On the Rational Retraction Index

Philippe Paradis

Thesis submitted to the Faculty of Graduate and Postdoctoral Studies
in partial fulfillment of the requirements for the degree of Master of Science in
Mathematics [1]

Department of Mathematics and Statistics
Faculty of Science
University of Ottawa

[1]The M.Sc. program is a joint program with Carleton University, administered by the Ottawa-Carleton Institute of Mathematics and Statistics

Abstract

If X is a simply connected CW complex, then it has a unique (up to isomorphism) minimal Sullivan model $(\Lambda V, d)$. There is an important rational homotopy invariant, called the rational Lusternik–Schnirelmann of X, denoted $\mathrm{cat}_0(X)$, which has an algebraic formulation in terms of $(\Lambda V, d)$. We study another such numerical invariant called the *rational retraction index* of X, denoted $\mathrm{r}_0(X)$, which is defined in terms of $(\Lambda V, d)$ and satisfies

$$0 \leq \mathrm{r}_0(X) \leq \mathrm{cat}_0(X).$$

It was introduced by Cuvilliez et al. [4] as a tool to estimate the rational Lusternik–Schnirelmann category of the total space of a fibration.

In this thesis we compute the rational retraction index on a range of rationally elliptic spaces, including for example spheres, complex projective space, the biquotient $\mathrm{Sp}(1) \setminus \mathrm{Sp}(3) \,/\, \mathrm{Sp}(1) \times \mathrm{Sp}(1)$, the homogeneous space $\mathrm{Sp}(3)/\mathrm{U}(3)$ and products of these. In particular, we focus on formal spaces and formulate a conjecture to answer a question posed in the original article of Cuvilliez et al.,

"If X is formal, what invariant of the algebra $H^*(X;\mathbb{Q})$ is $\mathrm{r}_0(X)$?"

Acknowledgements

I would like to express my sincere gratitude to my thesis supervisor, Paul-Eugène Parent, for his excellent guidance and helpfulness, as well as for being a great and inspiring teacher for many years. I would like to thank Barry Jessup for his advice and very helpful and detailed comments which undoubtedly made my thesis much better. I would also like to acknowledge NSERC and the FQRNT for financial support.

Contents

Introduction

This thesis is centered around the branch of mathematics called *rational homotopy theory*. There exists a notion of equivalence between two simply connected topological spaces X and Y called "rational homotopy equivalence," weaker than homotopy equivalence, denoted

$$X \simeq_{\mathbb{Q}} Y.$$

We say that $X \simeq_{\mathbb{Q}} Y$ if there exists a continuous map $f : X \to Y$ which induces an isomorphism between higher rational homotopy groups ($n \geq 2$)

$$\pi_n(f) \otimes \mathbb{Q} : \pi_n(X) \otimes \mathbb{Q} \xrightarrow{\cong} \pi_n(Y) \otimes \mathbb{Q}.$$

Rational homotopy equivalence ignores the torsion subgroup in homotopy groups, because tensoring an abelian group G by $\mathbb{Q}$ corresponds to quotienting out by the torsion subgroup of G, in addition to turning G into a vector space over the rationals. This means that some homotopy information is lost through rational homotopy equivalence and this is why it is weaker than homotopy equivalence.

On the other hand, rational homotopy equivalence has the considerable *advantage* of simplifying the picture sufficiently such that topological spaces can be completely substituted by a corresponding "algebraic object" which encodes all the rational homotopy information of the space. This is the bijective correspondence between "minimal Sullivan models" and "rational homotopy types" and it is key to the power of rational homotopy theory.

This algebraic counterpart to a topological space, called a minimal Sullivan model, was discovered by Sullivan [36], who in turn proved the bijective correspondence (restricting ourselves to spaces which are simply connected CW complexes with rational homology of finite type and to minimal Sullivan models which are 1-connected and of finite type)

$$\left\{\begin{array}{c}\text{rational homotopy}\\ \text{types}\end{array}\right\} \overset{\cong}{\longleftrightarrow} \left\{\begin{array}{c}\text{isomorphism classes of}\\ \text{minimal Sullivan models}\\ \text{over } \mathbb{Q}\end{array}\right\}$$

The minimal Sullivan model associated to a space X via this correspondence is a commutative differential graded algebra of the form $(\Lambda V, d)$, where ΛV denotes the "free commutative graded algebra" on the graded rational vector space $V = \{V^n\}_{n\geq 2}$. The minimal Sullivan model $(\Lambda V, d)$ is related to the space X by the properties $(n \geq 0)$

$$H^n(\Lambda V, d) \cong H^n(X;\mathbb{Q}) \qquad \text{and} \qquad \mathrm{Hom}_{\mathbb{Z}}(V^n, \mathbb{Q}) \cong \pi_n(X) \otimes \mathbb{Q}.$$

All the rational homotopy information of a space is encoded in its corresponding minimal Sullivan model, which is a particularly manageable and well-behaved type of algebraic object, explaining the computational power of rational homotopy theory. In fact, since a minimal Sullivan model is "free commutative" over a set of generators, its multiplicative structure is particularly straightforward. Moreover, a minimal Sullivan model is "well-behaved" in the sense that the ground ring is the field $\mathbb{Q}$, which means that the model is comprised of a sequence of rational vector spaces connected by linear maps, reducing many computations to a linear algebra problem over the rationals.

A homotopy invariant of particular interest in rational homotopy is the *Lusternik–Schnirelmann category* of a space X, denoted $\mathrm{cat}(X)$, defined to be the least integer k such that X can be covered by $k+1$ open sets, each contractible in X. There exists a "rational approximation" to $\mathrm{cat}(X)$, formulated in terms of the minimal Sullivan model of X, which we call the *rational Lusternik–Schnirelmann category* of X and denote $\mathrm{cat}_0(X)$. The rational L.-S. category $\mathrm{cat}_0(X)$ approximates $\mathrm{cat}(X)$ in the sense that

$$\mathrm{cat}_0(X) \leq \mathrm{cat}(X).$$

The goal of this thesis is to investigate a rational homotopy invariant called the *rational retraction index* of a space X and denoted $\mathrm{r}_0(X)$. It is very closely related to the rational L.-S. category and it satisfies

$$0 \leq \mathrm{r}_0(X) \leq \mathrm{cat}_0(X).$$

In the original article where Cuvilliez et al. [4] introduced the rational retraction index, it was used as a tool to help provide a better upper bound on the rational L.-S. category of the total space of a fibration.

It is known that $\mathrm{r}_0(S^n) = 1$ and that $\mathrm{r}_0(\mathbb{C}P^n) = 1$. The goal of this thesis is to understand how to compute $\mathrm{r}_0(X)$ for many other spaces X, where the computations might be more involved. Each chapter, from chapter 1 through chapter 3, is built towards the idea of reaching this goal. We limit ourselves to computing $\mathrm{r}_0(X)$ for examples where X is a "rationally elliptic space," which means that $H^*(X;\mathbb{Q})$ and $\pi_*(X) \otimes \mathbb{Q}$ are both finite dimensional.

In addition to computing $\mathrm{r}_0(X)$ for new examples of spaces X, we also try to answer a question posed in the original article, namely

"If X is formal, what invariant of the algebra $H^*(X;\mathbb{Q})$ is $\mathrm{r}_0(X)$?"

where a *formal space* X is a space whose minimal Sullivan model is completely determined by the cohomology algebra $H^*(X;\mathbb{Q})$.

We formulate at the end of this thesis a conjecture that, if verified, would answer this question.

In chapter 1, we fix the notation and terminology and recall basic concepts and definitions related to homotopy theory, including CW complexes and homotopy groups.

In chapter 2, we explain all the algebraic machinery required for rational homotopy theory. We also give examples of how to find the minimal Sullivan model of a topological space.

In chapter 3, we focus on the rational Lusternik–Schnirelmann category of a space, which is defined in terms of the minimal Sullivan model $(\Lambda V, d)$ of X, as follows. For any integer $m \geq 1$, the notation $\Lambda^{>m}V$ stands for the subspace consisting of all words in ΛV of length greater than m. It is a differential ideal of ΛV and so there is a projection map onto the quotient $\Lambda V/\Lambda^{>m}V$

$$\pi_m : (\Lambda V, d) \longrightarrow (\Lambda V/\Lambda^{>m}V, \bar{d}).$$

It will be shown that it is always possible to add generators in $(\Lambda V, d)$ and to extend the differential d to those new generators to obtain a relative Sullivan model of the form $(\Lambda(V \oplus V'), d')$ such that there exists a map

$$\varphi_m : (\Lambda(V \oplus V'), d') \longrightarrow (\Lambda V/\Lambda^{>m}V, \bar{d})$$

satisfying

(i) $\varphi_m \circ \lambda_m = \pi_m$, where $\lambda_m : (\Lambda V, d) \hookrightarrow (\Lambda(V \oplus V'), d')$ denotes the inclusion,

(ii) φ_m induces an isomorphism in cohomology.

The *rational Lusternik–Schnirelmann category* of X, denoted $\mathrm{cat}_0(X)$, is the least positive integer m such that the inclusion λ_m admits a retraction $\rho : (\Lambda(V \oplus V'), d') \to (\Lambda V, d)$, where a retraction ρ is a map of commutative differential graded algebras satisfying $\rho \circ \lambda_m = \mathrm{id}_{(\Lambda V, d)}$.

One of the most important properties of $\mathrm{cat}_0(X)$ is the additivity formula (for simply connected CW complexes with rational homology of finite type)

$$\mathrm{cat}_0(X \times Y) = \mathrm{cat}_0(X) + \mathrm{cat}_0(Y),$$

which is false in general for $\mathrm{cat}(X)$.

There exist a few other invariants that can be computed from a minimal Sullivan model which "approximate" the L.-S. category of X. Those invariants are easier to compute and provide a good approximation for many important classes of spaces. In particular, for rationally elliptic spaces, $\mathrm{cat}_0(X)$ reduces to another invariant $\mathrm{e}_0(X)$, the "rational Toomer invariant." Also, in the case of formal spaces, $\mathrm{cat}_0(X)$ reduces to the computation of the cohomology cup-length.

Finally, in chapter 4, we introduce the rational retraction index, $\mathrm{r}_0(X)$. The definition of $\mathrm{r}_0(X)$ is related to that of $\mathrm{cat}_0(X)$, such that it is necessary to first compute $\mathrm{cat}_0(X)$ before computing $\mathrm{r}_0(X)$ becomes possible. We start by listing the basic properties of $\mathrm{r}_0(X)$. Afterwards, we show detailed computations of $\mathrm{r}_0(X)$ for different examples of rationally elliptic spaces X. Finally, we list all spaces for which the rational retraction index is known to us and we formulate our conjecture for formal spaces.

Chapter 1

Topological preliminaries

In this section, we will fix some notation from topology and review concepts from homotopy theory. It is assumed that the reader is already familiar with the fundamental group, singular homology and singular cohomology.

1.1 Notation

Throughout this thesis, a *space* will always refer to a normal Hausdorff path-connected topological space.

The notation S^n stands for the n-sphere $S^n = \{x \in \mathbb{R}^{n+1} \mid ||x|| = 1\}$. In particular, $S^0 = \{-1, 1\}$.

The notation D^n stands for the n-disk $D^n = \{x \in \mathbb{R}^n \mid ||x|| \leq 1\}$.

The notation I stands for the unit interval $[0, 1] \subset \mathbb{R}$.

We denote the disjoint union of two spaces X and Y by $X \coprod Y$.

1.2 Basic definitions

A *pointed space* is a pair (X, x_0) where X is a space and $x_0 \in X$. We call the point x_0 the *base point* of (X, x_0). We say that a *map of pointed spaces* $(X, x_0) \to (Y, y_0)$

is a continuous map $f : X \to Y$ such that $f(x_0) = y_0$.

Consider two continuous maps $f, g : X \to Y$. We say that f is *homotopic* to g (denoted $f \sim g$) if there exists a continuous map $H : X \times I \to Y$ such that $H(x, 0) = f(x)$ and $H(x, 1) = g(x)$ for all $x \in X$. In this case, H is called a *homotopy* from f to g.

A continuous map $f : X \to Y$ is called a *homotopy equivalence* if there exists a continuous map $g : Y \to X$ such that $gf \sim \mathrm{id}_X$ and $fg \sim \mathrm{id}_Y$. In this case, we say that X and Y have the same *homotopy type*.

If $f, g : (X, x_0) \to (Y, y_0)$ are two maps of pointed spaces, then we say that f is *pointed homotopic* to g if there exists a continuous map $H : X \times I \to Y$ such that $H(x, 0) = f(x)$, $H(x, 1) = g(x)$ and $H(x_0, t) = y_0$ for all $x \in X$, $t \in I$. In this case, H is called a *pointed homotopy* from f to g. Two maps being pointed homotopic is an equivalence relation on the set of all maps of pointed spaces $(X, x_0) \to (Y, y_0)$ and we denote by $[(X, x_0), (Y, y_0)]$ the set of all equivalence classes under this relation.

A subspace U of a topological space X is said to be *contractible in* X if the inclusion map $i : U \hookrightarrow X$ is homotopic to a constant map $U \to \{x_0\}$ for some point $x_0 \in X$.

The *suspension* of a space X, denoted ΣX, is the quotient space $(X \times I)/\sim$, where $(x, 0) \sim (x', 0)$ and $(x, 1) \sim (x', 1)$ for any $x, x' \in X$, equipped with the quotient topology.

The *suspension* of a pointed space (X, x_0), denoted $\Sigma(X, x_0)$, or just ΣX by abuse of notation, is the quotient space $(X \times I)/\sim$, where $(x, 0) \sim (x', 0)$, $(x, 1) \sim (x', 1)$ for any $x, x' \in X$ and $(x_0, t) \sim (x_0, t')$ for any $t, t' \in I$.

Given two spaces X and Y, with a continuous map $f : A \subset Y \to X$, we define the *adjunction space* (also "the space obtained by attaching Y to X along f"), denoted $X \cup_f Y$, to be the quotient

$$X \cup_f Y = \left(X \coprod Y\right) \Big/ \sim$$

under the relation $a \sim f(a)$ (for any $a \in A$), equipped with the quotient topology. The map f is called the *attaching map*.

1.3 CW complexes

In this thesis, we will always be working with spaces which have the homotopy type of a CW complex.

Definition 1.1. A *relative CW complex* (X, A) is a space X and a space A such that X is the union $X = \bigcup_n X^n$, where

(1) $X^0 = A \coprod Y$ where Y is a discrete space,

(2) For each $n \geq 0$,
$$X^{n+1} = X^n \cup_f \left(\coprod_{\alpha \in J} D_\alpha^{n+1} \right)$$
for some attaching map
$$f : \coprod_{\alpha \in J} S_\alpha^n \to X^n$$
(viewing S^n as a subset of D^{n+1} under the correspondence $\partial D^{n+1} \cong S^n$).

where (X, A) is given the "weak topology", meaning that $U \subset X$ is closed if and only if $U \cap X^n$ is closed for every $n \geq 0$.

Definition 1.2. (1) The subspace X^n is called the *n-skeleton* of (X, A).

(2) If $A = \emptyset$, then $(X, \emptyset)$ is called a *CW complex*.

(3) When a space is a relative CW complex, we often say that it "admits a cell decomposition."

(4) For $n \geq 1$, the image e_α^n of D_α^n in the adjunction space is called a *cell of dimension n* (or *n-cell*). A *cell of dimension* 0 is the image of a point of Y in the adjunction space.

(5) If there exists a cell decomposition such that for some $n \geq 0$, $X = X^n$, then the relative CW complex (X, A) is said to be *finite dimensional*. In this case, the least integer $n \geq 0$ such that X admits a cell decomposition with $X = X^n$ is called the *dimension* of X.

Example 1.3. (1) *The circle S^1.*

Consider the disk $D^1 = [-1, 1]$. Take $X^0 = \{\star\}$, a one-point set. Consider an attaching map

$$f : \partial D^1 = \{-1, 1\} \longrightarrow \{\star\}.$$

We can see that the corresponding 1-skeleton is $X^1 = D^1 \cup_f \star \cong S^1$. Hence, S^1 is a CW complex of dimension at most 1, with one cell of dimension 1 and one cell of dimension 0.

(2) *The disk D^2.*

Build X^0 and X^1 as in the previous example and define the attaching map $f : \partial D^2 \cong S^1 \to X^1 = S^1$ to be the identity map on S^1. The corresponding 2-skeleton is

$$X^2 = D^2 \cup_{\mathrm{id}} X^1 \cong D^2.$$

This exhibits the disk D^2 as a finite dimensional CW complex with one cell in each of the dimensions 0, 1 and 2.

(3) More generally, the n-sphere and the n-disk are CW complexes of dimension n.

Remark. The category of spaces having the homotopy type of CW complexes is quite vast as it encompasses all finite dimensional smooth manifolds (see [31] for the proof).

1.4 The complex projective space

There is a space that we will encounter often in examples, which we call the n-*dimensional complex projective space* and is denoted by $\mathbb{C}P^n$.

The space $\mathbb{C}P^n$ can be pictured as the collection of all lines through the origin in $\mathbb{C}^{n+1}$. Formally, we construct $\mathbb{C}P^n$ as the quotient $(\mathbb{C}^{n+1}\backslash\{0\})/\sim$, under the relation $(z_1, \dots, z_{n+1}) \sim (\lambda z_1, \dots, \lambda z_{n+1})$ for any $\lambda \in \mathbb{C}\backslash\{0\}$.

The space $\mathbb{C}P^n$ is a CW complex of dimension $2n$, with one cell in each even dimension between 0 and $2n$, which we write as $\mathbb{C}P^n = e_0 \cup e_2 \cup \dots \cup e_{2n}$.

Moreover, $\mathbb{C}P^n$ has a particularly simple singular homology sequence. If we fix a commutative ring R, then

$$H_i(\mathbb{C}P^n; R) \cong \begin{cases} R & \text{if } i = 0,\, 2,\, 4,\, \dots,\, 2n, \\ 0 & \text{otherwise.} \end{cases}$$

Its singular cohomology is isomorphic to its singular homology and the multiplicative structure is the truncated polynomial ring $H^*(\mathbb{C}P^n; R) \cong R[x]/(x^{n+1})$, with x in degree 2.

The *infinite dimensional complex projective space* $\mathbb{C}P^\infty = \bigcup_{k=0}^{\infty} \mathbb{C}P^k$ is an infinite dimensional CW complex with one cell in each even dimension. Its cohomology is the polynomial ring $R[x]$ with x in degree 2.

1.5 Homotopy groups

In this section, we define higher homotopy groups.

Let X be any space, fix $x_0 \in X$ and fix $n \geq 1$. Fix some arbitrary base point in the n-sphere that we shall denote by $\star$. We now have pointed spaces (X, x_0) and $(S^n, \star)$.

Definition 1.4. The *n-th homotopy group* of X with base point x_0 is defined as the set of all pointed homotopy equivalence classes

$$\pi_n(X, x_0) = [(S^n, \star), (X, x_0)].$$

Remark. In the case $n = 0$, the definition of $\pi_0(X, x_0)$ still applies verbatim, except that $\pi_0(X, x_0)$ is not a group in general. We call $\pi_0(X, x_0)$ the *set of path-connected components* of X.

For $n \geq 1$, we can define a group operation on $\pi_n(X, x_0)$ as follows. Consider $[f], [g] \in \pi_n(X, x_0)$ and use the homeomorphism $\Sigma(S^{n-1}, \star) \cong (S^n, \star)$ to define $[f]+[g]$ to be the map represented by $f + g : (\Sigma S^{n-1}, \star) \to (X, x_0)$, where

$$(f+g)(x,t) = \begin{cases} f(x, 2t) & \text{if } 0 \leq t \leq \frac{1}{2}, \\ g(x, 2t-1) & \text{if } \frac{1}{2} \leq t \leq 1 \end{cases}$$

for any $x \in S^{n-1}$ and $t \in I$.

This operation is well defined and makes $\pi_n(X, x_0)$ into a group.

Remarks. (1) We will always assume a space to be path-connected. Since $\pi_n(X, x_0) \cong \pi_n(X, y_0)$ if x_0 and y_0 are in the same path-component, it follows that we can ignore the base point. Thus, from now on we will always write $\pi_n(X)$ for the n-th homotopy group of X, ignoring the choice of a base point.

(2) For $n \geq 2$, $\pi_n(X)$ is always an abelian group (see [20], Chapter 4).

(3) If $\varphi : X \to Y$ is a continuous map, then there is an induced map $\pi_n(\varphi) : \pi_n(X) \to \pi_n(Y)$, defined by $[f] \mapsto [\varphi \circ f]$. In fact, π_n is a functor from the category of topological spaces and continuous maps to the category of groups and group homomorphisms (for $n \geq 1$).

1.6 Homotopy theory

Basic definitions.

(1) Suppose that we are working over a field F. The singular homology $H_*(X;F)$ of a space is said to be of *finite type* if $H_*(X;F)$ is finite dimensional in every degree. Similarly, the singular cohomology of a space is of *finite type* if it is finite dimensional in every degree.

(2) A space X is *simply connected* if it is path-connected and $\pi_1(X) = 0$.

(3) More generally, a space X is said to be 0-*connected* if it is path-connected and it is said to be r-*connected* ($r \geq 1$) if it is path-connected and $\pi_i(X) = 0$ for $1 \leq i \leq r$. In particular, 1-connected just means simply connected.

The following result due to Whitehead [40] justifies why we restrict ourselves to spaces having the homotopy type of a CW complex.

Theorem 1.5 (Whitehead)**.** *Let X and Y be path-connected CW complexes and let $f : X \to Y$ be a continuous map such that the induced homomorphism*

$$\pi_n(f) : \pi_n(X) \to \pi_n(Y)$$

is an isomorphism for every $n \geq 1$. Then, f is a homotopy equivalence.

The original article contains a proof for finite dimensional CW complexes. For an arbitrary CW complex, see for example the proof in [20], Theorem 4.5.

Next, we introduce two types of spaces which are particularly useful in homotopy theory.

Definition 1.6. If G is a group and n a positive integer, then a space X is said to be an *Eilenberg–Mac Lane space* $K(G, n)$ if $\pi_n(X) \cong G$ and $\pi_i(X)$ is trivial for $i \neq n$.

Examples. The sphere S^1 is a $K(\mathbb{Z},1)$ space. The infinite complex projective space $\mathbb{C}P^\infty$ is a $K(\mathbb{Z},2)$ space. For any group G, there exists a CW complex which is a $K(G,1)$ space. Moreover, for any abelian group G and any $n>1$, there exists a CW complex which is $K(G,n)$ space. See [6] for more details.

Definition 1.7. If G is an abelian group and n a positive integer, then a space X is called a *Moore space* $M(G,n)$ if it satisfies $H_n(X;\mathbb{Z})\cong G$ and $\widetilde{H}_i(X;\mathbb{Z})=0$ for $i\neq n$.

Examples. The n-sphere is a Moore space $M(\mathbb{Z},n)$ for $n\geq 1$.

Chapter 2

Algebraic preliminaries

In this chapter we introduce the algebraic structures that are required to understand rational homotopy theory. Throughout this section, the ground field will be $\mathbb{Q}$.

Recall that every space is assumed to have the homotopy type of a CW complex.

2.1 Graded vector spaces

Definition. A *graded vector space* is a family $V = \{V^i\}_{i\geq 0}$ of vector spaces (over the field $\mathbb{Q}$), indexed by the non-negative integers. Elements $v \in V^i$ are said to have *degree* i and we denote this by $|v| = i$. A graded vector space V is *concentrated in degree* $i \in I$ ($I \subset \mathbb{N}$) if $V^i = 0$ for every $i \notin I$, and in this case, we would write $V = \{V^i\}_{i\in I}$. A graded vector space V is said to be of *finite type* if each V^i is finite dimensional. A graded vector space V is *finite dimensional* if each V^i is finite dimensional and if $V^i = 0$ for all but finitely many i's.

Now, we will need to introduce some notation for graded vector spaces. Define V^+ to be the graded vector space $\{V^i\}_{i\geq 1}$. Similarly, if $k \geq 0$, define $V^{\geq k}$ to be the graded vector space $\{V^i\}_{i\geq k}$. The graded vector spaces $V^{>k}$, $V^{\leq k}$ and $V^{<k}$ are defined analogously. We also define $V^{\text{even}} = \{V^{2i}\}_{i\geq 0}$ and $V^{\text{odd}} = \{V^{2i+1}\}_{i\geq 0}$.

Examples. (i) The field of rational numbers $\mathbb{Q}$ can be regarded as a graded vector space concentrated in degree 0.
(ii) The rational singular homology $H_n(X;\mathbb{Q})$ of a topological space X is a sequence of vector spaces over $\mathbb{Q}$ and hence is a graded vector space.

Remark. It is important to note that graded vector spaces lack additional graded structure. For example, given a graded vector space V, there is no addition defined for an element of V^i and an element of V^j if $i \neq j$. There is an alternative point of view where the graded vector space is seen as the direct sum $V = \bigoplus V^i$ (and hence addition of vectors is always formally defined). This is also why elements of V^i are referred to as *homogeneous elements of degree* i (in contrast with a formal sum of elements of different degrees).

Basic constructions. Typical constructions from linear algebra usually carry over to graded linear algebra in a natural way. We illustrate many useful constructions that will come up later.

(i) A *subspace* of a graded vector space $U \subset V$ is a graded vector space $\{U^i\}_{i\geq 0}$ such that U^i is a subspace of V^i for each $i \geq 0$.

(ii) Given a subspace U of a graded vector space V, the *quotient* of V by U is the graded vector space $V/U = \{V^i/U^i\}_{i\geq 0}$.

(iii) Given two graded vector spaces V and W, the *direct sum* of V and W is the graded vector space $V \oplus W = \{V^i \oplus W^i\}_{i\geq 0}$.

(iv) Given two graded vector spaces V and W, the *tensor product* of V and W is the graded vector space

$$V \otimes W = \left\{ \bigoplus_{j+k=i} V^j \otimes W^k \right\}_{i\geq 0}.$$

In particular, this means that an element $v \otimes w \in V^j \otimes W^k$ has degree $|v \otimes w| = j + k$.

Definition. A *linear map of degree* n from a graded vector space V to a graded vector space W is a family of linear maps $f_i : V^i \to W^{i+n}$ (for each $i \geq 0$).

Definition. A *differential* on a graded vector space V is a linear map $d : V \to V$ of degree 1 such that $d_{n+1} \circ d_n = 0$ for any $n \geq 0$.

Definition. Any graded vector space V equipped with a differential d has an associated *cohomology algebra* $H(V, d)$ defined by $H^n(V, d) = \ker d_n / \operatorname{Im} d_{n-1}$ for each $n \geq 1$ and by $H^0(V, d) = \ker d_0$. The elements of $\ker d_n$ are called *n-cocycles* and the elements of $\operatorname{Im} d_n$ are called *n-coboundaries*.

2.2 Commutative differential graded algebras

Definition. A *graded algebra* A is a graded vector space equipped with a linear map $A \otimes A \to A$ of degree zero, called multiplication and denoted by $x \otimes y \mapsto xy$, together with an identity element $1 \in A^0$, such that for all $x, y, z \in A$,

$$(xy)z = x(yz) \qquad \text{and} \qquad 1x = x1 = x.$$

A *morphism* of graded algebras $\varphi : A \to B$ is a linear map of degree zero such that $\varphi(xy) = \varphi(x)\varphi(y)$ for all $x, y \in A$ and such that $\varphi(1) = 1$.

A graded algebra A is *commutative* if

$$xy = (-1)^{|x||y|} yx \quad \text{for all homogeneous elements } x, y \in A.$$

In this case we will call A a "cga" (short for commutative graded algebra).

A *derivation of degree* k in a graded algebra A is a morphism $d : A \to A$ of degree k such that

$$d(xy) = (dx)y + (-1)^{k|x|}x(dy) \qquad \text{for all } x, y \in A.$$

This is essentially the graded version of the Leibniz product rule from differential calculus.

If A is a graded algebra, then a *left ideal* I of A is a graded subspace of A such that if $x \in A$ and $y \in I$, then $xy \in I$. A *right ideal* is defined analogously and we simply refer to two-sided ideal as *ideals*.

Example. Given two graded algebras A and B, the tensor product $A \otimes B$ admits a graded algebra structure. We define the multiplication as

$$(a \otimes b) \cdot (a' \otimes b') = (-1)^{|b||a'|} aa' \otimes bb' \qquad a \in A, \; b \in B.$$

Note: The sign convention ensures that the tensor product of two commutative graded algebras is still commutative.

Definition. A *commutative differential graded algebra* (*cdga* for short) is a commutative graded algebra A equipped with a differential $d : A \to A$ which is also a derivation. We usually denote the pair as (A, d). A *morphism* of cdga's $f : (A, d) \to (B, d)$ is a graded algebra morphism $f : A \to B$ such that $fd = df$.

Whenever (A, d) is a cdga, $\operatorname{Im} d$ is an ideal of the subalgebra $\ker d$, so it follows that its cohomology $H(A, d)$ is also a cga with the obvious multiplication (that is if $a, a' \in A$ are two cocycles, then define $[a][a'] = [aa']$).

Definition. A morphism $f : (A, d) \to (B, d)$ between to cdga's is called a *quasi-isomorphism* if the induced map in cohomology $H(f) : H(A, d) \to H(B, d)$, $[x] \mapsto$

$[f(x)]$ is an isomorphism. We denote the arrow by $(A,d) \xrightarrow{\simeq} (B,d)$ to indicate a quasi-isomorphism.

Example. Given two graded cdga's (A,d) and (B,d), the tensor product $A \otimes B$ admits a cdga structure with the multiplication given earlier and with differential

$$d(a \otimes b) = (da) \otimes b + (-1)^{|a|} a \otimes (db) \qquad a \in A,\ b \in B.$$

2.3 Free commutative graded algebras

Let V be a graded vector space. The *tensor algebra* TV is the graded vector space

$$TV = \bigoplus_{k=0}^{\infty} T^k V \quad \text{where} \quad T^0 V = \mathbb{Q},\ T^k V = \underbrace{V \otimes \cdots \otimes V}_{k \text{ times}} \quad (k \geq 1).$$

It is a graded algebra. Multiplication is defined as follows: if $x \in T^k V$, $y \in T^l V$, then $xy = x \otimes y \in T^{k+l} V$. The identity is $1 \in T^0 V$. Elements in $T^k V$ are said to have *word length* k.

Next, consider the ideal $I \subset TV$ generated by all elements of the form $x \otimes y - (-1)^{|x||y|} y \otimes x$ (for all $x, y \in V$).

Definition. The following quotient

$$\Lambda V = TV/I$$

is a commutative graded algebra, called the *free commutative graded algebra* (free cga) on V.

Denote by $\Lambda^k V$ the image of the natural projection $T^k V \longrightarrow \Lambda V$. Elements of $\Lambda^k V$ are said to have *word length* k.

Next, we introduce a very useful proposition.

Proposition 2.1. *Let V be a graded vector space and A a commutative graded algebra.*

(i) Any linear map $V \to V$ of degree k extends to a unique derivation $\Lambda V \to \Lambda V$ of degree k.

(ii) Any linear map $V \to A$ of degree 0 extends to a unique morphism of cga's $\Lambda V \to A$.

Remarks. (i) Whereas $\Lambda^k V$ stands for all elements of word length k, it is not to be confused with the notation $(\Lambda V)^k$, which stands for all elements in ΛV of degree k.

(ii) We often write $\Lambda(v_1, v_2, \ldots)$ instead of ΛV if $\{v_1, v_2, \ldots\}$ is a basis of V.

(iii) Consider $v \in V^k$. If k is odd, then $v^2 = 0$. This follows from the commutativity of ΛV, which says that $v^2 = (-1)^{k^2} v^2 = -v^2$. Hence $2v^2 = 0$, and so $v^2 = 0$.

(iv) If V and W are graded vector spaces, then there is a canonical isomorphism of cga's

$$\Lambda(V \oplus W) \cong \Lambda V \otimes \Lambda W$$

specified by $v \mapsto v \otimes 1$ for any $v \in V$ and $w \mapsto 1 \otimes w$ for any $w \in W$. We will frequently make implicit use of this isomorphism.

(v) Recall that $V^{\text{even}} = \{V^{2n}\}_{n \geq 0}$ and $V^{\text{odd}} = \{V^{2n+1}\}_{n \geq 0}$. It follows that

$$\Lambda V \cong \text{Symm}(V^{\text{even}}) \otimes \text{Exterior}(V^{\text{odd}})$$

where $\text{Symm}(V^{\text{even}})$ is the symmetric algebra on V^{even} and $\text{Exterior}(V^{\text{odd}})$ is the exterior algebra on V^{odd}.

Note. *The differential on ΛV.*

If there is a differential d defined on ΛV, then d is completely characterized by its values on V. Indeed, the restriction of d to V is a linear map $V \to \Lambda V$ of degree

1, so it extends uniquely to a derivation $\Lambda V \to \Lambda V$ of degree 1 by Theorem 2.1, determining the values of d on ΛV.

Definition 2.2. If such a map d is provided, we call the cdga $(\Lambda V, d)$ the *free commutative differential graded algebra* (*free cdga*) on V, with differential d.

Remark. It is not true that any linear map $d : V \to \Lambda V$ of degree 1 extends to a differential on ΛV. It is necessary to check that the extension $d : \Lambda V \to \Lambda V$ satisfies $d^2 = 0$. However, it is easy to check that if d is a derivative of odd degree, d^2 is a derivation of even degree, and so one would only need to verify that $d^2 = 0$ on V.

If $(\Lambda V, d)$ is a free cdga, then the differential d can always be written as a sum $d = d_0 + d_1 + d_2 + \cdots$ of derivations d_i of degree 1, where d_i increases the length of a word by exactly i. If we restrict d_i to $\Lambda^k V$, this gives rise to linear maps $d_i : \Lambda^k V \to \Lambda^{k+i} V$.

Definition 2.3. Let $(\Lambda V, d)$ be a free cdga, then we call d_0 the *linear part* of the differential and d_1 the *quadratic part* of the differential. We say that a free cdga $(\Lambda V, d)$ is *quadratic* if $d = d_1$.

Example 2.4. Consider a graded vector space V with basis $\{a, b\}$ such that $a \in V^2$ and $b \in V^5$. Now define a linear map d (of degree 1) by $da = 0$ and $db = a^3$. It follows that d extends uniquely to a derivation $d : \Lambda V \to \Lambda V$. Moreover, since $d(da) = d(db) = 0$, as indicated in the remark above, we can deduce that $d^2 = 0$ on ΛV by induction on the word length together with the Leibniz product rule. This makes d into a differential on ΛV, as desired, so that $(\Lambda V, d)$ is a free cdga.

Because $|b| = 5$ is odd, it follows that $b^2 = 0$. Therefore, a basis for $(\Lambda V, d)$ would be

$$1,\ b,\ a,\ ab,\ a^2,\ a^2 b,\ a^3,\ a^3 b,\ \ldots$$

and the differential on such a basis can easily be computed using the Leibniz product rule. We get

$$d(1) = 0,\ d(b) = a^3,\ da = 0,\ d(ab) = a^4,\ d(a^2) = 0,\ d(a^2b) = a^5,\ d(a^3) = 0,\ \ldots$$

It is now possible to calculate the cohomology $H(\Lambda(a,b),d)$. We see that a and a^2 are cocycles which are not coboundaries, but for $n \geq 3$, every cocycle a^n is a coboundary, since $d(a^{n-3}b) = a^n$. Moreover, the unit $1 \in \Lambda(a,b)$ is always a cocycle which is not a coboundary, but other than that there are no other cocycles in $(\Lambda(a,b),d)$. So,

$$H(\Lambda(a,b),d) = \mathbb{Q}\cdot[1] \oplus \mathbb{Q}\cdot[a] \oplus \mathbb{Q}\cdot[a^2].$$

2.4 Sullivan models

Now, we introduce one of the most important concepts in rational homotopy theory, the notion of a Sullivan algebra (or Sullivan model). Basically, Sullivan algebras are the "minimal models" of cdga's, in the sense that all the homotopy information in a cdga can be encoded in a corresponding (unique up to isomorphism) Sullivan algebra.

We will see that two cdga's are considered to be equivalent if there exists a morphism between them that induces an isomorphism in cohomology. Also, as long as a cdga has trivial homology in degree 0, then we will see (Theorem 2.7) that there exists a unique minimal Sullivan algebra (up to isomorphism) equivalent to this cdga.

Definition. A *Sullivan algebra* is a free commutative differential graded algebra $(\Lambda V, d)$ such that

(i) $V = \{V^p\}_{p\geq 1}$,

(ii) $V = \bigcup_{k=0}^{\infty} V(k)$, where $V(0) \subset V(1) \subset \cdots$ is an increasing sequence of graded subspaces such that

$$d|_{V(0)} = 0 \qquad \text{and} \qquad d : V(k) \longrightarrow \Lambda V(k-1), \quad k \geq 1.$$

A Sullivan algebra is *minimal* if $d(V) \subset \Lambda^{\geq 2}V$.

Remarks. (1) If $V^1 = 0$ and $d(V) \subset \Lambda^{\geq 2}V$, then $(\Lambda V, d)$ is automatically a minimal Sullivan algebra. Indeed, in this case we can define $V(k) = V^{\leq k}$ and property (ii) will be satisfied, because if d increases word length by at least 1 (and degree by 1) and every element of V has degree 2 or greater, then clearly $d(V^{\leq k}) \subset \Lambda(V^{\leq k-1})$.
(2) A Sullivan algebra is minimal if and only if $d_0 = 0$, where d_0 is the linear part of d.

Example. Consider the algebra $(\Lambda(x, y, z), d)$ with $|x| = |y| = |z| = 1$ and $dx = yz$, $dy = zx$ and $dz = xy$. In this case $(\Lambda(x, y, z), d)$ is *not* a Sullivan algebra because it fails to satisfy condition (ii).

Example. The algebra $(\Lambda(x, y), d)$ with $|x| = 2$, $|y| = 5$, $dx = 0$ and $dy = x^3$ is an example of a minimal Sullivan algebra.

Example. *The quadratic part of a minimal Sullivan algebra.*

If $(\Lambda V, d)$ is a minimal Sullivan algebra, then $(\Lambda V, d_1)$ is also a minimal Sullivan algebra. Indeed, because $(\Lambda V, d)$ is minimal, the differential d decomposes as $d = d_1 + d_2 + d_3 + \cdots$ (since $d_0 = 0$). Now, because d_1 increases word length by 1, it follows that d_1^2 increases word length by exactly 2. However, a quick argument shows that $d^2 - d_1^2$ increases word length by at least 3. Since $d^2 = 0$, this implies that $d_1^2 = 0$, since d_1^2 cannot both increase word length by exactly 2 and by more than 2 at the same time, unless it is the zero map.

Hence, whenever $(\Lambda V, d)$ is minimal, the quadratic part d_1 of d is a differential. In this case, $(\Lambda V, d_1)$ is clearly a minimal Sullivan algebra.

Those minimal Sullivan algebras whose differential is entirely quadratic constitute an important class of Sullivan algebras for our purposes, since they will turn out to be the trivial case when it comes to computing the rational retraction index, as we will see in chapter 4. Therefore, we make the following

Definition. A minimal Sullivan algebra $(\Lambda V, d)$ is said to be *quadratic* if $d = d_1$.

Sullivan algebras are particularly tractable types of cdga's with the property that for every cdga (A, d) with $H^0(A, d) = \mathbb{Q}$, there is an associated Sullivan algebra connected to (A, d) by a quasi-isomorphism.

Definition 2.5. Let (A, d) be a cdga. A *Sullivan model* for (A, d) is a quasi-isomorphism $m : (\Lambda V, d) \xrightarrow{\simeq} (A, d)$ from a Sullivan algebra $(\Lambda V, d)$.

Remark. Even though a Sullivan model is a map $m : (\Lambda V, d) \xrightarrow{\simeq} (A, d)$, we often just refer to $(\Lambda V, d)$ as the Sullivan model, by abuse of notation, when no confusion will arise.

Example 2.6. Let's see how we can construct a Sullivan model in practice. For example, consider the cohomology of the complex projective plane $(A, d) = (H^*(\mathbb{C}P^2; \mathbb{Q}), 0)$ as a cdga equipped with the zero differential. Recall that $H^*(\mathbb{C}P^2; \mathbb{Q}) \cong \mathbb{Q}[x]/(x^3)$, with $x \in H^2(\mathbb{C}P^2; \mathbb{Q})$. That is $A^0 = \mathbb{Q} \cdot 1$, $A^2 = \mathbb{Q} \cdot x$, $A^4 = \mathbb{Q} \cdot x^2$, $A^i = 0$ for any $i \neq 0, 2, 4$ and $d = 0$.

Construct $(\Lambda V, d)$ as follows. First, introduce a generator $a \in V^2$ of degree 2 and define $da = 0$. Next, introduce a generator $b \in V^5$ of degree 5 and define $db = a^3$. This defines a Sullivan algebra $(\Lambda V, d) = (\Lambda(a, b), d)$. Now, define a Sullivan model $m : (\Lambda(a, b), d) \to (A, d)$ by $m(a) = x$ and $m(b) = 0$. To check that this extends to a well-defined morphism of cdga's, it is sufficient to check that $m(da) = dm(a)$ and that $m(db) = dm(b)$, which is true. Next, we have to check that m is a quasi-isomorphism. Computing a basis for the cohomology of $(\Lambda V, d)$ we obtain 1, $[a]$ and $[a^2]$. Because m must preserve multiplication, it follows that $H(m)(1) = 1$, $H(m)([a]) = [m(a)] = x$, $H(m)([a^2]) = [m(a^2)] = x^2$. Hence, $H(m)$ maps a basis to a basis and is therefore an isomorphism.

Observe that $(\Lambda(a, b), d)$ is minimal.

2.5 Relative Sullivan models

Next, we will introduce relative Sullivan models, which serve as "minimal models" of morphisms between cdga's.

Definition. Let $\varphi : (A, d) \to (B, d)$ be a morphism of cdga's and suppose that $H^0(A) = \mathbb{Q}$. A *relative Sullivan model* for φ is a quasi-isomorphism of cdga's of the form

$$m : (A \otimes \Lambda V, \delta) \xrightarrow{\simeq} (B, d)$$

where $m|_A = \varphi$ and satisfying the following conditions

(i) δ extends d in the sense that $\delta = d$ on A,

(ii) $1 \otimes V = V = \{V^p\}_{p \geq 1}$,

(iii) $V = \bigcup_{k=0}^{\infty} V(k)$, where $V(0) \subset V(1) \subset \cdots$ is an increasing sequence of graded subspaces such that

$$d : V(0) \to A \quad \text{and} \quad d : V(k) \to A \otimes \Lambda V(k-1), \quad k \geq 1.$$

Definition. A relative Sullivan model $(A \otimes \Lambda V, \delta)$ is *minimal* if

$$\delta(A \otimes \Lambda V) \subset A^+ \otimes \Lambda V + A \otimes \Lambda^{\geq 2} V.$$

Remark. (1) Whereas m is called a relative Sullivan model for φ, we call $(A \otimes \Lambda V, \delta)$ a *relative Sullivan algebra.*

(2) We will frequently abuse notation by using the symbol d to stand for both δ and d, when there is no danger of confusion.

(3) Given a cdga (B, d), a relative Sullivan model for the unique morphism $(\mathbb{Q}, 0) \to (B, d)$ is just a Sullivan model for (B, d). Therefore, any result for relative Sullivan models also applies to Sullivan models as a particular case.

Theorem 2.7 (Existence and uniqueness of relative Sullivan models)**.**
Suppose $\varphi : (A,d) \longrightarrow (B,d)$ is a morphism of cdga's. If $H^0(A) = \mathbb{Q}$, $H^0(B) = \mathbb{Q}$ and $H^1(\varphi)$ is injective, then φ has a minimal relative Sullivan model

$$m : (A \otimes \Lambda V, \delta) \xrightarrow{\simeq} (B,d).$$

Moreover, if $m' : (A \otimes \Lambda W, \delta') \xrightarrow{\simeq} (B,d)$ is also a minimal relative Sullivan model for φ, then there is an isomorphism

$$(A \otimes \Lambda V, \delta) \cong (A \otimes \Lambda W, \delta').$$

We refer the reader to Theorem 6.1 and Theorem 6.2 of [17] or to Theorem 14.12 of [10] for a proof. Although we do not give the proof of this theorem here, we will however illustrate its application with the next example.

Remark. When the morphism is $\varphi : (\mathbb{Q}, 0) \rightarrow (B,d)$, then $H^1(\varphi)$ is automatically injective. Thus, a cdga (B,d) has a unique minimal Sullivan model as long as $H^0(B,d) = \mathbb{Q}$.

Example. We will construct a relative Sullivan model for the following morphism. Fix $x \in \mathbb{Q}\backslash\{0\}$ and consider

$$\begin{aligned}
(\Lambda(a_2, b_5), da = 0, db = a^3) &\xrightarrow{\varphi} (\Lambda(\alpha_2, \beta_3), d\alpha = 0, d\beta = \alpha^2) \\
a &\longmapsto x\alpha \\
b &\longmapsto x^3\alpha\beta
\end{aligned}$$

Note that the subscripts represent the degree of each element here. As can be verified, this is a morphism of cdga's. To construct a relative Sullivan model, we need to extend $(\Lambda(a_2, b_5), d)$ to a cdga of the form $(\Lambda(a_2, b_5) \otimes \Lambda V, d)$ such that the cohomology $H(\Lambda(a_2, b_2) \otimes \Lambda V, d)$ is isomorphic to $H(\Lambda(\alpha_2, \beta_3), d)$ (and we also need to extend φ

to a morphism $m : (\Lambda(a_2, b_5) \otimes \Lambda V, d) \xrightarrow{\simeq} (\Lambda(\alpha_2, \beta_3), d)$ such that m induces this isomorphism in cohomology).

Start by defining m on $\Lambda(a, b)$ by $m|_{\Lambda(a,b)} = \varphi$. The first step is then to compute the cohomology of $(\Lambda(\alpha_2, \beta_3), d)$. A straightforward computation reveals that a basis in cohomology is 1, $[\alpha]$. Now, on the other hand, we already computed in Example 2.4 that the cohomology of $(\Lambda(a_2, b_5), d)$ is given by the basis 1, $[a]$, $[a^2]$. Hence, to obtain an isomorphism in cohomology it is necessary to "kill off" the cocycle a^2 (by making it into a coboundary). Thus, first introduce a generator $s \in V^3$ with differential $ds = a^2$. Define $m(s) = x^2\beta$ (this is necessary in order for m to satisfy $dm = md$ and be a morphism of cdga's). At this point, having "killed off" a^2, we have isomorphic cohomologies up to at least degree 4. However, if we compute the cohomology of $(\Lambda(a_2, b_5) \otimes \Lambda(s_3), d)$, we find out that there is a new cocycle (which is not a coboundary) in degree 5, namely $b - as$. Indeed, $d(b - as) = a^3 - a^3 = 0$ and $[b - as]$ is not 0 in cohomology. Hence, we need to "kill off" $b - as$ by introducing $t \in V^4$ with $dt = b - as$. We define $m(t) = 0$ and this guarantees that m is a morphism of cdga's. Now, it follows that we have isomorphic cohomologies up to at least degree 5.

However, a computation of the cohomology reveals that $H(\Lambda(a, b) \otimes \Lambda(s, t), d) = \mathbb{Q} \cdot [1] \oplus \mathbb{Q} \cdot [a]$. Hence, we have constructed a cdga with cohomology isomorphic to $H(\Lambda(\alpha, \beta), d)$, as desired, and such that m induces this isomorphism (because $H(m)([1]) = [m(1)] = [1]$ and $H(m)([a]) = [m(a)] = [\alpha]$). Furthermore, $d(\Lambda(a, b) \otimes \Lambda(s, t)) \subset \Lambda^+(a, b) \otimes \Lambda(s, t)$. Hence, m is a minimal relative Sullivan model of φ.

Remark. It is not guaranteed in general that this process stops after a finite number of steps. In practice there will often be an infinite number of generators to be added.

Example. *The acyclic closure of a free cdga.*

Consider a free cdga $(\Lambda V, d)$. There is a unique morphism $\varphi : (\Lambda V, d) \longrightarrow (\mathbb{Q}, 0)$. A relative Sullivan model $(\Lambda V \otimes \Lambda Z, d)$ for φ is called an *acyclic closure* of $(\Lambda V, d)$, because it has the obvious property that

$$H^n(\Lambda V \otimes \Lambda Z, d) = \begin{cases} \mathbb{Q} & \text{if } n = 0, \\ 0 & \text{if } n \neq 0. \end{cases}$$

We introduce some new notation: define $\overline{V}$, called the *suspension* of V, by $\overline{V}^i = V^{i+1}$ (denote by $\bar{v} \in \overline{V}^i$ the element $v \in V^{i+1}$).

An acyclic closure of $(\Lambda V, d)$ can always be chosen of the form $(\Lambda V \otimes \Lambda\overline{V}, d)$ (see [10] Chapter 14, section (b), Example 1). Moreover, the linear part d_0 of d, when restricted to $\overline{V}$, is an isomorphism $d_0 : \overline{V} \xrightarrow{\cong} V$. This means that for any generator $\bar{v} \in \overline{V}$, the differential has the form $d(\bar{v}) = v + \omega$ for some $\omega \in \Lambda^{\geq 2}(V \oplus \overline{V})$.

The reason why $(\Lambda V \otimes \Lambda\overline{V}, d)$ is an acyclic closure of $(\Lambda V, d)$ follows from a topological argument. We refer the reader to [12], Example 2.66, for the proof.

Next, we will introduce the very important "lifting property," a technical result that is used in many proofs.

Theorem 2.8 (Lifting property)**.** *Suppose that $(A \otimes \Lambda V, d)$ is a relative Sullivan algebra and suppose that we are given the following commutative square*

$$\begin{tikzcd}
(A, d) \arrow[r, "\alpha"] \arrow[d, "i"] & (B, d) \arrow[d, "\eta"', "\simeq"] \\
(A \otimes \Lambda V, d) \arrow[r, "\psi"] \arrow[ur, dotted, "\varphi"] & (C, d).
\end{tikzcd}$$

Suppose further that i is the inclusion map and η is a surjective quasi-isomorphism. Then, there exists a morphism $\varphi : (A \otimes \Lambda V, d) \to (B, d)$ such that $\varphi i = \alpha$ and $\eta\varphi = \psi$ (we say that φ is a "lift" of ψ through η).

The reader is referred to Lemma 14.4 in [10] for a proof.

2.6 Rational homotopy theory

In this section, we briefly explain some aspects of rational homotopy theory and then give the results that will be relevant to our study.

Definition. We say that a simply connected space X is *rational* if its homotopy groups $\pi_*(X)$ are vector spaces over $\mathbb{Q}$.

Here a remark is in order about what it means for a group to be a "vector space over $\mathbb{Q}$." First, since X is simply connected and higher homotopy groups ($n \geq 2$) are always abelian, it follows that the groups $\pi_*(X)$ are actually abelian groups in each degree. A vector space is not that different from an abelian group in the sense that if we forget about scalar multiplication, a vector space along with its addition is an abelian group. Hence, an abelian group merely lacks a scalar multiplication by rational numbers (satisfying the vector space axioms) to become a rational vector space. As it turns out, an abelian group G admits a rational vector space structure if and only if G is divisible and torsion-free. In this case, the scalar multiplication by rationals is uniquely determined, as is easy to verify.

Moreover, tensoring an abelian group G by the abelian group of rational numbers $\mathbb{Q}$ will "kill off" the torsion in G. In fact, the tensor product $G \otimes_{\mathbb{Z}} \mathbb{Q}$ is always torsion-free[1] and divisible (where $\otimes_{\mathbb{Z}}$ stands for the tensor product of $\mathbb{Z}$-modules).

It follows from the remarks above that if X is a rational space, then $\pi_*(X) \otimes \mathbb{Q} \cong \pi_*(X)$.

Definition. A *rationalization* of X is a continuous map $\varphi : X \to X'$ where X' is simply connected and rational and such that

$$\pi_n(\varphi) \otimes \mathbb{Q} : \pi_n(X) \otimes \mathbb{Q} \longrightarrow \pi_n(X') \otimes \mathbb{Q} \cong \pi_n(X')$$

[1] If $g \in G$ is torsion, i.e. $ng = 0$ for some positive integer n, then in $G \otimes_{\mathbb{Z}} \mathbb{Q}$ we have $g \otimes 1 = g \otimes \frac{n}{n} = (ng) \otimes \frac{1}{n} = 0 \otimes \frac{1}{n} = 0$. It follows from this that $G \otimes \mathbb{Q}$ is torsion-free. Moreover, $G \otimes \mathbb{Q}$ becomes a rational vector space with an obvious scalar multiplication by rationals: if $a \in \mathbb{Q}$ and $g \otimes b \in G \otimes \mathbb{Q}$, then define $a \cdot (g \otimes b) = g \otimes (ab)$.

is an isomorphism for each $n \geq 1$.

Every simply connected space admits a rationalization:

Theorem 2.9. *If X is a simply connected space, then there exists a relative CW complex $(X_{\mathbb{Q}}, X)$ with no 0-cells and no 1-cells such that the inclusion $\varphi : X \to X_{\mathbb{Q}}$ is a rationalization. Moreover, given any continuous map $f : X \to Z$ for some simply connected rational space Z, there exists a map $\tilde{f} : X_{\mathbb{Q}} \to Z$ making the following diagram*

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[dr, "\varphi"'] & Z \\ & X_{\mathbb{Q}} \arrow[u, "\tilde{f}"'] \end{tikzcd}$$

commute.

The construction of the rationalization $X_{\mathbb{Q}}$ of a space X is originally due to Sullivan [35] (inspired by earlier developments from Serre [34]). However, we refer the reader to [10], Theorem 9.7, for a proof.

Definition 2.10. The *rational homotopy type* of a simply connected space X is the homotopy type of a rationalization $X_{\mathbb{Q}}$ of X. If two simply connected spaces X and Y have the same rational homotopy type, we write $X \simeq_{\mathbb{Q}} Y$. This is an equivalence relation, which we call *rational homotopy equivalence*.

The rational homotopy type is well defined and we present here the reason.

Proposition 2.11. *All rationalizations of X have the same homotopy type.*

Proof. If $\varphi : X \to X_{\mathbb{Q}}$ and $\varphi' : X \to X'_{\mathbb{Q}}$ are two rationalizations of X, then by Theorem 2.9, there exists a continuous map $g : X_{\mathbb{Q}} \to X'_{\mathbb{Q}}$ such that $g \circ \varphi = \varphi'$. Next, using the functoriality of π_n and of the tensor product, we get the equation

$(\pi_n(g) \otimes \mathbb{Q}) \circ (\pi_n(\varphi) \otimes \mathbb{Q}) = \pi_n(\varphi') \otimes \mathbb{Q}$. However, $\pi_n(\varphi) \otimes \mathbb{Q}$ and $\pi_n(\varphi') \otimes \mathbb{Q}$ are by hypothesis assumed to be isomorphisms, therefore $\pi_n(g) \otimes \mathbb{Q}$ must be an isomorphism as well. Now, if $\pi_n(g) \otimes \mathbb{Q}$ is an isomorphism, this means that $\pi_n(g)$ is also an isomorphism (since $X_{\mathbb{Q}}$ and $X'_{\mathbb{Q}}$ are rational spaces, thus $\pi_n(X_{\mathbb{Q}}) \otimes \mathbb{Q} \cong \pi_n(X_{\mathbb{Q}})$ and $\pi_n(X'_{\mathbb{Q}}) \otimes \mathbb{Q} \cong \pi_n(X'_{\mathbb{Q}})$). Therefore, we conclude that $X_{\mathbb{Q}}$ and $X'_{\mathbb{Q}}$ have the same homotopy type, according to Whitehead's theorem (Theorem 1.5). ■

2.7 From topology to algebra

Now, we will see how the transition from topological spaces to commutative differential graded algebras is achieved, so that we can make use of the algebraic tools introduced earlier.

First, let's fix some category theory notation, since we will be using functors. Denote by $\textsc{Top}$ the category of spaces and continuous maps and denote by $\mathrm{CDGA}_{\mathbb{Q}}$ the category of cdga's over the field $\mathbb{Q}$ and morphisms of cdga's.

Given a space X, we can construct a cdga $\mathcal{A}_{PL}(X)$, called the *cdga of piecewise-linear de Rham forms* on X. Moreover, if $f : X \to Y$ is any continuous map, then there is a morphism of cdga's $\mathcal{A}_{PL}(f) : \mathcal{A}_{PL}(Y) \to \mathcal{A}_{PL}(X)$. This extends to a contravariant functor from spaces to commutative differential graded algebras

$$\mathcal{A}_{PL} : \textsc{Top} \longrightarrow \mathrm{CDGA}_{\mathbb{Q}}.$$

The construction of $\mathcal{A}_{PL}$ is originally due to Sullivan [36].

The idea behind the construction of $\mathcal{A}_{PL}(X)$ for a space X is inspired by the construction of the de Rham complex of smooth forms on a manifold M. Indeed, suppose M is a smooth manifold. Let $\sigma : \Delta^k \to M$ be a smooth k-simplex and let $\omega \in \Omega^l(M)$ be a smooth differential form of degree l on M. If $\sigma_j : \Delta^{k-1} \to \Delta^k$ is the j-th face of σ, then there is a collection of smooth forms on Δ^{k-1}: namely $\sigma_j^*\sigma^*\omega$ for each j, satisfying some compatibility criteria. The construction of $\mathcal{A}_{PL}$ is a piecewise-linear analogue of this, in the sense that if X is a space, then the elements of $\mathcal{A}_{PL}(X)^l$ are functions which assign to each singular k-simplex of X a "piecewise-linear" l-form on Δ^k, $k \geq 0$, satisfying some compatibility criteria.

For the interested reader, all the details of the construction of $\mathcal{A}_{PL}$ can be found in chapter 10 of [10].

The $\mathcal{A}_{PL}$ functor has the following important property.

Theorem 2.12 ([10], Corollary 10.10)**.** *For any space X, the cohomology of $\mathcal{A}_{PL}(X)$ is isomorphic to the rational singular cohomology of X, i.e.*

$$H^*(X;\mathbb{Q}) \cong H(\mathcal{A}_{PL}(X)).$$

Conversely, there is a *spatial realization functor* (also contravariant)

$$\langle \quad \rangle : \mathrm{CDGA}_{\mathbb{Q}} \longrightarrow \text{CW complexes}.$$

It is obtained by composition of Sullivan's simplicial realization functor "$\mathrm{CDGA}_{\mathbb{Q}} \longrightarrow$ simplicial sets" (introduced in [36]) and Milnor's realization functor "simplicial sets $\longrightarrow$ CW complexes" (introduced in [32], alternatively see Chapter III of [30]).

This functor has the following two important properties

Theorem 2.13. *If $(\Lambda V, d)$ is a Sullivan algebra which is 1-connected and of finite type, then there always exists a quasi-isomorphism*

$$m : (\Lambda V, d) \xrightarrow{\simeq} \mathcal{A}_{PL}(\langle\, (\Lambda V, d)\,\rangle).$$

Moreover, for each $n \geq 1$, there is a canonical vector space isomorphism

$$\pi_n(\langle\, (\Lambda V, d)\,\rangle) \cong \mathrm{Hom}_{\mathbb{Z}}(V^n, \mathbb{Q}).$$

If we restrict ourselves to spaces which are simply connected CW complexes with rational homology of finite type and to 1-connected Sullivan algebras of finite type, then these two functors combined together give rise to the following bijective correspondence

$$\left\{\begin{array}{c}\text{rational homotopy types of simply}\\ \text{connected CW complexes with}\\ \text{rational homology of finite type}\end{array}\right\} \xleftrightarrow{\cong} \left\{\begin{array}{c}\text{isomorphism classes of minimal}\\ \text{Sullivan algebras over } \mathbb{Q} \text{ which}\\ \text{are 1-connected and of finite type}\end{array}\right\}$$

The fact that this is a bijection can be deduced from the properties of the functors stated in Theorem 2.12 and Theorem 2.13.

The power of rational homotopy theory lies precisely in the above bijection. It reduces all topological computations in rational homotopy theory to computations on an algebraic object, the minimal Sullivan algebra.

Having introduced the $\mathcal{A}_{PL}$ functor, we can now make the crucial definition that relates Sullivan algebras to topological spaces.

Definition 2.14. A *Sullivan model* of a path-connected space X is a Sullivan model for the cdga $\mathcal{A}_{PL}(X)$, in other words, it is a quasi-isomorphism

$$m : (\Lambda V, d) \xrightarrow{\simeq} \mathcal{A}_{PL}(X)$$

for some Sullivan algebra $(\Lambda V, d)$.

Remarks. (i) It follows from Theorem 2.7 that if X is simply connected, then there is a unique (up to isomorphism) minimal Sullivan model $(\Lambda V, d) \xrightarrow{\simeq} \mathcal{A}_{PL}(X)$ for X. We will often just refer to $(\Lambda V, d)$ as the model of X, by abuse of terminology and notation.

(ii) If $(\Lambda V, d)$ is a Sullivan model of X, then Theorem 2.12 implies that

$$H(\Lambda V, d) \cong H^*(X; \mathbb{Q}).$$

(iii) If $(\Lambda V, d)$ is a Sullivan model of X and X is simply connected, then Theorem 2.13 implies that

$$\mathrm{Hom}_{\mathbb{Z}}(V^n, \mathbb{Q}) \cong \pi_n(X) \otimes \mathbb{Q}.$$

Definition 2.15. If $f : X \to Y$ is a continuous map, then a relative Sullivan model for $\mathcal{A}_{PL}(f) : \mathcal{A}_{PL}(Y) \to \mathcal{A}_{PL}(X)$ is called a *relative Sullivan model* for f.

Next, we introduce a class of spaces which will turn out to be important in chapter 4.

Definition 2.16. A space X is called *coformal* if it has a minimal Sullivan model which is quadratic.

Remark. We make a remark here about the previous definition. Even though the minimal Sullivan model of a space is unique up to isomorphism, in general not every minimal Sullivan model of a coformal space will be quadratic, because the property of being quadratic is not preserved under isomorphisms of cdga's. For example, consider the minimal model $(\Lambda(a, b, x), d)$ with $|a| = 2$, $|b| = 4$, $|x| = 5$ and $dx = ab$. Construct a morphism $\varphi : (\Lambda(a, b, x), d) \to (\Lambda(\hat{a}, \hat{b}, \hat{x}), \hat{d})$, where $|\hat{a}| = |a| = 2$, $|\hat{b}| = |b| = 4$, $|\hat{x}| = |x| = 5$, by defining $\varphi(x) = \hat{x}$, $\varphi(a) = \hat{a}$ and $\varphi(b) = \hat{b} - \hat{a}^2$. Then, it is immediate that φ is a linear isomorphism. In order for φ to be a morphism of cdga's, then it must commute with the differentials: $\varphi(dx) = \hat{d}\varphi(x) = \hat{d}(\hat{x})$. Yet,

$$\varphi(dx) = \varphi(ab) = \hat{a}(\hat{b} - \hat{a}^2).$$

Therefore, it follows that we can define the differential $\hat{d}$ by $\hat{d}(\hat{x}) = \hat{a}(\hat{b} - \hat{a}^2)$ and this will make φ into an isomorphism of cdga's. Yet, $(\Lambda(\hat{a}, \hat{b}, \hat{x}), \hat{d})$ is not quadratic.

Example 2.17. *The minimal Sullivan model of* $\mathbb{C}P^n$.

We know that $H(\mathcal{A}_{PL}(\mathbb{C}P^n)) \cong H^*(\mathbb{C}P^n;\mathbb{Q}) \cong \mathbb{Q}[x]/(x^{n+1})$, where x has degree 2. So, necessarily there exists elements $\alpha \in \mathcal{A}_{PL}(\mathbb{C}P^n)^2$ and $\beta \in \mathcal{A}_{PL}(\mathbb{C}P^n)^{2n+1}$ such that $x = [\alpha]$ and $d\beta = \alpha^{n+1}$. Define $(\Lambda V, d)$ by introducing $a \in V^2$ and $b \in V^{2n+1}$, with $da = 0$ and $db = a^{n+1}$. It follows that the map $m : (\Lambda(a,b),d) \to \mathcal{A}_{PL}(\mathbb{C}P^n)$ defined by $m(a) = \alpha$ and $m(b) = \beta$ is a quasi-isomorphism, so m is the minimal Sullivan model of $\mathbb{C}P^n$.

2.8 Formal spaces

In general, it is possible for two non-isomorphic Sullivan algebras to have isomorphic cohomology. This means that in general a Sullivan algebra contains "information" that is not contained in its cohomology algebra. However, there are certain Sullivan algebras which are completely characterized by their cohomology, which we introduce now.

Definition 2.18. A Sullivan algebra $(\Lambda V, d)$ is *formal* if there exists a quasi-isomorphism

$$\varphi : (\Lambda V, d) \xrightarrow{\simeq} (H(\Lambda V, d), 0)$$

where $(H(\Lambda V, d), 0)$ is the cga $H(\Lambda V, d)$ equipped with the 0 differential.

Definition 2.19. A simply connected space X with minimal Sullivan model $(\Lambda V, d)$ is *formal* if $(\Lambda V, d)$ is formal. Equivalently, X is formal if there exists a quasi-isomorphism

$$\varphi : (\Lambda V, d) \xrightarrow{\simeq} (H^*(X;\mathbb{Q}), 0).$$

Examples. By taking together Example 2.6, where we computed the minimal Sullivan model for $(H^*(\mathbb{C}P^2;\mathbb{Q}), 0)$, and Example 2.17, where we computed the minimal

Sullivan model of $\mathcal{A}_{PL}(\mathbb{C}P^n)$, and found that they were identical, it follows that $\mathbb{C}P^2$ is formal.

In fact, any sphere and any complex projective space is a formal space. Moreover, the wedge of two formal spaces is again formal and the product of two formal spaces is formal if one of the factors has rational homology of finite type (see [10], Chapter 12, Section (c)). Any compact Khäler manifold is also formal (see [5], section 5).

2.9 Bigraded structures

Sometimes, it will be useful to consider algebraic structures that are indexed by two degrees. A *bigraded vector space* is a family $V = \{V^{i,j}\}_{i,j\in\mathbb{N}}$ of vector spaces $V^{i,j}$. Elements of $V^{i,j}$ are said to have *bidegree* (i,j). A *morphism* $f : V \to W$ of bidegree (k,l) between bigraded vector spaces is a collection of linear maps $f_{i,j} : V^{i,j} \to W^{i+k,j+l}$.

If V and W are two bigraded vector spaces, then $V \otimes W$ is also a bigraded vector space with bigradation

$$(V \otimes W)^{i,j} = \bigoplus_{\substack{p+r=i \\ q+s=j}} V^{p,q} \oplus W^{r,s}.$$

Bigradations naturally come up when working with free cdga's, because in addition to the usual degree, there is also the word length that can be viewed as another degree. Suppose that V is a graded vector space and recall that $\Lambda^k V$ stands for the set of all elements of word length k in ΛV. We can define a bigradation on ΛV by

$$(\Lambda V)^{p,q} := (\Lambda^p V)^{p+q}.$$

In this case, we call the integer p the *filtration degree* (or just word length), the integer $p + q$ the *topological degree* and the integer q the *complementary degree.*

Example. Consider the space $(\Lambda V, d) = (\Lambda(a, b, x, y), d)$, $|a| = 2$, $|b| = 6$, $|x| = 7$, $|y| = 11$, $da = db = 0$, $dx = a^4 - 2ab$ and $dy = b^2$, which happens to be a minimal Sullivan model for the homogeneous space $\mathrm{Sp}(3)/\mathrm{U}(3)$ (see [26]). In this case, the element abx for example has filtration degree 3 and topological degree 15, hence its bidegree is $(3, 12)$.

Observe that in this case, the differential d is *not* a morphism of bigraded vector spaces. Indeed, the differential d maps x to $a^4 - 2ab$, but this latter element is a linear combination of a^4 which has word length 4 and ab which has word length 2. Hence, there is no well-defined choice of bidegree for the differential. However, if we restrict our attention to the quadratic part d_1 of d (i.e. the map $d_1 a = d_1 b = 0$, $d_1 x = -2ab$ and $d_1 y = b^2$), then this map is a morphism of bigraded vector spaces of bidegree $(1, 0)$. Indeed, we see for example that y has bidegree $(1, 10)$ and that $d_1 y = b^2$ has bidegree $(2, 10)$ and similarly x has bidegree $(1, 6)$ and $d_1 x = -2ab$ has bidegree $(2, 6)$.

Indeed, in any case where $(\Lambda V, d)$ is quadratic, then, d is a bigraded map.

2.10 Modules

In this section, we introduce the concept of (A, d)-modules over a cdga (A, d). This will be useful to us sometimes as an alternative to cdga's, since sometimes we do not need the full multiplicative structure of a cdga and the "simpler" structure of an (A, d)-module is preferable.

We will introduce the notion of a *semifree* (A, d)-module, which is analogous to a Sullivan model, as well as the notion of a semifree resolution of an (A, d)-module, analogous to a relative Sullivan model. For a general reference on (A, d)-modules, the reader is referred to ([10], chapter 6) or ([24], chapter 16).

Throughout this section, fix a cdga (A, d).

Definition. A *left (A, d)-differential module* (or just *left (A, d)-module*) is a graded vector space $M = \{M^i\}_{i\in\mathbb{N}}$ over $\mathbb{Q}$ along with a linear map $A \otimes M \to M$ of degree zero (denoted $x \otimes m \mapsto x \cdot m$) and a differential d in M such that

$$x \cdot (ym) = (xy) \cdot m \quad \text{and} \quad 1 \cdot m = m \qquad \text{for all } x, y \in A, m \in M$$

and

$$d(x \cdot m) = dx \cdot m + (-1)^{|x|} x \cdot dm.$$

There is an analogous definition for *right* (A, d)*-module*. However, since (A, d) is commutative, the two definitions are equivalent and we will always work with left (A, d)-modules and simply refer to a left (A, d)-module as an (A, d)-module.

Definition. Let M and N be two (A, d)-modules. A *linear map of* (A, d)*-modules of degree* k is a linear map $f : M \to N$ of degree k such that

$$f(x \cdot m) = (-1)^{k|x|} x \cdot f(m)$$

and

$$df = (-1)^{|k|} fd.$$

Moreover, a *morphism of (A, d)-modules* is a linear map of (A, d)-modules of degree 0.

Remark. If (M, d) is an (A, d)-module, then $H(M, d)$ is an $H(A, d)$-module via $[x] \cdot [m] = [x \cdot m]$. Moreover, a morphism $f : (M, d) \to (N, d)$ between two (A, d)-modules induces a map $H(f) : H(M, d) \to H(N, d)$ and this map is automatically a morphism of $H(A, d)$-modules.

Example. Let $(A \otimes \Lambda V, d)$ be a relative Sullivan model for a morphism $(A, d) \to (B, d)$. The relative Sullivan model can be viewed as an (A, d)-module in the obvious way (simply define $a \cdot (x \otimes y) = (ax) \otimes y$ for $a \in A$, $x \otimes y \in A \otimes \Lambda V$ and forget about the cdga structure on ΛV, viewing it simply as a graded module with a differential).

Definition. Let (M, d) and (N, d) be (A, d)-modules. The *tensor product* $M \otimes_A N$ of M and N is defined as

$$M \otimes_A M' = (M \otimes M')\big/ \sim$$

where we quotient by the submodule spanned by elements of the form $am \otimes n - (-1)^{\deg a \deg m} m \otimes an$ for any $a \in A$, $m \in M$ and $n \in N$. It is made into an (A, d)-module $(M \otimes_A N, d)$ with multiplication defined by $a \cdot (m \otimes_A n) = am \otimes_A n$ and differential defined by

$$d(m \otimes_A n) = (dm) \otimes_A n + (-1)^{\deg m} m \otimes_A dn.$$

Definition. An (A, d)-module (M, d) is (A, d)-*free* if $M \cong A \otimes V$ for some free graded module V (that is, V^i is free for each $i \in \mathbb{N}$).

Definition. An (A, d)-module (M, d) is *semifree* if

$$M = \bigcup_{k=0}^{\infty} M(k) \quad \text{where } M(0) \subset M(1) \subset \ldots \subset M(k) \subset \ldots$$

and each $M(k)$ is a sub (A, d)-module of M, such that $M(0)$ and each quotient $M(k)/M(k-1)$ is (A, d)-free on a basis $\{v_\alpha\}$ of cocycles.

Definition. Let (Q, d) be an (A, d)-module. An (A, d)-*semifree resolution* of (Q, d) is an (A, d)-semifree module (M, d) along with a quasi-isomorphism $m : (M, d) \xrightarrow{\simeq} (Q, d)$ of (A, d)-modules.

Just like Sullivan models, semifree modules and semifree resolutions have an existence and uniqueness property as well as a lifting property, which we introduce below. The similarity with Sullivan models will be exploited in chapters 3 and 4, where we will see that in a particular case a relative Sullivan model can be replaced by a semifree resolution, which has a simpler structure and hence can make computations easier.

We refer the reader to ([10], Proposition 6.6) for a proof of the following theorem.

Theorem 2.20 (Existence of semifree resolutions)**.** *Given any (A, d)-module (Q, d), there exists an (A, d)-semifree resolution $(M, d) \xrightarrow{\simeq} (Q, d)$.*

Next is the lifting property for (A, d)-modules.

Theorem 2.21 (Lifting property [1], p.9)**.** *Consider the following commutative square of morphisms of (A, d)-modules*

$$\begin{tikzcd}
(M', d) \arrow[r, "\theta"] \arrow[d, "i"] & (P, d) \arrow[d, "\eta"', "\simeq"] \\
(M, d) \arrow[r, "\psi"] \arrow[ur, dotted, "\varphi"] & (B, d).
\end{tikzcd}$$

where $i : (M', d) \to (M, d)$ is an inclusion, the quotient M/M' is semifree and η is a surjective quasi-isomorphism of (A, d)-modules. Then, there exists a morphism $\varphi : (M, d) \to (P, d)$ such that $\varphi \circ \eta = \psi$ and $\varphi \circ i = \theta$ (we say that φ is a "lift" of ψ over η).

Chapter 3

The Lusternik–Schnirelmann category

3.1 Introduction

We will now introduce a homotopy invariant called the Lusternik–Schnirelmann category of a space. It is a numerical invariant defined for any topological space. It was originally introduced in [29], where it was only defined for manifolds and it was proven that this invariant was a lower bound for the number of critical points admitted by any smooth function $f : M \to \mathbb{R}$ on a closed manifold M (under the additional assumptions that M is a paracompact C^2-Banach manifold and that $f : M \to \mathbb{R}$ is a C^2-function that is bounded below. See [3], Theorem 1.15 for details).

The minimal models of rational homotopy theory have been highly successful tools for studying a lower bound of the Lusternik–Schnirelmann category, namely the rational Lusternik–Schnirelmann category (introduced in [8]). This latter invariant is essential to our main object of study, the rational retraction index of a space.

Recall that given a topological space X, a subspace U of X is said to be *contractible in* X if the inclusion map $i : U \hookrightarrow X$ is homotopic to a constant map

$U \to x_0$.

Definition 3.1 (Lusternik & Schnirelmann [29])**.** Let X be a topological space. The *Lusternik–Schnirelmann category* of X, denoted $\operatorname{cat} X$, is the least integer m such that X can be covered by $m+1$ open subsets each contractible in X. If there exists no such covering, we say that $\operatorname{cat} X = \infty$.

Remark. An open cover of a space X such that each subset is contractible in X is called a *categorical cover* of X. Also, note that our definition of the Lusternik–Schnirelmann category of X is 1 less than the definition from the original article [29].

The Lusternik–Schnirelmann category (henceforth "the category" or the "L.-S. category") of a space is a homotopy invariant ([10], Proposition 27.2). The category of any contractible space is 0 and the category of a circle (or in fact any n-sphere for $n \geq 1$) is 1. To see this, observe that the category of the n-sphere must be greater than or equal to 1 since this space is not contractible. A cover of the n-sphere using two contractible open sets is obtained by choosing as the first open set the sphere minus the "north pole" and as the second open set the sphere minus the "south pole". This is a categorical cover of cardinality 2, hence the category of the n-sphere is 1.

Another interesting example is the topologist's sine curve (Figure 3.1), which has infinite category. Recall that the topologist's sine curve is defined to be the union of the graph of the function $\sin(1/x)$ over the interval $(0,1]$ together with the vertical line segment $\{0\} \times [-1,1]$, with the subspace topology induced by the standard topology of $\mathbb{R}^2$. This space has infinite category because it admits no categorical cover. In fact, no neighborhood of the point $(0,0)$ is contractible.

Yet another interesting example is $\mathbb{C}P^\infty = \bigcup_{k=0}^{\infty} \mathbb{C}P^k$, which also has infinite category. Note however that contrary to the topologist's sine curve, $\mathbb{C}P^\infty$ has the homotopy type of a CW complex. To compute the category of $\mathbb{C}P^\infty$, we need to introduce the following homotopy invariant.

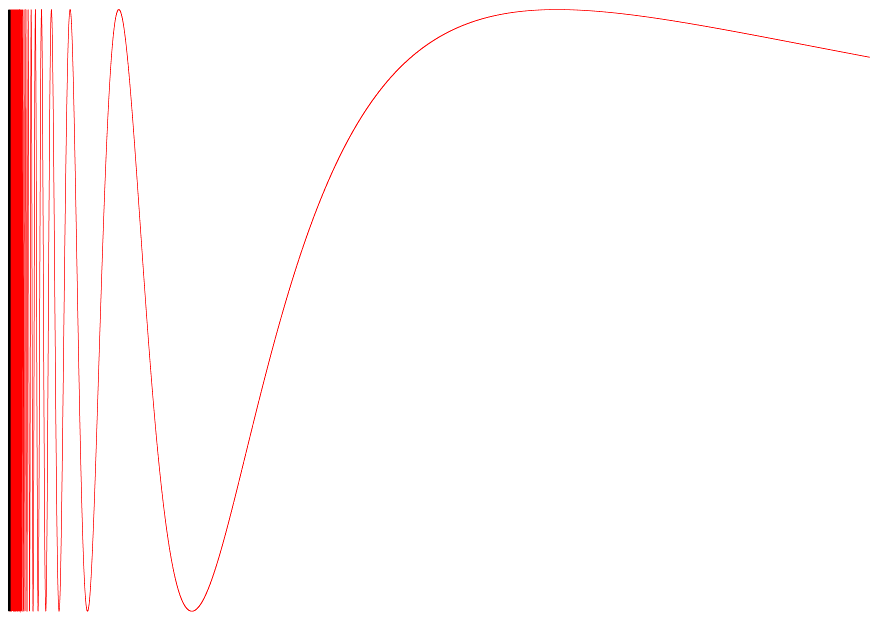

Figure 3.1: The topologist's sine curve

Definition 3.2. The *cup-length*, $\operatorname{cup}(X)$, of a path-connected space X is the greatest integer n such that there exist cohomology classes $\omega_1, \ldots, \omega_n \in \widetilde{H}^*(X; \mathbb{Q})$ with cup-product $\omega_1 \cup \cdots \cup \omega_n \neq 0$. If no such integer exists, then $\operatorname{cup}(X) = \infty$.

Theorem 3.3. *If X is a path-connected normal space, then*

$$\operatorname{cup}(X) \leq \operatorname{cat}(X).$$

For a proof of this theorem, the reader is referred to [10], Proposition 27.14. Note that any CW complex is a normal space.

Now, the cohomology of $\mathbb{C}P^\infty$ is isomorphic to the polynomial algebra in one variable $\mathbb{Q}[x]$, with x in degree 2 (see [10], Chapter 15, section (b), Example 5). Therefore, there is no bound on the length of a non-zero product of cohomology classes, i.e. $\operatorname{cup}(X) = \infty$. Hence, by the theorem above (since $\mathbb{C}P^\infty$ is a path-connected CW complex) $\operatorname{cat}(X) = \infty$.

There is also a useful upper bound to the category, in the case where the space admits a cell decomposition. Recall from chapter 1 that a space is r-connected ($r \geq 1$) if $\pi_i(X, x_0) = 0$ for $1 \leq i \leq r$ and a space is 0-connected if it is path-connected.

Theorem 3.4. *Let X be a $(q-1)$-connected CW complex of dimension n (assuming $q \geq 1$), then*

$$\mathrm{cat}(X) \leq \frac{n}{q}.$$

We refer the reader to [10], Proposition 27.5, for a proof.

Example. Consider the n-dimensional complex projective space $\mathbb{C}P^n$. It is a simply connected (i.e. 1-connected) CW complex of dimension $2n$, hence by Theorem 3.4,

$$\mathrm{cat}(\mathbb{C}P^n) \leq \frac{2n}{2} = n.$$

Because the rational cohomology of $\mathbb{C}P^n$ is the truncated polynomial ring $\mathbb{Q}[x]/(x^{n+1})$, where $x \in H^2(\mathbb{C}P^n; \mathbb{Q})$, it follows that $\mathrm{cup}(\mathbb{C}P^n) = n$. Hence, by Theorem 3.3,

$$n = \mathrm{cup}(\mathbb{C}P^n) \leq \mathrm{cat}(\mathbb{C}P^n).$$

Therefore, $\mathrm{cat}(\mathbb{C}P^n) = n$. ∎

The reader is referred to [10] or [3] for a comprehensive and modern treatment of the Lusternik–Schnirelmann category.

3.2 The rational Lusternik–Schnirelmann category

Albeit simple to define, the Lusternik–Schnirelmann category has been found to be difficult to compute in general, except for the simplest spaces, and one usually has to resort to computing approximations to the category (see for example the introduction of [16]).

Throughout the rest of this thesis, we will not be concerned directly with $\operatorname{cat} X$, instead we will consider the rational counterpart to $\operatorname{cat} X$, namely the category of the rationalization $X_{\mathbb{Q}}$ of X. We now give the definition

Definition 3.5. Let X be a simply connected space. The *rational L.-S. category* of a space X, denoted $\operatorname{cat}_0 X$, is the least integer m such that $X \simeq_{\mathbb{Q}} Y$ and $\operatorname{cat} Y = m$ for some simply connected space Y.

Remark. It is immediate from the definition that $\operatorname{cat}_0 X$ is an invariant of rational homotopy type.

Theorem 3.6. *If X is a simply connected CW complex, then*

(i) $\operatorname{cat}_0 X = \operatorname{cat} X_{\mathbb{Q}}$,

(ii) $\operatorname{cat} X_{\mathbb{Q}} \leq \operatorname{cat} X$.

We refer the reader to [8] for the proof of (i) and to [39] for the proof of (ii).

The study of cat_0 began with Toomer [38], [39]. Next, Lemaire and Sigrist [28] showed how to compute an approximation to cat_0 from the Quillen model of a space (from the Quillen approach [33] to rational homotopy theory). However, it is only with Félix et al. [8] and [9] that the relationship of $\operatorname{cat}_0 X$ with Sullivan's minimal models was formulated (see next chapter).

Remark. The hypothesis that X is simply connected is necessary for part (ii) of Theorem 3.6. For example, consider the non-simply connected space $X = S^1$. As we explained earlier, $\operatorname{cat} S^1 = 1$. However, the category of the rationalization $S^1_{\mathbb{Q}}$ of S^1 is $\operatorname{cat} S^1_{\mathbb{Q}} = 2 > 1 = \operatorname{cat} S^1$.

To see this, observe that $S^1_{\mathbb{Q}}$ has the rational homotopy type of $K(\mathbb{Q}, 1)$ (the Eilenberg–Mac Lane space, see chapter 1). Any space with category 1 has as its fundamental group a free group (see [3], exercise 1.21). However, $\pi_1(S^1_{\mathbb{Q}}) = \mathbb{Q}$ is not free as a group. This implies that $\mathrm{cat}(S^1_{\mathbb{Q}}) > 1$. To prove that $\mathrm{cat}(S^1_{\mathbb{Q}}) = 2$, it remains to show that $\mathrm{cat}(S^1_{\mathbb{Q}}) \leq 2$. Now, $S^1_{\mathbb{Q}}$ is 0-connected and it is a CW complex of dimension 2 (see the construction of $S^1_{\mathbb{Q}}$ as the "infinite telescope" for more details, in [10], chapter 9, section (a)). Thus, by Theorem 3.4,

$$\mathrm{cat}(S^1_{\mathbb{Q}}) \leq 2/1 = 2.$$

Hence, $\mathrm{cat}(S^1_{\mathbb{Q}}) = 2$. So we see that $\mathrm{cat}(X_{\mathbb{Q}}) \leq \mathrm{cat}(X)$ does not hold for non-simply connected spaces in general.

3.3 The L.-S. category of a Sullivan algebra

Our interest in $\mathrm{cat}_0 X$ lies in the fact that it can be characterized in an algebraic way from the minimal Sullivan model of a space. In other words, one can in principle compute the rational category of a space from its minimal Sullivan model. This is not too surprising, since all the rational homotopy type information about a simply connected space is encoded in its minimal Sullivan model (see section 2.7).

We will begin by introducing the notion of the L.-S. category $\mathrm{cat}_0(\Lambda V, d)$ of a Sullivan algebra $(\Lambda V, d)$. If X is a simply connected space, with minimal Sullivan model $\phi : (\Lambda V, d) \xrightarrow{\simeq} \mathcal{A}_{PL}(X)$, we will see that it satisfies $\mathrm{cat}_0(\Lambda V, d) = \mathrm{cat}_0 X$, where the invariant $\mathrm{cat}_0(\Lambda V, d)$ was first introduced in [8].

Suppose that $(\Lambda V, d)$ is any Sullivan algebra and fix an integer $m \geq 1$. The subspace $\Lambda^{>m} V$, consisting of words of length greater than m, is a differential ideal of $(\Lambda V, d)$. Hence, we can take the quotient cdga $\Lambda V/\Lambda^{>m} V$ with differential $\bar{d}(\bar{x}) = \overline{d(x)}$. There is an associated projection map

$$\pi_m : (\Lambda V, d) \longrightarrow (\Lambda V/\Lambda^{>m} V, \bar{d})$$

that sends x to its equivalence class $\bar{x}$. In practice, we will often omit the bar notation for simplicity. Now, recall from Theorem 2.7 that this map admits a minimal relative Sullivan model. That is, we can factor π_m as

$$\begin{tikzcd}
(\Lambda V, d) \arrow[r, "\pi_m"] \arrow[dr, "\lambda_m"'] & (\Lambda V/\Lambda^{>m} V, \bar{d}) \\
& (\Lambda V \otimes \Lambda Z_m, \delta) \arrow[u, "\simeq"', "h_m"]
\end{tikzcd}$$

such that $(\Lambda V \otimes \Lambda Z_m, \delta)$ is a relative Sullivan algebra and h_m is a quasi-isomorphism.

Definition 3.7. The *L.-S. category*, $\mathrm{cat}_0(\Lambda V, d)$, is the least integer m (or ∞) such that the inclusion map λ_m above admits a retraction ρ which is a morphism of cdga's. Recall that a retraction is a morphism $\rho : (\Lambda V \otimes \Lambda Z, \delta) \longrightarrow (\Lambda V, d)$ such that $\rho \circ \lambda_m = \mathrm{id}_{(\Lambda V, d)}$.

We refer the reader to [8] and [9] for the original articles introducing $\mathrm{cat}_0(\Lambda V, d)$ and to [10] for a recent treatment.

Of course, this definition is related to the rational category of a space, under certain hypotheses, via its minimal model. It is Félix and Halperin [8] who showed

Theorem 3.8. *If $(\Lambda V, d)$ is a Sullivan model for a simply connected space X with rational homology of finite type, then*

$$\mathrm{cat}_0(\Lambda V, d) = \mathrm{cat}_0 X.$$

It is possible to "relax the conditions in the definition of $\mathrm{cat}_0(\Lambda V, d)$". Recall from section 2.10 that the cdga's $(\Lambda V, d)$ and $(\Lambda V \otimes \Lambda Z, d)$ can be regarded as $(\Lambda V, d)$-modules. So, for example, we might merely require the retraction ρ to be a morphism of $(\Lambda V, d)$-module, an apparently weaker condition.

This is useful, because computing $\mathrm{cat}_0(\Lambda V, d)$ amounts to determining the existence or non-existence of a retraction for $(\Lambda V, d) \longrightarrow (\Lambda V \otimes \Lambda Z_m, d)$ and this can be rather difficult to do as the retraction needs to be an algebra map and preserve not only the differential but also the multiplicative structure. On the other hand, showing the existence or non-existence of a $(\Lambda V, d)$-module retraction happens to be often much easier to achieve, since the multiplicative structure of a $(\Lambda V, d)$-module is simpler than that of a cdga.

This led Halperin and Lemaire [18] to formulate the following

Definition 3.9. The *module L.-S. category*, $\mathrm{mcat}_0(\Lambda V, d)$, is the least integer m (or ∞) such that the inclusion map λ_m above admits a retraction ρ which is a morphism of $(\Lambda V, d)$-modules.

Finally, we could require the retraction ρ to merely be a linear map, an even weaker condition that completely ignores the multiplicative structure of our models. Indeed Toomer [38] formulated the following invariant (under a different formulation, using spectral sequences, see [10] chapter 29, section (g) for details of the equivalence).

Definition 3.10. The *rational Toomer invariant*, $\mathrm{e}_0(\Lambda V, d)$, is the least integer m (or ∞) such that $H(\pi_m)$ is injective. This is equivalent to the fact that λ_m admits a linear retraction.

Remark. The module category and the rational Toomer invariant are both rational homotopy invariants. As it turns out, they are lower bounds for the category, which are easier to compute in practice than the category. In particular, computing $e_0(\Lambda V, d)$ is the most straightforward in practice (at least for spaces with cohomology of finite dimension), as it can be achieved by computing the cohomology of the cdga $(\Lambda V, d)$ and of its quotients $(\Lambda V/\Lambda^{>m}V, \bar{d})$ for different values of $m \geq 1$ and checking for every m if the projection map induced in cohomology is injective.

Finally, another related invariant is the cup-length of the cohomology of $(\Lambda V, d)$.

Definition 3.11. The *cup-length* of a Sullivan algebra $(\Lambda V, d)$ is the greatest integer n such that there exist $\omega_1, \ldots, \omega_n \in H^+(\Lambda V, d)$ with $\omega_1 \cdots \omega_n \neq 0$.

It is clear that if X is any space with associated minimal Sullivan model $(\Lambda V, d)$, then $\mathrm{cup}(X) = \mathrm{cup}(\Lambda V, d)$.

Proposition 3.12. *If $(\Lambda V, d)$ is any Sullivan algebra, then*

$$\mathrm{cup}(\Lambda V, d) \leq \mathrm{e}_0(\Lambda V, d) \leq \mathrm{mcat}_0(\Lambda V, d) \leq \mathrm{cat}_0(\Lambda V, d).$$

Moreover,

Proof. The inequality $\mathrm{mcat}_0(\Lambda V, d) \leq \mathrm{cat}_0(\Lambda V, d)$ is obvious from the definition.

For the inequality $\mathrm{e}_0(\Lambda V, d) \leq \mathrm{mcat}_0(\Lambda V, d)$, suppose that $\mathrm{mcat}_0(\Lambda V, d) = m$ and hence (using the notation of the diagram above) the inclusion λ_m admits a certain retraction ρ of $(\Lambda V, d)$-modules. It follows from the functoriality of the homology functor that because $\rho \circ \lambda_m = \mathrm{id}_{(\Lambda V, d)}$, then $H(\rho) \circ H(\lambda_m) = \mathrm{id}_{H(\Lambda V, d)}$ and thus $H(\rho)$ is injective. Now, $H(\pi_m) = H(h) \circ H(\rho)$ and $H(h)$ is an isomorphism because h is a quasi-isomorphism. Thus, $H(\pi_m)$ is the composite of two injective functions and must be injective as well. Hence, $\mathrm{e}_0(\Lambda V, d) \leq \mathrm{mcat}_0(\Lambda V, d)$, as desired.

Finally, to prove that $\mathrm{cup}(\Lambda V, d) \leq e_0(\Lambda V, d)$, proceed by contradiction. Let $n = \mathrm{cup}(\Lambda V, d)$ and $m = e_0(\Lambda V, d)$ and suppose towards a contradiction that $m < n$. So, by assumption there is a product $\omega_1 \cdots \omega_n \neq 0$, with each $\omega_i \in H^+(\Lambda V, d)$. Moreover, each ω_i is represented by a cocycle $\alpha_i \in \Lambda^{\geq 1} V$. In particular, $\alpha_1 \cdots \alpha_n \in \Lambda^{\geq n} V$. Our second assumption is that $H(\pi_m)$ is injective. Observe however that $\alpha_1 \cdots \alpha_n$ is sent to $\pi_m(\alpha_1 \cdots \alpha_n) = 0$ in the quotient $(\Lambda V / \Lambda^{>m} V, \bar{d})$, since $m < n$. Thus, $H(\pi_m)(\omega_1 \cdots \omega_n) = [\pi_m(\alpha_1 \cdots \alpha_n)] = 0$, contradicting the injectivity of $H(\pi_m)$. So we conclude that $\mathrm{cup}(\Lambda V, d) \leq e_0(\Lambda V, d)$, as desired. ■

Surprisingly, the above theorem can be improved. As it turns out, the existence of a $(\Lambda V, d)$-module retraction in the diagram above guarantees the existence of a cdga retraction, so that in the end, despite the apparent weakness of the module category, one really is computing the category. It was Hess who proved [21] the following

Theorem 3.13 (Hess). *If* $(\Lambda V, d)$ *is any Sullivan algebra, then*

$$\mathrm{mcat}_0(\Lambda V, d) = \mathrm{cat}_0(\Lambda V, d).$$

The theorem above is of crucial importance in the theory of the rational category. Its proof involves understanding very intrinsic details of the structure of the model of the projection map $\pi_m : (\Lambda V, d) \longrightarrow (\Lambda V/\Lambda^{>m}V, \bar{d})$. We will study some of the the structure of this model in details in section 3.7.

3.4 The rational L.-S. category of a product

In this section, we consider the category of a topological product. First of all, consider two spaces X and Y. Bassi proved [2] (see also [13] or [3]) that if X and Y are path-connected and $X \times Y$ is a normal space, then

$$\mathrm{cat}(X \times Y) \leq \mathrm{cat}(X) + \mathrm{cat}(Y).$$

Note that CW complexes are normal. It is known that this inequality can be strict, for example the following product of Moore spaces (see chapter 1 for a definition) satisfies

$$\mathrm{cat}(M(\mathbb{Z}/2, 2) \times M(\mathbb{Z}/3, 2)) < \mathrm{cat}\, M(\mathbb{Z}/2, 2) + \mathrm{cat}\, M(\mathbb{Z}/3, 2).$$

This is a consequence of the fact that Moore spaces always have category equal to 1 (for a proof, see [3], Example 1.33), and using the Künneth theorem reveals that the product $M(\mathbb{Z}/2, 2) \times M(\mathbb{Z}/3, 2)$ has homology isomorphic to $\mathbb{Z}/2 \oplus \mathbb{Z}/3$ in degree 2 and trivial homology in all other positive degrees. Hence, $M(\mathbb{Z}/2,2) \times M(\mathbb{Z}/3,2)$ is the Moore space $M(\mathbb{Z}/2 \oplus \mathbb{Z}/3, 2)$. Hence

$$1 = \mathrm{cat}(M(\mathbb{Z}/2,2) \times M(\mathbb{Z}/3,2)) < \mathrm{cat}\, M(\mathbb{Z}/2,2) + \mathrm{cat}\, M(\mathbb{Z}/3,2) = 2.$$

In 1971, Ganea [15] formulated the conjecture

$$\mathrm{cat}(X \times S^n) = \mathrm{cat}(X) + 1$$

for any finite dimensional CW complex X and any $n \geq 1$. The reasoning behind this conjecture was based on the fact that all known counter-examples to the equality $\mathrm{cat}(X \times Y) = \mathrm{cat}(X) + \mathrm{cat}(Y)$ involved spaces that had different torsion in their homology groups as in the above example. Because the homology of S^n is torsion-free, Ganea made the conjecture. However, 27 years later, Iwase [22], [23] found counter-examples to the conjecture without different torsion in homology. Hence, the lack of torsion is only one possible factor causing Ganea's conjecture to fail.

Yet, the situation is different in the rational world, where by default every space is torsion-free (both in homology and homotopy, after passing to the rationalization of a space or equivalently, to its minimal Sullivan model). Jessup [25] first showed that Ganea's conjecture is true when replacing cat by mcat_0, the module L.-S. category. Hess later showed that $\mathrm{mcat}_0 = \mathrm{cat}_0$ (recall Theorem 3.13). Combining this with Jessup's result, it follows that Ganea's conjecture holds in the rational case, that is for any simply connected space X and $n \geq 2$, we have

$$\mathrm{cat}_0(X \times S^n) = \mathrm{cat}_0(X) + 1.$$

By considering the dual $(\Lambda V, d)^\sharp$ as a $(\Lambda V, d)$-module, Félix, Halperin and Lemaire finally proved that in fact, cat_0 is additive.

Theorem 3.14. [11] *If X and Y are simply connected CW complexes with rational homology of finite type, then*

$$\operatorname{cat}_0(X \times Y) = \operatorname{cat}_0 X + \operatorname{cat}_0 Y.$$

Because of the importance of this result, we will restrict our study to simply connected CW complexes with rational homology of finite type in chapter 4.

3.5 Elliptic spaces

3.5.1 Definition

Recall that a graded vector space is said to be *finite dimensional* if it is finite dimensional in each degree and if it is non-zero only in finitely many degrees.

Definition 3.15. A Sullivan algebra $(\Lambda V, d)$ is *elliptic* if $H(\Lambda V, d)$ and V are both finite dimensional.

Definition 3.16. A simply connected topological space X is *rationally elliptic* if $H^*(X; \mathbb{Q})$ and $\pi_*(X) \otimes \mathbb{Q}$ are both finite dimensional.

Proposition 3.17. *Consider a simply connected topological space X with associated minimal Sullivan model $(\Lambda V, d)$. Then, X is rationally elliptic if and only if $(\Lambda V, d)$ is elliptic.*

Proof. This is a direct consequence of the isomorphisms

$$H(\Lambda V, d) \cong H^*(X; \mathbb{Q}) \qquad \text{and} \qquad V \cong \operatorname{Hom}_{\mathbb{Z}}(\pi_*(X), \mathbb{Q})$$

described in chapter 2. ■

Examples. Important examples of rationally elliptic spaces include spheres, finite complex projective space, homogeneous spaces and products of rationally elliptic spaces.

3.5.2 Poincaré duality algebras

A related and useful notion is that of a "Poincaré duality algebra", which we introduce now.

Definition 3.18. Let $A = \{A_i\}_{0\leq i\leq n}$ be a finite dimensional commutative graded algebra such that $A^0 = \mathbb{Q}$. Let $\omega : A^n \to \mathbb{Q}$ be any linear form. We say that A is a *Poincaré duality algebra* with *fundamental class* ω if the following maps

$$\begin{aligned}\langle\ ,\ \rangle :\ & A^{n-p} \times A^p \longrightarrow \mathbb{Q} \\ & \qquad (a,b) \longmapsto \langle a,b\rangle = \omega(ab), \qquad a \in A^{n-p},\ b \in A^p\end{aligned}$$

are non-degenerate bilinear maps.

In particular, there are isomorphisms $A^{n-p} \cong A^p$ for $0 \leq p \leq n$. Thus, $\dim A^n = 1$ and any nonzero element in A^n is called a *top cohomology class.*

Now consider the following very useful formula for the degree of a top cohomology class:

Theorem 3.19. [19] *Suppose that X is a simply connected rationally elliptic space with associated minimal Sullivan model $(\Lambda V, d)$, such that its cohomology $H^*(X;\mathbb{Q})$ satisfies Poincaré duality. Let n be the largest integer such that $H^n(X;\mathbb{Q}) \neq 0$. Let $\{x_i\}$ be a basis for V^{even} and let $\{y_j\}$ be a basis for V^{odd}. Then,*

$$n = \sum_i (1 - |x_i|) + \sum_j |y_j|.$$

Moreover, if Poincaré duality is satisfied, computation of the L.-S. category is reduced to the computation of the Toomer invariant. The reader is referred to [10], Theorem 38.4, for the proof of the following theorem.

Theorem 3.20. *Suppose that X is a simply connected space such that its cohomology algebra satisfies Poincaré duality, then*

$$\mathrm{cat}_0(X) = \mathrm{e}_0(X).$$

The theorem above also apply to any rationally elliptic space, because the cohomology of any rationally elliptic space satisfies Poincaré duality, as guaranteed by the next result.

Theorem 3.21. *Let X be a rationally elliptic space. Then, its cohomology $H^*(X;\mathbb{Q})$ is a Poincaré duality algebra. Similarly, if $(\Lambda V, d)$ is an elliptic Sullivan algebra, then $H(\Lambda V, d)$ is a Poincaré duality algebra.*

For a proof, we refer the reader to [19] (see also [10], Theorem 38.3).

3.5.3 Pure Sullivan algebras

Here, we introduce another concept related to rationally elliptic spaces. It is not generally easy to compute the cohomology of a space or its related minimal Sullivan model in all degrees. Hence, we need simpler criteria. Such criteria are available in the particular case where a Sullivan model is "pure".

Definition 3.22. Let $(\Lambda V, d)$ be a Sullivan algebra. Define $Q = V^{\mathrm{even}}$ and $P = V^{\mathrm{odd}}$, so that $\Lambda V = \Lambda(Q \oplus P) \cong \Lambda Q \otimes \Lambda P$. We say that $(\Lambda V, d)$ is *pure* if V is finite dimensional, $d(Q) = 0$ and $d(P) \subset \Lambda Q$. A simply connected space X is *pure* if its associated minimal Sullivan model is pure.

Examples. The minimal Sullivan models $(\Lambda(a_{2n+1}), 0)$ and $(\Lambda(a_{2n}, x_{4n-1}),\ da = 0,$ $dx = a^2)$ of an odd sphere S^{2n+1} and of an even sphere S^{2n}, respectively, are pure Sullivan algebras. The minimal Sullivan model $(\Lambda(\alpha_2, \beta_{2n+1}), d\alpha = 0, d\beta = \alpha^{n+1})$ of the complex projective space $\mathbb{C}P^n$ is pure.

Example. Consider the minimal Sullivan algebra $(\Lambda V, d) = (\Lambda(x_3, y_3, z_5), d)$, where the index represent the degree of each generator and where d is defined by $d(x_3) = d(y_3) = 0$ and $d(z_5) = x_3y_3$. This Sullivan algebra is not pure, because the condition $d(P) \subset \Lambda Q$ is not satisfied. However, a simple calculation shows that $H(\Lambda(x, y, z), d) = \mathbb{Q} \cdot [x] \oplus \mathbb{Q} \cdot [y] \oplus \mathbb{Q} \cdot [xz] \oplus \mathbb{Q} \cdot [yz] \oplus \cdot \mathbb{Q}[xyz])$. Hence, this Sullivan algebra is elliptic, showing that elliptic algebras are not necessarily pure.

Now, we introduce a characterization of ellipticity.

Theorem 3.23. [27] *Let $(\Lambda V, d)$ be a pure Sullivan algebra, where $Q = V^{even}$ and $P = V^{odd}$. Then, $H(\Lambda V, d)$ is finite dimensional if and only if the following quotient*

$$\Lambda Q/\Lambda Q \cdot d(P)$$

is finite dimensional.

For a proof, the reader is also referred to [10], Proposition 32.1.

Example 3.24. Consider the minimal Sullivan algebra $(\Lambda V, d) = (\Lambda(a, b, x, y, z), d)$, $|a| = |b| = 2$, $|x| = 3$ and $|y| = |z| = 5$ with differential given by $d(a) = d(b) = 0$, $d(x) = a^2$, $d(y) = b^3$ and $d(z) = ab^2$. We will use the theorem above to prove that this Sullivan algebra is elliptic.

In this case, $Q = V^{\text{even}}$ has basis $\{a, b\}$ and $P = V^{\text{odd}}$ has basis $\{x, y, z\}$. So, since $dx = a^2$, $dy = b^3$ and $dz = ab^2$, it follows that $d(P)$ has basis $\{a^2, b^3, ab^2\}$. Thus, we compute that

$$\Lambda Q/\Lambda Q \cdot d(P) = \mathbb{Q} \cdot [a] \oplus \mathbb{Q} \cdot [b] \oplus \mathbb{Q} \cdot [ab].$$

So, because $\Lambda Q/\Lambda Q \cdot d(P)$ is finite dimensional, then $H(\Lambda V, d)$ is also finite dimensional. This means that $(\Lambda V, d)$ is elliptic in this case.

Finally, we make a remark regarding the case when a Sullivan algebra $(\Lambda V, d)$ is not pure. Assuming that V is finite dimensional, there is a construction that associate to $(\Lambda V, d)$ a pure Sullivan algebra $(\Lambda V, d_\sigma)$. Moreover, there is a theorem that says that $(\Lambda V, d)$ is elliptic if and only if $(\Lambda V, d_\sigma)$ is elliptic. So it suffices to check that $(\Lambda V, d_\sigma)$ is elliptic, using Theorem 3.23, to prove that $(\Lambda V, d)$ is also elliptic. Since most of the examples we will be working with are pure Sullivan algebras, we do not include the construction of $(\Lambda V, d_\sigma)$ here, but the interested reader is referred to [10], Proposition 32.4.

3.6 Formal spaces

Recall from chapter 2 that we defined a space X with associated minimal Sullivan model $(\Lambda V, d)$ to be formal if there exists a quasi-isomorphism of cdga's

$$\varphi : (\Lambda V, d) \xrightarrow{\simeq} (H^*(X; \mathbb{Q}), 0).$$

Computing the rational L.-S. category is particularly simple for formal spaces, because of the following result (for a proof, see [10], Chapter 29, section (b), Example 4).

Theorem 3.25. *If X is a formal space, then* $\mathrm{cup}(X) = \mathrm{e}_0(X) = \mathrm{cat}_0(X)$.

Examples. Sphere and complex projective spaces are formal, and a product of formal spaces or a wedge of formal spaces is also formal (see [10], Chapter 12, section (c)).

3.7 A model of the projection $\pi_m : (\Lambda V, d) \to (\Lambda V/\Lambda^{>m}V, \bar{d})$

Throughout this section, fix a minimal model $(\Lambda V, d)$ and fix a positive integer m. Denote by π_m the associated projection map $(\Lambda V, d) \longrightarrow (\Lambda V/\Lambda^{>m}V, \bar{d})$, where $\bar{d}$ is the map $\bar{d}(\bar{x}) = \overline{d(x)}$. We will revisit a construction originally due to Félix et al. [9], who refined the model of π_m. We will see how it is always possible to construct a semifree resolution

$$\zeta_m : (\Lambda V \otimes (\mathbb{Q} \oplus M), \delta) \xrightarrow{\simeq} (\Lambda V/\Lambda^{>m}V, \bar{d})$$

that fits into the following diagram of $(\Lambda V, d)$-modules

$$\begin{tikzcd}
(\Lambda V, d) \arrow[r, "\pi_m"] \arrow[dr, "\lambda_m"'] & (\Lambda V/\Lambda^{>m}V, \bar{d}) \\
& (\Lambda V \otimes (\mathbb{Q} \oplus M), \delta), \arrow[u, "\simeq"', "\zeta_m"]
\end{tikzcd}$$

such that $\zeta_m \circ \lambda_m = \pi_m$ and satisfying

(i) the map ζ_m can always be chosen such that $\zeta_m(M) = 0$,

(ii) every $(\Lambda V, d)$-module in the diagram above admits a bigradation,

(iii) every map in the diagram above is a morphism of bigraded $(\Lambda V, d)$-modules of bidegree $(0, 0)$.

Recall that the inclusion λ_m in the diagram above admits a retraction of $(\Lambda V, d)$-modules if and only if $\mathrm{cat}_0(\Lambda V, d) \leq m$. That is, there exists a map $\rho_m : (\Lambda V \otimes (\mathbb{Q} \oplus M), \delta) \to (\Lambda V, d)$ such that $\rho_m \circ \lambda_m = \mathrm{id}_{(\Lambda V, d)}$ if and only if $\mathrm{cat}_0(\Lambda V, d) \leq m$.

3.7.1 The construction of M and δ

We start by describing an algorithm for computing M and the differential δ. The exposition is based on [10] (Chapter 29, section (f), pp.396-399), but we include it here for convenience and because it will be used in all the computations of the next chapter. The reader is also referred to the original article [9] for a slightly different approach. The construction of M and δ are carried out in multiple steps.

First step. *The quadratic part. Construction of a model for* $(\Lambda V, d_1) \to (\Lambda V/\Lambda^{>m}V, \bar{d}_1)$.

The first step involves restricting our attention to the quadratic part d_1 of the differential d and constructing a model for the projection $(\Lambda V, d_1) \longrightarrow (\Lambda V/\Lambda^{>m}V, \bar{d}_1)$. We will construct a $(\Lambda V, d_1)$-semifree resolution $(\Lambda V \otimes (\mathbb{Q} \oplus M), d_1)$ for $(\Lambda V/\Lambda^{>m}V, \bar{d}_1)$. The module M is constructed using an induction argument.

Define $M^0 = 0$ and then assume that $M^{<n}$ is already defined and that d_1 has been extended onto $M^{<n}$, satisfying $d_1(M^{<n}) \subset \Lambda^{>m}V \oplus (\Lambda^+V \otimes M^{<n-1})$. Consider

$$H^{n+1}(\Lambda^{>m}V \oplus (\Lambda^+V \otimes M^{<n}), d_1)$$

and suppose that $\{[\omega_i]\}_{i\in I}$ is a basis for the above cohomology. Then, for each $[\omega_i]$, introduce a corresponding basis element for M^n called s_i. In other words, we define

$$M^n := \operatorname{span}\{s_i\}_{i\in I}.$$

Moreover, for each $i \in I$, define

$$d_1(s_i) := \omega_i.$$

Clearly, this makes d_1 into a differential. This completes the induction step and the definition of M.

Now, we can define a morphism of $(\Lambda V, d_1)$-modules

$$\xi : (\Lambda V \otimes (\mathbb{Q} \oplus M), d_1) \longrightarrow (\Lambda V/\Lambda^{>m}V, \bar{d}_1)$$

by setting $\xi(M) = 0$ and $\xi(1) = 1$. Then, ξ is a quasi-isomorphism of $(\Lambda V, d_1)$-modules: indeed, to show surjectivity of $H(\xi)$, suppose $[\bar{x}] \in H(\Lambda V/\Lambda^{>m}V, \bar{d}_1)$ is a cocycle, then by definition $d_1 x \in \Lambda^{>m}V$. Hence, from the inductive definition of M

above, there exists $s \in \Lambda V \otimes (\mathbb{Q} \oplus M)$ such that $d_1 s = d_1 x$. Thus $x - s$ is a cocycle in $(\Lambda V \otimes (\mathbb{Q} \oplus M), d_1)$ and $H(\xi)([x - s]) = [x]$.

Next, we show injectivity of $H(\xi)$. If $H(\xi)[x] = H(\xi)[y]$, then $\xi(x - y) \in \Lambda^{>m} V$. Because $\xi(M) = 0$, this means that $x - y \in \Lambda^{>m} V \oplus (\Lambda V \otimes M)$. If we let n be the degree of $x - y$, then we can write $x - y = a + b + c$, where $a \in \Lambda^{>m} V$, $b \in \Lambda^{+} V \otimes M$ and $c \in M^n$. However, we assumed x and y are cocycles, so in particular $d_1(x - y) = 0$. In other words, $d_1(a + b + c) = 0$. So $d_1(c) = d_1(-a - b)$. Now, by the inductive definition of d_1 on M^n above, we can infer that $c = 0$, because $d_1(-a - b)$ is a coboundary. Hence, we can conclude that $x - y = a + b$, i.e. $x - y \in \Lambda^{>m} V \oplus (\Lambda^{+} V \otimes M)$. Therefore, by the inductive definition of M above, there exists $s \in \Lambda V \otimes (\mathbb{Q} \oplus M)$ such that $d_1 s = x - y$. In other words, $[x] = [y]$ in the cdga $(\Lambda V \otimes (\mathbb{Q} \oplus M), d_1)$.

Second step. *Bigradation of M.*

Now, we explain how M can be viewed as a bigraded module. Start by bigrading ΛV by defining $(\Lambda V)^{p,q} = (\Lambda^p V)^{p+q}$. It follows that d_1 is homogeneous of bidegree $(1, 0)$ in ΛV. Moreover, the inductive definition of M above induces a bigradation in M. There is always a unique choice of bidegree (p, q) for elements of M that will make it such that d_1 is homogeneous of bidegree $(1, 0)$ on M. From this, it is clear in particular that $M = \bigoplus_{k \geq m} M^{k,*}$. So, the $(\Lambda V, d_1)$-differential module $\Lambda V \otimes (\mathbb{Q} \oplus M)$ also comes equipped with an induced bigradation, $(\Lambda V \otimes (\mathbb{Q} \oplus M))^{p,q}$. We call p the filtration degree or word-length and q the complementary degree.

As we noted, by the definition of the bigradation on M it is clear that $M = \bigoplus_{p \geq m} M^{p,*}$. Surprisingly, however, it turns out that $M^{p,*} = 0$ for any $p > m$. That is, $M = M^{m,*}$ (see [10], Lemma 29.11, for a proof):

Theorem 3.26. $M = M^{m,*}$

Third step. *Perturbation of d_1 to δ.*

At this point we have constructed a model M and a differential d_1. The next step involves perturbing d_1 into a differential δ, making $(\Lambda V \otimes (\mathbb{Q} \oplus M), \delta)$ into a $(\Lambda V, d)$-semifree resolution for the projection map onto the quotient $(\Lambda V/\Lambda^{>m}, \bar{d})$.

The new differential δ will be defined on M by induction on the topological degree. Moreover, at each step of the induction it will satisfy

$$\delta - d_1 : [\Lambda V \otimes (\mathbb{Q} \oplus M)]^{p,*} \longrightarrow [\Lambda V \otimes (\mathbb{Q} \oplus M)]^{\geq p+2,*}.$$

Before defining δ, however, we need the following lemma.

Lemma 3.27. *Every δ-cocyle of degree $n+2$ in $[\Lambda V \otimes (\mathbb{Q} \oplus M^{<n})]^{\geq m+3,*}$ is the δ-coboundary of an element in $[\Lambda V \otimes (\mathbb{Q} \oplus M^{<n})]^{\geq m+2,*}$.*

Proof. Let $z \in [\Lambda V \otimes (\mathbb{Q} \oplus M^{<n})]^{\geq m+3,*}$ be a δ-cocycle of degree $n+2$. We can decompose z as a sum $z = \sum_{i=r}^{n+2} x_i$, where each x_i has filtration degree i, for some $r \geq m+3$ (the filtration degree i is at most $n+2$ for degree reasons alone). Next, we can decompose the differential δ as $\delta = d_1 + D$, where D is the non-quadratic extension of d_1 (i.e. D increases filtration degree by at least 2). Then, keeping in mind that d_1 increases filtration degree by exactly 1, we can write the following

$$0 = \delta z = \delta\left(\sum_{i=r}^{n+2} x_i\right) = \underbrace{\delta_1 x_r}_{\substack{\text{filtration}\\ \text{degree}\\ r+1}} + \underbrace{Dx_r + \delta\left(\sum_{i=r+1}^{n+2} x_i\right)}_{\text{filtration degree } \geq r+2}$$

which implies that $\delta_1 x_r = 0$. However, by construction $H(\Lambda V \otimes (\mathbb{Q} \oplus M), d_1)$ is concentrated in filtration degrees strictly less than m, because $H(\Lambda V \otimes (\mathbb{Q} \oplus M), d_1) \cong H(\Lambda V/\Lambda^{>m} V, \bar{d}_1)$. Hence, x_r being a cocycle must also be a coboundary, so write $x_r = d_1 x'$ with x' of filtration degree $r-1$. In particular, $x' \in (\Lambda^{\geq 2} V \otimes M)^{n+1} \oplus \Lambda V$ because the filtration degree of x' is $r-1 \geq m+2$. We can see however (for degree reasons) that $(\Lambda^{\geq 2} V \otimes M)^{n+1} \oplus \Lambda V \subset \Lambda V \otimes (\mathbb{Q} \oplus M^{<n})$.

Now, $z - dx' = \sum_{i=r+1}^{n+2} x_i$ with x_i of filtration degree i. We can therefore iterate the same argument, at each step merely increasing the filtration degree, until $z - du = 0$, for some $u \in [\Lambda V \otimes (\mathbb{Q} \oplus M^{<n})]^{\geq m+2,*}$. ■

Now, we are ready to define δ by induction. Define $\delta = d$ on ΛV. Now, for the induction hypothesis, assume that δ is already defined on $M^{<n}$ and that $\delta - d_1$ increases filtration degree by at least 2. Then, fix $w \in M^n$ (note that when there is only one index n, it always refers to the topological degree). Recall that $d_1 w \in \Lambda V \otimes (\mathbb{Q} \oplus M^{<n})$. Clearly, $\delta d_1 w$ is a δ-cocyle of degree $n+2$. Moreover, $\delta d_1 w \in [\Lambda V \otimes (\mathbb{Q} \oplus M^{<n})]^{\geq m+3,*}$. This can be seen if one writes $\delta d_1 = (\delta - d_1)d_1$ (which we can do since $d_1^2 = 0$) and then notice that d_1 increases word length by 1 and $\delta - d_1$ increases word length by at least 2. So δd_1 increases filtration degree by at least 3 and ω has filtration degree m since $M = M^{m,*}$, hence δd_1 has filtration degree $m+3$ or greater.

Now, by the lemma above, there exists some $u \in [\Lambda V \otimes (\mathbb{Q} \oplus M^{<n})]^{\geq m+2,*}$ such that $\delta d_1 w = \delta u$. Therefore, we define δ on M^n by

$$\delta w := d_1 w - u$$

as w runs through a basis of M^n. Clearly, $\delta\delta w = 0$, making δ into a differential. This completes the induction step and so, δ can be defined on all of M with the property that

$$\delta - d_1 : [\Lambda V \otimes (\mathbb{Q} \oplus M)]^{p,*} \longrightarrow [\Lambda V \otimes (\mathbb{Q} \oplus M)]^{\geq p+2,*}.$$

and such that $\delta = d$ on ΛV.

Fourth step. *Semifree resolution of* $(\Lambda V/\Lambda^{>m}V, \bar{d})$.

Define a morphism of $(\Lambda V, d)$-differential modules

$$\zeta_m : (\Lambda V \otimes (\mathbb{Q} \oplus M), d) \longrightarrow (\Lambda V/\Lambda^{>m}V, \bar{d})$$

by $\zeta_m|_{\Lambda V} = \pi_m$ and $\zeta_m(M) = 0$. Then, a spectral sequence argument can be used to prove the following (see [10], step 2 of p.398)

Proposition 3.28. *The map ζ_m is a $(\Lambda V, d)$-semifree resolution of $(\Lambda V/\Lambda^{>m}V, \bar{d})$*

This completes the construction.

3.7.2 Relationship with L.-S. category

Next, we see how this semifree resolution relates to the L.-S. category of $(\Lambda V, d)$. We proceed as follows. First, construct a relative Sullivan model for the projection map π_m,

$$\begin{tikzcd}
(\Lambda V, d) \arrow[r, "\pi_m"] \arrow[dr, "\lambda_m"'] & (\Lambda V/\Lambda^{>m}V, \bar{d}) \\
& (\Lambda V \otimes \Lambda Z, \delta) \arrow[u, "\simeq"', "h"]
\end{tikzcd}$$

for some fixed positive integer m.

Lemma 3.29. *The relative Sullivan model $(\Lambda V \otimes \Lambda Z, \delta)$ can be chosen such that $Z = \bigoplus_{p \geq 0} Z_p$ and such that*

(i) $(\Lambda V \otimes (\mathbb{Q} \oplus Z_0), \delta)$ is a $(\Lambda V, d)$-semifree resolution of $(\Lambda V/\Lambda^{>m}V, \bar{d})$,

(ii) the inclusion of $(\Lambda V \otimes (\mathbb{Q} \oplus Z_0), \delta)$ in $(\Lambda V \otimes \Lambda Z, d)$ is a quasi-isomorphism.

Outline of proof. The idea begins with setting $Z_0 = M$, where M is the $(\Lambda V, d)$-differential module defined above, and then extend Z_0 to $Z = \bigoplus_{p \geq 0} Z_p$. However, we refer the reader to [10], Proposition 29.10, for the full proof. ■

Finally, we need the following result:

Theorem 3.30. *Let* $(\Lambda V, d)$ *be a minimal Sullivan model. Fix an integer* m *and consider the* $(\Lambda V, d)$*-semifree resolution for the quotient* $(\Lambda V/\Lambda^{>m}V, \bar{d})$*, which was constructed in this section and fits into the following commutative diagram*

$$\begin{tikzcd}
(\Lambda V, d) \arrow[r, "\pi_m"] \arrow[dr, "\lambda_m"'] & (\Lambda V/\Lambda^{>m}V, \bar{d}) \\
 & (\Lambda V \otimes (\mathbb{Q} \oplus M), \delta) \arrow[u, "\zeta_m"', "\simeq"]
\end{tikzcd}$$

Then, λ_m *admits a retraction* ρ *of* $(\Lambda V, d)$*-differential modules if and only if* $\mathrm{cat}_0(\Lambda V, d) \leq m$.

Proof. Suppose that $\mathrm{cat}_0(\Lambda V, d) \leq m$. This means that in the following commutative diagram

$$\begin{tikzcd}
(\Lambda V, d) \arrow[r, "\pi_m"] \arrow[dr, "i"'] & (\Lambda V/\Lambda^{>m}V, \bar{d}) \\
 & (\Lambda V \otimes \Lambda Z, \delta) \arrow[u, "h"', "\simeq"]
\end{tikzcd}$$

the inclusion i admits a retraction $\rho : (\Lambda V \otimes \Lambda Z, \delta) \to (\Lambda V, d)$ of cdga's. Next, construct the $(\Lambda V, d)$-semifree resolution as in the earlier diagram. Then, we have a commutative square

$$\begin{tikzcd}
(\Lambda V, d) \arrow[r, "i"] \arrow[d, "\lambda_m"] & (\Lambda V \otimes \Lambda Z, \delta) \arrow[d, "h", "\simeq"'] \\
(\Lambda V \otimes (\mathbb{Q} \oplus M), \delta) \arrow[r, "\zeta_m", "\simeq"'] & (\Lambda V/\Lambda^{>m}V, \bar{d}).
\end{tikzcd}$$

Hence, because h is a surjective quasi-isomorphism, by the lifting property (for

$(\Lambda V, d)$-differential modules, see Theorem 2.21), there exists a lift of ζ_m to a $(\Lambda V, d)$-differential map $\alpha : (\Lambda V \otimes (\mathbb{Q} \oplus M), \delta) \to (\Lambda V \otimes \Lambda Z, \delta)$. Therefore, define $\rho' = \rho \circ \alpha$. It is then straightforward to verify that ρ' is a retraction of λ_m, as desired.

The converse implication is a consequence of the corollary on p.395 of [10]. ■

3.8 An example (construction of M and δ)

In this section, we will see a concrete example of a perturbation of d_1 to δ, using the algorithm of section 3.7.

Consider again (from the example at the end of section 2.9) the Sullivan model $(\Lambda V, d) = (\Lambda(a, b, x, y), d)$, $|a| = 2$, $|b| = 6$, $|x| = 7$, $|y| = 11$, $da = db = 0$, $dx = a^4 - 2ab$ and $dy = b^2$. Notice that $(\Lambda V, d)$ is a pure model with $Q = V^{\text{even}} = \text{span}\{a, b\}$ and $P = V^{\text{odd}} = \text{span}\{x, y\}$ and that the quotient

$$\Lambda Q / \Lambda Q \cdot d(P)$$

is finite dimensional, because any basis element in ΛQ has the form $a^m b^n$, which is zero in the quotient algebra if either $n \geq 2$ or $m \geq 7$. Indeed, if $n \geq 2$, then $a^m b^n = a^m b^{n-2} dy$. If $m > 7$, then because $a^7 = a^3 dx + 2bdx + 2ady$, we can write $a^m b^n = a^{m-7} b^n (a^3 dx + 2bdx + 2ady)$. In either case, $a^m b^n \in \Lambda Q \cdot d(P)$. Therefore, this Sullivan model is elliptic.

Now, a straightforward computation of the cohomology of $(\Lambda V, d)$ reveals that $\mathrm{e}_0(\Lambda V, d) = 6$. Indeed, the top cohomology class is in degree 12 (recall Theorem 3.19), so for degree reasons alone, the cocycle of greatest word length in $(\Lambda V, d)$ is a^6. Moreover, it can be shown that a^6 is a cocycle which is not a coboundary, hence $\mathrm{e}_0(\Lambda V, d) = 6$. Then, by ellipticity (see Theorem 3.20) we know that $\mathrm{cat}_0(\Lambda V, d) = \mathrm{e}_0(\Lambda V, d) = 6$.

Throughout this section, we will fix $m = 6$ and consider the projection $(\Lambda V, d) \longrightarrow$

$(\Lambda V/\Lambda^{>6}V, d)$.

3.8.1 Construction of M and extension of d_1

The first step in constructing M is to consider the quadratic model $(\Lambda V, d_1)$, where $d_1 a = d_1 b = 0$, $d_1 x = -2ab$ and $d_1 y = b^2$. Recall the inductive definition of M:

$$M^n \cong H^{n+1}(\Lambda^{>6}V \oplus (\Lambda^+ V \otimes M^{<n}), d_1).$$

Observe that a^7 is the only linearly independent word of smallest degree (i.e. 14) in $\Lambda^{>6}V$. Hence, $M^{<13} = 0$ and

$$M^{13} \cong \mathbb{Q} \cdot [a^7] \cong \mathbb{Q} \cdot s$$

for some generator $s \in M^{13}$ (as a basis element for M^{13}) satisfying $d_1 s = a^7$. Note that the bidegree on s is defined to be $(6, 7)$ (we take the "word length" degree of s to be 6, then automatically the complementary degree has to be 7). This is necessary in order for d_1 to remain a morphism of bidegree $(1, 0)$, because the image of s is a^7, which has bidegree $(7, 7)$. Of course, the fact that the filtration degree of s is $6 = \mathrm{cat}_0(\Lambda V, d)$ is what we expect since it is guaranteed by Theorem 3.26. The next step is to calculate

$$M^{14} \cong H^{15}(\Lambda^{>6}V \oplus (\Lambda^+ V \otimes M^{13}), d_1).$$

A somewhat tedious computation will reveal that

$$M^{14} = 0$$

and similarly, we can calculate that $M^{15} = M^{16} = 0$. However, we find that

$$\begin{aligned} M^{17} &\cong H^{18}(\Lambda^{>6}V \oplus (\Lambda^+V \otimes M^{13}), d_1) \\ &= \mathbb{Q} \cdot [a^6 b]. \end{aligned}$$

Hence, we introduce a generator $t \in M^{17}$ and define $d_1 t = a^6 b$. Again, because $a^6 b$ has length 7, we need to define the bidegree of t to be $(6, 10)$. The next step is to compute

$$\begin{aligned} M^{18} &\cong H^{19}(\Lambda^{>6}V \oplus (\Lambda^+V \otimes (M^{13} \oplus M^{17})), d_1) \\ &= \mathbb{Q} \cdot [a^6 x + 2bs] \oplus \mathbb{Q} \cdot [bs - at]. \end{aligned}$$

Therefore, we introduce two generators $u, u' \in M^{18}$, with differentials $d_1 u = a^6 x + 2bs$ and $d_1 u' = bs - at$. If we keep going, we find generators $v \in M^{19}$, $w \in M^{21}$, etc. In general, there is no reason for this process to end so that M will often be infinite dimensional.

3.8.2 Perturbation of d_1 to δ

We perturb d_1 to δ using the algorithm described in the previous section. We assume that δ extends d, so that $\delta = d$ on ΛV. Then, we define δ on M by induction over the topological degree in M. The algorithm says that we need to compute $\delta(d_1(m))$ for every generator $m \in M$ and see if the result can be written as the image under δ of an element of word length strictly greater than 6.

First, define $\delta = d$ on ΛV. Next, we start with the generator s.

$$\delta(d_1 s) = \delta(a^7) = d(a^7) = 0.$$

Hence, we do not need to perturb d_1 on s. So, define $\delta s = d_1 s$. Next, compute

$$\delta(d_1 t) = \delta(a^6 b) = d(a^6 b) = 0.$$

Hence, we just define $\delta t = d_1 t$. Next, compute

$$\begin{aligned}
\delta(d_1 u) &= \delta(a^6 x + 2bs) = a^6(a^4 - 2ab) + 2ba^7 \\
&= a^{10} - 2a^7 b + 2a^7 b \\
&= a^{10} \\
&= \delta(a^3 s)
\end{aligned}$$

Here, we find that $\delta(d_1 u) = \delta(a^3 s)$, so we have to perturb d_1 by defining $\delta u = d_1 u - a^3 s = a^6 x + 2bs - a^3 s$.

The process continues henceforth by induction, such that δ becomes a differential on $\Lambda V \otimes M$ with $\delta = d$ on ΛV.

3.9 The isomorphism that describes M

Let $(\Lambda V, d)$ be a minimal model with $\mathrm{cat}_0(\Lambda V, d) = m$ and consider the usual diagram for a semifree resolution of the projection map:

$$\begin{tikzcd}
(\Lambda V, d) \arrow[r, "\pi"] \arrow[dr] & (\Lambda V/\Lambda^{>m} V, \bar{d}) \\
& (\Lambda V \otimes (\mathbb{Q} \oplus M), \delta) \arrow[u, "h", "\simeq"']
\end{tikzcd}$$

Our aim in this section is to understand an isomorphism that describes explicitly the model M and its differential δ. Recall from chapter 2 that $\overline{V}$ stands for the suspension of V, defined as $\overline{V}^i = V^{i+1}$. We can now state the following

Theorem 3.31. *There exists an isomorphism*

$$(\mathbb{Q} \oplus M)^n \cong H^{n+1}(\Lambda^{>m}V \otimes \Lambda\overline{V}, d)$$

for any integer $n \geq 1$.

Proof. We will need to understand exactly how this isomorphism is constructed. There are basically three important steps and in fact we will exhibit this isomorphism as the composition of three isomorphisms.

The first step is the connecting homomorphism

$$\partial^\sharp : H^n(\Lambda V/\Lambda^{>m}V \otimes \Lambda\overline{V}, d) \xrightarrow{\cong} H^{n+1}(\Lambda^{>m}V \otimes \Lambda\overline{V}, d),$$

obtained from the short exact sequence

$$0 \longrightarrow \Lambda^{>m}V \otimes \Lambda\overline{V} \longrightarrow \Lambda V \otimes \Lambda\overline{V} \longrightarrow \Lambda V/\Lambda^{>m}V \otimes \Lambda\overline{V} \longrightarrow 0.$$

The second step in the construction is the isomorphism induced by the following quasi-isomorphism:

$$h \otimes id_{\Lambda V \otimes \Lambda\overline{V}} : (\Lambda V \otimes (\mathbb{Q} \oplus M) \otimes_{\Lambda V} [\Lambda V \otimes \Lambda\overline{V}], \delta) \xrightarrow{\simeq} (\Lambda V/\Lambda^{>m}V \otimes_{\Lambda V} [\Lambda V \otimes \Lambda\overline{V}], d).$$

But observe that we have the following two isomorphisms of $(\Lambda V, d)$-modules:

$$(\Lambda V \otimes (\mathbb{Q} \oplus M) \otimes_{\Lambda V} [\Lambda V \otimes \Lambda\overline{V}], \delta) \cong ((\mathbb{Q} \oplus M) \otimes [\Lambda V \otimes \Lambda\overline{V}], \delta),$$
$$(\Lambda V/\Lambda^{>m}V \otimes_{\Lambda V} [\Lambda V \otimes \Lambda\overline{V}], d) \cong (\Lambda V/\Lambda^{>m}V \otimes \Lambda\overline{V}, d).$$

So we can see the map $h \otimes id_{\Lambda V \otimes \Lambda \overline{V}}$ as a map:

$$h \otimes id_{\Lambda V \otimes \Lambda \overline{V}} : ((\mathbb{Q} \oplus M) \otimes [\Lambda V \otimes \Lambda \overline{V}], \delta) \xrightarrow{\simeq} (\Lambda V / \Lambda^{>m} V \otimes \Lambda \overline{V}, d).$$

Finally, the third and last step is to consider the short exact sequence (exact in all positive degrees)

$$0 \longrightarrow (\Lambda V \otimes \Lambda \overline{V}, d) \longrightarrow ((\mathbb{Q} \oplus M) \otimes [\Lambda V \otimes \Lambda \overline{V}], \delta) \xrightarrow{\alpha} (\mathbb{Q} \oplus M, 0) \longrightarrow 0,$$

which is a well-defined sequence if we assume that $\delta(M) \subset \Lambda V^+ \otimes (\mathbb{Q} \oplus M)$. It follows that the map

$$\alpha : ((\mathbb{Q} \oplus M) \otimes [\Lambda V \otimes \Lambda \overline{V}], \delta) \xrightarrow{\simeq} (\mathbb{Q} \oplus M, 0)$$

is a quasi-isomorphism.

Thus, to be very explicit, we could describe the isomorphism stated in the theorem by the composition

$$\partial^\sharp \circ H(h \otimes id_{\Lambda V \otimes \Lambda \overline{V}}) \circ H(\alpha)^{-1} : (\mathbb{Q} \oplus M)^n \longrightarrow H^{n+1}(\Lambda^{>m} V \otimes \Lambda \overline{V}, d).$$

■

Remark. As it turns out, this process also allows us to determine the differential δ on M. This is achieved in the second step of the above proof. This will be illustrated in the examples of the next chapter.

Chapter 4

The rational retraction index

4.1 Definition and basic properties

In this section we introduce the rational retraction index of a space, a numerical homotopy invariant. It is the main object of study of this thesis. It was introduced in [4] and it is closely related to the rational L.-S. category of a space. The authors of the article [4] used this new invariant to give new upper bounds for the L.-S. category of the total space of a fibration.

Let X be a simply connected CW complex with rational homology of finite type and let $(\Lambda V, d)$ be a minimal Sullivan model for X. Fix an integer m and recall from chapter 3, section 3.7 that the projection map onto the quotient $(\Lambda V/\Lambda^{>m}V, \bar{d})$ admits a model of the form

$$\begin{tikzcd}
(\Lambda V, d) \arrow[rr, "\pi"'] \arrow[drr, "\lambda"'] & & (\Lambda V/\Lambda^{>m}V, \overline{d}) \\
& & (\Lambda V \otimes (\mathbb{Q} \oplus M_m), \delta) \arrow[u, "\simeq"', "h_m"]
\end{tikzcd}$$

where M_m is a module and h_m is a quasi-isomorphism of $(\Lambda V, d)$-differential modules which is a model of the quotient map π, in the sense that $h_m \circ \lambda = \pi$ and that h_m is a quasi-isomorphism, which can always be chosen such that $h_m(M_m) = 0$. Moreover, recall from chapter 3 that the rational L.-S. category of X, $\operatorname{cat}_0 X$, is the least integer m such that λ admits a retraction of $(\Lambda V, d)$-differential modules. That is, a morphism of $(\Lambda V, d)$-differential modules $\rho : (\Lambda V \otimes (\mathbb{Q} \oplus M_m), \delta) \to (\Lambda V, d)$ such that $\rho \circ \lambda = \mathrm{id}_{(\Lambda V, d)}$.

Definition (Cuvilliez et al. [4])**.** The *rational retraction index* of X, denoted $\mathrm{r}_0(X)$, is the largest integer r, not exceeding $\operatorname{cat}_0 X$, such that there exists a semifree model with $h(M_{\operatorname{cat}_0 X}) = 0$ and such that there exists a retraction ρ with $\rho(M_{\operatorname{cat}_0 X}) \subset \Lambda^{\geq r} V$.

Remark. It follows directly from the definition that $\mathrm{r}_0(X) = 0$ whenever X is contractible and that otherwise $1 \leq \mathrm{r}_0(X) \leq \operatorname{cat}_0(X)$.

Let X be a simply connected CW complex with rational homology of finite type. We list below some of the most important properties of the rational retraction index (we refer the reader to the original article for a proof, see [4], Propositions 1–4).

Theorem 4.1. *If $X = X_1 \times \cdots \times X_k$ is a product, then*

$$\mathrm{r}_0(X) \geq \mathrm{r}_0(X_1) + \cdots + \mathrm{r}_0(X_k).$$

In particular, if no X_i is contractible ($i = 1, \ldots, k$), then $\mathrm{r}_0(X) \geq k$.

Theorem 4.2. *If $n \geq 2$, then $\mathrm{r}_0(X \times S^n) = \mathrm{r}_0(X) + 1$.*

Theorem 4.3. *If X is coformal, then $\mathrm{r}_0(X) = \operatorname{cat}_0(X)$.*

Theorem 4.4. *If $H^*(X;\mathbb{Q})$ is a Poincaré duality algebra such that $H^*(X;\mathbb{Q})$ requires at least two generators, then $\mathrm{r}_0(X) \geq 2$.*

Basic examples. (1) It is immediate that for any space with rational category 1, the retraction index is also 1. In particular, the retraction index of a sphere is 1.

(2) The complex projective plane satisfies $\mathrm{r}_0(\mathbb{C}P^n) = 1$. We compute here as an illustrative example that $\mathrm{r}_0(\mathbb{C}P^2) = 1$. Of course, we know that $\mathrm{r}_0(\mathbb{C}P^2) \geq 1$, as is true for any non-contractible space, hence it suffices to show that $\mathrm{r}_0(\mathbb{C}P^2) < 2$. To do so, we have to show that there exist no model M and no retraction ρ for M such that $\rho(M) \subset \Lambda^{\geq 2}V$.

The Sullivan model of $\mathbb{C}P^2$ is $(\Lambda V, d) = (\Lambda(x, y), d)$ with $|x| = 2$, $|y| = 5$, $dx = 0$ and $dy = x^3$. Moreover, we know that $\mathrm{cat}_0(\mathbb{C}P^2) = 2$. The first step is to restrict our attention to the quadratic part d_1 of the differential. In this case, $d_1 = 0$. Now, construct a model for the projection map π, as in the diagram below

$$\begin{tikzcd}
(\Lambda(x,y),0) \arrow[r, "\pi"'] \arrow[dr, "\lambda"'] & (\Lambda(x,y)/\Lambda^{>2}(x,y),0) \\
& (\Lambda(x,y)\otimes(\mathbb{Q}\oplus M),d_1) \arrow[u, "\simeq"', "h"]
\end{tikzcd}$$

where h is a semifree resolution and M is constructed using the algorithm of section 3.7. Observe that $H^+(\Lambda^{>2}(x, y), d) = H^{\geq 6}(\Lambda^{>2}(x, y), d)$, so it follows that $M = M^{\geq 5}$. Indeed, we compute

$$M^5 = H^6(\Lambda^{>2}(x, y)) \cong \mathbb{Q} \cdot [x^3] \cong \mathbb{Q} \cdot s$$

with $d_1 s = x^3$. It will not be necessary for our purpose to compute M^n for $n \geq 6$. The next step however is to perturb d_1 to δ such that δ extends d in $\Lambda V \otimes (\mathbb{Q} \oplus M)$. But because $dd_1 s = dx^3 = 0$, it follows that we can take $\delta = d_1$ on M^6.

Next, we have to show that no retraction $\rho : (\Lambda(x, y) \otimes (\mathbb{Q} \oplus M), \delta) \to (\Lambda(x, y), d)$ satisfies $\rho(M) \subset \Lambda^{\geq 2}(x, y)$. Assume ρ to be an arbitrary retraction. Thus, ρ must commute with the differential, so

$$d\rho(s) = \rho(\delta s) = \rho(x^3) = x^3.$$

It follows that $\rho(s) = y$. That is, $r = 1$ is the largest integer such that $\rho(M) \subset \Lambda^{\geq r}V$. Hence $\mathrm{r}_0(\mathbb{C}P^2) = 1$.

(3) Fix a positive integer n and choose any integer $1 \leq r \leq n$. For any such r, there exists a simply connected CW complex X of dimension $2n$ such that $\mathrm{cat}_0(X) = n$ and $\mathrm{r}_0(X) = r$, namely

$$X = \mathbb{C}P^{n-r+1} \times \underbrace{S^2 \times \ldots \times S^2}_{r-1}.$$

In this sense, the rational retraction index can be seen as lifting the degeneracy of the L.-S. category for this class of CW complexes.

Indeed, using additivity of the rational category on we see that $\mathrm{cat}_0(X) = (n-r+1)+r-1 = n$. Moreover, using Theorem 4.2, we compute $\mathrm{r}_0(X) = \mathrm{r}_0(\mathbb{C}P^n)+r-1 = 1+r-1 = r$.

4.2 Computation of the retraction index

In this section we consider two elliptic spaces and illustrate in detail the steps required to calculate their retraction index. We also observe a connection between the retraction index and the top cohomology class. We will wrap up this information at the end of the chapter and make a conjecture based on those two examples and others.

4.2.1 Our first example

Here we compute the rational retraction index of the model $(\Lambda V, d) = (\Lambda(a, b, y_1, y_2, y_3), d)$ with $|a| = |b| = 2$, $|y_1| = 3$ and $|y_2| = |y_3| = 5$ with differential given by $d(a) = d(b) = 0$, $d(y_1) = a^2$, $d(y_2) = b^3$ and $d(y_3) = ab^2$. This minimal Sullivan model is elliptic, as we showed before with example 3.24.

Before we begin, we need to calculate $\mathrm{cat}_0(\Lambda V, d)$. Since $(\Lambda V, d)$ is elliptic, we can use the fact that $\mathrm{cat}_0(\Lambda V, d) = \mathrm{e}_0(\Lambda V, d)$ (see Theorem 3.20). A computation of the cohomology of $(\Lambda V, d)$ reveals that

$$H^2(\Lambda V, d) = \mathbb{Q} \cdot [a] \oplus \mathbb{Q} \cdot [b],$$

$$H^4(\Lambda V, d) = \mathbb{Q} \cdot [ab] \oplus \mathbb{Q} \cdot [b^2],$$

$$H^7(\Lambda V, d) = \mathbb{Q} \cdot [-ay_2 + by_3] \oplus \mathbb{Q} \cdot [-ay_3 + b^2 y_1],$$

$$H^9(\Lambda V, d) = \mathbb{Q} \cdot [-a^2 y_2 + aby_3] \oplus \mathbb{Q} \cdot [-aby_2 + b^2 y_3],$$

$$H^{11}(\Lambda V, d) = \mathbb{Q} \cdot [-a^2 by_2 + ab^2 y_3],$$

and cohomology in any other degree is trivial (recall from the top cohomology class degree formula, stated as Theorem 3.19, that cohomology stops after degree $5+5+3-(1+1) = 11$). A computation that we shall omit shows that there are no non-trivial cocycles of word length 5 or greater, hence $\mathrm{e}_0(\Lambda V, d) = 4$ and so $\mathrm{cat}_0(\Lambda V, d) = 4$.

Now we recall the definition of the *rational retraction index*, $\mathrm{r}_0(\Lambda V, d)$, of the model $(\Lambda V, d)$. First, consider the diagram

$$\begin{tikzcd}
(\Lambda V, d) \arrow[r, "\pi"'] \arrow[dr, "\lambda"'] & (\Lambda V/\Lambda^{>4}V, d) \\
& (\Lambda V \otimes (\mathbb{Q} \oplus M), \delta) \arrow[u, "\simeq", "h"']
\end{tikzcd}$$

where h is a relative model of the projection map π. The rational retraction index, $\mathrm{r}_0(\Lambda V, d)$, is the largest integer r, with $r \leq \mathrm{cat}_0(\Lambda V, d)$, such that there exists a relative model h of π as above satisfying $h(M) = 0$ and such that there exists a retraction $\rho : (\Lambda V \oplus (\mathbb{Q} \otimes M), \delta) \to (\Lambda V, d)$ with $\rho(M) \subset \Lambda^{\geq r} V$.

Now, we proceed to find a lower bound for $\mathrm{r}_0(\Lambda V, d)$. More specifically, we will show that $\mathrm{r}_0(\Lambda V, d) \geq 3$. To do so, we will construct a relative model M as above using the algorithm of section 3.7. Therefore, the first step is to construct a relative model for the quadratic part $(\Lambda V, d_1)$, as follows:

$$\begin{tikzcd}
(\Lambda V, d_1) \arrow[r, "\pi"'] \arrow[dr, tail, "\lambda"'] & (\Lambda V/\Lambda^{>4}V, d_1) \\
& (\Lambda V \otimes (\mathbb{Q} \oplus M), \delta_1) \arrow[u, "\simeq"', "h"]
\end{tikzcd}$$

Now, recall that we can see that $M = M^{4,\geq 9}$. Moreover, $(\Lambda V)^9 \oplus (\Lambda V)^{\geq 11} \subset \Lambda^{\geq 3}V$ and $H^{>11}(\Lambda V, d) = 0$, so it is clear that it suffices for our purpose to show that there exists a retraction ρ such that $\rho(M^{10}) \subset \Lambda^{\geq 3}V$. Now, we will lay out the details of the construction of $M^{\leq 10}$.

Step 1. "Define $M^9 \xrightarrow{\cong} H^{10}(\Lambda^{>4}V, d_1)$." We can compute that a basis of cocycles in degree 10 is given by $a^i b^{5-i}$ for $i = 0, \ldots, 5$. Hence, we may define

$$M^9 := \bigoplus_{i=0}^{5} \mathbb{Q} \cdot s_i$$

with $\delta_1(s_i) = a^i b^{5-i}$.

Step 2. "Define $M^{10} \xrightarrow{\cong} H^{11}(\Lambda^{>4}V \oplus (\Lambda^+ V \otimes M^9), d_1)$." We find that the cohomology has dimension 10 and a basis of cocycles is given by:

$$as_{i+1} - a^i b^{4-i} y_1 \qquad (i = 0, 1, 2, 3, 4),$$

$$bs_{i+2} - a^i b^{4-i} y_1 \qquad (i = 0, 1, 2, 3),$$

$$as_0 - bs_1.$$

Hence, in particular M^{10} will have a generator t such that $\delta_1(t) = as_0 - bs_1$.

The process can be continued. All the other generators of M have degree ≥ 11. At this point, we must make M into a relative model for $(\Lambda V, d) \longrightarrow (\Lambda V/\Lambda^{>4}V, d)$. To do so, we need to perturb δ_1 to δ, where δ extends d on $\Lambda V \otimes (\mathbb{Q} \oplus M)$. A quick computation shows that actually $\delta = \delta_1$ on $M^{\leq 10}$ (and this is all we will need).

Now, we can construct a retraction $\rho : (\Lambda V \otimes (\mathbb{Q} \oplus M), \delta) \to (\Lambda V, d)$. Define $\rho(s_0) = b^2 y_2$ (since it is a solution to $d(\rho(s_0)) = b^5$). Define $\rho(s_1) = aby_2$. It is interesting to note that this is not the only valid choice for $\rho(s_1)$. We will show that after having made that choice, ρ can be extended to all of M and that moreover, $\rho(M^{10}) \subset \Lambda^{\geq 3}V$, as desired.

To show that ρ can be extended to all of M, it suffices to compute the cohomology of $(\Lambda V, d)$, as we will see. We can define ρ by induction. First, assume that ρ has been defined on $M^{<n}$ and suppose that $u \in M^n$. To define $\rho(u)$ one needs to find a solution to the equation $d\rho(u) = \rho(\delta u)$. Clearly, δu is a cocycle in $\Lambda V \otimes (\mathbb{Q} \oplus M^{<n})$, so $\rho(\delta u)$ is a cocycle in $(\Lambda V, d)$ of degree $n+1$, because ρ preserves cocycles. Hence, ρ can be defined for u if and only if $\rho(\delta u)$ is a coboundary. It suffices now to observe that for our particular model, $\rho(\delta u)$ is always a coboundary as u runs through a basis of M^n. Indeed, the only non-trivial cocycle in $H^{\geq 10}(\Lambda V, d)$ is $[-a^2by_2 + ab^2y_3] \in H^{11}(\Lambda V, d)$. Thus, we can infer that there is no $u \in M$ such that $\rho(\delta u)$ is a non-trivial cocycle. Hence, ρ can be extended to all of M.

Finally, we show that $\rho(M^{10}) \subset \Lambda^{\geq 3}V$. First, notice that the only word of length 2 in $(\Lambda V)^{10}$ is y_2y_3. Also, notice that 9 of the 10 generators of M^{10} are "killing" off linear combinations that contain y_1. Hence, if u is one of those generators, then $\rho(u)$ cannot be a multiple of y_2y_3, because $d\rho(u)$ is either 0 or contains y_1 in one of its words. As far as the generator t is concerned, we compute that $\rho(\delta t) = \rho(as_0 - bs_1) = ab^2y_2 - ab^2y_2 = 0$, therefore we can define $\rho(t) = 0$. Hence $\rho(M^{10}) \subset \Lambda^{\geq 3}V$. This proves that $\mathrm{r}_0(\Lambda V, d) \geq 3$.

Finally, we have to show that $\mathrm{r}_0(\Lambda V, d) \leq 3$. In order to do so, we will suppose $(\Lambda V \otimes (\mathbb{Q} \oplus M), \delta)$ to be an arbitrary $(\Lambda V, d)$-semifree resolution of the projection map and we will consider the acyclic closure $(\Lambda V \otimes \Lambda\overline{V}, d)$ of $(\Lambda V, d)$. We compute that $\Lambda\overline{V} = \Lambda(\bar{a}, \bar{b}, \bar{y_1}, \bar{y_2}, \bar{y_3})$ with $d\bar{a} = a$, $d\bar{b} = b$, $d\bar{y_1} = y_1 - a\bar{a}$, $d\bar{y_2} = y_2 - b^2\bar{b}$ and $d\bar{y_3} = y_3 - b^2\bar{a}$.

Now, recall the following construction from section 3.9. Take the isomorphism

$$(\mathbb{Q} \oplus M)^n \cong H^n(\Lambda V/\Lambda^{>4}V \otimes \Lambda\overline{V}, d),$$

along with the short exact sequence

$$0 \to (\Lambda^{>4}V \otimes \Lambda\overline{V}, d) \to (\Lambda V \otimes \Lambda\overline{V}, d) \to (\Lambda V/\Lambda^{>4}V \otimes \Lambda\overline{V}, d) \to 0.$$

Combining these gives rise to the isomorphism

$$(\mathbb{Q} \oplus M)^n \cong H^{n+1}(\Lambda^{>4}V \otimes \Lambda\overline{V}, d).$$

We can use this to determine M^9. Observe that $H^{10}(\Lambda^{>4}V \otimes \Lambda\overline{V}, d) = \operatorname{span}\{a^i b^{5-i} : 0 \leq i \leq 5\}$, hence M^9 can be written as $M^9 = \operatorname{span}\{s_i : 0 \leq i \leq 5\}$ with $\delta s_i = a^i b^{5-i}$ (by choosing an appropriate basis $\{s_i\}$). It follows in particular that any retraction ρ must satisfy $\rho(s_0) = b^2 y_2 + \phi$, where ϕ is any linear combination of cocycles in $(\Lambda V)^9$. Observe that for degree reasons alone, $\phi \in \Lambda^{\geq 3}V$. Thus, any relative model M and any retraction ρ satisfy $\rho(M) \subset \Lambda^{\geq 3}V$, however $\rho(M) \not\subseteq \Lambda^{\geq 4}V$. Hence, $\mathrm{r}_0(\Lambda V, d) \leq 3$.

Therefore, we conclude that $\mathrm{r}_0(\Lambda V, d) = 3$.

4.2.2 Example two

We will now consider the model $(\Lambda V, d) = (\Lambda(a, b, x, y), d)$ with $|a| = |b| = 4$, $|x| = 7$ and $|y| = 11$, with differential $da = db = 0$, $dx = a^2 + ab + b^2$, and $dy = a^3$. This is a model for the biquotient $\mathrm{Sp}(1) \setminus \mathrm{Sp}(3) \,/\, (\mathrm{Sp}(1) \times \mathrm{Sp}(1))$ (see [12], Example 3.52). A computation of the cohomology of this model (that we will omit) shows that $\mathrm{e}_0(\Lambda V, d) = 3$, hence $\mathrm{cat}_0(\Lambda V, d) = 3$.

We will now compute the rational retraction index of $(\Lambda V, d)$. First, we will find a lower bound for $\mathrm{r}_0(\Lambda V, d)$. For this, we construct using the same algorithm as above a $(\Lambda V, d)$-semifree resolution

$$\begin{tikzcd}
(\Lambda V, d) \arrow[r, "\pi"'] \arrow[dr, "\lambda"'] & (\Lambda V/\Lambda^{>3}V, d) \\
 & (\Lambda V \otimes (\mathbb{Q} \oplus M), \delta) \arrow[u, "h", "\simeq"']
\end{tikzcd}$$

Note first that we have $M = M^{3,\geq 15}$. Indeed, $M = M^{3,*}$ by Theorem 3.26 and the topological degree has to be at least 15 simply because every cocycle in $(\Lambda^{>3}V, d)$ has degree 16 or higher. Observe moreover that any word of degree 15 or greater in ΛV must have word length 2 or more. Then, it follows at once that for any retraction $\rho : (\Lambda V \otimes (\mathbb{Q} \oplus M), \delta) \longrightarrow (\Lambda V, d)$

$$\rho(M) = \rho(M^{\geq 15}) \subset (\Lambda V)^{\geq 15} \subset \Lambda^{\geq 2} V.$$

Hence, we know that $\mathrm{r}_0(\Lambda V, d) \geq 2$.

Moreover, we can compute that

$$M^{15} \cong \bigoplus_{i=0}^{4} \mathbb{Q} \cdot s_i$$

with $\delta s_i = a^i b^{4-i}$. Then, any arbitrary retraction $\rho : (\Lambda V \otimes (\mathbb{Q} \oplus M), \delta) \longrightarrow (\Lambda V, d)$ will need to satisfy $\rho(s_4) = ay \in \Lambda^2 V$ (this is the only solution to $d\rho(s_4) = a^4$). Therefore, $\rho(M) \nsubseteq \Lambda^{\geq 3} V$. Hence, $\mathrm{r}_0(\Lambda V, d) \leq 2$

We conclude, as desired, that $\mathrm{r}_0(\Lambda V, d) = 2$.

4.3 Retraction index of a product

Consider the model $(\Lambda V \otimes \Lambda W, d) = (\Lambda(a, b, y_1, y_2, y_3) \otimes \Lambda(c, e, v, w), d)$, where $|a| = |b| = 2$, $|y_1| = 3$, $|y_2| = |y_3| = 5$, $da = db = 0$, $dy_1 = a^2$, $dy_2 = b^3$, $dy_3 = ab^2$, $|c| = 4$, $|e| = 4$, $|v| = 7$, $|w| = 11$, $dc = de = 0$, $dv = c^2 + ce + e^2$ and $dw = c^3$. This model is of course the product of the two models considered in the previous section (section 4.2). We know that $\mathrm{cat}_0(\Lambda V, d) = 4$ and that $\mathrm{cat}_0(\Lambda W, d) = 3$, hence $\mathrm{cat}_0(\Lambda V \otimes \Lambda W, d) = 7$. Moreover, we know that $\mathrm{r}_0(\Lambda V, d) = 3$ and $\mathrm{r}_0(\Lambda W, d) = 2$, hence we know using the inequality for a product that $\mathrm{r}_0(\Lambda V \otimes \Lambda W, d) \geq 5$. We prove the reverse inequality, showing that $\mathrm{r}_0(\Lambda V \otimes \Lambda W) = 5$.

Consider an arbitrary $(\Lambda V \otimes \Lambda W, d)$-semifree resolution $h : (\Lambda(V \oplus W) \otimes (\mathbb{Q} \oplus M), \delta) \longrightarrow (\Lambda(V \oplus W)/\Lambda^{>7}(V \oplus W), d)$ of the projection map and let $\rho : (\Lambda(V \oplus W) \otimes (\mathbb{Q} \oplus M), \delta) \longrightarrow (\Lambda(V \oplus W), d)$ be an arbitrary retraction.

Observe that the elements $b^3c^2e^3$ and $b^2c^3e^3$ both get "killed off", that is there must be $s \in M^{25}$ and $t \in M^{27}$ such that $\delta s = b^3c^2e^3$ and $\delta t = b^2c^3e^3$. This is immediate using the isomorphisms $M^{25} \cong H^{26}(\Lambda^{>7}(V \oplus W) \otimes \Lambda\overline{(V \oplus W)}, d)$ and $M^{27} \cong H^{28}(\Lambda^{>7}(V \oplus W) \otimes \Lambda\overline{(V \oplus W)}, d)$, respectively.

Next, consider the following isomorphism (see section 3.9)

$$M^{28} \cong H^{29}(\Lambda^{>7}(V \oplus W) \otimes \Lambda\overline{(V \oplus W)}, d).$$

Notice that

$$[b^3c^2e^3\bar{c} - b^2c^3e^3\bar{b}] \ \in H^{29}(\Lambda^{>7}(V \oplus W) \otimes \Lambda\overline{(V \oplus W)}, d).$$

We will use the construction of the isomorphism to show that this element is in correspondence to some $m \in M^{28}$ with $\delta m = tb - sc$.

Step 1. Use the connecting homomorphism (recall its construction from the Snake lemma) to get a correspondence

$$[b^3c^2e^3\bar{c} - b^2c^3e^3\bar{b}] \xleftrightarrow{\partial^\sharp} [b^2c^2e^3\bar{b}\bar{c}].$$

Step 2. The map $H(h \otimes id_{\Lambda V \otimes \Lambda \bar{V}})$ gives a correspondence

$$[b^2c^2e^3\bar{b}\bar{c}] \longleftrightarrow [b^2c^2e^3\bar{b}\bar{c} - s\bar{c} + t\bar{b} + m]$$

for some $m \in M^{28}$. That is, there must be a generator $m \in M^{28}$ with differential $\delta m = tb - sc$ for the isomorphism to be possible. This step is the most important step because it gives us the differential of m, which is what we are interested in.

Step 3. Finally the map $H(\alpha)$ sends anything with a factor in $\Lambda\overline{V \oplus W}$ to 0 and hence gives the correspondence

$$[b^2c^2e^3\bar{b}\bar{c} - s\bar{c} + t\bar{b} + m] \longleftrightarrow m.$$

Finally, all that remains to do is to compute the retraction. Notice that the choice of cocycles $b^3c^2e^3$ and $b^2c^3e^3$ was not arbitrary. Indeed, those cocycles were specifically chosen because the solutions $d(c^2e^3y_2) = b^3c^2e^3$ and $d(b^2e^3w) = b^2c^3e^3$ are unique, hence the retraction ρ is uniquely determined on s and t. That is, we must have $\rho(s) = c^2e^3y_2$ and $\rho(t) = b^2e^3w$. Thus,

$$\begin{aligned}\rho(\delta m) = \rho(tb - sc) &= b^3e^3w - c^3e^3y_2 \\ &= d(e^3y_2w).\end{aligned}$$

Hence, the retraction must satisfy $\rho(m) = e^3y_2w + \phi$, where ϕ is some linear combination of cocycles. Necessarily, $\phi \in \Lambda^{\geq 5}(V \oplus W)$, because we already know that $\mathrm{r}_0(\Lambda V \otimes \Lambda W, d) \geq 5$. Therefore, we can conclude that $\rho(M) \not\subseteq \Lambda^{\geq 6}(V \oplus W)$, yet $\rho(M) \subset \Lambda^{\geq 5}(V \oplus W)$. Since ρ was an arbitrary retraction, we can conclude that

$\mathrm{r}_0(\Lambda(V \oplus W), d) \leq 5$, as desired.

4.4 Retraction index and top cohomology class for formal elliptic spaces

Below we list examples of formal spaces for which the rational retraction index was computed. All of those spaces are rationally elliptic, hence their cohomology algebra satisfy Poincaré duality. We observe a certain relationship between the rational retraction index and the top class of the cohomology algebra. Namely, the rational retraction index always corresponds to the maximal number of linearly independent cohomology classes such that the top cohomology class can be written as a product of powers of these.

4.4.1 Sphere

The sphere S^n has cohomology $1, [a]$ and its top class is $[a]$. There is only one generator for the top class of the cohomology and we have already seen that $\mathrm{r}_0(S^n) = 1$.

4.4.2 Complex projective space

The complex projective space $\mathbb{C}P^n$ has cohomology 1, $[a]$, $[a]^2$, $\ldots$, $[a]^n$ with top class $[a]^n$. Its top class can be written as a power of one generator. Moreover, we mentioned before that $\mathrm{r}_0(\mathbb{C}P^n) = 1$.

4.4.3 Product of complex projective spaces

The product $\mathbb{C}P^n \times \mathbb{C}P^n$ has cohomology 1, $[a]$, $[b]$, $[a]^2$, $[a][b]$, $[b]^2$, $\ldots$, $[a]^n[b]^n$. Its top class is $[a]^n[b]^n$ which can only be written using powers of 2 distinct generators and it can be shown that $\mathrm{r}_0(\mathbb{C}P^n \times \mathbb{C}P^n) = 2$.

4.4.4 Biquotient $\mathrm{Sp}(1)\backslash\mathrm{Sp}(3)/\mathrm{Sp}(1)\times\mathrm{Sp}(1)$

The biquotient $\mathrm{Sp}(1)\setminus\mathrm{Sp}(3)\ /\ (\mathrm{Sp}(1)\times\mathrm{Sp}(1))$ of category 3 has a cohomology basis 1, $[a]$, $[b]$, $[a]^2$, $[a][b]$, $[a]^2[b]$ with top class $[a]^2[b]$. Its top class can only be written as a product involving exactly 2 distinct generators and we have shown before that $\mathrm{r}_0(\mathrm{Sp}(1)\backslash\mathrm{Sp}(3)/\mathrm{Sp}(1)\times\mathrm{Sp}(1)) = 2$.

4.4.5 Homogeneous space $\mathrm{Sp}(3)/\mathrm{U}(3)$

The homogeneous space $\mathrm{Sp}(3)/\mathrm{U}(3)$ of category 6 has a minimal model $(\Lambda(a_2, b_6, x_7, y_{11}), d)$ with $da = db = 0$, $dx = a^4 - 2ab$ and $dy = b^2$. A basis for its cohomology is $1, [a]$, $[a]^2$, $[a]^3$, $[b]$, $[a][b]$, $[a]^2[b]$, $[a]^3[b]$. Its top class is $[a]^3[b]$ or equivalently $\frac{1}{2}[a]^6$. Hence, the top class can be written using up to 2 distinct generators and it can be shown that $\mathrm{r}_0(\mathrm{Sp}(3)/\mathrm{U}(3)) = 2$.

4.4.6 Non-formal example

The next example is *not* formal, however, its retraction index still matches the pattern observed. Consider the minimal Sullivan algebra $(\Lambda V, d) = (\Lambda(a, b, y_1, y_2, y_3), d)$ with $|a| = |b| = 2$, $|y_1| = 3$, $|y_2| = |y_3| = 5$, $da = db = 0$, $dy_1 = a^2$, $dy_2 = b^3$ and $dy_3 = ab^2$. The category of this model is 4 and it has the following cohomology algebra:

$$\begin{array}{rl}
\text{deg } 2: & [a], [b], \\
\text{deg } 4: & [a][b], [b]^2, \\
\text{deg } 7: & [-ay_2 + by_3], [-ay_3 + b^2y_1], \\
\text{deg } 9: & [a][-ay_2 + by_3], [b][-ay_2 + by_3], \\
\text{deg } 11: & [a][b][-ay_2 + by_3].
\end{array}$$

The top class is $[a][b][-ay_2 + by_3] = [b]^2[-ay_3 + b^2y_1]$. Thus, the top class can be written either as a product of powers of 2 distinct generators or as a product of 3 distinct generators. However, what is important is that this means that the top class can be written using *up to* 3 distinct generators. We have shown previously that $\mathrm{r}_0(\Lambda V, d) = 3$.

4.5 Conjecture for formal elliptic spaces

Based on the examples above, we make a conjecture.

Let X be any rationally elliptic formal space (assuming also that X is a simply connected CW complex). It follows that the rational L.-S. category of X is finite, so fix $n = \mathrm{cat}_0(X) = \mathrm{cup}(X)$. Moreover, since X is rationally elliptic, its rational cohomology $H^*(X;\mathbb{Q})$ satisfies Poincaré duality and so admits a top cohomology class ω (which can be written as a product of at most n cohomology classes).

Conjecture. Let k be the greatest integer such that there exist k linearly independent cohomology classes $\alpha_1, \ldots, \alpha_k \in \tilde{H}^*(X;\mathbb{Q})/(\tilde{H}^+(X;\mathbb{Q}) \cdot \tilde{H}^+(X;\mathbb{Q}))$ such that $\omega = \alpha_1^{l_1} \cdots \alpha_k^{l_k}$ for some positive integers $l_1, \ldots, l_k$. Then,

$$\boxed{\mathrm{r}_0(X) = k.}$$

4.6 Non-formal counter-examples to the conjecture

Here we list non-formal examples for which the conjecture fails. Note however that the integer predicted by the conjecture is still a lower bound for the retraction index in the examples below (as well as for the non-formal example considered above).

First counter-example. Consider the Sullivan algebra $(\Lambda(x, y, z), d)$ with $|x| = |y| = 3$, $|z| = 5$, $dx = dy = 0$ and $dz = xy$. Then a basis for its cohomology is 1, $[x]$, $[y]$, $[xz]$, $[yz]$, $[x][yz]$. The cup-length is 2, the category is 3. Since the model is quadratic, it follows that the retraction index must equal the category, hence it is 3 as well. However, the top class is $[x][yz]$ and hence the conjecture predicts incorrectly the retraction index to be 2.

Second counter-example. Consider the Sullivan algebra $(\Lambda(a, b, x, y, z), d)$, $|a| = |b| = 2$, $|x| = |y| = |z| = 3$, $da = db = 0$, $dx = a^2$, $dy = b^2$ and $dz = ab$. A basis for cohomology is 1, $[a]$, $[b]$, $[-bx + az]$, $[-ay + bz]$, $[b][-bx + az]$. The category is 3 and hence (the model being quadratic), the rational retraction index is 3. However, the top class is $[b][-bx + az]$ and there is no way to write the top class as a product of 3 generators. Hence the conjecture would predict incorrectly a retraction index of 2.

Non-quadratic counter-example. Consider the Sullivan algebra $(\Lambda(a, b, x, y, z), d)$, $|a| = |b| = 2$, $|x| = |y| = 5$, $|z| = 3$, $da = db = 0$, $dx = a^3$, $dy = b^3$ and $dz = ab$. A basis for cohomology is 1, $[a]$, $[b]$, $[a]^2$, $[b]^2$, $[-bx + a^2z]$, $[-ay + b^2z]$, $[b][-bx + a^2z]$, $[a][-ay + b^2z]$, $[b]^2[-bx + a^2z]$. The category is 4 and the rational retraction index is 3. However, the top class is $[b]^2[-bx + a^2z] = [a]^2[-ay + b^2z]$ and hence the conjecture predicts incorrectly a retraction index of 2, since the top class can only be written using up to 2 distinct cohomology generators.

For each non-formal model, however, there is an associated formal model that we obtain by taking the minimal Sullivan model of the non-formal model's cohomology. For each of the previous three counter-examples, we will see that the retraction index of the associated formal model satisfies the conjecture.

First counter-example (continued). Consider again $(\Lambda(x, y, z), d)$ with $|x| = |y| = 3$, $|z| = 5$, $dx = dy = 0$ and $dz = xy$. Denote by $(\Lambda W, d)$ the minimal Sullivan model of $(H^*(\Lambda(x, y, z), d), 0)$. Of course, the cup-length of $(\Lambda W, d)$ is still

2. Since $(\Lambda W, d)$ is formal, it follows therefore that $\mathrm{cat}_0(\Lambda W, d) = \mathrm{cup}(\Lambda W, d) = 2$. Moreover, the cohomology $H^*(\Lambda W, d)$ satisfies Poincaré duality and requires at least 2 generators, hence $\mathrm{r}_0(\Lambda W, d) \geq 2$. Thus, $\mathrm{r}_0(\Lambda W, d) = 2$. This is what was predicted by the conjecture, from the cohomology of the model.

Second counter-example (continued). Let $(\Lambda(a, b, x, y, z), d)$ stand for the Sullivan algebra introduced in the second counter-example above. The formal minimal Sullivan model $(\Lambda W, d)$ of $(H(\Lambda(a, b, x, y, z), d), 0)$ also has cohomology satisfying Poincaré duality and requiring at least two generators, so we get $\mathrm{r}_0(\Lambda W, d) \geq 2$. Hence, either $\mathrm{r}_0(\Lambda W, d) = 2$ or $\mathrm{r}_0(\Lambda W, d) = 3$. More investigation would be required to determine if $(\Lambda W, d)$ satisfies the conjecture or not.

Third counter-example (continued). Consider the Sullivan algebra $(\Lambda(a, b, x, y, z), d)$ of the third counter-example above, with a basis in cohomology: 1, $[a]^2$, $[b]^2$, $[-bx + a^2z]$, $[-ay + b^2z]$, $[b][-bx + a^2z]$, $[a][-ay + b^2z]$, $[b]^2[-bx + a^2z]$. The corresponding formal model, i.e. the minimal Sullivan model of $(H(\Lambda(a, b, x, y, z, d), 0)$, is found to have the form $(\Lambda W, d) = (\Lambda(a_2, b_2, c_3, e_5, f_5, \ldots), d)$. The differential is given by $da = 0$, $db = 0$, $dc = ab$, $de = a^3$, $df = b^3$. Because the category of this formal model is just the cup-length, which is equal to 3, it is straightforward to check that $M = M^{\geq 7}$ and that in particular there is an element $s \in M^7$ with $\delta s = a^4$. This means that every retraction ρ will have to satisfy $\rho(s) = ae$. Therefore, $\mathrm{r}_0(\Lambda W, d) \leq 2$.

Notice that the cohomology algebra requires at least two generators. Hence, Theorem 4.4 implies that $\mathrm{r}_0(\Lambda W, d) \geq 2$. Therefore, $\mathrm{r}_0(\Lambda W, d) = 2$, which is what the conjecture predicted based on the cohomology algebra.

Bibliography

[1] Luchezar Avramov and Stephen Halperin. Through the looking glass: a dictionary between rational homotopy theory and local algebra. In *Algebra, algebraic topology and their interactions (Stockholm, 1983)*, volume 1183 of *Lecture Notes in Math.*, pages 1–27. Springer, Berlin, 1986.

[2] Achille Bassi. Su alcuni nuovi invarianti delle varietà topologiche. *Ann. Mat. Pura Appl.*, 16(1):275–297, 1937.

[3] Octav Cornea, Gregory Lupton, John Oprea, and Daniel Tanré. *Lusternik-Schnirelmann category*, volume 103 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2003.

[4] Maxence Cuvilliez, Yves Félix, Barry Jessup, and Paul-Eugène Parent. Rational Lusternik-Schnirelmann category of fibrations. *J. Pure Appl. Algebra*, 174(2):117–133, 2002.

[5] Pierre Deligne, Phillip Griffiths, John Morgan, and Dennis Sullivan. Real homotopy theory of Kähler manifolds. *Invent. Math.*, 29(3):245–274, 1975.

[6] Samuel Eilenberg and Saunders MacLane. Relations between homology and homotopy groups of spaces. *Ann. of Math. (2)*, 46:480–509, 1945.

[7] J.-H. Eschenburg. Cohomology of biquotients. *Manuscripta Math.*, 75(2):151–166, 1992.

[8] Yves Felix and Stephen Halperin. Rational L.-S. category and its applications. *Transactions of the American Mathematical Society*, 273(1):pp. 1–37, 1982.

[9] Yves Félix, Stephen Halperin, Carl Jacobsson, Clas Löfwall, and Jean-Claude Thomas. The radical of the homotopy Lie algebra. *Amer. J. Math.*, 110(2):301–322, 1988.

[10] Yves Félix, Stephen Halperin, and Jean-Claude Thomas. *Rational homotopy theory*, volume 205 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 2001.

[11] Yves Félix, Steve Halperin, and Jean-Michel Lemaire. The rational LS category of products and of Poincaré duality complexes. *Topology*, 37(4):749–756, 1998.

[12] Yves Félix, John Oprea, and Daniel Tanré. *Algebraic models in geometry*, volume 17 of *Oxford Graduate Texts in Mathematics*. Oxford University Press, Oxford, 2008.

[13] Ralph H. Fox. On the Lusternik-Schnirelmann category. *Ann. of Math. (2)*, 42:333–370, 1941.

[14] John B. Friedlander and Stephen Halperin. An arithmetic characterization of the rational homotopy groups of certain spaces. *Invent. Math.*, 53(2):117–133, 1979.

[15] Tudor Ganea. Some problems on numerical homotopy invariants. In *Symposium on Algebraic Topology (Battelle Seattle Res. Center, Seattle Wash., 1971)*, pages 23–30. Lecture Notes in Math., Vol. 249. Springer, Berlin, 1971.

[16] Sonia Ghorbal and Barry Jessup. Estimating the rational LS-category of elliptic spaces. *Proc. Amer. Math. Soc.*, 129(6):1833–1842 (electronic), 2001.

[17] S. Halperin. Lectures on minimal models. *Mém. Soc. Math. France (N.S.)*, (9-10):261, 1983.

[18] S. Halperin and J.-M. Lemaire. Notions of category in differential algebra. In *Algebraic topology—rational homotopy (Louvain-la-Neuve, 1986)*, volume 1318 of *Lecture Notes in Math.*, pages 138–154. Springer, Berlin, 1988.

[19] Stephen Halperin. Finiteness in the minimal models of Sullivan. *Trans. Amer. Math. Soc.*, 230:173–199, 1977.

[20] Allen Hatcher. *Algebraic topology*. Cambridge University Press, Cambridge, 2002.

[21] K. Hess. A proof of ganea's conjecture for rational spaces. *Topology*, 30:205–214, 1991.

[22] Norio Iwase. Ganea's conjecture on Lusternik-Schnirelmann category. *Bull. London Math. Soc.*, 30(6):623–634, 1998.

[23] Norio Iwase. A_∞-method in Lusternik-Schnirelmann category. *Topology*, 41(4):695–723, 2002.

[24] I. M. James, editor. *Handbook of algebraic topology*. North-Holland, Amsterdam, 1995.

[25] Barry Jessup. Rational L-S category and a conjecture of Ganea. *J. Pure Appl. Algebra*, 65(1):57–67, 1990.

[26] Vitali Kapovitch. A note on rational homotopy of biquotients. `http://www.math.toronto.edu/vtk/biquotient.pdf`, preprint.

[27] J. L. Koszul. Sur un type d'algèbres différentielles en rapport avec la transgression. In *Colloque de topologie (espaces fibrés), Bruxelles, 1950*, pages 73–81. Georges Thone, Liège, 1951.

[28] Jean-Michel Lemaire and Franis Sigrist. Sur les invariants d'homotopie rationnelle liés à la L.S. catégorie. *Commentarii Mathematici Helvetici*, 56:103–122, 1981. 10.1007/BF02566201.

[29] L. Lusternik and L. Schnirelmann. *Méthodes topologiques dans les problèmes variationels*. Number 188. Paris, Hermann, 1934.

[30] J. Peter May. *Simplicial objects in algebraic topology*. Van Nostrand Mathematical Studies, No. 11. D. Van Nostrand Co., Inc., Princeton, N.J.-Toronto, Ont.-London, 1967.

[31] J. Milnor. *Morse theory*. Based on lecture notes by M. Spivak and R. Wells. Annals of Mathematics Studies, No. 51. Princeton University Press, Princeton, N.J., 1963.

[32] John Milnor. The geometric realization of a semi-simplicial complex. *Ann. of Math. (2)*, 65:357–362, 1957.

[33] Daniel Quillen. Rational homotopy theory. *Ann. of Math. (2)*, 90:205–295, 1969.

[34] Jean-Pierre Serre. Groupes d'homotopie et classes de groupes abéliens. *Ann. of Math. (2)*, 58:258–294, 1953.

[35] Dennis Sullivan. *Geometric topology. Part I*. Massachusetts Institute of Technology, Cambridge, Mass., 1971. Localization, periodicity, and Galois symmetry, Revised version.

[36] Dennis Sullivan. Infinitesimal computations in topology. *Inst. Hautes Études Sci. Publ. Math.*, (47):269–331 (1978), 1977.

[37] Floris Takens. The Lusternik-Schnirelman categories of a product space. *Compositio Math.*, 22:175–180, 1970.

[38] Graham Hilton Toomer. Lusternik-Schnirelmann category and the Moore spectral sequence. *Math. Z.*, 138:123–143, 1974.

[39] Graham Hilton Toomer. Topological localization, category and cocategory. *Canad. J. Math.*, 27:319–322, 1975.

[40] J. H. C. Whitehead. Combinatorial homotopy. I. *Bull. Amer. Math. Soc.*, 55:213–245, 1949.