Certain resolvable directed cycle decompositions of directed graphs

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Abstract

In this thesis, we address two problems in cycle decompositions. In the first problem, we resolve the last outstanding case of the directed Oberwolfach problem with tables of uniform length. More specifically, we address the case with two tables of equal odd length. To do so, we construct a resolvable decomposition of the complete symmetric directed graph into two directed cycles of the same odd length. This affirms a conjecture of Burgess and Šajna (2014).

In the second problem addressed in this thesis, we partially solve a conjecture on directed cycle decompositions of products of directed graphs. This conjecture stipulates that, given two directed graphs G and H that both admit a decomposition into directed hamiltonian cycles, the wreath (lexicographic) product of G with H can also be decomposed into directed hamiltonian cycles. This conjecture has been shown to be true when $|V(G)|$ is odd and $|V(H)| > 2$ by Ng (1998). In this thesis, we assume that $|V(G)|$ is even and show that this conjecture is true if $|V(H)|$ is odd and $|V(H)| > 3$, or $|V(H)|$ is even, $|V(H)| > 2$, and G is not a directed cycle. In addition, we show that, if G is a directed cycle, where $|V(G)| > 2$, and H is either a directed m-cycle with $m \geqslant 4$ even or H is the complete symmetric directed graph on m vertices such that $m \geqslant 3$, then the aforementioned conjecture is also true. Lastly, we show that this conjecture is false when G is a directed cycle of even length and H is a directed cycle of length 2 or 3.

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Chapter 1

Introduction

In this thesis, we will be addressing two cycle decomposition problems. This type of problems generally lies at the intersection of graph theory and design theory, and can thus be solved using methods from either fields. We choose to formulate these problems as graph-theoretic problems. Therefore, we begin by providing key definitions in graph theory, followed by those in cycle decompositions.

1.1 Graph Theory

For standard graph-theoretic definitions not found in this thesis we direct the reader to [20].

Let G be a graph; we denote its vertex set as $V(G)$ and its edge set as $E(G)$. If vertices x and y are adjacent, we write $x \sim y$ and denote the corresponding edge as xy. All graphs in this thesis are simple graphs. If D is a directed graph (digraph for short), we denote its vertex set as $V(D)$ and its arc set as $A(D)$; an arc whose tail is x and head is y is written (x, y). A digraph is *strict* if it does not contain two arcs with the same tail and the same head, and it does not contain an arc whose head is also its tail. All digraphs in this thesis are strict. The symbols K_n and $\overline{K}_n$ will denote the complete graph on n vertices and the edgeless graph on n vertices, respectively. The *complete multipartite graph* with parts of sizes $m_1, m_2, \ldots, m_n$, denoted $K_{m_1,m_2,\ldots,m_t}$, is the graph whose vertex set is the disjoint union of n sets (parts) of vertices of sizes $m_1, m_2, \ldots, m_n$, respectively; two vertices are adjacent if and only if they are in distinct parts. We shall denote the *complete equipartite graph* $K_{m,m,\ldots,m}$ with n

(a) The complete graph K_4. (b) The complete symmetric digraph K_4^*.

Figure 1.1: The complete graph K_4 and its symmetric directed counterpart.

parts of size m as $K_{n[m]}$. In this thesis, we primarily concern ourselves with digraphs, in particular, symmetric digraphs. A digraph is said to be *symmetric* if, for every arc $(x, y) \in A(D)$, there is also the arc $(y, x) \in A(D)$. Starting with a graph G, we can construct its *symmetric directed* counterpart by replacing each edge $\{x, y\}$ of G with arcs (x, y) and (y, x); the resulting digraph is denoted G^*. We say that G is the *underlying graph* of G^*. The *complete symmetric digraph*, denoted K_n^*, is the digraph on n vertices such that for any two distinct vertices x and y, we have $(x, y), (y, x) \in A(K_n^*)$. See Figure 1.1 for a simple example of a complete symmetric digraph. Lemmas 1.1 and 1.2 summarize some basic properties of K_n and K_n^* that follow directly from their definitions.

Lemma 1.1. *Given positive integers n and m, the following hold:*

(S1) $|E(K_n)| = \binom{n}{2}$*;*

(S2) *K_n is $(n-1)$-regular.*

Lemma 1.2. *Consider a graph G and its symmetric directed counterpart G^*. Then*

(S1) $|A(G^*)| = 2|E(G)|$*;*

(S2) *The in-degree of each vertex of G^* is equal to its out-degree.*

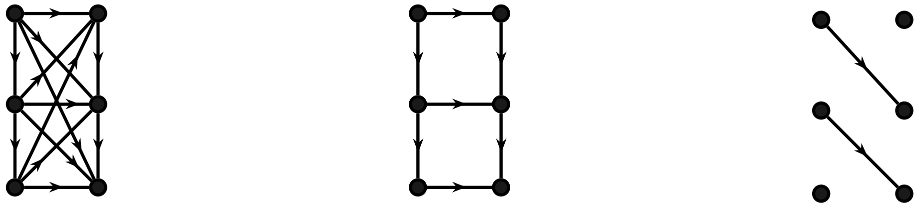
We now define a class of graphs and a class of digraphs frequently used in cycle decomposition problems. Let m be a positive integer and $C \subseteq \{1, 2, \ldots, \lfloor \frac{m}{2} \rfloor\}$. The *circulant of order m* with *connection set C*, denoted $X(m, C)$, is the graph with vertex set $V = \mathbb{Z}_m$ and edge set $E = \{\{x, y\} \,|\, \min\{x - y, m - (x - y)\} \in C\}$, with $x - y$ evaluated modulo m. Let $D \subseteq \{1, 2, \ldots, m - 1\}$. The *directed circulant of* *order* m with *connection set* D, denoted $\vec{X}(m, D)$, has vertex set $V = \mathbb{Z}_m$ and arc set $A = \{(x, y) \,|\, y - x \in D\}$, with $y - x$ evaluated modulo m.

In a simple graph G, a *walk* of length n is a sequence of vertices $W = (v_0, v_1, v_2, \ldots, v_n)$ such that, for all $i \in \{0, 1, \ldots, n{-}1\}$, we have that $v_i \sim v_{i+1}$. Vertices v_0 and v_n are the *endpoints* of W. The walk W is a *closed walk* if $v_n = v_0$ and $n \geqslant 1$. If W has no repeated vertices, it is a *path* of length n, denoted P_n. A *cycle* of length n, denoted C_n, is a closed walk of length n with no repeated vertices except for the first and the last vertex. Furthermore, given a strict digraph D, a *directed walk* of length n of D is a sequence of vertices $(v_0, v_1, v_2, \ldots, v_n)$ such that $(v_i, v_{i+1}) \in A(D)$ for all $i \in \{0, \ldots, n{-}1\}$. A *directed cycle* $\vec{C}_n$ is a directed walk of length n with no repeated vertices except that $v_0 = v_n$. A *directed path* (*dipath*) $\vec{P}_n$ is a directed walk of length n with no repeated vertices. We shall denote the length of a dipath P as $\operatorname{len}(P)$. The first vertex of a dipath P is known as its *source* and is denoted $s(P)$, while the last vertex of P is known as its *terminal* and is denoted $t(P)$. The set of *inner vertices* of a dipath P is the set $V(P) \backslash \{s(P), t(P)\}$. The *concatenation* of dipaths $P = (v_1, v_2, \ldots, v_n)$ and $Q = (v_n, v_{n+1}, \ldots, v_m)$ is the directed walk $PQ = (v_1, v_2, \ldots, v_{n-1}, v_n, v_{n+1}, \ldots, v_m)$. Note that each [directed] path and cycle of a [di]graph G corresponds to a unique sub[di]graph of G.

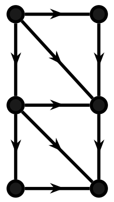
There are many ways in which we can combine two graphs to form a third graph. The *union* of graphs G and H is the graph with vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H)$. Graphs G and H are *disjoint* if $V(G) \cap V(H) = \emptyset$ and *edge-disjoint* if $E(G) \cap E(H) = \emptyset$. The disjoint union of m graphs, each isomorphic to G, is denoted mG. Analogous definitions apply to digraphs.

We can also construct a third graph from two graphs using binary operations known as graph products. Because the focus of this thesis is on digraphs, we shall define all graph products for digraphs. If G and H are digraphs, then the *wreath product of* G *with* H (also known as the *lexicographic product*), denoted $G \wr H$, is the digraph on vertex set $V(G) \times V(H)$ such that $((g_1, h_1), (g_2, h_2)) \in A(G \wr H)$ if and only if $(g_1, g_2) \in A(G)$ or $g_1 = g_2$ and $(h_1, h_2) \in A(H)$. Observe that the wreath product is not commutative. The *direct product of* G *and* H, denoted $G \times H$, is the digraph on vertex set $V(G) \times V(H)$ such that $((g_1, h_1), (g_2, h_2)) \in A(G \times H)$ if and only if $(g_1, g_2) \in A(G)$ and $(h_1, h_2) \in A(H)$. Next, the *Cartesian product of* G *and* H, denoted $G \square H$, is the digraph with vertex set $V(G) \times V(H)$ such that

$((g_1, h_1), (g_2, h_2)) \in A(G\Box H)$ if and only if $g_1 = g_2$ and $(h_1, h_2) \in A(H)$, or $h_1 = h_2$ and $(g_1, g_2) \in A(G)$. Lastly, the *strong product of G and H*, written $G \boxtimes H$, is the digraph with vertex set $V(G) \times V(H)$ such that $((g_1, h_1), (g_2, h_2)) \in A(G \boxtimes H)$ if and only if $g_1 = g_2$ and $(h_1, h_2) \in A(H)$, $h_1 = h_2$ and $(g_1, g_2) \in A(G)$, or $(g_1, g_2) \in A(G)$ and $(h_1, h_2) \in A(H)$. See Figure 1.2 for a very simple example of each type of graph product.



(a) The digraph $\vec{P}_1 \wr \vec{P}_2$. (b) The digraph $\vec{P}_1\Box\vec{P}_2$. (c) The digraph $\vec{P}_1 \times \vec{P}_2$



(d) The digraph $\vec{P}_1 \boxtimes \vec{P}_2$.

Figure 1.2: Illustration of $\vec{P}_1 \otimes \vec{P}_2$ for relevant graph products $\otimes$.

We now proceed with some further graph-theoretic definitions. If all vertices of a graph G are of even degree, then G is said to be *even*. Moreover, if all vertices of a graph G are of degree k, then G is *k-regular*. A *k-factor* of G is a k-regular spanning subgraph. The graph $K_n - I$ is the complete graph on n vertices, where n is even, with the edges of a 1-factor, I, removed. A 2-factor of G is a spanning subgraph that is a disjoint union of cycles. A *C_m-factor* is a 2-factor of G that is the disjoint union of m-cycles, while a *$(C_{m_1}, C_{m_2}, \ldots, C_{m_t})$-factor* is a 2-factor that is the disjoint union of t cycles of lengths $m_1, m_2, \ldots, m_t$, respectively. A *bipartite 2-factor* of G is a 2-factor comprised of disjoint cycles of even lengths. A *hamiltonian* cycle of G is a spanning cycle; the graph G is *hamiltonian* if it contains a hamiltonian cycle.

Now, we extend the definitions introduced in the previous paragraph to digraphs. If H^* is a symmetric digraph with underlying k-regular graph H, then H^* is a k-

regular digraph. A *directed 2-factor* of digraph G is a spanning subdigraph comprised of disjoint directed cycles. Furthermore, a $\vec{C}_m$*-factor* is a directed 2-factor of G in which all components are directed m-cycles. Moreover, a $(\vec{C}_{m_1}, \vec{C}_{m_2}, \ldots, \vec{C}_{m_t})$*-factor* of G is a directed 2-factor that is the disjoint union of t directed cycles of lengths $m_1, m_2, \ldots, m_t$, respectively. A *bipartite directed 2-factor* of digraph G is a directed 2-factor comprised of disjoint directed cycles of even lengths. A *directed hamiltonian path* of a digraph G is a spanning subdigraph of G that is also a dipath, and a *directed hamiltonian cycle* of G is a spanning subdigraph of G that is also a directed cycle. If G contains a directed hamiltonian cycle, then G is said to be *hamiltonian.*

1.2 Cycle decompositions

We now proceed by introducing several important definitions pertaining to cycle decomposition problems. First, a *decomposition* of a graph G is a set $\{H_1, H_2, \ldots, H_s\}$ of pairwise edge-disjoint subgraphs of G such that $E(G) = E(H_1) \cup E(H_2) \cup \ldots \cup E(H_s)$. If such a decomposition exists, we write $G = H_1 \oplus H_2 \oplus \ldots \oplus H_s$. If all H_i are isomorphic to the same graph H, then $\{H_1, H_2, \ldots, H_s\}$ is called an H*-decomposition* of G; in that case, we write $H|G$ and say that G is H*-decomposable.*

Let $\mathcal{D} = \{H_1, H_2, \ldots, H_s\}$ be an H-decomposition of G. The decomposition $\mathcal{D}$ is said to be *resolvable* if it can be partitioned into subsets such that the copies of H in the same subset partition the set of vertices of G; each subset is a *resolution class* of $\mathcal{D}$. If $\{H_{i_1}, H_{i_2}, \ldots, H_{i_k}\}$ is a resolution class of $\mathcal{D}$, then the subgraph $H_{i_1} \cup H_{i_2} \cup \ldots \cup H_{i_k}$ is called an H*-factor.* An H*-factorization* is a decomposition into H-factors, and corresponds to a resolvable H-decomposition. A $(C_{m_1}, C_{m_2}, \ldots, C_{m_t})$*-factorization* of G is a decomposition of G into $(C_{m_1}, C_{m_2}, \ldots, C_{m_t})$-factors. A *hamiltonian decomposition* of G is a decomposition comprising of hamiltonian cycles of G. Lastly, a graph is *hamiltonian decomposable* if it admits a hamiltonian decomposition. We conclude by stating that all definitions above naturally generalize to digraphs.

In this thesis, we are concerned with $(\vec{C}_{m_1}, \vec{C}_{m_2}, \ldots, \vec{C}_{m_t})$-factorizations of various digraphs including K_n^*. Of course, not every [di]graph admits a [directed] cycle decomposition. For instance, Veblen's Theorem stipulates that a graph admits a decomposition into cycles if and only if it is even. In the case of digraphs, the necessary and sufficient condition is that the in-degree of each vertex is equal to its out-degree.

Further conditions are needed if we impose restrictions on the length of the cycles in our decomposition. For instance, since a C_k-decomposition of G is a partition of its edge set into the edge sets of copies of C_k, it must be that k divides $|E(G)|$. Moreover, if this decomposition is to be resolvable, then k must also divide $|V(G)|$. Applied to K_n, these conditions translate to the following lemma.

Lemma 1.3. *If K_n admits a C_m-decomposition, then n is odd, $m|\frac{n(n-1)}{2}$, and $m \leqslant n$; if K_n admits a C_m-factorization, then additionally $m|n$. Lastly, if K_n admits a $(C_{m_1}, C_{m_2}, \ldots, C_{m_t})$-factorization, then $n = m_1 + m_2 + \ldots + m_t$.*

In the lemma above, the condition $m \leqslant n$ is needed for K_n to contain a copy of C_m. Observe that the equality $n = m_1 + m_2 + \ldots + m_t$ also implies that $m_i \leqslant n$ for all $i \in \{1, 2, \ldots, t\}$. Lastly, we consider necessary conditions for the existence of a certain decomposition of K_n^* into directed cycles. By Lemma 1.2, the out-degree equals the in-degree for all vertices of K_n^*. As such, one less necessary condition is required.

Lemma 1.4. *If K_n^* admits a $\vec{C}_m$-decomposition, then $m|n(n-1)$ and $m \leqslant n$; if K_n^* admits a $\vec{C}_m$-factorization, then additionally $m|n$. Lastly, if K_n^* admits a $(\vec{C}_{m_1}, \vec{C}_{m_2}, \ldots, \vec{C}_{m_t})$-factorization, then $n = m_1 + m_2 + \ldots + m_t$.*

The conditions given in Lemmas 1.3 and 1.4 above are commonly referred to as the obvious necessary conditions. The difficulty lies in determining whether these are in fact sufficient. This is precisely what we accomplish in this thesis with the conditions given in Lemma 1.4 for certain values of n and pairs (m_1, m_2).

We conclude with a simple example to illustrate some of the aforementioned concepts.

Example 1.5. In Figure 1.3, we give a simple example of a C_4-factorization of the graph $K_{2[4]}$. In this 2-factorization, denoted $\mathcal{F}$, we have two C_4-factors, given in Figures 1.3b and 1.3c respectively.

$\square$

We remark that a $\vec{C}_4$-factorization of $K_{2[4]}^*$ can easily be obtained from a C_4-factorization of $K_{2[4]}$. That is, for each cycle in the C_4-factorization of $K_{2[4]}$, we obtain

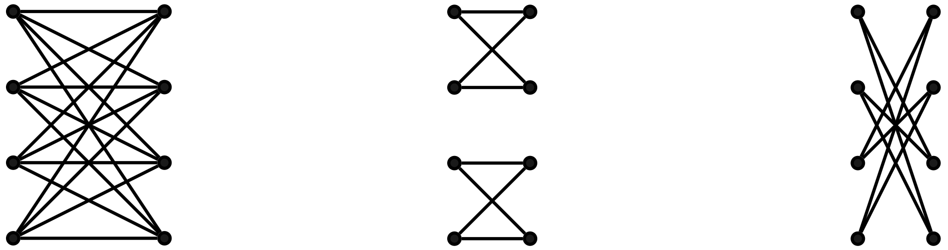

(a) The graph $K_{2[4]}$. (b) First C_4-factor of $\mathcal{F}$. (c) Second C_4-factor of $\mathcal{F}$.

Figure 1.3: A C_4-factorization of $K_{2[4]}$.

two directed cycles, one for each direction. In fact, if G admits a $(C_{m_1}, C_{m_2}, \ldots, C_{m_t})$-factorization, then G^* admits a $(\vec{C}_{m_1}, \vec{C}_{m_2}, \ldots, \vec{C}_{m_t})$-factorization. However, the converse does not always hold, chiefly because the underlying graph of a symmetric digraph may not satisfy the hypothesis of Veblen's Theorem or satisfy the other necessary conditions. For instance, if the in-degree of each vertex of G^* is odd, then the underlying graph is regular of odd degree and thus does not even admit a cycle decomposition. Furthermore, if G^* admits a $\vec{C}_m$-factorization, then $2 \cdot |E(G)|$ is divisible by m. However, this does not imply that $|E(G)|$ is divisible by m, especially if m is a composite integer. If m is even and $|E(G)|$ is odd, then clearly $m \nmid |E(G)|$ and thus G does not admit a C_m-factorization.

1.3 History

Cycle decomposition problems have a rich history dating back to the 1800s. In order to contextualize this thesis' results, we give a brief overview of some of the different types of cycle decomposition problems and their current states. For a comprehensive collections of known results (as of 2007), we direct the reader to [26].

1.3.1 Cycle decompositions of K_m

One of the earliest attempts to decompose graphs into cycles was made in 1835 by Plücker [55] when he discovered a decomposition of K_9 into copies of C_3 and claimed that a C_3-decomposition of K_n exists only if $n \equiv 3 \pmod 6$. He later revised his claim to include the case $n \equiv 1 \pmod 6$ [56]. One should note that, at the time, this investigation was not formulated as a cycle decomposition problem but rather as a geometrical problem. In 1844, Hesse [35] attempted to prove that the sufficient conditions for the existence of a C_3-decomposition of K_n were sufficient. Kirkman also began studying this problem in 1847 and showed that in fact the necessary conditions were sufficient in [42]. Unfortunately for Hesse and Kirkman, Steiner [64] also constructed C_3-decompositions of K_n for all $n \equiv 1, 3 \pmod 6$ and, although Steiner began working on this problem in 1853, these famous decompositions are now known as Steiner triple systems. Later, in 1892, Walecki [49] then showed that K_m is hamiltonian decomposable when m is odd and that $K_m - I$ is hamiltonian decomposable when m is even. In 1964, further advances were made on cycle decompositions of complete graphs. Kotzig [43] and Rosa [58] jointly showed that a C_m-decomposition of K_n exists when $n \equiv 1 \pmod{2m}$ and m is even. Furthermore, Rosa [59], proved that $C_5|K_n$ and $C_7|K_n$ under the necessary conditions. Jackson [39] then showed that the graph K_n admits a C_m-decomposition when $n \equiv 1 \pmod{2m}$ and m is odd. An important reduction step was made by Hoffman et al. in 1989 when they were able to show that if n and m are odd and a C_m-decomposition of K_n exists for all $n \in [m, 3m)$ that satisfy the necessary conditions, then a C_m-decomposition of K_n exists for all n satisfying the necessary conditions. Then, in 2001, Alspach and Gavlas [5] completely solved this problem when n and m are of the same parity. However, their original results contained a small error. A correction was then made in 2021 [8]. Šajna [60] completed the solution by solving the case in which m and n are of different parity. An alternative C_m-decomposition of K_n can be found in [22] whenever m is odd, n satisfies the necessary conditions, and $n < 3m$. The C_m-decomposition of K_n constructed in [22] satisfies additional constraints.

1.3.2 The Oberwolfach problem

Soon after the introduction of Steiner triple systems, the notion of a resolvable system was introduced by Kirkman in 1847 [42]. Kirkman posed the following problem: fifteen schoolgirls are to walk three abreast once a day for seven days; can they be arranged so that no two shall walk abreast twice? This problem is equivalent to finding a C_3-factorization of K_{15}. In that same year, a solution was independently found by Cayley and Kirkman. Kirkman generalized this problem to K_n. In that case, we seek a C_3-factorization of K_n. By Lemma 1.3 the necessary conditions require that $n \equiv 3 \pmod 6$. In 1961, Lu showed that this necessary condition sufficed. Unfortunately, his work remained unpublished until 1990 [48]. In the meantime, in 1973, Ray-Chaudhuri and Wilson [57] independently solved this problem and were initially credited with the result.

Posed by Ringel in 1967 in [32], the Oberwolfach problem generalizes the Kirkman schoolgirl problem by extending it to any 2-factor. The original Oberwolfach problem poses the following question. Given a conference with $n = 2r+1$ attendees, can these attendees be seated at t round tables, each seating $m_1, m_2, \ldots, m_t$, respectively, for r consecutive nights so that every participant sits beside every other participant exactly once? This question can be formulated as a cycle decomposition problem as follows. If $m_1+m_2+\ldots+m_t = n$, does the graph K_n admit a $(C_{m_1}, C_{m_2}, \ldots, C_{m_t})$-factorization? When all tables are of the same length m, we are then seeking a C_m-factorization of K_n. Of course, we cannot consider the problem as stated for $2r$ participants since the conditions of Veblen's theorem fail. However, Huang et al. [38] extended this problem to the case $n = 2r$ by removing a 1-factor from K_n. Alspach and Häggkvist [7] solved the uniform table case for tables of even length, showing that a C_m-factorization of $K_n - I$ exists when m is even, $m \geqslant 4$, and $m|n$. In 1989, Alspach et al. [9] then solved the uniform table case for m odd and $n \neq 4m$. Lastly, in 1991, Hoffman and Schellenberg [36] solved the remaining case where m is odd and $n = 4m$.

As for the Oberwolfach problem with tables of varying lengths, the necessary conditions are known to be sufficient for all $n \leqslant 60$ [3,27,29,30,61] except that there are no solutions when we have 2 or 4 tables of size 3, one table of size 4 and one of size 5, and two tables of size 3 and one table of size 5. Aside from these four exceptions, the necessary conditions are conjectured to be sufficient for all n. Recently, some notable advances have been made. For instance, the case in which all tables are of even size was completely solved by Häggkvist [34] when $n \equiv 2 \pmod 4$ and by Bryant and Danziger [21] when $n \equiv 0 \pmod 4$. In 2004, Gvozdjak [33] solved the two-table case when both tables are of odd lengths. Then, in 2013, Traetta [69] completed the solution to the two-table case of the Oberwolfach problem. Note that [69] gives a complete solution to the two-table case. In 2024, Traetta [70] shows that the Oberwolfach problem has a solution when we consider seating arrangements with one table of sufficiently large size. An explicit lower bound is given on the length of this table. Lastly, in 2021, the necessary conditions were shown to be sufficient for large enough n by Glock et al. [31]. Although Glock et al. do not provide a lower bound on n, their results imply that no infinite family of cases for which the obvious necessary conditions are not sufficient exists.

1.3.3 Directed cycle decompositions

Naturally, the question of whether K_n admits a C_m-decomposition can be extended to K_n^*. In the first chapter of his doctoral thesis, Bermond [14] conjectured that K_n^* admits a $\vec{C}_m$-decomposition if and only if $m|n(n-1)$ and $(n,m) \notin \{(6,3),(4,4),(6,6)\}$. He then affirmed his conjecture for $m \in \{10,12,14\}$. Bermond and Faber [16] then showed that K_n^* admits a $\vec{C}_m$-decomposition when $m \in \{4,6,8,16\}$, and also when $m|(n-1)$ and m is even, except when $(m,n) \in \{(8,8),(6,6)\}$. Furthermore, Sotteau [63] showed that K_n^* admits a $\vec{C}_m$-decomposition when m is odd, $m \geqslant 5$, and $n \equiv 0,1 \pmod{m}$. Lastly, in 2003, Alspach et al. [6] completely proved the conjecture of Bermond.

In [24], Burgess and Šajna introduced the directed Oberwolfach problem. In this variation of the original Oberwolfach problem, we ask whether it is possible to seat n participants at t round tables of lengths $m_1, m_2, \dots, m_t$, where $m_1 + m_2 + \ldots + m_t = n$, so that each participant is seated to the right of every other participant exactly once over the course of $n-1$ nights. In the language of cycle decomposition, this is equivalent to the existence of a $(\vec{C}_{m_1}, \vec{C}_{m_2}, \dots, \vec{C}_{m_t})$-factorization of K_n^* when $m_1 + m_2 + \ldots + m_t = n$. Much of the existing literature on the directed Oberwolfach problem pertains to the case with tables of uniform length. In that case, one aims to show that K_n^* admits a $\vec{C}_m$-factorization when $m|n$. A $\vec{C}_m$-factorization of K_n^* has been shown to exist for $m = 3$ and $n \neq 6$ [18], $m = 4$ and $n \neq 4$ [2, 13], and when n is odd [24] assuming the necessary conditions are satisfied. Tillson [68] showed that there exists a $\vec{C}_m$-factorization of K_m^* when m is even and $m \geqslant 8$. Burgess and Šajna [24] showed that, for all even $n \geqslant 6$, a $\vec{C}_m$-factorization of K_n^* exists when $m|n$, m is even, and $(n,m) \notin \{(6,3),(4,4),(6,6)\}$. Although they did not solve the case m odd and n even, Burgess and Šajna [24] showed that, to address that case, it suffices to construct a $\vec{C}_m$-factorization of K_{2m}^*. They then conjectured that K_{2m}^* admits a $\vec{C}_m$-factorization for all odd $m \geqslant 5$. This conjecture can be viewed as the last outstanding case of the directed Oberwolfach problem with tables of uniform length. In an attempt to prove this conjecture, Burgess et al. [23] showed that K^*_{2m} admits a $\vec{C}_m$-factorization for all odd m such that $5 \leqslant m \leqslant 49$.

We now discuss results on the directed Oberwolfach problem with cycles of varying length. Thus far, the only result on this more general case can be found in [40] and [37]. In [40], Kadri and Šajna used a recursive approach to obtain several infinite families of solutions. One of their key results is a near-complete solution to the two-table case of the directed Oberwolfach problem with cycles of varying lengths. Namely, Kadri and Šajna construct a $(\vec{C}_{m_1}, \vec{C}_{m_2})$-factorization of K^*_n when $m_1 + m_2 = n$ and $m_1 < m_2$ except when $m_1 \in \{4, 6\}$, m_2 is even, and $n \geqslant 14$. Horsley and Lacaze-Masmonteil [37] then completed the solution to the two-table case of the directed Oberwolfach problem by constructing a $(\vec{C}_{m_1}, \vec{C}_{m_2})$-factorization of K^*_n when $m_1 + m_2 = n$, $m_1 \in \{4, 6\}$, m_2 is even, and $n \geqslant 14$.

1.3.4 Cycle decompositions of products of graphs

Graph products give rise to graphs with particularly interesting structure. Generally, one is interested in determining whether a particular graph-theoretic property is inherited from G and H by the product $G \otimes H$. In the context of cycle decompositions, one assumes that G and H admit a certain type of decomposition and then investigates whether $G \otimes H$ also admits this type of decomposition.

Laskar was one of the first researchers to look at cycle decompositions of graph products. In her paper [45], she proved that $C_n \wr \overline{K}_m$ admits a hamiltonian decomposition, which Bermond [15] then used to construct a hamiltonian decomposition of $K_{n[m]}$ that is simpler than the one constructed by Laskar and Auerbach in [46]. Laskar [45] also constructed a hamiltonian decomposition of $C_r \wr C_m$ when m is odd or r is even.

In 1978, Bermond [15] made several conjectures regarding hamiltonian decompositions of graph products that have since garnered the attention of several researchers. Baranyai and Szász [12] settled one of the conjectures of Bermond by proving that, if G and H are hamiltonian decomposable graphs, then $G \wr H$ is also hamiltonian decomposable. In 1995, Muthusamy and Paulraja [51] partially settled a more general question first raised by Alspach et al. [4]: if G admits a decomposition into hamiltonian cycles and a single 1-factor and H is a hamiltonian decomposable graph, is $G \wr H$ a hamiltonian decomposable graph? Muthusamy and Paulraja [51] answered this question in the affirmative if an additional condition on G is imposed. As for the wreath products of digraphs, in [52], Ng affirmed the following conjecture for $|V(G)|$ is odd and $|V(H)|> 2$: if G and H are hamiltonian decomposable digraphs, then $G \wr H$ is hamiltonian decomposable. This conjecture remains open for the case in which $|V(G)|$ is even.

Regarding the direct product, Bermond [15] proved that $C_r \times C_m$ is hamiltonian decomposable when at least one of r and m is odd. Bermond then showed that, if G and H are hamiltonian decomposable, and at least one of $|V(G)|$ and $|V(H)|$ is odd, then $G \times H$ is hamiltonian decomposable. Balakrishnan et al. [10] have showed that $K_n \times K_m$ is hamiltonian decomposable for all $n, m \geqslant 3$ provided that either n or m is odd. As for digraphs, Paulraja and Sivasankar [54] have shown that the digraph $K_n^* \times K_m^*$ is hamiltonian decomposable when $n \geqslant 4$ is even, and $m \geqslant 5$ or $m = 3$.

In 1991, Stong [65] partially settled another conjecture of Bermond [15] which stipulates that, if G and H are hamiltonian decomposable graphs, then so is $G \Box H$. Regarding the Cartesian product of digraphs, Keating [41] has shown that $\vec{C}_n \Box \vec{C}_m$ is hamiltonian decomposable if and only if there exist positive integers s_1 and s_2 such that $\gcd(m, n) = s_1 + s_2$ and $\gcd(mn, s_1 s_2) = 1$. In 2006, Stong [66] also showed that $K_2^* \Box K_2^* \Box K_2^* \ldots \Box K_2^*$, which is the Cartesian product of r copies of K_2^*, is hamiltonian decomposable.

As for the strong product of two graphs, Fan and Liu [28] and Zhou [71] jointly showed that, if G and H are hamiltonian decomposable graphs, then $G \boxtimes H$ is also hamiltonian decomposable.

Of course, the study of cycle decompositions of graph products is not limited to decompositions into hamiltonian cycles. For example, in 2017, Bogdanowicz [19] showed that $\vec{C}_t | \vec{C}_r \Box \vec{C}_p$ when the obvious necessary conditions are satisfied. An example of a result on 2-factorization of graph products can be found in [53], in which Paulraja and Kumar show that $K_p \times K_r$ admits a C_t-factorization when t is even and the obvious necessary conditions are satisfied.

1.4 Overview of the thesis

In this thesis we address two problems in cycle decompositions. First, in Chapter 2, we survey current results on each of these problems. In addition, we also review certain well-known methods in cycle decompositions that are central to our investigation.

In Chapter 3, we complete the solution to the directed Oberwolfach problem with two tables of uniform odd size. Namely, we construct a $\vec{C}_m$-factorization of K_{2m}^* when $m \geqslant 11$ is odd. This proves a conjecture of Burgess and Šajna [24], thereby resolving the last open case of the directed Oberwolfach problem with tables of uniform length.

In Chapter 4, we examine the following conjecture: if G and H are hamiltonian decomposable digraphs, then $G \wr H$ is hamiltonian decomposable. We concentrate on the case in which $|V(G)|$ is even and show that this conjecture is true when $|V(H)|$ is odd and $|V(H)| > 3$, or $|V(H)| > 2$ is even and G is not a directed cycle. We show that this conjecture is also true when $G = \vec{C}_n$ and $H = \vec{C}_m$, n and m are even, and $n, m \geqslant 4$. In addition, we show that this conjecture is true when $G = \vec{C}_n$ and $H = K_m^*$ when $n \geqslant 4$ is even and $m \geqslant 3$. Lastly, we show that this conjecture is false when $G = \vec{C}_n$ with n even and $H \in \{\vec{C}_2, \vec{C}_3\}$.

Chapter 2

Preliminaries

For this thesis, we will address two open problems in cycle decomposition. Both of these problems are discussed in greater detail in this chapter. We present a more detailed history of each problem along with its current state. Moreover, we replicate certain proofs in order to showcase common techniques in the area of cycle decompositions. These methods are similar to those used in this thesis.

2.1 The directed Oberwolfach problem

As first discussed in Chapter 1, a directed variant of the Oberwolfach problem was first introduced by Burgess and Šajna et al. in [24]. In this variation, we treat the question of existence of a $(\vec{C}_{m_1}, \vec{C}_{m_2}, \ldots, \vec{C}_{m_t})$-factorization of K_n^* when the obvious necessary conditions are satisfied. In [40], Kadri and Šajna point out that, if n is odd, a $(\vec{C}_{m_1}, \vec{C}_{m_2}, \ldots, \vec{C}_{m_t})$-factorization of K_n^* can be obtained from a $(C_{m_1}, C_{m_2}, \ldots, C_{m_t})$-factorization of K_n. We replace each $(C_{m_1}, C_{m_2}, \ldots, C_{m_t})$-factor in this 2-factorization of K_n with two $(\vec{C}_{m_1}, \vec{C}_{m_2}, \ldots, \vec{C}_{m_t})$-factors oriented in opposite directions. Therefore, current constructive solutions to the original Oberwolfach problem give rise to the following corollary.

Corollary 2.1. *[3, 9, 27, 29, 30, 33, 49, 69] Let* $3 \leqslant m_1 \leqslant m_2 \leqslant \ldots \leqslant m_t$ *be integers such that* $m_1 + m_2 + \ldots + m_t = n$ *is odd. If* $(m_1, m_2, \ldots, m_t) \notin \{(4,5), (3,3,5)\}$*, then* K_n^* *admits a* $(\vec{C}_{m_1}, \vec{C}_{m_2}, \ldots, \vec{C}_{m_t})$*-factorization in each of the following cases:*

(S1) $m_1 = m_2 = \ldots = m_t$;

(S2) *$n \leqslant 60$;*

(S3) *$t = 2$.*

Observe that we cannot apply the same reasoning when n is even because solutions to the original Oberwolfach problem correspond to a $(C_{m_1}, C_{m_2}, \ldots, C_{m_t})$-factorization of $K_n - I$. Therefore, researchers generally concentrate on cases of the directed Oberwolfach problem in which n is even.

We will now further discuss current results on the directed Oberwolfach problem with tables of uniform length and thus, on $\vec{C}_m$-factorization of K_n^*. For readers with a background in design theory, we remark that a $\vec{C}_m$-factorization of K_n^* is also known as a resolvable Mendelsohn design with blocks of size m [26]. If a $\vec{C}_m$-factorization of K_n^* exists, the obvious necessary conditions stipulate that $n = \alpha m$ for some positive integer α. It is evident that these conditions are sufficient when $m = 2$ since such a decomposition corresponds to a 1-factorization of the graph $K_{2\alpha}$, which is widely known to exists [26]. As with many cycle decomposition problems, researchers began to tackle the problem of existence of a $\vec{C}_m$-factorization of K_n^* by looking at small cycle lengths. In [18], it was shown that the necessary conditions suffice for $m = 3$ except when $n = 6$, as stated in the theorem below.

Theorem 2.2. *[18] The graph $K_{3\alpha}^*$ admits a $\vec{C}_3$-factorization if and only if $\alpha \neq 2$.*

It is not uncommon for the necessary conditions to suffice for all but a finite number of cases, as seen in Theorem 2.2.

In 1990, Bennett and Zhang [13] showed that the necessary conditions are sufficient for $m = 4$ except when $n \in \{4, 12\}$. Regarding $n = 4$, Bennett and Zhang showed that no $\vec{C}_4$-factorization of K_4^* exists. However, they could not confirm the existence or non-existence of a $\vec{C}_4$-factorization of K_{12}^*. This missing case was then filled by Adams and Bryant [2]. These two results yield the following theorem.

Theorem 2.3. *[2, 13] Let α be a positive integer. The directed graph $K_{4\alpha}^*$ admits a $\vec{C}_4$-factorization if and only if $\alpha \neq 1$.*

In 2002, Abel et al. proved the following.

Theorem 2.4. *[1] The digraph $K_{5\alpha}^*$ admits a $\vec{C}_5$-factorization for all $\alpha \geqslant 215$.*

Note that Abel et al. [1] were investigating the existence of resolvable perfect Mendelsohn designs with blocks of size 5. These can be viewed as $\vec{C}_5$-factorizations of $K^*_{5\alpha}$ with further constraints.

We now proceed to the case of the directed Oberwolfach problem with a single table. In this case, we aim to determine when K^*_n is hamiltonian decomposable. Any decomposition into directed hamiltonian cycles is a directed 2-factorization; each directed cycle in such a decomposition is a directed 2-factor. Therefore, a hamiltonian decomposable digraph admits a directed 2-factorization comprised of directed hamiltonian cycles. Corollary 2.1 implies that K^*_n admits a $\vec{C}_n$-factorization for all odd $n \geqslant 3$. In 1980, Tillson [68] showed that for n even and $n \geqslant 8$, the digraph K^*_n is hamiltonian decomposable. The necessary conditions of Theorem 2.5 follow from [16].

Theorem 2.5. *[16, 68] Let n be an even integer. The digraph K^*_n is hamiltonian decomposable if and only if $n \not\in \{4, 6\}$.*

No other progress was made on the directed Oberwolfach problem until 2012 when Burgess and Šajna [24] used recent advances [47] to make substantial progress on the uniform table case. The following theorem, proved by Liu [47], is key to Burgess and Šajna's investigation and is also a fundamental result in the field of cycle decompositions.

Theorem 2.6. *[47] The graph $K_{n[m]}$ admits a C_t-factorization if and only if $t|mn$, $m(n-1)$ is even, t is even when $n = 2$, and $(m, n, t) \not\in \{(2, 3, 3), (6, 3, 3), (2, 6, 3), (6, 2, 6)\}$.*

Now, we are ready to summarize the work of Burgess and Šajna [24] on the uniform table case of the directed Oberwolfach problem. The case in which we have an odd number of tables is completely solved as is the case with an even number of tables of even length, as stated in Theorem 2.9 below. Before we proceed with the proof of Theorem 2.9, we prove the following two lemmas which are used by Burgess and Šajna to simplify the proofs of Theorems 2.9 and 2.11. We note that we will also refer to Lemmas 2.7 and 2.8 below in Chapters 3 and 4.

Lemma 2.7. *[24] Let H and F be digraphs such that H admits an F-factorization, and n a positive integer. Then nH admits an F-factorization.*

Proof: Let $nH = \bigcup_{i=0}^{n-1} H_i$ where $H_0, \ldots, H_{n-1}$ are pairwise disjoint and isomorphic to H. For each i, let $\{\mathcal{F}_1^i, \mathcal{F}_2^i, \ldots, \mathcal{F}_k^i\}$ be the set of F-factors of an F-factorization of H_i. For $j \in \{1, \ldots, k\}$, define $F_j = \bigcup_{i=0}^{n-1} \mathcal{F}_j^i$. Then F_j is a subdigraph of nH with vertex set $V(nH)$ and is a disjoint union of copies of F because the n copies of H are pairwise disjoint. An arc of nH that belongs to $A(H_i)$ and $\mathcal{F}_j^i$ is in $A(F_j)$. Since $\{F_j^i \mid i = 0, 1, \ldots, n-1 \text{ and } j = 0, 1, \ldots, k\}$ is a decomposition of nH, it follows that F_t and F_r are edge-disjoint when $t \neq r$ and thus, the set $\{F_1, F_2, \ldots, F_k\}$ is an F-factorization of nH. Therefore, the digraph nH admits an F-factorization, as desired.

∎

Lemma 2.8. *[24] Let $\{H_0, H_1, \ldots, H_k\}$ be a decomposition of a digraph G into spanning subdigraphs and let F be a digraph such that each H_i admits an F-factorization. Then G admits an F-factorization.*

Proof: For each i, let $\mathcal{D}_i$ be an F-factorization of H_i. Then, define $\mathcal{D} = \bigcup_{i=0}^{k} \mathcal{D}_i$. Since each $\mathcal{D}_i$ is a set of F-factors, so is $\mathcal{D}$. Hence, $\mathcal{D}$ is an F-factorization of G since the set $\{A(H_0), A(H_1), \ldots, A(H_k)\}$ partitions $A(G)$. ∎

Now, we give the proof of one of the main results of [24]. We mainly focus on the general construction using Lemmas 2.7 and 2.8 because we shall use them in a similar fashion in Chapter 3. Hence, we leave out one specific case, namely the case in which tables are of size 6. Although this specific case is itself interesting, it is not directly related to the investigation conducted in this thesis.

Theorem 2.9. *[24] Let $m \geqslant 5$. If m is even, or α and m are both odd, then $K_{\alpha m}^*$ admits a $\vec{C}_m$-factorization if and only if $(\alpha, m) \neq (1, 6)$.*

Proof: We consider four cases.

Case 1: m and α are both odd. Then, αm is odd and thus, $K_{\alpha m}^*$ admits a $\vec{C}_m$-factorization by (S1) of Corollary 2.1.

Case 2: m is even and $m > 6$. The digraph $K_{\alpha m}^*$ can be decomposed as follows:

$$\begin{aligned} K^*_{\alpha m} &= (\overline{K}_\alpha \wr K^*_m) \oplus (K^*_\alpha \wr \overline{K}_m) \\ &= \alpha K^*_m \oplus K^*_{\alpha[m]}. \end{aligned}$$

That is, the digraph $K^*_{\alpha m}$ is decomposed into α disjoint copies of K^*_m and one copy of $K^*_{\alpha[m]}$ that contains all arcs between these α copies of K^*_m. Since K^*_m admits a $\vec{C}_m$-factorization by Theorem 2.5, Lemma 2.7 implies that αK^*_m admits a $\vec{C}_m$-factorization. Moreover, from Theorem 2.6, we know that $K^*_{\alpha[m]}$ also admits a $\vec{C}_m$-factorization. Therefore, Lemma 2.8 implies that there exists a $\vec{C}_m$-factorization of $K^*_{\alpha m}$, as desired.

Case 3: $m = 6$ and α is even. This case is particularly challenging because K^*_6 does not admit a $\vec{C}_6$-decomposition by Theorem 2.5, and $K_{6,6}$ does not admit a C_6-factorization by Theorem 2.6. Instead, Burgess and Šajna construct a $\vec{C}_6$-factorization of K^*_{12}. This $\vec{C}_6$-factorization of K^*_{12} is then used to find a $\vec{C}_6$-factorization of $K^*_{6\alpha}$ for all even $\alpha \geqslant 2$. See [24] for a full proof.

Case 4: $m = 6$ and α is odd. An approach that is similar to Case 2 is taken. This time, the digraph $K^*_{\alpha m}$ is decomposed into one copy of $6K^*_\alpha$ and one copy of $K^*_{m[\alpha]}$. ∎

The proof of Theorem 2.9 allows us to see how known cycle decompositions can be used to form new cycle decompositions and thus solve problems that were once thought of as difficult.

When α is divisible by 4 and m is odd, Burgess and Šajna also provide a solution, as seen in Theorem 2.10 below.

Theorem 2.10. *[24] Suppose that $m \geqslant 5$ is an odd integer and that $\alpha \equiv 0 \pmod 4$. Then $K^*_{\alpha m}$ admits a $\vec{C}_m$-factorization.*

The case in which the number of tables of odd size is congruent to 2 modulo 4 remains unsolved for most m values. However, Burgess and Šajna [24] were able to take an important reduction step.

Theorem 2.11. *[24] Suppose that $m \geqslant 5$ is odd and that α is even. If there exists a $\vec{C}_m$-factorization of K^*_{2m}, then there exists a $\vec{C}_m$-factorization of $K^*_{\alpha m}$.*

Proof: Suppose that $\alpha = 2\beta$, where β is an integer. Then,

$$K^*_{2\beta m} = (\beta K^*_{2m}) \oplus K^*_{\beta[2m]}.$$

From the given hypothesis and Lemma 2.7, it follows that βK_{2m}^* admits a $\vec{C}_m$-factorization. Moreover, $2m(\beta-1)$ is even. Therefore, by orienting a C_m-factorization of $K_{\beta[2m]}$ from Theorem 2.6, we obtain a $\vec{C}_m$-factorization of $K_{\beta[2m]}^*$. Thus $K_{2\beta m}^*$ admits a $\vec{C}_m$-factorization by Lemma 2.8. ∎

Therefore, in order to complete the solution of the directed Oberwolfach problem with tables of uniform length, it suffices to construct a $\vec{C}_m$-factorization of K_{2m}^* for m odd. Unfortunately, this particular problem is itself difficult. Little progress had been made at the start of the thesis. In fact, solutions were only known to exist for all odd m such that $5 \leqslant m \leqslant 49$, as per Theorem 2.12 proven in 2018.

Theorem 2.12. *[23] If m is odd and $5 \leqslant m \leqslant 49$, then K_{2m}^* admits a $\vec{C}_m$-factorization.*

Despite the fact that solutions have been hard to find, it is conjectured that the necessary conditions are in fact sufficient for all odd $m \geqslant 5$.

Conjecture 2.13. *[24] If m is a positive odd integer, then the digraph K_{2m}^* admits a $\vec{C}_m$-factorization if and only if $m \geqslant 5$.*

In Chapter 3, we completely resolve Conjecture 2.13 using methods that are similar to those introduced in this section and the next section.

We conclude this section by discussing existing results on the directed Oberwolfach problem with tables of varying lengths. There exist very few results regarding this more general case of the directed Oberwolfach problem. In [62], Shanabi and Šajna construct a $(\vec{C}_{m_1}, \vec{C}_{m_2}, \ldots, \vec{C}_{m_t})$-factorization of K_n^* when $m_1+m_2+\ldots+m_t = n$, $(m_1, m_2, \ldots, m_t) = (2, 2, \ldots, 2, 3)$, and $n \equiv 1, 3, 7 \pmod{8}$. In [40], Kadri and Šajna obtain far more general results using a recursive approach. One of the key results of [40] is a near-complete solution to the directed Oberwolfach problem with two tables formulated in Theorem 2.14 below.

Theorem 2.14. *[40] Let m_1 and m_2 be integers such that $2 \leqslant m_1 \leqslant m_2$ and $m_1 + m_2 = n$. Then K_n^* admits a $(\vec{C}_{m_1}, \vec{C}_{m_2})$-factorization if and only if $(m_1, m_2) \neq (3, 3)$, with a possible exception in the case that $m_1 \in \{4, 6\}$, m_2 is even, and $n \geqslant 14$.*

Remark 2.15. The case of Theorem 2.14 in which $m_1 = m_2$, and m_1 and m_2 are odd was resolved by using results that appear in Chapter 3 of this thesis.

Lastly, in [37], Horsley and Lacaze-Masmonteil completed the solution to the directed Oberwolfach problem with two tables as follows.

Theorem 2.16. *[37] The digraph K_n^* admits a $(\vec{C}_{m_1}, \vec{C}_{m_2})$-factorization when $m_1 + m_2 = n$, $n \geqslant 14$, $m_1 \in \{4, 6\}$, and m_2 is even.*

2.2 Decomposition of K_n into bipartite 2-factors

Over the course of the Oberwolfach problem's rich history, researchers have developed numerous techniques to construct solutions. In this section, we will be focussing on a particular constructive method that was first introduced by Häggkvist in [34] and later used in [21]. We do so because the methods used in Chapter 3 were inspired from those of [34] and [21]. Refer to [25] for a detailed survey of other constructive methods for the Oberwolfach problem and its variants.

Both [34] and [21] use the same approach to jointly resolve the Oberwolfach problem with cycles of even lengths. We first note that the authors of [34] and [21] investigate the graph K_{2m} since the cycles in each 2-factor are of even length. The main idea is to first decompose K_m into spanning k-regular subgraphs with k small. In [34], where m is odd, these subgraphs are all hamiltonian cycles. In [21], where m is assumed to be even, these spanning subgraphs are all hamiltonian cycles except for one subgraph, G_1, which is the union of a hamiltonian cycle with a carefully chosen 1-factor of K_m. These decompositions of K_m are used to decompose K_{2m} into spanning subgraphs of the form $C_m \wr \overline{K}_2$ and $C_m \wr K_2$ when m is odd [34], and $C_m \wr \overline{K}_2$ and $G_1 \wr K_2$ when m is even [21]. The crux of [34] and [21] is to demonstrate that, for any combination of even integers $m_1, m_2, \ldots, m_t$ such that $m_1 + m_2 + \ldots + m_t = 2m$, $4 \leqslant m_1 \leqslant m_2 \leqslant \ldots \leqslant m_t$, the graph $C_m \wr \overline{K}_2$ admits a $(C_{m_1}, C_{m_2}, \ldots, C_{m_t})$-factorization, and $C_m \wr K_2$ or $G_1 \wr K_2$ admits a decomposition into one 1-factor and two or three $(C_{m_1}, C_{m_2}, \ldots, C_{m_t})$-factors, respectively.

We will now describe Häggkvist's approach in further detail starting with a crucial result in cycle decomposition. Note that the notation used in this thesis to describe results of [34] differ from the original notation.

Lemma 2.17. *[34] Let $m \geqslant 2$ and $4 \leqslant m_1 \leqslant m_2 \leqslant \ldots \leqslant m_t$ be even integers such that $m_1 + m_2 + \ldots + m_t = 2m$. The graph $C_m \wr \overline{K}_2$ admits a $(C_{m_1}, C_{m_2}, \ldots, C_{m_t})$-factorization.*

Lemma 2.17 has been used so often in the literature that it is commonly referred to as Häggkvist's Lemma. To make use of Lemma 2.17, Häggkvist needed Theorem 2.18 given below. Theorem 2.18 is another fundamental result in cycle decompositions and a key ingredient in the proof of Theorem 2.19.

Theorem 2.18. *[49] If n is odd, then K_n is hamiltonian decomposable. If n is even, then $K_n - I$ is hamiltonian decomposable.*

Häggkvist uses Lemma 2.17 and Theorem 2.18, to obtain the following solution to the Oberwolfach problem. We point out that Theorem 2.19 corresponds to Corollary 2 in [34].

Theorem 2.19. *[34] Let $m \geqslant 4$ be odd and $4 \leqslant m_1 \leqslant m_2 \leqslant \ldots \leqslant m_t$ be even integers such that $m_1 + m_2 + \ldots + m_t = 2m$. The graph $K_{2m} - I$ admits a $(C_{m_1}, C_{m_2}, \ldots, C_{m_t})$-factorization.*

Proof: To prove the desired statement, we will show that K_{2m} admits a decomposition into one 1-factor and $m-1$ $(C_{m_1}, C_{m_2}, \ldots, C_{m_t})$-factors.

By Theorem 2.18, the graph K_m admits a decomposition into $\frac{m-1}{2}$ copies of C_m. Furthermore, we note that $C_m \wr K_2 = (C_m \wr \overline{K}_2) \oplus (\overline{K}_m \wr K_2)$. We then see that

$$\begin{aligned} K_{2m} &= K_m \wr K_2 \\ &= (C_m \oplus C_m \oplus \ldots \oplus C_m) \wr K_2 \\ &= (C_m \wr K_2) \oplus (C_m \wr \overline{K}_2) \oplus (C_m \wr \overline{K}_2) \oplus \ldots \oplus (C_m \wr \overline{K}_2) \\ &= (C_m \wr \overline{K}_2) \oplus (\overline{K}_m \wr K_2) \oplus (C_m \wr \overline{K}_2) \oplus \ldots \oplus (C_m \wr \overline{K}_2). \end{aligned}$$

Notice that $\overline{K}_m \wr K_2 = mK_2$. Therefore, the subgraph $I = mK_2$ is a 1-factor of K_{2m}. By Lemma 2.17, each of the $\frac{m-1}{2}$ copies of $C_m \wr \overline{K}_2$ in the decomposition of K_{2m} is a spanning subgraph of K_{2m} that admits a $(C_{m_1}, C_{m_2}, \ldots, C_{m_t})$-factorization comprised of two $(C_{m_1}, C_{m_2}, \ldots, C_{m_t})$-factors. Therefore, we see that K_{2m} admits a decomposition into $m-1$ $(C_{m_1}, C_{m_2}, \ldots, C_{m_t})$-factors and one 1-factor, as desired. ∎

It is more complicated to prove the analogous statement to Theorem 2.19 for m is even because a key ingredient to Häggkvist's proof of Theorem 2.19 is a decomposition of K_m into hamiltonian cycles. Yet, by Veblen's Theorem, it is known that K_m is not hamiltonian decomposable when m is even. To address this particular case of the Oberwolfach problem, Bryant and Danziger [21] decompose K_{2m} into spanning subgraphs that fall into two particular isomorphism classes. To describe one of these isomorphism classes, we introduce the following class of graphs.

Definition 2.20. Let $m \geqslant 4$ be an even integer. The graph $Y(m, \{1, 3^e\})$ is the graph with vertex set $\mathbb{Z}_m$ and edge set

$$\{\{i, i+1\} \mid i \in \mathbb{Z}_m\} \cup \{\{i, i+3\} \mid i \in \mathbb{Z}_m \text{ is even}\},$$

with $i + 1$ and $i + 3$ evaluated modulo m.

The edges in $\{\{i, i+3\} \mid i \in \mathbb{Z}_m \text{ is even}\}$ form a 1-factor of K_m. Bryant and Danziger [21] then proceed to prove the following proposition.

Proposition 2.21. *[21] Let $m \geqslant 4$ be an even integer and let $4 \leqslant m_1 \leqslant m_2 \leqslant \ldots \leqslant m_t$ be even integers such that $m_1 + m_2 + \ldots + m_t = 2m$. The graph $Y(m, \{1, 3^e\}) \wr K_2$ admits a decomposition into one 1-factor and three $(C_{m_1}, C_{m_2}, \ldots, C_{m_t})$-factors.*

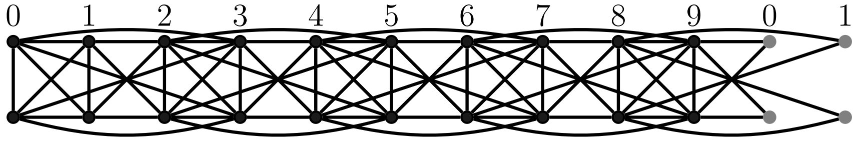


Figure 2.1: The graph $Y(10, \{1, 3^e\}) \wr K_2$.

See Figure 2.1 for an illustration of $Y(10, \{1, 3^e\}) \wr K_2$. The proof of Proposition 2.21 is the main achievement of [21]. In conjunction with Lemma 2.17, it allows Bryant and Danziger to obtain their main result formulated in Theorem 2.22 below.

Theorem 2.22. *[21] Let $m \geqslant 4$ be even and $4 \leqslant m_1 \leqslant m_2 \leqslant \ldots \leqslant m_t$ be even integers such that $m_1 + m_2 + \ldots + m_t = 2m$. The graph $K_{2m} - I$ admits a $(C_{m_1}, C_{m_2}, \ldots, C_{m_t})$-factorization.*

To prove Theorem 2.22, Bryant and Danziger first show that K_{2m} admits a decomposition into one copy of $Y(m, \{1, 3^e\}) \wr K_2$ and $\frac{m-4}{2}$ copies of $C_m \wr \overline{K}_2$. Then, they use Proposition 2.21 and Lemma 2.17 to obtain a decomposition of K_{2m} into one 1-factor and $m-1$ $(C_{m_1}, C_{m_2}, \ldots, C_{m_t})$-factors.

One of the main purposes of this thesis is to adapt the method discussed in this section to the directed Oberwolfach problem. Interestingly, we are successful in adapting this method, which generally works for bipartite 2-factors, to resolve Conjecture 2.13, which considers a case of the directed Oberwolfach problem with cycles of odd length.

2.3 Directed hamiltonian cycle decompositions of the wreath product of digraphs

We conclude this chapter by presenting the second problem addressed in this thesis which investigates cycle decomposition of products of graphs. As seen in Chapter 1, this research area has its own rich history. One of the earliest result on this particular type of problems can be found in [45] where Laskar showed that the graphs $C_n \wr C_m$ and $C_n \wr \overline{K}_t$ are hamiltonian decomposable. In 1978, Bermond [15] conjectured that, if G and H are both hamiltonian decomposable, then so is $G \wr H$. This conjecture was settled by Baranyai and Szász [12] in 1981. An analogous conjecture for digraphs was also made. This conjecture is the focus of this section and is thus formally stated below.

Conjecture 2.23. *If G and H are strict hamiltonian decomposable digraphs such that $G \neq \overline{K}_n$, then $G \wr H$ is also hamiltonian decomposable.*

Note that the digraph $\overline{K}_n$ is considered to be a hamiltonian decomposable digraph. If $G = \overline{K}_n$, then $G \wr H \cong nH$. Clearly, the digraph nH is not hamiltonian decomposable.

The origin of Conjecture 2.23 is unknown. In [52], Ng incorrectly attributes this conjecture to Alspach et al. [4]. Bermond [15], who made the initial conjecture for graphs, did suggest that the analogous problem for directed graph could be considered but made no formal conjecture. In [12], Baranyai and Szász claimed that their construction resolved Conjecture 2.23 when $|V(G)|$ and $|V(H)|$ are odd but stated that they were unsure if the statement was true for other cases. Ng [52] later pointed out that this claim of Baranyai and Szász's was false when H admits a decomposition into more than $|V(G)|+2$ directed hamiltonian cycles.

In 1998, Ng [52] solved a large case of Conjecture 2.23. This paper of Ng's was part of the very successful undergraduate research program run by Joseph Gallian, which produced over 240 papers, all by undergraduate students, over the span of 30 years. The remainder of this section is spent summarizing the results and methods of [52].

Ng approaches this problem in two steps. First, he constructs a directed hamiltonian decomposition for the digraph $\vec{C}_s \wr \overline{K}_r$ with $r > 2$. Then, using this decomposition, he constructs a directed hamiltonian decomposition for $\vec{C}_s \wr H$ where H is any strict digraph admitting a directed hamiltonian decomposition, s is odd, and $|V(H)| > 2$. These two decompositions are then used to form the desired directed hamiltonian decomposition of $G \wr H$ where G is any digraph of odd order that also admits a directed hamiltonian decomposition.

Before we proceed, we note that, if G and H are both hamiltonian decomposable, then both are regular digraphs. If G is k-regular, then a directed hamiltonian decomposition of G comprises of k directed cycles.

There are two lemmas, namely Lemma 2.24 and 2.25 below, that are key to Ng's investigation.

Lemma 2.24. *[52] If $r > 2$, then $\vec{C}_s \wr \overline{K}_r$ is hamiltonian decomposable.*

Ng [52] breaks down the proof of Lemma 2.24 into four cases according to the congruency of s modulo 4. The cases with s odd are key to the construction given in Ng's proof of Lemma 2.25 stated below.

Lemma 2.25. *[52] If s is an odd integer, $|V(H)| > 2$, and H is a strict hamiltonian decomposable digraph, then $\vec{C}_s \wr H$ is also hamiltonian decomposable.*

Suppose that $|V(H)| = r$. The crux of the proof of Lemma 2.25 is using a 2-factorization of $\vec{C}_s \wr \overline{K}_r$. Ng takes the union of one directed hamiltonian cycle of this decomposition, and pairs it with s pairwise disjoint copies of $\vec{C}_r$ arising from a directed hamiltonian decomposition of H. This results in a subdigraph of $\vec{C}_s \wr H$. Each of these copies of $\vec{C}_r$ is embedded in a distinct copy of H. We direct the reader to

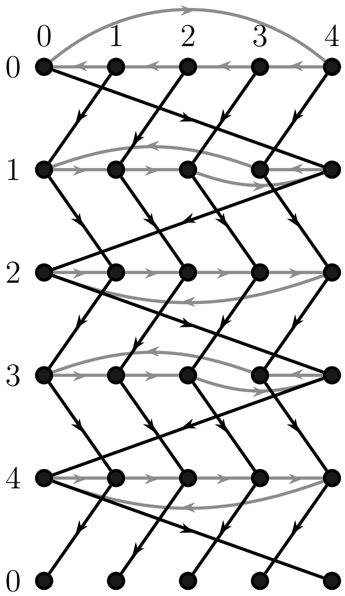


Figure 2.2: The union of one directed hamiltonian cycle in a directed hamiltonian decomposition of $\vec{C}_5 \wr \overline{K}_5$ and 5 copies of $\vec{C}_5$.

Figure 2.3 to help understand this approach. In Figure 2.3, we have a subdigraph of $\vec{C}_5 \wr H$, where $|V(H)| = 5$. We drew each copy of $\vec{C}_5$ in pink, while the arcs belonging to a directed hamiltonian cycle of $\vec{C}_5 \wr \overline{K}_5$ are drawn in black. The five copies of $\vec{C}_5$ need not arise from the same 5-cycle, nor the same hamiltonian decomposition of H. The digraph in Figure 2.3 is then denoted L. (Note that L is not necessarily the digraph $\vec{C}_s \wr \vec{C}_r$.) Ng shows that L admits a decomposition into two directed hamiltonian cycles. This approach was inspired by the work of Baranyai and Szász for the undirected case [12].

Lemmas 2.24 and 2.25 can then be combined to prove the crown jewel of Ng's paper stated in Theorem 2.26 below.

Theorem 2.26. *[52] Let G and H be two strict digraphs such that $|V(G)|$ is odd, $G \neq \overline{K}_n$, and $|V(H)| > 2$. If G and H are hamiltonian decomposable, then $G \wr H$ is hamiltonian decomposable.*

Proof: Suppose that all vertices of G have out-degree k, $|V(G)| = s$, and $|V(H)| = r$. By assumption, the digraph G can be decomposed into k directed hamiltonian cycles. Therefore, the digraph $G \wr H$ can be decomposed into one copy of $\vec{C}_s \wr H$ and

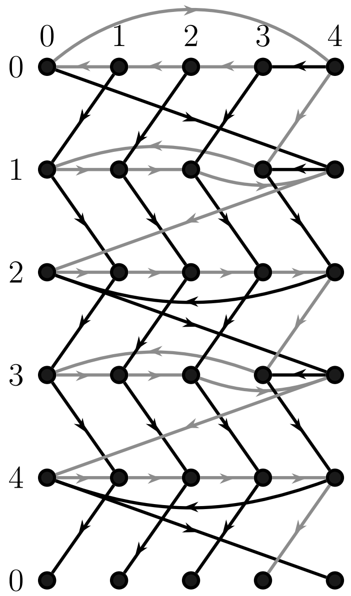


Figure 2.3: A decomposition of $\vec{C}_5 \wr \overline{K}_5$ into two directed hamiltonian cycles drawn in grey and black, respectively.

$k-1$ copies of $\vec{C}_s \wr \overline{K}_r$. By Lemmas 2.25 and 2.24, respectively, digraphs $\vec{C}_s \wr H$ and $\vec{C}_s \wr \overline{K}_r$ are hamiltonian decomposable. In conclusion, Lemma 2.8 implies that $G \wr H$ is hamiltonian decomposable. ∎

Therefore, we see that Conjecture 2.23 has been settled when the order of G is odd and $|V(H)| > 2$. In Chapter 4, we will address the majority of the remaining cases of Conjecture 2.23.

Chapter 3

Completing the solution of the directed Oberwolfach problem with cycles of equal length

In order to resolve the directed Oberwolfach Problem for an even number of tables of odd uniform length, Theorem 2.11 implies that it suffices to show that K_{2m}^* admits a $\vec{C}_m$-factorization for all odd $m \geqslant 5$. In this chapter, we construct a $\vec{C}_m$-factorization of K_{2m}^* for all odd $m \geqslant 11$. Our construction, in conjunction with Theorem 2.12, resolves Conjecture 2.13. Consequently, our results, along with results from [1, 2, 13, 16, 18, 23, 24, 68], completely resolve the directed Oberwolfach problem with tables of uniform length. Lastly, the main results of this chapter have appeared in the Journal of Combinatorial Designs [44].

Before we proceed, we give a brief outline of this chapter. In Section 3.1, we strategically decompose K_{2m}^* into particular spanning subdigraphs. These spanning subdigraphs fall into one of three isomorphism classes. Therefore, this step can be viewed as a reduction step since it narrows down our problem to the existence of a $\vec{C}_m$-factorization of three particular classes of digraphs, each requiring four or five $\vec{C}_m$-factors. We then show that each of these subdigraphs admits a $\vec{C}_m$-factorization for all m such that $m \equiv 1, 5 \pmod{6}$ and $m \geqslant 11$. Lemma 2.8 then implies that K_{2m}^* admits a $\vec{C}_m$-factorization for all m such that $m \equiv 1$ or $5 \pmod{6}$ and $m \geqslant 11$. Lastly, in Section 3.5, we use this result, along with results of [23], to obtain a $\vec{C}_m$-factorization of K_{2m}^* when $m \equiv 3 \pmod{6}$ and $m \neq 3$. Unfortunately, the approach taken in this chapter cannot be used to construct a $\vec{C}_m$-factorization of K^*_{2m} for $m \in \{5, 7\}$. For these values, we rely on Theorem 2.12.

3.1 Reduction step

In this section, our objective is to show that K^*_{2m} can be decomposed into $\frac{m-3}{2}$ spanning subdigraphs that are 4-regular or 9-regular. One of these subdigraphs falls into one isomorphism class while the remaining $\frac{m-5}{2}$ subdigraphs fall into another class. In turn, we can further decompose the 9-regular digraph into two spanning subdigraphs, meaning that it suffices to construct a $\vec{C}_m$-factorization for three classes of digraphs.

Before we proceed, we introduce some terminology related to the three classes of digraphs discussed in this chapter. This will also enable us to better describe the $\vec{C}_m$-factorizations given in the following sections.

Definition 3.1. Let m be an odd integer and let

$$\begin{aligned} H_{2m} &= \vec{X}(m, \{\pm 1\}) \wr \overline{K}_2, \\ L_{2m} &= \vec{X}(m, \{1, 3\}) \wr \overline{K}_2, \text{and} \\ G_{2m} &= \vec{X}(m, \{1, 3\}) \wr K^*_2. \end{aligned}$$

Notation 3.2. Let m be an odd integer and let $S \subseteq \mathbb{Z}_m$. We assume that

$$\begin{aligned} &V(\overline{K}_2) = V(K^*_2) = \{x, y\} \text{ and} \\ &V(\vec{X}(m, S) \wr \overline{K}_2) = V(\vec{X}(m, S) \wr K^*_2) = \{x_a, y_b \mid a, b \in \mathbb{Z}_m\} \end{aligned}$$

where $x_a = (a, x)$ and $y_b = (b, y)$.

Notation 3.2 is stated in more general terms than Definition 3.1 because, in Section 3.2, we will be briefly investigating digraphs of the form $\vec{X}(m, S) \wr \overline{K}_2$ and $\vec{X}(m, S) \wr K^*_2$, where $S = \{\pm 1\}$ or $S = \{\pm 1, \pm 2\}$.

If an arc is of the form (x_a, x_b), (x_a, y_b), (y_a, y_b), or (y_a, x_b), then this arc is of *difference* $b - a$. Differences are computed modulo m. Arcs of the form (x_i, x_{i+d}) and (y_i, y_{i+d}) are called arcs of *pure* x*-difference* d and arcs of *pure* y*-difference* d, respectively. Moreover, arcs of the form (x_i, y_{i+d}) and (y_i, x_{i+d}) are called arcs of *mixed* x*-difference* d or *mixed* y*-difference* d, respectively. In addition, for each $i \in \mathbb{Z}_m$, arcs of the form (x_i, y_i) and (y_i, x_i) are called *vertical arcs*. Observe that a directed odd cycle of H_{2m}, L_{2m} or G_{2m} must contain an even number of arcs of mixed difference and thus, an odd number of arcs of pure difference.

Let $W = (v_0, v_1, \ldots, v_n)$ be a walk in $\vec{X}(m,S) \wr \overline{K}_2$ or $\vec{X}(m,S) \wr K_2^*$, with consecutive arcs of difference $d_0, d_1, \ldots, d_{n-1}$. If $r = d_0 + d_1 + \ldots + d_{n-1}$, then we say that the arcs of W *sum* to r. If W is a closed walk of $\vec{X}(m,S) \wr \overline{K}_2$ or $\vec{X}(m,S) \wr K_2^*$, then $r \equiv 0 \pmod{m}$. A *type-k* cycle of $\vec{X}(m,S) \wr \overline{K}_2$ or $\vec{X}(m,S) \wr K_2^*$ is a directed m-cycle whose arcs sum to km or $-km$.

In Example 3.3 below, we give simple examples illustrating some of the definitions introduced above.

Example 3.3. In Figure 3.1, we illustrate the digraph L_{22}. Note that, in Figure 3.1, we will assume edges are arcs oriented from left to right.

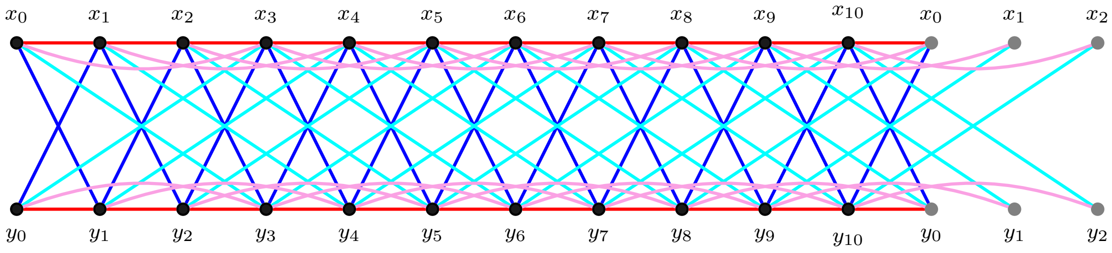


Figure 3.1: The digraph L_{22}. Edges are arcs oriented from left to right.

We point out that edges drawn in red in Figure 3.1 correspond to arcs of pure x-difference 1 and arcs of pure y-difference 1. Edges drawn in pink correspond to arcs of pure x-difference 3 and arcs of pure y-difference 3. Meanwhile, edges drawn in dark blue correspond to arcs of mixed x-difference 1 and mixed y-difference 1. Finally, edges drawn in light blue correspond to arcs of mixed x-difference 3 and mixed y-difference 3.

Next, we let

$$W = (x_0, x_1, y_2, y_3, y_4, x_7).$$

Observe that W has no repeated vertices and is thus a dipath. Arcs of W sum to $t = 1 + 1 + 1 + 1 + 3 = 7$. Below, we give an example of a type-1 directed cycle and a type-3 directed cycle of L_{22}:

$$C^1 = (y_0, x_1, x_2, y_3, x_4, y_7, x_{10}, y_0);$$
$$C^3 = (x_0, y_3, x_6, y_9, x_1, y_4, x_7, y_{10}, x_2, y_5, x_8, x_0).$$

The arcs of C^1 sum to 11 and the arcs of C^3 sum to 33. □

In the proof of Lemma 3.6, we show that K^*_{2m} admits a decomposition into one copy of L_{2m}, one copy of G_{2m}, and $\frac{m-5}{2}$ copies of H_{2m}. To do so, we first give a decomposition of K^*_m in Lemma 3.5. To obtain this decomposition, we make use of Theorem 3.4, which is a special case of results of Bermond et al. [17] on 2-factorizations of Cayley graphs.

Theorem 3.4. *[17] Let $S \subseteq \{1, 2, \ldots, \lfloor\frac{m-1}{2}\rfloor\}$ such that $\gcd(S \cup \{m\}) = 1$. The graph $X(m, S)$ is hamiltonian decomposable.*

Lemma 3.5. *Let m be an odd integer such that $m \geqslant 5$. The digraph K^*_m admits a decomposition into one copy of $\vec{X}(m, \{\pm1, \pm3\})$ and $\frac{m-3}{2}$ copies of $\vec{X}(m, \{\pm1\})$.*

Proof: We begin by strategically decomposing the graph K_m. Then, we replace each edge in that decomposition by a pair of arcs, one for each direction. This then yields the desired decomposition of K^*_m.

If $m = 5$, then $K^*_5 = X(m, \{1, 3\})$. Otherwise, if $m \equiv 1 \pmod 4$ and $m > 5$, then

$$K_m = X(m, \{1, 3\}) \oplus X(m, \{2, 4\}) \oplus X(m, \{5, 6\}) \oplus \ldots \oplus X(m, \{\tfrac{m-3}{2}, \tfrac{m-1}{2}\}).$$

If $m \equiv 3 \pmod 4$, then

$$K_m = X(m, \{1, 3\}) \oplus X(m, \{2\}) \oplus X(m, \{4, 5\}) \oplus \ldots \oplus X(m, \{\tfrac{m-3}{2}, \tfrac{m-1}{2}\}).$$

If m is odd, then $X(m, \{2\}) \cong C_m$. In addition, since all 4-regular spanning subgraphs in both decompositions of K_m satisfy the hypothesis of Theorem 3.4, it is implied that K_m can be decomposed into one copy of $X(m, \{1, 3\})$ and $\frac{m-5}{2}$ hamiltonian cycles. Each hamiltonian cycle of K_m is isomorphic to $X(m, \{1\})$. It follows that K^*_m admits a decomposition into one copy of $\vec{X}(m, \{\pm1, \pm3\})$ and $\frac{m-5}{2}$ copies of $\vec{X}(m, \{\pm1\})$. ▮

We then use Lemma 3.5 to prove the following lemma.

Lemma 3.6. *If each of H_{2m}, L_{2m}, and G_{2m} admits a $\vec{C}_m$-factorization, then K_{2m}^* also admits a $\vec{C}_m$-factorization.*

Proof: Assume that H_{2m}, L_{2m}, and G_{2m} each admits a $\vec{C}_m$-factorization. Below, we show that K_{2m}^* admits a decomposition into one copy of L_{2m}, one copy of G_{2m}, and $\frac{m-5}{2}$ copies of H_{2m}. Lemma 3.5 implies that

$$\begin{aligned}
K_{2m}^* &= K_m^* \wr K_2^* \\
&= \Big((\vec{X}(m,\{\pm1,\pm3\}) \oplus \vec{X}(m,\{\pm1\}) \oplus \ldots \oplus \vec{X}(m,\{\pm1\})\Big) \wr K_2^* \\
&= \Big(\vec{X}(m,\{\pm1,\pm3\}) \wr K_2^*\Big) \oplus \Big(\vec{X}(m,\{\pm1\}) \wr \overline{K}_2\Big) \oplus \ldots \oplus \Big(\vec{X}(m,\{\pm1\}) \wr \overline{K}_2\Big) \\
&= \Big(\vec{X}(m,\{\pm1,\pm3\}) \wr K_2^*\Big) \oplus H_{2m} \oplus \ldots \oplus H_{2m} \\
&= L_{2m} \oplus G_{2m} \oplus H_{2m} \oplus \ldots \oplus H_{2m}.
\end{aligned}$$

Since each of H_{2m}, L_{2m}, and G_{2m} is a spanning subdigraph of K_{2m}^*, Lemma 2.8 implies that K_{2m}^* admits a $\vec{C}_m$-factorization. ▮

In Section 3.3, we will demonstrate that the hypothesis of Lemma 3.6 is satisfied when $m \equiv 1$ or $5 \pmod 6$ and $m \geqslant 11$.

3.2 Simpler but unsuccessful approaches

The attentive reader may have noticed that the decomposition given in the proof of Lemma 3.6 can be further simplified. The purpose of this section is to consider two simpler constructions and to explain why these two constructions do not work. The reader solely interested in the construction of a $\vec{C}_m$-factorization of K_{2m}^* can skip this section.

With our first attempt at this problem, we took a similar approach as Häggkvist [34], which is described in the proof of Theorem 2.19. In our case, we aim to show that one can decompose K_{2m}^* into one copy of $\vec{X}(m,\{\pm1\}) \wr K_2^*$ and $\frac{m-3}{2}$ copies of $\vec{X}(m,\{\pm1\}) \wr \overline{K}_2$.

Lemma 3.7. *Let $m \geqslant 5$ be odd. If the digraphs $\vec{X}(m,\{\pm1\}) \wr K_2^*$ and $\vec{X}(m,\{\pm1\}) \wr \overline{K}_2$ both admit a $\vec{C}_m$-factorization, then K_{2m}^* also admits a $\vec{C}_m$-factorization.*

Proof: By Theorem 2.18, the graph K_m admits a decomposition into hamiltonian cycles. Therefore, the digraph K_m^* admits a decomposition into $\frac{m-1}{2}$ copies of $\vec{X}(m, \{\pm 1\})$. By a similar argument as the proof of Lemma 3.6, it follows that K_{2m}^* admits a decomposition into spanning subdigraphs $\vec{X}(m, \{\pm 1\}) \wr K_2^*$ and $\vec{X}(m, \{\pm 1\}) \wr \overline{K}_2$. Therefore, by Lemma 2.8 and the given hypothesis, it follows that K_{2m}^* admits a $\vec{C}_m$-factorization. ▮

Lemma 3.7 implies that it suffices to show that the digraphs $\vec{X}(m, \{\pm 1\}) \wr K_2^*$ and $\vec{X}(m, \{\pm 1\}) \wr K_2^*$ both admit a $\vec{C}_m$-factorization. It is tempting to take this particular route instead of the proposed approach given in the previous section. Unfortunately, it can be shown that, when $m \geqslant 11$, the digraph $\vec{X}(m, \{\pm 1\}) \wr K_2^*$ does not admit a $\vec{C}_m$-factorization.

Proposition 3.8. *If $m \geqslant 11$, then $\vec{X}(m, \{\pm 1\}) \wr K_2^*$ does not admit a $\vec{C}_m$-factorization.*

Proof: Suppose that $m \geqslant 11$ and that $\vec{X}(m, \{\pm 1\}) \wr K_2^*$ admits a $\vec{C}_m$-factorization. Then this $\vec{C}_m$-factorization is comprised of 10 cycles. Given that $m \geqslant 11$, we have at least 22 vertical arcs. Therefore, by the Pigeonhole Principle, at least one of our 10 cycles, call it C, contains at least 3 vertical arcs. We argue that this leads to a contradiction by showing that C must have repeated vertices other than its endpoints.

First, suppose that C is a type-0 directed cycle. Without loss of generality, suppose that C starts with x_0 and that it contains the following three vertical arcs: (x_0, y_0), (x_j, y_j), and (x_t, y_t) for $0 < j < t < m$. Note that the choice of direction of each arc is made without loss of generality. An analogous argument can be made if one or more of these three arcs is of the form (y_i, x_i). Again, without loss of generality, we also assume that $C = (x_0, y_0, x_1, \ldots, x_0)$; an analogous argument holds if we choose our third vertex to be one of $\{y_1, x_{-1}, y_{-1}\}$. By way of contradiction, suppose that x_t and x_j appear in the following order:

$$C = (x_0, y_0, x_1, \ldots, x_t, \ldots, x_j, y_j, \ldots, x_0).$$

Because $j < t$, the directed cycle C has a repeated vertex, namely x_j or y_j. This is because one of these two vertices must appear on the dipath from x_0 to x_t. Therefore, as endpoint of their respective vertical arc, our vertices x_t and x_j must appear in the following order :

$$C = (x_0, y_0, x_1, \ldots, x_j, y_j \ldots, x_t, y_t, \ldots, x_0).$$

However, recall that C is a type-0 directed cycle. Consequently, the dipath from x_t to x_0 must contain one of x_j or y_j, a contradiction.

Therefore, if C contains three vertical arcs, then C must be a type-1 directed cycle. Note that C cannot be a type-k directed cycle for $k \geqslant 2$ since each arc is of difference 1 or 0. As a result, the arcs of C must sum to m or $-m$. However, if C has at least three vertical arcs, then C has at most $m-3$ arcs of difference 1 and at most $m-3$ arcs of difference -1. Consequently, the absolute value of the sum of the arcs of C is at most $m-3$, a contradiction. In conclusion, if $m \geqslant 11$, then $\vec{X}(m, \{\pm 1\}) \wr K_2^*$ does not admit a $\vec{C}_m$-factorization. ▮

Of course, Proposition 3.8 does not imply that K_{2m}^* does not admit a $\vec{C}_m$-factorization. It simply means that we need to look for a decomposition of K_{2m}^* into slightly more complicated spanning subdigraphs.

In the next approach, we decompose K_{2m}^* into three types of digraphs. In Section 3.1, we construct L_{2m} and G_{2m} using arcs of difference 1 and 3. Simpler digraphs could be constructed by using arcs of difference 1 and 2. The proof of Lemma 3.9, given below, is similar to the proof of Lemmas 3.5 and 3.6 and is thus omitted.

Lemma 3.9. *Let $m \geqslant 5$ be odd. If each of $\vec{X}(m, \{\pm 1\}) \wr \overline{K}_2$, $\vec{X}(m, \{1, 2\}) \wr \overline{K}_2$, and $\overline{X}(m, \{1, 2\}) \wr K_2^*$ admits a $\vec{C}_m$-factorization, then K_{2m}^* also admits a $\vec{C}_m$-factorization.*

However, it can be shown that the digraph $\vec{X}(m, \{1, 2\}) \wr \overline{K}_2$ does not admit a $\vec{C}_m$-factorization when $m \not\equiv 0 \pmod 3$, as stated in Proposition 3.12 below. Before we can prove Proposition 3.12, we first show that, if $\vec{X}(m, \{1, 2\}) \wr \overline{K}_2$ were to admit a $\vec{C}_m$-factorization, then cycles in this decomposition must satisfy certain properties. These properties are given in Lemmas 3.10 and 3.11 below.

Lemma 3.10. *Let $m \geqslant 5$ be odd. A directed cycle of length m of $\vec{X}(m, \{1, 2\}) \wr \overline{K}_2$ cannot contain both arcs of difference 1 and arcs of difference 2.*

Proof: Suppose that C is a directed cycle of type-1 or type-2 that contains arcs of difference 1 and 2. Note that C cannot be of type-0 since all arcs are of positive difference and it cannot be of type $k > 2$ since it is comprised of m arcs of difference 1 or 2.

Let D be the sum of the arc differences of C. We know that $D < 2m$ because not all arcs in C are of difference 2. Yet, we also know that $m < D$ because C contains at least one arc of difference 2. However, we know that $D = m$ or $D = 2m$ since C is of type-1 or type-2, a contradiction. Therefore, each directed cycle of length m of G must contain only arcs of difference 1 or only arcs of difference 2. ∎

Lemma 3.11. *Let $m \geqslant 5$ be odd. If $\vec{X}(m, \{1, 2\}) \wr \overline{K}_2$ admits a $\vec{C}_m$-factorization, denoted $\mathcal{F}$, then each $\vec{C}_m$-factor of $\mathcal{F}$ consists of one type-1 cycle and one type-2 cycle.*

Proof: Suppose that $\mathcal{F}$ is a $\vec{C}_m$-factorization of $\vec{X}(m, \{1, 2\}) \wr \overline{K}_2$. As a result of Lemma 3.10, the decomposition $\mathcal{F}$ must consists of four type-1 directed cycles and four type-2 directed cycles. A type-1 directed cycles only contains arcs of difference 1 and a type-2 directed cycle only contains arcs of difference 2. Since each $\vec{C}_m$-factor consists of two cycles, it suffices to prove that $\mathcal{F}$ does not have a $\vec{C}_m$-factor comprised of two type-1 cycles. We do so by way of contradiction.

Suppose that $\mathcal{F}$ has a $\vec{C}_m$-factor that contains two type-1 cycles, denoted C^0 and C^1. First, we point out that for all i, if C^0 contains x_i (y_i), then C^1 contains y_i (x_i) . To see why this is true, note that a type-1 directed cycle of $\vec{X}(m, \{1, 2\}) \wr \overline{K}_2$ must contain exactly one of x_i and y_i for all i. Since C^0 and C^1 are disjoint and their union spans $\vec{X}(m, \{1, 2\}) \wr \overline{K}_2$, the claim follows. As a result, if C^0 contains the arc (x_i, y_{i+1}), then C^1 contains the arc (y_i, x_{i+1}). On the other hand, if C^0 contains the arc (x_i, x_{i+1}), then C^1 must contain the arc (y_i, y_{i+1}). Note that, since C^0 and C^1 are cycles of odd length, each contains an odd number of arcs of pure difference.

Next, let C^2 be the third type-1 cycle of $\mathcal{F}$. If C^0 contains the arc (x_i, x_{i+1}), then C^1 contains the arc (y_i, y_{i+1}). This means that C^2 contains one of (x_i, y_{i+1}) and (y_i, x_{i+1}). A symmetric argument holds if C^0 were to contain (y_i, y_{i+1}) instead. By a similar logic, if C^0 contains (x_i, y_{i+1}) or (y_i, x_{i+1}), then C^2 contains one of (x_i, x_{i+1}) and (y_i, y_{i+1}). As a result, there exists a bijection between the set of arcs of pure difference of C^0 and the set of arcs of mixed difference of C^2. Therefore, the cycle

C^2 contains an odd number of arcs of mixed difference, which is a contradiction. In summary, a $\vec{C}_m$-factor of $\mathcal{F}$ cannot be comprised of two type-1 directed cycles.

In conclusion, if $\vec{X}(m,\{1,2\}) \wr \overline{K}_2$ admits a $\vec{C}_m$-factorization $\mathcal{F}$, then each $\vec{C}_m$-factor of $\mathcal{F}$ consists of one type-1 cycle and one type-2 cycle. ∎

We now use Lemmas 3.10 and 3.11 to show that $\vec{X}(m,\{1,2\}) \wr \overline{K}_2$ is not hamiltonian decomposable when $m \not\equiv 0 \pmod 3$.

Proposition 3.12. *Let $m \geqslant 5$ be an odd integer such that $m \not\equiv 0 \pmod 3$. The digraph $\vec{X}(m,\{1,2\}) \wr \overline{K}_2$ does not admit a $\vec{C}_m$-factorization.*

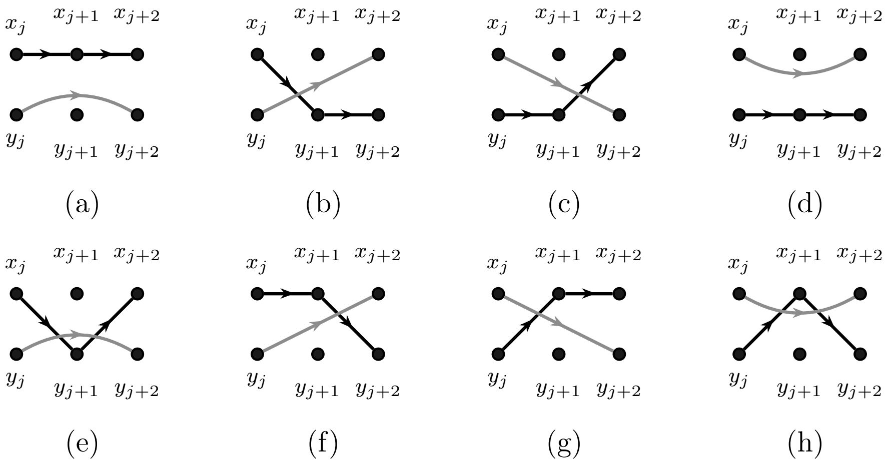
Proof: Suppose that $\vec{X}(m,\{1,2\}) \wr \overline{K}_2$ admits a $\vec{C}_m$-factorization $\mathcal{F}$. By Lemma 3.11, we know that each $\vec{C}_m$-factor consists of one type-1 cycle and one type-2 cycle. For each $i \in \{0,2,4,6\}$, let $\{C^i, C^{i+1}\}$ be a $\vec{C}_m$-factor of $\mathcal{F}$ such that each C^i is a type-1 cycle.

If C^i contains a subdipath (x_j, x_{j+1}, x_{j+2}), then C^{i+1} must contain vertex y_j and the arc of difference two with tail y_j. This arc can only be (y_j, y_{j+2}) since C^0 and C^1 are disjoint. See Figure 3.2a for an illustration of this configuration. An analogous reasoning applies to the other seven possible dipaths of length two of C^0 with source x_j or y_j.

Let C^0 and C^2 be the two type-1 cycles that pass through x_0, and let C^4 and C^6 be the two type-1 cycles that pass through y_0. Without loss of generality, we assume that $C^0 = (x_0, x_1, x_2, x_3, \ldots, x_0)$ and $C^2 = (y_0, y_1, z_2, \ldots, z_{m-2}, y_0)$. An analogous reasoning can be applied to all other cases by switching x_i and y_i as needed. Our objective is to show that C^0 and C^2 intersect at x_3 or y_3.

First, we suppose that C^0 and C^2 intersect at one of x_2 and y_2. In Figures 3.3b and 3.3c we illustrate these two cases. We can then see that, in the case of Figure 3.3b, the type-2 cycles C^1 and C^3 must both contain the arc (y_0, y_2). In the case of Figure 3.3c, we see that C^1 and C^3 must both contain the arc (y_0, x_2). In both cases, this leads to a contradiction. Therefore, the two type-1 cycles C^0 and C^2 cannot intersect at x_2 or y_2. As a result, we know that we must have the following configuration:

$$C^0 = (x_0, x_1, x_2, x_3, \ldots, x_0) \text{ and } C^2 = (x_0, y_1, y_2, \ldots, x_0).$$



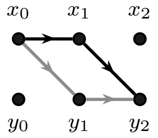
(a) (b) (c) (d)

(e) (f) (g) (h)

Figure 3.2: All possible subdipaths of length 2 (black) of type-1 cycle C^0 in $\vec{X}(m, \{1, 2\}) \wr \overline{K}_2$ and the corresponding arc of difference 2 (grey) of C^1.

(a) First arc of C^0 (black) and C^2 (grey).

(b) Cycles C^0(black) and C^2 (grey) intersect at x_2.



(c) Cycles C^0(black) and C^2 (grey) intersect at y_2.

Figure 3.3: Possible subdipaths of C^0(black) and C^2 (grey).

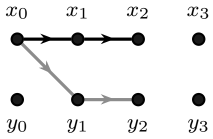


Figure 3.4: First two arcs of C^0 (black) and C^2 (grey) of Case 1.

This configurations is illustrated in Figure 3.4.

Suppose that $C^2 = (x_0, y_1, y_2, y_3, \ldots, x_0)$. This configuration is illustrated in

Figure 3.5. Therefore, without loss of generality, the other two type-1 cycles must start as: $C^4 = (y_0, y_1, x_2, y_3, \ldots, y_0)$ and $C^6 = (y_0, x_1, y_2, x_3, \ldots, y_0)$. This means that C^1 and C^7 both contain the arc (y_1, y_3), a contradiction. In conclusion, if $C^0 = (x_0, x_1, x_2, x_3, \ldots, x_0)$ then $C^2 = (x_0, y_1, y_2, x_3, \ldots, x_0)$. Hence, we see that C^0 and C^2 intersect at x_3.

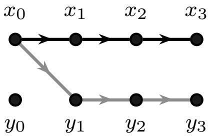


Figure 3.5: First three arcs of C^0 (black) and C^2 (grey).

Therefore, it must be that C^0 and C^2 intersect at one of x_3 and y_3. We point out that this implies that C^4 and C^6 must also intersect at one of x_3 and y_3. By applying the same argument repeatedly, since $m \not\equiv 0 \pmod 3$, we have that C^0 and C^2 intersect at one of x_i and y_i for all $i \in \mathbb{Z}_m$. This is a contradiction because C^0 and C^2 cannot intersect at x_2 nor y_2. ∎

The digraph $\vec{X}(m, \{1,2\}) \wr \overline{K}_2$ is an example of a digraph that satisfies the obvious necessary conditions for a digraph to admit a $\vec{C}_m$-factorization but that does not admit a $\vec{C}_m$-factorization.

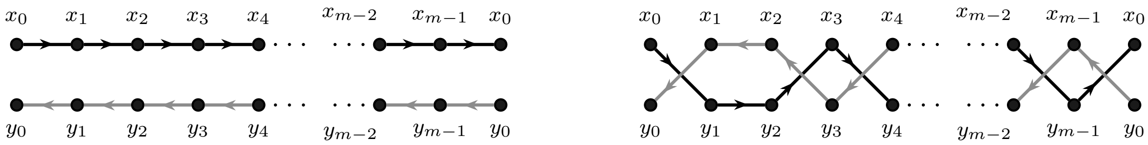
3.3 Decomposing H_{2m} and L_{2m}

In this section, we show that H_{2m} and L_{2m}, as defined in Definition 3.1, admit a $\vec{C}_m$-factorization when $m \equiv 1$ or $5 \pmod 6$ and $m \geqslant 11$. First, we construct a $\vec{C}_m$-factorization of the digraph H_{2m} in the proof of Proposition 3.13 for all odd $m \geqslant 3$. Then, for $m \equiv 1$ or $5 \pmod 6$ and $m \geqslant 11$, we construct a $\vec{C}_m$-factorization of L_{2m} in the proof of Theorem 3.14.

Proposition 3.13. *Let $m \geqslant 3$ be an odd integer. The digraph H_{2m} admits a $\vec{C}_m$-factorization.*

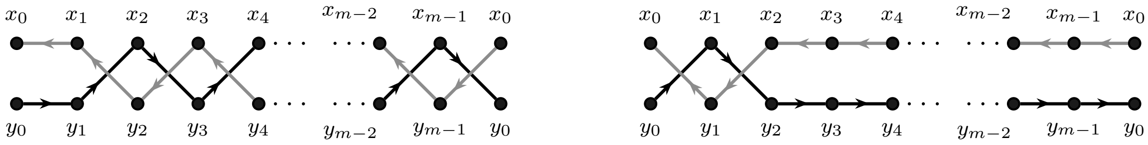
Proof: In order to prove the desired statement, we construct eight directed cycles of length m in H_{2m}:

$$C^0 = (x_0, x_1, x_2, x_3, \ldots, x_{m-3}, x_{m-2}, x_{m-1}, x_0);$$
$$C^1 = (y_0, y_{m-1}, y_{m-2}, y_{m-3}, \ldots, y_5, y_4, y_3, y_2, y_1, y_0);$$
$$C^2 = (x_0, y_1, y_2, x_3, y_4, x_5, y_6, x_7 \ldots, y_{m-3}, x_{m-2}, y_{m-1}, x_0);$$
$$C^3 = (y_0, x_{m-1}, y_{m-2}, x_{m-3}, \ldots, y_5, x_4, y_3, x_2, x_1, y_0);$$
$$C^4 = (y_0, y_1, x_2, y_3, x_4, \ldots, x_{m-3}, y_{m-2}, x_{m-1}, y_0);$$
$$C^5 = (x_0, y_{m-1}, x_{m-2}, y_{m-3}, \ldots, x_5, y_4, x_3, y_2, x_1, x_0);$$
$$C^6 = (y_0, x_1, y_2, y_3, y_4, y_5, y_6, y_7 \ldots, y_{m-3}, y_{m-2}, y_{m-1}, y_0);$$
$$C^7 = (x_0, x_{m-1}, x_{m-2}, x_{m-3}, \ldots, x_5, x_4, x_3, x_2, y_1, x_0).$$



(a) Directed cycles C^0 (black) and C^1 (grey).

(b) Directed cycles C^2 (black) and C^3 (grey).



(c) Directed cycles C^4 (black) and C^5 (grey).

(d) Directed cycles C^6 (black) and C^7 (grey).

Figure 3.6: The directed cycles C^i.

In Figure 3.6, we illustrate each directed cycle constructed above. It can be verified that, for each even i, the directed cycles C^i and C^{i+1} are disjoint. Therefore, for each $i \in \{0, 2, 4, 6\}$, $F_i = C^i \cup C^{i+1}$ is a $\vec{C}_m$-factor. Furthermore, we point out that each arc of H_{2m} occurs precisely once in these four $\vec{C}_m$-factors. Consequently, the set $\{F_0, F_2, F_4, F_6\}$ is a $\vec{C}_m$-factorization of H_{2m}, as desired. ∎

Now, we proceed with the construction of a $\vec{C}_m$-factorization of L_{2m}. We also show that, when $m \equiv 3 \pmod 6$, the digraph L_{2m} does not admit a $\vec{C}_m$-factorization. As for sufficiency, we consider two cases, one for each remaining congruency class of m modulo 6. Each $\vec{C}_m$-factor will consist of one type-1 directed cycle and one type-3 directed cycle.

Before we proceed with our solution, we make the following remark pertaining to the digraphs drawn in the figures of the proofs of Propositions 3.14 and 3.24 stated below. Although the edges of these figures are not oriented, it is understood that all non-vertical edges are directed from left to right. Vertical arcs are then oriented accordingly to produce directed cycles.

Proposition 3.14. *Let* $m \geqslant 11$ *be an odd integer. The digraph* L_{2m} *admits a* $\vec{C}_m$*-factorization if and only if* $m \not\equiv 3 \pmod 6$*.*

Proof: First assume that $m \equiv 3 \pmod 6$. Let C be a directed m-cycle of L_{2m} containing an arc of difference 3. Observe that C cannot be a type-1 directed m-cycle. Now suppose that C is a type-2 directed m-cycle comprised of k_1 arcs of difference 1 and k_2 arcs of difference 3. Then $k_1 + k_2 = m$ and $k_1 + 3k_2 = 2m$, implying that $k_1 \not\equiv k_2 \pmod 2$ and $k_1 \equiv k_2 \pmod 2$, respectively — a contradiction. Hence, if C contains an arc of difference 3, then all arcs of C are of difference 3. However, if all arcs of C are of difference 3 then C has a repeated vertex — also a contradiction. It follows that L_{2m} does not admit a $\vec{C}_m$-factorization.

Conversely, assume that $m \equiv 1, 5 \pmod 6$. We construct a $\vec{C}_m$-factorization of L_{2m} with four $\vec{C}_m$-factors, each consisting of one type-1 directed m-cycle and one type-3 directed m-cycle.

Case 1: $m \equiv 1 \pmod 6$. Then $m = 13 + 6k$ for some $k \geqslant 0$. We construct four $\vec{C}_m$-factors. In order to keep track of each arc, we partition the set of vertices into two sets:

$$V_0 = \{x_0, x_1, \ldots, x_{12}, y_0, y_1, \ldots, y_{12}\} \text{ and } V_1 = \{x_{13}, x_{14}, \ldots, x_{m-1}, y_{13}, y_{14}, \ldots, y_{m-1}\}.$$

We point out that every arc of L_{2m} has a tail in exactly one of these two sets. Also, note that if $k = 0$, then V_1 is empty.

To construct the first $\vec{C}_m$-factor, we first define the following four dipaths:

$$\begin{aligned} W_0 &= (y_0, y_1, y_2, x_3, x_4, x_5, y_6, y_7, x_8, x_9, x_{10}, y_{11}, x_{12}, y_{13}); \\ X_0 &= (x_2, y_5, y_8, x_{11}, x_{14}); \\ Y_0 &= (x_1, y_4, x_7, y_{10}, x_{13}); \\ Z_0 &= (x_0, y_3, x_6, y_9, y_{12}, x_{15}). \end{aligned}$$

See Figure 3.7 for an illustration of the dipaths W_0, X_0, Y_0, and Z_0. We see that these four dipaths are pairwise disjoint and only contain arcs with a tail in V_0. Observe that W_0 is of length 13, that Z_0 is of length 5, and that X_0 and Y_0 are both of length 4.

Next, we address the special case $m = 13$. We make the following observations:

$$y_{13} = y_0,\ x_{14} = x_1,\ x_{13} = x_0, \text{ and } x_{15} = x_2.$$

As a result, we see that W_0 is in fact a directed cycle of length 13 and that we can concatenate X_0, Y_0, and Z_0 as follows: $X_0Y_0Z_0$. We then see that $X_0Y_0Z_0$ is in fact a directed cycle of length 13. Therefore, the two cycles W_0 and $X_0Y_0Z_0$ form a $\vec{C}_{13}$-factor, F_0, of L_{26}.

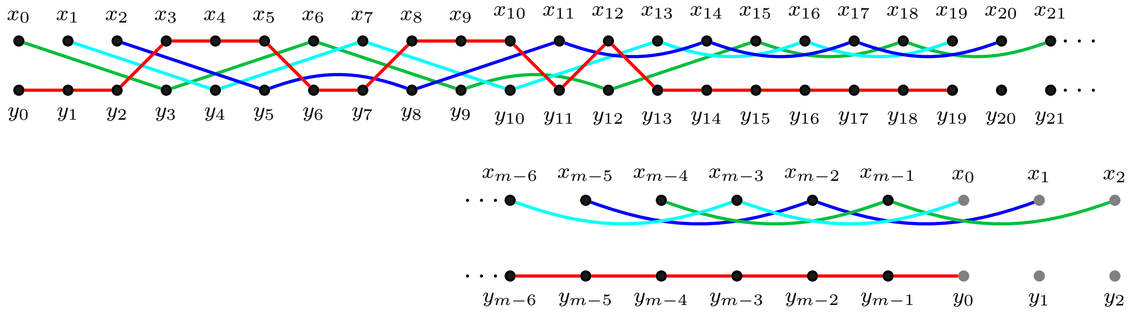


Figure 3.7: The dipaths W_0R_0 (red), X_0S_0 (dark blue), Y_0T_0 (light blue), and Z_0U_0 (green).

Next, we consider the general case in which $m > 13$. We do so by constructing the following four dipaths:

$$\begin{aligned} R_0 &= (y_{13}, y_{14}, y_{15}, \ldots, y_{m-2}, y_{m-1}, y_0); \\ S_0 &= (x_{14}, x_{17}, x_{20}, \ldots, x_{m-5}, x_{m-2}, x_1); \\ T_0 &= (x_{13}, x_{16}, x_{19}, \ldots, x_{m-6}, x_{m-3}, x_0); \\ U_0 &= (x_{15}, x_{18}, x_{21}, \ldots, x_{m-4}, x_{m-1}, x_2). \end{aligned}$$

See Figure 3.7 for an illustration of the first six arcs of R_0 and the first two arcs of each dipath in $\{S_0, T_0, U_0\}$. The dipath R_0 is of length $m - 13$ while the dipaths in $\{S_0, T_0, U_0\}$ are each of length $2k$. We point out that dipaths in $\{R_0, S_0, T_0, U_0\}$ only contain arcs with a tail in V_1 and are pairwise disjoint. Next, we form the following directed walks:

$$C^0 = W_0R_0 \text{ and } C^1 = X_0S_0Y_0T_0Z_0U_0.$$

It can be verified that C^0 and C^1 are of length $m = 13 + 6k$ and both are disjoint. Moreover, they both have no repeated vertices except for their endpoints. This means that $F_0 = C^0 \cup C^1$ is a $\vec{C}_m$-factor of L_{2m}.

We proceed by building our second $\vec{C}_m$-factor F_1. First, we build four dipaths:

$$\begin{aligned}
W_1 &= (x_0, x_1, x_2, y_3, x_4, y_5, y_6, x_7, x_8, y_9, y_{10}, x_{11}, y_{12}, x_{13});\\
X_1 &= (y_2, x_5, y_8, y_{11}, y_{14});\\
Y_1 &= (y_1, y_4, y_7, x_{10}, y_{13});\\
Z_1 &= (y_0, x_3, x_6, x_9, x_{12}, y_{15}).
\end{aligned}$$

See Figure 3.8 for an illustration of the dipaths W_1, X_1 Y_1, Z_1. Observe that these four dipaths are pairwise disjoint.

If $m = 13$, then W_1 and $X_1Y_1Z_1$ form a $\vec{C}_{13}$-factor, F_1, of L_{26}.

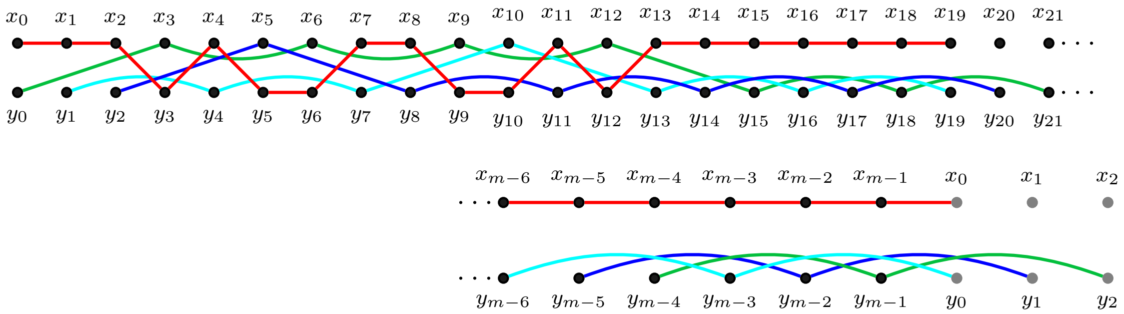


Figure 3.8: The dipaths W_1R_1 (red), X_1S_1 (dark blue), Y_1T_1 (light blue), and Z_1U_1 (green).

Now, suppose that $m > 13$. We then build four dipaths as follows:

$$\begin{aligned}
R_1 &= (x_{13}, x_{14}, x_{15}, \ldots, x_{m-2}, x_{m-1}, x_0);\\
S_1 &= (y_{14}, y_{17}, y_{20}, \ldots, y_{m-5}, y_{m-2}, y_1);\\
T_1 &= (y_{13}, y_{16}, y_{19}, \ldots, y_{m-6}, y_{m-3}, y_0);\\
U_1 &= (y_{15}, y_{18}, y_{21}, \ldots, y_{m-4}, y_{m-1}, y_2).
\end{aligned}$$

Figure 3.8 illustrates the dipaths R_1, S_1, T_1, and U_1, respectively. We see that R_1 is of length $m-13$ and that dipaths in $\{S_1, T_1, U_1\}$ are of length $2k$ each. Moreover, note that R_1, S_1, T_1, U_1 are pairwise disjoint. Lastly, we see that their endpoints allow us to form the following pair of directed walks:

$$C^2 = W_1 R_1 \text{ and } C^3 = X_1 S_1 Y_1 T_1 Z_1 U_1.$$

Then, we see that C^2 and C^3 have no repeated vertices except for their endpoints, meaning that they are both directed cycles of length m. Since they are also disjoint, this means $F_1 = C^2 \cup C^3$ is a $\vec{C}_m$-factor of L_{2m}.

Now, we form our third $\vec{C}_m$-factor, F_2. We do so by first forming four dipaths as follows:

$$\begin{aligned}
W_2 &= (x_0, y_1, x_2, x_3, y_4, x_5, x_6, x_7, y_8, y_9, x_{10}, x_{11}, x_{12}, x_{13});\\
X_2 &= (y_2, y_5, x_8, y_{11}, x_{14});\\
Y_2 &= (x_1, x_4, y_7, y_{10}, y_{13});\\
Z_2 &= (y_0, y_3, y_6, x_9, y_{12}, y_{15}).
\end{aligned}$$

See Figure 3.9 for an illustration of the dipaths W_2, X_2, Y_2, and Z_2. Note that these four dipaths are pairwise disjoint.

If $m = 13$, then W_2 and $X_2 Y_2 Z_2$ are disjoint directed cycles of length 13. Therefore, they form a $\vec{C}_{13}$-factor, F_2, of L_{26}.

As for the case $m > 13$, we construct the following four dipaths:

$$\begin{aligned}
R_2 &= (x_{13}, y_{14}, x_{15}, y_{16}, \ldots, x_{m-2}, y_{m-1}, x_0);\\
S_2 &= (x_{14}, y_{17}, x_{20}, y_{23}, \ldots, x_{m-5}, y_{m-2}, x_1);\\
T_2 &= (y_{13}, x_{16}, y_{19}, x_{22}, \ldots, y_{m-6}, x_{m-3}, y_0);\\
U_2 &= (y_{15}, x_{18}, y_{21}, x_{24}, \ldots, y_{m-4}, x_{m-1}, y_2).
\end{aligned}$$

See Figure 3.9 for an illustration of R_2, S_2, T_2, and U_2. We point out that dipaths in $\{R_2, S_2, T_2, U_2\}$ are pairwise disjoint. The dipath R_2 is of length $m-13$ while dipaths

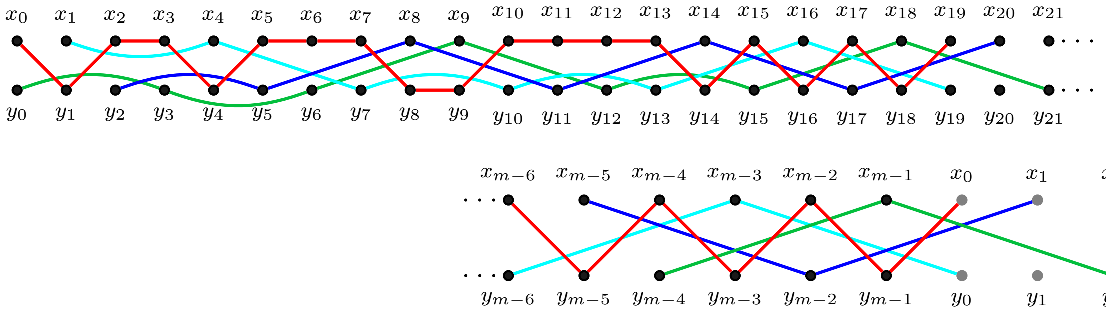


Figure 3.9: The dipaths W_2R_2 (red), X_2S_2 (dark blue), Y_2T_2 (light blue), and Z_2U_2 (green).

S_2, T_2, U_2 are of length $2k$. Next, the endpoints of dipaths in $\{R_2, S_2, T_2, U_2\}$ allow us to form the following two walks:

$$C^4 = W_2R_2 \text{ and } C^5 = X_2S_2Y_2T_2Z_2U_2.$$

We see that C^4 and C^5 are both of length m and have no repeated vertices except for their endpoints. Therefore, both C^4 and C^5 are directed cycles. Moreover, they are both disjoint meaning that $F_2 = C^4 \cup C^5$ is a $\vec{C}_m$-factor of L_{2m}.

Lastly, we build our fourth $\vec{C}_m$-factor F_3. We start with the following four dipaths:

$$\begin{aligned}
W_3 &= (y_0, x_1, y_2, y_3, y_4, y_5, x_6, y_7, y_8, x_9, y_{10}, y_{11}, y_{12}, y_{13});\\
X_3 &= (x_2, x_5, x_8, x_{11}, y_{14});\\
Y_3 &= (y_1, x_4, x_7, x_{10}, x_{13});\\
Z_3 &= (x_0, x_3, y_6, y_9, x_{12}, x_{15}).
\end{aligned}$$

In Figure 3.10, we illustrate the dipaths W_3, X_3, Y_3, and Z_3. Observe that these are pairwise disjoint.

If $m = 13$, then W_3 and $X_3Y_3Z_3$ form a $\vec{C}_{13}$-factor, F_3, of L_{26}.

Now, we move on to the general case $m > 13$. We form the following four dipaths:

$$\begin{aligned}
R_3 &= (y_{13}, x_{14}, y_{15}, x_{16}, \ldots, y_{m-2}, x_{m-1}, y_0);\\
S_3 &= (y_{14}, x_{17}, y_{20}, x_{23}, \ldots, y_{m-5}, x_{m-2}, y_1);\\
T_3 &= (x_{13}, y_{16}, x_{19}, y_{22}, \ldots, x_{m-6}, y_{m-3}, x_0);\\
U_3 &= (x_{15}, y_{18}, x_{21}, y_{24}, \ldots, x_{m-4}, y_{m-1}, x_2).
\end{aligned}$$

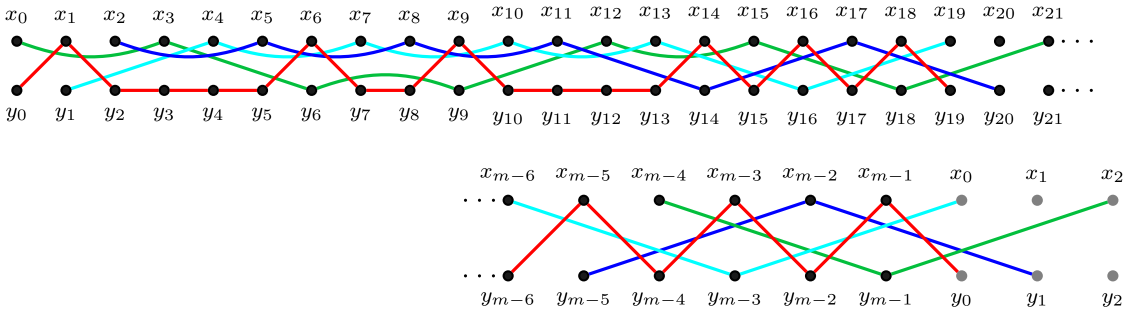


Figure 3.10: The dipaths W_3R_3 (red), X_3S_3 (dark blue), Y_3T_3 (light blue), and Z_3U_3 (green).

Figure 3.10 illustrates the dipaths R_3, S_3, T_3, and U_3. Observe that the length of R_3 is $m - 13$ while the length of each of the last three dipaths is $2k$. Furthermore, we point out that dipaths in $\{R_3, S_3, T_3, U_3\}$ are pairwise disjoint. Next, the endpoints of these four dipaths allow us to form the following two walks:

$$C^6 = W_3R_3 \text{ and } C^7 = X_3S_3Y_3T_3Z_3U_3.$$

We point out that C^6 and C^7 are disjoint directed walks of length m. Furthermore, we see that they have no repeated vertices with the exception of their endpoints. Therefore, the digraph $F_3 = C^6 \cup C^7$ is a $\vec{C}_m$-factor of L_{2m}.

In summary, we have formed four $\vec{C}_m$-factors F_0, F_1, F_2, and F_3. We must now verify that these four $\vec{C}_m$-factor are arc-disjoint. First, it can be verified that the set of dipaths $\{W_i, X_i, Y_i, Z_i \mid i = 0, 1, 2, 3\}$ contains precisely one copy of each arc with a tail in V_0. Meanwhile, every arc with a tail in V_1 occurs precisely once in $\{R_i, S_i, T_i, U_i \mid i = 0, 1, 2, 3\}$. Therefore, we see that $\{F_0, F_1, F_2, F_3\}$ is a $\vec{C}_m$-factorization of L_{2m}, as desired.

Case 2: $m \equiv 5 \pmod 6$. First, we give a separate construction for the case $m = 11$ below:

$$\begin{aligned}
C^0 &= (y_0, x_1, x_2, x_3, y_4, y_5, x_6, x_7, x_8, y_9, x_{10}, y_0);\\
C^1 &= (x_0, y_3, y_6, x_9, y_1, x_4, y_7, y_{10}, y_2, x_5, y_8, x_0);\\
C^2 &= (x_0, y_1, x_2, y_3, y_4, x_5, x_6, y_7, y_8, x_9, y_{10}, x_0);\\
C^3 &= (y_0, x_3, y_6, y_9, x_1, x_4, x_7, x_{10}, y_2, y_5, x_8, y_0);\\
C^4 &= (x_0, x_1, y_2, x_3, x_4, x_5, y_6 . y_7, x_8, x_9, x_{10}, x_0);\\
C^5 &= (y_0, y_3, x_6, y_9, y_1, y_4, x_7, y_{10}, x_2, y_5, y_8, y_0);\\
C^6 &= (y_0, y_1, y_2, y_3, x_4, y_5, y_6, x_7, y_8, y_9, y_{10}, y_0);\\
C^7 &= (x_0, x_3, x_6, x_9, x_1, y_4, y_7, x_{10}, x_2, x_5, x_8, x_0).
\end{aligned}$$

Each directed walk above has no repeated vertices except for its endpoints. Therefore, these eight directed walks are in fact directed 11-cycles. It can easily be verified that for $i \in \{0, 2, 4, 6\}$, $F_i = C^i \cup C^{i+1}$ is a $\vec{C}_{11}$-factor in L_{22}. Moreover, $\vec{C}_m$-factors in $\mathcal{F} = \{F_0, F_2, F_4, F_6\}$ are pairwise arc-disjoint. Therefore, $\mathcal{F}$ is a $\vec{C}_{11}$-factorization of L_{22}.

We now consider the case $m > 11$. Then $m = 11 + 6k$ for some $k \geqslant 1$. We construct a $\vec{C}_m$-factorization in the same fashion as in Case 1 by reusing some of the dipaths constructed in Case 1. We will form four arc-disjoint $\vec{C}_m$-factors which we shall denote as F_0, F_1, F_2, and F_3. Each $\vec{C}_m$-factor will consist of one type-1 cycle and one type-3 cycle. Since this construction is similar to that of Case 1, we omit certain details.

First, for each $i \in \{0, 1, 2, 3\}$, we consider the set of dipaths $\{W_i, X_i, Y_i, Z_i\}$ constructed in Case 1. This means that, for each i, the lengths of the dipaths in $\{X_i, Y_i, Z_i\}$ sum to 13.

Below, we define a set of 16 dipaths $\{R'_i, S'_i, T'_i, U'_i \mid i = 0, 1, 2, 3\}$. This set of 16 dipaths is constructed in a similar way as the set $\{R_i, S_i, T_i, U_i \mid i = 0, 1, 2, 3\}$ in Case 1. However, we note that, for each i, the dipath R'_i is of length $m - 13$, dipath U'_i is of length $2k$, and dipaths S'_i and T'_i are both of length $2k - 1$. We also point out that the last vertex of each dipath in $\{S'_i, T'_i, U'_i \mid i = 0, 1, 2, 3\}$ will be different from the last vertex of R_i, S_i, T_i, and U_i. Refer to Figures 3.7-3.10 for an illustration of $\{R_i, S_i, T_i, U_i \mid i = 0, 1, 2, 3\}$.

First, we build a set of four dipaths that will be used in the construction of F_0:

$$\begin{aligned}
R_0' &= (y_{13}, y_{14}, y_{15}, \ldots, y_{m-2}, y_{m-1}, y_0);\\
S_0' &= (x_{14}, x_{17}, x_{20}, \ldots, x_{m-6}, x_{m-3}, x_0);\\
T_0' &= (x_{15}, x_{18}, x_{21}, \ldots, x_{m-5}, x_{m-2}, x_1);\\
U_0' &= (x_{13}, x_{16}, x_{19}, \ldots, x_{m-4}, x_{m-1}, x_2).
\end{aligned}$$

It is easy to see that these four dipaths are pairwise disjoint. Then, we form the four dipaths that will be used to build F_1:

$$\begin{aligned}
R_1' &= (x_{13}, x_{14}, x_{15}, \ldots, x_{m-2}, x_{m-1}, x_0);\\
S_1' &= (y_{14}, y_{17}, y_{20}, \ldots, y_{m-6}, y_{m-3}, y_0);\\
T_1' &= (y_{15}, y_{18}, y_{21}, \ldots, y_{m-5}, y_{m-2}, y_1);\\
U_1' &= (y_{13}, y_{16}, y_{19}, \ldots, y_{m-4}, y_{m-1}, y_2).
\end{aligned}$$

Similarly, we see that these four dipaths are pairwise disjoint. Next, we construct the four dipaths that will be used to build F_2:

$$\begin{aligned}
R_2' &= (x_{13}, y_{14}, x_{15}, y_{16}, \ldots, x_{m-2}, y_{m-1}, x_0);\\
S_2' &= (x_{14}, y_{17}, x_{20}, y_{23}, \ldots, y_{m-6}, x_{m-3}, y_0);\\
T_2' &= (y_{15}, x_{18}, y_{21}, x_{24}, \ldots, x_{m-5}, y_{m-2}, x_1);\\
U_2' &= (y_{13}, x_{16}, y_{19}, x_{22}, \ldots, y_{m-4}, x_{m-1}, y_2).
\end{aligned}$$

Again, these four dipaths are pairwise disjoint. Lastly, we give below the four dipaths that will be used to build F_3:

$$\begin{aligned}
R_3' &= (y_{13}, x_{14}, y_{15}, x_{16}, \ldots, y_{m-2}, x_{m-1}, y_0);\\
S_3' &= (y_{14}, x_{17}, y_{20}, x_{23}, \ldots, x_{m-6}, y_{m-3}, x_0);\\
T_3' &= (x_{15}, y_{18}, x_{21}, y_{24}, \ldots, y_{m-5}, x_{m-2}, y_1);\\
U_3' &= (x_{13}, y_{16}, x_{19}, y_{22}, \ldots, x_{m-4}, y_{m-1}, x_2).
\end{aligned}$$

For each $i \in \{0, 1, 2, 3\}$, we examine the endpoints of the dipaths W_i, X_i, Y_i, Z_i and R_i', S_i', T_i', U_i' and observe that the following concatenations yield directed cycles:

$$\begin{aligned}
C^0 &= W_0R_0'; \quad C^4 = X_0S_0'Z_0T_0'Y_0U_0';\\
C^1 &= W_1R_1'; \quad C^5 = X_1S_1'Z_1T_1'Y_1U_1';\\
C^2 &= W_2R_2'; \quad C^6 = X_2S_2'Z_2T_2'Y_2U_2';\\
C^3 &= W_3R_3'; \quad C^7 = X_3S_3'Z_3T_3'Y_3U_3'.
\end{aligned}$$

For $i \in \{0,1,2,3\}$, it can be verified that $F_i = C^i \cup C^{i+4}$ is a $\vec{C}_m$-factor of L_{2m}. Furthermore, it can also be verified that each arc of L_{2m} occurs exactly once in $\{F_0, F_1, F_2, F_3\}$. Consequently, the set $\{F_0, F_1, F_2, F_3\}$ is a $\vec{C}_m$-factorization of L_{2m}. ∎

Observe that Proposition 3.14 implies that Lemma 3.6 is vacuous for $m \equiv 3 \pmod 6$. In Section 3.5, we show that, if $m \equiv 3 \pmod 6$ and $m \neq 3$, then a $\vec{C}_m$-factorization of K_{2m}^* can be constructed from a $\vec{C}_{m_1}$-factorization of $K_{2m_1}^*$ where $m_1 \equiv 1, 5 \pmod 6$ or $m_1 = 9$.

3.4 Decomposing G_{2m}

It remains to show that G_{2m} admits a $\vec{C}_m$-factorization when $m \equiv 1$ or $5 \pmod 6$ and $m \geqslant 11$. This digraph requires a more complicated construction. In order to simplify this construction, we first prove a set of three lemmas. Before we do so, we establish some notation used throughout this section.

Notation 3.15. Let $m = p + 12k$ for some non-negative integer k and $p \in \{11, 13, 17, 19\}$. We let

$$V_0 = \{x_0, x_1, \dots x_{p-1}\} \cup \{y_0, y_1, \dots, y_{p-1}\} \text{ and}$$

$$V_i = \{x_{p+12(i-1)}, x_{p+12(i-1)+1}, \dots, x_{p+12i-1}\} \cup \{y_{p+12(i-1)}, y_{p+12(i-1)+1}, \dots, y_{p+12i-1}\},$$

for $i = 1, \dots, k$. Then $V(G_{2m}) = \bigcup_{i=0}^{k} V_i$.

Observe that V_0 contains $2p$ vertices while each V_i, for $i \in \{1, 2, \dots, k\}$, contains 24 vertices. Next, we define a function on $V(G_{2m})$ that projects $V(G_{2m})$ onto itself.

Definition 3.16. Let $m = p + 12k$ for some positive integer k and p odd. Define a function $\rho : V(G_{2m}) \to V(G_{2m})$ as follows: $\rho(x_i) = x_{i+1}$ and $\rho(y_i) = y_{i+1}$ with addition done modulo m.

In the following example, we apply ρ to a dipath of G_{2m}.

Example 3.17. Suppose that $m = 47$ where $p = 11$ and $k = 3$. Let

$$P = (y_{15}, x_{16}, y_{17}, x_{18}, y_{21}, x_{22}, y_{23}, x_{24}, y_{27})$$

be a dipath of G_{2m}. Then we have that

$$\rho^{24}(P) \quad = \quad (y_{39}, x_{40}, y_{41}, x_{42}, y_{45}, x_{46}, y_0, x_1, y_4).$$

□

We now build a $\vec{C}_m$-factorization for G_{2m}. Our construction, given in the proof of Theorem 3.24, is elaborate. First, we consider four cases, one for each applicable congruency class modulo 12. In each case, we build five $\vec{C}_m$-factors. Three of these five $\vec{C}_m$-factors are comprised of two type-2 cycles each, and two of these $\vec{C}_m$-factors are comprised of two type-1 cycles each. To simplify the construction of these five $\vec{C}_m$-factors, we introduce a pair of lemmas below, namely Lemmas 3.19 and 3.22, that reduce the construction of each of these $\vec{C}_m$-factors to the existence of a specific set of dipaths and directed cycles of fixed lengths.

Before we proceed with these two lemmas, we further explain their purpose along with that of Lemma 3.23. All odd integers m such that $m \equiv 1, 5 \pmod 6$ and $m \geqslant 11$ can be written as $m = p + 12k$ with $p \in \{11, 13, 17, 19\}$ and k a non-negative integer. Lemma 3.19 implies that a $\vec{C}_m$-factor of G_{2m} can be constructed for all odd integers of the form $p + 12k$ using a set of four dipaths whose lengths sum to $2p$, and 4 dipaths of length 12, satisfying certain properties. Such a set of dipaths will be known as a type-2 basic set, defined in Definition 3.18. Lemma 3.22 implies that a $\vec{C}_m$-factor of G_{2m} can also be constructed from two dipaths of length p and two dipaths of length 12, with specific properties. This set of dipaths will be known as a type-1 basic set of dipaths and is defined in Definition 3.21. Lemma 3.23 then states that, if we can generate three type-2 basic sets of dipaths and two type-1 basic sets of dipaths such that these sets are pairwise arc-disjoint, then we can form a $\vec{C}_m$-factorization of G_{2m}. Therefore, if we are successful in constructing three type-2 basic sets of dipaths and two type-1 basic sets of dipaths that satisfy the hypothesis of Lemma 3.23 for each $p \in \{11, 13, 17, 19\}$, then we have a $\vec{C}_m$-factorization of G_{2m} for all odd integers $m \equiv 1, 5 \pmod 6$ and $m \geqslant 11$.

First, we define a type-2 basic sets of dipaths.

Definition 3.18. Let $m = p + 12k$ for some non-negative integer k and $p \in \{11, 13, 17, 19\}$. Let $\{W, X, Y, Z, Q, R, S, T\}$ be a set of dipaths of G_{2m}. The 8-tuple (W, X, Y, Z, Q, R, S, T) is a *type-2 basic set of dipaths of* G_{2m} if it satisfies the following properties.

(B1) Dipaths in $\{Q, R, S, T\}$ are pairwise disjoint. If $k \geqslant 1$, then dipaths in $\{W, X, Y, Z\}$ are pairwise disjoint; otherwise WX and YZ are disjoint type-2 directed cycles.

(B2) $s(X) = \rho^{-p}(t(W))$, $s(W) = \rho^{-p}(t(X))$, $s(Z) = \rho^{-p}(t(Y))$, and $s(Y) = \rho^{-p}(t(Z))$.

(B3) $\text{len}(W)+\text{len}(X) = \text{len}(Y)+\text{len}(Z) = p$; $\text{len}(Q)+\text{len}(R) = \text{len}(S)+\text{len}(T) = 12$ if $k \geqslant 1$, and $\text{len}(Q) = \text{len}(R) = \text{len}(S) = \text{len}(T) = 0$ otherwise.

(B4) Each of W, X, Y, and Z has its source and internal vertices in V_0, and its terminus x_t or y_t for some $t \in \{p, p+1, p+2\}$.

(B5) $t(W) = s(Q), t(X) = s(R), t(Y) = s(S)$, and $t(Z) = s(T)$.

(B6) If $k \geqslant 1$ and $P \in \{Q, R, S, T\}$ such that $s(P) = x_t$ (or y_t), then $t(P) = x_{t+12}$ (or y_{t+12} respectively). Moreover, all internal vertices of P are in V_1.

In Lemma 3.19 below, we prove that a type-2 basic set of dipaths can be used to construct two type-2 directed cycles that form a $\vec{C}_m$-factor of G_{2m}.

Lemma 3.19. *Let $m = p + 12k$ for some non-negative integer k and $p \in \{11, 13, 17, 19\}$. Suppose that $(W, X, Y, Z, P_0^0, P_0^1, Q_0^0, Q_0^1)$ is a type-2 basic set of dipaths of G_{2m}. For each $i \in \{0, 1\}$ and $j \in \{1, 2, \ldots, k-1\}$, let $P_j^i = \rho^{12j}(P_0^i)$ and $Q_j^i = \rho^{12j}(Q_0^i)$. Then*

$$C^0 = WP_0^0P_1^0 \ldots P_{k-1}^0XP_0^1P_1^1 \ldots P_{k-1}^1 \text{ and } C^1 = YQ_0^0Q_1^0 \ldots Q_{k-1}^0ZQ_0^1Q_1^1 \ldots Q_{k-1}^1$$

are type-2 directed m-cycles, and $C^0 \cup C^1$ is a $\vec{C}_m$-factor of G_{2m}.

Proof: First, consider the case $k = 0$. Observe that $V(G_{2m}) = V_0$ and by property (B3) of Definition 3.18, dipaths $P_0^0, P_0^1, Q_0^0, Q_0^1$ are of length 0. Properties (B1) and (B3) jointly imply that $WX \cup YZ$ is a $\vec{C}_m$-factor of G_{2m} consisting of two type-2 directed m-cycles.

Assume that $k \geqslant 1$. Without loss of generality, by properties (B4)-(B6) we may assume that P_0^0 is an (x_t, x_{t+12})-dipath for some $t \in \{p, p+1, p+2\}$ and all of its internal vertices in V_1. Hence, the dipath P_j^0 is an $(x_{t+12j}, x_{t+12(j+1)})$-dipath and all of its internal vertices are in V_{j+1}. Therefore, for all $0 \leqslant i < j \leqslant k-1$, dipaths in P_i^0 and P_j^0 are disjoint if $|i - j| > 1$ and share only vertex x_{t+12j} if $j = i + 1$. Therefore, the concatenation

$$I_0 = P_0^0 P_1^0 \dots P_{k-1}^0$$

is a well-defined (x_t, x_{t+12k})-dipath.

An analogous observation holds for P_0^1 and dipath

$$I_1 = P_0^1 P_1^1 \dots P_{k-1}^1.$$

If x_t is the terminus of W, then by (B2), vertex $x_{t-p} = x_{t+12k}$ is the source of X. As a result, the concatenation WI_0X is possible. Analogously, we show that XI_1W is well-defined.

Since W and X, as well as P_0^0 and P_0^1, are disjoint by (B1), and the sets of internal vertices of W, I_0, X, and I_1 are pairwise disjoint, it follows that $C^0 = WI_0XI_1$ is a directed cycle. Since $\text{len}(W) + \text{len}(X) = p$ and $\text{len}(P_0^0) + \text{len}(P_0^1) = 12$ by (B3), it follows that C^0 is a directed m-cycle.

Analogously, we show that C^1 is a directed m-cycle. Property (B1) then implies that C^0 and C^1 are disjoint. Therefore, the digraph $C^0 \cup C^1$ is a $\vec{C}_m$-factor of G_{2m}.

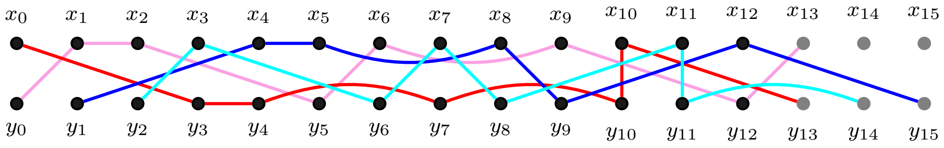
Lastly, we see that the arc differences in $I_0 \cup I_1$ sum to $24k$ and that the arc differences in $W \cup X$ sum to $2p$. It follows that C^0 is a type-2 directed m-cycle. Similarly, we can show that C^1 is also a type-2 directed m-cycle. ∎

We proceed with an example of the construction given in Lemma 3.19

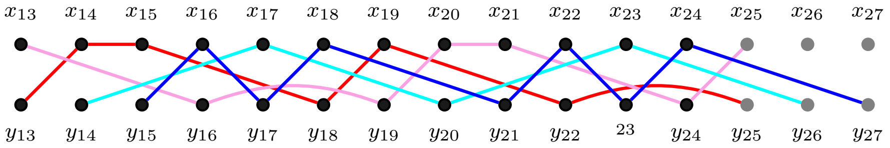
Example 3.20. Let $m = 13 + 12 \cdot 2$. Below, we construct a type-2 basic set of eight dipaths of G_{74}:

$$\begin{aligned}
W &= (y_0, x_1, x_2, y_5, x_6, x_9, y_{12}, x_{13});\\
X &= (x_0, y_3, y_4, y_7, y_{10}, x_{10}, y_{13});\\
P_0^0 &= (x_{13}, y_{16}, y_{19}, x_{20}, x_{21}, y_{24}, x_{25});\\
P_0^1 &= (y_{13}, x_{14}, x_{15}, y_{18}, x_{19}, y_{22}, y_{25});\\
Y &= (y_1, x_4, x_5, x_8, y_9, x_{12}, y_{15});\\
Z &= (y_2, x_3, y_6, x_7, y_8, x_{11}, y_{11}, y_{14});\\
Q_0^0 &= (y_{15}, x_{16}, y_{17}, x_{18}, y_{21}, x_{22}, y_{23}, x_{24}, y_{27});\\
Q_0^1 &= (y_{14}, x_{17}, y_{20}, x_{23}, y_{26}).
\end{aligned}$$

In Figure 3.11, we illustrate these eight dipaths.



(a) The directed paths W (pink), X (red), Y (dark blue), and Z (light blue).



(b) The directed paths P_0^0 (pink), Q_0^0 (dark blue), P_0^1 (red), and Q_0^1 (light blue).

Figure 3.11: A type-2 basic set of dipaths.

We will now verify that $(W, X, Y, Z, P_0^0, P_0^1, Q_0^0, Q_0^1)$ is a type-2 basic set of dipaths. First, we note that $P_0^0, P_0^1, Q_0^0, Q_0^1$ are pairwise disjoint. We also see that W, X, Y, Z are pairwise disjoint; this means that (B1) is satisfied since $k = 2$.

Next, we have

$$\rho^{-13}(x_{13}) = x_0 = s(X), \qquad \rho^{-13}(y_{13}) = y_0 = s(W),$$
$$\rho^{-13}(y_{15}) = y_2 = s(Z), \quad \text{and } \rho^{-13}(y_{14}) = y_1 = s(Y).$$

As a result, Condition (B2) is satisfied.

The dipaths P_0^0 and P_0^1 are both of length 6. Meanwhile, the dipaths Q_0^0 and Q_0^1 are of length 8 and 4, respectively. Consequently, we have

$$\operatorname{len}(P_0^0) + \operatorname{len}(P_0^1) = \operatorname{len}(Q_0^0) + \operatorname{len}(Q_0^1) = 12.$$

Lastly, we have $\operatorname{len}(W) + \operatorname{len}(X) = 13$ and $\operatorname{len}(Y) + \operatorname{len}(Z) = 13$. In conclusion, (B3) is satisfied.

We also see that $s(W), s(X), s(Y), s(Z) \in V_0$, where V_0 is given in Notation 3.15, and that the internal vertices of W, X, Y, Z are also in V_0. Additionally, $t(W), t(X), t(Y), t(Z) \in \{x_{13}, x_{14}, x_{15}, y_{13}, y_{14}, y_{15}\}$. Hence (B4) is satisfied.

Observe that

$$t(W) = x_{13} = s(P_0^0), \quad t(X) = y_{13} = s(P_0^1),$$
$$t(Y) = y_{15} = s(Q_0^0), \quad \text{and } t(Z) = y_{14} = s(Q_0^1).$$

This means that (B5) is satisfied.

Lastly, we see that $s(P_0^0) = x_{13}$ and $t(P_0^0) = x_{25}$ which means that $t(P_0^0) = x_{13+12}$. Additionally, all internal vertices of P_0^0 are in V_1. A similar observation holds for P_0^1, Q_0^0, and Q_0^1. As a result, (B6) is satisfied.

In conclusion, the 8-tuple $(W, X, Y, Z, P_0^0, P_0^1, Q_0^0, Q_0^1)$ is a type-2 basic set of dipaths because Properties (B1)-(B6) of Definition 3.18 are satisfied.

As in the proof of Lemma 3.19, we now construct four additional dipaths:

$$\begin{aligned}
P_1^0 &= \rho^{12}(P_0^0) = (x_{25}, y_{28}, y_{31}, x_{32}, x_{33}, y_{36}, x_0);\\
P_1^1 &= \rho^{12}(P_0^1) = (y_{25}, x_{26}, x_{27}, y_{30}, x_{31}, y_{34}, y_0);\\
Q_1^0 &= \rho^{12}(Q_0^0) = (y_{27}, x_{28}, y_{29}, x_{30}, y_{33}, x_{34}, y_{35}, x_{36}, y_2);\\
Q_1^1 &= \rho^{12}(Q_0^1) = (y_{26}, x_{29}, y_{32}, x_{35}, y_1).
\end{aligned}$$

We see that $t(P_1^0) = s(X)$ and that $t(Q_1^0) = s(Z)$. Therefore, we can concatenate these dipaths as follows:

$$\begin{aligned}
C^0 &= W P_0^0 P_1^0 X P_0^1 P_1^1\\
&= (y_0, x_1, x_2, y_5, x_6, x_9, y_{12}, x_{13}, y_{16}, y_{19}, x_{20}, x_{21}, y_{24}, x_{25}, y_{28}, y_{31}, x_{32}, x_{33}, y_{36}, x_0,\\
&\quad y_3, y_4, y_7, y_{10}, x_{10}, y_{13}, x_{14}, x_{15}, y_{18}, x_{19}, y_{22}, y_{25}, x_{26}, x_{27}, y_{30}, x_{31}, y_{34}, y_0);\\
C^1 &= Y Q_0^0 Q_1^0 Z Q_0^1 Q_1^1\\
&= (y_1, x_4, x_5, x_8, y_9, x_{12}, y_{15}, x_{16}, y_{17}, x_{18}, y_{21}, x_{22}, y_{23}, x_{24}, y_{27}, x_{28}, y_{29}, x_{30}, y_{33},\\
&\quad x_{34}, y_{35}, x_{36}, y_2, x_3, y_6, x_7, y_8, x_{11}, y_{11}, y_{14}, x_{17}, y_{20}, x_{23}, y_{26}, x_{29}, y_{32}, x_{35}, y_1).
\end{aligned}$$

One can then verify that C^0 and C^1 have no repeated vertices, except for their endpoints, and that both are of length 37. Moreover, we see that C^0 and C^1 are disjoint. In conclusion, the digraph $C^0 \cup C^1$ is a $\vec{C}_{37}$-factor of G_{74}. □

Lemma 3.19 implies that it suffices to construct a set of eight dipaths of fixed lengths with five specific properties to construct a $\vec{C}_m$-factor of G_{2m}. These six properties are easy to verify and thus, Lemma 3.19 greatly simplifies the constructions given in the proof of Theorem 3.24 below.

Similarly, we can build a $\vec{C}_m$-factor using two type-1 cycles. First, we define a set of four dipaths with particular properties.

Definition 3.21. Let $m = p + 12k$ with k a non-negative integer and $p \in \{11, 13, 17, 19\}$. Let $\{X, Y, R, S\}$ be a set of dipaths or directed cycles of G_{2m}. The 4-tuple (X, Y, R, S) is called a *type-1 basic set of dipaths* if it satisfies the following properties.

(D1) R and S are disjoint dipaths. If $k \geqslant 1$, then X are Y are disjoint dipaths; otherwise, they are disjoint type-1 directed cycles.

(D2) $s(X) = \rho^{-p}(t(X))$ and $s(Y) = \rho^{-p}(t(Y))$.

(D3) $\operatorname{len}(X) = \operatorname{len}(Y) = p$; $\operatorname{len}(R) = \operatorname{len}(S) = 12$ if $k \geqslant 1$ and $\operatorname{len}(R) = \operatorname{len}(S) = 0$ if $k = 0$.

(D4) Each of X and Y has its source and internal vertices in V_0, and its terminus is x_t or y_t for some $t \in \{p, p+1, p+2\}$.

(D5) $t(X) = s(R)$ and $t(Y) = s(S)$.

(D6) If $k \geqslant 1$, and $P \in \{R, S\}$ such that $s(P) = x_t$ (or y_t), then $t(P) = x_{t+12}$ (or y_{t+12}, respectively). Moreover, all internal vertices of P are in V_1.

A basic set of type-1 dipaths can also be used to form a $\vec{C}_m$-factor of G_{2m}. The proof of Lemma 3.19 below is similar to that of Lemma 3.22. For that reason, we omit certain details.

Lemma 3.22. *Let $m = p + 12k$ with k a non-negative integer and $p \in \{11, 13, 17, 19\}$. Suppose that (X, Y, P_0, Q_0) is a type-1 basic set of dipaths of G_{2m}. For each $j \in \{1, 2, \dots, k-1\}$, let $P_j = \rho^{12j}(P_0)$ and $Q_j = \rho^{12j}(Q_0)$. Then*

$$C^0 = XP_0P_1 \dots P_{k-1} \text{ and } C^1 = YQ_0Q_1 \dots Q_{k-1}$$

are type-1 directed m-cycles, and $C^0 \cup C^1$ is a $\vec{C}_m$-factor of G_{2m}.

Proof: First, assume that $k = 0$. Then, properties (D1) and (D3) of Definition 3.21 jointly imply that $X \cup Y$ is a $\vec{C}_m$-factor of G_{2m} consisting of two type-1 directed m-cycles.

Let $k \geqslant 1$. Without loss of generality, by properties (D4)-(D6), we may assume that P_0 is an (x_t, x_{t+12})-dipath for some $t \in \{p, p+1, p+2\}$ and all of its internal vertices are in V_1. Hence P_j is an $(x_{t+12j}, x_{t+12(j+1)})$-dipath with all of its internal vertices in V_{j+1}. As a result, for all $0 \leqslant i < j \leqslant k-1$, dipaths P_i and P_j are disjoint if $|i - j| > 1$, and share only vertex x_{t+12j} if $j = i + 1$. Therefore, the concatenation

$$I_0 = P_0P_1 \dots P_{k-1}$$

is a well-defined (x_t, x_{t+12k}) dipath.

Similarly, we can construct dipath I_1 as follows:

$$I_1 = Q_0 Q_1 \dots Q_{k-1}.$$

Next, by (D5), it follows that XI_0 is a well-defined concatenation. Furthermore, by (D2) vertex $x_{t-p} = x_{t+12k}$ is also the source of X. Therefore, the concatenation $I_0 X$ is also well-defined. Since the sets of internal vertices of X and I_0 are disjoint, it follows that $C^0 = XI_0$ is a directed cycle. Since $\text{len}(X) = p$ and $\text{len}(I_0) = 12k$ by (D3), the directed cycle C^0 is of length m. Analogously, it can be shown that C^1 is also a directed m-cycle.

Property (D1) implies that C^0 and C^1 are disjoint. Therefore, the digraph $C^0 \cup C^1$ is a $\vec{C}_m$-factor of G_{2m}.

Lastly, we see that the arc differences in I_0 and I_1 sum to $12k$. Moreover, the arc differences in X and Y sum to p. It follows that C^0 and C^1 are both type-1 directed m-cycles. ∎

Using three type-2 basic sets and two type-1 basic sets satisfying certain properties, we can construct a $\vec{C}_m$-factorization of G.

Lemma 3.23. *Let $m = p + 12k$ with k a non-negative integer and $p \in \{11, 13, 17, 19\}$. If G_{2m} admits three type-2 basic sets of dipaths and two type-1 basic sets of dipaths such that the dipaths and directed cycles in these five sets are pairwise arc-disjoint, then G_{2m} admits a $\vec{C}_m$-factorization*

Proof: Let A_1, A_2, and A_3 be type-2 basic sets of dipaths, and A_4 and A_5 be type-1 basic sets of dipaths such that the 32 dipaths and directed cycles in $S = A_1 \cup A_2 \cup A_3 \cup A_4 \cup A_5$ are pairwise arc-disjoint.

By Lemma 3.19, each A_i for $i \in \{1, 2, 3\}$ gives rise to a $\vec{C}_m$-factor F_i of G_{2m} consisting of two type-2 directed m-cycles. These $\vec{C}_m$-factors are constructed as described in the proof of Lemma 3.19. In addition, by Lemma 3.22, each A_i for $i \in \{4, 5\}$ gives rise to a $\vec{C}_m$-factor F_i consisting of two type-1 directed m-cycles. Likewise, these are constructed as in the proof of Lemma 3.22. It remains to show that the F_i, for $i \in \{1, 2, \dots, 5\}$, are pairwise arc-disjoint.

Suppose that an arc $a = (x_r, x_s)$ of G_{2m}, where $x_r, x_s \in V_j$ for some $j \in \{0, \dots, k\}$, occurs twice in the $\vec{C}_m$-factors $F_1, \dots, F_5$. By the construction of $F_1, \dots, F_5$ from Lemmas 3.19 and 3.22, it follows that $j \geqslant 2$, and that $a' = (x_{r-12(j-1)}, x_{s-12(j-1)})$ also occurs twice in $F_1, \dots, F_5$. However, we see that a' has a tail in V_1, and thus appears twice in S, a contradiction.

An analogous argument applies if a is of the form (y_r, y_s), (y_r, x_s), and (x_r, y_s). Therefore, each arc of G_{2m} occurs at most once in $F_1, \dots, F_5$. Since G_{2m} is of degree five, it follows that every arc occurs exactly once. Therefore, the set $\{F_1, \dots, F_5\}$ is a $\vec{C}_m$-factorization of G_{2m}. ∎

In the proof of Proposition 3.24 below, we consider four cases, one for each $p \in \{11, 13, 17, 19\}$. In each case, we construct three type-2 basic sets of dipaths and two type-1 basic sets. Then, we verify that the hypothesis of Lemma 3.23 is satisfied by these five sets of dipaths. This in turns implies that we have a $\vec{C}_m$-factorization of G_{2m} for all $m \equiv 1, 5 \pmod 6$ and $m \geqslant 11$.

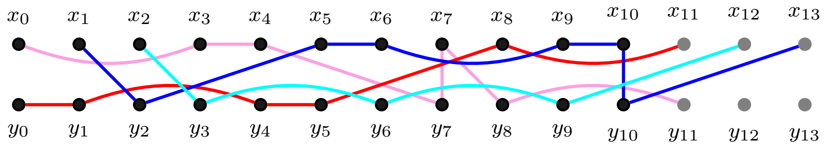
Proposition 3.24. *Let $m = p+12k$ with k a non-negative integer and $p \in \{11, 13, 17, 19\}$. The digraph G_{2m} admits a $\vec{C}_m$-factorization.*

Proof: Throughout this proof, we shall refer to notation introduced in Notation 3.15. In each case, we construct three type-2 basic sets of dipaths L_0, L_1, and L_2, and two type-1 basic sets of dipaths L_3 and L_4. In each case, it can then be verified that the dipaths in $L_0 \cup L_1 \cup L_2 \cup L_3 \cup L_4$ are pairwise arc-disjoint, thereby satisfying the hypothesis of Lemma 3.23.

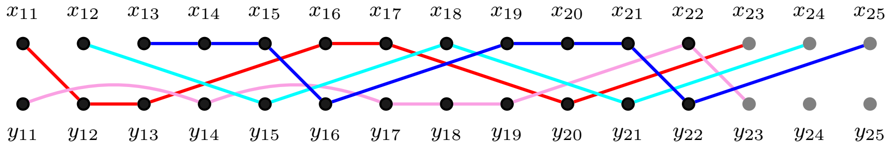
Case 1: $p = 11$. Let $L_0 = (W_0, X_0, Y_0, Z_0, Q_0, R_0, S_0, T_0)$ where

$$\begin{aligned}
W_0 &= (x_0, x_3, x_4, y_7, x_7, y_8, y_{11});\\
X_0 &= (y_0, y_1, y_4, y_5, x_8, x_{11});\\
Y_0 &= (x_1, y_2, x_5, x_6, x_9, x_{10}, y_{10}, x_{13});\\
Z_0 &= (x_2, y_3, y_6, y_9, x_{12});\\
Q_0 &= (y_{11}, y_{14}, y_{17}, y_{18}, y_{19}, x_{22}, y_{23});\\
R_0 &= (x_{11}, y_{12}, y_{13}, x_{16}, x_{17}, y_{20}, x_{23});\\
S_0 &= (x_{13}, x_{14}, x_{15}, y_{16}, x_{19}, x_{20}, x_{21}, y_{22}, x_{25});\\
T_0 &= (x_{12}, y_{15}, x_{18}, y_{21}, x_{24}).
\end{aligned}$$

In Figure 3.12, we illustrate these eight dipaths. It can be verified that properties (B1)-(B6) of Definition 3.18 are satisfied by L_0.



(a) The dipaths W_0 (pink), X_0 (red), Y_0 (dark blue), and Z_0 (light blue).



(b) The dipaths Q_0 (pink), R_0 (red), S_0 (dark blue), and T_0 (light blue).

Figure 3.12: The type-2 basic set of dipaths L_0 for $p = 11$.

We now build a second type-2 basic set of dipaths below, this time denoted as $L_1 = (W_1, X_1, Y_1, Z_1, Q_1, R_1, S_1, T_1)$:

$$\begin{aligned}
W_1 &= (x_0, y_1, x_4, y_5, y_8, y_9, x_9, x_{12}); \\
X_1 &= (x_1, y_4, x_7, y_{10}, x_{11}); \\
Y_1 &= (y_0, y_3, x_3, x_6, y_6, y_7, x_{10}, y_{13}); \\
Z_1 &= (y_2, x_2, x_5, x_8, y_{11}); \\
Q_1 &= (x_{12}, y_{12}, y_{15}, x_{15}, x_{18}, y_{18}, y_{21}, x_{21}, x_{24}); \\
R_1 &= (x_{11}, y_{14}, x_{17}, x_{20}, x_{23}); \\
S_1 &= (y_{13}, x_{13}, y_{16}, x_{16}, y_{19}, x_{19}, y_{22}, x_{22}, y_{25}); \\
T_1 &= (y_{11}, x_{14}, y_{17}, y_{20}, y_{23}).
\end{aligned}$$

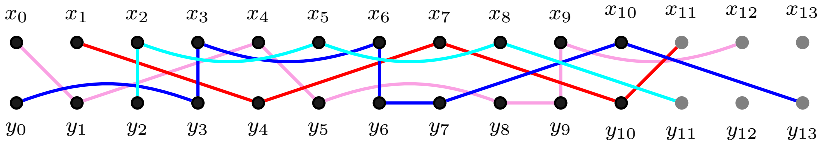
All dipaths in L_1 are illustrated in Figure 3.13.

Next, we construct another type-2 basic set of dipaths, this time denoted $L_2 = (W_2, X_2, Y_2, Z_2, Q_2, R_2, S_2, T_2)$:

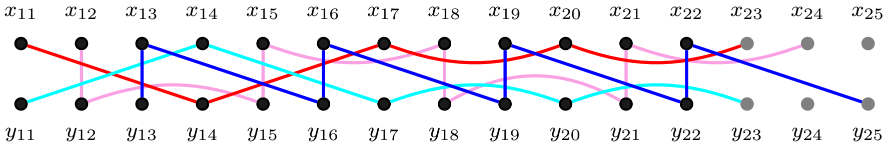
$$\begin{aligned}
W_2 &= (x_0, y_3, x_6, y_9, y_{10}, y_{11}); \\
X_2 &= (y_0, x_3, y_4, y_7, x_8, y_8, x_{11}); \\
Y_2 &= (y_1, x_1, x_4, x_7, x_{10}, x_{13});
\end{aligned}$$

$$\begin{aligned}
Z_2 &= (x_2, y_2, y_5, x_5, y_6, x_9, y_{12});\\
Q_2 &= (y_{11}, x_{12}, y_{13}, y_{16}, y_{17}, x_{20}, y_{23});\\
R_2 &= (x_{11}, x_{14}, x_{17}, x_{18}, y_{19}, y_{22}, x_{23});\\
S_2 &= (x_{13}, y_{14}, y_{15}, x_{16}, x_{19}, y_{20}, y_{21}, x_{22}, x_{25});\\
T_2 &= (y_{12}, x_{15}, y_{18}, x_{21}, y_{24}).
\end{aligned}$$

Figure 3.14 contains an illustration of all dipaths in L_2.

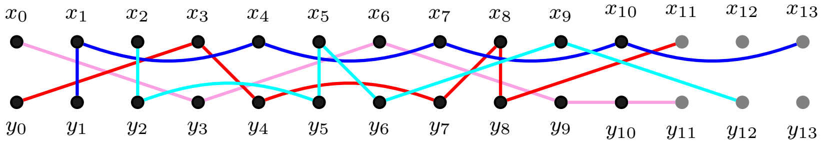


(a) The dipaths W_1 (pink), X_1 (red), Y_1 (dark blue), and Z_1 (light blue).

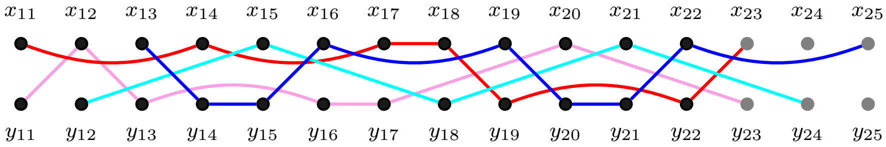


(b) The dipaths Q_1 (pink), R_1 (red), S_1 (dark blue), and T_1 (light blue).

Figure 3.13: The type-2 basic set of dipaths L_1 for $p = 11$.



(a) The dipaths W_2 (pink), X_2 (red), Y_2 (dark blue), and Z_2 (light blue).



(b) The dipaths Q_2 (pink), R_2 (red), S_2 (dark blue), and T_2 (light blue).

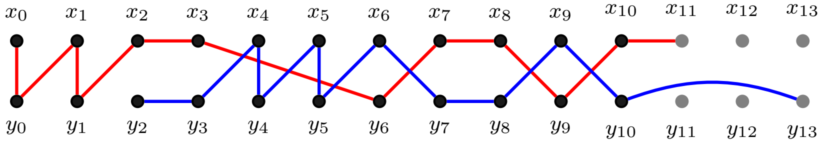
Figure 3.14: The type-2 basic set of dipaths L_2 for $p = 11$.

For $k = 0$, we replace each dipath in $\{Q_i, R_i, S_i, T_i \mid i = 0, 1, 2\}$ with a dipath of length 0 with the same source. It can be verified that the 8-tuples L_0, L_1, and L_2 satisfy conditions (B1)-(B6) of Definition 3.18, and thus are type-2 basic sets of dipaths.

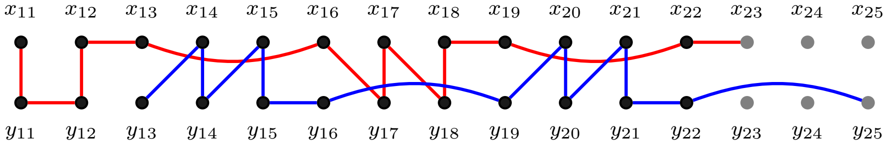
We now form a type-1 basic set of dipaths $L_3 = (X_3, Y_3, R_3, S_3)$:

$$\begin{aligned} X_3 &= (x_0, y_0, x_1, y_1, x_2, x_3, y_6, x_7, x_8, y_9, x_{10}, x_{11}); \\ Y_3 &= (y_2, y_3, x_4, y_4, x_5, y_5, x_6, y_7, y_8, x_9, y_{10}, y_{13}); \\ R_3 &= (x_{11}, y_{11}, y_{12}, x_{12}, x_{13}, x_{16}, y_{17}, x_{17}, y_{18}, x_{18}, x_{19}, x_{22}, x_{23}); \\ S_3 &= (y_{13}, x_{14}, y_{14}, x_{15}, y_{15}, y_{16}, y_{19}, x_{20}, y_{20}, x_{21}, y_{21}, y_{22}, y_{25}). \end{aligned}$$

Refer to Figure 3.15 for an illustration of X_3, Y_3, R_3, and S_3.



(a) The dipaths X_3 (red) and Y_3 (dark blue).



(b) The dipaths R_3 (red) and S_3 (dark blue).

Figure 3.15: The type-1 basic set of dipaths L_3 for $p = 11$.

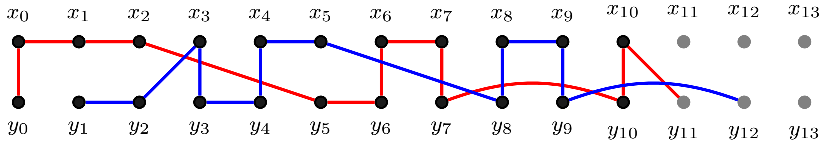
Lastly, we form a second type-1 basic set $L_4 = (X_4, Y_4, R_4, S_4)$:

$$\begin{aligned} X_4 &= (y_0, x_0, x_1, x_2, y_5, y_6, x_6, x_7, y_7, y_{10}, x_{10}, y_{11}); \\ Y_4 &= (y_1, y_2, x_3, y_3, y_4, x_4, x_5, y_8, x_8, x_9, y_9, y_{12}); \\ R_4 &= (y_{11}, x_{11}, x_{12}, x_{15}, x_{16}, y_{16}, x_{17}, y_{17}, x_{18}, x_{21}, x_{22}, y_{22}, y_{23}); \\ S_4 &= (y_{12}, x_{13}, y_{13}, y_{14}, x_{14}, y_{15}, y_{18}, x_{19}, y_{19}, y_{20}, x_{20}, y_{21}, y_{24}). \end{aligned}$$

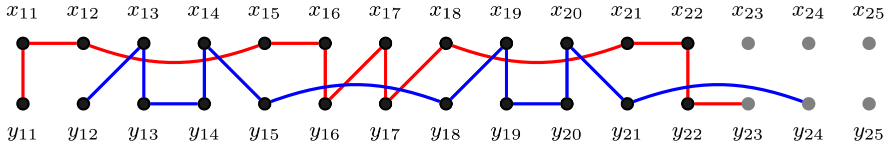
See Figure 3.16 for an illustration of the dipaths in L_4.

Again, for $k = 0$, we replace each dipath in $\{R_i, S_i \mid i = 3, 4\}$ with a dipath of length 0 with the same source. Similarly, it can be verified that quadruples L_3 and L_4 satisfy conditions (B1)-(B6) of Definition 3.21, and thus are type-1 basic sets of dipaths.

It is tedious, but straightforward, to verify that, for each vertex x_i (y_i) in V_0, each of the five arcs with tail x_i (y_i) appears in a dipath in $\{W_j, X_j, Y_j, Z_j \mid j = 0, 1, 2\} \cup \{X_j, Y_j \mid j = 3, 4\}$ exactly once. An analogous claim holds for vertices x_i (y_i) in V_1 and the set of dipaths $\{Q_j, R_j, S_j, T_j \mid j = 0, 1, 2\} \cup \{R_j, S_j \mid j = 3, 4\}$. It



(a) The dipaths X_4 (red) and Y_4 (dark blue).



(b) The dipaths R_4 (red) and S_4 (dark blue).

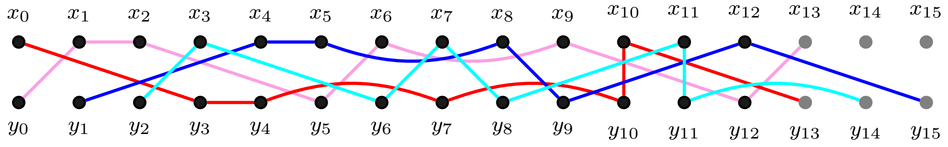
Figure 3.16: The type-1 basic set of dipaths L_4 for $p = 11$.

follows that the dipaths in $L_0 \cup \ldots \cup L_4$ are pairwise arc-disjoint and thus satisfy the hypothesis of Lemma 3.23. As a result, the digraph G_{2m} admits a $\vec{C}_m$-factorization.

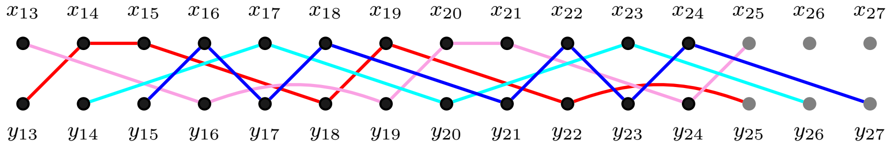
Case 2: $p = 13$. Let $L_0 = (W_0, X_0, Y_0, Z_0, Q_0, R_0, S_0, T_0)$ where

$$\begin{aligned}
W_0 &= (y_0, x_1, x_2, y_5, x_6, x_9, y_{12}, x_{13});\\
X_0 &= (x_0, y_3, y_4, y_7, y_{10}, x_{10}, y_{13});\\
Y_0 &= (y_1, x_4, x_5, x_8, y_9, x_{12}, y_{15});\\
Z_0 &= (y_2, x_3, y_6, x_7, y_8, x_{11}, y_{11}, y_{14});\\
Q_0 &= (x_{13}, y_{16}, y_{19}, x_{20}, x_{21}, y_{24}, x_{25});\\
R_0 &= (y_{13}, x_{14}, x_{15}, y_{18}, x_{19}, y_{22}, y_{25});\\
S_0 &= (y_{15}, x_{16}, y_{17}, x_{18}, y_{21}, x_{22}, y_{23}, x_{24}, y_{27});\\
T_0 &= (y_{14}, x_{17}, y_{20}, x_{23}, y_{26}).
\end{aligned}$$

See Figure 3.17 for an illustration of the dipaths in L_0.



(a) The dipaths W_0 (pink), X_0 (red), Y_0 (dark blue), and Z_0 (light blue).



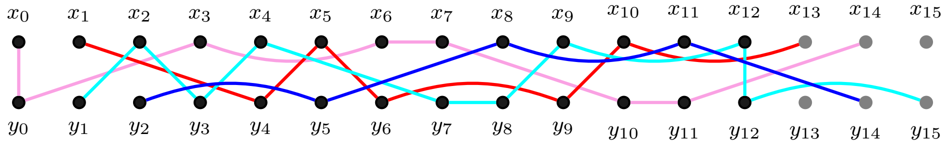
(b) The dipaths Q_0 (pink), R_0 (red), S_0 (dark blue), and T_0 (light blue).

Figure 3.17: The type-2 basic set of dipaths L_0 for $p = 13$.

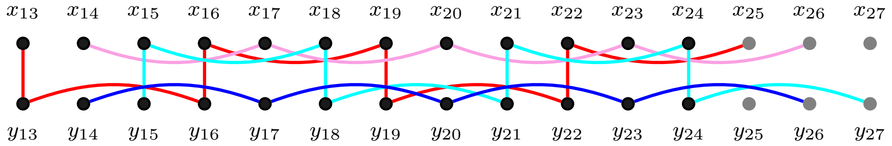
Next, we build a second type-2 basic set of dipaths denoted as $L_1 = (W_1, X_1, Y_1, Z_1, Q_1, R_1, S_1, T_1)$:

$$\begin{aligned}
W_1 &= (x_0, y_0, x_3, x_6, x_7, y_{10}, y_{11}, x_{14});\\
X_1 &= (x_1, y_4, x_5, y_6, y_9, x_{10}, x_{13});\\
Y_1 &= (y_2, y_5, x_8, x_{11}, y_{14});\\
Z_1 &= (y_1, x_2, y_3, x_4, y_7, y_8, x_9, x_{12}, y_{12}, y_{15});\\
Q_1 &= (x_{14}, x_{17}, x_{20}, x_{23}, x_{26});\\
R_1 &= (x_{13}, y_{13}, y_{16}, x_{16}, x_{19}, y_{19}, y_{22}, x_{22}, x_{25});\\
S_1 &= (y_{14}, y_{17}, y_{20}, y_{23}, y_{26});\\
T_1 &= (y_{15}, x_{15}, x_{18}, y_{18}, y_{21}, x_{21}, x_{24}, y_{24}, y_{27}).
\end{aligned}$$

See Figure 3.18, for an illustration of the dipaths in L_1.



(a) The directed paths W_1 (pink), X_1 (red), Y_1 (dark blue), and Z_1 (light blue).



(b) The directed paths Q_1 (pink), R_1 (red), S_1 (dark blue), and T_1 (light blue).

Figure 3.18: The type-2 basic set of dipaths L_1 for $p = 13$.

Now, we build our third type-2 basic set of dipaths, $L_2 = (W_2, X_2, Y_2, Z_2, Q_2, R_2, S_2, T_2)$:

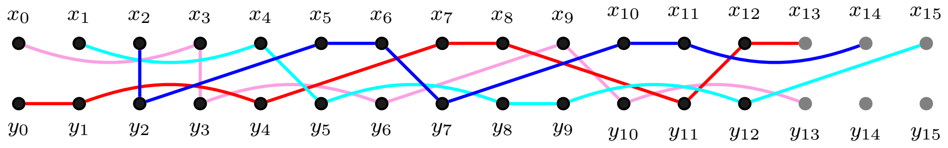
$$\begin{aligned}
W_2 &= (x_0, x_3, y_3, y_6, x_9, y_{10}, y_{13});\\
X_2 &= (y_0, y_1, y_4, x_7, x_8, y_{11}, x_{12}, x_{13});\\
Y_2 &= (x_2, y_2, x_5, x_6, y_7, x_{10}, x_{11}, x_{14});\\
Z_2 &= (x_1, x_4, y_5, y_8, y_9, y_{12}, x_{15});\\
Q_2 &= (y_{13}, y_{14}, y_{15}, x_{18}, x_{19}, x_{22}, y_{25});\\
R_2 &= (x_{13}, x_{16}, y_{19}, y_{20}, y_{21}, x_{24}, x_{25});\\
S_2 &= (x_{14}, y_{17}, x_{20}, y_{23}, x_{26});\\
T_2 &= (x_{15}, y_{16}, x_{17}, y_{18}, x_{21}, y_{22}, x_{23}, y_{24}, x_{27}).
\end{aligned}$$

Refer to Figure 3.19 for an illustration of the dipaths in L_2.

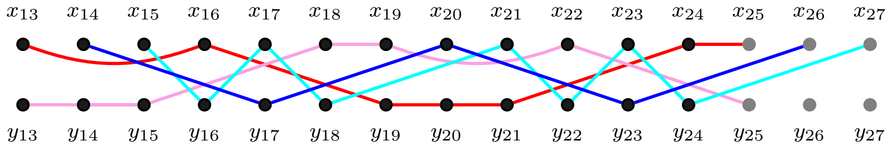
For $k = 0$, we replace each dipath in $\{Q_i, R_i, S_i, T_i \mid i = 0, 1, 2\}$ with a dipath of length 0 with the same source. It can be verified that the 8-tuples L_0, L_1, and L_2 satisfy conditions (B1)-(B6) of Definition 3.18, and thus are type-2 basic sets of dipaths.

Next, we construct a type-1 basic set of dipaths, denoted $L_3 = (X_3, Y_3, R_3, S_3)$:

$$\begin{aligned}
X_3 &= (y_0, x_0, y_1, x_1, y_2, y_3, x_6, y_6, y_7, x_7, x_{10}, y_{11}, y_{12}, y_{13});\\
Y_3 &= (x_2, x_3, x_4, y_4, y_5, x_5, y_8, x_8, x_9, y_9, y_{10}, x_{11}, x_{12}, x_{15});\\
R_3 &= (y_{13}, x_{13}, y_{14}, x_{14}, y_{15}, y_{18}, y_{19}, x_{19}, y_{20}, x_{20}, y_{21}, y_{24}, y_{25});\\
S_3 &= (x_{15}, x_{16}, y_{16}, y_{17}, x_{17}, x_{18}, x_{21}, x_{22}, y_{22}, y_{23}, x_{23}, x_{24}, x_{27}).
\end{aligned}$$



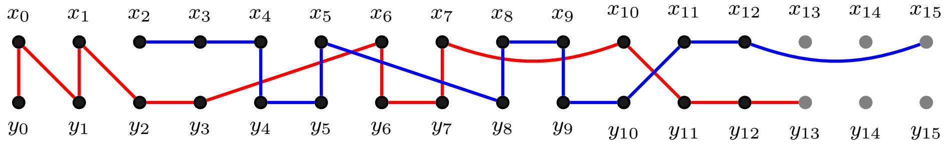
(a) The directed paths W_2 (pink), X_2 (red), Y_2 (dark blue), and X_2 (light blue).



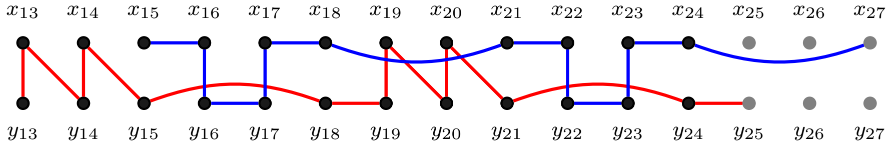
(b) The directed paths Q_2 (pink), R_2 (red), S_2 (dark blue), and T_2 (light blue).

Figure 3.19: The type-2 basic set of dipaths L_2 for $p = 13$.

In Figure 3.20, we illustrate all four dipaths in L_3.



(a) The directed paths X_3 (red) and Y_3 (dark blue).



(b) The directed paths R_3 (red) and S_3 (dark blue).

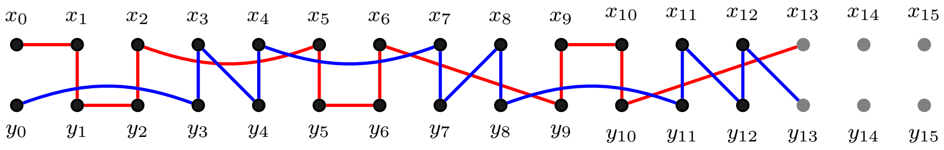
Figure 3.20: The type-1 basic set of dipaths L_3 for $p = 13$.

Lastly, we form another type-1 basic set $L_4 = (X_4, Y_4, R_4, S_4)$:

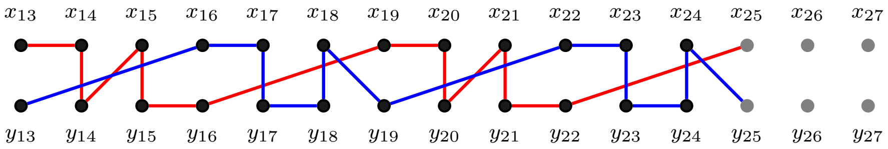
$$\begin{aligned} X_4 &= (x_0, x_1, y_1, y_2, x_2, x_5, y_5, y_6, x_6, y_9, x_9, x_{10}, y_{10}, x_{13}); \\ Y_4 &= (y_0, y_3, x_3, y_4, x_4, x_7, y_7, x_8, y_8, y_{11}, x_{11}, y_{12}, x_{12}, y_{13}); \\ R_4 &= (x_{13}, x_{14}, y_{14}, x_{15}, y_{15}, y_{16}, x_{19}, x_{20}, y_{20}, x_{21}, y_{21}, y_{22}, x_{25}); \\ S_4 &= (y_{13}, x_{16}, x_{17}, y_{17}, y_{18}, x_{18}, y_{19}, x_{22}, x_{23}, y_{23}, y_{24}, x_{24}, y_{25}). \end{aligned}$$

Figure 3.21 illustrates the four dipaths in L_4.

Again, for $k = 0$, we replace each dipath in $\{R_i, S_i \mid i = 3, 4\}$ with a dipath of length 0 with the same source. Similarly, it can be verified that quadruples L_3 and



(a) The dipaths X_4 (red) and Y_4 (dark blue).



(b) The dipaths R_4 (red) and S_4 (dark blue).

Figure 3.21: The type-1 basic set of dipaths L_4 for $p = 13$.

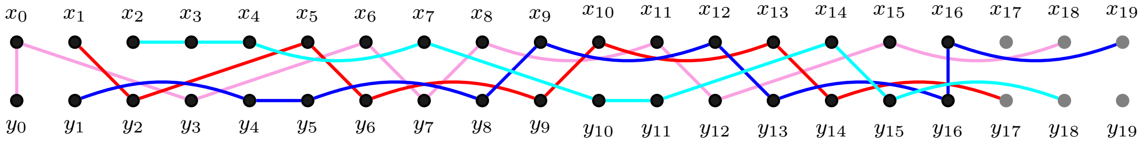
L_4 satisfy conditions (B1)-(B6) of Definition 3.21, and thus are type-1 basic sets of dipaths.

Once again, it is tedious but straightforward to verify that $\{L_0, L_1, L_2, L_3, L_4\}$ satisfies the hypothesis of Lemma 3.23.

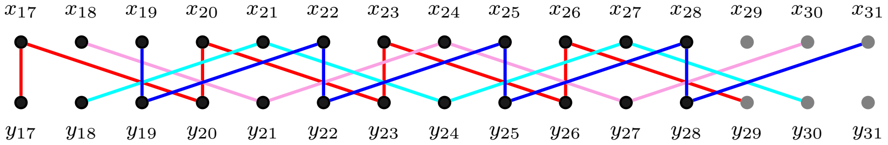
Case 3: $p = 17$. Let $L_0 = (W_0, X_0, Y_0, Z_0, Q_0, R_0, S_0, T_0)$ where

$$\begin{aligned}
W_0 &= (y_0, x_0, y_3, x_6, y_7, x_8, x_{11}, y_{12}, x_{15}, x_{18});\\
X_0 &= (x_1, y_2, x_5, y_6, y_9, x_{10}, x_{13}, y_{14}, y_{17});\\
Y_0 &= (y_1, y_4, y_5, y_8, x_9, x_{12}, y_{13}, y_{16}, x_{16}, x_{19});\\
Z_0 &= (x_2, x_3, x_4, x_7, y_{10}, y_{11}, x_{14}, y_{15}, y_{18});\\
Q_0 &= (x_{18}, y_{21}, x_{24}, y_{27}, x_{30});\\
R_0 &= (y_{17}, x_{17}, y_{20}, x_{20}, y_{23}, x_{23}, y_{26}, x_{26}, y_{29});\\
S_0 &= (x_{19}, y_{19}, x_{22}, y_{22}, x_{25}, y_{25}, x_{28}, y_{28}, x_{31});\\
T_0 &= (y_{18}, x_{21}, y_{24}, x_{27}, y_{30}).
\end{aligned}$$

We illustrate the dipaths in L_0 in Figure 3.22.



(a) The dipaths W_0 (pink), X_0 (red), Y_0 (dark blue), and Z_0 (light blue).



(b) The dipaths Q_0 (pink), R_0 (red), S_0 (dark blue), and T_0 (light blue).

Figure 3.22: The type-2 basic set of dipaths L_0 for $p = 17$.

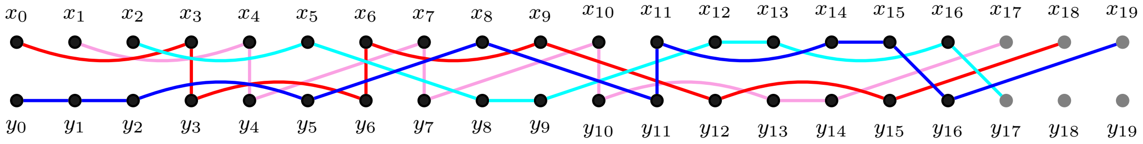
Next, we build a type-2 basic set of dipaths below, this time denoted as $L_1 = (W_1, X_1, Y_1, Z_1,\ Q_1, R_1, S_1, T_1)$:

$$\begin{aligned}
W_1 &= (x_1, x_4, y_4, x_7, y_7, x_{10}, y_{10}, y_{13}, y_{14}, x_{17});\\
X_1 &= (x_0, x_3, y_3, y_6, x_6, x_9, y_{12}, y_{15}, x_{18});\\
Y_1 &= (y_0, y_1, y_2, y_5, x_8, y_{11}, x_{11}, x_{14}, x_{15}, y_{16}, x_{19});\\
Z_1 &= (x_2, x_5, y_8, y_9, x_{12}, x_{13}, x_{16}, y_{17});\\
Q_1 &= (x_{17}, y_{18}, y_{19}, y_{22}, x_{23}, y_{24}, y_{25}, y_{28}, x_{29});\\
R_1 &= (x_{18}, x_{21}, x_{24}, x_{27}, x_{30});\\
S_1 &= (x_{19}, x_{20}, y_{21}, x_{22}, x_{25}, x_{26}, y_{27}, x_{28}, x_{31});\\
T_1 &= (y_{17}, y_{20}, y_{23}, y_{26}, y_{29}).
\end{aligned}$$

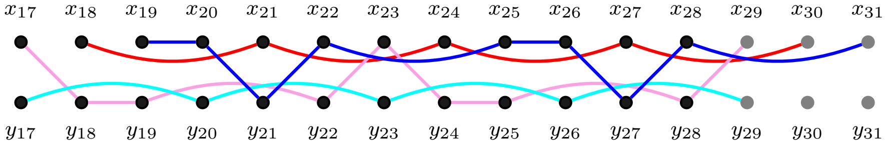
See Figure 3.23 for an illustration of all eight dipaths in L_0.

Below, we build a third type-2 basic set of dipaths, denoted $L_2 = (W_2, X_2, Y_2, Z_2, Q_2, R_2, S_2, T_2)$:

$$\begin{aligned}
W_2 &= (y_0, x_3, x_6, x_7, y_8, y_{11}, y_{14}, x_{14}, x_{17});\\
X_2 &= (x_0, x_1, y_4, x_5, x_8, y_9, y_{12}, x_{13}, y_{16}, y_{17});\\
Y_2 &= (y_1, x_2, y_5, y_6, x_9, x_{10}, y_{13}, x_{16}, y_{19});
\end{aligned}$$



(a) The dipaths W_1 (pink), X_1 (red), Y_1 (dark blue), and Z_1 (light blue).



(b) The dipaths Q_1 (pink), R_1 (red), S_1 (dark blue), and T_1 (light blue).

Figure 3.23: The type-2 basic set of dipaths L_1 for $p = 17$.

$$\begin{aligned}
Z_2 &= (y_2, y_3, x_4, y_7, y_{10}, x_{11}, x_{12}, y_{15}, x_{15}, y_{18});\\
Q_2 &= (x_{17}, x_{18}, x_{19}, y_{22}, y_{23}, x_{26}, x_{29});\\
R_2 &= (y_{17}, x_{20}, x_{23}, x_{24}, x_{25}, y_{28}, y_{29});\\
S_2 &= (y_{19}, y_{20}, x_{21}, x_{22}, y_{25}, y_{26}, x_{27}, x_{28}, y_{31});\\
T_2 &= (y_{18}, y_{21}, y_{24}, y_{27}, y_{30}).
\end{aligned}$$

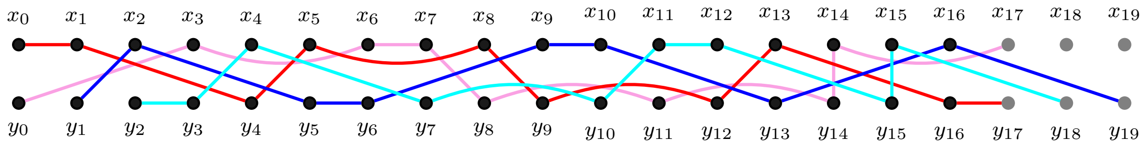
In Figure 3.24, we illustrate the eight dipaths in L_2.

For $k = 0$, we replace each dipath in $\{Q_i, R_i, S_i, T_i \mid i = 0, 1, 2\}$ with a dipath of length 0 with the same source. It can be verified that the 8-tuples L_0, L_1, and L_2 satisfy conditions (B1)-(B6) of Definition 3.18, and thus are type-2 basic sets of dipaths.

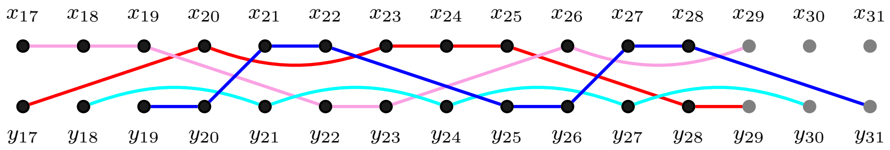
We now form a type-1 basic set of dipaths $L_3 = (X_3, Y_3, R_3, S_3)$:

$$\begin{aligned}
X_3 &= (x_0, y_0, x_1, y_1, x_4, y_5, x_5, x_6, y_6, x_7, x_{10}, x_{11}, y_{11}, y_{12}, x_{12}, x_{15}, x_{16}, x_{17});\\
Y_3 &= (y_2, x_2, y_3, x_3, y_4, y_7, y_8, x_8, x_9, y_9, y_{10}, x_{13}, y_{13}, x_{14}, y_{14}, y_{15}, y_{16}, y_{19});\\
R_3 &= (x_{17}, y_{17}, x_{18}, y_{18}, x_{19}, x_{22}, x_{23}, y_{23}, x_{24}, y_{24}, x_{25}, x_{28}, x_{29});\\
S_3 &= (y_{19}, x_{20}, y_{20}, y_{21}, x_{21}, y_{22}, y_{25}, x_{26}, y_{26}, y_{27}, x_{27}, y_{28}, y_{31}).
\end{aligned}$$

Refer to Figure 3.25 for an illustration of X_3, Y_3, R_3, and S_3.

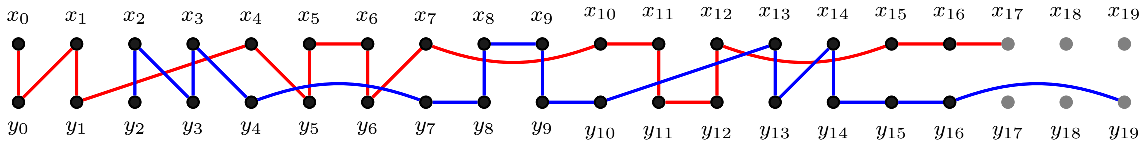


(a) The dipaths W_2 (pink), X_2 (red), Y_2 (dark blue), and Z_2 (light blue).

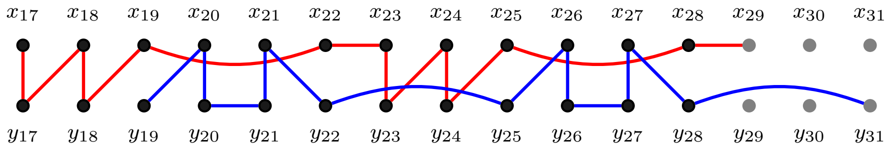


(b) The dipaths Q_2 (pink), R_2 (red), S_2 (dark blue), and T_2 (light blue).

Figure 3.24: The type-2 basic set of dipaths L_2 for $p = 17$.



(a) The dipaths X_3 (red) and Y_3 (dark blue).



(b) The dipaths R_3 (red) and S_3 (dark blue).

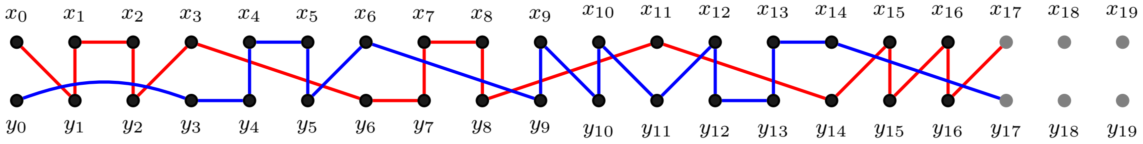
Figure 3.25: The type-1 basic set of dipaths L_3 for $p = 17$.

Lastly, we form another type-1 basic set $L_4 = (X_4, Y_4, R_4, S_4)$:

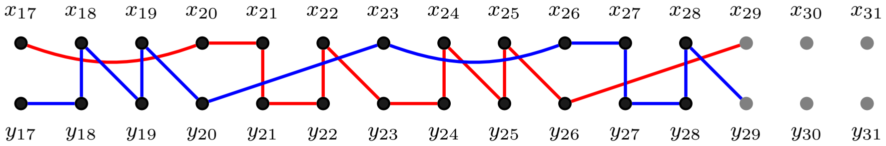
$$\begin{aligned}
X_4 &= (x_0, y_1, x_1, x_2, y_2, x_3, y_6, y_7, x_7, x_8, y_8, x_{11}, y_{14}, x_{15}, y_{15}, x_{16}, y_{16}, x_{17});\\
Y_4 &= (y_0, y_3, y_4, x_4, x_5, y_5, x_6, y_9, x_9, y_{10}, x_{10}, y_{11}, x_{12}, y_{12}, y_{13}, x_{13}, x_{14}, y_{17});\\
R_4 &= (x_{17}, x_{20}, x_{21}, y_{21}, y_{22}, x_{22}, y_{23}, y_{24}, x_{24}, y_{25}, x_{25}, y_{26}, x_{29});\\
S_4 &= (y_{17}, y_{18}, x_{18}, y_{19}, x_{19}, y_{20}, x_{23}, x_{26}, x_{27}, y_{27}, y_{28}, x_{28}, y_{29}).
\end{aligned}$$

In Figure 3.26, we give an illustration of X_4, Y_4, R_4, and S_4.

Again, for $k = 0$, we replace each dipath in $\{R_i, S_i \mid i = 3, 4\}$ with a dipath of length 0 with the same source. Similarly, it can be verified that quadruples L_3 and L_4 satisfy conditions (B1)-(B6) of Definition 3.21, and thus are type-1 basic sets of dipaths.



(a) The dipaths X_4 (red) and Y_4 (dark blue).



(b) The dipaths R_4 (red) and S_4 (dark blue).

Figure 3.26: The type-1 basic set of dipaths L_4 for $p = 17$.

Again, it is laborious but routine to verify that $\{L_0, L_1, L_2, L_3, L_4\}$ satisfies the hypothesis of Lemma 3.23.

<u>Case 4</u>: $p = 19$. Let $L_0 = (W_0, X_0, Y_0, Z_0, Q_0, R_0, S_0, T_0)$ where

$$\begin{aligned}
W_0 &= (y_0, y_1, x_4, y_4, y_7, x_{10}, x_{13}, y_{14}, y_{17}, x_{20});\\
X_0 &= (x_1, y_2, y_5, y_6, x_7, y_{10}, x_{11}, y_{12}, y_{13}, y_{16}, y_{19});\\
Y_0 &= (x_0, y_3, x_3, x_6, y_9, x_9, x_{12}, y_{15}, x_{18}, x_{21});\\
Z_0 &= (x_2, x_5, y_8, x_8, y_{11}, x_{14}, x_{15}, x_{16}, x_{17}, y_{18}, x_{19});\\
Q_0 &= (x_{20}, y_{21}, x_{22}, y_{23}, x_{26}, y_{27}, x_{28}, y_{29}, x_{32});\\
R_0 &= (y_{19}, y_{22}, y_{25}, y_{28}, y_{31});\\
S_0 &= (x_{21}, x_{24}, x_{27}, x_{30}, x_{33});\\
T_0 &= (x_{19}, y_{20}, x_{23}, y_{24}, x_{25}, y_{26}, x_{29}, y_{30}, x_{31}).
\end{aligned}$$

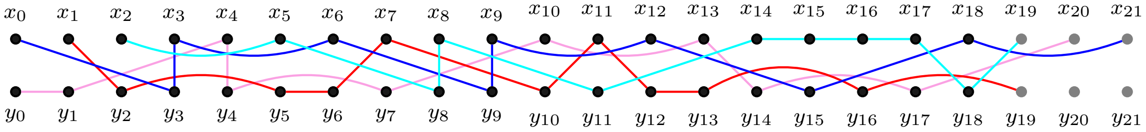
In Figure 3.27, we illustrate all eight dipaths in L_0.

We build our second type-2 basic set of dipaths below, this time denoted as $L_1 = (W_1, X_1, Y_1, Z_1, Q_1, R_1, S_1, T_1)$:

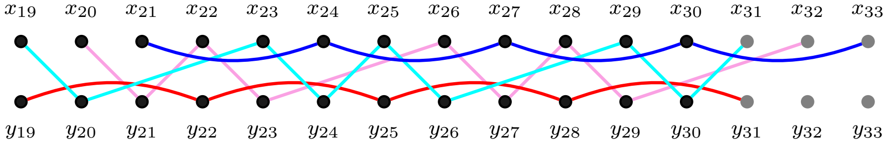
$$\begin{aligned}
W_1 &= (y_2, x_2, y_5, x_8, y_9, y_{12}, x_{12}, y_{13}, x_{16}, y_{17}, x_{17}, x_{20});\\
X_1 &= (x_1, x_4, x_7, y_8, x_{11}, x_{14}, y_{15}, y_{18}, y_{21});\\
Y_1 &= (x_0, y_1, y_4, x_5, y_6, x_9, x_{10}, y_{11}, y_{14}, x_{15}, x_{18}, y_{19});\\
Z_1 &= (y_0, x_3, y_3, x_6, y_7, y_{10}, x_{13}, y_{16}, x_{19});
\end{aligned}$$

$$\begin{aligned} Q_1 &= (x_{20}, x_{21}, y_{22}, x_{23}, x_{26}, x_{27}, y_{28}, x_{29}, x_{32}); \\ R_1 &= (y_{21}, y_{24}, y_{27}, y_{30}, y_{33}); \\ S_1 &= (y_{19}, y_{20}, y_{23}, x_{24}, x_{25}, x_{28}, y_{31}); \\ T_1 &= (x_{19}, x_{22}, y_{25}, y_{26}, y_{29}, x_{30}, x_{31}). \end{aligned}$$

Figure 3.28 illustrates all eight dipaths of L_1.

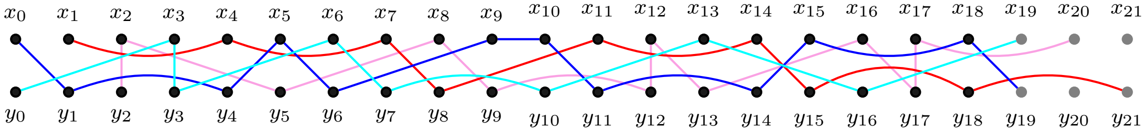


(a) The dipaths W_0 (pink), X_0 (red), Y_0 (dark blue), and Z_0 (light blue).

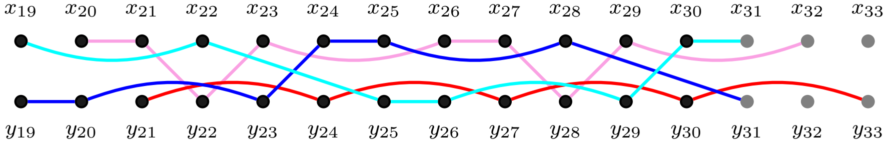


(b) The dipaths Q_0 (pink), R_0 (red), S_0 (dark blue), and T_0 (light blue).

Figure 3.27: The type-2 basic set of dipaths L_0 for $p = 19$.



(a) The dipaths W_1 (pink), X_1 (red), Y_1 (dark blue), and Z_1 (light blue).



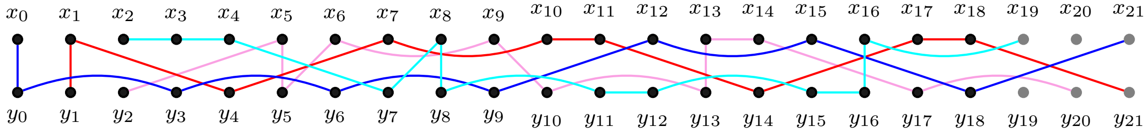
(b) The dipaths Q_1 (pink), R_1 (red), S_1 (dark blue), and T_1 (light blue).

Figure 3.28: The type-2 basic set of dipaths L_1 for $p = 19$.

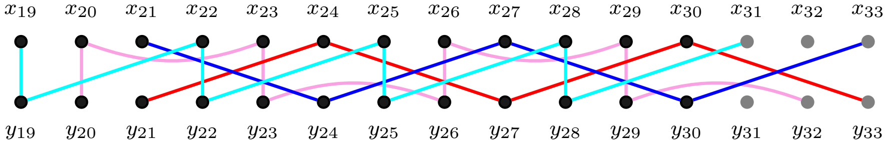
We build our third type-2 basic set of dipaths denoted as $L_2 = (W_2, X_2, Y_2, Z_2, Q_2, R_2, S_2, T_2)$:

$$\begin{aligned}
W_2 &= (y_2, x_5, y_5, x_6, x_9, y_{10}, y_{13}, x_{13}, x_{14}, y_{17}, y_{20});\\
X_2 &= (y_1, x_1, y_4, x_7, x_{10}, x_{11}, y_{14}, x_{17}, x_{18}, y_{21});\\
Y_2 &= (x_0, y_0, y_3, y_6, y_9, x_{12}, x_{15}, y_{18}, x_{21});\\
Z_2 &= (x_2, x_3, x_4, y_7, x_8, y_8, y_{11}, y_{12}, y_{15}, y_{16}, x_{16}, x_{19});\\
Q_2 &= (y_{20}, x_{20}, x_{23}, y_{23}, y_{26}, x_{26}, x_{29}, y_{29}, y_{32});\\
R_2 &= (y_{21}, x_{24}, y_{27}, x_{30}, y_{33});\\
S_2 &= (x_{21}, y_{24}, x_{27}, y_{30}, x_{33});\\
T_2 &= (x_{19}, y_{19}, x_{22}, y_{22}, x_{25}, y_{25}, x_{28}, y_{28}, x_{31}).
\end{aligned}$$

Refer to Figure 3.29 for an illustration of all eight dipaths in L_2.



(a) The dipaths W_2 (pink), X_2 (red), Y_2 (dark blue), and Z_2 (light blue).



(b) The dipaths Q_2 (pink), R_2 (red), S_2 (dark blue), and T_2 (light blue).

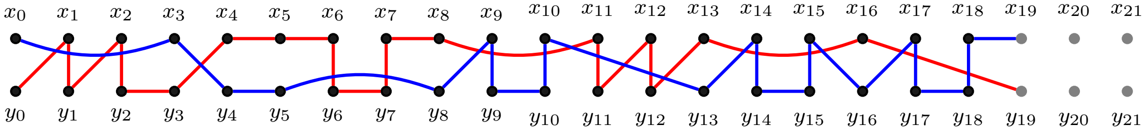
Figure 3.29: The type-2 basic set of dipaths L_2 for $p = 19$.

For $k = 0$, we replace each dipath in $\{Q_i, R_i, S_i, T_i \mid i = 0, 1, 2\}$ with a dipath of length 0 with the same source. It can be verified that the 8-tuples L_0, L_1, and L_2 satisfy conditions (B1)-(B6) of Definition 3.18, and thus are type-2 basic sets of dipaths.

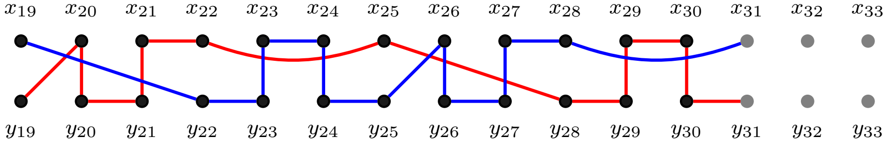
We proceed by forming a type-1 basic set of dipaths $L_3 = (X_3, Y_3, R_3, S_3)$:

$$\begin{aligned}
X_3 &= (y_0, x_1, y_1, x_2, y_2, y_3, x_4, x_5, x_6, y_6, y_7, x_7, x_8, x_{11}, y_{11}, x_{12}, y_{12}, x_{13}, x_{16}, y_{19});\\
Y_3 &= (x_0, x_3, y_4, y_5, y_8, x_9, y_9, y_{10}, x_{10}, y_{13}, x_{14}, y_{14}, y_{15}, x_{15}, y_{16}, x_{17}, y_{17}, y_{18}, x_{18}, x_{19});\\
R_3 &= (y_{19}, x_{20}, y_{20}, y_{21}, x_{21}, x_{22}, x_{25}, y_{28}, y_{29}, x_{29}, x_{30}, y_{30}, y_{31});\\
S_3 &= (x_{19}, y_{22}, y_{23}, x_{23}, x_{24}, y_{24}, y_{25}, x_{26}, y_{26}, y_{27}, x_{27}, x_{28}, x_{31}).
\end{aligned}$$

In Figure 3.30, we give an illustration of all four dipaths in L_3.



(a) The dipaths X_3 (red) and Y_3 (dark blue).



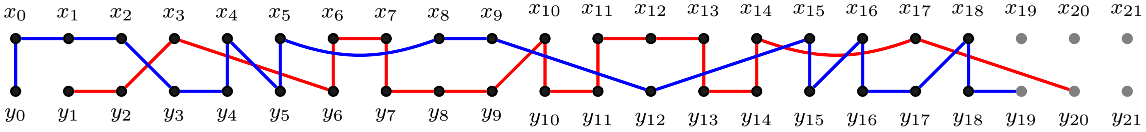
(b) The dipaths R_3 (red) and S_3 (dark blue).

Figure 3.30: The type-1 basic set of dipaths L_3 for $p = 19$.

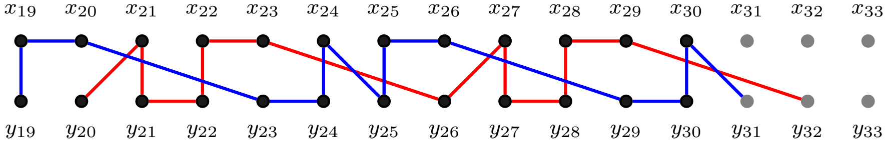
Lastly, we build our second type-1 basic set $L_4 = (X_4, Y_4, R_4, S_4)$:

$$\begin{aligned} X_4 &= (y_1, y_2, x_3, y_6, x_6, x_7, y_7, y_8, y_9, x_{10}, y_{10}, y_{11}, x_{11}, x_{12}, x_{13}, y_{13}, y_{14}, x_{14}, x_{17}, y_{20}); \\ Y_4 &= (y_0, x_0, x_1, x_2, y_3, y_4, x_4, y_5, x_5, x_8, x_9, y_{12}, x_{15}, y_{15}, x_{16}, y_{16}, y_{17}, x_{18}, y_{18}, y_{19}); \\ R_4 &= (y_{20}, x_{21}, y_{21}, y_{22}, x_{22}, x_{23}, y_{26}, x_{27}, y_{27}, y_{28}, x_{28}, x_{29}, y_{32}); \\ S_4 &= (y_{19}, x_{19}, x_{20}, y_{23}, y_{24}, x_{24}, y_{25}, x_{25}, x_{26}, y_{29}, y_{30}, x_{30}, y_{31}). \end{aligned}$$

Figure 3.31 illustrates the four dipaths in L_4.



(a) The dipaths X_4 (red) and Y_4 (dark blue).



(b) The dipaths R_4 (red) and S_4 (dark blue)

Figure 3.31: The type-1 basic set of dipaths L_4 for $p = 19$.

Again, for $k = 0$, we replace each dipath in $\{R_i, S_i \mid i = 3, 4\}$ with a dipath of length 0 with the same source. As before, it can be verified that quadruples L_3 and L_4 satisfy conditions (B1)-(B6) of Definition 3.21, and thus are type-1 basic sets of dipaths.

One can then verify that $\{L_0, L_1, L_2, L_3, L_4\}$ satisfies the hypothesis of Lemma 3.23.

In conclusion, we see that the hypothesis of Lemma 3.23 is satisfied for all odd $m \geqslant 11$ such that $m \equiv 1$ or $5 \pmod 6$ and thus, the digraph G_{2m} admits a $\vec{C}_m$-factorization. ▮

3.5 Conclusion

We now use this chapter's technical lemmas to prove our main theorem. In Sections 3.3 and 3.4, we constructed a $\vec{C}_m$-factorization of H_{2m}, L_{2m}, and G_{2m} for all $m \equiv 1$ or $5 \pmod 6$ such that $m \geqslant 11$. However, in Proposition 3.14, we showed that L_{2m} does not admit a $\vec{C}_m$-factorization when $m \equiv 3 \pmod 6$. We circumvent this case using Lemma 3.25 below. This lemma reduces Conjecture 2.13 to the case m is odd and $m \not\equiv 0 \pmod 3$ or $m = 9$.

Lemma 3.25. *Let m be odd. If K^*_{2m} admits a $\vec{C}_m$-factorization, then $K^*_{2(3m)}$ admits a $\vec{C}_{3m}$-factorization.*

Proof: Assume that K^*_{2m} admits a $\vec{C}_m$-factorization. Let $F_1, F_2, \ldots, F_{2m-1}$ be the corresponding $\vec{C}_m$-factors, so for each $k \in \{1, 2, \ldots, 2m-1\}$, we have $F_k \cong 2\vec{C}_m$. Then, we can obtain the following decomposition of $K^*_{2(3m)}$ into spanning subdigraphs:

$$\begin{aligned} K^*_{2(3m)} &= K^*_{2m} \wr K^*_3 \\ &= (F_1 \oplus F_2 \oplus \ldots \oplus F_{2m-1}) \wr K^*_3 \\ &= (F_1 \wr K^*_3) \oplus (F_2 \wr \overline{K}_3) \oplus \ldots \oplus (F_{2m-1} \wr \overline{K}_3). \end{aligned}$$

Observe that

$$\begin{aligned} F_1 \wr K^*_3 &\cong (2\vec{C}_m) \wr K^*_3 \\ &\cong 2(\vec{C}_m \wr K^*_3), \end{aligned}$$

and that

$$\begin{aligned} F_i \wr \overline{K}_3 &\cong (2\vec{C}_m) \wr \overline{K}_3 \\ &\cong 2(\vec{C}_m \wr \overline{K}_3). \end{aligned}$$

for $i > 1$. Lemma 2.25 implies that $\vec{C}_m \wr K_3^*$ is hamiltonian decomposable. This means that $\vec{C}_m \wr K_3^*$ admits a $\vec{C}_{3m}$-factorization by Lemma 2.25. Therefore, Lemma 2.7 implies that $2(\vec{C}_m \wr K_3^*)$ also admits a $\vec{C}_{3m}$-factorization. Similarly, one can show that $2(\vec{C}_m \wr \overline{K}_3)$ admits a $\vec{C}_{3m}$-factorization using Lemmas 2.7 and 2.24. In conclusion, Lemma 2.8 implies that $K^*_{2(3m)}$ admits a $\vec{C}_{3m}$-factorization. ∎

Observe that Lemma 3.25 is vacuous when $m = 3$ because K_6^* does not admit a $\vec{C}_3$-factorization [18].

Now, we proceed with the proof of our main theorem, Theorem 3.26.

Theorem 3.26. *If $m \geqslant 5$ and m is odd, then K^*_{2m} admits a $\vec{C}_m$-factorization.*

Proof: If $m \equiv 1$ or $5 \pmod 6$ and $m \geqslant 11$, the statement follows from Lemma 3.6, Propositions 3.13, 3.14, and 3.24.

Next, we consider the case $m \equiv 3 \pmod 6$ or $5 \leqslant m \leqslant 9$. If $m = 3^r t$ for some $t \equiv 1$ or $5 \pmod 6$, $t \geqslant 11$, and $r \geqslant 1$, then a $\vec{C}_t$-factorization of K^*_{2t} exists by the above, and the existence of a $\vec{C}_m$-factorization of K^*_{2m} is established by a repeated application of Lemma 3.25.

Otherwise, we have $m = 3^r t$ for $t \in \{5, 7, 9\}$. If $r = 0$, then a $\vec{C}_t$-factorization of K^*_{2t} exists by Theorem 2.12 [23], and if $r \geqslant 1$, then a repeated application of Lemma 3.25 applied to this result can be used to show existence of a $\vec{C}_m$-factorization of K^*_{2m}. ∎

Lastly, we combine Theorems 2.11 and 3.26 to obtain a general solution to the directed Oberwolfach problem with an even number of tables of uniform odd length.

Theorem 3.27. *If $m \geqslant 5$ is odd and α is even, then $K^*_{\alpha m}$ admits a $\vec{C}_m$-factorization.*

In conjunction with results from [1,2,13,16,18,23,24,68], Theorem 3.27 completes the solution to the directed Oberwolfach problem with tables of uniform length. This result is stated in the form of Theorem 3.28 below.

Theorem 3.28. *Let m and α be positive integers. The digraph $K^*_{\alpha m}$ admits a $\vec{C}_m$-factorization if and only if $(\alpha, m) \notin \{(1,6),(1,4),(2,3)\}$.*

In conclusion, we have completed the proof of Conjecture 2.13. The implications of this chapter's results are far-reaching as exhibited by Theorem 3.28. In addition, results from this chapter were instrumental in the completion of the solution of the two-table case of the directed Oberwolfach problem. Recently, Kadri and Šajna [40] and Horsley and Lacaze-Masmonteil [37] jointly obtained a complete solution of the two-table case of the directed Oberwolfach problem with tables of varying lengths. Namely, in [37], the methods developed in this chapter were successfully adapted to address certain cases of the directed Oberwolfach problem with tables of varying lengths.

Chapter 4

Hamiltonian decompositions of the wreath product of two hamiltonian decomposable digraphs

The study of cycle decompositions of graphs (digraphs) arising from graph products has a rich history, as suggested by the vast amount of literature on this topic [11, 12, 15, 19, 41, 46, 50, 52–54, 65, 66, 71]. As shown in the proof of Lemma 3.25, which was instrumental in proving Theorem 3.28, these decompositions can be used to solve fundamental cycle decomposition problems. For that reason, many problems pertaining to cycle decompositions of products of graphs are fundamental problems themselves. In this chapter, we will be studying the problem of existence of a decomposition of the wreath product of two digraphs into directed hamiltonian cycles. Namely, we aim to address a specific case of the following fundamental problem in cycle decompositions.

Problem 4.1. *Given two graphs (digraphs) G and H that are both hamiltonian decomposable, is $G \otimes H$, where $\otimes$ is one of the four graph products defined in Chapter 1, also hamiltonian decomposable?*

The undirected version of Problem 4.1 has been settled for the wreath (lexicographic) product [12] and the strong product [28, 71]. In [12], and [28] and [71], it was shown that, if G and H are hamiltonian decomposable graphs, then $G \wr H$ and $G \boxtimes H$, respectively, are also hamiltonian decomposable. Problem 4.1 has also been partially settled for the Cartesian product [65]. Regarding the categorical (direct) product, Bermond [15] has shown that, if G and H are hamiltonian decomposable and at least one of $|V(G)|$ and $|V(H)|$ is odd, then $G \times H$ is hamiltonian decomposable.

Very little progress has been made on the directed version of Problem 4.1, in part due to the fact that there exist infinite families of hamiltonian decomposable digraphs G and H for which $G \otimes H$ is not hamiltonian decomposable. For instance, the digraph $\vec{C}_n \square \vec{C}_m$ is hamiltonian decomposable if and only if there exist positive integers s_1 and s_2 such that $\gcd(m, n) = s_1 + s_2$ and $\gcd(mn, s_1 s_2) = 1$ [41]. As for the wreath product, we have the following conjecture, first stated in Chapter 2 and restated here for convenience.

Conjecture 2.23. *If G and H are strict hamiltonian decomposable digraphs such that $G \neq \overline{K}_n$, then $G \wr H$ is also hamiltonian decomposable.*

In [52], Ng identifies two exceptions to Conjecture 2.23. Namely, Ng shows that Conjecture 2.23 does not hold when $G = \vec{C}_n$, with n odd, and $H \in \{\overline{K}_2, K_2^*\}$. Additionally, Ng [52] affirms Conjecture 2.23 when $|V(G)|$ is odd and $|V(H)| > 2$, and claims without providing it to have a proof for the case when $|V(G)|$ and $|V(H)|$ are even and H admits a decomposition into an even number of directed hamiltonian cycles. Ng's main result was originally presented in Chapter 2 and is restated below for completeness.

Theorem 2.26. [52] *Let G and H be two strict digraphs such that $|V(G)|$ is odd, $G \neq \overline{K}_n$, and $|V(H)| > 2$. If G and H are hamiltonian decomposable, then $G \wr H$ is hamiltonian decomposable.*

In this chapter, we will be examining Conjecture 2.23 for the case $|V(G)|$ is even. Theorem 4.2, given below, summarizes our contributions to Conjecture 2.23.

Theorem 4.2. *Let G and H be two strict hamiltonian decomposable digraphs such that $G \neq \overline{K}_n$. Let $|V(G)| = n$, where n is even, $|V(H)| = m$, and let c be the number of cycles in a hamiltonian decomposition of H. The digraph $G \wr H$ is hamiltonian decomposable in each of the following cases:*

(S1) *m is odd, $(m, c) \neq (3, 1)$, and $(n, m, c) \neq (2, 3, 2)$;*

(S2) *m and c are even;*

(S3) *m is even, $3 \leqslant c \leqslant m-3$ is odd, and $G \neq \vec{C}_n$;*

(S4) *m is even, $c = m-1$, and $(n,m) \neq (2,2)$;*

(S5) *m is even, $m \geqslant 4$, $c = 1$, and $n \geqslant 4$.*

If $G = \vec{C}_n$ and $(m,c) \in \{(2,1),(3,1)\}$ or $(n,m,c) = (2,3,2)$, then $G \wr H$ is not hamiltonian decomposable.

If $(n,m,c) = (2,3,2)$, then $G = K_2^*$ and $H = K_3^*$. Note that $K_2^* \wr K_3^* = K_6^*$. Theorem 3.28 implies that K_6^* is not hamiltonian decomposable. In addition, we identify two non-trivial exceptions to Conjecture 2.23. Namely, in Propositions 4.10 and 4.19, we demonstrate that Conjecture 2.23 is not true for $G = \vec{C}_n$, where n is even, and $H \in \{\vec{C}_2, \vec{C}_3\}$.

Now, we discuss some of Ng's results from [52] because we will be taking a similar approach. In [52], Ng showed that, if G and H are hamiltonian decomposable digraphs on n and m vertices, respectively, it suffices to show that $\vec{C}_n \wr \overline{K}_m$ and $\vec{C}_n \wr H$ are hamiltonian decomposable. Ng then proceeded to show that $\vec{C}_n \wr \overline{K}_m$ is hamiltonian decomposable for all $m > 2$.

Lemma 4.3. *[52] Let n be a positive integer. If $m > 2$, then $\vec{C}_n \wr \overline{K}_m$ is hamiltonian decomposable.*

Lemma 4.3 allowed Ng to take the following reduction step which is implicitly proved in the proof of Proposition 1 of [52].

Lemma 4.4. *[52] Let G and H be two strict hamiltonian decomposable digraphs such that $|V(G)| = n$ and $|V(H)| > 2$. If $\vec{C}_n \wr H$ is hamiltonian decomposable, then $G \wr H$ is also hamiltonian decomposable.*

Theorem 2.26 and Lemma 4.4 allow us to concentrate on the case of Conjecture 2.23 where $G = \vec{C}_n$ with n even. Therefore, we aim to establish when $\vec{C}_n \wr H$ is hamiltonian decomposable if H is a strict hamiltonian decomposable digraph and n is even.

4.1 Preliminaries

The aim of this section is to introduce some terminology and notation that is used in the constructions given in this chapter. Note that some of the definitions introduced below are similar to those introduced in earlier chapters. However, the context in which these are used is different. For that reason, we repeat some of these definitions.

Notation 4.5. Let H be a digraph on m vertices that is hamiltonian decomposable. Let $V(H) = \mathbb{Z}_m$ and $V(\vec{C}_n) = \mathbb{Z}_n$. Then, $V(\vec{C}_n \wr H) = \{(x, y) \mid x \in \mathbb{Z}_n, y \in \mathbb{Z}_m\}$, and we write shortly x_y for (x, y). For each $i \in \mathbb{Z}_n$, we let $V_i = \{i_0, i_1, \ldots, i_{m-1}\}$.

Definition 4.6. An arc of $\vec{C}_n \wr H$ is of *difference* d if it is of the form $(i_j, (i+1)_{j+d})$ for some $i \in \mathbb{Z}_n$, with addition of the indices done modulo m. If an arc is of the form (i_{j_1}, i_{j_2}) where $(j_1, j_2) \in A(H)$, then we call it a *horizontal arc*.

To describe our decompositions of $\vec{C}_n \wr H$ into directed hamiltonian cycles, we use permutations from the symmetric group S_m. In this chapter, we adopt the following convention regarding the product of two permutations. If $\pi, \sigma \in S_m$, then $i^{\pi\sigma} = (i^\pi)^\sigma$. Furthermore, a cycle of a permutation is written as $(a_0, a_1, a_2, \ldots, a_{m-1})$ instead of the standard $(a_0\ a_1\ a_2\ \ldots\ a_{m-1})$ for ease of notation in later computations. Lastly, we note that the identity permutation will be denoted as id.

Notation 4.7. The cyclic group $\langle(0, 1, 2, 3, \ldots, m-1)\rangle$ is denoted Π_m. We have that $\Pi_m = \{\pi_i \mid i = 0, 1, \ldots, m-1\}$, where $\pi_i = (0, 1, 2, 3, \ldots, m-1)^i$.

For $\sigma \in S_m$, we let $T(\sigma)$ denote the number of cycles of σ in its disjoint cycle notation, including cycles of length 1. If $\sigma \in S_m$, then $1 \leqslant T(\sigma) \leqslant m$.

An n-tuple $(\sigma_0, \sigma_1, \ldots, \sigma_{n-1})$ of permutations in S_m corresponds to a spanning subdigraph F of $\vec{C}_n \wr \overline{K}_m$ with arc set

$$A(F) = \{(i_j, (i+1)_{j^{\sigma_i}}) \mid j \in \mathbb{Z}_m \text{ and } i \in \mathbb{Z}_n\}.$$

We then write $F = (\sigma_0, \sigma_1, \ldots, \sigma_{n-1})$. It is easy to see that F is in fact a directed 2-factor of $\vec{C}_n \wr \overline{K}_m$. Lastly, a permutation σ_i of $(\sigma_0, \sigma_1, \ldots, \sigma_{n-1})$ corresponds to the following set of m arcs:

$$\{(i_j, (i+1)_{j^{\sigma_i}}) \mid j \in \mathbb{Z}_m\}.$$

It is easy to see that the number of directed cycles in F equals $T(\sigma_0\sigma_1\ldots\sigma_{n-1})$. Hence, the directed 2-factor $F = (\sigma_0, \sigma - 1, \ldots, \sigma_{n-1})$ is a hamiltonian cycle if and only if $T(\sigma_0\sigma_1\ldots\sigma_{n-1}) = 1$.

Note that a directed 2-factor of $\vec{C}_n \wr \overline{K}_m$ described as $F = (\sigma_0, \sigma_1, \ldots, \sigma_{n-1})$ could be described using different permutations by simply relabeling vertices of $\overline{K}_m$. This would give rise to a new n-tuple of permutations such that $F = (\mu_0, \mu_1, \ldots, \mu_{n-1})$ where μ_i is necessarily a conjugate of σ_i.

We now give an example of some of the concepts defined above.

Example 4.8. Consider the directed 2-factor F of $\vec{C}_2 \wr \overline{K}_7$ illustrated in Figure 4.1. We can express F as (μ, σ) where $\mu = (0,3,5,6)(1,2)$ and $\sigma = \pi_1$. Arcs with tails in V_0 are of the form $(0_j, 1_{j^\mu})$ and arcs with tail in V_1 are of the form $(1_j, 0_{j^\sigma})$. We see that $\mu\sigma = (0,4,5)(1,3,6)(2)$. This means that $T(\mu\sigma) = 3$ and thus, our directed 2-factor F comprises of three disjoint directed cycles.

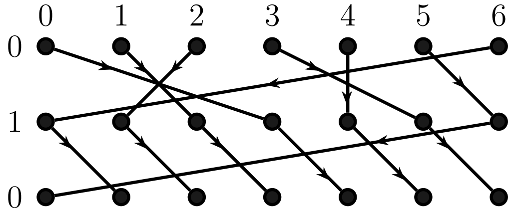


Figure 4.1: A directed 2-factor of $\vec{C}_2 \wr \overline{K}_7$.

□

We conclude this section by discussing a property of the wreath product of two digraphs that is key to our approach. Note that the subdigraph of $\vec{C}_n \wr H$ induced by the set of vertices V_i, which we will call H_i, is necessarily isomorphic to H.

Definition 4.9. Let H_i be the subdigraph of $\vec{C}_n \wr H$ induced by vertex set V_i. An *embedding of* H *into* V_i is an isomorphism $\phi : H \to H_i$.

In the next section, we will be embedding $H \in \{\vec{C}_m, \vec{P}_m\}$ into each V_i, specifying the image of $\vec{C}_m$ (or $\vec{P}_m$) by the chosen isomorphism instead of the isomorphism itself. We will be using the notation $v_0\, v_1\, \ldots\, v_{m-1}$ and $v_0\, v_1\, \ldots\, v_{m-1}\, v_0$ to refer to a dipath and a directed cycle of order m, respectively, instead of the tuples $(v_0, v_1, \ldots, v_{m-1})$

and $(v_0, v_1, \ldots, v_{m-1}, v_0)$ that were used in Chapters 2 and 3. We do so to avoid conflict with the notation used for permutations.

4.2 Decomposition of $G \wr \vec{C}_m$ into directed hamiltonian cycles

In this section and Section 4.4, we show that two specific digraphs are generally hamiltonian decomposable, namely the digraphs $\vec{C}_n \wr \vec{C}_m$ and $\vec{C}_n \wr K_m^*$. We then use Lemma 4.4 to show that, if G is hamiltonian decomposable, then $G \wr \vec{C}_m$ and $G \wr K_m^*$ are also hamiltonian decomposable. We address these two particular cases on their own because the methods used to construct hamiltonian decompositions for these two digraphs are different from those used for the more general case.

In this section, we will be focussing on the digraph $\vec{C}_n \wr \vec{C}_m$. We partition our investigation into two cases: m is even and m is odd.

4.2.1 The case m is even

Our objective is to show that $\vec{C}_n \wr \vec{C}_m$ is hamiltonian decomposable when n and m are even and $n, m \geqslant 4$. In addition, we show that Conjecture 2.23 is not true when $G = \vec{C}_n$, with n even, and $H = \vec{C}_2$. We first prove this statement below.

Proposition 4.10. *Let n be a positive even integer. The digraph $\vec{C}_n \wr \vec{C}_2$ is not hamiltonian decomposable.*

Proof: Note that a 2-factorization of $\vec{C}_n \wr \vec{C}_2$ is comprised of three directed 2-factors. Suppose that $\mathcal{D} = \{C^0, C^1, C^2\}$ is a decomposition of $\vec{C}_n \wr \vec{C}_2$ into directed hamiltonian cycles.

We first show that all horizontal arcs are contained in just two directed hamiltonian cycles of $\mathcal{D}$. Without loss of generality, suppose that C^0 contains the horizontal arc $(0_0, 0_1)$ and that $C^0 = 0_0\, 0_1 \ldots 0_0$. We claim that C^0 contains n horizontal arcs. Suppose that there exists some $i \in \mathbb{Z}_n$, $i \neq 0$, such that C^0 contains neither arc (i_0, i_1) nor (i_1, i_0). Without loss of generality, suppose that i_0 appears before i_1 in C^0. Observe that the sequence of vertices of C^0 is not decreasing in the first coordinate. The dipath of C^0 with source i_0 and terminal i_1 does not contain the arc (i_0, i_1), and hence contains at least one of 0_0 and 0_1. This implies that C^0 has a repeated vertex (0_0 or 0_1) other than its endpoints, a contradiction. Therefore, the directed cycle C^0 contains at least n horizontal arcs, and thus exactly n horizontal arcs of $\vec{C}_n \wr \vec{C}_2$. By a similar reasoning, we can show that C^1 contains the n horizontal arcs that do not appear in C^0.

It follows from the above property that $A(C^2)$ contains an even number of arcs of difference 1. In addition, $(i_0, (i+1)_1) \in A(C^2)$ if and only if $(i_1, (i+1)_0) \in A(C^2)$. Similarly, we see that $(i_1, (i+1)_1) \in A(C^2)$ if and only if $(i_0, (i+1)_0) \in A(C^2)$. Consequently, the digraph C^2 is actually the disjoint union of two directed cycles of length n, a contradiction. ∎

We now show that $\vec{C}_n \wr \vec{C}_m$ is hamiltonian decomposable when n and m are even and $n, m \geqslant 4$, or $n = 2$ and $m = 4$. Because our general approach does not work for the case $m = 4$, we first address this case on its own in Lemma 4.11 below.

Lemma 4.11. *Let n be an even integer. The digraph $\vec{C}_n \wr \vec{C}_4$ is hamiltonian decomposable.*

Proof: We will consider two cases.

Case 1: $n = 2$. The digraph $\vec{C}_4$ is embedded into V_0 and V_1 as follows: $0_0\, 0_1\, 0_2\, 0_3\, 0_0$ and $1_0\, 1_1\, 1_2\, 1_3\, 1_0$, respectively. Using this embedding, we construct five directed hamiltonian cycles:

$$C^0 = 0_0\, 1_0\, 0_3\, 1_2\, 0_1\, 1_1\, 0_2\, 1_3\, 0_0;$$
$$C^1 = 0_0\, 1_2\, 0_2\, 1_0\, 0_1\, 1_3\, 0_3\, 1_1\, 0_0;$$
$$C^2 = 0_0\, 0_1\, 1_0\, 1_1\, 0_3\, 1_3\, 0_2\, 1_2\, 0_0;$$
$$C^3 = 0_0\, 1_3\, 0_1\, 0_2\, 1_1\, 1_2\, 0_3\, 1_0\, 0_0; \text{and}$$
$$C^4 = 0_0\, 1_1\, 0_1\, 1_2\, 1_3\, 1_0\, 0_2\, 0_3\, 0_0.$$

It can easily be verified that each arc of $\vec{C}_2 \wr \vec{C}_4$ is used by exactly one directed cycle in $D = \{C^0, C^1, C^2, C^3, C^4\}$. As a result, the set D is a decomposition of $\vec{C}_2 \wr \vec{C}_4$ into directed hamiltonian cycles.

Case 2: $n \geqslant 4$. For $i \in \{5, 7, \ldots, n-1\}$, $\vec{C}_4$ is embedded into V_i as follows: $i_0\, i_3\, i_2\, i_1\, i_0$. Otherwise, for all other values of i, $\vec{C}_4$ is embedded into V_i as follows: $i_0\, i_1\, i_2\, i_3\, i_0$. We now construct the following 16 dipaths:

$U_0 = 0_0\,1_2\,2_0\,3_1\,4_1;$

$U_1 = 0_1\,1_3\,2_2\,3_2\,4_2;$

$U_2 = 0_2\,1_1\,2_3\,3_3\,4_3;$

$U_3 = 0_3\,1_0\,2_1\,3_0\,4_0;$

$W_0 = 0_0\,1_3\,1_0\,2_2\,2_3\,3_1\,4_2;$

$W_1 = 0_2\,0_3\,1_2\,2_1\,3_3\,3_0\,4_1;$

$W_2 = 0_1\,1_1\,2_0\,3_2\,4_0;$

$X_0 = 0_3\,0_0\,1_1\,2_2\,3_3\,4_2;$

$X_1 = 0_2\,1_2\,1_3\,2_0\,2_1\,3_2\,4_1;$

$X_2 = 0_1\,1_0\,2_3\,3_0\,3_1\,4_3;$

$Y_0 = 0_0\,0_1\,1_2\,2_2\,3_0\,4_2;$

$Y_1 = 0_2\,1_0\,1_1\,2_1\,3_1\,3_2\,4_3;$

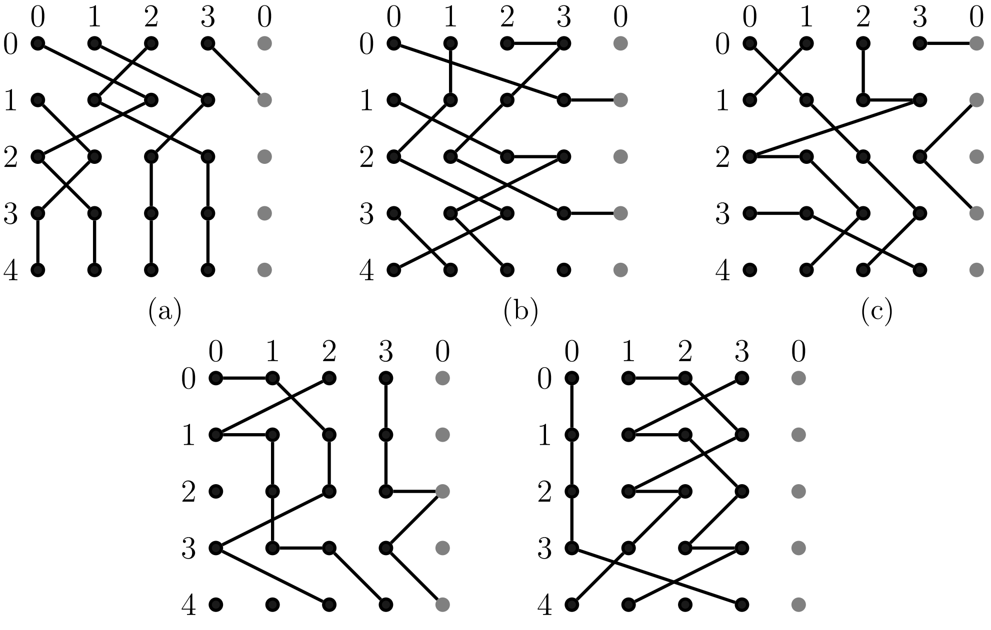
$Y_2 = 0_3\,1_3\,2_3\,2_0\,3_3\,4_0;$

$Z_0 = 0_0\,1_0\,2_0\,3_0\,4_3;$

$Z_1 = 0_3\,1_1\,1_2\,2_3\,3_2\,3_3\,4_1;$

$Z_2 = 0_1\,0_2\,1_3\,2_1\,2_2\,3_1\,4_0.$

Refer to Figure 4.2 for an illustration of these 16 dipaths. Note that, in Figures 4.2 and 4.3, it it assumed that non-horizontal arcs are oriented downwards. Horizontal arcs are then oriented to produce a dipath. All dipaths U_i are of length 4; likewise for W_2 and Z_0. Next, we note that W_0, W_1, X_1, Y_1, Z_1, and Z_2 are each of length 6. Lastly, dipaths X_0, X_2, Y_0, and Y_2 are each of length 5.



(a) (b) (c)

(d) (e)

Figure 4.2: Illustration of the dipaths in $\{U_i \mid i = 0, 1, 2, 3\}$ in (a), $\{W_i \mid i = 0, 1, 2\}$ in (b), $\{X_i \mid i = 0, 1, 2\}$ in (c), $\{Y_i \mid i = 0, 1, 2\}$ in (d), and $\{Z_i \mid i = 0, 1, 2\}$ in (e).

Observe that, if $n = 4$, then $4_j = 0_j$. Therefore, if $n = 4$, then we can form the following five cycles:

$$\begin{aligned}
C^0 &= U_0U_1U_2U_3;\\
C^1 &= W_0W_1W_2;\\
C^2 &= X_0X_1X_2;\\
C^3 &= Y_0Y_1Y_2;\\
C^4 &= Z_0Z_1Z_2.
\end{aligned}$$

Observe that each C^k is of length 16 and thus forms a directed hamiltonian cycle of $\vec{C}_4 \wr \vec{C}_4$. Moreover, it can be verified that directed cycles in $D = \{C^0, C^1, C^2, C^3, C^4\}$ are pairwise arc-disjoint. This means that D is a decomposition of $\vec{C}_4 \wr \vec{C}_4$ into directed hamiltonian cycles.

Otherwise, if $n \geqslant 6$, we also form the following 16 dipaths:

$$\begin{aligned}
&L_0 = 4_1\,5_3\,6_1\,7_3\,8_1 \ldots (n-2)_1\,(n-1)_3\,0_1;\\
&L_1 = 4_2\,5_0\,6_2\,7_0\,8_2 \ldots (n-2)_2\,(n-1)_0\,0_2;\\
&L_2 = 4_3\,5_1\,6_3\,7_1\,8_3 \ldots (n-2)_3\,(n-1)_1\,0_3;\\
&L_3 = 4_0\,5_2\,6_0\,7_2\,8_0 \ldots (n-2)_0\,(n-1)_2\,0_0;
\end{aligned}$$

$$\begin{aligned}
&M_0 = 4_2\,4_3\,5_3\,5_2\,6_2\,6_3\;\,7_3\,7_2\;\ldots (n-2)_2\,(n-2)_3\,(n-1)_3\,(n-1)_2\,0_0;\\
&M_1 = 4_1\,5_0\,6_1\,7_0\,8_1\,9_0\;\ldots (n-2)_1\,(n-1)_0\,0_1;\\
&M_2 = 4_0\,5_1\,6_0\,7_1\,8_0\,9_1\;\ldots (n-2)_0\,(n-1)_1\,0_0;
\end{aligned}$$

$$\begin{aligned}
&N_0 = 4_2\,5_1\,6_2\,7_1\,8_2\,9_1\;\ldots (n-2)_2\,(n-1)_1\,0_2;\\
&N_1 = 4_1\,5_2\,6_1\,7_2\,8_1\,9_2\;\ldots (n-2)_1\,(n-1)_2\,0_1;\\
&N_2 = 4_3\,4_0\,5_0\,5_3\,6_3\,6_0\;\,7_0\,7_3\;\ldots (n-2)_3\,(n-2)_0\,(n-1)_0\,(n-1)_3\,0_3;
\end{aligned}$$

$$\begin{aligned}
&O_0 = 4_2\,5_3\,6_2\,7_3\,8_2\,9_3\;\ldots (n-2)_2\,(n-1)_3\,0_2;\\
&O_1 = 4_3\,5_2\,6_3\,7_2\,8_3\,9_2\;\ldots (n-2)_3\,(n-1)_2\,0_3;\\
&O_2 = 4_0\,4_1\,5_1\,5_0\,6_0\,6_1\;\,7_1\,7_0\;\ldots (n-2)_0\,(n-2)_1\,(n-1)_1\,(n-1)_0\,0_0;
\end{aligned}$$

$P_0 = 4_3\,5_0\,6_3\,7_0\,8_3\,9_0\,\ldots(n-2)_3\,(n-1)_0\,0_3;$

$P_1 = 4_1\,4_2\,5_2\,5_1\,6_1\,6_2\;7_2\,7_1\,\ldots(n-2)_1\,(n-2)_2\,(n-1)_2\,(n-1)_1\,0_0;$

$P_2 = 4_0\,5_3\,6_0\,7_3\,8_0\,9_3\,\ldots(n-2)_0\,(n-1)_3\,0_0.$

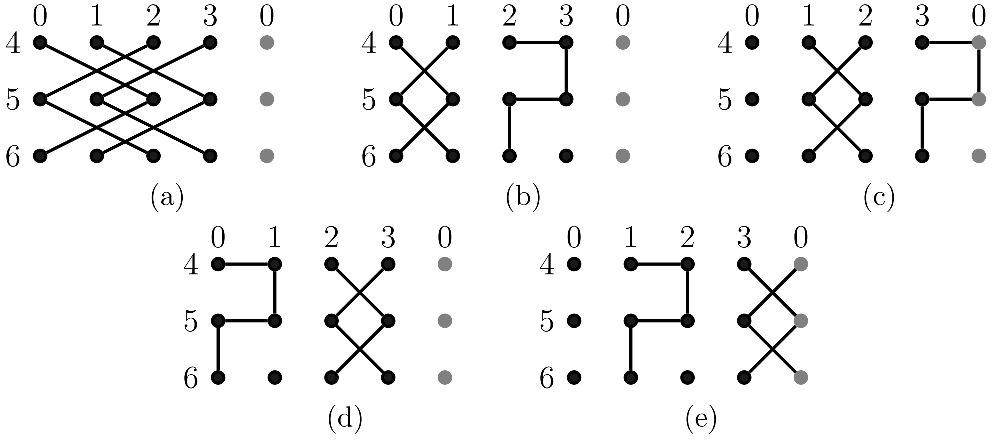


Figure 4.3: Illustration of the dipaths $\{L_i \mid i = 0,1,2,3\}$ in (a), $\{M_i \mid i = 0,1,2\}$ in (b), $\{N_i \mid i = 0,1,2\}$ in (c), $\{O_i \mid i = 0,1,2\}$ in (d), and $\{P_i \mid i = 0,1,2\}$ in (e) when $n = 6$.

Refer to Figure 4.3 for an illustration of these 16 dipaths for $n = 6$.

All L_i are of length $n-4$. Furthermore, dipaths M_0, N_2, O_2, and P_1 are of length $2(n-4)$. The remaining dipaths are of length $n-4$. We can then form the following directed walks:

$$C^0 = U_0L_0U_1L_1U_2L_2U_3L_3;$$

$$C^1 = W_0M_0W_1M_1W_2M_2;$$

$$C^2 = X_0N_0X_1N_1X_2N_2;$$

$$C^3 = Y_0O_0Y_1O_1Y_2O_2;$$

$$C^4 = Z_0P_0Z_1P_1Z_2P_2.$$

Let $D = \{C^0, C^1, C^2, C^3, C^4\}$. It is tedious but straightforward to verify that each directed walk of D is a directed cycle of length $4n$ and that these directed cycles are pairwise arc-disjoint. Therefore, the set D is a decomposition of $\vec{C}_n \wr \vec{C}_4$ into directed hamiltonian cycles. ∎

We now proceed with the more general case in which m and n are even, $m \geqslant 6$, and $n \geqslant 4$. To construct the desired decompositions of $\vec{C}_n \wr \vec{C}_m$, we first form two directed hamiltonian cycles, C^0 and C^1, that jointly use all horizontal arcs and all arcs of difference 0. Then, we make the observation that $\vec{C}_n \wr \vec{C}_m = C^0 \oplus C^1 \oplus (\vec{C}_n \times K_m^*)$. In other words, the spanning subdigraph of $\vec{C}_n \wr \vec{C}_m$ that contains all arcs of difference d, for $1 \leqslant d \leqslant m-1$, is precisely the direct product $\vec{C}_n \times K_m^*$. We can then use the following result of Paulraja and Sivasankar given as Theorem 2.6 in [54].

Theorem 4.12. *[54] Let $n \geqslant 4$ be an even integer, $m \geqslant 3$, and $m \neq 4$. The digraph $\vec{C}_n \times K_m^*$ is hamiltonian decomposable.*

Note that Theorem 4.12, which is key to the proof of Proposition 4.14 below, does not apply to the case $n = 2$. This means that we are unable to prove that $\vec{C}_2 \wr \vec{C}_m$ is hamiltonian decomposable for m even and $m \geqslant 6$ because we could not extend Theorem 4.12 to $n = 2$. Note that Lemma 4.11 implies that $\vec{C}_2 \wr \vec{C}_4$ is hamiltonian decomposable.

Using a computer search, we have verified that $\vec{C}_2 \times K_m^*$ is indeed hamiltonian decomposable for all even $6 \leqslant m \leqslant 16$. We refer the reader to Appendix B for a list of these solutions. This observation gives rise to the following conjecture.

Conjecture 4.13. *Let $m \geqslant 6$ be an even integer. The digraph $\vec{C}_2 \times K_m^*$ is hamiltonian decomposable.*

Although we were unable to prove Conjecture 4.13, we still prove Conjecture 2.23 for $G = \vec{C}_n$ where $n \geqslant 4$ is even, and $H = \vec{C}_m$ where $m \geqslant 4$ is even. We do so in Proposition 4.14 below.

Proposition 4.14. *Let $n \geqslant 4$ and $m \geqslant 4$ be even integers. The digraph $\vec{C}_n \wr \vec{C}_m$ is hamiltonian decomposable.*

Proof: We will construct a decomposition of $\vec{C}_n \wr \vec{C}_m$ into $m+1$ directed hamiltonian cycles. Note that we may assume that $m \geqslant 6$ since $\vec{C}_n \wr \vec{C}_4$ was shown to be hamiltonian decomposable in Lemma 4.11 above.

If i is even or $i = n-1$, we embed $\vec{C}_m$ into V_i as follows: $i_0\, i_1\, \ldots\, i_{m-1}\, i_0$. If i is odd and $i \leqslant n-3$, then we embed $\vec{C}_m$ into V_i as follows: $i_0\, i_{m-1}\, i_{m-2}\, \ldots\, i_1\, i_0$.

First, we construct two directed hamiltonian cycles. For each $j \in \mathbb{Z}_m$, we construct the following dipath of length $2n$:

$$P_j = 0_j\, 0_{j+1}\, 1_{j+1}\, 1_j\, 2_j\, 2_{j+1}\, \dots\, (n-2)_j\, (n-2)_{j+1}\, (n-1)_{j+1}\, (n-1)_{j+2}\, 0_{j+2}.$$

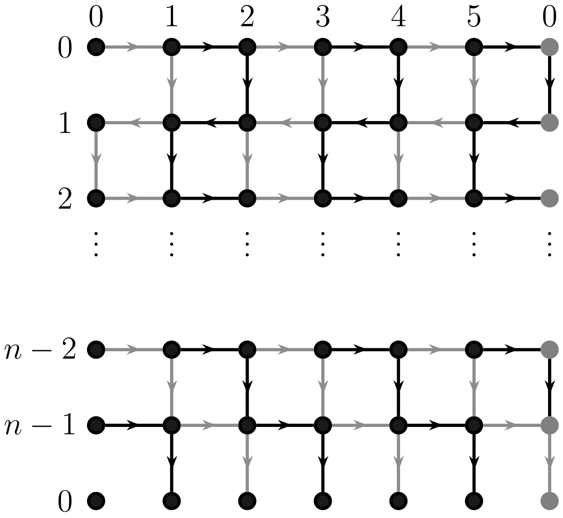


Figure 4.4: The two directed hamiltonian cycles C^0 (grey) and C^1 (black) of $\vec{C}_n \wr \vec{C}_6$.

We point out that $t(P_j) = s(P_{j+2}) = 0_{j+2}$ for all $j \in \mathbb{Z}_m$, and that P_j and P_{j+2} have no other vertices in common. Otherwise, if $|j-k| > 2$, then P_j and P_k are disjoint. We can concatenate these dipaths as follows:

$$C^0 = P_0 P_2 \dots P_{m-4} P_{m-2} \text{ and } C^1 = P_1 P_3 P_5 \dots P_{m-3} P_{m-1}.$$

In Figure 4.4, we illustrate C^0 and C^1 when $m = 6$. Each of C^0 and C^1 is a directed cycle of length nm. Moreover, if $i \neq j$, then P_i and P_j are arc-disjoint. This means that C^0 and C^1 are also arc-disjoint.

The directed cycles C^0 and C^1 jointly use all arcs of difference 0 and all horizontal arcs exactly once. Since $\vec{C}_n \wr \vec{C}_m = C^0 \oplus C^1 \oplus (\vec{C}_n \times K_m^*)$ and $\vec{C}_n \times K_m^*$ is a spanning subdigraph of $\vec{C}_n \wr \vec{C}_m$, it follows from Theorem 4.12 and Lemma 2.8 that $\vec{C}_n \wr \vec{C}_m$ also admits a decomposition into directed hamiltonian cycles. ∎

4.2.2 The case m is odd and $m \geqslant 5$

We now proceed with the case of Conjecture 2.23 where $G = \vec{C}_n$ and $H = \vec{C}_m$, where n is even, $m \geqslant 5$, and m is odd. We take a similar approach as Proposition 4.14.

Proposition 4.15. *Let n be an even integer and $m \geqslant 5$ be an odd integer. The digraph $\vec{C}_n \wr \vec{C}_m$ is hamiltonian decomposable.*

Proof: To construct a directed hamiltonian decomposition of $\vec{C}_n \wr \vec{C}_m$, we consider two cases. In both cases, we use the following embedding of $\vec{C}_m$ into each V_i. If i is even or $i = n-1$, we embed $\vec{C}_m$ into V_i as follows: $i_0\, i_1\, i_2 \ldots i_{m-1}\, i_0$. If i is odd and $i \leqslant n-3$, then we embed $\vec{C}_m$ into V_i as follows: $i_0\, i_{m-1} \ldots i_2\, i_1\, i_0$.

Case 1: $m \equiv 1 \pmod 4$. First, we construct two specific directed hamiltonian cycles of $\vec{C}_n \wr \vec{C}_m$.

If $n = 2$, we let

$$C^0 = 0_0\, 0_1\, 1_1\, 1_2\, 0_2\, 0_3\, 1_3\, 1_4 \ldots 0_{m-3}\, 0_{m-2}\, 1_{m-2}\, 0_{m-1}\, 1_{m-1}\, 1_0\, 0_0;$$

$$C^1 = 0_1\, 0_2\, 1_2\, 1_3\, 0_3\, 0_4\, 1_4\, 1_5 \ldots 0_{m-4}\, 0_{m-3}\, 1_{m-3}\, 1_{m-2}\, 1_{m-1}\, 0_{m-2}\, 0_{m-1}\, 0_0\, 1_0\, 1_1\, 0_1.$$

Otherwise, if $n \geqslant 4$, we construct a set of m dipaths. For each j such that $0 \leqslant j \leqslant m-5$, we construct the following dipath:

$$P_j = 0_j\, 0_{j+1}\, 1_{j+1}\, 1_j\, 2_j\, 2_{j+1} \ldots (n-2)_j\, (n-2)_{j+1}\, (n-1)_{j+1}\, (n-1)_{j+2}\, 0_{j+2}.$$

Next, we build the following four dipaths:

$$\begin{aligned} P_{m-4} = {} & 0_{m-4}\, 0_{m-3}\, 1_{m-3}\, 1_{m-4}\, 2_{m-4}\, 2_{m-3} \ldots (n-2)_{m-4}\, (n-2)_{m-3}\, (n-1)_{m-3} \\ & (n-1)_{m-2}\, (n-1)_{m-1}\, 0_{m-2}; \\ P_{m-3} = {} & 0_{m-3}\, 0_{m-2}\, 1_{m-2}\, 1_{m-3}\, 2_{m-3}\, 2_{m-2} \ldots (n-2)_{m-3}\, (n-2)_{m-2} \\ & (n-1)_{m-2}\, 0_{m-1}; \\ P_{m-2} = {} & 0_{m-2}\, 0_{m-1}\, 0_0\, 1_0\, 1_{m-1}\, 1_{m-2}\, 2_{m-2}\, 2_{m-1}\, 2_0 \ldots (n-2)_{m-2}\, (n-2)_{m-1} \\ & (n-2)_0\, (n-1)_0\, (n-1)_1\, 0_1; \\ P_{m-1} = {} & 0_{m-1}\, 1_{m-1}\, 2_{m-1} \ldots (n-2)_{m-1}\, (n-1)_{m-1}\, (n-1)_0\, 0_0. \end{aligned}$$

Dipaths P_j for $0 \leqslant j \leqslant m-5$ are of length $2n$. As for the remaining four dipaths, we see that $\mathrm{len}(P_{m-4}) = 2n+1$, $\mathrm{len}(P_{m-3}) = 2n-1$, $\mathrm{len}(P_{m-1}) = 3n-1$, and $\mathrm{len}(P_{m-1}) = n+1$. Observe that $t(P_j) = s(P_{j+2}) = 0_{j+2}$ for $0 \leqslant j \leqslant m-3$, $t(P_{m-2}) = s(P_1) = 0_1$, and $s(P_0) = t(P_{m-1}) = 0_0$. Otherwise, for $0 \leqslant j \leqslant m-3$,

P_j and P_{j+2} have no other vertices in common. Furthermore, if $|j-k|> 2$, then P_j and P_k are disjoint except for P_0 and P_{m-1}, and P_1 and P_{m-2}, where each pair shares exactly one vertex. We then concatenate the dipaths as follows:

$$C^0 = P_0P_2P_4 \dots P_{m-3}P_{m-1} \text{ and } C^1 = P_1P_3P_5 \dots P_{m-4}P_{m-2}.$$

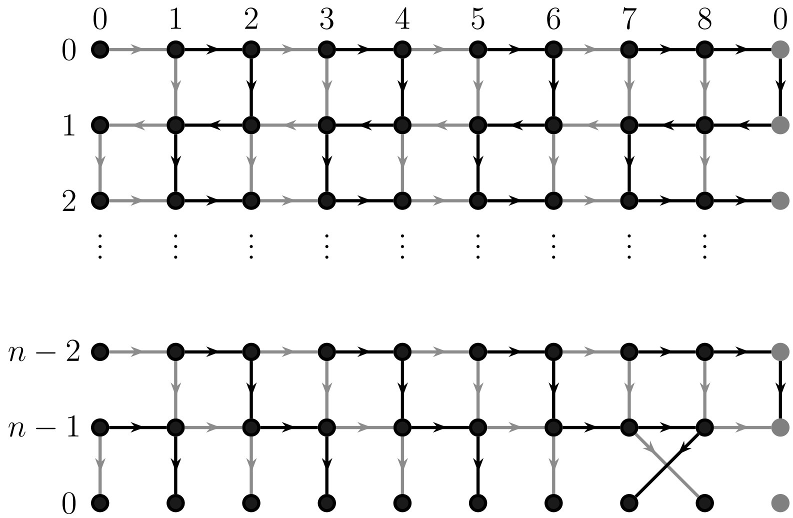


Figure 4.5: The two directed hamiltonian cycles C^0 (grey) and C^1 (black) of $\vec{C}_n \wr \vec{C}_9$.

Refer to Figure 4.5 for an illustration of C^0 and C^1 when $m = 9$. In both cases ($n = 2$ and $n \geqslant 4$), C^0 and C^1 jointly use all horizontal arcs and all arcs of difference 0 except for arcs $((n-1)_{m-2}, 0_{m-2})$ and $((n-1)_{m-1}, 0_{m-1})$. Moreover, the directed walks C^0 and C^1 have no repeated vertices other than their respective endpoints. Therefore, both C^0 and C^1 are directed hamiltonian cycles of $\vec{C}_n \wr \vec{C}_m$. Lastly, we note that C^0 and C^1 are arc-disjoint.

Next, we use permutations to construct $m-1$ directed hamiltonian cycles of $\vec{C}_n \wr \vec{C}_m$. We first define the following two permutations:

$$\mu = (0,1)(2,3)(4,5)\dots(m-3,m-2);$$
$$\sigma = (1,2)(3,4)(5,6)\dots(m-4,m-3)(m-1,0).$$

We will be considering the following set of arcs associated with μ and σ:

$$\{((n-1)_j, 0_{j^\mu}) \mid j \in \mathbb{Z}_m\} \text{ and } \{((n-1)_j, 0_{j^\sigma}) \mid j \in \mathbb{Z}_m\}, \text{ respectively.}$$

These two sets are disjoint, disjoint from $A(C^0)$ and $A(C^1)$, and contain the arcs of difference 0 missing from C^0 and C^1. Refer to Figure 4.6 for an illustration of the arcs corresponding to μ and σ when $m = 9$.

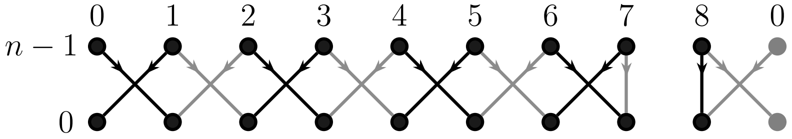


Figure 4.6: Illustration of μ (black) and σ (grey) when $m = 9$.

Recall that $\pi_i = (0, 1, 2, \ldots, m-1)^i$. If $n = 2$, we construct the following directed 2-factors: $C^2 = (\pi_2, \mu)$, and $C^3 = (\pi_3, \sigma)$. Observe that

$$\pi_2\mu = (0, 3, 4, 7, 8, 11, 12, 15, 16, \ldots, m-5, m-2, 1, 2, 5, 6, 9, 10, \ldots, m-3, m-1);$$

$$\pi_3\sigma = (0, 4, 8, 12, \ldots, m-5, m-2, 2, 6, \ldots, m-3, m-1, 1, 3, 5, 7, \ldots, m-4).$$

Otherwise, if $n > 2$, we construct the following directed 2-factors:

$$C^2 = (\pi_1, \pi_{-1}, \pi_1, \pi_{-1}, \ldots, \pi_1, \pi_{-1}, \pi_2, \mu);$$

$$C^3 = (\pi_{-1}, \pi_1, \pi_{-1}, \pi_1, \ldots, \pi_{-1}, \pi_1, \pi_3, \sigma).$$

We point out that

$$\pi_1\pi_{-1}\pi_1\pi_{-1} \ldots \pi_1\pi_{-1}\pi_2\mu = \pi_2\mu;$$

$$\pi_{-1}\pi_1\pi_{-1}\pi_1 \ldots \pi_{-1}\pi_1\pi_3\sigma = \pi_3\sigma.$$

Note that $T(\pi_2\mu) = T(\pi_3\sigma) = 1$, which implies that C^2 and C^3 are directed hamiltonian cycles for all even $n \geqslant 2$. Furthermore, note that each arc of difference 0 in $\vec{C}_n \wr \vec{C}_m$ appears precisely once in $\{C^0, C^1, C^2, C^3\}$. Next, we define $m-4$ directed 2-factors. For each $i \in \mathbb{Z}_m$ such that $i \notin \{0, \pm 1, -2\}$, we create the following directed 2-factor:

$$C^{i+2} = (\pi_i, \pi_{-i}, \ldots, \pi_i, \pi_{-i}, \pi_{-i+1}, \pi_i).$$

We also let $C^m = (\pi_{-2}, \pi_2, \ldots, \pi_{-2}, \pi_2, \pi_1, \pi_{-2})$. Note that

$$T(\pi_i\pi_{-i} \ldots \pi_i\pi_{-i}\pi_{-i+1}\pi_i) = T(\pi_1) = 1 \text{ and } T(\pi_{-2}\pi_2 \ldots \pi_{-2}\pi_2\pi_1\pi_{-2}) = T(\pi_{-1}) = 1.$$

Therefore, each directed 2-factor C^i, where $0 \leqslant i \leqslant m$, is a directed hamiltonian cycle. Lastly, it can be verified that each arc of each difference $d \in \mathbb{Z}_m$ and each horizontal arc appears exactly once in $D = \{C^0, C^1, \ldots, C^m\}$. Therefore, the set D is a directed hamiltonian decomposition of $\vec{C}_n \wr \vec{C}_m$.

Case 2: $m \equiv 3 \pmod 4$. Hence $m \geqslant 7$. First, we construct the following six specific dipaths:

$$\begin{aligned}
P_0 &= 0_0\, 0_1\, 0_2\, 1_2\, 1_1\, 1_0\, 2_0\, 2_1\, 2_2\, 3_2\, \ldots (n-2)_0;\\
P'_0 &= (n-2)_0\, (n-2)_1\, (n-2)_2\, (n-1)_2\, (n-1)_3\, (n-1)_4\, 0_3;\\
P_1 &= 0_1\, 1_1\, 2_1\, \ldots\, (n-2)_1;\\
P'_1 &= (n-2)_1\, (n-1)_1\, (n-1)_2\, 0_2;\\
P_2 &= 0_2\, 0_3\, 1_3\, 1_2\, 2_2\, \ldots\, (n-2)_2;\\
P'_2 &= (n-2)_2\, (n-2)_3\, (n-1)_3\, 0_4.
\end{aligned}$$

For j odd and $3 \leqslant j \leqslant m-2$, we construct:

$$\begin{aligned}
P_j &= 0_j\, 0_{j+1}\, 0_{j+2}\, 1_{j+2}\, 1_{j+1}\, 1_j\, 2_j\, 2_{j+1}\, 2_{j+2}\, 3_{j+2}\, 3_{j+1}\, 3_j\, \ldots\, (n-2)_j;\\
P'_j &= (n-2)_j\, (n-2)_{j+1}\, (n-2)_{j+2}\, (n-1)_{j+2}\, 0_{j+3}.
\end{aligned}$$

For j even and $4 \leqslant j \leqslant m-1$, we construct:

$$\begin{aligned}
P_j &= 0_j\, 1_j\, 2_j \ldots\, (n-2)_j;\\
P'_j &= (n-2)_j\, (n-1)_j\, (n-1)_{j+1}\, (n-1)_{j+2}\, 0_{j+1}.
\end{aligned}$$

If $n = 2$, then $\mathrm{len}(P_j) = 0$ for all $j \in \mathbb{Z}_m$. When j is even and $j \geqslant 4$, $\mathrm{len}(P_j) = n-2$ and $\mathrm{len}(P'_j) = 4$. If j is odd and $j \geqslant 3$, then $\mathrm{len}(P_j) = 3n-6$ and $\mathrm{len}(P'_j) = 4$. As for the remaining dipaths, we see that $\mathrm{len}(P_0) = 3n-6$, $\mathrm{len}(P'_0) = 6$, $\mathrm{len}(P_1) = n-2$, $\mathrm{len}(P'_1) = 3$, $\mathrm{len}(P_2) = 2n-4$, and $\mathrm{len}(P'_2) = 3$.

First, we consider the case $n = 2$. If j is odd and $3 \leqslant j \leqslant m-2$, then $t(P'_j) = s(P'_{j+3}) = 0_{j+3}$. Similarly, if j is even and $4 \leqslant j \leqslant m-1$, then $t(P'_j) = s(P'_{j+1}) = 0_{j+1}$. We also point out that $t(P'_0) = s(P'_3) = 0_3$, $t(P'_1) = s(P'_2) = 0_2$, $t(P'_2) = s(P'_4) = 0_4$, $t(P'_{m-1}) = s(P'_0) = 0_0$, and that $t(P'_{m-2}) = s(P'_1) = 0_1$. We then form the following directed walks:

$$C^0 = P'_0 P'_3 P'_6 P'_7 P'_{10} P'_{11} \ldots P'_{m-4} P'_{m-1};\; C^1 = P'_1 P'_2 P'_4 P'_5 P'_8 P'_9 \ldots P'_{m-3} P'_{m-2}.$$

Next, consider the case $n \geqslant 4$. If j is odd and $3 \leqslant j \leqslant m-2$, then $t(P'_j) = s(P_{j+3}) = 0_{j+3}$. Similarly, if j is even and $4 \leqslant j \leqslant m-1$, then $t(P'_j) = s(P_{j+1}) = 0_{j+1}$. We also point out that $t(P'_0) = s(P_3) = 0_3$, $t(P'_1) = s(P_2) = 0_2$, $t(P'_2) = s(P_4) = 0_4$, $t(P'_{m-1}) = s(P_0) = 0_0$, and that $t(P'_{m-2}) = s(P_1) = 0_1$. Lastly, we see that $t(P_j) = s(P'_j)$ for all j. Note that dipaths in $\{P_j, P'_j \mid j = 0, 3, 6, 7, 10, \ldots, m-4, m-1\}$ have no other vertices in common. Similarly, dipaths in $\{P_j, P'_j \mid j = 1, 2, 4, 5, 8, \ldots, m-$

$3, m-2\}$ also have no other vertices in common. Consequently, we form the following directed walks:

$$C^0 = P_0P'_0P_3P'_3P_6P'_6P_7P'_7P_{10}P'_{10}P_{11}P'_{11}\dots P_{m-4}P'_{m-4}P_{m-1}P'_{m-1};$$

$$C^1 = P_1P'_1P_2P'_2P_4P'_4P_5P'_5P_8P'_8P_9P_9\dots P_{m-3}P'_{m-3}P_{m-2}P'_{m-2}.$$

Refer to Figure 4.7 for an illustration of C^0 and C^1 when $m = 7$ and $n \geqslant 4$. We see that both directed walks have no repeated vertices except for their respective endpoints, and that both are of length nm. Therefore, both C^0 and C^1 are directed hamiltonian cycles. Lastly, we point out that C^0 and C^1 are arc-disjoint.

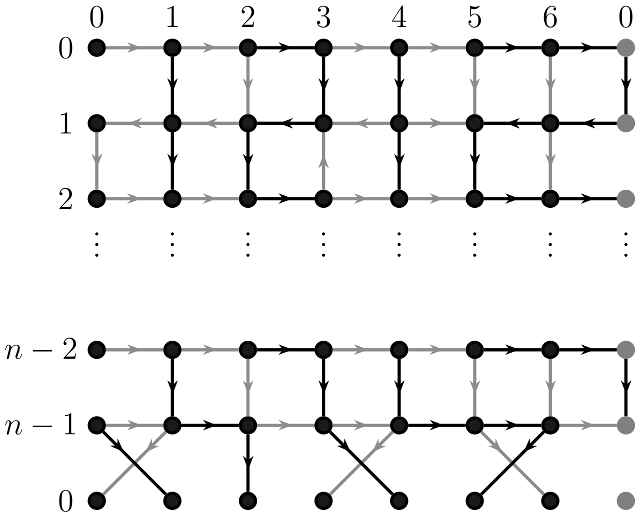


Figure 4.7: The directed hamiltonian cycles C^0 (grey) and C^1 (black) of $\vec{C}_n \wr \vec{C}_7$.

Next, similarly to Case 1, we construct $m-1$ directed hamiltonian cycles using permutations. First, we define the following two permutations:

$$\mu = (1,2)(4,5);$$

$$\sigma = (2,3)(6,7)(8,9)\dots(m-3,m-2)(m-1,0).$$

We will be considering the following set of arcs associated with μ and σ:

$$\{((n-1)_j, 0_{j^\mu}) \mid j \in \mathbb{Z}_m\} \text{ and } \{((n-1)_j, 0_{j^\sigma}) \mid j \in \mathbb{Z}_m\}, \text{ respectively.}$$

These two sets are disjoint, disjoint from $A(C^0)$ and $A(C^1)$, and contain the arcs of difference 0 missing from C^0 and C^1. Refer to Figure 4.8 for an illustration of μ and σ when $m = 7$.

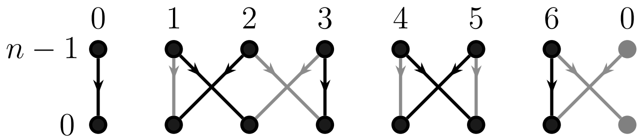


Figure 4.8: Illustration of μ (black) and σ (grey) when $m = 7$.

If $n = 2$, we construct the following directed 2-factors: $C^2 = (\pi_2, \mu)$, $C^3 = (\pi_3, \sigma)$. Observe that

$$\pi_2\mu = (0, 1, 3, 4, 6, 8, 10, \dots, m-1, 2, 5, 7, \dots, m-2);$$

$$\pi_3\sigma = (0, 2, 5, 9, 13, 17, \dots, m-2, 1, 4, 6, 8, 10, \dots, m-1, 3, 7, \dots, m-4).$$

If $n \geqslant 4$, we construct the following directed 2-factors in the same fashion as Case 1:

$$C^2 = (\pi_1, \pi_{-1}, \pi_1, \pi_{-1}, \dots, \pi_1, \pi_{-1}, \pi_2, \mu);$$

$$C^3 = (\pi_{-1}, \pi_1, \pi_{-1}, \pi_1, \dots, \pi_{-1}, \pi_1, \pi_3, \sigma).$$

Observe that

$$\pi_1\pi_{-1}\pi_1\pi_{-1}\dots\pi_1\pi_{-1}\pi_2\mu = \pi_2\mu;$$

$$\pi_{-1}\pi_1\pi_{-1}\pi_1\dots\pi_{-1}\pi_1\pi_3\sigma = \pi_3\sigma.$$

We then note that $T(\pi_2\mu) = T(\pi_3\sigma) = 1$. Therefore, both C^2 and C^3 are directed hamiltonian cycles for all even $n \geqslant 2$. Next, we construct $m-4$ directed 2-factors. For each $i \in \mathbb{Z}_m$ and $i \notin \{0, \pm 1, -2\}$, we let :

$$C^{i+2} = (\pi_i, \pi_{-i}, \dots, \pi_i, \pi_{-i}, \pi_{-i+1}, \pi_i).$$

Furthermore, we let $C^m = (\pi_{-2}, \pi_2, \dots, \pi_{-2}, \pi_2, \pi_1, \pi_{-2})$. Note that

$$T(\pi_i\pi_{-i}\dots\pi_i\pi_{-i}\pi_{-i+1}\pi_i) = T(\pi_1) = T(\pi_{-2}\pi_2\dots\pi_{-2}\pi_2\pi_1\pi_{-2}) = T(\pi_{-1}) = 1.$$

Therefore, each directed 2-factor C^i, where $0 \leqslant i \leqslant m$, is a directed hamiltonian cycle. Lastly, it can be verified that each arc of each difference $d \in \mathbb{Z}_m$, and each horizontal arc, appears precisely once in $D = \{C^0, C^1, \dots, C^m\}$. Therefore, the set D is a directed hamiltonian decomposition of $\vec{C}_n \wr \vec{C}_m$. ∎

Proposition 4.15 does not apply to the case $m = 3$. This is due to the fact that $\vec{C}_n \wr \vec{C}_3$ is not hamiltonian decomposable when n is even. The proof of this claim is fairly involved and is thus given in its own section.

4.3 The digraph $\vec{C}_n \wr \vec{C}_3$ is not hamiltonian decomposable

In this section, we will show that $\vec{C}_n \wr \vec{C}_3$ is not hamiltonian decomposable, for all even $n \geqslant 2$. This is another non-trivial exception to Conjecture 2.23.The proof of this statement is involved and requires two cases. In Subsection 4.3.1, we will first introduce key notation and show that we can partition the set of all 2-factorizations of $\vec{C}_n \wr \vec{C}_3$ into two particular sets. Then, in Subsections 4.3.2 and 4.3.3, we show that neither set contains a directed 2-factorization of $\vec{C}_n \wr \vec{C}_3$ that is also a hamiltonian decomposition.

4.3.1 Preliminaries

We begin by introducing the following assumption made throughout this section.

Assumption 4.16. In $\vec{C}_n \wr \vec{C}_3$, we embed $\vec{C}_3$ into V_j as $j_0\, j_1\, j_2\, j_0$ for all $j \in \mathbb{Z}_n$ and we let $C_3^j = j_0\, j_1\, j_2\, j_0$.

We now demonstrate that all 2-factors of $\vec{C}_n \wr \vec{C}_3$ must contain the same number of horizontal arcs from each C_3^j.

Lemma 4.17. *Let F be a directed 2-factor of $\vec{C}_n \wr \vec{C}_3$. Then there exists an integer k such that $0 \leqslant k \leqslant 3$ and $|A(F) \cap A(C_3^j)| = k$ for all $j \in \mathbb{Z}_n$.*

Proof: Suppose that F is a directed 2-factor of $\vec{C}_n \wr \vec{C}_3$ such that $|A(F) \cap A(C_3^j)|$ is not constant. Then, without loss of generality, we may assume that, for some $j \in \mathbb{Z}_n$, F contains k_1 horizontal arcs from C_3^j and k_2 horizontal arcs from C_3^{j+1} and that $0 \leqslant k_1 < k_2 \leqslant 3$. See Figure 4.9 for an example of this configuration when $k_1 = 1$ and $k_2 = 2$.

If F uses k_1 horizontal arcs from C_3^j, then F contains $3 - k_1$ non-horizontal arcs with tail in V_j. By our assumption, F also uses k_2 horizontal arcs from C_3^{j+1}. This means that F must contain $3 - k_2$ non-horizontal arcs with head in V_{j+1} and these must be the non-horizontal arcs with tail in V_j. Since $3 - k_1 \neq 3 - k_2$, we have a contradiction. Therefore, there exists some constant k such that $|A(F) \cap A(C_3^j)| = k$ for all $j \in \mathbb{Z}_n$, as desired. ∎

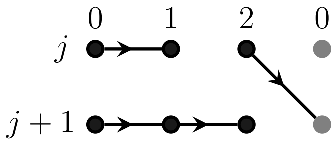


Figure 4.9: The 2-factor F contains $k_1 = 1$ arc from C_3^j and $k_2 = 2$ arcs from C_3^{j+1}.

As a result of Lemma 4.17, we can introduce the following definition.

Definition 4.18. Let $0 \leqslant k \leqslant 3$. A *type-$k$* 2-factor of $\vec{C}_n \wr \vec{C}_3$ is a directed 2-factor that contains k horizontal arcs from each C_3^j. A *type-I* 2-factorization of $\vec{C}_n \wr \vec{C}_3$ is comprised of three type-1 2-factors and one type-0 2-factor. A *type-II* 2-factorization of $\vec{C}_n \wr \vec{C}_3$ is comprised of one type-2 2-factor, one type-1 2-factor, and two type-0 2-factors.

In Definition 4.18, we described all possible 2-factorizations of interest. If a directed 2-factorization of $\vec{C}_n \wr \vec{C}_3$ contains a type-3 2-factor, then this 2-factorization is not hamiltonian since a type-3 2-factor is necessarily the digraph $n\vec{C}_3$.

We now state this section's main result.

Proposition 4.19. *Let n be a positive even integer. The digraph $\vec{C}_n \wr \vec{C}_3$ is not hamiltonian decomposable.*

Proof: As per the observation below Definition 4.18, a directed 2-factorization of $\vec{C}_n \wr \vec{C}_3$ that is also a hamiltonian decomposition must be a type-I or a type-II 2-factorization. Propositions 4.44 and 4.47, respectively, show that $\vec{C}_n \wr \vec{C}_3$ admits neither a type-I nor a type-II 2-factorization that is also a hamiltonian decomposition. ∎

In Subsection 4.3.2, we show that all type-I 2-factorizations of $\vec{C}_n \wr \vec{C}_3$ are not hamiltonian decompositions. Then, in Subsection 4.3.3, we show that all type-II 2-factorizations of $\vec{C}_n \wr \vec{C}_3$ are not hamiltonian decompositions.

Before we proceed, we give some further notation and definitions used to describe the constructions given in Subsections 4.3.2 and 4.3.3.

Notation 4.20. By L_j, we denote the subdigraph of $\vec{C}_n \wr \vec{C}_3$ induced by the set of vertices $V_j \cup V_{j+1}$ where $j \in \{0, 1, 2, \ldots, n-2\}$; the subdigraph L_{n-1} is the subdigraph of $\vec{C}_n \wr \vec{C}_3$ induced by $V_{n-1} \cup V_0$.

See Figure 4.10 for an illustration of L_j.

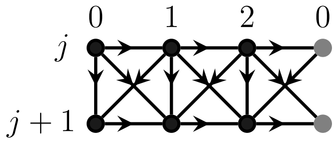


Figure 4.10: The subdigraph L_j of $\vec{C}_n \wr \vec{C}_3$.

Definition 4.21. Let $\mathcal{F} = \{F_0, F_1, F_2, F_3\}$ be a type-I or a type-II 2-factorization of $\vec{C}_n \wr \vec{C}_3$. By $F_i[j]$, where $i \in \{0, 1, 2, 3\}$, we denote the subdigraph of $\vec{C}_n \wr \vec{C}_3$ with vertex set $V(L_j)$, where L_j is given in Definition 4.20, and arc set $A(F_i) \cap A(L_j)$.

If F_i is a type-1 directed 2-factor of $\vec{C}_n \wr \vec{C}_3$, it follows that $F_i[j]$ contains two horizontal arcs and two non-horizontal arcs of $\vec{C}_n \wr \vec{C}_3$. Furthermore, $F_i[j]$ is the disjoint union of two dipaths that both contain precisely one non-horizontal arc.

4.3.2 Type-I 2-factorization of $\vec{C}_n \wr \vec{C}_3$

Our objective is to demonstrate that $\vec{C}_n \wr \vec{C}_3$ does not admit a type-I 2-factorization that is also a hamiltonian decomposition. We do so by first showing that each type-I 2-factorization of $\vec{C}_n \wr \vec{C}_3$ can be described by an n-tuple of certain elements of a particular group. Then, we will show that this 2-factorization is a hamiltonian decomposition only if the product of these n group elements is one of two specific elements. Lastly, we show that, for all even n, no n-tuple satisfies this necessary condition.

First, we introduce the following definition.

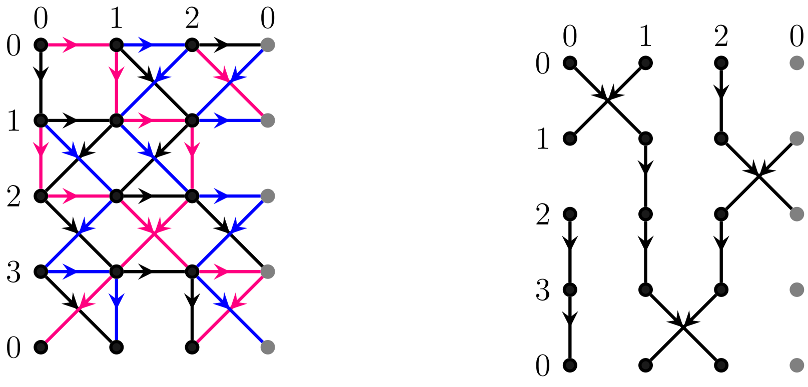
Definition 4.22. Let $j \in \mathbb{Z}_n$ and $i \in \mathbb{Z}_3$. A type-1 directed 2-factor of $\vec{C}_n \wr \vec{C}_3$ that contains the arc (j_i, j_{i+1}) is called a *(j, i)-rooted 2-factor.*

We will make the following assumption for the rest of this subsection.

Assumption 4.23. Let $\mathcal{F}$ be a type-I 2-factorization of $\vec{C}_n \wr \vec{C}_3$ comprised of four 2-factors F_0, F_1, F_2, and F_3. For each $i \in \{0, 1, 2\}$, we may assume that F_i is the type-1 $(0, i)$-rooted 2-factor of $\mathcal{F}$, and that F_3 is the type-0 2-factor of $\mathcal{F}$. We henceforth denote the type-1 2-factorization $\mathcal{F}$ as a 4-tuple (F_0, F_1, F_2, F_3).

Next, we illustrate the concepts introduced thus far by first giving an example of a type-I 2-factorization. We will refer to Example 4.24 throughout this section to illustrate definitions introduced over the course of our investigation.

Example 4.24. In this example, we construct a type-I 2-factorization of $\vec{C}_4 \wr \vec{C}_3$ which we call $\mathcal{F}_e$. See Figure 4.11 for this 2-factorization. For each $(a_1, a_2) \in \{(0,0), (1,1), (2,0), (3,2)\}$, F_0 is the (a_1, a_2)-rooted 2-factor of $\mathcal{F}_e$.



(a) The three type-1 2-factors of $\mathcal{F}_e$: F_0 (pink), F_1 (blue), and F_2 (black).

(b) The type-0 2-factor of $\mathcal{F}_e$.

Figure 4.11: The type-I 2-factorization $\mathcal{F}_e$ of $\vec{C}_4 \wr \vec{C}_3$.

Note that $\mathcal{F}_e$ is not a hamiltonian decomposition of $\vec{C}_4 \wr \vec{C}_3$ because F_0, F_2, and F_3 are not directed hamiltonian cycles. □

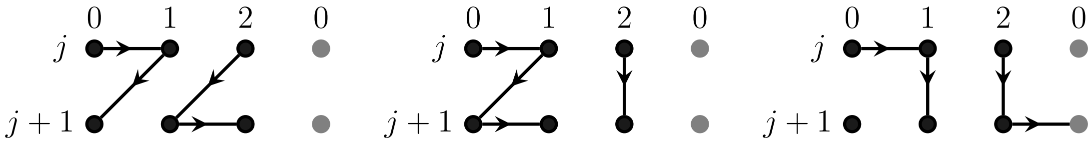
We now proceed with our main objective. First, we list all possible subdigraphs $F_i[j]$, defined in Definition 4.21, where $i \in \{0, 1, 2\}$. Define

$$D_j = \{F_i[j] \mid i \in \mathbb{Z}_3 \text{ and } F_i \text{ is a type-1 2-factor}\}.$$

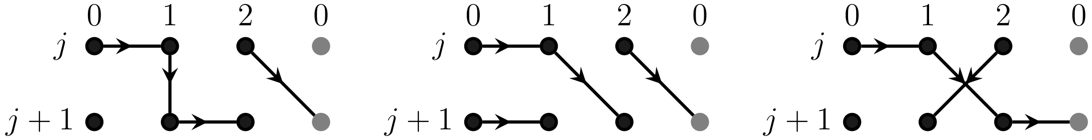
Elements of D_j are subdigraphs comprised of pairs of dipaths whose lengths sum to four. There are 18 distinct subdigraphs in D_j. We first list all elements of D_j that contain the arc (j_0, j_1). These are listed as the union of two dipaths as follows:

$$\begin{array}{llcl}
M_0^0[j] & = j_0\, j_1\, (j+1)_0 & \cup & j_2\, (j+1)_1\, (j+1)_2; \\
M_1^0[j] & = j_0\, j_1\, (j+1)_0\, (j+1)_1 & \cup & j_2\, (j+1)_2; \\
M_2^0[j] & = j_0\, j_1\, (j+1)_1 & \cup & j_2\, (j+1)_2\, (j+1)_0; \\
M_3^0[j] & = j_0\, j_1\, (j+1)_1\, (j+1)_2 & \cup & j_2\, (j+1)_0; \\
M_4^0[j] & = j_0\, j_1\, (j+1)_2 & \cup & j_2\, (j+1)_0\, (j+1)_1; \\
M_5^0[j] & = j_0\, j_1\, (j+1)_2\, (j+1)_0 & \cup & j_2\, (j+1)_1.
\end{array}$$

In Figure 4.12, we illustrate the six elements of D_j given above. These are the only six subdigraphs $F_i[j]$ that contain the arc (j_0, j_1).



(a) The digraph $M_0^0[j]$. (b) The digraph $M_1^0[j]$. (c) The digraph $M_2^0[j]$.



(d) The digraph $M_3^0[j]$. (e) The digraph $M_4^0[j]$. (f) The digraph $M_5^0[j]$.

Figure 4.12: All possible subdigraphs $F_i[j]$ of a type-I 2-factor of $\vec{C}_n \wr \vec{C}_3$ containing the arc (j_0, j_1).

Using these six elements of D_j, we construct the remaining 12 elements of D_j by using the following permutation of the elements of $V(\vec{C}_n \wr \vec{C}_3)$.

Definition 4.25. Define a permutation $\rho : V(\vec{C}_n \wr \vec{C}_3) \to V(\vec{C}_n \wr \vec{C}_3)$ as follows: $\rho(j_k) = j_{k+1}$ with addition of the indices done modulo 3. Given a dipath of $\vec{C}_n \wr \vec{C}_3$, $P = v_1\, v_2\, \ldots\, v_\ell$, we let $\rho(P) = \rho(v_1)\, \rho(v_2)\, \ldots\, \rho(v_\ell)$. In addition, if S is a subdigraph of $\vec{C}_n \wr \vec{C}_3$ such that $S = P^0 \cup P^1 \cup \ldots \cup P^r$, where P^s is a dipath in $\vec{C}_n \wr \vec{C}_3$, then we let $\rho(S) = \rho(P^0) \cup \rho(P^1) \cup \ldots \cup \rho(P^r)$.

Using Definition 4.25, we construct all elements of D_j as follows:

$$D_j = \{M_\ell^k[j] = \rho^k(M_\ell^0[j]) \mid k = 0, 1, 2 \text{ and } \ell = 0, 1, \ldots, 5\}.$$

Observe that, for each $k \in \{0,1,2\}$, the set $\{M_\ell^k[j] \mid \ell = 0,1,2,3,4,5\}$ contains all subdigraphs $F_i[j]$ of D_j that contain the horizontal arc (j_k, j_{k+1}). It is straightforward to verify that D_j contains no other elements.

Given a type-I 2-factorization (F_0, F_1, F_2, F_3), each set $\{F_0[j], F_1[j], F_2[j]\}$ corresponds to a unique triple of the form $\{M_{\ell_0}^0[j], M_{\ell_1}^1[j], M_{\ell_2}^2[j]\}$, where $\ell_0, \ell_1, \ell_2 \in \{0,1,2,3,4,5\}$. Our next objective is to list all possible triples $\{M_{\ell_0}^0[j], M_{\ell_1}^1[j], M_{\ell_2}^2[j]\}$ such that digraphs in each triple are pairwise arc-disjoint. To do so, we create a graph with vertex set D_j. Two vertices are joined by an edge if and only if the two corresponding digraphs are arc-disjoint. To find the desired set of all triples, it suffices to find all cliques of size 3 in this graph. We formally define this graph below.

Definition 4.26. By H_3^j, we denote the graph with vertex set $V(H_3^j) = D_j$ and edge set $E(H_3^j) = \{\{M_1, M_2\} \mid M_1, M_2 \in D_j \text{ are arc-disjoint}\}$.

See Figure 4.13 for an illustration of H_3^j. Note that, for each $k \in \{0,1,2\}$, the set $\{M_\ell^k[j] \mid \ell = 0,1,2,3,4,5\}$ is an independent set of H_3^j. This means that this graph is tripartite and that all cliques of size 3 are maximum cliques. Using a computer, we find the set of all triples $\{M_{\ell_0}^0[j], M_{\ell_1}^1[j], M_{\ell_1}^2[j]\}$ of pairwise arc-disjoint digraphs, which we denote as $\mathcal{K}_j$. The elements of $\mathcal{K}_j$ are listed in the first column of Table 4.1.

For each $T \in \mathcal{K}_j$, the digraph obtained by taking the union of all three digraphs in T, which we denote as D, contains 12 arcs. This means that there are exactly three arcs in $A(L_j)$ that are not contained in $A(D)$. It is routine to verify that these three arcs are non-horizontal and pairwise vertex-disjoint for each triple in $\mathcal{K}_j$, and can thus be described as a permutation in S_3. This set of three arcs corresponds to the digraph $F_3[j]$. In the third column of Table 4.1, we give the permutation that describes $F_3[j]$, which we simply denote by $F_3[j]$.

Each type-I 2-factorization of $\vec{C}_n \wr \vec{C}_3$ can be described by a unique n-tuple $(t_0, t_1, \ldots, t_{n-1})$, where $t_j \in \mathcal{K}_j$. Our objective is to show that no n-tuples $(t_0, t_1, \ldots, t_{n-1})$, where $t_j \in \mathcal{K}_j$, give rise to decomposition of $\vec{C}_n \wr \vec{C}_3$ into directed hamiltonian cycles.

Definition 4.27. Let $\mathcal{F} = (F_0, F_1, F_2, F_3)$ be a type-I 2-factorization of $\vec{C}_n \wr \vec{C}_3$. The n-tuple of elements $(t_0, t_1, \ldots, t_{n-1})$, where $t_j = \{F_0[j], F_1[j], F_2[j]\}$, is the *spine of* $\mathcal{F}$.

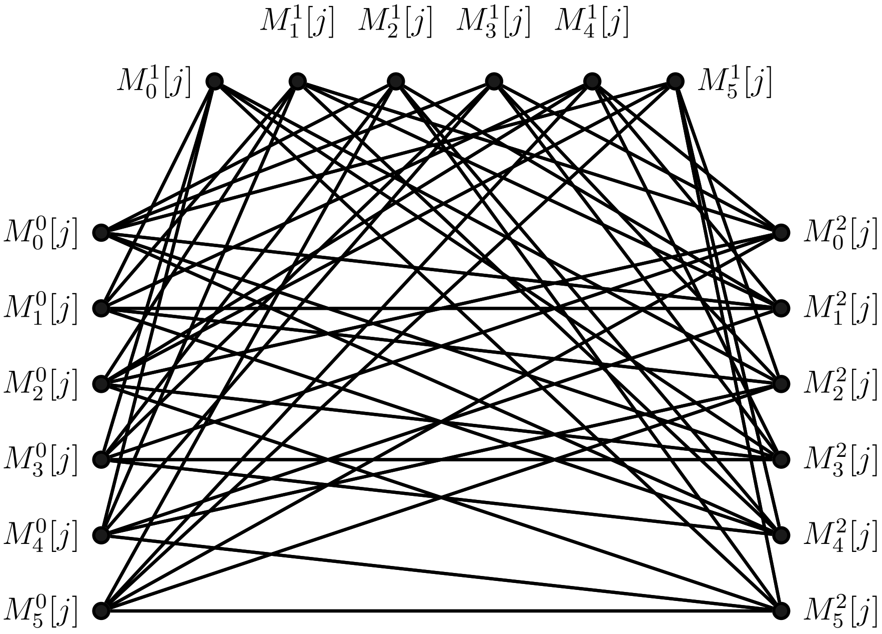


Figure 4.13: The graph H_3^j.

Example 4.28. In the 2-factorization $\mathcal{F}_e$ given in Figure 4.11, we give the corresponding triple in $\mathcal{K}_j$ for each set $\{F_0[j], F_1[j], F_2[j]\}$:

$$\begin{aligned}
\{F_0[0], F_1[0], F_2[0]\} &= \{M_3^0[0], M_0^1[0], M_3^2[0]\};\\
\{F_0[1], F_1[1], F_2[1]\} &= \{M_0^0[1], M_2^1[1], M_4^2[1]\};\\
\{F_0[2], F_1[2], F_2[2]\} &= \{M_5^0[2], M_4^1[2], M_0^2[2]\};\\
\{F_0[3], F_1[3], F_2[3]\} &= \{M_3^0[3], M_3^1[3], M_3^2[3]\}.
\end{aligned}$$

Moreover, $\mathcal{F}_e$ has spine $(T_{21}, T_9, T_{22}, T_2)$. □

To show that not all elements from $\mathcal{K}_0 \times \mathcal{K}_1 \times \ldots \times \mathcal{K}_{n-1}$ will give rise to a 2-factorization of $\vec{C}_n \wr \vec{C}_3$, we introduce the following permutation of the elements of $\mathbb{Z}_3$.

Definition 4.29. Let $\mathcal{F} = (F_0, F_1, F_2, F_3)$ be a type-I 2-factorization of $\vec{C}_n \wr \vec{C}_3$ with spine $(t_0, t_1, \ldots, t_{n-1})$. In addition, for $j \in \mathbb{Z}_n$ and $i \in \{0, 1, 2\}$, let $k_{(j,i)}$ be the element of $\mathbb{Z}_3$ such that F_i is the $(j, k_{(j,i)})$-rooted 2-factor. Observe that $\{k_{(j,0)}, k_{(j,1)}, k_{(j,2)}\} = \mathbb{Z}_3$. For each $j \in \mathbb{Z}_n$, we let

	Element of $\mathcal{K}_j$	σ_{T_s}	$F_3[j]$	Element of $\Gamma_{\mathcal{F}}$
T_1	$(M_0^0[j], M_5^1[j], M_4^2[j])$	$(0,1)(2)$	id	$((1,0,1),((0,1)(2), id))$
T_2	$(M_3^0[j], M_3^1[j], M_3^2[j])$	$(0,1,2)$	$(0,2,1)$	$((0,0,0),((0,1,2),(0,2,1)))$
T_3	$(M_2^0[j], M_4^1[j], M_0^2[j])$	$(0,2)(1)$	$(1,2)(0)$	$((1,1,1),((0,2)(1),(1,2)(0)))$
T_4	$(M_1^0[j], M_1^1[j], M_1^2[j])$	id	$(0,1,2)$	$((0,0,0),(id,(0,1,2)))$
T_5	$(M_1^0[j], M_0^1[j], M_2^2[j])$	$(1,2)(0)$	$(0,1,2)$	$((0,1,1),((1,2)(0),(0,1,2)))$
T_6	$(M_3^0[j], M_3^1[j], M_0^2[j])$	$(0,1,2)$	$(1,2)(0)$	$((0,0,1),((0,1,2),(1,2)(0)))$
T_7	$(M_1^0[j], M_1^1[j], M_4^2[j])$	id	$(0,2)(1)$	$((0,0,1),(id,(0,2)(1)))$
T_8	$(M_5^0[j], M_5^1[j], M_5^2[j])$	$(0,2,1)$	id	$((0,0,0),((0,2,1), id))$
T_9	$(M_0^0[j], M_2^1[j], M_4^2[j])$	$(0,1)(2)$	$(0,2)(1)$	$((1,1,1),((0,1)(2),(0,2)(1)))$
T_{10}	$(M_0^0[j], M_3^1[j], M_3^2[j])$	$(0,1,2)$	$(0,2)(1)$	$((1,0,0),((0,1,2),(0,2)(1)))$
T_{11}	$(M_5^0[j], M_2^1[j], M_5^2[j])$	$(0,2,1)$	$(0,2)(1)$	$((0,1,0),((0,2,1),(0,2)(1)))$
T_{12}	$(M_3^0[j], M_2^1[j], M_4^2[j])$	$(0,1)(2)$	$(0,2,1)$	$((0,1,1),((0,1)(2),(0,2,1)))$
T_{13}	$(M_4^0[j], M_0^1[j], M_2^2[j])$	$(1,2)(0)$	$(0,1)(2)$	$((1,1,1),((1,2)(0),(0,1)(2)))$
T_{14}	$(M_0^0[j], M_2^1[j], M_1^2[j])$	$(0,1)(2)$	$(0,1,2)$	$((1,1,0),((0,1)(2),(0,1,2)))$
T_{15}	$(M_4^0[j], M_3^1[j], M_2^2[j])$	$(1,2)(0)$	$(0,2,1)$	$((1,0,1),((1,2)(0),(0,2,1)))$
T_{16}	$(M_2^0[j], M_5^1[j], M_5^2[j])$	$(0,2,1)$	$(1,2)(0)$	$((1,0,0),((0,2,1),(1,2)(0)))$
T_{17}	$(M_4^0[j], M_1^1[j], M_1^2[j])$	id	$(0,1)(2)$	$((1,0,0),(id,(0,1)(2)))$
T_{18}	$(M_1^0[j], M_4^1[j], M_1^2[j])$	id	$(1,2)(0)$	$((0,1,0),(id,(1,2)(0)))$
T_{19}	$(M_2^0[j], M_1^1[j], M_0^2[j])$	$(0,2)(1)$	$(0,1,2)$	$((1,0,1),((0,2)(1),(0,1,2)))$
T_{20}	$(M_4^0[j], M_0^1[j], M_5^2[j])$	$(1,2)(0)$	id	$((1,1,0),((1,2)(0), id))$
T_{21}	$(M_3^0[j], M_0^1[j], M_3^2[j])$	$(0,1,2)$	$(0,1)(2)$	$((0,1,0),((0,1,2),(0,1)(2)))$
T_{22}	$(M_5^0[j], M_4^1[j], M_0^2[j])$	$(0,2)(1)$	id	$((0,1,1),((0,2)(1), id))$
T_{23}	$(M_5^0[j], M_5^1[j], M_2^2[j])$	$(0,2,1)$	$(0,1)(2)$	$((0,0,1),((0,2,1),(0,1)(2)))$
T_{24}	$(M_2^0[j], M_4^1[j], M_3^2[j])$	$(0,2)(1)$	$(0,2,1)$	$((1,1,0),((0,2)(1),(0,2,1)))$

Table 4.1: All triples in $\mathcal{K}_j$.

$$\sigma_{t_j} : \mathbb{Z}_3 \to \mathbb{Z}_3, \quad (k_{(j,i)})^{\sigma_{t_j}} = k_{(j+1,i)}$$

for all $i \in \{0,1,2\}$. We point out that $\{k_{(j+1,0)}, k_{(j+1,1)}, k_{(j+1,2)}\} = \mathbb{Z}_3$ for all $j \in \mathbb{Z}_n$ because F_0, F_1, and F_2 are arc-disjoint. Therefore, the function σ_{t_j} is a permutation of $\mathbb{Z}_3$ and is called the *permutation of the horizontal arcs by* t_j.

In the second column of Table 4.1, we give the corresponding permutation of the horizontal arcs by T_s for each $s \in \{1,2,\ldots,24\}$.

Example 4.30. The 2-factorization $\mathcal{F}_e$ of Figure 4.11 has spine $(T_{21}, T_9, T_{22}, T_2)$ with the corresponding 4-tuple of permutations of horizontal arcs: $((012),(01),(02),(012))$.

□

Using Definition 4.29, we will now derive a necessary condition on n-tuples $(t_0, t_1, \ldots, t_{n-1})$, where $t_j \in \mathcal{K}_j$, that correspond to a 2-factorization of $\vec{C}_n \wr \vec{C}_3$. This necessary condition is given in Corollary 4.32. First, we prove the following lemma.

Lemma 4.31. *Let $\mathcal{F} = (F_0, F_1, F_2, F_3)$ be a type-I 2-factorization of $\vec{C}_n \wr \vec{C}_3$ with spine $(t_0, t_1, \ldots, t_{n-1})$, $j \in \mathbb{Z}_n$. For each $j \in \mathbb{Z}_n$, let*

$$\mu_j = \sigma_{t_0}\sigma_{t_1}\ldots\sigma_{t_j}.$$

Then, for $i \in \{0,1,2\}$, the type-1 2-factor F_i is the $(j+1, i^{\mu_j})$-rooted 2-factor of $\mathcal{F}$.

Proof: We prove the statement for F_0. Analogous reasoning applies to F_1 and F_2. For each $j \in \mathbb{Z}_n$, F_0 is the $(j, k_{(j,0)})$-rooted 2-factor of $\mathcal{F}$. Note that $k_{(0,0)} = 0$ since, by Assumption 4.45, F_0 is the $(0,0)$-rooted 2-factor of $\mathcal{F}$. We prove the statement by way of induction on j.

Base case: $j = 1$. Then $\mu_0 = \sigma_{t_0}$. By Definition 4.29, we have that $0^{\sigma_{t_0}} = k_{(1,0)}$. The statement follows because F_0 is the $(1, k_{(1,0)})$-rooted 2-factor.

Induction step: Assume that F_0 is the $(j, 0^{\mu_{j-1}})$-rooted 2-factor of $\mathcal{F}$. By Definition 4.29, this means that $0^{\mu_{j-1}} = k_{(j,0)}$. Therefore,

$$0^{\mu_j} = (0^{\mu_{j-1}})^{\sigma_{t_j}} = k_{(j,0)}^{\sigma_{t_j}} = k_{(j+1,0)}.$$

Consequently, the directed 2-factor F_0 is a $(j+1, 0^{\mu_j})$-rooted 2-factor, as claimed. ∎

Corollary 4.32. *Let $\mathcal{F} = (F_0, F_1, F_2, F_3)$ be a type-I 2-factorization of $\vec{C}_n \wr \vec{C}_3$ with spine $(t_0, t_1, \ldots, t_{n-1})$, and let $\mu = \sigma_{t_0}\sigma_{t_1}\ldots\sigma_{t_{n-1}}$. Then $\mu = id$.*

Proof: It follows from Lemma 4.31 that, for $i \in \{0,1,2\}$, F_i is the $(0,i)$-rooted 2-factor and the $(0, i^{\mu})$-rooted 2-factor of $\mathcal{F}$. If $\mu \neq id$, then we have a contradiction. ∎

Corollary 4.32 is the first step towards our main objective, which is to show that no n-tuples $(t_0, t_1, \ldots, t_{n-1})$ will give rise to a hamiltonian type-I 2-factorization of $\vec{C}_n \wr \vec{C}_3$ when n is even. To that end, we now introduce another function. Note that this function applies to any type-1 2-factor.

Definition 4.33. Let $\mathcal{F} = \{F_0, F_1, F_2, F_3\}$ be a directed 2-factorization of $\vec{C}_n \wr \vec{C}_3$. If $F_i \in \mathcal{F}$ is a type-1 2-factor and both dipaths of $F_i[j]$ are of length two, then F_i is said to be *switched at* j. We then define the following function:

$$s_j[F_i] = \begin{cases} 1, & \text{if } F_i \text{ is switched at } j; \\ 0, & \text{otherwise.} \end{cases}$$

The type-1 2-factor F_i is switched at j precisely when the dipath of $F_i[j]$ that contains the horizontal arc of C_3^j does not contain the horizontal arc of C_3^{j+1}. For a type-I 2-factorization $\mathcal{F} = (F_0, F_1, F_2, F_3)$ with $i \in \{0, 1, 2\}$, the type-1 2-factor F_i is switched at j if and only if

$$F_i[j] \in \{\rho^k(M_0^0[j]), \rho^k(M_2^0[j]), \rho^k(M_4^0[j]) \mid k = 0, 1, 2\}.$$

Refer to Figures 4.12a, 4.12c, and 4.12e for an illustration of $M_0^0[j]$, $M_2^0[j]$, and $M_4^0[j]$, respectively.

Example 4.34. In the 2-factorization $\mathcal{F}_e$, given in Figure 4.11, the 2-factor F_0 is switched at 1 and 3. We also see that F_1 is switched at 0,1, and 2. Lastly, observe that F_2 is switched at 1 and 2. □

We now use Definition 4.33 to show that, if a type-I 2-factor of $\vec{C}_n \wr \vec{C}_3$ is a directed hamiltonian cycle, then we have the following necessary condition.

Lemma 4.35. *Let $\mathcal{F} = \{F_0, F_1, F_2, F_3\}$ be a directed 2-factorization of $\vec{C}_n \wr \vec{C}_3$. A type-1 2-factor $F_i \in \mathcal{F}$ is a directed hamiltonian cycle of $\vec{C}_n \wr \vec{C}_3$ only if $\sum_{j=0}^{n-1} s_j[F_i]$ is odd.*

Proof: Assume that F_i is a type-1 2-factor that contains the arc $(0_i, 0_{i+1})$. If F_i is a directed hamiltonian cycle, then it is not difficult to see that F_i can be expressed as the concatenation of two dipaths P_i and Q_i such that

(C1) $F_i = P_i Q_i$;

(C2) $s(P_i) = 0_i$, $t(P_i) = 0_{i+2}$, and P_i contains the horizontal arc $(0_i, 0_{i+1})$;

(C3) $s(Q_i) = 0_{i+2}$, $t(Q_i) = 0_i$, and $0_{i+1} \notin V(Q_i)$.

Because F_i is a type-1 2-factor, dipaths P_i and Q_i must jointly use exactly one horizontal arc from each C_3^j for each $j \in \mathbb{Z}_n$. By (C2), P_i is the dipath that contains the horizontal arc of C_3^0. For $j \in \{0, 1, \ldots, n-2\}$, if F_i is switched at j and P_i contains a horizontal from C_3^j, then Q_i must be the dipath that contains a horizontal arc from C_3^{j+1} and vice versa.

Suppose that $\sum_{j=0}^{n-1} s_j[F_i]$ is even. If $s_{n-1}[F_i] = 1$, then $\sum_{j=0}^{n-2} s_j[F_i]$ is odd and Q_i must be the dipath that contains the horizontal arc of F_i from C_3^{n-1}. However, since F_i is switched at $n-1$, and F_i contains the arc $(0_i, 0_{i+1})$, then $t(P_i) = 0_i$, a contradiction. Similarly, if $s_{n-1}[F_i] = 0$, then $\sum_{j=0}^{n-2} s_j[F_i]$ is even and P_i must be the dipath of $\mathcal{F}$ that contains the horizontal arc from C_3^{n-1} and thus $t(P_i) = 0_i$ since F_i is not switched at $n-1$. In conclusion, it follows that $\sum_{j=0}^{n-1} s_j[F_i]$ is odd. ∎

We will now describe each element of $\mathcal{K}_j$ using an element of a group constructed in Definition 4.37 below. To construct this group, we will be using the semi-direct product of two particular groups. Therefore, we first define the semi-direct product of two groups in Definition 4.36 below. Readers that are well-versed in group theory will recognize that we have in fact defined the external (or outer) semi-direct product.

Definition 4.36. Let Γ_1 and Γ_2 be two groups and let $\phi : \Gamma_2 \to \mathrm{Aut}(\Gamma_1)$ where ϕ is a group homomorphism. We let $\phi(k) = \phi_k$ for each $k \in \Gamma_2$. The *semi-direct product* of Γ_1 with Γ_2, denoted $\Gamma_1 \rtimes_\phi \Gamma_2$, is the group with elements $\{(h, k) \mid h \in \Gamma_1, k \in \Gamma_2\}$ and binary operation

$$(h, k)(h', k') = (h\phi_k(h'), kk').$$

Now, we introduce the group that will be used to describe the elements of $\mathcal{K}_j$.

Definition 4.37. Let $\Gamma_1 = \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 = \mathbb{Z}_2^3$, where $\mathbb{Z}_2$ is the additive group on two elements and $\oplus$ denotes the direct sum of two groups, and let $\Gamma_2 = S_3 \oplus S_3$. Furthermore, for $(a_0, a_1, a_2) \in \mathbb{Z}_2^3$ and $(\sigma, \gamma) \in S_3 \oplus S_3$, we define $\phi : \Gamma_2 \to \mathrm{Aut}(\Gamma_1)$ by

$$\phi_{(\sigma,\ \gamma)} : \mathbb{Z}_2^3 \to \mathbb{Z}_2^3,$$
$$\phi_{(\sigma,\ \gamma)}(a_0, a_1, a_2) = (a_{0^\sigma}, a_{1^\sigma}, a_{2^\sigma}).$$

Then, we define $\Gamma_{\mathcal{F}} = \Gamma_1 \rtimes_\phi \Gamma_2$.

In Definition 4.37, it is straightforward to verify that $\phi_{(\sigma,\gamma)} \in \mathrm{Aut}(\Gamma_1)$ and that ϕ is a group homomorphism.

Remark 4.38. If $g_a = ((a_0, a_1, a_2), (\sigma_1, \gamma_1))$ and $g_b = ((b_0, b_1, b_2), (\sigma_2, \gamma_2))$ such that $g_a, g_b \in \Gamma_{\mathcal{F}}$, then

$$g_a g_b = ((a_0 + b_{0^{\sigma_1}}, a_1 + b_{1^{\sigma_1}}, a_2 + b_{2^{\sigma_1}}), (\sigma_1\sigma_2, \gamma_1\gamma_2)),$$

with addition done in $\mathbb{Z}_2$.

Example 4.39. Let $g_1 = ((0, 1, 0), ((0, 1, 2), (0, 2)))$ and $g_2 = ((1, 0, 0), ((0, 1), (0, 1, 2)))$ be two elements of $\Gamma_{\mathcal{F}}$. We now compute $g_1 g_2$. First, we note that $\phi_{((0,1,2),(0,2)}(1, 0, 0) = (0, 0, 1)$. Then, we see that

$$g_1 \cdot g_2 = (((0, 1, 0) + (0, 0, 1)), ((1, 2)(0), (1, 2)(0))) = ((0, 1, 1), ((1, 2)(0), (1, 2)(0))).$$

□

Definition 4.40. Let (F_0, F_1, F_2, F_3) be a type-I 2-factorization $\mathcal{F}$ of $\vec{C}_n \wr \vec{C}_3$ with spine $(t_0, t_1, \ldots, t_{n-1})$. For each $j \in \mathbb{Z}_m$, let $f_j : \mathcal{K}_j \to \Gamma_{\mathcal{F}}$ such that

$$f_j(t_j) = ((a_0, a_1, a_2), (\sigma_{t_j}, F_3[j]))$$

where $a_i = 1$ if the (j, i)-rooted 2-factor is switched at j; otherwise $a_i = 0$.

Definition 4.41. By $\mathcal{S}$, we denote the set of elements of $\Gamma_{\mathcal{F}}$ of the form $f_j(T)$ for some $T \in \mathcal{K}_j$.

In the fourth column of Table 4.1, we give the element of $\mathcal{S}$ corresponding to each triple in $\mathcal{K}_j$.

Example 4.42. For the 2-factorization $\mathcal{F}_e$, we see that it can be described by the following elements of $\mathcal{S}$:

$$\begin{aligned} f_0(T_{21}) &= ((0, 1, 0), ((0, 2, 1), (0, 1))); \\ f_1(T_9) &= ((1, 1, 1), ((0, 1), (0, 2))); \\ f_2(T_{22}) &= ((0, 1, 1), ((0, 2), id))); \\ f_3(T_2) &= ((0, 0, 0), ((0, 1, 2), (0, 2, 1))). \end{aligned}$$

We now give necessary conditions for a type-I 2-factorization of $\vec{C}_n \wr \vec{C}_3$ with spine $(t_0, t_1, \ldots, t_{n-1})$, where $t_j \in \mathcal{K}_j$, to be a hamiltonian decomposition.

Lemma 4.43. *Let* $\mathcal{F} = (F_0, F_1, F_2, F_3)$ *be a type-I 2-factorization of* $\vec{C}_n \wr \vec{C}_3$ *with spine* $(t_0, t_1, \ldots, t_{n-1})$ *such that*

$$(f_0(t_0), f_1(t_1), \ldots, f_{n-1}(t_{n-1})) = ((h_0, k_0), (h_1, k_1), (h_2, k_2), \ldots, (h_{n-1}, k_{n-1})).$$

Let $(g_1, g_2) = (h_0, k_0)(h_1, k_1) \ldots (h_{n-1}, k_{n-1})$*. Then* $\mathcal{F}$ *is a hamiltonian decomposition of* $\vec{C}_n \wr \vec{C}_3$ *only if*

$$(g_1, g_2) \in \{((1,1,1), (id, (012))), ((1,1,1), (id, (021)))\}.$$

Proof: We assume that $\mathcal{F} = (F_0, F_1, F_2, F_3)$ is a hamiltonian type-I 2-factorization of $\vec{C}_n \wr \vec{C}_3$. Let $F_3[j] = \gamma_j$, $k_j = (\sigma_{t_j}, \gamma_j)$, and let $g_2 = (\mu, \gamma) \in S_3 \oplus S_3$. We see that

$$\mu = \sigma_{t_0}\sigma_{t_1} \ldots \sigma_{t_{n-1}}, \text{ and } \gamma = \gamma_0\gamma_1 \ldots \gamma_{n-1}.$$

Corollary 4.32 implies that $\mathcal{F}$ is a 2-factorization of $\vec{C}_n \wr \vec{C}_3$ only if $\mu = id$. In addition, since $\gamma_j = F_3[j]$, it follows that F_3 is a directed hamiltonian cycle only if γ is a permutation comprised of a single cycle. There are exactly two permutations in S_3 comprised of a single cycle. Therefore, $\gamma \in \{(012), (021)\}$. In conclusion, $g_2 = (\mu, \gamma) \in \{(id, (012)), (id, (021))\}$.

Next, we show that $g_1 = (1, 1, 1)$. Let $h_j = (x_0^j, x_1^j, x_2^j) \in \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$, and let

$$\mu_{j-1} = \sigma_{t_0}\sigma_{t_1} \ldots \sigma_{t_{j-1}} \text{ and } \omega_{j-1} = \gamma_0\gamma_1 \ldots \gamma_{j-1}$$

where $j \in \{1, 2, \ldots, n-1\}$. We see that

$$\begin{aligned} g_1 = \; & h_0 + \sum_{j=1}^{n-1} \phi_{(\mu_{j-1}, \omega_{j-1})}(h_j) \\ = \; & h_0 + \phi_{(\mu_0, \omega_0)}(h_1) + \phi_{(\mu_1, \omega_1)}(h_2) + \phi_{(\mu_2, \omega_2)}(h_3) + \ldots + \phi_{(\mu_{n-2}, \omega_{n-2})}(h_{n-1}). \end{aligned}$$

We claim that $\phi_{(\mu_{j-1}, \omega_{j-1})}(h_j) = (s_j[F_0], s_j[F_1], s_j[F_2])$. First, we point out that $\phi_{(\mu_{j-1}, \omega_{j-1})}(h_j) = (x^j_{0^{\mu_{j-1}}}, x^j_{1^{\mu_{j-1}}}, x^j_{2^{\mu_{j-1}}})$. Recall that, by Lemma 4.31, F_i is the $(j, i^{\mu_{j-1}})$-rooted 2-factor of $\mathcal{D}$. If F_i is switched at j, then $x^j_{i^{\mu_j}} = 1$ by Definition 4.40 and $s_j[F_i] = 1$ by Definition 4.33. Consequently, $x^j_{i^{\mu_{j-1}}} = 1 = s_j[F_i]$. Otherwise,

$x_{i^{\mu_{j-1}}} = 0 = s_j[F_i]$. It follows that $(x^j_{0^{\mu_{j-1}}}, x^j_{1^{\mu_{j-1}}}, x^j_{2^{\mu_{j-1}}}) = (s_j[F_0], s_j[F_1], s_j[F_2])$, as desired.

We then see that

$$\begin{aligned} g_1 = & \ (x^0_0 + \sum_{j=1}^{n-1} x^j_{0^{\mu_{j-1}}}, x^0_1 + \sum_{j=1}^{n-1} x^j_{1^{\mu_{j-1}}}, x^0_2 + \sum_{j=1}^{n-1} x^j_{2^{\mu_{j-1}}}) \\ = & \ (x^0_0 + \sum_{j=1}^{n-1} s_j[F_0], x^0_1 + \sum_{j=1}^{n-1} s_j[F_1], x^0_2 + \sum_{j=1}^{n-1} s_j[F_2]) \\ = & \ (\sum_{j=0}^{n-1} s_j[F_0], \sum_{j=0}^{n-1} s_j[F_1], \sum_{j=0}^{n-1} s_j[F_2]). \end{aligned}$$

Since F_0, F_1, and F_2 are directed hamiltonian cycles, Lemma 4.35 implies that $\sum_{j=0}^{n-1} s_j[F_i]$ is odd for each $i \in \{0,1,2\}$. Because $g_1 \in \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$, then $g_1 = (1,1,1)$.

In summary, $\mathcal{F}$ is a hamiltonian 2-factorization only if

$$(g_1, g_2) \in \{((1,1,1),(id,(012))),((1,1,1),(id,(021)))\}.$$

∎

We will now show that no type-I 2-factorization of $\vec{C}_n \wr \vec{C}_3$ is hamiltonian. To do so, we will use Lemma 4.43 by showing that the product of no n elements of $\mathcal{S}$, where n is even and $\mathcal{S}$ is defined in Definition 4.41, yields one of the two elements in $\{((1,1,1),(id,(012))),((1,1,1),(id,(021)))\}$.

Proposition 4.44. *Let n be an even integer. The digraph $\vec{C}_n \wr \vec{C}_3$ does not admit a type-I hamiltonian 2-factorization.*

Proof: Let $\mathcal{F} = (F_0, F_1, F_2, F_3)$ be a type-I 2-factorization of $\vec{C}_n \wr \vec{C}_3$ with spine $(t_0, t_1, \ldots, t_{n-1})$ where

$$(f(t_0), f(t_1), \ldots, f(t_{n-1})) = ((h_0,k_0),(h_1,k_1),(h_2,k_2),\ldots,(h_{n-1},k_{n-1})).$$

Note that $(h_j, k_j) \in \mathcal{S}$ for each $j \in \mathbb{Z}_n$. We prove the statement by showing that, for all even n, the product of any n elements of $\mathcal{S}$ is not an element in

$$A = \{((1,1,1),(id,(012))),((1,1,1),(id,(021)))\}.$$

First, we start with the case $n = 2$. We compute the set of elements

$$U_2 = \{(h_0, k_0)(h_1, k_1) \mid (h_0, k_0), (h_1, k_1) \in \mathcal{S}\}.$$

Refer to Appendix A for the computation of all 576 products. We then see that the set U_2 contains 126 distinct elements. All of these elements can also be found in Section A.1 of Appendix A. Observe that no element of U_2 is in A. By Lemma 4.43, the digraph $\vec{C}_2 \wr \vec{C}_3$ does not admit a type-I 2-factorization.

Next, we consider the case $n = 4$. It suffices to construct the set:

$$U_4 = \{(h'_0, k'_0)(h'_1, k'_1) \mid (h'_0, k'_0), (h'_1, k'_1) \in U_2\}.$$

We see that U_4 is the set of the products of all pairs of elements of U_2. See Section A.3 of Appendix A for the set U_4, which contains 144 elements. Observe that U_4 and A are disjoint. By Lemma 4.43, the digraph $\vec{C}_4 \wr \vec{C}_3$ does not admit a type-I 2-factorization.

For $n \geqslant 6$ and n even, we let

$$U_n = \{(h'_0, k'_0)(h'_1, k'_1) \mid (h'_0, k'_0) \in U_2 \text{ and } (h'_1, k'_1) \in U_{n-2}\}.$$

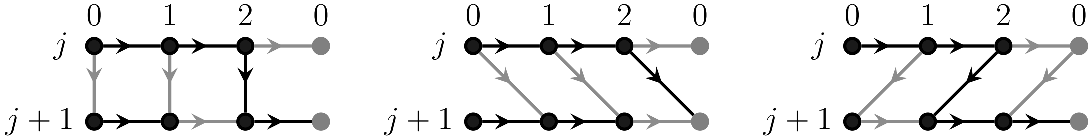
Our last step is to show that $U_n = U_4$ for all even $n \geqslant 4$. To do so, we have computed all 18144 possible products of the form $(h'_0, k'_0)(h'_1, k'_1)$, where $(h'_0, k'_0) \in U_2$ and $(h'_1, k'_1) \in U_4$, by computer. We have found that $U_4 = U_6$. It follows that $U_4 = U_6 = U_n$ for all $n \geqslant 8$ and n even. Since U_6 does not contain any element of A, Lemma 4.43 implies that, for all even $n \geqslant 6$, $\vec{C}_n \wr \vec{C}_3$ does not admit a type-I 2-factorization that is also a hamiltonian decomposition. ∎

4.3.3 Type-II 2-factorization of $\vec{C}_n \wr \vec{C}_3$

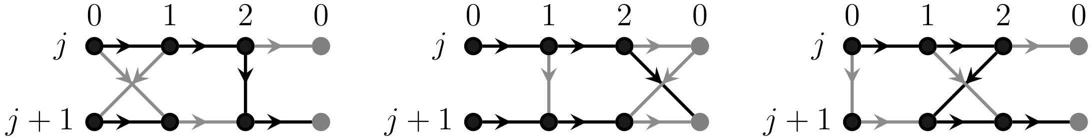
Our objective is to demonstrate that no type-II 2-factorization of $\vec{C}_n \wr \vec{C}_3$ is a hamiltonian decomposition. Our approach is considerably simpler than that of Subsection 4.3.2. We will begin by introducing the following assumption which is analogous to Assumption 4.45.

Assumption 4.45. Let $\mathcal{F} = \{F_0, F_1, F_2, F_3\}$ be a type-II 2-factorization of $\vec{C}_n \wr \vec{C}_3$ comprised of four 2-factors. We shall assume that F_0 is the type-2 2-factor of $\mathcal{F}$, F_1 the type-1 2-factor, and F_2 and F_3 the type-0 2-factors of $\mathcal{F}$. We then write (F_0, F_1, F_2, F_3).

In Figure 4.15, we list all six possible pairs $(F_0[j], F_1[j])$ such that $F_1[j]$ contains the arc (j_2, j_0). These six pairs form the set $\{M_\ell[j] \mid \ell = 0, 1, \ldots, 5\}$. Then, for each $\ell \in \{1, 2, \ldots, 6\}$, we let $M_{\ell+6k}[j] = \rho^k(M_\ell)$, where ρ is defined in Definition 4.25 and $k \in \{1, 2\}$. This gives rise to a set of 18 pairs; it is routine to verify that no other pair exists.



(a) The pair $M_1[j]$. (b) The pair $M_2[j]$. (c) The pair $M_3[j]$.



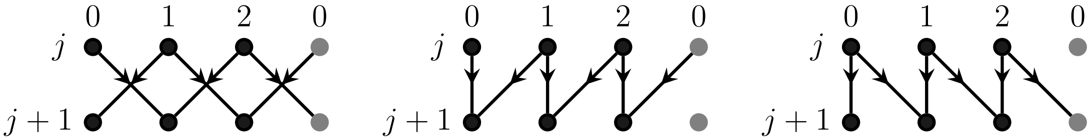
(d) The pair $M_4[j]$. (e) The pair $M_5[j]$. (f) The pair $M_6[j]$.

Figure 4.14: All possible pairs of digraphs $(F_0[j], F_1[j])$ such that $F_1[j]$ contains the arc (j_2, j_0).

We are particularly interested in F_1. Observe that, if $(F_0[j], F_1[j]) \in \{M_\ell[j] \mid \ell \equiv 1, 2, 3 \pmod 6\}$, then the type-1 2-factor F_1 is switched at j, as defined in Definition 4.33. Otherwise, if $(F_0[j], F_1[j]) \in \{M_\ell[j] \mid \ell \equiv 0, 4, 5 \pmod 6\}$, then F_1 is not switched at j. Whether or not F_1 is switched as j dictates the set of permutations that can be used to describe the corresponding type-0 2-factors F_2 and F_3, as stated in Lemma 4.46 below.

Lemma 4.46. *Let $\mathcal{F} = (F_0, F_1, F_2, F_3)$ be a type-II directed 2-factorization of $\vec{C}_n \wr \vec{C}_3$. If F_1 is switched at j, then $F_2[j], F_3[j] \in \{(012), (021), id\}$. Otherwise, $F_2[j], F_3[j] \in \{(01), (02), (12)\}$.*

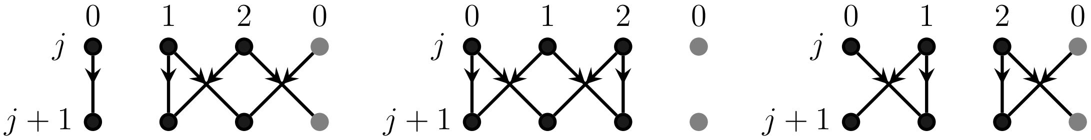
Proof: For all possible pairs $F_0[j], F_1[j])$, there exist precisely six distinct sets of leftover arcs $L_t = A(L_j) - (A(F_0[j]) \cup A(F_1[j])$ where $t \in \{1, 2, \ldots, 6\}$. This can be verified by using Figure 4.14.



(a) The set L_1. (b) The set L_2. (c) The set L_3.

Figure 4.15: The three sets of leftover arcs when $F_1[j]$ is switched.

Let L_1, L_2, and L_3 be the three sets of leftover arcs that correspond to a pair $(F_0[j], F_1[j])$ in which $F_1[j]$ is switched at j. See Figure 4.15 for an illustration of these three sets. Each of L_1, L_2, and L_3 admits exactly one partition into two sets of three vertex-disjoint arcs. Such set of three disjoint arcs corresponds to a permutation in $\{(012), (021), id\}$. Consequently, $F_2[j], F_3[j] \in \{(012), (021), id\}$



(a) The set L_4. (b) The set L_5. (c) The set L_6.

Figure 4.16: The three sets of leftover arcs when $F_1[j]$ is not switched.

Similarly, let L_4, L_5, and L_6 be the three sets of arcs that correspond to a pair $M_\ell = (F_0[j], F_1[j])$ in which $F_1[j]$ is not switched at j. These three sets are illustrated in Figure 4.16. Each of L_4, L_5, and L_6 admits precisely one partition into two sets of three vertex-disjoint arcs. In this case, each such set of three arcs corresponds to a permutation in $\{(01), (02), (12)\}$. Hence, we see that $F_2[j], F_3[j] \in \{(01), (02), (12)\}$. ▮

We will now use Lemma 4.46 to prove this subsection's main result stated below.

Proposition 4.47. *Let n be an even integer. The digraph $\vec{C}_n \wr \vec{C}_3$ does not admit a type-II 2-factorization that is also a hamiltonian decomposition.*

Proof: Let $\mathcal{F} = (F_0, F_1, F_2, F_3)$ be a type-II 2-factorization. In this proof, we will describe F_2 and F_3 as the following product of n permutations of S_3 where $F_2[j] = \sigma_j$ and $F_3[j] = \mu_j$:

$$\sigma = \sigma_0\sigma_1 \dots \sigma_{n-1}, \text{ and } \mu = \mu_0\mu_1 \dots \mu_{n-1}.$$

If F_2 and F_3 are directed hamiltonian cycles, then $\sigma, \mu \in \{(012), (021)\}$. Observe that (012) and (021) are even permutations.

By Lemma 4.46, if F_1 is switched at j, then $\sigma_j, \mu_j \in \{(012), (021), id\}$. This means that σ_j and μ_j are even permutations. Recall that, by Lemma 4.35, if F_1 is a type-1 directed hamiltonian cycle, then $\sum_{j=0}^{n-1} s_j(F_1)$ must be odd. This means that the set $\{\sigma_j \mid j \in \mathbb{Z}_n\}$ contains an odd number of even permutations and thus, an odd number of odd permutations. Therefore, the permutation σ is odd; likewise for μ. This is a contradiction, meaning that $\mathcal{F}$ cannot be a type-II hamiltonian 2-factorization. ∎

4.3.4 Summary

Propositions 4.10 and 4.19 imply that Conjecture 2.23 is not true for $G = \vec{C}_n$, where n is even, and $H \in \{\vec{C}_2, \vec{C}_3\}$. Moreover, although Propositions 4.10 and 4.19 do not imply that Conjecture 2.23 is false for $G \neq \vec{C}_n$ and $H \in \{\vec{C}_2, \vec{C}_3\}$, they do imply that it will be particularly difficult to determine if $G \wr \vec{C}_2$ and $G \wr \vec{C}_3$ are hamiltonian decomposable. This is because our general approach, which is similar to that of Ng [52] and Baranyai and Szász [12], heavily relies on the fact that, in general, the digraph $\vec{C}_n \wr H$ is hamiltonian decomposable.

We now summarize our results on the decomposition of $\vec{C}_n \wr \vec{C}_m$ into directed hamiltonian cycles given in this section and the previous section.

Theorem 4.48. *Let n be an even integer. The digraph $\vec{C}_n \wr \vec{C}_m$ is hamiltonian decomposable in each of the following cases:*

(S1) *m is odd, and $m \geqslant 5$;*

(S2) *m is even, $m \geqslant 4$, and $n \geqslant 4$;*

(S3) *$m = 4$ and $n = 2$.*

Furthermore, if $m \in \{2, 3\}$, then $\vec{C}_n \wr \vec{C}_m$ is not hamiltonian decomposable.

Proof:

(S1) The result follows from Proposition 4.15.

(S2) The result follows from Proposition 4.14.

(S3) The result follows from Lemma 4.11.

Lastly, Propositions 4.10 and 4.19, respectively, imply that $\vec{C}_n \wr \vec{C}_2$ and $\vec{C}_n \wr \vec{C}_3$ are not hamiltonian decomposable for all even n. ∎

Lastly, we point out that Lemma 4.4, in conjunction with Theorem 4.48, implies Corollary 4.49 below.

Corollary 4.49. *Let G be a strict hamiltonian decomposable digraph of even order n such that $G \neq \overline{K}_n$. The digraph $G \wr \vec{C}_m$ is hamiltonian decomposable in each of the following cases:*

(A1) *m is odd, and $m \geqslant 5$;*

(A2) *m is even, $m \geqslant 4$, and $n \geqslant 4$;*

(A3) *$m = 4$ and $n = 2$.*

Furthermore, if $G = \vec{C}_n$ and $m \in \{2, 3\}$, then $G \wr \vec{C}_m$ is not hamiltonian decomposable.

4.4 Decompositions of $G \wr K_m^*$ into directed hamiltonian cycles

We now consider the case in which H can be decomposed into $m-1$ directed hamiltonian cycles. In this case, the digraph H must necessarily be the complete symmetric digraph since H is strict. In order to construct a decomposition of $\vec{C}_n \wr K_m^*$ into directed hamiltonian cycles, we will use a decomposition of K_m^* into hamiltonian dipaths. The question of existence of these decompositions is resolved in [67] when m is even and in [68] when m is odd.

Theorem 4.50. *[67] Let m be an even integer. The graph K_m admits a decomposition into hamiltonian paths.*

We then have the following corollary.

Corollary 4.51. *Let m be an even integer. The digraph K_m^* admits a decomposition into hamiltonian dipaths.*

It is known that K_m does not admit a decomposition into hamiltonian paths when m is odd. Consequently, a different approach is needed in this case.

Theorem 4.52. *[68] Let $m \geqslant 7$ be an odd integer. The digraph K_m^* admits a decomposition into hamiltonian dipaths.*

To prove Theorem 4.52, Tillson considers two cases. In the case $m \equiv 3 \pmod 4$, Tillson gives an explicit construction. As for the case $m \equiv 1 \pmod 4$, Tillson first shows that K_{m+1}^* admits a decomposition D into directed hamiltonian cycles. Then, by deleting a vertex of K_{m+1}^*, and thus deleting a vertex from each directed hamiltonian cycle in D, he obtains a decomposition of K_m^* into hamiltonian dipaths.

The following lemma tells us that all decompositions of K_m^* into hamiltonian dipaths have a particular property. This property is key to our construction of a decomposition of $\vec{C}_n \wr K_m^*$ into directed hamiltonian cycles.

Lemma 4.53. *If D is a decomposition of K_m^* into hamiltonian dipaths, then no two distinct dipaths in D have the same source or the same terminus.*

Proof: Suppose that dipaths $P_i, P_j \in D$, where $i \neq j$, both have the same source. Since there are m dipaths in D, it follows that there exists a vertex $u \in V(K_m^*)$ such that no dipath in D has u as a source. Since vertex u is not a source, then u has an in-neighbour in each of the m dipaths in D. Because D is a decomposition and K_m^* is strict, this means that u has m distinct in-neighbours, a contradiction. An analogous argument holds when $P_i, P_j \in D$ share a terminus v. ∎

In Proposition 4.54 and Lemma 4.55 below, we construct a directed hamiltonian decomposition of $\vec{C}_n \wr K_m^*$. Recall that K_4^* and K_6^* are not hamiltonian decomposable by Theorem 3.28. However, the proof of Proposition 4.54 also applies to the case $m =$

6. In addition, Lemma 4.55 implies that $\vec{C}_n \wr K_4^*$ is also hamiltonian decomposable. This means that digraphs G and H need not both be hamiltonian decomposable for $G \wr H$ to be hamiltonian decomposable.

Proposition 4.54. *Let n be an even integer and $m \geqslant 6$. The digraph $\vec{C}_n \wr K_m^*$ is hamiltonian decomposable.*

Proof: First, we consider the case $n = 2$. In that case, we see that $\vec{C}_2 \wr K_m^* \cong K_{2m}^*$. It follows from Theorem 3.28 that K_{2m}^* is hamiltonian decomposable since $m \geqslant 6$.

Next, we consider the case $n \geqslant 4$. By Corollary 4.51 and Theorem 4.52 there exists a decomposition $D = \{P_1, P_2, \ldots, P_m\}$ of K_m^* into hamiltonian dipaths. For each $P_j \in D$, assume that $P_j = v_1^j\, v_2^j \ldots v_m^j$. Using D, we can obtain a second decomposition of K_m^* into directed hamiltonian dipaths, $D' = \{P_1', P_2', \ldots, P_m'\}$, where $P_j' = v_m^j\, v_{m-1}^j \ldots v_1^j$.

For each $P_j \in D$, we then construct the following directed cycle of $\vec{C}_n \wr K_m^*$:

$$C^j = 0_{v_1^j}\, 0_{v_2^j} \ldots 0_{v_m^j}\, 1_{v_m^j}\, 1_{v_{m-1}^j}\, 1_{v_{m-2}^j} \ldots 1_{v_1^j}\, 2_{v_1^j}\, 2_{v_2^j}\, 2_{v_3^j} \ldots (n-1)_{v_2^j}\, (n-1)_{v_1^j}\, 0_{v_1^j}.$$

The directed cycle C^j is in fact a directed hamiltonian cycle of $\vec{C}_n \wr K_m^*$. The directed cycle C^j contains the following arcs of difference 0:

$$\{(0_{v_m^j}, 1_{v_m^j}), (1_{v_1^j}, 2_{v_1^j}), \ldots, ((n-1)_{v_1^j}, 0_{v_1^j})\}.$$

By Lemma 4.53, v_1^j is the source of exactly one dipath in D, and v_m^j is the terminal of exactly one dipath in D; likewise for D' in which v_m^j and v_1^j will appear as the source and terminal, respectively, exactly one dipath in D'. It follows that each arc of difference 0 appears exactly once in $F = \{C^1, C^2, \ldots, C^m\}$. Furthermore, since D and D' are decomposition of K_m^* into hamiltonian dipaths, each horizontal arc of $\vec{C}_n \wr K_m^*$ also appears exactly once in F. Otherwise, no arc of difference $d > 0$ appears in F. As a result, we have that $\vec{C}_n \wr K_m^* = C^1 \oplus C^2 \ldots \oplus C^m \oplus (\vec{C}_n \times K_m^*)$. By Theorem 4.12, the digraph $\vec{C}_n \times K_m^*$ is hamiltonian decomposable. Therefore, Lemma 2.8 implies that $\vec{C}_n \wr K_m^*$ is hamiltonian decomposable. ∎

Note that, because K_m^* does not admit a decomposition into hamiltonian dipaths when $m \in \{3, 5\}$, Proposition 4.54 does not consider the case n is even and $m \in \{3, 5\}$.

In addition, when $m = 4$, Theorem 4.12 does not apply to the digraph $\vec{C}_n \times K_m^*$. Therefore, Proposition 4.54 also does not consider the case n is even and $m = 4$. In Lemma 4.55 below, we address these three omissions.

Lemma 4.55. *Let n be an even integer and $m \in \{3, 4, 5\}$. The digraph $\vec{C}_n \wr K_m^*$ is hamiltonian decomposable if and only if $(n, m) \neq (2, 3)$.*

Proof: We consider three cases, one for each $m \in \{3, 4, 5\}$.
Case 1: $m = 3$. Since $\vec{C}_2 \wr K_3^* = K_6^*$, it follows from Theorem 3.28 that $\vec{C}_2 \wr K_3^*$ is not hamiltonian decomposable.

Conversely, let $n \geqslant 4$. We begin by constructing 9 dipaths of $\vec{C}_n \wr K_3^*$ as follows:

$X_0 = 0_2\, 0_0\, 0_1\, 1_0\, 1_1\, 1_2\, 2_2\, 2_1\, 2_0\, 3_0\, 3_2\, 3_1\, 4_2;$

$X_1 = 0_1\, 1_2\, 1_1\, 2_1\, 2_2\, 3_1\, 4_0;$

$X_2 = 0_0\, 0_2\, 1_0\, 2_0\, 3_2\, 3_0\, 4_1;$

$X_3 = 0_1\, 0_2\, 1_2\, 2_1\, 3_0\, 4_0;$

$X_4 = 0_0\, 1_1\, 1_0\, 2_2\, 2_0\, 3_1\, 3_2\, 4_1;$

$X_5 = 0_1\, 0_0\, 1_0\, 1_2\, 2_0\, 2_1\, 3_2\, 4_2;$

$X_6 = 0_2\, 1_1\, 2_2\, 3_0\, 3_1\, 4_1;$

$X_7 = 0_2\, 0_1\, 1_1\, 2_0\, 2_2\, 3_2\, 4_0;$

$X_8 = 0_0\, 1_2\, 1_0\, 2_1\, 3_1\, 3_0\, 4_2.$

Refer to Figure 4.17 for an illustration of the 9 dipaths above.

(a) Illustration of X_0 (pink), X_1 and X_2 (black), and X_3 and X_4 (blue).

(b) Illustration of X_5 and X_6 (pink), and X_7 and X_8 (black).

Figure 4.17: Key dipaths in the construction of a directed hamiltonian decomposition of $\vec{C}_n \wr K_3^*$.

Subcase 1.1: $n = 4$. Then $4_2 = 0_2$, meaning that $C^0 = X_0$ is a directed cycle of length 12. Furthermore, we see that $t(X_1) = 0_0$, $s(X_2) = 0_0$, $t(X_2) = 0_1$ and $s(X_1) = 0_1$, and that X_1 and X_2 have no other vertices in common. Therefore, we can obtain the following cycle of length 12: $C^1 = X_1X_2$. A similar reasoning can be applied to obtain the following three directed cycles of length 12: $C^2 = X_3X_4$, $C^3 = X_5X_6$, and $C^4 = X_7X_8$. Lastly, each arc of $\vec{C}_4 \wr K_3^*$ appears exactly once in $D = \{C^0, C^1, C^2, C^3, C^4\}$. Therefore, the set D is a directed hamiltonian decomposition of $\vec{C}_4 \wr K_3^*$.

Subcase 1.2: $n \geqslant 6$. Let $\mathbb{I} = \{4, 6, \dots, n-2\}$. For each $i \in \mathbb{I}$, we construct the following set of 9 dipaths:

$M^i = i_2\, i_0\, i_1\, (i+1)_2\, (i+1)_1\, (i+1)_0\, (i+2)_2$;

$N_0^i = i_0\, i_2\, (i+1)_1\, (i+2)_0$;

$N_1^i = i_1\, (i+1)_0\, (i+1)_2\, (i+2)_1$;

$O_0^i = i_0\, (i+1)_1\, (i+1)_2\, (i+2)_0$;

$O_1^i = i_1\, i_2\, (i+1)_0\, (i+2)_1$;

$P_0^i = i_2\, (i+1)_2\, (i+2)_2$;

$P_1^i = i_1\, i_0\, (i+1)_0\, (i+1)_1\, (i+2)_1$;

$Q_0^i = i_0\, (i+1)_2\, (i+1)_0\, (i+2)_0$;

$Q_1^i = i_2\, i_1\, (i+1)_1\, (i+2)_2$.

Refer to Figure 4.18 for an illustration of these 9 dipaths when $i = 4$. It is straightforward to verify that, for each $i \in \mathbb{I}$, dipaths in $\{M^i, N_0^i, N_1^i, \dots, Q_1^i\}$ are pairwise arc-disjoint.

(a) Illustration of M^4 (pink), N_0^4 and N_1^4 (black), and O_0^4 and O_1^4 (blue).

(b) Illustration of P_0^4 and P_1^4 (pink), and Q_0^4 and Q_1^4 (black).

Figure 4.18: Key dipaths in the construction of a directed hamiltonian decomposition of $\vec{C}_n \wr K_3^*$ for $n \geqslant 6$.

We now use these dipaths to construct five directed hamiltonian cycles of $\vec{C}_n \wr K_3^*$. First, observe that each M^i is a dipath of length 6. In addition, note that $t(M^i) = s(M^{i+2})$ and that M^i and M^{i+2} have no other vertices in common. If $|i-j| > 2$, then dipaths M^i and M^j are vertex-disjoint. Lastly, we see that $s(M^4) = t(X_0) = 4_2$, $t(M^{n-2}) = s(X_0) = 0_2$, and that X_0 has no other vertices in common with dipaths in

$\{M^4, M^6, \ldots, M^{n-2}\}$. This means that we can construct the following directed cycle of length $3n$: $C^0 = X_0 M^4 M^6 M^8 \ldots M^{n-2}$.

Next, we observe that, for each $i \in \mathbb{I}$, $\operatorname{len}(N_0^i) + \operatorname{len}(N_1^i) = 6$, $t(N_0^i) = s(N_0^{i+2})$, $t(N_1^i) = s(N_1^{i+2})$, $t(N_0^{n-2}) = s(X_2)$, and $t(N_1^{n-2}) = s(X_1)$. We also note that N_0^i and N_0^{i+2} have no other vertices in common and that N_0^i and N_1^j are vertex-disjoint for all $i, j \in \mathbb{I}$. If $|j - i| > 2$, then N_0^i and N_0^j are also vertex-disjoint; likewise for N_1^i and N_1^j. Dipaths X_1 and X_2 are vertex-disjoint and have no other vertices in common with dipaths in $\{N_0^i, N_1^i \mid i \in \mathbb{I}\}$. This means that we can form the following directed cycle of length $3n$:

$$C^1 = X_1 N_0^4 N_0^6 \ldots N_0^{n-2} X_2 N_1^4 N_1^6 \ldots N_1^{n-2}.$$

By applying a similar logic, we can construct the following three directed cycles of length $3n$:

$$C^2 = X_3 O_0^4 O_0^6 \ldots O_0^{n-2} X_4 O_1^4 O_1^6 \ldots O_1^{n-2};$$
$$C^3 = X_5 P_0^4 P_0^6 \ldots P_0^{n-2} X_6 P_1^4 P_1^6 \ldots P_1^{n-2};$$
$$C^4 = X_7 Q_0^4 Q_0^6 \ldots Q_0^{n-2} X_8 Q_1^4 Q_1^6 \ldots Q_1^{n-2}.$$

Observe that the dipaths in $\cup_{i=4}^{n-2}\{M^i, N_0^i, N_1^i, \ldots, Q_1^i\}$ are pairwise arc-disjoint. Since the dipaths in $\{X_0, X_1, \ldots, X_8\}$ are also pairwise arc-disjoint, this implies that the directed cycles in $D = \{C^0, C^1, C^2, C^3, C^4\}$ are pairwise arc-disjoint. Consequently, the set D is a directed hamiltonian decomposition of $\vec{C}_n \wr K_3^*$.

Case 2: $m = 4$. Observe that, if $n = 2$, then $\vec{C}_n \wr K_4^* = K_8^*$. The statement follows from Theorem 3.28.

Therefore, we consider the case $n \geqslant 4$. We begin by constructing three directed cycles of length $4n$ as follows:

$$C^0 = 0_3\, 0_2\, 0_1\, 0_0\, 1_0\, 1_1\, 1_2\, 1_3\, 2_3\, 2_2\, 2_1 \ldots (n-1)_0\, (n-1)_1\, (n-1)_2\, (n-1)_3\, 0_3;$$
$$C^1 = 0_1\, 0_3\, 0_0\, 0_2\, 1_2\, 1_0\, 1_3\, 1_1\, 2_1\, 2_3\, 2_0 \ldots (n-1)_2\, (n-1)_0\, (n-1)_3\, (n-1)_1\, 0_1;$$
$$C^2 = 0_2\, 0_0\, 0_3\, 0_1\, 1_1\, 1_3\, 1_0\, 1_2\, 2_2\, 2_0\, 2_3 \ldots (n-1)_1\, (n-1)_3\, (n-1)_0\, (n-1)_2\, 0_2.$$

Observe that C^0, C^1, and C^2 are pairwise arc-disjoint. Let $S = A(C^0) \cup A(C^1) \cup A(C^2)$. We see that all arcs of difference 0 are contained in S except for arcs in the set

$$\{(i_3, (i+1)_3), ((i+1)_0, (i+2)_0) \mid i \in \mathbb{Z}_n \text{ and } i \text{ even}\}.$$

Furthermore, the set S also contains all horizontal arcs except for arcs in the set

$$\{(i_0,i_1),(i_1,i_2),(i_2,i_3) \mid i \in \mathbb{Z}_n \text{ and } i \text{ is even}\} \cup$$

$$\{(i_3,i_2),(i_2,i_1),(i_1,i_0) \mid i \in \mathbb{Z}_n \text{ and } i \text{ is odd}\}.$$

Lastly, note that S does not contain any arc of difference $d > 0$.

We now construct four directed hamiltonian cycles. To do so, we first construct the following 13 dipaths:

$$\begin{array}{llll} U_0 = 0_1\,0_2\,0_3\,1_2\,1_1\,1_0\,2_0\,; & U_1 = 0_0\,1_3\,2_1; & & \\ W_0 = 0_0\,0_1\,1_3\,1_2\,2_3; & W_1 = 0_3\,1_0\,2_2; & W_2 = 0_2\,1_1\,2_0; & \\ X_0 = 0_0\,1_1\,2_3; & X_1 = 0_3\,1_3\,2_2; & X_2 = 0_2\,1_0\,2_1; & X_3 = 0_1\,1_2\,2_0; \\ Y_0 = 0_0\,1_2\,2_1; & Y_1 = 0_1\,1_0\,2_3\,; & Y_2 = 0_3\,1_1\,2_2; & Y_3 = 0_2\,1_3\,2_0. \end{array}$$

Refer to Figure 4.19 for an illustration of these 13 dipaths.

(a) Illustration of U_0 and U_1 (black), and W_j for each $j \in \{0,1,2\}$ (grey).

(b) Illustration of X_j for each $j \in \{0,1,2,3\}$ (black), and Y_j for each $j \in \{0,1,2,3\}$ (grey).

Figure 4.19: Key dipaths in the construction of a directed hamiltonian decomposition of $\vec{C}_n \wr K_4^*$.

Let $\mathbb{I} = \{2,4,6,\ldots,n-2\}$. For each $i \in \mathbb{I}$, we construct the following set of 13 dipaths:

$$M_0^i = i_0\,(i+1)_3\,(i+2)_0;$$

$$M_1^i = i_1\,i_2\,i_3\,(i+1)_2\,(i+1)_1\,(i+1)_0\,(i+2)_1;$$

$$N_0^i = i_3\,(i+1)_3\,(i+1)_2\,(i+2)_3; \qquad N_1^i = i_2\,(i+1)_1\,(i+2)_2;$$

$$N_2^i = i_0\,i_1\,(i+1)_0\,(i+2)_0;$$

$$O_0^i = i_3\,(i+1)_0\,(i+2)_3; \qquad O_1^i = i_2\,(i+1)_3\,(i+2)_2;$$
$$O_2^i = i_1\,(i+1)_2\,(i+2)_1; \qquad O_3^i = i_0\,(i+1)_1\,(i+2)_0;$$

$$P_0^i = i_1\,(i+1)_3\,(i+2)_1; \qquad P_1^i = i_3\,(i+1)_1\,(i+2)_3;$$
$$P_2^i = i_2\,(i+1)_0\,(i+2)_2; \qquad P_3^i = i_0\,(i+1)_2\,(i+2)_0.$$

Refer to Figure 4.20 for an illustration of these 13 dipaths when $i = 4$.

(a) Illustration of M_0^2 and M_1^2 (black), and N_j^2 for each $j \in \{0,1,2\}$ (grey).

(b) Illustration of O_j^2 for each $j \in \{0,1,2,3\}$ (black), and P_j^2 for each $j \in \{0,1,2,3\}$ (grey).

Figure 4.20: Key dipaths in the construction of a directed hamiltonian decomposition of $\vec{C}_n \wr K_4^*$ for $n \geqslant 4$.

Observe that $\text{len}(M_0^i) + \text{len}(M_1^i) = 8$ for all $i \in \mathbb{Z}_n$. Moreover, we see that $t(M_0^i) = s(M^{i+2})$ and $t(M_1^i) = s(M_1^{i+2})$ for all $i \in \mathbb{I} - \{n-2\}$. Furthermore, we see that $s(M_0^2) = t(U_0)$, $t(M_0^{n-2}) = s(U_1)$, $s(M_1^2) = t(U_1)$, $t(M_1^{n-2}) = s(U_0)$ If $|i-j|> 2$, then M_0^i and M_0^j are also disjoint; likewise for M_1^i and M_1^j. Lastly, for all $i, j \in \mathbb{I}$, M_0^i and M_1^j are disjoint. Therefore, we can construct the following directed cycle of length $4n$:

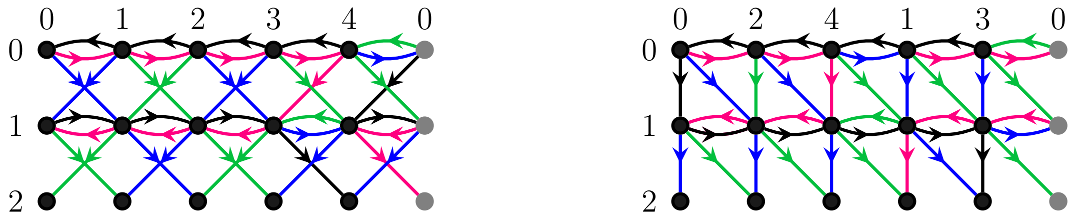
$$C^4 = U_0 M_0^2 M_0^4 \ldots M_0^{n-2} U_1 M_1^2 M_1^4 \ldots M_1^{n-2}.$$

By a similar logic, we construct the following three directed cycles of length $4n$:

$$C^5 = W_0 N_0^2 N_0^4 \ldots N_0^{n-2} W_1 N_1^2 N_1^4 \ldots N_1^{n-2} W_2 N_2^2 N_2^4 \ldots N_2^{n-2};$$
$$C^6 = X_0 O_0^2 O_0^4 \ldots O_0^{n-2} X_1 O_1^2 O_1^4 \ldots O_1^{n-2} X_2 O_2^2 O_2^4 \ldots O_2^{n-2} X_3 O_3^2 O_3^4 \ldots O_3^{n-2};$$
$$C^7 = Y_0 P_0^2 P_0^4 \ldots P_0^{n-2} Y_1 P_1^2 P_1^4 \ldots P_1^{n-2} Y_2 P_2^2 P_2^4 \ldots P_2^{n-2} Y_3 P_3^2 P_3^4 \ldots P_3^{n-2}.$$

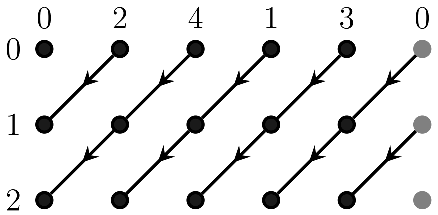
We point out that $S' = A(C^3) \cup A(C^4) \cup A(C^5) \cup A(C^6)$ contains all horizontal arcs and all arcs of difference 0 that do not appear in S. Furthermore, each arc of difference $d > 0$ appears exactly once in $\{C^3, C^4, C^5, C^6\}$. This means that $D = \{C^0, C^1, \ldots, C^6\}$ is a directed hamiltonian decomposition of $\vec{C}_n \wr K_4^*$.

Case 3: $m = 5$. Observe that, if $n = 2$, then $\vec{C}_n \wr K_5^* = K_{10}^*$. The statement follows from Theorem 3.28.



(a) Illustration of L_0 (pink), L_1 (black), U_j for each $j \in \{0, 1, 2, 3\}$ (blue), and W_j for each $j \in \{0, 1, 2, 3\}$ (green).

(b) Illustration of L_2 (pink), L_3 (black), X_j for each $j \in \{0, 1, 2, 3\}$ (blue), and Y_j for each $j \in \{0, 1, 2, 3\}$ (green).



(c) Illustration of Z_j for each $j \in \{0, 1, 2, 3, 4\}$.

Figure 4.21: Key dipaths in the construction of a directed hamiltonian decomposition of $\vec{C}_n \wr K_5^*$.

We now construct a decomposition of $\vec{C}_n \wr K_5^*$ into 9 directed hamiltonian cycles for $n \geqslant 4$. First, we construct the following set of 25 dipaths:

$L_0 = 0_0\, 0_1\, 0_2\, 0_3\, 0_4\, 1_3\, 1_2\, 1_1\, 1_0\, 1_4\, 2_0;$

$L_1 = 0_4\, 0_3\, 0_2\, 0_1\, 0_0\, 1_4\, 1_0\, 1_1\, 1_2\, 1_3\, 2_4;$

$L_2 = 0_1\, 0_3\, 0_0\, 0_2\, 0_4\, 1_4\, 1_2\, 1_0\, 1_3\, 1_1\, 2_1\,;$

$L_3 = 0_3\, 0_1\, 0_4\, 0_2\, 0_0\, 1_0\, 1_2\, 1_4\, 1_1\, 1_3\, 2_3;$

$U_0 = 0_4\, 0_0\, 1_1\, 2_2;$ $U_1 = 0_2\, 1_3\, 1_4\, 2_3;$

$U_2 = 0_3\, 1_2\, 2_1;$ $U_3 = 0_1\, 1_0\, 2_4;$

$X_0 = 0_4\, 0_1\, 1_1\, 2_3;$ $X_1 = 0_3\, 1_3\, 1_0\, 2_0;$

$X_2 = 0_0\, 1_2\, 2_2\,;$ $X_3 = 0_2\, 1_4\, 2_4;$

$$W_0 = 0_0\,0_4\,1_0\,2_1; \quad W_1 = 0_1\,1_2\,2_3; \qquad Y_0 = 0_0\,0_3\,1_0\,2_2; \quad Y_1 = 0_2\,1_2\,2_4;$$

$$W_2 = 0_3\,1_4\,1_3\,2_2; \quad W_3 = 0_2\,1_1\,2_0; \qquad Y_2 = 0_4\,1_1\,1_4\,2_1; \quad Y_3 = 0_1\,1_3\,2_0;$$

$$Z_0 = 0_0\,1_3\,2_1\,;\; Z_1 = 0_1\,1_4\,2_2;$$

$$Z_2 = 0_2\,1_0\,2_3;\; Z_3 = 0_3\,1_1\,2_4;$$

$$Z_4 = 0_4\,1_2\,2_0.$$

See Figure 4.21 for an illustration of these 25 dipaths.

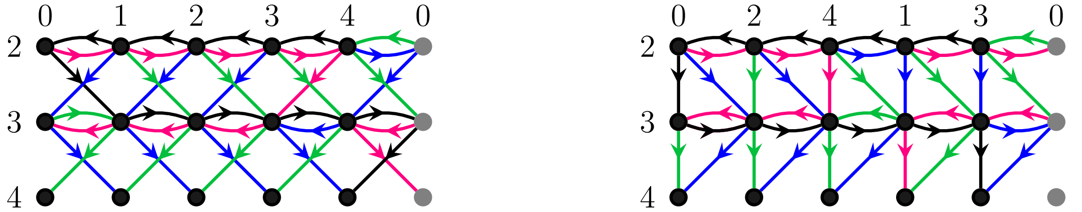
Next, let $\mathbb{I} = \{2, 4, \ldots, n-2\}$. For each $i \in \mathbb{I}$, we form the following set of four dipaths:

$$M_0^i = i_0\,i_1\,i_2\,i_3\,i_4\,(i+1)_3\,(i+1)_2\,(i+1)_1\,(i+1)_0\,(i+1)_4\,(i+2)_0;$$

$$M_1^i = i_4\,i_3\,i_2\,i_1\,i_0\,(i+1)_1\,(i+1)_2\,(i+1)_3\,(i+1)_4\,(i+1)_0\,(i+2)_4;$$

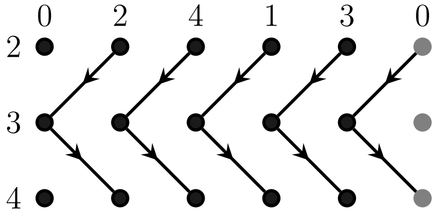
$$M_2^i = i_1\,i_3\,i_0\,i_2\,i_4\,(i+1)_4\,(i+1)_2\,(i+1)_0\,(i+1)_3\,(i+1)_1\,(i+2)_1\,;$$

$$M_3^i = i_3\,i_1\,i_4\,i_2\,i_0\,(i+1)_0\,(i+1)_2\,(i+1)_4\,(i+1)_1\,(i+1)_3\,(i+2)_3.$$



(a) Illustration of M_0^2 (pink), M_1^2 (black), N_j^2 for each $j \in \{0,1,2,3\}$ (blue), and O_j^2 for each $j \in \{0,1,2,3\}$ (green).

(b) Illustration of M_2^2 (pink), M_3^2 (black), P_j^2 for each $j \in \{0,1,2,3\}$ (blue), and Q_j^2 for each $j \in \{0,1,2,3\}$ (green).



(c) Illustration of R_j^2 for each $j \in \{0,1,2,3,4\}$.

Figure 4.22: Key dipaths in the construction of a directed hamiltonian decomposition of $\vec{C}_n \wr K_5^*$ for $n \geqslant 4$.

See Figure 4.22 for an illustration of these four dipaths for $i = 2$. First, we note that, for $i \in \{2, 4, \ldots, n-4\}$, the dipath M_0^i is of length 10 and that M_0^i and M_0^{i+2} share precisely one vertex; namely $(i+2)_0 = t(M_0^i) = s(M_0^{i+2})$. Otherwise, if $|i-j| > 2$, then M_0^i and M_0^j have no vertices in common. Furthermore, we see that $t(L_0) = s(M_0^2) = 2_0$ and that $s(L_0) = t(M_0^{n-2}) = 0_0$. Lastly, we see that L_0 has no other vertices in common with M_0^2 and M_0^{n-2} and that L_0 is vertex-disjoint with M_0^i for $i \in \{4, 6, \ldots, n-4\}$. Therefore, we are able to form the following directed cycle of length $5n$: $C^0 = L_0 M_0^2 M_0^4 M_0^6 \ldots M_0^{n-2}$.

By a similar logic, we form the following three cycles of length $5n$:

$$C^1 = L_1 M_1^2 M_1^4 M_1^6 \ldots M_1^{n-2};$$

$$C^2 = L_2 M_2^2 M_2^4 M_2^6 \ldots M_2^{n-2};$$

$$C^3 = L_3 M_1^3 M_3^4 M_3^6 \ldots M_3^{n-2}.$$

Next, for each $i \in \mathbb{I}$, we construct a set of 21 dipaths as follows:

$N_0^i = i_2\,(i+1)_1\,(i+2)_2;$

$N_1^i = i_3\,(i+1)_2\,(i+2)_3;$

$N_2^i = i_1\,(i+1)_0\,(i+2)_1;$

$N_3^i = i_4\,i_0\,(i+1)_4\,(i+1)_3\,(i+2)_4;$

$O_0^i = i_1\,(i+1)_2\,(i+2)_1;$

$O_1^i = i_3\,(i+1)_4\,(i+2)_3;$

$O_2^i = i_2\,(i+1)_3\,(i+2)_2;$

$O_3^i = i_0\,i_4\,(i+1)_0\,(i+1)_1\,(i+2)_0;$

$P_0^i = i_3\,(i+1)_3\,(i+1)_0\,(i+2)_3;$

$P_1^i = i_0\,(i+1)_2\,(i+2)_0;$

$P_2^i = i_2\,(i+1)_4\,(i+2)_2;$

$P_3^i = i_4\,i_1\,(i+1)_1\,(i+2)_4,$

$Q_0^i = i_2\,(i+1)_2\,(i+2)_2;$

$Q_1^i = i_4\,(i+1)_1\,(i+1)_4\,(i+2)_4;$

$Q_2^i = i_1\,(i+1)_3\,(i+2)_1;$

$Q_3^i = i_0\,i_3\,(i+1)_0\,(i+2)_0;$

$R_0^i = i_1\,(i+1)_4\,(i+2)_1;$

$R_1^i = i_2\,(i+1)_0\,(i+2)_2;$

$R_2^i = i_3\,(i+1)_1\,(i+2)_3;$

$R_3^i = i_4\,(i+1)_2\,(i+2)_4;$

$R_4^i = i_0\,(i+1)_3\,(i+2)_0.$

See Figure 4.22 for an illustration of these dipaths for $i = 2$. We first point out that, for each $i \in \mathbb{I}$, we have $\mathrm{len}(N_0^i) + \mathrm{len}(N_1^i) + \mathrm{len}(N_2^i) + \mathrm{len}(N_3^i) = 10$. Furthermore, for each $i \in \mathbb{I} - \{n-2\}$, we see that $t(N_0^i) = s(N_0^{i+2})$, $t(N_1^i) = s(N_1^{i+2})$, $t(N_2^i) = s(N_2^{i+2})$, and $t(N_3^i) = s(N_3^{i+2})$. If $|i-j| > 2$ then N_r^i and N_r^j have no common vertices for each $r \in \{0, 1, 2, 3\}$. Moreover, if $r_1 \neq r_2$, then $N_{r_1}^i$ and $N_{r_2}^j$ have no common vertices for all $i, j \in \mathbb{I}$. Observe that: $t(U_0) = s(N_0^2)$, $t(N_0^{n-2}) = s(U_1)$, $t(U_1) = s(N_1^2)$,

$t(N_1^{n-2}) = s(U_2)$, $t(U_2) = s(N_2^2)$, $t(N_2^{n-2}) = s(U_3)$, $t(U_3) = s(N_3^2)$, and $t(N_3^{n-2}) = s(U_0)$. Otherwise, dipaths in $\{U_0, U_1, U_2, U_3\}$ have no other common vertices with dipaths in $\cup_{i\in\mathbb{I}}\{N_0^i, N_1^i, N_2^i, N_3^i\}$. As a result, we can form the following directed cycle of length $5n$:

$$C^4 = U_0N_0^2N_0^4\dots N_0^{n-2}U_1N_1^2N_1^4\dots N_1^{n-2}U_2N_2^2N_2^4\dots N_2^{n-2}U_3N_3^2N_3^4\dots N_3^{n-2}.$$

By applying similar reasoning, we obtain the following four directed cycles of length $5n$:

$$\begin{aligned}
C^5 &= W_0O_0^2O_0^4\dots O_0^{n-2}W_1O_1^2O_1^4\dots O_1^{n-2}W_2O_2^2O_2^4\dots O_2^{n-2}W_3O_3^2O_3^4\dots O_3^{n-2};\\
C^6 &= X_0P_0^2P_0^4\dots P_0^{n-2}X_1P_1^2P_1^4\dots P_1^{n-2}X_2P_2^2P_2^4\dots P_2^{n-2}X_3P_3^2P_3^4\dots P_3^{n-2};\\
C^7 &= Y_0Q_0^2Q_0^4\dots Q_0^{n-2}Y_1Q_1^2Q_1^4\dots Q_1^{n-2}Y_2Q_2^2Q_2^4\dots Q_2^{n-2}Y_3Q_3^2Q_3^4\dots Q_3^{n-2};\\
C^8 &= Z_0R_0^2R_0^4\dots R_0^{n-2}Z_1R_1^2R_1^4\dots R_1^{n-2}Z_2R_2^2R_2^4\dots R_2^{n-2}Z_3R_3^2R_3^4\dots R_3^{n-2}\\
&\qquad Z_4R_4^2R_4^4\dots R_4^{n-2}.
\end{aligned}$$

Let

$$S = \cup_{i\in\mathbb{I}}\{M_0^i, M_1^i, M_2^i, M_3^i, N_0^i, \dots, R_2^i\} \cup \{L_0, L_1, L_2, L_3, U_1, \dots, Z_4\}.$$

It is routine to verify that the dipaths in S are pairwise arc-disjoint. Therefore, the directed cycles in $D = \{C^0, C^1, \dots, C^8\}$ are also pairwise-arc disjoint and thus, the set D is a decomposition of $\vec{C}_n \wr K_5^*$ into directed hamiltonian cycles. ∎

We conclude this section with a corollary that generalizes this section's results.

Corollary 4.56. *Let G be a strict hamiltonian decomposable digraph of even order n such that $G \neq \overline{K}_n$ and let $m \geqslant 3$. The digraph $G \wr K_m^*$ is hamiltonian decomposable if and only if $(n, m) \neq (2, 3)$.*

Proof: Lemma 4.4, in conjunction with Proposition 4.54 and Lemma 4.55, implies sufficiency. Conversely, we point out that there exists exactly one strict hamiltonian decomposable digraph on two vertices, namely $G = \vec{C}_2 = K_2^*$. This means that, if $(n, m) = (2, 3)$, then $G \wr K_m^* = K_6^*$. Theorem 3.28 then implies that $G \wr K_m^*$ is not hamiltonian decomposable. ∎

4.5 Reduction

We now proceed with our general approach, which is similar to that of [52]. The aim of this section is to show that, if n is even and $|V(H)|= m$, the digraph $\vec{C}_n \wr H$ is hamiltonian decomposable when $\vec{C}_2 \wr \overline{K}_m$ admits a particular directed 2-factorization. We first introduce several key definitions.

Definition 4.57. Let $S = \{\sigma_0, \sigma_1, \dots, \sigma_{m-1}\}$ be a set of m permutations of S_m acting on $\mathbb{Z}_m$. The set S is a *regular permutation set of order* m if $j^{\sigma_{k_1}} \neq j^{\sigma_{k_2}}$ for all $j \in \mathbb{Z}_m$ and $k_1, k_2 \in \mathbb{Z}_m$ such that $k_1 \neq k_2$.

See Example 4.58 below for a very simple example of a regular permutation set.

Example 4.58. Below, we give an example of a regular permutation set of order 13:

$$\begin{array}{rll}
S = \{ & (0,1,12,2,3,4,5,6,7,8,9,10,11), & (0,2,4,6,12,8,10)(1,3,5,7,9,11),\\
& (0,12,3,6,9)(1,4,7,10)(2,5,8,11), & (0,4,8)(1,5,12,9)(2,6,10)(3,7,11),\\
& (0,5,10,3,8,1,6,11,12,4,9,2,7), & (0,6)(1,7)(2,8)(3,9)(4,12,10)(5,11),\\
& (0,7,2,9,4,11,6,1,8,3,10,12,5), & (0,8,4)(1,9,5)(2,10,6)(3,12,11,7),\\
& (0,9,12,6,3)(1,10,7,4)(2,11,8,5), & (0,10,8,6,4,2,12)(1,11,9,7,5,3),\\
& (0,11,10,9,8,12,7,6,5,4,3,2,1), & (0,3,11,4,10,5,9,6,8,7,12,1,2),\\
& id\}. &
\end{array}$$

It is laborious yet straightforward to verify that $j^{\sigma_{k_1}} \neq j^{\sigma_{k_2}}$ for all $j \in \mathbb{Z}_{13}$ and all $\sigma_{k_1}, \sigma_{k_2} \in S$ such that $k_1 \neq k_2$. $\square$

Definition 4.59. A permutation $\sigma \in S_m$ is an $(m-1)$-*stabilizer* if $(m-1)^\sigma = m-1$.

Note that a regular permutation set contains a unique $(m-1)$-stabilizer. We now introduce a method for constructing a new regular permutation set from an existing one.

Lemma 4.60. *Let* $S = \{\sigma_0, \sigma_1, \dots, \sigma_{m-1}\}$ *be a regular permutation set of order* m *and* $\mu \in S_m$. *Then the set* $\mu \cdot S = \{\mu\sigma_0, \mu\sigma_1, \dots, \mu\sigma_{m-1}\}$ *is also a regular permutation set of order* m.

Proof: We will prove the statement by way of contradiction. First, we construct the set $\mu \cdot S = \{\mu\sigma_i \mid i = 0, 1, \dots, m-1\}$. Let $\gamma_i, \gamma_k \in \mu \cdot S$ where $\gamma_i = \mu\sigma_i$, $\gamma_k = \mu\sigma_k$, and $\sigma_i \neq \sigma_k$. Suppose that $j^{\gamma_i} = j^{\gamma_k}$ for some $j \in \mathbb{Z}_m$, and let $\ell = j^\mu$. Then, we see that

$$\ell^{\sigma_i} = \ell^{\mu^{-1}\gamma_i} = j^{\gamma_i} = j^{\gamma_k} = \ell^{\mu^{-1}\gamma_k} = \ell^{\sigma_k}.$$

It follows that $\sigma_i \neq \sigma_k$ and $\sigma_i, \sigma_k \in S$; a contradiction. Therefore, the set $\mu \cdot S$ is indeed a regular permutation set, as defined in Definition 4.57. ∎

We now proceed with this section's main objective, which is to give a detailed description of our general approach. This approach follows closely that of Ng [52]. Note that our terminology differs from the original terminology used in [52]. This is due to the fact that we explain why this method gives rise to the desired decompositions of $\vec{C}_n \wr H$ in far more detail than [52].

We begin with Lemma 4.64 in which we show that a carefully chosen subdigraph of $\vec{C}_n \wr \vec{C}_m$ is hamiltonian decomposable. To prove Lemma 4.64, we will require the following definition.

Definition 4.61. Let $\sigma \in S_m$ be such that σ is not an $(m-1)$-stabilizer. The *truncation* of σ, denoted $\hat{\sigma}$, is the permutation

$$\hat{\sigma} = \sigma\,(m-1, (m-1)^{\sigma}).$$

See Example 4.62 below for a very simple example of the truncation of a permutation.

Example 4.62. Let $\sigma = (0,1,2,3,4,5,6,7)$ such that $\sigma \in S_8$. Then, we see that $7^{\sigma} = 0$. Therefore, $\hat{\sigma} = (0,1,2,3,4,5,6,7)(7,0) = (0,1,2,3,4,5,6)(7)$. □

It follows from Definition 4.61 that, if σ is not an $(m-1)$-stabilizer, then $(m-1)^{\hat{\sigma}} = m-1$, so $\hat{\sigma}$ is an $(m-1)$-stabilizer.

We will also require the following definition.

Definition 4.63. Let $F = (\sigma_0, \sigma_1, \ldots, \sigma_{n-1})$ be a directed 2-factor of $\vec{C}_n \wr \overline{K}_m$.

1. If $T(\sigma_0\sigma_1 \ldots \sigma_{n-1}) = 1$, then F is called a *hamiltonian n-tuple.*
2. If no permutation in F is an $(m-1)$-stabilizer and $T(\hat{\sigma}_0\hat{\sigma}_1 \ldots \hat{\sigma}_{n-1}) = 2$, then F is called a *truncated hamiltonian n-tuple.*

Lemma 4.64. *Let* $F = (\sigma_0, \sigma_1, \ldots, \sigma_{n-1})$ *be a directed 2-factor of* $\vec{C}_n \wr \vec{C}_m$ *such that* σ_i *is not an* $(m-1)$*-stabilizer for all* $i \in \mathbb{Z}_n$*, and assume that* $(\hat{\sigma}_0, \hat{\sigma}_1, \ldots, \hat{\sigma}_{n-1})$ *is a truncated hamiltonian* n*-tuple. Moreover, let* Γ *be a spanning subdigraph of* $\vec{C}_n \wr \vec{C}_m$ *obtained from* F *by adjoining all horizontal arcs of* $\vec{C}_n \wr \vec{C}_m$*. Then* Γ *is hamiltonian decomposable.*

Proof: Every vertex in Γ has out-degree 2 and thus, any hamiltonian decomposition of Γ consists of two directed hamiltonian cycles.

Before we proceed, we first specify how we embed $\vec{C}_m$ into each V_i of $\vec{C}_n \wr \vec{C}_m$. The digraph $\vec{C}_m$ is embedded into V_i such that $(i_{m-1}, i_{(m-1)^{\sigma_{i-1}}})$ is a horizontal arc in V_i. Observe that, in the embedding of $\vec{C}_m$ into V_i, we then have a dipath of length $m-1$ with source $i_{(m-1)^{\sigma_{i-1}}}$ and terminal i_{m-1}. We will call this dipath P_i.

We now form a spanning subdigraph of Γ, which we call C^0, such that

$$A(C^0) = \cup_{i=0}^{n-1}\{((i-1)_{m-1}, i_{(m-1)^{\sigma_{i-1}}})\} \bigcup (\cup_{i=0}^{n-1} A(P_i)).$$

Refer to Figure 4.23 for an illustration of C^0 in $\vec{C}_3 \wr \vec{C}_7$. Observe that there exists an arc in C^0 from $t(P_{i-1})$ to $s(P_i)$ for each $i \in \mathbb{Z}_n$, namely the arc $((i-1)_{m-1}, i_{(m-1)^{\sigma_{i-1}}})$. Therefore, the digraph C^0 is a directed hamiltonian cycle of Γ.

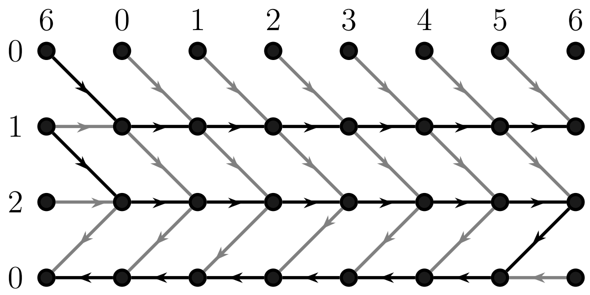
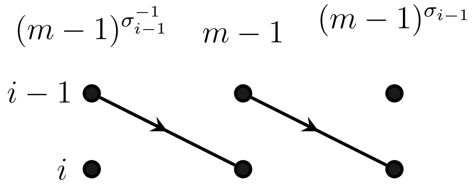


Figure 4.23: A decomposition of the subdigraph Γ of $\vec{C}_3 \wr \vec{C}_7$ into two directed hamiltonian cycles: C^0 (black) and C^1 (grey).

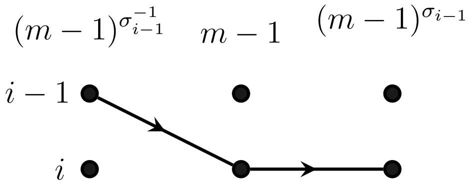
Next, we let C^1 be the spanning digraph of Γ such that $A(C^1) = A(\Gamma) - A(C^0)$. We will show that C^1 is also a directed hamiltonian cycle. First, for each $i \in \mathbb{Z}_n$, we form the following set of arcs:

$$A_i = \{((i-1)_j, i_{j^{\sigma_{i-1}}}) \mid j \in \mathbb{Z}_m/\{m-1\}\} \cup \{(i_{m-1}, i_{(m-1)^{\sigma_{i-1}}})\}.$$

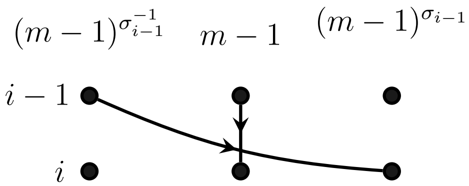
The digraph C^1 is a spanning digraph comprised of disjoint directed cycles. It remains to show that C^1 is indeed a directed hamiltonian cycle.



(a) The arcs corresponding to σ_i.



(b) To obtain A_i, we replace the arc $((i-1)_{m-1}, i_{(m-1)^{\sigma_{i-1}}})$ with $(i_{m-1}, i_{(m-1)^{\sigma_{i-1}}})$.



(c) The arcs corresponding to the permutation $\hat{\sigma}_i$.

Figure 4.24: Illustration of the truncation of σ_i.

In order to show that C^1 is a directed hamiltonian cycle, we will first show that the permutation of elements of $\mathbb{Z}_m/\{m-1\}$ by the arcs in A_i and by $\hat{\sigma}_{i-1}$ is the same. The set A_i is the set of arcs corresponding to σ_{i-1} with tail in V_{i-1} and head in V_i and with the arc $((i-1)_{m-1}, i_{(m-1)^{\sigma_{i-1}}})$ replaced by the arc $(i_{m-1}, i_{(m-1)^{\sigma_{i-1}}})$. See Figures 4.24a and 4.24b for an illustration of this replacement. Observe that $\hat{\sigma}_{i-1}$ fixes $m-1$ and maps $(m-1)^{\sigma_{i-1}^{-1}}$ to $(m-1)^{\sigma_{i-1}}$. Otherwise, the permutation $\hat{\sigma}_{i-1}$ maps j to $j^{\sigma_{i-1}}$ when $j \notin \{m-1, (m-1)^{\sigma_{i-1}^{-1}}\}$.

For each $i \in \mathbb{Z}_n$, we obtain B_i from A_i by replacing the two arcs in Figure 4.24b with the two arcs in Figure 4.24c. Observe that B_i is precisely the set of arcs corresponding to $\hat{\sigma}_i$. Let $\hat{F} = (\hat{\sigma}_0, \hat{\sigma}_1, \ldots, \hat{\sigma}_{n-1})$. That is, $\hat{F}$ is a directed 2-factor of $\vec{C}_n \wr \overline{K}_m$ with arc set $\cup_{i=0}^{n-1} B_i$. By assumption, $\hat{F}$ consists of two disjoint directed cycles, one of which is a directed n-cycle with vertex set $\{i_{m-1} \mid i \in \mathbb{Z}_n\}$. It follows that the arc set

$$\cup_{i=0}^{n-1}(B_i - \{(i-1)_{m-1}, i_{m-1}\})$$

gives rise to a directed $(mn-n)$-cycle, and hence $\cup_{i=0}^{n-1} A_i$ gives rise to a directed (mn)-cycle of $\vec{C}_n \wr \vec{C}_m$.

Therefore, the subdigraph C^1 is a directed hamiltonian cycle. In conclusion, the set $\{C^0, C^1\}$ is a decomposition of Γ into directed hamiltonian cycles. ▮

Our next objective is to use Lemma 4.64 to show that $\vec{C}_n \wr H$ is hamiltonian decomposable by using a carefully constructed directed 2-factorization of $\vec{C}_n \wr \overline{K}_m$. In Definition 4.65 below, we use Definition 4.63 to define a particular directed 2-factorization of $\vec{C}_n \wr \overline{K}_m$.

Definition 4.65. Let $\{S_1, S_2, \ldots, S_n\}$ be a set of n regular permutation sets of order m and let

$$D = \{(\sigma_{(k,1)}, \sigma_{(k,2)}, \ldots, \sigma_{(k,n)}) \mid k \in \mathbb{Z}_m, \sigma_{(k,j)} \in S_j \text{ for all } j = 1, 2, \ldots, n\}$$

be a directed 2-factorization of $\vec{C}_n \wr \overline{K}_m$. Furthermore, assume that $0 \leqslant c \leqslant m-2$. The set D is called a *c-twined 2-factorization of* $\vec{C}_n \wr \overline{K}_m$ if there exists a partition of D into two sets D_T and D_H such that:

1. D_T contains c truncated hamiltonian n-tuples;
2. D_H contains $m-c$ hamiltonian n-tuples.

Recall that each regular permutation set contains a unique $(m-1)$-stabilizer. As a result, a c-twined 2-factorization of $\vec{C}_n \wr \overline{K}_m$ contains exactly n $(m-1)$-stabilizers, none of which can be part of a truncated hamiltonian n-tuple. However, a hamiltonian n-tuple cannot contain n $(m-1)$-stabilizers. Consequently, we must have at least two hamiltonian n-tuples in a c-twined 2-factorization of $\vec{C}_n \wr \overline{K}_m$. Hence the requirement in Definition 4.65 that $c \leqslant m-2$.

In Proposition 4.66, we show that a c-twined 2-factorization of $\vec{C}_n \wr \overline{K}_m$ gives rise to a decomposition of $\vec{C}_n \wr H$ into directed hamiltonian cycles when $|V(H)| = m$ and H admits a decomposition into c directed hamiltonian cycles, with $0 \leqslant c \leqslant m-2$.

Proposition 4.66. *Let H be a digraph on m vertices that admits a decomposition into c directed hamiltonian cycles, with $0 \leqslant c \leqslant m-2$. If $\vec{C}_n \wr \overline{K}_m$ admits a c-twined 2-factorization, then $\vec{C}_n \wr H$ is hamiltonian decomposable.*

Proof: Assume that $\{S_1, S_2, \ldots, S_n\}$ is a set of n regular permutation sets of order m and

$$D = \{(\sigma_{(k,1)}, \sigma_{(k,2)}, \ldots, \sigma_{(k,n)}) \mid k \in \mathbb{Z}_m, \sigma_{(k,j)} \in S_j \text{ for all } j = 1, 2, \ldots, n\}$$

is a c-twined 2-factorization of $\vec{C}_n \wr \overline{K}_m$. Furthermore, let

$$\begin{aligned} D_T &= \{(\sigma_{(k,1)}, \sigma_{(k,2)}, \ldots, \sigma_{(k,n)}) \mid k = 0, 1, \ldots, c-1\} \text{ and} \\ D_H &= \{(\sigma_{(k,1)}, \sigma_{(k,2)}, \ldots, \sigma_{(k,n)}) \mid k = c, c+1, \ldots, m-1\} \end{aligned}$$

form a partition of D such that D_T is comprised of c truncated hamiltonian n-tuples and D_H is comprised of $m-c$ hamiltonian n-tuples. For each $i \in \mathbb{Z}_n$, we embed H into each V_i so that the set of c out-neighbours of i_{m-1} is

$$\{i_{(m-1)^{\sigma_{(0,i-1)}}}, i_{(m-1)^{\sigma_{(1,i-1)}}}, \ldots, i_{(m-1)^{\sigma_{(c-1,i-1)}}}\}.$$

Let H_i be the image of the embedding of H into V_i described above, and let D_i be the decomposition of H_i into directed hamiltonian cycles inherited from H.

Next, for each $L_k \in D_T$, where $L_k = (\sigma_{(k,1)}, \sigma_{(k,2)}, \ldots, \sigma_{(k,n)})$, we construct a spanning subdigraph of $\vec{C}_n \wr H$, which we call Γ_k. The subdigraph Γ_k is comprised of the directed 2-factor described by L_k. In addition, the digraph Γ_k also contains the directed hamiltonian cycle in D_i that contains the arc $(i_{m-1}, i_{(m-1)^{\sigma_{(k,i-1)}}})$. Because L_k is a truncated hamiltonian n-tuple, it follows from the proof of Lemma 4.64 that Γ_k is hamiltonian decomposable.

If $k_1 \neq k_2$, then Γ_{k_1} and Γ_{k_2} are arc-disjoint. Digraphs Γ_{k_1} and Γ_{k_2} do not share a horizontal arc because because, for each $i \in \mathbb{Z}_m$, they contain different directed hamiltonian cycles of D_i.

For each $k \in \{c, \ldots, m-1\}$, let $C^k = (\sigma_{(k,1)}, \sigma_{(k,2)}, \ldots, \sigma_{(k,n)}) \in D_H$. Then C^k is a directed hamiltonian cycle of $\vec{C}_n \wr \overline{K}_m$, and hence of $\vec{C}_n \wr H$. Since D is a directed 2-factor of $\vec{C}_n \wr \overline{K}_m$, each non-horizontal arc of $\vec{C}_n \wr H$ appears precisely once in

$$\{\Gamma_0, \Gamma_1, \ldots, \Gamma_{c-1}\} \cup \{C^0, C^1, \ldots, C^{m-1}\}.$$

Furthermore, each horizontal arc of $\vec{C}_n \wr H$ appears in a subdigraph in $\{\Gamma_0, \Gamma_1, \ldots, \Gamma_{c-1}\}$ exactly once. Therefore, we see that

$$\vec{C}_n \wr H = \Gamma_0 \oplus \Gamma_1 \oplus \ldots \oplus \Gamma_{c-1} \oplus C^c \oplus C^{c+1} \oplus \ldots \oplus C^{m-1}.$$

Since each subdigraph in $\{\Gamma_0, \Gamma_1, \ldots, \Gamma_{c-1}\}$ is a spanning subdigraph of $\vec{C}_n \wr H$ that is hamiltonian decomposable, Lemma 2.8 implies that $\vec{C}_n \wr H$ is hamiltonian decomposable. ∎

By Proposition 4.66, to show that $\vec{C}_n \wr H$ is hamiltonian decomposable when $|V(H)| = m$ and c is the number of directed hamiltonian cycles in a decomposition of H, it suffices to find a c-twined 2-factorization of $\vec{C}_n \wr \overline{K}_m$. In Lemma 4.67, we reduce the latter problem to the case $n = 2$.

Lemma 4.67. *Let $0 \leqslant c \leqslant m-2$ and $n \geqslant 4$ be even. If $\vec{C}_2 \wr \overline{K}_m$ admits a c-twined 2-factorization, then $\vec{C}_n \wr \overline{K}_m$ also admits a c-twined 2-factorization.*

Proof: Let D be a c-twined 2-factorization of $\vec{C}_2 \wr \overline{K}_m$. Then there exist two regular permutation sets of order m, call them S_1' and S_2', such that $S_1' = \{\sigma_0, \sigma_1, \ldots, \sigma_{m-1}\}$ and $S_2' = \{\mu_0, \mu_1, \ldots, \mu_{m-1}\}$, and $D = \{(\sigma_i, \mu_i) \mid i = 0, 1, \ldots, m-1\}$. Without loss of generality, we may assume that $D_T = \{(\sigma_i, \mu_i) \mid i = 0, 1, \ldots, c-1\}$ is the subset of D comprised of c truncated hamiltonian pairs and that $D_H = \{(\sigma_i, \mu_i) \mid i = c, c+1, \ldots, m-1\}$ is the subset of D that contains $m-c$ hamiltonian pairs.

Our objective is to construct a c-twined 2-factorization of $\vec{C}_n \wr \overline{K}_m$. To do so, we will be constructing a set of m n-tuples. First, we give the n regular permutation sets with which we intend to build our directed 2-factorization. Recall that $\Pi_m = \{\pi_i \mid i \in \mathbb{Z}_m\}$ and $\pi_i = (0, 1, 2 \ldots, m-1)^i$. For $i \in \{1, \ldots, n-3\}$, we let $S_i = \Pi_m$. Furthermore, we let $S_{n-1} = S_1'$, and $S_n = S_2'$. We then construct the following set of n-tuples:

$$D_T' = \{(\pi_{i+1}, \pi_{-i-1}, \ldots, \pi_{i+1}, \pi_{-i-1}, \sigma_i, \mu_i) \mid i = 0, 1, \ldots, c-1\}.$$

Observe that no n-tuple of D_T' contains an $(m-1)$-stabilizer. Furthermore, we see that, since (μ_i, σ_i) is a truncated hamiltonian pair, we have

$$T(\hat{\pi}_{i+1}\hat{\pi}_{-i-1} \ldots \hat{\pi}_{i+1}\hat{\pi}_{-i-1}\hat{\sigma}_i\hat{\mu}_i) = T(\hat{\sigma}_i\hat{\mu}_i) = 2.$$

As a result, each n-tuple of D'_T is a truncated hamiltonian n-tuple.

Next, we construct the following set of n-tuples:

$$D'_H = \{(\pi_{i+1}, \pi_{-i-1}, \ldots, \pi_{i+1}, \pi_{-i-1}, \sigma_i, \mu_i) \mid i = c, c+1, \ldots, m-1\}.$$

Observe that

$$T(\pi_{i+1}\pi_{-i-1}\pi_{i+1}\pi_{-i-1}\ldots\sigma_i\mu_i) = T(\sigma_i\mu_i) = 1.$$

Therefore, each n-tuple in D'_H is a hamiltonian n-tuple.

We now point out that, for each $j \in \{1, 2, \ldots, n\}$, each element of S_j appears in exactly one n-tuple of $D' = D'_H \cup D'_T$. Since each S_j is a regular permutation set of order m, it follows that D' is a directed 2-factorization of $\vec{C}_n \wr \overline{K}_m$. In addition, note that D'_T contains c truncated hamiltonian n-tuples, the set D'_H contains $m-c$ hamiltonian n-tuples, and that D'_T and D'_H are disjoint. Therefore, the set D' is a c-twined 2-factorization of $\vec{C}_n \wr \overline{K}_m$. ∎

We summarize the implications of Proposition 4.66 and Lemma 4.67 in Corollary 4.68 below.

Corollary 4.68. *Let H be a strict digraph that admits a decomposition into c directed hamiltonian cycles, and n be an even integer. If the digraph $\vec{C}_2 \wr \overline{K}_m$ admits a c-twined 2-factorization, then $\vec{C}_n \wr H$ is hamiltonian decomposable.*

The implications of Corollary 4.68 are as follows. To show that $\vec{C}_n \wr H$ is hamiltonian decomposable, where $|V(H)| = m$ and H admits a decomposition into c directed hamiltonian cycles, it suffices to construct two regular permutation sets from which we can form a c-twined factorization of $\vec{C}_2 \wr \overline{K}_m$ for all $2 \leqslant c \leqslant m-2$. Recall that, if $c \in \{1, m-1\}$, then $H \in \{\vec{C}_m, K^*_m\}$. These extremal cases are addressed in Sections 4.2–4.4 and are thus not considered in our general approach. Lastly, note that the case $c = 0$ is addressed in [52].

We now discuss some differences between our approach and that of Ng [52]. Our constructions are more elaborate than those of Ng [52] for the following reason. In [52], Ng constructs a directed 2-factorization D of $\vec{C}_n \wr \overline{K}_m$ for n odd by constructing a set of m directed 2-factors described by n-tuples of permutations from S_m. This set of n-tuples is strategically constructed such that $m-2$ of these tuples are both hamiltonian and truncated hamiltonian while the remaining two n-tuples are hamiltonian. This allows Ng to produce a c-twined 2-factorization of $\vec{C}_n \wr \overline{K}_m$ for all $0 \leqslant c \leqslant m-2$, thereby obtaining a decomposition of $\vec{C}_n \wr H$ into directed hamiltonian cycles. When n is even, the same approach cannot work because an n-tuple cannot be both hamiltonian and truncated hamiltonian as shown below.

Lemma 4.69. *Let $F = (\sigma_0, \sigma_1, \ldots, \sigma_{n-1})$ be a directed 2-factor of $\vec{C}_n \wr \overline{K}_m$ where n is an even integer. If F is a hamiltonian n-tuple, then F is not a truncated hamiltonian n-tuple.*

Proof: Suppose that $F = (\sigma_0, \sigma_1, \ldots, \sigma_{n-1})$ is both a hamiltonian n-tuple and a truncated hamiltonian n-tuple. Moreover, suppose that F is comprised of n_o odd permutations and n_e even permutations. Since n is even, it follows that n_o and n_e are of the same parity. Next, we let

$$\gamma = \sigma_0\sigma_1 \ldots \sigma_{n-1} \text{ and } \gamma' = \hat{\sigma}_0\hat{\sigma}_1 \ldots \hat{\sigma}_{n-1}.$$

We point out that, if σ_i is an odd permutation, then $\hat{\sigma}_i$ is even. Similarly, if σ_i is even, then $\hat{\sigma}_i$ is odd. This means that γ' is the product of n_o even permutations and n_e odd permutations. Therefore, if n_o and n_e are odd, then both γ and γ' are odd permutations. Similarly, if n_o and n_e are even, then γ and γ' are even. As a result, permutations γ and γ' are of the same parity.

From the hypothesis, it follows that $T(\gamma) = 1$ and $T(\gamma') = 2$. The permutation γ is an element of S_m comprised of a single cycle of length m, while γ' is comprised of one cycle of length one and one cycle of length $m-1$. This means that γ can be expressed as the product of $m-1$ transpositions while γ' can be expressed as the product of $m-2$ transpositions. Therefore, permutations γ and γ' are of distinct parity; a contradiction. ∎

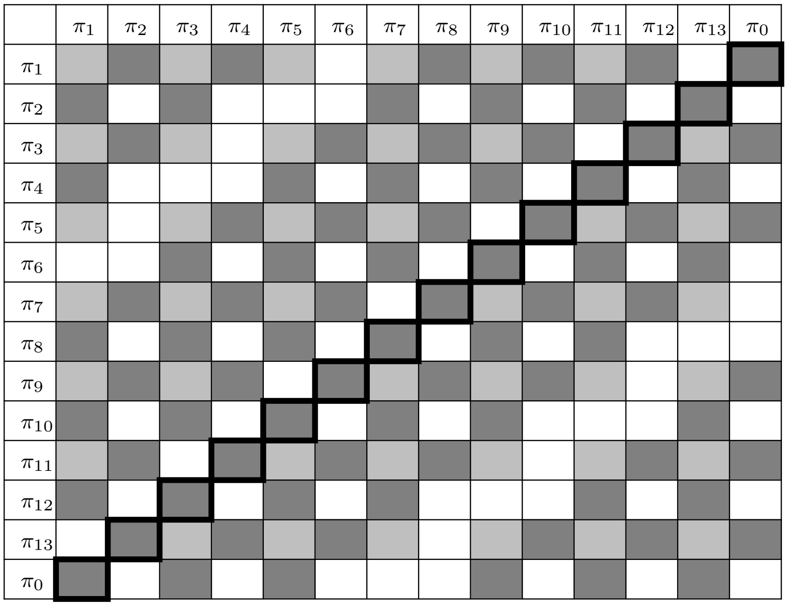
As a result of Lemma 4.69, our constructions are more elaborate than those of [52]. In the next section, we will be illustrating some of these constructions for small values of m. To do so, we will be using an $m \times m$ array comprised of coloured squares. Colours are used to indicate whether a pair of permutation is hamiltonian, truncated hamiltonian, or neither. These arrays are defined in Definition 4.70 below.

Definition 4.70. Let $S_1 = \{\mu_0, \mu_1, \ldots, \mu_{m-1}\}$ and $S_2 = \{\sigma_0, \sigma_1, \ldots, \sigma_{m-1}\}$ be two regular permutation sets of order m and let the elements of S_1 and S_2 be ordered as follows: $(\mu_0, \mu_1, \ldots, \mu_{m-1})$ and $(\sigma_0, \sigma_1, \ldots, \sigma_{m-1})$. The *hamiltonian table of* $S_1 \times S_2$ is an $m \times m$ array with entries defined as follows:

$a_{i,j} =$ ■ if (μ_i, σ_j) is a hamiltonian pair;

$a_{i,j} =$ ░ if (μ_i, σ_j) is a truncated hamiltonian pair;

$a_{i,j} =$ □ otherwise.



	π_1	π_2	π_3	π_4	π_5	π_6	π_7	π_8	π_9	π_{10}	π_{11}	π_{12}	π_{13}	π_0
π_1	░	■	░	■	░	□	░	■	░	■	░	■	□	**■**
π_2	■	□	■	□	□	□	■	□	■	□	■	□	**■**	□
π_3	░	■	░	□	░	■	░	■	░	■	□	**■**	░	■
π_4	■	□	□	□	■	□	■	□	■	□	**■**	□	■	□
π_5	░	□	░	■	░	■	░	■	□	**■**	░	■	░	■
π_6	□	□	■	□	■	□	■	□	**■**	□	■	□	■	□
π_7	░	■	░	■	░	■	□	**■**	░	■	░	■	░	□
π_8	■	□	■	□	■	□	**■**	□	■	□	■	□	□	□
π_9	░	■	░	■	□	**■**	░	■	░	■	░	□	░	■
π_{10}	■	□	■	□	**■**	□	■	□	■	□	□	□	■	□
π_{11}	░	■	□	**■**	░	■	░	■	░	□	░	■	░	■
π_{12}	■	□	**■**	□	■	□	■	□	□	□	■	□	■	□
π_{13}	□	**■**	░	■	░	■	░	□	░	■	░	■	░	■
π_0	**■**	□	■	□	■	□	□	□	■	□	■	□	■	□

Figure 4.25: The hamiltonian table of $\Pi_{14} \times \Pi_{14}$ with a 0-twined 2-factorization (in black).

Refer to Figure 4.25 for the hamiltonian table of $\Pi_{14} \times \Pi_{14}$.

Remark 4.71. Note that, since $n = 2$, Lemma 4.69 implies that the entries of a hamiltonian table are well-defined since a pair cannot be both hamiltonian and truncated hamiltonian. A directed 2-factorization of $\vec{C}_2 \wr \overline{K}_m$ constructed from two regular permutation sets S_1 and S_2 corresponds to a set D of m entries of the hamiltonian table of $S_1 \times S_2$ that contains exactly one entry from each row and column. This

2-factorization is a c-twined 2-factorization precisely when D contains c light grey squares (truncated hamiltonian pairs) and $m - c$ dark grey squares (hamiltonian pairs).

In Figure 4.25, we present a 0-twined 2-factorization of $\vec{C}_{14} \wr \overline{K}_2$. Note that all hamiltonian tables given in this chapter are generated by computer.

4.6 The case c is even and $0 \leqslant c \leqslant m - 2$

In this section, we will be constructing a c-twined 2-factorization of $\vec{C}_2 \wr \overline{K}_m$ for c even and all $m \geqslant 4$. We partition our investigation into two cases. First, we address the case in which m is even in Proposition 4.72. Then, in Propositions 4.80–4.82, we settle the case m is odd.

4.6.1 The case m is even

Before we proceed, we give an outline of Proposition 4.72, which is also similar to that of Propositions 4.80–4.82. First, we strategically construct two regular permutation sets of order m. Using these two families, we will first construct a 0-twined 2-factorization of $\vec{C}_2 \wr \overline{K}_m$ by forming a set of m hamiltonian pairs. To show that each pair is hamiltonian, we will compute the product of the two permutations and show that this product is a permutation with a single cycle. Next, we will construct a c-twined 2-factorization for each even c such that $2 \leq c \leq m - 4$. To do so, we will construct a set of c truncated hamiltonian pairs that we will call D_T. To prove that a pair in D_T is a truncated hamiltonian pair, we will compute the product of the truncations of the two permutations in this pair. Lastly, we will produce a set of $m - c$ pairs that we will call D_H. The set D_H will be disjoint from D_T and is constructed so that $D = D_H \cup D_T$ is a 2-factorization of $\vec{C}_2 \wr \overline{K}_m$. The set D will then be the desired c-twined 2-factorization of $\vec{C}_2 \wr \overline{K}_m$. Lastly, we address the case $c = m-2$ by using the same approach as in the case $2 \leq c \leq m-4$ but using different regular permutation sets.

Proposition 4.72. *Let m and c be even integers such that $m \geqslant 4$ and $0 \leq c \leq m-2$. The digraph $\vec{C}_2 \wr \overline{K}_m$ admits a c-twined 2-factorization.*

Proof: We will partition our constructions into three cases. To construct a c-twined 2-factorization for the first two cases, we will construct a set of m pairs by using the following two sets of permutations:

$$S_1 = (1, m-2) \cdot \Pi_m = \{\mu_i = (1, m-2)\pi_i \mid i = 0, 1, \ldots, m-1\};$$
$$S_2 = \Pi_m = \{\pi_0, \pi_1, \ldots, \pi_{m-1}\}.$$

Note that Π_m is a regular permutation set. By Lemma 4.60, the set S_1 is also a regular permutation set. Refer to Notation 4.7 for the definition of $\pi_i \in \Pi_m$. All m pairs constructed in this proof will lie in the set $S_1 \times S_2$. See Figure 4.26 for the hamiltonian table of $S_1 \times S_2$ when $m = 14$. Note that μ_0 and π_0 are the $(m-1)$-stabilizers of S_1 and S_2, respectively. Otherwise, if $i \neq 0$, then

$$(m-1)^{\mu_i} = (m-1)^{\pi_i} = i - 1.$$

It follows that, if $i \neq 0$, then

$$\hat{\mu}_i = \mu_i(m-1, i-1) \text{ and } \hat{\pi}_i = \pi_i\,(m-1, i-1).$$

Case 1: $c = 0$. We construct a set of m hamiltonian pairs. First, we form a set of $\frac{m}{2}$ pairs, denoted D_{H_1}, as follows:

$$D_{H_1} = \{(\mu_i, \pi_{m-i-2}) \mid i \in \mathbb{Z}_m \text{ and } i \text{ is odd}\}.$$

Observe that each permutation $\mu_i \in S_1$ with i odd is used by precisely one pair of D_{H_1}. Additionally, each permutation $\pi_j \in S_2$ with j odd appears in exactly one pair of D_{H_1}. Below, we show that each of these pairs is hamiltonian:

$$\begin{aligned} \mu_i\,\pi_{m-i-2} &= (1, m-2)\,\pi_i\,\pi_{-i-2} \\ &= (1, m-2)\pi_{-2} \\ &= (0, m-2, m-1, m-3, m-5, \ldots, 3, 1, m-4, m-6, \ldots, 2). \end{aligned}$$

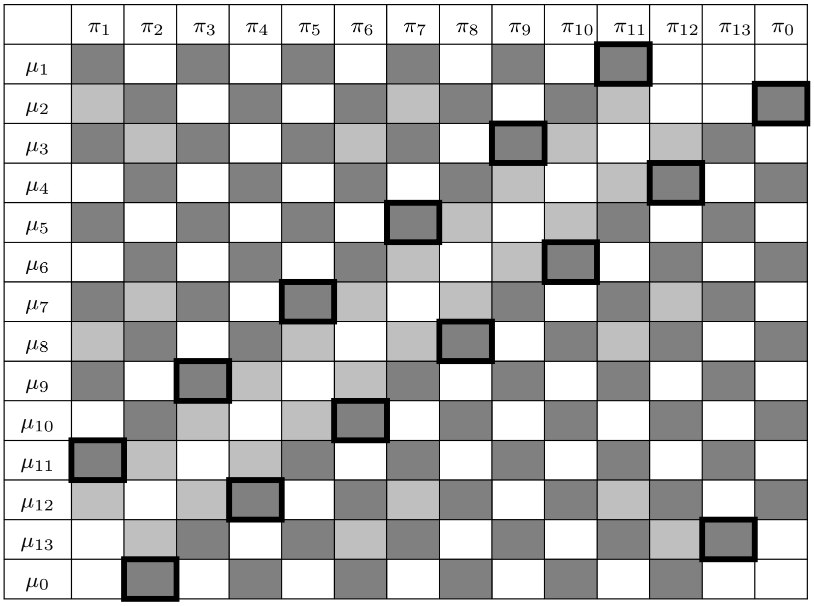
Next, we form a second set of $\frac{m}{2}$ pairs, denoted D_{H_2}:

$$D_{H_2} = \{(\mu_i, \pi_{m-i+2}) \mid i \in \mathbb{Z}_m \text{ and } i \text{ is even}\}.$$

We point out that D_{H_2} uses each permutation $\mu_i \in S_1$ and each $\pi_j \in S_2$, where i and j are even, exactly once. We now show that each of the pairs in D_{H_2} is hamiltonian:

$$\begin{aligned}\mu_i\pi_{m-i+2} &= (1,m-2)\,\pi_i\,\pi_{-i+2}\\ &= (1,m-2)\pi_2\\ &= (0,2,4,6,\ldots,m-2,3,5,7,\ldots,m-1,1).\end{aligned}$$

Next, we let $D = D_{H_1} \cup D_{H_2}$. Refer to Figure 4.26 for an illustration of D when $m = 14$. Each permutation of $S_1 \cup S_2$ appears in exactly one pair of D. Therefore, the set D is a directed 2-factorization of $\vec{C}_2 \wr \overline{K}_m$. Moreover, it follows that D contains m hamiltonian pairs and thus, the set D is a 0-twined 2-factorization of $\vec{C}_2 \wr \overline{K}_m$.



	π_1	π_2	π_3	π_4	π_5	π_6	π_7	π_8	π_9	π_{10}	π_{11}	π_{12}	π_{13}	π_0
μ_1	■		■		■		■		■		▣			
μ_2	▒	■		■		■	▒	■		■	▒			▣
μ_3	■	▒	■		■	▒	■		▣	▒		▒	■	
μ_4		■		■		■		■	▒		▒	▣		■
μ_5	■		■		■		▣	▒		▒	■		■	
μ_6		■		■		■	▒		▒	▣		■		■
μ_7	■	▒	■		▣	▒		▒	■		■	▒	■	
μ_8	▒	■		■	▒		▒	▣		■	▒	■		■
μ_9	■		▣	▒		▒	■		■		■		■	
μ_{10}		■	▒		▒	▣		■		■		■		■
μ_{11}	▣	▒		▒	■		■		■		■		■	
μ_{12}	▒		▒	▣		■	▒	■		■	▒	■		■
μ_{13}		▒	■		■	▒	■		■		■	▒	▣	
μ_0		▣		■		■		■		■		■		

Figure 4.26: Case 1: hamiltonian table of $S_1 \times S_2$ for $m = 14$ with a 0-twined 2-factorization in black.

Case 2: $2 \leqslant c \leqslant m-4$. Let $c = 2t$ where $1 \leqslant t \leqslant \frac{m-4}{2}$. We will construct two disjoint sets D_T and D_H such that D_T contains c truncated hamiltonian pairs and D_H contains $m-c$ hamiltonian pairs. Assume that $\mathbb{I} = \{3, 5, 7, \ldots, m-3\}$ and let M_t be a subset of size t of $\mathbb{I}$. Observe that $\mathbb{I}$ contains $\frac{m-4}{2}$ elements. We then let

$$D_T = \{(\mu_i, \pi_{m-i+1}), (\mu_{i+1}, \pi_{m-i-2}) \mid i \in M_t\}.$$

Below, we show that (μ_i, π_{m-i+1}) and (μ_{i+1}, π_{m-i-2}) are truncated hamiltonian pairs for all $i \in \mathbb{I}$:

$$\begin{aligned}
\hat{\mu}_i\hat{\pi}_{m-i+1} &= (1, m-2)\pi_i(m-1, i-1)\,\pi_{-i+1}(m-1, m-i)\\
&= (0, 1, m-i, m-i+1, \ldots, m-2, 2, 3, 4, \ldots, m-i-1)(m-1);\\
\hat{\mu}_{i+1}\hat{\pi}_{m-i-2} &= (1, m-2)\,\pi_{i+1}\,(m-1, i)\,\pi_{-i-2}\,(m-1, m-i-3)\\
&= (0, m-i-3, m-i-4, m-i-5, \ldots, 2, 1, m-3, m-4, m-5,\\
&\quad \ldots, m-i-2, m-2)(m-1).
\end{aligned}$$

Therefore, the set D_T is comprised of $2t$ truncated hamiltonian pairs.

We then construct D_H as follows:

$$D_H = \{(\mu_i, \pi_{m-i-2}), (\mu_{i+1}, \pi_{m-i+1}) \mid i \text{ odd and } i \in \mathbb{Z}_m \backslash M_t\}.$$

The set D_H contains $m - 2t$ pairs. In Case 1 above, we defined a set D of m hamiltonian pairs. Observe that $D_H \subseteq D$. This means that all $m - 2t$ pairs in D_H are hamiltonian pairs.

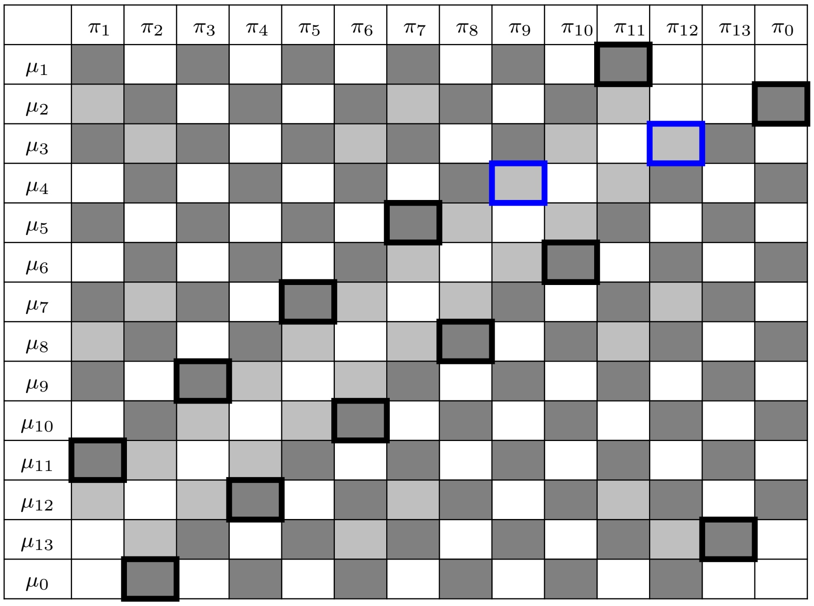


Figure 4.27: Case 2: hamiltonian table of $S_1 \times S_2$ for $m = 14$ with a 2-twined 2-factorization in black (hamiltonian pairs) and blue (truncated hamiltonian pairs).

Let $D' = D_T \cup D_H$. See Figures 4.27 and 4.28 for an illustration of D' when $m = 14$, and $M_t = \{3\}$ and $M_t = \mathbb{I}$, respectively. Notice that D_T and D_H are disjoint. Furthermore, each permutation of $S_1 \cup S_2$ appears in exactly one pair of D' meaning

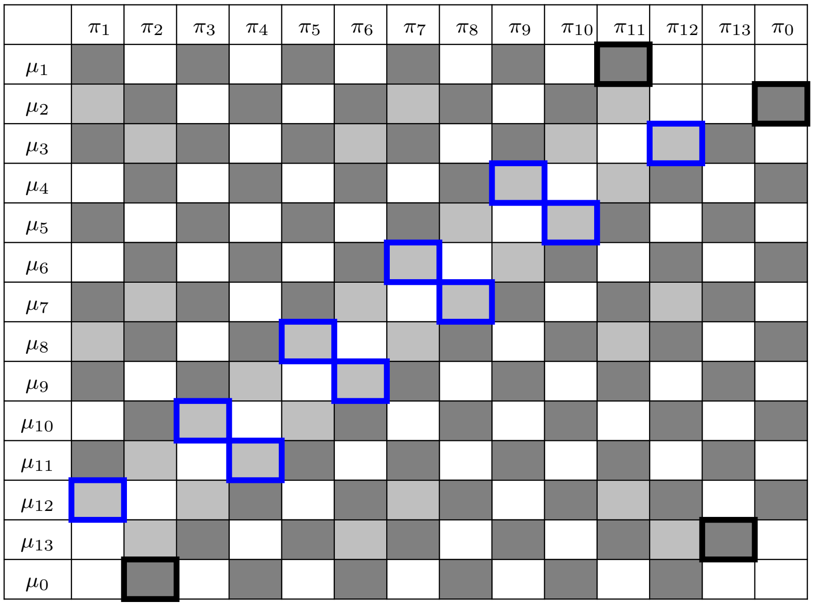


Figure 4.28: Case 2: hamiltonian table of $S_1 \times S_2$ for $m = 14$ with a 10-twined 2-factorization in black (hamiltonian pairs) and blue (truncated hamiltonian pairs).

that D' is a directed 2-factorization of $\vec{C}_2 \wr \overline{K}_m$. In conclusion, the set D' is a c-twined 2-factorization of $\vec{C}_2 \wr \overline{K}_m$.

Case 3: $c = m - 2$. We construct a set of m pairs that contains $m - 2$ truncated hamiltonian pairs and two hamiltonian pairs. We use the following two sets of permutations:

$$S_1 = (0, m-1) \cdot \Pi_m = \{\omega_i = (0, m-1)\pi_i \mid i = 0, 1, \dots, m-1\};$$

$$S_2 = \Pi_m = \{\pi_0, \pi_1, \dots, \pi_{m-1}\}.$$

Since Π_m is a regular permutation set, Lemma 4.60 implies that S_1 below is also a regular permutation set. Permutations ω_{m-1} and π_0 are the $(m-1)$-stabilizers of S_1 and S_2, respectively. Otherwise, if $i \neq m - 1$, then

$$\hat{\omega}_i = \omega_i(m-1, i).$$

If $i \neq 0$, then

$$\hat{\pi}_i = \pi_i\,(m-1, i-1).$$

First, we form the following set:

$$D_H = \{(\omega_{m-1}, \pi_{m-1}), (\omega_{m-2}, \pi_0)\}.$$

Below, we prove that the two pairs of D_H are hamiltonian pairs:

$$\begin{aligned}
\omega_{m-1}\pi_{m-1} &= (0, m-1)\pi_{m-1}\pi_{m-1} \\
&= (0, m-1)\pi_{-2} \\
&= (0, m-3, m-5, m-7, \ldots, 1, m-1, m-2, m-4, \ldots, 2); \\
\omega_{m-2}\pi_0 &= (0, m-1)\,\pi_{m-2}\,id \\
&= (0, m-3, m-5, m-7, \ldots, 1, m-1, m-2, m-4, \ldots, 2).
\end{aligned}$$

Therefore, both pairs in D_H are hamiltonian.

Next, we form the following set of $m-2$ pairs:

$$D_T = \{(\omega_i, \pi_{m-i-1}) \mid i \in \mathbb{Z}_m/\{0, m-2, m-1\}\} \cup \{(\omega_0, \pi_1)\}.$$

First, we show that (ω_0, π_1) is a truncated hamiltonian pair. Note that $\hat{\omega}_0 = id$. Therefore, we have that

$$\begin{aligned}
\hat{\omega}_0\hat{\pi}_1 &= \hat{\pi}_1 = (0, 1, \ldots, m-1)(m-1, 0) \\
&= (0, 1, 2, \ldots, m-2)(m-1).
\end{aligned}$$

Below, we show that the remaining pairs of D_T are also truncated hamiltonian pairs

$$\begin{aligned}
\hat{\omega}_i\hat{\pi}_{m-i-1} &= (0, m-1)\,\pi_i\,(m-1, i)\;\pi_{-i-1}\,(m-1, -i-2) \\
&= (0, m-2, m-3, \ldots, 2, 1)(m-1).
\end{aligned}$$

In conclusion, all pairs in D_T are truncated hamiltonian pairs.

Let $D = D_H \cup D_T$. Observe that each permutation of S_1 and S_2 appears in precisely one pair of D. As a result, the set D is a directed 2-factorization of $\vec{C}_2 \wr \overline{K}_m$. Moreover, note that D_H and D_T are disjoint. This means that D is an $(m-2)$-twined 2-factorization of $\vec{C}_2 \wr \overline{K}_m$. ∎

4.6.2 The case m is odd

Our next objective is to show that $\vec{C}_2 \wr \overline{K}_m$ admits a c-twined 2-factorization when $m \geqslant 5$ is odd, c is even, and $2 \leqslant c \leqslant m-2$. To do so, we first construct a particular regular permutation set of order m. In order to construct this regular permutation set, we will be needing elements from the following set.

Definition 4.73. Let γ_1 be the permutation $(0, 1, \ldots, m-2)(m-1)$ in S_m. We call the permutation group $G_{m-1} = \langle \gamma_1 \rangle$ on $\mathbb{Z}_m$ the *truncated cyclic group of order* $m-1$. In addition, for any $i \in \mathbb{Z}$, we let $\gamma_i = \gamma_1^i$.

Observe that G_{m-1} contains $m-1$ permutations. Note that $T(\gamma_i) = 2$ if and only if i is relatively prime with $m-1$.

Lemma 4.74. *Let m be an odd integer such that $m \geqslant 5$, and let G_{m-1} be the truncated cyclic group of order $m-1$. Then*

(S1) $\gamma_{m-i} = \gamma_{-i+1}$ *for all* $i \in \mathbb{Z}$*;*

(S2) *If* $i \in \{\pm 1, \pm 2, \ldots, \pm(m-2)\}$ *and* $s \in \mathbb{Z}_m$*, then*

$$s^{\gamma_i} = \begin{cases} s+i, & \text{if } 0 \leqslant s+i < m-1; \\ s+i-(m-1), & \text{if } s+i \geqslant m-1; \\ s+i+(m-1), & \text{if } s+i < 0; \\ m-1, & \text{if } s = m-1. \end{cases}$$

Proof: **(S1)** Observe that $\gamma_1^m = \gamma_1$. It follows that

$$\gamma_{m-i} = \gamma_1^{m-i} = \gamma_1^m \gamma_1^{-i} = \gamma_1^{-i+1} = \gamma_{-i+1}.$$

(S2) Next, we aim to compute s^{γ_i}. Clearly, $(m-1)^{\gamma_i} = m-1$. Therefore, we assume that $s \in \{0, \ldots, m-2\}$, and let $t = s^{\gamma_i}$. By definition, $t = s + i \pmod{m-1}$. Since $-(m-2) \leqslant s+i \leqslant 2(m-2)$, we know that $t \in \{s+i-(m-1), s+i, s+i+(m-1)\}$ and the result follows. ∎

The two properties of G_{m-1} established in Lemma 4.74 will be key in constructing the desired c-twined 2-factorizations of $\vec{C}_2 \wr \overline{K}_m$. To that end, we now use the elements of G_{m-1} to construct the following set of m permutations for all odd $m \geqslant 5$.

Construction 4.75. Let $m = 2k + 1$ where $k \geqslant 2$, and recall Definition 4.73. We construct the set of permutations $\mathcal{F}_m = \{\sigma_0, \sigma_1, \ldots, \sigma_{m-1}\}$ as follows.

If i is odd, then $i = 2j + 1$, where $0 \leqslant j \leqslant k - 1$. Then, we let

$$\sigma_i = \gamma_i(m - 1, j + 2).$$

If i is even and $i \notin \{0, m - 3, m - 1\}$, then $i = 2j$, where $1 \leqslant j \leqslant k - 2$. Then, we let

$$\sigma_i = \gamma_i(m - 1, k + j + 1).$$

Moreover, we let

$$\sigma_{m-3} = \gamma_{m-3}(m - 1, 0);$$

$$\sigma_0 = id.$$

Lastly, we define σ_{m-1} as follows. For $a \in \mathbb{Z}_m$, we let

$$a^{\sigma_{m-1}} = \begin{cases} m - a + 1, & \text{if } 3 \leqslant a \leqslant k; \\ m - a + 2, & \text{if } k + 2 \leqslant a \leqslant m - 2; \\ 3, & \text{if } a = 0; \\ 2, & \text{if } a = 1; \\ 0, & \text{if } a = 2; \\ m - 1, & \text{if } a = k + 1; \\ 1, & \text{if } a = m - 1. \end{cases}$$

We note that, if $m \geqslant 7$, then σ_{m-1} can be written in cycle notation as follows:

$$\sigma_{m-1} = (0, 3, m - 2, 4, m - 3, 5, \ldots, k + 3, k, k + 2, k + 1, m - 1, 1, 2).$$

If $m = 5$, then

$$\sigma_4 = (0, 3, 4, 1, 2).$$

Hence σ_{m-1} is comprised of a single cycle.

Lastly, we note that, if $\sigma_i \in \mathcal{F}_m$ such that $i \notin \{0, m-1\}$, then it follows from the construction of $\mathcal{F}_m$ that $\hat{\sigma}_i = \gamma_i$. This property is key in all remaining constructions.

□

Below we give an example of Construction 4.75.

Example 4.76. In this example, we construct $\mathcal{F}_{15}$. First, we list the elements of G_{14}:

$$\gamma_0 = id;$$

$$\gamma_1 = (0,1,2,3,4,5,6,7,8,9,10,11,12,13)(14);$$

$$\gamma_2 = (0,2,4,6,8,10,12)(1,3,5,7,9,11,13)(14);$$

$$\gamma_3 = (0,3,6,9,12,1,4,7,10,13,2,5,8,11)(14);$$

$$\gamma_4 = (0,4,8,12,2,6,10)(1,5,9,13,3,7,11)(14);$$

$$\gamma_5 = (0,5,10,1,6,11,2,7,12,3,8,13,4,9)(14);$$

$$\gamma_6 = (0,6,12,4,10,2,8)(1,7,13,5,11,3,9)(14);$$

$$\gamma_7 = (0,7)(1,8)(2,9)(3,10)(4,11)(5,12)(6,13)(14);$$

$$\gamma_8 = (0,8,2,10,4,12,6)(1,9,3,11,5,13,7)(14);$$

$$\gamma_9 = (0,9,4,13,8,3,12,7,2,11,61,10,5)(14);$$

$$\gamma_{10} = (0,10,6,2,12,8,4)(1,11,7,3,13,9,5)(14);$$

$$\gamma_{11} = (0,11,8,5,2,13,10,7,4,1,12,9,6,3)(14);$$

$$\gamma_{12} = (0,12,10,8,6,4,2)(1,13,11,9,7,5,3)(14);$$

$$\gamma_{13} = (0,13,12,11,10,9,8,7,6,5,4,3,2,1)(14).$$

Note that $m = 2k+1$ where $k = 7$. Below, we list each of the 15 permutations in $\mathcal{F}_{15}$:

$$\sigma_0 = id;$$

$$\sigma_1 = \gamma_1(14,2) = (14,2,3,4,5,6,7,8,9,10,11,12,13,0,1);$$

$$\sigma_2 = \gamma_2(14,9) = (14,9,11,13,1,3,5,7)(2,4,6,8,10,12,0);$$

$$\sigma_3 = \gamma_3(14,3) = (14,3,6,9,12,1,4,7,10,13,2,5,8,11,0);$$

$$\sigma_4 = \gamma_4(14,10) = (14,10,0,4,8,12,2,6)(1,5,9,13,3,7,11);$$

$$\sigma_5 = \gamma_5(14,4) = (14,4,9,0,5,10,1,6,11,2,7,12,3,8,13);$$

$$\sigma_6 = \gamma_6(14,11) = (14,11,3,9,1,7,13,5)(2,8,0,6,12,4,10);$$

$$\sigma_7 = \gamma_7(14,5) = (14,5,12)(1,8)(2,9)(3,10)(4,11)(6,13)(0,7);$$

$$\sigma_8 = \gamma_8(14,12) = (14,12,6,0,8,2,10,4)(1,9,3,11,5,13,7);$$
$$\sigma_9 = \gamma_9(14,6) = (14,6,1,10,5,0,9,4,13,8,3,12,7,2,11);$$
$$\sigma_{10} = \gamma_{10}(14,13) = (14,13,9,5,1,11,7,3)(2,12,8,4,0,10,6);$$
$$\sigma_{11} = \gamma_{11}(14,7) = (14,7,4,1,12,9,6,3,0,11,8,5,2,13,10);$$
$$\sigma_{12} = \gamma_{12}(14,0) = (14,0,12,10,8,6,4,2)(1,13,11,9,7,5,3);$$
$$\sigma_{13} = \gamma_{13}(14,8) = (14,8,7,6,5,4,3,2,1,0,13,12,11,10,9);$$
$$\sigma_{14} = (0,3,13,4,12,5,11,6,10,7,9,8,14,1,2).$$

□

Now, we prove that $\mathcal{F}_m$ is indeed a regular permutation set of order m.

Lemma 4.77. *Let $m \geqslant 5$ be an odd integer. The set of permutation $\mathcal{F}_m$ is a regular permutation set of order m.*

Proof: It suffices to show that, for each $a \in \mathbb{Z}_m$, we have

$$\{a^{\sigma_i} \mid i = 0,1,\dots,m-1\} = \mathbb{Z}_m.$$

Case 1: $3 \leqslant a \leqslant k$. Using Construction 4.75, it can be verified that $a^{\sigma_i} = a^{\gamma_i}$ except if $i = m-1$ or $i = 2j$, where $j = k-a+1$. In the first exception, we have that $a^{\sigma_{m-1}} = m-a+1$, and in the second exception, $a^{\sigma_{2j}} = m-1$. Since $\{a^{\gamma_i} \mid i = 0,1,\dots,m-2\} = \{0,1,\dots,m-2\}$ and $a^{\gamma_i} = m-a+1$ for $i = 2(k-a+1)$, we conclude that $\{a^{\sigma_i} \mid i = 0,1,\dots,m-1\} = \mathbb{Z}_m$.
Case 2: $k+2 \leqslant a \leqslant m-2$. Similarly to Case 1, we can use Construction 4.75 to verify that $a^{\sigma_i} = a^{\gamma_i}$ except if $i = m-1$ or $i = 2j+1$, where $j = m-a$. In the first exception, we have that $a^{\sigma_{m-1}} = m-a+2$, and in the second exception, $a^{\sigma_{2j+1}} = m-1$. Since $\{a^{\gamma_i} \mid i = 0,1,\dots,m-2\} = \{0,1,\dots,m-2\}$ and $a^{\gamma_i} = m-a+2$ for $i = 2(m-a)+1$, we conclude that $\{a^{\sigma_i} \mid i = 0,1,\dots,m-1\} = \mathbb{Z}_m$.
Case 3: $a = 0$. Using Construction 4.75, it can be verified that $0^{\sigma_i} = 0^{\gamma_i}$ except if $i = 3$ or $i = m-1$. In the first exception, we see that $0^{\sigma_3} = m-1$ and in the second exception $0^{\sigma_{m-1}} = 3$.
Case 4: $a = 1$. Similarly it can be verified that $1^{\sigma_i} = 1^{\gamma_i}$ except if $i = 1$ or $i = m-1$. In the first exception, we see that $1^{\sigma_1} = m-1$ and in the second exception $1^{\sigma_{m-1}} = 2$.
Case 5: $a = 2$. It can be verified that $2^{\sigma_i} = 2^{\gamma_i}$ except if $i = m-3$ or $i = m-1$. In the first exception, we see that $2^{\sigma_{m-3}} = m-1$ and in the second exception $2^{\sigma_{m-1}} = 0$.

Case 6: $a = k+1$. It can be verified that $(k+1)^{\sigma_i} = (k+1)^{\gamma_i}$ except if $i = m-1$. In this one exception, we see that $(k+1)^{\sigma_{m-1}} = m-1$.

Case 7: $a = m-1$. We see that, if $i = 2j$, where $1 \leqslant j \leqslant k-2$, Construction 4.75 implies that $(m-1)^{\sigma_i} = k+j+1$. Furthermore, if $i = 2j+1$, where $0 \leqslant j \leqslant k-1$, then Construction 4.75 implies that $(m-1)^{\sigma_i} = j+2$. We also point out that $(m-1)^{\sigma_{m-3}} = 0$, $(m-1)^{\sigma_0} = m-1$, and $(m-1)^{\sigma_{m-1}} = 1$. In conclusion, we see that $\{a^{\sigma_i} \mid i = 0, 1, \ldots, m-1\} = \mathbb{Z}_m$.

In summary, we have established that, for each $a \in \mathbb{Z}_m$, we have $\{a_i^{\sigma} \mid i = 0, 1, \ldots, m-1\} = \mathbb{Z}_m$. Therefore, the set $\mathcal{F}_m$ is a regular permutation set of order m. ∎

To construct the desired c-twined 2-factorization of $\vec{C}_2 \wr \overline{K}_m$, we will use permutations of $\mathcal{F}_m$ of the form σ_{m-i+t} where $t \in \{0, 1, \pm 2, -3, -4\}$. Our next objective is to simplify our computations by first computing $(m-1)^{\sigma_{m-i+t}}$. We note that Construction 4.75 and Lemma 4.74 imply that $\sigma_{m-i+t} = \gamma_{-i+t+1}\left(m-1, f(i,k,t)\right)$ where $f(i,k,t) = (m-1)^{\sigma_{m-i+t}}$. In Lemma 4.78, we compute $f(i,k,t)$.

Lemma 4.78. *Let $m = 2k+1$ where $k \geqslant 2$, and $\sigma_i \in \mathcal{F}_m$, where $i \in \mathbb{Z}_m$ such that $i \notin \{0, m-1\}$. Furthermore, let $t \in \{0, 1, \pm 2, -3, -4\}$ such that $t < i$ and $t - i \notin \{-3, -1, -m\}$. Then*

$$\sigma_{m-i+t} = \gamma_{-i+t+1}\left(m-1, f(i,k,t)\right)$$

where

1. *$f(i,k,t) = m - j + \frac{t+1}{2}$ if $i = 2j$ where $1 \leqslant j \leqslant k-2$ and t is odd;*
2. *$f(i,k,t) = k - j + \frac{t}{2} + 2$ if $i = 2j$ where $1 \leqslant j \leqslant k-2$ and t is even;*
3. *$f(i,k,t) = k - j + \frac{t+3}{2}$ if $i = 2j+1$ where $0 \leqslant j \leqslant k-1$ and t is odd, and $j \neq k-1$ or $t \neq -3$;*
4. *$f(i,k,t) = m - j + \frac{t}{2}$ if $i = 2j+1$ where $0 \leqslant j \leqslant k-1$ and t is even, and $j \neq k-1$ or $t \neq -4$.*

Proof: We first note that Lemma 4.74 and Construction 4.75 imply that, if $m-i+t \notin \{0, m-1\}$, then $\sigma_{m-i+t} = \gamma_{-i+t+1}(m-1, f(i,k,t))$ where $f(i,k,t) = (m-1)^{\sigma_{m-i+t}}$. Note that, if $t-i \notin \{-3,-1,-m\}$, then $m-i+t \notin \{0, m-1\}$. To compute $f(i,k,t)$, we must consider four cases.

Case 1: $i = 2j$ where $1 \leqslant j \leqslant k-2$ and t is odd. Recall that $m = 2k+1$. Therefore, we see that

$$m-i+t = m-2j+t = 2k+1-2j+t = 2\left(k-j+\frac{t+1}{2}\right).$$

Let $j_1 = k-j+\frac{t+1}{2}$. Next, we aim to show that $1 \leqslant j_1 \leqslant k-2$. We do so by way of contradiction. Suppose that $j_1 \geqslant k-1$. It follows that

$$k-j+\frac{t+1}{2} \geqslant k-1 \quad\Rightarrow\quad -2j+t+1 \geqslant -2 \quad\Rightarrow\quad -i+t+3 \geqslant 0 \quad\Rightarrow\quad t+3 \geqslant i.$$

Since $t+3 \geqslant i$, $i > t$, and $t-i$ is odd, then $t-i \in \{-1,-3\}$. This contradicts our hypothesis. Consequently $j_1 \leqslant k-2$.

We now show that $j_1 \geqslant 1$. Note that $k-j \geqslant 2$ and $-1 \leqslant \frac{t+1}{2} \leqslant 1$. This means that $k-j+\frac{t+1}{2} = j_1 \geqslant 1$.

In summary, we see that $m-i+t = 2j_1$ and $1 \leqslant j_1 \leqslant k-2$. Construction 4.75 implies that $\sigma_{m-i+t} = \gamma_{-i+t+1}(m-1, k+j_1+1)$. Consequently, we see that

$$f(i,k,t) = k+k-j+\frac{t+1}{2}+1 = 2k+1-j+\frac{t+1}{2} = m-j+\frac{t+1}{2}.$$

Case 2: $i = 2j$ where $1 \leqslant j \leqslant k-2$ and t is even. Then

$$m-i+t = m-2j+t = 2k+1-2j+t = 2\left(k-j+\frac{t}{2}\right)+1.$$

Let $j_1 = k-j+\frac{t}{2}$. We now aim to show that $0 \leqslant j_1 \leqslant k-1$. Suppose that $j_1 \geqslant k$. It follows that

$$k-j+\frac{t}{2} \geqslant k \quad\Rightarrow\quad -2j+t \geqslant 0 \quad\Rightarrow\quad t \geqslant i.$$

Since $i > t$ by hypothesis, we have a contradiction. Hence $j_1 \leqslant k-1$.

Furthermore, we note that $k-j \geqslant 2$ and that $-2 \leq \frac{t}{2} \leq 1$. As a result, we see that $j_1 \geqslant 0$.

In summary, we have $0 \leqslant j_1 \leqslant k-1$. Since $m-i+t = 2j_1+1$ and $0 \leqslant j_1 \leqslant k-1$, Construction 4.75 then implies that $\sigma_{m-i+t} = \gamma_{-i+t+1}(m-1, j_1+2)$. This means that

$$f(i,k,t) = k - j + \frac{t}{2} + 2.$$

Case 3: $i = 2j + 1$ where $0 \leqslant j \leqslant k-1$ and t is odd, and $j \neq k-1$ or $t \neq -3$. We see that

$$m - i + t = 2k + 1 - 2j - 1 + t = 2k - 2j + t = 2\left(k - j + \frac{t-1}{2}\right) + 1.$$

Let $j_1 = k - j + \frac{t-1}{2}$. Now, we aim to show that $0 \leqslant j_1 \leqslant k-1$. Suppose that $j_1 \geqslant k$. It follows that

$$k - j + \frac{t-1}{2} \geqslant k \quad \Rightarrow \quad -2j + t - 1 \geqslant 0 \quad \Rightarrow \quad t \geqslant i.$$

This contradicts our hypothesis. Therefore, $j_1 \leqslant k-1$.

Next, we show that $j_1 \geqslant 0$. First, we note that $k - j \geqslant 1$ and that $-2 \leq \frac{t-1}{2} \leq 0$. If $j_1 < 0$, then $\frac{t-1}{2} = -2$ and $k - j = 1$. In that case, we have $t = -3$ and $j = k-1$. This contradicts our hypothesis. Hence $j_1 \geqslant 0$.

In summary, it follows that $m - i + t = 2j_1 + 1$ and $0 \leqslant j_1 \leqslant k-1$, Construction 4.75 implies that $\sigma_{m-i+t} = \gamma_{-i+t+1}(m-1, j_1+2)$. This means that

$$f(i,k,t) = k - j + \frac{t-1}{2} + 2 = k - j + \frac{t+3}{2}.$$

Case 4: $i = 2j + 1$ where $0 \leqslant j \leqslant k-1$ and t is even, and $j \neq k-1$ or $t \neq -4$. This means that

$$m - i + t = m - (2j+1) + t = 2k + 1 - 2j - 1 + t = 2k - 2j + t = 2\left(k - j + \frac{t}{2}\right).$$

Assume $j_1 = k - j + \frac{t}{2}$. Now, we aim to show that $j_1 \leqslant k-2$. We do so by way of contradiction. Suppose that $j_1 \geqslant k-1$. It follows that

$$k - j + \frac{t}{2} \geqslant k - 1 \quad \Rightarrow \quad -2j + t \geqslant -2 \quad \Rightarrow \quad t + 3 \geqslant i.$$

If $t + 3 \geqslant i$ and $i > t$, then $t - i \in \{-1, -3\}$ since $t - i$ is odd. This contradicts our hypothesis. Hence $j_1 \leqslant k-2$.

Next, we aim to show that $j_1 \geqslant 0$. Note that $k - j \geqslant 1$ and that $-2 \leqslant \frac{t}{2} \leqslant 1$. If $j_1 < 0$, then $\frac{t}{2} = -2$ and $k - j = 1$, so $t = -4$ and $j = k-1$. This also contradicts our hypothesis, which means that $j_1 \geqslant 0$.

In summary, we have $m - i + t = 2j_1$ and $0 \leqslant j_1 \leqslant k - 2$, Construction 4.75 implies that $\sigma_{m-i+t} = \gamma_{-i+t+1}\,(m-1, k + j_1 + 1)$. This means that

$$f(i,k,t) = k + j_1 + 1 = k + k - j + \frac{t}{2} + 1 = m - j + \frac{t}{2}.$$

∎

Next, we illustrate the computations performed in the proof of Lemma 4.78 with the following example.

Example 4.79. We consider the regular permutation set $\mathcal{F}_{15}$ given in Example 4.76. Observe that $m = 2 \cdot 7 + 1$.

Let $i = 3$. Then, we see that $i = 2j + 1$ for $j = 1$. Next, we make the following computations using Lemma 4.78:

$$\sigma_{m-i+1} = \gamma_{-3+2}\left(14, 7 - 1 + \frac{1+3}{2}\right) = \gamma_{13}\,(14, 8) = \sigma_{13};$$

$$\sigma_{m-i-2} = \gamma_{-3-1}\left(14, 15 - 1 + \frac{-2}{2}\right) = \gamma_{10}\,(14, 13) = \sigma_{10}.$$

Next, let $i = 8$ so that $i = 2j$ where $j = 4$:

$$\sigma_{m-i+1} = \gamma_{-8+2}\left(14, 15 - 4 + \frac{1+1}{2}\right) = \gamma_{8}\,(14, 12) = \sigma_{8};$$

$$\sigma_{m-i-2} = \gamma_{-8+(-1)}\left(14, 7 - 4 + \frac{-2}{2} + 2\right) = \gamma_{5}\,(14, 4) = \sigma_{5}.$$

□

We now construct a c-twined 2-factorization of $\vec{C}_2 \wr \overline{K}_m$ for m is odd, $m \geqslant 5$, and c is even. Our construction is partitioned into two cases, one for each congruency class of m modulo 4.

The case $m \equiv 1 \pmod 4$. We now proceed with the proof of Proposition 4.80, in which we construct a c-twined 2-factorization of $\vec{C}_2 \wr \overline{K}_m$ for c is even and $m \equiv 1 \pmod 4$ using two regular permutation sets S_1 and S_2 such that $S_1 = S_2 = \mathcal{F}_m$. In this proof, we consider two cases. In the first case, we construct a set comprised of two truncated hamiltonian pairs and $m-2$ hamiltonian pairs. Then, in the second case, we take $m - c$ hamiltonian pairs from Case 1 and show that the remaining permutations of S_1 and S_2 can be used to form $2 \leqslant c \leqslant m-3$ truncated hamiltonian pairs. We thus obtain the desired c-twined 2-factorization of $\vec{C}_2 \wr \overline{K}_m$.

Proposition 4.80. *Let $m \equiv 1 \pmod 4$ such that $m \geqslant 5$, and c be an even integer such that $2 \leq c \leq m-3$. The digraph $\vec{C}_2 \wr \overline{K}_m$ admits a c-twined 2-factorization.*

Proof: Let $m = 2k+1$ where k is even. In this proof, it is understood that, for each $i \in \mathbb{Z}_{m-1}$, the permutation $\gamma_i \in G_{m-1}$ is as defined in Definition 4.73. We will construct a c-twined 2-factorization of $\vec{C}_2 \wr \overline{K}_m$ by using the following two sets of m permutations:

$$S_1 = S_2 = \mathcal{F}_m = \{\sigma_0, \sigma_1, \ldots, \sigma_{m-1}\}.$$

Observe that both sets are regular permutation sets by Lemma 4.77. Refer to Construction 4.75 for the definition of each $\sigma_i \in \mathcal{F}_m$. In each case, the m pairs constructed are elements of $S_1 \times S_2$. See Figure 4.29 for the hamiltonian table of $S_1 \times S_2$ when $m = 13$. We point out that σ_0 is the $(m-1)$-stabilizer of S_1 and S_2. Otherwise, if $i \notin \{0, m-1\}$, then $\hat{\sigma}_i = \gamma_i$ where γ_i is defined in Definition 4.73.

Case 1: $c = 2$. We form $m-2$ hamiltonian pairs and two truncated hamiltonian pairs. We begin by forming the set

$$D_T = \{(\sigma_{m-2}, \sigma_2), (\sigma_{m-3}, \sigma_3)\}.$$

We then show that (σ_{m-2}, σ_2) and (σ_{m-3}, σ_3) are truncated hamiltonian pairs. By Lemma 4.74, we have $\gamma_{m-2} = \gamma_{-1}$ and $\gamma_{m-3} = \gamma_{-2}$. Therefore, we have

$$\hat{\sigma}_{m-2}\hat{\sigma}_2 = \gamma_{-1}\gamma_2 = \gamma_1;$$

$$\hat{\sigma}_{m-3}\hat{\sigma}_3 = \gamma_{-2}\gamma_3 = \gamma_1.$$

Since $T(\gamma_1) = 2$ and $T(\gamma_{-1}) = 2$, the pairs (σ_{m-2}, σ_2) and (σ_{m-3}, σ_3) are truncated hamiltonian pairs.

In our next step, we form a set D_{H_1} of $\frac{m-5}{2} + 2$ pairs, and show that each of these pairs is a hamiltonian pair:

$$D_{H_1} = \{(\sigma_i, \sigma_{m-i+1}) \mid i \equiv 0, 3 \pmod 4 \text{ and } 3 \leqslant i \leqslant m-5\} \cup \{(\sigma_0, \sigma_{m-1}), (\sigma_{m-1}, \sigma_0)\}.$$

The sets D_{H_1} and D_T jointly use each $\sigma_i \in S_1$ such that $i \equiv 0, 3 \pmod 4$ precisely once as the first component of a pair. Additionally, the sets D_{H_1} and D_T jointly use each $\sigma_r \in S_2$ such that $r \equiv 2, 3 \pmod 4$ exactly once as the second component of a pair. Below, we verify that each pair of D_{H_1} is a hamiltonian pair.

Recall that $\sigma_0 = id$. Since σ_{m-1} is a permutation with a single cycle by Construction 4.75, we see that (σ_0, σ_{m-1}) and (σ_{m-1}, σ_0) are hamiltonian pairs.

We now show that $(\sigma_i, \sigma_{m-i+1})$ is a hamiltonian pair when $m \geqslant 9$ and $i = 4$. This means that $k \geqslant 4$:

$$\begin{aligned}\sigma_4\sigma_{m-3} &= \gamma_4\,(m-1, k+3)\,\gamma_{-2}\,(m-1, 0)\\ &= (0, 2, 4, \ldots, m-3, m-1, k+1, k+3, \ldots, m-2, 1, 3, \ldots, k-1).\end{aligned}$$

Next, we show that pairs of the form $(\sigma_i, \sigma_{m-i+1})$ are hamiltonian pairs for all $i \equiv 0 \pmod 4$ and $8 \leqslant i \leqslant m-5$. Observe that $i = 2j$ where j is even and $4 \leqslant j \leqslant k-2$. Note that, since i is even, Lemma 4.78 implies that:

$$\sigma_{m-i+1} = \gamma_{-i+2}\,(m-1, m-j+1).$$

Lastly, we remind the reader that k is even. This means that $k+j+1$ is odd and $m-j-1$ is even. These observations jointly imply that

$$\begin{aligned}\sigma_i\sigma_{m-i+1} &= \gamma_i\,(m-1, k+j+1)\,\gamma_{-i+2}\,(m-1, m-j+1)\\ &= (0, 2, 4, \ldots, m-j-1, m-1, k-j+3, k-j+5, \ldots, m-2, 1, 3, \ldots,\\ &\qquad k-j+1, m-j+1, m-j+3, \ldots, m-3).\end{aligned}$$

We proceed with the case $i \equiv 3 \pmod 4$. First, we verify that the pairs (σ_3, σ_{m-2}) and (σ_7, σ_{m-6}) are hamiltonian pairs:

$$\begin{aligned}\sigma_3\sigma_{m-2} &= \gamma_3\,(m-1, 3)\,\gamma_{-1}\,(m-1, k+1)\\ &= (0, k+1, k+3, \ldots, m-2, 1, 3, 5, \ldots, k-1, m-1, 2, 4, 6, \ldots, m-3);\\ \sigma_7\sigma_{m-6} &= \gamma_7\,(m-1, 5)\,\gamma_{-5}\,(m-1, k-1)\\ &= (0, 2, 4, \ldots, m-3, k-1, k+1, k+3, \ldots, m-2, 1, 3, \ldots, k-3, m-1).\end{aligned}$$

We now consider the case $i \equiv 3 \pmod 4$ where $i = 2j+1$ and $5 \leqslant j \leqslant k-2$ and j is odd. Note that, since i is odd, Lemma 4.78 implies that

$$\sigma_{m-i+1} = \gamma_{-i+2}\,(m-1, k-j+2).$$

Since $i = 2j+1$, Lemma 4.74 implies that $(m-j)^{\gamma_i} = j+2$ and that $(j+2)^{\gamma_{-i+2}} = m-j+2$. Then, we see that

$$\begin{aligned}\sigma_i\sigma_{m-i+1} &= \gamma_i\,(m-1, j+2)\,\gamma_{-i+2}\,(m-1, k-j+2)\\ &= (0, 2, 4, \ldots, m-j, k-j+2, k-j+4, k-j+6, \ldots, m-2, 1, 3, \ldots,\\ &\qquad k-j, m-1, m-j+2, m-j+4, \ldots, m-3).\end{aligned}$$

Therefore, the set D_{H_1} is a set of hamiltonian pairs.

Next, we form a set containing $\frac{m-3}{2}$ pairs:

$$D_{H_2} = \{(\sigma_i, \sigma_{m-i-3}) \mid i \equiv 1, 2 \pmod 4 \text{ and } 1 \leqslant i \leqslant m-4\}.$$

Observe that D_{H_2} and D_T jointly contain all $\sigma_i \in S_1$ such that $i \equiv 1, 2 \pmod 4$ as the first component of a pair. Similarly, the sets D_{H_2} and D_T contain all $\sigma_r \in S_2$ such that $r \equiv 0, 1 \pmod 4$ as the second component of a pair. Next, we show that each pair of D_{H_2} is hamiltonian.

If $i \equiv 1 \pmod 4$, then $i = 2j+1$ where j is even and $0 \leqslant j \leqslant k-2$. Since i is odd, and $i \neq m-2$, Lemma 4.78 implies that

$$\sigma_{m-i-3} = \gamma_{-i-2}\,(m-1, k-j).$$

Furthermore, recall that Lemma 4.74 implies that $(j+2)^{\gamma_{-i-2}} = m-j-2$ and $(m-j)^{\gamma_i} = j+2$. We then see that

$$\begin{aligned}
\sigma_i\sigma_{m-i-3} &= \gamma_i\,(m-1, j+2)\,\gamma_{-i-2}\,(m-1, k-j) \\
&= (0, m-3, m-5, \ldots, k-j+2, m-1, m-j-2, m-j-4, \ldots, 3, 1, \\
&\qquad m-2, m-4, \ldots, m-j, k-j, k-j-2, k-j-4, \ldots, 4, 2).
\end{aligned}$$

If $i \equiv 2 \pmod 4$, then $i = 2j$ where j is odd and $1 \leqslant j \leqslant k-2$. Since i is even, Lemma 4.78 implies that

$$\sigma_{m-i-3} = \gamma_{-i-2}\,(m-1, m-j-1).$$

Recall that k is even. Since j is odd, this means that $k-j+1$ is even and $m-j-1$ is odd. We then see that

$$\begin{aligned}
\sigma_i\sigma_{m-i-3} &= \gamma_i\,(m-1, k+j+1)\,\gamma_{-i-2}\,(m-1, m-j-1) \\
&= (0, m-3, m-5, m-7, \ldots, k-j+1, m-j-1, m-j-3, \ldots, 3, \\
&\qquad 1, m-2, m-4, \ldots, m-j+1, m-1, k-j-1, k-j-3, \ldots, 4, 2).
\end{aligned}$$

In conclusion, the set D_{H_2} is a set of hamiltonian pairs.

Observe that each permutation in S_1 and S_2 appears precisely once in the set $D = D_T \cup D_{H_1} \cup D_{H_2}$ as the first and second component, respectively, in a pair. Therefore the set D is a 2-factorization of $\vec{C}_2 \wr \overline{K}_m$. Refer to Figure 4.29 for the hamiltonian table of $S_1 \times S_2$ and an illustration of D when $m = 13$. We also note that D_T contains two truncated hamiltonian pairs and $D_H = D_{H_1} \cup D_{H_2}$ contains $m-2$ hamiltonian pairs. Observe that D_T and D_H are disjoint. Therefore, the set D is a 2-twined 2-factorization of $\vec{C}_2 \wr \overline{K}_m$.

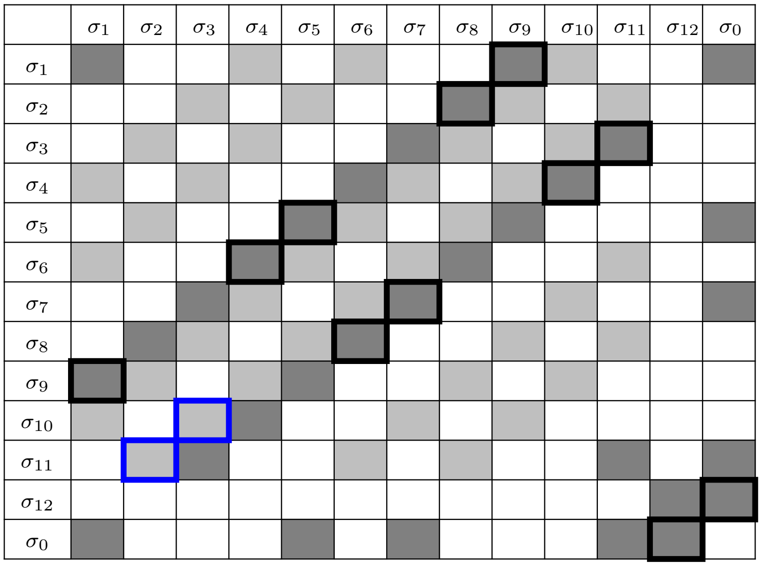


Figure 4.29: Case 1: hamiltonian table of $S_1 \times S_2$ for $m = 13$ with a 2-twined 2-factorization in black (hamiltonian pairs) and blue (truncated hamiltonian pairs).

Case 2: $4 \leqslant c \leqslant m-3$. Let $c = 2+2t$, where $1 \leqslant t \leqslant \frac{m-5}{2}$, and let $\mathbb{I} = \{i \mid i \equiv 1, 2 \pmod 4), 1 \leqslant i \leqslant m-5\}$. Moreover, let M_t be a subset of order t of $\mathbb{I}$. Note that $\mathbb{I}$ contains $\frac{m-5}{2}$ elements.

We proceed by constructing the following set:

$$D'_{T_1} = \{(\sigma_i, \sigma_{m-i-2}), (\sigma_{i+3}, \sigma_{m-i-3}) \mid i \equiv 1 \pmod 4) \text{ and } i \in M_t\} \cup \{(\sigma_{m-3}, \sigma_3)\}.$$

See Case 1 for proof that (σ_{m-3}, σ_3) is a truncated hamiltonian pair.

Below, we verify that all other pairs of D'_{T_1} are truncated hamiltonian pairs. Construction 4.75 implies that $\hat{\sigma}_i = \gamma_i$ for all $i \in \{0, 1, \ldots, m-2\}$. Additionally, Lemma 4.74 implies that $\gamma_{m-i-2} = \gamma_{-i-1}$ and $\gamma_{m-i-3} = \gamma_{-i-2}$. Therefore, we have that

$$\begin{aligned} \hat{\sigma}_i \hat{\sigma}_{m-i-2} &= \gamma_i \ \gamma_{-i-1} &&= \gamma_{-1}; \\ \hat{\sigma}_{i+3} \hat{\sigma}_{m-i-3} &= \gamma_{i+3} \ \gamma_{-i-2} &&= \gamma_1. \end{aligned}$$

Since $T(\gamma_1) = T(\gamma_{-1}) = 2$, all pairs in D'_{T_1} are truncated hamiltonian.

Then, we form the set

$$\begin{aligned} D'_{H_1} &= \{(\sigma_i, \sigma_{m-i-3}), (\sigma_{i+3}, \sigma_{m-i-2}) \mid i \equiv 1 \pmod 4 \text{ and } i \in \mathbb{I}/M_t\} \cup \\ &\quad \{(\sigma_0, \sigma_{m-1}), (\sigma_{m-1}, \sigma_0), (\sigma_{m-4}, \sigma_1)\}. \end{aligned}$$

Let D_H be the set of hamiltonian pairs constructed in Case 1. We see that $(\sigma_{m-4}, \sigma_1) \in D_H$. In addition, if $s = i + 3$, then $(\sigma_{i+3}, \sigma_{m-i-2}) = (\sigma_s, \sigma_{m-s+1})$. This means that $D'_{H_1} \subseteq D_H$. Therefore, all pairs of D'_{H_1} are hamiltonian pairs. Next, we form the set:

$$D_{c_1} = D'_{T_1} \cup D'_{H_1}.$$

Observe that D_{c_1} contains all $\sigma_i \in S_1$ such that $i \equiv 0, 1 \pmod 4$ as the first permutation in a pair and all $\sigma_r \in S_2$ such that $r \equiv 1, 2 \pmod 4$, except for σ_2, as the second permutation in a pair.

Next, we construct the following set:

$$D'_{T_2} = \{(\sigma_i, \sigma_{m-i}), (\sigma_{i+1}, \sigma_{m-i-3}) \mid i \equiv 2 \pmod 4 \text{ and } i \in M_t\} \cup \{(\sigma_{m-2}, \sigma_2)\}.$$

See Case 1 for a proof that (σ_{m-2}, σ_2) is a truncated hamiltonian pair. We now show that all other pairs of D'_{T_2} are truncated hamiltonian. Lemma 4.74 and Construction 4.75 imply that

$$\begin{aligned} \hat{\sigma}_i \hat{\sigma}_{m-i} &= \gamma_i \ \gamma_{-i+1} &&= \gamma_1; \\ \hat{\sigma}_{i+1} \hat{\sigma}_{m-i-3} &= \gamma_{i+1} \ \gamma_{-i-2} &&= \gamma_{-1}. \end{aligned}$$

Since $T(\gamma_1) = T(\gamma_{-1}) = 2$, all pairs in D'_{T_2} are truncated hamiltonian. .

Then, we form the set

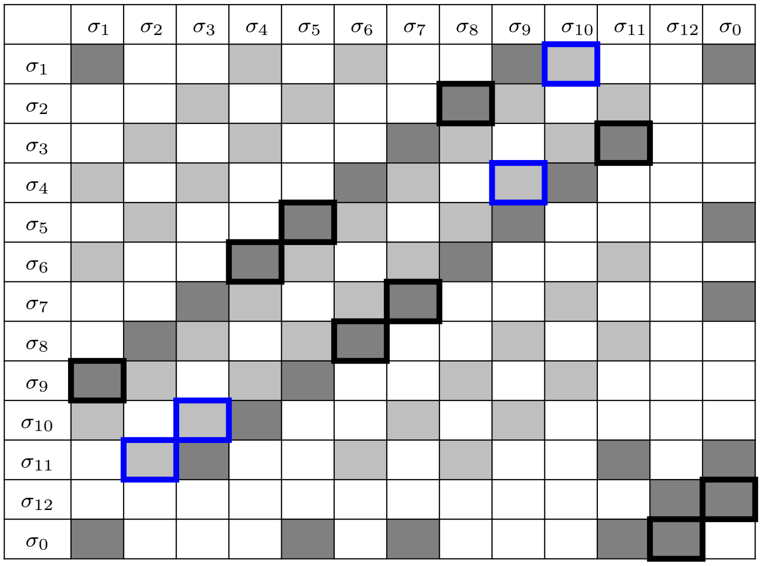
$$D'_{H_2} = \{(\sigma_i, \sigma_{m-i-3}), (\sigma_{i+1}, \sigma_{m-i}) \mid i \equiv 2 \pmod 4 \text{ and } i \in \mathbb{I} \backslash M_t\}.$$

Observe that $D'_{H_2} \subseteq D_H$. It follows that all pairs of D'_{H_2} are hamiltonian pairs. Next, we form the set:

$$D_{c_2} = D'_{T_2} \cup D'_{H_2}.$$

We point out that D_{c_2} contains all $\sigma_i \in S_1$ such that $i \equiv 2, 3 \pmod 4$, except for σ_{m-3}, as the first permutation of a pair. Similarly, the set D_{c_2} contains all $\sigma_r \in S_2$ such that $r \equiv 0, 3 \pmod 4$, except for σ_3, as the second permutation of a pair. Note that $\sigma_2 \in S_2$ appears in D_{c_2} as the second permutation in a pair while $\sigma_{m-3} \in S_1$ and $\sigma_3 \in S_2$ appear in D_{c_1} as the first and second permutation in a pair, respectively.

Let $D_c = D_{c_1} \cup D_{c_2}$. Observe that each permutation of S_1 and S_2 appears exactly once in D_c as the first and the second permutation of a pair, respectively. This means that D_c is a 2-factorization of $\vec{C}_2 \wr \overline{K}_m$. Refer to Figures 4.30 and 4.31 for an illustration of D_c when $m = 13$, and $M_t = \{1\}$ and $M_t = \{1, 2, 5, 6\}$, respectively.

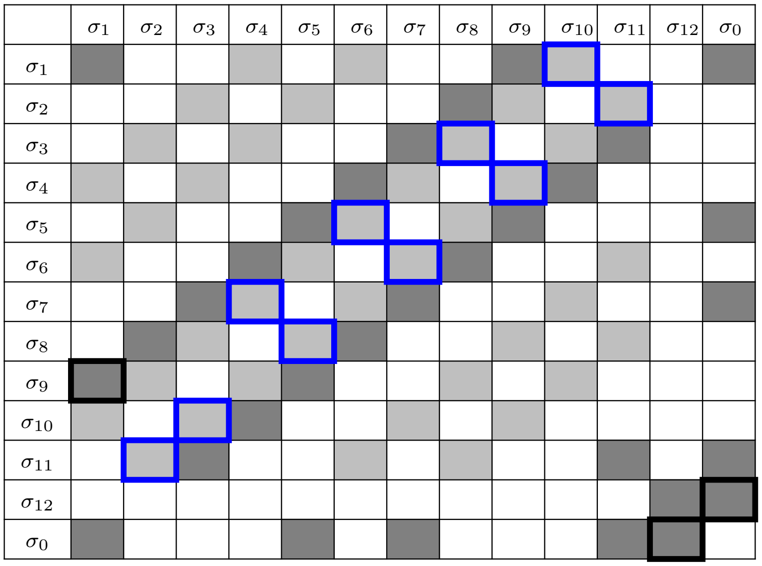


	σ_1	σ_2	σ_3	σ_4	σ_5	σ_6	σ_7	σ_8	σ_9	σ_{10}	σ_{11}	σ_{12}	σ_0
σ_1	▓			░		░			▓	[░]			▓
σ_2			░		░			[▓]	░		░		
σ_3		░		░			▓	░		░	[▓]		
σ_4	░		░			▓	░		[░]	▓			
σ_5		░			[▓]	░		░	▓				▓
σ_6	░			[▓]	░		░	▓			░		
σ_7			▓	░		░	[▓]			░			▓
σ_8		▓	░		░	[▓]			░		░		
σ_9	[▓]	░		░	▓			░		░			
σ_{10}	░		[░]	▓			░		░				
σ_{11}		[░]	▓			░		░			▓		▓
σ_{12}												▓	[▓]
σ_0	▓				▓		▓				▓	[▓]	

Figure 4.30: Case 2: hamiltonian table of $S_1 \times S_2$ for $m = 13$ with a 4-twined 2-factorization in black (hamiltonian pairs) and blue (truncated hamiltonian pairs).

The set $D'_T = D'_{T_1} \cup D'_{T_2}$ contains $2t + 2$ truncated hamiltonian pairs; the set $D'_H = D'_{H_1} \cup D'_{H_2}$ contains $m - 2t - 2$ hamiltonian pairs. In addition, we see that D'_H and D'_T are disjoint and thus partition D_c. Consequently, the set D_c is a c-twined 2-factorization of $\vec{C}_2 \wr \overline{K}_m$. ▮

The case $m \equiv 3 \pmod 4$. This case is partitioned into two subcases: $m \equiv 7$ or $11 \pmod{12}$, and $m \equiv 3 \pmod{12}$. In the former case, our construction depends on the fact that m is relatively prime with 3 while in the latter case, our construction relies on the fact that $m - 1$ is relatively prime with 3. Therefore, two distinct constructions are needed. Below, we start with the case $m \equiv 7$ or $11 \pmod{12}$.



	σ_1	σ_2	σ_3	σ_4	σ_5	σ_6	σ_7	σ_8	σ_9	σ_{10}	σ_{11}	σ_{12}	σ_0
σ_1	■			▒		▒			■	▒			■
σ_2			▒		▒			■	▒		▒		
σ_3		▒		▒			■	▒		▒	■		
σ_4	▒		▒			■	▒		▒	■			
σ_5		▒			■	▒		▒	■				■
σ_6	▒			■	▒		▒	■			▒		
σ_7			■	▒		▒	■			▒			■
σ_8		■	▒		▒	■			▒		▒		
σ_9	■	▒		▒	■			▒		▒			
σ_{10}	▒		▒	■			▒		▒				
σ_{11}		▒	■			▒		▒			■		■
σ_{12}												■	■
σ_0	■				■		■				■	■	

Figure 4.31: Case 2: hamiltonian table of $S_1 \times S_2$ for $m = 13$ with a 10-twined 2-factorization in black (hamiltonian pairs) and blue (truncated hamiltonian pairs).

Proposition 4.81. *Let* $m \equiv 7$ *or* $11 \pmod{12}$ *such that* $m \geqslant 7$*, and let* c *be an even integer such that* $0 \leq c \leq m-3$. *The digraph* $\vec{C}_2 \wr \overline{K}_m$ *admits a* c*-twined 2-factorization.*

Proof: Let $m = 2k+1$ where k is odd. In this proof, it is understood that, for each $i \in \mathbb{Z}_{m-1}$, the permutation $\gamma_i \in G_{m-1}$ is defined as in Definition 4.73. We will construct the desired set of pairs by using the following two sets of permutations:

$$\begin{aligned} S_1 &= \gamma_1 \cdot \mathcal{F}_m = \{\mu_i = \gamma_1\sigma_i \mid i = 0, 1, \ldots, m-1\}; \\ S_2 &= \gamma_{-1} \cdot \mathcal{F}_m = \{\tau_i = \gamma_{-1}\sigma_i \mid i = 0, 1, \ldots, m-1\}. \end{aligned}$$

Note that, by Lemmas 4.60 and 4.77, these are regular permutation sets. Refer to Construction 4.75 for the definition of each $\sigma_i \in \mathcal{F}_m$. In addition, see Figure 4.32 for the hamiltonian table of $S_1 \times S_2$ when $m = 19$. Observe that $\mu_0 = \gamma_1$ and $\tau_0 = \gamma_{-1}$, and that μ_0 and τ_0 are the $(m-1)$-stabilizers of S_1 and S_2, respectively.

Case 1: $c = 0$. We start by forming the following set of three pairs:

$$D_{H_s} = \{(\mu_0, \tau_1), (\mu_{m-2}, \tau_0), (\mu_{m-1}, \tau_{m-1})\}.$$

Below, we show that (μ_0, τ_1) and (μ_{m-2}, τ_0) are hamiltonian pairs:

$$\begin{aligned}
\mu_0\,\tau_1 &= \gamma_1\,\gamma_{-1}\,\gamma_1\,(m-1,2)\\
&= \gamma_1\,(m-1,2)\\
&= (0,1,m-1,2,3,4,\ldots,m-3,m-2);\\
\mu_{m-2}\,\tau_0 &= \gamma_1\,\gamma_{-1}\,(m-1,k+1)\,\gamma_{-1}\\
&= (m-1,k+1)\,\gamma_{-1}\\
&= (0,m-2,m-3,m-4,\ldots,k+1,m-1,k,k-1,k-2,k-3,\ldots,1).
\end{aligned}$$

Next, we show that (μ_{m-1},τ_{m-1}) is a hamiltonian pair. We rely on the fact that m is relatively prime with 3. We consider two subcases because the resulting product depends on the congruency class of m modulo 12.

If $m \equiv 7 \pmod{12}$, then $k \equiv 0 \pmod 3$ and thus

$$\begin{aligned}
\mu_{m-1}\,\tau_{m-1} &= \gamma_1\,(0,3,m-2,4,m-3,5,m-4,\ldots,k+3,k,k+2,k+1,\\
&\quad m-1,1,2)\gamma_{-1}\,(0,3,m-2,4,m-3,5,m-4,\ldots,k+3,k,k+2,\\
&\quad k+1,m-1,1,2)\\
&= (0,2,5,8,11,\ldots,k-1,m-1,3,6,9,\ldots,k,1,4,7,10,\ldots,k+1,\\
&\quad k+2,k+3,k+4,\ldots,m-2).
\end{aligned}$$

If $m \equiv 11 \pmod{12}$, then $k \equiv 2 \pmod 3$ and thus

$$\begin{aligned}
\mu_{m-1}\,\tau_{m-1} &= \gamma_1\,(0,3,m-2,4,m-3,5,m-4,\ldots,k+3,k,k+2,k+1,\\
&\quad m-1,1,2)\gamma_{-1}\,(0,3,m-2,4,m-3,5,m-4,\ldots,k+3,k,k+2,\\
&\quad k+1,m-1,1,2)\\
&= (0,2,5,8,\ldots,k,1,4,7,\ldots,k-1,m-1,3,6,9,\ldots,k+1,\\
&\quad k+2,k+3,k+4,\ldots,m-2).
\end{aligned}$$

Next, we form the following set of $\frac{m-3}{2}$ pairs:

$$D_{H_1} = \{(\mu_i,\tau_{m-i+1}) \mid i \equiv 0,3 \pmod 4 \text{ and } 3 \leqslant i \leqslant m-3\}.$$

Observe that D_{H_1} and D_{H_s} jointly use all $\mu_i \in S_1$ such that $i \equiv 0,3 \pmod 4$ exactly once and that D_{H_1} and D_{H_s} jointly use all $\tau_r \in S_2$ such that $r \equiv 0,1 \pmod 4$ precisely once. Next, we demonstrate that D_{H_1} is a set of hamiltonian pairs.
If $m = 7$ and $i = 4$, then

$$\begin{aligned}
\mu_4\,\tau_4 &= \gamma_1\,\gamma_4\,(6,0)\,\gamma_{-1}\,\gamma_4\,(6,0)\\
&= (0,2,4,6,3,5,1).
\end{aligned}$$

If $m \geqslant 11$ and $i = 4$, then

$$\begin{aligned}\mu_4\,\tau_{m-3} &= \gamma_1\,\gamma_4\,(m-1,k+3)\,\gamma_{-1}\,\gamma_{-2}\,(m-1,0)\\ &= (0,2,4,\ldots,m-3,m-1,k,k+2,k+4,\ldots,m-2,\\ &\quad 1,3,\ldots,k-2).\end{aligned}$$

If $i \equiv 0 \pmod 4$ and $i \notin \{0,4\}$, then $i = 2j$ where j is even and $4 \leqslant j \leqslant k-1$. Since i is even, Lemma 4.78 implies that

$$\tau_{m-i+1} = \gamma_{-1}\,\sigma_{m-i+1} = \gamma_{-1}\;\gamma_{-i+2}\,(m-1,m-j+1) = \gamma_{-i+1}\,(m-1,m-j+1).$$

Note that, since k is odd and j is even, it follows that $m-j-1$ is even and $k-j$ is odd. We then see that, if $i \neq m-3$, we have

$$\begin{aligned}\mu_i\,\tau_{m-i+1} &= \gamma_1\,\gamma_i\,(m-1,k+j+1)\,\gamma_{-i+1}\,(m-1,m-j+1)\\ &= (0,2,4,\ldots,m-j-1,m-1,k-j+2,k-j+4,k-j+6,\ldots,\\ &\quad m-2,1,3,\ldots,k-j,m-j+1,m-j+3,\ldots,m-3).\end{aligned}$$

If $i = m-3$, then

$$\begin{aligned}\mu_{m-3}\,\tau_4 &= \gamma_1\,\gamma_{-2}\,(m-1,0)\,\gamma_3\,(m-1,k+3)\\ &= (0,2,4,\ldots,k+1,m-1,3,5,7,\ldots,m-2,1,k+3,k+5,\ldots,m-3).\end{aligned}$$

If $i = 3$, then

$$\begin{aligned}\mu_3\,\tau_{m-2} &= \gamma_1\,\gamma_3\,(m-1,3)\,\gamma_{-1}\,\gamma_{-1}\,(m-1,k+1)\\ &= (0,2,4,\ldots,k-1,m-1,1,3,5,7,\ldots,m-2,k+1,k+3,\ldots,m-3).\end{aligned}$$

If $i \equiv 3 \pmod 4$ and $i \neq 3$, then $i = 2j+1$, where j is odd, and $2 \leqslant j \leqslant k-2$. Since i is odd, Lemma 4.78 implies that

$$\tau_{m-i+1} = \gamma_{-1}\,\sigma_{m-i+1} = \gamma_{-1}\;\gamma_{-i+2}\,(m-1,k-j+2) = \gamma_{-i+1}\,(m-1,k-j+2).$$

Note that Lemma 4.74 implies that, since $i = 2j+1$, we have $(m-j-1)^{\gamma_{i+1}} = j+2$ and $(j+2)^{\gamma_{-i+1}} = m-j+1$. Lastly, we note that $k-j$ is even and that $m-j+1$ is odd. We then see that

$$\begin{aligned}\mu_i\,\tau_{m-i+1} &= \gamma_1\,\gamma_i\,(m-1,j+2)\,\gamma_{-i+1}\,(m-1,k-j+2)\\ &= (0,2,4,\ldots,k-j,m-1,m-j+1,m-j+3,\ldots,m-2,1,3,5,\\ &\quad \ldots,m-j-1,k-j+2,k-j+4,\ldots,m-3).\end{aligned}$$

In conclusion, all pairs in D_{H_1} are hamiltonian pairs.

Next, we form a set of $\frac{m-3}{2}$ pairs as follows:

$$D_{H_2} = \{(\mu_i, \tau_{m-i-3}) \mid i \equiv 1, 2 \pmod 4 \text{ and } 1 \leqslant i \leqslant m-5\}.$$

Observe that D_{H_2} and D_{H_s} jointly use all $\mu_i \in S_1$ such that $i \equiv 1, 2 \pmod 4$ exactly once and that D_{H_2} and D_{H_s} jointly use all $\tau_r \in S_2$ such that $r \equiv 2, 3 \pmod 4$ precisely once. Now, we demonstrate that D_{H_2} is a set of hamiltonian pairs.

If $i \equiv 1 \pmod 4$ and $i \neq m-2$, then $i = 2j+1$ where j is even and $0 \leqslant j \leqslant k-2$. Since i is odd and $i \neq m-2$, Lemma 4.78 implies that

$$\tau_{m-i-3} = \gamma_{-1}\,\sigma_{m-i-3} = \gamma_{-1}\ \gamma_{-i-2}\,(m-1, k-j) = \gamma_{-i-3}\,(m-1, k-j).$$

Note that, since $i = 2j+1$, Lemma 4.74 implies that $(m-j-1)^{\gamma_{i+1}} = j+2$ and that $(j+2)^{\gamma_{-i-3}} = m-j-3$. In addition, we point out that $k-j$ is odd and $m-j-1$ is even. We then see that

$$\begin{aligned}\mu_i\,\tau_{m-i-3} &= \gamma_1\,\gamma_i\,(m-1, j+2)\,\gamma_{-i-3}\,(m-1, k-j)\\ &= (0, m-3, m-5, \ldots, m-j-1, k-j, k-j-2, \ldots, 1, m-2, m-4,\\ &\qquad \ldots, k-j+2, m-1, m-j-3, m-j-5, m-j-7, \ldots, 4, 2).\end{aligned}$$

If $i \equiv 2 \pmod 4$ then $i = 2j$, where j is odd and $1 \leqslant j \leqslant k-2$. Since i is even, Lemma 4.78 implies that

$$\tau_{m-i-3} = \gamma_{-1}\,\sigma_{m-i-3} = \gamma_{-1}\ \gamma_{-i-2}\,(m-1, m-j-1) = \gamma_{-i-3}\,(m-1, m-j-1).$$

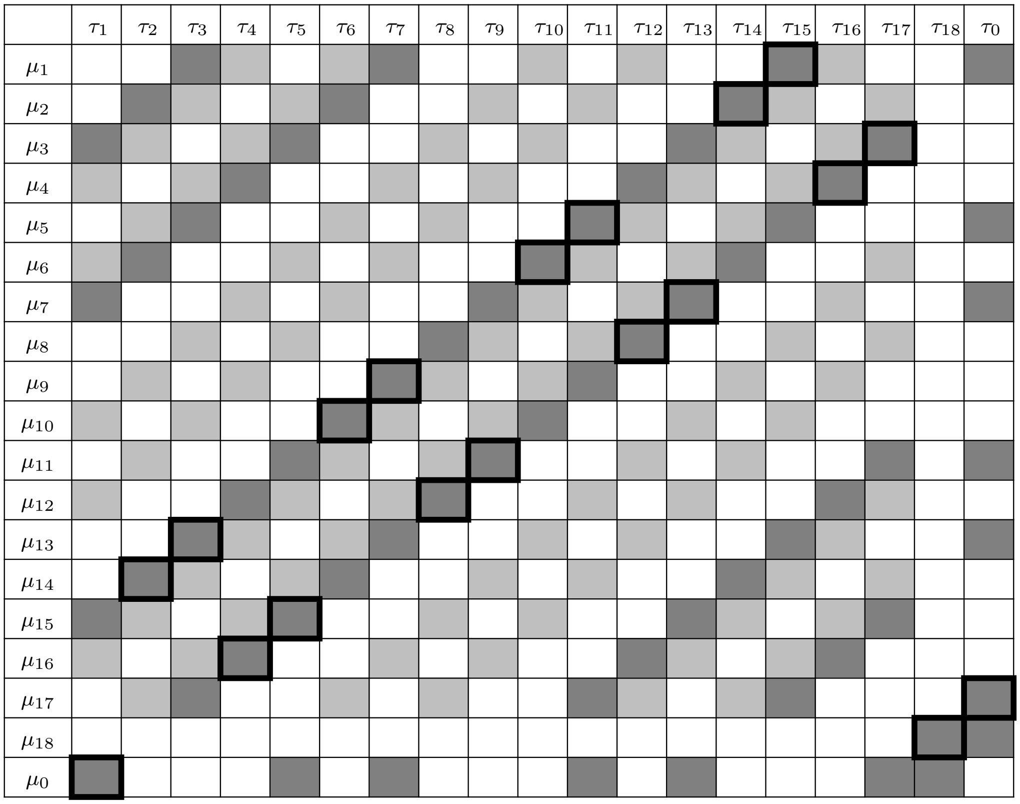
Note that, since $i = 2j$, Lemma 4.74 implies that $(k-j)^{\gamma_{i+1}} = k+j+1$. In addition, we point out that $k-j$ is even and $m-j-1$ is odd. We then see that

$$\begin{aligned}\mu_i\,\tau_{m-i-3} &= \gamma_1\gamma_i\,(m-1, k+j+1)\,\gamma_{-i-3}\,(m-1, m-j-1)\\ &= (0, m-3, m-5, \ldots, k-j, m-j-1, m-j-3, \ldots, 1, m-2, m-4,\\ &\qquad \ldots, m-j+1, m-1, k-j-2, k-j-4, \ldots, 4, 2).\end{aligned}$$

In conclusion, each pair of D_{H_2} is a hamiltonian pair.

Let $D = D_{H_s} \cup D_{H_1} \cup D_{H_2}$. Refer to Figure 4.32 for an illustration of D when $m = 19$. Observe that each permutation of S_1 and S_2 appears exactly once in D. Moreover, all pairs of D are hamiltonian pairs. Therefore, the set D is a 0-twined 2-factorization of $\vec{C}_2 \wr \overline{K}_m$.

<u>Case 2</u>: $2 \leqslant c \leqslant m-3$. Let $c = 2t$ where $1 \leqslant t \leqslant \frac{m-3}{2}$. Let $\mathbb{I} = \{i \mid i \equiv 1, 2 \pmod 4 \text{ and } 1 \leqslant i \leqslant m-3\}$ and let M_t be a subset of $\mathbb{I}$ of size t. Note that $\mathbb{I}$ contains $\frac{m-3}{2}$ elements. We first form the following set of pairs:



	τ_1	τ_2	τ_3	τ_4	τ_5	τ_6	τ_7	τ_8	τ_9	τ_{10}	τ_{11}	τ_{12}	τ_{13}	τ_{14}	τ_{15}	τ_{16}	τ_{17}	τ_{18}	τ_0
μ_1			▓	░		░	▓			░		░			▣	░			▓
μ_2		▓	░		░	▓			░		░			▣	░		░		
μ_3	▓	░		░	▓			░		░			▓	░		░	▣		
μ_4	░		░	▓			░		░			▓	░		░	▣			
μ_5		░	▓			░		░			▣	░		░	▓				▓
μ_6	░	▓			░		░			▣	░		░	▓			░		
μ_7	▓			░		░			▓	░		░	▣			░			▓
μ_8			░		░			▓	░		░	▣			░		░		
μ_9		░		░			▣	░		░	▓			░		░			
μ_{10}	░		░			▣	░		░	▓			░		░				
μ_{11}		░			▓	░		░	▣			░		░			▓		▓
μ_{12}	░			▓	░		░	▣			░		░			▓	░		
μ_{13}			▣	░		░	▓			░		░			▓	░			▓
μ_{14}		▣	░		░	▓			░		░			▓	░		░		
μ_{15}	▓	░		░	▣			░		░			▓	░		░	▓		
μ_{16}	░		░	▣			░		░			▓	░		░	▓			
μ_{17}		░	▓			░		░			▓	░		░	▓				▣
μ_{18}																		▣	▓
μ_0	▣				▓		▓				▓		▓				▓	▓	

Figure 4.32: Case 1: hamiltonian table of $S_1 \times S_2$ for $m = 19$ with a 0-twined 2-factorization in black.

$$D_{T_1} = \{(\mu_i, \tau_{m-i-2}), (\mu_{i+3}, \tau_{m-i-3}) \mid i \equiv 1 \pmod 4 \text{ and } i \in M_t\}.$$

Below, we show that all pairs in D_{T_1} are truncated hamiltonian pairs.

First, we point out that $(m-1)^{\mu_i} = (m-1)^{\tau_i} = (m-1)^{\sigma_i}$ for all $i \in \{0, 1, \ldots, m-1\}$. As a result, if $i \notin \{0, m-1\}$, we see that

$$\hat{\mu}_i = \gamma_1 \gamma_i = \gamma_{i+1} \text{ and } \hat{\tau}_i = \gamma_{-1}\gamma_i = \gamma_{i-1}.$$

This means that

$$\begin{aligned} \hat{\mu}_i \hat{\tau}_{m-i-2} &= \gamma_{i+1}\gamma_{m-i-3} &&= \gamma_{i+1}\gamma_{-i-2} &&= \gamma_{-1}; \\ \hat{\mu}_{i+3}\hat{\tau}_{m-i-3} &= \gamma_{i+4}\gamma_{m-i-4} &&= \gamma_{i+4}\,\gamma_{-i-3} &&= \gamma_1. \end{aligned}$$

Hence, all pairs of D_{T_1} are truncated hamiltonian pairs.

Next, we form the set

$$\begin{aligned} D'_{H_1} &= \{(\mu_i, \tau_{m-i-3}), (\mu_{i+3}, \tau_{m-i-2}) \mid i \equiv 1 \pmod 4 \text{ and } i \in \mathbb{I}\backslash M_t\} \cup \\ &\quad \{(\mu_0, \tau_1), (\mu_{m-2}, \tau_0)\}. \end{aligned}$$

Let D be the 0-twined 2-factorization constructed in Case 1. We see that $D'_{H_1} \subseteq D$. It follows from Case 1 that all pairs of D'_{H_1} are hamiltonian pairs. Next, we form the set

$$D_{c_1} = D_{T_1} \cup D'_{H_1}.$$

Observe that D_{c_1} contains all $\mu_i \in S_1$ such that $i \equiv 0, 1 \pmod 4$ and all $\tau_r \in S_2$ such that $r \equiv 0, 3 \pmod 4$.

We proceed by forming the following set of pairs:

$$D_{T_2} = \{(\mu_i, \tau_{m-i}), (\mu_{i+1}, \tau_{m-i-3}) \mid i \equiv 2 \pmod 4 \text{ and } i \in M_t\}.$$

Below, we show that each pair in D_{T_2} is truncated hamiltonian:

$$\begin{aligned} \hat{\mu}_i \hat{\tau}_{m-i} &= \gamma_{i+1}\, \gamma_{m-i-1} &&= \gamma_{i+1}\, \gamma_{-i} &&= \gamma_1; \\ \hat{\mu}_{i+1} \hat{\tau}_{m-i-3} &= \gamma_{i+2}\, \gamma_{m-i-4} &&= \gamma_{i+2}\, \gamma_{-i-3} &&= \gamma_{-1}. \end{aligned}$$

In conclusion, all pairs of D_{T_2} are truncated hamiltonian pairs.

Next, we form the set

$$D'_{H_2} = \{(\mu_i, \tau_{m-i-3}), (\mu_{i+1}, \tau_{m-i}) \mid i \equiv 2 \pmod 4 \text{ and } i \in \mathbb{I}\backslash M_t\} \cup \{(\mu_{m-1}, \tau_{m-1})\}.$$

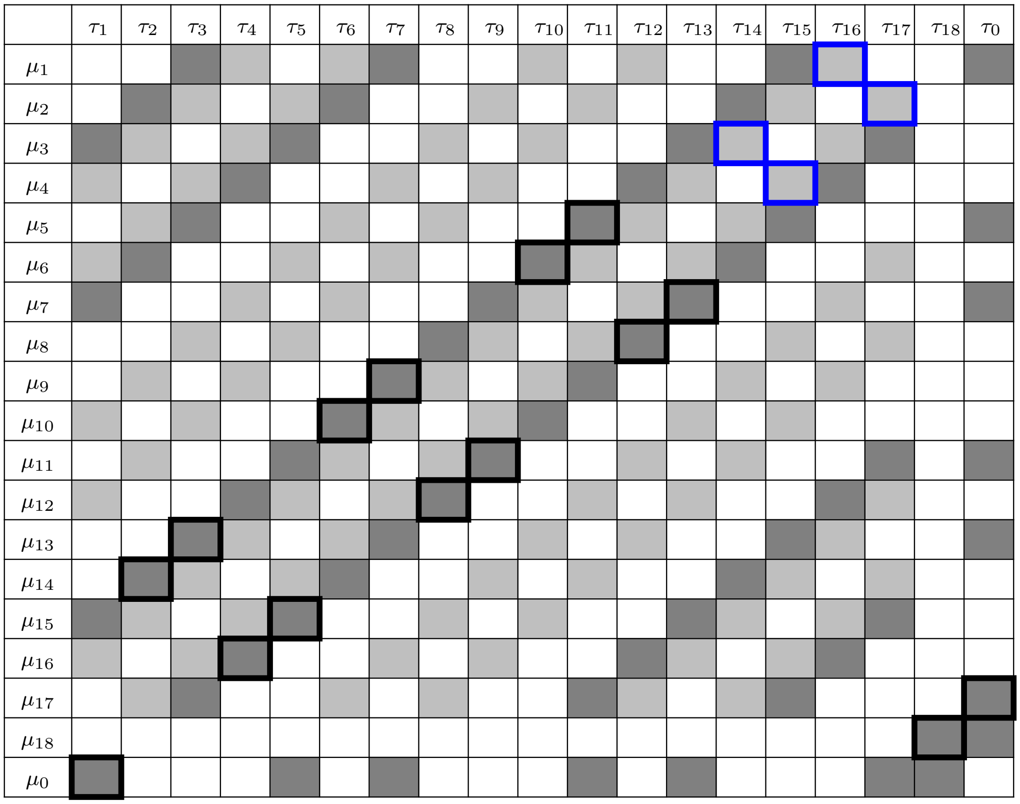
We point out that $D'_{H_2} \subseteq D$. It follows from the proof of Case 1 that all pairs of D'_{H_2} are hamiltonian pairs. Next, we form the set:

$$D_{c_2} = D_{T_2} \cup D'_{H_2}.$$

Observe that D_{T_2} and D'_{H_2} are disjoint. Furthermore, observe that D_{c_2} contains all $\mu_i \in S_1$ such that $i \equiv 2, 3 \pmod 4$ and all $\tau_r \in S_2$ such that $r \equiv 1, 2 \pmod 4$ except for τ_1. Lastly, note that τ_1 appears in one pair of D_{c_1}.

We now form the set $D' = D_{c_1} \cup D_{c_2}$. We point out that all permutations of S_1 and S_2 appear in D' exactly once. Consequently, the set D' is a 2-factorization of $\vec{C}_2 \wr \overline{K}_m$. In Figure 4.33, we illustrate D' for $m = 19$ and $M_t = \{1, 2\}$.

Note that D' can be partitioned into two sets: $D_T = D_{T_1} \cup D_{T_2}$, which contains $2t$ truncated hamiltonian pairs; and $D'_H = D'_{H_1} \cup D'_{H_2}$, which contains $m - 2t$ hamiltonian pairs. Therefore, we see that D' is indeed a c-twined 2-factorization of $\vec{C}_2 \wr \overline{K}_m$. ∎



	τ_1	τ_2	τ_3	τ_4	τ_5	τ_6	τ_7	τ_8	τ_9	τ_{10}	τ_{11}	τ_{12}	τ_{13}	τ_{14}	τ_{15}	τ_{16}	τ_{17}	τ_{18}	τ_0
μ_1			▓	░		░	▓			░		░			▓	░			▓
μ_2		▓	░		░	▓			░		░			▓	░		░		
μ_3	▓	░		░	▓			░		░			▓	░		░	▓		
μ_4	░		░	▓			░		░			▓	░		░	▓			
μ_5		░	▓			░		░			▓	░		░	▓				▓
μ_6	░	▓			░		░			▓	░		░	▓			░		
μ_7	▓			░		░			▓	░		░	▓			░			▓
μ_8			░		░			▓	░		░	▓			░		░		
μ_9		░		░			▓	░		░	▓			░		░			
μ_{10}	░		░			▓	░		░	▓			░		░				
μ_{11}		░			▓	░		░	▓			░		░			▓		▓
μ_{12}	░			▓	░		░	▓			░		░			▓	░		
μ_{13}			▓	░		░	▓			░		░			▓	░			▓
μ_{14}		▓	░		░	▓			░		░			▓	░		░		
μ_{15}	▓	░		░	▓			░		░			▓	░		░	▓		
μ_{16}	░		░	▓			░		░			▓	░		░	▓			
μ_{17}		░	▓			░		░			▓	░		░	▓				▓
μ_{18}																		▓	▓
μ_0	▓				▓		▓				▓		▓				▓	▓	

Figure 4.33: Case 2: hamiltonian table of $S_1 \times S_2$ for $m = 19$ with a 4-twined 2-factorization in black (hamiltonian pairs) and blue (truncated hamiltonian pairs).

Next, we consider the case $m \equiv 3 \pmod{12}$. Although we use the same regular permutation sets as in the proof of Proposition 4.81, the constructions given in the proof of Proposition 4.82 differ from those of Proposition 4.81 because m is not relatively prime with 3. Instead, we will use the fact that $m-1$ is relatively prime with 3.

Proposition 4.82. *Let $m \equiv 3 \pmod{12}$ such that $m \geqslant 15$, and let c be an even integer such that $2 \leqslant c \leqslant m-3$. The digraph $\vec{C}_2 \wr \overline{K}_m$ admits a c-twined 2-factorization.*

Proof: Let $m = 2k+1$ where k is odd. In this proof, it is understood that, for each $i \in \mathbb{Z}_{m-1}$, the permutation $\gamma_i \in G_{m-1}$ is as defined in Definition 4.73. We will be using the following two sets of permutations:

$$\begin{aligned} S_1 = \gamma_1 \cdot \mathcal{F}_m &= \{\mu_i = \gamma_1\sigma_i \mid i = 0, 1, \ldots, m-1\}; \\ S_2 = \gamma_{-1} \cdot \mathcal{F}_m &= \{\tau_i = \gamma_{-1}\sigma_i \mid i = 0, 1, \ldots, m-1\}. \end{aligned}$$

Lemmas 4.60 and 4.77 imply that S_1 and S_2 are regular permutation sets. See Construction 4.75 for the definition of each $\sigma_i \in \mathcal{F}_m$. Refer to Figure 4.34 for the hamiltonian table of $S_1 \times S_2$ when $m = 15$. Notice that μ_0 and τ_0 are the $(m-1)$-stabilizers of S_1 and S_2, respectively.

Case 1: $c = 2$. We first form a set of two pairs as follows:

$$D_T = \{(\mu_1, \tau_2), (\mu_2, \tau_1)\}.$$

We now show that (μ_1, τ_2) and (μ_2, τ_1) are truncated hamiltonian pairs. We see that

$$\begin{aligned} \hat{\mu}_1\hat{\tau}_2 &= \gamma_2\,\gamma_1 &= \gamma_3; \\ \hat{\mu}_2\hat{\tau}_1 &= \gamma_3\,\gamma_0 &= \gamma_3. \end{aligned}$$

Since $m-1$ is relatively prime with 3, it follows that $T(\gamma_3) = 2$ and thus (μ_1, τ_2) and (μ_2, τ_1) are truncated hamiltonian.

Next, we form a set of $m-2$ pairs:

$$D_H = \{(\mu_i, \tau_{m-i+1}) \mid 3 \leqslant i \leqslant m-2\} \cup \{(\mu_0, \tau_{m-1}), (\mu_{m-1}, \tau_0)\}.$$

Below, we show that D_H is a set of hamiltonian pairs. First, we show that (μ_0, τ_{m-1}) and (μ_{m-1}, τ_0) are hamiltonian pairs:

$$\begin{aligned} \mu_0\,\tau_{m-1} &= \gamma_1\gamma_{-1}\sigma_{m-1} \\ &= \sigma_{m-1}; \\ \mu_{m-1}\,\tau_0 &= \gamma_1\sigma_{m-1}\gamma_{-1}. \end{aligned}$$

Note that $\gamma_1\sigma_{m-1}\gamma_{-1}$ is a conjugate of σ_{m-1} so $T(\sigma_{m-1}) = T(\gamma_1\sigma_{m-1}\gamma_{-1})$, and by Construction 4.75, $T(\sigma_{m-1}) = 1$. Therefore, we see that (μ_0, τ_{m-1}) and (μ_{m-1}, τ_0) are hamiltonian pairs.

We now aim to show that all pairs of the form $(\mu_i, \tau_{m-i+1}) \in D_H$ are hamiltonian pairs. First, we address three special cases, namely $i \in \{4, 5, m-3\}$:

$$\begin{aligned}\mu_4\,\tau_{m-3} &= \gamma_5(m-1,k+3)\gamma_{-3}\,(m-1,0)\\ &= (0,2,4,\ldots,m-3,m-1,k,k+2,\ldots,m-2,1,3,5,,\ldots,k-2);\end{aligned}$$

$$\begin{aligned}\mu_5\,\tau_{m-4} &= \gamma_6(m-1,4)\gamma_{-4}\,(m-1,k)\\ &= (0,2,4,\ldots,m-3,k,k+2,k+4,\ldots,m-2,1,3,,\ldots,k-2,m-1);\end{aligned}$$

$$\begin{aligned}\mu_{m-3}\,\tau_4 &= \gamma_{-1}(m-1,0)\gamma_3\,(m-1,k+3)\\ &= (0,2,4,\ldots,k+1,m-1,3,5,\ldots,m-2,k+3,k+5,\ldots,m-3).\end{aligned}$$

In conclusion, the pairs (μ_4,τ_{m-3}), (μ_5,τ_{m-4}), and (μ_{m-3},τ_4) are hamiltonian pairs.

We now proceed with the more general case. Note that when i is even and $i=2j$, Lemma 4.78 implies that

$$\tau_{m-i+1}=\gamma_{-1}\sigma_{m-i+1}=\gamma_{-1}\gamma_{-i+2}(m-1,m-j+1)=\gamma_{-i+1}(m-1,m-j+1).$$

When i is odd and $i=2j+1$, Lemma 4.78 implies that

$$\tau_{m-i+1}=\gamma_{-1}\sigma_{m-i+1}=\gamma_{-1}\gamma_{-i+2}(m-1,k-j+2)=\gamma_{-i+1}(m-1,k-j+2).$$

These two observations are key in the following computations.

If $i\equiv 0 \pmod 4$ and $8\leqslant i\leqslant m-7$, then $i=2j$ where j is even and $4\leqslant j\leqslant k-3$. Lastly, note that $m-j-1$ is even and $k-j+2$ is odd. Therefore, we see that

$$\begin{aligned}\mu_i\,\tau_{m-i+1} &= \gamma_{i+1}\,(m-1,k+j+1)\,\gamma_{-i+1}\,(m-1,m-j+1)\\ &= (0,2,4,\ldots,m-j-1,m-1,k-j+2,k-j+4,\ldots,m-2,1,3,\ldots,\\ &\qquad k-j,m-j+1,m-j+3,\ldots,m-3).\end{aligned}$$

If $i\equiv 1 \pmod 4$ and $9\leqslant i\leqslant m-2$, then $i=2j+1$ where j is even and $4\leqslant j\leqslant k-1$. Furthermore, Lemma 4.74 implies that $(m-j-1)^{\gamma_{i+1}}=j+2$ and that $(j+2)^{\gamma_{-i+1}}=m-j+1$. Consequently

$$\begin{aligned}\mu_i\,\tau_{m-i+1} &= \gamma_{i+1}\,(m-1,j+2)\,\gamma_{-i+1}\,(m-1,k-j+2)\\ &= (0,2,4,\ldots,m-j-1,k-j+2,k-j+4,\ldots,m-2,1,3,\ldots,k-j,\\ &\qquad m-1,m-j+1,m-j+3,\ldots,m-3).\end{aligned}$$

If $i\equiv 2 \pmod 4$ and $6\leqslant i\leqslant m-5$, then $i=2j$ where j is odd and $3\leqslant j\leqslant k-3$. We then have

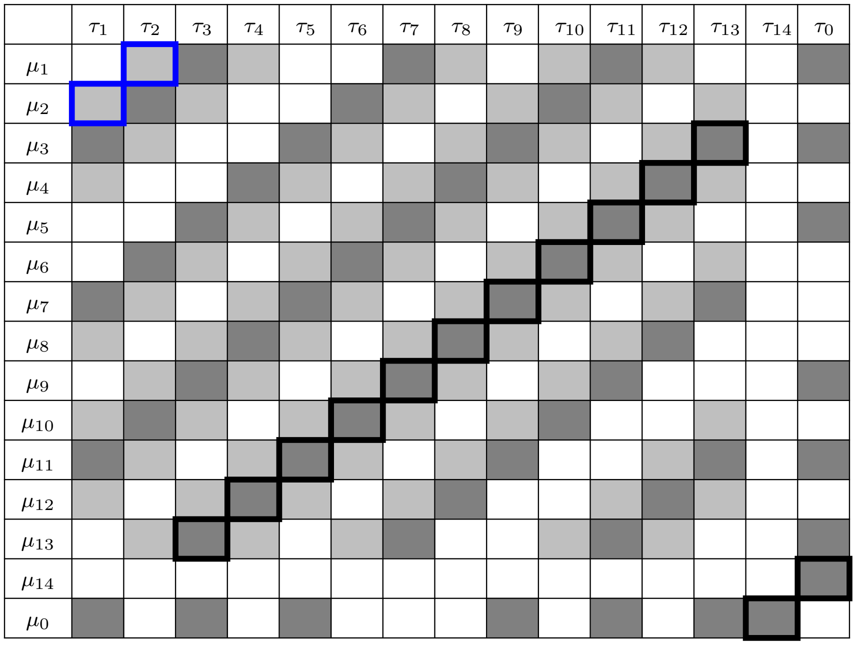
$$\begin{aligned} \mu_i\,\tau_{m-i+1} &= \gamma_{i+1}\,(m-1,k+j+1)\,\gamma_{-i+1}\,(m-1,m-j+1) \\ &= (0,2,4,\ldots,k-j,m-j+1,m-j+3,\ldots,m-2,1,3,\ldots, \\ &\quad\; m-j-1,m-1,k-j+2,k-j+4,\ldots,m-3). \end{aligned}$$

If $i \equiv 3 \pmod 4$ and $3 \leqslant i \leqslant m-4$, then $i = 2j+1$ where j is odd and $1 \leqslant j \leqslant k-2$. Note that Lemma 4.74 implies that $(j+2)^{\gamma_{-i+1}} = m-j+1$ and $(m-j-1)^{\gamma_{i+1}} = j+2$. Therefore

$$\begin{aligned} \mu_i\,\tau_{m-i+1} &= \gamma_{i+1}\,(m-1,j+2)\gamma_{-i+1}\,(m-1,k-j+2) \\ &= (0,2,4,\ldots,k-j,m-1,m-j+1,\ldots,m-2,1,3,5,\ldots, \\ &\quad\; m-j-3,m-j-1,k-j+2,k-j+4,\ldots,m-3). \end{aligned}$$

In summary, all pairs in D_H are hamiltonian pairs.

Next, we form the set $D = D_T \cup D_H$. Refer to Figure 4.34 for an illustration of D when $m = 15$. Observe that D contains m pairs and that each permutation of S_1 and S_2 appears exactly once in D. In addition, the sets D_H and D_T form a partition of D. Therefore, the set D is a 2-twined 2-factorization of $\vec{C}_2 \wr \overline{K}_m$.



	τ_1	τ_2	τ_3	τ_4	τ_5	τ_6	τ_7	τ_8	τ_9	τ_{10}	τ_{11}	τ_{12}	τ_{13}	τ_{14}	τ_0
μ_1		▒	■	▒			■	▒		▒	■	▒			■
μ_2	▒	■	▒			■	▒		▒	■	▒		▒		
μ_3	■	▒			■	▒		▒	■	▒		▒	■		■
μ_4	▒			■	▒		▒	■	▒		▒	■	▒		
μ_5			■	▒		▒	■	▒		▒	■	▒			■
μ_6		■	▒		▒	■	▒		▒	■	▒		▒		
μ_7	■	▒		▒	■	▒		▒	■	▒		▒	■		
μ_8	▒		▒	■	▒		▒	■	▒		▒	■			
μ_9		▒	■	▒		▒	■	▒		▒	■				■
μ_{10}	▒	■	▒		▒	■	▒		▒	■			▒		
μ_{11}	■	▒		▒	■	▒		▒	■			▒	■		■
μ_{12}	▒		▒	■	▒		▒	■			▒	■	▒		
μ_{13}		▒	■	▒		▒	■			▒	■	▒			■
μ_{14}															■
μ_0	■		■		■				■		■		■	■	

Figure 4.34: Case 1: hamiltonian table of $S_1 \times S_2$ for $m = 15$ with a 2-twined 2-factorization in black (hamiltonian pairs) and blue (truncated hamiltonian pairs).

Case 2: $4 \leqslant c \leqslant m-3$. Let $c = 2t+2$ where $1 \leqslant t \leqslant \frac{m-5}{2}$. Assume that $\mathbb{I} = \{i \mid i \equiv 1 \pmod 2 \text{ and } 3 \leqslant i \leqslant m-4\}$ and let M_t be a subset of size t of $\mathbb{I}$. We point out that $\mathbb{I}$ contains $\frac{m-5}{2}$ elements. We form the following set of $2t+2$ pairs:

$$D'_T = \{(\mu_i, \tau_{m-i}), (\mu_{i+1}, \tau_{m-i+1}) \mid i \in M_t\} \cup \{(\mu_1, \tau_2), (\mu_2, \tau_1)\}.$$

Below, we show that D'_T is a set of truncated hamiltonian pairs.

First, observe that, if $i \notin \{0, m-1\}$, $s \in \{0,1\}$, and $m-i+s \neq m-1$, then Lemma 4.74 and Construction 4.75 jointly imply that

$$\hat{\mu}_i = \gamma_{i+1}, \text{ and } \hat{\tau}_{m-i+s} = \gamma_{-i+s}.$$

It follows that, for all $i \in M_t$, we have

$$\begin{array}{lllll} \hat{\mu}_i \hat{\tau}_{m-i} & = & \gamma_{i+1}\,\gamma_{-i} & = & \gamma_1; \\ \hat{\mu}_{i+1} \hat{\tau}_{m-i+1} & = & \gamma_{i+2}\,\gamma_{-i+1} & = & \gamma_3. \end{array}$$

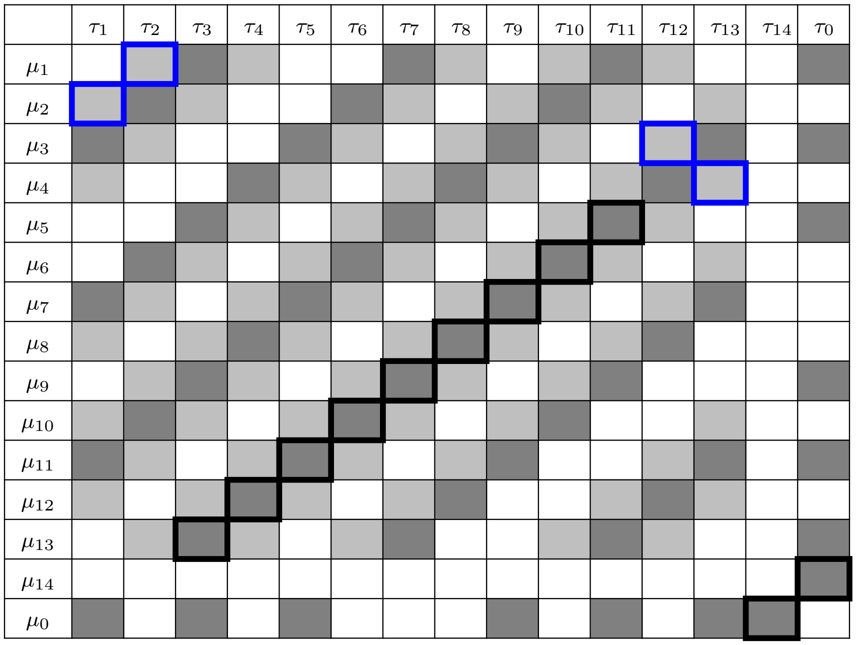
If $m-1$ is relatively prime with 3, then $T(\gamma_3) = 2$ and thus $(\mu_{i+1}, \tau_{m-i+1})$ is a truncated hamiltonian pair. Refer to Case 1 for a proof that (μ_1, τ_2) and (μ_2, τ_1) are truncated hamiltonian pairs. In conclusion, all pairs in D'_T are truncated hamiltonian pairs.

Next, we form the following set:

$$D'_H = \{(\mu_i, \tau_{m-i+1}) \mid 3 \leqslant i \leqslant m-2 \text{ and } i, i-1 \notin M_t\} \cup \{(\mu_0, \tau_{m-1}), (\mu_{m-1}, \tau_0)\}.$$

Observe that $D'_H \subseteq D_H$, where D_H is defined in Case 1. Hence D'_H contains $m-2t-2$ hamiltonian pairs. Next, we form the set $D' = D'_T \cup D'_H$. Refer to Figure 4.35 for an illustration of D' when $m = 15$ and $M_t = \{3\}$. Observe that each permutation of S_1 and S_2 appears in exactly one pair of D'. Moreover, the sets D'_T and D'_H partition D'. Therefore, the set D' is a c-twined 2-factorization of $\vec{C}_2 \wr \overline{K}_m$. ∎

We now summarize the implications of Propositions 4.72, 4.80, 4.81, and 4.82 in Theorem 4.83 below.



	τ_1	τ_2	τ_3	τ_4	τ_5	τ_6	τ_7	τ_8	τ_9	τ_{10}	τ_{11}	τ_{12}	τ_{13}	τ_{14}	τ_0
μ_1		▒	■	▒			■	▒		▒	■	▒			■
μ_2	▒	■	▒			■	▒		▒	■	▒		▒		
μ_3	■	▒			■	▒		▒	■	▒		▒	■		■
μ_4	▒			■	▒		▒	■	▒		▒	■	▒		
μ_5			■	▒		▒	■	▒		▒	■	▒			■
μ_6		■	▒		▒	■	▒		▒	■	▒		▒		
μ_7	■	▒		▒	■	▒		▒	■	▒		▒	■		
μ_8	▒		▒	■	▒		▒	■	▒		▒	■			
μ_9		▒	■	▒		▒	■	▒		▒	■				■
μ_{10}	▒	■	▒		▒	■	▒		▒	■			▒		
μ_{11}	■	▒		▒	■	▒		▒	■			▒	■		■
μ_{12}	▒		▒	■	▒		▒	■			▒	■	▒		
μ_{13}		▒	■	▒		▒	■			▒	■	▒			■
μ_{14}															■
μ_0	■		■		■				■		■		■	■	

Figure 4.35: Case 2: hamiltonian table of $S_1 \times S_2$ for $m = 15$ with a 4-twined 2-factorization in black (hamiltonian pairs) and blue (truncated hamiltonian pairs).

Theorem 4.83. *Let H be a strict digraph such that $|V(H)| = m$ and $m \geqslant 4$. Assume that n is an even integer and that H admits a decomposition into c directed hamiltonian cycles where c is even and $2 \leqslant c \leqslant m-2$. The digraph $\vec{C}_n \wr H$ is hamiltonian decomposable.*

Proof: Propositions 4.72, 4.80, 4.81, and 4.82 jointly imply that the digraph $\vec{C}_2 \wr \overline{K}_m$ admits a c-twined 2-factorization for all $m \geqslant 4$ and even c such that $2 \leqslant c \leqslant m-2$. The statement then follows from Corollary 4.68. ∎

We will now use Theorem 4.83 in conjunction with the fact that $\vec{C}_n \wr \vec{C}_m$ is hamiltonian decomposable when $m \notin \{2, 3\}$ and $(n, m) \neq (2, 2r)$, where $r \geqslant 3$, to prove Theorem 4.84 below.

Theorem 4.84. *Let G and H be strict hamiltonian decomposable digraphs such that $|V(G)| = n$ where n is even and $G \neq \overline{K}_n$, and $|V(H)| = m$ where $m \geqslant 4$. Furthermore, assume that H admits a decomposition into c directed hamiltonian cycles where $2 \leqslant$*

$c \leqslant m-2$. The digraph $G \wr H$ is hamiltonian decomposable in each of the following cases:

(S1) *c is even;*

(S2) *c is odd and $G \neq \vec{C}_n$.*

Proof:

(S1) Assume that $2 \leqslant c \leqslant m-2$. If c is even, then Theorem 4.83 in conjunction with Lemma 4.4 implies that $G \wr H$ is hamiltonian decomposable.

(S2) Next, we assume that c is odd and that $G \neq \vec{C}_n$. This means that G admits a decomposition into at least two directed hamiltonian cycles. Let $H = \Gamma \oplus \vec{C}_m$ where Γ is the digraph obtained from H by removing the arcs of a single directed hamiltonian cycle. Hence, the digraph Γ admits a decomposition into $c-1$ directed hamiltonian cycles, where $c-1$ is even. By our assumption, we have that $G = \vec{C}_n \oplus \vec{C}_n \oplus \ldots \oplus \vec{C}_n$. It follows from the properties of the wreath product that

$$\begin{aligned} G \wr H &= (\vec{C}_n \oplus \vec{C}_n \oplus \ldots \oplus \vec{C}_n) \wr (\Gamma \oplus \vec{C}_m) \\ &= (\vec{C}_n \wr \Gamma) \oplus (\vec{C}_n \wr \vec{C}_m) \oplus (\vec{C}_n \wr \overline{K}_m) \oplus \ldots \oplus (\vec{C}_n \wr \overline{K}_m). \end{aligned}$$

If $n = 2$, then $G = \vec{C}_2$. Since $G \neq \vec{C}_n$, we may assume that $n \geqslant 3$. Theorems 4.48 and 4.83 then imply that $\vec{C}_n \wr \vec{C}_m$ and $\vec{C}_n \wr \Gamma$ are hamiltonian decomposable, respectively. Lastly, Lemma 4.3 implies that $\vec{C}_n \wr \overline{K}_m$ is hamiltonian decomposable. Since all subdigraphs in the above decomposition of $G \wr H$ are spanning sudigraphs, Lemma 2.8 implies that $G \wr H$ is hamiltonian decomposable. ▮

In the next section, we will improve Theorem 4.84 by showing that $\vec{C}_n \wr H$ is hamiltonian decomposable when H admits a decomposition into $3 \leqslant c \leqslant m-2$ directed hamiltonian cycles where c and m are odd.

4.7 The case m odd and $c \geqslant 3$ odd

In this section, we affirm Conjecture 2.23 when $m \geqslant 5$ and c are odd and $3 \leqslant c \leqslant m-2$, thereby improving Theorem 4.84. To do so, we consider five cases. In the first two cases, addressed in Propositions 4.85 and 4.86 below, we create two regular permutation sets that allow us to form up to $\frac{m+1}{2}$ truncated pairs when $m \equiv 1 \pmod 4$, $\frac{m-5}{2}$ truncated hamiltonian pairs when $m \equiv 3 \pmod 8$, and up to $\frac{m-1}{2}$ truncated hamiltonian pairs when $m \equiv 7 \pmod 8$.

4.7.1 The case c is small

Proposition 4.85 addresses the case $m \equiv 1 \pmod 4$ and Proposition 4.86 addresses the case $m \equiv 3 \pmod 4$. The constructions given in the proofs of both propositions are similar. First, we create a 3-twined 2-factorization. Then, we consider the case $c = 3 + 2t$ where $1 \leqslant t \leqslant \frac{m-5}{4}$ when $m \equiv 1 \pmod 4$, $1 \leqslant t \leqslant \frac{m-11}{4}$ when $m \equiv 3 \pmod 8$, and $1 \leqslant t \leqslant \frac{m-7}{4}$ when $m \equiv 7 \pmod 8$ and construct a $(2t+3)$-twined 2-factorization of $\vec{C}_2 \wr \overline{K}_m$ in each case.

Proposition 4.85. *Let $m \equiv 1$ (mod 4) where $m \geqslant 5$, and let c be an odd integer such that $3 \leqslant c \leqslant \frac{m+1}{2}$. The digraph $\vec{C}_2 \wr \overline{K}_m$ admits a c-twined 2-factorization.*

Proof: Let $m = 2k+1$, where k is even. In this proof, it is understood that, for each $i \in \mathbb{Z}_{m-1}$, the permutation $\gamma_i \in G_{m-1}$ is as defined in Definition 4.73. We construct our c-twined 2-factorization using the following two sets of permutations:

$$S_1 = (1, m-1) \cdot \mathcal{F}_m = \{\mu_i = (1, m-1)\sigma_i \mid i = 0, 1, \ldots, m-1\};$$
$$S_2 = \mathcal{F}_m = \{\sigma_0, \sigma_1, \ldots, \sigma_{m-1}\}.$$

Lemmas 4.60 and 4.77 imply that S_1 and S_2 are regular permutation sets. Refer to Construction 4.75 for the definition of each $\sigma_i \in \mathcal{F}_m$. Moreover, see Figure 4.36 for the hamiltonian table of $S_1 \times S_2$ when $m = 13$. Observe that μ_1 and σ_0 are the $(m-1)$-stabilizers of S_1 and S_2 respectively. We also point out that, if $i \notin \{1, m-2, m-1\}$, then

$$(m-1)^{\mu_i} = 1^{\sigma_i} = i + 1.$$

Moreover, we see that

$$(m-1)^{\mu_{m-2}} = 1^{\sigma_{m-2}} = 0, \text{ and } (m-1)^{\mu_{m-1}} = 1^{\sigma_{m-1}} = 2.$$

Case 1: $c = 3$. We construct a set that contains $m-3$ hamiltonian pairs and three truncated hamiltonian pairs. We begin by forming the following set of three pairs:

$$D_T = \{(\mu_0, \sigma_1), (\mu_{m-2}, \sigma_{m-2}), (\mu_{m-1}, \sigma_{m-1})\}.$$

We claim that D_T is a set of truncated hamiltonian pairs. Below, we verify this claim.

First, if $m = 5$, then

$$\begin{aligned} \mu_4 &= (0, 3, 4, 2)(1); \\ \hat{\mu}_4 &= (0, 3, 2)(1)(4). \end{aligned}$$

Otherwise, we have that

$$\begin{aligned} \mu_{m-1} &= (1, m-1)\,(0, 3, m-2, 4, m-3, \ldots, k, k+2, k+1, m-1, 1, 2) \\ &= (0, 3, m-2, 4, m-3, \ldots, k, k+2, k+1, m-1, 2)(1); \end{aligned}$$

$$\begin{aligned} \hat{\mu}_{m-1} &= (1, m-1)\,(0, 3, m-2, 4, m-3, \ldots, k, k+2, k+1, m-1, 1, 2)\,(m-1, 2) \\ &= (0, 3, m-2, 4, m-3, \ldots, k, k+2, k+1, 2)(1)(m-1). \end{aligned}$$

Then, for all $m \equiv 1 \pmod 4$, we have that

$$\begin{aligned} \hat{\mu}_0 &= \gamma_0 = id; \\ \hat{\mu}_{m-2} &= (1, m-1)\,\gamma_{-1}\,(m-1, k+1)(m-1, 0) \\ &= (0, m-2, m-3, \ldots, k+3, k+2)(1, k+1, k, k-1, \ldots, 2)(m-1). \end{aligned}$$

Therefore, if $m = 5$, then

$$\begin{aligned} \hat{\mu}_4\hat{\sigma}_4 &= (0, 3, 2)(1)(4)\,(0, 3, 1, 2)(4) \\ &= (0, 1, 2, 3)(4). \end{aligned}$$

Otherwise, we have that

$$\begin{aligned} \hat{\mu}_{m-1}\hat{\sigma}_{m-1} &= (0, 3, m-2, 4, m-3, 5, \ldots, k, k+2, k+1, 2)(1)(m-1)\,(0, 3, m-2, \\ &\qquad 4, \ldots, k, k+2, k+1, 1, 2)(m-1) \\ &= (0, m-2, m-3, m-4, \ldots, k+2, 1, 2, 3, 4, 5, \ldots, k, k+1)(m-1). \end{aligned}$$

Recall that $\hat{\sigma}_i = \gamma_i$ for $i \notin \{0, m-1\}$. We then see that

$$\hat{\mu}_0\hat{\sigma}_1 = id\,\gamma_1 = \gamma_1.$$

To show that the pair $(\mu_{m-2}, \sigma_{m-2})$ is truncated hamiltonian, we use the fact that k is even. This means that

$$\begin{aligned} \hat{\mu}_{m-2}\hat{\sigma}_{m-2} &= (0, m-2, \ldots, k+3, k+2)(1, k+1, k, k-1, \ldots, 2)\gamma_{-1} \\ &= (0, m-3, m-5, \ldots, k+2, m-2, m-4, m-6, \ldots, 3, \\ &\qquad 1, k, k-2, k-4, \ \ldots, 2)(m-1). \end{aligned}$$

Therefore, we see that all pairs in D_T are truncated hamiltonian pairs. Next, we construct $\frac{m-3}{2}$ pairs as follows:

$$D_{H_1} = \{(\mu_i, \sigma_{m-i-2}) \mid i \text{ odd and } 1 \leqslant i \leqslant m-4\}.$$

Observe that D_T and D_{H_1} jointly use each permutation $\mu_i \in S_1$ such that i is odd exactly once. Similarly, the sets D_T and D_{H_1} jointly use each permutation $\sigma_r \in S_2$ such that r is even except for σ_0. Below, we show that D_{H_1} is a set of hamiltonian pairs.

First, we consider the case $i = 1$. We see that

$$\begin{aligned} \mu_1\sigma_{m-3} &= (1, m-1)\,\gamma_1\;(m-1, 2)\,\gamma_{-2}\,(m-1, 0) \\ &= (0, m-2, m-3, \ldots, 2, 1, m-1). \end{aligned}$$

We will now make several observations that will allow us to compute the product of μ_i with σ_{m-i-2}. If i is odd and $1 \leqslant i \leqslant m-4$, then $i = 2j+1$ where $1 \leqslant j \leqslant k-2$. Since i is odd, Lemma 4.78 implies that

$$\sigma_{m-i-2} = \gamma_{-i-1}\,(m-1, m-j-1).$$

Moreover, Lemma 4.74 implies that $(m-j)^{\gamma_i} = j+2$ and $(j+2)^{\gamma_{-i-1}} = m-j-1$. As a result, we see that

$$\begin{aligned} \mu_i\sigma_{m-i-2} &= (1, m-1)\,\gamma_i\;(m-1, j+2)\,\gamma_{-i-1}\,(m-1, m-j-1) \\ &= (0, m-2, m-3, \ldots, m-j, m-j-1, m-j-2, \ldots, 2, 1, m-1). \end{aligned}$$

In conclusion, each pair of D_{H_1} is a hamiltonian pair.

Next, if $m \geqslant 9$, we form a set of $\frac{m-3}{2}$ pairs as follows:

$$D_{H_2} = \{(\mu_i, \sigma_{m-i}) \mid i \text{ even and } 4 \leqslant i \leqslant m-3\} \cup \{(\mu_2, \sigma_0)\}.$$

If $m = 5$, then D_{H_2} contains one pair. Observe that D_{H_2} and D_T jointly use each permutation $\mu_i \in S_1$ where i is even exactly once. Similarly, the sets D_{H_2} and D_T jointly use each permutation $\sigma_r \in S_2$ such that r is odd exactly once. In addition, note that σ_0 appears in D_{H_2}. Below, we verify that D_{H_2} is a set of hamiltonian pairs.

First, we show that (μ_2, σ_0) is a hamiltonian pair. Note that we assume that $m \geqslant 9$ and thus $k \geqslant 4$:

$$\begin{aligned} \mu_2\,\sigma_0 &= (1, m-1)\,\sigma_2\,id = (1, m-1)\gamma_2(m-1, k+2) \\ &= (0, 2, 4, \ldots, k, m-1, 3, 5, \ldots, m-2, 1, k+2, k+4, \ldots, m-3). \end{aligned}$$

If i is even and $4 \leqslant i \leqslant m-5$, then $i = 2j$ where $2 \leqslant j \leqslant k-2$. Lemma 4.78 implies that

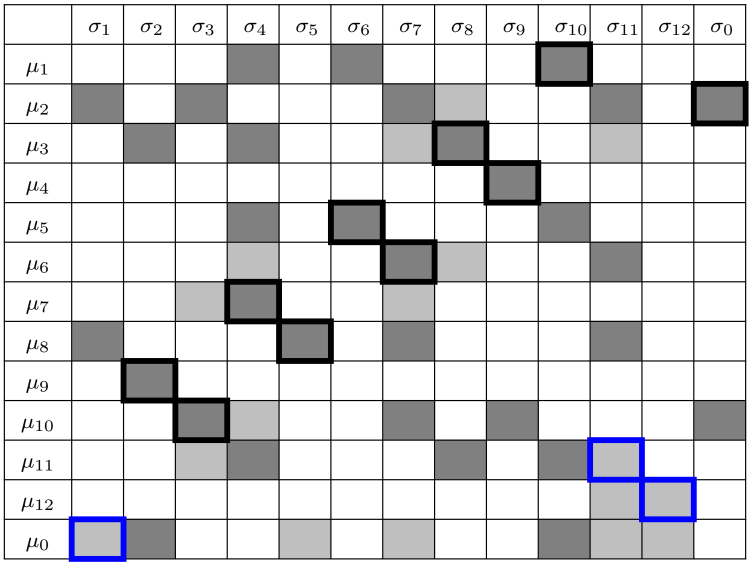
$$\sigma_{m-i} = \gamma_{-i+1}\,(m-1, k-j+2).$$

Since $i \neq m-3$, we see that

$$\begin{aligned}\mu_i\sigma_{m-i} &= (1, m-1)\,\gamma_i\,(m-1, k+j+1)\,\gamma_{-i+1}\,(m-1, k-j+2)\\ &= (0, 1, m-1, 2, 3, 4, \ldots, m-2).\end{aligned}$$

Lastly, if $i = m-3$ then

$$\begin{aligned}\mu_{m-3}\sigma_3 &= (1, m-1)\,\gamma_{-2}\,(m-1, 0)\,\gamma_3\,(m-1, 3)\\ &= (0, 1, m-1, 2, 3, 4, \ldots, m-2).\end{aligned}$$



	σ_1	σ_2	σ_3	σ_4	σ_5	σ_6	σ_7	σ_8	σ_9	σ_{10}	σ_{11}	σ_{12}	σ_0
μ_1				▓		▓				▣			
μ_2	▓		▓				▓	░			▓		▣
μ_3		▓		▓			░	▣			░		
μ_4									▣				
μ_5				▓		▣				▓			
μ_6				░			▣	░			▓		
μ_7			░	▣			░						
μ_8	▓				▣		▓				▓		
μ_9		▣											
μ_{10}			▣	░			▓		▓				▓
μ_{11}			░	▓				▓		▓	▢		
μ_{12}											░	▢	
μ_0	▢	▓			░		░			▓	░	░	

Figure 4.36: Case 1: hamiltonian table of $S_1 \times S_2$ for $m = 13$ with a 3-twined 2-factorization in black (hamiltonian pairs) and blue (truncated hamiltonian pairs).

Let $D = D_T \cup D_{H_1} \cup D_{H_2}$. Each permutation in S_1 and S_2 is used by exactly one pair of D thus implying that D is a 2-factorization of $\vec{C}_2 \wr \overline{K}_m$. Refer to Figure 4.36 for an illustration of D when $m = 13$. In summary, we have one set of three truncated hamiltonian pairs, D_T, and one set of $m-3$ hamiltonian pairs defined as $D_H = D_{H_1} \cup D_{H_2}$. Observe that D_T and D_H are disjoint. Therefore, the set D is a 3-twined 2-factorization of $\vec{C}_2 \wr \overline{K}_m$.

<u>Case 2</u>: $5 \leqslant c \leqslant \frac{m+1}{2}$. Let $c = 3 + 2t$, so $1 \leqslant t \leqslant \frac{m-5}{4}$. Let $\mathbb{I} = \{i \mid i \equiv 3 \pmod 4$ and $3 \leqslant i \leqslant m-6\}$ and let M_t be a subset of size t of $\mathbb{I}$. We point out that

$\mathbb{I}$ contains $\frac{m-5}{4}$ elements. We first form a set D'_T of c truncated hamiltonian pairs. Note that D_T is defined in Case 1. We have

$$D'_T = \{(\mu_i, \sigma_{m-i-3}), (\mu_{i+3}, \sigma_{m-i-2}) \mid i \in M_t\} \cup D_T.$$

Observe that D'_t contains $3 + 2t$ pairs. Next, we show that each pair of D'_T is a truncated hamiltonian pair. Refer to Case 1 for a proof that pairs in D_T are truncated hamiltonian.

First we look at the case $i = 3$:

$$\begin{aligned}\hat{\mu}_3\hat{\sigma}_{m-6} &= (1, m-1)\,\gamma_3\,(m-1, 3)\,(m-1, 4)\,\gamma_{-5}\\ &= (0, m-2, m-4, \ldots, 1, m-3, m-5, \ldots, 2)(m-1);\end{aligned}$$

$$\begin{aligned}\hat{\mu}_6\hat{\sigma}_{m-5} &= (1, m-1)\,\gamma_6\,(m-1, k+4)\,(m-1, 7)\,\gamma_{-4}\\ &= (0, 2, 4, 6, \ldots, k-2, 3, 5, 7, \ldots, m-2, 1, k, k+2,\\ &\quad k+4, \ldots, m-3)(m-1).\end{aligned}$$

Now, we aim to show that the remaining pairs of D'_T are truncated hamiltonian pairs. Note that

$$\hat{\mu}_i = \mu_i\,(m-1, i+1) \text{ and that } \hat{\mu}_{i+3} = \mu_{i+3}\,(m-1, i+4) \text{ if } i \in M_t.$$

We begin with pairs of the form (μ_i, σ_{m-i-3}). Since $i \equiv 3 \pmod 4$ and $3 \leqslant i \leqslant m-6$, then $i = 2j+1$ where j is odd and $1 \leqslant j \leqslant k-3$. Moreover, we note that Lemma 4.74 and Construction 4.75 jointly imply that

$$\hat{\sigma}_{m-i-3} = \gamma_{-i-2}.$$

Lastly, Lemma 4.74 implies that $(m-j)^{\gamma_i} = j+2$ and $(j+2)^{\gamma_{-i-2}} = m-j-2$. Consequently, we see that

$$\begin{aligned}\hat{\mu}_i\hat{\sigma}_{m-i-3} &= (1, m-1)\,\gamma_i\,(m-1, j+2)\,(m-1, i+1)\,\gamma_{-i-2}\\ &= (0, m-3, m-5, \ldots, m-j, m-2, m-4, \ldots, 3, 1, m-j-2,\\ &\quad m-j-4, m-j-6, \ldots, 2)(m-1).\end{aligned}$$

Hence, all pairs of D'_T of the form (μ_i, σ_{m-i-3}) are truncated hamiltonian.

Next, we show that pairs of the form $(\mu_{i+3}, \sigma_{m-i-2})$ are truncated hamiltonian. First, note that

$$\hat{\sigma}_{m-i-2} = \gamma_{-i-1}.$$

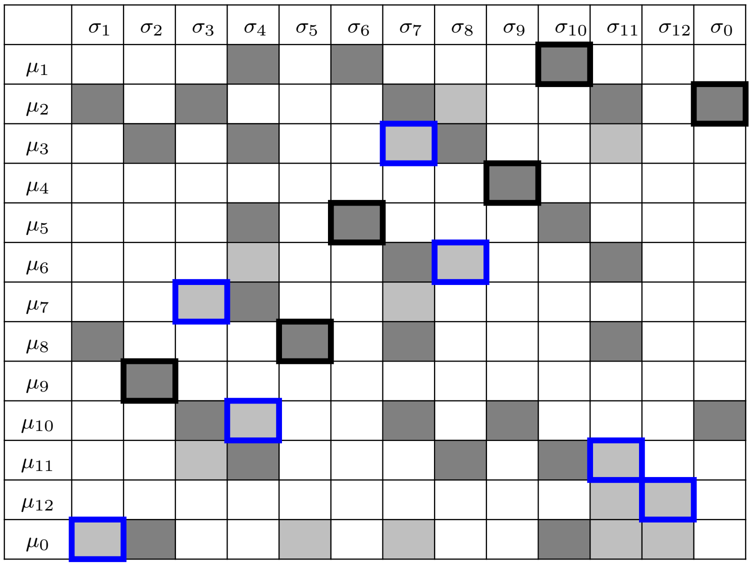
Furthermore, we note that $i+3 = 2j+4 = 2(j+2)$. Construction 4.75 then implies that $\sigma_{i+3} = \gamma_{i+3}\,(m-1, k+j+3)$. This means that $\mu_{i+3} = (1, m-1)\gamma_{i+3}\,(m-1, k+j+3)$. Therefore

$$\begin{aligned}
\hat{\mu}_{i+3}\hat{\sigma}_{m-i-2} &= (1, m-1)\,\gamma_{i+3}\,(m-1, k+j+3)\,(m-1, i+4)\,\gamma_{-i-1} \\
&= (0, 2, 4, 6, \ldots, k-j-1, 3, 5, \ldots, m-2, 1, k-j+1, k-j+3, \\
&\quad\; k-j+5, \ldots, m-3)(m-1).
\end{aligned}$$

In conclusion, all pairs of D'_T are truncated hamiltonian pairs.

Next we form the following set of $m-2t-3$ pairs:

$$\begin{aligned}
D'_H = \; & \{(\mu_i, \sigma_{m-i-2}) \mid i \text{ odd}, i \notin M_t, \text{and } 3 \leqslant i \leqslant m-4\} \cup \\
& \{(\mu_i, \sigma_{m-i}) \mid i \text{ even}, i-3 \notin M_t, \text{and } 4 \leqslant i \leqslant m-3\} \cup \{(\mu_2, \sigma_0)\}.
\end{aligned}$$



	σ_1	σ_2	σ_3	σ_4	σ_5	σ_6	σ_7	σ_8	σ_9	σ_{10}	σ_{11}	σ_{12}	σ_0
μ_1				▓		▓				[▓]			
μ_2	▓		▓				▓	░			▓		[▓]
μ_3		▓		▓			[░]	▓			░		
μ_4									[▓]				
μ_5				▓		[▓]				▓			
μ_6				░			▓	[░]			▓		
μ_7			[░]	▓			░						
μ_8	▓				[▓]		▓				▓		
μ_9		[▓]											
μ_{10}			▓	[░]			▓		▓				▓
μ_{11}			░	▓				▓		▓	[░]		
μ_{12}											░	[░]	
μ_0	[░]	▓			░		░			▓	░	░	

Figure 4.37: Case 2: hamiltonian table of $S_1 \times S_2$ for $m = 13$ with a 7-twined 2-factorization in black (hamiltonian pairs) and blue (truncated hamiltonian pairs).

Assume that D_H is the set of hamiltonian pairs given in Case 1. Then, we see that $D'_H \subseteq D_H$. It follows from Case 1 that all pairs of D'_H are hamiltonian pairs. Let $D' = D'_T \cup D'_H$ and refer to Figure 4.37 for an illustration of D' when $m = 13$ and $M_t = \{3, 7\}$. It is tedious but straightforward to verify that each permutation of S_1

and S_2 appears in precisely one pair of D'. As a result, the set D' is a 2-factorization of $\vec{C}_2 \wr \overline{K}_m$. Note that D'_T and D'_H are disjoint. Consequently, the set D' is a c-twined 2-factorization of $\vec{C}_2 \wr \overline{K}_m$. ∎

Proposition 4.86. *Let $m \geqslant 7$ and c be odd. The digraph $\vec{C}_2 \wr \overline{K}_m$ admits a c-twined 2-factorization in each of the following cases:*

(S1) *$m \equiv 3 \pmod 8$ and $3 \leqslant c \leqslant \frac{m-5}{2}$;*

(S2) *$m \equiv 7 \pmod 8$ and $3 \leqslant c \leqslant \frac{m-1}{2}$.*

Proof: Let $m = 2k+1$ where k is odd. In this proof, it is understood that, for each $i \in \mathbb{Z}_{m-1}$, permutation $\gamma_i \in G_{m-1}$ is as defined in Definition 4.73. We will be using the following two sets of permutations:

$$S_1 = (0, m-1) \cdot \mathcal{F}_m = \{\mu_i = (0, m-1)\sigma_i \mid i = 0, 1, \ldots, m-1\};$$
$$S_2 = \mathcal{F}_m = \{\sigma_0, \sigma_1, \ldots, \sigma_{m-1}\}.$$

Lemmas 4.60 and 4.77 imply that S_1 and S_2 are regular permutation sets. See Construction 4.75 for the definition of each $\sigma_i \in \mathcal{F}_m$. Refer to Figure 4.38 for the hamiltonian table of $S_1 \times S_2$ when $m = 15$. Observe that μ_3 and σ_0 are the $(m-1)$-stabilizers of S_1 and S_2 respectively. We also point out that $(m-1)^{\mu_i} = 0^{\sigma_i} = i$ if $i \notin \{3, m-1\}$ and $(m-1)^{\mu_{m-1}} = 3$. This means that

$$\hat{\mu}_i = \mu_i(m-1, i) \text{ if } i \notin \{3, m-1\}, \text{ and } \hat{\mu}_{m-1} = \mu_{m-1}(m-1, 3).$$

We also point out that

$$\hat{\sigma}_i = \gamma_i \text{ for } i \notin \{0, m-1\}.$$

<u>Case 1</u>: $m \equiv 3 \pmod 4$ and $c = 3$. We begin by forming the following set of three pairs:

$$D_T = \{(\mu_{m-2}, \sigma_{m-2}), (\mu_{m-1}, \sigma_{m-1}), (\mu_0, \sigma_1)\}.$$

We now show that $(\mu_{m-2}, \sigma_{m-2})$, $(\mu_{m-1}, \sigma_{m-1})$, and (μ_0, σ_1) are truncated hamiltonian pairs. Note that $\hat{\mu}_0 = id$ and that Lemma 4.74 implies that $\gamma_{m-2} = \gamma_{-1}$:

$$\begin{aligned}
\hat{\mu}_{m-2}\hat{\sigma}_{m-2} &= (0, m-1)\gamma_{-1}\,(m-1, k+1)(m-1, m-2)\,\gamma_{-1} \\
&= (0, k, k-2, \ldots, 3, 1, m-2, m-4, \ldots, k+2, m-3, m-5, \ldots, \\
&\quad k+1, k-1, \ldots, 2)(m-1);
\end{aligned}$$

$$\begin{aligned}
\hat{\mu}_{m-1}\hat{\sigma}_{m-1} &= (0, m-1)\ (0, 3, m-2, 4, \ldots, k+2, k+1, m-1, 1, 2)\,(m-1, 3) \\
&\quad (0, 3, m-2, 4, m-3, 5, \ldots, k+2, k+1, m-1, 1, 2)\,(m-1, 1) \\
&= (0, 2, 3, 4, 5, 6, \ldots, k+1, m-2, m-3, \ldots, k+2, 1)(m-1); \\
\hat{\mu}_0\hat{\sigma}_1 &= \gamma_1.
\end{aligned}$$

Next, we form a set of $\frac{m-5}{2} + 1$ pairs as follows:

$$D_{H_1} = \{(\mu_i, \sigma_{m-i-2}) \mid i \text{ is odd and } 1 \leqslant i \leqslant m-4\} \cup \{(\mu_2, \sigma_0)\}.$$

Observe that D_T and D_{H_1} jointly use each permutation $\mu_i \in S_1$ such that i is odd exactly once. Moreover, the sets D_T and D_{H_1} jointly use each permutation $\sigma_r \in S_2$, such that r is even, precisely once. Below, we show that each pair of D_{H_1} is a hamiltonian pair, starting with the pairs (μ_1, σ_{m-3}) and (μ_2, σ_0):

$$\begin{aligned}
\mu_1\,\sigma_{m-3} &= (0, m-1)\,\gamma_1\,(m-1, 2)\,\gamma_{-2}\,(m-1, 0) \\
&= (0, m-1, m-2, m-3, m-4, \ldots, 1); \\
\mu_2\,\sigma_0 &= (0, m-1)\,\gamma_2\,(m-1, k+2)id \\
&= (0, k+2, k+4, \ldots, m-2, 1, 3, \ldots, k, m-1, 2, 4, 6, \ldots, m-3).
\end{aligned}$$

If i is odd and $3 \leqslant i \leqslant m-4$, then $i = 2j+1$ where $1 \leqslant j \leqslant k-2$. Lemma 4.78 implies that

$$\sigma_{m-i-2} = \gamma_{-i-1}\,(m-1, m-j-1).$$

Moreover, Lemma 4.74 implies that $(j+2)^{\gamma_{-i-1}} = m-j-1$ and $(m-j)^{\gamma_i} = j+2$. We see that

$$\begin{aligned}
\mu_i\,\sigma_{m-i-2} &= (0, m-1)\,\gamma_i\,(m-1, j+2)\,\gamma_{-i-1}\,(m-1, m-j-1) \\
&= (0, m-1, m-2, m-3, m-4, \ldots, m-j, m-j-1, \\
&\quad m-j-2, \ldots, 1).
\end{aligned}$$

Hence, all pairs of D_{H_1} are hamiltonian pairs.

For our next step, we form a set of $\frac{m-5}{2}$ pairs as follows:

$$D_{H_2} = \{(\mu_i, \sigma_{m-i}) \mid i \text{ is even and } 4 \leqslant i \leqslant m-3\}.$$

Observe that D_T and D_{H_2} jointly use each permutation $\mu_i \in S_1$ such that i is even exactly once except for σ_2. Furthermore, the sets D_T and D_{H_1} jointly use each permutation $\sigma_r \in S_2$ such that r is odd precisely once. Below, we show that each pair of D_{H_2} is a hamiltonian pair.

If i is even and $4 \leqslant i \leqslant m-5$, then $i = 2j$ and $2 \leqslant j \leqslant k-2$. Lemma 4.78 implies that

$$\sigma_{m-i} = \gamma_{-i+1}\,(m-1, k-j+2).$$

Then

$$\begin{aligned} \mu_i\,\sigma_{m-i} &= (0, m-1)\,\gamma_i\,(m-1, k+j+1)\,\gamma_{-i+1}\,(m-1, k-j+2) \\ &= (0, m-1, 1, 2, 3, \ldots, k-j+1, k-j+2, k-j+3, \ldots, m-3, m-2). \end{aligned}$$

If $i = m-3$, then

$$\begin{aligned} \mu_{m-3}\,\sigma_3 &= (0, m-1)\,\gamma_{-2}\,(m-1, 0)\,\gamma_3\,(m-1, 3) \\ &= (0, m-1, 1, 2, 3, 4, \ldots, m-3, m-2). \end{aligned}$$

Therefore, the set D_{H_2} is a set of hamiltonian pairs.

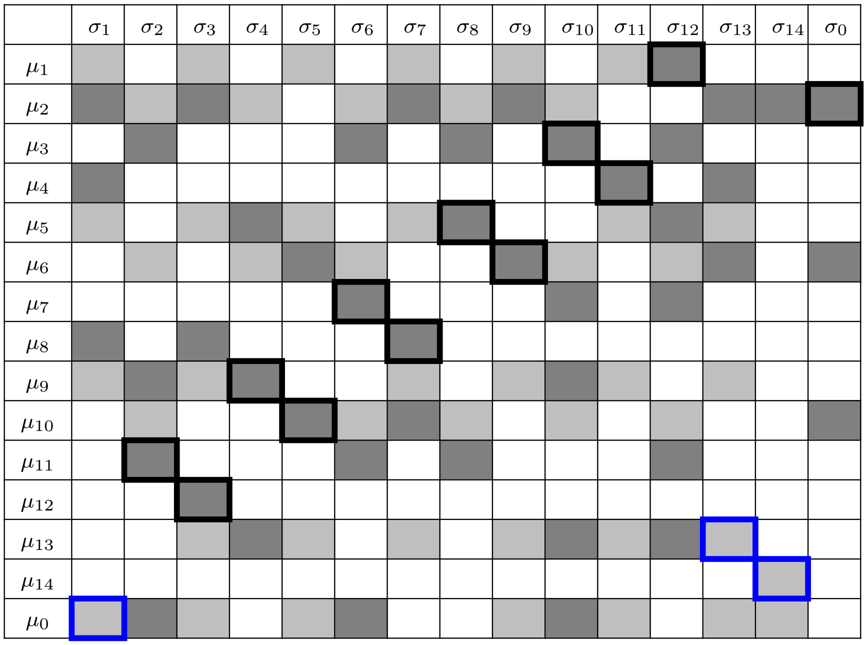
Let $D = D_T \cup D_{H_1} \cup D_{H_2}$ and let $D_H = D_{H_1} \cup D_{H_2}$. Note that D_H and D_T are disjoint. Refer to Figure 4.38 for an illustration of D when $m = 15$. Observe that each permutation of S_1 and S_2 is used by precisely one pair of D. As a result, the set D is a 3-twined 2-factorization of $\vec{C}_2 \wr \overline{K}_m$.

<u>Case 2</u>: $m \equiv 3 \pmod 8$ and $5 \leqslant c \leqslant \frac{m-5}{2}$. Let $c = 3 + 2t$ where $1 \leqslant t \leqslant \frac{m-11}{4}$. Let $\mathbb{I} = \{i \mid i \equiv 5, 6 \pmod 8 \text{ and } 5 \leqslant i \leqslant m-13\}$ and let M_t be a subset of size t of $\mathbb{I}$. Observe that $\mathbb{I}$ contains $\frac{m-11}{4}$ elements.

We will form two sets of truncated pairs D_{T_1} and D_{T_2}. Let

$$D_{T_1} = \{(\mu_i, \sigma_{m-i-5}), (\mu_{i+5}, \sigma_{m-i-2}) \mid i \in M_t \text{ and } i \equiv 5 \pmod 8\}.$$

We show that all pairs of D_{T_1} are truncated hamiltonian pairs. Since $m \equiv 3 \pmod 8$ and $m = 2k+1$, then $k \equiv 1 \pmod 4$. Additionally, if $i \equiv 5 \pmod 8$, then $i = 2j+1$ where $j \equiv 2 \pmod 4$ and $2 \leq j \leq k-7$. We also remind the reader that $(m-1)^{\mu_i} = 0^{\sigma_i} = i$ if $i \notin \{3, m-1\}$. It follows that



	σ_1	σ_2	σ_3	σ_4	σ_5	σ_6	σ_7	σ_8	σ_9	σ_{10}	σ_{11}	σ_{12}	σ_{13}	σ_{14}	σ_0
μ_1	░		░		░		░		░		░	▣			
μ_2	▓	░	▓	░		░	▓	░	▓	░			▓	▓	▣
μ_3		▓				▓		▓		▣		▓			
μ_4	▓										▣		▓		
μ_5	░		░	▓	░		░	▣			░	▓	░		
μ_6		░		░	▓	░			▣	░		░	▓		▓
μ_7						▣				▓		▓			
μ_8	▓		▓				▣								
μ_9	░	▓	░	▣			░		░	▓	░		░		
μ_{10}		░			▣	░	▓	░		░		░			▓
μ_{11}		▣				▓		▓				▓			
μ_{12}			▣												
μ_{13}			░	▓	░		░		░	▓	░	▓	▤		
μ_{14}														▤	
μ_0	▤	▓	░		░	▓			░	▓	░		░	░	

Figure 4.38: Case 1: hamiltonian table of $S_1 \times S_2$ for $m = 15$ with a 3-twined 2-factorization in black (hamiltonian pairs) and blue (truncated hamiltonian pairs).

$$\hat{\mu}_i = (0, m-1)\, \sigma_i (m-1, i) \text{ if } i \notin \{3, m-1\}.$$

Lastly, Lemma 4.74 implies that $\gamma_{m-i-5} = \gamma_{-i-4}$ and that $(j+2)^{\gamma_{-i-4}} = m-j-4$ and $(m-j)^{\gamma_i} = j+2$. Then we see that

$$\begin{aligned} \hat{\mu}_i \hat{\sigma}_{m-i-5} &= (0, m-1)\, \sigma_i\, (m-1, i)\, \gamma_{-i-4} \\ &= (0, m-1)\, \gamma_i\, (m-1, j+2)\, (m-1, i)\, \gamma_{-i-4} \\ &= (0, m-j-4, m-j-8, \ldots, 5, 1, m-4, m-8, m-12, \ldots, 7, 3, \\ &\quad m-2, m-6, \ldots, m-j, m-5, m-9, \ldots, 2, m-3, m-7, \\ &\quad \ldots, 8, 4)(m-1); \end{aligned}$$

Next, we will show that pairs of form $(\mu_{i+5}, \sigma_{m-i-2}) \in D_{T_1}$ are truncated hamiltonian pairs. To do so, we note that $i+5 = 2j+6 = 2(j+3)$. Construction 4.75 then implies that $\sigma_{i+5} = \gamma_{i+5}(m-1, k+j+4)$. We then see that

$$\begin{aligned}
\hat{\mu}_{i+5}\hat{\sigma}_{m-i-2} &= (0, m-1)\,\sigma_{i+5}\,(m-1, i+5)\,\gamma_{-i-1} \\
&= (0, m-1)\,\gamma_{i+5}\,(m-1, k+j+4)\,(m-1, i+5)\,\gamma_{-i-1} \\
&= (0, k-j+2, k-j+6, k-j+10, \dots, m-2, 3, 7, \dots, m-4, 1, \\
&\quad\; 5, 9, 13, \dots, k-j-2, 4, 8, 12, \dots, m-3, 2, 6, \dots, m-5)(m-1).
\end{aligned}$$

Therefore, all pairs of D_{T_1} are truncated hamiltonian pairs.

Next, we let

$$D_{T_2} = \{(\mu_i, \sigma_{m-i-5}), (\mu_{i+3}, \sigma_{m-i}) \mid i \in M_t \text{ and } i \equiv 6 \pmod 8\}.$$

We now show that all pairs of D_{T_2} are truncated hamiltonian pairs. Recall that $k \equiv 1 \pmod 4$. In addition, if $i \equiv 6 \pmod 8$, then $i = 2j$ where $j \equiv 3 \pmod 4$ and $3 \le j \le k-6$. We then see that

$$\begin{aligned}
\hat{\mu}_i\hat{\sigma}_{m-i-5} &= (0, m-1)\,\gamma_i\,(m-1, k+j+1)\,(m-1, i)\,\gamma_{-i-4} \\
&= (0, k-j-3, k-j-7, \dots, 3, m-2, m-6, \dots, 5, 1, m-4, m-8, \\
&\quad\; \dots, k-j+1, m-5, m-9, \dots, 2, m-3, m-7, \dots, 8, 4)(m-1).
\end{aligned}$$

Now, we aim to show that pairs of the form $(\mu_{i+3}, \sigma_{m-i}) \in D_{T_2}$ are truncated hamiltonian pairs. We start with the case $i = 6$. Note that $(m-4)^{\gamma_9} = 6$. Therefore, we have that

$$\begin{aligned}
\hat{\mu}_9\hat{\sigma}_{m-6} &= (0, m-1)\,\gamma_9\,(m-1, 6)\,(m-1, 9)\,\gamma_{-5} \\
&= (0, 1, 5, 9, 13, \dots, m-2, 3, 7, \dots, m-4, 4, 8, 16, \dots, m-3, 2, 6, \\
&\quad\; \dots, m-5)(m-1).
\end{aligned}$$

Now, we show that pairs of the form $(\mu_{i+3}, \sigma_{m-i}) \in D_{T_2}$ are truncated hamiltonian pairs for $10 \leqslant i \leqslant m-13$. First, we point out that $i+3 = 2j+3 = 2(j+1)+1$. Construction 4.75 implies that $\sigma_{i+3} = \gamma_{i+3}(m-1, j+3)$. Lastly note that Lemma 4.74 implies that $(j+3)^{\gamma_{-i+1}} = m-j+3$ and $(m-j-1)^{\gamma_{i+3}} = j+3$. We then have

$$\begin{aligned}
\hat{\mu}_{i+3}\hat{\sigma}_{m-i} &= (0, m-1)\,\gamma_{i+3}\,(m-1, j+3)\,(m-1, i+3)\,\gamma_{-i+1} \\
&= (0, m-j+3, m-j+7, m-j+11, \dots, m-4, 1, 5, 9, 13, \dots, m-2, \\
&\quad\; 3, 7, \dots, m-j-1, 4, 8, \dots, m-3, 2, 6, \dots, m-5)(m-1).
\end{aligned}$$

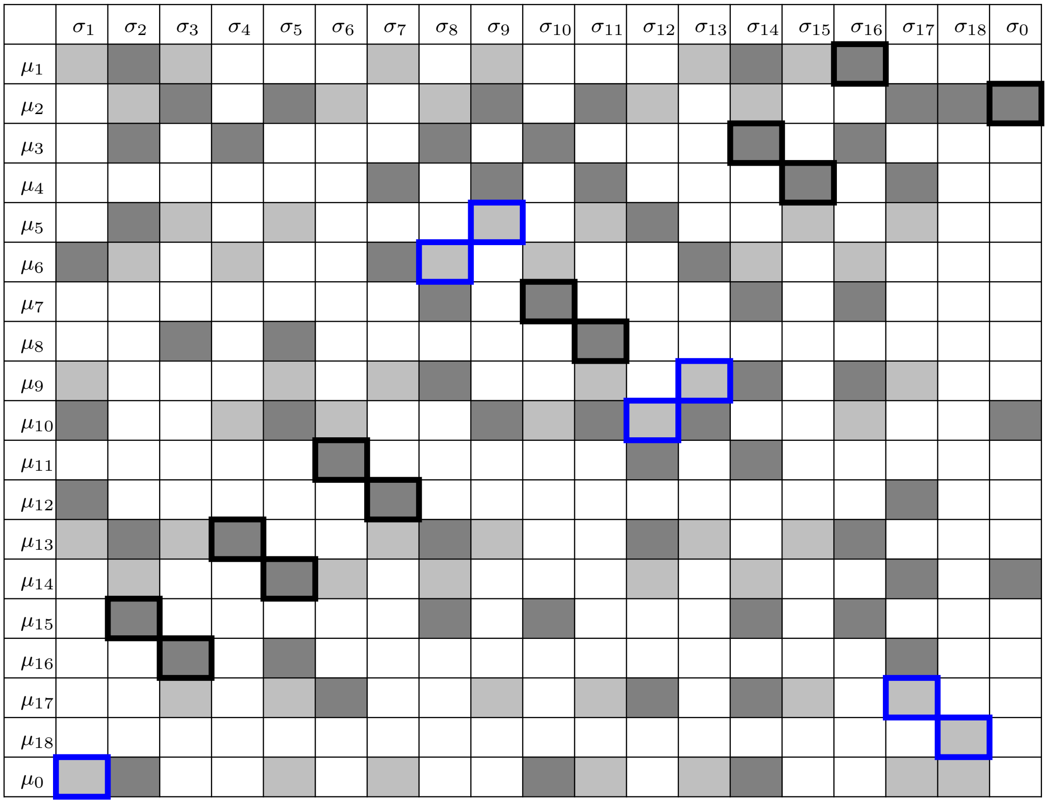
In conclusion, all pairs in D_{T_2} are truncated hamiltonian.

We let D_T be the set of three truncated hamiltonian pairs defined in Case 1 and let

$$D'_T = D_{T_1} \cup D_{T_2} \cup D_T.$$

Observe that D'_T contains $2t+3$ truncated hamiltonian pairs. Next we form the following set of $m-2t-3$ pairs:

$$\begin{aligned} D'_H = \quad & \{(\mu_i, \sigma_{m-i-2}) \mid i, i-3 \notin M_t, i \text{ odd, and } 1 \leqslant i \leqslant m-4\} \cup \\ & \{(\mu_i, \sigma_{m-i}) \mid i, i-5 \notin M_t, i \text{ even, and } 4 \leqslant i \leqslant m-3\} \cup \{(\mu_2, \sigma_0)\}. \end{aligned}$$



	σ_1	σ_2	σ_3	σ_4	σ_5	σ_6	σ_7	σ_8	σ_9	σ_{10}	σ_{11}	σ_{12}	σ_{13}	σ_{14}	σ_{15}	σ_{16}	σ_{17}	σ_{18}	σ_0
μ_1	░	▓	░				░		░				░	▓	░	▣			
μ_2		░	▓		▓	░		░	▓		▓	░		░			▓	▓	▣
μ_3		▓		▓				▓		▓				▣		▓			
μ_4							▓		▓		▓				▣		▓		
μ_5		▓	░		░				▢		░	▓			░		░		
μ_6	▓	░		░			▓	▢		░			▓	░		░			
μ_7								▓		▣				▓		▓			
μ_8			▓		▓						▣								
μ_9	░				░		░	▓			░		▢	▓		▓	░		
μ_{10}	▓			░	▓	░			▓	░	▓	▢	▓			░			▓
μ_{11}						▣						▓		▓					
μ_{12}	▓						▣										▓		
μ_{13}	░	▓	░	▣			░	▓	░			▓	░		░	▓			
μ_{14}		░			▣	░		░				░		░			▓		▓
μ_{15}		▣						▓		▓				▓		▓			
μ_{16}			▣		▓												▓		
μ_{17}			░		░	▓			░		░	▓		▓	░		▢		
μ_{18}																		▢	
μ_0	▢	▓			░		░			▓	░		░	▓			░	░	

Figure 4.39: Case 2: hamiltonian table of $S_1 \times S_2$ for $m = 19$ with a 7-twined 2-factorization in black (hamiltonian pairs) and blue (truncated hamiltonian pairs).

Let D_H refer to the set of hamiltonian pairs from Case 1. Observe that $D'_H \subseteq D_H$, implying that all pairs in D'_H are hamiltonian. Then, we let $D' = D'_T \cup D'_H$. Refer to Figure 4.39 for an illustration of D' when $m = 19$ and $M_t = \{5, 6\}$. It is laborious yet straightforward to verify that each permutation of S_1 and S_2 appears in exactly one pair of D'. Moreover, the sets D'_T and D'_H are disjoint and thus form a partition of D'. As a result, the set D' is a c-twined 2-factorization of $\vec{C}_2 \wr \overline{K}_m$.

<u>Case 3</u>: $m \equiv 7 \pmod 8$ and $5 \leqslant c \leqslant \frac{m-1}{2}$. Let $c = 2t+3$ where $1 \leqslant t \leqslant \frac{m-7}{4}$. Assume

that $\mathbb{I} = \{i \mid i \equiv 5, 6 \pmod 8 \text{ and } 5 \leqslant i \leqslant m-7\}$ and let M_t be a subset of size t of $\mathbb{I}$. Observe that $\mathbb{I}$ contains $\frac{m-7}{4}$ elements. We will form two sets of truncated pairs, which we call D_{T_3} and D_{T_4}, such that $D_{T_3} \cup D_{T_4}$ contains $2t$ pairs.

First, we form D_{T_3} as follows:

$$D_{T_3} = \{(\mu_i, \sigma_{m-i-5}), (\mu_{i+5}, \sigma_{m-i-2}) \mid i \in M_t \text{ and } i \equiv 5 \pmod 8\}.$$

We show that all pairs of D_{T_3} are truncated hamiltonian. See Case 2 for a description of $\hat{\mu}_i$ and for a proof that pairs of the form $(\mu_i, \sigma_{m-i-5}) \in D_{T_3}$ are truncated hamiltonian.

Next, we aim to show that pairs of the form $(\mu_{i+5}, \sigma_{m-i-2}) \in D_{T_3}$ are truncated hamiltonian pairs. Since $m \equiv 7 \pmod 8$ and $m = 2k+1$, then $k \equiv 3 \pmod 4$. If $i \equiv 5 \pmod 8$, then $i = 2j+1$ where $j \equiv 2 \pmod 4$ and $2 \leq j \leq k-5$. Therefore, we see that

$$\begin{aligned}\hat{\mu}_{i+5}\hat{\sigma}_{m-i-2} &= (0, m-1)\,\gamma_{i+5}\,(m-1, k+j+4)\,(m-1, i+5)\,\gamma_{-i-1}\\ &= (0, k-j+2, k-j+6, k-j+10, \ldots, m-4, 1, 5, 9, \ldots, m-2,\\ &\qquad 3, 7, \ldots, k-j-2, 4, 8, 12, \ldots, m-3, 2, 6, \ldots, m-5)(m-1).\end{aligned}$$

Consequently, all pairs of D_{T_3} are truncated hamiltonian pairs.

Next, we create the following set:

$$D_{T_4} = \{(\mu_i, \sigma_{m-i-5}), (\mu_{i+3}, \sigma_{m-i}) \mid i \in M_t \text{ and } i \equiv 6 \pmod 8\}.$$

Now, we show that all pairs of D_{T_4} are truncated hamiltonian pairs. Proof that pairs of the form $(\mu_{i+3}, \sigma_{m-i}) \in D_{T_4}$ are truncated hamiltonian follows from Case 2.

Hence, it remains to show that pairs of the form $(\mu_i, \sigma_{m-i-5}) \in D_{T_4}$ are truncated hamiltonian pairs. Recall that $k \equiv 3 \pmod 4$. Moreover, if $i \equiv 6 \pmod 8$ and $i = 2j$, then $j \equiv 3 \pmod 4$ and $3 \leq j \leq k-4$. Consequently, we see that

$$\begin{aligned}\hat{\mu}_i\hat{\sigma}_{m-i-5} &= (0, m-1)\,\gamma_i\,(m-1, k+j+1)\,(m-1, i)\,\gamma_{-i-4}\\ &= (0, k-j-3, k-j-7, \ldots, 1, m-4, m-8, \ldots, 3, m-2, m-6, \ldots,\\ &\qquad k-j+1, m-5, m-9, \ldots, 2, m-3, m-7, \ldots, 8, 4)(m-1).\end{aligned}$$

In conclusion, all pairs of D_{T_4} are truncated hamiltonian pairs.

We let D_T be the set of three truncated hamiltonian pairs defined in Case 1 and let

$$D'_T = D_{T_3} \cup D_{T_4} \cup D_T.$$

Observe that D'_T contains $2t+3$ truncated hamiltonian pairs.

Next we form the following set of $m-2t-3$ pairs:

$$\begin{aligned} D'_H = \ & \{(\mu_i, \sigma_{m-i-2}) \mid i, i-3 \notin M_t, i \text{ odd, and } 1 \leqslant i \leqslant m-4\}\} \cup \\ & \{(\mu_i, \sigma_{m-i}) \mid i, i-5 \notin M_t, i \text{ even, and } 4 \leqslant i \leqslant m-3\} \cup \{(\mu_2, \sigma_0)\}. \end{aligned}$$

Let D_H be the set of hamiltonian pairs from Case 1. Observe that $D'_H \subseteq D_H$. Therefore, all pairs in D'_H are hamiltonian. Then, we let $D' = D'_T \cup D'_H$. Refer to Figure 4.40 for an illustration of D' when $m = 15$ and $M_t = \{5, 6\}$. It is laborious yet straightforward to verify that each permutation of S_1 and S_2 appears in exactly one pair of D'. In addition, the sets D'_T and D'_H are disjoint and thus form a partition of D'. As a result, the set D' is a c-twined 2-factorization of $\vec{C}_2 \wr \overline{K}_m$. ∎

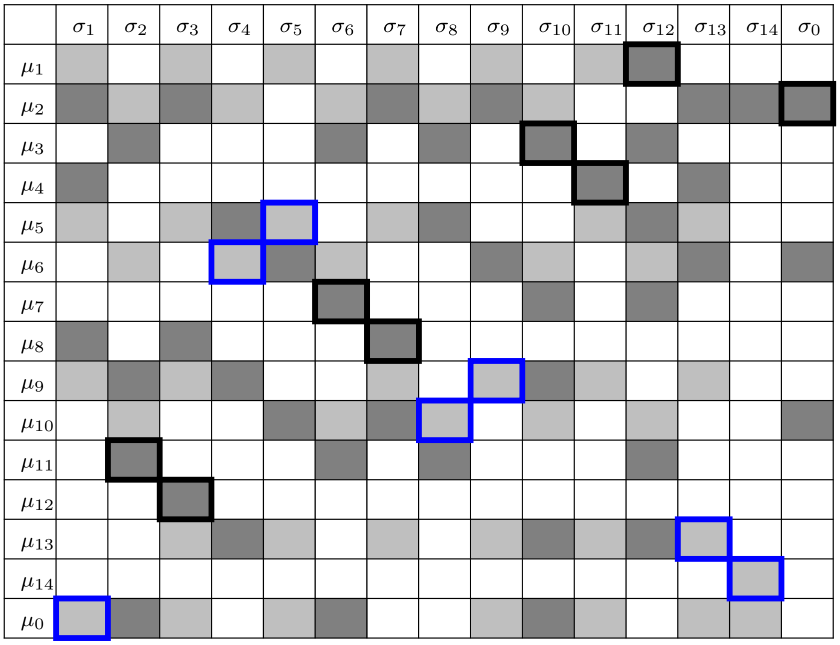


Figure 4.40: Case 3: hamiltonian table of $S_1 \times S_2$ for $m = 15$ with a 7-twined 2-factorization in black (hamiltonian pairs) and blue (truncated hamiltonian pairs).

4.7.2 The case c is large

We now proceed with the construction of a c-twined 2-factorization of $\vec{C}_2 \wr \overline{K}_m$ when $\frac{m-5}{2} \leqslant c \leqslant m-2$. In fact, the constructions given in Propositions 4.87–4.89 apply to $\lceil \frac{m}{3} \rceil + 5 \leqslant c \leqslant m-2$. Therefore, there exists some overlap with the constructions given in Propositions 4.85 and 4.86.

To address the case $\frac{m-5}{2} \leqslant c \leqslant m-2$, we consider three cases according to the congruency class of m modulo 6. The most complicated construction is given for $m \equiv 1 \pmod 6$ because, in this case, $m-1$ is not relatively prime with 3. As for the cases $m \equiv 3$ or $5 \pmod 6$, we rely on the fact that $m-1$ is relative prime with 3 to obtain the desired c-twined 2-factorizations of $\vec{C}_2 \wr \overline{K}_m$.

Proposition 4.87. *Let $m \equiv 1 \pmod 6$ such that $m \geqslant 7$, and let c be an odd integer such that $\frac{m+2}{3} \leq c \leq m-2$. The digraph $\vec{C}_2 \wr \overline{K}_m$ admits a c-twined 2-factorization.*

Proof: Let $m = 2k+1$. If $m \equiv 1 \pmod{12}$, then k is even; otherwise, if $m \equiv 7 \pmod{12}$, then k is odd. In both cases, we see that $k \equiv 0 \pmod 3$. In this proof, it is understood that, for each $i \in \mathbb{Z}_{m-1}$, permutation $\gamma_i \in G_{m-1}$ is as defined in Definition 4.73. We will be using the following two sets of permutations:

$$S_1 = (m-1, 1, 2, 3) \cdot \mathcal{F}_m = \{\mu_i = (m-1, 1, 2, 3)\sigma_i \mid i = 0, 1, \ldots, m-1\};$$

$$S_2 = \mathcal{F}_m = \{\sigma_0, \sigma_1, \ldots, \sigma_{m-1}\}.$$

Lemmas 4.60 and 4.77 imply that S_1 and S_2 are regular permutation sets. Refer to Construction 4.75 for the definition of each $\sigma_i \in \mathcal{F}_m$. See Figure 4.41 for the hamiltonian table of $S_1 \times S_2$ when $m = 13$. Observe that μ_1 and σ_0 are the $(m-1)$-stabilizers of S_1 and S_2, respectively. Furthermore, we point out that $(m-1)^{\mu_i} = 1^{\sigma_i}$. This means that

$$(m-1)^{\mu_i} = i+1 \text{ when } i \notin \{1, m-1\} \text{ and } (m-1)^{\mu_{m-1}} = 2.$$

<u>Case 1</u>: $c = \frac{m+2}{3}$. We form a set of m pairs that can be partitioned into one set of $\frac{m+2}{3}$ truncated hamiltonian pairs and one set of $\frac{2m-2}{3}$ hamiltonian pairs. We start by forming a set D_s of four pairs, two of which will be truncated hamiltonian pairs and two of which will be hamiltonian pairs:

$$D_s = \{(\mu_1, \sigma_2), (\mu_2, \sigma_0), (\mu_{m-1}, \sigma_{m-1}), (\mu_0, \sigma_1)\}.$$

Below, we show that (μ_1, σ_2) is a hamiltonian pair. We consider two cases.

If $m = 7$, then

$$\begin{aligned} \mu_1\sigma_2 &= (6,1,2,3)\,\gamma_1\,(6,2)\gamma_2(6,5) \\ &= (0,3,4,1,6,5,2). \end{aligned}$$

Otherwise, if $m \geqslant 13$, then

$$\begin{aligned} \mu_1\sigma_2 &= (m-1,1,2,3)\,\gamma_1\,(m-1,2)\gamma_2(m-1,k+2) \\ &= (0,3,4,7,10,\ldots,m-3,1,5,8,\ldots,k-1,m-1,k+2,k+5, \\ &\quad \ldots,m-2,2,6,9,\ldots,m-4). \end{aligned}$$

We now show that (μ_2, σ_0) is a hamiltonian pair. The resulting product depends on the congruency classes of m modulo 12. If $m \equiv 1 \pmod{12}$ and $m = 2k+1$, then k is even and thus

$$\begin{aligned} \mu_2\sigma_0 &= (m-1,1,2,3)\,\gamma_2\,(m-1,k+2)\,id \\ &= (0,2,5,7,\ldots,m-2,1,4,6,8,\ldots,k,m-1, \\ &\quad 3,k+2,k+4,k+6,\ldots,m-3). \end{aligned}$$

If $m = 7$, then

$$\begin{aligned} \mu_2\sigma_0 &= (6,1,2,3)\,\gamma_2\,(6,5)\,id \\ &= (0,2,6,3,5,1,4). \end{aligned}$$

If $m \equiv 7 \pmod{12}$, $m \geqslant 19$, and $m = 2k+1$, then k is odd and thus

$$\begin{aligned} \mu_2\sigma_0 &= (m-1,1,2,3)\,\gamma_2\,(m-1,k+2)\,id \\ &= (0,2,5,7,9,\ldots,k,m-1,3,k+2,k+4, \\ &\quad k+6,\ldots,m-2,1,4,6,\ldots,m-3). \end{aligned}$$

Next, we show that $(\mu_{m-1}, \sigma_{m-1})$ is a truncated hamiltonian pair. First, we compute the expression for $\hat{\mu}_{m-1}$:

$$\begin{aligned} \hat{\mu}_{m-1} &= (m-1,1,2,3)\,(0,3,m-2,4,m-3,\ldots,k,k+2,k+1, \\ &\quad m-1,1,2)\,(m-1,2) \\ &= (0,3,1)\,(2,m-2,4,m-3,5,\ldots,k,k+2,k+1)(m-1). \end{aligned}$$

We then see that

$$\begin{aligned}\hat{\mu}_{m-1}\hat{\sigma}_{m-1} &= (0,3,1)\,(2,m-2,4,m-3,5,\ldots,k+3,k,k+2,k+1)(m-1)(0,3,\\ &\quad m-2,4,\ldots,k,k+2,k+1,1,2)(m-1)\\ &= (0,m-2,m-3,m-4,\ldots,k+3,k+2,1,3,2,4,5,\\ &\quad \ldots,k,k+1)(m-1);\end{aligned}$$

Below, we show that (μ_0, σ_1) is also a truncated hamiltonian pair. Note that $\hat{\mu}_0 = (1,2,3)$. This means that

$$\begin{aligned}\hat{\mu}_0\hat{\sigma}_1 &= (1,2,3)\,\gamma_1\\ &= (0,1,3,2,4,5,6,\ldots,m-2)(m-1).\end{aligned}$$

In summary, the set D_s is comprised of two hamiltonian pairs and two truncated hamiltonian pairs.

Next, we construct a set of $\frac{m-4}{3}$ pairs denoted D_{T_1}:

$$D_{T_1} = \{(\mu_i, \sigma_{m-i+1}) \mid i \equiv 2,3 \pmod 6 \text{ and } 3 \leqslant i \leqslant m-4\}$$

Since $m \equiv 1 \pmod 6$, it follows that D_{T_1} contains exactly $\frac{m-1}{3} - 1$ pairs. Moreover, we point out that each permutation $\mu_i \in S_1$ such that $i \equiv 2,3 \pmod 6$ appears in exactly one pair of $D_{T_1} \cup D_s$. In addition, each permutation $\sigma_r \in S_2$ such that $r \equiv 0,5 \pmod 6$ also appears in exactly one pair of $D_{T_1} \cup D_s$.

Next, we show that each pair of D_{T_1} is truncated hamiltonian. Before we proceed, we point out that, if $i \notin \{1, m-1\}$, then $\hat{\mu}_i = \mu_i(m-1, i+1)$. Furthermore, if i is odd and $i = 2j+1$, where $1 \leqslant j \leqslant k-1$, then

$$\mu_i = (m-1,1,2,3)\sigma_i = (m-1,1,2,3)\gamma_i(m-1,j+2).$$

Similarly, if $i \neq 0$ and i is even, then $i = 2j$, where $1 \leqslant j \leqslant k-1$. We see that

$$\mu_i = (m-1,1,2,3)\sigma_i = (m-1,1,2,3)\gamma_i(m-1,k+j+1).$$

We now prove that all pairs of D_{T_1} are truncated hamiltonian. We will consider seven subcases.

Subcase 1.1: $i = m-5$. Then

$$\begin{aligned}\hat{\mu}_{m-5}\hat{\sigma}_6 &= (m-1,1,2,3)\,\gamma_{-4}(m-1,m-2)\,(m-1,m-4)\,\gamma_6\\ &= (0,2,3,5,7,9,\ldots,m-2,1,4,6,8,\ldots,m-3)(m-1).\end{aligned}$$

Subcase 1.2: $i \equiv 2 \pmod{12}$, $14 \leqslant i \leqslant m-11$, and $m \equiv 1 \pmod{12}$. Then $i = 2j$, where j is odd, and k is even. This also means that $1 \leqslant j \leqslant k-5$. Therefore

$$\begin{aligned} \hat{\mu}_i\hat{\sigma}_{m-i+1} &= (m-1,1,2,3)\,\gamma_i\,(m-1,k+j+1)\,(m-1,i+1)\,\gamma_{-i+2} \\ &= (0,2,5,7,\dots,m-2,1,4,6,8,\dots,k-j+1,3,k-j+3,k-j+5, \\ &\qquad \dots,m-3)(m-1). \end{aligned}$$

Subcase 1.3: $i \equiv 2 \pmod{12}$, $14 \leqslant i \leqslant m-17$, and $m \equiv 7 \pmod{12}$. Then $i = 2j$, where j is odd, and k is odd. This also means that $1 \leqslant j \leqslant k-8$. Consequently,

$$\begin{aligned} \hat{\mu}_i\hat{\sigma}_{m-i+1} &= (m-1,1,2,3)\,\gamma_i\,(m-1,k+j+1)\,(m-1,i+1)\,\gamma_{-i+2} \\ &= (0,2,5,7,\dots,k-j+1,3,k-j+3,k-j+5,\dots,m-2,1,4,6,8, \\ &\qquad \dots,m-3)(m-1) \end{aligned}$$

Subcase 1.4: $i \equiv 3 \pmod{12}$, $3 \leqslant i \leqslant m-5$. Then $i = 2j+1$ where j is odd and $1 \leqslant j \leqslant k-2$. Furthermore, observe that $(m-j)^{\gamma_i} = j+2$ and $(j+2)^{\gamma_{-i+2}} = m-j+2$. We then see that

$$\begin{aligned} \hat{\mu}_i\hat{\sigma}_{m-i+1} &= (m-1,1,2,3)\,\gamma_i\,(m-1,j+2)\,(m-1,i+1)\,\gamma_{-i+2} \\ &= (0,2,5,7,\dots,m-2,1,4,6,\dots,m-j,3,m-j+2,m-j+4,\dots, \\ &\qquad m-3)(m-1). \end{aligned}$$

Subcase 1.5: $i \equiv 8 \pmod{12}$, $8 \leqslant i \leqslant m-17$, and $m \equiv 1 \pmod{12}$. Then $i = 2j$, where j is even, and k is even. This also means that $4 \leqslant j \leqslant k-8$. Computation of $\hat{\mu}_i\hat{\sigma}_{m-i+1}$ is the same as Case 1.3.

Subcase 1.6: $i \equiv 8 \pmod{12}$, $8 \leqslant i \leqslant m-11$, and $m \equiv 7 \pmod{12}$. Then $i = 2j$, where j is even, and k is odd. This also means that $4 \leqslant j \leqslant k-5$. The computation of $\hat{\mu}_i\hat{\sigma}_{m-i+1}$ is the same as Case 1.2.

Subcase 1.7: $i \equiv 9 \pmod{12}$ and $9 \leqslant i \leqslant m-4$. Then $i = 2j+1$ where j is even and $4 \leqslant j \leqslant k-2$. Recall that Lemma 4.74 implies that $(m-j)^{\gamma_i} = j+2$ and $(j+2)^{\gamma_{-i+2}} = m-j+2$. We then see that

$$\begin{aligned} \hat{\mu}_i\hat{\sigma}_{m-i+1} &= (m-1,1,2,3)\,\gamma_i\,(m-1,j+2)\,(m-1,i+1)\,\gamma_{-i+2} \\ &= (0,2,5,7,\dots,m-j,3,m-j+2,m-j+4,\dots,m-2,1,4,6,8,\dots, \\ &\qquad m-3)(m-1). \end{aligned}$$

Therefore, all pairs of D_{T_1} are truncated hamiltonian pairs .

Next, we define a set denoted D_{H_1}:

$$D_{H_1} = \{(\mu_i, \sigma_{m-i}) \mid i \equiv 0, 4 \pmod 6 \text{ and } 4 \leqslant i \leqslant m-3\}.$$

Since $m \equiv 1 \pmod 6$, it follows that D_{H_1} contains exactly $\frac{m-1}{3} - 1$ pairs. Observe that D_{H_1} and D_s jointly use each permutation $\mu_i \in S_1$ where $i \equiv 0, 4 \pmod 6$ precisely once. Furthermore, the sets D_{H_1} and D_s also jointly use each $\sigma_r \in S_2$ such that $r \equiv 1, 3 \pmod 6$ exactly once.

Below, we prove that each pair in D_{H_1} is a hamiltonian pair. If $i \equiv 0, 4 \pmod 6$ and $4 \leqslant i \leqslant m-3$, then $i = 2j$ where $2 \leqslant j \leqslant k-2$. Lemma 4.78 implies that

$$\sigma_{m-i} = \gamma_{-i+1}\,(m-1, k-j+2).$$

We then see that, if $i \neq m-3$, then

$$\begin{aligned} \mu_i\sigma_{m-i} &= (m-1,1,2,3)\,\gamma_i\,(m-1,k+j+1)\;\gamma_{-i+1}\,(m-1,k-j+2) \\ &= (0,1,3,m-1,2,4,5,6,\ldots,k-j+1,k-j+2,k-j+3,\ldots,m-2). \end{aligned}$$

If $i = m-3$, then

$$\begin{aligned} \mu_{m-3}\sigma_3 &= (m-1,1,2,3)\,\gamma_{-2}\,(m-1,0)\;\gamma_3\,(m-1,3) \\ &= (0,1,3,m-1,2,4,5,6,\ldots,m-2). \end{aligned}$$

As a result, all pairs in D_{H_1} are hamiltonian pairs.

Next, we form a set denoted D_{H_2}:

$$D_{H_2} = \{(\mu_i, \sigma_{m-i+2}) \mid i \equiv 1, 5 \pmod 6 \text{ and } 5 \leqslant i \leqslant m-2\}.$$

We see that D_{H_2} contains $\frac{m-4}{3}$ pairs. Moreover, it follows that each permutation $\mu_i \in S_1$ such that $i \equiv 1, 5 \pmod 6$ appears exactly once in $D_{H_2} \cup D_s$. In addition, each permutation $\sigma_r \in S_2$ such that $r \equiv 2, 4 \pmod 6$ also appears precisely once in $D_{H_2} \cup D_s$.

Next, we prove that each pair of D_{H_2} is a hamiltonian pair.

If $i = 5$, then

$$\begin{aligned} \mu_5\sigma_{m-3} &= (m-1,1,2,3)\,\gamma_5\,(m-1,4)\;\gamma_{-2}\,(m-1,0) \\ &= (0,3,2,6,9,\ldots,m-4,m-1,4,7,\ldots,m-3,1,5,8,\ldots,m-2). \end{aligned}$$

If $i = 7$, then

$$\begin{aligned} \mu_7\sigma_{m-5} &= (m-1,1,2,3)\,\gamma_7\,(m-1,5)\;\gamma_{-4}\,(m-1,m-2) \\ &= (0,3,1,5,8,\ldots,m-5,m-1,4,7,\ldots,m-3,m-2,2 \\ &\quad\; 6,9,\ldots,m-4). \end{aligned}$$

If $i \equiv 1$ or $5 \pmod 6$ and $11 \leqslant i \leqslant m-2$, then $i = 2j+1$ and $5 \leqslant j \leqslant k-1$. Lemma 4.78 implies that

$$\sigma_{m-i+2} = \gamma_{-i+3}\,(m-1, m-j+1).$$

If $i \equiv 1 \pmod 6$ and $11 \leqslant i \leqslant m-2$, then $j \equiv 0 \pmod 3$. Moreover, Lemma 4.74 implies that $(j+2)^{\gamma_{-i+3}} = m-j+3$ and $(m-j)^{\gamma_i} = j+2$. We then see that

$$\begin{aligned}\mu_i\sigma_{m-i+2} &= (m-1,1,2,3)\,\gamma_i\,(m-1,j+2)\;\gamma_{-i+3}\,(m-1,m-j+1)\\ &= (0,3,m-j+3,m-j+6,m-j+9,\dots,m-3,1,5,8,\dots,\\ &\quad m-j-2,m-1,4,7,10,\dots,m-j,m-j+1,m-j+4,\dots,m-2,\\ &\quad 2,6,9,\dots,m-4).\end{aligned}$$

If $i \equiv 5 \pmod 6$ and $11 \leqslant i \leqslant m-2$, then $i = 2j+1$ where $j \equiv 2 \pmod 3$ and $5 \leqslant j \leqslant k-1$. Therefore,

$$\begin{aligned}\mu_i\sigma_{m-i+2} &= (m-1,1,2,3)\,\gamma_i\,(m-1,j+2)\;\gamma_{-i+3}\,(m-1,m-j+1)\\ &= (0,3,m-j+3,m-j+6,\dots,m-2,2,6,9,\dots,m-j-2,m-1,\\ &\quad 4,7,10,\dots,m-3,1,5,8,11,\dots,m-j,m-j+1,m-j+4,\\ &\quad m-j+7,\dots,m-4).\end{aligned}$$

Therefore, each pair in D_{H_2} is a hamiltonian pair.

Next, we let $D = D_s \cup D_{T_1} \cup D_{H_1} \cup D_{H_2}$. See Figure 4.41 for an illustration of D when $m = 11$. It is straightforward to verify that each permutation of S_1 appears in exactly one pair of D; a similar observation holds for each permutation of S_2.

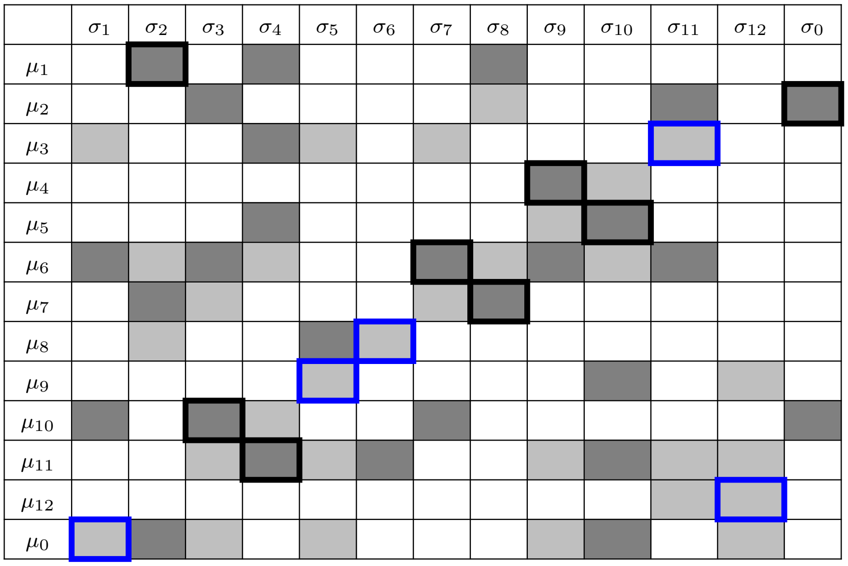
Let

$$\begin{aligned}D_T &= D_{T_1} \cup \{(\mu_{m-1},\sigma_{m-1}),(\mu_0,\sigma_1)\};\\ D_H &= D_{H_1} \cup D_{H_2} \cup \{(\mu_1,\sigma_2),(\mu_2,\sigma_0)\}.\end{aligned}$$

The set D_T contains exactly $\frac{m+2}{3}$ truncated hamiltonian pairs and the set D_H contains $2 \cdot \frac{m-1}{3}$ hamiltonian pairs. Furthermore, the sets D_T and D_H are disjoint and thus partition D. Therefore, the set D is a c-twined 2-factorization of $\vec{C}_2 \wr \overline{K}_m$.

<u>Case 2</u>: $\frac{m+8}{3} \leqslant c \leqslant m-2$. Let $c = \frac{m+8}{3} + 2t$ where $2 \leqslant 2t \leqslant \frac{2m-8}{3}$. Let $\mathbb{I} = \{i \mid i \equiv 0, 4 \pmod 6$ and $4 \leqslant i \leqslant m-3\}$ and M_t be a subset of size t of $\mathbb{I}$. Observe that $\mathbb{I}$ contains $\frac{m-4}{3}$ elements. We then form the following set:

$$D_{T_2} = \{(\mu_i,\sigma_{m-i+1}),(\mu_{i+1},\sigma_{m-i}) \mid i \in M_t\}.$$



	σ_1	σ_2	σ_3	σ_4	σ_5	σ_6	σ_7	σ_8	σ_9	σ_{10}	σ_{11}	σ_{12}	σ_0
μ_1		■		■				■					
μ_2			■					▒			■		■
μ_3	▒			■	▒		▒				▒		
μ_4									■	▒			
μ_5				■					▒	■			
μ_6	■	▒	■	▒			■	▒	■	▒	■		
μ_7		■	▒				▒	■					
μ_8		▒			■	▒							
μ_9					▒					■		▒	
μ_{10}	■		■	▒			■						■
μ_{11}			▒	■	▒	■			▒	■	▒	▒	
μ_{12}											▒	▒	
μ_0	▒	■	▒		▒				▒	■		▒	

Figure 4.41: Case 1: hamiltonian table of $S_1 \times S_2$ for $m = 13$ with a 5-twined 2-factorization in black (hamiltonian pairs) and blue (truncated hamiltonian pairs).

Below, we demonstrate that each pair of D_{T_2} is a truncated hamiltonian pair. First, we show that, for all $i \in M_t$, all pairs of the form $(\mu_{i+1}, \sigma_{m-i})$ are truncated hamiltonian pairs. Note that, if $i \in \mathbb{I}$ and $4 \leqslant i \leqslant m-4$, then $(m-1)^{\mu_{i+1}} = i+2$. Therefore,

$$\hat{\mu}_{i+1} = \mu_{i+1}\,(m-1, i+2).$$

If $i = m-3$, then $\mu_{i+1} = \mu_{m-2}$ and $(m-1)^{\mu_{m-2}} = 0$. Therefore,

$$\hat{\mu}_{m-3} = \mu_{m-2}\,(m-1, 0).$$

First, we consider the case $i = 4$. Lemma 4.74 implies that $\gamma_{m-4} = \gamma_{-3}$. We then see that

$$\begin{aligned} \hat{\mu}_5 \hat{\sigma}_{m-4} &= (m-1, 1, 2, 3)\,\gamma_5\,(m-1, 4)\,(m-1, 6)\gamma_{-3} \\ &= (0, 2, 5, 7, 9, \ldots, m-2, 3, 1, 4, 6, \ldots, m-3)(m-1). \end{aligned}$$

Next, we consider the case $i = m-3$. In this case, we see that $\sigma_{m-i} = \sigma_3$. Moreover, Lemma 4.74 implies that $\gamma_{m-2} = \gamma_{-1}$. We consider two subcases. If $m \equiv 1 \pmod{12}$, then k is even. This means that

$$\begin{aligned}
\hat{\mu}_{m-2}\hat{\sigma}_3 &= (m-1,1,2,3)\,\gamma_{m-2}\,(m-1,k+1)\,(m-1,0)\gamma_3\\
&= (m-1,1,2,3)\,\gamma_{-1}\,(m-1,k+1)\,(m-1,0)\gamma_3\\
&= (0,2,5,7,9,\ldots,m-2,1,4,6,8,\ldots,k+2,3,k+4,k+6,\\
&\quad \ldots,m-3)(m-1).
\end{aligned}$$

If $m \equiv 7 \pmod{12}$, then k is odd. Hence

$$\begin{aligned}
\hat{\mu}_{m-2}\hat{\sigma}_3 &= (m-1,1,2,3)\,\gamma_{-1}\,(m-1,k+1)\,(m-1,0)\gamma_3\\
&= (0,2,5,7,9,\ldots,k+2,3,k+4,k+6,\ldots,m-2,1,4,6,8,\\
&\quad \ldots,m-3)(m-1).
\end{aligned}$$

We now proceed with the case $i \equiv 0$ or $4 \pmod{12}$, and $12 \leqslant i \leqslant m-7$. In that case, we have $i = 2j$ where j is even and $6 \leq j \leq k-2$. Moreover, Lemma 4.74 implies that $(m-j)^{\gamma_i} = j+1$ and $(j+2)^{\gamma_{-i+1}} = m-j+2$. Therefore,

$$\begin{aligned}
\hat{\mu}_{i+1}\hat{\sigma}_{m-i} &= (m-1,1,2,3)\,\gamma_{i+1}\,(m-1,j+2)\,(m-1,i+2)\gamma_{-i+1}\\
&= (0,2,5,7,9,\ldots,m-j,3,m-j+2,m-j+4,\ldots,m-2,\\
&\quad 1,4,6,\ldots,m-3)(m-1).
\end{aligned}$$

If $i \equiv 6$ or $10 \pmod{12}$, and $6 \leqslant i \leqslant m-7$, then $i = 2j$ where j is odd and $3 \leq j \leq k-2$. Then

$$\begin{aligned}
\hat{\mu}_{i+1}\hat{\sigma}_{m-i} &= (m-1,1,2,3)\,\gamma_{i+1}\,(m-1,j+2)\,(m-1,i+2)\gamma_{-i+1}\\
&= (0,2,5,7,9,\ldots,m-2,1,4,6,\ldots,m-j,3,m-j+2,\\
&\quad m-j+4,\ldots,m-3)(m-1).
\end{aligned}$$

As a result, all pairs of the form (μ_{i+1},σ_{m-i}), where $i \in M_t$, are truncated hamiltonian pairs.

Next, we show that, all pairs of the form (μ_i,σ_{m-i+1}) in D_{T_2} are truncated hamiltonian pairs. To do so, we consider five cases.

Case 2.1: $i = m-3$. Then

$$\begin{aligned}
\hat{\mu}_{m-3}\hat{\sigma}_4 &= (m-1,1,2,3)\,\gamma_{-2}\,(m-1,0)\,(m-1,m-2)\gamma_4\\
&= (0,2,5,7,9,\ldots,m-2,1,3,4,6,\ldots,m-3)(m-1).
\end{aligned}$$

Case 2.2: $i \equiv 0$ or $4 \pmod{12}$, and $4 \leqslant i \leqslant m-9$, and $m \equiv 1 \pmod{12}$. Then $i = 2j$, where j is even, and k is even. We then see that $2 \leqslant j \leqslant k-4$. Therefore

$$\begin{aligned}\hat{\mu}_i\hat{\sigma}_{m-i+1} &= (m-1,1,2,3)\,\gamma_i\,(m-1,k+j+1)\,(m-1,i+1)\gamma_{-i+2}\\ &= (0,2,5,7,9,\ldots,k-j+1,3,k-j+3,k-j+5,\ldots,m-4,m-2,1,\\ &\quad 4,6,8,\ldots,m-3)(m-1).\end{aligned}$$

Case 2.3: $i \equiv 0$ or $4 \pmod{12}$, $4 \leqslant i \leqslant m-7$, and $m \equiv 7 \pmod{12}$. Then $i = 2j$, where j is even, and k is odd. In addition, we have $2 \leq j \leq k-3$. As a result,

$$\begin{aligned}\hat{\mu}_i\hat{\sigma}_{m-i+1} &= (m-1,1,2,3)\,\gamma_i\,(m-1,k+j+1)\,(m-1,i+1)\gamma_{-i+2}\\ &= (0,2,5,7,9,\ldots,m-2,1,4,6,8,\ldots,k-j+1,3,\\ &\quad k-j+3,k-j+5,\ldots,m-3)(m-1).\end{aligned}$$

Case 2.4: $i \equiv 6$ or $10 \pmod{12}$, $6 \leqslant i \leqslant m-7$, and $m \equiv 1 \pmod{12}$. Then $i = 2j$ where j is odd, k is even, and $3 \leqslant j \leqslant k-3$. The computation of $\hat{\mu}_i\hat{\sigma}_{m-i+1}$ is the same as Case 2.3.

Case 2.5: $i \equiv 6$ or $10 \pmod{12}$, $6 \leqslant i \leqslant m-9$, and $m \equiv 7 \pmod{12}$. Then $i = 2j$ where j is odd, k is odd, and $3 \leq j \leq k-4$. The computation of $\hat{\mu}_i\hat{\sigma}_{m-i+1}$ is the same as Case 2.1.

In conclusion, all pairs of the form (μ_i, σ_{m-i+1}), where $i \in M_t$, are truncated hamiltonian pairs. This means that all pairs of D_{T_2} are truncated hamiltonian pairs.

Let D_T be the set of truncated hamiltonian pairs given in Case 1. We then form the following set:

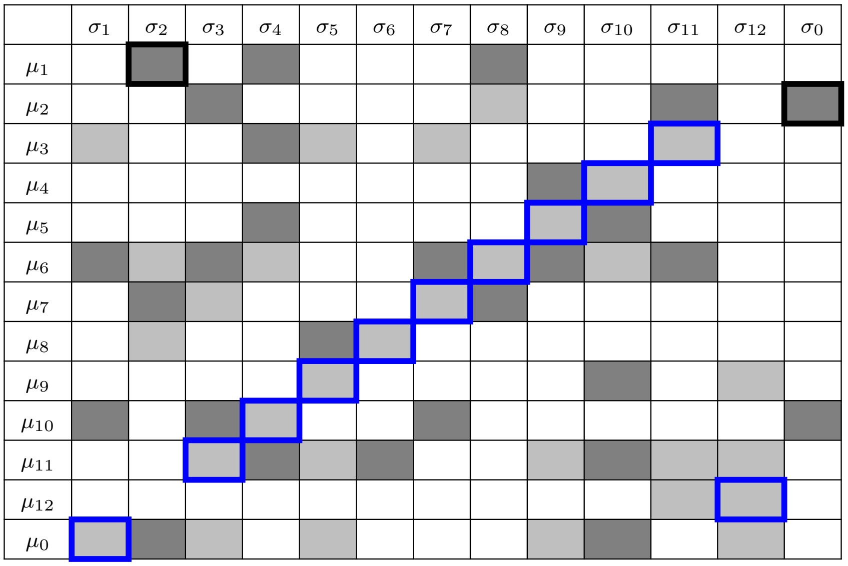
$$D'_T = D_{T_2} \cup D_T.$$

Note that each permutation in S_1 and S_2 appears at most once in D'_T, and that D_T and D_{T_2} are disjoint. This means that D'_T contains $\frac{m+2}{3} + 2t$ truncated hamiltonian pairs. Since $2t$ is at most $2\frac{m-4}{3} = \frac{2m-8}{3}$, then D'_T contains up to $m-2$ truncated hamiltonian pairs.

Next, we form the following three sets of pairs as follows:

$$\begin{aligned}D'_{H_1} &= \{(\mu_i, \sigma_{m-i}) \mid i \notin M_t,\ i \equiv 0,4 \pmod{6}, \text{and } 4 \leqslant i \leqslant m-3\};\\ D'_{H_2} &= \{(\mu_i, \sigma_{m-i+2}) \mid i-1 \notin M_t, i \equiv 1,5 \pmod{6},\ \text{and } 5 \leqslant i \leqslant m-2\};\\ D'_{H_s} &= \{(\mu_1, \sigma_2), (\mu_2, \sigma_0)\}.\end{aligned}$$

Let D_H be the set of hamiltonian pairs constructed in Case 1 and let $D'_H = D'_{H_1} \cup D'_{H_2} \cup D'_{H_s}$. Observe that $D'_H \subseteq D_H$. Therefore, it follows from Case 1 that all pairs of D'_H are hamiltonian.



	σ_1	σ_2	σ_3	σ_4	σ_5	σ_6	σ_7	σ_8	σ_9	σ_{10}	σ_{11}	σ_{12}	σ_0
μ_1		▓		▓				▓					
μ_2			▓					░			▓		▓
μ_3	░			▓	░		░				░		
μ_4									▓	░			
μ_5				▓					░	▓			
μ_6	▓	░	▓	░			▓	░	▓	░	▓		
μ_7		▓	░				░	▓					
μ_8		░			▓	░							
μ_9					░					▓		░	
μ_{10}	▓		▓	░			▓						▓
μ_{11}			░	▓	░	▓			░	▓	░	░	
μ_{12}											░	░	
μ_0	░	▓	░		░				░	▓		░	

Figure 4.42: Case 2: hamiltonian table of $S_1 \times S_2$ for $m = 13$ with an 11-twined 2-factorization in black (hamiltonian pairs) and blue (truncated hamiltonian pairs).

The set D'_H is disjoint from D'_T. Let $D' = D'_T \cup D'_H$. Refer to Figure 4.42 for an illustration of D' when $m = 13$ and $M_t = \mathbb{I}$. One can verify that each permutation of S_1 and S_2 appears precisely once in D'. Therefore, the set D' is a directed 2-factorization of $\vec{C}_2 \wr \overline{K}_m$. In addition, since D'_T and D'_H are disjoint, the set D' is also a c-twined 2-factorization of $\vec{C}_2 \wr \overline{K}_m$. ▮

Proposition 4.88. *Let $m \equiv 3 \pmod 6$ such that $m \geqslant 9$, and let c be an odd integer such that $\frac{m}{3} + 4 \leqslant c \leqslant m - 2$. The digraph $\vec{C}_2 \wr \overline{K}_m$ admits a c-twined 2-factorization.*

Proof: Let $m = 2k + 1$. If $m \equiv 3 \pmod 6$, then $k \equiv 1 \pmod 3$. In this proof, it is understood that, for each $i \in \mathbb{Z}_{m-1}$, permutation $\gamma_i \in G_{m-1}$ is as defined in Definition 4.73. We will construct a c-twined 2-factorization of $\vec{C}_2 \wr \overline{K}_m$ by using the following two sets of m permutations:

$$S_1 = (1,2,3,4) \cdot \mathcal{F}_m = \{\mu_i = (1,2,3,4)\sigma_i \mid i = 0,1,\ldots,m-1\};$$

$$S_2 = \mathcal{F}_m = \{\sigma_0, \sigma_1, \ldots, \sigma_{m-1}\}.$$

Lemmas 4.60 and 4.77 imply that S_1 and S_2 are regular permutation sets. See Construction 4.75 for a description of each $\sigma_i \in \mathcal{F}_m$. Refer to Figure 4.43 for the hamiltonian table of $S_1 \times S_2$ when $m = 15$. Observe that μ_0 and σ_0 are the $(m-1)$-stabilizers of S_1 and S_2, respectively. We also point out that

$$(m-1)^{\mu_i} = (m-1)^{\sigma_i} \text{ and } \hat{\sigma}_i = \gamma_i.$$

Therefore, if $i \notin \{0, m-1\}$, then

$$\hat{\mu}_i = (1,2,3,4)\,\gamma_i.$$

<u>Case 1</u>: $c = \frac{m}{3} + 4$. We start by constructing of a set of five pairs:

$$D_s = \{(\mu_{m-4}, \sigma_{m-4}), (\mu_{m-2}, \sigma_{m-1}), (\mu_{m-1}, \sigma_{m-2}), (\mu_{m-3}, \sigma_0), (\mu_0, \sigma_{m-3})\}.$$

Below, we show that each of $(\mu_{m-4}, \sigma_{m-4}), (\mu_{m-2}, \sigma_{m-1})$, and $(\mu_{m-1}, \sigma_{m-2})$ is a truncated hamiltonian pair:

$$\begin{aligned}
\hat{\mu}_{m-4}\hat{\sigma}_{m-4} &= (1,2,3,4)\,\gamma_{-3}\,\gamma_{-3} \\
&= (0, m-7, m-13, \ldots, 8, 2, m-4, m-10, \ldots, 5, m-2, m-8, \\
&\quad \ldots, 7, 1, m-5, m-11, \ldots, 4, m-6, m-12, \ldots, 9, 3, \\
&\quad m-3, m-9, \ldots, 6)(m-1);
\end{aligned}$$

$$\begin{aligned}
\hat{\mu}_{m-2}\hat{\sigma}_{m-1} &= (1,2,3,4)\,\gamma_{-1}\,(0, 3, m-2, 4, \ldots, k, k+2, k+1, 1, 2) \\
&= (0, 4, 3, m-2, 5, m-3, 6, \ldots, k, k+3, k+1, k+2, 1, 2)(m-1);
\end{aligned}$$

$$\begin{aligned}
\hat{\mu}_{m-1}\hat{\sigma}_{m-2} &= (1,2,3,4)\,(0, 3, m-2, 4, m-3, 5, \ldots, k, k+2, k+1, 1, 2)\gamma_{-1} \\
&= (0, 2, m-3, 4, 1, m-2, 3, m-4, 5, m-5, 6, m-6, 7, \ldots, \\
&\quad k+2, k, k+1)(m-1).
\end{aligned}$$

Next, we show that (μ_{m-3}, σ_0) and (μ_0, σ_{m-3}) are hamiltonian pairs:

$$\begin{aligned}
\mu_0\sigma_{m-3} &= (1,2,3,4)\,id\,\gamma_{-2}\,(m-1, 0) \\
&= (0, m-3, m-5, \ldots, 4, m-2, m-4, m-6, \ldots, 3, 2, 1, m-1);
\end{aligned}$$

$$\begin{aligned}
\mu_{m-3}\sigma_0 &= (1,2,3,4)\,\gamma_{-2}\,(m-1, 0)\,id \\
&= \mu_0\sigma_{m-3}.
\end{aligned}$$

We now proceed by forming a set of $\frac{m}{3}+1$ pairs as follows:

$$\begin{aligned} D_{T_1} = \; & \{(\mu_i, \sigma_{m-i-5}), (\mu_{i+1}, \sigma_{m-i-4}) \mid i \equiv 1 \pmod 6 \text{ and } 1 \leqslant i \leqslant m-8\} \cup \\ & \{(\mu_{m-6}, \sigma_1), (\mu_{m-5}, \sigma_2)\}. \end{aligned}$$

Since $m \equiv 3 \pmod 6$, then D_{T_1} contains $\frac{m}{3}+1$ pairs. Observe that D_{T_1} and D_s jointly use each $\mu_i \in S_1$ such that $i \equiv 1, 2 \pmod 6$ exactly once. Similarly, the sets D_{T_1} and D_s jointly use each $\sigma_r \in S_2$ such that $r \equiv 3, 4 \pmod 6$ precisely once. Below, we show that each pair of D_{T_1} is truncated hamiltonian.

First, we show that pairs $(\mu_i, \sigma_{m-i-5}) \in D_{T_1}$ are truncated hamiltonian pairs. We consider two subcases. If $m \equiv 1 \pmod 4$, it follows that

$$\begin{aligned} \hat{\mu}_i \hat{\sigma}_{m-i-5} &= (1,2,3,4)\, \gamma_i\, \gamma_{-i-4} \\ &= (0, m-5, m-9, \ldots, 4, m-4, m-8, \ldots, 5, 1, m-3, m-7, \ldots, 2, \\ &\quad\; m-2, m-6, \ldots, 3)(m-1). \end{aligned}$$

If $m \equiv 3 \pmod 4$, it follows that

$$\begin{aligned} \hat{\mu}_i \hat{\sigma}_{m-i-5} &= (1,2,3,4)\, \gamma_{-4} \\ &= (0, m-5, m-9, \ldots, 2, m-2, m-6, \ldots, 5, 1, m-3, m-7, \ldots, 4, \\ &\quad\; m-4, m-8, \ldots, 3)(m-1). \end{aligned}$$

Next, we verify that each pair of the form $(\mu_{i+1}, \sigma_{m-i-4})$ is a truncated hamiltonian pair. We see that

$$\begin{aligned} \hat{\mu}_{i+1} \hat{\sigma}_{m-i-4} &= (1,2,3,4)\, \gamma_{i+1}\, \gamma_{-i-3} \\ &= (0, m-3, m-5, \ldots, 6, 4, m-2, m-4, \ldots, 7, 5, 3, 2, 1)(m-1). \end{aligned}$$

In conclusion, each pair in D_{T_1} is a truncated hamiltonian pair.

Our next objective is to form a set of $\frac{2m}{3}-6$ pairs:

$$D_{H_1} = \{(\mu_i, \sigma_{m-i-4}) \mid i \equiv 0, 3, 4, 5 \pmod 6 \text{ and } 3 \leqslant i \leqslant m-9\}.$$

Since $m \equiv 3 \pmod 6$, it follows that D_{H_1} contains $\frac{2m}{3}-6$ pairs. Observe that D_{H_1} and D_s jointly use each $\mu_i \in S_1 \backslash \{\mu_{m-5}, \mu_{m-6}\}$ such that $i \equiv 0, 3, 4, 5 \pmod 6$ exactly once. Note that each of μ_{m-5} and μ_{m-6} appears in exactly one pair of D_{T_1}. Similarly, the sets D_{H_1} and D_s jointly use each $\sigma_r \in S_2 \backslash \{\sigma_1, \sigma_2\}$ such that $r \equiv 0, 1, 2, 5 \pmod 6$ exactly once. Once again, we note that each of σ_1 and σ_2 appears in exactly one pair of D_{T_1}.

Next, we demonstrate that D_{H_1} is a set of hamiltonian pairs. First, we make the following observations. If i is even, $2 \leqslant i \leqslant m-9$, and $i = 2j$, then $1 \leqslant j \leqslant k-4$, and Lemma 4.78 implies that

$$\sigma_{m-i-4} = \gamma_{-i-3}\,(m-1, k-j).$$

If i is odd, $1 \leqslant i \leqslant m-6$, and $i = 2j+1$, then $0 \leqslant j \leqslant k-3$ and Lemma 4.78 implies that

$$\sigma_{m-i-4} = \gamma_{-i-3}\,(m-1, m-j-2).$$

Lastly, we recall that $k \equiv 1 \pmod 3$.

If $i \equiv 0 \pmod 6$ and $6 \leq i \leq m-9$, then $i = 2j$ where $j \equiv 0 \pmod 3$ and $3 \leqslant j \leqslant k-4$. We then see that

$$\begin{aligned}\mu_i\sigma_{m-i-4} &= (1,2,3,4)\,\gamma_i\,(m-1,k+j+1)\gamma_{-i-3}(m-1,k-j)\\ &= (0,m-4,m-7,\ldots,k-j+1,k-j,k-j-3,k-j-6,\ldots,7,4,\\ &\quad m-3,m-6,\ldots,3,1,m-2,m-5,\ldots,k-j+3,m-1,k-j-2,\\ &\quad k-j-5,\ldots,2).\end{aligned}$$

If $i \equiv 4 \pmod 6$ and $4 \leq i \leq m-11$, then $i = 2j$ where $j \equiv 2 \pmod 3$ and $2 \leq j \leq k-6$:

$$\begin{aligned}\mu_i\sigma_{m-i-4} &= (1,2,3,4)\,\gamma_i\,(m-1,k+j+1)\gamma_{-i-3}(m-1,k-j)\\ &= (0,m-4,m-7,\ldots,k-j+3,m-1,k-j-2,k-j-5,\ldots,3,1,\\ &\quad m-2,m-5,\ldots,4,m-3,m-6,\ldots,k-j+1,k-j,\\ &\quad k-j-3,\ldots,2).\end{aligned}$$

If $i = 3$, then

$$\begin{aligned}\mu_3\sigma_{m-7} &= (1,2,3,4)\,\gamma_3\,(m-1,3)\,\gamma_{-6}\,(m-1,m-3)\\ &= (0,m-3,m-6,\ldots,6,3,1,m-2,m-5,\ldots,4,m-1,m-4,\\ &\quad m-7,\ldots,5,2).\end{aligned}$$

If $i \equiv 3 \pmod 6$ and $9 \leq i \leq m-12$, then $i = 2j+1$ where $j \equiv 1 \pmod 3$ and $4 \leq j \leq k-6$. Lemma 4.74 implies that $(m-j)^{\gamma_i} = j+2$ and $(j+2)^{\gamma_{-i-3}} = m-j-3$. We see that

$$\begin{aligned}\mu_i\sigma_{m-i-4} &= (1,2,3,4)\,\gamma_i\,(m-1,j+2)\gamma_{-i-3}(m-1,m-j-2)\\ &= (0,m-4,m-7,\ldots,m-j,m-j-2,m-j-5,\ldots,6,3,1,m-2,\\ &\quad m-5,\ldots,4,m-3,\ldots,m-j+1,m-1,m-j-3,\\ &\quad m-j-6,\ldots,5,2)\end{aligned}$$

If $i \equiv 5 \pmod 6$ and $5 \leq i \leq m-10$, then $i = 2j+1$ where $j \equiv 2 \pmod 3$ and $2 \leqslant j \leqslant k-5$:

$$\begin{aligned}\mu_i\sigma_{m-i-4} &= (1,2,3,4)\,\gamma_i\,(m-1,j+2)\gamma_{-i-3}(m-1,m-j-2)\\ &= (0,m-4,m-7,\ldots,m-j+1,m-1,m-j-3,m-j-6,\ldots,\\ &\quad 7,4,m-3,m-6,\ldots,3,1,m-2,m-5,\ldots,m-j,m-j-2,\\ &\quad m-j-5,\ldots,5,2).\end{aligned}$$

In summary, all pairs of D_{H_1} are hamiltonian pairs.

Next, we form a set of m pairs:

$$D = D_s \cup D_{T_1} \cup D_{H_1}.$$

Observe that each permutation of S_1 and S_2 appears exactly once in D. This means that D is a directed 2-factorization of $\vec{C}_2 \wr \overline{K}_m$. Refer to Figure 4.43 for an illustration of D when $m = 15$.

Next, we let

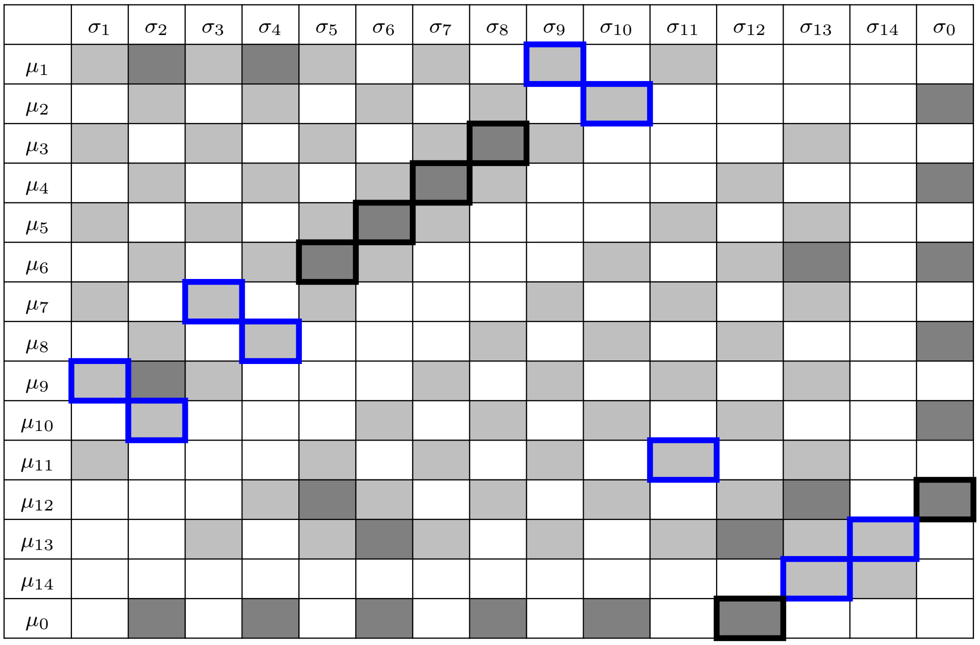
$$\begin{aligned}D_T &= D_{T_1} \cup \{(\mu_{m-4},\sigma_{m-4}),(\mu_{m-2},\sigma_{m-1}),(\mu_{m-1},\sigma_{m-2})\};\\ D_H &= D_{H_1} \cup \{(\mu_{m-3},\sigma_0),(\mu_0,\sigma_{m-3})\}.\end{aligned}$$

Observe that D_T and D_H form a partition of D. The set D_T contains $\frac{m}{3}+4$ truncated hamiltonian pairs while D_H contains $\frac{2m}{3}-4$ hamiltonian pairs. Therefore, the set D is an $(\frac{m}{3}+4)$-twined 2-factorization of $\vec{C}_2 \wr \overline{K}_m$.

Case 2: $\frac{m}{3}+6 \leqslant c \leqslant m-2$. Let $c = \frac{m}{3}+4+2t$ where $2 \leqslant 2t \leqslant \frac{2m}{3}-6$. Assume that $\mathbb{I} = \{i \mid i \equiv 3,5 \pmod 6 \text{ and } 3 \leqslant i \leqslant m-10\}$ and let M_t be a subset of size t of $\mathbb{I}$. Note that $\mathbb{I}$ contains $\frac{m}{3}-3$ elements. We now form the following set of $2t$ pairs:

$$D_{T_2} = \{(\mu_i,\sigma_{m-i-5}),(\mu_{i+1},\sigma_{m-i-4}) \mid i \in M_t\}.$$

That (μ_i,σ_{m-i-5}) and $(\mu_{i+1},\sigma_{m-i-4})$ are truncated hamiltonian pairs follows from the proof that pairs in D_{T_1}, where D_{T_1} is defined in Case 1, are truncated hamiltonian pairs. Given the set D_T from Case 1, we let



	σ_1	σ_2	σ_3	σ_4	σ_5	σ_6	σ_7	σ_8	σ_9	σ_{10}	σ_{11}	σ_{12}	σ_{13}	σ_{14}	σ_0
μ_1	░	▓	░	▓	░		░		░		░				
μ_2		░		░		░		░		░					▓
μ_3	░		░		░		░	▓	░				░		
μ_4		░		░		░	▓	░				░			▓
μ_5	░		░		░	▓	░				░		░		
μ_6		░		░	▓	░				░		░	▓		▓
μ_7	░		░		░				░		░		░		
μ_8		░		░				░		░		░			▓
μ_9	░	▓	░				░		░		░		░		
μ_{10}		░				░		░		░		░			▓
μ_{11}	░				░		░		░		░		░		
μ_{12}				░	▓	░		░		░		░	▓		▓
μ_{13}			░		░	▓	░		░		░	▓	░	░	
μ_{14}													░	░	
μ_0		▓		▓		▓		▓		▓		▓			

Figure 4.43: Case 1: hamiltonian table of $S_1 \times S_2$ for $m = 15$ with a 9-twined 2-factorization in black (hamiltonian pairs) and blue (truncated hamiltonian pairs).

$$D'_T = D_T \cup D_{T_2}.$$

We point out that D_T and D_{T_2} are disjoint and that each permutation of S_1 and S_2 appears at most once in D'_T. As a result, D'_T contains $\frac{m}{3} + 4 + 2t$ pairs.

Next, we form the following set of $m - 2t$ pairs:

$$D'_H = \{(\mu_i, \sigma_{m-i-4}) \ \mid i, i-1 \not\in M_t \text{ and } 3 \leqslant i \leqslant m-9\} \cup \{\mu_{m-3}, \sigma_0), (\mu_0, \sigma_{m-3})\}.$$

Consider the set D_H from Case 1. Observe that $D'_H \subseteq D_H$. Therefore, each pair of D'_H is a hamiltonian pair. We point out that D'_H and D'_T are disjoint and that each permutation in S_1 and S_2 appears in precisely one pair of $D' = D'_T \cup D'_H$. Refer to Figure 4.44 for an illustration of D' for $m = 15$ and $M_t = \{3\}$. In conclusion, the set D' is a c-twined 2-factorization of $\vec{C}_2 \wr \overline{K}_m$. ∎

Proposition 4.89. *Let $m \equiv 5 \pmod 6$ such that $m \geqslant 11$, and let c be an odd integer such that $\frac{m+1}{3} + 5 \leq c \leq m-2$. The digraph $\vec{C}_2 \wr \overline{K}_m$ admits a c-twined 2-factorization.*

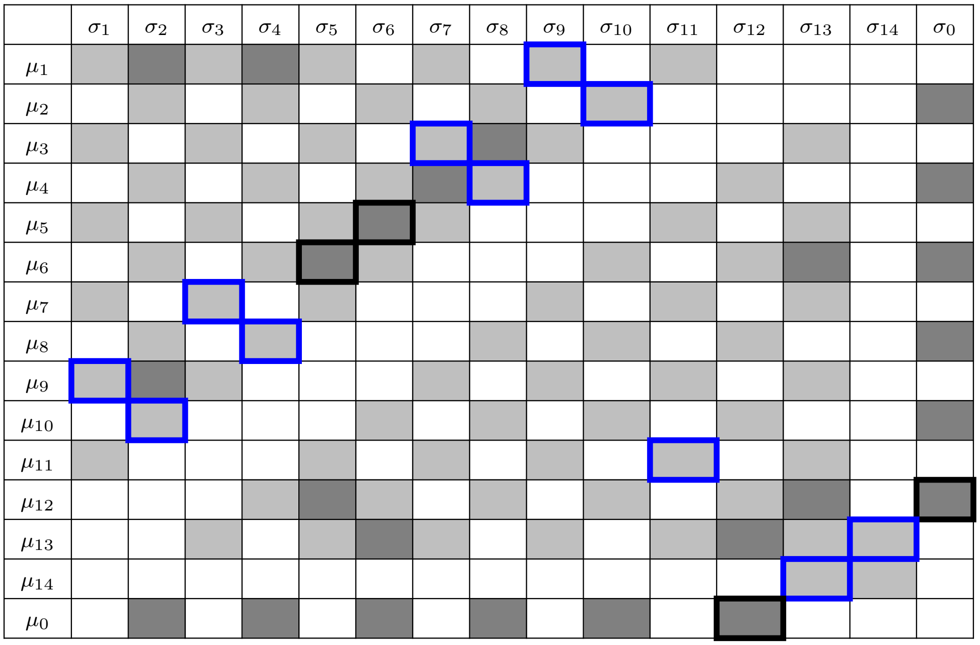


Figure 4.44: Case 2: hamiltonian table of $S_1 \times S_2$ for $m = 15$ with an 11-twined 2-factorization in black (hamiltonian pairs) and blue (truncated hamiltonian pairs).

Proof: Let $m = 2k + 1$ so that $k \equiv 2 \pmod 3$. In this proof, it is understood that, for each $i \in \mathbb{Z}_{m-1}$, the permutation $\gamma_i \in G_{m-1}$ is as defined in Definition 4.73. We will construct a c-twined 2-factorization of $\vec{C}_2 \wr \overline{K}_m$ by using the following two sets of m permutations:

$$S_1 = (1,2,3,4) \cdot \mathcal{F}_m = \{\mu_i = (1,2,3,4)\sigma_i \mid i = 0,1,\ldots,m-1\};$$
$$S_2 = \mathcal{F}_m = \{\sigma_0, \sigma_1, \ldots, \sigma_{m-1}\}.$$

Lemmas 4.60 and 4.77 imply that S_1 and S_2 are regular permutation sets. See Construction 4.75 for a description of each $\sigma_i \in \mathcal{F}_m$. Moreover, refer to Figure 4.45 for the hamiltonian table of $S_1 \times S_2$ when $m = 17$. Observe that μ_0 and σ_0 are the $(m-1)$-stabilizers of S_1 and S_2, respectively. We also point out that $(m-1)^{\mu_i} = (m-1)^{\sigma_i}$. Therefore, if $i \notin \{0, m-1\}$, then

$$\hat{\mu}_i = (1,2,3,4)\,\gamma_i.$$

<u>Case 1</u>: $c = \frac{m+1}{3} + 5$. We will construct a set of pairs that contains $\frac{m+1}{3} + 5$ truncated

hamiltonian pairs and $\frac{2m-1}{3}-5$ hamiltonian pairs. First, we construct a set of five pairs. Let

$$D^s=\{(\mu_{m-4},\sigma_{m-4}),(\mu_{m-2},\sigma_{m-1}),(\mu_{m-1},\sigma_{m-2}),(\mu_{m-3},\sigma_0),(\mu_0,\sigma_{m-3})\}.$$

Observe that $D^s=D_s$, where D_s is constructed in Case 1 of Proposition 4.88. Computations are similar with the only difference being the congruency class of m modulo 6. In fact, the products $\hat{\mu}_{m-2}\hat{\sigma}_{m-1}$, $\hat{\mu}_{m-1}\hat{\sigma}_{m-2}$, $\hat{\mu}_{m-3}\hat{\sigma_0}$, and $\hat{\mu}_0\hat{\sigma}_{m-3}$ are the same as in Case 1 of Proposition 4.88. Consequently, it suffices to show that (μ_{m-4},σ_{m-4}) is a truncated hamiltonian pair:

$$\begin{aligned}\hat{\mu}_{m-4}\hat{\sigma}_{m-4} &= (1,2,3,4)\,\gamma_{-3}\,\gamma_{-3}\\ &= (1,2,3,4)\gamma_{-6}\\ &= (0,m-7,m-13,\ldots,10,4,m-6,m-12,\ldots,5,m-2,m-8,\\ &\quad \ldots,9,3,m-3,m-9,\ldots,8,2,m-4,m-10,\ldots,7,1,m-5,\\ &\quad m-11,\ldots,6)(m-1).\end{aligned}$$

Hence, all pairs of D^s are truncated hamiltonian pairs.

We proceed with the construction of D_{T_1} as follows:

$$\begin{aligned}D_T &= \{(\mu_i,\sigma_{m-i-5}),(\mu_{i+1},\sigma_{m-i-4})\mid i\equiv 2 \pmod 6 \text{ and } 2\leqslant i\leqslant m-15\}\cup\\ &\quad \{(\mu_i,\sigma_{m-i-5}),(\mu_{i+1},\sigma_{m-i-4})\mid i\in\{m-10,m-8,m-6\}\}.\end{aligned}$$

Since $m\equiv 5 \pmod 6$, the set D_T contains $\frac{m+1}{3}+2$ pairs. Observe that D_T and D_s jointly use each $\mu_i\in S_1$ where $i\equiv 2,3 \pmod 6$ exactly once. Similarly, the sets D_T and D_s jointly use each $\sigma_r\in S_2$ where $r\equiv 4,5 \pmod 6$ precisely once.

Proof that all pairs of D_T are truncated hamiltonian can be found in Case 1 of the proof of Proposition 4.88 where we showed that elements of a set called D_{T_1} were truncated hamiltonian.

At this stage, we have formed $\frac{m+1}{3}+5$ truncated hamiltonian pairs and two hamiltonian pairs. Next, we form a set of $\frac{2m-1}{3}-7$ pairs as follows:

$$D_H=\{(\mu_i,\sigma_{m-i-4})\mid i\equiv 0,1,4,5 \pmod 6 \text{ and } 1\leqslant i\leqslant m-11\}$$

Observe that D_T, D_H, and D_s jointly use each $\mu_i\in S_1$ such that $i\equiv 0,1,4,5 \pmod 6$ exactly once. Similarly, the sets D_T, D_H, and D_s jointly use each $\sigma_r\in S_2$ such that $r\equiv 0,1,2,3 \pmod 6$. Below, we show that each pair of D_H is hamiltonian.

Before we proceed, we point out that, when $i = 2j$ and $4 \leqslant i \leqslant m-11$, we have $2 \leqslant j \leqslant k-5$ and Lemma 4.78 implies that

$$\sigma_{m-i-4} = \gamma_{-i-3}\,(m-1, k-j).$$

In addition, if $i = 2j+1$ and $1 \leqslant i \leqslant m-10$, then $0 \leqslant j \leqslant k-5$ and Lemma 4.78 implies that

$$\sigma_{m-i-4} = \gamma_{-i-3}\,(m-1, m-j-2).$$

Lastly, we remind the reader that $k \equiv 2 \pmod 3$.

We now prove that each pair of D_H is hamiltonian, starting with the case $i = 1$:

$$\begin{aligned} \mu_1\sigma_{m-5} &= (1,2,3,4)\gamma_1\,(m-1,2)\,\gamma_{-4}(m-1,m-2) \\ &= (0, m-4, m-7, \ldots, 4, m-2, m-5, \ldots, 6, 3, 1, m-1, m-3, m-7, \\ &\quad \ldots, 5, 2). \end{aligned}$$

If $i \equiv 1 \pmod 6$ and $7 \leq i \leq m-16$, then $i = 2j+1$ where $j \equiv 0 \pmod 3$ and $3 \leq j \leq k-8$. Note that Lemma 4.74 implies that $(m-j)^{\gamma_i} = j+2$ and $(j+2)^{\gamma_{-i-3}} = m-j-3$. This implies that

$$\begin{aligned} \mu_i\sigma_{m-i-4} &= (1,2,3,4)\,\gamma_i\,(m-1, j+2)\gamma_{-i-3}(m-1, m-j-2) \\ &= (0, m-4, m-7, \ldots, 4, m-3, m-6, \ldots, m-j, m-j-2, m-j-5, \\ &\quad \ldots, 6, 3, 1, m-2, m-5, \ldots, m-j+1, m-1, m-j-3, \\ &\quad m-j-6, \ldots, 5, 2). \end{aligned}$$

If $i = 5$, we have

$$\begin{aligned} \mu_5\sigma_{m-9} &= (1,2,3,4)\,\gamma_5\,(m-1,4)\gamma_{-8}(m-1,m-4) \\ &= (0, m-1, m-5, m-8, \ldots, 6, 3, 1, m-2, m-4, m-7, \ldots, 7, 4, m-3, \\ &\quad m-6, \ldots, 5, 2). \end{aligned}$$

If $i \equiv 5 \pmod 6$ and $11 \leq i \leq m-12$, then $i = 2j+1$ where $j \equiv 2 \pmod 3$ and $5 \leq j \leq k-6$. Note that Lemma 4.74 implies that $(m-j)^{\gamma_i} = j+2$ and $(j+2)^{\gamma_{-i-3}} = m-j-3$. We then have

$$\begin{aligned} \mu_i\sigma_{m-i-4} &= (1,2,3,4)\,\gamma_i\,(m-1, j+2)\gamma_{-i-3}(m-1, m-j-2) \\ &= (0, m-4, m-7, \ldots, m-j+1, m-1, m-j-3, m-j-6, \ldots, 6, \\ &\quad 3, 1, m-2, m-5, \ldots, m-j, m-j-2, m-j-5, \ldots, 7, 4, \\ &\quad m-3, m-6, \ldots, 5, 2). \end{aligned}$$

If $i \equiv 0 \pmod 6$ and $6 \le i \le m-11$, then $i = 2j$ where $j \equiv 0 \pmod 3$ and $3 \le j \le k-5$. We then see that

$$\begin{aligned} \mu_i\sigma_{m-i-4} &= (1,2,3,4)\,\gamma_i\,(m-1,k+j+1)\gamma_{-i-3}(m-1,k-j) \\ &= (0,m-4,m-7,\dots,4,m-3,m-6,\dots,k-j+3,m-1,k-j-2, \\ &\quad k-j-5,\dots,3,1,m-2,m-5,\dots,k-j+1,k-j, \\ &\quad k-j-3,\dots,5,2). \end{aligned}$$

If $i \equiv 4 \pmod 6$ and $4 \leqslant i \le m-13$, then $i = 2j$ where $j \equiv 2 \pmod 3$ and $2 \leqslant j \leqslant k-6$. Therefore

$$\begin{aligned} \mu_i\sigma_{m-i-4} &= (1,2,3,4)\,\gamma_i\,(m-1,k+j+1)\gamma_{-i-3}(m-1,k-j) \\ &= (0,m-4,m-7,\dots,k-j+1,k-j,k-j-3,k-j-6,\dots,6,3,1, \\ &\quad m-2,m-5,\dots,k-j+3,m-1,k-j-2,k-j-5,\dots,7,4, \\ &\quad m-3,m-6,\dots,5,2). \end{aligned}$$

In conclusion, each of the $\frac{2m-1}{3} - 7$ pairs of D_H is a hamiltonian pair.

Let

$$\begin{aligned} D'_H &= D_H \cup \{(\mu_{m-3},\sigma_0),(\mu_0,\sigma_{m-3})\}; \\ D'_T &= D_T \cup \{(\mu_{m-4},\sigma_{m-4}),(\mu_{m-2},\sigma_{m-1}),(\mu_{m-1},\sigma_{m-2})\}; \\ D &= D'_T \cup D'_H. \end{aligned}$$

The set D contains m pairs. See Figure 4.45 for an illustration of D when $m = 17$. Each permutation of S_1 and S_2 appears exactly once in D. In addition, the sets D'_H and D'_T are disjoint and thus partition D. Therefore, the set D is an $(\frac{m+1}{3}+5)$-twined 2-factorization of $\vec{C}_2 \wr \overline{K}_m$.

Case 2: $\frac{m+1}{3} + 7 \leqslant c \leqslant m-4$. Let $c = \frac{m+1}{3} + 5 + 2t$ where $2 \leqslant 2t \leqslant \frac{2m-1}{3} - 9$. Moreover, let $\mathbb{I} = \{i \mid i \equiv 0,4 \pmod 6 \text{ and } 4 \leqslant i \leqslant m-13\}$ and let M_t be a subset of size t of $\mathbb{I}$. Observe that $\mathbb{I}$ contains $\frac{m-2}{3} - 4$ elements.

We first form a set of $2t$ pairs as follows:

$$D_{T_2} = \{(\mu_i,\sigma_{m-i-5}),(\mu_{i+1},\sigma_{m-i-4}) \mid i \in M_t\}.$$

That (μ_i,σ_{m-i-5}) and $(\mu_{i+1},\sigma_{m-i-4})$ are truncated hamiltonian pairs follows from the proof that pairs in D_{T_1} from Case 1 of Proposition 4.88 are truncated hamiltonian pairs.

Given the set D'_T from Case 1, we let

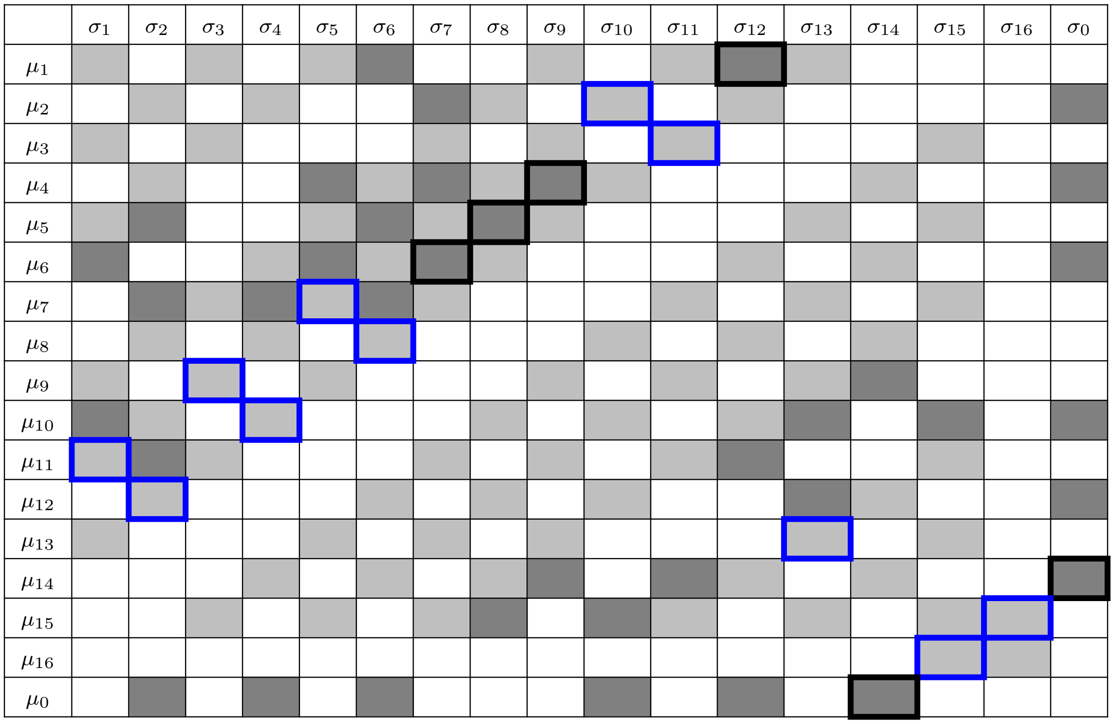


Figure 4.45: Case 1: hamiltonian table of $S_1 \times S_2$ for $m = 17$ with an 11-twined 2-factorization in black (hamiltonian pairs) and blue (truncated hamiltonian pairs).

$$D_{T_3} = D'_T \cup D_{T_2}.$$

We point out that D'_T and D_{T_2} are disjoint and that each permutation of S_1 and S_2 appears at most once in D'_T. This means that D_{T_3} contains $\frac{m+1}{3} + 5 + 2t$ pairs.

Next, we form the following set of pairs:

$$\begin{aligned} D_{H_2} \;\; = \;\; & \{(\mu_i, \sigma_{m-i-4}) \;\mid i, i-1 \notin M_t, i \equiv 0,1,4,5 \pmod 6), \text{and } 1 \leqslant i \leqslant m-11\} \\ & \cup \{\mu_{m-3}, \sigma_0), (\mu_0, \sigma_{m-3})\}. \end{aligned}$$

Consider the set D'_H from Case 1. Observe that $D_{H_2} \subseteq D'_H$. Therefore, each pair of D_{H_2} is hamiltonian. We point out that D_{H_2} and D_{T_3} are disjoint and that each permutation in S_1 and S_2 appears in precisely one pair of $D' = D_{T_3} \cup D_{H_2}$. Refer to Figure 4.46 for an illustration of D' where $M_t = \{4\}$. In conclusion, the set D' is a c-twined 2-factorization of $\vec{C}_2 \wr \overline{K}_m$.

Case 3: $c = m - 2$. We form the following set of $m - 5$ pairs:

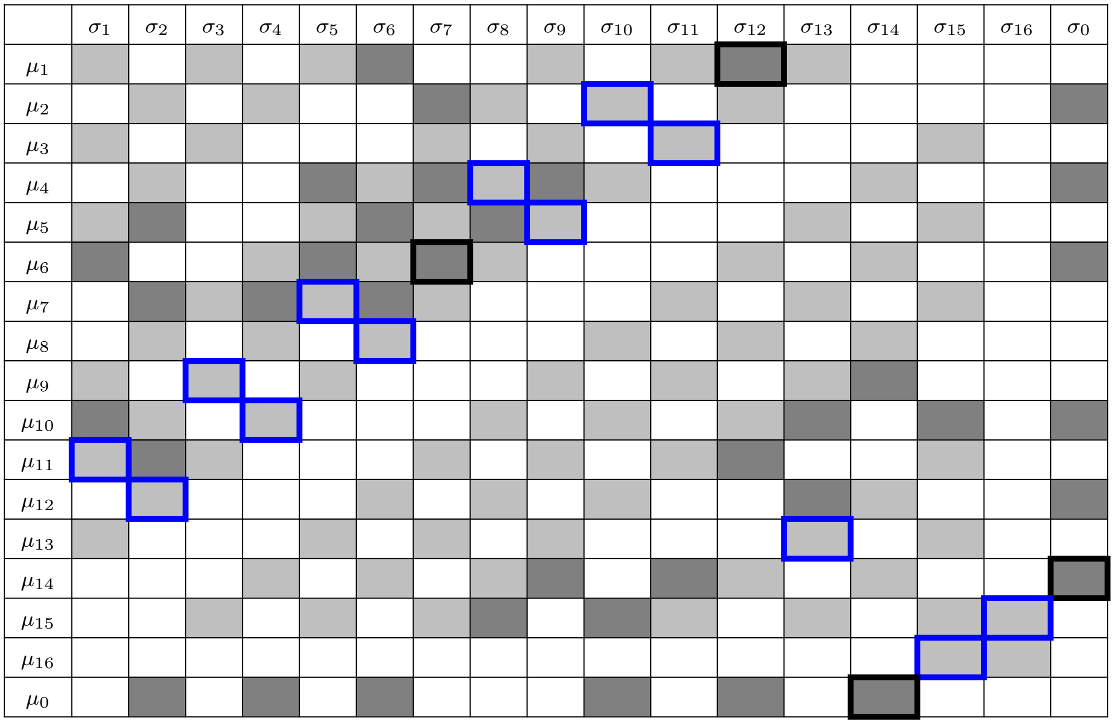
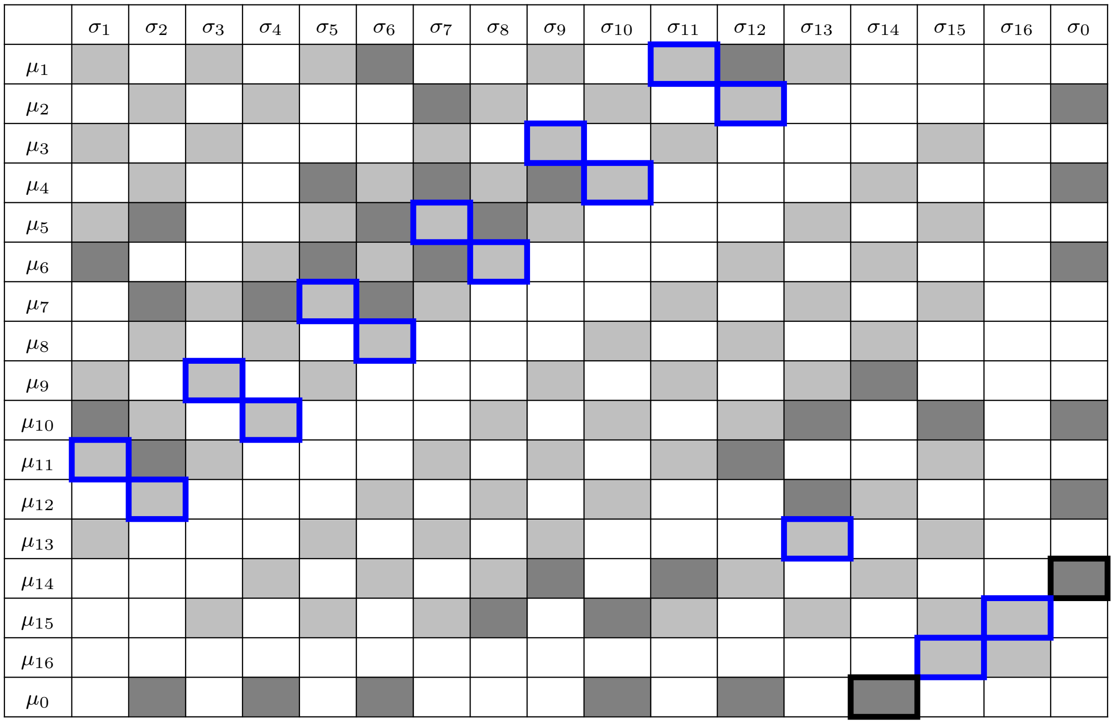


Figure 4.46: Case 2: hamiltonian table of $S_1 \times S_2$ for $m = 17$ with a 13-twined 2-factorization in black (hamiltonian pairs) and blue (truncated hamiltonian pairs).

$$D_T = \{(\mu_i, \sigma_{m-i-5}), (\mu_{i+1}, \sigma_{m-i-4}) \mid i \text{ is odd and } 1 \leqslant i \leqslant m-6\}.$$

Proof that each pair of D_T is truncated hamiltonian can be found in Case 1 of the proof of Proposition 4.88 when we showed that pairs in D_{T_1} were truncated hamiltonian pairs.

Let D^s be set of pairs defined in Case 1. Recall that D^s contains three truncated hamiltonian pairs and two hamiltonian pairs, as shown in Case 1. Therefore, the set $D = D_T \cup D^s$ contains $m-2$ truncated hamiltonian pairs and two hamiltonian pairs. We point out that D contains m pairs and that D_T and D^s are disjoint. One can verify that each permutation of S_1 and S_2 appears in precisely one pair of D. In conclusion, D is an $(m-2)$-twined 2-factorization of $\vec{C}_2 \wr \overline{K}_m$. See Figure 4.47 for an illustration of D when $m = 17$. ∎



	σ_1	σ_2	σ_3	σ_4	σ_5	σ_6	σ_7	σ_8	σ_9	σ_{10}	σ_{11}	σ_{12}	σ_{13}	σ_{14}	σ_{15}	σ_{16}	σ_0
μ_1	▨		▨		▨	■			▨		▨	■	▨				
μ_2		▨		▨			■	▨		▨		▨					■
μ_3	▨		▨				▨		▨		▨				▨		
μ_4		▨			■	▨	■	▨	■	▨				▨			■
μ_5	▨	■			▨	■	▨	■	▨				▨		▨		
μ_6	■			▨	■	▨	■	▨				▨		▨			■
μ_7		■	▨	■	▨	■	▨				▨		▨		▨		
μ_8		▨		▨		▨				▨		▨		▨			
μ_9	▨		▨		▨				▨		▨		▨	■			
μ_{10}	■	▨		▨				▨		▨		▨	■		■		■
μ_{11}	▨	■	▨				▨		▨		▨	■			▨		
μ_{12}		▨				▨		▨		▨			■	▨			■
μ_{13}	▨				▨		▨		▨				▨		▨		
μ_{14}				▨		▨		▨	■		■	▨		▨			■
μ_{15}			▨		▨		▨	■		■	▨		▨		▨	▨	
μ_{16}															▨	▨	
μ_0		■		■		■				■		■		■			

Figure 4.47: Case 3: hamiltonian table of $S_1 \times S_2$ for $m = 17$ with a 15-twined 2-factorization in black (hamiltonian pairs) and blue (truncated hamiltonian pairs).

We note that, when $m = 11$, neither of Propositions 4.86 and 4.89 gives rise to a c-twined 2-factorization of $\vec{C}_2 \wr \overline{K}_m$ for $c \in \{5, 7\}$. We remedy this omission in Lemma 4.90 below.

Lemma 4.90. *Let* $m = 11$ *and* $c \in \{5, 7\}$. *The digraph* $\vec{C}_2 \wr \overline{K}_m$ *admits a* c*-twined 2-factorization.*

Proof: We consider two cases, one for each $c \in \{5, 7\}$. Since we will be using the regular permutation set $\mathcal{F}_{11}$ to construct the desired c-twined 2-factorizations for both cases, we list all elements of $\mathcal{F}_{11}$ below to facilitate the verification of the computations:

$$\sigma_0 = id;$$

$$\sigma_1 = (10, 2, 3, 4, 5, 6, 7, 8, 9, 0, 1);$$

$$\begin{aligned}
\sigma_2 &= (10,7,9,1,3,5)(0,2,4,6,8);\\
\sigma_3 &= (10,3,6,9,2,5,81,4,7,0);\\
\sigma_4 &= (10,8,2,6,0,4)(1,5,9,3,7);\\
\sigma_5 &= (10,4,9)(1,6)(2,7)(3,8)(0,5);\\
\sigma_6 &= (10,9,5,1,7,3)(0,6,2,8,4);\\
\sigma_7 &= (10,5,2,9,6,3,0,7,4,1,8);\\
\sigma_8 &= (10,0,8,6,4,2)(1,9,7,5,3);\\
\sigma_9 &= (10,6,5,4,3,2,1,0,9,8,7);\\
\sigma_{10} &= (0,3,9,4,8,5,7,6,10,1,2).
\end{aligned}$$

<u>Case 1</u>: $c = 5$. Let

$$S_1 = \mathcal{F}_{11} = \{\sigma_0, \sigma_1, \ldots, \sigma_{10}\};$$

$$S_2 = (0,10) \cdot \mathcal{F}_{11} = \{\mu_i = (0,10)\sigma_i \mid i = 0,1,\ldots,10\}.$$

First, we construct the following set of five truncated hamiltonian pairs:

$$D_T = \{(\sigma_1,\mu_0),(\sigma_5,\mu_1),(\sigma_8,\mu_6),(\sigma_9,\mu_9),(\sigma_{10},\mu_{10})\}.$$

Below, we show that all pairs of D_T are truncated hamiltonian:

$$\begin{aligned}
\hat{\sigma}_1\hat{\mu}_0 &= (1,2,3,4,5,6,7,8,9,0)(10);\\
\hat{\sigma}_5\hat{\mu}_1 &= (1,7,3,9,5,2,8,4,0,6)(10);\\
\hat{\sigma}_8\hat{\mu}_6 &= (1,5,6,0,4,8,2,9,3,7)(10);\\
\hat{\sigma}_9\hat{\mu}_9 &= (1,6,4,2,0,8,9,7,5,3)(10);\\
\hat{\sigma}_{10}\hat{\mu}_{10} &= (1,0,9,8,7,3,4,5,6,2)(10).
\end{aligned}$$

Next, we form a set of six hamiltonian pairs:

$$D_H = \{(\sigma_2,\mu_7),(\sigma_3,\mu_8),(\sigma_4,\mu_5),(\sigma_6,\mu_3),(\sigma_7,\mu_4),(\sigma_0,\mu_2)\}.$$

Below, we show that all pairs of D_H are hamiltonian pairs:

$$\begin{aligned}
\sigma_2\mu_7 &= (10,4,3,2,1,0,9,8,5,7,6);\\
\sigma_3\mu_8 &= (10,1,2,3,4,5,6,7,0,8,9);\\
\sigma_4\mu_5 &= (10,3,2,1,0,9,8,7,6,4,5);\\
\sigma_6\mu_3 &= (10,2,1,0,9,8,7,6,5,4,3);\\
\sigma_7\mu_4 &= (10,9,0,1,2,3,8,4,5,6,7);\\
\sigma_0\mu_2 &= (10,2,4,6,8,0,7,9,1,3,5).
\end{aligned}$$

Hence $D = D_T \cup D'_H$ is a 5-twined 2-factorization.
Case 2: $c = 7$. Let

$$S_1 = (10, 1, 2, 3, 4, 5) \cdot \mathcal{F}_{11} = \{\mu_i = (10, 1, 2, 3, 4, 5)\sigma_i \mid i = 0, 1, \dots, 10\};$$
$$S_2 = \mathcal{F}_{11} = \{\sigma_0, \sigma_1, \dots, \sigma_{10}\}.$$

First, we construct the following set of seven truncated hamiltonian pairs:

$$D_T = \{(\mu_2, \sigma_6), (\mu_3, \sigma_1), (\mu_5, \sigma_9), (\mu_6, \sigma_8), (\mu_7, \sigma_7), (\mu_9, \sigma_5), (\sigma_{10}, \mu_{10})\}.$$

See below for proof that all pairs of D_T are truncated hamiltonian pairs:

$$\begin{aligned}
\hat{\mu}_2\hat{\sigma}_6 &= (1, 0, 8, 6, 4, 9, 7, 5, 3, 2)(10);\\
\hat{\mu}_3\hat{\sigma}_1 &= (1, 6, 0, 5, 4, 9, 3, 8, 2, 7)(10);\\
\hat{\mu}_5\hat{\sigma}_9 &= (1, 6, 0, 4, 9, 5, 3, 8, 2, 7)(10);\\
\hat{\mu}_6\hat{\sigma}_8 &= (1, 6, 0, 4, 9, 3, 8, 2, 5, 7)(10);\\
\hat{\mu}_7\hat{\sigma}_7 &= (1, 6, 0, 4, 9, 3, 8, 5, 2, 7)(10);\\
\hat{\mu}_9\hat{\sigma}_5 &= (1, 6, 0, 4, 9, 3, 8, 2, 7, 5)(10);\\
\hat{\mu}_{10}\hat{\sigma}_{10} &= (1, 3, 5, 2, 4, 6, 0, 9, 8, 7)(10).
\end{aligned}$$

Next, we form a set of four hamiltonian pairs:

$$D_H = \{(\mu_1, \sigma_4), (\mu_4, \sigma_0), (\mu_8, \sigma_3), (\mu_0, \sigma_2)\}.$$

Below, we show that all pairs of D_H are hamiltonian pairs:

$$\begin{aligned}
\mu_1\sigma_4 &= (10, 8, 3, 9, 4, 0, 5, 6, 1, 7, 2);\\
\mu_4\sigma_0 &= (10, 5, 8, 2, 7, 1, 6, 0, 4, 9, 3);\\
\mu_8\sigma_3 &= (10, 2, 4, 6, 7, 8, 9, 0, 1, 3, 5);\\
\mu_0\sigma_2 &= (10, 3, 6, 8, 0, 2, 5, 7, 9, 1, 4).
\end{aligned}$$

Then $D = D_T \cup D_H$ is the desired 7-twined 2-factorization. ∎

In Theorem 4.91, we summarize this section's results.

Theorem 4.91. *Let H be a digraph such that $|V(H)| = m$ is odd and $m \geqslant 5$. Furthermore, assume that H is a strict digraph that admits a decomposition into c directed hamiltonian cycles where c is odd and $3 \leqslant c \leqslant m - 2$. The digraph $\vec{C}_n \wr H$ is hamiltonian decomposable.*

Proof: We consider three cases, one for each congruency class of m modulo 6. In each case, we demonstrate that Propositions 4.85–4.89 and Lemma 4.90 jointly give rise to a c-twined 2-factorization of $\vec{C}_2 \wr \overline{K}_m$ for all possible odd c such that $3 \leqslant c \leqslant m-2$.

Case 1: $m \equiv 1 \pmod 6$.

Subcase 1.1: $m \equiv 1 \pmod 4$. Then $\vec{C}_2 \wr \overline{K}_m$ admits a c-twined 2-factorization if $c \leqslant \frac{m+1}{2}$ or $c \geqslant \frac{m+2}{3}$ by Propositions 4.85 and 4.87, respectively. Since $\frac{m+2}{3} \leqslant \frac{m+1}{2}$ for all applicable odd m, the result follows.

Subcase 1.2: $m \equiv 3 \pmod 4$. Then $\vec{C}_2 \wr \overline{K}_m$ admits a c-twined 2-factorization if $c \leqslant \frac{m-5}{2}$ or $c \geqslant \frac{m+2}{3}$ by Propositions 4.86 and 4.87, respectively. If $\frac{m-5}{2} < c < \frac{m+2}{3}$, then $m = 7$ and $1 < c < 3$, thus contradicting the assumption that c is odd.

Case 2: $m \equiv 3 \pmod 6$.

Subcase 2.1: $m \equiv 1 \pmod 4$. Then $\vec{C}_2 \wr \overline{K}_m$ admits a c-twined 2-factorization if $c \leqslant \frac{m+1}{2}$ or $c \geqslant \frac{m}{3}+4$ by Propositions 4.85 and 4.88, respectively. If $\frac{m+1}{2} < c < \frac{m}{3}+4$, then $m = 9$, and $5 < c < 7$, a contradiction.

Subcase 2.2: $m \equiv 3 \pmod 8$. Then $\vec{C}_2 \wr \overline{K}_m$ admits a c-twined 2-factorization if $c \leqslant \frac{m-5}{2}$ or $c \geqslant \frac{m}{3}+4$ by Propositions 4.86 and 4.88, respectively. If $\frac{m-5}{2} < c < \frac{m}{3}+4$, then $m = 27$, and $11 < c < 13$, a contradiction..

Subcase 2.3: $m \equiv 7 \pmod 8$. Then $\vec{C}_2 \wr \overline{K}_m$ admits a c-twined 2-factorization if $c \leqslant \frac{m-1}{2}$ or $c \geqslant \frac{m}{3}+4$ by Propositions 4.86 and 4.88, respectively. If $\frac{m-1}{2} < c < \frac{m}{3}+4$, then $m = 15$, and $7 < c < 9$, a contradiction.

Case 3: $m \equiv 5 \pmod 6$.

Subcase 3.1: $m \equiv 1 \pmod 4$. For $m \geqslant 11$, $\vec{C}_2 \wr \overline{K}_m$ admits a c-twined 2-factorization if $c \leqslant \frac{m+1}{2}$ or $c \geqslant \frac{m+1}{3} + 5$ by Propositions 4.85 and 4.89, respectively. If $\frac{m+1}{2} < c < \frac{m+1}{3} + 5$, then $m = 17$ and $9 < c < 11$, a contradiction. If $m = 5$, Proposition 4.85 gives the desired 3-twined 2-factorization.

Subcase 3.2: $m \equiv 3 \pmod 8$. Then $\vec{C}_2 \wr \overline{K}_m$ admits a c-twined 2-factorization if $c \leqslant \frac{m-5}{2}$ or $c \geqslant \frac{m+1}{3}+5$, by Propositions 4.86 and 4.89, respectively. If $\frac{m-5}{2} < c < \frac{m+1}{3}+5$ then $m = 11$ and $3 < c < 9$. Lemma 4.90 gives a c-twined 2-factorization of $\vec{C}_2 \wr \overline{K}_{11}$ for $c \in \{5, 7\}$.

Subcase 3.3: $m \equiv 7 \pmod 8$. Then $\vec{C}_2 \wr \overline{K}_m$ admits a c-twined 2-factorization if $c \leqslant \frac{m-1}{2}$ or $c \geqslant \frac{m+1}{3}+5$ by Propositions 4.86 and 4.89, respectively. If $\frac{m-1}{2} < c < \frac{m+1}{3}+5$, then $m = 23$, and $11 < c < 13$, a contradiction.

In summary, there exists a c-twined 2-factorization of $\vec{C}_2 \wr \overline{K}_m$ for all odd $m \geqslant 5$ and all $3 \leqslant c \leqslant m-2$. Corollary 4.68 then implies that, with these conditions, $\vec{C}_n \wr H$ is hamiltonian decomposable. ▮

4.8 Conclusion

We now conclude this chapter by summarizing our results. We do so by giving a proof of Theorem 4.2. For convenience, we first restate Theorem 4.2 below.

Theorem 4.2. *Let G and H be two strict hamiltonian decomposable digraphs such that $G \neq \overline{K}_n$. Let $|V(G)|= n$, where n is even, $|V(H)|= m$, and let c be the number of cycles in a hamiltonian decomposition of H. The digraph $G \wr H$ is hamiltonian decomposable in each of the following cases:*

(S1) *m is odd, $(m, c) \neq (3, 1)$, and $(n, m, c) \neq (2, 3, 2)$;*

(S2) *m and c are even;*

(S3) *m is even, c is odd, $3 \leqslant c \leqslant m-3$, and $G \neq \vec{C}_n$;*

(S4) *m is even, $c = m-1$, and $(n, m) \neq (2, 2)$;*

(S5) *m is even, $m \geqslant 4$, $c = 1$, and $n \geqslant 4$.*

If $G = \vec{C}_n$ and $(m, c) \in \{(2, 1), (3, 1)\}$ or $(n, m, c) = (2, 3, 2)$, then $G \wr H$ is not hamiltonian decomposable.

Proof:

(S1) The result follows from Corollary 4.49 when $c = 1$, and from Corollary 4.56 when $c = m-1$. For $2 \leqslant c \leqslant m-2$, the result follows from Theorem 4.84 when c is even or $G \neq \vec{C}_n$, and Theorem 4.91 when c is odd and $G = \vec{C}_n$.

(S2) The result follows from Theorem 4.84.

(S3) The result follows from Theorems 4.84.

(S4) The result follows from Corollary 4.56.

(S5) The result follows from Corollary 4.49.

Lastly, Propositions 4.10 and 4.19 state that $\vec{C}_n \wr \vec{C}_2$ and $\vec{C}_n \wr \vec{C}_3$, respectively, are not hamiltonian decomposable, and Theorem 3.28 implies that $G \wr H$ is not hamiltonian decomposable when $(n, m) = (2, 2)$ and $(n, m, c) = (2, 3, 2)$. ∎

Theorem 4.2 is a significant contribution to Conjecture 2.23. However, there remain three important open cases:

1. $G = \vec{C}_n$ where n is even, m is even, and H admits a decomposition into c directed hamiltonian cycles where c is odd and $3 \leqslant c \leqslant m - 3$. We believe that a similar approach as that taken in Sections 4.6 and 4.7 will give the desired decompositions. Therefore, one needs to construct two regular permutation sets of order m that yield a c-twined 2-factorization of $\vec{C}_2 \wr \overline{K}_m$.
2. $G = \vec{C}_2$ and $H = \vec{C}_m$ where m is even. To resolve this particular case, it suffices to prove Conjecture 4.13. Note that we have shown Conjecture 4.13 to be true for $6 \leqslant m \leqslant 16$ (see Appendix B). We have also shown that Conjecture 2.23 is true for $m = 4$ in Lemma 4.11. Therefore, we believe that Conjecture 2.23 holds when $G = \vec{C}_2$ and $H = \vec{C}_m$ where $m \geqslant 4$ is even.

   To resolve Conjecture 4.13, it would suffice to construct a set D of m pairs of permutations from the set $S_1 \times S_2$, where S_1 and S_2 are regular permutation sets of order m, that satisfy the following conditions

   (a) Each permutation of S_1 and S_2 appears precisely once in D;
   (b) $(id, id) \in D$;
   (c) D contains $m - 1$ hamiltonian pairs.

   It is not too hard to see that the $m - 1$ hamiltonian pairs of D give rise to the desired hamiltonian decomposition of $\vec{C}_2 \times K_m^*$.
3. $H = \vec{C}_m$ where $m \in \{2, 3\}$ and $G \neq \vec{C}_n$. Since we know that $\vec{C}_n \wr \vec{C}_m$ is not hamiltonian decomposable when $m \in \{2, 3\}$ and n is even, a new approach will be needed to settle Conjecture 2.23 for this case.

Appendix A

Computations for the proof of Proposition 4.44

A.1 Elements of U_2

$$
\begin{array}{rlll}
U_2 = & \{((1,0,0),((2),(12))), & ((0,0,1),((012),(02))), & ((0,1,0),((2),(12))), \\
& ((1,0,0),((012),(2)(01))), & ((1,0,0),((12),(02))), & ((0,0,0),((2),(021))), \\
& ((0,0,1),((12),(12))), & ((1,0,0),((02),(2)(01))), & ((0,1,0),((2),(2)(01))), \\
& ((0,0,1),((2),(02))), & ((0,1,0),((012),(12))), & ((1,1,1),((02),(12))), \\
& ((1,1,0),((2),(021))), & ((0,1,0),((12),(12))), & ((0,1,0),((021),(02))), \\
& ((0,0,1),((021),(2)(01))), & ((1,0,0),((021),(12))), & ((1,1,0),((2)(01),(021))), \\
& ((1,1,0),((2),(012))), & ((0,1,0),((02),(2)(01))), & ((0,0,1),((021),(12))), \\
& ((0,0,1),((02),(02))), & ((1,0,0),((021),(2)(01))), & ((1,0,0),((2)(01),(02))), \\
& ((1,1,1),((2)(01),(2)(01))), & ((1,1,0),((02),(021))), & ((1,1,0),((021),(021))), \\
& ((1,1,0),((012),(012))), & ((1,1,0),((2)(01),(012))), & ((0,0,1),((2)(01),(02))), \\
& ((1,1,0),((12),(012))), & ((1,0,0),((12),(12))), & ((1,1,0),((02),(012))), \\
& ((0,1,1),((02),(2))), & ((0,1,0),((2)(01),(02))), & ((0,1,0),((12),(2)(01))), \\
& ((0,0,1),((012),(2)(01))), & ((1,1,0),((12),(021))), & ((0,1,0),((012),(2)(01))), \\
& ((1,1,1),((02),(2)(01))), & ((1,0,1),((012),(021))), & ((1,1,1),((2)(01),(12))), \\
& ((1,1,0),((021),(012))), & ((1,1,0),((012),(021))), & ((1,0,0),((02),(02))), \\
& ((0,1,1),((2),(2))), & ((0,1,1),((021),(2))), & ((1,0,1),((12),(021))), \\
& ((0,0,0),((021),(021))), & ((1,0,1),((012),(012))), & ((0,0,1),((12),(02))),
\end{array}
$$

$$\begin{gathered}
((1,0,1),((2),(2))),\\
((0,1,1),((2)(01),(2))),\\
((0,0,0),((021),(012))),\\
((1,0,1),((2)(01),(2))),\\
((1,0,0),((2)(01),(2)(01))),\\
((1,0,1),((02),(2))),\\
((0,1,1),((2)(01),(021))),\\
((1,1,1),((12),(2)(01))),\\
((0,0,1),((02),(2)(01))),\\
((0,1,1),((012),(012))),\\
((1,0,1),((2)(01),(012))),\\
((0,1,0),((021),(2)(01))),\\
((0,1,1),((2),(012))),\\
((1,0,1),((012),(2))),\\
((0,1,0),((021),(12))),\\
((0,0,1),((12),(2)(01))),\\
((1,0,1),((021),(021))),\\
((1,0,0),((2)(01),(12))),\\
((0,0,1),((02),(12))),\\
((1,0,0),((012),(12))),\\
((0,1,1),((02),(012))),\\
((1,0,1),((2)(01),(021))),\\
((0,1,0),((02),(12))),\\
((1,0,1),((2),(021))),
\end{gathered}$$

$$\begin{gathered}
((1,1,1),((12),(12))),\\
((0,1,1),((012),(2))),\\
((1,0,0),((2),(02))),\\
((0,0,0),((2),(2))),\\
((0,0,1),((2)(01),(12))),\\
((1,0,1),((021),(2))),\\
((1,0,1),((12),(012))),\\
((1,0,1),((02),(012))),\\
((1,0,0),((021),(02))),\\
((0,1,1),((2),(021))),\\
((0,0,0),((021),(2))),\\
((0,1,1),((12),(012))),\\
((1,0,1),((021),(012))),\\
((1,1,1),((2)(01),(02))),\\
((0,1,1),((02),(021))),\\
((0,1,1),((12),(021))),\\
((1,0,1),((02),(021))),\\
((0,1,0),((2)(01),(12))),\\
((1,1,1),((02),(02))),\\
((0,1,1),((012),(021))),\\
((1,1,0),((021),(2))),\\
((0,0,0),((012),(021))),\\
((1,1,0),((2)(01),(2))),\\
((1,1,1),((12),(02))),
\end{gathered}$$

$$\begin{gathered}
((1,0,0),((012),(02))),\\
((0,0,0),((012),(2))),\\
((0,0,1),((012),(12))),\\
((0,1,0),((02),(02))),\\
((0,1,1),((12),(2))),\\
((1,0,0),((12),(2)(01))),\\
((0,1,0),((2),(02))),\\
((0,1,0),((2)(01),(2)(01))),\\
((0,1,1),((021),(021))),\\
((0,1,1),((2)(01),(012))),\\
((1,0,1),((12),(2))),\\
((0,0,1),((2),(12))),\\
((1,0,1),((2),(012))),\\
((0,0,1),((2),(2)(01))),\\
((0,1,0),((12),(02))),\\
((0,0,0),((012),(012))),\\
((0,0,1),((2)(01),(2)(01))),\\
((0,0,1),((021),(02))),\\
((0,1,0),((012),(02))),\\
((0,1,1),((021),(012))),\\
((1,1,0),((02),(2))),\\
((1,0,0),((02),(12))),\\
((1,1,0),((012),(2))),\\
((1,0,0),((2),(2)(01)))\}.
\end{gathered}$$

A.2 Computations of the elements of U_2

	((1, 0, 1), ((2)(01), (2)))	((0, 0, 0), ((012), (021)))	((1, 1, 1), ((02), (12)))	((0, 0, 0), ((2), (012)))	((0, 1, 1), ((12), (012)))
((1, 0, 1), ((2)(01), (2)))	((1, 1, 0), ((2), (2)))	((0, 1, 1), ((02), (021)))	((0, 1, 0), ((012), (12)))	((1, 0, 1), ((2)(01), (012)))	((1, 0, 1), ((021), (012)))
((0, 0, 0), ((012), (021)))	((1, 0, 1), ((12), (021)))	((0, 0, 0), ((021), (012)))	((1, 1, 1), ((2)(01), (2)(01)))	((0, 0, 0), ((012), (2)))	((0, 1, 1), ((02), (2)))
((1, 1, 1), ((02), (12)))	((0, 1, 0), ((021), (12)))	((1, 1, 1), ((12), (02)))	((0, 0, 0), ((2), (2)))	((1, 1, 1), ((02), (2)(01)))	((1, 0, 0), ((012), (2)(01)))
((0, 0, 0), ((2), (012)))	((1, 0, 1), ((2)(01), (012)))	((0, 0, 0), ((012), (2)))	((1, 1, 1), ((02), (02)))	((0, 0, 0), ((2), (021)))	((0, 1, 1), ((12), (021)))
((0, 1, 1), ((12), (012)))	((0, 0, 0), ((012), (012)))	((1, 1, 0), ((2)(01), (2)))	((0, 0, 1), ((021), (02)))	((0, 1, 1), ((12), (021)))	((0, 0, 0), ((2), (021)))
((0, 0, 1), ((012), (12)))	((1, 0, 0), ((12), (12)))	((0, 1, 0), ((021), (02)))	((0, 1, 1), ((2)(01), (2)))	((0, 0, 1), ((012), (2)(01)))	((0, 0, 1), ((02), (2)(01)))
((0, 0, 1), ((2), (02)))	((1, 0, 0), ((2)(01), (02)))	((0, 1, 0), ((012), (2)(01)))	((0, 1, 1), ((02), (012)))	((0, 0, 1), ((2), (12)))	((0, 0, 1), ((12), (12)))
((0, 0, 0), ((021), (2)))	((1, 0, 1), ((02), (2)))	((0, 0, 0), ((2), (021)))	((1, 1, 1), ((12), (12)))	((0, 0, 0), ((021), (012)))	((0, 1, 1), ((2)(01), (012)))
((1, 1, 1), ((2)(01), (02)))	((0, 1, 0), ((2), (02)))	((1, 1, 1), ((02), (2)(01)))	((0, 0, 0), ((012), (012)))	((1, 1, 1), ((2)(01), (12)))	((1, 0, 0), ((021), (12)))
((1, 0, 0), ((012), (02)))	((1, 1, 1), ((12), (02)))	((0, 0, 1), ((021), (2)(01)))	((1, 1, 0), ((2)(01), (012)))	((1, 0, 0), ((012), (12)))	((1, 1, 1), ((02), (12)))
((0, 1, 0), ((021), (02)))	((0, 0, 1), ((02), (02)))	((1, 0, 0), ((2), (2)(01)))	((1, 0, 1), ((12), (012)))	((0, 1, 0), ((021), (12)))	((0, 1, 0), ((2)(01), (12)))
((0, 1, 1), ((2)(01), (021)))	((0, 0, 0), ((2), (021)))	((1, 1, 0), ((02), (012)))	((0, 0, 1), ((012), (2)(01)))	((0, 1, 1), ((2)(01), (2)))	((0, 0, 0), ((021), (2)))
((1, 1, 1), ((12), (2)(01)))	((0, 1, 0), ((012), (2)(01)))	((1, 1, 1), ((2)(01), (12)))	((0, 0, 0), ((021), (021)))	((1, 1, 1), ((12), (02)))	((1, 0, 0), ((2), (02)))
((1, 1, 0), ((2)(01), (012)))	((0, 1, 1), ((2), (012)))	((1, 0, 1), ((02), (2)))	((1, 0, 0), ((012), (02)))	((1, 1, 0), ((2)(01), (021)))	((1, 1, 0), ((021), (021)))
((1, 0, 1), ((12), (021)))	((1, 1, 0), ((012), (021)))	((0, 1, 1), ((2)(01), (012)))	((0, 1, 0), ((021), (2)(01)))	((1, 0, 1), ((12), (2)))	((1, 0, 1), ((2), (2)))
((1, 0, 0), ((021), (12)))	((1, 1, 1), ((02), (12)))	((0, 0, 1), ((2), (02)))	((1, 1, 0), ((12), (2)))	((1, 0, 0), ((021), (2)(01)))	((1, 1, 1), ((2)(01), (2)(01)))
((1, 0, 0), ((2), (2)(01)))	((1, 1, 1), ((2)(01), (2)(01)))	((0, 0, 1), ((012), (12)))	((1, 1, 0), ((02), (021)))	((1, 0, 0), ((2), (02)))	((1, 1, 1), ((12), (02)))
((0, 1, 0), ((2), (12)))	((0, 0, 1), ((2)(01), (12)))	((1, 0, 0), ((012), (02)))	((1, 0, 1), ((02), (2)))	((0, 1, 0), ((2), (2)(01)))	((0, 1, 0), ((12), (2)(01)))
((1, 0, 1), ((02), (012)))	((1, 1, 0), ((021), (012)))	((0, 1, 1), ((12), (2)))	((0, 1, 0), ((2), (02)))	((1, 0, 1), ((02), (021)))	((1, 0, 1), ((012), (021)))
((1, 1, 0), ((12), (2)))	((0, 1, 1), ((012), (2)))	((1, 0, 1), ((2)(01), (021)))	((1, 0, 0), ((021), (12)))	((1, 1, 0), ((12), (012)))	((1, 1, 0), ((2), (012)))
((0, 1, 0), ((012), (2)(01)))	((0, 0, 1), ((12), (2)(01)))	((1, 0, 0), ((021), (12)))	((1, 0, 1), ((2)(01), (021)))	((0, 1, 0), ((012), (02)))	((0, 1, 0), ((02), (02)))
((0, 1, 1), ((02), (2)))	((0, 0, 0), ((021), (2)))	((1, 1, 0), ((12), (021)))	((0, 0, 1), ((2), (12)))	((0, 1, 1), ((02), (012)))	((0, 0, 0), ((012), (012)))
((0, 0, 1), ((021), (2)(01)))	((1, 0, 0), ((02), (2)(01)))	((0, 1, 0), ((2), (12)))	((0, 1, 1), ((12), (021)))	((0, 0, 1), ((021), (02)))	((0, 0, 1), ((2)(01), (02)))
((1, 1, 0), ((02), (021)))	((0, 1, 1), ((021), (021)))	((1, 0, 1), ((12), (012)))	((1, 0, 0), ((2), (2)(01)))	((1, 1, 0), ((02), (2)))	((1, 1, 0), ((012), (2)))

	((0, 0, 1), ((012), (12)))	((0, 0, 1), ((2), (02)))	((0, 0, 0), ((021), (2)))	((1, 1, 1), ((2)(01), (02)))	((1, 0, 0), ((012), (02)))
((1, 0, 1), ((2)(01), (2)))	((0, 1, 0), ((02), (12)))	((1, 0, 0), ((2)(01), (02)))	((1, 1, 0), ((12), (2)))	((1, 0, 0), ((2), (02)))	((1, 1, 1), ((02), (02)))
((0, 0, 0), ((012), (021)))	((0, 0, 1), ((021), (2)(01)))	((0, 0, 1), ((012), (12)))	((0, 0, 0), ((2), (021)))	((1, 1, 1), ((12), (12)))	((1, 0, 0), ((021), (12)))
((1, 1, 1), ((02), (12)))	((1, 1, 0), ((12), (2)))	((1, 1, 0), ((02), (021)))	((1, 1, 1), ((2)(01), (12)))	((0, 0, 0), ((021), (021)))	((0, 1, 1), ((12), (021)))
((0, 0, 0), ((2), (012)))	((0, 0, 1), ((012), (02)))	((0, 0, 1), ((2), (2)(01)))	((0, 0, 0), ((021), (012)))	((1, 1, 1), ((2)(01), (2)(01)))	((1, 0, 0), ((012), (2)(01)))
((0, 1, 1), ((12), (012)))	((1, 1, 1), ((2)(01), (02)))	((0, 1, 0), ((12), (2)(01)))	((1, 0, 1), ((02), (012)))	((0, 1, 0), ((012), (2)(01)))	((0, 1, 0), ((2)(01), (2)(01)))
((0, 0, 1), ((012), (12)))	((0, 1, 1), ((021), (2)))	((0, 0, 0), ((012), (021)))	((1, 0, 0), ((2), (12)))	((1, 1, 0), ((12), (021)))	((1, 1, 0), ((021), (021)))
((0, 0, 1), ((2), (02)))	((0, 1, 1), ((012), (012)))	((0, 0, 0), ((2), (2)))	((1, 0, 0), ((021), (02)))	((1, 1, 0), ((2)(01), (2)))	((1, 1, 0), ((012), (2)))
((0, 0, 0), ((021), (2)))	((0, 0, 1), ((2), (12)))	((0, 0, 1), ((021), (02)))	((0, 0, 0), ((012), (2)))	((1, 1, 1), ((02), (02)))	((1, 0, 0), ((2), (02)))
((1, 1, 1), ((2)(01), (02)))	((1, 1, 0), ((02), (012)))	((1, 1, 0), ((2)(01), (2)))	((1, 1, 1), ((12), (02)))	((0, 0, 0), ((2), (2)))	((0, 1, 1), ((02), (2)))
((1, 0, 0), ((012), (02)))	((0, 0, 0), ((021), (012)))	((1, 0, 1), ((012), (2)))	((0, 1, 0), ((2), (02)))	((1, 0, 1), ((12), (2)))	((1, 0, 1), ((021), (2)))
((0, 1, 0), ((021), (02)))	((1, 0, 1), ((2), (012)))	((0, 1, 1), ((021), (2)))	((0, 0, 1), ((012), (02)))	((0, 1, 1), ((02), (2)))	((0, 0, 0), ((2), (2)))
((0, 1, 1), ((2)(01), (021)))	((1, 1, 1), ((02), (2)(01)))	((0, 1, 0), ((2)(01), (12)))	((1, 0, 1), ((12), (021)))	((0, 1, 0), ((2), (12)))	((0, 1, 0), ((02), (12)))
((1, 1, 1), ((12), (2)(01)))	((1, 1, 0), ((2)(01), (021)))	((1, 1, 0), ((12), (012)))	((1, 1, 1), ((02), (2)(01)))	((0, 0, 0), ((012), (012)))	((0, 1, 1), ((2)(01), (012)))
((1, 1, 0), ((2)(01), (012)))	((1, 0, 0), ((02), (02)))	((1, 1, 1), ((2)(01), (2)(01)))	((0, 1, 1), ((12), (012)))	((0, 0, 1), ((2), (2)(01)))	((0, 0, 1), ((02), (2)(01)))
((1, 0, 1), ((12), (021)))	((0, 1, 0), ((2)(01), (2)(01)))	((1, 0, 0), ((12), (12)))	((1, 1, 0), ((02), (021)))	((1, 0, 0), ((012), (12)))	((1, 1, 1), ((2)(01), (12)))
((1, 0, 0), ((021), (12)))	((0, 0, 0), ((2), (2)))	((1, 0, 1), ((021), (021)))	((0, 1, 0), ((012), (12)))	((1, 0, 1), ((02), (021)))	((1, 0, 1), ((2), (021)))
((1, 0, 0), ((2), (2)(01)))	((0, 0, 0), ((012), (021)))	((1, 0, 1), ((2), (012)))	((0, 1, 0), ((021), (2)(01)))	((1, 0, 1), ((2)(01), (012)))	((1, 0, 1), ((012), (012)))
((0, 1, 0), ((2), (12)))	((1, 0, 1), ((012), (2)))	((0, 1, 1), ((2), (021)))	((0, 0, 1), ((021), (12)))	((0, 1, 1), ((2)(01), (021)))	((0, 0, 0), ((012), (021)))
((1, 0, 1), ((02), (012)))	((0, 1, 0), ((12), (02)))	((1, 0, 0), ((02), (2)(01)))	((1, 1, 0), ((2)(01), (012)))	((1, 0, 0), ((021), (2)(01)))	((1, 1, 1), ((12), (2)(01)))
((1, 1, 0), ((12), (2)))	((1, 0, 0), ((2)(01), (12)))	((1, 1, 1), ((12), (02)))	((0, 1, 1), ((02), (2)))	((0, 0, 1), ((012), (02)))	((0, 0, 1), ((2)(01), (02)))
((0, 1, 0), ((012), (2)(01)))	((1, 0, 1), ((021), (021)))	((0, 1, 1), ((012), (012)))	((0, 0, 1), ((2), (2)(01)))	((0, 1, 1), ((12), (012)))	((0, 0, 0), ((021), (012)))
((0, 1, 1), ((02), (2)))	((1, 1, 1), ((12), (12)))	((0, 1, 0), ((02), (02)))	((1, 0, 1), ((2)(01), (2)))	((0, 1, 0), ((021), (02)))	((0, 1, 0), ((12), (02)))
((0, 0, 1), ((021), (2)(01)))	((0, 1, 1), ((2), (021)))	((0, 0, 0), ((021), (012)))	((1, 0, 0), ((012), (2)(01)))	((1, 1, 0), ((02), (012)))	((1, 1, 0), ((2), (012)))
((1, 1, 0), ((02), (021)))	((1, 0, 0), ((12), (2)(01)))	((1, 1, 1), ((02), (12)))	((0, 1, 1), ((2)(01), (021)))	((0, 0, 1), ((021), (12)))	((0, 0, 1), ((12), (12)))

	$((0,1,0),((021),(02)))$	$((0,1,1),((2)(01),(021)))$	$((1,1,1),((12),(2)(01)))$	$((1,1,0),((2)(01),(012)))$	$((1,0,1),((12),(021)))$
$((1,0,1),((2)(01),(2)))$	$((1,0,0),((12),(02)))$	$((0,0,0),((2),(021)))$	$((0,0,1),((021),(2)(01)))$	$((1,0,1),((2),(012)))$	$((0,1,1),((021),(021)))$
$((0,0,0),((012),(021)))$	$((0,1,0),((2),(12)))$	$((0,1,1),((12),(012)))$	$((1,1,1),((02),(02)))$	$((1,1,0),((12),(2)))$	$((1,0,1),((02),(012)))$
$((1,1,1),((02),(12)))$	$((1,0,1),((2)(01),(021)))$	$((1,0,0),((021),(02)))$	$((0,0,0),((012),(012)))$	$((0,0,1),((021),(2)(01)))$	$((0,1,0),((012),(02)))$
$((0,0,0),((2),(012)))$	$((0,1,0),((021),(2)(01)))$	$((0,1,1),((2)(01),(2)))$	$((1,1,1),((12),(12)))$	$((1,1,0),((2)(01),(021)))$	$((1,0,1),((12),(2)))$
$((0,1,1),((12),(012)))$	$((1,1,1),((02),(2)(01)))$	$((1,1,0),((012),(2)))$	$((1,0,0),((2),(12)))$	$((0,1,1),((012),(021)))$	$((1,1,0),((2),(2)))$
$((0,0,1),((012),(12)))$	$((1,1,0),((2),(021)))$	$((0,1,0),((12),(02)))$	$((1,0,1),((02),(012)))$	$((1,1,1),((12),(2)(01)))$	$((1,1,1),((02),(02)))$
$((0,0,1),((2),(02)))$	$((1,1,0),((021),(2)))$	$((0,1,0),((2)(01),(2)(01)))$	$((1,0,1),((12),(021)))$	$((1,1,1),((2)(01),(12)))$	$((1,1,1),((12),(2)(01)))$
$((0,0,0),((021),(2)))$	$((0,1,0),((012),(02)))$	$((0,1,1),((02),(021)))$	$((1,1,1),((2)(01),(2)(01)))$	$((1,1,0),((02),(012)))$	$((1,0,1),((2)(01),(021)))$
$((1,1,1),((2)(01),(02)))$	$((1,0,1),((12),(2)))$	$((1,0,0),((2),(2)(01)))$	$((0,0,0),((021),(021)))$	$((0,0,1),((2),(12)))$	$((0,1,0),((021),(2)(01)))$
$((1,0,0),((012),(02)))$	$((0,0,0),((2),(2)))$	$((0,0,1),((12),(2)(01)))$	$((0,1,1),((02),(021)))$	$((1,0,0),((12),(12)))$	$((0,0,1),((02),(2)(01)))$
$((0,1,0),((021),(02)))$	$((0,1,1),((012),(2)))$	$((1,1,1),((02),(2)(01)))$	$((1,1,0),((2)(01),(021)))$	$((0,1,0),((02),(12)))$	$((1,0,0),((2)(01),(2)(01)))$
$((0,1,1),((2)(01),(021)))$	$((1,1,1),((12),(12)))$	$((1,1,0),((2),(012)))$	$((1,0,0),((021),(02)))$	$((0,1,1),((2),(2)))$	$((1,1,0),((021),(012)))$
$((1,1,1),((12),(2)(01)))$	$((1,0,1),((02),(012)))$	$((1,0,0),((012),(12)))$	$((0,0,0),((2),(2)))$	$((0,0,1),((012),(02)))$	$((0,1,0),((2),(12)))$
$((1,1,0),((2)(01),(012)))$	$((0,0,1),((12),(2)(01)))$	$((1,0,1),((2),(2)))$	$((0,1,0),((021),(12)))$	$((0,0,0),((2),(021)))$	$((0,0,0),((021),(2)))$
$((1,0,1),((12),(021)))$	$((1,0,0),((02),(12)))$	$((0,0,0),((012),(012)))$	$((0,0,1),((2),(02)))$	$((1,0,1),((012),(2)))$	$((0,1,1),((2),(012)))$
$((1,0,0),((021),(12)))$	$((0,0,0),((012),(021)))$	$((0,0,1),((02),(02)))$	$((0,1,1),((2)(01),(012)))$	$((1,0,0),((02),(2)(01)))$	$((0,0,1),((2)(01),(02)))$
$((1,0,0),((2),(2)(01)))$	$((0,0,0),((021),(012)))$	$((0,0,1),((2)(01),(12)))$	$((0,1,1),((12),(2)))$	$((1,0,0),((2)(01),(02)))$	$((0,0,1),((12),(12)))$
$((0,1,0),((2),(12)))$	$((0,1,1),((021),(021)))$	$((1,1,1),((2)(01),(02)))$	$((1,1,0),((12),(012)))$	$((0,1,0),((2)(01),(2)(01)))$	$((1,0,0),((12),(02)))$
$((1,0,1),((02),(012)))$	$((1,0,0),((2)(01),(2)(01)))$	$((0,0,0),((021),(2)))$	$((0,0,1),((012),(12)))$	$((1,0,1),((021),(021)))$	$((0,1,1),((012),(2)))$
$((1,1,0),((12),(2)))$	$((0,0,1),((02),(02)))$	$((1,0,1),((012),(021)))$	$((0,1,0),((2),(2)(01)))$	$((0,0,0),((012),(012)))$	$((0,0,0),((2),(021)))$
$((0,1,0),((012),(2)(01)))$	$((0,1,1),((2),(012)))$	$((1,1,1),((12),(12)))$	$((1,1,0),((02),(2)))$	$((0,1,0),((12),(02)))$	$((1,0,0),((02),(12)))$
$((0,1,1),((02),(2)))$	$((1,1,1),((2)(01),(02)))$	$((1,1,0),((021),(021)))$	$((1,0,0),((012),(2)(01)))$	$((0,1,1),((021),(012)))$	$((1,1,0),((012),(021)))$
$((0,0,1),((021),(2)(01)))$	$((1,1,0),((012),(012)))$	$((0,1,0),((02),(12)))$	$((1,0,1),((2)(01),(2)))$	$((1,1,1),((02),(02)))$	$((1,1,1),((2)(01),(12)))$
$((1,1,0),((02),(021)))$	$((0,0,1),((2)(01),(12)))$	$((1,0,1),((021),(012)))$	$((0,1,0),((012),(02)))$	$((0,0,0),((021),(2)))$	$((0,0,0),((012),(012)))$

	$((1,0,0),((021),(12)))$	$((1,0,0),((2),(2)(01)))$	$((0,1,0),((2),(12)))$	$((1,0,1),((02),(012)))$	$((1,1,0),((12),(2)))$
$((1,0,1),((2)(01),(2)))$	$((0,1,0),((12),(12)))$	$((0,0,1),((2)(01),(2)(01)))$	$((1,1,1),((2)(01),(12)))$	$((0,0,0),((012),(012)))$	$((0,0,0),((021),(2)))$
$((0,0,0),((012),(021)))$	$((1,0,0),((2),(2)(01)))$	$((1,0,0),((012),(02)))$	$((0,1,0),((012),(2)(01)))$	$((1,0,1),((2)(01),(2)))$	$((1,1,0),((02),(021)))$
$((1,1,1),((02),(12)))$	$((0,1,1),((2)(01),(2)))$	$((0,1,1),((02),(012)))$	$((1,0,1),((02),(2)))$	$((0,1,0),((2),(2)(01)))$	$((0,0,1),((012),(12)))$
$((0,0,0),((2),(012)))$	$((1,0,0),((021),(02)))$	$((1,0,0),((2),(12)))$	$((0,1,0),((2),(02)))$	$((1,0,1),((02),(021)))$	$((1,1,0),((12),(012)))$
$((0,1,1),((12),(012)))$	$((0,0,1),((02),(02)))$	$((1,1,1),((12),(12)))$	$((0,0,1),((12),(02)))$	$((0,1,1),((021),(021)))$	$((1,0,1),((2),(012)))$
$((0,0,1),((012),(12)))$	$((0,0,0),((2),(2)))$	$((1,0,1),((012),(012)))$	$((0,1,1),((012),(2)))$	$((0,0,1),((2)(01),(2)(01)))$	$((1,0,0),((02),(12)))$
$((0,0,1),((2),(02)))$	$((0,0,0),((021),(012)))$	$((1,0,1),((2),(021)))$	$((0,1,1),((2),(012)))$	$((0,0,1),((02),(12)))$	$((1,0,0),((12),(02)))$
$((0,0,0),((021),(2)))$	$((1,0,0),((012),(12)))$	$((1,0,0),((021),(2)(01)))$	$((0,1,0),((021),(12)))$	$((1,0,1),((12),(012)))$	$((1,1,0),((2)(01),(2)))$
$((1,1,1),((2)(01),(02)))$	$((0,1,1),((12),(012)))$	$((0,1,1),((2)(01),(021)))$	$((1,0,1),((2)(01),(012)))$	$((0,1,0),((012),(12)))$	$((0,0,1),((021),(02)))$
$((1,0,0),((012),(02)))$	$((1,1,0),((2),(012)))$	$((0,0,0),((012),(021)))$	$((1,1,0),((012),(012)))$	$((1,0,0),((2)(01),(12)))$	$((0,1,0),((02),(02)))$
$((0,1,0),((021),(02)))$	$((1,0,1),((012),(012)))$	$((1,1,0),((021),(021)))$	$((0,0,0),((021),(012)))$	$((1,1,1),((12),(12)))$	$((1,1,1),((2)(01),(02)))$
$((0,1,1),((2)(01),(021)))$	$((0,0,1),((12),(2)(01)))$	$((1,1,1),((2)(01),(02)))$	$((0,0,1),((2)(01),(2)(01)))$	$((0,1,1),((012),(2)))$	$((1,0,1),((021),(021)))$
$((1,1,1),((12),(2)(01)))$	$((0,1,1),((02),(021)))$	$((0,1,1),((12),(2)))$	$((1,0,1),((12),(021)))$	$((0,1,0),((021),(02)))$	$((0,0,1),((2),(2)(01)))$
$((1,1,0),((2)(01),(012)))$	$((1,1,1),((12),(02)))$	$((0,1,0),((2)(01),(12)))$	$((1,0,0),((2)(01),(02)))$	$((1,1,0),((012),(021)))$	$((0,1,1),((021),(012)))$
$((1,0,1),((12),(021)))$	$((0,1,0),((02),(2)(01)))$	$((0,0,1),((12),(02)))$	$((1,1,1),((12),(2)(01)))$	$((0,0,0),((021),(2)))$	$((0,0,0),((2),(021)))$
$((1,0,0),((021),(12)))$	$((1,1,0),((012),(2)))$	$((0,0,0),((021),(012)))$	$((1,1,0),((021),(2)))$	$((1,0,0),((12),(2)(01)))$	$((0,1,0),((2)(01),(12)))$
$((1,0,0),((2),(2)(01)))$	$((1,1,0),((021),(021)))$	$((0,0,0),((2),(2)))$	$((1,1,0),((2),(021)))$	$((1,0,0),((02),(02)))$	$((0,1,0),((12),(2)(01)))$
$((0,1,0),((2),(12)))$	$((1,0,1),((021),(2)))$	$((1,1,0),((2),(012)))$	$((0,0,0),((2),(2)))$	$((1,1,1),((02),(2)(01)))$	$((1,1,1),((12),(12)))$
$((1,0,1),((02),(012)))$	$((0,1,0),((2)(01),(02)))$	$((0,0,1),((02),(12)))$	$((1,1,1),((02),(02)))$	$((0,0,0),((2),(021)))$	$((0,0,0),((012),(012)))$
$((1,1,0),((12),(2)))$	$((1,1,1),((02),(12)))$	$((0,1,0),((12),(2)(01)))$	$((1,0,0),((12),(12)))$	$((1,1,0),((021),(012)))$	$((0,1,1),((2),(2)))$
$((0,1,0),((012),(2)(01)))$	$((1,0,1),((2),(021)))$	$((1,1,0),((012),(2)))$	$((0,0,0),((012),(021)))$	$((1,1,1),((2)(01),(02)))$	$((1,1,1),((02),(2)(01)))$
$((0,1,1),((02),(2)))$	$((0,0,1),((2)(01),(12)))$	$((1,1,1),((02),(2)(01)))$	$((0,0,1),((02),(12)))$	$((0,1,1),((2),(012)))$	$((1,0,1),((012),(2)))$
$((0,0,1),((021),(2)(01)))$	$((0,0,0),((012),(021)))$	$((1,0,1),((021),(2)))$	$((0,1,1),((021),(021)))$	$((0,0,1),((12),(02)))$	$((1,0,0),((2)(01),(2)(01)))$
$((1,1,0),((02),(021)))$	$((1,1,1),((2)(01),(2)(01)))$	$((0,1,0),((02),(02)))$	$((1,0,0),((02),(2)(01)))$	$((1,1,0),((2),(2)))$	$((0,1,1),((012),(021)))$

	((0, 1, 0), ((012), (2)(01)))	((0, 1, 1), ((02), (2)))	((0, 0, 1), ((021), (2)(01)))	((1, 1, 0), ((02), (021)))
((1, 0, 1), ((2)(01), (2)))	((0, 0, 1), ((02), (2)(01)))	((1, 1, 0), ((012), (2)))	((1, 1, 1), ((12), (2)(01)))	((0, 1, 1), ((012), (021)))
((0, 0, 0), ((012), (021)))	((0, 1, 0), ((021), (02)))	((0, 1, 1), ((2)(01), (021)))	((0, 0, 1), ((2), (02)))	((1, 1, 0), ((2)(01), (012)))
((1, 1, 1), ((02), (12)))	((1, 0, 1), ((12), (012)))	((1, 0, 0), ((2), (12)))	((1, 1, 0), ((2)(01), (012)))	((0, 0, 1), ((2), (02)))
((0, 0, 0), ((2), (012)))	((0, 1, 0), ((012), (12)))	((0, 1, 1), ((02), (012)))	((0, 0, 1), ((021), (12)))	((1, 1, 0), ((02), (2)))
((0, 1, 1), ((12), (012)))	((1, 0, 0), ((2)(01), (12)))	((1, 0, 1), ((021), (012)))	((1, 0, 0), ((02), (12)))	((0, 0, 0), ((021), (2)))
((0, 0, 1), ((012), (12)))	((0, 0, 0), ((021), (012)))	((1, 1, 1), ((2)(01), (12)))	((1, 0, 1), ((2), (012)))	((0, 1, 0), ((2)(01), (02)))
((0, 0, 1), ((2), (02)))	((0, 0, 0), ((012), (021)))	((1, 1, 1), ((02), (02)))	((1, 0, 1), ((021), (021)))	((0, 1, 0), ((02), (2)(01)))
((0, 0, 0), ((021), (2)))	((0, 1, 0), ((2), (2)(01)))	((0, 1, 1), ((12), (2)))	((0, 0, 1), ((012), (2)(01)))	((1, 1, 0), ((12), (021)))
((1, 1, 1), ((2)(01), (02)))	((1, 0, 1), ((02), (021)))	((1, 0, 0), ((012), (02)))	((1, 1, 0), ((12), (021)))	((0, 0, 1), ((012), (2)(01)))
((1, 0, 0), ((012), (02)))	((0, 1, 1), ((021), (021)))	((0, 1, 0), ((2)(01), (02)))	((0, 1, 1), ((2), (021)))	((1, 1, 1), ((2)(01), (2)(01)))
((0, 1, 0), ((021), (02)))	((1, 1, 0), ((2), (021)))	((0, 0, 1), ((12), (02)))	((0, 0, 0), ((012), (021)))	((1, 0, 0), ((12), (2)(01)))
((0, 1, 1), ((2)(01), (021)))	((1, 0, 0), ((02), (02)))	((1, 0, 1), ((012), (021)))	((1, 0, 0), ((12), (02)))	((0, 0, 0), ((012), (012)))
((1, 1, 1), ((12), (2)(01)))	((1, 0, 1), ((2)(01), (2)))	((1, 0, 0), ((021), (2)(01)))	((1, 1, 0), ((02), (2)))	((0, 0, 1), ((021), (12)))
((1, 1, 0), ((2)(01), (012)))	((1, 1, 1), ((02), (12)))	((0, 0, 0), ((012), (012)))	((0, 1, 0), ((12), (12)))	((1, 0, 1), ((012), (2)))
((1, 0, 1), ((12), (021)))	((0, 0, 1), ((2)(01), (02)))	((1, 1, 0), ((021), (021)))	((1, 1, 1), ((02), (02)))	((0, 1, 1), ((021), (012)))
((1, 0, 0), ((021), (12)))	((0, 1, 1), ((2), (012)))	((0, 1, 0), ((12), (12)))	((0, 1, 1), ((012), (012)))	((1, 1, 1), ((12), (02)))
((1, 0, 0), ((2), (2)(01)))	((0, 1, 1), ((012), (2)))	((0, 1, 0), ((02), (2)(01)))	((0, 1, 1), ((021), (2)))	((1, 1, 1), ((02), (12)))
((0, 1, 0), ((2), (12)))	((1, 1, 0), ((012), (012)))	((0, 0, 1), ((02), (12)))	((0, 0, 0), ((021), (012)))	((1, 0, 0), ((02), (02)))
((1, 0, 1), ((02), (012)))	((0, 0, 1), ((12), (12)))	((1, 1, 0), ((2), (012)))	((1, 1, 1), ((2)(01), (12)))	((0, 1, 1), ((2), (2)))
((1, 1, 0), ((12), (2)))	((1, 1, 1), ((2)(01), (2)(01)))	((0, 0, 0), ((021), (2)))	((0, 1, 0), ((02), (2)(01)))	((1, 0, 1), ((021), (021)))
((0, 1, 0), ((012), (2)(01)))	((1, 1, 0), ((021), (2)))	((0, 0, 1), ((2)(01), (2)(01)))	((0, 0, 0), ((2), (2)))	((1, 0, 0), ((2)(01), (12)))
((0, 1, 1), ((02), (2)))	((1, 0, 0), ((12), (2)(01)))	((1, 0, 1), ((2), (2)))	((1, 0, 0), ((2)(01), (2)(01)))	((0, 0, 0), ((2), (021)))
((0, 0, 1), ((021), (2)(01)))	((0, 0, 0), ((2), (2)))	((1, 1, 1), ((12), (2)(01)))	((1, 0, 1), ((012), (2)))	((0, 1, 0), ((12), (12)))
((1, 1, 0), ((02), (021)))	((1, 1, 1), ((12), (02)))	((0, 0, 0), ((2), (021)))	((0, 1, 0), ((2)(01), (02)))	((1, 0, 1), ((2), (012)))

A.3 Elements of U_4

$$
\begin{array}{rllll}
U_4 = & \{((1,0,0),((02),(12))), & ((1,1,0),((012),(021))), & ((1,0,0),((2)(01),(2)(01))), \\
& ((0,1,1),((2),(2))), & ((0,1,0),((12),(12))), & ((1,0,1),((2)(01),(021))), \\
& ((1,1,1),((12),(2)(01))), & ((1,1,1),((02),(2)(01))), & ((0,0,1),((12),(12))), \\
& ((0,1,1),((021),(2))), & ((1,1,0),((2)(01),(021))), & ((1,0,1),((021),(021))), \\
& ((0,0,0),((12),(2))), & ((0,1,1),((2)(01),(2))), & ((1,1,1),((12),(12))), \\
& ((1,1,0),((012),(012))), & ((1,0,0),((012),(2)(01))), & ((0,1,0),((2),(2)(01))), \\
& ((0,0,1),((12),(2)(01))), & ((1,1,0),((021),(2))), & ((1,0,1),((02),(012))), \\
& ((1,1,0),((02),(2))), & ((1,0,1),((012),(021))), & ((1,1,0),((12),(012))), \\
& ((1,0,0),((021),(12))), & ((0,1,0),((021),(12))), & ((0,0,1),((021),(12))), \\
& ((0,1,0),((012),(12))), & ((1,0,1),((12),(021))), & ((0,0,1),((02),(12))), \\
& ((1,0,1),((02),(021))), & ((0,1,0),((2)(01),(12))), & ((1,0,1),((02),(2))), \\
& ((1,0,0),((012),(02))), & ((1,1,1),((2),(02))), & ((1,1,1),((2)(01),(2)(01))), \\
& ((0,0,1),((021),(2)(01))), & ((0,0,1),((2),(2)(01))), & ((1,1,1),((012),(02))), \\
& ((1,0,1),((12),(012))), & ((1,1,0),((021),(012))), & ((0,1,0),((2),(12))), \\
& ((1,0,0),((12),(12))), & ((1,1,0),((02),(012))), & ((0,0,1),((2)(01),(2)(01))), \\
& ((1,0,1),((012),(012))), & ((0,0,0),((12),(021))), & ((1,1,0),((021),(021))), \\
& ((1,0,1),((12),(2))), & ((0,0,1),((2)(01),(12))), & ((1,1,1),((012),(2)(01))), \\
& ((0,1,1),((2)(01),(021))), & ((0,1,0),((02),(12))), & ((0,0,0),((021),(012))), \\
& ((1,0,1),((2)(01),(2))), & ((0,1,1),((2),(012))), & ((1,1,1),((2)(01),(12))), \\
& ((1,0,1),((012),(2))), & ((1,0,0),((2)(01),(12))), & ((1,0,0),((02),(2)(01))), \\
& ((0,0,1),((2),(12))), & ((0,0,1),((12),(02))), & ((1,1,0),((012),(2))), \\
& ((0,1,1),((2)(01),(012))), & ((0,0,0),((021),(021))), & ((1,1,0),((02),(021))), \\
& ((0,0,0),((2),(021))), & ((1,0,1),((021),(2))), & ((1,0,0),((2),(12))), \\
& ((0,0,0),((2)(01),(012))), & ((0,0,1),((02),(2)(01))), & ((1,1,0),((12),(2))), \\
& ((0,0,0),((12),(012))), & ((0,0,1),((012),(12))), & ((0,1,0),((12),(02))), \\
& ((1,0,0),((02),(02))), & ((0,1,1),((012),(012))), & ((1,1,0),((2)(01),(2))),
\end{array}
$$

$$\begin{array}{lll}
((0,0,0),((012),(2))), & ((0,0,0),((2),(012))), & ((1,0,0),((021),(2)(01))), \\
((0,1,1),((021),(021))), & ((1,1,1),((2),(2)(01))), & ((0,1,1),((2),(021))), \\
((1,1,1),((02),(12))), & ((0,0,1),((021),(02))), & ((0,1,0),((012),(02))), \\
((0,1,1),((12),(012))), & ((1,1,1),((021),(02))), & ((1,1,0),((2),(2))), \\
((0,1,0),((2)(01),(02))), & ((0,0,0),((02),(012))), & ((1,0,0),((021),(02))), \\
((0,1,0),((021),(02))), & ((0,0,0),((2)(01),(021))), & ((0,1,1),((12),(021))), \\
((1,0,0),((12),(2)(01))), & ((0,0,1),((012),(2)(01))), & ((0,1,0),((012),(2)(01))), \\
((0,1,1),((02),(2))), & ((0,1,1),((021),(012))), & ((1,1,1),((12),(02))), \\
((0,1,1),((012),(021))), & ((0,0,0),((02),(2))), & ((1,0,1),((2),(021))), \\
((1,1,1),((012),(12))), & ((1,0,0),((12),(02))), & ((0,1,1),((02),(021))), \\
((0,1,0),((2),(02))), & ((0,0,1),((02),(02))), & ((1,0,1),((2),(2))), \\
((0,0,0),((02),(021))), & ((1,0,0),((012),(12))), & ((1,1,1),((2),(12))), \\
((0,1,1),((02),(012))), & ((0,1,0),((12),(2)(01))), & ((1,0,0),((2),(02))), \\
((1,1,0),((2),(021))), & ((0,0,0),((012),(021))), & ((0,1,1),((012),(2))), \\
((1,1,0),((2)(01),(012))), & ((0,0,0),((2)(01),(2))), & ((1,0,1),((021),(012))), \\
((0,1,0),((02),(02))), & ((1,0,1),((2),(012))), & ((1,1,1),((2)(01),(02))), \\
((1,1,1),((021),(2)(01))), & ((1,0,0),((2)(01),(02))), & ((0,1,0),((2)(01),(2)(01))), \\
((0,0,1),((2),(02))), & ((0,1,1),((12),(2))), & ((1,1,0),((2),(012))), \\
((1,1,1),((02),(02))), & ((0,0,0),((2),(2))), & ((0,0,0),((012),(012))), \\
((0,0,0),((021),(2))), & ((1,0,0),((2),(2)(01))), & ((1,0,1),((2)(01),(012))), \\
((1,1,1),((021),(12))), & ((1,1,0),((12),(021))), & ((0,1,0),((021),(2)(01))), \\
((0,0,1),((2)(01),(02))), & ((0,1,0),((02),(2)(01))), & ((0,0,1),((012),(02)))\}.
\end{array}$$

Appendix B

Partial results on Conjecture 4.13

The notation used in this appendix differs from that used in Chapter 4.

Notation B.1. Let $V(\vec{C}_2 \times K_m^*) = \mathbb{Z}_{2m}$. Furthermore, let $(a,b),(b,a) \in A(\vec{C}_2 \times K_m^*)$ if and only if $a \in \{0,1,\ldots,m-1\}$, $b \in \{m, m-1, \ldots, 2m-1\}$, and $b-a \not\equiv 0 \pmod{m}$.

Below, we give a directed hamiltonian decomposition of $\vec{C}_2 \times K_m^*$ for each $m \in \{6,8,10,12,14,16\}$, thereby providing partial results to Conjecture 4.13.

$\underline{m=6}$:

$$\begin{aligned}
\{&(0,8,3,7,5,10,1,9,2,6,4,11),(0,11,2,9,4,8,1,6,3,10,5,7),\\
&(0,9,1,11,3,6,5,8,4,7,2,10),(0,7,4,6,2,11,1,10,3,8,5,9),\\
&(0,10,2,7,3,11,4,9,5,6,1,8)\}.
\end{aligned}$$

$\underline{m=8}$:

$$\begin{aligned}
\{&(0,13,3,12,1,15,2,9,6,8,4,11,5,10,7,14),\\
&(0,14,1,12,6,15,4,13,2,8,3,9,7,10,5,11),\\
&(0,12,3,10,6,11,1,14,7,9,2,13,4,8,5,15),\\
&(0,15,5,12,2,14,4,9,3,8,1,13,7,11,6,10),\\
&(0,10,3,14,5,9,4,15,1,11,2,12,7,8,6,13),\\
&(0,11,4,10,1,8,7,12,5,14,2,15,3,13,6,9),\\
&(0,9,5,8,2,11,7,13,1,10,4,14,3,15,6,12)\}.
\end{aligned}$$

$\underline{m = 10}$:

$$
\begin{aligned}
\{&(0, 16, 4, 19, 8, 11, 5, 10, 6, 13, 1, 12, 9, 15, 3, 18, 7, 14, 2, 17),\\
&(0, 13, 9, 17, 5, 18, 3, 19, 1, 16, 8, 15, 4, 10, 7, 12, 6, 11, 2, 14),\\
&(0, 15, 9, 18, 2, 13, 7, 10, 4, 17, 8, 16, 1, 19, 3, 11, 6, 14, 5, 12),\\
&(0, 18, 9, 10, 3, 12, 1, 17, 4, 15, 2, 11, 8, 13, 6, 19, 5, 14, 7, 16),\\
&(0, 14, 3, 17, 1, 18, 4, 13, 8, 12, 5, 16, 9, 11, 7, 15, 6, 10, 2, 19),\\
&(0, 12, 4, 18, 6, 17, 9, 14, 1, 15, 8, 19, 7, 11, 3, 16, 2, 10, 5, 13),\\
&(0, 19, 6, 15, 1, 10, 8, 17, 2, 16, 3, 14, 9, 13, 5, 11, 4, 12, 7, 18),\\
&(0, 17, 6, 12, 3, 15, 7, 13, 2, 18, 1, 14, 8, 10, 9, 16, 5, 19, 4, 11),\\
&(0, 11, 9, 12, 8, 14, 6, 18, 5, 17, 3, 10, 1, 13, 4, 16, 7, 19, 2, 15)\}.
\end{aligned}
$$

$\underline{m = 12}$:

$$
\begin{aligned}
\{&(0, 20, 11, 15, 4, 23, 7, 14, 6, 12, 3, 22, 9, 17, 2, 19, 5, 16, 10, 13, 8, 18, 1, 21),\\
&(0, 23, 10, 12, 6, 13, 3, 20, 4, 18, 7, 16, 1, 22, 11, 17, 8, 21, 2, 15, 5, 19, 9, 14),\\
&(0, 13, 2, 23, 4, 21, 11, 19, 3, 18, 5, 14, 8, 12, 7, 15, 10, 17, 9, 16, 6, 22, 1, 20),\\
&(0, 17, 10, 15, 6, 20, 9, 13, 11, 21, 1, 18, 3, 14, 7, 12, 2, 16, 8, 23, 5, 22, 4, 19),\\
&(0, 18, 2, 13, 7, 21, 8, 14, 10, 20, 1, 15, 11, 16, 5, 12, 4, 17, 3, 19, 6, 23, 9, 22),\\
&(0, 19, 10, 21, 6, 17, 11, 18, 8, 15, 9, 20, 5, 23, 3, 16, 7, 22, 2, 12, 1, 14, 4, 13),\\
&(0, 22, 8, 19, 2, 17, 1, 16, 11, 12, 10, 23, 6, 14, 5, 20, 3, 21, 7, 13, 9, 18, 4, 15),\\
&(0, 15, 2, 20, 7, 23, 1, 12, 8, 17, 4, 14, 9, 19, 11, 22, 6, 16, 3, 13, 5, 21, 10, 18),\\
&(0, 21, 4, 20, 6, 15, 1, 17, 7, 18, 9, 12, 11, 14, 3, 23, 2, 22, 5, 13, 10, 19, 8, 16),\\
&(0, 14, 11, 13, 4, 12, 5, 18, 10, 16, 9, 15, 8, 22, 7, 20, 2, 21, 3, 17, 6, 19, 1, 23),\\
&(0, 16, 2, 18, 11, 20, 10, 14, 1, 19, 4, 22, 3, 12, 9, 23, 8, 13, 6, 21, 5, 15, 7, 17)\}.
\end{aligned}
$$

$\underline{m = 14}$:

$$
\begin{aligned}
\{&(0, 21, 13, 17, 4, 16, 10, 14, 8, 24, 12, 20, 3, 18, 5, 26, 9, 25, 6, 27, 2, 22, 1, 19, 11, 15, 7, 23),\\
&(0, 23, 1, 14, 6, 24, 4, 22, 7, 25, 8, 15, 9, 20, 10, 16, 3, 27, 12, 17, 13, 18, 2, 26, 5, 21, 11, 19),\\
&(0, 19, 4, 17, 10, 20, 11, 14, 1, 26, 13, 15, 6, 25, 2, 18, 12, 22, 9, 16, 5, 24, 8, 21, 3, 23, 7, 27),\\
&(0, 17, 8, 23, 12, 24, 1, 21, 9, 19, 13, 25, 10, 27, 4, 14, 3, 16, 11, 26, 2, 15, 5, 20, 7, 18, 6, 22),
\end{aligned}
$$

$$(0,25,9,24,11,23,5,17,12,14,4,19,6,15,10,18,13,20,1,27,3,22,2,21,8,16,7,26),$$
$$(0,20,8,17,1,25,7,16,6,26,3,14,2,19,10,22,4,23,13,24,9,18,11,27,5,15,12,21),$$
$$(0,15,8,20,5,18,7,14,10,23,6,17,9,26,1,22,13,16,4,24,2,27,11,21,12,19,3,25),$$
$$(0,22,5,23,3,24,7,19,8,25,12,27,9,15,13,26,10,21,4,20,2,14,11,18,1,17,6,16),$$
$$(0,27,6,14,12,23,2,20,9,17,7,15,3,19,1,24,5,22,11,16,13,21,10,25,4,26,8,18),$$
$$(0,16,1,23,10,17,2,25,3,21,5,27,7,24,6,19,9,14,13,22,12,18,8,26,11,20,4,15),$$
$$(0,18,9,22,3,20,13,19,2,23,8,27,10,26,6,21,1,16,12,15,4,25,5,14,7,17,11,24),$$
$$(0,26,7,22,6,18,10,19,12,25,13,23,4,21,2,24,3,15,11,17,5,16,8,14,9,27,1,20),$$
$$(0,24,13,14,5,25,1,18,3,26,4,27,8,19,7,20,12,16,9,21,6,23,11,22,10,15,2,17)\}.$$

$\underline{m=16}$:

$$\{(0,17,8,19,7,28,2,23,5,24,9,30,11,31,13,20,15,27,6,29,1,18,14,26,12,16,4,21,10,25,3,22),$$
$$(0,19,12,23,9,31,14,18,13,30,8,22,5,17,10,16,15,28,4,27,2,21,1,29,11,24,7,26,3,20,6,25),$$
$$(0,29,15,18,10,21,11,17,7,22,3,25,14,24,4,19,13,16,8,26,6,31,1,28,9,20,12,30,2,27,5,23),$$
$$(0,26,1,31,4,23,11,22,13,19,2,17,12,21,14,29,9,27,3,18,6,20,10,24,15,16,7,30,5,25,8,28),$$
$$(0,25,15,21,7,18,12,27,1,20,13,23,3,24,2,29,5,19,14,16,10,22,9,26,4,30,6,17,11,28,8,31),$$
$$(0,21,6,28,14,17,15,22,2,16,1,24,10,27,4,29,3,30,7,25,13,26,11,23,12,20,5,31,9,19,8,18),$$
$$(0,28,7,16,12,25,6,23,2,26,15,24,13,21,4,18,3,27,8,29,14,31,11,19,1,22,10,30,9,17,5,20),$$
$$(0,27,9,23,10,31,6,26,5,30,13,22,15,20,11,29,4,24,1,21,12,18,8,16,2,28,3,17,14,25,7,19),$$
$$(0,24,11,25,10,20,8,30,4,16,5,18,9,21,15,19,6,27,7,29,12,22,14,28,1,26,2,31,3,23,13,17),$$
$$(0,31,5,28,10,18,11,26,14,21,3,29,7,17,6,30,12,19,9,16,13,27,15,25,1,23,4,22,8,20,2,24),$$
$$(0,22,1,30,3,26,9,24,5,27,13,25,12,31,8,17,4,28,15,23,6,16,11,18,7,21,2,20,14,19,10,29),$$
$$(0,18,4,31,12,29,10,23,1,27,14,22,11,21,8,25,5,26,7,20,9,28,13,24,3,16,6,19,15,17,2,30),$$
$$(0,23,8,27,10,19,4,25,2,22,12,17,3,31,7,24,14,20,1,16,9,18,5,29,6,21,13,28,11,30,15,26),$$
$$(0,30,1,19,5,16,3,21,9,22,4,17,13,18,15,29,2,25,11,20,7,31,10,28,6,24,12,26,8,23,14,27),$$
$$(0,20,3,28,5,22,7,27,12,24,6,18,1,25,4,26,13,31,2,19,11,16,14,23,15,30,10,17,9,29,8,21)\}.$$

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