

On Iwase's Construction of a Counterexample to Ganea's Conjecture

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ABSTRACT

In 1971, Ganea put forth a conjecture that the LS category of the Cartesian product of a topological space X with a sphere S^n is always exactly 1 higher than the LS category of X by itself. Several special cases of this conjecture were proved in the years following, however the question remained open until 1998 when Iwase produced not just one, but infinitely many counterexamples. In this thesis, we study the methods implemented by Iwase, culminating in the construction of his counterexample.

DEDICATIONS

To the unit circle, S^1 , for always being a round when I need it.

ACKNOWLEDGEMENTS

I have been very fortunate in my education to encounter many amazing and influential teachers. First and foremost, I would like to thank my parents, for encouraging me to be inquisitive and never stop asking questions. Next I would like to thank the first teacher I ever had – my kindergarten teacher, Lynn Barkley, who believed in me from the time I was four, and was never shy about letting me know it.

I would also like to give thanks to my high school physics teacher, Richard Taylor, who got me hooked on physics and set me on this path in the first place; to my history teacher Mario Polisenà, for teaching me the importance of a well-told story, and that the presentation of the story is every bit as important as its content; and to my graphic design teacher, Irv Osterer, for teaching me that a presentation is well-designed when it is as simple as it can be, and no simpler.

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I also thank my coach, John Clarke, for his patience with me while I completed this thesis, and for enforcing the lesson that often the best way through a tough problem is to grit your teeth and put in the work.

Lastly, I would like to thank my colleague, Cameron Reuther, without whose help I would never have known that "homotopy" is pronounced "homotopy" and not "homotopy".

The subject which we focus on in this study began in 1934, when two mathematicians, Lusternik and Schnirelmann, introduced a new kind of invariant of topological spaces [LS34]. This invariant, now known as the Lusternik-Schnirelmann category, or LS category for short, was invented to estimate a lower bound on the number of critical points real-valued functions on a given manifold M could take on. The LS category was calculated by building coverings of the manifold consisting of closed sets, each contractible in M . The category of M , according to them, was the number of sets in the smallest such covering, and was denoted $\text{cat } M$.

Over time, other mathematicians modified this definition to suit their needs. More recently, it has been more or less agreed upon that the category is to be calculated with open contractible (in M) sets, rather than closed, and that the category of M would be the smallest integer k for which there is a covering consisting of $k+1$ such open sets. Ralph Fox [Fox41] was, in particular, a large proponent of this definition, and showed that for many spaces on which there was active study, these two definitions are equivalent, except that the value of newer definition is shifted down one from the original. Hence contractible spaces have category 0, rather than 1.

An interesting property of the LS category is that, given two spaces, the category of the product is always less than or equal to the sum of the categories of the two factor spaces. In fact, in common cases, it was exactly equal. A natural question arose – does equality always hold?

An example was quickly found which showed that equality did not hold in general. However, this example relied on the manipulation of algebraic properties of spaces rather than taking a geometric inspiration, as one might wish. This led a Romanian mathematician, Tudor Ganea, to put forward a conjecture which would perplex a generation of mathematicians. If one of the factors had to be a sphere, he wondered, would equality then always hold?

In 1998 a Japanese mathematician, Norio Iwase accomplished an extraordinary feat. He surprised topologists around the world by constructing a counterexample [Iwa98] to Ganea's famous conjecture set forward nearly thirty years earlier, in 1971 [Gan71] – a conjecture which many thought to be true, though none had been able to prove it. To truly understand this conjecture and its counterexample, we must first go back to the basics.

This thesis consists of two parts. In the first part, we introduce and explore several ideas and areas which are necessary for the understanding of Ganea’s conjecture and Iwase’s counterexample. In the second part, we apply these results to construct this counterexample from scratch.

First, we talk about the notion of homotopies between maps and the group structure which they induce on maps from spheres S^n to a given space X , otherwise known as the homotopy groups of X , $\pi_n(X)$. An important use of homotopy groups is found in Hopf invariants. A fortunate result from [Cor95] affords us a canonical decomposition of the homotopy groups of X , in the form

$$\pi_r(S^n \vee S^n) = \pi_r(S^n) \oplus \pi_r(S^n) \oplus \pi_r(\Omega S^n * \Omega S^n),$$

where ΩX represents the space of loops in X . We use this decomposition to define the Hopf invariant. This invariant is of particular importance, as it allows us to see that $\pi_3(S^2)$ is nontrivial, a result which itself is nontrivial. We construct a particularly special map $\eta : S^3 \rightarrow S^2$. We call η the Hopf map, or Hopf fibration, after its discoverer. It was the first constructed example of a map $S^3 \rightarrow S^2$ which was not homotopically trivial. In fact, η is a generator of the group $\pi_3(S^2)$, which we now know is isomorphic to \mathbb{Z} . This map is at the heart of our study, and is the crux of some of our most important results.

With this map in our pocket, we move on to study the properties of the LS category, as well as two slightly more modern (but equivalent) definitions of the category of a space, introduced by Ganea and Whitehead, respectively. We review them here briefly.

Ganea’s view of category involves a series of homotopy pushouts and pullbacks to form new spaces $G^n X$, known as the Ganea spaces of X . Starting from X , we consider the homotopy fibre of the inclusion of the basepoint of X , which gives us the loop space ΩX . We then look at the cofibre of the induced map $\Omega X \rightarrow *$, which gives $\Sigma \Omega X$. This is taken to be $G^1 X$ and comes equipped with its own canonical map into X , the evaluation $p_1 : G^1 X \rightarrow X$. We then repeat this process, taking the fibre of the map p_1 , followed by the cofibre of the new induced map, giving us $G^2 X$ and $p_2 : G^2 \rightarrow X$, and so on in this fashion. Ganea then said that $\text{cat } X$ is the smallest integer k for which p_k admits a section.

Whitehead’s definition relies instead on the fat wedge, $X^{[n]}$, which is the subspace X^n of all points at least one of whose coordinates is the basepoint $*$. Whitehead asserts that the category of X is the smallest integer k such that the diagonal map $\Delta_{k+1} : X \rightarrow X^{k+1}$ can be lifted into a map $h : X \rightarrow X^{[k+1]}$ so that the diagram

$$\begin{array}{ccc} & & X^{[k+1]} \\ & \nearrow h & \downarrow \\ X & \xrightarrow{\Delta_{k+1}} & X^{k+1} \end{array}$$

commutes up to homotopy. We then show that the definitions from Ganea and Whitehead are equivalent to one another, and indeed equivalent to the original definition involving open

coverings. We then introduce some important properties of the LS category, focusing mainly on how it pertains to CW complexes, including the inequality which lies at the very heart of this study, that for any spaces X and Y , $\text{cat}(X \times Y) \leq \text{cat} X + \text{cat} Y$.

From here we move into the second part of the thesis, in which we focus specifically on Ganea's conjecture, its motivation, and its resolution. Now, as mentioned above, an example to attest to the fact that $\text{cat}(X \times Y)$ and $\text{cat} X + \text{cat} Y$ are not, in general, the same, is not so hard to find. The example presented in this paper comes in the form of Moore spaces. A Moore space, $M(G, n)$ is a topological space whose n^{th} homology is the abelian group G , and whose other homology groups are trivial. The spaces, which we construct in the paper, are $X = M(\mathbb{Z}_p, 2)$ and $Y = M(\mathbb{Z}_q, 2)$, where p and q are any positive coprime integers. Now, due to the interaction of torsion groups with the tensor product, since p and q are coprime, we have an isomorphism in homology modules $H_*(X \times Y) \cong H_*(X \vee Y)$. This also means that there is a homotopy equivalence $X \times Y \simeq X \vee Y$, and so the category of the product is the same as the category of the wedge. This is easily found to be 1, strictly less than the sum of $\text{cat} X$ and $\text{cat} Y$.

While this example is valid, it feels thoroughly unsatisfying. This example is motivated purely by manipulations involving algebra and torsion groups, whereas we, ideally, would like an example motivated by geometry and topology. This leads us finally to Ganea's conjecture. To avoid such examples as Moore spaces, we restrict one of the factors to simply be a sphere, and ask whether it is true, given any CW complex X and any sphere S^n , that $\text{cat}(X \times S^n) = \text{cat} X + 1$.

This drove many topologists on a search for a conclusion to the issue, trying to prove or disprove the conjecture. Some great strides were made along the way. The work of Hess [Hes91] and Jessup [Jes90], for example, proved that in fact for rational spaces, Ganea's conjecture *does* hold. However even with the rational picture figured out, nobody could prove the conjecture's validity for general spaces. It was not until 1998 that we found an answer to the problem, when Iwase [Iwa98] threaded the needle, so to speak. Not only did he find an example to counter Ganea's conjecture, but his example was, impressively, a two-cell CW complex. Moreover, he not only found one such example, but *infinitely many* – one for each odd prime, in fact. We conclude this thesis with a detailed analysis of the construction of these examples.

Remark

From time to time, there will be remarks that contain vital information for the reader. When such a remark arises, it will be contained in a box like this one.

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Part I

Basic Homotopy Theory

0.1 CW Complexes

One of the most prevalent types of topological spaces are manifolds. Manifolds, roughly speaking, are topological spaces which can be constructed by “stitching together” open pieces of Euclidean space. That is to say, a manifold M is locally homeomorphic to \mathbb{R}^n at each point. The reason we are so interested in manifolds in particular is that these are precisely the type of spaces which arise in the study of physics, and so they are the spaces with which we are most naturally familiar. Indeed it is conjectured that the universe itself is a manifold, as described by Einstein’s general theory of relativity.

Formally, a manifold is a space M together with a collection of pairs (U_i, φ_i) , where φ_i is a homeomorphism from U_i to some open subspace of \mathbb{R}^n , with the additional condition that $\bigcup U_i = M$ and $\varphi_i \varphi_j^{-1} : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is continuous, where defined, for all i and j . This carries with it an entire field of study, but for our purposes, it is enough to know that every manifold can be represented as another very special type of topological space, known as a *CW complex*.

A CW complex is a space which is built inductively out of closed n -disks, $e^n = \{x \in \mathbb{R}^n \mid \|x\| \leq 1\}$, also referred to as n -cells. Evidently, 0-cells are points, 1-cells are closed line segments, 2-cells are disks, and so on. To do this, we take any number of 0-cells, and attach to them some number of 1-cells via maps known as *attaching maps*. For illustrative purposes, let us construct the circle S^1 as a CW complex. To do this, we take exactly one 0-cell and exactly one 1-cell, and define an attaching map which sends the end points of e^1 to the point

e^0 , as shown below.



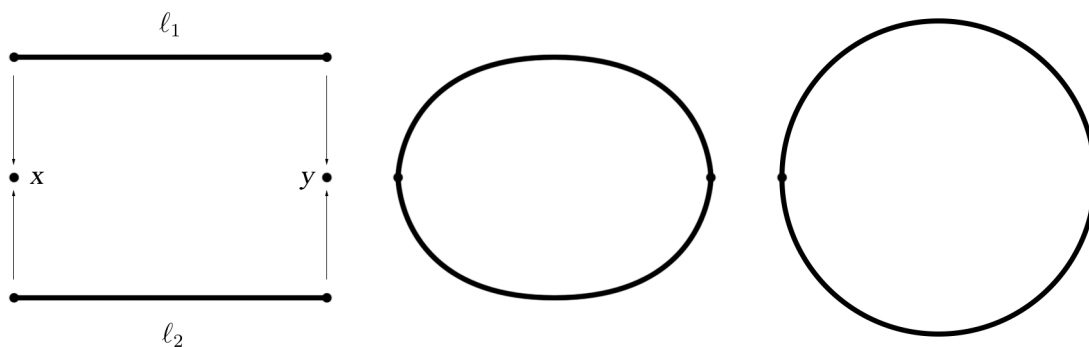
Formally, what is happening is that the space we create by "attaching" the end points to e^0 via the map φ is the quotient space $X = e^0 \cup_{\varphi} e^1$. Here, $X \cup_f Y$ denotes the quotient space $X \cup Y / \sim$, where \sim is the equivalence relation generated by $x \sim y$ if $f(x) = y$ (that is, $x \sim y$ if $x = y$ or if $f(x) = y$).

In general, the 1-cells are attached to the existing 0-cells, the 2-cells are attached to the existing 1-cells and 0-cells, the 3-cells attached to the existing 0-cells, 1-cells, and 2-cells, and so on. This collection of 0-cells up to n -cells is called the n -skeleton of the CW complex. For a CW complex X , its 0-skeleton, denoted X_0 , is a collection of points. 1-cells are attached to X_0 to form the 1-skeleton, and so on, so that n -cells are attached to X_{n-1} to form the n -skeleton X_n . We will restrict ourselves to CW complexes of *finite type*, which means that in each dimension, we attach only finitely many n -cells to X_{n-1} . If n is the smallest integer among all CW decompositions such that $X = X_n$, then X is said to be n -dimensional. If no such n exists, it is said to be infinite dimensional. This leads us to the full definition.

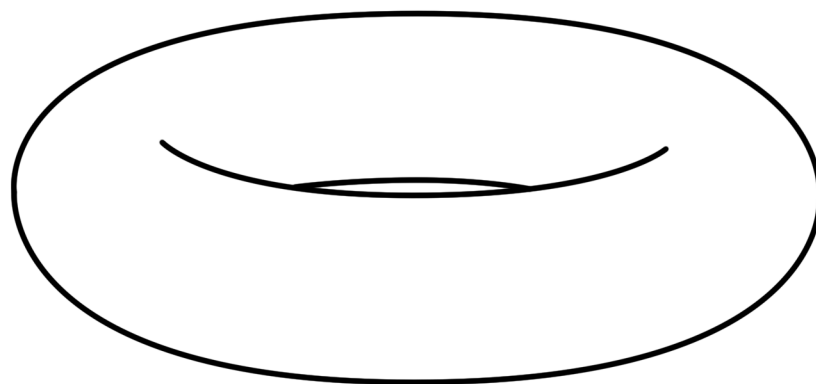
Definition 0.1 A space X is a *CW complex of finite type* if there is a sequence of subspaces $\{X_n\}_{n \geq 0}$ such that

- $X_0 = \{e_1^0, \dots, e_{m_0}^0\}$ is a collection of points.
- X_n is the quotient space $X_{n-1} \cup_{\varphi_1} e_1^n \cup \dots \cup_{\varphi_{m_n}} e_{m_n}^n$ where each φ_i is a continuous map $\varphi_i : \partial e_i^n \rightarrow X_{n-1}$, and
- $X = \bigcup X_n$.

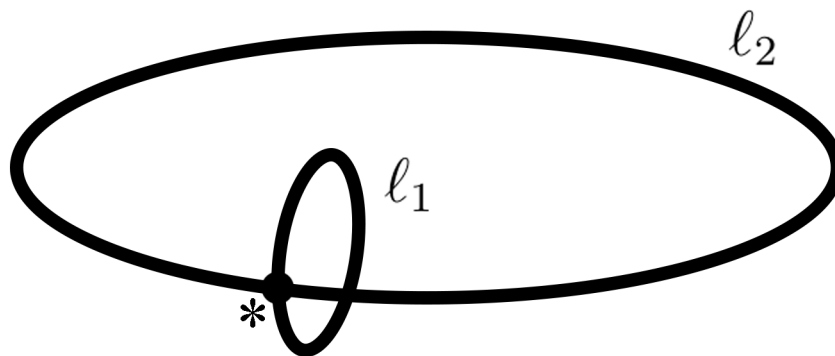
It is worth noting that the CW complex structure of a space is not unique. For example, we could also build the circle S^1 out of two 0-cells, x and y , and two 1-cells, ℓ_1 and ℓ_2 , like so:



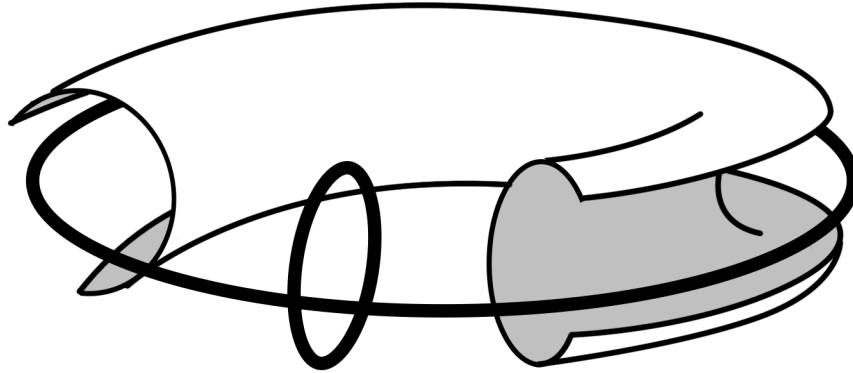
For a less trivial example, let us see that the torus, T , has a CW complex structure.



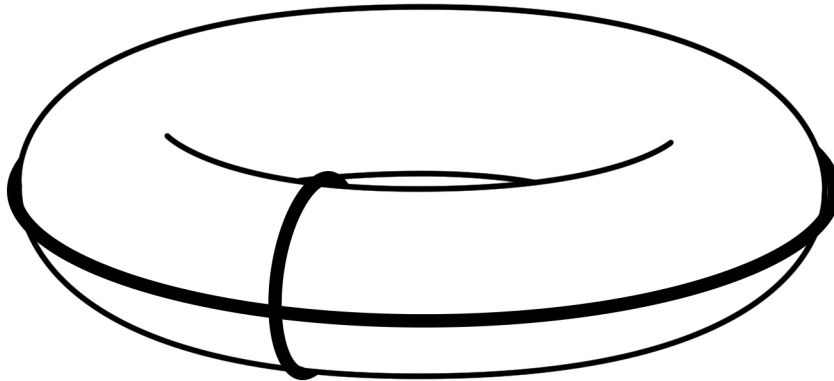
We can begin by taking a single point, $*$, and attaching to it two 1-cells, ℓ_1 and ℓ_2 , creating a wedge of circles.



Then we attach a 2-cell, d , viewing it as the square I^2 and attaching it as seen below.



This leaves us with the torus with which we are familiar.



Now, recall that $T = S^1 \times S^1$. Let us place a CW complex structure on each circle, such that the first consists of a 0-cell, x , and a 1-cell, ℓ_x , whose boundary is mapped to x , while the second is a 0-cell, y , and a similarly attached 1-cell, ℓ_y . It is interesting to see that for each m -cell in the first circle and n -cell in the second, there is a corresponding $m + n$ -cell in the product, T . That is, the 0-cell, $*$, in T , can be viewed as the product of the 0-cells in the circles, $x \times y$. Similarly, ℓ_1 can be viewed as $x \times \ell_y$ while ℓ_2 can be viewed as $\ell_x \times y$, and the 2-cell $d = \ell_x \times \ell_y$. This leads us to an important theorem from Whitehead ([Whi78, p.50])

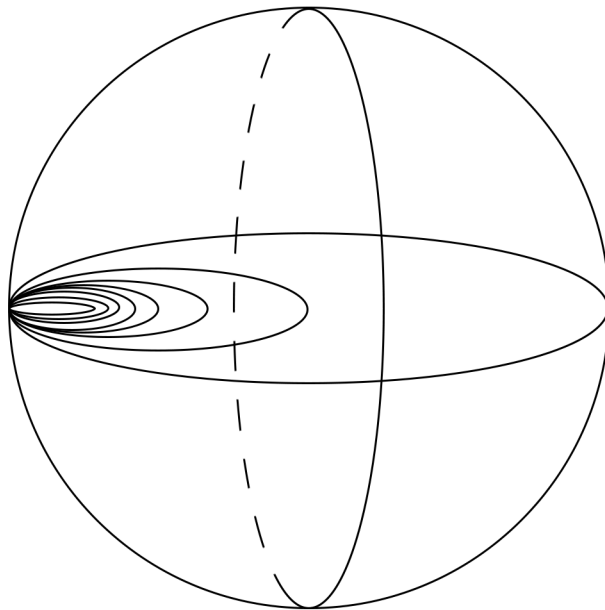
Theorem 0.2 Let X and Y be CW complexes with skeleta $\{X_n\}$ and $\{Y_m\}$ respectively. Then $X \times Y$ is a CW complex with skeleta $Z_n = \bigcup_{i=0}^n X_i \times Y_{n-i}$.

This follows from a slightly more general version of the theorem, but first we introduce another idea related to CW complexes that will be of use to us, that of a relative CW complex. A relative CW complex is similar to a CW complex, with the exception that we allow it to have an extra skeleton, X_{-1} , which need not have a cellular decomposition.

Definition 0.3 A pair (X, A) is a *relative CW complex* if there is a sequence of subspaces $\{X_n\}_{n \geq -1}$ such that

- $A \subseteq X$ is closed in X ,
- $A = X_{-1}$,
- $X_n = X_{n-1} \cup_{\varphi_1} e_1^n \cup \cdots \cup_{\varphi_{m_n}} e_{m_n}^n$ where each $\varphi_i : \partial e_i^n \rightarrow X_{n-1}$ is continuous, and
- $X = \bigcup X_n$.

An example of a relative CW complex can be found by taking A to be the Hawaiian earring $A = \{x \in \mathbb{R}^2 \mid \|x - (\frac{1}{n}, 0)\| = \frac{1}{n} \text{ for some } n \in \mathbb{N}\}$ with the subspace topology inherited from \mathbb{R}^2 , and attaching two 2-cells to the outermost ring, encasing the earring in a ball. While the Hawaiian earring does not itself possess a CW complex structure (indeed, CW complexes are locally contractible, where the Hawaiian earring is clearly not as no neighbourhood of the origin is contractible), X is formed by attaching cells to A , and so (X, A) does have a relative CW complex structure.



We now generalize Theorem 0.2. ([Whi78, p. 50])

Theorem 0.4 Let (X, A) and (Y, B) be relative CW complexes with skeleta $\{X_n\}$ and $\{Y_m\}$ respectively. Then $(X, A) \times (Y, B) = (X \times Y, A \times Y \cup X \times B)$ is a relative CW complex with skeleta $Z_n = \bigcup_{i=0}^n X_i \times Y_{n-i}$.

Theorem 0.2 follows as a direct result of Theorem 0.4, as every CW complex can be written as a relative CW complex with $A = \emptyset$.

Remark

From here on out, when we talk about a space (or topological space) X , we will be referring to a CW complex of finite type, unless otherwise stated.

0.2 Introduction to Homology and Cohomology

A powerful tool for studying topological spaces algebraically is found in homology theory. We will need the use of some results from homology and so we present a brief overview of the topic. A more complete introduction can be found in [Hat01].

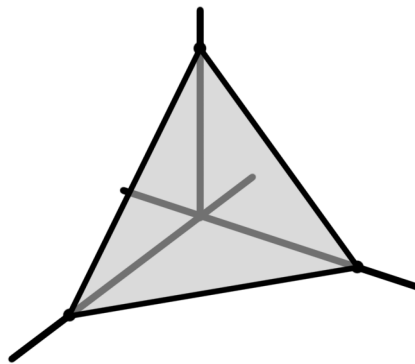
Generally speaking, a homology theory begins with a functor $C_* : \text{Top} \rightarrow \text{DGM}$, where DGM is the category of differential graded modules over some ring R . So, given a space X , we have a sequence of modules C_n , $n \in \mathbb{N}$, with boundary homomorphisms $\partial_n : C_n \rightarrow C_{n-1}$

$$\cdots \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} \cdots \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0} 0$$

such that $\partial_n \partial_{n+1} = 0$ for all n . We define two families of submodules, $Z_n(X) = \ker \partial_n$, which we call the cycles, and $B_n(X) = \text{im } \partial_{n+1}$, which we call the boundaries. The condition $\partial^2 = 0$ implies that $B_n(X) \subseteq Z_n(X)$ for each n . The n^{th} homology module of X is then defined to be the quotient module

$$H_n(X) = Z_n(X) / B_n(X).$$

A common definition of homology involves maps from the standard n -simplices into X . A standard n -simplex, denoted Δ^n , is the convex hull of the standard basis of \mathbb{R}^{n+1} . Algebraically, that is $\Delta^n = \{(x_0, \dots, x_n) \in \mathbb{R}^{n+1} \mid \sum x_i = 1 \text{ and each } x_i \geq 0\}$, as depicted below for $n = 2$.



A singular n -simplex is a continuous map $\sigma : \Delta^n \rightarrow X$. After fixing a ring, R , C_n is defined to be the free R -module generated by the singular n -simplices. In this case, the boundary operator ∂_n is defined on an n -simplex by

$$\partial_n \sigma = \sum_{i=0}^n (-1)^i \hat{\sigma}_i$$

where $\hat{\sigma}_i$ is the restriction of σ to the subspace $\{(x_0, \dots, x_n) \in \Delta^n \mid x_i = 0\}$, called the i^{th} face of Δ^n , and then extended linearly. It can be shown that this meets the restriction that $\partial^2 = 0$, and generates what is known as the *singular homology* of X .

Now, homology is indeed $H_* : \text{Top} \rightarrow \text{GM}$, with GM the category of graded modules. In this view, for a map $f : X \rightarrow Y$, the induced map f_* is defined by $f_n([\sigma]) = [f \circ \sigma]$. It is trivial to check that $(fg)_* = f_*g_*$ and that $\mathbb{1}_{X*} = \mathbb{1}_{H_*(X)}$.

Dual to this is singular *cohomology*. For each C_n , define C^n to be the linear dual module $C_n^\vee = \text{Hom}(C_n, R)$. Similarly replace each ∂_n by its dual homomorphism $d_n : C^{n-1} \rightarrow C^n$, to obtain what is known as a cochain complex.

$$0 \xrightarrow{d_0} C^0 \xrightarrow{d_1} C^1 \xrightarrow{d_2} \dots \xrightarrow{d_n} C^n \xrightarrow{d_{n+1}} \dots$$

Now we have $d_{n+1}d_n = 0$ for each n , and so we define two new families of submodules, as above – the cocycles, $Z^n(X) = \ker d_{n+1}$, and the coboundaries, $B^n(X) = \text{im } d_n$. The n^{th} cohomology module of X is then the quotient

$$H^n(X) = Z^n(X) / B^n(X).$$

From the construction of H_* , cohomology may also be viewed as a (this time contravariant) functor $H_* : \text{Top} \rightarrow \text{GM}$.

One of the important features of cohomology is the existence of a natural product of cohomology classes, known as the cup product. The Künneth theorem ([Kün23]) gives us an injection

$$H^i(X) \otimes H^j(Y) \hookrightarrow H^{i+j}(X \times Y).$$

Taking the special case $Y = X$, we then have a map $H^*X \otimes H^*X \hookrightarrow H^*(X \times X)$. All we require now to have a product in H^*X is a map $H^*(X \times X) \rightarrow H^*X$. As H^* is contravariant, a map $X \rightarrow X \times X$ induces a map $H^*(X \times X) \rightarrow H^*X$, which we are in search of. An obvious candidate is the diagonal map, $\Delta : X \rightarrow X \times X$. Indeed, in composing these two maps we get the cup product

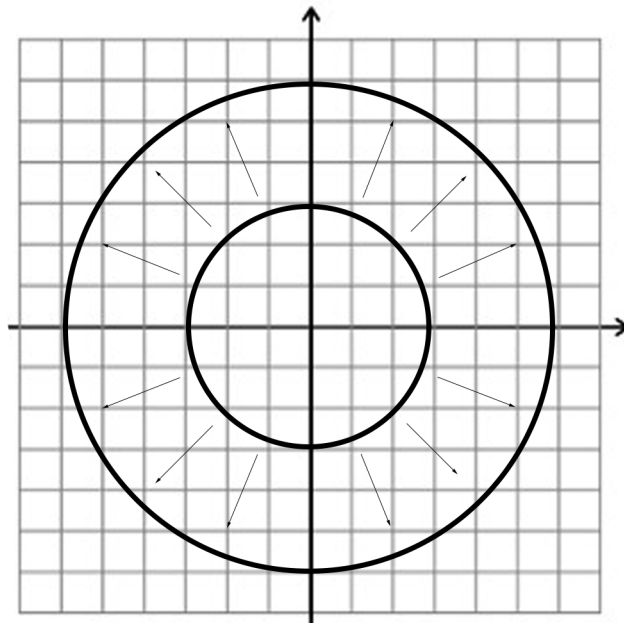
$$\smile : H^*X \otimes H^*X \hookrightarrow H^*(X \times X) \xrightarrow{\Delta^*} H^*X.$$

Proposition 0.5 [Bre93] When endowed with the cup product, the graded module H^*X becomes a commutative graded algebra.

SECTION 1

HOMOTOPY THEORY

Consider two maps $f : \mathbb{R} \rightarrow \mathbb{R}^2$ and $g : \mathbb{R} \rightarrow \mathbb{R}^2$ defined by $f(x) = (\cos x, \sin x)$ and $g(x) = (2 \cos x, 2 \sin x)$. Both $\text{im } f$ and $\text{im } g$ are circles in \mathbb{R}^2 , as can be seen below.



Now, intuitively, these maps seem very similar – all one needs to do to obtain g from f is to “stretch” f outward from the origin. This notion is formalized by homotopy theory.

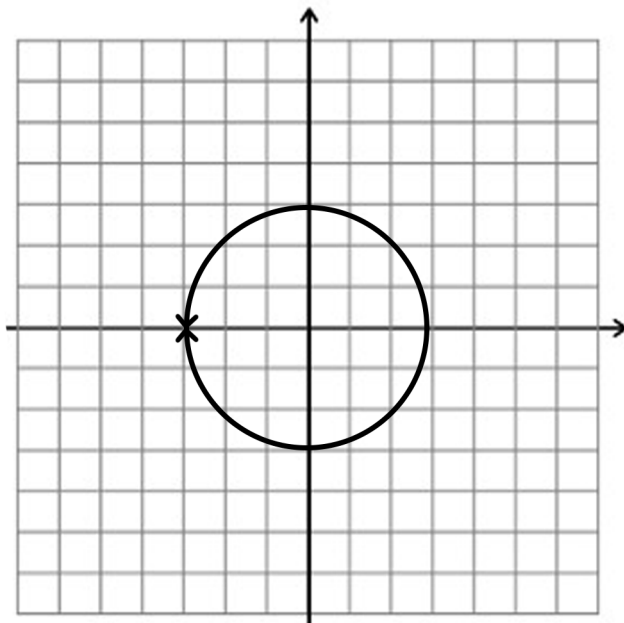
Given two maps $f : X \rightarrow Y$ and $g : X \rightarrow Y$, we define a *homotopy* between these two maps to be another map $H : X \times I \rightarrow Y$ such that

- H is continuous,
- $H(x, 0) = f(x)$ for all x , and

- $H(x, 1) = g(x)$ for all x .

If such an H exists, f and g are said to be *homotopic*. This is denoted $f \simeq g$. In the above example, we could take, for instance, $H : \mathbb{R} \times I \rightarrow \mathbb{R}^2$ defined by $H(x, t) = ((1+t) \cos x, (1+t) \sin x)$. It is easy to see that this H satisfies the criteria listed above, and so H is a homotopy from f to g .

Suppose that f and g are two maps from X to Y , and $H : f \simeq g$. If there is a subspace A of X for which $H(a, t) = f(a) = g(a)$ for all $t \in I$ and $a \in A$, we say that f is *homotopic to g relative to A* , and we denote this $f \simeq_A g$. An important example of this is when the maps f and g are paths in Y . Consider two paths in \mathbb{R}^2 , $p(s) = (\cos(\pi s), \sin(\pi s))$ and $q(t) = (\cos(-\pi s), \sin(-\pi s))$. Both $p, q : [0, 1] \rightarrow \mathbb{R}^2$ are paths from $(1, 0)$ to $(-1, 0)$, taking opposite directions around the unit circle, like so:



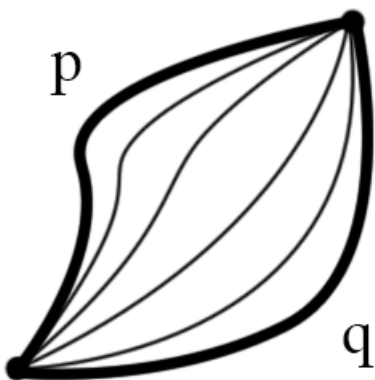
Formally, a *path* between two points x and y in a space X is a continuous map $p : [0, 1] \rightarrow X$ with $p(0) = x$ and $p(1) = y$.

For each s , let $\ell_s(t)$ be the straight line path from $p(s)$ to $q(s)$, $\ell_s(t) = (1-t)p(s) + tq(s)$. Then we can define $H : I^2 \rightarrow \mathbb{R}^2$ by $H(s, t) = \ell_s(t)$. Evidently, H is continuous. Further $H(s, 0) = p(s)$, $H(s, 1) = q(s)$, so H is a homotopy from p to q . The reason for this example, though, is that for all t , $H(0, t) = p(0) = q(0) = (1, 0)$, and $H(1, t) = p(1) = q(1) = (-1, 0)$, making H a homotopy from p to q relative to $\{0, 1\}$.

Consider, however, the same two paths, this time in $\mathbb{R}^2 \setminus \{(0, 0)\}$. The same homotopy constructed above no longer works, as $H(\frac{1}{2}, \frac{1}{2}) = (0, 0)$ is not in this space.

In fact, no homotopy exists which takes p to q , relative to $\{0, 1\}$, in our new codomain. Note that for each $t \in I$, for a fixed homotopy $H(s, t)$ which takes p to q , H simply defines a new

path $p_t(s) = H(s, t)$. Informally, what H does is morph p into q by pushing it from one place to another in the codomain, as depicted below.



However to push p along $\mathbb{R}^2 \setminus \{(0, 0)\}$ to form q , one would have to “jump over” the origin, and thus make our homotopy discontinuous. For a more rigorous proof of this, if p and q were homotopic, then the integrals $\int_p f(z)dz$ and $\int_q f(z)dz$ should give the same result for the function $f : \mathbb{C} \rightarrow \mathbb{C}$ defined by $f(z) = \frac{1}{z}$ by Cauchy’s integral theorem ([Rud00]). However,

$$\int_p f(z)dz = \int_p \frac{1}{z} dz = \int_0^1 \frac{1}{e^{i\pi t}} \cdot i\pi e^{i\pi t} dt = i\pi$$

while

$$\int_q f(z)dz = \int_q \frac{1}{z} dz = \int_0^1 \frac{1}{e^{-i\pi t}} \cdot (-i\pi) e^{-i\pi t} dt = -i\pi.$$

Thus, homotopies between paths inherently carry information about the structure of spaces they are in. \mathbb{R}^2 must somehow be fundamentally different from $\mathbb{R}^2 \setminus \{(0, 0)\}$.

1.1 Homotopy Groups

To measure this precisely, we restrict ourselves to those paths which begin and end at the same point, which we call *loops*. For a space X , define $\Omega(X, *)$ to be the space of loops on X which begin and end at the point $*$. In this case we call $*$ the *basepoint* of X . We can construct a binary operation on $\Omega(X, *)$ by taking two loops α and β , and constructing a new loop

$$\alpha \star \beta(s) = \begin{cases} \alpha(2s), & 0 \leq s \leq \frac{1}{2} \\ \beta(2s - 1), & \frac{1}{2} \leq s \leq 1 \end{cases}$$

which traces out the loop α , and then the loop β . This operation is not quite associative, as, for a third loop γ , a small calculation reveals that

$$(\alpha \star \beta) \star \gamma(s) = \begin{cases} \alpha(4s), & 0 \leq s \leq \frac{1}{4} \\ \beta(4s - 1), & \frac{1}{4} \leq s \leq \frac{1}{2} \\ \gamma(2s - 1), & \frac{1}{2} \leq s \leq 1 \end{cases}$$

while

$$\alpha \star (\beta \star \gamma)(s) = \begin{cases} \alpha(2s), & 0 \leq s \leq \frac{1}{2} \\ \beta(4s - 2), & \frac{1}{2} \leq s \leq \frac{3}{4} \\ \gamma(4s - 3), & \frac{3}{4} \leq s \leq 1 \end{cases}$$

so indeed $(\alpha \star \beta) \star \gamma \neq \alpha \star (\beta \star \gamma)$ in most cases. However these two loops *are* homotopic via the homotopy

$$H(s, t) = \begin{cases} \alpha\left(\frac{4s}{t+1}\right), & 0 \leq s < \frac{1}{4}t + \frac{1}{2} \\ \beta(4s - t - 1), & \frac{1}{4}t + \frac{1}{4} \leq s < \frac{1}{4}t + \frac{1}{2} \\ \gamma\left(\frac{4s-t-2}{2-t}\right), & \frac{1}{4}t + \frac{1}{2} \leq s \leq 1 \end{cases}$$

which is continuous, by the Pasting Lemma ([Hat01]).

Similarly, there are no two loops $\alpha \star \beta = \alpha$. However, if c_* is the constant loop at $*$, $c_*(s) = x_0$, then for all loops α , $\alpha \star c_*$ is homotopic to α , with the homotopy

$$H(s, t) = \begin{cases} \alpha\left(\frac{2s}{t+1}\right), & 0 \leq s \leq \frac{1}{2}t + \frac{1}{2} \\ c_*\left(\frac{2s-t-1}{1-t}\right), & \frac{1}{2}t + \frac{1}{2} < s \leq 1 \end{cases}.$$

Finally, if we define a path $\bar{\alpha}(s) = \alpha(1 - s)$, then we see that $\alpha \star \bar{\alpha}$ homotopic to c_* , as is $\bar{\alpha} \star \alpha$, via the homotopies

$$H(s, t) = \begin{cases} \alpha\left(\frac{2s}{1-t}\right), & 0 \leq s \leq \frac{1}{2} - \frac{1}{2}t \\ c_*\left(\frac{2s-1+t}{2t}\right), & \frac{1}{2} - \frac{1}{2}t \leq s \leq \frac{1}{2}t + \frac{1}{2} \\ \bar{\alpha}\left(\frac{2s-t-1}{1-t}\right), & \frac{1}{2}t + \frac{1}{2} \leq s \leq 1 \end{cases}$$

and

$$H(s, t) = \begin{cases} \bar{\alpha}\left(\frac{2s}{1-t}\right), & 0 \leq s \leq \frac{1}{2} - \frac{1}{2}t \\ c_*\left(\frac{2s-1+t}{2t}\right), & \frac{1}{2} - \frac{1}{2}t \leq s \leq \frac{1}{2}t + \frac{1}{2} \\ \alpha\left(\frac{2s-t-1}{1-t}\right), & \frac{1}{2}t + \frac{1}{2} \leq s \leq 1 \end{cases}$$

respectively. So this operation has properties similar to those of a group operation. However we are held back by the restriction that these properties only work up to homotopy. To work around this, define a relation \sim on $\Omega(X, *)$ by saying that $\alpha \sim \beta$ if α is homotopic to β relative to $\{0, 1\}$.

This relation is reflexive, as for all α , $H(s, t) = \alpha(s)$ is a homotopy $H : \alpha \simeq_{\{0,1\}} \alpha$. Moreover, if $H : \alpha \simeq_{\{0,1\}} \beta$ for some loops α and β , then $\tilde{H}(s, t) = H(s, 1 - t)$ is a homotopy $\tilde{H} : \beta \simeq_{\{0,1\}} \alpha$, so the relation is symmetric. Lastly, if $H : \alpha \simeq_{\{0,1\}} \beta$ and $J : \beta \simeq_{\{0,1\}} \gamma$, then G defined by

$$G(s, t) = \begin{cases} H(s, 2t), & 0 \leq t \leq \frac{1}{2} \\ H(s, 2t - 1), & \frac{1}{2} \leq t \leq 1 \end{cases}$$

is a homotopy between α and γ , so $\alpha \sim \gamma$. Thus \sim is an equivalence relation.

Define a new set $\pi_1(X, *)$ to be the set of all equivalence classes of loops in $\Omega(X, *)$, and define $[\alpha] \star [\beta] = [\alpha \star \beta]$. By the calculations presented above, this operation is well defined, and $\pi_1(X, *)$ is a group with the operation \star . We call this group the *fundamental group* or *1st homotopy group* of X with basepoint $*$.

Remark 1.1 Note that if x_0 and y_0 are two points within the same path component of X , then $\pi_1(X, x_0) \cong \pi_1(X, y_0)$. With this in mind, if the space X in question is path connected, or if the basepoint $*$ is understood, we will write $\pi_1(X)$ and refer simply to the fundamental group of X .

To extend this idea, let us work in the category of pointed pairs of topological spaces, Top_2^* . An object here is a pair (X, A) , where A is a closed subspace of X with $*$ \in A . This is also sometimes written as a triple $(X, A, *)$. A morphism between two pairs (X, A) , and (Y, B) is a pointed map $f : X \rightarrow Y$ (that is, $f(*) = *$), such that $f(A) \subseteq B$.

Note that any loop γ in X is simply a morphism in Top_2^* , $(I, \{0, 1\}) \rightarrow (X, *)$. Further, we can factor γ through the pair $(S^1, *)$, so that we have a commutative diagram

$$\begin{array}{ccc} (I, \{0, 1\}) & \xrightarrow{\quad\quad\quad} & (X, *) \\ & \searrow & \nearrow \\ & (S^1, *) & \end{array}$$

where the bottom-left map is the quotient map $I \rightarrow I / \{0, 1\} \cong S^1$. In this way, we can consider loops in X to be pointed maps $(S^1, *) \rightarrow (X, *)$. Now that we have this result for S^1 , the next step is to generalize it to all spheres. Indeed, we note that for every n , there is a similar commutative diagram

$$\begin{array}{ccc} (I^n, \partial I^n) & \xrightarrow{\quad\quad\quad} & (X, *) \\ & \searrow & \nearrow \\ & (S^n, *) & \end{array}$$

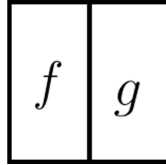
where the bottom left arrow is the quotient map $I^n \rightarrow I^n / \partial I^n \cong D^n / S^{n-1} \cong S^n$. In this way, we can consider maps $(I^n, \partial I^n) \rightarrow (X, *)$ to be pointed maps $S^n \rightarrow X$, and vice versa.

We then construct the n th homotopy group of X out of the homotopy classes of such maps. Of course, for this to in fact be a group we must have a group operation. The operation in this case is as follows. For maps $f, g \in \pi_n(X)$, define $f \star g \in \pi_n(X)$ by

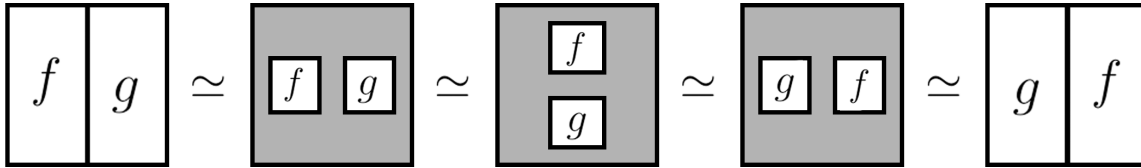
$$(f \star g)(x) = (f \vee g)\psi(x)$$

where $\psi : S^n \rightarrow S^n \vee S^n$ is the pinch map, which is the quotient map which takes S^n to $S^n \vee S^n$ by collapsing the equator to a point, and rescaling. We can picture this operation, utilizing the fact that we can think of maps in $\pi_n(X)$ as a map $(I^n, \partial I^n) \rightarrow (X, *)$. Given a two maps $f, g \in \pi_n(X)$, we consider them as maps $(I^n, \partial I^n)$, and apply each to a cube I^n , and attach these cubes along a side, This has the effect of applying a single, new map to a single cube, as

shown for $n = 2$ in the diagrams below, adapted from [Hat01].



Interestingly, this picture allows us to explain why $\pi_n(X)$ is abelian for $n > 1$, but not, in general, for $n = 1$. Since each map is trivial on the edge of each cube, shown below as solid black lines, we may contract the image away from the edge, filling the rest of the space with the trivial map, shown below in gray, and freely move the images as we see fit. Given this, we can see that $f \star g$ and $g \star f$ are indeed the same map, up to homotopy.



Similar diagrams show that this operation is associative. Inverse are obtained by switching the orientation of the sphere. Hence the inverse of $f(x) = (f_1(x), f_2(x), f_3(x), \dots, f_n(x))$ can be obtained as $f^{-1}(x) = (f_2(x), f_1(x), f_3(x), \dots, f_n(x))$ (or by switching any pair of coordinates).

1.2 Relative Homotopy Groups

Now, rather than maps of the form $(I^n, \partial I^n) \rightarrow (X, *)$, let us allow ourselves more generality and consider maps $(I^n, \partial I^n) \rightarrow (X, A)$. Homotopy classes of these maps, together with the same operations explored above, make up the *relative homotopy groups* $\pi_n(X, A)$. Here, we insist that two maps in the same class be not only homotopic, but homotopic relative to A .

Note that in the case when $A = *$, $\pi_n(X, A)$ reduces to the regular homotopy group, $\pi_n(X)$.

1.3 Functorial Properties of π_n

Categorically speaking, for each n , π_n can be viewed as a covariant functor $\pi_n : \text{Top}_2^* \rightarrow \text{Grp}$, where Grp is the category whose objects are groups, and whose arrows are group homomorphisms.

Take a map $f : (X, A) \rightarrow (Y, B)$. Then for $n \in \mathbb{N}$, we define a map $\pi_n(f) : \pi_n(X, A) \rightarrow \pi_n(Y, B)$ as follows. For a map $\varphi : (I^n, \partial I^n) \rightarrow (X, A)$, define a map $\psi = f\varphi : (I^n, \partial I^n) \rightarrow (Y, B)$, and set $\pi_n(f)[\varphi] = [\psi]$.

To see that this is a functor, note that for all $\varphi \in \pi_n(X)$, $\pi_n(\mathbf{1}_X)[\varphi] = [\mathbf{1}_X\varphi] = [\varphi]$, so $\pi_n(\mathbf{1}_X) = \mathbf{1}_{\pi_n(X)}$. Further for any map $g : Y \rightarrow Z$, $\pi_n(gf)[\varphi] = [gf\varphi] = \pi_n(g)[f\varphi] = \pi_n(g)[\pi_n(f)[\varphi]]$, so $\pi_n(gf) = \pi_n(g)\pi_n(f)$.

1.4 Features of Top_2^*

For convenience, when we are talking about the pair $(X, *)$, we will often suppress the basepoint and simply talk about the space X . Now, as we are concerned with pointed spaces and pointed maps, some familiar constructions on topological spaces must take on some amount of extra structure in order to maintain well-defined basepoints. We provide a list of some of these constructions and a description their analogues in Top_2^* which will be used throughout this thesis. A more thorough treatment can be found in [Swi75].

Definition 1.2 One of the most commonly used constructions for us will be the *wedge sum*, or topological sum. The wedge of the spaces X and Y , denoted $X \vee Y$, is the subspace of $X \times Y$ consisting of all points which contain the basepoint in at least one coordinate.

The wedge sum offers a form of commutativity and associativity, so that

$$X \vee Y \cong Y \vee X$$

and

$$(X \vee Y) \vee Z \cong X \vee (Y \vee Z).$$

and so we will often suppress parentheses and refer to $X \vee Y \vee Z$, the subspace of $X \times Y \times Z$ of all points, all but one of whose coordinates are the basepoint.

Definition 1.3 From time to time, we will make use of the smash product. The *smash product* of two spaces X and Y , denoted $X \wedge Y$, is the quotient space

$$X \wedge Y = X \times Y / X \vee Y.$$

Conveniently, for CW complexes, X , Y , and Z , we again have commutativity and associativity, so that

$$X \wedge Y \cong Y \wedge X$$

and

$$(X \wedge Y) \wedge Z \cong X \wedge (Y \wedge Z)$$

and so we will again often suppress parentheses and refer to $X \wedge Y \wedge Z$.

Definition 1.4 The *reduced cone* of a space X is the quotient

$$CX = X \times I / X \times \{0\} \cup * \times I = X \times I / X \vee I = X \wedge I$$

where I is the unit interval $[0, 1]$. When we talk about CX , we will often simply refer to it as the cone of X .

An important fact to know about the cone is that for any sphere S^n , $CS^n \cong D^{n+1}$. Moreover, for any space X , its cone CX is contractible.

Definition 1.5 The *reduced suspension* of a space X is

$$\Sigma X = X \times I / X \times \{0\} \cup * \times I \cup X \times \{1\} = CX / X \times \{1\}$$

where I is the unit interval. As with the cone, we will often simply refer to this as the suspension of X .

Note that for any sphere S^n , $\Sigma S^n \cong S^{n+1}$. Further, we can write $\Sigma X = X \wedge S^1$, and it follows that $S^n \wedge S^m = S^{n+m}$.

Definition 1.6 We will also make use of the *reduced join* of two spaces, denoted $X * Y$. This space is somewhat more involved to construct.

First, let us construct a space

$$X \star Y = X \times Y \times I / \sim$$

where \sim is the equivalence relation generated by $(x, y, 0) \sim (x, y', 0)$ and $(x, y, 1) \sim (x', y, 1)$ for all $x, x' \in X$ and all $y, y' \in Y$. The reduced join is then

$$X * Y = X \star Y / X \star * \cup * \star Y.$$

As always, we will often refer to the reduced join simply as the join of X and Y . Some properties of the join which will become useful for us include

- $X * Y \cong CX \times Y \cup_{X \times Y} X \times CY$,
- $X * Y \cong \Sigma(X \wedge Y)$, and consequently
- $S^n * S^m \cong S^{n+m+1}$.

1.5 Fibrations and Cofibrations

Two important types of maps which we will use and study are fibrations and cofibrations. Roughly speaking, fibrations describe what it means for one space to be parametrized by another. Formally, we have the following.

Definition 1.7 A continuous map $p : E \rightarrow B$ is said to satisfy the *homotopy lifting property* with respect to X if for any homotopy $f : X \times I \rightarrow B$, and for any map $f_0 : X \rightarrow E$ lifting $f_0 = f|_{X \times \{0\}} : X \rightarrow B$ through E , there exists a homotopy $\tilde{f} : X \times I \rightarrow E$ making the following diagram commute.

$$\begin{array}{ccc} X & \xrightarrow{\tilde{f}_0} & E \\ \downarrow \scriptstyle X \times \{0\} & \nearrow \tilde{f} & \downarrow p \\ X \times I & \xrightarrow{f} & B \end{array}$$

Further, a map which satisfies the homotopy lifting property with respect to every space is called a *fibration*.

Definition 1.8 Dual to fibrations, a map $i : A \rightarrow X$ is said to be a *cofibration* if it satisfies the *homotopy extension property* with respect to all spaces Y . The homotopy extension property can be stated dually to the homotopy lifting property, as follows.

$$\begin{array}{ccc} Y & \xleftarrow{\tilde{f}_0} & X \\ \uparrow & \swarrow \tilde{f} & \uparrow i \\ Y^I & \xleftarrow{f} & A \end{array}$$

SECTION 2

HOMOTOPY COMMUTATIVE DIAGRAMS

2.1 Pushouts and Pullbacks

An essential tool to understand in order to reach our goal is that of homotopy commutative diagrams. Of particular importance will be homotopy pullbacks and pushouts, introduced by Mather [Mat76]. Here we borrow definitions from the treatment of this subject given by Doeraene [Doe98].

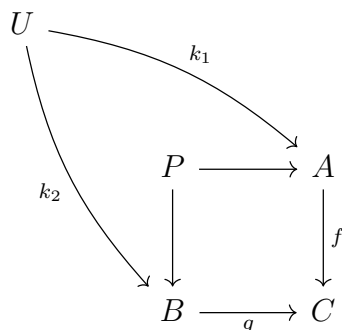
Recall that a commutative diagram in a category C is a collection of objects and morphisms in C such that all compositions of arrows with the same beginning and end point result in the same map. For example, the square

$$\begin{array}{ccc} P & \xrightarrow{g'} & A \\ f' \downarrow & & \downarrow f \\ B & \xrightarrow{g} & C \end{array}$$

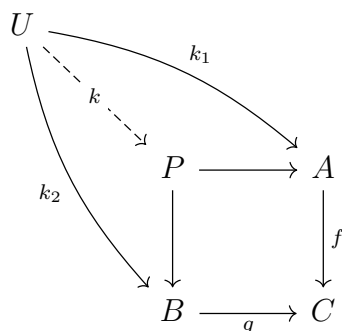
is commutative if $g'f = f'g$. For our purposes, we will not require that the maps be exactly equal, but merely homotopic to one another. To this end, we say that the above diagram is homotopy commutative, or that it commutes up to homotopy, if there exists a homotopy $H : g'f \sim f'g$.

Definition 2.1 P is said to be the *pullback* of $f : A \rightarrow C$ and $g : B \rightarrow C$ if, for any U and

maps $k_1 : U \rightarrow A$ and $k_2 : U \rightarrow B$ such that



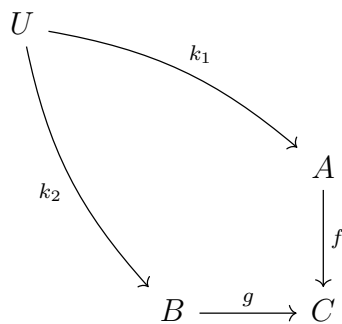
commutes, there exists a unique map $k : U \rightarrow P$, referred to as the *whisker map*, such that the following diagram also commutes.



Definition 2.2 Extending this idea to the language of homotopy commutative diagrams, we say that a homotopy commutative diagram

$$\begin{array}{ccc} P & \xrightarrow{g'} & A \\ f' \downarrow & & \downarrow f \\ B & \xrightarrow{g} & C \end{array}$$

equipped with a homotopy $H : g'f \sim f'g$ is a *homotopy pullback* if for any homotopy commutative diagram



equipped with its own homotopy $G : k_1 f \sim k_2 g$, there is a unique (up to homotopy) map $k : U \rightarrow P$ and homotopies $J : k g' \sim k_1$ and $K : k f' \sim k_2$ such that the diagram

$$\begin{array}{ccccc}
 U & & & & \\
 \searrow^{k_1} & & & & \\
 & \dashrightarrow^k & & & \\
 & & P & \xrightarrow{g'} & A \\
 & & \downarrow f' & & \downarrow f \\
 & & B & \xrightarrow{g} & C \\
 \searrow^{k_2} & & & &
 \end{array}$$

together with the homotopies $H, G, J,$ and K above also commutes up to homotopy. That is to say, $G \sim gK \star Hk \star fJ$, where, by $gK \star Hk \star fJ$ we refer to the homotopy

$$(gK \star Hk \star fJ)(x, t) = \begin{cases} gK(x, 1 - 3t), & 0 \leq t \leq \frac{1}{3} \\ H(k(x), 3t - 1), & \frac{1}{3} \leq t \leq \frac{2}{3} \\ fJ(x, 3t - 2), & \frac{2}{3} \leq t \leq 1 \end{cases}$$

taking gk_2 to fk_1 .

Definition 2.3 Similarly, given a diagram

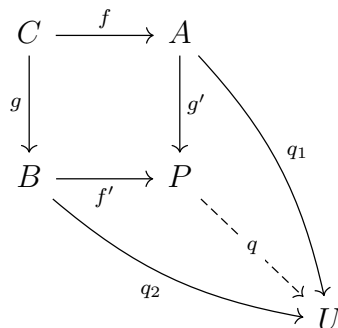
$$\begin{array}{ccc}
 C & \xrightarrow{f} & A \\
 \downarrow g & & \downarrow g' \\
 B & \xrightarrow{f'} & P
 \end{array}$$

equipped with a homotopy $H : fg' \sim gf'$, we say that P is the *homotopy pushout* of $f : C \rightarrow A$ and $g : C \rightarrow B$ if, for any U and maps $q_1 : A \rightarrow U$ and $q_2 : B \rightarrow U$ such that the diagram

$$\begin{array}{ccc}
 C & \xrightarrow{f} & A \\
 \downarrow g & & \searrow^{q_1} \\
 B & & U \\
 & \searrow^{q_2} &
 \end{array}$$

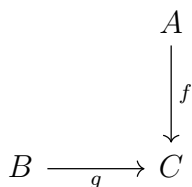
is homotopy commutative with the homotopy $G : fq_1 \sim gq_2$, there exists a unique (again, up

to homotopy) map $q : P \rightarrow U$ and homotopies $J : qf' \sim q_2$ and $K : qg' \sim q_1$ so that

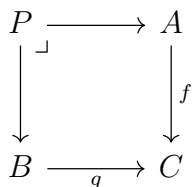


together with the homotopies H , G , J , and K commutes, so that $G \sim Jg \star qH \star Kf$.

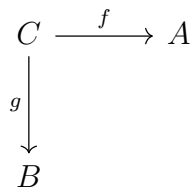
Notation 2.4 If P is the pullback of two maps $f : A \rightarrow C$ and $g : B \rightarrow C$, we will often say that it is the pullback of the diagram



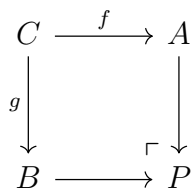
and denote it by the following.



Similarly, if P is the pushout of two maps $f : C \rightarrow A$ and $g : C \rightarrow B$, we will often say that it is the pushout of the diagram



and denote it diagrammatically as follows.



Another common convention is to write the pullback of

$$\begin{array}{ccc} & & A \\ & & \downarrow \\ B & \longrightarrow & C \end{array}$$

by $P = A \times_C B$, and the pushout of

$$\begin{array}{ccc} C & \longrightarrow & A \\ \downarrow & & \\ B & & \end{array}$$

by $P = A \amalg_C B$.

2.2 Comparing Pullbacks and Homotopy Pullbacks

Let us look at an example of a homotopy pullback. Consider the diagram

$$\begin{array}{ccc} & & * \\ & & \uparrow i \\ * & \xrightarrow{i} & S^1 \end{array}$$

where the arrows $i : * \hookrightarrow S^1$ are the inclusion of the basepoint. Taking inspiration from the standard construction of the pullback in Top , we might make a first guess and say that the homotopy pullback of this diagram is $P = \{(a, b) \in * \times * \mid i(a) = i(b)\} \cong *$. The resulting diagram, equipped with any homotopy $H : I \times * \rightarrow S^1$, is certainly homotopy commutative (indeed it commutes in the regular sense). Now, however, let us consider for example the homotopy commutative diagram

$$\begin{array}{ccc} \Omega S^1 & \xrightarrow{p} & * \\ \downarrow p & & \downarrow i \\ * & \xrightarrow{i} & S^1 \end{array}$$

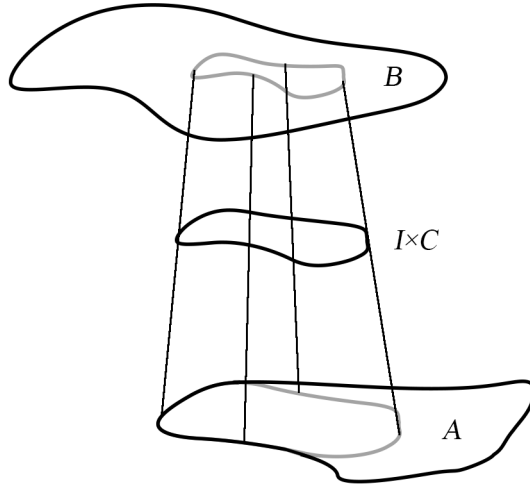
is

$$\begin{aligned}
 P &= \{a, \gamma, b \in * \times (S^1)^I \times * \mid i(a) = \gamma(0) \text{ and } i(b) = \gamma(1)\} \\
 &= \{(*, \gamma, *) \in * \times (S^1)^I \times * \mid \gamma(0) = * \text{ and } \gamma(1) = *\} \\
 &\cong \{\gamma \in (S^1)^I \mid \gamma(0) = * \text{ and } \gamma(1) = *\} \\
 &= \Omega S^1.
 \end{aligned}$$

Similarly, the familiar construction of the pushout in the category Top is insufficient to determine the homotopy pushout of two maps $f : C \rightarrow A$ and $g : C \rightarrow B$. However, a standard construction parallel to that of the homotopy pushout presented above can be stated as ([Doe98])

$$A \amalg_C B = A \cup (I \times C) \cup B / \sim$$

where \sim is the relation $f(c) \sim (c, 0)$, and $g(c) \sim (c, 1)$. Pictorially, this is the result of gluing either end of the cylinder of C to its image in A and B , respectively.



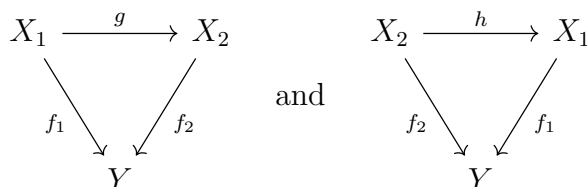
2.3 Replacing Maps in Diagrams

The methods of computing homotopy pushouts and pullbacks above are convenient, but lack geometric inspiration. Ideally we would like the ability to intuitively find pullbacks and pushouts motivated by the geometry of the maps and spaces involved.

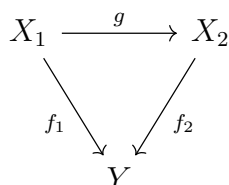
To accomplish this, we will require the ability to replace maps with maps which are in some sense equivalent to the original. Our answer to this problem comes by defining the slice category, $\text{Top} \downarrow Y$ whose objects are maps $f : X \rightarrow Y$, and whose morphisms are triangles

$$\begin{array}{ccc}
 X_1 & \xrightarrow{g} & X_2 \\
 & \searrow f_1 & \swarrow f_2 \\
 & & Y
 \end{array}$$

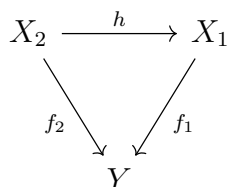
which commute in Top . One might try to define homotopy in this category by saying that $f_1 \simeq f_2$ if there is a homotopy equivalence $g : X_1 \rightarrow X_2$ with homotopy inverse $h : X_2 \rightarrow X_1$ such that



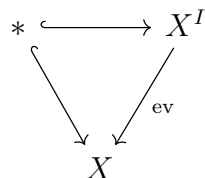
commute in Top . Of course, we would like to relax the condition so that these triangles need only commute up to homotopy. The problem here is that for a triangle



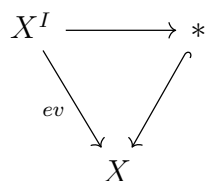
in $\text{Top} \downarrow Y$, we have no guarantee that the corresponding triangle



is in $\text{Top} \downarrow Y$ as well. For example, consider the triangle



in $\text{Top} \downarrow X$, where $\text{ev} : X^I \rightarrow X$ is the evaluation map. Clearly $* \hookrightarrow X^I$ admits a homotopy inverse, however the corresponding triangle



is *not* in $\text{Top} \downarrow X$ as it does not commute in Top . The solution to this is to construct *weak equivalences*, and work in the resulting quotient category, $ho(\text{Top} \downarrow Y)$, as in [Mac98, p. 51].

g for which the diagram

$$\begin{array}{ccc} (S^1)^I & \xrightarrow{g} & * \\ \text{ev} \searrow & & \swarrow i \\ & S^1 & \end{array}$$

commutes up to homotopy. Thus we can consider, instead, the diagram

$$\begin{array}{ccc} & (S^1)^I & \\ & \downarrow \text{ev} & \\ * & \xrightarrow{i} & S^1 \end{array}$$

and the pullback, in the traditional sense, of this diagram is

$$\begin{aligned} P &= \{(a, b) \in * \times (S^1)^I \mid i(a) = \text{ev}(b)\} \\ &= \{(*, \gamma) \in * \times (S^1)^I \mid * = \gamma(1)\} \\ &\cong \{\gamma \in (S^1)^I \mid \gamma(0) = * \text{ and } \gamma(1) = *\} \\ &= \Omega S^1. \end{aligned}$$

This is interesting. Although the maps are equivalent, $i : * \hookrightarrow S^1$ produced an incorrect result when used in the traditional construction of the pullback, while $\text{ev} : (S^1)^I \rightarrow S^1$ performed perfectly and gave us the correct answer. What is it about ev that differs so much from i that it gives us the desired space? Could this result be applied more generally so that we can obtain other pullbacks and pushouts in the traditional sense simply by finding the correct replacement map?

The answer, here, is that ev is a *fibration*. By replacing at least one of the maps in question by an equivalent fibration, one unifies the homotopy pullback and the pullback in the familiar sense. Now, however, there is still the question of if we can *always* do this.

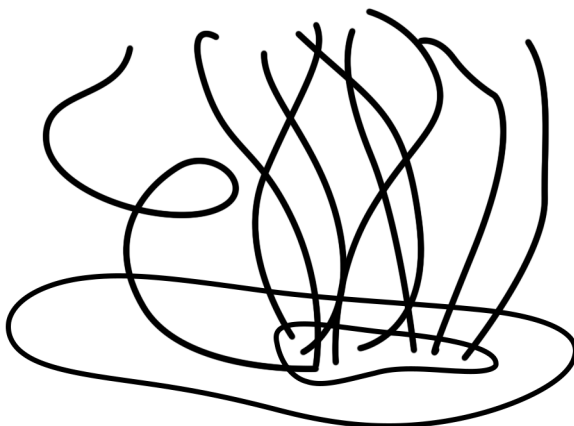
Theorem 2.6 Every map $f : X \rightarrow Y$ is, up to homotopy, a fibration.

The method to this proof is to actually construct the fibration which is homotopy equivalent to f . Helpfully, doing so also gives us insight into the geometry of the issue.

Our goal here is to replace X by a homotopy equivalent space P_f , just as in our example above we replaced $*$ by a homotopy equivalent space $(S^1)_*$, and thereby replaced f by a homotopic map $P_f \rightarrow Y$ which is a fibration.

Define P_f to be the space $P_f = \{(x, \alpha) \in X \times Y^I \mid f(x) = \alpha(0)\}$. P_f is known as the *mapping path space* of f . Less formally, P_f is the space of paths in Y which begin in the image of f . Visually, one can picture this space as the space $f(X)$ with a collection of spaghetti-like

strings coming off of each point, similar to seaweed in a fish tank, each representing one such path, like so.



Now, since each line segment has the homotopy type of a single point, it is not hard to see that P_f should, intuitively, be homotopy equivalent to X . We shall show this shortly.

We then define a map known as the *mapping path fibration*, $p : P_f \rightarrow Y$, by $p(x, \alpha) = \alpha(1)$. The desired result follows immediately from the following lemma.

Lemma 2.7 Let $f : X \rightarrow Y$. Then for P_f and $p : P_f \rightarrow Y$ as defined above,

1. there exists a homotopy equivalence $h : P_f \rightarrow X$ such that the diagram

$$\begin{array}{ccc}
 P_f & \xrightarrow{h} & X \\
 & \searrow p & \swarrow f \\
 & & Y
 \end{array}$$

is homotopy commutative, and

2. the map $p : P_f \rightarrow Y$ is a fibration.

Proof:

Define $h(x, \alpha) = x$. First we show that h is a homotopy equivalence. To see this, define a map $g : X \rightarrow P_f$ by $g(x) = (x, c_{f(x)})$, where, recall, $c_{f(x)}$ is the constant loop at $f(x)$. Then $fg(x) = f(x, c_{f(x)}) = x$, so that $fg = \mathbb{1}_X$, and $gf(x, \alpha) = g(x) = (x, c_{f(x)})$, so $gf \simeq \mathbb{1}_{P_f}$ via the homotopy $G((x, \alpha), s) = (x, \alpha_s)$ where α_s is the path $\alpha_s(t) = \alpha(st)$. So h is indeed a homotopy equivalence.

To see that the diagram commutes, note that for all $(x, \alpha) \in P_f$, $fh(x, \alpha) = f(x)$, while, $p(x, \alpha) = \alpha(1)$. Define $H : P_f \times I \rightarrow Y$ by $H((x, \alpha), s) = \alpha_s(1)$. Then

$$H((x, \alpha), 0) = \alpha_0(1) = \alpha(0) = f(x) = fh(x, \alpha)$$

while

$$H((x, \alpha), 1) = \alpha_1(1) = \alpha(1) = p(x, \alpha)$$

so that $p \simeq fh$, as required.

It remains to show that p is a fibration. Let Z be a space, let $q : Z \rightarrow P_f$, and let $H : Z \times I \rightarrow Y$ be a homotopy such that the following diagram commutes ([Hat01, p. 407]).

$$\begin{array}{ccc} Z \times \{0\} & \xrightarrow{q} & P_f \\ \downarrow & & \downarrow p \\ Z \times I & \xrightarrow{H} & Y \end{array}$$

Here, we identify Z with $Z \times \{0\} \subset Z \times I$. For $z \in Z$, let us write $q(z) = (q_1(z), q_2(z)) \in P_f$. Then due to the commutativity of the diagram, we have that $H(z, 0) = pq(z) = p(q_1(z), q_2(z)) = q_2(z)[1]$. That is, the starting point of the path $H(z, *)$ is the ending point of $q_2(z)$. So then defining

$$\tilde{H}(z, s) = \begin{cases} (g_1(z), g_2(a)[(1+s)t]), & 0 \leq t \leq \frac{1}{1+s} \\ H(z, (1+s)t - 1), & \frac{1}{1+s} \leq t \leq 1 \end{cases}$$

the following diagram commutes.

$$\begin{array}{ccc} Z \times \{0\} & \xrightarrow{q} & P_f \\ \downarrow & \nearrow \tilde{H} & \downarrow p \\ Z \times I & \xrightarrow{H} & Y \end{array}$$

Thus $p : P_f \rightarrow Y$ has the homotopy lifting property, and so it is a fibration.

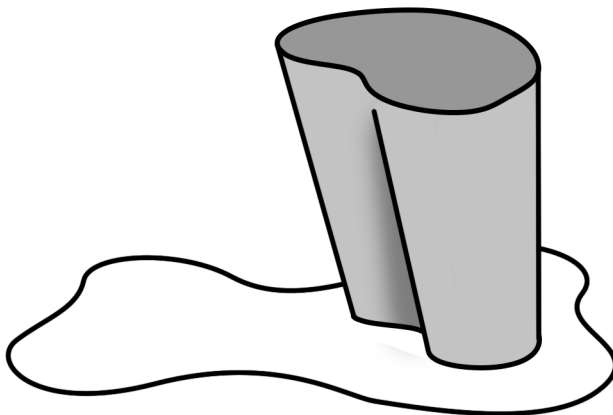
□

There is a similar result, dual to this. We omit the details of the proof, but offer the statement and explanation of the theorem here.

Theorem 2.8 Every map $f : X \rightarrow Y$ is, up to homotopy, a cofibration.

The method to this proof is, similarly, to construct the cofibration in question. To do this, we must first construct the object dual to that of the mapping path space, which we call the *mapping cylinder* of f . The mapping cylinder of f is the space $M_f = Y \cup_f (X \times I)$, where we identify X with $X \times \{1\}$. Visually, this the same as taking the image of f in Y and extruding

it upward along the unit interval to form a kind of top hat shape, like so.



It is easy to see that this space is homotopy equivalent to Y . Now, the inclusion $k : X \hookrightarrow M_f$ is a cofibration.

Now that we have these results, we note that for any diagram

$$\begin{array}{ccc} & X & \\ & \downarrow f & \\ Z & \xrightarrow{g} & Y \end{array}$$

finding the homotopy pullback becomes a matter of replacing one of the maps (in this example, f by an equivalent fibration, giving us the diagram

$$\begin{array}{ccc} & P_f & \\ & \downarrow p & \\ Z & \longrightarrow & Y \end{array}$$

and taking the pullback in the traditional sense. Similarly, given a diagram of the form

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow g & & \\ Z & & \end{array}$$

to find its homotopy pushout, we simply replace f (or g) by an equivalent cofibration, to obtain the following diagram,

$$\begin{array}{ccc} X & \xrightarrow{k} & M_f \\ \downarrow g & & \\ Z & & \end{array}$$

and take the traditional pushout.

As a final note, until this point we have been working in the category Top_2^* , with objects pointed pairs of CW complexes and morphisms continuous pointed maps between those pairs. However, knowing now that each map is homotopy equivalent to a fibration and a cofibration, we can switch over to a new category, which we will call hTop_2^* , where the objects are again topological pairs of CW complexes, but the morphisms this time are homotopy classes of continuous maps. Indeed from now on we will do just that.

Remark

From this point forward, when we say a diagram commutes we mean that it commutes up to homotopy. Similarly, when we reference a pullback or pushout, we mean the homotopy pullback or homotopy pushout, respectively. As always, for convenience, we will often confuse maps with their homotopy classes and shorten the pair $(X, *)$ to X when no confusion can occur.

2.4 Homotopy Fibres and Cofibres

Homotopy fibres and cofibres are specific, but important, example of homotopy pullbacks and pushouts, respectively. Given a map $f : X \rightarrow Y$, we define the homotopy fibre of f , F , to be the pullback of f along the inclusion of the basepoint $i : * \hookrightarrow Y$.

$$\begin{array}{ccc} F & \longrightarrow & X \\ \downarrow \lrcorner & & \downarrow f \\ * & \xrightarrow{i} & Y \end{array}$$

Dual to this, the homotopy cofibre of f is the pushout of f along the constant map $j : X \rightarrow *$.

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow j & & \downarrow \\ * & \longrightarrow & C \end{array}$$

Now, one would hope that, if we are to give these names to these objects, if f is a fibration, then the homotopy fibre has the homotopy type of the fibre of f (and that a similar statement holds true for a cofibration).

Proposition 2.9 Let $f : X \rightarrow Y$ be a fibration. Then the homotopy fibre F and the fibre \tilde{F} of f have the same homotopy type.

Proof:

The homotopy fibre of f can be written as follows.

$$\begin{aligned} F &= \{(x, \alpha, *) \in X \times Y^I \times * \mid \alpha(0) = f(x), \alpha(1) = *\} \\ &\cong \{(x, \alpha) \in X \times Y^I \mid \alpha(0) = f(x), \alpha(1) = *\} \end{aligned}$$

We require a homotopy equivalence between this space and the fibre $\tilde{F} = \{x \in X \mid f(x) = *\}$. Define $h : \tilde{F} \rightarrow F$ by $h(x) = (x, c_*, *)$. As f is a fibration, given $(x, \alpha, *) \in F$, there is a unique lifting of α to a path $\tilde{\alpha}_x$ in X starting at x . Let $z_x = \tilde{\alpha}_x(1)$. It is clear that $z_x \in \tilde{F}$, as $f(z_x) = f(\tilde{\alpha}_x(1)) = [f \circ \tilde{\alpha}_x](1) = \alpha(1) = *$, and so we define a second map $g : F \rightarrow \tilde{F}$ by $g((x, \alpha, *)) = z_x$. Then for all $x \in \tilde{F}$,

$$\begin{aligned} g \circ h(x) &= g(h(x)) = g(x, c_*, *) \\ &= z_x \\ &= x \end{aligned}$$

so that $g \circ h = \mathbb{1}_{\tilde{F}}$. Now define $H : F \times I \rightarrow F$ by $H((x, \alpha, *), t) = (p(t), \alpha_t, *)$ where p is a path in X from x to z_x , and α_t is the path in Y defined by $\alpha_t(s) = \alpha(t + s(1 - t))$. Then

$$H((x, \alpha, *), 0) = (p(0), \alpha_0, *) = (x, \alpha, *) = \mathbb{1}_F(x, \alpha, *)$$

and

$$H((x, \alpha, *), 1) = (p(1), \alpha_1, *) = (z_x, c_*, *) = h(z_x) = h \circ g(x, \alpha, *)$$

so that $h \circ g \simeq \mathbb{1}_F$, and $F \simeq \tilde{F}$. A similar result applies to the cofibre and homotopy cofibre of f .

□

It would be prudent, now to stop and compute a few basic examples of fibres and cofibres, to get a feel for the tools in use in this section. We have already seen an example of a homotopy fibre which will become useful later. This is the homotopy fibre of the inclusion of the basepoint, $i : * \hookrightarrow X$. The homotopy fibre being the pullback of the diagram

$$\begin{array}{ccc} & * & \\ & \downarrow i & \\ * & \xrightarrow{i} & X \end{array}$$

we first attempt to replace one of these arrows by an equivalent fibration. The mapping path space of i is

$$P_i = \{(*, \alpha) \in * \times X^I \mid \alpha(0) = i(*) = *\} \cong \{\alpha \in X^I \mid \alpha(0) = *\}$$

so we exchange the above diagram for the one below.

$$\begin{array}{ccc} & & P_i \\ & & \downarrow p \\ * & \xleftarrow{i} & X \end{array}$$

So the homotopy fibre we are seeking is simply the pullback

$$F = \{(*, (*, \alpha)) \in * \times P_i \mid p(*, \alpha) = i(*) = *\} = \{(*, (*, \alpha)) \mid \alpha(0) = *, \alpha(1) = *\} \cong \Omega X.$$

$$\begin{array}{ccc} \Omega X & \longrightarrow & * \\ \downarrow \lrcorner & & \downarrow i \\ * & \xleftarrow{i} & X \end{array}$$

A similarly useful example of a cofibre is the pushout of the diagram below.

$$\begin{array}{ccc} X & \xrightarrow{j} & * \\ \downarrow j & & \\ * & & \end{array}$$

To do this, we replace one copy of j by an equivalent cofibration, this time with the mapping cylinder,

$$M_j = * \cup_j (X \times I) = CX$$

where CX represents the cone of X , giving us a new diagram.

$$\begin{array}{ccc} X & \xrightarrow{j} & * \\ \downarrow & & \\ CX & & \end{array}$$

So the homotopy cofibre is then $C = CX \cup_j * = \Sigma X$.

$$\begin{array}{ccc} X & \xrightarrow{j} & * \\ \downarrow j & & \downarrow \\ * & \longrightarrow & \Sigma X \end{array}$$

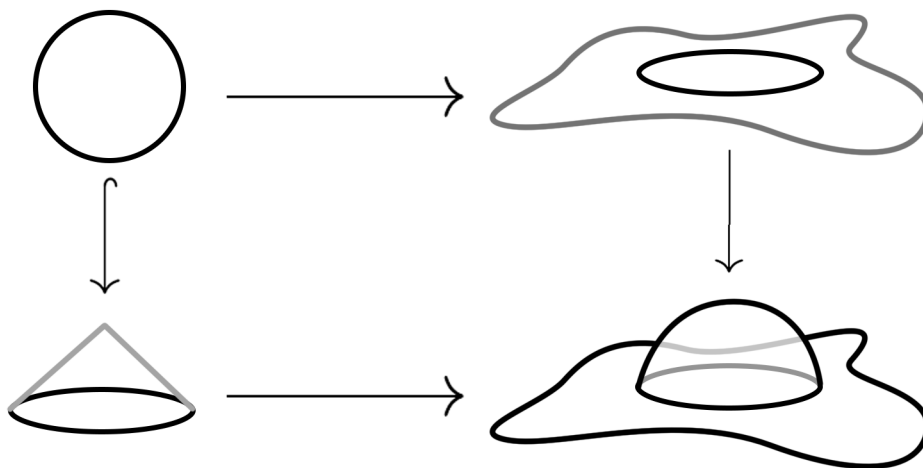
A last example of homotopy cofibres that is important to see for the work ahead is that of a map $f : S^n \rightarrow X$ for some space X . If we would like to consider the pushout of

$$\begin{array}{ccc} S^n & \xrightarrow{f} & X \\ \downarrow j & & \\ * & & \end{array}$$

we, as usual, replace one of maps with a cofibration to the mapping cylinder. While it is valid to choose either of the maps, we will, for this example, choose $j : S^n \rightarrow *$. As above, the mapping cylinder is $M_j = C(S^n) \cong D^{n+1}$, and so we obtain the following diagram.

$$\begin{array}{ccc} S^n & \xrightarrow{f} & X \\ \downarrow k & & \\ D^{n+1} & & \end{array}$$

Doing this we see, interestingly, that the homotopy cofibre of f is $X \cup_f D^{n+1}$. Notice how familiar this looks from our overview on CW complexes. This is exactly the space we get when we attach an $n + 1$ -cell to X via the attaching map $f : S^n \rightarrow X$. This is depicted for the case when $n = 1$ here.



Remark

Whenever we talk about the fibre or cofibre of a map, we will refer to that map's homotopy fibre or homotopy cofibre, respectively.

2.5 The Whitehead Bracket

Similar to the cup product in cohomology, which arises naturally from the geometry of spaces, there is a product in homotopy which also arises from the geometry of the space. That is, given $f \in \pi_m(X)$ and $g \in \pi_n(X)$, we can define a map $[f, g] \in \pi_{m+n-1}(X)$ as follows.

We note that, referring to Theorem 0.2, $S^m \times S^n$ is obtained by attaching an $m+n$ -cell to the wedge $S^m \vee S^n$. By the last example in the above section, this is the same as saying that $S^m \times S^n$ is the following pushout.

$$\begin{array}{ccc}
 S^{m+n-1} & \longrightarrow & S^m \vee S^n \\
 \downarrow & & \downarrow \\
 * & \longrightarrow & S^m \times S^n
 \end{array}$$

We then take the composition $S^{m+n-1} \rightarrow S^m \vee S^n \xrightarrow{f \vee g} X$, where the first map is the attaching map, and denote it $[f, g]$. This is known as the *Whitehead product* or *Whitehead bracket*. We will refer to this briefly in our section on the Hopf invariant.

SECTION 3

EXACT SEQUENCES

3.1 Exact Sequences of Groups

Definition 3.1 An *exact sequence* of groups is a sequence G_n , $n \in I \subseteq \mathbb{Z}$, and homomorphisms $\varphi_n : G_n \rightarrow G_{n+1}$ such that $\ker \varphi_{n+1} = \text{im } \varphi_n$.

There are some useful special cases of this. For example, a sequence of the form

$$0 \rightarrow A \xrightarrow{f} B$$

is exact if and only if f is injective. Similarly, a sequence of the form

$$A \xrightarrow{f} B \rightarrow 0$$

is evidently exact if and only if f is surjective.

Putting these together, it becomes apparent that a sequence of the form

$$0 \rightarrow A \xrightarrow{f} B \rightarrow 0$$

is exact if and only if f is an isomorphism.

Definition 3.2 A *short exact sequence* is an exact sequence of the form

$$0 \rightarrow A \xrightarrow{f} B \xrightarrow{g} C \rightarrow 0$$

In a short exact sequence, therefore, it must be the case that f is injective and g is surjective.

Now, in the case when there is a homomorphism $h : C \rightarrow B$ such that $gh = \mathbb{1}_C$, the sequence is called *split* (or is said to split). In this case, it follows via the splitting lemma ([Hat01]) that there is an isomorphism $B \cong A \oplus C$.

On the other hand, a *long exact sequence* is simply an exact sequence consisting of infinitely many groups and homomorphisms.

3.2 Exact Sequences of Spaces

Definition 3.3 Now, in hTop_2^* , we say a sequence

$$(X, A) \xrightarrow{f} (Y, B) \xrightarrow{g} (Z, C)$$

is *exact* if for each (W, D) , the sequence induced by the functor $[(W, D), -]$,

$$[(W, D), (X, A)] \xrightarrow{f \circ -} [(W, D), (Y, B)] \xrightarrow{g \circ -} [(W, D), (Z, C)]$$

is exact in the category of pointed sets, Set^* . Note, we often refer to these maps as f_* and g_* , respectively. Conversely, such a sequence is *coexact* if for each (W, D) the induced sequence

$$[(Z, C), (W, D)] \xrightarrow{- \circ g} [(Y, B), (W, D)] \xrightarrow{- \circ f} [(X, A), (W, D)]$$

is exact in Set^* . Similarly, we refer to these maps as f^* and g^* , respectively.

Lemma 3.4 The map $g \circ f : (X, A) \rightarrow (Z, C)$ is nullhomotopic if and only if g has an extension $h : (Y \cup_f CX, B \cup_{f|_A} CA) \rightarrow (Z, C)$.

Proof:

Suppose $g \circ f$ is nullhomotopic. Let $H : X \times I \rightarrow Z$ be the homotopy taking it to the constant map, c_* . Note, then that $H(X \vee I) = H(X \times \{0\} \cup * \times I) = *$, and so H clearly induces a map $h_X : X \times I / X \vee I = CX \rightarrow Z$. Then the map

$$h(w) = \begin{cases} g(w), & w \in Y \\ h_X(x, t), & w = (x, t) \in CX \end{cases}$$

is the desired map.

Conversely, suppose g extends to $h : (Y \cup_f CX, B \cup_{f|_A} CA) \rightarrow (Z, C)$. Define $H : X \times I \rightarrow Z$ to be the composition

$$X \times I \xrightarrow{q} X \times I / X \vee I = CX \hookrightarrow Y \cup_f CX \xrightarrow{h} Z$$

where q is the quotient map. Then $H(x, 0) = h(x, 0) = *$ for all x . Further, $H(x, 1) = h(x, 1) = h(f(x)) = g(f(x)) = g \circ f(x)$, whence $g \circ f$ is nullhomotopic as required. □

Theorem 3.5 Every cofibration sequence

$$(X, A) \xrightarrow{f} (Y, B) \xrightarrow{i} (Y \cup_f CX, B \cup_{f|_A} CA)$$

is coexact.

Proof:

Note that since i has an extension to $Y \cup_f CX$ (namely, the identity map $\mathbb{1}_{Y \cup_f CX}$), by Lemma 3.4, $i \circ f$ is nullhomotopic. Thus $f^* \circ i^* = (i \circ f)^* = *$ is the constant map, and so $\text{im } i^* \subseteq \ker f^*$.

Now suppose $h \in [(Y, B), (W, D)]$ is in the kernel, $\ker f^* = (f^*)^{-1}(*)$. Then $h \circ f$ is nullhomotopic. By Lemma 3.4, there is an extension h' of h to $(Y \cup_f CX, B \cup_{f|_A} CA)$, so that $i^*h' = h' \circ i = h$. Thus $h \in \text{im } i^*$, whence $\text{im } i^* = \ker f^*$, and the sequence is exact, as desired. \square

Dually, we have the following.

Theorem 3.6 Every fibration sequence

$$(P_f, P_{f|_A}) \rightarrow (X, A) \xrightarrow{f} (Y, B)$$

is exact.

3.3 The Barratt-Puppe Sequence

Consider a cofibration sequence $X \xrightarrow{f} Y \xrightarrow{i} Z = Y \cup_f CX$. We would like to find a space A and a map $\partial : Z \rightarrow A$ with which to append this sequence such that each consecutive triplet is a cofibration. Further, if possible, we would like to find a space B such that the same is true of the sequence $X \rightarrow Y \rightarrow Z \rightarrow A \rightarrow B$. Working in HTop^* , this is easy enough. Our first guess would simply be to take A to be the cofibre of i , $(Y \cup_f CX) \cup_i CY$, and do the same for B to get $B = ((Y \cup_f CX) \cup_i CY) \cup_{\partial} C(Y \cup_f CX)$. This gives us the desired result, however these expressions are monstrous and difficult to deal with.

Proposition 3.7 The spaces A and B above can be identified as ΣX and ΣY , respectively.

Note that $A = (Y \cup_f CX) \cup_i CY$ is simply $CY \cup_f CX$. Now, CY has the homotopy type of a point, and so we have that $CY \cup_f CX = CY \cup_f CX / CY$. Note that this also collapses $Y = Y \times \{0\}$ to a point, and thus collapses $X = X \times \{0\}$ along with it, and so indeed $A = \Sigma X$. By a similar argument, $B = C(Y \cup_f CX) \cup_i CY = CZ \cup_i CY$, and since CZ has the homotopy type of a point, $B = CZ \cup_i CY = CZ \cup_i CY / CZ = \Sigma Y$. Moreover the desired map $\Sigma X \rightarrow \Sigma Y$ is simply the suspension Σf .

Lemma 3.8 Suppose that $X \xrightarrow{f} Y \xrightarrow{i} Z = Y \cup_f CX$ is a cofibration sequence. Then so is $\Sigma X \xrightarrow{\Sigma f} \Sigma Y \xrightarrow{\Sigma i} \Sigma Z$.

Proof:

Note that

$$\Sigma Z = \Sigma(Y \cup_f CX) = \Sigma Y \cup_{\Sigma f} \Sigma CX = \Sigma Y \cup_{\Sigma f} C\Sigma X$$

and so it follows readily that

$$\Sigma X \xrightarrow{\Sigma f} \Sigma Y \xrightarrow{\Sigma i} \Sigma Z = \Sigma Y \cup_{\Sigma f} C\Sigma X$$

is a cofibration sequence. \square

Theorem 3.9 [Barratt-Puppe] For any $f : X \rightarrow Y$, there is a long coexact sequence of the form

$$X \xrightarrow{f} Y \rightarrow Z \rightarrow \Sigma X \xrightarrow{f} \Sigma Y \rightarrow \Sigma Z \rightarrow \Sigma^2 X \rightarrow \cdots \rightarrow \Sigma^n X \rightarrow \Sigma^n Y \rightarrow \Sigma^n Z \rightarrow \cdots$$

such that every consecutive triplet is a cofibration sequence.

The proof of this theorem follows immediately from Proposition 3.7 and Lemma 3.8.

□

Dual to this, we have the following:

Theorem 3.10 [Swi75] For any $f : E \rightarrow B$, there is a long exact sequence of the form

$$\cdots \rightarrow \Omega^n F \rightarrow \Omega^n E \rightarrow \Omega^n B \rightarrow \cdots \rightarrow \Omega^2 B \rightarrow \Omega F \rightarrow \Omega E \rightarrow \Omega B \rightarrow F \rightarrow E \xrightarrow{f} B$$

such that every consecutive triplet is a fibration sequence.

The above can be easily generalized to pairs of spaces so that we have a similar coexact sequence

$$(X, A) \xrightarrow{f} (Y, B) \rightarrow (Z, C) \rightarrow (\Sigma X, \Sigma A) \rightarrow \cdots \rightarrow (\Sigma^n X, \Sigma^n A) \rightarrow (\Sigma^n Y, \Sigma^n B) \rightarrow (\Sigma^n Z, \Sigma^n C) \rightarrow \cdots$$

and a similar exact sequence

$$\cdots \rightarrow (\Omega^n F, \Omega^n F') \rightarrow (\Omega^n E, \Omega^n E') \rightarrow (\Omega^n B, \Omega^n B') \rightarrow \cdots \rightarrow (F, F') \rightarrow (E, E') \rightarrow (B, B')$$

of pointed pairs of spaces.

3.4 Long Exact Sequence of a Fibration

Consider a fibration sequence $F \hookrightarrow E \xrightarrow{f} B$. By the above, we know that we can extend this to the long exact sequence

$$\cdots \Omega^n F \rightarrow \Omega^n E \rightarrow \Omega^n B \rightarrow \cdots \rightarrow \Omega^2 B \rightarrow \Omega F \rightarrow \Omega E \rightarrow \Omega B \rightarrow F \rightarrow E \xrightarrow{f} B$$

which in turn produces a long exact sequence

$$\cdots \rightarrow [W, \Omega^n F] \rightarrow [W, \Omega^n E] \rightarrow [W, \Omega^n B] \rightarrow \cdots \rightarrow [W, \Omega B] \rightarrow [W, F] \rightarrow [W, E] \rightarrow [W, B]$$

for each space W . But this sequence is exactly the sequence

$$\cdots \rightarrow [\Sigma^n W, F] \rightarrow [\Sigma^n W, E] \rightarrow [\Sigma^n W, B] \rightarrow \cdots \rightarrow [\Sigma W, B] \rightarrow [W, F] \rightarrow [W, E] \rightarrow [W, B].$$

Taking the particular case when $W = S^0$, we have

$$\cdots \rightarrow [S^n, F] \rightarrow [S^n, E] \rightarrow [S^n, B] \rightarrow \cdots \rightarrow [S^1, B] \rightarrow [S^0, F] \rightarrow [S^0, E] \rightarrow [S^0, B]$$

which gives us a long exact sequence of pointed sets

$$\cdots \rightarrow \pi_n(F) \rightarrow \pi_n(E) \rightarrow \pi_n(B) \rightarrow \cdots \rightarrow \pi_1(E) \rightarrow \pi_1(B) \rightarrow \pi_0(F) \rightarrow \pi_0(E) \rightarrow \pi_0(B).$$

3.5 Long Exact Sequence of Relative Homotopy Groups

Consider, now, the map $f : (S^0, S^0) \hookrightarrow (D^1, S^0)$. The cofibre of this map is

$$(D^1 \cup_f CS^0, S^0 \cup_f CS^0) = (D^1 \cup_f D^1, S^0 \cup_f D_1) = (S^1, D^1) = (S^1, *),$$

giving us the cofibration sequence

$$(S^0, S^0) \xrightarrow{f} (D^1, S^0) \rightarrow (S^1, *)$$

and hence, a long exact sequence

$$(S^0, S^0) \rightarrow (D^1, S^0) \rightarrow (S^1, *) \rightarrow \cdots \rightarrow (\Sigma^n S^0, \Sigma^n S^0) \rightarrow (\Sigma^n D^1, \Sigma^n S^0) \rightarrow (\Sigma^n S^1, \Sigma^n *) \rightarrow \cdots .$$

This is just the sequence

$$(S^0, S^0) \rightarrow (D^1, S^0) \rightarrow (S^1, *) \rightarrow \cdots \rightarrow (S^n, *) \rightarrow (S^n, S^n) \rightarrow (D^{n+1}, S^n) \rightarrow (S^{n+1}, *) \rightarrow \cdots .$$

Then for a pair of spaces, (X, A) , we get the following

$$\begin{aligned} & [(S^{n+1}, *), (X, A)] \rightarrow [(D^{n+1}, S^n), (X, A)] \rightarrow [(S^n, S^n), (X, A)] \rightarrow [(S^n, *), (X, A)] \rightarrow \cdots \\ & \cdots \rightarrow [(S^1, S^1), (X, A)] \rightarrow [(S^1, *), (X, A)] \rightarrow [(D^1, S^0), (X, A)] \rightarrow [(S^0, S^0), (X, A)] \end{aligned}$$

Now, note that $[(S^n, *), (X, A)] = \pi_n(X)$, $[(S^n, S^n), (X, A)] = \pi_n(A)$, and $[(D^{n+1}, S^n), (X, A)] = \pi_{n+1}(X, A)$. Thus, the above sequence reduces to

$$\cdots \rightarrow \pi_{n+1}(X, A) \rightarrow \pi_n(A) \rightarrow \pi_n(X) \rightarrow \pi_n(X, A) \rightarrow \cdots \rightarrow \pi_1(A) \rightarrow \pi_1(X) \rightarrow \pi_1(X, A) \rightarrow \pi_0(A)$$

giving us an exact sequence of pointed sets.

SECTION 4

THE HOPF INVARIANT AND $\pi_3(S^2)$

4.1 Complex Projective Space

With the homology and cohomology modules of every sphere known precisely, it seems natural to ask whether it is equally simple to compute the homotopy groups of all spheres. One might make a first guess that

$$\pi_k(S^n) = \begin{cases} \mathbb{Z}, & k = n \\ 0, & k \neq n \end{cases}$$

similar to the homology case, but this guess turns out to be naive. Consider the complex projective space $\mathbb{C}P^1$. This is the space $\mathbb{C}^2 \setminus \{0\} / \sim$ where \sim is the equivalence relation $z \sim w$ if and only if there exists a $\lambda \in \mathbb{C}$ such that $z = \lambda w$. Equivalently, since $S^3 \subseteq \mathbb{C}^2$, we can describe $\mathbb{C}P^1$ as the space S^3 / \sim where \sim is the relation defined by $z \sim w$ if and only if there is a $\lambda \in \{\lambda \in \mathbb{C} \mid |\lambda| = 1\} = S^1$ such that $z = \lambda w$. Moreover, we know that $\mathbb{C}P^1 \cong S^2$ ([GH81]), so the quotient map $S^3 \rightarrow S^3 / \sim = \mathbb{C}P^1 \cong S^2$ is a map which, for lack of a more rigorous term, collapses copies of circles on S^3 to a point in S^2 .

Moving on to a more rigorous description, define on $S^{2n+1} \subset \mathbb{C}^{n+1}$ an equivalence relation $z \sim w$ if and only if $\exists \lambda \in S^1$ such that $z = \lambda w$, and denote the resulting quotient map $S^{2n+1} \rightarrow S^{2n+1} / \sim$ by γ_n (that is, $\gamma_n(z) = [z]$, the equivalence class of z). Then $S^{2n+1} / \sim = \mathbb{C}P^n$ and we have the following.

$$S^{2n+1} \xrightarrow{\gamma_n} S^{2n+1} / \sim = \mathbb{C}P^n$$

Proposition 4.1 $\mathbb{C}P^n$ can be constructed inductively from the following pushout.

$$\begin{array}{ccc} S^{2n-1} & \xrightarrow{\gamma_{n-1}} & \mathbb{C}P^{n-1} \\ \downarrow & & \downarrow \\ D^{2n} & \longrightarrow & \mathbb{C}P^n \end{array}$$

That is, γ_{n-1} is the attaching map of the $2n$ -cell in $\mathbb{C}P^n$ (and so $\mathbb{C}P^n = \mathbb{C}P^{n-1} \cup_{\gamma_{n-1}} e^{2n}$).

To see this, we think of D^{2n} as contained in \mathbb{C}^n and note that the above diagram commutes when the map $D^{2n} \rightarrow \mathbb{C}P^n$ is the map $\omega = (z_1, \dots, z_n) \mapsto [\sqrt{1 - |\omega|^2}, z_1, \dots, z_n]$ and the map $\mathbb{C}P^{n-1} \hookrightarrow \mathbb{C}P^n$ is the inclusion $[z_1, \dots, z_n] \mapsto [0, z_1, \dots, z_n]$. Now suppose we have a space Y and maps $f : D^{2n} \rightarrow Y$ and $g : \mathbb{C}P^{n-1} \rightarrow Y$ such that the following diagram commutes.

$$\begin{array}{ccc}
 S^{2n-1} & \xrightarrow{\gamma_{n-1}} & \mathbb{C}P^{n-1} \\
 \downarrow & & \downarrow \\
 D^{2n} & \longrightarrow & \mathbb{C}P^n \\
 & \searrow f & \downarrow g \\
 & & Y
 \end{array}$$

We construct a map $k : \mathbb{C}P^n \rightarrow Y$ as follows: for $[z_0, z_1, \dots, z_n] \in \mathbb{C}P^n$, let $\lambda \in S^1$ such that $z_0 = \lambda|z_0|$. Then $[z_0, z_1, \dots, z_n] = [|z_0|, \bar{\lambda}z_1, \dots, \bar{\lambda}z_n]$. As this is a point in $\mathbb{C}P^n = S^{2n+1}/\sim$, it has unit norm, and so

$$\sqrt{|z_0|^2 + |\bar{\lambda}z_1|^2 + \dots + |\bar{\lambda}z_n|^2} = 1 \quad \Rightarrow \quad |z_0| = \sqrt{1 - (|\bar{\lambda}z_1|^2 + \dots + |\bar{\lambda}z_n|^2)} = \sqrt{1 - |\omega|^2},$$

where $\omega = (\bar{\lambda}z_1, \dots, \bar{\lambda}z_n)$. Then the desired map k is defined as $k([z_0, z_1, \dots, z_n]) = f(\bar{\lambda}z_1, \dots, \bar{\lambda}z_n)$. In this case, if q is the map $D^{2n} \rightarrow \mathbb{C}P^n$ and p the map $\mathbb{C}P^{n-1} \rightarrow \mathbb{C}P^n$, then

$$\begin{aligned}
 kq(\omega) &= kq(z_1, \dots, z_n) \\
 &= k([\sqrt{1 - |\omega|^2}, z_1, \dots, z_n]) \\
 &= k([1 \cdot \sqrt{1 - |\omega|^2}, z_1, \dots, z_n]) \\
 &= k([\sqrt{1 - |\omega|^2}, \bar{1}z_1, \dots, \bar{1}z_n]) \\
 &= k([\sqrt{1 - |\omega|^2}, z_1, \dots, z_n]) \\
 &= f(z_1, \dots, z_n)
 \end{aligned}$$

and, as the above diagram commutes,

$$\begin{aligned}
 kp([z_1, \dots, z_n]) &= k([0, z_1, \dots, z_n]) \\
 &= k([\sqrt{1 - 1^2}, z_1, \dots, z_n]) \\
 &= f(z_1, \dots, z_n) \\
 &= g([z_1, \dots, z_n])
 \end{aligned}$$

so $kq = f$ and $kp = g$, thus the following diagram commutes.

$$\begin{array}{ccc}
 S^{2n-1} & \xrightarrow{\gamma_{n-1}} & \mathbb{C}P^{n-1} \\
 \downarrow & & \downarrow \\
 D^{2n} & \longrightarrow & \mathbb{C}P^n \\
 & \searrow f & \swarrow g \\
 & & Y
 \end{array}$$

(Note: A dashed arrow labeled k points from $\mathbb{C}P^n$ to Y .)

So we see that $\mathbb{C}P^n$ possesses the universal property for $S^{2n-1} \hookrightarrow D^{2n}$ and γ_{n-1} , identifying $\mathbb{C}P^n$ as the pushout of the diagram

$$\begin{array}{ccc}
 S^{2n-1} & \xrightarrow{\gamma_{n-1}} & \mathbb{C}P^{n-1} \\
 \downarrow & & \\
 D^{2n} & &
 \end{array}$$

as required. □

Consider now the map $\gamma_1 : S^3 \rightarrow S^2$. It remains to show that this map is not homotopically trivial. To do this, we will use a few results from [Bre93]. Note that, if it were, then we would have $\mathbb{C}P^2 \simeq S^2 \vee S^4$. To see that this is not true, we look at $H^*(\mathbb{C}P^2)$ and $H^*(S^2 \vee S^4)$ respectively. While it is true that

$$H^m(\mathbb{C}P^2) = H^m(S^2 \vee S^4) = \begin{cases} \mathbb{Z}, & m = 0, 2, 4 \\ 0, & \text{else} \end{cases}$$

we can distinguish between the two by means of their respective cup products. If α is a generator of $H^2(\mathbb{C}P^2)$ and β is a generator of $H^2(S^2 \vee S^4)$, α^2 is a generator of $H^4(\mathbb{C}P^2)$ while $\beta^2 = 0$. We can generalize this notion even further.

Definition 4.2 Let $f : S^{2n-1} \rightarrow S^n$. Then f induces a CW complex of the form $S^n \cup_f e^{2n}$. This will have cohomology modules

$$H^m(S^n \cup_f e^{2n}) = \begin{cases} \mathbb{Z}, & m = 0, n, 2n \\ 0, & \text{else} \end{cases}$$

and so let α be a generator of $H^n(S^n \cup_f e^{2n})$ and let β be a generator of $H^{2n}(S^n \cup_f e^{2n})$. Since $\alpha^2 \in H^{2n}(S^n \cup_f e^{2n})$, let $k \in \mathbb{Z}$ such that $\alpha^2 = k\beta$. We define $|k|$ to be the *Hopf invariant* of f and denote it by $h(f)$.

4.2 Hopf Invariants

While the Hopf invariant had a relatively simple beginning, over time it was expanded and generalized — from maps $S^{2n-1} \rightarrow S^n$ to maps $S^r \rightarrow S^n$ and then to maps $S^r \rightarrow X$ for any space X .

We have already seen Hopf's original definition for a map $\alpha : S^{2n-1} \rightarrow S^n$ in terms of cohomology. We would now like to take the next step in generalizing the invariant.

We show in section 7.1 the existence of a homotopy equivalence

$$\Omega(X \vee Y) \simeq \Omega X \times \Omega Y \times \Omega(\Omega X * \Omega Y)$$

which induces an isomorphism

$$\pi_r(X \vee Y) = \pi_r(X) \oplus \pi_r(Y) \oplus \pi_r(\Omega X * \Omega Y)$$

for $r > 1$. More specifically, taking $X = Y = S^n$, we have a canonical decomposition

$$\pi_r(S^n \vee S^n) = \pi_r(S^n) \oplus \pi_r(S^n) \oplus \pi_r(\Omega S^n * \Omega S^n)$$

which brings us to our generalized definition of the Hopf invariant, and connects it to the original.

Definition 4.3 Let $\alpha : S^r \rightarrow S^n$. The Hopf invariant of α , denoted $\mathcal{H}(\alpha)$, is the image of α under the composition

$$\pi_r(S^n) \rightarrow \pi_r(S^n \vee S^n) \rightarrow \pi_r(\Omega S^n * \Omega S^n)$$

where the left map is the map p_* induced by the pinch map, while the right map is the canonical projection onto the relevant summand in the above decomposition of $\pi_r(S^n \vee S^n)$.

Now, we will require the following lemma.

Lemma 4.4 A pushout diagram of the form

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \downarrow & & \downarrow \\ * & \longrightarrow & C \end{array}$$

induces a homotopy commutative diagram

$$\begin{array}{ccc}
 A & \xrightarrow{f} & B \\
 \downarrow & & \downarrow \\
 * & \longrightarrow & C \\
 \downarrow & & \downarrow \\
 \Sigma A & \longrightarrow & \Sigma A \\
 \downarrow & & \downarrow \\
 \Sigma A & \xrightarrow{\Sigma f} & \Sigma B
 \end{array}$$

Proof:

Taking the cofibres of the vertical maps of the original diagram, we get the following.

$$\begin{array}{ccc}
 A & \xrightarrow{f} & B \\
 \downarrow & & \downarrow \\
 * & \longrightarrow & C \\
 \downarrow & & \downarrow \\
 \Sigma A & \longrightarrow & D
 \end{array}$$

Now, since D is the result of a pushout

$$\begin{array}{ccc}
 B & \longrightarrow & * \\
 \downarrow & & \downarrow \\
 C & \longrightarrow & D
 \end{array}$$

we can append it to the first pushout diagram to obtain a new diagram.

$$\begin{array}{ccccc}
 A & \xrightarrow{f} & B & \longrightarrow & * \\
 \downarrow & & \downarrow & & \downarrow \\
 * & \longrightarrow & C & \longrightarrow & D
 \end{array}$$

So by the prism lemma ([Doe98]),

$$\begin{array}{ccc}
 A & \longrightarrow & * \\
 \downarrow & & \downarrow \\
 * & \longrightarrow & D
 \end{array}$$

is also a pushout, whence $D = \Sigma A$. Now, we extend this diagram further, using the proceeding arrows of the relevant coexact Puppe sequences, giving us the commutative diagram

$$\begin{array}{ccc}
 A & \xrightarrow{f} & B \\
 \downarrow & & \downarrow \\
 * & \longrightarrow & C \\
 \downarrow & & \downarrow \\
 \Sigma A & \longrightarrow & \Sigma A \\
 \downarrow & & \downarrow \\
 \Sigma A & \xrightarrow{\Sigma f} & \Sigma B
 \end{array}$$

as required. □

Theorem 4.5 Let $\alpha : S^{2n-1} \rightarrow S^n$, and let ι_{2n-1} be a generator of $\pi_{2n-1}(S^{2n-1}) = \mathbb{Z}$. Then $\mathcal{H}(\alpha) = \pm h(\alpha)\iota_{2n-1}$.

Proof:

First we must identify $\pi_{2n-1}(\Omega S^n * \Omega S^n)$. Let us recall one small fact that will be of use. Note that for a given space X , $\Sigma X \cong X \wedge S^1$. Moreover, the smash product is commutative and associative, up to homeomorphism. Lastly, the topological join is related to suspension by $X * Y \cong \Sigma(X \wedge Y)$. From this, we obtain the following.

$$\begin{aligned}
 X * Y &\cong \Sigma(X \wedge Y) \cong (X \wedge Y) \wedge S^1 \\
 &\cong X \wedge (Y \wedge S^1) \cong X \wedge \Sigma Y \\
 &\cong (X \wedge S^1) \wedge Y \cong \Sigma X \wedge Y
 \end{aligned}$$

Next, from [Jam55] we get a decomposition of ΩS^n as a CW complex with one cell in each dimension which is a multiple of $n - 1$ so that

$$\Omega S^n \simeq S^{n-1} \cup_{\varphi_2} e^{2n-2} \cup_{\varphi_3} e^{3n-3} \cup \dots$$

where φ_i is the appropriate attaching map of the ni -cell, $i > 0$. Further, James provides us with a homotopy equivalence

$$\Sigma \Omega S^n \simeq \bigvee_{k=1}^{\infty} S^{k(n-1)+1}$$

which allows us to identify $\pi_{2n-1}(\Omega S^n * \Omega S^n)$ as follows.

$$\begin{aligned}
\pi_{2n-1}(\Omega S^n * \Omega S^n) &= \pi_{2n-1}(\Sigma(\Omega S^3 \wedge \Omega S^3)) \\
&= \pi_{2n-1}(\Omega S^3 \wedge \Sigma(\Omega S^3)) \\
&= \pi_{2n-1}(\Omega S^3 \wedge (S^n \vee S^{2n-1} \vee S^{3n-2} \vee \dots)) \\
&= \pi_{2n-1}(\Omega S^3 \wedge \Sigma(S^{n-1} \vee S^{2n-2} \vee S^{3n-3} \vee \dots)) \\
&= \pi_{2n-1}(\Sigma \Omega S^3 \wedge (S^{n-1} \vee S^{2n-2} \vee \dots)) \\
&= \pi_{2n-1}(\Sigma(S^{n-1} \vee S^{2n-2} \vee \dots) \wedge (S^{n-1} \vee S^{2n-2} \vee \dots)) \\
&= \pi_{2n-1}(\Sigma((S^{n-1} \vee S^{2n-2} \vee \dots) \wedge (S^{n-1} \vee S^{2n-2} \vee \dots))) \\
&= \pi_{2n-1}(\Sigma(S^{2n-2} \vee S^{3n-3} \vee S^{3n-3} \vee S^{4n-4} \vee \dots)) \\
&= \pi_{2n-1}(S^{2n-1} \vee S^{3n-2} \vee S^{3n-2} \vee S^{4n-3} \vee \dots) \\
&= \pi_{2n-1}(S^{2n-1})
\end{aligned}$$

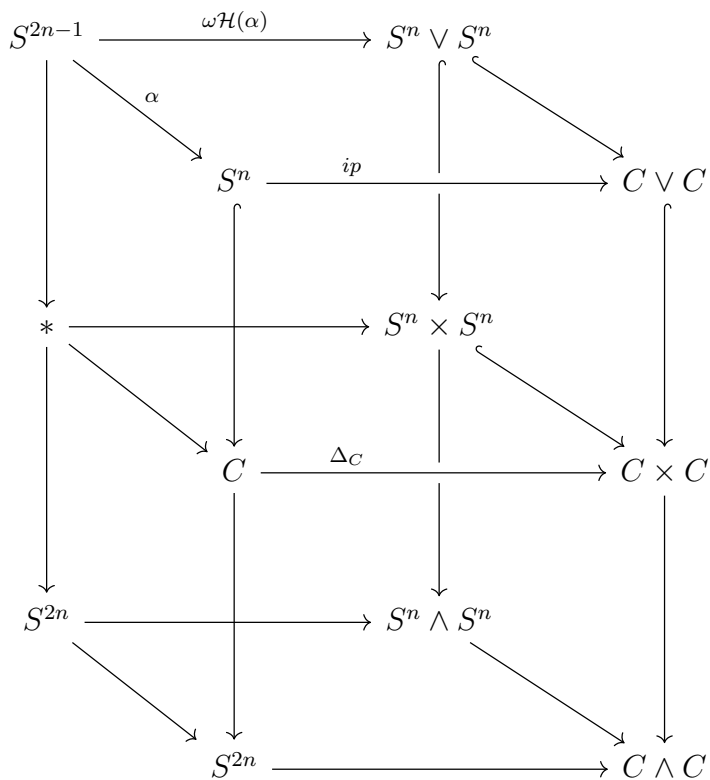
As a point of interest, this same argument shows that $\pi_r(\Omega S^n * \Omega S^n) = \pi_r(S^{2n-1})$ for all $r \leq 3n - 3$.

Now that we know $\mathcal{H}(\alpha) \in \pi_{2n-1}(S^{2n-1})$, let $C = S^n \cup_\alpha e^{2n}$ be the cofibre of $S^{2n-1} \xrightarrow{\alpha} S^n$, and consider the following diagram.

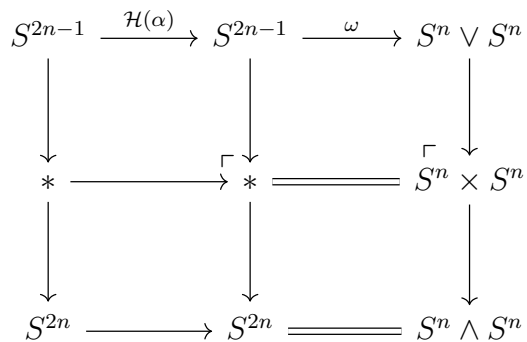
$$\begin{array}{ccccc}
S^{2n-1} & \xrightarrow{\omega \mathcal{H}(\alpha)} & S^n \vee S^n & & \\
\downarrow & \searrow \alpha & \downarrow & \searrow & \\
& & S^n & \xrightarrow{ip} & C \vee C \\
& & \downarrow & & \downarrow \\
& & * & \xrightarrow{\quad} & S^n \times S^n \\
& & \downarrow & \searrow & \downarrow \\
& & C & \xrightarrow{\Delta_C} & C \times C
\end{array}$$

Here, ω is the attaching map $S^{2n-1} \rightarrow S^n \vee S^n$, discussed earlier in the context of the Whitehead bracket, p is the pinch map, and $i : S^n \vee S^n \rightarrow C \vee C$ the inclusion. It is clear for all but the top face that the faces are commutative. To see the commutativity of the top face, recall that $\pi_{2n-1}(S^n \vee S^n) = \pi_{2n-1}(S^n) \oplus \pi_{2n-1}(S^n) \oplus \pi_{2n-1}(S^{2n-1})$. In particular, $\omega \mathcal{H}(\alpha)$ can be decomposed as $(\iota, \iota, \mathcal{H}(\alpha))$, where ι is the fundamental class of $\pi_n(S^n)$, while $ip\alpha$ is represented by $(\alpha, \alpha, \mathcal{H}(\alpha))$, so the difference between $\omega \mathcal{H}(\alpha)$ and $ip\alpha$ is exactly $(\alpha, \alpha, 0)$. However, this element is trivial in $\pi_{2n-1}(C \vee C)$ as α is homotopically trivial in C by its construction, and so the top face indeed commutes.

Now we extend to include the cofibres of the vertical maps, forming the following lattice.



To aid with our calculations, we will amend this diagram by factoring each of the horizontal arrows.



Note that each of the top squares remains a pushout, and so the result discussed above holds true. We apply it to our diagram to obtain

$$\begin{array}{ccccc}
S^{2n-1} & \xrightarrow{\mathcal{H}(\alpha)} & S^{2n-1} & \xrightarrow{\omega} & S^n \vee S^n \\
\downarrow & & \downarrow & & \downarrow \\
* & \xlongequal{\quad} & * & \xrightarrow{\quad} & S^n \times S^n \\
\downarrow & & \downarrow & & \downarrow \\
S^{2n} & \xrightarrow{\quad} & S^{2n} & \xlongequal{\quad} & S^n \wedge S^n \\
\parallel & & \parallel & & \downarrow \\
S^{2n} & \xrightarrow{\Sigma\mathcal{H}(\alpha)} & S^{2n} & \xrightarrow{\Sigma\omega} & S^{n+1} \vee S^{n+1}
\end{array}$$

which can in turn be used in tandem with our original diagrams, as well as the connecting map $C \rightarrow S^{2n}$ of the Puppe sequence of $S^{2n-1} \rightarrow S^n \rightarrow C$ and the reduced diagonal $\bar{\Delta}_C : C \rightarrow C \wedge C$, to get the following commutative diagram.

$$\begin{array}{ccc}
C & \xrightarrow{\bar{\Delta}_C} & C \wedge C \\
\downarrow & & \uparrow \\
S^{2n} & \xrightarrow{\Sigma\mathcal{H}(\alpha)} & S^n \wedge S^n
\end{array}$$

Now we apply the cohomology functor to this diagram, and obtain the following.

$$\begin{array}{ccc}
H^*C & \xleftarrow{\bar{\Delta}_C^*} & H^*(C \wedge C) \\
\uparrow & & \downarrow \\
H^*S^{2n} & \xleftarrow{\Sigma\mathcal{H}(\alpha)_*} & H^*(S^n \wedge S^n)
\end{array}$$

Reversing the horizontal direction and localizing to the relevant degrees for ease of reading, we obtain

$$\begin{array}{ccccc}
\bigoplus_{i+j=2n} \tilde{H}^i C \otimes \tilde{H}^j C = H^n C \otimes H^n C & \longrightarrow & H^{2n}(C \wedge C) & \longrightarrow & H^{2n} C \\
& & \downarrow & & \uparrow \\
& & H^{2n}(S^n \wedge S^n) & \xrightarrow{\Sigma\mathcal{H}(\alpha)_*} & H^{2n} S^{2n}
\end{array}$$

where we append the Kunneth map $H^n C \otimes H^n C \xrightarrow{\cong} H^{2n}(C \wedge C)$ ([Cor95, p. 170]) to the upper left corner, so that the composition of maps along the top is the cup product. Lastly, we realize

that $H^{2n}(S^{2n}) = \pi_{2n}(S^{2n}) = \pi_{2n}(\Sigma S^{2n-1}) = \pi_{2n-1}(S^{2n-1})$, and add this.

$$\begin{array}{ccccc}
& & \frown & & \\
& & \text{---} & & \\
H^n C \otimes H^n C & \xrightarrow{\cong} & H^{2n}(C \wedge C) & \xrightarrow{\quad} & H^{2n} C \\
& & \parallel & & \parallel \\
& & H^{2n}(S^n \wedge S^n) & \xrightarrow{\Sigma \mathcal{H}(\alpha)_*} & H^{2n}(S^{2n}) \\
& & \parallel & & \parallel \\
& & \pi_{2n}(S^n \wedge S^n) & \xrightarrow{\Sigma \mathcal{H}(\alpha)_*} & \pi_{2n}(S^{2n}) \\
& & \parallel & & \parallel \\
& & \pi_{2n-1}(S^{2n-1}) & \xrightarrow{\mathcal{H}(\alpha)_*} & \pi_{2n-1}(S^{2n-1}) \\
& & & & \parallel \\
& & & & \Sigma_*
\end{array}$$

From here, we simply follow the path of the generator, $e_n \otimes e_n$, of $H^n C \otimes H^n C$ through the diagram to $H^{2n}(C)$. Following the path along the top, we see that $e_n \otimes e_n \mapsto e_n \smile e_n = \pm h\alpha e_{2n}$, where e_{2n} is a generator of $H^{2n} C$. On the other hand, if we follow the path through the bottom map, noting that all maps between $H^n C \otimes H^n C$ and $\pi_{2n-1}(S^{2n-1})$ are isomorphisms, we see that $e_n \otimes e_n$ is mapped to a generator of $\pi_{2n-1}(S^{2n-1})$. Without loss of generality, assume it is mapped to ι_{2n-1} . Then, as $\pi_{2n-1}(S^{2n-1}) = \mathbb{Z}$, $\mathcal{H}(\alpha)_*$ takes ι_{2n-1} to $k\iota_{2n-1}$. Following back up the right side of the diagram, ι_{2n-1} is again mapped to a generator, so that $k\iota_{2n-1} \mapsto ke_{2n}$. As the diagram is commutative, we have that $k = h(\alpha)$, whence $\mathcal{H}(\alpha) = \pm h(\alpha)\iota_{2n-1}$.

4.3 Homotopy Groups of S^2

Since (S^3, S^2, η, S^1) , where η denotes the Hopf map, is a fibre bundle, ([Swi75, p. 87]), the Hopf map is a fibration by [Whi78, p. 33], and the related fibration sequence

$$S^1 \hookrightarrow S^3 \xrightarrow{\eta} S^2$$

induces a long exact sequence in homotopy.

$$\cdots \rightarrow \pi_n(S^1) \rightarrow \pi_n(S^3) \rightarrow \pi_n(S^2) \rightarrow \pi_{n-1}(S^1) \rightarrow \cdots \rightarrow \pi_2(S^2) \rightarrow \pi_1(S^1) \rightarrow \pi_1(S^3) \rightarrow \pi_1(S^2)$$

However, $\pi_n(S^1) = 0$ whenever $n \neq 1$. Thus, for $n > 2$, the sequence

$$\pi_n(S^1) \rightarrow \pi_n(S^3) \rightarrow \pi_n(S^2) \rightarrow \pi_{n-1}(S^1)$$

reduces to

$$0 \rightarrow \pi_n(S^3) \rightarrow \pi_n(S^2) \rightarrow 0$$

from which we get an isomorphism $\pi_n(S^3) \cong \pi_n(S^2)$ for $n \geq 3$.

In particular, $\pi_3(S^2) = \pi_3(S^3) = \mathbb{Z}$. Further, as we have already seen, its Hopf invariant is a generator of $\pi_3(\Omega S^2 * \Omega S^2) = \pi_3(S^3)$, and so η itself is a generator of $\pi_3(S^2)$.

SECTION 5

THE LUSTERNIK-SCHNIRELMANN CATEGORY

In 1934, Lazar Lusternik and Lev Schnirelmann introduced the notion of the category of a manifold, M . This definition originally used closed coverings of the manifold, defining the category, $\text{cat } M$ of M to be the smallest integer k such that there was a closed cover consisting of k sets, each of which was contractible in M . This, of course, extends easily to arbitrary topological spaces.

In 1941, Ralph Fox proposed a change to this definition, using open contractible sets rather than closed, which we present here.

Definition 5.1 The *Lusternik-Schnirelmann category* of a space X (henceforth referred to as the LS category of X , or simply $\text{cat } X$) is the smallest integer n such that there exist contractible (in X) open sets U_1, \dots, U_{n+1} such that $\mathcal{U} = \{U_1, \dots, U_{n+1}\}$ is an open cover of X . Such an open cover is called a *categorical covering* of X .

This definition does not lend itself well to many of the computations in our main line of study. However it is a convenient one to use for some smaller results along the way. As a first example, it follows immediately from this definition that $\text{cat } X = 0$ if and only if X is contractible. For a more interesting example, we have the following.

Proposition 5.2 For any $n \geq 1$, $\text{cat } S^n = 1$.

Proof:

Define subsets U_1 and U_2 of S^n by $U_1 = S^n \setminus \{(1, 0, \dots, 0)\}$ and $U_2 = S^n \setminus \{(-1, 0, \dots, 0)\}$. Then it is clear that both U_1 and U_2 are homeomorphic to the open n -disk, and thus contractible, and that $U_1 \cup U_2 = S^n$. Thus $\mathcal{U} = \{U_1, U_2\}$ is a categorical open covering of S^n , whence $\text{cat } S^n \leq 1$. As S^n is not contractible, we are left with $\text{cat } X = 1$.

More recently, different notions of category were introduced by George Whitehead and Tudor Ganea, respectively. These definitions of the category of a space can be more easily utilized for our purposes, and so we introduce them here.

5.1 The Whitehead Category

Definition 5.3 Let X be a topological space, and n be a positive integer. Define a space $X^{[n]}$ to be the subspace of X^n consisting of the points at least one of whose coordinates is the base point, $*$, of X . That is,

$$X^{[n]} = \{(x_1, \dots, x_n) \in X^n \mid x_i = * \text{ for at least one } i\}$$

$X^{[n]}$ is called the n -fold fat wedge.

Definition 5.4 Now consider topological pairs (and for our purposes specifically, CW pairs) (X, A) and (Y, B) . A map $f : (X, A) \rightarrow (Y, B)$ is said to be *compressible* if it is homotopic relative to A to a map of X into B . Such a homotopy is called a *compression*.

With these definitions, we are ready to bring to light Whitehead's definition of the LS category.

Definition 5.5 We say that the category, $\text{cat } X$, of a space X is less than or equal to n (or that X has category less than or equal to n) if the $n + 1$ -fold diagonal map $\Delta_{n+1} : (X, *) \rightarrow (X^{n+1}, X^{[n+1]})$ is compressible. That is, there exists a map $h : X \rightarrow X^{[n+1]}$ such that the following diagram commutes (up to homotopy).

$$\begin{array}{ccc} & & X^{[n+1]} \\ & \nearrow h & \downarrow \\ X & \xrightarrow{\Delta_{n+1}} & X^{n+1} \end{array}$$

Of course we would now like to say that the category of X is equal to the smallest n such that $\text{cat } X \leq n$. However, for this to make sense would require that if the category of X is less than or equal to n , then it is also less than or equal to $n + 1$ – a statement which is not immediately obvious.

Lemma 5.6 If $\text{cat } X \leq n$, then $\text{cat } X \leq n + 1$ as well.

Suppose that $\text{cat } X \leq n$. Then there is a homotopy $H : X \times I \rightarrow X^{n+1}$ taking the $n + 1$ -fold diagonal map Δ_{n+1} to a map $h : X \rightarrow X^{[n+1]}$. Now we consider the $n + 2$ -fold diagonal map. As $\Delta_{n+2} = (\Delta_{n+1}, \mathbb{1}_X)$, define a homotopy $G : X \times I \rightarrow X^{n+2}$ by $G(x, t) = (H(x, t), x) = (H(x, t), \mathbb{1}_X(x))$. Then for all t , $G(*, t) = (H(*, t), *) = (\Delta_{n+1}(*), *) = \Delta_{n+2}(*)$, while for all $x \in X$, $G(x, 0) = (H(x, 0), x) = (\Delta_{n+1}(x), x) = \Delta_{n+2}(x)$, and $G(x, 1) = (H(x, 1), x) = (h(x), x) \in X^{[n+2]}$. Thus G is a compression of Δ_{n+2} into $X^{[n+2]}$, whence $\text{cat } X \leq n + 1$, as desired.

□

So, finally, we may define the *LS*-category of X to be the least nonnegative integer n such that $\text{cat } X \leq n$. If no such n exists then we shall say that $\text{cat } X = \infty$. Next we introduce the definition of category constructed by Ganea.

5.2 Ganea's Fibre-Cofibre Construction

Take a space X , and consider the homotopy fibre of the inclusion of the basepoint $* \hookrightarrow X$. As seen in section 2.4, this is defined by the diagram

$$\begin{array}{ccc} P & \longrightarrow & * \\ \downarrow \lrcorner & & \downarrow \\ * & \longrightarrow & X \end{array}$$

which, as we have seen, gives $P = \Omega X$, the loop space of X . We then take the homotopy cofibre of the induced map $\Omega X \rightarrow *$. Again, we have seen in section 2.4 that this is described by the diagram

$$\begin{array}{ccc} \Omega X & \longrightarrow & * \\ \downarrow & & \downarrow \\ * & \longrightarrow & P \end{array} \quad \lrcorner$$

giving $P = \Sigma \Omega X$. We denote this $G^1 X$ and refer to it as the *1st Ganea space of X* .

If we then repeat this process, we can look at the homotopy fibre of the projection $p_1 : G^1 X \rightarrow X$, which we will call the *1st homotopy fibre of X* and denote it as $F^1 X$. Thus we have the pullback diagram

$$\begin{array}{ccc} F^1 X & \longrightarrow & G^1 X \\ \downarrow \lrcorner & & \downarrow p_1 \\ * & \longrightarrow & X \end{array}$$

and can then construct the *2nd Ganea space* in the same way as the first.

$$\begin{array}{ccc} F^1 X & \longrightarrow & G^1 X \\ \downarrow & & \downarrow \\ * & \longrightarrow & G^2 X \end{array} \quad \lrcorner$$

Thus, more generally, we define $G^0 X = *$, and define the *m th Ganea space of X* iteratively as follows, using what is known as the *Ganea fibre-cofibre construction*.

$$\begin{array}{ccc} F^m X & \longrightarrow & G^m X \\ \downarrow \lrcorner & & \downarrow p_m \\ * & \longrightarrow & X \end{array} \quad \begin{array}{ccc} F^m X & \longrightarrow & G^m X \\ \downarrow & & \downarrow \\ * & \longrightarrow & G^{m+1} X \end{array} \quad \lrcorner$$

Putting all of the above together, we obtain the following diagram.

$$\begin{array}{ccccccc}
 \Omega X & & F^1 X & & F^2 X & & \dots & & F^m X & & \dots \\
 \downarrow & & \downarrow & & \downarrow & & & & \downarrow & & \\
 * & \longrightarrow & \Sigma \Omega X & \longrightarrow & G^2 X & \longrightarrow & \dots & \longrightarrow & G^m X & \longrightarrow & \dots \\
 \downarrow & & \nearrow^{p_1} & \nearrow^{p_2} & \nearrow^{p_m} & & & & & & \\
 X & & & & & & & & & &
 \end{array}$$

Finally, we say that $\text{cat } X \leq m$ if p_m admits a section (that is, there exists a $\sigma : X \rightarrow G^m X$ such that $p_m \circ \sigma \simeq \mathbb{1}_X$). We further define $\text{cat } X$ to be the smallest such integer. If no such integer exists, we say that $\text{cat } X = \infty$.

Remark 5.7 It is shown in [Por66] that $F^m X$ is the $(m+1)$ -fold join of the loop space of X . That is to say $F^1 X = \Omega X * \Omega X$, $F^2 X = \Omega X * \Omega X * \Omega X$, and so on in this fashion. We will denote the map $F^m X \rightarrow G^m X$ induced by the pullback in the definition of $F^m X$ by h_m . We will use this in later sections.

5.3 Bringing It All Together

These three notions of category are useful in their own way for our purposes, but we must first show that they are equivalent in order to use them properly. To that end, we establish the following.

Lemma 5.8 If a path connected, normal space X has a categorical covering $\mathcal{U} = \{U_1, \dots, U_{n+1}\}$, then it has a categorical covering $\mathcal{V} = \{V_1, \dots, V_{n+1}\}$ such that each V_i is contractible in X to the basepoint $*$ of X . Such a covering is called a *based categorical covering*.

Proof:

Let $\{U_1, \dots, U_{n+1}\}$ be any categorical covering of X . By normality, we can find for each i an open set W_j such that $W_j \subset \overline{W}_j \subset U_j$. Further, we can find a neighbourhood N of $*$ which is contractible to $*$ relative to $*$. Relabeling if necessary so that $*$ $\in U_i, i = 1, \dots, k$ and $*$ $\notin U_i, i = k+1, \dots, n+1$, we use N to construct a new neighbourhood of $*$, namely

$$\tilde{N} = N \cap U_1 \cap \dots \cap U_k \cap (X \setminus \overline{W}_{k+1}) \cap \dots \cap (X \setminus \overline{W}_{n+1})$$

so that for all $j > k$, $\tilde{N} \cap W_j = \emptyset$. Again by normality, we choose yet another neighbourhood M of $*$ with $M \subset \overline{M} \subset \tilde{N} \subset U_j$ for all $j \leq k$. We now define $\mathcal{V} = \{V_i\}_{i=1}^{n+1}$ by

$$V_j = \begin{cases} (U_j \cap (X \setminus \overline{M})) \cup M, & j \leq k \\ W_j \cup \tilde{N}, & j > k \end{cases}$$

making \mathcal{V} the desired (based) categorical covering.

□

Lemma 5.9 [Whi78] A space X has a categorical covering consisting of $n + 1$ sets if and only if the $(n + 1)$ -fold diagonal map Δ_{n+1} is compressible into $X^{[n+1]}$.

Proof:

Suppose that $\mathcal{U} = \{U_1, \dots, U_{n+1}\}$ is a categorical covering of X . By the above, we may assume without loss of generality that \mathcal{U} is a based categorical covering. Thus, for each i , let H_i be the homotopy such that $H_i(x, 0) = x$ for all $x \in X$ and $H_i(u, 1) = *$ for all $u \in U_i$. Define a new homotopy $H(x, t) : X \times I \rightarrow X^{n+1}$ by $H(x, t) = (H_1(x, t), H_2(x, t), \dots, H_{n+1}(x, t))$. Then $H(x, 0) = \Delta_{n+1}(x)$ for all $x \in X$. Further, as \mathcal{U} is a covering of X , for all $x \in X$, there is a j such that $x \in U_j$. But then we have

$$H(x, 1) = (H_1(x, 1), \dots, H_j(x, 1), \dots, H_{n+1}(x, 1)) = (H_1(x, 1), \dots, *, \dots, H_{n+1}(x, 1)) \in X^{[n+1]}$$

and so $H(x, t)$ is the desired compression of Δ_{n+1} .

Now suppose that Δ_{n+1} is compressible into $X^{[n+1]}$. Let h be the lift of Δ_{n+1} through $X^{[n+1]}$ so that $\Delta_{n+1} \simeq jh$, where $j : X^{[n+1]} \hookrightarrow X^{n+1}$ is the inclusion, and let $H : X \times I \rightarrow X^{n+1}$ be that compression. Let $p_i : X^{n+1} \rightarrow X$ be the i th coordinate projection. Take the composition $p_i H : X \times I \rightarrow X$. We have $p_i H(x, 0) = x$ and $p_i H(x, 1) = p_i jh$. Take a neighbourhood N of the basepoint which is contractible to $*$, and define a set $U_i = (p_i jh)^{-1}(N)$. As $jh(X) \subseteq X^{[n+1]}$, we have $X \subseteq (jh)^{-1}(X^{[n+1]})$. Further, we have $X^{[n+1]} = \bigcup_i p_i^{-1}(*)$. Putting these together, we get

$$X \subseteq (jh)^{-1}(X^{[n+1]}) = (jh)^{-1}\left(\bigcup_i p_i^{-1}(*)\right) = \bigcup_i (p_i jh)^{-1}(*) = \bigcup_i (p_i jh)^{-1}(N) = \bigcup_i U_i \subseteq X$$

whence $\mathcal{U} = \{U_1, \dots, U_{n+1}\}$ is an open covering of X . Moreover, if $G : N \times I \rightarrow X$ is the homotopy contracting N to the basepoint, define a new homotopy $J_i : U_i \times I \rightarrow X$ by

$$J_i(u, t) = \begin{cases} p_i H(u, 2t), & 0 \leq t \leq \frac{1}{2} \\ G(h_i(u), 2t - 1), & \frac{1}{2} \leq t \leq 1 \end{cases}$$

so that $J_i(u, 0) = H(u, 0) = u$ and $J_i(u, 1) = G(h_i(u), 1) = *$, whence \mathcal{U} is indeed a categorical covering of X as required.

□

Lemma 5.10 The n -fold diagonal map $\Delta_{n+1} : X \rightarrow X^{n+1}$ is compressible into $X^{[n+1]}$ if and only if the projection $p_m : G^m X \rightarrow X$ of the m th Ganea space onto X admits a section.

Proof:

This proof is less direct than the one above. To see this equivalence, let us first define some

pullbacks. Let (X, A) and (Y, B) be CW pairs with $i : A \hookrightarrow X$ and $j : B \hookrightarrow Y$ the inclusions. Also let $f : Z \rightarrow X$ and $g : Z \rightarrow Y$ for some space Z . We define the following.

$$\begin{array}{cccc}
\Omega_i \longrightarrow * & \Omega_j \longrightarrow * & \Omega_{i,f} \longrightarrow A & \Omega_{j,g} \longrightarrow B \\
\downarrow \lrcorner & \downarrow \lrcorner & \downarrow \lrcorner & \downarrow \lrcorner \\
A \xrightarrow{i} X & B \xrightarrow{j} Y & Z \xrightarrow{f} X & Z \xrightarrow{g} Y
\end{array}$$

More concretely, that is,

$$\begin{aligned}
\Omega_i &= \{(a, \alpha) \in A \times X^I \mid \alpha(0) = * \text{ and } \alpha(1) = a\} \\
&\cong \{\alpha \in X^I \mid \alpha(0) = * \text{ and } \alpha(1) \in A\} \\
\Omega_j &= \{(b, \beta) \in B \times Y^I \mid \beta(0) = * \text{ and } \beta(1) = b\} \\
&\cong \{\beta \in Y^I \mid \beta(0) = * \text{ and } \beta(1) \in B\} \\
\Omega_{i,f} &= \{(z, \alpha) \in Z \times X^I \mid f(z) = \alpha(0), \alpha(1) \in A\} \\
\Omega_{j,g} &= \{(z, \beta) \in Z \times Y^I \mid g(z) = \beta(0), \beta(1) \in B\}
\end{aligned}$$

respectively. Now if we have maps $i \times j : A \times B \hookrightarrow X \times Y$, $k : A \times Y \cup X \times B \hookrightarrow X \times Y$, and $(f, g) = (f \times g)\Delta_Z : Z \rightarrow X \times Y$, we can similarly define the following.

$$\begin{aligned}
\Omega_{i \times j} &= \{(\alpha, \beta) \in X^I \times Y^I \mid \alpha(0) = *, \beta(0) = *, \alpha(1) \in A, \beta(1) \in B\} \\
&= \Omega_i \times \Omega_j \\
\Omega_k &= \{(\alpha, \beta) \in X^I \times Y^I \mid \alpha(0) = *, \beta(0) = *, (\alpha(1), \beta(1)) \in A \times Y \cup X \times B\} \\
\Omega_{i \times j, (f, g)} &= \{(z, \alpha, \beta) \in Z \times X^I \times Y^I \mid f(z) = \alpha(0), g(z) = \beta(0), (\alpha, \beta) \in \Omega_{i \times j}\} \\
\Omega_{k, (f, g)} &= \{(z, \alpha, \beta) \in Z \times X^I \times Y^I \mid f(z) = \alpha(0), g(z) = \beta(0), (\alpha, \beta) \in \Omega_k\}
\end{aligned}$$

There are very natural projections from $\Omega_{i \times j, (f, g)}$ onto $\Omega_{i, f}$ and $\Omega_{j, g}$, which we will call p_X and p_Y respectively. These are defined by $p_X(z, \alpha, \beta) = (z, \alpha)$ and $p_Y(z, \alpha, \beta) = (z, \beta)$.

Now we claim that $\Omega_{k, (f, g)}$, the pull back of $(f, g) : Z \rightarrow X \times Y$ and $k : A \times Y \cup X \times B \hookrightarrow X \times Y$, has the same homotopy type as the pushout of p_X and p_Y , and so the following diagram commutes.

$$\begin{array}{ccccc}
\Omega_{i \times j, (f, g)} & \xrightarrow{p_X} & \Omega_{i, f} & & \\
p_Y \downarrow & & \downarrow & & \\
\Omega_{j, g} & \longrightarrow & \Omega_{k, (f, g)} & \longrightarrow & A \times Y \cup X \times B \\
& & \downarrow \lrcorner & & \downarrow k \\
& & Z & \xrightarrow{(f, g)} & X \times Y
\end{array}$$

To see this, we define spaces $W_0, W_1, W_2 \subseteq W = \Omega_{k,(f,g)}$ by

$$\begin{aligned} W_0 &= \{(z, \alpha, \beta) \in W \mid \alpha(0) = f(z), \beta(0) = g(z), \alpha(1) \in A, \beta(1) \in B\} = \Omega_{i \times j, (f,g)} \\ W_1 &= \{(z, \alpha, \beta) \in W \mid \beta(1) \in B\} \supset \{(z, c_{f(z)}, \beta) \in W \mid \beta(0) = g(z), \beta(1) \in B\} \cong \Omega_{j,g} \\ W_2 &= \{(z, \alpha, \beta) \in W \mid \alpha(1) \in A\} \supset \{(z, \alpha, c_{g(z)}) \in W \mid \alpha(0) = f(z), \alpha(1) \in A\} \cong \Omega_{i,f} \end{aligned}$$

where c_x is the constant path at x . It is clear then that $W = W_1 \cup W_2$ while $W_0 = W_1 \cap W_2$. Now define $D : W_1 \times I \rightarrow \Omega_{j,g}$ by $D((z, \alpha, \beta), t) \rightarrow (z, \alpha_t, \beta)$ where $\alpha_t(s) = \alpha(s(1-t))$. Then

- $D((z, \alpha, \beta), 0) = (z, \alpha_0, \beta) = (z, \alpha, \beta)$ for all $(z, \alpha, \beta) \in W_1$.
- $D(z, \alpha, \beta), 1) = (z, \alpha_1, \beta) = (z, c_{\alpha(0)}, \beta) \in \Omega_{j,g}$ for all $(z, \alpha, \beta) \in W_1$, and
- $D((z, c_{f(z)}, \beta), 1) = (z, c_{f(z)}, \beta)$ for all $(z, c_{f(z)}, \beta) \in \Omega_{j,g}$.

Thus $\Omega_{j,g}$ is a deformation retract of W_1 and so $W_1 \simeq \Omega_{j,g}$. Similarly, $\Omega_{i,f}$ is a deformation retract of W_2 and $W_2 \simeq \Omega_{i,f}$. So, following our standard construction, the homotopy pushout, P of the diagram

$$\begin{array}{ccc} \Omega_{i \times j, (f,g)} & \xrightarrow{p_X} & \Omega_{i,f} \\ p_Y \downarrow & & \\ & & \Omega_{j,g} \end{array}$$

is

$$\begin{aligned} P &= \Omega_{i,f} \sqcup I\Omega_{i \times j, (f,g)} \sqcup \Omega_{j,g} / \{p_X(z, \alpha, \beta) \sim ((z, \alpha, \beta), 1) \text{ and } p_Y(z, \alpha, \beta) \sim ((z, \alpha, \beta), 0)\} \\ &\simeq \Omega_{i,f} \cup_{p_X} \Omega_{i \times j, (f,g)} \cup_{p_Y} \Omega_{j,g} \\ &\simeq W_1 \cup W_2 \\ &= W \\ &= \Omega_{k,(f,g)} \end{aligned}$$

as required. Next, we use this result to establish that for all $m \in \mathbb{N}$, the m th Ganea space $G^m X$ is, up to homotopy, the pullback of the following diagram.

$$\begin{array}{ccc} & & X^{[m+1]} \\ & & \downarrow \\ X & \xrightarrow{\Delta_{m+1}} & X^{m+1} \end{array}$$

Note that for $m = 0$, we have

$$X^{[m+1]} = X^{[1]} = \{(x_1) \in X^1 \mid \text{at least one } x_i = *\} = *$$

as well as $X^{m+1} = X^1 = X$ and $\Delta_{m+1} = \Delta_1 = \mathbb{1}_X$. So we have that

$$G^m X = G^0 X = * \begin{array}{c} \xrightarrow{\quad} * \\ \lrcorner \quad \downarrow \\ \downarrow \quad \downarrow \\ X \quad \xlongequal{\quad} X \end{array}$$

showing the result for $m = 0$. Now, we suppose the result holds true for some $m \in \mathbb{N}$. Then, using the above result, we consider the case $Z = X$, $Y = X^{m+1}$, $f = \mathbb{1}_X$, $g = \Delta_{m+1}$, $A = \{*\}$, and $B = X^{[m+1]}$. This gives $X \times Y = X^{m+2}$ and $A \times Y \cup X \times B = \{*\} \times X^{m+1} \cup X \times X^{[m+1]} = X^{[m+2]}$. So then we have that $\Omega_{k,(f,g)}$ is given by the following diagram.

$$\Omega_{k,(f,g)} \begin{array}{c} \xrightarrow{\quad} X^{[m+2]} \\ \lrcorner \quad \downarrow \\ \downarrow \quad \downarrow \\ X \quad \xrightarrow{\Delta_{m+2}} X^{m+2} \end{array}$$

Meanwhile, by definition, $\Omega_{i \times j,(f,g)}$, $\Omega_{i,f}$, and $\Omega_{j,g}$ are given by the three diagrams

$$\begin{array}{ccc} \Omega_{i \times j,(f,g)} \xrightarrow{\quad} \Omega_{i,f} & \Omega_{i,f} \xrightarrow{\quad} A & \Omega_{j,g} \xrightarrow{\quad} B \\ \downarrow \lrcorner \downarrow & \downarrow \lrcorner \downarrow i & \downarrow \lrcorner \downarrow j \\ \Omega_{j,g} \xrightarrow{\quad} Z & Z \xrightarrow{f} X & Z \xrightarrow{g} Y \end{array}$$

respectively. Now, as $A = *$, $Z = X$, $f = \mathbb{1}_X$, $B = X^{[m+1]}$, $Y = X^{m+1}$, and $g = \Delta_{m+1}$, we can identify $\Omega_{i,f}$ and $\Omega_{j,g}$ as

$$\Omega_{i,f} = * \begin{array}{c} \xrightarrow{\quad} * \\ \lrcorner \quad \downarrow \\ \downarrow \quad \downarrow \\ X \quad \xlongequal{\quad} X \end{array} \quad \Omega_{j,g} = G^m X \begin{array}{c} \xrightarrow{\quad} X^{[m+1]} \\ \lrcorner \quad \downarrow \\ \downarrow \quad \downarrow \\ X \quad \xrightarrow{\Delta_{m+1}} X^{m+1} \end{array}$$

respectively. Putting these together, we can then identify $\Omega_{i \times j,(f,g)}$ via the diagram

$$\Omega_{i \times j,(f,g)} = F^m X \begin{array}{c} \xrightarrow{\quad} * \\ \lrcorner \quad \downarrow \\ \downarrow \quad \downarrow \\ G^m X \quad \xrightarrow{\quad} X \end{array}$$

to be the m th homotopy fibre of X . Thus $\Omega_{k,(f,g)}$ has the homotopy type of the pushout of

$$\begin{array}{ccc} F^m X & \longrightarrow & * \\ \downarrow & & \\ G^m X & & \end{array}$$

which, by definition, is $G^{m+1}X$. Thus, we have the following diagram,

$$\begin{array}{ccccc} F^m X & \longrightarrow & * & & \\ \downarrow & & \downarrow & & \\ G^m X & \longrightarrow & G^{m+1} X & \longrightarrow & X^{[m+2]} \\ & & \downarrow & \lrcorner & \downarrow \\ & & X & \longrightarrow & X^{m+2} \end{array}$$

showing the result for $m + 1$ and thus for all $m \in \mathbb{N}$.

Now we finally have the tool essential to proving our lemma. Suppose that Δ_{m+1} is compressible into $X^{[m+1]}$, so that there is an $h : X \rightarrow X^{[m+1]}$ making the following diagram commute.

$$\begin{array}{ccc} & X^{[m+1]} & \\ & \nearrow h & \downarrow \\ X & \xrightarrow{\Delta_{m+1}} & X^{m+1} \end{array}$$

We invoke the universal property of $G^m X$ as the pullback of $X^{[m+1]} \hookrightarrow X^{m+1}$ and Δ_{m+1} , taking $U = X$, $k_1 = \mathbb{1}_X$, and $k_2 = h$, and obtain the following commutative diagram.

$$\begin{array}{ccccc} X & & & & \\ & \searrow h & & & \\ & & X^{[m+1]} & & \\ & & \downarrow & & \\ & & X^{m+1} & & \\ & \nearrow \Delta_{m+1} & & & \\ & & X & & \\ & \nearrow \sigma & & & \\ & & G^m X & & \\ & & \downarrow & & \\ & & X & & \end{array}$$

The whisker map, σ , is the desired section of p_m .

Lastly, we suppose that p_m admits a section. Then there is a $\sigma : X \rightarrow G^m X$ such that

$$\begin{array}{ccc} G^m X & \longrightarrow & X^{[m+1]} \\ \sigma \nearrow & \downarrow p_m & \downarrow \\ X = X & \xrightarrow{\Delta_{m+1}} & X^{m+1} \end{array}$$

commutes. If f is the map $G^m X \rightarrow X^{[m+1]}$, then $h = f \circ \sigma$ is the desired compression of Δ_{m+1} into $X^{[m+1]}$.

Thus, as required, Δ_{m+1} is compressible into $X^{[m+1]}$ if and only if $p_m : G^m X \rightarrow X$ admits a section. □

This brings us to the culmination of this section, in the form of the following theorem.

Theorem 5.11 The definitions of LS category introduced by Fox, Whitehead, and Ganea are equivalent.

Proof:

The proof of this follows immediately from lemmas 5.9 and 5.10 above.

5.4 Properties of the LS Category

Proposition 5.12 If X dominates Y (that is, if there is a function $f : X \rightarrow Y$ which admits a section), then $\text{cat } X \geq \text{cat } Y$.

Proof:

Let $f : X \rightarrow Y$ and $g : Y \rightarrow X$ such that $f \circ g \simeq \mathbb{1}_Y$, and suppose that $\text{cat } X = n$. Consider the diagonal map $\Delta_{n+1}^Y : (Y, *) \rightarrow (Y^{n+1}, Y^{[n+1]})$. As we know the map $\Delta_{n+1}^X : (X, *) \rightarrow (X^{n+1}, X^{[n+1]})$ is compressible, let $H : X \times I \rightarrow X^{n+1}$ be a compression of it to a map $h : X \rightarrow X^{[n+1]}$. Define a homotopy $G : Y \times I \rightarrow Y^{n+1}$ by $G(y, t) = f^{n+1} \circ H((g(y), t))$. Then for all y , $G(y, 0) = f^{n+1}(H(g(y), 0)) = f^{n+1}\Delta_{n+1}^X(g(y)) = \Delta_{n+1}^Y \circ f \circ g(y)$ and $G(y, 1) = f^{n+1} \circ H(g(y), 1) = f^{n+1}h(g(y)) \in Y^{[n+1]}$. So G is a compression of $f^{n+1} \circ \Delta_{n+1} \circ g = \Delta_{n+1} \circ f \circ g$ into $Y^{[n+1]}$.

Now recall that $f \circ g \simeq \mathbb{1}_Y$, and so there is a homotopy $J : f \circ g \simeq \mathbb{1}_Y$. Thus, combining G and J ,

$$\Delta_{n+1}^Y = \Delta_{n+1}^Y \circ \mathbb{1}_Y \simeq \Delta_{n+1}^Y \circ f \circ g$$

is compressible, and so $\text{cat } Y \leq n = \text{cat } X$, as required. □

It follows immediately from this result that if two spaces X and Y are homotopy equivalent, then $\text{cat } X = \text{cat } Y$ as if $f : X \rightarrow Y$ and $g : Y \rightarrow X$ are homotopy inverses of one another, $f \circ g \simeq \mathbb{1}_Y$, whence $\text{cat } X \geq \text{cat } Y$, while $g \circ f \simeq \mathbb{1}_X$, whence $\text{cat } Y \geq \text{cat } X$. Thus we conclude that $\text{cat } X = \text{cat } Y$. In particular, this means that the category of a space is a topological invariant of that space.

Proposition 5.13 Let X and Y be spaces. Then $\text{cat}(X \times Y) \leq \text{cat } X + \text{cat } Y$.

A proof of this can be found in [Cor95, p. 18].

Proposition 5.14 Let X and Y be spaces. Then $\text{cat}(X \vee Y) = \max\{\text{cat } X, \text{cat } Y\}$.

Proof:

Let $\text{cat } X = n$, $\text{cat } Y = m$, and take based categorical covers $\mathcal{U} = \{U_1, \dots, U_{n+1}\}$ and $\mathcal{V} = \{V_1, \dots, V_{m+1}\}$ of X and Y respectively. As each U_i and V_j contains the base point, U_i and V_j are not open in $X \vee Y$. However the union of any pair $U_i \cup V_j$ is open in $X \vee Y$. As such, assume without loss of generality that $n \leq m$, and define $\mathcal{W} = \{W_k\}$ by

$$W_k = \begin{cases} U_k \cup V_k, & 1 \leq k \leq n+1 \\ U_{n+1} \cup V_k, & n+1 \leq k \leq m+1 \end{cases}.$$

Then each W_k is contractible to $*$ via the contracting map

$$J_k(w, t) = \begin{cases} H_k(w, t), & w \in X, 1 \leq k \leq n+1 \\ H_{n+1}(w, t), & w \in X, k > n+1 \\ G_k(w, t), & w \in Y \end{cases}$$

where $H_k : U_k \times I \rightarrow X$ contracts U_k to $*$ in X and $G_k : V_k \times I \rightarrow Y$ contracts V_k to $*$ in Y . Thus \mathcal{W} is a based categorical covering of $X \vee Y$ consisting of $m+1$ open sets, and so $\text{cat}(X \vee Y) \leq \max\{\text{cat } X, \text{cat } Y\}$.

Now define $q : X \vee Y \rightarrow X$ to be the canonical quotient map collapsing Y to a point. Then q admits a section, namely the canonical inclusion $i : X \hookrightarrow X \vee Y$ (indeed, $q \circ i = \mathbb{1}_X$). Then $X \vee Y$ dominates X and so $\text{cat } X \leq \text{cat}(X \vee Y)$. Of course, this same argument applies to show that $\text{cat } Y \leq \text{cat}(X \vee Y)$, whence $\max\{\text{cat } X, \text{cat } Y\} \leq \text{cat}(X \vee Y)$, giving the desired equality $\text{cat}(X \vee Y) = \max\{\text{cat } X, \text{cat } Y\}$. □

Theorem 5.15 [Cor95] Let X be a space. Define the cuplength of X , denoted $\text{cup } X$, to be the maximal wordlength of elements of $H^*(X)$. That is, $\text{cup } X$ is the largest integer m such that there are $a_1, \dots, a_m \in H^*(X)$ such that $a_1 a_2 \cdots a_m \neq 0$. Then $\text{cup } X \leq \text{cat } X$.

Proof:

Suppose $\text{cat } X = n$ and let $\mathfrak{A} = \{A_1, \dots, A_{n+1}\}$ be a categorical covering of X . Recall that for each i , we have a long exact sequences of cohomology modules, as follows,

$$\cdots \longrightarrow H^{k-1}(A_i) \longrightarrow H^k(X, A_i) \xrightarrow{q_i^*} H^k(X) \xrightarrow{j_i^*} H^k(A_i) \longrightarrow H^{k+1}(X, A_i) \longrightarrow \cdots$$

where j_i^* is the map induced by $j_i : A_i \hookrightarrow X$ and q_i^* is the map induced by $q_i : X \hookrightarrow (X, A_i)$. As the inclusion j_i is homotopically trivial for each i , $j_i^* = 0$. So, by exactness, q_i^* is surjective. Now for $i \in \{1, \dots, n+1\}$, choose $a_i \in H^k(X)$, and for each a_i let $b_i \in H^k(X, A_i)$ such that $q_i^*(b_i) = a_i$.

Recall that for $b_s \in H^*(X, A_s)$ and $b_t \in H^*(X, A_t)$, the product $b_s b_t$ is defined in $H^*(X, A_s \cup A_t)$. Moreover $q^*(b_s b_t) = q_s^*(b_s) q_t^*(b_t)$. The product $b_1 \cdots b_{n+1}$ lies in $H^*(X, \bigcup A_i) = H^*(X, X) = 0$, whence $b_1 \cdots b_{n+1} = 0$. Finally, this gives

$$a_1 \cdots a_{n+1} = q_1^*(b_1) \cdots q_{n+1}^*(b_{n+1}) = q^*(b_1 \cdots b_{n+1}) = q^*(0) = 0$$

from which we see that for any product $a_1 \cdots a_m$ of elements in $H^*(X)$, if $m > \text{cat } X$, $a_1 \cdots a_m = 0$, whence $\text{cup } X \leq \text{cat } X$.

Example 5.16 To put this to use, let us calculate the LS category of the torus. Since $T = S^1 \times S^1$, we have by the above that $\text{cat } T \leq \text{cat } S^1 + \text{cat } S^1 = 1 + 1 = 2$. Moreover, we have that $H^*T \cong H^*S^1 \otimes H^*S^1 = \Lambda(a, b)$, with $|a| = |b| = 1$, where $\Lambda(a, b)$ is the exterior algebra on the generators a and b . Since both a and b are of odd degree, $a^2 = b^2 = 0$, so the longest word we can produce in H^*T is ab with wordlength 2. Thus $\text{cup } T = 2$, and so $2 = \text{cup } T \leq \text{cat } T \leq \text{cat } S^1 + \text{cat } S^1 = 2$, whence $\text{cat } T = 2$.

More generally, for the n -dimensional torus T^n , as it is the product of n copies of S^1 , we have that

$$\text{cat } T^n \leq \underbrace{\text{cat } S^1 + \cdots + \text{cat } S^1}_{n \text{ times}} = n.$$

Similarly, $H^*T^n \cong \underbrace{H^*S^1 \otimes \cdots \otimes H^*S^1}_{n \text{ times}} = \Lambda(a_1, \dots, a_n)$ with $|a_i| = 1$ for all i , so again we have $a_i^2 = 0$ for all i , meaning the longest word we can form is $a_1 a_2 \cdots a_n$, giving $\text{cup } T^n = n$, whence $\text{cat } T^n = n$.

Proposition 5.17 Let X be a space with $\text{cat } X \leq n$, and let $f : S^r \rightarrow X$. Let $Y = X \cup_f e^{r+1}$ be the cofibre of f . Then $\text{cat } Y \leq n + 1$.

Proof:

Note that, if c is the vertex of the cone $C(S^r) = e^{r+1}$, then $Y \setminus \{c\} \simeq X$, and so $\text{cat } Y \setminus \{c\} = \text{cat } X \leq n$. As such, let $\{U_0, \dots, U_n\}$ be a categorical covering of $Y \setminus \{c\}$. Now define $U_{n+1} = C(S^r) \setminus S^r = e^{r+1} \setminus S^r$. Clearly U_{n+1} is open and contractible in Y . Moreover $\bigcup_{i=0}^{n+1} U_i = Y$. Thus $\{U_0, \dots, U_n, U_{n+1}\}$ is a categorical covering of Y , and $\text{cat } Y \leq n + 1$.

This result can be generalized further. That is, if A is any space and $f : A \rightarrow X$, then if $Y = X \cup_f C(A)$, $\text{cat } Y \leq n + 1$ by the same argument presented above.

Lastly, the LS category has an interesting interaction with the Hopf invariant that we will need to use. We have the following from [BH60].

Theorem 5.18 Let X be a $(q-1)$ -connected CW complex with $\text{cat } X \leq n$. Let $Y = X \cup_{\beta} e^{r+1}$ for $r \geq q \geq 2$. If $\dim X \leq (n+1)q - 2$, then $\text{cat } Y \leq n$ if and only if $\mathcal{H}(\beta) = 0$.

As a particular case of this, we have the following.

Corollary 5.19 Let $\beta : S^r \rightarrow S^q$. If $r \geq q \geq 2$, then $\text{cat}(S^q \cup_{\beta} e^{r+1}) \leq 1$ if and only if $\mathcal{H}(\alpha) = 0$.

Part II
Ganea's Conjecture

SECTION 6

GANEVA'S CONJECTURE

We saw above that for arbitrary spaces X and Y , $\text{cat}(X \times Y) \leq \text{cat} X + \text{cat} Y$. A natural question then arises – when does equality hold?

Constructing an example where equality does not hold, it turns out, is not particularly difficult. Consider the Moore space, denoted $M(G, n)$, whose n^{th} homology group is the group G and whose other homology groups are trivial. Recall we can construct the Moore space $M(\mathbb{Z}_m, n)$ as the cofibre of a degree m map $f_m : S^n \rightarrow S^n$, as follows.

$$\begin{array}{ccc} S^n & \xrightarrow{f_m} & S^n \\ \downarrow & & \downarrow \\ * & \longrightarrow & S^n \cup_{f_m} e^{n+1} \end{array}$$

Now, recall that for a space X , and for $k \geq 1$, $H_{k+1}(\Sigma X) = H_k(X)$. Thus we have the following relation between Moore spaces.

$$\Sigma M(G, n) = M(G, n + 1)$$

Note that the map $f_m : S^1 \rightarrow S^1$ defined by $z \mapsto z^m$ is a degree m map, and so $S^1 \cup_{f_m} e^2 = M(\mathbb{Z}_m, 1)$. Thus, as a simply connected Moore space is completely characterized by its homology, $M(\mathbb{Z}_m, 2) = \Sigma M(\mathbb{Z}_m, 1)$. Since $M(\mathbb{Z}_m, 2)$ is a suspension, its LS category is 1.

Consider, for instance, $M(\mathbb{Z}_2, 2)$ and $M(\mathbb{Z}_3, 2)$. As above, these are each of category 1. Further, we can identify the homology of their product as follows.

$$H_k(M(\mathbb{Z}_2, 2) \times M(\mathbb{Z}_3, 2)) \cong \bigoplus_{i+j=k} H_i(M(\mathbb{Z}_2, 2)) \otimes H_j(M(\mathbb{Z}_3, 2)) = \begin{cases} \mathbb{Z}_2 \oplus \mathbb{Z}_3, & k = 2 \\ \mathbb{Z}_2 \otimes \mathbb{Z}_3, & k = 4 \\ 0, & \text{else} \end{cases}$$

Note, however, that $\mathbb{Z}_2 \otimes \mathbb{Z}_3 = 0$, as

$$1 \otimes 2 = 1 \otimes (2 \cdot 1) = 2 \otimes 1 = 0 \otimes 1 = 0$$

and

$$1 \otimes 1 = 1 \otimes (2 \cdot 2) = 2 \otimes 2 = 0 \otimes 2 = 0$$

implying that

$$H_k(M(\mathbb{Z}_2, 2) \times M(\mathbb{Z}_3, 2)) = \begin{cases} \mathbb{Z}_2 \oplus \mathbb{Z}_3 & k = 2 \\ 0, & \text{else} \end{cases}$$

leading to the interesting observation that the inclusion $\kappa : M(\mathbb{Z}_2, 2) \vee M(\mathbb{Z}_3, 2) \hookrightarrow M(\mathbb{Z}_2, 2) \times M(\mathbb{Z}_3, 2)$ induces an isomorphism $\kappa_* : H_*(M(\mathbb{Z}_2, 2) \vee M(\mathbb{Z}_3, 2)) \xrightarrow{\cong} H_*(M(\mathbb{Z}_2, 2) \times M(\mathbb{Z}_3, 2))$.

Recall, as a consequence of Whitehead's theorem and Hurewicz's theorem, that for simply connected CW complexes X and Y , if a map $f : X \rightarrow Y$ induces an isomorphism in homology, then f is a homotopy equivalence. Thus, by the above, $M(\mathbb{Z}_2, 2) \times M(\mathbb{Z}_3, 2) \simeq M(\mathbb{Z}_2, 2) \vee M(\mathbb{Z}_3, 2)$, and we have:

$$\begin{aligned} \text{cat}(M(\mathbb{Z}_2, 2) \times M(\mathbb{Z}_3, 2)) &= \text{cat}(M(\mathbb{Z}_2, 2) \vee M(\mathbb{Z}_3, 2)) \\ &= \max\{\text{cat } M(\mathbb{Z}_2, 2), \text{cat } M(\mathbb{Z}_3, 2)\} \\ &= \max\{1, 1\} \\ &= 1 \\ &< 2 \\ &= \text{cat } M(\mathbb{Z}_2, 2) + \text{cat } M(\mathbb{Z}_3, 2) \end{aligned}$$

giving an example of strict inequality. What's more, this method does not produce just one such example. Note that the above arguments work for any coprime numbers p and q , building the Moore spaces $M(\mathbb{Z}_p, 2)$ and $M(\mathbb{Z}_q, 2)$.

However this is not a very satisfying solution to the problem, being derived from algebraic calculations, rather than being inspired geometrically.

This led Ganea to produce his famous conjecture. What would happen if one of the pieces in the product were not allowed to have only torsion in its homology? Perhaps, more simply, if we restricted to the case of spheres, would the LS category then behave as expected? This question leads us to the heart of this thesis.

Ganea's Conjecture

For any CW complex X , and for any sphere S^n , $\text{cat}(X \times S^n) = \text{cat } X + 1$.

6.1 The Rational Case – The Hess-Jessup Theorem

Definition 6.1 Define a *rational space* to be a simply connected CW complex, X , whose homotopy groups are all vector spaces over the rational numbers, or, equivalently, if the homomorphism $\mathbb{1}_{\pi_n(X)} \otimes \mathbb{Q} : \pi_n(X) \rightarrow \pi_n(X) \otimes \mathbb{Q}$ defined by $\omega \mapsto \omega \otimes 1$ is an isomorphism for all n .

For any simply connected space X , there is a rational space, denoted X_0 , and a map $X \rightarrow X_0$ which induces an isomorphism $\pi_*(X_0) \cong \pi_*(X) \otimes \mathbb{Q}$. Moreover, this space is unique up to homotopy equivalence. The space X_0 is called the rationalization of X .

Definition 6.2 Let X be a simply connected space. We define the *rational LS category* of X , denoted $\text{cat}_0 X$, by

$$\text{cat}_0(X) = \text{cat}(X_0).$$

The work of Jessup [Jes90] and Hess [Hes91] showed

Theorem 6.3 Let X be a simply connected space. Then for all $n \geq 1$,

$$\text{cat}_0(X \times S^n) = \text{cat}_0 X + 1.$$

This is a (very) brief overview of this topic. A more full treatment can be found in [FHT95]. However, it is interesting to see this result, as it gives a bit of historical context for the problem. With results like this one pointing to the veracity of Ganea's conjecture, the discovery of a family of counterexamples came as quite a surprise.

SECTION 7

IN SEARCH OF AN ATTACHING MAP

7.1 A Small Miracle

In his 1998 article, Iwase constructed not just one counter example to Ganea's conjecture, but one for each odd prime p . We replicate his construction with the the help of a result from [Cor95, p. 300] we used earlier, during the construction of the Hopf invariant.

Definition 7.1 An H -group is a topological space X together with a map $\mu : X \times X \times X$ such that

- the diagrams

$$\begin{array}{ccc}
 H & \xrightarrow{i_1} & H \times H \\
 \searrow \mathbb{1}_H & & \swarrow \mu \\
 & & H
 \end{array}$$

and

$$\begin{array}{ccc}
 H & \xrightarrow{i_2} & H \times H \\
 \searrow \mathbb{1}_H & & \swarrow \mu \\
 & & H
 \end{array}$$

commute up to homotopy, where $i_1(x) = (x, *)$ and $i_2(x) = (*, x)$,

- the diagram

$$\begin{array}{ccc}
 X \times X \times X & \xrightarrow{\mathbb{1}_X \times \mu} & X \times X \\
 \mu \times \mathbb{1}_X \downarrow & & \downarrow \mu \\
 X \times X & \xrightarrow{\mu} & X
 \end{array}$$

commutes up to homotopy, and

- for each $x \in X$, there is a $y \in X$ such that $\mu(x, y) = * = \mu(y, x)$.

Proposition 7.2 Let $F \xrightarrow{i} E \xrightarrow{p} B$ be a fibre sequence such that E and B are H -groups, and p is an H -group homomorphism. If p admits a section, s , so that $p \circ s \simeq \mathbf{1}_B$, then $E \simeq B \times F$.

Proof:

We show this by direct verification. Define a map $i + s : B \times F \rightarrow E$ by $(s \cdot i)(b, f) = s(b) \cdot f$. It is readily verified that this is a homotopy equivalence with homotopy inverse $\sigma : E \rightarrow B \times F$ defined by $\sigma(e) = (p(e), (s \circ p(e))^{-1} \cdot e)$.

□

Corollary 7.3 Let X and Y be spaces. There is a homotopy equivalence

$$\Omega(X \vee Y) \simeq \Omega X \times \Omega Y \times \Omega(\Omega X * \Omega Y).$$

To see this, consider the fibre sequence $F \xrightarrow{i} X \vee Y \xrightarrow{j} X \times Y$. Then

$$\Omega F \xrightarrow{\Omega i} \Omega(X \vee Y) \xrightarrow{\Omega j} \Omega(X \times Y) = \Omega X \times \Omega Y$$

is also a fibre sequence. Now, $\Omega(X \vee Y)$ and $\Omega(X \times Y)$ are H -groups. We simply need to observe that $\Omega(i_X) \cdot \Omega(i_Y) : \Omega X \times \Omega Y \rightarrow \Omega(X \vee Y)$ is a homotopy section of $\Omega j : \Omega(X \vee Y) \rightarrow \Omega(X \times Y)$, where $i_X : X \rightarrow X \vee Y$ and $i_Y : Y \rightarrow X \vee Y$ are the canonical inclusions. So Proposition 7.2 gives us a homotopy equivalence

$$\Omega(X \vee Y) \simeq \Omega(X \times Y) \times \Omega F \simeq \Omega X \times \Omega Y \times \Omega F.$$

Lastly, we have from [Por66] that the fibre of the inclusion $X \vee Y \hookrightarrow X \times Y$ is $\Omega X * \Omega Y$.

□

This implies the decomposition of the homotopy groups of $X \vee Y$ used in the construction of the Hopf invariant, as for all $r > 1$, we get the following.

$$\begin{aligned}
\pi_r(X \vee Y) &= \pi_{r-1}(\Omega(X \vee Y)) \\
&= \pi_{r-1}(\Omega X \times \Omega Y \times \Omega(\Omega X * \Omega Y)) \\
&= \pi_{r-1}(\Omega X) \oplus \pi_{r-1}(\Omega Y) \oplus \pi_{r-1}(\Omega(\Omega X * \Omega Y)) \\
&= \pi_r(X) \oplus \pi_r(Y) \oplus \pi_r(\Omega X * \Omega Y)
\end{aligned}$$

To build our first candidate for an attaching map, we begin by selecting a generator, α of $\pi_9(S^3) = \mathbb{Z}_3$ (Table 9). Next, we define $\beta : S^9 \rightarrow S^2$ by $\beta = \eta \circ \alpha$, where $\eta : S^3 \rightarrow S^2$ is the Hopf fibration.

Proposition 7.4 The Hopf invariant of α is 0.

Proof:

Since $\mathcal{H}(\alpha) \in \pi_9(\Omega S^3 * \Omega S^3)$, we begin by finding a more familiar expression for $\pi_9(\Omega S^3 * \Omega S^3)$. To do this, we refer to [Jam55], whose results about reduced product spaces give us a homotopy equivalence

$$\Sigma \Omega S^{n+1} \simeq \bigvee_{k=1}^{\infty} S^{kn+1}.$$

So,

$$\begin{aligned}
\pi_9(\Omega S^3 * \Omega S^3) &= \pi_9(\Sigma(\Omega S^3 \wedge \Omega S^3)) \\
&= \pi_9(\Omega S^3 \wedge \Sigma \Omega S^3) \\
&= \pi_9(\Omega S^3 \wedge (S^3 \vee S^5 \vee S^7 \vee S^9 \vee \dots)) \\
&= \pi_9(\Omega S^3 \wedge (S^3 \vee S^5 \vee S^7 \vee S^9)) \\
&= \pi_9(\Omega S^3 \wedge (\Sigma(S^2 \vee S^4 \vee S^6 \vee S^8))) \\
&= \pi_9(\Sigma \Omega S^3 \wedge (S^2 \vee S^4 \vee S^6 \vee S^8)) \\
&= \pi_9((\Sigma(S^2 \vee S^4 \vee S^6 \vee S^8)) \wedge (S^2 \vee S^4 \vee S^6 \vee S^8)) \\
&= \pi_9(\Sigma((S^2 \wedge S^2) \vee (S^2 \wedge S^4) \vee (S^4 \wedge S^2) \vee \dots \vee (S^8 \wedge S^8))) \\
&= \pi_9(S^5 \vee S^7 \vee S^7 \vee S^9 \vee S^9 \vee S^9 \vee \dots) \\
&= \pi_9(S^5 \vee S^7 \vee S^7 \vee S^9 \vee S^9 \vee S^9).
\end{aligned}$$

To identify this group further, we must make use of Corollary 7.3. Note that for an n -connected space X and an m -connected space Y , $\Omega X * \Omega Y$ is $(m+n)$ -connected, as

$$\begin{aligned}
\text{conn } \Omega X * \Omega Y &= \text{conn } \Sigma(\Omega X \wedge \Omega Y) \\
&= \text{conn } \Omega X \wedge \Omega Y + 1 \\
&\geq \text{conn } \Omega X + \text{conn } \Omega Y + 2 \\
&= \text{conn } X - 1 + \text{conn } Y - 1 + 2 \\
&= \text{conn } X + \text{conn } Y.
\end{aligned}$$

Now, $X = S^5 \vee S^7 \vee S^7 \vee S^9 \vee S^9$ is 4-connected, while $Y = S^9$ is 8-connected, so that $\Omega(\Omega X * \Omega Y)$ is 11-connected. Thus, $\pi_9(\Omega X * \Omega Y) = 0$, and by Corollary 7.3,

$$\pi_9(\Omega S^3 * \Omega S^3) = \pi_9(S^5 \vee S^7 \vee S^7 \vee S^9 \vee S^9 \vee S^9) = \pi_9(S^5 \vee S^7 \vee S^7 \vee S^9 \vee S^9) \oplus \pi_9(S^9).$$

We repeat this process, noting that $S^5 \vee S^7 \vee S^7 \vee S^9$ is 4-connected and S^9 is 8-connected, so that

$$\pi_9(S^5 \vee S^7 \vee S^7 \vee S^9 \vee S^9) \oplus \pi_9(S^9) = \pi_9(S^5 \vee S^7 \vee S^7 \vee S^9) \oplus \pi_9(S^9) \oplus \pi_9(S^9).$$

Carrying on this way, we get

$$\pi_9(S^5 \vee S^7 \vee S^7 \vee S^9 \vee S^9 \vee S^9) = \pi_9(S^5 \vee S^7) \oplus \pi_9(S^7) \oplus \pi_9(S^9) \oplus \pi_9(S^9) \oplus \pi_9(S^9)$$

and, by some small miracle, S^5 is 4-connected while S^7 is 6-connected, so that $\Omega S^5 * \Omega S^7$ is 10-connected, just barely highly connected enough so that

$$\pi_9(\Omega S^3 * \Omega S^3) = \pi_9(S^5) \oplus \pi_9(S^7) \oplus \pi_9(S^7) \oplus \pi_9(S^9) \oplus \pi_9(S^9) \oplus \pi_9(S^9).$$

We now refer to the table in Section 9 to see that

$$\pi_9(\Omega S^3 * \Omega S^3) = \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z},$$

and therefore has no element of order 3. Thus, as \mathcal{H} is a homomorphism, it becomes apparent that $\mathcal{H}(\alpha) = 0$, as claimed. □

Proposition 7.5 Let h be the composition $S^r \xrightarrow{f} S^q \xrightarrow{g} S^p$, such that $\mathcal{H}(f) = 0$. Then $\mathcal{H}(h) = \mathcal{H}(g) \circ f$.

This follows directly from Corollary 6.23 in [Cor95].

Now, by Proposition 7.5, since $\mathcal{H}(\alpha) = 0$,

$$\mathcal{H}(\beta) = \mathcal{H}(\eta \circ \alpha) = \mathcal{H}(\eta) \circ \alpha = \mathbb{1}_{S^3} \circ \alpha = \alpha \neq 0.$$

However, referring again to our table we see that $\pi_{11}(S^5) = \mathbb{Z}_2$ has no element of order 3. Thus, $\Sigma^2 \mathcal{H}(\beta) = 0$.

7.2 The Hilton Milnor Theorem

While we were able to replicate Iwase's steps in the low dimensional case when $p = 3$, we were not able to do it in the general case. However, while we cannot follow his steps exactly, we were able to reproduce his findings of the existence of a counter example to Ganea's conjecture for each odd prime. We present an alternative argument here.

We have from [Ser53] that for each odd prime p , $\pi_{4p-3}(S^3) = \mathbb{Z}_p$. Let $\alpha_p \in \pi_{4p-3}(S^3)$ be a

generator.

In [Hil55], Hilton showed that the loop space on a wedge of spheres is homotopy equivalent to some product of loop space of spheres. This result was generalized by Milnor in [Mil56] to the loop spaces of wedges of suspensions. We use a special case of this result here.

Theorem 7.6 Let X_1, \dots, X_n be finite, connected CW complexes. Then there is a homotopy equivalence

$$\Omega(\Sigma X_1 \vee \dots \vee \Sigma X_n) \simeq \prod_{i=1}^{\infty} \Omega \Sigma Y_i,$$

where each $Y_i = \bigwedge (X_1^{(i_1)}, \dots, X_n^{(i_n)})$ is the smash product of i_j copies of X_j , $1 \leq j \leq n$, for some n -tuple (i_1, \dots, i_n) .

Proposition 7.7 For each p , $\mathcal{H}(\alpha_p) = 0$.

Proof:

Recall that $\mathcal{H}(\alpha_p)$ lives in $\pi_{4p-3}(\Omega S^3 * \Omega S^3)$. By arguments similar to those above, we can express $\pi_{4p-3}(\Omega S^3 * \Omega S^3)$ as follows.

$$\begin{aligned} \pi_{4p-3}(\Omega S^3 * \Omega S^3) &= \pi_{4p-3}(\Omega S^3 \wedge \Sigma \Omega S^3) \\ &= \pi_{4p-3}((\Omega S^3) \wedge (\Sigma(S^2 \vee S^4 \vee S^6 \vee \dots))) \\ &= \pi_{4p-3}((\Sigma(S^2 \vee S^4 \vee S^6 \vee \dots)) \wedge (S^2 \vee S^4 \vee S^6 \vee \dots)) \\ &= \pi_{4p-3}(\Sigma((S^2 \vee S^4 \vee S^6 \vee \dots) \wedge (S^2 \vee S^4 \vee S^6 \vee \dots))) \\ &= \pi_{4p-3}(\Sigma(S^4 \vee S^6 \vee S^6 \vee S^8 \vee S^8 \vee S^8 \vee \dots)) \\ &= \pi_{4p-3}(\Sigma(S^4 \vee S^6 \vee S^6 \vee \dots \vee S^{4p-4})) \end{aligned}$$

From Theorem 7.6, this is simply the infinite sum $\bigoplus_{i=1}^{\infty} \pi_{4p-3}(\Sigma Y_i)$, where each Y_i is given by

$$\begin{aligned} Y_i &= \bigwedge ((S^{n_1})^{(i_1)}, (S^{n_2})^{(i_2)}, \dots, (S^{n_m})^{(i_m)}) \\ &= S^{i_1 n_1} \wedge S^{i_2 n_2} \wedge \dots \wedge S^{i_m n_m} \\ &= S^{i_1 n_1 + \dots + i_m n_m}. \end{aligned}$$

Define, for each i , an integer k_i such that $\Sigma Y_i = S^{i_1 n_1 + \dots + i_m n_m + 1} = S^{k_i}$. Then we have

$$\pi_{4p-3}(\Omega S^3 * \Omega S^3) = \bigoplus_{i=1}^{\infty} \pi_{4p-3}(S^{k_i}).$$

It is important to note that each k_i is an odd integer with $k_i \geq 5$, since each n_i is even. Thus, we apply the following result from Serre.

Proposition 7.8 [Ser53] Let $k \geq 3$ be an odd integer, and $p \neq 2$ be a prime. Then the p -primary summand of $\pi_r(S^k)$ is as follows.

$$\pi_r(S^k; p) = \begin{cases} 0, & r < k + 2p - 3 \\ \mathbb{Z}_p, & r = k + 2p - 3 \\ 0, & k + 2p - 3 < r < k + 4p - 6 \end{cases}$$

In particular, taking $r = 4p - 3$, $\pi_{4p-3}(S^k; p) = 0$ as long as

$$4p - 3 < k + 4p - 6 \quad \Rightarrow \quad k > 3$$

and

$$4p - 3 \neq k + 2p - 3 \quad \Rightarrow \quad k \neq 2p$$

which is exactly the case. Thus, $\pi_{4p-3}(\Omega S^3 * \Omega S^3)$ has no element of order p , whence $\mathcal{H}(\alpha_p) = 0$. Now, referring to Proposition 7.5, the map $\beta_p = \eta \circ \alpha_p$ has a Hopf invariant

$$\mathcal{H}(\beta_p) = \mathcal{H}(\eta \circ \alpha_p) = \mathcal{H}(\eta) \circ \alpha_p = \mathbb{1}_{S^3} \circ \alpha_p = \alpha_p$$

which is clearly nonzero. However, we have from [Tod62] that $\pi_{4p-1}(S^5)$ has no p -torsion, and so clearly $\Sigma^2 \mathcal{H}(\beta_p) = 0$.

□

Remark

In his article, Iwase seems to decompose the group $\pi_{4p-3}(S^5 \vee S^7 \vee S^7 \vee \dots \vee S^{4p-3})$ by replacing wedge sums of spaces by direct sums of homotopy groups, a much stronger statement than the one we have used above. Note, it is true that this group will be the direct product of some number of $(4p - 3)$ -homotopy groups of spheres of odd dimension between 5 and $4p - 3$, by Hilton. However I was unable to replicate the result that the configuration of spheres remains the same as the configuration of the wedge.

This statement happens to be true for exactly the case $p = 3$, however it seems to have been a happy accident due to the fact that the connectivity of $\Omega S^5 * \Omega S^7$ is at least 10 – just high enough to have a trivial homotopy group in degree 9. However, we were unable to reproduce this result, even for $p = 5$, as for all other odd primes, $4p - 3 > \text{conn } \Omega S^5 * \Omega S^7$.

SECTION 8

LS CATEGORY OF $Q_p \times S^n$

Now that we have our spaces, $Q_p = S^2 \cup_{\beta_p} e^{4p-2}$, it remains to calculate their categories, and the categories of $Q_p \times S^n$.

Proposition 8.1 The category of the space Q_p is 2, for any odd prime p .

Proof:

By Proposition 5.17, $\text{cat } Q_p \leq \text{cat } S^2 + 1 = 2$. Further S^2 is a 1-connected CW complex with $\text{cat } S^2 \leq 1$, and $\dim S^2 = 2 \leq (1+1)(2) - 2 = 2$. Thus, by corollary 5.19, $\text{cat } Q_p \leq 1$ if and only if $\mathcal{H}(\beta_p) = 0$. However, as we have just shown, $\mathcal{H}(\beta_p) = \alpha_p \neq 0$. Thus $\text{cat } Q_p > 1$, whence $\text{cat } Q_p = 2$. □

We now calculate the category of $Q_p \times S^n$. We will need a few results to aid in this.

Lemma 8.2 The following diagram commutes up to homotopy.

$$\begin{array}{ccccc}
 S^{4p-3} & \xrightarrow{\beta_p} & S^2 & \xrightarrow{i} & Q_p \\
 \downarrow \alpha_p & & \downarrow \Sigma\Omega i \Sigma j & & \parallel \\
 S^1 * S^1 & & & & \\
 \downarrow (\Omega i * \Omega i)(j * j) & & & & \\
 \Omega Q_p * \Omega Q_p & \xrightarrow{h_1^{Q_p}} & \Sigma \Omega Q_p & \xrightarrow{p_1^{Q_p}} & Q_p \\
 & & \searrow i_1^{Q_p} & & \nearrow e_2^{Q_p} \\
 & & G^2 Q_p & &
 \end{array}$$

Here, $i : S^2 \rightarrow Q_p$ is the canonical inclusion, $j : S^1 \hookrightarrow S^1 \cup S^2 \cup S^3 \cup \dots \simeq \Omega S^2$ is the inclusion of the bottom cell, and $h_1^X : \Omega X * \Omega X = F^1 X \rightarrow \Sigma \Omega X = G^1 X$ is the attaching map of $C(F^1 X)$

onto G^1X to form G^2X . Further, p_1^X is the canonical projection map of G^1X onto X , i_1^X the canonical inclusion of G^1X in G^2X , and e_2^X the canonical inclusion of G^2X into $G^\infty X \simeq X$.

Remark

In order to help avoid getting bogged down by cumbersome notation, note that $(\Omega i * \Omega i)(j * j) : S^1 * S^1 \hookrightarrow \Omega Q_p * \Omega Q_p$ and $\Sigma \Omega i \Sigma j : S^2 \hookrightarrow \Sigma \Omega Q_p$ are simply inclusions.

Proof:

The commutativity of the right half of the diagram is clear. To see the commutativity of the leftmost square, consider the following diagram from [Gan67].

$$\begin{array}{ccccc}
 S^1 * S^1 & \xrightarrow{\omega} & S^2 \vee S^2 & \hookrightarrow & S^2 \times S^2 \\
 \downarrow (\Omega i * \Omega i)(j * j) & & \downarrow i \vee i & & \downarrow i \times i \\
 \Omega Q_p * \Omega Q_p & \xrightarrow{q^{Q_p}} & Q_p \vee Q_p & \hookrightarrow & Q_p \times Q_p
 \end{array}$$

where q^X is the map induced by the pullback

$$\begin{array}{ccc}
 \Omega X * \Omega X & \xrightarrow{q^X} & X \vee X \\
 \downarrow \lrcorner & & \downarrow \\
 * & \hookrightarrow & X \times X
 \end{array}$$

and ω is the attaching map $\omega : S^3 \rightarrow S^2 \vee S^2$ forming $S^2 \times S^2$.

From [Whi78, p. 494], we have $\mu^2 \eta \simeq (\eta \vee \eta) \mu^3 + \omega$, where μ^k is the pinch map on S^k , $\mu^k : S^k \rightarrow S^k \vee S^k$. So then $\mu^2 \beta_p = \mu^2 \eta \alpha_p \simeq (\eta \vee \eta) \mu^3 \alpha_p + \omega \alpha_p$. Since α_p is a co-Hopf map, we then get that $\mu^2 \beta_p \simeq (\eta \alpha_p \vee \eta \alpha_p) \mu^{4p-3} + \omega \alpha_p \simeq (\beta_p \vee \beta_p) \mu^{4p-3} + \omega \alpha_p$, and finally that

$$(i \vee i) \mu^2 \beta_p \simeq (i \beta_p \vee i \beta_p) \mu^{4p-3} + (i \vee i) \omega \alpha_p \simeq q^{Q_p} (\Omega i * \Omega i)(j * j) \alpha_p$$

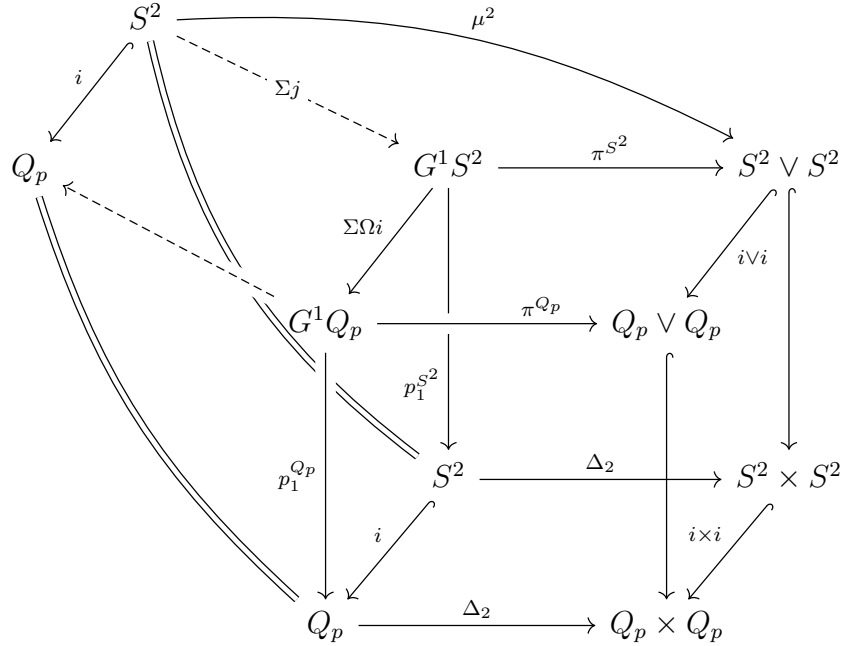
as $(i \beta_p \vee i \beta_p) \mu^{4p-3}$ is the trivial element in $\pi_{4p-3}(Q_p \vee Q_p)$. We have seen already that $\text{cat } S^k = 1$ for all k , so that there exists a map represented by the dashed line making the following pullback diagram commute.

$$\begin{array}{ccc}
 S^2 & \xrightarrow{\mu^2} & S^2 \vee S^2 \\
 \downarrow \Sigma j & \searrow & \downarrow \\
 G^1 S^2 & \xrightarrow{\pi^{S^2}} & S^2 \vee S^2 \\
 \downarrow p_1^{S^2} & \lrcorner & \downarrow \\
 S^2 & \xrightarrow{\Delta_{S^2}} & S^2 \times S^2
 \end{array}$$

Thus, $(i \vee i)\mu^2 = (i \vee i)\pi^{S^2}\Sigma j = \pi^{Q_p}\Sigma\Omega i\Sigma j$, and so, putting all of the above together,

$$\pi^{Q_p}\Sigma\Omega i\Sigma j\beta_p = (i \vee i)\mu^2\beta_p = q^{Q_p}(\Omega i * \Omega i)(j * j)\alpha_p.$$

For clarity, as a visual aid, we add a layer to the above diagram, raising it into Q_p via the inclusion $i : S^2 \rightarrow Q_p$.



Now note that we have the following commutative diagram.

$$\begin{array}{ccc} \Omega Q_p * \Omega Q_p & \xlongequal{\quad} & \Omega Q_p * \Omega Q_p \\ \downarrow h_1^{Q_p} & & \downarrow q^{Q_p} \\ G^1Q_p & \xrightarrow{\pi^{Q_p}} & Q_p \vee Q_p \\ \downarrow p_1^{Q_p} & & \downarrow \\ Q_p & \xrightarrow{\Delta_2} & Q_p \times Q_p \end{array}$$

From this we see that $q^{Q_p} = \pi^{Q_p}h_1^{Q_p}$, giving us the identity

$$\pi^{Q_p}\Sigma\Omega i\Sigma j\beta_p = q^{Q_p}(\Omega i * \Omega i)(j * j)\alpha_p = \pi^{Q_p}h_1^{Q_p}(\Omega i * \Omega i)(j * j)\alpha_p.$$

Moreover, we also have that

$$p_1^{Q_p}\Sigma\Omega i\Sigma j\beta_p = ip_1^{S^2}\Sigma j\beta_p = i\mathbb{1}_{S^2}\beta_p = *$$

and

$$p_1^{Q_p}h_1^{Q_p}(\Omega i * \Omega i)(j * j)\alpha_p = *$$

as

$$\Omega Q_p * \Omega Q_p \xrightarrow{h_1^{Q_p}} \Sigma \Omega Q_p \xrightarrow{p_1^{Q_p}} Q_p$$

is a fibration sequence so that $p_1^{Q_p} h_1^{Q_p}$ is trivial. Thus, we have the identity

$$p_1^{Q_p} \Sigma \Omega i \Sigma j \beta_p = * = P_1^{Q_p} h_1^{Q_p} (\Omega i * \Omega i)(j * j) \alpha_p$$

as well.

Now that we have the equations

$$\begin{aligned} p_1^{Q_p} \Sigma \Omega i \Sigma j \beta_p &= p_1^{Q_p} h_1^{Q_p} (\Omega i * \Omega i)(j * j) \alpha_p, \\ \pi^{Q_p} \Sigma \Omega i \Sigma j \beta_p &= \pi^{Q_p} h_1^{Q_p} (\Omega i * \Omega i)(j * j) \alpha_p, \end{aligned}$$

let us consider the pullback diagram defining the space $G^1 Q_p$.

$$\begin{array}{ccc} G^1 Q_p & \xrightarrow{\pi^{Q_p}} & Q_p \vee Q_p \\ \downarrow p_1^{Q_p} & \lrcorner & \downarrow \\ Q_p & \xrightarrow{\Delta_2} & Q_p \times Q_p \end{array}$$

The above equations together say that in the extended diagram

$$\begin{array}{ccccc} S^{4p-3} & \xrightarrow{\alpha_p} & S^3 & \xlongequal{\quad} & S^1 * S^1 \\ \downarrow \beta_p & & & & \downarrow (\Omega i * \Omega i)(j * j) \\ & & \Omega Q_p * \Omega Q_p & \xlongequal{\quad} & \Omega Q_p * \Omega Q_p \\ & & \downarrow h_1^{Q_p} & & \downarrow q^{Q_p} \\ S^2 & \xrightarrow{\Sigma \Omega i \Sigma j} & G^1 Q_p & \xrightarrow{\pi^{Q_p}} & Q_p \vee Q_p \\ & & \downarrow p_1^{Q_p} & & \downarrow \\ & & Q_p & \xrightarrow{\Delta_2} & Q_p \times Q_p \end{array}$$

all paths in the diagram from S^{4p-3} to $Q_p \vee Q_p$ commute, as do all paths from S^{4p-3} to Q_p commute. That is, more poignantly, the diagram

$$\begin{array}{ccc}
 S^{4p-3} & \xrightarrow{q^{Q_p}(\Omega i * \Omega i)(j * j)\alpha_p} & Q_p \vee Q_p \\
 \searrow \sigma & \swarrow \pi^{Q_p} & \downarrow \\
 & G^1 Q_p & Q_p \vee Q_p \\
 \searrow \pi^{Q_p} \Sigma \Omega i \Sigma j \beta_p & \downarrow p_1^{Q_p} & \downarrow \\
 & Q_p & Q_p \times Q_p \\
 & \xrightarrow{\Delta_2} &
 \end{array}$$

commutes without the dashed arrow. Invoking the universal property of $G^1 Q_p$ as a pullback, then, there exists a map, σ , unique up to homotopy, which makes this diagram commute. However our equations suggest that both $h_1^{Q_p}(\Omega i * \Omega i)(j * j)\alpha_p$ and $\Sigma \Omega i \Sigma j \beta_p$ are suitable choices for σ . Thus, we must have by uniqueness that

$$\Sigma \Omega i \Sigma j \beta_p = h_1^{Q_p}(\Omega i * \Omega i)(j * j)\alpha_p$$

whereby we obtain the commutativity of the original diagram, which we show again here for the reader's convenience.

$$\begin{array}{ccccc}
 S^{4p-3} & \xrightarrow{\beta_p} & S^2 & \xrightarrow{i} & Q_p \\
 \downarrow \alpha_p & & \downarrow \Sigma \Omega i \Sigma j & & \parallel \\
 S^1 * S^1 & & & & \\
 \downarrow (\Omega i * \Omega i)(j * j) & & & & \\
 \Omega Q_p * \Omega Q_p & \xrightarrow{h_1^{Q_p}} & \Sigma \Omega Q_p & \xrightarrow{p_1^{Q_p}} & Q_p \\
 & & \searrow i_1^{Q_p} & \nearrow e_2^{Q_p} & \\
 & & G^2 Q_p & &
 \end{array}$$

□

Remark

Here we have again deviated from Iwase's prescription. Iwase used the fact that q^{Q_p} induces a split monomorphism in homotopy to show that $\Sigma \Omega i \Sigma j \beta_p = h_1^{Q_p}(\Omega i * \Omega i)(j * j)\alpha_p$. However, we are aware of no result which gives this implication, and so elected to find an alternate argument to show commutativity.

Now, let $H : S^{4p-3} \times I \rightarrow G^1Q_p$ be the homotopy deforming $\Sigma\Omega i\Sigma j\beta_p$ to $h_1^{Q_p}(\Omega i * \Omega i)(j * j)\alpha_p = h_1^{Q_p}|_{S^1 * S^1}\alpha_p$, and $\tilde{\chi} : (C(\Omega Q_p * \Omega_p), \Omega Q_p * \Omega Q_p) \rightarrow (G^2Q_p, G^1Q_p)$ the characteristic map of the mapping cylinder $G^2Q_p = G^1Q_p \cup_{h_1^{Q_p}} C(\Omega Q_p * \Omega Q_p)$. Then $\tilde{\chi}|_{C(S^1 * S^1)}C(\alpha_p) : D^{4p-2} \rightarrow D^4 \rightarrow G^2Q_p$ in combination with $h_1^{Q_p}|_{S^1 * S^1}\alpha_p$ defines for us a map $\lambda : Q_p \rightarrow G^2Q_p$.

Now, let $n \geq 2$. Note that (Q_p, S^2) and $(S^n, *)$ are relative CW complexes, so by Theorem 0.4, so is $(Q_p \times S^n, Q_p \times * \cup S^2 \times S^n)$, and moreover $Q_p \times S^n$ can be obtained by attaching the $(4p - 2 + n)$ -cell $C(S^{4p-3} * S^{n-1})$ to $Q_p \times * \cup S^2 \times S^n$ via the map $\varphi : S^{4p-3} * S^{n-1} = D^{4p-2} \times S^{n-1} \cup S^{4p-3} \times D^n \rightarrow Q_p \times * \cup S^2 \times S^n$ defined by the equations

$$\begin{aligned}\varphi|_{D^{4p-2} \times S^{n-1}} &= \chi \times *, \\ \varphi|_{S^{4p-3} \times D^n} &= \beta \times \chi_n\end{aligned}$$

where χ is the characteristic map $\chi : (D^{4p-2}, S^{4p-3}) \rightarrow (Q_p, S^2)$ of the top cell of Q_p , and $\chi_n : (D^n, S^{n-1}) \rightarrow (S^n, *)$ is the relative homeomorphism. Similarly, (G^2Q_p, G^1Q_p) is a relative CW complex and so $G^2Q_p \times S^n$ can be constructed as the mapping cylinder of the map

$$\psi : (\Omega Q_p * \Omega Q_p) * S^{n-1} = C(\Omega Q_p * \Omega Q_p) \times S^{n-1} \cup (\Omega Q_p * \Omega Q_p) \times D^n \rightarrow G^2Q_p \times * \cup G^1Q_p \times S^n$$

which is similarly defined by the equations

$$\begin{aligned}\psi|_{C(\Omega Q_p * \Omega Q_p) * S^{n-1}} &= \hat{\chi} \times *, \\ \psi|_{(\Omega Q_p * \Omega Q_p) \times D^n} &= h_1^{Q_p} \times \chi_n\end{aligned}$$

where, this time, $\hat{\chi}$ is the characteristic map $\hat{\chi} : (C(\Omega Q_p * \Omega Q_p), \Omega Q_p * \Omega Q_p) \rightarrow (G^2Q_p, G^1Q_p)$.

Then we define a new map $Q_p \times S^n \rightarrow G^2Q_p \times S^n$ by

$$\lambda \times * \cup (\Sigma\Omega i\Sigma j) \times \mathbb{1}_{S^n}.$$

This, with the map φ defined above, tells us that the following diagram, without the dashed arrow, commutes.

$$\begin{array}{ccccc} S^{4p-3} * S^{n-1} & \xrightarrow{\varphi} & Q_p \times * \cup S^2 \times S^n & \hookrightarrow & Q_p \times S^n \\ \downarrow \alpha_p * \mathbb{1}_{S^{n-1}} & & \downarrow \lambda \times * \cup (\Sigma\Omega i\Sigma j) \times \mathbb{1}_{S^n} & & \downarrow \lambda \times \mathbb{1}_{S^n} \\ (S^1 * S^1) * S^{n-1} & & & & \\ \downarrow (\Omega i * \Omega i)(j * j) * \mathbb{1}_{S^{n-1}} & & \downarrow & & \\ (\Omega Q_p * \Omega Q_p) * S^{n-1} & \xrightarrow{\psi} & G^2Q_p \times * \cup G^1Q_p \times S^n & \hookrightarrow & G^2Q_p \times S^n \xrightarrow{e_2^{Q_p} \times \mathbb{1}_{S^n}} Q_p \times S^n \end{array}$$

Now we will need to make use of the following result from [Cor95, pp. 178-179].

Lemma 8.3 Let $S^n \xrightarrow{\beta} X \xrightarrow{\rho} Y = X \cup_{\beta} e^{n+1}$ be a cofibration and let $p : Z \rightarrow Y$ be a map such that $\pi_{r+1}(p)$ is surjective. Let $\varphi : X \rightarrow Z$ be a map such that $\varphi \circ \alpha = 0$, and the following diagram, without the dashed arrow, commutes.

$$\begin{array}{ccccc}
 S^n & \xrightarrow{\beta} & X & \xrightarrow{\rho} & Y \\
 & & \searrow \varphi & & \uparrow p \\
 & & & & Z \\
 & & \searrow 0 & & \uparrow \sigma
 \end{array}$$

Then there exists a map $\sigma : Y \rightarrow Z$, represented by the dashed arrow, such that $\sigma \circ \rho = \varphi$ and $p \circ \sigma = \mathbb{1}_Y$.

Now, $\alpha_p * \mathbb{1}_{S^{n-1}} = \pm \Sigma(\alpha_p \wedge \mathbb{1}_{S^{n-1}}) = \pm \Sigma(\Sigma^{n-1} \alpha_p) = \pm \Sigma^n \alpha_p$. Thus, for $n \geq 2$, $\alpha_p * \mathbb{1}_{S^{n-1}} = \pm \Sigma^n \alpha_p = 0$. Now we consider a subdiagram of the above.

$$\begin{array}{ccccc}
 S^{4p-3} * S^{n-1} & \xrightarrow{\varphi} & Q_p \times * \cup S^2 \times S^n & \xrightarrow{\quad} & Q_p \times S^n \\
 & & \searrow \lambda \times * \cup (\Sigma \Omega i \Sigma j) \times \mathbb{1}_{S^n} & & \uparrow e_2^{Q_p} \times \mathbb{1}_{S^n} \\
 & & & & G^2 Q_p \times S^n \\
 & & & & \uparrow k \\
 & & & & G^2 Q_p \times * \cup G^1 Q_p \times S^n
 \end{array}$$

By the commutativity of the larger diagram as well as the fact that $\alpha_p * \mathbb{1}_{S^{n-1}} = 0$, we indeed have that $(\lambda \times * \cup (\Sigma \Omega i \Sigma j) \times \mathbb{1}_{S^n}) \varphi = \psi(((\Omega i * \Omega i)(j * j)) * \mathbb{1}_{S^{n-1}})(\alpha_p * \mathbb{1}_{S^{n-1}}) = 0$. Moreover, $S^{4p-3} * S^{n-1} = S^{4p-3+n}$, so we rewrite our diagram as follows.

$$\begin{array}{ccccc}
 S^{4p-3+n} & \xrightarrow{\varphi} & Q_p \times * \cup S^2 \times S^n & \xrightarrow{\quad} & Q_p \times S^n \\
 & & \searrow \lambda \times * \cup (\Sigma \Omega i \Sigma j) \times \mathbb{1}_{S^n} & & \uparrow e_2^{Q_p} \times \mathbb{1}_{S^n} \\
 & & & & G^2 Q_p \times S^n \\
 & & \searrow 0 & & \uparrow k \\
 & & & & G^2 Q_p \times * \cup G^1 Q_p \times S^n
 \end{array}$$

Now, consider that the map

$$Q_p \times * \rightarrow G_2 Q_p \times * \cup G^1 Q_p \times S^n \hookrightarrow G_2 Q_p \times S^n \hookrightarrow Q_p \times S^n$$

induces a surjection on $\pi_*(Q_p)$, while the map

$$S^2 \times S^n \rightarrow G_2 Q_p \times * \cup G^1 Q_p \times S^n \hookrightarrow G_2 Q_p \times S^n \hookrightarrow Q_p \times S^n$$

induces a surjection on $\pi_*(S^n)$. Thus, the inclusion $k : G^2Q_p \times * \cup G^1Q_p \times S^n \hookrightarrow G^2Q_p \times S^n$ induces a surjection in homotopy, and we apply Lemma 8.3 to obtain a map $\sigma : Q_p \times S^n \rightarrow G^2Q_p \times * \cup G^1Q_p \times S^n$ such that the diagram

$$\begin{array}{ccccc}
 S^{4p-3+n} & \xrightarrow{\varphi} & Q_p \times * \cup S^2 \times S^n & \xrightarrow{\quad} & Q_p \times S^n \\
 & & \searrow^{\lambda \times * \cup (\Sigma\Omega_i \Sigma_j) \times \mathbb{1}_{S^n}} & & \uparrow^{e_2^{Q_p} \times \mathbb{1}_{S^n}} \\
 & & & & G^2Q_p \times S^n \\
 & & & & \uparrow^k \\
 & & & & G^2Q_p \times * \cup G^1Q_p \times S^n \\
 & \searrow^0 & & & \swarrow^{\sigma}
 \end{array}$$

commutes and $(e_2^{Q_p} \times \mathbb{1}_{S^n})k \circ \sigma = \mathbb{1}_{Q_p \times S^n}$. Thus, $G^2Q_p \times * \cup G^1Q_p \times S^n$ dominates $Q_p \times S^n$. Since $\text{cat}(G^2Q_p \times *) \leq \text{cat} G^2Q_p + \text{cat} * = 2$, $\text{cat}(G^1Q_p \times S^n) \leq \text{cat} G^1Q_p + \text{cat} S^n = 2$, $* \subset S^n$, and $G^1Q_p \subset G^2Q_p$,

$$2 = \text{cat} Q_p \leq \text{cat}(Q_p \times S^n) \leq \text{cat}(G^2Q_p \times * \cup G^1Q_p \times S^n) \leq 2$$

proving

Theorem

For all $n \geq 2$, and for all odd primes p , $\text{cat}(Q_p \times S^n) = \text{cat} Q_p = 2$.

8.1 Rational Category of $Q_p \times S^n$

A good question to ask at this point is where exactly this example fails in the rational case. This turns out to be a fairly simple exercise. Note that $\pi_9(S_0^3) = \pi_9(S^3) \otimes \mathbb{Q} = \mathbb{Z}_3 \otimes \mathbb{Q} = 0$. Thus, the attaching map β_p is rationally trivial, and if we were to repeat this process rationally, we would obtain

$$Q_p = S^2 \cup_* e^{4p-2} = S^2 \vee S^{4p-2}$$

so that $\text{cat}_0(Q_p) = 1$, while $\text{cat}_0(Q_p \times S^n) = 2 = \text{cat}_0(Q_p) + 1$, in compliance with Ganea's conjecture for rational spaces.

SECTION 9

TABLE OF HOMOTOPY GROUPS OF SPHERES

	S^1	S^2	S^3	S^4	S^5	S^6	S^7	S^8
π_1	\mathbb{Z}	0	0	0	0	0	0	0
π_2	0	\mathbb{Z}	0	0	0	0	0	0
π_3	0	\mathbb{Z}	\mathbb{Z}	0	0	0	0	0
π_4	0	\mathbb{Z}_2	\mathbb{Z}_2	\mathbb{Z}	0	0	0	0
π_5	0	\mathbb{Z}_2	\mathbb{Z}_2	\mathbb{Z}_2	\mathbb{Z}	0	0	0
π_6	0	\mathbb{Z}_{12}	\mathbb{Z}_{12}	\mathbb{Z}_2	\mathbb{Z}_2	\mathbb{Z}	0	0
π_7	0	\mathbb{Z}_2	\mathbb{Z}_2	$\mathbb{Z} \times \mathbb{Z}_{12}$	\mathbb{Z}_2	\mathbb{Z}_2	\mathbb{Z}	0
π_8	0	\mathbb{Z}_2	\mathbb{Z}_2	\mathbb{Z}_2^2	\mathbb{Z}_{24}	\mathbb{Z}_2	\mathbb{Z}_2	\mathbb{Z}
π_9	0	\mathbb{Z}_3	\mathbb{Z}_3	\mathbb{Z}_2^2	\mathbb{Z}_2	\mathbb{Z}_{24}	\mathbb{Z}_2	\mathbb{Z}_2
π_{10}	0	\mathbb{Z}_{15}	\mathbb{Z}_{15}	$\mathbb{Z}_{24} \times \mathbb{Z}_3$	\mathbb{Z}_2	0	\mathbb{Z}_{24}	\mathbb{Z}_2
π_{11}	0	\mathbb{Z}_2	\mathbb{Z}_2	\mathbb{Z}_{15}	\mathbb{Z}_2	\mathbb{Z}	0	\mathbb{Z}_{24}
π_{12}	0	\mathbb{Z}_2^2	\mathbb{Z}_2^2	\mathbb{Z}_2	\mathbb{Z}_{30}	\mathbb{Z}_2	0	0
π_{13}	0	$\mathbb{Z}_{12} \times \mathbb{Z}_2$	$\mathbb{Z}_{12} \times \mathbb{Z}_2$	\mathbb{Z}_2^3	\mathbb{Z}_2	\mathbb{Z}_{60}	\mathbb{Z}_2	0
π_{14}	0	$\mathbb{Z}_{84} \times \mathbb{Z}_2^2$	$\mathbb{Z}_{84} \times \mathbb{Z}_2^2$	$\mathbb{Z}_{120} \times \mathbb{Z}_{12} \times \mathbb{Z}_2$	\mathbb{Z}_2^3	$\mathbb{Z}_{24} \times \mathbb{Z}_2$	\mathbb{Z}_{120}	\mathbb{Z}_2
π_{15}	0	\mathbb{Z}_2^2	\mathbb{Z}_2^2	$\mathbb{Z}_{84} \times \mathbb{Z}_2^5$	$\mathbb{Z}_{72} \times \mathbb{Z}_2$	\mathbb{Z}_2^3	\mathbb{Z}_2^3	$\mathbb{Z} \times \mathbb{Z}_{120}$

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