Université d'Ottawa · University of Ottawa

OPTIMAL CONTROL
OF SYSTEMS GOVERNED BY B-EVOLUTIONS

By
Sebti Kerbal

A Thesis
submitted to the School of Graduate Studies and Research
in partial fulfillment of the requirements
for the degree of
Doctor of Philosophy in Mathematics[1]

[1]The Ph.D. Program is a joint program with Carleton University, administered by the Ottawa-Carleton Institute of Mathematics and Statistics

0-612-36777-0

Abstract

In this thesis the question of optimal control of systems governed by B-evolutions is considered. First, a necessary conditions of optimality for a Lagrange problem subject to a semilinear systems is derived. Two cases are carried out: In the first case the Fréchet differentiability of the cost integrand with respect to the control variable is assumed and in the second only the continuity. Then these results are applied to a linear quadratic regulator problem. Two results are presented one of which contains control cost and the other does not. The optimal control law is given by the solution of an appropriate operator Riccati differential equations. Finally, using the calculus of variation a result is proved on the existence of optimal controls.

Acknowledgments

The author thanks his supervisor, Prof. N.U. Ahmed who directed and guided him in the study of optimal control for systems governed by B-evolutions, which has led to the author's thesis.

The author wishes to express his sincere appreciation to Professors S. Biswas, J. Pandey and B. Dionne for their helpful suggestions.

Special thanks are due to Professor R. Vaillancourt for his help, suggestions and discussions.

The author is very thankfull to the staff of the Department of Mathematics and Statistics, University of Ottawa for their kindness and support.

Finally the author wishes to hightlight and thanks his wife Khadra for her loving support and encouragment.

Dedication

I dedicate this work to my mother, my father and my wife.

Contents

Chapter 1

Introduction

1.1 A Brief Literature Review

Dynamic systems which describe processes in science, engineering, physics and economics are usually very complex and the identification of mathematical models is difficult. Since the introduction of the state space approach, systems theory has become very popular in the construction of dynamical models for a variety of areas in science and engineering systems.

Systems governed by partial differential equations can then be modeled by a differential equation in infinite dimensional space. Those systems are called distributed parameter systems. In order to treat, in a mathematically unified manner, a variety of physical systems modeled by ordinary or partial differential equations, integro-differential equations and functional equations, a good deal of research has focused on control systems defined on infinite dimensional spaces. This is where semigroup theory plays a central role and provides a unified and powerful tool for the study of a wide class of systems (see [1], [32]).

Semigroups in infinite dimensional spaces are direct extensions of exponentials of matrices in finite dimensional Euclidean spaces, where system operators (matrices in finite dimensional space) are called generators of semigroups. Using the powerful semigroup representation, many systems theory concepts such as controllability and optimality have also been studied and analyzed [1], [3]. Control problems are mainly concerned with the study and discussion of the following questions related to any posed problem:

(i) Existence of solutions.

(ii) Uniqueness of solutions.

(iii) Continuous dependence of solutions on the initial data and parameters.

(iv) Necessary conditions of optimality or sufficient conditions of optimality (the most difficult question).

Necessary conditions for optimal control of linear difference and linear ordinary differential equations (in the engineering terminology called lumped parameters systems) were derived by Pontryagin *et al.* [17] (see also [20]–[23], [18], [19]).

Butkovskii [21] was the first to discuss the optimal control problems for distributed parameter systems. The maximum principle as a set of necessary conditions for optimal control of distributed parameter systems has been studied by many authors (see [3], [18], [31]) and the references therein. Since it is well known that the maximum principle may be false for distributed parameter systems (see [24]), there are many papers that give some conditions ensuring that the maximum principle remains true. We refer the readers to [31], [32], [33], and [34] for some results of this aspect.

During the last ten years, a class of systems governed by partial differential equations where a dynamic boundary is involved have been extensively studied by several authors [5], [7], [8], [9], [10], [15], [16]. See also the references therein. The abstract model of this class of systems involves two different Banach spaces. In fact, problems of this kind may be treated by considering a dynamical process in which cause and effect are represented in different spaces.

In the literature, substantial and very interesting work has been done on the question concerning existence, uniqueness, and various properties of solutions. To the knowledge of the author, it appears that Sauer and Van Dalsen [6], [7], [9], [15], [16] did the pioneering work on B-evolution. More recently, Ahmed and Kerbal [26] did a preliminary step in studying the stochastic problems involving B-evolutions and developed interesting results on the question of existence of solutions and their regularity properties for a large class of stochastic equations governed by B-evolutions involving two different Hilbert spaces. However it seems that no attention has been given to the development and study of the question of optimal control for systems involving B-evolutions. A preliminary step for boundary control of distributed systems involving dynamic boundary conditions have been done by Kerbal and Ahmed [12], [27] who considered the question of the necessary conditions of optimality for semilinear systems. In [27], strong condition on the operator (considered as a pair of the state and its trace) is used to have an inverse on the whole space. However in [12] a weaker conditions is used, assuming the operator to have an inverse on its range. The obtained result has been applied to a linear quadratic regulator problem (see [26]).

1.2 Research Objective and Motivation

Mathematical models for infinite-dimensional control systems can be written abstractly as a semilinear equation in the form:

$$\dot{x} = Ax + f(x, u)$$

where A generates a linear or nonlinear semigroup on a Banach space X and f is a nonlinear operator from $X \times U$ to X. A more general class, however, is given by the following model:

$$(d/dt)(Bx) = Ax + f(Bx, u)$$

where A and B are operators from a Banach space X to another Banach space Y and f is a nonlinear map from $Y \times U$ to another Banach space Y.

In addition to covering all classical boundary and distributed control problems, the class includes a new set of problems called systems governed by B-evolutions in which the boundary conditions are determined by an evolution equation. For further examples, see chapter 3.

The control problem of those systems has not been studied before. This gives the author an initial motivation to consider the question of the optimal control. Control theory for classical cases ($X = Y$, $B = I$ identity operator) and their abstract versions have been studied extensively in the literature (see [3], [14], [37], [38], [39] and many others, and the references therein).

In [38], the author gives necessary conditions of optimality for a semilinear control problem in terms of a semigroup of operators which, in general, is not available.

In this thesis we consider a Lagrange problem subject to a semilinear systems governed by B-evolutions. We derive necessary conditions of optimality in terms of the available data, that is in terms of the given operators A and B. Then we apply these results to a quadratic regulator problem, and present an algorithm for computing the optimal policy for each case. Finally, we present a result on the existence of optimal control.

1.3 Thesis Outline and Contributions

The thesis is organized as follows:

Chapter 2 gives a brief review of the literature for semigroup theory, their characterization and their application to dynamical systems.

In chapter 3, as a motivation we present few examples of systems involving dynamic boundary conditions arising in a variety of physical problems. We convert those problem to an abstract form involving two different Banach spaces and present known basic results from B-evolutions theory. Then the main results of the problems treated in this thesis are presented as follows.

In chapter 4 we consider a Lagrange problem subject to semilinear systems. Assuming the existence of an optimal control, we derive necessary conditions of optimality for two cases:

Case A: We assume that the cost integrand is Fréchet differentiable in the control variable.

Case B: We assume that the cost is merely continuous in the control variable and

Fréchet differentiable in the state. We then prove Pontryagin's type necessary conditions of optimality using the well-known Ekland variational principle.

In the above results we have used the case of the holomorphic B-evolutions of type L.

In chapter 5 we apply the results of chapter 4 to a linear quadratic regulator problem. Two cases are presented, one of which contains control cost and the other does not. The first one uses the standard coercivity condition and the second uses controllability. The optimal control law is given by the solution of appropriate operator Riccati differential equations. Using these results we derive an optimal control law for our original problem under a deterministic perturbation.

In chapter 6 we use the calculus of variation, a classical technique to prove the existence of optimal control for a Lagrange problem subject to a semilinear systems.

Finally, concluding remarks and suggestions for further research are presented in chapter 7.

The original contributions of the thesis include:

(i) Derivation of the necessary conditions of optimality for a semilinear control problem.

(ii) Application of the results obtained in part (i) to a linear regulator problem. We consider two cases: The first one contains control cost and the second one does not.

(iii) Proof of the existence of optimal control.

In part **(i)**, the necessary conditions of optimality as a set of equations and inequality expressed in terms of the given data. This generalizes the results of Li and Yong [38], who give necessary conditions of optimality in terms of a semigroup of operators which in general is not available and hence can not be used to construct an algorithm for computing optimal controls. In fact if the operator B is the identity our problem reduces to the classical one.

In part **(ii)** we generalize the case of finite dimensional space studied by Ahmed and Mouadeb [2], who derived an optimal control law determined by the solution of a parameterized family of matrix Riccati differential equations. Here we extend this result to infinite dimensional problem.

Chapter 2

Basic Properties of Semigroup of Operators

2.1 Introduction

In many scientific and engineering problems, the system is modeled by partial differential equations, integral equations, functional differential equations or coupled ordinary and partial differential equations. Semigroup theory provides a unified and powerful tool to investigate these interesting problems known as distributed parameter systems. In fact, they can be described as linear or nonlinear (including semilinear and quasilinear systems) dynamical systems on infinite dimensional Banach spaces using a semigroup approach, or more generally, the evolution operator approach. Also, semigroup theory has been widely used in the study of control theory for systems governed by differential equations on Banach space. In this chapter we shall give some basic and well-known definitions and properties of a strongly continuous and analytic semigrous of operators.

2.2 Basic Semigroup Properties

In this entire chapter we denote by X a Banach space, $\mathcal{L}(X)$ ($\mathcal{L}_u(X)$) the space of bounded (unbounded) linear operators on X satisfying the following properties:

(i) $T(0) = I_X$ (I_X is the identity operator on X)
(ii) $T(t+s) = T(t)T(s)$ for every $t,\ s \geq 0$ (the semigroup property).

The main purpose of this section is to describe the class of linear systems in terms of semigroup of operators which we should use later. It is of interest from the theoretical (since the generator uniquely determine the semigroup) and practical point of view as well (the generator is the given object, not the semigroup) to treat the characterization of semigroup in terms of its generator. Therefore, it is useful to have conditions on the generator itself giving information on the solutions (which might not be known explicitly). Consider the abstract Cauchy problem given by

$$\begin{cases} \dot{x}(t) = Ax(t),\ t \geq 0 \\ x(0) = x_0 \end{cases}$$

where A is a linear operator on X defined by

$$D(A) = \{x \in X : \lim_{t \downarrow 0} \frac{T(t)x - x}{t} \ \textit{exists}\}$$

and

$$Ax = \lim_{t \downarrow 0} \frac{T(t)x - x}{t} \ \textit{for}\ x \in D(A).$$

We say that A is the infinitesimal generator of the semigroup $T(t)$ and $D(A)$ is the domain of A.
In case A is variable (i.e. time varying), the operator A is a generator of an evolution operator $E(t,s)$, $0 \leq s \leq t < \infty$ with values $E(t,s) \in \mathcal{L}(X)$.
Clearly the operator $A: \ D(A) \subseteq X \longrightarrow X$ is linear but not necessarily bounded unless $D(A)$ is all of X.

Definition 2.2.1. *A semigroup of bounded linear operators* $T(t)$, $t \geq 0$, *is uniformly continuous if*

$$\lim_{t \downarrow 0} \|T(t) - I_X\| = 0. \tag{1}$$

From semigroup properties (*i*) and (*ii*), it is not difficult to see that (1) implies that $T(t)$ is uniformly continuous for all $t \geq 0$.

Theorem 2.2.1. [35] *A linear generator A is the infinitesimal generator of a uniformly continuous semigroup $T(t)$ if and only if A is a bounded linear operator on X.*

Based on Theorem 2.2.1, if A is unbounded linear operator on X, the semigroup $T(t)$, $t \geq 0$ generated by A is not uniformly continuous. Therefore, we introduce the following definition

Definition 2.2.2. (Strongly continuous semigroup) *A semigroup of bounded linear operator on X (i.e. verifying the conditions (i), (ii)) which satisfies the property:*

$$\|T(t)x - x\|_X \to 0, \quad \text{as } t \downarrow 0 \text{ for every } x \in X. \tag{2}$$

is called a strongly continuous semigroup or in short C_0-semigroup.

Recall that if A is a linear, not necessarily bounded operator in X, the resolvant set $\rho(A)$ of A is the set of all complex numbers λ for which $(\lambda I_X - A)$ is invertible, i.e. $(\lambda I_X - A)^{-1}$ is a bounded linear operator in X. The operator $R(\lambda, A) \equiv (\lambda - A)^{-1}$ for $\lambda \in \rho(A)$ is a bounded linear operator and is called the resolvant of A. The following theorem is due to Hille and Yosida which gives a complete characterization of linear operators generating C_0-semigroups.

Theorem 2.2.2. [Hille-Yosida] *A linear operator A is the infinitesimal of a C_0-semigroup $T(t)$ satisfying*

$$\|T(t)\| \leq M \exp(\omega t), \ M \geq 1, \ \omega \in R,$$

if and only if

(i) *A is closed and $D(A)$ is dense in X.*

(ii) *The resolvant set $\rho(A)$ of A contains the ray (ω, ∞) and*

$$\|R(\lambda, A)^n\| \leq M/(\Re(\lambda) - \omega)^n \quad \text{for } \Re(\lambda) > \omega, \quad n = 1, 2, \ldots.$$

We now describe another class of semigroups called analytic semigroup which plays an important role in the study of diffusion, heat transfer and many hydrodynamic problem [1]. These classes of semigroups are also known as holomorphic and be characterized by a particularly simple condition.

Definition 2.2.3. (Analytic semigroup) *Let*

$$\triangle = \{z \in C : \ |\arg z| < \delta, \ \delta > 0\},$$

and for $z \in \triangle$, let $T(z)$ be a bounded linear operator on a Hilbert space X. The family $T(z)$, $z \in \triangle$ is an analytic semigroup in $\triangle$ if

(i) *$z \mapsto T(z)$ is analytic in $\triangle$.*

(ii) *$T(0) = I$ and $\lim_{z \to 0} T(z)x = x$ for every $x \in X$ (in the strong sense).*

(iii) *$T(z_1 + z_2) = T(z_1)T(z_2)$ for $z_1, z_2 \in \triangle$.*

A semigroup $T(t)$ will be called analytic if it is analytic in some sector $\triangle$ containing the nonnegative real axis.
The following theorem is due to Arent *et al.* which gives a characterization of linear operator generating an analytic semigroup.

Theorem 2.2.3. [4] *Let A be a densely defined operator on X. Then A is the generator of an analytic semigroup of angle $\delta \in (0, \pi)$ if and only if there exist $M > 0$ and $\gamma \geq 0$ such that whenever $\lambda \in \rho(A)$ we have,*

$$\|R(\lambda, A)\| \leq (M/|\lambda|) \quad \text{for } \Re(\lambda) > 0, \quad |\lambda| \geq \gamma.$$

Compared with Hille-Yosida theorem, Theorem 2.2.3 gives a simple criterion for an operator to be the generator of an analytic semigroup. Merely the resolvent and not its powers have to be estimated.
A complete characterization of analytic semigroup is given in the following theorem.

Theorem 2.2.4. [1] *Let A be the infinitesimal generator of a uniformly bounded C_0-semigroup $T(t)$, $t \geq 0$, with $0 \in \rho(A)$. Then the following statements are equivalent:*

(i) *$T(t)$ can be extended to an analytic semigroup from the negative real line to a sector around it, given by,*

$$\Gamma_\delta \equiv \{z \in C : |\arg z| < \delta\} \quad \textit{for some } \delta > 0,$$

and $\|T(z)\|$ is uniformly bounded on every closed subsector

$$\Gamma_{\delta'} \subset \Gamma_\delta, \quad \delta' < \delta.$$

(ii) *There exists a constant $c > 0$ such that, for every $\sigma > 0$, and $\tau \neq 0$,*

$$\|R(\sigma + i\tau, A)\|_{\mathcal{L}(X)} \leq (c/|\tau|).$$

(iii) *There exist $0 < \delta < (\pi/2)$ and $M \geq 1$, such that*

$$\rho(A) \supset \sum \equiv \{\gamma \in C \ : |\arg \gamma| < (\pi/2) + \delta\} \bigcup \{0\}$$

and

$$\|R(\lambda, A)\| \leq (M/|\lambda|) \quad \textit{for all } \lambda \in \sum \backslash \{0\}.$$

(iv) *$T(t)$ is differentiable for $t > 0$ and there exists a constant $M_1 > 0$, such that*

$$\|AT(t)\| \leq (M_1/t) \quad \textit{for } t > 0.$$

So far, we have considered time-invariant systems, that is, systems with generating operators being time independent. For completeness, we include here some well-known results with time-varying generating operators. Consider the homogeneous initial value problem

$$(d/dt)x = A(t)x, t > 0 \tag{3}$$

in the Banach space X with $A(t)$, $t \geq 0$, being generally unbounded operator with domain and range contained in X.

Definition 2.2.4. *An operator valued function $U(t,s)$ with values in $\mathcal{L}(X)$, defined on the triangle $\Delta \equiv \{0 \leq s \leq t \leq a < \infty\}$, is said to be a transition operator (or an evolution operator) corresponding to the family $\{A(t),\ t \geq 0\}$ for the system* (3) *if*

(i): $U(s,s) = I,\ s \geq 0$

(ii): $U(t,s) = U(t,\tau)U(\tau,s)$ for $0 \leq s \leq \tau \leq t < \infty$,

(iii): $U(t,s)$ is continuous on the triangle Δ in the strong operator topology in $\mathcal{L}(X)$.

In general the abstract Cauchy problem

$$\begin{cases} (d/dt)x = A(t)x,\ t > 0, \\ x(0) = x_0, \end{cases} \tag{4}$$

has a unique solution $x(t) = T(t)\xi,\ t \geq 0$, with $x(t) \in D(A)$, provided $\xi \in D(A)$. If $\xi \in D(A)$ and f is any strongly continuously differentiable function with values $f(t) \in X$, then the inhomogeneous problem

$$\begin{cases} (d/dt)x = Ax + f(t)\ ,t > 0, \\ x(0) = x_0, \end{cases} \tag{5}$$

has a unique continuously differentiable solution x given by

$$x(t) = T(t)\xi + \int_0^t T(t-s)f(s)\,ds, \tag{6}$$

with $x(t) \in D(A)$ for all $t \geq 0$.
A solution satisfying these conditions is called a strong solution.
For control problems these conditions are rather too strong. Moreover, in general, we do not expect the controls, for example $f(t)$, to be even continuous. Thus there is a need for a broader definition and this is provided by the so called mild solution.

Any function $x : I \mapsto X$ having the integral representation (6) is called a mild solution of the problem (5). Thus we have the following general result [1]:

Theorem 2.2.5. *Suppose A is the generator of a C_0-semigroup $T(t),\ t \geq 0$, in X and let $x_0 \in X$ and $f \in L^p(I,X)$, $1 \leq p \leq \infty$. Then the evolution equation (5) has a* *unique mild solution x given by*

$$x(t) = T(t)x_0 + \int_0^t T(t-s)f(s)\,ds. \tag{7}$$

In this case $x(t)$ does not necessarily belong to $D(A)$. In the case of a time varying system. that is, $A = A(t)$, the mild solution x is given by

$$x(t) = U(t,0)x_0 + \int_0^t U(t,s)f(s)\,ds, \tag{8}$$

where the evolution operator $U(t,\tau) \in \mathcal{L}(X)$, $0 \le \tau \le t \le T$ solves the abstract Cauchy problem (4). That is,

$$x(t) = U(t,0)x_0,\ t \in I,\ with\ x \in C(I,X). \tag{9}$$

Let us consider the following semilinear evolution equation written as an Abstract Cauchy problem

$$\begin{cases} (d/dt)x &= A(t)x + f(t,x), \quad t > 0, \\ x(0) &= x_0. \end{cases} \tag{10}$$

In view of (8), a mild solution of (10) is given by a solution of the integral equation

$$x(t) = U(t,0)x_0 + \int_0^t U(t,s)f(s,x(s))\,ds, \tag{11}$$

if one exists. Defining the operator G by

$$(Gx)(t) \equiv U(t,0)x_0 + \int_0^t U(t,s)f(s,x(s))\,ds. \tag{12}$$

one then looks for a fixed point for G, that is, an element x such that $x = Gx$. Using a priori estimates, on can establish the existence of a solution, by use of a suitable fixed point theorem, for example Banach, Shauder or Leray-Shauder fixed point theorems.

A control problem associated to the above system can be written as follows:

$$\begin{cases} (d/dt)x &= A(t)x + f(t,x), \quad t > 0, \\ x(0) &= x_0, \end{cases} \tag{13}$$

where the control u is taking value in a Banach or Hilbert space U.
Evolution equations of the above problems are often used to model differential equations of parabolic or hyperbolic type (see for example [1]). The dynamical process is, in such cases a semigroup of operator $T(t) : X \to X$ determined by its infinitesimal generator A.

Chapter 3

Basic Results for B-evolution

3.1 Introduction

Many physical systems governed by partial differential equations with Dirichlet, Neumann, mixed or moving boundary conditions can be modeled by ordinary differential equations in infinite dimensional space as discussed in the previous chapter. Among those, there is a more general class of systems with applications in multi-phase problems in physics and engineering [5], [7],[8], [9], [10], [12], [16], [26]. Those systems are governed by partial differential equation involving dynamic boundary conditions. For illustration of the above problems we present here a few examples.

3.2 Examples

Example 3.2.1

Consider the following heat transfer equation with dynamic boundary condition. Let $\Omega \subset \mathbb{R}^n, (n = 1, 2, 3)$ be an open bounded domain with smooth boundary which consists of two parts $\partial\Omega \equiv \Gamma_0 \cup \Gamma_1$. The material (e.g. fluid) in the interior of the domain receives heat energy through the boundary Γ_1 from an external source distributed on the exterior of the boundary layer Γ_1. Taking the dynamics of heat source into account, the problem can be modeled as follows:

$$\begin{cases} (\frac{\partial}{\partial t})T(t,\xi) &= \operatorname{div}(k(\xi)\ \nabla T) + v \cdot \nabla T + f(t,\xi,T(t,\xi)),\ t > 0,\ \xi \in \Omega \\ T(t,\xi)|_{\Gamma_0} &= 0, \\ (\frac{\partial}{\partial t})(T(t,\xi)|_{\Gamma_1}) &= -\beta D_\nu T(t,\xi)|_{\Gamma_1} + g(t,\xi,T(t,\xi)|_{\Gamma_1},u), \\ T(0,\xi) &= T_0(\xi),\quad \xi \in \Omega,\quad T(0,\xi)|_{\Gamma_1} = T_1(\xi),\quad \xi \in \Gamma_1. \end{cases} \tag{14}$$

Here T denotes the space-time temperature distribution in the interior of the domain. $k : \bar{\Omega} \mapsto [0,\infty)$, represents the thermal conductivity which equals a constant $K(> 0)$ in Ω and $\beta(> 0)$ on the boundary $\partial\Omega$. The constant β represents the thermal conductivity of the material that constitutes the boundary layer Γ_1. The quantity $v \equiv v(t,\xi) \in \mathbb{R}^3$ denotes the transport velocity of the material and f represents the internal heat source possibly nonlinear. The function g represents a nonlinear heat transfer characteristic of the boundary, $u \equiv u(t,\xi)$ denotes the (control) temperature of the external source on the part Γ_1 and D_ν denotes the outward normal derivative. Here we have a dynamic boundary condition.

Define the trace operator γ_i, for $i = 0,1$ as $\gamma_i\phi \equiv \phi|_{\Gamma_i}$ and the formal differential operators

$$L\phi = \operatorname{div}(k\ \nabla\phi) + v.\nabla\phi$$

$$M\phi = -\beta D_\nu \phi.$$

We take $X \equiv L_2(\Omega)$ and $Y \equiv L_2(\Omega) \times L_2(\Gamma_1)$ with the norm topology on Y given by

$$\|y\|_Y \equiv (\|y_1\|^2_{L_2(\Omega)} + \|y_2\|^2_{L_2(\Gamma_1)})^{1/2},\quad y = \{y_1, y_2\} \in Y.$$

The operators A and B are defined as follows:

$$\begin{aligned} D(A) &\equiv \{\phi \in H^2(\Omega) : \gamma_0\phi = 0\} \\ A\phi &= \{L\phi, M\phi\}, \phi \in D(A) \end{aligned}$$

and

$$\begin{aligned} D(B) &\equiv \{\phi \in H^1(\Omega) : \gamma_0\phi = 0\} \\ B\phi &\equiv \{\phi, \gamma_1\phi\},\quad \phi \in D(B). \end{aligned}$$

Defining $x(t) \equiv T(t, \cdot)$ and, for a fixed control $u \in L_2(I, L_2(\Gamma_1))$,

$$F(t, y) \equiv \{f(t, \cdot, y_1), g(t, y_2, u(t, \cdot))\},$$

and $y_0 \equiv \{T_0, T_1\}$, the heat transfer equation (14) can be written as an abstract B-evolution in the two Hilbert spaces $\{X, Y\}$

$$\begin{cases} (d/dt)Bx &= Ax + F(t, Bx), \quad t \geq 0, \\ Bx|_{t=0} &= y_0. \end{cases}$$

Example 3.2.2 (Structural Vibration)
A simplified mathematical model for the space shuttle mast-antenna system can be described by the following system of coupled partial and ordinary differential equations [13] :

$$\begin{cases} V_{tt} + D^4V + f_1(V_t) = f_2, \quad t > 0, \quad \Omega = (0, 1), \\ V(t, 0) = DV(t, 0) = 0, \\ V_{tt}(t, 1) - D^3V(t, 1) = f_3, \\ (DV)_{tt}(t, 1) + D^2V(t, 1) = f_4, \end{cases} \tag{15}$$

where the first two equations describe the transverse vibration of the mast mounted on the shuttle body and the last two describe the dynamics of the antenna attached to the free end of the mast giving the lateral and angular displacements. Here we have used $D^k \equiv \partial^k/\partial\xi^k$ to denote the partial derivative of order k with respect to the spatial variable on Ω. Defining the state variable

$$z \equiv z(t, \xi), t > 0, \quad \xi \in \Omega \quad \text{as} \quad z \equiv \{z_1, z_2\} \equiv \{V, V_t\}$$

the system can be written as a first-order equation. Without going through the routine details, we introduce the two Hilbert spaces $\{X, Y\}$

$$X \equiv L_2(\Omega) \times L_2(\Omega), \quad Y \equiv L_2(\Omega) \times L_2(\Omega) \times L_2(\Omega) \times \mathbb{R}^2$$

with the natural inner products and norm topologies and the operators $\{A, B\}$ as follows

$$D(A) \equiv \dot{H}^4(\Omega) \times \dot{H}^2(\Omega) \subset X,$$

$$\begin{aligned} \mathring{H}^m(\Omega) &\equiv \{\phi \in H^m : \phi(0) = D\phi(0) = 0\}, \quad m = 2, 4, \\ Az &\equiv \{z_2, -D^4 z_1, D^3 z_1(1), -D^2 z_1(1)\}, \quad \text{for } z \in D(A), \\ D(B) &\equiv H^2(\Omega) \times H^2(\Omega) \subset X, \\ Bz &\equiv \{z_1, z_2, D^2 z_1, z_2(1), Dz_2(1)\}, \quad \text{for } z \in D(B). \end{aligned}$$

Note that by the well-known Sobolev embedding theorem, $H^m \hookrightarrow C^k$ for $m > (n/2) + k$ and hence the point values in the definitions of A and B are well defined. For stability of the system, for example suppression of vibration, often the input forces f_3 and f_4 are provided as feedback controls based on boundary velocity [13] as described by

$$f_3 \equiv f_3(y_4), \qquad f_4 \equiv f_4(y_5).$$

Define the nonlinear operator

$$F(t, y) \equiv \{0, -f_1(y_2) + f_2(t), 0, f_3(y_4), f_4(y_5)\},$$

where $f_2 \equiv f_2(t, \xi)$, $\xi \in \Omega$, describes the distributed force acting on the mast. Thus the system (15) can be written as an abstract differential equation on the two Hilbert spaces $\{X, Y\}$ as

$$\begin{cases} (d/dt)Bx &= Ax + F(t, Bx), \quad t > 0, \\ Bx|_{t=0} &= y_0, \end{cases}$$

where

$$x(t) \equiv \{z_1(t, \xi), z_2(t, \xi), \ \xi \in \Omega\}$$

and

$$y_0 = \{z_1(0, \xi), z_2(0, \xi), D^2 z_1(0, \xi), z_2(0, 1), Dz_2(0, 1), \xi \in \Omega\}.$$

Evolution equations of the form

$$(d/dt)x = Ax, \quad t > 0, \tag{16}$$

with initial condition x_0 in Banach space X, are often used to model differential equations of parabolic or hyperbolic type. As discussed in the previous chapter, the underlying dynamical process is, in such cases, a semigroup of operators $T(t) : X \to X$ determined by its infinitesimal generator A.

Partial differential equations with dynamic boundary conditions may be modeled as an abstract ordinary differential equations in two Banach spaces as follows (see the above examples):

$$\begin{cases} (d/dt)Bx &= Ax, \quad t > 0, \\ Bx|_{t=0} &= y_0, \end{cases} \tag{17}$$

ith A and B operators from a Banach space X to a Banach space Y. oreover, the natural initial state y_0 appears to be in the space Y.

roblems of this kind may be treated by considering dynamical ocesses in which cause and effect are represented in different spaces.

Formally by setting $y = Bx$ and assuming the operator B to have an inverse on its range, the problem (18) is reduced to the standard problem:

$$\begin{cases} (d/dt)y &= AB^{-1}y, \quad t > 0, \\ y(0) &= y_0. \end{cases} \tag{18}$$

By virtue of the Hille-Yoshida theorem on the characterization of a generator of semigroup, the operator AB^{-1} should be closed in order to generate a C_0 semigroup. But the operators A and B, even linear, are generally not closed. Hence the operator AB^{-1} is not necessarily closed and fails to generate a C_0-semigroup.

It was found in [15] that systems of equations such as (18) could be treated with the aid of so-called B-evolutions of operators.
In the following section we present some facts from the theory of the so called B-evolution. For more details see [15], [16] and [7], [8], [9].

3.3 Basic Results for B-evolution

For any pair of Banach spaces X, Y, respectively with norms, $\|.\|_X$, $\|.\|_Y$, $\mathcal{L}(X,Y)$ and $\mathcal{L}_{ub}(X,Y)$ will denote the class of bounded and unbounded linear operators from

X to Y respectively. Let I=[0.T], $T < \infty$, and for $1 \leq p < \infty$, $L^p(I, X)$, endowed with its usual p-norm $\|\cdot\|_p$, denote the class of all p-th power Lebesgue–Bochner integrable functions on I with values in X.

As the theory of B-evolutions has been developed only during the past few years, we shall list, for the sake of completeness, the basic definitions and properties of B-evolutions, as introduced in [15], [16].
Let B be a linear operators with domain $D(B) \subset X$, and range $R(B) \subset Y$.

Definition 3.3.1. *A family $\{S(t) : \ t > 0\}$ of bounded linear operators defined on Y is called a B-evolution if*

$$S(t)(Y) \subset D(B), \quad \textit{for all } t > 0$$

and

$$S(t+s) = S(s)BS(t), \quad \textit{for all } s, t > 0.$$

From the definition, it follows that the family $\{E(t) : \quad t \geq 0\}$ of linear operators in Y, defined by

$$E(t) = BS(t), \quad \text{for all } t > 0,$$

satisfies the semigroup property

$$E(t+s) = E(s)E(t), \quad \text{for all } s, t \geq 0.$$

$E(t)$ is called the associated semigroup. The B-evolution $S(t)$ is called strongly continuous if $E(t)$ is a semigroup of class C_0.

Definition 3.3.2. *A strongly continuous, uniformly bounded B-evolution is said to be of type L if*

$$P_B(\lambda)y \triangleq \int_0^\infty e^{-\lambda t} S(t)y \, dt \in D(B),$$

for all $y \in Y$ and a complex number λ, with $\Re(\lambda) > 0$, and

$$BP_B(\lambda)y = \int_0^\infty e^{-\lambda t} BS(t)y \, dt.$$

Definition 3.3.3. *A B-evolution $S(t)$ of type L is called holomorphic if the associated semigroup $E(t)$ is holomorphic.*

Definition 3.3.4. *The infinitesimal generator A of B-evolution $S(t)$ is defined by*

$$D(A) \quad = \quad \{x \in D(B): \quad Ax = \lim_{h \to 0} h^{-1}(BS(h) - B)x \quad exist\}.$$

It is clear from the last definition that $D(A) \subset D(B)$.

Remark 3.3.1. *The infinitesimal generator A of B-evolution is not necessarily closed or densely defined.*

A useful results due to Sauer [15] are given by the following lemmas:

Lemma 3.3.1. ([16], Theorem 2.1, p. 289). *Let $S(t)$ be a strongly continuous B-evolution. Then*
(a): for $x \in D(A)$, $S(t)Bx \in D(A)$, and

$$AS(t)Bx = BS(t)Ax = (d/dt)BS(t)Bx. \ \ for \ \ t > 0$$

(b): if A_Y is the infinitesimal generator of $E(t)$ then $x \in D(A)$ if and only if $Bx \in D(A_Y)$ and for such x

$$Ax = A_Y Bx.$$

(c): $B(D(A))$ is dense in Y.
(d): for $y \in Y$, the mapping $t \to S(t)y$ is right continuous.

Lemma 3.3.2 . ([15], Theorem 2.3, p. 290) *Let $S(t)$, $t > 0$, be a strongly continuous uniformly bounded B-evolution. Then*
(a): for each $y \in Y$, $t \to S(t)y$, $t > 0$, is strongly continuous with values in X.
(b): there exists an operator $C \in \mathcal{L}(Y, X)$ such that
$Cy = \lim_{t \to 0^+} S(t)y$ for each $y \in Y$; and $S(t)y = CE(t)y$, $t > 0$.

(c): C restricted to the range of B is the right inverse of B.

Lemma 3.3.3. ([15], Theorem 5.1, p. 296) *The pair $< A_0, B_0 >$, where A_0 and B_0 are suitable restrictions of A and B respectively, is the generating pair of a B-evolution $S(t)$, $t > 0$, of type L if and only if :*
(a): B_0 has a bounded inverse on its range $R(B_0) \subset Y$,
(b): $A_0 B_0^{-1}$ generates a uniformly bounded C_0-semigroup $E(t)$for $t > 0$, in Y,
(c): The bounded linear operator C which is the strong limit of $C_n \equiv (B_0-(1/n)A_0)^{-1}$ is invertible on $\cup_{t>0} E(t)(Y) + R(B_0)$

In case $E(t)$, $t > 0$, is holomorphic semigroup in Y or B_0 is closeable the last condition is superfluous. In this case the pair $< A_0, B_0 >$ coincides with the pair $< A, B >$.

The following assumptions will be used:

Assumptions:
(I) The domain $D(A(t)) = D(A)$ is independent of t.

(II) There exists a number $\epsilon > 0$ such that for all λ in $\Theta \equiv \{\lambda \in C : \quad \lambda \neq 0, \quad -(\epsilon+\pi/2) < \arg\lambda < \epsilon+\pi/2\}$ the resolvents $(\lambda B - A(t))^{-1}$ and $B(\lambda B - A(t))^{-1}$are strongly continuous in t, with respect to the norm topologies of $\mathcal{L}(Y, X)$ and $\mathcal{L}(Y)$ respectively. The continuity in t is uniform in λ on every compact subset of Θ.

(III) There exist positive constants M and N such that for $\lambda \in \Theta$ and $t \in [0, T]$

$$\begin{aligned} \|(\lambda B - A(t))^{-1}\|_{\mathcal{L}(Y,X)} &\leq M/ \mid \lambda \mid, \\ \|B(\lambda B - A(t))^{-1}\|_{\mathcal{L}(Y)} &\leq N/ \mid \lambda \mid . \end{aligned}$$

(IV) B is injective and has a bounded inverse on its range $R(B)$

(V) There exists a constant $\widetilde{N} > 0$ such that

$$\|(A(\tau)B^{-1})^{-1}\|_{\mathcal{L}(Y)} \leq \widetilde{N}, \quad \tau \in [0, T].$$

(VI) There exists a constant $K > 0$ such that for $t, s, \tau \in [0, T]$, $0 < \alpha < 1$,

$$\|(A(t) - A(s))A^{-1}(\tau)\|_{\mathcal{L}(Y)} \leq K \mid t - s \mid^{\alpha},$$

A characterization of a generating pair of operators is given by the following lemma:

Lemma 3.3.4. ([7]) *Under the assumptions* **(I)** $-$ **(VI)** *the pair* $\{A(\cdot), B\}$ *generates a holomorphic B-evolution* V *of type* L *with values* $\{V(t, s), t > s \geq 0\} \in \mathcal{L}(Y, X)$ *satisfying the following properties:*

(P1): $V(t, s)$ *is continuous on* $0 \leq s < t \leq T$ *in the strong operator topology of* $\mathcal{L}(Y, X)$

(P2): $E(t, s) \equiv BV(t, s)$ *is uniformly bounded in* $\mathcal{L}(Y)$ *i.e. there exists a constant,* $C_T > 0$ *such that*

$$\|BV(t, s)\|_{\mathcal{L}(Y)} \leq C_T \quad for \quad 0 \leq s < t \leq T.$$

(P3): $BV(t, s)$, *is continuously differentiable in the strong topology of* $\mathcal{L}(Y)$ *on* $[0, T)$ *and*

$$\begin{cases} (\partial/\partial t)BV(t, s) = A(t)V(t, s) \\ (\partial/\partial s)BV(t, s) = -BV(t, s)A(s)B^{-1} \end{cases}$$

(P4): BV *satisfy the following evolution property :*

$$\begin{cases} BV(t, s)BV(s, \tau) = BV(t, \tau) \quad \text{for } 0 \leq \tau < s \leq T, \\ BV(t, t) = I. \end{cases}$$

Chapter 4

Necessary Optimality Conditions

4.1 Introduction

In this chapter we consider Lagrange type control problem for systems involving dynamic boundary conditions that is, with boundary operators containing time derivatives. Assuming the existence of optimal controls, B-evolutions theory is used to present necessary conditions of optimality. The result is illustrated by an example from heat transfer problem and also an algorithm for computing optimal controls is presented.

4.2 Motivation

For motivation let us consider a heat transfer problem arising in a nuclear reactor. Let Ω be a bounded open subset of $\mathbb{R}^3$ modeling the interior of an annular tube of finite length with smooth boundary which consists of two parts: Γ_1, the inner boundary, and Γ_2, the outer boundary. The coolant (e.g. heavy water) in the annular region Ω receives heat energy from the heat produced by nuclear reaction inside the fuel rods surrounded by the boundary layer Γ_1. The corresponding control system model can be described as follows:

$$\begin{cases} (\frac{\partial}{\partial t})T(t,\xi) = \operatorname{div}(k(\xi)\nabla T) + v\cdot\nabla T, \xi \in \Omega, t > 0, \\ T(t,\xi)|_{\Gamma_2} = h(t,\xi), \xi \in \Gamma_2, t \geq 0, \\ (\partial/\partial t)(T(t,\xi)|_{\Gamma_1}) = \triangle_{\Gamma_1}T(t,\xi) - \beta D_\nu(T(t,\xi)|_{\Gamma_1}) + g(\xi, T(t,\xi)|_{\Gamma_1}, u(t,\xi)), \quad t > 0, \\ T(0,\xi) = T_0(\xi), \xi \in \Omega \\ T(0,\xi)|_{\Gamma_1} = T_1(\xi), \xi \in \Gamma_1. \end{cases} \tag{19}$$

The quantity T denotes the space-time temperature distribution in the interior of the domain, $k : \overline{\Omega} \mapsto \mathbb{R}^+$, represents the thermal conductivity which satisfies

$$k(\xi) = \begin{cases} 0 & \text{on } \Gamma_2 \\ K & \text{on } \Omega \\ \beta & \text{on } \Gamma_1. \end{cases}$$

The constant $\beta(> 0)$ represents the thermal conductivity of the material that constitutes the boundary layer Γ_1. The quantity $v = v(t,\xi) \in \mathbb{R}^3$ denotes the transport velocity of the fluid in Ω and $u = u(t,\xi)$ (the control) temperature of the outer surface Γ_1 of the fuel rod (due to nuclear reaction). The Laplace-Beltrami operator $\triangle_{\Gamma_1}$ represents the rate at which thermal energy is transferred within Γ_1 and D_ν denotes the outward normal derivative. The function g represents a convective heat transfer given by $g = \alpha(T_r - T|_{\Gamma_1})$, where the parameter α is the heat transfer coefficient due to convection and T_r denotes the surface temperature of the fuel rod which is the control u.

For this example the integrand may be taken as

$$l(T(t), u(t)) \equiv \int_\Omega |T(t,\xi) - T^d(t,\xi)|^2 d\xi + \int_{\Gamma_1} |u(t,\xi)|^2 d\xi$$

and the cost functional to be minimized subject to the system (19) is given by

$$J(u) = \int_I l(T(t), u(t))\, dt.$$

That is we like to keep the temperature distribution at some preassigned value T^d.

The necessary conditions of optimality for the above example will be presented in section 5 once the general theory has been developed.

The abstract mathematical model for the example (19) can be written in two Banach spaces as follows:

$$(P_f)\begin{cases} \frac{d}{dt}Bx(t) = A(t)x(t) + f(Bx(t), u(t)), & t \in I \\ (Bx)(0) = z_0 \end{cases}$$

where A and B are linear unbounded operators between two Banach spaces. f is a map with values in a Banach space and u is a suitable function representing the control actions taking values in another Banach space.

In addition to covering all classical boundary and distributed control problems. the system (P_f) includes a new set of such problems in which the boundary conditions are determined also by an evolution equation (see example above).

The control problems of systems governed by B-evolutions has not been studied before. Control theory for classical cases ($X = Y$, $B = I$ identity operator) and their abstract versions have been studied extensively in the literature (see [37]. [38]. [14]. [3] and many others. and the references therein).

In this chapter we consider a class of semilinear problem governed by B-evolutions as described above by the system (P_f). We derive necessary conditions of optimality in terms of the available data. that is in terms of the given operators A and B and the cost integrand l.

4.3 Preparatory Results

For the development of necessary conditions of optimality. we need some preliminary results.

Here we consider the case of holomorphic B-evolution of type L. In this case the operators A and B are not necessarily closed but the pair $\{A(t), B\}$ from $D(A) \cap D(B) \subset X$ to $Y \times Y$ is closed. It follows from the Lemma 3.3.1, that the Cauchy problem

$$\begin{cases} (d/dt)Bx = A(t)x \\ s - \lim_{t \to 0^+}(Bx) = z_0 \end{cases}$$

has a unique solution given by

$$x(t) = V(t,0)z_0, \quad t > 0. \tag{20}$$

As a consequence of Lemma 3.3.1, it follows that $A(t)B^{-1}$ with domain $B(D(A)) \subset Y$ is the infinitesimal generator of the transition operator $E(t,s) \equiv BV(t,s), t > s \geq 0$. In this case the system (P_f) can be written as

$$\begin{cases} \dot{z} = A(t)B^{-1}z + f(z,u) \\ z(0) = z_0 \end{cases} \tag{21}$$

and call $x(t) \equiv Cz(t)$, $t \geq 0$, as the generalized solution of the system (P_f) where z is the mild solution of equation (21) and $C \in \mathcal{L}(Y,X)$ is the operator given by Lemma 3.3.2.

Thus in the case of holomorphic B-evolution the problem (P_f) is related to the classical problem (21). In the homogeneous case (i.e. $f \equiv 0$) the solution of the problem (P_0) is given by $x(t) = B^{-1}z(t)$ since $z(t) = E(t,0)z_0 \in R(B)$ and $C|_{R(B)} = B^{-1}$. In the nonhomogeneous case, $z(t)$ may not be in the range of the operator B and hence the definition of generalized solution $x(t) = Cz(t)$ makes sense.
For simplicity of notation we have written $s - \lim_{t \to 0^+}(Bx) = Bx(t)\mid_{t=0} = z(0)$.

In order to study the control problem given by the system (22) we introduce the class of admissible controls as follows:

Admissible controls:

Let Λ be a closed, bounded and convex subset of U. For admissible controls, we choose the set

$$\mathcal{U}_{ad} \equiv \{u: \ \textit{strongly measurable and} \ u(t) \in \Lambda \ \textit{a.e.}\}.$$

Occasionally, we use the notation $x(u)$ to denote the solution, of the system (P_f) corresponding to $u \in \mathcal{U}_{ad}$.

In the following lemma we present an a priori bound and existence result.

Lemma 4.3.1. *Suppose the following assumptions hold*

(b1): *The pair* $\{A(t), B\}$ *is the generating pair of a holomorphic B-evolution* V *of type* L.

(b2): $f : Y \times U \to Y$ *is a map such that* f *is locally lipshitz in* Y *i.e. for each* $0 < r < \infty$, *and* $z_0 \in Y$, *there exists a positive constant* $K_r \equiv K_r(z_0)$ *such that*

$$\|f(z_1, u) - f(z_2, u)\|_Y \leq K_r \|z_1 - z_2\|_Y$$

for all $z_1, z_2 \in B_r(z_0) \equiv \{z \in Y : \|z(t) - z_0\|_Y \leq r\}$ *and* $u \in \Lambda$ *and satisfies the growth condition*

$$\|f(z, u)\|_Y \leq K(1 + \|z\|_Y),$$

for some $K > 0$, $z \in Y$, $u \in \Lambda$. *Then*

(i): *There exist finite positive numbers* $\widetilde{M}$ *such that:*
$\sup\{\|x(u)(t)\|_X; t \in I, \quad u \in \mathcal{U}_{ad}\} \leq \widetilde{M}$,

(ii): *for each* $z_0 \in Y$, *the problem* (P_f) *has a unique generalized solution* $x \in C(I, X)$ *and this is given by* $x = Cz$ *where* z *is the solution of the integral equation*

$$z(t) = E(t, 0)z_0 + \int_0^t E(t, s)f(z(s), u(s))\, ds, \quad t \in I. \tag{22}$$

Proof: By virtue of Lemma 3.3.4, $A(t)B^{-1}$ is the infinitesimal generator of uniformly bounded holomorphic evolution operator $E(t, s)$ in Y, $0 \leq s < t \leq T$. Using the variation of constants formula we obtain the following integral equation

$$z(t) = E(t, 0)z_0 + \int_0^t E(t, s)f(z(s), u(s))\, ds, \quad t \in I. \tag{23}$$

Since the controls are contained in a bounded set and f satisfies the growth condition (assumption **(b2)**), using Gronwall's inequality and the integral equation above one can easily verify that there exists an $M > 0$ such that

$$\sup\{\|z(t)\|_Y, t \in I\} \leq M. \tag{24}$$

Where $M \equiv M(\|z_0\|_Y, T)$ is a positive constant dependent on the parameters shown. Thus the map $u \mapsto z$ from $\mathcal{U}_{ad}$ to $L^\infty(I;Y)$ is bounded. Since C is bounded. there exists a constant $\widetilde{M} > 0$ such that

$$\sup\{\|x(t)\|_X, t \in I\} \leq \widetilde{M}. \tag{25}$$

That is the map $u \mapsto x$ from $\mathcal{U}_{ad}$ to $L^\infty(I;X)$ is bounded. This justifies the conclusion(**i**).

To prove (**ii**), let $z \in C(I,Y)$ satisfying $z(0) = z_0$ and for some $0 < r < \infty$ $z(t) \in B_r(z_0)$ for all $t \in I$. Define the operator Q by

$$(Qz)(t) = E(t,0)z_0 + \int_0^t E(t,s)f(z(s),u(s))\,ds, \ \ for \ \ t \in I. \tag{26}$$

Using the strong continuity of $E(t,s)$ on $\Delta \equiv \{0 \leq s \leq t \leq T\}$ in $\mathcal{L}(Y)$, the assumption (**b2**), the estimate (**P2**) of Lemma 3.3.4 and the above integral equation, one can show that $(Qz)(t) \in B_r(z_0)$. Further, $t \mapsto (Qz)(t)$ is continuous Y-valued function on I. Define

$$\Sigma_r \equiv \left\{ z \in C([0,\sigma],Y) : z(0) = z_0 \ \ and \sup_{t\in[0,\sigma]} \|z(t) - z_0\|_Y \leq r \right\}.$$

The set Σ_r, furnished with the natural metric topology

$$\rho(z_1, z_2) \equiv \sup_{t\in[0,\sigma]} \|z_1(t) - z_2(t)\|_Y, \ \ z_1, z_2 \in \Sigma_r,$$

is a complete metric space and Q maps Σ_r to Σ_r.

Under the property (**P2**) and assumption (**b2**), one can show that Q is a contraction in Σ_r and hence from the Banach fixed point theorem it follows that Q has a unique fixed point $z \in \Sigma_r$. Since $C \in \mathcal{L}(Y,X)$, it follows that the Cauchy problem (P_f) has unique generalized solution $x = Cz(t) \in C(I;X)$. This completes the proof of the lemma.

Remark 4.3.1. *By virtue of the a priori bound, all solutions* $\{x(u)\}$ *of the system* (P_f) *lie in a closed ball* $B_{\widetilde{M}}(X) \equiv \{\xi \in X : \|\xi\| \leq \widetilde{M}\}$.

For the necessary conditions of optimality we shall give the following result on continuous dependence of solutions on controls.

Lemma 4.3.1'. Suppose
(A1): assumptions of lemma 4.3.1 hold.
(A2): $f : Y \times U \longrightarrow Y$ is Frechet differentiable with respect to $z \in Y$ and $u \in U$ with respective Frechet derivatives $f_1 : Y \times U \longrightarrow \mathcal{L}(Y)$, $f_2 : Y \times U \longrightarrow \mathcal{L}(U, Y)$ being continuous and bounded on bounded subsets of $Y \times U$.
Let z^0 and $z(u^\epsilon)$ be the solution of the system (21) corresponding to control u^0 and u^ϵ respectively and $x^0 = Cz^0$ and $x(u^\epsilon) = Cz(u^\epsilon)$ be the generalized solution of (P_f) corresponding to control u^0 and u^ϵ respectively. Then if $u^\epsilon \equiv u^0 + \epsilon(u - u^0) \longrightarrow u^0$, the solution $x(u^\epsilon) \longrightarrow x^0$ in $C(I; X)$ and $z(u^\epsilon) \longrightarrow z^0$ in $C(I; Y)$.

Proof: Define $z^\epsilon \equiv z(u^\epsilon) - z^0$. Then z^ϵ satisfies the following equation

$$\begin{aligned} z^\epsilon &= \int_0^1 E(t,s)\Big(f(z(u^\epsilon)(s), u^\epsilon(s)) - f(z^0(s), u^0(s))\Big)ds \\ &= \int_0^1 E(t,s)\Big(f(z(u^\epsilon)(s), u^\epsilon(s)) - f(z^0(s), u^\epsilon(s))\Big)ds \\ &\quad + \int_0^1 E(t,s)\Big(f(z^0, u^0) - f(z^0, u^\epsilon)\Big)ds \end{aligned}$$

By virtue of Lagrange formula applied to the function f, one can rewrite the above equation in the following form

$$\begin{aligned} (z^\epsilon)(t) &= \int_0^1 E(t,s)F_1^\epsilon(s)(z^\epsilon)(s)ds \\ &\quad + \int_0^1 E(t,s)F_2^\epsilon(s)(u^\epsilon - u^0)(s)ds \end{aligned} \tag{27}$$

where

$$F_1^\epsilon(s) \equiv \int_0^1 f_1\Big(z^0(s) + \theta(z(u^\epsilon)(s) - z^0(s)), u^0(s)\Big)d\theta, \quad s \in I$$

and

$$F_2^\epsilon(s) \equiv \int_0^1 f_2\Big(z^0(s), u^0 + \theta(u^\epsilon(s) - u^0(s))\Big)d\theta, \quad s \in I.$$

Clearly under the assumption **(A2)** and the a priori bound (see Remark 3.1), there exists a constant $\bar{C} > 0$ such that

$$\sup\{\| F_1^\epsilon(t) \|_{\mathcal{L}(Y)}, \ \| F_2^\epsilon(t) \|_{\mathcal{L}(U;Y)}, \ t \in I\} \ \leq \ \bar{C}.$$

Further, as $\epsilon \to 0$, the operators $F_1^\epsilon(t)$ and $F_2^\epsilon(t)$ converge to $F_1^0(t)$ and $F_2^0(t) \equiv f_2(z^0(t), u^0)$ in the uniform operator topology for almost all $t \in I$. Thus $t \longmapsto F_1^\epsilon(t)$ and $t \longmapsto F_2^\epsilon(t)$ are uniformly measurable operator valued functions taking values from $\mathcal{L}(Y)$ and $\mathcal{L}(U; Y)$ respectively.
By virtue of the property (**P2**) (see Lemma 3.3.4) , it follows from (27) that

$$\begin{aligned} \| (z^\epsilon)(t) \|_Y &\leq C_T\bar{C}\int_0^t \| (z^\epsilon)(s) \|_Y \, ds \\ &\quad + C_T\bar{C}\int_0^t \| (u^\epsilon - u^0)(s) \|_Z \, ds. \end{aligned}$$

Hence, by Gronwall's inequality we obtain

$$\sup_{t\in I} \| (z^\epsilon)(t) \|_Y \leq \bar{C}C_T \exp(C_T\bar{C}T) \int_0^T \| (u^\epsilon - u^0)(s) \|_Z \, ds.$$

Since $C \in \mathcal{L}(Y, X)$, the result follows from the last inequality by letting ϵ go to zero.

Let Y^* denote the dual of the Banach space Y and Y_w^* the space Y^* endowed with the w^* topology and $C(I; Y_w^*)$ the topological space of w^*-continuous Y^*-valued functions defined on the interval I=[0,T]. Let $< \cdot, \cdot >_{Y^*,Y}$ denote the duality pairing between Y^* and Y. Let A^* denote the adjoint of the operator A. For the study of control problem we shall need the following Cauchy problem called the adjoint equation

$$\begin{cases} (d/dt)\psi(t) + (A(t)B^{-1} + F_1^0)^*\psi(t) = -g(t) \quad a.e. \\ \psi(T) = 0 \in Y^* \end{cases} \tag{28}$$

Lemma 4.3.2. *Let $g \in L^1(I, Y^*)$ and $F_1^0(t)$ the operator as defined above. Then the adjoint problem* (28) *has a unique mild solution $\psi \in C(I; Y_w^*)$, which satisfies equation* (28) *also in the weak sense.*

Proof: Let $E^*(t, s)$, $0 \leq s \leq t < \infty$, denote the adjoint of the operator $E(t, s)$ and write (28) as a Volterra integral equation

$$\psi(t) = \int_t^T E^*(\theta, t)g(\theta)d\theta + \int_t^T E^*(\theta, t)(F_1^0)^*(\theta)\psi(\theta)d\theta \tag{29}$$

Define the operator $\tilde{Q}$ by

$$\begin{aligned} (\tilde{Q}\psi)(t) &= \int_t^T E^*(\theta, t)g(\theta)d\theta + \int_t^T E^*(\theta, t)(F_1^0)^*(\theta)\psi(\theta)d\theta \\ &\equiv h_1(t) + h_2(t). \end{aligned} \tag{30}$$

We show that $\tilde{Q}$ has a fixed point in $L^\infty(I;Y^*)$ and any such solution is actually w^*-continuous. First we show that $\tilde{Q}$ maps $L^\infty(I;Y^*)$ to $L^\infty(I;Y^*)$.
Since the operator $E(t,\theta)$, $0 \le \theta \le t \le T$ is bounded and $g \in L^1(I;Y^*)$ it is easy to verify that $h_1 \in L^\infty(I;Y^*)$. Since the evolution operator $E(\theta,t)$ and the operator $F_1^0(t)$, $0 \le t \le \theta \le T$ are bounded and uniformly measurable, their adjoint are also bounded and measurable. Hence the integral in (30) is well defined in the Bochner sense and moreover for $\psi \in L^\infty(I;Y^*)$, $h_2 \in L^\infty(I;Y^*)$. Hence $\tilde{Q}$ maps $L^\infty(I;Y^*)$ into itself.
Since the operators $E^*(\theta,t)$ and $(F_1^0)^*(\theta)$ are bounded on $0 \le t \le \theta \le T$, there exists a constant $\widetilde{K} > 0$ so that

$$\|(\tilde{Q}\psi_1 - \tilde{Q}\psi_2)(t)\|_{Y^*} \;\le\; \widetilde{K}\int_t^T \|(\psi_1 - \psi_2)(\theta)\|_{Y^*} d\theta \tag{31}$$

for all $\psi_1, \psi_2 \in L^\infty(I;Y^*)$ and $t \in I$. Substuting (31) into itself, at the nth iteration, we obtain

$$\|(\tilde{Q}^n\psi_1 - \tilde{Q}^n\psi_2)(t)\|_{Y^*} \;\le\; (\widetilde{K}T)^n/n! \int_t^T \|(\psi_1 - \psi_2)(\theta)\|_{Y^*} d\theta.$$

Taking the supremum norm in the last inequality we obtain

$$d(\tilde{Q}^n\psi_1, \tilde{Q}^n\psi_2) \;\le\; \alpha_n d(\psi_1, \psi_2),$$

where

$$d(y_1; y_2) \;\equiv\; ess\sup\{\|y_1(t) - y_2(t)\|_{Y^*} \quad t \in I\}$$

and

$$\alpha_n \;=\; (\widetilde{K}T)^n T/n!.$$

Then one can choose a positive integer n_0 such that $\alpha_{n_0} < 1$. Hence, for $n > n_0$, $\tilde{Q}^n$ is a contraction on $L^\infty(I;Y^*)$. It follows from Banach fixed point theorem that $\tilde{Q}^n$ and hence, $\tilde{Q}$ has a unique fixed point in $L^\infty(I;Y^*)$ and therefore equation (29) has a unique mild solution $\psi \in L^\infty(I;Y^*)$.

We show that $\psi \in C(I; Y_w^*)$ and that it is also a weak solution. Taking $\eta \in Y$, it follows from (29) that

$$\begin{aligned} < \psi(t), \eta >_{Y^*,Y} &= \int_t^T < g(\theta), E(\theta, t)\eta >_{Y,Y^*} d\theta \\ &+ \int_t^T < \psi(\theta), F_1^0(\theta)E(\theta, t)\eta >_{Y^*,Y} d\theta \\ &\equiv I_1(t) + I_2(t). \end{aligned} \tag{32}$$

Since $t \mapsto E(\theta, t)$ is strongly continuous in $\mathcal{L}(Y)$ on $[0, \theta]$ and $g \in L^1(I; Y^*)$, it follows that $t \mapsto I_1(t)$ is continuous on I. Further, since $E(\theta, t)$ is strongly continuous in $\mathcal{L}(Y)$, for $\psi \in L^\infty(I; Y^*)$, $t \mapsto I_2(t)$ is also continuous on I. Thus $t \mapsto \psi \in C(I; Y_w^*)$. Replacing η by z for $z \in B(D(A))$, it follows from (32) that

$$\begin{aligned} < \psi(t), z >_{Y^*,Y} &= \int_t^T < g(\theta), E(\theta, t)z >_{Y^*,Y} d\theta \\ &+ \int_t^T < (F_1^0)^*(\theta)\psi(\theta), E(\theta, t)z >_{Y^*,Y} d\theta \end{aligned}$$

Using the differentiation property **(P3)**(see Lemma 3.3.4) it is not difficult to verify that

$$\begin{aligned} &\frac{d}{dt} < \psi(t), z >_{Y^*,Y} + < \psi(t), F_1^0(t)z >_{Y^*,Y} + \\ &\int_t^T < E^*(\theta, t)(F_1^0)^*(\theta)\psi(\theta) + E^*(\theta, t)g(\theta), A(t)B^{-1}z >_{Y^*,Y} d\theta \\ &= - < g(t), z >_{Y^*,Y} . \end{aligned}$$

Thus, for all $z \in B(D(A))$,

$$\begin{aligned} \frac{d}{dt} < \psi(t), z >_{Y^*,Y} &+ < \psi(t), F_1^0(t)z >_{Y^*,Y} + < \psi(t), A(t)B^{-1}z >_{Y^*,Y} \\ &= - < g(t), \xi >_{X^*,X} \end{aligned} \tag{33}$$

for almost $t \in I$. Clearly by (29) $\psi(T) = 0$. Thus, ψ as defined above, solves the problem (28) also in the weak sense.

4.4 Necessary Optimality Conditions

In this section we present our main results on the necessary conditions of optimality for the following Lagrange problem

(P): find $u^0 \in \mathcal{U}_{ad}$ such that $J(z^0, u^0) \leq J(z, u)$, for all $u \in \mathcal{U}_{ad}$

where

$$J(z, u) \equiv \int_I l(Cz, u)\, dt.$$

Here z denotes the solution of the system (22) corresponding to the control $u \in \mathcal{U}_{ad}$.

In what follows we shall assume that the optimal control problem **(P)** has a solution, that is, there exists an admissible state-control pair (z^0, u^0) such that

$$J(z^0, u^0) \leq J(z, u), \quad for \quad all \quad u \in \mathcal{U}_{ad}. \tag{34}$$

We consider two cases: The cases where the cost integrand l is Fréchet differentiable and merely continuous in the control variable.

Case A : l Fréchet differentiable in control variable

Theorem 4.4.1. *Let $(z^0, u^0) \in C(I, Y) \times L^\infty(I, U)$ be any state-control pair associated with system* (22) *and suppose the following conditions hold :*

(d1): *$l : X \times U \longrightarrow R$ such that $l(w, \varsigma)$ is continuously Fréchet differentiable in w and ς with Fréchet derivatives denoted by l_1, and l_2 respectively. Further $C^* l_1^0 \in L^1(I; Y^*)$ and $l_2^0 \in L^1(I, U^*)$ along the pair (z^0, u^0).*

(d2): *f satisfies assumptions (***b2***) of Lemma 4.3.1 and (***A2***).*

Then, in order that (z^0, u^0) be the optimal pair, it is necessary that there exists $\psi \in C(I; Y_w^*)$ so that the triple (z^0, u^0, ψ) satisfies the following equations and inequalities:

(1) : $(d/dt)z^0 = A(t)B^{-1}z^0 + f(z^0, u^0)$, $z^0 = z(0)$.

(2) : $(d/dt)\psi(t) + (A(t)B^{-1})^*\psi + (F_1(t))^*\psi = -C^* l_1^0(t), \quad \psi(T) = 0.$

(3): $\int_I < l_2^0 + F_2^0\psi, (u - u^0) >_{U^*, U} dt \geq 0$, *for all* $u \in \mathcal{U}_{ad}$;

where

$$l_i^0(t) \equiv l_i(Cz^0(t), u^0(t)), \quad for\ i = 1, 2.$$

Proof: Let (z^0, u^0) be the optimal pair for the problem (21). By convexity of $\mathcal{U}_{ad}$, for $u \in \mathcal{U}_{ad}$, $u^\epsilon \equiv u^0 + \epsilon(u - u^0) \in \mathcal{U}_{ad}$, for $0 \le \epsilon \le 1$. According to Lemma 4.4.1, the state equation (21) has a unique mild solution $z(u^\epsilon)$ corresponding to the control u^ϵ and by definition of optimality (34) we have

$$\int_I l(Cz(u^\epsilon)(t), u^\epsilon(t))\, dt - \int_I l(Cz^0(t), u^0(t))\, dt \ge 0 \tag{35}$$

Define $y^\epsilon = (z(u^\epsilon) - z^0)/\epsilon$. Since $z(u^\epsilon)$ and z^0 are a mild solutions of the state equation (21) then one obtain

$$\begin{aligned} z(u^\epsilon)(t) &= \int_0^t E(t,s)F_1^\epsilon(s)z(u^\epsilon)(s)\, ds \\ &\quad + \int_0^t E(t,s)F_2^\epsilon(s)(u^\epsilon)(s)\, ds. \end{aligned}$$

and

$$\begin{aligned} z(u^0)(t) &= \int_0^t E(t,s)F_1^0(s)z^0(s)\, ds \\ &\quad + \int_0^t E(t,s)F_2^0(s)(u^0)(s)\, ds. \end{aligned}$$

Substructing $z(u^\epsilon)$ from $z^0$and dividing the difference by ϵ one can verify that y^ϵ satisfies the integral equation

$$\begin{aligned} y^\epsilon(t) &= \int_0^t E(t,s)F_1^\epsilon(s)y^\epsilon(s)\, ds \\ &\quad + \int_0^t E(t,s)F_2^\epsilon(s)(u(s) - u^0(s))\, ds. \end{aligned} \tag{36}$$

By virtue of assumption (**A2**) and once more applying dominated convergence theorem one can justify taking ϵ to zero in the above equation to obtain

$$\begin{aligned} y(t) &= \int_0^t E(t,s)F_1^0(s)y(s)\, ds \\ &\quad + \int_0^t E(t,s)F_2^0(s)(u(s) - u^0(s))\, ds \end{aligned} \tag{37}$$

Since a linear Volterra integral equation has a unique solution (see Theorem 2.4.3 of [10]) y is a mild solution of equation

$$\begin{cases} \frac{d}{dt}y(t) = (AB^{-1} + F_1^0)(t)y + F_2^0(t)(u(t) - u^0(t)) \\ (y)(0) = 0. \end{cases} \tag{38}$$

Note that y is the Gâteaux differential of z in the direction $u - u^0$. One can rewrite the inequality (35) as follows :

$$\int_I \Big(l(Cz(u^\epsilon), u^\epsilon(t)) - l(Cz^0(t), u^\epsilon(t) \Big) \, dt + \int_I \Big(l(Cz^0(t), u^\epsilon(t)) - l(Cz^0(t), u^0(t) \Big) \, dt \geq 0.$$

By use of the hypothesis **(d1)**, one obtains the following inequality

$$\int_I < C^* l_1^0, z >_{Y^*,Y} \, dt + \int_I < l_2^0, u - u^0 >_{U^*,U} \, dt \geq 0. \tag{39}$$

By virtue of **(d1)**, $C^* l_1^0 \in L_1(I, Y^*)$, and hence, by Lemma 4.3.2, the adjoint equation

$$\begin{cases} \frac{d}{dt}\psi(t) + (A(t)B^{-1})^*\psi + (F_1^0)^*(t)\psi = -C^* l_1^0 \\ \psi(T) = 0 \end{cases} \tag{40}$$

has a unique weak solution $\psi \in C(I, Y_w^*)$.

Since the solution $y(t)$ need not belong to $B(D(A))$,following the same technique as in [6], we use the Yosida approximation of the identity, $J_n(t) = nR(n, A(t)B^{-1})$ where $R(\lambda, A(t)B^{-1})$ is the resolvant of the operator AB^{-1} corresponding to $\lambda \in \rho(AB^{-1})$. It is well known (see [7]) that $J_n(t) \to J$ (Identity operator in Y) as $n \to \infty$ in the strong operator topology of $\mathcal{L}(Y)$ uniformly with respect to $t \in I$ and for any $z \in Y$, $J_n z \in B(D(A))$ for $n \in \rho(AB^{-1})$.

Now we regularize equation (38) as follows :

$$\begin{cases} \frac{d}{dt}(y_n) = A(t)B^{-1}y_n + J_n(t)F_1^0(t)y_n + J_n(t)F_2^0(t)(u - u^0) \\ y_n(0) = 0. \end{cases} \tag{41}$$

Equation (41) has a unique strong solution y_n with $y_n(t) \in B(D(A))$ for almost all $t \in I$ provided $n \in \rho(AB^{-1})$. Since a strong solution is obviously a mild solution, the y_n satisfies the following integral equation:

$$\begin{aligned} y_n(t) &= \int_0^t E(t,s) J_n F_1^0(s) y_n(s) \, ds \\ &\quad + \int_0^t E(t,s) J_n F_2^0(s)(u(s) - u^0(s)) \, ds. \end{aligned} \tag{42}$$

Using Gronwall inequality it is easy to verify that $y_n \rightarrow y$ in the usual topology of $C(I,Y)$. Hence it follows from equation (40) that

$$\begin{aligned}
\int_I \langle & C^* l_1^0, z\rangle_{Y^*,Y}\, dt \\
&= \lim_{n\to\infty} \int_I \langle C^* l_1^0, z_n\rangle_{Y^*,Y}\, dt \\
&= \lim_{n\to\infty} \int_I \langle -\frac{d}{dt}(\psi) - ((AB^{-1})^* + (F_1^0)^*)\psi, z_n\rangle_{Y^*,Y}\, dt \\
&= \lim_{n\to\infty} \int_I \langle \psi, \frac{d}{dt}(z_n) - AB^{-1}z_n - (J_n F_1^0)z_n - F_1^0 z_n + J_n F_1^0 z_n\rangle_{Y^*,Y}\, dt \\
&= \lim_{n\to\infty} \int_I \langle \psi, J_n F_2^0(u-u^0) - F_1^0 z_n + J_n F_1^0 z_n\rangle_{Y^*,Y}\, dt \\
&= \int_I \langle \psi, F_2^0(u-u^0)\rangle_{Y^*,Y}\, dt. \qquad (43)
\end{aligned}$$

Here we have used the strong convergence of J_n to J and uniform convergence of z_n to z and the following estimate.

$$\|J_n F_1^0 z_n - F_1^0 z_n\|_Y \leq \|J_n F_1^0(z_n - z)\|_Y + \|J_n(F_1^0 z) - F_1^0 z\|_Y + \|F_1^0(z - z_n)\|_Y.$$

Combining (39) and (43), we have

$$\int_I \langle (F_2^0)^*\psi + l_2^0, u - u^0\rangle_{U^*,U}\, dt \geq 0. \qquad (44)$$

This proves inequality **(3)** and completes the proof of Theorem 4.4.1.

Remark4.4.1 Note that the results of the Theorem 4.4.1 are in abstract form, but one can study the integration on Manifolds to derive necessary conditions of optimality in the explicit form.

For our specific practical system and in the derivation of the adjoint equation we have been faced with the study of the Laplace-Beltrami operator. In other words the study of the integration by parts of the Laplacian on the boundary of a given domain.

Case B : l merely continuous in control variable

In the above result we assumed that l is Fréchet differentiable in the control variable. In case $l(x, u)$ is merely continuous in u and Fréchet differentiable in $x = Cz$ and $\Lambda \subset U$ is a closed bounded convex set, we can prove Pontryagin type necessary conditions of optimality using the well-known Ekland variational principle. Define

$$\mathcal{M} \equiv \{u : I \to U, \textit{ strongly measurable}: \; u(t) \in \Lambda \textit{ a.e.}\}$$

with the topology induced by the metric

$$\rho(u, v) \equiv \lambda\{t \in I : \; u(t) \neq v(t)\},$$

where λ denotes the Lebesgue measure. Since Λ is a closed subset of a Banach space, the set $\mathcal{M}$, with the metric ρ as defined above, is a complete metric space.

We need the continuous dependence of solutions on control.

Lemma 4.4.1. *Suppose the assumptions* (**A1**) *and* (**A2**) *hold and* $\mathcal{U}_{ad} = \mathcal{M}$. *Then for the semilinear system* (22) *the mapping*

$$u \longrightarrow z(u)$$

is continuous from $\mathcal{M}$ *to* $C(I, Y)$ *in the respective metric topologies and further there exists a constant* β *such that*

$$\|z(u) - z(v)\|_{C(I,Y)} \leq \beta\rho(u, v)$$

for all $u, v \in \mathcal{M}$.

Proof: Let $z(u)$ and $z(v)$ denote the solutions corresponding to u and v respectively. Let $\sigma \equiv \{t \in I : u(t) \neq v(t)\}$. We have

$$\begin{aligned} z(t, u) - z(t, v) &= \int_0^t E(t, s)[f(z(s, u), u(s)) - f(z(s, v), v(s))]\, ds \\ &= \int_0^t E(t, s) \int_0^1 f_1(z(s, v) + \theta(z(s, u) - z(s, v)), u(s)) d\theta(z(t, u) - z(t, v))\, ds \\ &+ \int_0^t E(t, s)[f(z(s, v), u(s)) - f(z(s, v), v(s))]\, ds. \end{aligned}$$

It follows from our assumptions that there exist constants a, b, such that

$$\|z(t,u)-z(t,v)\|_Y \leq a\int_0^t \|z(s,u)-z(s,v)\|_Y\,ds + b\rho(u,v).$$

Thus the assertion follows from Gronwall inequality.

Theorem 4.4.2. *Suppose the assumptions of the Lemma 4.3.1 hold and further $u \to l(Cz,u)$ is merely continuous and $z \to l(Cz,u)$ is continuously Fréchet differentiable with Fréchet derivative denoted by l_1. Further $C^*l_1 \in L^1(I;Y^*)$. Then the optimality conditions* **(1)**,**(2)**,**(3)** *of Theorem 4.4.1 hold and* **(3)** *is replaced by* **(3)′***:*

$$\begin{aligned}(3)': \; l(Cz^0(t),u^0(t)) \;&+\; <\psi(t), f(z^0(t),u^0(t))>_{Y^*,Y}\\ &\leq\; l(Cz^0(t),v) + <\psi(t), f(z^0(t),v)>_{Y^*,Y}\end{aligned}$$

for all $v\in\Lambda$.

Proof: Since u^0 is optimal, again by the inequality (34), we have

$$\int_I l(Cz(t,u),u)\,dt - \int_I l(Cz^0,u^0)\,dt \geq 0 \;\;\forall\;\; u\in\mathcal{M}.$$

For any measurable set $\sigma\in I$ and $v\in\Lambda$, define

$$u^\sigma(t) = \begin{cases} u^0(t) & t\in I\backslash\sigma \\ v(t) & t\in\sigma. \end{cases}$$

Let z^σ be the solution of the system (22) corresponding to u^σ. Then

$$\begin{aligned}\int_I l(Cz^\sigma,u^\sigma)\,dt \;&-\; \int_I l(Cz^0,u^0)\,dt = \int_\sigma l(Cz^\sigma,v)\,dt - \int_\sigma l(Cz^0,u^0)\,dt\\ &+\; \int_{I\backslash\sigma}[l(Cz^\sigma,u^0)-l(Cz^0,u^0)]\,dt \geq 0. \qquad (45)\end{aligned}$$

By virtue of Fréchet differentiability of l, we have

$$\begin{aligned}\int_{I\backslash\sigma}[l(Cz^\sigma,u^0) \;&-\; l(Cz^0,u^0)]\,dt = \int_{I\backslash\sigma}\langle C^*l_1(Cz^0,u^0), z^\sigma - z^0\rangle_{Y^*,Y}\,dt + o(\lambda(\sigma))\\ &=\; \int_I\langle C^*l_1(Cz^0,u^0), z^\sigma - z^0\rangle_{Y^*,Y}\,dt + o(\lambda(\sigma)), \qquad (46)\end{aligned}$$

where $o(\cdot)$ stands for small order of approximation.

Hence expression (45) reduces to

$$\int_\sigma l(Cz^0, u^0)\,dt \leq \int_\sigma l(Cz^\sigma, v) + \int_I \langle C^* l_1(Cz^0, u^0), z^\sigma - z^0\rangle_{Y^*,Y}\,dt + o(\lambda(\sigma)). \quad (47)$$

Using the adjoint equation (2) of Theorem 4.4.1 and following similar arguments as in that Theorem, one can verify that

$$\begin{aligned}
&\int_I \langle C^* l_1(Cz^0, u^0), z^\sigma - z^0\rangle_{Y^*,Y}\,dt \\
&\quad = \int_I \langle \psi, f(z^\sigma, u^\sigma) - f(z^0, u^0) - F_1^0(t)(z^\sigma - z^0)\rangle_{Y^*,Y}\,dt \\
&\quad = \int_I \langle \psi, f(z^0, u^\sigma) - f(z^0, u^0)\rangle_{Y^*,Y}\,dt + o(\lambda(\sigma)) \\
&\quad = \int_\sigma \langle \psi, f(z^0, v)\rangle_{Y^*,Y}\,dt - \int_\sigma \langle \psi, f(z^0, u^0)\rangle_{Y^*,Y}\,dt + o(\lambda(\sigma)).
\end{aligned}$$

Thus the expression (42) reduces to

$$\begin{aligned}
&\int_\sigma l(Cz^0, u^0)\,dt + \int_\sigma \langle \psi, f(z^0, u^0)\rangle_{Y^*,Y}\,dt \\
&\quad \leq \int_\sigma l(Cz^\sigma, v)\,dt + \int_\sigma \langle \psi, f(z^0, v)\rangle_{Y^*,Y}\,dt + o(\lambda(\sigma)).
\end{aligned} \quad (48)$$

Let t be any Lebesgue density point of u^0 and σ any measurable set containing t shrinking to the one point set $\{t\}$ as $\lambda(\sigma) \to 0$. Dividing (48) by $\lambda(\sigma)$ and letting it converges to zero, we obtain the inequality (3)$'$. This completes the proof.

4.5 An example

In this section, we work out in details an example (Heat transfer) of boundary control problem illustrating the applicability of our results.

Example 4.5.1 (Heat transfer)

In this example we consider the heat transfer problem as stated in the Motivation section. In order to formulate and treat problem (19) as two-space evolution equation, we introduce the following notations:

$H^m(\Omega)$ denotes the standard Sobolev space and $\gamma_i\phi \equiv \phi\,|_{\Gamma_i}$ $(i = 1, 2)$ denotes the trace operator.

Let L, M denote the formal differential operators

$$\begin{aligned} L\phi &= \operatorname{div}(k\nabla\phi) + v\cdot\nabla\phi \\ M\phi &= \Delta_{\Gamma_1}\phi - \beta D_\nu \phi. \end{aligned}$$

We take $X \equiv L^2(\Omega)$ and $Y \equiv L^2(\Omega) \times L^2(\Gamma_1)$, with the norm topology on Y given by $\|y\|_Y = \left(\|y^1\|^2_{L^2(\Omega)} + \|y^2\|^2_{L^2(\Gamma_1)}\right)^{1/2}$, for $y = \{y^1, y^2\} \in Y$.

The operators A and B are defined as follows:

$$\begin{aligned} D(A) &\equiv \{\phi \in H^2(\Omega) : L\phi \in L^2(\Omega), M\phi \in L^2(\Gamma_1) \text{ and } \gamma_2\phi = h\} \\ A\phi &\equiv \{L\phi, M\phi\}, \phi \in D(A) \end{aligned}$$

and

$$D(B) \equiv \{\phi \in H^2(\Omega) : \gamma_2\phi = h\}$$
$$B\phi \equiv (\phi, \gamma_1\phi), \phi \in D(B).$$

The range of the operator B is given by

$$R(B) \equiv \{(\psi_1, \psi_2) \in Y : \gamma_1\psi_1 = \psi_2 a.e.\}.$$

Defining $x(t) \equiv T(t,\cdot)$ the control $u \in L^2(I; L^2(\Gamma_1))$, $f(y, u(t)) \equiv \{0, \alpha(u - y^2)\}$ and $z_0 \equiv \{T_0, T_1\}$,the heat transfer equation (19) can be written as an abstract B-evolution in two Banach spaces $\{X, Y\}$ as follows :

$$(Q_f) \begin{cases} \frac{d}{dt}(Bx)(t) = A(t)x(t) + f(Bx(t), u(t)), \\ Bx|_{t=0} = z_0. \end{cases}$$

The integrand $l(v, w) \equiv \int_\Omega |v(\xi)|^2 d\xi + \int_{\Gamma_1} |w(\eta)|^2 d\eta$ maps $L^2(\Omega) \times L^2(\Gamma_1)$ to R, for $v \in L^2(\Omega)$, $w \in L^2(\Gamma_1)$ and the cost functional is given by

$$J(u) \equiv \int_I (l(x(t), u(t))\, dt.$$

Following standard procedure as in [9] one can verify that the operators A and B satisfy the following properties:

(i) $R(B)$. range of B, is dense in Y

(ii) A is closed

(iii) B is injective and has a bounded inverse on $R(B) \subset Y$.

we shall show part (ii) only. In fact we show that the graph of A defined by

$$\{(\phi, \psi) : \ \phi \in L^2(\Omega) \ and \ \psi = A\phi \in L^2(\Omega) \times L^2(\Gamma_1)\}$$

is closed. In fact, let $\phi_n \in D(A)$ and suppose that

$$\phi_n \ \longrightarrow \ \phi \ in \ L^2(\Omega) \tag{49}$$

and

$$A\phi_n \ \longrightarrow \ \psi \ in \ L^2(\Omega) \times L^2(\Gamma_1). \tag{50}$$

We want to show that $\phi \in D(A)$ and $A\phi = \psi$.

Since $\phi_n \in D(A)$ then $\{\phi_n\} \subset H^2(\Omega)$ and $\phi_n|_{\Gamma_0} = h$. Also, it follows from (50), that

$$L\phi_n \longrightarrow \phi_1 \ in \ L^2(\Omega) \tag{51}$$

and

$$M\phi_n \longrightarrow \psi_2 \ in \ L^2(\Gamma_1). \tag{52}$$

Thus $\{L\phi_n\}$ and $\{M\phi_n\}$ are Cauchy sequences in $L^2(\Gamma_1)$ respectively. Also, it follows from (49) that $\{\phi_n\}$ is a Cauchy sequence in $H^2(\Omega)$ (see [11]. Theorem 18.1. p. 68) and hence $\lim_{n\to\infty} \phi_n = \phi \in H^2(\Omega)$. Since $L\phi_n \longrightarrow \phi_1$ in $L^2(\Omega)$ and $C_0^\infty(\Omega)$ is dense in $L^2(\Omega)$, then

$$\{L\phi_n, \eta\} \longrightarrow \{\psi_1, \eta\} \ for \ all \ \eta \in C_0^\infty(\Omega). \tag{53}$$

Finally

$$\{\psi_1, \eta\} = \lim_n \{L\phi_n, \eta\} = \{\lim_n L\phi_n, \eta\} = \{L\phi, \eta\} \ for \ all \ \eta \in C_0^\infty(\Omega). \tag{54}$$

Thus $\psi_1 = L\phi$. Similarly for M, one can show that $\psi_2 = M\phi$ and hence $A\phi \equiv \{L\phi, M\phi\}$. This Completes the proof of part (ii).

Since the operator A is closed and B^{-1} is bounded, then $AB^{-1} : B(D(A)) \equiv R(B) \subset Y \mapsto Y$ is a closed densely defined linear operator.
For λ on the positive real axis and $\psi \in R(B)$ we have

$$\begin{aligned}\langle(\lambda B - A)B^{-1}\psi, \psi\rangle_Y &= \lambda\|\psi\|^2 - \langle AB^{-1}\psi, \psi\rangle_Y \\ &= \lambda\|\psi\|^2 - \langle A\phi, \psi\rangle_Y, \;\; \phi \in D(B) \end{aligned} \tag{55}$$

where ϕ is given by $\psi = B\phi$.

From the expression of the operator A as defined previously we obtain

$$\begin{aligned}\langle A\phi, \psi\rangle_Y &\equiv \langle A\phi, B\phi\rangle_Y = \langle\{L\phi, M\phi\}, \{\phi, \gamma_1\phi\}\rangle_Y \\ &= \langle L\phi, \phi\rangle_{L^2(\Omega)} + \langle M\phi, \phi\rangle_{L^2(\Gamma_1)}\end{aligned}$$

Substituting the expression of the operators L and M in the above equality and integrating by part we obtain

$$\langle A\phi, \psi\rangle_Y = -K\| \nabla \phi\|^2_{L^2(\Omega)} + \langle v \nabla \phi, \phi\rangle_{L^2(\Omega)} + \langle \triangle_L\phi, \phi\rangle_{L^2(\Gamma_1)} \tag{56}$$

$$\leq -K\| \nabla \phi\|^2_{L^2(\Omega)} + \|v\|_{L^\infty(\Omega)}\| \nabla \phi\|\|\phi\|_{L^2(\Omega)} + \langle \triangle_L\phi, \phi\rangle_{L^2(\Gamma_1)}. \tag{57}$$

From the last inequality use the Cauchy inequality

$$ab \leq \epsilon a^2 + (1/4\epsilon)b^2$$

and choose $\epsilon = K/2\|v\|_{L^\infty}$ to easily verify that

$$\langle A\phi, \psi\rangle_Y \leq -K\| \nabla \phi\|^2_{L^2(\Omega)} + (\|v\|^2_{L^\infty}/2K)\|\phi\|^2_{L^2(\Omega)}. \tag{58}$$

Finally, using this and that $\|\psi\|^2_Y = \|B\phi\|^2_Y \geq \|\phi\|^2_{L^2(\Omega)}$, one obtains from equation (55) the following inequality

$$\langle(\lambda B - A)B^{-1}\psi, \psi\rangle_Y \geq (\lambda - (\|v\|^2_{L^\infty}/2K))\|\psi\|^2_Y. \tag{59}$$

Hence for λ real ($\Re\lambda > \omega = \|v\|^2_{L^\infty}/2K$) we have

$$\|(\lambda B - A)B^{-1}\psi\|_Y \geq (\lambda - \omega)\|\psi\|^2_Y, \; for\ each\ \psi \in R(B) \tag{60}$$

and

$$\|(\lambda B - A)B^{-1}\psi\|_Y \geq |\lambda|\|\psi\|_Y, \ for\ \lambda \in C,\ \Re(\lambda) > 0. \tag{61}$$

Note that the quantity $\omega(> 0)$ is dependent on the L^∞ bound of v and the material constant K.

For λ complex and $\psi \in R(B)$, we next consider the following expression

$$\mathcal{Q} \equiv \langle(\lambda B - A)B^{-1}\psi, \psi\rangle_Y = \lambda\|\psi\|_Y^2 - \langle AB^{-1}\psi, \psi\rangle_Y. \tag{62}$$

Since $\psi = B\phi$, $\phi \in D(B)$, we have

$$\mathcal{K} = -\langle AB^{-1}\psi, \psi\rangle_Y = \langle A\phi, B\phi\rangle_Y = K\| \nabla \phi\|^2_{L^2(\Omega)} - \langle \triangle_L \phi, \phi\rangle_{L^2(\Gamma_1)} \geq 0. \tag{63}$$

Hence $\mathcal{Q} = \lambda\|\psi\|_Y^2 + \mathcal{K}$, with $\mathcal{K} \geq 0$. This gives rise to

$$\begin{aligned} |\mathcal{Q}|^2 &= |\lambda|^2\|\psi\|_Y^4 + \mathcal{K}^2 + 2\mathcal{K}\Re(\lambda)\|\psi\|_Y^2 \\ &= |\lambda|^2\|\psi\|_Y^4 + \mathcal{K}^2 + 2\mathcal{K}|\lambda|\cos\theta\|\psi\|_Y^2, \end{aligned} \tag{64}$$

where $\theta = \arg\lambda$. From the relation $2ab \leq a^2 + b^2$ we obtain the estimate

$$|\mathcal{Q}|^2 \geq |\lambda|^2\|\psi\|_Y^4 \sin^2\theta. \tag{65}$$

Using the following estimate :

$$|\langle(\lambda B - A)B^{-1}\psi, \psi\rangle_Y| \leq \|(\lambda B - A)B^{-1}\psi\|_Y\|\psi\|_Y,$$

it is clear that $\mathcal{Q}$ satisfy the following inequality

$$|\mathcal{Q}| \leq \|(\lambda B - A)B^{-1}\psi\|_Y\|\psi\|_Y.$$

Using this fact and the estimate (65) it follows that that there exists a constant $M \geq 1$, and $0 < \delta < (\pi/2)$ such that for λ in the sectoriel domain

$$\Sigma_{\omega,\delta} = \{\lambda \in C, \Re(\lambda) > \omega, -(\pi/2 + \delta) < \arg\lambda < (\pi/2 + \delta)\}$$

the operator $(\lambda B - A)B^{-1}$ has a bounded inverse satisfying

$$\|B(\lambda B - A)^{-1}\|_{\mathcal{L}(Y)} \leq M/(\Re\lambda - \omega), \Re(\lambda) > \omega.$$

Hence AB^{-1} generates an holomorphic semigroup (see [33], [1] Theorem 3.2.7, p. 82]). Thus by Lemma 3.3.4, the pair $\{A, B\}$ is the generating pair of a holomorphic B-evolution of type L.

For the set of controls we define

$$\Lambda \equiv \{\omega \in U \equiv L^2(\Gamma_1) : \omega = T_r|_{\Gamma_1} and \quad 0 \leq \omega(\xi) \leq \gamma_d\}$$

and for admissible controls take $\mathcal{U}_{ad} \equiv \{w \in L^\infty(I, U) : w(t) \in \Lambda \ a.e.\}$
Clearly Λ is a closed bounded convex subset of U. We note that f as defined above is locally Lipschitz, satisfies the growth conditions and maps $Y \times U$ to Y. One can also verify the the integrand l satisfy the hypothesis of Theorem 4.4.1. Thus all the assumptions of the main theorem are satisfied. and hence the necessary conditions of optimality holds.

4.6 Computational Algorithm

Based on the necessary conditions of optimality given by Theorem 4.4.1 (**case A**), we can compute the optimal solution of our original problem:

(Q) Find $u^0 \in \mathcal{U}_{ad}$ such that $J(x^0, u^0) \leq J(x, u)$ for all $u \in \mathcal{U}_{ad}$.
subject to the following system

$$(P_f)\begin{cases} \frac{d}{dt}Bx(t) = A(t)x(t) + f(Bx(t), u(t)), & t \in I \\ (Bx)(0) = z_0 \end{cases}$$

where

$$J(x, u) \equiv \int_I l(x, u)\, dt.$$

Here $x = Cz$ denotes the generalized solution of the system (P_f) and z the solution of the system (21) corresponding to the control $u \in \mathcal{U}_{ad}$.

The computation of the optimal solution of the problem (**Q**) can be done by constructing an algorithm for computing the optimal solution of the problem (**P**) (see section 4.4). For this purpose, we require the duality maps.

The map $\nu: \ U^* \mapsto U$ denotes the duality map. That is, for $\xi \in U^*$

$$\nu_1(\xi) \equiv \{\eta \in U : \ \langle \xi, \eta \rangle_{U^*,U} = \|\xi\|^2_{U^*} = \|\eta\|^2_U\}.$$

For Fréchet differentiable l, we can define $D_u J(\cdot) \equiv l_2^0 + F_2^0 \psi$. The inequality (**3**) of Theorem 4.4.1 is then equivalent to the following inequality

$$\langle D_u J, u - u^0 \rangle = \int_I \langle l_2^0 + F_2^0 \psi, u - u^0 \rangle_{U^*,U} \, dt \geq 0$$

for all $u \in \mathcal{U}_{ad}$.

Now the algorithm may be stated as follows:

Algorithm :

Step 1. Suppose that the n-th stage, the control is given by $u^n \in U$.

Step 2. Use u^n to determine $\{z^n, \psi^n\}$ where z^n is the solution of equation (21) corresponding to u^n and ψ^n is the solution of the adjoint equation (**2**) of Theorem 4.4.1.

Step 3. Compute $D_u J(u^n)$

Step 4. Define

$$u^{n+1} \equiv u^n - \epsilon \tilde{u}, \quad \epsilon > 0,$$

choosing ϵ sufficiently small so that

$$\begin{aligned} J(u^{n+1}, z^{n+1}) &\equiv J(u^n, z^n) - \epsilon \langle D_u J(u^n, z^n), \tilde{u}^n \rangle + 0(\epsilon) \\ &= J(u^n, z^n) - \epsilon \|D_u J(u^n, z^n)\|^2_{U^*} + 0(\epsilon) \leq J(u^n, z^n). \end{aligned}$$

Step 5. Solve the state equation corresponding to u^n and compute $J(u^{n+1}, z^{n+1})$ using the following expression:

$$J(u^{n+1}, z^{n+1}) = \int_I l(Cz^{n+1}(t), u^{n+1}(t))\, dt.$$

If $|J(u^{n+1}, z^{n+1}) - J(u^n, z^n)| < \delta$ for some preassigned small positive number δ. stop. otherwise go back to step 2 with new control u^{n+1}.

Chapter 5

Linear Quadratic Regulator Problem

5.1 Introduction

We consider the linear, quadratic regulator problem for a class of infinite dimensional systems involving dynamic boundary conditions. Two results are presented one of which contains control cost and the other does not. The first one uses the standard coercivity condition and second uses controllability. The optimal control law is given by the solution of appropriate operator Riccati differential equations.

Many physical systems, with dynamic boundary conditions, have applications in multi-phase problems in physics and engineering [6], [8], [12], [14]. These include heat transfer, Navier-Stokes equations and structural vibration.

The abstract mathematical model for these systems can be written in two Banach spaces as follows:

$$(P_c)\begin{cases} (d/dt)Bx(t) = A(t)x(t) + Du(t), & t \in I \equiv [0,T] \\ \lim_{t\to 0^+}(Bx)(t) = z_0 \end{cases}$$

where A and B are linear unbounded operators between two Banach spaces, D is a bounded linear operator between two Banach spaces and u is a suitable function representing the control actions.

In addition to covering all classical boundary and distributed control problems, the system (P_c) includes a new set of such problems in which the boundary conditions are determined also by an evolution equation.

In this chapter we use the results obtained in chapter 4 to derive the necessary condition of optimality for a quadratic cost functional subject to the system (P_c). We show that the associated Riccati equation possess a unique solution in the weak sense.

Control theory for classical cases $(X = Y, B = I)$ and their abstract versions have been studied extensively in the literature (see [3], [14] and the references therein).

5.2 Preparatory Results

From the previous results (see chapter 4) we know that in the case of holomorphic B-evolution of type L the operators A and B are not necessarily closed but the pair $\{A(t), B\}$ is closed when considered as a map from $D(A) \cap D(B) \subset X$ to $Y \times Y$. Then it follows from a result due to Sauer that the Cauchy problem

$$\begin{cases} (d/dt)Bx = A(t)x, t > s, \\ \lim_{t \to s^+}(Bx(t)) = \xi \end{cases} \tag{66}$$

has a unique solution given by

$$x(t) = V(t, s)\xi, t \geq s. \tag{67}$$

In other words the Cauchy problem (66) admits an evolution operator $V(t, s), 0 \leq s \leq t < \infty$.

Further, in case of B-evolution of type L, it follows from [15, Theorem 5.1, p. 296] that $A(t)B^{-1}$ with domain $B(D(A)) \equiv \mathcal{D} \subset Y$ is the infinitesimal generator of a transition operator $E(t, s) \equiv BV(t, s), t > s \geq 0$ satisfying the properties **(P1)**$-$**(P4)** as stated in chapter 3.

Thus the problem reduces to the classical case and the system (P_c) can be written as

$$\begin{cases} \dot{z} = A(t)B^{-1}z + Du(t) \\ z(0) = z_0. \end{cases} \tag{68}$$

The solution of equation (68) is given by

$$z(t) = E(t,0)z_0 + \int_0^t E(t,s)Du(s)\,ds, 0 \leq t < \infty.$$

Let $C \in \mathcal{L}(Y,X)$ denote the continuous extension of the operator B^{-1} from its range $R(B)$ to Y. For details see ([15], Theorem 2.3, p. 290). Using this operator one can define $x(t) \equiv Cz(t)$, $t > 0$, as a generalized solution of the original problem. Clearly this is a true solution of the original problem (P_c) if $R(B)$ is invariant with respect to the family $E(t,s)$, $0 \leq s < t < \infty$, and $D \in \mathcal{L}(U, R(B))$.

Our objective is to find optimal linear control law (optimal regulator) that minimizes the quadratic cost functional given by:

$$J(u) = (1/2)\int_0^T \{(Q(t)Cz(t), Cz(t))_X + (R(t)u,u)_U\}\,dt. \tag{69}$$

To solve this problem we make use of the necessary conditions of optimality given in Chapter 4.

In the following lemma we present a result on perturbation theory for evolution operators.

Lemma 5.2.1. *Suppose the assumption* (**A**) *holds and let* $K(t)$, $t \geq 0$ *be a family of strongly measurable bounded linear operators from* Y *to* Y. *Then the perturbed system*

$$\begin{cases} (d/dt)z = A(t)B^{-1}z + K(t)z, \ t \geq s \\ z(s) = \xi \end{cases} \tag{70}$$

has a unique mild solution $z(\cdot) \in C((s,T],Y)$ *for every* $s < T$.

Proof. Write the equation (70) as a linear Volterra integral equation

$$z(t) = E(t,s)\xi + \int_s^t E(t,\tau)K(\tau)z(\tau)d\tau.$$

Since E and K are bounded strongly measurable operator valued functions, it is easy to show by classical argument that this equation has a unique solution $z(\cdot) \in C((s,T],Y)$ for every $T < \infty$. Thus there exists an evolution operator $\tilde{E}(t,s)$, $0 \leq s < t \leq T < \infty$ so that $z(t) = \tilde{E}(t,s)\xi$.

5.3 Optimal Regulator Rroblem

For the necessary conditions of optimality, we need the following assumptions on the given data of the cost functional as defined by the expression (69).
From here on X, Y and U are Hilbert spaces.
(A1): The map $t \to Q(t)$ is continuous with values from the class a positive self-adjoint operators in X.

(A2): The operator $R \in \mathcal{L}(I, \mathcal{L}(U))$ and for each t, $R(t)$ is a positive sel-adjoint operator and satisfies the following inequality:

$$< R(t)v, v >\geq \mu\|v\|_U^2 \quad a.e. \ for \ some \ \mu > 0.$$

Theorem 5.3.1. *Suppose the assumptions* (**A**), (**A1**) *and* (**A2**) *hold. Then the necessary conditions of optimality are given by:*

(i) : $(d/dt)z = A(t)B^{-1}z + Du(t),\ z_0 = z(0).$

(ii) : $(d/dt)p(t) + (A(t)B^{-1})^*p(t) = -C^*Q(t)Cz,\ a.e.\ p(T) = 0$

(iii) : $u(t) = -R^{-1}(t)D^*p(t), t \in I$

where z and p are the weak solutions of the above problems.

Proof. Suppose there exists an optimal control u such that $J(u) \leq J(v)$, $\forall v \in \mathcal{U}_{ad} \equiv L^2(I, U)$. Then it follows from Theorem 4.4.1 that u is characterized by:

$$J'(u, u - v) = \tag{71}$$
$$\int_0^T (D^*(t)\{\int_t^T E^*(\theta, t)C^*Q(t)Cz(\theta)d\theta\} + R(t)u(t), u(t) - v(t))_U\, dt \geq 0$$

$\forall v \in \mathcal{U}_{ad}$,

where J' denotes the Gâteaux derivative of J at u in the direction $u - v$. Since there is no constraint on the control, we have

$$D^*(t)\{\int_t^T E^*(\theta, t)C^*Q(t)Cz(\theta)d\theta\} + R(t)u(t) = 0. \tag{72}$$

Define

$$p(t) \equiv P(t)\eta \equiv \int_t^T E^*(s,t)C^*Q(s)Cz(s,t,\eta)\,ds,\ 0 < t \leq s < T \tag{73}$$

where $z(s,t,\eta) = \tilde{E}(s,t)\eta, s \geq t,$ is the solution trajectory of the system (68) starting at time t from an arbitrary position $\eta \in \mathcal{D}$. Clearly this defines a family of bounded linear operators $P(t), t \geq 0,$ in Y. Then, for any $\xi \in \mathcal{D}$,

$$(p(t),\xi)_Y = \int_t^T (C^*Q(s)Cz(s,t,\eta), E(s,t)\xi)_Y\,ds$$

is well defined and furthermore one can verify that the function $t \mapsto (p(t),\xi)_Y$, is absolutely continuous. Thus we have

$$(d/dt)(p(t),\xi)_Y =$$
$$-(C^*Q(t)Cz(t,\eta),\xi)_Y - \int_t^T (C^*Q(s)Cz(s,\eta), E(s,t)A(t)B^{-1}\xi)_Y\,ds.$$

Consequently, for all $\xi \in \mathcal{D}$,

$$(d/dt)(p(t),\xi)_Y + (p(t), A(t)B^{-1}\xi)_Y = -(C^*Q(t)Cz(t,\eta),\xi)_Y \tag{74}$$

for almost all $t \in I$ and $p(T) = 0$. Thus, $p(t)$ as defined by equation (73) is the weak solution of the problem (ii). It follows from the expression (72) that the optimal control has the form

$$u(t) = -R^{-1}(t)D^*p(t). \tag{75}$$

For the linear quadratic cost problem the necessary condition is also a sufficient condition and hence u as given above is optimal. This completes the proof of the theorem.

As $s \to t$ we have $\lim_{s\to t} z(s,t,\eta) = \eta$. Since η is arbitrary we can choose $\eta = z(t)$ a point on the solution trajectory of the system equation (i). Substituting $p(t) = P(t)z(t)$ in equation (75), the state equation (i) can be written in the feedback form as follows:

$$(d/dt)z = \{A(t)B^{-1} - DR^{-1}(t)D^*P(t)\}z,\ \ z(0) = z_0. \tag{76}$$

Since the operators D, R^{-1} and P are bounded, $K(t) \equiv DR^{-1}(t)D^*P(t)$ is also bounded, and hence by virtue of Lemma 5.2.1, $\mathcal{A}(t) \equiv A(t)B^{-1} - DR^{-1}(t)D^*P(t)$ is the generator of a transition operator $\tilde{E}$

In the following Theorem we derive the Riccati equation in the weak form.

Theorem 5.3.2. *The operator P given by equation (73) satisfies the following differential equation*

$$\begin{cases} (d/dt)P(t) + P(t)A(t)B^{-1} + (A(t)B^{-1})^*P(t) = \\ \quad P(t)DR^{-1}(t)D^*P(t) - C^*Q(t)C, \\ P(T) = 0, \end{cases} \tag{77}$$

in the weak sense, that is, for all $\xi, \eta \in \mathcal{D}$

$$\begin{cases} (d/dt)(P(t)\eta, \xi)_Y + (P(t)A(t)B^{-1}\eta, \xi)_Y + (P(t)\eta, A(t)B^{-1}\xi)_Y = \\ \quad (P(t)DR^{-1}(t)D^*P(t)\eta, \xi)_Y - (C^*Q(t)C\eta, \xi)_Y, \\ (P(T)\eta, \xi)_Y = 0 \ \forall \eta, \xi \in \mathcal{D}. \end{cases} \tag{78}$$

Proof. For any $t \in I$ and $s \geq t$, let $z(s, t, \eta) \equiv \tilde{E}(s, t)\eta$ denote the solution of equation (76) starting from state $\eta \in \mathcal{D}$ at time $t \in I$. Scalar multiplying equation (73) by $\xi \in \mathcal{D}$, and differentiating this with respect to t while using the properties of the evolution operators E and $\tilde{E}$, we obtain equation (78) which is the weak form of equation (77).

5.4 Optimal Regulator Without Control Cost

In this section we consider the terminal control problem. The problem is to find a feedback control law u that minimizes the functional

$$J(u) = 1/2(MCz(T), Cz(T)) \tag{79}$$

over the Hilbert space $L^2(I, U)$, subject to the condition that z satisfies the system equation (P_c) as defined in the section 5.1. We call this problem **(P2)**. Due to absence of the cost of control, this is a nonstandard problem, in the sense that $J(u)$ may not be coercive and, hence, without control constraints, the problem may have no solution. In the case of finite dimensional space, the problem (**P2**) was studied by Ahmed and Mouadeb [2]. The authors derived an optimal control law determined by the solution of a parameterized family of matrix Riccati differential equations. Here we extend this result to infinite dimensional problem.

Before we can present the necessary conditions of optimality, it is essential to prove the existence of an optimal control. In the following result we present sufficient conditions for the existence.

Lemma 5.4.1. *Suppose the basic assumption* (**A**) *holds and the pair* $\{A(t)B^{-1}, D\}$ *is strongly controllable and* M *is a strictly positive self adjoint operator in Hilbert space* X. *Then there exists an optimal control* $u \in L^2(I, U)$ *for the problem* (**P2**).

Proof. Corresponding to the initial state $z_0 \in Y$ and control $u \in L^2(I, U)$, the state z at time T is given by

$$z(T) \;=\; E(T,0)z_0 + \int_0^T E(T,s)Du(s)\,ds.$$

Substituting this expression in the terminal cost given by equation (79), one can easily verify that

$$J(u) \;=\; \mathcal{C} + \tilde{\ell}(u) + (Nu, u) \tag{80}$$

where

$$\mathcal{C} \;=\; 1/2(E^*(T,0)C^*MCE(T,0)z_0, z_0) \tag{81}$$

$$\tilde{\ell}(u) \;\equiv\; \int_0^T (D^*E^*(T,\theta)C^*MCE(T,0)z_0, u(\theta))_U d\theta, \tag{82}$$

$$(Nu, u) \;\equiv\; (1/2)\int_{I\times I} (K(s,\theta)u(\theta), u(s))_U d\theta\, ds = \|Lu\|_X^2, \tag{83}$$

where the kernel K is given by

$$K(s,\theta) \;=\; D^*E^*(T,s)C^*MCE(T,\theta)D, \tag{84}$$

and the operator L is given by

$$Lu = \int_I \sqrt{M}E(T,\theta)Du(\theta)d\theta. \quad (85)$$

Note that K is a symmetric kernel and L is a bounded linear operator from $L^2(I,U)$ to X. It follows from the strong controllability of the pair $\{AB^{-1}, D\}$ and strict positivity of the operator M that there exists a number $\delta > 0$ so that

$$(Nu, u)_{L^2(I,U)} \geq \delta\|u\|^2_{L^2(I,U)}.$$

Therefore, the functional $J(u)$ given by equation (80) is coercive, that is,

$$\lim_{\|u\|\to\infty}\left(\frac{J(u)}{\|u\|}\right) = \infty. \quad (86)$$

Now let $\{u_n\}$ be a minimizing sequence for the functional J, that is

$$\lim_n J(u_n) = \inf_{u\in L^2} J(u) = m. \quad (87)$$

Since $J(0) = C < \infty$ and $J(u) \geq 0$ for all u, it follows that $0 \leq m < \infty$. Since m is finite and J is coersive, the sequence $\{u_n\}$ is bounded sequence from $L^2(I,U)$. A bounded subset of $L^2(I,U)$ is relatively weakly compact, and therefore the sequence $\{u_n\}$ has a subsequence denoted again by $\{u_n\}$ that converges weakly to an element $u_0 \in L^2(I,U)$. Then it follows from the weak lower semicontinuity of the quadratic functional J that

$$J(u_0) \leq \underline{\lim}_n J(u_n) = \lim_n J(u_n) = m. \quad (88)$$

Since $u_0 \in L^2$, $J(u_0) \geq m$ also. Combining these, we have $J(u_0) = m$, proving that u_0 is optimal. This proves the existence of an optimal control $u^0 \in L^2(I,U)$, thereby completing the proof.

The optimal regulator for the problem (**P2**) is given by the following theorem.

Theorem 5.4.2. *Consider the problem* (**P2**) *and suppose the assumptions of Lemma 5.4.1 hold. Then the optimal linear regulator has the form*

$$u(z) \equiv -\alpha D^* P(t)z, \ \alpha > 0 \quad (89)$$

where P satisfies the modified operator Riccati differential equation.

$$\begin{cases} (d/dt)(P(t)\eta,\xi)_Y + (P(t)\eta, A(t)B^{-1}\xi)_Y + (A(t)B^{-1}\eta, P(t)\xi)_Y \\ \qquad\qquad = \alpha(P(t)DD^*P(t)\eta,\xi)_Y, t \in I \\ (P(T)\eta,\xi) = (C^*MC\eta,\xi), \forall\ \xi,\eta \in \mathcal{D}. \end{cases} \tag{90}$$

Proof. The proof follows from similar arguments as in the finite dimensional case [2].

Remark 5.4.1. Since the range of B is dense in Y, and restriction of C to $R(B)$ coincides with B^{-1}, the differential Riccati equation (90) reduces to:

$$\begin{cases} (d/dt)(P(t)\eta,\xi)_Y + (P(t)\eta, A(t)B^{-1}\xi)_Y + (A(t)B^{-1}\eta, P(t)\xi)_Y \\ \qquad\qquad = \alpha(P(t)DD^*P(t)\eta,\xi)_Y,\ t \in I, \\ (P(T)\eta,\xi)\ = ((B^{-1})^*MB^{-1}\eta,\xi), \forall\ \xi,\eta \in R(B). \end{cases} \tag{91}$$

The same conclusion holds for the system (78). Further, it is evident that if $X = Y$ and B is the identity operator, then these equations reduce to the classical ones [3], [14].

Remark 5.4.2. Note that equation (91) depends on an unknown parameter $\alpha \geq 0$. For a fixed α one can solve for $P = P_\alpha$ and substitute in equation (89) to obtain u_α. To find the optimal control law one must compute the cost functional given by equation (79) giving $J(u_\alpha)$. By repeating this process one can find the optimal control law. For further details see [2].

Remark 5.4.3. In order to solve the equation (91) (in the weak sense) one requires the inverse of the operator B on its range. Any computational program designed to solve these equations must also include a subprogram for computing the B^{-1}.

Consider the following problem: Find a feedback control law u that minimizes the functional given by the expression (69) over the Hilbert space $L^2(I,U)$, subject to the condition that x satisfies the system equation (P_c) perturbed by a process $w(t)$

taking values from Y. That is, the system is given by:

$$(Q_c)\begin{cases} (d/dt)Bx(t) = A(t)x(t) + Du(t) + w(t), & t \in I \equiv [0,T] \\ \lim_{t\to 0^+}(Bx)(t) = z_0 \end{cases}$$

The process w is a deterministic perturbation. We call this problem (**P3**). We have the following result:

Theorem 5.4.3 *Suppose the assumptions of Theorem 5.3.1 hold. Then the optimal control for the problem* (**P3**) *is given by*

$$u(t) = -R^{-1}(t)D^*(P(t)z(t) + \Gamma(t)) \tag{92}$$

where P is given by equation (3.7) and Γ satisfies the following differential equation

$$\begin{cases} (d/dt)\Gamma(t) + (A(t)B^{-1} - DR^{-1}D^*P)^*\Gamma(t) + P(t)w(t) = 0, \\ \Gamma(T) = 0, \end{cases} \tag{93}$$

in the weak sense.

Proof: We present here an outline of the proof. Following the same argument as in the first part of the proof in Theorem 5.3.1, one can show that the feedback control is given by equation (92). Substituting the obtained control in the state equation of the system (Q_c) and following the same techniques as in the proof of Theorem 5.3.1 one can arrive to a coupled system where the operators P and Γ satisfy equation (93) and (77) in the weak sense respectively.

Note that if the disturbance $w(t)$ is identically zero over the time interval $[0,T]$, then $\Gamma(t)$ is also identically zero.

5.5 Conclusions

The obtained results represent an important preliminary step for boundary control of distributed systems involving dynamic boundary conditions. In this work B-evolution concept was used where an important special case of holomorphic B-evolutions has been considered allowing the pair of operators $\{A, B\}$ to be closed.

Assuming the operator B to have an inverse on its range we converted the original problem to a classical one involving the spatial operator AB^{-1} which is not closed. since the operators A and B are not necessarily closed.
With the help of the closed pair of operators and the results obtained in [2]. [12], necessary conditions for optimality have been derived for a class of infinite dimensional linear systems.

We note also that control problems for systems involving dynamic boundary has not been yet studied by means of B-evolutions theory or either by classical theory. Our results extend the finite dimensional case studied by Ahmed and Mouadeb [2] and also cover a general class of boundary control problems. In fact if the operator B is the identity the problem is reduced to a standard class of control problems involving non dynamic boundary conditions.

Chapter 6

Existence of Optimal Controls

6.1 Introduction

In this chapter we study the optimal control of systems governed by a semilinear systems. We establish the existence of optimal solutions for Lagrange problem by means of calculus of variations. For such Problems, the existence of optimal controls depends on the properties of admissible trajectories, control sets, and the cost functionals.

6.2 Basic Assumptions and Notations

Let $I = [0, T]$, $T < \infty$, H a separable Hilbert space and Y a dense subspace of H, carrying the structure of a separable,reflexive, Banach space, which embeds in H continuously. Identifying H with its dual, we have $Y \hookrightarrow H \hookrightarrow Y^*$, with all embeddings being continuous and dense. We will assume that all the embeddings are compact. By $\|\cdot\|_Y$ (resp. $|\cdot|_H$, $\|\cdot\|_{Y^*}$) we will denote the norm of Y (resp. of H, X^*). Also by $\langle\cdot,\cdot\rangle$, we will denote the duality brackets for the pair (Y, Y^*) and by $(\cdot,\cdot)$, the inner product of H. Let $W^{p,q}(I) = \{v \in L^p(I, Y) \ : \dot{u} \in L^q(I, Y^*)\}$. The derivative in this definition is understood in the sense of vector-valued distributions. Then $W^{p,q}$, is a separable, reflexive Banach space with the norm

$\|v\|_{W^{p,q}} = \left(\|v\|_{L^p(I,Y)} + \|\dot{v}\|_{L^q(I,Y^*)}\right)$. Recall that $W^{p,q}(I)$ embeds into $C(I,H)$ continuously (see Ahmed and Teo [3]). So every equivalence class in $W^{p,q}(I)$ has a unique representative in $C(I,H)$. Further, since we have assumed that Y embeds into H compactly, we have that $W^{p,q}(I)$ embeds into $L^p(I,H)$, compactly too. For any pair of Banach space X, Y, $\mathcal{L}(X,Y)$ and $\mathcal{L}_{ub}(X,Y)$ will denote the class of bounded and unbounded linear operators from X to Y respectively.

6.3 Existence of Optimal Controls

Let (Y,H,Y^*) be an evolution triple, with $Y \hookrightarrow H$ compactly (hence $H \hookrightarrow Y^*$), E a separable, reflexive Banach space, modeling the control space and suppose $t \mapsto U(t)$ is a measurable multifunction on I where the controls are taking values. Denote by $\mathcal{U}_{ad} = \{u : \ u(t) \in U(t) \quad a.e.\}$ the admissible set of controls. The control system we wish to consider is governed by the nonlinear evolution equation

$$\begin{cases} (d/dt)Bx(t) = A(t)x(t) + f(t, Bx(t), u(t)). \ u(t) \in U(t) \quad a.e. \\ \lim_{t \to 0^+}(Bx)(t) = z_0 \in Y \end{cases} \tag{94}$$

where $A(t)$ and B are linear unbounded operators between two Banach spaces. f is a map with values in a Banach space and and u is a suitable function representing the control actions.

We shall consider the case of holomorphic B-evolution of type L. In this case the operators A and B are not necessarily closed but the pair $\{A(t), B\}$ is closed when considered as a map from $D(A) \cap D(B) \subset X$ to $Y \times Y$. It follows from a result due to Sauer [15] that the Cauchy problem

$$\begin{cases} (d/dt)Bx = A(t)x, t > s, \\ \lim_{t \to s^+}(Bx(t)) = \xi \end{cases} \tag{95}$$

has a unique solution given by

$$x(t) = V(t,s)\xi, t \geq s. \tag{96}$$

In other words the Cauchy problem (95) admits an evolution operator $V(t,s), 0 \leq s \leq t < \infty$.

Further, in case of B-evolution of type L, it follows from [15, Theorem 5.1, p. 296] that $A(t)B^{-1}$ with domain $D(A(t)B^{-1} \equiv \mathcal{D} \subset Y$ is the infinitesimal generator of a transition operator $E(t,s) \equiv BV(t,s), t > s \geq 0$. Thus the problem reduces to the classical case and the system (94) can be written as

$$\begin{cases} \dot{z} = A(t)B^{-1}z + f(t,z,u) \\ z(0) = z_0. \end{cases} \tag{97}$$

We consider the following Lagrange type optimal control problem:

$$(P)\begin{cases} J(z,u) = \int_0^T \ell(t, Cz, u(t))\, dt \to \inf = m \\ \text{subject to the following state and control constraints :} \\ \dot{z} = A(t)B^{-1}z + f(t,z,u),\ z(0) = z_0 \in Y,\ u(t) \in U(t) \quad \text{a.a. } t \in I. \end{cases}$$

The operator $C \in \mathcal{L}(Y, X)$ denote the continuous extension of the operator B^{-1} from its range $R(B)$ to Y.

By an admissible 'state-control' pair for the problem (P), we understand a pair of a state trajectory $z \in C(I,Y)$ and of control function $u \in L^\infty(I,E)$ so that $z \in W^{p,q}(I)$ and both functions z, u satisfy the constraints of problem (P). An admissible 'state-control' pair $\{z,u\}$ is said to be 'optimal', if $J(z,u) = m$.
We will be using the following notation:

$$P_{f(c)}(E) = \{\Gamma \subseteq E : \ non-empty,\ closed\ (convex)\}$$

$$P_{wk(c)}(E) = \{\Gamma \subseteq E : \ non-empty,\ weakly\ compact\ (convex)\}$$

A multifunction (set valued function) $F : I \to P_f(E)$ is said to be measurable, if for every $v \in E$, the R_+-valued function $t \to d(v, F(t)) = \inf\{\|v - w\| : w \in F(t)\}$ is measurable. By S_F^1 we will denote the set of selectors of $F(\cdot)$ that belong in the Lebesgue-Bochner space $L^1(I,E)$; i.e.

$$S_F^1 = \{f \in L^1(I,E) : \ f(t) \in F(t) \quad a.e.\}$$

One can check that for a measurable multifunction $F : I \to P_f(E)$, S_F^1 is nonempty if and only if $t \to \inf\{\|v\| : \ v \in F(t)\} \in L_+^1$ and such a multifunction is called 'integrably bounded'. For further details we refer to Wagner [36].

To establish the existence of an optimal pair for the problem (P). We will need the following hypothesis on the data:

H(AB^{-1}) :

1. The pair $\{A, B\}$ is the generating pair of a holomorphic B-evolution of type L in Y.

2. $\langle A(t)B^{-1}\eta, \eta\rangle_{Y^*,Y} \leq 0$ for all $\eta \in \mathcal{D}$, $t \in I$.

H(f) :

1. The function $f : I \times Y \times E \to Y^*$ is demicontinuous in the sense that whenever $t_n \to t$ in I, $z_n \xrightarrow{s} z$ in Y, and $u_n \xrightarrow{w} u$ in E, $\langle f(t_n, z_n, u_n), \eta\rangle_{Y^*,Y} \to \langle f(t, z, u), \eta\rangle_{Y^*,Y}$ for each $\eta \in Y$.

2. $\langle f(t, y, v) - f(t, z, v), y - z\rangle_{Y^*,Y} \leq 0$ for all $y, z \in Y$ and $v \in U(t)$, $t \in I$.

3. $\|f(t, z, v)\|_{Y^*} \leq h(t) + \alpha\|z\|_Y^{p/q}$ for $h \in L^q(I, R_+)$, $\alpha > 0$, and $(z, v) \in Y \times U(t)$, $t \in I$.

4. $\langle f(t, z, v), z\rangle_{Y^*,Y} \leq h_1(t) - \beta\|z\|_Y^p$ for $h_1 \in L^1_+(I, R)$, $\beta > 0$, and $(z, v) \in Y \times U(t)$, $t \in I$.

H(U) :

$U : I \to P_{wkc}(E)$ is a measurable multifunction so that
$t \to \sup\{\|u\|_E : u \in U(t)\} \equiv g(t)$, $g \in L^\infty_+$.

H(ℓ) :

$\ell : I \times H \times E \to R \cup \{\infty\}$ is an integrand so that

1. $(t, e, u) \to \ell(t, e, u)$ is Borel measurable,
2. $(e, u) \to \ell(t, e, u)$ if l.s.c.,
3. $u \to \ell(t, e, u)$ is convex,
4. $\phi(t) - \lambda(\|e\|_H + \|u\|_E) \leq \ell(t, e, u)$ a.e. with $\phi \in L^1$, $\lambda > 0$.

Finally since our cost functional is $R \cup \{\infty\}$ valued, we will need the following feasibility hypothesis.

$(\mathbf{H_0})$: there exists an admissible 'state-control' pair $\{z, u\}$ so that $J(z, u) < \infty$.

Before studying the problem of existence of optimal controls, we will start by deriving some a priori bounds for the admissible trajectories of problem (P).

Denote by $\mathcal{Y} = \{z_u : u \in \mathcal{U}_{ad}\}$ the set of solution trajectories of the evolution equation of problem (P) corresponding to the admissible set of controls as defined above, where each $z_u \in L^p(I, Y)$.

Lemma 6.3.1. (A priori estimates). *Under the assumptions* $\mathbf{H(AB^{-1})}$ *and* $\mathbf{H(f)4}$, *the set* $\mathcal{Y}$ *is a bounded subset of* $L^p(I, Y) \cap L^\infty(I, H)$.

Proof. Let z_u be any solutions trajectory of the evolution equation in problem (P), corresponding to an admissible control $u(\cdot) \in L^\infty(I, E)$. Then

$$< \dot{z}_u(t), z_u(t) >_{Y,Y^*} = < A(t)B^{-1}z_u(t), z_u(t) >_{Y,Y^*} + < f(t, z_u(t), u(t)), z_u(t) >_{Y,Y^*}$$

Since $z_u \in L^p(I, Y)$ and $\dot{z}_u \in L^q(I, Y^*)$, then $z_u \in W^{p,q}(I)$. Recall that $W^{p,q}(I)$ embeds into $C(I, H)$ continuously, it follows that $z_u \in C(I, H)$ and further, it follows from proposition 23.23 (iv), p.422 of Zeidler [30], that

$$< \dot{z}_u(t), z_u(t) >_{Y,Y^*} = (\dot{z}_u(t), z_u(t))_H = (1/2)(d/dt)(|z_u(t)|_H^2). \tag{98}$$

Using assumption $\mathbf{H(AB^{-1})2}$ and $\mathbf{H(f)4}$, we have

$$(1/2)(d/dt)(|z_u(t)|_H^2) \leq 2h_1(t) - \beta|z_u(t)|_Y^p \quad a.e. \tag{99}$$

Integrating the above inequality, we have, for each $t \in I$,

$$|z_u(t)|_H^2 + 2\beta \int_0^t |z_u(\theta)|_Y^p d\theta \leq |z_0|_H^2 + 2\int_0^t h_1(\theta)d\theta. \tag{100}$$

Since the right-hand side of the above inequality is independent of $u \in \mathcal{U}_{ad}$, it follows that

$$z_u \in L^\infty(I, H) \cap L^p(I, Y). \tag{101}$$

Lemma 6.3.2. *Suppose that $D((AB^{-1})^*)$, contained in $L^p(I,Y)$, is a set of category II, assumptions of Lemma 6.3.1 and* **H(f)4** *hold. Then the set $Z = \{\dot{z}_u : u \in \mathcal{U}_{ad}\}$ is a bounded subset of $L^q(I,Y^*)$.*

Proof: Let $((\cdot,\cdot))_0$ denote the duality brackets for the pair $(L^p(I,Y), L^q(I,Y^*))$ (i.e. if $y \in L^p(I,Y)$, $z \in L^q(I,Y^*)$, then $((\cdot,\cdot))_0 = \int_0^T \langle y(t), z(t)\rangle_{Y,Y^*}$). Let z_u be a solution of the evolution equation (97) corresponding to the control $u \in \mathcal{U}_{ad}$, and let $\xi \in D((AB^{-})^*)$. Then

$$\int_I \langle \dot{z}_u(t), \xi(t)\rangle_{Y^*,Y}\, dt$$

is defined and the following equality hol ds.

$$\int_I \langle \dot{z}_u(t), \xi(t)\rangle_{Y^*,Y}\, dt =$$
$$\int_I \{\langle z_u(t), (A(t)B^{-1})^*\xi(t)\rangle_{Y,Y^*} + \langle f(t, z_u(t), u(t)), \xi(t)\rangle_{Y^*,Y}\}\, dt.$$

By applying Holder's inequality, to the last integral on the right-hand side and using the assumption **H(f)3** we obtain,

$$|\int_I \langle \dot{z}_u(t), \xi(t)\rangle_{Y^*,Y}\, dt| \le$$
$$\|(AB^{-1})^*\xi\|_{L^q(I,Y^*)}\|z_u\|_{L^p(I,Y)} + \{\|h\|_{L^q(I,R)} + \alpha(\|z_u\|^{p/q}_{L^p(I,Y)})\} \times \|\xi\|_{L^p(I,Y)}.$$

By the a priori estimates (see Lemma 6.3.1) of the solution trajectory z_u of the evolution equation in problem (P), $\sup\{\|z_u\|_{L^p(I,Y)} : z_u \in Y\} \equiv b$ is finite. Thus, there exists a constant b_ξ, dependent on ξ, such that

$$|\int_I \langle \dot{z}_u(t), \xi(t)\rangle_{Y^*,Y}\, dt| \le$$
$$b\|(AB^{-1})^*\xi\|_{L^q(I,Y^*)}\|z_u\|_{L^p(I,Y)} + \{\|h\|_{L^q(I,R)} + \alpha b^{p/q}\} \times \|\xi\|_{L^p(I,Y)} \equiv b_\xi$$

for all $u \in \mathcal{U}_{ad}$. Consequently,

$$\sup_{u\in\mathcal{U}} |\int_I \langle \dot{z}_u(t), \xi(t)\rangle_{Y^*,Y}\, dt| \le b_\xi \; for\ each\ \xi \in D((AB^{-1})^*). \tag{102}$$

This implies that the set $\{\dot{z}_u : u \in \mathcal{U}_{ad}\}$ is a family of bounded linear functionals on $D((AB^{-1})^*)$. By hypothesis, $D((AB^{-1})^*)$ is a set of category II; thus, it follows from the uniform boundedness principle that there exists a constant $c > 0$ such that

$$\sup \|\dot{z}_u\|_{L^q(I,Y^*)} \leq c. \tag{103}$$

This ends the proof of the lemma.

With the help of the above preparatory results we can now prove our main results.

Theorem 6.3.1. *If hypothesis* $\mathbf{H(AB^{-1})}$, $\mathbf{H(f)}$, $\mathbf{H(U)}$, $\mathbf{H(\ell)}$, $\mathbf{H_0}$ *hold, then problem* (P) *admits an optimal pair.*

Proof: From Lemma 6.3.1 and Lemma 6.3.2, we deduce that $\{z_u\}$ is bounded in $L^p(I,Y)$ while $\{\dot{z}_u\}$ is bounded in $L^q(I,Y^*)$. Hence the set $\mathcal{X} = \{z_u : z_u \in \mathcal{Y}\}$ is bounded subset of the reflexive Banach space $W^{p,q}(I)$. So $\mathcal{X}$ is relatively compact subset of $W^{p,q}(I)$. Now let $\{(z^n, u^n)\}_{n\geq 1}$ be a minimizing sequence of admissible 'state-control' pairs for the problem (P); i.e.
$\lim_{n\to\infty} J(z^n, u^n) = \inf\{J(z_u, u),\ for\ admissible\ 'state-control'\ pair\ \{z_u, u\}\} \equiv m$. Here z^n denotes the solution of the system equation (97), corresponding to the control u^n. Since $\{z^n\}_{n\geq 1} \subseteq \mathcal{X}$, by passing to a subsequence if necessary, we may assume that $z^n \xrightarrow{w} y$ in $W^{p,q}(I)$ and that $y = z_u$. But recall that $W^{p,q}(I)$ embeds compactly into $L^p(I,H)$. So $z^n \xrightarrow{s} z_u$ in $L^p(I,H)$ and since $W^{p,q}(I)$ embeds into $C(I,H)$ continuously, by considering the continuous versions of these functions and by passing to a further subsequence if necessary, we can have $z^n(t) \xrightarrow{s} z_u(t)$ in H for all $t \in I$. Furthermore, from hypothesis $H(U)$ and proposition 3.1 of [29] we have $S_U^\infty \equiv \{u \in L^\infty(I,E) : u(t) \in U(t) \quad a.e.\}$ is w_*-compact in $L^\infty(I,E)$. So we may assume that $u_n \xrightarrow{w_*} u$ in $L^\infty(I,E)$. Then invoking Theorem 2.1 of Balder [28], we conclude that $J(z_u, u)$ is strong-w_* l.s.c., i.e. $J(z_u, u) \leq \underline{\lim} J(z^n, u_n) = m$, whenever $z^n \xrightarrow{s} z_u$ in $L^1(I,H)$ and $u_n \xrightarrow{w_*} u$ in $L^\infty(I,Y)$.

It suffices to show that $\{z_u, u\}$ is an admissible 'state-control' pair for the problem (P). Let $\tilde{f}$ denote the Nemitsky operators of f; i.e. $\tilde{f}(z,u)(t) = f(t,, z(t), u(t))$ a.e. To this end, we have

$$((\dot{z}^n, z^n - z_u))_0 + ((AB^{-1}z^n, z^n - z_u))_0 = ((\tilde{f}(z^n, u_n), z^n - z_u))_0. \tag{104}$$

From integration by parts formula for functions in $W^{p,q}(I)$ (see for example [30], proposition 23.23, p.422), we have

$$((\dot{z}^n - \dot{z}_u, z^n - z_u))_0 = (1/2)\|z^n(T) - z_u(T)\|_H^2.$$

This implies that

$$((\dot{z}^n, z^n - z_u))_0 = (1/2)\|z^n(T) - z_u(T)|_H^2 + ((\dot{z}_u, z^n - z_u))_0.$$

and hence

$$\lim_{n\to\infty}((\dot{z}^n, z^n - z_u))_0 = 0.$$

By virtue of hypothesis **H(f)1** we have

$$\lim_{n\to\infty}((\tilde{f}(z^n, u_n), z^n - z_u))_0 = 0.$$

Thus, taking the limit as $n \to \infty$ in the equation (104), we obtain

$$\lim_{n\to\infty}((AB^{-1}z^n, z^n - z_u))_0 = 0.$$

Since $\{z^n\} \in \mathcal{X}$ and $\{\dot{z}^n\} \in \mathcal{Z}$ and both $\mathcal{X}$ and $\mathcal{Z}$ are bounded subsets of $L^p(I, Y)$ and $L^q(I, Y^*)$, respectively, and f satisfies assumption **H(f)3**, it is clear that $AB^{-1}zn$ is a bounded sequence from $L^q(I, Y^*)$. Thus, there exists a subsequence of the sequence $AB^{-1}z^n$, again denoted by $AB^{-1}z^n$, and an element $y \in L^q(I, Y^*)$ such that $AB^{-1}z^n \to y$ weakly in $L^q(I, Y^*)$. Thus, $y = AB^{-1}z_u$ is defined a.e. on I and $y(t) \in Y^*$ a.e. So if $\eta \in L^p(I, Y)$, we have,

$$\begin{aligned} ((\dot{z}^n, \eta))_0 &= ((AB^{-1}z^n, \eta))_0 + ((\tilde{f}(z^n, u_n), \eta))_0. \\ &\to ((\dot{z}_u, \eta))_0 + ((AB^{-1}z_u, \eta))_0 = ((\tilde{f}(z_u, u), \eta))_0 \; as \; n \to \infty. \end{aligned}$$

Since $\eta \in L^p(I, Y)$ was arbitrary, we deduce that

$$\begin{cases} \dot{z}_u = AB^{-1}z_u + f(t, z_u, u) & a.e. \\ z_0 = 0. \end{cases}$$

This complete the proof of the theorem.

Chapter 7

Conclusions and Future Research

7.1 Conclusions

The results obtained in this thesis represent an important preliminary step for boundary control of distributed systems involving dynamic boundary conditions. In this work B-evolution concept was used where an important special case of holomorphic B-evolutions has been considered allowing the pair of operators $\{A, B\}$ to be closed.

Assuming the operator B to have an inverse on its range we converted the original problem to a classical one involving the spatial operator AB^{-1} which is not closed. since the operators A and B are not necessarily closed. With the help of the closed pair of operators, necessary conditions for optimality and existence of optimal controls have been derived for semilinear problems and applied to a linear quadratic regulator problem. Also an algorithm for computing optimal controls has been presented.

7.2 Suggestions for Future Research

The question of existence of solutions for systems governed by B-evolutions has been studied by several authors [5], [10], [15], [16], [7], [8], [9]. In Brill [5], it assumed that the operator B is dominant, the pair $\{A, B\}$ closed, and B is bijective with compact inverse; in Favini [10], the operator A is dominant and the pair$\{A, B\}$ closed; Sauer [15], [16] and Van Dalsen [7], [8], [9], the pair $\{A, B\}$ is closed. The last assumption is the (weakest) most general one so far. In dealing with semilinear control problems considered in this thesis, it is assumed as in Sauer and Van Dalsen, that $\{A, B\}$ is a generating pair of a holomorphic B-evolution of type L. This allows us to convert the problem to a classical one, where we are dealing directly with the operator AB^{-1} as a generator of a C_0 semigroup.

Further research could be conducted in the following direction:
As pointed out, in the case of holomorphic B-evolution of type L, the problem can be converted to a classical one. We do not know at this time how to avoid this case for the semilinear problem. We suggest, instead to deal directly with the original system by considering the pair $\{A, B\}$ and see if it generates a C_0-semigroup. This question is totally open and excellent problem from both theoretical and practical point of view.

Bibliography

[1] N.U. Ahmed, *Semigroup theory with applications to systems and control*, Pitman Res. Notes in Math. Ser., vol. 246, Longman Scientific and Technical and John Wiley, London, New York, 1991.

[2] N.U. Ahmed and A.R. Mouadeb, *Optimal regulator for linear systems with no control cost*, Dynamics and Control **1** (1991), 341–355.

[3] N.U. Ahmed and K.L. Teo, *Optimal control of distributed parameter systems*, North-Holland, Amsterdam, 1981.

[4] W. Arent, A. Grabosh, G. Greinert, U. Groh, H.P. Lotz, U. Moustakes, R. Nagel, F. Neubrander and U. Schlotterbeck, *One-parameter semigroups of positive operators*, (R. Nagel, ed.), Lecture Notes in Mathematics, vol. 1184, Springer-Verlag, New York, 1996.

[5] H. Brill, *A semilinear evolution equation in a Banach space*, J. Differential Equations, **24** (1977), 412–425.

[6] M. Grobbelaar-Van Dalsen, *Die Teorie van nie-stasionere evolusies geassosieer met dinamies-gekoppelde Randwaardeprobleme*, Doctoral Thesis, Pretoria University, Pretoria, 1978.

[7] M. Grobbelaar-Van Dalsen, *Evolution problems involving non-stationary operators between two Banach spaces I. Existence and uniqueness theorems*, Quaestiones Mathematicae **8**(2) (1985), 97–129.

[8] M. Grobbelaar-Van Dalsen, *On the solvability of the boundary-Value problem for the elastic beam with attached load*, Mathematical Models and Methods in Applied Sciences, **4**(1) (1994), 89–105.

[9] M. Grobbelaar-Van Dalsen, *Semilinear evolution equations and fractional powers of a closed pair of operators*, Proc. of the Royal Soc. of Edinburgh. **105 A** (1987), 101–115.

[10] A. Favini, *Laplace transform method for a class of degenerate evolution problems*, Rend. Mat. **12** (1979), 511–536.

[11] A. Freidman, *Partial differential equations*, Holt, New York, 1969.

[12] S. Kerbal and N.U. Ahmed, *Optimal boundary control of distributed systems involving dynamic boundary conditions*, Mathematical Problems in Engineering **3** (1998), 387–411.

[13] P. Li, S.S. Lim, G. Vukovich and N.U. Ahmed, *Stability and robustness analysis of boundary control of flexible space structure*, Int. J. Systems Sci., **26**(10) (1995), 1759–1776.

[14] J.L. Lions, *Optimal control of systems governed by partial differential equations*, Springer-Verlag, Berlin, 1971.

[15] N. Sauer, *Linear evolution equations in two Banach spaces*, Proceedings of the Royal Society of Edinburgh, **91A** (1982), 287–303.

[16] N. Sauer, *Dynamical processes associated with dynamic boundary conditions for partial differential equations*, Proc. Intern. Conf. on Theory and Applications of Differential Equations, Ohio University, **2** (1988), 374–378.

[17] L.S. Pontryagin, V.G. Boltyanskii, R.V. Gamkrelidze and E.F. Mischenko, *Mathematical theory of optimal processes*, John Willey New York, 1962.

[18] L.S. Pontryagin, *The maximum principle in the theory of optimal processes*, 1st Congress IFAC, Moscow, 1960.

[19] J. Warga, *Optimal control of differential and functional equations*, Academic Press, New York,1972.

[20] L.O. Berkovitz, *Optimal control theory*, Springer-Verlag, New York, 1974.

[21] A. G. Butkovskii, *Maximum principle of optimal control for distributed parameters systems*, Automat. Telemekh., **22** (1961), 1288–1301. (In Russian)

[22] F. H. Clarke, *The maximum principle with minimum hypothesis*, SIAM J. Control Optim., **14** (1976), 1078–1091.

[23] F. H. Clarke, *Optimization and nonsmooth analysis*, John Wiley, New York, 1993.

[24] A.V. Balakrishnan, *Applied functional analysis*, Springer-Verlag, New York, 1976.

[25] N.U. Ahmed and S. Kerbal, *Stochastic systems governed by B-evolutions on Hilbert spaces*, Proceedings of the Royal Society of Edinburgh, **127 A** (1997), 903–920.

[26] N.U. Ahmed and S. Kerbal, *Linear quadratic regulator for systems governed by B-Evolutions*, Dynamics of Continuous, Discrete and Impulsive Systems, (to appear)

[27] S. Kerbal and N.U. Ahmed, *Necessary Conditions of Optimality for Systems Governed By B-evolution*, proceedings of Dynamical Systems and Applications, 2(1996), 293-300.

[28] E. Badler,*Necessary and sufficient conditions of L^1-strong-weak Lower Semicontinuity of integral functionals*, Non. Anal., TMA 11, (1987), 1399-1404.

[29] N.S. Papageorgiou, *On the theory of Banach space-valued integrable multifunctions Part1: Integration and Conditional expectation*, J. Multiv. Anal., 17(1985), 185-206.

[30] Z. Zeidler, *Nonlinear Functional Analysis and its Applications II*. Springer, New York 1990.

[31] N.U. Ahmed, *Elements of finite dimensional systems and control theory*, John Wiley, New York, 1988.

[32] N.U. Ahmed and S.K. Biswas, *Mathematical modelling and control of large space structures with multiple appendages*, Mathematical and Computer Modelling, 10(1988), 891–900.

[33] N.U. Ahmed and P. Li, *Stabilizing controllers for perturbed distributed parameter systems*, Proceedings of the 28th IEEE-CDC, Tampa, Florida, (1989), 1446–457.

[34] S.K. Biswas and N.U. Ahmed, *Modelling of flexible space craft and their stabilization*, International J. of Systems Sciences, **6** (1985), 535–551.

[35] A. Pazy, *Semigroup of linear operators and applications to partial differential equations*, Springer-Verlag, New York, 1983.

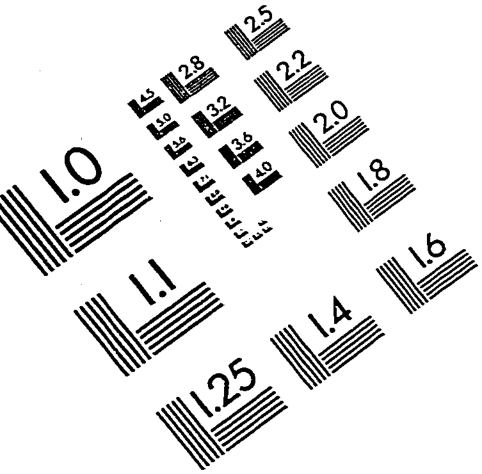
[36] D. Wagner, *Survey of measurable selection theorem*, SIAM J. Control and Optim., 15(1997), 859–903.

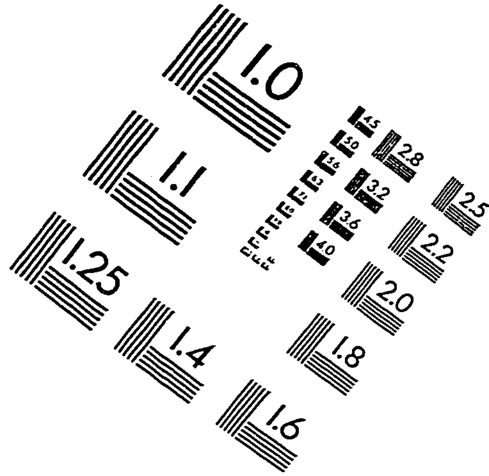
[37] H.O. Fattorini, *Relaxation theorems, differential inclusions, and Filippov's theorem for relaxed control in semilinear infinite dimensional systems*, Journal of Differential Equations, 1112(1994), 131–153.

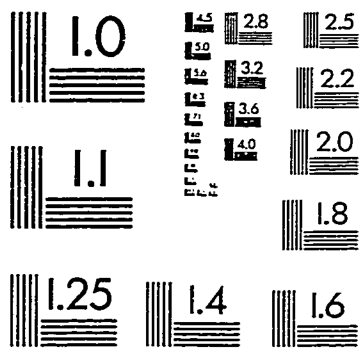
[38] X. Li and J. Yong, *Necessary conditions for optimal control of distributed parameter systems*, SIAM J. Control and Optimization, **29** (1991), 895–908.

[39] N.U. Ahmed and S. Kerbal, *On approximate controllability for semilinear systems*, Nonlinear World, **2** (1995), 53–67.

IMAGE EVALUATION TEST TARGET (QA-3)

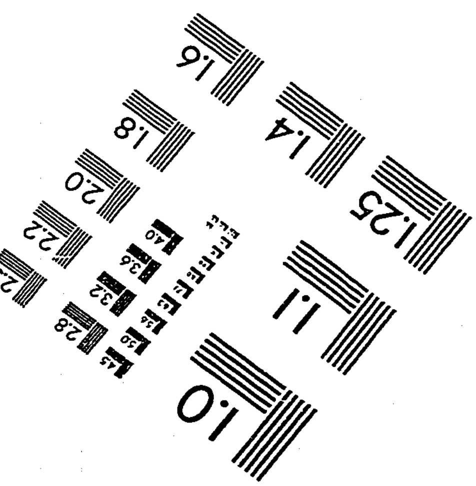


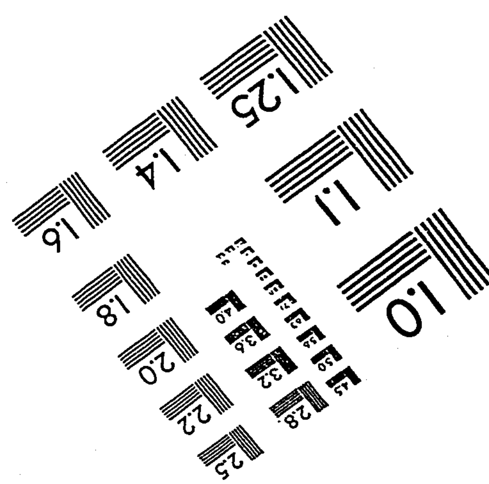




150mm

6"





APPLIED IMAGE, Inc
1653 East Main Street
Rochester, NY 14609 USA
Phone: 716/482-0300
Fax: 716/288-5989