ABIAN'S ORDER RELATION AND C(X)

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ABSTRACT

In this thesis Abian's order relation for commutative semiprime rings, defined by $a \leq b$ if, and only if $ab = a^2$, is studied for the ring of all continuous functions from X to ℝ, where X is a completely regular topological space. Also, the concept of a Baer ring is considered and those spaces X are characterized for which C(X) is Baer. Moreover partial answers to the questions "when is C(X) orthogonally complete?" and "when does C(X) have an orthogonal completion?" are given. It is also shown that C(X), where X is a locally connected space with certain properties, does not have an orthogonal completion, but if a topological space X has a basis of clopen sets then Q(X) is the orthogonal completion of C(X). Finally, it is shown that if X is locally connected then C(X) is conditionally complete.

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ABIAN'S ORDER RELATION AND C(X)

Introduction

The usual order relation in a Boolean ring R extends to commutative semiprime rings when expressed as: $x \le y$ if, and only if, $xy = x^2$. This was first studied by A. Abian [1] and is an order relation, with which R acquires the structure of a partially ordered semigroup. Abian used this order relation to characterize direct products of fields among commutative semiprime rings. One of the characteristic properties of a direct product of a fields is that it is orthogonally complete, i.e., every orthogonal set in R has a supremum in R with respect to Abian's order. Later on the same order relation was studied by W. D. Burgess and R. Raphael ([2] and [3]) and they have shown that if R is regular then its complete ring of quotients, Q(R), is its completion; in the sense that, if $R \subset S$ are rings, then S is called an orthogonal extension of R if every element of S is the supremum of an orthogonal set of R, further if S is an orthogonal extension so that S is orthogonally complete, then S is called an orthogonal completion of R. Moreover they showed that a Baer ring has an orthogonal completion.

The purpose of this thesis is to study the Abian's order relation for the ring of all continuous functions from X to $\mathbb{R}$, where X is a completely regular topological space. For C(X), the order relation is defined in the same way i.e., for all $f, g \in C(X)$; $f \le g$ if, and only if, $fg = f^2$.

The first chapter deals with commutative semiprime rings and presents some of the notions that are used in the later chapters i.e., annihilator ideals, complete ring of quotients, regular rings and Baer rings. Most of the details can be found in [8].

In the second chapter, some properties of regular and Baer rings are given, when the ring under consideration is C(X). Proposition 2.3. ([6. Ex. 4N]) shows that C(X) is regular if, and only if, X is a p-space. Further we have characterized those spaces X for which C(X) is Baer, by showing that C(X) is Baer if, and only if, closure of union of cozero sets is clopen and if, and only if, X is extremely disconnected.

The third chapter begins by the definition of Abian's order relation for a commutative semiprime ring R. Further an explicit description of the supremum of an orthogonal family of continuous functions in C(X) is given, with the help of which, it is shown in Thm. 3.4. that is if X has a basis of clopen sets, then Q(X) is the orthogonal completion of C(X). It is shown in Cor. 3.8. that if X is locally connected then C(X) is conditionally complete. Moreover it is shown in Thm. 3.9. that if X is connected and locally connected then either C(X) is orthogonally complete or C(X) has no orthogonal completion. Finally a partial answer to the question; when does C(X) not have an orthogonal completion, is given by Thm. 3.14. which states that if X is locally connected and contains an open subset U of more than one point such that U is connected and $\overline{U}$ is metrizable then C(X) does not have an orthogonal completion.

The results 2.5., 3.4., 3.8., 3.9. and 3.14 are original, having been developed from outlines suggested by W. D. Burgess.

CHAPTER 1 Introduction: Commutative rings

§1. Description of Commutative Semiprime rings

In this thesis, all rings are assumed to be commutative with 1. The material of this chapter is quite standard and is included for the convenience of the reader, cf., [8. ch. 2].

A proper ideal P in a ring R is called prime if, and only if, for all elements a and b, $ab \in P$ implies $a \in P$ or $b \in P$. The intersection of all prime ideals of a ring R is called the prime radical of R, and is denoted by rad R. An element $r \in R$ is called nilpotent if $r^n = 0$ for some $n > 0$. The prime radical of a ring R consists of all nilpotent elements of R [8. p. 29].

Definition. A ring R is called semiprime if its prime radical is 0, that is if it has no non-zero nilpotent elements.

For any ring R, R/rad R is an example of a semiprime ring. Specific examples are integral domains and product of integral domains.

A ring R is called a subdirect product of a family of rings $\{S_i : i \in I\}$ if there is a monomorphism $k: R \to S = \Pi S_i$, such that $\pi_i \circ k$ is onto for all $i \in I$, where $\pi_i : S \to S_i$ canonically. A well known result about subdirect products is the following [8. p. 30].

Proposition 1.1. R is a subdirect product of the rings S_i, $i \in I$ if, and only if, $S_i \cong R/K_i$, K_i is an ideal of R, and

$\bigcap_{i \in I} K_i = 0.$

Immediate consequences of the above result are the following.

Corollary 1.2. A ring is a subdirect product of integral domains if, and only if, it is semiprime.

Proof. If R is a subdirect product of the family of integral domains $\{S_i : i \in I\}$ then $S_i \cong R/K_i$, where K_i is the kernal of $\pi_i \cdot k$, and $\cap K_i = 0$. Since S_i is an integral domain so is R/K_i and, hence, K_i is a prime ideal [8. p. 28].

Conversely, let R be semiprime and $\{P_i : i \in I\}$ the family of prime ideals of R. Then $\bigcap_{i \in I} P_i = 0$. Hence R/P_i is an integral domain. Put $K_i = R/P_i$, and thus R is a subdirect product of a family of integral domains.

Q.E.D.

Corollary 1.3. A ring is semiprime if, and only if, it is isomorphic to a subring of a direct product of integral domains.

This follows immediately from corollary 1.2.

Further we notice that a subring of a semiprime ring is semiprime, since a nilpotent element of the subring is surely a nilpotent element of the whole ring.

Corollary 1.4. A ring is semiprime if, and only if, it is isomorphic to a subring of a direct product of fields.

Proof. Any integral domain can be embedded in a field [7. p. 56].

Q.E.D.

Definition. An ideal I in a ring R is called large if it has non zero intersection with every non-zero ideal.

For any ring R, R itself is a large ideal. Also in the ring of integers, all ideals generated by a non-zero elements are large.

Definition. An ideal D in a ring R is called dense if, for all $r \in R$, $rD = 0$ implies $r = 0$. In other words, if the annihilator of D in R is 0.

Some properties of dense ideals are:

(1) R is dense.

(2) If D is dense and $D \subset D'$ then D' is dense.

(3) If D and D' are dense, so is DD'.

(4) If $R \neq 0$, then 0 is not dense.

(5) Every dense ideal is large.

A characterization of semiprime rings in terms of large and dense ideals is given by the following lemma.

Lemma 1.5. R is semiprime if, and only if, every large ideal is dense.

Proof. Assume R is semiprime, L a large ideal of R and $a \neq 0$. Since $L \cap (a) \neq 0$, (where (a) stands for the ideal generated by a) we have $0 \neq (L \cap (a))^2 \subset La$, which shows that L is dense.

Conversely, assume every large ideal is dense, and let I be an ideal such that $I^2 = 0$. Then for any a, we have $aI \subset I^*$, where $I^* = \{b \in R: bI = 0\}$, the annihilator of I. Hence if $(a) \cap I^* = 0$ then $aI \subset (a) \cap I^* = 0$. This shows that I^* is large. Therefore I^* is dense. Since $II^* = 0$ then $I = 0$. Hence R is semiprime.

Q.E.D.

§2. The Complete ring of quotients

By a fraction, in a ring R we mean an element $f \in \mathrm{Hom}_R(D, R)$, where D is any dense ideal (the relationship with the usual use of "fraction" will be noted later). Thus f is a group homomorphism of D into R such that $f(dr) = (fd)r$, for any $d \in D$ and $r \in R$. We define $-f \in \mathrm{Hom}_R(D, R)$ by $(-f)d = -(fd)$. Fractions $0, 1 \in \mathrm{Hom}_R(R, R)$ are defined by $0r = 0$; $1r = r$, for all $r \in R$. Addition and multiplication of fractions are defined by:

$(f_1 + f_2)d = f_1d + f_2d$, $f_1 + f_2 \in \mathrm{Hom}(D_1 \cap D_2, R)$; D_1, D_2 are domains of f_1 and f_2 respectively;

$(f_1f_2)d = f_1(f_2d)$; $f_1f_2 \in \mathrm{Hom}_R(f_2^{-1}D_1, R)$, where

$f^{-1}{}_2 D_1 = \{r \in R: f_2r \in D_1\}$ (this is dense since $D_2D_1 \subset f_2^{-1}D_1$).

With these definitions of addition and multiplication, fractions form an additive abelian semigroup $(F, 0, +)$ with 0, and an abelian semigroup $(F, 1, \cdot)$ with 1.

If f and g are fractions of R, $f \Theta g$ means f and g agree on the intersection of their domains; that is, $fd = gd$ for all $d \in D(f) \cap D(g)$ (where $D(f)$, $D(g)$ denotes the domains of f and g respectively). In fact $f \Theta g$ if, and only if, f and g coincide on some dense ideal. Assume f and g agree on a dense ideal D, take $d \in D(f) \cap D(g)$ and $d' \in D$, then $(fd)d' = f(dd') = g(dd') = (gd)d'$. Hence $(fd - gd)D = 0$. It follows that $fd = gd$, for all $d \in D(f) \cap D(g)$ and $f \Theta g$. Obviously Θ is a congruence on the system $(F, 1, 0, +, \cdot, -)$.

Proposition 1.6. If R is a ring, the system $(F, 1, 0, +, \cdot, -)/\Theta$ is also a commutative ring.

Proof. Ref. [8. ch. 2].

Definition. $(F, 1, 0, +, \cdot, -)/\Theta$ extends R and is called its complete ring of quotients, denoted by $Q(R)$.

The following property of the complete ring of quotients will be useful later.

If $R \subseteq S$ are rings, such that for all $0 \neq s \in S$, $s(s^{-1}R) \neq 0$, where $s^{-1}R = \{r \in R: sr \in R\}$, then there is a monomorphism $S \to Q(R)$ leaving elements of R fixed [8. p. 40]. Also $Q(Q(R)) = Q(R)$.

It is interesting to note that the family of abelian groups $\mathrm{Hom}_R(D, R)$, D a dense ideal, is a directed system with the homomorphism defined by restriction i.e., $f \to f|D_2$ where $f \in \mathrm{Hom}_R(D_1, R)$ and $D_2 \subset D_1$. The abelian group $\varinjlim_{D \text{ dense}} \mathrm{Hom}_R(D, R)$

has a multiplication induced by composition. It can be seen that this construction also yields $Q(R)$.

Definition. An element $a \in R$ is termed a non-zero-divisor if $ar \neq 0$, for all $0 \neq r \in R$, and is called a zero-divisor if $sa = 0$ for some $s \neq 0$.

With any non-zero-divisor d of R, associate the dense ideal dR. If $r \in R$, we have a classical fraction $r/d \in \mathrm{Hom}(dR, R)$ defined by $(r/d)(ds) = rs$, for any $s \in R$. Two such fractions r_1/d_1 and r_2/d_2 are equivalent if, and only if, $r_1d_2 = r_2d_1$. The equivalence classes $\Theta(r/d)$, $r \in R$ and d not a zero-divisor form a subring of $Q(R)$, which is called the classical ring of quotients of R and is denoted by $Q_{c\ell}(R)$. Two equivalence classes $\Theta(r_1/d_1)$ and $\Theta(r_2/d_2)$ are same if, and only if, r_1/d_1 agrees with r_2/d_2 on D_1D_2, hence if, and only if, $r_1d_2 = r_2d_1$.

If R is an integral domain, then $Q_{c\ell}(R) = Q(R)$. But, if $R \subset \prod_{N} \mathbb{C}$ ($\mathbb{C}$ the complex field) and $R = \{f\colon f \text{ is almost everywhere real}\}$, then $Q_{c\ell}(R) = R$ and $Q(R) = \prod \mathbb{C}$. Thus we see that $Q_{c\ell}(R)$ is not always equal to $Q(R)$.

Definition. A ring R is called regular if for all $r \in R$ there exists at least one $r' \in R$ such that $rr'r = r$.

Fields, and products of fields, are examples of regular rings. Note that in a regular ring every non-zero divisor is invertible and so if R is regular, $R = Q_{c\ell}(R)$. But the ring

$R \subset \Pi\mathbb{C}$ where $R = \{f\colon f \text{ is almost everywhere real}\}$ is regular, and $Q(R)$ is larger than R. Thus for regular rings $Q(R)$ is not necessarily $Q_{c\ell}(R)$.

Now we shall see precisely how complete ring of quotients look in the case of semiprime rings by the following theorem.

Theorem 1.7. If R is a ring then $Q(R)$ is regular if, and only if, R is semiprime.

Proof. Ref. [8. p. 42].

With any subset K of a ring R, associate its annihilator K^*. Then K^* is an ideal of R and an ideal K is dense if, and only if, $K^* = 0$. For any ideal K in a semiprime ring R we have $K \cap K^* = 0$ and $K + K^*$ is dense.

In any (commutative) ring R if $K \subset J$ then $J^* \subset K^*$, moreover $K \subset K^{**}$ and $K^{***} = K^*$.

Definition. The ideals of the form K^* are called annihilator ideals.

Thus I is an annihilator ideal if, and only if, $I = K^*$ for some subset K of R; i.e., $I^{**} = K^{***} = K^* = I$.

The annihilator ideals in a semiprime ring form a complete Boolean algebra $B^*(R)$, with intersection as infimum and $*$ as complementation; also for a semiprime ring R, $B^*(R)$ is isomorphic to $B^*(Q(R))$, the isomorphism being defined by $K \to K \cap R$, where K is an annihilator ideal in $Q(R)$ [8. p. 43].

Definition. An element a of a ring R is said to be idempotent if $a^2 = a$.

Another useful class of rings is defined below.

Definition. A ring R is called Baer if all of its annihilator ideals are direct summands, i.e., are principal ideals generated by idempotents.

Fields, products of fields, integral domains, products of integral domains are examples of Baer rings.

If R is Baer, then it is semiprime for, if not, then there exists an $x \in R$ and an integer $n > 1$ such that $x^n = 0$ and $x^{n-1} \neq 0$. Thus $x \in (x^{n-1})^* = eR$, where e is an idempotent of R. But, then $x = ex$ and so $0 = ex^{n-1} = (ex) \cdot x^{n-2} = x^{n-1}$; a contradiction.

A Baer ring R and its complete ring of quotients are related by the following proposition.

Proposition 1.8. Let R be a Baer ring and Q(R) its complete ring of quotients. Then all idempotents of Q(R) are in R.

Proof. Ref. [9. Lemma 1.6].

CHAPTER II Introduction: Rings of Continuous Functions

§1. Zero sets and cozero sets

Let X be a topological space and C(X) the ring of all continuous functions from X to ℝ (the real field). Under the pointwise operations, C(X) becomes a ring, with zero and unity elements being the constant functions 0 and 1, respectively. Obviously C(X) is semiprime.

A Hausdorff space X is called completely regular if, for any neighborhood U of a point x, there exists a function f in C(X) such that f vanishes outside U but not at x.

For any topological space X, there exists a completely regular space Y and a continuous mapping τ of X onto Y such that the mapping $g \to g \cdot \tau$ is an isomorphism of C(Y) onto C(X) [6. p. 41]. So, while studying C(X), we assume without any loss of generality that the space X is completely regular. If a subset S of X is dense in X (one whose closure is X) then the homomorphism $f \to f|_S$ from C(X) into C(S) is a monomorphism; since two continuous functions on X, which coincide on a dense subset of X, coincide on X [10. p. 48].

From now on all spaces are assumed to be completely regular.

In studying relations between topological properties of a space X and algebraic properties of C(X), it is natural to look at the subsets of the form $f^{-1}(r) = \{x \in X: f(x) = r\}$, where $f \in C(X)$ and $r \in \mathbb{R}$. If we replace r by 0, then we have a subset $f^{-1}(0)$ of X which is called the zero set of f and is denoted by $Z_X(f)$ or $Z(f)$. Any set that is a zero set of some function in $C(X)$ is called a zero set in X. Obviously zero sets are closed. In addition every zero set is a G_δ (i.e., countable intersection of open sets), since $\{0\}$ is a G_δ in $\mathbb{R}$. Evidently, $Z(f) = Z(|f|) = Z(f^n)$, for all $n \in N$), $Z(0) = X$ and $Z(1) = \emptyset$. Furthermore, $Z(fg) = Z(f) \cup Z(g)$, $Z(f^2 + g^2) = Z(|f| + |g|) = Z(f) \cap Z(g)$, and countable intersections of zero sets are zero sets [6. p. 16].

The cozero set of f, denoted by $\mathrm{coz}_X(f)$, or simply by $\mathrm{coz}(f)$, is the complement of $Z(f)$. Hence cozero sets are open. Every set of the form $\{x: f(x) \geq 0\}$ is a zero set. Since $\{x: f(x) \geq 0\} = Z(f - |f|)$. Likewise $\{x: f(x) \leq 0\} = Z(f + |f|)$. Thus the open sets, pos $f = \{x: f(x) > 0\}$ and neg $f = \{x: f(x) < 0\}$ are cozero sets. But if the function is an idempotent e, then the zero set and cozero set are both open and closed. This follows since $Z(e) = \{x: e(x) = 0\} = e^{-1}(-1, 1)$.

Conversely, if a subset Y of X is both closed and open then define a function $e: X \to \mathbb{R}$ by the following:

$$e(x) = \begin{cases} 0 & \text{for } x \text{ in } Y \\ 1 & \text{otherwise.} \end{cases}$$

Obviously, e is an idempotent and is continuous on X. Also $Z(e) = Y$ and $\mathrm{coz}(e) = {\sim}Y$ (the symbol ~ denotes complement of Y). Thus Y is the zero set of an idempotent.

§2. Description of $Q(C(X))$

Let $\mathcal{S}$ be a family of non void subsets of X, which is closed under finite intersections. We consider the direct limit $\lim\limits_{s \in \mathcal{S}} C(S)$ of the rings $C(S)$, with respect to the restriction homomorphism ($f \to f|S_2$ where $f \in C(S_1)$ and $S_2 \subset S_1$). When $S = \nu_0(X)$, the family of dense open sets, all these homomorphisms are one-to-one and $\lim\limits_{V \in \nu_0} C(V)$ may be thought of as $\bigcup\limits_{V \in \nu_0(X)} C(V)$, where we identify $f_1 \in C(V_1)$ with $f_2 \in C(V_2)$ whenever f_1 and f_2 agree on $V_1 \cap V_2$.

If $0 \neq h \in \lim\limits_{V \in \nu_0(X)} C(V)$, then h may be thought of as in $C(V)$ for some $V \in \nu_0(X)$. Thus $h(h^{-1}C(X)) \neq 0$, hence $C(V)$ can be considered as a subring of $Q(C(X))$. Hence $\lim\limits_{V \in \nu_0(X)} C(V)$ can also be considered as a subring of $Q(C(X))$. In fact we will see that they are equal. Explicitly if $h \in C(V)$, then $D = h^{-1}C(X)$ is a dense ideal in $C(X)$. A relationship between dense ideals of $C(X)$ and dense ideals of X is given by the following theorem [5. p. 11].

<u>Theorem 2.1.</u> If D is an ideal in $C(X)$, then the following are equivalent:

(1) D is dense in $C(X)$.

(2) For all $g \in C(X)$, $\operatorname{coz} D \subset Z(g)$ implies $g = 0$. where $\operatorname{coz} D = \cup\{\operatorname{coz} f\colon f \in D\}$.

(3) $\operatorname{coz} D$ is dense.

Proof. (1) $\Longrightarrow$ (2). Suppose $x \in \operatorname{coz} f$ for some $f \in D$ then $x \in \operatorname{coz} D$ and $x \in Z(g)$. Thus $g(x)f(x) = 0$; i.e., $gf = 0$ for $f \in D$. But D is dense in $C(X)$, hence $g = 0$.

(2) $\Longrightarrow$ (3). Let $x \in \sim\overline{\operatorname{coz} D}$, then by complete

regularity there exists an $h \in C(X)$, such that $h(x) \neq 0$ and $h(\overline{\text{coz } D}) = 0$; this implies $\overline{\text{coz } D} \subset Z(h)$ which further implies $\text{coz } D \subset Z(h)$. Hence by (2) we get $h = 0$, a contradiction. So $X = \overline{\text{coz } D}$.

(3) $\Longrightarrow$ (2). Let $\text{coz } D \subset Z(g)$, therefore $\overline{\text{coz } D} = X \subset Z(g)$, for all g in $C(X)$, which implies that g = [illegible]

(2) $\Longrightarrow$ (1). Suppose $gD = 0$ for some $g \in C(X)$. So $gf = 0$ for some g in $C(X)$ and for all f in D. If $x \in \text{coz } D$, that is $x \in \text{coz } f$, for some f in D, then $f(x) \neq 0$. But $gf = 0$ implies $g(x) = 0$. Hence $\text{coz } D \subset Z(g)$, which implies that $g = 0$.

Q.E.D.

Lemma 2.2. Let A be a subring of $C(X)$; for any ideal D in A, we have $\text{Hom}_A(D, A) \subset C(\text{coz } D)$.

Proof. [5. p. 13].

For convenience, we write $Q(X)$ instead of $Q(C(X))$. By [5. Cor. 1.9], $Q(X) = \lim\limits_{D \in \mathcal{D}_0} \text{Hom}(D, C(X))$, where $\mathcal{D}_0$ stands for the family of all dense ideals in $C(X)$. We have

$$
\begin{aligned}
Q(X) = \lim_{D \in \mathcal{D}_0} \text{Hom}(D, C(X)) &\subset \lim_{D \in \mathcal{D}_0} C(\text{coz } D) && \text{(By lemma 2.2.)} \\
&= \lim_{V \in \nu_0(X)} C(V) && \text{(By Theorem 2.1.)} \\
&\subset Q(X)
\end{aligned}
$$

Thus $Q(X)$ is the ring of all equivalence classes of continuous functions on the dense open set in X. Also $Q(X)$ is regular since C(X) is semiprime.

§3. Regular and Baer rings

Regular and Baer rings have been defined in an earlier section. In this section we shall give some properties of regular and Baer rings, when the ring under consideration is C(X).

For any ring R, every maximal ideal in R is prime. When conversely, every prime ideal in C(X) is maximal, X is called a p-space.

Every discrete space is a p-space, since if X is discrete then $C(X) \cong \prod_{x \in X} R$ and every prime ideal is maximal.

Definition. An ideal I in C(X) is called a Z-ideal if $Z(f) \in Z[I]$ implies $f \in I$, where $Z[I] = \{Z(f): f \in I\}$.

Proposition 2.3. The following are equivalent [6. Ex. 4J].

(1) X is a p-space.

(2) For all $p \in X$, $M_p = O_p$. Where $M_p = \{f \in C(X): f(p) = 0\}$ and $O_p = \{f \in C(X): Z(f)$ is a neighborhood of $p\}$.

(3) Every zero set is open.

(4) Every G_δ is open.

(5) Every ideal in C(X) is a Z-ideal.

(6) Every ideal is an intersection of prime ideals.

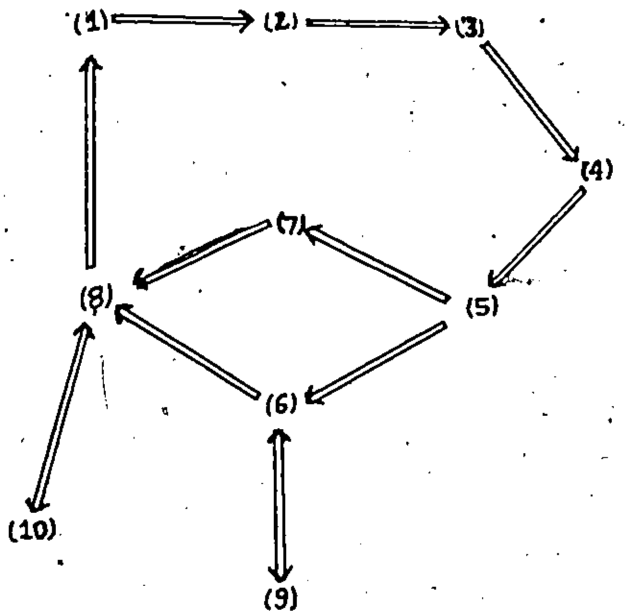
(7) For every $f, g \in C(X)$, the ideal (f, g) is the principal ideal $(f^2 + g^2)$. Where (f, g) denotes the ideal generated by f and g.

(8) For every $f \in C(X)$, there exists $f_0 \in C(X)$, such that $f^2 f_0 = f$. That is $C(X)$ is a regular ring.

(9) Every ideal is an intersection of maximal ideals.

(10) Every principal ideal is generated by an idempotent.

Proof. The following implications will be proved.



(1) $\Longrightarrow$ (2). If $O_p \neq M_p$, then O_p is contained in a prime ideal that is not maximal [6. 4I.3]; but this contradicts the fact that X is a p-space. Hence for all $p \in X$, $M_p = O_p$.

(2) $\Longrightarrow$ (3). Let $Z(f)$ be a zero set in X and $p \in Z(f)$, then $f \in M_p = O_p$, implies that $Z(f)$ is a neighborhood of p. But p was an arbitrary point of $Z(f)$ and $Z(f)$ was an arbitrary zero set. Hence every zero set is open.

(3) $\Longrightarrow$ (4). A G_δ-set B has the form $\cap U_n$, $n \in \mathbf{N}$ where each U_n is open. If $s \in B$ then $s \in U_n$ for all $n \in \mathbf{N}$. By complete regularity there is a zero set F_n such that $s \in F_n \subset U_n$. Hence $s \in \bigcap_{n\in\mathbf{N}} F_n \subset B$. Since each F_n is a zero set, therefore $\bigcap_{n\in\mathbf{N}} F_n$ is a zero set [6. p. 16]. Hence B is open.

(4) $\Longrightarrow$ (5). Let I be an ideal in C(X) and $Z(f) \in Z[I]$, $Z(f)$, being a zero set, is a G_δ, hence open. Now if $Z(g) \subset Z(f)$ (for some $g \in I$) define a function $h: X \to R$ by:

$$h(x) = \begin{cases} f(x)/g(x) & \text{for } x \in \operatorname{coz}(g) \\ 0 & \text{otherwise.} \end{cases}$$

Then h is a continuous function on X and $f = gh$. Hence if $g \in I$ with $Z(g) \subset Z(f)$ then there exists $h \in C(X)$ with $gh = f \in I$, which implies that I is a Z-ideal.

(5) $\Longrightarrow$ (6). Every Z-ideal is the intersection of prime ideals, by [6. Thm. 2.8]. But every ideal in C(X) is a Z-ideal. Hence every ideal is an intersection of prime ideals.

(5) $\Longrightarrow$ (7). $Z(f^2 + g^2) = Z(f) \cap Z(g) \subset Z(f)$, implies $f \in (f^2 + g^2)$; similarly, $g \in (f^2 + g^2)$. Thus $(f, g) \subseteq (f^2 + g^2)$. Also $f^2 + g^2 \in (f, g)$. Hence $(f^2 + g^2) = (f, g)$.

(6) $\Longrightarrow$ (8). Let us assume that $f \in C(X)$ and $I = f^2C(X)$; i.e., I is the ideal generated by f^2. By (6) every ideal is an intersection of prime ideals, so let $I = \cap P_i$.

Since $f^2 \in I$ therefore $f^2 \in P_i$ for all i, this implies that $f \in P_i$ for all i, hence $f \in I$. But $I = f^2C(X)$, therefore there exists an $f_0 \in C(X)$ such that $f = f^2 f_0$. Hence C(X) is regular.

(7) $\Longrightarrow$ (8). Set $g = 0$ in (7).

(8) $\Longrightarrow$ (1). Let $f \in C(X)$ and $f \notin P$, P a prime ideal in C(X). Consider the function $1 - ff_0$, where $f^2 f_0 = f$: $f(1 - ff_0) = f - f^2 f_0 = f - f = 0 \in P$. But $f \notin P$, therefore $1 - ff_0 \in P$. Hence P is a maximal ideal.

(6) $\Longrightarrow$ (9). This follows immediately, since (6) $\Longrightarrow$ (8) $\Longrightarrow$ (1).

(9) $\Longrightarrow$ (6). This is obvious since every maximal ideal is prime.

(10) $\Longrightarrow$ (8). Let $f \in C(X)$ and $fC(X) = eC(X)$ where $e^2 = e$. Therefore $f = eg$ for some $g \in C(X)$ and $fh = e$ for some $h \in C(X)$. So $ef = e^2 g = eg = f$, and $fhf = ef = f$. Hence C(X) is regular.

(8) $\Longrightarrow$ (10). Suppose C(X) is regular and fC(X) a principal ideal generated by f. Since $f \in C(X)$ and C(X) is regular, there exists $g \in C(X)$ such that $fgf = f$. Also $(fg)^2 = (fg)(fg) = (fgf)g = fg$; hence fg is an idempotent. Call it e. Since $fgf = ef = f$, so $fC(X) \subseteq eC(X)$. Also $e = fg$ implies $eC(X) \subseteq fC(X)$. Hence $fC(X) = eC(X)$.

Q.E.D.

Thus C(X) is regular, if and only if, X is a p-space. Here are some properties of p-spaces: (1) Every subspace of a p-space is a p-space. (2) Every quotient space of a p-space is a p-space. (3) A finite product of p-spaces is a p-space.

Definition. A space X is said to be extremally disconnected if every open set has an open closure. Equivalently, for any two disjoint open sets U and V in X, $\overline{U} \cap \overline{V} = \emptyset$ [4. p. 257].

Definition. A space X is said to be basically disconnected if every cozero set has an open closure.

Any extremally disconnected space is basically disconnected but the converse fails. An example for such a space is non discrete p-space, and this will be discussed in detail later.

While studying p-spaces, it is not inappropriate to study the structure of a base for p-spaces. The following proposition shows that clopen (closed as well as open) sets form a base for p-spaces.

Proposition 2.4. Every p-space and, more generally, every basically disconnected space has a base of open-and-closed sets [6. Ex. 4K.8].

Proof. Let U be an open set in X and x an arbitrary point of U. By complete regularity of X, there exists a function $f \in C(X)$ such that $f(x) = 1$ and $f|_{\sim U} = 0$. Thus $x \in \operatorname{coz} f \subseteq U$. Consider the continuous function $f - 1/2$ and let $g = (f - 1/2) \vee 0$ i.e., $g = \sup\{f - 1/2, 0\}$. With this choice of g, we see that $\operatorname{coz} g \subset \operatorname{coz} f$. Also

$\overline{\text{coz } g} = f^{-1}([1/2, \infty)) \subset U$. Thus $x \in \text{coz } g \subset \overline{\text{coz } g} \subset U$. But $\overline{\text{coz } g}$ is clopen. Hence clopen sets form a base for basically disconnected spaces.

Q.E.D.

Now we go over to Baer rings and characterize those spaces X for which C(X) is Baer.

Notation ∂V denotes boundary of V, defined by $\partial V = \overline{V} \cap (\overline{\sim V})$.

Proposition 2.5. The following are equivalent:

(1) C(X) is Baer.

(2) The closure of a union of cozero sets is clopen.

(3) X is extremally disconnected.

Proof. (1) $\Longrightarrow$ (2). Let C(X) be Baer, and $\{f_\alpha\}_{\alpha \in \Lambda}$ a family of functions in C(X). Further assume that $I = (f_\alpha)$. Then $I^* = eC(X)$ for some idempotent e. For $f \in I$, $fe = 0$. If $f(x) \neq 0$, for some $x \in X$, then $e(x) = 0$. Thus $\text{coz } f \subset Z(e)$ for all $f \in I$. Let $Y = \underset{\alpha}{\cup} \text{coz } f_\alpha$. Since $\text{coz } f_\alpha \subset Z(e)$ for all α, therefore $Y \subset Z(e)$ which further implies that $\overline{Y} \subset \overline{Z(e)} = Z(e)$.

There are two possibilities.

(1) $\overline{Y} = Z(e)$ (2) $\overline{Y} \neq Z(e)$

If $\overline{Y} = Z(e)$, then $\overline{Y}$ is clopen. For the second case, let $x \in Z(e)$ such that $x \notin \overline{Y}$. By complete regularity of X; there exists a $g \in C(X)$ such that $g|_{\overline{Y}} = 0$ and $g(x) = 1$. Consider the function $g^2 + e$. Now $g^2 + e$ is non zero at x and on coz e, and is zero on $\overline{Y}$. Thus $\overline{Y}$ is contained in the zero set of $g^2 + e$.

Also $\text{coz}(g^2 + e) \supseteq \text{coz}(e)$. Therefore, $\overline{Y} \subset Z(g^2 + e) \subset Z(e)$. But $Z(e)$ is the smallest zero set containing $\overline{Y}$. To show this, let $\overline{Y} \subset Z(f) \subset Z(e)$, for some function $f \in C(X)$; then $f_\alpha f = 0$ for all α. Hence $f = ef$ and so $Z(f) = Z(e) \cup Z(f) \supseteq Z(e)$. So the possibility that $\overline{Y} \neq Z(e)$ has been eliminated. Hence $\overline{Y}$ is clopen.

(2) $\Longrightarrow$ (1). Let $Y = \bigcup_{\alpha} \text{coz } f_\alpha$ and suppose $\overline{Y}$ is clopen; this implies that $\overline{Y}$ is the zero set of some idempotent, let it be e. Then $\overline{Y} = Z(e) = \overline{\bigcup_{\alpha} \text{coz } f_\alpha}$, and $Y = \bigcup_{\alpha\in\Lambda} \text{coz } f_\alpha \subset Z(e)$; this implies that $f_\alpha e = 0$ for all $\alpha\in\Lambda$. Hence $eC(X) \subseteq I^*$ where $I = (f_\alpha)_{\alpha\in\Lambda}$.

Conversely, $f \in I^*$ implies $f_\alpha f = 0$ for all α, which further implies that, $\text{coz } f_\alpha \subset Z(f)$ for all α. Hence $\bigcup_{\alpha} \text{coz } f_\alpha = Y \subset Z(f)$. But $Z(e)$ is the smallest closed set containing Y. Therefore $Y \subset Z(e) \subset Z(f)$. Then $ef = f$ since for $x \in Z(f)$, $(ef)(x) = f(x) = 0$ and for $x \in \text{coz } f$, $e(x)f(x) = 1 \cdot f(x) = f(x)$. Therefore $I^* = eC(X)$ and $C(X)$ is Baer.

(1) $\Longrightarrow$ (3). Let U be an open set in X then $U \cap \sim\overline{U} = \emptyset$. Let $y = U \cup \sim\overline{U}$, V is open. Moreover, V is dense in X; indeed if $x \in X$ and $x \notin V$ then $x \in \partial U$. Thus every neighborhood N of x meets U and, hence, V. So V is dense open in X. Now U open implies $\sim\overline{U}$ is closed in V, moreover $\sim\overline{U}$ is open so U is closed in V. Thus U and $\sim\overline{U}$ are disjoint clopen subsets of V. Define a function $e: V \to \mathbb{R}$ as follows:

$$e(x) = \begin{cases} 1 & \text{for } x \in U \\ 0 & \text{for } x \in \sim\overline{U}. \end{cases}$$

Then e is a continuous function and is an idempotent; therefore $e \in Q(X)$, but all idempotents of $Q(X)$ are in $C(X)$ [9. Lemma 1.6]. Hence e is an idempotent in $C(X)$. Hence there exists an idempotent $f \in C(X)$ such that f and e coincide on some dense open subset H of X. Since V is dense open in X and $H \subset V$, H is dense open in V. Now f and e are two functions which coincide on H, and H is dense open in V, so f and e coincide on $V = U \cup \sim\overline{U}$ [10. p. 48]. Also $U \subset \text{coz } f$ and $\sim\overline{U} \subset Z(f)$; it follows that $U \subset \overline{\text{coz } f} \subset \overline{U}$. Hence $\text{coz } f = \overline{U}$ is clopen.

(3) $\Longrightarrow$ (2). Let $\{f_\alpha\}_{\alpha\in\Lambda}$ be a family of functions in $C(X)$ and consider $\text{coz } f_\alpha$ for all α. Each $\text{coz } f_\alpha$ is open and so is $\bigcup_{\alpha\in\Lambda} \text{coz } f_\alpha$. This implies that $\overline{\bigcup_{\alpha\in\Lambda} \text{coz } f_\alpha}$ is open. Hence $C(X)$ is Baer by (2).

Q.E.D.

As mentioned earlier, extremally disconnected spaces are basically disconnected, but the converse fails. Here is an example of a space X which is a p-space but $C(X)$ is not Baer. This also serves as an example of a space which is a p-space but not extremally disconnected. So it would follow that not every p-space is extremally disconnected.

Example Ref. [6. Ex. 4N]. Let S be an uncountable space in which all points are isolated except for a distinguished point s, a neighborhood of s being any set containing s whose complement is countable.

S is a non-discrete p-space. We first see that $\{s\}$ is not a zero set. If it were then let $\{s\} = Z(f)$, for some function $f \in C(S)$ and let F be a neighborhood filter converging to s. The filter base $f(F)$ converges to 0, so for all $n > 0$ there exists an $F_n \in F$ such that $f(F_n) \subset (-1/n, 1/n)$. Without loss of generality we can choose the F_n to form a descending chain. Since $\cap F_n = \sim(\bigcup_1^\infty (\sim F_n))$, and each $\sim F_n$ is countable, this implies $\bigcup_1^\infty (\sim F_n)$ is countable. Hence $\cap F_n$ is uncountable. Further $f\big|_{\cap F_n} = 0$. (i.e., $\cap F_n \subset Z(f) = \{s\}$) implies $\{s\}$ is uncountable, which is a contradiction. Hence $\{s\}$ is not a zero set. But every subset of $S \backslash \{s\}$ is clopen and, hence, a zero set. Let $Z(f)$ be a zero set containing s, we shall show that $Z(f)$ is open by showing that coz f is countable. Let $Z(f) = \bigcap_{n \in \mathbf{N}} U_n$ (since a zero set is G_δ) where each U_n is open, $s \in Z(f)$ implies that $s \in U_n$ for all $n \in \mathbf{N}$. Also coz $f = \bigcup_{n \in \mathbf{N}} \sim U_n$, but each U_n is an open set containing s, hence $\sim U_n$ is countable for each $n \in \mathbf{N}$. Thus coz f is countable. Hence $Z(f)$ is open and so every zero set in the space S is clopen.

To complete the proof that S is a p-space, the only thing which remains to be proved is that S is completely regular. Let a, b be two distinct points of S, if neither of them is s then $\{a\} \cap \{b\} = \emptyset$. But if one of them is s say b, and H is a neighborhood of s, then H is a set containing s and ~H is countable. If $a \notin H$, then a and s have disjoint open neighborhoods, but if $a \in H$ then consider $G = H \setminus \{a\}$, G is a set containing s and ~G is countable; thus G is a neighborhood of the point s not containing a. Hence, as before a and s have disjoint open sets containing them which implies that S is a Hausdorff space.

Let M be a closed set in S and $a \notin M$, there are two possibilities either $s \in M$ or $s \notin M$. If $s \in M$, then define $g: S \to \mathbb{R}$, by $g(a) = 0$ and $g(x) = 1$ for $x \neq a$. Then g is obviously continuous and separates a and M. If $s \notin M$, then $s \in {\sim}M$ implies M is countable and is clopen. Define $h: S \to \mathbb{R}$ by

$$h(x) = \begin{cases} 0 & \text{for } x \text{ in } M \\ 1 & \text{otherwise.} \end{cases}$$

Then h is continuous and separates s from M. Thus S is completely regular. Hence S is a non-discrete p-space.

Now we show that C(S) is not Baer. Divide S into two parts S_1 and S_2 both uncountable and $s \in S_1$. Then s is not in the interior of S_1; for if, it were, then there would exist an open set H containing s in S_1, implying that ~H is countable. This contradicts the fact that S_2 is uncountable.

Let $U = S_1 - \{s\}$; U is the union of uncountably many open sets. Hence U is open and $U \subset S_1$ which implies $\overline{U} \subset \overline{S}_1 = S_1$ (since ${\sim}S_1 = S_2$ is open). Also $S_1 \subset \overline{U}$, since every point of S_1, different from s, has a neighborhood having non empty intersection with U and neighborhoods of s have countable complement. Hence $S_1 = \overline{U}$, but S_1 is closed (since $\sim S_1 = S_2$ is open) so $C(S)$ is not Baer.

Now we give an example of a space X which is extremally disconnected but not a p-space. Before giving the example here are some definitions.

A filter F on a set S is a non empty collection of non empty subsets of S with the properties:

(1) If $A, B \in F$ then $A \cap B \in F$.

(2) If $A \in F$ and $A \subset B$ then $B \in F$.

If F and G are filters on X then F is said to be finer than G if, and only if, $F \supset G$. Maximal filters are called ultrafilters. A filter F is called free if, and only if, $\bigcap_{A \in F} A = \emptyset$.

Example Ref. [6. Ex. 4M]. Let $\mathcal{U}$ be a free ultrafilter on $\mathbf{N}$, let $\Sigma = \mathbf{N} \cup \{\sigma\}$ $\sigma \notin \mathbf{N}$. Define a topology on Σ by: all points of $\mathbf{N}$ are isolated and the neighborhoods of σ are the sets $U \cup \{\sigma\}$ for $U \in \mathcal{U}$. Then Σ is extremally disconnected but not a p-space.

To prove the above statement we first show the following.

(1) $\mathbf{N}$ is a dense subspace of Σ.

(2) Every set containing σ is closed; hence every subset of Σ is open or closed.

(3) Every closed set is a zero set.

(4) The space Σ is completely regular.

Proof. (1) $\sigma \in \Sigma$, and any neighborhood of σ is of the form $U \cup \{\sigma\}$, for $U \in \mathcal{U}$. Also $(U \cup \{\sigma\}) \cap \mathbf{N} = U \cap \mathbf{N} \neq \emptyset$, implies $\sigma \in \overline{\mathbf{N}}$. Hence $\mathbf{N}$ is dense in Σ.

(2) Suppose H is a subset of Σ containing σ, and $\sigma \neq a \in \overline{H}$, then $\{a\} \cap H \neq \emptyset$ which implies that $a \in H$. Hence H is closed.

Proper subsets of Σ are of the form: $A \subseteq \mathbf{N}$, $\{\sigma\}$ and $\{A \cup \{\sigma\}: A \subset \mathbf{N}\}$. Subsets of $\mathbf{N}$ are open and $\{\sigma\} = \sim\mathbf{N}$ is closed. If $A \in \mathcal{U}$, then $A \cup \{\sigma\}$ is open, also as we have seen, all subsets containing σ are closed. Hence every subset of Σ is open or closed.

(3) We first show that $\{\sigma\}$ is a zero set. Define $f: \Sigma \to \mathbb{R}$ by:

$f(\sigma) = 0$ and $f(n) = 1/n$ for all $n \in \mathbf{N}$. It must be shown that f is continuous; i.e., given $\varepsilon > 0$ there exists an $A \in \mathcal{U}$ such that for all $n \in A$ $1/n < \varepsilon$.

Let $N_0 = \{n \in \mathbf{N}: n \geq n_0\}$. If $N_0 \in \mathcal{U}$, then we are finished. If $N_0 \notin \mathcal{U}$ then we proceed as follows: $\sim N_0$ is finite, so there are only finitely many subsets of $\sim N_0$; let them be X_1, X_2, $\ldots$, $X_{2^{n_0}}$. Since $\mathcal{U}$ is a free ultrafilter, there exists $C_1 \in \mathcal{U}$ such that C_1 does not contain X_1, similarly there exists $C_2 \in \mathcal{U}$ such that C_2 does not contain X_2 and so on. Since filters are closed under finite intersection, therefore $\cap C_i \in \mathcal{U}$ let $A = \cap C_i$. We have $A \in \mathcal{U}$ such that $A \subseteq N_0$. So we have found an $A \in \mathcal{U}$ which has the required property. Hence f is continuous.

(4) For Σ to be completely regular, Σ must be a

Hausdorff space. To see this, let n, m be two distinct points of Σ, if neither of them is σ, then $\{n\}$ and $\{m\}$ are disjoint open sets. But if $n = \sigma$ then $U \in \mathcal{U}$ can be chosen in such a way that $m \notin U$. Hence $(U \cup \{\sigma\}) \cap \{m\} = \emptyset$. So Σ is a Hausdorff space. Also, each closed set is a zero set. Hence Σ is completely regular.

Let us come back to the proofs of the results referred to in the example. As we have seen, N is the only proper open subset of Σ which is not closed, also N is dense in Σ which implies that Σ is extremally disconnected.

To show that Σ is not a p-space, we will show that the Z-ideal O_σ is not maximal. If it were then $O_\sigma = M_\sigma$ and so σ would be a p- point [6. p. 63]. Also C(N) is regular. Thus, for all $p \in \Sigma$ we would have $M_p = O_p$, this would imply that every zero set is open ($2 \Longrightarrow 3$. Prop. 2.3.). Contradicting the fact that $\{\sigma\}$ is closed. Hence $O_\sigma \neq M_\sigma$. But O_σ is a Z-ideal, therefore O_σ is contained in a prime ideal that is not maximal. Hence Σ is not a p-space.

Let T be the topological sum of Σ and S (both defined earlier), $T = \Sigma \dot{\cup} S$ (disjoint union of Σ and S). Since Σ and S are both basically disconnected so is T. Also $C(T) \cong C(\Sigma) \times C(S)$. We notice that C(T) is not Baer nor is it regular, since C(S) is not Baer and $C(\Sigma)$ is not regular. Hence T is basically disconnected, but it is neither a p-space nor is it an extremally disconnected space.

CHAPTER III Abian's order relation and C(X)

§1. Abian's order relation

Definition. [1]. Let R be a (commutative) semiprime ring, for $x, y \in R$, x is less than or equal to y with respect to Abian's order, denoted by $x \leq y$, if and only if, $xy = x^2$.

Lemma 3.1. The order relation "$\leq$" is a partial order and makes R into a partially ordered multiplicative semigroup.

Proof. Since $xx = x^2$ it follows that $x \leq x$. Thus $\leq$ is reflexive.

Moreover, if $x \leq y$ and $y \leq x$, then $xy = x^2$ and $yx = y^2$, so that $(x - y)^2 = x^2 - xy - yx + y^2 = 0$. But then $x - y = 0$, thus $x = y$ and therefore $\leq$ is antisymmetric.

Furthermore, if $x \leq y$ and $y \leq z$, then $xy = x^2$ and $yz = y^2$ so that $x^2z = xyz = xy^2 = x^2y = x^3$. Thus, $x^2z^2 = x^3z$ and $x^3z = x^4$ so that $x^2z^2 - x^3z - x^3z + x^4 = (xz - x^2)^2 = 0$. But then $xz = x^2$. Hence $x \leq z$ and therefore, $\leq$ is transitive.

Finally, let $x \leq y$, for all $x, y \in R$; and consider $(x^2z^2 - xyz^2)^2$ for all $z \in R$.

$$\begin{aligned}(x^2z^2 - xyz^2)^2 &= x^4z^4 - x^3yz^4 - x^3yz^4 + x^2y^2z^4.\\ &= x^4z^4 - x^2xyz^4 - x^2xyz^4 + x^2y^2z^4.\\ &= x^4z^4 - x^4z^4 - x^4z^4 + x^4z^4.\\ &= 0\end{aligned}$$

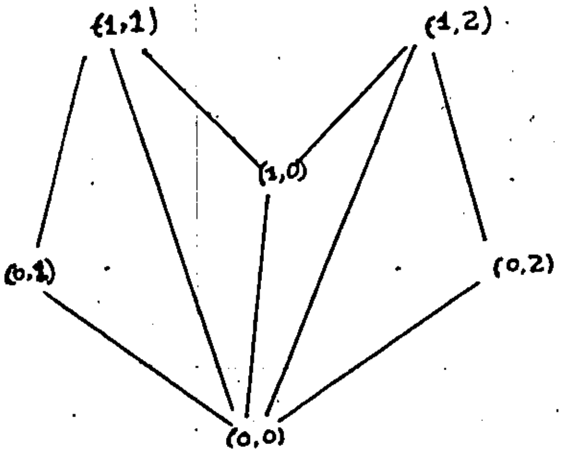
Hence $x^2z^2 = xyz^2$, thus $xz \leq yz$ for all $x, y, z \in R$.

Q.E.D.

From now onwards "order relation" means "Abian's order relation".

Lemma 3.1. generalizes the corresponding result for Boolean rings. In general, this order relation does not turn all rings R into a lattice. In the case of an integral domain, if two distinct elements a and b are related by Abian's order i.e., $a \le b$, then $a = 0$. So all nonzero points are unrelated, unless they are equal.

Consider the product $Z_2 \times Z_3$, elements of $Z_2 \times Z_3$ are (0, 0), (0, 1), (0, 2), (1, 0), (1, 1) and (1, 2). Following diagram shows the relationship between the elements of $Z_2 \times Z_3$, and we note that not all elements are related by the order relation, and not every pair of elements has supremum.



Hence $Z_2 \times Z_3$ is not a lattice.

Note If R and S are semiprime rings then a homomorphism $f: R \to S$ preserves the order relation.

For C(X) the order relation is defined in the same way; that is, $f \le g$ if, and only if, $fg = f^2$ for all $f, g \in C(X)$.

There is little connection between Abian's order and the natural order relation in $C(X)$. However, if $f \leq g$ (Abian's order) then $|f| \leq |g|$ (natural order).

For the remainder of this thesis, all rings will be assumed to be (commutative) semiprime.

Let R be a (semiprime) ring equipped with the order relation "$\leq$". An element s is called an <u>upper bound</u> of a subset $H \subseteq R$ if, and only if, $r \leq s$ for all $r \in H$; and s will be called the <u>supremum</u> if $s \leq t$ for every upper bound t of H. "Lower bound" and "infimum" may be defined analogously but are of little use.

Let R be a ring, then a set X in R is said to be <u>orthogonal</u> if $ab = 0$ for all $a \neq b$ in X and R is <u>orthogonally complete</u> if every orthogonal set in R has a supremum with respect to "$\leq$" in R.

Integral domains and products of integral domains are orthogonally complete rings whereas $C([0, 1])$ is not orthogonally complete (see Theorem 3.14. below).

<u>Proposition 3.2.</u> $\prod_\alpha R_\alpha$ is orthogonally complete if, and only if each R_α is orthogonally complete.

Proof. Let $\prod R_\alpha$ be orthogonally complete H_β an orthogonal subset of R_β. Embed H_β in $\prod R_\alpha$ by making all components, other than β^{th}, equal to zero. Then H_β can be considered as an orthogonal subset of $\prod R_\alpha$, and since $\prod R_\alpha$ is orthogonally complete it follows that r = supremum $H_\beta \in \Pi R_\alpha$. Since the projection map is a homomorphism, it preserves the order. Hence r = supremum H_β implies that $p_\beta(r) = r_\beta$ is an upper bound of $p_\beta(H_\beta) = H_\beta$. Let b_β be another upper bound for H_β in R_β so $h_\beta \leq b_\beta$ for all $h_\beta \in H_\beta$. As above embed b_β into ΠR_α and clearly b_β is an upper bound of H_β in ΠR_α. But r is the supremum in ΠR_α of H_β, therefore $r \leq b_\beta$ which implies that $r_\beta \leq b_\beta$. Hence r_β is the supremum in R_β of H_β and thus R_β is orthogonally complete.

Conversely, let each R_β be orthogonally complete and H be an orthogonal set in ΠR_α. Therefore $p_\beta(H) = H_\beta$ is an orthogonal subset of R_β, but R_β is orthogonally complete, therefore supremum $H_\beta = r_\beta \in R_\beta$. Suppose $h \in H$, therefore $p_\beta(h) = h_\beta \in H_\beta$; but $h_\beta \leq r_\beta$ for all β, this implies that $h \leq r$ where $r \in \Pi R_\alpha$ has β-component r_β. Thus r is an upper bound of H. If s is another upper bound of H, then $h \leq s$ for all h in H, so $h_\beta \leq s_\beta$ for all β. But r_β is the supremum of H_β. Therefore $r_\beta \leq s_\beta$ for all β; this implies that $r \leq s$. Hence r is the supremum of H in ΠR_α, so ΠR_α is orthogonally complete.

Q.E.D.

If $R \subset S$ are rings, then S is called orthogonal extension of R if every element of S is the supremum of an orthogonal set of R. Further, if S is an orthogonal extension so that S is orthogonally complete, then S is called an orthogonal completion of R [2. Definition 9.].

Not all rings are orthogonally complete nor do all rings have orthogonal completions; but if orthogonal completions exist, they are unique [2. remark after Thm. 12]. It is shown in [2. Thm. 14 and Thm. 18] that regular rings and Baer rings have orthogonal completions. It is also shown in [2. Thm. 12] that any orthogonal extension of R lies in the orthogonally complete ring Q(R), and that if R has an orthogonal completion S, then S is the set of all suprema in Q(R) of orthogonal sets in R. In the last section of this chapter it will be shown that if X is connected and locally connected then either C(X) is orthogonally complete or C(X) has no orthogonal completion.

Now we give the explicit description of the supremum of an orthogonal family of functions in C(X) by giving the following proposition.

<u>Proposition 3.3</u>. Let $\{f_\alpha\}$ be an orthogonal family of functions in C(X) with supremum $\overline{q} \in Q(X)$. Then the function q defined by

$$q(x) = \begin{cases} f_\alpha(x) \text{ on coz } f_\alpha \text{ for all } \alpha \\ 0 \text{ on } \sim\overline{\cup \text{coz } f_\alpha}. \end{cases}$$

represents $\overline{q}$.

Proof. Let $H = (\cup \text{coz } f_\alpha) \cup (\sim\overline{\cup \text{coz } f_\alpha})$. Clearly H is dense open in X. Now q is defined on a dense open subset H of X. Let (a, b) be an open interval of $\mathbb{R}$, then we see that

$$q^{-1}(a, b) = \begin{cases} \bigcup_{\alpha} f_\alpha^{-1}(a, b) & \text{if } 0 \notin (a, b). \\ q^{-1}(a, 0) \cup q^{-1}(0, b) \cup q^{-1}(\{0\}) & \text{if } 0 \in (a, b). \end{cases}$$

If $0 \notin (a, b)$ then $q^{-1}(a, b)$ is open. Similarly $q^{-1}(a, 0)$ and $q^{-1}(0, b)$ are open. Also $q^{-1}(0) = \sim\overline{\cup \operatorname{coz} f_\alpha}$ is open in H. Hence $q: H \to \mathbb{R}$ is continuous.

Now we show that q actually represents the supremum of the family $\{f_\alpha\}$. Let $x \in H$, so $x \in \cup \operatorname{coz} f_\alpha$ or $x \in \sim\overline{\cup \operatorname{coz} f_\alpha}$. If $x \in \cup \operatorname{coz} f_\alpha$, then $q(x)\, f_\alpha(x) = f_\alpha(x)\, f_\alpha(x) = f_\alpha^2(x)$. On the other hand if $x \in \sim\overline{\cup \operatorname{coz} f_\alpha}$, then $q(x)\, f_\alpha(x)$ and $f_\alpha^2(x)$ are both zero. Thus for $x \in H$, $qf_\alpha = f_\alpha^2$, for all α; i.e., $f_\alpha \le \overline{q}$ for all α. Hence $\overline{q}$ is an upper bound for the family $\{f_\alpha\}$.

Let $\overline{h}$ be another upper bound for the family $\{f_\alpha\}$, with representative h. Then $f_\alpha \le \overline{h}$ for all α, therefore $f_\alpha h - f_\alpha^2$ is zero on a dense open set V_α for each α. Hence h coincides with f_α on $\operatorname{coz} f_\alpha \cap V_\alpha$.

Let $x \in H \cap D(h)$ and consider $(qh)(x)$, $(qh)(x) = q(x)h(x) = q(x)f_\alpha(x) = q^2(x)$, if $x \in \operatorname{coz} f_\alpha \cap V_\alpha$, and $q(x)h(x) = 0 = q^2(x)$ if $x \in \sim\overline{\cup \operatorname{coz} f_\alpha} \cap V_\alpha$. Thus $qh = q^2$ on the dense open set $\bigcup_\alpha (\operatorname{coz} f_\alpha \cap V_\alpha) \cup \sim\overline{\cup \operatorname{coz} f_\alpha}$, and $\overline{q} \le \overline{h}$. This implies $\overline{q}$ is the supremum of the family $\{f_\alpha\}$.

Q.E.D.

In the other direction, we have the following theorem.

Theorem 3.4. Let X have a basis of clopen sets, then if $\overline{q} \in Q(X)$, $\overline{q}$ is the supremum of an orthogonal family in $C(X)$. Hence, $Q(X)$ is the orthogonal completion of $C(X)$.

Proof. Let $\bar{q} \in Q(X)$ be represented by $q \in C(V)$, for some dense open set V in X. Since V is open in X and clopen sets form basis for X, therefore there exists a clopen set U, in V. Thus we have a family of disjoint clopen sets in V.

If there is a chain of families of disjoint clopen sets in V,

$$\ldots\ldots \subseteq B_{\alpha-1} \subseteq B_{\alpha} \subseteq B_{\alpha+1} \subseteq \ldots\ldots$$

then it has an upper bound (union of all the B_α, s). Hence by Zorn's lemma [10. p. 10] there is a maximal family $\mathcal{B}$ of disjoint clopen sets in V. Note that $\bigcup_{U \in \mathcal{B}} U = Y$ is dense in V. If not, then there would exist an open set in V disjoint from Y; hence there would exist a clopen set W in V disjoint from Y making $\mathcal{B} \cup \{W\}$ a family of clopen sets in V properly containing $\mathcal{B}$. This contradicts the fact that $\mathcal{B}$ is maximal. Hence Y is dense in V, and therefore dense in X. For each U in $\mathcal{B}$ define:

$$q_U(x) = \begin{cases} q(x), & \text{for } x \text{ in } U. \\ 0 & \text{otherwise.} \end{cases}$$

Obviously, $q_U \in C(X)$ (because U is clopen and q is continuous). The family $\{q_U\}$ is orthogonal since the cozero sets are disjoint. Let $h = q|_Y$, so $h \in C(Y)$ and $\bar{h} = \bar{q}$. Hence by proposition 3.3. $\bar{h} = \bar{q}$ is the supremum of the orthogonal family $\{q_U\}$.

Q.E.D.

§2. Conditional Completeness

Definition. An ordered set A is said to be conditionally complete if every non-empty subset which possesses an upper (lower) bound also possesses a supremum (infimum) in A.

The real line and the partially ordered set P(S), S any set, are examples of conditionally complete sets whereas the rational line is not conditionally complete.

We are interested in those topological spaces X for which C(X) is conditionally complete, and a partial answer will be given below. Before giving the main theorem, we give a few definitions and lemmas.

<u>Definition</u>. A space X is called <u>connected</u> if it can not be written as the union of two disjoint closed or open sets; equivalently, X and ∅ are the only subsets which are clopen.

Hence in a connected space the boundary of any open set different from ∅ and X is non empty.

<u>Definition</u>. A space X is said to be <u>locally connected</u> if for each $p \in X$ and each neighborhood U of p, there is a connected neighborhood V of p such that $p \in V \subseteq U$.

<u>Note</u> A connected space need not be locally connected, and a locally connected space is not always connected, [10. p. 199].

<u>Lemma 3.5.</u> $\prod_{\alpha \in \Lambda} R_\alpha$ is conditionally complete if, and only if, each R_α is conditionally complete.

Proof. Let ΠR_α be conditionally complete, and H_β a non-empty subset of R_β, further assume that b_β is an upper bound of H_β. Embed H_β in ΠR_α by making all components, other than β^{th}, equal to zero. Then H_β can be considered as a subset of ΠR_α, but ΠR_α is conditionally complete. Hence H_β, considered as a subset of ΠR_α, has a supremum, let it be ℓ. The projection maps are homomorphisms which implies that they preserve order. Hence ℓ = supremum H_β implies that $p_\beta(\ell) = \ell_\beta$ is an upper bound of H_β. Let b_β be another upper bound of H_β, so for all $h_\beta \in H_\beta$, $h_\beta \leq b_\beta$. As above embed b_β into ΠR_α and clearly b_β is an upper bound of H_β in ΠR_α. But ℓ is the supremum in ΠR_α for H_β, hence $\ell \leq b_\beta$, this implies that $\ell_\beta \leq b_\beta$. Thus ℓ_β is the supremum for H_β. Hence R_β is conditionally complete.

Conversely, assume that each R_α is conditionally complete and H a non-empty subset of ΠR_α; further, assume that b is an upper bound for H. Since H is a subset of ΠR_α, therefore $p_\beta(H) = H_\beta$ is a subset of R_β. Let $h \in H$, therefore $h \leq b$ for all $h \in H$ implies that $h_\beta \leq b_\beta$ for all β which further implies that b_β is an upper bound of H_β. But R_β is conditionally complete, therefore H_β has a supremum k_β in R_β. Suppose $h \in H$, so $h_\beta \in H_\beta$ and therefore $h_\beta \leq k_\beta$ for all β, let $k \in \Pi R_\alpha$ such that β-component is k_β. Clearly k is an upper bound of H, also b is an upper bound of H, hence for all $h \in H$ $h \leq b$, which implies that $h_\beta \leq b_\beta$ for all β. But k_β is the supremum of H_β, therefore $k_\beta \leq b_\beta$, for all β. Thus $k \leq b$, which implies that k is the supremum for H. Hence ΠR_α is conditionally complete.

Q.E.D.

Lemma 3.6. If N is a connected subset of X and N meets an open set U and ~U, then N contains a boundary point of U.

Proof. If not, then N $\cap$ Interior U and N $\cap$ Exterior U = N $\cap$ Interior (~U) are two disjoint open subsets of N, whose union is N, contradicting the fact that N is connected. Hence N contains at least one point of the boundary of U.

Q.E.D.

Theorem 3.7. If X is connected and locally connected, then C(X) is conditionally complete.

Proof. Let M = $\{f_\alpha\}$ be a non empty subset of C(X) such that M has an upper bound, and let h be one of the upper bounds for M. So $f_\alpha \leq h$ for all α which implies that f_α coincides with h on coz f_α. Define a function g as follows:

$$g(x) = \begin{cases} h^2(x)/f_\alpha(x) & \text{on coz } f_\alpha \text{ for each } \alpha \\ 0 & \text{on } \cap Z(f_\alpha). \end{cases}$$

Assume for the moment that g is continuous. Clearly $Z(g) = (\bigcap_\alpha Z(f_\alpha)) \cup Z(h)$. But $f_\alpha \leq h$ implies that $Z(h) \subseteq Z(f_\alpha)$ for all α. Hence $Z(g) = \bigcap_\alpha Z(f_\alpha)$. With this choice of g, $f_\alpha \leq g$ for all α. To show this, let $x \in X$ and consider $(f_\alpha g)(x)$. If $x \in$ coz f_α for some α then $(f_\alpha g)(x) = f_\alpha(x)g(x) = f_\alpha(x)h^2(x)/f_\alpha(x) = h^2(x) = f_\alpha{}^2(x)$.

And if $x \notin$ coz f_α for any α then $x \in Z(f_\alpha)$ for all α, so $f_\alpha(x)g(x)$ and $f_\alpha{}^2(x)$ are both zero. Thus $f_\alpha \leq g$ for all α, Also $g \leq k$, where k is any upper bound for M. We see that if $x \in \operatorname{coz} f_\alpha$ then $(gk)(x) = g(x)k(x) = h^2(x)/_{f_\alpha(x)} k(x) =$ $= h^2(x) = f_\alpha{}^2(x) = g^2(x)$ and $g(x)k(x) = g^2(x) = 0$ if $x \in \underset{\alpha}{\cap} Z(f_\alpha)$. Hence $g \leq k$.

Now we prove the continuity of g. Obviously g is continuous on $\operatorname{coz} g$ and on $\sim\overline{\operatorname{coz} g}$. Let $x \in \overline{\operatorname{coz} g} \setminus \operatorname{coz} g$, therefore every neighborhood N of x meets $\operatorname{coz} g$, hence it meets at least one $\operatorname{coz} f_\alpha$ for some α. N can be chosen to be connected (because the space is locally connected). Now $N \cap \operatorname{coz} f_\alpha \neq \emptyset$ for some α and $N \cap Z(f_\alpha) \neq \emptyset$, so by lemma 3.6, N contains at least one point y of $\partial(\operatorname{coz} f_\alpha)$ and at this point h is zero. To show this, let U be a connected neighborhood of y such that $f_\alpha(U) \subseteq (-\varepsilon, \varepsilon)$, then $h(U) \cap (-\varepsilon, \varepsilon) \neq \emptyset$. Since h is continuous, therefore $h(y) = 0$. Now $0 \in h(N)$, say $h(y) = 0$, for $y \in N$; and h is continuous, therefore $h(x) = 0$. Since h is continuous, therefore given $\epsilon > 0$ there exists a neighborhood H of x such that for all y in H, $|h(y)| < \varepsilon$. Also $|g(y)| < \varepsilon$ for all y in H since g coincides with h on $\operatorname{coz} g$ and outside of $\operatorname{coz} g$, $g(x) = 0$, this implies that g is continuous at x. Hence $C(X)$ is conditionally complete.

Q.E.D.

An immediate corollary is:

Corollary 3.8. If X is locally connected, then $C(X)$ is conditionally complete.

Proof. X locally connected implies that its components are clopen [10. p. 200]. Therefore $X = \dot{\bigcup}_{\alpha} A_{\alpha}$ (disjoint union of components) and so $C(X) \cong \Pi C(A_{\alpha})$. But, A_{α} is both connected and locally connected, therefore (by Theorem 3.7.) $C(A_{\alpha})$ is conditionally complete, for all α. Hence $\Pi C(A_{\alpha})$ is conditionally complete (by lemma 3.5.), i.e., $C(X)$ is conditionally complete.

Q.E.D.

§3. Local connectedness and orthogonal completions

As mentioned earlier in §1. of this chapter, neither all rings are orthogonally complete nor do all rings have orthogonal completions. If the ring under consideration is $C(X)$, then the following questions could be asked.

(1) When is $C(X)$ orthogonally complete?

(2) When does $C(X)$ have an orthogonal completion?

The following theorems will provide us with partial answers to the above questions.

Theorem 3.9. If X is connected and locally connected then one of the following holds.

(1) $C(X)$ is orthogonally complete,

(2) $C(X)$ has no orthogonal completion.

Proof. Suppose $C(X)$ is not orthogonally complete, and $\overline{q} \in Q(X)$, $\overline{q} \notin C(X)$, is the supremum of an orthogonal family $\{f_{\alpha}\}$ of more than one non-zero function in $C(X)$. So $\overline{q}$ has values arbitrarily close to zero. Indeed let $q \in C(V)$ be the function of proposition 3.3 which represents the supremum of $\{f_\alpha\}$, where $V = (\underset{\alpha}{\cup}\text{coz } f_\alpha) \cup \sim\overline{\cup\text{coz } f_\alpha}$ is dense open in X. Then some coz f_α is a proper non empty open set of X. Now $\partial(\text{coz } f_\alpha) \neq \emptyset$ (since the space is connected) and for $x \in \partial(\text{coz } f_\alpha)$, every neighborhood N of x is such that $V \cap N \cap \text{coz } f_\alpha \neq \emptyset$. On this intersection q coincides with f_α. For N take $f_\alpha^{-1}(-\varepsilon, \varepsilon)$, thus q has values arbitrarily close to zero.

Next, $\overline{q} = \sup\{f_\alpha\}$ implies $|\overline{q}| = \sup\{|f_\alpha|\}$ (by prop. 3.3.). Suppose $\overline{q} \notin C(X)$, we shall show that $|\overline{q}| \notin C(X)$ and hence we may suppose that $\overline{q}$ is non-negative.

Suppose that $|q| \in C(V)$ can be extended to $h \in C(X)$ and define h' by:

$$h'(x) = \begin{cases} q(x) & x \in D(q) \\ 0 & \text{otherwise.} \end{cases}$$

Clearly h' is continuous on D(q). For $x \notin D(q) = V$, $x \notin \underset{\alpha}{\cup}\text{coz } f_\alpha$ and $x \in \overline{\cup\text{coz } f_\alpha}$, therefore any connected neighborhood N of x meets $\underset{\alpha}{\cup}\text{coz } f_\alpha$, hence it meets at least one coz f_α; so (by lemma 3.6.) N contains at least one boundary point of coz f_α and at that point $h(x) = h'(x) = 0$ (as in theorem 3.7.). Therefore given $\varepsilon > 0$ there exists a neighborhood H of x such that for all y in H, $|h(y)| < \varepsilon$. But $|q|$ coincides with h on V, therefore for all y in $H \cap V$, $0 \leq |q(y)| < \varepsilon$; i.e., $0 \leq |h'(y)| < \varepsilon$, and for $y \in H \cap \sim V$, $h'(y) = 0$. Hence h' is continuous, thus q and hence $\overline{q}$ can be extended to $h' \in C(X)$, a contradiction. Thus $|\overline{q}| \notin C(X)$, and so $\overline{q}$ can be assumed to be non-negative.

Now $\overline{q} + 1 \in Q(X)$ is bounded away from zero and thus can not be the supremum of an orthogonal family in C(X). But $\overline{q}+1$ is in any ring between C(X) and Q(X) containing $\overline{q}$. Hence C(X) has no orthogonal completion.

Q.E.D.

An immediate corollary is:

Corollary 3.10. If X is locally connected then one of the following holds,

(1) C(X) is orthogonally complete.

(2) C(X) has no orthogonal completion.

Proof. X locally connected implies that its components are clopen [10. p. 200]. Therefore $X = \dot{\cup}_{\alpha} A_{\alpha}$ (disjoint union of components), and $C(X) \cong \Pi C(A_{\alpha})$. Since A_{α} is both connected and locally connected for all α, therefore either $C(A_{\alpha})$ is orthogonally complete or $C(A_{\alpha})$ has no orthogonal completion (Thm. 3.9.). If each $C(A_{\alpha})$ is orthogonally complete then C(X) is orthogonally complete (prop. 3.2.); on the other hand if some $C(A_{\alpha})$ has no orthogonal completion then C(X) has no orthogonal completion [3]. Hence the corollary follows.

Q.E.D.

Note. No cases of (1) are known except where X is discrete but many cases of (2) are known as theorem 3.14. will show.

Lemma 3.11. In any metric space (X, d), an open ball is a cozero set.

Proof. Suppose $B_{x,r}$ is an open ball in (X, d), where $d: X \times X \to \mathbb{R}$, carrying (x, y) to $d(x, y)$. Fix x, so we get another function $d_x: X \to \mathbb{R}$ defined by $d_x(y) = d(x, y)$. Obviously d_x is continuous. Also $B_{x,r} = \{y \in X: d(x, y) < r\} =$ $= \{y \in X: d_x(y) < r\} = \{y \in X: (d_x - \overline{r})(y) < 0\}$ ($\overline{r}$ is the constant function with value r). So $B_{x,r} = \mathrm{coz}((d_x - \overline{r}) \wedge 0)$, where $(d_x - \overline{r}) \wedge 0$ denotes the infimum of $d_x - \overline{r}$ and 0.

Q.E.D.

<u>Proposition 3.12.</u> If X is metrizable and connected then there exists a sequence of disjoint cozero sets $\{\mathrm{coz}\, f_i\}$ such that $\bigcup_i \overline{\mathrm{coz}\, f_i}$ is not closed. X is assumed to have more than one point.

Proof. Let $x \in X$, and $\overline{B}_{x,1}$, $\overline{B}_{x,1/2}$ be two concentric closed balls of radii 1 and 1/2 respectively. We may assume that $d(x, y) = 1$ for some $y \in X$ and then $\overline{B}_{x,1} - \overline{B}_{x,1/2} \neq \emptyset$. Otherwise $B_{x,1/2+\varepsilon}$ and $\sim\overline{B}_{x,1/2+\varepsilon}$ would be two disjoint open balls with union X, contradicting the fact that X is connected. Let $y_1 \in \overline{B}_{x,1} - \overline{B}_{x,1/2}$ and choose $r_1 = 1/2 \min\{d(x, y_1) - 1/2, 1 - d(x, y_1)\}$. With this choice of r_1, $1/2 < d(x, y_1) < 1$. Also if $d(y_1, z) < r_1$ then $1/2 < d(x, z) < 1$. Since $d(x, z)$ $\leq d(x, y_1) + d(y_1, z) < d(x, y_1) + r_1 < d(x, y_1) + 1/2(1 - d(x, y_1))$ $< 1/2d(x, y_1) + 1/2 < 1$, and $d(x, y_1) \leq d(x, z) + d(z, y_1)$ $< d(x, z) + r_1 < d(x, z) + 1/2(d(x, y_1) - 1/2)$. So $1/2 <$ $< 1/2d(x, y_1) + 1/4 < d(x, z) < 1$. Thus we have a closed ball $\overline{B}_{y_1,r_1}$ in $\overline{B}_{x,1} - \overline{B}_{x,1/2}$. As before start with $\overline{B}_{x,1/2}$ and $\overline{B}_{x,1/3}$, here also $\overline{B}_{x,\,1/2} - \overline{B}_{x,\,1/3} \neq \emptyset$, so choose $y_2 \in \overline{B}_{x,\,1/2} - \overline{B}_{x,\,1/3}$ and $r_2 = 1/2\min\{d(x, y_2) - 1/3, 1/2 - d(x, y_2)\}$. As before $\overline{B}_{y_2,\,r_2}$ is a closed ball in $\overline{B}_{x,\,1/2} - \overline{B}_{x,\,1/3}$ and $\overline{B}_{y_1,\,r_1} \cap \overline{B}_{y_2,\,r_2} = \emptyset$. By continuing this process we get a family of closed balls $\{\overline{B}_{y_i,\,r_i}\}_{i\in I}$, where each $B_{y_i,\,r_i} = \operatorname{coz}((d_{y_i} - \overline{r}_i) \wedge 0) = \operatorname{coz} f_i$ (say). But $\cup B_{y_i,\,r_i}$ is not closed because $x \notin B_{y_i,\,r_i}$ for any i. So we have a sequence of disjoint cozero sets $\{\operatorname{coz} f_i\}$ such that $\cup\overline{\operatorname{coz} f_i}$ is not closed.

Q.E.D.

Now we give a theorem which describes some spaces for which C(X) does not have an orthogonal completion. First we give a lemma.

<u>Lemma 3.13.</u> If $X \neq \{x\}$ is connected, locally connected and metrizable, then C(X) has no orthogonal completion.

Proof. X is metrizable and connected, therefore there exists a sequence of disjoint cozero sets $\{\operatorname{coz}((d_{y_i} - \overline{r}_i) \wedge 0)\}$ such that $\overline{\cup\operatorname{coz}((d_{y_i} - \overline{r}_i) \wedge 0)}$ is not closed (Prop. 3.12.), where $\operatorname{coz}((d_{y_i} - \overline{r}_i) \wedge 0) = B_{y_i,\,r_i}$ for all i. Since cozero sets are disjoint, therefore $\{(d_{y_i} - \overline{r}_i) \wedge 0\}$ is an orthogonal family. Let $g_i = 1/r_i((d_{y_i} - \overline{r}_i) \wedge 0)$ for all i. Since $\operatorname{coz} g_i = \operatorname{coz}((d_{y_i} - \overline{r}_i) \wedge 0)$ and $\{(d_{y_i} - \overline{r}_i) \wedge 0\}$ is an orthogonal family, therefore $\{g_i\}$ is an orthogonal family. So we have an orthogonal family $\{g_i\}$ and a sequence $(y_i) \to x$, $y_i \in \operatorname{coz} g_i$. Suppose that supremum of the family $\{g_i\}$ exists and let it be g. Now g coincides with g_i on $\operatorname{coz} g_i$ and is continuous at each point of the sequence (y_i), for g to be continuous at x, g(x) should be -1, but g(x) = 0 (because $x \notin \overline{\cup \operatorname{coz}((d_{y_i} - r_i) \wedge 0)}$; this implies that $x \in$ Interior of $Z(g_i)$, for all i). This implies that $g \notin C(X)$. Hence C(X) has no orthogonal completion (by Thm. 3.9.).

Q.E.D.

<u>Theorem 3.14.</u> If X is a locally connected space which contains an open subset U of more than one point such that U is connected and $\overline{U}$ is metrizable, then C(X) does not have an orthogonal completion.

Proof. Let $B_{x,r}$ be an open ball in U with centre at x and radius r>0, further $B_{x,\, r/2}$ is another ball such that $\overline{B_{x,\, r/2}} \subset U$. $\overline{B_{x,\, r/2}}$ is closed in U, hence in $\overline{U}$. Also U is connected, therefore $\overline{U}$ is connected [10. p. 193]. Now $\overline{U}$ is connected and metrizable, therefore there exists a family of open balls $\{B_n\}$ and $\{g_n\}$ the corresponding family of functions in $\overline{B}_{x,\, r/2}$ as in (Prop. 3.12.). Let g be defined by:

$$g(x) = \begin{cases} g_i(x), & x \in \operatorname{coz} g_i \\ 0 & \text{otherwise.} \end{cases}$$

Note that $g \notin C(\bar{U})$ because g is not continuous at x. Now each B_n is open in U and U is open in X therefore each B_n is open in X. For each n, define a function $\bar{g}_n : X \to \mathbb{R}$ as follows

$$\bar{g}_n(x) = \begin{cases} g_n(x) & \text{for } x \in \bar{U} \\ 0 & \text{otherwise.} \end{cases}$$

Obviously $\bar{g}_n$ is continuous on $\sim\bar{U}$ and U. Let $p \in \bar{U} - U$. Since $\bar{B}_n$ is a closed ball in $\bar{U}$ and $\bar{U}$ is closed in X, therefore $\bar{B}_n$ is closed in X. Now $\sim\bar{B}_n$ is open in X and $p \in \sim\bar{B}_n$, since $\bar{B}_n \subset U$. This implies that $\bar{g}_n$ is continuous at p. Since coz g_n = coz $\bar{g}_n$ for all n and $\{g_n\}$ is an orthogonal family, hence $\{\bar{g}_n\}$ is an orthogonal family in X. Extend g to $\bar{g}$ as follows, define $\bar{g} : X \to \mathbb{R}$ by

$$\bar{g}(x) = \begin{cases} g(x) & \text{for } x \text{ in } \bar{U} \\ 0 & \text{otherwise.} \end{cases}$$

Now $\bar{g}$ = supremum $\{g_n\}$ (in $Q(X)$) but $\bar{g} \notin C(X)$, hence $C(X)$ is not orthogonally complete. Thus $C(X)$ has no orthogonal completion (Corollary 3.10.).

Q.E.D.

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