uOttawa

L'Université canadienne
Canada's university

Guennadi Kondratiev
AUTEUR DE LA THÈSE / AUTHOR OF THESIS

Ph.D. (Mathematics)
GRADE / DEGREE

Department of Mathematics and Statistics
FACULTÉ, ÉCOLE, DÉPARTEMENT / FACULTY, SCHOOL, DEPARTMENT

Duality, Manifolds, Structures
TITRE DE LA THÈSE / TITLE OF THESIS

Philip Scott
DIRECTEUR (DIRECTRICE) DE LA THÈSE / THESIS SUPERVISOR

Richard Blute
CO-DIRECTEUR (CO-DIRECTRICE) DE LA THÈSE / THESIS CO-SUPERVISOR

EXAMINATEURS (EXAMINATRICES) DE LA THÈSE / THESIS EXAMINERS

Inna Bumagin	Paul-Eugène Parent
Pieter Hofstra	James Stasheff

Gary W. Slater
Le Doyen de la Faculté des études supérieures et postdoctorales / Dean of the Faculty of Graduate and Postdoctoral Studies

DUALITY, MANIFOLDS, STRUCTURES

G. V. Kondratiev

A Thesis submitted to the
Faculty of Graduate and Postdoctoral Studies
University of Ottawa
in partial fulfillment of the requirements for the
Ph.D. degree in the

Ottawa-Carleton Institute for Graduate Studies and Research in Mathematics and Statistics

Library and
Archives Canada

Published Heritage
Branch

395 Wellington Street
Ottawa ON K1A 0N4
Canada

Bibliothèque et
Archives Canada

Direction du
Patrimoine de l'édition

395, rue Wellington
Ottawa ON K1A 0N4
Canada

Your file Votre référence
ISBN: 978-0-494-49370-0
Our file Notre référence
ISBN: 978-0-494-49370-0

NOTICE:
The author has granted a non-exclusive license allowing Library and Archives Canada to reproduce, publish, archive, preserve, conserve, communicate to the public by telecommunication or on the Internet, loan, distribute and sell theses worldwide, for commercial or non-commercial purposes, in microform, paper, electronic and/or any other formats.

The author retains copyright ownership and moral rights in this thesis. Neither the thesis nor substantial extracts from it may be printed or otherwise reproduced without the author's permission.

AVIS:
L'auteur a accordé une licence non exclusive permettant à la Bibliothèque et Archives Canada de reproduire, publier, archiver, sauvegarder, conserver, transmettre au public par télécommunication ou par l'Internet, prêter, distribuer et vendre des thèses partout dans le monde, à des fins commerciales ou autres, sur support microforme, papier, électronique et/ou autres formats.

L'auteur conserve la propriété du droit d'auteur et des droits moraux qui protège cette thèse. Ni la thèse ni des extraits substantiels de celle-ci ne doivent être imprimés ou autrement reproduits sans son autorisation.

In compliance with the Canadian Privacy Act some supporting forms may have been removed from this thesis.

While these forms may be included in the document page count, their removal does not represent any loss of content from the thesis.

Conformément à la loi canadienne sur la protection de la vie privée, quelques formulaires secondaires ont été enlevés de cette thèse.

Bien que ces formulaires aient inclus dans la pagination, il n'y aura aucun contenu manquant.

Canada

Abstract

Concrete duality is a key tool in the development of modern algebraic geometry. It gives general directions, vistas of the subject, as well as concrete machinery (local and global techniques). The sort of higher order duality proposed in this thesis reflects still more of the close relationship of the associated algebraic and geometric categories. By higher order here, we mean duality with respect to n-categorical structure.

It is significant that all famous dualities (by Stone, Pontryagin, Gelfand-Naimark, Grothendieck) can be obtained in a similar categorical fashion. This was proved by H.-E. Porst and W. Tholen [P-Th]. Once the structure of concrete duality was clarified, people were encouraged to discover new dualities (e.g., see [Luk]). What is the practical sense of that? On a basic level, duality gives concrete "functional representations" of objects via "functions" on their duals. More precisely, we have a duality between geometric and algebraic categories, so that each object is presented by a functional space on its dual. For example, in this sense, the solution spaces of algebraic or differential equations become visible . On a more abstract level it brings sense and often ease of calculation to categorical constructions via their duals (for example, complicated algebraic colimits often become clear via geometric limits).

In this work a concept of higher order concrete duality is introduced and a criterion of existence of natural such dualities is given. One of the aims was to strictly prove that the extension of Gelfand-Naimark duality over 2-cells (homotopy classes of homotopies) is a 2-duality in a proper categorical sense. For that, a notion of infinity category is introduced in the first chapter and developed to cover such usual categorical topics as representability, the Yoneda lemma, and adjunctions for this infinite-dimensional environment. We include a discussion of the distinction between weak (up to equivalence) versus strict (up to isomorphism).

For pointed objects of an infinity category, homotopy-like groups are introduced (which coincide in special cases with the usual ones). The advantage of these objects is that they are defined more categorically (i.e. internally) and so have more chance to be preserved by functors. For example, the usual homotopy groups are very rarely preserved by functors.

In the second chapter a concept of category of manifolds over a Grothendieck site is introduced. It is shown that usual manifolds, fibre bundles, and foliations fit into this scheme. The construction of such categories makes use of stacks.

The third and final chapter contains examples of concrete duality of first and second order. Some of them are new at least in their categorical formulation (Vinogradov duality, duality for differential equations, Gelfand-Naimark 2-duality, Pontryagin-Lukacs duality). They provide a framework in which each concrete theory has a duality which we successfully develop.

The main results of this work are presented in two papers in the electronic (non refereed) website `xxx.lanl.gov`.

Content

0. Introduction

> *"One should jump over not just*
>
> *theorems but whole theories as well"*
>
> V.I. Arnold

This work is an analysis of concepts being used in modern geometry. In particular, it considers from a categorical viewpoint manifolds and associated structures. The basic categorical construction we use is *duality*. Our viewpoint is that manifolds and structures provide the ontology, while duality is the *vita vitalis* of geometric theories.

0.1. Why duality is effective.

Duality is one of the fundamental recurring ideas in all of mathematics, and category theory provides the appropriate framework for defining and analyzing the idea that "opposite structures can reflect one another". Some of the most famous theorems in mathematics are duality theorems. We are thinking in particular of Pontryagin duality, Gelfand -Naimark duality and Stone duality.

The computational power of such dualities increases since one can choose that side which works in the simplest and most effective manner in a given situation. Most mathematicians, when they develop practical techniques, use some kind of duality (such as distributions in functional analysis or flows in differential topology) which simplifies the main ideas and formulations. It would probably not be a sufficient reason for using duality if everything was exactly mirrored. Some properties are not preserved under categorical equivalence. This gives rise to an additional dimension for those new constructions which are not reducible to either of the opposites. Notions such as schemes arise in this way. Historically, the abstract concept of duality was introduced much later than numerous (famous) concrete examples. A deep categorical analysis of first order duality was given in [P-Th].

0.2. Not everything is preserved under duality. Bifurcation theory.

As an example, this phenomenon is well-known in the qualitative theory of differential equations when small changes of parameter cause catastrophes in the solution space (under the general duality of differential equations to their solution spaces). This is the subject of bifurcation theory. The same phenomenon holds for algebraic equations and for any type of equations and their deformations in the previous sense.

0.3. Development of Modern Geometry.

Duality plays a central role in the principal steps of the development of Modern Geometry. It is now a standard tool to talk about spaces which are unknown but which are well representable by their dual objects. All the development of modern algebraic geometry can be regarded as a sequence of extensions of algebraic duals, which can be seen from the following diagram:

$$\begin{tikzcd}
\mathbf{AlgVar}^{op} \arrow[r, "\sim"] \arrow[d, hook] & \mathbf{FinGenComAlg} \arrow[d, hook] & \mathbf{StieSubBn}^{op} \arrow[r, hook] \arrow[d, "\sim"'] & \mathbf{SolSpDiffEq}^{op} \arrow[d, "\sim"'] \\
\mathbf{AffSchemes}^{op} \arrow[r, "\sim"] & \mathbf{ComAlg} \arrow[r, hook] & \mathbf{AntiComALg} \arrow[r, hook] \arrow[d, hook] & \mathbf{DiffAntiComAlg} \arrow[d, hook] \\
\mathbf{Diff}^{op} \arrow[r, "\sim"] \arrow[u, hook] \arrow[d, hook] & \mathbf{FinGenSmoothComAlg} \arrow[u, hook] \arrow[d, hook] & \mathbf{NonComALg} \arrow[r, hook] & \mathbf{DiffNonComAlg} \arrow[d, "\sim"'] \\
\mathbf{SmAffSchemes}^{op} \arrow[r, "\sim"] & \mathbf{SmComAlg} & \mathbf{NonComSp}^{op} \arrow[r, hook] \arrow[u, "\sim"] & \mathbf{SolSpNCDiffEq}^{op}
\end{tikzcd}$$

It is still a compact diagram, some steps are omitted, some extensions are not unique (e.g., for algebraic geometry, it is better to regard commutative algebras as anticommutative ones concentrated in degree 0. But for algebraic topology it is natural to regard graded commutative algebras as graded anticommutative ones with degrees of all elements doubled). One of the key ideas of this thesis is that the ∞-category setting allows us to expand this diagram in a new (homotopical) dimension. So that (monoidal) ∞-categories give an appropriate framework for

Homotopical Algebraic Geometry.

0.4. Low-dimensional and ∞-dimensional Approaches to Homotopy Theory.

Higher dimensional functors preserve "homotopy" invariants but not in a canonical way, i.e. they usually do not preserve π_*, H_*, H^*, etc. This is because these "homotopy" invariants are not formulated **internally** in a category. For example, regard the classifying space functor B : **wTopGrp** $\to$ **Top**. In **wTopGrp** (category of weak topological groups) there are two (noncomparable in general) 2-categorical structures: when 2-cells are conjugations and when they are homotopy classes of homotopies; [the last structure is weaker if we restrict ourselves to a subcategory of path connected weak topological groups]. We note:

Proposition 0.4.1. *The classifying space functor* B : **wTopGrp** $\to$ **Top** *is*

- *a 2-functor with respect to conjugations in* **wTopGrp**,
- *a 2-functor and 2-equivalence (not 1-equivalence) with respect to homotopy classes of homotopies in* **wTopGrp**. *Its quasiinverse is the loop space functor* Ω : **Top** $\to$ **wTopGrp**. □

One would expect that there are many relations between conjugation invariants and homotopy invariants for **wTopGrp** and **Top**. Indeed, there are some such, but they are not straightforward. For example, $H^*(BG)$ is a conjugation-invariant commutative anticommutative algebra. We would expect that it is a subalgebra (or, maybe a quotient algebra) of **AdInvPol**($\mathfrak{g}$) (the algebra of polynomials on the Lie algebra $\mathfrak{g}$ invariant under conjugations) but this is not true in general, although **AdInvPol**($\mathfrak{g}$) is isomorphic to $H^*(BG)$ for compact Lie groups G (Chern-Weil homomorphism). The relation between $H^*(G)$ and $H^*(BG)$ is rather complicated: it is given by the Eilenberg-Moore spectral sequence $H^*(G) \otimes H^*(BG) \to 0$. Why does such a nice equivalence B : **wTopGrp** $\xrightarrow{\sim}$ **Top** : Ω give such complicated relations between homotopy invariants? Because these homotopy invariants are not defined internally.

The typical definitions of homotopy invariants, such as the functor $\pi_n(X) = [S^n, X]$, are not invariant under 2(and higher order)-functors, because the n-spheres S^n are not traditionally determined categorically. One of the goals of this thesis is to introduce a new notion of homotopy group, which are invariant under higher-order categorical equivalences Our reinterpretation of homotopy is as follows:

Definition 0.4.1.

- For an ∞-category $\mathbf{C}$, $I \in Ob\,\mathbf{C}$, and a point $x : I \to X$ (**formal**) **homotopy groups** of X are defined as follows $\tilde{\pi}_n^I(X, x) := \mathbf{Aut}(e^{n-1}x)/\sim$ (where e^{n-1} is $n-1$ times application of the identity operation e).
- When functors $\mathbf{Aut}(e^{n-1}(-))/\sim$ are **representable** the representing objects $\tilde{S}^n$ are called (**formal**) **spheres** (in this case we have $\tilde{\pi}_n^I(X, x) := (\mathbf{Aut}(e^{n-1}(x))/\sim) \xrightarrow{\sim} [\tilde{S}^n, X]$). $\square$

For ∞-**Top** when $I = \mathbf{1}$ there is a homomorphism of the usual homotopy groups into our formal ones $\pi_n(X, x) \to \tilde{\pi}_n^I(X, x)$ (induced by the quotient map $I^n/(I^{n-1} \times 0) \cup (I^{n-1} \times 1) \twoheadrightarrow S^n$). For a category ∞-$\mathbf{TopMan}_b$ of topological manifolds with boundary and homotopies relative to the boundary, formal homotopy groups coincide with the usual ones $\tilde{\pi}_n^I = \pi_n$ when $I = \mathbf{1}$. For ∞-$\mathbf{Top}_*$ (pointed spaces and maps) $\tilde{\pi}_n^{\mathbf{1}}(X, x) = [\mathbf{1}, X]$ are trivial for all n although $[S^n, X]$ gives the usual homotopy groups.

Both functors B and Ω preserve the homotopy type of $\mathbf{1}$. So, $\tilde{\pi}_n^{\mathbf{1}}(G) = \tilde{\pi}_n^{\mathbf{1}}(BG)$. But these groups are trivial and they give no information (if we change $\mathbf{1}$ to a more complicated object $I \in Ob\,\mathbf{Top}$ then the information can be very nontrivial).

Proposition 0.4.2. *If $F : \mathcal{A} \to \mathcal{B}$ is an ∞-equivalence between* **full** *topological subcategories of ∞-$\mathbf{TopMan}_b$ such that $\mathbf{1} \sim F(\mathbf{1})$ then F preserves the usual homotopy groups.* $\square$

0.5. Concrete Duality.

The underlying philosophy of our theory of concrete duality is that the world is nonlinear and opposites converge rather than diverge.

Definition 0.5.1. Two n-categories $\mathcal{A}$ and $\mathcal{B}$ are called **concretely dual** if there exists a "schizophrenic" object D living in both of theses categories such that hom-functors $\mathcal{A}(-,D)$: $\mathcal{A} \to n$-$\mathbf{Cat}$ and $\mathcal{B}(-,D) : \mathcal{B} \to n$-$\mathbf{Cat}$ factor through the other category, i.e.

$$\begin{tikzcd}
\mathcal{A} \arrow[r, "F"] \arrow[dr, "\mathcal{A}(-{,}D)"'] & \mathcal{B} \arrow[d, "V"] \\
& n\text{-}\mathbf{Cat}
\end{tikzcd}$$

and

$$\begin{tikzcd}
\mathcal{B} \arrow[r, "G"] \arrow[dr, "\mathcal{B}(-{,}D)"'] & \mathcal{A} \arrow[d, "U"] \\
& n\text{-}\mathbf{Cat}
\end{tikzcd}$$

where F, G are equivalences quasiinverse to each other. $\square$

The higher order the duality is, the more (homotopy) invariants are preserved.

If $\mathcal{A} \hookrightarrow n$-$\mathbf{Top}$ then the forgetful functor $U : \mathcal{A} \to n$-$\mathbf{Cat}$ is usually the composite of inclusion and the n-groupoid functor $\mathcal{A} \hookrightarrow n\text{-}\mathbf{Top} \xrightarrow{n\text{-}\mathbf{Top}(\mathbf{1},-)} n\text{-}\mathbf{Cat}$ [by Grothendieck's hypothesis ∞-$\mathbf{Top}(\mathbf{1},-) : \infty$-$\mathbf{Top} \to \infty$-$\mathbf{Cat}$ is an equivalence with its image].

The above factorization (lifting) of hom-functors is frequently **initial.** For first order categories it was proven by Porst and Tholen [P-Th] that initial means maximal and any other concrete duality factors through the initial (natural) one; for higher order categories, the analogous statement has not been proven yet. We hope that the structures introduced here will be useful in extending this result to the higher order case.

Proposition 0.5.1.

- *Every (weak) duality (adjunction)* $\mathcal{A} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{B}$ *is concrete (over $\mathcal{C}$) if there are* **representable** *forgetful functors $U : \mathcal{A} \to \mathcal{C}$ and $V : \mathcal{B} \to \mathcal{C}$. The dualizing object D is both FI and GJ in a sense to be made precise, where I, J are representing objects for U, V.*
- *If $\mathcal{A}$ and $\mathcal{B}$ have representable forgetful functors over $\mathcal{C}$ and a dualizing object D such that the corresponding hom-functors $\mathcal{A}(-,D)$, $\mathcal{B}(-,D)$ satisfy the* **initial lifting condition** *(essen-*

tially, the arrow $f^n : VX \to \mathcal{A}(Y,D)$ *is a* $\mathcal{B}$*-arrow iff the composite* $VX \xrightarrow{f^n} \mathcal{A}(Y,D) \xrightarrow{ev_{x^n}} \mathcal{A}(I,D)$ *is a* $\mathcal{B}$*-arrow* $\forall x^n : I \to Y$*, and similarly for* $\mathcal{B}(-,D)$*) then there exists a concrete dual adjunction between* $\mathcal{A}$ *and* $\mathcal{B}$ *which is natural and strict.*

- *Concrete natural duality is a strict adjunction. [Higher order duality need not be an adjunction at all]* □

Point 2 of the above proposition is a generalization (for n-categories) of the Porst-Tholen theorem about concrete duality for first order categories.

The main and most interesting interplay for duality is between algebra and geometry. Certain complicated colimits in algebraic categories are often easily viewed via duality as geometric limits (e.g. the notion of tensor product of algebras is more understandable via the notion of product of manifolds).

Examples of **well-known dualities** are those between algebraic varieties and finitely-generated commutative algebras, between affine schemes and commutative rings (Grothendieck), compact abelian groups and abelian groups (Pontryagin), Boolean algebras and Boolean spaces (Stone), commutative C^*-algebras and compact Hausdorff spaces (Gelfand-Naimark), and others.

In my thesis several new examples of concrete duality are introduced. These include duality for differential equations (introducing anticommutative geometry of solution spaces), Vinogradov duality (formalizing the well-known duality between modules of linear differential operators and jet modules of sections), Gelfand-Naimark 2-duality (extending the usual one to homotopy classes of homotopies), Pontryagin-Lukacs duality (Lukacs' extension of Pontryagin duality to locally precompact abelian groups).

0.6. Manifolds.

Manifolds give an example of geometric objects organized similarly at each neighbourhood without any singularities. They are essentially **unions of simple pieces** compatible on intersections. Respectively, **dually**, they are **limits** of certain simple "algebras" [such duality theorems should be formulated explicitly]. If we study spaces via their dual category, we usually do not come to manifolds (singularities always arise). Nevertheless, manifolds are usually regular dense pieces of such spaces and singularities are often intersections of them.

Since different kinds of objects (ordinary manifolds, locally trivial fibre bundles and foliations) turn out to be manifold-like structures, an attempt to provide a universal framework and to define universal properties of them was undertaken in my thesis. For why do we need to redo the same properties in different areas of mathematics if they can be studied in a unifying way?

Definition 0.6.1. For a given fibration over a Grothendieck site $$\begin{tikzcd}\mathbf{E} \arrow[d, "p"'] \\ (\mathbf{B},\tau)\end{tikzcd}$$ a (localizable) functor $$\begin{tikzcd}\mathbf{E\text{-}Man} \arrow[d, "F"'] \\ (\mathbf{B},\tau)\end{tikzcd}$$ (from a category bigger than $\mathbf{E}$) is called a **category of manifolds over B of type E** if objects and morphisms of F locally look like (in a sense to be formulated later) objects and morphisms of p. ☐

Remark. By the localization property, a maximal such F is unique up to equivalence. Existence follows from the construction (see further). ☐

For example, one can take $\mathbf{B} = \mathbf{Top}$, τ to be all open coverings, $\mathbf{E}$ to be all Euclidean spaces with all smooth maps between them, p is forgetful, then $\mathbf{E\text{-}Man} = \mathbf{Diff}$, F is forgetful.

Construction of (a nonmaximal) **E-Man**

1. Factor a given functor Φ through a free fibration $$\begin{tikzcd}\mathbf{E} \arrow[r] \arrow[dr, "\Phi"'] & 1/\Phi \arrow[d, "dom"] \\ & (\mathbf{B},\tau)\end{tikzcd}$$ (objects of $1/\Phi$ are like charts on objects of $\mathbf{B}$ with values in $\Phi(\mathbf{E})$, but there are more such charts than necessary).

2. Make a stack $\begin{tikzcd}\tilde{\mathbf{E}} \arrow[d] & 1/\Phi \arrow[l, hook'] \arrow[dl, "dom"] \\ \mathbf{B} & \end{tikzcd}$ from the free fibration (link together compatible charts into atlases, but there will be still more objects there than proper manifolds).
3. Take a full subcategory $\begin{tikzcd}\mathbf{E}\text{-}\mathbf{Man} \arrow[r, hook] \arrow[d] & \tilde{\mathbf{E}} \arrow[dl] \\ \mathbf{B} & \end{tikzcd}$ consisting of objects $(B, \mathcal{E})$ which locally look like charts of a certain class $\mathcal{M}$ (i.e. there exists a sieve $s \in \mathbf{B}(-, B)$ such that $\forall f \in s$ $\mathbf{Cart}_f(\mathcal{E}) \in \mathcal{M}$) [for usual manifolds $\mathcal{M}$ is a class of topological embeddings of open subspaces].

□

This work has three chapters. In the first chapter, an elementary theory of ∞-categories is introduced (including representability, the Yoneda lemma and adjunction) with applications to concrete duality. There are many theories of infinity categories in the literature. We especially cite the work of J. Baez and his coauthors [Baez1,Baez2], T. Leinster [Lei], E. Cheng and A. Lauda [C-L], and work of Batanin [Bat]. This thesis is not primarily a study of infinity categories; rather, we find our version of infinity category to be an appropriate framework for higher-order concrete duality. The definition we consider here is appropriate for the concrete applications we have in mind. The main result here is a generalization (to ∞-categories) of a theorem by Porst and Tholen [P-Th] about the existence of natural strict concrete duality under initial lifting conditions.

In the second chapter, notions of (almost) structures and manifolds are analyzed and a construction of the category of manifolds over a Grothendieck site is given via stacks. Such manifolds cover all the usual manifolds in mathematics, fiber bundles and foliations. Another result of this chapter is an enrichment construction with generalized elements in a presheaf category over a category endowed with binary products and a forgetful functor preserving them. That process is an abstraction of what is usually meant by continuous families of continuous functions or smooth families of smooth functions in appropriate categories; i.e. it allows us to see hom-sets as (almost) objects and transfer to them appropriate techniques (like tangent functors).

The third chapter is about applications and examples of these newly introduced concepts. New examples of first and second order concrete duality are presented (such as duality for differential equations, Vinogradov duality, Gelfand-Naimark 2-duality).

Throughout the work we take as a convention that a displayed diagram asserts its own commutativity if the opposite is not stated explicitly.

Acknowledgment. The author expresses gratitude for support from his supervisors Prof. P. Scott and Prof. R. Blute, and also Profs. W. Tholen, W. Rossman, P. Hofstra and Doctors M. Weber and J. Egger.

1. ∞-categories

In this chapter a workable notion of (strict) ∞-category (over a category of big sets **SET**) is introduced. Our categories are slightly stricter than usual ones [C-L, Lei] by the requirement that $\sim$ (see definition 1.1.2) is an equivalence relation (which however automatically holds for strict n-categories). A version of weak ∞-categories (without a coherence principle) is defined as well. We do not focus on coherence issues in the weak case, as it is only introduced as a stepping-stone towards ∞-category.

1.1. Categories, functors, natural transformations, modifications.

Definition 1.1.1.

- **∞-precategory** is a (big) set L endowed with

  (1) a grading $L = \coprod\limits_{n\geq 0} L^n$

  (2) unary operations $d, c : \coprod\limits_{n\geq 1} L^n \to \coprod\limits_{n\geq 0} L^n$, $deg(d) = deg(c) = -1$, $dc = d^2$, $cd = c^2$

  (3) a unary operation $e : \coprod\limits_{n\geq 0} L^n \to \coprod\limits_{n\geq 0} L^n$, $deg(e) = 1$, $de = 1$, $ce = 1$

  (4) partial binary operations $\circ_k$, $k = 1, 2, ...$, of degree 0. $f \circ_k g$ is determined iff $d^k f = c^k g$

  such that each **hom-set** $L(a, a') := \{f \in L \mid \exists\, k \in \mathbb{N}\ d^k f = a,\ c^k f = a'\}$, $deg(a) = deg(a')$, inherits all properties (1)-(4).

- $\forall\, a, a', a'' \in L^m$ there are maps $\mu_{a,a',a''} : \coprod\limits_{n\geq 0} L^n(a', a'') \times L^n(a, a') \to L(a, a'')$ such that if the bottom composite is determined then

$$\begin{tikzcd}
\coprod\limits_{n\geq 0} L^n(a',a'') \times L^n(a,a') \arrow[r, "\mu_{a,a',a''}"] & L(a,a'') \\
L^n(a',a'') \times L^n(a,a') \arrow[u, "i\times i"] \arrow[r, "\circ_{n+1}"'] & L^n(a,a'') \arrow[u, "i"']
\end{tikzcd}$$

  $\mu_{a,a',a''}$ are called **horizontal composites** on level $deg(a)$; all composites inside of $L(a, a')$ are **vertical**. □

Remarks.

- Our definition of ∞-**precategory** coincides with what Penon calls a *magma*; essentially it is a reflexive globular set with all possible binary composites [Lei].
- If $\alpha^n, \beta^n \in L^n, n > 0$, such that $d\alpha^n \neq d\beta^n$ or $c\alpha^n \neq c\beta^n$, then $L(\alpha^n, \beta^n) = \emptyset$ (because of $d^2 = dc$, $c^2 = cd$). So, $\mu_{a,a',a''}$ can be the empty map $\emptyset : \emptyset \to \emptyset$.
- It is convenient to use a letter with appropriate superscript, like x^m, α^k, etc., as an element (or sometimes as a variable) with domain L^m, L^k, etc. respectively (or with domain $L^m(a,b), L^k(x,y)$, etc.) Also, the grading can be taken to range over $\mathbb{Z}$ under the assumption that $L^{-m} := \emptyset$, $m > 0$.
- Call elements $a \in L^0$ of degree 0 **objects** of L, elements $f^n \in L^n(a,a')$, $a, a' \in L^0$, **arrows of degree** $n+1$ **from** a **to** a'.
- Denote **horizontal composites** by $*$, and extend it over arrows **of different degrees** by the rule $* : L(b,c) \times L(a,b) \to L(a,c) : (g^n, f^m) \mapsto \mu_{a,b,c}(e^{max(m,n)-n} g^n, e^{max(m,n)-m} f^m) =: g^n * f^m$ $(f^m \in L^m(a,b), g^n \in L^n(b,c))$. $\square$

The following definition of equivalence is given “coinductively” (see [J-R])

Definition 1.1.2. For $a, b \in L^n$ $a \sim b$ iff $\exists$ $a \underset{g}{\overset{f}{\rightleftarrows}} b$ such that $e(a) \sim g \circ_1 f$ and $f \circ_1 g \sim e(b)$ (it means that there exists an $f \in L^0(a,b)$, $g \in L^0(b,a)$ and two infinite sequences of arrows of higher order, one in $L(a,a)$ and the other in $L(b,b)$; all this data we will call *arrows representing the given equivalence*). $\square$

$\sim$ is reflexive and symmetric, but may be not transitive.

Lemma 1.1.1. *If L is an ∞-precategory such that*

$\circ_1$ *is weakly associative:* $f \circ_1 (g \circ_1 h) \sim (f \circ_1 g) \circ_1 h$ *(for composable arrows),*

$\circ_1$ *satisfies the weak unit law:* $\forall f \in \coprod_{n \geq 1} L^n \begin{cases} f \circ_1 edf \sim f \\ ecf \circ_1 f \sim f \end{cases}$,

$\sim$ *is compatible with* $\circ_1$, *i.e.* $(f \sim g) \,\&\, (h \sim k) \Rightarrow (f \circ_1 h) \sim (g \circ_1 k)$ *(for composable arrows),*

$\sim$ is transitive in higher orders: i.e. there exists $m > 0$ such that if $\sim$ is transitive for $\coprod_{n\geq m} L^n$, then $\sim$ is transitive in all orders.

PROOF. Let $a \underset{g}{\overset{f}{\rightleftarrows}} b \underset{g'}{\overset{f'}{\rightleftarrows}} c$ be the given equivalences, i.e. $ea \sim g \circ_1 f$, $eb \sim f \circ_1 g$, $eb \sim g' \circ_1 f'$, $ec \sim f' \circ_1 g'$. Then $a \underset{g \circ_1 g'}{\overset{f' \circ_1 f}{\rightleftarrows}} c$ is the required equivalence since $ea \sim g \circ_1 f \sim g \circ_1 (eb \circ_1 f) \sim g \circ_1 ((g' \circ_1 f') \circ_1 f) \sim (g \circ_1 g') \circ_1 (f' \circ_1 f)$ and similarly $ec \sim (f' \circ_1 f) \circ_1 (g \circ_1 g')$.

□

Remarks.

- Transitivity in higher orders trivially holds for n-categories (starting from level n), taking $\sim$ as the identity. For proper ∞-categories it is better to make the assumption "$\sim$ is transitive in all orders" from the beginning.
- This lemma shows that although transitivity of $\sim$ is not automatic for ∞-precategories, it is indeed consistent with (weak) associativity, the unit law, and compatibility of $\sim$ with composites. □

Definition 1.1.3. An ∞-precategory L with relation $\sim$ as above is called a (weak) **∞-category** iff

- $\sim$ is transitive: $\alpha \sim \beta \sim \gamma \Rightarrow \alpha \sim \gamma$,
- $\sim$ is compatible with all composites: $(f \sim g)\,\&\,(h \sim k) \Rightarrow (f \circ_n h) \sim (g \circ_n k)$ (when they are defined),
- horizontal composites preserve properties (1)-(2) and weakly preserve properties (3)-(4) of ∞-precategories in the following sense:
  (1) grading $deg_{L(a,a'')}(\mu_{a,a',a''}(f,g)) = deg_{L(a',a'')}(f) = deg_{L(a,a')}(g)$
  (2) $\mu_{a,a',a''}(df,dg) = d\mu_{a,a',a''}(f,g)$, $\mu_{a,a',a''}(cf,cg) = c\mu_{a,a',a''}(f,g)$
  (3) $\mu_{a,a',a''}(ef,eg) \sim e\mu_{a,a',a''}(f,g)$

(4) $\mu_{a,a',a''}(f \circ_k f', g \circ_k g') \sim \mu_{a,a',a''}(f,g) \circ_k \mu_{a,a',a''}(f',g')$ ("interchange law")

- each $\circ_k, k \in \mathbb{N}$, is **weakly associative**: $(f \circ_k g) \circ_k h \sim f \circ_k (g \circ_k h)$ (for composable elements),
- The **weak unit law** holds: $e^k c^k f \circ_k f \sim f$, $f \circ_k e^k d^k f \sim f$ (when all operations are defined).

☐

Remarks.

I t is instructive to see what goes wrong if we attempt to consider a bicategory as an instance of this definition. One would think that we could obtain an example by defining $\sim$ on 1-cells as isomorphism of 1-cells and as equality for 2-cells. However, the problem lies in the horizontal composition of 2-cells which would be required to be strictly associative, whereas in general the horizontal composite of 2-cells is not. In particular, one horizontal composite might be $f \otimes (g \otimes h)$ whereas the other might be $(f \otimes g) \otimes h$.

- By lemma 1.1.1, for n-categories, the transitivity condition on $\sim$ follows from the others.
- Hom-sets in an ∞-category L are ∞-categories themselves, and horizontal composites $* : L(b,c) \times L(a,b) \to L(a,c)$, are ∞-functors.
- Since strict functors preserve the equivalences $\sim$ for categories in which horizontal composites preserve identity and composites strictly, the compatibility condition on $\sim$ with composites holds automatically. ☐

A category is called **strict** if the associativity and unit laws hold for elements (not just for $\sim$-equivalence classes) and horizontal composites preserve identities and composites strictly. Note that $\sim$ still makes sense for strict categories.

Proposition 1.1.1. *In a strict ∞-category L, arrows of degree n (i.e., L^n) form a 1-category with objects L^0, arrows L^n, domain function d^n, codomain function c^n. Observe that $d, c : L^n \to L^{n-1}$ are 1-functors.* ☐

Lemma 1.1.2.

- *In the strict ∞-category L $e^k(f \circ_n g) = e^k f \circ_{n+k} e^k g$ (when either side is defined).*
- *$\sim$ is preserved under $\sim$, i.e., if $a \xrightarrow[f]{\sim} a'$ is an equivalence with $a' \xrightarrow[g]{\sim} a$, its quasiinverse (i.e. $ea \sim g \circ f$, $ea' \sim f \circ g$), and if $f' \sim f$ then g is quasiinverse of f' as well.*
- *A quasiinverse is determined up to $\sim$, i.e. if* $$\begin{tikzcd} a \arrow[r, bend left, "f"] & b \arrow[l, bend left=15, "g"'] \arrow[l, bend left=50, "g'"] \end{tikzcd}$$ *and $g' \circ_1 f \sim ea \sim g \circ_1 f$ and $f \circ_1 g' \sim eb \sim f \circ_1 g$ then $g' \sim g$.*
- *All $n+1$ composites in $\mathbf{End}(e^n a) := L^0(e^n a, e^n a)$, $n \geq 0$ coincide up to equivalence $\sim$.*

PROOF.

- Assume $f, g \in L^m$, $m \geq n$. Then $f \circ_n g = \mu_{d^n g, c^n g, c^n f}(f, g)$, which preserves e.
- $ea \sim g \circ f \sim g \circ f'$, $ea' \sim f \circ g \sim f' \circ g$.
- $g' = g' \circ_1 eb \sim g' \circ_1 f \circ_1 g \sim g \circ_1 f \circ_1 g \sim g \circ_1 eb = g$.
- $f \circ_{n+1} g = \mu_{a,a,a}(f, g) \sim \mu_{a,a,a}(f \circ_k e^{n+1}a, e^{n+1}a \circ_k g) \sim \mu_{a,a,a}(f, e^{n+1}a) \circ_k \mu_{a,a,a}(e^{n+1}a, g) \sim f \circ_k g$, $1 \leq k \leq n+1$. □

Definition 1.1.4. An arrow $(f : a \longrightarrow a') \in L^0(a, a')$, $deg(a) = deg(a') = m \geq 0$, is called

- **monic** if $\forall g, h : z \longrightarrow a$ if $f \circ_1 g \sim f \circ_1 h$ then $g \sim h$
- **epic** if $\forall g', h' : a' \longrightarrow w$ if $g' \circ_1 f \sim h' \circ_1 f$ then $g' \sim h'$
- an **equivalence** if there exists $f' : a' \longrightarrow a$ such that $edf \sim f' \circ_1 f$ and $edf' \sim f \circ_1 f'$ □

Proposition 1.1.2. *For composable arrows*

- *If f, g are monics then $f \circ_1 g$ is monic. If $f \circ_1 g$ is monic then g is monic*
- *If f, g are epics then $f \circ_1 g$ is epic. If $f \circ_1 g$ is epic then f is epic*
- *If f, g are equivalences then $f \circ_1 g$ is an equivalence* □

Proposition 1.1.3. *All arrows representing equivalence $a \sim b$ are equivalences.* □

Definition 1.1.5. An **∞-functor** $F : L \longrightarrow L'$ is a function which strictly preserves the following properties (1)-(2) of precategories:

(1) if $a \in L^n$ then $F(a) \in L'^n$

(2) $F(da) = dF(a), F(ca) = cF(a)$

and weakly preserves the following properties (3)-(4):

(3) $F(ea) \sim eF(a)$

(4) $F(a \circ_k b) \sim F(a) \circ_k F(b)$ □

Remark.

- We do not require the functor F to preserve equivalences $\sim$ because it is not automatic and can be too restrictive. However, the functors preserving $\sim$ are very important (e.g., see point 1.2).
- The inverse map F' for a bijective weak functor F is not a functor, in general. If F preserves $\sim$ then to say the inverse map F' is a (weak) functor is equivalent to saying F' preserves $\sim$. The inverse of a strict functor is always a strict functor. □

Lemma 1.1.3.

- *Strict functors preserve equivalences* $\sim$ *.*
- *If functor* $F : L \rightarrow L'$ *is such that each restriction on hom-sets* $F_{a,b} : L(a,b) \rightarrow L'(F(a), F(b))$, $a, b \in L^0$, *preserves equivalences* $\sim$*, then* F *preserves equivalences* $\sim$ *.*
- *If* $F : L \rightarrow L'$ *is an embedding (injective map) such that* $\forall a, b \in L^0$ $F_{a,b} : L(a,b) \rightarrow L'(F(a), F(b))$ *is a strict isomorphism and inverse* F' *to codomain restriction of* $\begin{tikzcd} F : L \arrow[r, shift right, "F|_{Im(F)}"'] & Im(F) \arrow[l, dashed, shift right, "F'"'] \arrow[r, hook] & L' \end{tikzcd}$ *is a functor, then* F *reflects* $\sim$ *.*

PROOF.

- Each arrow presenting a given equivalence $x \sim y$ is between a domain and a codomain which are constructed in a certain way only by composites and identity operations from arrows of smaller degree presenting the given equivalence and from elements x and y. A strict functor keeps the structure of the domains and codomains of arrows presenting the equivalence $x \sim y$. So, the image of arrows presenting an equivalence $x \sim y$ will be a family of arrows presenting an equivalence $F(x) \sim F(y)$.

- For arrows of degree > 0 equivalences are preserved by assumption. Let $\begin{tikzcd} a \arrow[r, bend left, "f"] \arrow[r, phantom, "\approx"] & b \arrow[l, bend left, "g"] \end{tikzcd}$, $a, b \in L^0$, be an equivalence for objects in L, i.e. $ea \sim g \circ_1 f$, $eb \sim f \circ_1 g$. Then there are two opposite arrows $\begin{tikzcd} F(a) \arrow[r, bend left, "F(f)"] & F(b) \arrow[l, bend left, "F(g)"] \end{tikzcd}$. By assumption, $F(ea) \sim F(g \circ_1 f)$, $F(eb) \sim F(f \circ_1 g)$. So, $eF(a) \sim F(ea) \sim F(g \circ_1 f) \sim F(g) \circ_1 F(f)$ and $eF(b) \sim F(eb) \sim F(f \circ_1 g) \sim F(f) \circ_1 F(g)$. Therefore, $\begin{tikzcd} F(a) \arrow[r, bend left, "F(f)"] \arrow[r, phantom, "\approx"] & F(b) \arrow[l, bend left, "F(g)"] \end{tikzcd}$ is an equivalence.
- The inverse of a strict isomorphism is a strict isomorphism, i.e. preserves equivalences. So, F' is a functor which preserves equivalences in all hom-sets and, consequently, preserves all equivalences. Preservation of equivalences for F' is exactly reflection of equivalences for F. $\square$

Lemma 1.1.4.

- $x = y$ *iff* $ex \sim ey$ *[in particular,* $=$ *is definable via* $\sim$*].*
- *Functors preserving* $\sim$ *strictly preserve all composites* $\circ_k$, $k \geq 1$.
- *Functors weakly preserving* e^2 *strictly preserve* e, *i.e.* $e^2F(a) \sim F(e^2a) \Rightarrow eF(a) = F(ea)$.
- **Quasiequal** *functors (i.e.* $F(f^n) \sim G(f^n)$ *for all* $f^n \in L^n$, $n \geq 0$*) are equal.*

PROOF.

- $x = y \Rightarrow ex = ey \Rightarrow ex \sim ey$. Conversely, $ex \sim ey \Rightarrow dex = dey \Rightarrow x = y$.
- Sufficient to prove $eF(f \circ_k g) \sim e(F(f) \circ_k F(g))$, but it holds $eF(f \circ_k g) \sim F(e(f \circ_k g)) \sim$ (F preserves $\sim$) $F((ef) \circ_{k+1} (eg)) \sim F(ef) \circ_{k+1} F(eg) \sim eF(f) \circ_{k+1} eF(g) \sim e(F(f) \circ_k F(g))$.
- $e^2F(a) \sim F(e^2a) \Rightarrow de^2F(a) = dF(e^2a) \Rightarrow eF(a) = F(ea)$.
- Again, it is sufficient to prove $eF(f^n) \sim eG(f^n)$.

  $eF(f^n) \sim F(ef^n) \sim$ (by assumption) $G(ef^n) \sim eG(f^n)$. $\square$

Corollary. ∞*-categories in the sense of definition 1.1.3 are almost strict*, namely, with strict associativity, identity, and interchange laws.

PROOF. Strict associativity and strict identity laws hold because, by the axioms, the functors $L(x,y)\times L(y,z)\times L(z,t)\to L(x,t) : (f^n,g^n,h^n)\mapsto (h^n*g^n)*f^n$ and $L(x,y)\times L(y,z)\times L(z,t)\to L(x,t) : (f^n,g^n,h^n)\mapsto h^n*(g^n*f^n)$, $deg(x)=deg(y)=deg(z)=deg(t)$, are quasiequal, and, respectively, functors $L(x,y)\to L(x,y) : f\mapsto f$ and $L(x,y)\to L(x,y) : f\mapsto ey*f$, $deg(x)=deg(y)$ (similarly for the right identity), are quasiequal. The strict interchange law holds because the functor $L(x,y)\times L(y,z) : (f,g)\mapsto g*f$ preserves $\sim$. ☐

Definition 1.1.6. For two given functors F, G , an ∞-**natural transformation** $\alpha : F\to G$ is a function $\alpha : L^0\to {L'}^1 : a\mapsto (\ F(a)\xrightarrow{\alpha(a)} G(a)\ )$ such that

$$\mu_{F(a),F(b),G(b)}(e^k\alpha(b),F(f))\sim \mu_{F(a),G(a),G(b)}(G(f),e^k\alpha(a))$$

for all $f\in L^k(a,b)$, $k=0,1,\ldots$ ☐

Definition 1.1.7. For two given functors F, G and two natural transformations $F\overset{\alpha}{\underset{\beta}{\rightrightarrows}} G$ an ∞-**modification** $\lambda : \alpha\to\beta$ is a function $\lambda : L^0\to {L'}^2 : a\mapsto (\ \alpha(a)\xrightarrow{\lambda(a)}\beta(a)\ )$ such that

$$\mu_{F(a),F(b),G(b)}(e^k\lambda(b),F(f))\sim \mu_{F(a),G(a),G(b)}(G(f),e^k\lambda(a))$$

for all $f\in L^{k+1}(a,b)$, $k=0,1,\ldots$ ☐

Analogously, modifications of higher order are introduced. We call modifications 1-modifications, natural transformations 0-modifications.

Definition 1.1.8. Given two functors F, G, two 0-modifications $F\overset{\alpha_1^0}{\underset{\alpha_2^0}{\rightrightarrows}} G$,

two 1-modifications $\alpha_1^0\overset{\alpha_1^1}{\underset{\alpha_2^1}{\rightrightarrows}}\alpha_2^0$,..., two $n-1$-modifications $\alpha_1^{n-2}\overset{\alpha_1^{n-1}}{\underset{\alpha_2^{n-1}}{\rightrightarrows}}\alpha_2^{n-2}$
∞-n-**modification** $\alpha^n : \alpha_1^{n-1}\to\alpha_2^{n-1}$ is a function $\alpha^n : L^0\to {L'}^{n+1}$:
$a\mapsto (\ \alpha_1^{n-1}(a)\xrightarrow{\alpha^n(a)}\alpha_2^{n-1}(a)\ )$ such that

$$\mu_{F(a),F(b),G(b)}(e^k\alpha^n(b),F(f))\sim \mu_{F(a),G(a),G(b)}(G(f),e^k\alpha^n(a))$$

for all $f \in L^{k+n}(a,b)$, $k = 0, 1, \ldots$ □

Corollary. *All n-modifications in the sense of Definition 1.1.8 are strict, i.e. all naturality squares commute strictly.*

PROOF. By the conditions in Definition 1.1.8, two functors $\alpha^n(b) * F(-) : L^{\geq n}(a,b) \to L'^{\geq n}(F(a), G(b))$ and $G(-) * \alpha^n(a) : L^{\geq n}(a,b) \to L'^{\geq n}(F(a), G(b))$ are quasiequal and, so, equal. □

Definition 1.1.9. ∞-**CAT** is an ∞-category consisting of

- A graded set $C = \coprod_{n \geq 0} C^n$, where C^0 are categories, C^1 functors, C^n $(n-2)$-modifications
- if $\alpha^n : \alpha_1^{n-1} \to \alpha_2^{n-1} \in C^n$ then $d\,\alpha^n = \alpha_1^{n-1}$, $c\,\alpha^n = \alpha_2^{n-1}$
- $e\,\alpha^n \in C^{n+1}$ is the map $L^0 \to L'^{(n+1)} : a \mapsto e(\alpha^n(a))$
- for given two n-modifications α_1^n, α_2^n such that $d^k \alpha_1^n = c^k \alpha_2^n$

$$\alpha_1^n \circ_k \alpha_2^n := \begin{cases} a \mapsto (\alpha_1^n(a) \circ_k \alpha_2^n(a)) & \text{if } \ k < n+2 \\ a \mapsto (\alpha_1^n(F'(a)) \circ_{(n+1)} G(\alpha_2^n(a))) & \text{if } \ k = n+2, \ F' = c^{(n+1)}\alpha_2^n, \ G = d^{(n+1)}\alpha_1^n \end{cases}$$

The first composite works when $\alpha_1^n, \alpha_2^n \in \infty$-**CAT**$(L, L')$, the second when $\alpha_1^n \in \infty$-**CAT**(L', L'') $\alpha_2^n \in \infty$-**CAT**(L, L'), where L, L', L'' are categories. □

Lemma 1.1.5. *In* ∞-**CAT** *there are two ways of taking horizontal composites (and they are equal):* $\alpha^n * \beta^n := \alpha^n F' \circ_{n+1} G\beta^n = G'\beta^n \circ_{n+1} \alpha^n F$ *(where* $F := d^{n+1}\beta^n$, $F' := c^{n+1}\beta^n$, $G := d^{n+1}\alpha^n$, $G' := c^{n+1}\alpha^n$*).*

PROOF follows from the naturality square for α^n

$$\begin{tikzcd} G'F(a) \arrow[r, "G'(\beta^n(a))"] & G'F'(a) \\ GF(a) \arrow[u, "\alpha^n(F(a))"] \arrow[r, "G(\beta^n(a))"'] & GF'(a) \arrow[u, "\alpha^n(F'(a))"'] \end{tikzcd}$$

□

Proposition 1.1.4. *Categories, functors, natural transformations, modifications, etc. form the* ∞*-category* ∞-**CAT** *of* ∞*-categories.* □

PROOF is similar to that for 2-**CAT**.

- Horizontal composites preserve grading (obvious).

- d, c, e are preserved for a similar reason, e.g., take d: $(d\alpha^n) * (d\beta^n)(a) := (d\alpha^n)(F'(a)) \circ_n G(d\beta^n(a)) = d(\alpha^n(F'(a))) \circ_n d(G(\beta^n(a))) = d(\alpha^n(F'(a)) \circ_{n+1} G(\beta^n(a))) = d((\alpha^n * \beta^n)(a)) = (d(\alpha^n * \beta^n))(a)$ (where $F' := c^{n+1}\beta^n,\ G := d^{n+1}\alpha^n$).

- (interchange law) $$\begin{tikzcd} L \arrow[r, bend left=80, "F"] \arrow[r, bend left=25, "F'"'] \arrow[r, bend right=25, "F''"] \arrow[r, bend right=80, "F'''"'] \arrow[r, phantom, shift left=4ex, "\Downarrow\gamma^n"] \arrow[r, phantom, shift right=4ex, "\Downarrow\delta^n"] & L' \arrow[r, bend left=80, "G"] \arrow[r, bend left=25, "G'"'] \arrow[r, bend right=25, "G''"] \arrow[r, bend right=80, "G'''"'] \arrow[r, phantom, shift left=4ex, "\Downarrow\alpha^n"] \arrow[r, phantom, shift right=4ex, "\Downarrow\beta^n"] & L'' \end{tikzcd}$$ (by condition $d^k\delta^n = c^k\gamma^n$, $d^k\beta^n = c^k\alpha^n$, $k < n+2$, all F's and G's are functors)

  $(\beta^n \circ_k \alpha^n) * (\delta^n \circ_k \gamma^n)(a) := (\beta^n \circ_k \alpha^n)(F'''(a)) \circ_{n+1} G((\delta^n \circ_k \gamma^n)(a)) = (\beta^n(F'''(a)) \circ_k \alpha^n(F'''(a))) \circ_{n+1} (G(\delta^n(a)) \circ_k G(\gamma^n(a))) =$

$$\begin{cases}(\beta^n(F'''(a)) \circ_{n+1} G(\delta^n(a))) \circ_k (\alpha^n(F'''(a)) \circ_{n+1} G(\gamma^n(a))) = (*) \\ \beta^n(F'''(a)) \circ_{n+1} \big(\alpha^n(F'''(a)) \circ_{n+1} G(\delta^n(a))\big) \circ_{n+1} G(\gamma^n(a)) = (**)\end{cases}$$

$$\begin{cases}(*) = ((\beta^n * \delta^n) \circ_k (\alpha^n * \gamma^n))(a) \quad \text{if } k < n+1 \text{ (in this case } G = G'', G' = G''', F = F'', F' = F''') \\ (**) = \beta^n(F'''(a)) \circ_{n+1} \big(G'(\delta^n(a)) \circ_{n+1} \alpha^n(F''(a))\big) \circ_{n+1} G(\gamma^n(a)) = (**)\end{cases}$$

$$\begin{cases} \\ (**) = ((\beta^n * \delta^n) \circ_{n+1} (\alpha^n * \gamma^n))(a) \quad \text{if } k = n+1 \text{ (in this case } F' = F'', G' = G'') \end{cases}$$

- The associativity law for vertical composites and the identity law hold essentially because of the componentwise definition of vertical composites. The associativity law for horizontal composites is due to the interchange law and lemma 1.1.5. ☐

Definition 1.1.10. A category L is called an **∞-n-category** if $L^{j+1} = e(L^j)$ for $j \geq n$. ☐

A quotient $L/\sim$ is not a category in general since $\sim$ is not compatible with e. However, if we take the quotient only on a fixed level n and make all higher arrows identities we get ∞-n-category $L^{(n)}$, n-th approximation of L. Generally there are no functors $L^{(n)} \hookrightarrow L$, $L \twoheadrightarrow L^{(n)}$ (except for the last surjection if L is a weak ∞-$(n+1)$-category and all $(n+1)$-

arrows are isomorphisms's).

1.1.a. Weak categories, functors, natural transformations, modifications.

As we saw above, using a weak language (substituting $\sim$ for $=$) does not give a weak category theory. The only advantage was that we could deal with $\sim$ instead of $=$, which is important for the classification problem (that still makes sense for strict ∞-categories). All known definitions of weak categories [C-L, Lei, Koc] are nonelementary (at least, they use functors, natural transformations, operads, monads just for the very definition). Probably, this is a fundamental feature of weak categories. To introduce them we also need the whole universe ∞-**PreCat** of ∞-precategories.

Definition 1.1.a.1. ∞-**PreCat** consists of

- ∞-**precategories** (definition 1.1.1) together with $\sim$-relation in each [$\sim$ may be not transitive],
- ∞-**functors** (definition is like 1.1.5 for ∞-categories), i.e. functions $F : L \to L'$ of degree 0 preserving d and c strictly, and e and $\circ_k$, $k \geq 1$, weakly,
- **lax** ∞-n-**modifications**, $n \geq 0$, i.e. **total** maps $\alpha^n : L \to L'$ (with variable degree on different elements, but $\leq n+1$, more precisely, the induced map $\mathbb{N} \to \mathbb{N} : deg(x) \mapsto (deg(\alpha^n(x)) - deg(x))$ is an antimonotone map, decreasing by 1 at each step from $n+1$ at $deg(x) = 0$ to 1 at $deg(x) = n$ and remaining constant 1 after) being defined for a given sequence of two functors $F, G : L \to L'$, two 0-modifications (natural transformations) $\alpha_1^0, \alpha_2^0 : F \to G$, ..., two $(n-1)$-modifications $\alpha_1^{n-1}, \alpha_2^{n-1} : \alpha_1^{n-2} \to \alpha_2^{n-2}$ as $\alpha^n :=$

$$\left\{\begin{array}{ll}
(\alpha^n(x) : \alpha_1^{n-1}(x) \to \alpha_2^{n-1}(x)) \in L'^n(F(x), G(x)) & x \in L^0 \\
\alpha^n(x) := \alpha^n(e^{n+1-k}x) \in L'^{n+1}(F(d^k x), G(c^k x)) & \underset{0<k<n+1}{x \in L^k} \\
(\alpha^n(x) : \alpha^n(c^{n+1}x) \circ_{n+1} F(x) \to G(x) \circ_{n+1} \alpha^n(d^{n+1}x)) \in & x \in L^{n+1} \\
\qquad\qquad\qquad\qquad \in L'^{n+1}(F(d^{n+1}x), G(c^{n+1}x)) & \\
(\alpha^n(x) : \alpha^n(cx) \circ_1 (e\alpha^n(c^{n+2}x) \circ_{n+2} F(x)) \to & x \in L^{n+2} \\
(G(x) \circ_{n+2} e\alpha^n(d^{n+2}x)) \circ_1 \alpha^n(dx)) \in L'^{n+2}(F(d^{n+2}x), G(c^{n+2}x)) & \\
\alpha^n(x) : \alpha^n(cx) \circ_1 (e\alpha^n(c^2x) \circ_2 (e^2\alpha^n(c^{n+3}x) \circ_{n+3} F(x))) \to & x \in L^{n+3} \\
((G(x) \circ_{n+3} e^2\alpha^n(d^{n+3}x)) \circ_2 e\alpha^n(d^2x)) \circ_1 \alpha^n(dx) \in L'^{n+3}(F(d^{n+3}x), G(c^{n+3}x)) & \\
\dots & \\
\alpha^n(x) : & x \in L^{n+m} \\
\alpha^n(cx) \circ_1 \cdots \circ_{m-2} (e^{m-2}\alpha^n(c^{m-1}x) \circ_{m-1} (e^{m-1}\alpha^n(c^{n+m}x) \circ_{n+m} F(x) \underbrace{)\dots)}_{m-1} \to & \\
\underbrace{(\dots(}_{m-1} G(x) \circ_{n+m} e^{m-1}\alpha^n(d^{n+m}x)) \circ_{m-1} e^{m-2}\alpha^n(d^{m-1}x)) \circ_{m-2} \cdots \circ_1 \alpha^n(dx) \in & \\
\qquad\qquad\qquad\qquad \in L'^{n+m}(F(d^{n+m}x), G(c^{n+m}x)) & \\
\dots &
\end{array}\right.$$

$$d\alpha^n := \alpha_1^{n-1},\ c\alpha^n := \alpha_2^{n-1} \quad [(d\alpha^n)(x) \neq d(\alpha^n(x)),\ (c\alpha^n)(x) \neq c(\alpha^n(x)) \text{ if } deg(x) > 0]. \qquad \square$$

Remarks.

- ∞-n-modifications look terrible but they are the weakest form of naturality (infinite sequences of naturality squares arising by considering naturality squares given by equations $e_1(x) \sim e_2(x)$ which express $\sim$-naturality in x. This leads to an infinite sequence of naturality squares). To deal with such entities a kind of operad is needed.
- To give an n-modification α^n is the same as to give a map $\alpha^n \big|_{L^0} : L^0 \to L'$ of degree $n+1$ and $\forall a, b \in L^0$ a natural transformation $\nu^{\alpha^n}_{a,b} : \alpha^n(b) * F(-) \to G(-) * \alpha^n(a) : L^{\geq n}(a, b) \to L'^{\geq n}(F(a), G(b))$, where $F = d^{n+1}\alpha^n$, $G = c^{n+1}\alpha^n$.
- When $\alpha^n(x)$, $deg(x) > 0$, are all identities (of the required types) ∞-n-modifications are called **strict**. They are the usual modifications and composable as in definition 1.1.9 when the universe ∞-**CAT** is strict (in that case strict modifications are weak as well). In a weak universe ∞-**CAT** strict modifications need not to be weak (i.e. to be modifications at all).
- ∞-**PreCAT** is not an ∞-precategory itself because there are no identities and composites

for weak n-modifications. The problem here with identities and composites is not clear, for example if they exist at all without making either naturality condition or ∞-categories stricter.

- In general, these two sides "categories and functors" and "n-modifications" form a strange pair. If we weaken one of these sides, the other one becomes stricter (under condition that ∞-**CAT** is a (let it be very weak) **category**). So, the following **hypothesis** holds:

  There is no ∞-**CAT** *with* **simultaneously weak** *categories, functors, and n-modification.*

  For example, if we want weak modifications and want them to be composable we need to introduce several axioms on categories, one of which is like '$\forall a, b \in L^0$ *and* $\forall$ *functors* $F, G : L \longrightarrow L'$ *if* $\exists$ *natural transformations* $\alpha : f_1 * F(-) \longrightarrow G(-) * g_1 : L^{\geq n}(a,b) \longrightarrow L'^{\geq n}(F(a), G(b))$ *and* $\beta : f_2 * F(-) \longrightarrow G(-) * g_2 : L^{\geq n}(a,b) \longrightarrow L'^{\geq n}(F(a), G(b))$ *and* $n+1$*-cells* f_1, f_2 *and* g_1, g_2 *are* $\circ_k$*-composable then* $\exists$ *a natural transformation (k-composite)* $\gamma : (f_1 \circ_k f_2) * F(-) \longrightarrow G(-) * (g_1 \circ_k g_2) : L^{\geq n}(a,b) \longrightarrow L'^{\geq n}(F(a), G(b))$'. But such axioms make very special categories. From the other side, if we want categories to be weak we need to make stricter (maybe, strict) n-modifications in order that they would be composable. The problem is in existence of composites (and units) for weak n-modifications.

- Instead of lax n-modifications we could use modifications with $\alpha^n(x)$ being $\sim$ for $deg(x) > 0$ in L'. In both cases in order to make horizontal composites (at least, $F * \alpha^n := F \circ_{\mathbf{SET}} \alpha^n$) we need functors preserving composites (or composites and $\sim$), i.e. 'weak modifications' $\Rightarrow$ 'strict functors'.

- If the above hypothesis was true it would be nice, e.g. a universe where ∞-**Top** lives would contain only strict n-modifications. ☐

Definition 1.1.a.2. A **weak** ∞**-category** L is an ∞-precategory (see definition 1.1.1) such that all composites below are well-defined.

- $\sim$ is transitive $x \sim y \sim z \Rightarrow x \sim z$,

- horizontal composites $*$ strictly preserve properties (1)-(2) of precategories

  (1) $deg(x*y) = deg(x) = deg(y)$ if $deg(x) = deg(y)$ (interchange law for degree)

  (2) $d(x*y) = (dx)*(dy)$, $c(x*y) = (cx)*(cy)$ if $deg(x) = deg(y)$ (interchange law for domain and codomain)

  and weakly preserve properties (3)-(4) of precategories

  (3) $e(x*y) \sim (ex)*(ey)$ if $deg(x) = deg(y)$ (interchange law for identity)

  (4) $(x \circ_k y)*(z \circ_k t) \sim (x*z) \circ_k (y*t)$ if $deg(x) = deg(y) = deg(z) = deg(t)$ (interchange law for composites) [$\circ_k$ has smaller 'deepness' k than the given $* = \circ_n$, $n > k$],

- **(weak associativity)**

  $\forall x, y, z, t \in L^n$ for two functors $l_{x,y,z,t} : L(x,y) \times L(y,z) \times L(z,t) \to L(x,t) : (f,g,h) \mapsto (h*g)*f$ and $r_{x,y,z,t} : L(x,y) \times L(y,z) \times L(z,t) \to L(x,t) : (f,g,h) \mapsto h*(g*f)$ $\exists$ natural transformation $\alpha_{x,y,z,t} : l_{x,y,z,t} \to r_{x,y,z,t}$,

- **(weak unit)**

  $\forall x, y \in L^n$ and functors $u^l_{x,y} : L(x,y) \to L(x,y) : f \mapsto ey*f$ and $u^r_{x,y} : L(x,y) \to L(x,y) : f \mapsto f*ex$ $\exists$ natural transformations $\epsilon^l_{x,y} : u^l_{x,y} \to Id$ and $\epsilon^r_{x,y} : Id \to u^r_{x,y}$. □

Remarks.

- We do not introduce a universe ∞-**CAT** with weak categories, functors and n-modifications because there are no (at least, obvious) units and composites for n-modifications (however, identity natural transformations exist if only the vertical composites of natural transformations are defined, for if $F : L \to L'$ is a functor take $(eF)(a) := e(F(a))$, $a \in L^0$ and by the weak unit law $\forall a, b \in L^0$ $\exists$ a natural transformation $\nu_{a,b} : e(F(b))*F(-) \to F(-)*e(F(a)) : L^{\geq 0}(a,b) \to L'^{\geq 0}(F(a), F(b))$, take $\nu_{a,b} := (\epsilon^{u^r}_{F(a),F(b)} \circ_1 \epsilon^{u^l}_{F(a),F(b)}) * F := (\epsilon^{u^r}_{F(a),F(b)} \circ_1 \epsilon^{u^l}_{F(a),F(b)}) \circ_{\mathbf{SET}} F)$. **The problem** is what are the weakest conditions on categories, functors and n-modifications in order that they form a category. Maybe there are several independent such conditions and, so, several categories ∞-**CAT** with weakest entities.

- To keep a usual form of (weak) associativity and (weak) unit we could introduce relations $\underset{k}{\sim}$ for elements of images of two functors $F, G : L \to L'$ connected by a natural transformation $\alpha : F \to G$, namely, $x \underset{k}{\sim} y$ if $\exists\ z \in L^k$ such that $x = F(z)$, $y = G(z)$. These relations are not reflexive, symmetric or transitive. Then we could write associativity and unit laws as $(x \circ_k y) \circ_k z \underset{k-1}{\sim} x \circ_k (y \circ_k z)$ and $e^k c^k x \circ_k x \underset{k-1}{\sim} x$, $x \underset{k-1}{\sim} x \circ_k e^k d^k x$. Under assumption that composites and units exist in an ∞-**CAT** we could choose more sensible piece of ∞-**CAT** with categories in which $\underset{0}{\sim} \equiv \sim$ and all $\underset{k}{\sim}$ are symmetric and transitive by the requirement that $\alpha_{x,y,z,t}$, $\epsilon^l_{x,y}$, $\epsilon^r_{x,y}$ are equivalences. ☐

Examples

1. **2-Top** is a strict ∞-2-category with 2-cells, as homotopy classes of homotopies, and just identities in higher order ($\sim$ on the level of objects means homotopy equivalence of spaces, on the level of 1-arrows homotopies of maps, and on the level ≥ 2 coincidence). 2-**Cat** is similar.
2. It is widely believed that ∞-**Top** is a (weak) ∞-category with homotopies between homotopies as higher order cells. It is hoped that this notion of (weak) ∞-category (as above) will be useful in clarifying this issue. Assuming this, we can give two further examples, as follows.
3. ∞-**Diff** is an ∞-category of differentiable manifolds in the same way as ∞-**Top**.
4. ∞-**TopALg** is an ∞-category of topological algebras in the same way as ∞-**Top** where each instance of homotopy is a homomorphism of topological algebras.
5. ∞-**Compl** is an ∞-category of (co)chain complexes with (algebraic) homotopies for homotopies as higher order cells (see [Lei]).
6. For a 1-category A, A_{equiv} is a strict ∞-2-category such that $A^0_{equiv} = A^0$, $A^1_{equiv} = \Big\{ f \in$

$$A \,\Bigg|\, \begin{tikzcd} \bullet \arrow[r, "\exists H", "\sim"'] & \bullet \\ \bullet \arrow[u, "f"] \arrow[r, "\sim", "\forall h"'] & \bullet \arrow[u, "f"'] \end{tikzcd} \Bigg\}, \; A^2_{equiv} = \Bigg\{ \text{isomorphisms's} \,\Bigg|\, \forall f, g \in A^1_{equiv} \ \exists! \ f \xrightarrow[\sim]{\gamma} g \ \text{ iff } \begin{tikzcd} \bullet \arrow[r, "\exists H", "\sim"'] & \bullet \\ \bullet \arrow[u, "f"] \arrow[r, "\sim", "\forall h"'] & \bullet \arrow[u, "g"'] \end{tikzcd} \Bigg\}.$$

A_{equiv} contains all equivariant maps $f : X \to Y$ with respect to a group homomorphism $\rho : \mathbf{Aut}(X) \to \mathbf{Aut}(Y)$.

7. The (weak) **covariant** ∞-**Hom**-functor $L(a, -) : L \to \infty\text{-}\mathbf{CAT}$:

$$\begin{cases} b \mapsto L(a,b) & b \in L^0 \\ (f : b \to b') \mapsto (L(a,f) : g \mapsto \mu(e^k f, g)) & f \in L^0(b,b'), \ g \in L^k(a,b) \\ (\alpha : f \to f') \mapsto (L(a,\alpha) : x \mapsto \mu(\alpha, ex) & \alpha \in L^1(b,b'), \ x \in L^0(a,b) \\ (\delta : \alpha \to \alpha') \mapsto (L(a,\delta) : x \mapsto \mu(\delta, e^2 x)) & \delta \in L^2(b,b'), \ x \in L^0(a,b) \\ \ldots & \\ (\alpha^n : \alpha_1^{(n-1)} \to \alpha_2^{(n-1)}) \mapsto (L(a,\alpha^n) : x \mapsto \mu(\alpha^n, e^n x)) & \alpha^n \in L^n(b,b'), \ x \in L^0(a,b) \\ \ldots & \end{cases}$$

8. The **opposite category** L^{op} is an ∞-category such that
   - $(L^{op})^n = L^n$, $n \geq 0$
   - $d^{op}(\alpha^n) = \begin{cases} d(\alpha^n) & \text{if } n \geq 2 \\ c(\alpha^n) & \text{if } n = 1 \end{cases}$ $\qquad c^{op}(\alpha^n) = \begin{cases} c(\alpha^n) & \text{if } n \geq 2 \\ d(\alpha^n) & \text{if } n = 1 \end{cases}$
   - $e^{op} = e$
   - $\beta^n \circ_k^{op} \alpha^n = \begin{cases} \beta^n \circ_k \alpha^n & \text{if } \ \alpha^n, \beta^n \in L^n, \ k < n \\ \alpha^n \circ_k \beta^n & \text{if } \ \alpha^n, \beta^n \in L^n, \ k = n \end{cases}$ (for composable elements)

9. The (weak) **contravariant** ∞-**Hom**-functor $L(-, b) : L^{op} \to \infty\text{-}\mathbf{CAT}$:

$$\begin{cases} a \mapsto L(a,b) & a \in L^0 \\ (f : a \to a') \mapsto (L(f,b) : g \mapsto \mu(g, e^k f)) & f \in L^0(a,a'), \ g \in L^k(a',b) \\ (\alpha : f \to f') \mapsto (L(\alpha,b) : x \mapsto \mu(ex, \alpha) & \alpha \in L^1(a,a'), \ x \in L^0(a',b) \\ (\delta : \alpha \to \alpha') \mapsto (L(\delta,b) : x \mapsto \mu(e^2 x, \delta)) & \delta \in L^2(a,a'), \ x \in L^0(a',b) \\ \ldots & \\ (\alpha^n : \alpha_1^{(n-1)} \to \alpha_2^{(n-1)}) \mapsto (L(\alpha^n,b) : x \mapsto \mu(e^n x, \alpha^n)) & \alpha^n \in L^n(a,a'), \ x \in L^0(a',b) \\ \ldots & \end{cases}$$

10. The **Yoneda embedding** $\mathbf{Y} : L \to \infty\text{-}\mathbf{CAT}^{L^{op}} : \alpha \mapsto L(-,\alpha), \ \ \alpha \in L$, where L is an ∞-category.

11. **Set** is simultaneously an object and a full subcategory of ∞-**CAT**.

12. A (big) set $L_\sim := \coprod_{n\geq 0} L^n_\sim$, where $L^n_\sim$ are defined recursively as $L^0_\sim := L^0$ and $L^n_\sim$ are all equivalences from L^n with domain and codomain in $L^{n-1}_\sim$, is a subcategory of L. Similarly, $L_{k\sim} := \coprod_{n\geq 0} L^n_{k\sim}$, $k \geq 0$, where $L^n_{k\sim} := \begin{cases} L^n & n \leq k \\ \text{equivalences from } L^n \text{ with dom and codom in } L^{n-1}_{k\sim} & n > k \end{cases}$, is a subcategory of L. From this point $L_\sim = L_{0\sim}$. Such categories are most important for the classification problem (up to $\sim$). Sometimes, 'invariants' can be constructed only for $L_\sim$ (see point 1.2.1).

13. **Higher order concepts** can **simplify** proof of first order facts. E.g., each strict 2-functor $\Phi : 2\text{-}\mathbf{CAT} \to 2\text{-}\mathbf{CAT}$, where $2\text{-}\mathbf{CAT}$ is the usual strict category of categories, functors, and natural transformations, preserves adjunction $\left(\text{indeed, triangle identities } \begin{cases} \varepsilon G \circ G\eta = 1_G \\ F\varepsilon \circ \eta F = 1_F \end{cases}\right.$ are respected by $\Phi \left.\begin{cases} \Phi(\varepsilon)\Phi(G) \circ \Phi(G)\Phi(\eta) = 1_{\Phi(G)} \\ \Phi(F)\Phi(\varepsilon) \circ \Phi(\eta)\Phi(F) = 1_{\Phi(F)} \end{cases}\right)$. It gives short proofs of the following results.

a) *Right adjoints preserve limits (left adjoints preserve colimits).*

PROOF.

$$\begin{tikzcd}[row sep=huge, column sep=huge]
\mathbf{A}^{\mathbf{I}} \arrow[r, "F^{\mathbf{I}}"'] \arrow[d, bend right=60, "colim"'] \arrow[d, bend left=60, "lim"] & \mathbf{B}^{\mathbf{I}} \arrow[l, bend right=30, "G^{\mathbf{I}}"', "\perp"] \arrow[d, bend right=60, "colim"'] \arrow[d, bend left=60, "lim"] \\
\mathbf{A} \arrow[u, "\Delta", "\dashv\quad\dashv"'] \arrow[r, "F"'] & \mathbf{B} \arrow[u, "\Delta", "\dashv\quad\dashv"'] \arrow[l, bend right=30, "G"', "\perp"]
\end{tikzcd}$$

where $(-)^{\mathbf{I}} \equiv 2\text{-}\mathbf{CAT}(\mathbf{I}, -) : 2\text{-}\mathbf{CAT} \to 2\text{-}\mathbf{CAT}$ is a hom-2-functor.

Now, $G^{\mathbf{I}} \circ \Delta = \Delta \circ G$ (obvious). Taking right adjoints of both sides completes the proof $lim \circ F^{\mathbf{I}} \simeq F \circ lim$ (for colimits the same argument works $F^{\mathbf{I}} \circ \Delta = \Delta \circ F \Rightarrow colim \circ G^{\mathbf{I}} \simeq G \circ colim$). ☐

b) *Each* 1-**Cat**-*valued presheaf admits a sheafification (*1-**Cat** *is a category of small categories and functors between them).*

PROOF. 1-**Cat**-valued presheaf on $\mathbf{C}$ is the same as an internal category object in $\mathbf{Set}^{\mathbf{C}^{op}}$. There is an adjoint situation $\begin{tikzcd} \mathbf{Sh}(\mathbf{C}) \arrow[r, hook] & \mathbf{Set}^{\mathbf{C}^{op}} \arrow[l, bend right, "\perp"'] \end{tikzcd}$ in **LEX**, where $\mathbf{LEX} \hookrightarrow 2\text{-}\mathbf{CAT}$ is a 2-category of finitely complete categories, functors preserving finite limits, and (arbitrary) natural transformations. There is a 2-functor $\mathbf{CAT}(-) : \mathbf{LEX} \to 2\text{-}\mathbf{CAT}$ assigning to each category in **LEX** the category of its internal category objects and to each functor and natural transformation the induced ones. Then $\exists$ an adjunction

$$\begin{tikzcd}\mathbf{CAT}(\mathbf{Sh}(\mathbf{C})) \arrow[r] & \mathbf{CAT}(\mathbf{Set}^{\mathbf{C}^{op}}) \arrow[l, bend right, "\perp"']\end{tikzcd}$$

which means that each 1-**Cat**-valued presheaf can be sheafified by the top curved arrow. $\square$

We introduce the following intuition to help the reader understand our approach.

In spite of its complicated structure, each ∞-category and even ∞-**CAT** itself, has a regular structure which is repeated for certain arbitrary small pieces. Such pieces are, of course, the hom-sets $L(a,b)$ which inherit all properties (1)-(4), associativity and identity laws, and each piece of which still has the same structure. In particular, $L(a,b)(c,d) = L(c,d)$. An ∞-functor restricted to such a piece is again an ∞-functor. Moreover, each ∞-category can be regarded as a hom-set of a little bit bigger category if we formally attach two distinct elements $\alpha, \beta \in L^{-1}$ with their identities of higher order $e^n(\alpha)$, $e^n(\beta)$, $n \geq 1$ (such that $d(L^0) = \alpha$, $c(L^0) = \beta$ and composites with these identities of other elements hold strictly). Other natural pieces of L which inherit all properties and are ∞-categories are $L^{\geq n}$, $L^{\geq n}(a,b)$ (elements of degree not lower than n).

1.1.1. Notes on Coherence Principle.

This principle is an axiom to deal with the equivalence relation $\sim$. It is not logically necessary for higher order category theory itself. There can be categories in which it does not hold.

Coherence Principle. For a given set of cells $\{a_i\}_I$ and a given set of base equivalences $\{t_j(\{a_i\}_I) \sim s_j(\{a_i\}_I)\}_J$ for any two constructions $F_1(\{a_i\}_I)$ and $F_2(\{a_i\}_I)$ and any two derived equivalences $\varepsilon_i^0 : F_1(\{a_i\}_I) \sim F_2(\{a_i\}_I)$, $i = 1,2$ there are derived equivalences $\varepsilon_m^1 : \varepsilon_1^0 \sim \varepsilon_2^0$, $m \in M^1$, such that for any two of them $\varepsilon_{m_1}^1$, $\varepsilon_{m_2}^1$ there are derived equivalences $\varepsilon_m^2 : \varepsilon_{m_1}^1 \sim \varepsilon_{m_2}^1$, $m \in M^2$ again such that for any pair of them $\varepsilon_{m_1}^2$, $\varepsilon_{m_2}^2$ there are derived equivalences of higher order, etc. $\square$

Here constructions mean application of composites, functors, natural transformations,.. to $\{a_i\}_I$. Derived equivalences mean equivalences obtained from base ones by virtue of the categorical axioms.

1.2. (m,n)-invariants

Definition 1.2.1.

- **Equivalence** $x^k \sim y^k,\ x^k, y^k \in L^k,\ k \geq 0$, is called **of degree** $l,\ deg(\sim) := l,\ l \geq 0$, if all arrows representing it (starting from order $k+l+1$ and higher) are identities and for $l > 0$ there is at least one nonidentity arrow on level $k+l$. If there is no such $l \in \mathbb{N},\ deg(\sim) := \infty$. Denote $\sim$ of degree l by $\sim_l$.
- A pair of equivalent elements $x^k \sim y^k,\ k \geq 0$, is called **of degree** $l,\ deg(x^k \sim y^k) := l, l \geq 0$, if the lowest degree of equivalences existing between x^k and y^k is l.
- An ∞-**category** L is called **of degree l**, $deg(L) = l,\ l \geq 0$, if for any pair of equivalent objects $a \sim a',\ a, a' \in L^0$, there exists an equivalence $a \sim_k a'$ of degree $k \leq l$ and there exists at least one pair of equivalent objects from L of degree l.
- A **Functor** $F : L \to L'$ is called (m,n)**-invariant** if F preserves equivalences $\sim$, $m = deg(L)$, $0 \leq n \leq deg(L')$ and F maps every pair of equivalent objects of degree $\leq m$ to a pair of equivalent objects of degree $\leq n$, i.e. $deg(a \sim a') \leq m \Rightarrow deg(F(a) \sim F(a')) \leq n$, and boundary n is actually achieved on a pair of equivalent objects of L. □

Remarks.

- (m,n)-invariants are important for the classification problem (up to $\sim$). If $n < m$ an (m,n)-invariant decreases complexity of the equivalence relation, i.e. partially resolves it.
- There can be trivial invariants which do not distinguish anything and do not carry any information such as constant functors $c : L \to L'$ (although they are $(deg(L), 0)$-invariants).

□

Examples

1. $deg(ea) = 0$; $deg(f : a \xrightarrow[isomorphisms]{\sim} a') = 1$; $deg(\mathbf{Set}) = 1$; $deg(\infty\text{-}\mathbf{Top}) = 2$; $deg(\infty\text{-}\mathbf{CAT}) = \infty$ (?).
2. Homology and cohomology functors $H_*, H^* : \infty\text{-}\mathbf{Top} \to \mathbf{Ab}$ (trivially extended over higher order cells) are $(2,1)$-invariants.
3. $\tilde{\pi}_n^I/\!\sim : L^*_{1\sim} \to \mathbf{Grp}$ is an $(\infty, 1)$-invariant (see proposition 1.2.1.2).
4. Let X be a smooth manifold with Lie group action $\rho : G \times X \to X$, L be a category with L^0, the set of submanifolds of X, $L^1(a,b) := \{(a,g,b) \in L^0 \times G \times L^0 \mid \rho(g,a) = b\}$, $L^n := eL^{n-1}$ for $n \geq 2$, L' be a category with $L'^0 := C^\infty(X, \mathbb{R})$ (smooth functions), $L'^1(f,h) := \{(f,g,h) \in L'^0 \times G \times L'^0 \mid f \circ \rho(g^{-1}, -) = h\}$, $L'^n := eL'^{n-1}$ for $n \geq 2$. If $F : L \to L'$ is a construction (functor) assigning invariant functions to objects from L then F is a $(1,0)$-invariant.
5. Each equivalence $L \xrightarrow{\sim} L'$ is $(deg(L), deg(L'))$-invariant with $deg(L) = deg(L')$.

1.2.1. Homotopy groups associated to ∞-categories.

Let L be an ∞-category in which $*$ strictly preserves e and $\sim$ (i.e. $*$ is a strict functor). Denote by $eqL := \{f \in L \mid \exists g.\ edf \sim g \circ_1 f, edg \sim f \circ_1 g\}$ the subset of equivalences of the ∞-category L. It may not be a category (because it is not closed under d, c, in general).

Definition 1.2.1.1. Assume, $L(I,-) : L \to \infty\text{-}\mathbf{CAT}$, $x \in L^0(I,a)$. Then $\tilde{\pi}_n^I(a,x) :=$

$$\begin{cases} (L^0(I,a), x) & \text{if n=0} \\ \mathbf{Aut}_{L(I,a)}(e^{n-1}x) := eqL(I,a)(e^{n-1}x, e^{n-1}x) \cap (L(I,a))^0(e^{n-1}x, e^{n-1}x) = & \\ = eqL(e^{n-1}x, e^{n-1}x) \cap L^{n+1} & \text{if } n > 0 \end{cases}$$

are (weak) **homotopy groups** of object a at point x with representing object $I \in L^0$. □

$\tilde{\pi}_0^I(a,x)$ or $\tilde{\pi}_0^I(a,x)/\sim$ are just pointed sets, $\tilde{\pi}_n^I(a,x)/\sim, n > 0$ are strict groups.

Remarks.

- If $L = \infty$-**Top**, $I = \mathbf{1}$ then $\tilde{\pi}_n^I(X,x) = [I^n/(I^{n-1} \times 0) \cup (I^{n-1} \times 1), X]$. The quotient map $(I^{n-1} \times 0) \cup (I^{n-1} \times 1) \twoheadrightarrow S^n$ induces a group homomorphism $\pi_n(X,x) \to \tilde{\pi}_n^I(X,x)$.

  If $L = \infty$-**TopMan**$_b$ (the infinity category of topological manifolds with boundary as objects, and homotopies relative to the boundary as higher order cells), $I = \mathbf{1}$ then $\tilde{\pi}_n^I(X,x) = [I^n/(I^{n-1} \times 0) \cup (I^{n-1} \times 1), X] \operatorname{rel}(\partial I^n) = [S^n, X] = \pi_n(X,x)$ (i.e. formal homotopy groups coincide with the usual ones).

- In the case when functors $\tilde{\pi}_n^I$ are **representable** (by certain cogroup objects $\tilde{S}_I^n$) we call the representing objects $\tilde{S}_I^n$ **(formal) spheres**. It makes sense to define (as usual) $\tilde{\pi}_n^I(a) := [\tilde{S}_I^n, a]$, but these two definitions will not always be equivalent. The first one is more internal, and the only external parameter is I.

- Any ∞-functor $F : L \to L'$, preserving $\sim$, induces (weak) **group homomorphisms** $F_{I,a} : \tilde{\pi}_n^I(a) \to \tilde{\pi}_n^{FI}(Fa)$. So, for example, every ∞-equivalence between full subcategories of ∞-**TopMan**$_b$, preserving the homotopy type of $\mathbf{1}$, will preserve the (usual) homotopy groups.

□

Definition 1.2.1.2. For a map $f : a \to b$ such that $f \circ x = y$, $x \in L^0(I,a)$, $y \in L^0(I,b)$ the **induced map** $f_* \equiv \tilde{\pi}_n^I(f) : \tilde{\pi}_n^I(a,x) \to \tilde{\pi}_n^I(b,y)$ is determined by restriction of the functor $L(I,f)$:

$$\begin{cases} L^0(I,a) \to L^0(I,b) : x' \mapsto f \circ_1 x' & \text{if } n = 0 \\ \mathbf{Aut}_{L(I,a)}(e^{n-1}x) \to \mathbf{Aut}_{L(I,b)}(e^{n-1}y) : g \mapsto \mu_{I,a,b}(e^n f, g) & \text{if } n > 0 \end{cases}$$

□

Remark. To be correctly defined, induced maps $\tilde{\pi}_n^I(f)$ for $n > 1$ need commutativity of $*$ with e. The first two "groups" $\tilde{\pi}_0^I(a,x), \tilde{\pi}_1^I(a,x)$ always make sense and depend functorially on objects. □

Proposition 1.2.1.1 (homotopy invariance of homotopy groups). *If* $x : I \to a$, $f \sim$

$f' \in L^0(a,b)$ such that $f \circ_1 x \sim f' \circ_1 x$ is a trivial equivalence (all arrows for $\sim$ are identities) then $\tilde{\pi}^I_n(f)/\sim\, = \tilde{\pi}^I_n(f')/\sim\, : \tilde{\pi}^I_n(a,x)/\sim\, \to \tilde{\pi}^I_n(b, f \circ x)/\sim$.

PROOF is immediate. □

Proposition 1.2.1.2. *$\tilde{\pi}^I_n/\sim\colon L^*_{1\sim} \to$ **Grp** is an $(\infty,1)$-invariant, where $L^*_{1\sim} := \coprod_{n \geq 0} L^{*n}_{1\sim}$,*

$$L^{*n}_{1\sim} := \begin{cases} L^{*n} \textit{ (pointed objects and maps)} & n = 0,1 \\ \textit{equivalences from } L^n \textit{ with dom and codom in } L^{*(n-1)}_{1\sim} & n > 1 \end{cases} .$$

PROOF. The partial functor $\tilde{\pi}^I_n/\sim\colon L^{*0} \coprod L^{*1} \to$ **Grp** is trivially extendable starting from equivalences on level 2 (because of proposition 1.2.1.1). □

Example (Fundamental Group)

Let 2-**Top** be the usual **Top** with homotopy classes of homotopies as 2-cells. Define the **fundamental groupoid** 2-functor as the representable $\Pi(-) := Hom_{2\text{-}\mathbf{Top}}(1,-) : 2\text{-}\mathbf{Top} \to 2\text{-}\mathbf{Cat}$:

$$\begin{cases} X \to \Pi(X) & Ob\,(\Pi(X)) \text{ are its points, } Ar\,(\Pi(X)) \text{ are homotopy classes of paths} \\ (X \xrightarrow{f} Y) \mapsto \Pi(f) & \text{transformation of fundamental groupoids, } \Pi(f) : \begin{cases} x \mapsto f(x) \\ [\gamma] \mapsto [f \circ \gamma] \end{cases} \\ (f \xrightarrow{[H]} f') \mapsto \Pi([H]) & \text{nat. trans. } \Pi([H]) = \{[H] * i_x\}_{x \in X} : Hom_{2\text{-}\mathbf{Top}}(1,f) \xrightarrow{\sim} Hom_{2\text{-}\mathbf{Top}}(1,f') \end{cases}$$

(where $\{[H] * i_x\}_{x \in X} = \{[H(x,-)]\}_{x \in X}$ are homotopy classes of paths between $f(x)$ and $f'(x)$ natural in $x \in X$).

$\pi_1(X,x_0) := \mathbf{Aut}_{\Pi(X)}(x_0) \hookrightarrow \Pi(X)$ is the **fundamental group** of the space X at point $x_0 \in X$, $\pi_1((X,x_0) \xrightarrow{f} (Y,y_0)) := \mathbf{Aut}_{\Pi(X)}(x_0) \xrightarrow{\Pi(f)} \mathbf{Aut}_{\Pi(Y)}(y_0)$.

Proposition 1.2.1.3.

- *If $[H] : f \xrightarrow{\sim} f' : X \to Y$ is a 2-cell in* 2-**Top** *then $\pi_1(f')([\gamma]) = [H(x_0,-)] \circ \pi_1(f)([\gamma]) \circ [H(x_0,-)]^{-1}$ for all $[\gamma] \in \pi_1(X,x_0)$.*
- *In the case $[H] : f \xrightarrow{\sim} f' : (X,x_0) \to (Y,y_0)$ is a pointed 2-cell ($[H(x_0,-)] = 1_{f(x_0)} : f(x_0) \to f(x_0) = f'(x_0)$) then $\pi_1(f) = \pi_1(f')$.*

PROOF follows from the naturality square

$$\begin{tikzcd} f(x_0) \arrow[r, "{[H(x_0,-)]}", "\sim"'] \arrow[d, "\Pi(f)({[\gamma]})"'] & f'(x_0) \arrow[d, "\Pi(f')({[\gamma]})"] \\ f(x_0) \arrow[r, "\sim", "{[H(x_0,-)]}"'] & f'(x_0) \end{tikzcd}$$

□

1.3. Representable ∞-functors

Definition 1.3.1. ∞-categories L and L' are **equivalent** if $L \sim L'$ in ∞-**CAT**. □

If equivalence $L \sim L'$ is given by functors $\begin{tikzcd} L \arrow[r, bend left, "F", "\sim"'] & L' \arrow[l, bend left, "G", "\sim"'] \end{tikzcd}$ then $\forall a \in L^0$ $a \sim G \circ F(a)$, $\forall b \in L'^0$ $b \sim F \circ G(b)$ naturally in a and b.

Definition 1.3.2. ∞-functor $F : L \to L'$ is (weakly)

- **faithful** if $\forall a, a' \in L^0$ $\forall f^n, g^n \in L^n(a,a')$ $F(f^n) \sim F(g^n) \Rightarrow f^n \sim g^n$,
- **full** if $\forall a, a' \in L^0$ $\forall h^n \in L'^n(F(a), F(a'))$ $\exists f^n \in L^n(a,a')$ such that $F(f^n) \sim h^n$,
- **surjective on objects** if $\forall b \in L'^0$ $\exists a \in L^0$ such that $F(a) \sim b$. □

Unlike first order equivalence, there is no simple criterion of higher order equivalence.

Proposition 1.3.1. *If the functor* $L \xrightarrow[\sim]{F} L'$ *is an equivalence then* F *is (weakly) faithful, full and surjective on objects.*

PROOF. "$\Rightarrow$" Regard the diagram (where G is a quasiinverse of F)

$$\begin{tikzcd} a \arrow[r, bend left, "e^n\rho_a", "\sim"'] \arrow[d, "f^n"'] & G \circ F(a) \arrow[l, bend left, "e^n\theta_a", "\sim"'] \arrow[d, "G(F(f^n))"] \\ a' \arrow[r, bend left, "e^n\rho_{a'}", "\sim"'] & G \circ F(a') \arrow[l, bend left, "e^n\theta_{a'}", "\sim"'] \end{tikzcd}$$

where: $f^n \in L^n(a,a')$, $e^n\rho_a \in L^n(a, G(F(a)))$, $e^n\theta_a \in L^n(G(F(a)), a)$, $n \geq 0$.

Take $f^n, g^n : a \to a' \in L^n(a,a')$ such that $F(f^n) \sim F(g^n)$. Then $f^n \sim e^n\theta_{a'} \circ_{n+1} G(F(f^n)) \circ_{n+1} e^n\rho_a \sim e^n\theta_{a'} \circ_{n+1} G(F(g^n)) \circ_{n+1} e^n\rho_a \sim g^n$, i.e., F is faithful (G is faithful by symmetry).

Take $\alpha^n : F(a) \to F(a') \in L'^n(F(a), F(a'))$. Then $\beta^n := e^n\theta_{a'} \circ_{n+1} G(\alpha^n) \circ_{n+1} e^n\rho_a : a \to a' \in L^n(a, a')$ is such that $G(F(\beta^n)) \sim G(\alpha^n)$. So, $F(\beta^n) \sim \alpha^n$ because G is faithful. Therefore, F is full (G is full by symmetry).

F and G are obviously surjective on objects. ☐

Remark. The inverse direction "$\Leftarrow$" for the above proposition works only partially. Namely, for each $b \in L'^0$ choose $G(b) \in L^0$ and equivalence $b \underset{\theta_b}{\overset{\rho_b}{\underset{\sim}{\overset{\sim}{\rightleftarrows}}}} F(G(b))$ (which is possible since F is surjective on objects), moreover, if $b = F(a)$ choose $G(b) = a$, $\rho_b = eb$, $\theta_b = e(F(G(b))) = eb$. For each $f^n : b \to b' \in L'^n(b, b')$ choose an element $G(f^n) \in L^n(G(b), G(b'))$ such that $e^n\rho_{b'} \circ_{n+1} f^n \circ_{n+1} e^n\theta_b \sim F(G(f^n))$ (which is possible since F is fully faithful). Then $G : L' \to L$ is obviously a (weak) functor. $a = G(F(a))$ is natural in a by construction, but $b \sim F(G(b))$ is natural in b for only first order arrows ρ_b, θ_b presenting $\sim$. So, F should be somehow 'naturally surjective on objects' which does not make sense yet when the functor G is not defined. ☐

Definition 1.3.3. An ∞-functor $F : L \to L'$ is called

- an **isomorphism** if it is a bijection (on sets L, L') and the inverse map is a functor,
- a **quasiisomorphism** if there exists a functor $G : L' \to L$ such that $\forall a^n \in L^n\ G(F(a^n)) \sim a^n$ and $\forall b^n \in L'^n\ F(G(b^n)) \sim b^n,\ n \geq 0$. ☐

Proposition 1.3.2. *The notions of (functor) isomorphism and quasiisomorphism coincide.*

Proof. Each isomorphism is a quasiisomorphism. Conversely, if $L \underset{G}{\overset{F}{\rightleftarrows}} L'$ is a quasiisomorphism then $\forall a^n \in L^n,\ n \geq 0,\ G(F(ea^n)) \sim ea^n$. So, $d(G(F(ea^n))) = dea^n$, i.e. $G(F(dea^n)) = dea^n$ and $G(F(a^n)) = a^n$ (instead of d, c could be used). The same, $\forall b^n \in L'^n$, $n \geq 0,\ F(G(b^n)) = b^n$. ☐

Denote (quasi)isomorphism (equivalence) relation by $\simeq$.

Examples (isomorphic ∞-categories)

1. Assume, $f^n \xrightarrow[\alpha]{\simeq} g^n$ are isomorphic elements of degree n (in a strict category L) then $L(f^n, f^n) \simeq L(g^n, g^n)$ are isomorphic ∞-categories. Indeed, there is an isomorphism $F : L(f^n, f^n) \to L(g^n, g^n) : x \mapsto \alpha * (x * \alpha^{-1})$, where $*$ means a horizontal composite. F is a functor. Its inverse is $G : L(g^n, g^n) \to L(f^n, f^n) : y \mapsto \alpha^{-1} * (y * \alpha)$. [For α just an equivalence, it is not true]
2. ∞-**CAT**$(L(-, a), F) \simeq F(a)$ (see below, the Yoneda Lemma). ☐

Definition 1.3.4. Two n-modifications $\alpha^n, \beta^n : L \to \infty$-**CAT**, $n \geq 0$, are called **quasiequivalent** of depth k, $0 \leq k \leq n+1$, (denote it by $\alpha^n \approx_k \beta^n$) if their corresponding components are quasiequivalent of depth $k-1$, i.e. $\forall a \in L^0$ $\alpha^n(a) \approx_{k-1} \beta^n(a)$. $\approx_0$ means $\sim$ by definition. [In other words, $\alpha^n \approx_k \beta^n$ if all their components of components on depth k are equivalent, i.e. $\alpha^n \approx_0 \beta^n$ if they are equivalent $\alpha^n \sim \beta^n$; $\alpha^n \approx_1 \beta^n$ if their components are equivalent $\forall a \in L^0$ $\alpha^n(a) \sim \beta^n(a)$; $\alpha^n \approx_2 \beta^n$ if components of all components are equivalent; etc.]. If $\alpha^n, \beta^n : L \to L'$ are proper n-modifications (living in ∞-**CAT**) for them only $\approx_0$ and $\approx_1$ make sense. ☐

Lemma 1.3.1.

- *$\approx_k$ is an equivalence relation.*
- *$\approx_{k_1} \Rightarrow \approx_{k_2}$ if $k_1 \leq k_2$.*
- *If $\alpha^n \approx_k \beta^n$ then $d\alpha^n = d\beta^n$, $c\alpha^n = c\beta^n$.*
- *If $(L_1, \approx_{k_1})$, $(L_2, \approx_{k_2})$ are two ∞-categories (not necessarily proper, i.e. living in ∞-**CAT**) for which given equivalence relations make sense for all elements, and $F : L_1 \to L_2$, $G : L_2 \to L_1$ are maps (not necessarily functors) such that $\forall l_1 \in L_1$ $G(F(l_1)) \approx_{k_1} l_1$ and $\forall l_2 \in L_2$ $F(G(l_2)) \approx_{k_2} l_2$, and F, G both preserve d (or c) then F, G are bijections inverse to each other.*

- *For* $L, L' \in Ob\,(\infty\text{-}\mathbf{CAT})$ *and* $a \in L^0$ *the map* $ev_a : \infty\text{-}\mathbf{CAT}(L, L') \to L' : f^n \mapsto f^n(a)$ *is a strict functor. [Similar statement holds when* L, L' *are not proper, e.g.* $\infty\text{-}\mathbf{CAT}$*, but we need to formulate it for a bigger universe containing* $\infty\text{-}\mathbf{CAT}$*]*

Proof. The first two statements are obvious. The third one follows from the fact $x \sim y \Rightarrow dx = dy,\ cx = cy$ and that d, c are taken componentwise. The fourth statement follows by the same argument as in the proof of proposition 1.3.2. The last statement holds because, again, all operations in $\infty\text{-}\mathbf{CAT}(L, L')$ are taken componentwise. □

Remark. For the proof of the Yoneda lemma, a double evaluation functor is needed. For two functors $F, G : L \to \infty\text{-}\mathbf{CAT}$ take the restriction of the evaluation functor ev_a on the hom-set between F and G, i.e. $ev_{a\,F,G} : \infty\text{-}\mathbb{CAT}(L, \infty\text{-}\mathbf{CAT})(F, G) \to \infty\text{-}\mathbf{CAT}(F(a), G(a)) : f^n \mapsto f^n(a)$, where $\infty\text{-}\mathbb{CAT}$ is a bigger (and weaker) universe containing $\infty\text{-}\mathbf{CAT}$ as an object. Now, take a second evaluation functor $ev_x : \infty\text{-}\mathbf{CAT}(F(a), G(a)) \to G(a) : g^n \mapsto g^n(x),\ x \in (F(a))^0$. Then the double evaluation functor is the composite $ev_x \circ_1 ev_{a\,F,G} : \infty\text{-}\mathbb{CAT}(L, \infty\text{-}\mathbf{CAT})(F, G) \to G(a) : f^n \mapsto f^n(a)(x)$. It is a strict functor. □

$\infty\text{-}\mathbf{CAT}$-valued functors, natural transformations and modifications live now in a bigger universe $\infty\text{-}\mathbb{CAT}$, and we do not yet have for them appropriate definitions.

Definition 1.3.5. $\infty\text{-}\mathbf{CAT}$-valued functors, natural transformations and modifications are introduced in a similar way as the usual ones by changing all occurrences of $\sim$ with (one degree weaker relation) $\approx_1$, i.e.

- a map $F : L \to \infty\text{-}\mathbf{CAT}$ of degree 0 is a **functor** if F strictly preserves d and c, $Fdx = dFx,\ Fcx = cFx$, and weakly up to $\approx_1$ preserves e and composites, $Fex \approx_1 eFx,\ F(x \circ_k y) \approx_1 F(x) \circ_k F(y)$,
- For a given sequence of two functors $F, G : L \to \infty\text{-}\mathbf{CAT}$, …, two $(n-1)$-modifications $\alpha_1^{n-1}, \alpha_2^{n-1} : \alpha_1^{n-2} \to \alpha_2^{n-2}$ strict (or weak) n-**modification** $\alpha^n : \alpha_1^{n-1} \to \alpha_2^{n-1}$ is a map

$\alpha^n : L^0 \to \infty\text{-}\mathbf{CAT}^{n+1}$ such that $\forall a, b \in L^0$ $\alpha^n(b) * F(-) \approx_1 G(-) * \alpha^n(a) : L^{\geq n}(a,b) \to L'^{\geq n}(F(a), G(b))$ (components of values of functors are equivalent). □

Definition 1.3.6. A covariant (contravariant) functor $F : L \to \infty\text{-}\mathbf{CAT}$ is

- **weakly representable** if $\exists a \in L^0$ such that $L(a,-) \sim F$ $(L(-,a) \sim F)$. It means there is an equivalence of two ∞-categories $L(a,b) \sim F(b)$ $(L(b,a) \sim F(b))$ natural in b,
- **strictly representable** if there exists $a \in L^0$ such that $L(a,-) \simeq F$ $(L(-,a) \simeq F)$, i.e. $\forall b \in L^0$ $\exists$ an isomorphism $L(a,b) \simeq F(b)$ $(L(b,a) \simeq F(b))$ natural in b. □

Lemma 1.3.2. *For given representable $L(-,a) : L^{op} \to \infty\text{-}\mathbf{CAT}$ and functor $F : L^{op} \to \infty\text{-}\mathbf{CAT}$*

- *all natural transformations $\tau^0 : L(-,a) \to F$ are of the form $\forall\, b \in Ob\, L$ the b-component is a functor $\tau^0_b : L(b,a) \to F(b)$, $\tau^0_b(f^m) \sim F(f^m)(\tau^0_a(ea))$, $f^m \in L^m(b,a)$,*
- *all n-modifications $\tau^n : L(-,a) \to F$, $n \geq 1$, are of the form $\forall\, b \in Ob\, L$ the b-component is a $(n-1)$-modification $\tau^n_b : L(b,a) \to F(b)$, $\tau^n_b(f^0) \sim F(f^0)(\tau^n_a(ea))$, $f^0 \in L^0(b,a)$.*

Proof : follows from the naturality square

$$\begin{tikzcd}
a & L(a,a) \arrow[r, "\tau^n_a"] \arrow[d, "L(f^m{,}a)"'] & F(a) \arrow[d, "F(f^m)"] \\
b \arrow[u, "f^m"] & L(b,a) \arrow[r, "\tau^n_b"'] & F(b)
\end{tikzcd}$$

$n \geq 0$ □

Lemma 1.3.3. *For a given n-cell $\beta^n \in (F(a))^n$, $n \geq 0$, n-modification $\tau^n : L(-,a) \to F$ such that $\tau^n_a(ea) = \beta^n$ exists and unique up to $\approx_2$.*

Proof:. Uniqueness follows from lemma 1.3.2, existence from the definition of n-modification $\tau^n_b(f^m) := F(f^m)(\beta^n)$ (for $n > 0$, $m = 0$ only) and the naturality square showing correctness of the definition

$$\begin{tikzcd}
b & L(b,a) \arrow[r, "\tau^n_b"] \arrow[d, "L(g^k{,}a)"'] & F(b) \arrow[d, "F(g^k)"] \\
c \arrow[u, "g^k"] & L(c,a) \arrow[r, "\tau^n_c"'] & F(c)
\end{tikzcd}$$

$(\mu_{c,b,a}(f^m, g^k) := \mu_{c,b,a}(e^{max(m,k)-m} f^m, e^{max(m,k)-k} g^k))$ □

Corollary 1. All n-modifications $\tau^n : L(-,a) \to F$, $n \geq 0$, have strict form $\tau_b^n(f^0) = F(f^0)(\tau_a^n(ea))$, $f^0 \in L^0(b,a)$. □

Corollary 2 (criterion of representability). A ∞-**CAT**-valued presheaf $F : L^{op} \to \infty$-**CAT** is

- **strictly representable** (with representing object $a \in L^0$) iff there exists an object $\beta^0 \in (F(a))^0$ such that $\forall \gamma^n \in (F(b))^n$, $n \geq 0$, $\exists!$ n-arrow $(f^n : b \to a) \in L^n(b,a)$ with $\gamma^n = F(f^n)(\beta^0)$,
- **weakly representable** (with representing object $a \in L^0$) iff there exists an object $\beta^0 \in (F(a))^0$ such that $\forall b \in Ob\, L$ the functor $L(b,a) \to F(b) : f^n \mapsto F(f^n)(\beta^0)$ is an equivalence of categories.

(Similar statements hold for a covariant presheaf $F : L \to \infty$-**CAT**) □

Proposition 1.3.3 (Yoneda Lemma). *For the functor $F : L^{op} \to \infty$-**CAT** and the object $a \in L^0$, there is a strict isomorphism $\infty\text{-}\mathbb{CAT}(L(-,a),F) \simeq F(a)$ natural in a and F.*

Proof. Strict functoriality of the correspondence $\tau^n \mapsto \tau_a^n(ea)$ is straightforward (because it is a double evaluation functor). The map $\beta^n \mapsto F(-)(\beta^n)$ is quasiinverse to the first map (with respect to $\approx_2$ and $=$ equivalence relations in $\infty\text{-}\mathbb{CAT}(L(-,a),F)$ and $F(a)$ respectively), and it strictly preserves d and c. So, these both maps are strict isomorphisms.

Naturality is given by

$$\begin{tikzcd}
a & F \arrow[d, "\alpha^k"'] & \infty\text{-}\mathbb{CAT}(L(-,a),F) \arrow[r, "\simeq"] \arrow[d, "\infty\text{-}\mathbb{CAT}(L(-,f^m),\alpha^k)"'] & F(a) \arrow[d, "\alpha^k(f^m)"] \\
b \arrow[u, "f^m"] & G & \infty\text{-}\mathbb{CAT}(L(-,b),G) \arrow[r, "\simeq"'] & G(b)
\end{tikzcd}$$

(where $\alpha^k(f^m) := \mu_{F(a),F(b),G(b)}(e^{max(k,m)-k}\alpha_b^k, e^{max(k,m)-m+1}F(f^m))$, $k,m \geq 0$) □

Remark. The Yoneda lemma for ∞-categories is similar to the one for first order categories with the difference that elements $\beta^n \in (F(a))^n$ of degree n now determine higher degree arrows (n-modifications) $\beta^n : L(-,a) \to F$ in a ∞-**CAT**-valued presheaf category. □

Proposition 1.3.4 (Yoneda embedding). *There is a Yoneda embedding* $\mathbf{Y}: L \to \infty\text{-}\mathbf{CAT}^{L^{op}}$: $\alpha \mapsto L(-,\alpha)$, $\alpha \in L$, *which is an extension of the isomorphisms from the Yoneda lemma determined on hom-sets* $L(a,b)$, $a,b \in L^0$. *The Yoneda embedding preserves and reflects equivalences* $\sim$.

PROOF. By the Yoneda isomorphism for a given $f^n \in L^n(a,b)$, the corresponding n-modification is $L(-,b)(f^n) : L(-,a) \to L(-,b)$ which is the same as $L(-,f^n) : L(-,a) \to L(-,b)$; i.e. the functor $\mathbf{Y}: L \to \infty\text{-}\mathbf{CAT}^{L^{op}}$: $\alpha \mapsto L(-,\alpha)$, $\alpha \in L$, locally coincides with isomorphisms from the Yoneda lemma. By lemma 1.1.3 this functor preserves and reflects equivalences $\sim$. ☐

Remark. Under the assumption that the category ∞-**CAT** of **weak** categories, functors and n-modifications exists, all the above reasons remain essentially the same, i.e. the Yoneda lemma and embedding seem to hold in a weak situation. ☐

1.4. (Co)limits

Definition 1.4.1. An ∞-**graph** is a graded set $G = \coprod_{n\geq 0} G^n$ with two unary operations $d,c : \coprod_{n\geq 1} G^n \to \coprod_{n\geq 0} G^n$ of degree -1 such that $d^2 = dc$, $c^2 = cd$. ☐

Definition 1.4.2. An ∞-**diagram** $D : G \to L$ from ∞-graph G to ∞-category L is a function of degree 0 which preserves operations d, c. ☐

Proposition 1.4.1. *All diagrams from* G *to* L, *natural transformations, modifications form an* ∞*-category* $\mathbf{Dgrm}_{G,L}$ *in the same way as the functor category* ∞-$\mathbf{CAT}(L', L)$. ☐

For a given object $a \in L^0$ the **constant diagram** to a is $\Delta(a) : G \to L : g \mapsto e^n a$ if $g \in G^n$. $\Delta : L \to \mathbf{Dgrm}_{G,L}$ is an ∞-functor.

Denote $\{e\}\alpha := \{\alpha, e\alpha, e^2\alpha, ..., e^n\alpha, ...\}$, $\alpha \in L$.

Definition 1.4.3. Diagram $D : G \to L$ has

- a **limit** if the functor $\mathbf{Dgrm}_{G,L}(\Delta(-),D) : L^{op} \to \infty\text{-}\mathbf{CAT}$ is representable.

  If $\nu : L(-,a) \xrightarrow{\sim} \mathbf{Dgrm}_{G,L}(\Delta(-),D)$ is an equivalence then

  $\nu_a(\{e\}ea) \subset \mathbf{Dgrm}_{G,L}(\Delta(a),D)$ is called a **limit cone** over D, a is its **vertex** (or diagram **limit** $lim\, D$), $\nu_a(ea)$ are its **edges**, $\nu_a(e^k a)$, $k > 1$, are identities

- **colimit** if functor $\mathbf{Dgrm}_{G,L}(D,\Delta(-)) : L \to \infty\text{-}\mathbf{CAT}$ is representable.

  If $\nu : L(a,-) \xrightarrow{\sim} \mathbf{Dgrm}_{G,L}(D,\Delta(-))$ is the equivalence then

  $\nu_a(\{e\}ea) \subset \mathbf{Dgrm}_{G,L}(D,\Delta(a))$ is called **colimit cocone** over D, a is its **vertex** (or diagram **colimit** $colim\, D$), $\nu_a(ea)$ are its **edges**, $\nu_a(e^k a)$, $k > 1$, are identities □

Remark. The conditions on equivalence ν in the above definition can be strengthened. If it is a (natural) isomorphism then (co)limits are called **strict** and as a rule they are different from **weak** ones [Borl].

Proposition 1.4.2. *For strict (co)limits the following is true*

- $$\begin{tikzcd} L \arrow[r, "\top", "\Delta"'] & \mathbf{Dgrm}_{G,L} \arrow[l, bend right=60, "lim"'] \arrow[l, bend left=60, "colim", "\top"'] \end{tikzcd}$$

- *Strict right adjoints preserve limits (strict left adjoints preserve colimits).*

PROOF.

- It is immediate from definition 1.4.3 and proposition 1.5.1.
- The argument is the same as for first order categories (see example 13.a, point 1.1) [the essential thing is that a strict adjunction is determined by (triangle) identities which are preserved under ∞-functors]. □

Examples

1. (strict binary products in 2-**Top** and 2-**CAT**) They coincide with '1-dimensional' products.

The mediating 2-cell arrow is given componentwise

$$\begin{tikzcd}
& A \\
C \arrow[ur, bend left, "f"{name=F}] \arrow[ur, "f'"'{name=F1}] \arrow[r, bend left=15, "<f{,}g>"{name=U}] \arrow[r, bend right=15, "<f'{,}g'>"'{name=L}] \arrow[dr, "g"{name=G}] \arrow[dr, bend right, "g'"'{name=G1}] & A \times B \arrow[u, "p_1"'] \arrow[d, "p_2"] \\
& B
\arrow[Rightarrow, from=F, to=F1, "\alpha"]
\arrow[Rightarrow, from=U, to=L, "<\alpha{,}\beta>"]
\arrow[Rightarrow, from=G, to=G1, "\beta"]
\end{tikzcd}$$

2. ("equalizer" of a 2-cell in 2-**CAT**) [Borl] For a given 2-cell $\begin{tikzcd} \mathbf{A} \arrow[r, bend left, "F"{name=U}] \arrow[r, bend right, "G"'{name=D}] & \mathbf{B} \arrow[Rightarrow, from=U, to=D, "\alpha"] \end{tikzcd}$ in 2-**CAT** its strict limit is a subcategory $\mathbf{E} \hookrightarrow \mathbf{A}$ such that $F(A) = G(A)$ and $\alpha_A = 1_{F(A)} : F(A) \to G(A)$ (on objects), and $F(f) = G(f)$ (on arrows).

3. (strict and weak pullbacks in 2-**CAT**) [Borl] Let $\mathcal{P}$ be a "2-dimensional" graph $1 \xrightarrow{x} 0 \xleftarrow{y} 2$ with trivial 2-cells, $F : \mathcal{P} \to$ 2-**CAT** be a 2-functor. Then its limit is a pullback diagram in 2-**CAT** $$\begin{tikzcd} F(1) \times_{F(0)} F(2) \arrow[r, "p_2"] \arrow[d, "p_1"'] \arrow[dr, "p_3"] & F(2) \arrow[d, "F(y)"] \\ F(1) \arrow[r, "F(x)"'] & F(0) \end{tikzcd}.$$ When the limit is taken **strictly** $F(1) \times_{F(0)} F(2)$ coincides with the "1-dimensional" pullback, i.e. $F(1) \times_{F(0)} F(2) \hookrightarrow F(1) \times F(2)$ is a subcategory consisting of objects (A, B), $A \in Ob\,F(1)$, $B \in Ob\,F(2)$, $F(x)(A) = F(y)(B)$ and arrows (f, g), $f \in Ar\,F(1)$, $g \in Ar\,F(2)$, $F(x)(f) = F(y)(g)$. When the limit is taken **weakly** $F(1) \times_{F(0)} F(2)$ is not a subcategory of product $F(1) \times F(2)$. It consists of 5-tuples (A, B, C, f, g), $A \in Ob\,F(1)$, $B \in Ob\,F(2)$, $C \in Ob\,F(0)$, $f : F(x)(A) \xrightarrow{\sim} C$, $g : F(y)(B) \xrightarrow{\sim} C$ are isomorphisms, with arrows (a, b, c), $a : A \to A'$, $b : B \to B'$, $c : C \to C'$ such that $c \circ f = f' \circ F(x)(a)$, $c \circ g = g' \circ F(y)(b)$. Projections p_1, p_2, p_3 are obvious. The pullback square commutes up to isomorphisms $f : F(x) \circ p_1 \Rightarrow p_3$, $g : F(y) \circ p_2 \Rightarrow p_3$. $\square$

1.5. Adjunction

Definition 1.5.1. The situation $\begin{tikzcd} L \arrow[r, bend left, "F"] \arrow[r, phantom, "\perp"] & L' \arrow[l, bend left, "G"] \end{tikzcd}$ (where L, L' are ∞-categories, F, G are ∞-

functors) is called

- **weak ∞-adjunction** if there is an equivalence $L(-,G(+)) \sim L'(F(-),+) : L^{op} \times L' \to \infty$-**CAT** (i.e. $L(a,G(b)) \sim L'(F(a),b)$ natural in $a \in L^0$, $b \in L'^0$),
- **strict ∞-adjunction** if there is an isomorphism $L(-,G(+)) \simeq L'(F(-),+) : L^{op} \times L' \to \infty$-**CAT** (i.e. $L(a,G(b)) \simeq L'(F(a),b)$ natural in $a \in L^0$, $b \in L'^0$). □

Proposition 1.5.1. *The following are equivalent*

1. $\begin{tikzcd} L \arrow[r, bend left, "F"] \arrow[r, phantom, "\perp"] & L' \arrow[l, bend left, "G"] \end{tikzcd}$ *is a strict ∞-adjunction*
2. $\forall b \in L'^0$ $L'(F(-),b)$ *is strictly representable*
3. $\forall a \in L^0$ $L(a,G(-))$ *is strictly representable*

PROOF.

- 1. $\Longrightarrow$ 2., 3. is immediate
- 2. $\Longrightarrow$ 1. From the criterion of strict representability (see point 1.3) it follows that $\forall\, b \in L'^0$ there exists a "universal element" $(\beta^0_b : F(G(b)) \to b) \in L'^0(F(G(b)),b)$ such that $\forall\,(f^n : F(c) \to b) \in L'^n(F(c),b)$ $\exists!$ n-arrow $(g^n : c \to G(b)) \in L^n(c,G(b))$ with $f^n = \mu_{F(c),F(G(b)),b}(e^n\beta^0_b, F(g^n))$

$$\begin{tikzcd} G(b) & F(G(b)) \arrow[r, "e^n\beta^0_b"] & b \\ c \arrow[u, dashed, "\exists! g^n"] & F(c) \arrow[u, "F(g^n)"] \arrow[ur, "\forall f^n"'] & \end{tikzcd}$$

Consequently, $\forall\,(f^n : b' \to b) \in L'^n(b',b)$ the diagram holds

$$\begin{tikzcd} G(b) & F(G(b)) \arrow[r, "e^n\beta^0_b"] & b \\ G(b') \arrow[u, dashed, "G(f^n)"] & F(G(b')) \arrow[u, "F(G(f^n))"] \arrow[r, "e^n\beta^0_{b'}"'] & b' \arrow[u, "f^n"'] \end{tikzcd}$$

It shows that assignment $Ob\,L' \ni b \mapsto G(b) \in Ob\,L$ is extendable to a functor $G : L' \to L$ (in an essentially unique way) and that $\beta^0 : FG \to 1_{L'}$ is a natural transformation (**counit** ε of the adjunction $F \dashv G$).

Isomorphism $\varphi_{c,b} : L'(F(c), b) \to L(c, G(b))$ such that

$$\begin{tikzcd} F(G(b)) \arrow[r, "e^n\beta^0_b"] & b \\ F(c) \arrow[u, "F(\varphi_{c,b}(f^n))"] \arrow[ur, "f^n"'] & \end{tikzcd}$$

is natural in $c \in Ob\, L$, $b \in Ob\, L'$ because of the naturality square

$$\begin{tikzcd} c & b \arrow[d, "f^n"'] & L'(F(c), b) \arrow[r, "\varphi_{c,b}"] \arrow[d, "L'(F(g^n), f^n)"'] & L(c, G(b)) \arrow[d, "L(g^n, G(f^n))"] \\ c' \arrow[u, "g^n"] & b' & L'(F(c'), b') \arrow[r, "\varphi_{c',b'}"'] & L(c', G(b')) \end{tikzcd}$$

(indeed, $\forall h^n \in L'^n(F(c), b)\;\; G(f^n) * \varphi_{c,b}(h^n) * g^n \sim \varphi_{c',b'}(f^n * h^n * F(g^n))$, where $*$ is the horizontal composite, since $e^n\beta^0_{b'} * F(G(f^n) * \varphi_{c,b}(h^n) * g^n) \sim f^n * e^n\beta^0_b * F(\varphi_{c,b}(h^n)) * F(g^n) \sim f^n * h^n * F(g^n)$)

- 3. $\Longrightarrow$ 1. is similar to 2. $\Longrightarrow$ 1. □

Remark. The analogous statement for a weak ∞-adjunction is not true. In the above proof "universal elements" were used in an essential way. □

Definition 1.5.2. For a given **strict** ∞-adjunction $\begin{tikzcd} L \arrow[r, bend left, "F"] \arrow[r, phantom, "\perp"] & L' \arrow[l, bend left, "G"] \end{tikzcd}$

- universal elements $\varepsilon_b : F(G(b)) \to b$ representing functors $L'(F(-), b)$ ($b \in Ob\, L'$ is a parameter) form a natural transformation $\varepsilon : FG \to 1_{L'}$ which is called the **counit** of the adjunction,
- Universal elements $\eta_a : a \to G(F(a))$ representing functors $L(a, G(-))$ ($a \in Ob\, L$ is a parameter) form a natural transformation $\eta : 1_L \to GF$ which is called the **unit** of the adjunction.

□

Remark. For a **weak** ∞-adjunction no useful unit and counit exist. □

Proposition 1.5.2.

- *For both weak and strict adjunctions: the composition of left adjoints is a left adjoint (the composition of right adjoints is a right adjoint).*
- *For a weak (strict) adjunction, a right or left adjoint is determined uniquely up to equivalence $\sim$ (up to isomorphism $\simeq$).*

PROOF.

- If $\begin{tikzcd} L \arrow[r, bend left, "F"] \arrow[r, phantom, "\perp"] & L' \arrow[l, bend left, "G"] \arrow[r, bend left, "F'"] \arrow[r, phantom, "\perp"] & L'' \arrow[l, bend left, "G'"] \end{tikzcd}$ then $L''(F'Fl, l'') \sim L'(Fl, G'l'') \sim L(l, GG'l'')$ (composite of natural equivalences). [For a strict adjunction the same reason works]
- Assume, $L'(a, G'b) \sim L(Fa, b) \sim L'(a, Gb)$ are natural equivalences then $L'(-, G'b) \sim L'(-, Gb)$ is a natural transformation (equivalence) natural in b. Then, by the Yoneda embedding, $G'b \sim Gb$ naturally in b, i.e. $G' \sim G$. [Again, changing $\sim$ with $\simeq$ still works]. □

Proposition 1.5.3. *For a strict adjunction* $\begin{tikzcd} L \arrow[r, bend left, "F"] \arrow[r, phantom, "\perp"] & L' \arrow[l, bend left, "G"] \end{tikzcd}$ *the Kan definition and the definition via "unit-counit" coincide, i.e. the following are equivalent*

- $\varphi_{a,b} : L(a, G(b)) \simeq L'(F(a), b) : \varphi^*_{a,b}$ *natural in* $a \in L^0$, $b \in L^{\prime 0}$,
- $\exists$ *natural transformations* $\eta : 1_L \to GF$ *and* $\varepsilon : FG \to 1_{L'}$ *satisfying the triangle identities* $\varepsilon F \circ_1 F\eta = 1_F$ *and* $G\varepsilon \circ_1 \eta G = 1_G$.

PROOF. For a strict adjunction, the same proof as for first order categories works.

- Universal elements η_a, ε_b for functors $L(a, G(-)), L'(F(-), b)$ mean that they are images of $1_{F(a)}, 1_{G(b)}$ under functors $\varphi^*_{a,F(a)}, \varphi_{G(b),b}$, i.e.

$$\begin{tikzcd} FGFa \arrow[r, "\varepsilon_{Fa}"] & Fa & Gb \arrow[r, "\eta_{Gb}"] \arrow[dr, "1_{Gb}"'] & GFGb \arrow[d, dashed, "G\varepsilon_b"] \\ Fa \arrow[u, dashed, "F\eta_a"] \arrow[ur, "1_{Fa}"'] & & & Gb \end{tikzcd}$$

(strict equalities)

- Define maps $\begin{cases} \varphi_{a,b}(f^n) := e^n(\varepsilon_b) \circ_{n+1} F(f^n), & f^n \in L^n(a, G(b)) \\ \varphi^*_{a,b}(g^n) := G(g^n) \circ_{n+1} e^n(\eta_a), & g^n \in L'^n(F(a), b) \end{cases}$

They are functors $\begin{cases} \varphi_{a,b} := \varepsilon_b * F(-) : L(a, G(b)) \to L'(F(a), b) \\ \varphi^*_{a,b} := G(-) * \eta_a : L'(F(a), b) \to L(a, G(b)) \end{cases}$ and inverses to each other: $\varphi^*_{a,b}(\varphi_{a,b}(f^n)) = \varphi^*_{a,b}(e^n\varepsilon_b \circ_{n+1} F(f^n)) = G(e^n\varepsilon_b \circ_{n+1} F(f^n)) \circ_{n+1} e^n\eta_a = e^nG(\varepsilon_b) \circ_{n+1} (GF(f^n) \circ_{n+1} e^n\eta_a) = e^nG(\varepsilon_b) \circ_{n+1} (e^n\eta_{G(b)} \circ_{n+1} f^n) = e^{n+1}G(b) \circ_{n+1} f^n = f^n$, and similar, $\varphi_{a,b}(\varphi^*_{a,b}(g^n)) = g^n$.

Naturality (e.g., of $\varphi_{a,b}$) follows from the square

$$\begin{tikzcd} L(a, G(b)) \arrow[r, "\varphi_{a,b}"] \arrow[d, "L(x^m, G(y^m))"'] & L'(F(a), b) \arrow[d, "L'(F(x^m), y^m)"] \\ L(a', G(b')) \arrow[r, "\varphi_{a',b'}"'] & L'(F(a'), b') \end{tikzcd}$$

$(\varphi_{a',b'}(L(x^m, G(y^m))(f^n)) = \varphi_{a',b'}(G(y^m) * f^n * x^m) = \varepsilon_{b'} * FG(y^m) * F(f^n) * F(x^m) = y^m * \varepsilon_b * F(f^n) * F(x^m) = L'(F(x^m), y^m)(\varphi_{a,b}(f^n))$, where $n = 0$ or $m = 0)$. ☐

Examples of higher order adjunctions

1. Every usual 1-adjunction $\begin{tikzcd} A \arrow[r, bend left] \arrow[r, phantom, "\perp"] & B \arrow[l, bend left] \end{tikzcd}$ is an ∞-1-adjunction for trivial ∞-extensions of A and B.
2. Gelfand-Naimark dual 1-adjunction $\begin{tikzcd} \mathbf{C^*Alg}^{op} \arrow[r, shift left] \arrow[r, phantom, "\perp"] & \mathbf{CHTop} \arrow[l, shift left] \end{tikzcd}$ is extendable to ∞-2-adjunction (see chapter 3).
3. *Quillen theorem* [Mac]. *Let* Δ *be a category of finite linearly ordered sets,* $\mathbf{Set}^{\Delta^{op}}$ *the category of simplicial sets,* $Ho(\mathbf{Top}) := (2\text{-}\mathbf{Top})^{(1)}$, $Ho(\mathbf{Set}^{\Delta^{op}}) := (2\text{-}\mathbf{Set}^{\Delta^{op}})^{(1)}$. *Then*

$$\begin{tikzcd}
\Delta \arrow[d, tail, "\text{Yoneda}"] \arrow[dr, tail] & \\
\mathbf{Set}^{\Delta^{op}} \arrow[r, dashed] \arrow[r, phantom, "\perp"'] \arrow[d] & \mathbf{Top} \arrow[l, bend left] \arrow[d] \\
Ho(\mathbf{Set}^{\Delta^{op}}) \arrow[r] \arrow[r, phantom, "\perp"'] & Ho(\mathbf{Top}) \arrow[l, bend left]
\end{tikzcd}$$

☐

So, the top adjunction is actually a 2-adjunction (or ∞-2-adjunction).

All the above adjunctions are strict.

1.6. Concrete duality for ∞-categories

Duality preserves all categorical properties. It is significant that concrete duality for ∞-categories behaves the same as for 1-categories.

Definition 1.6.1.

- **Duality** is an equivalence $L^{op} \sim L'$.

- A **Concrete duality** over $\mathbb{B} \hookrightarrow \infty$-**CAT** is a duality $\begin{tikzcd} L^{op} \arrow[r, bend left, "G"] \arrow[r, phantom, "\approx"] & L' \arrow[l, bend left, "F"] \end{tikzcd}$ such that there exist (faithful) forgetful functors $U : L \to \mathbb{B}$, $V : L' \to \mathbb{B}$ and objects $\tilde{A} \in L^0$, $\tilde{B} \in L'^0$ such that
  - $U(\tilde{A}) \sim V(\tilde{B})$,
  - $V \circ_1 G \sim L(-,\tilde{A})$, $U \circ_1 F^{op} \sim L'(-,\tilde{B})$

$$\begin{tikzcd} L^{op} \arrow[r, "G"] \arrow[dr, "{L(-,\tilde{A})}"'] & L' \arrow[d, "V"] \\ & \mathbb{B} \end{tikzcd} \qquad \begin{tikzcd} L'^{op} \arrow[r, "F^{op}"] \arrow[dr, "{L'(-,\tilde{B})}"'] & L \arrow[d, "U"] \\ & \mathbb{B} \end{tikzcd}$$

Representing objects $\tilde{A} \in L^0$, $\tilde{B} \in L'^0$ are called **dualizing** or **schizophrenic objects** for the given concrete duality[P-Th].

[for a **concrete dual adjunction** the definition is similar] □

Proposition 1.6.1 (representable forgetfuls ⇒ concrete dual adjunction). *Let (L,U), (L',V) be (weakly) dually adjoint ∞-categories* $\begin{tikzcd} L^{op} \arrow[r, bend left, "G"] \arrow[r, phantom, "\top"] & L' \arrow[l, bend left, "F"] \end{tikzcd}$ *with representable forgetful functors $U \sim L(A_0,-) : L \to \mathbb{B}$, $V \sim L'(B_0,-) : L' \to \mathbb{B}$ (where $\mathbb{B} \hookrightarrow \infty$-**CAT** is a subcategory). Then this adjunction is concrete over $\mathbb{B}$ with dualizing object $(\tilde{A},\tilde{B})$, where $\tilde{A} := F(B_0)$, $\tilde{B} := G(A_0)$, i.e.*

- $U(\tilde{A}) \sim V(\tilde{B})$
- $V \circ_1 G \sim L(-,\tilde{A})$, $U \circ_1 F^{op} \sim L'(-,\tilde{B})$

$$\begin{tikzcd} L^{op} \arrow[r, "G"] \arrow[dr, "{L(-,\tilde{A})}"'] & L' \arrow[d, "V"] \\ & \mathbb{B} \end{tikzcd} \qquad \begin{tikzcd} L'^{op} \arrow[r, "F^{op}"] \arrow[dr, "{L'(-,\tilde{B})}"'] & L \arrow[d, "U"] \\ & \mathbb{B} \end{tikzcd}$$

PROOF.

- $U(\tilde{A}) = UF(B_0) \sim L(A_0,FB_0) \sim L'(B_0,GA_0) \sim VGA_0 = V\tilde{B}$
- $VG(-) \sim L'(B_0,G(-)) \sim L(-,FB_0) = L(-,\tilde{A})$ (and similar, $UF(-) \sim L'(-,\tilde{B})$) □

Remarks.

- Concrete duality as above should be called **weak**. **Strict** variants of definition 1.6.1 and proposition 1.6.1 also exist (by changing $\sim$ to isomorphism $\simeq$ and weak dual adjunction to the strict one).

- (Weak or strict) concrete duality (dual adjunction) is given essentially by hom-functors which admit lifting along forgetful functors (to obtain proper values). Representing objects of these functors have equivalent (or isomorphic) underlying objects.
- For the usual 1-dimensional categories $\mathbb{B} = \mathbf{Set} \hookrightarrow \infty\text{-}\mathbf{CAT}$ (∞-1-subcategory). For dimension n, as a rule, $\mathbb{B} = n\text{-}\mathbf{Cat} \hookrightarrow \infty\text{-}\mathbf{CAT}$ (∞-n-subcategory of small $(n-1)$-categories). □

1.6.1. Natural and non natural duality.

Definition 1.6.1.1.

- For hom-set $L(A, \tilde{A})$ and element $(x : A_0 \to A) \in L^0(A_0, A)$ the **evaluation functor** at the point x is $ev_{A,x} := L(x, \tilde{A}) : L(A, \tilde{A}) \to L(A_0, \tilde{A})$ $(ev_{A,x} \in \mathbb{B}^1 \hookrightarrow \infty\text{-}\mathbf{CAT}^1)$.

  Similarly, the **evaluation** $(n-1)$**-modification** ev_{A,x^n}, $n = 1, 2, \ldots$, for $x^n \in L^n(A_0, A)$ is $L(x^n, \tilde{A}) \in \mathbb{B}^n(L(A, \tilde{A}), L(A_0, \tilde{A}))$.
- For a forgetful functor $V : L' \to \mathbb{B}$ an arrow $f^n : V(Y) \to V(Y') \in \mathbb{B}^n(V(Y), V(Y'))$ is called an L'**-arrow** if $\exists \Phi^n : Y \to Y' \in L'^n(Y, Y')$ such that $V(\Phi^n) = f^n$.
- A lifting of hom-functor $V \circ G \sim L(-, \tilde{A})$ is called **initial** [A-H-S] if $\forall A \in L^0$ $\forall Y \in L'^0$ $\forall f^n : V(Y) \to L(A, \tilde{A}) \in \mathbb{B}^n(V(Y), L(A, \tilde{A}))$ f^n is an L'-arrow iff $\forall (x^n : A_0 \to A) \in L^n(A_0, A)$ $ev_{A,x^n} \circ_{n+1} f^n : V(Y) \to L(A_0, \tilde{A}) \in \mathbb{B}^n(V(Y), L(A_0, \tilde{A}))$ is an L'-arrow.
- If liftings of hom-functors $V \circ G \sim L(-, \tilde{A})$, $U \circ F \sim L'(-, \tilde{B})$ are both initial, then the concrete dual adjunction $\begin{tikzcd} L^{op} \arrow[r, bend left, "G"] \arrow[r, phantom, "\top"] & L' \arrow[l, bend left, "F"] \end{tikzcd}$, if it exists, is called **natural** [Hof, P-Th], and otherwise, non-natural. □

Even if $U\tilde{A} \sim V\tilde{B}$ and $\forall A \in L^0, B \in L'^0$ $\mathbb{B}$-objects $L(A, \tilde{A})$, $L'(B, \tilde{B})$ can be lifted to L', L, the hom-functors $L(-, \tilde{A})$, $L'(-, \tilde{B})$ need not (which happens only if lifting of the assignments $A \mapsto L(A, \tilde{A})$, $B \mapsto L'(B, \tilde{B})$ can be extended functorially over all cells).

We introduce the following concept. **The initial lifting condition for the evaluation**

cones

$\{ev_{A,x^n} \in \mathbb{B}^n(L(A,\tilde{A}), L(A_0,\tilde{A}))\}^{n\in\mathbb{N}}_{x^n\in L^n(A_0,A)}, \{ev_{B,y^n} \in \mathbb{B}^n(L'(B,\tilde{B}), L'(B_0,\tilde{B}))\}^{n\in\mathbb{N}}_{y^n\in L'^n(B_0,B)}$

consists of the following requirements:

- hom-categories of the form $L(A,\tilde{A})$, $L'(B,\tilde{B}) \in Ob(\mathbb{B})$ can be lifted to L', L
- evaluation cones

  $\{ev_{A,x^n} \in \mathbb{B}^n(L(A,\tilde{A}), L(A_0,\tilde{A}))\}^{n\in\mathbb{N}}_{x^n\in L^n(A_0,A)}, \{ev_{B,y^n} \in \mathbb{B}^n(L'(B,\tilde{B}), L'(B_0,\tilde{B}))\}^{n\in\mathbb{N}}_{y^n\in L'^n(B_0,B)}$ can be lifted to $\{ev_{A,x^n} \in L'^n(G(A),\tilde{B})\}^{n\in\mathbb{N}}_{x^n\in L^n(A_0,A)}$, $\{ev_{B,y^n} \in L^n(F(B),\tilde{A})\}^{n\in\mathbb{N}}_{y^n\in L'^n(B_0,B)}$ in L', L
- $\forall f^n \in \mathbb{B}^n(VX, L(A,\tilde{A}))$ f^n is L'-arrow iff $\forall x^n \in L^n(A_0,A)$ $\mu(ev_{A,x^n}, f^n) \in \mathbb{B}^n(VX, L(A_0,\tilde{A}))$ is L'-arrow (and, symmetrically, $\forall g^n \in \mathbb{B}^n(UY, L'(B,\tilde{B}))$ g^n is an L-arrow iff $\forall y^n \in L'^n(B_0,B)$ $\mu(ev_{B,y^n}, g^n) \in \mathbb{B}^n(UY, L'(B_0,\tilde{B}))$ is an L-arrow) $\square$

In the following proof, we denote lifted evaluation maps by $ev_{A,x}$ (or something similar) and underlying evaluation maps in $\mathbb{B}$ by $|ev_{A,x}|$.

Proposition 1.6.1.1. *If two strict ∞-categories L, L' concrete over $\mathbb{B} \hookrightarrow \infty$-***CAT** *with representable (strictly faithful) forgetful functors $U = L(A_0,-)$, $V = L'(B_0,-)$ have objects $\tilde{A} \in L^0$, $\tilde{B} \in L'^0$ such that*

- *$U\tilde{A} \sim V\tilde{B}$*
- *the hom-functors $L(-,\tilde{A}) : L^{op} \to \mathbb{B}$, $L'(-,\tilde{B}) : L'^{op} \to \mathbb{B}$ satisfy the* **initial lifting condition for evaluation cones**

  then there exists a natural **strict** *concrete dual adjunction* $$\begin{tikzcd} L^{op} \arrow[r, bend left, "G"] \arrow[r, phantom, "\top"] & L' \arrow[l, bend left, "F"] \end{tikzcd}$$ $L(A,FB) \underset{\text{nat. isomorphisms}}{\simeq}$ $L'(B,GA)$ $$\begin{tikzcd} L^{op} \arrow[r, "G"] \arrow[dr, "L(-,\tilde{A})"'] & L' \arrow[d, dashed, "V"] \\ & \mathbb{B} \end{tikzcd} \qquad \begin{tikzcd} L'^{op} \arrow[r, "F^{op}"] \arrow[dr, "L'(-,\tilde{B})"'] & L \arrow[d, "U"] \\ & \mathbb{B} \end{tikzcd}$$ *with $(\tilde{A},\tilde{B})$ its schizophrenic object.*

PROOF.

- $L(A,\tilde{A})$, $L'(B,\tilde{B})$ are lifted to L', L by the assumed condition.

- Let $f^n \in L^n(A, A')$, then $L(f^n, \tilde{A}) : L(A', \tilde{A}) \to L(A, \tilde{A})$ is an L'-arrow since $ev_{A,a^n} \circ_{n+1} L(f^n, \tilde{A}) := L(a^n, \tilde{A}) \circ_{n+1} L(f^n, \tilde{A}) = L(f^n \circ_{n+1} a^n, \tilde{A}) =: ev_{A', f^n \circ_{n+1} a^n}$, which is liftable $\forall a^n \in L^n(A_0, \tilde{A})$. Therefore, $L(f^n, \tilde{A})$ is an L'-arrow, and similarly, $L'(g^n, \tilde{B})$ is an L-arrow, i.e., there exist maps $$\begin{tikzcd} L^{op} \arrow[r, bend left, "G"] & L' \arrow[l, bend left, "F"] \end{tikzcd}$$, which are obviously functorial.

  Why do they give an adjunction?

- (unit and counit) 1-arrow (unit) $\eta_B : B \to GFB$ is given by $|\eta_B| =: V\eta_B : |B| \to |GFB| : b \mapsto [ev_{B,b} : FB \to \tilde{A}]$, $b \in |B| = L'(B_0, B)$, $|GFB| = L(FB, \tilde{A})$, $|ev_{B,b}| : |FB| \to |\tilde{A}|$, $|FB| = L'(B, \tilde{B})$, $|\tilde{A}| = L(A_0, \tilde{A}) \sim L'(B_0, \tilde{B})$. Why can $|\eta_B|$ be lifted to L'? Take the composite with evaluation maps $|ev_{FB,c}| \circ_1 |\eta_B|(b) = |ev_{FB,c}|(ev_{B,b}) = |ev_{B,b}|(c) = |c|(b)$, where $c \in |FB|^0 = L'^0(B, \tilde{B}) = L^0(A_0, FB)$, $b \in |B|^n$. So, $|ev_{FB,c}| \circ_1 |\eta_B| = |c|$ is an L'-arrow. Therefore, $|\eta_B|$ is an L'-arrow. The counit is given symmetrically $\varepsilon_A \to FGA$, $|\varepsilon_A| : |A| \to |FGA| : a \mapsto [ev_{A,a} : GA \to \tilde{B}]$, $|A| = L(A_0, A)$, $|FGA| = L'(GA, \tilde{B})$, $|ev_{A,a}| : |GA| \to |\tilde{B}|$, $|GA| = L(A, \tilde{A})$, $|\tilde{B}| = L'(B_0, \tilde{B}) \sim L(A_0, \tilde{A})$. By the same argument $|\varepsilon_A|$ is an L-arrow.

- (triangle identities) $G\varepsilon_A \circ_1 \eta_{GA} = 1_{GA}$, $F\eta_B \circ_1 \varepsilon_{FB} = 1_{FB}$. It is sufficient to prove them for underlying maps. Since forgetful functors are faithful this follows.

  $|G\varepsilon_A| \circ_1 |\eta_{GA}| \overset{?}{=} |1_{GA}|$, where $|\eta_{GA}| : |GA| \to |GFGA|$, $|GA| = L(A, \tilde{A})$, $|GFGA| = L(FGA, \tilde{A})$, $\varepsilon_A : A \to FGA$, $|G\varepsilon_A| : |GFGA| \to |GA|$.

  Take $(f^n : A \to \tilde{A}) \in |GA| = L^n(A, \tilde{A})$, $a^m \in |A| = L^m(A_0, A)$. Two cases are possible
  $$\begin{cases} (a)\ (f^n, n > 0)\ \&\ (a^0) : ||G\varepsilon_A| \circ_1 |\eta_{GA}|(f^n)|(a^0) = |L(\varepsilon_A, \tilde{A})(ev_{GA,f^n})|(a^0) = |ev_{GA,f^n \circ_{n+1}} \\ (b)\ (f^0)\ \&\ (a^n, n \geq 0) : ||G\varepsilon_A| \circ_1 |\eta_{GA}|(f^0)|(a^n) = |L(\varepsilon_A, \tilde{A})(ev_{GA,f^0})|(a^n) = |ev_{GA,f^0 \circ_1} \end{cases}$$
  $$\begin{cases} (a)\ e^n\varepsilon_A|(a^0) = |ev_{GA,f^n}| \circ_{n+1} e^n|\varepsilon_A|(a^0) = |ev_{GA,f^n}|(ev_{A,e^n a^0}) = |ev_{A,e^n a^0}|(f^n) = |f^n|(a^0) \\ (b)\quad \varepsilon_A|(a^n) = |ev_{GA,f^0}| \circ_1 |\varepsilon_A|(a^n) = |ev_{GA,f^0}|(ev_{A,a^n}) = |ev_{A,a^n}|(f^0) = |f^0|(a^n) \end{cases}$$
  $$\begin{cases} (a) =: \mu^L_{A_0,A,\tilde{A}}(f^n, e^n a^0) = ||1_{GA}|(f^n)|(a^0) \\ (b) =: \mu^L_{A_0,A,\tilde{A}}(e^n f^0, a^n) = ||1_{GA}|(f^0)|(a^n) \end{cases}$$
  The second triangle identity holds similarly.

- (naturality of η_B, ε_A) Again, it is sufficient to prove naturality for underlying maps

$$\begin{tikzcd}
{|B|} \arrow[r, "|\eta_B|"] \arrow[d, "|f|"'] & {|GFB|} \arrow[d, "|GFf|=L(Ff,\tilde{A})"] \\
{|B'|} \arrow[r, "|\eta_{B'}|"'] & {|GFB'|}
\end{tikzcd}$$

Two cases are $\begin{cases} (a)\ (b^n \in |B|^n, n \geq 0)\ \&\ (f^0 \in L'^0(B,B')) \\ (b)\ (b^0 \in |B|^0)\ \&\ (f^n \in L'^n(B,B')) \end{cases}$

(a)

$$\begin{tikzcd}
{|B|} \arrow[r, "|\eta_B|"] \arrow[d, "|f^0|"'] & {|GFB|} \arrow[d, "|GFf^0|=L(Ff^0,\tilde{A})"] \\
{|B'|} \arrow[r, "|\eta_{B'}|"'] & {|GFB'|}
\end{tikzcd}
\qquad
\begin{tikzcd}
b^n \arrow[r, mapsto] \arrow[dd, mapsto] & ev_{B,b^n} \arrow[d, mapsto] \\
 & ev_{B,b^n} \circ_{n+1} e^n(Ff^0) \arrow[d, Rightarrow, "="] \\
{|f^0|(b^n)} \arrow[r, mapsto] & ev_{B',|f^0|(b^n)}
\end{tikzcd}$$

(b)

$$\begin{tikzcd}
{|B|} \arrow[r, "e^n|\eta_B|"] \arrow[d, "|f^n|"'] & {|GFB|} \arrow[d, "|GFf^n|=L(Ff^n,\tilde{A})"] \\
{|B'|} \arrow[r, "e^n|\eta_{B'}|"'] & {|GFB'|}
\end{tikzcd}
\qquad
\begin{tikzcd}
b^0 \arrow[r, mapsto] \arrow[dd, mapsto] & ev_{B,e^n b^0} \arrow[d, mapsto] \\
 & ev_{B,e^n b^0} \circ_{n+1} (Ff^n) \arrow[d, Rightarrow, "="] \\
{|f^n|(b^0)} \arrow[r, mapsto] & ev_{B',|f^n|(b^0)}
\end{tikzcd}$$

(recall $|f^n|(b^0) \equiv \mu(f^n, e^n b^0)$, $|f^0|(b^n) \equiv \mu(e^n f^0, b^n)$)

Why $\begin{cases} (a)\ \ ev_{B,b^n} \circ_{n+1} e^n(Ff^0) = ev_{B',|f^0|(b^n)} \\ (b)\ \ ev_{B,e^n b^0} \circ_{n+1} (Ff^n) = ev_{B',|f^n|(b^0)} \end{cases}$?

Take underlying maps:

$$\begin{cases} (a)\ \ |ev_{B,b^n}| \circ_{n+1} e^n |Ff^0|(h^n) = |ev_{B,b^n}|(h^n \circ_{n+1} e^n f^0) = |h^n \circ_{n+1} e^n f^0|(b^n) = \\ (b)\ \ |ev_{B,e^n b^0}| \circ_{n+1} |Ff^n|(h^0) = |ev_{B,e^n b^0}|(e^n h^0 \circ_{n+1} f^n) = |e^n h^0 \circ_{n+1} f^n|(e^n b^0) = \end{cases}$$

$$\begin{cases} (a)\ \ = |h^n| \circ_{n+1} |e^n f^0|(b^n) = |ev_{B',|f^0|(b^n)}|(h^n), \quad h^n \in L'^n(B', \tilde{B}) \\ (b)\ \ = e^n|h^0| \circ_{n+1} |f^n|(e^n b^0) = |ev_{B',|f^n|(b^0)}|(h^0), \quad h^0 \in L'^0(B', \tilde{B}) \end{cases}$$

(the types of the above arrows are $Ff : FB' \to FB$, $ev_{B,b} : FB \to \tilde{A}$ (L-map), $ev_{B',|f|(b)} : FB' \to \tilde{A}$ (L-map), $|ev_{B,b}| : L'(B, \tilde{B}) \to |\tilde{B}| = L'(B_0, \tilde{B})$, $|ev_{B',|f|(b)}| : L'(B', \tilde{B}) \to |\tilde{B}| = L'(B_0, \tilde{B})$, $|Ff| : L'(B', \tilde{B}) \to L'(B, \tilde{B})$, $|Ff| = L'(f, \tilde{B})$).

Therefore, η_B is natural. Similarly, ε_A is natural.

- (isomorphisms-functors $\begin{tikzcd} L(A,FB) \arrow[r, bend left, "\theta_{A,B}"] & L'(B,GA) \arrow[l, bend left, "\theta^*_{A,B}"] \end{tikzcd}$)

Define $\begin{cases} \theta_{A,B}(f^n) := G(f^n) \circ_{n+1} e^n(\eta_B), & f^n \in L^n(A,FB) \\ \theta^*_{A,B}(g^n) := F(g^n) \circ_{n+1} e^n(\varepsilon_A), & g^n \in L^{\prime n}(B,GA) \end{cases}$

Let $g^n \in L^{\prime n}(B,GA)$. Then $\theta_{A,B}(\theta^*_{A,B}(g^n)) := G(Fg^n \circ_{n+1} e^n(\varepsilon_A)) \circ_{n+1} e^n(\eta_B) = e^n(G\varepsilon_A) \circ_{n+1} GFg^n \circ_{n+1} e^n(\eta_B) \underset{\text{nat. of } \eta_B}{=} e^n(G\varepsilon_A) \circ_{n+1} e^n(\eta_{GA}) \circ_{n+1} g^n \underset{\text{triangle id.}}{=} e^n(1_{GA}) \circ_{n+1} g^n = e^{n+1}(GA) \circ_{n+1} g^n = g^n$. Similarly, $\theta^*_{A,B}(\theta_{A,B}(f^n)) = f^n$, $f^n \in L^n(A,FB)$. $\theta_{A,B}$, $\theta^*_{A,B}$ are obviously functors. So, they are isomorphisms.

- (naturality of $\theta_{A,B}$, $\theta^*_{A,B}$) We need to prove the diagram

$$\begin{tikzcd}
A & B & L(A,FB) \arrow[r, "e^n\theta_{A,B}"] \arrow[d, "L(x^n,Fy^n)"'] & L'(B,GA) \arrow[d, "L'(y^n,Gx^n)"] \\
A' \arrow[u, "x^n"] & B' \arrow[u, "y^n"] & L(A',FB') \arrow[r, "e^n\theta_{A',B'}"] & L'(B',GA')
\end{tikzcd}$$

commutes.

$$L'(y^n,Gx^n) \circ_{n+1} e^n\theta_{A,B} \overset{?}{=} e^n\theta_{A',B'} \circ_{n+1} L(x^n,Fy^n)$$

Two cases are: $\begin{cases} (a) \ \ (f^0 \in L(A,FB)) \ \& \ (x^n, y^n, n > 0) \\ (b) \ \ (f^n \in L(A,FB), n \geq 0) \ \& \ (x^0, y^0) \end{cases}$

$$\begin{tikzcd}
 & f^0 \arrow[r, mapsto] \arrow[dd, mapsto] & e^nG(f^0) \circ_{n+1} e^n(\eta_B) \arrow[d, mapsto] \\
 & & Gx^n \circ_{n+1} (e^nG(f^0) \circ_{n+1} e^n(\eta_B)) \circ_{n+1} y^n \arrow[d, Rightarrow, "="', "?"] \arrow[dd, Rightarrow, bend left=60, "= \ (\eta_B \text{ is nat.})"] \\
(a) & Fy^n \circ_{n+1} e^nf^0 \circ_{n+1} x^n \arrow[r, mapsto] & G(Fy^n \circ_{n+1} e^nf^0 \circ_{n+1} x^n) \circ_{n+1} e^n(\eta_{B'}) \arrow[d, Rightarrow, "="'] \\
 & & Gx^n \circ_{n+1} e^nGf^0 \circ_{n+1} GFy^n \circ_{n+1} e^n(\eta_{B'})
\end{tikzcd}$$

$$\begin{tikzcd}
 & f^n \arrow[r, mapsto] \arrow[dd, mapsto] & G(f^n) \circ_{n+1} e^n(\eta_B) \arrow[d, mapsto] \\
 & & e^nGx^0 \circ_{n+1} (G(f^n) \circ_{n+1} e^n(\eta_B)) \circ_{n+1} e^ny^0 \arrow[d, Rightarrow, "="', "?"] \arrow[dd, Rightarrow, bend left=60, "= \ (\eta_B \text{ is nat.})"] \\
(b) & e^nFy^0 \circ_{n+1} f^n \circ_{n+1} e^nx^0 \arrow[r, mapsto] & G(e^nFy^0 \circ_{n+1} f^n \circ_{n+1} e^nx^0) \circ_{n+1} e^n(\eta_{B'}) \arrow[d, Rightarrow, "="'] \\
 & & e^nGx^0 \circ_{n+1} Gf^n \circ_{n+1} e^nGFy^0 \circ_{n+1} e^n(\eta_{B'})
\end{tikzcd}$$

Therefore, L and L' are concretely dually adjoint. This correspondence is natural (by condition) and strict ($\theta_{A,B}$ and $\theta^*_{A,B}$ are isomorphisms). ☐

Corollary. Concrete natural duality is a **strict** adjunction. ☐

Well-known dualities [P-Th, Bel, A-H-S]

All dualities below are of first order, natural [P-Th], and obtained by restriction of appropriate dual adjunctions.

1. $\mathbf{Vec}_k$ is dually equivalent to itself $\begin{tikzcd} \mathbf{Vec}_k^{op} \arrow[r, bend left, "\mathbf{Vec}_k(-,k)"] \arrow[r, phantom, "\perp"] & \mathbf{Vec}_k \arrow[l, bend left, "\mathbf{Vec}_k(-,k)"] \end{tikzcd}$, where $\mathbf{Vec}_k$ is a category of vector spaces over field k
2. $\mathbf{Set}^{op} \sim$ **Complete Atomic Boolean Algebras**
3. $\mathbf{Bool}^{op} \sim$ **Boolean Spaces** (Stone duality), where **Bool** is a category of Boolean rings (every element is idempotent). It is obtained from the dual adjunction $\begin{tikzcd} \mathbf{CRing} \arrow[r, bend left, "\mathbf{CRing}(-,\mathbf{2})"] \arrow[r, phantom, "\perp"] & \mathbf{Top} \arrow[l, bend left, "\mathbf{Top}(-,\mathbf{2})"] \end{tikzcd}$, where $\mathbf{2}$ is two-element ring and discrete topological space. $\mathbf{CRing}(A, \mathbf{2}) \hookrightarrow \mathbf{2}^A$ (subspace in Tychonoff topology)
4. $\hom(-, \mathbb{R}/\mathbb{Z}) : \mathbf{CompAb}^{op} \sim \mathbf{Ab}$ (Pontryagin duality), where **CompAb**, **Ab** are categories of compact abelian groups and abelian groups respectively
5. $\hom(-, \mathbb{C}) : \mathbf{C^*Alg}^{op} \sim \mathbf{CHTop}$ (Gelfand-Naimark duality), where **C*Alg**, **CHTop** are categories of commutative $\mathbb{C}^*$-algebras and compact Hausdorff spaces. $\mathbf{C^*Alg}(A, \mathbb{C}) \hookrightarrow \mathbb{C}^A$ (subspace in Tychonoff topology)

2. Manifolds, structures

In this chapter, the general notion of manifold is introduced. It covers all the usual manifolds in mathematics. Essentially, it is a way of obtaining a larger fibration over a site which locally coincides with a given one. Also the theory of group objects and their actions is developed, and the problem of lifting of a group action onto "structures over a base manifold" is considered.

2.1. Fibrations and cofibrations

(Co)fibrations play the role of structures over objects in a given category which can be transported along morphisms. Transport systems are called (co)cartesian morphisms. The situation is very similar to fibrations with connection in Differential Geometry.

Definition 2.1.1. For a functor $\begin{tikzcd}\mathbf{E} \arrow[d, "p"'] \\ \mathbf{B}\end{tikzcd}$

- a morphism $f : B' \to p(E)$ has a **cartesian lifting** $\tilde{f} : E' \to E \in Ar\,\mathbf{E}$ if $\forall f' : E'' \to E$ such that $p(f')$ factors through f (i.e., $\exists g : p(E'') \to B'$ such that $p(f') = f \circ g$) f' itself uniquely factors through $\tilde{f}$ over the base factorization (i.e., $\exists!\, \tilde{g} : E'' \to E'$ such that $f' = \tilde{f} \circ \tilde{g}$ and $p(\tilde{g}) = g$) [Jac]

$$\begin{tikzcd}
& E'' \arrow[dr, dashed, "\exists!\,\tilde{g}"'] \arrow[drr, "\forall f'"] & & \\
p(E'') \arrow[drr, "\forall g"'] \arrow[drrr, "p(f')"] & & E' \arrow[r, "\tilde{f}"'] & E \\
& & B' \arrow[r, "f"'] & p(E)
\end{tikzcd}$$

- a morphism $f : p(E) \to B'$ has a **cocartesian lifting** $\tilde{f} : E \to E' \in Ar\,\mathbf{E}$ if $\forall f' : E \to E''$ such that $p(f')$ factors through f (i.e., $\exists g : B' \to p(E'')$ such that $p(f') = g \circ f$) f' itself uniquely factors through $\tilde{f}$ over the base factorization (i.e., $\exists!\, \tilde{g} : E' \to E''$ such that $f' = \tilde{g} \circ \tilde{f}$

and $p(\tilde{g}) = g$) [Jac]

$$\begin{tikzcd}
 & & E'' \\
E \arrow[r, "\tilde{f}"'] \arrow[urr, "\forall f'"] & E' \arrow[ur, dashed, "\exists ! \tilde{g}"'] & \\
 & & p(E'') \\
p(E) \arrow[r, "f"'] \arrow[urr, "p(f')"] & B' \arrow[ur, "\forall g"'] &
\end{tikzcd}$$

□

Remark. (Co)cartesian morphisms if they exist are unique up to vertical isomorphism ($v : E \to E' \in Ar\,\mathbf{E}$ is **vertical** if $p(v) = 1_B$ for some $B \in Ob\,\mathbf{B}$).

Definition 2.1.2. A functor $$\begin{tikzcd} \mathbf{E} \arrow[d, "p"'] \\ \mathbf{B} \end{tikzcd}$$ is called [Jac, Str2]

- a **fibration** if for each $f : B' \to p(E) \in Ar\,\mathbf{B}$ there exists a cartesian lifting $\tilde{f} : E' \to E \in Ar\,\mathbf{E}$
- a **cofibration** if for each $f : p(E) \to B' \in Ar\,\mathbf{B}$ there exists a cocartesian lifting $\tilde{f} : E \to E' \in Ar\,\mathbf{E}$
- a **bifibration** if it is both fibration and cofibration □

The subcategory $\mathbf{E}_B := p^{-1}(B, 1_B) \hookrightarrow \mathbf{E}$ is called the **fiber** over B. $\mathbf{E}_B$ consists of all vertical morphisms over B.

Examples

1. For a category $\mathbf{C}$ with pullbacks, the **codomain fibration** is $\begin{tikzcd} \mathbf{C}^{\to} \arrow[d, "\mathrm{cod}"'] \\ \mathbf{C} \end{tikzcd}$, where $\mathrm{cod}\left(\begin{tikzcd} \bullet \arrow[r, "\Phi"] \arrow[d] & \bullet \arrow[d] \\ \bullet \arrow[r, "f"'] & \bullet \end{tikzcd}\right) = f$, and the cartesian lifting is a pullback square.
2. For a category $\mathbf{C}'$ with pushouts, the **domain cofibration** is $\begin{tikzcd} \mathbf{C}'^{\to} \arrow[d, "\mathrm{dom}"'] \\ \mathbf{C}' \end{tikzcd}$, where $\mathrm{dom}\left(\begin{tikzcd} \bullet \arrow[r, "\Phi"] \arrow[d] & \bullet \arrow[d] \\ \bullet \arrow[r, "f"'] & \bullet \end{tikzcd}\right) = \Phi$, and the cocartesian lifting is a pushout square.

3. [Kom] Let **Rng–Mod** denote the category of left modules with variable rings. In other words $Ob\,(\mathbf{Rng{-}Mod})$ are pairs (R, M) where: R is a ring, M is an R-module. $Ar\,(\mathbf{Rng{-}Mod})$ are pairs $(\varphi : R_1 \to R_2, f : M_1 \to M_2)$ such that $\forall r \in R_1, m \in M_1 \quad f(r \cdot m) = \varphi(r) \cdot f(m)$. Then 'projection on first component' $\mathbf{Rng{-}Mod} \xrightarrow{p_1} \mathbf{Rng}$ is a bifibration. If $\varphi : R_1 \to R_2 \in Ar\,\mathbf{Rng}$, then the **direct image** of (R_1, M_1) over φ is $(R_2, R_2 \otimes_{R_1} M_1)$. The cocartesian lifting of φ is $(\varphi, \bar{\varphi})$ with $\bar{\varphi} : M_1 \to R_2 \otimes_{R_1} M_1 : m \mapsto 1 \otimes m$.

$$\begin{tikzcd}
 & & (R_3, M_3) \\
(R_1, M_1) \arrow[rru, "(\psi\circ\varphi, f)"] \arrow[r, "(\varphi, \bar{\varphi})"'] & (R_2, R_2 \otimes_{R_1} M_1) \arrow[ru, dashed, "(\psi, \psi\cdot f)"'] &
\end{tikzcd}$$

$$\begin{tikzcd}
 & & R_3 \\
R_1 \arrow[rru, "\psi\circ\varphi"] \arrow[r, "\varphi"'] & R_2 \arrow[ru, "\psi"'] &
\end{tikzcd}$$

where: $\psi \cdot f := \mu \circ (\psi \otimes f) : R_2 \otimes_{R_1} M_1 \xrightarrow{\psi \otimes f} R_3 \otimes_{R_1} M_3 \xrightarrow{\mu} M_3$, μ is module multiplication, $\psi(r_1 \cdot r_2) \cdot f(m) = (\psi \cdot f)(r_1 \cdot r_2 \otimes m) = \psi(r_1) \cdot (\psi \cdot f)(r_2 \otimes m) = \psi(r_1) \cdot (\psi(r_2) \cdot f(m))$.

Inverse image of (R_1, M_1) over $\alpha : R \to R_1$ is (R, M_1) with action $(r, m) \mapsto \alpha(r) \cdot m$. Cartesian lifting of α is $(\alpha, 1_{M_1})$.

$$\begin{tikzcd}
(R', M') \arrow[rrd, "(\alpha\circ\beta, f)"] \arrow[rd, dashed, "(\beta, f)"'] & & \\
 & (R, M_1) \arrow[r, "(\alpha, 1_{M_1})"'] & (R_1, M_1)
\end{tikzcd}$$

$$\begin{tikzcd}
R' \arrow[rrd, "\alpha\circ\beta"] \arrow[rd, "\beta"'] & & \\
 & R \arrow[r, "\alpha"'] & R_1
\end{tikzcd}$$

4. (Differential Equations)

In the following, we assume the reader is familiar with the theory of jet bundles, as in [Sau].

$$\mathbf{Set} \xleftarrow{p_1} \mathbf{Top} \xleftarrow{p_2} \mathbf{Diff} \xleftarrow{p_3} \mathbf{Diff}^{\twoheadrightarrow} \xleftarrow{p_4} \mathbf{Jet}^{\infty}(\mathbf{Diff}^{\twoheadrightarrow}) \xleftarrow{p_5} \mathbf{Sub}(\mathbf{Jet}^{\infty}(\mathbf{Diff}^{\twoheadrightarrow}))$$

where $\mathbf{Diff}^{\twoheadrightarrow}$ is the full subcategory of $\mathbf{Diff}^{\rightarrow}$ consisting of surjective submersions (surjective maps with surjective differential at each point), $\mathbf{Jet}^{\infty}(\mathbf{Diff}^{\twoheadrightarrow})$ is the corresponding category of ∞-jet-bundles, $\mathbf{Sub}(\mathbf{Jet}^{\infty}(\mathbf{Diff}^{\twoheadrightarrow}))$ is the category of subobjects of ∞-jet-bundles (differential relations),

$p_1 : \mathbf{Top} \to \mathbf{Set}$ is a fibration ($\forall$ arrow $S' \xrightarrow{u} p_1(T)$ $\exists$ Cartesian 'completion'
$$\begin{tikzcd} T' \arrow[r, "v"] & T \\ S' \arrow[r, "u"] & p_1(T) \end{tikzcd}$$
, T' has initial topology w.r.t. u),

$p_2 : \mathbf{Diff} \to \mathbf{Top}$ is not a fibration if differentiable manifolds are regarded in the usual sense (as locally Euclidean), but it is a fibration if differentiable manifolds are topological spaces endowed with a subsheaf of continuous functions closed under smooth operations,

$p_3 : \mathbf{Diff}^{\twoheadrightarrow} \to \mathbf{Diff}$ is a codomain fibration since the pullback of a surjective submersion is a surjective submersion,

$p_4 : \mathbf{Jet}^{\infty}(\mathbf{Diff}^{\twoheadrightarrow}) \to \mathbf{Diff}^{\twoheadrightarrow}$ is not a fibration (if we admit arbitrary fibre bundle arrows between objects in $\mathbf{Jet}^{\infty}(\mathbf{Diff}^{\twoheadrightarrow})$), but it is a structure over $\mathbf{Diff}^{\twoheadrightarrow}$ (see 2.1.1),

$p_5 : \mathbf{Sub}(\mathbf{Jet}^{\infty}(\mathbf{Diff}^{\twoheadrightarrow})) \to \mathbf{Jet}^{\infty}(\mathbf{Diff}^{\twoheadrightarrow})$ is a "subobject" fibration.

Lemma 2.1.1. *Every functor $F : \mathbf{C} \to \mathbf{B}$ factors through a* **free fibration**

$$\begin{tikzcd} \mathbf{C} \arrow[r, "i"] \arrow[dr, "F"'] & 1/F \arrow[d, "\mathrm{dom}"] \\ & \mathbf{B} \end{tikzcd}$$

where $i : \mathbf{C} \to 1/F : \begin{cases} C \mapsto 1_{F(C)} & C \in Ob\,\mathbf{C} \\ (f : C \to C') \mapsto (F(f), f) & f \in Ar\,\mathbf{C} \end{cases}$

$\square$

Taking a (co)cartesian morphism over $f : B' \to B \in Ar\,\mathbf{B}$ depends on the choice of an object $E \in Ob\,\mathbf{E}_B$ (respectively, $E' \in Ob\,\mathbf{E}_{B'}$) and the choice of an arrow $\tilde{f}_E : E' \to E$ (respectively, $\tilde{f}_{E'} : E' \to E$) in the isomorphism class.

Lemma 2.1.2.

- *For a (co)fibration* $\begin{tikzcd} \mathbf{E} \arrow[d, "p"'] \\ \mathbf{B} \end{tikzcd}$, *an arrow* $f : B' \to B \in Ar\,\mathbf{B}$, *and a choice* $\tilde{f}_E : E' \to E \ \ \forall E \in Ob\,\mathbf{E}_B$ *(respectively,* $\tilde{f}'_E : E' \to E \ \ \forall E' \in Ob\,\mathbf{E}_{B'}$*) there is a functor* $\mathbf{Cart}_f : \mathbf{E}_B \to \mathbf{E}_{B'}$:

$$\left\{ \begin{array}{lll} E \mapsto d(\tilde{f}_E) & & E \in Ob\,\mathbf{E}_B \\ (g : E_1 \to E) \mapsto \mathbf{Cart}_f(g) \quad s.t. & \begin{tikzcd} E'_1 \arrow[r, "\tilde{f}_{E_1}"] \arrow[d, dashed, "\exists!\,\mathbf{Cart}_f(g)"'] & E_1 \arrow[d, "g"] \\ E' \arrow[r, "\tilde{f}_E"'] & E \end{tikzcd} & g \in Ar\,\mathbf{E}_B \end{array} \right.$$

respectively, $\mathbf{Cart}_f : \mathbf{E}_{B'} \to \mathbf{E}_B$:

$$\left\{ \begin{array}{lll} E' \mapsto c(\tilde{f}_{E'}) & & E' \in Ob\,\mathbf{E}_{B'} \\ (g : E'_1 \to E') \mapsto \mathbf{Cart}_f(g) \quad s.t. & \begin{tikzcd} E'_1 \arrow[r, "\tilde{f}_{E'_1}"] \arrow[d, "g"'] & E_1 \arrow[d, dashed, "\exists!\,\mathbf{Cart}_f(g)"] \\ E' \arrow[r, "\tilde{f}_{E'}"'] & E \end{tikzcd} & g \in Ar\,\mathbf{E}_{B'} \end{array} \right.$$

- *For any two such choices of (co)cartesian morphisms* $\tilde{f}_E$, $\forall E \in \mathbf{E}_B$ *and* $\tilde{\tilde{f}}_E$, $\forall E \in \mathbf{E}_B$ *the corresponding functors* $\mathbf{Cart}_f$ *and* $\widetilde{\mathbf{Cart}}_f$ *are isomorphic.*

PROOF is straightforward. See [Jac]. □

Another equivalent description of (co)fibrations is via (co)contravariant pseudofunctors $\mathbf{B} \to 2\text{-}\mathbf{CAT}$.

Proposition 2.1.1.

- *For each choice of (co)cartesian liftings for all arrows* $f : B' \to p(E)$ *(or respectively,* $f : p(E) \to B'$*) in the base category of a (co)fibration* $\begin{tikzcd} \mathbf{E} \arrow[d, "p"'] \\ \mathbf{B} \end{tikzcd}$ *there is a corresponding pseudofunctor*

$$F_p : \mathbf{B} \to \mathbf{CAT} : \begin{cases} B \mapsto \mathbf{E}_B & B \in Ob\,\mathbf{B} \\ f \mapsto \mathbf{Cart}_f & f \in Ar\,\mathbf{B} \end{cases}$$

- *Conversely, for a given (co)contravariant pseudofunctor* $F : \mathbf{B} \to 2\text{-}\mathbf{CAT}$ *there is a (co)fibration*

$$\begin{tikzcd}\mathbf{E}_F \arrow[d, "p_F"'] \\ \mathbf{B}\end{tikzcd}$$ *(***Grothendieck construction***).*

$$\begin{cases} Ob\,\mathbf{E}_F \textit{ are pairs } \binom{E}{B}, & E \in Ob\,F(B),\ \ B \in Ob\,\mathbf{B} \\ Ar\,\mathbf{E}_F \textit{ are pairs } \binom{h}{f} \in \mathbf{E}_F\left(\binom{E}{B}, \binom{E'}{B'}\right) & h \in F(B)(E, F(f)(E')),\ \ f \in \mathbf{B}(B, B') \end{cases}$$

$$1_{\binom{E}{B}} := \begin{pmatrix} E \xrightarrow{\sim} F(1_B)(E) \\ 1_B \end{pmatrix},$$

for $\binom{u}{f} : \binom{E}{B} \to \binom{E'}{B'})$ *and* $\binom{v}{g} : \binom{E'}{B'}) \to \binom{E''}{B''}$ *the composite*

$$\binom{v}{g} \circ \binom{u}{f} := \binom{w}{g \circ f} \textit{ where } w : E \xrightarrow{u} F(f)(E') \xrightarrow{F(f)(v)} F(f)(F(g)(E'')) \xrightarrow{\sim} F(g \circ f)(E'')$$

or respectively,

$$\begin{cases} Ob\,\mathbf{E}_F \textit{ are pairs } \binom{E}{B}, & E \in Ob\,F(B),\ \ B \in Ob\,\mathbf{B} \\ Ar\,\mathbf{E}_F \textit{ are pairs } \binom{h}{f} \in \mathbf{E}_F((E,B),(E',B')) & h \in F(B')(F(f)(E), E'),\ \ f \in \mathbf{B}(B, B') \end{cases}$$

$$1_{\binom{E}{B}} := \begin{pmatrix} F(1_B)(E) \xrightarrow{\sim} E \\ 1_B \end{pmatrix},$$

for $\binom{u}{f} : \binom{E}{B} \to \binom{E'}{B'}$ *and* $\binom{v}{g} : \binom{E'}{B'} \to \binom{E''}{B''}$ *the composite*

$$\binom{v}{g} \circ \binom{u}{f} := \binom{w}{g \circ f} \textit{ where } w : F(g \circ f)(E) \xrightarrow{\sim} F(g)(F(f)(E)) \xrightarrow{F(g)(u)} F(g)(E') \xrightarrow{v} E'',$$

p_F is the projection onto the bottom component in both cases.

- *The two processes above are (weakly) inverse to each other.*

PROOF is straightforward. See [Jac].

(It is essential for the Grothendieck construction that every (co)fibration $\begin{tikzcd}\mathbf{E} \arrow[d, "p"'] \\ \mathbf{B}\end{tikzcd}$ is equivalent to (co)fibration $\begin{tikzcd}p \arrow[d, "p_2"'] \\ \mathbf{B}\end{tikzcd}$ where $\begin{cases} Ob\,p := \left\{ \binom{E}{p(E)} \;\middle|\; E \in Ob\,\mathbf{E} \right\} \\ Ar\,p := \left\{ \binom{f}{p(f)} \;\middle|\; f \in Ar\,\mathbf{E} \right\} \end{cases}$, p_2 is the projection onto the bottom component, and that every morphism in $\mathbf{E}$ factors through a (co)cartesian one) ☐

2.1.1. Almost-structures.

We now give an abstract notion of structure on objects of a category. It correctly captures such varying examples as manifold structures on a set, algebraic structures on a set, and foliations on a manifold. It also provides a convenient means of discussing fibre and jet bundles.

Definition 2.1.1.1. A **structure of type E on (objects of) category B** is a functor $\begin{tikzcd}\mathbf{E} \arrow[d, "p"'] \\ \mathbf{B}\end{tikzcd}$ which is

- faithful
- admits lifting of isomorphisms's of type $f : B' \xrightarrow{\sim} p(E)$ (or, the same, $f : p(E) \xrightarrow{\sim} B'$)
- each fiber $\mathbf{E}_B$ is skeletal ☐

Lemma 2.1.1.1. *Let* $\begin{tikzcd}\mathbf{E} \arrow[d, "p"'] \\ \mathbf{B}\end{tikzcd}$ *be a structure on* $\mathbf{B}$*,* $E', E'' \in Ob(\mathbf{E}_B)$ *for some* $B \in Ob(\mathbf{B})$*.*
The following are equivalent

a) $E' = E''$

b) $\forall E \in Ob\,\mathbf{E}$ $p(\mathbf{E}(E, E')) = p(\mathbf{E}(E, E''))$

c) $\forall E \in Ob\,\mathbf{E}$ $p(\mathbf{E}(E', E)) = p(\mathbf{E}(E'', E))$

d) $\forall E \in Ob\,\mathbf{E}_B$ $\mathbf{E}_B(E, E') = \emptyset$ *iff* $\mathbf{E}_B(E, E'') = \emptyset$

e) $\forall E \in Ob\,\mathbf{E}_B$ $\mathbf{E}_B(E', E) = \emptyset$ *iff* $\mathbf{E}_B(E'', E) = \emptyset$

PROOF. $a) \Rightarrow b), c), d), e)$ is obvious.

$b) \Rightarrow a)$ Take $E = E'$ and $E = E''$ then $\exists f : E' \to E''$ and $\exists g : E'' \to E'$ such that $p(f) = 1_B = p(g)$. So, $p(f \circ g) = p(g \circ f) = 1_B$. By faithfulness of p and the skeletal condition on $\mathbf{E}_B$, f and g are trivial isomorphisms's.

$c) \Rightarrow a)$ is the same as $b) \Rightarrow a)$.

$d) \Rightarrow a)$ Take $E = E'$ and $E = E''$ then $\exists f : E' \to E''$, f is vertical, and $\exists g : E'' \to E'$, g is vertical. So, $f \circ g = 1_{E''}$, $g \circ f = 1_{E'} \Rightarrow E' \simeq E'' \Rightarrow E' = E''$.

$e) \Rightarrow a)$ is the same as $d) \Rightarrow a)$. □

Proposition 2.1.1.1. *For the structure* $\begin{array}{c}\mathbf{E}\\ {\scriptstyle p}\big\downarrow\\ \mathbf{B}\end{array}$ *on* $\mathbf{B}$*, each fiber* $\mathbf{E}_B$ *is a poset with vertical cartesian morphisms just identities.*

Proof. $\mathbf{E}_B$ is a preorder since $\exists$ at most one morphism between objects (p is faithful). $\mathbf{E}_B$ is a partial order since all isomorphic objects are the same (skeletal condition). Cartesian lift of isomorphisms is isomorphisms, so the cartesian lift $\widetilde{1_B}$ is an identity (since $\mathbf{E}_B$ is a partial order).

□

Proposition 2.1.1.2.

- *The pullback of a fibration is a fibration,*
- *The pullback of a structure (on* $\mathbf{B}$*) is a structure (on* $\mathbf{A}$*)*

$$\begin{tikzcd}
F^*\mathbf{E} \arrow[r, "\bar{F}"] \arrow[d, "p'"'] & \mathbf{E} \arrow[d, "p"] \\
\mathbf{A} \arrow[r, "F"'] & \mathbf{B}
\end{tikzcd}$$

Proof.

- Pullbacks in **CAT** exist and they are certain subcategories of direct products. Cartesian morphisms are preserved under pullbacks which is seen from the following diagram

$$\begin{tikzcd}
(A'',E'') \arrow[dr, "(w,\beta)"'] \arrow[drr, "(u\circ w,\alpha)"] & & \\
& (A',E') \arrow[r, "(u,v)"'] & (A,E)
\end{tikzcd}
\qquad
\begin{tikzcd}
E'' \arrow[dr, "\beta"'] \arrow[drr, "\alpha"] & & \\
& E' \arrow[r, "v"'] & E
\end{tikzcd}$$

$$\begin{tikzcd}
A'' \arrow[dr, "\forall w"] & & \\
& A' \arrow[r, "u"] & A
\end{tikzcd}
\qquad
\begin{tikzcd}
FA'' \arrow[dr, "Fw"] & & \\
& FA' \arrow[r, "Fu"] & FA
\end{tikzcd}$$

(where: $p(E) = F(A)$, $p(\alpha) = F(u \circ w)$, v is cartesian over $F(u)$).

- If $\begin{array}{c}\mathbf{E}\\ {\scriptstyle p}\big\downarrow\\ \mathbf{B}\end{array}$ admits lifting of isomorphisms's of type $B' \xrightarrow{\sim} p(E)$ then $\begin{array}{c}F^*\mathbf{E}\\ {\scriptstyle p'}\big\downarrow\\ \mathbf{A}\end{array}$ admits lifting of isomorphisms's of the same type (obvious). If p is faithful then p' is faithful (assume, $p'(u,v) = p'(u,v_1) = u$ then $F(u) = p(v) = p(v_1)$, so, $v = v_1$). Fibres of $\begin{array}{c}F^*\mathbf{E}\\ {\scriptstyle p'}\big\downarrow\\ \mathbf{A}\end{array}$ are skeletal (assume, $(1_A, v) : (A, E') \xrightarrow{\sim} (A, E)$ is an isomorphisms in $(F^*\mathbf{E})_A$ then $v : E' \xrightarrow{\sim} E$ is an isomorphisms in $\mathbf{E}_{F(A)}$, so, $E' = E$, and $(A, E') = (A, E)$). □

Special cases of a pullback are "fiber" and "intersection of structures":

$$\begin{tikzcd}
\mathbf{E}_B \arrow[r] \arrow[d] & \mathbf{E} \arrow[d, "p"] \\
1 \arrow[r, "B"'] & \mathbf{B}
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
\mathbf{E}_1 \wedge \mathbf{E} \arrow[r] \arrow[d] \arrow[dr, "\pi"] & \mathbf{E} \arrow[d, "p"] \\
\mathbf{E}_1 \arrow[r, "p_1"'] & \mathbf{B}
\end{tikzcd}$$

Remark. The notion of *structure on objects* of a category was introduced in [Kom] in order to deal with the usual structures in Differential Geometry like smooth structures or fibre bundles on topological spaces. However, it turned out to be too weak in one sense (no inverse and direct images) and too strict (skeletal fibres) in another. An appropriate framework was created with the theory of (co)fibrations. Nevertheless, a weaker notion of almost-structure emphasizes the direct connection with the main structure, which is the case of importance especially when almost-structure is introduced onto hom-sets.

Proposition 2.1.1.3. *For each structure* $\begin{array}{c}\mathbf{E}\\ {\scriptstyle p}\big\downarrow\\ \mathbf{B}\end{array}$ *of type* $\mathbf{E}$ *on objects of category* $\mathbf{B}$

- *there is an embedding* $i_p : \begin{array}{c}\mathbf{E}\\ {\scriptstyle p}\big\downarrow\\ \mathbf{B}\end{array} \hookrightarrow \mathbf{Set}^{\mathbf{E}^{op}} : \begin{cases} \begin{pmatrix} E \\ p(E) \end{pmatrix} \mapsto p(\mathbf{E}(-,E)) & \textit{on objects} \\ \begin{pmatrix} v \\ p(v) \end{pmatrix} \mapsto p(\mathbf{E}(-,v)) & \textit{on arrows} \end{cases}$
- $p(\mathbf{E}(-,E)) \hookrightarrow \mathbf{B}(p(-), p(E)) : \mathbf{E}^{op} \to \mathbf{Set}$ *(hom-subfunctor)*

PROOF. Functoriality is obvious. Injectivity follows from Lemma 2.1.1.1. □

Remark. This means that every structure $\mathbf{E}$ on objects in $\mathbf{B}$ is **faithfully** representable by a specific subcategory of $\mathbf{B}$-hom-subfunctors (in which it is sufficient to take arrows of only simple type $f \circ -$).

A reasonable question is whether an object $E \in Ob\,\mathbf{E}$ can be recovered from a functor $F \hookrightarrow \mathbf{B}(p(-), p(E))$? If it can it is unique but the answer is no, in general. Even when it is impossible, the subfunctor $F \hookrightarrow \mathbf{B}(p(-), p(E))$ behaves like an object in $\mathbf{E}$.

Definition 2.1.1.2.

- An arbitrary subfunctor $F \hookrightarrow \mathbf{B}(p(-), B) : \mathbf{E}^{op} \to \mathbf{Set}$ is called an **almost-E structure** over object $B \in Ob\,\mathbf{B}$.
- The category $\begin{array}{c} A\mathbf{E} \\ \downarrow \\ \mathbf{B} \end{array}$ with objects $\begin{pmatrix} F \\ B \end{pmatrix}$, $B \in Ob\,\mathbf{B}$, $F \hookrightarrow \mathbf{B}(p(-), B)$ and morphisms $\begin{pmatrix} f \circ - \\ f \end{pmatrix} \equiv \begin{pmatrix} \mathbf{B}(p(-), f) \\ f \end{pmatrix}$, $f : B \to B'$ is called a **category of almost-E structures over B**.
- An **almost-E costructure over** $B \in Ob\,\mathbf{B}$ is a subfunctor $F' \hookrightarrow \mathbf{B}(B, p(-)) : \mathbf{E} \to \mathbf{Set}$ [almost-costructures are not dual to almost-structures, they behave all together in a covariant way].
- Category $\begin{array}{c} A'\mathbf{E} \\ \downarrow \\ \mathbf{B} \end{array}$ with objects $\begin{pmatrix} F' \\ B \end{pmatrix}$, $B \in Ob\,\mathbf{B}$, $F' \hookrightarrow \mathbf{B}(B, p(-))$ and morphisms $f : \begin{pmatrix} F' \\ B \end{pmatrix} \to \begin{pmatrix} F'_1 \\ B_1 \end{pmatrix}$, if $f : B \to B_1 \in Ar\,\mathbf{B}$ and $\forall E_1 \in Ob\,\mathbf{E}$, $\forall g \in \mathbf{B}(B_1, p(E_1))$. $g \circ f \in \mathbf{B}(B, p(E_1))$,

  is called a **category of almost-E costructures over B**. □

Example

Take $\mathbf{Poly}(E_1, E_2, ..., E_n; -) \hookrightarrow \mathbf{Set}(p(E_1 \times \cdots \times E_n), p(-)) : \mathbf{Vect} \to \mathbf{Set}$, the subfunctor of multilinear maps. Then $\mathbf{Poly}(+, +, ..., +; -) : \mathbf{Vect}^n \to \mathbf{Set}^{\mathbf{Vect}}$ determines a subcategory of almost-**Vect** costructures over $(p \circ (\times^n))(\mathbf{Vect}^n) \hookrightarrow \mathbf{Set}$ ($p : \mathbf{Vect} \to \mathbf{Set}$ is forgetful).

Proposition 2.1.1.4. *For a structure of type* **E** *on* **B** $\begin{tikzcd} \mathbf{E} \arrow[d, "p"'] \\ \mathbf{B} \end{tikzcd}$

- $\begin{tikzcd} A\mathbf{E} \arrow[d] \\ \mathbf{B} \end{tikzcd}$ *is a fibration;*
- $\begin{tikzcd} \mathbf{E} \arrow[d, "p"'] & A\mathbf{E} \arrow[d] \\ \mathbf{B} \arrow[r, phantom, "\hookrightarrow"] & \mathbf{B} \end{tikzcd}$ *is a subcategory*

PROOF.

- If $\begin{pmatrix} F \\ B \end{pmatrix} \in Ob \begin{tikzcd} A\mathbf{E} \arrow[d] \\ \mathbf{B} \end{tikzcd}$ and $f : B' \to B$ take

$f^*F := \{g : p(X) \to B' \mid X \in Ob\,\mathbf{E},\ f \circ g \in F(X) \subset \mathbf{B}(p(X), B)\}$. Then $f^*F \hookrightarrow \mathbf{B}(p(-), B')$ is a subfunctor, and $\begin{pmatrix} f^*F \\ B' \end{pmatrix} \xrightarrow{\binom{f\circ -}{f}} \begin{pmatrix} F \\ B \end{pmatrix}$ is cartesian over f

$$\begin{tikzcd}
F'' \arrow[rd, dashed, "k\circ -"'] \arrow[rrd, "f\circ k\circ -"] & & \\
B'' \arrow[rd, "k"'] \arrow[rrd, "f\circ k"] & f^*F \arrow[r, "f\circ -"] & F \\
& B' \arrow[r, "f"] & B
\end{tikzcd}$$

- The assignment $\begin{cases} \begin{pmatrix} E \\ p(E) \end{pmatrix} \mapsto p(\mathbf{E}(-, E)) \hookrightarrow \mathbf{B}(p(-), p(E)) & \text{on objects} \\ \begin{pmatrix} v \\ p(v) \end{pmatrix} \mapsto p(\mathbf{E}(-, v)) = \mathbf{B}(p(-), p(v)) & \text{on arrows} \end{cases}$

gives the required embedding. □

2.2. Enrichment with generalized elements in hom-sets

Definition 2.2.1. A 1-category **C** is **enriched in the tensor category** $(\mathcal{V}, I, \otimes)$ [Kel, Bor2]

if

- $\forall x, y \in \mathrm{Ob}\,\mathbf{C}\quad \mathbf{C}(x, y) \in \mathrm{Ob}\mathcal{V}$
- $\forall x, y, z \in \mathrm{Ob}\,\mathbf{C}\quad \mu_{x,y,z} : \mathbf{C}(y, z) \otimes \mathbf{C}(x, y) \to \mathbf{C}(x, z) \in \mathrm{Ar}\mathcal{V}$
- $\forall x, y, z, w \in \mathrm{Ob}\,\mathbf{C}$

$$\begin{tikzcd}
(\mathbf{C}(z,w)\otimes\mathbf{C}(y,z))\otimes\mathbf{C}(x,y) \arrow[r, "\sim"] \arrow[d, "\mu\otimes 1"'] & \mathbf{C}(z,w)\otimes(\mathbf{C}(y,z)\otimes\mathbf{C}(x,y)) \arrow[r, "1\otimes\mu"] & \mathbf{C}(z,w)\otimes\mathbf{C}(x,z) \arrow[d, "\mu"] \\
\mathbf{C}(w,y)\otimes\mathbf{C}(x,y) \arrow[rr, "\mu"'] & & \mathbf{C}(x,w)
\end{tikzcd}$$

- $\forall x \in \operatorname{Ob}\mathbf{C}\ \exists u_x : I \to \mathbf{C}(x,x) \in \operatorname{Ar}\mathcal{V}$ such that

$$\begin{tikzcd}
\mathbf{C}(x,y) & I\otimes\mathbf{C}(x,y) \arrow[l, "\sim"'] \arrow[d, "u_x\otimes 1"] \\
& \mathbf{C}(x,x)\otimes\mathbf{C}(x,y) \arrow[ul, "\mu"]
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbf{C}(z,x) & \mathbf{C}(z,x)\otimes I \arrow[l, "\sim"'] \arrow[d, "1\otimes u_x"] \\
& \mathbf{C}(z,x)\otimes\mathbf{C}(x,x) \arrow[ul, "\mu"]
\end{tikzcd}$$

□

Generalized elements in hom-sets are just parametrized families of arrows in the same way as continuous or smooth families of maps are generalized elements. They form an almost-**C** structure on the hom-sets of the category **C** [Kom].

Definition 2.2.2. Assume that **C** has binary products and $|\text{-}| : \mathbf{C} \to \mathbf{Set}$ is a faithful functor. Then the **Set**-map $f : |Z| \to \mathbf{C}(X,Y)$, $Z \in \operatorname{Ob}\mathbf{C}$ is called a **generalized** (or acceptable) **element** of $\mathbf{C}(X,Y)$ with domain $|Z|$ if the arrow $\tilde{f}\circ\gamma_Z$ can be lifted to **C**

$$\begin{tikzcd}
\mathbf{C}(X,Y)\times|X| \arrow[r, "ev"] & {|Y|} \\
{|Z|\times|X|} \arrow[u, "f\times 1"] \arrow[ur, "\tilde{f}"'] & \\
{|Z\times X|} \arrow[u, "\gamma_Z"] &
\end{tikzcd}$$

where: γ_Z is the mediating arrow to product, $ev(g,x) := |g|(x)$ □

Denote by $\mathcal{G}(|Z|,\mathbf{C}(X,Y)) \hookrightarrow \mathbf{Set}(|Z|,\mathbf{C}(X,Y))$ the subset of generalized elements of $\mathbf{C}(X,Y)$ with domain $|Z|$.

Proposition 2.2.1. *The assignment* $Z \mapsto \mathcal{G}(|Z|,\mathbf{C}(X,Y))$ *is extendable to a functor* $\mathcal{G}(|\text{-}|,\mathbf{C}(X,Y)) : \mathbf{C}^{op} \to \mathbf{Set}$.

PROOF. Assume, $\alpha : Z' \to Z$ is an arrow, $f \in \mathcal{G}(|Z|, \mathbf{C}(X,Y))$ is a generalized element. We need to show that $f \circ \alpha \in \mathcal{G}(|Z'|, \mathbf{C}(X,Y))$ is a generalized element as well, i.e., that $\exists h : Z' \times X \to Y$ such that $|h| = ev \circ (f \times 1) \circ (\alpha \times 1) \circ \gamma_{Z'} = \tilde{f} \circ (\alpha \times 1) \circ \gamma_{Z'}$.

Since $\gamma_Z : |Z \times X| \to |Z| \times |X|$ is natural in Z

$$\begin{tikzcd}
{|Z' \times X|} \arrow[r, "\gamma_{Z'}"] \arrow[d, "|\alpha \times 1|"'] & {|Z'| \times |X|} \arrow[d, "|\alpha| \times |1|"] & \\
{|Z \times X|} \arrow[r, "\gamma_Z"'] & {|Z| \times |X|} \arrow[r, "\tilde{f}"'] & {|Y|}
\end{tikzcd}$$

the requirement will be $\exists h : Z' \times X \to Y$ such that $|h| = \tilde{f} \circ \gamma_Z \circ |\alpha \times 1|$. By assumption on f, $\exists g : Z \times X \to Y$ such that $|g| = \tilde{f} \circ \gamma_Z$. So, take $h := g \circ (\alpha \times 1)$. □

Proposition 2.2.2. *If* $|\text{-}| : \mathbf{C} \to \mathbf{Set}$ *is a faithful functor which preserves binary products, then the category* $\mathbf{C}$ *is enriched with generalized elements in the presheaf category* $\mathbf{Set}^{\mathbf{C}^{op}}$.

PROOF.

- $\forall X, Y \in Ob\,\mathbf{C}\ \ (\mathcal{G}(|\text{-}|, \mathbf{C}(X,Y)) : \mathbf{C}^{op} \to \mathbf{Set}) \in Ob(\mathbf{Set}^{\mathbf{C}^{op}})$
- $\forall X, Y, Z \in Ob\,\mathbf{C}$ take $\mu_{X,Y,Z} : \mathcal{G}(|\text{-}|, \mathbf{C}(Y,Z)) \otimes \mathcal{G}(|\text{-}|, \mathbf{C}(X,Y)) \to \mathcal{G}(|\text{-}|, \mathbf{C}(X,Z))$ such that $\forall W \in Ob\,\mathbf{C}\ \ \mu_{X,Y,Z;W}(f,g) := \mu^{\mathbf{C}}_{X,Y,Z} \circ < f, g >$ where: $\mu^{\mathbf{C}}_{X,Y,Z} : \mathbf{C}(Y,Z) \times \mathbf{C}(X,Y) \to \mathbf{C}(X,Z)$ is the composite in $\mathbf{C}$, $< f, g >: |W| \to \mathbf{C}(Y,Z) \times \mathbf{C}(X,Y)$ is the mediating arrow to product. $\mu_{X,Y,Z;W}$ is natural in W since $(\mu^{\mathbf{C}}_{X,Y,Z} \circ < f, g >) \circ |h| = \mu^{\mathbf{C}}_{X,Y,Z} \circ < f \circ |h|, g \circ |h| >$ for $h : W' \to W$.

Why can $ev \circ ((\mu^{\mathbf{C}} \circ < f, g >) \times 1) \circ \gamma$ be lifted to $\mathbf{C}$? By the above condition,

$$\begin{tikzcd}
{\mathbf{C}(Y,Z) \times |Y|} \arrow[r, "ev"] & {|Z|} \\
{|W| \times |Y|} \arrow[u, "f \times 1"] \arrow[ur, "\tilde{f}"'] & \\
{|W \times Y|} \arrow[u, "\gamma"] \arrow[uur, "\text{lifted}"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
{\mathbf{C}(X,Y) \times |X|} \arrow[r, "ev"] & {|Y|} \\
{|W| \times |X|} \arrow[u, "g \times 1"] \arrow[ur, "\tilde{g}"'] & \\
{|W \times X|} \arrow[u, "\gamma"] \arrow[uur, "\text{lifted}"'] &
\end{tikzcd}$$

It is sufficient to take $\gamma = 1$.

$$\begin{tikzcd}
\mathbf{C}(X,Z) \times |X| \arrow[rr, dotted, "ev"] & & {|Z|} \\
\mathbf{C}(Y,Z) \times \mathbf{C}(X,Y) \times |X| \arrow[u, dotted, "\mu^{\mathbf{C}} \times 1_{|X|}"] \arrow[rr, "1_{\mathbf{C}(Y,Z)} \times ev"] & & \mathbf{C}(Y,Z) \times |Y| \arrow[u, "ev"'] \\
 & & {|W| \times |W| \times |X|} \arrow[u, "{f \times \tilde{g} = (f \times 1_{|Y|}) \circ (1_{|W|} \times \tilde{g})}"] \arrow[llu, "f \times g \times 1_{|X|}"] \arrow[uu, bend right=60, "\tilde{f} \circ (1_{|W|} \times \tilde{g})"'] \\
 & & {|W| \times |X|} \arrow[u, "{<1_{|W|}, 1_{|W|}> \times 1_{|X|}}"'] \arrow[lluu, dotted, "{<f,g> \times 1_{|X|}}"] \\
 & & {|W \times X|} \arrow[u, dotted, "\gamma=1"']
\end{tikzcd}$$

The required dotted arrows $ev \circ ((\mu^{\mathbf{C}} \circ < f, g >) \times 1_{|X|}) \circ \gamma$ can be lifted to $\mathbf{C}$ since the rightmost way $\tilde{f} \circ (1_{|W|} \times \tilde{g}) \circ (< 1_{|W|}, 1_{|W|} > \times 1_{|X|}) \circ 1_{|W \times X|}$ can be lifted (for take liftings for $\tilde{f}, \tilde{g}$ and identities for identities).

- (associativity) $\forall f, g, h$ such that $f : |W| \to \mathbf{C}(Z, Z')$, $g : |W| \to \mathbf{C}(Y, Z)$, $h : |W| \to \mathbf{C}(X, Y)$ $\mu^{\mathbf{C}}_{X,Z,Z'} \circ < f, \mu^{\mathbf{C}}_{X,Y,Z} \circ < g, h >>= \mu^{\mathbf{C}}_{X,Y,Z'} \circ < \mu^{\mathbf{C}}_{Y,Z,Z'} \circ < f, g >, h >$ because the equality holds at each point $w \in |W|$.
- (identities) $\forall X \in Ob\,\mathbf{C}$ take $u_{X;W} : \mathbf{1} \to \mathbf{Set}(|W|, \mathbf{C}(X,X)) : * \mapsto (w \mapsto 1_X)$. It is natural in W and $\forall f, g$ such that $f : |W| \to \mathbf{C}(X,Y)$, $g : |W| \to \mathbf{C}(Z,X)$ the equalities hold $\mu^{\mathbf{C}} \circ < f, u_{X;W} > (w) = \mu^{\mathbf{C}}(f(w), 1_X) = f(w)$, $\mu^{\mathbf{C}} \circ < u_{X;W}, g > (w) = \mu^{\mathbf{C}}(1_X, g(w)) = g(w)$ $w \in |W|$. □

Corollary (refinement of Proposition 2.2.2). *Under the above assumptions (*$\mathbf{C}$ *has binary products,* $|\text{-}| : \mathbf{C} \to \mathbf{Set}$ *is a faithful functor preserving binary products) hom-sets of* $\mathbf{C}$ *are enriched with an almost-*$\mathbf{C}$ *structure of generalized elements.*

PROOF is immediate (because presheaves of generalized elements are of the specific form

$\mathcal{G}(|\text{-}|, \mathbf{C}(X,Y)) \hookrightarrow \mathbf{Set}(|\text{-}|, \mathbf{C}(X,Y))$ and μ is actually postcomposite $\mu^{\mathbf{C}} \circ -$). □

We mean further that $\mathbf{C}$ is $A\mathbf{C}$-category if this **specific enrichment** with generalized elements is given. Moreover, we call $\mathbf{D}$ is an $A\mathbf{C}$-category if it is enriched with presheaves of generalized elements with domains in $\mathbf{C}$.

Remark. All the usual concrete categories, like **Top**, **Grp**, **Rng**, etc. carry corresponding almost-structures (which in some cases can be strict).

Example

Proposition 2.2.3. *If X is a locally compact topological space (so that the family $\mathcal{T}$ of topologies on $\mathbf{Top}(X,Y)$ for which the evaluation map $ev : \mathbf{Top}(X,Y) \times |X| \to |Y|$ is continuous is not empty and contains a minimal element, the compact-open topology on $\mathbf{Top}(X,Y)$) then $\tau \in \mathcal{T}$ is compact-open iff $\forall Z \in Ob\,\mathbf{Top}$ each generalized element $f : |Z| \to \mathbf{Top}(X,Y)$ is continuous.*

Proof. " $\Rightarrow$ " Consider the diagram

$$\begin{tikzcd}
\mathbf{Top}(X,Y) \times |X| \arrow[r, "ev"] & {|Y|} \\
{|Z| \times |X|} \arrow[u, "f\times 1"] \arrow[ur, "\tilde{f}"'] &
\end{tikzcd}$$

We want to show that $f : |Z| \to \mathbf{Top}(X,Y)$ is continuous (with compact-open topology in $\mathbf{Top}(X,Y)$) if $\tilde{f} : |Z| \times |X| \to |Y|$ is continuous, i.e., that $\forall z \in |Z|$ $\forall$ (subbase) compact-open set $U^K \ni f(z)$ $\exists$ nbhd $V \ni z$ such that $f(V) \subset U^K$. It is equivalent that $\forall z \in |Z|\, \forall U^K \ni f(z)\, \exists V \ni z$ such that $\tilde{f}(V \times K) \subset U$. Since $\tilde{f}$ is continuous $\forall (z,x) \in \{z\} \times K$ and $\forall$ open $U \ni \tilde{f}(z,x)$ $\exists$ open nbhd $V_z \times W_x \ni (z,x)$ such that $\tilde{f}(V_z \times W_x) \subset U$. $\bigcup_{x \in K} W_x \supset K$ (open cover). So, by compactness of K, $\exists W_{x_1}, \dots, W_{x_n}$, such that $\bigcup_{i=1}^{n} W_{x_i} \supset K$. Take $V := \bigcap_{i=1}^{n} (V_i)_z$, where $(V_i)_z$ corresponds to W_{x_i} (i.e., $(V_i)_z$ is open, $(V_i)_z \ni z$, $\tilde{f}((V_i)_z \times W_{x_i}) \subset U$). Then $\tilde{f}(V \times K) \subset U$.

" $\Leftarrow$ " Take $Z = \mathbf{Top}(X,Y)$ with compact-open topology. Take $\mathbf{Top}(X,Y)$ itself (on the top of the diagram) with non minimal $\tau \in \mathcal{T}$, $f := 1 \in Ar\,\mathbf{Set}$. Then $1 : \mathbf{Top}(X,Y) \to \mathbf{Top}(X,Y)$ is an admissible generalized element, since ev is continuous, but 1 is not continuous. ☐

Remark. Therefore, for a locally compact space X, the almost-**Top** structure $\mathcal{G}(|Z|, \mathbf{Top}(X,Y))$ coincides with the compact-open topology, i.e., is actually a **Top** structure.

If we agree that the functor $\mathcal{G}(|\text{-}|, \mathbf{C}(X,Y)) : \mathbf{C}^{op} \to \mathbf{Set}$ reflects essential properties of **C**-hom-sets, we immediately get a unique (up to isomorphism) extension of each functor $F : \mathbf{C} \to \mathbf{D}$, i.e., we deal with **C**-hom-sets as with **C**-objects. In this way, for example, the tangent or jet functor can be introduced directly on $\mathbf{Aut}(X)$, $X \in Ob\,\mathbf{Diff}$ to give rise to a calculus on $\mathbf{Aut}(X)$. The possibility of such an extension follows from the fact that each presheaf is a certain (canonical) colimit of representables [Mac, M-M].

Proposition 2.2.4.

- *The Yoneda embedding* $y : \mathbf{C} \to \mathbf{Set}^{\mathbf{C}^{op}}$ *is a universal cocompletion of* $\mathbf{C}$, *i.e.,* $\forall F : \mathbf{C} \to \mathbf{E}$, *where* $\mathbf{E}$ *is cocomplete,* $\exists!$ *(up to isomorphisms) cocontinuous* $\bar{F} : \mathbf{Set}^{\mathbf{C}^{op}} \to \mathbf{E}$ *such that*

$$\begin{tikzcd}
\mathbf{Set}^{\mathbf{C}^{op}} \arrow[r, dashed, "\bar{F}"] & \mathbf{E} \\
\mathbf{C} \arrow[u, "y"] \arrow[ur, "F"'] &
\end{tikzcd}$$

- $\bar{F}(P) = Colim(\int_{\mathbf{C}} P \xrightarrow{\pi} \mathbf{C} \xrightarrow{F} \mathbf{E})$, *where* $P \in Ob\,\mathbf{Set}^{\mathbf{C}^{op}}$, $\int_{\mathbf{C}} P$ *is a category of elements of* P, π *is the natural projection.*

- $$\begin{tikzcd}
\mathbf{Cat} \arrow[r, bend left, "\mathbf{Set}^{(-)^{op}}"] \arrow[r, phantom, "\perp"] & \mathbf{Cocomp} \arrow[l, bend left, "\text{forgetful}"]
\end{tikzcd}$$, *the adjunction between* **Cat** *and a full subcategory of cocomplete categories* **Cocomp** *with Yoneda embedding* $y_{\mathbf{C}} : \mathbf{C} \to \mathbf{Set}^{\mathbf{C}^{op}}$ *as a unit.*

- *Each functor* $F : \mathbf{C} \to \mathbf{D}$ *admits a unique (up to isomorphisms) cocontinuous extension* $F : \mathbf{Set}^{\mathbf{C}^{op}} \to \mathbf{Set}^{\mathbf{D}^{op}}$ *such that*

$$\begin{tikzcd}
\mathbf{Set}^{\mathbf{C}^{op}} \arrow[r, dashed, "F"] & \mathbf{Set}^{\mathbf{D}^{op}} \\
\mathbf{C} \arrow[u, "y_{\mathbf{C}}"] \arrow[r, "F"] & \mathbf{D} \arrow[u, "y_{\mathbf{D}}"']
\end{tikzcd}$$

Proof. See [M-M]. ☐

For example, if $T : \mathbf{Diff} \to \mathbf{Diff}$ is a tangent functor, $\mathbf{Diff}(X,Y)$ is a presheaf on **Diff** (hom-set enriched as above) then $T(\mathbf{Diff}(X,Y)) = \int_{\mathbf{Diff}} \mathbf{Diff}(X,Y) \xrightarrow{\pi} \mathbf{Diff} \xrightarrow{T} \mathbf{Diff} \xrightarrow{y} \mathbf{Set}^{\mathbf{Diff}^{op}}$.

2.3. General manifolds

Definition 2.3.1. The functor $\begin{tikzcd} \mathbf{E} \arrow[d, "F"'] \\ \mathbf{B} \end{tikzcd}$ is called a **fibration with respect to** a class of arrows $\mathcal{C} \subset Ar\,\mathbf{B}$ if $\forall f : B' \to F(E) \in \mathcal{C}$ $\exists \tilde{f} : E' \to E \in Ar\,\mathbf{E}$ such that $\tilde{f}$ is over f and $\tilde{f}$ is cartesian.

□

Definition 2.3.2. [M-M]

- A Grothendieck **pretopology** τ_0 on a category **B** with pullbacks is a family of **coverings** (or **sieves**) τ_{0B} for each object $B \in Ob\,\mathbf{B}$ (elements of a covering are just arrows with codomain B) such that
  - if $f : B' \xrightarrow{\sim} B$ is an isomorphisms then $\{f\} \in \tau_{0B}$ is an one-element covering
  - if $g : B'' \to B$ is an arrow and $\mathfrak{c} \in \tau_{0B}$ then pullback family $\mathbf{plbk}_g(\mathfrak{c}) \in \tau_{0B''}$
  - (coverings are composable) if $\mathfrak{c} \in \tau_{0B}$ and $\forall B' \in d(\mathfrak{c})$ there is given a covering $\mathfrak{c}_{B'} \in \tau_{0B'}$ then $\mathfrak{c} \circ \bigcup\limits_{B' \in d(\mathfrak{c})} \mathfrak{c}_{B'} \in \tau_{0B}$
- A Grothendieck **topology** τ on a category **B** (not necessarily with pullbacks) is a family of hom-subfunctors τ_B for each object $B \in Ob\,\mathbf{B}$ such that
  - $\mathbf{B}(-,B) \in \tau_B$
  - if $f : B' \to B$ and $\mathfrak{t} \in \tau_B$ then the **inverse image** $\big(f^*(\mathfrak{t}) : X \mapsto (f^*\mathfrak{t})(X,B') \subset \mathbf{B}(X,B')\big) \in \tau_{B'}$ $(h \in f^*(\mathfrak{t})(X,B')$ iff $f \circ h \in \mathfrak{t}(X,B))$
  - if $\mathfrak{s} \hookrightarrow \mathbf{B}(-,B)$ is any hom-subfunctor such that $\forall f : B' \to B$ $f^*(\mathfrak{s}) \in \tau_{B'}$ then $\mathfrak{s} \in \tau_B$ □

Every topology is a pretopology if **B** has pullbacks, and every pretopology generates a topology [M-M]. The category **B** with (pre)topology is called a **site** $(\mathbf{B}, \tau_{(0)})$.

Definition 2.3.3. [Kom] The functor

$$\begin{tikzcd} \mathbf{E} \arrow[d, "F"'] \\ (\mathbf{B}, \tau_0) \end{tikzcd}$$

is called **local** if it is a fibration with respect to all elements of all coverings $\bigcup\limits_{B \in Ob\,\mathbf{B}} \bigcup \tau_{0B}$. ☐

Definition 2.3.4. For a given functor to a site

$$\begin{tikzcd} \mathbf{E} \arrow[d, "F"'] \\ (\mathbf{B}, \tau_0) \end{tikzcd}$$

the smallest local functor

$$\begin{tikzcd} \mathbf{E\text{-}Man} \arrow[d, "p_F"'] & \mathbf{E} \arrow[l, hook'] \arrow[dl, "F"] \\ (\mathbf{B}, \tau_0) & \end{tikzcd}$$

is called an **E-manifold structure over B**. It means

- $\forall X \in Ob\,\mathbf{E\text{-}Man}$ $\exists$ a covering $\mathfrak{c}_{p_F(X)} = \{i : B_i \to p_F(X)\}_{i \in I} \in \tau_{0 p_F(X)}$ such that there are inverse images $i^*(X) \in Ob\,\mathbf{E}$ (i.e., **E** contains isomorphic representatives of the inverse images).

- $\forall f : X' \to X \in Ar\,\mathbf{E\text{-}Man}$ $\exists$ coverings $\mathfrak{c}'_{p_F(X')} = \{i' : B'_{i'} \to p_F(X')\} \in \tau_{0 p_F(X')}$ and $\mathfrak{c}_{p_F(X)} = \{i : B_i \to p_F(X)\} \in \tau_{0 p_F(X)}$ such that $\forall i \in \mathfrak{c}_{p_F(X)}$ $\exists i' \in \mathfrak{c}'_{p_F(X')}$ such that

$$\begin{tikzcd} B'_{i'} \arrow[r, "\exists\, \varphi"] \arrow[d, "i'"'] & B_i \arrow[d, "i"] \\ p_F(X') \arrow[r, "p_F(f)"'] & p_F(X) \end{tikzcd} \quad \text{and over it} \quad \begin{tikzcd} i'^*(X') \arrow[r, dashed, "\exists!\, \Phi"] \arrow[d, "\tilde{i'}"'] & i^*(X) \arrow[d, "\tilde{i}"] \\ X' \arrow[r, "f"'] & X \end{tikzcd}$$

where $\tilde{i'}, \tilde{i}$ are cartesian, $p_F(\Phi) = \varphi$, $\Phi \in Ar\,\mathbf{E}$ (arrows are locally in **E**).

- **E-Man** is maximal with respect to the above two properties. ☐

Examples

1. **Set** as a manifold structure

$$\begin{tikzcd} \mathbf{Set} \arrow[d, "1"'] & \mathbf{Set}_{inj} \arrow[l, hook'] \arrow[dl, hook] \\ (\mathbf{Set}, \tau_0) & \end{tikzcd}$$

where $\mathbf{Set}_{inj}$ is a category of sets with injective maps only, τ_0 is a pretopology consisting of all families of injective maps with common codomains.

2. **Differentiable** manifolds

$$\begin{tikzcd} C^r\text{-}\mathbf{Man}_k \arrow[d, hook] & \mathtt{Triv}C^r\text{-}\mathbf{Man}_k \arrow[l, hook'] \arrow[dl, hook] \\ (\mathbf{Top}, \tau_0) & \end{tikzcd}$$

where $k = \mathbb{R}$ or $\mathbb{C}$, τ_0 consists of all open coverings, $r = 0, 1, ..., \infty$ (or ω for complex manifolds),

$$\begin{cases} Ob\,(\mathtt{Triv}C^r\text{-}\mathbf{Man}_k) = \{k^0, k^1, ..., k^n, ...\} \\ Ar\,(\mathtt{Triv}C^r\text{-}\mathbf{Man}_k) = C^r\text{-maps} \end{cases}$$

3. Locally trivial **fibre bundles**

$$\begin{tikzcd}
\mathbf{Bn}\,(\mathcal{E},p) \arrow[d] & \mathbf{Bn}_0\,(\mathcal{E},p) \arrow[l, hook'] \arrow[dl] \\
(\mathbf{Man}^{\rightarrow},\tau_0) &
\end{tikzcd}$$

(For more details, the reader is referred to Section 2.4.)

4. **Foliations** over **Man**

$$\begin{tikzcd}
\mathbf{Fol}\,(\mathcal{E},p) \arrow[d] & \mathbf{Fol}_0\,(\mathcal{E},p) \arrow[l, hook'] \arrow[dl] \\
(\mathbf{Man},\tau_0) &
\end{tikzcd}$$

associated to an A**Man**-functor sequence $\mathcal{E} \xrightarrow{p} \mathbf{Man} \xrightarrow{\pi} \mathbf{Top}$ (see 2.4), where: **Man** is a category of manifolds (of type $\mathcal{E}'$) over **Top**, τ_0 are all 'open coverings' of objects in **Man**, $\mathbf{Fol}_0\,(\mathcal{E},p) \equiv \mathbf{Bn}_0\,(\mathcal{E},p)$ is a category of trivial foliations ('direct products') with leaves in $\mathcal{E}$, the projection functor

$$\begin{tikzcd}
\mathbf{Fol}_0(\mathcal{E},p) \arrow[d] \\
(\mathbf{Man},\tau_0)
\end{tikzcd}$$

is the first (top) component of projection

$$\begin{tikzcd}
\mathbf{Bn}_0\,(\mathcal{E},p) \arrow[d] \\
(\mathbf{Man}^{\rightarrow},\tau_0)
\end{tikzcd}$$

(τ_0 is different in these two cases and corresponding categories of manifolds are glued differently).

5. **E**-manifolds over **Top**. Let

$$\begin{tikzcd}
\mathbf{E} \arrow[d, "p"'] \\
(\mathbf{Top},\tau_0)
\end{tikzcd}$$

be a local structure on **Top** with τ_0, all open coverings.

- A **local E-map** on a topological space X is a pair $\begin{pmatrix} E \\ U \end{pmatrix} \in Ob \begin{tikzcd} \mathbf{E} \arrow[d] \\ (\mathbf{Top},\tau_0) \end{tikzcd}$, U is open.
- A family $\left\{\begin{pmatrix} E_i \\ U_i \end{pmatrix}\right\}_{i\in I}$ is **compatible** iff $\forall (i,j) \in I^2\ E_i|_{U_i\cap U_j} \xrightarrow[\varphi]{\sim} U_j|_{U_i\cap U_j}$, φ is a vertical isomorphisms.
- An **E-atlas** $\mathcal{A}$ on X is a compatible family $\left\{\begin{pmatrix} E_i \\ U_i \end{pmatrix}\right\}_{i\in I}$ such that $\bigcup\limits_{i\in I} U_i = X$.
- Two **E**-atlases $\mathcal{A}$ and $\mathcal{A}'$ are **equivalent** iff $\mathcal{A} \cup \mathcal{A}'$ is still an **E**-atlas on X (so, there exist maximal atlases, call them $\mathcal{A}_{max}$).
- The above 'equivalence' on atlases is not transitive in general. So, there can be different

maximal atlases containing a given one. But, it is transitive if $\forall \begin{pmatrix}E\\U\end{pmatrix}, \begin{pmatrix}E'\\U\end{pmatrix} \in Ob \begin{array}{c}\mathbf{E}\\ {\scriptstyle p}\big\downarrow\\ \mathbf{Top}\end{array}$ and $\forall$ open covering $\bigcup_{i\in I} U_i \supset U$ $E|_{U_i} \xrightarrow[\text{vert}]{\sim} E'|_{U_i}$ (for all $i \in I$) implies $E \xrightarrow[\text{vert}]{\sim} E'$.

- A topological space X together with an **E**-atlas $\mathcal{A}$ on it is called an **E-manifold**, i.e., $(X, \mathcal{A}) \in Ob\,\mathbf{E\text{-}Man}$.

- An **arrow** in **E-Man** is $f : (X, \mathcal{A}) \to (Y, \mathcal{B})$ such that $f : X \to Y$ is continuous and $\forall \begin{pmatrix}E\\U\end{pmatrix} \in \mathcal{A}, \begin{pmatrix}E'\\V\end{pmatrix} \in \mathcal{B}$ if $U \cap f^{-1}(V) \neq \emptyset$ then $f|_{U\cap f^{-1}(V)} : U \cap f^{-1}(V) \to V$ admits (unique) lifting $\bar{f}|_{U\cap f^{-1}(V)} : E|_{U\cap f^{-1}(V)} \to E' \in Ar\,\mathbf{E}$.

2.4. Fibre bundles

Locally trivial fibre bundles give an important example of general manifolds over $\mathbf{Man}^{\to}$ [Kom].

Definition 2.4.1. The category of **trivial fibre bundles** $\mathbf{Bn}_0\,(\mathcal{E}, p)$ over $\mathbf{Man}^{\to}$ with **typical fibres** in a category $\mathcal{E}$ consists of the following data

- $\mathcal{E} \xrightarrow{p} \mathbf{Man} \xrightarrow{\pi} \mathbf{Top}$, where $\mathcal{E}$ and **Man** are A**Man**-categories, p is A**Man**-functor, π preserves binary products [i.e., **Man** is enriched in $\mathbf{Set}^{\mathbf{Man}^{op}}$ with presheaves of generalized elements $\mathcal{G}(|-|, \mathbf{Man}(A, A'))$ for each hom-set $\mathbf{Man}(A, A')$, $\mathcal{E}$ is enriched in $\mathbf{Set}^{\mathbf{Man}^{op}}$ with subfunctors $\mathcal{H}(|-|, \mathcal{E}(E, E')) \hookrightarrow \mathbf{Set}(|-|, \mathcal{E}(E, E'))$ for each hom-set $\mathcal{E}(E, E')$, $p_{E,E';X} : \mathcal{H}(|X|, \mathcal{E}(E.E')) \to \mathcal{G}(|X|, \mathbf{Man}(p(E), p(E')))$: $f \mapsto p_{E,E'} \circ f$ is natural in $X \in Ob\,\mathbf{Man}$, $p_{E,E'} : \mathcal{E}(E, E') \to \mathbf{Man}(p(E), p(E'))$ is the restriction of functor p on the hom-set]

- $Ob\,\mathbf{Bn}_0\,(\mathcal{E}, p) := \{(X, E) \,|\, X \in Ob\,\mathbf{Man}, E \in Ob\,\mathcal{E}\}$;
  $Ar\,\mathbf{Bn}_0\,(\mathcal{E}, p) := \{(X, E) \xrightarrow{(f,\Phi)} (X', E') \,|\, f : X \to X',\ \Phi \in \mathcal{H}(|X|, \mathcal{E}(E, E'))\}$

- functor

$$\begin{tikzcd} \mathbf{Bn}_0(\mathcal{E},p) \arrow[d, "p_0"'] \\ \mathbf{Man}^{\to} \end{tikzcd} : \begin{cases} (X,E) \mapsto \quad X \times p(E) \xrightarrow{p_1} X & (X,E) \in Ob\,\mathbf{Bn}_0(\mathcal{E},p) \\ (f,\Phi) \mapsto \begin{tikzcd} X \times p(E) \arrow[r, "<f\circ p_1{,}\phi>"] \arrow[d, "p_1"'] & X' \times p(E') \arrow[d, "p_1"] \\ X \arrow[r, "f"'] & X' \end{tikzcd} & (f,\Phi) \in Ar\,\mathbf{Bn}_0(\mathcal{E},p) \end{cases}$$

where $\phi := ev \circ ((p_{E,E'} \circ \Phi) \times 1_{|p(E)|}),\ \ p_{E,E'} \circ \Phi \in \mathcal{G}(|X|, \mathbf{Man}(p(E),p(E')))$ □

Definition 2.4.2. The category of **locally trivial fibre bundles** $\mathbf{Bn}(\mathcal{E},p)$ over the site $(\mathbf{Man}^{\to}, \tau_0)$, where τ_0 is the set of pullbacks of all open coverings of codomains (i.e., if $q : Y \to X \in Ob\,\mathbf{Man}^{\to}$ and $\{U_i\}_{i\in I}$ is an open covering of X then $\{q^{-1}(U_i) \xrightarrow{q|_{U_i}} U_i\}_{i\in I}$ is a covering of q), is a manifold structure of type $\mathbf{Bn}_0(\mathcal{E},p)$ over $\mathbf{Man}^{\to}$. □

A usual way of constructing new fibre bundles from old ones is by fibrewise operations. Let $\mathcal{E} \xrightarrow{p} \mathbf{Man} \xrightarrow{\pi} \mathbf{Top}$ and $\mathcal{E}' \xrightarrow{p'} \mathbf{Man} \xrightarrow{\pi} \mathbf{Top}$ be two sequences generating categories of fibre bundles $\mathbf{Bn}(\mathcal{E},p)$ and $\mathbf{Bn}(\mathcal{E}',p')$ of types $\mathcal{E}$ and $\mathcal{E}'$, respectively, $F : \mathcal{E} \to \mathcal{E}'$ be an A**Man**-functor. Then there exists a corresponding functor $\mathbf{Bn}(F) : \mathbf{Bn}(\mathcal{E},p) \to \mathbf{Bn}(\mathcal{E}',p')$.

Denote by A**Man-CAT** an 1-category such that

$$\begin{cases} Ob\,(A\mathbf{Man\text{-}CAT}) \ni (\mathcal{E},p), & \text{if } \mathcal{E} \text{ is } A\mathbf{Man}\text{-category},\ p : \mathcal{E} \to \mathbf{Man} \text{ is } A\mathbf{Man}\text{-functor} \\ Ar\,(A\mathbf{Man\text{-}CAT}) \ni (F : (\mathcal{E},p) \to (\mathcal{E}',p')), & \text{if } F : \mathcal{E} \to \mathcal{E}' \text{ is } A\mathbf{Man}\text{-functors} \end{cases}$$

and by $\mathbf{Bn}_0$ and **Bn** subcategories of 1-**CAT** consisting of categories of trivial and locally trivial fibre bundles with fibres of a fixed type (i.e., of categories like $\mathbf{Bn}_0(\mathcal{E},p)$ and $\mathbf{Bn}(\mathcal{E},p)$) and functors preserving atlases as arrows (see 2.5, remarks). Of course, $\mathbf{Bn}_0(\mathcal{E},p) \hookrightarrow \mathbf{Bn}(\mathcal{E},p)$.

Proposition 2.4.1. *There are functors* $\mathbf{Bn}_0(-) : A\mathbf{Man\text{-}CAT} \to \mathbf{Bn}_0 \hookrightarrow 1\text{-}\mathbf{CAT}$:

$$\begin{cases} (\mathcal{E},p) \mapsto \mathbf{Bn}_0(\mathcal{E},p) & \textit{on objects} \\ (F : \mathcal{E} \to \mathcal{E}') \mapsto \mathbf{Bn}_0(F) : \begin{cases} (X,E) \mapsto (X, F(E)) & (X,E) \in Ob\,(\mathbf{Bn}_0(\mathcal{E},p)) \\ (f,\Phi) \mapsto (f, F_{E,E'} \circ \Phi) & (f,\Phi) \in Ar\,(\mathbf{Bn}_0(\mathcal{E},p)) \end{cases} & \textit{on arrows} \end{cases}$$

and $\mathbf{Bn}(-) : A\mathbf{Man\text{-}CAT} \to \mathbf{Bn} \hookrightarrow 1\text{-}\mathbf{CAT}$, *such that* $\mathbf{Bn}(-) = \mathbf{Man}(\mathbf{Bn}_0(-))$ *(see 2.5, remarks) (i.e., to each fibrewise functor there corresponds an actual functor on fibre bundles).*

Proof. The given assignment for $\mathbf{Bn}_0(-)$ is obviously functorial. If $\mathcal{A} := \left\{ \begin{array}{c} U_i \times p(E_i) \\ \downarrow \\ U_i \end{array} \right\}_{i\in I}$ is an $\mathcal{E}$-atlas for $\begin{array}{c} X \\ \downarrow \\ B \end{array} \in Ob(\mathbf{Man}^{\to})$ then $\mathcal{A}' := \left\{ \begin{array}{c} U_i \times p'(F(E_i)) \\ \downarrow \\ U_i \end{array} \right\}_{i\in I}$ is an $\mathcal{E}'$-atlas for $\begin{array}{c} X' \\ \downarrow \\ B \end{array} := \left| \mathbf{Bn}(F)\left(\begin{array}{c} X \\ \downarrow \\ B \end{array}, \mathcal{A} \right) \right| \in Ob(\mathbf{Man}^{\to})$ (i.e., essentially, $\mathcal{A}'$ is a compatible family of arrows, if $\mathcal{A}$ is compatible, and can be glued to an arrow $\begin{array}{c} X' \\ \downarrow \\ B \end{array}$). So, $\mathbf{Bn}_0(F)$ and $\mathbf{Bn}(F)$ preserve atlases. □

Remark. Similarly, there can be defined fibrewise functors of more than one variables (e.g., $\mathbf{Bn}(\mathcal{E},p) \times \mathbf{Bn}(\mathcal{E}',p') \longrightarrow \mathbf{Bn}(\mathcal{E}'',p'')$ induced by $F : \mathcal{E} \times \mathcal{E}' \longrightarrow \mathcal{E}''$, an $A\mathbf{Man}$-functor). In this way the usual fibrewise operations like $\times$, $\oplus$, $\otimes$, etc., are introduced. □

2.5. Stacks and construction of general manifolds

Stacks give an example of relative higher order Category Theory. They are functors from Grothendieck sites to n-**CAT** with deep connections between local and global properties. n-categories form an $(n+1)$-category, so that (forgetting set-theoretical difficulties) $\mathrm{Hom}_{(n+1)\text{-}\mathbf{CAT}}(\mathbf{C}, n\text{-}\mathbf{CAT})$ is an n-category.

Definition 2.5.1. Let $(\mathbf{B},\tau)$ be a site ($\mathbf{B}$ is a 1-category), $F : \mathbf{B}^{op} \longrightarrow n\text{-}\mathbf{CAT}$ a (weak) functor.

- For a sieve $i_s : s \hookrightarrow \mathbf{B}(-,B)$ $(n-1)$-category $\mathbf{Desc}(s,F) := \mathrm{Hom}_{(n+1)\text{-}\mathbf{CAT}}(s,F)$ is called **descent data** for the functor F and sieve s.
- $\forall B \in Ob(\mathbf{B})$ and $\forall$ sieve $i_s : s \hookrightarrow \mathbf{B}(-,B)$ there is an induced (**restriction**) functor $i_s^* : \mathrm{Hom}_{(n+1)\text{-}\mathbf{CAT}}(\mathbf{B}(-,B),F) \longrightarrow \mathrm{Hom}_{(n+1)\text{-}\mathbf{CAT}}(s,F)$. If i_s^* is full and faithful ($\forall B$ and $\forall s$)

then F is called a **prestack**. If, moreover, it is an equivalence F is called a **stack**, i.e., F is a **prestack** iff $\forall B, s$ $\mathrm{Hom}_{(n+1)\text{-}\mathbf{CAT}}(s, F) \xleftarrow[\text{full, faith.}]{i_s^*} \mathrm{Hom}_{(n+1)\text{-}\mathbf{CAT}}(\mathbf{B}(-, B), F) \xrightarrow[\text{Yoneda}]{\sim} F(B)$ and a **stack** iff i_s^* is an equivalence. ☐

For $n = 2$ there is another definition of stack via matching families [Moe, Vis].

Denote by $\mathtt{PreSt}(\mathbf{B}^{op}, n\text{-}\mathbf{CAT})$, $\mathtt{St}(\mathbf{B}^{op}, n\text{-}\mathbf{CAT}) \hookrightarrow \mathrm{Hom}_{(n+1)\text{-}\mathbf{CAT}}(\mathbf{B}^{op}, n\text{-}\mathbf{CAT})$ the full subcategories of prestacks and stacks respectively.

Proposition 2.5.1. *Both inclusions*

$$\begin{tikzcd}
\mathtt{St}(\mathbf{B}^{op}, 2\text{-}\mathbf{CAT}) \arrow[r, hook, bend right, "i_0"'] \arrow[r, phantom, "\perp"] & \mathtt{PreSt}(\mathbf{B}^{op}, 2\text{-}\mathbf{CAT}) \arrow[l, bend right, "\Phi"'] \arrow[r, hook, bend right, "i_1"'] \arrow[r, phantom, "\perp"] & Hom_{3\text{-}\mathbf{CAT}}(\mathbf{B}^{op}, 2\text{-}\mathbf{CAT}) \arrow[l, bend right, "\Psi"']
\end{tikzcd}$$

have left adjoints.

PROOF. See [Moe, Vis]. ☐

Construction of manifolds of type E over a site $(\mathbf{B}, \tau)$

We now give an explicit construction for building manifolds of type $\mathbf{E}$ over a site $(\mathbf{B}, \tau)$.

1. Factor the (1)-functor $F : \mathbf{E} \to (\mathbf{B}, \tau)$ through a free fibration (see **Lemma 2.1.1**) $$\begin{tikzcd} \mathbf{E} \arrow[r, "i"] \arrow[dr, "F"'] & 1/F \arrow[d, "\mathrm{dom}"] \\ & \mathbf{B} \end{tikzcd}$$
2. For the fibration $\begin{tikzcd} 1/F \arrow[d, "dom"'] \\ \mathbf{B} \end{tikzcd}$ form a corresponding (weak) functor $\hat{F} : \mathbf{B}^{op} \to 1\text{-}\mathbf{CAT}$ and complete it to a stack $(\Phi \circ \Psi)\hat{F} : \mathbf{B}^{op} \to 1\text{-}\mathbf{CAT}$ with respect to topology τ.
3. Get back (by the Grothendieck construction) from the stack $(\Phi \circ \Psi)\hat{F} : \mathbf{B}^{op} \to 1\text{-}\mathbf{CAT}$ to a fibration $$\begin{tikzcd} \tilde{\mathbf{E}} \arrow[d, "p"'] & 1/F \arrow[l, hook'] \arrow[dl, "dom"] \\ \mathbf{B} & \end{tikzcd}$$
4. Choose a (correct) class of arrows $\mathcal{M}$ in $\mathbf{B}$ representing 'embeddings of simple pieces into manifolds'.
5. Take a full subcategory $\mathbf{E}\text{-}\mathbf{Man} \hookrightarrow \tilde{\mathbf{E}}$ consisting of all objects $(B, \mathcal{E}) \in Ob(\tilde{\mathbf{E}})$ such that

$\exists$ a sieve $s \hookrightarrow \mathbf{B}(-,B)$ (depending on B) and $\forall f \in s$ $\mathbf{Cart}_f(\mathcal{E}) = ((\Phi \circ \Psi)\hat{F}(f))(\mathcal{E}) : (df \to F(E)) \in \mathcal{M}$ for some $E \in Ob(\mathbf{E})$. Then

$$\begin{tikzcd}
\mathbf{E}\text{-}\mathbf{Man} \arrow[r, hook] \arrow[dr, "p_F"'] & \tilde{\mathbf{E}} \arrow[d, "p"] & 1/F \arrow[l, hook'] \arrow[dl, "dom"] \\
 & \mathbf{B} &
\end{tikzcd}$$

is the required category of manifolds of type $\mathbf{E}$ over base site $(\mathbf{B}, \tau)$. $\square$

Remarks.

- The categories **E-Man** depend on the choice of the class $\mathcal{M}$ (different $\mathcal{M}$'s may give different categories), so $\mathcal{M}$ is an additional parameter. For the cases of the usual manifolds (smooth real or complex) $\mathcal{M}$ is always the class of topological embeddings of open subspaces.
- An object $(B, \mathcal{E})$ in **E-Man** consists of a base object B and an 'atlas' $\mathcal{E}$, where $\mathcal{E}$ is a class of compatible charts $(U \to F(E)) \in \mathcal{M}$, $U \in Ob(\mathrm{Im}F)$, $E \in \mathbf{E}$. All arrows are represented by vertical arrows for the stack completion of $1/F$.
- $\mathrm{Im}(p_F) \supset \mathrm{Im}(F)$.
- The resulting category of manifolds $\begin{tikzcd} \mathbf{E}\text{-}\mathbf{Man} \arrow[d, "p_F"] \\ \mathbf{B} \end{tikzcd}$ is not usually a fibration.
- Denote by $\mathbf{Man}_0 \hookrightarrow \mathbf{1}\text{-}\mathbf{CAT}$ a category consisting of a subcategory of **E-Man** of trivial manifolds of type **E** for each type **E** and functors "mapping **E**-atlases to **E**$'$-atlases"; similarly, denote by $\mathbf{Man} \hookrightarrow \mathbf{1}\text{-}\mathbf{CAT}$ a category consisting of **E-Man** for each type **E** and functors "mapping **E**-atlases to **E**$'$-atlases". Then there exists a "manifoldification" functor $\mathbf{Man}(-) : \mathbf{Man}_0 \to \mathbf{Man}$. The inclusion functor $Ob(\mathbf{Man}_0) \ni \mathbf{E}\text{-}\mathbf{Man}_0 \hookrightarrow \mathbf{E}\text{-}\mathbf{Man} \in Ob(\mathbf{Man})$ is natural in **E** (and itself preserves atlases). $\square$

Proposition 2.5.2. *For every prestack $F \in Ob\,(\mathtt{PreSt}(\mathbf{B}^{op}, 2\text{-}\mathbf{CAT}))$ there is a* **local equivalence** *(i.e. a natural transformation whose components are full, faithful, and locally surjective functors, see* [Moe]*) $w : F \to \Phi(F)$, where $\Phi : \mathtt{PreSt}(\mathbf{B}^{op}, 2\text{-}\mathbf{CAT}) \to \mathtt{St}(\mathbf{B}^{op}, 2\text{-}\mathbf{CAT})$ is the stack completion functor.*

PROOF. See [Moe, Vis] □

Remark. Meaning of proposition 2.5.2 is that objects and morphisms of a stack completion locally look like objects and morphisms of the original fibered category. □

Examples

In Example 5, p. 72, we showed how to view the category of smooth manifolds as fibred over **Top**. We will now sketch how the abstract construction of manifolds described above results in the same fibred category.

1 (**single manifold**). Let $\mathbf{E}$ be the category whose objects are $k^n, n = 0, 1, ...,$ ($k = \mathbb{R}$ or $\mathbb{C}$) and all smooth (analytic) local isomorphisms and let $\mathcal{M}$ be the set of embeddings of open subspaces. Let $\mathbf{B}$ be the union of the following: the category of open subsets of a space X with inclusion arrows, the category $\mathbf{E}$, and arrows from $\mathcal{M}$ with codomains in $\mathbf{E}$, and let $F : \mathbf{E} \to \mathbf{B}$ be the inclusion functor. The assignment $X \subset U \mapsto \{f : U \to k^n \,|\, n = 0, 1, ..., f \in \mathcal{M}\}$ gives a prestack on X. If X is a manifold, then $(X, \mathcal{E}) \in Ob\,(\mathbf{E}\text{-}\mathbf{Man})$, where $\mathcal{E}$ is an atlas on X in the usual sense.

2 (**smooth manifolds**). Let $\mathbf{B} = \mathbf{Top}$; τ be the set of all open coverings; $\mathbf{E}$ consists of $\mathbb{R}^n$, $n = 0, 1, 2, \ldots$, the set of all open subspaces of $\mathbb{R}^n$, and all smooth maps between them; $p : \mathbf{E} \to \mathbf{B}$ be the inclusion functor. Then

- the functor corresponding to $1/\Phi$ assigns to each topological space $X \in Ob\,\mathbf{Top}$ a category of charts on X, $\{f : X \to E \mid E \in Ob\,\mathbf{E}\}$ with suitable commutative triangles as arrows [$1/\Phi$ is a prestack since for a given covering $\mathcal{U} = \bigcup_i U_i$ of a space X and given two charts $f : X \to E_1$, $g : X \to E_2$ existence of compatible (smooth) arrows $\varphi_i : f|_{U_i} \to g|_{U_i}$ implies existence and uniqueness of a total (smooth) arrow $\varphi : f \to g$ because compatibility here means $\varphi_i = \varphi_j\ \forall i, j$, and so $\varphi = \varphi_i$, i.e. $V \mapsto Hom(f|_V, g|_V)$ is a sheaf],
- total category $\tilde{\mathbf{E}}$ of the stack completion consists of all atlases (compatible charts) of all

topological spaces (see proposition 2.5.2),

- for the class $\mathcal{M}$ of open embeddings the objects of **E-Man** are atlases over those topological spaces which are differentiable manifolds in usual sense.

2.6. Lifting problem for a group action

Let **Grp** be a category of groups, $(-)_{\mathbf{Grp}} : \mathbf{1\text{-}CAT} \to \mathbf{1\text{-}CAT}$ be a functor which assigns (in an obvious way) to each category without group action a category with groups actions, namely,

$$\begin{cases} \mathcal{C} \mapsto \mathbf{Grp}\text{-}\mathcal{C} & \mathcal{C} \in Ob\,(\mathbf{1\text{-}CAT}) \\ (F : \mathcal{C} \to \mathcal{C}') \mapsto (\mathbf{Grp}\text{-}F : \mathbf{Grp}\text{-}\mathcal{C} \to \mathbf{Grp}\text{-}\mathcal{C}') & F \in Ar\,(\mathbf{1\text{-}CAT}) \end{cases} \qquad \text{where:}$$

$\mathbf{Grp}\text{-}\mathcal{C}$ consists of triples (G, C, ρ) ($G \in Ob(\mathbf{Grp})$, $C \in Ob(\mathcal{C})$, $\rho : G \to \mathbf{Aut}(C)$ is a group homomorphism) as objects, and pairs $(\sigma : G \to G', f : C \to C') : (G, C, \rho) \to (G', C', \rho')$ (s.t. $\forall g \in G \;\; \rho'(\sigma(g)) \circ f = f \circ \rho(g)$) as arrows,

$$\mathbf{Grp}\text{-}F : \begin{cases} (G, C, \rho) \mapsto (G, F(C), F_{C,C} \circ \rho) & (G, C, \rho) \in Ob\,(\mathbf{Grp}\text{-}\mathcal{C}) \\ (\sigma, f) \mapsto (\sigma, F(f)) & (\sigma, f) \in Ar\,(\mathbf{Grp}\text{-}\mathcal{C}) \end{cases}$$

[$(\sigma, F(f))$ is an (equivariant) arrow in $\mathbf{Grp}\text{-}\mathcal{C}'$ because $F(\rho'(\sigma(g))) \circ F(f) = F(f) \circ F(\rho(g) \;\; \forall g \in G$]

Proposition 2.6.1. *If* $\begin{array}{c}\mathbf{E}\\ \big\downarrow {\scriptstyle p}\\ \mathbf{B}\end{array}$ *is a* **structure over B** *(i.e., all isomorphisms of type* $(B' \xrightarrow{\sim} p(E)) \in Ar\,\mathbf{B}$ *can be lifted to isomorphisms* $(E' \xrightarrow{\sim} E) \in Ar\,\mathbf{E}$*) then* $\begin{array}{c}\mathbf{Grp}\text{-}\mathbf{E}\\ \big\downarrow {\scriptstyle \mathbf{Grp}\text{-}p}\\ \mathbf{Grp}\text{-}\mathbf{B}\end{array}$ *is a structure over* **Grp-B**.

Proof. If $(\varphi, f) : (G', B', \rho') \xrightarrow{\sim} (G, p(E), p \circ \rho)$ is an isomorphisms then $\exists \begin{pmatrix} E' \\ B' \end{pmatrix} \xrightarrow[\hat{f}]{\sim} \begin{pmatrix} E \\ B \end{pmatrix}$, isomorphisms, because $\begin{array}{c}\mathbf{E}\\ {\scriptstyle p}\big\downarrow \\ \mathbf{B}\end{array}$ is a structure over **B**. Regard the diagram (of group homomorphisms)

$$\begin{tikzcd}
\mathbf{Aut_E}(E') \arrow[ddd, "p"'] & & & \mathbf{Aut_E}(E) \arrow[lll, "\hat{f}^{-1}\circ_{\mathbf{E}} - \circ_{\mathbf{E}}\hat{f}"'] \arrow[ddd, "p"] \\
& & & \\
& G' \arrow[uul, dashed, "\rho''"'] \arrow[r, "\varphi", "\sim"'] \arrow[ddl, "\rho'"] & G \arrow[uur, "\rho"] \arrow[ddr, "p\circ\rho"'] & \\
\mathbf{Aut_B}(B') \arrow[u, phantom, "(**)"] & & (*) & \mathbf{Aut_B}(p(E)) \arrow[lll, "f^{-1}\circ_{\mathbf{B}} - \circ_{\mathbf{B}} f"]
\end{tikzcd}$$

$$\rho''(g') := \hat{f}^{-1} \circ_{\mathbf{E}} \rho(\varphi(g')) \circ_{\mathbf{E}} \hat{f}$$

$(*)$ commutes because $f \circ_{\mathbf{B}} \rho'(g') = (p \circ \rho)(\varphi(g')) \circ_{\mathbf{B}} f$ by equivariance condition.

$(**)$ commutes because $p(\rho''(g')) = f^{-1} \circ_{\mathbf{B}} p(\rho(\varphi(g'))) \circ_{\mathbf{B}} f = \rho'(g')$.

So, $\exists$ isomorphisms $\begin{pmatrix}(G', E', \rho'') \\ (G', B', \rho')\end{pmatrix} \xrightarrow[\sim]{\binom{(\varphi,\hat{f})}{(\varphi,f)}} \begin{pmatrix}G, E, \rho \\ G, B, p\circ\rho\end{pmatrix}$, i.e. $\begin{tikzcd}\mathbf{Grp\text{-}E} \arrow[d, "\mathbf{Grp}\text{-}p"'] \\ \mathbf{Grp\text{-}B}\end{tikzcd}$ is a structure over **Grp-B**. □

There is a commutative diagram in **1-CAT** $\begin{tikzcd}\mathbf{E} \arrow[d, "p"'] & \mathbf{Grp\text{-}E} \arrow[l] \arrow[d, "\mathbf{Grp}\text{-}p"] \\ \mathbf{B} & \mathbf{Grp\text{-}B} \arrow[l]\end{tikzcd}$ (where horizontal arrows forget group actions). So, there exists a forgetful fiber functor $\mathbf{E}_B \longleftarrow \mathbf{Grp\text{-}E}_{(G,B,\rho)}$.

Definition 2.6.1. For a given G-action $(G, B, \rho) \in Ob\,(\mathbf{Grp\text{-}B}_B)$, an object $E \in Ob\,(\mathbf{E}_B)$ **admits the lifting of a G-action** if $\exists\ (G, E, \rho') \in Ob\,(\mathbf{Grp\text{-}E}_{(G,B,\rho)})$, i.e., $\begin{tikzcd}E \arrow[d, mapsto, "p"'] & (G, E, \rho') \arrow[l, mapsto] \arrow[d, mapsto, "\mathbf{Grp}\text{-}p"] \\ B & (G, B, \rho) \arrow[l, mapsto]\end{tikzcd}$ (essentially, $\rho = p \circ \rho'$). □

The **Lifting problem** for a G-action $\rho : G \to \mathbf{Aut_B}(B)$ is equivalent to completion of the diagram of group homomorphisms with exact row

$$\begin{tikzcd}
1 \arrow[r] & \mathbf{Aut}_{\mathbf{E}_B}(E) \arrow[r, tail] & \mathbf{Aut_E}(E) \arrow[r, "p"] & \mathbf{Aut_B}(B) \\
& & G \arrow[u, dashed, "\rho'\ ?"] \arrow[ur, "\rho"'] &
\end{tikzcd}$$

where $\mathbf{Aut}_{\mathbf{E}_B}(E)$ are vertical automorphisms of E over B.

For single element $g \in \mathbf{Aut_B}(B)$ there is a simple criterion of existence of $g' \in \mathbf{Aut_E}(E)$ such that $p(g') = g$ (see [Kom]).

Proposition 2.6.2. *For a fibration* $\begin{array}{c}\mathbf{E}\\ {\scriptstyle p}\big\downarrow\\ \mathbf{B}\end{array}$ *(or structure over* $\mathbf{B}$*)* $g \in \mathbf{Aut}_{\mathbf{B}}(B)$ *can be lifted to* $g' \in \mathbf{Aut}_{\mathbf{E}}(E)$ *iff* $\mathbf{Cart}_g(E) \xrightarrow{\sim} E$ *(***vertical** *isomorphisms).*

PROOF. '$\Longleftrightarrow$'

$$\begin{tikzcd}
E \arrow[dr, "\sim", "vertical"'] \arrow[drr, "g'", "\sim"'] & & \\
& \mathbf{Cart}_g(E) \arrow[r, "\sim", "\tilde{g}"'] & E \\
& B \arrow[r, "\sim", "g"'] & B
\end{tikzcd}$$

☐

Proposition 2.6.3. *If* $\begin{array}{c}\mathbf{E}\\ \big\downarrow\\ \mathbf{B}\end{array}$ *is a structure on* $\mathbf{B}$ *(or (co)fibration with* **unique** *(co)Cartesian lifting), and* $(G, B, \rho) \in Ob(\mathbf{Grp}\text{-}\mathbf{B})$*, then* $\exists$ *a representation*

$$\begin{tikzcd}
G \arrow[r, "\mathcal{R}"] \arrow[dr, "\rho"'] & \mathbf{Aut}_{\mathbf{1\text{-}CAT}}(\mathbf{E}_B) \\
& \mathbf{Aut}_{\mathbf{B}}(B) \arrow[u]
\end{tikzcd}$$

$$\text{where } \mathcal{R}(g) : \left\{\begin{array}{ll}
E \mapsto \mathbf{Cart}_{\rho(g)}(E) & E \in Ob(\mathbf{E}_B) \\
\begin{tikzcd}
E \arrow[dd, "f"'] & & \mathbf{Cart}_{\rho(g)}(E) \arrow[ll, "\widetilde{\rho(g)}"'] \arrow[dd, dashed, "\mathcal{R}(g)(f)"] \\
& \mapsto & \\
E' & & \mathbf{Cart}_{\rho(g)}(E') \arrow[ll, "\widetilde{\rho(g)}"]
\end{tikzcd} & f \in Ar(\mathbf{E}_B)
\end{array}\right.$$

PROOF is straightforward. ☐

Corollary. If $E \in Ob(\mathbf{E}_B)$ is such that $\forall g \in G$ $\mathbf{Cart}_{\rho(g)}(E) = E$ then the Cartesian lifting $\rho(g) \mapsto \widetilde{\rho(g)}$ lifts an action $(G, B, \rho) \in Ob(\mathbf{Grp}\text{-}\mathbf{B})$ to an action $(G, E, \tilde{\rho}) \in Ob(\mathbf{Grp}\text{-}\mathbf{E}_{(G,B,\rho)})$.

☐

Example (Covering Space)

A covering space is a (co)fibration $\begin{array}{c}E\\ {\scriptstyle p}\big\downarrow\\ B\end{array}$ over the groupoid B with unique (co)cartesian lifting in which all morphisms are (co)cartesian. Moreover, the representation $\mathbf{Aut}(b) \to \mathbf{Aut}(E_b)$,

$b \in Ob\,(B)$ (induced by (co)cartesian lifting) is transitive on objects of E_b.

Proposition 2.6.4. *For a covering space* $\begin{tikzcd} E \arrow[d, "p"'] \\ B \end{tikzcd}$ *over a connected groupoid* B

- $\mathbf{Aut}\left(\begin{tikzcd} E \arrow[d, "p"'] \\ B \end{tikzcd}\right) \simeq \mathbf{Aut}(E_b)$ *(where* $g \in \mathbf{Aut}(E_b)$ *iff* $g \circ f^* = f^* \circ g,\ f^* \equiv co\mathbf{Cart}_f,\ \forall f \in \mathbf{Aut}(b)$*),*
- $\mathbf{Aut}(E_b) \simeq W(\mathbf{Stab}(e)) \simeq N(\mathbf{Stab}(e))/\mathbf{Stab}(e)$ *(where* $\mathbf{Stab}(e) \hookrightarrow \mathbf{Aut}(b)$ *is the stabilizer of an object* $e \in Ob\,(E_b)$*,* $N(\mathbf{Stab}(e))$*,* $W(\mathbf{Stab}(e))$ *are its normalizer and Weil group respectively).*

PROOF.

- An automorphism g of the covering space p is given by family of fiberwise functors g_b, $b \in Ob\,(B)$, such that $f^* \circ g_b = g_{b'} \circ f^*$, $f^* \equiv co\mathbf{Cart}_f$, $\forall\,(f : b \to b') \in Ar\,(B)$. Take $g_b \in \mathbf{Aut}(E_b)$ and define $g_{b'} := h^* \circ g_b \circ (h^*)^{-1}$ for some $h : b \to b'$. Then $g_{b'}$ is well-defined (if $h_1 : b \to b'$ is another morphism then $h^* \circ g_b \circ (h^*)^{-1} = h_1^* \circ g_b \circ (h_1^*)^{-1}$ since $(h_1^{-1} \circ h)^* \circ g_b = (h_1^*)^{-1} \circ h^* \circ g_b = g_b \circ (h_1^*)^{-1} \circ h^* = g_b \circ (h_1^{-1} \circ h)^*$, $h_1^{-1} \circ h \in \mathbf{Aut}(b)$), and it is an automorphism of the covering space p (if $f : b' \to b''$ then $f^* \circ g_{b'} = g_{b''} \circ f^*$ since $f^* \circ h^* \circ g_b = g_{b''} \circ f^* \circ h^*$, $f \circ h : b \to b''$).
- See [May]. □

2.6.1. Lifting of a groupoid action for a sheaf.

Definition 2.6.1.1. Let $(\mathbf{Top}, \tau_0)$ be a site for all open coverings on topological spaces.

- A **Set**-valued presheaf $P : \mathbf{Top}^{op} \to \mathbf{Set}$ is a **sheaf** iff $\forall$ sieve $S \hookrightarrow \mathbf{B}(-, B)$ and $\forall$ natural transformation $f : S \to P$ $\exists!$ $\hat{f} : \mathbf{B}(-, B) \to P$ such that $$\begin{tikzcd} S \arrow[r, hook] \arrow[dr, "\forall f"'] & \mathbf{B}(-,B) \arrow[d, dashed, "\exists! \hat{f}"] \\ & P \end{tikzcd}$$

- A **Cat**-valued presheaf $P : \mathbf{Top}^{op} \to \mathbf{Cat}$ is a **sheaf** iff its object and morphism parts are sheaves, i.e., $\mathbf{Top}^{op} \xrightarrow{P} \mathbf{Cat} \xrightarrow{Ob} \mathbf{Set}$ and $\mathbf{Top}^{op} \xrightarrow{P} \mathbf{Cat} \xrightarrow{Mor} \mathbf{Set}$ are **Set**-valued sheaves.
- For presheaf $P : \mathbf{Top}^{op} \to \mathbf{Cat}$, space $X \in Ob(\mathbf{Top})$ and sieve $S \hookrightarrow \mathbf{Top}(-, X)$ matching family of objects $\tilde{E} : S \to Ob \circ P$ (nat. trans.) (or matching family of arrows $\tilde{f} : S \to Mor \circ P$ (nat. trans.)) **has a germ** $\mathbf{germ}_x(\tilde{E})$ (respectively, $\mathbf{germ}_x(\tilde{f})$) at point $x \in X$ iff $\exists\, Colim_{s \in S,\, Im(s) \ni x}(\tilde{E}(s)) =: \mathbf{germ}_x(\tilde{E})$ (respectively, $Colim_{s \in S,\, Im(s) \ni x}(\tilde{f}(s)) =: \mathbf{germ}_x(\tilde{f})$) (if the germ exists, it is unique up to isomorphisms and does not depend on the choice of sieve S).
- An **Etale space** is $E := \coprod_{x \in X} \mathbf{germ}_x(\tilde{E})$ (respectively, $f := \coprod_{x \in X} \mathbf{germ}_x(\tilde{f})$) (depending on two variables: "point" $x \in X$ and "matching family" $\tilde{E}$ or $\tilde{f}$) with topology generated by basic open sets $\big(U, \{\mathbf{germ}_x(\tilde{E}) \,|\, x \in U\}\big)$ (or, $\big(U, \{\mathbf{germ}_x(\tilde{f}) \,|\, x \in U\}\big)$), U is open in X. There is a natural continuous projection $p : E \to X : (x, \mathbf{germ}_x(\tilde{E})) \mapsto x$ (respectively, $p : f \to X : (x, \mathbf{germ}_x(\tilde{f})) \mapsto x$) which is a local homeomorphism. $\square$

Lemma 2.6.1.1. *Every fibration is a cofibration with respect to isomorphisms's (every cofibration is a fibration with respect to isomorphisms's).*

PROOF. Let $\begin{tikzcd} \mathbf{E} \arrow[d, "p"'] \\ \mathbf{B} \end{tikzcd}$ be a fibration, and $p(E) \xrightarrow{f} B'$ be an isomorphisms in $\mathbf{B}$. Then $\tilde{f} := \big(\widetilde{f^{-1}}\big)^{-1} : E \to E'$ (where $\tilde{}$ on the right is a cartesian lifting) is a cocartesian lifting of f (obvious). $\square$

Corollary. For a (co)fibration $\begin{tikzcd} \mathbf{E} \arrow[d, "p"'] \\ \mathbf{B} \end{tikzcd}$ for each isomorphisms $(f : B \xrightarrow{\sim} B') \in Ar\,\mathbf{B}$, the inverse image $\mathbf{Cart}_f : \mathbf{E}_{B'} \to \mathbf{E}_B : s_{B'} \mapsto f^*(s_{B'})$ and direct image $\mathbf{Cart}_f : \mathbf{E}_B \to \mathbf{E}_{B'} : s_B \mapsto f_*(s_B)$ (where s_B is a 'section' (object or morphism) over B) exist. $\square$

Definition 2.6.1.2.

- Let $X \in Ob\,(\mathbf{Top})$ be a space. The **groupoid of local homeomorphisms** of X is a subcategory $\mathbf{Gr}_X \hookrightarrow \mathbf{Top}$ such that $\begin{cases} Ob\,(\mathbf{Gr}_X) & \text{are open subsets in } X \\ Ar\,(\mathbf{Gr}_X) & \text{are isomorphisms's in } \mathbf{Top} \ \ \text{(between open subsets in } X) \end{cases}$

  The (Nonfull) subcategory $\mathbf{Gr}_{X,x} \hookrightarrow \mathbf{Gr}_X$ with objects $U \ni x$ and morphisms $f : U \to V, f(x) = x$,

  is called the **groupoid of local homeomorphisms** of X **with fixed point** $x \in X$.

- X is **transitive** with respect to $\mathbf{Gr}_X$ if $\forall x, y \in X\ \exists U, V \in Ob(\mathbf{Gr}_X)$, $(f : U \to V) \in Ar\,(\mathbf{Gr}_X)$ such that $U \ni x, V \ni y$, $f(x) = y$.

- For a (co)fibration $\begin{tikzcd}\mathbf{E} \arrow[d, "p"'] \\ \mathbf{Top}\end{tikzcd}$ with unique (co)cartesian lifting and space $X \in Ob\,(\mathbf{Top})$ two actions of $\mathbf{Gr}_X$ on local sections over X are defined:

  The **left action** $\forall f \in \mathbf{Gr}_X(U, V)$ $\ f^* \equiv \mathbf{Cart}_f : \mathbf{E}_V \to \mathbf{E}_U : s_V \mapsto f^* s_V$

  right action $\forall f \in \mathbf{Gr}_X(U, V)$ $\ f_* \equiv co\mathbf{Cart}_f : \mathbf{E}_U \to \mathbf{E}_V : s_U \mapsto f_* s_U$

  (where s_V, s_U are objects or morphisms).

- To each of the actions f^*, f_* (on local sections of $\mathbf{E}_X$) there correspond respectively left and right actions of $\mathbf{Gr}_{X,x}$ on $\{\mathbf{germ}_x(\tilde{s}) \mid \tilde{s}$ is a matching family of local sections of $\mathbf{E}_X\}$. If $\mathfrak{s} = \mathbf{germ}_x(s_U)$ is a germ at point x presented by a local section s_U (i.e., $\mathfrak{s} = \underset{U \supset V \ni x}{Colim}(s_U|_V)$) then if $(f : W \to V) \in \mathbf{Gr}_{X,x}(W, V)$

  left action $f^*\mathfrak{s} := \mathbf{germ}_x((f|_{f^{-1}(U\cap V)})^*(s_U|_{U\cap V}))$

  right action $f_*\mathfrak{s} := \mathbf{germ}_x((f|_{U\cap W})_*(s_U|_{U\cap W}))$.

- For a subgroupoid $\mathbf{G} \hookrightarrow \mathbf{Gr}_X$, $\binom{s}{X} \in \left(\begin{tikzcd}\mathbf{E} \arrow[d, "p"'] \\ \mathbf{Top}\end{tikzcd}\right)$ is **G-invariant** if $\forall (f : U \to V) \in Ar\,(\mathbf{G})$

  $f^*(s_V) = s_U$ (or $f_*(s_U) = s_V$) (where: $s_U := s|_U := (i : U \hookrightarrow X)^* s$, $\begin{tikzcd}\mathbf{E} \arrow[d, "p"'] \\ \mathbf{B}\end{tikzcd}$ is a fibration with unique cartesian lifting (or a local structure with respect to inclusions of open sets), s is a section (object or morphism) over X). In other words, $\mathbf{G}$-invariant sections **admit a lifting** of the groupoid $\mathbf{G}$.

- The $\mathbf{germ}_{x,y}(f)$ of a map $(f : U \to V) \in Ar\,(\mathbf{Gr}_X)$, such that $f(x) = y$, is an equivalence class of maps $\{\, g \in Ar\,(\mathbf{Gr}_X) \,|\, g(x) = y,\ \exists$ opens $W_x \ni x, W_y \ni y,$ such that $\exists$ the same restrictions $f\,\underset{W_x W_y}{|} = g\,\underset{W_x,W_y}{|} \in Ar\,(\mathbf{Gr}_X)\,\}$. Assume, $\mathfrak{s}_x = \mathbf{germ}_x(s_{U_1})$, $\mathfrak{s}'_y = \mathbf{germ}_y(s'_{V_1})$. Then
left action $(\mathbf{germ}_{x,y}(f))^*\mathfrak{s}'_y := \mathbf{germ}_x((f\underset{f^{-1}(V\cap V_1)}{|})^*(s'\underset{V\cap V_1}{|}))$
right action $(\mathbf{germ}_{x,y}(f))_*\mathfrak{s}_x := \mathbf{germ}_y((f\underset{(U\cap U_1)}{|})_*(s\underset{U\cap U_1}{|}))$. ☐

Lemma 2.6.1.2.

- *Let X be a topological space, $\mathbf{G} \hookrightarrow \mathbf{Gr}_X$ be a subgroupoid, $\mathbf{G}_x \hookrightarrow \mathbf{G}$ be a subgroupoid of pointed maps with fixed point $x \in X$. Then $\forall x, y \in X$ and $\forall f \in Ar\,(\mathbf{G})$, s.t. $f(x) = y$, $\mathbf{germ}_{x,x}(\mathbf{G}_x) = \mathbf{germ}_{y,x}(f^{-1}) \cdot \mathbf{germ}_{y,y}(\mathbf{G}_y) \cdot \mathbf{germ}_{x,y}(f)$ (for a certain unique composite $\cdot$ of germs of maps).*
- *If $\mathfrak{s}_x = \mathbf{germ}_x(s_U) \in S_x \subset S$ is a point of an etale space $\begin{tikzcd} S \arrow[d] \\ X \end{tikzcd}$ (corresponding to objects or morphisms over X for a fibration $\begin{tikzcd} \mathbf{E} \arrow[d, "p"'] \\ \mathbf{Top} \end{tikzcd}$) then $\mathfrak{s}_x$ is $\mathbf{G}_x$-invariant iff it is $\mathbf{germ}_{x,x}(\mathbf{G}_x)$-invariant.*
- *If $\mathbf{G}$ is transitive on X and $\mathfrak{s}_x$ is $\mathbf{G}_x$-invariant, then $\forall f, g \in Ar\,(\mathbf{G})$, s.t. $f(x) = y,\ g(x) = y$, there is a unique induced germ at point y $(\mathbf{germ}_{x,y}(f))_*\mathfrak{s}_x = (\mathbf{germ}_{x,y}(g))_*\mathfrak{s}_x$ and this germ is $\mathbf{G}_y$-invariant (or, respectively, $(\mathbf{germ}_{y,x}(f^{-1}))^*\mathfrak{s}_x = (\mathbf{germ}_{y,x}(g^{-1}))^*\mathfrak{s}_x$ is a unique $\mathbf{G}_y$-invariant germ at point y), i.e. $\mathfrak{s}_x$ can be distributed in a unique way over all X (to give rise a section $s : X \to S$ of etale space $\begin{tikzcd} S \arrow[d] \\ X \end{tikzcd}$ consisting of invariant germs at each point).* ☐

Proposition 2.6.1.1. *For sheaf $P : \mathbf{Top}^{op} \to \mathbf{CAT}$, space $X \in Ob\,(\mathbf{Top})$, and transitive* *groupoid* $\mathbf{G} \hookrightarrow \mathbf{Gr}_X$ $\mathbf{G}$*-invariant sections over* X *are in bijective correspondence with a subset of* $\mathbf{G}_x$*-invariant germs (of local sections) for a fixed point* $x \in X$.

Proof. To each $\mathbf{G}$-invariant section over X, there corresponds a $\mathbf{G}_x$-invariant germ of this section at point x. Conversely, by lemma 2.6.1.2, each $\mathbf{G}_x$-invariant germ generates a section of the corresponding etale space. When this section is continuous there is a global section over X of sheaf P (which is locally invariant). □

2.7. Equivalence, groups, actions

Let $\mathcal{R}$ be a category of sets with a given equivalence relation for each set. We introduce the following functors:

- **forgetful** $p : \mathcal{R} \to \mathbf{Set} : (A, R) \mapsto A$
- **quotient** $Q : \mathcal{R} \to \mathbf{Set} : (A, R) \mapsto A/R$
- **inclusion** $\Delta : \mathbf{Set} \to \mathcal{R} : A \mapsto (A, \Delta_A)$, $\Delta_A := \{(a, a) \mid a \in A\}$

  such that $\begin{tikzcd} \mathcal{R} \arrow[r, bend left, "Q"] \arrow[r, phantom, "\perp"] & \mathbf{Set} \arrow[l, bend left, "\Delta"] \end{tikzcd}$, i.e. $\mathbf{Set}(Q(A,R), B) \underset{nat.isomorphisms}{\xrightarrow{\sim}} \mathcal{R}((A,R), \Delta(B)) : f \mapsto f \circ \pi$, where $A \overset{\pi}{\twoheadrightarrow} A/R$ is the canonical projection (so, the quotient object $Q(A,R)$ represents the functor $\mathcal{R}((A,R), \Delta(-)) : \mathbf{Set} \to \mathbf{Set}$).

In the next definition, equivalence relations on objects are introduced as usual via relations on hom-sets.

Definition 2.7.1.

- A functor $R : \mathbf{C}^{op} \to \mathcal{R}$ is called an **equivalence relation on object** $C \in Ob\,\mathbf{C}$ iff

$$\begin{tikzcd} \mathbf{C}^{op} \arrow[r, "R"] \arrow[dr, "\mathbf{C}(-,C)"'] & \mathcal{R} \arrow[d, "p"] \\ & \mathbf{Set} \end{tikzcd}$$

  (i.e. usual equivalence relations are introduced on hom-sets $\mathbf{C}(C', C)$, $C' \in Ob\,\mathbf{C}$ and they are preserved under precomposition $- \circ f$, $f : C'' \to C'$).

- Let $\mathbf{C}_{\mathcal{R}}$ be a category such that

$Ob\,(\mathbf{C}_{\mathcal{R}})$ are pairs (C, R), $C \in Ob\,\mathbf{C}$, R is an equivalence relation on C,

$Ar\,(\mathbf{C}_{\mathcal{R}})$ are maps $(C, R) \xrightarrow{(f,F)} (C', R')$, where $(f : C \to C') \in Ar\,\mathbf{C}$ and $F : R \Rightarrow R'$ is a natural transformation of equivalence relations such that

$$\begin{tikzcd}
\mathbf{C}^{op} \arrow[r, bend left, "R"] \arrow[r, bend right, "R'"'] \arrow[r, phantom, "\Downarrow F"] \arrow[rr, bend left=60, "\mathbf{C}(-{,}C)"] \arrow[rr, bend right=60, "\mathbf{C}(-{,}C')"'] & \mathcal{R} \arrow[r, "p"'] & \mathbf{Set}
\end{tikzcd}$$

$p_F = \mathbf{C}(-, f)$ (this means that $(C, R) \xrightarrow{(f,F)} (C', R')$ is a morphism in $\mathbf{C}_{\mathcal{R}}$ iff $f : C \to C'$ is an arrow in $\mathbf{C}$ and $f \circ -$ preserves equivalence relation, i.e. if $g_1 \sim_R g_2$ then $f \circ g_1 \sim_{R'} f \circ g_2$ for $g_1, g_2 : X \to C$). $\square$

$\mathbf{C}_{\mathcal{R}}$ is an analogue of $\mathcal{R}$ for arbitrary category $\mathbf{C}$. Again, there are the following functors:

- **forgetful** $p : \mathbf{C}_{\mathcal{R}} \to \mathbf{C} : (C, R) \mapsto C$
- **inclusion** $\Delta : \mathbf{C} \to \mathbf{C}_{\mathcal{R}} : C \mapsto (C,\ \Delta \circ \mathbf{C}(-, C))$, where $\Delta : \mathbf{Set} \to \mathcal{R} : A \mapsto (A, \Delta_A)$, $\Delta_A := \{(a, a) \,|\, a \in A\}$
- **quotient** $Q : \mathbf{C}_{\mathcal{R}} \to \mathbf{C} : (C, R) \mapsto C/R$ which is a left adjoint to $\Delta : \mathbf{C} \to \mathbf{C}_{\mathcal{R}}$, i.e. $\begin{tikzcd} \mathbf{C}_{\mathcal{R}} \arrow[r, bend left, "Q"] \arrow[r, phantom, "\perp"] & \mathbf{C} \arrow[l, bend left, "\Delta"] \end{tikzcd}$ or $\mathbf{C}(Q(C, R),\, C') \xrightarrow[nat.isomorphisms]{\sim} \mathbf{C}_{\mathcal{R}}((C, R),\, \Delta(C'))$ (the quotient object $C/R := Q(C, R)$ represents functor $\mathbf{C}_{\mathcal{R}}((C, R), \Delta(-)) : \mathbf{C} \to \mathbf{Set}$ which means that $\exists$ an arrow $\pi : (C, R) \to \Delta(Q(C, R))$ such that $\forall f : (C, R) \to \Delta(C')$ $\exists! \hat{f} : Q(C, R) \to C'$ with $f = \Delta(\hat{f}) \circ \pi$, in other words, the quotient map $\pi : C \to Q(C, R)$ is a common coequalizer of all equivalent arrows $f \sim_R g$ with arbitrary domain and codomain C and, in particular, is always an epimorphism).

The quotient functor may not exist for the whole category $\mathbf{C}_{\mathcal{R}}$, but there always exists a (maximal) full subcategory $\mathbf{C}_{\mathcal{R}Q} \hookrightarrow \mathbf{C}_{\mathcal{R}}$ for which $\begin{tikzcd} \mathbf{C}_{\mathcal{R}Q} \arrow[r, bend left, "Q"] \arrow[r, phantom, "\perp"] & \mathbf{C} \arrow[l, bend left, "\Delta"] \end{tikzcd}$ (indeed, $\mathbf{C}_{\mathcal{R}Q}$ is always non empty since $Q \circ \Delta(C) = C$, i.e. $\Delta(C) \in Ob\,(\mathbf{C}_{\mathcal{R}Q})$).

If $\mathbf{C}$ is a concrete category with representable underlying functor $U := \mathbf{C}(I, -)$, then to each equivalence relation $R : \mathbf{C}^{op} \to \mathcal{R}$ on the object C with quotient map $\pi : (C, R) \to$

$\Delta(Q(C,R))$ there corresponds an ordinary equivalence relation on $\mathbf{C}(I,C)$ with quotient map $\pi \circ - : \mathbf{C}(I,C) \to \mathbf{C}(I, Q(C,R))$, and, conversely, to an ordinary equivalence relation on $\mathbf{C}(I,C)$ with quotient map $\pi \circ - : \mathbf{C}(I,C) \to \mathbf{C}(I,C')$ there corresponds a maximal "saturated" equivalence relation $R : \mathbf{C}^{op} \to \mathcal{R}$ on object C with quotient map $\pi : C \to C' \equiv Q(C,R)$ such that $f \sim_R g$ iff $\pi \circ f = \pi \circ g$. In general, the notion of equivalence relation on hom-sets is weaker than the usual one. That is, in a Set-like category C (i.e. C concrete over Set) with nonrepresentable forgetful functor, the partition of hom-sets $hom(Z,X), Z \in Ob(C)$ does not imply a partition on the object X. So, equivalence relations defined via hom-sets could induce a weaker notion than the usual notion of equivalence relation.

Let $C \in Ob\,\mathbf{C}$, $\sigma : G \to \mathbf{Aut_C}(C)$, then G also acts on hom-sets $\mathbf{C}(C',C)$, $C' \in Ob\,\mathbf{C}$, $G \times \mathbf{C}(C',C) \to \mathbf{C}(C',C) : \begin{cases} (g,f) \mapsto \sigma(g) \circ f & \text{left action} \\ (g,f) \mapsto \sigma(g^{-1}) \circ f & \text{right action} \end{cases}$, i.e. $\exists$ a functor $\Sigma : \mathbf{C}^{op} \to$ G-**Set** such that

$$\begin{tikzcd} \mathbf{C}^{op} \arrow[r, "\Sigma"] \arrow[dr, "\mathbf{C}(-,C)"'] & G\text{-}\mathbf{Set} \arrow[d, "p"] \\ & \mathbf{Set} \end{tikzcd}$$

(it means that all hom-sets $\mathbf{C}(C',C)$, $C' \in Ob\,\mathbf{C}$, are regarded with the given G-action).

There are functors

- $r : G\text{-}\mathbf{Set} \to \mathcal{R} : \begin{cases} (X,G,\sigma) \mapsto (X,R_\sigma) & \text{on objects} \\ ((X,G,\sigma) \xrightarrow{f} (X',G,\sigma')) \mapsto ((X,R_\sigma) \xrightarrow{f} (X',R_{\sigma'})) & \text{on arrows} \end{cases}$

  (where R_σ is an equivalence relation on X such that $(x,y) \in R_\sigma$ iff $\exists\, g \in G \; y = \sigma(g)x$)

  r is a functor over **Set**, i.e.

$$\begin{tikzcd} G\text{-}\mathbf{Set} \arrow[r, "r"] \arrow[dr, "p"'] & \mathcal{R} \arrow[d, "p"] \\ & \mathbf{Set} \end{tikzcd}$$

- $r : G\text{-}\mathbf{C} \to \mathbf{C}_{\mathcal{R}} : \begin{cases} (C,G,\sigma) \mapsto (C,R_\sigma) & \text{on objects} \\ ((C,G,\sigma) \xrightarrow{f} (C',G,\sigma')) \mapsto ((C,R_\sigma) \xrightarrow{f} (C',R_{\sigma'})) & \text{on arrows} \end{cases}$

  (where $R_\sigma := r \circ \Sigma$ is an equivalence relation on object C corresponding to σ,

$$\begin{tikzcd} \mathbf{C}^{op} \arrow[r, "R_\sigma"] \arrow[dr, "\mathbf{C}(-,C)"'] & \mathcal{R} \arrow[d, "p"] \\ & \mathbf{Set} \end{tikzcd}$$

)

r is a functor over $\mathbf{C}$, i.e.

$$\begin{tikzcd} G\text{-}\mathbf{C} \arrow[r, "r"] \arrow[dr, "p"'] & \mathbf{C}_{\mathcal{R}} \arrow[d, "p"] \\ & \mathbf{C} \end{tikzcd}$$

Let $G\text{-}\mathbf{C}_Q := r^{-1}(\mathbf{C}_{\mathcal{R}Q})$. Then there exists a quotient functor $G\text{-}\mathbf{C}_Q \xrightarrow{r} \mathbf{C}_{\mathcal{R}Q} \xrightarrow{Q} \mathbf{C}$. We denote it again by Q, and $Q \circ r\,(C, G, \sigma)$ by C/G.

For an arbitrary functor $F : \mathbf{C} \to \mathbf{D}$ we have $G\text{-}F : G\text{-}\mathbf{C} \to G\text{-}\mathbf{D}$ such that

$$\begin{tikzcd} \mathbf{C} \arrow[r, "F"] & \mathbf{D} \\ G\text{-}\mathbf{C} \arrow[u, "p"] \arrow[r, "G\text{-}F"'] & G\text{-}\mathbf{D} \arrow[u, "p"'] \end{tikzcd}$$

but F need not preserve quotients, i.e. for the diagram

$$\begin{tikzcd} G\text{-}\mathbf{C}_Q \arrow[r, dotted, "G\text{-}F"] \arrow[d, "Q"'] \arrow[dr, phantom, "\cong"] & G\text{-}\mathbf{D}_Q \arrow[d, "Q"] \\ \mathbf{C} \arrow[r, "F"'] & \mathbf{D} \end{tikzcd}$$

the dotted arrow may not exist and the natural isomorphism may not hold. If the above diagram holds (up to isomorphisms) then $F : \mathbf{C} \to \mathbf{D}$ **preserves quotients** (of category G-$\mathbf{C}$). In this case $F(C/G) \cong F(C)/G$. The quotient C/G is called **universal** [Kom] if $\forall C' \in Ob\,\mathbf{C}$

$$\begin{tikzcd} C' \times C \arrow[d, "\pi"'] \arrow[dr, "1\times p"] & \\ (C' \times C)/G \arrow[r, "\sim"] & C' \times (C/G) \end{tikzcd}$$

(C' is with trivial G-action).

Proposition 2.7.1. *Let* $\begin{tikzcd} \mathbf{E} \arrow[d, "p"'] \\ \mathbf{B} \end{tikzcd}$ *be a structure on* $\mathbf{B}$ *such that* p *preserves quotients in the category* G-$\mathbf{E}$, *and let* $(E, G, \sigma) \in Ob\,(G\text{-}\mathbf{E})$ *be an object such that* E/G *exists with* $\pi : E \to E/G$ *the canonical projection, then* $\begin{pmatrix} E/G \\ p(E/G) \end{pmatrix} = (p(\pi))_* \begin{pmatrix} E \\ p(E) \end{pmatrix}$ *is a direct image of* $\begin{pmatrix} E \\ p(E) \end{pmatrix}$.

Proof. We need to prove that $\begin{pmatrix} E \\ p(E) \end{pmatrix} \xrightarrow{\begin{pmatrix} \pi \\ p(\pi) \end{pmatrix}} \begin{pmatrix} E/G \\ p(E/G) \end{pmatrix}$ is cocartesian. Take $\begin{pmatrix} u \\ v \end{pmatrix} : \begin{pmatrix} E \\ p(E) \end{pmatrix} \to \begin{pmatrix} E' \\ p(E') \end{pmatrix}$ such that $v = k \circ p(\pi)$ for some $k : p(E/G) \to p(E')$, i.e. $\forall\, f \sim_G f' : B \to p(E)$ $\;v \circ f = v \circ f'$.

Assume, $h \sim_G h' : E_1 \to E$ then $p(h) \sim_G p(h') : p(E_1) \to p(E)$ (because, if $h' = \sigma(g) \circ h$ then $p(h') = p(\sigma(g)) \circ p(h)$). So, $v \circ p(h) = v \circ p(h')$, $p(u) \circ p(h) = p(u) \circ p(h')$, $u \circ h = u \circ h'$ (p is faithful), i.e. u coequalizes all $\sim_G$-equivalent arrows (in R_σ).

Therefore, $u = \hat{u} \circ \pi$ for a unique $\hat{u} : E/G \to E'$.

$\big(v = p(u) = p(\hat{u}) \circ p(\pi) = k \circ p(\pi)\big) \Rightarrow \big(p(\hat{u}) = k\big)$ by universality of $p(\pi)$.

Finally, $\exists! \begin{pmatrix} \hat{u} \\ k \end{pmatrix} : \begin{pmatrix} E/G \\ p(E/G) \end{pmatrix} \to \begin{pmatrix} E' \\ p(E') \end{pmatrix}$ such that $\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \hat{u} \\ k \end{pmatrix} \circ \begin{pmatrix} \pi \\ p(\pi) \end{pmatrix}$, i.e. $\begin{pmatrix} \pi \\ p(\pi) \end{pmatrix}$ is cocartesian. □

2.7.1. Group objects, subgroups, quotient objects.

Definition 2.7.1.1. Let $\mathbf{C}$ be a category with binary products and terminal object 1.

- $G \in Ob\,\mathbf{C}$ is called a **group object** if $\exists$ maps $m : G \times G \to G,\ e : 1 \to G,\ inv : G \to G$ such that the following group-like diagrams hold

$$\begin{tikzcd}
G \times G \times G \arrow[r, "1\times m"] \arrow[d, "m\times 1"'] & G \times G \arrow[d, "m"] \\
G \times G \arrow[r, "m"'] & G
\end{tikzcd}
\qquad
\begin{tikzcd}
G \times 1 \arrow[r, "1\times e"] \arrow[dr, "p_1"'] & G \times G \arrow[d, "m"] & 1 \times G \arrow[l, "e\times 1"'] \arrow[dl, "p_2"] \\
 & G &
\end{tikzcd}
\qquad
\begin{tikzcd}
G \times G \arrow[r, "1\times inv"] & G \times G \arrow[d, "m"] & G \times G \arrow[l, "inv\times 1"'] \\
G \arrow[u, "\Delta"] \arrow[r, "eo!"'] & G & G \arrow[l, "eo!"] \arrow[u, "\Delta"']
\end{tikzcd}$$

- A subobject $K \rightarrowtail G$ of a group object G is called a **subgroup** (object) if there exist maps $m_K : K \times K \to K,\ e_K : 1 \to K,\ inv_K : K \to K$ such that

$$\begin{tikzcd}
K \times K \arrow[r, tail] \arrow[rrr, bend left, "m_K"] & G \times G \arrow[r, "m"'] & G & K \arrow[l, tail]
\end{tikzcd}$$

$$\begin{tikzcd}
1 \arrow[r, "e_K"'] \arrow[rr, bend left, "e"] & K \arrow[r, tail] & G
\end{tikzcd}
\qquad
\begin{tikzcd}
K \arrow[r, tail] \arrow[rrr, bend left, "inv_K"] & G \arrow[r, "inv"'] & G & K \arrow[l, tail]
\end{tikzcd}$$

- For two elements $f, g : 1 \to G$ **multiplication** $f \cdot g : 1 \to G$ is $\begin{tikzcd} 1 \arrow[r, "<f{,}g>"'] \arrow[rr, bend left, "f\cdot g"] & G \times G \arrow[r, "m"'] & G \end{tikzcd}$

- The **Right shift** $R_g : G \to G$ (by element $g : 1 \to G$) is $\begin{tikzcd} G \arrow[r, "\sim"'] \arrow[rrr, bend left, "R_g"] & G \times 1 \arrow[r, "1\times g"'] & G \times G \arrow[r, "m"'] & G \end{tikzcd}$

  The **Left shift** $L_g : G \to G$ (by element $g : 1 \to G$) is $\begin{tikzcd} G \arrow[r, "\sim"'] \arrow[rrr, bend left, "L_g"] & 1 \times G \arrow[r, "g\times 1"'] & G \times G \arrow[r, "m"'] & G \end{tikzcd}$

□

Proposition 2.7.1.1. *For a group object $G \in Ob\,\mathbf{C}$*

- $\mathbf{C}(1, G)$ *is a group,*
- $\exists$ *(anti)representation* $\mathbf{C}(1, G) \to \mathbf{Aut}_{\mathbf{C}}(G) : g \mapsto R_g$ *(by right shifts) and representation* $\mathbf{C}(1, G) \to \mathbf{Aut}_{\mathbf{C}}(G) : g \mapsto L_g$ *(by left shifts).*

PROOF.

- It follows immediately from group object axioms ($e : 1 \to G$ is the identity, $inv \circ g : 1 \to G$ is the inverse of $g : 1 \to G$).
- $R_e = 1_G : G \to G$ (obvious)

  $R_f \circ R_g = R_{g \circ f}$ follows from the diagram

$$\begin{tikzcd}
G \arrow[r, "\sim"] \arrow[dr, "\sim"'] & G \times 1 \arrow[r, dotted, "1\times g"] & G \times G \arrow[r, dotted, "m"] & G \arrow[r, dotted, "\sim"] & G \times 1 \arrow[r, "1\times f"] & G \times G \arrow[r, "m"] & G \\
 & G \times 1 \arrow[rrrr, "1\times(g\cdot f)"] \arrow[drr, "1\times<g,f>"] \arrow[dl, dotted, "1\times<1,1>"'] & & & & G \times G \arrow[ur, "m"'] & \\
G \times 1 \times 1 \arrow[rrr, "1\times g\times f"] \arrow[drrr, dotted, "(m\circ(1\times g))\times 1"'] & & & G \times G \times G \arrow[urr, "1\times m"] \arrow[rrr, "m\times 1"] & & & G \times G \arrow[uu, "m"'] \\
 & & & G \times 1 \arrow[urrr, "1\times f"'] & & &
\end{tikzcd}$$

Two dotted paths $G \times 1 \xrightarrow{1\times g} G \times G \xrightarrow{m} G \xrightarrow[\sim]{p_1^{-1}} G \times 1$ and

$G \times 1 \xrightarrow{1\times <1,1>} G \times 1 \times 1 \xrightarrow{(m\circ(1\times g))\times 1} G \times 1$ are equal since their composites with projections $p_1 : G\times 1 \xrightarrow{\sim} G$ and $p_2 \equiv ! : G\times 1 \to 1$ are equal, indeed, $\begin{cases} p_1 \circ (p_1^{-1} \circ m \circ (1 \times g)) = m \circ (1 \times g) \\ p_2 \circ (p_1^{-1} \circ m \circ (1 \times g)) = ! \end{cases}$

and $\begin{cases} p_1 \circ (((m \circ (1 \times g)) \times 1) \circ (1\times < 1, 1 >)) = p_1 \circ (((m \circ (1 \times g)) \times 1) \circ (< 1, ! > \times 1)) = * \\ p_2 \circ (((m \circ (1 \times g)) \times 1) \circ (1\times < 1, 1 >)) = ! \end{cases}$

$$\begin{cases} * = p_1 \circ ((m \circ (1 \times g) \circ < 1, ! >) \times 1) = m \circ (1 \times g) \circ < 1, ! > \circ p_1 = m \circ (1 \times g) \circ 1_{G\times 1} = * \\ \qquad * = m \circ (1 \times g) \end{cases}$$

The proof for left shift L_g is similar. □

Corollary. If the $K \rightarrowtail G$ is a subgroup (object) of G then $\mathbf{C}(1, K) \xrightarrow{\sigma} \mathbf{Aut}_{\mathbf{C}}(G) : k \mapsto R_k$ is a (right) action (G, K, σ) on G (by right shifts from K). If the quotient $Q(G, K, \sigma) \in Ob\,\mathbf{C}$ exists it is called a **quotient object** G/K (under right action of K). □

Proposition 2.7.1.2. *Let $p : G \to G/K$ be a quotient map s.t. $\mathbf{C}(1, p) : \mathbf{C}(1, G) \to \mathbf{C}(1, G/K)$ is surjective. Then*

- *$L_g : G \to G$ induces a* **Set**-*map $\bar{L}'_g : \mathbf{C}(1, G/K) \xrightarrow{\sim} \mathbf{C}(1, G/K)$*
- *If G/K is universal then $\exists\, \bar{L}_g : G/K \xrightarrow{\sim} G/K$ in $\mathbf{C}$ s.t. $\mathbf{C}(1, \bar{L}_g) = \bar{L}'_g$ and*

$$\begin{tikzcd}
G \arrow[r, "L_g", "\sim"'] \arrow[d, "p"'] & G \arrow[d, "p"] \\
G/K \arrow[r, "\sim", "\bar{L}_g"'] & G/K
\end{tikzcd}$$

PROOF.

- Claim 1: $\bar{L}'_g : px \mapsto pL_g x$ is well-defined and isomorphisms

$$\begin{tikzcd}
G \arrow[rr, "L_g"] \arrow[dd, "p"'] & & G \arrow[dd, "p"] \\
& 1 \arrow[ul, "x"'] \arrow[ur, "L_g x"] \arrow[dl, "px"] \arrow[dr, dashed, "pL_g x"] & \\
C/K & & G/K
\end{tikzcd}$$

Proof of Claim 1: If $px = px'$ then $\exists k \in \mathbf{C}(1, K)$ such that $x' = R_k x = x \cdot k$. Then $L_g x' = L_g(x \cdot k) = g \cdot (x \cdot k) = (g \cdot x) \cdot k = (L_g x) \cdot k = R_k(L_g x)$, i.e. $pL_g x' = pL_g x$.

$\bar{L}'_{g^{-1}} \circ \bar{L}'_g(px) = \bar{L}'_{g^{-1}}(pL_g x) = p(L_{g^{-1}} L_g x) = px$. □

- Claim 2: $G \times G \xrightarrow{1 \times p} G \times G/K$ is a quotient map of $(G \times G, K, < 1, \sigma >) \in Ob\,\mathbf{C}\text{-}K$, where $\mathbf{C}\text{-}K$ is a category of right actions of $\mathbf{C}(1, K)$ on objects of $\mathbf{C}$, $< 1, \sigma >: \mathbf{C}(1, K) \to \mathbf{Aut}_{\mathbf{C}}(G \times G) : k \mapsto 1 \times R_k$.

Proof of Claim 2 follows immediately from the definition of universal quotient. □

Claim 3: $\forall\, k : 1 \to G \;\; m \circ (1 \times R_k) = R_k \circ m$.

Proof of Claim 3 follows from the diagram

$$\begin{tikzcd}
G \times G \arrow[r, dotted, "\sim", "1 \times p_1^{-1}"'] \arrow[d, dotted, "m"'] \arrow[rrr, bend left, "1 \times R_k"] & G \times G \times 1 \arrow[r, "1 \times 1 \times k"'] \arrow[d, dotted, "m \times 1"'] & G \times G \times G \arrow[r, "1 \times m"'] \arrow[d, "m \times 1"'] & G \times G \arrow[r, "m"] & G \\
G \arrow[r, dotted, "\sim", "p_1^{-1}"'] \arrow[rrrru, bend right, "R_k"'] & G \times 1 \arrow[r, "1 \times k"'] & G \times G \arrow[rru, "m"'] & &
\end{tikzcd}$$

The dotted paths $p_1^{-1} \circ m$, $(m \times 1) \circ (1 \times p_1^{-1}) : G \times G \to G \times 1$ are equal since their composites with projections $p_1 : G \times G \to G$, $p_2 = ! : G \times G \to 1$ are equal: $\begin{cases} p_1 \circ (p_1^{-1} \circ m) = m \\ p_2 \circ (p_1^{-1} \circ m) = ! \end{cases}$ and

$$\begin{cases} p_1 \circ (m \times 1) \circ (1 \times p_1^{-1}) = m \times p_{G \times G} \circ (1 \times p_1^{-1}) = m \circ 1_{G \times G} = m \\ p_2 \circ (m \times 1) \circ (1 \times p_1^{-1}) = ! \end{cases}$$

[where $p_{G \times G} : G \times G \times 1 \to G \times G$ is the projection, $p_{G \times G} \circ (1 \times p_1^{-1}) = 1_{G \times G}$ since $p_{G \times G} \circ (1 \times p_1^{-1}) \circ < x, y >= p_{G \times G} \circ < x, y, ! >=< x, y >$]. □

Claim 4: $\exists !$ map $\bar{m} : G \times G/K \to G/K$ such that

$$\begin{tikzcd}
G \times G \arrow[r, "m"] \arrow[d, "1 \times p"'] & G \arrow[d, "p"] \\
G \times G/K \arrow[r, dashed, "\exists ! \bar{m}"'] & G/K
\end{tikzcd}$$

Proof of Claim 4: Take $< f,g >\sim_K< f',g' >: X \to G\times G$ (where $G\times G$ is equipped with the action $\mathbf{C}(1,K)\ni k\mapsto 1\times R_k \in \mathbf{Aut}_{\mathbf{C}}(G\times G)$) then $< f',g' >= (1\times R_k)\circ < f,g >$ for some $k\in \mathbf{C}(1,K)$.

$p\circ m\circ < f',g' >= p\circ m\circ(1\times R_k)\circ < f,g >= p\circ R_k\circ m\circ < f,g >= p\circ m\circ < f,g >$ $[p\circ R_k = p$ by definition of quotient $p]$. So, $p\circ m$ coequalizes $\sim_K$-equivalent arrows to $G\times G$. Therefore, $\exists!\,\bar{m} : G\times G/K \to G/K$ filling out the above diagram. ☐

Now, define $\bar{L}_g : G/K \to G/K$ as $G/K \xrightarrow{\sim} 1\times G/K \xrightarrow{g\times 1} G\times G/K \xrightarrow{\bar{m}} G/K$. It works since

$$\begin{tikzcd}
G \arrow[r, "\sim"] \arrow[rrr, bend left, "L_g"] \arrow[d, "p"'] & 1\times G \arrow[r, "g\times 1"] \arrow[d, "1\times p"'] & G\times G \arrow[r, "m"] \arrow[d, "1\times p"] & G \arrow[d, "p"] \\
G/K \arrow[r, "\sim"'] \arrow[rrr, bend right, "\bar{L}_g"'] & 1\times G/K \arrow[r, "g\times 1"'] & G\times G/K \arrow[r, "\bar{m}"'] & G/K
\end{tikzcd}$$

and $\mathbf{C}(1,\bar{L}_g)(px) = \bar{L}_g\circ p\circ x = p\circ L_g\circ x = pL_gx = \bar{L}'_g(px)$, i.e. $\mathbf{C}(1,\bar{L}_g) = \bar{L}'_g$. ☐

2.7.2. C-group actions.

Definition 2.7.2.1.

- Let G be a group object in $\mathbf{C}$, $X\in Ob\,\mathbf{C}$, then the $\mathbf{C}$-map $\rho : G\times X\to X$ is a (left) **group action** on X if
$$\begin{tikzcd}
G\times G\times X \arrow[r, "1\times\rho"] \arrow[d, "m\times 1"'] & G\times X \arrow[d, "\rho"] \\
G\times X \arrow[r, "\rho"'] & X
\end{tikzcd}
\qquad
\begin{tikzcd}
1\times X \arrow[r, "e\times 1"] \arrow[dr, "p_2"'] & G\times X \arrow[d, "\rho"] \\
& X
\end{tikzcd}$$

- The **Left shift** $L^X_g : X\to X$ (by $g\in\mathbf{C}(1,G)$) is the composite
$$\begin{tikzcd}
X \arrow[r, "p_2^{-1}", "\sim"'] \arrow[drr, "L^X_g"'] & 1\times X \arrow[r, "g\times 1"] & G\times X \arrow[d, "\rho"] \\
& & X
\end{tikzcd}$$

- If $K \overset{i_1}{\rightarrowtail} G$ is a subgroup of G, $Y \overset{i_2}{\rightarrowtail} X$ is a subobject of X then K **stabilizes** Y if $\exists f : K\times Y\to Y$ such that
$$\begin{tikzcd}
G\times X \arrow[r, "\rho"] & X \\
K\times Y \arrow[r, "f"'] \arrow[u, tail, "i_1\times i_2"] & Y \arrow[u, tail, "i_2"']
\end{tikzcd}$$
☐

Lemma 2.7.2.1. *Let* $Y \overset{i}{\rightarrowtail} X$ *be a subobject of object* X *with* G*-action* $\rho : G\times X\to X$.

the assignment $\mathbf{Stab}_Y : Ob\,\mathbf{C} \to Ob\,\mathbf{Set} : Z \mapsto \mathbf{Stab}_Y(Z) \subset \mathbf{C}(Z,G)$ *such that* $(x : Z \to G) \in$ $\mathbf{Stab}_Y(Z)$ *iff* $\exists\,\rho_x : Z \times Y \to Y$ *such that*

$$\begin{tikzcd}
G \times X \arrow[r, "\rho"] & X \\
G \times Y \arrow[u, tail, "1\times i"] & Y \arrow[u, tail, "i"'] \\
Z \times Y \arrow[u, "x\times 1"] \arrow[ur, dashed, "\exists\rho_x"'] &
\end{tikzcd}$$

is functorial (hom-subfunctor).

Proof. For $(f : W \to Z) \in Ar\,\mathbf{C}$ define $\mathbf{Stab}_Y(f) : \mathbf{Stab}_Y(Z) \to \mathbf{Stab}_Y(W) : x \mapsto x \circ f$ (as precomposition with f). This is correct since if $x \in \mathbf{Stab}_Y(Z)$ then $x \circ f \in \mathbf{Stab}_Y(W)$ which can be seen from the diagram

$$\begin{tikzcd}
G \times X \arrow[r, "\rho"] & X \\
Z \times Y \arrow[u, "x\times i"] \arrow[r, "\rho_x"] & Y \arrow[u, tail, "i"'] \\
W \times Y \arrow[u, "f\times 1"] \arrow[ur, dashed, "\rho_{x\circ f}"'] &
\end{tikzcd}$$

Functorial properties of $\mathbf{Stab}_Y$ are obvious.

So, $\exists$ functor $\mathbf{Stab}_Y : \mathbf{C}^{op} \to \mathbf{Set}$ and $\mathbf{Stab}_Y \hookrightarrow \mathbf{C}(-,G)$ is a hom-subfunctor. □

Definition 2.7.2.2. If $\mathbf{Stab}_Y : \mathbf{C}^{op} \to \mathbf{Set}$ is representable then we denote its representing object by $\mathtt{Stab}_Y \in Ob\,\mathbf{C}$ and call it a **stabilizer** of $Y \rightarrowtail X$ (for the group G acting on X).

□

Proposition 2.7.2.1. *Let* $\mathbf{Stab}_Y : \mathbf{C}^{op} \to \mathbf{Set}$ *be represented by* $\mathtt{Stab}_Y \in Ob\,\mathbf{C}$. *Then*

- $\mathtt{Stab}_Y \overset{j}{\rightarrowtail} G$ *is a subobject of the group* G *(but not necessarily a group object itself),*
- j *is the universal element of the functor* $\mathbf{Stab}_Y$,
- *each element in* $\mathbf{Stab}_Y(Z)$ *has the form* $j \circ x$ *for a unique* $x : Z \to \mathtt{Stab}_Y$, *and all elements of this form* ($\forall x : Z \to \mathtt{Stab}_Y$) *are in* $\mathbf{Stab}_Y(Z)$ *[in other words,* $(z \in_Z G) \,\&\, (z \in \mathbf{Stab}_Y(Z)) \Leftrightarrow (z \in_Z \mathtt{Stab}_Y)$*],*
- *There exists a unique* $\rho_j : \mathtt{Stab}_Y \times Y \to Y$ *such that*

$$\begin{tikzcd}
G \times X \arrow[r, "\rho"] & X \\
\mathtt{Stab}_Y \times Y \arrow[u, "j\times i"] \arrow[r, "\rho_j"'] & Y \arrow[u, tail, "i"']
\end{tikzcd}$$

and ρ_j *is universal*

among arrows with a similar property, i.e. $\forall \rho_{x'} : Z \times Y \to Y$ *such that*
$$\begin{tikzcd}
G \times X \arrow[r, "\rho"] & X \\
Z \times Y \arrow[u, "x' \times i"] \arrow[r, "\rho_{x'}"'] & Y \arrow[u, tail, "i"']
\end{tikzcd}$$

$\exists!\, x : Z \to \mathtt{Stab}_Y$ *such that* $x' = j \circ x$ *and*
$$\begin{tikzcd}
\mathtt{Stab}_Y \times Y \arrow[r, "\rho_j"] & Y \\
Z \times Y \arrow[u, "x \times 1"] \arrow[ur, "\rho_{x'}"']
\end{tikzcd}$$

PROOF. The first three points follow from the Yoneda Lemma (j is the universal element of the representation $\mathbf{C}(-, \mathtt{Stab}_Y) \xrightarrow{\sim} \mathbf{Stab}_Y$ corresponding (under the Yoneda embedding) to the monic (natural transformation)
$$\begin{tikzcd}
\mathbf{C}(-, \mathtt{Stab}_Y) \arrow[r, "\sim"] \arrow[rr, tail, bend right] & \mathbf{Stab}_Y \arrow[r, hook] & \mathbf{C}(-, G)
\end{tikzcd}$$
). The fourth point follows from the definition and above properties of the functor $\mathbf{Stab}_Y$ and that $Y \overset{i}{\rightarrowtail} X$ is monic. □

Lemma 2.7.2.2. *A subobject* $H \rightarrowtail G$ *of a group object* $G \in Ob\,\mathbf{C}$ *is itself a group object iff* $\forall Z \in Ob\,\mathbf{C}$ *there are induced group operations in the hom-set* $\mathbf{C}(Z, H)$ *such that*
$$\begin{tikzcd}
G \times G \arrow[r, "m"] & G \\
H \times H \arrow[u, tail] & H \arrow[u, tail] \\
Z \arrow[u, "<x{,}y>"] \arrow[ur, dashed, "\exists\, m(<x{,}y>)"']
\end{tikzcd}
\qquad
\begin{tikzcd}
G \arrow[r, "inv"] & G \\
H \arrow[u, tail] & H \arrow[u, tail] \\
Z \arrow[u, "x"] \arrow[ur, dashed, "\exists\, inv(x)"']
\end{tikzcd}
\qquad
\begin{tikzcd}
1 \arrow[r, "e"] & G \\
Z \arrow[u, "!"] \arrow[r, dashed, "\exists\, e(!)"'] & H \arrow[u, tail]
\end{tikzcd}$$

PROOF is obvious in both directions. □

Proposition 2.7.2.2.

- $\mathtt{Stab}_Y \rightarrowtail G$ *is always a submonoid of the group object* $G \in Ob\,\mathbf{C}$.
- $\mathtt{Stab}_Y \rightarrowtail G$ *is a subgroup of the group object* $G \in Ob\,\mathbf{C}$ *if* $\forall Z \in Ob\,\mathbf{C}$ $\forall x \in \mathbf{Stab}_Y(Z)$ *the corresponding map* $\rho_x : Z \times Y \to Y$ *as in the diagram*
$$\begin{tikzcd}
G \times X \arrow[r, "\rho"] & X \\
Z \times Y \arrow[u, "x \times i"] \arrow[r, dashed, "\rho_x"'] & Y \arrow[u, tail, "i"']
\end{tikzcd}$$
is surjective in the second argument, i.e. $\forall t : T \to Z$ *the map* $\mathbf{C}(T, Y) \ni s \mapsto \rho_x \circ < t, s > \in \mathbf{C}(T, Y)$ *is surjective [it holds in the classical case in* **Set***].*

PROOF.

- $$\begin{tikzcd}
 & G\times X \arrow[dr, "\rho"] & \\
G\times G\times X \arrow[ur, "m\times 1"] \arrow[r, "1\times\rho"] & G\times X \arrow[r, "\rho"] & X \\
Z\times Z\times Y \arrow[u, "x\times y\times i"] \arrow[r, "1\times\rho_y"'] & Z\times Y \arrow[u, "x\times i"] \arrow[r, "\rho_x"'] & Y \arrow[u, hook, "i"'] \\
Z\times Y \arrow[u, "<1,1>\times 1"] \arrow[urr, dashed, "\rho_{m(<x,y>)}"'] & &
\end{tikzcd}$$

  i.e. $m(<x,y>) \in \mathbf{Stab}_Y(Z)$ if $x,y \in \mathbf{Stab}_Y(Z)$

  (similarly, $m(<x,y>) \in_Z \mathtt{Stab}_Y$ if $x,y \in_Z \mathtt{Stab}_Y$)

  $$\begin{tikzcd}
G\times X \arrow[r, "\rho"] & X \\
1\times X \arrow[u, "e\times 1"] \arrow[r, "p_2"] & X \arrow[u, "1"'] \\
1\times Y \arrow[u, "1\times i"] \arrow[r, "p_2"] & Y \arrow[u, hook, "i"'] \\
Z\times Y \arrow[u, "!\times 1"] \arrow[ur, dashed, "p_2=\rho_{e(!)}"'] &
\end{tikzcd}$$

  i.e. $e(!) := e\circ ! \in \mathbf{Stab}_Y(Z)$ (similarly, $e(!) \in_Z \mathtt{Stab}_Y$)

- In general, for $x \in_Z$ Stab$_Y$ $inv(x) \in_Z G$, but $inv(x) \notin_Z \mathtt{Stab}_Y$.

  $$\begin{tikzcd}
G\times X \arrow[drr, bend left, "\rho"] & & \\
G\times G\times X \arrow[u, "m\times 1"'] \arrow[r, "1\times\rho"] & G\times X \arrow[r, "\rho"] & X \\
Z\times Z\times Y \arrow[u, "inv(x)\times x\times i"'] \arrow[r, "1\times\rho_x"'] & Z\times Y \arrow[u, "inv(x)\times i"'] \arrow[r, dashed, "?", "\rho_{inv(x)}"'] & Y \arrow[u, hook, "i"'] \\
Z\times Y \arrow[u, "<1,1>\times 1"'] \arrow[uuu, bend left=60, "(e\circ !)\times i"] \arrow[urr, bend right, "p_2=\rho_{e(!)}"'] & & \\
T \arrow[u, "<t,s>"] & &
\end{tikzcd}$$

  Why does $\rho_{inv(x)}$ exist?

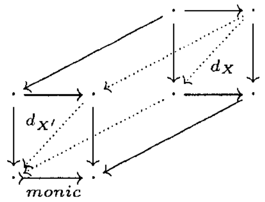
Lemma 2.7.2.3. *If a square* $\begin{tikzcd}
Hom(-,A) \arrow[r, two heads, "f"] \arrow[d, "h"'] & Hom(-,B) \arrow[d, "g"] \arrow[dl, dashed, "\exists\, d"] \\
Hom(-,C) \arrow[r, tail, "k"'] & Hom(-,D)
\end{tikzcd}$ *of natural transformations of representables commutes, where* f *and* k *are respectively componentwise surjective and compo-*

nentwise injective maps, then there exists a (unique) diagonal d making the 2 triangles commute.

PROOF OF LEMMA.

$$\begin{tikzcd}
Hom(X,A) \arrow[rr, two heads, "f_X"] \arrow[dd, "h_X"'] & & Hom(X,B) \arrow[dl, "g'_X"'] \arrow[dd, "g_X"] \\
 & Im(k) \arrow[dr, hook] & \\
Hom(X,C) \arrow[rr, tail, "k_X"'] \arrow[ur, "k'_X", "\sim"'] & & Hom(X,D)
\end{tikzcd}$$

k_X, g_X factor through $Im(k)$ since k_X is injective and f_X is surjective. Define the diagonal $d_X := (k'_X)^{-1} \circ g'_X$. The arrows d_X (X is a parameter) form a natural transformation which can be seen from the diagram



□

So, apply the above lemma to the square

$$\begin{tikzcd}
\mathbf{C}(-,Z\times Y) \arrow[r, two heads, "(1\times\rho_x)\circ(<1,1>\times 1)"] \arrow[d, "p_2"'] & \mathbf{C}(-,Z\times Y) \arrow[d, "\rho\circ(inv(x)\times i)"] \arrow[dl, dashed, "\exists\, \rho_{inv(x)}"] \\
\mathbf{C}(-,Y) \arrow[r, tail, "i"'] & \mathbf{C}(-,X)
\end{tikzcd}$$

(The top arrow is componentwise surjective since $(1 \times \rho_x) \circ (< 1,1 > \times 1) \circ < t,s >= (1 \times \rho_x) \circ < t,t,s >=< t, \rho_x \circ < t,s >>$, and, so that, $\forall < m,l >: T \rightarrow Z \times Y$ $\exists$ its preimage $< t,s >: T \rightarrow Z \times Y$ with $t = m$ and s is a solution of the equation $\rho_x \circ < m,s >= l$ [which exists because ρ_x is surjective in the second argument]. The bottom arrow is componentwise injective since i is monic.)

Consequently, $\exists (!)\, \rho_{inv(x)} : Z \times Y \rightarrow Y$ such that

$$\begin{tikzcd}
G\times X \arrow[r, "\rho"] & X \\
Z\times Y \arrow[u, "inv(x)\times i"] \arrow[r, "\rho_{inv(x)}"'] & Y \arrow[u, tail, "i"']
\end{tikzcd}$$

i.e. if $x \in_Z \mathtt{Stab}_Y$ then $inv(x) \in_Z \mathtt{Stab}_Y$. □

Lemma 2.7.2.4. *If $L_g : X \xrightarrow{\sim} X$ is a left shift, and $Y, Z \rightarrowtail X$ are subobjects of X*

such that an induced isomorphism $\overline{L}_g : Y \xrightarrow{\sim} Z$ *exists, i.e. such that the diagram*

$$\begin{tikzcd}
X \arrow[r, "L_g"] & X \\
Y \arrow[u, "i_Y"] \arrow[r, dashed, "\sim", "\exists \overline{L}_g"'] & Z \arrow[u, "i_Z"']
\end{tikzcd}$$

commutes, then $\exists$ *an induced map (isomorphisms)* $\overline{L_g \circ R_{g^{-1}}} : \mathtt{Stab}_Y \xrightarrow{\sim} \mathtt{Stab}_Z$*, corresponding to* $\overline{L}_g : Y \xrightarrow{\sim} Z$*, such that the following diagram commutes*

$$\begin{tikzcd}
G \times X \arrow[rrr, "\rho"] & & & X & \\
& G \times X \arrow[ul, "(L_g \circ R_{g^{-1}}) \times L_g"'] \arrow[rrr, "\rho"] & & & X \arrow[ul, "L_g"'] \\
\mathtt{Stab}_Z \times Z \arrow[uu, "j_Z \times i_Z"] \arrow[rrr, dotted, "\rho_{j_Z}"] & & & Z \arrow[uu, dotted, "i_Z"'] & \\
& \mathtt{Stab}_Y \times Y \arrow[ul, "\overline{L_g \circ R_g^{-1}} \times \overline{L}_g"] \arrow[uu, "j_Y \times i_Y"] \arrow[rrr, "\rho_{j_Y}"'] & & & Y \arrow[uu, "i_Y"'] \arrow[ul, dotted, "\overline{L}_g"', "\sim"]
\end{tikzcd}$$

PROOF. The only difficulty is to show that the left hand square commutes, namely,

$$\begin{tikzcd}
G & G \arrow[l, "L_g \circ R_{g^{-1}}"'] \\
\mathtt{Stab}_Z \arrow[u, "j_Z"] & \mathtt{Stab}_Y \arrow[u, "j_Y"'] \arrow[l, dashed, "?"', "\overline{L_g \circ R_{g^{-1}}}"]
\end{tikzcd}$$

For this, it is sufficient to show that $(L_g \circ R_{g^{-1}}) \circ j_Y \in_{\mathtt{Stab}_Y} \mathtt{Stab}_Z$, i.e. that $(L_g \circ R_{g^{-1}}) \circ j_Y \in \mathbf{Stab}_Z(\mathtt{Stab}_Y)$, or that there exists $\rho' : \mathtt{Stab}_Y \times Z \to Z$ such that

$$\begin{tikzcd}
G \times X \arrow[r, "\rho"] & X \\
\mathtt{Stab}_Y \times Z \arrow[u, "((L_g \circ R_{g^{-1}}) \circ j_Y) \times i_Z"] \arrow[r, dashed, "\rho'"'] & Z \arrow[u, "i_Z"']
\end{tikzcd}$$

Indeed, $((L_g \circ R_{g^{-1}}) \circ j_Y) \times (L_g \circ i_Y) = ((L_g \circ R_{g^{-1}}) \circ j_Y) \times (i_Z \circ \overline{L}_g) = (((L_g \circ R_{g^{-1}}) \circ j_Y) \times i_Z) \circ (1 \times \overline{L}_g)$, then $\rho \circ (((L_g \circ R_{g^{-1}}) \circ j_Y) \times i_Z) \circ (1 \times \overline{L}_g) = i_Z \circ \overline{L}_g \circ \rho_{j_Y}$, and so, $\rho \circ (((L_g \circ R_{g^{-1}}) \circ j_Y) \times i_Z) = i_Z \circ \overline{L}_g \circ \rho_{j_Y} \circ (1 \times \overline{L}_g)^{-1}$, i.e. $\rho' := \overline{L}_g \circ \rho_{j_Y} \circ (1 \times \overline{L}_g)^{-1}$. Therefore, $\forall x \in_T \mathtt{Stab}_Y$ $L_g \circ R_{g^{-1}}(x) \in_T \mathtt{Stab}_Z$, i.e. $\exists$ the induced map $\overline{L_g \circ R_{g^{-1}}} : \mathtt{Stab}_Y \to \mathtt{Stab}_Z$. ☐

Proposition 2.7.2.3.

- *Two objects* $(G, \mathbf{C}(1, \mathtt{Stab}_Y), \sigma_1)$ *and* $(G, \mathbf{C}(1, \mathtt{Stab}_Z), \sigma_2)$ *from* **C-Grp** *(the category of right group actions on objects from* **C***) are (equivariantly) isomorphic if* $\exists g \in \mathbf{C}(1, G)$ *and an induced isomorphism (as in lemma 2.7.2.4)* $\overline{L}_g : Y \xrightarrow{\sim} Z$*. The required isomorphism has form* $(G, \mathbf{C}(1, \mathtt{Stab}_Y), \sigma_1) \xrightarrow[\sim]{(L_g \circ R_{g^{-1}},\, \overline{L_g \circ R_{g^{-1}}} \circ -)} (G, \mathbf{C}(1, \mathtt{Stab}_Z), \sigma_2)$.

- $G/\mathtt{Stab}_Y \simeq G/\mathtt{Stab}_Z$ *(if these quotients exist).*

PROOF.

- It is necessary to prove that $\forall g : 1 \to G$ and $k : 1 \to \mathtt{Stab}_Y$ $L_g \circ R_{g^{-1}} \circ R_k = R_{g\cdot k \cdot g^{-1}} \circ L_g \circ R_{g^{-1}}$. It follows from two facts $R_{g\cdot k\cdot g^{-1}} = R_{g^{-1}} \circ R_k \circ R_g$ (antihomomorphism) and commutativity of left and right shifts $L_{g_1} \circ R_{g_2} = R_{g_2} \circ L_{g_1}$ [this last fact follows from the associativity axiom $\forall < t, r, s >: T \to G \times G \times G$ $m(t, m(r, s)) = m(m(t, r), s)$, and so, $L_{g_1} \circ R_{g_2} \circ t = m(g_1 \circ !, m(t, g_2 \circ !)) = m(m(g_1 \circ !, t), g_2 \circ !) = R_{g_2} \circ L_{g_1} \circ t$].
- Isomorphic objects in **C**-$\mathbf{Grp}_Q$ have isomorphic quotients in **C** since $Q : \mathbf{C}\text{-}\mathbf{Grp}_Q \to \mathbf{C}$ is a functor. So, $Q(G, \mathbf{C}(1, \mathtt{Stab}_Y), \sigma_1) \simeq Q(G, \mathbf{C}(1, \mathtt{Stab}_Z), \sigma_2)$. □

Definition 2.7.2.3. An object $X \in Ob\,\mathbf{C}$ with a group action $\rho : G \times X \to X$ such that $\forall x : 1 \to X$ both $\mathtt{Stab}_x$ and $G/\mathtt{Stab}_x$ exist, and $G/\mathtt{Stab}_x$ is universal, is called **homogeneous** if $\exists$ an isomorphism $f : G/\mathtt{Stab}_x \xrightarrow{\sim} X$ such that

$$\begin{tikzcd}
G \times (G/\mathtt{Stab}_x) \arrow[r, "\rho'"] \arrow[d, "1\times f"', "\sim"] & G/\mathtt{Stab}_x \arrow[d, "\sim"', "f"] \\
G \times X \arrow[r, "\rho"'] & X
\end{tikzcd}$$

(for an $x : 1 \to X$)

where ρ' is defined from

$$\begin{tikzcd}
G \times G \arrow[r, "m"] \arrow[d, "1\times p"'] & G \arrow[d, "p"] \\
G \times (G/\mathtt{Stab}_x) \arrow[r, dashed, "\exists!\,\rho'"'] & G/\mathtt{Stab}_x
\end{tikzcd}$$

($1 \times p$ and p are quotient maps). □

Proposition 2.7.2.4. *If X is a homogeneous object (with G-action $\rho : G \times X \to X$) and $\mathbf{C}(1, p) : \mathbf{C}(1, G) \to \mathbf{C}(1, G/\mathtt{Stab}_x)$ is surjective, where $G \overset{p}{\twoheadrightarrow} G/\mathtt{Stab}_x$ is a quotient map, then*

- *$\mathbf{C}(1, G)$ acts transitively on $\mathbf{C}(1, X)$, i.e. $\forall x, y : 1 \to X$ $\exists g : 1 \to G$ such that $y = L_g \circ x$,*
- *definition of homogeneous object $X \underset{f^{-1}}{\xrightarrow{\sim}} G/\mathtt{Stab}_x$ does not depend on the choice of $x : 1 \to X$.*

PROOF.

- $\forall a', b' : 1 \to G/\mathtt{Stab}_x$ $\exists a, b : 1 \to G$ s.t. $pa = a'$, $pb = b'$, and $\exists g : 1 \to G$ s.t. $b = L_g a$. By proposition 2.7.1.2, $\bar{L}'_g(a') = \bar{L}'_g(pa) = pL_g(a) = pb = b'$ (where $\bar{L}'_g$ is the induced left shift on $\mathbf{C}(1, G/\mathtt{Stab}_x)$). So, $\mathbf{C}(1, G)$ acts transitively on set $\mathbf{C}(1, G/\mathtt{Stab}_x)$, and consequently, on $\mathbf{C}(1, X)$.

- Regard the diagram

$$\begin{tikzcd}
 & & G \times G \arrow[r, "m"] \arrow[d, dotted, "1\times p_y"'] & G \arrow[d, "p_y"] \\
G \times G \arrow[urr, "(L_g\circ R_{g^{-1}})\times(L_g\circ R_{g^{-1}})"] \arrow[r, "m"] \arrow[d, "1\times p_x"'] & G \arrow[urr, "L_g\circ R_{g^{-1}}"'] \arrow[d, "p_x"] & G \times (G/\mathtt{Stab}_y) \arrow[r, dotted, "\rho''"'] & G/\mathtt{Stab}_y \\
G \times (G/\mathtt{Stab}_x) \arrow[r, "\rho'"'] \arrow[d, "1\times f"', "\sim"] & G/\mathtt{Stab}_x \arrow[d, "f", "\sim"'] & & \\
G \times X \arrow[r, "\rho"'] \arrow[uurr, dotted, bend right, "(L_g\circ R_{g^{-1}})\times\alpha", "\sim"'] & X \arrow[uurr, dashed, "\exists!\,\alpha"', "\sim"] & &
\end{tikzcd}$$

[α exists as a mediating arrow because $f \circ p_x$ is a quotient map of $(G, \mathbf{C}(1, \mathtt{Stab}_x), \sigma_1)$, and $p_y \circ (L_g \circ R_{g^{-1}})$ coequalizes $\sim_{\sigma_1}$-equivalent arrows ($L_g \circ R_{g^{-1}}$ is equivariant, and p_y is a quotient map of $(G, \mathbf{C}(1, \mathtt{Stab}_y), \sigma_2)$), essentially, $\alpha = Q(L_g \circ R_{g^{-1}})$. The bottom square commutes because $(1 \times f) \circ (1 \times p_x)$ is a quotient map, and so, epi]

Therefore,

$$\begin{tikzcd}
G \times (G/\mathtt{Stab}_y) \arrow[r, "\rho''"] & G/\mathtt{Stab}_y \\
G \times X \arrow[u, "(L_g\circ R_{g^{-1}})\times\alpha", "\sim"'] \arrow[r, "\rho"] & X \arrow[u, "\sim", "\alpha"'] \\
G \times X \arrow[u, "(L_{g^{-1}}\circ R_g)\times L_{g^{-1}}", "\sim"'] \arrow[r, "\rho"'] \arrow[uu, bend left=80, "\sim", "1\times(\alpha\circ L_{g^{-1}})"'] & X \arrow[u, "\sim", "L_{g^{-1}}"']
\end{tikzcd}$$

i.e. $L_g \circ \alpha^{-1}$ is the required isomorphism (by definition 2.7.2.3). □

3. Applications. First and second order concrete duality

3.1. Duality and Invariant Theory

We are now going to discuss the most important applications of our ideas. Specifically, we want to consider various higher-order notions of concrete duality. For the sort of duality we have in mind, we mention the following result; it is stated in the framework of [P-Th], although it is not explicitly stated there as such. It is a formulation of the standard result in geometry that orbit spaces and the corresponding algebra of invariants are essentially the same.

Proposition 3.1.1. Let $\mathbf{K}$ be **Set**, **Top** or $\mathbf{Diff}^+$ (spectra of smooth completion (see **3.2**) of commutative algebras with Zariski topology), G a group. Then there exists a concrete natural dual adjunction

$$\begin{tikzcd}\mathbf{ComAlg}^{op} \arrow[r, shift left, "F"] \arrow[r, phantom, "\top"] & G\text{-}\mathbf{K} \arrow[l, shift left, "H"]\end{tikzcd}$$

with k ($\mathbb{R}$ or $\mathbb{C}$), its schizophrenic object, such that $k \in Ob\, G\text{-}\mathbf{K}$ has trivial action of G, and $F \circ H : G\text{-}\mathbf{K} \to G\text{-}\mathbf{K}$ is a functor "taking the quotient space generated by the equivalence relation $x \sim y$ iff $x, y \in$ Closure(the same orbit)" (it is essentially the orbit space). $\square$

Definition 3.1.1.

- The adjoint object $\mathcal{A}_X = HX$ for an object X in G-$\mathbf{K}$ is called the **algebra of invariants**.
- If $U : G\text{-}\mathbf{K} \to G\text{-}\mathbf{K}$ is an endofunctor then $\mathcal{A}_{U(X)}$ is called the **algebra of** U**-invariants** of the object X. $\square$

Remarks.

- For $U = (-)^n$, the n-fold Cartesian product, $\mathcal{A}_{U(X)}$ is the **algebra of** n **point invariants**.
- For $\mathbf{K} = \mathbf{Diff}$, $U = \mathbf{Jet}^n$, $\mathbf{Jet}^n(X) := \{j_0^n f \mid f \in \mathbf{Diff}(k, X)\}$, the set of all n-jets of all maps from k to X at point 0 (with a certain manifold structure obtained from local trivializations), we get **differential invariants**.
- The functor $U = \mathbf{Jet}^\infty : \mathbf{Diff} \to \mathbf{Diff}^+$ does not fit into the above scheme, but everything

is still correct if $U : G\text{-}\mathbf{K} \to G\text{-}\mathbf{K}_1$ is an extension to $G\text{-}\mathbf{K}_1$, a category concretely adjoint to **ComAlg**.

- G can be, of course, $\mathbf{Aut}(X)$.

According to Klein's Erlangen Program, every group acting on a space determines a geometry and, conversely, every geometry hides a group of transformations. Properties of geometric objects which are invariant under all transformations are called *geometric* (or invariant or absolute) for the given G-space and a class of geometric objects.

The equivalence problem [Car1, Car2, Vas, Olv, Gar] consists of a G-space X and two "geometric objects" S_1, S_2 of the same type on the space X. It is required to determine if these two objects can be mapped to one another by an element of G. An approach is to find a (complete) system of invariants of each object.

3.1.1. Classification of covariant geometric objects.

By *covariant geometric objects* we mean objects like submanifolds, foliations or systems of differential equations, i.e., objects which behave contravariantly (!) from the categorical viewpoint and which can be described by a **differential ideal** I ($dI \subset I$) in $\Lambda(X)$, the exterior differential algebra of X.

Proposition 3.1.1.1. *Let G be a Lie-like group (i.e., there exists an algebra of invariant forms on G)* [A-V-L, Car1, Car2]*. Then any G-equivariant map $\sigma : G \to X$ (G is given with left shift action and X is a left G-space) produces a system of invariants of the differential ideal $I \subset \Lambda(X)$ (with generators of degree 0 and 1) in the following way:*

- *Take the image $\bar{\Lambda}_{inv} := Im\,(\Lambda_{inv}(G) \hookrightarrow \Lambda(G) \twoheadrightarrow \Lambda(G)/\sigma^*(I))$, where $\Lambda_{inv}(G)$ is a subalgebra of left-invariant forms on G, $\sigma^* : \Lambda(X) \to \Lambda(G)$ is the induced map of exterior differential algebras, $\sigma^*(I)$ is the smallest differential ideal in $\Lambda(G)$ containing the image of I under σ^*.*
- *Take the module $\Lambda^0(G) \cdot \bar{\Lambda}^1_{inv}$ generated by 1-forms in $\bar{\Lambda}_{inv}$ over $\Lambda^0(G)$. There is an open*

set $\mathcal{O} \subset G$ *and a basis* $\{\omega^{\alpha}_{inv}\}_{\alpha\in A} \subset \bar{\Lambda}^1_{inv}$ *for the module* $\Lambda^0(G)\cdot\bar{\Lambda}^1_{inv}$ *restricted to* $\mathcal{O}$*, i.e.,* $\forall\,\omega^i_{inv} \in \bar{\Lambda}^1_{inv}\ \exists!$ *functions* $f^i_\alpha \in C^\infty(\mathcal{O})$ *such that* $\omega^i_{inv} = \sum\limits_\alpha f^i_\alpha \omega^\alpha_{inv}$*. Form set* $J_0 := \{f^i_\alpha\}$*.*

- *Take the expansion of differentials* $df^i_\alpha = \sum\limits_\beta f^i_{\alpha\beta}\omega^\beta_{inv}$ *(over* $\mathcal{O}$*). Form the set* $J_1 := \{f^i_{\alpha\beta}\}$*.*
- *Continue this process to get* $J_2 := \{f^i_{\alpha\beta\gamma}\}, \ldots, J_n := \{f^i_{\alpha_1\ldots\alpha_{n+1}}\}\ldots$ *Form the set* $J := \bigcup\limits_n J_n$*. Its elements are relative invariants of the differential ideal* $I \subset \Lambda(X)$*.*
- *Take the algebra* $\mathcal{A}_J \subset C^\infty(\mathcal{O})$*, generated by* J*, and take its smooth completion* $\overline{\mathcal{A}_J}$ *(see* **3.2***). Then the ideal* $\mathbf{Rel}(\mathcal{A}_J) \rightarrowtail \overline{\mathbf{Alg}(J)} \twoheadrightarrow \overline{\mathcal{A}_J}$ *, of all relations of* $\mathcal{A}_J$*, gives absolute invariants of the differential ideal* $I \subset \Lambda(X)$*, where* $\overline{\mathbf{Alg}(J)}$ *is the smooth completion of the free algebra generated by* J*.*

PROOF follows from the diagrams

$$\begin{tikzcd}
G \arrow[r, "l_g"] \arrow[d, "\sigma"'] & G \arrow[d, "\sigma"] \\
X \arrow[r, "l_g"'] & X
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
\Lambda_{inv}(G) & \Lambda_{inv}(G) \arrow[l, "id"'] \\
\Lambda(X) \arrow[u, "\sigma^*"] & \Lambda(X) \arrow[l, "l_g^*"] \arrow[u, "\sigma^*"']
\end{tikzcd}$$

and equations $\omega^i_{inv} = \sum\limits_\alpha f^i_\alpha \omega^\alpha_{inv}\ mod(\sigma^*(I))$. □

Remark. G-$\mathbf{Diff}(G, X)$ is in 1-1-correspondence with all sections of the orbit space X_G. So, if X is homogeneous then it is exactly the set of all points of X and $\sigma : G \to X = G \xrightarrow{\sim} G \times \{x_0\} \xrightarrow{1\times i_{x_0}} G \times X \xrightarrow{\rho} X$ is a G-equivariant map corresponding to the point $x_0 \in X$, where ρ is the given G-action on X.

The following result can be found in [Lap]. Although not well-known, it is a fundamental classification of analytic geometric objects.

Proposition 3.1.1.2 (Exterior differential algebra associated to a group of analytic automorphisms). *Let* X *be an analytic* n*-dimensional manifold,* $\mathbf{An}(X)$*, its group of automorphisms,* $H^\infty(X) := \{j^\infty_0 f \mid f \in \mathbf{Diff}(k^n, X),\ X \text{ is analytic},\ Jacobian(f) \neq 0\}$*, the* ∞*-frame bundle over* X *(with a usual topology and manifold structure). Then there is an exterior differen-*

tial k-algebra $\Lambda_{inv}(H^\infty(X))$ *of invariant forms on* $H^\infty(X)$ *freely generated by elements of degree 1 obtained by the following process:*

- $\omega^i := x^i_j dx^j$ *are any "shift" forms on* X
- ω^i_j *are the most general solutions of Maurer-Cartan equations* $d\omega^i = \omega^i_j \wedge \omega^j$
- ω^i_{jk} *are the most general solutions of Maurer-Cartan equations* $d\omega^i_j = \omega^i_k \wedge \omega^k_j + \omega^i_{jk} \wedge \omega^k$
- $\omega^i_{jkl}, \cdots, \omega^i_{i_1\ldots i_n}, \cdots$

All forms are symmetric in the lower indices. They characterize the underlying space of $\mathbf{An}(X)$ *uniquely up to analytic isomorphisms.* □

Remark. At each point $x_0 \in X$, $\omega^i = 0$, and forms $\bar{\omega}^i_{i_1\ldots i_n} := \omega^i_{i_1\ldots i_n}\big|_{\omega^i=0}$, $n \geq 1$, are free generators of the exterior differential algebra of the **differential group** acting simply transitively on each fiber of $H^\infty(X)$.

3.1.2. Classification of smooth embeddings into a Lie group.

This is often the last step of smooth classification of geometric objects [Car2, Fin, Kob]. The process of finding differential invariants is similar to that in Proposition 3.1.1.1. The following is essentially in [Vas0, Vas, Lap].

Proposition 3.1.2.1. *For a smooth embedding* $f : X \to G$ *of a smooth manifold* X *into a Lie group* G*, a complete system of differential invariants of* f *can be obtained in the following way:*

- *Im*$(f^* : \Lambda^1_{inv}(G) \to \Lambda(X))$ *is locally free, so, has as its basis* $\omega^i_{inv}, i = 1, \ldots, n$, $n = dim(X)$, *near each point.*
- *Coefficients of linear combinations* $\omega^I_{inv} = \sum_{i=1}^{n} a^I_i \omega^i_{inv}$, $I = n+1, \ldots, dim(G)$, *are differential invariants of first order (of the map* f*).*
- *Coefficients of differentials of invariants of first order* $da^I_i = \sum_{j=1}^{n} a^I_{ij}\omega^j_{inv}$ *are differential invariants of second order (of the map* f*).*

- *... Coefficients of differentials of invariants of* $(k-1)$ *order* $da^I_{i_1 \ldots i_{k-1}} = \sum_{i_k=1}^{n} a^I_{i_1 \ldots i_k} \omega^{i_k}_{inv}$ *are differential invariants of order* k *...*

Such calculated invariants characterize an orbit $G \cdot f$ *uniquely up to "changing the parameter space"* $X \xrightarrow{\sim} X'$. □

3.2. Tangent functor for smooth algebras

This is an example of the dual construction for the main functor of Differential Geometry (which suggests how it can be extended over spectra of commutative algebras).

Let $T : \mathbf{Diff} \to \mathbf{Diff}$ be the **tangent** functor on the category of real ∞-smooth manifolds. In local coordinates it is of the form $\begin{cases} X \to TX : (x^i) \to (x^i, \xi^j) & X \in Ob\,\mathbf{Diff} \\ f \to Tf : (f^i(x)) \to (f^i(x), \frac{\partial f^j}{\partial x^k}\xi^k) & f \in Ar\,\mathbf{Diff} \end{cases}$

$\mathbf{Diff} \hookrightarrow \mathbb{R}\text{-}\mathbf{Alg}^{op}$ is a subcategory of the opposite of the category of real commutative algebras. Working in **Diff**, it is hard (if possible at all) to give a coordinate-free characterization of T. The question is how to characterize the image in $\mathbb{R}$-**Alg**?

Definition 3.2.1. Let $\mathcal{A} \in Ob\,\mathbb{R}\text{-}\mathbf{Alg}$.

- $\rho : \mathcal{A} \to \mathbf{Top}(\mathbf{Spec}_{\mathbb{R}}(\mathcal{A}), \mathbb{R})$ is called **functional representation** homomorphism of $\mathcal{A}$, where $\mathbf{Spec}_{\mathbb{R}}(\mathcal{A}) = \mathbb{R}\text{-}\mathbf{Alg}(\mathcal{A}, \mathbb{R})$ has the initial topology with respect to all functions $\rho(a),\ a \in \mathcal{A}$, $\rho(a)(f) := ev(f,a) := |f|(a)$.
- $\mathcal{A}$ is called **smooth** if $\forall a_1, a_2, ..., a_n \in \mathcal{A}$ and $\forall f : \mathbb{R}^n \to \mathbb{R} \in C^\infty(\mathbb{R}^n)$ the composite $f \circ < \rho(a_1), \rho(a_2), ..., \rho(a_n) > \in \ \mathrm{Im}\,(\rho)$. □

Denote by $\mathbb{R}$-Sm-**Alg** $\hookrightarrow$ $\mathbb{R}$-**Alg** full subcategory of smooth algebras.

Lemma 3.2.1. $\mathbb{R}$-Sm-**Alg** $\hookrightarrow$ $\mathbb{R}$-**Alg** *is a reflective subcategory, i.e. the inclusion has a left adjoint* Sm : $\mathbb{R}$-**Alg** $\to$ $\mathbb{R}$-Sm-**Alg**, **smooth completion** *of* $\mathbb{R}$*-algebras.*

Proof. Just take for each $\mathbb{R}$-algebra $\mathcal{A}$ $\mathbb{R}$-algebra $\mathrm{Sm}(\mathcal{A})$ of all terms $\{f(a_1, ..., a_n) \mid f : \mathbb{R}^n \to$

$\mathbb{R}$, $a_1, \ldots, a_n \in \mathcal{A}\}$ (all smooth operations are admitted). Each morphism f from an $\mathbb{R}$-algebra $\mathcal{A}$ to a smooth algebra $\mathcal{B}$ is uniquely extendable to $\tilde{f} : \mathtt{Sm}(\mathcal{A}) \to \mathcal{B}$. □

Let **Sym-Alg** be the category of symmetric partial differential algebras. *Ob* (**Sym-Alg**) are graded commutative algebras over commutative $\mathbb{R}$-algebras with a differential $d : \mathcal{A}^0 \to \mathcal{A}^1$ of degree 1 determined only on elements of degree 0 (d is $\mathbb{R}$-linear and satisfies the Leibniz rule). *Ar* (**Sym-Alg**) are graded degree 0 algebra homomorphisms which respect d.

Lemma 3.2.2. *There is an adjunction* $\begin{tikzcd} \mathbb{R}\text{-}\mathbf{Alg} \arrow[r, bend left, "\mathbf{Sym}"] \arrow[r, phantom, "\perp"] & \mathbf{Sym}\text{-}\mathbf{Alg} \arrow[l, bend left, "p_0"] \end{tikzcd}$,

where: p_0 *is the projection onto the* 0*-component* $\begin{cases} p_0(\mathcal{A}) := \mathcal{A}^0 \\ p_0(\mathcal{A} \xrightarrow{f} \mathcal{B}) := (\mathcal{A}^0 \xrightarrow{f^0} \mathcal{B}^0) \end{cases}$,

Sym *is the functor forming the graded symmetric algebra over the module of differentials of the given algebra*

$$\begin{cases} \mathbf{Sym}(\mathcal{C}) := \mathbf{Sym}(\Lambda^1(\mathcal{C})) \\ \mathbf{Sym}(\mathcal{C} \xrightarrow{h} \mathcal{D}) := (\mathbf{Sym}(\mathcal{C}) \xrightarrow{\tilde{h}} \mathbf{Sym}(\mathcal{D})) \\ \tilde{h}(\sum c_{i^1 \ldots i^k}(dc_1)^{i_1} \cdots (dc_k)^{i_k}) := \sum h(c_{i^1 \ldots i^k})(dh(c_1))^{i_1} \cdots (dh(c_k))^{i_k} \end{cases}$$ □

Lemma 3.2.3.

$$\begin{tikzcd} \mathbb{R}\text{-}\mathbf{Alg} \arrow[r, "\mathtt{Sm}"] \arrow[dr, "\mathbf{Spec}_{\mathbb{R}}"'] & \mathbb{R}\text{-}\mathtt{Sm}\text{-}\mathbf{Alg} \arrow[d, "\mathbf{Spec}_{\mathbb{R}}"] \\ & \mathbf{Top} \end{tikzcd}$$

(smooth completion does not change spectrum).

PROOF. $\forall \alpha : \mathcal{A} \to \mathbb{R}\ \exists!$ an extension $\tilde{\alpha} : \mathtt{Sm}(\mathcal{A}) \to \mathbb{R} : f(a_1, \ldots, a_n) \mapsto f(\alpha(a_1), \ldots, \alpha(a_n))$. And conversely, each such $\tilde{\alpha}$ is restricted uniquely to α. Initial topology on $\mathbb{R}\text{-}\mathbf{Alg}((\mathtt{Sm})(\mathcal{A}), \mathbb{R})$ does not change because new functions are functionally (continuously) dependent on old ones. □

Remark. With the Zariski topology on spectra, the smooth completion yields the same set with a weaker topology. For $C^\infty(X)$, $X \in Ob\,\mathbf{Diff}$ the Zariski and initial topologies coincide.

Proposition 3.2.1. • *The tangent functor* $T : \mathbb{R}\text{-}\mathtt{Sm}\text{-}\mathbf{Alg} \to \mathbb{R}\text{-}\mathtt{Sm}\text{-}\mathbf{Alg}$ *is equal to the composite* $\mathbb{R}\text{-}\mathtt{Sm}\text{-}\mathbf{Alg} \hookrightarrow \mathbb{R}\text{-}\mathbf{Alg} \xrightarrow{\mathbf{Sym}} \mathbf{Sym}\text{-}\mathbf{Alg} \xrightarrow{U} \mathbb{R}\text{-}\mathbf{Alg} \xrightarrow{\mathtt{Sm}} \mathbb{R}\text{-}\mathtt{Sm}\text{-}\mathbf{Alg}$, *where* U *forgets the differential* d *and grading.*

- *To the canonical projection* $$\begin{tikzcd} TX \arrow[d, "p_X"'] \\ X \end{tikzcd}$$ *there corresponds a canonical embedding* $$\begin{tikzcd} T(C^\infty(X)) \\ C^\infty(X) \arrow[u, "i_{C^\infty(X)}"] \end{tikzcd}$$.

PROOF.

- If $X \in \mathit{Ob}\,\mathbf{Diff}$ $TX \sim \mathbf{Spec}_{\mathbb{R}}(U \circ \mathbf{Sym}(C^\infty(X))) \sim \mathbf{Spec}_{\mathbb{R}}(\mathtt{Sm} \circ U \circ \mathbf{Sym}(C^\infty(X)))$.
- This is immediate. $\square$

Remark. It is reasonable to define T on $\mathbb{R}$-**Alg** as $T := U \circ \mathbf{Sym}$ and transfer it to spectra via duality $\begin{tikzcd} \mathbb{R}\text{-}\mathbf{Alg}^{op} \arrow[r, bend left, "F"] \arrow[r, phantom, "\perp"] & \mathbf{Spec}_{\mathbb{R}} \arrow[l, bend left, "G"] \end{tikzcd}$ (as $F \circ T^{op} \circ G$).

3.3. Vinogradov duality

Let K be a commutative ring, A a commutative algebra over K, A-**Mod** $\hookrightarrow$ K-**Mod** be the categories of modules over A and K respectively.

Definition 3.3.1. [V-K-L] For $P, Q \in \mathit{Ob}\,(A\text{-}\mathbf{Mod})$

- K-linear maps

  $l(a) := a\ \cdot\ -, r(a) := -\ \cdot\ a, \delta(a) := l(a) - r(a) : K\text{-}\mathbf{Mod}(P, Q) \to K\text{-}\mathbf{Mod}(P, Q)$ are called **left, right multiplications** and **difference operator** (by element $a \in A$),
- A K-linear map $\Delta : P \to Q$ is a **differential operator of order** $\leq r$ if $\forall a_0, a_1, \ldots, a_r \in A$

  $\delta_{a_0, a_1, \ldots, a_r}(\Delta) = 0$, where $\delta_{a_0, a_1, \ldots, a_r} := \delta_{a_0} \circ \delta_{a_1} \circ \cdots \circ \delta_{a_r}$. $\square$

Lemma 3.3.1.

- *If* $\Delta_1 \in K\text{-}\mathbf{Mod}(P, Q), \Delta_2 \in K\text{-}\mathbf{Mod}(Q, R)$ *are differential operators of order* $\leq r$ *and* $\leq s$ *respectively, then* $\Delta_2 \circ \Delta_1 : K\text{-}\mathbf{Mod}(P, R)$ *is a differential operator of order* $\leq r + s$,
- $\forall a \in A$, $P \in \mathit{Ob}\,(A\text{-}\mathbf{Mod})$ *module multiplication (by* a*)* $l_a : P \to P : p \mapsto ap$ *is a differential operator of order* 0. $\square$

The differential operators between A-modules form the arrows of a category A-**Diff**, such that A-**Mod** $\hookrightarrow$ A-**Diff** $\hookrightarrow$ K-**Mod**, and the first two categories have the same objects. A-**Diff** is enriched in $(K\text{-}\mathbf{Mod}, \otimes_K)$ and enriched in two different ways in $(A\text{-}\mathbf{Mod}, \otimes_K)$, except that composition is not an A-module map. Module multiplication for the first enrichment A-**Diff** in $(A\text{-}\mathbf{Mod}, \otimes_K)$ is given by $A \times A\text{-}\mathbf{Diff}(P,Q) \to A\text{-}\mathbf{Diff}(P,Q) : (a,\Delta) \mapsto l_a \circ \Delta$, for the second enrichment by $A \times A\text{-}\mathbf{Diff}(P,Q) \to A\text{-}\mathbf{Diff}(P,Q) : (a,\Delta) \mapsto \Delta \circ l_a$. Denote A-**Diff** with left module multiplication in hom-sets $l_a \circ -$ by the same name A-**Diff** and with right multiplication in hom-sets $- \circ l_a$ by $A\text{-}\mathbf{Diff}^+$.

Proposition 3.3.1.

- $\forall P,Q \in Ob\,(A\text{-}\mathbf{Mod})$ $A\text{-}\mathbf{Diff}(P,Q) = \bigcup\limits_{s=0}^{\infty} \mathbf{Diff}_s(P,Q)$, $A\text{-}\mathbf{Diff}^+(P,Q) = \bigcup\limits_{s=0}^{\infty} \mathbf{Diff}_s^+(P,Q)$ *are filtered A-modules by submodules of differential operators of order* $\leq s$, $s = 0,1,\ldots,$
- $\forall P \in Ob\,(A\text{-}\mathbf{Mod})$ $A\text{-}\mathbf{Diff}(P,P)$ *is an associative K-algebra.* $\square$

Proposition 3.3.2.

- $\mathbf{Diff}_s(P,-)$, $\mathbf{Diff}_s^+(-,P) : A\text{-}\mathbf{Mod} \to A\text{-}\mathbf{Mod}$ *are A-linear functors,*
- $\forall P \in Ob\,(A\text{-}\mathbf{Mod})$ *functor* $\mathbf{Diff}_s^+(-,P)$ *is representable by object* $\mathbf{Diff}_s^+(P) := \mathbf{Diff}_s^+(A,P)$, *i.e.* $\forall Q \in Ob\,(A\text{-}\mathbf{Mod})$ $A\text{-}\mathbf{Mod}(Q, \mathbf{Diff}_s^+(P)) \xrightarrow{\sim} \mathbf{Diff}_s^+(Q,P)$,
- $\forall P \in Ob\,(A\text{-}\mathbf{Mod})$ *functor* $\mathbf{Diff}_s(P,-)$ *is representable by object* $\mathbf{Jet}^s(P) := A \otimes_K P \bmod \mu^{s+1}$, *where* μ^{s+1} *is a submodule of* $A \otimes_K P$ *generated by elements* $\delta^{a_0} \circ \cdots \circ \delta^{a_{s+1}}(a \otimes p)$ $[\delta^b(a \otimes p) := ab \otimes p - a \otimes bp]$, *i.e.* $\forall Q \in Ob\,(A\text{-}\mathbf{Mod})$ $A\text{-}\mathbf{Mod}(\mathbf{Jet}^s(P), Q) \xrightarrow{\sim} \mathbf{Diff}_s(P,Q)$,
- *inclusion* $A\text{-}\mathbf{Mod} \hookrightarrow A\text{-}\mathbf{Diff}^+$ *is an (enriched) left adjoint with counit* $ev : \mathbf{Diff}^+(P) \to P : \Delta \mapsto \Delta(1)$, *i.e.* $\forall \Delta \in \mathbf{Diff}^+(Q,P)$ $\exists! f_\Delta \in A\text{-}\mathbf{Mod}(Q, \mathbf{Diff}^+(P))$ *such that*

$$\begin{tikzcd}
\mathbf{Diff}^+(P) \arrow[r, "ev"] & P \\
Q \arrow[u, dashed, "f_\Delta"] \arrow[ur, "\Delta"'] &
\end{tikzcd}$$

and this correspondence is A-linear, $f_\Delta : q \mapsto (a \mapsto \Delta(aq))$,

- *inclusion* $A\text{-}\mathbf{Mod} \hookrightarrow A\text{-}\mathbf{Diff}$ *is an (enriched) right adjoint with unit* $j^\infty : P \to \mathbf{Jet}^\infty(P) : p \mapsto$

$1 \otimes p \bmod \mu^{\infty}$ $[\mu^{\infty} := \bigcap\limits_{s=0}^{\infty} \mu^{s}]$, *i.e.* $\forall \Delta \in \mathbf{Diff}(P, Q)$ $\exists! f^{\Delta} \in A\text{-}\mathbf{Mod}(\mathbf{Jet}^{\infty}(P), Q)$ *such that*

$$\begin{tikzcd}
P \arrow[r, "j^{\infty}"] \arrow[dr, "\Delta"'] & \mathbf{Jet}^{\infty}(P) \arrow[d, dashed, "f^{\Delta}"] \\
& Q
\end{tikzcd}$$

and this correspondence is A-linear, $f^{\Delta} : (a \otimes p) \bmod \mu^{\infty} \mapsto a\Delta(p)$,

- *subcategory* A-$\mathbf{Mod}$ *is reflective and coreflective in* A-$\mathbf{Diff}$ *(enriched in* K-$\mathbf{Mod}$*).* □

$\forall s \in \mathbb{N}$ introduce two full subcategories of A-$\mathbf{Mod}$:

- A-$\mathbf{Mod}$-$\mathbf{Diff}_s$, consisting of all A-modules of type $\mathbf{Diff}_s(P, A)$, $P \in Ob\,(A\text{-}\mathbf{Mod})$,
- A-$\mathbf{Mod}$-$\mathbf{Jet}^s$, consisting of all A-modules of type $\mathbf{Jet}^s(P)$, $P \in Ob\,(A\text{-}\mathbf{Mod})$.

Proposition 3.3.3 (Vinogradov Duality). *For a commutative algebra* A *there is a concrete natural dual adjunction*

$$\begin{tikzcd}
A\text{-}\mathbf{Mod}\text{-}\mathbf{Diff}_s^{op} \arrow[r, bend left] \arrow[r, phantom, "\perp"] & A\text{-}\mathbf{Mod}\text{-}\mathbf{Jet}^s \arrow[l, bend left]
\end{tikzcd}$$

, $s \in \mathbb{N}$, *obtained by restriction of*

$$\begin{tikzcd}
A\text{-}\mathbf{Mod}^{op} \arrow[r, bend left, "A\text{-}\mathbf{Mod}(-{,}A)"] \arrow[r, phantom, "\perp"] & A\text{-}\mathbf{Mod} \arrow[l, bend left, "A\text{-}\mathbf{Mod}(-{,}A)"]
\end{tikzcd}$$

. *A is a schizophrenic object.* □

Remarks.

- The above duality theorem is not stated explicitly in [V-K-L] but the result is implicitly there.
- The above proposition states a formal analogue of duality between differential operators and jets over a fixed manifold X. Geometric modules of sections of vector bundles over X correspond to modules P over $C^{\infty}(X)$ with the property $\bigcap\limits_{x \in X} \mu_x P = 0$, where μ_x is a maximal ideal at point $x \in X$. Functors $\mathbf{Diff}_s(-, A)$ and $\mathbf{Jet}^s(-)$ preserve the module property to be geometric [V-K-L].
- This duality is an alternative (algebraic) way to introduce jet-bundles in Geometry (instead of the classical approach due to Grothendieck and Ehresmann as equivalence classes of maps which tangent of order s at a point). When $A = C^{\infty}(X)$ and P is a geometric module realizable as a vector bundle $V(P)$ over X, then $\mathbf{Jet}^s(P)$ is realizable as $\mathbf{Jet}^s(V(P))$ over X

in the classical sense [V-K-L, Vin1, Vin2]. □

3.4. Duality for differential equations

Proposition 3.4.1. *Let* **UAlg** *be a category of universal algebras with a representable forgetful functor. Then every topological algebra* $\mathfrak{A}$ *is a schizophrenic object (see* [P-Th]*), and so yields a natural dual adjunction between* **UAlg** *and* **Top**.

Proof.

- The initial topology on $\mathbf{UAlg}(\mathrm{B},\mathfrak{A})$ gives the initial lifting with respect to evaluation maps $ev_{B,b} : \mathbf{UAlg}(B,\mathfrak{A}) \to |\mathfrak{A}|,\ \ b \in |B|$.
- The algebra of continuous functions $\mathbf{Top}(X,\mathfrak{A})$ is initial with respect to the evaluation maps $ev_{X,x} : \mathbf{Top}(X,\mathfrak{A}) \to |\mathfrak{A}|$, $x \in |X|$ (which are obviously homomorphisms) since operations in $\mathbf{Top}(X,\mathfrak{A})$ are pointwise and each arrow $f \in \mathbf{Top}(X,\mathfrak{A})$ is completely determined by all its values $ev_{X,x}(f) = |f|(x),\ \ x \in |X|$. Hence, if $g : |B| \to \mathbf{Top}(X,\mathfrak{A})$ is a **Set**-map such that $\forall x \in |X|$ $ev_{X,x} \circ g$ is a homomorphism, $(\omega_n(ev_{X,x} \circ g)b_1, ..., (ev_{X,x} \circ g)b_n = ev_{X,x} \circ g\omega_n b_1, ..., b_n = ev_{X,x}\omega_n gb_1, ..., gb_n$, where ω_n is an n-ary operation. The first equality holds because $ev_{X,x} \circ g$ is a homomorphism, the second equality because $ev_{X,x}$ is a homomorphism), then g is a homomorphism since two maps whose values coincide at each point coincide themselves. □

Corollary. *Take* **UAlg**= k-Λ-**Alg**, *the category of exterior differential algebras over a field* k *(*$\mathbb{R}$ *or* $\mathbb{C}$*). These are thought of as presenting "generalized differential equations". Take* $\mathfrak{A} = \Lambda(C^{\infty}(\mathbb{R}^n))$ *or* $\Lambda(C^{\omega}(\mathbb{C}^n))$ *(which acts as a parameter space) with a topology not weaker than* jet^{∞}. *Then there exists a natural dual adjunction* $k\text{-}\Lambda\text{-}\mathbf{Alg}^{op} \underset{\longrightarrow}{\overset{\longleftarrow}{\perp}} \mathbf{Top}$ *(between differential equations and their solution spaces).* □

Remark. If we regard the category k-Λ-**Alg** whose forgetful functor is representable, we will get a lot of extra "points" which do not have geometric sense. Only graded maps of degree 0 to $\mathfrak{A}$ have geometric sense (they present integral manifolds of dimension not bigger than n). In this case, the representation of exterior differential algebras, when it exists, will not be via their solution spaces but via much bigger spaces. If we restrict k-Λ-**Alg** to only graded morphisms of degree 0 then the forgetful functor is not representable. But the notion of "schizophrenic object" still makes sense and the theorem for natural dual adjunction [P-Th] still holds. So, there is a representation of exterior differential algebras via their usual solution spaces. $\square$

We denote concrete subcategories of **Top** dual to categories k-**Alg** (algebras over k) and k-Λ-**Alg** (exterior differential algebras over k with graded degree 0 morphisms) by `alg`-**Sol** and `diff`-**Sol** respectively, i.e., $k\text{-}\mathbf{Alg}^{op} \sim \texttt{alg-}\mathbf{Sol}$, $k\text{-}\Lambda\text{-}\mathbf{Alg}^{op} \sim \texttt{diff-}\mathbf{Sol}$. In particular, `alg`-**Sol** contains all algebraic and all smooth k-manifolds ($k = \mathbb{R}$ or $\mathbb{C}$), `diff`-**Sol** contains all spaces of the form $\texttt{alg-}\mathbf{Sol}(k^n, X)$ (with representing object $\mathfrak{A} = \Lambda(C^\infty(k^n))$).

Lemma 3.4.1 (rough structure of `diff`-**Sol**).

- *Ob* (`diff`-**Sol**) *are pairs* $(X, \coprod_{i=1}^{n} \mathcal{F}_i)$ *where* $X := k\text{-}\Lambda\text{-}\mathbf{Alg}(D,k) = k\text{-}\mathbf{Alg}(D,k) \in Ob\,(\texttt{alg-}\mathbf{Sol})$, $\mathcal{F}_i \subset \texttt{alg-}\mathbf{Sol}(k^i, X)$, $1 \leq i \leq n$ *[*$\mathcal{F}_i$ *are not arbitrary subspaces of* $\texttt{alg-}\mathbf{Sol}(k^i, X)$*].*
- *Ar* (`diff`-**Sol**) *are pairs* $(f, \coprod_{i=1}^{n} \texttt{alg-}\mathbf{Sol}(k^i, f)) : (X, \coprod_{i=1}^{n} \mathcal{F}_i) \to (X', \coprod_{i=1}^{n} \mathcal{F}'_i)$ *where* $f : X \to X' \in Ar\,(\texttt{alg-}\mathbf{Sol})$, $\texttt{alg-}\mathbf{Sol}(k^i, f) : \mathcal{F}_i \to \mathcal{F}'_i$, $1 \leq i \leq n$. $\square$

Proposition 3.4.2. *There are the following adjunctions*

- $$\begin{tikzcd} k\text{-}\mathbf{Alg} \arrow[r, bend left, "\Lambda_k"] \arrow[r, phantom, "\perp"] & k\text{-}\Lambda\text{-}\mathbf{Alg} \arrow[l, bend left, "p_0"] \end{tikzcd}$$ *where* Λ_k *is the* **free exterior differential algebra functor**, p_0 *is the projection onto the subalgebra of degree-*0 *elements,*
- $$\begin{tikzcd} \texttt{alg-}\mathbf{Sol} \arrow[r, bend left, "\hom(k^n,-)"] \arrow[r, phantom, "\top"] & \texttt{diff-}\mathbf{Sol} \arrow[l, bend left, "b"] \end{tikzcd}$$ *where* b *is the base space functor,*

such that

$$\begin{tikzcd}[column sep=huge, row sep=huge]
k\text{-}\Lambda\text{-}\mathbf{Alg}^{op} \arrow[r, bend left, "F", "\sim"'] \arrow[d, bend left, "p_0^{op}"] \arrow[d, phantom, "\vdash"] & \mathtt{diff}\text{-}\mathbf{Sol} \arrow[l, bend left, "F'", "\sim"'] \arrow[d, bend left, "b"] \arrow[d, phantom, "\vdash"] \\
k\text{-}\mathbf{Alg}^{op} \arrow[u, bend left, "\Lambda_k^{op}"] \arrow[r, bend left, "G", "\sim"'] & \mathtt{alg}\text{-}\mathbf{Sol} \arrow[l, bend left, "G'", "\sim"'] \arrow[u, bend left, "\hom(k^n,-)"]
\end{tikzcd}$$

PROOF.

- $$\begin{tikzcd}
k\text{-}\Lambda\text{-}\mathbf{Alg}(\Lambda_k(A),D) \arrow[r, "\sim"] & k\text{-}\mathbf{Alg}(A,p_0(D)) & \text{(natural in A and D)} \\
\rho \arrow[u, "\in"] \arrow[r, mapsto, "\sim"] & \rho_0 \arrow[u, "\in"'] &
\end{tikzcd}$$
  where ρ_0 is the 0-component of graded degree 0 homomorphism $\rho = \bigoplus_{i\geq 0} \rho_i$.
- $$\begin{tikzcd}
\mathtt{diff}\text{-}\mathbf{Sol}(S,\hom(k^n,X)) \arrow[r, "\sim"] & \mathtt{alg}\text{-}\mathbf{Sol}(b(S),X) & \text{(natural in S and X)} \\
f \arrow[u, "\in"] \arrow[r, mapsto, "\sim"] & f \arrow[u, "\in"'] &
\end{tikzcd}$$
  where: S is a pair $(b(S), \coprod_{i=1}^{n} \mathcal{F}_i)$, $\mathcal{F}_i \subset \hom(k^i, b(S))$, $1 \leq i \leq n$, right $f : b(S) \to X$ is a usual map, and left $f := (f, \coprod_{i=1}^{n} \hom(k^i, f)) : (b(S), \coprod_{i=1}^{n} \mathcal{F}_i) \to (X, \coprod_{i=1}^{n} \hom(k^i, X))$.

The above square of adjunctions is immediate. ☐

3.4.1. Cartan involution.

For systems in Cartan involution (as defined below) a (single) solution can be calculated recursively beginning from smallest 0 dimension. By Cartan's theorem [BC3G, Car1, Fin, Vas] every system can be made into such a form by a sufficient number of differential prolongations [BC3G, Car1, Fin, Vas]. There is a cohomological criterion for systems to be in involution.

Definition 3.4.1.1. Let $\mathcal{A} \in Ob\,(k\text{-}\Lambda\text{-}\mathbf{Alg})$, $\mathfrak{A}_n$ be $\Lambda_{\mathbb{R}}(C^\infty(\mathbb{R}^n))$ or $\Lambda_{\mathbb{C}}(C^\omega(\mathbb{C}^n))$, $n \geq 0$.

- Any (differential homomorphism of degree 0) $\rho : \mathcal{A} \to \mathfrak{A}_n$ is called an **integral manifold** of $\mathcal{A}$ (of dimension not bigger than n).

- $deg\,(\rho : \mathcal{A} \to \mathfrak{A}_n) = m,\ 0 \le m \le n,$ iff ρ can be factored through a $\gamma : \mathcal{A} \to \mathfrak{A}_m$, i.e.,
$$\begin{tikzcd} \mathcal{A} \arrow[r, dotted, "\gamma"] \arrow[dr, "\rho"'] & \mathfrak{A}_m \arrow[d, dotted] \\ & \mathfrak{A}_n \end{tikzcd}$$
and m is the smallest such number.

- $deg\,(\mathcal{A}) = n$ iff maximal degree of integral manifolds of $\mathcal{A}$ is n.

- $\mathcal{A}$, $deg(\mathcal{A}) = n$, is in **Cartan involution** iff for each m-dimensional integral manifold $\rho : \mathcal{A} \to \mathfrak{A}_m$, $m < n$, there exists an $(m+1)$-dimensional integral manifold $\beta : \mathcal{A} \to \mathfrak{A}_{m+1}$ which **contains** ρ, i.e.,
$$\begin{tikzcd} \mathcal{A} \arrow[r, dotted, "\exists\,\beta"] \arrow[dr, "\forall\,\rho"'] & \mathfrak{A}_{m+1} \arrow[d, dotted] \\ & \mathfrak{A}_m \end{tikzcd}$$
□

Remarks.

- $\mathfrak{A}_0$ is just k ($\mathbb{R}$ or $\mathbb{C}$) with the trivial differential. $\rho : \mathcal{A} \to \mathfrak{A}_0$ corresponds to a point $b\,(\rho) : b\,(\mathcal{A}) \to k$. Each point of $\mathcal{A}$ is a 0-dimensional integral manifold.

- The original Cartan definition was for classical algebras (quotient algebras of $\Lambda_{\mathbb{R}}(C^{\omega}(\mathbb{R}^N))$) and in terms of 'infinitesimal integral elements' (nondifferential homomorphisms of degree 0 $f : \mathcal{A} \to \Lambda_k(d\tau^1, ..., d\tau^N)$) [BC3G, Car1, Fin]. For that case, the two definitions coincide.

- By a number of differential prolongations (adding new jet-variables with obvious relations), every classical system can be put into Cartan involution form (E. Cartan's theorem).

- The integration step (constructing an integral manifold of 1 higher dimension) is done by the method of "Cauchy characteristics". □

Proposition 3.4.1.1. *Let $\mathcal{A}$ be a quotient algebra of $\Lambda_{\mathbb{R}}(C^{\omega}(\mathbb{R}^N))$, $deg(\mathcal{A}) = n$, corresponding to a system of differential equations*
$$\begin{tikzcd} \mathcal{E}^q \arrow[r, tail] \arrow[dr] & X \equiv b(F(\mathcal{A})) \arrow[d, two heads, "\pi"] \arrow[r, equal] & \mathbf{Jet}^q(\mathbb{R}^{n+k}) \arrow[dl, two heads, "\pi"] \\ & \mathbb{R}^n & \end{tikzcd}$$
, $dim(X) = N$.

Then $\mathcal{A}$ is in Cartan involution iff the following **Spencer δ-complex** *is acyclic:*

$$0 \longrightarrow g^{(r)} \xrightarrow{\delta} g^{(r-1)} \otimes \Lambda^1(\mathbb{R}^n) \xrightarrow{\delta} g^{(r-2)} \otimes \Lambda^2(\mathbb{R}^n) \xrightarrow{\delta} \cdots$$

$$\cdots \xrightarrow{\delta} g^{(r-n)} \otimes \Lambda^n(\mathbb{R}^n) \longrightarrow 0$$

where $g^{(r)} := T(\mathbf{Jet}^r(\mathcal{E}^q)) \bigcap V\pi^{q+r}_{q+r-1} \hookrightarrow S_{q+r}(T_*\mathbb{R}^n) \otimes V\pi$ *is* r*-th prolongation of symbol* g, $\pi^{q+r}_{q+r-1} : \mathbf{Jet}^{q+r}(\mathbb{R}^{n+k}) \to \mathbf{Jet}^{q+r-1}(\mathbb{R}^{n+k})$ *is a natural projection of jet-bundles,* V *is the "vertical" subbundle,* S_p *is the* p*-th symmetric power,*
$\delta(\alpha_1 \cdots \alpha_{q+r-l} \otimes v \otimes \beta_1 \wedge \cdots \wedge \beta_l) := \sum_{i=1}^{q+r-l} \alpha_1 \cdots \hat{\alpha}_i \cdots \alpha_{q+r-l} \otimes v \otimes \alpha_i \wedge \beta_1 \wedge \cdots \wedge \beta_l$, v *is a section of* $V\pi$.

PROOF. See [A-V-L, Sei, Vin2, V-K-L, Ver] $\square$

The original Cartan involutivity test was in terms of certain dimensions of "infinitesimal integral elements". The above theorem is due to J.P. Serre [A-V-L, La-Se].

3.5. Gelfand-Naimark 2-duality

Let $\mathbf{C^*Alg^{op}} \underset{G}{\overset{F}{\rightleftarrows}} \mathbf{CHTop}$ ($\perp$) be the usual Gelfand-Naimark duality between commutative C^*-algebras and compact Hausdorff spaces. Both categories are strict 2-categories with homotopy classes of homotopies as 2-cells (homotopy of C^*-algebras is a homotopy in **Top** each instance of which is a C^*-algebra homomorphism). A reasonable question to ask is: can it be extended to a 2-duality? The answer is yes.

By definition

$$\begin{tikzcd}
\mathbf{C^*Alg}(A,B) \times |A| \arrow[r, "ev"] & {|B|} & \mathbf{C^*Alg}(B,\mathbb{C}) \times \mathbf{C^*Alg}(A,B) \arrow[r, "c_{A,B,\mathbb{C}}"] & \mathbf{C^*Alg}(A,\mathbb{C}) \\
{|I| \times |A|} \arrow[u, "f\times 1"] \arrow[ur, "\bar{f}"'] & & \mathbf{C^*Alg}(B,\mathbb{C}) \times |I| \arrow[u, "1\times f"] \arrow[ur, "F(\bar{f})"'] &
\end{tikzcd}$$

So that, if $f : |I| \times |A| \to |B|$ is a homotopy in $\mathbf{C^*Alg}$, then its image in **CHTop** is $F(\bar{f}) : |F(B)| \times |I| \to |F(A)|$ (where $| \ |$ denotes the underlying set or map).

We need to prove that such an extension of F preserves 2-categorical structure (the proof for G is dual).

Preserving homotopies

Lemma 3.5.1. *If B is locally compact, then* $\mathbf{Top}(B,C) \times \mathbf{Top}(A,B) \xrightarrow{c_{A,B,C}} \mathbf{Top}(A,C)$ *is continuous* (with *compact-open* topology in all hom-sets).

PROOF is standard. Let $f = g \circ h = c_{A,B,C}(g,h)$. Take U^K to be a (subbase) nbhd of f. It is sufficient to show that $\exists$ (subbase) neighbourhoods $U_1^{K^1} \ni g$, $U_2^{K^2} \ni h$, s.t. $U_1^{K^1} \circ U_2^{K^2} = c_{A,B,C}(U_1^{K^1}, U_2^{K^2}) \subset U^K$. Take $U_1 = U$, $K_2 = K$, K_1 a compact nbhd of $h(K)$, s.t. $K_1 \subset g^{-1}(U)$ (K_1 exists by local compactness of B), $U_2 = \mathrm{int}(K_1)$. □

Corollary. *If A is locally compact then $ev_{A,B} : \mathbf{Top}(A,B) \times |A| \to |B|$ is continuous.*

PROOF. Each space A is homeomorphic to $\mathbf{Top}(1,A)$ (with compact-open topology), and $ev_{A,B}$ corresponds to $c_{1,A,B}$. □

Lemma 3.5.2. • *The initial topology on $|F(A)| = \mathbf{C^*Alg}(A,\mathbb{C})$ w.r.t. evaluation maps $\forall a \in A$ $\mathbf{C^*Alg}(A,\mathbb{C}) \times 1 \xrightarrow[1\times a]{} \mathbf{C^*Alg}(A,\mathbb{C}) \times |A| \xrightarrow[ev]{} |\mathbb{C}|$ is point-open (i.e. the topology of pointwise convergence).*

• *The initial topology on $|G(X)| = \mathbf{CHTop}(X,\mathbb{C})$ w.r.t. evaluation maps $\forall x \in X$ $\mathbf{CHTop}(X,\mathbb{C}) \times 1 \xrightarrow[1\times x]{} \mathbf{CHTop}(X,\mathbb{C}) \times |X| \xrightarrow[ev]{} |\mathbb{C}|$ is compact-open.*

PROOF. See [P-Th], [Joh], [Eng]. □

Lemma 3.5.3. *If $\mathcal{A}, \mathcal{B} \subset$ **LCTop** are naturally concretely dual subcategories of locally compact spaces (let D be a dualizing object) then if $\mathcal{A}(X,D)$ has compact-open topology then the initial topology of $|X| \cong \mathcal{B}(\mathcal{A}(X,D),D)$ is compact-open as well.*

PROOF. The evaluation map $ev : \mathcal{A}(X,D) \times |X| \to |D|$ is continuous (since X is locally compact and $\mathcal{A}(X,D)$ has the compact-open topology). This implies that the initial (point-open) topology on $|X| \cong \mathcal{B}(\mathcal{A}(X,D),D)$ is actually compact-open (by assumption, the topology of $|X|$ is initial w.r.t. all maps $'f' : |X| \xrightarrow{\sim} 1 \times |X| \xrightarrow{f \times 1} \mathcal{A}(X,D) \times |X| \xrightarrow{ev} |D|$. It means that the topology on $|X| \cong \mathcal{B}(\mathcal{A}(X,D),D)$ is point-open since the subbasic open sets in point-open and initial topologies are the same $U^{'f'} := \{x \in |X| \mid 'f'(x) \in \underset{open}{U} \subset D\} = 'f'^{-1}(U))$.

We need to show that $\{x \in |X| \mid \forall f \in \underset{compact}{K} \subset \mathcal{A}(X,D).\ \ 'f'(x) \in \underset{open}{U} \subset D\} = \bigcap_{f \in K} 'f'^{-1}(U)$ is open in the point-open topology on $|X|$.

Take $x \in \bigcap_{f \in K} 'f'^{-1}(U)$, then $ev\,(K,x) \subset U$. By continuity of ev, $\forall y \in K.\ \exists \underset{open}{V_y} \ni y$. $\exists \underset{open}{W_y} \ni x$, s.t. $ev\,(V_y, W_y) \subset U$. $\underset{\substack{y \in K \\ open \\ covering}}{\bigcup} V_y \supset K$, so, by compactness, $\bigcup_{j=1,\dots,n} V_{y_j} \supset K$. Therefore, $ev\,(V_{y_j}, \bigcap_{j=1,\dots,n} W_{y_j}) \subset U$, $ev\,(\bigcup_{j=1,\dots,n} V_{y_j}, \bigcap_{j=1,\dots,n} W_{y_j}) \subset U$, $ev\,(K, \bigcap_{j=1,\dots,n} W_{y_j}) \subset U$, i.e., x is internal. ☐

Corollary. *Gelfand-Naimark duality preserves homotopies.*

PROOF. $|A| = \mathbf{CHTop}(X,\mathbb{C})$ has the compact-open topology. $|X| = \mathbf{C^*Alg}(A,\mathbb{C})$ has the point-open topology, so, by **Lemma 3.5.3** the compact-open topology.

Multiplication $c_{A,B,\mathbb{C}}$ is continuous (since all hom-sets have compact-open topology). Therefore, $F(\bar{f})$ is continuous.

[In the other direction $G : \mathbf{CHTop} \to \mathbf{C^*Alg}$, there are no problems because $\mathbf{CHTop}(X,\mathbb{C})$ has the compact-open topology. See also [Loo]]. ☐

Preserving homotopy relation between homotopies

Let $\bar{\bar{f}} : |I| \times |I| \times |A| \to |B|$ be continuous, s.t. $\bar{\bar{f}}(0,t,a) = \bar{f}_0(t,a)$, $\bar{\bar{f}}(1,t,a) = \bar{f}_1(t,a)$.

$$\begin{tikzcd}
\mathbf{C^*Alg}(A,B)\times{|A|} \arrow[r, "ev"] & {|B|} \\
{|I|\times|I|\times|A|} \arrow[u, "\bar{\bar{f}}^T\times 1_{|A|}"] \arrow[ur, "\bar{\bar{f}}"'] & \\
1\times{|I|\times|A|} \arrow[u, bend left, "0\times 1_{|I|\times|A|}"] \arrow[u, bend right, "1\times 1_{|I|\times|A|}"'] & \\
{|I|\times|A|} \arrow[u, "<!,1_{|I|}>\times 1_{|A|}", "\sim"'] \arrow[uuur, bend right=20, "\bar{f}_0"] \arrow[uuur, bend right=30, "\bar{f}_1"']
\end{tikzcd}$$

$$\begin{tikzcd}
\mathbf{C^*Alg}(B,\mathbb{C})\times\mathbf{C^*Alg}(A,B) \arrow[r, "c_{A,B,\mathbb{C}}"] & \mathbf{C^*Alg}(A,\mathbb{C}) \\
\mathbf{C^*Alg}(B,\mathbb{C})\times{|I|\times|I|} \arrow[u, "1\times\bar{\bar{f}}^T"] \arrow[ur, "F(\bar{\bar{f}})"] & \\
\mathbf{C^*Alg}(B,\mathbb{C})\times{|I|} \arrow[u, bend left, "1\times((0\times 1_{|I|})\circ<!,1_{|I|}>)"] \arrow[u, bend right, "1\times((1\times 1_{|I|})\circ<!,1_{|I|}>)"'] \arrow[uur, bend right=20, "F(\bar{f}_0)"] \arrow[uur, bend right=30, "F(\bar{f}_1)"']
\end{tikzcd}$$

So, $F(\bar{\bar{f}})$ is a homotopy from $F(\bar{f}_0)$ to $F(\bar{f}_1)$. $F(\bar{\bar{f}})$ is continuous since $c_{A,B,\mathbb{C}}$ is continuous in compact-open topology. $\mathbf{C^*Alg}(B,\mathbb{C})$ has compact-open topology by **Lemma 3.5.3**.

Preserving unit 2-cells i_f

$$\begin{tikzcd}
\mathbf{C^*Alg}(A,B)\times{|A|} \arrow[r, "ev"] & {|B|} \\
1\times{|A|} \arrow[u, "f'\times 1"] \arrow[r, "\sim", "p_2"'] & {|A|} \arrow[u, "f"'] \\
{|I|\times|A|} \arrow[u, "!\times 1"] \arrow[ur, "p_2"']
\end{tikzcd}$$

$$\begin{tikzcd}
\mathbf{C^*Alg}(B,\mathbb{C})\times\mathbf{C^*Alg}(A,B) \arrow[r, "c_{A,B,\mathbb{C}}"] & \mathbf{C^*Alg}(A,\mathbb{C}) \\
\mathbf{C^*Alg}(B,\mathbb{C})\times 1 \arrow[u, "1\times f'"] \arrow[ur, "F(f\circ p_2)"] \arrow[r, "\sim", "p_1"'] & \mathbf{C^*Alg}(B,\mathbb{C}) \arrow[u, "-\circ f"'] \\
\mathbf{C^*Alg}(B,\mathbb{C})\times{|I|} \arrow[u, "1\times !"] \arrow[uur, "F(i_f)"] \arrow[ur, "p_1"']
\end{tikzcd}$$

So, if $i_f = f\circ p_2\circ(!\times 1_{|A|}) = f\circ p_2$, then $F(i_f) = F(f)\circ p_1 = i_{F(f)}$.

Preserving composites $i_g * \bar{f} : |I|\times|A| \xrightarrow{\bar{f}} |B| \xrightarrow{g} |C|$ and $\bar{f} * i_h : |I|\times|A'| \xrightarrow{1\times h} |I|\times|A| \xrightarrow{\bar{f}} |B|$

$$\begin{tikzcd}
\mathbf{C^*Alg}(A,C) \times |A| \arrow[r, "ev"] & {|C|} \\
\mathbf{C^*Alg}(A,B) \times |A| \arrow[r, "ev"] \arrow[u, "(g\circ -)\times 1"] & {|B|} \arrow[u, "g"'] \\
{|I| \times |A|} \arrow[u, "f\times 1"] \arrow[ur, "\bar{f}"'] &
\end{tikzcd}$$

$$\begin{tikzcd}
\mathbf{C^*Alg}(C,\mathbb{C}) \times |I| \arrow[d, "1\times(\mathbf{C^*Alg}(A,g)\circ f)"] \arrow[dr, "F(g\circ\bar{f})"] \arrow[ddd, bend right=60, "\mathbf{C^*Alg}(g,\mathbb{C})\times 1"'] & \\
\mathbf{C^*Alg}(C,\mathbb{C}) \times \mathbf{C^*Alg}(A,C) \arrow[r, "c_{A,C,\mathbb{C}}"'] & \mathbf{C^*Alg}(A,\mathbb{C}) \\
\mathbf{C^*Alg}(B,\mathbb{C}) \times \mathbf{C^*Alg}(A,B) \arrow[r, "cA{,}B{,}\mathbb{C}"] & \mathbf{C^*Alg}(A,\mathbb{C}) \arrow[u, "1", "\sim"'] \\
\mathbf{C^*Alg}(B,\mathbb{C}) \times |I| \arrow[u, "1\times f"] \arrow[ur, "F(\bar{f})"'] &
\end{tikzcd}$$

$g \circ \bar{f}$ is a homotopy corresponding to $\mathbf{C^*Alg}(A,g) \circ f$. The outer perimeter of the right diagram commutes because of the definition of $F(\bar{f})$, $F(g \circ \bar{f})$ and the associativity law (if $(s,t) \in \mathbf{C^*Alg}(C,\mathbb{C}) \times |I|$ then $s \circ (g \circ f(t)) = (s \circ g) \circ f(t))$. So, $F(g \circ \bar{f}) = F(\bar{f}) \circ (F(g) \times 1_{|I|})$, i.e., $F(i_g * \bar{f}) = F(\bar{f}) * i_{F(g)}$.

$$\begin{tikzcd}
\mathbf{C^*Alg}(A,B) \times |A| \arrow[r, "ev"] & {|B|} \\
{|I| \times |A|} \arrow[u, "f\times 1"] \arrow[ur, "\bar{f}"] & \\
{|I| \times |A'|} \arrow[u, "1\times h"] \arrow[uur, "\bar{f}\circ(1\times h)"'] \arrow[d, "(\bar{f}\circ(1\times h))^T\times 1"'] \arrow[dr, "\begin{smallmatrix}\bar{f}\circ(1\times h)=\\ ev\circ(f\times 1)\circ(1\times h)=\\ ev\circ(f\times h)\end{smallmatrix}"] & \\
\mathbf{C^*Alg}(A',B) \times |A'| \arrow[r, "ev"'] & {|B|} \arrow[uuu, "1", "\sim"']
\end{tikzcd}$$

$$\begin{tikzcd}
\mathbf{C^*Alg}(B,\mathbb{C}) \times \mathbf{C^*Alg}(A,B) \arrow[r, "c_{A,B,\mathbb{C}}"] & \mathbf{C^*Alg}(A,\mathbb{C}) \arrow[dd, "\mathbf{C^*Alg}(h,\mathbb{C})"] \\
\mathbf{C^*Alg}(B,\mathbb{C}) \times |I| \arrow[u, "1\times f"] \arrow[ur, "F(\bar{f})"] \arrow[d, "1\times(\bar{f}\circ(1\times h))^T"'] \arrow[dr, "F(\bar{f}\circ(1\times h))"] & \\
\mathbf{C^*Alg}(B,\mathbb{C}) \times \mathbf{C^*Alg}(A',B) \arrow[r, "c_{A',B,\mathbb{C}}"'] & \mathbf{C^*Alg}(A',\mathbb{C})
\end{tikzcd}$$

The right internal triangle of the right diagram commutes since if $(g,t) \in \mathbf{C^*Alg}(B,\mathbb{C}) \times |I|$ then $\mathbf{C^*Alg}(h,\mathbb{C}) \circ c_{A,B,\mathbb{C}} \circ (1 \times f)(g,t) = (g \circ f(t)) \circ h = g \circ (f(t) \circ h) = c_{A',B,\mathbb{C}}(g, f(t) \circ h) = c_{A',B,\mathbb{C}}(g, (\bar{f} \circ (1 \times h))^T(t)) = c_{A',B,\mathbb{C}} \circ (1 \times (\bar{f} \circ (1 \times h))^T)(g,t)$. So, $F(\bar{f} * i_h) = F(\bar{f} \circ (1 \times h)) = F(h) \circ F(\bar{f}) = i_{F(h)} * F(\bar{f})$.

Preserving vertical composites

We need to show that if $\bar{f} : \bar{f} \circ i_0 \simeq \bar{f} \circ i_1$ and $\bar{g} : \bar{g} \circ i_0 \simeq \bar{g} \circ i_1$ are homotopies in $\mathbf{C^*Alg}$ s.t. $\bar{f} \circ i_1 = \bar{g} \circ i_0$ then $F(\bar{g} \odot \bar{f}) = F(\bar{g}) \odot F(\bar{f})$.

By definition, the vertical composite $\bar g \odot \bar f$ is

$$\begin{tikzcd}
{|A| \times |[0,\frac{1}{2}]|} \arrow[dr, hook, "1\times i"'] & {|A| \times |I|} \arrow[l, "1\times\alpha"', "\sim"] \arrow[dr, "\bar f"] & \\
 & {|A| \times |I|} \arrow[r, dashed, "\exists!\,\bar g\odot\bar f"] & {|B|} \\
{|A| \times |[\frac{1}{2},1]|} \arrow[ur, hook', "1\times j"] & {|A| \times |I|} \arrow[l, "1\times\beta"', "\sim"] \arrow[ur, "\bar g"'] &
\end{tikzcd}$$

$$\begin{tikzcd}
{\mathbf{C^*Alg}(A,B) \times |A|} \arrow[rr, "ev"] & & {|B|} \\
{|I| \times |A|} \arrow[u, "(\bar g\odot\bar f)^T\times 1"] \arrow[urr, "\bar g\odot\bar f"'] & & \\
{|[\frac{1}{2},1]| \times |A|} \arrow[u, hook', "j\times 1"] & {|[0,\frac{1}{2}]| \times |A|} \arrow[ul, hook', "i\times 1"'] & \\
{|I| \times |A|} \arrow[uuu, bend left, "f\times 1"] \arrow[ur, "\alpha\times 1"'] \arrow[uuurr, "\bar f"'] & & \\
{|I| \times |A|} \arrow[uuuu, bend left=40, "g\times 1"] \arrow[uu, bend left, "\beta\times 1"] \arrow[uuuurr, bend right=40, "\bar g"'] & &
\end{tikzcd}$$

$$\begin{tikzcd}
{\mathbf{C^*Alg}(B,\mathbb{C}) \times \mathbf{C^*Alg}(A,B)} \arrow[rr, "c_{A,B,\mathbb{C}}"] & & {\mathbf{C^*Alg}(A,\mathbb{C})} \\
{\mathbf{C^*Alg}(B,\mathbb{C}) \times |I|} \arrow[u, "1\times(\bar g\odot\bar f)^T"] \arrow[urr, "F(\bar g\odot\bar f)"] & & \\
{\mathbf{C^*Alg}(B,\mathbb{C}) \times |I|} \arrow[u, "1\times(j\circ\beta)"] \arrow[uurr, "F(\bar g)"'] & & {\mathbf{C^*Alg}(B,\mathbb{C}) \times |I|} \arrow[ull, "1\times(i\circ\alpha)"'] \arrow[uu, "F(\bar f)"']
\end{tikzcd}$$

By uniqueness $f \equiv \bar f^T = (\bar g \odot \bar f)^T \circ i \circ \alpha$, $g \equiv \bar g^T = (\bar g \odot \bar f)^T \circ j \circ \beta$.

So, $\begin{cases} F(\bar{g}\odot\bar{f})\circ(1\times(i\circ\alpha))=F(\bar{f}) \\ F(\bar{g}\odot\bar{f})\circ(1\times(j\circ\beta))=F(\bar{g}) \end{cases}$. It means $F(\bar{g}\odot\bar{f})=F(\bar{g})\odot F(\bar{f})$.

Preserving horizontal composites $$\begin{tikzcd} A \arrow[r, bend left, "f_0"] \arrow[r, bend right, "f_1"'] \arrow[r, phantom, "\bar{f}\Downarrow"] & B \arrow[r, bend left, "g_0"] \arrow[r, bend right, "g_1"'] \arrow[r, phantom, "\bar{g}\Downarrow"] & C \end{tikzcd}$$

$\bar{g}*\bar{f}:=(\bar{g}*i_{f_1})\odot(i_{g_0}*\bar{f})\simeq(i_{g_1}*\bar{f})\odot(\bar{g}*i_{f_0})$ (homotopic homotopies).

$$F(\bar{g}*\bar{f})=F(\bar{g}*i_{f_1})\odot F(i_{g_0}*\bar{f})=(i_{F(f_1)}*F(\bar{g}))\odot(F(\bar{f})*i_{F(g_0)})\simeq F(\bar{f})*F(\bar{g}).$$

The following Proposition 3.5.1 completes the proof of Gelfand-Naimark 2-duality $$\begin{tikzcd} \mathbf{C^*Alg^{op}} \arrow[r, bend left, "F"] \arrow[r, phantom, "\perp"] & \mathbf{CHTop} \arrow[l, bend left, "G"] \end{tikzcd}$$.

Proposition 3.5.1. *If* $\begin{tikzcd} \mathbf{C} \arrow[r, bend left, "F"] & \mathbf{D} \arrow[l, bend left, "G"] \end{tikzcd}$ *are two strict* n*-categories and two strict* n*-functors such that the restriction* $\begin{tikzcd} \mathbf{C}^{\leq 1} \arrow[r, bend left, "F^{\leq 1}"] \arrow[r, phantom, "\perp"] & \mathbf{D}^{\leq 1} \arrow[l, bend left, "G^{\leq 1}"] \end{tikzcd}$ *is an adjunction with unit* $\eta : 1_{\mathbf{C}^{\leq 1}} \to G^{\leq 1}F^{\leq 1}$ *and counit* $\varepsilon : F^{\leq 1}G^{\leq 1} \to 1_{\mathbf{D}^{\leq 1}}$ *which are still natural transformations for the extension (i.e.* $\eta : 1_{\mathbf{C}} \to GF$ *and* $\varepsilon : FG \to 1_{\mathbf{D}}$ *are natural transformations) then the extended situation* $\begin{tikzcd} \mathbf{C} \arrow[r, bend left, "F"] \arrow[r, phantom, "\perp"] & \mathbf{D} \arrow[l, bend left, "G"] \end{tikzcd}$ *is a strict adjunction.*

PROOF. A strict adjunction is completely determined by its "unit-counit" (proposition 1.5.3). $\eta : 1_{\mathbf{C}} \to GF$ and $\varepsilon : FG \to 1_{\mathbf{D}}$ are natural transformations and satisfy triangle identities $\varepsilon F \circ_1 F\eta = 1_F$ and $G\varepsilon \circ_1 \eta G = 1_G$ (because, e.g. $\varepsilon F = \varepsilon F^{\leq 1}$, $1_F = 1_{F^{\leq 1}}$ (set-theoretically), etc.)

□

Corollary. Any 1-adjunction between a category of topological algebras and a subcategory of topological spaces is a 2-adjunction if it can be extended functorially over 2-cells in such a way that each instance of the image of a homotopy is the image of this instance of the preimage-homotopy.

PROOF. Under the given conditions, the unit and counit of the 1-adjunction are automatically natural transformations for the extension. E.g., take the unit η, the naturality square

$$\begin{tikzcd} A \arrow[r, "\eta_A"] \arrow[d, "f^1"'] & GFA \arrow[d, "GFf^1"] \\ B \arrow[r, "\eta_B"'] & GFB \end{tikzcd}$$

, where $f^1 : A \times I \to B$ is a homotopy, commutes because each instance of it commutes (since η is a unit of the 1-adjunction), i.e. $\forall t \in I \; \eta_B \circ f^1(-,t) = GF(f^1(-,t)) \circ \eta_A$, it means $\eta_B \circ f^1 = GF(f^1) \circ (\eta_A \times I)$, i.e. $\eta_B * f^1 = GF(f^1) * \eta_A$. ☐

☐

Remark. There are 'forgetful' functors $\mathbf{C^*Alg} \to$ 2-**Set** and **CHTop** $\to$ 2-**Set** (where 2-**Set** is the usual **Set** with just one isomorphisms-2-cell for each pair of maps with the same domain and codomain) but they are not faithful and forget too much for 2-**Set** to be an underlying category for Gelfand-Naimark 2-duality. ☐

Proposition 3.5.2. • *Gelfand-Naimark 2-duality is concrete over* 2-**Cat** *(*2-**Cat** *is the usual 2-category of (small) categories, functors and natural transformations), i.e. there exist (faithful) forgetful functors* $U : \mathbf{C^*Alg} \to 2\text{-}\mathbf{Cat}$ *and* $V : \mathbf{CHTop} \to 2\text{-}\mathbf{Cat}$ *such that*

$$\begin{tikzcd} \mathbf{C^*Alg}^{\mathrm{op}} \arrow[r, "F"] \arrow[dr, "\mathbf{C^*Alg}(-{,}\mathbb{C})"'] & \mathbf{CHTop} \arrow[d, "V"] \\ & 2\text{-}\mathbf{Cat} \end{tikzcd}$$

and

$$\begin{tikzcd} \mathbf{CHTop}^{\mathrm{op}} \arrow[r, "G^{op}"] \arrow[dr, "\mathbf{CHTop}(-{,}\mathbb{C})"'] & \mathbf{C^*Alg} \arrow[d, "U"] \\ & 2\text{-}\mathbf{Cat} \end{tikzcd}$$

where U and V are composites of the inclusion and fundamental groupoid functors ($U : \mathbf{C^*Alg} \hookrightarrow 2\text{-}\mathbf{Top} \xrightarrow{2\text{-}\mathbf{Top}(1,-)} 2\text{-}\mathbf{Cat}$ *and* $V : \mathbf{CHTop} \hookrightarrow 2\text{-}\mathbf{Top} \xrightarrow{2\text{-}\mathbf{Top}(1,-)} 2\text{-}\mathbf{Cat}$*).*

- *This duality is natural, i.e. lifting of hom-functors* $\mathbf{C^*Alg}(-,\mathbb{C})$, $\mathbf{CHTop}(-,\mathbb{C})$ *along V and U is initial ($\mathbb{C}$ is the schizophrenic object of the duality).* ☐

Remark. 2-duality allows us to transfer (co)homology theories from one side of a duality to the other. Under the reasonable assumption that K-theory is determined in a universal way, we could get **M. Atiyah theorem** that *K-groups of commutative C^*-algebras and compact Hausdorff spaces coincide.* The problem, however, is that K-groups were not determined by a universal mapping property. But, there is a theorem by J. Cuntz [Weg] that K-theory is universally determined on a large subcategory of C^*-algebras. □

3.6. Differential algebras as a dual to Lie calculus

For Lie groups there is an equivalent alternative calculus via exterior differential algebras. For Lie groups of transformations, it turns out to be more powerful than via Lie algebras. It was developed by E. Cartan and after him by the Russian School in Differential Geometry, mainly, by A.M. Vasiliev [Vas0, Vas].

Definition 3.6.1.

- The exterior differential algebra $\Lambda \in Ob\,(k\text{-}\Lambda\text{-}\mathbf{Alg})$, $k = \mathbb{C}$ or $\mathbb{R}$, is called **linear** if it finitely generated by elements of degree 1 (with possible linear (resolvable) relations between them over k).
- The exterior differential algebra $\Lambda \in Ob\,(k\text{-}\Lambda\text{-}\mathbf{Alg})$, $k = \mathbb{C}$ or $\mathbb{R}$, is called **quasilinear** if it is finitely generated by elements of degree 0 and 1 with relations between either elements of degree 0 or linear relations on elements of degree 1 with coefficients in Λ^0.
- A smooth map $f : X \to Y$ is called **quasialgebraic** if there exist quasilinear subalgebras $\Lambda_1 \hookrightarrow k\text{-}\Lambda(X)$ and $\Lambda_2 \hookrightarrow k\text{-}\Lambda(Y)$ such that $f^*(\Lambda_2) := k\text{-}\Lambda(f)(\Lambda_2) \hookrightarrow \Lambda_1$. □

Quasialgebraic maps admit an effective description. All homomorphisms of Lie groups are quasialgebraic.

Proposition 3.6.1. *There are equivalences* **locLieGrp** $\sim$ **LieAlg** $\sim$ $k\text{-}\Lambda\text{-}\mathbf{ALg}^{op}_{lin}$ *(local Lie groups* $\sim$ *Lie algebras* $\sim$ *(opposite of the category of) linear exterior differential algebras).* □

Lemma 3.6.1.

- *The functor* $\mathbf{Diff}^{op} \xrightarrow{C^\infty} \mathbf{ComAlg} \xrightarrow{k\text{-}\Lambda} k\text{-}\Lambda\text{-}\mathbf{Alg} \xrightarrow{compl} k\text{-}\Lambda\text{-}\mathbf{Alg}_{compl}$ *is monoidal with respect to Cartesian product* $\times$ *in* **Diff** *and exterior product* $\wedge$ *in* $k\text{-}\Lambda\text{-}\mathbf{Alg}_{compl}$, *where* $k\text{-}\Lambda$ *is* *a free exterior differential algebra functor over k, compl is a smooth (or analytic) completion of exterior differential algebras.*

- *Analogously, the functor* $\mathbf{LieGrp}^{op} \xrightarrow{k\text{-}\Lambda_{inv}} k\text{-}\Lambda\text{-}\mathbf{Alg}$ *(assigning the algebra of (left)invariant forms) is monoidal with respect to cartesian product* $\times$ *in* **LieGrp** *and exterior product* $\wedge$ *in* k-Λ-**Alg**.

□

Remarks.

- The smooth (analytic) completion functor *compl* : $k\text{-}\Lambda\mathbf{Alg} \to k\text{-}\Lambda\text{-}\mathbf{Alg}_{compl}$ is a left adjoint to the inclusion (of the subcategory of smooth (analytic) exterior differential algebras) $k\text{-}\Lambda\text{-}\mathbf{Alg}_{compl} \hookrightarrow k\text{-}\Lambda\text{-}\mathbf{Alg}$ (it is given essentially by the smooth (analytic) completion of the algebra of coefficients of an exterior differential algebra).
- The exterior product $\wedge$ in $k\text{-}\Lambda\text{-}\mathbf{Alg}_{compl}$ is bigger than in k-Λ-**Alg** and is equal to the smooth (analytic) completion of (the usual algebraic) exterior product in k-Λ-**Alg**. □

Definition 3.6.2.

- The exterior differential algebra $\mathcal{A}$ is called **smoothly realizable** if there exists a manifold $Y \in Ob\,\mathbf{Diff}$ and an embedding $\mathcal{A} \hookrightarrow \Lambda(Y)$. It is **fully** smoothly realizable if $\mathcal{A}^1$ (locally) generates T^*Y.
- A **Geometric triple** is a (locally trivial) fibre bundle $(G \times Y \to X) \in Ar\,\mathbf{Diff}$, equivariant with respect to a (left) action of (Lie group) G on X and $\rho : G \times G \times Y \to G \times Y : (g, h, y) \mapsto (gh, y)$ an action on $G \times Y$. A **morphism of geometric triples** is a morphism of fibre bundles

$$\begin{tikzcd} G_1 \times Y_1 \arrow[r, "\sigma \times F"] \arrow[d, "\rho_1"'] & G_2 \times Y_2 \arrow[d, "\rho_2"] \\ X_1 \arrow[r, "f"'] & X_2 \end{tikzcd}$$

where $\sigma : G_1 \to G_2$ is a Lie group homomorphism. A geometric triple $\rho : G \times Y \to X$ is **local** if G is a local Lie group and X is a local G-space (admits a local group of transformations).

- An **algebraic triple** is an exterior product of two differential algebras $\mathcal{A} \wedge \mathcal{B}$, where $\mathcal{A}$ is linear, with a differential ideal $I \subset \mathcal{A} \wedge \mathcal{B}$ generated by elements of degree 1. A **morphism of algebraic triples** $(\mathcal{A}_1 \wedge \mathcal{B}_1, I_1) \to (\mathcal{A}_2 \wedge \mathcal{B}_2, I_2)$ is a differential homomorphism $\alpha \wedge \beta : \mathcal{A}_1 \wedge \mathcal{B}_1 \to \mathcal{A}_2 \wedge \mathcal{B}_2$ such that the differential ideal generated by the image $\alpha \wedge \beta\,(I_1)$ is I_2. An algebraic triple $(\mathcal{A} \wedge \mathcal{B}, I)$ is **smoothly realizable** if $\mathcal{B}$ is a smoothly realizable algebra. □

Lie groups of transformations are particular cases of geometric triples when $X = Y$ and the projection $\rho : G \times Y \to X$ coincides with the action of G on X.

Proposition 3.6.2. [Vas] *The smooth manifold X admits a left action of the finite dimensional Lie group G iff there exists a smooth manifold Y, smoothly realizable algebra $\mathcal{B} \hookrightarrow \Lambda(Y)$, and differential ideal $I \subset \Lambda_{inv}(G) \wedge \mathcal{B}$ generated by 1-forms such that the foliation in $G \times Y$ determined by I is a (locally trivial) fibre bundle $G \times Y \to X$ with the base X.* □

Proposition 3.6.3. locGeomTriple $\sim$ **realAlgTriple**^{op} *(local geometric triples $\sim$ (opposite to) smoothly realizable algebraic triples).* □

Remark. By proposition 3.6.2. Lie groups of transformations are in duality with a certain full subcategory of **realAlgTriple**.

3.7. An extension of Pontryagin duality

The following is a new and recent example of a concrete duality, due to G.Lukacs [Luk]. The extension is natural with the same dualizing object $\mathbb{R}/\mathbb{Z}$, and establishes a concrete duality for abelian locally precompact groups.

Definition 3.7.1.

- The set X in a topological group G is called **precompact** if $\forall U \ni e$ (neighbourhood of identity) $\exists$ a finite subset $F \subset G$ such that $X \subset FU$.
- The group G is **locally precompact** if it contains a precompact neighbourhood of the identity.

□

(Locally) precompact groups are very closed to (locally) compact ones. Namely, their two-sided uniformity completions give (locally) compact groups, and conversely, dense subgroups of (locally) compact groups are (locally) precompact.

Proposition 3.7.1 (Pontryagin-Lukacs). *There are the following natural dualities*

$$\begin{tikzcd}
\mathbf{Ab}^{op} \arrow[r, bend left, "\sim"] \arrow[d, hook] & \mathbf{CompAb} \arrow[l, bend left, "\sim"] \arrow[d, hook] \\
\mathbf{locCompAb}^{op} \arrow[r, bend left, "\sim"] \arrow[d, hook] & \mathbf{locCompAb} \arrow[l, bend left, "\sim"] \arrow[d, hook] \\
\mathbf{locPreCompAb}^{op} \arrow[r, bend left, "\sim"] & \mathbf{locCompAb}^{\Rightarrow} \arrow[l, bend left, "\sim"]
\end{tikzcd}$$

where **locCompAb**$^{\Rightarrow}$ *is a category of dense embeddings of locally compact abelian groups into compact abelian groups (with commutative squares in* **locCompAb** *as arrows).* □

Remarks.

- The main idea of this extension is that every locally precompact group G can be represented as a dense injective **locCompAb**-morphism $G_d \to compl(G)$, where G_d is the same group with discrete topology, and $compl(G)$ is its completion with respect to two-sided uniformity on G. After that, the usual Pontryagin duality is used [Luk].
- The dualizing object in **locCompAb**$^{\Rightarrow}$ is
$$\begin{tikzcd}
\mathbb{R}/\mathbb{Z} \arrow[d, "id"] \\
\mathbb{R}/\mathbb{Z}
\end{tikzcd}$$
. □

Bibliography

[A-H-S] J. Adamek, H. Herrlich, G.E. Strecker, *Abstract and Concrete Categories. The Joy of Cats*, online edition, 2004.

[A-V-L] D.V. Alekseevskiy, A.M. Vinogradov, V.V. Lychagin, *Main Ideas and Concepts of Differential Geometry*, Moscow, 1988. (Russian)

[Baez1] J. Baez and J. Dolan, *Higher-dimensional algebra and topological quantum field theory*, Jour. Math. Phys. 36, 6073-6105, 1995.

[Baez2] J. Baez, *An introduction to n-categories*, in 7th Conference on Category Theory and Computer Science, eds. E. Moggi and G. Rosolini, Springer Lecture Notes in Computer Science vol. 1290, Springer, Berlin, 1997.

[Bat] M.A. Batanin, *Monoidal Globular Categories as a Natural Environment for the Theory of Weak n-Categories*, School of MPCE, Macquarie University, North Ryde, NSW 2109, Australia.

[Bel] J.L. Bell, *Toposes and Local Set Theories: An Introduction*, Clarendon Press, Oxford, 1988.

[Bi-Cr] R.L. Bishop, R.J. Crittenden, *Geometry of Manifolds*, Academic Press, New York and London, 1964.

[Bor1] F. Borceux, *Handbook of Categorical Algebra 1. Basic Category Theory*, Cambridge University Press, 1994.

[Bor2] F. Borceux, *Handbook of Categorical Algebra 2. Categories and Structures*, Cambridge University Press, 1994.

[Bor3] F. Borceux, *Handbook of Categorical Algebra 3. Categories of Sheaves*, Cambridge University Press, 1994.

[Bou] N. Bourbaki, *Algebre Homologique*, Masson, 1980. (translated into Russian)

[Bru] U. Bruzzo, *Introduction to Algebraic Topology and Algebraic Geometry*, International School for Advanced Studies, Trieste, 2002.

[BC3G] R.L. Bryant, S.S. Chern, R.B. Gardner, H.L. Goldschmidt, P.A. Griffiths, *Exterior Differential Systems*, Springer-Verlag, 1991.

[Car1] E. Cartan, *Exterior Differential Systems and their Geometric Applications*, Moscow State University, 1962. (translated into Russian)

[Car2] E. Cartan, *Moving Frame Method and Theory of Groups of Transformations*, Moscow State University, 1963. (translated into Russian)

[C-C-L] S.S. Chern, W.H.Chen, K.S. Lam, *Lectures on Differential Geometry*, World Scientific, 2000.

[Cl-D] D.M. Clark, B.A. Davey, *Natural Dualities for the Working Algebraist*, Cambridge University Press, 1997.

[C-L] E. Cheng, A. Lauda, *Higher-Dimensional Categories: an illustrated guide book*, University of Cambridge, 2004.

[D-N-F] B.A. Dubrovin, S.P. Novikov, A.T. Fomenko, *Modern Geometry*, Moscow, 1979. (Russian)

[Eng] R. Engelking, *General Topology*, 1977.

[ELOS] L.E. Evtushik, U.G. Lumiste, N.M. Ostianu, A.P. Shirokov, *Differential Geometric Structures on Manifolds*, Moscow, 1979. (Russian)

[Fin] S.P. Finikov, *The Method of Cartan's Exterior Forms*, Moscow, 1948. (Russian)

[Gar] R.B. Gardner, *The Method of Equivalence and Its Applications*, Society for Industrial and Applied Mathematics, 1989.

[Gol] R. Goldblatt, *Topoi: The Categorial Analysis of Logic*, North-Holland, 1984.

[Hir] F. Hirzebruch, *Topological Methods in Algebraic Geometry*, Springer-Verlag, 1966.

[Jac] B. Jacobs, *Categorical Logic and Type Theory*, Elsevier, North-Holland, 2001.

[J-R] B. Jacobs and Jan Rutten, *A Tutorial on (Co)Algebras and (Co)Induction*, Bulletin of the European Association for Theoretical Computer Science, v. 62, pp. 222–259, 1997.

[Joh] P.T. Johnstone, *Stone spaces*, Cambridge University Press, 1982.

[Kel] G.M. Kelly, *Basic Concepts of Enriched Category Theory*, Reprints in TAC, No. 10, 2005.

[Kob] S. Kobayashi, *Groups of Transformations in Differential Geometry*, Moscow, 1986. (translated into Russian)

[Koc] J. Kock, *Weak Identity Arrows in Higher Categories*, IMRP International Mathematics Research Papers, Volume 2006, Article ID 69163, 2006, pp. 1-54.

[Kom] B.P. Komrakov, *Structures on Manifolds and Homogeneous Spaces*, Minsk, 1978. (Russian)

[Kra] I.S. Krasil'shchik, *Calculus over Commutative Algebras: a Concise User Guide*, The Diffiety Institute Preprint Series, 1996.

[L-S] J. Lambek, P.J. Scott, *Introduction to Higher Order Categorical Logic*, Cambridge University Press, 1986.

[La-Se] L.A. Lambe and W.M. Seiler, *Differential Equations, Spencer Cohomology, and Computing Resolutions*, online article.

[Lap] G.F. Laptev, *The Main Infinitesimal Structures of Higher Order on a Smooth Manifold*, Proceedings of Geometric Seminar, vol. 1, Moscow, 1966, pp. 139-190. (Russian)

[Lei] T. Leinster, *Higher Operads, Higher Categories*, Cambridge University Press, 2003.

[Lic] A. Lichnerowicz, *Theorie Globale des Connexions et des Groupes d'Holonomie*, Roma, Edizioni Cremonese, 1955. (translated into Russian)

[Loo] L.H. Loomis, *An Introduction to Abstract Harmonic Analysis*, D. Van Nostrand Company, 1953.

[Luk] G. Lukacs, *Duality Theory of Locally Precompact Groups*, Category Theory Octoberfest, Ottawa University, 2006.

[Mac] S. MacLane, *Categories for the Working Mathematician*, Springer-Verlag, 1971.

[M-M] S. MacLane, I. Moerdijk, *Sheaves in Geometry and Logic*, Springer-Verlag, 1992.

[M-T] I. Madsen, J. Tornehave, *From Calculus to Cohomology*, Cambridge University Press, 1996.

[Man] U.I. Manin, *Lectures in Algebraic Geometry. Part 1. Affine Schemes*, Moscow State University, 1970. (Russian)

[May] J.P. May, *A Concise Course in Algebraic Topology*, The University of Chicago Press, 1999.

[M-S] J.W. Milnor, J.D. Stasheff, *Characteristic Classes*, Princeton, 1974.

[Moe] I. Moerdijk, *Introduction to the Language of Stacks and Gerbes*, University of Utrecht, arXiv:math.AT/0212266v1, 2002.

[Nar] R. Narasimhan, *Analysis on Real and Complex Manifolds*, North-Holland, 1968.

[Nes] J. Nestruev, *Smooth manifolds and observables*, Moscow, 2003. (Russian)

[Olv] P.J. Olver, *Equivalence, Invariants and Symmetry*, Cambridge University Press, 1995.

[P-Th] H.-E. Porst, W. Tholen, *Concrete dualities*, Category Theory at Work (H. Herrlich. H.-E. Porst, eds.), Heldermann Verlag, Berlin, 1991, pp. 111-136.

[Pos] M.M. Postnikov, *Lectures in Algebraic Topology. Foundations of Homotopy Theory*, Moscow, 1984. (Russian)

[Sat] H. Sato, *Algebraic Topology: An Intuitive Approach*, American Mathematical Society, 1996.

[Sau] D.J. Saunders, *The Geometry of Jet Bundles*, Cambridge University Press, 1989.

[Sei] W.M. Seiler, *Differential Equations, Spencer Cohomology, and Pommaret Bases*, Heidelberg University, online lecture notes.

[Sch] J.T. Schwartz, *Differential Geometry and Topology*, New York, 1968.

[Ste] S. Sternberg, *Lectures on Differential Geometry*, Prentice Hall, 1964.

[Str1] T. Streicher, *Introduction to Category Theory and Categorical Logic*, online lecture notes, 2003.

[Str2] T. Streicher, *Fibred Categories a la Jean Benabou*, online lecture notes, 1999.

[Strt] R. Street, *Categorical and Combinatorial Aspects of Descent Theory* (2003), talk at ICIAM.

[S-W] R. Sulanke, P. Wintgen, *Differential Geometry and Fiber Bundles*, Berlin, 1972. (translated into Russian)

[Vas0] A.M. Vasiliev, *Differential Algebra as a Technique in Differential Geometry*, Proceedings of Geometric Seminar, vol. 1, Moscow, 1966. (Russian)

[Vas] A.M. Vasiliev, *Theory of Differential-Geometric Structures*, Moscow State University, 1987. (Russian)

[Ver] A.M. Verbovetskii, *Geometry of Finite Jets and Differential Equations: additional chapters*, Moscow Independent University, 1999. (Russian)

[Vin1] A.M. Vinogradov, *The Logic Algebra for the Theory of Linear Differential Operators* (1972), Dokl. Akad. Nauk SSSR (Soviet Math. Dokl.), vol. 13, No. 4. (Russian)

[Vin2] A.M. Vinogradov, *Geometry of Nonlinear Differential Equations*, Moscow, 1979. (Russian)

[V-K-L] A.M. Vinogradov, I.C. Krasil'schik, V.V. Lychagin, *Introduction to Geometry of Nonliniear Differential Equations*, Moscow, 1986. (Russian)

[Vis] A. Vistoli, *Notes on Grothendieck Topologies, Fibered Categories and Descent Theory*, Bologna, Italy, online lecture notes.

[Weg] N.E. Wegge-Olsen, *K-Theory and C^*-Algebras, A Friendly Approach*, Oxford University Press, 1993.

[Wel] R. Wells, *Calculus on Complex Manifolds*, 1976.