TRACE FORMULAS, INVARIANT BILINEAR FORMS AND DYNKIN INDICES OF LIE ALGEBRA REPRESENTATIONS OVER RINGS

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Contents

Abstract

The trace form gives a connection between the representation ring and the space of invariant bilinear forms of a Lie algebra L. This thesis reviews the definition of the trace of an endomorphism of a finitely generated projective module over a commutative ring R. We then use this to look at the trace form of a finitely generated projective representation of a Lie algebra L over R and its representation ring. While doing so, we prove a few trace formulas which are useful in the theory of the Dynkin index, an invariant introduced by Dynkin in 1952 to study homomorphisms between simple Lie algebras.

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Dedication

I dedicate this thesis to my parents and my sister, for their love and encouragement.

Introduction

In the theory of finite-dimensional complex semisimple Lie algebras, invariant bilinear forms play an important role. Perhaps, the most important invariant bilinear form is the Killing form κ, defined by $\kappa(x, y) = \mathrm{tr}(\mathrm{ad}(x)\,\mathrm{ad}(y))$. As an example, the Killing form is used to establish the root space decomposition of a semisimple Lie algebra over the complex numbers.

The Killing form is a special case of the notion of trace form. It is the trace form associated to the adjoint representation. More precisely, given a finite-dimensional representation ρ of a Lie algebra L, the trace form τ_ρ associated to ρ is defined by $\tau_\rho(x, y) = \mathrm{tr}(\rho(x)\rho(y))$. We then see that the trace form establishes a connection between the representation ring and the space of invariant bilinear forms.

Over a commutative ring R, the trace of an endomorphim of an R-module is in general not defined. However, one can defined the trace for an endomorphism of a finitely generated projective R-module using the technique of descent. This can be found in [KO] and [Alm].

Over a field, the representation ring $\mathscr{R}(L)$ of a Lie algebra L, as an abelian group, is defined to be the Grothendieck group generated by the finite-dimensional representations of L where the addition is defined by $[V]+[W] := [V \oplus W]$ for $[V], [W] \in \mathscr{R}(L)$. This group has the structure of unital ring with multiplication $[V] \cdot [W] := [V \otimes W]$ for $[V], [W] \in \mathscr{R}(L)$.

The analogue over a commutative ring is to consider finitely generated projective L-modules. In this framework, we can then define trace forms of finitely generated projective representations and obtain a connection between the corresponding representation ring and the corresponding space of invariant bilinear forms.

In order to study representations, using the aforementioned connection, one can study the space of invariant bilinear forms. We also review this space in the thesis.

The space of invariant bilinear forms sometimes happens to be free of rank 1. In this case, the situation becomes much simpler and one can define the Dynkin index associated to a finitely generated projective representation. More precisely, fix an invariant bilinear form γ, $\tau_\rho = \mathrm{dyn}(\rho)\gamma$ for some $\mathrm{dyn}(\rho) \in R$. The *Dynkin index* of the representation ρ is defined to be $\mathrm{dyn}(\rho)$.

Dynkin defines the notion of the Dynkin index of a Lie algebra representation over $\mathbb{C}$ in his 1952 paper *Semisimple subalgebras of semisimple Lie algebras*. A part of his original aim is to study representations of a complex simple Lie algebra. The Dynkin index has appeared in various contexts. It has been introduced in the theory of G-bundles over a curve by Faltings [F], Kumar-Narasimhan [KN], Kumar-Narasimhan-Ramanathan [KNR], Laszlo-Sorger [LS]. It has appeared as an invariant in the theory of algebraic group, which can be found in *Cohomological invariants in Galois cohomology* [GMS] by Garibaldi, Merkujev and Serre. It also appears in the recent preprint by Garibaldi and Zainoulline [GZ] on the γ-filtration and the Rost invariant.

Let $\mathfrak{g}$ be a complex simple Lie algebra. There exists a unique invariant bilinear form γ such that $\gamma(h_\theta, h_\theta) = 2$ for any long root θ. The Dynkin index of a representation ρ then measures the difference between γ and the trace form τ_ρ.

An interesting result is that the Dynkin index of a representation of a complex simple Lie alebra is an integer ([Dyn, Thm. 2.2] and [KN, Prop. 4.2 and Rem. 4.3]). This result leads to the notion of the Dynkin index of a complex simple Lie algebra $\mathfrak{g}$, defined to be the gcd of the Dynkin indices of its finite-dimensional representations.

Another very interesting result as well as the original motivation of this thesis states that there exists a (non-unique) fundamental representation whose Dynkin index coincides with the Dynkin index of $\mathfrak{g}$, see [KN, Prop. 4.7] and [LS, Prop. 2.6]. The only known proof of this fact is a case-by-case direct computation: "Unfortunately, our proof is case by case."- Kumar [KN, proof of Prop 4.7]. In this thesis, while we do not have a classification-free and conceptual proof of this result, we have partial results that allow us to provide a conceptual proof in many cases.

The thesis has two main aims. The first aim is to study the connection given by trace form between the representation ring and the space of invariant bilinear forms of a Lie algebra defined over a commutative ring R. The second aim is to generalize the Dynkin index in a natural way and to provide a conceptual proof of [KN, Prop. 4.7] in many cases using some trace formulas.

Chapter 1 is simply a review of many concepts that we will need in subsequent chapters. We review a part of module theory, projective modules, flat modules, faithfully flat modules. This serves as background since, in later chapters, we want to talk about finitely generated projective modules of a Lie algebra L over a commutative ring R.

In Chapter 2, we review the definition and properties of the trace of an endomorphism of a finitely generated projective module over a commutative ring. As references, we use [KO] and [Alm]. We provide more details than these sources. We also provide some trace formulas that will give us the Dynkin index of exterior powers and symmetric powers of a representation in Chapter 5.

Chapter 3 is a review of the representation ring of a Lie algebra over a commutative ring. We provide some results inspired by [B:Lie, VIII, §7].

Chapter 4 deals with invariant bilinear forms. As reference, we use [NPPS]. Using the trace form, we establish a map from the representation ring to the space of invariant bilinear forms. Also, we show that the space of invariant bilinear forms of simply-connected Chevalley orders is free of rank 1 and exhibit a basis which is compatible with Dynkin's choice.

Chapter 5 is about the Dynkin index. We summarize some key results found in [Dyn] and [KN]. Finally, we apply some of our results in previous chapters (e.g., the trace formulas from Chapter 2) to give the proof of [KN, Prop. 4.7] in many cases: simple Lie algebras of type A, B, C, D, E_6, G_2. Unfortunately, our approach does not work on the remaining types.

Here, we give a summary of the new results of this thesis. In §2.2, for $f \in \operatorname{End}_R(P)$ where P is a finitely generated projective R-module, we define $f^{\wedge n} \in \operatorname{End}_R(\bigwedge^n P)$, $f^{\circ n} \in \operatorname{Sym}^n P$ and then compute their traces. These results are given in Proposition 2.2.12, Corollary 2.2.13, Proposition 2.2.14 and Proposition 2.2.15. These are later used in Chapter 5 to determine the Dynkin indices of symmetric and exteriors powers of representations.

In §4.3.2, we prove that the space of invariant bilinear forms on simply-connected Chevalley orders is free of rank 1 (as stated in [NPPS]). These results are Theorem

4.3.26, Corollary 4.3.27, Corollary 4.3.29, Corollary 4.3.30 and Corollary 4.3.33.

In chapter 5, we apply our new results in §2.2 to prove new formulas for Dynkin indices of symmetric and exterior powers of representations in Proposition 5.1.2. In §5.3, we use our new formulas from Proposition 5.1.2 to give the proof of [KN, Prop. 4.7] for Lie algebras of type A, B, C, D, E_6, G_2.

Chapter 1

Background

Throughout this chapter, R denotes a unital associative commutative ring, M denotes an R-module, $\operatorname{Spec} R$ denotes the set of prime ideals of R (also called the prime spectrum of R), R-**alg** denotes the category of commutative associative unital R-algebras. Throughout this thesis, unless stated otherwise, all R-algebras are assumed to be commutative, associative and unital. By convention, $\mathbb{N} = \{0, 1, 2, \dots\}$.

1.1 Localization of modules

Localization is a technique from commutative algebra. It is often used to reduce the study to the case of local rings. Here, we present the process of localization of a module, in particular, the process of localization of a ring. For this section, our reference is [B:CA, Chap II, §2]. We present some proofs so that the reader can get a feeling of the material.

Definition 1.1.1. Let R be a commutative ring. A subset U of R is called *multiplicative* if all finite products of elements of U belong to U. By convention, the multiplicative identity 1_R, being the empty product, lies in U. We allow U to contain zero divisors.

Example 1.1.2. Let R be a ring. Then

1. For all $r \in R$, $\{r^n : n \in \mathbb{N}\}$ is a multiplicative subset.

2. For $\mathfrak{p} \in \operatorname{Spec} R$, the set $R \backslash \mathfrak{p}$ is multiplicative.

3. If U and T are multiplicative subsets, then so is $UT := \{ut : u \in U, t \in T\}$.

4. An arbitrary intersection of multiplicative subsets of R is clearly a multiplicative subset of R.

Lemma 1.1.3. *Let U be a multiplicative subset of R. We define a relation on $M \times U$ by $(m, u) \sim (m', u')$ if there exists $v \in U$ such that $v(u'm - um') = 0$. Then $\sim$ is an equivalence relation and the set of equivalence classes $(M \times U)/\sim$ has the structure of an R-module. Denote by m/u the equivalence class of (m, u), the operations of $(M \times U)/\sim$ are*

$$m/u + m'/u' = (u'm + um')/mm' \quad \textit{and} \quad r \cdot (m/u) = (rm)/u.$$

Definition 1.1.4. Let U be a multiplicative subset of R. The set of equivalence classes $(M \times U)/\sim$ with the above structure of an R-module is called the *localization of M at U*, denoted by $U^{-1}M$. In particular, $U^{-1}R$ is called the localization of the ring R where R is considered as a left R-module in the obvious way.

Notation 1.1.5. For convenience, we denote the localization of M at $\{r^n : n \in \mathbb{N}\}$ by M_r and denote the localization of M at $R \backslash \mathfrak{p}$ by $M_{\mathfrak{p}}$.

The localization process is closely related to scalar extension.

Proposition 1.1.6. [B:CA, II, §2.2, Prop. 3]. *If U is a multiplicative subset of R then the map $M \otimes_R U^{-1}R \longrightarrow U^{-1}M$, $m \otimes \frac{r}{u} \mapsto \frac{rm}{u}$, is an isomorphism of R-modules.*

Lemma 1.1.7. [B:CA, II, §2.3, Prop. 7]. *If M is an R-module, then we have:*

(a) *If U and T are multiplicative subsets of R, then $(UT)^{-1}M \simeq T'^{-1}(U^{-1}M)$ as R-modules where T' is the image of T under the canonical map $R \longrightarrow U^{-1}R$, $r \mapsto \frac{r}{1}$.*

(b) *Let $f, g \in R$. Then $M_{fg} = (\{f^m g^n : m, n \in \mathbb{N}\})^{-1}M$ and $M_{fg} \simeq (M_f)_{g/1}$ as R-modules.*

Proof. (a). Define the map $j : (UT)^{-1}M \longrightarrow T'^{-1}(U^{-1}M)$ by $\frac{m}{st} \mapsto \frac{m/s}{t/1}$. We show that j is well-defined. Suppose that $m/(st) = m'/(s't')$. So, there exists $uv \in UT$

such that $uv(s't'm - stm') = 0$. We claim that $\frac{m/s}{t/1} = \frac{m'/s'}{t'/1}$. Consider $v/1 \in T'$. We have that

$$\frac{v}{1}\left(\frac{t'm}{s} - \frac{tm'}{s'}\right) = \frac{uv(s't'm - stm')}{uss'} = 0.$$

Thus, j is well-defined. Next, one has to show that f is indeed R-linear. We have that

$$\begin{aligned} j\left(r\frac{m}{st} + \frac{m'}{s't'}\right) &= j\left(\frac{rs't'm + stm'}{ss'tt'}\right) = \frac{(rs't'm + stm')/ss'}{tt'} \\ &= \frac{rt'm/s}{tt'} + \frac{tm'/s'}{tt'} = \frac{rm/s}{t} + \frac{m'/s'}{t'} = j\left(r\frac{m}{st}\right) + j\left(\frac{m'}{s't'}\right). \end{aligned}$$

Hence, j is a well-defined R-linear map. Finally, we show that j is indeed an isomorphism. First, we prove injectivity. Let $m/(st) \in \ker(j)$. So, $\frac{m/s}{t/1} = 0$. Thus, there exists $v/1 \in T'$ such that $\frac{vm}{s} = 0$. So, $uvm = 0$ in M for some $u \in S$. Then $m/(st) = 0$. Thus, j is injective. Now, let $\frac{m/s}{t} \in T'^{-1}(U^{-1}M)$. Clearly, $j(m/(st)) = \frac{m/s}{t}$. So, j is surjective. Therefore, $(UT)^{-1}M \simeq T'^{-1}(U^{-1}M)$ as R-modules.

(b). Set $S = \{(fg)^m : m \in \mathbb{N}\}$, $T = \{f^m g^n : m, n \in \mathbb{N}\}$, $F = \{f^n : n \in \mathbb{N}\}$ and $G = \{g^m : m \in \mathbb{N}\}$.

Next, S and T satisfy part (c) of [B:CA, II, §2.3, Prop. 8], which is equivalent to saying that the homomorphism $S^{-1}R \longrightarrow T^{-1}R$ is an isomorphism.

Next, $T = FG$. It follows from part (a) that $G'^{-1}(F^{-1}R) \simeq T^{-1}R \simeq S^{-1}R$ as R-modules. Thus, by repeatedly applying Proposition 1.1.6,

$$\begin{aligned} S^{-1}M \simeq S^{-1}R \otimes_R M \simeq G'^{-1}(F^{-1}R) \otimes_R M \simeq G^{-1}R \otimes_R F^{-1}R \otimes_R M \\ \simeq G^{-1}R \otimes_R F^{-1}M = G'^{-1}(F^{-1}M). \end{aligned}$$

We are done □

Proposition 1.1.8. [B:CA, Ch II, §2.7, Prop 18(i)] *Let U be a multiplicative subset of R. Let M and N be two R-modules. Then the $U^{-1}R$-modules $U^{-1}M \otimes_R N$, $M \otimes_R U^{-1}N$, $(U^{-1}M) \otimes_{U^{-1}R} (U^{-1}N)$ and $U^{-1}(M \otimes_R N)$ are canonically isomorphic.*

Proposition 1.1.9. *Let U, T be multiplicative subsets of R. Then*

$$\psi : U^{-1}R \otimes_R T^{-1}R \longrightarrow (UT)^{-1}R, \quad \frac{r}{u} \otimes \frac{s}{t} \mapsto \frac{rs}{ut},$$

is an isomorphism of rings with inverse $\varphi : (UT)^{-1}R \longrightarrow U^{-1}R \otimes_R T^{-1}R$, $\frac{r}{ut} \mapsto \frac{r}{u} \otimes \frac{1}{t}$. As a consequence, $(UT)^{-1}M \simeq U^{-1}M \otimes_R T^{-1}R \simeq T^{-1}M \otimes U^{-1}R$.

Proof. We define the map $\widetilde{\psi} : U^{-1}R \times T^{-1}R$, $\left(\frac{r}{u}, \frac{s}{t}\right) \mapsto \frac{rs}{ut}$. We check that the map is well-defined, R-balanced and the induced map ψ is a ring homomorphism. Suppose that $\left(\frac{r}{u}, \frac{s}{t}\right) = \left(\frac{r'}{u'}, \frac{s'}{t'}\right)$. So, there exists $v \in U$ and $w \in T$ such that $v(u'r - ur') = 0$ and that $w(t's - ts') = 0$. One has that

$$\begin{aligned} vw(u't'rs - utr's') &= vwu't'rs - vwt'sur' + vwt'sur' - vwutr's' \\ &= vwt's(u'r - ur') + vwur'(t's - ts') = 0. \end{aligned}$$

It follows that $\widetilde{\psi}$ is well-defined. Now, for $r, r', s, s' \in R$, $u, u' \in U$ and $t, t' \in T$, we have that:

$$\begin{aligned} \widetilde{\psi}\left(\frac{r}{u} + \frac{r'}{u'}, \frac{s}{t}\right) &= \psi\left(\frac{u'r + ur'}{uu'}, \frac{s}{t}\right) = \frac{(u'r + ur')s}{uu't} \\ &= \frac{rs}{ut} + \frac{r's}{u't} = \widetilde{\psi}\left(\frac{r}{u}, \frac{s}{t}\right) + \widetilde{\psi}\left(\frac{r'}{u'}, \frac{s}{t}\right), \\ \widetilde{\psi}\left(\frac{r}{u}, \frac{s}{t} + \frac{s'}{t'}\right) &= \widetilde{\psi}\left(\frac{r}{u}, \frac{t's + ts'}{tt'}\right) = \frac{r(t's + ts')}{utt'} \\ &= \frac{rs}{ut} + \frac{rs'}{ut'} = \widetilde{\psi}\left(\frac{r}{u}, \frac{s}{t}\right) + \widetilde{\psi}\left(\frac{r}{u}, \frac{s'}{t'}\right), \\ \widetilde{\psi}\left(\frac{r}{u}s, \frac{r'}{u'}\right) &= \frac{(rs)r'}{uu'} = \frac{r(sr')}{uu'} = \widetilde{\psi}\left(\frac{r}{u}, s\frac{r'}{u'}\right). \end{aligned}$$

Therefore, $\widetilde{\psi}$ is R-balanced. Hence, it induces a map $\psi : U^{-1}R \otimes_R T^{-1}R \longrightarrow UT^{-1}R$ such that $\widetilde{\psi} = \psi \circ \otimes$, meaning $\widetilde{\psi}\left(\frac{r}{u}, \frac{s}{t}\right) = \psi\left(\frac{r}{u} \otimes \frac{s}{t}\right)$.

Next, we check that the map φ from the statement of the Proposition is well-defined. Suppose that $\frac{r}{ut} = \frac{r'}{u't'}$. Then there is $vw \in UT$ so that $vw(u't'r - utr') = 0$. It follows that $\frac{rt'}{u} = \frac{r't}{u'}$. We see that $\frac{r}{u} \otimes \frac{1}{t} = \frac{rt'}{u} \otimes \frac{1}{tt'}$ and that $\frac{r'}{u'} \otimes \frac{1}{t'} = \frac{r't}{u'} \otimes \frac{1}{tt'}$. It follows that φ is well-defined.

Next, for $r, s \in R$, $u \in U$ and $t \in T$,

$$\begin{aligned} \psi\varphi\left(\frac{r}{ut}\right) &= \psi\left(\frac{r}{u} \otimes \frac{1}{t}\right) = \frac{r}{ut}, \\ \varphi\psi\left(\frac{r}{u} \otimes \frac{s}{t}\right) &= \varphi\left(\frac{rs}{ut}\right) = \frac{rs}{u} \otimes \frac{1}{t} = \frac{r}{u} \otimes \frac{s}{t}. \end{aligned}$$

It follows that ψ is a ring isomorphism. Now, by Proposition 1.1.8, one has that

$$(UT)^{-1}M = M \otimes_R (UT)^{-1}R \simeq M \otimes_R U^{-1}R \otimes_R T^{-1}R \simeq U^{-1}M \otimes_R T^{-1}R.$$

Moreover, $M \otimes_R U^{-1}R \otimes_R T^{-1}R \simeq M \otimes_R T^{-1}R \otimes_R U^{-1}R$ and we are done. □

Corollary 1.1.10. *Let $f, g \in R$. Let M be an R-module. Then $M_{fg} \simeq M_f \otimes_R R_g \simeq M_g \otimes_R R_f$. In particular, if M_f is a free R-module then M_{fg} is free as an R_g-module.*

Proof. The first assertion is simply a direct application of Proposition 1.1.9. The second assertion follows immediately. $\square$

Lemma 1.1.11. *Let U, T be multiplicative subsets of R. Let M be an R-module and let $\alpha \in \operatorname{End}_R(M)$. Consider the map ψ of Proposition* 1.1.9. *Then the map $\operatorname{Id}_M \otimes \psi : M \otimes_R U^{-1}R \otimes_R T^{-1}R \longrightarrow M \otimes_R (UT)^{-1}R$ satisfies*

$$(\operatorname{Id}_M \otimes \psi)^{-1} \circ (\alpha \otimes \operatorname{Id}_{(UT)^{-1}R}) \circ (\operatorname{Id}_M \otimes \psi) = \alpha \otimes \operatorname{Id}_{U^{-1}R} \otimes \operatorname{Id}_{T^{-1}R}\,.$$

Proof. One has the following computation:

$$\begin{aligned}&(\alpha \otimes \operatorname{Id}_{(UT)^{-1}R}) \circ (\operatorname{Id}_M \otimes \psi) \left(m \otimes \frac{r}{u} \otimes \frac{s}{t}\right) = (\alpha \otimes \operatorname{Id}_{(UT)^{-1}R}) \left(m \otimes \frac{rs}{ut}\right) = \\ &= \alpha(m) \otimes \frac{rs}{ut} = (\operatorname{Id}_M \otimes \psi) \circ (\alpha \otimes \operatorname{Id}_{U^{-1}R} \otimes \operatorname{Id}_{T^{-1}R}) \left(m \otimes \frac{r}{u} \otimes \frac{s}{t}\right).\end{aligned}$$

This completes the proof. $\square$

The next proposition relates localization and scalar extension in the case of R-algebras.

Proposition 1.1.12. [B:CA, II, §2.2, Prop 6]. *Let $(S, \rho) \in R$-**alg** where $\rho : R \longrightarrow S$ is a ring homomorphism. Let U be a multiplicative subset of R. Then the map $\psi : U^{-1}R \otimes_R S \longrightarrow \rho(U)^{-1}S$ induced by $U^{-1}R \times S \longrightarrow \rho(U)^{-1}S$, $\left(\frac{r}{u}, s\right) \mapsto \frac{\rho(r)s}{\rho(u)}$ is a ring isomorphism with inverse $\varphi : \rho(U)^{-1}S \longrightarrow U^{-1}R \otimes_R S$, $\frac{s}{\rho(u)} \mapsto \frac{1}{u} \otimes s$.*

Proof. We define a map $\lambda : U^{-1}R \times S \longrightarrow \rho(U)^{-1}S$, by $\left(\frac{r}{u}, s\right) \mapsto \frac{\rho(r)s}{\rho(u)}$. We show that λ is well-defined and R-balanced. Suppose that $r/u = r'/u'$. Then there exists $u'' \in U$ such that $u''(u'r - ur') = 0$. Then $\rho(u'')\rho(u'r - ur')s = 0$. It follows that $\frac{\rho(r)s}{u} = \frac{\rho(r')s}{u'}$.

Next, one has that

$$\begin{aligned}\lambda\left(\frac{r}{u} + \frac{r'}{u'}, s\right) &= \frac{\rho(u'r + ur')s}{\rho(uu')} = \frac{\rho(u'r)s}{\rho(uu')} + \frac{\rho(ur')s}{\rho(uu')} = \lambda\left(\frac{r}{u}, s\right) + \lambda\left(\frac{r'}{u'}, s\right), \\ \lambda\left(\frac{r}{u}, s + s'\right) &= \frac{\rho(r)(s + s')}{\rho(u)} = \frac{\rho(r)s}{\rho(u)} + \frac{\rho(r)s'}{\rho(u)} = \lambda\left(\frac{r}{u}, s\right) + \lambda\left(\frac{r'}{u'}, s\right), \\ \lambda\left(\frac{rr'}{u}, s\right) &= \frac{\rho(rr')s}{u} = \frac{\rho(r)(\rho(r')s)}{u} = \lambda\left(\frac{r}{u} \otimes \rho(r')s\right).\end{aligned}$$

Thus, λ is R-balanced. Thus, it induces an R-linear map $\psi : U^{-1}R \otimes_R S \longrightarrow \rho(U)^{-1}S$ such that $\lambda = \psi \circ \otimes$ where $\otimes : U^{-1}R \times S \longrightarrow U^{-1}R \otimes_R S$, $\left(\frac{r}{u}, s\right) \mapsto \frac{r}{u} \otimes s$. We claim that ψ is moreover a ring homomorphism. It is enough to check this on generators.

$$\psi\left(\left(\frac{r}{u} \otimes s\right)\left(\frac{r'}{u'} \otimes s'\right)\right) = \frac{rr'ss'}{\rho(uu')} = \frac{rs}{\rho(u)}\frac{r's'}{\rho(u')} = \psi\left(\frac{r}{u} \otimes s\right)\psi\left(\frac{r'}{u'} \otimes s'\right).$$

Thus, ψ is a ring homomorphism. Next, we check that the map φ is well-defined. Suppose that $\frac{s}{\rho(u)} = \frac{s'}{\rho(u')}$. Thus, there exists $u'' \in U$ such that $\rho(u'')(\rho(u')s - \rho(u)s') = 0$. Then

$$0 = \frac{1}{uu'u''} \otimes \rho(u'')(\rho(u')s - \rho(u)s') = \frac{1}{uu'} \otimes (\rho(u')s - \rho(u)s') = \frac{1}{u} \otimes s - \frac{1}{u'} \otimes s'.$$

Hence, φ is well-defined. Now, it is left to check that ψ and φ are mutually inverses. We have that

$$\begin{aligned}
\psi\varphi\left(\frac{s}{\rho(u)}\right) &= \psi\left(\frac{1}{u} \otimes s\right) = \frac{s}{\rho(u)}, \\
\varphi\psi\left(\frac{r}{u} \otimes s\right) &= \varphi\left(\frac{\rho(r)s}{\rho(u)}\right) = \frac{1}{u} \otimes \rho(r)s = \frac{r}{u} \otimes s.
\end{aligned}$$

This completes the proof. □

The next result provides us a way to extract global properties from local properties of a ring.

Proposition 1.1.13. [B:CA, Ch II, §3.2, Cor 2 of Thm. 1]. *Let R be a ring and let M be an R-module. For a maximal ideal $\mathfrak{m}$ of R, let $\lambda_{\mathfrak{m}} : M \longrightarrow M_{\mathfrak{m}}$ be the canonical homomorphism. Then the homomorphism $M \longrightarrow \prod_{\mathfrak{m} \in \mathrm{Max}(R)} M$, $x \mapsto (\lambda_{\mathfrak{m}}(x))$, is injective, where* $\mathrm{Max}(R)$ *is the set of maximal ideals of R .*

1.2 Flat and faithfully flat modules

Definition 1.2.1. An R-module is M is called *flat* if whenever $N' \xrightarrow{\alpha} N \xrightarrow{\beta} N''$ is exact, the sequence $N' \otimes_R M \xrightarrow{\alpha\otimes 1} N \otimes_R M \xrightarrow{\beta\otimes 1} N'' \otimes_R M$ is also exact.

Example 1.2.2. We give several examples of flat modules.

1. [B:CA, II, §3.4, Prop. 13]. If R is a ring and U is a multiplicative subset of R, then $U^{-1}R$ is a flat R-module.

2. [B:CA, I, §2.3, Prop 2]. Let $(M_i)_{i\in I}$ be a family of R-modules. Then $\bigoplus_{i\in I} M_i$ is flat if and only if each M_i is flat.

Proposition 1.2.3. [B:CA, I, §3.1, Prop 1]. *For an R-module M, the following are equivalent:*

(a) $N' \xrightarrow{\alpha} N \xrightarrow{\beta} N''$ *is exact if and only if* $N' \otimes_R M \xrightarrow{\alpha\otimes 1} N \otimes_R M \xrightarrow{\beta\otimes 1} N'' \otimes_R M$ *is exact.*

(b) *M is flat and, for all R-module N, if $N \otimes_R M = 0$, then $N = 0$.*

(c) *M is flat and if $N' \xrightarrow{\alpha} N$ is such that $\alpha \otimes 1$ is injective, then α is injective.*

(d) *M is flat and if $N' \xrightarrow{\alpha} N$ is such that $\alpha \otimes 1 = 0$, then $\alpha = 0$.*

(e) *M is flat and if $M \neq 0$, then $\mathfrak{m}M \neq M$ for all maximal ideal $\mathfrak{m}$ of R.*

Definition 1.2.4. An R-module M is called *faithfully flat* if it satisfies one of the equivalent conditions of Proposition 1.2.3.

Definition 1.2.5. An R-algebra (S, ρ) is called *flat* (resp. *faithfully flat*) if it is flat (resp. faithfully flat) as an R-module.

Remark 1.2.6. (i) Faithfully flat modules are flat. However, the converse is not true. $\mathbb{Q}$ is flat (and faithful) but not faithfully flat as a $\mathbb{Z}$-module.

(ii) $\bigoplus_{\mathfrak{p}\in \mathrm{Spec}(\mathbb{Z})} \mathbb{Z}_{\mathfrak{p}}$ is faithfully flat as a $\mathbb{Z}$-module. However, each of its summands is flat but not faithfully flat as $\mathbb{Z}$-module.

Proposition 1.2.7. [B:CA, I, §3.2, Prop 4]. *Let R be a commutative ring. Let M and N be R-modules. Assume that M is faithfully flat. Then N is faithfully flat if and only if $M \otimes_R N$ is a faithfully flat R-module.*

1.3 Brief facts from faithfully flat descent theory

In this section, we consider a faithfully flat extension S/R, i.e., S is an associative, commutative R-algebra which is faithfully flat as R-module. We present several important facts that we will use later on to define the trace of an endomorphism of a finitely generated projective R-module. Our reference is [KO, Ch. II].

Remark 1.3.1. Given a ring extension S/R, one has two natural homomorphism

$$\epsilon_1 : S \longrightarrow S \otimes_R S, \qquad s \mapsto s \otimes 1,$$
$$\epsilon_2 : S \longrightarrow S \otimes_R S, \qquad s \mapsto 1 \otimes s.$$

For faithfully flat extension, these two homomorphisms will help us understand the process of descent.

Proposition 1.3.2. [KO, II, §2, Prop. 2.1]. *Let S/R be a faithfully flat ring extension. Let N be an R-module. Consider the maps* $\operatorname{Id}_N \otimes 1 : N \longrightarrow N \otimes_R S$ *and* $\operatorname{Id}_N \otimes \epsilon_i : N \otimes_R S \longrightarrow N \otimes_R S \otimes_R S$. *Then* $\operatorname{Id}_N \otimes 1$ *is injective and* $\operatorname{im}(\operatorname{Id}_N \otimes 1) = \ker(\operatorname{Id}_N \otimes \epsilon_1 - \operatorname{Id}_N \otimes \epsilon_2)$.

The above proposition tells how we can recover an R-module N by knowing $N \otimes_R S$ for a faithfully flat extension S/R. Essentially, this is the key to "faithfully flat descent". In section 2, we will use faithfully flat descent to define the trace of an endomorphism of a finitely generated projective module (which will be introduced later).

1.4 Zariski topology and covering

In this section, we introduce the concept of a covering of a ring. For this, our references are [KO, Chap I, §5] and [B:CA, II]. Let $\operatorname{Spec}(R)$ be the prime spectrum of R. For a subset E of R, denote by $V(E)$ the set of elements of $\operatorname{Spec} R$ that contain E. We note that if $I = \langle E \rangle$ is the ideal generated by E, then $V(I) = V(E)$.

Proposition 1.4.1. [B:CA, II, §4.3]. *The set* $\operatorname{Spec}(R)$ *has the structure of a topological space, where the closed sets are the $V(E)$ for some $E \subseteq R$.*

Definition 1.4.2. The above topology on $\operatorname{Spec}(R)$ is called the *Zariski topology*. For $r \in R$, the complement of $V(r) = V(\{r\})$, denoted U_r, is called a *distinguished open set* of $\operatorname{Spec}(R)$. Thus, $U_r = \{\mathfrak{p} \in \operatorname{Spec}(R) : r \notin \mathfrak{p}\}$. Note that the distinguished open sets form a base of the (Zariski) topology.

Remark 1.4.3. The process of localization is able to tell us some information about the prime spectrum. Namely,

1. Let $r \in R$. Consider the map $\lambda : R \longrightarrow R_r$, $s \mapsto s/1$. Then the maps $U_r \longrightarrow \operatorname{Spec}(R_r)$, $\mathfrak{p} \mapsto \lambda(\mathfrak{p})$, and $\operatorname{Spec}(R_r) \longrightarrow U_r$, $\mathfrak{q} \mapsto \lambda^{-1}\mathfrak{q}$ are mutually inverse bijections.

2. A family $\{U_{r_i}\}_{i\in I}$ covers $\operatorname{Spec}(R)$ if and only if $\langle r_i : i \in I\rangle = \langle 1\rangle = R$.

   Proof. Suppose that $\langle r_i : i \in I\rangle = R$. Let $\mathfrak{p} \in \operatorname{Spec}(R)$. Since $\mathfrak{p}$ is prime, there exists i such that $r_i \notin \mathfrak{p}$. It follows that $\mathfrak{p} \in U_{r_i}$. Thus, the sets U_{r_i} cover $\operatorname{Spec}(R)$. Conversely, suppose that $I = \langle r_i : i \in I\rangle$ is a proper ideal of R. Thus, $I \subseteq \mathfrak{m}$ for some maximal ideal $\mathfrak{m}$ of R. It follows that $r_i \in \mathfrak{m}$ for all i. Hence, $\mathfrak{m} \notin U_{r_i}$ for all i. Thus, $\bigcup_i U_{r_i} \subsetneq \operatorname{Spec}(R)$. □

Definition 1.4.4. For a finite covering $\{U_{r_i} : i = 1, \ldots, n\}$ of $\operatorname{Spec}(R)$, the ring $S = \prod_{i=1}^n R_{r_i}$ is called a *Zariski covering* of R.

Lemma 1.4.5. [B:CA, II, §5.1, Prop 3]. *Let R be a commutative ring and let $S = \prod_{i=1}^n R_{f_i}$ be a Zariski covering of R. Then S/R is a faithfully flat ring extension.*

Proof. Since S is a finite product of localizations, it is flat by Example 1.2.2. Now, let $\mathfrak{m}$ be a maximal ideal of R. Since $R = \langle f_1, \ldots, f_n\rangle$, there exists i such that $f_i \notin \mathfrak{m}$. Suppose that $\mathfrak{m}R_{f_i} = R_{f_i}$. Then $x\frac{y}{f_i^s} = 1$ for some $x \in \mathfrak{m}$, $y \in R$ and $s \in \mathbb{N}$. Thus, there exists f_i^t such that $f_i^t(xy - f_i^s) = 0$. It follows that $f_i^t xy = f_i^{s+t}$. If $s + t = 0$, then $x \in R^* \cap \mathfrak{m}$, a contradiction. If $s + t \neq 0$, then since $f_i^t xy \in \mathfrak{m}$, we see that $f_i \in \mathfrak{m}$, a contradiction. It follows that $\mathfrak{m}S \neq S$. Therefore, by Proposition 1.2.3(e), S/R is indeed a faithfully flat ring extension. □

Next, we define a more general concept of covering for a ring.

Definition 1.4.6. A *faithfully flat covering* of R is a finite family $R_1, \ldots, R_n$ where $R_i \in R$-**alg** such that $\bigoplus_{i=1}^n R_i$ is a faithfully flat R-module.

In particular, note that a Zariski covering is a faithfully flat covering.

1.5 Projective modules

1.5.1 Basic definitions

In this part, we will introduce the notion of projective modules, which is a generalization of the notion of free modules. For more details/proofs, the reader can consult [KO, I].

Proposition 1.5.1. [KO, I, Lem 1.1]. *Let P be an R-module. The following are equivalent:*

(a) *P is a direct summand of a free module.*

(b) *Every short exact sequence $0 \longrightarrow M \longrightarrow N \xrightarrow{g} P \longrightarrow 0$ splits, i.e., there exists a homomorphism of R-module $f : P \longrightarrow N$ such that $g \circ f = \mathrm{Id}_P$.*

(c) *For all surjective map $f : M \longrightarrow N$, the map*

$$\mathrm{Hom}(1, f) : \mathrm{Hom}(P, M) \longrightarrow \mathrm{Hom}(P, N), \quad g \mapsto f \circ g,$$

is surjective, i.e., the functor $\mathrm{Hom}(P, -)$ is right exact.

(d) *There is a set $\{x_i : i \in I\} \subseteq P$ where I is an index set, and a subset $\{\phi_i : i \in I\}$ of $P^* = \mathrm{Hom}_R(P, R)$ satisfying*

  (i) *For all $x \in P$, $\phi_i(x) = 0$ for all but finitely many $i \in I$,*

  (ii) *For all $x \in P$, $x = \sum_{i \in I} \phi_i(x) x_i$.*

Definition 1.5.2. Let P be an R-module. Then P is called *projective* if P satisfies one of the equivalent conditions of Proposition 1.5.1. A subset S of P satisfying part (d) of Proposition 1.5.1 is called a *projective basis*.

Remark 1.5.3. [B:CA, I, §2.4, Ex. 1]. Every projective module is flat.

Example 1.5.4. We list several examples of projective modules:

(i) Any free module is projective. So, the notion of a projective module is a generalization of the notion of a free module.

Proof. This is clear since a free module is clearly a direct summand of itself. $\square$

(ii) If M is a projective R-module and S is an R-algebra, then $S \otimes_R M$ is projective as an S-module. This means that projective modules respect scalar extension.

Proof. Since M is projective, there exists an R-module N and an index set I such that $M \oplus N = \bigoplus_{i \in I} R_i$ where $R_i = R$. Hence,

$$(S \otimes_R M) \oplus (S \otimes_R N) \simeq S \otimes_R (M \oplus N) = S \otimes_R \bigoplus_{i \in I} R_i \simeq \bigoplus_i S_i,$$

where $S_i = S$ and the last isomorphism comes from the fact that tensor product commutes with direct sum. $\square$

(iii) [B:A, II, §5.3, after Thm. 2]. If P and N are projective R-modules, then so is $P \otimes_R N$.

Non-Example 1.5.5. (Un)fortunately, not all modules are projective. For example:

(i) The abelian group $\mathbb{Q}$ is not a projective $\mathbb{Z}$-module.

Proof. Here, we used the fact that a projective module over a principal ideal domain must be free ([B:A, VII, §3.1, Cor. 3]). We assume for a contradiction that $\mathbb{Q}$ is a projective $\mathbb{Z}$-module, hence free. It follows that $\operatorname{Hom}_{\mathbb{Z}}(\mathbb{Q}, \mathbb{Z}) \neq \{0\}$. However, we claim that $\operatorname{Hom}_{\mathbb{Z}}(\mathbb{Q}, \mathbb{Z}) = 0$. Let $\varphi : \mathbb{Q} \longrightarrow \mathbb{Z}$ be a $\mathbb{Z}$-linear map. For $m, n \in \mathbb{Z}$, one has that $m\varphi(1) = \varphi(m) = \varphi\left(\frac{m}{n}\right) n$. We assume m and n to be relatively prime. It follows that n divides $\varphi(1)$. Thus, $\varphi(1)$ has infinitely many divisors, so it must be 0. Thus, $\varphi = 0$. $\square$

(ii) Submodules of a projective module need not be projective.

Proof. The ring $\mathbb{Z}/4\mathbb{Z}$, as a module over itself, is free, whence projective. Now, we have the following: $U := \{0, 2\}$ is a submodule isomorphic to $\mathbb{Z}/2\mathbb{Z}$. We claim that U is not projective. Consider the map $\varphi : \mathbb{Z}/4\mathbb{Z} \longrightarrow \mathbb{Z}/2\mathbb{Z}$ given by $x \mapsto x \mod 2$. This is a surjective homomorphism of $\mathbb{Z}/4\mathbb{Z}$-modules. Suppose that $\mathbb{Z}/2\mathbb{Z}$ is projective as a $\mathbb{Z}/4\mathbb{Z}$-module. Then, by Proposition 1.5.1(b), there exists a $\mathbb{Z}/4\mathbb{Z}$-linear map $\psi : \mathbb{Z}/2\mathbb{Z} \longrightarrow \mathbb{Z}/4\mathbb{Z}$ such that $\varphi\psi = \mathrm{Id}_{\mathbb{Z}/2\mathbb{Z}}$. Thus, one has that $1 = \varphi(\psi(1)) = \psi(1) \mod 2$. It follows that $\psi(1) \in \{1, 3\}$. Thus, one has that $0 = \psi(1+1) = \psi(1)+\psi(1) = 2$, a contradiction. Therefore, $U \simeq \mathbb{Z}/2\mathbb{Z}$ is not a projective $\mathbb{Z}/4\mathbb{Z}$-module. □

In analogy to the theory of vector spaces, the modules which are generated by a finite set of elements are generally more manageable.

Definition 1.5.6. An R-module M is called *finitely generated* if there exist a free module F_0 of finite rank and an exact sequence $F_0 \longrightarrow M \longrightarrow 0$, and is called *finitely presented* if there exist free modules F_0, F_1 of finite ranks and an exact sequence $F_1 \longrightarrow F_0 \longrightarrow M \longrightarrow 0$.

Remark 1.5.7. We have several relations between the notions of projective modules, finitely generated modules and finitely presented modules.

(i) If an R-module M is finitely presented, then it is clearly finitely generated.

(ii) [KO, I,§2, Ex. 3] If an R-module M is projective and finitely generated, then it is finitely presented.

Proof. Since M is finitely generated projective, there exist a free module F_0 of finite rank and a submodule N of F_0 such that $F_0 = M \oplus N$. Thus, we can consider the projection $\pi : F_0 \longrightarrow M$. Moreover, $\ker \pi = N \simeq F_0/M$, whence is of finite type. There exists a free module F_1 of finite rank and a surjective map $F_1 \longrightarrow N$. Thus, this map induces $\alpha : F_1 \longrightarrow F_0$. We see that $F_1 \longrightarrow F_0 \longrightarrow M \longrightarrow 0$ is exact. Therefore, M is finitely presented. □

Lemma 1.5.8. *Suppose that M and N are finitely generated projective R-modules. Then $M \otimes_R N$ is aso finitely generated projective.*

Proof. There exist m, n and R-modules M' and N' such that $M \oplus M' \simeq R^m$ and $N \oplus N' \simeq R^n$. Then, by distributivity of the tensor product over direct sum, we have that $R^{mn} \simeq R^m \otimes_R R^n \simeq (M \otimes_R N) \oplus L$ where L is the rest of the terms. Therefore, $M \otimes_R N$ is finitely generated projective. □

Lemma 1.5.9. *Let $f : M \longrightarrow N$ be an R-module homomorphism between finitely generated projective R-modules. Then the following are equivalent:*

(i) *f is bijective,*

(ii) *the iduced map $f_{\mathfrak{m}} : M_{\mathfrak{m}} \longrightarrow N_{\mathfrak{m}}$ is bijective for all maximal ideals $\mathfrak{m}$ of R,*

(iii) *the iduced map $f[\mathfrak{m}] : M/\mathfrak{m}M \longrightarrow N/\mathfrak{m}N$ is bijective for all maximal ideals $\mathfrak{m}$ of R.*

Proof. (i) $\Longleftrightarrow$ (ii) is [B:CA, II,§3.3, Thm. 1]. We need to show (ii) $\Longleftrightarrow$ (iii). First, $\mathfrak{m}R_{\mathfrak{m}}$ is the radical of the local ring $R_{\mathfrak{m}}$ and $R_{\mathfrak{m}}/\mathfrak{m}R_{\mathfrak{m}} \simeq R/\mathfrak{m}$. Thus,

$$M_{\mathfrak{m}}/\mathfrak{m}M_{\mathfrak{m}} \simeq M \otimes (R_{\mathfrak{m}}/\mathfrak{m}R_{\mathfrak{m}}) \simeq M \otimes R/\mathfrak{m} \simeq M/\mathfrak{m}M.$$

We then have the following commutative diagram.

$$\begin{tikzcd}
M_{\mathfrak{m}}/\mathfrak{m}M_{\mathfrak{m}} \arrow[r, "\simeq"] \arrow[d, "\overline{f_{\mathfrak{m}}}"'] & M/\mathfrak{m}M \arrow[d, "f[\mathfrak{m}]"] \\
N_{\mathfrak{m}}/\mathfrak{m}N_{\mathfrak{m}} \arrow[r, "\simeq"'] & N/\mathfrak{m}N
\end{tikzcd}$$

Therefore, by [B:CA, II,§3.2, Cor of Prop 6], for each maximal ideal $\mathfrak{m}$, $f_{\mathfrak{m}}$ is bijective if and only if the induced map $f[\mathfrak{m}]$ is bijective. □

Next, we quote a result involving the bidual of a module. Recall that there exists a canonical map $c_M : M \longrightarrow M^{**}$ for an R-module M, given by $m \mapsto (\varphi \mapsto \varphi(m))$ for $\varphi \in M^*$.

Lemma 1.5.10. [B:A, II, §2.7, Cor. 4]. *Let P be a finitely generated projective R-module. Then the canonical map $c_P : P \longrightarrow P^{**}$ is an isomorphism.*

1.5.2 Some nice identities

So far, we have introduced the basic definitions of projective modules, finitely generated modules and finitely presented modules. Next, we will state several interesting results that we will need. The first proposition generalizes a result on finite-dimensional vector spaces. From now on, if $S \in R$-**alg**, we denote $M_S = M \otimes_R S$.

Proposition 1.5.11. [KO, I, Lem 4.3(a)], [B:A, II, §4.2, Cor.]. *Let P be a finitely generated projective R-module. Then the map*

$$\theta_P : P \otimes_R P^* \longrightarrow \mathrm{End}_R(P), \quad x \otimes f \mapsto (y \mapsto f(y)x),$$

is an isomorphism of R-modules.

Lemma 1.5.12. *Let R be a ring. Then the map $\beta : \mathrm{End}_R(R) \longrightarrow R$, $f \mapsto f(1)$ is an isomorphism of R-modules.*

Proof. First, we check that β is R-linear. One has that $\beta(f + rg) = (f + rg)(1) = f(1) + rg(1) = \beta(f) + r\beta(g)$. Thus, β is R-linear. Now, let $f \in \ker(\beta)$. So, $f(1) = 0$. Thus, $f(r) = f(r \cdot 1) = rf(1) = 0$ for all $r \in R$. Thus, $f = 0$. Hence, β is injective. It is left to show that it is surjective. Let $r \in R$. Then $r \cdot \mathrm{Id}_R$ does the job. $\square$

Proposition 1.5.13. (special case of [B:A, II, §5.1, Prop 3]). *Let R be a ring and let S be an R-algebra. Let M be an R-module. Then there exists a unique S-homomorphism $\mu : M_S \otimes_S (M^*)_S \longrightarrow (M \otimes_R M^*)_S$, sending $(m \otimes 1) \otimes (\varphi \otimes 1)$ to $m \otimes \varphi \otimes 1$. Moreover, this map μ is an isomorphism.*

Lemma 1.5.14. [B:A, II, §5.1, Prop 2]. *Let S and T be R-algebras. The map $\nu : (M \otimes_R S) \otimes_S (T \otimes_R S) \longrightarrow M \otimes_R T \otimes_R S$, $(m \otimes s_1) \otimes (t \otimes s_2) \mapsto m \otimes t \otimes s_1 s_2$ is an isomorphism of S-modules.*

Lemma 1.5.15. *Let N be an R-module. Then the map $\alpha_N : \mathrm{Hom}_R(R^m, N) \longrightarrow N^m$, $f \mapsto (f(e_i))_{i=1}^m$, where the e_i's form the canonical basis of R^m, is an isomorphism of R-modules.*

Proof. First, we show that α_N is a homomorphism of R-modules. One has that

$$\begin{aligned}\alpha_N(f + sg) &= ((f + sg)(e_i))_{i=1}^m = (f(e_i) + sg(e_i))_{i=1}^m \\ &= (f(e_i))_{i=1}^m + s(g(e_i))_{i=1}^m = \alpha_N(f) + s\alpha_N(g).\end{aligned}$$

Given $(n_1, \ldots, n_m) \in N^m$, we can define an R-linear map from the canonical basis of R^m to N by $e_i \mapsto n_i$. We know that this map will extend uniquely to a linear map $\alpha^N\big((n_i)_{i=1}^m\big) : R^m \longrightarrow N$, $e_i \mapsto n_i$ for all i. It it sufficient to show that that α^N and

α_N are mutually inverse.

$$\begin{aligned}\Big(\alpha^N\big(\alpha_N(f)\big)\Big)(e_j) &= \Big(\alpha^N\big(f(e_i)\big)_{i=1}^m\Big)(e_j) = f(e_j),\\ \alpha_N\Big(\alpha^N\big((n_i)_{i=1}^m\big)\Big) &= \Big(\alpha^N\big((n_i)_{i=1}^m\big)(e_j)\Big)_{j=1}^m = (n_j)_{j=1}^m.\end{aligned}$$

This completes the proof. □

Lemma 1.5.16. [B:A, II, §3.7, Prop 7]. *Let S_i, $i = 1, 2$, be two R-algebras. Let N_i, $i = 1, 2$, be S_i-modules. Then the map $\mu : N_1^{m_1} \otimes_R N_2^{m_2} \longrightarrow (N_1 \otimes_R N_2)^{m_1 m_2}$, $(n_i^{(1)}) \otimes (n_j^{(2)}) \mapsto (n_i^{(1)} \otimes n_j^{(2)})$ is an isomorphism.*

Proposition 1.5.17. [KO, I, Lem 4.1], [B:A, II, §4.4, Prop. 4]. *Let R be a ring and let S_i, $i = 1, 2$, be R-algebras. Let M_i and N_i be S_i-modules. Then the canonical R-linear map*

$$\gamma : \operatorname{Hom}_{S_1}(M_1, N_1) \otimes_R \operatorname{Hom}_{S_2}(M_2, N_2) \longrightarrow \operatorname{Hom}_{S_1 \otimes_R S_2}(M_1 \otimes_R M_2, N_1 \otimes_R N_2)$$

induced by the map $(f_1, f_2) \mapsto f_1 \otimes f_2$, where $(f_1 \otimes f_2)(m_1 \otimes m_2) = f_1(m_1) \otimes f_2(m_2)$, is an isomorphism if each M_i is a finitely generated projective S_i-module.

Proof. To prove the proposition, we need several steps.

Step 1: γ is well-defined. First, we need to show that the map $f_1 \otimes f_2$ is indeed an element of $\operatorname{Hom}_{S_1 \otimes S_2}(M_1 \otimes M_2, N_1 \otimes N_2)$. Given f_1 and f_2, we know that the induced map $f_1 \otimes f_2$ is R-linear. We need to show in addition that $f_1 \otimes f_2$ is also $S_1 \otimes_R S_2$-linear. Additivity is clear from R-linearity, it is left to show scalar multiplicativity.

$$\begin{aligned}(f_1 \otimes f_2)((s_1 \otimes s_2)(m_1 \otimes m_2)) &= (f_1 \otimes f_2)((s_1 m_1) \otimes (s_2 m_2))\\ &= f_1(s_1 m_1) \otimes f_2(s_2 m_2)\\ &= (s_1 f_1(m_1)) \otimes (s_1 f_2(m_2))\\ &= (s_1 \otimes s_2)(f_1 \otimes f_2)(m_1 \otimes m_2).\end{aligned}$$

It follows that $f_1 \otimes f_2 \in \operatorname{Hom}_{S_1 \otimes S_2}(M_1 \otimes M_2, N_1 \otimes N_2)$.

Next, we show that γ is well-defined, i.e., we have to show that the map

$$\operatorname{Hom}_{S_1}(M_1, N_1) \otimes_R \operatorname{Hom}_{S_2}(M_2, N_2) \longrightarrow \operatorname{Hom}_{S_1 \otimes_R S_2}(M_1 \otimes_R M_2, N_1 \otimes_R N_2),$$

induced by $(f_1, f_2) \mapsto f_1 \otimes f_2$ is R-balanced. Again, additivity is clear from S_i-linearity of f_i $(i = 1, 2)$ and from bilinearity of the tensor product.

$$\begin{aligned}((f_1 r) \otimes f_2)(m_1 \otimes m_2) &= (r f_1(m_1)) \otimes f_2(m_2) = f_1(r m_1) \otimes f_2(m_2)\\ &= (f_1 \otimes f_2)((r m_1) \otimes m_2) = (f_1 \otimes f_2)(m_1 \otimes (r m_2))\\ &= (f_1(m_1) \otimes (r f_2(m_2))) = (f_1 \otimes r f_2)(m_1 \otimes m_2).\end{aligned}$$

It follows that the map is R-balanced. Therefore, γ is a well-defined R-linear map.

Step 2: Special case $M_i = S_i^{m_i}$. Assume that $M_i = S_i^{m_i}$ where $i = 1, 2$ and $m_i \in \mathbb{N}$. We claim that the following diagram is commutative,

$$\begin{tikzcd}
\mathrm{Hom}_{S_1}(S_1^{m_1}, N_1) \otimes_R \mathrm{Hom}_{S_2}(S_2^{m_2}, N_2) \arrow[r, "\gamma"] \arrow[d, "\alpha_{N_1} \otimes \alpha_{N_2}"', "\simeq"] & \mathrm{Hom}_{S_1 \otimes_R S_2}((S_1 \otimes S_2)^{m_1 m_2}, N_1 \otimes_R N_2) \arrow[d, "\simeq"', "\alpha_{N_1 \otimes_R N_2}"] \\
N_1^{m_1} \otimes_R N_2^{m_2} \arrow[r, "\mu"'] & (N_1 \otimes_R N_2)^{m_1 m_2}
\end{tikzcd}$$

where the α's are as in Lemma 1.5.15 and μ is as in Lemma 1.5.16. Indeed,

$$\begin{aligned}(\alpha_{N_1 \otimes_R N_2})\gamma(f_1 \otimes f_2) &= (\gamma(f_1 \otimes f_2)(e_i^{(1)} \otimes e_j^{(2)})) = (f_1(e_i^{(1)}) \otimes f_2(e_j^{(2)})),\\ \mu(\alpha_{N_1} \otimes \alpha_{N_2})(f_1 \otimes f_2) &= \mu(\alpha_{N_1}(f_1) \otimes \alpha_{N_2}(f_2)) = ((f_1(e_i^{(1)})) \otimes (f_2(e_j^{(2)}))\\ &= (f_1(e_i^{(1)}) \otimes f_2(e_j^{(2)})).\end{aligned}$$

Thus, the claim is proved. It follows that $\gamma = \alpha^{-1}_{N_1 \otimes_R N_2} \mu(\alpha_{N_1} \otimes \alpha_{N_2})$ is an isomorphism.

Step 3: General case. Suppose that M_i is a finitely generated projective over S_i $(i = 1, 2)$. Then one can write $M_i \oplus M_i' = S_i^{m_i}$ for some S_i-module M_i' and some integer m_i. Now, one has the following,

$$\begin{aligned}\mathrm{Hom}_{S_1}(M_1 \oplus M_1', N_1) \otimes_R \mathrm{Hom}_{S_2}(M_2 \oplus M_2', N_2) &= \mathrm{Hom}_{S_1}(M_1, N_1) \otimes_R \mathrm{Hom}_{S_2}(M_2, N_2)\\ &\oplus \mathrm{Hom}_{S_1}(M_1, N_1) \otimes_R \mathrm{Hom}_{S_2}(M_2', N_2)\\ &\oplus \mathrm{Hom}_{S_1}(M_1', N_1) \otimes_R \mathrm{Hom}_{S_2}(M_2, N_2)\\ &\oplus \mathrm{Hom}_{S_1}(M_1', N_1) \otimes_R \mathrm{Hom}_{S_2}(M_2', N_2),\\ \mathrm{Hom}_{S_1 \otimes S_2}((S_1 \otimes S_2)^{m_1 m_2}, N_1 \otimes_R N_2) &= \mathrm{Hom}_{S_1 \otimes S_2}(M_1 \otimes M_2, N_1 \otimes N_2)\\ &\oplus \mathrm{Hom}_{S_1 \otimes S_2}(M_1 \otimes M_2', N_1 \otimes N_2)\\ &\oplus \mathrm{Hom}_{S_1 \otimes S_2}(M_1' \otimes M_2, N_1 \otimes N_2)\\ &\oplus \mathrm{Hom}_{S_1 \otimes S_2}(M_1' \otimes M_2', N_1 \otimes N_2).\end{aligned}$$

By definition of γ, we see that

$$\begin{aligned}
&\gamma(\mathrm{Hom}_{S_1}(M_1, N_1) \otimes_R \mathrm{Hom}_{S_2}(M_2, N_2)) \subseteq \mathrm{Hom}_{S_1 \otimes S_2}(M_1 \otimes M_2, N_1 \otimes N_2),\\
&\gamma(\mathrm{Hom}_{S_1}(M_1, N_1) \otimes_R \mathrm{Hom}_{S_2}(M_2', N_2)) \subseteq \mathrm{Hom}_{S_1 \otimes S_2}(M_1 \otimes M_2', N_1 \otimes N_2),\\
&\gamma(\mathrm{Hom}_{S_1}(M_1', N_1) \otimes_R \mathrm{Hom}_{S_2}(M_2, N_2)) \subseteq \mathrm{Hom}_{S_1 \otimes S_2}(M_1' \otimes M_2, N_1 \otimes N_2),\\
&\gamma(\mathrm{Hom}_{S_1}(M_1', N_1) \otimes_R \mathrm{Hom}_{S_2}(M_2', N_2)) \subseteq \mathrm{Hom}_{S_1 \otimes S_2}(M_1' \otimes M_2', N_1 \otimes N_2).
\end{aligned}$$

Since γ is an isomorphism by the special case, we see that $\gamma|\big((\mathrm{Hom}_{S_1}(M_1, N_1) \otimes_R \mathrm{Hom}_{S_2}(M_2, N_2)\big)$ is also an isomorphism. □

The following result is a corollary of the previous proposition, together with an explicit isomorphism.

Proposition 1.5.18. [B:A, II, §5.4, Prop 8]. *Let $S \in R$-**alg**. Let P be a finitely generated projective R-module. Then the map*

$$\upsilon : \mathrm{Hom}_R(P, R) \otimes_R S \longrightarrow \mathrm{Hom}_S(P \otimes_R S, R \otimes_R S),\ \varphi \otimes s \mapsto (m \otimes s' \mapsto \varphi(m) \otimes ss'),$$

is an isomorphism of S-modules.

1.5.3 The rank of a finitely generated projective module

For free module of finite type over a commutative ring, the notion of rank is well-defined. It is indeed possible to define a similar notion for finitely generated projective modules. The following Proposition/Theorem by Kaplansky tells us that the key point is to look at localizations of our module.

Proposition 1.5.19. [KO, I, Cor. 2.6]. *Let R be a local ring. Every finitely generated projective R-module is free.*

The proof of Proposition 1.5.19 uses non-obvious results from commutative algebra, e.g., Nakayama's Lemma (see [KO, I, Cor. 2.3]).

To extend the notion of rank, the idea is that we can localize R and then extend the scalar to the localization. We can then look at the rank of the corresponding free module of finite type. This means that we can obtain "local ranks" for our module.

Definition 1.5.20. An R-module M is called *locally free* if there exists a Zariski covering $\prod_{i=1}^{n} R_{f_i}$ such that $M_{f_i} := M \otimes_R R_{f_i}$ is free of finite type for all i.

Lemma 1.5.21. [KO, I, Lem 5.2]. *Let P be an R-module. Then the following are equivalent:*

(a) *P is finitely generated projective.*

(b) *P is finitely presented and, for all $\mathfrak{q} \in \operatorname{Spec}(R)$, $P_{\mathfrak{q}} = P \otimes_R R_{\mathfrak{q}}$ is a free $R_{\mathfrak{q}}$-module.*

(c) *P is finitely presented and, for $\mathfrak{m} \in \operatorname{Max}(R)$, $P_{\mathfrak{m}} = P \otimes_R R_{\mathfrak{m}}$ is a free $R_{\mathfrak{m}}$-module.*

(d) *For all $\mathfrak{m} \in \operatorname{Max}(R)$, there exists $f \in \mathfrak{m}$ such that P_f is a free R_f-module of finite type.*

(e) *P is locally free of finite rank.*

(f) *P is finitely generated and, for each $\mathfrak{p} \in \operatorname{Spec}(R)$, the $R_{\mathfrak{p}}$-module $P_{\mathfrak{p}}$ is free, say of rank $r_{\mathfrak{p}}$, the function $\operatorname{Spec}(R) \longrightarrow \mathbb{Z}$, $\mathfrak{p} \mapsto r_{\mathfrak{p}}$ is locally constant in the topological space* $\operatorname{Spec}(R)$*.*

Definition 1.5.22. Given a finitely generated projective module P over R. For $\mathfrak{p} \in \operatorname{Spec}(R)$, the rank of the free module $P \otimes_R R_{\mathfrak{p}}$ is called the *local rank* of P at $\mathfrak{p}$. We denote it by $r_{\mathfrak{p}}$. The module P is said to have *constant rank* if its local ranks are all equal.

Remark 1.5.23. Even if M is locally free of constant rank, M need not be free. An example is the the $\mathbb{Z}/6\mathbb{Z}$-algebra $\mathbb{Z}/2\mathbb{Z}$ with the obvious $\mathbb{Z}/6\mathbb{Z}$-action and algebra structure.

Proposition 1.5.24. [B:CA, II, §5.3, p.142]. *Let P and N be finitely generated projective R-modules. Then, for all $\mathfrak{p} \in \operatorname{Spec}(R)$,*

$$\begin{aligned}
\operatorname{rk}_{\mathfrak{p}}(P \oplus N) &= \operatorname{rk}_{\mathfrak{p}}(P) + \operatorname{rk}_{\mathfrak{p}}(N), \\
\operatorname{rk}_{\mathfrak{p}}(P \otimes_R N) &= \operatorname{rk}_{\mathfrak{p}}(P) \cdot \operatorname{rk}_{\mathfrak{p}}(N), \\
\operatorname{rk}_{\mathfrak{p}}(\operatorname{Hom}_R(P, N)) &= \operatorname{rk}_{\mathfrak{p}}(P) \cdot \operatorname{rk}_{\mathfrak{p}}(N), \\
\operatorname{rk}_{\mathfrak{p}}(P^*) &= \operatorname{rk}_{\mathfrak{p}}(P), \\
\operatorname{rk}_{\mathfrak{p}}\left(\bigwedge\nolimits^{k} P\right) &= \binom{\operatorname{rk}_{\mathfrak{p}}(P)}{k}.
\end{aligned}$$

1.5.4 Structure of a projective module

Review 1.5.25. We present an important concept that helps us to understand the structure of a finitely generated projective module. Let P be a finitely generated projective R-module. We equip $\mathbb{Z}$ with the discrete topology and consider the map $\mathrm{rk} : \mathrm{Spec}(R) \longrightarrow \mathbb{Z}, \mathfrak{p} \mapsto \mathrm{rk}_{\mathfrak{p}}(P)$. One can show that rk is continuous. For $i \in \mathbb{Z}$, consider $\mathrm{rk}^{-1}(i) = \{\mathfrak{p} \in \mathrm{Spec}(R) : \mathrm{rk}_{\mathfrak{p}}(P) = i\}$. Thus, $\mathrm{Spec}(R) = \bigcup_{i \in \mathbb{Z}} \mathrm{rk}^{-1}(i)$. Moreover, since $\{i\}$ is both open and closed, so is $\mathrm{rk}^{-1}(i)$. Now, since $\mathrm{Spec}(R)$ is quasi-compact, there exists an $n \in \mathbb{N}$ such that $\mathrm{Spec}(R) = \bigcup_{i=0}^{n} \mathrm{rk}^{-1}(i)$. By [B:CA, II, §4.3, Prop 15], there exists a unique family $\varepsilon_i \in R$, $i = 1, \dots, n$, such that the ε_i are orthogonal idempotents and that $\varepsilon_1 + \cdots + \varepsilon_n = 1$.

Now, one can show that R and P decompose as $R = R_1 \times \cdots \times R_n$ where $R_i = \varepsilon_i R$ and $P = P_1 \oplus \cdots \oplus P_n$ where $P_i = \varepsilon_i P$. Moreover, it can be shown that R_i is an ideal of R, that ε_i is its multiplicative identity and that P_i is finitely generated projective of constant rank over R_i (see [KO, Lem. 6.3]). Moreover, every $f \in \mathrm{End}_R(P)$ leaves every P_i invariant.

Lemma 1.5.26. *Let* $R = R_1 \oplus \cdots \oplus R_n$ *as in Review* 1.5.25. *Let* $\mathfrak{p}$ *be a prime ideal of* R. *Then* $\mathfrak{p}$ *is of the form* $\mathfrak{p} = I_1 \oplus \cdots \oplus I_n$ *where* I_i *is prime for exactly one* i *and* $I_j = R_j$ *for* $j \neq i$.

Proof. First, we show that if $I = I_1 \oplus \cdots \oplus I_n$ where I_i is prime for exactly one i and $I_j = R_j$ for $j \neq i$, then I is a prime ideal. Without loss of generality, assume that $i = 1$. Clearly, I is an ideal of R. Now, if $(a_1 + \cdots + a_n)(b_1 + \cdots + b_n) \in I$, then it follows that $a_1 \in I_1$ or $b_1 \in I_1$. Hence, I is a prime ideal.

Conversely, let $\mathfrak{p}$ be a prime ideal of R. Since $\mathfrak{p}$ is prime ideal, there exists ε_i such that $\varepsilon_i \notin \mathfrak{p}$. For all $j \neq i$, since $\varepsilon_i \varepsilon_j = 0 \in \mathfrak{p}$, $\varepsilon_j \in \mathfrak{p}$, whence $R_j \subseteq \mathfrak{p}$. Now, as an R-module, we have that $\mathfrak{p} = (\mathfrak{p} \cap R_1) \oplus (\mathfrak{p} \oplus R_n) = (\mathfrak{p} \cap R_i). \bigoplus_{j \neq i} R_j$. Clearly, $\mathfrak{p} \cap R_1$ is a proper ideal, it is prime since $\mathfrak{p}$ is a prime ideal. □

1.6 Some facts about symmetric and exterior powers of free modules

We refer the reader to [FH, Appendix B] and [B:A, III, §6 and §7] for more detailed treatments of multilinear algebra. We present a brief review of the subject and mention a few results that we will need later on.

Let M be an R-module. Recall that the tensor algebra of M is $T(M) = \bigoplus_{i\in\mathbb{N}} M^{\otimes i}$ where $M^{\otimes 0} = R$ and $M^{\otimes i} = M \otimes \cdots \otimes M$ (i times) for $i > 0$. The *symmetric algebra* $\operatorname{Sym}(M)$ is defined to be the quotient of $T(M)$ by the 2-sided ideal I generated by the elements $x \otimes y - y \otimes x$ where $x, y \in M$. The ideal I is graded. Then $\operatorname{Sym}(M) = \bigoplus_{i\in\mathbb{N}} \operatorname{Sym}^i(M)$ where $\operatorname{Sym}^i(M) = M^{\otimes i}/(I \cap M^{\otimes i})$.

Similarly, the *exterior* algebra $\bigwedge(M)$ is defined to be the quotient of $T(M)$ by the 2-sided ideal J generated by the elements $x \otimes x$ where $x \in M$. This ideal J is graded. Hence, $\bigwedge M = \bigoplus_{i\in\mathbb{N}} \bigwedge^i M$ where $\bigwedge^i M = T^i(M)/(J \cap M^{\otimes i})$. Finally, note that $\operatorname{Sym}^0 M = R = \bigwedge^0 M$.

We use the following convention

$$\binom{m}{-1} = 0, \quad \binom{m}{0} = 1 \quad \text{for } m \in \mathbb{Z},\ m \geq -1, \text{ and } \quad \binom{a}{b} = 0 \quad \text{if } 0 \leq a < b.$$

Proposition 1.6.1. [FH, Appendix B.2, Formula B.2]. *Let M and N be free R-modules of finite rank. Then* $\operatorname{Sym}^n(M \oplus N) \simeq \bigoplus_{k=0}^{n} \left(\operatorname{Sym}^k M \otimes \operatorname{Sym}^{n-k} N\right)$ *and* $\bigwedge^n(M \oplus N) \simeq \bigoplus_{k=0}^{n} \left(\bigwedge^k M \otimes \bigwedge^{n-k} N\right)$.

Lemma 1.6.2. *Let $a, n \in \mathbb{N}$. Then* $\sum_{l=0}^{n} \binom{a+l}{l} = \binom{a+1+n}{n}$.

Proof. We use induction on n. For $n = 0$, $\binom{a}{0} = 1 = \binom{a+1}{0}$. Thus, the base case holds.

Induction hypothesis: For some $n \geq 0$, $\sum_{l=0}^{n} \binom{a+l}{l} = \binom{a+1+n}{n}$.

Inductive step: Consider $\sum_{l=0}^{n+1} \binom{a+l}{l}$. One has that

$$\sum_{l=0}^{n+1} \binom{a+l}{l} = \sum_{l=0}^{n} \binom{a+l}{l} + \binom{a+n+1}{n+1} = \binom{a+1+n}{n} + \binom{a+n+1}{n+1},$$

where the second equality follows from induction hypothesis. Now, using the identity $\binom{n-1}{l} + \binom{n-1}{l-1} = \binom{n}{l}$, one sees that $\sum_{l=0}^{n+1} \binom{a+l}{l} = \binom{a+n+2}{n+1} = \binom{(a+1)+(n+1)}{n+1}$. The result follows by induction. □

From Proposition 1.6.1, we can deduce a nice known result.

Lemma 1.6.3. [B:A, III, §6.6, Thm. 1]. *Let M be a free R-module of rank r. Then, for $n \in \mathbb{N}$, one has that* $\dim(\operatorname{Sym}^n M) = \binom{n+r-1}{n}$.

Proof. We proceed by induction on r. For $r = 1$, then $\operatorname{Sym}^n(M) \simeq \operatorname{Sym}^n(R) \simeq R$ since $\operatorname{Sym}^n(R)$ only has one vector in the basis. Thus, the base case is proved.

Induction hypothesis: For some $r \geq 1$, $\dim(\operatorname{Sym}^l(R^r)) = \binom{r-1+l}{l}$ for all l.

Induction step: Consider $\dim(\operatorname{Sym}^n(R^{r+1})) = \dim(\operatorname{Sym}^n(R^r \oplus R))$. By Proposition 1.6.1, $\operatorname{Sym}^n(R^r \oplus R) = \bigoplus_{l=0}^{n} \operatorname{Sym}^l(R^r) \otimes \operatorname{Sym}^{n-l}(R) \simeq \bigoplus_{l=0}^{n} \operatorname{Sym}^l(R^r) \otimes R$. It follows that

$$\begin{aligned}\dim(\operatorname{Sym}^n(R^r \oplus R)) &= \sum_{l=0}^{n} \dim \operatorname{Sym}^l(R^r) = \sum_{l=0}^{n} \binom{r-1+l}{l} \\ &= \binom{r+n}{n} = \binom{(r+1)+n-1}{n},\end{aligned}$$

where the first equality uses the fact that $\dim_R(R) = 1$, the second equality uses induction hypothesis and the third equality follows from Lemma 1.6.2. The result now follows by induction. □

The next corollary is a technical result that we will need later on. It is likely known but we did not find a reference.

Corollary 1.6.4. *Let $r, n \in \mathbb{N}$ be such that $n \geq 2$. Then*

$$\sum_{i=2}^{n} \binom{i}{2}\binom{r+n-i-2}{n-i} = \binom{r+n-1}{n-2} = \sum_{i=2}^{n} \binom{i+1}{3}\binom{r+n-i-3}{n-i}.$$

Proof. Let F be a field. We have the following

$$\begin{aligned}\sum_{i=2}^{n} \binom{i}{2}\binom{r+n-i-2}{n-i} &= \sum_{i=2}^{n} \dim \operatorname{Sym}^{i-2}(F^3) \dim \operatorname{Sym}^{n-i}(F^{r-1}) \\ &= \sum_{k=0}^{n-2} \dim \operatorname{Sym}^{k}(F^3) \dim \operatorname{Sym}^{(n-2)-k}(F^{r-1}) \\ &= \dim \operatorname{Sym}^{n-2}(F^{r+2}) = \binom{r+n-1}{n-2}.\end{aligned}$$

For the second equality, we have that

$$\begin{aligned}\sum_{i=2}^{n}\binom{i+1}{3}\binom{r+n-i-3}{n-i} &= \sum_{i=2}^{n}\dim\operatorname{Sym}^{i-2}(F^4)\dim\operatorname{Sym}^{n-k}(F^{r-2})\\ &= \sum_{k=0}^{n-2}\dim\operatorname{Sym}^{k}(F^4)\dim\operatorname{Sym}^{(n-2)-k}(F^{r-2})\\ &= \dim\operatorname{Sym}^{n-2}(F^{r+2}) = \binom{r+n-1}{n-2}. \qquad \square\end{aligned}$$

The next few propositions relate these constructions to the scalar extension process.

Proposition 1.6.5. [B:A, III, §6.4, Prop 7], [B:A, III, §7.5, Prop. 8]. *Let R be a ring, let S be an R-algebra. Let M be an R-module. Then the maps*

$$\nu : \left(\bigwedge^n M\right)\otimes_R S \longrightarrow \bigwedge^n (M\otimes_R S),\ (m_1\wedge\cdots\wedge m_n)\otimes s\mapsto (m_1\otimes 1)\wedge\cdots\wedge(m_n\otimes s),$$

$$\nu' : (\operatorname{Sym}^n M)\otimes_R S \longrightarrow \operatorname{Sym}^n(M\otimes_R S),\ (m_1\circ\cdots\circ m_n)\otimes s\mapsto (m_1\otimes 1)\circ\cdots\circ(m_n\otimes s),$$

are isomorphisms of S-modules.

Notation 1.6.6. Let M and N be R-modules and let $f\in\operatorname{Hom}_R(M,N)$. Recall that $T(M)$ and $T(N)$ are the tensor algebras of M and N respectively. Then f induces a map from $T(M)$ to $T(N)$, sending $m_1\otimes\cdots\otimes m_n$ to $f(m_1)\otimes\cdots\otimes f(m_n)$. Let us denote $\bigwedge^n(f) : \bigwedge^n M \longrightarrow \bigwedge^n N$ the corresponding, and similarly the map $\operatorname{Sym}^n(f) : \operatorname{Sym}^n(M) \longrightarrow \operatorname{Sym}^n(N)$.

We have the following well-known fact.

Proposition 1.6.7. [B:A, III, §6, §7]. *Let R be a ring, let S be an R-algebra. Let M be an R-module. Let ν,ν' be as in Proposition 1.6.5. Let $f\in\operatorname{End}_R(M)$. Then*

(a) *The following diagram is commutative*

$$\begin{tikzcd}(\bigwedge^n M)\otimes_R S \arrow[r, "\nu"] \arrow[d, "(\bigwedge^n f)\otimes \mathrm{Id}_S"'] & \bigwedge^n(M\otimes_R S) \arrow[d, "\bigwedge^n(f\otimes \mathrm{Id}_S)"] \\ (\bigwedge^n N)\otimes_R S \arrow[r, "\nu"'] & \bigwedge^n(N\otimes_R S)\end{tikzcd}$$

where ν is the canonical map.

(b) *The following diagram is commutative*

$$\begin{tikzcd}
(\operatorname{Sym}^n M) \otimes_R S \arrow[r, "\nu'"] \arrow[d, "\left(\operatorname{Sym}^n f\right)\otimes \mathrm{Id}_S"'] & \operatorname{Sym}^n(M \otimes_R S) \arrow[d, "\operatorname{Sym}^n(f\otimes \mathrm{Id}_S)"] \\
(\operatorname{Sym}^n N) \otimes_R S \arrow[r, "\nu'"'] & \operatorname{Sym}^n(N \otimes_R S)
\end{tikzcd}$$

where ν is the canonical map.

Proof. For (a), let $(m_1 \wedge \cdots \wedge m_n) \otimes s \in \bigwedge^n M \otimes_R S$. One has that

$$\begin{aligned}
\left(\bigwedge^n (f \otimes \mathrm{Id}_S) \circ \nu\right)((m_1 \wedge \cdots \wedge m_n) \otimes s) &= \bigwedge^n (f \otimes \mathrm{Id}_S)((m_1 \otimes 1) \wedge \cdots \wedge (m_n \otimes s)) \\
&= (f(m_1) \otimes 1) \wedge \cdots \wedge (f(m_n) \otimes 1)
\end{aligned}$$

and that

$$\begin{aligned}
\left(\nu' \circ \left(\bigwedge^n (f) \otimes \mathrm{Id}_S\right)\right)((m_1 \wedge \cdots \wedge m_n) \otimes s) &= \nu'((f(m_1) \wedge \cdots \wedge f(m_n)) \otimes s) \\
&= (f(m_1) \otimes 1) \wedge \cdots \wedge (f(m_n) \otimes 1).
\end{aligned}$$

Thus, the diagram of (a) is commutative. The proof for (b) is totally similar. □

Chapter 2

Trace formulas

2.1 Trace of endomorphisms of free modules and applications

In this section, we present several results on the trace of endomorphisms on free modules and related facts. Note that all facts presented in this section are standard. Unless stated otherwise, R still denotes a commutative ring, M denotes an R-module, $S, T \in R$-**alg**.

Lemma 2.1.1. (i) *Let* $\nu : M \longrightarrow N$ *be an isomorphism of free* R*-modules of finite rank, and let* $f \in \operatorname{End}_R(N)$*. Then* $\operatorname{tr}(f) = \operatorname{tr}(\nu^{-1} f \nu)$*.*

(ii) *Let* N *be a free* R*-module and let* $f, g \in \operatorname{End}_R(N)$*. Then* $\operatorname{tr}(fg) = \operatorname{tr}(gf)$*.*

This lemma has the following immediate corollary (using Proposition 1.6.7).

Corollary 2.1.2. *Let* M *be a finitely generated projective* R*-module. Assume that* $M \otimes_R S$ *is free of finite rank as an* S*-module. Then we have* $\operatorname{tr}\big((\bigwedge^n f) \otimes \operatorname{Id}_S\big) = \operatorname{tr}\big(\bigwedge^n (f \otimes \operatorname{Id}_S)\big)$ *and* $\operatorname{tr}\big((\operatorname{Sym}^n f) \otimes \operatorname{Id}_S\big) = \operatorname{tr}\big(\operatorname{Sym}^n (f \otimes \operatorname{Id}_S)\big)$*.*

The next two lemmas are general results that will allow us to use Lemma 2.1.1 to study the trace.

Lemma 2.1.3. *Let $f \in \operatorname{End}_R(M)$, let $\nu : (M\otimes_R S)\otimes_S(T\otimes_R S) \longrightarrow M\otimes_R T\otimes_R S$, as in Lemma 1.5.14, be the canonical isomorphism. Then the following diagram commutes.*

$$\begin{tikzcd}
(M \otimes_R S) \otimes_S (T \otimes_R S) \arrow[r, "\nu"] \arrow[d, "(f\otimes \mathrm{Id}_S)\otimes \mathrm{Id}_{T\otimes S}"'] & M \otimes_R T \otimes_R S \arrow[d, "f\otimes \mathrm{Id}_T \otimes \mathrm{Id}_S"] \\
(M \otimes_R S) \otimes_S (T \otimes_R S) \arrow[r, "\nu"'] & M \otimes_R T \otimes_R S
\end{tikzcd}$$

Proof. Let $(m \otimes s_1) \otimes (t \otimes s_2) \in (M \otimes_R S) \otimes_S (T \otimes_R S)$. One has that

$$\begin{aligned}
((f \otimes \mathrm{Id}_T \otimes \mathrm{Id}_S) \circ \nu)((m \otimes s_1) \otimes (t \otimes s_2)) &= (f \otimes \mathrm{Id}_T \otimes \mathrm{Id}_S)(m \otimes t \otimes s_1 s_2) \\
&= f(m) \otimes t \otimes s_1 s_2. \\
(\nu \circ (f \otimes \mathrm{Id}_S) \otimes \mathrm{Id}_{T\otimes S})((m \otimes s_1) \otimes (t \otimes s_2)) &= \nu((f(m) \otimes s_1) \otimes (t \otimes s_2) \\
&= f(m) \otimes t \otimes s_1 s_2.
\end{aligned}$$

□

Lemma 2.1.4. *Let $M_1, \dots, M_n$ be R-modules. Let $f_i \in \operatorname{End}_R(M_i)$ for $i = 1, \dots, n$. Consider the canonical S-module isomorphism*

$$\nu : M_1 \otimes_R \cdots \otimes_R M_n \otimes_R S \longrightarrow (M_1 \otimes_R S) \otimes_S \cdots \otimes_S (M_n \otimes_R S),$$

$m_1 \otimes \cdots \otimes m_n \otimes s \mapsto s((m_1 \otimes 1) \otimes \cdots \otimes (m_n \otimes 1))$. *Then the following diagram commutes.*

$$\begin{tikzcd}
M_1 \otimes_R \cdots \otimes_R M_n \otimes_R S \arrow[r, "\nu"] \arrow[d, "f_1\otimes\cdots\otimes f_n\otimes \mathrm{Id}_S"'] & (M_1 \otimes_R S) \otimes_S \cdots \otimes_S (M_n \otimes_R S) \arrow[d, "(f_1\otimes \mathrm{Id}_S)\otimes\cdots\otimes(f_n\otimes \mathrm{Id}_S)"] \\
M_1 \otimes_R \cdots \otimes_R M_n \otimes_R S \arrow[r, "\nu"'] & (M_1 \otimes_R S) \otimes_S \cdots \otimes_S (M_n \otimes_R S)
\end{tikzcd}$$

Proof. Let $m_1 \otimes \cdots \otimes m_n \otimes s \in M_1 \otimes_R \cdots \otimes_R M_n \otimes_R S$. Denote $f_{i,S} = f_i \otimes \mathrm{Id}_S$. One has the following

$$\begin{aligned}
\big((f_{1,S} \otimes \cdots \otimes f_{n,S}) \circ \nu\big)(m_1 \otimes \cdots m_n \otimes s) &= s\Big(\big(f_1(m_1) \otimes 1\big) \otimes \cdots \otimes \big(f(m_n) \otimes 1\big)\Big), \\
\big(\nu \circ (f_1 \otimes \cdots \otimes f_n \otimes \mathrm{Id}_S)\big)(m_1 \otimes \cdots \otimes m_n \otimes s) &= \nu\big(f(m_1) \otimes \cdots \otimes f(m_n) \otimes s\big) \\
&= s\Big(\big(f_1(m_1) \otimes 1\big) \otimes \cdots \otimes \big(f(m_n) \otimes 1\big)\Big),
\end{aligned}$$

completing the proof. □

Lemma 2.1.5. *Let $g \in \operatorname{End}_S(M \otimes_R S)$. Assume that that $M \otimes_R S$ is a free S-module of finite rank r. Then $M\otimes_R S\otimes_R T$ is a free $S\otimes_R T$-module of finite rank r. Moreover,* $\operatorname{tr}(g) \otimes 1_T = \operatorname{tr}(g \otimes \mathrm{Id}_T)$.

Proof. By assumption, there exist $n_1, \dots, n_r \in M \otimes_R S$ such that $M \otimes_R S \simeq \bigoplus_{i=1}^r Sn_i$ where $Sn_i \simeq S$ as S-modules. Since the tensor product commutes with direct sum, one sees that $(M \otimes_R S) \otimes_R T \simeq \bigoplus_{i=1}^r (Sn_i \otimes_R T) \simeq \bigoplus_{i=1}^r (S \otimes_R T)$ as $S \otimes_R T$-modules. So, $(M \otimes_R S) \otimes_R T$ is a free $S \otimes_R T$-module and $n_i \otimes 1_T$, $1 \leq i \leq r$, is an $S \otimes_R T$ basis of $(M \otimes_R S) \otimes_R T$.

Suppose that $g(n_i) = \sum_j \alpha_{ji} n_j$ with $\alpha_{ji} \in S$. Hence,

$$\mathrm{tr}(g) \otimes 1_T = \left(\sum_i \alpha_{ii} \right) \otimes 1_T = \sum_i (\alpha_{ii} \otimes 1_T).$$

Now, $(g \otimes \mathrm{Id}_T)(n_i \otimes 1_T) = g(n_i) \otimes 1_T = \left(\sum_j \alpha_{ji} n_j \right) \otimes 1_T = \sum_j (\alpha_{ji} \otimes 1_T)(n_j \otimes 1_T)$. It follows that $\mathrm{tr}(g \otimes \mathrm{Id}_T) = \sum_i (\alpha_{ii} \otimes 1_T)$. The result follows. $\square$

Notation 2.1.6. Let $\tau_M : M \otimes_R M^* \longrightarrow R$, $x \otimes \phi \mapsto \phi(x)$. Assume that the map $\theta_M : M \otimes M^* \longrightarrow \mathrm{End}_R(M)$, $x \otimes f \mapsto (y \mapsto f(y)x)$ is an isomorphism, e.g. if M is finitely generated projective. In this case, we denote $t_M = \tau_M \theta_M^{-1} : \mathrm{End}_R(M) \longrightarrow R$.

Lemma 2.1.7. *Let M be a free R-module of finite rank, let $f \in \mathrm{End}_R(M)$. Then the map t_M is precisely the trace map, i.e., $t_M(f) = \mathrm{tr}(f)$.*

Proof. Choose a basis $\{m_1, \dots, m_n\}$ of M and a dual basis $\{\phi_1, \dots, \phi_n\}$ so that $\phi_i(m_j) = \delta_{ij}$. So, $\{m_i \otimes \phi_j\}$ is a basis for $M \otimes_R M^*$. Since θ is an isomorphism, we may write $\theta^{-1}(f) = \sum_{i,j} a_{ij}(m_i \otimes \phi_j)$ for $a_{ij} \in R$. It follows that $f = \sum_{i,j} a_{ij} \theta(m_i \otimes \phi_j)$. Next, $f(m_i) = \sum_{i,j} a_{ij} \theta(m_i \otimes \phi_j)(m_k) = \sum_{i,j} a_{ij} \phi_j(m_k) m_i = \sum_i a_{ik} m_k$. Hence, only a_{ii} constibutes to the trace. Therefore, $\mathrm{tr}(f) = \sum_i a_{ii}$.

Next, $t_M(f) = \tau_M \theta_M^{-1}(f) = \sum_{i,j} a_{ij} \tau_M(m_i \otimes \phi_j) = \sum_{i,j} a_{ij} \delta_{ij} = \sum_i a_{ii} = \mathrm{tr}(f)$. $\square$

Proposition 2.1.8. *Let μ be as in Proposition* 1.5.13 *and ν be as in Proposition* 1.5.18*. Let P be a finitely generated projective R-module. Then the following diagram of S-modules is commutative:*

$$\begin{tikzcd}
P \otimes_R P^* \otimes_R S \arrow[rr, "\mu", "\simeq"'] \arrow[d, "\theta_P \otimes \mathrm{Id}_S"', "\simeq"] \arrow[rrd, "\tau_P \otimes \mathrm{Id}_S"] & & (P \otimes_R S) \otimes_S (P^* \otimes_R S) \arrow[rr, "\mathrm{Id}_{P_S} \otimes \upsilon", "\simeq"'] & & (P \otimes_R S) \otimes_S (P \otimes_R S)^* \arrow[d, "\simeq"', "\theta_{P_S}"] \arrow[lld, "\tau_{P_S}"'] \\
\mathrm{End}_R(P) \otimes_R S \arrow[rr, "t_P \otimes \mathrm{Id}_S"'] & & R \otimes_R S = S & & \mathrm{End}_S(P \otimes_R S) \arrow[ll, "t_{P_S}"]
\end{tikzcd}$$

Proof. Clearly, the two "outer triangles" in the diagram commute by definition. It is enough to check commutativity of the middle triangle. Let $m\otimes\varphi\otimes s \in P\otimes_R P^*\otimes_R S$. For $n\otimes t\in P\otimes_R S$, one has the following:

$$\begin{aligned}
\tau_{P_S}(\mathrm{Id}_{P_S}\otimes\upsilon)\mu(m\otimes\varphi\otimes s) &= \tau_{P_S}(\mathrm{Id}_{P_S}\otimes\upsilon)\big((m\otimes s)\otimes(\varphi\otimes 1)\big)\\
&= \tau_{P_S}((m\otimes s)\otimes\upsilon(\varphi\otimes 1))\\
&= \upsilon(\varphi\otimes 1)(m\otimes s)\\
&= \varphi(m)\otimes s\\
&= (\tau_P\otimes\mathrm{Id}_S)(m\otimes\varphi\otimes s).
\end{aligned}$$

Therefore, the diagram is commutative, as desired. □

Corollary 2.1.9. *Let P be a finitely generated projective R-module and let f be an element of $\mathrm{End}_R(P)$. Then $t_P(f)\otimes 1_S = t_{P_S}(f\otimes\mathrm{Id}_S)$.*

Proof. We know that $\theta_P : P\otimes_R P^* \longrightarrow \mathrm{End}_R(P)$ is an isomorphism, we may write $f=\theta_P\big(\sum_{i=1}^l x_i\otimes\phi_i\big)$. So, $f(m)=\sum_{i=1}^l \phi_i(m)x_i$. By commutativity of the diagram in Proposition 2.1.8,

$$\begin{aligned}
t_P(f)\otimes 1_S = t_P\theta_P\left(\sum_{i=1}^l x_i\otimes\phi_i\right)\otimes 1_S &= (\tau_P\otimes\mathrm{Id}_S)\left(\sum_{i=1}^l (x_i\otimes\phi_i\otimes 1_S)\right)\\
&= t_{P_S}\theta_{P_S}(\mathrm{Id}_{P_S}\otimes\nu)\mu\left(\sum_{i=1}^l x_i\otimes\phi_i\otimes 1_S\right)\\
&= t_{P_S}\theta_{P_S}\left(\sum_{i=1}^l (x_i\otimes 1)\otimes\nu(\phi_i\otimes 1)\right).
\end{aligned}$$

It is left to show that $\theta_{P_S}\left(\sum_{i=1}^l (x_i\otimes 1)\otimes\nu(\phi_i\otimes 1)\right) = f\otimes\mathrm{Id}_S$. For $m\otimes s\in P\otimes_R S$,

$$(f\otimes\mathrm{Id}_S)(m\otimes s) = f(m)\otimes s = \theta_P\left(\sum_{i=1}^l x_i\otimes\phi_i\right)(m)\otimes s = \sum_{i=1}^l \phi_i(m)x_i\otimes s,$$

$$\begin{aligned}
\theta_{P_S}\left(\sum_{i=1}^l (x_i\otimes 1)\otimes\nu(\phi_i\otimes 1)\right)(m\otimes s) &= \sum_{i=1}^l \nu(\phi_i\otimes 1)(m\otimes s)(x_i\otimes 1)\\
&= \sum_{i=1}^l (\phi_i(m)s)(x_i\otimes 1)\\
&= \sum_{i=1}^l \phi_i(m)x_i\otimes s.
\end{aligned}$$

□

2.2 Generalization of the definition of the trace

In this section, we define the trace of an endomorphism of a finitely generated projective R-module P. Our references are [KO] and [Alm]. None of the results are truly new results, although the existing literature does not provide as many details as are given here.

The next proposition will help us to define a generalization of the trace.

Proposition 2.2.1. *Let M be an R-module, let $f \in \mathrm{End}_R(M)$ and let $(R_i)_{i=1}^n$ be a faithfully flat covering (see Definition 1.4.6) such that each $M_i = M \otimes_R R_i$ is free of finite rank as R_i-module. Then there is a unique $r \in R$ such that $r \otimes 1_{R_i} = \mathrm{tr}(f \otimes_R \mathrm{Id}_{R_i})$ holds for all i, $1 \leq i \leq n$. Moreover, $r \in R$ is independent of the faithfully flat covering, i.e., if $(S_j)_{j=1}^m$ is another faifully flat covering such that each $M \otimes_R S_j$ is free of finite rank, then $r \otimes 1_{S_j} = \mathrm{tr}(f \otimes \mathrm{Id}_{S_{g_i}})$, $1 \leq j \leq m$.*

Proof. Let $S = \bigoplus_{i=1}^n R_i$. Recall that we have two linear maps $\epsilon_1 : S \longrightarrow S \otimes_R S$, $s \mapsto s \otimes 1_S$, and $\epsilon_2 : S \longrightarrow S \otimes_R S$, $s \mapsto 1_S \otimes s$. By faithful flatness, we can identify $R = \{x \in S : \epsilon_1(x) = \epsilon_2(s)\}$. Now, since $S \otimes_R S = \bigoplus_{i,j} R_i \otimes_R R_j$, an element $(r_i)_{i=1}^n$ of S, $r_i \in R_i$, defines an element $r \in R$ if and only if $r_i \otimes 1_{R_j} = 1_{R_i} \otimes r_j$ in $R_i \otimes_R R_j$. In this case, $r \in R$ is unique by injectivity of the map $R \longrightarrow R \otimes_R S$ in Proposition 1.3.2. We let $r_i = \mathrm{tr}(f \otimes \mathrm{Id}_{R_i})$, which exists since M_i is free of finite rank. Then, by Lemma 2.1.5 and by symmetry in i and j,

$$r_i \otimes 1_{R_j} = \mathrm{tr}(f \otimes \mathrm{Id}_{R_i}) \otimes 1_{R_j} = \mathrm{tr}(f \otimes \mathrm{Id}_{R_i \otimes_R R_j}) = 1_{R_i} \otimes \mathrm{tr}(f \otimes \mathrm{Id}_{R_j}) = 1_{R_i} \otimes r_j.$$

Then, there exists $r \in R$ such that $r \otimes 1_{R_i} = \mathrm{tr}(f \otimes \mathrm{Id}_{R_i})$.

Suppose that (S_j) is another faithfully flat covering as in the statement of the proposition. By Proposition 1.2.7, $(\bigoplus_{i=1}^n R_i) \otimes_R \left(\bigoplus_{j=1}^m S_j\right) \simeq \bigoplus_{i,j} R_i \otimes_R S_j$ is faithfully flat as an R-module. Hence, the family T_{ij}, $T_{ij} = R_i \otimes_R S_j$, is a faithfully flat covering. By Lemma 2.1.5, $M \otimes_R T_{ij} = (M \otimes_R R_i) \otimes_R S_j$ is a free T_{ij}-module of finite rank. Next,

$$r \otimes 1_{T_{ij}} = (r \otimes 1_{R_i}) \otimes 1_{S_j} = \mathrm{tr}(f \otimes \mathrm{Id}_{R_i}) \otimes 1_{S_j} = \mathrm{tr}(f \otimes \mathrm{Id}_{T_{ij}}),$$

where third equality follows from the fact that $M \otimes_R R_i$ is a free R_i-module and from Lemma 2.1.5. Hence, by uniqueness, the element of R determined by the covering

(T_{ij}) is r. Exchanging the role of (R_i) and (S_j), it follows that r is also given by the covering (S_j). ☐

Definition 2.2.2. Let P be a finitely generated projective R-module. Then there exists a faithfully flat covering (R_i) of R such that $P \otimes_R R_i$ is free of finite rank. For any $f \in \mathrm{End}_R(P)$, we define the *trace* of f, denoted $\mathrm{tr}(f)$, as the unique element of R determined in Proposition 2.2.1.

The next corollary tells us that the trace behaves well under base change, as one would expect from the free case.

Corollary 2.2.3. [Alm, Prop. 1.3(i)]. *Let P be a finitely generated projective R-module, $f \in \mathrm{End}_R(P)$, $S \in R$-**alg**. Then, $\mathrm{tr}(f) \otimes 1_S = \mathrm{tr}(f \otimes \mathrm{Id}_S)$. In particular, as a direct consequence, $\mathrm{tr}(f \otimes \mathrm{Id}_{R_\mathfrak{p}}) = \mathrm{tr}(f) \otimes 1_{R_\mathfrak{p}}$ for all $\mathfrak{p} \in \mathrm{Spec}(R)$.*

Proof. We fix a faithfully flat covering (R_i) of R with $M \otimes_R R_i$ free of finite rank as R_i-modules. Then $P \otimes_R R_i \otimes_R S$ is a free $R_i \otimes_R S$-module with finite rank. We consider $R_i \otimes_R S$ as an S-algebra; we obtain that $\bigoplus_{i=1}^n (R_i \otimes_R S) \simeq (\bigoplus_{i=1}^n R_i) \otimes_R S$ is a faithfully flat S-module. Hence, $R_i \otimes_R S$ is a faithfully flat covering of S. Moreover, $(P \otimes_R S) \otimes_S (S \otimes_R R_i) \simeq P \otimes_R S \otimes_R R_i$ is a free $S \otimes_R R_i$-module of finite rank.

By faithfully flat descent of elements in S with respect to the covering $(R_i \otimes_R S)$, we need to show $(\mathrm{tr}(f) \otimes_R 1_S) \otimes_S (1_{R_i \otimes_R S}) = \mathrm{tr}(f \otimes_R \mathrm{Id}_S) \otimes_S 1_{R_i \otimes_R S}$.

By definition of $\mathrm{tr}(f \otimes_R \mathrm{Id}_S)$, $\mathrm{tr}(f \otimes_R \mathrm{Id}_S) \otimes 1_{R_i \otimes_R S} = \mathrm{tr}((f \otimes_R \mathrm{Id}_S) \otimes_S \mathrm{Id}_{R_i \otimes_R S})$. By Lemma 2.1.3, under the canonical isomorphism $(P \otimes_R S) \otimes_S (R_i \otimes_R S) \simeq (P \otimes_R R_i) \otimes_R S$ as $R_i \otimes_R S$-modules, the map $f \otimes \mathrm{Id}_S \otimes_R \mathrm{Id}_{R_i \otimes S}$ is conjugate to $f \otimes_R \mathrm{Id}_{R_i} \otimes_R \mathrm{Id}_S$. Hence,

$$\begin{aligned}
\mathrm{tr}((f \otimes_R \mathrm{Id}_S) \otimes_S \mathrm{Id}_{R_i \otimes_R S}) &= \mathrm{tr}((f \otimes_R \mathrm{Id}_{R_i}) \otimes_R \mathrm{Id}_S) = \mathrm{tr}(f \otimes_R \mathrm{Id}_{R_i}) \otimes_R 1_S \\
&= \mathrm{tr}(f) \otimes_R 1_{R_i} \otimes_R 1_S = (\mathrm{tr}(f) \otimes_R 1_S) \otimes_R 1_{R_i} \\
&= (\mathrm{tr}(f) \otimes_R 1_S) \otimes_S 1_{R_i \otimes_R S},
\end{aligned}$$

where the fourth and fifth equalities are true when the modules are viewed as S-modules. ☐

Now we have obtained a definition of the trace of endomorphisms of finitely generated projective module. The next proposition will tell us that this definition gives us the usual trace in the case of a free module of finite rank. Part (a) says that our definition coincides with Bourbaki's definition in [B:A, II, §4.3].

Proposition 2.2.4. *Let P be a finitely generated projective R-module, $f \in \operatorname{End}_R(P)$.*

(a) $\operatorname{tr}(f) = t_P(f)$ *where t_P is defined as in Notation* 2.1.6 *with* $\tau_P(m \otimes \varphi) = \varphi(m)$, *and* $\theta_P(m \otimes \varphi)(m') = \varphi(m')m$.

(b) *Let N be an R-module such that $P \oplus N$ is free of finite rank. Extend f to* $\widetilde{f} \in \operatorname{End}_R(M \oplus N)$ *by* $\widetilde{f}(m \oplus n) = f(m)$. *Then* $\operatorname{tr}(f) = \operatorname{tr}(\widetilde{f})$.

Proof. (a) For $\mathfrak{p} \in \operatorname{Spec}(R)$, by Corollary 2.1.9 and Lemma 2.1.7, we see that $t_P(f) \otimes 1_{R_\mathfrak{p}} = t_{P_\mathfrak{p}}(f \otimes \operatorname{Id}_{R_\mathfrak{p}}) = \operatorname{tr}(f \otimes \operatorname{Id}_{R_\mathfrak{p}})$. Since this is true for all $\mathfrak{p} \in \operatorname{Spec}(R)$, it follows from Proposition 1.1.13 that $\operatorname{tr}(f) = t_P(f)$, as desired.

(b) We have the following commutative diagram.

$$\begin{tikzcd}
\operatorname{End}_R(M \oplus N) \arrow[dr, "t_{M\oplus N}"'] & & (M \oplus N) \otimes (M \oplus N)^* \arrow[ll, "\theta_{M\oplus N}"'] \arrow[dl, "\tau_{M\oplus N}"] \\
& R &
\end{tikzcd}$$

Next, since $(M \oplus N)^* \simeq M^* \oplus N^*$, we have that

$$(M \oplus N) \otimes (M \oplus N)^* \simeq (M \otimes M^*) \oplus (M \otimes N^*) \oplus (N \otimes M^*) \oplus (N \otimes N^*).$$

By construction, one sees that $\theta_{M\oplus N}^{-1}(\widetilde{f}) = \theta_M^{-1}(f) \in M \otimes M^* \subseteq (M \oplus N) \otimes (M \oplus N)^*$ and $\tau_{M\oplus N}|(M \otimes M^*) = \tau_M$. Therefore, by part (a),

$$\operatorname{tr}(f) = \tau_M \circ \theta_M^{-1}(f) = \tau_{M\oplus N} \circ \theta_{M\oplus N}^{-1}(\widetilde{f}) = \operatorname{tr}(\widetilde{f}),$$

completing the proof. □

Remark 2.2.5. Before going any further, let us note that an equivalent definition of the trace of an endomorphism of a finitely generated projective module over a commutative ring was introduced in [Alm]. Consider the canonical isomorphisms

$$\begin{tikzcd}
\operatorname{End}_R(P^*) \arrow[r, "\xi"] & \operatorname{Hom}_R(P \otimes_R P^*, R) \arrow[r, "\eta"] & \operatorname{Hom}_R(\operatorname{End}_R(P), R)
\end{tikzcd}$$

where, for $x \in P$, $\alpha \in P^*$, $\varphi \in \operatorname{End}_R(P^*)$, $\psi \in \operatorname{Hom}_R(P \otimes_R P^*, R)$ and $f \in \operatorname{End}_R(P)$, $\xi(\varphi)(x \otimes \alpha) = \varphi(\alpha)(x)$ and $\eta(\psi)(f) = \psi(\theta_P^{-1}(f))$ where $\theta_P : P \otimes_R P^* \longrightarrow \operatorname{End}_R(P)$ is our usual isomorphism.

Then the *trace* of an endomorphism $f \in \operatorname{End}_R(P)$ is defined to be $\big(\eta\xi(\operatorname{Id}_{P^*})\big)(f)$. In fact, we can say that the equality in in Prop 2.2.4(a) is the same as the definition of the trace in [Alm] and [B:A, II].

Lemma 2.2.6. *Let $f \in \operatorname{End}_R(P)$. Then, using the notation of Proposition* 2.2.4*,* $\operatorname{tr}(f) = \big(\eta\xi(\operatorname{Id}_{P^*})\big)(f)$.

Proof. Consider the isomorphism $\theta_P : P \otimes_R P^* \longrightarrow \operatorname{End}_R(P)$. We may write $\theta_P^{-1}(f) = \sum_i m_i \otimes \phi_i$. Then, by Proposition 2.2.4,

$$\operatorname{tr}(f) = t_P(f) = \tau_P \theta^{-1}(f) = \tau_P\left(\sum_i m_i \otimes \phi_i\right) = \sum_i \phi_i(m_i).$$

Finally, $\big(\eta\xi(\operatorname{Id}_{P^*})\big)(f) = \xi(\operatorname{Id}_{P^*})\left(\sum_i m_i \otimes \phi_i\right) = \sum_i \operatorname{Id}_{P^*}(\phi_i)(m_i) = \sum_i \phi_i(m_i)$. □

Before studying properties of the trace, we show how the trace interact with the decomposition of a finitely generated projective R-module into modules of constant rank (see Review 1.5.25).

Lemma 2.2.7. *Let $\varepsilon_1 \in R$ be an idempotent. Put $\varepsilon_2 = 1 - \varepsilon_1$ and $R_i = \varepsilon_i R$, whence $R = R_1 \oplus R_2$ is a direct sum of rings. Let P be a finitely generated projective R-module with $\operatorname{Ann}_R(P) = R_2$. We denote by P_1 the R_1-module obtained by restricting scalars to R_1. Then*

- (a) *P_1 is a finitely generated projective R_1-module.*
- (b) *Any $f \in \operatorname{End}_R(P)$ is R_1-linear and $\operatorname{tr}_{R_1}(f) = \operatorname{tr}_R(f)$.*

Proof. (a) Let N be an R-module such that $P \oplus N = R^n$. Then $N = N_1 \oplus N_2$ with $N_i = \varepsilon_i N$. Now, $P_1 \oplus N_1 = \varepsilon_1 P \oplus \varepsilon_1 N = \varepsilon_1(P \oplus N) = \varepsilon_1 R^n = R_1^n$. This shows that P_1 is a finitely generated projective R_1-module.

(b) Since $\operatorname{Ann}_R(P) = R_2$, $\operatorname{End}_R(P) = \operatorname{End}_{R_1}(P_1)$ as abelian groups. Next, we note that $\operatorname{tr}_R(f) \in R_1$. This is true since $\varepsilon_1 \operatorname{tr}_R(f) = \operatorname{tr}_R(\varepsilon_1 f) = \operatorname{tr}_R(f)$. We also have that $P \otimes_R P^* = P_1 \otimes_{R_1} P_1^*$. Consider the isomorphisms $\theta : P \otimes_R P^* \longrightarrow R$ and $\theta_1 : P_1 \otimes_{R_1} P_1^* \longrightarrow R_1$. For $p \otimes_R \varphi = p \otimes_{R_1} \varphi \in P \otimes_R P^* = P_1 \otimes_{R_1} P_1$, we have that $\theta(p \otimes_R \varphi) = \varphi(p) = \theta_1(p \otimes_{R_1} \varphi)$. Therefore, we have that $\theta = \theta_1$. The result follows. □

The following proposition is a generalization of Lemma 2.1.1.

Proposition 2.2.8. [Alm, Prop. 1.3(iv)]. *Let M and N be isomorphic R-modules with isomorphism $\nu : M \longrightarrow N$ and let $f \in \operatorname{End}_R(N)$. Suppose that M and N are projective. Then* $\operatorname{tr}(f) = \operatorname{tr}(\nu^{-1} f \nu)$.

Proof. Since ν is an isomorphism, so is $\nu \otimes \operatorname{Id}_{R_\mathfrak{p}}$ for $\mathfrak{p} \in \operatorname{Spec}(R)$. Thus, we see that $\operatorname{tr}(f \otimes \operatorname{Id}_{R_\mathfrak{p}}) = \operatorname{tr}((\nu \otimes \operatorname{Id}_{R_\mathfrak{p}})^{-1}(f \otimes \operatorname{Id}_{R_\mathfrak{p}})(\nu \otimes \operatorname{Id}_{R_\mathfrak{p}})) = \operatorname{tr}((\nu^{-1} f \nu) \otimes \operatorname{Id}_{R_\mathfrak{p}})$ where the first equality follows from Lemma 2.1.1 and the second equality follows from Lemma 2.1.4. Then, the result follows from Corollary 2.2.3 and Proposition 1.1.13. □

Proposition 2.2.9. *Let P be a finitely generated projective R-module. Then the map* $\operatorname{tr} : \operatorname{End}_R(P) \longrightarrow R$ *is R-linear and satisfies the following properties:*

(a) [Alm, Prop. 2.1]. *For all $f, g \in \operatorname{End}_R(P)$,* $\operatorname{tr}(fg) = \operatorname{tr}(gf)$.

(b) (special case of [Alm, Prop. 1.5]). *Let $0 \longrightarrow P_1 \longrightarrow P \longrightarrow P_2 \longrightarrow 0$ be an exact sequence of finitely generated projective R-modules. Let $h \in \operatorname{End}_R(P)$ and $h_i \in \operatorname{End}_R(P_i)$ $(i = 1, 2)$ be such that the following diagram*

$$\begin{tikzcd}
0 \arrow[r] & P_1 \arrow[r, "f"] \arrow[d, "h_1"'] & P \arrow[r, "g"] \arrow[d, "h"'] & P_2 \arrow[r] \arrow[d, "h_2"'] & 0 \\
0 \arrow[r] & P_1 \arrow[r, "f"'] & P \arrow[r, "g"'] & P_2 \arrow[r] & 0
\end{tikzcd}$$

commutes. Then $\operatorname{tr}(h) = \operatorname{tr}(h_1) + \operatorname{tr}(h_2)$.

(c) [Alm, Prop. 3.2]. *Let N be an R-module such that $P \oplus N$ is free of finite rank, and observe that $(P \oplus N)^* \simeq P^* \oplus N^*$ is free of finite rank. Let f be an endomorphism of P. Consider the transpose $f^t : P^* \longrightarrow P^*$, defined by $f^t(\lambda) = \lambda \circ f$. Then, $\widetilde{f^t} = (\widetilde{f})^t$ where $\widetilde{f^t}$ is defined as in Proposition* 2.2.4. *In particular,* $\operatorname{tr}(f) = \operatorname{tr}(f^t)$.

(d) [Alm, Thm. 2.2(i)]. *Let $\{\varepsilon_i\}_{i=1}^n$ be the orthogonal idemptotents of Review* 1.5.25. *Suppose that r_i is the constant rank of P_i. Then* $\operatorname{tr}_R(\operatorname{Id}_P) = \sum_{i=1}^n r_i \varepsilon_i$.

Proof. Linearity of tr comes from the fact that the map $t_P : \operatorname{End}_R(P) \longrightarrow R$ of Proposition 2.2.4 is linear.

(a). Let f, g be endomorphisms of P. Let $\mathfrak{p} \in \operatorname{Spec}(R)$. We have that

$$\begin{aligned}\operatorname{tr}(fg) \otimes_R 1_{\mathfrak{p}} &= \operatorname{tr}((fg) \otimes \operatorname{Id}_{R_{\mathfrak{p}}}) = \operatorname{tr}((f \otimes \operatorname{Id}_{R_{\mathfrak{p}}})(g \otimes \operatorname{Id}_{R_{\mathfrak{p}}})) \\ &= \operatorname{tr}((g \otimes \operatorname{Id}_{R_{\mathfrak{p}}})(f \otimes \operatorname{Id}_{R_{\mathfrak{p}}})) = \operatorname{tr}(gf \otimes \operatorname{Id}_{R_{\mathfrak{p}}}) = \operatorname{tr}(gf) \otimes 1_{\mathfrak{p}},\end{aligned}$$

where the first and the fifth equalities follows from Corollary 2.2.3 the second and the fourth equalites follow from Lemma 2.1.4 and the third equality uses Lemma 2.1.1(b). By uniqueness of the trace, we conclude that $\operatorname{tr}(fg) = \operatorname{tr}(gf)$.

(b). First we consider the special case where the following diagram

$$\begin{tikzcd}
0 \arrow[r] & P_1 \arrow[r, "f"] \arrow[d, "h_1"'] & P \arrow[r, "\pi"] \arrow[d, "h"'] & P/f(P_1) \arrow[r] \arrow[d, "\overline{h}"'] & 0 \\
0 \arrow[r] & P_1 \arrow[r, "f"'] & P \arrow[r, "\pi"'] & P/f(P_1) \arrow[r] & 0
\end{tikzcd}$$

is commutative, where $\overline{h}$ is induced by h on the quotient $P/f(P_1)$. We show, in this case, that $\operatorname{tr}(h) = \operatorname{tr}(h_1) + \operatorname{tr}(\overline{h})$.

We begin by proving the case where P, P_1 and $P/i(P_1)$ are free modules of finite rank. In this case, we can find a basis $\{v_1, \ldots, v_n\}$ of P such that, for some m, $\{v_1, \ldots, v_m\}$ is a basis of $f(P_1)$ and $\{\pi(v_{m+1}), \ldots, \pi(v_n)\}$ is a basis of $P/f(P_1)$. The result is then immediate.

Now, consider the case where P_1, P and $P/f(P_1)$ are finitely generated projective. Let $\mathfrak{p} \in \operatorname{Spec}(R)$. Since $R_{\mathfrak{p}}$ is a flat R-module, the sequence $0 \longrightarrow P_{1,\mathfrak{p}} \xrightarrow{f_{\mathfrak{p}}} P_{\mathfrak{p}} \xrightarrow{\pi_{\mathfrak{p}}} (P/f(P_1))_{\mathfrak{p}} \longrightarrow 0$ is exact. First, we see that, $h_{\mathfrak{p}}$ leaves $f_{\mathfrak{p}}(P_{1,\mathfrak{p}})$ invariant since h leaves $f(P_1)$ invariant. It follows from the free case that $\operatorname{tr}(h_{\mathfrak{p}}) = \operatorname{tr}(h_{\mathfrak{p}}|f_{\mathfrak{p}}(P_{1,\mathfrak{p}})) + \operatorname{tr}(\overline{h_{\mathfrak{p}}})$ where $\overline{h_{\mathfrak{p}}}$ is the map induced by $h_{\mathfrak{p}}$ on $P_{\mathfrak{p}}/f(P_1)_{\mathfrak{p}}$.

Note that the map $\nu : P_{\mathfrak{p}}/f(P_1)_{\mathfrak{p}} \longrightarrow (P/f(P_1))_{\mathfrak{p}}$, $p \otimes s + f(P_1)_{\mathfrak{p}} \mapsto (p + f(P_1)) \otimes s$ is an isomorphism of $R_{\mathfrak{p}}$-modules. We claim that $h_{\mathfrak{p}}|f_{\mathfrak{p}}(P_{1,\mathfrak{p}})$ is conjugate to $h_{1,\mathfrak{p}}$ and that $\overline{h_{\mathfrak{p}}}$ is conjugate to $\overline{h}_{\mathfrak{p}}$. For the first part of the claim, note that $f : P_1 \longrightarrow f(P_1)$ is invertible. Hence, $h|f(P_1) = fh_1f^{-1}$. Therefore, $h_{\mathfrak{p}}|f_{\mathfrak{p}}(P_{1,\mathfrak{p}})$ is conjugate to $h_{1,\mathfrak{p}}$.

For the second part of the claim, we prove that $\nu\overline{h_{\mathfrak{p}}} = \overline{h}_{\mathfrak{p}}\nu$:

$$\begin{aligned}\nu\overline{h_{\mathfrak{p}}}(p \otimes s + f(P_1)_{\mathfrak{p}}) &= \nu(h_{\mathfrak{p}}(p \otimes s) + f(P_1)_{\mathfrak{p}}) = \nu(h(p) \otimes s + f(P_1)_{\mathfrak{p}}) \\ &= (h(p) + f(P_1)) \otimes s, \\ \overline{h}_{\mathfrak{p}}\nu(p \otimes s + f(P_1)_{\mathfrak{p}}) &= \overline{h}_{\mathfrak{p}}((p + f(P_1)_{\mathfrak{p}}) \otimes s) = (h(p) + f(P_1)) \otimes s.\end{aligned}$$

Thus, $\mathrm{tr}(\overline{h_{\mathfrak{p}}}) = \mathrm{tr}(\overline{h}_{\mathfrak{p}})$. It follows that $\mathrm{tr}(h_{\mathfrak{p}}) = \mathrm{tr}(h_{1,\mathfrak{p}}) + \mathrm{tr}(\overline{h}_{\mathfrak{p}})$. By Corollary 2.2.3, we have that $\mathrm{tr}(h) \otimes 1_{R_{\mathfrak{p}}} = (\mathrm{tr}(h_1) + \mathrm{tr}(\overline{h})) \otimes 1_{R_{\mathfrak{p}}}$. Therefore, $\mathrm{tr}(h) = \mathrm{tr}(h_1) + \mathrm{tr}(\overline{h})$.

Now, we show the general case. Suppose that the following diagram

$$\begin{tikzcd}
0 \arrow[r] & P_1 \arrow[r, "f"] \arrow[d, "h_1"'] & P \arrow[r, "g"] \arrow[d, "h"'] & P_2 \arrow[r] \arrow[d, "h_2"'] & 0 \\
0 \arrow[r] & P_1 \arrow[r, "f"'] & P \arrow[r, "g"'] & P_2 \arrow[r] & 0
\end{tikzcd}$$

is commutative. We claim that $\mathrm{tr}(h) = \mathrm{tr}(h_1) + \mathrm{tr}(h_2)$. Let $\overline{h}$ be the map induced by h on the quotient $P/f(P_1)$. Then, by the special case, $\mathrm{tr}(h) = \mathrm{tr}(h_1) + \mathrm{tr}(\overline{h})$.

We claim that $\overline{h}$ and h_2 are conjugate. First, the map $\alpha : P/f(P_1) \longrightarrow P_2$, $p + f(P_1) \mapsto g(p)$, is an R-module isomorphism. We claim that $\alpha\overline{h} = h_2\alpha$:

$$h_2\alpha\big(p + P/f(P_1)\big) = h_2\big(g(p)\big) = gh(p) = \alpha\big(h(p)\big) = \alpha\Big(\overline{h}\big(p + f(P_1)\big)\Big)$$

It follows that $\mathrm{tr}(\overline{h}) = \mathrm{tr}(h_2)$. Therefore, $\mathrm{tr}(h) = \mathrm{tr}(h_1) + \mathrm{tr}(h_2)$.

(c). Suppose that $P \oplus N = F$ where F is a free module of finite rank. Then, it follows that $F^* \simeq P^* \oplus N^*$. We consider $\widetilde{f} \in \mathrm{End}_R(F)$, its transpose $(\widetilde{f})^t \in \mathrm{End}_R(F^*)$, $f^t \in \mathrm{End}_R(P^*)$, $\widetilde{(f^t)} \in \mathrm{End}_R(F^*)$. Let $\alpha \oplus \beta \in F^*$ where $\alpha \in P^*$ and $\beta \in N^*$, let $m \oplus n \in F$. One has that

$$\begin{aligned}(\widetilde{f})^t(\alpha \oplus \beta)(m \oplus n) &= (\alpha \oplus \beta)(\widetilde{f}(m \oplus n)) = (\alpha \oplus \beta)(f(m) \oplus 0) \\ &= \alpha(f(m)) \oplus 0 = \widetilde{(f^t)}(\alpha \oplus \beta)(m \oplus n).\end{aligned}$$

For the second assertion, one uses that

$$\mathrm{tr}(f) = \mathrm{tr}(\widetilde{f}) = \mathrm{tr}((\widetilde{f})^t) = \mathrm{tr}(\widetilde{(f^t)}) = \mathrm{tr}(f^t),$$

where the second equality comes from that fact that our claim is well-known for endomorphisms of free modules of finite rank.

(d). First, define $\mathrm{Id}_i = \pi_{P_j}$ where π_{P_j} is the canonical projection. Observe that $\mathrm{Id}\,|P_i = \mathrm{Id}_i$. By repeatedly applying part (b) and Lemma 2.2.7,

$$\mathrm{tr}_R(\mathrm{Id}_P) = \sum_{i=1}^{n} \mathrm{tr}_R(\mathrm{Id}_{P_i}) = \sum_{i=1}^{n} \mathrm{tr}_{R_i}(\mathrm{Id}_{P_i}) = \sum_{i=1}^{n} (r_i \cdot 1_{P_i}) = \sum_{i=1}^{n} r_i\varepsilon_i,$$

since $1_{P_i} = \varepsilon_i$ in R. □

Lemma 2.2.10. *Let $P_1, \ldots, P_n$ be finitely generated and projective R-modules and let $f_i \in \operatorname{End}_R(P_i)$ for each i.*

(a) *The trace of $f_1 \otimes \cdots \otimes f_n \in \operatorname{End}_R(P_1 \otimes \cdots \otimes P_n)$, induced by the assignment*

$$v_1 \otimes \cdots \otimes v_n \mapsto f_1(v_1) \otimes \cdots \otimes f_n(v_n),$$

is $\operatorname{tr}(f_1) \cdots \operatorname{tr}(f_n)$.

(b) *Let $f_i, g_i \in \operatorname{End}_R(P_i)$. Put $\widehat{r_i} = \prod_{k \neq i} \operatorname{tr}(\operatorname{Id}_{P_k})$ and put $\widehat{r_{ij}} = \prod_{k \neq i,j} \operatorname{tr}(\operatorname{Id}_{P_k})$. Define $\boldsymbol{f}^{\otimes n} \in \operatorname{End}_R(P_1 \otimes \cdots \otimes P_n)$ by*

$$\boldsymbol{f}^{\otimes n}(v_1 \otimes \cdots \otimes v_n) = \sum_{i=1}^{n} v_1 \otimes \cdots \otimes v_{i-1} \otimes f_i(v_i) \otimes v_{i+1} \otimes \cdots \otimes v_n.$$

Then

$$\operatorname{tr}(\boldsymbol{f}^{\otimes n}) = \sum_{i=1}^{n} \widehat{r_i} \operatorname{tr}(f_i), \tag{1}$$

$$\operatorname{tr}(\boldsymbol{f}^{\otimes n} \boldsymbol{g}^{\otimes n}) = \sum_{i=1}^{n} \widehat{r_i} \operatorname{tr}(f_i g_i) + \sum_{i \neq j} \widehat{r_{ij}} \operatorname{tr}(f_i) \operatorname{tr}(g_j). \tag{2}$$

Proof. (a). The formula is easily seen to hold for endomorphisms of free modules of finite rank. Now, for all $\mathfrak{p} \in \operatorname{Spec}(R)$, by Corollary 2.2.3, we know that

$$\begin{aligned}
\operatorname{tr}(f_1 \otimes \cdots \otimes f_n) \otimes 1_{R_\mathfrak{p}} &= \operatorname{tr}\big((f_1 \otimes \cdots \otimes f_n) \otimes \operatorname{Id}_{R_\mathfrak{p}}\big) \\
&= \operatorname{tr}\big((f_1 \otimes \operatorname{Id}_{R_\mathfrak{p}}) \otimes \cdots \otimes (f_n \otimes \operatorname{Id}_{R_\mathfrak{p}})\big) \\
&= \operatorname{tr}(f_1 \otimes \operatorname{Id}_{R_\mathfrak{p}}) \cdots \operatorname{tr}(f_n \otimes \operatorname{Id}_{R_\mathfrak{p}}) \\
&= \big(\operatorname{tr}(f_1) \cdots \operatorname{tr}(f_n)\big) \otimes 1_{R_\mathfrak{p}}
\end{aligned}$$

where the third equality follows from the fact that $P \otimes_R R_\mathfrak{p}$ is free of finite rank. The result follows.

(b). Formula (1) follows immediately from part (a). For the proof of the second formula, put $h_i = f_i g_i$, define $\boldsymbol{h}$ analogously to $\boldsymbol{f}$ and note

$$\begin{aligned}
\boldsymbol{f}^{\otimes n} \boldsymbol{g}^{\otimes n} = \boldsymbol{h}^{\otimes n} + \sum_{i<j} (&\operatorname{Id} \otimes \cdots \otimes \operatorname{Id} \otimes f_i \otimes \operatorname{Id} \cdots \otimes \operatorname{Id} \otimes g_j \otimes \operatorname{Id} \otimes \cdots \otimes \operatorname{Id} \\
&+ \operatorname{Id} \otimes \cdots \otimes \operatorname{Id} \otimes g_i \otimes \operatorname{Id} \cdots \otimes \operatorname{Id} \otimes f_j \otimes \operatorname{Id} \otimes \cdots \otimes \operatorname{Id}).
\end{aligned}$$

Then, (2) follows from (1) part (a). □

Notation 2.2.11. For $f \in \operatorname{End}_R(M)$, one can show that the map $\boldsymbol{f}^{\otimes n}$ from Lemma 2.2.10 induces well-defined maps on the levels of symmetric algebra and exterior algebra. From now on, we denote by $f^{\wedge n}$ the map induced on $\bigwedge^n(M)$ by $\boldsymbol{f}^{\otimes n}$, and by $f^{\circ n}$ the map induced by $\boldsymbol{f}^{\otimes n}$ on $\operatorname{Sym}^n(M)$.

Proposition 2.2.12. *Let P be a finitely generated projective R-module of constant rank r and let $f, g \in \operatorname{End}_R(P)$. Then, for $n \in \mathbb{N}$, $1 \leq n \leq r$,*

$$\operatorname{tr}(f^{\wedge n}) = \binom{r-1}{n-1} \operatorname{tr}(f), \tag{3}$$

$$\operatorname{tr}(f^{\wedge n} g^{\wedge n}) = \binom{r-2}{n-2} \operatorname{tr}(f) \operatorname{tr}(g) + \binom{r-2}{n-1} \operatorname{tr}(fg), \tag{4}$$

$$\operatorname{tr}(f^{\circ n}) = \binom{r+n-1}{n-1} \operatorname{tr}(f), \tag{5}$$

$$\operatorname{tr}(f^{\circ n} g^{\circ n}) = \binom{r+n-1}{n-2} \operatorname{tr}(f) \operatorname{tr}(g) + \binom{r+n}{n-1} \operatorname{tr}(fg). \tag{6}$$

Proof. The usual localization process shows that it is sufficient to prove the result in case of a free R-module P of rank r. Thus, we choose and fix a basis $\{v_1, \ldots, v_r\}$ of P.

Hence, $\{v_{i_1} \wedge \cdots \wedge v_{i_n} : i_1 < \cdots < i_n\}$ is a basis of $\bigwedge^n(P)$. We define $f_{ij} \in R$ by $f(v_i) = \sum_{j=1}^n f_{ji} v_j$ $(1 \leq i \leq n)$. We have

$$f^{\wedge n}(v_{i_1} \wedge \cdots \wedge v_{i_n}) = (f_{i_1 i_1} + \cdots + f_{i_n i_n}) v_{i_1} \wedge \cdots \wedge v_{i_n} + u$$

where u is a linear combination of basis vectors different from $v_{i_1} \wedge \cdots \wedge v_{i_n}$ which therefore does not contribute to the trace. Formula (3) then says

$$\sum_{i_1 < \cdots < i_n} (f_{i_1 i_1} + \cdots + f_{i_n i_n}) = \binom{r-1}{n-1} (f_{11} + \cdots + f_{rr}),$$

which can be proven by a simple counting argument: For a fixed i, the coefficient f_{ii} appears precisely in those terms of the left hand side which are obtained by choosing $n-1$ numbers in $\{1, \cdots, r\} \backslash \{i\}$ and then arranging them in increasing order.

For the proof of (4), we write $f^{\wedge n} \circ g^{\wedge n} = (f \circ g)^{\wedge n} + h$ where $h \in \operatorname{End}_R(\bigwedge^n P)$ acts by

$$\begin{aligned} h(u_1 \wedge \cdots \wedge u_n) = \sum_{i<j} (&u_1 \wedge \cdots \wedge f(u_i) \wedge \cdots \wedge g(u_j) \wedge \cdots \wedge u_n \\ &+ u_1 \wedge \cdots \wedge g(u_i) \wedge \cdots \wedge f(u_j) \wedge \cdots \wedge u_n). \end{aligned}$$

Note $h = 0$ if $n = 1$. It will be sufficient to prove

$$\operatorname{tr}(h) = \binom{r-2}{n-2}\big(\operatorname{tr}(f)\operatorname{tr}(g) - \operatorname{tr}(fg)\big), \tag{7}$$

since then (4) follows from (3):

$$\begin{aligned}\operatorname{tr}(f^{\wedge n}\circ g^{\wedge n}) &= \left(\binom{r-1}{n-1} - \binom{r-2}{n-2}\right)\operatorname{tr}(f\circ g) + \binom{r-2}{n-1}\operatorname{tr}(f)\operatorname{tr}(g) \\ &= \binom{r-2}{n-1}\operatorname{tr}(f\circ g) + \binom{r-2}{n-1}\operatorname{tr}(f)\operatorname{tr}(g).\end{aligned}$$

To prove (7), we let (g_{ij}) be the matrix representing g. We then get, with the notation above,

$$\begin{aligned}h(v_{i_1}\wedge\cdots\wedge v_{i_n}) &= \sum_{s<t} v_{i_1}\wedge\cdots\wedge f(v_{i_s})\wedge\cdots\wedge g(v_{i_t})\wedge\cdots\wedge v_{i_n} \\ &\qquad + v_{i_1}\wedge\cdots\wedge g(v_{i_s})\wedge\cdots\wedge f(v_{i_t})\wedge\cdots\wedge v_{i_n} \\ &= \sum_{s<t}(f_{i_si_s}g_{i_ti_t} + g_{i_si_s}f_{i_ti_t} - f_{i_ti_s}g_{i_si_t} - g_{i_ti_s}f_{i_si_t})(v_{i_1}\wedge\cdots\wedge v_{i_n} + w)\end{aligned}$$

where w is a linear combination of basis vectors distinct from $v_{i_1}\wedge\cdots\wedge v_{i_n}$ and does therefore not contribute to the trace.

Hence, defining for $i < j$, $h_{ij} = f_{ii}g_{jj} + f_{jj}g_{ii} - f_{ji}g_{ij} - f_{ij}g_{ji}$, we find

$$\operatorname{tr}(h) = \sum_{1\le i_1<\cdots<i_n\le r}\left(\sum_{i_s<i_t} h_{i_si_t}\right) = \sum_{1\le i<j\le r} m_{ij}h_{ij},$$

where m_{ij} counts the number of times the pair tuple (i, j) occurs in some type $(i_1, \cdots, i_n)$ with $1 \le i_1 < \cdots < i_n \le r$. We use a counting argument: Once (i, j) is fixed, we can choose $n-2$ numbers among the $r-2$ numbers $\{1, \cdots, r\}\backslash\{i, j\}$. Hence, $\operatorname{tr}(h) = \binom{r-2}{n-2}\sum_{1\le i<j\le r} h_{ij}$ follows. To finish the proof of (4), it remains to observe

$$\begin{aligned}\operatorname{tr}(f)\operatorname{tr}(g) - \operatorname{tr}(f\circ g) &= \sum_{i,j}(f_{ii}g_{jj} - f_{ij}g_{ji}) \\ &= \sum_{i<j}((f_{ii}g_{jj} - f_{ij}g_{ji}) + (f_{jj}g_{ii} - f_{ji}g_{ij})) = \sum_{i<j} h_{ij}.\end{aligned}$$

Finally, we show (5) and (6). We know that $\{v_{i_1}\circ\cdots\circ v_{i_n} : i_1 \le \cdots \le i_n\}$ is a basis of $\operatorname{Sym}^n(P)$ where $v_{i_1}\circ\cdots\circ v_{i_n}$ denotes the image of $v_{i_1}\otimes\cdots\otimes v_{i_n}$ by the canonical projection from $P^{\otimes n}$ to $\mathrm{Sym}^n(P)$. One has that

$$f^{\circ n}(v_{i_1}\circ\cdots v_{i_n}) = (f_{i_1i_1}+\cdots+f_{i_ni_n})(v_{i_1}\circ\cdots\circ v_{i_n}) + w$$

where w is a linear combination of basis vectors different from $v_{i_1}\circ\cdots\circ v_{i_n}$, whence does not contribute to the trace. Now, formula (5) becomes

$$\sum_{i_1\le\cdots\le i_n}(f_{i_1i_1}+\cdots+f_{i_ni_n}) = \binom{r+n-1}{n-1}(f_{11}+\cdots+f_{rr}).$$

This can be shown by a counting argument. First, by symmetry, we note that the total numbers of f_{ii} and of f_{jj} are equal. It means we can count the total number of all f_{ii}'s in all basis elements of $\mathrm{Sym}^n(P)$ and divide it by the number of variables. This number is $\frac{n\dim\mathrm{Sym}^n(P)}{r}$. One can easily show that $\frac{n\dim\mathrm{Sym}^n(P)}{r}=\binom{r+n-1}{n-1}$. Indeed, by Lemma 1.6.3,

$$\frac{n\dim\mathrm{Sym}^n(P)}{r} = \frac{n}{r}\binom{r+n-1}{n} = \frac{n}{r}\frac{(r+n-1)!}{n!(r-1)!} = \frac{(r+n-1)!}{(n-1)!r!} = \binom{r+n-1}{n-1}.$$

This completes the proof of (5).

For the proof of (6), we write $f^{\circ n}\circ g^{\circ n} = (fg)^{\circ n}+h$ where $h\in\mathrm{End}_R(\mathrm{Sym}^n(P))$ acts by

$$\begin{aligned}h(u_1\circ\cdots\circ u_n) = \sum_{i<j}(&u_1\circ\cdots\circ f(u_i)\circ\cdots\circ g(u_j)\circ\cdots\circ u_n\\ &+u_1\circ\cdots\circ g(u_i)\circ\cdots\circ f(u_j)\circ\cdots\circ u_n).\end{aligned}$$

Note $h=0$ if $n=1$. It will be sufficient to prove

$$\mathrm{tr}(h) = \binom{r+n-1}{n-2}\big(\mathrm{tr}(f)\,\mathrm{tr}(g)+\mathrm{tr}(fg)\big), \tag{8}$$

since then (6) follows from (5),

$$\begin{aligned}\mathrm{tr}(f^{\circ n}g^{\circ n}) &= \binom{r+n-1}{n-2}\mathrm{tr}(f)\,\mathrm{tr}(g) + \left(\binom{r+n-1}{n-2}+\binom{r+n-1}{n-1}\right)\mathrm{tr}(fg)\\ &= \binom{r+n-1}{n-2}\mathrm{tr}(f)\,\mathrm{tr}(g) + \binom{r+n}{n-1}\mathrm{tr}(fg).\end{aligned}$$

Now, let (g_{ij}) be the matrix representing g. We have that:

$$h(v_{i_1} \circ \cdots \circ v_{i_n}) = \sum_{s<t} (f_{i_s i_s} g_{i_t i_t} + f_{i_t i_t} g_{i_s i_s} + (1 - \delta_{i_s i_t})(f_{i_s i_t} g_{i_t i_s} + f_{i_t i_s} g_{i_s i_t})) + w',$$

where w' is a linear combination of other basis elements, hence doesn't contribute to the trace. Thus, defining for $i < j$, $h_{ij} = f_{ii} g_{jj} + f_{jj} g_{ii} + (1 - \delta_{ij})(f_{ij} g_{ji} + f_{ji} g_{ij})$, we find

$$\mathrm{tr}(h) = \sum_{1 \le i_1 \le \cdots \le i_n \le r} \left(\sum_{i_s \le i_t} h_{i_s i_t} \right) = \sum_{1 \le i \le j \le r} m_{ij} h_{ij},$$

where m_{ij} counts the number of times the pair (i, j) occurs in some vector $(i_1, \ldots, i_n)$ with $1 \le i_1 \cdots \le i_n \le r$. We are going to count h_{ij} where $i = j$ and where $i < j$. By symmetry, we see that $m_{ii} = m_{jj}$ for all i, j and that $m_{ij} = m_{kl}$ for all $i \ne j$ and $k \ne l$. Hence, it is enough to find m_{11} and m_{12}.

We will show that $m_{11} = \binom{r+n-1}{n-2} = m_{12}$. We have 3 cases:

Case 1: $r = 1$. In this case, there is one vector in the basis of $\mathrm{Sym}^n(P)$, namely $v_1 \circ \cdots \circ v_1$. Moreover, one sees that the number of times the pair $(1, 1)$ appears in $(1, \ldots, 1)$ is $\binom{n}{2} = \binom{n}{n-2}$.

Case 2: $r = 2$. In this case, there are exactly n vectors in the basis of $\mathrm{Sym}^n(P)$. They are $v_1 \circ v_2 \circ \cdots \circ v_2, \ldots, v_1 \circ \cdots \circ v_1 \circ v_2$. Hence, one sees that

$$m_{11} = \sum_{i=1}^{n} \binom{i}{2} = \binom{n+1}{3} = \binom{2+n-1}{n-2}.$$

Secondly, we find m_{12} in this case. The number of times the pair $(1, 2)$ appears in each tuple $(1, \ldots, 1, 2, \ldots, 2)$ of i 1's and $n - i$ 2's is $i(n - i)$. Hence, one has that

$$m_{12} = \sum_{i=1}^{n} i(n-i) = \binom{n+1}{3}.$$

This completes the proof of case 2.

Case 3: $r \ge 3$. In this case, there are more tuples that have i 1's. So, we need to count how many of those tuples are there. Since there are $n - i$ remaining slots and $n - 1$ number to choose from, the number of n-tuples that have i 1's is exactly the dimension of the space of homogeneous polynomials of degree $n - i$ in $r - 1$ variables, namely $\binom{r+n-i-2}{n-i}$. Thus, by Corollary 1.6.4,

$$m_{11} = \sum_{i=1}^{n} \binom{i}{2} \binom{r+n-i-2}{n-i} = \binom{r+n-1}{n-2}.$$

Next, we find m_{12}. To do this, we need to know how many n-tuples that contain i 1's and 2's, i.e., i is the sum of the number of 1's and the number of 2's. Since there are $n-i$ remaining slots and $n-2$ numbers to choose from, the number of such n-tuples is exactly the dimension of the space of homogeneous polynomials of degree $n-i$ in $n-2$ variables, namely $\binom{r+n-i-3}{n-i}$. Moreover, the number of times the pair (1,2) appearing in a tuple containing i 1's and 2's is the product of the number of 1's and the number of 2's. Thus, by Corollary 1.6.4,

$$\begin{aligned} m_{12} &= \sum_{i=1}^{n} \left(\sum_{k=1}^{i} k(i-k) \right) \binom{r+n-i-3}{n-i} = \\ &= \sum_{i=1}^{n} \binom{i+1}{3} \binom{r+n-i-3}{n-i} = \binom{r+n-1}{n-2}. \end{aligned}$$

It follows that $\operatorname{tr}(h) = \binom{r+n-1}{n-2} \sum_{1 \le i \le j \le r} h_{ij}$. Finally, it is enough to note that

$$\operatorname{tr}(fg) + \operatorname{tr}(f)\operatorname{tr}(g) = \sum_{i=1}^{n} h_{ii} + \sum_{i<j} h_{ij} = \sum_{i \le j} h_{ij}.$$

This completes the proof. □

We have the following immediate generalization.

Corollary 2.2.13. *Let R be a ring. Let M be a finitely generated projective R-module. Let $f, g \in \operatorname{End}_R(M)$. Then, $\operatorname{tr}(f^{\wedge n})$, $\operatorname{tr}(f^{\wedge n} g^{\wedge n})$, $\operatorname{tr}(f^{\circ n})$ and $\operatorname{tr}(f^{\circ n} g^{\circ n})$ are the unique elements satisfying, for all $\mathfrak{p} \in \operatorname{Spec}(R)$,*

$$\begin{aligned} \operatorname{tr}(f^{\wedge n}) \otimes 1_{R_\mathfrak{p}} &= \binom{r_\mathfrak{p}-1}{n-1} \operatorname{tr}(f) \otimes 1_{R_\mathfrak{p}}, \\ \operatorname{tr}(f^{\wedge n} g^{\wedge n}) \otimes 1_{R_\mathfrak{p}} &= \left(\binom{r_\mathfrak{p}-2}{n-2} \operatorname{tr}(f)\operatorname{tr}(g) + \binom{r_\mathfrak{p}-2}{n-1} \operatorname{tr}(fg) \right) \otimes 1_{R_\mathfrak{p}}, \\ \operatorname{tr}(f^{\circ n}) \otimes 1_{R_\mathfrak{p}} &= \binom{r_\mathfrak{p}+n-1}{n-1} \operatorname{tr}(f) \otimes 1_{R_\mathfrak{p}}, \\ \operatorname{tr}(f^{\circ n} g^{\circ n}) \otimes 1_{R_\mathfrak{p}} &= \left(\binom{r_\mathfrak{p}+n-1}{n-2} \operatorname{tr}(f)\operatorname{tr}(g) + \binom{r_\mathfrak{p}+n}{n-1} \operatorname{tr}(fg) \right) \otimes 1_{R_\mathfrak{p}}. \end{aligned}$$

Proof. The formulas follow immediately from Proposition 2.2.12 and Corollary 2.1.2. □

Proposition 2.2.14. *Let P be a finitely generated projective R-module and let $P = P_1 \oplus \cdots \oplus P_s$ be a decomposition as in Review 1.5.25. For $n \in \mathbb{N}_+$, the following hold*

(a) $P^{\otimes n} = \bigoplus_{i=1}^{s} P_i^{\otimes n}$, $\bigwedge^n P = \bigoplus_{i=1}^{s} \bigwedge^n P_i$ *and* $\operatorname{Sym}^n P = \bigoplus_{i=1}^{s} \operatorname{Sym}^n P_i$.

(b) *An* $f \in \operatorname{End}_R(P)$ *leaves every* P_i *invariant. Setting* $f_i = f|P_i$, *we have, with the notation of Lemma* 2.2.10 *and Proposition* 2.2.12, *for* $1 \leq i \leq s$,

  (i) $\boldsymbol{f}^{\otimes n}(P_i^{\otimes n}) \subseteq P_i^{\otimes n}$, $\boldsymbol{f}^{\otimes n}|P_i^{\otimes n} = \boldsymbol{f}_i^{\otimes n}$, *whence* $\operatorname{tr}(\boldsymbol{f}^{\otimes n}) = \sum_{i=1}^{s} \operatorname{tr}(\boldsymbol{f}_i^{\otimes n})$.

  (ii) $f^{\wedge n}(P_i^{\wedge n}) \subseteq P_i^{\wedge n}$, $f^{\wedge n}|P_i^{\wedge n} = f_i^{\wedge n}$, *whence* $\operatorname{tr}(f^{\wedge n}) = \sum_{i=1}^{s} \operatorname{tr}(f_i^{\wedge n})$.

  (iii) $f^{\circ n}(P_i^{\circ n}) \subseteq P_i^{\circ n}$, $f^{\circ n}|P_i^{\circ n} = f_i^{\circ n}$, *whence* $\operatorname{tr}(f^{\circ n}) = \sum_{i=1}^{s} \operatorname{tr}(f_i^{\circ n})$.

Proposition 2.2.15. *Let* P *be a finitely generated projective* R*-module. Let* $P = P_1 \oplus \cdots \oplus P_s$ *where each* P_i *is finitely generated projective of constant rank* r_i. *Set* $f_i := f|P_i$ *as before. Then, for* $n \in \mathbb{N}$,

$$\begin{aligned}
\operatorname{tr}(f^{\wedge n}) &= \sum_{i=1}^{s} \binom{r_i - 1}{n-1} \operatorname{tr}(f_i)\varepsilon_i,\\
\operatorname{tr}(f^{\wedge n} g^{\wedge n}) &= \sum_{i=1}^{s} \left(\binom{r_i - 2}{n-2} \operatorname{tr}(f_i)\operatorname{tr}(g_i) + \binom{r_i - 2}{n-1} \operatorname{tr}(f_i g_i) \right) \varepsilon_i,\\
\operatorname{tr}(f^{\circ n}) &= \sum_{i=1}^{s} \binom{r_i + n - 1}{n-1} \operatorname{tr}(f_i)\varepsilon_i,\\
\operatorname{tr}(f^{\circ n} g^{\circ n}) &= \sum_{i=1}^{s} \left(\binom{r_i + n - 1}{n-2} \operatorname{tr}(f_i)\operatorname{tr}(g_i) + \binom{r_i + n}{n-1} \operatorname{tr}(f_i g_i) \right) \varepsilon_i.
\end{aligned}$$

Proof. We begin by showing the first formula. By Proposition 2.2.14, one sees that

$$f^{\wedge n} = f_1^{\wedge n} + \cdots + f_s^{\wedge n}.$$

By Proposition 2.2.14, $\operatorname{tr}_R(f) = \operatorname{tr}_{R_1}(f_1) + \cdots + \operatorname{tr}_{R_s}(f_s)$. So, we see that

$$\operatorname{tr}_R(f) = \operatorname{tr}_R(f_1^{\wedge n}) + \cdots + \operatorname{tr}_R(f_s^{\wedge n}) = \varepsilon_1 \operatorname{tr}_{R_1}(f_1^{\wedge n}) + \cdots + \varepsilon_s \operatorname{tr}_{R_s}(f_s^{\wedge n}).$$

Thus, the result follows by Proposition 2.2.12.

Now, we show the second equation. Recall from Proposition 2.2.14 that

$$f^{\wedge n} = \sum_{i=1}^{s} f_i^{\wedge n} \text{ and } g^{\wedge n} = \sum_{i=1}^{s} g_i^{\wedge n}.$$

We see that $f^{\wedge n} g^{\wedge n} = \sum_{i,j} f_i^{\wedge n} g_j^{\wedge n} = \sum_{i=1}^{s} f_i^{\wedge n} g_i^{\wedge n}$. Then, again, the result follows from Proposition 2.2.12 and Proposition 2.2.14.

The last two equations are proved in a similar fashion. □

Chapter 3

Representation ring

3.1 A brief review of representation theory

In this section, we review some concepts from representation theory. We denote by R a commutative ring and by L a R-Lie algebra. If $\rho : L \longrightarrow \mathfrak{gl}(M)$ is an L-module, we sometimes write this as (M, ρ).

The first lemma gives us several natural constructions of L-modules from old ones. These results are well-known in the literature.

Lemma 3.1.1. (a) *Let (M, ρ) and (N, μ) be L-modules. Then $M \otimes_R N$ has the structure of an L-module, given by the map $\rho \otimes \mu : L \longrightarrow \mathfrak{gl}_R(M \otimes_R N)$, defined by $(\rho \otimes \mu)(l) = \rho(l) \otimes \mathrm{Id} + \mathrm{Id} \otimes \mu(l)$ for $l \in L$.*

(b) *The ring R is an L-module with the trivial action given by the zero map from L to $\mathfrak{gl}_R(R) \simeq R$. Moreover, given an L-module (M, ρ), the canonical map $\nu : (M \otimes_R R, \rho \otimes 0) \longrightarrow (M, \rho)$ of R-modules is an isomorphism of L-modules.*

(c) *Suppose that $f : (M, \rho) \longrightarrow (M', \rho')$ is a homomorphism of L-modules. Let (N, μ) be another L-module. Then the map $f \otimes \mathrm{Id}_N : M \otimes_R N \longrightarrow M' \otimes_R N$ is a homomorphism of L-modules.*

(d) *Let (M, ρ) and (N, μ) be L-modules. The canonical isomorphism of R-modules $\nu : (M \otimes_R N, \rho \otimes \mu) \longrightarrow (N \otimes_R M, \mu \otimes \rho)$ is also an isomorphism of L-modules.*

(e) *Let (M,ρ), (N,μ) and (N',μ') be L-modules. Then the canonical isomorphism of R-modules $\nu : M \otimes_R (N \oplus N') \longrightarrow (M \otimes_R N) \oplus (M \otimes_R N')$ is also an isomorphism of L-modules.*

(f) *Let $(M,\rho), (N,\mu), (N',\mu')$ be L-modules. Then the canonical isomorphism of R-modules $\nu : \big((M \otimes_R N) \otimes_R N, (\rho \otimes \mu) \otimes \mu'\big) \longrightarrow \big(M \otimes_R (N \otimes_R N'), \rho \otimes (\mu \otimes \mu')\big)$ is an isomorphism of L-modules.*

Proof. (a) is well known.

In the proof of (b)-(f), we will use without further mention that a linear map is an L-module map as soon as it satisfies the L-module condition on a spanning set.

(b) It suffices to show that ν is an isomorphism of L-modules. Now, recall that $\nu(m \otimes r) = mr$. For $l \in L$, $m \in M$ and $r \in R$, we have the following

$$\nu\big((\rho \otimes 0)(l)(m \otimes r)\big) = \nu\Big(\big(\rho(l)(m)\big) \otimes r\Big) = r\rho(l)(m),$$
$$(\rho \otimes 0)(l)\nu(m \otimes r) = (\rho \otimes 0)(l)(rm) = \rho(x)(rm) = r\rho(l)(m).$$

Hence, ν is a homomorphism of L-modules.

(c) For $l \in L$, $m \in M$ and $r \in R$, we have the following

$$\begin{aligned}(f \otimes \mathrm{Id}_N)\Big((\rho \otimes \mu)(l)(m \otimes n)\Big) &= (f \otimes \mathrm{Id}_N)\Big(\big(\rho(l)(m)\big) \otimes n + m \otimes \big(\rho(l)(n)\big)\Big) \\ &= f\big(\rho(l)(m)\big) \otimes n + f(m) \otimes \big(\mu(l)(n)\big) \\ &= \Big(\rho(l)\big(f(m)\big)\Big) \otimes n + f(m) \otimes \Big(\mu(l)(n)\Big) \\ &= (\rho \otimes \mu)(l)\big(f(m) \otimes n\big) \\ &= (\rho \otimes \mu)(l)\Big((f \otimes \mathrm{Id}_N)(m \otimes n)\Big)\end{aligned}$$

Thus, $f \otimes \mathrm{Id}_N$ is a homomorphism of L-modules.

(d) For $l \in L$, $m \in M$ and $r \in R$, we have the following

$$\begin{aligned}\nu\big((\rho \otimes \mu)(l)(m \otimes n)\big) &= \nu\Big(\big(\rho(l)(m)\big) \otimes n + m \otimes \big(\mu(l)(n)\big)\Big) \\ &= n \otimes \big(\rho(l)(m)\big) + \big(\mu(l)(n)\big) \otimes m = (\mu \otimes \rho)(l)\nu(m \otimes n).\end{aligned}$$

Therefore, ν is an isomorphism of L-modules.

(e) Let $m \otimes (n+n') \in M \otimes_R (N \oplus N')$ and let $l \in L$. We have that

$$\begin{aligned}\nu\Big(\big(\rho \otimes (\mu+\mu')\big)(l)\big(m \otimes (n+n')\big)\Big) &= \nu\Big(\big(\rho(l)(m)\big) \otimes (n+n') \\ &\qquad + m \otimes \big(\mu(l)(n) + \mu'(l)(n')\big)\Big) \\ &= \big(\rho(l)(m)\big) \otimes n + \big(\rho(l)(m)\big) \otimes n' \\ &\qquad + m \otimes \big(\mu(l)(n)\big) + m \otimes \big(\mu'(l)(n')\big) \\ &= (\rho \otimes \mu + \rho \otimes \mu')(l)\nu\big(m \otimes (n+n')\big).\end{aligned}$$

Therefore, since the tensors $m \otimes (n+n')$ span $M \otimes_R (N \oplus N')$, ν is an isomorphism of L-modules.

(f) For $m \in M, n \in N, n' \in N'$, we have that

$$\begin{aligned}\nu\Big(\big((\rho \otimes \mu) \otimes \mu'\big)(l)\big((m \otimes n) \otimes n'\big)\Big) &= \nu\Big(\big((\rho \otimes \mu)(l)(m \otimes n)\big) \otimes n' \\ &\qquad + (m \otimes n) \otimes \big(\mu'(n')\big)\Big) \\ &= \nu\Big(\big(\rho(l)(m) \otimes n\big) \otimes n' + \big(m \otimes \mu(l)(n)\big) \otimes n' \\ &\qquad + (m \otimes n) \otimes \mu'(l)(n')\Big) \\ &= \big(\rho(l)(m)\big) \otimes (n \otimes n') + m \otimes \Big(\big(\mu(l)(n)\big) \otimes n'\Big) \\ &\qquad + m \otimes \Big(n \otimes \big(\mu'(l)(n')\big)\Big) \\ &= \big(\rho \otimes (\mu \otimes \mu')\big)(l)\nu\big((m \otimes n) \otimes n'\big).\end{aligned}$$

Hence, since the tensors $(m \otimes n) \otimes n'$ span $(M \otimes_R N) \otimes_R N'$, ν is an isomorphism of L-modules. $\square$

Lemma 3.1.2. (a) *Let (M, ρ) be L-module and let N be an R-module. Then $\mathrm{Hom}_R(M, N)$ has the structure of an L-module with respect to the operation*

$$\widetilde{\rho}(l)(f) := \rho(l) \circ f - f \circ \rho(l)$$

for $l \in L, f \in \mathrm{Hom}_R(M, N)$. In particular, (M^*, ρ^*) has the structure of an L-module.

(b) *Let $f : (M, \rho) \longrightarrow (N, \mu)$ be an L-module homomorphism. Define a map $f^t : N^* \longrightarrow M^*$ by $f^t(\psi) = \psi \circ f$. Then $f^t : (N^*, \mu^*) \longrightarrow (M^*, \rho^*)$ is a homomorphism of L-modules.*

(c) *Let (M,ρ) be an L-module. Then the canonical map $c_M : (M,\rho) \longrightarrow (M^{**},\rho^{**})$ of R-modules is also a homomorphism of L-modules.*

(d) *Let (M,ρ) and (N,μ) be L-modules. Then the canonical map of R-modules*

$$\gamma : (M^* \otimes_R N^*, \rho^* \otimes \mu^*) \longrightarrow \big((M \otimes_R N)^*, (\rho \otimes \mu)^*\big)$$

is also a homomorphism of L-modules.

Proof. (a) and (b) are well-known.

(c) For $l \in L, m \in M, \varphi \in M^*$, we have that:

$$\Big(\rho^{**}(l)\big(c_M(m)\big)\Big)(\varphi) = -\big(c_M(m) \circ \rho^*(l)\big)(\varphi) = c_M(m)\big(\varphi \circ \rho(l)\big)$$
$$= \varphi\big(\rho(l)(m)\big) = c_M\big(\rho(l)(m)\big)(\varphi).$$

Therefore, c_M is a homomorphism of L-modules.

(d) By Proposition 1.5.17, we know that the map $\gamma : M^* \otimes_R N^* \longrightarrow (M \otimes_R N)^*$ induced by $(\varphi,\psi) \mapsto \varphi \otimes \psi$ is a homomorphism of R-modules. It is left to check that γ is a homomorphism of L-modules. For $l \in L, \varphi \in M^*, \psi \in N^*$, we have that

$$\begin{aligned}
\gamma\Big(\big((\rho^* \otimes \mu^*)(l)\big)(\varphi \otimes \psi)\Big)(m \otimes n) &= \gamma\Big(\big(\rho^*(l)(\varphi)\big) \otimes \psi + \varphi \otimes \big(\mu^*(l)(\psi)\big)\Big)(m \otimes n) \\
&= -\gamma\Big(\big(\varphi \circ \rho(l)\big) \otimes \psi + \varphi \otimes \big(\psi \circ \mu(l)\big)\Big)(m \otimes n) \\
&= -\Big(\big(\varphi \circ \rho(l)\big)(m) \otimes \psi(n) \\
&\qquad + \varphi(m) \otimes \big(\psi \circ \mu(l)\big)(n)\Big) \\
&= -\gamma(\varphi \otimes \psi)\big((\rho \otimes \mu)(l)(m \otimes n)\big) \\
&= \Big((\rho \otimes \mu)^*(l)\big(\gamma(\varphi \otimes \psi)\big)\Big)(m \otimes n).
\end{aligned}$$

Therefore, γ is a homomorphism of L-modules. □

Note that, unless specified otherwise, we will always use the L-module structures defined above.

3.2 Basic definitions and properties

In this section, we present the concept of the representation ring of a Lie algebra. As in 3.1, we will denote by R a commutative ring, and by L an arbitrary R-Lie algebra.

Definition 3.2.1. An L-module P is called *finitely generated projective* if P is finitely generated projective as an R-module.

Definition 3.2.2. The set of isomorphism classes of finitely generated projective L-modules is denoted by $\mathscr{P}(L)$. Moreover, we denote by $\mathrm{cl}(P)$ the isomorphism class of a finitely generated projective L-module P.

Definition 3.2.3. Consider the free abelian group $\mathscr{F}(L)$ generated by $\mathscr{P}(L)$. Let $\mathscr{E}(L)$ be the subgroup of $\mathscr{F}(L)$ generated by the elements $\mathrm{cl}(P)-\mathrm{cl}(P_1)-\mathrm{cl}(P_2)$ where $0 \longrightarrow P_1 \longrightarrow P \longrightarrow P_2 \longrightarrow 0$ is an exact sequence of L-modules. We call the quotient group $\mathscr{R}(L) := \mathscr{F}(L)/\mathscr{E}(L)$ the *representation ring* of L. For $\mathrm{cl}(M,\rho) \in \mathscr{F}(L)$, we denote by $[M,\rho]$ the corresponding element of $\mathscr{R}(L)$.

We will show that $\mathscr{R}(L)$ has the structure of a ring. The next lemma is quite useful. It is essentially [B:A, VIII, $2^{\text{ème}}$ ed., §11.2, Prop. 4(b)], see also [B:Lie, VIII, §7.6, Lem. 3].

Lemma 3.2.4. *Let G be a group. Let $\varphi : \mathscr{P}(L) \longrightarrow G$ be a map satisfying $\varphi(\mathrm{cl}(P)) = \varphi(\mathrm{cl}(P_1))+\varphi(\mathrm{cl}(P_2))$ whenever the sequence of L-modules $0 \longrightarrow P_1 \longrightarrow P \longrightarrow P_2 \longrightarrow 0$ is exact. Let $\widetilde{\varphi} : \mathscr{F}(L) \longrightarrow G$ be the unique group homomorphism extending φ. Then there exists a unique group homomorphism $\psi : \mathscr{R}(L) \longrightarrow G$ such that $\psi([P]) = \widetilde{\varphi}(\mathrm{cl}(P))$.*

Proof. The existence of $\widetilde{\varphi}$ follows from the fact that $\mathscr{F}(L)$ is the free group generated by $\mathscr{P}(L)$. From group theory, it is sufficient to show that $\mathscr{E}(L) \subseteq \ker(\varphi)$. So, consider an exact sequence $0 \longrightarrow P_1 \longrightarrow P \longrightarrow P_2 \longrightarrow 0$ and $\mathrm{cl}(P) - \mathrm{cl}(P_1) - \mathrm{cl}(P_2)$. Then $\varphi\big(\,\mathrm{cl}(P) - \mathrm{cl}(P_1) - \mathrm{cl}(P_2)\big) = \varphi\big(\,\mathrm{cl}(P)\big) - \varphi\big(\,\mathrm{cl}(P_1)\big) - \varphi\big(\,\mathrm{cl}(P_2)\big) = 0$. Again, by group theory, there exists a group homomorphism $\psi : \mathscr{F}(L)/\mathscr{E}(L) \longrightarrow G$ so that $\psi([P]) = \widetilde{\varphi}(\mathrm{cl}(P))$. □

Proposition 3.2.5. (special case of [B:A, VIII, $2^{\text{ème}}$ ed., §11.7, Prop. 9]). *There exists a unique bilinear product $- \cdot -$ on $\mathscr{R}(L)$ satisfying $[P]\cdot[N] = [P \otimes_R N]$ where P and N are finitely generated projective L-modules. With respect to this product, $\mathscr{R}(L)$ is a unital commutative ring. The class $[R]$ is the multiplicative identity.*

Proof. We define a map $\widetilde{m} : \mathscr{F}(L) \times \mathscr{F}(L) \longrightarrow \mathscr{R}(L)$ by $(\mathrm{cl}(P), \mathrm{cl}(N)) \longrightarrow [P \otimes_R N]$. This map is well-defined since $P \otimes_R N$ is also a finitely generated projective L-module, by Example 1.5.4 (iii). Moreover, by definition, $\widetilde{m}$ is bilinear.

Now, we claim that $\widetilde{m}(\mathscr{E}(L) \times \mathscr{F}(L)) = 0 = \widetilde{m}(\mathscr{F}(L) \times \mathscr{E}(L))$. Indeed, consider an exact sequence $0 \longrightarrow (P_1, \rho_1) \xrightarrow{f} (P, \rho) \xrightarrow{g} (P_2, \rho_2) \longrightarrow 0$ and the corresponding element $\mathrm{cl}(P) - \mathrm{cl}(P_1) - \mathrm{cl}(P_2) \in \mathscr{F}(L)$. Let (N, μ) be another finitely generated projective L-module. The sequence

$$0 \longrightarrow P_1 \otimes_R N \xrightarrow{f \otimes \mathrm{Id}_N} P \otimes_R N \xrightarrow{g \otimes \mathrm{Id}_N} P_2 \otimes_R N \longrightarrow 0$$

of R-modules is exact since projective modules are flat. Moreover, it is also an exact sequence of L-modules by Lemma 3.1.1(c). It follows that $\widetilde{m}(\mathscr{E}(L) \times \mathscr{F}(L)) = 0$. Similarly, one has that $\widetilde{m}(\mathscr{F}(L) \times \mathscr{E}(L)) = 0$. Hence, by Lemme 3.2.4, $\widetilde{m}$ induces well-defined maps $\widetilde{m}' : \mathscr{R}(L) \times \mathscr{F}(L) \longrightarrow \mathscr{R}(L)$. Similarly, $\widetilde{m}'(\mathscr{R}(L) \times \mathscr{E}(L)) = 0$. Therefore, $\widetilde{m}$ induces a unique $\mathbb{Z}$-bilinear map $m : \mathscr{R}(L) \times \mathscr{R}(L) \longrightarrow \mathscr{R}(L)$ satisfying $m([P], [N]) = [P \otimes_R N]$.

Now, consider the L-module $(R, 0)$. By Lemma 3.1.1(b), we see that $[P] = [P \otimes_R R]$ for all L-module P. Finally, by parts (d), (e) and (f) of Lemma 3.1.1, we see that $\mathscr{R}(L)$ is a ring with multiplicative identity $[R]$. □

Lemma 3.2.6. *Let M be an L-module. Let $0 = M_0 \subseteq \cdots \subseteq M_l = M$ be a sequence of L-submodules of M so that M_i/M_{i-1} is a finitely generated projective L-module for $i = 1, \ldots, l$. Then $[M] = \sum_{i=1}^{l} [M_i/M_{i-1}]$.*

Proof. We know that $0 \longrightarrow M_{i-1} \longrightarrow M_i \longrightarrow M_i/M_{i-1} \longrightarrow 0$ is an exact sequence of finitely generated projective L-modules. It follows that $[M_i/M_{i-1}] = [M_i] - [M_{i-1}]$. Therefore, since $M_0 = 0$ and $M_l = M$, $[M] = \sum_{i=1}^{l}([M_i] - [M_{i-1}]) = \sum_{i=1}^{l}[M_i/M_{i-1}]$. □

Lemma 3.2.7. [B:A, VIII, 2$^{\text{ème}}$ ed., §11.3, Lemme 1]. *Let R be a field. For a simple L-module S and a finite dimensional L-module M, let $l_S(M) = [M : S]$. Let $0 \longrightarrow M \longrightarrow N \longrightarrow P \longrightarrow 0$ be an exact sequence of finite dimensional L-modules. As a consequence, l_S induces a $\mathbb{Z}$-linear map $\varphi_S : \mathscr{R}(L) \longrightarrow \mathbb{Z}$, $[E] \mapsto l_S(E)$.*

Proof. Since M, N and P are finite dimensional, they have composition series. Let Σ_M be a composition series of M and let Σ_P be a composition series of N/M. Then concatenating Σ_M and Σ_P gives a composition series for N. The second assertion is a direct application of Lemma 3.2.4 □

Proposition 3.2.8. *Let $f : L \longrightarrow L'$ be a homomorphism of R-Lie algebras.*

(a) *If (M', ρ') is an L'-module then $(M', \rho \circ f)$ is an L-module, which is finitely generated projective if M' is so. The map $(M', \rho') \mapsto (M', \rho' \circ f)$ preserves isomorphism classes and exact sequences of modules, hence induces a morphism of rings $\mathscr{R}(f) : \mathscr{R}(L') \longrightarrow \mathscr{R}(L)$.*

(b) *$\mathscr{R}$ is a contravariant functor from the category of R-Lie algebras to the category of commutative rings.*

Proof. (a) The first assertion is clear from the fact that composition of homomorphisms is a homomorphism. Next, we show that $(M', \rho') \mapsto (M', \rho' \circ f)$ preserves isomorphism classes of modules. Suppose $(M', \rho') \simeq (N', \mu')$ with L'-module isomorphism $\varphi : (M', \rho') \mapsto (N', \mu')$. We prove that $\varphi : (M', \rho' \circ f) \longrightarrow (N', \mu' \circ f)$ is a homomorphism of L-modules. For $x \in L$ and $n \in N'$, we have that

$$\varphi\big((\rho' \circ f)(x)(n)\big) = \varphi\big(\rho'(f(x))(n)\big) = \rho'(f(x))\varphi(n) = (\rho' \circ f)(x)\varphi(n).$$

The result follows. The statement about exact sequences is also clear. So, by Lemma 3.2.4, we have a map $\mathscr{R}(L') \longrightarrow \mathscr{R}(L)$. Moreover, for L'-modules (M, ρ) and (N, μ), we have the following

$$\mathscr{R}(f)[M \otimes N, \rho \otimes \mu] = [M \otimes N, (\rho \otimes \mu) \circ f] = [M, \rho \circ f] \cdot [N, \mu \circ f].$$

It follows that $\mathscr{R}(f)$ is a homomorphism of rings.

(b) This is clear from the description of $\mathscr{R}$ in part (a) and we omit the details. □

We prove some properties of $\mathscr{R}(L)$. Let $\mathrm{U}(L)$ be the envelopping algebra of L, and denote by $\mathrm{U}(L)^*$ is R-dual. In part (d), we use the fact that $U(L)$ is cocommutative coalgebra and that $U(L)^*$ is a ring. The ring structure is as follows. Let $c : U(L) \longrightarrow U(L) \otimes U(L)$ be the coalgebra map. If $\varphi, \psi \in U(L)^*$ and if $u = \sum_i u_i \otimes v_i$, then $(\varphi \cdot \psi)(u) := \sum_i \varphi(u_i)\psi(v_i)$ is the ring multiplication in $U(L)^*$.

Proposition 3.2.9. (a) *There exists a unique ring automorphism τ of $\mathscr{R}(L)$ of period 2 satisfying $\tau([P]) = [P^*]$.*

(b) *For $\mathfrak{p} \in \mathrm{Spec}(R)$, there exists a unique ring homomorphism $r_{\mathfrak{p}} : \mathscr{R}(L) \longrightarrow \mathbb{Z}$ associating to any finitely generated projective L-module P its $\mathfrak{p}$-rank $\mathrm{rk}_{\mathfrak{p}}(P)$.*

(c) *Let $\mathcal{F}_{loc}$ be the ring of locally constant functions* $\mathrm{Spec}(R) \longrightarrow \mathbb{Z}$. *There exists a unique ring homomorphism* $\dim : \mathscr{R}(L) \longrightarrow \mathcal{F}_{loc}$ *assigning to a finitely generated projective L-module P the function* $\mathfrak{p} \mapsto \mathrm{rk}_{\mathfrak{p}}(P)$.

(d) *For every finitely generated projective L-module (ρ, P), the map* $\mathrm{Tr}_P : \mathrm{U}(L) \longrightarrow R$, $u \mapsto \mathrm{tr}_P(\rho(u))$ *is an element of* $\mathrm{U}(L)^*$. *The map* $P \mapsto \mathrm{Tr}_P$ *extends to a ring homomorphism* $\mathrm{Tr} : \mathscr{R}(L) \longrightarrow \mathrm{U}(L)^*$.

(e) *If $R = k$ is a field, then the additive group $(\mathscr{R}(L), +) \simeq \bigoplus_{\mathscr{S}_L} \mathbb{Z}$ where $\mathscr{S}_L$ is the set of isomorphism classes of simple finite-dimensional L-modules. Moreover, using the notation of Lemma 3.2.7, $[M] = \sum_{S \in \mathscr{S}_L} l_S(M)[S]$.*

Proof. (a). Define $\varphi : \mathscr{F}(L) \longrightarrow \mathscr{R}(L)$ by $[P] \mapsto [P^*]$. We show that

$$\varphi([P]) = \varphi(\mathrm{cl}(P_1)) + \varphi(\mathrm{cl}(P))$$

whenever there is an exact sequence $0 \longrightarrow P_1 \longrightarrow P \longrightarrow P_2 \longrightarrow 0$ of L-modules.

Let $0 \longrightarrow P_1 \xrightarrow{f} P \xrightarrow{g} P_2 \longrightarrow 0$ be an exact sequence of finitely generated projective L-modules. Since $0 \longrightarrow P_1 \xrightarrow{f} P \xrightarrow{g} P_2 \longrightarrow 0$ is split exact as R-modules by projectivity of P_2, the corresponding dual sequence of R-modules

$$0 \longrightarrow P_2^* \xrightarrow{g^t} P^* \xrightarrow{f^t} P_1^* \longrightarrow 0$$

is split exact by [B:A, II, §2.1, Prop. 1]. Moreover, by Lemma 3.1.2(b), the maps f^t and g^t are homomorphisms of L-modules. It follows that

$$\varphi([P]) - \varphi(\mathrm{cl}(P_1)) - \varphi(\mathrm{cl}(P)) = [P^*] - [P_1^*] - [P_2^*] = 0.$$

By Lemma 3.2.4, φ induces a unique group homomorphism $\tau : \mathscr{R}(L) \longrightarrow \mathscr{R}(L)$ satisfying $\tau([P]) = [P^*]$. Since $(P \otimes_R N)^* \simeq P^* \otimes_R N^*$ as L-modules for finitely generated projective R-modules P and N (Proposition 1.5.17), it follows that τ is a ring homomorphism. Finally, since $P^{**} \simeq P$ as L-modules (Lemma 1.5.10), it follows that τ is a ring automorphism of period 2.

(b). For each $\mathfrak{p} \in \mathrm{Spec}(R)$, define the map $\varphi : \mathscr{P}(L) \longrightarrow \mathbb{Z}$, $\mathrm{cl}(P) \mapsto \mathrm{rk}_{R_{\mathfrak{p}}}(P)$. We claim that whenever we have an exact sequence $0 \longrightarrow P_1 \longrightarrow P \longrightarrow P_2 \longrightarrow 0$ in $\mathscr{P}(L)$, then we have $\varphi(\mathrm{cl}(P)) = \varphi(\mathrm{cl}(P_1)) + \varphi(\mathrm{cl}(P_2))$.

Suppose that one has an exact sequence $0 \longrightarrow P_1 \longrightarrow P \longrightarrow P_2 \longrightarrow 0$ of finitely generated projective L-modules. Since $R_{\mathfrak{p}}$ is flat,

$$0 \longrightarrow P_1 \otimes_R R_{\mathfrak{p}} \longrightarrow P \otimes_R R_{\mathfrak{p}} \longrightarrow P_2 \otimes_R R_{\mathfrak{p}} \longrightarrow 0$$

is an exact sequence of free R-modules of finite rank.

So, $P \otimes_R R_{\mathfrak{p}} \simeq (P_1 \otimes_R R_{\mathfrak{p}}) \oplus (P_2 \otimes_R R_{\mathfrak{p}})$ as $R_{\mathfrak{p}}$-modules. Thus, $\varphi(\mathrm{cl}(P)) = \mathrm{rk}_{\mathfrak{p}}(P) = \mathrm{rk}_{\mathfrak{p}}(P_1) + \mathrm{rk}_{\mathfrak{p}}(P_2) = \varphi(\mathrm{cl}(P_1)) + \varphi(\mathrm{cl}(P_2))$. Hence, by Lemma 3.2.4, φ induces a unique group homomorphism $r_{\mathfrak{p}} : \mathscr{R}(L) \longrightarrow \mathbb{Z}$ satisfying $r_{\mathfrak{p}}([P]) = \varphi(\mathrm{cl}(P))$. It is left to check that $r_{\mathfrak{p}}$ is also a ring homomorphism. By Proposition 1.5.24, we know that $\mathrm{rk}_{\mathfrak{p}}(P \otimes_R N) = \mathrm{rk}_{\mathfrak{p}}(P) \cdot \mathrm{rk}_{\mathfrak{p}}(N)$ for finitely generated projective R-modules P and N. Since $\mathrm{rk}_{\mathfrak{p}}(R) = 1$, it follows that $\mathrm{rk}_{\mathfrak{p}}$ is a ring homomorphism.

(c). We define a map $\varphi : \mathscr{P}(L) \longrightarrow \mathcal{F}_{loc}$ by $\mathrm{cl}(P) \mapsto (\mathrm{rk}(P) : \mathfrak{p} \mapsto \mathrm{rk}_{\mathfrak{p}}(P))$. We show that φ preserves exact sequences. Let $0 \longrightarrow P_1 \longrightarrow P \longrightarrow P_2 \longrightarrow 0$ be an exact sequence in $\mathscr{P}(L)$. We claim that $\mathrm{rk}(P) = \mathrm{rk}(P_1) + \mathrm{rk}(P_2)$. But this is clear since by Proposition 1.5.24, $\mathrm{rk}_{\mathfrak{p}}(P) = \mathrm{rk}_{\mathfrak{p}}(P_1) + \mathrm{rk}_{\mathfrak{p}}(P_2)$ for all $\mathfrak{p} \in \mathrm{Spec}(R)$. Therefore, by Lemma 3.2.4, φ induces a unique group homomorphism $\psi : \mathscr{R}(L) \longrightarrow \mathcal{F}_{loc}$ satisfying $\psi([P]) = \mathrm{rk}(P)$. It is left to show that ψ is a ring homomorphism. But, again, this is clear since by Proposition 1.5.24, $\mathrm{rk}_{\mathfrak{p}}(P \otimes_R N) = \mathrm{rk}_{\mathfrak{p}}(P) \cdot \mathrm{rk}_{\mathfrak{p}}(N)$ for finitely generated projective L-modules P and N and for $\mathfrak{p} \in \mathrm{Spec}(R)$.

(d). Clearly, given a finitely generated projective L-module P, tr_P is a linear functional since the trace map is R-linear. Define a map $\varphi : \mathscr{P}(L) \longrightarrow \mathrm{U}(L)^*$, $[P] \mapsto \mathrm{Tr}_P$. Once again, we show that φ preserves exact sequences.

Let $0 \longrightarrow P_1 \xrightarrow{f} P \xrightarrow{g} P_2 \longrightarrow 0$ be an exact sequence where P, P_1, P_2 are finitely generated and projective as L-modules. We claim that $\mathrm{Tr}_P = \mathrm{Tr}_{P_1} + \mathrm{Tr}_{P_2}$, i.e., $\mathrm{tr}_P(\rho(u)) = \mathrm{tr}_{P_1}(\rho_1(u)) + \mathrm{tr}_{P_2}(\rho_2(u))$ for all $u \in \mathrm{U}(L)$. As f, g are homomorphisms of L-modules, it follows that the diagram

$$\begin{tikzcd}
0 \arrow[r] & P_1 \arrow[r, "f"] \arrow[d, "\rho_1(u)"'] & P \arrow[r, "g"] \arrow[d, "\rho(u)"'] & P_2 \arrow[r] \arrow[d, "\rho_2(u)"'] & 0 \\
0 \arrow[r] & P_1 \arrow[r, "f"'] & P \arrow[r, "g"'] & P_2 \arrow[r] & 0
\end{tikzcd}$$

is commutative. Hence, by Proposition 2.2.9(b), $\mathrm{tr}(\rho(u)) = \mathrm{tr}(\rho_1(u)) + \mathrm{tr}(\rho_2(u))$. Therefore, φ preserves exact sequences.

By Lemma 3.2.4, there exists a group homomorphism $\mathrm{Tr} : \mathscr{R}(L) \longrightarrow \mathrm{U}(L)^*$ satisfying the equation $\mathrm{Tr}([P]) = \mathrm{Tr}_P$ for $[P] \in \mathscr{R}(L)$. It is left to show that Tr is a ring homomorphism. Let P and N be finitely generated projective L-modules. Let $u \in U(L)$ and let c be the coproduct of $U(L)$. Assume that $c(u) = \sum_i u_i \otimes u_i'$. By definition of the $U(L)$-module $P \otimes_R N$, one has that $u_{P\otimes_R N} = \sum_i (u_i)_P \otimes (u_i')_N$. Hence,

$$\mathrm{Tr}_{P\otimes_R N}(u) = \sum_i \mathrm{tr}(u_i)_P \,\mathrm{tr}(u_i')_N = \sum_i \mathrm{tr}_P(u_i)\,\mathrm{tr}_N(u_i') = (\mathrm{Tr}_P \otimes \mathrm{Tr}_N)(c(u)).$$

It follows that $\mathrm{Tr}_P \,\mathrm{Tr}_N = \mathrm{tr}_{P\otimes_R N}$. Therefore, Tr is a ring homomorphism

(e). Note that when $R = k$ is a field, being finitely generated projective is equivalent to being finite-dimensional. We claim that the set of isomorphism classes corresponding to simple modules form a basis for $\mathscr{R}(L)$.

Let $[P] \in \mathscr{R}(L)$. Since P is finite-dimensional, it has a composition series $[P_0, \ldots, P_l]$, i.e., submodules P_i satisfying $0 = P_0 \subset P_1 \subset \cdots \subset P_l = P$ and P_i/P_{i-1} simple for $i = 1, \ldots, l$. Now, the simple modules P_i/P_{i-1} are finite-dimensional, whence $\mathrm{cl}(P_i/P_{i-1}) \in \mathscr{P}(L)$. By Lemma 3.2.6, $[P] = \sum_{i=1}^{l} [P_i/P_{i-1}]$. Therefore, the set of isomorphism classes of simple modules form a generating set of $\mathscr{R}(L)$. Hence, $[P] = \sum_{S\in\mathscr{S}_L} l_S(P)[S]$. It is left to show linear independence.

By Lemma 3.2.7, for each $S \in \mathscr{S}_L$, there exists a homomorphism $\varphi_S : \mathscr{R}(L) \longrightarrow \mathbb{Z}$, $[E] \mapsto l_S(E)$. We have that $l_S(S) = 1$ and $l_S(S') = 0$ for all simple module $S' \not\cong S$. Therefore, $\{[S] : S \in \mathscr{S}_L\}$ is linearly independent. This completes the proof. □

Remark 3.2.10. Note that part (e) of the previous proposition is known; it can be found in [B:Lie, VIII, §7.6].

Proposition 3.2.11. [B:Lie, VIII, §7, Exercice 9]. *There exists a unique ring homomorphism* $\varphi : \mathscr{R}(\mathfrak{a}_1) \otimes_{\mathbb{Z}} \mathscr{R}(\mathfrak{a}_2) \longrightarrow \mathscr{R}(\mathfrak{a}_1 \times \mathfrak{a}_2)$ *satisfying* $\varphi([P_1] \otimes [P_2]) = [P_1 \otimes_R P_2]$. *If* $R = k$ *is a field of characteristic 0 and* $\mathfrak{a}_1$ *or* $\mathfrak{a}_2$ *is finite-dimensional split semisimple, then* φ *is an isomorphism.*

Proof. Define $\psi : \mathscr{F}(\mathfrak{a}_1) \times \mathscr{F}(\mathfrak{a}_2) \longrightarrow \mathscr{R}(\mathfrak{a}_1 \times \mathfrak{a}_2)$, $(\mathrm{cl}(M_1), \mathrm{cl}(M_2)) \mapsto [M_1 \otimes M_2]$. We claim that $\psi(\mathscr{E}(\mathfrak{a}_1) \times \mathscr{F}(\mathfrak{a}_2)) = 0 = \psi(\mathscr{F}(\mathfrak{a}_1) \times \mathscr{E}(\mathfrak{a}_2))$.

If $0 \longrightarrow (L, \rho) \longrightarrow (M, \mu) \longrightarrow (N, \nu)$ is an exact sequence of $\mathfrak{a}_1$-modules, then the sequence $0 \longrightarrow L \otimes P \longrightarrow M \otimes P \longrightarrow N \otimes P$ must also be exact since N

is flat. The claim then follows. So, we have a unique group homomorphism from $\mathscr{R}(\mathfrak{a}_1) \times \mathscr{F}(\mathfrak{a}_2)$ to $\mathscr{R}(\mathfrak{a}_1 \times \mathfrak{a}_2)$. Similarly, we obtain a unique group homomorphism $\mathscr{F}(\mathfrak{a}_1) \times \mathscr{R}(\mathfrak{a}_2) \longrightarrow \mathscr{R}(\mathfrak{a}_1 \times \mathfrak{a}_2)$.

Therefore, we have a unique group homomorphism $\mathscr{R}(\mathfrak{a}_1) \times \mathscr{R}(\mathfrak{a}_2) \longrightarrow \mathscr{R}(\mathfrak{a}_1 \times \mathfrak{a}_2)$. Moreover, this map is bilinear. Therefore, it induces a unique ring homomorphism $\psi : \mathscr{R}(\mathfrak{a}_1) \times \mathscr{R}(\mathfrak{a}_2) \longrightarrow \mathscr{R}(\mathfrak{a}_1 \times \mathfrak{a}_2)$, $([M_1], [M_2]) \mapsto [M_1 \otimes M_2]$.

Now, assume that either $\mathfrak{a}_1$ or $\mathfrak{a}_2$ is finite-dimensional split semisimple. We put $\mathfrak{a} = \mathfrak{a}_1 \times \mathfrak{a}_2$ and recall that $U(\mathfrak{a}) = U(\mathfrak{a}_1) \otimes_R U(\mathfrak{a}_2)$. Without loss of generality, we assume that $\mathfrak{a}$ is finite-dimensional split semisimple.

Claim: If S_i are simple finite-dimensional $\mathfrak{a}_i$-modules, $i = 1, 2$, then $S_1 \otimes_k S_2$ is a simple $\mathfrak{a}$-module.

Indeed, by [B:Lie, VIII, §6.2. Prop. 3], S_i is absolutely simple, i.e., its commutant is 1-dimensional. Hence, by [B:A, VIII, $2^{\text{ème}}$ ed., §12.2, Cor. du Thm. 2], $S_1 \otimes_R S_2$ is a simple $\mathfrak{a}$-module.

Now, let S be a finite-dimensional $\mathfrak{a}$-module. By [B:A, VIII, $2^{\text{ème}}$ ed., §12.1, Prop. 2] and the claim, we get that $S = S_1 \otimes_R S_2$ for simple finite-dimensional $\mathfrak{a}_i$-modules S_i. By the claim, the map $\mathscr{S}_{\mathfrak{a}_1} \times \mathscr{S}_{\mathfrak{a}_2} \longrightarrow \mathscr{S}_{\mathfrak{a}}$, $\big(\mathrm{cl}(S_1), \mathrm{cl}(S_2)\big) \mapsto \mathrm{cl}(S_1 \otimes_R S_2)$, is well-defined and surjective. The map is also injective since, for $S = S_1 \otimes S_2$, the $\mathfrak{a}_i$-module S is isotypic of type S_i ([B:A, VIII, $2^{\text{ème}}$ ed., §12.1, Prop. 2]), proving that the class of $[S_i]$ only depends on $[S_1 \otimes S_2]$. By Proposition 3.2.9(e), $\{[S_1] \otimes [S_2] : S_i \in \mathscr{S}_{\mathfrak{a}_i}\}$ is a basis of the $\mathbb{Z}$-module $\mathscr{R}(\mathfrak{a}_1) \otimes \mathscr{R}(\mathfrak{a}_2)$. It is bijectively mapped onto a basis of $\mathscr{R}(\mathfrak{a})$, proving the Proposition. $\square$

Theorem 3.2.12. [B:Lie, VIII, §7.7, Thm. 2]. *Let $\mathfrak{g}$ be a split semisimple Lie algebra over a field k of characteristic zero. We denote by P its weight lattice (viewed as linear forms on $\mathfrak{h}^*$ for $\mathfrak{h}$ a splitting Cartan subalgebra of $\mathfrak{g}$. For a finite-dimensional $\mathfrak{g}$-module V, we let $\mathrm{ch}(V)$ be its character. Then the map $V \mapsto \mathrm{ch}(V)$ induces a ring homomorphism $\mathrm{ch} : \mathscr{R}(\mathfrak{g}) \longrightarrow \mathbb{Z}[P]^W$ where $\mathbb{Z}[P]$ is the group ring of P and $\mathbb{Z}[P]^W$ is the ring of invariants under the action of the Weyl group W of $\mathfrak{g}$ on $\mathbb{Z}[P]$, i.e., $\mathbb{Z}[P] = \bigoplus_{\lambda \in P} e^{\lambda}$, $w \cdot e_{\lambda} = e^{w\lambda}$ for $\lambda \in P$, and $w \in W$.*

The next few results investigate the behaviour of the representation ring under a base ring extension of the Lie algebra.

Lemma 3.2.13. *Let $S \in R$-**alg**. Then:*

(a) *$L \otimes_R S$ has the structure of a S-Lie algebra given by $[x \otimes s, y \otimes s'] := [x, y] \otimes ss'$ for $x, y \in L$ and $s, s' \in S$.*

(b) *Let L_1, L_2 be two R-Lie algebras. Let $f : L_1 \longrightarrow L_2$ be a homomorphism of R-Lie algebras. Then $f \otimes \mathrm{Id}_S : L_1 \otimes_R S \longrightarrow L_2 \otimes_R S$ is a homomorphism of S-Lie algebras.*

(c) *Let (M, ρ) be an L-module. Then the maps $\rho \otimes \mathrm{Id}_S : L \otimes_R S \longrightarrow \mathfrak{gl}_R(M) \otimes_R S$ and $\nu : \mathrm{End}_R(M) \otimes_R S \longrightarrow \mathrm{End}_S(M \otimes_R S)$, $f \otimes s \mapsto (m \otimes t \mapsto f(m) \otimes st)$, are homomorphism of S-Lie algebras.*

(d) *If (P, ρ) is a finitely generated projective L-module, then $(P \otimes_R S, \nu \circ (\rho \otimes \mathrm{Id}_S))$ is finitely generated projective $L \otimes_R S$-module, where ν is as in part* (c).

(e) *Let $(M, \rho), (N, \mu)$ are finitely generated projective L-module. Then the canonical map of S-modules $(M_S \otimes_R N_S, \rho_S \otimes \mu_S) \longrightarrow ((M \otimes_R N) \otimes_R S, (\rho \otimes \mu) \otimes \mathrm{Id}_S)$ is also an isomorphism of L_S-modules.*

Proof. (a). Bilinearity of the bracket in $L \otimes_R S$ follows from bilinearity of the Lie bracket of L, of the distributivity of the product in S and of the bilinearity of the tensor product. On simple tensors, the Jacobi identity follows from the Jacoby indentity of the Lie bracket of L, thus is true for all elements of $L \otimes_R S$. Therefore, $L \otimes_R S$ is a S-Lie algebra.

(b). For $f \in \mathrm{End}_R(M), x, y \in L$ and $s, s' \in S$, we have the following:

$$\begin{aligned}(f \otimes \mathrm{Id}_S)([x \otimes s, y \otimes s']) &= (f \otimes \mathrm{Id}_S)([x, y] \otimes ss') = f([x, y]) \otimes ss' \\ &= [f(x), f(y)] \otimes ss' = [f(x) \otimes s, f(y) \otimes s'] \\ &= [(f \otimes \mathrm{Id}_S)(x \otimes s), (f \otimes \mathrm{Id}_S)(y \otimes s')].\end{aligned}$$

(c). The first assertion follows from (b). For $f, g \in \mathrm{End}_R(M), m \in M$, $s, s', t \in S$,

$$\begin{aligned}\nu([f \otimes s, g \otimes s'])(m \otimes t) &= \nu([f, g] \otimes ss')(m \otimes t) = [f, g](m) \otimes ss't \\ &= (fg - gf) \otimes ss't, \\ [\nu(f \otimes s), \nu(g \otimes s')](m \otimes t) &= (\nu(f \otimes s)\nu(g \otimes s') - \nu(g \otimes s')\nu(f \otimes s))(m \otimes t) \\ &= fg(m) \otimes ss't - gf(m) \otimes s'st = (fg - gf) \otimes ss't.\end{aligned}$$

(d). From Example 1.5.4(ii), we know that $P \otimes_R S$ is a finitely generated projective S-module. Hence, it is sufficient to show that $P \otimes_R S$ is an $L \otimes_R S$-module. However, this is clear from part (b) and (c) since $\rho \otimes \mathrm{Id}_S$ and ν are Lie algebra homomorphisms.

(e). We know that the canonical map $\nu : M_S \otimes_R N_S \longrightarrow (M \otimes_R N) \otimes_R S$ is an isomorphism of S-modules. It suffices to show that it is a homomorphism of L-modules. For $l \in L, m \in M, n \in N$ and $s, s', t \in S$,

$$\begin{aligned}
\nu\Big((\rho_S \otimes \mu_S)(l \otimes t)\big((m \otimes s) \otimes (n \otimes s')\big)\Big) &= \nu\big(\rho_S(l \otimes t)(m \otimes s) \otimes (n \otimes s') + \\
&\qquad + (m \otimes s) \otimes \mu_S(l \otimes t)(n \otimes s')\big) \\
&= \nu\big(\rho(l)(m) \otimes ts \otimes (n \otimes s') \\
&\qquad + (m \otimes s) \otimes \mu(l)(n) \otimes ts'\big) \\
&= \big(\rho(l)(m) \otimes n\big) \otimes tss' \\
&\qquad + \big(m \otimes \mu(l)(n)\big) \otimes tss' \\
&= \big(\rho(l)(m) \otimes n + m \otimes \mu(l)(n)\big) \otimes tss', \\
(\rho \otimes \mu)_S(l \otimes t)\nu\big((m \otimes s) \otimes (n \otimes s')\big) &= \big((\rho \otimes \mu)(l) \otimes t\big)\big((m \otimes n) \otimes ss'\big) \\
&= \big(\rho(l)(m) \otimes n + m \otimes \mu(l)(n)\big) \otimes tss'.
\end{aligned}$$

We are done. ☐

Lemma 3.2.14. *Let S be a flat R-algebra. Then there exists a unique ring map $\psi : \mathscr{R}(L) \longrightarrow \mathscr{R}(L \otimes_R S)$ satisfying $\psi(\pi_L([(P, \rho)])) = \pi_{L\otimes_R S}([(P \otimes_R S, \nu \circ (\rho \otimes \mathrm{Id}_S))])$.*

Proof. Define $\varphi : \mathscr{P}(L) \longrightarrow \mathscr{R}(L \otimes_R S)$, $[(P, \rho)] \mapsto \pi_{L\otimes_R S}([(P \otimes_R S, \nu \circ (\rho \otimes \mathrm{Id}_S)])$.

Suppose that $0 \longrightarrow P_1 \longrightarrow P \longrightarrow P_2 \longrightarrow 0$ is an exact sequence of L-modules. Since S is flat, the sequence $0 \longrightarrow P_1 \otimes_R S \longrightarrow P \otimes_R S \longrightarrow P_2 \otimes_R S \longrightarrow 0$ is an exact sequence of S-modules. Moreover, by Lemma 3.2.13(b), it is also an exact sequence of $L \otimes_R S$-module. It follows that φ satisfies the condition of Lemma 3.2.4. Therefore, there exists a unique group homomorphism $\psi : \mathscr{R}(L) \longrightarrow \mathscr{R}(L \otimes_R S)$ satisfying $\psi\pi_L = \varphi$. It is left to check that it is also a ring homomorphism.

$$\begin{aligned}
\psi([(M, \rho)] \cdot [(N, \mu)]) &= \psi([(M \otimes_R N, \rho \otimes \mu)]) = [((M \otimes_R N) \otimes_R S, (\rho \otimes \mu) \otimes \mathrm{Id}_S)] \\
&= [(M_S \otimes N_S, \rho_S \cdot \mu_S)] = [(M_S, \rho_S)] \cdot [(N_S, \mu_S)] \\
&= \psi([(M, \rho)]) \cdot \psi([(N, \mu)]).
\end{aligned}$$

Therefore, ψ is a ring homomorphism. We are done. ☐

Chapter 4

Invariant bilinear forms

4.1 Bilinear maps

Before mentioning invariant bilinear forms, we will briefly present some general results on general bilinear forms which can be found in [NPPS].

In this section, R denotes a unital associative commutative ring, M is an R-module.

Notation 4.1.1. For an R-module M, we will denote by $\mathscr{L}^2_R(M)$ the R-module of R-bilinear forms.

Remark 4.1.2. We have the following commutative diagram of R-module isomorphisms.

$$\begin{tikzcd}
\operatorname{Hom}_R(M \otimes_R M, R) \arrow[rr, "\mathrm{x}", "\simeq"'] \arrow[dr, "\simeq", "\mathrm{y}"'] & & \mathscr{L}^2_R(M) \\
& \operatorname{Hom}_R(M, \operatorname{Hom}_R(M, R)) \arrow[ur, "\simeq", "\mathrm{z}"'] &
\end{tikzcd}$$

where: for $\varphi \in \operatorname{Hom}_R(M \otimes_R M, R)$, $m_1, m_2 \in M$, $\psi \in \operatorname{Hom}_R(M, \operatorname{Hom}_R(M, R))$, $(\mathrm{x}(\varphi))(m_1, m_2) = \varphi(m_1 \otimes m_2)$, $(\mathrm{y}(\varphi))(m_1)(m_2) = \varphi(m_1 \otimes m_2)$ and $\mathrm{z}(\psi)(m_1, m_2) = \psi(m_1)(m_2)$.

Definition 4.1.3. One calls $\kappa \in \mathscr{L}^2_R(M)$ *nondegenerate* (resp. *nonsingular*) if $\mathrm{z}^{-1}(\kappa)$ is injective (resp. bijective).

Remark 4.1.4. Equivalently, a bilinear form $\kappa \in \mathscr{L}^2(M)$ is nondegenerate if and only if $\kappa(x, y) = 0$ for all $y \in M$ implies that $x = 0$. Indeed, assume that κ is nondegenerate. If $\kappa(x, y) = 0$ for all $y \in M$, then $\kappa(x, -) = \kappa(0, -)$. Since κ is nondegenerate, $x = 0$. Conversely, assume that $\kappa(x, y) = 0$ for all $y \in M$ implies that $x = 0$. If $\kappa(x, -) = \kappa(y, -)$, then $\kappa(x - y, -) = 0$. It follows that $x = y$.

The following result is standard.

Lemma 4.1.5. *Let M be a free R-module with basis $\{m_1, \ldots, m_n\}$. Let κ be a bilinear form on M. Let A be the matrix of κ with respect to the basis $\{m_i\}$. Then*

(i) *κ is nondegenerate if and only if* $\det A$ *is not a zero divisor in R.*

(ii) *κ is nonsingular if and only if* $\det A$ *is invertible in R.*

Proof. First, that A^T defines an endomorphism of M by $A^T(m_j) = \sum_{i=1}^{n} \kappa(m_j, m_i)m_i$ where A^T denotes the transpose of A. We start the proof by showing that κ is nondegenerate (resp. nonsingular) if and only if A^T, as an endomorphism of M, is injective (resp. bijective).

Suppose that A^T is not injective. There exist $m \neq 0$ such that $0 = A^T(m) = \sum_{i=1}^{n} \kappa(m, m_i)m_i$. Since $\{m_i\}$ is a basis of M, $\kappa(m, m_i) = 0$ for all i. It follows that κ is degenerate. Conversely, suppose that κ is degenerate, then $\kappa(m, m_i) = 0$ for all i. Therefore, $A^T(m) = 0$.

Suppose that κ is nonsingular. We show that A^T is surjective. Let $m' = \sum_i \alpha_i m_i$, $\alpha_i \in R$, be an element of M. Define a linear map $\varphi : M \longrightarrow R$ by $\varphi(m_i) = \alpha_i$. Then, since κ is nonsingular, $\varphi = \kappa(m, -)$ for some unique $m \in M$. Hence, $m' = A^T(m)$. Therefore, A^T is bijective. Conversely, suppose that A^T is bijective. Let $\varphi \in \mathrm{Hom}_R(M, R)$. Let $m' = \sum_i \varphi(m_i)m_i$. There exists m such that $A^T(m) = \sum_i \kappa(m, m_i)m_i = m'$. It follows that $\varphi = \kappa(m, -)$. Therefore, κ is nonsingular.

Both parts follow from the fact that $\det(A^T) = \det(A)$ and from [B:A, III, §8.2, Prop. 3 and Thm. 1]. □

Remark 4.1.6. If R is a field, then $\kappa \in \mathscr{L}_R^2(M)$ nonsingular implies that M is finite-dimensional. For a finite-dimensional R-vector space M, κ is nonsingular if and only if κ is nondegenerate.

Example 4.1.7. Consider $\mathbb{Z}$ as a module over itself. We define a bilinear form on $\mathbb{Z}$ as $\beta(a,b) := 2ab$ for $a,b \in \mathbb{Z}$. It is clear from the definition that this form is nondegenerate. Consider $\mathrm{z}^{-1}(\beta) \in \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}, \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z},\mathbb{Z}))$. Suppose that there is $m \in \mathbb{Z}$ so that $\mathrm{z}^{-1}(\beta)(m) = \mathrm{Id}_{\mathbb{Z}}$. Hence, for all $n \in \mathbb{Z}$,

$$\mathrm{z}^{-1}(\beta)(m)(n) = \beta(m,n) = 2mn = n.$$

This is impossible. Therefore, β is nondegenerate but is singular.

Let S be an R-algebra given by the structure map $\alpha : R \longrightarrow S$. Viewing S as an R-module via α, we can form the tensor product $M \otimes_R S$. Since this tensor product is determined by the choice of α, by convention, we denote it by $M \otimes_\alpha N$ instead if α is important (only). Recall, moreover, that if A is an R-algebra, then $A \otimes_\alpha S$ has a structure of an S-algebra given by the product $(a_1 \otimes s_1)(a_2 \otimes s_2) = (a_1a_2) \otimes (s_1s_2)$.

We denote by κ_α the S-bilinear form $(M \otimes_\alpha S) \times (M \otimes_\alpha S) \longrightarrow S$ uniquely determined by $\kappa_\alpha(m_1 \otimes s_1, m_2 \otimes s_2) = \alpha\big(\kappa(m_1,m_2)\big)s_1s_2$ for $m_i \in M, s_i \in S$. We call κ_α the *base change of* κ *to* S *using* α. If α is not important we write κ_S instead of κ_α.

Let $f : M \longrightarrow N$ be an R-linear map of R-modules. Then any $\kappa \in \mathscr{L}^2_R(N)$ gives rise to $f^*\kappa \in \mathscr{L}^2_R(M)$ defined by the following

$$(f^*\kappa)(m_1,m_2) = \kappa(fm_1, fm_2) \qquad (m_i \in M),$$

and hence to an R-linear map $f^* : \mathscr{L}^2_R(N) \longrightarrow \mathscr{L}^2_R(M)$. Denoting by R-**mod** the category of R-modules, it follows that

$$M \mapsto \mathscr{L}^2_R(M), \qquad f \mapsto f^* = \mathscr{L}^2_R(f)$$

defines a contravariant functor $\mathscr{L}^2_R : R\text{-}\mathbf{mod} \longrightarrow R\text{-}\mathbf{mod}$.

We collect some useful results in the following proposition.

Proposition 4.1.8. [NPPS, Lem. 4.2 and Lem. 6.16].

(a) (Transitivity of base change). *Let* $\kappa \in \mathscr{L}^2_R(M)$, *let* $S \in R$-**alg** *be given by the structure map* $\alpha : R \longrightarrow S$ *and let* $T \in S$-**alg** *given by the structure map* $\beta : S \longrightarrow T$ *and let* $\rho : (M \otimes_\alpha S) \otimes_\beta T \longrightarrow M \otimes_{\beta\circ\alpha} T$ *be the canonical isomorphism. Then* $\rho^*(\kappa_{\beta\circ\alpha}) = (\kappa_\alpha)_\beta$.

(b) *Let $f : M \longrightarrow N$ be an R-linear map of R-modules, let $S \in R$-**alg** with structure map $\alpha : R \longrightarrow S$. Then, for $\lambda \in \mathscr{L}^2_R(N)$, we have $(f^*(\lambda))_\alpha = (f \otimes \mathrm{Id}_S)^*(\lambda_\alpha)$.*

(c) *Let $\kappa, \kappa' \in \mathscr{L}^2_R(M)$. Then, $\kappa = \kappa'$ if and only if $\kappa_\alpha = \kappa'_\alpha$ for some faithfully flat extension $S \in R$-**alg** where $\alpha : R \longrightarrow S$ is the structure map.*

(d) *Assume that M is a finitely presented R-module, and let $S \in R$-**alg** be a flat R-module. If $\kappa \in \mathscr{L}^2_R(M)$ is nondegenerate (resp. nonsingular), then κ_S is nondegenerate (resp. nonsingular). Then converse holds in both cases if S/R is faithfully flat.*

(e) *Let M be a finitely generated projective R-module and let $\kappa \in \mathscr{L}^2_R(M)$. Then κ is nonsingular if and only if κ_K is nondegenerate for all $K \in R$-**alg** which are fields.*

Proof. (a). For $m_i \in M, s_i \in S, t_i \in T$ (for $i = 1, 2$), one has the following:

$$\begin{aligned}
\rho^*(\kappa_{\beta\circ\alpha})\big((m_1 \otimes s_1) \otimes t_1, (m_2 \otimes s_2) \otimes t_2\big) &= \kappa_{\beta\circ\alpha}\Big(\rho\big((m_1 \otimes s_1) \otimes t_1)\big), \\
&\qquad \rho\big((m_2 \otimes s_2) \otimes t_2)\big)\Big) \\
&= \kappa_{\beta\circ\alpha}\big(m_1 \otimes \beta(s_1)t_1, m_2 \otimes \beta(s_2)t_2\big) \\
&= (\beta \circ \alpha)\big(\kappa(m_1, m_2)\big)\beta(s_1)\beta(s_2)t_1t_2, \\
\big((\kappa_\alpha)_\beta\big)\big((m_1 \otimes s_1) \otimes t_1, (m_2 \otimes s_2) \otimes t_2\big) &= \beta\big(\kappa_\alpha(m_1 \otimes s_1, m_2 \otimes s_2)\big)t_1t_2 \\
&= \beta\Big(\alpha\big(\kappa(m_1, m_2)\big)s_1s_2\Big)t_1t_2 \\
&= (\beta \circ \alpha)\big(\kappa(m_1, m_2)\big)\beta(s_1)\beta(s_2)t_1t_2.
\end{aligned}$$

Since the elements $(m \otimes s) \otimes t$ span $(M \otimes_\alpha S) \otimes_\beta T$, it follows that $\rho^*(\kappa_{\beta\circ\alpha}) = (\kappa_\alpha)_\beta$.

(b). For $m_i \in M$ and $s_i \in S$, we have that

$$\begin{aligned}
\big(f^*(\lambda)\big)_\alpha(m_1 \otimes s_1, m_2 \otimes s_2) &= \alpha\big(f^*(\lambda)(m_1, m_2)\big)s_1s_2 = \alpha\Big(\lambda\big(f(m_1), f(m_2)\big)\Big)s_1s_2 \\
(f \otimes \mathrm{Id}_S)^*(\lambda_\alpha)(m_1 \otimes s_1, m_2 \otimes s_2) &= \lambda_\alpha(f(m_1) \otimes s_1, f(m_2) \otimes s_2) \\
&= \alpha(\lambda(f(m_1), f(m_2)))s_1s_2.
\end{aligned}$$

Since the elements $m \otimes s$ span $M \otimes_\alpha S$, it follows that $(f^*(\lambda))_\alpha = (f \otimes \mathrm{Id}_S)^*(\lambda_\alpha)$.

(c). If $\kappa = \kappa'$, then it is clear that $\kappa_\alpha = \kappa'_\alpha$ for all extension $\alpha : R \longrightarrow S$. It is left to prove the other implication. Suppose that $\kappa_\alpha = \kappa'_\alpha$ for some faithfully flat extension $S \in R$-**alg** with structure map $\alpha : R \longrightarrow S$. Then

$$(\mathrm{x}^{-1}(\kappa))_S = \mathrm{x}^{-1}(\kappa_\alpha) = \mathrm{x}^{-1}(\kappa'_\alpha) = (\mathrm{x}^{-1}(\kappa'))_S.$$

Now, by faithfully flat descent, $\mathrm{x}^{-1}(\kappa) = \mathrm{x}^{-1}(\kappa')$, whence $\kappa = \kappa'$.

(d). Since M is finitely presented and S/R is flat, the canonical map

$$\gamma : \mathrm{Hom}_R(M, R) \otimes_R S \longrightarrow \mathrm{Hom}_S(M \otimes_R S, S)$$

is an isomorphism. Also, we have the following commutative diagram.

$$\begin{tikzcd}
M \otimes_R S \arrow[rr, "\mathrm{z}^{-1}(\kappa)\otimes \mathrm{Id}_S"] \arrow[dr, "\mathrm{z}^{-1}(\kappa_S)"'] & & \mathrm{Hom}_R(M, R) \otimes_R S \arrow[dl, "\gamma"] \\
& \mathrm{Hom}_S(M \otimes_R S, S) &
\end{tikzcd}$$

Hence, $\mathrm{z}^{-1}(\kappa_S)$ is injective (resp. bijective) if and only if $\mathrm{z}^{-1}(\kappa) \otimes \mathrm{Id}_S$ is so. Then the result follows from standard properties of flat and faithfully flat extensions.

(e). This follows from Lemma 1.5.9. □

4.2 Invariant bilinear forms

4.2.1 Definition

In this subsection, we review some results from the theory of invarant bilinear forms on Lie algebras. Unless stated otherwise, in this subsection, R denotes a commutative ring and L is a R-Lie algebra.

Definition 4.2.1. A bilinear form $\beta : L \times L \longrightarrow R$ is *invariant* if $\beta([l_1, l_2], l_3) = \beta(l_1, [l_2, l_3])$ holds for all $l_i \in L$. Note that the set of invariant bilinear forms is an R-module, denoted by $\mathrm{IBF}(L)$.

Next, we give a standard example of an invariant bilinear form.

Example 4.2.2. Let $\rho : L \longrightarrow \mathfrak{gl}(V)$ be a finitely generated projective representation. The form $\tau_\rho(x,y) := \mathrm{tr}(\rho(x)\rho(y))$ is an invariant bilinear form. For $x, y, z \in L$,

$$\begin{aligned}
\tau_\rho([x,y],z) &= \mathrm{tr}\left(\rho([x,y])\rho(z)\right) = \mathrm{tr}\left([\rho(x),\rho(y)]\rho(z)\right) \\
&= \mathrm{tr}\left(\rho(x)\rho(y)\rho(z) - \rho(y)\rho(x)\rho(z)\right) \\
&= \mathrm{tr}\left(\rho(x)\rho(y)\rho(z)\right) - \mathrm{tr}\left(\rho(y)\rho(x)\rho(z)\right) \\
&= \mathrm{tr}\left(\rho(x)\rho(y)\rho(z)\right) - \mathrm{tr}\left(\rho(x)\rho(z)\rho(y)\right) \qquad \text{since } \mathrm{tr}(AB) = \mathrm{tr}(BA), \\
&= \mathrm{tr}\left(\rho(x)\rho([x,y])\right) = \tau_\rho(x,[y,z]).
\end{aligned}$$

Lemma 4.2.3. IBF *is a contravariant functor from the category of R-Lie algebras to the category of R-modules. More precisely, for $f : L \longrightarrow L'$ is a homomorphism of Lie algebras, $\big(\mathrm{IBF}(f)(\kappa)\big)(x,y) = \kappa\big(f(x), f(y)\big)$ for $\kappa \in \mathrm{IBF}(L')$, $x, y \in L$.*

Proof. The proof is straightforward and we omit it. □

Lemma 4.2.4. *Let S be an R-algebra with structure map $\alpha : R \longrightarrow S$. Then $\xi : \mathrm{IBF}_R(L) \longrightarrow \mathrm{IBF}_S(L_S)$, $\kappa \mapsto \Big(\kappa_\alpha : (l_1 \otimes s_1, l_2 \otimes s_2) \mapsto \alpha\big(\kappa(l_1,l_2)\big)s_1s_2\Big)$ is a homomorphism of R-modules.*

Proof. The proof is again straightforward. □

We need a way to compute these bilinear forms explicitly. To do so, consider the following R-modules:

$$\mathrm{ibf}(L) := \mathrm{span}_R\{[l_1,l_2]\otimes l_3 - l_1 \otimes [l_2,l_3] : l_i \in L\}, \quad \mathbf{IBF}(L) := L \otimes_R L/\,\mathrm{ibf}(L).,$$

We let $l_1 \otimes l_2 \mapsto \overline{l_1 \otimes l_2}$ denote the canonical quotient map. Then, for any map $\varphi \in \mathrm{Hom}_R(\mathbf{IBF}(L), R)$, the bilinear form $\beta_\varphi(l_1,l_2) = \varphi(\overline{l_1 \otimes l_2})$ is invariant and the map $\mathrm{Hom}_R(\mathbf{IBF}(L), R) \longrightarrow \mathrm{IBF}(L)$, $\varphi \mapsto \beta_\varphi$ is an isomorphism of R-modules by [NPPS, (3.6)].

The following two results provide us with an understanding of invariant bilinear forms in a special case.

Lemma 4.2.5. *Let $\mathfrak{g}$ be a finite-dimensional Lie algebra over a field $\mathbb{K}$. Then, for any field extension $\mathbb{F}$ of $\mathbb{K}$, $\mathrm{IBF}(\mathfrak{g}) \otimes_{\mathbb{K}} \mathbb{F} \simeq \mathrm{IBF}(\mathfrak{g} \otimes_{\mathbb{K}} \mathbb{F})$.*

Proof. We have the following

$$\begin{aligned}
\operatorname{IBF}(\mathfrak{g}) \otimes_{\mathbb{K}} \mathbb{F} &\simeq \operatorname{Hom}_{\mathbb{K}}(\mathbf{IBF}(\mathfrak{g}), \mathbb{K}) \otimes_{\mathbb{K}} \mathbb{F} \quad \text{by the above isomorphisms} \\
&\simeq \operatorname{Hom}_{\mathbb{F}}(\mathbf{IBF}(\mathfrak{g}) \otimes_{\mathbb{K}} \mathbb{F}, \mathbb{F}) \quad \text{by finite-dimensionality} \\
&\simeq \operatorname{Hom}_{\mathbb{F}}(\mathbf{IBF}(\mathfrak{g} \otimes_{\mathbb{K}} \mathbb{F}), \mathbb{F}) \quad \text{[NPPS, Prop. 4.3]} \\
&\simeq \operatorname{IBF}(\mathfrak{g} \otimes_{\mathbb{K}} \mathbb{F}).
\end{aligned}$$

□

Recall the notion of the centroid of a Lie algebra over R:

$$\operatorname{Ctd}(L) = \{\chi \in \operatorname{End}_R(L) : \chi[l_1, l_2] = [\chi(l_1), l_2] \text{ for all } l_i \in L\}.$$

The centroid always contains $R \operatorname{Id}_L$ and is an associative R-algebras. A simple R-algebras is called *central-simple* if $R \longrightarrow \operatorname{Ctd}(L)$, $r \mapsto r \operatorname{Id}_L$ is an isomorphism. It is known that a simple Lie algebra over a field $\mathbb{K}$ is central-simple if and only if $L \otimes_{\mathbb{K}} \mathbb{F}$ is simple for every field extension $\mathbb{F}/\mathbb{K}$. A finite-dimensional algebra L over $\mathbb{K}$ is central-simple if and only if $L \otimes_{\mathbb{K}} \overline{\mathbb{K}}$ is simple for the algebraic closure $\overline{\mathbb{K}}$ of $\mathbb{K}$. In particular, every split simple Lie algebra over a field of characteristic zero is central-simple.

Corollary 4.2.6. *Let $\mathfrak{g}$ be a finite-dimensional central-simple Lie algebra over a field $\mathbb{K}$ of characteristic zero. Then* $\operatorname{IBF}(\mathfrak{g})$ *is one-dimensional.*

Proof. Since $\mathfrak{g}$ is central-simple, we know that $\mathfrak{g} \otimes_{\mathbb{K}} \overline{\mathbb{K}}$ is simple. It is known from the classical theory that $\operatorname{IBF}(\mathfrak{g} \otimes_{\mathbb{K}} \overline{\mathbb{K}})$ is one-dimensional by [Hum, Ex 6.6]. It follows from Lemma 4.2.5 that $\operatorname{IBF}(\mathfrak{g})$ is one-dimensional. □

In general, of course, the space of invariant bilinear forms is of higher dimensions.

Example 4.2.7. If $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$ and if κ_1 and κ_2 are invariant bilinear forms on $\mathfrak{g}_1$ and $\mathfrak{g}_2$. These forms κ_i induce invariant bilinear forms κ_i' on $\mathfrak{g}$ (extending by zero). We see that the κ_i's are linearly independent, whence $\operatorname{IBF}(\mathfrak{g})$ is not one-dimensional.

Next, as another example, we compute the invariant bilinear forms for Heisenberg Lie algebras. We denote by $\mathfrak{h}_n$ the $(2n+1)$-dimensional Heisenberg Lie algebra over a ring R. Recall that this Lie algebra is spanned by $\{P_1, \dots, P_n, Q_1, \dots, Q_n, C\}$ with the following relations:

$$[P_i, P_j] = [Q_i, Q_j] = [P_i, C] = [Q_i, C] = 0, \qquad [P_i, Q_j] = \delta_{ij} C.$$

Lemma 4.2.8. $\mathrm{ibf}(\mathfrak{h}_n) = \mathrm{span}_R\{P_i \otimes C, C \otimes P_i, Q_i \otimes C, C \otimes Q_i, C \otimes C : i = 1, \ldots, n\}$.

Proof. We have the following computations:

$$\begin{aligned}
[P_i, P_j] \otimes Q_k - P_i \otimes [P_j, Q_k] &= -\delta_{jk} P_i \otimes C, \\
[Q_i, Q_j] \otimes P_k - Q_i \otimes [Q_j, P_k] &= -\delta_{jk} Q_i \otimes C, \\
[P_i, Q_j] \otimes P_k - P_i \otimes [Q_j, P_k] &= \delta_{ij} C \otimes P_k + \delta_{jk} P_i \otimes C \\
[P_i, Q_j] \otimes C - P_i \otimes [Q_j, C] &= \delta_{ij} C \otimes C.
\end{aligned}$$

The other computations either give analogous results or zero. Since we have computed all generators of $\mathrm{ibf}(\mathfrak{h}_n)$, the result follows. ☐

As we can see from the lemma, the space $\mathrm{IBF}(\mathfrak{h}_n)$ is definitely not one-dimensional. The following proposition can be regarded as a corollary.

Proposition 4.2.9. *We have the following*

(a) $\mathbf{IBF}(\mathfrak{h}_n) \simeq \bigoplus_{1 \leq i,j \leq n} \big(R(P_i \otimes Q_j) \oplus R(Q_i \otimes P_j) \oplus R(P_i \otimes P_j) \oplus R(Q_i \otimes Q_j)\big)$.

(b) $\mathrm{IBF}(\mathfrak{h}_n) \simeq \mathscr{L}^2\big(\bigoplus_{1 \leq i \leq n}(RP_i \oplus RQ_i)\big)$.

Proof. This follows immediately from Lemma 4.2.8. ☐

4.2.2 Invariant bilinear forms and representation ring

In this subsection, we relate the notions of invariant bilinear forms and representation ring. This relation gives us a way to study modules of a Lie algebra using invariant bilinear forms. Unless stated otherwise, R is a commutative ring and L is a R-Lie algebra.

Proposition 4.2.10. *Associating to a finitely generated projective L-module $\rho : L \longrightarrow \mathfrak{gl}(V)$ the trace form τ_ρ defines a group homomorphism*

$$\tau = \tau_L : \mathscr{R}(L) \longrightarrow \mathrm{IBF}(L), \quad [V] \mapsto \tau_\rho.$$

Proof. First, recall that, for a representation (V, ρ) of L, the trace form τ_ρ is invariant by Example 4.2.2. The trace form τ_ρ only depends on $\mathrm{cl}(V)$ and thus gives rise to a well-defined map $\tau' : \mathscr{F}(L) \longrightarrow \mathrm{IBF}(L)$, $\mathrm{cl}(V, \rho) \mapsto \tau_\rho$.

Assume that $0 \longrightarrow (V_1, \rho_1) \longrightarrow (V, \rho) \longrightarrow (V_2, \rho_2) \longrightarrow 0$ is an exact sequence of finitely generated projective L-modules. We claim that $\tau_\rho = \tau_{\rho_1} + \tau_{\rho_2}$. It suffices to show this after localizing at $\mathfrak{p} \in \operatorname{Spec}(R)$. Equivalently, we may assume that V_1, V and V_2 are free R-modules.

We then choose an R-basis of V, containing an R-basis of V_1. It follows that $\rho(x)$ for $x \in L$ can be represented by a matrix of the form

$$\begin{pmatrix} \rho_1(x) & * \\ 0 & \widetilde{\rho_2}(x) \end{pmatrix}$$

where $x \mapsto \widetilde{\rho_2}(x)$ is a representation isomorphic to ρ_2. Hence, we see that $\operatorname{tr}\big(\rho(x)\rho(y)\big) = \operatorname{tr}\big(\rho_1(x)\rho_1(y)\big) + \operatorname{tr}\big(\widetilde{\rho_2}(x)\widetilde{\rho_2}(y)\big)$. Thus, $\tau_\rho = \tau_{\rho_1} + \tau_{\rho_2}$. Then, by Lemma 3.2.4, τ' descends to a well-defined group homomorphism $\tau : \mathscr{R}(L) \longrightarrow \operatorname{IBF}(L)$. $\square$

Proposition 4.2.11. *Assume that L is perfect.*

(a) *If V and W are finitely generated projective L-modules of constant rank r_V and r_W respectively, then*

$$\begin{aligned} \tau([V] \cdot [W]) &= r_W \tau([V]) + r_V \tau([W]), \\ \tau([\textstyle\bigwedge^n V]) &= \binom{r_V - 2}{n-1} \tau([V]), \\ \tau([\operatorname{Sym}^n V]) &= \binom{r+n}{n-1} \tau([V]). \end{aligned}$$

(b) *If the representation ring $\mathscr{R}(L)$ is finitely generated as a ring by finitely generated projective representations ρ_i, $1 \le i \le l$, of constant rank, then image* $\operatorname{im}(\tau)$ *is generated by τ_{ρ_i}, $1 \le i \le l$, as an abelian group.*

Proof. Let $x \in L$. Since L is perfect, we may write $x = \sum_i [x_i, y_i]$ for $x_i, y_i \in L$. Hence,

$$\operatorname{tr}\big(\rho(x)\big) = \sum_i \operatorname{tr}\big(\rho([x_i, y_i])\big) = \sum_i \operatorname{tr}\big([\rho(x), \rho(y)]\big) = 0.$$

It follows that $\rho_V(L) \subseteq \mathfrak{sl}(V) = \{f \in \operatorname{End}_R(V) : \operatorname{tr}(f) = 0\}$ $\big($which is a subalgebra of $\mathfrak{gl}(V)\big)$ for any finitely generated projective L -module (V, ρ_V). The formulas in (a) then follow from Lemma 2.2.10 and Proposition 2.2.12, namely,

$$\begin{aligned}
\tau_{\rho_V \otimes \rho_W}(x,y) &= \operatorname{tr}\left(\rho_{V\otimes W}(x)\rho_{V\otimes W}(y)\right) = \operatorname{tr}\left(\left(\rho_V(x)\rho_W(x)\right)^{\otimes 2}\left(\rho_V(y)\rho_W(y)\right)^{\otimes 2}\right)\\
&= r_W \operatorname{tr}(\rho_V(x)\rho_V(y)) + r_V \operatorname{tr}(\rho_W(x)\rho_W(y)) = (r_W\tau_{\rho_V} + r_V\tau_{\rho_W})(x,y)\\
\tau_{\rho_V^{\wedge n}}(x,y) &= \operatorname{tr}\left(\rho(x)^{\wedge n}\rho(y)^{\wedge n}\right) = \binom{r-2}{n-1}\operatorname{tr}(\rho(x)\rho(y)) = \binom{r-2}{n-1}\tau_{\rho_V}(x,y),\\
\tau_{\rho_V^{\circ n}}(x,y) &= \operatorname{tr}\left(\rho(x)^{\circ n}\rho(y)^{\circ n}\right) = \binom{r+n}{n-1}\operatorname{tr}(\rho(x)\rho(y)) = \binom{r+n}{n-1}\tau_{\rho_V}(x,y).
\end{aligned}$$

(b) This part follows from the first formula in (a). ☐

We already know from Proposition 3.2.8 and Lemma 4.2.3 that $\mathscr{R}$ and IBF are contravariant functors from the category of R-Lie algebras to the category of R-modules. The next lemma relates these two functors.

Lemma 4.2.12. *The map* $\tau = (\tau_L)$, *where* $\tau_L : \mathscr{R}(L) \longrightarrow \mathrm{IBF}(L)$ *is defined in Proposition* 4.2.10, *is a natural transformation from* $\mathscr{R}$ *to* IBF.

Proof. Suppose $f : L \longrightarrow L'$ is a Lie algebra homomorphism. Consider the following diagram

$$\begin{tikzcd}
\mathscr{R}(L') \arrow[r, "\tau_{L'}"] \arrow[d, "\mathscr{R}(f)"'] & \mathrm{IBF}(L') \arrow[d, "\mathrm{IBF}(f)"] \\
\mathscr{R}(L) \arrow[r, "\tau_L"'] & \mathrm{IBF}(L)
\end{tikzcd}$$

We check that the above diagram is commutative. Indeed, for $[(V,\rho)] \in \mathscr{R}(L')$, we have

$$\begin{aligned}
(\mathrm{IBF}(f)\circ\tau'_L)([V,\rho])(l_1,l_2) &= \mathrm{IBF}(f)(\tau_\rho)(l_1,l_2) = \tau_\rho(f(l_1),f(l_2))\\
(\tau_L\circ\mathscr{R}(f))(([V,\rho])(l_1,l_2) &= \tau([V,\rho\circ f])(l_1,l_2) = \operatorname{tr}(\rho(f(l_1)),\rho(f(l_2)))\\
&= \tau_\rho(f(l_1),f(l_2)).
\end{aligned}$$

Therefore, τ is a natural transformation. ☐

Lemma 4.2.13. *Let* $\rho : L \longrightarrow \mathfrak{gl}_R(V)$ *be a finitely generated projective* L*-module with associated trace form* τ_ρ, *and let* S *be an* R*-algebra. Then the base change* $(\tau_\rho)_S$

of τ_ρ *to* S *coincides with the trace form of* $\rho_S : L_S \longrightarrow \mathfrak{gl}(V_S)$*, i.e.,* $(\tau_\rho)_S = \tau_{\rho_S}$*. As a consequence, if* S/R *is flat, then the following diagram is commutative*

$$\begin{tikzcd}
\mathscr{R}(L) \arrow[r, "\tau_L"] \arrow[d, "\psi"'] & \mathrm{IBF}(L) \arrow[d, "\xi"] \\
\mathscr{R}(L_S) \arrow[r, "\tau_{L_S}"'] & \mathrm{IBF}(L_S)
\end{tikzcd}$$

where ψ *is defined as in Lemma* 3.2.14 *and* ξ *is defined as in Lemma* 4.2.4.

Proof. Since both forms are S-bilinear, it suffices to show

$$(\tau_\rho)_S(x \otimes 1, y \otimes 1) = \tau_{\rho_S}(x \otimes 1, y \otimes 1)$$

for $x, y \in L$. This follows from Corollary 2.2.3

$$\begin{aligned}(\tau_{\rho_S})(x \otimes 1, y \otimes 1) &= \mathrm{tr}(\rho_S(x \otimes 1)\rho_S(y \otimes 1)) = \mathrm{tr}\left((\rho(x) \otimes \mathrm{Id}_S)(\rho(y) \otimes \mathrm{Id}_S)\right) \\ &= \mathrm{tr}(\rho(x)\rho(y) \otimes \mathrm{Id}_S) = \mathrm{tr}(\rho(x)\rho(y)) \otimes 1 = (\tau_\rho)_S(x \otimes 1, y \otimes 1).\end{aligned}$$

This completes the proof. □

4.3 Invariant bilinear forms on Chevalley orders

4.3.1 A brief review of Chevalley orders

In this section, we denote by $(\mathfrak{g}, \mathfrak{h})$ a split reductive Lie algebra over $\mathbb{Q}$. We denote by $\Delta = \Delta(\mathfrak{g}, \mathfrak{h})$ the corresponding root system. Our reference for this section is [B:Lie, VIII, §12]. Let $\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta} \mathfrak{g}_\alpha$ be the root space decomposition of $\mathfrak{g}$. Recall that, for $\alpha \in \Delta$, there exists a unique $h_\alpha \in \mathfrak{h}$ such that $\alpha(h_\alpha) = 2$, $h_\alpha \in [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]$.

Definition 4.3.1. Let V be a $\mathbb{Q}$-vector space. We call a free $\mathbb{Z}$-submodule $\mathscr{V}$ of V a *lattice* if the canonical $\mathbb{Q}$-linear map $\alpha_{\mathscr{V},V} : \mathscr{V} \otimes_{\mathbb{Z}} \mathbb{Q} \longrightarrow V$ is an isomorphism of $\mathbb{Q}$-vector spaces. Moreover, if V is a $\mathbb{Q}$-algebra, then an *order* $\mathscr{V}$ of V is a lattice of V that is also a $\mathbb{Z}$-algebra.

Remark 4.3.2. Let V be a $\mathbb{Q}$-algebra and let $\mathscr{V}$ be an order in V. Then the map $\alpha_{\mathscr{V},V} : \mathscr{V} \otimes_{\mathbb{Z}} \mathbb{Q} \longrightarrow V$ is an isomorphism of $\mathbb{Q}$-algebras.

Remark 4.3.3. Let V and W be $\mathbb{Q}$-vector spaces. Let $\mathscr{V}$ be lattice of V and $\mathscr{W}$ be a lattice of W. We have that $(\mathscr{V} \otimes_{\mathbb{Z}} \mathscr{W}) \otimes_{\mathbb{Z}} \mathbb{Q} \simeq (\mathscr{V} \otimes_{\mathbb{Z}} \mathbb{Q}) \otimes_{\mathbb{Q}} (\mathscr{W} \otimes_{\mathbb{Z}} \mathbb{Q}) \simeq V \otimes_{\mathbb{Q}} W$. Then, since these isomorphisms are canonical, $\mathscr{V} \otimes_{\mathbb{Z}} \mathscr{W}$ is a lattice of $V \otimes_{\mathbb{Q}} W$.

Definition 4.3.4. A lattice $\mathscr{H}$ in $\mathfrak{h}$ is called *permissible* (relatively to $\mathfrak{g}$) if, for all $\alpha \in \Delta$, $h_\alpha \in \mathscr{H}$ and $\alpha(\mathscr{H}) \subseteq \mathbb{Z}$.

Now, we choose a permissible lattice $\mathscr{H}$ in $\mathfrak{h}$ and, for all $\alpha \in \Delta$, a lattice $\mathscr{G}_\alpha \subseteq \mathfrak{g}_\alpha$. Next, set $\mathscr{G} = \mathscr{H} \oplus \sum_{\alpha\in\Delta} \mathscr{G}_\alpha$. Let $\mathscr{U}$ be the subalgebra of $\mathrm{U}(\mathfrak{g})$ spanned by the elements $\binom{h}{n}$ and $x^{(n)}$ for $h \in \mathfrak{h}, n \in \mathbb{N}, x \in \mathscr{G}_\alpha, \alpha \in \Delta$ where

$$x^{(n)} = \frac{x^n}{n!} \quad \text{and} \quad \binom{h}{n} = \frac{h(h-1)\cdots(h-n+1)}{n!}.$$

Also, for $\alpha \in \Delta$ and $x \in \mathfrak{g}_\alpha$, set $w_\alpha(x) = \exp(\operatorname{ad} x)\exp(\operatorname{ad} y)\exp(\operatorname{ad} x)$ where y is the unique element of $\mathfrak{g}_{-\alpha}$ such that $[x, y] = h_\alpha$.

Definition 4.3.5. [B:Lie, VIII, §2.4]. Given a split Lie algebra $(\mathfrak{g}, \mathfrak{h})$, a *Chevalley system* of $(\mathfrak{g}, \mathfrak{h})$ is a family $(x_\alpha)_{\alpha\in\Delta}$ such that

(i) $x_\alpha \in \mathfrak{g}_\alpha$ for all $\alpha \in \Delta$;

(ii) $[x_\alpha, x_{-\alpha}] = h_\alpha$ for all $\alpha \in \Delta$;

(iii) the linear map from $\mathfrak{g}$ to $\mathfrak{g}$ that is $-\mathrm{Id}$ on $\mathfrak{h}$ and sends x_α to $x_{-\alpha}$ is an automorphism of $\mathfrak{g}$.

Theorem 4.3.6. [B:Lie, VIII, §12.7, Thm. 2]. *Let $\mathscr{H} \subseteq \mathfrak{h}$ be a permissible lattice. The following conditions are equivalent:*

(a) *There exists a Chevalley system $(x_\alpha)_{\alpha\in\Delta}$ of $(\mathfrak{g}, \mathfrak{h})$ such that $\mathscr{G}_\alpha = \mathbb{Z}x_\alpha$, $\alpha \in \Delta$.*

(b) *$\mathscr{U} \cap \mathfrak{g} = \mathscr{G}$ and $[\mathscr{G}_\alpha, \mathscr{G}_{-\alpha}] = \mathbb{Z}h_\alpha$ for all $\alpha \in \Delta$.*

(c) *For all $\alpha \in \Delta, x \in \mathscr{G}_\alpha, n \in \mathbb{N}$, the endomorphism $(\operatorname{ad} x)^n/n!$ of $\mathfrak{g}$ leaves $\mathscr{G}$ invariant.*

(d) *For all $\alpha \in \Delta$ and for all bases x of $\mathscr{G}_\alpha$, $w_\alpha(x)$ leaves $\mathscr{G}$ invariant, i.e., $w_\alpha(x)$ maps $\mathscr{G}_\beta$ to $\mathscr{G}_{s_\alpha\beta}$ for $\alpha, \beta \in \Delta$ where s_α is the corresponding reflection.*

Definition 4.3.7. If $\mathscr{G}$ satisfies one of the equivalent conditions of Theorem 4.3.6, then it is called a *Chevalley order* of $(\mathfrak{g}, \mathfrak{h})$. It then follows that $\mathscr{G}$ is an order of $\mathfrak{g}$ ([B:Lie, VIII, §12.7, Thm. 3]).

Example 4.3.8. Consider $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{Q})$. Let $\{h, e, f\}$ be the standard basis of $\mathfrak{sl}_2$. The only two Chevalley orders of $\mathfrak{g}$ (up to isomorphism) are $\mathscr{G}_1 = \mathfrak{sl}_2(\mathbb{Z}) = \mathrm{span}_{\mathbb{Z}}\{h, e, f\}$ and $\mathscr{G}_2$ with basis $\{\frac{1}{2}h, e, f\}$.

Proof. Let $\mathscr{G}$ be a Chevalley order of $\mathfrak{g}$. So, we may write $\mathscr{G} = \mathscr{H} \oplus \sum_\alpha \mathscr{G}^\alpha$ where $\mathscr{H}$ is a permissible lattice. First, we want to know what $\mathscr{H}$ can be. Since $\mathscr{H}$ is permissible, $h \in \mathscr{H}$. Moreover, since $\mathfrak{h}$ is of dimension 1, $\mathscr{H}$ must be of rank 1. Hence, $\mathscr{H} = \mathrm{span}_{\mathbb{Z}}\{\frac{m}{n}h\}$ where $\gcd(m, n) = 1$. Moreover, we have that $\alpha\big(\frac{m}{n}h\big) = 2\frac{m}{n} \in \mathbb{Z}$. It follows that $n = 1$ or $n = 2$. If $n = 1$, then we see that $\mathscr{H} = \mathbb{Z}h$. If $n = 2$, then we see that $\mathscr{H} = \mathbb{Z}\big(\frac{1}{2}h\big)$. This is what we expect.

Let $x \in \mathfrak{g}_\alpha$ and $y \in \mathfrak{g}_{-\alpha}$ be the corresponding Chevalley basis. We have that $x = \lambda e$ and $y = \mu f$ where $\lambda, \mu \in \mathbb{Q}$. We have that

$$h = [x, y] = \lambda\mu[e, f] = \lambda\mu h.$$

It follows that $\lambda = \mu = \pm 1$. So, we may assume $\lambda = \mu = 1$ after changing x and y by $-x$ and $-y$. So, the only possible Chevalley orders are $\mathscr{G}_1$ and $\mathscr{G}_2$. Since $\{e, f\}$ is a Chevalley system, we conclude that the only Chevalley orders of $\mathfrak{g}$ are $\mathscr{G}_1$ and $\mathscr{G}_2$. $\square$

Definition 4.3.9. Let M be a $\mathfrak{g}$-module. Then a lattice $\mathscr{M}$ of M is called *admissible* if $\mathscr{U}(\mathscr{M}) \subseteq \mathscr{M}$, or equivalently, $\mathscr{M}$ is stable under the actions of $\binom{h}{n}$ and of $x^{(n)}$ for $h \in \mathfrak{h}$, $x \in \mathscr{G}^\alpha$, $\alpha \in \Delta$, $n \in \mathbb{N}$.

Lemma 4.3.10. *Let M be a $\mathfrak{g}$-module and let $\mathscr{M}$ be an admissible lattice of M. Let $\mathscr{N}$ be a $\mathscr{G}$-module such that $\mathscr{M} \simeq \mathscr{N}$ as $\mathscr{G}$-modules. Then there exists a $\mathfrak{g}$-module N such that $\mathscr{N} \subseteq N$ is an admissible lattice and that $M \simeq N$ as $\mathfrak{g}$-modules.*

Proof. Let $\varphi : \mathscr{M} \longrightarrow \mathscr{N}$ be an isomorphism of $\mathscr{G}$-modules. Consider the $\mathbb{Q}$-vector space $N := \mathscr{N} \otimes_{\mathbb{Z}} \mathbb{Q}$. Note that N has the induced structure of a $\mathfrak{g}$-module. Moreover, clearly, $\mathscr{N}$ is a lattice in N.

For $y \in \mathscr{U}$ and $n \in \mathscr{N}$, $y \cdot n = y \cdot \varphi(\varphi^{-1}(n)) = \varphi(y \cdot \varphi^{-1}(n)) \in \mathscr{N}$ since $\mathscr{U}(\mathscr{M}) \subseteq \mathscr{M}$. Therefore, $\mathscr{N}$ is admissible.

Finally, we need to show that $M \simeq N$ as $\mathfrak{g}$-modules. Since $\mathscr{M} \simeq \mathscr{N}$ as $\mathscr{G}$-modules, $\mathscr{M} \otimes_{\mathbb{Z}} \mathbb{Q} \simeq \mathscr{N} \otimes_{\mathbb{Z}} \mathbb{Q}$ as $\mathfrak{g}$-modules. Consider the canonical isomorphism $\alpha : \mathscr{M} \otimes_{\mathbb{Z}} \mathbb{Q} \longrightarrow M$, $m \otimes q \mapsto qm$, of $\mathbb{Q}$-vector spaces. We check that α is a $\mathfrak{g}$-module isomomorphism. For $x \in \mathfrak{g}$ and $m \otimes q \in \mathscr{M} \otimes_{\mathbb{Z}} \mathbb{Q}$.

$$\alpha(x \cdot (m \otimes q)) = \alpha((x \cdot m) \otimes q) = q(x \cdot m)$$

$$x \cdot \alpha(m \otimes q) = x \cdot (qm) = q(x \cdot m)$$

since the action of $\mathfrak{g}$ is $\mathbb{Q}$-linear. The result follows. □

In general, the structure of $\mathscr{H}$ can be complicated. It is useful to restrict the study to Chevalley orders with a special type of $\mathscr{H}$.

Definition 4.3.11. A Chevalley order $\mathscr{G}$ is called *simply-connected* if

$$\mathscr{H} = \operatorname{span}\{h_\alpha : \alpha \in \Delta\}.$$

Simply-connected Chevalley orders are much easier to understand since $\mathscr{H}$ has a simpler structure.

Equivalently, $\mathscr{H} = \bigoplus_{\beta \in \Pi} \mathbb{Z} h_\beta$ where Π is a root basis of Δ. The justification of the term "simply-connected" is that a Chevalley order of a split semisimple Lie algebra gives rise to a Chevalley-Demazure group scheme $\mathbb{G}$ which is simply-connected if and only if $\mathscr{G}$ is so.

Example 4.3.12. Let $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{Q})$. Let $\mathscr{G}_1$ and $\mathscr{G}_2$ be as in Example 4.3.8. One sees that $\mathscr{G}_1$ is simply-connected but $\mathscr{G}_2$ is not.

Definition 4.3.13. Let $\mathscr{G}$ be a simply-connected Chevalley order. We denote by $\mathscr{R}_{\mathrm{adm}}(\mathscr{G})$ the submodule of $\mathscr{R}(\mathscr{G})$ consisting of admissible representations. Note that this definition is in line with Lemma 4.3.10.

Lemma 4.3.14. *Let $\mathscr{V}$ and $\mathscr{W}$ be admissible representations of a $\mathscr{G}$. Then $\mathscr{V} \otimes_{\mathbb{Z}} \mathscr{W}$ is admissible. As a consequence, $\mathscr{R}_{adm}(\mathscr{G})$ is a subring of $\mathscr{R}(\mathscr{G})$.*

Proof. For all $y \in \mathscr{U}$ and $v \otimes w \in \mathscr{V} \otimes_{\mathbb{Z}} \mathscr{W}$, $y \cdot (v \otimes w) = (y \cdot v) \otimes w + v \otimes (y \cdot w)$. Since $\mathscr{V}$ and $\mathscr{W}$ be admissible representations of a $\mathscr{G}$, it follows immediately that $\mathscr{V} \otimes_{\mathbb{Z}} \mathscr{W}$ is admissible. □

The next lemma gives us the relation between the representation ring of a split semisimple Lie algebra and that of the associated simply-connected Chevalley order.

Lemma 4.3.15. *Let* $(\mathfrak{g}, \mathfrak{h})$ *be a split semisimple Lie algebra over* $\mathbb{Q}$. *Let* $\mathscr{G}$ *be a simply-connected Chevalley order associated to* $(\mathfrak{g}, \mathfrak{h})$, *whence* $\mathscr{G} \cap \mathfrak{h} = \mathbb{Z}[\Delta^\vee]$ *for* $\Delta^\vee \subseteq \mathfrak{h}$ *the dual root system of* $(\mathfrak{g}, \mathfrak{h})$ *Observe that* $\mathscr{G} \otimes_{\mathbb{Z}} \mathbb{Q} = \mathfrak{g}$, *whence we have the base change map* $\psi : \mathscr{R}(\mathscr{G}) \longrightarrow \mathscr{R}(\mathfrak{g})$ *since* $\mathbb{Q}$ *is a flat* $\mathbb{Z}$*-module. This map has a section* σ *in the category of rings. In particular,* ψ *is surjective. Moreover,* $\sigma\big(\mathscr{R}(\mathfrak{g})\big) \subseteq \mathscr{R}_{\mathrm{adm}}(\mathscr{G})$.

Proof. By [B:Lie, VIII, §12.8, Cor.], every finite-dimensional representation (V, ρ) of $\mathfrak{g}$ contains an admissible lattice $\mathscr{E}$ which is, by restriction, in a finitely generated projective (in fact free) representation $\rho|\mathscr{E}$ of $\mathscr{G}$. Since $\mathscr{E}$ is a lattice, we have $\rho_{\mathscr{E}} \otimes_{\mathbb{Z}} \mathrm{Id}_{\mathbb{Q}} = \rho$. We know that $\mathscr{R}(\mathfrak{g}) \simeq \bigoplus_{\mathscr{S}_{\mathfrak{g}}} \mathbb{Z}$ as abelian groups where $\mathscr{S}_{\mathfrak{g}}$ is the set of isomorphism classes of finite-dimensional simple representations of $\mathfrak{g}$. For every $(V, \rho) \in \mathscr{S}_{\mathfrak{g}}$, choose an admissible lattice, say $\mathscr{E}_{(V,\rho)}$. The map $[(V, \rho)] \mapsto [\mathscr{E}_{(V,\rho)}, \rho|\mathscr{E}]$ extends to a ring homomorphism $\sigma : (\mathscr{R}(\mathfrak{g}), +) \longrightarrow (\mathscr{R}(\mathscr{G}), +)$ such that $\psi \circ \sigma = \mathrm{Id}_{\mathscr{R}(\mathfrak{g})}$. Since each $\mathscr{E}_{(V,\rho)}$ is admissible and since $\mathscr{R}_{\mathrm{adm}}(\mathscr{G})$ is a subring (Lemma 4.3.14), we see that $\sigma\big(\mathscr{R}(\mathfrak{g})\big) \subseteq \mathscr{R}_{\mathrm{adm}}(\mathscr{G})$. ☐

Consider the Lie algebra $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{Q})$. We know that $\mathscr{G} = \mathfrak{sl}_2(\mathbb{Z})$ is a simply-connected Chevalley order of $\mathfrak{g}$.

Lemma 4.3.16. *Let* $V = \mathbb{Q}^2$ *be the natural* $\mathfrak{g}$*-module where* $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{Q})$. *Let* $\mathscr{V}$ *be a lattice in* V *which is also* $\mathscr{G}$*-module. Then,*

(i) $\mathscr{V}$ *is admissible and* $\mathscr{V} = q\mathbb{Z}^2$ *for some* $q \in \mathbb{Q}^\times$. *Moreover,* $\mathscr{V}$ *is isomorphic to* $\mathbb{Z}^2$ *as* $\mathfrak{sl}_2(\mathbb{Z})$*-modules.*

(ii) *The fiber of the base change map* $\psi : \mathscr{R}(\mathscr{G}) \longrightarrow \mathscr{R}(\mathfrak{sl}_2(\mathbb{Q}))$ *of* $[\mathbb{Q}^2]$ *consists of one element.*

Proof. (i) From the action of $\mathfrak{sl}_2(\mathbb{Z})$, one sees that $(a, b) \in \mathscr{V} \implies (b, 0), (0, a) \in \mathscr{V}$. From this, $\mathscr{V}$ is automatically admissible.

Now, by [B:CA, VII, §4.1, Prop. 3], $\mathscr{V} \cap (\mathbb{Q}, 0)$ is a lattice in $(\mathbb{Q}, 0)$. Thus, $\mathscr{V} \cap (\mathbb{Q}, 0)$ is a rank one $\mathbb{Z}$-submodule. So, there exists $q \in \mathbb{Q}^\times$ such that $\mathscr{V} \cap (\mathbb{Q}, 0) = q(\mathbb{Z}, 0)$.

Now, by the action, $q(0,\mathbb{Z}) \subseteq \mathscr{V}$, whence $q(\mathbb{Z},\mathbb{Z}) \subseteq \mathscr{V}$. We claim that the other inclusion also holds. Let $(a,b) \in \mathscr{V}$. By the action, $(b,0),(0,a) \in \mathscr{V}$. We see that $b \in q\mathbb{Z}$. Similarly, $\mathscr{V} \cap (0,\mathbb{Q}) = q'\mathbb{Z}$ for some $q' \in \mathbb{Q}^\times$. Again, admissiblity implies that $(q',0) \in \mathscr{V}$, whence $q' \in q\mathbb{Z}$. It follows that $a \in q\mathbb{Z}$. This shows that $\mathscr{V} = q\mathbb{Z}^2$. The map $\mathscr{V} \longrightarrow \mathbb{Z}^2$, $(a,b) \mapsto q(a,b)$, is an isomorphism of $\mathscr{G}$-modules.

(ii) A finitely generated projective $\mathfrak{sl}_2(\mathbb{Z})$-module is free since all finitely generated projective modules over a PID are free. Hence, the rank 2 $\mathfrak{sl}_2(\mathbb{Z})$-modules are all lattices in $\mathbb{Q}^2$ invariant under the natural $\mathfrak{sl}_2(\mathbb{Z})$-action. By part (i), all such lattices are admissible; moreover, they are all $\mathfrak{sl}_2(\mathbb{Z})$-isomorphic. Hence, the fiber of the base change $\psi : \mathscr{R}(\mathscr{G}) \longrightarrow \mathscr{R}(\mathfrak{sl}_2(\mathbb{Q}))$ over $[\mathbb{Q}^2]$ consists of one element. □

The next lemma classifies lattices of $\mathbb{Q}^3$ which are $\mathfrak{sl}_2(\mathbb{Z})$-modules.

Lemma 4.3.17. *Let $V = \mathbb{Q}^3$ be the $\mathfrak{sl}_2(\mathbb{Q})$-module with the natural action. Let $\mathscr{V}$ be a lattice of V which is also $\mathscr{V}$ is also a $\mathfrak{sl}_2(\mathbb{Z})$-module such that $\mathscr{V} = \mathscr{V}_{-2} \oplus \mathscr{V}_0 \oplus \mathscr{V}_2$ where $\mathscr{V}_i = \mathscr{V} \cap V_i$ (for example, if $\mathscr{V}$ is admissible, see* [Hum, Thm. 27.1]*). Then, up to isomorphism, $\mathscr{V}$ is $\mathbb{Z}^3$, $(\mathbb{Z},2\mathbb{Z},\mathbb{Z})$, $(2\mathbb{Z},2\mathbb{Z},\mathbb{Z})$ or $(\mathbb{Z},2\mathbb{Z},2\mathbb{Z})$. The first two are admissible, while the last two are not.*

Proof. Let $V = V_{-2} \oplus V_0 \oplus V_2$ be the decomposition of V into weight spaces. We see that $V_2 = (\mathbb{Q},0,0)$, $V_0 = (0,\mathbb{Q},0)$ and $V_{-2} = (0,0,\mathbb{Q})$.

We have that $\mathscr{V} = \mathscr{V}_2 \oplus \mathscr{V}_0 \oplus \mathscr{V}_{-2}$ where $\mathscr{V}_i = \mathscr{V} \cap V_i$. Since $\mathscr{V}_2$ is a lattice in V_i, we have that $\mathscr{V}_2 = q_2(\mathbb{Z},0,0)$ for some $q_2 \in \mathbb{Q}$. Similarly, $\mathscr{V}_0 = q_0(0,\mathbb{Z},0)$ and $\mathscr{V}_{-2} = q_{-2}(0,0,\mathbb{Z})$ for some $q_0, q_{-2} \in \mathbb{Q}$.

Recall that $e(a,b,c) = (b,2c,0)$ and $f(a,b,c) = (0,2a,b)$. Hence, it follows that $2q_2\mathbb{Z} \subseteq q_0\mathbb{Z} \subseteq q_{-2}\mathbb{Z}$ and that $2q_{-2}\mathbb{Z} \subseteq q_0\mathbb{Z} \subseteq q_2\mathbb{Z}$. Write $q_0 = mq_2$ and $q_2 = \frac{nq_0}{2}$ for some $m,n \in \mathbb{N}$. Thus, $q_0 = \frac{mnq_0}{2}$, whence $mn = 2$. Similarly, if we write $q_0 = m'q_{-2}$ and $q_{-2} = \frac{n'q_0}{2}$, then $m'n' = 2$. So, we have four possibilities.

If $q_0 = q_2 = q_{-2}$, then $\mathscr{V}$ isomorphic to $\mathbb{Z}^3$. If $q_2 = q_0 = 2q_{-2}$, then $\mathscr{V} \simeq (2\mathbb{Z},2\mathbb{Z},\mathbb{Z})$. If $2q_2 = q_0 = q_{-2}$, then $\mathscr{V} \simeq (\mathbb{Z},2\mathbb{Z},2\mathbb{Z})$. If $2q_2 = q_0 = 2q_{-2}$, then $\mathscr{V} \simeq (\mathbb{Z},2\mathbb{Z},\mathbb{Z})$.

We have that $\frac{e^2}{2}(a,b,c) = (c,0,0)$ and that $\frac{f^2}{2}(a,b,c) = (0,0,a)$. Simply by looking at the formulas, we see that the $\mathbb{Z}^3$ and $(\mathbb{Z},2\mathbb{Z},\mathbb{Z})$ are admissible. On the other hand,

$\frac{e^2}{2}(0,0,1) = (1,0,0) \notin (2\mathbb{Z}, 2\mathbb{Z}, \mathbb{Z})$ and $\frac{f^2}{2}(1,0,0) = (0,0,1) \notin (\mathbb{Z}, 2\mathbb{Z}, 2\mathbb{Z})$. Therefore, $(2\mathbb{Z}, 2\mathbb{Z}, \mathbb{Z})$ and $(\mathbb{Z}, 2\mathbb{Z}, 2\mathbb{Z})$ are not admissible.

Now, assume that $\varphi : \mathscr{U} \longrightarrow \mathscr{V}$ is an isomorphism of $\mathscr{G}$-modules where $\mathscr{U}$ and $\mathscr{V}$ are lattices in V which decompose into weight spaces. One observes that h acts diagonally on both $\mathscr{U}$ and $\mathscr{V}$. It follows that φ is diagonal with respect to the standard basis of $\mathbb{Q}^3$, i.e., $\varphi(a,b,c) = (\varphi_1 a, \varphi_2 b, \varphi_3 c)$ for $\varphi_i \in \mathbb{Q}^\times$. One has that

$$\begin{aligned}\varphi\big(e(a,b,c)\big) &= \varphi\big((b,2c,0)\big) = (\varphi_1 b, \varphi_2 2c, 0),\\ e\varphi\big((a,b,c)\big) &= e\big((\varphi_1 a, \varphi_2 b, \varphi_3 c)\big) = (\varphi_2 b, 2\varphi_3 c, o).\end{aligned}$$

It follows that $\varphi_1 = \varphi_2 = \varphi_3$. The result follows. □

Conjecture 4.3.18. *With the same setting as in Lemma* 4.3.17*, the fiber of* $[\mathbb{Q}^3]$ *under the base change map has at least 4 elements.*

To prove this conjecture, one might want to consider the 4 representations of Lemma 4.3.17 and prove that they are pairwise distinct on the level of representation ring. Note that they are pairwise non-isomorphic but could be the same on the level of representation ring.

4.3.2 Computation of invariant bilinear forms on simply-connected Chevalley orders

As we have seen in Corollary 4.2.6, the space of invariant bilinear forms of a finite-dimensional split simple Lie algebra over a field of characteristic zero is one-dimensional. However, in general, we do not know the structure of this space. Our goal in this section is to compute the space of invariant bilinear forms on a particularly nice type of Lie algebras, namely simply-connected Chevalley orders. For this section, our references are [Hum, VII,§25] and [B:Lie, VIII,§12].

Let Δ be an (irreducible) root system and let $Q(\Delta)$ be the corresponding weight lattice. We let $\Delta^e = \Delta \cup \{0\}$. Let $\mathscr{G}$ be a Chevalley order. One knows that $\mathscr{G}$ has a natural Δ-grading, $\mathscr{G} = \bigoplus_{\alpha\in\Delta} \mathscr{G}_\alpha$ with $\mathscr{H} = \mathscr{G}_0$ and $[\mathscr{G}_\alpha, \mathscr{G}_\beta] \subseteq \mathscr{G}_{\alpha+\beta}$ with $\mathscr{G}_{\alpha+\beta} = 0$ if $\alpha + \beta \notin \Delta$. It follows that $\mathscr{G} \otimes_{\mathbb{Z}} \mathscr{G}$ has a grading as a $\mathbb{Z}$-module by $Q(\Delta)$, namely, $\mathscr{G} \otimes \mathscr{G} = \bigoplus_q (\mathscr{G} \otimes \mathscr{G})_q$ where $(\mathscr{G} \otimes \mathscr{G})_q = \bigoplus_{\alpha+\beta=q} \mathscr{G}_\alpha \otimes \mathscr{G}_\beta$.

Notation 4.3.19. We denote by $\{x_\alpha\}_{\alpha\in\Delta}$ the Chevalley generators of $\mathscr{G}$.

$$x_{\alpha\beta\gamma} := [x_\alpha, x_\beta] \otimes x_\gamma - x_\alpha \otimes [x_\beta, x_\gamma].$$

We also denote by $\overline{x \otimes y}$ for the image of $x \otimes y$ under the canonical projection from $\mathscr{G} \otimes \mathscr{G}$ onto $\mathbf{IBF}(\mathscr{G})$. We also denote by x_0 a generic element of $\mathscr{H}$. Then, for $\alpha, \beta, \gamma \in \Delta^e$, we define $x_{\alpha\beta\gamma}$ in a similar fashion.

Note that $\mathbf{IBF}(\mathscr{G})$ is a graded submodule of $(\mathscr{G} \otimes \mathscr{G})/\operatorname{ibf}(\mathscr{G})$. Now, $\mathbf{IBF}(\mathscr{G}) = \bigoplus_{q\in Q(R)} \mathbf{IBF}(\mathscr{G})_q$ with respect to the $Q(\Delta)$-grading. Our approach is to study the space of invariant bilinear forms using the root system. We are going to use the following notations:

Notation 4.3.20. Let $\alpha, \beta \in \Delta$. $S(\beta, \alpha)$ denotes the root string of α though β, i.e., $S(\alpha, \beta) = (\alpha + \mathbb{Z}\beta) \cap \Delta$.

(i) $\alpha \top \beta \iff \langle \alpha, \beta^\vee \rangle = 1 = \langle \beta, \alpha^\vee \rangle \iff \angle(\alpha, \beta) = \pi/3$. We say that α and β are *collinear.*

(ii) $\alpha \perp_w \beta \iff \alpha \perp \beta$ and $\alpha \pm \beta \in \Delta$.

Example 4.3.21. In the root system B_2, if α and β are orthogonal short roots, then $\alpha \perp_w \beta$.

Lemma 4.3.22. *Let $\alpha, \beta \in \Delta$, $\langle \alpha, \beta^\vee \rangle \geq 0$. Then exactly one of the following cases occurs*

(i) *α, β are short collinear roots in G_2.*

(ii) *$\alpha \perp_w \beta$.*

(iii) *$\alpha + \beta \notin \Delta$.*

Proof. It suffices to show that the assumption $\alpha + \beta \in \Delta$ and $\alpha \not\perp_w \beta$ implies (i). But this is immediate from $\langle \alpha+\beta, \alpha^\vee \rangle = 2 + \langle \beta, \alpha^\vee \rangle \leq 3$ and $\langle \alpha+\beta, \beta^\vee \rangle = 2 + \langle \alpha, \beta^\vee \rangle$. $\square$

We need to understand $[x_\alpha, x_\beta]$ and also the root strings between two roots.

Lemma 4.3.23. *Let Δ be a root system and let $\alpha, \beta \in \Delta$.*

(a) *Assume that α and β are collinear.*

  (i) *If $\Delta \not\simeq G_2$ or if α is a long root in G_2, then $S(\beta,\alpha) = \{\alpha-\beta, \alpha\}$ and $[x_\alpha, x_\beta] = 0$.*

  (ii) *If α is a short root in G_2, then $S(\beta,\alpha) = \{\alpha - 2\beta, \alpha-\beta, \alpha, \alpha+\beta\}$ for short root α and $[x_\alpha, x_\beta] = \pm 3x_{\alpha+\beta}$.*

(b) *Assume that $\alpha \perp_w \beta$. Then $\mathbb{R}\alpha \oplus \mathbb{R}\beta \cap \Delta \simeq B_2$ and $S(\beta,\alpha) = \{\alpha-\beta, \alpha, \alpha+\beta\}$ and $S(\beta,\alpha) = \{\beta-\alpha, \beta, \beta+\alpha\}$, so $[x_\alpha, x_\beta] = \pm 2x_{\alpha+\beta}$ and $[x_\beta, x_\alpha] = \pm 2x_{\alpha+\beta}$.*

(c) *Assume that Δ is irreducible and let $\gamma \in \Delta$. Then $\langle \gamma, \Delta^\vee \rangle \subseteq 2\mathbb{Z}$ if and only if $\Delta \simeq C_l$, $l \geq 1$, and γ is a long root. Otherwise, there exists $\alpha \in \Delta$ such that $\langle \gamma, \alpha^\vee \rangle = 1$.*

Proof. For both parts (a) and (b), one can consider the subsystem $S = \Delta \cap (\mathbb{R}\alpha \oplus \mathbb{R}\beta)$, which is an irreducible root system of rank 2. The claims then follow by inspection and by [Hum, Theorem 25.2]: $[x_\alpha, x_\beta] = \pm(r+1)x_{\alpha+\beta}$ if α, β are linearly independent with $\alpha+\beta \in \Delta$ and $r = \max\{n \in \mathbb{N} : \beta - r\alpha \in \Delta\}$ while $[x_\alpha, x_\beta] = 0$ if $\alpha + \beta \notin \Delta$.

(c) If γ is a long root in C_l, then it is not hard to see that the result holds. Conversely, if $\Delta \simeq A_1$ or $\Delta \simeq G_2$, then the result follows by inspection. If γ is a short root, then $\langle \gamma, \alpha^\vee \rangle \in \{0, \pm 1\}$ for all roots α. We are left to deal with the case where γ is a long root. We only need to look at the non-simply laced cases. In general, one knows from the classification that $\langle \gamma, \alpha^\vee \rangle \in \{0, \pm 2\}$ for all short and non-long roots α.

The long roots of F_4 form an D_4-subsystem, so there exists $\alpha \in \Delta_{\text{long}}$ so that $\langle \gamma, \alpha^\vee \rangle = 1$. The long roots of B_l is a subsystem of type D_l, hence there exists $\alpha \in \Delta_{\text{long}}$ so that $\langle \gamma, \alpha^\vee \rangle = 1$. Therefore, Δ is of type C and γ is a long root. $\square$

The following lemma gives us some information about the case $q = 0$. We use the terminology that in a simply-laced root system all roots are long.

Lemma 4.3.24. *Let $\mathscr{G}$ be a simply-connected Chevalley order. Then*

(a) *If $\Delta \neq C_l$ $(l \geq 1)$, then $\mathbf{IBF}(\mathscr{G})_0 = \mathbb{Z}(\overline{x_\alpha \otimes x_{-\alpha}})$ for some long root α. In this case, $\mathbf{IBF}(\mathscr{G})_0$ is a free $\mathbb{Z}$-module.*

(b) *$\Delta \simeq C_l$, $l \geq 1$, $\mathbf{IBF}(\mathscr{G})_0 = \mathbb{Z}(\overline{x_\alpha \otimes x_{-\alpha}}) \oplus \operatorname{span}\{\overline{x_\alpha \otimes x_{-\alpha}} - \overline{x_\beta \otimes x_{-\beta}} : \beta$ long$\}$.*

(i) $\mathbb{Z}(\overline{x_\alpha \otimes x_{-\alpha}})$ *is a free submodule.*

(ii) $2(\overline{x_\alpha \otimes x_{-\alpha}} - \overline{x_\beta \otimes x_{-\beta}}) = 0$, *whence* $\operatorname{span}\{\overline{x_\alpha \otimes x_{-\alpha}} - \overline{x_\beta \otimes x_{-\beta}} : \beta \text{ } \textit{long}\}$ *is the torsion submodule of* $\mathbf{IBF}(\mathscr{G})_0$.

Proof. We begin the proof by proving that $\mathbf{IBF}(\mathscr{G})_0 = \operatorname{span}\{\overline{x_\alpha \otimes x_{-\alpha}} : \alpha \text{ long}\}$.

By definition, $\mathbf{IBF}(\mathscr{G})_0$ is the image of $(\mathscr{G} \otimes \mathscr{G})_0 = \bigoplus_{\alpha \in \Delta} \mathscr{G}_\alpha \otimes \mathscr{G}_{-\alpha}$ under the projection $(\mathscr{G} \otimes \mathscr{G})_0 \longrightarrow (\mathscr{G} \otimes \mathscr{G})_0 / \operatorname{ibf}(\mathscr{G})_0$ where $\operatorname{ibf}(\mathscr{G})_0 = \{x_{\alpha\beta\gamma} : \alpha + \beta + \gamma = 0\}$.

For $\alpha = 0 \neq \beta$, we obtain $x_{0,\beta,-\beta} = \beta(h) x_\beta \otimes x_{-\beta} - h \otimes h_\beta$ for arbitrary $h \in \mathscr{H}$. Since $Q(\Delta^\vee) \subseteq \mathscr{H} \subseteq \mathscr{P}(\Delta^\vee)$ and $|\mathscr{P}(\Delta^\vee)/Q(\Delta^\vee)| < \infty$, knowing $\mathscr{G}_\alpha \otimes \operatorname{span}\{h_\alpha\}$ is enough to calculate $\mathbf{IBF}$, $\overline{\mathscr{H} \otimes \mathscr{H}} \subseteq \sum_{0 \neq \alpha \in \Delta} \overline{\mathscr{G}_\alpha \otimes \mathscr{G}_{-\alpha}} = \mathbf{IBF}(\mathscr{G})_0$.

To finish the proof of part (a), we may assume that Δ is not simply laced. Let $\alpha \in \Delta$ be a short root. Since Δ is irreducible, α embeds into an irreducible rank 2 subsystem Ψ which is not simply laced. Thus, Ψ is either of type B_2 or G_2.

If $S \simeq B_2$, then there exists a short root γ so that $\gamma \perp \alpha$ and that $\beta = \gamma + \alpha$ is a long root. Thus, one sees that $x_{-\alpha} = \pm[x_\gamma, x_{-\beta}]$. Hence, with $\varepsilon = \pm 1$, we have that

$$x_\alpha \otimes x_{-\alpha} = \varepsilon x_\alpha \otimes [x_\gamma, x_{-\beta}] = \varepsilon(x_\alpha \otimes [x_\gamma, x_{-\beta}] - [x_\alpha, x_\gamma] \otimes x_{-\beta}) + \varepsilon[x_\alpha, x_\gamma] \otimes x_{-\beta}.$$

Hence, $\overline{x_\alpha \otimes x_{-\alpha}} = 2\varepsilon \overline{x_\beta \otimes x_{-\beta}}$ since $[x_\alpha, x_\gamma] = \pm 2 x_\beta$. Next, if $S \simeq G_2$, then there exists γ so that $\gamma \top \alpha$ and that $\beta = \alpha + \gamma$ is a long root. From

$$x_\alpha \otimes x_{-\alpha} = \varepsilon x_\alpha \otimes [x_\gamma, x_{-\beta}] = \varepsilon(x_\alpha \otimes [x_\gamma, x_{-\beta}] - [x_\alpha, x_\gamma] \otimes x_{-\beta}) + \varepsilon[x_\alpha, x_\gamma] \otimes x_{-\beta},$$

it follows that $\overline{x_\alpha \otimes x_{-\alpha}} = 3\varepsilon \overline{x_\beta \otimes x_{-\beta}}$ since $[x_\alpha, x_\gamma] = \pm 3 x_\beta$. Therefore, we have that $\mathbf{IBF}(\mathscr{G})_0 = \operatorname{span}\{\overline{x_\alpha \otimes x_{-\alpha}} : \alpha \text{ long root}\}$.

For the proof of part (a), suppose that Δ_{long} is connected, i.e., Δ is not of type C_l. Then $\Delta_{(\text{long})} = \{\alpha \in \Delta : \alpha \text{ long root}\}$ is a subsystem of rank at least 2. We need the following claim:

Claim: Any two roots of $\alpha, \beta \in \Delta_{\text{long}}$ are connected by roots $\alpha = \alpha_1, \alpha_2, \ldots, \alpha_n = \beta$ satisfying $\alpha_i \top \alpha_{i+1}$ for $1 \leq i < n$. By the classification of root systems, Δ_{long} is an irreducible root subsystem of rank at least 2. If α, β embeds into an A_2-subsystem, then the result follows from inspection. Otherwise, $\alpha \perp \beta$ and there exists $\gamma \in \Delta_{\text{long}}$ such that $\alpha \not\perp \gamma \not\perp \beta$. In this case, $(\mathbb{R}\alpha + \mathbb{R}\gamma) \cap \Delta_{\text{long}}$ and $(\mathbb{R}\beta + \mathbb{R}\gamma) \cap \Delta_{\text{long}}$ are A_2 subsystem. Then, the claim follows from the first part (by concatenating two chains).

To prove the first assertion of (b), by the claim, it now suffices to show that $\overline{\mathscr{G}_\alpha \otimes \mathscr{G}_{-\alpha}} = \overline{\mathscr{G}_\beta \otimes \mathscr{G}_{-\beta}}$ for $\alpha \top \beta$. In this case, $\gamma = \beta - \alpha \in \Delta$, there are $\varepsilon_1, \varepsilon_2 \in \{\pm 1\}$ such that

$$x_{\alpha,\gamma,-\beta} = \varepsilon_1 x_\beta \otimes x_{-\beta} - \varepsilon_2 x_\alpha \otimes x_{-\alpha},$$

proving our first claim.

Since $\mathbb{Z}$ is a PID and $\mathbf{IBF}(\mathscr{G})_0$ is cyclic, it is free if and only if it is torsion-free. Suppose that it is a torsion module. Hence, there exists $m \in \mathbb{Z}$ such that $m\,\mathbf{IBF}(\mathscr{G})_0 = 0$. It follows that $\mathrm{IBF}(\mathscr{G})_0 = \mathrm{Hom}_{\mathbb{Z}}(\mathbf{IBF}(\mathscr{G})_0, \mathbb{Z}) = \{0\}$. So, any $f \in \mathrm{Hom}_{\mathbb{Z}}(\mathbf{IBF}(\mathscr{G})_0, \mathbb{Z})$ satisfies $mf\big(\mathbf{IBF}(\mathscr{G})_0\big) = \{0\}$. On the other hand, since $\mathscr{G}$ is free of finite rank, it has a Killing form κ, which is not a torsion element since $\kappa_{\mathbb{Q}} = \kappa \otimes \mathrm{Id}_{\mathbb{Q}} = \kappa_{\mathscr{G} \otimes_{\mathbb{Z}} \mathbb{Q}} \neq 0$, a contradiction.

To prove part (b), we know that the long roots of Δ are of the form $\pm 2\varepsilon_i$. We claim that $2\overline{x_{2\varepsilon_i} \otimes x_{-2\varepsilon_i}} = \pm 2\overline{x_{2\varepsilon_j} \otimes x_{-2\varepsilon_j}}$ for all i, j. We know that it is true if $i = j$. Suppose that $i \neq j$. Then $2\varepsilon_i$ and $2\varepsilon_j$ can be embedded into a rank 2 root system of type C_2 such that $2\varepsilon_i \perp_s 2\varepsilon_j$. We have the following

$$\begin{aligned}
[x_{2\varepsilon_i}, x_{-\varepsilon_i+\varepsilon_j}] \otimes x_{-\varepsilon_i-\varepsilon_j} - x_{2\varepsilon_i} \otimes [x_{-\varepsilon_i+\varepsilon_j}, x_{-\varepsilon_i-\varepsilon_j}] &= x_{\varepsilon_i+\varepsilon_j} \otimes x_{-(\varepsilon_i+\varepsilon_j)} \pm 2x_{2\varepsilon_i} \otimes x_{-2\varepsilon_i} \\
[x_{2\varepsilon_j}, x_{\varepsilon_i-\varepsilon_j}] \otimes x_{-\varepsilon_i-\varepsilon_j} - x_{2\varepsilon_j} \otimes [x_{\varepsilon_i-\varepsilon_j}, x_{-\varepsilon_i-\varepsilon_j}] &= x_{\varepsilon_i+\varepsilon_j} \otimes x_{-(\varepsilon_i+\varepsilon_j)} \pm 2x_{2\varepsilon_j} \otimes x_{-2\varepsilon_j}.
\end{aligned}$$

It follows that $2\overline{x_{2\varepsilon_i} \otimes x_{-2\varepsilon_i}} = \pm 2\overline{x_{2\varepsilon_j} \otimes x_{-2\varepsilon_j}}$ for all i, j. Hence, for long roots α and β, $2\overline{x_\alpha \otimes x_{-\alpha}} = \pm 2\overline{x_\beta \otimes x_{-\beta}}$. We claim however that $2\overline{x_\alpha \otimes x_{-\alpha}} = 2\overline{x_\beta \otimes x_{-\beta}}$. To prove this, we have the following useful computations.

$$\begin{aligned}
[x_\alpha, x_{-\alpha}] \otimes h_\alpha - x_\alpha \otimes [x_{-\alpha}, h_\alpha] &= h_\alpha \otimes h_\alpha - 2x_\alpha \otimes x_{-\alpha}, \\
[x_{-\alpha}, x_\alpha] \otimes h_\alpha - x_{-\alpha} \otimes [x_\alpha, h_\alpha] &= -h_\alpha \otimes h_\alpha + 2x_{-\alpha} \otimes x_\alpha.
\end{aligned}$$

Hence, $\overline{h_\alpha \otimes h_\alpha} = 2\overline{x_\alpha \otimes x_{-\alpha}} = 2\overline{x_{-\alpha} \otimes x_\alpha}$.

Since $2\overline{x_\alpha \otimes x_{-\alpha}} = \pm 2\overline{x_\beta \otimes x_{-\beta}}$, we have that $\overline{h_\alpha \otimes h_\alpha} = \pm \overline{h_\beta \otimes h_\beta}$. Next, consider the Killing form κ of $\mathscr{G}$. We see that $\kappa(h_\alpha, h_\alpha) = \pm \kappa(h_\beta, h_\beta)$. However, for all $h \in \mathscr{H}$, we have that $\kappa(h, h) = \sum_{\delta \in \Delta} \delta(h)^2 \geq 0$. It follows that $\kappa(h_\alpha, h_\alpha) = \kappa(h_\beta, h_\beta)$. Therefore, we must have that $2\overline{x_\alpha \otimes x_{-\alpha}} = 2\overline{x_\beta \otimes x_{-\beta}}$ for all long roots α and β.

This shows that $\mathbf{IBF}(\mathscr{G})_0 = \mathbb{Z}(\overline{x_\alpha \otimes x_{-\alpha}}) + \mathrm{span}\{\overline{x_\alpha \otimes x_{-\alpha}} - \overline{x_\beta \otimes x_{-\beta}} : \beta \text{ long}\}$. Next, $\mathrm{span}\{\overline{x_\alpha \otimes x_{-\alpha}} - \overline{x_\beta \otimes x_{-\beta}} : \beta \text{ long}\}$ is torsion since $2(\overline{x_\beta \otimes x_{-\beta}} - \overline{x_\alpha \otimes x_{-\alpha}}) = 0$.

Let κ be the Killing form of $\mathscr{G}$. Since $\mathrm{IBF}(\mathscr{G}) \simeq \mathrm{Hom}(\mathbf{IBF}(\mathscr{G}), \mathbb{Z})$, $\kappa(x, y) = \varphi(\overline{x \otimes y})$ for some $\varphi \in \mathrm{Hom}(\mathbf{IBF}(\mathscr{G}, \mathbb{Z}))$. Note that $\kappa_{\mathbb{Q}}$ is the Killing form of $\mathscr{G} \otimes_{\mathbb{Z}} \mathbb{Q}$, it follows that $\kappa(x_\alpha, x_{-\alpha}) = \varphi(\overline{x_\alpha \otimes x_{-\alpha}}) \neq 0$. It follows that $\overline{x_\alpha \otimes x_{-\alpha}}$ is not torsion.

Therefore, $\mathbf{IBF}(\mathscr{G})_0 = \mathbb{Z}(\overline{x_\alpha \otimes x_{-\alpha}}) \oplus \mathrm{span}\{\overline{x_\alpha \otimes x_{-\alpha}} - \overline{x_\beta \otimes x_{-\beta}} : \beta \text{ long}\}$ where we have that $\mathbb{Z}(\overline{x_\alpha \otimes x_{-\alpha}})$ is the free part and the rest is the torsion part. □

We will address the question: For which α, β does $\mathscr{G}_\alpha \otimes \mathscr{G}_\beta \subseteq \mathrm{ibf}(\mathscr{G})$ hold? The following lemma tells us the answer.

Lemma 4.3.25. *Let Δ be an irreducible root system and $\mathscr{G}$ be a corresponding Chevalley order. Then,*

(a) *For all $\alpha \in \Delta$, $\mathscr{G}_\alpha \otimes \mathscr{H} \subseteq \mathrm{ibf}(\mathscr{G})$ and $\mathscr{H} \otimes \mathscr{G}_\alpha \subseteq \mathrm{ibf}(\mathscr{G})$.*

(b) *For $\alpha, \beta \in \Delta$ with $\alpha + \beta \neq 0$, we have*

  (i) *$2\mathscr{G}_\alpha \otimes \mathscr{G}_\beta \subseteq \mathrm{ibf}(\mathscr{G})$ if α, β are long roots in C_l.*

  (ii) *$\mathscr{G}_\alpha \otimes \mathscr{G}_\beta \subseteq \mathrm{ibf}(\mathscr{G})$ otherwise.*

Proof. (a). We will show that $\mathscr{H} \otimes \mathscr{G}_\alpha \subseteq \mathrm{ibf}(\mathscr{G})$. The other inclusion follows by symmetry. For $\alpha, \beta \in \Delta$, observe that

$$[h_\alpha, h] \otimes x_\beta - h_\alpha \otimes [h, x_\beta] = -\beta(h) h_\alpha \otimes x_\beta \in \mathrm{ibf}(\mathscr{G}).$$

In particular, $-\beta(h_\gamma) h_\alpha \otimes x_\beta = -\langle \beta, \gamma^\vee \rangle h_\alpha \otimes x_\beta \in \mathrm{ibf}(\mathscr{G})$ for any $\gamma \in \Delta^\vee$. By Lemma 4.3.23(c), we see that $\mathscr{H} \otimes x_\beta \subseteq \mathrm{ibf}(\mathscr{G})$ unless $\beta \in C_l$ is a long root, $l \geq 1$. From now on, we assume that β is a long root in C_l. One knows that, for $\alpha \in \Delta$ with $\alpha + \beta \notin \Delta$,

$$x_{-\alpha,\alpha,\beta} = -h_\alpha \otimes x_\beta \in \mathrm{ibf}(\mathscr{G}),$$

whence $(\mathrm{span}_{\mathbb{Z}}\{h_\alpha : \alpha + \beta \notin \Delta\}) \otimes x_\beta \subseteq \mathrm{ibf}(\mathscr{G})$. It thus suffices to show that $\mathrm{span}_{\mathbb{Z}}\{h_\alpha : \alpha + \beta \notin \Delta\} = \mathscr{H}$. By Lemma 4.3.22, for $\langle \alpha, \beta^\vee \rangle \geq 0$, $\alpha + \beta \notin \Delta \iff \alpha \not\perp_w \beta$. On the other hand, for a long root in C_l, $l \geq 1$, we never have $\alpha \perp_w \beta$, so that $\mathrm{span}_{\mathbb{Z}}\{h_\alpha : \alpha + \beta \notin \Delta\} = \mathrm{span}_{\mathbb{Z}}\{h_\alpha : \langle \alpha, \beta^\vee \rangle \geq 0\}$, which is $\mathscr{H}$ because $h_{-\alpha} = -h_\alpha$ for all $\alpha \in \Delta$.

(b). We consider various cases:

(I). Assume that $1 = \langle \alpha+\beta, \gamma^\vee \rangle$ for some $\gamma \in \Delta$. Then $x_{\alpha 0 \beta} = -(\alpha+\beta)(h) x_\alpha \otimes x_\beta$ implies $x_\alpha \otimes x_\beta \in \operatorname{ibf}(\mathscr{G})$.

Since $\langle \alpha+\beta, \alpha^\vee \rangle = 1 \iff \langle \beta, \alpha^\vee \rangle = -1$, $\langle \alpha+\beta, \beta^\vee \rangle = 1 \iff \langle \alpha, \beta^\vee \rangle = -1$ and $\alpha + \beta \neq 0$, it follows that this case covers all possibilities such that $\langle \alpha, \beta^\vee \rangle < 0$. We can also use this to deal with the case $\alpha \top \beta$ for α, β short roots in G_2, because the long roots in G_2 form an A_2 subsystem, so that there exists γ such that $\langle \alpha+\beta, \gamma^\vee \rangle = 1$ e.g., $\gamma = 2\beta - \alpha$. This shows our claim in the first case.

(II). $\langle \alpha, \beta^\vee \rangle \geq 0$ where α, β are not short roots in G_2 with $\alpha \top \beta$. We have the following two subcases:

(IIi). $\alpha \perp_w \beta$. Then $\beta - \alpha \in \Delta$ and the root string is $S(\alpha, \beta-\alpha) = \{\beta-\alpha, \beta, \beta+\alpha\}$, whence $[x_\alpha, x_{\beta-\alpha}] = \pm x_\beta$. Thus

$$x_{\alpha,\alpha,\beta-\alpha} = -x_\alpha \otimes [x_\alpha, x_{\beta-\alpha}] = \pm x_\alpha \otimes x_\beta$$

shows that $x_\alpha \otimes x_\beta \in \operatorname{ibf}(\mathscr{G})$.

(IIii). $\alpha \not\perp_w \beta$, $\alpha + \beta \notin \Delta$. Here we use

$$x_{\alpha\beta 0} = \beta(h) x_\alpha \otimes x_\beta \quad \text{and} \quad x_{0\alpha\beta} = \alpha(h) x_\alpha \otimes x_\beta$$

to see that $x_\alpha \otimes x_\beta \in \operatorname{ibf}(\mathscr{G})$ as soon as $1 \in \langle \alpha, \Delta \rangle$ or $1 \in \langle \beta, \Delta \rangle$. By Lemma 4.3.23(c), this is the case unless α, β are long roots in C_l, $l \geq 1$. In this case, we get, using $x_{\alpha\beta 0} = \beta(h) x_\alpha \otimes x_\beta$, that $\beta(h_\beta) x_\alpha \otimes x_\beta = 2 x_\alpha \otimes x_\beta \in \operatorname{ibf}(\mathscr{G})$. □

The following theorem summarizes what we have done so far.

Theorem 4.3.26. *Let Δ be an irreducible finite root system and let $\mathscr{G}$ be a Chevalley order of the split simple Lie algebra with root system Δ.*

(a) *If $\Delta \neq C_l$, $l \geq 1$, then $\mathbf{IBF}(\mathscr{G}) = \mathbf{IBF}(\mathscr{G})_0 = \mathbb{Z}\overline{x_\alpha \otimes x_{-\alpha}}$ is a free $\mathbb{Z}$-module.*

(b) *If $\Delta = C_l$, $l \geq 1$, then $\mathbf{IBF}(\mathscr{G}) = \mathbf{IBF}(\mathscr{G})_{free} \oplus \mathbf{IBF}(\mathscr{G})_{tor}$ where*

  (i) *$\mathbf{IBF}(\mathscr{G})_{free} = \mathbb{Z}(\overline{x_\alpha \otimes x_{-\alpha}})$ is free of rank 1 for any fixed long root α, and*

  (ii) *$\mathbf{IBF}(\mathscr{G})_{tor}$ is spanned by the elements $\overline{x_\alpha \otimes x_\beta}$, $\alpha, \beta \in \Delta_{long}, \alpha + \beta = 0$, and $\overline{x_\beta \otimes x_{-\beta}} - \overline{x_\alpha \otimes x_{-\alpha}}$ for $\beta \in \Delta_{long}$. Moreover, $\mathbf{IBF}(\mathscr{G})_{tor}$ satisfies $2\,\mathbf{IBF}(\mathscr{G})_{tor} = 0$.*

Next, we will go back to the setting of invariant bilinear forms. Our goal is to obtain a normalized form similar to that of [NPPS, Ex. 6.3] and [NPPS, Rem. 6.4].

Corollary 4.3.27. *Let $\mathscr{G}$ be a simply-connected Chevalley order. Then there exists a unique $\gamma \in \mathrm{IBF}(\mathscr{G})$ satisfying $\gamma(h_\alpha, h_\alpha) = 2$ for every long root α.*

(i) $\mathrm{IBF}(\mathscr{G}) = \mathbb{Z}\gamma$ *is free of rank 1 with basis* γ,

(ii) γ *is symmetric and nondegenerate.*

Proof. By Theorem 4.3.24, for a long root α, $\mathbf{IBF}(\mathscr{G}) = \mathbb{Z}(\overline{x_\alpha \otimes x_{-\alpha}}) \oplus T$ where $\mathbb{Z}(\overline{x_\alpha \otimes x_{-\alpha}})$ is a free submodule and T is a torsion module. Since we have that $\mathrm{IBF}(\mathscr{G}) \simeq \mathrm{Hom}_\mathbb{Z}(\mathbf{IBF}(\mathscr{G}), \mathbb{Z})$, $\mathrm{IBF}(\mathscr{G}) \simeq \mathrm{Hom}_\mathbb{Z}(\mathbb{Z}(\overline{x_\alpha \otimes x_{-\alpha}}), \mathbb{Z})$ is free of rank 1 with a basis corresponding to the linear form $\overline{x_\alpha \otimes x_{-\alpha}} \mapsto 1$. For any long root β, since $2\overline{x_\alpha \otimes x_{-\alpha}} = 2\overline{x_\beta \otimes x_{-\beta}}$, $2\gamma(x_\alpha, x_{-\alpha}) = 2\gamma(x_\beta, x_{-\beta})$. Therefore, we have that $\gamma(x_\beta, x_{-\beta}) = \gamma(x_\alpha, x_{-\alpha}) = 1$. Since $\overline{h_\alpha \otimes h_\alpha} = 2\overline{x_\alpha \otimes x_{-\alpha}}$, we have that $\gamma(h_\beta, h_\beta) = 2\gamma(x_\beta, x_{-\beta}) = 2$. From this, we also see that γ is nondegenerate.

Consider the Killing form of $\mathscr{G}$. Since κ is symmetric and is a multiple of γ, it follows that γ is also symmetric. This completes the proof. □

Remark 4.3.28. Corollary 4.3.27 has another proof: Since $\mathscr{G}$ is free of finite rank, $\mathbf{IBF}(\mathscr{G})$ is a finitely generated $\mathbb{Z}$-module, which we can therefore write as $\mathbf{IBF}(\mathscr{G}) = F \oplus T$ where F is a free submodule and T is a torsion submodule.

Now, $\mathrm{IBF}(\mathscr{G}) \simeq \mathrm{Hom}_\mathbb{Z}(\mathbf{IBF}(\mathscr{G}), \mathbb{Z}) \simeq \mathrm{Hom}_\mathbb{Z}(F, \mathbb{Z})$ is the dual of a free module F of rank r, whence also free of rank r. By [NPPS],

$$\mathbf{IBF}(\mathscr{G}) \otimes_\mathbb{Z} \mathbb{Q} \simeq \mathbf{IBF}(\mathscr{G} \otimes_\mathbb{Z} \mathbb{Q}) \simeq F \otimes_\mathbb{Z} \mathbb{Q}$$

has dimension r, which equals to 1 by Lemma 4.2.5. This argument does however not determine a basis γ. But one can at this point invoke [GN, Prop. 4] which proves the existence of γ as claimed. However, this approach does not determine $\mathbf{IBF}(\mathscr{G})$ and hence cannot be used to determine $\mathrm{IBF}(\mathscr{G} \otimes R)$, $R \in \mathbb{Z}$-**alg**, which we are able to do in the next corollary.

Next, we discuss the behaviour of $\mathbf{IBF}(\mathscr{G})$ and $\mathrm{IBF}(\mathscr{G})$ under base change.

Corollary 4.3.29. *Let $R \in \mathbb{Z}$-**alg**, let $\mathscr{G}$ be a simply-connected Chevalley order of a split simple Lie algebra with root system Δ. Put $L = \mathscr{G} \otimes_\mathbb{Z} R$ which we consider as an R-algebra in the obvious way. Then*

(a) *If* $\Delta \not\equiv C_l$, $l \geq 1$, *then* $\mathbf{IBF}(L) = R(\overline{x_\alpha \otimes x_{-\alpha}})$ *is free of rank 1 where* α *is any long root in* Δ.

(b) *If* $\Delta = C_l$, $l \geq 1$, *then* $\mathbf{IBF}(L) = \mathbf{IBF}(L)_{\mathit{free}} \oplus \mathbf{IBF}(L)_{\mathit{tor}}$ *where* $\mathbf{IBF}(L)_{\mathit{free}} = R(\overline{x_\alpha \otimes x_{-\alpha}})$ *for any fixed long root* α *and*

$$\mathbf{IBF}(L)_{\mathit{tor}} = \bigoplus_{\beta \ \mathit{long}} (R \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z})(\overline{x_\beta \otimes x_{-\beta}} - \overline{x_\alpha \otimes x_{-\alpha}}).$$

In particular, if $2R = 0$ *or if* $2 \in R^\times$, *then* $\mathbf{IBF}(L)$ *is free of rank 1.*

Proof. The proof is simply an application of the following fact: $\mathbf{IBF}(\mathscr{G} \otimes_{\mathbb{Z}} R) \simeq \mathbf{IBF}(\mathscr{G}) \otimes_{\mathbb{Z}} R$ by [NPPS, Prop. 4.3(b)]. □

Corollary 4.3.30. *Consider the setting of Corollary* 4.3.29. *Assume that* $\gamma \in \mathrm{IBF}(\mathscr{G})$ *is as in Corollary* 4.3.27. *Let* γ_R *be the bilinear form obtained from base change.*

(a) $\gamma_R \in \mathrm{IBF}_R(L)$ *is symmetric and non-zero.*

(b) *If* $\Delta \not\simeq C_l$, $l \geq 1$, *then* $\mathrm{IBF}_R(L) = R\gamma_R$ *is free of rank 1 with basis* γ_R.

(c) *If* $\Delta = C_l$, $l \geq 1$, *then* $\mathrm{IBF}_R(L) \simeq R\gamma_R \oplus \mathrm{Hom}_{\mathbb{Z}}(T, R_2)$ *where we have that* $T = \bigoplus_{\alpha,\beta \ \mathit{long}, \alpha+\beta\neq 0}(\mathbb{Z}/2\mathbb{Z})(\overline{x_\alpha \otimes x_\beta}) \oplus \bigoplus_{\beta \ \mathit{long}}(\mathbb{Z}/2\mathbb{Z})(\overline{x_\alpha \otimes x_{-\alpha}} - \overline{x_\beta \otimes x_{-\beta}})$, R_2 *is the 2-torsion part; in particular,* $\mathrm{IBF}(\mathscr{G}) = \mathbb{Z}\gamma$ *if* $R_2 = 0$.

Proof. In part (a), it is clear that $\gamma_R \in \mathrm{IBF}_R(L)$ and that γ_R is symmetric. Moreover, γ_R is nonzero since $\gamma(x_\alpha, x_{-\alpha}) = 1$ for any long root α. Next, write $\mathbf{IBF}(\mathscr{G}) = \mathbb{Z}(\overline{x_\alpha \otimes x_{-\alpha}}) \oplus T$ where T is torsion. To show (b) and (c), we have the following

$$\begin{aligned}
\mathrm{IBF}_R(L) &\simeq \mathrm{Hom}_R(\mathbf{IBF}_R(\mathscr{G} \otimes_{\mathbb{Z}} R), R) \simeq \mathrm{Hom}_R(\mathbf{IBF}(\mathscr{G} \otimes_{\mathbb{Z}} R), R) \\
&\simeq \mathrm{Hom}_{\mathbb{Z}}(\mathbf{IBF}(\mathscr{G}), \mathrm{Hom}_R(R, R)) \simeq \mathrm{Hom}_{\mathbb{Z}}(\mathbf{IBF}(\mathscr{G}), R) \\
&= \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}(\overline{x_\alpha \otimes x_{-\alpha}}), R) \oplus \mathrm{Hom}_{\mathbb{Z}}(T, R).
\end{aligned}$$

We have that $T = 0$ if $\Delta \not\simeq C_l$. Since T is 2-torsion, any $\varphi \in \mathrm{Hom}_{\mathbb{Z}}(T, R)$ will map T to R_2. This completes the proof. □

Remark 4.3.31. Suppose that R is a $\mathbb{Z}$-algebra. Let $\{m_1, \ldots, m_n\}$ be a basis of $\mathscr{G}$. We see that $\{m_1 \otimes 1_R, \ldots, m_n \otimes 1_R\}$ is an R-basis of $\mathscr{G} \otimes_{\mathbb{Z}} R$. It follows that the matrices representing γ and γ_R are the same. Therefore, $\det_R(\gamma_R) = \det_{\mathbb{Z}}(\gamma) \otimes_{\mathbb{Z}} 1_R$.

Remark 4.3.32. The existence of the form γ is known from [GN, §5]. Indeed, denoting by κ the Killing form of $\mathscr{G}$, it is shown in loc. cit. that $\kappa = 2h^\vee\gamma$ where $h^\vee$ is the dual Coxeter number as given in the table below [GN, §2]. Moreover, the determinant of γ is also calculated in [GN, §5].

Type of $\mathscr{G}$	A_n	B_n	C_n	D_n	G_2	F_4	E_6	E_7	E_8
$h^\vee$	$n+1$	$2n-1$	$n+1$	$2n-2$	4	9	12	18	30
$\det(\gamma)$	$n+1$	2^{n+2}	2^{n^2}	4	3^7	2^{26}	3	2	1

Corollary 4.3.33. *Let $\mathscr{G}$ be a simply-connected Chevalley order, γ the normalized form of Corollary 4.3.27, R a $\mathbb{Z}$-algebra and let γ_R be the invariant bilinear form obtained by the base change. Then*

(a) *γ_R is nondegenerate if and only if* $\det(\gamma)1_R$ *is not a zero divisor in R.*

(b) *γ_R is nonsingular if and only if* $\det(\gamma)1_R \in R^\times$

Proof. Both parts follow from Remark 4.3.31 and Lemma 4.1.5. □

Example 4.3.34. The simply-connected Chevalley order $\mathscr{G}$ of classical types $\Delta = A_n, B_n, C_n$ and D_n have been determined in [B:Lie, VIII, §13 (IX)]. In all cases, $\mathscr{G}$ is an integer subalgebra of some $\mathfrak{gl}_m(\mathbb{Z})$, the $\mathbb{Z}$-Lie algebra of integer $(m \times m)$-matrices, and $\gamma = \mathrm{tr}\,|\mathscr{G}$ where tr is the usual trace form of $\mathfrak{gl}_m(\mathbb{Z})$. Indeed, it is obvious that tr is an invariant bilinear form. Hence, by uniqueness of Corollary 4.3.27, it suffices to verify $\mathrm{tr}(h_\alpha^2) = 2$ for any long root α. Using [B:Lie, VIII, §13 (II)], this can be done case-by-case.

Example 4.3.35. Recall from [NPPS, 3.6] that an R-Lie algebra L and $\gamma \in \mathrm{IBF}_R(L)$ satisfies the IBF-principle if the associated R-linear map $\overline{\gamma} : \mathbf{IBF}(L) \longrightarrow R$, $\overline{x \otimes y} \mapsto \gamma(x, y)$ is an R-module isomorphism. In this case $\mathrm{IBF}(L)$ is free of rank 1 with basis $\{\gamma\}$. By [NPPS, Lemma 4.6] (the assumption that B be finitely presented is not necessary), the IBF-principle then holds for $L_S = L \otimes_R S$ whenever S/R is flat, and in that case, $\mathrm{IBF}_S(L_S)$ is free of rank 1 with basis $\{\gamma_S\}$. Moreover, if S/R is faithfully flat and if (L_S, γ_S) satisfies the IBF-principle, the so does (L, γ).

Corollary 4.3.36. *Let $\mathscr{G}$ be a simply-connected Chevalley order of type Δ. Let*

$$k = \begin{cases} \mathbb{Z} & if \Delta \neq C_l, l \geq 1 \\ \mathbb{Z}\left[\frac{1}{2}\right] & if \Delta = C_l, l \geq 1. \end{cases}$$

Then $\mathscr{G}_k = \mathscr{G} \otimes_{\mathbb{Z}} k$ *satisfies the IBF-principle. Hence, for any flat* $R \in k\text{-}\mathbf{alg}$*, the Lie algebra* $\mathscr{G}_R = (\mathscr{G} \otimes_{\mathbb{Z}} k) \otimes_k R$ *satisfies the IBF-principle with respect to* γ_R*.*

Proof. The first assertion follows from Corollary 4.3.29. The second assertion is a direct consequence of Example 4.3.35. $\square$

Chapter 5

The Dynkin index

5.1 Setting, definition and properties

In this section, we define the notion of the Dynkin index of a Lie algebra representation. This is a way to study modules of a Lie algebra using invariant bilinear forms. In this section, unless stated otherwise R is a commutative ring and L is a R-Lie algebra. We also assume that $\mathrm{IBF}_R(L) = R\gamma$ is free of rank 1 with a fixed basis $\{\gamma\}$.

Definition 5.1.1. Let $\operatorname{dyn} : \mathscr{R}(L) \longrightarrow R$ be the composite of τ and the coordinate map. The *Dynkin index* of a representation (V, ρ) is defined to be $\operatorname{dyn}([V, \rho])$, sometimes denoted by $\operatorname{dyn}(\rho)$ or $\operatorname{dyn}(V)$ if there is no confusion.

The following properties are immediate consequences of Proposition 4.2.11. Note that the first two equalities can be found in [Dyn, p. 110], [KN, Lem 4.5].

Proposition 5.1.2. *Assume that L is perfect. Suppose that V and W are finitely generated projective L-modules of constant rank r_V and r_W respectively. Then*

$$\begin{aligned}
\operatorname{dyn}(\rho_V \oplus \rho_W) &= \operatorname{dyn}(\rho_V) + \operatorname{dyn}(\rho_W), \\
\operatorname{dyn}(\rho_V \otimes \rho_W) &= r_V \operatorname{dyn}(\rho_W) + r_W \operatorname{dyn}(\rho_V), \\
\operatorname{dyn}(\rho^{\wedge n}) &= \binom{r_V - 2}{n - 1} \operatorname{dyn}(\rho), \\
\operatorname{dyn}(\rho^{\circ n}) &= \binom{r_V + n}{n - 1} \operatorname{dyn}(\rho).
\end{aligned}$$

We call the map $c_L : \mathrm{IBF}(L) \longrightarrow R$, $r\gamma \mapsto r$ the *coordinate map.* We prove that the coordinate map respects base change.

Lemma 5.1.3. *Let S be a R-algebra with structure map $\alpha : R \longrightarrow S$ and suppose that $\mathrm{IBF}_S(L_S)$ is free with basis $\{\gamma_S\}$. Then the following diagram is commutative.*

$$\begin{tikzcd}
\mathrm{IBF}_R(L) \arrow[r, "c_L"] \arrow[d, "\xi"'] & R \arrow[d, "\alpha"] \\
\mathrm{IBF}_S(L_S) \arrow[r, "c_{L_S}"'] & S
\end{tikzcd}$$

where $\xi : \kappa \mapsto \Big(\kappa_\alpha : (l_1 \otimes s_1, l_2 \otimes s_2) \mapsto \alpha\big(\kappa(l_1, l_2)\big)s_1 s_2\Big)$*, as defined in Lemma* 4.2.4.

Proof. Since $\xi(\gamma) = \gamma_S$, the result follows immediately. ☐

The following proposition tells us how the Dynkin index behaves under scalar extension.

Proposition 5.1.4. *Let S be an R-algebra with structure map $\alpha : R \longrightarrow S$ and suppose that $\mathrm{IBF}_S(L \otimes_R S)$ is free with basis $\{\gamma_S\}$. Then, for any finitely generated projective L-module (V, ρ), $\alpha(\mathrm{dyn}(\rho)) = \mathrm{dyn}_{\gamma_S}(\rho_S)$.*

Proof. By Lemma 4.2.13 and Lemma 5.1.3, we see that the following diagram

$$\begin{tikzcd}
\mathscr{R}(L) \arrow[r, "\tau_L"] \arrow[d, "\psi"'] & \mathrm{IBF}(L) \arrow[r, "c_L"] \arrow[d, "\xi"] & R \arrow[d, "\alpha"] \\
\mathscr{R}(L_S) \arrow[r, "\tau_{L_S}"'] & \mathrm{IBF}(L_S) \arrow[r, "c_{L_S}"'] & S
\end{tikzcd}$$

is commutative. Since $\mathrm{dyn} = c_L \circ \tau_L$ and $\mathrm{dyn}_{\gamma_S} = \tau_{L_S} \circ c_{L_S}$, the result follows. ☐

Remark 5.1.5. Note that Proposition 5.1.4 says that, in the case α is injective (e.g. if S/R if faithfully flat), we can calculate $\mathrm{dyn}\,\rho$ by determining $\mathrm{dyn}_{\rho_S}(\rho_S)$. It is therefore of interest to have a class of Lie algebras satisfying the assumption on $\mathrm{IBF}_S(L_S)$ for all S or at least all faithfully flat S.

Definition 5.1.6. Assume that $\mathrm{dyn}(\mathscr{R}(L)) \subseteq \mathbb{Z}$. We define the *Dynkin index of L* to be $\gcd\{m_V : [V] \in \mathscr{R}(L)\}$. We will denote it by d_L.

5.2 Computation of Dynkin index

5.2.1 Lie algebra over a field of characteristic zero

In this subsection, we replace R by a field $\mathbb{K}$ of characteristic zero and consider a split simple Lie algebra $\mathfrak{g}$ over $\mathbb{K}$. In this case, we know that $\mathrm{IBF}(\mathfrak{g})$ is one-dimensional. Namely, $\mathrm{IBF}(\mathfrak{g}) = \mathbb{K} \cdot \kappa$ where κ is the Killing form. For our purpose, it will be more appropriate to use the normalized bilinear form γ satisfying $\gamma(t_\alpha, t_\alpha) = 2$ for any long root α. Here we denote by t_α the unique vector in a splitting Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ representing the root α, i.e., $\gamma(t_\alpha, h) = \alpha(h)$ holds for all $h \in \mathfrak{h}$. Such a normalization can always be achieved by multiplyig κ by a suitable scalar. Of course, $\mathrm{IBF}(\mathfrak{g}) = \mathbb{K} \cdot \gamma$. Since $\mathfrak{g}$ is simple, it is perfect. Hence, clearly, Proposition 5.1.2 holds in this case.

The next proposition is a review. Actually, it provides the technique to prove the fact that the Dynkin indices are invariant under automorphisms (we leave this to the reader).

Proposition 5.2.1. *Let $s = s_0 s_1 \in \mathrm{Aut}(L)$ where s_0 is an inner automorphism and s_1 is a Dynkin diagram automorphism. Consider an irreducible finite-dimensional representation $\rho : L \longrightarrow \mathfrak{gl}(V)$ of highest weight λ. Then $(V, \rho \circ s)$ is a finite-dimensional irreducible representation of L of highest weight $s_1 \cdot \lambda$.*

Proof. It is sufficient to prove the following claims.

Claim 1: If $s = s_0$ is an inner automorphism then $(V, \rho \circ s) \cong (V, \rho)$.

Since $s_0 \in \mathrm{Aut}_0(L)$, by [B:Lie, VII, §7.1, Prop. 2], there exists $S \in \mathrm{GL}(V)$ such that $(s(x))_V = S x_V S^{-1}$. For any weight μ, one has that:

$$\begin{aligned}(\rho \circ s)(h) S V_\mu = \rho(s(h)) S V_\mu &= S \rho(h) S^{-1} S V_\mu \quad \text{by the above identity,} \\ &= S \rho(h) V_\mu = \mu(h) S V_\mu .\end{aligned}$$

It follows that SV_μ in $(V, \rho \circ s)$ has the same weights as V_μ in (V, ρ). Thus $(V, \rho \circ s)$ and (V, ρ) are isomorphic as L-modules.

Claim 2: If $s = s_1 \in \mathrm{Aut}(R, B)$ then $(V, \rho \circ s)$ is a finite-dimensional irreducible representation of L with highest weight $s \cdot \lambda$.

Since s leaves H invariant and corresponds to a Dynkin diagram automorphism, one sees that it also acts on the dominant integral weights. Denote $s^* : P^+ \longrightarrow P^+$

the action of s on the dominant integral weights. We have

$$\begin{aligned}(\rho \circ s)(h)V_\mu &= \rho(s(h))V_\mu = (\mu \circ s)(h)V_\mu \quad (\text{since } s(H) = H)\\ &= s^*(\mu)(h)V_\mu.\end{aligned}$$

We see from this that the action on the dominant weights is given by transposition. So, every weight of $(V, \rho \circ s)$ is of the form $s^*(\mu) = s \cdot \mu = \mu \circ s$ for some weight μ of (V, ρ).

Next, we will show that if $\mu_1 \prec \mu_2$ then $s \cdot \mu_1 \prec s \cdot \mu_2$. Let $\mathcal{B} = \{\alpha_i\}$ be a base of the root system. Write $\mu_1 = \sum_i k_i \alpha_i$ and $\mu_2 = \sum_i l_i \alpha_i$. So, $\mu_2 - \mu_1 = \sum_i (k_i - l_i)\alpha_i$ is a sum of positive roots. In particular, for each i, $k_i - l_i$ is a nonnegative integer. Now, since s leaves the base invariant, write $s \cdot \alpha_i = \alpha_{s \cdot i}$. Moreover, $s \cdot u_2 - s \cdot \mu_1 = \sum_i (k_i - l_i)\alpha_{s \cdot i}$, which is also a sum of positive roots. Therefore, the highest weight of $(V, \rho \circ s)$ is $s \cdot \lambda$, as desired. $\square$

Next, there are several formulas allowing us to explicitly compute the Dynkin index.

Theorem 5.2.2. [Dyn, Lem 2.5], [KN, Lem 4.4]. *The Dynkin index of an irreducible representation $\rho : \mathfrak{g} \longrightarrow \mathfrak{gl}(V)$ of highest weight Λ is given by*

$$\operatorname{dyn}(\rho) = \frac{\dim \rho}{\dim \mathfrak{g}} \gamma(\Lambda, \Lambda + 2\delta),$$

where δ is half the sum of the positive roots.

Theorem 5.2.3. [KN]. *Let $\rho : \mathfrak{g} \longrightarrow \mathfrak{gl}(V)$ be a representation. We use the setting above and for a fixed root α and we put*

$$\mathfrak{s} = \mathfrak{g}_\alpha \oplus [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] \oplus \mathfrak{g}_{-\alpha} \simeq \mathfrak{sl}_2(\mathbb{K}),$$

and $h_\alpha \in [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] \subseteq \mathfrak{h}$ satisfying $\alpha(h_\alpha) = 2$. Let $\mathcal{P}(V)$ be the set of weights of the $\mathfrak{s}$-module V under $\rho|\mathfrak{s}$, and let n_λ be the multiplicity of $\lambda \in \mathcal{P}(V)$. Then,

$$\operatorname{dyn}(\rho) = \frac{1}{2} \sum_{\lambda \in \mathcal{P}(V)} n_\lambda \langle \lambda, h_\alpha \rangle^2 \in \mathbb{Z}.$$

Moreover, if $V = \bigoplus_i V(m_i)$ is the decomposition of V under $\rho|\mathfrak{s}$ into irreducible $\mathfrak{sl}_2$ modules, then

$$\operatorname{dyn}(\rho) = \sum_i \binom{m_i + 2}{3}.$$

In particular, one sees that the Dynkin indices are integers. Using the above formulas, one can compute the Dynkin indices for any finite-dimensional representations of $\mathfrak{g}$. The following tables of Dynkin indices for fundamental representations can be found in [KN] and [LS].

Type of $\mathfrak{g}$	Dynkin indices of the fundamental representations
A_n	$\mathrm{dyn}(\varpi_i) = \binom{n-1}{i-1}$ for $i = 1, \ldots, n$
B_n $(n \geq 3)$	$\mathrm{dyn}(\varpi_i) = 2\binom{2n-1}{i-1}$ for $i = 1, \ldots, n-1$ and $d_{\varpi_n} = 2^{n-2}$
C_n	$\mathrm{dyn}(\varpi_i) = \binom{2n-2}{i-1} - \binom{2n-2}{i-3}$ for $i = 1, \ldots, n$
D_n $(n \geq 4)$	$\mathrm{dyn}(\varpi_i) = 2\binom{2n-2}{i-1}$, $i = 1, \ldots, n-2$; $\mathrm{dyn}(\varpi_{n-1}) = \mathrm{dyn}(\varpi_n) = 2^{n-3}$

	G_2	F_4	E_6	E_7	E_8
$\mathrm{dyn}(\varpi_1)$	2	18	6	36	1500
$\mathrm{dyn}(\varpi_2)$	8	882	24	360	85500
$\mathrm{dyn}(\varpi_3)$		126	150	4680	5292000
$\mathrm{dyn}(\varpi_4)$		6	1800	297000	8345660400
$\mathrm{dyn}(\varpi_5)$			150	17160	141605100
$\mathrm{dyn}(\varpi_6)$			6	648	1778400
$\mathrm{dyn}(\varpi_7)$				12	14700
$\mathrm{dyn}(\varpi_8)$					60

We want to compute the Dynkin index of $\mathfrak{g}$. The following theorem is used in [LS, proof of Proposition 2.6] and [KN, proof of Propposition 4.7]. We provide its proof here.

Theorem 5.2.4. [LS, proof of Proposition 2.6], [KN, proof of Propposition 4.7]. *Let $V(\varpi_i)$ be the irreducible representations with highest weight ϖ_i where the ϖ_i are fundamental weights of $\mathfrak{g}$. Then, $d_{\mathfrak{g}} = \gcd(\{\mathrm{dyn}(V(\varpi_i)) : 1 \leq i \leq l\})$.*

Proof. Since U is completely reducible, by Proposition 5.1.2, it suffices to show this in the case where U is irreducible, say $U = V(\lambda)$ where $\lambda \in \Lambda^+$. Thus $\lambda = \sum_{i=1}^{l} m_i \varpi_i$ where $m_1, m_2, \ldots, m_l \in \mathbb{N}$. If $m_i = 0$ for all i, $U = V(0) = \mathbb{K}$ is the trivial representation. Moreover, $m_{V(0)} = 0$, and there is nothing to prove. Hence, we can assume that $\sum_i m_i \geq 2$, since if we have $\sum_i m_i = 1$, λ is one of the fundamental weights, there is also nothing to prove.

Also, we know that

$$W := \bigotimes_i V(\varpi_i)^{\otimes m_i} \cong V(\lambda) \oplus \left(\bigoplus_{\mu \in \Lambda^+, \mu \prec \lambda} V_\mu^{\oplus n_\mu} \right). \tag{9}$$

for suitable n_μ in $\mathbb{N}$. We claim that $d_{\mathfrak{g}} \mid \operatorname{dyn}(U)$. We are going to prove the result by induction on the cardinality of $S_\lambda = \{\mu \in \Lambda^+ : \mu \prec \lambda\}$.

If $|S_\lambda| = 1$, then λ is a *minimal* weight (see [B:Lie, VIII, §7.3]), whence $\lambda = 0$ or λ is a fundamental weight. We can therefore assume that $|S_\lambda| \geq 1$ and that the claim is true for $V(\eta)$, $\eta \in \Lambda^+, |S_\eta| \leq |S_\lambda|$.

Consider the decomposition (9). By Weyl's complete reducibility theorem, we know that $\bigoplus_{\mu \in \Lambda^+, \mu \prec \lambda} V_\mu^{\oplus n_\mu}$ decomposes into $V(\mu_1) \oplus \cdots \oplus V(\mu_l)$ for some l. Moreover, one knows that $\mu_i \prec \lambda$, whence $|S_{\mu_i}| < |S_\lambda|$. It follows by induction that $d \mid \operatorname{dyn}(U)$.

□

Theorem 5.2.5. [KN, Prop. 4.7], [LS, Prop. 2.6]. *There exists a (non-unique) fundamental representation ρ_0 such that $d_{\mathfrak{g}} = \operatorname{dyn}(\rho_0)$. The Dynkin index of $\mathfrak{g}$ and the fundamental representation ρ_0 are shown in the following table (using the notation of* [B:Lie, V]*).*

Type of $\mathfrak{g}$	$d_{\mathfrak{g}}$	λ *s.t* $d_{\mathfrak{g}} = \operatorname{dyn}(V(\lambda))$
A_l $(l \geq 1)$	1	ϖ_1 *or* ϖ_k
B_l $(l \geq 3)$	2	ϖ_1
C_l $(l \geq 1)$	1	ϖ_1
D_4	2	ϖ_1, ϖ_3, ϖ_4
D_l $(l \geq 5)$	2	ϖ_1
E_6	6	ϖ_1 *or* ϖ_6
E_7	12	ϖ_7
E_8	60	ϖ_8
F_4	6	ϖ_4
G_2	2	ϖ_1

The published proof of the known fact is "unfortunately case-by-case" (as quoted from [KN], first line of proof). Then, the proof can be completed by calculating the indices using Theorem 5.2.2.

While we do not have a classification-free proof of this theorem, we can at least give a new proof with substantially fewer calculations. Our approach uses the following result empirically found by Adams and J. H. Conway and proved by Guillot in [Gui].

Definition 5.2.6. Let D be a connected Dynkin diagram. An *arm of length k* of D is a subdiagram with nodes $v_1, \dots, v_k$ such that

(i) v_i is connected to v_{i+1}, $1 \leq i \leq k-1$ by a signle bond, and

(ii) the only edges in D connecting to some node v_i, $1, \leq i \leq k$ are those in (i).

Theorem 5.2.7. J.H. Conway, [A], [Gui]. *Let $\mathfrak{g}$ be a finite-dimensional split simple Lie algebra over $\mathbb{K}$. Let $A_1 \cup \dots \cup A_s$ be a partition of the Dynkin diagram D of $\mathfrak{g}$ into arms of lengths $l_1, \dots, l_s$ respectively ($1 \leq s \geq 3$) as explained above. Also, let ρ_i ($1 \leq i \leq s$) be the fundamental representations with highest weight correspondind to the first node in A_i. Therefore, $\mathscr{R}(L) \simeq \mathbb{Z}[\rho_1, \dots, \bigwedge^{l_1} \rho_1, \dots, \rho_s, \dots, \bigwedge^{l_s} \rho_s]$.*

As a consequence of this theorem, we have the following corollary.

Corollary 5.2.8. *With the notation as the above theorem, the Dynkin index of L is the gcd of the ρ_i's, $1 \leq i \leq s$, as in Theorem 5.2.7.*

5.2.2 Simply connected Chevalley order

By Lemma 4.3.15 and Proposition 5.1.4, we have the following commutative diagram

$$\begin{tikzcd}
\mathscr{R}(\mathscr{G}) \arrow[r, "\tau_{\mathscr{G}}"] \arrow[d, "\psi"'] & \mathbb{Z} \arrow[d, "i"] \\
\mathscr{R}(\mathfrak{g}) \arrow[r, "\tau_{\mathfrak{g}}"'] & \mathbb{Z} \subseteq \mathbb{Q}
\end{tikzcd}$$

where i denotes the inclusion map. So, we see that $\tau_{\mathscr{G}} = \tau_{\mathfrak{g}} \circ \psi$.

Lemma 4.3.15 also tells us that ψ has a section in the category of abelian groups (in particular, surjective). Similar to the previous subsection, we have the following definition.

Definition 5.2.9. Let $\mathscr{G}$ be a simply-connected Chevalley order. The *Dynkin index of $\mathscr{G}$* is defined to be $\gcd\{m_V : [V] \in \mathscr{R}(\mathscr{G})\}$.

Proposition 5.2.10. $d_{\mathscr{G}} = d_{\mathfrak{g}}$.

Proof. By Lemma 5.1.4, $\mathrm{dyn}(V) = \mathrm{dyn}(V \otimes_{\mathbb{Z}} \mathbb{Q})$ for all $\mathscr{G}$-module V. The result follows from this and the fact that ψ is surjective (Lemma 4.3.15). $\square$

The representation theory of simply-connected Chevalley orders goes beyond the scope of this thesis. However, we can at least look at the case of admissible lattices.

Proposition 5.2.11. *Let V be an irreducible module for $\mathfrak{g}$ and let $\mathscr{V}$ be an admissible lattice in V. Then,* $\mathrm{dyn}(V) = \mathrm{dyn}(\mathscr{V})$.

Proof. Again, this simply follows from the fact that $\mathrm{dyn}(V) = \mathrm{dyn}(V \otimes_{\mathbb{Z}} \mathbb{Q})$. $\square$

Example 5.2.12. Let $\mathscr{G} = \mathfrak{sl}_2(\mathbb{Z})$. Let $\mathscr{U} = a\mathbb{Z}^2 \subseteq \mathbb{Z}^2$ be a $\mathscr{G}$-module with the natural action. Since $h^2 = \mathrm{Id}$, we see that $\mathrm{tr}(h^2) = 2$. Hence, $\mathrm{dyn}(\mathscr{U}) = 1$. So, it follows that $\mathrm{dyn}(\mathscr{G}) = 1 = \mathrm{dyn}(\mathfrak{sl}_2(\mathbb{Q}))$.

Let $\mathscr{V} = (a\mathbb{Z}, b\mathbb{Z}, c\mathbb{Z}) \subseteq \mathbb{Z}^3$ be a $\mathscr{G}$-module with the natural action. Since $h^2(m_1, m_2, m_3) = (4m_1, 0, 4m_2)$, $\mathrm{tr}(h^2) = 8$. Hence, $\mathrm{dyn}(\mathscr{V}) = 4$.

5.3 Application

Through case-by-case consideration, we will exhibit the fact that $d_{\mathfrak{g}} = \mathrm{dyn}(\rho)$ for some fundamental representation ρ, [KN, Prop. 4.7]. We will follow the notation of [B:Lie]. We will denote an arm of a Dynkin diagram by enumerating the vertices $v_1, \ldots, v_n$.

5.3.1 Type A_n

The result follows directly from Theorem 5.2.7.

5.3.2 Type B_n ($n \geq 3$)

We recall that one has a finite-dimensional vector space V with a nondegenerate symmetric bilinear form Ψ on V. Then the set of endomorphisms x of V satisfying $\Psi(xv, v') + \Psi(v, xv') = 0$ for all $v, v' \in V$ is a Lie subalgebra of $\mathfrak{sl}(V)$ which is semisimple for $\dim V \neq 2$. For $n \geq 3$, this Lie algebra is a simple Lie algebra of type B_n.

We will show that the Dynkin index of the standard representation $V = V(\varpi_1)$ divides the indices of others. We may partition the Dynkin diagram of B_n as follows: $\varpi_1 \cdots \varpi_{n-1}$, ϖ_n. It follows from Theorem 5.2.7 that the Dynkin index of V divides those of $V(\varpi_k)$ for $1 \leq k \leq n-1$. Next, we look at the spin representation.

Spin representation. The spin representation cannot be realized from the exterior powers of the standard module. So, our approach will not work here. Instead, we will simply compute the traces of the actions of a particular element of our Lie algebra $\mathfrak{g}$ on the standard module and on the spin module. We then compare them and use uniqueness up to a scalar of the nondegenerate symmetric bilinear form. First, let $\{e_0, e_1, \ldots, e_n, e_{-1}, \ldots, e_{-n}\}$ be a Witt basis of V with respect to Ψ. In this case, one assumes that $\Psi(e_0, e_0) = -2$, $\Psi(e_i, e_{-i}) = 1$ for $i = 1, \ldots, n$. Otherwise, the form Ψ evaluated on the basis vectors gives zero.

Consider $H_1 = E_{1,1} - E_{-1,-1}$, which is an element of a Cartan subalgebra of $\mathfrak{g}$, as considered in our reference. Now, let σ be the standard representation of $\mathfrak{g}$. From the reference, one has that the action of H_1 on the standard module is given by $\sigma(H_1)(e_1) = e_1$, $\sigma(H_1)(e_{-1}) = -e_{-1}$, and $\sigma(H_1)(e_j) = 0$ for all $j \neq \pm 1$. It follows that $\operatorname{tr}_V(\sigma(H_1)^2) = 2$.

Now, let ρ be the spin representation of $\mathfrak{g}$. We know that

$$\{e_{-i_1} \wedge \cdots \wedge e_{-i_k} \mid i_1 < \cdots < i_k\}$$

is a basis of the spin module of $\mathfrak{g}$ and that its dimension is 2^n. Now, from [B:Lie, VIII, §13.2 (IV)], we know that

$$\rho(H_1)(e_{-i_1} \wedge \cdots \wedge e_{-i_k}) = \begin{cases} -\frac{1}{2} e_{-i_1} \wedge \cdots \wedge e_{-i_k} & \text{if } 1 \in \{i_1, \ldots, i_k\}, \\ \frac{1}{2} e_{-i_1} \wedge \cdots \wedge e_{-i_k} & \text{if } 1 \notin \{i_1, \ldots, i_k\}. \end{cases}$$

It follows from this that $\operatorname{tr}(\rho(H_1))^2 = \frac{1}{4} \cdot 2^n = 2^{n-2}$. So $\operatorname{tr}(\rho(H_1)^2) = 2^{n-3} \operatorname{tr}_V(\sigma(H_1)^2)$. Hence, since nondegenerate symmetric invariant bilinear forms on L are unique up to a scalar, it follows that $\operatorname{tr}(\rho(x)\rho(y)) = 2^{n-3} \operatorname{tr}_V(\sigma(x)\sigma(y))$ for all $x, y \in L$. Therefore, $\operatorname{dyn}(V(\varpi_l)) = 2^{n-3} \operatorname{dyn}(V(\varpi_1))$.

5.3.3 Type C_n $(n \geq 2)$

Again, we will show that the Dynkin index of the natural representation $V = V(\varpi_1)$ divides that of others. We partition the Dynkin digram of type C_n into arms: $\varpi_1 \cdots \varpi_{n-1}$, ϖ_n. By Theorem 5.2.7, one sees that the Dynkin index of V divides $\operatorname{dyn}(V(\varpi_k))$ for $1 \leq k \leq n-1$. It is left to look at $V(\varpi_n)$.

We consider a vector space V of dimension $2n$, equipped with a nondegenerate symplectic form. Let $\{e_1, \cdots, e_n, e_{-1}, \cdots, e_{-n}\}$ be a symplectic basis of V

Let us look at endomorphisms X_+ and X_- of $\bigwedge V$ whose actions are: for $u \in V$, $X_- u = \sum_{i=1}^n (e_i \wedge e_{-i}) \wedge u$ and $X_+ u = \sum_{i=1}^n (e_i^* \wedge e_{-i}^*) \wedge u$.

Let H be the endomorphism of $\bigwedge V$ which acts by multiplying by $(n-r)$ on $\bigwedge^r V$ where $0 \leq r \leq n$. We denote E_r the subspace of primitive elements of V in $\bigwedge^r V$. So, $E_r = \operatorname{Ker} X_+ \cap \bigwedge^r V$. From the theory, the restriction of X_- to $\bigwedge^r V$ is injective. Moreover, we have:

$$\bigwedge^r V = E_r \oplus X_- \left(\bigwedge^{r-2} V \right).$$

Now, we claim that the map $X_- : \bigwedge^{r-2} V \longrightarrow \bigwedge^r V$ is a homomorphism of C_l-modules. This is true since X_- commutes with the action $\bigwedge \phi(x)$ for $x \in L$ where we have $\bigwedge \phi(x) = 1 \oplus \phi(x) \oplus \phi^2(x) \oplus \ldots$

Hence, we can write

$$\bigwedge^r V = E_r \oplus \bigwedge^{r-2} V.$$

Moreover, we know from our reference that the restriction of $\bigwedge^r \phi$ to E_r is the fundamental representation of highest weight ϖ_r. Hence,

$$\operatorname{dyn}(V(\varpi_r)) = \operatorname{dyn}\left(\bigwedge^r V\right) - \operatorname{dyn}\left(\bigwedge^{r-2} V\right) = \left(\binom{2l-2}{r-1} - \binom{2l-2}{r-3}\right) \operatorname{dyn}(V(\varpi_1)).$$

5.3.4 Type D_n $(n \geq 4)$

Here, we use [B:Lie, §13.4] as our reference. The material in this section is largely similar to what we did for type B_n. As in type B_n, V is a vector space of dimension 2n and Ψ is a nondegenerate symmetric bilinear form on $\mathfrak{g}$. When $n \geq 4$, the corresponding Lie algebra of skew-symmetric endomorphisms is of type D_n.

As before, we will show that the Dynkin index of the standard representation $V = V(\varpi_1)$ divides the indices of other fundamental representations. We partition the Dynkin digram of type D_n into three arms: $\varpi_1 \cdots \varpi_{n-2}$, ϖ_{n-1}, ϖ_n. It follows from Theorem 5.2.7 that the Dynkin index of V divides $\operatorname{dyn}(V(\varpi_k))$ for $1 \leq k \leq n-2$. It is left to look at the spin representations.

Spin representations. Let $\{e_1, \dots, e_n, e_{-1}, \dots, e_{-n}\}$ be a Witt basis of V. More precisely, one has that $\{e_{-1}, \dots, e_{-n}\}$ is the dual basis of $\{e_1, \dots, e_n\}$ relative to the bilinear form Ψ. Using the notation from our reference, consider an element H_1 of a Cartan subalgebra H of L. Let σ be the standard representation of $\mathfrak{g}$, let ρ_+ and ρ_- be the spin representations of L as in the reference.

Now, one knows that $\sigma(H_1)(e_1) = e_1$, $\sigma(H_1)(e_{-1}) = -e_{-1}$ and $\sigma(H_1)(e_j) = 0$ for all $j \neq \pm 1$. It follows that $\operatorname{tr}_V(\sigma(H_1)^2) = 2$.

Let us now consider the representation ρ^+. From the theory, we know that a basis for ρ^+ is $\{e_{-i_1}, \dots, e_{-i_k} \mid i_1 < \cdots < i_k \text{ and } k \text{ is even}\}$. Moreover, from [B:Lie, VIII, §13.4], one knows that the action is given by

$$\rho^+(H_1)(e_{-i_1} \wedge \cdots \wedge e_{-i_k}) = \begin{cases} -\frac{1}{2} e_{-i_1} \wedge \cdots \wedge e_{-i_k} & \text{if } 1 \in \{i_1, \dots, i_k\}, \\ \frac{1}{2} e_{-i_1} \wedge \cdots \wedge e_{-i_k} & \text{if } 1 \notin \{i_1, \dots, i_k\}. \end{cases}$$

Thus, $\operatorname{tr}(\rho^+(H_1)^2) = \frac{1}{4} \cdot 2^{n-1} = 2^{n-3}$. So, $\operatorname{tr}(\rho^+(H_1)^2) = 2^{n-4} \operatorname{tr}_V(\sigma(H_1)^2)$. Therefore, using the same argument as for type B_n, one has $\operatorname{dyn}(\rho^+) = 2^{n-4} \operatorname{dyn}(\sigma)$. Similarly, $\operatorname{dyn}(\rho^-) = 2^{n-4} \operatorname{dyn}(\sigma)$. Therefore, $\operatorname{dyn}(V(\varpi_{n-1})) = \operatorname{dyn}(V(\varpi_n)) = 2^{n-4} \operatorname{dyn}(V(\varpi_1))$.

Remark 5.3.1. In the special case where $n = 4$, we may use Proposition 5.2.1 to see that $\operatorname{dyn}(V(\varpi_1)) = \operatorname{dyn}(V(\varpi_3)) = \operatorname{dyn}(V(\varpi_4))$. Thus, the result follows.

5.3.5 Type G_2

For type G_2, we need the decomposition of $V(\varpi_1) \wedge V(\varpi_1)$ into irreducible components. The set of weights of $V(\varpi_1)$, denoted $\mathrm{P}(V(\varpi_1))$, is

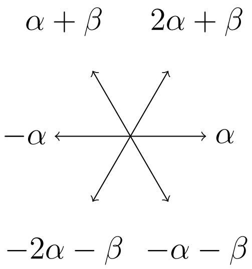
$$\mathrm{P}(V(\varpi_1)) = \{0, 6 \text{ short roots of } G_2\}.$$

Also, it is known that $V(\varpi_2) \cong G_2$ as adjoint modules.

Let us write $(V(\varpi_1) = \mathbb{K}e_0 \oplus \left(\bigoplus_{i=1}^{6} \mathbb{K}e_i\right)$. Hence,

$$V(\varpi_1) \wedge V(\varpi_1) = \bigoplus_{0 \leq i \leq j \leq 6} \mathbb{K}(e_i \wedge e_j)$$

Then $e_i \wedge e_j$ has weight $\mu_1 + \mu_j$ if each e_i has weight μ_i. The set of weights of $V(\varpi_1)$ consists of 0 and the 6 short roots of G_2. It can be visualized as follows



The sum of any two "adjacent" roots gives a long root. By *adjacent*, we mean two roots that form an angle of 60°. Also, the sum of the roots that have another root in between gives that same root in between. And evidently, the sum of opposite roots gives 0.

Then $\mathrm{P}(V(\varpi_1) \wedge V(\varpi_1))$ contains 6 long roots from the sum of adjacent roots, contains 0 with multiplicity 3 from the sum of opposite roots. Moreover, each of the short roots appear twice, once from the sum of itself and 0, another time from the sum of the two roots that are adjacent to it. Those roots (with multiplicities) are weights of $V(\varpi_1) \wedge V(\varpi_1)$. We summarize the result in the following table

Roots of G_2	Multiplicity	Roots of G_2	Multiplicity
0	3	β	1
α	2	$-\beta$	1
$-\alpha$	2	$3\alpha + \beta$	1
$\alpha + \beta$	2	$-3\alpha - \beta$	1
$-\alpha - \beta$	2	$3\alpha + 2\beta$	1
$2\alpha + \beta$	2	$-3\alpha - 2\beta$	1
$-2\alpha - \beta$	2		

So $\bigwedge^2 V(\varpi_1) = V(\varpi_1) \oplus V(\varpi_2)$, $\mathrm{dyn}(V(\varpi_2)) = \mathrm{dyn}\left(\bigwedge^2 V(\varpi_1)\right) - \mathrm{dyn}(V(\varpi_1))$. We see that the right side is a multiple of $\mathrm{dyn}(V(\varpi_1))$.

5.3.6 Type E_6

We show that the Dynkin index of $V = V(\varpi_6)$ divides that of others. By Proposition 5.2.1, we see that it is enough to look at the arms $\varpi_6\varpi_5\varpi_4$ and ϖ_2. By Theorem 5.2.7, we see that the Dynkin index of V divides $m_{V(\varpi_k)}$ for $k \in \{1,3,4,5,6\}$. It is left to look at the case where $k = 2$.

By comparing with the tables, we see that what we've gotten is indeed correct. It is left for us to look at the index of the adjoint representation $V(\varpi_2)$. By direct computation using program LiE, we obtain $\bigwedge^2 V(\varpi_2) = V(\varpi_4) \oplus V(\varpi_2)$. It follows that $\operatorname{dyn}\left(\bigwedge^2 V(\varpi_2)\right) = \operatorname{dyn}(\varpi_4) + \operatorname{dyn}(\varpi_2)$.

We know that $\dim V(\varpi_2) = 78$. Substituting the value of $\operatorname{dyn}(\varpi_4)$ (in term of $\operatorname{dyn}(\varpi_1)$) and using the formula (2.2.12), $\binom{78-2}{2-1}\operatorname{dyn}(\varpi_2) = 300 \cdot \operatorname{dyn}(\varpi_1) + \operatorname{dyn}(\varpi_2)$. Therefore, $\operatorname{dyn}(\varpi_2) = \frac{300}{75} \cdot \operatorname{dyn}(\varpi_1) = 4 \cdot \operatorname{dyn}(\varpi_1)$. The result follows.

5.4 Observation

In this section, we observe a result which is, in spirit, quite similar to Theorem 5.2.7.

Observation 5.4.1. *Consider a Dynkin diagram D (not of type D_n, $n \geq 6$, and E_8). Let $v_0 \cdots v_s$ be an arm in D. We put* dyn_i *for the Dynkin index of the fundamental module corresponding to the vertex v_i. Then* dyn_0 *divides* dyn_i *for all $1 \leq i \leq s$.*

Numerical evidence For type A_n, from the table of Dynkin indices, we see that $\operatorname{dyn}(V(\varpi_1)) = \binom{n-1}{1-1} = 1$. Hence, for type A_n, Observation 5.4.1 is clear. For type B_n, we see that $\operatorname{dyn}(V(\varpi_1)) = 2\binom{2n-1}{i-1} = 2$. We see that, for $1 \leq i \leq n-1$, $\operatorname{dyn}(V(\varpi_i))$ is divisible by 2. So, Observation 5.4.1 is clear from this. For type C_n, we see that $\operatorname{dyn}(V(\varpi_1)) = \binom{2n-2}{1-1} - \binom{2n-2}{1-3} = 1$. Hence, Observation 5.4.1 follows. One can check that it holds for D_4, D_5, D_6 and all exceptional types except E_8 using the tables.

Remark 5.4.2. One can check that the observation is not true for D_n, $n \geq 7$, and E_8. For $\Delta = E_8$, $\operatorname{dyn}(\varpi_1) = 1500$ does not divide $\operatorname{dyn}(\varpi_4) = 8345660400$. It is possible to prove that the observation is not true for D_n, $n \geq 7$, and we leave it to the reader. One way to do it is to consider $2\binom{2n-2}{n-3}$ and count the number of 2 in the expression.

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