LIMIT ORBITS IN REAL REDUCTIVE LIE ALGEBRAS

by

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ABSTRACT

Let G *be a real reductive group,* X *a semisimple element of the Lie algebra* $\mathfrak{g}$ *of* G *. We define the* limit set

$$\lim_{t \to 0^+} t\mathrm{G} \cdot X := \{Y \mid Y = \lim_{i \to \infty} t_i \cdot Ad(g_i) \cdot X, \ g_i \in \mathrm{G}, \ t_i \to 0^+\}$$

The problem considered in this thesis is to determine when this set is the closure of a single G *- orbit (necessarily nilpotent).*

For complex groups this problem has been solved by W. Borho and H. Kraft [2] *. For real groups, a conjecture of D. Barbasch and D. Vogan* [1] *states that* $\lim_{t \to \infty} t\mathrm{G} \cdot X$ *is the closure of exactly one real nilpotent orbit where* X *is an* elliptic *semisimple element.*

In this thesis, we do not solve the above problem in general, but we prove a result which provides a reduction to certain Levi subalgebras. This reduction leads to a complete solution in some special cases. In particular, for complex groups we recover a result of W. Borho and H. Kraft.

CHAPTER I. PARABOLIC SUBALGEBRAS AND LIMIT ORBITS IN REAL REDUCTIVE LIE ALGEBRAS

§ 1.1 *PRELIMINARIES AND DEFINITIONS*

First of all, we give some well- known definitions and notations [22] . Let A be a linear transformation of a real vector space V. We say that A is

a) ***semisimple*** if A is diagonalizable over C (i.e. on the complexification V_C) .

b) ***elliptic*** if A is diagonalizable over C with imaginary eigenvalues.

c) ***hyperbolic*** if A is diagonalizable over C with real eigenvalues.

d) ***nilpotent*** if A has only the zero as eigenvalues (i.e. $A^k = 0$ for some positive integer k) .

Any semisimple A can be uniquely written in the form $A = A_e + A_h$ with A_e elliptic, A_h hyperbolic, and

$$A_e \cdot A_h = A_h \cdot A_e .$$

Let $\mathrm{GL}(n, \mathrm{C})$ denote the group of all invertible elements of $n \times n$ matrices $\mathrm{M}_n(\mathrm{C})$ over C, $f_1, \dots, f_m$ be complex polynomials on $\mathrm{M}_n(\mathrm{C})$ such that each f_j is real valued on $\mathrm{M}_n(\mathrm{R})$ and such that the set of simultaneous zeros of the f_j in $\mathrm{GL}(n, \mathrm{C})$ is a subgroup G_C of $\mathrm{GL}(n, \mathrm{C})$. Then G_C is called an ***affine algebraic group defined over*** R . The subgroup

$$\mathrm{G}_R = \mathrm{G}_C \cap \mathrm{GL}(n, \mathrm{R})$$

is called the group of ***real points.*** If in addition, $g^* \in \mathrm{G}_C$ for $g \in \mathrm{G}_C$ ($g^* = \bar{g}^t$) , then G_C is called a ***symmetric subgroup*** of $\mathrm{GL}(n, \mathrm{C})$. We define an automorphism θ of G_R by

$$\theta(g) = (g^{-1})^* .$$

Let G_C be a symmetric subgroup of $\mathrm{GL}(n, \mathrm{C})$ with real points G_R .

By ***real reductive group*** G we will mean a finite covering of an open subgroup G_0 of G_R . We will identify the Lie algebra $\mathbf{g}$ of G with that of G_R , thus we can define on $\mathbf{g}$ an involutive automorphism θ given by

$$\theta(Y) = -Y^*,$$

This automorphism is usually called a ***Cartan involution.***

Let G be a real reductive group, g be its Lie algebra, a real reductive Lie algebra .

An element $X \in \mathbf{g}$ is called ***semisimple*** (***elliptic, hyperbolic, nilpotent,*** etc) if $ad(X)$ is.

Any semisimple element $X \in \mathbf{g}$ can be written as $X = X_e + X_h$, with $X_e \in \mathbf{g}$ elliptic, $X_h \in \mathbf{g}$ hyperbolic, $[X_e, X_h] = 0$. This decomposition is unique modulo the centre of g .

Lemma 1.1 *Let* g *be a Lie algebra over* C (*or* R) , n *a nilpotent subalgebra, and* $X \in \mathbf{g}$ *an element so that*

$$ad(X)\cdot \mathbf{n} \subseteq \mathbf{n}$$

and

$$ad(X) : \mathbf{n} \rightarrow \mathbf{n}$$

is a semisimple, invertible linear transformation. Then the map

$$\phi : Y \rightarrow exp(ad(Y))\cdot X$$

is a bijection of n *onto*

$$exp(ad(\mathbf{n}))\cdot X = X + \mathbf{n} .$$

Proof. Let g be a linear Lie algebra over C (the real case follows from this) , n a nilpotent subalgebra. For any integer $k \geq 0$ let

$$\mathbf{n}^k = [\mathbf{n}, [\mathbf{n}, \cdots [\mathbf{n}, \mathbf{n}]]$$

be the ideal spanned by k–fold brackets. There is an integer $k \geq 1$, such that $\mathbf{n}^k = 0$, $\mathbf{n}^{k-1} \neq 0$. Since

$$ad(X) : \mathbf{n} \rightarrow \mathbf{n}$$

is a semisimple, invertible linear transformation, n can be written as a direct sum of its eigenspaces under $ad(X)$. So we can chose an element $Z \in \mathbf{n}^{k-1}$, i.e. Z is in the centre of n , and Z is one of the eigenvectors of $ad(X)$, $Z \neq 0$, such that

$$ad(X)\cdot Z \neq 0, \quad ad(X)\cdot Z \in \mathbf{C}Z .$$

Consider the nilpotent algebra $\mathbf{n}/\mathbf{C}Z$. The induced map

$$ad(X): \ \mathfrak{n}/\mathbf{C}Z \to \mathfrak{n}/\mathbf{C}Z$$

is also a semisimple linear transformation. We claim that this transformation is also invertible. If there is an element

$$\overline{Y} = Y + \mathbf{C}Z \in \mathfrak{n}/\mathbf{C}Z$$

such that

$$ad(X)\cdot\overline{Y} = \overline{0}, \quad \text{i.e.} \quad ad(X)\cdot Y \in \mathbf{C}Z,$$

then $Y \in \mathbf{C}Z$ since $ad(X)$ is invertible on $\mathfrak{n}$ and

$$ad(X)\cdot Z \in \mathbf{C}Z, \quad ad(X)\cdot Z \neq 0.$$

So $\overline{Y} = \overline{0}$ and hence $ad(X)$ is also invertible on $\mathfrak{n}/\mathbf{C}Z$.

By using induction on $dim\ \mathfrak{n}$, we have that

$$\overline{Y} \to exp(ad(\overline{Y}))\cdot X$$

is a bijection of $\mathfrak{n}/\mathbf{C}Z$ onto $exp(ad(\mathfrak{n}/\mathbf{C}Z))\cdot X$.

By induction on $dim\ \mathfrak{n}$, we can choose an element Z as above in $\mathfrak{n}$, so that assertion holds for $\mathfrak{n}/\mathbf{C}Z$. Then

$$exp(ad(\mathfrak{n}/\mathbf{C}Z))\cdot X = X + \mathfrak{n}/\mathbf{C}Z.$$

Since

$$ad(X)\cdot Z \neq 0, \quad ad(X)\cdot Z \in \mathbf{C}Z,$$

and

$$exp(ad(\mathbf{C}Z))\cdot X = X + \mathbf{C}Z$$

we have

$$exp(ad(\mathfrak{n}))\cdot X = X + \mathfrak{n}.$$

Thus we have a bijection of $\mathfrak{n}$ onto

$$exp(ad(\mathfrak{n}))\cdot X = X + \mathfrak{n}. \qquad \square$$

Since

$$Ad(exp(\mathfrak{n}))\cdot X = exp(ad(\mathfrak{n}))\cdot X$$

we can write

$$Ad(exp(\mathfrak{n}))\cdot X = X + \mathfrak{n}.$$

We need a well-known result due to Jacobson-Morozov:

Theorem 1.2 (Jacobson-Morozov) *Let* g *be a semisimple Lie algebra and* X *be a nonzero nilpotent element of* g *. Then there exist elements* $H, Y \in \mathrm{g}$ *with*

$$[H, X] = 2X\ ,\ [H, Y] = -2Y\ ,\ [X, Y] = \mathrm{H}\ . \qquad \square$$

In fact, this theorem holds over arbitrary basefields of characteristic 0, and under some restrictions in characteristic p as well. In particular, it holds for real Lie algebras. The proof can be found in [7] (page 37) .

§ 1.2 *LIMIT ORBITS IN REAL REDUCTIVE LIE ALGEBRAS*

We shall use the notation

$$g \cdot X := \mathrm{Ad}(g)X$$

for the adjoint action of $g \in \mathrm{G}$ on $X \in \mathrm{g}$. Let $\mathcal{N}$ denote the set of all nilpotent elements of g and let $\mathrm{R}_+^\times$ denote the positive real numbers. For any $X \in \mathrm{g}$, we define

$$\lim_{t \to 0^+}(t\mathrm{G} \cdot X) := \{Y \in \mathrm{g}\ \mid\ Y = \lim_{k \to \infty} t_k g_k \cdot X\ ,\ \text{some } g_k \in \mathrm{G}\ ,\ t_k > 0 \text{ with } t_k \to 0\}\ .$$

The next lemma will show that the elements in the set $\lim_{t \to 0^+}(t\mathrm{G} \cdot X)$ are nilpotent elements contained in the set $\mathcal{N}$.

Lemma 1.3 *Let* G *be a real reductive group,* g *be the Lie algebra of* G *. For any element* $X \in \mathrm{g}$,

$$\lim_{t \to 0^+}(t\mathrm{G} \cdot X) = \overline{(\mathrm{R}_+^\times \mathrm{G} \cdot X)} \bigcap \mathcal{N}\ .$$

Proof. From definition we know that

$$\lim_{t \to 0^+}(t\mathrm{G} \cdot X)\ \subseteq\ \overline{(\mathrm{R}_+^\times \mathrm{G} \cdot X)}\ .$$

We show that every element in $\lim_{t \to 0^+}(t\mathrm{G} \cdot X)$ is nilpotent. Let $f(\lambda)$ be the characteristic polynomial of X . Then

$$\begin{aligned} f(\lambda) &= det(\lambda I - X) \\ &= \prod_{j=1}^{n} (\lambda - \xi_j) \\ &= \sum_{j=1}^{n} a_j(\xi_1, ..., \xi_n) \cdot \lambda^j \end{aligned}$$

where $\xi_1, ..., \xi_n$ are the eigenvalues of X, and $a_j(\xi_1, ..., \xi_n)$, $j = 1, 2, ..., n$, are the coefficients. Then the characteristic polynomial of tX is

$$\begin{aligned} g(\lambda) &= det(\lambda I - tX) \\ &= \prod_{j=1}^{n} (\lambda - t\xi_j) \\ &= \sum_{j=1}^{n} a_j(t\xi_1, ..., t\xi_n) \cdot \lambda^j \\ &= \sum_{j=1}^{n} t^{n-j} a_j(\xi_1, ..., \xi_n) \cdot \lambda^j . \end{aligned}$$

By the *Cayley-Hamilton* theorem, we have

$$\begin{aligned} 0 &= g(tX) \\ &= \sum_{j=1}^{n} t^{n-j} a_j(\xi_1, ..., \xi_n) \cdot (tX)^j \end{aligned}$$

so we have

$$\begin{aligned} 0 &= ad(g_k) \cdot 0 = ad(g_k) \cdot g(tX) \\ &= g_k \cdot \Big(\sum_{j=1}^{n} t^{n-j} a_j(\xi_1, ..., \xi_n) \cdot (tX)^j \Big) \cdot g_k^{-1} \\ &= \sum_{j=1}^{n} t^{n-j} a_j(\xi_1, ..., \xi_n) \cdot (t\,ad(g_k) \cdot X)^j, \end{aligned}$$

for any sequence $\{g_k\} \subset \mathbf{G}$.

Now let $t = t_k$ and $t_k \to 0$ where $k \to \infty$. For any

$$Y = \lim_{k \to \infty} (t_k g_k \cdot X) \in \lim_{t \to 0^+} (t\mathrm{G} \cdot X),$$

we have

$$\begin{aligned} 0 &= \lim_{k \to \infty} g(t_k X) \\ &= \lim_{k \to \infty} \Big(\sum_{j=1}^{n} t_k^{n-j} a_j(\xi_1, ..., \xi_n) \cdot (t_k g_k \cdot X)^j \Big) \\ &= \lim_{k \to \infty} \Big(\sum_{j=1}^{n-1} t_k^{n-j} a_j(\xi_1, ..., \xi_n) \cdot (t_k g_k \cdot X)^j + (t_k g_k \cdot X)^n \Big) \end{aligned}$$

since $\lim_{k \to \infty} \sum_{j=1}^{n-1} t_k^{n-j} a_j(\xi_1, ..., \xi_n) = 0$ and $\lim_{k \to \infty} (t_k g_k \cdot X)^j = Y^j$, we have

$$0 = \lim_{k \to \infty} g(t_k X)$$

$$= \lim_{k\to\infty}\Big(\sum_{j=1}^{n-1} t_k^{n-j} a_j(\xi_1, \ldots, \xi_n)\cdot(t_k g_k\cdot X)^j + (t_k g_k\cdot X)^n\Big)$$

$$= \lim_{k\to\infty}(t_k g_k\cdot X)^n$$

$$= Y^n$$

So we have $Y^n = 0$, i.e. $Y\in \mathcal{N}$. Thus

$$\lim_{t\to 0^+}(t\mathrm{G}\cdot X) \subseteq \mathcal{N}$$

and we have

$$\lim_{t\to 0^+}(t\mathrm{G}\cdot X) \subseteq (\overline{\mathrm{R}_+^\times \mathrm{G}\cdot X}) \cap \mathcal{N} .$$

We have to prove the other inclusion.

If $X\in \mathcal{N}$, then obviously

$$(\overline{\mathrm{R}_+^\times \mathrm{G}\cdot X}) \subset \mathcal{N}$$

and so

$$(\overline{\mathrm{R}_+^\times \mathrm{G}\cdot X}) \cap \mathcal{N} = \overline{\mathrm{R}_+^\times \mathrm{G}\cdot X} .$$

By Theorem 1.2, there are H and Y in $\mathfrak{g}$ such that $\{X, H, Y\}$ is an sl_2-*triple*. Using the method in [16], we define elements $g_k = exp(kH)$ in G and $t_k = exp(-2k)$ in R. We have

$$\lim_{k\to\infty} t_k g_k\cdot X = \lim_{k\to\infty} exp(-2k)\cdot\{Ad(exp(kH)\cdot X)\}$$

$$= \lim_{k\to\infty} exp(-2k)\cdot\{exp(ad(kH))\cdot X)\}$$

$$= \lim_{k\to\infty} exp(-2k)\cdot\{exp(2k)\cdot X)\}$$

$$= X .$$

So $X \in \lim_{t\to 0^+}(t\mathrm{G}\cdot X)$ and

$$\lim_{t\to 0^+}(t\mathrm{G}\cdot X) \supseteq \mathrm{R}_+^\times \mathrm{G}\cdot X .$$

Since the set $\lim_{t\to 0^+}(t\mathrm{G}\cdot X)$ is closed, we have

$$\lim_{t\to 0^+}(t\mathrm{G}\cdot X) \supseteq \overline{\mathrm{R}_+^\times \mathrm{G}\cdot X}$$

$$= (\overline{\mathrm{R}_+^\times \mathrm{G}\cdot X}) \cap \mathcal{N} .$$

If $X \notin \mathcal{N}$, then some eigenvalues of X are not zero, that is $\xi_j \neq 0$ for some j. Let

$$Y \in \overline{\mathbb{R}_+^\times G \cdot X} .$$

By definition

$$Y = \lim_{k \to \infty} t_k g_k \cdot X , \quad g_k \in G , \quad t_k > 0 .$$

If $\lim_{k \to \infty} t_k \neq 0$, we can pass to a subsequence and assume $t_k \to t_0$, $t_0 > 0$. Then

$$\begin{aligned} det(\lambda I - Y) &= \lim_{k \to \infty} det(\lambda I - t_k X) \\ &= \lim_{k \to \infty} \prod_{j=1}^{n} (\lambda - t_k \xi_j) \\ &= \prod_{j=1}^{n} (\lambda - t_0 \xi_j), \end{aligned}$$

for some j, $t_0 \xi_j \neq 0$. That is, $Y \notin \mathcal{N}$. Thus if

$$Y \in (\overline{\mathbb{R}_+^\times G \cdot X}) \bigcap \mathcal{N},$$

we must have

$$Y = \lim_{k \to \infty} t_k g_k \cdot X \text{ for } t_k > 0 \text{ and } t_k \to 0 ,$$

i.e.

$$Y \in \lim_{t \to 0^+} (tG \cdot X) .$$

So we get the result

$$\lim_{t \to 0^+} (tG \cdot X) \supseteq (\overline{\mathbb{R}_+^\times G \cdot X}) \bigcap \mathcal{N}$$

and

$$\lim_{t \to 0^+} (tG \cdot X) = (\overline{\mathbb{R}_+^\times G \cdot X}) \bigcap \mathcal{N} . \qquad \square$$

§ 1.3 *PARABOLIC SUBGROUPS AND PARABOLIC SUBALGEBRAS*

We need some further definitions and facts concerning reductive groups ([10], [22]). Let G be a real reductive group with Lie algebra $\mathfrak{g}$. A subalgebra of a complex reductive Lie algebra is ***parabolic*** if and only if it contains a ***Borel subalgebra*** (maximal solvable subalgebra).

A subalgebra $\mathfrak{p}$ of a real reductive Lie algebra is ***parabolic*** if and only if $\mathfrak{p}_C$ (the complexification of $\mathfrak{p}$) is parabolic in $\mathfrak{g}_C$. In fact, any such $\mathfrak{p}$ is conjugate to a unique "standard parabolic subalgebra" in the sense of [22]. Let

$$\mathrm{P} = \{g \in \mathrm{G} \mid Ad(g)\cdot \mathrm{p} \subseteq \mathrm{p}\},$$

Then P is a ***parabolic subgroup*** of G .

Any semisimple element $X \in \mathrm{g}$ is contained in a ***Cartan subalgebra*** h of g .

A ***nil ideal*** of a Lie algebra g is an ideal n such that $ad(Y)$ is nilpotent for all $Y \in \mathrm{n}$. We know that every Lie algebra contains a unique maximal nil ideal which is called the ***nilradical*** .

Let Δ be the set of simple roots of Φ_+ , where Φ_+ is a set of positive root of g , Δ' a subset of Δ . Define $\mathrm{p}(\Delta')$ to be the subalgebra of g generated by all

$$\mathrm{g}^{\mu} = \{Y \in \mathrm{g} \mid [H, Y] = \mu(H)\cdot Y \text{ for all } H \in \mathrm{a}\}, \quad \mu \in \Delta \text{ or } \mu \in \Delta'$$

along with h . Let $<\Delta'>$ denote the subroot system of Φ generated by Δ' and put

$$<\Delta'>^+ = <\Delta'> \bigcap \Phi_+ .$$

Then $\mathrm{p}(\Delta')$ (standard parabolic subalgebra) has the following properties [10] .

(a) $\mathrm{p}(\Delta')$ is a parabolic subalgebra of g containing a Borel subalgebra b .

(b) Every parabolic subalgebra of g containing b is the form $\mathrm{p}(\Delta')$, for some $\Delta' \subseteq \Delta$.

(c) Every parabolic subalgebra of g is conjugate to one of the form $\mathrm{p}(\Delta')$, for some $\Delta' \subseteq \Delta$.

(d) Let

$$\mathrm{l} = \mathrm{h} \oplus \sum_{\mu \in <\Delta'>} \mathrm{g}^{\mu},$$

$$\mathrm{n} = \sum_{\mu \in \Phi_+ \backslash <\Delta'>^+} \mathrm{g}^{\mu}.$$

The subalgebra l is called a ***Levi subalgebra*** of g and is reductive; n is the nilradical of $\mathrm{p}(\Delta')$. There is a direct sum decomposition $\mathrm{p}(\Delta') = \mathrm{l} \oplus \mathrm{n}$. Put Levi subgroup

$$\mathrm{L} = \{g \in \mathrm{G} \mid Ad(g)\cdot \mathrm{l} \subseteq \mathrm{l}\}$$

and N denote the ***connected subgroup*** of G with Lie algebra n .

We need another two lemmas to get the main result of this chapter. The proof of these lemmas can be found in [22] (page51) .

Lemma 1.4 *Let* $P = LN$ *be a parabolic subgroup of* G. *Then the map*

$$\phi : L \times N \to P, \qquad (l, n) \mapsto ln$$

is a surjective diffeomorphism. $\square$

Let G_0 be the connected subgroup of G, and let

$$K = \{ g \in G_0 \mid \theta(g) = g \}.$$

Then K is a compact subgroup of G and

$$K = G_0 \cap O(n)$$

where $O(n)$ is the orthogonal group of $GL(n, \mathbb{R})$.

Lemma 1.5 $G = KP$. $\square$

From these two lemmas, we get that

$$G = KLN.$$

§ 1.4 *LIMIT ORBITS IN LEVI SUBALGEBRAS*

Using previous lemmas and proposition we can get a result on limit orbits for a general real reductive group G by reduction to real reductive group L.

Theorem 1.6 *Let* X *be a semisimple element of* $\mathfrak{g}$. *Suppose* $\mathfrak{p} = \mathfrak{l} + \mathfrak{n}$ *is a parabolic subalgebra of* $\mathfrak{g}$ *so that* $X \in \mathfrak{l}$ *and*

$$ad(X) : \mathfrak{n} \to \mathfrak{n}$$

is invertible. Then

$$\lim_{t \to 0^+} (tG \cdot X) = K \cdot \{ \lim_{t \to 0^+} (tL \cdot X) + \mathfrak{n} \}.$$

Proof. From Lemma 1.1 and Lemma 1.3, we get that

$$\begin{aligned} K \cdot \{ \lim_{t \to 0^+} (tL \cdot X) + \mathfrak{n} \} &= K \cdot \{ (\overline{\mathbb{R}_+^\times L \cdot X} \cap \mathcal{N}) + \mathfrak{n} \} \\ &= K \cdot \overline{\{ \mathbb{R}_+^\times L \cdot X} + \mathfrak{n} \} \cap \mathcal{N} \end{aligned}$$

$$= \overline{\mathbf{K}\cdot\{\mathbf{R}_+^\times \mathbf{L}\cdot X + \mathfrak{n}\}} \cap \mathcal{N}$$

$$\subseteq \overline{\mathbf{R}_+^\times \mathbf{G}\cdot X} \cap \mathcal{N}\,.$$

Thus

$$\lim_{t\to 0^+}(t\mathbf{G}\cdot X) \supseteq \mathbf{K}\cdot\{\lim_{t\to 0^+}(t\mathbf{L}\cdot X) + \mathfrak{n}\}$$

We have $\mathbf{G} = \mathbf{KP} = \mathbf{KLN}$, by Lemma 1.4 and Lemma 1.5 Consider a sequence $\{g_j\}$ in $\mathbf{G}$, for every $g_j \in \mathbf{G}$ write $g_j = k_j l_j n_j$, where $k_j \in \mathbf{K}$, $l_j \in \mathbf{L}$, $n_j \in \mathbf{N}$. Using Lemma 1.1 and $\mathbf{L}\cdot\mathfrak{n} = \mathfrak{n}$, we have

$$\begin{aligned} g_j\cdot t_jX &= k_j\cdot l_j\cdot n_j\cdot t_jX \\ &= k_j\cdot l_j\cdot(t_jX + Y_j') \\ &= k_j\cdot(l_j\cdot t_jX + Y_j'') \end{aligned}$$

where Y_j', $Y_j'' \in \mathfrak{n}$, and $t_j \to 0$. Since $\mathbf{K}$ is compact and $\mathfrak{n}$ is closed, we can choose subsequences of $\{k_j\}$ which converges in $\mathbf{K}$. Then $l_j\cdot t_jX + Y_j''$ converges in $\mathfrak{l}\oplus\mathfrak{n}$, hence $l_j\cdot t_jX$ and Y_j'' converge in $\mathfrak{l}$ and $\mathfrak{n}$ respectively. Thus we have

$$\begin{aligned} \lim_{j\to\infty} g_j\cdot t_jX &= \lim_{j\to\infty} k_j\cdot(l_j\cdot t_jX + Y_j'') \\ &\in \mathbf{K}\cdot\{\lim_{j\to\infty}(l_j\cdot t_jX) + \mathfrak{n}\} \\ &\subseteq \mathbf{K}\cdot\{\lim_{j\to\infty}(\mathbf{L}\cdot t_jX) + \mathfrak{n}\}\,. \end{aligned}$$

Thus we get the other inclusion

$$\lim_{t\to 0^+}(t\mathbf{G}\cdot X) \subseteq \mathbf{K}\cdot\{\lim_{t\to 0^+}(t\mathbf{L}\cdot X) + \mathfrak{n}\}\,.$$

So

$$\lim_{t\to 0^+}(t\mathbf{G}\cdot X) = \mathbf{K}\cdot\{\lim_{t\to 0^+}(t\mathbf{L}\cdot X) + \mathfrak{n}\}\,. \qquad \square$$

If X is in the centre of $\mathfrak{l}$, that is

$$\mathfrak{l} \subseteq C_{\mathfrak{g}}(X) = \{Y \in \mathfrak{g} \mid [Y, X] = 0\}$$

we have the following simple result.

Corollary 1.7. *In the previous proposition, suppose X is contained in the centre of $\mathfrak{l}$. Then*

$$\lim_{t\to 0^+}(t\mathbf{G}\cdot X) = \mathbf{G}\cdot\mathfrak{n}\,.$$

Proof. Since X is contained in the centre of $\mathfrak{l}$, $\mathbf{L} \cdot X = X$. We know that $\mathbf{G} = \mathbf{KP}$ and $\mathbf{P} \cdot \mathfrak{n} = \mathfrak{n}$. So

$$\begin{aligned} \lim_{t \to 0^+} (t\mathbf{G} \cdot X) &= \mathbf{K} \cdot \{ \lim_{t \to 0^+} (t\mathbf{L} \cdot X) + \mathfrak{n} \} \\ &= \mathbf{K} \cdot \{ \lim_{t \to 0^+} (tX) + \mathfrak{n} \} \\ &= \mathbf{K} \cdot \mathfrak{n} \\ &= \mathbf{K} \cdot \mathbf{P} \cdot \mathfrak{n} \\ &= \mathbf{G} \cdot \mathfrak{n} . \end{aligned}$$ $\square$

In the next chapter we shall analyze this set $\mathbf{G} \cdot \mathfrak{n}$.

CHAPTER II. LIMIT ORBITS FOR DIAGONALIZABLE ELEMENTS

Under the conditions in the Corollary 1.7, the conjecture stated in the abstract says that for an (elliptic) semisimple element X of $\mathbf{g}$ the set

$$\lim_{t \to 0^+}(t\mathrm{G} \cdot X) = \mathrm{G} \cdot \mathbf{n}$$

is the closure of exactly one real nilpotent orbit. By the results of Theorem 1.6 and Corollary 1.7, the problem can be reduced from the set $\lim_{t \to 0^+}(t\mathrm{G} \cdot X)$ to the set $\mathrm{K} \cdot \{ \lim_{t \to 0^+}(t\mathrm{L} \cdot X) + \mathbf{n}\}$ and the set $\mathrm{G} \cdot \mathbf{n}$.

In this chapter, we will prove the reduction of limit sets for *diagonalizable* elements X (i.e. semisimple elements in the complex case, hyperbolic in the real case). In the complex case we further prove that there is exactly one nilpotent G-orbit in the Lie algebra $\mathbf{g}$ of G meeting $\lim_{t \to 0^+}(t\mathrm{G} \cdot X)$ in an open dense set. This result has been proved by W. Borho and H. Kraft [2] using algebraic geometry and invariant theory. Our proof is quite different.

In the last section of this chapter, we will give some results on limit orbits in $\mathrm{gl}(n, \mathrm{C})$.

§ 2.1 *LIMIT ORBITS AND PARABOLIC SUBALGEBRAS*

In this section $\mathbf{g}$ will denote a real *or* complex reductive Lie algebra. We recall the $X \in \mathbf{g}$ is *diagonalizable* if $\mathrm{ad}(X)$ is diagonalizable over the ground field $\mathbf{R}$ or $\mathbf{C}$ of $\mathbf{g}$. For a semisimple element X in a real Lie reductive algebra $\mathbf{g}$ which is contained in a Levi factor $\mathbf{l}$ of a parabolic subalgebra $\mathbf{p} = \mathbf{l} + \mathbf{n}$ we proved Theorem 1.6 under the condition that

$$ad(X): \mathbf{n} \to \mathbf{n}$$

is invertible and we proved the Corollary 1.7 under the condition that X is contained in the centre of $\mathbf{l}$. In the case of diagonalizable element X these conditions are satisfied by the next theorem.

Theorem 2.1 *Let* X *be any diagonalizable element of* $\mathbf{g}$. *Then there is a parabolic subalgebra* $\mathbf{p} = \mathbf{l} + \mathbf{n}$ *so that*

$$\mathbf{l} = C_{\mathbf{g}}(X) = \{Y \in \mathbf{g} \mid [Y, X] = 0\} .$$

For any such $\mathbf{p}$, *the linear transformation*

$$ad(X) : \mathbf{n} \to \mathbf{n}$$

is invertible. Furthermore

$$\lim_{t \to 0^+} (t\mathbf{G} \cdot X) = \mathbf{G} \cdot \mathbf{n} .$$

Proof. Let

$$\mathbf{g} = \sum_{\lambda \in \mathbf{C}} \mathbf{g}_\lambda$$

where

$$\mathbf{g}_\lambda = \{Y \in \mathbf{g} \mid ad(X) \cdot Y = \lambda \cdot Y\}$$

is the decomposition of $\mathbf{g}$ into eigenspaces under $ad(X)$. Let $\Lambda \in \mathbf{C}$ be the set of non-zero eigenvalues of $ad(X)$ and put

$$\Lambda_+ = \{\lambda \in \Lambda \mid \text{either } Re(\lambda) > 0 \text{ or else } Re(\lambda) = 0 \text{ and } Im(\lambda) > 0\} .$$

Then we have

$$\Lambda = \Lambda_+ \bigcup (-\Lambda_+) .$$

If λ , $\nu \in \Lambda_+$ and $\lambda + \nu \in \Lambda$ then $\lambda + \nu \in \Lambda_+$. Let

$$\mathbf{n} = \sum_{\lambda \in \Lambda_+} \mathbf{g}_\lambda ,$$

$$\mathbf{l} = \mathbf{g}_0 = C_{\mathbf{g}}(X) .$$

Then $\mathbf{p} = \mathbf{l} + \mathbf{n}$ is a parabolic subalgebra and by the definition of $\mathbf{n}$ and $\mathbf{l}$, the linear transformation

$$ad(X) : \mathbf{n} \to \mathbf{n}$$

is invertible.

Thus, for any diagonalizable element $X \in \mathbf{g}$, we can find a parabolic subalgebra $\mathbf{p} = \mathbf{l} + \mathbf{n}$ of $\mathbf{g}$ so that its Levi factor $\mathbf{l}$ is the centralizer of this semisimple element X . So the conditions in Proposition 1.6 and Corollary 1.7 are satisfied. Thus we have

$$\lim_{t \to 0^+} (t\mathbf{G} \cdot X) = \mathbf{G} \cdot \mathbf{n} . \qquad \square$$

Corollary 2.2. *If* X *is a hyperbolic element in a* real *semisimple Lie algebra, then there is a parabolic subalgebra* $\mathfrak{p} = \mathfrak{l} + \mathfrak{n}$ *of* $\mathfrak{g}$ *so that* $\mathfrak{l} = C_{\mathfrak{g}}(X)$. *For any such* $\mathfrak{p}$

$$\lim_{t \to 0^+} (t\mathrm{G} \cdot X) = \mathrm{G} \cdot \mathfrak{n} \, .$$

Proof. This is a restatement of the theorem in the real case. $\square$

We now turn to the complex case. The following result is well-known [7] , we briefly outline the proof, because of its importance here.

In the complex case the previous theorem leads to a complete solution of the problem mentioned in above. The proof is based on the following well-known result [7] , for which we give only an outline of the proof.

Theorem 2.3. *Let* G *be complex reductive group,* $\mathfrak{g}$ *its Lie algebra, and* $\mathfrak{p} = \mathfrak{l} + \mathfrak{n}$ *a parabolic subalgebra of* $\mathfrak{g}$. *Then there is a (necessarily unique and nilpotent)* G-*orbit* $\mathcal{O}$ *so that* $\mathcal{O} \bigcap \mathfrak{n}$ *is open and dense in* $\mathfrak{n}$. *Furthermore,*

$$G \cdot \mathfrak{n} = \overline{\mathcal{O}} \, .$$

Proof. We know that

$$\mathrm{G} \cdot \mathfrak{n} \subseteq \mathcal{N} \, .$$

Consequently $\mathrm{G} \cdot \mathfrak{n}$ is a finite union of orbits, at least one of which must have the same dimension as $\mathrm{G} \cdot \mathfrak{n}$ itself, and thus be open [7] . Since $\mathrm{G} \cdot \mathfrak{n}$ is irreducible as variety, it follows that $\mathrm{G} \cdot \mathfrak{n}$ has a unique orbit $\mathcal{O}$ which is also dense. Since any other orbit meeting $\mathrm{G} \cdot \mathfrak{n}$ lies in the closure of $\mathcal{O}$, this orbit is also the unique one of its dimension meeting $\mathrm{G} \cdot \mathfrak{n}$ [7] . So we get

$$\mathrm{G} \cdot \mathfrak{n} = \overline{\mathcal{O}} \, . \qquad \square$$

The $\mathcal{O}$ in the theorem above is called the nilpotent orbit of $\mathfrak{g}$ ***induced*** from $\mathfrak{p} = \mathfrak{l} + \mathfrak{n}$ (or from $\mathfrak{l}$) , also called *Richardson orbit.* It is sometimes denoted $Ind_P^G\{0\}$, where P is the group corresponding to $\mathfrak{p}$. More generally, for any nilpotent L-orbit O in l one can define [7] [14] similarly an induced orbit

$$\mathcal{O} = Ind\,_{\mathrm{P}}^{\mathrm{G}}\{O\}$$

by the condition that $\mathcal{O} \bigcap (O + \mathrm{n})$ is open and dense in $O + \mathrm{n}$. Then [7] [14]

$$\overline{\mathcal{O}} = \mathrm{G}\cdot(\overline{O} + \mathrm{n}) .$$

Using this theorem and Theorem 2.1 above we will obtain the result that $\lim_{t\to 0^+}(t\mathrm{G}\cdot X)$ is the closure of only one nilpotent orbit .

Theorem 2.4 *Let* G *be a complex reductive group,* g *its Lie algebra,* $X \in \mathrm{g}$ *any semisimple element. Then there is a parabolic subalgebra* $\mathrm{p} = \mathrm{l} + \mathrm{n}$ *with* $\mathrm{l} = C_{\mathrm{g}}(X)$ *and*

$$\lim_{t\to 0^+}(t\mathrm{G}\cdot X) = \mathrm{G}\cdot\mathrm{n}$$

is the closure of a single nilpotent orbit $\mathcal{O}$.

Proof. By the theorem 2.1 there is a parabolic subalgebra $\mathrm{p} = \mathrm{l} + \mathrm{n}$ with $\mathrm{l} = C_{\mathrm{g}}(X)$ and by theorem 1.6 and corollary 1.7, we have

$$\lim_{t\to 0^+}(t\mathrm{G}\cdot X) = \mathrm{G}\cdot\mathrm{n} .$$

By theorem 2.3 we have

$$\mathrm{G}\cdot\mathrm{n} = \overline{\mathcal{O}} .$$

So

$$\lim_{t\to 0^+}(t\mathrm{G}\cdot X) = \mathrm{G}\cdot\mathrm{n} = \overline{\mathcal{O}}$$

is the closure of a single nilpotent orbit. □

§ 2.2 *LIMIT ORBITS IN* gl(n , C) .

In this section we take $\mathrm{G} = \mathrm{GL}(n, \mathrm{C})$, $\mathrm{g} = \mathrm{gl}(n, \mathrm{C})$. We define the orbit and the parabolic subalgebra associated to a partition. Then we will give the relationship between the partition of the limit orbit in the complex Lie algebra gl(n, C) and the parabolic subalgebra mentioned in the last section.

Let X be a semisimple element of g . By Theorem 2.1, there is a parabolic subalgebra $\mathbf{p} = \mathbf{p}(X)$ with Levi decomposition $\mathbf{p} = \mathbf{l} + \mathbf{n}$ so that

$$\mathbf{l} = C_{\mathbf{g}}(X) = \{Y \in \mathbf{g} \mid [Y, X] = 0\} .$$

We call $\mathbf{p}$ a parabolic subalgebra *associated to* X . We give an explicit description of this subalgebra.

A ***partition*** of n is a finite sequence $[d_1, d_2, \dots, d_k]$ of positive integers so that

$$d_1 \geq d_2 \geq \dots \geq d_k > 0$$
$$d_1 + d_2 + \dots + d_k = n .$$

We denote the set of all partitions of n by $P(n)$. For $d = [d_1, d_2, \dots, d_k]$ in $P(n)$, let X_d be the nilpotent element of $\mathbf{g}$ in Jordan normal form:

$$X_d = diag[J_{d_1}, J_{d_2}, \dots, J_{d_k}]$$
$$= \begin{bmatrix} J_{d_1} & 0 & \cdots & 0 \\ 0 & J_{d_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & J_{d_k} \end{bmatrix}$$

where

$$J_{d_j} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix}$$

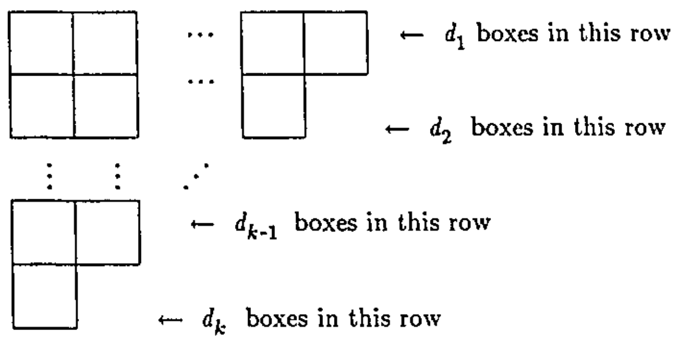
$$j = 1, 2, \dots, k$$

is the $d_j \times d_j$ ***Jordan block.*** Let

$$\mathcal{O}_d = \mathrm{G} \cdot X_d$$

be the nilpotent orbit of X_d. If $d \neq p$ then $\mathcal{O}_d$ and $\mathcal{O}_p$ are disjoint, by the uniqueness of the Jordan normal form.

The partition $d = [d_1, d_2, \dots, d_k]$ is associated to a ***Young diagram*** [8]



with d_i boxes in the ith row. Given a partition $d = [d_1, d_2, ..., d_k] \in P(n)$, define a new partition

$$d' = [d'_1, d'_2, ..., d'_l]$$

called the ***dual partition*** (***conjugate partition***) of d, by

$$d'_j = |\{ i \mid d_i \geq j \}|$$

and $l = d_1$. The dual partition is obtained by interchanging rows and columns in the Young diagram of partition d, i.e. reflecting the diagram in the 45°.

Given two different partitions

$$d = [d_1, d_2, ..., d_k], \text{ and } p = [p_1, p_2, ..., p_l]$$

of n, we say that $d \geq p$, if the following condition holds:

$$d_1 \geq p_1$$

$$d_1 + d_2 \geq p_1 + p_2$$

$$\cdots\cdots\cdots$$

$$d_1 + d_2 + \cdots + d_k \geq p_1 + p_2 + \cdots + p_l,$$

i.e.

$$\sum_{1 \leq j \leq k} d_j \geq \sum_{1 \leq j \leq k} p_j,$$

for $1 \leq k \leq n$. This is equivalent to

$$\sum_{1 \leq j \leq k} d'_j \leq \sum_{1 \leq j \leq k} p'_j,$$

for $1 \leq k \leq n$.

The element X is conjugate to a diagonal matrix with d_1 entries λ_1,

d_2 entries $\lambda_2, \cdots, d_k$ entries λ_k, where $\lambda_1, \lambda_2, \cdots, \lambda_k$ are the distinct eigenvalues, $d_1, d_2, \ldots, d_k$ are their multiplicities and we assume

$$d_1 \geq d_2 \geq \ldots \geq d_k > 0 .$$

We call $d = d(X) = [d_1, d_2, \ldots, d_k]$ the partition *corresponding* to X. Thus X is conjugate to

$$\begin{bmatrix} B_{d_1} & 0 & \cdots & 0 \\ 0 & B_{d_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & B_{d_k} \end{bmatrix}$$

where

$$B_{d_j} = \begin{bmatrix} \lambda_j & 0 & 0 & \cdots & 0 & 0 \\ 0 & \lambda_j & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_j & 0 \\ 0 & 0 & 0 & \cdots & 0 & \lambda_j \end{bmatrix}$$

$$j = 1, 2, \ldots, k$$

are $d_j \times d_j$ matrices.

The parabolic subalgebra $\mathfrak{p}(d)$ *associated* to $d = [d_1, d_2, \ldots, d_k]$, is defined as follows. Let $\{V_i\}$ $i = 1, 2, \ldots, k$ be the flag of subspaces of n dimensional space $V = \mathbb{C}^n$ with basis $\{e_1, e_2, \ldots, e_n\}$, put $V_0 = \{0\}$ and

$$V_i = \bigoplus_{s=1}^{m_i} \mathbb{C}\, e_s ,$$

where

$$m_i = \sum_{j=1}^{i} d_j$$

$i = 1, 2, \ldots, k$. Define

$$\mathfrak{p}(d) := \{Y \in \mathfrak{g} \mid Y \cdot (V_i) \subseteq V_i ,\ 1 \leq i \leq k\} .$$

We have

$$\mathfrak{p}(d) = \mathfrak{l}(d) + \mathfrak{n}(d)$$

where $\mathfrak{l}(d)$ and $\mathfrak{n}(d)$ are defined as follows.

$$\text{Elements of } \mathfrak{l}(d) : \begin{bmatrix} M_{d_1} & 0 & 0 & \cdots & 0 \\ 0 & M_{d_2} & 0 & \cdots & 0 \\ 0 & 0 & M_{d_3} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & M_{d_k} \end{bmatrix}$$

where M_{d_j} $j = 1, 2, \ldots, k$ are $d_j \times d_j$ matrices in $\mathrm{GL}(d_j, \mathrm{R})$.

$$\text{Elements of } \mathfrak{n}(d) : \begin{bmatrix} O_{d_1} & * & * & \cdots & * \\ 0 & O_{d_2} & * & \cdots & * \\ 0 & 0 & O_{d_3} & \cdots & * \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & O_{d_k} \end{bmatrix}$$

where O_{d_j} $j = 1, 2, \ldots, k$ are the $d_j \times d_j$ zero matrices in $\mathrm{GL}(d_j, \mathrm{R})$. Thus $\mathfrak{p}(d) = \mathfrak{l}(d) + \mathfrak{n}(d)$ consists of elements of the following form.

$$\text{Elements of } \mathfrak{p}(d) : \begin{bmatrix} M_{d_1} & * & * & \cdots & * \\ 0 & M_{d_2} & * & \cdots & * \\ 0 & 0 & M_{d_3} & \cdots & * \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & M_{d_k} \end{bmatrix}.$$

Before we give the main result in $\mathrm{GL}(n, \mathrm{C})$, we have to recall two theorems. Their proofs can be found in [7], [9].

Lemma 2.5 *Let $\mathcal{O}_d$ and $\mathcal{O}_f$ be the nilpotent orbits in $\mathfrak{gl}(n, \mathrm{C})$ associated to the partitions d and f, respectively, and let $X \in \mathcal{O}_d$, $Y \in \mathcal{O}_f$. Then $d \geq f$ if and only if*

$$rank(X^i) \geq rank(Y^i)$$

for all $i \geq 0$. $\square$

Theorem 2.6 *Let* $\mathfrak{g} = \mathfrak{gl}(n, \mathbb{C})$ *and let* d , f *be two partitions with associated nilpotent orbits* $\mathcal{O}_d$, $\mathcal{O}_f$ *in* $\mathfrak{g}$. *Then*

$$\mathcal{O}_d > \mathcal{O}_f \text{ i.e. } \overline{\mathcal{O}}_d \supset \overline{\mathcal{O}}_f$$

if and only if $d > f$. □

The following result is due to H. Kraft, H. Ozeki and M. Wakimoto [12] [15] . We include an outline of the proof.

Theorem 2.7 *Let* $G = GL(n, \mathbb{C})$, $\mathfrak{g} = \mathfrak{gl}(n, \mathbb{C})$. *Let* $X \in \mathfrak{g}$ *be any semisimple element,* $\mathfrak{p} = \mathfrak{l} + \mathfrak{n}$ *a parabolic subalgebra with* $\mathfrak{l} = C_{\mathfrak{g}}(X)$. *Let* $d = [d_1, \cdots, d_k]$ *be the partition* $\mathfrak{p}(d)$ *associated to* X . *Then*

$$\lim_{t \to 0^+} (tG \cdot X) = G \cdot \mathfrak{n} = \overline{\mathcal{O}}_{d'}$$

where d' *is the dual partition of* d .

Proof. Let $\mathfrak{p} = \mathfrak{p}(d)$, $\mathfrak{n} = \mathfrak{n}(d)$ and V_i be defined as above. By definition,

$$\mathfrak{n} = \{ Y \in \mathfrak{g} \mid Y \cdot (V_i) \subseteq V_{i-1},\ 1 \le i \le k \} .$$

So we get that for any element $Y \in \mathfrak{n}$,

$$Y^i \cdot (V_i) = V_0 = \{0\}.$$

Hence we have

$$\begin{aligned} rank(Y^i) &\le n - dim(V_i) \\ &= n - \sum_{j=1}^{i} d_j . \end{aligned}$$

We claim that there is an element $Y_0 \in \mathfrak{n}$ such that equality in the inequation above holds for all i. Let

$$W_i = \bigoplus_{j=s_{i-1}+1}^{s_i} \mathbb{C} e_j \text{ where } s_i = \sum_{k=1}^{i} d_k \text{ if } j > 0 \text{ and } s_0 = 0 .$$

Then V is the direct sum of the W_i and we have $dim(W_i) = d_i \ge d_{i+1} = dim(W_{i+1})$. Hence, we can construct a nilpotent matrix $Y_0 \in \mathfrak{n}$ sending W_1 to 0 and embedding each W_i with $i > 0$ into W_{i-1}. Thus we have

$$rank(Y_0^i) = n - \sum_{j=1}^{i} dj,$$

so this proves the claim.

By Lemma 2.5 and Theorem 2.6, the orbit $\mathcal{O}_{Y_0}$ through Y_0 strictly dominates any other nilpotent orbit meeting $\mathbf{n}$, i.e. every orbit in $\mathbf{G} \cdot \mathbf{n}$ is contained in the the orbit $\overline{\mathcal{O}_{Y_0}}$. So $\mathcal{O}_{Y_0}$ must be the Richardson orbit in $\mathbf{G} \cdot \mathbf{n}$ and we have

$$\mathbf{G} \cdot \mathbf{n} = \overline{\mathcal{O}_{Y_0}}.$$

By the formula

$$rank(Y_0^i) = n - \sum_{j=1}^{i} dj,$$

we know that the kernel of Y_0^i has dimension equal to the sum of the first i parts of the dual partition $d = [d_1, \cdots, d_k]$, i.e.

$$dim(Ker(Y_0^i)) = \sum_{j=1}^{i} d_j.$$

It also has the same dimension as that of the i^{th} power of any nilpotent matrix with partition $d' = [d'_1, d'_2, ..., d'_l]$, so the partition of $\mathcal{O}_{Y_0}$ must be d' [7].

Since $\mathbf{G} \cdot \mathbf{n}$ has a unique orbit which is dense as in Theorem 2.3, we have

$$\lim_{t \to 0^+} (t\mathbf{G} \cdot X) = \mathbf{G} \cdot \mathbf{n} = \overline{\mathcal{O}_{d'}}$$

where $\overline{\mathcal{O}_{d'}} = \overline{\mathcal{O}_{Y_0}}$. □

CHAPTER III. SOME RESULTS FOR REAL REDUCTIVE ALGEBRAS

In this chapter, we will return to the real case. We will prove a reduction result on limit orbits $\lim_{t\to 0^+}(t\mathrm{G}\cdot X)$ for any semisimple element X in the Lie algebra g of G. For $\mathrm{G} = \mathrm{GL}(n, \mathrm{R})$ we will describe explicitly the parabolic subalgebras entering into this reduction.

§ 3.1 *A REDUCTION RESULT FOR LIMIT ORBITS IN A REAL REDUCTIVE ALGEBRA*

Let $\mathbf{g}$ denote a real reductive Lie algebra, G denote the real reductive group corresponding to $\mathbf{g}$. In chapter II, we proved that

$$\lim_{t\to 0^+}(t\mathrm{G}\cdot X) = \mathrm{G}\cdot \mathbf{n}$$

if X is any *hyperbolic* semisimple element in a real reductive Lie algebra $\mathbf{g}$. If $\mathbf{g} = \mathrm{gl}(n, \mathrm{C})$ and X is any semisimple element in $\mathrm{gl}(n, \mathrm{C})$ with associated partition of d, we proved that

$$\lim_{t\to 0^+}(t\mathrm{G}\cdot X) = \mathrm{G}\cdot \mathrm{n} = \overline{\mathrm{O}}_{d'}$$

where $\mathrm{G} = \mathrm{GL}(n, \mathrm{C})$ and d' is the dual partition of d. For a general semisimple element we shall prove the theorem which in some cases permits a reduction to the simple case.

We shall need more notation and further facts about real reductive groups, which can be found in [22]. For $Y, Z \in \mathbf{g}$, let

$$B(Y, Z) = tr(YZ),$$

For $Y \in \mathbf{g}$ let

$$\theta(Y) = -Y^* \in \mathbf{g}$$

and

$$\langle Y, Z\rangle = -B(Y, \theta(Z)).$$

Then B is non-degenerate and $\langle\cdot,\cdot\rangle$ is a positive definite inner product on $\mathbf{g}$. Set $\mathbf{k}$ equal to the $+1$ eigenspace for θ in $\mathbf{g}$ and set $\mathbf{s}$ equal to the -1 eigenspace of θ in $\mathbf{g}$. Then the decomposition $\mathbf{g} = \mathbf{k} \oplus \mathbf{s}$ is called the ***Cartan decomposition*** of $\mathbf{g}$. One has that $\mathbf{k}$ is the Lie algebra of a compact subgroup of G and $[\mathbf{k}, \mathbf{s}] \subset \mathbf{s}$, $[\mathbf{s}, \mathbf{s}] \subset \mathbf{k}$.

Let $\mathbf{a}$ be a subspace of $\mathbf{s}$ that is maximal subject to the condition that it is an abelian subalgebra of $\mathbf{g}$. If $H \in \mathbf{a}$ then H is diagonalizable since H is self-adjoint. Thus $ad(H)$ is also diagonalizable. If $\mu \in \mathbf{a}^*$ we set

$$\mathbf{g}^{\mu} = \{Y \in \mathbf{g} \mid [H, Y] = \mu(H) \cdot Y \text{ for all } H \in \mathbf{a}\}.$$

Set

$$\Phi = \Phi(\mathbf{g}, \mathbf{a}) = \{\mu \in \mathbf{a}^* \mid \mu \neq 0 \text{ and } \mathbf{g}^{\mu} \neq 0\}.$$

Let $\mathbf{m}^0 = \mathbf{k} \cap \mathbf{g}^0$. Clearly $\mathbf{a} = \mathbf{s} \cap \mathbf{g}^0$, $\mathbf{g}^0 = \mathbf{m}^0 + \mathbf{a}$, and

$$\mathbf{g} = \mathbf{m}^0 \oplus \mathbf{a} \ (\bigoplus_{\mu \in \Phi} \mathbf{g}^{\mu}).$$

Let

$$\mathbf{a}' = \{H \in \mathbf{a} \mid \mu(H) \neq 0, \ \mu \in \Phi(\mathbf{g}, \mathbf{a})\}.$$

Fix $H_0 \in \mathbf{a}'$, put

$$\Phi_+ = \Phi_+(\mathbf{g}, \mathbf{a}) = \{\mu \in \Phi(\mathbf{g}, \mathbf{a}) \mid \mu(H_0) > 0\},$$

and set

$$\mathbf{n}_0 = \bigoplus_{\mu \in \Phi_+} \mathbf{g}^{\mu}, \quad \overline{\mathbf{n}_0} = \theta(\mathbf{n}_0).$$

Then $\mathbf{n}_0$ and $\overline{\mathbf{n}_0}$ are subalgebras of $\mathbf{g}$ and we have

$$\mathbf{g} = \overline{\mathbf{n}_0} \oplus \mathbf{m}^0 \oplus \mathbf{a} \oplus \mathbf{n}_0.$$

The following decomposition of $\mathbf{g}$ is called the ***Iwasawa decomposition***:

$$\mathbf{g} = \mathbf{k} \oplus \mathbf{a} \oplus \mathbf{n}_0.$$

Let $\Phi(\mathbf{g}_C, \mathbf{h})$ be the root system of $\mathbf{g}_C$ relative to a Cartan subalgebra $\mathbf{h}$ where $\mathbf{h} \supset \mathbf{a}$. It is obvious that

$$\Phi(\mathbf{g}, \mathbf{a}) = \Phi(\mathbf{g}_C, \mathbf{h})\,|_{\mathbf{a}} - \{0\}.$$

Let $\mathbf{k}'$ be a maximal abelian subalgebra of $\mathbf{m}^0$. Since the elements of $\Phi(\mathbf{g}_C, \mathbf{h})$ are real valued on $\mathbf{a}$ and take pure imaginary value on $\mathbf{k}'$, it follows that

$$\mathbf{h}_{\mathbb{R}} = (i\mathbf{k}' + \mathbf{a}) \cap [\mathbf{g}_C, \mathbf{g}_C].$$

Let H_1 be an element of $\mathbf{a}' \cap \mathbf{h}_R$. Let $H_1, \cdots, H_r$ be a basis of $\mathbf{h}_R$. We order $\Phi(\mathbf{g}_C, \mathbf{h})$ lexicographically relative to this basis. Let R denote the corresponding positive root system of $\Phi(\mathbf{g}_C, \mathbf{h})$. Let R_0 be the set of all $\mu \in \Phi(\mathbf{g}, \mathbf{a})$ such that $\mu(H_1) > 0$. Then R_0 is a system of positive roots for $\Phi(\mathbf{g}, \mathbf{a})$. Then it is clear that

$$R|_{\mathbf{a}} - \{0\} = R_0 .$$

Let Δ (resp. Δ_0) be the corresponding system of simple roots for R (resp. R_0). Set

$$F_0 = \{\alpha \in \Delta \mid \alpha|_{\mathbf{a}} = 0\} .$$

Then

$$(\Delta - F_0)|_{\mathbf{a}} = \Delta_0 .$$

Theorem 3.3 *Let X be any semisimple element of $\mathbf{g}$. Then there is a parabolic subalgebra $\mathbf{p} = \mathbf{l} + \mathbf{n}$ of $\mathbf{g}$ so that $\mathbf{l} \supseteq C_{\mathbf{g}}(X)$ and the centre $\mathbf{z}$ of $C_{\mathbf{g}}(X)$ consists of elements which are elliptic in $\mathbf{l}$. Furthermore*

$$\lim_{t\to 0^+}(t\mathrm{G}\cdot X) = \mathrm{K}\cdot\{\lim_{t\to 0^+}(t\mathrm{L}\cdot X) + \mathbf{n}\} .$$

Proof. Let X be any semisimple element in $\mathbf{g}$, and let $\mathbf{z}$ be the centre of $C_{\mathbf{g}}(X)$. Then $\mathbf{z}$ has a Cartan decomposition

$$\mathbf{z} = t_Z \oplus a_Z ,$$

so that t_Z consists of elliptic elements in $\mathbf{g}$ and $a_Z \subseteq \mathbf{a}$. Let

$$\mathbf{l} = \{Y \in \mathbf{g} \mid [Y, a_Z] = 0\} ,$$

let R_Z be the subset of positive roots for $\Phi(\mathbf{g}, \mathbf{a})$ whose restriction to a_Z is non-zero. Set

$$\mathbf{n} = \bigoplus_{\mu \in R_Z} \mathbf{g}^{\mu} .$$

Then

$$\mathbf{p} = \mathbf{l} + \mathbf{n}$$

is a parabolic subalgebra of $\mathbf{g}$ and $\mathbf{l} \supseteq C_{\mathbf{g}}(X)$ since a_Z is contained in the centre of $C_{\mathbf{g}}(X)$. Also, the centre $\mathbf{z}$ of $C_{\mathbf{g}}(X)$ consists of elements which are elliptic in $\mathbf{l}$ since

$$\mathbf{l} = \{Y \in \mathbf{g} \mid [Y, a_Z] = 0\} ,$$

and

$$\mathbf{z} = t_Z \oplus a_Z , \quad a_Z|_{\mathbf{l}} = 0 .$$

We know that $ad(X)\cdot\mathbf{n} \subseteq \mathbf{n}$ since $ad(X)\cdot\mathbf{n}_c \subseteq \mathbf{n}_c$, where $\mathbf{n}_c$ is the complexification of $\mathbf{n}$. So we have a map

$$ad(X) : \mathbf{n} \to \mathbf{n} .$$

Since $ker(ad(X)) \subseteq C_{\mathfrak{g}}(X) \subseteq \mathfrak{l}$, and $\mathfrak{l} \cap \mathfrak{n} = \{0\}$, we have

$$ker(ad(X) \mid_{\mathfrak{n}}) = \{0\} .$$

It follows that the map

$$ad(X) : \mathfrak{n} \to \mathfrak{n}$$

is invertible. By Theorem 1.6, we have the final result

$$\lim_{t \to 0^+} (t\mathrm{G} \cdot X) = \mathrm{K} \cdot \{ \lim_{t \to 0^+} (t\mathrm{L} \cdot X) + \mathfrak{n} \} . \qquad \square$$

§ 3.2 *PARABOLIC SUBALGEBRAS IN* gl(n, R)

In this section we give an explicit description of the parabolic subalgebras introduced in the previous section for the case when $\mathrm{G} = \mathrm{GL}(n, \mathrm{R})$.

Let $X \in \mathfrak{gl}(n, \mathrm{R})$ be a semisimple element. After conjugation by $\mathrm{GL}(n, \mathrm{R})$, we may assume that X is of the type

$$X = diag[R_1, ..., R_k, C_1, ..., C_l]$$

$$= \begin{bmatrix} R_1 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & R_k & 0 & \cdots & 0 \\ 0 & \cdots & 0 & C_1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & C_l \end{bmatrix}$$

where R_i is a diagonal $m_i \times m_i$ matrix,

$$R_i = diag[r_i, \cdots, r_i] = \begin{bmatrix} r_i & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & r_i \end{bmatrix} ,$$

$r_i \in \mathrm{R}$, $i = 1, 2, \cdots, k$, and C_j is $2n_j \times 2n_j$ matrix,

$$C_j = diag\{ \begin{bmatrix} \alpha_j & \beta_j \\ -\beta_j & \alpha_j \end{bmatrix}, \dots , \begin{bmatrix} \alpha_j & \beta_j \\ -\beta_j & \alpha_j \end{bmatrix} \}$$

$$= \begin{bmatrix} \alpha_j & \beta_j & 0 & \cdots & 0 \\ -\beta_j & \alpha_j & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & \alpha_j & \beta_j \\ 0 & \cdots & 0 & -\beta_j & \alpha_j \end{bmatrix}$$

$\alpha_j, \beta_j \in \mathrm{R}$, $j = 1, 2, \cdots, l$. We have

$$\sum_{i=1}^{k} m_i + 2\sum_{j=1}^{l} n_j = n$$

and $r_i \neq r_j$, $(\alpha_i, \beta_i) \neq (\alpha_j, \beta_j)$ if $i \neq j$.

We identify the space R^{2n_j} on which C_j acts with C^{n_j} so that C_j acts by multiplication with $c_j = \alpha_j + i\beta_j$. Thus we write

$$X = r_1 \mathrm{I}_{m_1} \times \cdots \times r_k \mathrm{I}_{m_k} \times c_1 \mathrm{I}_{n_1} \times \cdots \times c_l \mathrm{I}_{n_l}$$

where I_t is $t \times t$ identity matrix, $t = m_1, \cdots, m_k, n_1, \cdots, n_l$, $c_j = \alpha_j + i\beta_j$ $j = 1, \cdots l$. We have

$$C_{\mathrm{g}}(X) \cong \mathrm{gl}(m_1, \mathrm{R}) \times \cdots \times \mathrm{gl}(m_k, \mathrm{R}) \times \mathrm{gl}(n_1, \mathrm{C}) \times \cdots \times \mathrm{gl}(n_l, \mathrm{C}).$$

Its elements are of the form

$$\mathrm{diag}(A_1, \cdots, A_k, B_1, \cdots, B_l) = \begin{bmatrix} A_1 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & A_k & 0 & \cdots & 0 \\ 0 & \cdots & 0 & B_1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & B_l \end{bmatrix}$$

where $A_i = \mathrm{gl}(m_i, \mathrm{R})$ $i = 1, \ldots, k$, B_j is $2n_j \times 2n_j$ matrix,

$$B_j = \begin{bmatrix} b_{1,1} & \cdots & b_{1,n_j} \\ \vdots & \cdots & \vdots \\ b_{n_j,1} & \cdots & b_{n_j,n_j} \end{bmatrix}, \quad b_{x,y} = \begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix};$$

$\alpha\,,\,\beta\,\in\,\mathbf{R}\,;\;\; x,\,y=1\,,\,\dots\,,\,n_j\,,\,j=1\,,\,\dots\,,\,l\,.$ Then the centre $\mathfrak{z}$ of $C_{\mathfrak{g}}(X)$ is

$$\mathfrak{z} \cong \mathbf{R}\cdot\mathrm{I}_{m_1} \times \cdots \times \mathbf{R}\cdot\mathrm{I}_{m_k} \times \mathbf{C}\cdot\mathrm{I}_{n_1} \times \cdots \times \mathbf{C}\cdot\mathrm{I}_{n_l}.$$

Its elements are of the form

$$\mathrm{diag}(X_1,\,\dots,\,X_k,\,Y_1,\,\dots,\,Y_l) = \begin{bmatrix} X_1 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & X_k & 0 & \cdots & 0 \\ 0 & \cdots & 0 & Y_1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & Y_l \end{bmatrix}$$

where $X_i = \mathbf{R}\cdot\mathrm{I}_{m_i}$ $i=1\,,\,\dots\,,\,m_k\,,$ Y_j is $2n_j \times 2n_j$ matrix,

$$Y_j = \mathrm{diag}\left\{\begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix},\cdots,\begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix}\right\} = \begin{bmatrix} \alpha & \beta & 0 & \cdots & 0 \\ -\beta & \alpha & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & \alpha & \beta \\ 0 & \cdots & 0 & -\beta & \alpha \end{bmatrix}$$

$\alpha\,,\,\beta\,\in\,\mathbf{R}\,,\,j=1\,,\,\dots\,,\,l\,.$ $\mathfrak{z}$ has the Cartan decomposition

$$\mathfrak{z} = \mathfrak{t}_Z \oplus \mathfrak{a}_Z$$

The subalgebra $\mathfrak{t}_Z$ is

$$\mathfrak{t}_Z \cong 0 \times \cdots \times 0 \times i\mathbf{R}\cdot\mathrm{I}_{n_1} \times \cdots \times i\mathbf{R}\cdot\mathrm{I}_{n_l}$$

with elements of the form

$$\mathrm{diag}(0,\,\dots,\,0,\,Y_1,\,\dots,\,Y_l) = \begin{bmatrix} 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & t_1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & t_l \end{bmatrix}$$

where t_j is $2n_j \times 2n_j$ matrix,

$$t_j = diag\left\{\begin{bmatrix} 0 & \beta \\ -\beta & 0 \end{bmatrix}, \cdots, \begin{bmatrix} 0 & \beta \\ -\beta & 0 \end{bmatrix}\right\} = \begin{bmatrix} 0 & \beta & 0 & \cdots & 0 \\ -\beta & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 & \beta \\ 0 & \cdots & 0 & -\beta & 0 \end{bmatrix},$$

$\beta \in \mathrm{R}$, $j = 1, \cdots, l$.

The subalgebra $\mathfrak{a}_Z$ is

$$\mathfrak{a}_Z \cong \mathrm{R}\cdot\mathrm{I}_{m_1} \times \cdots \times \mathrm{R}\cdot\mathrm{I}_{m_k} \times \mathrm{R}\mathrm{I}_{n_1} \times \cdots \times \mathrm{R}\mathrm{I}_l$$

with elements of the form

$$\mathrm{diag}\{\mathrm{R}\cdot\mathrm{I}_{m_1}, \cdots, \mathrm{R}\cdot\mathrm{I}_{m_k}, \mathrm{R}\cdot\mathrm{I}_{n_1}, \cdots, \mathrm{R}\cdot\mathrm{I}_{n_l}\}.$$

The parabolic subalgebra $\mathfrak{p} = \mathfrak{l} + \mathfrak{n}$ of Theorem 3.3 is therefore

$$\mathfrak{p} = \begin{bmatrix} l_1 & \cdots & * & * & \cdots & * \\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & l_k & * & \cdots & * \\ 0 & \cdots & 0 & l_1' & \cdots & * \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & l_l' \end{bmatrix}$$

where $l_i = \mathrm{gl}(m_i, \mathrm{R})$ $i = 1, \dots, k$, $l_j' = \mathrm{gl}(2n_j, \mathrm{R})$, $j = 1, \dots, l$.
The Levi subalgebra $\mathfrak{l}$ of $\mathfrak{p}$ is the centralizer in $\mathfrak{g}$ of $\mathfrak{a}_Z$, i.e.

$$\mathfrak{l} = \mathrm{gl}(m_1, \mathrm{R}) \times \cdots \times \mathrm{gl}(m_k, \mathrm{R}) \times \mathrm{gl}(2n_1, \mathrm{R}) \times \cdots \times \mathrm{gl}(2n_l, \mathrm{R}) .$$

Theorem 3.3 provides a description of $\lim_{t \to 0+} (t\mathrm{G}\cdot X)$ in terms of $\lim_{t \to 0+} (t\mathrm{L}\cdot X)$, hence a certain reduction from G to L, which is a strictly smaller group unless $k, l \in \{0, 1\}$, i.e. X has only one eigenvalue. However, this reduction by itself does not provide information on the relation between the orbits of G and of L in these limit sets, and hence does not provide a complete solution of the conjecture of Barbasch and Vogan in [1].

APPENDIX. AN EXAMPLE: LIMIT ORBITS IN $sl(2, \mathrm{R})$

We will give an example to illustrate the conjecture mentioned at the end of the last chapter for the Lie algebra

$$sl(2, \mathrm{R}) = \{X \in \mathrm{M}_2(\mathrm{R}) \mid tr(X) = 0\} .$$

The corresponding group is

$$SL(2, \mathrm{R}) = \{ g \in \mathrm{M}_2(\mathrm{R}) \mid det(g) = 1\} .$$

Let E_1, E_2, E_3 be the basis for $sl(2, \mathrm{R})$ defined by

$$E_1 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad E_3 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

Define a bilinear form on $sl(2, \mathrm{R})$ by

$$\langle X, Y\rangle = \xi_1\eta_1 + \xi_2\eta_2 - \xi_3\eta_3$$

where

$$X = \xi_1 E_1 + \xi_2 E_2 + \xi_3 E_3$$
$$Y = \eta_1 E_1 + \eta_2 E_2 + \eta_3 E_3 .$$

Then we have

$$\begin{aligned}\langle X, X\rangle &= \xi_1^2 + \xi_2^2 - \xi_3^2 \\ &= -det(X) \\ &= \tfrac{1}{2}\mathrm{tr}(X^2) .\end{aligned}$$

The complement of the cone $\langle X, X\rangle = 0$ in $sl(2, \mathrm{R})$ is the disjoint union of the following three connected sets.

(1) $\langle X, X\rangle > 0$

(2) $\langle X, X\rangle < 0, \ \langle X, E_3\rangle > 0$

(3) $\langle X, X\rangle < 0, \ \langle X, E_3\rangle < 0$.

Consider $SL(2, \mathrm{R})$ acting on $sl(2, \mathrm{R})$ by the adjoint representation. We know that the following conditions are equivalent.

(a) g maps each of the three sets in (1), (2), (3) into itself.

(b) g maps the set in (2) into itself.

(c) $\langle Ad(g)\cdot E_3, E_3\rangle < 0$.

Let $\mathbf{g} = sl(2, \mathbf{R})$, $\mathbf{G} = SL(2, \mathbf{R})$, put
$S = \{X = \xi_1 E_1 + \xi_2 E_2 + \xi_3 E_3 \in sl(2, \mathbf{R}) \mid \det(X) = \langle X, X\rangle = 0, \text{ and } \xi_3 \geq 0\}$.
So S is the upper cone of $\xi_1^2 + \xi_2^2 - \xi_3^2 = 0$. Let

$$X_0 = \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix} = \tfrac{1}{2}(-E_2 + E_3)\,.$$

We claim that

$$\lim_{t\to 0^+}(t\mathbf{G}\cdot E_3) = S = \overline{\mathbf{G}\cdot X_0}\,.$$

First we prove $\lim_{t\to 0^+}(t\mathbf{G}\cdot E_3) \subseteq S$. Let $Y = \eta_1 E_1 + \eta_2 E_2 + \eta_3 E_3$ be an element of $\lim_{t\to 0^+}(t\mathbf{G}\cdot E_3)$, say

$$Y = \lim_{k\to\infty} g_k \cdot t_k E_3$$

where $t_k \to 0^+$. Then we have

$$det(Y) = det(\lim_{k\to\infty} g_k \cdot t_k E_3) = \lim_{k\to\infty} t_k \cdot det(E_3) = 0$$

and by the property (c) above

$$-\eta_3 = \langle Y, E_3\rangle = \lim_{k\to\infty}\langle g_k \cdot t_k E_3, E_3\rangle \leq 0$$

so we have $Y \in S$ and $\lim_{t\to 0^+}(t\mathbf{G}\cdot E_3) \subseteq S$.

Second we prove $S \subseteq \overline{\mathbf{G}\cdot X_0}$. We know that every non-zero element Y in $sl(2, \mathbf{R})$ for which $det(Y) = 0$ is conjugate to the element

$$\pm X_0 = \begin{bmatrix} 0 & \mp 1 \\ 0 & 0 \end{bmatrix}$$

under $SL(2, \mathbf{R})$, and every element in S is conjugate to X_0. So we have

$$S\backslash\{0\} \subseteq \mathrm{G}\cdot X_0 \text{ and } S \subseteq \overline{\mathrm{G}\cdot X_0}\,.$$

At last we prove $\overline{\mathrm{G}\cdot X_0} \subseteq \lim_{t\to 0^+}(t\mathrm{G}\cdot E_3)$. It is enough to prove that X_0 belongs to $\lim_{t\to 0^+}(t\mathrm{G}\cdot E_3)$. Let $g_k = exp(kE_1)$. Since

$$ad(E_1)\cdot E_3 = -2E_2\,,\quad ad(E_1)\cdot E_2 = -2E_3\,,$$

we have

$$\begin{aligned}
g_k\cdot E_3 &= Ad(exp(kE_1))\cdot E_3 = exp(ad(kE_1))\cdot E_3\\
&= \sum_{n=0}^{\infty}\frac{(ad(kE_1))^{2n}}{(2n)!}E_3 + \sum_{n=0}^{\infty}\frac{(ad(kE_1))^{2n+1}}{(2n+1)!}E_3\\
&= \sum_{n=0}^{\infty}\frac{(-2k)^{2n}}{(2n)!}E_3 + \sum_{n=0}^{\infty}\frac{(-2k)^{2n+1}}{(2n+1)!}E_2\\
&= \sum_{n=0}^{\infty}\frac{(2k)^{2n}}{(2n)!}E_3 - \sum_{n=0}^{\infty}\frac{(2k)^{2n+1}}{(2n+1)!}E_2\\
&= cosh(2k)E_3 - sinh(2k)E_2\\
&= \begin{bmatrix} 0 & -e^{2k} \\ e^{-2k} & 0 \end{bmatrix}.
\end{aligned}$$

Let $t_k = exp(-2k)$, then

$$\begin{aligned}
\lim_{k\to\infty} g_k\cdot t_kE_3 &= \lim_{k\to\infty} exp(-2k)\cdot Ad(exp(kE_1))\cdot E_3\\
&= \lim_{k\to\infty} exp(-2k)\cdot\begin{bmatrix} 0 & -e^{2k} \\ e^{-2k} & 0 \end{bmatrix}\\
&= \lim_{k\to\infty}\begin{bmatrix} 0 & -1 \\ e^{-4k} & 0 \end{bmatrix} = X_0\,.
\end{aligned}$$

So $X_0 \in \lim_{t\to 0^+}(t\mathrm{G}\cdot E_3)$ and $\overline{\mathrm{G}\cdot X_0} \subseteq \lim_{t\to 0^+}(t\mathrm{G}\cdot E_3)$. Thus we proved the claim

$$\lim_{t\to 0^+}(t\mathrm{G}\cdot E_3) = S = \overline{\mathrm{G}\cdot X_0}\,.$$

It follows that the set $\lim_{t\to 0^+}(t\mathrm{G}\cdot E_3)$ is the closure of exactly one real nilpotent orbit $\mathrm{G} \cdot X_0$.

BIBLIOGRAPHY

[1] Dan Barbasch and David Vogan. ***Weyl Group Representations and Nilpotent Orbits.*** Representations of Reductive Groups. P. Trombi, Editor, Birkhäuser, (1983) , 21-33.

[2] W. Borho and H. Kraft. ***Über Bahnen und Deformationen bei linearen Aktionen reduktiver Gruppen.*** Comm. Math. Helv. 54 (1979) , 61-104.

[3] N. Burgoyne and R. Cushman. ***Conjugacy Classes in Linear Groups.*** Journal of Algebra 44 (1977) , 339-362.

[4] N. Burgoyne and R. Cushman. ***Normal forms for Linear Hamiltonian Systems.*** Celest. Mech. 8 (1974) , 435-443.

[5] N. Burgoyne and R. Cushman. ***The Decomposition of a Linear Mapping.*** Linear Algebra Appl. 8 (1974) , 515-519.

[6] Roger W. Carter. ***Finite Groups of Lie Type,*** *Conjugacy classes and complex Characters.* A Wiley-Interscience Publication, Chichester, 1985.

[7] David H. Collingwood and William M. McGovern. ***Nilpotent Orbits in Semisimple Lie Algebras.*** Van Nostrand Reinhold, New York, 1993.

[8] William Fulton and Joe Harris. ***Representation Theory,*** *A First Course, Graduate Texts in Mathematics 129.* Springer-Verlag, New York, 1991.

[9] W. Hesselink. ***Singularities in the Nilpotent Scheme of a Classical Group.*** Trans. Am. Math. Soc. 222 (1976) , 1-32.

[10] James E. Humphreys. *Introduction to Lie Algebras and Representation Theory, Graduate texts in Mathematics 9.* Springer-Verlag, New York, 1972.

[11] James E. Humphreys. *Linear Algebraic Groups.* Springer-Verlag, New York, 1975, 1981.

[12] H. Kraft. *Paramatrisierung von Konjugationsklassen in* sl_n . Math. Ann. 234 (1978) , 209-220.

[13] Hanspeter Kraft and Claudio Procesi. *Closures of conjugacy Classes of Matrices are Namal.* Inventiones math. 53 (1979) , 227-247.

[14] G. Lusztig and N. Spaltenstein. *Induced Unipotent Classes.* J. London Math. Soc. 19 (1979) , 41-52.

[15] H. Ozeki and M. Wakimoto. *On Polarizations of Certain Homogeneous Spaces.* Hiroshima Math. J. 2 (1972) , 445-482.

[16] W. Rossmann. *Limit Orbits in Reductive Lie Algebras.* Duke Mathematics Journal, Vol. 49, No. 1, March 1982.

[17] W. Rossmann. *Lie groups, An Introduction Through Linear Groups.* Preprint.

[18] T. Springer. *Linear Algebraic Groups, Progress in Mathematics 9.,* Birkhäuser, Boston, 1981.

[19] T. A. Springer and R. Steinberg. *Conjugacy Classes.* Lecture Notes in Mathematics, Vol. 131 , Springer, Berlin, 1970.

[20] V.S. Varadarajan. *Lie Groups, Lie Algebras, and their Representations, Graduate Texts in Mathematics 102.* Springer-Verlag, New York, 1984.

[21] G. E. Wall. ***On the Conjugacy Classes in the Unitary, Symplectic and Orthogonal groups.*** J. Austral. Math. Soc. 3 (1963) , 1-62.

[22] Nolan R. Wallach. ***Real Reductive Groups*** I, II. Academic Press, 1988.

[23] H. Zassenhaus. ***On a Normal Form of the Orthogonal Transformation.*** Canad. Math. Bull. 1 (1958) , 31-40 , 101-111 , 183-191.