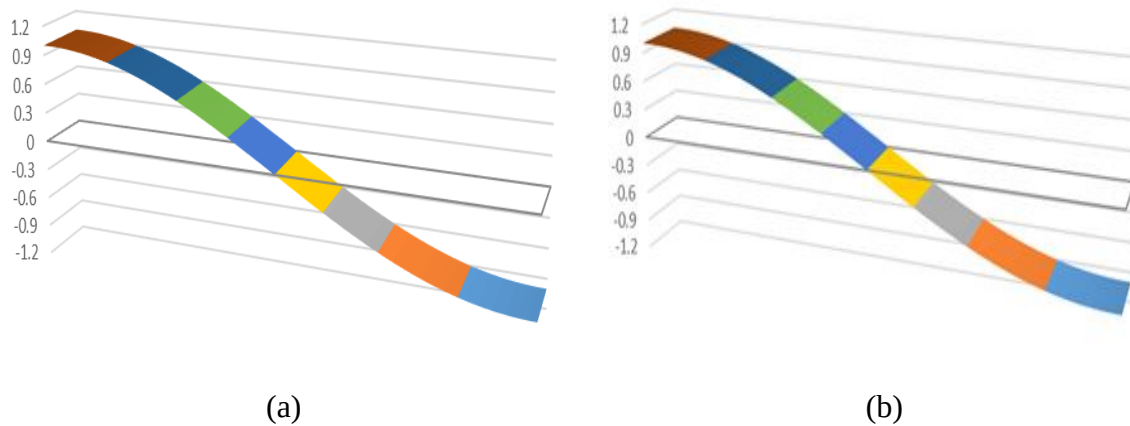


Figure 3.15- In-plane buckling displacement functions based on, (a) Abaqus (v_b), (b) approximation ($\tilde{v}_b$)

3.7.4.3 Buckling transverse displacement function

The nodal transverse buckling displacements based on the first buckling mode w_b as predicted by the FEA solution are extracted (Figure 3.16 (a)). The displacements are then visualized to provide insight on proposing suitable approximating displacement function w_b with single degree of freedom which satisfies all the boundary conditions applied on the plate.

$$w_b = A_{10} f_{12}(x, y) \quad (3.34)$$

where

$$f_{12}(x, y) = \sin(\pi y/a) \quad (3.35)$$

in which $a = 4 \text{ m}$.

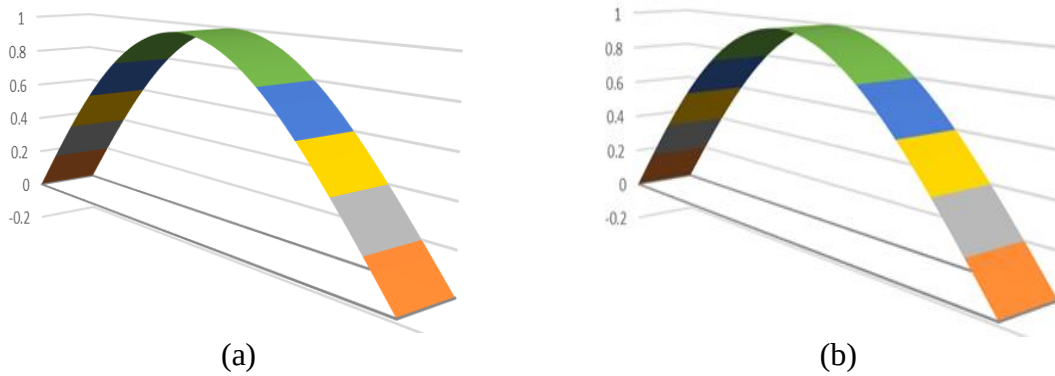


Figure 3.16- Transverse displacement function based on (a) w_b Abaqus (w_b) - (b) approximation (w_b)

3.7.5 Comparison of results

From the displacement functions $\tilde{v}_b$ and w_b proposed in Eqs.(3.32) and (3.34), by substituting into the variational expression developed in Chapter 2 (Section 2.4.9) and setting the variation of second variation of total potential energy to zero, the critical load is determined.

The buckling load as predicted by the energy solution and that based on the Abaqus 3D model are provided in Table 3.15. It is observed that the energy solution overestimates the buckling pressure of the system only by 0.92%, another indication of the accuracy of the variational expression developed.

Table 3.15- Comparison of critical load between 3D FEA and Energy solution for Example 5

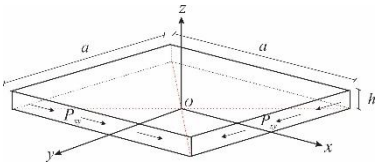
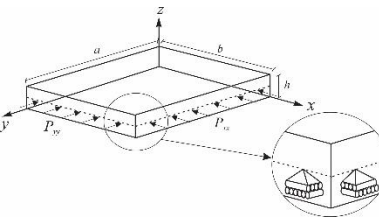
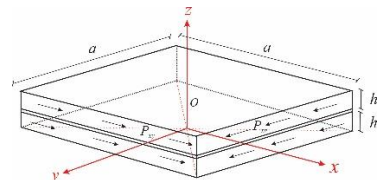
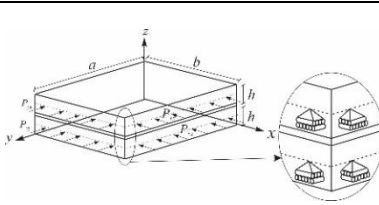
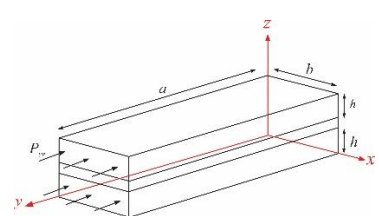
3D FEA (MPa)	Energy formulation (MPa)	Percentage difference
$P_{yy,cr} = 444.7$	$P_{yy,cr} = 448.8$	0.92 %

3.8 Summary and conclusions

This chapter has provided an assessment of the validity of the total potential energy expressions developed in Chapter 2 for a single plate and for two steel plates bonded together through an adhesive layer. Five examples have been investigated. Table 3.16 provides a summary of problems

and the results. All buckling predictions were observed to be between 1-5% higher than those based on the 3D FEA solutions, suggesting the validity of the variational principles developed in Chapter 2. The validated variational principles can thus be adopted to develop finite element formulations for the problem. This task will be developed in Chapter 4.

Table 3.16- Summary of examples considered in Chapter 3

Examples	Dimensions of plate (m)	Critical pressure based on 3D FEA (MPa)	Critical pressure based on Energy solution (MPa)	Percentage difference
	$4 \times 4 \times 0.1$ soft layer thickness 25 mm	1699	1743	2.52 %
	$6 \times 4 \times 0.1$ soft layer thickness 25 mm	156.4	163.0	4.01 %
	$4 \times 4 \times 0.1$ soft layer thickness 25 mm	1644	1671	1.61 %
	$6 \times 4 \times 0.1$ soft layer thickness 25 mm	178.9	181.8	1.60 %
	$4 \times 0.5 \times 0.2$ soft layer thickness 100 mm	444.7	448.8	0.92 %

4 Finite Element Formulation

4.1 General

The objective of this chapter is to develop a general finite element formulation for the buckling of a steel plate bonded to two GFRP plates of equal thickness through adhesive layers. The composite system is assumed to be subjected to biaxial pressures and shear that are uniform along the x and y coordinates. Figure 4.1 provides an arbitrary edge line of the composite system and pre-buckling displacements of each plate. The applied in-plane biaxial pressures and shear are assumed to induce identical pre-buckling displacements (i.e., $u_{1p} = u_{3p} = u_{5p}$ and $v_{1p} = v_{3p} = v_{5p}$) between all layers. The critical pressure-shear combination which induces buckling of the composite system is sought.

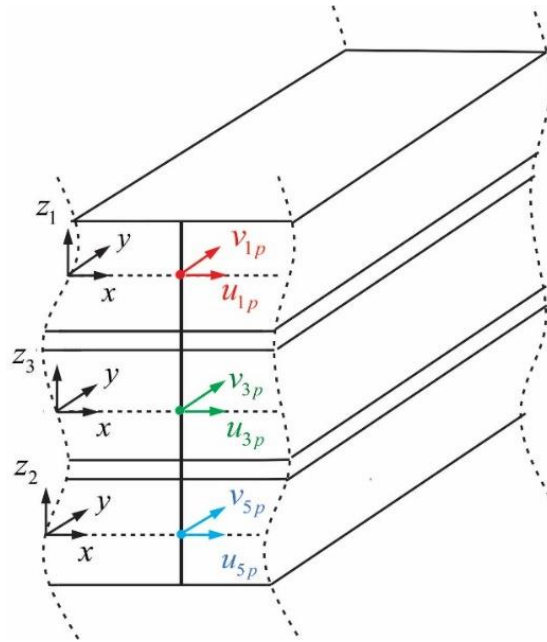


Figure 4.1- An arbitrary edge line of the composite system and the pre-buckling displacements of the plates

4.2 Pre-buckling Strains

Since the displacement fields within the layers are postulated to be equal, the corresponding pre-buckling strains (i.e., $\epsilon_{x,p}$, $\epsilon_{y,p}$ and $\gamma_{xy,p}$) within the plate are also equal and uniform within the thickness of the composite system.

4.3 Buckling analysis

The second variation of the total potential energy developed in Eq. 2.41 is expressed in terms of nodal displacements through interpolation functions. A four noded non-conforming element is developed (Section 4.3.2) with five degrees of freedom per node (Section 4.3.3). Bilinear interpolation is adopted to interpolate the in-plane displacement fields (Section 4.3.4) and incomplete cubic interpolation functions are adopted to relate the transverse displacement within the element to the nodal displacements and rotations (Section 4.3.5), and the neutral stability condition is then expressed by setting to zero the variation of the second variation of the discretized form of the total potential energy (Section 4.3.6).

4.3.1 Second variation of total potential energy

The second variation of total potential energy as given by Eq. 2.41 takes the form

$$\begin{aligned} \bar{\bar{\Pi}}_t = \iint \{ & \bar{A}_1 \bar{u}_{1b}^2 + \bar{A}_2 \bar{u}_{1b}'^2 + \bar{A}_3 \bar{u}_{1b}^2 + \bar{A}_4 \bar{v}_{1b}^2 + \bar{A}_5 \bar{v}_{1b}'^2 + \bar{A}_6 \bar{v}_{1b}^2 + \bar{A}_7 \bar{v}_{1b}'^2 \\ & + \bar{A}_8 \bar{w}_b'^2 + \bar{A}_9 \bar{w}_b^2 + \bar{A}_{10} \bar{w}_b'^2 + \bar{A}_{11} \bar{w}_b^2 + (\bar{A}_6 + \bar{A}_7 + \bar{A}_8) \bar{w}_b'^2 \\ & + \bar{A}_9 \bar{w}_b'' \bar{w}_b + \bar{A}_{10} \bar{u}_{1b} \bar{w}_b' + \bar{A}_{11} \bar{v}_{1b} \bar{w}_b + \bar{A}_{12} \bar{u}_{1b}' \bar{v}_{1b} + \bar{A}_{13} \bar{u}_{1b} \bar{v}_{1b}' \\ & + (\bar{A}_{13} + \bar{A}_{14}) \bar{u}_{1b} \bar{w}_b' + (\bar{A}_{13} + \bar{A}_{14}) \bar{v}_{1b} \bar{w}_b' \\ & + \lambda (\bar{A}_{15} \bar{w}_b'^2 + \bar{A}_{16} \bar{w}_b^2 + \bar{A}_{17} \bar{w}_b \bar{w}_b') \} dx dy \end{aligned} \quad (4.1)$$

in which coefficients $\bar{A}_1$ to $\bar{A}_{17}$ have been defined in Table 2.5 in terms of the physical and mechanical properties of the composite system.

4.3.2 Element Geometry

A shell element (Figure 4.2), with four nodes and dimension $2l_x \times 2l_y$ is developed. The origin of right-handed coordinate system is assumed at the centroid of the shell element.

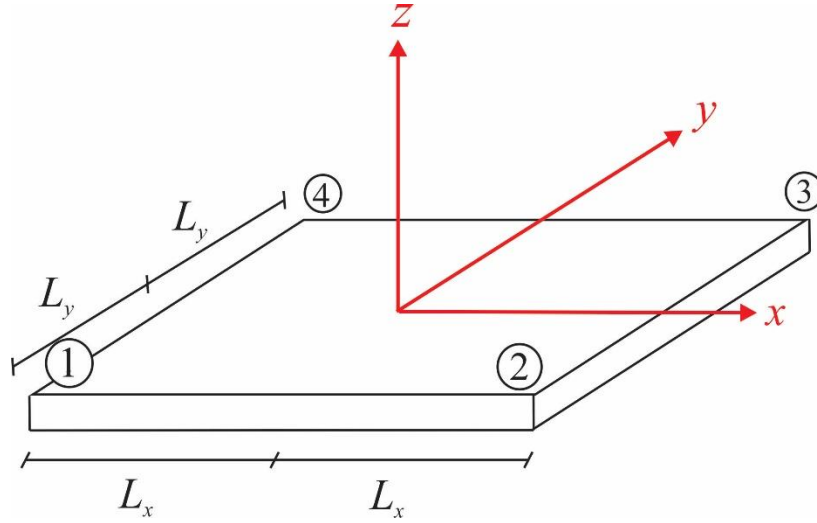


Figure 4.2- Geometry of developed shell element

4.3.3 Nodal Degrees of Freedom

Based on the assumptions made in Chapter 2, both GFRP plates have equal and opposite in-plane buckling displacement while the transverse buckling displacements and the rotations about x and y direction are identical for all layers of the composite system. Thus, it is assumed that shell element has five degrees of freedom at each node $(u_b^j, v_b^j, w_b^j, w_b'^j, \dot{w}_b^j)$, $j=1,2,3,4$ and a total of twenty independent degrees of freedom (Figure 4.3).

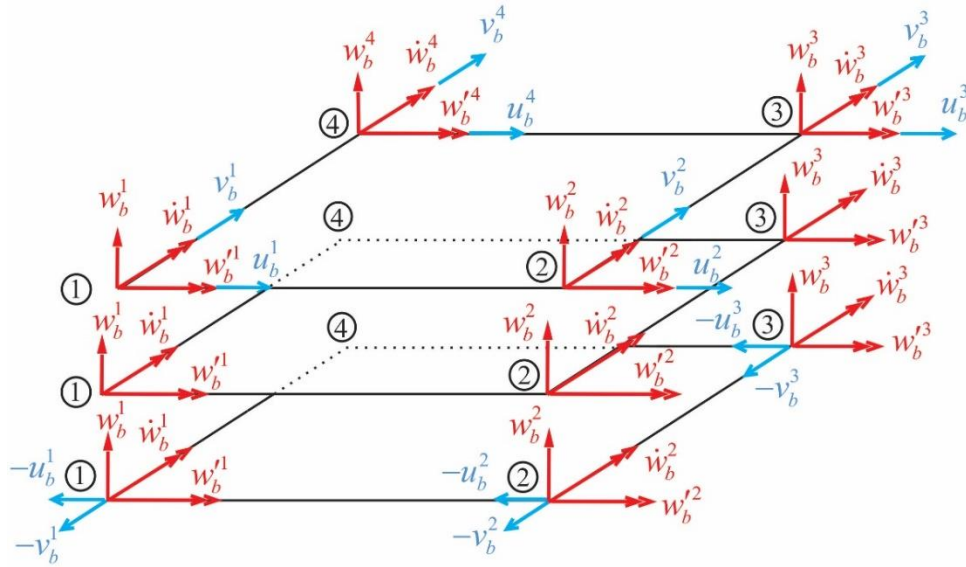


Figure 4.3- 20 independent degrees of freedom of developed shell element

4.3.4 Interpolation of in-plane buckling displacements for the GFRP plates

In-plane buckling displacement fields $\bar{u}_b, \bar{v}_b$ for the GFRP plates are written in terms of nodal displacements and shape functions

$$\bar{u}_b = \langle L \rangle_{1 \times 4} \{u_b^j\}_{4 \times 1} = \langle L_1 \quad L_2 \quad L_3 \quad L_4 \rangle \begin{Bmatrix} u_b^1 \\ u_b^2 \\ u_b^3 \\ u_b^4 \end{Bmatrix} \quad (4.2)$$

$$\bar{v}_b = \langle L \rangle_{1 \times 4} \{v_b^j\}_{4 \times 1} = \langle L_1 \quad L_2 \quad L_3 \quad L_4 \rangle \begin{Bmatrix} v_b^1 \\ v_b^2 \\ v_b^3 \\ v_b^4 \end{Bmatrix} \quad (4.3)$$

in which L_i is the bilinear shape functions as

$$\begin{aligned} L_1 &= (1/4)(1 - \xi - \eta + \xi\eta) \\ L_2 &= (1/4)(1 + \xi - \eta - \xi\eta) \\ L_3 &= (1/4)(1 + \xi + \eta + \xi\eta) \\ L_4 &= (1/4)(1 - \xi + \eta - \xi\eta) \end{aligned} \quad (4.4)$$

and $\xi = \frac{x}{L_x}$, $\eta = \frac{y}{L_y}$.

4.3.5 Interpolation of Transverse Displacements

According to the Kirchhoff plate bending theory, the transverse displacement $\bar{w}_b$ along the z direction and associated rotations $\dot{w}_b'$ and $\dot{w}_b$ are expressed in terms of nodal displacements as

$$\bar{w}_b = \langle H \rangle_{1 \times 12} \{w_b^j\}_{12 \times 1} \quad (4.5)$$

where the terms of $\{w_b^j\}_{12 \times 1}$ related to the rotations are

$$w_b^{(3j-1)} = w_b'^{(3j-2)} \quad , \quad w_b^{3j} = \dot{w}_b^{(3j-2)} \quad j = 1, 2, 3, 4 \quad (4.6)$$

and $\langle H \rangle_{1 \times 12}$ denotes the shape function matrix as follows

$$\begin{aligned}
H_1 &= (1/16)(4 - 6\xi - 6\eta + 7\xi\eta + 2\xi^3 + 2\eta^3 - \xi^3\eta - 2\xi\eta^3) \\
H_4 &= (1/16)(4 + 6\xi - 6\eta - 7\xi\eta - 2\xi^3 + 2\eta^3 + \xi^3\eta + 2\xi\eta^3) \\
H_7 &= (1/16)(4 + 6\xi + 6\eta + 7\xi\eta - 2\xi^3 - 2\eta^3 - \xi^3\eta - 2\xi\eta^3) \\
H_{10} &= (1/16)(4 - 6\xi + 6\eta - 7\xi\eta + 2\xi^3 - 2\eta^3 + \xi^3\eta + 2\xi\eta^3)
\end{aligned} \tag{4.7}$$

where $\xi = \frac{x}{L_x}$ and $\eta = \frac{y}{L_y}$ and the remaining shape functions are

$$H_{(3j-1)} = H'_{(3j-2)} \quad , \quad H_{3j} = \dot{H}_{(3j-2)} \quad j = 1, 2, 3, 4 \tag{4.8}$$

4.3.6 Condition of neutral stability

From Eqs. (4.2), (4.3) and (4.5), by substituting into Eq.(4.1), one obtains

$$\begin{aligned}
\bar{\Pi}_t = \iint_A & \left\{ \bar{A}_1 \langle u_b^j \rangle^T \{L\} \langle L \rangle^T \{u_b^j\} + \bar{A}_2 \langle u_b^j \rangle^T \{L'\} \langle L' \rangle^T \{u_b^j\} + \bar{A}_3 \langle u_b^j \rangle^T \{\dot{L}\} \langle \dot{L} \rangle^T \{u_b^j\} \right. \\
& + \bar{A}_1 \langle v_b^j \rangle^T \{L\} \langle L \rangle^T \{v_b^j\} + \bar{A}_2 \langle v_b^j \rangle^T \{\dot{L}\} \langle \dot{L} \rangle^T \{v_b^j\} + \bar{A}_3 \langle v_b^j \rangle^T \{L'\} \langle L' \rangle^T \{v_b^j\} \\
& + \bar{A}_4 \langle w_b^j \rangle^T \{H_w'\} \langle H_w' \rangle^T \{w_b^j\} + \bar{A}_4 \langle w_b^j \rangle^T \{\dot{H}_w\} \langle \dot{H}_w \rangle^T \{w_b^j\} \\
& + \bar{A}_5 \langle w_b^j \rangle^T \{H_w''\} \langle H_w'' \rangle^T \{w_b^j\} + \bar{A}_5 \langle w_b^j \rangle^T \{\ddot{H}_w\} \langle \ddot{H}_w \rangle^T \{w_b^j\} \\
& + (\bar{A}_6 + \bar{A}_7 + \bar{A}_8) \langle w_b^j \rangle^T \{\dot{H}_w'\} \langle \dot{H}_w' \rangle^T \{w_b^j\} + \bar{A}_9 \langle w_b^j \rangle^T \{H_w''\} \langle \ddot{H}_w \rangle^T \{w_b^j\} \\
& + \bar{A}_{10} \langle u_b^j \rangle^T \{L\} \langle H_w' \rangle^T \{w_b^j\} + \bar{A}_{10} \langle v_b^j \rangle^T \{L\} \langle \dot{H}_w \rangle^T \{w_b^j\} \\
& + \bar{A}_{11} \langle u_b^j \rangle^T \{L'\} \langle \dot{L} \rangle^T \{V_n\} + \bar{A}_{12} \langle u_b^j \rangle^T \{\dot{L}\} \langle L' \rangle^T \{v_b^j\} \\
& + (\bar{A}_{13} + \bar{A}_{14}) \langle u_b^j \rangle^T \{\dot{L}\} \langle \dot{H}_w' \rangle^T \{w_b^j\} + (\bar{A}_{13} + \bar{A}_{14}) \langle v_b^j \rangle^T \{L'\} \langle \dot{H}_w' \rangle^T \{w_b^j\} \\
& \left. + \lambda \left(\bar{A}_{15} \langle w_b^j \rangle^T \{H_w'\} \langle H_w' \rangle^T \{w_b^j\} + \bar{A}_{16} \langle w_b^j \rangle^T \{\dot{H}_w\} \langle \dot{H}_w \rangle^T \{w_b^j\} + \bar{A}_{17} \langle w_b^j \rangle^T \{\dot{H}_w\} \langle H_w' \rangle^T \{w_b^j\} \right) \right\} dx dy \tag{4.9}
\end{aligned}$$

Taking variation of the second variation of total potential energy developed in Eq. (4.9) with respect to the nodal displacements results in

$$\begin{aligned}
\overline{(\Pi_t)} = & \iint_A \left\{ 2\bar{A}_1 \overline{\langle u_b^j \rangle^T} \{L\} \langle L \rangle^T \{u_b^j\} + 2\bar{A}_2 \overline{\langle u_b^j \rangle^T} \{L'\} \langle L' \rangle^T \{u_b^j\} + 2\bar{A}_3 \overline{\langle u_b^j \rangle^T} \{\dot{L}\} \langle \dot{L} \rangle^T \{u_b^j\} \right. \\
& + 2\bar{A}_4 \overline{\langle v_b^j \rangle^T} \{L\} \langle L \rangle^T \{v_b^j\} + 2\bar{A}_2 \overline{\langle v_b^j \rangle^T} \{\dot{L}\} \langle \dot{L} \rangle^T \{v_b^j\} + 2\bar{A}_3 \overline{\langle v_b^j \rangle^T} \{L'\} \langle L' \rangle^T \{v_b^j\} \\
& + 2\bar{A}_4 \overline{\langle w_b^j \rangle^T} \{H_w'\} \langle H_w' \rangle^T \{w_b^j\} + 2\bar{A}_4 \overline{\langle w_b^j \rangle^T} \{\dot{H}_w\} \langle \dot{H}_w \rangle^T \{w_b^j\} \\
& + 2\bar{A}_5 \overline{\langle w_b^j \rangle^T} \{H_w''\} \langle H_w'' \rangle^T \{w_b^j\} + 2\bar{A}_5 \overline{\langle w_b^j \rangle^T} \{\ddot{H}_w\} \langle \ddot{H}_w \rangle^T \{w_b^j\} \\
& + 2(\bar{A}_6 + \bar{A}_7 + \bar{A}_8) \overline{\langle w_b^j \rangle^T} \{H_w'\} \langle \dot{H}_w' \rangle^T \{w_b^j\} + 2\bar{A}_9 \overline{\langle w_b^j \rangle^T} \{H_w''\} \langle \ddot{H}_w \rangle^T \{w_b^j\} \\
& + \bar{A}_{10} \overline{\langle u_b^j \rangle^T} \{L\} \langle H_w' \rangle^T \{w_b^j\} + \bar{A}_{10} \overline{\langle w_b^j \rangle^T} \{H_w'\} \langle L \rangle^T \{u_b^j\} \\
& + \bar{A}_{10} \overline{\langle v_b^j \rangle^T} \{L\} \langle \dot{H}_w \rangle^T \{w_b^j\} + \bar{A}_{10} \overline{\langle w_b^j \rangle^T} \{\dot{H}_w\} \langle L \rangle^T \{v_b^j\} \\
& + \bar{A}_{11} \overline{\langle u_b^j \rangle^T} \{L'\} \langle \dot{L} \rangle^T \{v_b^j\} + \bar{A}_{11} \overline{\langle v_b^j \rangle^T} \{\dot{L}\} \langle L' \rangle^T \{u_b^j\} \\
& + \bar{A}_{12} \overline{\langle u_b^j \rangle^T} \{\dot{L}\} \langle L' \rangle^T \{v_b^j\} + \bar{A}_{12} \overline{\langle v_b^j \rangle^T} \{L'\} \langle \dot{L} \rangle^T \{u_b^j\} \\
& + (\bar{A}_{13} + \bar{A}_{14}) \overline{\langle u_b^j \rangle^T} \{\dot{L}\} \langle \dot{H}_w' \rangle^T \{w_b^j\} + (\bar{A}_{13} + \bar{A}_{14}) \overline{\langle w_b^j \rangle^T} \{\dot{H}_w'\} \langle \dot{L} \rangle^T \{u_b^j\} \\
& + (\bar{A}_{13} + \bar{A}_{14}) \overline{\langle v_b^j \rangle^T} \{L'\} \langle \dot{H}_w' \rangle^T \{w_b^j\} + (\bar{A}_{13} + \bar{A}_{14}) \overline{\langle w_b^j \rangle^T} \{\dot{H}_w'\} \langle L' \rangle^T \{v_b^j\} \\
& + \lambda \left(2\bar{A}_{15} \overline{\langle w_b^j \rangle^T} \{H_w'\} \langle H_w' \rangle^T \{w_b^j\} + 2\bar{A}_{16} \overline{\langle w_b^j \rangle^T} \{\dot{H}_w\} \langle \dot{H}_w \rangle^T \{w_b^j\} + \bar{A}_{17} \overline{\langle w_b^j \rangle^T} \{\dot{H}_w\} \langle H_w' \rangle^T \{w_b^j\} \right. \\
& \left. \left. + \bar{A}_{17} \overline{\langle w_b^j \rangle^T} \{H_w'\} \langle \dot{H}_w \rangle^T \{w_b^j\} \right) \right\} dx dy
\end{aligned}
\tag{4.10}$$

By setting the variation of second variation of total potential energy to zero and grouping the coefficients of the same field, one obtains

$$\begin{aligned}
\delta \langle u_b^j \rangle & \left([\bar{C}_1]_{4 \times 4} \{u_b^j\}_{4 \times 1} + [\bar{C}_2]_{4 \times 4} \{v_b^j\}_{4 \times 1} + [\bar{C}_3]_{4 \times 12} \{w_b^j\}_{12 \times 1} \right) = 0 \\
\delta \langle v_b^j \rangle & \left([\bar{C}_4]_{4 \times 4} \{u_b^j\}_{4 \times 1} + [\bar{C}_5]_{4 \times 4} \{v_b^j\}_{4 \times 1} + [\bar{C}_6]_{4 \times 12} \{w_b^j\}_{12 \times 1} \right) = 0 \\
\delta \langle w_b^j \rangle & \left([\bar{C}_7]_{12 \times 4} \{u_b^j\}_{4 \times 1} + [\bar{C}_8]_{12 \times 4} \{v_b^j\}_{4 \times 1} + [\bar{C}_9]_{12 \times 12} \{w_b^j\}_{12 \times 1} + \lambda [\bar{D}_1]_{12 \times 12} \{w_b^j\}_{12 \times 1} \right) = 0
\end{aligned}
\tag{4.11}$$

in which the following matrices have been defined by performing integrations with respect to x and y numerically using MATLAB.

$$\begin{aligned}
[\bar{C}_1]_{4 \times 4} &= \iint_A \left(2\bar{A}_1 \{L\} \langle L \rangle + 2\bar{A}_2 \{L'\} \langle L' \rangle + 2\bar{A}_3 \{\dot{L}\} \langle \dot{L} \rangle \right) dx dy \\
[\bar{C}_2]_{4 \times 4} &= \iint_A \left(\bar{A}_{11} \{L'\} \langle \dot{L} \rangle + \bar{A}_{12} \{\dot{L}\} \langle L' \rangle \right) dx dy \\
[\bar{C}_3]_{4 \times 12} &= \iint_A \left[(\bar{A}_{13} + \bar{A}_{14}) \{\dot{L}\} \langle \dot{H}' \rangle + \bar{A}_{10} \{L\} \langle H' \rangle \right] dx dy \\
[\bar{C}_4]_{4 \times 4} &= [\bar{C}_2]_{4 \times 4}^T = \iint_A \left(\bar{A}_{11} \{\dot{L}\} \langle L' \rangle + \bar{A}_{12} \{L'\} \langle \dot{L} \rangle \right) dx dy \\
[\bar{C}_5]_{4 \times 4} &= \iint_A \left(2\bar{A}_1 \{L\} \langle L \rangle + 2\bar{A}_2 \{\dot{L}\} \langle \dot{L} \rangle + 2\bar{A}_3 \{L'\} \langle L' \rangle \right) dx dy \\
[\bar{C}_6]_{4 \times 12} &= \iint_A \left[(\bar{A}_{13} + \bar{A}_{14}) \{L'\} \langle \dot{H}' \rangle + \bar{A}_{10} \{L\} \langle \dot{H} \rangle \right] dx dy \\
[\bar{C}_7]_{12 \times 4} &= [\bar{C}_3]_{4 \times 12}^T = \iint_A \left[(\bar{A}_{13} + \bar{A}_{14}) \{\dot{H}'\} \langle \dot{L} \rangle + \bar{A}_{10} \{H'\} \langle L \rangle \right] dx dy \\
[\bar{C}_8]_{12 \times 4} &= [\bar{C}_6]_{4 \times 12}^T = \iint_A \left(\bar{A}_{13} \{\dot{H}'\} \langle L' \rangle + \bar{A}_{14} \{H'\} \langle L' \rangle + \bar{A}_{10} \{\dot{H}\} \langle L \rangle \right) dx dy \\
[\bar{C}_9]_{12 \times 12} &= \iint_A \left[2\bar{A}_5 \{H''\} \langle H'' \rangle + 2\bar{A}_5 \{\ddot{H}\} \langle \ddot{H} \rangle + (2\bar{A}_6 + 2\bar{A}_7 + 2\bar{A}_8) \{\dot{H}'\} \langle \dot{H}' \rangle \right. \\
&\quad \left. + \bar{A}_9 \{H''\} \langle \ddot{H} \rangle + \bar{A}_9 \{\ddot{H}\} \langle H'' \rangle + 2\bar{A}_4 \{H'\} \langle H' \rangle + 2\bar{A}_4 \{\dot{H}\} \langle \dot{H} \rangle \right] dx dy \\
[\bar{D}_1]_{12 \times 12} &= \iint_A \left(2\bar{A}_{15} \{H'\} \langle H' \rangle + \bar{A}_{17} \{H'\} \langle \dot{H} \rangle + \bar{A}_{17} \{\dot{H}\} \langle H' \rangle + 2\bar{A}_{16} \{\dot{H}\} \langle \dot{H} \rangle \right) dx dy
\end{aligned}$$

By consolidating Eq.(4.11) into a single matrix equation, one recovers the following relation

$$\left([K]_{20 \times 20} + \lambda [K_g]_{20 \times 20} \right) \begin{Bmatrix} \{U_n\}_{4 \times 1} \\ \{V_n\}_{4 \times 1} \\ \{W_n\}_{12 \times 1} \end{Bmatrix}_{20 \times 1} = 0 \quad (4.12)$$

where $[K]_{20 \times 20}$ is the stiffness matrix and $[K_g]_{20 \times 20}$ is the geometric stiffness matrix defined as

$$[K]_{20 \times 20} = \begin{bmatrix} [\bar{C}_1]_{4 \times 4} & [\bar{C}_2]_{4 \times 4} & [\bar{C}_3]_{4 \times 12} \\ [\bar{C}_4]_{4 \times 4} & [\bar{C}_5]_{4 \times 4} & [\bar{C}_6]_{4 \times 12} \\ [\bar{C}_7]_{12 \times 4} & [\bar{C}_8]_{12 \times 4} & [\bar{C}_9]_{12 \times 12} \end{bmatrix} \quad [K_g]_{20 \times 20} = \begin{bmatrix} [0]_{4 \times 4} & [0]_{4 \times 4} & [0]_{4 \times 12} \\ [0]_{4 \times 4} & [0]_{4 \times 4} & [0]_{4 \times 12} \\ [0]_{12 \times 4} & [0]_{12 \times 4} & [\bar{D}_1]_{12 \times 12} \end{bmatrix}$$

4.4 Closing Remarks

The above element stiffness relations are assembled to form the structure stiffness matrix, and the relevant boundary equations are enforced and the resulting eigenvalue problem is solved for the unknown load multiplier λ . The above procedure has been implemented under MATLAB Appendix A3). Comparison of the results obtained from the solution developed in this chapter and those based on 3D FEA in Abaqus which will be presented in Chapter 5.

5 VERIFICATION AND PARAMETRIC STUDY

5.1 General

This chapter provides an assessment of the validity of the finite element formulation developed in Chapter 4 for composite plates. Towards this goal, two examples are considered in Sections 5.2 and 5.3. These are (1) a steel-GFRP flange fixed at one end and free at the other three edges subjected to uniaxial normal pressure, (2) a steel-GFRP plate fully fixed at all four edges and subjected to pure shear. Section 5.4 subsequently provides the results of a parametric study aimed at investigating the effect of GFRP thickness, adhesive thickness, and shear modulus of adhesive on the buckling capacity of the composite system introduced in Examples 1 and 2.

5.2 Example 1- Reinforced flange

5.2.1 Statement of the problem

The reference member (Figure 5.1) is W 310×74 steel beam ($E_s = 200\text{ GPa}$, $\nu_s = 0.3$, $F_y = 350\text{ MPa}$). A 0.3075 m long segment is considered to assess the local buckling strength of the flange. The beam flange meets class 3 requirements (Handbook of Steel Construction, CSA S16-09) i.e.,

$$\frac{b_f}{2t_s} = \frac{205}{2 \times 16.3} = 6.29 < \frac{200}{\sqrt{F_y}} = \frac{200}{\sqrt{350}} = 10.69 \quad (6.1)$$

which signifies that the strength of the flange is dictated by its yield strength rather than its local buckling strength. Due to corrosion, it is assumed that the flange thickness has been reduced to almost half of the initial thickness (i.e., $t_{sr} = 8\text{ mm}$). The flange of the corroded section is then categorized as a Class 4, i.e.,

$$\frac{b_t}{2t_{sr}} = \frac{205}{2 \times 8.0} = 12.81 > \frac{200}{\sqrt{350}} = 10.69 \quad (6.2)$$

signifying that the strength of the corroded flange is now governed by local buckling. Two GFRP plates are proposed to strengthen the corroded flange and restore its local buckling capacity. GFRP

plate dimensions are $b_f/2 \times L = 0.1025\text{ m} \times 0.3075\text{ m}$ and thicknesses $t_g = 7\text{ mm}$ with typical material properties $E_g = 42\text{ GPa}$, $\nu_g = 0.3$ (Section 1.2.1.2, Table 1.2). Both plates are bonded to the steel flange through two adhesive layers with thicknesses $t_a = 1\text{ mm}$ and shear modulus $G_a = 200\text{ MPa}$, which falls in the practical range of shear modulus values (Section 1.2.1.3, Table 1.3) of $G_a = 100 - 400\text{ MPa}$

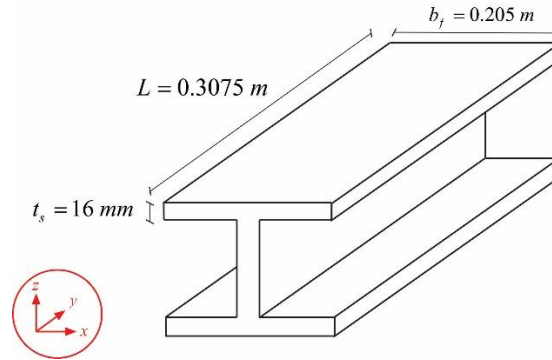


Figure 5.1- A W 310×74 steel beam (steel 350W)

The resulting composite flange is assumed to be subjected to a total reference normal pressure $P_{yy} = 1\text{ KPa}$. In line with the assumptions and formulation introduced in Chapter 4, all three plates are assumed to undergo identical pre-buckling displacements. It is required to find the critical pressure λP_{yy} at which the composite flange would undergo local buckling. The problem is to be solved using the finite element formulation developed in Chapter 4 and a 3D finite element formulation under Abaqus to provide a basis to assess the solution developed in the present study.

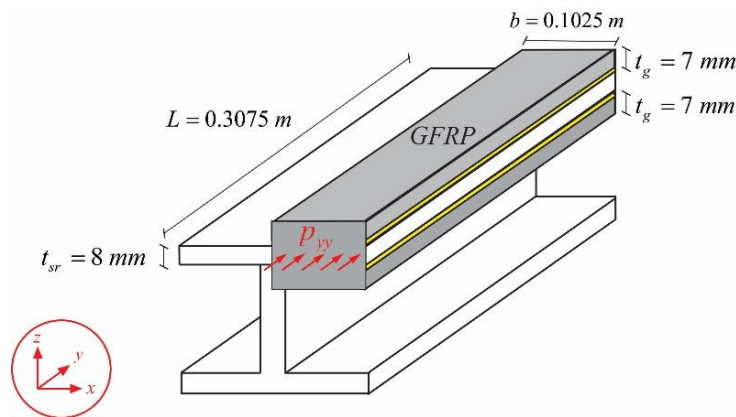


Figure 5.2- Application of GFRP plate for reinforcing flange, Example 1

5.2.2 Solution based on Present Study

5.2.2.1 Mesh study

Table 5.1 provides a mesh study for the present finite element solution. The first two columns of Table 5.1 provide the number of subdivision along x and y direction (n_x and n_y), the third column is total number of degrees of freedom, and the forth column provides the critical pressure obtained by the present study. As observed, the solution based on 16×48 -element is observed to be in agreement with that based on the 15×45 -element solution within four significant digits. Thus, convergence is deemed to be achieved for the 15×45 -element mesh.

Table 5.1- Mesh study for the present finite element solution for reinforced flange

Number of elements		# of DOFs	Critical pressure $P_{yy(Cr)}$ (MPa)
n_x	n_y		
5	15	480	1567
6	18	665	1566
7	21	880	1565
8	24	1125	1565
9	27	1400	1564
10	30	1705	1564
15	45	3680	1564
16	48	4165	1564

5.2.3 Abaqus 3D Model

5.2.3.1 Boundary conditions

For the edge of an arbitrary surface $M_1M_6M_5M_{10}$ normal to the x axis (Figure 5.3), the following constrains are applied

$$v_{M_1} = -v_{M_6}, v_{M_2} = -v_{M_7}, v_{M_3} = -v_{M_8}, v_{M_4} = -v_{M_9}, v_{M_5} = -v_{M_{10}} \quad (6.3)$$

These constrains are applied for all the surfaces parallel to surface $M_1M_6M_5M_{10}$. Besides, all the nodes of surface $O_1O_4O_5O_8$ are restrained along x direction and from rotation about the x and y axws. No constraints are applied on surface $O_2O_3O_6O_7$. All constrains were applied using the *EQUATION feature in Abaqus.

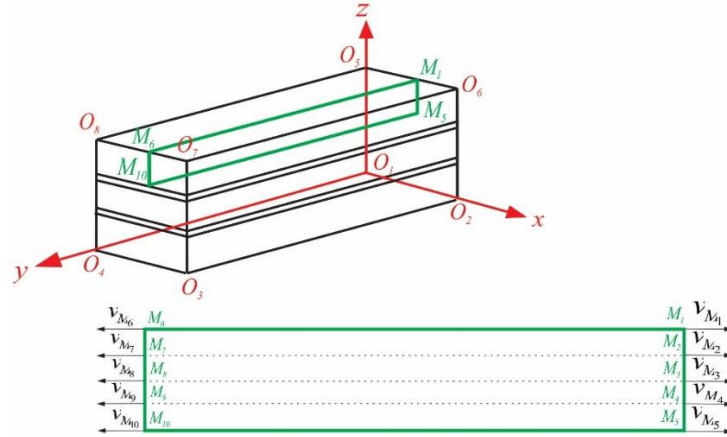


Figure 5.3 – The reinforced flange of Example 1 and in-plane displacements of the edges of an arbitrary surface

5.2.3.2 Mesh study

Table 5.2 provides the results of a mesh sensitivity study for the 3D FEA in Abaqus. The first two columns of Table 5.2 provide number of subdivision along x and y direction (n_x and n_y), the third column is total number of degrees of freedom, and the forth column provides the critical pressure obtained from 3D FEA in Abaqus. Four elements are taken across each plate thickness and one element is taken across the adhesive layer. Along the plate sides, meshes based on 20×60 , 40×120 , 50×150 , 70×210 , 80×240 , 100×300 and 160×480 elements are investigated. The solution based on the 160×480 -element mesh is observed to be in agreement with that based on the 100×300 -element solution, within four significant digits. Thus, convergence is deemed to be achieved for the 100×300 -element mesh.

Table 5.2- Mesh study for 3D FEA in Abaqus for reinforced flange

Number of elements		# of DOFs	Critical pressure $P_{yy(Cr)}$ (MPa)
n_x	n_y		
20	60	403200	2026
40	120	1612800	1950
50	150	2520000	1931
70	210	4939200	1853
80	240	6451200	1841
100	300	10080000	1839
160	480	25804800	1839

5.2.4 Comparison of result

The critical pressure obtained from the present finite element solution and that based on the 3D FEA in Abaqus are found to be in agreement within 15 % (Table 5.3). Unlike the energy based solution presented in Chapter 3, the present finite element solution is observed to underestimate the buckling capacity of the composite system. This is a result of the fact that the FEA developed in the present study is non-conforming, i.e., the gradient of the transverse displacements in the direction normal to the element edges is discontinuous, resulting in a more flexible representation of stiffness of the composite system. Figure 5.4 depicts the outcome of the mesh study both present finite element formulation and for the 3D FEA solution in Abaqus. By comparing the number of degrees of freedom of the converged mesh based the present solution (Table 5.1) and those of the converged 3D FEA model in Abaqus (Table 5.2), the computational effort needed under the present solution is four orders less than that required by the Abaqus model.

Table 5.3- Comparison of critical pressure values $P_{yy(cr)}$ (MPa)

Method	3D FEA (Abaqus)	Present Solution	Percentage difference
Mesh	100×300	15×45	
$P_{yy(Cr)}$ (MPa)	1839	1564	15%

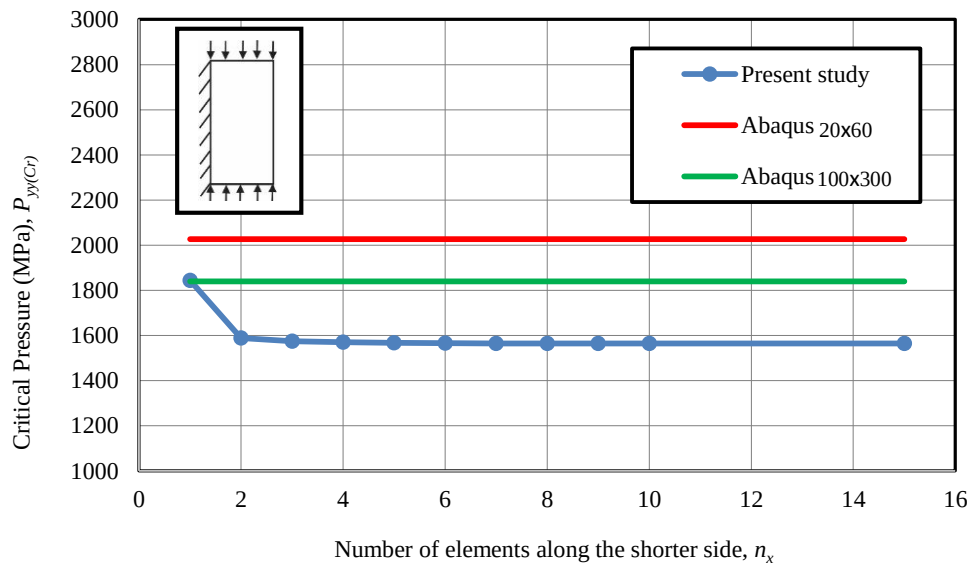


Figure 5.4- Comparison of the mesh study for the present finite element solution and 3D FEA in Abaqus

5.3 Example 2- Fully fixed square plate under pure shear

5.3.1 Statement of the problem

An existing steel plate ($E_s = 200 \text{ GPa}$ and $\nu_s = 0.3$) with dimension $a \times a = 2 \text{ m} \times 2 \text{ m}$ and thickness $t_s = 14 \text{ mm}$ is to be reinforced with two GFRP plates ($E_g = 42 \text{ GPa}$ and $\nu_g = 0.3$) (Chapter 1, Section 1.2.1.2, Table 1.2)) with dimensions $a \times a = 2 \text{ m} \times 2 \text{ m}$ and thicknesses $t_g = 7 \text{ mm}$ through two adhesive layers ($G_a = 100 \text{ MPa}$ and $G_a = 400 \text{ MPa}$ (Chapter 1, Section 1.2.1.2, Table 1.2)) with thicknesses $t_a = 1 \text{ mm}$ (Figure 5.5). The resulting composite system is assumed to be subjected to a total reference pure shear traction $P_{xy} = 1 \text{ KPa}$. In line with the assumptions and formulation introduced in chapter 4, all three plates are assumed to undergo identical pre-buckling displacements. All edges of the composite system are assumed to be fully fixed relative to rotations about x and y . It is required to find the critical pressure λP_{xy} at which the composite system would buckle out of its own plane. The problem is to be solved using the finite element formulation developed in Chapter 4 and using a 3D finite element model under Abaqus.

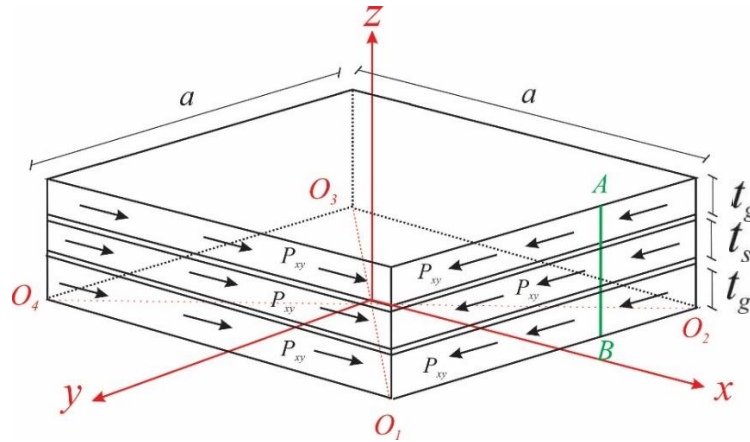


Figure 5.5- A steel plate reinforced with two GFRP plates through two adhesive layers subjected to pure shear traction

5.3.2 Solution based on present study

5.3.2.1 Mesh study

Table 5.4 provides the results of a mesh sensitivity study for the finite element solution based on the present study. The first two columns of Table 5.4 provide number of subdivisions along x and

y direction (n_x and n_y), the third column is total number of degrees of freedom, and the forth column provides the critical pressure obtained from 3D FEA in Abaqus. The solution based on the 30×30 -element mesh is observed to agree with that based on the 25×25 -element solution within four significant digits. Thus, convergence is deemed to be achieved for the 25×25 -element mesh.

Table 5.4- Mesh study for present finite element solution for composite system under pure shear

Number of elements		# of DOFs	Critical pressure $P_{xy(Cr)}$ (MPa)	
n_x	n_y		$G_a = 100 \text{ MPa}$	$G_a = 400 \text{ MPa}$
5	5	180	177.7	257.5
10	10	605	149.8	159.9
15	15	1280	142.6	148.9
20	20	2205	142.4	148.1
25	25	3380	142.3	147.9
30	30	4805	142.3	147.9

5.3.3 Abaqus 3D model

5.3.3.1 Boundary Conditions

For a generic edge line of the plate such as line AB (Figure 5.5), all the nodes are constrained to move equally along the x and y directions in order to restrain the edges against rotations about x and y . Also, because all in-plane displacements at the mid-surface of the top and bottom plates must be equal and opposite (Assumption made in Chapter 2, Section 2.4.2). Both requirements are met only if all mid-surface displacements at the edges vanish. To apply symmetry condition about diagonals O_1O_3 and O_2O_4 (Figure 5.5) the following constrains are enforced to the nodes O_1, O_2, O_3, O_4 and all the nodes through thickness of the composite system.

$$u_{O_1} = v_{O_1} \quad , \quad u_{O_3} = v_{O_3} \quad , \quad u_{O_2} = -u_{O_4} \quad , \quad v_{O_2} = -v_{O_4} \quad (6.4)$$

All constrains are enforced by using the *EQUATION keyword in Abaqus.

5.3.3.2 Mesh study

Table 5.5 provides the results of a mesh sensitivity study for the 3D FEA in Abaqus. The first two columns of Table 5.5 provide number of subdivision along x and y direction (n_x and n_y), the third

column is total number of degrees of freedom, and the forth column provides the critical pressure obtained from 3D FEA in Abaqus. Four elements are taken across each plate thickness and one element is taken across the adhesive layer. Along the plate sides, meshes based on the 40, 80, 100, 120, 240 and 250 elements are investigated. The solution based on 250×250 -element mesh is observed to be in agreement with that based on the 240×240 -element solution, within four significant digits. Thus, convergence is deemed to be achieved for 240×240 -element mesh.

Table 5.5- Mesh study for 3D FEA in Abaqus for composite system under pure shear

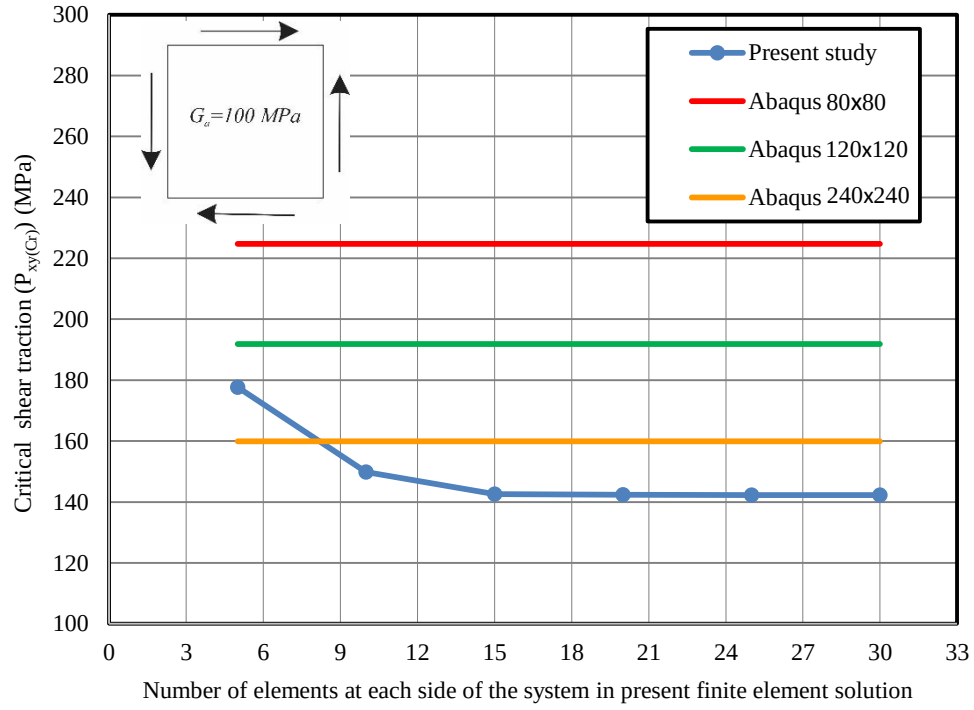
Number of elements		# of DOFs	Critical pressure $P_{xy(Cr)}$ (MPa)	
n_x	n_y		$G_a = 100 \text{ MPa}$	$G_a = 400 \text{ MPa}$
40	40	537600	401.5	409.9
80	80	2150400	224.7	254.7
100	100	3360000	202.7	225.6
120	120	4838400	191.9	199.4
240	240	19353600	159.9	169.6
250	250	21000000	159.9	169.6

5.3.4 Comparison of result

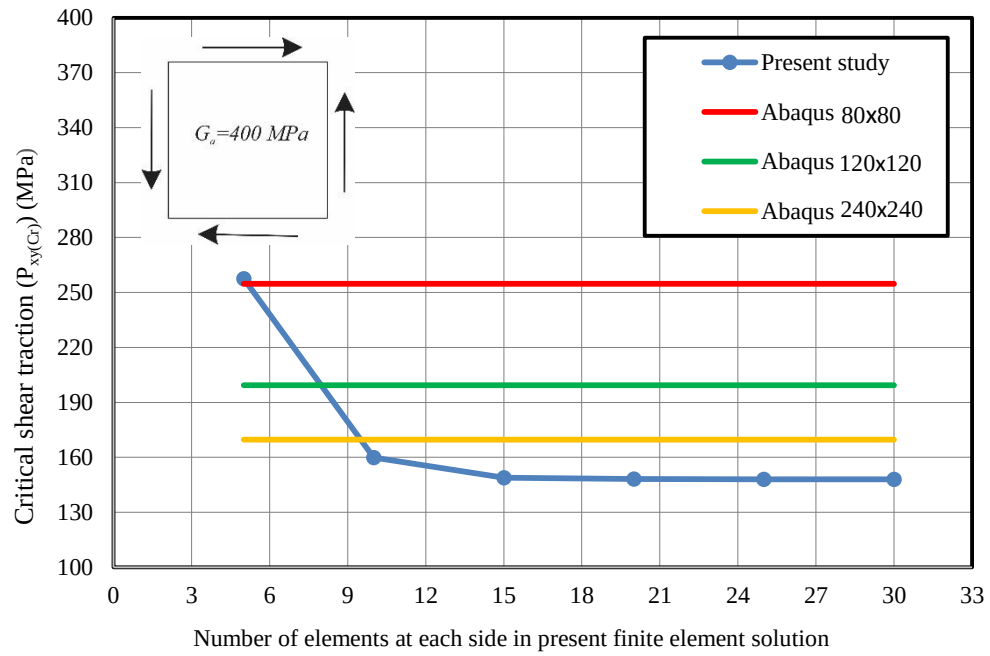
The critical pressure obtained from present finite element solution and that based on the 3D FEA in Abaqus are found to be in agreement within 11-13% difference (Table 5.6). Figure 5.6 (a), (b) represent the mesh study for both present finite element solution and 3D FEA (Abaqus) in the case where $G_a = 100 \text{ MPa}$ and $G_a = 400 \text{ MPa}$. Again by comparing the number of degrees of freedom of the converged mesh based the present solution (Table 5.1) and those of the converged 3D FEA model in Abaqus (Table 5.2), the computational effort needed under the present solution is four orders less than that required by the Abaqus model.

Table 5.6- Comparison of critical pressure values $P_{xy(Cr)}$ (MPa) for Example 2

Method	3D FEA (Abaqus)	Present Solution	Percentage difference
Mesh	240×240	25×25	
$P_{xy(Cr)}$ (MPa) ($G_a = 100 \text{ MPa}$)	159.9	142.3	11.0 %
$P_{xy(Cr)}$ (MPa) ($G_a = 400 \text{ MPa}$)	169.6	147.8	12.8 %



(a)



(b)

Figure 5.6- Comparison of mesh study for the present finite element solution and 3D FEA (Abaqus) in case
(a) $G_s = 100 \text{ MPa}$, (b) $G_s = 400 \text{ MPa}$

5.4 Parametric Study

This section aims at investigating the effect of three parameters on the critical pressure of composite systems introduced in Example 1 (Section 5.2) and Example 2 (Section 5.3), namely (1) the thickness of the GFRP plate, (2) that of the adhesive, and (3) the shear modulus of the adhesive. Towards this goal, the normalized critical line loads (the ratio of critical line loads of the composite system to the critical line load of a single steel plate) versus the relevant normalized properties will be investigated

5.4.1 Composite flange under compressive strains

In the composite system introduced in Example 1 (Section 5.2) normal strains have been identically applied to all materials while in the single steel case. In order to normalize all the critical pressure values of Example 1, the critical line load for the composite system ($P_{yy(cr)}(t_{sr} + 2t_g)$, $t_{sr} = 8mm$, $7mm \leq t_g \leq 22mm$) and the critical line load for the flange before corrosion ($P_0 t_s$, $t_s = 16mm$) are divided by the critical line load for the corroded flange ($P_r t_{sr}$, $t_{sr} = 8mm$). In all three cases, normalized critical line load $\left[P_{yy(cr)}(t_{sr} + 2t_g) \right] / (P_r t_{sr})$ versus the relevant normalized property are depicted. For the present problem, one has $P_0 t_s = 1.07 \times 10^5 (KN/m)$ and $P_r t_{sr} = 0.1524 \times 10^5 (KN/m)$.

5.4.1.1 Effect of GFRP thickness

In the FEA series in this section, all the parameters of the reference system are kept constant (i.e., $G_a = 200 MPa$, $t_a = 1 mm$) except the GFRP thickness which has been varied in the range i.e., $7 mm \leq t_g \leq 22 mm$ to investigate the effect of GFRP thickness on the buckling pressure of the composite flange under compressive stress (Table A.3.1 in Appendix A.3). Figure 5.7 depicts the normalized critical line load $\left[P_{yy(cr)}(t_{sr} + 2t_g) \right] / (P_r t_{sr})$ versus the normalized GFRP thickness (t_g / t_{sr}). It is observed that increasing the GFRP thickness within the practical range ($7 mm \leq t_g \leq 22 mm$) significantly increases the buckling pressure of the composite system (by a factor of 724%). It is also observed that the critical pressure obtained from the present study and that based on the 3D FEA in Abaqus agree within -11% to 15% for the range of GFRP thicknesses of $7 mm \leq t_g \leq 22 mm$. For low GFRP thicknesses, the present non-conforming FEA formulation

tends to underestimate the buckling pressure of the composite system. This effect tends to decrease as (t_g/t_{sr}) increases. For higher GFRP thicknesses $(t_g/t_{sr}) \geq 2$, the effect of transverse shear deformation becomes significant and the present model, which omits the transverse shear deformation effects, tend to overestimate the buckling strength of the composite system.

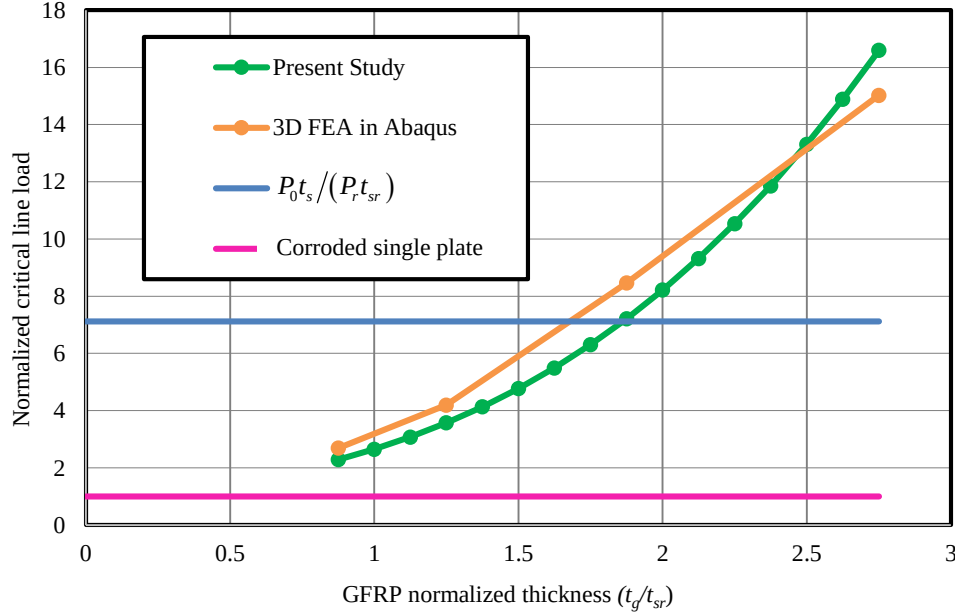


Figure 5.7- Effect of GFRP thickness on the critical line load of the composite flange under normal strains ($G_a = 200 \text{ MPa}$, $t_a = 1 \text{ mm}$)

5.4.1.2 Effect of adhesive thickness

In the FEA run in this section, all the properties of the reference system are kept constant (i.e., $G_a = 200 \text{ MPa}$, $t_g = 7 \text{ mm}$) except adhesive thickness which varies in the range i.e., $0.5 \text{ mm} \leq t_a \leq 4 \text{ mm}$ to investigate the effect of adhesive thickness on the buckling pressure of the composite flange (Table A.3.2 in Appendix A.3). Figure 5.8 depicts the normalized critical line load $[P_{yy(cr)}(t_{sr} + 2t_g)] / (P_r t_{sr})$ versus the adhesive normalized thickness (t_a/t_{sr}) . It is observed that increasing the adhesive thickness in a range i.e., $0.5 \text{ mm} \leq t_a \leq 4 \text{ mm}$ decreases the buckling pressure of the composite system by 53% (A similar observation has been made in Pham and Mohareb (1) (2014), in which the thicker adhesive layer has been associated with a non-flexible system). This effect can be categorized as a moderate effect compared to the effects of the other

parameters. Also, it is observed that for a practical range of adhesive thickness ($0.5\text{mm} \leq t_a \leq 2\text{mm}$), the critical pressure obtained from the present study and that based on the 3D FEA in Abaqus agree within 9-24%. As the thickness of the adhesive increases beyond the practical range, it is found that the present solution increasingly underestimate the critical line load compared to the Abaqus solution. To find out the possible reason for this difference, an investigation was conducted in Abaqus by comparing the ratio of mid-height transverse shear stress to the critical buckling pressure of the composite at 5 mm from the plate edge. Three cases were considered for the adhesive thicknesses were $t_a = 0.5, 1.8, 3.8\text{ mm}$. It was observed the above defined normalized shear stress increased with the thickness of adhesive, from 0.3977 to 0.4496 to 0.4689. It is recalled that the present study neglects the transverse shear stresses. Thus the growing difference between the predictions based on the present theory and that based on Abaqus can be attributed, at least in part, due to the neglect in transverse shear deformation.

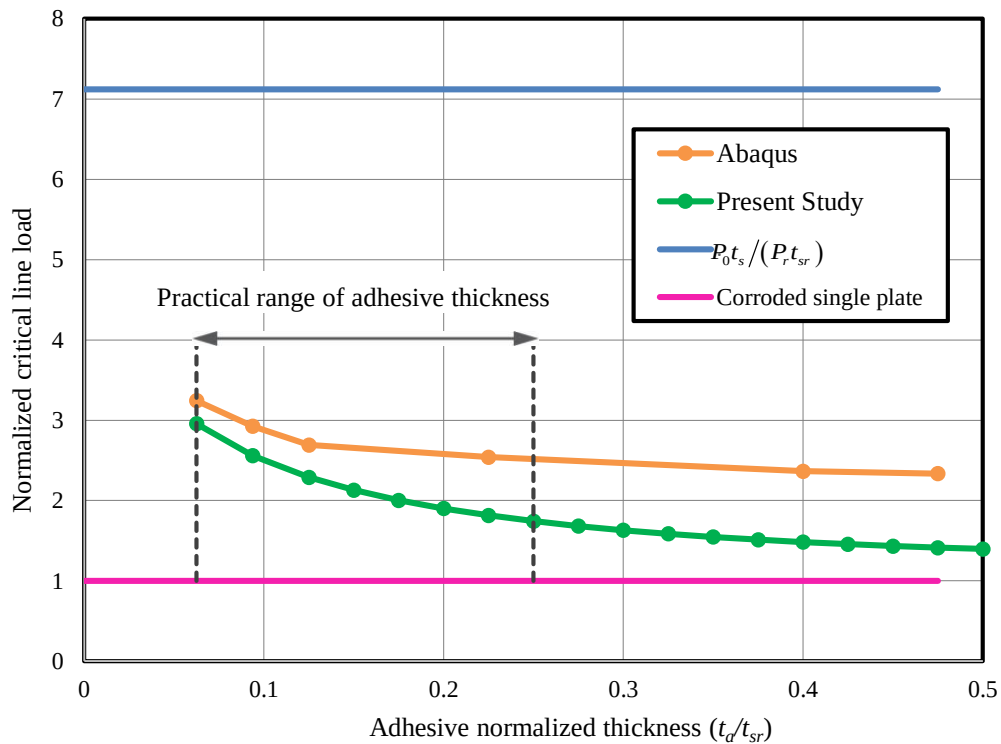


Figure 5.8- Effect of adhesive thickness on critical line load ($Ga = 200\text{ MPa}$, $t_g = 7\text{ mm}$)

5.4.1.3 Effect of shear modulus of adhesive

In this parametric study, all the properties of the reference system are kept constant (i.e., $t_a = 1\text{ mm}$, $t_g = 7\text{ mm}$) except the shear modulus of the adhesive which has been varied in the practical range i.e., $100\text{ MPa} \leq G_a \leq 400\text{ MPa}$ to investigate the effect of shear modulus of the adhesive on the buckling pressure of the composite system (Table A.3.3 in Appendix A.3). Figure 5.9 depicts the normalized critical line load $\left[P_{yy(cr)} (t_{sr} + 2t_g) \right] / (P_r t_{sr})$ versus the adhesive normalized modulus (G_a / E_s) . It is observed that increasing the shear modulus of adhesive in a practical range $100\text{ MPa} \leq G_a \leq 400\text{ MPa}$ increases the buckling pressure of the composite system by 40%. The present example suggests that the shear modulus of adhesive has moderate influence the buckling pressure of the composite system compared to the effect of GFRP and adhesive thickness. It is also observed that for a practical range of modulus ($100\text{ MPa} \leq G_a \leq 400\text{ MPa}$), the critical pressure obtained from the present study and that based on the 3D FEA in Abaqus agree within 14-15% difference.

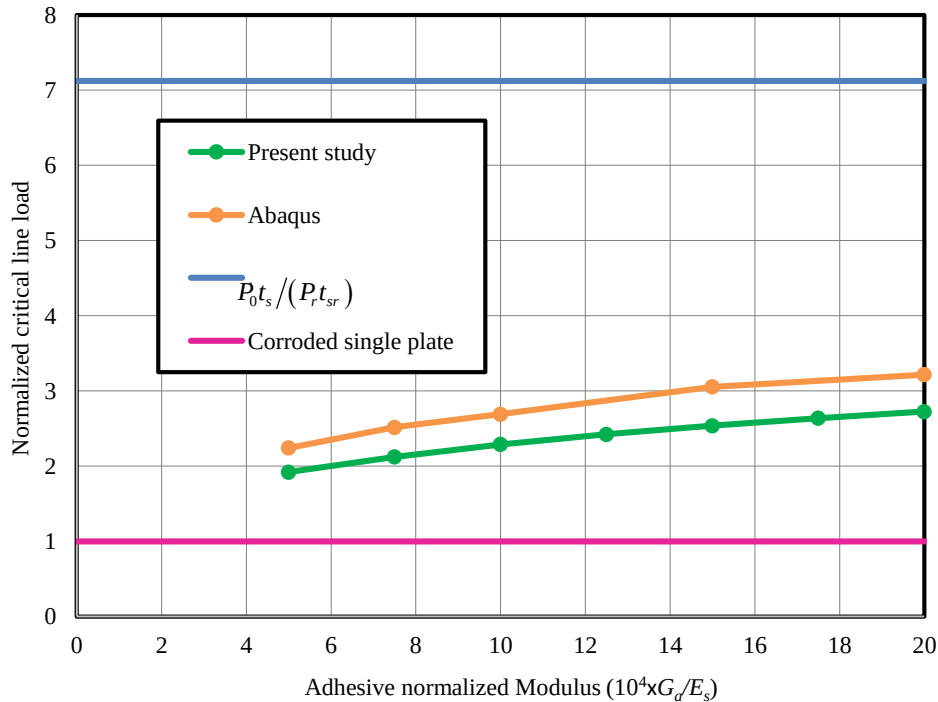


Figure 5.9- Effect of adhesive shear modulus on critical line load ($t_a = 1\text{ mm}$, $t_g = 7\text{ mm}$)

5.4.2 Composite plate under pure shear

In the composite system introduced in Example 2 (Section 5.3) the shear strains have been identically applied to all layers. To evaluate the effects of GFRP and adhesive thickness and the shear modulus of the adhesive on the buckling pressure of the system, the critical line load for the composite system is normalized with respect to the critical line pressure of the steel plate $\left(\left[P_{xy(cr)} (t_s + 2t_g) \right] / (P_0 t_s) \right)$, $t_s = 14mm$, $7mm \leq t_g \leq 13mm$. In the present problem, one has $P_0 t_s = 0.01949 \times 10^5 (KN/m)$. The effect of various parameters on the normalized line pressure will be investigated in the following sub-sections.

5.4.2.1 Effect of GFRP thickness

In the parametric FEA runs in this section, all the parameters of the reference system are kept constant (i.e., $G_a = 100 MPa$, $t_a = 1 mm$) except the GFRP thickness which has been varied in the range $(7mm \leq t_g \leq 13mm)$ to investigate the effect of the GFRP thickness on the critical pressure of the composite plate under pure shear (Table A.3.4 in Appendix A.3). Figure 5.10 depicts the normalized critical line load $\left[P_{xy(cr)} (t_s + 2t_g) \right] / (P_0 t_s)$ versus the normalized GFRP thickness (t_g/t_s) . It is observed that increasing the thickness of GFRP plates in the range $7mm \leq t_g \leq 13mm$ significantly improved the buckling capacity of the system two to three folds.

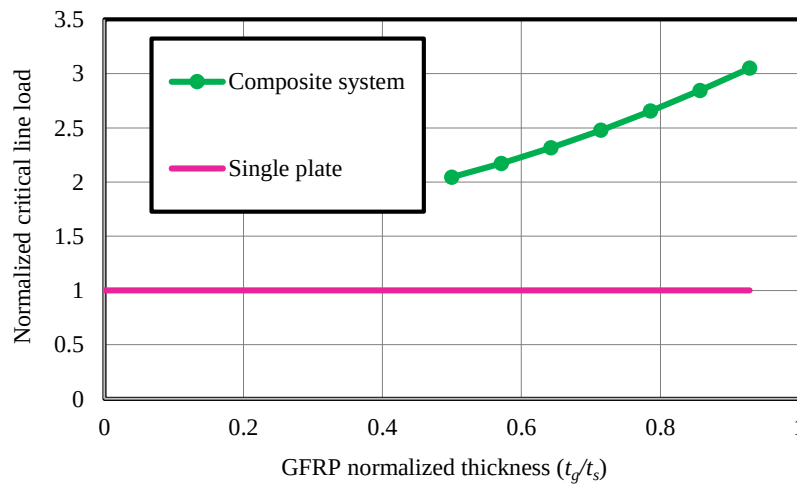


Figure 5.10- Effect of GFRP thickness on the critical line load ($G_a = 100 MPa$, $t_a = 1 mm$)

5.4.2.2 Effect of adhesive thickness

In this parametric study, all the parameters of the reference system are kept constant (i.e., $G_a = 100 \text{ MPa}$, $t_g = 7 \text{ mm}$) except the adhesive thickness which was varied in the range $0.5 \text{ mm} \leq t_a \leq 4 \text{ mm}$ to investigate effect of the adhesive thickness on the buckling pressure of the composite system under pure shear (Table A.3.5 in Appendix A.3). Figure 5.11 depicts the normalized critical line load $\left[P_{xy(cr)}(t_s + 2t_g) \right] / (P_0 t_s)$ versus the adhesive normalized thickness (t_a/t_s) . An increase of the adhesive thickness from 0.5mm to 4mm is observed to correspond to a 37% decrease in the capacity. A similar observation was made in Pham and Mohareb (2014)a , in which a thicker adhesive layer has been associated with a more flexible response.

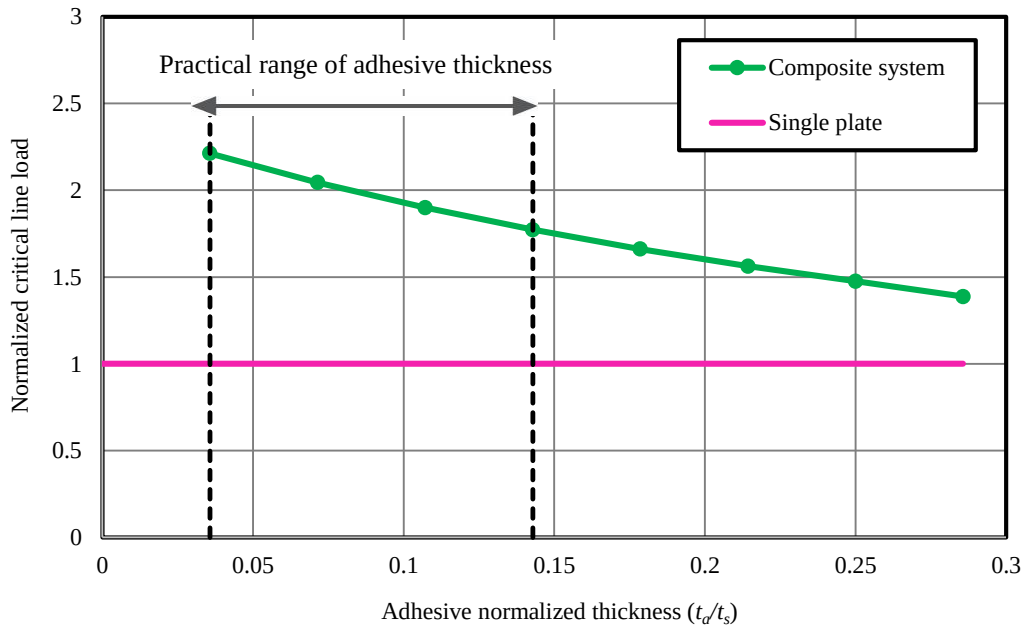


Figure 5.11- Effect of adhesive thickness on critical line load ($G_a = 100 \text{ MPa}$, $t_g = 7 \text{ mm}$)

5.4.2.3 Effect of shear modulus of adhesive

In this parametric study, all the parameters of the reference system are kept constant and identical to those of the reference case, i.e., $(t_a = 1 \text{ mm}$, $t_g = 7 \text{ mm})$ except the shear modulus of adhesive which has been varied in the practical range $(100 \text{ MPa} \leq G_a \leq 400 \text{ MPa})$ in order to investigate the effect of shear modulus of adhesive on the critical pressure of the composite system (Table

A.3.6 in Appendix A.3). Figure 5.12 depicts the normalized critical line load $\left[P_{xy(cr)} (t_s + 2t_g) \right] / (P_0 t_s)$ versus the adhesive normalized modulus (G_a/E_s) . As the shear modulus of the adhesive increases, the critical line load is observed to very slightly increase. Only a again in buckling strength of 4% is achieved by increasing the shear modulus value from 100 MPA to 400 MPa.

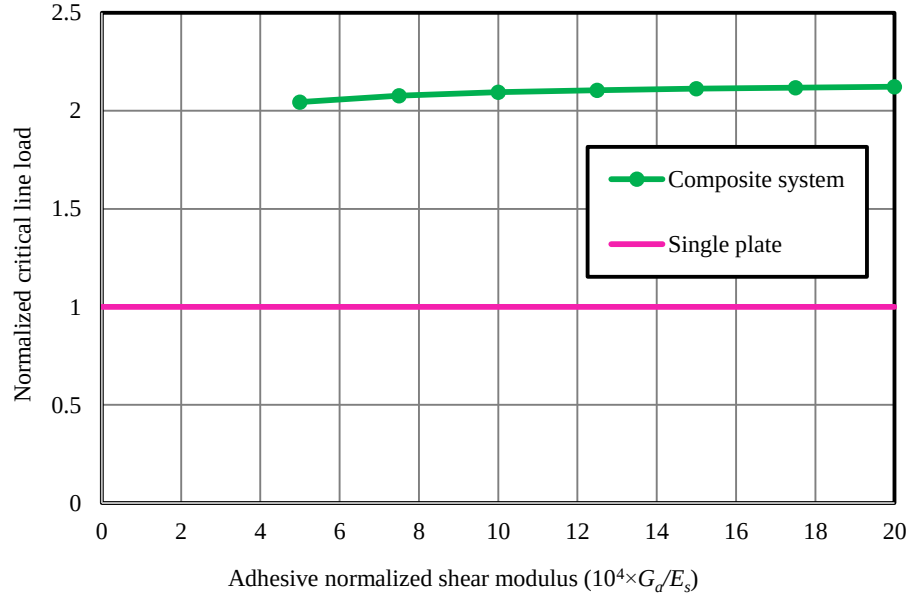


Figure 5.12- Effect of shear modulus of the adhesive on critical line load ($t_a = 1 \text{ mm}$, $t_g = 7 \text{ mm}$)

5.4.3 Summary and Conclusions

A summary of the effect of geometric and material parameters on the buckling capacity of the composite system are summarized in Table 5.7. For the composite flange under normal stresses and composite plate under pure shear, the following conclusions can be drawn

- (1) The GFRP thickness has the most influence on the buckling capacity of the composite system. An increase in the thickness of the GFRP is found to significantly improve the buckling capacity of the system.
- (2) Reducing the thickness of adhesive layer within the practical range is associated with a moderate increase in the buckling capacity of the system.
- (3) The shear modulus of the adhesive has the least influence on the buckling capacity of the system compared to the previous parameters

Table 5.7- Effect of different parameters on the buckling capacity of composite system

Parameter	Effect on the buckling capacity of the system by increasing the parameter		Level of effectiveness
	Increase	Decrease	
GFRP thickness	✓	-	High
Adhesive thickness	-	✓	Moderate
Shear modulus of adhesive	✓	-	Moderate (flange under normal stress) Low (plate under shear)

6 SUMMARY, CONCLUSIONS AND RECOMMENDATIONS

6.1 Summary

The present study has focused on the buckling analysis of steel plates symmetrically reinforced by GFRP plates through adhesive layers. Towards this goal, the thesis has made the following contributions:

- (1) A variational principle was developed for the buckling analysis of single plates, two plates bonded together through an adhesive layer and a single steel plate reinforced with two GFRP plates through adhesive layers. The latter two theories account for the effect of transverse shear strains in the adhesive layers.
- (2) The validity of variational principles for single plates and two plates bonded through an adhesive layer were assessed by developing 3D models in Abaqus and determining the buckling stresses and associated modes shapes. The Abaqus mode shapes were then used to develop approximate buckling displacement functions through regression analysis. The approximate functions were then substituted into the variational principle and the stationary condition were evoked to determine the critical loads. The buckling loads based on the present variational principle were observed to be in agreement with those based on 3D FEA as predicted by Abaqus within a 5% percentage difference.
- (3) A finite element formulation was then developed for the case of steel plate reinforced with two GFRP plates based on the variational principle developed.
- (4) The validity of the finite element formulation was assessed through comparisons with results based on Abaqus 3D solutions.
- (5) The validated finite element was then used to conduct parametric studies to investigate the local buckling capacity of practical retrofitting problems for steel plates symmetrically reinforced with GFRP plates.

6.2 Conclusions

Based on the present study, for a half flange symmetrically reinforced with two GFRP plates under uniaxial normal pressure (Section 5.2) the following conclusions can be drawn

- (1) The computational and modeling effort needed under the present study is significantly less than that of the 3D FEA model. For the Abaqus model 10,080,000 degrees of freedom were needed for convergence while the present solution attained convergence with 3,680 degrees of freedom.
- (2) For the range of GFRP thickness ($7mm \leq t_g \leq 22mm$), $G_a = 200 MPa$ and $t_a = 1mm$, the critical pressure obtained from the present study and that based on the 3D FEA in Abaqus were found to agree within -11% to 15% difference.
- (3) For the practical range of adhesive thicknesses $0.5 \leq t_a \leq 2.0mm$, $G_a = 200 MPa$ and $t_g = 7mm$, the critical pressure obtained from the present study and that based on the 3D FEA in Abaqus were found to agree within 9-24% difference. However, for composite systems with adhesive layer thicknesses $2mm \leq t_a \leq 4mm$, the percentage difference between the results based on present study and the 3D FEA in Abaqus is found to increase to 40% due to neglect in transverse shear deformation in the present study.
- (4) For the range of practical shear modulus of adhesive $100 MPa \leq G_a \leq 400 MPa$, $t_a = 1mm$ and $t_g = 7mm$, the critical pressure obtained from the present study and that based on the 3D FEA in Abaqus were found to agree within 14-15% difference.
- (5) It was observed that increasing the thickness of the GFRP plates within the range $7mm \leq t_g \leq 22mm$ improves the buckling capacity of the system by about seven-folds.
- (6) Increasing the thickness of adhesive layers from $0.5mm$ to $4mm$ was observed to decrease the buckling capacity of the system by 53%.
- (7) Increasing the shear modulus of adhesive within the practical range ($100 MPa \leq G_a \leq 400 MPa$) increased the buckling capacity of the system by 40%.
- (8) Reinforcing the corroded 8mm flange with two 15 mm thick GFRP plates results in restoring the whole buckling capacity of the un-corroded 16mm thick flange.

Based on the present study, for a plate reinforced with two GFRP plates under pure shear traction (Section 5.3) the following conclusions can be drawn

- (1) The computational and modeling effort needed under the present study is significantly less than that of the 3D FEA model. For the Abaqus model 19,353,600 degrees of freedom were needed for convergence while the present solution attained convergence with 3,380 degrees of freedom.
- (2) For the physical and mechanical properties $t_g = 7mm$, $G_a = 100 MPa$ and $t_a = 1mm$, the critical pressure obtained from the present study and that based on the 3D FEA in Abaqus were found to agree within 11% difference.
- (3) The critical pressure obtained from the present study and that based on the 3D FEA in Abaqus were found to agree within 12.9% difference for $t_g = 7mm$, $G_a = 400 MPa$ and $t_a = 1mm$.
- (4) It was observed that increasing the thickness of the GFRP plates from $7mm$ to $13mm$ improves the buckling capacity of the system by 50%.
- (5) Increasing the thickness of adhesive layers from $0.5mm$ to $4mm$ was observed to decrease the buckling capacity of the system by 37%.
- (6) Increasing the shear modulus of the adhesive within the range $(100 MPa \leq G_a \leq 400 MPa)$ increased the buckling capacity of the system by 4%.

6.3 Recommendations for future research

- (1) The present study developed a variational principle and finite element formulation for the plates reinforced with identical GFRP plates in terms of physical and mechanical properties. The boundary conditions and pre-buckling deformations were assumed to be identical for all layers. It can be of practical interest to develop a finite element formulation for the case where the boundary conditions and loading are distinct in all layers.
- (2) The present non-conforming formulations accounts for in-plane shear strains within the five layers. In most cases, the results were observed to be in good agreement with Abaqus, with a few cases to depart from those based on the Abaqus 3D model. It is of interest to incorporate into the formulation the effect of transverse shear strains in the GFRP and steel layers to achieve better agreement with 3D solutions. Also, it can be of interest to develop a conforming FEA formulation to see if it is possible to achieve a better agreement.

- (3) The variational principle developed was used to develop a finite element formulation. The principle can be equally adopted to develop other closed form solutions (e.g., Navier type solutions or Levy type solutions, etc.) and other numerical solutions (e.g., finite difference solutions).

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APPENDICES

A.1-2 DETAILS OF THE FORMULATION

A.3- RESULTS

A.4 - MATLAB PROGRAM

A.5 -TYPICAL ABAQUS INPUT FILE

APPENDIX A.1- Second variation of total potential energy for two plates bonded through a soft layer

The goal of this appendix is to show the details of formulating the second variation of total potential energy for two single plates bonded through a soft layer. From the strain-displacement relations in Eq. (2.36), Table 2.2, and Table 2.3, by substituting into Eq. (2.36), the total potential energy of the composite system takes the form

$$\begin{aligned}
 \Pi_t^* = & \sum_{i=1}^2 \frac{1}{2} \left[\iiint_V \frac{E_s}{1-\nu_s^2} (\lambda \varepsilon_{x,p} + \varepsilon_{x,b})_i^2 dx dy dz + \iiint_V \frac{E_s}{1-\nu_s^2} (\lambda \varepsilon_{y,p} + \varepsilon_{y,b})_i^2 dx dy dz \right] \\
 & + \sum_{i=1}^2 \frac{1}{2} \iiint_V \frac{2\nu_s E_s}{1-\nu_s^2} (\lambda \varepsilon_{x,p} + \varepsilon_{x,b})_i (\lambda \varepsilon_{y,p} + \varepsilon_{y,b})_i dx dy dz \\
 & + \sum_{i=1}^2 \frac{1}{2} \iiint_V \frac{E_s}{2(1+\nu_s)} (\lambda \gamma_{xy,p} + \gamma_{xy,b})_i^2 dx dy dz \\
 & + \iiint_V \frac{1}{2} G_a \left[(\lambda \gamma_{xy,p} + \gamma_{xy,b})_{i=3}^2 + (\lambda \gamma_{xz,p} + \gamma_{xz,b})_{i=3}^2 + (\lambda \gamma_{yz,p} + \gamma_{yz,b})_{i=3}^2 \right] dx dy dz \\
 & - \sum_{i=1}^3 \left\{ \int_0^a \int_0^b h_i \left[P_x u'_p + P_y \dot{v}_p + P_{xy} (\dot{u}_p + v'_p) \right] dx dy \right\}
 \end{aligned} \tag{A.1.1}$$

By taking the second variation of total potential energy of the whole composite system (Eq. (A.1.1)) with respect to the buckling displacement fields based on the identity developed in (Hashemian (2014))

$$\delta^2 \left(\int_V \varepsilon_{x,b} \varepsilon_{y,b} dV \right) = \int_V 2 (\delta \varepsilon_{x,b}) (\delta \varepsilon_{y,b}) + \varepsilon_{x,b} \delta^2 \varepsilon_{y,bn} + \varepsilon_{y,b} \delta^2 \varepsilon_{x,bn} dV$$

results in

$$\begin{aligned}
\overline{\Pi}_t^* = & \iiint_V \left\{ \frac{E}{1-\nu^2} \left[2(\bar{\varepsilon}_{x,bl})^2 + 2\varepsilon_{x,p} \bar{\bar{\varepsilon}}_{x,bn} \right] \right\} dx dy dz + \iiint_V \left\{ \frac{E}{1-\nu^2} \left[2(\bar{\varepsilon}_{y,bl})^2 + 2\varepsilon_{y,p} \bar{\bar{\varepsilon}}_{y,bn} \right] \right\} dx dy dz \\
& + \iiint_V \left\{ \frac{2\nu E}{1-\nu^2} \left[2(\bar{\varepsilon}_{x,bl})(\bar{\varepsilon}_{y,bl}) + \varepsilon_{x,p} \bar{\bar{\varepsilon}}_{y,bn} + \varepsilon_{y,p} \bar{\bar{\varepsilon}}_{x,bn} \right] \right\} dx dy dz \\
& + \iiint_V \frac{E}{2(1+\nu)} \left[2(\bar{\gamma}_{xy,bl})^2 + 2\gamma_{xy,p} \bar{\bar{\gamma}}_{xy,bn} \right] dx dy dz \\
& + \frac{1}{2} \iiint_V \left\{ G_a \left[2(\bar{\gamma}_{xy,bl})^2 + 2\gamma_{xy,p} \bar{\bar{\gamma}}_{xy,bn} \right] + G_a \left[2(\bar{\gamma}_{xz,bl})^2 + 2\gamma_{xz,p} \bar{\bar{\gamma}}_{xz,bn} \right] \right\} dx dy dz \\
& + \frac{1}{2} \iiint_V \left\{ G_a \left[2(\bar{\gamma}_{yz,bl})^2 + 2\gamma_{yz,p} \bar{\bar{\gamma}}_{yz,bn} \right] \right\} dx dy dz
\end{aligned} \tag{A.1.2}$$

From Table 2.2 and Table 2.3, by substituting the strains by associated displacements into Eq. (A.1.2), the second variation of the total potential energy $\overline{\Pi}_t^*$ takes the form

$$\begin{aligned}
\overline{\Pi}_t^* = & \iiint_V \frac{E_s}{1-\nu_s^2} \left[2(-2z\bar{w}_b''\bar{u}_{1b}' + \bar{u}_{1b}'^2 + z^2\bar{w}_b''^2) + 2\lambda u_p' \bar{w}_b'^2 \right] dx dy dz \\
& + \iiint_V \frac{E_s}{1-\nu_s^2} \left[2(-2z\bar{w}_b \bar{v}_{1b}' + \bar{v}_{1b}'^2 + z^2\bar{w}_b'^2) + 2\lambda \dot{v}_p \bar{w}_b'^2 \right] dx dy dz \\
& + \iiint_V \frac{2\nu_s E_s}{1-\nu_s^2} \left[2(-z\bar{w}_b'' + \bar{u}_{1b}')(-z\bar{w}_b + \bar{v}_{1b}') + \lambda u_p' \bar{w}_b'^2 + \lambda \dot{v}_p \bar{w}_b'^2 \right] dx dy dz \\
& + \iiint_V \frac{E_s}{2(1+\nu_s)} \left[2(-2z\bar{w}_b' + \bar{u}_{1b}' + \bar{v}_{1b}')^2 + 4\lambda(\dot{u}_p + \dot{v}_p) \bar{w}_b' \bar{w}_b \right] dx dy dz \\
& + \iiint_V G_a \left[-2z\bar{w}_b' + z \left(\frac{2\bar{u}_{1b} - h_1 \bar{w}_b'}{h_3} \right) + z \left(\frac{2\bar{v}_{1b} - h_1 \bar{w}_b'}{h_3} \right) \right]^2 dx dy dz \\
& + \iiint_V G_a \left[\left(\frac{2\bar{u}_{1b} - h_1 \bar{w}_b'}{h_3} \right) + \bar{w}_b' \right]^2 dx dy dz + \iiint_V G_a \left[\left(\frac{2\bar{v}_{1b} - h_1 \bar{w}_b'}{h_3} \right) + \bar{w}_b \right]^2 dx dy dz \\
& + \iiint_V 2\lambda G_a \bar{w}_b' \bar{w}_b (\dot{u}_p + \dot{v}_p) dx dy dz
\end{aligned} \tag{A.1.3}$$

From Eq. (A.1.3), by performing integration with respect to z , one obtains

$$\begin{aligned}
\overline{\Pi^*}_t = & \iint_A \frac{E_s}{1-\nu_s^2} \left[2 \left(\bar{u}'_{1b}{}^2 h_1 + \frac{h_1^3}{12} \bar{w}''^2 \right) + 2\lambda h_1 u'_p \bar{w}'^2 \right] dx dy \\
& + \iint_A \frac{E_s}{1-\nu_s^2} \left[2 \left(\bar{v}'_{1b}{}^2 h_1 + \frac{h_1^3}{12} \bar{w}_b''^2 \right) + 2\lambda h_1 \dot{v}_p \bar{w}_b'^2 \right] dx dy \\
& + \iint_A \frac{2\nu_s E_s}{1-\nu_s^2} \left[2 \left(\frac{h_1^3}{12} \bar{w}_b'' \bar{w}_b + h_1 \bar{u}'_{1b} \bar{v}'_{1b} \right) + \lambda h_1 (u'_p) \bar{w}_b'^2 + \lambda h_1 (\dot{v}_p) \bar{w}_b'^2 \right] dx dy dz \\
& + \iint_A \frac{E_s}{2(1+\nu_s)} \left[2 \left(\frac{h_1^3}{3} \bar{w}_b'^2 + \bar{u}_{1b}{}^2 h_1 + \bar{v}_{1b}{}^2 h_1 + 2h_1 \bar{u}_{1b} \bar{v}'_{1b} \right) + 4\lambda h_1 (\dot{u}_p + v'_p) \bar{w}_b' \bar{w}_b \right] dx dy \\
& + \iint_A G_a \left[\frac{h_3^3}{3} \bar{w}_b'^2 + \frac{h_3}{12} (4\bar{u}_{1b}{}^2 - 4h_1 \bar{u}_{1b} \bar{w}_b' + h_1^2 \bar{w}_b'^2) \right] dx dy \\
& + \iint_A G_a \left[\frac{h_3}{12} (4\bar{v}_{1b}'^2 - 4h_1 \bar{v}'_{1b} \bar{w}_b' + h_1^2 \bar{w}_b'^2) \right] dx dy \\
& + \iint_A G_a \left[\frac{h_3}{6} (4\bar{u}_{1b} \bar{v}'_{1b} - 2h_1 \bar{w}_b' \bar{u}_{1b} - 2h_1 \bar{w}_b' \bar{v}'_{1b} + h_1^2 \bar{w}_b'^2) \right] dx dy \\
& + \iint_A G_a \left[-\frac{h_3^2}{3} (2\bar{u}_{1b} \bar{w}_b' - h_1 \bar{w}_b'^2) \right] dx dy \\
& + \iint_A G_a \left[-\frac{h_3^2}{3} (2\bar{v}_{1b} \bar{w}_b' - h_1 \bar{w}_b'^2) \right] dx dy \\
& + \iint_A G_a \left[\left(\frac{4\bar{u}_{1b}^2 - 4h_1 \bar{u}_{1b} \bar{w}_b' + h_1^2 \bar{w}_b'^2}{h_3} \right) + h_3 \bar{w}_b'^2 + 2(2\bar{u}_{1b} \bar{w}_b' - h_1 \bar{w}_b'^2) \right] dx dy \\
& + \iint_A G_a \left[\left(\frac{4\bar{v}_{1b}^2 - 4h_1 \bar{v}_{1b} \bar{w}_b' + h_1^2 \bar{w}_b'^2}{h_3} \right) + h_3 \bar{w}_b'^2 + 2(2\bar{v}_{1b} \bar{w}_b' - h_1 \bar{w}_b'^2) \right] dx dy \\
& + \iint_A 2G_a h_3 \lambda (\dot{u}_{1p} + v'_{1p}) \bar{w}_b' \bar{w}_b dx dy
\end{aligned}
\tag{A.1.4}$$

APPENDIX A.2- Second variation of total potential energy of steel plate reinforced with two GFRP plates through adhesive layers

The goal of this appendix is to perform second variation of total potential energy of steel plate reinforced with two GFRP plates through adhesive layers. From the strain-displacement relations (i.e., Eqs. (2.43), (2.44), (2.45) and (2.46)), by substituting into Eq. (2.54), one obtains

$$\begin{aligned}
 \Pi_t^* = & \left\{ \frac{1}{2} \left[\iiint_V \frac{E_g}{1-\nu_g^2} (\lambda \varepsilon_{x,p} + \varepsilon_{x,b})_i^2 dx dy dz_1 + \iiint_V \frac{E_g}{1-\nu_g^2} (\lambda \varepsilon_{y,p} + \varepsilon_{y,b})_i^2 dx dy dz_1 \right. \right. \\
 & + \iiint_V \frac{2\nu_g E_g}{1-\nu_g^2} (\lambda \varepsilon_{x,p} + \varepsilon_{x,b})_i (\lambda \varepsilon_{y,p} + \varepsilon_{y,b})_i dx dy dz_1 \\
 & \left. + \iiint_V \frac{E_g}{2(1+\nu_g)} (\lambda \gamma_{xy,p} + \gamma_{xy,b})_i^2 dx dy dz_1 \right] \\
 & - \left[\int_0^a \int_0^b h_i \left[P_x u'_p + P_y \dot{v}_p + P_{xy} (\dot{u}_p + v'_p) \right] dx dy \right] \Bigg\}_{i=1} \\
 & + \left\{ \frac{1}{2} \left[\iiint_V \frac{E_g}{1-\nu_g^2} (\lambda \varepsilon_{x,p} + \varepsilon_{x,b})_i^2 dx dy dz_2 + \iiint_V \frac{E_g}{1-\nu_g^2} (\lambda \varepsilon_{y,p} + \varepsilon_{y,b})_i^2 dx dy dz_2 \right. \right. \\
 & + \iiint_V \frac{2\nu_g E_g}{1-\nu_g^2} (\lambda \varepsilon_{x,p} + \varepsilon_{x,b})_i (\lambda \varepsilon_{y,p} + \varepsilon_{y,b})_i dx dy dz_2 \\
 & \left. + \iiint_V \frac{E_g}{2(1+\nu_g)} (\lambda \gamma_{xy,p} + \gamma_{xy,b})_i^2 dx dy dz_2 \right] \\
 & - \left[\int_0^a \int_0^b h_i \left[P_x u'_p + P_y \dot{v}_p + P_{xy} (\dot{u}_p + v'_p) \right] dx dy \right] \Bigg\}_{i=5} \\
 & + \left\{ \frac{1}{2} \left[\iiint_V \frac{E_s}{1-\nu_s^2} (\lambda \varepsilon_{x,p} + \varepsilon_{x,b})_i^2 dx dy dz_i + \iiint_V \frac{E_s}{1-\nu_s^2} (\lambda \varepsilon_{y,p} + \varepsilon_{y,b})_i^2 dx dy dz_i \right. \right. \\
 & + \iiint_V \frac{2\nu_s E_s}{1-\nu_s^2} (\lambda \varepsilon_{x,p} + \varepsilon_{x,b})_i (\lambda \varepsilon_{y,p} + \varepsilon_{y,b})_i dx dy dz_i + \iiint_V \frac{E_s}{2(1+\nu_s)} (\lambda \gamma_{xy,p} + \gamma_{xy,b})_i^2 dx dy dz_i \Bigg] \\
 & \left. - \left[\int_0^a \int_0^b h_i \left[P_x u'_p + P_y \dot{v}_p + P_{xy} (\dot{u}_p + v'_p) \right] dx dy \right] \right\}_{i=3}
 \end{aligned}$$

$$\begin{aligned}
& + \sum_{i=2,4} \left\{ \iiint_V \frac{1}{2} G_a \left[\left(\lambda \gamma_{xy,p} + \gamma_{xy,b} \right)_i^2 + \left(\lambda \gamma_{xz,p} + \gamma_{xz,b} \right)_i^2 + \left(\lambda \gamma_{yz,p} + \gamma_{yz,b} \right)_i^2 \right] dx dy dz_3 \right. \\
& \quad \left. - \int_0^a \int_0^b h_i \left[P_x u'_p + P_y v'_p + P_{xy} \left(\dot{u}_p + v'_p \right) \right] dx dy \right\}
\end{aligned} \tag{A.2.1}$$

Taking the second variation of the total potential energy of the whole composite system (Eq. (A.2.1)) with respect to the buckling displacement fields based on (Hashemian (2014))

$$\left[\delta^2 \left(\int_V \varepsilon_{x,b} \varepsilon_{y,b} dV \right) = \int_V 2 \left(\delta \varepsilon_{x,bl} \right) \left(\delta \varepsilon_{y,bl} \right) + \varepsilon_{x,b} \delta^2 \varepsilon_{y,bn} + \varepsilon_{y,b} \delta^2 \varepsilon_{x,bn} \right]$$

one obtains,

$$\begin{aligned}
\overline{\overline{\Pi}}^*_t = & \iiint_V \left\{ \frac{E_g}{1-\nu_g^2} \left[2 \left(\overline{\varepsilon}_{x,bl} \right)^2 + 2 \varepsilon_{x,p} \overline{\overline{\varepsilon}}_{x,bn} \right] \right\} dx dy dz + \iiint_V \left\{ \frac{E_g}{1-\nu_g^2} \left[2 \left(\overline{\varepsilon}_{y,bl} \right)^2 + 2 \varepsilon_{y,p} \overline{\overline{\varepsilon}}_{y,bn} \right] \right\} dx dy dz \\
& + \iiint_V \left\{ \frac{2\nu_g E_g}{1-\nu_g^2} \left[2 \left(\overline{\varepsilon}_{x,bl} \right) \left(\overline{\varepsilon}_{y,bl} \right) + \varepsilon_{x,p} \overline{\overline{\varepsilon}}_{y,bn} + \varepsilon_{y,p} \overline{\overline{\varepsilon}}_{x,bn} \right] \right\} dx dy dz \\
& + \frac{1}{2} \left\langle \iiint_V \frac{E_g}{2(1+\nu_g)} \left[2 \left(\overline{\gamma}_{xy,bl} \right)^2 + 2 \gamma_{xy,p} \overline{\overline{\gamma}}_{xy,bn} \right] dx dy dz \right. \\
& + \iiint_V \left\{ \frac{E_s}{1-\nu_s^2} \left[2 \left(\overline{\varepsilon}_{x,bl} \right)^2 + 2 \varepsilon_{x,p} \overline{\overline{\varepsilon}}_{x,bn} \right] \right\} dx dy dz + \iiint_V \left\{ \frac{E_s}{1-\nu_s^2} \left[2 \left(\overline{\varepsilon}_{y,bl} \right)^2 + 2 \varepsilon_{y,p} \overline{\overline{\varepsilon}}_{y,bn} \right] \right\} dx dy dz \\
& + \iiint_V \left\{ \frac{2\nu_s E_s}{1-\nu_s^2} \left[2 \left(\overline{\varepsilon}_{x,bl} \right) \left(\overline{\varepsilon}_{y,bl} \right) + \varepsilon_{x,p} \overline{\overline{\varepsilon}}_{y,bn} + \varepsilon_{y,p} \overline{\overline{\varepsilon}}_{x,bn} \right] \right\} dx dy dz \\
& + \iiint_V \frac{E_s}{2(1+\nu_s)} \left[2 \left(\overline{\gamma}_{xy,bl} \right)^2 + 2 \gamma_{xy,p} \overline{\overline{\gamma}}_{xy,bn} \right] dx dy dz \Bigg\rangle \\
& + \iiint_V \left\{ G_a \left[2 \left(\overline{\gamma}_{xy,bl} \right)^2 + 2 \gamma_{xy,p} \overline{\overline{\gamma}}_{xy,bn} \right] + G_a \left[2 \left(\overline{\gamma}_{xz,bl} \right)^2 + 2 \gamma_{xz,p} \overline{\overline{\gamma}}_{xz,bn} \right] \right\} dx dy dz \\
& + \iiint_V \left\{ G_a \left[2 \left(\overline{\gamma}_{yz,bl} \right)^2 + 2 \gamma_{yz,p} \overline{\overline{\gamma}}_{yz,bn} \right] \right\} dx dy dz
\end{aligned} \tag{A.2.2}$$

From Table 2.5, Table 2.6 and Table 2.7, by expressing the strains in terms of the displacement fields, and substituting into Eq. (A.2.2) and performing integration with respect to z , $\overline{\overline{\Pi}}^*_t$ will be

$$\begin{aligned}
\bar{\Pi}_{Total} = & \iint_A \frac{E_g}{1-\nu_g^2} \left[2 \times \left(\bar{u}_{1b}'^2 h_1 + \frac{h_1^3}{12} \bar{w}_b'^2 \right) + 2\lambda h_1 u_p' \bar{w}_b'^2 \right] dx dy \\
& + \iint_A \frac{E_g}{1-\nu_g^2} \left[2 \times \left(\bar{v}_{1b}'^2 h_1 + \frac{h_1^3}{12} \bar{w}_b'^2 \right) + 2\lambda h_1 v_p' \bar{w}_b'^2 \right] dx dy \\
& + \iint_A \frac{2\nu E_g}{1-\nu_g^2} \left[2 \left(\frac{h_1^3}{12} \bar{w}_b'' \bar{w}_b + h_1 \bar{u}_{1b}' \bar{v}_{1b}' \right) + \lambda h_1 (u_p') \bar{w}_b'^2 + \lambda h_1 (v_p') \bar{w}_b'^2 \right] dx dy dz \\
& + \iint_A \frac{E_g}{2(1+\nu_g)} \left[2 \times \left(\frac{h_1^3}{3} \bar{w}_b'^2 + \bar{u}_{1b}'^2 h_1 + \bar{v}_{1b}'^2 h_1 + 2h_1 \bar{u}_{1b}' \bar{v}_{1b}' \right) + 4\lambda h_1 (\dot{u}_p + v_p') \bar{w}_b' \bar{w}_b \right] dx dy \\
& + \frac{1}{2} \iint_A \frac{E_s}{1-\nu_s^2} \left(\frac{h_3^3}{6} \bar{w}_b'^2 + 2h_3 \lambda u_p' \bar{w}_b'^2 \right) dx dy + \frac{1}{2} \iint_A \frac{E_s}{1-\nu_s^2} \left(\frac{h_3^3}{6} \bar{w}_b'^2 + 2\lambda h_3 v_p' \bar{w}_b'^2 \right) dx dy \\
& + \frac{1}{2} \iint_A \frac{2\nu_s E_s}{1-\nu_s^2} \left(\frac{h_3^3}{6} \bar{w}_b'' \bar{w}_b + \lambda h_3 u_p' \bar{w}_b'^2 + \lambda h_3 v_p' \bar{w}_b'^2 \right) dx dy \\
& + \frac{1}{2} \iint_A \frac{E_s}{2(1+\nu_s)} \left[\frac{2h_3^3}{3} \bar{w}_b'^2 + 4\lambda h_3 (\dot{u}_p + v_p') \bar{w}_b' \bar{w}_b \right] dx dy \\
& + \iint 2G_a \left\{ \frac{h_2^3}{3} \bar{w}_b'^2 + h_2 \left[\frac{\bar{u}_{1b}'^2}{4} + \left(\frac{h_3 - h_1}{4} \right)^2 \bar{w}_b'^2 \right] + h_2 \left[\frac{\bar{v}_{1b}'^2}{4} + \left(\frac{h_3 - h_1}{4} \right)^2 \bar{w}_b'^2 \right] \right. \\
& + \frac{h_2}{48} \left(4\bar{u}_{1b}'^2 + (h_1 + h_3)^2 \bar{w}_b'^2 - 4\bar{u}_{1b}' \bar{w}_b' (h_1 + h_3) \right) + \frac{h_2}{48} \left(4\bar{v}_{1b}'^2 + (h_1 + h_3)^2 \bar{w}_b'^2 - 4\bar{v}_{1b}' \bar{w}_b' (h_1 + h_3) \right) \\
& - \frac{h_a^2}{6} \left(2\bar{u}_{1b}' \bar{w}_b' - h_g \bar{w}_b'^2 - h_s \bar{w}_b'^2 \right) - \frac{h_a^2}{6} \left(2\bar{v}_{1b}' \bar{w}_b' - h_g \bar{w}_b'^2 - h_s \bar{w}_b'^2 \right) \\
& + h_a \left[\left(\frac{h_s - h_g}{4} \right) \bar{u}_{1b}' \bar{w}_b' + \frac{\bar{u}_{1b}' \bar{v}_{1b}'}{2} + \left(\frac{h_s - h_g}{4} \right) \bar{v}_{1b}' \bar{w}_b' + 2 \left(\frac{h_s - h_g}{4} \right)^2 \bar{w}_b'^2 \right] \\
& + \frac{h_a}{24} \left[4\bar{u}_{1b}' \bar{v}_{1b}' - 2\bar{u}_{1b}' (h_g + h_s) \bar{w}_b' - 2\bar{v}_{1b}' (h_g + h_s) \bar{w}_b' + (h_g + h_s)^2 \bar{w}_b'^2 \right] \\
& + \left[\frac{4\bar{u}_{1b}'^2 + (h_1 + h_3)^2 \bar{w}_b'^2 - 4\bar{u}_{1b}' (h_1 + h_3) \bar{w}_b'}{4h_2} \right] + h_2 \bar{w}_b'^2 + (2\bar{u}_{1b}' \bar{w}_b' - h_1 \bar{w}_b'^2 - h_3 \bar{w}_b'^2) \\
& + \left[\frac{4\bar{v}_{1b}'^2 + (h_1 + h_3)^2 \bar{w}_b'^2 - 4\bar{v}_{1b}' (h_1 + h_3) \bar{w}_b'}{4h_2} \right] + h_2 \bar{w}_b'^2 + (2\bar{v}_{1b}' \bar{w}_b' - h_1 \bar{w}_b'^2 - h_3 \bar{w}_b'^2) \\
& \left. + \iint_A 2h_3 \lambda (\dot{u}_{1p} + v_{1p}') \bar{w}_b' \bar{w}_b dx dy \right\}
\end{aligned}$$

(A.2.3)

APPENDIX A.3-Tables related to parametric studies in Chapter 5

A.3.1- Effect of GFRP thickness on the critical pressure of composite flange (Section 5.4.1.1)

Table A.3.1 provides the numeric values for runs 1 through 16 of the parametric study investigated in Section 5.4.1.1 where the GFRP thickness was varied.

Table A.3.1- Effect of GFRP thickness on the critical pressure ($t_a = 1mm, G_a = 200 MPa$)

Run number	t_g/t_{sr}	$\left[P_{yy(cr)}(t_{sr} + 2t_g) \right] / (P_r t_{sr})$ Based on Present Study	$\left[P_{yy(cr)}(t_{sr} + 2t_g) \right] / (P_r t_{sr})$ Based on Abaqus
1 (Reference case)	0.875	2.290	2.693
2	1	2.654	-
3	1.125	3.078	-
4	1.25	3.572	4.197
5	1.375	4.133	-
6	1.5	4.772	-
7	1.625	5.496	-
8	1.75	6.308	-
9	1.875	7.211	8.464
10	2	8.216	-
11	2.125	9.319	-
12	2.25	10.536	-
13	2.375	11.860	-
14	2.5	13.314	-
15	2.625	14.890	-
16	2.75	16.599	15.025

A.3.2- Effect of adhesive thickness on the critical pressure of composite flange (Section 5.4.1.2)

Table A.3.2 provides the numeric values for runs 1 through 18 of the parametric study investigated in Section 5.4.1.2 in which the adhesive thickness is varied.

Table A.3.2-Effect of adhesive thickness on critical pressure ($t_g = 7mm$, $G_a = 200 MPa$)

Run number	t_a / t_{sr}	$\left[P_{yy(cr)} (t_{sr} + 2t_g) \right] / (P_r t_{sr})$ Based on Present Study	$\left[P_{yy(cr)} (t_{sr} + 2t_g) \right] / (P_r t_{sr})$ Based on Abaqus
1	0.0625	2.956	3.243
2	0.09375	2.558	2.925
3 (Reference Case)	0.125	2.290	2.692
4	0.15	2.130	-
5	0.175	2.003	-
6	0.2	1.900	-
7	0.225	1.815	2.541
8	0.25	1.743	-
9	0.275	1.682	-
10	0.3	1.630	-
11	0.325	1.586	-
12	0.35	1.546	-
13	0.375	1.512	-
14	0.4	1.482	2.366
15	0.425	1.456	-
16	0.45	1.433	-
17	0.475	1.413	2.336
18	0.5	1.396	-

A.3.3 Effect of Adhesive shear modulus on critical pressure of composite flange (Section 5.4.1.3)

Table A.3.3 provides the results of runs 1 through 7 in which the shear modulus of adhesive was varied.

Table A.3.3 - Effect of adhesive shear modulus on the critical pressure ($t_g = 7 mm$, $t_a = 1mm$)

Run number	$10000 \times G_a / E_s$	$\left[P_{yy(cr)} (t_{sr} + 2t_g) \right] / (P_r t_{sr})$ Based on Present Study	$\left[P_{yy(cr)} (t_{sr} + 2t_g) \right] / (P_r t_{sr})$ Based on Abaqus
1	5	1.923	2.243
2	7.5	2.124	2.518
3 (Reference Case)	10	2.290	2.693
4	12.5	2.424	-
5	15	2.539	3.056
6	17.5	2.639	-
7	20	2.726	3.218

A.3.4 Effect of GFRP thickness on the critical pressure of composite plate under pure shear (Section 5.4.2.1)

Table A.3.4 provides runs 1 through 7 of the parametric study investigated in Section 5.4.1.1 in which the GFRP thickness is varied from the minimum to maximum range.

Table A.3.4- Effect of GFRP thickness on the critical pressure ($t_a = 1mm, G_a = 100 MPa$)

Run number	t_g/t_{sr}	$\left[P_{yy(cr)} (t_{sr} + 2t_g) \right] / (P_r t_{sr})$ Based on Present Study
1 (Reference case)	0.5	2.045
2	0.571	2.172
3	0.643	2.316
4	0.714	2.477
5	0.786	2.655
6	0.857	2.845
7	0.928	3.049

A.3.5 Effect of adhesive thickness on the critical pressure of a composite plate under pure shear (Section 5.4.2.2)

Table A.3.5 provides the results of runs 1 through 8 of the parametric study investigated in Section 5.4.2.2 in which the adhesive thickness was varied

Table A.3.5-Effect of adhesive thickness on critical pressure ($t_g = 7mm, G_a = 100 MPa$)

Run number	t_a/t_{sr}	$\left[P_{yy(cr)} (t_{sr} + 2t_g) \right] / (P_r t_{sr})$ Based on Present Study
1	0.03571	2.212
2(Reference Case)	0.07143	2.045
3	0.1071	1.899
4	0.1428	1.773
5	0.1786	1.661
6	0.2143	1.563
7	0.2500	1.476
8	0.2857	1.387

A.3.6 Effect of shear modulus of adhesive on critical pressure of composite plate under pure shear (Section 5.4.2.3)

Table A.3.6 the results of runs 1 through 7 in which the shear modulus of adhesive was varied.

Table A.3.6 - Effect of adhesive shear modulus on the critical pressure ($t_g = 7 \text{ mm}, t_a = 1 \text{ mm}$)

Run number	G_a/E_s	$\left[P_{yy(cr)} (t_{sr} + 2t_g) \right] / (P_r t_{sr})$ Based on Present Study
1	0.0005	2.045
2	0.00075	2.077
3 (Reference Case)	0.001	2.095
4	0.00125	2.106
5	0.0015	2.113
6	0.00175	2.119
7	0.002	2.123

APPENDIX A.4- Finite element MATLAB Code

This appendix contains the codes have been used in MATLAB to implement the finite element formulation developed in Chapter 4. The code is consist of a main program which calls several functions. All the input data are collected from an excel sheet and are read by MATLAB. Table A.4.1 is a sample of input data related to Section 5.4.1.2 which evaluate the effect of adhesive thickness on the buckling capacity of reinforced flange under uniaxial normal pressure. All the parameters are inputted in to the cells of a table in sheet 2 of a spreadsheet named 'input.xlsx'. Eighteen model are inputted in the excel file but only four models are presented in Table A.4.1 as a sample. However, more than 1000 models can be inputted and read by the program. The second and third row of Table A.4.1 is related to the length of the composite system along x and y direction. The fourth and fifth row shows the number of elements along x and y . The mechanical properties and thicknesses of the layers are tabulated in rows 6 to12. The pre-buckling in-plane strains are inputted in rows 13 to 16 of the table. The main program and all the functions are listed below.

Table A.4.1- Sample of input data file related to Section 5.4.1.2

1	A		B	C	D	E
	Number of Runs=	18	Model #1	Model #2	Model #3	Model #4
2	L_x		0.1025	0.1025	0.1025	0.1025
3	L_y		0.3075	0.3075	0.3075	0.3075
4	n_x		15	15	15	15
5	n_y		45	45	45	45
6	ν		0.3	0.3	0.3	0.3
7	E_s		200000000	200000000	200000000	200000000
8	E_g		420000000	420000000	420000000	420000000
9	G_a		200000	200000	200000	200000
10	t_s		0.008	0.008	0.008	0.008
11	t_g		0.007	0.007	0.007	0.007
12	t_a		0.0005	0.00075	0.001	0.0012
13	u_p		3.106×10^{-9}	3.106×10^{-9}	3.106×10^{-9}	3.105×10^{-9}
14	v_d		-1.005×10^{-8}	-1.005×10^{-8}	-1.005×10^{-8}	-1.004×10^{-8}
15	u_d		0	0	0	0
16	v_p		0	0	0	0

```

clc
clear all

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%% Reading Input Data from an Excel File %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
NDFN = 5;          % Number of Degree of Freedom for each Node

file_name='input.xlsx';
Run=xlsread(file_name,2,'B1');
Results=zeros(Run,1);
x=findcolumn(Run);

Lxt=xlsread(file_name, 2, ['C2:' char(x) '2']);
Lyt=xlsread(file_name, 2, ['C3:' char(x) '3']);
nxt=xlsread(file_name, 2, ['C4:' char(x) '4']);
nyt=xlsread(file_name, 2, ['C5:' char(x) '5']);
vt=xlsread(file_name, 2, ['C6:' char(x) '6']);
Est=xlsread(file_name, 2, ['C7:' char(x) '7']);
Egt=xlsread(file_name, 2, ['C8:' char(x) '8']);
Gat=xlsread(file_name, 2, ['C9:' char(x) '9']);
hst=xlsread(file_name, 2, ['C10:' char(x) '10']);
hgt=xlsread(file_name, 2, ['C11:' char(x) '11']);
hat=xlsread(file_name, 2, ['C12:' char(x) '12']);
upt=xlsread(file_name, 2, ['C13:' char(x) '13']);
vdt=xlsread(file_name, 2, ['C14:' char(x) '14']);
udt=xlsread(file_name, 2, ['C15:' char(x) '15']);
vpt=xlsread(file_name, 2, ['C16:' char(x) '16']);

for RC=1:Run

Lx=Lxt(1,RC);
Ly=Lyt(1,RC);
nx=nxt(1,RC);
ny=nyt(1,RC);
v=vt(1,RC);
Es=Est(1,RC);
Eg=Egt(1,RC);
Ga=Gat(1,RC);
hs=hst(1,RC);
hg=hgt(1,RC);
ha=hat(1,RC);
up=upt(1,RC);
vd=vdt(1,RC);
ud=udt(1,RC);
vp=vpt(1,RC);

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
Determine element dimensions, number of nodes and number of supports
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
a=Lx/(2*nx);          % Half-length of element in x direction
b=Ly/(2*ny);          % Half-length of element in y direction
NN=(nx+1)*(ny+1); % Number of nodes

```

```

NS=(nx+ny+2)*2-4; % Number of supports
sup=zeros(NS,6);
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%% Determine boundary conditions %%%%%%%%%%
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% Front edge
for i=1:nx+1
    sup(i,1)=i; sup(i,2)=1; sup(i,3)=1; sup(i,4)=1; sup(i,5)=1; sup(i,6)=1;
end

% Left edge
j=0;
for i=nx+2:2:NS-(nx+1)
    sup(i,1)=nx+2+j*(nx+1); sup(i,2)=1; sup(i,3)=1; sup(i,4)=1; sup(i,5)=1;
    sup(i,6)=1;
    j=j+1;
end

% Right edge
j=1;
for i=nx+3:2:NS-(nx+1)
    sup(i,1)=nx+1+j*(nx+1); sup(i,4)=0; sup(i,5)=0; sup(i,6)=0;
    j=j+1;
end

% Back edge
for i=1:nx+1
    sup(NS-nx+i-1,1)=NN-nx+i-1; sup(NS-nx+i-1,2)=1; sup(NS-nx+i-1,3)=1;
    sup(NS-nx+i-1,4)=1; sup(NS-nx+i-1,5)=1; sup(NS-nx+i-1,6)=1;
end
NR=0;
for i=1:NS
    for j=2:(NDFN+1)
        if sup(i,j)==1
            NR=NR+1;
        end
    end
end
NDFT=NDFN*NN-NR;

%Calculate Element Stiffness Matrix
[k,g]=Calculate_member_stiffness_matrix(Es,Eg,v,Ga,hg,hs,ha,a,b,up,vd,ud,vp);
% Identify nodes with supports %%%%%%%%%%
DFC=Build_DFC(NDFN,NN,NS,NDFT, sup);

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%% Calculate Total Stiffness Matrix %%%%%%%%%%
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
K=zeros(NDFT);
G=zeros(NDFT);
ii=0;
for j=1:ny
    for i=1:nx
        ii=ii+1;
        N1=ii+(j-1);
        N2=N1+1;
    end
end

```

```

N3=N2+nx+1;
N4=N3-1;

K=Calculate_K(NDFN, NDFT, N1, N2, N3, N4, k, K,DFC);
G=Calculate_K(NDFN, NDFT, N1, N2, N3, N4, g, G,DFC);

end
end
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%% Calculate Eigenvalues and Eigen vectors %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
[V,D]=eig(K,-G);
[D2, IX]=sort(diag(D));
for i=1:NDFT
    if D2(i,1)>0

        lambda(RC,1)=D2(i,1)
        jj=IX(i,1);
        break
    end
end
end

icount=0;
Disp=zeros(NDFN*NN,1);
for i=1:NDFN*NN
    if DFC(i,1)<=NDFT
        icount=icount+1;
        Disp(i,1)=V(icount,jj);
    else
        Disp(i,1)=0;
    end
end
end

for i=3:5:NDFN*NN

    Wb((i-3)/5+1,1)=Disp(i,1);
end

for i=1:5:NDFN*NN

    Ub((i-1)/5+1,1)=Disp(i,1);
end
for i=2:5:NDFN*NN

    Vb((i-2)/5+1,1)=Disp(i,1);
end
i1=0;
for i=1:ny+1
    for j=1:nx+1
        i1=i1+1;
        wb(i,j)=Wb(i1,1);
        ub(i,j)=Ub(i1,1);
        vb(i,j)=Vb(i1,1);
    end
end
end

```

```

        D2t(:,RC)=D2;
        wbt(:, :,RC)=wb;
        ubt(:, :,RC)=ub;
        vbt(:, :,RC)=vb;
end
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%% End of the main program %%%%%%%%%%

```

```

function [x]=findcolumn(RC)
% This function reads the input data from an excel file

a=24;
b=26;
if RC<=a
    x=RC+66; end
if RC>a && RC<=a+b
    RCC=RC-a; x=[65 64+RCC]; end
if RC>a+b && RC<=a+2*b
    RCC=RC-a-b; x=[66 64+RCC]; end
if RC>a+2*b && RC<=a+3*b
    RCC=RC-a-2*b; x=[67 64+RCC]; end
if RC>a+3*b && RC<=a+4*b
    RCC=RC-a-3*b; x=[68 64+RCC]; end
if RC>a+4*b && RC<=a+5*b
    RCC=RC-a-4*b; x=[69 64+RCC]; end
if RC>a+5*b && RC<=a+6*b
    RCC=RC-a-5*b; x=[70 64+RCC]; end
if RC>a+6*b && RC<=a+7*b
    RCC=RC-a-6*b; x=[71 64+RCC]; end
if RC>a+7*b && RC<=a+8*b
    RCC=RC-a-7*b; x=[72 64+RCC]; end
if RC>a+8*b && RC<=a+9*b
    RCC=RC-a-8*b; x=[73 64+RCC]; end
if RC>a+9*b && RC<=a+10*b
    RCC=RC-a-9*b; x=[74 64+RCC]; end
if RC>a+10*b && RC<=a+11*b
    RCC=RC-a-10*b; x=[75 64+RCC]; end
if RC>a+11*b && RC<=a+12*b
    RCC=RC-a-11*b; x=[76 64+RCC]; end
if RC>a+12*b && RC<=a+13*b
    RCC=RC-a-12*b; x=[77 64+RCC]; end
if RC>a+13*b && RC<=a+14*b
    RCC=RC-a-13*b; x=[78 64+RCC]; end
if RC>a+14*b && RC<=a+15*b
    RCC=RC-a-14*b; x=[79 64+RCC]; end
if RC>a+15*b && RC<=a+16*b
    RCC=RC-a-15*b; x=[80 64+RCC]; end
if RC>a+16*b && RC<=a+17*b
    RCC=RC-a-16*b; x=[81 64+RCC]; end
if RC>a+17*b && RC<=a+18*b
    RCC=RC-a-17*b; x=[82 64+RCC]; end
if RC>a+18*b && RC<=a+19*b
    RCC=RC-a-18*b; x=[83 64+RCC]; end
if RC>a+19*b && RC<=a+20*b
    RCC=RC-a-19*b; x=[84 64+RCC]; end
end
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%% End of the function %%%%%%%%%%%%%%%

```

```

function [DFC]=Build_DFC(NDFN,NN,NS,NDFT, SUP)
DFC=zeros(NDFN*NN,1);
j=0;
k=NDFT;
for i=1:NN
    icount=0;
    for i1=1:NS
        if SUP(i1,1)==i
            icount=1;
            for i2=1:NDFN
                i3=(i-1)*NDFN+i2;
                if SUP(i1,i2+1)==1
                    k=k+1;
                    DFC(i3)=k;
                else
                    j=j+1;
                    DFC(i3)=j;
                end
            end
        end
    end
    if icount==0
        for i2=1:NDFN
            i3=(i-1)*NDFN+i2;
            j=j+1;
            DFC(i3)=j;
        end
    end
end
end
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%% End of the function %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

```

```

function[k,g]=Calculate_member_stiffness_matrix(Es,Eg,v,Ga,hg,hs,ha,a,b,up,vd
,ud,vp)
% function that calculate the member stiffness matrix
syms x y
%material properties of the layers
B1=0.25-0.25*(x/a)-0.25*(y/b)+0.25*(x*y/(a*b));
B2=0.25+0.25*(x/a)-0.25*(y/b)-0.25*(x*y/(a*b));
B3=0.25+0.25*(x/a)+0.25*(y/b)+0.25*(x*y/(a*b));
B4=0.25-0.25*(x/a)+0.25*(y/b)-0.25*(x*y/(a*b));

B=[B1;B2;B3;B4];

B1p=diff(B1,x);
B1d=diff(B1,y);
B2p=diff(B2,x);
B2d=diff(B2,y);
B3p=diff(B3,x);
B3d=diff(B3,y);
B4p=diff(B4,x);
B4d=diff(B4,y);

Bp=[B1p;B2p;B3p;B4p];
Bd=[B1d;B2d;B3d;B4d];

[Hw1,Hw2,Hw3,Hw4,Htetax1,Htetax2,Htetax3,Htetax4,Htetay1,Htetay2,Htetay3,Htet
ay4] = wfunction(a,b);
%H=[Hw1;Htetax1;Htetay1;Hw2;Htetax2;Htetay2;Hw3;Htetax3;Htetay3;Hw4;Htetax4;H
tetay4;];

Hw1p=diff(Hw1,x);
Hw2p=diff(Hw2,x);
Hw3p=diff(Hw3,x);
Hw4p=diff(Hw4,x);
Htetax1p=diff(Htetax1,x);
Htetax2p=diff(Htetax2,x);
Htetax3p=diff(Htetax3,x);
Htetax4p=diff(Htetax4,x);
Htetay1p=diff(Htetay1,x);
Htetay2p=diff(Htetay2,x);
Htetay3p=diff(Htetay3,x);
Htetay4p=diff(Htetay4,x);

Hp=[Hw1p; Htetax1p; Htetay1p; Hw2p; Htetax2p; Htetay2p; Hw3p; Htetax3p;
Htetay3p; Hw4p; Htetax4p; Htetay4p];

Hw1d=diff(Hw1,y);
Hw2d=diff(Hw2,y);
Hw3d=diff(Hw3,y);
Hw4d=diff(Hw4,y);
Htetax1d=diff(Htetax1,y);
Htetax2d=diff(Htetax2,y);
Htetax3d=diff(Htetax3,y);
Htetax4d=diff(Htetax4,y);
Htetay1d=diff(Htetay1,y);

```

```
Htetay2d=diff(Htetay2,y);
Htetay3d=diff(Htetay3,y);
Htetay4d=diff(Htetay4,y);
```

```
Hd=[Hw1d; Htetax1d; Htetay1d; Hw2d; Htetax2d; Htetay2d; Hw3d; Htetax3d;
Htetay3d; Hw4d; Htetax4d; Htetay4d];
```

```
Hw1pp=diff(Hw1p,x);
Hw2pp=diff(Hw2p,x);
Hw3pp=diff(Hw3p,x);
Hw4pp=diff(Hw4p,x);
Htetax1pp=diff(Htetax1p,x);
Htetax2pp=diff(Htetax2p,x);
Htetax3pp=diff(Htetax3p,x);
Htetax4pp=diff(Htetax4p,x);
Htetay1pp=diff(Htetay1p,x);
Htetay2pp=diff(Htetay2p,x);
Htetay3pp=diff(Htetay3p,x);
Htetay4pp=diff(Htetay4p,x);
```

```
Hpp=[Hw1pp; Htetax1pp; Htetay1pp; Hw2pp; Htetax2pp; Htetay2pp; Hw3pp;
Htetax3pp; Htetay3pp; Hw4pp; Htetax4pp; Htetay4pp];
```

```
Hw1dd=diff(Hw1d,y);
Hw2dd=diff(Hw2d,y);
Hw3dd=diff(Hw3d,y);
Hw4dd=diff(Hw4d,y);
Htetax1dd=diff(Htetax1d,y);
Htetax2dd=diff(Htetax2d,y);
Htetax3dd=diff(Htetax3d,y);
Htetax4dd=diff(Htetax4d,y);
Htetay1dd=diff(Htetay1d,y);
Htetay2dd=diff(Htetay2d,y);
Htetay3dd=diff(Htetay3d,y);
Htetay4dd=diff(Htetay4d,y);
```

```
Hdd=[Hw1dd; Htetax1dd; Htetay1dd; Hw2dd; Htetax2dd; Htetay2dd; Hw3dd;
Htetax3dd; Htetay3dd; Hw4dd; Htetax4dd; Htetay4dd];
```

```
Hw1pd=diff(Hw1d,x);
Hw2pd=diff(Hw2d,x);
Hw3pd=diff(Hw3d,x);
Hw4pd=diff(Hw4d,x);
Htetax1pd=diff(Htetax1d,x);
Htetax2pd=diff(Htetax2d,x);
Htetax3pd=diff(Htetax3d,x);
Htetax4pd=diff(Htetax4d,x);
Htetay1pd=diff(Htetay1d,x);
Htetay2pd=diff(Htetay2d,x);
Htetay3pd=diff(Htetay3d,x);
Htetay4pd=diff(Htetay4d,x);
```

```
Hpd=[Hw1pd; Htetax1pd; Htetay1pd; Hw2pd; Htetax2pd; Htetay2pd; Hw3pd;
Htetax3pd; Htetay3pd; Hw4pd; Htetax4pd; Htetay4pd];
```

$BB=B*B'$;
 $BpBp=Bp*Bp'$;
 $BdBd=Bd*Bd'$;
 $BpBd=Bp*Bd'$;
 $BdBp=Bd*Bp'$;

$BHp=B*Hp'$;
 $BHd=B*Hd'$;
 $BpHp=Bp*Hp'$;
 $BdHp=Bd*Hp'$;
 $HpB=Hp*B'$;
 $HdB=Hd*B'$;
 $HpdB=HpdBp'$;
 $HpdBd=HpdBd'$;

$HpHp=Hp*Hp'$;
 $HdHd=Hd*Hd'$;
 $HdHp=Hd*Hp'$;
 $HpHd=Hp*Hd'$;
 $HddHdd=Hdd*Hdd'$;
 $HppHpp=Hpp*Hpp'$;
 $HpdBp=Bp*Hp'$;
 $HppHdd=Hpp*Hdd'$;
 $HddHpp=Hdd*Hpp'$;

$A1=2*Ga/ha$; $A2=2*Eg*hg/(1-v^2)$;
 $A3=2*Ga*(hs^2/4/ha+(1/3/ha^2)*(hs/2+ha)^3-(hs/2)^3-hs/2/ha^2*((hs/2+ha)^2-(hs/2)^2))+Eg*hg/(1+v)$;
 $A4=2*Ga/ha$; $A5=2*Eg*hg/(1-v^2)$;
 $A6=2*Ga*(hs^2/4/ha+(1/3/ha^2)*(hs/2+ha)^3-(hs/2)^3-hs/2/ha^2*((hs/2+ha)^2-(hs/2)^2))+Eg*hg/(1+v)$;
 $A7=2*Ga*((hg+hs)^2/4/ha+ha-(hs+hg))$; $A8=2*Ga*((hg+hs)^2/4/ha+ha-(hs+hg))$;
 $A9=Eg*hg^3/6/(1-v^2)+Es*hs^3/12/(1-v^2)$;
 $A10=Eg*hg^3/6/(1-v^2)+Es*hs^3/12/(1-v^2)$;
 $A11=2*Ga*(ha*(hg*hs/2/ha+hs^2/2/ha+hs)^2+(1/3)*((hs/2+ha)^3-(hs/2)^3)*((hg+hs)/ha+2)^2)$;
 $A12=2*Ga*(-((hs/2+ha)^2-(hs/2)^2)*(hg*hs/2/ha+hs^2/2/ha+hs)*((hg+hs)/ha+2))$;
 $A13=Eg*hg^3/3/(1+v)+Es*hs^3/6/(1+v)$; $A14=v*Eg*hg^3/3/(1-v^2)+v*Es*hs^3/6/(1-v^2)$;
 $A15=2*Ga*(2-(hg+hs)/ha)$; $A16=2*Ga*(2-(hg+hs)/ha)$; $A17=4*v*Eg*hg/(1-v^2)$;
 $A18=2*Ga*(2/3/ha^2*((hs/2+ha)^3-(hs/2)^3)+hs^2/2/ha-hs/ha^2*((hs/2+ha)^2-(hs/2)^2))+2*Eg*hg/(1+v)$;
 $A19=2*Ga*(-2/3*((hs/2+ha)^3-(hs/2)^3)*((hg+hs)/ha^2+2/ha)-hs*(hg*hs/2/ha+hs^2/2/ha+hs))$;
 $A20=2*Ga*(hs/2/ha*((hg+hs)/ha+2)*((hs/2+ha)^2-(hs/2)^2)+((hs/2+ha)^2-(hs/2)^2)*(hg*hs/2/ha+hs^2/2/ha+hs/ha))$;
 $A21=A19$; $A22=A20$;
 $Bb1=(2*Eg*hg*up+2*v*Eg*hg*vd+Es*hs*up+v*Es*hs*vd)/(1-v^2)$;
 $Bb2=(2*Eg*hg*vd+2*v*Eg*hg*up+Es*hs*vd+v*Es*hs*up)/(1-v^2)$;
 $Bb3=(ud+vp)*(4*Ga*ha+(2*Eg*hg+Es*hs)/(1+v))$;

$C1=2*A1*BB+2*A2*BpBp+2*A3*BdBd$;
 $C2=A17*BpBd+A18*BdBp$;
 $C3=(A19+A20)*BdHp+B15*BHp$;

$C4=A17*BdBp+A18*BpBd;$
 $C5=2*A4*BB+2*A5*BdBd+2*A6*BpBp;$
 $C6=(A21+A22)*BpHp d+A16*BHD;$

 $C7=(A19+A20)*Hp dBd+A15*HpB;$
 $C8=A21*Hp dBp+A22*Hp dBp+A16*HdB;$
 $C9=2*A9*HppHpp+2*A10*HddHdd+2*(A11+A12+A13)*Hp dHp d+A14*HppHdd+A14*HddHpp+2*A7$
 $*HpHp+2*A8*HdHd;$

 $D1=2*Bb1*HpHp+Bb3*HpHd+Bb3*HdHp+2*Bb2*HdHd;$

 $k(1,1)=C1(1,1); \quad k(1,2)=C2(1,1); \quad k(1,3)=C3(1,1); \quad k(1,4)=C3(1,2);$
 $k(1,5)=C3(1,3); \quad k(1,6)=C1(1,2); \quad k(1,7)=C2(1,2); \quad k(1,8)=C3(1,4);$
 $k(1,9)=C3(1,5); \quad k(1,10)=C3(1,6); \quad k(1,11)=C1(1,3); \quad k(1,12)=C2(1,3);$
 $k(1,13)=C3(1,7); \quad k(1,14)=C3(1,8); \quad k(1,15)=C3(1,9); \quad k(1,16)=C1(1,4);$
 $k(1,17)=C2(1,4); \quad k(1,18)=C3(1,10); \quad k(1,19)=C3(1,11); \quad k(1,20)=C3(1,12);$

 $k(2,1)=C4(1,1); \quad k(2,2)=C5(1,1); \quad k(2,3)=C6(1,1); \quad k(2,4)=C6(1,2);$
 $k(2,5)=C6(1,3); \quad k(2,6)=C4(1,2); \quad k(2,7)=C5(1,2); \quad k(2,8)=C6(1,4);$
 $k(2,9)=C6(1,5); \quad k(2,10)=C6(1,6); \quad k(2,11)=C4(1,3); \quad k(2,12)=C5(1,3);$
 $k(2,13)=C6(1,7); \quad k(2,14)=C6(1,8); \quad k(2,15)=C6(1,9); \quad k(2,16)=C4(1,4);$
 $k(2,17)=C5(1,4); \quad k(2,18)=C6(1,10); \quad k(2,19)=C6(1,11); \quad k(2,20)=C6(1,12);$

 $k(3,1)=C7(1,1); \quad k(3,2)=C8(1,1); \quad k(3,3)=C9(1,1); \quad k(3,4)=C9(1,2);$
 $k(3,5)=C9(1,3); \quad k(3,6)=C7(1,2); \quad k(3,7)=C8(1,2); \quad k(3,8)=C9(1,4);$
 $k(3,9)=C9(1,5); \quad k(3,10)=C9(1,6); \quad k(3,11)=C7(1,3); \quad k(3,12)=C8(1,3);$
 $k(3,13)=C9(1,7); \quad k(3,14)=C9(1,8); \quad k(3,15)=C9(1,9); \quad k(3,16)=C7(1,4);$
 $k(3,17)=C8(1,4); \quad k(3,18)=C9(1,10); \quad k(3,19)=C9(1,11); \quad k(3,20)=C9(1,12);$

 $k(4,1)=C7(2,1); \quad k(4,2)=C8(2,1); \quad k(4,3)=C9(2,1); \quad k(4,4)=C9(2,2);$
 $k(4,5)=C9(2,3); \quad k(4,6)=C7(2,2); \quad k(4,7)=C8(2,2); \quad k(4,8)=C9(2,4);$
 $k(4,9)=C9(2,5); \quad k(4,10)=C9(2,6); \quad k(4,11)=C7(2,3); \quad k(4,12)=C8(2,3);$
 $k(4,13)=C9(2,7); \quad k(4,14)=C9(2,8); \quad k(4,15)=C9(2,9); \quad k(4,16)=C7(2,4);$
 $k(4,17)=C8(2,4); \quad k(4,18)=C9(2,10); \quad k(4,19)=C9(2,11); \quad k(4,20)=C9(2,12);$

 $k(5,1)=C7(3,1); \quad k(5,2)=C8(3,1); \quad k(5,3)=C9(3,1); \quad k(5,4)=C9(3,2);$
 $k(5,5)=C9(3,3); \quad k(5,6)=C7(3,2); \quad k(5,7)=C8(3,2); \quad k(5,8)=C9(3,4);$
 $k(5,9)=C9(3,5); \quad k(5,10)=C9(3,6); \quad k(5,11)=C7(3,3); \quad k(5,12)=C8(3,3);$
 $k(5,13)=C9(3,7); \quad k(5,14)=C9(3,8); \quad k(5,15)=C9(3,9); \quad k(5,16)=C7(3,4);$
 $k(5,17)=C8(3,4); \quad k(5,18)=C9(3,10); \quad k(5,19)=C9(3,11); \quad k(5,20)=C9(3,12);$

 $k(6,1)=C1(2,1); \quad k(6,2)=C2(2,1); \quad k(6,3)=C3(2,1); \quad k(6,4)=C3(2,2);$
 $k(6,5)=C3(2,3); \quad k(6,6)=C1(2,2); \quad k(6,7)=C2(2,2); \quad k(6,8)=C3(2,4);$
 $k(6,9)=C3(2,5); \quad k(6,10)=C3(2,6); \quad k(6,11)=C1(2,3); \quad k(6,12)=C2(2,3);$
 $k(6,13)=C3(2,7); \quad k(6,14)=C3(2,8); \quad k(6,15)=C3(2,9); \quad k(6,16)=C1(2,4);$
 $k(6,17)=C2(2,4); \quad k(6,18)=C3(2,10); \quad k(6,19)=C3(2,11); \quad k(6,20)=C3(2,12);$

 $k(7,1)=C4(2,1); \quad k(7,2)=C5(2,1); \quad k(7,3)=C6(2,1); \quad k(7,4)=C6(2,2);$
 $k(7,5)=C6(2,3); \quad k(7,6)=C4(2,2); \quad k(7,7)=C5(2,2); \quad k(7,8)=C6(2,4);$
 $k(7,9)=C6(2,5); \quad k(7,10)=C6(2,6); \quad k(7,11)=C4(2,3); \quad k(7,12)=C5(2,3);$
 $k(7,13)=C6(2,7); \quad k(7,14)=C6(2,8); \quad k(7,15)=C6(2,9); \quad k(7,16)=C4(2,4);$
 $k(7,17)=C5(2,4); \quad k(7,18)=C6(2,10); \quad k(7,19)=C6(2,11); \quad k(7,20)=C6(2,12);$

$k(8,1)=C7(4,1);$ $k(8,2)=C8(4,1);$ $k(8,3)=C9(4,1);$ $k(8,4)=C9(4,2);$
 $k(8,5)=C9(4,3);$ $k(8,6)=C7(4,2);$ $k(8,7)=C8(4,2);$ $k(8,8)=C9(4,4);$
 $k(8,9)=C9(4,5);$ $k(8,10)=C9(4,6);$ $k(8,11)=C7(4,3);$ $k(8,12)=C8(4,3);$
 $k(8,13)=C9(4,7);$ $k(8,14)=C9(4,8);$ $k(8,15)=C9(4,9);$ $k(8,16)=C7(4,4);$
 $k(8,17)=C8(4,4);$ $k(8,18)=C9(4,10);$ $k(8,19)=C9(4,11);$ $k(8,20)=C9(4,12);$

$k(9,1)=C7(5,1);$ $k(9,2)=C8(5,1);$ $k(9,3)=C9(5,1);$ $k(9,4)=C9(5,2);$
 $k(9,5)=C9(5,3);$ $k(9,6)=C7(5,2);$ $k(9,7)=C8(5,2);$ $k(9,8)=C9(5,4);$
 $k(9,9)=C9(5,5);$ $k(9,10)=C9(5,6);$ $k(9,11)=C7(5,3);$ $k(9,12)=C8(5,3);$
 $k(9,13)=C9(5,7);$ $k(9,14)=C9(5,8);$ $k(9,15)=C9(5,9);$ $k(9,16)=C7(5,4);$
 $k(9,17)=C8(5,4);$ $k(9,18)=C9(5,10);$ $k(9,19)=C9(5,11);$ $k(9,20)=C9(5,12)$

$k(10,1)=C7(6,1);$ $k(10,2)=C8(6,1);$ $k(10,3)=C9(6,1);$ $k(10,4)=C9(6,2);$
 $k(10,5)=C9(6,3);$ $k(10,6)=C7(6,2);$ $k(10,7)=C8(6,2);$ $k(10,8)=C9(6,4);$
 $k(10,9)=C9(6,5);$ $k(10,10)=C9(6,6);$ $k(10,11)=C7(6,3);$ $k(10,12)=C8(6,3);$
 $k(10,13)=C9(6,7);$ $k(10,14)=C9(6,8);$ $k(10,15)=C9(6,9);$ $k(10,16)=C7(6,4);$
 $k(10,17)=C8(6,4);$ $k(10,18)=C9(6,10);$ $k(10,19)=C9(6,11);$ $k(10,20)=C9(6,12);$

$k(11,1)=C1(3,1);$ $k(11,2)=C2(3,1);$ $k(11,3)=C3(3,1);$ $k(11,4)=C3(3,2);$
 $k(11,5)=C3(3,3);$ $k(11,6)=C1(3,2);$ $k(11,7)=C2(3,2);$ $k(11,8)=C3(3,4);$
 $k(11,9)=C3(3,5);$ $k(11,10)=C3(3,6);$ $k(11,11)=C1(3,3);$ $k(11,12)=C2(3,3);$
 $k(11,13)=C3(3,7);$ $k(11,14)=C3(3,8);$ $k(11,15)=C3(3,9);$ $k(11,16)=C1(3,4);$
 $k(11,17)=C2(3,4);$ $k(11,18)=C3(3,10);$ $k(11,19)=C3(3,11);$ $k(11,20)=C3(3,12);$

$k(12,1)=C4(3,1);$ $k(12,2)=C5(3,1);$ $k(12,3)=C6(3,1);$ $k(12,4)=C6(3,2);$
 $k(12,5)=C6(3,3);$ $k(12,6)=C4(3,2);$ $k(12,7)=C5(3,2);$ $k(12,8)=C6(3,4);$
 $k(12,9)=C6(3,5);$ $k(12,10)=C6(3,6);$ $k(12,11)=C4(3,3);$ $k(12,12)=C5(3,3);$
 $k(12,13)=C6(3,7);$ $k(12,14)=C6(3,8);$ $k(12,15)=C6(3,9);$ $k(12,16)=C4(3,4);$
 $k(12,17)=C5(3,4);$ $k(12,18)=C6(3,10);$ $k(12,19)=C6(3,11);$ $k(12,20)=C6(3,12);$

$k(13,1)=C7(7,1);$ $k(13,2)=C8(7,1);$ $k(13,3)=C9(7,1);$ $k(13,4)=C9(7,2);$
 $k(13,5)=C9(7,3);$ $k(13,6)=C7(7,2);$ $k(13,7)=C8(7,2);$ $k(13,8)=C9(7,4);$
 $k(13,9)=C9(7,5);$ $k(13,10)=C9(7,6);$ $k(13,11)=C7(7,3);$ $k(13,12)=C8(7,3);$
 $k(13,13)=C9(7,7);$ $k(13,14)=C9(7,8);$ $k(13,15)=C9(7,9);$ $k(13,16)=C7(7,4);$
 $k(13,17)=C8(7,4);$ $k(13,18)=C9(7,10);$ $k(13,19)=C9(7,11);$ $k(13,20)=C9(7,12);$

$k(14,1)=C7(8,1);$ $k(14,2)=C8(8,1);$ $k(14,3)=C9(8,1);$ $k(14,4)=C9(8,2);$
 $k(14,5)=C9(8,3);$ $k(14,6)=C7(8,2);$ $k(14,7)=C8(8,2);$ $k(14,8)=C9(8,4);$
 $k(14,9)=C9(8,5);$ $k(14,10)=C9(8,6);$ $k(14,11)=C7(8,3);$ $k(14,12)=C8(8,3);$
 $k(14,13)=C9(8,7);$ $k(14,14)=C9(8,8);$ $k(14,15)=C9(8,9);$ $k(14,16)=C7(8,4);$
 $k(14,17)=C8(8,4);$ $k(14,18)=C9(8,10);$ $k(14,19)=C9(8,11);$ $k(14,20)=C9(8,12);$

$k(15,1)=C7(9,1);$ $k(15,2)=C8(9,1);$ $k(15,3)=C9(9,1);$ $k(15,4)=C9(9,2);$
 $k(15,5)=C9(9,3);$ $k(15,6)=C7(9,2);$ $k(15,7)=C8(9,2);$ $k(15,8)=C9(9,4);$
 $k(15,9)=C9(9,5);$ $k(15,10)=C9(9,6);$ $k(15,11)=C7(9,3);$ $k(15,12)=C8(9,3);$
 $k(15,13)=C9(9,7);$ $k(15,14)=C9(9,8);$ $k(15,15)=C9(9,9);$ $k(15,16)=C7(9,4);$
 $k(15,17)=C8(9,4);$ $k(15,18)=C9(9,10);$ $k(15,19)=C9(9,11);$ $k(15,20)=C9(9,12);$

$k(16,1)=C1(4,1);$ $k(16,2)=C2(4,1);$ $k(16,3)=C3(4,1);$ $k(16,4)=C3(4,2);$
 $k(16,5)=C3(4,3);$ $k(16,6)=C1(4,2);$ $k(16,7)=C2(4,2);$ $k(16,8)=C3(4,4);$
 $k(16,9)=C3(4,5);$ $k(16,10)=C3(4,6);$ $k(16,11)=C1(4,3);$ $k(16,12)=C2(4,3);$
 $k(16,13)=C3(4,7);$ $k(16,14)=C3(4,8);$ $k(16,15)=C3(4,9);$ $k(16,16)=C1(4,4);$
 $k(16,17)=C2(4,4);$ $k(16,18)=C3(4,10);$ $k(16,19)=C3(4,11);$ $k(16,20)=C3(4,12);$

$k(17,1)=C4(4,1);$ $k(17,2)=C5(4,1);$ $k(17,3)=C6(4,1);$ $k(17,4)=C6(4,2);$
 $k(17,5)=C6(4,3);$ $k(17,6)=C4(4,2);$ $k(17,7)=C5(4,2);$ $k(17,8)=C6(4,4);$
 $k(17,9)=C6(4,5);$ $k(17,10)=C6(4,6);$ $k(17,11)=C4(4,3);$ $k(17,12)=C5(4,3);$
 $k(17,13)=C6(4,7);$ $k(17,14)=C6(4,8);$ $k(17,15)=C6(4,9);$ $k(17,16)=C4(4,4);$
 $k(17,17)=C5(4,4);$ $k(17,18)=C6(4,10);$ $k(17,19)=C6(4,11);$ $k(17,20)=C6(4,12);$

$k(18,1)=C7(10,1);$ $k(18,2)=C8(10,1);$ $k(18,3)=C9(10,1);$ $k(18,4)=C9(10,2);$
 $k(18,5)=C9(10,3);$ $k(18,6)=C7(10,2);$ $k(18,7)=C8(10,2);$ $k(18,8)=C9(10,4);$
 $k(18,9)=C9(10,5);$ $k(18,10)=C9(10,6);$ $k(18,11)=C7(10,3);$ $k(18,12)=C8(10,3);$
 $k(18,13)=C9(10,7);$ $k(18,14)=C9(10,8);$ $k(18,15)=C9(10,9);$ $k(18,16)=C7(10,4);$
 $k(18,17)=C8(10,4);$ $k(18,18)=C9(10,10);$ $k(18,19)=C9(10,11);$ $k(18,20)=C9(10,12);$

$k(19,1)=C7(11,1);$ $k(19,2)=C8(11,1);$ $k(19,3)=C9(11,1);$ $k(19,4)=C9(11,2);$
 $k(19,5)=C9(11,3);$ $k(19,6)=C7(11,2);$ $k(19,7)=C8(11,2);$ $k(19,8)=C9(11,4);$
 $k(19,9)=C9(11,5);$ $k(19,10)=C9(11,6);$ $k(19,11)=C7(11,3);$ $k(19,12)=C8(11,3);$
 $k(19,13)=C9(11,7);$ $k(19,14)=C9(11,8);$ $k(19,15)=C9(11,9);$ $k(19,16)=C7(11,4);$
 $k(19,17)=C8(11,4);$ $k(19,18)=C9(11,10);$ $k(19,19)=C9(11,11);$ $k(19,20)=C9(11,12);$

$k(20,1)=C7(12,1);$ $k(20,2)=C8(12,1);$ $k(20,3)=C9(12,1);$ $k(20,4)=C9(12,2);$
 $k(20,5)=C9(12,3);$ $k(20,6)=C7(12,2);$ $k(20,7)=C8(12,2);$ $k(20,8)=C9(12,4);$
 $k(20,9)=C9(12,5);$ $k(20,10)=C9(12,6);$ $k(20,11)=C7(12,3);$ $k(20,12)=C8(12,3);$
 $k(20,13)=C9(12,7);$ $k(20,14)=C9(12,8);$ $k(20,15)=C9(12,9);$ $k(20,16)=C7(12,4);$
 $k(20,17)=C8(12,4);$ $k(20,18)=C9(12,10);$ $k(20,19)=C9(12,11);$ $k(20,20)=C9(12,12);$

$g(3,3)=D1(1,1);$ $g(3,4)=D1(1,2);$ $g(3,5)=D1(1,3);$ $g(3,8)=D1(1,4);$
 $g(3,9)=D1(1,5);$ $g(3,10)=D1(1,6);$ $g(3,13)=D1(1,7);$ $g(3,14)=D1(1,8);$
 $g(3,15)=D1(1,9);$ $g(3,18)=D1(1,10);$ $g(3,19)=D1(1,11);$ $g(3,20)=D1(1,12);$

$g(4,3)=D1(2,1);$ $g(4,4)=D1(2,2);$ $g(4,5)=D1(2,3);$ $g(4,8)=D1(2,4);$
 $g(4,9)=D1(2,5);$ $g(4,10)=D1(2,6);$ $g(4,13)=D1(2,7);$ $g(4,14)=D1(2,8);$
 $g(4,15)=D1(2,9);$ $g(4,18)=D1(2,10);$ $g(4,19)=D1(2,11);$ $g(4,20)=D1(2,12);$

$g(5,3)=D1(3,1);$ $g(5,4)=D1(3,2);$ $g(5,5)=D1(3,3);$ $g(5,8)=D1(3,4);$
 $g(5,9)=D1(3,5);$ $g(5,10)=D1(3,6);$ $g(5,13)=D1(3,7);$ $g(5,14)=D1(3,8);$
 $g(5,15)=D1(3,9);$ $g(5,18)=D1(3,10);$ $g(5,19)=D1(3,11);$ $g(5,20)=D1(3,12);$

$g(8,3)=D1(4,1);$ $g(8,4)=D1(4,2);$ $g(8,5)=D1(4,3);$ $g(8,8)=D1(4,4);$
 $g(8,9)=D1(4,5);$ $g(8,10)=D1(4,6);$ $g(8,13)=D1(4,7);$ $g(8,14)=D1(4,8);$
 $g(8,15)=D1(4,9);$ $g(8,18)=D1(4,10);$ $g(8,19)=D1(4,11);$ $g(8,20)=D1(4,12);$

$g(9,3)=D1(5,1);$ $g(9,4)=D1(5,2);$ $g(9,5)=D1(5,3);$ $g(9,8)=D1(5,4);$
 $g(9,9)=D1(5,5);$ $g(9,10)=D1(5,6);$ $g(9,13)=D1(5,7);$ $g(9,14)=D1(5,8);$
 $g(9,15)=D1(5,9);$ $g(9,18)=D1(5,10);$ $g(9,19)=D1(5,11);$ $g(9,20)=D1(5,12);$

$g(10,3)=D1(6,1);$ $g(10,4)=D1(6,2);$ $g(10,5)=D1(6,3);$ $g(10,8)=D1(6,4);$
 $g(10,9)=D1(6,5);$ $g(10,10)=D1(6,6);$ $g(10,13)=D1(6,7);$ $g(10,14)=D1(6,8);$
 $g(10,15)=D1(6,9);$ $g(10,18)=D1(6,10);$ $g(10,19)=D1(6,11);$ $g(10,20)=D1(6,12);$

$g(13,3)=D1(7,1);$ $g(13,4)=D1(7,2);$ $g(13,5)=D1(7,3);$ $g(13,8)=D1(7,4);$
 $g(13,9)=D1(7,5);$ $g(13,10)=D1(7,6);$ $g(13,13)=D1(7,7);$ $g(13,14)=D1(7,8);$
 $g(13,15)=D1(7,9);$ $g(13,18)=D1(7,10);$ $g(13,19)=D1(7,11);$ $g(13,20)=D1(7,12);$

```

g(14,3)=D1(8,1);    g(14,4)=D1(8,2);    g(14,5)=D1(8,3);    g(14,8)=D1(8,4);
g(14,9)=D1(8,5);    g(14,10)=D1(8,6);    g(14,13)=D1(8,7);    g(14,14)=D1(8,8);
g(14,15)=D1(8,9);g(14,18)=D1(8,10); g(14,19)=D1(8,11);    g(14,20)=D1(8,12);

g(15,3)=D1(9,1);    g(15,4)=D1(9,2);    g(15,5)=D1(9,3);    g(15,8)=D1(9,4);
g(15,9)=D1(9,5);    g(15,10)=D1(9,6);    g(15,13)=D1(9,7);    g(15,14)=D1(9,8);
g(15,15)=D1(9,9);g(15,18)=D1(9,10); g(15,19)=D1(9,11);    g(15,20)=D1(9,12);

g(18,3)=D1(10,1);    g(18,4)=D1(10,2);    g(18,5)=D1(10,3);    g(18,8)=D1(10,4);
g(18,9)=D1(10,5);    g(18,10)=D1(10,6);    g(18,13)=D1(10,7);    g(18,14)=D1(10,8);
g(18,15)=D1(10,9);    g(18,18)=D1(10,10);g(18,19)=D1(10,11);g(18,20)=D1(10,12);

g(19,3)=D1(11,1);    g(19,4)=D1(11,2);    g(19,5)=D1(11,3);    g(19,8)=D1(11,4);
g(19,9)=D1(11,5);    g(19,10)=D1(11,6);    g(19,13)=D1(11,7);    g(19,14)=D1(11,8);
g(19,15)=D1(11,9);    g(19,18)=D1(11,10);    g(19,19)=D1(11,11);g(19,20)=D1(11,12);

g(20,3)=D1(12,1);    g(20,4)=D1(12,2);    g(20,5)=D1(12,3);    g(20,8)=D1(12,4);
g(20,9)=D1(12,5);    g(20,10)=D1(12,6);    g(20,13)=D1(12,7);    g(20,14)=D1(12,8);
g(20,15)=D1(12,9);    g(20,18)=D1(12,10);    g(20,19)=D1(12,11);g(20,20)=D1(12,12);

for i=1:20
    for j=1:20
        k(i,j)=(int(int(k(i,j),x,-a,a),y,-b,b));
    end
end
for i=1:20
    for j=1:20
        g(i,j)=(int(int(g(i,j),x,-a,a),y,-b,b));
    end
end
end
%%%%%%%% End of function %%%%%%%%%

```

```

function [K] = Calculate_K(NDFN, NDFT, N1, N2, N3, N4, k, K,DFC)
% Function that stores the pertinent elements of the member global
% stiffness matrix [kg] in the structure stiffness matrix [K]
for i = 1:(4*NDFN)
    if i <= NDFN
        i1 = (N1-1)*NDFN + i;
    elseif i>NDFN && i<=2*NDFN
        i1 = (N2-1)*NDFN + (i - NDFN);
    elseif i>2*NDFN && i<=3*NDFN
        i1 = (N3-1)*NDFN + (i - 2*NDFN);
    else
        i1 = (N4-1)*NDFN + (i - 3*NDFN);
    end
    n1 = DFC(i1);
    if n1 <= NDFT
        for j = 1:(4*NDFN)
            if j <= NDFN
                i1 = (N1-1)*NDFN + j;
            elseif j>NDFN && j<=2*NDFN
                i1 = (N2-1)*NDFN + (j - NDFN);
            elseif j>2*NDFN && j<=3*NDFN
                i1 = (N3-1)*NDFN + (j - 2*NDFN);
            else
                i1 = (N4-1)*NDFN + (j - 3*NDFN);
            end
            n2=DFC(i1);
            if n2 <= NDFT
                K(n1,n2) = K(n1,n2) + k(i,j);
            end
        end
    end
end
end
end
end

```

APPENDIX A.5- Abaqus Input File

The following is a sample input file read by Abaqus to model a flange reinforced with GFRP through adhesive layers subjected to uniaxial normal pressure (Section 5.2.3).

```
*HEADING
Steel flange reinforced with GFRP plates through adhesive layers
SI units (kg, m,)
*PARAMETER
#####
L1=0.1025    # Short dimension of plate
L2=0.3075    # Long dimension of plate
t1=0.007     # thickness of the top GFRP
Eg=0.42E8    # modulus of elasticity of GFRP
Vg=0.3       # Poisson's ratio of GFRP
t2=0.001     # thickness of top adhesive layer
Ea=520000    # modulus of elasticity of adhesive
Va=0.3       # Poisson's ratio of adhesive
t3=0.008     # thickness of steel layer
Es=2.0E8     # modulus of elasticity of steel
Vs=0.3       # Poisson's ratio of steel
t4=0.001     # thickness of bottom adhesive layer
t5=0.007     # thickness of bottom GFRP plate
n1=20        # number of elements in x direction
n2=60        # number of elements in y direction
n3=4         # number of elements through thickness of top GFRP plate
n4=1         # number of elements through thickness of top adhesive layer
n5=4         # number of elements through thickness of the steel plate
n6=1         # number of elements through thickness of bottom adhesive layer
n7=4         # number of elements through thickness of bottom GFRP plate
h2=t1+t2
h4=t1+t2+t3
h5=t1+t2+t3+t4
h6=t1+t2+t3+t4+t5
#####
### Increments for nodal numbers
nL1=1        # Increment in node number in x direction
nL2=201      # Increment in node number in y direction
nt1=90000    # Increment in node number in z direction of top GFRP plate
nt2=90000    # Increment in node number in z direction of top adhesive layer
nt3=90000    # Increment in node number in z direction of steel plate
nt4=90000    # Increment in node number in z direction of bottom adhesive layer
nt5=90000    # Increment in node number in z direction of bottom GFRP plate
### Increment in element number
elx1=1       # Increment in element number in the x direction
ely1=n1      # Increment in element number in the y direction
elz1=n1*n2   # Increment in element number in the z direction
#####
### Numbering of generated nodes
A1=100
A2=A1+n1
A3=A1+nL2*n2
A4=A3+n1
A5=A1+n3*nt1
```

$A6=A5+n1$
 $A7=A5+nL2*n2$
 $A8=A7+n1$
 $A9=A5+n4*nt2$
 $A10=A9+n1$
 $A11=A9+nL2*n2$
 $A12=A11+n1$
 $A13=A9+n5*nt3$
 $A14=A13+n1$
 $A15=A13+nL2*n2$
 $A16=A15+n1$
 $A17=A13+n6*nt4$
 $A18=A17+n1$
 $A19=A17+nL2*n2$
 $A20=A19+n1$
 $A21=A17+n7*nt5$
 $A22=A21+n1$
 $A23=A21+nL2*n2$
 $A24=A23+n1$
 $A25=A1+nt1$
 $A26=A2+nt1$
 $A27=A25+nt1$
 $A28=A26+nt1$
 $A29=A27+nt1$
 $A30=A28+nt1$
 $A31=A3+nt1$
 $A32=A4+nt1$
 $A33=A31+nt1$
 $A34=A32+nt1$
 $A35=A33+nt1$
 $A36=A34+nt1$
 $A37=A17+nt1$
 $A38=A18+nt1$
 $A39=A37+nt1$
 $A40=A38+nt1$
 $A41=A39+nt1$
 $A42=A40+nt1$
 $A43=A19+nt1$
 $A44=A20+nt1$
 $A45=A43+nt1$
 $A46=A44+nt1$
 $A47=A45+nt1$
 $A48=A46+nt1$
 $A49=A1+2*nt1+nL1$
 $A50=A2+2*nt1-nL1$
 $A51=A3+2*nt1+nL1$
 $A52=A4+2*nt1-nL1$
 $A53=A17+2*nt1+nL1$
 $A54=A18+2*nt1-nL1$
 $A55=A19+2*nt1+nL1$
 $A56=A20+2*nt1-nL1$
 $A57=A9+2*nt1+nL1$
 $A58=A10+2*nt1-nL1$
 $A59=A11+2*nt1+nL1$
 $A60=A12+2*nt1-nL1$
 $A61=A9+2*nt1$
 $A62=A10+2*nt1$

```

A63=A11+2*nt1
A64=A12+2*nt1
A65=A25
A66=A26
A67=A27
A68=A28
A69=A29
A70=A30
A71=A5
A72=A6
A73=A9
A74=A10
A75=A9+nt1
A76=A10+nt1
A77=A75+nt1
A78=A76+nt1
A79=A75+2*nt1
A80=A76+2*nt1
A81=A13
A82=A14
A83=A17
A84=A18
A85=A17+nt1
A86=A18+nt1
A87=A17+2*nt1
A88=A18+2*nt1
A89=A17+3*nt1
A90=A18+3*nt1
A91=A1
A92=A2
A93=A21
A94=A22
A95=A65+nL2*n2
A96=A66+nL2*n2
A97=A67+nL2*n2
A98=A68+nL2*n2
A99=A69+nL2*n2
A100=A70+nL2*n2
A101=A71+nL2*n2
A102=A72+nL2*n2
A103=A73+nL2*n2
A104=A74+nL2*n2
A105=A75+nL2*n2
A106=A76+nL2*n2
A107=A77+nL2*n2
A108=A78+nL2*n2
A109=A79+nL2*n2
A110=A80+nL2*n2
A111=A81+nL2*n2
A112=A82+nL2*n2
A113=A83+nL2*n2
A114=A84+nL2*n2
A115=A85+nL2*n2
A116=A86+nL2*n2
A117=A87+nL2*n2
A118=A88+nL2*n2
A119=A89+nL2*n2

```

$A_{120}=A_{90}+nL_2*n_2$
 $A_{121}=A_{91}+nL_2*n_2$
 $A_{122}=A_{92}+nL_2*n_2$
 $A_{123}=A_{93}+nL_2*n_2$
 $A_{124}=A_{94}+nL_2*n_2$
 $A_{125}=A_1+(n_1/2)*nL_1+(n_2/2)*nL_2$
 $A_{126}=A_5+(n_1/2)*nL_1+(n_2/2)*nL_2$
 $A_{127}=A_9+(n_1/2)*nL_1+(n_2/2)*nL_2$
 $A_{128}=A_{13}+(n_1/2)*nL_1+(n_2/2)*nL_2$
 $A_{129}=A_{17}+(n_1/2)*nL_1+(n_2/2)*nL_2$
 $A_{130}=A_{21}+(n_1/2)*nL_1+(n_2/2)*nL_2$
 $A_{651}=A_{25}+nL_2$
 $A_{661}=A_{26}+nL_2$
 $A_{671}=A_{27}+nL_2$
 $A_{681}=A_{28}+nL_2$
 $A_{691}=A_{29}+nL_2$
 $A_{701}=A_{30}+nL_2$
 $A_{711}=A_5+nL_2$
 $A_{721}=A_6+nL_2$
 $A_{731}=A_9+nL_2$
 $A_{741}=A_{10}+nL_2$
 $A_{751}=A_9+nt_1+nL_2$
 $A_{761}=A_{10}+nt_1+nL_2$
 $A_{771}=A_{751}+nt_1$
 $A_{781}=A_{761}+nt_1$
 $A_{791}=A_{751}+2*nt_1$
 $A_{801}=A_{761}+2*nt_1$
 $A_{811}=A_{13}+nL_2$
 $A_{821}=A_{14}+nL_2$
 $A_{831}=A_{17}+nL_2$
 $A_{841}=A_{18}+nL_2$
 $A_{851}=A_{17}+nt_1+nL_2$
 $A_{861}=A_{18}+nt_1+nL_2$
 $A_{871}=A_{17}+2*nt_1+nL_2$
 $A_{881}=A_{18}+2*nt_1+nL_2$
 $A_{891}=A_{17}+3*nt_1+nL_2$
 $A_{901}=A_{18}+3*nt_1+nL_2$
 $A_{911}=A_1+nL_2$
 $A_{921}=A_2+nL_2$
 $A_{931}=A_{21}+nL_2$
 $A_{941}=A_{22}+nL_2$
 $A_{951}=A_{651}+nL_2*n_2-2*nL_2$
 $A_{961}=A_{661}+nL_2*n_2-2*nL_2$
 $A_{971}=A_{671}+nL_2*n_2-2*nL_2$
 $A_{981}=A_{681}+nL_2*n_2-2*nL_2$
 $A_{991}=A_{691}+nL_2*n_2-2*nL_2$
 $A_{1001}=A_{701}+nL_2*n_2-2*nL_2$
 $A_{1011}=A_{711}+nL_2*n_2-2*nL_2$
 $A_{1021}=A_{721}+nL_2*n_2-2*nL_2$
 $A_{1031}=A_{731}+nL_2*n_2-2*nL_2$
 $A_{1041}=A_{741}+nL_2*n_2-2*nL_2$
 $A_{1051}=A_{751}+nL_2*n_2-2*nL_2$
 $A_{1061}=A_{761}+nL_2*n_2-2*nL_2$
 $A_{1071}=A_{771}+nL_2*n_2-2*nL_2$
 $A_{1081}=A_{781}+nL_2*n_2-2*nL_2$
 $A_{1091}=A_{791}+nL_2*n_2-2*nL_2$
 $A_{1101}=A_{801}+nL_2*n_2-2*nL_2$

```

A1111=A811+nL2*n2-2*nL2
A1121=A821+nL2*n2-2*nL2
A1131=A831+nL2*n2-2*nL2
A1141=A841+nL2*n2-2*nL2
A1151=A851+nL2*n2-2*nL2
A1161=A861+nL2*n2-2*nL2
A1171=A871+nL2*n2-2*nL2
A1181=A881+nL2*n2-2*nL2
A1191=A891+nL2*n2-2*nL2
A1201=A901+nL2*n2-2*nL2
A1211=A911+nL2*n2-2*nL2
A1221=A921+nL2*n2-2*nL2
A1231=A931+nL2*n2-2*nL2
A1241=A941+nL2*n2-2*nL2
A131=A9+nt1
A132=A10+nt1
A133=A131+nt1
A134=A132+nt1
A135=A133+nt1
A136=A134+nt1
A137=A11+nt1
A138=A12+nt1
A139=A137+nt1
A140=A138+nt1
A141=A139+nt1
A142=A140+nt1
B1=A1
B2=A1+nt1
B3=A1+2*nt1
B4=A1+3*nt1
B5=A5
B6=A9
B7=A9+nt1
B8=A9+2*nt1
B9=A9+3*nt1
B10=A13
B11=A17
B12=A17+nt1
B13=A17+2*nt1
B14=A17+3*nt1
B15=A21
B16=B1+n2*nL2
B17=B2+n2*nL2
B18=B3+n2*nL2
B19=B4+n2*nL2
B20=B5+n2*nL2
B21=B6+n2*nL2
B22=B7+n2*nL2
B23=B8+n2*nL2
B24=B9+n2*nL2
B25=B10+n2*nL2
B26=B11+n2*nL2
B27=B12+n2*nL2
B28=B13+n2*nL2
B29=B14+n2*nL2
B30=B15+n2*nL2
#####

```

```

### Nodes of master element of top GFRP
NMEG11=A1+nt1
NMEG12=NMEG11+nL1
NMEG13=NMEG12+nL2
NMEG14=NMEG13-nL1
NMEG15=A1
NMEG16=A1+nL1
NMEG17=NMEG16+nL2
NMEG18=NMEG17-nL1
### Nodes of master element of top adhesive layer
NMEAL11=A5+nt2
NMEAL12=NMEAL11+nL1
NMEAL13=NMEAL12+nL2
NMEAL14=NMEAL13-nL1
NMEAL15=A5
NMEAL16=NMEAL15+nL1
NMEAL17=NMEAL16+nL2
NMEAL18=NMEAL17-nL1
### Nodes of master element of the steel plate
NMESL1=A9+nt3
NMESL2=NMESL1+nL1
NMESL3=NMESL2+nL2
NMESL4=NMESL3-nL1
NMESL5=A9
NMESL6=A9+nL1
NMESL7=NMESL6+nL2
NMESL8=NMESL7-nL1
### Nodes of master element of bottom adhesive layer
NMEAL21=A13+nt4
NMEAL22=NMEAL21+nL1
NMEAL23=NMEAL22+nL2
NMEAL24=NMEAL23-nL1
NMEAL25=A13
NMEAL26=NMEAL25+nL1
NMEAL27=NMEAL26+nL2
NMEAL28=NMEAL27-nL1
### Nodes of master element of bottom GFRP plate
NMEG21=A17+nt5
NMEG22=NMEG21+nL1
NMEG23=NMEG22+nL2
NMEG24=NMEG23-nL1
NMEG25=A17
NMEG26=A17+nL1
NMEG27=NMEG26+nL2
NMEG28=NMEG27-nL1
### Number of first element of each layer
NFEA1=n3*elz1+1
NFES=(n3+n4)*elz1+1
NFEA2=(n3+n4+n5)*elz1+1
NFEG2=(n3+n4+n5+n6)*elz1+1
### Number of last element
NLE=(n3+n4+n5+n6+n7)*elz1
###FIRST AND LAST ELEMENT NUMBER of each layer
e1=1
e2=n1
e3=n1*n2+1
e4=e3+n1-1

```

```

e5=2*n1*n2+1
e6=e5+n1-1
e7=3*n1*n2+1
e8=e7+n1-1
e9=e2
e10=n1*n2
e11=e9+n1*n2
e12=2*n1*n2
e13=e11+n1*n2
e14=3*n1*n2
e15=e13+n1*n2
e16=4*n1*n2
e17=e10-n1+1
e18=e10
e19=e17+n1*n2
e20=e12
e21=e19+n1*n2
e22=e14
e23=e21+n1*n2
e24=e16
e25=e1
e26=e17
e27=e3
e28=e19
e29=e5
e30=e21
e31=e7
e32=e23
e33=10*n1*n2+1
e34=e33+n1-1
e35=11*n1*n2+1
e36=e35+n1-1
e37=12*n1*n2+1
e38=e37+n1-1
e39=13*n1*n2+1
e40=e39+n1-1
e41=e34
e42=11*n1*n2
e43=e36
e44=12*n1*n2
e45=e38
e46=13*n1*n2
e47=e40
e48=14*n1*n2
e49=e42-n1+1
e50=e42
e51=e44-n1+1
e52=e44
e53=e46-n1+1
e54=e46
e55=e48-n1+1
e56=e48
e57=e33
e58=e57+(n1*n2-n1)
e59=e35
e60=e59+(n1*n2-n1)
e61=e37

```

```

e62=e61+(n1*n2-n1)
e63=e39
e64=e63+(n1*n2-n1)
e65=5*n1*n2+1
e66=e65+n1-1
e67=6*n1*n2+1
e68=e67+n1-1
e69=7*n1*n2+1
e70=e69+n1-1
e71=8*n1*n2+1
e72=e71+n1-1
e73=e66
e74=6*n1*n2
e75=e68
e76=7*n1*n2
e77=e70
e78=8*n1*n2
e79=e72
e80=9*n1*n2
e81=e74-n1+1
e82=e74
e83=e76-n1+1
e84=e76
e85=e78-n1+1
e86=e78
e87=e80-n1+1
e88=e80
e89=e65
e90=e81
e91=e67
e92=e83
e93=e69
e94=e85
e95=e71
e96=e87
#####
**
** Model definition
*NODE
<A1>,0,0,0
<A2>, <L1>, 0, 0
<A3>, 0, <L2>, 0
<A4>, <L1>, <L2>, 0
<A5>,0,0,-<t1>
<A6>, <L1>, 0, -<t1>
<A7>, 0, <L2>, -<t1>
<A8>, <L1>, <L2>, -<t1>
<A9>,0,0,-<h2>
<A10>, <L1>, 0, -<h2>
<A11>, 0, <L2>, -<h2>
<A12>, <L1>, <L2> , -<h2>
<A13>,0,0,-<h4>
<A14>, <L1>, 0, -<h4>
<A15>, 0, <L2>, -<h4>
<A16>, <L1>, <L2> , -<h4>
<A17>,0,0,-<h5>
<A18>, <L1>, 0, -<h5>

```

```

<A19>, 0, <L2>, -<h5>
<A20>, <L1>, <L2> , -<h5>
<A21>,0,0,-<h6>
<A22>, <L1>, 0, -<h6>
<A23>, 0, <L2>, -<h6>
<A24>, <L1>, <L2> , -<h6>
**
** Top node set at front surface of top GFRP
*NGEN, NSET=TFSTG
<A1>, <A2>, <nL1>
** Top node set at back surface of top GFRP
*NGEN, NSET=TBSTG
<A3>, <A4>, <nL1>
** Bottom node set at front surface of top GFRP
*NGEN, NSET=BFSTG
<A5>, <A6>, <nL1>
** Bottom node set at back surface of top GFRP
*NGEN, NSET=BBSTG
<A7>, <A8>, <nL1>
** Bottom node set at front surface of top adhesive
*NGEN, NSET=BFSTA
<A9>, <A10>, <nL1>
** Bottom node set at back surface of top adhesive
*NGEN, NSET=BBSTA
<A11>, <A12>, <nL1>
** Bottom node set at front surface of steel
*NGEN, NSET=BFSS
<A13>, <A14>, <nL1>
** Bottom node set at back surface of steel
*NGEN, NSET=BBSS
<A15>, <A16>, <nL1>
** Bottom node set at front surface of bottom adhesive
*NGEN, NSET=BFSBA
<A17>, <A18>, <nL1>
** Bottom node set at back surface of bottom adhesive
*NGEN, NSET=BBSBA
<A19>, <A20>, <nL1>
** Bottom node set at front surface of bottom GFRP
*NGEN, NSET=BFSBG
<A21>, <A22>, <nL1>
** Bottom node set at back surface of bottom GFRP
*NGEN, NSET=BBSBG
<A23>, <A24>, <nL1>
** Top node set at left surface of top GFRP
*NGEN, NSET=TLSTG
<A1>, <A3>, <nL2>
** Top node set at right surface of top GFRP
*NGEN, NSET=TRSTG
<A2>, <A4>, <nL2>
** Bottom node set at left surface of top GFRP
*NGEN, NSET=BLSTG
<A5>, <A7>, <nL2>
** Bottom node set at right surface of top GFRP
*NGEN, NSET=BRSTG
<A6>, <A8>, <nL2>
** Bottom node set at left surface of top adhesive
*NGEN, NSET=BLSTA

```

```

<A9>, <A11>, <nL2>
** Bottom node set at right surface of top adhesive
*NGEN, NSET=BRSTA
<A10>, <A12>, <nL2>
** Bottom node set at left surface of steel
*NGEN, NSET=BLSS
<A13>, <A15>, <nL2>
** Bottom node set at right surface of steel
*NGEN, NSET=BRSS
<A14>, <A16>, <nL2>
** Bottom node set at left surface of bottom adhesive
*NGEN, NSET=BLSBA
<A17>, <A19>, <nL2>
** Bottom node set at right surface of bottom adhesive
*NGEN, NSET=BRBBA
<A18>, <A20>, <nL2>
** Bottom node set at left surface of bottom GFRP
*NGEN, NSET=BLSBG
<A21>, <A23>, <nL2>
** Bottom node set at right surface of bottom GFRP
*NGEN, NSET=BRBGB
<A22>, <A24>, <nL2>
*NFILL, NSET=TOP1
TFSTG, TBSTG, <n2>, <nL2>
*NFILL, NSET=BOTTOM1
BFSTG, BBSTG, <n2>, <nL2>
*NFILL, NSET=GFRP1
TOP1, BOTTOM1, <n3>, <nt1>
*NFILL, NSET=BOTTOM2
BFSTA, BBSTA, <n2>, <nL2>
*NFILL, NSET=adhesive layer 1
BOTTOM1, BOTTOM2, <n4>, <nt2>
*NFILL, NSET=BOTTOM3
BFSS, BBSS, <n2>, <nL2>
*NFILL, NSET=steel layer
BOTTOM2, BOTTOM3, <n5>, <nt3>
*NFILL, NSET=BOTTOM4
BFSBA, BBSBA, <n2>, <nL2>
*NFILL, NSET=adhesive layer 2
BOTTOM3, BOTTOM4, <n6>, <nt4>
*NFILL, NSET=BOTTOM5
BFSBG, BBSBG, <n2>, <nL2>
*NFILL, NSET=GFRP2
BOTTOM4, BOTTOM5, <n7>, <nt5>
** TOP GFRP PLATE ELEMENT SET
*ELEMENT, TYPE=C3D8, ELSET=GFRP1
1, <NMEG11>, <NMEG12>, <NMEG13>, <NMEG14>, <NMEG15>, <NMEG16>, <NMEG17>, <NMEG18>
*ELGEN, ELSET=GFRP1
1, <n1>, <nL1>, <elx1>, <n2>, <nL2>, <ely1>, <n3>, <nt1>, <elz1>
*solid SECTION, ELSET=GFRP1, MATERIAL=GFRP
*MATERIAL, NAME=GFRP
*ELASTIC
<Eg>, <Vg>
** TOP ADHESIVE LAYER ELEMENT SET
*ELEMENT, TYPE=C3D8, ELSET=all
<NFEA1>, <NMEAL11>, <NMEAL12>, <NMEAL13>, <NMEAL14>, <NMEAL15>, <NMEAL16>, <NMEAL17>
, <NMEAL18>

```

```

*ELGEN,ELSET=a11
<NFEA1>,<n1>,<nL1>,<elx2>,<n2>,<nL2>,<ely2>
*solid SECTION,ELSET=a11,MATERIAL=ADHESIVE
*MATERIAL,NAME=ADHESIVE
*ELASTIC
<Ea>,<Va>
** STEEL PLATE ELEMENT SET
*ELEMENT,TYPE=C3D8,ELSET=s1
<NFES>,<NMESL1>,<NMESL2>,<NMESL3>,<NMESL4>,<NMESL5>,<NMESL6>,<NMESL7>,<NMESL8
>
*ELGEN,ELSET=s1
<NFES>,<n1>,<nL1>,<elx3>,<n2>,<nL2>,<ely3>,<n5>,<nt3>,<elz3>
*solid SECTION,ELSET=s1,MATERIAL=STEELL
*MATERIAL,NAME=STEELL
*ELASTIC
<Es>,<Vs>
** BOTTOM ADHESIVE LAYER ELEMENT SET
*ELEMENT,TYPE=C3D8,ELSET=a12
<NFEA2>,<NMEAL21>,<NMEAL22>,<NMEAL23>,<NMEAL24>,<NMEAL25>,<NMEAL26>,<NMEAL27>
,<NMEAL28>
*ELGEN,ELSET=a12
<NFEA2>,<n1>,<nL1>,<elx2>,<n2>,<nL2>,<ely2>
*solid SECTION,ELSET=a12,MATERIAL=ADHESIVEE
*MATERIAL,NAME=ADHESIVEE
*ELASTIC
<Ea>,<Va>
** BOTTOM GFRP PLATE ELEMENT SET
*ELEMENT,TYPE=C3D8,ELSET=GFRP2
<NFEG2>,<NMEG21>,<NMEG22>,<NMEG23>,<NMEG24>,<NMEG25>,<NMEG26>,<NMEG27>,<NMEG2
8>
*ELGEN,ELSET=GFRP2
<NFEG2>,<n1>,<nL1>,<elx1>,<n2>,<nL2>,<ely1>,<n7>,<nt1>,<elz1>
*solid SECTION,ELSET=GFRP2,MATERIAL=GFRPP
*MATERIAL,NAME=GFRPP
*ELASTIC
<Eg>,<Vg>
*ELSET, ELSET=totalplate, GENERATE
1,<NLE>,<elx1>
*ELSET, ELSET=topgfrpfront, GENERATE
<e1>,<e2>,1
<e3>,<e4>,1
<e5>,<e6>,1
<e7>,<e8>,1
*ELSET, ELSET=topgfrpright, GENERATE
<e9>,<e10>,<n1>
<e11>,<e12>,<n1>
<e13>,<e14>,<n1>
<e15>,<e16>,<n1>
*ELSET, ELSET=topgfrpback, GENERATE
<e17>,<e18>,1
<e19>,<e20>,1
<e21>,<e22>,1
<e23>,<e24>,1
*ELSET, ELSET=topgfrpleft, GENERATE
<e25>,<e26>,<n1>
<e27>,<e28>,<n1>
<e29>,<e30>,<n1>

```

```

<e31>,<e32>,<n1>
*ELSET, ELSET=bottomgfrpfront, GENERATE
<e33>,<e34>,1
<e35>,<e36>,1
<e37>,<e38>,1
<e39>,<e40>,1
*ELSET, ELSET=bottomgfrpright, GENERATE
<e41>,<e42>,<n1>
<e43>,<e44>,<n1>
<e45>,<e46>,<n1>
<e47>,<e48>,<n1>
*ELSET, ELSET=bottomgfrpback, GENERATE
<e49>,<e50>,1
<e51>,<e52>,1
<e53>,<e54>,1
<e55>,<e56>,1
*ELSET, ELSET=bottomgfrpleft, GENERATE
<e57>,<e58>,<n1>
<e59>,<e60>,<n1>
<e61>,<e62>,<n1>
<e63>,<e64>,<n1>
*ELSET, ELSET=midsteelfront, GENERATE
<e65>,<e66>,1
<e67>,<e68>,1
<e69>,<e70>,1
<e71>,<e72>,1
*ELSET, ELSET=midsteelright, GENERATE
<e73>,<e74>,<n1>
<e75>,<e76>,<n1>
<e77>,<e78>,<n1>
<e79>,<e80>,<n1>
*ELSET, ELSET=midsteelback, GENERATE
<e81>,<e82>,1
<e83>,<e84>,1
<e85>,<e86>,1
<e87>,<e88>,1
*ELSET, ELSET=midsteelleft, GENERATE
<e89>,<e90>,<n1>
<e91>,<e92>,<n1>
<e93>,<e94>,<n1>
<e95>,<e96>,<n1>
**
** Generating all the lines of the surfaces around the composite system
*NSET,NSET=ab,generate
<A27>,<A28>,<nL1>
*NSET,NSET=bc,generate
<A28>,<A34>,<nL2>
*NSET,NSET=ad,generate
<A27>,<A33>,<nL2>
*NSET,NSET=dc,generate
<A33>,<A34>,<nL1>
*NSET,NSET=eh,generate
<A39>,<A45>,<nL2>
*NSET,NSET=fg,generate
<A40>,<A46>,<nL2>
*NSET,NSET=ef,generate
<A39>,<A40>,<nL1>

```

```

*NSET, NSET=hg, generate
<A45>, <A46>, <nL1>
*NSET, NSET=lf1, GENERATE
<A25>, <A26>, <nL1>
*NSET, NSET=lf2, GENERATE
<A29>, <A30>, <nL1>
*NSET, NSET=lb1, GENERATE
<A31>, <A32>, <nL1>
*NSET, NSET=lb2, GENERATE
<A35>, <A36>, <nL1>
*NSET, NSET=lr1, GENERATE
<A26>, <A32>, <nL2>
*NSET, NSET=lr2, GENERATE
<A30>, <A36>, <nL2>
*NSET, NSET=ll1, GENERATE
<A25>, <A31>, <nL2>
*NSET, NSET=ll2, GENERATE
<A29>, <A35>, <nL2>
*NSET, NSET=lf3, GENERATE
<A37>, <A38>, <nL1>
*NSET, NSET=lf4, GENERATE
<A41>, <A42>, <nL1>
*NSET, NSET=lb3, GENERATE
<A43>, <A44>, <nL1>
*NSET, NSET=lb4, GENERATE
<A47>, <A48>, <nL1>
*NSET, NSET=lr3, GENERATE
<A38>, <A44>, <nL2>
*NSET, NSET=lr4, GENERATE
<A42>, <A48>, <nL2>
*NSET, NSET=ll3, GENERATE
<A37>, <A43>, <nL2>
*NSET, NSET=ll4, GENERATE
<A41>, <A47>, <nL2>
*NSET, NSET=line1, GENERATE
<A65>, <A66>, <nL1>
*NSET, NSET=line2, GENERATE
<A67>, <A68>, <nL1>
*NSET, NSET=line3, GENERATE
<A69>, <A70>, <nL1>
*NSET, NSET=line4, GENERATE
<A71>, <A72>, <nL1>
*NSET, NSET=line5, GENERATE
<A73>, <A74>, <nL1>
*NSET, NSET=line6, GENERATE
<A75>, <A76>, <nL1>
*NSET, NSET=line7, GENERATE
<A77>, <A78>, <nL1>
*NSET, NSET=line8, GENERATE
<A79>, <A80>, <nL1>
*NSET, NSET=line9, GENERATE
<A81>, <A82>, <nL1>
*NSET, NSET=line10, GENERATE
<A83>, <A84>, <nL1>
*NSET, NSET=line11, GENERATE
<A85>, <A86>, <nL1>
*NSET, NSET=line12, GENERATE

```

```

<A87>,<A88>,<nL1>
*NSET, NSET=line13, GENERATE
<A89>,<A90>,<nL1>
*NSET, NSET=line14, GENERATE
<A91>,<A92>,<nL1>
*NSET, NSET=line15, GENERATE
<A93>,<A94>,<nL1>
*NSET, NSET=line16, GENERATE
<A95>,<A96>,<nL1>
*NSET, NSET=line17, GENERATE
<A97>,<A98>,<nL1>
*NSET, NSET=line18, GENERATE
<A99>,<A100>,<nL1>
*NSET, NSET=line19, GENERATE
<A101>,<A102>,<nL1>
*NSET, NSET=line20, GENERATE
<A103>,<A104>,<nL1>
*NSET, NSET=line21, GENERATE
<A105>,<A106>,<nL1>
*NSET, NSET=line22, GENERATE
<A107>,<A108>,<nL1>
*NSET, NSET=line23, GENERATE
<A109>,<A110>,<nL1>
*NSET, NSET=line24, GENERATE
<A111>,<A112>,<nL1>
*NSET, NSET=line25, GENERATE
<A113>,<A114>,<nL1>
*NSET, NSET=line26, GENERATE
<A115>,<A116>,<nL1>
*NSET, NSET=line27, GENERATE
<A117>,<A118>,<nL1>
*NSET, NSET=line28, GENERATE
<A119>,<A120>,<nL1>
*NSET, NSET=line29, GENERATE
<A121>,<A122>,<nL1>
*NSET, NSET=line30, GENERATE
<A123>,<A124>,<nL1>
*NSET, NSET=line31, GENERATE
<A651>,<A951>,<nL2>
*NSET, NSET=line32, GENERATE
<A671>,<A971>,<nL2>
*NSET, NSET=line33, GENERATE
<A691>,<A991>,<nL2>
*NSET, NSET=line34, GENERATE
<A711>,<A1011>,<nL2>
*NSET, NSET=line35, GENERATE
<A731>,<A1031>,<nL2>
*NSET, NSET=line36, GENERATE
<A751>,<A1051>,<nL2>
*NSET, NSET=line37, GENERATE
<A771>,<A1071>,<nL2>
*NSET, NSET=line38, GENERATE
<A791>,<A1091>,<nL2>
*NSET, NSET=line39, GENERATE
<A811>,<A1111>,<nL2>
*NSET, NSET=line40, GENERATE
<A831>,<A1131>,<nL2>

```

```

*NSET, NSET=line41, GENERATE
<A851>,<A1151>,<nL2>
*NSET, NSET=line42, GENERATE
<A871>,<A1171>,<nL2>
*NSET, NSET=line43, GENERATE
<A891>,<A1191>,<nL2>
*NSET, NSET=line44, GENERATE
<A911>,<A1211>,<nL2>
*NSET, NSET=line45, GENERATE
<A931>,<A1231>,<nL2>
*NSET, NSET=fronttopgfrp
lf1
ab
lf2
*NSET, NSET=frontbottomgfrp
lf3
ef
lf4
*NSET, NSET=backtopgfrp
lb1
dc
lb2
*NSET, NSET=backbottomgfrp
lb3
hg
lb4
*NSET, NSET=righttopgfrp
lr1
bc
lr2
*NSET, NSET=rightbottomgfrp
lr3
fg
lr4
*NSET, NSET=lefttopgfrp
ll1
ad
ll2
*NSET, NSET=leftbottomgfrp
ll3
eh
ll4
*equation
2
line1,2,1,line7,2,-1
2
line2,2,1,line7,2,-1
2
line3,2,1,line7,2,-1
2
line4,2,1,line7,2,-1
2
line5,2,1,line7,2,-1
2
line6,2,1,line7,2,-1
2
line8,2,1,line7,2,-1

```

```

2
line9,2,1,line7,2,-1
2
line10,2,1,line7,2,-1
2
line11,2,1,line7,2,-1
2
line12,2,1,line7,2,-1
2
line13,2,1,line7,2,-1
2
line14,2,1,line7,2,-1
2
line15,2,1,line7,2,-1
2
line16,2,1,line22,2,-1
2
line17,2,1,line22,2,-1
2
line18,2,1,line22,2,-1
2
line19,2,1,line22,2,-1
2
line20,2,1,line22,2,-1
2
line21,2,1,line22,2,-1
2
line23,2,1,line22,2,-1
2
line24,2,1,line22,2,-1
2
line25,2,1,line22,2,-1
2
line26,2,1,line22,2,-1
2
line27,2,1,line22,2,-1
2
line28,2,1,line22,2,-1
2
line29,2,1,line22,2,-1
2
line30,2,1,line22,2,-1
2
line7,2,1,line22,2,1
*boundary
lefttopgfrp,3
leftbottomgfrp,3
fronttopgfrp,3
frontbottomgfrp,3
backtopgfrp,3
backbottomgfrp,3
line31,1
line32,1
line33,1
line34,1
line35,1
line36,1

```

```
line37,1
line38,1
line39,1
line40,1
line41,1
line42,1
line43,1
line44,1
line45,1
*step
*buckle
5,
*DLOAD
midsteelfront,P3,1
midsteelback,P5,1
topgfrpfront,P3,1
topgfrpback,P5,1
bottomgfrpfront,P3,1
bottomgfrpback,P5,1
*output,field
*node output
u,
*element output
s,
*node file,global=yes
u,
*end step
```