Finding a Peter-Weyl Basis for Small Representations of O_N^+

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Abstract

We review the basic theory of compact quantum groups. We study thire quantum Peter-Weyl theory and the Woronowicz-Tannaka-Krein theorem. In the last chapter we discuss the orthogonal quantum group O_N^+ and find an orthogonal basis of the invariant subspace for $n = 2, 3$ and a partial result for $n = 4$.

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1 Introduction

1.1 Motivation

When first studying a mathematical object, a common idea is to create new objects from given ones. As an example, for a vector space V there is always the dual vector space V^* of linear maps from V to the base field $\mathbb{F}$ or the cartesian product $V \times V$. We can also, given two groups G and H, construct the direct product $G \times H$. These new examples usually have a very simple description and so a natural question to ask is whether all objects be arise via these constructions. The answer is often yes if the object isn't too complex, see for example the fundamental theorem of finite dimensional vector spaces or the fundamental theorem of finite abelian groups.

Due to the interdisciplinary nature of mathematics, the next step would be to create entirely new objects of a different kind from an old one. An ambitious, but reasonable question is to ask whether one can completely classify both objects in this manner. For instance, given a locally compact Hausdorff topological space X we can create $C_0(X)$, the commutative C^*-algebra of continous complex valued functions on X vanishing at infinity. Are all commutative C^*-algebras of this form? That is, given a commutative C^*-algebra C, is it true that $C = C_0(X)$ for some locally compact space X? Remarkably, Gelfand's theorem asserts that this is indeed the case. Conversly, it tells us that all locally compact spaces are in fact the spectrum of some C^*-algebra . Therefore, one could think of commutative C^*-algebra as being equivalent, or dual, to locally comapct Hausdorff space.

However the notion of equivalence is a bit more subtle than it appears at first glance. Indeed, in mathematics we not only care about mathematical objects, but also the structure preserving maps between them. If locally compact spaces and commutative C^*-algebras are truly equivalent, then any morphism between two topological spaces X and Y should give rise to a morphism between $C_0(X)$ and $C_0(Y)$ and vice versa. That is, we are not only interested in equivalences of structures, but equivalences of categories with specified morphisms. Gelfand's theorem is not unique in this regard, in fact there are many examples of an algebra-geometry correspondance. Two other well known examples of this duality are the Serre-Swan theorem and Hilberts Nullstellensatz.

Once this duality has been established between two categories, we must ask again ask what new objects can emerge from this equivalence. At first it would seem that no new concepts could arise, but this is not necessarily the case. For example, we know how commutative C^*-algebras arise, but Gelfand's theorem says nothing about noncommutative variants. Due to the aforementionned duality, one could think of general noncommutative C^*-algebras as continuous functions on some noncommutative topological space. Of course, noncommutative topological spaces don't actually exist so this notion is purely philosophical. This thinking extends to other areas of geometry and algebra. First, consider a geometric object of some sort and the continuous algebra of functions on them. If we then deform or quantize this algebra in some manner, we think of this algebra as the 'continuous functions on some quantum space'.

The reason we are being so vague is because, once a duality has been established, there are many ways to deform the algebraic object in question. Take for example where our geometric object is a group. The quantized notion should of course be called a quantum group. However there is no universal definition of a quantum group. Furthermore, because of the duality in the non deformed case, there is no distinction made between the algebras and the space they act on. Indeed, quantum groups aren't groups, but associative algebras!

In this paper, we study quantum groups in the sense of Woronowicz, also called compact quantum groups. In this first section, we give a brief overview of Gelfand's theorem as well as the axioms for compact quantum groups. The similarity between compact groups and compact quantum groups will become evident as we study the quantum Peter-Weyl theorem and the Woronowicz-Tannaka-Krein duality theorem in section 2 and 3 respectively. Finally in the last section, we examine O_N^+, the orthgonal quantum group and find an invariant orthogonal basis for all N when the dimension of the representation $n = 2, 3$ and a partial result for $n = 4$.

1.2 Gelfand's Theorem

A classical theorem of Gelfand's state that the category of locally compact Hausdorff topological spaces LCH is naturally equivalent to the opposite category of commutative C^*-algebras CC^{*op}. The morphisms in the first category are continuous proper maps and the morphisms in the second category are proper *-homomorphisms. Recall that a continuous map $f : X \to Y$ is called proper if for any compact $K \subset Y$, $f^{-1}(K) \subset X$ is compact and that a *-homomorphism $g : A \to B$ is called proper if for any approximate identity $\{e_i\} \in A$, $\{g(e_i)\}$ is an approximate identity in B.

The functor $C_0 : LCH \to CC^{*op}$ is defined as follows. For any object $X \in LCH$, $C_0(X)$ is the commutative C^*-algebra of continuous complex valued functions that vanish at infinity. A function $f : X \to \mathbb{C}$ is said to vanush at infinity if for all $\epsilon > 0$ there exists a compact $K \subset X$ such that $||f(x)|| < \epsilon$ for all $x \notin K$. Addition and multiplication is done pointwise, the norm is the sup norm and the * operation is pointwise complex conjugation. If $f : X \to Y$ is a continuous and proper map, for $g \in C_0(Y)$,

$$C_0(f) = f^* : C_0(Y) \to C_0(X), f^*(g) = gf$$

One then shows that this indeed a proper *-homomorphism.

On the other hand, if $A \in CC^{*op}$ then $S(A)$ is the set of characters on A, which is simply a non zero algebra homomorphism from A to the complex numbers. This is locally compact space under the topology of pointwise convergence. Given a proper *-homomorphism $g : A \to B$, then for $f \in S(A)$, we have

$$S(g) = g^* : S(B) \to S(A), g^*(f) = fg$$

One shows tha g^* is in fact proper and continuous. To complete the proof and show that these are in fact natural isomorphisms of categories, one then demonstrates that there is a natural isomorphism

$$X \to S(C_0(X)), \quad A \to C_0(S(A))$$

$$x \to ev_x, \quad a \to \hat{a}$$

where $ev_x(f) = f(x)$ is the evalution at x and $\hat{a}(g) = g(a)$ is the Gelfand transform.

One takeaway from Gelfand's theorem is that topological invariants of locally compact spaces correspond to invariants of commutative C^*-algebras and vice versa. More than that however, if gives an algebraic counterpart to topological notions and vice-versa.

Topology	**Algebra**
Compact	Unital
1-point compactification	Unitization
Stone-Čech compactification	Multiplier algebra
Injection	Surjection
Surjection	Injection
Cartesian Product	Tensor product

1.3 Compact Quantum Groups

We would like to add extra structure to our topological space and see where Gelfand's duality theorem takes us. Namely, let G be a compact Hausdorff topological group. Since the multiplication map $m : G \times G \to G$ is continuous, it induces a morphism on $C_0(G) = C(G)$ (since G is compact). However, because the functor C_0 is contravariant the induced map, called the comultiplication, is of the form $\Delta : C(G) \to C(G) \otimes_{min} C(G)$ where

$\otimes_{min}$ is the minimal tensor product. From now on $\otimes$ will always denote the minimal tensor product unless stated otherwise. More specifically, if $f \in C(G)$, then

$$\Delta : C(G) \to C(G) \otimes C(G), (\Delta f)(x, y) = f(xy)$$

where we've identified $C(G \times G)$ with $C(G) \otimes C(G)$.

The first thing we notice is that Δ is a unital *-homomorphism. Furthermore, due to the associativity of the multiplication, the comultiplication is coassociative in the sense that $(\Delta \otimes id)\Delta = (id \otimes \Delta)\Delta$. In addition, a non obvious property is that the sets

$$(C(G) \otimes id)\Delta C(G) \quad \text{and} \quad (id \otimes C(G))\Delta C(G)$$

are linearly dense in $C(G) \otimes C(G)$. This is called the cancelation property. To see that this is true, note that $(C(G) \otimes id)\Delta C(G)$ is a unital subalgebra that seperates points in $C(G) \otimes C(G) \cong C(G \times G)$ spanned by all functions of the form $(x, y) \to f_1(xy)f_2(x)$ and similarly for $(id \otimes C(G))\Delta C(G)$. The Stone-Weirstrass theorem then implies the cancelation property.

We would like to describe this cancelation property for, say the circle group $\mathbb{T}$. The continuous functions on the circle are spanned by functions of the form $f_n(x) = e^{i2\pi nx}$. Thus we would like to know if

$$(f_n \otimes id)\Delta(f_{n'})(x, y) = e^{i2\pi((n+n')x+n'y)}$$

is dense in $C(\mathbb{T} \times \mathbb{T})$. But this is certainly the case, because by baisc Fourier analysis this space is spanned by functions of the form $(x, y) \to f_n(x)f_{n'}y$. We can easily attain these functions due to the cancelation property of the group. This construciton remains valid groups that are the duals of discrete groups. Of course,$C(G)$ is always a commutative C^*-algebra . In the spirit of quantization, the example above motivates the following definition.

Definition 1.1. A compact quantum groups is a pair (A, Δ) where A is a unital C^*-algebra and $\Delta : A \to A \otimes A$ is a unital *-homomorphism such that

1. Δ is coassociative: $(\Delta \otimes id)\Delta = (id \otimes \Delta)\Delta$
2. $(A \otimes id)\Delta A$ and $(id \otimes A)\Delta A$ are linearly dense in $A \otimes A$

If we really wanted to used this definition, we would like to see if the duality still holds in the case of commutative compact quantum groups. That is, given a compact quantum group (A, Δ) with A commutative, do we have $(A, \Delta) = (C(G), \Delta)$ for some compact group G. This is indeed the case.

Gelfand's theorem defines the comultiplication and the coassociativity of the compact quantum group implies the associativity of the group. This turns the space into a compact semigroup. The cancelation property of the compact quantum group actually implies cancelation in the semigroup.

Suppose $xy = xz$. Then we have, for any $f, g \in A = C(G)$

$$g(x)f(xy) = g(x)f(xz)$$

$$(g \otimes id)(\Delta f)(x, y) = (g \otimes id)(\Delta f)(x, z)$$

Consider the evaluation function $ev_{(x,y)} : C(G \times G) \to \mathbb{C}$, $ev_{(x,y)}(h) = h(x, y)$. Since the set above is dense in $C(G \times G)$, this means that $ev_{(x,y)} = ev_{(x,z)}$ which implies y = z. Thus G is a compact semigroup with cancelation property. It is known that a compact semigroup with cancellation is in fact a group. Thus, any commutative compact quantum group is of the form $(C(G), \Delta)$.

2 Peter-Weyl Theory

2.1 Haar State

The following two sections are based on the excellent books by Neshvyev and Tuset [1],the notes by Woronowicz [2], and the notes by Maes and Van Daele [3]. Any missing proof in the following two chapters can be found there.

The representation theory of compact groups is fairly well understood. A major role is played by the Haar measure, which is the unique (up to scalar) measure that is translation invariant. A similar role is played by the Haar state in the theory of compact quantum groups, which is also unique and translation invariant. In his original paper, Woronowicz assumed seperability of the C^*-algebra A to assure the existence of a faithful state. This condition was later removed by Van Daele. The proof of the existence requires three steps.

Theorem 2.1. *Let ω be a state on A. Then there is a state φ such that $\omega * \varphi = \varphi * \omega = \varphi$ where the convolution product for bounded linear functions is defined as $\omega * \varphi = (\omega \otimes \varphi)\Delta$*

Proof. Define

$$\varphi_n = \frac{1}{n}(\omega + \omega^{*2} + \ldots + \omega^{*n})$$

Then φ_n is a state for all n because A has an identity. By the Banach-Aloaglu theorem, this has a *-weak accumulation point φ. Now since

$$||\varphi_n * \omega - \varphi_n|| = \frac{1}{n}||\omega^{*(n+1)} - \omega|| \leq \frac{2}{n}$$

we get $\varphi * \omega = \varphi$. Similarly, $\omega * \varphi = \varphi$ ☐

Theorem 2.2. *Let (A, Δ) be a compact quantum group and ω, φ two states such that $\omega * \varphi = *\varphi$. Then if $\rho \in A^*$ such that $0 \leq \rho \leq \omega$, then $\rho * \varphi = \rho(1)\varphi$*

Finally, we get to the more important result for which compactness is essential.

Theorem 2.3. *Let (A, Δ) be a compact quantum group. Then there is a state h such that $\rho * h = h * \rho = \rho(1)h$ for all $\rho \in A^*$*

Proof. For any positive linear functional $\omega \geq 0$, define

$$K_\omega = \{h \in A^* | h \text{ is a state and } \omega * h = \omega(1)h\}$$

This is a compact subset of A^* in the weak *-topology. It is non-empty by the first theorem, and by the second theorem we have that $K_\omega \subseteq K_\rho$ when $0 \leq \rho \leq \omega$. Thus $K_{\omega_1+\omega_2} \subseteq K_{\omega_1} \cap K_{\omega_2}$ for any positive linear functionals ω_1, ω_2. Thus by compactness the intersection of all such ω is non empty.

☐

2.2 Representation Theory

We will now discuss the representation theory of compact quantum groups. As in the compact case, the Haar state will be very useful in this theory. We must first define what exactly is a representation. As usual let (A, Δ) be a compact quantum gorup and let H be a finite dimensional Hilbert space.

Definition 2.4. A representation of A on H is an invertible element $U \in B(H) \otimes A$ such that

$$(id \otimes \Delta)(U) = U_{12}U_{13}$$

The reason why it a representation is defined in this way isn't very complicated. Writing $U = \sum_{i,j} E_{ij} \otimes u_{ij}$, where E_{ij} are the matrix units, then the condition that $(id \otimes \Delta)(U) = U_{12}U_{13}$ is equivalent to saying

$$\Delta(u_{ij}) = \sum_{k=1}^{n} u_{ik} \otimes u_{kj}$$

If U is a unitary element of $B(H) \otimes A$, then we say that the representation is unitary. Of course, we can also define the direct sum and tensor product of representaitons.

Definition 2.5. The direct sum of two finite dimensional representaitons U and V is the representaion $U \oplus V$ on the space $H_U \oplus H_V$. The tensor product, $U \otimes V$ is the representation on $H_U \otimes H_V$ defined by $U \otimes V = U_{12}V_{13}$

Definition 2.6. If U and V are two finite dimensional representations of a compact quantum group, an operator $T : H_U \to H_V$ is an intertwiner between U and V if

$$(T \otimes 1)U = V(T \otimes 1)$$

The space of intertwiners between U and V is denoted $\mathrm{Hom}(U, V)$. Two representations are said to be equivalent if $\mathrm{Hom}(U, V)$ contains an invertible element. A represenation is said to be irreducible if $\mathrm{Hom}(U, U) = \mathrm{End}(U) = \mathbb{C}$. The power of the Haar functional is evident here, due to the following theorem

Theorem 2.7. *Any finite dimensional representation U is equivalent to a unitary one*

Proof. Define

$$Q = (id \otimes h)(U^*U) \in B(H)$$

Now since U is invertible, $U^*U \geq \epsilon 1$ for some $\epsilon > 0$. So Q is a positive invertible element of $B(H)$. We have

$$(id \otimes \Delta)(U^*U) = (U^*U)_{12}(U^*U)_{13} = U_{13}^*U_{12}^*U_{12}U_{13}$$

The last inequality isn't obvious, but a simple computation can be done to show that it is true. Applying $(id \otimes h \otimes id)$ to both sides and remembering that $(id \otimes h)\Delta(\cdot) = h(\cdot)1$, we have

$$Q \otimes 1 = U^*(Q \otimes 1)U$$

Now define

$$V = (Q^{\frac{1}{2}} \otimes 1)U(Q^{-\frac{1}{2}} \otimes 1)$$

Thus V is a finite dimensional representation. Multiplying both sides on the right by $(Q^{\frac{1}{2}} \otimes 1)$ shows that $Q^{\frac{1}{2}} \in \mathrm{Hom}(U, V)$. Furthermore, a simple computation plus the previous result gives $V^*V = 1$. So V is indeed a unitary representation. $\square$

In the classical case there is also an additional "natural" representation induced by the Haar measure called the right regular representation. Let $x, y \in G$ with G a compact group and let $f \in L^2(G)$ where $L^2(G)$ is the vector space of (equivalence classes) of square integrable functions with respect to the normalized Haar measure. Then the right regular representation $u_r : G \to B(L^2(G))$ is given by

$$(u_r(x)f)(y) = f(yx)$$

If (A, Δ) is a compact quantum group, we would like to construct a representation on that is "natural". We will first examine the case where $A = C(G)$ and see if we can generalize. Because the Haar state in this case is given as

$$h(f) = \int_G f(s)d\mu(s)$$

where μ is the normalized Haar measure of the underlying group, the space $L^2(G)$ is actually the Hilbert space obtained using the GNS construction with respect to the Haar state. Furthermore, if $g \in L^2(G)$, we have (in some sense)

$$(g(u_r(x)f)(y)) = g(y)f(yx) = (g \otimes id)(\Delta f)(y, x)$$

Of course, comultiplication is only defined on $C(G)$ and in general not $L^2(G)$. Furthermore, when dealing with the general case, we have to speak about the representation of A with respect to h as well as a cyclic vector ξ. In any case, this previous example gives the motivation for what the right regular representation of a general compact quantum group should be.

Theorem 2.8. *Let (A, Δ) be a compact quantum group, h its Haar state, π_h the representation of the underlying C^* algebra A obtained via the GNS construction applied to the Haar state h, H the Hilbert space of π and $\xi \in H$ the cyclic vector. Furthermore, let K be another Hilbert space and suppose that A acts faithfully and non-degenerately on K. Then there is a unitary representation U of A on $M(B_0(H) \otimes A)$ such that*

$$U(\pi_h(a)\xi \otimes k) = \Delta(a)(\xi \otimes k)$$

for all $a, a' \in A$ where $B_0(H)$ are the compact operators on H and $M(B_0(H) \otimes A)$ is the multiplier algebra of $B_0(H) \otimes A$.

Proof. We'll first show that U defines an isometry. Let $a_1, ..., a_n \in A$ and $k_1, ..., k_n \in K$. Then

$$||\sum_i \Delta(a_i)\xi \otimes k_i||^2 = \sum_{i,j} \langle \Delta(a_j^* a_i)\xi \otimes k_i, \xi \otimes k_j \rangle$$

$$= \sum_{i,j} \langle (h \otimes id)\Delta(a_j^* a_i)k_i, k_j \rangle$$

$$= \sum_{i,j} h(a_j^* a_i) \langle k_i, k_j \rangle$$

$$= || \sum_i \pi_h(a_i) \xi \otimes k_i ||^2$$

So U is indeed an isometry. Note that the image of U contains all elements of the form $\Delta(a)(1 \otimes b)(\xi \otimes k)$. By the cancelation property $\Delta(a)(1 \otimes b)$ is dense in $A \otimes A$. Because ξ is cyclic and A acts non degenerately on K, this implies that the range of U is dense. Thus, it is unitary.

We would also need to prove that $U \in M(B_0(H) \otimes A)$, but but we will skip this proof. The idea is to show that $u(x \otimes 1)$ and $u(1 \otimes x)$ are in $B_0(H) \otimes A$ for all finite rank operators x and use they fact that these are dense in $B_0(H)$.

Now we would like to show that U is in fact a representation. Let $a \in A$ and $\omega \in A'$ and define

$$\omega * a = (\omega \otimes id)\Delta(a) \in A$$

Then, by definition of U,

$$(id \otimes \omega)(U)\pi_h(a)\xi = (\omega * a)\xi$$

Thus, for $\nu \in A'$

$$(id \otimes \omega \otimes \nu)(id \otimes \Delta)(U)\pi_h(a)\xi = (id \otimes \omega * \nu)(U)\pi_h(a)\xi = ((\omega * \nu) * a)\xi_h$$

and

$$(id \otimes \omega \otimes \nu)(U_{12}U_{13})\pi_h(a)\xi = (id \otimes \omega)(W)(id \otimes \nu)(W)\pi_h(a)\xi = (\omega * (\nu * a))\xi$$

Now because ω, ν were arbitrary and by coassociativity, $(id \otimes \Delta)(W) = W_{12}W_{13}$. So U is indeed a unitary representation. □

This representation U is called the right regular representation of A.

2.3 Quantum Peter-Weyl Theorem

One of the more remarkable theorems in the theory of compact groups in the Peter-Weyl theorem, which essetianly breaks down into three parts. Each part has its analogue in the case of compact quantum groups.

Theorem 2.9. *Let ρ be unitary representation of a compact group G on some Hilbert space H. Then H splits into an orthogonal direct sum of fintie dimensional irreducible representations of G.*

The compact quantum group case is almost the same, word for word.

Theorem 2.10. *Every unitary representation of a compact quantum groups decomposes into a direct sum of finite dimensional irreducible unitary representations.*

This theorem is then used to make sense of the second theorem.The second part of the Peter-Weyl theorem is a statement about matrix coefficiants which are functions of the form $\phi : G \to \mathbb{C}, \varphi = L \circ \pi$ where π is a finite dimensional representation on a vector space $V$and L is a linear functional on $\mathrm{End}(V)$.

Theorem 2.11. *Let G be a compact group. Then the matrix coefficiants are dense in $C(G)$.*

Proof. The matrix coefficiants form a *-subalgebra of $C(G)$. Indeed it is closed under addition, multiplication and the $*$ operation because the direct sum, tensor product and dual representation are all representations respectively. Matrix coeffeciants clearly seperates points, so by Stone-Weirstrass, the theorem is shown. $\square$

The counterpart in the quantum case is the following.

Theorem 2.12. *Let $(u^\alpha)_{\alpha \in Irr}$ be a complete family of unitary representative of classes of all irreducible representations of A. And let A_0 be the subspace spanned by the matrix elements $(u^\alpha_{i,j})_{\alpha \in Irr}$ is a dense *-subalgebra of A. Furthermore the set $\{u^\alpha_{ij} \mid \alpha \in \hat{A}, 1 \leq i, j \leq dim(H_\alpha)\}$ is a basis of A*

The proof of the theorem is somewhat similar to that in the classical case with the exception of two things. The first is that we haven't introduced the contragradient representation and the second is that we can't use the Stone-Weirstrass theorem. When showing that any finite dimensional representation is equivalent to a unitary one, we used the fact that

$$Q = (id \otimes h)(UU^*) \to Q \otimes 1 = U^*(Q \otimes 1)U$$

All we did was apply $(id \otimes \Delta)$ and then $(id \otimes h \otimes id)$. In the same vein, if U, V are representations and $S : H_U \to H_V$ and we define

$$T = (id \otimes h)(V^*(S \otimes 1)U) \in B(H_U, H_V)$$

then

$$T \otimes 1 = V^*(T \otimes 1)U$$

We also have the following theorem, which we will not prove.

Theorem 2.13. *Let U be a finite dimensional representaiton of a CQG on a Hilbert space H. Then U decomposes into a direct sum of irreducible representations.*

With that in mind, we can prove the following theorem

Theorem 2.14. *Any unitary representation $U \in M(B_0(H) \otimes A)$ acting on a Hilbert space decomposes into a direction sum of finite dimensional irreducible unitary representations.*

Proof. By the previous theorem, it suffices to show that U decomposes into fintie dimensional representations. Let $S \in B_0(H)$ and let B be the subspcae of $B_0(H)$ which are operators of the form

$$T = (id \otimes h)(U^*(S \otimes 1)U)$$

By the previous remark, $T \in \mathrm{End}(U)$. Take an increasing net (x_λ) of positive elements of $B_0(H)$ such that $sup(x_\lambda) = 1$. Then we have that

$$\sup((id \otimes h)(U^*(x_\lambda \otimes 1)U)) = (id \otimes h)(U^*U) = h(1)1 = 1$$

By the definition of the strong operator topology, it following that B is a set of compact operators that acts non degenerately on H.

Pick a positive element of B. Consider the spectral projection P of T corresponding to a stricly positive eigenvalue. It also intertwines U and furthermore is finite dimensional. The existence of one such projection is enough. Indeed, we have that $H = PH \oplus (1-P)H$, U intertwines PH and $(1-P)H$ and furthermore that $U = U_P \oplus U_{1-P}$. Using transfinite induction on $(1-P)H$ gives us the required result. ☐

Remember that the classical Peter-Weyl theorem asserts if G is a compact group,the matrix coefficiants are dense in $C(G)$. The proof is really just showing that matrix coefficiants form a *-sugalgebra of $C(G)$ and using the Stone-Weirstrass theorem. The proof is very similar in the quantum case, except that we don't get to use the Stone-Weirstrass theorem, so we must be a bit more clever. Before we get started with the proof, we must introduce the concept of the contragradient reprsentation.

Definition 2.15. Let H^* be the dual space of H and let $t : B(H) \to B(H^*)$ be the map that sends an operator to it's dual. The contragradient representation to a representation U is defined as

$$U^c = (t \otimes 1)(U^{-1}) \in B(H^*) \otimes A$$

Note that the definition of the adjoint depends on the choice of matrix units, but different choices give equivalent reprsentation. Furthermore, in the quantum case, the adjoint of a unitary representation isn't necessarily a unitary representation. However if U is irreducible, then U^* will also be irreducible. We can now prove the second part of the quantum Peter-Weyl theorem.

Theorem 2.16. *Let $G = (A, \Delta)$ be a compact quantum group. Then $Pol(G) = \{u^{\alpha}_{ij} : \alpha \in Irr(\hat{G}), 1 \leq i, j \leq n_{\alpha}\}$ is a dense *-subalgebra of A.*

Proof. Pol(G) is closed under addition, multiplication and the * operation because the direct sum, tensor product, and adjoint represenations of reprsentations are all representaitons respectively.

The density is the really tricky part. First we show that the set

$$(\omega \otimes id)(U_R)|\omega \in B_0(H)$$

is dense in A, where U_R is the right regular representation. Let $a, b \in A$ and for $\xi_1, \xi_2 \in H_h$ we define the linear functional ω_{ξ_1,ξ_2} as

$$\omega_{\xi_1,\xi_2}(x) = \langle x\xi_1, \xi_2\rangle \forall x \in B(H_h)$$

We have

$$\langle((\omega_{a\xi_0,b\xi_0} \otimes 1)U)k_1, k_2\rangle = \langle U(a\xi_0 \otimes k_1), b\xi_0 \otimes k_2\rangle$$

$$= \langle\Delta(a)(\xi_0 \otimes k_1), b\xi_0 \otimes k_2\rangle$$

$$= \langle(h \otimes id)((b^* \otimes 1)\Delta(a))k_1, k_2\rangle$$

for any $k_1, k_2 \in K$. Thus

$$(\omega_{a\xi_0,b\xi_0} \otimes 1)U = (h \otimes id)((b^* \otimes 1)\Delta(a))$$

By the axioms, we have that $(b^* \otimes id)\Delta(a)$ is dense in $A \otimes A$, so clearly the set $(\omega_{a\xi_0,b\xi_0} \otimes 1)U$ is dense in A. Now since linear functionals of the form $\omega_{a\xi_0,b\xi_0}$ are dense in $B_0'(H)$, it follows that $(\omega \otimes id)(U_R)|\omega \in B_0(H)$ is dense in A.

□

We also need the following proof, which I will not prove. It states that any irreducible unitary representation is contained in the right regular representation. Once we have that, we can go on to prove the main result. Let $L^2(G) = H_h$ be the Hilbert space induced by the Haar state. By the first quantum Peter-Weyl theorem, $L^2(G)$ and U decompose into a direct sum

$$L^2(G) = \bigoplus_{\alpha \in \mathrm{Irr}(G)} H_{\alpha} \text{ and } U = \bigoplus_{\alpha \in \mathrm{Irr}(G)} u^{\alpha}$$

Let $n(\alpha) = \dim H_{\alpha}$ and $\{\xi^{\alpha}_1, ..., \xi^{\alpha}_{n(\alpha)}\}$ be an orthonormal basis of H_{α}. Define $\omega^{\alpha,\beta}_{i,j} \in B_0(L^2(G))$ by $\omega^{\alpha,\beta}_{i,j}(x) = \langle x\xi^{\alpha}_i, \xi^{\beta}_j\rangle \forall x \in B_0(L^2(G))$. Then we have

$$(\omega_{i,j}^{\alpha,\beta} \otimes id)U = \begin{cases} (\omega_{i,j}^{\alpha} \otimes id)(u^{\alpha}) & \text{if } \alpha = \beta \\ 0 \text{ if } \alpha \neq \beta \end{cases}$$

by the first part of the theorem, we have that $(\omega_{i,j}^{\alpha,\beta} \otimes id)U$ is dense in A. But $(\omega_{i,j}^{\alpha,\beta} \otimes id)u^{\alpha}$ are exactly the matrix coefficiants.

In the representation theory of compact groups, any representation defines a contragradient, or dual, representation. This also has an analogue in the case of compact quantum groups.
Note that if U is unitary, and is written as a matrix with respect to an orthonormal basis of H_U, then $U^c = \{u_{ij}^*\}$. It is not trivial to prove that U^c is in fact a representation, that is to say invertible and is compatible with the comultiplication. For now, we'll take for granted that that is in fact the case.

Let U be an irreducible finite dimensional unitary representation of a compact quantum group. Define the operators

$$Q_l = (id \otimes h)(U^c U^{c*}), \quad Q_r = (id \otimes h)(U^c U^{c*}) \in B(H^*)$$

They play an important role in the third part of the quantum Peter-Weyl theorem.

Theorem 2.17. *Q_l and Q_r are invertible operators.*

Proof. First note that $Q_l \neq 0$ because

$$Tr(Q_l) = (Tr \otimes h)(U^c U^{c*}) = (Tr \otimes h)(UU^*) = \text{dim} U$$

where I used the fact that the trace of a matrix and its transpose is equal. Now let $p \in B(H^*)$ be the projection onto the kernel of Q_l. By an earlier lemma, we have

$$Q_l \otimes 1 = U^c(Q_l \otimes 1)U^{c*}$$

Thus

$$(p \otimes 1)U^c(Q_l \otimes 1)U^{c*}(p \otimes 1) = 0$$

$$((Q_l^{1/2} \otimes 1)U^{c*}(p \otimes 1))^*(Q_l^{1/2} \otimes 1)U^c(p \otimes 1) = 0$$

Hence $(Q_l^{1/2} \otimes 1)(t \otimes 1)U(p \otimes 1) = 0$. So

$$(t(p) \otimes 1)U(t(Q_l^{1/2}) \otimes 1) = 0$$

There is a small lemma that I haven't proved that says that if U is an irreducible representation U, then for any non-zero $X, Y \ inB(H)$, $(X \otimes$

1)$U(Y \otimes 1) \neq 0$. Since in this case, $Q_l \neq 0$, then this means $p = 0$ and so Q_l is invertible. Similarly, one show that Q_r is invertible

□

Clearly the operators defined above are positive by the Haar state. Now suppose that U is a unitary fintie dimensional representation of a compact quantum group. The following lemma gives us important information about these operators.

Theorem 2.18. *Let Q_r and Q_l be defined as above. Then*

i) $t(Q_l) \in Hom(U^{cc}, U)$ and $t(Q_r) \in Hom(U, U^{cc})$
ii) Q_rQ_l is a scalar multiple of the identity operator
iii) $Hom(U, U^{cc})$ is one dimensional and spanned by a unique positive invertible operator F such that $Tr(F) = Tr(F^{-1})$

Proof. i) As we previously mentioned, by definition of Q_r, we have

$$U^{c*}(Q_r \otimes 1)U^c = (Q_r \otimes 1)$$

By multiplying both sides by $(U^c)^{-1}$ we get

$$U^{c*}(Q_r \otimes 1) = (Q_r \otimes 1)(U_c)^{-1}$$

$$(t \otimes 1)(U)(Q_r \otimes 1) = (Q_r \otimes 1)(U^c)^{-1}$$

where we have crucially used the fact that U is unitary, so $U^{-1} = U^*$. Now applying $(t \otimes 1)$ to both sides and remembering that the transpose is antimultiplicative and $t^2 = 1$, we get

$$(t(Q_r) \otimes 1)U = U^{cc}(t(Q_r) \otimes 1)$$

So indeed, we have $t(Q_r) \in \mathrm{Hom}(U, U^{cc})$. Showing that $t(Q_l) \in \mathrm{Hom}(U^{cc}, U)$ is done in a similar manner.

ii) We have

$$(t(Q_r) \otimes 1)U = U^{cc}(t(Q_r) \otimes 1)$$

Multiplying by $(t(Q_r) \otimes 1)$ on the left gives

$$(t(Q_l) \otimes 1)(t(Q_r) \otimes 1)U = (t(Q_l) \otimes 1)U^{cc}(t(Q_r) \otimes 1)$$

Since $t(Q_l) \in \mathrm{Hom}(U^{cc}, U)$ we can replace the first part of the left hand side with $U(t(Q_l) \otimes 1)$. thus

$$(t(Q_l)t(Q_r) \otimes 1)U = U(t(Q_l)t(Q_r)) \otimes 1)$$

So $t(Q_l)t(Q_r) \in End(U)$ and since U is irreducible, $t(Q_l)t(Q_r)$ is a scalar multiple of the identity.

iii) $t(Q_r)$ is one such positive operator. Suppose there was another linearl idependant operator $R \in \mathrm{Hom}(U, U^{cc})$. Then using the same reasoning as in ii), $t(Q_l)R$ would have to a scalar multiple of the identity, which is to say a sclar multiple of $t(Q_r)$. Clearly, the condition $\mathrm{Tr}(F) = \mathrm{Tr}(F^{-1})$ fixes this operator.

□

We would like to have a relation between this F, F^{-1} and Q_r, Q_l. We know that they are scalar multiples of each other. As we previously proved that $\mathrm{Tr}t(Q_r) = \mathrm{Tr}t(Q_l) = \dim_U$, and $j(Q_l)j(Q_r) = \lambda$ for some scalalr λ. Remembering that $\mathrm{Tr}F = \mathrm{Tr}F^{-1}$, then we have

$$Q_r = \frac{\dim U}{\mathrm{Tr}F} t(F) \quad Q_l = \frac{\dim U}{\mathrm{Tr}F} t(F^{-1})$$

We're now ready to prove the quantum Peter-Weyl orthogonality relations. Let U be a irreducible finite dimensional unitary representation. We've prove that for any $T \in B(H)$, we have $(id \otimes h)(U^*(T \otimes 1)U) \in \mathrm{End}(U)$. Because U is irreducible, we have

$$\lambda Id_{B(H_U)} = (id \otimes h)(U(T \otimes 1)U^*)$$

Taking the trace on both sides and using the fact that $Tr(XYZ) = Tr(YZX) = Tr(t(X)t(Z)t(Y)$

$$\lambda \cdot \dim U = (\mathrm{Tr} \otimes h)(U(T \otimes 1)U^*)$$

$$= (\mathrm{Tr} \otimes h)((t \otimes 1)U(t \otimes 1)U^*(t(T) \otimes 1)$$

Because U is unitary, we have $U = (U^*)^{-1}$, and $U^* = U^{-1}$ so

$$\lambda \cdot \dim U = (\mathrm{Tr} \otimes h)(U^{c*}U^c(t(T) \otimes 1))$$

$$= \mathrm{Tr}(Q(r)t(T)) = \mathrm{Tr}(t(Q_r)T)$$

So

$$\lambda = \mathrm{Tr}\frac{t(Q_r)T}{\dim U} = \mathrm{Tr}\frac{FT}{\mathrm{Tr}F}$$

from which we deduce

$$\mathrm{Tr}\frac{FT}{\mathrm{Tr}F}\, Id = (id \otimes h)(U(T \otimes 1)U^*)$$

Similarly, we can show that

$$\mathrm{Tr}\frac{F^{-1}T}{\mathrm{Tr}F}\ Id = (id \otimes h)(U^*(T \otimes 1)U), \forall T \in B(H_U)$$

Set the operator $T = E_{lj}$ where E_{lj} are the canonical matrix units. Thus, if we write the representation U in matrix form with repsect to an orthonormal basis, we get

$$h(u_{kl}u_{ij}^*) = \frac{\delta_{ki}F_{lj}}{\mathrm{TrF}} \qquad h(u_{ij}^*u_{kl}) = \frac{\delta_{jl}(F^{-1})_{ki}}{\mathrm{TrF}}$$

So now that we've tackled the quantum Petet-Weyl theory for compact quantum groups, we must ask: what is next? Naturally, we look back to the classical case. For compact groups, what comes next is the Tannak-Krein duality theorem.

3 Woronowicz-Tannaka-Krein Duality

3.1 Classical Tannaka-Krein

Let G be a locally compact abelian topological group. A character of G is a continuous homomorphism $\chi : G \to \mathbb{T}$ where $\mathbb{T}$ is the circle group. The set of all such characters can be made into a topological group, called the dual group and is denoted $G^\wedge$. The group operation is defined by pointwise multiplication of characters and the topology is that of uniform convergence on compact subsets. The dual group has its own dual $(G^\wedge)^\wedge$, called the bidual. It is obvious that the evaluation map at $g \in G$ defined by $\mathrm{ev}_g(\chi) = \chi(g)$ is a character on $G^\wedge$. The Pontryagin duality theorem states that the evaluation map $\mathrm{ev} : G \to (G^\wedge)^\wedge$, $\mathrm{ev}(g) = \mathrm{ev}_g$ is an isomorphism of topological groups.

The idea of Tannaka-Krein duality is to extend the notion of Pontryagin duality to non-abelian compact topological groups. Characters are in fact one dimensional unitary representations of G. If we allow G to be noncommutative, one would then expect that its dual object be the set of equivalence classes of irreducible unitary representations with the tensor product taking the place of the product of characters. However, irreducible representations of G fail to be a group, due to the fact that the tensor products of two irreducible representations isn't necessarily irreducible.

As it turns out, one needs to consider the tensor category $\Pi(G)$ of a all fintie dimensional representations of G, where the morphisms are the intertwiners of representations. We also have the forgetful functor $F : \Pi(G) \to \mathrm{Vect}_{\mathbb{C}}$ where $\mathrm{Vect}_{\mathbb{C}}$ denotes the category of fintie dimensional complex vector spaces. Recall that a *natural transformation* $\tau : F \to F$ is a family of maps $\tau_V : V \to V$ indexed by $V \in \Pi(G)$, called the component at V, such that the square

$$\begin{tikzcd}
F(V) \arrow[r, "F(\tau_v)"] \arrow[d, "F(h)"'] & F(V) \arrow[d, "F(h)"] \\
F(W) \arrow[r, "F(\tau_w)"'] & F(W)
\end{tikzcd}$$

commutes for all morphisms $h : V \to W$ of representations. We then put a topology on the set of natural transformations by letting it be the coarsest topology possible such that each of the projections $\mathrm{End}(F) \to \mathrm{End}(V)$ given by $\tau \to \tau_V$ are continuous $\forall\ V \in \Pi(G)$. There is the *conjugation operator* $\mathrm{End}(F) \to \mathrm{End}(F)$ with $\overline{\tau}$ defined as $\overline{\tau}_V(x) = \overline{\tau_{\overline{V}}(\overline{x})}$, where $\overline{V}$ is the conjugate vector space of V. A natural transformation is said to be *self-conjugate* if $\overline{\tau} = \tau$. Furthermore, a natural transformation is said to be *tensor-preserving* if it is the identity of the trivial representation of G and $\tau_{V \otimes W} = \tau_V \otimes \tau_W$.

The set $\mathcal{T}(G)$ of all tensor-preserving, self-conjugate natural transformations of F is a closed subset of $\mathrm{End}(F)$.In addition $\mathcal{T}(G)$ is in fact a compact group whenever G is, with the group operation being the obvious one. Each element $g \in G$ gives such a natural transformation $\pi(g) \in \mathcal{T}(G)$ whose V component is $\pi_V(g) : V \to V$. Tannaka's theorem states that the map $\pi : G \to \mathcal{T}(G)$ is a isomorphisms of topological groups.

Conversly, Krein's theorem states which categories can arise as the dual object of a compact group. Let Π be a category of fintie dimensional vector spaces endowed with a tensor product and involution. There are three necessary and sufficient conditions that make Π a dual object of a compact group G. First, there exists a unique up to isomorphism object I such that $I \otimes A \approx A$ for all objects A. Furthermore, every object A can be decomposed in a sum of minimal objects. Finally, if A and B are two minimal objects, then the space $\mathrm{Hom}_\Pi(A, B)$ are one dimensional when they are isomorphic, or it is zero.

Let $\mathcal{C}$ be a category as described above. with the forgetful functor F. Define a group $G \subseteq \Pi_X B(F(X))$ and a representation $\Pi_X : \eta \to \eta_X$ as before. Now each object $X \in \mathcal{C}$ and functional $\omega \in B(F(X))^*$ defines a matrix coefficiant

$$f_\omega^X : G \to \mathbb{C},\ \ f_\omega^X = \omega \circ \pi_X$$

As we've seen before, functions of this form are in fact a *-subalgebra of

$C(G)$. We can then re-use our entire knowledge (Stone-Weirstrass, Peter-Weyl, etc..) to better understand how this category works. One uses this to show that $\mathcal{C} \to \mathrm{Rep}_{f.d.}(G)$, $X \to \pi_X$ is essentialy surjective on objects.

Now one wishes to extend this duality to compact quantum groups. We appear to have almost everything we need. The category of representations of a compact quantum group certainly form a category with intertwiners as morphisms. It is semi-simple with every representation being the direct sum of fintie dimensional irreducible represenations. Furthermore, there is a distinguished unit object, the trivial representation on $\mathbb{C}$, and a natural tensor product. So what more do we need to prove the Woronowicz Tannak-Krein duality? The classical theorem uses *-algebras quite heavily, as we've just seen. Seeing as quantum groups are a generalized version of groups, we actually need a more complicated notion of a *-algebra called a Hopf *-algebra.

3.2 Hopf *-algebras

Definition 3.1. A Hopf *-algebra is a unital *- algebra equipped with three aditional structures.
i) A unital * homomorphism $\Delta : A \to A \odot A$ such that $(\Delta \odot id)\Delta = (id \odot \Delta)\Delta$

ii) Linear maps $\epsilon : A \to \mathbb{C}$, $S : A \to A$ such that

$$(\epsilon \otimes id)\Delta = (id \otimes \epsilon)\Delta = id, \quad and \quad m(S \otimes id)\Delta = m(id \otimes S)\Delta = \epsilon(\cdot)1_A$$

for all $a \in A$ where $m : A \odot A \to A$ is the multiplication map and $\odot$ is the algebraic tensor product.

The theory of Hopf *-algebras is already quite established and from the axioms, one can deduce some of the important properties of the maps ϵ and S, called the counit and anitpode respectively. For example, ϵ is always a *-homomoprhism and S is an antihomomorphism. We won't prove any of these things, but they will be obvious when working with concrete examples.

Theorem 3.2. *For any compact quantum group G, (Pol(G), Δ_0) is a Hopf *-algebra where Δ_0 is the restriction of the comultiplication to Pol(G).*

Proof. For any finite dimensional representation of U of G, we define the counit and antipode as

$$(id \otimes \epsilon)U = 1 \quad and \quad (id \otimes S)U = U^*$$

To show that these maps exists, choose equivalence classes of irreducible unitary representations U_α of U. The matrix coefficiants u^{α}_{ij}, with respect to a fixed orthonormal basis of H_{U_α}, form a basis of Pol(G). Thus we can define on these matrix coefficants $\epsilon(u^\alpha_{ij}) = \delta_{ij}$ and $S(u^\alpha_{ij}) = u^{\alpha *}_{ji}$. Since every finite dimensional representation U can be decomposed into a direct sum of irreducible unitray representations, it is not hard to see that we in fact have $(id \otimes \epsilon)U = 1$ and $(id \otimes S)U = U^*$.

By applying $(id \otimes \epsilon \otimes id)$ to $(id \otimes \Delta)U = U_{12}U_{13}$ we get $(\epsilon \otimes id)\Delta$ and by applying $(id \otimes id \otimes \epsilon)$ we get $(id \otimes \epsilon)\Delta = id$. Applying $(id \otimes m)(id \otimes S \otimes id)$ to $(id \otimes \Delta)U$, we get

$$(id \otimes m)(id \otimes S \otimes id)(id \otimes \Delta)U = (id \otimes m)(U^*_{12}U_{13}) = U^*U = (id \otimes \epsilon(\cdot)1_A)(U)$$

Therefore $m(S \otimes id)\Delta = \epsilon(\cdot)1_A$. Similarly $m(id \otimes S) = \epsilon(\cdot)1_A$. ☐

Hopf algebras, much like quantum groups, also have a representation or corepresentation theory.

Definition 3.3. A corepresentation of a Hopf *-algebra $(\mathcal{A}, \Delta)$ on a Hilbert space H is a linear map $\delta : H \to H \otimes A$ such that

$$(\delta \otimes id)\delta = (id \otimes \Delta)\delta \quad and \quad (id \otimes \epsilon)\delta = id$$

The corepresentation is said to be unitary if

$$\langle \delta(\xi), \delta(\zeta) \rangle = \langle \xi, \zeta \rangle \ 1_A \, \forall a, b \in H$$

where the inner product on $H \otimes \mathcal{A}$ is defined as $\langle \xi \otimes a, \zeta \otimes b \rangle = \langle \xi, \zeta \rangle b^* a \in \mathcal{A}$. A subspace $K \subset H$ is called invariant under δ if $\delta(K) \subset K \otimes \mathcal{A}$. The corepresentation is said to be irreducible if H admits no proper invariant subspaces

The condition that $(\delta \otimes id)\delta = (id \otimes \Delta)\delta$ is a compatibility relation between the corepresentation and the comultiplication. It resembles the compatibility condition for a representation of a compact quantum group and that is indeed the case. In fact for every fintie dimensional corepresentation $\delta : H \to H \otimes \mathcal{A}$ there is a uniquely determined element $U \in B(H) \otimes \mathcal{A}$ such that $\delta(\xi) = U(\xi \otimes 1)$. Then we have

$$(id \otimes \Delta)\delta(\xi) = (\delta \otimes id)\delta(\xi)$$

$$(id \otimes \Delta)U(\xi \otimes 1) = (\delta \otimes id)U(\xi \otimes 1)$$

$$(id \otimes \Delta)U(\xi \otimes 1) = U_{12}U_{13}(\xi \otimes 1)$$

Also, we have

$$(id \otimes \epsilon)\delta(\xi) = (id \otimes \epsilon)U(\xi \otimes 1) = id$$

So $(id \otimes \Delta)U = U_{12}U_{13}$. Conversly, any element $U \in B(H) \otimes \mathcal{A}$ that satisfies these properties defines a corepresentation. By applying $id \otimes m(id \otimes S)$ and $id \otimes m(S \otimes id)$ to $(id \otimes \Delta)U = U_{12}U_{13}$ and remembering the axioms of Hopf *-algebras, we get

$$(id \otimes m(S \otimes id))(id \otimes \Delta)U = (id \otimes m(S \otimes id))(U_{12}U_{13})$$

$$id \otimes \epsilon(U) = (id \otimes m(S \otimes id))(U_{12}U_{13})$$

So U is invertible and we have $id \otimes S(U) = U^{-1}$. It is easy to see that that δ being unitary is equivalent to $U^*U = 1$, and since U is invertible, this is equivalent to U being unitary. In particular, a finite dimensional corepresentation of $(\mathrm{Pol}(G), \Delta)$ is the same as a finite dimensional representation of G. Furthermore irreducibility of δ means that there is no non scalar operator $T \in B(H)$ such that $T \otimes 1$ commutes with U. This is because any eigenspace of an such an operator would be an invariant subspace of the operator part of U, leading to a contradiction. It can also be shown that if U is invariant in the sense of compact quantum groups, then δ is irreducible.

As previously mentionned the ultimate goal is to relate Hopf *-algebras generated by matrix coefficiants to compact quantum groups. This next theorem, as we will see, is actually equivalent to the cancelation property of a compact quantum group

Theorem 3.4. *For any corepresentation $\delta : H \to H \otimes \mathcal{A}$, we have $H \otimes \mathcal{A} = \delta(H) \otimes (1 \otimes \mathcal{A})$*

Proof. Consider the map $r(\xi \otimes a)\delta(\xi)(1 \otimes a)$. If we think of corepresentations as representations, then we have $r(\xi \otimes a) = U(\xi \otimes 1)(1 \otimes a)$. Since every corepresentation of a Hopf *-algebra has an inverse, we can easily find an inverse of r, which means that its surjective. This proves the claim.

□

We can also use this theorem to show that if δ is a unitary corepresentation and $K \subset H$ is an invariant subspace, then $K^\perp$ is also invariant. Take $\zeta \in K$, $\xi \in K^\perp$, and $a \in \mathcal{A}$ then we have

$$\langle \delta(\xi), \delta(\zeta)(id \otimes a) \rangle = \langle \xi, \zeta \rangle a^* = 0$$

Now since $\delta(K)(1 \otimes \mathcal{A}) = K \otimes \mathcal{A}$, we have $\langle \delta(\xi), \zeta \otimes 1 \rangle = 0$ for all $\zeta \in K$, which means $\delta(\xi) = K^\perp \otimes \mathcal{A}$. If follows that any finite dimensional unitary corepresentation decomposes into a direct sum of finite dimensional irreducible unitary corepresentation.

The following proof will be essentiall when discussing Tannaka-Krein duality for compact quantum groups.

Theorem 3.5. *Let $(\mathcal{A}, \Delta)$ be a Hopf *-algebra such that $\mathcal{A}$ is generated by matrix coeffeciants of finite dimensional corepresentations. Then $(\mathcal{A}, \Delta) = (Pol(G), \Delta)$ for some compact quantum group G.*

Proof. The proof can be split into 3 parts. It is fairly involved and rather lengthy. We present the main ideas and leave the technicl details aside. The first step is to construct a faithful state h on $\mathcal{A}$ that will play the role of the Haar state.

Step 1: There exists a unique linear functional $h : \mathcal{A} \to \mathbb{C}$ satysfying the same properties as the Haar state. Namely, $h(1) = 1, (id \otimes h)\Delta(a) = (id \otimes h)\Delta(a) = h(a)1$ Let $\{U_\alpha\}_{\alpha \in Irr(\mathcal{A})}$ be a set of pairwise non equivalent representatives of equivalences classes of finite dimensional irreducible unitary corepresentations of $\mathcal{A}$. Since any finite dimensional corepresentation decomposes into a direct sum of irreducible unitary corepresentations, matrix coefficients of U_α form a basis in $\mathcal{A}$. Define a linear functional h such that $h(1) = 1$ and $(id \otimes h)(U_\alpha) = 0$ if U_α is not 1. Since $(id \otimes \Delta)(U_\alpha) = (U_\alpha)_{12}(U_\alpha)_{13}$, h has the desired properties.

Step 2: For every α there exists a positive invertible operator $Q_\alpha \in B(\bar{H}_\alpha)$ such that

$$(id \otimes h)(U_\alpha^*(T \otimes 1)U_\beta) = \delta_{\alpha,\beta} \frac{Tr(T \cdot t(Q_\alpha))}{\mathrm{Tr}(t(Q_\alpha))} \;\forall\, T \in B(H_{U_\beta}, H_{U_\alpha})$$

Note that these are the same orthogonality relations for compact quantum groups. It is important to note that, when we proved these relations a few weeks ago, we didn't use the positivity of the Haar state. This means that we can use that proof, with $Q_\alpha = (id \otimes h)(U_\alpha^{*c} U_\alpha^c)$ and $U_\alpha^c = (t \otimes 1)U_\alpha^{-1} = (t \otimes S)U_\alpha$. All that is left to show is that Q_α is positive and invertible for every α. We skip this proof, because it is not terribly enlightening.

Step 3: $h(a^*a) > 0$ for every non-zero $a \in \mathcal{A}$. Let u_{ij}^α be the matrix coeffeciants of U_α written in bases such that the operators $t(Q_\alpha)$ are diagonal. Just as in the quantum group case, we can take an operator $T = E_{ij}$ to be a matrix units and we obtain the relation

$$h(u_{kl}^\alpha u_{ij}^{\beta *}) = \delta_{\alpha\beta}\delta_{ki} \frac{t(Q_\alpha)_{lj}}{\mathrm{Tr}(t(Q_\alpha))}$$

It is obvious that matrix coeffeciants for an orthonormal basis with repsect to the sequilinear form $\langle a|b \rangle = h(b^*a)$ and furthermore $\langle u_{kl}^\alpha | u_{ij}^\alpha \rangle$ is strickly positive.

We now have everything we need to prove the theorem. $\mathcal{A}$ acts on the pre-Hilbert space $\mathcal{A}$ via left multiplication with an inner product $\langle a|b\rangle = h(b^*a)$. This is a representation by bounded operators, since $\mathcal{A}$ is generated by matrix coefficiants of unitary corepresentations. A similar argument shows that there exists a envolopping C^*-algebra A. Thus this representation extends to a faithful representation on the Hilbert space completion of $\mathcal{A}$, which can be viewed as a subalgebra of A. This should remind one of the GNS construction for an arbitrary C^*-algebra . However, the key difference is that since $h(a^*a) > 0$, we don't have to quotient our space, which is why $\mathcal{A}$ is a genuine subalgebra of A. By the universal property, the map Δ extends to a unital *-homomorphism $\Delta : A \to A \otimes A$.

By a previous theorem (with $\delta = \Delta$), we have $(1 \otimes \mathcal{A})\Delta(\mathcal{A}) = \mathcal{A} \otimes \mathcal{A}$ and similarly $(\mathcal{A} \otimes 1)\Delta(\mathcal{A}) = \mathcal{A} \otimes \mathcal{A}$. It follows that (A, Δ) has the cancelation property and is thus a compact quantum group. Evidently, we have $\mathcal{A} \subset \mathrm{Pol}(A)$. But since $\mathcal{A}$ is dense in $\mathrm{Pol}(A)$ and spanned by matrix coefficiants of irreducible unitary representations, by the orthogonality relations we conclude that $\mathcal{A} = \mathrm{Pol}(G)$. ☐

3.3 C*-Tensor Categories

As we saw, Krein's theorem tells us which categories can arise as dual objects to compact groups. Thus, for the quantum compact group case, our first task is to define what categories we will be working with.

Definition 3.6. A category $\mathcal{C}$ is called a C^*-category if for every object U, V, W, $\mathrm{Hom}(U, V)$ is a complex Banach space, the composition $(S, T) \to ST$ is bilinear, and $\|ST\| \leq \|S\|\|T\|$ for every $S \in \mathrm{Hom}(V, W), T \in \mathrm{Hom}(U, V)$. Furthermore, there exists a antilinear involutive contravariant functor $* : \mathcal{C} \to \mathcal{C}$ which is the identity on objects, such that if $S \in \mathrm{Hom}(U, V)$ then $S^* \in \mathrm{Hom}(V, U)$, and $\|T^*T\| = \|T\|^2$.

One can add even more structure to a C^* category, the most obvious of which is to add a tensor structure that is compatible with all the operations above.

Definition 3.7. A C^* category $\mathcal{C}$ is called a C^* tensor category if it is equipped with a bilinear bifunctor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$, a unit object $\mathbb{1}$, natural unitary isomorphisms $\alpha_{U,V,W} : (U \otimes V) \otimes W \to U \otimes (V \otimes W)$, $\lambda_U : \mathbb{1} \otimes U \to U$, and $\rho_U : U \otimes \mathbb{1} \to U$ such that the pentagon diagram

$$\begin{tikzcd}
 & ((U \otimes V) \otimes W) \otimes X \arrow[dl, "\alpha_{1,2,3} \otimes 1_X"'] \arrow[dr, "\alpha_{12,3,4}"] & \\
(U \otimes (V \otimes W)) \otimes X \arrow[d, "\alpha_{1,23,4}"'] & & (U \otimes V) \otimes (W \otimes X) \arrow[d, "\alpha_{1,2,34}"] \\
U \otimes ((V \otimes W) \otimes X) \arrow[rr, "1_U \otimes \alpha_{2,3,4}"'] & & U \otimes (V \otimes (W \otimes X))
\end{tikzcd}$$

commutes and such that the triangle diagram

$$\begin{tikzcd}
(U \otimes \mathbb{1}) \otimes V \arrow[rr, "\alpha_{1,2,3}"] \arrow[dr, "\rho_U \otimes 1_V"'] & & U \otimes (\mathbb{1} \otimes V) \arrow[dl, "1_U \otimes \lambda"] \\
 & U \otimes V &
\end{tikzcd}$$

commutes. Finally, we require that for any morphisms S and T the equality $(S \otimes T)^* = S^* \otimes T^*$ holds.

The conditions imposed on tensor categories arise in a very natural way. Indeed, the unit object $\mathbb{1}$ and natural isomorphisms $\alpha_{U,V,W}$, called the associativty morphisms, simply state that the tensor product turns $\mathcal{C}$ into a monoid. In fact, tensor categories are often called monoidal categories for exactly this reasons. For our purposes, we still need to impose extra conditions on the categories we are working with.

We assume that our C^* tensor category $\mathcal{C}$ is closed under finite direct sum, and that $\mathrm{End}\mathbb{1} = \mathbb{C}1$ which is to say that the unit object is simple. Furthermore, we assume that our category is semisimple, that is, one for which every object is the direct sum of simple objects. A category is called strict if $(U \otimes V) \otimes W = U \otimes (V \otimes W), \mathbb{1} \otimes U = U \otimes \mathbb{1} = U$, and α, λ, and ρ are the identity morphisms. By a results of Mac Lane, any monoidal category can be strictified. Thus without any loss of generality we will assume that $\mathcal{C}$ is strict.Now that we've defined the categories of interest, we now need to define the functors that will preserve all of the internal structure.

Definition 3.8. Suppose we have two C^* tensor categories $\mathcal{C}$ and $\mathcal{C}'$. A tensor functor $F : \mathcal{C} \to \mathcal{C}'$ is called a tensor functor if it is linear on morphisms, along with an isomorphism $F_0 : \mathbb{1}' \to F(\mathbb{1})$ and natural isomorphisms $F_2 : F(U) \otimes F(V) \to F(U \otimes V)$ such that the diagram

$$\begin{tikzcd}
(F(U) \otimes F(V)) \otimes F(W) \arrow[r, "F_2 \otimes 1_{F(W)}"] \arrow[d, "\alpha'_{1,2,3}"'] & F(U \otimes V) \otimes F(W) \arrow[r, "F_2"] & F((U \otimes V) \otimes W) \arrow[d, "F(\alpha_{1,2,3})"] \\
F(U) \otimes (F(V) \otimes F(W)) \arrow[r, "1_{F(U)} \otimes F_2"'] & F(U) \otimes F(V \otimes W) \arrow[r, "F_2"'] & F(U \otimes (V \otimes W))
\end{tikzcd}$$

and the diagrams

$$\begin{tikzcd}
F(\mathbb{1}) \otimes F(U) \arrow[r, "F_2"] & F(\mathbb{1} \otimes U) \arrow[d, "F(\lambda)"] \\
\mathbb{1} \otimes F(U) \arrow[u, "F_0 \otimes 1_{F(U)}"] \arrow[r, "\lambda'"'] & F(U)
\end{tikzcd}
\qquad
\begin{tikzcd}
F(U) \otimes F(\mathbb{1}) \arrow[r, "F_2"] & F(U \otimes \mathbb{1}) \arrow[d, "F(\rho')"] \\
F(U) \otimes \mathbb{1}' \arrow[u, "1_{F(U)} \otimes F_0"] \arrow[r, "\rho'"'] & F(U)
\end{tikzcd}$$

commutes. Futhermore, we say that a tensor functor is unitary if F_0 and F_2 are unitary and $F(T)^* = F(T^*)$ for all morphisms T. It should be noted that although we've assumed the existence of F_0, one can simply assume $F(\mathbb{1}) \cong \mathbb{1}'$. In any case, for our purposes, there will always be an obvious choice of F_0.

Definition 3.9. A natural isomorphism $\eta : F \to G$ between two tensor functors $F : \mathcal{C} \to \mathcal{C}'$, $G : \mathcal{C} \to \mathcal{C}'$ is said to be monoidal if the diagrams

$$\begin{tikzcd}
F(U) \otimes F(V) \arrow[r, "F_2"] \arrow[d, "\eta \otimes \eta"'] & F(U \otimes V) \arrow[d, "\eta"] \\
G(U) \otimes G(V) \arrow[r, "G_2"'] & G(U \otimes V)
\end{tikzcd}
\qquad
\begin{tikzcd}
 & \mathbb{1}' \arrow[dl, "F_0"'] \arrow[dr, "G_0"] & \\
F(\mathbb{1}) \arrow[rr, "\eta"'] & & G(\mathbb{1})
\end{tikzcd}$$

commute.

Definition 3.10. Two C^* tensor categories $\mathcal{C}$ and $\mathcal{C}'$ are said to be monoidally equivalent if there exists tensor functors $F : \mathcal{C} \to \mathcal{C}'$ and $G : \mathcal{C}' \to \mathcal{C}$ such that FG and GF are monoidally isomorphic to the identity functors. If the isomorphisms $FG \cong Id_{\mathcal{C}'}$ and $GF \cong Id_{\mathcal{C}}$ are in addition unitary, we say that $\mathcal{C}$ and $\mathcal{C}'$ are unitarly monoidally equivalent.

At first glance it would seem that we have everything we need to tackle the Woronowicz-Tannaka-Krein duality. Let $\mathrm{Rep}_{u,fd}(G)$ be the category of finite dimensional unitary representations of a compact quantum group G. It is obviously a strict C^* tensor category, where the morphisms are intertwiners of representations, the tensor product is the tensor product of representations and the unit object is the trivial representation. Furthermore, since every finite dimensional representaiton can be decomposed into a direct sum of finite dimensional irreducible unitary representations, $\mathrm{Rep}_{u,fd}(G)$ is semisimple. However, a crucial component in proving all of these things has been U^c, the contragradient represention. We therefor need a notion of duality in a category.

Definition 3.11. An object $\overline{U}$ in a strict C^* tensor category $\mathcal{C}$ is said to be a conjugate of U if there exists morphisms $R : \mathbb{1} \to \overline{U} \otimes U$ and $\overline{R} : \mathbb{1} \to U \otimes \overline{U}$ such that

$$U \xrightarrow{Id \otimes R} U \otimes \overline{U} \otimes U \xrightarrow{\overline{R}^* \otimes Id} U \quad \text{and} \quad \overline{U} \xrightarrow{Id \otimes \overline{R}} \overline{U} \otimes U \otimes \overline{U} \xrightarrow{R^* \otimes Id} \overline{U}$$

are the identity morphisms. The identities $(\overline{R}^* \otimes Id)(Id \otimes R) = Id$ and $(R^* \otimes Id)(Id \otimes \overline{R})$ are called the conjugate equations. If every object in $\mathcal{C}$ has a conjugate, $\mathcal{C}$ is said to be a rigid C^* tensor category.

Note that due to the symmetry, if $\overline{U}$ is a conjugate of U, then U is as conjugate of $\overline{U}$. Furthermore, conjugate objects can be defined on non strict tensor categories, but the conjugate equations would involve the associativity morphisms. One important fact about conjugates is that, if they exist, they are unique up to isomorphism.

Theorem 3.12. *Let U be an object in a C^* tensor category $\mathcal{C}$ and suppose there exists two conjugate objects $\overline{U}$ and $\overline{U}'$ such that the maps $(R, \overline{R})$ and $(R', \overline{R}')$ solve the conjugate equations. Then $\overline{U}$ and $\overline{U}'$ are isomorphic.*

Proof. Define a morphism $T = (id_{\overline{U}} \otimes \overline{R}'^*)(R \otimes id_{\overline{U}'}) \in \mathrm{Mor}(\overline{U}', \overline{U})$. We claim that it is invertible with inverse $S = (id_{\overline{U'}} \otimes \overline{R}^*)(R \otimes id_{\overline{U}})$. Computation gives

$$\begin{aligned} TS &= (id \otimes \overline{R}'^*)(R \otimes id)(id \otimes \overline{R}^*)(R \otimes id) \\ &= (id \otimes \overline{R}'^*)(id \otimes id \otimes id \otimes \overline{R}^*)(R \otimes id \otimes id \otimes id)(R' \otimes id) \\ &= (id \otimes \overline{R}^*)(id \otimes \overline{R}'^* \otimes id \otimes id)(id \otimes id \otimes R' \otimes id)(R \otimes id) \\ &= (id \otimes \overline{R}^*)(R \otimes id) = id \end{aligned}$$

In a similar manner, one can show that $ST = id$. So $\overline{U}$ and $\overline{U}'$ are isomoprhic. The maps $(R, \overline{R})$ and $(R', \overline{R}')$ are also related.

$$(T^{-1} \otimes id)R = (id \otimes \overline{R}^* \otimes id)(R' \otimes id \otimes id)R = (id \otimes \overline{R}^* \otimes id)(id \otimes id \otimes R)R' = R'$$

So $R' = (T^{-1} \otimes id)R$ and one can do a similar computation to show that $\overline{R}' = (id \otimes T^*)\overline{R}$. $\square$

Theorem 3.13. *If an object U has a conjugate, with $(R, \overline{R})$ solving the conjugate equations, then the map $Fr : Hom(U \otimes V, W) \to Hom(V, \overline{U} \otimes W)$ given by*

$$Fr(T) = (id_{\overline{U}} \otimes T)(R \otimes id_V)$$

is an isomorphism with inverse $Fr^{-1}(S) = (\overline{R}^* \otimes id_W)(id_U \otimes S)$. This is called the Frobenius reciprocity.

Proof. A simple computation gives

$$Fr^{-1}(Fr(T)) = (\overline{R}^* \otimes id_W)(id_U \otimes (id_{\overline{U}} \otimes T)(R \otimes id_V))$$

$$= (\overline{R}^* \otimes id_W)(id_U \otimes id_{\overline{U}} \otimes T)(id_U \otimes R \otimes id_V)$$

Because we can identify $U \otimes \mathbb{1} \otimes V \cong \mathbb{1} \otimes U \otimes V$, we have

$$Fr^{-1}(Fr(T)) = (\overline{R}^* \otimes id_W)(id_{\overline{U}} \otimes id_U \otimes T)(R \otimes id_U \otimes id_V)$$
$$= (\overline{R}^* \otimes id_U)(id_{\overline{U}} \otimes R)T = T$$

□

3.4 Woronowicz's Duality Theorem

Everything we've mentionned about C^* tensor categories with conjugates has to with the inherent properties. But if we wish to really characterize them as duals of compact quantum groups, we need a way to relate these abstract categories to a concrete one.

Definition 3.14. A tensor functor $F : \mathcal{C} \to \text{Hilb}_{fd}$, from a tensor C^*-tensor category to the category of finite dimensional Hilbert spaces, is called a fiber functor if it is faithful, and exact.

For our considerations, the requirment that F is exact is unnecesary. Indeed, since we assume that every object is the direct sum of simple ones, every exact sequence splits, so every linear functor $F : \mathcal{C} \to \text{Hilb}_{fd}$ is exact. For the same reason, a linear functor is faithful if and only if the image of every simple ojbect is non-zero. This is always true however if F is a fiber functor. This is because for any non-zero object U, the unit object embeds into $\overline{U} \otimes U$, thus $\mathcal{C}$ is a subobject $F(\overline{U} \otimes U) \cong F(\overline{U} \otimes F(U))$ and so so $F(U) \neq 0$. Thus if $\mathcal{C}$ is a semi-simple C^* tensor category with conjugates, a fiber functor is simply a tensor functor $F : \mathcal{C} \to \text{Hilb}_{fd}$.

If G be a compact quantum group and $\text{Rep}_{u,fd}(G)$ its category of fintie dimensional unitary representation. As we've previously stated, it is a semi-simple C^*-tensor category with conjugates. For U a finite dimensional unitary representaiton, we define a unitary fiber functor $F(U) = H_U$ while the action of F_2 and F on morphisms is the identity. This is called the canonical fiber functor. We are now ready to state the precise Woronowicz-Tannaka-Krein duality theorem.

Theorem 3.15. *Let $\mathcal{C}$ be a semi-simple C^*-tensor category with conjugates and $F : \mathcal{C} \to Hilb_{fd}$ a unitary fiber functor. Then there exists a compact quantum group G and a unitary monoidal equivalence $E : \mathcal{C} \to Rep_{u,fd}(G)$ such that F is naturally unitarily monoidally isomorphic to the composition of the canonical fiber functor $Rep_{u,fd}(G) \to Hilb_{fd}$ with E.*

The proof will, as on would expect, come in several parts. The plan is to use $\mathrm{End}(F)$ to construct a Hopf *-algebra generated by finite dimensional corepresentations which, by an earlier theorem, gives us $\mathrm{Pol}(G)$. More specifically, this Hopf *-algebra will turn out to be a subalgebra of $\mathrm{End}(F)^*$. Do a multiplicaiton, comultiplicaiton and antipode we will need to equip $\mathrm{End}(F)$ with some additional structure.

We assume that $\mathcal{C}$ is strict, that $F(\mathbb{1}) = \mathbb{C}$ and F_0 is the identity map. Under our assumptions, $\mathrm{End}(F)$ has a fairly straightforward description. Pick representatives U_α of equivalences classes of simple objects. Since $\mathcal{C}$ is semi-simple, an element $\eta \in \mathrm{End}(F)$ is completely determined by it's action on $F(U_\alpha)$. Therefore

$$\mathrm{End}(F) \cong \prod_\alpha B(F(U_\alpha))$$

Furthermore, $\mathrm{End}(F)$ is a *-algebra, where the relevant operations are done component wise. That is, if $\eta, \tau \in \mathrm{End}(F)$, then $(\eta\tau)_U = \eta_U \tau_U$ and so on.

We can also define a sort of comultiplication structure on $\mathrm{End}(F)$. More specifically we define a *-homomorphism $\delta : \mathrm{End}(F) \to \mathrm{End}(F^{\otimes 2})$ such that $\delta(\eta)$ is defined using the commutative diagram

$$\begin{tikzcd}
F(U) \otimes F(V) \arrow[r, "\delta(\eta)_{U,V}"] \arrow[d, "F_2"'] & F(U) \otimes F(V) \arrow[d, "F_2"] \\
F(U \otimes V) \arrow[r, "\eta_{U\otimes V}"'] & F(U \otimes V)
\end{tikzcd}$$

More explicitly, since F_2 is unitary, $\delta(\eta)_{U,V} = F_2^* \eta_{U\otimes V} F_2$. The homomorphisms $id \otimes \delta$ and $\delta \otimes id$ extend to homomorphisms $\mathrm{End}(F^{\otimes 2}) \to \mathrm{End}(F^{\otimes 3})$. Then the coassociativity of δ is obvious from the definition of a tensor functor. As we will see, this δ will allow us to define a multiplication on $\mathrm{End}(F)^*$ whereas we will use the multiplication on $\mathrm{End}(F)$ to define the comultiplication on $\mathrm{End}(F)^*$. For $T \in \mathrm{Hom}(U, V \otimes W)$, let Θ be the map defined by

$F_2^* F(T) : F(U) \to F(V) \otimes F(W)$. If $(R, \overline{R})$ solve the conjugate equations for U and $\overline{U}$, then one can easily verify that $(\Theta(R), \Theta\overline{R})$ solves the conjugate equations for $F(U)$ and $F(\overline{U})$

Theorem 3.16. *For every $\eta \in End(F)$, there exists a unique element $\eta^\vee \in End(F)$ such that if $(R, \overline{R})$ solves the conjugate equations for U and $\overline{U}$ then $(\eta^\vee)_{\overline{U}} = (\eta_U)^\vee$ where $(\eta_U)^\vee$ is computed using the solution $\Theta(R), \Theta(\overline{R})$ to the conjugate equations for $F(U)$ and $F(\overline{U})$.*

Proof. First, we must define what $(\eta^\vee)_U$ is. Recall that by Frobenius reciprocity, there exists an isomorphism between $\text{End}(U) \to \text{End}(\overline{U})$: $T \to T^\vee$ uniquely defined by

$$(id \otimes T)R = (T^\vee \otimes id)R$$

Thus the defition of $(\eta^\vee)_U$ is

$$(id \otimes \eta_U)\Theta(R_U) = ((\eta^\vee)_U \otimes id)\Theta(R_U)$$

The problem in having a well defined natural transformation $\eta^\vee$ is that, a priori, it depends on the choice of solutions to the conjugate equation. It turns out that this is not the case. Recall, that for any other solution $(R', \overline{R}')$ to the conjugate equation is necessarily of the form $R' = (id \otimes T^*)R, \overline{R}' = (T^{-1} \otimes id)\overline{R}$. If we then use $\Theta(R') = (id \otimes F(T^*))\Theta(R)$ to define $(\eta^\vee)_U$,

$$(id \otimes \eta_U)(id \otimes F(T^*))\Theta(R) = ((\eta^\vee)_U \otimes id)(id \otimes F(T^*))\Theta(R)$$

η_U commutes with $F(T^*)$ by the naturality of η. Thus

$$(id \otimes F(T^*)\eta_U)\Theta(R) = ((\eta^\vee)_U \otimes F(T^*))\Theta(R)$$

Applying $(id \otimes F(T^*)^{-1})$ to both sides gives us our original definition of $(\eta^\vee)_U$. Furthermore, using the same naturality argument, it is easy to see if two objects U and U' are isomorphic, then $(\eta^\vee)_U = (\eta^\vee)_{U'}$. We now have a well defined collection of morphisms such that if $\overline{U}$ is conjugate to U, then $(\eta_{\overline{U}})^\vee = (\eta^\vee)_U$ All that is left to prove is the naturality of these maps.

Any morphisms $S : \overline{U}_2 \to \overline{U}_1$ equals $T^\vee$ for some morphism $T : U_1 \to U_2$ for a fixed choice of solutions to the conjugate equations. Since $F(T)\eta_{U_1} = \eta_{U_2}F(T)$, we get

$$F(T)^\vee(\eta_{U_2})^\vee = (\eta_{U_1})^\vee F(T)^\vee$$

$$F(T^\vee)(\eta^\vee)_{\overline{U_2}} = (\eta^\vee)_{\overline{U_1}} F(T^\vee)$$

$$F(S)(\eta^\vee)_{\overline{U_2}} = (\eta^\vee)_{\overline{U_1}} F(S)$$

where $F(T)^\vee$ is computed using $\Theta(R_1)$ and $\Theta(R_2)$. So $\eta^\vee$ is indeed a natural transformation. $\square$

We now have all the ingredients to form a Hopf *-algebra. Let $\mathcal{A}$ be the subspace of $\mathrm{End}(F)^*$ such that a function $a \in \mathcal{A}$ only depends on a finite number of operators η_U. Thus, since $End(F) \cong \prod_\alpha B(F(U_\alpha))$ then $\mathcal{A} \cong \oplus_\alpha B(F(U_\alpha))^*$. Addition is done pointwise and if $a, b \in \mathcal{A}$ we define a multiplication by

$$ab = (a \otimes b)\delta$$

If $\alpha \in B(F(U_\alpha))^*$ and $\beta \in B(F(U_\beta))^*$, then $\alpha\beta \in \oplus_\gamma B(F(U_\gamma))$ where gamma ranges over a finite sum of indices such that $\mathrm{Hom}(U_\gamma, U_\alpha \otimes U_\beta) \neq 0$. Since δ is coassociative, then the multiplication is associative. Furthermore, $\mathcal{A}$ is unital with $1(\eta) = \eta_{\mathbb{1}} \in \mathbb{C}$. We define a comultiplication $\delta : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$ by

$$\Delta(a)(\omega \otimes \eta) = a(\omega\eta)$$

By the associativity of multiplicaiton, Δ is coassociative. Define linear maps $\epsilon : \mathcal{A} \to \mathbb{C}$ and $S : \mathcal{A} \to \mathcal{A}$ by

$$\epsilon(a) = a(1) \quad \text{and} \quad S(a)(\eta) = a(\eta^\vee)$$

Theorem 3.17. *$\mathcal{A}$ is a Hopf algebra with counit ϵ and antipode S*

Since we are only dealing with finite sums, it will be useful to introduce Sweedler's sumless notation where we write $\delta(\eta) = \eta_{(1)} \otimes \eta_{(2)}$.

Proof. The identities $(id \otimes \epsilon)\Delta(a) = (\epsilon \otimes id)\Delta(a) = a$ is easily verifiable by using a test $\eta \in \mathrm{End}(F)$.

$$(id \otimes \epsilon)\Delta(a)(\eta) = a(\eta 1) = a(\eta)$$

and similarly for $(\epsilon \otimes id)\Delta(a)$. The identity $m(id \otimes S)\Delta(a) = \epsilon(a)1$ for all a is equivalent to

$$\eta_{(1)}\eta^\vee_{(2)} = \eta_{\mathbb{1}} 1 \;\text{ for all }\; \eta \in \mathrm{End}(F)$$

First, fix $(R, \overline{R})$ for $U, \overline{U}$. Recall that by definition of $T^\vee$, $(1 \otimes T)\Theta(R) = (T^\vee \otimes 1)\Theta(R)$.

$$((\eta_{(1)})_{\overline{U}}(\eta^\vee_{(2)})_{\overline{U}} \otimes 1)\Theta(R) = \delta(\eta)_{\overline{U},U}\Theta(R) = F_2^*\eta_{\overline{U}\otimes U}F(R) = \Theta(R)\eta_{\mathbb{1}}.$$

So $(\eta_{(1)}\eta^\vee_{(2)})_{\overline{U}} = \eta_{\mathbb{1}}1$. In a similar manner, one proves the identity $m(S \otimes id)\Delta(a) = \epsilon(a)1$ by using the identity $\Theta(\overline{R})^*(T \otimes 1) = \Theta(\overline{R})^*(1 \otimes T^\vee)$. $\square$

Now define a linear map $*: \mathcal{A} \to \mathcal{A}$ by

$$a^*(\eta) = \overline{a}(\eta^\vee) = \overline{a(\eta^{\vee *})}$$

One checks that $\eta^{\vee * \vee *} = \eta$, which means * is involutive. Furthermore, one verifies that Δ is *-preserving. This turns $\mathcal{A}$ into a Hopf *-algebra .

Recall that what we ultimately want is a Hopf *-algebra generated by matrix coeffeciants of finite dimensional unitary representations. We need a way to represent our Hopf *-algebra $\mathcal{A}$. For every object $U \in \mathcal{C}$, we define an element $X^U \in B(F(U)) \otimes \mathrm{End}(F)^*$ by requiring that

$$(id \otimes ev_\eta)X^U = \eta_U \quad \forall \eta \in \mathrm{End}(F)$$

where $ev_\eta(a) = a(\eta)$. It is obvious that $X^U \in B(F(U)) \otimes \mathcal{A}$. We must now check that X^U is in fact a corepresentation. In fact, we have more than that.

Theorem 3.18. *X^U are unitary corepresentations of the Hopf-* algebra $(\mathcal{A}, \Delta)$. If $T \in Hom(U, V)$, then $(F(T) \otimes id)X^U = X^V(F(T) \otimes id)$. Finally, $(F_2 \otimes id)X^U_{13}X^V_{23} = X^{U \otimes V}(F_2 \otimes id)$.*

Proof. The identity $(id \otimes \epsilon)(X^U) = 1$ is obvious using the fact that $\epsilon = ev_1$. Checking the compatibility condition with the comultiplication, we get

$$\begin{aligned}(id \otimes ev_\omega \otimes ev_\eta)(id \otimes \Delta)(X^U) &= (id \otimes ev_{\omega \otimes \eta})(X^U) \\ &= \omega_U \eta_U \\ &= (id \otimes ev_\omega \otimes ev_\eta)X^U_{12}X^U_{13}\end{aligned}$$

Thus $(id \otimes \Delta)(X^U) = X^U_{12}X^U_{13}$. So X^U are indeed corepresentations and thus, as we saw earlier, they are invertible with $(id \otimes S)X^U = (X^U)^{-1}$. So

$$\begin{aligned}(id \otimes ev_\eta)((X^U)^{-1}) &= (id \otimes ev_{\eta^\vee})X^U = (\eta^\vee)_U \\ = (\eta^{\vee *})^*_U &= ((id \otimes ev_{\eta^{\vee *}})(X^U))^* = (id \otimes ev_\eta)((X^U)^*)\end{aligned}$$

It follows that X^U are in fact unitary corepresentations. Applying $(id \otimes ev_\eta)$ to $(F(T) \otimes id)X^U$ and using the naturality of η gives

$$\begin{aligned}(id \otimes ev_\eta)(F(T) \otimes id)X^U &= F(T)\eta_U \\ = \eta_V F(T) &= (id \otimes ev_\eta)X^V(F(T) \otimes 1)\end{aligned}$$

thus $(F(T) \otimes 1)X^U = X^V(F(T) \otimes 1)$. This is rather a natural condition. It shows that the morphisms in $\mathcal{C}$ are morphisms (intertwiners) in a category of representations. Applying $(id \otimes id \otimes ev_\eta)$ to $(F_2 \otimes id)X^U_{13}X^V_{23}$ gives

$$F_2(id \otimes id \otimes ev_\eta)(X^U_{13}X^V_{23}) = F_2(id \otimes id \otimes ev_{\delta(\eta)})(X^U_{13}X^V_{24})$$

$$= F_2\delta(\eta)_{U,V} = \eta_{U\otimes V}F_2 = (id \otimes id \otimes ev_{U\otimes V})X^{U\otimes V}(F_2 \otimes id)$$

Thus $\mathcal{A}$ is generated by unitary finite dimensional corepresentations of a Hopf *-algebra .

□

We are now in fact in position to complete the proof of Woronowicz-Tannaka-Krein.

Proof. By a theorem earlier $(\mathcal{A}, \Delta) = (\mathrm{Pol}(G), \Delta)$ for some compact quuantum group G. Define the tensor functor $E : \mathcal{C} \to \mathrm{Rep}_{u,fd}(G)$ by letting $E(U) = X^U$ for objects $U \in \mathcal{C}$ and $E(T) = F(T)$ on morphisms and $E_2 = F_2$. Then F is equal to the composition to the canonical fiber functor on $\mathrm{Rep}_{u,fd}$ and E. By the discussion earlier, to show that E is a unitary monoidal equivalence one must check that the representations X^{U_α} of G are irreducible, pairwise nonequivalent, and that they exhaust the equivalences classes of irreducible representations of G. However, all of these properties follow immediately from the fact that the matrix coeffeciants of the corepresentations X^{U_α} form a basis of $\mathcal{A}$. □

The main takeaway from the Woronowicz-Tannaka-Krein theorem is that a compact quantum group can be recovered its Hom spaces. However, one must be careful to choose the right fiber functor F. Indeed, a C^* tensor category can have more than one fiber functor which leads to different compact quantum groups. In fact, it is entirely possible that two non isomorphic compact quantum groups gives rise to the same representation category.

4 Orthogonal Quantum Group

4.1 Compact Quantum Matrix Groups

Throughout this paper, we've discussed quantum groups as abstractly as possible without making reference to any property that can't be deduced from the axioms. However even with the quantum Woronowicz-Tannaka-Krein theorem, from which one can constrcut a compact quantum group with a C^* tensor category with conjugates, most concrete examples of compact quantum groups are either duals of discrete groups or deformations of classical groups. That isn't to say that these examples aren't interesting in their own right. In fact, much of the theory developed here in just a generalization of the theory developped by Woronowicz [4] for a special type of compact quantum groups which he calls compact quantum pseudogroups. However, we will refer to them as compact quantum matrix groups.

Definition 4.1. Let A be a unital C^*-algebra, u a $N \times N$ matrix with entries in A and $\mathcal{A}$ the C^* algebra generated by entries of u. Then (A, u) is compact quantum matrix group if

1. $\mathcal{A}$ is dense in A
2. There exists a C^* homomoprhism $\Delta : A \to A \otimes A$ such that

   $$\Delta(u_{ij}) = \sum_{k=1}^{N} u_{ik} \otimes u_{kj}$$

   for all $1 \leq i, j \leq N$.
3. There exists an linear multiplicative mapping $S : \mathcal{A} \to \mathcal{A}$ such that

   $$S(S(a^*)^*) = a$$

   for every $a \in \mathcal{A}$ and

   $$\sum_{k=1}^{N} S(u_{ik})u_{kj} = \delta_{ij}1$$
   $$\sum_{k=1}^{N} u_{ik}S(u_{kj}) = \delta_{ij}1$$

One easily verifies the comultiplication is coassociative and that the cancelation property is satisfied, so that compact quantum matrix groups are in fact compact quantum groups. But their representation theory is somewhat simplified considering that u is a finite dimensional representation of itself, the so called fundamental representation.

Recall that in section 2, we proved that for a compact quantum group G, the *-subalgebra generated by matrix coeffeciants of irreducible finite dimensional unitary representation $\mathrm{Pol}(G)$ is a dense *-subalgebra. In the case of a compact matrix pseudogroup (A, u), it then follows by the properties of $\mathcal{A}$ that every finite dimensional representation of (A, u) can be found in tensor powers and the direct sums of the matrix u and its conjugate representation $\overline{u}$. From a categorical point of view, this means that under the operation of tensor power and direct sum $\mathrm{Rep}_{u,fd}(A)$ is generated by $\{u, \overline{u}\}$. What Woronowicz essentially proved in his orginal article [5] is that given a C^* tensor category $\mathcal{C}$ with conjugates such that $\mathcal{C}$ is generated by distinguised objects $\{r, \overline{r}\}$, then $\mathcal{C}$ is equivalent to $\mathrm{Rep}_{u,fd}(G)$ for a compact quantum matrix group G. What this allows one to do is study a compact quantum matrix groups via their representation categories which are often times much better understood and easier to work with.

4.2 O_N^+ and its Representation Category

If we wish to study the compact quantum matrix groups as a whole, the first obvious step is to study the quantized version of classical matrix groups. However we must be very careful when described what the quantized version of a classical group should be. Consider for example O_N, the group of orthogonal $N \times N$ matrices. Recall that a matrix $g = \{g_{ij}\}$ is called orthogonal if $\{g_{ij}^{-1}\} = \{g_{ji}\}$ and $\{\overline{g}_{ij}\} = \{g_{ij}\}$. We know that $C(O_N)$, the commutative C^*-algebra of continous complex valued functions on O_N is a compact quantum group. Let $v_{ij} \in C(O_N)$ be the function that assigns to g its ij-th coordinate, that is $v_{ij}(g) = g_{ij}$. It is obvious that the matrix $v = \{v_{ij}\}$ is also orthogonal. By the Stone-Weirstrass theorem, the C^*-algebra generated by the entries are dense in $C(O_N)$. To sum up, if we denote C^*_{com} by the universal commutative C^*-algebra , we have

$$C(O_N) = C^*_{com}(v_{ij} | v \text{ is an orthogonal } N \times N \text{ matrix})$$

What this tells us is that all the important information about O_N lies in its coordinates. This may seem obvious enough, but it allowed Wang to give the definition of the orthogonal quantum group [8].

Definition 4.2. The orthogonal quantum group O_N^+ also called the free orthogonal group, is defined as

$$O_N^+ = C^*(v_{ij} | u \text{ is an orthogonal } N \times N \text{ matrix})$$

as is equipped with the coproduct, counit and antipode Δ, ϵ, S

$$\Delta(v_{ij}) = \sum_{k=0}^{N} v_{ik} \otimes v_{kj}$$

$$\epsilon(v_{ij}) = \delta_{ij}$$

$$S(v_{ij}) = v_{ji}$$

There is alot to be said about O_N^+ and indeed it has been studied extensively. Let's jump straight to its representation category. As mentionned earlier, all irreducible representations can be found in tensor powers of the fundamental representation. However, the general theory doesn't tell us how such to decompose such a tensor power. In the case of O_N^+, the problem was solved by Banica [7]. As it turns out, it is very similar to the representation theory of $SU(2)$.

Theorem 4.3. *For a fixed $N \geq 2$, the irreducible representations of O_N^+ are labbeled by the non negative integers*

$$\{v^n = \{v_{ij}\} | 1 \leq i, j \leq d_n^N\}_{n \in \mathbb{N}}$$

which have the following properties.

1. $v^0 = 1_{O_N^+}$ *and* v^1 *is the fundamental representation*

2. For $N \geq 2$, $d_n^N = U_n(N/2)$ *where* U_n *is the* n*-th Chebyshev polynomial of the second kind.*

3. $\overline{v^n} \cong v^n$ *for all* $n \in \mathbb{N}$

4. The family $\{v^n\}_{n\in\mathbb{N}}$ *satisfy the fusion rules*

$$v^r \otimes v^s \cong \bigoplus_{t=0}^{min\{r,s\}} v^{r+s-2t}$$

While this theorem gives us important insight into the structure of O_N^+, dealing with tensor powers of a matrix is messy enough without the added difficulty that the elements themselves don't commute. However, thanks to the Tannaka-Krein-Woronowicz theorem, there is a combinatoral tool that helps us in the study of O_N^+ called Temperley-Lieb diagrams.

Definition 4.4. Let k, l be two natural numbers with the same parity. The set of Temperley-Lieb diagrams $D(k,l)$ consists of diagrams formed by a row of k upper points and l lower points with $(k+l)/2$ non-crossing strings joining pairs of points. Diagrams are taken up to planar isotopy, and if k, l don't have the same parity, then $D(k,l) = \emptyset$.

Temperley-Lieb diagrams can in fact be used to define the tensor categories we are interested in. Objects in this category are natural numbers and morphisms between them are precisely the Temperley-Lieb diagrams between points. Composition of morphisms is given by horizontal concatenation, the tensor product by vertical concatenation and conjugation by the upside down turning of diagram.

Consider the vector space on which v acts, namely $V = \mathbb{C}^N$ and let $e_1, ..., e_N$ denote its standard basis. Each diagram $p \in D(k,l)$ defines a linear map $V^{\otimes k} \to V^{\otimes l}$

$$p(e_{i_1} \otimes \ldots \otimes e_{i_k}) = \sum_{j_1 \cdots j_l} \begin{pmatrix} i_1 \ldots i_k \\ p \\ j_1 \ldots j_l \end{pmatrix} e_{j_1} \otimes \ldots \otimes e_{j_l}$$

where the middle symbol is 1 if all strings of p joins pairs of equal indices and 0 otherwise. Linear maps corresponding to different diagrams can be shown to be linear independant provided $N \geq 2$. This gives us an embedding

$$TL(k,l) \subset \mathrm{Hom}(V^{\otimes k}, V^{\otimes l})$$

where $TL(k,l)$ is the vector space spanned by the diagrams in $D(k,l)$. As it turns out this vector spaces gives us exactly the morphisms in the representation category of O_N^+. We give here a proof of Banica and Collins [6].

Theorem 4.5. *We have equality of vector spaces*

$$TL(k,l) = Hom(v^{\otimes k}, v^{\otimes l})$$

where $Hom(v^{\otimes k}, v^{\otimes l})$ *is the subalgebra of* $Hom(V^{\otimes k}, V^{\otimes l})$ *of morphisms of* O_N^+.

Proof. Let $\xi \in V^{\otimes 2}$ be the vector

$$\xi = \sum_k e_k \otimes e_k$$

The fact that v is orthogonal is equivalent to the fact that it is fixed by $v^{\otimes 2}$. That is,

$$v^{\otimes 2}(\xi \otimes 1) = \xi \otimes 1$$

Now let $E : \mathbb{C} \to V^{\otimes 2}$ be the linear map

$$E(1) = \xi$$

The condition that $v^{\otimes 2}$ fixes ξ means that

$$E \in \mathrm{Hom}(1, v^{\otimes 2})$$

But in terms of Temperley Lieb diagrams, E is nothing but the semi circle in $D(0,2)$:

$$E = \cap$$

Thus, O_N^+ is the universal C^*-algebra generated by entries of a $N \times N$ orthogonal matrix v such that

$$\cap \in \mathrm{Hom}(1, v^{\otimes 2})$$

Using the Woronowicz-Tannaka-Krein duality we obtain

$$< \cap >= \{TL(k,l)|k,l\} = \{\mathrm{Hom}(v^{\otimes k}, v^{\otimes l})\}$$

Where $< \cap >$ is the category generated by $\cap$ via the composition, tensor product and conjugation. The reason we have $< \cap >= \{TL(k,l)|k,l\}$ is because, every such Temperley-Lieb diagram can be obtained in such a way (this isn't immediately obvious, but it can be shown).

□

4.3 Finding an orthogonal invariant basis

Now that we know more about the structure of the representations of O_N^+, we would like to find an orthogonal basis for the invariant subspaces ofv^n. Since we can view the v^n as acting on $V^{\otimes n}$, the invariant subspaces will be in $V^{\otimes n}$, the dimension of which is, as we previously saw,given by $d_n^N = U_n(N/2)$. But this doesn't really give us additional information about when a vector is or isn't in the subspace. We need to study the situation in greater depth. Let's start with the simplest case, when $n = 2$.

Banica's theorem tells us that $v^{\otimes 2} = v^1 \otimes v^1 = v^0 \oplus v^2$ where the invariant subspace of v^0 is of dimension 1. We've already seen that the vector

$$\xi = \sum_k e_k \otimes e_k$$

is fixed by $v^{\otimes 2}$, so its span is the invariant subspace of v^0. Therefore the invariant subspace of v^2 is the one orthogonal to the vector ξ. Let us be more explicit by first introducing new notation. Any vector $\varphi \in V^{\otimes n}$ can be written in the form

$$\varphi = \sum_{i_1 \dots i_n} a_{i_1 \dots i_n} e_{i_1} \otimes \dots \otimes e_{i_n}$$

For $1 \leq r < s \leq n$, let $\langle \varphi \rangle_{r,s}$ be defined as

$$\langle \varphi \rangle_{r,s} = \sum_{i_1 \dots i_n} \langle e_r, e_s \rangle a_{i_1 \dots i_n} e_{i_1} \otimes \dots \otimes e_{r-1} \otimes e_{r+1} \dots e_{s-1} \otimes e_{s+1} \dots \otimes e_{i_n}$$

where $\langle \cdot, \cdot \rangle$ denotes the usual inner product of $\mathbb{C}^N$.

Returning to the discussion at hand, the inner product between $\varphi \in V^{\otimes 2}$ and ξ is, by the induced inner product on tensor spaces,

$$\begin{aligned} \langle \varphi, \xi \rangle &= \sum_{i,j,k} a_{ij} \langle e_i, e_k \rangle \langle e_j, e_k \rangle = \sum_{i,j,k} a_{ij} \delta_{ik} \delta_{jk} \\ &= \sum_{i,j} a_{ij} \delta_{ij} = \sum_{i,j} a_{ij} \langle e_i, e_j \rangle = \langle \varphi \rangle_{1,2} \end{aligned}$$

We now have a simlpe criterion for a vector to be in v^2.

Now let's examine the case where $n = 3$. Consider the action of the diagram $\cap| \in TL(1,3)$ on the canonical basis vectors $e_i \in V = \mathbb{C}^N$.

$$\cap|(e_i) = \sum_{j_1, j_2, j_3} \begin{pmatrix} i \\ \cap| \\ j_1 j_2 j_3 \end{pmatrix} e_{j_1} \otimes e_{j_2} \otimes e_{j_3} = \sum_k e_k \otimes e_k \otimes e_i = \xi \otimes e_i$$

Now since $\cap|\in \mathrm{Hom}(v, v^{\otimes 3})$, it is easy to see that $v^{\otimes 3}$ fixes the N dimensional subspace spanned by the vectors of the form $\xi \otimes e_i$, $1 \leq i \leq N$. Using the same reasoning for the diagram $|\cap \in \mathrm{Hom}(v, v^{\otimes 3})$, we see that $v^{\otimes 3}$ also fixes the N dimensional subspace spanned the vectors $e_i \otimes \xi$, $1 \leq i \leq N$. Furthermore, these two subspaces are obviously orthogonal. Since, by Using the recursion relation, $v^{\otimes 3} = v^1 \oplus v^1 \oplus v^3$, this implies that the invariant vectors of v^3 are orthogonal to the two previously mentionned subspaces. Using the same computing as the $n = 2$ case, a vector $\varphi \in v^3$ if and only if

$$\langle \varphi \rangle_{1,2} = \langle \varphi \rangle_{2,3} = 0$$

One can use recursion to show that this condition generalizes to any n. That is, a vector φ is in the invariant subspace of v^n if and only if

$$\langle \varphi \rangle_{j,j+1} = 0 \ \ \forall\, 1 \leq j \leq n-1$$

This result shouldn't be terribly surprising. After all, the diagram $\cap$ generates our category and its action on vectors will, in some sense, result in a vector whose j and $j+1$ th tensor component is the same. What is perhaps surprising is that finding an orthogonal basis for the subspaces is trickier than it first appears. In what follows, we find such a basis for all N when $n = 2, 3$ and a partial result for $n = 4$.

The most obvious place to start looking for an invariant basis is trivial cases. Indeed, we immediately have $N(N-1)$ orthogonal basis vectors given by

$$e_i \otimes e_j, \quad i \neq j,$$

We therefore need to find $d_2^N - N(N-1) = N - 1$ more basis vectors orthogonal to these ones. A generating set of this subspace is given by

$$e_i \otimes e_i - e_{i+1} \otimes e_{i+1}, \quad 1 \leq i \leq N-1$$

However, these vectors are not mutually orthogonal. But since the inner product on this subspace is preserved by the identification of $e_i \otimes e_i$ with e_i we can do the Gram-Schmidt algorithm to obtain orthogonal vectors

$$\sum_{i=1}^{j} e_i \otimes e_i - j(e_{j+1} \otimes e_{j+1}),\ 1 \leq j \leq N-1$$

Of course, the Gram-Schmidt algorithm depends on ordering. For a given $1 \leq m \leq N$, there is always a different basis given by

$$E_{m,j} = \sum_{i=m}^{j} e_i \otimes e_i - j(e_{j+1} \otimes e_{j+1}),\ (m \leq j \leq N-m) \bmod N$$

where j, and everything else, is taken to be mod N. The important thing to notice is that for a fixed m, then the $N-2$ vectors $E_{(m+1),j}, j \neq m-1, j \neq m$ don't contain the term $e_m \otimes e_m$. This will be important for the following cases. Just as an example, we'll list this basis explicitely for $N = 3$

$$e_1 \otimes e_2,\ e_1 \otimes e_3,\ e_2 \otimes e_1,\ e_2 \otimes e_3,\ e_3 \otimes e_1,\ e_3 \otimes e_2$$

$$E_{1,1} = e_1 \otimes e_1 - e_2 \otimes e_2,\ E_{1,2} = e_1 \otimes e_1 + e_2 \otimes e_2 - 2e_3 \otimes e_3$$

Recall that what we want are vectors for which the 'inner product' between any two adjacent indices is zero. For $n = 3$, we have $N(N-1)^2$ trivial cases

$$e_i \otimes e_j \otimes e_k,\ \ i \neq j, j \neq k$$

This leaves us with $d_3^N - N(N-1)(N-1) = 2N^2 - 3N$ vectors. This is where our vectors $E_{m,j}$ come into play. We have an additional $2N(N-2)$ vectors given by

$$e_i \otimes E_{i+1,j}, \quad E_{i+1,j} \otimes e_i,$$

The vectors are mutually perpendicular by construction and furthermore they are obviously perpendicular to the trival cases. We can also see that they are in our subspace. The pseudo inner product on the first and second tensor is zero because, as previously noted, $E_{i+1,j}$ doesn't contain the term $e_i \otimes e_i$. We are then left with $2N^2 - 3N - 2N(N-2) = N$ vectors remaining. What's left are vectors which have all three tensor product the same index.

$$e_i \otimes (\sum_{j \neq i} e_j \otimes e_j) + (\sum_{j \neq i} e_j \otimes e_j) \otimes e_i + (N-1) e_i \otimes e_i \otimes e_i$$

where i varies from 1 to N. These vectors are mutually perperdicular to each other and furthermore $\langle \cdot, \cdot \rangle_{1,2} = \langle \cdot, \cdot \rangle_{2,3} = 0$. They are also perpendicualr to the trivial cases. Furthermore, we have

$$\langle e_i \otimes E_{i+1,j}, e_k \otimes (\sum_{l \neq k} e_l \otimes e_l) + (\sum_{l \neq k} e_l \otimes e_l) \otimes e_k + (N-1) e_k \otimes e_k \otimes e_k \rangle$$

$$= \delta_{ik} \langle E_{i+1,j}, \sum_{l \neq k} e_l \otimes e_l - (N-1) e_k \otimes e_k \otimes e_k \rangle = \delta_{ik} \langle E_{i+1,j}, E_{k+1,k} \rangle = 0$$

Thus we've completed the case $n = 3$.

The case $n = 4$ is trickier but, as always, there are the trivial cases, of which we have $N(N-1)(N-1)(N-1)$

$$e_i \otimes e_j \otimes e_k \otimes e_l, i \neq j, j \neq k, k \neq l$$

In the same spirit as the $n = 3$ case, we use, for a fixed j, the $N-2$ vectors $E_{j+1,l}$ to get $2N(N-1)(N-2)$ vectors of the form

$$e_i \otimes e_j \otimes E_{j+1,l}, \quad E_{j+1,l} \otimes e_j \otimes e_i, i \neq j$$

We could also put the $E_{j+1,l}$ vectors in the middle tensor like so

$$e_i \otimes E_{i+1,l} \otimes e_j$$

but we must be careful. Now there are only $(N-3)$ values for l to take if $i \neq j$ and $(N-2)$ if $i = j$. Hence, vectors of this type contribute $N^2(N-3)+N$ terms. It is easy to see why these vectors are mutually orthogonal and in our subspace. Taking all these into account, we have $d_4^N - N(N-1)^3 - 2N(N-1)(N-2) - N^2(N-3) - N = 3N^2 - 4N + 1$.

Now there is no advanced machinery being used in this process and that is perhaps why there is no obvious generalization to higher dimensions or indeed a complete solution when $n = 4$. In any case, just based on the definition of the dimension d_n^N, it is fairly obvious that the solution should involve recursion in some manner.

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