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THE UNITARY GROUP AS A COMPLETE INVARIANT FOR SIMPLE UNITAL AF ALGEBRAS

By

Andrew Booth

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Abstract

In 1955, Henry Dye showed that factors with isomorphic unitary groups are in a certain sense isomorphic, provided they are not of type I_{2n}. We establish a similar result for simple AF algebras, namely that the unitary group of a simple AF algebra is a complete invariant. Also, there appears to be an error in Dye's paper which allows us to strengthen his result so that it is valid for factors not of type I_2.

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Finally, I would like to thank my family and friends for being who they are.

Contents

Chapter 1

Introduction

1.0.0.1 Using [Dy, Theorem 2], M. Broise shows in [Be] the following result:

1.0.0.2 Theorem *Let M and N be factors not of type I_n (with n finite). Let φ be an isomorphism of the unitary group of M with the unitary group of N. Then there is a map $\Phi : M \to N$, which is either an isomorphism or an antiisomorphism, such that $\varphi(u) = \Phi(u)$ for every unitary u in M.*

In Chapter 3 we show a similar result (3.4.0.2) for simple AF algebras. We begin by sketching the proof of [Dy, Theorem 2], as the approach we take is similar to Dye's approach.

For those who decide to consult Dye's paper note that his matrices act on vectors from the right, i.e. are the transpose of the matrices we would normally associate to a linear transformation.

Part 1

1.0.0.3 Let $\mathfrak{A}$ and $\mathfrak{B}$ be unital C*-algebras. Denote by $\mathcal{P}(\mathfrak{A})$ the set of projections of $\mathfrak{A}$ and by $\mathcal{P}(\mathfrak{B})$ the set of projections of $\mathfrak{B}$. We say that a map

$$\theta : \mathcal{P}(\mathfrak{A}) \to \mathcal{P}(\mathfrak{B})$$

is a **projection orthoisomorphism** if it is a bijection which preserves orthogonality, in the sense that $pq = 0$ if and only if $\theta(p)\theta(q) = 0$, for any $p, q \in \mathcal{P}(\mathfrak{A})$.

The goal of the first part of [Dy] is to show that a projection orthoisomorphism between von Neumann algebras can be extended to an "isomorphism" between the von Neumann algebras (in the sense of [Dy, Theorem 1], see (1.0.0.6)). An orthoisomorphism preserves many properties of the projection structure of C*-algebras, including 0, I, orthocomplements, order, and commutativity of projections (cf. ([Dy, Lemma1]) or (3.3.0.1)).

1.0.0.4 Dye remarks here an orthoisomorphism is at least as strong as a lattice isomorphism, but that it is not known if the two are equivalent.

Note also that examining mappings between the projections of C*-algebras would be fruitless because of the short supply of projections. To overcome this problem to some degree, Dye considers matrix algebras over C*-algebras. As a culmination of this approach we have (in Dye's words)

> Kadison studies Jordan (or C^*-) isomorphisms and related mappings between C*-algebras. In the von Neumann algebra case (or more expressly, when matrix representations are available), he shows that any Jordan isomorphism is the direct sum of an isomorphism and an antiisomorphism. This develops as a natural consequence of a purely algebraic theorem, due to Jacobson and Rickart, to the effect that any Jordan homomorphism of a matrix ring is the direct sum of an (associative) homomorphism and antihomomorphism. We wish to show here that Kadison's theorem can also be proved by means of the preceding (elementary) theory of C^*-matrix algebras.
>
> Explicitly, we will prove that *any Jordan *-isomorphism of the* C^**-matrix algebra* $\mathbb{M}_n(\mathfrak{A})$ $(n \geq 3)$ *on a* C^**-algebra* $\mathfrak{B}$ *is the direct sum of a *-isomorphism and a *-anitiisomorphism.*
>
> —[Dy, p79]

1.0.0.5 The focus then transfers entirely to von Neumann algebras (with their abundance of projections), with the following end results.

1.0.0.6 Theorem ([Dy, Theorem 1]) *Let θ be a projection orthoisomorphism of the von Neumann algebra $\mathbb{M}_n(M)$ (M a von Neumann algebra) onto the von Neumann algebra N. Then there is a mapping Φ of $\mathbb{M}_n(M)$ on N which extends θ and is the direct sum of a *-isomorphism and a *-antiisomorphism.*

1.0.0.7 Corollary ([Dy, Corollary to Theorem 1]) *Let M be a von Neumann algebra with no direct summands of type I_2 in the large. Then any projection orthoisomorphism of M on a von Neumann algebra N is implemented by the direct sum of a *-isomorphism and a *-antiisomorphism.*

1.0.0.8 Following the above results is a comment that in the I_2 case the theorem fails, as well as a link to [K].

Part 2

The rest of the paper shifts attention to factors with the aim of proving [Dy, Theorem 2] (cf. (1.0.0.14)). The idea is to construct an orthoisomorphism from a map between the unitary groups, and then to use Theorem 1.

1.0.0.9 Recall (2.7.10.7) that a factor is a von Neumann algebra with trivial centre (i.e. whose centre consists of scalar multiples of the (multiplicative) identity).

For the rest of this section, let M and N be factors with unitary groups $\mathcal{U}(M)$ and $\mathcal{U}(N)$, and let $\varphi : M \to N$ be a group isomorphism. As φ sends self-adjoint unitaries to self-adjoint unitaries, it induces a natural map $\theta : \mathcal{P}(M) \to \mathcal{P}(N)$ given by

$$I - 2\theta(p) = \varphi(I - 2p),$$

or equivalently by

$$\theta(p) = \frac{I - \varphi(I - 2p)}{2}.$$

Then Dye shows that θ is a projection orthoisomorphism in all but pathological cases, namely the I_{2n} case with $n \geq 3$. Essentially, the critical point is to be able to divide projections appropriately.

1.0.0.10 We call a projection p an **n-fold divisor of** I if I can be written as the sum of n mutually othogonal projections, each equivalent to p. This equivalence can be taken as unitary equivalence (for example, using 2.7.12.9).

1.0.0.11 Lemma ([Dy, Lemma 12]) *The map θ preserves orthogonality among $(2k+1)$-fold divisors of I.*

1.0.0.12 Lemma ([Dy, Lemma 13]) *Let M be a factor not of type I_{2n} (with $n \geq 1$). Then θ is an orthoisomorphism.*

1.0.0.13 The proof of this key result is really a von Neumann algebra proof. Let p and q be two fixed orthogonal projections in M. To show that $\theta(p)\theta(q) = 0$, Dye divides the proof according to the type of the projections p and q.

1.0.0.14 Theorem ([Dy, Theorem 2]) *Let M and N be factors not of type I_{2n}, and let φ be a group isomorphism of their unitary groups $\mathcal{U}(M)$ and $\mathcal{U}(N)$. Then there exists a linear (or conjugate linear) *-isomorphism Φ of M on N which implements φ in the following sense: for some (possibly discontinuous) character χ of $\mathcal{U}(M)$, and for every unitary $u \in \mathcal{U}(M)$, we have $\varphi(u) = \chi(u)\Phi(u)$.*

Dye provides a potential weakening of [Dy, Lemma 13], in which there seems to be a mistake. We provide another counterexample in Appendix A, and we then use this to show (in Appendix B) that in fact [Dy, Theorem 2] holds also in the I_{2n} case provided $n > 1$.

Chapter 2

Background material

2.1 Ordered Sets

2.1.0.1 Definition *A* **pre-order** *on a set $\mathcal{I}$ is a relation $\leq$ which satisfies the following, for any $x, y, z \in \mathcal{I}$*

$$\begin{array}{ll} 1)\ x \leq x & \text{(reflexive)} \\ 2)\ x \leq y, y \leq z \Rightarrow x \leq z & \text{(transitive)} \end{array}$$

A pre-order is a **partial order** *if $\leq$ is antisymmetric, i.e. if $x \leq y$ and $y \leq x$ implies $x = y$.*

A set $\mathcal{I}$ with a pre-order $\leq$ is said to be a **pre-ordered set**, *or simply an ordered set, and will often be denoted $(\mathcal{I}, \leq)$. If $\leq$ is a partial order on $\mathcal{I}$, we say that $\mathcal{I}$ is a* **partially ordered set**.

If $x, y \in \mathcal{I}$ are such that $x \leq y$ or $y \leq x$ then we say that x and y are **comparable**. *A partial order on a set $\mathcal{I}$ in which any two elements are comparable is called a* **total order**, *and a set with a total order is called a* **totally ordered set**.

2.1.0.2 Some examples:

1. The set $\mathbb{Z}^+$of positive integers is a totally ordered set.

2. The set $\mathbb{Z}^2$ with the **product ordering** $(a, b) \leq (x, y)$ if and only if $a \leq x$ and $b \leq y$ is a partially ordered set which is not totally ordered.

3. The collection $\mathfrak{P}(X)$ of all subsets of X is a partially ordered set with respect to **inclusion**.

   $\mathfrak{P}(X)$ is also partially ordered with respect to the partial order $\supseteq$. We say that $\mathfrak{P}(X)$ is partially ordered with respect to **reverse inclusion**.

4. Let $(X, \leq)$ be a pre-ordered set. The any subset Y of X is also a pre-ordered set with respect to the relation $\leq$ restricted to $Y \times Y$.

5. The collection $\mathcal{O}(x)$ of all neighbourhoods of x in a topological space X is a partial order with respect to $\supseteq$. We say that $\mathcal{O}(x)$ is partially ordered with respect to **reverse inclusion**.

2.1.0.3 Let $(\mathcal{I}, \leq)$ be a partially ordered set, and let S be a subset of $\mathcal{I}$. An element $x \in \mathcal{I}$ is an **upper bound**, or a **majorant**, for S in $\mathcal{I}$ if for every $s \in S$, $s \leq x$. An element $x \in \mathcal{I}$ is a **lower bound**, or a **minorant**, for S in $\mathcal{I}$ if for every $s \in S$, $x \leq s$.

A partially ordered set $(\mathcal{I}, \leq)$ is said to be **upward filtering** or **upward directed** if every finite subset of $\mathcal{I}$ has an upper bound in $\mathcal{I}$, and is **downward directed** if every finite subset of $\mathcal{I}$ has a lower bound in $\mathcal{I}$. Note that $(\mathcal{I}, \leq)$ is upward filtering if and only if for every $x, y \in \mathcal{I}$, there exists $z \in \mathcal{I}$ such that $x \leq z$ and $y \leq z$, and there is a similar criterion for downward directed.

2.1.0.4 The pre-ordered sets in Examples (1),(2),(3), and (5) of (2.1.0.2) are all upward filtering. As a simple example of a partially ordered set which is not upward filtering, consider a set X which contains at least two elements. Define a partial order $\leq$ on this set by $x \leq y$ if and only if $x = y$. Then $(X, \leq)$ is not upward filtering. As another example, consider the set $\mathbb{Z}^2$ where $(a, b) \leq (x, y)$ if and only if $a \leq x$ (as integers) and $b = y$. Then $(0, 0)$ and $(0, 1)$ are not comparable, and anything comparable to $(0, 0)$ is not comparable to $(0, 1)$. It follows that $\leq$ is not upward filtering on $\mathbb{Z}^2$.

2.2 Topology

2.2.1 Nets and Filters

2.2.1.1 Let X be a set, $\mathfrak{P}(X)$ the collection of subsets of X. If $\mathfrak{F} \subseteq \mathfrak{P}(X)$, we say that $\mathfrak{F}$ is a **filter in X** (or just a **filter**) if

1. $A \in \mathfrak{F}, B \in \mathfrak{P}(X), A \subseteq B \Rightarrow B \in \mathfrak{F}$
2. $A, B \in \mathfrak{F} \Rightarrow A \cap B \in \mathfrak{F}$
3. $\emptyset \notin \mathfrak{F}$

It is clear that any finite intersection of sets in $\mathfrak{F}$ must be nonempty.

2.2.1.2 Examples of filters:

1. Let X be a topological space, and $x \in X$. Then the collection $\mathcal{O}(x)$ of all neighbourhoods of x is a filter in X.
2. Let X be a non-empty set and $Y \subseteq X$. The collection of all subsets of X which contain Y is a filter in X.
3. If X is an infinite set, let $\mathfrak{F}$ be the family of all subsets of X whose complement is finite. Then $\mathfrak{F}$ is a filter in X. If $X = \mathbb{Z}^+$ then we call $\mathfrak{F}$ the **Fréchet filter**.

2.2.1.3 Let $\mathfrak{F}$ and $\mathfrak{G}$ be filters on a set X. We say that $\mathfrak{F}$ is **finer** that $\mathfrak{G}$ if $\mathfrak{G} \subseteq \mathfrak{F}$. If also $\mathfrak{G} \neq \mathfrak{F}$ we say that $\mathfrak{F}$ is strictly finer than $\mathfrak{G}$.

2.2.1.4 If $\mathfrak{F}$ and $\mathfrak{G}$ are filters on a set X, we say that $\mathfrak{F}$ and $\mathfrak{G}$ are **comparable** if one is finer than the other. The family of all filters on X is a partially ordered set with respect to the order induced on $\mathfrak{P}(\mathfrak{P}(X))$ by $\subseteq$. Hence we say that $\mathfrak{G}$ is smaller than $\mathfrak{F}$ if $\mathfrak{G} \subseteq \mathfrak{F}$.

2.2.1.5 Proposition (**[BkT, chap 1 §6 n°3 prop 1]**) *Let $\mathfrak{G}$ be some collection of subsets of X. There is a filter $\mathfrak{F}$ on X containing $\mathfrak{G}$ if and only if no finite intersection of elements of $\mathfrak{G}$ is empty.*

It is clear that the intersection of any two filters containing $\mathfrak{G}$ again contains $\mathfrak{G}$. So if there is a filter containing $\mathfrak{G}$, there is a smallest filter containing $\mathfrak{G}$. This filter is called the filter generated by $\mathfrak{G}$.

2.2.1.6 Proposition ([BkT, chap 1 §6 n°4 prop 2]) *Let $\mathfrak{B}$ be a collection of subsets of X. Then $\mathfrak{F} = \{Y \subseteq X; \exists A \in \mathfrak{B} \text{with } A \subseteq Y\}$ is a filter in X if and only if*

1. *The intersection of any two elements of $\mathfrak{B}$ contains an element of $\mathfrak{B}$.*

2. *$\mathfrak{B}$ is not empty, and the empty set is not in $\mathfrak{B}$.*

2.2.1.7 We say that a collection of subsets $\mathfrak{B}$ of X that satisfies the conditions of Proposition 2.2.1.6 is a **filter basis** for the filter $\mathfrak{F} = \{Y \subseteq X; \exists A \in \mathfrak{B}\text{with } A \subseteq Y\}$, or that it is a **basis** for the filter $\mathfrak{F}$. Noting that any filter containing $\mathfrak{B}$ must also contain $\mathfrak{F}$, we see that $\mathfrak{F}$ is the filter generated by $\mathfrak{B}$. We call two filter bases **equivalent** if they generate the same filter.

2.2.1.8 ([BkT, chap 1 §6 n°3]) As an example, let $\mathcal{I}$ be a nonempty set with an upward filtering pre-order $\leq$. For any $j \in \mathcal{I}$, we define the **section** of $\mathcal{I}$ with respect to j to be the set $S(j) = \{i \in \mathcal{I}; j \leq i\}$. The collection $\mathfrak{B}$ of all sections of $\mathcal{I}$ is a filter basis. The filter generated by $\mathfrak{B}$ is called the **filter of sections** of the set $\mathcal{I}$ with respect to $\leq$. As particular cases we have

1. The Fréchet filter is the filter of sections on $(\mathbb{Z}^+, \leq)$.

2. Let $\mathfrak{F}$ be a filter in a set X. For $A, B \in \mathfrak{F}$, say that $A \leq B$ if $B \subseteq A$. Then $\mathfrak{F}$ is an upward filtering partially ordered set with respect to $\leq$. So we can define on $\mathfrak{F}$ the filter of sections, where in this case a section with respect to a set $A \in \mathfrak{F}$ is the collection $S(A) = \{B \in \mathfrak{F}; B \subseteq A\}$. This filter is called the **filter of sections of the filter** $\mathfrak{F}$.

2.2.1.9 If $\mathfrak{B}$ is a filter basis on X and if f is a map from X into Y, then $f(\mathfrak{B})$ is a filter basis on Y. If $\mathfrak{B}$ and $\mathfrak{B}'$ are filter bases on X and the filter generated by $\mathfrak{B}$ finer than the filter generated by $\mathfrak{B}'$, then the filter generated by $f(\mathfrak{B})$ is finer than the filter generated by $f(\mathfrak{B}')$.

2.2.1.10 Definition *Let X be a topological space, $\mathfrak{F}$ a filter in X, and $y \in X$. We say that $\mathfrak{F}$* **converges** *to y if $\mathfrak{F}$ is finer than the filter $\mathcal{O}(y)$ of neighbourhoods of y in X. If $\mathfrak{F}$ converges to y, we say that y is a* **limit point of** $\mathfrak{F}$ *(or just a* **limit of** $\mathfrak{F}$*). If $\mathfrak{B}$ is a filter basis then we say that x is a* **limit of** $\mathfrak{B}$*, or that $\mathfrak{B}$* **converges to** x*; if x is a limit of the filter generated by $\mathfrak{B}$. We say that x is an* **accumulation point** *of $\mathfrak{B}$ if x is an accumulation point of every element of $\mathfrak{B}$.*

If a filter $\mathfrak{F}$ converges to x so does any filter finer than $\mathfrak{F}$.

2.2.1.11 Let f be a function from a set X into a topological space Y, and let $\mathfrak{F}$ be a filter in X. We say that a point $y \in Y$ is a **limit point** (or just a **limit**) **of** f **with respect to** $\mathfrak{F}$ if the filter basis $f(\mathfrak{B})$ converges to y. In this case we write $\lim_{\mathfrak{F}} f = y$, or $\lim_{x,\mathfrak{F}} f(x) = y$, or even $\lim_x f(x) = y$ if no confusion will result.

2.2.1.12 Proposition ([BkT, Chapter III §7 n°3 prop 7]) *For $y \in Y$ to be a limit for f with respect to $\mathfrak{F}$, it is necessary and sufficient that for each neighbourhood V of y there is an $M \in \mathfrak{F}$ such that $f(M) \subseteq V$ (i.e. that $f^{-1}(V) \in \mathfrak{F}$ for all neighbourhood N of y).*

2.2.1.13 Note the similarity between this definition and the usual definition of convergence in topology. In particular, if X and Y are topological spaces and $\mathfrak{F} = \mathcal{O}(x)$ then Proposition 2.2.1.12 shows that $\lim_{\mathcal{O}(x)} f = y$ if and only if f is continuous at x and $f(x) = y$.

2.2.1.14 ([BkT, Chapter 1 §7 n°3 Example 2]) Let f be a function from an upwards filtering pre-ordered set $\mathcal{I}$ into a topological space X. If $x \in X$ is a limit point of f with respect to the filter of sections of $\mathcal{I}$ (see (2.2.1.8)), then we say that x is a **limit of** f **with respect to the ordered set** $\mathcal{I}$, and we write $x = \lim_{i \in \mathcal{I}} f(i)$. Similarily, if $x \in X$ is an accumulation point of f with respect to the filter of sections of $\mathcal{I}$, then we say that x is an **accumulation point of** f **with respect to the ordered set** $\mathcal{I}$.

2.2.1.15 As a familiar example of (2.2.1.14), consider the upward filtering set $\mathbb{N}$, and f a function from $\mathbb{N}$ into $\mathbb{R}$. Then we have the familiar case of a real sequence,

$x_n = f(n)$ for $n \in \mathbb{N}$. In this case, the limit of f with respect to the ordered set $\mathbb{N}$ is just the usual limit of the sequence (x_n), and a point in $\mathbb{R}$ is an accumulation point of f with respect to the ordered set $\mathbb{N}$ iff it is an accumulation point of the set $\{x_n; n \in \mathbb{N}\}$.

In particular, the pair $(\mathcal{I}, f)$ in (2.2.1.14) can be thought of as a sort of generalized sequence. Indeed, a pair $(\mathcal{I}, f)$ where $\mathcal{I}$ is an upward filtering pre-ordered set and f is a function from $\mathcal{I}$ into a set X is called a **net**, or a **generalized sequence**. See Section 2.2.2 for more details on nets and their relation to filters.

2.2.1.16 (**[BkT, Chapter IV §5 n°1]**) Let X be a set. We call a function $f : X \to \overline{\mathbb{R}} = \mathbb{R} \cup \{\infty\} \cup \{-\infty\}$ a **real-valued** function, or we say that f is **real-valued**. If f and g are two real valued functions we say that $f \leq g$ if $f(x) \leq g(x)$ for all $x \in X$. This gives a partial order relation on the set of real valued functions on X.

2.2.1.17 We say that a real-valued function on an upward filtering pre-ordered set $\mathcal{I}$ is

1. **increasing** if $j \leq i \Rightarrow f(j) \leq f(i)$
2. **decreasing** if $j \leq i \Rightarrow f(j) \geq f(i)$

If f is either increasing or decreasing we say that f is **monotone**.

2.2.1.18 Proposition *Let f and g be two real-valued functions on a set X, and let $\mathfrak{F}$ be a filter in X. If $\lim_{\mathfrak{F}} f$ and $\lim_{\mathfrak{F}} g$ exist and if, for every $A \in \mathfrak{F}$ there is an $x \in A$ such that $f(x) \leq g(x)$, then $\lim_{\mathfrak{F}} f \leq \lim_{\mathfrak{F}} g$.*

2.2.1.19 Theorem (**[BkT, Chapter IV §5 n°2 Theorem 1]**) *Let f and g be two real-valued functions on a set X, and let $\mathfrak{F}$ be a filter in X. If $\lim_{\mathfrak{F}} f$ and $\lim_{\mathfrak{F}} g$ exist and if $f \leq g$ then $\lim_{\mathfrak{F}} f \leq \lim_{\mathfrak{F}} g$.*

2.2.1.20 Theorem (**[BkT, Chapter IV §5 n°2 Theorem 2]**) *Let X be a partially ordered set, $\mathcal{I}$ an upwards filtering subset of X. Every monotone real valued function on $\mathcal{I}$ has a limit with respect to the filter of sections of $\mathcal{I}$. If f is increasing, this limit is the same as the least upper bound of the set $f(\mathcal{I}) \subseteq \overline{\mathbb{R}}$. If f is decreasing, this limit is the same as the greatest lower bound of the set $f(\mathcal{I}) \subseteq \overline{\mathbb{R}}$.*

The Limit Superior of a Real-valued Function with respect to a Filter

2.2.1.21 **([BkT, Chapter IV §7 n°4])** Let f be a real-valued function on X, and let $\mathfrak{F}$ be a filter in X. Then $\mathfrak{F}$ is upward filtering with respect to $\supseteq$ (see (2.2.1.8 n°2)). We define a function s from $\mathfrak{F}$ to $\overline{\mathbb{R}}$ by

$$s(A) = \sup_{x \in A} f(x), \text{ for all } A \in \mathfrak{F}.$$

The function s is decreasing with respect to $\mathfrak{F}$, since $A \subseteq B$ implies that $\sup_{x\in A} f(x) \leq \sup_{x \in B} f(x)$. It follows from Theorem 2.2.1.20 that s has a limit with respect to $\mathfrak{F}$.

2.2.1.22 **([BkT, IV §5 n°6])** We define the **limit superior of f with respect to $\mathfrak{F}$**, denoted $\limsup_{\mathfrak{F}} f$, to be the limit of the real-valued function $A \mapsto \sup_{x \in A} f(x)$ with respect to $\mathfrak{F}$. That is,

$$\limsup_{\mathfrak{F}} f = \lim_{A \in \mathfrak{F}} (\sup_{x \in A} f(x)).$$

Since the function $A \mapsto \sup_{x\in A} f(x)$ is decreasing with respect to $\mathfrak{F}$ then

$$\limsup_{\mathfrak{F}} f = \inf_{A \in \mathfrak{F}} \left(\sup_{x \in A} f(x) \right). \tag{2.2.1}$$

In particular, if $\mathfrak{F}$ is the filter of sections of an upward filtering ordered set $\mathcal{I}$ then (2.2.1) is equivalent to

$$\limsup_{\mathfrak{F}} f = \inf_{j \in \mathcal{I}} \left(\sup_{i \geq j} f(i) \right). \tag{2.2.2}$$

2.2.1.23 Proposition ([BkT, IV §5 n°6]) *The limit superior with respect to $\mathfrak{F}$ has the following properties:*

1. $f \leq g \Rightarrow \limsup_{\mathfrak{F}} f \leq \limsup_{\mathfrak{F}} g$
2. $c \in \mathbb{R}, c \geq 0 \Rightarrow \limsup_{\mathfrak{F}} c \cdot f = c \cdot \limsup_{\mathfrak{F}} f$
3. $\limsup_{\mathfrak{F}} (f + g) \leq \limsup_{\mathfrak{F}} f + \limsup_{\mathfrak{F}} g$

whenever both sides of the inequalities are defined.

2.2.2 Nets

In much the same way that filters are a generalizattion of neighbourhoods, nets are a generalization of sequences.

2.2.2.1 A **net** in a set X is a pair $(\mathcal{I}, \iota)$, where $\mathcal{I}$ is an upward filtering pre-ordered set and ι is a map from $\mathcal{I}$ into X. Typically we will write a net as $(x_i)_{i\in\mathcal{I}}$ where $x_i = \iota(i)$. The most important example of a net occurs when $\mathcal{I} = \mathbb{N}$, when we have a **sequence** in X. It is of course the usual notation for sequences that motivates the notation $(x_i)_{i\in\mathcal{I}}$. Indeed, nets are sometimes called **generalized sequences.**

2.2.2.2 (**[P2, 1.3.2]**) A **subnet** of a net $(\mathcal{I}, \iota)$ in X is a pair $(\mathcal{J}, \kappa)$ in X together with a map $h : \mathcal{J} \to \mathcal{I}$, such that $\kappa = \iota \circ h$, and such that for each $i \in \mathcal{I}$ there is a $j(i) \in \mathcal{J}$ such that $i \leq h(j)$ for every $j \geq j(i)$. In most cases we may choose h to be monotone [i.e. $j \leq k$ in $\mathcal{J}$ implies $h(j) \leq h(k)$ in $\mathcal{I}$], and then, in order to have a subnet it suffices to check that for each $i \in \mathcal{I}$ there is a $j \in \mathcal{J}$ with $i \leq h(j)$.

The definition of a subnet may sound intricate, but try to formulate a strictly correct definition of a subsequence!

2.2.2.3 Let $(x_i)_{i\in\mathcal{I}}$ be a net in a topological space X. Then $\mathcal{I}$ is an upward filtering pre-ordered set, and we know that the filter of sections (cf. 2.2.1.8) is a natural filter in such a set. If we denote by $\mathfrak{B}$ the collection of sections of $\mathcal{I}$, then $\iota(\mathfrak{B})$ is a filter basis in X. We say that $(x_i)_{i\in\mathcal{I}}$ **converges** to x if $\iota(\mathfrak{B})$ converges to x. We write this as $x = \lim_{i\in\mathcal{I}} x_i$, $x = \lim x_i$, or simply as $x_i \to x$. Also, a point x is an **accumulation point** of $(x_i)_{i\in\mathcal{I}}$ iff it is an accumulation point of $\iota(\mathfrak{B})$.

Conversely, let $\mathfrak{F}$ be a filter in X which converges to x. Let

$$\mathcal{I} = \{(x, A) \in X \times \mathfrak{F}\, ; x \in A\},$$

and define $(x, A) \leq (y, B)$ if $B \subseteq A$. Then $\leq$ is an upward filtering pre-order on $\mathcal{I}$. Let ι the map from $\mathcal{I}$ into X given by $\iota(i) = \iota(x, A) = x$. Denote $x_i = \iota(i)$. Then $(x_i)_{i\in\mathcal{I}}$ is a net in X which converges to x, and the accumulation points of $\mathfrak{F}$ and $(x_i)_{i\in\mathcal{I}}$ are the same.

So there is a natural correspondence between filters in X converging to x, and nets in X converging to x. However, we now formulate the concepts of convergence and accumulation points of nets more explicitly.

2.2.2.4 We say that a net $(\mathcal{I}, \iota)$ in a set X is **eventually** in a subset A of X, if there is an $i(A) \in \mathcal{I}$ such that $x_i \in A$ for every $i \geq i(A)$. We say that the net is **frequently** in A if for each $j \in \mathcal{I}$ there is an $i \geq j$ with $x_i \in A$. Note that (x_i) is not frequently in A iff (x_i) is eventually in $X \setminus A$.

A net $(\mathcal{I}, \iota)$ converges to a point x iff it is eventually in each A in $\mathcal{O}(x)$. A point $x \in X$ is an accumulation point for a net $(\mathcal{I}, \iota)$ if the net is frequently in every A in $\mathcal{O}(x)$.

Note that x is an accumulation point of a net if some subnet converges to x, and Proposition 2.2.2.6 gives the converse.

2.2.2.5 ([P2, 1.3.4]) Let $\mathfrak{B}$ be a system of subsets of X that is upward filtering under reverse inclusion. If a net $(\mathcal{I}, \iota)$ is frequently in every set B in $\mathfrak{B}$, there is a subnet $(\mathcal{J}, \kappa)$ that is eventually in every B in $\mathfrak{B}$.

2.2.2.6 Proposition ([P2, 1.3.5]) *For every accumulation point x of a net in a topological space there is a subnet that converges to x.*

2.2.2.7 Proposition ([P2, 1.3.6]) *A point x in a topological space belongs to the closure of a set Y iff there is a net in Y converging to x.*

2.2.2.8 The above propositions show that a topology is determined by the family of convergent nets on the space.

2.3 Direct Limits

In the following sections we discuss the inductive limit, which can be considered as an extension of the process of forming unions. It might help intuitively to think of the example $\mathcal{I} = \mathbb{N}$, but in general $\mathcal{I}$ need not be totally ordered.

2.3.0.1 Recall that the **disjoint union** of the collection of sets $(\mathfrak{A}_i)_{i\in\mathcal{I}}$ is the set

$$\coprod_{i\in\mathcal{I}} \mathfrak{A}_i = \{(a,i); a \in \mathfrak{A}_i, i \in \mathcal{I}\}.$$

For each i we have the canonical inclusion maps $\iota_i : \mathfrak{A}_i \to \coprod_{i\in\mathcal{I}} \mathfrak{A}_i$ given by $\iota_i(a) = (a,i)$ for any $a \in \mathfrak{A}_i, i \in \mathcal{I}$. Typically we will drop the i, and will refer simply to the map ι. It is clear that

$$\iota(\mathfrak{A}_i) \cap \iota(\mathfrak{A}_j) = \emptyset \text{ whenever } i \neq j$$

and that

$$\coprod_{i\in\mathcal{I}} \mathfrak{A}_i = \bigcup_{i\in\mathcal{I}} \iota(\mathfrak{A}_i).$$

For each $i \in \mathcal{I}$, the map ι is a bijection between $\mathfrak{A}_i$ and $\iota(\mathfrak{A}_i)$. By abuse of notation, we identify $\mathfrak{A}_i$ with the subset $\iota(\mathfrak{A}_i) \subset \coprod_{i\in\mathcal{I}} \mathfrak{A}_i$. For each $i \in \mathcal{I}$ we can define the map $\lambda_i : \iota(\mathfrak{A}_i) \subset \coprod_{i\in\mathcal{I}} \mathfrak{A}_i \to \mathcal{I}$ by $\lambda((a_i, i)) = i$. Since the $\iota(\mathfrak{A}_i)$ partition $\coprod_{i\in\mathcal{I}} \mathfrak{A}_i$, we also have a unique map $\lambda : \coprod_{i\in\mathcal{I}} \mathfrak{A}_i \to \mathcal{I}$ such that $\lambda|_{\iota(\mathfrak{A}_i)} = \lambda_i$. In particular, for any $x \in \coprod_{i\in\mathcal{I}} \mathfrak{A}_i$, $\lambda(x) = i$ if and only if i is the unique element of $\mathcal{I}$ for which there is some $a \in \mathfrak{A}_i$ with $\iota(a) = x$.

2.3.0.2 Definition ([L, Chapter III §9]) *Let C be a category, $\{\mathfrak{A}_i\}_{i\in\mathcal{I}}$ a collection of objects of C, $\{\Phi_{ij}\}_{i\in\mathcal{I}}$ a collection of morphisms of C, and $\mathcal{I}$ an upward filtering index set. If*

1. *for every $i, j \in \mathcal{I}$ with $i \geq j$ there is a map $\Phi_{ij} : \mathfrak{A}_j \to \mathfrak{A}_i$,*
2. *$\Phi_{ij} = \Phi_{ik} \circ \Phi_{kj}$ whenever $j < k < i$,*
3. *Φ_{ii} is the identity map on $\mathfrak{A}_i$,*

then we call $\{\mathfrak{A}_i, \Phi_{ij}\}$ **a directed system in the category** *C, or just a directed system, and we refer to the directed system $\{\mathfrak{A}_i, \Phi_{ij}\}$.*

2.3.0.3 Definition ([L, Chapter III §9]) *Let $\{\mathfrak{A}_i, \Phi_{ij}\}$ be a directed system in the category C. A* **direct limit for** $\{\mathfrak{A}_i, \Phi_{ij}\}$ **in the category** *C is an object $\mathfrak{A}$ in*

C together with morphisms $\Phi_i : \mathfrak{A}_i \to \mathfrak{A}$ such that $\Phi_j = \Phi_i \circ \Phi_{ij}$ for all $i \leq j$ which satisfy the following universal property: Let $\mathfrak{N}$ be an object in C and assume that to each $i \in \mathcal{I}$ there is a morphism Ψ_i such that $\Psi_i \circ \Phi_{ij} = \Psi_j$ for every $j < i$. Then there is exactly one morphism $\Xi : \mathfrak{A} \to N$ that leaves the diagram

$$\begin{tikzcd} \mathfrak{A}_j \arrow[r, "\Phi_j"] \arrow[d, "\Phi_{ij}"'] \arrow[dr, "\Psi_j"] & \mathfrak{A} \arrow[d, dotted, "\Xi"] \\ \mathfrak{A}_i \arrow[ur, "\Phi_i"] \arrow[r, "\Psi_i"'] & \mathfrak{N} \end{tikzcd}$$

commutative. When no confusion will result (i.e. when the Φ_{ij} are unambiguous) we will sometimes write $\mathfrak{A} = \varinjlim\{\mathfrak{A}_i\}$.

2.3.0.4 It is a useful fact that direct limits exist and are unique for many useful categories, including sets, algebraic structures, normed algebras, and C*-algebras. We give some explicit constructions below.

2.3.1 Direct Limits of Sets

2.3.1.1 Theorem *Let $\{\mathfrak{A}_i, \Phi_{ij}\}$ be a directed system. Then there exists a set $\mathfrak{A}$ and maps Φ_i such that*

1. *the diagram*

$$\begin{tikzcd} \mathfrak{A}_j \arrow[r, "\Phi_j"] \arrow[d, "\Phi_{ij}"'] & \mathfrak{A} \\ \mathfrak{A}_i \arrow[ur, "\Phi_i"'] & \end{tikzcd}$$

   commutes whenever $j \leq i$ and

2. *$\mathfrak{A} = \cup_{\mathcal{I}} \Phi_i(\mathfrak{A}_i)$*

Proof: Given a directed system $\{\mathfrak{A}_i, \Phi_{ij}\}$, let $\mathfrak{D} = \coprod_{i \in \mathcal{I}} \mathfrak{A}_i$. Denote by $\sim$ the equivalence relation on $\mathfrak{D}$ given by

$$(a_i, i) \sim (a_j, j) \iff \exists k \in \mathcal{I} \text{ with } i, j \leq k \text{ such that } \Phi_{ki}(a_i) = \Phi_{kj}(a_j)$$

and let $\mathfrak{A} = \mathfrak{D}/\sim$ be the quotient set. For $x \in \mathfrak{D}$, let $[x]$ denote its equivalence class with respect to $\sim$ and $\pi : \mathfrak{D} \to \mathfrak{A}$ be the quotient map.

It is clear that $\mathfrak{A}$ is non-empty if at least one of the $\mathfrak{A}_i$ is non-empty. For each $\mathfrak{A}_i$ we have a function $\Phi_i : \mathfrak{A}_i \to \mathfrak{A}$ given by

$$\Phi_i(a_i) = (\pi \circ \iota)(a_i) = [(a_i, i)],$$

and from the definition of $\sim$ we have

$$\Phi_j = \Phi_i \circ \Phi_{ij}, \text{ whenever } j \leq i.$$

Since $\coprod_{i \in \mathcal{I}} \mathfrak{A}_i = \bigcup_{i \in \mathcal{I}} \iota(\mathfrak{A}_i)$, and $\mathfrak{A} = \pi\left(\coprod_{i \in \mathcal{I}} \mathfrak{A}_i\right)$ it follows that

$$\mathfrak{A} = \bigcup_{i \in \mathcal{I}} \Phi_i(\mathfrak{A}_i).$$

Hence the result. ■

2.3.1.2 It is straightforward to check that $\{\mathfrak{A}, \{\Phi_i\}\}$ is a direct limit for $\{\mathfrak{A}_i, \Phi_{ij}\}$in the category of sets and mappings, and that any other direct limit for $\{\mathfrak{A}_i, \Phi_{ij}\}$is isomorphic to $\{\mathfrak{A}, \{\Phi_i\}\}$.

2.3.1.3 Lemma ([BkE, Chapter 3 §7 n°5 Lemma 1]) *Suppose that we have a directed system $\{\mathfrak{A}_i, \Phi_{ij}\}$ in the category of sets, and suppose that $\mathfrak{A} = \varinjlim \{\mathfrak{A}_i, \Phi_{ij}\}$ together with maps $\Phi_i : \mathfrak{A}_i \to \mathfrak{A}$ is its direct limit.*

1. *Let $\{x^{(k)}\}_{k=1}^n$ be a finite subset of $\mathfrak{A}$. Then there is an $i \in \mathcal{I}$ and a set $\{x_i^{(k)}\}_{k=1}^n \subset \mathfrak{A}_i$ such that $x^{(k)} = \Phi_i(x_i^{(k)})$ for every $1 \leq k \leq n$.*

2. *Let $\{x^{(k)}\}_{k=1}^n$ be a finite subset of $\mathfrak{A}_i$. If $\Phi_i(x^{(k)}) = \Phi_i(x^{(l)})$ for every pair $1 \leq k, l \leq n$ then there exists $i \leq j \in \mathcal{I}$ such that $\Phi_{ji}(x^{(k)}) = \Phi_{ji}(x^{(l)})$ for every $1 \leq k, l \leq n$.*

2.3.1.4 Theorem ([BkE, Chapter 3 §7 n°6 Proposition 6]) *Let $\{\mathfrak{A}_i, \Phi_{ij}\}$ be a directed system in the category of sets, with direct limit $\mathfrak{A}$.*

1. *for the map Ξ of (2.3.0.3) to be surjective is neccessary and sufficient that $\mathfrak{N} = \cup_{i \in \mathcal{I}} \Psi_i(\mathfrak{A}_i)$.*

2. *a neccessary and sufficient condition for Ξ to be injective is that for any $j \in \mathcal{I}$ and $x, y \in \mathfrak{A}_j$ such that $\Psi_j(x) = \Psi_j(y)$ there is an $i \in \mathcal{I}$ with $i \geq j$ such that $\Phi_{ij}(x) = \Phi_{ij}(y)$.*

2.3.2 Direct Limits of Algebraic Structures

2.3.2.1 In this section we will be interested in the direct limit of a directed system $\{\mathfrak{A}_i, \Phi_{ij}\}$ in which each $\mathfrak{A}_i$ is an algebraic structure, all in the same category (note that algebraic structures (cf. [C] or [BS]) include groups, algebras, and *-algebras, among others). Similar constructions are possible for objects which are not necessarily algebraic structures (e.g. scaled dimension groups (cf. 2.6.3.9)). Since in our case we are interested in various flavours of groups and algebras (namely groups, algebras, and *-algebras), we can assume that all the $\mathfrak{A}_i$ are non-empty.

2.3.2.2 Theorem ([W, Appendix L]) *Let $\{\mathfrak{A}_i, \Phi_{ij}\}$ be a directed system, in which each $\mathfrak{A}_i$ is an algebraic stucture in the same category, and each of the Φ_{ij} are morphisms in the appropriate category. Then there exists an algebraic structure $\mathfrak{A}$ in the same category as all the $\mathfrak{A}_i$ and morphisms $\Phi_i : \mathfrak{A}_i \to \mathfrak{A}$ such that*

1. *the diagram*

$$\begin{tikzcd} \mathfrak{A}_j \arrow[r, "\Phi_j"] \arrow[d, "\Phi_{ij}"'] & \mathfrak{A} \\ \mathfrak{A}_i \arrow[ur, "\Phi_i"'] & \end{tikzcd}$$

   commutes whenever $j < i$ and

2. $\mathfrak{A} = \cup_{\mathcal{I}} \Phi_i(\mathfrak{A}_i)$

It is straightforward to check that $\{\mathfrak{A}, \{\Phi_i\}\}$ is a direct limit for $\{\mathfrak{A}_i, \Phi_{ij}\}$ in the appropriate category, and that any other direct limit for $\{\mathfrak{A}_i, \Phi_{ij}\}$is isomorphic to $\{\mathfrak{A}, \{\Phi_i\}\}$. We also have that the map Ξ of (2.3.0.3) is surjective if and only if $\mathfrak{N} = \cup_{i \in \mathcal{I}} \Psi_i(\mathfrak{A}_i)$, and Ξ is injective if and only if for any $j \in \mathcal{I}$ and $x, y \in \mathfrak{A}_j$ such that $\Psi_j(x) = \Psi_j(y)$ there is an $i \in \mathcal{I}$ with $i \geq j$ such that $\Phi_{ij}(x) = \Phi_{ij}(y)$.

2.3.3 Direct Limits of Normed and C*-Algebras

So far we have constructed a direct limit $\mathfrak{A}$ of a directed system $\{\mathfrak{A}_i, \Phi_{ij}\}$ in which the $\mathfrak{A}_i$ are all algebraic structures in the same category and the Φ_{ij} are morphisms in the appropriate category. We did this by introducing operations on the direct limit $\mathfrak{A}$ of $\{\mathfrak{A}_i, \Phi_{ij}\}$ in the category of sets. This all worked very well because the Φ_{ij} preserved all of the essential properties of the $\mathfrak{A}_i$. We now look at the direct limit of normed and C*-algebras. It is not satisfactory to require that the Φ_{ij} preserve the norm, since this would require the maps to be injective. Since in general the maps between normed algebras do not preserve the norm (although see (2.7.7.4)), we will require an additional condition on the directed system in order to perform a similar construction (although note that the underlying set is no longer the direct limit of $\{\mathfrak{A}_i, \Phi_{ij}\}$ in the category of sets).

2.3.3.1 ([Pa, p42]) If $\{\mathfrak{A}_i, \Phi_{ij}\}$ is a directed system such that each $\mathfrak{A}_i$ is a normed algebra, each Φ_{ij} is a continuous algebra homomorphism, and

$$\limsup_{i \geq j} \|\Phi_{ij}\| < \infty \text{ for every } j \in \mathcal{I}, \tag{2.3.3}$$

then we call $\{\mathfrak{A}_i, \Phi_{ij}\}$ a **normed inductive system of normed algebras.**

2.3.3.2 Theorem ([Pa, p42]) *Let $\{\mathfrak{A}_i, \Phi_{ij}\}$ be a normed inductive system of normed algebras. Then there exists a normed algebra $\mathfrak{A}$ and there exist continuous morphisms $\Phi_i : \mathfrak{A}_i \to \mathfrak{A}$ such that*

1. *the diagram*

$$\begin{tikzcd} \mathfrak{A}_j \arrow[r, "\Phi_j"] \arrow[d, "\Phi_{ij}"'] & \mathfrak{A} \\ \mathfrak{A}_i \arrow[ur, "\Phi_i"'] & \end{tikzcd}$$

   commutes whenever $j < i$ and

2. *$\mathfrak{A} = \cup_{\mathcal{I}} \Phi_i(\mathfrak{A}_i)$*

Furthermore, $\mathfrak{A}$ is the unique direct limit for $\{\mathfrak{A}_i, \Phi_{ij}\}$ in the category of normed algebras. The map Ξ is surjective if and only if $\mathfrak{N} = \cup_{i\in\mathcal{I}}\Psi_i(\mathfrak{A}_i)$, and Ξ is injective if and only if for any $x, y \in \mathfrak{A}_j$ such that $\Psi_j(x) = \Psi_j(y)$, and for any $\varepsilon > 0$, there is an $i \in \mathcal{I}$ with $i \geq j$ such that $\|\Phi_{ij}(x) - \Phi_{ij}(y)\| < \varepsilon$.

Proof: Let $\{\mathfrak{A}_i, \Phi_{ij}\}$ be a normed inductive system of normed algebras, and let $\{\mathfrak{A}_{\text{temp}}, \{\Psi_i\}\}$ be the direct limit of the $\{\mathfrak{A}_i, \Phi_{ij}\}$ in the category of *-algebras. We define the function $\mathfrak{n} : \mathfrak{A}_{\text{temp}} \to \mathbb{R}^+$ by

$$\mathfrak{n}(a) = \limsup_{i\geq j} \|\Phi_{ij}(a_i)\|, \text{ where } a_i \in \mathfrak{A}_i \text{ such that } \Psi_i(a_i) = a.$$

It is immediate that the definition of $\mathfrak{n}$ does not depend on the choice of a_i, and from the properties of lim sup in Proposition 2.2.1.23 we have that $\mathfrak{n}$ is a seminorm on $\mathfrak{A}_{\text{temp}}$. Let

$$N = \{a \in \mathfrak{A}_{\text{temp}}; \mathfrak{n}(a) = 0\}.$$

Then N is a two-sided ideal of $\mathfrak{A}_{\text{temp}}$, and $\|a + N\| = \mathfrak{n}(a)$ is a norm on $\mathfrak{A}_{\text{temp}}/N$. We define $\mathfrak{A} = \mathfrak{A}_{\text{temp}}/N$ with this norm, and we define maps $\Phi_i : \mathfrak{A}_i \to \mathfrak{A}_{\text{temp}}$ by letting $\Phi_i(a)$ be the equivalence class of a in $\mathfrak{A}$. With these definitions, $\mathfrak{A}$ satisfies (1) and (2). The remaining assertions are straightforward. ∎

2.3.3.3 When all the $\mathfrak{A}_i$ are Banach algebras, the uniform boundedness principle shows that Equation (2.3.3) of (2.3.3.1) is satisfied if and only if

$$\limsup_{i\geq j} \|\Phi_{ij}(a_j)\| < \infty \; \forall\, j \in \mathcal{I}, a_j \in \mathfrak{A}_j. \tag{2.3.4}$$

When the $\mathfrak{A}_i$ are C*-algebras and the Φ_{ij} are C*-algebra morphisms, then Equation (2.3.4) is always satisfied, by Theorem 2.7.7.4. It follows that any directed system of C*-algebras and C*-algebra morphisms is a normed inductive system of normed algebras, and hence has a direct limit $\mathfrak{A}_\infty$ in the category of normed algebras. It is easy to see also that $\mathfrak{A}_\infty$ has a *-algebra structure, and that the C^*-identity $\|a^*a\| = \|a\|^2$ carries over to the direct limit. Hence we can construct the (unique) **C*-algebraic direct limit** $\mathfrak{A}$ of the directed system $\{\mathfrak{A}_i, \Phi_{ij}\}$ to be the closure of the normed direct limit

$$\mathfrak{A} = \overline{\mathfrak{A}_\infty},$$

which is clearly a C*-algebra. The map Ξ of (2.3.0.3) is surjective if and only if $\mathfrak{N} = \overline{\cup_{i\in\mathcal{I}}\Psi_i(\mathfrak{A}_i)}$, and Ξ is injective if and only if for any $\varepsilon > 0$ and $x, y \in \mathfrak{A}_j$ such that $\Psi_j(x) = \Psi_j(y)$ there is an $i \in \mathcal{I}$ with $i \geq j$ such that $\|\Phi_{ij}(x) - \Phi_{ij}(y)\| < \varepsilon$.

2.3.4 Further properties of the direct limit

2.3.4.1 Direct limits can be a little tricky at times. It may happen that $\varinjlim M_i \neq M$ even if $M_i = M$ for all i. Consider for example the case where $M = \mathbb{M}_2 \oplus \mathbb{M}_2$ the Φ_{ij} are given by $a \oplus b \mapsto a$. Then $\varinjlim M_i = \mathbb{M}_2$.

2.3.4.2 The following extremely useful lemma is used in the proof of Elliott's Theorem (2.10.3.3). It follows from the defintion of the direct limit (in the appropriate category), and from the appropriate construction ((2.3.1.1), (2.3.2.2), or (2.3.3.2)).

2.3.4.3 Lemma ([W, L.1.5]) *Let $\{\mathfrak{A}_i, \Phi_{ij}\}_{i,j\in\mathbb{N}}$ be a directed system in the category of sets, algebraic structures, or C*-algebras. Let $\mathfrak{A}$ be the direct limit of $\{\mathfrak{A}_i, \Phi_{ij}\}_{i,j\in\mathbb{N}}$ and let $\mathfrak{B}$ be the direct limit of $\{\mathfrak{B}_i, \Psi_{ij}\}_{i,j\in\mathbb{N}}$. Assume also that there is a sequence of indices*

$$n_1 \leq m_1 < n_2 \leq m_2 < n_3 \leq m_3 < \dots$$

such that for each k there are morphisms $\alpha_k : \mathfrak{A}_{n_k} \to \mathfrak{B}_{m_k}$ and $\beta_k : \mathfrak{B}_{m_k} \to \mathfrak{A}_{n_{k+1}}$ such that the diagram

$$\begin{tikzcd}
\mathfrak{A}_{n_1} \arrow[r] \arrow[dr, "\alpha_1"'] & \cdots \arrow[r] & \mathfrak{A}_{n_2} \arrow[r] \arrow[dr, "\alpha_2"'] & \cdots \arrow[r] & \cdots \arrow[r] & \mathfrak{A} \\
\mathfrak{B}_{n_1} \arrow[r] & \mathfrak{B}_{m_1} \arrow[ur, "\beta_1"'] \arrow[r] & \cdots \arrow[r] & \mathfrak{B}_{m_2} \arrow[ur, "\beta_2"'] \arrow[r] & \cdots \arrow[r] & \mathfrak{B}
\end{tikzcd}$$

commutes. Then there is an isomorphism between $\mathfrak{A}$ and $\mathfrak{B}$ commuting with the diagram.

2.3.4.4 Suppose that $\{\mathfrak{A}_i, \Phi_{ij}\}$ is a directed system of algebraic structures, all in the same category. We have seen (2.3.2.2) that the direct limit $\mathfrak{A}$ can be thought of as $\cup_{i\in\mathcal{I}}\Phi_i(\mathfrak{A}_i)$. Since $\mathfrak{B}_i = \Phi_i(\mathfrak{A}_i)$ is an object in the same category as $\mathfrak{A}_i$, we have that $\mathfrak{A}$ is also the increasing union of the directed system $\{\mathfrak{B}_i, \{\iota_{ij}\}\}$, where ι_{ij} is the appropriate inclusion map. Therefore, we will sometimes write the direct limit $\mathfrak{A}$ as $\mathfrak{A} = \cup_{i\in\mathcal{I}}\mathfrak{B}_i$.

Similarly, if $\{\mathfrak{A}_i, \Phi_{ij}\}$ is a directed system of C*-algebras, we will sometimes write $\mathfrak{A} = \overline{\cup_{i\in\mathcal{I}}\mathfrak{B}_i}$ (using (2.3.3.3) and (2.3.3.2)).

2.4 Bratteli Diagrams

2.4.0.1 Definition *A* **Bratteli Diagram** (V, E) *consists of a vertex set* V *and an edge set* E, *where the sets* V *and* E *can be written as a countable disjoint union of non-empty finite sets*

$$V = \coprod_{n=0}^{\infty} V_n, \qquad E = \coprod_{n=1}^{\infty} E_n$$

with the additional property that there exists a source function $s : E \to V$ *and a range function* $r : E \to V$ *such that*

1. $s(E_n) \subset V_{n-1}, r(E_n) \subset V_n$
2. $s^{-1}(v) \neq \emptyset, \forall\, v \in V$
3. $r^{-1}(v) \neq \emptyset, \forall\, v \in V \setminus V_0$

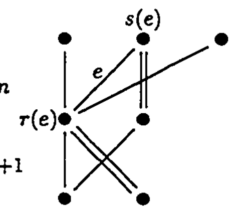
2.4.0.2 There is a natural graphical representation of a Bratteli diagram, obtained by identifying the Bratteli diagram with a subset of the plane. Indeed, let $|V_n| = m_n$. Then we identify the vertices $v_k^n, 1 \leq k \leq m_n$, in V_n with the points $(k, -n), \dots, (m_n, -n)$. We identify E_n with edges between V_{n-1} and V_n. In particular, for each edge $e \in E_n$ draw a line between $s(e) \in V_{n-1}$ and $r(e) \in V_n$.

We can represent the set E_n conveniently as a matrix $(a_{ij}) \in \mathbb{M}_{m_n \times m_{n-1}}(\mathbb{Z}^+)$, called the incidence matrix, in which

$$a_{ij} = |\{e \in E_n; s(e) = (j, -(n-1)), r(e) = (i, -n)\}|$$

is the number of edges joining v_j^{n-1} with v_i^n.

Level	Incidence Matrix		
$n-1$			V_{n-1}
	$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 0 \end{pmatrix}$	E_n	
n			V_n
	$\begin{pmatrix} 1 & 1 \\ 2 & 0 \end{pmatrix}$	E_{n+1}	
$n+1$			V_{n+1}



2.4.0.3 Let $k, l \in \mathbb{Z}^+, k < l$. We denote by $\boldsymbol{E_{k+1} \circ E_{k+2} \circ \ldots \circ E_l}$ the set of all paths from V_k to V_l, i.e. $E_{k+1} \circ \ldots \circ E_l$ is the set

$$\left\{ \begin{array}{ll} (e_i)_{i=k+1}^{l} \quad ; & e_i \in E_i \text{ when } k+1 \leq i \leq l, \\ & \text{and } r(e_i) = s(e_{i+1}) \text{ when } k+1 \leq i \leq l-1 \end{array} \right\}$$

Let $v_k \in V_k$, $v_l \in V_l$. A **path** from v_k to v_l is a sequence $(e_i)_{i=k+1}^{l} \in E_{k+1} \circ E_{k+2} \circ \ldots \circ E_l$ such that $s(e_{k+1}) = v_k$ and $r(e_l) = v_l$. We say that v_l is an **image** of v_k if there is a path from v_k to v_l.

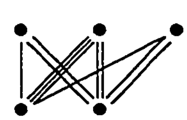
2.4.0.4 Given a Bratteli diagram (V, E) and a sequence

$$0 = m_0 < m_1 < m_2 < \ldots$$

we define the **telescoping** of (V, E) to (m_n) as (V', E'), where $V'_n = V_{m_n}$

For example, if we remove level n of the above diagram we get a telescoping to levels $n-1$ and $n+1$. The new diagram is shown below. Note that the new incidence matrix is the product of the two incidence matrices from the above diagram.

$\begin{pmatrix} 1 & 3 & 1 \\ 2 & 2 & 2 \end{pmatrix}$ $E_n \circ E_{n+1}$



V_{n-1}

V_n

2.4.0.5 There is a natural notion of isomorphism between Bratteli diagrams, indeed the Bratteli diagrams (V, E) and (V', E') are said to be **isomorphic** if there is a map $\phi : (V, E) \to (V', E')$ such that:

1. ϕ is a bijection between V_n and V'_n, for all $n \in \mathbb{N}$,

2. ϕ is a bijection between E_n and E'_n, for all $n \in \mathbb{N}$, such that $\phi(s(e)) = s(\phi(e))$ and $\phi(r(e)) = r(\phi(e))$, for all $e \in E$.

2.4.0.6 ([GPS, p69]) We let $\sim$ denote the equivalence relation on Bratteli diagrams generated by isomorphism and telescoping. It is not hard to show that $(V^1, E^1) \sim (V^2, E^2)$ if and only if there exists a Bratteli diagram (V, E) so that telescoping to odd levels $0 < 1 < 3 < \ldots$ yields a telescoping of either (V^1, E^1) or (V^2, E^2), and telescoping to even levels $0 < 2 < 4 \ldots$ yields a telescoping of the other. We see in Section 2.8.4 how to assign an AF-algebra to a Bratteli diagram. The AF-algebra $AF(V, E)$ associated to (V, E) only depends on the equivalence class of (V, E). In fact, by Bratteli's fundamental paper [Br], Theorem 2.7, we have that $AF(V, E) \simeq AF(V', E')$ if and only if $(V, E) \sim (V', E')$. We will also see (cf. Section 2.10.2) how to associate a Bratteli diagram to a dimension group which we denote by $K_0(V, E)$—the notation being motivated by the connection to K-theory. Elliott showed that [El] $(V, E) \sim (V', E')$ if and only if $K_0(V, E)$ is order isomorphic to $K_0(V', E')$ by a map sending the distinguished order unit of $K_0(V, E)$ to the distinguished order unit of $K_0(V', E')$.

2.5 The Grothendieck Group

The construction of the Grothendieck group is a generalization of the following procedure.

2.5.0.1 Given the nonnegative integers $\mathbb{Z}^+$, there is a natural way of constructing the integers $\mathbb{Z}$.

$$\mathbb{Z} = \{(p, q); p, q \in \mathbb{Z}^+\},$$

where we say that $(p, q) = (r, s)$ if $(p + s) = (q + r)$. More precisely, we define an equivalence relation $\equiv$ on $\mathbb{Z}^+ \times \mathbb{Z}^+$ by $(p, q) \equiv (r, s)$ if $p + s = q + r$. We identify $\mathbb{Z}$ with $\mathbb{Z}^+ \times \mathbb{Z}^+ / \equiv$ by identifying $[(p, q)] \in \mathbb{Z}^+ \times \mathbb{Z}^+ / \equiv$ with $p - q \in \mathbb{Z}$.

2.5.0.2 The goal of the Grothendieck group construction is to construct, for an arbitrary commutative semigroup $\mathcal{H}$, the "smallest" abelian group $\mathfrak{G}(\mathcal{H})$ containing $\mathcal{H}$. The group $\mathfrak{G}(\mathcal{H})$ so constructed is called the Grothendieck group of $\mathcal{H}$, or the enveloping group of $\mathcal{H}$. Using this notation, we have in (2.5.0.1) that $\mathfrak{G}(\mathbb{Z}^+) = \mathbb{Z}$.

Recall that a semigroup $\mathcal{H}$ is said to satisfy **cancellation** if

$$x + z = y + z \Rightarrow x = y \quad \text{for any } x, y, z \in \mathcal{H}.$$

Note in particular that $\mathbb{Z}^+$ satisfies cancellation, and this is exactly what made the relation $\equiv$ of (2.5.0.1) transitive.

We now consider the process of constructing $\mathfrak{G}(\mathcal{H})$, where $\mathcal{H}$ can be any abelian semigroup (in particular $\mathcal{H}$ need not satisfy cancellation).

2.5.0.3 Let $\mathcal{H}$ be an abelian (additive) semigroup. Denote by $\equiv$ the equivalence relation given on $\mathcal{H} \times \mathcal{H}$ by

$$\begin{aligned}(h_1, k_1) \equiv &(h_2, k_2) \\ &\iff \exists\, y_1, y_2 \in \mathcal{H} \text{ such that } (h_1 + y_1, k_1 + y_1) = (h_2 + y_2, k_2 + y_2) \\ &\iff \exists\, x \in \mathcal{H} \text{ such that } h_1 + k_2 + x = h_2 + k_1 + x\end{aligned}$$

and denote by $[a]$ the equivalence class in $\mathcal{H} \times \mathcal{H}$ of the element $a \in \mathcal{H} \times \mathcal{H}$.

We define the **Grothendieck group** for $\mathcal{H}$ to be

$$\mathfrak{G}(\mathcal{H}) := \mathcal{H} \times \mathcal{H} / \equiv$$

We can define addition in $\mathfrak{G}(\mathcal{H})$ by

$$[(h_1, k_1)] + [(h_2, k_2)] = [(h_1 + h_2, k_1 + k_2)].$$

Let $x \in \mathcal{H}$. Then $I := [(x, x)] \in \mathfrak{G}(\mathcal{H})$ is an identity element for $\mathfrak{G}(\mathcal{H})$, and $\mathfrak{G}(\mathcal{H})$ is an abelian group with identity.

2.5.0.4 Thinking of an element $(h,k) \in \mathfrak{G}(\mathcal{H})$ as a "formal difference" $h-k$, it is natural to represent $h \in \mathcal{H}$ in $\mathfrak{G}(\mathcal{H})$ as $[(h,0)]$. Indeed, we have the following:

2.5.0.5 Proposition ([W, p296]) *The map $\iota : \mathcal{H} \to \mathfrak{G}(\mathcal{H})$ defined by $\iota(h) = [(h+k,k)]$ is a group homomorphism.*

1. *ι is injective $\iff$ $\mathcal{H}$ has cancellation.*
2. *Every element of $\mathfrak{G}(\mathcal{H})$ can be written in the form $\iota(h)-\iota(k)$, for some $h,k \in \mathcal{H}$.*

2.5.0.6 The above proposition tells us that the elements of $\mathfrak{G}(\mathcal{H})$ do behave as "differences" of elements in $\mathcal{H}$ with elements of $\mathcal{H}$. For example, it is now quite clear that

$$\begin{aligned} \mathfrak{G}(\mathbb{Z}^+) &= \mathbb{Z} \\ \mathfrak{G}(\mathbb{R}^+) &= \mathbb{R} \\ \mathfrak{G}(\mathbb{Q}^+) &= \mathbb{Q} \end{aligned}$$

Also, if we consider $<\mathbb{N},\cdot>$, the semigroup of strictly positive integers with multiplication, it is clear that $\mathfrak{G}(<\mathbb{N},\cdot>) = \mathbb{Q}^+ \setminus \{0\}$.

The following theorem gives us a universal property for $\mathfrak{G}(\mathcal{H})$.

2.5.0.7 Theorem ([W, p296]) *Let $\mathcal{H}$ be a commutative semigroup and $\mathcal{S}$ be a commutative semigroup with neutral element and $\phi : \mathcal{H} \to \mathcal{S}$ a homomorphism of semigroups that maps $\mathcal{H}$ into invertible elements in $\mathcal{S}$. Then ϕ extends uniquely to a homomorphism $\psi : \mathfrak{G}(\mathcal{H}) \to \mathcal{S}$ such that*

$$\begin{tikzcd} \mathcal{H} \arrow[r, "\phi"] \arrow[dr, "\iota"'] & \mathcal{S} \\ & \mathfrak{G}(\mathcal{H}) \arrow[u, dashed, "\psi"'] \end{tikzcd}$$

As a corollary to Theorem 2.5.0.7 we obtain a universal property for Grothendieck groups:

2.5.0.8 Corollary (**[W, p297]**) *Let $\mathcal{H}_1, \mathcal{H}_2$ be commutative semigroups with neutral elements. Let $\phi : \mathcal{H}_1 \to \mathcal{H}_2$ be a homomorphism of semigroups. Then there is a unique homomorphism of groups $\psi : \mathfrak{G}(\mathcal{H}_1) \to \mathfrak{G}(\mathcal{H}_2)$ that makes*

$$\begin{tikzcd} \mathcal{H}_1 \arrow[r, "\phi"] \arrow[d, "\iota"'] & \mathcal{H}_2 \arrow[d, "\iota"] \\ \mathfrak{G}(\mathcal{H}_1) \arrow[r, dashed, "\psi"'] & \mathfrak{G}(\mathcal{H}_2) \end{tikzcd}$$

commute.

This universal property will be instrumental in defining the group $K_0(\mathfrak{A})$ associated to a C*-algebra $\mathfrak{A}$.

2.5.0.9 Proposition (**[W, p297]**) *Assume that $\mathcal{H}$ has an element ∞ with the property that*

$$h + \infty = \infty \qquad \forall\, h \in \mathcal{H}.$$

Then $\mathfrak{G}(\mathcal{H}) = 0$.

Proposition 2.5.0.9 is used in (2.9.0.6) and (2.9.0.13) to calculate $K_0(\mathbb{B})$.

2.6 Partially Ordered Abelian Groups

2.6.1 Ordered Groups

When we discuss K-theory for AF algebras we will see that two AF algebras are isomorphic if and only if their K_0-groups are isomorphic as scaled dimension groups. We follow [Go] in describing partially ordered abelian groups and dimension groups.

2.6.1.1 Definition *A* **partially ordered abelian group** *is an abelian group G with a specified partial order $\leq$ which is translation-invariant, i.e. for any $x, y, z \in G$ we have*

$$x \leq y \Rightarrow x + z \leq y + z.$$

A **totally ordered abelian group** *is a partially ordered abelian group in which $\leq$ is a total order.*

2.6.1.2 Some examples:

1. The (additive) group $\mathbb{Z}$ with the usual partial order $x \leq y$ if and only if $0 \leq y-x$ is a partially ordered abelian group(in fact, a totally ordered abelian group). This will be refered to as the **usual ordering** on $\mathbb{Z}$. Similarily, the additive groups $\mathbb{Q}$ and $\mathbb{R}$ with the usual orderings $\leq$ are totally ordered abelian groups.

2. The additive group $\mathbb{C}$ is a partially ordered abelian group with $z \leq w$ if and only if $w - z \in \mathbb{R}$ and $w - z \geq 0$ in $\mathbb{R}$.

3. We can define another ordering on $\mathbb{Z}$ by setting $x \leq y$ if and only if $y - x$ is non-negative and even. Then $\mathbb{Z}$ is a partially ordered abelian group (but not a totally ordered abelian group).

4. Let $x = (x_1, \ldots, x_n), y = (y_1, \ldots, y_n) \in \mathbb{Z}^n$. Say that $x \leq y$ if and only if $x_i \leq y_i$ for every $i = 1, \ldots, n$, where x_i and y_i are compared using the usual ordering on $\mathbb{Z}$. Then $\mathbb{Z}^n$ is a partially ordered abelian group. The ordering $\leq$ on $\mathbb{Z}^n$ is called the **simplicial ordering** on $\mathbb{Z}^n$. Unless otherwise specified, whenever we refer to $\mathbb{Z}^n$ as a partially ordered abelian group we assume $\mathbb{Z}^n$ has the product ordering.

5. The simplicial ordering on $\mathbb{Z}^n$ is a special case of a more general construction: Suppose that $\{G_i\}$ is some family of partially ordered abelian groups, and consider the group $G = \prod_i G_i$. Then the product ordering on G is given by $(x_i) \leq (y_i)$ if and only if $x_i \leq y_i$ for all i (i.e. the elements of G are compared "pointwise"). Then $(G, \leq)$ is a partially ordered abelian group. This will be the usual ordering we give to products, and unless otherwise stated we assume that any product is given this ordering. Another ordering on the product $G = \prod_i G_i$ is the **strict ordering**, denoted $\lll$, given by $(x_i) \lll (y_i)$ if and only if

   $$x_i < y_i \text{ for all } i \text{ or } (x_i) = (y_i).$$

6. Let F be the set of functions from a set $\mathcal{I}$ into a partially ordered abelian group G. Then F is isomorphic with $\prod_{\mathcal{I}} G$ via the map which sends a function $f \in F$ to $(f(i))_{i \in \mathcal{I}} \in \prod_{\mathcal{I}} G$. Hence any ordering on G induces an ordering on F. Considering $\prod_{\mathcal{I}} G$ as a partially ordered abelian group with its usual product ordering (as in (5)), this isomorphism induces an ordering on F, called the **pointwise ordering**. The **strict ordering** on F is the order induced by the strict order on $\prod_{\mathcal{I}} G$.

2.6.1.3 A **cone** in an abelian group G is any subset C of G such that $0 \in C$ and C is closed under addition. A cone C is called a **strict cone** if 0 is the only element $x \in C$ for which $-x \in C$.

2.6.1.4 Let G be a partially ordered abelian group. Let $x \in G$. We say that x is a **positive element** of G if $0 \leq x$, and we say that x is a **strictly positive element** of G if $0 < x$.

We denote by G^+ the set of all positive elements of G. Then G^+ is a strict cone, called the positive cone of $(G, \leq)$ (or just the positive cone of G). Conversely, given a cone C in G we can define a relation $\leq_C$ on G by

$$x \leq_C y \iff x - y \in C \text{ (for any } x, y \in G).$$

The relation $\leq_C$ is a translation invariant pre-order on G, and $\leq_C$ is a partial order if and only if C is a strict cone. Furthermore, if $(G, \leq)$ is a partially ordered abelian group then $\leq_{G^+}$ is the same relation as $\leq$. So $\leq$ and G^+ contain the same information. It is often convenient to define the ordering on a partially ordered abelian group by specifying the positive cone.

2.6.1.5 Note that the convention that G^+ is the set of positive elements of G may conflict slightly with some common notation. In particular, when we speak about $\mathbb{R}^+$ we mean the set $\{x \in \mathbb{R}; x \geq 0\}$ of positive elements of $\mathbb{R}$. Similarily for $\mathbb{Z}^+$ and $\mathbb{Q}^+$.

2.6.1.6 For each of the examples in (2.6.1.2), the positive cone is as follows:

1. $\mathbb{Z}^+$ is the non-negative integers, $\mathbb{Z}^+ = \{0, 1, 2, \ldots\}$. Similarily for $\mathbb{Q}^+$ and $\mathbb{R}^+$.

2. $\mathbb{C}^+ = \mathbb{R}^+$.

3. With this ordering, the positive cone of $\mathbb{Z}$ is $2\mathbb{Z}^+$, the non-negative even integers.

4. $(\mathbb{Z}^n)^+ = \{(x_i); x_i \in \mathbb{Z}^+\}$.

5. In the pointwise ordering, an element (x_i) of $\prod_i G_i$ is positive if and only if $x_i \geq 0$ for each $i \in \mathcal{I}$. In the strict ordering, (x_i) is positive if and only if $x_i = 0$ for every $i \in \mathcal{I}$ or $x_i > 0$ for every $i \in \mathcal{I}$.

6. In the pointwise order, the functions whose values are positive. In the strict order, the functions whose values are strictly positive, as well as the 0 function.

2.6.1.7 Proposition ([Go, 1.3]) *Let H be a subgroup of a partially ordered abelian group G. Then the following are equivalent:*

1. *H is an upward directed partially ordered set (cf. 2.1.0.3),*

2. *H is a downward directed partially ordered set,*

3. *H is generated (as a subgroup of G) by a subgroup of G^+,*

4. *All elements of H have the form $x - y$ for some $x, y \in H^+$.*

2.6.1.8 A **directed subgroup** of a partially ordered abelian group G is any subgroup H of G which satisfies the equivalent conditions of Proposition 2.6.1.7. If G is a directed subgroup of itself then we say that G is **directed**, or that G is a **directed abelian group**.

2.6.1.9 Definition *Let G be a partially ordered abelian group. We say that G is* **unperforated** *if for any $x \in G$ and any $n \in \mathbb{N}$ we have that nx is positive only if x is positive.*

2.6.1.10 Example If $\mathbb{Z}$ is made into a partially ordered abelian group with positive cone $\{0, 2, 4, \dots\}$, then $\mathbb{Z}$ is not unperforated.

2.6.1.11 We will be mostly concerned with directed, unperforated partially ordered abelian group, so we give an alternative description here in terms of the positive cone. This is the description given in [Ef], for example, in which directed, unperforated partially ordered abelian group are referred to simply as "ordered groups". We will not adopt this convention.

Let G be an abelian group together with a subset P of G, called the **positive cone** or **ordering**, such that

1. $P + P \subseteq P$,
2. $P - P = G$,
3. $P \cap (-P) = \{0\}$,
4. if $a \in G$ and $na \in P$ for some $n \in \mathbb{N}$, then $a \in P$.

Then G is a directed, unperforated partially ordered abelian group. Conversely, any directed, unperforated partially ordered abelian group satisfies (1–4) with $P = G^+$.

It is possible that G has many subsets which satisfy the above properties. If no ambiguity results, we will sometimes denote P by G^+.

2.6.1.12 Let G be a directed, unperforated partially ordered abelian group. If $a, b \in G$ and $na \leq nb$ for some $n \in \mathbb{N}$ then $a \leq b$. Indeed,

$$na \leq nb \Rightarrow 0 \leq nb - na = n(b-a) \Rightarrow n(b-a) \in G^+ \Rightarrow b - a \in G^+ \Rightarrow a \leq b.$$

So in particular, $na = 0$ implies that $a \leq 0$ and $0 \leq a$, so $a = 0$. Hence a directed, unperforated partially ordered abelian group is torsion free.

2.6.2 Homomorphisms and Direct Limits

2.6.2.1 Let G and H be partially ordered abelian groups. A **positive homomorphism** from G to H is a group homomorphism $f : G \to H$ such that

$$f(G^+) \subseteq H^+$$

Note that a homomorphism is positive if and only if it is order preserving (that is, if $x \leq y$ implies $f(x) \leq f(y)$, for any $x, y \in G$).

2.6.2.2 Of particular importance for us later will be ordered groups which have an order unit. Let G be a partially ordered abelian group, and let $u \in G^+$. We say that u is an **order unit** in G if, for any $x \in G$, there is an $n \in \mathbb{N}$ such that $x < nu$. We have the following elementary properties of order units:

1. If a partially ordered abelian group G contains an order unit, then G must be directed.
2. If u is an order unit in G, then $0 < u$.
3. $0 < u \leq nu$ for any $n \in \mathbb{N}$.

If G is a partially ordered abelian group with order unit u and H is a partially ordered abelian group with order unit v, we say that a positive homomorphism $f : G \to H$ is a **normalized positive homomorphism** if $f(u) = v$.

2.6.2.3 Example Any strictly positive element of $\mathbb{R}$ is an order unit.

2.6.2.4 By **the category of partially ordered abelian groups** we mean the category whose objects are all partially ordered abelian groups and whose morphisms are all positive homomorphisms between them. By **the category of partially ordered abelian groups with order unit** we mean the category whose objects are all pairs (G, u), where G is a partially ordered abelian group and u is an order unit in G, and whose morphisms are all normalized positive homomorphisms between these objects.

2.6.2.5 A (normalized) positive homomorphism which is bijective need not be an order isomorphism, as we need that both f and f^{-1} be (normalized) positive homomorphisms. For example, let G be any group with two different positive cones G_1^+ and G_2^+ with $G_1^+ \subset G_2^+$ and $G_1^+ \neq G_2^+$. Then the identity map $\mathrm{id} : (G, G_1^+) \to (G, G_2^+)$ is a bijective positive homomorphism which is not an order isomorphism.

2.6.2.6 There is a natural ordered group direct limit of a directed system of partially ordered abelian groups and positive homomorphisms. Indeed, we have the following two propositions from [Go].

2.6.2.7 Proposition ([**Go, 1.15**]) *Let $\{G_i, \Phi_{ij}\}$ be a directed system of partially ordered abelian groups and positive homomorphisms, indexed by a directed set $\mathcal{I}$. Let G be the abelian group direct limit of this system, and for each $i \in \mathcal{I}$ let $\Phi_i : G_i \to G$ be the natural map. Let*

$$G^+ = \bigcup_{i \in \mathcal{I}} \Phi_i(G^+).$$

Then G^+ is a strict positive cone for G, so the corresponding relation $\leq_{G^+}$ is a partial order on G, and G together with $\leq_{G^+}$ is a partially ordered abelian group. With this ordering, G together with the maps Φ_i is a direct limit for the given system in the category of partially ordered abelian groups.

2.6.2.8 Proposition ([**Go, 1.16**]) *Let $\{(G_i, u_i), \Phi_{ij}\}$ be a directed system of partially ordered abelian groups with order unit and normalized positive homomorphisms, indexed by a directed set $\mathcal{I}$. Let G be the abelian group direct limit of this system, and for each $i \in \mathcal{I}$ let $\Phi_i : G_i \to G$ be the natural map. Then there exists a unique order unit $u \in G$ such that $\Phi_i(u_i) = u$ for all $i \in \mathcal{I}$, and (G, u) together with the maps Φ_i is a direct limit for the given system in the category of partially ordered abelian groups with order unit.*

2.6.3 Dimension Groups

Interpolation

2.6.3.1 Proposition ([**Go, 2.1**]) *For a partially ordered abelian group G, the following are equivalent:*

1. *Given $x_1, x_2, y_1, y_2 \in G$ such that $x_i \leq y_j$ for all i, j, there exists an element $z \in G$ such that $x_i \leq z \leq y_j$ for all i, j.*

2. *Given $x, y_1, y_2 \in G^+$ such that $x \leq y_1 + y_2$ there exists $x_1, x_2 \in G^+$ such that $x = x_1 + x_2$ and $x_j \leq y_j$ for all j.*

3. *Given $x_1, x_2, y_1, y_2 \in G^+$ such that $x_1 + x_2 = y_1 + y_2$, there exist elements $z_{11}, z_{12}, z_{21}, z_{22} \in G^+$ such that $x_i = z_{i1} + z_{i2}$ for each i and $y_j = z_{1j} + z_{2j}$ for each j.*

2.6.3.2 A partially ordered abelian group G is said to satisfy the **Riesz interpolation property** provided G satisfies condition (1) of Proposition 2.6.3.1, and is said to satisfy the **Riesz decomposition properties** if conditions (2) and (3) of Proposition 2.6.3.1 hold in G.

Note that the class of groups which satisfy the Riesz interpolation property is closed under the formation of direct products and direct limits.

2.6.3.3 Proposition ([Go, 2.2]) *Let G be an interpolation group.*

1. *Given $x_i, y_j \in G$ such that $x_i \leq y_j$ for all $i = 1, \dots n, j = 1, \dots k$, there exists $z \in G$ such that $x_i \leq z \leq y_j$ for all i, j.*

2. *Given $x, y_1, \dots, y_k \in G^+$ such that $x \leq \sum_{j=1}^{k} y_j$ there exists $x_1, \dots, x_k \in G^+$ such that $x = \sum_{i=1}^{k} x_i$ and $x_j \leq y_j$ for all j.*

3. *Given $x_1, \dots, x_n, y_1, \dots, y_k \in G^+$ such that $\sum_i x_i = \sum_j y_j$, there exist $z_{ij} \in G^+$ such that :*

$$x_i = \sum_j z_{ij} \text{ and } y_j = \sum_i z_{ij}$$

   for all i, j.

Dimension Groups

2.6.3.4 Definition *A* **dimension group** *is a directed, unperforated partially ordered abelian group which satisfies the Riesz interpolation property.*

2.6.3.5 A **scale** on a dimension group G is a subset Γ of G^+ which satisfies

1. For each $a \in G^+$ there are $a_1, \dots, a_r \in \Gamma$ such that $a = a_1 + \dots + a_n$. (i.e. Γ generates G)

2. If $0 \leq a \leq b$ with $b \in \Gamma$, then $a \in \Gamma$. (Γ is hereditary)

3. Given $a, b \in \Gamma$, there is a $c \in \Gamma$ with $a, b \leq c$. (Γ is directed)

2.6.3.6 Let G be a dimension group, and let $\Gamma(G)$ be a scale on G. We call the pair $(G, \Gamma(G))$ a **scaled dimension group.**

Let $(G, \Gamma(G))$ and $(H, \Gamma(H))$ be scaled dimension groups. Let $f : G \to H$ be a positive homomorphism. We call f a **contraction** if $f(\Gamma(G)) \subseteq \Gamma(H)$. If f is invertible and if f^{-1} is also a contraction, then we call f an isomorphism of scaled dimension groups.

2.6.3.7 Considering again the examples in (2.6.1.2), we have:

1. $\mathbb{Z}$ with the usual ordering is a directed ordered group which satisfying the Riesz interpolation property, so it is a dimension group. Any hereditary subset of $\mathbb{Z}^+$ has the form $[0, n]$ for some $n \in \mathbb{Z}^+ \setminus \{0\}$. Conversely, for any $n \in \mathbb{Z}^+ \setminus \{0\}$ the set $[0, n]$ is a scale on $\mathbb{Z}$. Similarily for $\mathbb{Q}$ and $\mathbb{R}$.

2. The additive group $\mathbb{C}$ in which $z \leq w$ if and only if $w - z \in \mathbb{R}^+$ satisfies the Riesz interpolation property, but is not directed, so it is not an ordered group, and hence not a dimension group.

3. $\mathbb{Z}$ with the ordering $x \leq y$ if and only if $y - x$ is even and non-negative satisfies the Riesz interpolation property, but it is not unperforated and is not directed, and hence is not a dimension group.

4. $\mathbb{Z}^n$ with the simplicial ordering is an ordered group satisfies the Riesz interpolation property, so is a dimension group. The scales on $\mathbb{Z}^n$ are the sets $[0, p_1] \times \ldots \times [0, p_n]$ with $p_1, \ldots, p_n \in \mathbb{Z}^+ \setminus \{0\}$.

5. Suppose that $\{G_i\}$ is some family of partially ordered abelian groups indexed by $\mathcal{I}$, and let $G = \prod_i G_i$ with the product ordering. If each of the G_i are dimension groups, so is G. If the G_i are scaled dimension groups and if $\mathcal{I}$ is finite then G is a scaled dimension group with scale $\prod_{i \in \mathcal{I}} \Gamma(G_i)$.

6. Let F be the set of functions from a set $\mathcal{I}$ into a dimension group G, with the pointwise ordering. Then F is isomophic with $\prod_{\mathcal{I}} G$, and hence is a dimension group.

2.6.3.8 We will be most interested in dimension groups with an order unit. In this situation there is a natural scale to consider. Indeed, let G be a dimension group with order unit u. Then the set $[0,u] = \{x \in G; 0 \leq x \leq u\}$ is a scale on G. Unless otherwise specified, when we refer to the scale of a partially ordered abelian group G with order unit u, we assume that G is given the scale $\Gamma(G) = [0,u]$. If G is a partially ordered abelian group with order unit u and H is a partially ordered abelian group with order unit v, and if $f : G \to H$ is a normalized positive homomorphism, then f is a contraction.

2.6.3.9 As mentioned in Number (5) of (2.6.3.7), dimension groups are closed under direct products and scaled dimension groups are closed under direct sums. Also, the direct limit of a system $\{\mathfrak{A}_i, \Phi_{ij}\}$ of dimension groups and positive homomorphisms is a dimension group with positive cone $\bigcup_{i\in\mathcal{I}} \Phi_i(G_i^+)$. If the G_i are scaled dimension groups and the Φ_{ij} are contractions then G is a scaled dimension group with scale $\Gamma(G) = \bigcup_{i\in\mathcal{I}} \Gamma(G_i)$.

2.6.3.10 Let $(G, \Gamma(G))$ and $(H, \Gamma(H))$ be scaled dimension groups. We say that a map $f : \Gamma(G) \to \Gamma(H)$ is a **scale homomorphism** if whenever $a = b + c$, for some $a, b, c \in \Gamma(G)$, it follows that $f(a) = f(b) + f(c)$. If f is invertible and if f^{-1} is also a scale homomorphism then we call f a **scale isomorphism**.

2.6.3.11 Proposition ([Ef, Lemma 7.4]) *Any scale homomorphism $f : \Gamma(G) \to \Gamma(H)$ extends to a unique contraction $\tilde{f} : G \to H$. If f is a scale isomorphism, then $\tilde{f}$ is an isomorphism of scaled dimension groups.*

2.6.3.12 Consider $\mathbb{Z}^n$ with the simplicial ordering. Let $u = (u_i)_{i=1}^n$ be an element of $\mathbb{Z}^n$ such that $u_i > 0$ for every i. Then u is an order unit for $\mathbb{Z}^n$, and hence $[0,u]$ is a scale on $\mathbb{Z}^n$. Let $G = (\mathbb{Z}^n, u)$ and $H = (\mathbb{Z}^m, v)$ be partially ordered abelian groups with order units, so they are also scaled dimension groups. If $A \in M_{m\times n}(\mathbb{N})$ is an $(m \times n)$ matrix then A defines a homomorphism from G into H. If $Au \leq v$, then A is a scale homomorphism of $\mathbb{Z}^n$ with $\mathbb{Z}^m$, and A is a scale isomorphism if and only if $Au = v$.

Simplicial Groups

2.6.3.13 A **simplicial group** is a partially ordered abelian group which is isomorphic (as an ordered group) to $\mathbb{Z}^n$ for some nonnegative integer n. A **simplicial basis** for a simplicial group G is a basis $\{x_1, \ldots, x_n\}$ for G as a free abelian group such that also $G^+ = \sum \mathbb{Z}^+ x_i$. By convention, the empty set is considered to be a simplicial basis for the zero simplicial group. Note that since $\mathbb{Z}^n$ is a dimension group, any simplicial group is a dimension group. Also, since the direct limit of dimension groups with positive homomorphisms is a dimension group, the direct limit of simplicial groups and positive homomorphisms is a dimension group. The converse is also true as was shown by Effros, Handelman, and Shen (see [Go, p59] for a more complete description of these results).

2.6.3.14 Theorem ([Go, 3.17]) *Any countable dimension group is isomorphic to a direct limit of a countable sequence of simplicial groups (in the category of partially ordered abelian groups).*

2.6.3.15 Corollary ([Go, 3.18]) *Any countable dimension group with order unit is isomorphic to a direct limit of a countable sequence of simplicial groups with order unit (in the category of partially ordered abelian groups with order unit).*

2.6.3.16 Theorem ([Go, 3.19]) *Any dimension group is isomorphic to a direct limit of a directed system of simplicial groups (in the category of partially ordered abelian groups).*

2.6.3.17 Corollary ([Go, 3.20]) *Any dimension group with order unit is isomorphic to a direct limit of a directed system of simplicial groups with order unit (in the category of partially ordered abelian groups with order unit).*

2.6.4 Ideals

2.6.4.1 An **order convex subset** of a partially ordered set G is a set $H \subseteq G$ such that

$$(x, z \in H, y \in G, x \leq y \leq z) \Rightarrow y \in H.$$

A **convex subgroup** of a partially ordered abelian group G is a subgroup H of G which is a convex subset of G.

To show that a subgroup H of a partially ordered abelian group G is convex, it suffices to show that whenever $a \in G$ and $b \in H$ then

$$0 \leq a \leq b \Rightarrow a \in H.$$

Let $X \subseteq G$. Then there is a smallest subgroup $H \leq G$ such that $X \subseteq H$, called the **convex subgroup of G generated by X**.

2.6.4.2 A **ideal** of a partially ordered abelian group G is any directed convex subgroup of G. A partially ordered abelian group G is **simple** if G is nonzero and directed (so that G is a nonzero ideal of itself) and the only ideals of G are $\{0\}$ and G.

2.6.4.3 Proposition ([Go, 1.9]) *Let G be a partially ordered abelian group.*

1. *If X is a nonempty subset of G^+ and K is the convex subgroup of G generated by X, then K is an ideal of G.*
2. *If H is a directed subgroup of G and K is the convex subgroup of G generated by H^+, then K is the smallest ideal of G that contains H.*
3. *If H is a convex subgroup of G and*

$$K = \{x - y; x, y \in H^+\}$$

(so that K is the subgroup of G generated by H^+), then K is the largest ideal of G that is contained in H.

2.6.4.4 Later we see the construction of a covariant functor from the category of C*-algebras to the category of abelian groups. This functor will send an AF C*-algebra $\mathfrak{A}$ to a dimension group $K_0(\mathfrak{A})$, and will send an ideal of $\mathfrak{A}$ to an ideal of $K_0(\mathfrak{A})$. It follows that a simple AF C*-algebra gets sent to a simple dimension groups.

Note also that a simple dimension group G has many order units. Indeed, any nonzero element of G^+ is an order unit for G (see [Go, Lemma 14.1]).

2.7 C*-Algebra Basics

2.7.1 Banach Algebras

2.7.1.1 We begin with some definitions. A (complex)***-algebra** is an algebra $\mathfrak{A}$ over $\mathbb{C}$ with a conjugate linear involution * (called the adjoint) which is an anti-isomorphism. That is, for $\lambda \in \mathbb{C}, a, b \in \mathfrak{A}$ we have

$$(a+b)^* = a^* + b^* \tag{2.7.5}$$

$$(\lambda a)^* = \overline{\lambda} a^* \tag{2.7.6}$$

$$a^{**} = a \tag{2.7.7}$$

$$(ab)^* = b^* a^* \tag{2.7.8}$$

A subset $\mathfrak{X}$ of a *-algebra $\mathfrak{A}$ is said to be **self-adjoint**, or ***-invariant**, if $\mathfrak{X} = \mathfrak{X}^*$. A **Banach algebra** is a complex normed algebra which is complete (as a normed space) and such that

$$||ab|| \leq ||a|| ||b|| \text{ for all } a, b \in \mathfrak{A}.$$

A **Banach *-algebra** is a (complex)*-algebra which is also a Banach algebra. Notice that in general an involution on a Banach *-algebra is not necessarily an isometry. A **C*-algebra** is a Banach *-algebra with the additional norm condition

$$||a^* a|| = ||a||^2 \text{ for all } a \in \mathfrak{A}. \tag{2.7.9}$$

Condition (2.7.9) implies in particular that the adjoint map is isometric. Indeed,

$$||a||^2 = ||a^* a|| \leq ||a^*|| ||a||$$

so

$$||a|| \leq ||a^*|| \leq ||a^{**}|| = ||a||$$

and hence

$$||a|| = ||a^*||.$$

A **C*-subalgebra** of a C*-algebra $\mathfrak{A}$ is a norm closed *-invariant subalgebra of $\mathfrak{A}$. Clearly a C*-subalgebra of a C*-algebra is a C*-algebra.

2.7.1.2 Example Let $\mathfrak{A} = \mathbb{M}_n = \mathbb{M}_n(\mathbb{C})$. Then $\mathfrak{A}$ with the usual matrix addition, multiplication, and scalar multiplication is an algebra. Define an involution on $\mathfrak{A}$ by $a^* = \overline{a}^t$, and define a norm on a by

$$\|a\| = \sup\{\|ax\|; x \in \mathbb{C}^n, \|x\| = 1\}.$$

This norm is known as the operator norm on $\mathfrak{A}$. Since $\mathfrak{A}$ is finite dimensional it is closed. With these definitions, $\mathfrak{A}$ is a C*-algebra.

2.7.1.3 Example The algebra of all bounded operators $\mathbb{B}(\mathfrak{H})$ on a Hilbert space $\mathfrak{H}$ is a C*-algebra with the usual adjoint operation. Indeed, for any $a \in \mathbb{B}(\mathfrak{H})$, $\xi \in \mathfrak{H}$ such that $\|\xi\| = 1$, we have

$$\|a\xi\|^2 = (a\xi, a\xi) = (a^*a\xi, \xi) \leq \|a^*a\| \leq \|a^*\|\|a\|.$$

Hence $\|a\|^2 \leq \|a^*a\| \leq \|a^*\|\|a\|$ and $\|a\| \leq \|a^*\|$. As $a^{**} = a$, we get $\|a^*\| = \|a\|$ and $\|a\|^2 = \|a^*\|\|a\|$. So $\mathbb{B}(\mathfrak{H})$ satisfies equation (2.7.9).

2.7.1.4 Example A **concrete C*-algebra** is a C*-subalgebra of $\mathbb{B}(\mathfrak{H})$, for some Hilbert space $\mathfrak{H}$. So $\mathbb{B}(\mathfrak{H})$ is an example of a concrete C*-algebra. Another example of a concrete C*-algebra is the algebra $\mathfrak{K}$ of all compact operators on a separable Hilbert space.

2.7.1.5 Let $\mathfrak{A}$ be a C*-algebra, and let $X \subseteq \mathfrak{A}$. Denote by $C^*(X)$ the smallest C*-subalgebra of $\mathfrak{A}$ containing X. In particular, if $X = \{a\}$ for some $a \in \mathfrak{A}$, then we write $C^*(a)$ for $C^*(X)$, and $C^*(a)$ is the norm closure of all polynomials in a and a^* without constant term (see in particular (2.7.5.10) or (2.7.5.8)).

2.7.1.6 Example Let X be a locally compact Hausdorff space. Define $C(X)$ to be the set of all continuous functions f from X into $\mathbb{C}$, and define $C_0(X)$ to be the set of continuous functions f from X into $\mathbb{C}$ such that for every $\varepsilon > 0$ the set

$$\{x \in X; |f(x)| \geq \varepsilon\}$$

is compact. Note that $f \in C_0(X)$ if and only if f has an extension to a continuous function $\tilde{f}$ on the one-point compactification $X \cup \{\infty\}$ of X, such that $\tilde{f}(\infty) = 0$. We say that f **vanishes at infinity**.

The sets $C(X)$ and $C_0(X)$ form C*-algebras with pointwise operations, the sup-norm, and with complex conjugation as the involution. Indeed,

$$\|\bar{f}f\|_X = \sup_{x\in X} |\overline{f(x)}f(x)| = \sup_{x\in X} |f(x)|^2 = \|f\|_X^2,$$

shows the necessary norm condition.

2.7.1.7 Example Let $\mathcal{I}$ be an index set, and for each $i \in \mathcal{I}$ let $\mathfrak{A}_i$ be a C*-algebra. Then

$$\mathfrak{A} = \{(a_i) \in \prod_{i\in\mathcal{I}} \mathfrak{A}_i; \sup_{i\in\mathcal{I}} \|a_i\| < \infty\}$$

is a C*-algebra with pointwise operations and the sup-norm, and is called the product of the $\mathfrak{A}_i$.

2.7.1.8 A (C*-algebra) **homomorphism** of C*-algebras $\mathfrak{A}$ and $\mathfrak{B}$ is a map $\varphi : \mathfrak{A} \to \mathfrak{B}$ which is linear, multiplicative, and preserves the adjoint operation i.e. such that for any $a, b \in \mathfrak{A}$ and any $\lambda \in \mathbb{C}$,

$$\begin{aligned}
\varphi(\lambda a) &= \lambda \varphi(a) \\
\varphi(a+b) &= \varphi(a) + \varphi(b) \\
\varphi(ab) &= \varphi(a)\varphi(v) \\
\varphi(a^*) &= \varphi(a)^*
\end{aligned}$$

If also φ is bijective and if φ^{-1} is a homomorphism then we call φ a (C*-algebra) **isomorphism**, and we say that $\mathfrak{A}$ and $\mathfrak{B}$ are isomorphic. We say that φ is a **representation of $\mathfrak{A}$ on $\mathfrak{H}$** (or just a **representation**) if $\mathfrak{B} = \mathbb{B}(\mathfrak{H})$ for some Hilbert space $\mathfrak{H}$. A representation is said to be **faithful** if it is injective.

Note that the definition of a homomorphism makes *a priori* no restriction on what happer.s to the norm. We will see in (2.7.7.4) that any C*-algebra homomorphism is norm decreasing, and hence that a C*-algebra isomorphism is an isometry. Basically, this is because the norm on a C*-algebra is determined by its algebraic structure (2.7.5.3).

2.7.1.9 Our main interest lies with C*-algebras, but we will occasionally have use for *-algebras and Banach algebras. Since any result on Banach algebras applies *a fortiori* to C*-algebras, and since we may have a few uses for Banach algebras, we develop some Banach algebra tools which we will need for C*-algebras.

2.7.1.10 An **ideal** (more precisely a two-sided ideal) in a Banach algebra $\mathfrak{A}$ is a subspace $\mathfrak{I}$ such that $\mathfrak{A}\mathfrak{I} \subset \mathfrak{I}$ and $\mathfrak{I}\mathfrak{A} \subset \mathfrak{I}$. If only one of these inclusions is satisfied, we talk about a left or a right-ideal. Given an ideal $\mathfrak{I}$ in $\mathfrak{A}$, the Banach space quotient $\mathfrak{A}/\mathfrak{I}$ becomes an algebra with the product

$$(a + \mathfrak{I})(b + \mathfrak{I}) = ab + \mathfrak{I}.$$

If $\mathfrak{I}$ is a closed ideal in $\mathfrak{A}$, then $\mathfrak{A}/\mathfrak{I}$ is again a Banach algebra (by (2.7.7.2) it suffices to show that the norm is submultiplicative on $\mathfrak{A}/\mathfrak{I}$). Thus every closed ideal is the kernel of a continuous (even norm decreasing) morphism $q : \mathfrak{A} \to \mathfrak{A}/\mathfrak{I}$. Conversely, if $\Phi : \mathfrak{A} \to \mathfrak{B}$ is a continuous homomorphism between Banach algebras then $\ker \Phi$ is a closed ideal in $\mathfrak{A}$. A **maximal ideal** is a proper ideal which is contained in no larger proper ideal.

By an **ideal** of a C*-algebra we mean a norm-closed two-sided ideal. We will see in (2.7.7.1) that an ideal of a C*-algebra is *-invariant, so an ideal of a C*-algebra is a C*-algebra. An **essential ideal** of $\mathfrak{A}$ is an ideal $\mathfrak{I}$ in $\mathfrak{A}$ such that $\mathfrak{I} \cap \mathfrak{I}' \neq \{0\}$ whenever $\mathfrak{I}'$ is a nonzero ideal in $\mathfrak{A}$.

2.7.1.11 A **unit** in a Banach algebra is a multiplicative identity I such that $Ia = aI = a$ for every $a \in \mathfrak{A}$. A Banach algebra $\mathfrak{A}$ with a unit is called a **unital Banach algebra**, and $\mathfrak{A}$ is a **unital C*-algebra** if $\mathfrak{A}$ is a C*-algebra. Such a unit is unique and $\|I\| \geq 1$ (if $\mathfrak{A} \neq \{0\}$). In any unital C*-algebra, $\|I\| = 1$. Also, given a unital Banach algebra $\mathfrak{A}$, it is possible to find a norm on $\mathfrak{A}$ which is equivalent to the original norm on $\mathfrak{A}$, such that $\|I\| = 1$ (cf. [P2, 4.1.4]). So we lose nothing by assuming that $\|I\| = 1$, and this will be done from now on.

In a unital Banach algebra $\mathfrak{A}$ an element a is **invertible** if there are elements $b, c \in \mathfrak{A}$ such that $ba = ac = I$. Since

$$b = bI = bac = Ic = c,$$

we see that the left and right inverses for a are the same and that the inverse is unique. We write a^{-1} for the inverse of a. The set of invertible elements in $\mathfrak{A}$ is denoted by $\mathbf{GL}(\mathfrak{A})$.

2.7.1.12 Given a Banach algebra $\mathfrak{A}$, let $\mathcal{B}(\mathfrak{A})$ denote the C*-algebra of bounded linear operators on $\mathfrak{A}$, with the operator norm. We define the (left) regular representation $\rho : \mathfrak{A} \to \mathcal{B}(\mathfrak{A})$ by $\rho(a)b = ab$ for any $a, b \in \mathfrak{A}$. Then ρ is a norm decreasing homomorphism of Banach algebras. If $\mathfrak{A}$ is unital then

$$\|a\| = \|aI\| \leq \|\rho(a)I\|\|I\| \leq \|\rho(a)\|\|I\| \text{ and } \|\rho(a)\| \leq \|a\|$$

shows that ρ is an isometry. If $\mathfrak{A}$ has an approximate identity (2.7.6.8) then ρ is an isometry, even when $\mathfrak{A}$ is not unital.

2.7.1.13 If $\mathfrak{A}$ is non-unital Banach algebra, it can be embedded isometrically into a unital Banach algebra $\tilde{\mathfrak{A}}$ as a maximal ideal of codimension one. Specifically, we define $\tilde{\mathfrak{A}} = \mathfrak{A} \oplus \mathbb{C}$ with the product

$$(a, \alpha)(b, \beta) = (ab + \alpha b + \beta a, \alpha\beta) \tag{2.7.10}$$

and the submultiplicative norm

$$\|(a, \alpha)\| = \sup_{\|b\| \leq 1} \|ab + \alpha b\|. \tag{2.7.11}$$

With these definitions, $\tilde{\mathfrak{A}}$ is a unital Banach algebra with $I = (0, 1)$ which contains $\mathfrak{A}$ as an ideal of codimension one. The algebra $\tilde{\mathfrak{A}}$ is the **unitization of** $\mathfrak{A}$, and the process is of embedding $\mathfrak{A}$ into $\tilde{\mathfrak{A}}$ is known as **adjoining a unit to** $\mathfrak{A}$.

If $\mathfrak{A}$ is a non-unital C*-algebra then $\tilde{\mathfrak{A}}$, the unitization of $\mathfrak{A}$ as a Banach algebra, is also a C*-algebra with the adjoint operation

$$(a, \alpha)^* = (a^*, \overline{\alpha}). \tag{2.7.12}$$

If $\mathfrak{A}$ is unital, we will adopt the convention that $\mathfrak{A} = \tilde{\mathfrak{A}}$.

Note that if $\mathfrak{B}$ is a non-unital C*-subalgebra of a unital C*-algebra $\mathfrak{A}$, then the set

$$\mathfrak{B}' = \{b + tI_{\mathfrak{A}}; b \in \mathfrak{B}, t \in \mathbb{C}\} \subseteq \mathfrak{A}$$

is a C*-subalgebra of $\mathfrak{A}$. The map $b + tI_{\mathfrak{A}} \xmapsto{\varphi} (b, t)$ is a *-algebra homomorphism of $\mathfrak{B}'$ with $\tilde{\mathfrak{B}}$. Since the left regular representation is an isometry in $\mathfrak{A}$, we see that $\|b + tI_{\mathfrak{A}}\|$ is the same as the operator norm of $\|b + tI_{\mathfrak{A}}\|$ acting on $\mathfrak{A}$. In other words, φ is an isometric *-isomorphic of $\mathfrak{B}'$ with $\tilde{\mathfrak{B}}$. So we can think of $\tilde{\mathfrak{B}}$ as sitting inside of $\mathfrak{A}$. More generally, if $\mathfrak{B}$ is a non-unital C*-subalgebra of a C*-algebra $\mathfrak{A}$, then we can think of $\tilde{\mathfrak{B}}$ as being contained in $\tilde{\mathfrak{A}}$, with with $I_{\mathfrak{B}'} = I_{\tilde{\mathfrak{A}}}$.

If $\mathfrak{A}$ is a unital C*-algebra with unit $I_{\tilde{\mathfrak{A}}}$ and $\mathfrak{B}$ is a C*-subalgebra of $\mathfrak{A}$ with unit $I_{\tilde{\mathfrak{B}}}$, it does not follow that $I_{\tilde{\mathfrak{A}}} = I_{\tilde{\mathfrak{B}}}$ (see for example (2.7.12.2)). In this case, $\tilde{\mathfrak{B}} = \mathfrak{B}$ and we still have $\tilde{\mathfrak{B}} \subseteq \mathfrak{A} \subseteq \tilde{\mathfrak{A}}$.

So in any case, we can see $\tilde{\mathfrak{B}}$ as being contained in $\tilde{\mathfrak{A}}$.

2.7.1.14 We will see in Section 2.9 some use for the above construction, even when $\mathfrak{A}$ is unital. In particular, if $\mathfrak{A}$ is a C*-algebra, define $\mathfrak{A}^+ = \mathfrak{A} \oplus \mathbb{C}$ with pointwise addition, and with multiplication, norm, and * defined as in Equations (2.7.10), (2.7.11)), and ((2.7.12)). Then $\mathfrak{A}^+ = \tilde{\mathfrak{A}}$ if $\mathfrak{A}$ is non-unital, and $\mathfrak{A}^+ \simeq \mathfrak{A} \oplus \mathbb{C}$ with the pointwise operations and the product norm. In either case, $\mathfrak{A}^+/\mathfrak{A} \simeq \mathbb{C}$.

2.7.1.15 Lemma (**[P2, 4.1.7]**) *If a is an element in a unital Banach algebra $\mathfrak{A}$ with $\|a\| < 1$, then $I - a \in \mathbf{GL}(\mathfrak{A})$ and*

$$(I - a)^{-1} = \sum_{n=0}^{\infty} a^n.$$

2.7.1.16 As an immediate corollary, if $\mathfrak{A}$ is a unital Banach algebra and $a \in \mathfrak{A}$, we have that $r(a) \leq \|a\|$ (see the proof of [P2, 4.1.12]).

2.7.1.17 Proposition (**[P2, 4.1.8]**) *In a unital Banach algebra $\mathfrak{A}$ the multiplicative group $\mathbf{GL}(\mathfrak{A})$ is an open subset of $\mathfrak{A}$, and the map $a \mapsto a^{-1}$ is a homeomorphism of $\mathbf{GL}(\mathfrak{A})$.*

2.7.2 The Spectrum

Following [R] (or [Pa]), we will define the spectrum of an element $a \in \mathfrak{A}$ entirely in terms of $\mathfrak{A}$ rather then its unitization $\tilde{\mathfrak{A}}$.

2.7.2.1 Define the **quasi-product** on $\mathfrak{A}$ by

$$a \cdot b = a + b - ab.$$

Note that 0 acts as an identity element under quasi-multiplication. We denote the set of quasi-invertible elements of $\mathfrak{A}$ by $\mathbf{GL}_q(\mathfrak{A})$. We say that a is quasi-singular if it is not quasi-invertible.

2.7.2.2 Let $\mathfrak{A}$ be a Banach algebra. For every $a \in \mathfrak{A}$ we define the **spectrum** $\sigma_{\mathfrak{A}}(a)$ of a as the set of all $t \in \mathbb{C}$ such that $t^{-1}a$ is quasi-singular, plus 0 if a is not invertible (if $\mathfrak{A}$ is nonunital, no element in $\mathfrak{A}$ is invertible). When no confusion will result, we write simply $\sigma(a)$ for $\sigma_{\mathfrak{A}}(a)$.

2.7.2.3 We have

1. If $\mathfrak{A}$ is unital, this definition agrees with the usual one. Indeed, the usual definition is

$$\sigma_{\mathfrak{A}}(a) = \{t \in \mathbb{C}; tI - a \notin \mathbf{GL}(\mathfrak{A})\}.$$

If $t \neq 0$ the relations $I - t^{-1}a = t^{-1}(tI - a)$ and $(I - x)(I - y) = I - (x \cdot y)$ show that t is in the "usual" spectrum of a if and only if $t^{-1}a$ is quasi-singular. Also, 0 is in the "usual" spectrum if and only if a is not invertible. Notice that

$$\sigma_{\mathfrak{A}}(a) \cup \{0\} = \{t \in \mathbb{C}; tI - a \notin \mathbf{GL}(\tilde{\mathfrak{A}})\} = \sigma_{\tilde{\mathfrak{A}}}(a).$$

2. If $\mathfrak{A}$ is non unital, then

$$\sigma_{\mathfrak{A}}(a) = \{t \in \mathbb{C}; tI - a \notin \mathbf{GL}(\tilde{\mathfrak{A}})\} = \sigma_{\tilde{\mathfrak{A}}}(a).$$

2.7.2.4 We have from [R, 4.8.2] that if $\mathfrak{B}$ is a C*-algebra which contains $\mathfrak{A}$ as a sub-C*-algebra and if $a \in \mathfrak{A}$ then

1. $\sigma_{\mathfrak{A}}(a) \cup \{0\} = \sigma_{\mathfrak{B}}(a) \cup \{0\}$

2. If $\mathfrak{B}$ has an identity which also lies in $\mathfrak{A}$ then $\sigma_{\mathfrak{A}}(a) = \sigma_{\mathfrak{B}}(a)$.

2.7.2.5 Theorem ([R, 1.6.15]) *Let $\mathfrak{A}$ be a Banach algebra and let e be a proper idempotent in $\mathfrak{A}$ (that is $e \neq 0, I$, and $e^2 = e$). Then $e\mathfrak{A}e$ is a closed subalgebra of $\mathfrak{A}$ with e as an identity element and, for every $a \in e\mathfrak{A}e$, $\sigma_{\mathfrak{A}}(a) = \sigma_{e\mathfrak{A}e}(a) \cup \{0\}$.*

2.7.2.6 Most of the results in the following sections are stated for unital algebras, but they have obvious applications to non-unital C*-algebras by considering the unitization, especially in view of the previous discussion. The catch is that (2.7.2.4) uses the continuous functional calculus (2.7.5.8), and hence should properly be included only after this result.

2.7.2.7 The smallest number $r \geq 0$ such that $\sigma(a) \subset B(0, r)$ is called the **spectral radius** of a, and is denoted by $r(a)$. Thus,

$$r(a) = \sup\{|\lambda| \,;\, \lambda \in \sigma(a)\}.$$

The complement of $\sigma(a)$ is the **resolvent set**. On this set we can define the **resolvent** (function) $R(a, \lambda) = (\lambda I - a)^{-1}$. When the a is clear we write $R(\lambda)$ instead of $R(a, \lambda)$. In particular, the resolvent function is analytic on the resolvent set [Da, I.2.1].

2.7.2.8 [P2, Lemma 4.1.11] is the holomorphic functional calculus for Banach algebras, it can be paraphrased as:

> If $a \in \mathfrak{A}$ and $f(z) = \sum_{n=0}^{\infty} \alpha_n z^n$ is a holomorphic function in a region that contains the closed disk $B(0, \|a\|)$, then we can define $f(a) = \sum \alpha_n a^n \in \mathfrak{A}$. Moreover,
>
> $$f(\sigma(a)) \subseteq \sigma(f(a)),$$
>
> i.e. if $\lambda \in \sigma(a)$, then $f(\lambda) \in \sigma(f(a))$.

Note that this applies to any a if f is an entire function. If p is a polynomial then we also have (cf. [Da, I.2.1])

$$\sigma(p(a)) = p(\sigma(a)).$$

2.7.2.9 ([P2, 4.1.12]) For every $a \in \mathfrak{A}$ we have $r(a) \leq \inf \|a^n\|^{1/n}$. So in particular that $r(a) \leq \|a\|$, for any a in a unital Banach algebra.

2.7.2.10 Theorem ([P2, 4.1.13]) *For every element $a \in \mathfrak{A}$ in a unital Banach algebra $\mathfrak{A}$, the spectrum of a is a compact, nonempty subset of $\mathbb{C}$, and the spectral radius of a is the limit of the convergent sequence $(\|a^n\|^{1/n})$*

$$r(a) = \lim_{n\to\infty} \|a^n\|^{1/n}$$

2.7.2.11 Corollary (Gelfand-Mazur Theorem—cf. [P2, 4.1.14]) *If* $\mathbf{GL}(\mathfrak{A}) = \mathfrak{A}\setminus\{0\}$, *then* $\mathfrak{A} = \mathbb{C}$.

2.7.3 The Gelfand Transform

2.7.3.1 A **character of $\mathfrak{A}$** is a nonzero homomorphism $\gamma : \mathfrak{A} \to \mathbb{C}$. Note that a character of $\mathfrak{A}$ is surjective.

2.7.3.2 Proposition ([P2, 4.2.2]) *In a commutative unital Banach algebra $\mathfrak{A}$ there is a bijective correspondence, given by $\gamma \leftrightarrow \ker\gamma$, between the set $\hat{\mathfrak{A}}$ of characters of $\mathfrak{A}$ and the set $\mathcal{M}(\mathfrak{A})$ of maximal ideals in $\mathfrak{A}$. Every γ in $\hat{\mathfrak{A}}$ is automatically continuous, and every $\mathfrak{J}$ in $\mathcal{M}(\mathfrak{A})$ is closed. Finally, we have for each a in $\mathfrak{A}$ that*

$$\sigma(a) = \{\gamma(a); \gamma \in \hat{\mathfrak{A}}\}.$$

2.7.3.3 Theorem ([P2, 4.2.3]) *Given a commutative, unital Banach algebra $\mathfrak{A}$, the set $\hat{\mathfrak{A}}$ of characters has a compact Hausdorff topology, such that the map $\Gamma : \hat{\mathfrak{A}} \to \mathbb{C}$ [where we write $\Gamma(a) = \hat{a}$] defined by*

$$\Gamma(a)(\gamma) = \hat{a}(\gamma) = \gamma(a), \qquad a \in \mathfrak{A}, \gamma \in \hat{\mathfrak{A}},$$

is a norm decreasing homomorphism of $\mathfrak{A}$ onto a subalgebra of $C(\hat{\mathfrak{A}})$ that separates points in $\hat{\mathfrak{A}}$. For every $a \in \mathfrak{A}$ we have

$$\hat{a}(\hat{\mathfrak{A}}) = \sigma(a), \qquad \|\hat{a}\|_\infty = r(a).$$

Applying (2.7.3.3) to $\tilde{\mathfrak{A}}$ gives us (with some work)

2.7.3.4 Corollary ([Da, I.2.6]) *If $\mathfrak{A}$ is abelian but not unital then $\mathcal{M}(\mathfrak{A})$ is locally compact.*

The map Γ in Proposition 2.7.3.3 is the **Gelfand transform**. Its importance can hardly be overemphasized. In particular it allows us to develop a continuous functional calculus for C*-algebras (2.7.5.6).

2.7.3.5 Corollary (part of [Da, I.2.8]) *In a commutative unital Banach algebra $\mathfrak{A}$, a is invertible if and only if $\hat{a}$ is invertible, which is precisely when $\hat{a}$ does not vanish on $\mathcal{M}(\mathfrak{A})$.*

2.7.3.6 Example ([P2, 4.2.6]) Let $\ell^1(\mathbb{Z})$ be the Banach space

$$\ell^1(\mathbb{Z}) = \left\{ (a_n)_{-\infty}^{\infty}; a_n \in \mathbb{C} \text{ and } \sum_{-\infty}^{\infty} |a_n| < \infty \right\}$$

of doubly infinite summable sequences. Equipped with the convolution product

$$(ab)_n = \sum_{-\infty}^{\infty} a_k b_{n-k}$$

this is a unital, commutative Banach algebra. The element e in $\ell^1(\mathbb{Z})$, with $e_1 = 1$ and $e_n = 0$ for $n \neq 1$, is a generator $\ell^1(\mathbb{Z})$, since each element has the form

$$a = \sum_{-\infty}^{\infty} a_n e^n \qquad \text{(uniform convergence).}$$

a character χ is therefore determined by its value on e. Since $e \in \mathbf{GL}(\ell^1(\mathbb{Z}))$, we have

$$|<e,\chi>| \leq 1, \text{ and } |<e^{-1},\chi>| \leq 1,$$

as $\|\chi\| \leq 1$. Consequently, $\chi(e) \in \mathbb{S}^1$. Conversely, we see that for each $t \in \mathbb{S}^1$ we can define a χ in $\ell^1 \hat{(\mathbb{Z})}$ by

$$<a,\chi>= \sum_{-\infty}^{\infty} a_n t^n, a \in \mathfrak{A}.$$

This establishes a continuous map from $\hat{\mathfrak{A}}$ onto $\mathbb{S}^1$, and it can be shown that this map is in fact a homeomorphism. The Gelfand transform is therefore the map of $\ell^1(\mathbb{Z})$ into $C(\mathbb{S}^1)$ given by

$$\hat{a}(\chi) = \sum_{-\infty}^{\infty} a_n t^n, a \in \ell^1(\mathbb{Z}).$$

Identifying $\mathbb{S}^1$ with $\mathbb{R}/2\pi\mathbb{N}$ we see that the set $\Gamma(\ell^1(\mathbb{Z}))$ of Gelfand transforms in $C(\mathbb{S}^1)$ is the set of continuous, periodic functions on $[0, 2\pi]$ whose Fourier transforms are absolutely convergent.

Suppose that f is a continuous, periodic function on $[0, 2\pi]$ whose Fourier series is absolutely convergent, and suppose that $f(x) \neq 0$ for every $x \in [0, 2\pi]$. Then Wiener showed that the reciprocal function f^{-1} also has an absolutely convergent Fourier series. Gelfand theory makes this result quite obvious (a fact that helped considerably in making the theory acceptable to the mathematical community). Indeed, $f = \hat{a}$ for some a in $\ell^1(\mathbb{Z})$, and since $0 \notin \sigma(a)$ by assumptions, it follows that $a^{-1} \in \ell^1(\mathbb{Z})$ with $\widehat{(a^{-1})} = \hat{a}^{-1} = f^{-1}$.

2.7.4 The Stone-Weierstrass Theorem

Our current goal is the development of a continuous functional calculus for a Banach algebra $\mathfrak{A}$ (or at least for a well-behaved class of Banach algebras—the C*-algebras (see Section 2.7.5)). A continuous functional calculus is a collection of injective algebra homomorphisms of the form $\Phi : \mathcal{C} \to \mathfrak{A}$, where $\mathcal{C}$ is an algebra of continuous functions on some compact Hausdorff space. Clearly the image of any such homomorphism must be commutative, and hence in general it cannot be surjective. Nonetheless, it will be sufficient to give some very powerful tools. The Gelfand transform of (2.7.3.3) is our first step towards a functional calculus, as it gives us a homomorphism between $\mathfrak{A}$ and an algebra of continuous functions on the compact Hausdorff space $\mathcal{M}(\mathfrak{A})$. Although the Gelfand transform goes in the "wrong direction" there are important cases where it is an isometry, thus allowing us to use the inverse map. With this in mind, we see that Theorem 2.7.3.3 gives us one important piece of information about the size of $\Gamma(\mathfrak{A})$ in $C(\hat{\mathfrak{A}})$, namely that it separates points in $\hat{\mathfrak{A}}$. With certain additional conditions this implies that the algebra $\Gamma(\mathfrak{A})$ is dense in $C(\hat{\mathfrak{A}})$ (cf. (2.7.4.4)).

2.7.4.1 Definition ([P2, 4.3.1]) *Let $f : X \to \mathbb{C}$ be a function. We denote by $\overline{f} : X \to \mathbb{C}$ the function given by $\overline{f}(x) = \overline{f(x)}$, for all x in X. We say that a vector space of complex functions is* **self-adjoint** *if $\overline{f} \in \mathfrak{A}$ whenever $f \in \mathfrak{A}$.*

2.7.4.2 Lemma (**[P2, 4.3.2]**) *Let $\mathfrak{A}$ be a vector space of continuous, real-valued functions on a compact Hausdorff space X. If the supremum of f and g (denoted $f \vee g$) and the infimum of f and g (denoted $f \wedge g$) belong to $\mathfrak{A}$ for all f and g in $\mathfrak{A}$, then every continuous function on X that can be approximated from $\mathfrak{A}$ in every pair of points in X can in fact be approximated uniformly from $\mathfrak{A}$.*

2.7.4.3 Lemma (**[P2, 4.3.3]**) *If $\mathfrak{A}$ is a uniformly closed algebra of continuous, bounded, real-valued functions on a topological space X, then $\mathfrak{A}$ is stable under the lattice operations $f \vee g$ and $f \wedge g$ in $C(X)$.*

2.7.4.4 Theorem (Stone-Weierstrass Theorem—cf. [P2, 4.3.4]) *Let X be a compact Hausdorff space and $\mathfrak{A}$ be a self-adjoint subalgebra of $C(X)$ containing the constants and separating the points in X. Then $\mathfrak{A}$ is uniformly dense in $C(X)$.*

2.7.4.5 Corollary (**[P2, 4.3.5]**) *Let X be a locally compact Hausdorff space and $\mathfrak{A}$ be a self-adjoint subalgebra of $C_0(X)$ that separates points in X and does not vanish identically at any point of X. Then $\mathfrak{A}$ is uniformly dense in $C_0(X)$.*

2.7.5 More on C*-algebras

We now look at some more properties of C*-algebras, with the main results being the continuous functional calculus for commutative C*-algebras (2.7.5.6) and the continuous functional calculus for the C*-algebra generated by a normal element (2.7.5.8).

2.7.5.1 We say that an element a of a C*-algebra $\mathfrak{A}$ is **normal** if $a^*a = aa^*$. An element $a \in \mathfrak{A}$ is **self-adjoint** if $a = a^*$, and we denote the set of all self-adjoint elements of $\mathfrak{A}$ by $\mathfrak{A}_{\mathrm{sa}}$. We say that a is a **projection** if $a = a^* = a^2$, and we denote the set of all projections of $\mathfrak{A}$ by $\mathcal{P}(\mathfrak{A})$. Projections will be discussed in greater detail in Section 2.7.12. A **positive** element of $\mathfrak{A}$ is a self-adjoint element a such that $\sigma(a) \subseteq \mathbb{R}^+$. The set of positive elements of $\mathfrak{A}$ is denoted $\mathfrak{A}_+$.

An element $a \in \mathfrak{A}$ is a **partial isometry** if a^*a is a projection. See (2.7.12.4) for more on partial isometries.

If $\mathfrak{A}$ is a unital C*-algebra, then $a \in \mathfrak{A}$ is a **unitary** if $aa^* = I = a^*a$, and the set of all unitaries in $\mathfrak{A}$ is denoted by $\mathcal{U}(\mathfrak{A})$. A **symmetry** is a self-adjoint unitary.

2.7.5.2 Lemma ([P2, 4.3.11]) *If a is a normal element in a C*-algebra $\mathfrak{A}$, then $r(a) = \|a\|$.*

2.7.5.3 ([M, 2.1.2]) Let $\mathfrak{A}$ be a C*-algebra. If $a \in \mathfrak{A}$ is normal then $\|a\| = r(a)$ by (2.7.5.2), and if a is any element of $\mathfrak{A}$ then

$$\|a\|^2 = \|a^*a\| = r(a^*a).$$

Since the spectral radius is an algebraic property of $\mathfrak{A}$, the norm on $\mathfrak{A}$ is completely determined by the algebraic structure of $\mathfrak{A}$. In particular, there is only one norm making $\mathfrak{A}$ a C*-algebra.

2.7.5.4 Lemma ([P2, 4.3.12]) *If $\mathfrak{A}$ is a self-adjoint element in a C*-algebra $\mathfrak{A}$, then $\sigma(a) \subseteq \mathbb{R}$. If $\mathfrak{A}$ is unital and u is unitary in $\mathfrak{A}$, then $\sigma(u) \subseteq S^1$.*

2.7.5.5 Theorem 2.7.5.6 and Corollary 2.7.5.7 together make up what is known as the Gelfand-Naimark Theorem. These results are the basis for the following results on C*-algebras. Note that there is another Gelfand-Naimark Theorem, namely (2.7.8.10), which is also quite important.

Proposition (2.7.5.8) is known as the continuous functional calculus for a normal element. A more explicit formulation is (2.7.5.10). It is a powerful tool, and we will use it frequently. In particular, Theorems 2.7.5.6–2.7.5.8 give us cases of the continuous functional calculus we have been looking for.

2.7.5.6 Theorem ([P2, 4.3.13]) *Every commutative, unital C*-algebra $\mathfrak{A}$ is isometrically *-isomorphic to $C(\hat{\mathfrak{A}})$, where $\hat{\mathfrak{A}}$ is the compact Hausdorff space of characters of $\mathfrak{A}$.*

2.7.5.7 Corollary ([P2, 4.3.14]) *Every commutative, nonunital C*-algebra $\mathfrak{A}$ is isometrically *-isomorphic to $C_0(\hat{\mathfrak{A}})$, where $\hat{\mathfrak{A}}$ is the locally compact Hausdorff space of characters of $\mathfrak{A}$.*

2.7.5.8 Proposition ([P2, 4.3.15]) *Let a be a normal element in a unital C^*-algebra $\mathfrak{A}$ and denote by $C^*(a)$ the smallest C^*-subalgebra of $\mathfrak{A}$ that contains a and I. There is then an isometric *-isomorphism Φ of $C(\sigma(a))$ onto $C^*(a)$, such that $\Phi(1) = I$ and $\Phi(id) = a$.*

2.7.5.9 In each of the Theorems (2.7.5.6)–(2.7.5.8), the isomorphism referred to is the Gelfand transform. Note that under the isomorphism described in (2.7.5.8), the (not necessarily unital) C*-algebra generated by (a normal element) a of $\mathfrak{A}$ is taken onto $C_0(\sigma(a)\setminus\{0\})$ (see [Da, I.3.2]). Indeed, if we denote by $\mathfrak{B}$ the C*-algebra generated by a then by (2.7.5.8), there is an isomorphism of $\tilde{\mathfrak{B}} = C^*(a, I_{\mathfrak{A}})$ with $C(\sigma(a))$ which maps I to 1 and a to z. It follows that $\mathfrak{B}$ is isomorphic to the C*-subalgebra of $C(\sigma(a))$ which is generated by polynomials with no constant term, which is exactly $C_0(\sigma(a)\setminus\{0\})$ by Stone-Weierstrass (2.7.4.4).

We now obtain the **continuous functional calculus** for a normal element.

2.7.5.10 Corollary ([Da, I.3.3]) *If a is a normal element of a unital C^*-algebra and f is a continuous function on $\sigma(a)$, the element $f(a) \in \mathfrak{A}$ is defined as the image under the isomorphism $\Phi : C(\sigma(a)) \to C^*(a)$ of (2.7.5.8). If $0 \in \sigma(a)$ and $f(0) = 0$, then $f(a)$ is in the non-unital C^*-algebra generated by a and a^*. Moreover, $\sigma(f(a)) = f(\sigma(a))$. If g is continuous on $f(\sigma(a))$, then $g(f(a)) = (g \circ f)(a)$.*

2.7.5.11 Example ([HJ, 2.25]) As an immediate illustration of the power of (2.7.5.10), note that every element of a unital C*-algebra $\mathfrak{A}$ can be written as a linear combination of four unitaries in $\mathfrak{A}$.

For each $a \in \mathfrak{A}$, the elements $\frac{a+a^*}{2}$ and $\frac{a-a^*}{2i}$ are self-adjoint. So it suffices to show that any self-adjoint element b of norm 1 in $\mathfrak{A}$ can be written as a linear combination of two unitaries. Note that $\sigma(b) \subset \mathbb{R}$ by (2.7.5.4), and that $r(b) = 1$ by (2.7.5.2). So the function $f : \sigma(b) \to \mathbb{C}$ given by

$$f(x) = x + i\sqrt{1 - x^2}$$

is continuous. Applying (2.7.5.10) to f we get an element

$$u = f(b) = \Phi(f) \in \mathfrak{A}.$$

Note that

$$u^*u = f(b)^*f(b) = \Phi(f)^*\Phi(f) = \Phi(\overline{f}f) = \Phi(1) = I,$$

and similarily $uu^* = I$. So u is a unitary, and

$$u + u^* = \left(b + i\sqrt{1-b^2}\right) + \left(b - i\sqrt{1-b^2}\right) = 2b,$$

so

$$b = \frac{u+u^*}{2}.$$

2.7.5.12 We will see other uses for (2.7.5.10) in the next section, where in particular it will allow us to define square roots of positive elements, and to establish the existence of an approximate identity in any C*-algebra (see (2.7.6.9)).

2.7.6 Positive Elements

2.7.6.1 As mentioned in (2.7.5.1), an element $a \in \mathfrak{A}$ is said to be positive if $\sigma(a) \subseteq \mathbb{R}^+$, in particular a must be self-adjoint (this is analogous to the situation in $\mathbb{C}$ where we only try to order the real numbers). We see in (2.7.6.5) that the positive elements $\mathfrak{A}_+$ of $\mathfrak{A}$ form a strict cone, and hence give us a partial order on $\mathfrak{A}$.

Note that in light of (2.7.2.4), an element a in a C*-algebra $\mathfrak{A}$ is positive if and only if it is positive in every C*-algebra to which it belongs.

2.7.6.2 (**[Da, I.4.1]**) As an immediate corollary to the continuous functional calculus, we have that if a is a positive element of a C*-algebra then it has a unique positive square root. Indeed, the function $f(x) = \sqrt{x}$ is continuous on $\sigma(a) \subset \mathbb{R}^+$, and $f(a)^2 = a$.

2.7.6.3 (**[Da, I.4.2]**) As another corollary of the continuous functional calculus we have:

If a is self-adjoint element of a C*-algebra $\mathfrak{A}$, then there are positive elements a_+ and a_- in $\mathfrak{A}$ such that $a = a_+ - a_-$ and $a_+a_- = 0$.

2.7.6.4 Lemma ([Da, I.4.3]) *Let $\mathfrak{A}$ be a unital C^*-algebra. For a self-adjoint element $a \in \mathfrak{A}$, the following are equivalent:*

1. $\sigma(a) \subset \mathbb{R}^+$,
2. $a = b^2$ *for some* $b \in \mathfrak{A}_{sa}$
3. $\|\lambda I - a\| \leq \lambda$ *for all* $\lambda \geq \|a\|, \lambda \in \mathbb{R}$,
4. $\|\lambda I - a\| \leq \lambda$ *for some* $\lambda \geq \|a\|, \lambda \in \mathbb{R}$.

2.7.6.5 Theorem ([P1, 1.3.3]) *The set $\mathfrak{A}_+$ is a closed real cone in $\mathfrak{A}_{sa}$, and $x \in \mathfrak{A}_+$ if and only if $x = y^*y$ for some y in $\mathfrak{A}$.*

2.7.6.6 Corollary ([Da, I.4.6]) *If $a \leq b$ in $\mathfrak{A}_{sa}$ and $x \in \mathfrak{A}$, then we have that $x^*ax \leq x^*bx$.*

2.7.6.7 Lemma ([Da, I.4.7]) *If $0 \leq a \leq b$ are invertible, self-adjoint elements of $\mathfrak{A}_{sa}$, then $b^{-1} \leq a^{-1}$*

2.7.6.8 A net $(e_i)_{i \in \mathcal{I}}$ in the unit ball of a Banach algebra $\mathfrak{A}$ is an **approximate identity** if

$$\lim e_i a = \lim a e_i = a$$

for every $a \in \mathfrak{A}$. in a C*-algebra we can further stipulate that $0 \leq e_i, \|e_i\| \leq I$, and that the net $(e_i)_{i \in \mathcal{I}}$ be directed. We have the following result on approximate identities in C*-algebras.

2.7.6.9 Theorem ([Da, I.4.8]) *Every C*-algebra has an approximate identity.*

As an indication of their importance, the existence of approximate identities is the main tool in the proof of (2.7.7.1).

2.7.6.10 Corollary ([Da, I.4.9]) *If $\mathfrak{A}$ is a separable C*-algebra, then there is an increasing sequence $0 \leq e_1 \leq e_2 \leq \ldots$ of positive norm one elements which form an approximate identity for $\mathfrak{A}$.*

2.7.6.11 Example Consider $\mathfrak{A} = C_0(\mathbb{R})$. For each $n \in \mathbb{N}$, let

$$f_n(x) = \begin{cases} 1 & \text{if } |x| \le n \\ e^{n-|x|} & \text{if } |x| > n \end{cases}.$$

Then $\{f_n\}$ is an approximate identity for $C_0(\mathbb{R})$.

2.7.7 Ideals, Quotients, and Homomorphisms

Recall that an ideal of a C*-algebra is a norm closed two-sided ideal.

2.7.7.1 Lemma ([Da, I.5.1]) *Every ideal of a C*-algebra is self-adjoint.*

So every ideal of a C*-algebra is a C*-algebra. Recall the following Banach space theory result:

2.7.7.2 Proposition ([P2, 2.1.5]) *Let V be a normed vector space and N be a subspace of V. Denote by q the quotient map of V onto the linear space V/N of cosets $x + N, x \in X$. The definition*

$$\|q(x)\| = \inf\{\|x - y\|; y \in N\}$$

gives a seminorm on V/N. If N is closed in V, we actually have a norm; furthermore, if V is a Banach space then V/N is a Banach space.

For C*-algebras, we have

2.7.7.3 Theorem ([Da, I.5.4]) *If $\mathfrak{J}$ is an ideal of a C*-algebra $\mathfrak{A}$, then the quotient algebra $\mathfrak{A}/\mathfrak{J}$ is a C*-algebra.*

We obtain the following fundamental facts about homomorphisms of C*-algebras.

2.7.7.4 Theorem ([Da, I.5.5]) *Let Φ be a non-zero morphism of a C*-algebra $\mathfrak{A}$ into another C*-algebra $\mathfrak{B}$. Then $\|\Phi\| = 1$ and $\Phi(\mathfrak{A})$ is a C*-subalgebra of $\mathfrak{B}$. If Φ is injective, then it is isometric. So in general, Φ factors as $\dot{\Phi}q$ where $q : \mathfrak{A} \to \mathfrak{A}/\ker\Phi$ is the quotient map and $\dot{\Phi}$ is the induced isometric *-isomorphism of the quotient onto $\Phi(\mathfrak{A})$.*

$$
\begin{tikzcd}
\mathfrak{A} \arrow[r, "\Phi"] \arrow[d, "q"'] & \mathfrak{B} \\
\mathfrak{A}/\ker\Phi \arrow[ur, "\Phi"'] &
\end{tikzcd}
$$

2.7.7.5 Corollary ([Da, I.5.6]) *Suppose that $\mathfrak{I}$ is an ideal of a C*-algebra $\mathfrak{A}$, and that $\mathfrak{B}$ is a C*-subalgebra of $\mathfrak{A}$. Then $\mathfrak{B} + \mathfrak{I}$ is closed, and*

$$\mathfrak{B}/(\mathfrak{B} \cap \mathfrak{I}) \simeq (\mathfrak{B} + \mathfrak{I})/\mathfrak{I}$$

2.7.8 The Gelfand-Naimark Theorems

2.7.8.1 A **positive linear functional** on a C*-algebra $\mathfrak{A}$ is a linear functional φ on $\mathfrak{A}$ such that $\varphi(a) \geq 0$ whenever $a \geq 0$. A **state** is a positive linear functional of norm 1. We denote by $S_{\mathfrak{A}}$ the state space of $\mathfrak{A}$.

2.7.8.2 Let $\mathfrak{A}$ be a C*-subalgebra of $\mathbb{B}(\mathfrak{H})$ for some Hilbert space $\mathfrak{H}$. We say that a vector ξ is a **cyclic vector** for $\mathfrak{A}$ if the set

$$\{a\xi; a \in \mathfrak{A}\}$$

is dense in $\mathfrak{H}$.

2.7.8.3 Theorem (The GNS-Construction—cf. [Da, I.9.6]) *Let φ be a positive linear functional on $\mathfrak{A}$. Then there is a representation Φ_φ of $\mathfrak{A}$ on a Hilbert space $\mathfrak{H}$ and a vector $x_\varphi \in \mathfrak{H}$ which is a cyclic vector for $\Phi(\mathfrak{A})$ such that $\|x_\varphi\|^2 = \|\varphi\|$ and*

$$\varphi(a) = (\Phi_\varphi(a)x_\varphi, x_\varphi) \qquad \textit{for all } a \in \mathfrak{A}.$$

2.7.8.4 Note in particular that if a is in the kernel of Φ_φ then

$$\varphi(a) = (\Phi_\varphi(a)x_\varphi, x_\varphi) = (0x_\varphi, x_\varphi) = 0,$$

so $\ker\Phi_\varphi \subseteq \ker\varphi$.

2.7.8.5 (**[P2, 3.1.5]**) If $\{\mathfrak{H}_i; i \in \mathcal{I}\}$ is a family of Hilbert spaces, the **algebraic direct sum of the** $\mathfrak{H}_i$ (denoted by $\sum \mathfrak{H}_i$) consists of those elements of $\prod_{i\in\mathcal{I}} \mathfrak{H}_i$ which have only finitely many nonzero components. The algebraic direct sum of a family $\mathfrak{H}_i$ is a subspace of $\prod_{i\in\mathcal{I}} \mathfrak{H}_i$, on which we define the inner product

$$< x, y > = \sum_{i\in\mathcal{I}} < P_i x, P_i y > \text{ for any } x, y \in \sum \mathfrak{H}_i.$$

Taking the completion of $\sum \mathfrak{H}_i$ gives us a Hilbert space, called the **direct sum** of the $\mathfrak{H}_i$, denoted $\bigoplus_{i\in\mathcal{I}} \mathfrak{H}_i$. The elements of $\bigoplus \mathfrak{H}_i$ are the $x \in \prod \mathfrak{H}_i$ such that $\sum \|P_i x\|^2$ is finite. In particular, $P_i x = 0$ except for a countable number of i's.

2.7.8.6 For each φ in $S_{\mathfrak{A}}$, let $\mathfrak{H}_\varphi$ denote the Hilbert space of (2.7.8.3). For each subset F of $S_{\mathfrak{A}}$ we form the Hilbert space

$$\mathfrak{H}_F = \bigoplus_{\varphi\in F} \mathfrak{H}_\varphi.$$

and the representation

$$\Phi_F = \bigoplus_{\varphi\in F} \Phi_\varphi,$$

given by

$$\Phi_F(a)(x) = (\Phi_\varphi(a)P_\varphi x)_{\varphi\in F} \in \mathfrak{H}_F, \text{ for each } a \in \mathfrak{A} \text{ and each } x \in \mathfrak{H}_F.$$

Note that

$$\ker \Phi_F = \bigcap_{\varphi\in F} \ker \Phi_\varphi.$$

2.7.8.7 Theorem (**[M, Theorem 3.3.6]**) *If a is a normal element of a non-zero C^*-algebra $\mathfrak{A}$, then there is a state φ of $\mathfrak{A}$ such that $\|a\| = |\varphi(a)|$.*

We now combine the previous results to obtain the Gelfand-Naimark Theorem (2.7.8.10) (although see (2.7.5.5)).

2.7.8.8 We say that a subset F of $S_{\mathfrak{A}}$ is **separating** for $\mathfrak{A}$ if, for each x in $\mathfrak{A}_+$, $\varphi(x) = 0$ for all φ in F implies $x = 0$. Note that $S_{\mathfrak{A}}$ is separating for $\mathfrak{A}$ by (2.7.8.7).

2.7.8.9 Theorem ([P1, 3.7.4]) *If $\mathfrak{A}$ is a C^*-algebra and F is a separating family of states of $\mathfrak{A}$, then Φ_F is a faithful representation of $\mathfrak{A}$ into $\mathbb{B}(\mathfrak{H}_F)$.*

2.7.8.10 Corollary (Gelfand-Naimark Theorem—cf. [P1, 3.7.5]) *Any C^*-algebra $\mathfrak{A}$ has a faithful representation as an algebra of operators on a Hilbert space $\mathfrak{H}$. If $\mathfrak{A}$ is separable then $\mathfrak{H}$ can be chosen separable.*

2.7.9 Weak Topologies

The **weak operator topology** on $\mathbb{B}(\mathfrak{H})$ is the topology with a subbasis consisting of the sets

$$\mathcal{W}(T, x, y) = \{A \in \mathbb{B}(\mathfrak{H}); |((T - A)x, y)| < 1\}.$$

A net (T_i) converges **WOT** to an operator T (written $T_i \xrightarrow{WOT} T$) if and only if

$$\lim_i (T_i x, y) = (Tx, y) \quad \text{for all} \quad x, y \in \mathfrak{H}.$$

Analogously, the **strong operator topology**(SOT) on $\mathbb{B}(\mathfrak{H})$ is the topology with a subbasis consisting of the sets

$$\mathcal{S}(T, x) = \{A \in \mathbb{B}(\mathfrak{H}); \|(T - A)x\| < 1\}.$$

A net (T_i) converges SOT to an operator T (written $T_i \xrightarrow{SOT} T$) if and only if

$$\lim_i \|T_i x\| = \|Tx\| \quad \text{for all} \quad x \in \mathfrak{H}.$$

2.7.9.1 Proposition ([Da, I.6.1]) *The* WOT*-continuous and the* SOT*-continuous linear functionals on $\mathbb{B}(\mathfrak{H})$ coincide, and are given by*

$$f(A) = \sum_{i=1}^{n} (Ax_i, y_i), \qquad \text{for all } T \in \mathbb{B}(\mathfrak{H})$$

for a finite set of vectors $x_1, \ldots, x_n, y_1, \ldots, y_n$ in $\mathfrak{H}$.

2.7.9.2 Corollary ([Da, I.6.2]) *$\mathbb{B}(\mathfrak{H})$ has the same closed convex sets in the weak operator and strong operator topologies.*

2.7.9.3 Proposition ([Da, I.6.3]) *If $\mathfrak{H}$ is a separable Hilbert space, then the unit ball of $\mathbb{B}(\mathfrak{H})$ is metrizable in both the weak and strong operator topologies.*

2.7.9.4 ([P2, 4.6.1]) Note also that the unit ball of $\mathbb{B}(\mathfrak{H})$ is compact in the weak operator topology.

2.7.10 Density Theorems

2.7.10.1 If $\mathcal{S}$ is any subset of $\mathbb{B}(\mathfrak{H})$, let the **commutant** of $\mathcal{S}$ be

$$\mathcal{S}' = \{T \in \mathbb{B}(\mathfrak{H}); ST = TS \text{ for all } S \in \mathcal{S}\}.$$

It is straightforward to check that if $\mathcal{S}$ is self-adjoint, then $\mathcal{S}'$ is a self-adjoint, unital algebra.

The following **double commutant theorem** by von Neumann (1929) is a fundamental result in operator algebra theory.

2.7.10.2 Theorem ([P2, 4.6.7]) *For a self-adjoint unital subalgebra of $\mathbb{B}(\mathfrak{H})$, the following are equivalent:*

1. $\mathfrak{A} = \mathfrak{A}''$
2. *$\mathfrak{A}$ is weakly closed*
3. *$\mathfrak{A}$ is strongly closed*

2.7.10.3 A C*-subalgebra of $\mathbb{B}(\mathfrak{H})$ which contains the identity operator and satisfies any of the equivalent conditions of (2.7.10.2) is called a **von Neumann algebra**.

2.7.10.4 Corollary ([P2, 4.6.8]) *If $\mathfrak{A}$ is a self-adjoint, unital subalgebra of $\mathbb{B}(\mathfrak{H})$, its strong (=weak) closure in $\mathbb{B}(\mathfrak{H})$ is precisely $\mathfrak{A}''$.*

The following classical result is the Kaplansky Density Theorem. One proof can be found in [P1, 2.3.3].

2.7.10.5 Theorem (Kaplansky) *Let $\mathfrak{A}$ be a C^*-subalgebra of $\mathbb{B}(\mathfrak{H})$ with strong closure M. Then the unit ball $\mathfrak{A}^1$ of $\mathfrak{A}$ is strongly dense in the unit ball M^1 of M. Furthermore, $\mathfrak{A}^1_{\mathrm{sa}}$ (respectively $\mathfrak{A}^1_+$) is strongly dense in M^1_{sa} (respectively M^1_+). Finally, if $I \in \mathfrak{A}$ then the unitary group of $\mathfrak{A}$ is strongly dense in the unitary group of M.*

2.7.10.6 (**[Da, I.7.4]**) When $\mathfrak{H}$ is separable, the unit ball of $\mathbb{B}(\mathfrak{H})$ in the strong operator topology is metrizable by Proposition 2.7.9.3. So the nets in the Kaplansky Density Theorem can be replaced by sequences.

2.7.10.7 Recall that the centre of a C*-algebra $\mathfrak{A}$ is $\mathfrak{A}' \cap \mathfrak{A}$, and is denoted $\mathcal{Z}(\mathfrak{A})$. A **factor** is a von Neumann algebra $\mathfrak{A}$ with **trivial centre** (i.e. with $\mathcal{Z}(\mathfrak{A}) = \mathbb{C}$). Chapter 3 follows the idea of [Dy, Theorem 2], a result about factors.

2.7.11 Operator Theory

A partial isometry V of a Hilbert space $\mathfrak{H}$ has associated to it its initial projection $P = V^*V$ and its range projection $Q = VV^*$. Then V maps $P\mathfrak{H}$ isometrically onto $Q\mathfrak{H}$, and vanishes on $P^\perp\mathfrak{H}$. The **polar decomposition** of an operator $T \in \mathbb{B}(\mathfrak{H})$ is a factorization $T = VA$ where A is a positive operator and V is a partial isometry from $\overline{\mathrm{ran}(A)}$ to $\overline{\mathrm{ran}(T)}$.

2.7.11.1 Theorem (**[Da, I.8.1]**) *Every operator $T \in \mathbb{B}(\mathfrak{H})$ has a unique polar decomposition $T = UA$. The positive operator $A = |T| = (T^*T)^{1/2}$ lies in $C^*(T)$; and the partial isometry U belongs to the smallest von Neumann algebra in $\mathbb{B}(\mathfrak{H})$ which contains T and I. If T is invertible, then U is a unitary element of $C^*(T, I)$ (see (2.7.1.5) for the definition of $C^*(T, I)$).*

2.7.11.2 Note that a von Neumann algebra has an abundance of projections. If T is a normal operator in $\mathbb{B}(\mathfrak{H})$, the spectral projections of T are in any von Neumann algebra containing T. The spectral theorem tells us that we can "reconstruct" T as a "linear combination" of these projections. More precisely (cf. [P2, Proposition 4.5.8])

$$T = \int_{\sigma(T)} t\, dE(t),$$

where E is a projection valued measure on the Borel subsets of $\sigma(T)$.

2.7.12 Projections in C*-algebras

2.7.12.1 Recall (2.7.5.1) that a projection in a C*-algebra $\mathfrak{A}$ is a self-adjoint idempotent, and that the set of projections in $\mathfrak{A}$ is denoted by $\mathcal{P}(\mathfrak{A})$.

An element $p \in \mathfrak{A}$ is a projection if and only if p is a normal element of $\mathfrak{A}$ and $\sigma(p) \subseteq \{0,1\}$. Indeed, if a is a normal element of $\mathfrak{A}$ with $\sigma(a) \subseteq \{0,1\}$, then we can see a as the continuous function x on $\sigma(a)$ (by using the continuous functional calculus for a normal element). The function x takes on only the values 0 and 1 on $\sigma(a)$, so x is a projection. So a normal element with spectrum contained in $\{0,1\}$ is a projection. Conversely, any projection is normal with spectrum in $\{0,1\}$. In particular, projections are positive.

Let p and q be projections in $\mathfrak{A}$ such that $pq = 0$. Then

$$qp = q^*p^* = (pq)^* = 0^* = 0,$$

so in particular p and q commute. Projections $p, q \in \mathcal{P}(\mathfrak{A})$ are said to be **orthogonal** if $pq = 0 = qp$. If $\mathfrak{A}$ is a C*-algebra and $p \in \mathcal{P}(\mathfrak{A})$ then $I - p$ is a projection in $\tilde{\mathfrak{A}}$ which is orthogonal to p (the projection $I - p$ is called the **orthogonal complement of** p). Note that if p and q are projections in $\mathfrak{A}$ then $p \leq q$ in $\mathfrak{A}$ if and only if $pq = qp = p$, and in this case $q - p$ is a projection orthogonal to p. In particular, any projection p satisfies $0 \leq p \leq 1$.

2.7.12.2 Example Let $\mathfrak{A}$ be a C*-algebra, p a projection in $\mathfrak{A}$. Then pAp is a C*-algebra with multiplicative unit p. That $p\mathfrak{A}p$ is a *-algebra with multiplicative identity p is straightforward. To show that $p\mathfrak{A}p$ is closed, suppose that x is an accumulation point of $p\mathfrak{A}p$. Let (x_n) be a sequence in $p\mathfrak{A}p$ converging to some $x \in \mathfrak{A}$. Since $\|p\| \leq 1$ we see that the sequence (px_np) converges to pxp. But $px_np = x_n$ since $x_n \in p\mathfrak{A}p$ for all n. So $x = \lim_n px_np = pxp$, and hence $x \in p\mathfrak{A}p$. So $p\mathfrak{A}p$ is a C*-algebra.

2.7.12.3 Lemma (**[W, 5.1.3]**) *The sum of two projections is a projection precisely when they are orthogonal, or equivalently when their sum is less than the identity.*

2.7.12.4 Let v be a partial isometry (2.7.5.1) in a C*-algebra $\mathfrak{A}$. Then v^*v is a projection from the definition of a partial isometry. We now show that $x = vv^*$ is also a projection. Indeed,

$$\begin{aligned} x^2 &= vv^*vv^* \\ &= v(v^*v)v^* \\ &= v(v^*v)^2v^* && \text{since } v^*v \text{ is a projection} \\ &= x^3 \end{aligned}$$

Note that x is normal, so by the continuous functional calculus we can think of $x \in C(\sigma(x))$. Since $x^2 = x^3$ in $\mathfrak{A}$, then also $x^2 = x^3$ in $C(\sigma(x))$. It follows that $\sigma(x) \subseteq \{0, 1\}$, so the function x takes on only the values 0 and 1 on $\sigma(x)$. It follows that x is a projection. The projection $p = v^*v$ is called the **support projection** for a partial isometry v, and $q = vv^*$ the **range projection** for v. The following are useful equations on partial isometries:

$$v = vv^*v = vp = qv$$
$$v^* = v^*vv^* = pv^* = v^*q.$$

2.7.12.5 Definition (**[W, 5.2.1]**) *Projections p and q in a C^*-algebra are said to be*

- **equivalent**, *denoted $p \sim q$, when there is a partial isometry $v \in \mathfrak{A}$ such that $p = v^*v$ and $q = vv^*$;*
- **unitarily equivalent**, *denoted $p \sim_u q$, when there is a unitary $u \in \tilde{\mathfrak{A}}$ such that $p = u^*qu$;*
- **homotopic**, *denoted $p \sim_h q$ when p and q are connected by a norm continuous path of projections in $\mathfrak{A}$.*

2.7.12.6 Some remarks on equivalent projections:

1. Equivalent projections are also called **Murray-von Neumann equivalent**, and homotopy equivalence is sometimes called **strong equivalence**.

2. Clearly the trivial projection 0 is never equivalent to anything but 0. The projection I is never unitarily equivalent to anything but I, whereas it can easily be Murray-von Neumann equivalent to smaller projections via non-unitary isometries.

3. If $\mathfrak{J}$ is an ideal of $\mathfrak{A}$ and $p \in \mathfrak{J}$ is a projection which is equivalent to $q \in \mathfrak{A}$, then also $q \in \mathfrak{J}$.

4. If $\mathfrak{A} = \mathbb{B}(\mathfrak{H})$, then

$$p \sim q \iff p\mathfrak{H} \text{ isometrically isomorphic to } q\mathfrak{H} \iff \dim p\mathfrak{H} = \dim q\mathfrak{H}.$$

5. If p, q are equivalent projections, it does not follow that $I - p$ and $I - q$ are equivalent. Indeed, let $\mathfrak{H}$ be an infinite dimensional Hilbert space. Let U be a one-dimensional subspace of $\mathfrak{H}$, and let V be a two-dimensional subspace of $\mathfrak{H}$. Let p be the projection onto the orthogonal complement of U and let q be the projection onto the orthogonal complement of V. Since $\dim(p\mathfrak{H}) = \dim(U^{\perp}) = \dim(V^{\perp}) = \dim(q\mathfrak{H})$ we see that p and q are equivalent (from the comment in (4)). However, it is clear that $I - p$ is not equivalent to $I - q$. If $p \sim_u q$ then necessarily $I - p \sim_u I - q$, and the same holds for $\sim_h$.

2.7.12.7 Lemma ([W, 5.2.3]) *If p_1, p_2, q_1, and q_2 are projections in $\mathfrak{A}$ such that $p_1 \sim q_1$, $p_2 \sim q_2$, $p_1 \perp p_2$, $q_1 \perp q_2$, then $p_1 \oplus p_2 \sim q_1 \oplus q_2$.*

2.7.12.8 Lemma ([W, 5.2.4]) *If p and q are projections in $\mathfrak{A}$ and $z \in \tilde{\mathfrak{A}}$ is an invertible such that $q = zpz^{-1}$ then $p \sim_u q$*

2.7.12.9 Lemma ([W, 5.2.5]) *Let p and q be projections in $\mathfrak{A}$. Then $p \sim_u q$ if and only if $p \sim q$ and $I - p \sim I - q$.*

2.7.12.10 Proposition ([W, 5.2.6]) *If p and q are projections in $\mathfrak{A}$ such that $\|p - q\| < 1$, then they are homotopic. In fact, for each projection $p \in \mathfrak{A}$ there is a continuous* *map $q \mapsto u_q$ from the set of projections close to p into the group of unitaries such that $q = u_q p u_q^*$, with u_q homotopic to the identity when $\|p - q\| < 1$.*

2.7.12.11 Corollary ([W, 5.2.8]) *If (p_i) is a net of projection in $\mathfrak{A}$ converging to a projection p, and if $\|p - p_i\| < 1$ for all $i \in \mathcal{I}$, then there is a net (u_i) of unitaries in $\tilde{\mathfrak{A}}$ such that the net $(\|1 - u_i\|)$ converges to 0 and $p_i = u_i p u_i^*$ for each $i \in \mathcal{I}$.*

2.7.12.12 Let $p, q \in \mathcal{P}(\mathfrak{A})$. If p and q are homotopic, then they are unitarily equivalent. Indeed, if $\{p_t\}_{t\in[0,1]}$ is a norm continuous path of projections with $p_0 = p$ and $p_1 = q$ then there exist $0 = t_0 < t_1 < \ldots < t_n = 1$ such that $\|p_{t_j} - p_{t_{j-1}}\| < 1$ for every $1 \leq j \leq n$. The result now follows from (2.7.12.10).

2.7.12.13 Corollary ([W, 5.2.9]) *Let p and q be projections. Then p is homotopic to q if and only if there is a homotopy (u_λ) of unitaries such that $u_0 = 1$ and $p = u_1 q u_1^*$.*

2.7.12.14 Proposition ([W, 5.2.10]) *If p and q are projections in $\mathfrak{A}$, then*

$$p \sim_h q \Rightarrow p \sim_u q \Rightarrow p \sim q.$$

2.7.12.15 Proposition ([W, 5.2.12])

$$p \sim q \Rightarrow \begin{pmatrix} p & 0 \\ 0 & 0 \end{pmatrix} \sim_u \begin{pmatrix} q & 0 \\ 0 & 0 \end{pmatrix} \text{ and } p \sim_u q \Rightarrow \begin{pmatrix} p & 0 \\ 0 & 0 \end{pmatrix} \sim_h \begin{pmatrix} q & 0 \\ 0 & 0 \end{pmatrix}.$$

It follows that $p \sim q$ implies that $\operatorname{diag}(p, 0, 0, 0) \sim_h \operatorname{diag}(q, 0, 0, 0)$ in $\mathbb{M}_4(\mathfrak{A})$.

As elementary matrix operations leave homotopy classes invariant (see [W, 4.2.8] and [W, 4.2.9]), we obtain the following result.

2.7.12.16 Lemma ([W, 5.2.14]) *If p, q are projections, then*

$$\begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix} \text{ and } \begin{pmatrix} q & 0 \\ 0 & p \end{pmatrix}$$

are homotopic. More generally, a projection is homotopic to any projection constructed from it by elementary matrix operations.

2.7.12.17 Proposition ([W, 5.3.8]) *Let $\mathfrak{A}$ be a C^*-algebra (unital or non-unital). If p and q are equivalent projections for which $\|pq\| < 1$, then they are homotopic.*

2.7.13 Hereditary and Simple C*-subalgebras

2.7.13.1 Let $\mathfrak{A}$ be a C*-algebra, and let $\mathfrak{B}$ be a C^*-subalgebra of $\mathfrak{A}$. Recall that $\mathfrak{A}_+$ is the set of positive elements of $\mathfrak{A}$, and that an element $x \in \mathfrak{B}$ is positive in $\mathfrak{B}$ if and only if x is positive in $\mathfrak{A}$ (cf. (2.7.2.4). We say that $\mathfrak{B}$ is **hereditary** (or a **hereditary subalgebra of** $\mathfrak{A}$) if

$$[a \in \mathfrak{A}, b \in \mathfrak{B}, 0 \leq a \leq b] \Rightarrow a \in \mathfrak{B}.$$

2.7.13.2 Example Obviously, 0 and $\mathfrak{A}$ are hereditary C*-algebras of $\mathfrak{A}$, and any intersection of hereditary C*-algebras is one also.

2.7.13.3 Example ([M, Example 3.2.1]) Let p be a projection in the C*-algebra $\mathfrak{A}$. Then the C*-subalgebra $\mathfrak{B} = p\mathfrak{A}p$ of $\mathfrak{A}$ is hereditary (recall that $\mathfrak{B}$ is a C*-algebra by (2.7.12.2)).

We consider first the case where $\mathfrak{A}$ is a unital C*-algebra. If $pap \in \mathfrak{B}_+$, and $b \in \mathfrak{A}_+$ such that

$$0 \leq b \leq pap \qquad \text{in } \mathfrak{A},$$

then by (2.7.6.6)

$$0 \leq (I-p)b(I-p) \leq (I-p)pap(I-p) = 0 \text{ in } \mathfrak{A},$$

so $(I-p)b(I-p) = 0$. Hence

$$\begin{aligned} 0 &= \|0\| \\ &= \|(I-p)b(I-p)\| \\ &= \|(I-p)b^{1/2}b^{1/2}(I-p)\| \\ &= \|(b^{1/2}(I-p))^* b^{1/2}(I-p)\| \\ &= \|b^{1/2}(I-p)\|^2, \end{aligned}$$

so $b^{1/2}(I-p) = 0$. Thus

$$b(I-p) = b^{1/2}b^{1/2}(I-p) = 0,$$

and so also

$$(I-p)b = (b(I-p))^* = 0.$$

Therefore, $b = pbp \in p\mathfrak{A}p$.

If $\mathfrak{A}$ is not unital, then $\mathfrak{B} = p\mathfrak{A}p = p\tilde{\mathfrak{A}}p$ is a subalgebra of the unital C*-algebra $\tilde{\mathfrak{A}}$. So $\mathfrak{B}$ is hereditary in $\tilde{\mathfrak{A}}$, and hence also in $\mathfrak{A}$.

2.7.13.4 Theorem ([Da, I.5.3]) *Suppose that $\mathfrak{J}$ is an ideal of a C*-algebra $\mathfrak{A}$. If $j \in \mathfrak{J}$ is positive and $a^*a \leq j$, then $a \in \mathfrak{J}$. In particular, ideals are hereditary.*

2.7.13.5 Theorem ([M, Theorem 3.2.7]) *Let $\mathfrak{B}$ be a hereditary C*-subalgebra of a C*-algebra $\mathfrak{A}$, and let $\mathfrak{J}$ be a (closed) ideal of $\mathfrak{B}$. Then there exists a (closed) ideal $\mathfrak{I}$ of $\mathfrak{A}$ such that $\mathfrak{J} = \mathfrak{B} \cap \mathfrak{I}$.*

2.7.13.6 Definition *We say that a C*-algebra is* **simple** *if it has no proper nonzero ideals.*

2.7.13.7 Note that a C*-subalgebra of a simple C*-algebra need not be simple. As a trivial example, $\mathbb{M}_2 \oplus \mathbb{M}_2 \subset \mathbb{M}_4$. Clearly $\mathbb{M}_2 \oplus \mathbb{M}_2$ is not simple, even though $\mathbb{M}_4$ is simple. However, we do have the following result, which follows from (2.7.13.5).

2.7.13.8 Theorem ([M, Theorem 3.2.8]) *Every hereditary C*-subalgebra of a simple C*-algebra is simple.*

In particular, if $\mathfrak{A}$ is simple and $p \in \mathcal{P}(\mathfrak{A})$ then $p\mathfrak{A}p$ is simple by (2.7.13.3) and (2.7.13.8). This fact will be used in the proof of (3.1.0.5), which is a fundamental tool in Chapter 3.

2.7.13.9 Proposition *If $\mathfrak{A}$ is a simple unital C*-algebra then*

$$\mathcal{Z}(\mathfrak{A}) = \{\lambda I; \lambda \in \mathbb{C}\}.$$

Proof:

1. If $\mathcal{Z}(\mathfrak{A}) \neq \mathbb{C}$ then there is a self-adjoint element $a \in \mathcal{Z}(\mathfrak{A}) \setminus \{0\}$ which is not invertible.

   If not, then every nonzero element of $\mathcal{Z}(\mathfrak{A})$ is invertible, so $\mathcal{Z}(\mathfrak{A}) = \mathbb{C}$ by the Gelfand-Mazur theorem (2.7.2.11). So there is some $a \in \mathcal{Z}(\mathfrak{A}) \setminus \{0\}$ which is not invertible. It follows that a^*a is not invertible, and is nonzero since $\|a^*a\| = \|a\|^2 \neq 0$. Hence the result.

2. If there is a nonzero, self-adjoint a in $\mathcal{Z}(\mathfrak{A})$ which is not invertible, then $\mathfrak{A}$ is not simple.

   We have that

   $$\mathfrak{J} = \overline{a\mathfrak{A}}$$

   is an ideal of $\mathfrak{A}$. Indeed, $\mathfrak{J}$ is a proper ideal of $\mathfrak{A}$.

   Since a is not invertible, it follows that $I \notin a\mathfrak{A}$. We now show that $I \notin \overline{a\mathfrak{A}}$ either. Suppose not, then there is some $b \in \mathfrak{A}$ such that

   $$\|I - ab\| < 1.$$

   It follows that ab is invertible. But then for some $c \in \mathfrak{A}$,

   $$I = (ab)c = a(bc)$$

   and

   $$I = c(ab) = c(ba) = (cb)a$$

   so a is left and right invertible, and hence invertible. This is a contradiction, so $I \notin \overline{a\mathfrak{A}}$. Hence the result. ∎

2.8 AF Algebras

2.8.1 Finite Dimensional C*-algebras and Homomorphisms

2.8.1.1 In this chapter we define and examine approximately finite dimensional C*-algebras. As the name suggests, approximately finite dimensional C*-algebras have much in common with their finite dimensional counterparts. We begin with a closer look at the finite dimensional C*-algebras.

2.8.1.2 Lemma ([Da, III.1.1]) *A finite dimensional C*-algebra has a unit.*

This follows from the existence of approximate identities (cf. (2.7.6.9)) and the norm-compactness of the unit ball in a finite dimensional C*-algebra. The following result is more technical, it can be shown either using the Wedderburn structure theorem for semi-simple algebras (cf. [Da]) or by applying C*-algebra results directly (cf. [Da, III.1.2]).

2.8.1.3 Lemma ([Da, III.1.2]) *Every finite dimensional C*-algebra $\mathfrak{A}$ is isomorphic (as a C*-algebra) to a direct sum of full matrix algebras*

$$\mathfrak{A} \simeq \mathbb{M}_{n_1} \oplus \ldots \oplus \mathbb{M}_{n_k} \text{ for some } n_i, k \in \mathbb{N}.$$

2.8.2 Standard Homomorphisms

2.8.2.1 Consider the map $\Phi : \mathbb{M}_2 \rightarrow \mathbb{M}_{2k}$ given by

$$A \mapsto \underbrace{\begin{pmatrix} A & & \\ & \ddots & \\ & & A \end{pmatrix}}_{k \text{ copies of } A} = A \otimes I_k.$$

This map is clearly a homomorphism of $\mathbb{M}_2$ into $\mathbb{M}_{2k}$ (it is also unital and injective). This is an example of a standard homomorphism of finite dimensional C*-algebras (see below). Standard homomorphisms will be the basis for our construction of Bratteli diagrams for approximately finite dimensional C*-algebras (cf. Section 2.8.4).

2.8.2.2 Recall ((4) of (2.6.1.2)) that if

$$(n_1, \ldots, n_l), (p_1, \ldots, p_l) \in \mathbb{Z}^{l+},$$

we say that

$$(n_1, \ldots, n_l) \leq (p_1, \ldots, p_l) \text{ if and only if } n_i \leq p_i \text{ for all } 1 \leq i \leq l.$$

Let $\mathfrak{A} = \mathbb{M}_{m_1} \oplus \ldots \oplus \mathbb{M}_{m_k}$ and $\mathfrak{B} = \mathbb{M}_{n_1} \oplus \ldots \oplus \mathbb{M}_{n_l}$ be finite dimensional C*-algebras. Associate to $\mathfrak{A} = \mathbb{M}_{m_1} \oplus \ldots \oplus \mathbb{M}_{m_k}$ the vector $(m_1, \ldots, m_k) \in \mathbb{Z}^{k+}$, and to $\mathfrak{B} = \mathbb{M}_{n_1} \oplus \ldots \oplus \mathbb{M}_{n_l}$ the vector $(n_1, \ldots, n_l) \in \mathbb{Z}^{l+}$. Let $A = (a_{i,j})$ be an $(l \times k)$ matrix with entries in $\mathbb{Z}^+$ such that

$$A \begin{bmatrix} m_1 \\ \vdots \\ m_k \end{bmatrix} = \begin{bmatrix} p_1 \\ \vdots \\ p_l \end{bmatrix} \leq \begin{bmatrix} n_1 \\ \vdots \\ n_l \end{bmatrix}. \tag{2.8.13}$$

The **standard homomorphism** $\Phi : \mathfrak{A} \to \mathfrak{B}$ associated with A is defined as follows:

$$\Phi(X_1, \ldots, X_k) = \bigoplus_{i=1}^{l} \underbrace{\begin{pmatrix} X_1 \otimes I_{a_{i,1}} & & & \\ & \ddots & & \\ & & X_k \otimes I_{a_{i,k}} & \\ & & & 0_{n_i - p_i} \end{pmatrix}}_{\in \mathbb{M}_{n_i}} \in \bigoplus_{i=1}^{l} \mathbb{M}_{n_i},$$

for $(X_1, \ldots, X_k) \in \mathfrak{A}$. Note also that Φ is unital if and only if

$$A \begin{pmatrix} m_1 \\ \vdots \\ m_k \end{pmatrix} = \begin{pmatrix} n_1 \\ \vdots \\ n_l \end{pmatrix}.$$

Standard homomorphisms are also called canonical homomorphisms (in particular in [Ef]).

2.8.2.3 For any unitary u in a unital C*-algebra $\mathfrak{A}$, we define $\mathrm{Ad}\ u : \mathfrak{A} \to \mathfrak{A}$ to be the inner automorphism $(\mathrm{Ad}\ u)(x) := uxu^*$, for all $x \in \mathfrak{A}$.

Let $\mathfrak{A}$ and $\mathfrak{B}$ be unital C*-algebras. Homomorphisms (not necessarily unital) $\Phi, \Psi : \mathfrak{A} \to \mathfrak{B}$ are said to be **inner equivalent** if there is a unitary $u \in \mathcal{U}(\mathfrak{B})$ such that $\Psi = (\mathrm{Ad}\ u) \circ \Phi$.

2.8.2.4 Proposition *Let $\mathfrak{A}$ and $\mathfrak{B}$ be unital *-algebras. If $\Psi : \mathfrak{A} \to \mathfrak{B}$ is a (not necessarily unital) homomorphism, and if u is a unitary in $\mathfrak{A}$, then there is a unitary $w \in \mathfrak{B}$ such that*

$$\Psi \circ (\mathrm{Ad}\ u) = (\mathrm{Ad}\ w) \circ \Psi. \tag{2.8.14}$$

Consequently, homomorphisms $\Phi, \Psi : \mathfrak{A} \to \mathfrak{B}$ are inner equivalent if and only if there is a unitary $u \in \mathcal{U}(\mathfrak{A})$ and a unitary $v \in \mathcal{U}(\mathfrak{B})$ giving a commutative diagram:

$$\begin{tikzcd} \mathfrak{A} \arrow[r, "\Phi"] \arrow[d, "\mathrm{Ad}\ u"'] & \mathfrak{B} \arrow[d, "\mathrm{Ad}\ v"'] \\ \mathfrak{A} \arrow[r, "\Psi"] & \mathfrak{B} \end{tikzcd}$$

Proof:

For (2.8.14), note that $e = \Psi(I)$ is a projection in $\mathfrak{B}$ and

$$\Psi(u^*)\Psi(u) = \Psi(u)\Psi(u)^* = e.$$

So $w = \Psi(u) + (I - e)$ is a unitary in $\mathfrak{B}$, and

$$\Psi \circ (\mathrm{Ad}\ u) = (\mathrm{Ad}\ w) \circ \Psi.$$

This shows (2.8.14). We now show the equivalence.

If Ψ and Φ are inner equivalent then there is a unitary $u \in \mathcal{U}(\mathfrak{A})$ such that $\Psi = (\mathrm{Ad}\ u) \circ \Phi$, and we have the diagram:

$$\begin{tikzcd} \mathfrak{A} \arrow[r, "\Phi"] \arrow[d, "\mathrm{Ad}\ I"'] & \mathfrak{B} \arrow[d, "\mathrm{Ad}\ u"'] \\ \mathfrak{A} \arrow[r, "\Psi"] & \mathfrak{B} \end{tikzcd} .$$

Conversely, suppose that we have a commutative diagram

$$\begin{tikzcd} \mathfrak{A} \arrow[r, "\Phi"] \arrow[d, "\mathrm{Ad}\ u"'] & \mathfrak{B} \arrow[d, "\mathrm{Ad}\ v"'] \\ \mathfrak{A} \arrow[r, "\Psi"] & \mathfrak{B} \end{tikzcd} ,$$

By the first assertion, $\Psi \circ (\mathrm{Ad}\ u) = (\mathrm{Ad}\ w) \circ \Psi$ for some $w \in \mathcal{U}(\mathfrak{B})$, so

$$(\mathrm{Ad}\ w) \circ \Psi = \Psi \circ (\mathrm{Ad}\ u) = (\mathrm{Ad}\ v) \circ \Phi,$$

so

$$\begin{aligned}\Psi &= (\mathrm{Ad}\ w)^{-1} \circ (\mathrm{Ad}\ v) \circ \Phi \\ &= (\mathrm{Ad}\ w^*) \circ (\mathrm{Ad}\ v) \circ \Phi \\ &= (\mathrm{Ad}\ w^* v) \circ \Phi\end{aligned}$$

Hence Ψ and Φ are inner equivalent.

■

2.8.2.5 Lemma ([Ef, Lemma 2.2]) *Given $\mathfrak{A} = \mathbb{M}_{n_1} \oplus \ldots \oplus \mathbb{M}_{n_k}$ and $\mathfrak{B} = \mathbb{M}_{m_1} \oplus \ldots \oplus \mathbb{M}_{m_l}$, any homomorphism $\Phi : \mathfrak{A} \to \mathfrak{B}$ is inner equivalent to a unique standard homomorphism.*

Proof:

We now consider the case where $\varphi : \mathbb{M}_n \to \mathbb{M}_m$ is a unital C*-algebra homomorphism. We claim that there is some $k \in \mathbb{Z}^+$ such that $m = kn$, and that φ is unitarily equivalent to

$$A \mapsto A \otimes I_k \tag{2.8.15}$$

Indeed, let $\{e_{ij}\}$ be a system of matrix units in $\mathbb{M}_n$. Then $\{\varphi(e_{ii})\}$ are equivalent projections in $\mathbb{M}_m$ which sum to the identity, so they each have the same rank, and the sum of the ranks is m. It follows that $m = kn$ for some $k \in \mathbb{Z}^+$, and that each of the $\varphi(e_{ii})$ has rank k. Let $\{f_1^j\}$, be orthogonal minimal projections in $\mathbb{M}_m$ under $\varphi(e_{11})$, and for each j let x_1^j be a unit vector in $f_1^j \mathbb{C}^m$. Let $x_i^j = \varphi(e_{i1}) x_1^j$. We define an ordering of $\{x_i^j\}$ by letting $x_i^j \leq x_k^l$ if $j \leq l$ or if $j = l$ and $i \leq k$. With respect to the ordered basis $\{x_i^j\}$ of $\mathbb{C}^m$, the f_i^i are diagonal of rank one, and the map φ is represented as in (2.8.15). Hence the claim.

We now consider the case of a C*-algebra homomorphism $\varphi : \mathbb{M}_n \to \mathbb{M}_m$ which is not unital. Then $\varphi(I_n)$ is a projection in $\mathbb{M}_m$, and $\varphi(I_n) \mathbb{M}_m \varphi(I_n)$ is a C*-subalgebra of

$\mathbb{M}_m$ with unit $\varphi(I_n)$, and is isomorphic to $\mathbb{M}_{m'}$ for some $m' \in \mathbb{Z}^+$. It follows from the previous case that there is a $k \in \mathbb{Z}^+$ and a unitary $u \in \mathcal{U}(\mathbb{M}_{m'})$ such that $kn = m' \leq m$ and $\varphi = (\mathrm{Ad}\ u)\circ\psi$ for some standard homomorphism $\psi : \mathbb{M}_n \to \mathbb{M}_{m'} \subset \mathbb{M}_m$. Let $v = \varphi(I_n)u\varphi(I_n) + I_m - \varphi(I_n)$. Then $\varphi = (\mathrm{Ad}\ v)\circ\psi$, i.e. φ is inner equivalent to the standard homomorphism

$$A \mapsto \begin{pmatrix} A \otimes I_k & \\ & 0_{m-kn} \end{pmatrix}.$$

and we say that $\mathbb{M}_n$ is embedded in $\mathbb{M}_m$ with multiplicity k.

The remaining assertions are straightforward. ∎

2.8.2.6 Let $\mathfrak{A} = \mathbb{M}_{m_1} \oplus \ldots \oplus \mathbb{M}_{m_k}$ and $\mathfrak{B} = \mathbb{M}_{n_1} \oplus \ldots \oplus \mathbb{M}_{n_l}$ be finite dimensional C*-algebras and let $\Phi : \mathfrak{A} \to \mathfrak{B}$ be a homomorphism. To Φ we can associate a unique standard homomorphism, and therefore a unique matrix $A = (a_{ij}) \in \mathbb{M}_{l\times k}(\mathbb{Z}^+)$. This will sometimes be written $\Phi = (a_{ij})$. Note in particular that A has the property

$$A \begin{bmatrix} m_1 \\ \vdots \\ m_k \end{bmatrix} \leq \begin{bmatrix} n_1 \\ \vdots \\ n_l \end{bmatrix}. \tag{2.8.16}$$

and that Φ is unital if and only if we have equality in Equation (2.8.16).

2.8.3 AF Algebras

2.8.3.1 Definition *A C*-algebra $\mathfrak{A}$ is an* **approximately finite C*-algebra** *(or* **AF algebra***) if it is the C*-algebraic direct limit of a directed system $\{\mathfrak{A}_i, \Phi_{ij}\}_{i,j\in\mathbb{N}}$ of finite dimensional C*-algebras $\mathfrak{A}_i$ and C*-algebra homomorphisms $\{\Phi_{ij}\}$.*

In words, $\mathfrak{A}$ is an AF algebra if it is the C-algebraic direct limit of a (countable) sequence of finite dimensional C*-algebras and C*-algebra homomorphisms.*

2.8.3.2 Given a sequence of C*-algebras $\mathfrak{A}_i$, and homomorphisms $\Phi_{i+1,i} : \mathfrak{A}_i \to \mathfrak{A}_{i+1}$ we can construct a directed system by letting $\Phi_{ij} = \Phi_{i,i-1}\circ\ldots\circ\Phi_{j+1,j}$ whenever $i > j$, and $\Phi_{ii} = \mathrm{id}$. It is clear that $\Phi_{ij} = \Phi_{ik} \circ \Phi_{kj}$ whenever $i \geq k \geq j$. Letting $\mathfrak{A}$ be the C*-algebraic direct limit of this directed system we write $\mathfrak{A} = \varinjlim\{\mathfrak{A}_i, \Phi_{i+1,i}\}$, or

$$\mathfrak{A}_1 \xrightarrow{\Phi_{2,1}} \mathfrak{A}_2 \xrightarrow{\Phi_{3,2}} \cdots \longrightarrow \mathfrak{A} \tag{2.8.17}$$

2.8.3.3 Example For each $n \in \mathbb{N}$, let $\mathfrak{A}_n = \mathbb{M}_n(\mathbb{C})$, and let $\Phi_{n+1,n}$ be the embedding of $\mathfrak{A}_n$ into $\mathfrak{A}_{n+1}$ with multiplicity one, given by the standard homomorphism

$$\Phi_{n+1,n}(A) = \left(\begin{array}{c|c} \mathrm{A} & 0 \\ \hline 0 & 0 \end{array}\right) \in \mathbb{M}_{n+1}, \text{ for any } A \in \mathbb{M}_n.$$

Constructing maps Φ_{ij} as in (2.8.3.2), we obtain a directed system $\{\mathfrak{A}_i, \Phi_{ij}\}$. We can think of the C*-algebraic direct limit $\mathfrak{A}$ of this system as operating on the Hilbert space $\ell^2(\mathbb{N})$. Clearly $\mathfrak{A}$ is the closure of the finite rank operators in $\mathfrak{B}(\ell^2(\mathbb{N}))$, so $\mathfrak{A}$ is in fact the compact operators on $\ell^2(\mathbb{N})$ (cf. [P2, 3.3.3]). Note that there are compact operators which do not have finite rank (cf. [P2, 3.3.5]), and hence $\mathfrak{A} \neq \cup\Phi_i(\mathfrak{A}_i)$.

2.8.3.4 Suppose that $\{\mathfrak{A}_i, \Phi_{i+1,i}\}$ and $\{\mathfrak{B}_i, \Psi_{i+1,i}\}$ are two sequences of C*-algebras and C*-algebra homomorphisms, and that there is a sequence of indices

$$n_1 \leq m_1 < n_2 \leq m_2 < n_3 \leq m_3 < \ldots$$

so that for every $k \in \mathbb{N}$ there are homomorphisms $\alpha_k : \mathfrak{A}_{n_k} \to \mathfrak{B}_{m_k}$ and $\beta_k : \mathfrak{B}_{m_k} \to \mathfrak{A}_{n_{k+1}}$ such that the diagram

$$\begin{tikzcd}
\mathfrak{A}_{n_1} \arrow[r] \arrow[dr, "\alpha_1"'] & \cdots \arrow[r] & \mathfrak{A}_{n_2} \arrow[r] \arrow[dr, "\alpha_2"'] & \cdots \arrow[r] & \cdots \arrow[r] & \mathfrak{A} \\
\mathfrak{B}_{n_1} \arrow[r] & \mathfrak{B}_{m_1} \arrow[r] \arrow[ur, "\beta_1"'] & \cdots \arrow[r] & \mathfrak{B}_{m_2} \arrow[r] \arrow[ur, "\beta_2"'] & \cdots \arrow[r] & \mathfrak{B}
\end{tikzcd}$$

commutes. In this case we say that the systems $\{\mathfrak{A}_i, \Phi_{i+1,i}\}$ and $\{\mathfrak{B}_i, \Psi_{i+1,i}\}$ are **isomorphic**. A special case occurs when we have a commutative diagram:

$$\begin{tikzcd}
\mathfrak{A}_1 \arrow[r, "\Phi_{2,1}"] \arrow[d, "\theta_1"'] & \mathfrak{A}_2 \arrow[r, "\Phi_{3,2}"] \arrow[d, "\theta_2"'] & \cdots \\
\mathfrak{B}_1 \arrow[r, "\psi_{2,1}"] & \mathfrak{B}_2 \arrow[r, "\psi_{3,2}"] & \cdots
\end{tikzcd}$$

in which the θ_n are all isomorphisms. If two systems are isomorphic we have, by Lemma 2.3.4.3, that there is a C*-algebra isomorphism between $\mathfrak{A}$ and $\mathfrak{B}$ commuting with the diagram. Hence if systems are isomorphic, so are their limits. We see a converse of this for AF algebras in 2.8.5.6.

2.8.3.5 Let $\{\mathfrak{A}_i, \Phi_{i+1,i}\}$ and $\{\mathfrak{B}_i, \Psi_{i+1,i}\}$ be two sequences of C*-algebras and C*-algebra homomorphisms. If for each $n \in \mathbb{N}$ there are isomorphisms δ_n and γ_n such that

$$\begin{array}{ccc} \mathfrak{A}_n & \xrightarrow{\Phi_{n+1,n}} & \mathfrak{A}_{n+1} \\ {\scriptstyle\delta_n}\big\downarrow & & {\scriptstyle\gamma_n}\big\downarrow \\ \mathfrak{B}_n & \xrightarrow{\Psi_{n+1,n}} & \mathfrak{B}_{n+1} \end{array}$$

commutes, it does *not* follow that the systems $\{\mathfrak{A}_i, \Phi_{i+1,i}\}$ and $\{\mathfrak{B}_i, \Psi_{i+1,i}\}$ are isomorphic. Even in the special case where $\mathfrak{A}_n = \mathfrak{B}_n$ for each $n \in \mathbb{N}$, and with $\delta_n \in \mathrm{Aut}(\mathfrak{A}_n)$ and $\gamma_n \in \mathrm{Aut}(\mathfrak{A}_{n+1})$, it does not follow that the systems are isomorphic (indeed, their limits need not be isomorphic ...). The following result will be fundamental in linking AF algebras and Bratteli diagrams.

2.8.3.6 Lemma ([Ef, Lemma 2.1]) *Let $\mathfrak{A}_1, \mathfrak{A}_2, \ldots$ be unital *-algebras, and $\Phi_{n+1,n}$ and $\Psi_{n+1,n}$ homomorphisms (not neccessarily unital). If the systems*

$$\mathfrak{A}_1 \xrightarrow{\Phi_{2,1}} \mathfrak{A}_2 \xrightarrow{\Phi_{3,2}} \cdots$$

and

$$\mathfrak{A}_1 \xrightarrow{\Psi_{2,1}} \mathfrak{A}_2 \xrightarrow{\Psi_{3,2}} \cdots$$

are such that (for each n) $\Phi_{n+1,n}$ and $\Psi_{n+1,n}$ are inner equivalent, then

$$\varinjlim(\mathfrak{A}_n, \Phi_{n+1,n}) \simeq \varinjlim(\mathfrak{A}_n, \Psi_{n+1,n}).$$

Proof: The goal is to construct inner automorphisms $\theta_n : \mathfrak{A}_n \to \mathfrak{A}_n$ such that the diagram

$$\begin{array}{ccccccccc} \mathfrak{A}_1 & \xrightarrow{\Phi_{2,1}} & \mathfrak{A}_2 & \xrightarrow{\Phi_{3,2}} & \cdots & \xrightarrow{\Phi_{n,n-1}} & \mathfrak{A}_n & \xrightarrow{\Phi_{n+1,n}} & \cdots \\ {\scriptstyle\theta_1}\big\downarrow & & {\scriptstyle\theta_2}\big\downarrow & & & & {\scriptstyle\theta_n}\big\downarrow & & \\ \mathfrak{A}_1 & \xrightarrow{\Psi_{2,1}} & \mathfrak{A}_2 & \xrightarrow{\Psi_{3,2}} & \cdots & \xrightarrow{\Psi_{n,n-1}} & \mathfrak{A}_n & \xrightarrow{\Psi_{n+1,n}} & \cdots \end{array}$$

commutes. We proceed inductively. Let $\theta_1 = \mathrm{id}$. Suppose that we have defined inner automorphisms $\theta_1, \theta_2, \ldots, \theta_n$ such that the diagram commutes. Since $\Phi_{n+1,n}$ and $\Psi_{n+1,n}$ are inner equivalent, the same is true of $\Phi_{n+1,n}$ and $\Psi_{n+1,n} \circ \theta_n$ by Equation (2.8.14). It now follows from the definition of inner equivalence of $\Phi_{n+1,n}$ and $\Psi_{n+1,n} \circ \theta_n$ that there is an inner automorphism θ_{n+1} of $\mathfrak{A}_{n+1}$ such that $\Psi_{n+1,n} \circ \theta_n = \theta_{n+1} \circ \Phi_{n+1,n}$. Hence the diagram

$$\begin{tikzcd}
\mathfrak{A}_1 \arrow[r, "\Phi_{2,1}"] \arrow[d, "\theta_1"'] & \mathfrak{A}_2 \arrow[r, "\Phi_{3,2}"] \arrow[d, "\theta_2"'] & \ldots \arrow[r, "\Phi_{n,n-1}"] & \mathfrak{A}_n \arrow[r, "\Phi_{n+1,n}"] \arrow[d, "\theta_n"'] & \mathfrak{A}_{n+1} \arrow[d, "\theta_{n+1}"'] \\
\mathfrak{A}_1 \arrow[r, "\Psi_{2,1}"] & \mathfrak{A}_2 \arrow[r, "\Psi_{3,2}"] & \ldots \arrow[r, "\Psi_{n,n-1}"] & \mathfrak{A}_n \arrow[r, "\Psi_{n+1,n}"] & \mathfrak{A}_{n+1}
\end{tikzcd}$$

commutes. The result now follows from Lemma 2.3.4.3. ∎

2.8.3.7 The proof of Lemma 2.8.3.6 shows also that given, for every $n \in \mathbb{N}$, a commutative diagram

$$\begin{tikzcd}
\mathfrak{A}_n \arrow[r, "\Phi_{n+1,n}"] \arrow[d, "\mathrm{Ad}\, u_n"'] & \mathfrak{A}_{n+1} \arrow[d, "\mathrm{Ad}\, v_n"'] \\
\mathfrak{A}_n \arrow[r, "\Psi_{n+1,n}"] & \mathfrak{A}_{n+1}
\end{tikzcd}$$

then it is possible to find a morphism $\theta : \mathfrak{A} \to \mathfrak{A}$ and unitaries $w_n \in \mathcal{U}(\mathfrak{A}_n)$ such that

$$\begin{tikzcd}
\mathfrak{A}_1 \arrow[r, "\Phi_{2,1}"] \arrow[d, "\mathrm{Ad}\, w_1"'] & \mathfrak{A}_2 \arrow[r, "\Phi_{3,2}"] \arrow[d, "\mathrm{Ad}\, w_2"'] & \ldots \arrow[r] & \mathfrak{A} \arrow[d, "\theta"] \\
\mathfrak{A}_1 \arrow[r, "\Psi_{2,1}"] & \mathfrak{A}_2 \arrow[r, "\Psi_{3,2}"] & \ldots \arrow[r] & \mathfrak{A}
\end{tikzcd}$$

commutes.

2.8.3.8 Consider the system

$$\mathfrak{A}_1 \xrightarrow{\Phi_{2,1}} \mathfrak{A}_2 \xrightarrow{\Phi_{3,2}} \ldots$$

of finite dimensional C*-algebras and C*-algebra homomorphisms. Using the result of Lemma 2.8.1.3, for each $n \in \mathbb{N}$, we have an isomorphism $\theta_n : \mathfrak{A}_n \to \mathbb{M}_{m_{n,1}} \oplus \ldots \oplus \mathbb{M}_{m_{n,k_n}}$.

Setting $\Psi_{n+1,n} = \theta_{n+1} \circ \Phi_{n+1,n} \circ \theta_n^{-1}$, and letting $\mathfrak{B}$ denote the C*-algebraic direct limit $\mathfrak{B} = \varinjlim\{\mathbb{M}_{m_{n,1}} \oplus \ldots \oplus \mathbb{M}_{m_{n,k_n}}, \Psi_{n+1,n}\}$ we obtain (again using the result of Lemma 2.3.4.3) an isomorphism θ and a commutative diagram

$$\begin{tikzcd}
\mathfrak{A}_1 \arrow[r, "\Phi_{2,1}"] \arrow[d, "\theta_1"'] & \mathfrak{A}_2 \arrow[r, "\Phi_{3,2}"] \arrow[d, "\theta_2"'] & \ldots \arrow[r] & \mathfrak{A} \arrow[d, "\theta"] \\
\mathbb{M}_{m_{1,1}} \oplus \ldots \oplus \mathbb{M}_{m_{1,k_1}} \arrow[r, "\Psi_{2,1}"] & \mathbb{M}_{m_{2,1}} \oplus \ldots \oplus \mathbb{M}_{m_{2,k_2}} \arrow[r, "\Psi_{3,2}"] & \ldots \arrow[r] & \mathfrak{B}
\end{tikzcd}$$

Lemma 2.8.2.5 shows that for each $n \in \mathbb{N}$ there is a (unique) standard homomorphism $\xi_{n+1,n}$ which is inner equivalent to $\Psi_{n+1,n}$. Using (2.8.3.7) we see that there is an isomorphism $\omega : \mathfrak{B} \to \mathfrak{B}$, and $w_n \in \mathcal{U}(\mathbb{M}_{m_{n,1}} \oplus \ldots \oplus \mathbb{M}_{m_{n,k_n}})$ such that

$$\begin{tikzcd}
\mathfrak{A}_1 \arrow[r, "\Phi_{2,1}"] \arrow[d, "\theta_1"'] & \mathfrak{A}_2 \arrow[r, "\Phi_{3,2}"] \arrow[d, "\theta_2"'] & \ldots \arrow[r] & \mathfrak{A} \arrow[d, "\theta"] \\
\mathbb{M}_{m_{1,1}} \oplus \ldots \oplus \mathbb{M}_{m_{1,k_1}} \arrow[r, "\Psi_{2,1}"] \arrow[d, "\mathrm{Ad}\, w_1"'] & \mathbb{M}_{m_{2,1}} \oplus \ldots \oplus \mathbb{M}_{m_{2,k_2}} \arrow[r, "\Psi_{3,2}"] \arrow[d, "\mathrm{Ad}\, w_2"'] & \ldots \arrow[r] & \mathfrak{B} \arrow[d, "\omega"] \\
\mathbb{M}_{m_{1,1}} \oplus \ldots \oplus \mathbb{M}_{m_{1,k_1}} \arrow[r, "\xi_{2,1}"] & \mathbb{M}_{m_{2,1}} \oplus \ldots \oplus \mathbb{M}_{m_{2,k_2}} \arrow[r, "\xi_{3,2}"] & \ldots \arrow[r] & \mathfrak{B}
\end{tikzcd}$$

commutes.

This shows that

$$\varinjlim(\mathfrak{A}_n, \Phi_{n+1,n}) \simeq \varinjlim(\mathbb{M}_{m_{n,1}} \oplus \ldots \oplus \mathbb{M}_{m_{n,k_n}}, \xi_{n+1,n}).$$

This result allows us to focus exclusively on **standard systems**, i.e. on systems: $\{\mathfrak{A}_i, \Phi_{i+1,i}\}$ in which the $\mathfrak{A}_n$ are direct sums of full matrix algebras, and the $\Phi_{n+1,n}$ are standard homomorphisms.

The standard system $\{\mathbb{M}_{m_{n,1}} \oplus \ldots \oplus \mathbb{M}_{m_{n,k_n}}, \xi_{n+1,n}\}$ constructed as above will be called the standard system corresponding to the directed system $\{\mathfrak{A}_i, \Phi_{i+1,i}\}$.

2.8.4 Bratteli diagrams of AF Algebras

We now associate to a system of finite dimensional C*-algebras and C*-algebra homomorphisms a Bratteli diagram (see Section 2.4). We will do this by specifying the

V_n and the incidence matrices A_n, and hence specifying the Bratteli diagram (up to equivalence).

Let $\{\mathfrak{A}_i, \Phi_{i+1,i}\}$ be such a system, and let $\{\mathbb{M}_{m_{n,1}} \oplus \ldots \oplus \mathbb{M}_{m_{n,k_n}}, \xi_{n+1,n}\}$ be the correponding standard system. For each $n \in \mathbb{N}$, let A_n be the matrix corresponding to $\xi_{n+1,n}$. We now define the sets $V = \cup V_n$ and $E = \cup E_n$, as well as the source and range maps at level n. Let

$$V_n = \{(1, -n), \ldots, (k_n, -n)\}.$$

Let $E_n, r_n : E_n \to V_n$,and $s_n : E_n \to V_{n-1}$ be defined so as to correspond to the incidence matrix A_n. Then the set (E, V) is a Bratteli diagram, called the Bratteli diagram of the system $\{\mathfrak{A}_i, \Phi_{i+1,i}\}$. Note that sometimes the Bratteli diagram just constructed will be refered to as the Bratteli diagram of $\mathfrak{A}$. This is somewhat misleading as $\mathfrak{A}$ will be the direct limit of many directed systems, so there are many Bratteli diagrams corresponding to $\mathfrak{A}$. The main result of Bratteli's Thesis is that these Bratteli diagrams are all equivalent, but this is not obvious at this stage. What can be shown immediately is that a Bratteli diagram corresponds to a unique algebra.

2.8.4.1 Proposition *If two AF algebras $\mathfrak{A} = \varinjlim\{\mathfrak{A}_i, \Phi_{i+1,i}\}$ and $\mathfrak{B} = \varinjlim\{\mathfrak{B}_i, \Psi_{i+1,i}\}$ have the same Bratteli diagrams, they are isomorphic.*

2.8.4.2 Examples

1. We consider again the compact operators on a separable Hilbert space, as discussed in (2.8.3.3). Let $\mathfrak{A}_n = \mathbb{M}_n(\mathbb{C})$, and let $\Phi_{n+1,n}$ be the embedding of $\mathfrak{A}_n$ into $\mathfrak{A}_{n+1}$ by

$$\Phi_{n+1,n}(A) = \left(\begin{array}{c|c} \mathrm{A} & 0 \\ \hline 0 & 0 \end{array}\right) \in \mathfrak{A}_{n+1}, \text{ for any } A \in \mathbb{M}_n.$$

Since each $\mathfrak{A}_n$ is embedded in $\mathfrak{A}_{n+1}$ with multiplicity one, the standard homomorphism corresponding to the $\Phi_{n+1,n}$ are all given by the (1×1) matrix (1). So the corresponding canonical system can be written

$$\mathbb{M}_1 \xrightarrow{(1)} \mathbb{M}_2 \xrightarrow{(1)} \mathbb{M}_3 \xrightarrow{(1)} \ldots$$

The Bratteli diagram corresponding to this system is

2. The **CAR algebra** is obtained by taking the direct limit of the algebras $\mathfrak{A}_n = \mathbb{M}_{2^n}$ with connecting homomorphisms $\Phi_{n+1,n} : \mathfrak{A}_n \to \mathfrak{A}_{n+1}$ given by

$$\Phi_{n+1,n}(A) = A \otimes I_2.$$

Clearly the $\Phi_{n+1,n}$ are unital standard homomorphisms, so the corresponding canonical system is

$$\mathbb{M}_1 \xrightarrow{(2)} \mathbb{M}_2 \xrightarrow{(2)} \mathbb{M}_4 \longrightarrow \ldots .$$

The Bratteli diagram for this canonical system is

Let $\mathfrak{B}_k = \mathbb{M}_{2^k} \oplus \mathbb{M}_{2^k}$ and let $\Psi_{k+1,k} : \mathfrak{B}_k \to \mathfrak{B}_{k+1}$ be defined by

$$\Psi_{k+1,k}\left(B_1 \oplus B_2\right) = \begin{pmatrix} B_1 & \\ & B_2 \end{pmatrix} \oplus \begin{pmatrix} B_1 & \\ & B_2 \end{pmatrix}.$$

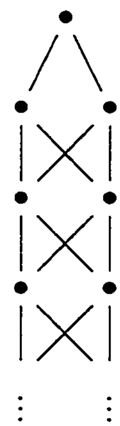
We have

$$\begin{tikzcd}
 & \mathfrak{B}_1 \arrow[rr, "\Psi_{2,1}"] \arrow[dr, "\iota"'] & & \mathfrak{B}_2 \arrow[rr] \arrow[dr, "\iota"'] & & \cdots \arrow[r] & \mathfrak{B} \\
\mathfrak{A}_1 \arrow[rr] & & \mathfrak{A}_2 \arrow[ur, "\beta"'] \arrow[rr, "\Phi_{2,1}"'] & & \mathfrak{A}_3 \arrow[ur, "\beta"'] \arrow[r] & \cdots \arrow[r] & \mathfrak{A}
\end{tikzcd}$$

for the obvious choices of ι and β. It follows, again from (2.3.4.3), that the algebra $\mathfrak{A}$ is also the direct limit of the $\mathfrak{B}_k$. The canonical system corresponding to the directed system $\{\mathfrak{B}_k, \Psi_{k+1,k}\}$ is

$$\mathbb{M}_1 \xrightarrow{\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}} \mathbb{M}_1 \oplus \mathbb{M}_1 \xrightarrow{\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}} \mathbb{M}_2 \oplus \mathbb{M}_2 \longrightarrow \ldots$$

with Bratteli diagram



2.8.5 Perturbations

2.8.5.1 The following results show that, in many ways, AF algebras are "almost finite dimensional". The next result, while valid in general C*-algebras, has strong consequences when $\mathcal{B}$ is AF. For example, when combined with (2.8.6.1) it shows that every projection in an AF algebra is finite dimensional.

2.8.5.2 Lemma ([Da, III.3.1]) *Given $\epsilon > 0$ and $n \in \mathbb{N}$, there is a positive real number $\delta = \delta(\epsilon, n)$ so that if $p_1, \ldots, p_n$ are pairwise orthogonal projections in a C*-algebra $\mathcal{D}$ and $\mathfrak{B}$ is a subalgebra of $\mathcal{B}$ such that*

$$\operatorname{dist}(p_i, \mathfrak{B}) = \inf\{||p_i - b||; b \in \mathfrak{B}\} < \delta \text{ for all } 1 \leq i \leq n,$$

then there exist pairwise orthogonal projections $q_i \in \mathfrak{B}$ such that

$$||p_i - q_i|| < \epsilon \text{ for all } 1 \leq i \leq n.$$

If $\sum_{i=1}^{n} p_i = I$, then we can arrange that $\sum_{i=1}^{n} q_i = I$ also.

2.8.5.3 Lemma ([Da, III.3.2]) *Given $\varepsilon > 0$ and $n \in \mathbb{N}$, there is a positive real number $\delta = \delta(\varepsilon, n)$ so that whenever $\mathfrak{A}$ and $\mathfrak{B}$ are C^*-subalgebras of a unital C^*-algebra B such that $\dim \mathfrak{A} \leq n$ and $\mathfrak{A}$ has a system of matrix units $\{e_{ij}^{(s)}\}$ satisfying* $\operatorname{dist}(e_{ij}^{(s)}, \mathfrak{B}) < \delta$, *then there is a unitary $u \in C^*(\mathfrak{A}, \mathfrak{B})$ with $\|u - I\| < \varepsilon$ such that $u\mathfrak{A}u^* \subseteq \mathfrak{B}$.*

2.8.5.4 Lemma ([Da, III.3.3]) *If in addition to the hypothesis of the preceding lemma, we have an algebra $\mathfrak{A}_1$ contained in $\mathfrak{A} \cap \mathfrak{B}$, then we may choose the unitary u to lie in the commutant of $\mathfrak{A}_1$.*

Using the preceding lemmas we can now show the equivalence of our definition of AF algebras and the following (original) definition given by Bratteli. Note in particular that this description of an AF algebra does not depend on the choice of a directed system.

2.8.5.5 Theorem ([Da, III.3.4]) *A C^*-algebra $\mathfrak{A}$ is AF if and only if it is separable and for all $\epsilon > 0$ and $x_1, \ldots, x_n \in \mathfrak{A}$ there exists a finite dimensional C^*-subalgebra $\mathfrak{B} \subset \mathfrak{A}$ such that* $\operatorname{dist}(x_i, \mathfrak{B}) < \epsilon$ *for $1 \leq i \leq n$.*

Moreover, if $\mathfrak{A}_1$ is a finite dimensional subalgebra of $\mathfrak{A}$ then we can choose $\mathfrak{B}$ so that it contains $\mathfrak{A}_1$.

Having used Lemma 2.3.4.3 many times, we now establish a converse for AF algebras.

2.8.5.6 Theorem ([Da, III.3.5]) *Suppose that $\mathfrak{A}$ is an AF algebra which is the closure of the increasing union of two chains $\mathfrak{A} = \overline{\cup_{n\geq 1}\mathfrak{A}_n} = \overline{\cup_{n\geq 1}\mathfrak{B}_n}$. Then for any $\epsilon > 0$, there is a unitary operator $w \in \mathfrak{A}^\sim$ with $\|w - I\| < \epsilon$ so that*

$$\cup_{n\geq 1}\mathfrak{A}_n = w\left(\cup_{n\geq 1}\mathfrak{B}_n\right)w^*.$$

In particular, there are subsequences m_i and n_i of $\mathbb{N}$ so that

$$\mathfrak{A}_{m_i} \subset w\mathfrak{B}_{n_i}w^* \subset \mathfrak{A}_{m_{i+1}} \text{for all } i \geq 1.$$

2.8.5.7 Corollary ([Da, III.3.6]) *If $\mathfrak{A} = \overline{\cup_{n\geq 1}\mathfrak{A}_n}$ and $\mathfrak{B} = \overline{\cup_{n\geq 1}\mathfrak{B}_n}$ are $*$-isomorphic AF algebras, then $\cup_{n\geq 1}\mathfrak{A}_n$ and $\cup_{n\geq 1}\mathfrak{B}_n$ are $*$-isomorphic as normed $*$-algebras.*

2.8.6 Some results on equivalence of projections

2.8.6.1 Proposition ([Da, IV.1.1]) *If p and q are projections in a C^*-algebra $\mathfrak{A}$ such that $\|p-q\| < 1$, then they are equivalent. Thus equivalence is a homotopy invariant.*

2.8.6.2 Corollary ([Da, IV.1.3]) *If $\mathfrak{A} = \overline{\cup_{n\geq 1}\mathfrak{A}_n}$ is an AF algebra, every projection in $\mathfrak{A}$ is equivalent to a projection in $\cup_{n\geq 1}\mathfrak{A}_n$. Moreover, if p and q in $\cup_{n\geq 1}\mathfrak{A}_n$ are equivalent in $\mathfrak{A}$, they are equivalent in $\cup_{n\geq 1}\mathfrak{A}_n$.*

2.8.6.3 Proposition *Let $\mathfrak{A} = \overline{\cup_{n\geq 1}\mathfrak{A}_n}$ be an AF algebra. Then equivalence implies unitary equivalence.*

Proof: Let $p, q \in \mathfrak{A}$ be equivalent. By (2.8.6.2) there is a $p_n \in \cup_{n\geq 1}\mathfrak{A}_n$ such that $p \sim p_n$, say $p_n \in \mathfrak{A}_n$. Similarily let $q_n \in \mathfrak{A}_m$ be such that $q_n \sim q$. Taking n and m larger if necessary, we can assume that p_n and q_n are in the same finite dimensional C*-algebra, which we again call $\mathfrak{A}_n$. Now, $p_n \sim p \sim q \sim q_n$, so $p_n \sim q_n$ in $\mathfrak{A}_n$. Assume that $\mathfrak{A}_n = \oplus_{i=1}^k \mathbb{M}_{n_i}$, and denote by I_{n_i} the identity element in $\mathbb{M}_{n_i}$. Then in $\mathfrak{A}_n$, $p_n \sim q_n$ if and only if rank $(p_n I_{n_i}) =$ rank $(q_n I_{n_i})$ for every i if and only if $p_n \sim_u q_n$. So $p_n \sim_u q_n$.

It remains to show that $p \sim_u p_n$. The following is the proof of [Ef, Corollary A8.3], which we reproduce here for consistency of notation. Since $\|p - p_n\| \leq 1$ then $\|(I-p) - (I-p_n)\| \leq 1$. Hence $I - p \sim I - p_n$. Let u be the partial isometry such that $u^*u = p$ and $uu^* = p_n$. Let v be the partial isometry such that $v^*v = I - p$ and $vv^* = I - p_n$. Then $w = u + v$ is a unitary and $wpw^* = p_n$.

2.8.7 Ideals of AF algebras

The finite dimensionality of the $\mathfrak{A}_n$ is not used in the proof of the following result, and hence the result may be used for more general inductive limits.

2.8.7.1 Lemma ([Da, III.4.1]) *Let $\mathcal{I}$ be an ideal of an AF algebra $\mathfrak{A} = \overline{\cup_{n\geq 1}\mathfrak{A}_n}$. Then*

$$\mathcal{I} = \overline{\cup_{n\geq 1}(\mathcal{I} \cap \mathfrak{A}_n)} = \overline{\mathcal{I} \cap (\cup_{n\geq 1}\mathfrak{A}_n)}$$

Consequently, $\mathcal{I}$ is AF.

2.8.7.2 Definition *A subset $\mathcal{S}$ of a Bratteli diagram is said to be* **directed** *if whenever $x \in \mathcal{S}$ then every image of x (2.4.0.3) is in $\mathcal{S}$. A subset $\mathcal{S}$ of a Bratteli diagram is said to be* **hereditary** *if, whenever $x \in V_n$ is a vertex and every image of x at level $n+1$ is in $\mathcal{S}$, then x is an element of $\mathcal{S}$.*

2.8.7.3 Theorem ([Da, III.4.2]) *Let $\mathfrak{A}$ be an AF algebra with a Bratteli diagram $\mathcal{B}$. Then the ideals of $\mathfrak{A}$ are in a one to one correspondence with directed hereditary subsets $\mathcal{S}$ of $\mathcal{B}$.*

2.8.7.4 Corollary ([Da, III.4.3]) *A unital AF algebra $\mathfrak{A}$ with Bratteli diagram $\mathcal{B}$ is simple if and only if for each $x \in \mathcal{B}$ there is an $n \in \mathbb{N}$ such for every node m at level n, there is a path starting at x and ending at m.*

2.8.7.5 By (2.8.7.4) we see that each of the examples in (2.8.4.2) are simple.

2.8.7.6 Theorem ([Da, III.4.4]) *If $\mathcal{I}$ is an ideal of an AF algebra $\mathfrak{A}$ corresponding to a subset $\mathcal{S}$ of the Bratteli diagram $\mathcal{B}$, then $\mathfrak{A}/\mathcal{I}$ is AF and corresponds to the diagram $\mathcal{B} \setminus \mathcal{S}$.*

2.8.7.7 Proposition *Let $\mathfrak{A}$ be an AF algebra, and let $p \in \mathcal{P}(\mathfrak{A})$. Then $p\mathfrak{A}p$ is AF.*

Proof: Let $b_1, \ldots, b_n \in p\mathfrak{A}p$. By Theorem 2.8.5.5 we see that there is a finite dimensional subalgebra $\mathfrak{B}$ of $\mathfrak{A}$ such that $\operatorname{dist}(b_i, \mathfrak{B}) < \epsilon$ for all i. Now $p\mathfrak{B}p$ is a finite dimensional subalgebra of $p\mathfrak{A}p$ and $\operatorname{dist}(pB_ip, p\mathfrak{B}p) \leq \operatorname{dist}(B_i, \mathfrak{B}) < \epsilon$. Hence $p\mathfrak{A}p$ is AF by Theorem 2.8.5.5. ∎

2.9 K-Theory for C*-algebras

2.9.0.1 Following the presentation of [W, Chapter 6], we will construct a covariant functor K_0 from the category of C*-algebras to the category of abelian groups. The group $K_0(\mathfrak{A})$ will consist of differences of equivalence classes of projections, though for technical reasons these projections will be taken from matrix rings over the algebra.

We will be using projections heavily. See (2.7.5.1) and Section 2.7.12 for notation, basic definitions, and properties.

Our first step is to define an addition on equivalence classes of projections. Let $\mathfrak{A}$ be a C*-algebra, and for each $n \in \mathbb{N}$, let $\mathbb{M}_n(\mathfrak{A})$ be the matrix algebra over $\mathfrak{A}$. We imbed $\mathbb{M}_n(\mathfrak{A})$ into $\mathbb{M}_{n+1}(\mathfrak{A})$ by means of the standard homomorphism

$$A \in \mathbb{M}_n(\mathfrak{A}) \longmapsto \left(\begin{array}{c|c} \mathrm{A} & \\ \hline & 0 \end{array}\right) \in \mathbb{M}_{n+1}(\mathfrak{A}). \tag{2.9.18}$$

and we denote the direct limit of this sequence by $\mathbb{M}_\infty(\mathfrak{A})$. We now begin our construction of K_0. To construct the group $K_0(\mathfrak{A})$, we first construct an abelian semigroup and then take the Grothendieck group to get an abelian group called $K_{00}(\mathfrak{A})$. The definition of $K_0(\mathfrak{A})$ follows shortly thereafter.

2.9.0.2 Let $p \in \mathcal{P}(\mathbb{M}_\infty(\mathfrak{A}))$. Define

$$[p] = \{q \in \mathcal{P}(\mathbb{M}_\infty(\mathfrak{A})); p \sim q\},$$

and

$$V(\mathfrak{A}) = \{[p]; p \in \mathcal{P}(\mathbb{M}_\infty(\mathfrak{A}))\}.$$

Note that any projection in $\mathcal{P}(\mathbb{M}_\infty(\mathfrak{A}))$ sits in some $\mathbb{M}_n(\mathfrak{A})$. Given projections p and q in $\mathcal{P}(\mathbb{M}_\infty(\mathfrak{A}))$, it is therefore possible to find a number $n \in \mathbb{N}$ so that p and q are in $\mathcal{P}(\mathbb{M}_n(\mathfrak{A}))$. Using (2.7.12.16), we see that

$$\begin{pmatrix} q & 0 \\ 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 0 & 0 \\ 0 & q \end{pmatrix}.$$

(where the sizes of the zeros are not necessarily the same). It follows that, given two equivalence classes $[p], [q] \in V(\mathfrak{A})$, we can always pick orthogonal representatives for $[p]$ and $[q]$. As we have matrices of arbitrary size available in $V(\mathfrak{A})$, there is a natural **addition** in $V(\mathfrak{A})$ given by

$$[p] + [q] = \left[\begin{pmatrix} p & \\ & q \end{pmatrix}\right] \in \mathcal{P}(\mathbb{M}_\infty(\mathfrak{A})),$$

which we will sometimes denote by $[p]+[q] = [p\oplus q]$. Note that if p', q' are projections in $\mathcal{P}(\mathbb{M}_\infty(\mathfrak{A}))$ with $p' \sim p$, $q' \sim q$ and $p'q' = 0$, then $[p] + [q] = [p' + q']$. Using (2.7.12.14)-(2.7.12.16), projections in $\mathcal{P}(\mathbb{M}_\infty(\mathfrak{A}))$ are equivalent if and only if they are unitary equivalent if and only if they are homotopic. So $[p]$ can be thought of as the equivalence class with respect to which ever type of equivalence is most convenient.

2.9.0.3 Proposition ([W, 6.1.3]) *For any C^*-algebra $\mathfrak{A}$, with the addition defined above, $V(\mathfrak{A})$ is an abelian semigroup with additive identity $0 = [0]$.*

Given any morphism $\alpha : \mathfrak{A} \to \mathfrak{B}$ of C^*-algebras, there is a corresponding **induced map** *$V(\alpha) : V(\mathfrak{A}) \to V(\mathfrak{B})$ given by*

$$V(\alpha)([(a_{ij})]) = [(\alpha(a_{ij}))].$$

The correspondence sending $\mathfrak{A}$ to $V(\mathfrak{A})$ and $\alpha \mapsto V(\alpha)$ is a covariant functor from the category of C^*-algebras to the category of abelian semigroups.

2.9.0.4 ([W, 6.1.4]) Some examples:

1. Let p and q be equivalent projections in $\mathbb{M}_\infty(\mathbb{C})$. There is a $n \in \mathbb{N}$ such that $p \sim q \in \mathcal{P}(\mathbb{M}_n)$. Since projections are equivalent in $\mathbb{M}_n$ precisely when they have the same rank, we see that

$$[p] = \{q \in \mathcal{P}(\mathbb{M}_\infty(\mathbb{C})); \text{rank } q = \text{rank } p \text{ in some } \mathbb{M}_n\}.$$

Identifying $[p]$ with the rank of p we see that

$$V(\mathbb{C}) = \mathbb{Z}^+.$$

2. Since $\mathbb{M}_m(\mathbb{M}_n) \simeq \mathbb{M}_{mn}$, we see that $\mathbb{M}_\infty(\mathbb{M}_n) \simeq \mathbb{M}_\infty(\mathbb{C})$. So also

$$V(\mathbb{M}_n) = \mathbb{Z}^+.$$

3. Let $\mathfrak{A}$ be a C*-algebra. Since $\mathbb{M}_m(\mathbb{M}_n(\mathfrak{A})) \simeq \mathbb{M}_{mn}(\mathfrak{A})$, we see that

$$V(\mathbb{M}_n(\mathfrak{A})) = V(\mathfrak{A}).$$

4. If $\mathfrak{H}$ is an infinite dimensional, separable Hilbert space,

$$\mathbb{M}_n(\mathbb{K}(\mathfrak{H})) \simeq \mathbb{K}(\mathfrak{H}^n) \simeq \mathbb{K}(\mathfrak{H}),$$

and similarily $\mathbb{M}_n(\mathbb{B}(\mathfrak{H})) \simeq \mathbb{B}(\mathfrak{H})$. Projections in $\mathbb{B}(\mathfrak{H})$ are equivalent precisely when their ranges have the same dimension (2.7.12.6), and since $\mathbb{B}(\mathfrak{H})$ has both finite and infinite dimensional projections we have

$$V(\mathbb{B}) = \mathbb{Z}^+ \cup \{\infty\}.$$

Similarily, projections in $\mathbb{K}(\mathfrak{H})$ are equivalent if and only if they have the same dimension (since $\mathbb{K}(\mathfrak{H}) \subseteq \mathbb{B}(\mathfrak{H})$ and the partial isometry implementing the equivalence is compact), and since all projections in $\mathbb{K}(\mathfrak{H})$ are finite dimensional, we have

$$V(\mathbb{K}) = \mathbb{Z}^+.$$

All finite rank projections in $\mathbb{B}(\mathfrak{H})$ are in $\mathbb{K}(\mathfrak{H})$ so $V(\mathbb{B}/\mathbb{K})$ has only two elements, namely the equivalence class of all projections of finite rank and the class containing I. Hence

$$V(\mathbb{B}/\mathbb{K}) = \{0, \infty\}.$$

2.9.0.5 Let $\mathfrak{A}$ be a C*-algebra. Define $K_{00}(\mathfrak{A})$ to be the Grothendieck group of $V(\mathfrak{A})$:

$$K_{00}(\mathfrak{A}) = \mathfrak{G}(V(\mathfrak{A})).$$

We can think of elements of $K_{00}(\mathfrak{A})$ as "differences" of equivalence classes of projections. When $\mathfrak{A}$ is unital, we will see in (2.9.0.12) that $K_0(\mathfrak{A})$ is exactly $K_{00}(\mathfrak{A})$.

2.9.0.6 From (2.9.0.4) and the definition of K_{00} we have

1. Since $V(\mathbb{C}) = \mathbb{Z}^+$ we see that $K_{00}(\mathbb{C})$ is $\mathbb{Z}$.

2. Similarily, $K_{00}(\mathbb{M}_n) = \mathbb{Z}$.

3. Since $V(\mathbb{M}_n(\mathfrak{A})) = V(\mathfrak{A})$ for any C*-algebra $\mathfrak{A}$, then also

$$K_{00}(\mathbb{M}_n(\mathfrak{A})) = K_{00}(\mathfrak{A}).$$

4. Let $\mathfrak{H}$ be a separable, infinite dimensional Hilbert space. Clearly $K_{00}(\mathbb{K}) = \mathbb{Z}$ since $V(\mathbb{K}) = \mathbb{Z}^+$. The infinity element in $V(\mathbb{B})$ and $V(\mathbb{B}/\mathbb{K})$ makes $K_{00}(\mathbb{B}) = 0 = K_{00}(\mathbb{B}/\mathbb{K})$ (cf. 2.5.0.9).

2.9.0.7 The functorial nature of V extends to K_{00}. Indeed, the universal property of Grothendieck groups (2.5.0.8) allows us to extend a morphism of semigroups $V(\alpha) : V(\mathfrak{A}) \to V(\mathfrak{B})$ into a morphism $K_{00}(\alpha) : K_{00}(\mathfrak{A}) \to K_{00}(\mathfrak{B})$.

$$\begin{tikzcd} V(\mathfrak{A}) \arrow[r, "V(\alpha)"] \arrow[d, "\iota_A"'] & V(\mathfrak{B}) \arrow[d, "\iota_B"] \\ K_{00}(\mathfrak{A}) \arrow[r, "K_{00}(\alpha)"'] & K_{00}(\mathfrak{B}) \end{tikzcd} \tag{2.9.19}$$

So K_{00} is a covariant functor from the category of C*-algebras to the category of abelian groups. We now use the functoriality of K_{00} to define $K_0(\mathfrak{A})$.

2.9.0.8 Definition *Let $\mathfrak{A}$ be a C*-algebra, and let $\mathfrak{A}^+$ be as in (2.7.1.14). Let $\pi : \mathfrak{A}^+ \to \mathfrak{A}^+/\mathfrak{A} = \mathbb{C}$ be the canonical quotient map. Then $K_0(\mathfrak{A})$ is the kernel of the homomorphism*

$$K_{00}(\pi) : K_{00}(\mathfrak{A}^+) \to K_{00}(\mathbb{C}) = \mathbb{Z}.$$

2.9.0.9 We delay examples of K_0 until (2.9.0.13).

2.9.0.10 Proposition ([W, 6.2.4]) *K_0 is a covariant functor from the category of C*-algebras to the category of abelian groups.*

The homomorphism induced by some morphism $\alpha : \mathfrak{A} \to \mathfrak{B}$ is given by

$$K_0(\alpha)([(x_{ij})] - [(y_{ij})]) = [(\alpha^+(x_{ij}))] - [(\alpha^+(y_{ij}))],$$

where the matrices $(x_{ij}), (y_{ij})$ are projections in $\mathbb{M}_\infty(\mathfrak{A}^+)$ and $\alpha^+ : \mathfrak{A}^+ \to \mathfrak{B}^+$ is the unital morphism defined by $\alpha^+(a + \lambda) = \alpha(a) + \lambda$.

2.9.0.11 Proposition **([W, 6.2.1])** *Let $\mathfrak{A}_1, \mathfrak{A}_2$ be C^*-algebras and consider their direct sum $\mathfrak{A}_1 \oplus \mathfrak{A}_2$ with the obvious projections $\pi_k : \mathfrak{A}_1 \oplus \mathfrak{A}_2 \to \mathfrak{A}_k, k = 1, 2$. The induced maps give rise to isomorphisms*

$$\begin{aligned} V(\pi_1 \oplus \pi_2) &: V(\mathfrak{A}_1 \oplus \mathfrak{A}_2) \to V(\mathfrak{A}_1) \oplus V(\mathfrak{A}_2) \\ K_{00}(\pi_1 \oplus \pi_2) &: K_{00}(\mathfrak{A}_1 \oplus \mathfrak{A}_2) \to K_{00}(\mathfrak{A}_1) \oplus K_{00}(\mathfrak{A}_2) \\ K_0(\pi_1 \oplus \pi_2) &: K_0(\mathfrak{A}_1 \oplus \mathfrak{A}_2) \to K_0(\mathfrak{A}_1) \oplus K_0(\mathfrak{A}_2). \end{aligned}$$

2.9.0.12 Proposition **([W, 6.2.2])** *For every C^*-algebra $\mathfrak{A}$, the split exact sequence*

$$0 \to \mathfrak{A} \to \mathfrak{A}^+ \underset{\iota}{\overset{\pi}{\rightleftarrows}} \mathbb{C} \to 0$$

induces a split exact sequence of groups

$$0 \to K_0(\mathfrak{A}) \to K_{00}(\mathfrak{A}^+) \underset{K_{00}(\iota)}{\overset{K_{00}(\pi)}{\rightleftarrows}} \mathbb{Z} \to 0$$

Thus,

$$K_{00}(\mathfrak{A}^+) \simeq K_0(\mathfrak{A}) \oplus \mathbb{Z}.$$

In particular, if $\mathfrak{A}$ has a unit then $K_0(\mathfrak{A}) = K_{00}(\mathfrak{A})$ is the Grothendieck group for the semigroup $V(\mathfrak{A})$.

2.9.0.13 **([W, 6.2.3])** We see from (2.9.0.12) and (2.9.0.6) that

$$K_0(\mathbb{M}_n) = K_0(\mathbb{C}) = \mathbb{Z}$$

and

$$K_0(\mathbb{B}) = K_0(\mathbb{B}/\mathbb{K}) = \{0\}.$$

It is more difficult to compute $K_0(\mathbb{K})$, but we will see in (2.9.0.19) that K_0 is insensitive to stabilization, and hence

$$K_0(\mathbb{K}) = K_0(\mathbb{C}) = \mathbb{Z}.$$

2.9.0.14 Lemma *Let a be a $(k \times k)$-matrix with entries in a C^*-algebra $\mathfrak{A}$. Then a commutes with $p_n, n \leq k$, precisely when a is a block diagonal matrix of the form* $\operatorname{diag}(a_1, a_2)$ *with $a_1 \in \mathbb{M}_n(\mathfrak{A}), a_2 \in \mathbb{M}_{k-n}(\mathfrak{A})$. If a is invertible then a_1 and a_2 are invertible. If a is unitary then a_1 and a_2 are unitary. Moreover,*

$$ap_n = p_n a = 0 \iff a = \operatorname{diag}(0, a_2), a_2 \in \mathbb{M}_{k-n}(\mathfrak{A}) \iff a_1 = 0.$$

2.9.0.15 Proposition (A Portrait of K_0—[W, 6.2.7]) *The following are some observations about K_0:*

1. *For any C^*-algebra $\mathfrak{A}$, $K_0(\mathfrak{A})$ is an abelian group.*
2. *The elements of $K_0(\mathfrak{A})$ can be visualized as formal differences*

$$[p] - [q]$$

   where $p, q \in \mathcal{P}(\mathbb{M}_k(\mathfrak{A}^+))$ for some $k \in \mathbb{N}$ and $p - q \in \mathbb{M}_k(\mathfrak{A})$.

   When $\mathfrak{A}$ is unital, p and q can be chosen in $\mathbb{M}_k(\mathfrak{A})$ rather than $\mathbb{M}_k(\mathfrak{A}^+)$.

3. *Each element of $K_0(\mathfrak{A})$ can be written*

$$[p] - [p_n]$$

   where $p \in \mathcal{P}(\mathbb{M}_k(\mathfrak{A}^+)), k \geq n$, and $p - p_n \in \mathbb{M}_k(\mathfrak{A})$.

4. *If $[p] - [q] = 0$ in $K_0(\mathfrak{A})$, then for an appropriate $m \leq n$,*

$$\operatorname{diag}(p, p_m) \sim_h \operatorname{diag}(q, p_m) \text{ in } \mathbb{M}_{k+n}(\mathfrak{A}^+)$$

2.9.0.16 Proposition (Continuity of K_0—[W, 6.2.9]) *If $\mathfrak{A}$ is the C^*-algebra inductive limit of a directed system $\{\mathfrak{A}_i, \Phi_{ij}\}$ of C^*-algebras, then the induced system $\{K_0(\mathfrak{A}_i), K_0(\Phi_{ij})\}$ is a directed system of groups and*

$$K_0(\mathfrak{A}) = K_0(\varinjlim \mathfrak{A}_i) \simeq \varinjlim K_0(\mathfrak{A}_i)$$

2.9.0.17 Lemma ([W, 6.2.10]) *Let $\mathfrak{A}$ be a C^*-algebra and denote by ι_{n1} the canonical embedding $\mathfrak{A} \hookrightarrow \mathbb{M}_n(\mathfrak{A})$ given by $a \mapsto \operatorname{diag}(a, 0)$. Then the induced map $K_0(\iota_{n1}) : K_0(\mathfrak{A}) \to K_0(\mathbb{M}_n(\mathfrak{A}))$ is an isomorphism.*

2.9.0.18 Let $\mathfrak{A}$ be a C*-algebra. We call $\mathfrak{A} \otimes \mathbb{K}$ the **stabilization** of $\mathfrak{A}$, and we say that C*-algebras $\mathfrak{A}$ and $\mathfrak{B}$ are **stably isomorphic** if

$$\mathfrak{A} \otimes \mathbb{K} \simeq \mathfrak{B} \otimes \mathbb{K}.$$

2.9.0.19 Corollary (Stability of K_0—[W, 6.2.11]) *The morphism of $\mathfrak{A}$ into $\mathfrak{A} \otimes \mathbb{K}$ sending a to $a \otimes e_{11}$, where e_{11} is a rank one projection in $\mathbb{K}$, induces an isomorphism $K_0(\mathfrak{A}) \simeq K_0(\mathfrak{A} \otimes \mathbb{K})$. In particular, if $\mathfrak{A} \otimes \mathbb{K} \simeq \mathfrak{B} \otimes \mathbb{K}$ (i.e. if $\mathfrak{A}$ and $\mathfrak{B}$ are* **stably isomorphic***), then $K_0(\mathfrak{A}) \simeq K_0(\mathfrak{B})$.*

2.9.0.20 That K_0 cannot distinguish between $\mathfrak{A}$ and $\mathfrak{A} \otimes \mathbb{K}$ should not be too surprising, since $\mathfrak{A} \otimes \mathbb{K}$ is the C^*-completion of $\mathbb{M}_\infty(\mathfrak{A})$.

2.9.1 Exact Sequences

2.9.1.1 Theorem (Half Exactness of K_0—[W, 6.3.2]) *Let $\mathfrak{J}$ be an ideal of $\mathfrak{A}$. The exact sequence $0 \to \mathfrak{J} \xrightarrow{\iota} \mathfrak{A} \xrightarrow{\pi} \mathfrak{A}/\mathfrak{J} \to 0$ induces a short exact sequence of K_0-groups:*

$$K_0(\mathfrak{J}) \xrightarrow{K_0(\iota)} K_0(\mathfrak{A}) \xrightarrow{K_0(\pi)} K_0(\mathfrak{A}/\mathfrak{J})$$

2.9.1.2 ([W, 6.3.3]) Note that in general $K_0(\iota)$ is not injective. Normally $\mathfrak{A}$ contains more partial isometries that $\mathfrak{J}$, so equivalence is easier in $\mathfrak{A}$ than in $\mathfrak{J}$ (making $K_0(\iota)$ potentially non-injective). As a concrete example, consider

$$0 \longrightarrow \mathbb{K} \longrightarrow \mathbb{B} \longrightarrow \mathbb{B}/\mathbb{K} \longrightarrow 0$$

for which the induced exact sequence at the K_0 level is

$$\mathbb{Z} \longrightarrow 0 \longrightarrow 0,$$

so $K_0(\iota)$ is not injective.

Even though surjectivity of $\pi : \mathfrak{A} \to \mathfrak{A}/\mathfrak{J}$ implies the surjectivity of the maps $\mathfrak{A}^+ \to (\mathfrak{A}/\mathfrak{J})^+$ and $\mathbb{M}_n(\mathfrak{A}^+) \to \mathbb{M}_n((\mathfrak{A}/\mathfrak{J})^+)$, still π need not lift projections in $\mathbb{M}_\infty((\mathfrak{A}/\mathfrak{J})^+)$ to projections in $\mathbb{M}_\infty((\mathfrak{A})^+)$. As an example, consider

$$\mathfrak{A} = C([0,1]), \mathfrak{J} = C((0,1)).$$

Then $\mathfrak{A}/\mathfrak{I} \simeq \mathbb{C} \oplus \mathbb{C}$ and we get the exact sequence $0 \to \mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z}$. So $K_0(\pi)$ need not be surjective in general.

Note however the special case in (2.9.0.11), and also that Theorem 2.9.1.1 is not true with K_{00} instead of K_0(see [W, 6.3.3] for details).

2.9.1.3 **([W, 6.5])** Here we give a short list of basic K-groups (see [W, Section 6.5] for a more complete table).

$\mathfrak{A}$	$K_0(\mathfrak{A})$
$\mathbb{C}$	$\mathbb{Z}$
$\mathbb{M}_n(\mathbb{C})$	$\mathbb{Z}$
$\mathbb{K}$	$\mathbb{Z}$
$\mathbb{B}$	0
$\mathbb{B}/\mathbb{K}$	0

2.10 K-Theory for AF Algebras

2.10.1 The Link to Scaled Dimension Groups

2.10.1.1 We will now look more carefully at $K_0(\mathfrak{A})$ when $\mathfrak{A}$ is an AF-algebra. In particular, we will show that if $\mathfrak{A}$ is an AF-algebra, then $K_0(\mathfrak{A})$ is a scaled dimension group.

2.10.1.2 We have seen that

$$K_0(\mathbb{C}) = \mathbb{Z} = K_0(\mathbb{M}_n),$$

and that

$$K_0(\mathfrak{A} \oplus \mathfrak{B}) = K_0(\mathfrak{A}) \oplus K_0(\mathfrak{B})$$

for C*-algebras $\mathfrak{A}$ and $\mathfrak{B}$. It follows that

$$K_0(\mathbb{M}_{m_1} \oplus \ldots \oplus \mathbb{M}_{m_n}) = \mathbb{Z}^n.$$

Let us denote by dim the map from $K_0(\mathbb{M}_{m_1} \oplus \ldots \oplus \mathbb{M}_{m_n})$ into $\mathbb{Z}^n$ which implements this identification.

Let $\mathfrak{A} = \mathbb{M}_{m_1} \oplus \ldots \oplus \mathbb{M}_{m_n}$, so in particular $K_0(\mathfrak{A}) = \mathbb{Z}^n$ is a simplicial group. Also, the image $\dim I_{\mathfrak{A}} \in \mathbb{Z}^n$ of $I_{\mathfrak{A}}$ is always a strictly positive element of $\mathbb{Z}^n$, and hence is an order unit for $K_0(\mathfrak{A})$. For various reasons it will be desirable to regard $K_0(\mathfrak{A})$ as a scaled dimension group with scale $[0, \dim I_{\mathfrak{A}}]$.

2.10.1.3 We have defined the group $K_0(\mathfrak{A})$ for an arbitrary C*-algebra as the Grothendieck group of $V(\mathfrak{A})$. It is desirable to keep track of the original semigroup $V(\mathfrak{A})$, and of which elements of $K_0(\mathfrak{A})$ come from $\mathfrak{A}$. One way to do this is to try to put an ordering on $K_0(\mathfrak{A})$ by taking the image $K_0(\mathfrak{A}_+)$ of $V(\mathfrak{A})$ in $K_0(\mathfrak{A})$ to be the positive cone and to define a "scale" $\Gamma(K_0(\mathfrak{A})) = \{x \in K_0(\mathfrak{A}); x = [p]$ for some projection $p \in \mathfrak{A}\}$ on $K_0(\mathfrak{A})$ (although even here we could lose information, since the map from $V(\mathfrak{A})$ into $K_0(\mathfrak{A})$ will be injective only if $V(\mathfrak{A})$ has cancellation). For finite dimensional and AF algebras the above ordering and scale make $K_0(\mathfrak{A})$ into a scaled dimension group (for more general C*-algebras the "ordering" can be much less well behaved (cf. [B, §6])).

2.10.1.4 Proposition ([W, 12.1.2]) *Let $\mathfrak{A}$ and $\mathfrak{B}$ be finite dimensional C*-algebras, and let $\Psi : \mathfrak{A} \to \mathfrak{B}$ be a C*-algebra homomorphism. Then the induced map $K_0(\Psi) : K_0(\mathfrak{A}) \to K_0(\mathfrak{B})$ is a morphism of scaled dimension groups. Also, Ψ is unital if and only if $K_0(\Psi)([I_{\mathfrak{A}}]) = [I_{\mathfrak{B}}]$.*

2.10.1.5 Note that we can also go in the other direction, i.e., any scale homomorphism of $K_0(\mathfrak{A})$ into $K_0(\mathfrak{B})$ corresponds to a matrix A with entries in $\mathbb{Z}^+$ satisfying Equation (2.8.13), and hence to a standard homomorphism of $\mathfrak{A}$ into $\mathfrak{B}$. This idea will be picked up again in (2.10.3.3).

2.10.1.6 Let $\mathfrak{A} = \varinjlim\{\mathfrak{A}_i, \Phi_{i+1,i}\}$ be an AF-algebra, where the $\mathfrak{A}_i$ are finite dimensional C*-algebras and the $\Phi_{i+1,i}$ are C*-algebra homomorphisms. Then

$$K_0(\mathfrak{A}) = \varinjlim\{K_0(\mathfrak{A}_i), K_0(\Phi_{i+1,i})\}$$

from (2.9.0.16). Since $K_0(\mathfrak{A}_i)$ is a scaled dimension group for every i, and since the $K_0(\Phi_{i+1,i})$ are contractive homomorphisms (cf. 2.6.3.6), it follows from (2.6.3.9) that

$K_0(\mathfrak{A})$ is a dimension group. This next result gives slightly more information.

2.10.1.7 Proposition *Let $\mathfrak{A}$ be an AF-algebra, and let $\mathfrak{A} = \varinjlim\{\mathfrak{A}_i, \Phi_{i+1,i}\}$. Then $K_0(\mathfrak{A}) = \varinjlim\{K_0(\mathfrak{A}_i), K_0(\Phi_{i+1,i})\}$ in the category of scaled dimension groups. Also, $\Gamma(K_0(\mathfrak{A})) = \cup\Gamma(K_0(\mathfrak{A}_n))$. If the $\Phi_{n+1,n}$ are unital then $K_0(\mathfrak{A})$ has an order unit $u = K_0(\Phi_n)([I_n])$ and $\Gamma(K_0(\mathfrak{A})) = [0, u]$.*

2.10.2 Bratteli Diagram and K_0 Groups of AF algebras

2.10.2.1 Let $\{\mathfrak{A}_i, \Phi_{i+1,i}\}$ be a canonical directed system, say

$$\mathfrak{A}_n = \mathbb{M}_{m_{n,1}} \oplus \ldots \oplus \mathbb{M}_{m_{n,k_n}},$$

and let $\mathfrak{A} = \varinjlim\{\mathfrak{A}_i, \Phi_{i+1,i}\}$ (so $\mathfrak{A}$ is an AF-algebra). Then

$$K_0(\mathfrak{A}) = \varinjlim\{K_0(\mathfrak{A}_i), K_0(\Phi_{i+1,i})\}$$

and

$$K_0(\mathfrak{A}_n) = \oplus_{i=1}^{k_n}\mathbb{Z}.$$

Set $V_n = \{(1, -n), \ldots, (k_n, -n)\}$. Let E_n as well as the source and range maps on E_n be given by the incidence matrix A_n, where A_n is the matrix corresponding to $K_0(\Phi_{n,n-1})$. If we set $V = \cup V_n$ and $E = \cup E_n$, then (V, E) is a Bratteli diagram. Notice that this is in fact the same Bratteli diagram that we associated to the canonical system $\varinjlim\{\mathfrak{A}_i, \Phi_{i+1,i}\}$. This leads to three interesting observations:

1. There are many Bratteli diagrams corresponding to $K_0(\mathfrak{A})$
2. Any countable dimension group is the direct limit of a (countable) sequence of simplicial groups (in the category of partially ordered abelian groups), by the result of Effros, Handelman, and Shen (2.6.3.14). So any countable dimension group is the K_0 group of some AF-algebra.
3. Since we can recover $\mathfrak{A}$ from a Bratteli diagram for $\mathfrak{A}$ (and hence from the Bratteli diagram for $K_0(\mathfrak{A})$), it should be possible to recover $\mathfrak{A}$ from $K_0(\mathfrak{A})$. This will be the focus of the next section.

2.10.3 Elliott's Theorem

One fundamental observation about AF-algebras is the close relationship between a directed system $\{\mathfrak{A}_i, \Phi_{i+1,i}\}$ of finite dimensional C*-algebras, and the corresponding sequence of scaled dimension groups $\{K_0(\mathfrak{A}_i), K_0(\Phi_{i+1,i})\}$. Indeed, given the scaled dimension group $\{K_0(\mathfrak{A}_i) \simeq \oplus_{i=1}^{k_n} \mathbb{Z}^{m_{n,i}}\}$, we can recover $\mathfrak{A}_n = \mathbb{M}_{m_{n,1}} \oplus \ldots \oplus \mathbb{M}_{m_{n,k_n}}$. Since the same matrix corresponds to both $\Phi_{i+1,i}$ and to $K_0(\Phi_{i+1,i})$, we can also recover (up to inner equivalence) $\Phi_{i+1,i}$ from $K_0(\Phi_{i+1,i})$. These survey remarks are made more precise in the following results, culminating in Elliott's theorem (2.10.3.3).

2.10.3.1 Lemma ([Da, IV.4.1]) *Suppose that $\mathfrak{A}$ and $\mathfrak{B}$ are finite dimensional C*-algebras and that $\psi : K_0(\mathfrak{A}) \to K_0(\mathfrak{B})$ is a contractive homomorphism. Then there is a *-homomorphism $\alpha : \mathfrak{A} \to \mathfrak{B}$ such that $K_0(\alpha) = \psi$, and α is unique up to unitary equivalence in $\mathfrak{B}$. Moreover, if ψ is unital then so is α.*

2.10.3.2 Lemma ([Da, IV.4.2]) *Suppose that $\mathfrak{A}$ is a finite dimensional C*-algebra, and that $\mathfrak{B} = \varinjlim\{\mathfrak{B}_i, \Psi_{i+1,i}\}$ is AF. Let the imbeddings of $\mathfrak{B}_n$ into $\mathfrak{B}$ be denoted by β_n. If $\psi : K_0(\mathfrak{A}) \to K_0(\mathfrak{B})$ is a contractive homomorphism, then there is an integer n and a *-homomorphism $\alpha : \mathfrak{A} \to \mathfrak{B}_n$ such that $K_0(\beta_n)K_0(\alpha) = \psi$. Moreover, α is unique up to unitary equivalence in $\mathfrak{B}$. If ψ is unital then so is α.*

2.10.3.3 Theorem (Elliott's Theorem—cf. [Da, IV.4.3]) *Let $\mathfrak{A}$ and $\mathfrak{B}$ be AF-algebras. Then $\mathfrak{A} \simeq \mathfrak{B}$ if and only if $K_0(\mathfrak{A}) \simeq K_0(\mathfrak{B})$ as scaled dimension groups. Moreover, given a scaled dimension group isomorphism $\rho : K_0(\mathfrak{A}) \to K_0(\mathfrak{B})$, there is a *-isomorphism $\alpha : \mathfrak{A} \to \mathfrak{B}$ such that $K_0(\alpha) = \rho$.*

2.10.3.4 Elliott's theorem will be used in the proof of (3.4.0.2).

Chapter 3

Results for Simple Unital AF Algebras

The goal of this chapter is to show the following result for simple unital AF algebras.

Theorem *Let $\mathfrak{A}$ and $\mathfrak{B}$ be simple, unital, AF algebras with isomorphic unitary groups. Then $\mathfrak{A}$ and $\mathfrak{B}$ are isomorphic as C^*-algebras.*

Let $\varphi : \mathcal{U}(\mathfrak{A}) \to \mathcal{U}(\mathfrak{B})$ be an (algebraic) isomorphism of groups. To establish the desired result it suffices, by Elliott's theorem, to show that there exists an isomorphism of the scaled dimension groups $K_0(\mathfrak{A})$ and $K_0(\mathfrak{B})$. This will be done as follows:

1. In Section 3.1 we obtain from the isomorphism φ a map $\theta_\varphi : \mathcal{P}(\mathfrak{A}) \to \mathcal{P}(\mathfrak{B})$ from the projections of $\mathfrak{A}$ to those of $\mathfrak{B}$, and we describe the first properties of θ_φ.

2. We construct (Section 3.2) from θ_φ an orthoisomorphism which preserves unitary equivalence. This is the bulk of the work.

3. We show that an orthoisomorphism which preserves unitary equivalence induces an isomorphism of the scaled dimension groups $K_0(\mathfrak{A})$ and $K_0(\mathfrak{B})$ (see Section 3.3).

3.1 The Construction of θ_φ

Let $\mathfrak{A}$ and $\mathfrak{B}$ be unital C*-algebras. In this section we obtain from a homomorphism $\varphi : \mathcal{U}(\mathfrak{A}) \to \mathcal{U}(\mathfrak{B})$ a map $\theta_\varphi : \mathcal{P}(\mathfrak{A}) \to \mathcal{P}(\mathfrak{B})$ from the projections of $\mathfrak{A}$ to those of $\mathfrak{B}$.

3.1.0.1 Any self-adjoint unitary u has eigenvalues contained in the set $\{-1, 1\}$, and hence can be written $u = I - 2p$ where p is the projection onto the eigenspace of u with eigenvalue -1. Let η be the map which sends a self-ajoint unitary $u = I - 2p$ to the projection p. Then η is a bijection from the set of self-adjoint unitaries to the set of projections, and there is exactly one map θ_φ which makes the diagram

$$\begin{array}{ccc} \mathcal{U}(A) \cap \mathfrak{A}_{\text{sa}} & \xrightarrow{\varphi} & \mathcal{U}(B) \cap \mathfrak{B}_{\text{sa}} \\ \eta \big\downarrow & & \eta \big\downarrow \\ \mathcal{P}(\mathfrak{A}) & \xrightarrow{\theta_\varphi} & \mathcal{P}(\mathfrak{B}) \end{array}$$

commute. Explicitly,

$$u = I - 2p \iff \varphi(u) = I - 2\theta_\varphi(p)$$

The map θ_φ will be denoted by θ when no confusion will result.

3.1.0.2 Recall that the symmetric difference $p\Delta q$ of two commuting projections p and q is defined by $p\Delta q = p + q - 2pq$, and is also a projection.

The following properties of the symmetric difference of commuting projections can be checked by direct computation:

$$p\Delta p = 0 \tag{3.1.20}$$

$$p\Delta(q\Delta r) = (p\Delta q)\Delta r \tag{3.1.21}$$

$$(I - p)\Delta q = I - p\Delta q \tag{3.1.22}$$

$$p(q\Delta r) = (pq)\Delta(pr) \tag{3.1.23}$$

The following lemma gives certain elementary properties of θ_φ, which will be used frequently.

3.1.0.3 Lemma *Let $\mathfrak{A}$ and $\mathfrak{B}$ be unital C^*-algebras, and let $\varphi : \mathcal{U}(\mathfrak{A}) \to \mathcal{U}(\mathfrak{B})$ be a group homomorphism with corresponding map $\theta : \mathcal{P}(\mathfrak{A}) \to \mathcal{P}(\mathfrak{B})$. Then*

1. *$\theta(upu^*) = \varphi(u)\theta(p)\varphi(u)^*$ for any unitary $u \in \mathcal{U}(\mathfrak{A})$, and $p \in \mathcal{P}(\mathfrak{A})$,*
2. *$\theta(0) = 0$,*
3. *If $p, q \in \mathcal{P}(\mathfrak{A})$ commute then so do $\theta(p)$ and $\theta(q)$,*
4. *$\theta(p\Delta q) = \theta(p)\Delta\theta(q)$ for any commuting projections $p, q \in \mathcal{P}(\mathfrak{A})$,*
5. *If $\mathfrak{A}$ and $\mathfrak{B}$ are simple unital C^*-algebras and φ is an isomorphism then we have that $\varphi(-I) = -I$,*
6. *If $\varphi(-I) = -I$ then we also have that $\theta(I) = I$ and $\theta(I-p) = I - \theta(p)$.*

Proof:

1. $\theta(upu^*) = \frac{I-\varphi(I-2upu^*)}{2} = \frac{I-\varphi(u)\varphi(I-2p)\varphi(u)^*}{2} = \varphi(u)\theta(p)\varphi(u)^*$.
2. $\theta(0) = \frac{I-\varphi(I-2(0))}{2} = \frac{I-\varphi(I)}{2} = 0$.
3. Since p and q commute the same is true of $I-2p$ and $I-2q$, so $\varphi(I-2p)$ commutes with $\varphi(I-2q)$. It follows that

$$\begin{aligned}\theta(p)\theta(q) &= \frac{(I-\varphi(I-2p))(I-\varphi(I-2q))}{4} \\ &= \frac{(I-\varphi(I-2q))(I-\varphi(I-2p))}{4} \\ &= \theta(q)\theta(p).\end{aligned}$$

4. If p and q are commuting projections then $p\Delta q$ is also a projection and

$$\begin{aligned}I - 2\theta(p\Delta q) &= \varphi(I - 2(p\Delta q)) \\ &= \varphi((I-2p)(I-2q)) \\ &= (I-2\theta(p))(I-2\theta(q)) \\ &= I - 2(\theta(p)\Delta\theta(q)).\end{aligned}$$

5. If φ is an isomorphism between simple C*-algebras $\mathfrak{A}$ and $\mathfrak{B}$, we show that $\varphi(-I) = -I$. Indeed, $-I$ is a central, self-adjoint unitary which is not I, so the same is true of $\varphi(-I)$. Since any central element of $\mathfrak{B}$ is a scalar multiple of the identity (2.7.13.9), and since $\varphi(-I) \neq I$ then $\varphi(-I)$ must be $-I$.

6. If $\varphi(-I) = -I$, then

$$\theta(I) = \frac{I - \varphi(I - 2I)}{2} = \frac{I - \varphi(-I)}{2} = \frac{2I}{2} = I.$$

   Since $\varphi(I - 2q) = I - 2\theta(q)$ for any projection $q \in \mathcal{P}(\mathfrak{A})$, we have

$$\begin{aligned} I - 2\theta(I - p) &= \varphi(I - 2(I - p)) \\ &= \varphi(-(I - 2p)) \\ &= \varphi(-I(I - 2p)) \\ &= \varphi(-I)\varphi(I - 2p) \\ &= -\varphi(I - 2p) \\ &= -(I - 2\theta(p)) \\ &= I - 2(I - \theta(p)). \end{aligned}$$

   So $\theta(I - p) = I - \theta(p)$.

■

We establish a functorial property of the map $\varphi \mapsto \theta_\varphi$, which will be used in the proof of (3.1.0.5).

3.1.0.4 Proposition *Let $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}$ be unital C*-algebras with unitary groups $\mathcal{U}(\mathfrak{A}), \mathcal{U}(\mathfrak{B}), \mathcal{U}(\mathfrak{C})$.*

1. *Suppose that $\varphi : \mathcal{U}(\mathfrak{A}) \to \mathcal{U}(\mathfrak{B})$ and $\psi : \mathcal{U}(\mathfrak{B}) \to \mathcal{U}(\mathfrak{C})$ are group homomorphisms with corresponding projection maps θ_φ and θ_ψ. Then*

$$\theta_{\psi\varphi} = \theta_\psi \theta_\varphi.$$

2. *If $\mathrm{id} : \mathcal{U}(\mathfrak{A}) \to \mathcal{U}(\mathfrak{A})$ is the identity map then θ_{id} is the identity map on projections in $\mathfrak{A}$.*

3. *If φ is a group isomorphism then $\theta_{\varphi^{-1}} = {\theta_\varphi}^{-1}$.*

Proof:

1. As $\varphi(I-2p) = I - 2\theta_\varphi(p)$ for all $p \in \mathcal{P}(\mathfrak{A})$, we have

$$\begin{aligned}\theta_\psi(\theta_\varphi(p)) &= \frac{I - \psi(I - 2\theta_\varphi(p))}{2} \\ &= \frac{I - \psi(\varphi(I-2p))}{2} \\ &= \theta_{\psi\varphi}(p).\end{aligned}$$

2. $\theta_{\mathrm{id}}(p) = \frac{I - \mathrm{id}(I-2p)}{2} = p$, for any $p \in \mathcal{P}(\mathfrak{A})$.

3. Follows immediately from (1) and (2).

■

The next proposition is a generalization of [Dy, Lemma 10], for simple unital C*-algebras. It will allow us to define in (3.2.1.2) a map $c : \mathcal{P}(\mathfrak{A}) \setminus \{0, I\} \to \mathrm{Aut}(\mathbb{S}^1)$ and establish some of the fundamental properties of this map. If $\mathfrak{C}$ is a unital C*-algebra and $p \in \mathcal{P}(\mathfrak{C})$, we will denote by

$$\mathfrak{C}_{\text{block } p} = \{x \in \mathfrak{C}; px = xp\} = \{p\}' \cap \mathfrak{C}.$$

We have

$$\mathfrak{C}_{\text{block } p} = p\mathfrak{C}p + (I-p)\mathfrak{C}(I-p).$$

3.1.0.5 Proposition *Let $\mathfrak{A}$ and $\mathfrak{B}$ be simple unital C*-algebras and let $\varphi : \mathcal{U}(\mathfrak{A}) \to \mathcal{U}(\mathfrak{B})$ be a group isomorphism. Then*

1. *The map φ restricted to $\mathcal{U}(\mathfrak{A}_{\text{block } p})$ is an isomorphism of $\mathcal{U}(\mathfrak{A}_{\text{block } p})$ with $\mathcal{U}(\mathfrak{B}_{\text{block } \theta_\varphi(p)})$.*

2. *For each fixed p in $\mathcal{P}(\mathfrak{A}) \setminus \{0, I\}$, there exist (possibly discontinuous) characters of the circle group $\mathbb{S}^1$ such that*

$$\varphi(\lambda p + I - p) = a_\varphi(\lambda)\theta_\varphi(p) + b_\varphi(\lambda)(I - \theta_\varphi(p)), \forall \lambda \in \mathbb{S}^1. \tag{3.1.24}$$

Proof:

1. We will show that φ maps $\mathcal{U}(\mathfrak{A}_{\text{block } p})$ isomorphically onto $\mathcal{U}(\mathfrak{B}_{\text{block } \theta_\varphi(p)})$. Note that

$$\mathcal{U}(\mathfrak{A}_{\text{block } p}) = \{u \in \mathcal{U}(\mathfrak{A}); pu = up\} = \{u \in \mathcal{U}(\mathfrak{A}); (I-2p)u = u(I-2p)\}.$$

Hence if $u \in \mathcal{U}(\mathfrak{A}_{\text{block } p})$, then $\varphi(u)\varphi(I-2p) = \varphi(I-2p)\varphi(u)$ and by definition of θ, $\varphi(u)(I-2\theta(p)) = (I-2\theta(p))\varphi(u)$. Therefore

$$\varphi(\mathcal{U}(\mathfrak{A}_{\text{block } p})) \subseteq \{\varphi(u) \in \mathcal{U}(\mathfrak{B}); \varphi(u)\theta(p) = \theta(p)\varphi(u)\} \subseteq \mathcal{U}(\mathfrak{B}_{\text{block } \theta_\varphi(p)}).$$

As φ is an isomorphism, it induces an isomorphism from $\mathcal{U}(\mathfrak{A}_{\text{block } p})$ onto $\mathcal{U}(\mathfrak{B}_{\text{block } \theta_\varphi(p)})$.

2. Let us fix a $p \in \mathcal{P}(\mathfrak{A})$. As $\mathfrak{B}$ is simple we have from (2.7.13.9) that

$$\mathcal{Z}(\mathfrak{B}_{\text{block } \theta_\varphi(p)}) = \{\mu\theta(p) + \nu(I-\theta(p)); \mu, \nu \in \mathbb{C}\}.$$

By (2.7.5.11) (or by the Russo-Dye theorem) we see that any element which commutes with all unitaries in $\mathfrak{A}$ is central in $\mathfrak{A}$, and we then get that

$$\mathcal{Z}(\mathcal{U}(\mathfrak{B}_{\text{block } \theta_\varphi(p)})) = \{\mu\theta(p) + \nu(I-\theta(p)); \mu, \nu \in \mathbb{S}^1\}.$$

As for any $\mu \in \mathbb{S}^1$ the unitary $\mu p + I - p$ belongs to $\mathcal{Z}(\mathcal{U}(\mathfrak{A}_{\text{block } p}))$, its image $\varphi(\mu p + I - p) \in \mathcal{Z}(\mathcal{U}(\mathfrak{B}_{\text{block } \theta_\varphi(p)}))$. Therefore, for each $\lambda \in \mathbb{S}^1$, there are numbers $a_\varphi(\lambda)$ and $b_\varphi(\lambda)$ in $\mathbb{S}^1$ such that

$$\varphi(\lambda p + I - p) = a(\lambda)\theta(p) + b(\lambda)(I - \theta(p)).$$

A straightforward computation shows that $a_\varphi : \mathbb{S}^1 \to \mathbb{S}^1, \lambda \mapsto a_\varphi(\lambda)$ and $b : \mathbb{S}^1 \to \mathbb{S}^1, \lambda \mapsto b_\varphi(\lambda)$ are characters.

∎

3.1.0.6 For the rest of this chapter we assume that φ is a fixed isomorphism between the unitary groups $\mathcal{U}(\mathfrak{A})$ and $\mathcal{U}(\mathfrak{B})$ of simple unital C*-algebra $\mathfrak{A}$ and $\mathfrak{B}$, and that $\theta = \theta_\varphi$ is the corresponding map between projections.

We make the following observations:

3.1.0.7 Let $\mathfrak{A}$, $\mathfrak{B}$, φ, and θ be as in (3.1.0.6). For commuting projections $p, q \in \mathcal{P}(\mathfrak{A})$ we denote

$$\mathcal{S}_{p,q} = \{\theta(p)\theta(q), \theta(p)\theta(I-q), \theta(I-p)\theta(q), \theta(I-p)\theta(I-q)\}.$$

By (3.1.0.3) the elements of $\mathcal{S}_{p,q}$ are mutually orthogonal projections, and form a partition of I. Indeed, we have

$$\begin{aligned}
&\theta(p)\theta(q) + \theta(p)\theta(I-q) + \theta(I-p)\theta(q) + \theta(I-p)\theta(I-q) \\
&= \theta(p)(\theta(q) + \theta(I-q)) + \theta(I-p)(\theta(q) + \theta(I-q)) \\
&= \theta(p)(\theta(q) + I - \theta(q)) + \theta(I-p)(\theta(q) + I - \theta(q)) \\
&= \theta(p) + \theta(I-p) = I.
\end{aligned} \tag{3.1.25}$$

The following gives information on the behaviour of θ with respect to various operations on projections. It will be used in several computations.

3.1.0.8 Proposition *Let $\mathfrak{A}$, $\mathfrak{B}$, φ, and θ be as in (3.1.0.6). Let $p, q \in \mathcal{P}(\mathfrak{A})$ be commuting projections and let $r = p\Delta q$. Then*

$$\theta(p)\theta(I-q) = \theta(p)\theta(r), \tag{3.1.26}$$

$$\theta(I-p)\theta(q) = \theta(q)\theta(r), \tag{3.1.27}$$

$$\theta(p)\theta(q) = \theta(p)\theta(I-r) = \theta(q)\theta(I-r) = \theta(p)\theta(q)\theta(I-r), \tag{3.1.28}$$

$$\theta(I-p)\theta(I-q)\theta(r) = 0, \tag{3.1.29}$$

$$\theta(r) = \theta(p)\theta(I-q) + \theta(I-p)\theta(q), \tag{3.1.30}$$

$$\theta(I-r) = \theta(p)\theta(q) + \theta(I-p)\theta(I-q). \tag{3.1.31}$$

Proof: By (3.1.0.2) and (3.1.0.3) we have

$$\begin{aligned}
\theta(p)\theta(r) &= \theta(p)(\theta(p)\Delta\theta(q)) \\
&= \theta(p)\Delta\theta(p)\theta(q) \\
&= \theta(p)(I\Delta\theta(q)) \\
&= \theta(p)(I - \theta(q)) \\
&= \theta(p)\theta(I-q),
\end{aligned}$$

so also $\theta(q)\theta(r) = \theta(I-p)\theta(q)$, by symmetry. This shows (3.1.26) and (3.1.27). Using this we get

$$\begin{aligned}\theta(p)\theta(I-r) &= \theta(p) - \theta(p)\theta(r) \\ &= \theta(p) - \theta(p)\theta(I-q) && \text{by (3.1.26)} \\ &= \theta(p)\left(I - \theta(I-q)\right) \\ &= \theta(p)\theta(q).\end{aligned}$$

Hence also $\theta(p)\theta(q) = \theta(q)\theta(I-r)$ by symmetry. Also,

$$\theta(p)\theta(q) = \theta(p)(\theta(p)\theta(q)) = \theta(p)(\theta(q)\theta(I-r)) = \theta(p)\theta(q)\theta(I-r).$$

So we have shown (3.1.28). Using equation (3.1.23) we have

$$\begin{aligned}&\theta(I-p)\theta(I-q)\theta(r) \\ &\quad = (\theta(I-p)\theta(I-q))(\theta(p)\Delta\theta(q)) \\ &\quad = [\theta(I-p)\theta(I-q)\theta(p)]\Delta[\theta(I-p)\theta(I-q)\theta(q)] \\ &\quad = 0\Delta 0 \\ &\quad = 0,\end{aligned}$$

which shows (3.1.29). By (3.1.26) and (3.1.27) we have that $\theta(p)\theta(I-q) \leq \theta(r)$ and $\theta(I-p)\theta(q) \leq \theta(r)$, and by (3.1.28) and (3.1.29) $\theta(p)\theta(q) \leq \theta(I-r)$ and $\theta(I-p)\theta(I-q) \leq \theta(I-r)$. Since

$$\theta(p)\theta(I-q) + \theta(I-p)\theta(q) + \theta(p)\theta(q) + \theta(I-p)\theta(I-q) = I$$

(see (3.1.0.7)) the last two assertions follow. ∎

3.1.0.9 Let $\mathfrak{A}$, $\mathfrak{B}$, φ, and θ be as in (3.1.0.6). Let p and q be two commuting projections of $\mathfrak{A}$ and let $r = p\Delta q$, and suppose that $\theta(p)\theta(r) = 0$. Then we have the rather peculiar consequence that $\theta(r) + \theta(p) = \theta(q)$. Indeed, by (3.1.0.8) $\theta(p)\theta(I-q) = 0$ and therefore $\theta(r) = (\theta(I-p))\theta(q) = \theta(q) - \theta(p)\theta(q)$. By (3.1.0.8),

$$\theta(p) = \theta(p)(\theta(r) + I - \theta(r)) = \theta(p)\theta(I-r) = \theta(p)\theta(q).$$

Hence $\theta(r) + \theta(p) = \theta(q)$. Similarily $\theta(q)\theta(r) = 0$ implies that $\theta(q) + \theta(r) = \theta(p)$.

3.2 Constructing an Orthoisomorphism

Our goal in this section is to construct an orthoisomorphisms $\theta'_\varphi : \mathcal{P}(\mathfrak{A}) \to \mathcal{P}(\mathfrak{B})$ which preserves unitary equivalence. We might hope that the map θ_φ would serve this purpose, but although it is a bijection which preserves unitary equivalence (by (3.1.0.3)) it need not be an orthoisomorphism. The idea will be to modify θ_φ to obtain a map $\theta'_\varphi : \mathcal{P}(\mathfrak{A}) \to \mathcal{P}(\mathfrak{B})$ which has the desired properties. The following finite dimensional example is instructive.

3.2.0.1 Example We find in $\mathfrak{A} = \mathbb{M}_{2n}$ some behaviour which recurs in AF algebras.

There exists (see Appendix A) an automorphism φ of the unitary group of $\mathbb{M}_{2n}$ such that

$$\theta_\varphi(p) = \begin{cases} p & \text{if } p \text{ is of even rank} \\ I - p & \text{if } p \text{ is of odd rank} \end{cases}$$

The map θ_φ is clearly not an orthoisomorphism. However we can construct from θ_φ the map θ'_φ:

$$\theta'_\varphi(p) = \begin{cases} \theta_\varphi(p) & \text{if } p \text{ is of even rank} \\ I - \theta_\varphi(p) & \text{if } p \text{ is of odd rank} \end{cases}$$

The map θ'_φ is an orthoisomorphisms which preserves unitary equivalence (in fact $\theta'_\varphi = \mathrm{id}$). ■

3.2.0.2 For a simple unital C*-algebra $\mathfrak{A}$, we define in (3.2.1.2) a map $c : \mathcal{P}(\mathfrak{A}) \setminus \{0, I\} \to \mathrm{Aut}(\mathbb{S}^1)$. If $\mathfrak{A}$ is also AF, we show in (3.2.2.6) that for a simple unital AF algebra the image of the map c consists of at most two elements. This allows us to construct a partition $\{\mathcal{P}_{c_p}, \mathcal{P}_{\bar{c}_p}\}$ of $\mathcal{P}(\mathfrak{A}) \setminus \{0, I\}$, where $\mathcal{P}_{c_p}$ is analogous to the projection of even rank in (3.2.0.1) and $\mathcal{P}_{\bar{c}_p}$ is analogous to the projections of odd rank. Indeed, in (3.2.3.14), (3.2.3.15), and (3.2.2.3), we show that if $p, q \in \mathcal{P}(\mathfrak{A}) \setminus \{0, I\}$

with $pq = 0$ then

$$\begin{aligned} p, q \in \mathcal{P}_{c_p} &\Rightarrow \theta_\varphi(p)\theta_\varphi(q) = 0 \\ p, q \in \mathcal{P}_{\bar{c}_p} &\Rightarrow \theta_\varphi(I-p)\theta_\varphi(I-q) = 0 \\ p \in \mathcal{P}_{c_p}, q \in \mathcal{P}_{\bar{c}_p} &\Rightarrow \theta(p)\theta(I-q) = 0 \end{aligned}$$

and

$$\begin{aligned} p, q \in \mathcal{P}_{c_p} &\Rightarrow p+q \in \mathcal{P}_{c_p} \\ p, q \in \mathcal{P}_{\bar{c}_p} &\Rightarrow p+q \in \mathcal{P}_{c_p} \\ p \in \mathcal{P}_{c_p}, q \in \mathcal{P}_{\bar{c}_p} &\Rightarrow p+q \in \mathcal{P}_{\bar{c}_p} \end{aligned}$$

From such a partition we obtain a map

$$\theta'_\varphi(p) = \begin{cases} \theta_\varphi(p) & \text{if } p \in \mathcal{P}_{c_p} \\ I - \theta_\varphi(p) & \text{if } p \in \mathcal{P}_{\bar{c}_p} \\ 0 & \text{if } p = 0 \\ I & \text{if } p = I \end{cases}$$

which is in fact an orthoisomorphisms which preserves unitary equivalence (see (3.2.2.9)).

3.2.1 The Map c

3.2.1.1 Denote by $\hat{\mathbb{S}}^1$ the group of characters of $\mathbb{S}^1$, endowed with the discrete topology.

$$\hat{\mathbb{S}}^1 = \{\chi : \mathbb{S}^1 \to \mathbb{S}^1; \chi(\mu\nu) = \chi(\mu)\chi(\nu)\}$$

with the pointwise product

$$(\chi\nu)(\mu) = \chi(\mu)\nu(\mu), \text{ for any } \chi, \nu \in \hat{\mathbb{S}}^1.$$

The inverse of $\chi \in \hat{\mathbb{S}}^1$ is the function $\overline{\chi} \in \hat{\mathbb{S}}^1$ defined by

$$\overline{\chi}(\mu) = \overline{\chi(\mu)}, \text{ for all } \mu \in \mathbb{S}^1.$$

The group $\mathrm{Aut}(\mathbb{S}^1)$ of (algebraic) automorphisms of $\mathbb{S}^1$ is clearly a subgroup of $\hat{\mathbb{S}}^1$.

3.2.1.2 Definition (Definition of c) *We have seen (3.1.0.5) that if $\mathfrak{A}$ and $\mathfrak{B}$ are simple unital C^*-algebras then any isomorphism $\varphi : \mathcal{U}(\mathfrak{A}) \to \mathcal{U}(\mathfrak{B})$ gives rise to a projection map θ_{φ}: such that for any $p \in \mathcal{P}(\mathfrak{A}) \setminus \{0, I\}$ there are elements $a_{\varphi,p}$ and $b_{\varphi,p}$ of $\hat{\mathbb{S}}^1$ such that*

$$\varphi(\mu p + I - p) = a_{\varphi,p}(\mu)\theta_{\varphi}(p) + b_{\varphi,p}(\mu)(I - \theta_{\varphi}(p)), \text{ for all } \mu \in \mathbb{S}^1.$$

Hence to any such φ and any $p \in \mathcal{P}(\mathfrak{A}) \setminus \{0, I\}$ we can associate the element

$$c_{\varphi}(p) = a_{\varphi,p}\bar{b}_{\varphi,p} \in \hat{\mathbb{S}}^1.$$

Since φ is now a fixed isomorphism (see 3.1.0.6), we will write c_p for $c_{\varphi}(p)$, a_p for $a_{\varphi,p}$, b_p for $b_{\varphi,p}$, and we consider the map $c : \mathcal{P}(\mathfrak{A}) \setminus \{0, I\} \to \hat{\mathbb{S}}^1$ given by $c(p) = c_p$.

We collect elementary properties of c in the following proposition.

3.2.1.3 Proposition *Let $\mathfrak{A}$, $\mathfrak{B}$, φ, and θ be as in (3.1.0.6). The map $c : \mathcal{P}(\mathfrak{A}) \setminus \{0, I\} \to \hat{\mathbb{S}}^1$ has the following properties:*

1. *If p is unitarily equivalent to q then $c_p = c_q$.*
2. *For any $p \in \mathcal{P}(\mathfrak{A}) \setminus \{0, I\}$, the map $c_p : \mathbb{S}^1 \to \mathbb{S}^1$ is a group isomorphism of $\mathbb{S}^1$. So in fact $c : \mathcal{P}(\mathfrak{A}) \setminus \{0, I\} \to Aut(\mathbb{S}^1)$.*
3. *For any $p \in \mathcal{P}(\mathfrak{A}) \setminus \{0, I\}$, we have $c_p = c_{I-p}$.*

Proof:

1. Let $u \in \mathcal{U}(\mathfrak{A})$ be such that $p = uqu^*$. Then

$$\begin{aligned}
&a_p(\mu)\theta(p) + b_p(\mu)\theta(I - p) \\
&\quad = \varphi(\mu p + I - p) \\
&\quad = \varphi(\mu uqu^* + I - uqu^*) \\
&\quad = \varphi(u)\varphi(\mu q + I - q)\varphi(u)^* \\
&\quad = \varphi(u)(a_q(\mu)\theta(q) + b_q(\mu)\theta(I - q))\varphi(u)^* \\
&\quad = a_q(\mu)\varphi(u)\theta(q)\varphi(u)^* + b_q(\mu)\varphi(u)\theta(I - q)\varphi(u)^* \\
&\quad = a_q(\mu)\theta(p) + b_q(\mu)\theta(I - p) && \text{by (1) of (3.1.0.3)}
\end{aligned}$$

So $a_p = a_q$ and $b_p = b_q$, hence $c_p = c_q$.

2. Let $p \in \mathcal{P}(\mathfrak{A}) \setminus \{0, I\}$. As $c_p \in \hat{\mathbb{S}}^1$, we only have to show that c_p is a bijection. Let $\mu \in \mathbb{S}^1$. By (3.1.0.5) applied to φ^{-1}, there exist $\alpha, \beta \in \mathbb{S}^1$ such that

$$\begin{aligned} \mu\theta(p) + I - \theta(p) &= \varphi(\varphi^{-1}(\mu\theta(p) + I - \theta(p))) \\ &= \varphi(\alpha p + \beta(I - p)) \end{aligned}$$

with $\gamma = \alpha\overline{\beta}$ we get

$$\begin{aligned} \mu\theta(p) + I - \theta(p) &= \varphi(\beta(\gamma p + I - p)) \\ &= \varphi(\beta) b_p(\gamma)(c_p(\gamma)\theta(p) + I - \theta(p)) \end{aligned}$$

and comparing coefficients gives that $c_p(\gamma) = \mu$. Since μ was arbitrary, this shows that c_p is onto. If $c_p(\mu) = 1$ for some $\mu \in \mathbb{S}^1$ then

$$\begin{aligned} \varphi(\mu p + I - p) &= a_p(\mu)\theta(p) + b_p(\mu)\theta(I - p) \\ &= b_p(\mu)[c_p(\mu)\theta(p) + \theta(I - p)] \\ &= b_p(\mu) I. \end{aligned}$$

Since $b_p(\mu)I$ is a central, $\varphi^{-1}(b_p I)$ belongs to $\mathcal{Z}(\mathcal{U}(\mathfrak{A}))$. From (2.7.13.9) it follows that $\mu p + I - p$ is a scalar, and therefore that $\mu = 1$. Hence c_p is injective.

3. Let $p \in \mathcal{P}(\mathfrak{A}) \setminus \{0, I\}$. Then

$$\begin{aligned} \varphi(\mu I) &= \varphi(\mu p + I - p)\varphi(p + \mu(I - p)) \\ &= [a_p(\mu)\theta(p) + b_p(\mu)\theta(I - p)][b_{I-p}(\mu)\theta(p) + a_{I-p}(\mu)(I - \theta(p))] \\ &= a_p(\mu)b_{I-p}(\mu)\theta(p) + b_p(\mu)a_{I-p}(\mu)(I - \theta(p)) \end{aligned}$$

It follows that $a_p(\mu)b_{I-p}(\mu) = \varphi(\mu I) = b_p(\mu)a_{I-p}(\mu)$, for any $\mu \in \mathbb{S}^1$. So $a_p b_{I-p} = b_p a_{I-p}$, and hence $c_p = c_{I-p}$.

■

3.2.2 The Fundamental Tools

In this section we develop the technical result (3.2.2.3), which is valid for any simple unital C*-algebra, and which will be used repeatedly. As a consequence of this result we obtain, under an additional assumption on the algebra $\mathfrak{A}$ (an assumption which will be satisfied if $\mathfrak{A}$ is AF), that the range of the map c has at most two elements (cf. (3.2.2.6)). This result allows us to construct a partition $\{\mathcal{P}_{c_p}, \mathcal{P}_{\bar{c}_p}\}$ of $\mathcal{P}(\mathfrak{A})\backslash\{0, I\}$. Moreover, if $\mathcal{P}_{c_p}$ and $\mathcal{P}_{\bar{c}_p}$ have the properties stated in (3.2.0.2), we then show that we can construct an orthoisomorphisms which preserves unitary equivalence.

For a simple unital AF algebra we will show in Section (3.2.3) that the properties of $\mathcal{P}_{c_p}$ and $\mathcal{P}_{\bar{c}_p}$ stated in (3.2.0.2) are always satisfied.

3.2.2.1 Lemma *Let $\mathfrak{A}$, $\mathfrak{B}$, φ, and θ be as in (3.1.0.6). Let $p, q \in \mathcal{P}(\mathfrak{A}) \setminus \{0, I\}$ be such that $pq = 0$ and $p + q = r \neq I$. Then*

$$b_p b_q c_p = b_r c_r \qquad \textit{if } \theta(p)\theta(I-q) \neq 0 \tag{3.2.32}$$

$$b_p b_q c_q = b_r c_r \qquad \textit{if } \theta(I-p)\theta(q) \neq 0 \tag{3.2.33}$$

$$b_p b_q c_p c_q = b_r \qquad \textit{if } \theta(p)\theta(q) \neq 0 \tag{3.2.34}$$

$$b_p b_q = b_r \qquad \textit{if } \theta(I-p)\theta(I-q) \neq 0. \tag{3.2.35}$$

Proof: Note that from (3.1.0.5) and (3.1.0.3) we have

$$\begin{aligned}
&\varphi(\mu p + I - p)\varphi(\mu q + I - q)\\
&= [a_p(\mu)\theta(p) + b_p(\mu)\theta(I-p)][a_q(\mu)\theta(q) + b_q(\mu)\theta(I-q)]\\
&= b_p(\mu)b_q(\mu)[c_p(\mu)\theta(p) + \theta(I-p)][c_q(\mu)\theta(q) + \theta(I-q)]\\
&= b_p(\mu)b_q(\mu)\begin{bmatrix} c_p(\mu)c_q(\mu)\theta(p)\theta(q) + c_p(\mu)\theta(p)\theta(I-q) \\ \qquad + c_q(\mu)\theta(I-p)\theta(q) + \theta(I-p)\theta(I-q)\end{bmatrix}.
\end{aligned}$$

for all $\mu \in \mathbb{S}^1$. By (3.1.0.8) and (3.1.0.5) we have

$$\begin{aligned}
&\varphi(\mu r + I - r)\\
&= a_r(\mu)\theta(r) + b_r(\mu)\theta(I-r)\\
&= b_r(\mu)[c_r(\mu)\theta(r) + \theta(I-r)],\\
&= b_r(\mu)[c_r(\mu)\theta(p)\theta(I-q) + c_r(\mu)\theta(I-p)\theta(q) + \theta(p)\theta(q) + \theta(I-p)\theta(I-q)],
\end{aligned}$$

for all $\mu \in \mathbb{S}^1$. Since p and q are orthogonal,

$$\varphi((\mu p + I - p)(\mu q + I - q)) = \varphi(\mu r + I - r).$$

Therefore, by identifying coefficients, we get

$$b_p b_q c_p = b_r c_r \qquad \text{if } \theta(p)\theta(I-q) \neq 0 \tag{3.2.36}$$

$$b_p b_q c_q = b_r c_r \qquad \text{if } \theta(I-p)\theta(q) \neq 0 \tag{3.2.37}$$

$$b_p b_q c_p c_q = b_r \qquad \text{if } \theta(p)\theta(q) \neq 0 \tag{3.2.38}$$

$$b_p b_q = b_r \qquad \text{if } \theta(I-p)\theta(I-q) \neq 0. \tag{3.2.39}$$

■

Using (3.2.2.1), we obtain more information on θ:

3.2.2.2 Proposition *Let $\mathfrak{A}$, $\mathfrak{B}$, φ, and θ be as in (3.1.0.6). Let $p, q \in \mathcal{P}(\mathfrak{A}) \setminus \{0, I\}$ be such that $pq = 0$ and $p + q = r \neq I$. Then exactly one element of $\mathcal{S}_{p,q}$ (as defined in (3.1.0.7)) is 0.*

Proof: If all the elements of $\mathcal{S}_{p,q}$ are nonzero then from (3.2.32), (3.2.33), and (3.2.35) we have $c_p = c_q = c_r$ in $\hat{\mathbb{S}}^1$ and from (3.2.34) $c_r^2 = 1$ for all $\mu \in \mathbb{S}^1$. This contradicts the surjectivity of c_r.

Let us check that at most one element of $\mathcal{S}_{p,q}$ is nonzero. Suppose that $\theta(p)\theta(q) = 0$. If $\theta(p)\theta(I-q) = \theta(p)(I - \theta(q)) = 0$ (respectively $\theta(q)\theta(I-p) = 0$) then $\theta(p) = 0$ (respectively $\theta(q) = 0$), which contradicts the injectivity of θ. If $\theta(I-p)\theta(I-q) = 0$ then $\theta(I-r) = 0$ by (3.1.0.8), which again contradicts the injectivity of θ.

Suppose now that $\theta(p)\theta(I-q) = 0$ (respectively $\theta(I-q)\theta(p) = 0$). If $\theta(I-p)\theta(q) = 0$ (respectively $\theta(p)\theta(I-q) = 0$), then $\theta(r) = 0$ by (3.1.0.8). As above, this is impossible. If $\theta(I-p)\theta(I-q) = 0$ then $\theta(I-q) = 0$ (respectively $\theta(I-p) = 0$) is again impossible. ■

Combining (3.2.2.1) and (3.2.2.2) we get the following (fundamental) proposition, on which the remaining results are based.

3.2.2.3 Proposition *Let $\mathfrak{A}$, $\mathfrak{B}$, φ, and θ be as in (3.1.0.6). Let $p, q \in \mathcal{P}(\mathfrak{A}) \setminus \{0, I\}$*

be such that $pq = 0$ and $p + q = r \neq I$. Then

$$\begin{aligned}
\theta(p)\theta(q) = 0 &\iff c_p = c_q = c_r \\
\theta(I-p)\theta(I-q) = 0 &\iff c_p = c_q = \bar{c}_r \\
\theta(I-p)\theta(q) = 0 &\iff c_p = \bar{c}_q = c_r \\
\theta(p)\theta(I-q) = 0 &\iff \bar{c}_p = c_q = c_r
\end{aligned}.$$

Proof: Recall that we have from (3.2.32)–(3.2.35) that

$$\begin{aligned}
b_p b_q c_p &= b_r c_r && \text{if } \theta(p)\theta(I-q) \neq 0 \\
b_p b_q c_q &= b_r c_r && \text{if } \theta(I-p)\theta(q) \neq 0 \\
b_p b_q c_p c_q &= b_r && \text{if } \theta(p)\theta(q) \neq 0 \\
b_p b_q &= b_r && \text{if } \theta(I-p)\theta(I-q) \neq 0
\end{aligned}$$

If $c_p = c_q = c_r$ then by (3.2.2.2) we have that exactly one element of $\mathcal{S}_{p,q}$ is 0, so three of (3.2.32)–(3.2.35) hold. It follows that $b_p b_q = b_r$, and therefore $\theta(p)\theta(q) = 0$ (if not $c_p^2 = 1$ is not surjective). Conversely, if $\theta(p)\theta(q) = 0$ then the other elements of $\mathcal{S}_{p,q}$ are non zero (3.2.2.2) so by (3.2.35),(3.2.32), and (3.2.33) we get $c_p = c_q = c_r$.

If $c_p = c_q = \bar{c}_r$ then by (3.2.2.2) and (3.2.32)–(3.2.35) we have $b_p b_q \neq b_r$ and therefore $\theta(I-p)\theta(I-q) = 0$. Conversely, if $\theta(I-p)\theta(I-q) = 0$ then by (3.2.35),(3.2.32), and (3.2.33), $c_p = c_q = \bar{c}_r$.

The other two equivalences are checked in the same way. ∎

3.2.2.4 Corollary *Let $\mathfrak{A}$, $\mathfrak{B}$, φ, and θ be as in (3.1.0.6). Let $p, q \in \mathcal{P}(\mathfrak{A}) \setminus \{0, I\}$ be such that $pq = 0$ and $p + q = r \neq I$. Then we have $c_p = c_q$ or $c_p = \bar{c}_q$. Also, if $c_p = c_q$ and $\theta(p)\theta(q) \neq 0$ then $\theta(p)\theta(q) = \theta(I-p)\Delta\theta(q)$.*

Proof: The first statement comes by inspecting the result of (3.2.2.3). Notice that

$$\begin{aligned}
\theta(I-p)\theta(I-q) &= (I - \theta(p))(I - \theta(q)) \\
&= I - \theta(p) - \theta(q) + \theta(p)\theta(q).
\end{aligned}$$

Together with (3.1.0.2) and (3.1.0.3) this gives

$$\begin{aligned}\theta(I-p)\Delta\theta(q) &= I - (\theta(p)\Delta\theta(q)) \\ &= I - \theta(p) - \theta(q) + 2\theta(p)\theta(q) \\ &= \theta(I-p)\theta(I-q) + \theta(p)\theta(q),\end{aligned}$$

from which the second statement follows immediately. ■

The above result tells us that if p is a nontrivial projection then any nontrivial projection q orthogonal to p has one of at most two possible images under c. The following result extends this observation to projections more loosely related to p. It will be used in the proof of (3.2.2.6).

3.2.2.5 Corollary *Let $\mathfrak{A}$, $\mathfrak{B}$, φ, and θ be as in (3.1.0.6). Let $p, q \in \mathcal{P}(\mathfrak{A}) \setminus \{0, I\}$ be commuting projections. Then $c_p = c_q$ or $c_p = \overline{c}_q$.*

Proof: The proof is in cases, depending on pq.

1. If $pq = 0$, then the result follows from (3.2.2.4).
2. If $pq = p$, then $p(I-q) = 0$, so by (3.2.2.4) we have that $c_p = c_{I-q}$ or $c_p = \overline{c}_{I-q}$. Since $c_{I-q} = c_q$ by (3) of (3.2.1.3), we get that $c_p = c_q$ or $c_p = \overline{c}_q$. The case $pq = q$ can be dealt with similarily.
3. If $0, p, q$, and pq are distinct then $p - pq, q - pq \in \mathcal{P}(\mathfrak{A}) \setminus \{0, I\}$ and

$$\begin{aligned} p(q-pq) &= 0 \\ (q-pq)(p-pq) &= 0 \\ (p-pq)q &= 0. \end{aligned}$$

   By (3.2.2.4) or (3) of (3.2.1.3) we have $c_p = c_q$ or $c_p = \overline{c}_q$.

■

The next result shows that, if for example $\mathfrak{A}$ is a simple unital AF algebra, the range of c has at most two elements.

3.2.2.6 Proposition *Let $\mathfrak{A}$, $\mathfrak{B}$, φ, and θ be as in (3.1.0.6), and suppose also that for each pair of projections p and q in $\mathcal{P}(\mathfrak{A})$ there exist unitarily equivalent subprojections $p_1 \leq p$ and $q_1 \leq q$. Then the image of $c : \mathcal{P}(\mathfrak{A}) \setminus \{0, I\} \rightarrow Aut(\mathbb{S}^1)$ (cf. (3.2.1.2) and (3.2.1.3)) contains at most two elements. More precisely, for any projection $p \in \mathcal{P}(\mathfrak{A}) \setminus \{0, I\}$ we have*

$$c(\mathcal{P}(\mathfrak{A}) \setminus \{0, I\}) \subseteq \{c_p, \bar{c}_p\}.$$

Proof: Let $p, q \in \mathcal{P}(\mathfrak{A}) \setminus \{0, I\}$. By assumption, there exist $p_i \leq p$ and $q_1 \leq q$ such that $p_1 \sim_u q_1$. By (3.2.1.3), $c_{p_1} = c_{q_1}$. Since p_1 commutes with p then $c_{p_1} = c_p$ or $c_{p_1} = \bar{c}_p$ by (3.2.2.5). Similarily, $c_{q_1} = c_q$ or $c_{q_1} = \bar{c}_q$. So we see that $c_q = c_p$ or $c_q = \bar{c}_p$. So fixing p, for any $q \in \mathcal{P}(\mathfrak{A}) \setminus \{0, I\}$ we have $c_q = c_p$ or $c_q = \bar{c}_p$. Hence $c(\mathcal{P}(\mathfrak{A}) \setminus \{0, I\}) \subseteq \{c_p, \bar{c}_p\}$. ■

We now come to the definition of the partiton of $\mathcal{P}(\mathfrak{A}) \setminus \{0, I\}$ that we have been looking for.

3.2.2.7 Let $\mathfrak{A}$, $\mathfrak{B}$, φ, and θ be as in (3.1.0.6), and suppose also that $\mathfrak{A}$ is such that $c(\mathcal{P}(\mathfrak{A}) \setminus \{0, I\}) = \{c_p, \bar{c}_p\}$ for some $p \in \mathcal{P}(\mathfrak{A}) \setminus \{0, I\}$. For any projection $p \in \mathcal{P}(\mathfrak{A}) \setminus \{0, I\}$, we define

$$\mathcal{P}_{c_p} = \{q \in \mathcal{P}(\mathfrak{A}) \setminus \{0, I\}; c_q = c_p\}$$

and

$$\mathcal{P}_{\bar{c}_p} = \{q \in \mathcal{P}(\mathfrak{A}) \setminus \{0, I\}; c_q = \bar{c}_p\}.$$

Clearly, $\mathcal{P}_{c_p}$ and $\mathcal{P}_{\bar{c}_p}$ form a partition of $\mathcal{P}(\mathfrak{A}) \setminus \{0, I\}$.

Here we show that from partition of $\mathcal{P}(\mathfrak{A}) \setminus \{0, I\}$ with the properties described in (3.2.0.2), one obtains an orthoisomorphisms which preserves unitary equivalence.

3.2.2.8 Definition *Let $\mathfrak{A}$, $\mathfrak{B}$, φ, and θ be as in (3.1.0.6), and suppose there is a $p \in \mathcal{P}(\mathfrak{A}) \setminus \{0, I\}$ such that $c(\mathcal{P}(\mathfrak{A}) \setminus \{0, I\}) \subseteq \{c_p, \bar{c}_p\}$. For any projection $q \in \mathcal{P}(\mathfrak{A})$*

define $\theta' : \mathcal{P}(\mathfrak{A}) \to \mathcal{P}(\mathfrak{B})$ *by*

$$\theta'(q) = \begin{cases} \theta(q) & \textit{if } q \in \mathcal{P}_{c_p} \\ I - \theta(q) & \textit{if } q \in \mathcal{P}_{\bar{c}_p} \\ I & \textit{if } q = I \\ 0 & \textit{if } q = 0. \end{cases}$$

3.2.2.9 Proposition *Let* $\mathfrak{A}$, $\mathfrak{B}$, φ, *and* θ *be as in (3.1.0.6). Suppose there is a* $p \in \mathcal{P}(\mathfrak{A}) \setminus \{0, I\}$ *such that* $c(\mathcal{P}(\mathfrak{A}) \setminus \{0, I\}) \subseteq \{c_p, \bar{c}_p\}$, *and that the corresponding partition (3.2.2.7) is as in (3.2.0.2). Then* θ' *is an orthoisomorphisms and*

$$\theta'(uqu^*) = \varphi(u)\theta'(q)\varphi(u)^*, \tag{3.2.40}$$

so θ' *preserves unitary equivalence.*

Proof: That θ' is a mapping is clear from the fact that $\mathcal{P}_{c_p}$ and $\mathcal{P}_{\bar{c}_p}$ partition $\mathcal{P}(\mathfrak{A}) \setminus \{0, I\}$, and that it is a bijection follows from (3) of (3.2.1.3). To show that θ' preserves orthogonality we consider the possible cases. Suppose that p_1 and p_2 are non-trivial orthogonal projections in $\mathcal{P}(\mathfrak{A}) \setminus \{0, I\}$.

1. If $p_1, p_2 \in \mathcal{P}_{c_p}$ then $\theta(p_1)\theta(p_2) = 0$ follows from (3.2.0.2) and (3.2.2.3).

2. If $p_1, p_2 \in \mathcal{P}_{\bar{c}_p}$ then by (3.2.0.2) we have $c_{p_1} = c_{p_2} = \bar{c}_{p_1+p_2}$, and by (3.2.2.3) we have $\theta(I - p_1)\theta(I - p_2) = 0$. So $\theta'(p_1)\theta'(p_2) = 0$, by (3.1.0.3) and the definition of θ'.

3. If $p_1 \in \mathcal{P}_{c_p}$ and $p_2 \in \mathcal{P}_{\bar{c}_p}$ then $c_{p_1+p_2} = \bar{c}_p$ by (3.2.0.2). So we have that $\bar{c}_{p_1} = c_{p_2} = c_{p_1+p_2}$. So $\theta(p_1)\theta(I - p_2) = 0$ by (3.2.2.3) and so from the definition of θ' we have $\theta'(p_1)\theta'(p_2) = \theta(p_1)\theta(I - p_2) = 0$.

So θ' preserves orthogonality. We now show (3.2.40). If $q = 0$ or if $q = I$ then the result is clear, so let $q \in \mathcal{P}(\mathfrak{A}) \setminus \{0, I\}$. If $q \in \mathcal{P}_{c_p}$ then $\theta' = \theta$ and the result is (1) of (3.1.0.3), and if $q \in \mathcal{P}_{\bar{c}_p}$ then so is uqu^* and we have by (1) of (3.1.0.3) that

$$\theta'(uqu^*) = I - \theta(uqu^*) = \varphi(u)(I - q)\varphi(u)^* = \varphi(u)\theta'(q)\varphi(u)^*.$$

∎

3.2.3 Properties of the Partition $\{\mathcal{P}_{c_p}, \mathcal{P}_{\bar{c}_p}\}$

To this point, except in (3.2.2.6), we have assumed only that $\mathfrak{A}$ and $\mathfrak{B}$ are simple unital C*-algebras. We now show that, under an additional assumption on $\mathfrak{A}$ (an assumption which will be satisfied for simple unital AF algebras), the partition constructed in (3.2.2.7) satisfies the properties stated in (3.2.0.2) (cf. (3.2.3.14) and (3.2.3.15)). To show these key results we need certain technical results (namely (3.2.3.1)–(3.2.3.7)), which are valid in any simple unital C*-algebra.

Decomposition Properties of c

Below we examine how c behaves with respect to decompositions of projections into sums of smaller projections. The main results are Proposition 3.2.3.5 and its corollaries which, under additional assumptions on the algebra, will yield an abundance of projections with the same image under c. These results will be used in the proofs of (3.2.3.14) and (3.2.3.15).

The following two lemmas give another link between c and orthogonality.

3.2.3.1 Lemma *Let $\mathfrak{A}$, $\mathfrak{B}$, φ, and θ be as in (3.1.0.6). Let $p_1, p_2, p_3 \in \mathcal{P}(\mathfrak{A}) \setminus \{0, I\}$ which are pairwise orthogonal, with $c_{p_1} = c_{p_2} = c_{p_3}$ and such that $\Sigma p_i = r \neq I$. If $\theta(p_1)\theta(p_3) \neq 0$, $\theta(p_2)\theta(p_3) \neq 0$, then also $\theta(p_1)\theta(p_2) \neq 0$.*

Proof: Suppose that $\theta(p_1)\theta(p_3) \neq 0$ and $\theta(p_2)\theta(p_3) \neq 0$, but that $\theta(p_1)\theta(p_2) = 0$. Let $p = p_3$, $q = p_1 + p_2$. Then $pq = 0$ and since $\theta(p_1)\theta(p_2) = 0$ we have by (3.2.2.3) that

$$c_q = c_{p_1+p_2} = c_{p_1} = c_{p_3} = c_p,$$

Case 1 Suppose $c_r = c_p$.

Then $c_p = c_q = c_r$. By Proposition (3.2.2.3), $\theta(p)\theta(q) = 0$. So,

$$
\begin{aligned}
0 &= \theta(p)\theta(q) \\
&= \theta(p_3)\theta(p_1 + p_2) \\
&= \theta(p_3)[\theta(p_1)\Delta\theta(p_2)] \\
&= [\theta(p_1)\theta(p_3)]\Delta[\theta(p_2)\theta(p_3)] \\
&= [\theta(I - p_3)\Delta\theta(p_1)]\Delta[\theta(I - p_3)\Delta\theta(p_2)] && \text{from (3.2.2.4)} \\
&= \theta(I - p_3)\Delta\theta(I - p_3)\Delta\theta(p_1)\Delta\theta(p_2) \\
&= \theta(p_1)\Delta\theta(p_2) \\
&= \theta(p_1 + p_2).
\end{aligned}
$$

But $p_1 + p_2 \neq 0 \Rightarrow \theta(p_1 + p_2) \neq 0$. From this contradiction we see that $c_r \neq c_p$.

Case 2 Suppose $c_r = \bar{c}_p$. Then $\theta(I - p)\theta(I - q) = 0$ follows from (3.2.2.3), and we have

$$
\begin{aligned}
0 &= \theta(I - p)\theta(I - q) \\
&= \theta(I - p_3)\theta(I - (p_1 + p_2)) \\
&= I - \theta(p_3) - \theta(p_1)\Delta\theta(p_2) + \theta(p_3)[\theta(p_1)\Delta\theta(p_2)] \\
&= I - \theta(p_3) - \theta(p_1)\Delta\theta(p_2) + \theta(p_1)\Delta\theta(p_2) && \text{as in Case 1} \\
&= I - \theta(p_3).
\end{aligned}
$$

Hence, $p_3 \neq I$, so $\theta(p_3) \neq I$, and hence $I - \theta(p_3) \neq 0$. It follows that $c_r \neq \bar{c}_p$.

■

3.2.3.2 Lemma *Let $\mathfrak{A}$, $\mathfrak{B}$, φ, and θ be as in (3.1.0.6). Let $p_1, p_2, p_3 \in \mathcal{P}(\mathfrak{A}) \setminus \{0, I\}$ which are pairwise orthogonal with $c_{p_1} = c_{p_2} = c_{p_3}$ and $\Sigma p_i = r \neq I$. If $\theta(p_1)\theta(p_2) = 0$ and $\theta(p_2)\theta(p_3) = 0$, then also $\theta(p_1)\theta(p_3) = 0$.*

Proof: Since

$$\theta(p_1)\theta(p_2) = 0 \Rightarrow \theta(p_2) \leq \theta(I - p_1)$$

and

$$\theta(p_2)\theta(p_3) = 0 \Rightarrow \theta(p_2) \leq \theta(I - p_3)$$

it follows from (2.7.12.1) that $\theta(p_2) \leq \theta(I - p_1)\theta(I - p_3)$. But since $c_{p_1} = c_{p_3}$ and $\theta(p_1)\theta(p_3) \neq 0$ then $\theta(I - p_1)\theta(I - p_3) = 0$ (from (3.2.2.3)). This forces $\theta(p_2) = 0$, a contradiction to the injectivity of θ. ■

The following result extends the preceding two. It will be used in the proofs of (3.2.3.4) and (3.2.3.13).

3.2.3.3 Lemma *Let $\mathfrak{A}$, $\mathfrak{B}$, φ, and θ be as in (3.1.0.6). Let $\{p_i\}_{i=1}^n$ be n projections of $\mathcal{P}(\mathfrak{A}) \setminus \{0, I\}$ which are pairwise orthogonal and such that $p_i + p_j + p_k \neq I$ for any $1 \leq i < j < k \leq n$ and $c_{p_1} = c_{p_2} = \ldots = c_{p_n}$. If $\theta(p_k)\theta(p_l) = 0$ for some k, l, then $\theta(p_i)\theta(p_j) = 0$ for all i, j.*

Proof: By induction. The cases $n = 1$ and $n = 2$ are trivial, and the case $n = 3$ follows immediately from (3.2.3.1) and (3.2.3.2), since $p_1 + p_2 + p_3 \neq I$ by hypothesis.

Suppose that the result is true for some m such that $m \geq 3$. We now prove it for $n = m + 1$. Let $\{p_i\}_{i=1}^{m+1}$, $p_i \in \mathcal{P}(\mathfrak{A}) \setminus \{0, I\}$ for all i be pairwise orthogonal and such that $c_{p_i} = c_{p_j}$ for all i, j. Suppose without loss of generality that $\theta(p_1)\theta(p_2) = 0$. The collection $\{p_i\}_{i=1}^m$ satisfies the induction hypothesis, and hence $\theta(p_i)\theta(p_j) = 0$ for all $1 \leq i \neq j \leq m$. So it remains to show that $\theta(p_{m+1})\theta(p_i) = 0$ for all $1 \leq i \leq m$. Since $m > 3$, both the collections $\{p_1\}_{i=2}^{m+1}$ and $\{p_i\} \cup \{p_i\}_{i=3}^{m+1}$ satisfy the induction hypothesis, so $\theta(p_{m+1})\theta(p_i) = 0$ for all $i \neq m + 1$.

Hence the result. ■

The following lemma will allow us, under additional assumptions on the algebra, to find many projections with the same image under c.

3.2.3.4 Lemma *Let $\mathfrak{A}$, $\mathfrak{B}$, φ, and θ be as in (3.1.0.6). Let $p_1, p_2, p_3 \in \mathcal{P}(\mathfrak{A}) \setminus \{0, I\}$ which are pairwise orthogonal with $c_{p_1} = c_{p_2} = c_{p_3}$ and such that $\Sigma p_i = r \neq I$. Then $c_{p_1} = c_r$.*

Proof: Suppose first of all that $c_{p_i+p_j} = c_{p_i}$ for some $i \neq j$. By (3.2.2.3) and (3.2.3.3), we have that $\theta(p_i)\theta(p_j)$ for all $i \neq j$. Hence

$$\begin{aligned}\theta(p_1+p_2)\theta(p_3) &= [\theta(p_1)\Delta\theta(p_2)]\theta(p_3)\\ &= [\theta(p_1)+\theta(p_2)]\theta(p_3)\\ &= \theta(p_1)\theta(p_3)+\theta(p_2)\theta(p_3)\\ &= 0.\end{aligned}$$

Using Proposition 3.2.2.3 we have $c_{(p_1+p_2)+p_3} = c_{p_3} = c_{p_1}$, and it follows that $c_r = c_{p_1+p_2+p_3} = c_{p_1}$.

Suppose now that for all $i \neq j$, $c_{p_i+p_j} = \bar{c}_{p_i}$. By (3.2.2.4), we obtain

$$\theta(p_i)\theta(p_j) = \theta(I-p_i)\Delta\theta(p_j) \text{ for all } i \neq j. \tag{3.2.41}$$

If $c_r = \bar{c}_{p_1}$ then by (3.2.2.4) we get that $\theta(p_1) + \theta(p_1+p_2+p_3) = \theta(p_2+p_3)$, so $\theta(p_1) \leq \theta(p_2+p_3)$ and hence

$$\begin{aligned}\theta(p_1) &= \theta(p_1)\theta(p_2+p_3)\\ &= \theta(p_1)\theta(p_2\Delta p_3)\\ &= [\theta(p_1)\theta(p_2)]\Delta[\theta(p_1)\theta(p_3)] && \text{by (3.1.0.2)}\\ &= [\theta(I-p_1)\Delta\theta(p_2)]\Delta[\theta(I-p_1)\Delta\theta(p_3)] && \text{by (3.2.41)}\\ &= \theta(I-p_1)\Delta\theta(I-p_1)\Delta\theta(p_2)\Delta\theta(p_3)\\ &= \theta(p_2)\Delta\theta(p_3)\\ &= \theta(p_2\Delta p_3)\\ &= \theta(p_2+p_3)\end{aligned}$$

which contradicts the injectivity of θ. So $c_r = c_{p_1}$. ■

We now extend (3.2.3.4).

3.2.3.5 Proposition *Let $\mathfrak{A}$, $\mathfrak{B}$, φ, and θ be as in (3.1.0.6). Let n be an odd integer and let $\{p_i\}_{i=1}^{n}$ be a family of pairwise orthogonal projections in $\mathcal{P}(\mathfrak{A}) \setminus \{0, I\}$ with $c_{p_i} = c_{p_j}$ for all i, j and such that $\Sigma p_i = r \neq I$. Then $c_{p_1} = c_r$.*

Proof: We proceed by induction. The case $n = 3$ is Proposition 3.2.3.4. Suppose the result holds for some odd integer $n \in \mathbb{Z}^+$ such that $n \geq 3$. Consider $\{p_i\}_{i=1}^{n+2}$ satisfying the hypotheses. Set $q_i = p_i$ if $i < n$ and set $q_n = p_n + p_{n+1} + p_{n+2}$. By (3.2.3.4) $c_{q_n} = c_{p_1}$, so $\{q_i\}_{i=1}^{n}$ is an odd family of projections such that $c_{q_i} = c_{p_1}$ for all i. Now apply the induction hypothesis to conclude that $c_{p_1} = c_{\Sigma q_i} = c_{\Sigma p_1}$. ∎

We obtain the following corollaries which, under additional assumptions on the algebra (see also (3.2.3.9)), will allow us to find continually "smaller" projections with the same image under c.

3.2.3.6 Corollary *Let $\mathfrak{A}$, $\mathfrak{B}$, φ, and θ be as in (3.1.0.6). Let $p \in \mathcal{P}(\mathfrak{A}) \setminus \{0, I\}$ such that $p = q_1 + q_2 + q_3$ for some pairwise orthogonal projections $q_1, q_2, q_3 \in \mathcal{P}(\mathfrak{A}) \setminus \{0, I\}$. Then $c_{q_i} = c_p$ for some i.*

Proof: Suppose not. Then $p = q_1 + q_2 + q_3$ for some pairwise orthogonal projections $q_1, q_2, q_3 \in \mathcal{P}(\mathfrak{A}) \setminus \{0, I\}$ and $c_{q_i} = \overline{c}_p$ for every i. By (3.2.3.5) it follows that $c_p = c_{q_1+q_2+q_3} = c_{q_1} = \overline{c}_p$. This contradiction establishes the result. ∎

3.2.3.7 Corollary *Let $\mathfrak{A}$, $\mathfrak{B}$, φ, and θ be as in (3.1.0.6). Let $p \in \mathcal{P}(\mathfrak{A}) \setminus \{0, I\}$ such that there exists an odd integer n and a finite set $\{q_i\}_{i=1}^{n}$ of pairwise orthogonal non-trivial projections with $p = \sum_{i=1}^{n} q_i$. Then $c_{q_i} = c_p$ for some i.*

Proof: By induction on n. The case $n = 1$ is trivial, and the case $n = 3$ is (3.2.3.6). Suppose the result is true for some odd $n > 3$. Consider $p = \sum_{i=1}^{n+2} q_i'$. Set $q_i = q_i'$ for $1 \leq i \leq n-1$ and set $q_n = q_n' + q_{n+1}' + q_{n+2}'$. The family $\{q_i\}_{i=1}^{n}$ satisfies the induction hypothesis, so for some j, $c_{q_j} = c_p$. If $1 \leq j \leq n-1$ then we have finished. If not then $c_{q_n} = c_p$, and by (3.2.3.6) applied to q_n we have $c_{q_i'} = c_p$ for some $n \leq i \leq n+2$. ∎

Oddly Decomposable C*-algebras

Now we define oddly decomposable algebras (3.2.3.8), and examine some of their properties. For these algebras:

1. Corollary 3.2.3.10 shows that the image of c has at most two elements, which allows us to construct the partition of $\mathcal{P}(\mathfrak{A}) \setminus \{0, I\}$ as in (3.2.2.7),
2. Proposition 3.2.3.14 and Corollary 3.2.3.15 show that the partition of $\mathcal{P}(\mathfrak{A}) \setminus \{0, I\}$ constructed in (3.2.2.7) satisfies the properties stated in (3.2.0.2).

The following rather odd definition gives the key property used in the proofs of this section.

3.2.3.8 Definition *Let $\mathfrak{A}$ be a C*-algebra. We say that $\mathfrak{A}$ is* **oddly decomposable** *if for every pair $p, q \in \mathcal{P}(\mathfrak{A}) \setminus \{0, I\}$ there is an odd integer $n \geq 3$ and a decomposition of q as a sum $q = \sum_{i=1}^{n} r_i$ of pairwise orthogonal projections $r_i \in \mathcal{P}(\mathfrak{A}) \setminus \{0, I\}$, such that each r_i is unitarily equivalent to some projection $r_i' < p$.*

Note that simple unital AF algebras are oddly decomposable (see (3.4.0.1)).

Having found that we can continually find smaller projections in an oddly decomposable algebra which all have the same image under c, we now look at the "other" image under c.

3.2.3.9 Proposition *Let $\mathfrak{A}$, $\mathfrak{B}$, φ, and θ be as in (3.1.0.6), and suppose also that $\mathfrak{A}$ is oddly decomposable. If there are projections $p, q \in \mathcal{P}(\mathfrak{A}) \setminus \{0, I\}$ such that $c_p \neq c_q$, then there is a projection $r \in \mathcal{P}(\mathfrak{A}) \setminus \{0, I\}$ for which $r < p$ and $c_r = c_q$.*

Proof: Since $\mathfrak{A}$ is oddly decomposable, there is an odd integer $n \geq 3$ and a decomposition of q as a sum $q = \sum_{i=1}^{n} r_i$ of pairwise orthogonal projections $r_i \in \mathcal{P}(\mathfrak{A}) \setminus \{0, I\}$, such that each r_i is unitarily equivalent to some projection $r_i' < p$. By (3.2.3.7), $c_{r_j} = c_q$ for some j. The projection $r = r_j'$ has the desired property. ∎

3.2.3.10 Corollary *Let $\mathfrak{A}$, $\mathfrak{B}$, φ, and θ be as in (3.1.0.6), and suppose also that $\mathfrak{A}$ is oddly decomposable. Then for any projection $p \in \mathcal{P}(\mathfrak{A}) \setminus \{0, I\}$ we have $c(\mathcal{P}(\mathfrak{A}) \setminus \{0, I\}) \subseteq \{c_p, \overline{c}_p\}$.*

Proof: A simple unital oddly decomposable C*-algebra satisfies the conditions of (3.2.2.6). ∎

3.2.3.11 Let $\mathfrak{A}$, $\mathfrak{B}$, φ, and θ be as in (3.1.0.6). Let $p, q \in \mathcal{P}(\mathfrak{A}) \setminus \{0, I\}$ be such that $pq = 0$ and $p+q = r \neq I$. If $c : \mathcal{P}(\mathfrak{A}) \setminus \{0, I\} \to \mathrm{Aut}(\mathbb{S}^1)$ is constant then $c_p = c_q = c_r$, so by (3.2.2.3) we have that $\theta(p)\theta(q) = 0$. So θ is an orthoisomorphism.

We now develop the key properties of the partition. The following definition will be convenient.

3.2.3.12 Definition *Let $\theta : \mathcal{P}(\mathfrak{A}) \to \mathcal{P}(\mathfrak{B})$ be any map. Let $p_1, p_2 \in \mathcal{P}(\mathfrak{A})$ be orthogonal projections. We say that θ* **preserves the orthogonality of p_1 and p_2** *if*

$$\theta(p_1)\theta(p_2) = 0.$$

Let $S = \{p_i\}$ be a set of projections. We say that θ **preserves orthogonality on** S *if θ preserves the orthogonality of p_i and p_j for any orthogonal $p_i, p_j \in S$. Similarily, we say that θ* **flips the orthogonality of p_1 and p_2** *if*

$$(I - \theta(p_1))(I - \theta(p_2)) = 0.$$

Let $S = \{p_i\}$ be a set of projections. We say that θ **flips orthogonality** *on S if θ flips the orthogonality of p_i and p_j for any orthogonal $p_i, p_j \in S$.*

Note that with this terminology we could describe Example 3.2.0.1 by saying that θ preserves orthogonality of the projections with even rank, and flips orthogonality of the projections with odd rank. Analogously, we have the following three results.

3.2.3.13 Lemma *Let $\mathfrak{A}$, $\mathfrak{B}$, φ, and θ be as in (3.1.0.6), and suppose also that $\mathfrak{A}$ is oddly decomposable. Let $p \in \mathcal{P}(\mathfrak{A}) \setminus \{0, I\}$. Suppose that p_1 and p_2 are two orthogonal projections of $\mathcal{P}_{c_p}$ such that $p_1 + p_2 \neq I$ and $\theta(p_1)\theta(p_2) = 0$. Then θ preserves orthogonality on $\mathcal{P}_{c_p}$.*

Proof: Let r and s be two orthogonal projections of $\mathcal{P}_{c_p}$. By (6) of (3.1.0.3) we can assume that $r + s \neq I$. By Lemma 3.2.3.3, it is enough to show that there exist two projections $y, z \in \mathcal{P}_{c_p}$ with $yz = 0$, $y + z < I - (r + s)$, and such that $\theta(y)\theta(z) = 0$.

Since $\mathfrak{A}$ is oddly decomposable, there is an odd integer $n \geq 3$ and a decomposition of $I - (p_1 + p_2) = \sum_{i=1}^{n} x_i'$ where the x_i' are pairwise orthogonal projections, each of which is unitarily equivalent to some $x_i < I - (r+s)$. Using (3.2.3.7), we obtain $c_{x_i} = c_{I-(p_1+p_2)} = c_p$ for some i. Set $x' = x_i$. Since $\mathfrak{A}$ is oddly decomposable, in particular $x = x_1 + x_2 + x_3$ for some projections $x_i \in \mathcal{P}(\mathfrak{A}) \setminus \{0, I\}$. By (3.2.3.7), $c_{x_i} = c_{x'}$ for some i. Set $y' = x_i$. Similarily, we can find a projection $z' < x' - y'$ with $c_{z'} = c_{x'}$. So we have $y', z' < x' < I - (p_1 + p_2)$ such that $yz = 0$ and $c_{y'} = c_{z'} = c_{x'}$. Hence

$$c_{y'} = c_{z'} = c_{x'} = c_x = c_p = c_{p_1} = c_{p_2}.$$

Applying (3.2.3.3) to the set $\{y', z', p_1, p_2\}$ yields $\theta(y')\theta(z') = 0$. Since x' is unitarily equivalent to x, there is a unitary $u \in \mathcal{U}(\mathfrak{A})$ such that $x = ux'u^*$. Setting $y = uy'u^*$ and $z = uz'u^*$, we have $y, z \in \mathcal{P}_{c_p}$, $yz = 0$, and $y, z < I - (r+s)$. So $\{r, s, x, y\}$ is a subset of $\mathcal{P}_{c_p}$ consisting of pairwise orthogonal projections with $c_r = c_s = c_x = c_y$ and $\theta(y)\theta(z) = 0$. Applying (3.2.3.3) to the set $\{r, s, x, y\}$, we conclude that also $\theta(r)\theta(s) = 0$.

So if θ preserves the orthogonality of two elements of $\mathcal{P}_{c_p}$ which do not sum to I, then θ preseves the orthogonality of *any* two elements in $\mathcal{P}_{c_p}$. ∎

3.2.3.14 Proposition *Let $\mathfrak{A}$, $\mathfrak{B}$, φ, and θ be as in (3.1.0.6), and suppose also that $\mathfrak{A}$ is oddly decomposable. Let p be a projection in $\mathcal{P}(\mathfrak{A}) \setminus \{0, I\}$. If $\mathcal{P}_{c_p} \neq \mathcal{P}(\mathfrak{A}) \setminus \{0, I\}$, then θ preserves orthogonality on $\mathcal{P}_{c_p}$ (respectively on $\mathcal{P}_{\bar{c}_p}$) and flips orthogonality on $\mathcal{P}_{\bar{c}_p}$ (respectively $\mathcal{P}_{c_p}$).*

Proof: Let $p_1, p_2 \in \mathcal{P}_{c_p}$ be such that $p_1 p_2 = 0$ and $p_1 + p_2 \neq I$. By (3.2.2.3), either $\theta(p_1)\theta(p_2) = 0$ or $\theta(I - p_1)\theta(I - p_2) = 0$. By (3.2.3.13) we have that θ either preserves or flips orthogonality on $\mathcal{P}_{c_p}$, and likewise on $\mathcal{P}_{\bar{c}_p}$. Let us show that θ does not preserve orthogonality on $\mathcal{P}_{\bar{c}_p}$ if it does so on $\mathcal{P}_{c_p}$.

Suppose that θ preserves orthogonality on both $\mathcal{P}_{c_p}$ and $\mathcal{P}_{\bar{c}_p}$. By (3.2.3.9) and the fact that $\mathfrak{A}$ is oddly decomposable, there exists a projection $q \in \mathcal{P}_{\bar{c}_p}$ such that $q < I - p$. Since the image of c contains at most two elements (3.2.3.10), we know that either $c_{I-(p+q)} = \bar{c}_p = c_q$ or $c_{I-(p+q)} = c_p = \bar{c}_q$. Interchanging p and q if necessary, we can assume that $c_{I-(p+q)} = \bar{c}_p$. Again use (3.2.3.9) to find $x < p$ such that $c_x = \bar{c}_p$.

Then

$$
\begin{aligned}
c_{p-x} &= c_{I-(p-x)} && \text{by (3.2.1.3)} \\
&= c_{x+q+(I-(p+q))} \\
&= c_q && \text{by (3.2.3.4)} \\
&= \overline{c}_p.
\end{aligned}
$$

So x and $p-x$ are orthogonal projections in $\mathcal{P}_{\overline{c}_p}$ such that

$$c_x = c_{p-x} = \overline{c}_{x+(p-x)} = \overline{c}_p.$$

But by hypothesis θ preserves orthogonality on $\mathcal{P}_{\overline{c}_p}$, so we have $c_x = c_{p-x} = c_{x+(p-x)}$. This implies that $c_p = \overline{c}_p$. This is impossible, so θ does not preserve orthogonality on $\mathcal{P}_{\overline{c}_p}$ if it does on $\mathcal{P}_{c_p}$.

Now let us show that θ does not flip orthogonality on $\mathcal{P}_{c_p}$ if it does so on $\mathcal{P}_{\overline{c}_p}$. Suppose that θ flips orthogonality on both $\mathcal{P}_{c_p}$ and $\mathcal{P}_{\overline{c}_p}$. We will find a projection $x \in \mathcal{P}_{c_p}$ with $x < p$ such that $c_x = c_{p-x} = c_p$, so $\theta(x)\theta(p-x) = 0$ and hence θ will preserves orthogonality on $\mathcal{P}_{c_p}$.

By (3.2.3.9) there is a projection $q \in \mathcal{P}_{\overline{c}_p}$ such that $q < I - p$. Without loss of generality suppose that $c_{I-(p+q)} = \overline{c}_p$. Again use (3.2.3.7) and the fact that $\mathfrak{A}$ is oddly decomposable to find $x < p$ such that $c_x = c_p$. If $c_{p-x} \neq c_x$ then by (3.2.3.10) we have $c_{p-x} = \overline{c}_x = c_q$, and so

$$
\begin{aligned}
c_x &= c_{I-x} && \text{by (3) of (3.2.1.3)} \\
&= c_{(p-x)+q+(I-(p+q))} \\
&= c_q && \text{by (3.2.3.4)} \\
&= \overline{c}_p \\
&= \overline{c}_x.
\end{aligned}
$$

So x and $p-x$ are orthogonal projections in $\mathcal{P}_{c_p}$ such that

$$c_x = c_{p-x} = c_{x+(p-x)} = c_p.$$

So θ does not flip orthogonality on both $\mathcal{P}_{c_p}$ and $\mathcal{P}_{\overline{c}_p}$. ■

3.2.3.15 Corollary *Let $\mathfrak{A}$, $\mathfrak{B}$, φ, and θ be as in (3.1.0.6), and suppose also that $\mathfrak{A}$ is oddly decomposable. Let $p \in \mathcal{P}(\mathfrak{A}) \setminus \{0, I\}$ be such that θ preserves orthogonality on $\mathcal{P}_{c_p}$ and flips orthogonality on $\mathcal{P}_{\bar{c}_p}$. Let p_1 and p_2 be orthogonal projections in $\mathcal{P}(\mathfrak{A}) \setminus \{0, I\}$ such that $p_1 + p_2 = r \neq I$. Then*

1. $p_1, p_2 \in \mathcal{P}_{c_p} \Rightarrow r \in \mathcal{P}_{c_p}$
2. $p_1 \in \mathcal{P}_{\bar{c}_p}$ and $p_2 \in \mathcal{P}_{\bar{c}_p} \Rightarrow r \in \mathcal{P}_{c_p}$
3. $p_1 \in \mathcal{P}_{c_p}$ and $p_2 \in \mathcal{P}_{\bar{c}_p} \Rightarrow r \in \mathcal{P}_{\bar{c}_p}$

Proof: Parts (1) and (2) follow directly from (3.2.2.3) and (3.2.3.12). For part (3), let $p_1 \in \mathcal{P}_{c_p}$, $p_2 \in \mathcal{P}_{\bar{c}_p}$, and suppose that $r \in \mathcal{P}_{c_p}$. Then by (3.2.1.3), $I - r \in \mathcal{P}_{c_p}$. So we have that $(I - r) + p_1 \in \mathcal{P}_{c_p}$ from part (1). But then $p_2 = I - (I - r + p_1) \in \mathcal{P}_{c_p}$, so $c_{p_2} = c_p$ by (3.2.1.3). This contradiction establishes the result. ∎

3.3 An Isomorphism of Scaled K_0-groups

In this section we show that from an orthoisomorphisms which preserves unitary equivalence we obtain an isomorphism of the scaled dimension groups $K_0(\mathfrak{A})$ and $K_0(\mathfrak{B})$, for AF algebras $\mathfrak{A}$ and $\mathfrak{B}$.

3.3.0.1 Lemma ([Dy, Lemma 1]) *Let θ be any projection orthoisomorphisms between unital C^*-algebras $\mathfrak{A}$ and $\mathfrak{B}$. Then θ has the following properties*

1. $\theta(0) = 0$.
2. $\theta(I) = I$.
3. $\theta(I - p) = I - \theta(p)$.
4. *For projections p and q in $\mathfrak{A}$, $p \leq q$ implies $\theta(p) \leq \theta(q)$.*
5. *If $pq = qp$ if and only if $\theta(p)\theta(q) = \theta(q)\theta(p)$.*
6. *If $p + q = r \in \mathcal{P}(\mathfrak{A})$ if and only if $\theta(p + q) = \theta(p) + \theta(q)$.*

In words, θ preserves 0, I, orthocomplements, order, commutativity, and sums of projections.

Proof:

1. 0 is the only projection which is orthogonal to every projection, so $\theta(0) = 0$.

2. I is the only projection which is not orthogonal to any non-zero projection, so $\theta(I_{\mathfrak{A}}) = I_{\mathfrak{B}}$.

3. The projection $I - p \in \mathcal{P}(\mathfrak{A})$ is characterized as the unique projection which is orthogonal to p and such that any projection r such that $r \perp p, (I-p)$ must be 0. So $\theta(I-p) = I - \theta(p)$.

4. Let $p, q \in \mathcal{P}(\mathfrak{A})$. Using part (3) we have

$$\begin{aligned} p \leq q &\iff p \perp (I - q) \\ &\iff \theta(p) \perp \theta(I - q) \\ &\iff \theta(p) \perp (I - \theta(q)) \\ &\iff \theta(p) \leq \theta(q). \end{aligned}$$

5. If $p, q \in \mathcal{P}(\mathfrak{A})$ are orthogonal projections then $p + q$ is the (unique) projection in $\mathcal{P}(\mathfrak{A})$ such that $p + q \geq p, q$ and such that $r \in \mathcal{P}(\mathfrak{A}), r \geq p, q \Rightarrow r \geq p + q$. The result follows from (4).

3.3.0.2 Proposition *Let $\mathfrak{A}$ and $\mathfrak{B}$ be AF algebras, and let $\tilde{\theta} : \mathcal{P}(\mathfrak{A}) \to \mathcal{P}(\mathfrak{B})$ be an orthoisomorphisms which preserves equivalence of projections. Then $\tilde{\theta}$ induces an isomorphism of scaled dimension groups $\tilde{\theta}_* : K_0(\mathfrak{A}) \to K_0(\mathfrak{B})$, and hence there is a C^*-algebra isomorphism $\Phi : \mathfrak{A} \to \mathfrak{B}$ such that $K_0(\Phi) = \tilde{\theta}_*$.*

Proof: Define $\tilde{\theta}_* : \Gamma(\mathfrak{A}) \to \Gamma(\mathfrak{B})$ by $\tilde{\theta}_*([p]) = [\theta'(p)]$ for all $p \in \mathcal{P}(\mathfrak{A})$. The map $\tilde{\theta}_*$ is well defined since θ' preserves unitary equivalence, and hence equivalence (by (2.8.6.3)). Suppose $x, y, z \in \Gamma(\mathfrak{A})$ with $x + y = z$. By definition of $\Gamma(\mathfrak{A})$ there are orthogonal projections $p, q \in \mathcal{P}(\mathfrak{A})$ such that $[p] = x, [q] = y$, and $[p + q] = z$. Since $\tilde{\theta}$ is an orthoisomorphisms it preserves sums (3.3.0.1), so

$$\begin{aligned}\tilde{\theta}_*(x) + \tilde{\theta}_*(y) &= [\theta'(p)] + [\theta'(q)] \\ &= [\theta'(p) + \theta'(q)] \\ &= [\theta'(p + q)] \\ &= \tilde{\theta}_*([p + q]) \\ &= \tilde{\theta}_*(x + y) \\ &= \tilde{\theta}_*(z)\end{aligned}$$

So $\tilde{\theta}_*$ is a scale homomorphism. Since the inverse of $\tilde{\theta}_*$ under composition is given by $\tilde{\theta}_*^{-1}[q] = [\tilde{\theta}^{-1}(q)]$ for any $q \in \mathcal{P}(\mathfrak{B})$, we see that $\tilde{\theta}_*^{-1}$ is also a scale homomorphism and hence $\tilde{\theta}_*$ is a scale isomorphism. It follows from (2.6.3.11) that $\tilde{\theta}_*$ extends to an isomorphism of $K_0(\mathfrak{A})$ with $K_0(\mathfrak{B})$, and from (2.10.3.3) that there is an isomorphism Φ of $\mathfrak{A}$ with $\mathfrak{B}$ with the stated property. ∎

3.4 Conclusion

3.4.0.1 Lemma *Simple unital infinite dimensional AF algebras are oddly decomposable.*

Proof: Let $\mathfrak{A}$ be a simple unital AF algebra, $\mathfrak{A} = \varinjlim\{\mathfrak{A}_i, \Phi_{ij}\}$ where $\mathfrak{A}_i = \oplus_k \mathbb{M}_{i,k}$. Denote by $I_{i,k}$ the unit of $\mathbb{M}_{i,k}$. Let $p, q \in \mathcal{P}(\mathfrak{A}) \setminus \{0, I\}$.

Assume first that $p, q \in \mathfrak{A}_i$ for some i, and that i is large enough so that $pI_{i,k} \neq 0$ for every k (cf. (2.8.7.4)). In particular, every minimal projection in $\mathfrak{A}_i$ is equivalent to some minimal projection which is under p. Since $q \in \mathfrak{A}_i$, we can write $q = \sum_{a=1}^{b} m_a$ for pairwise orthogonal minimal projections m_a, each of which is equivalent to a projection under p. If b is odd and $b \geq 3$ then we have the desired decomposition. If not, use the fact that $\mathfrak{A}$ is a simple AF algebra to decompose m_1 as the sum of two smaller projections in $\mathfrak{A}$. This increases the number of projections in the decompostion of q by 1. Continuing in this fashion if necessary, we obtain the desired decomposition of q.

Suppose now that p, q are any projections in $\mathcal{P}(\mathfrak{A}) \setminus \{0, I\}$. Since $\mathfrak{A}$ is AF, there is an integer i and projections $p', q' \in \mathcal{P}(\mathfrak{A}_i)$, with p unitarily equivalent to p' and q unitarily equivalent to q'. The result follows. ∎

Finally, we come to our main result.

3.4.0.2 Theorem *Let $\mathfrak{A}$, $\mathfrak{B}$, φ, and θ be as in (3.1.0.6), suppose also that $\mathfrak{A}$ and $\mathfrak{B}$ are AF algebras. Then $\mathfrak{A}$ and $\mathfrak{B}$ are isomorphic C^*-algebras.*

Proof: If $\mathfrak{A}$ finite dimensional, see [Dy, comments following Theorem 2]. If not, note that infinite dimensional AF algebras are oddly decomposable (3.4.0.1). If $\mathcal{P}_{c_p} = \mathcal{P}(\mathfrak{A}) \setminus \{0, I\}$ for some projection $p \in \mathcal{P}(\mathfrak{A}) \setminus \{0, I\}$ then the result follows from (3.2.2.3) and (3.3.0.2). Otherwise use (3.2.2.9) and (3.3.0.2). ∎

Appendix A

Example from [Dy]

Here we construct explicitly a group automorphism $\varphi : \mathcal{U}(\mathbb{M}_{2r}) \to \mathcal{U}(\mathbb{M}_{2r})$ $(r \geq 2)$ such that the corresponding map θ_φ is not an orthoisomorphism.

Let $G = \mathcal{U}(\mathbb{M}_{2r})$. Let K be the subgroup of G generated by the self-adjoint unitaries. Then K is a characteristic subgroup of G, and Dye shows (in [Dy], p87) that

$$K = \{u \in G; \det u = \pm 1\}.$$

We have

1. The map

$$\begin{aligned} \mathbb{S}^1 \times K &\xrightarrow{\pi} G \\ (\mu, k) &\longmapsto \mu k \end{aligned}$$

   is a surjective homomorphism with $\ker \pi = \{(\mu, \mu I); \mu \in \mathbb{S}^1, \mu^{4r} = 1\}$. So we have

$$G \simeq (\mathbb{S}^1 \times K)/\ker \pi.$$

2. The map

$$\begin{aligned} v : K &\to K \\ k &\mapsto k \det k \end{aligned}$$

   is an automorphism of K. If $\mu \in \mathbb{S}^1 \cap K$ then $v(\mu) = \mu^{2r+1}$.

3. Since the centre of G is $\mathbb{S}^1$, we have that $\mathbb{S}^1$ is a characteristic subgroup. Since $\mathbb{S}^1$ and K are characteristic subgroups of G, the map v extends to an automorphism φ of G if and only if there is a $\chi \in \mathrm{Aut}(\mathbb{S}^1)$ such that $\chi(\mu) = \mu^{2r+1}$ if $\mu \in \mathbb{S}^1$ and $\mu^{4r} = 1$. Therefore if and only if there is a $\chi \in \mathrm{Aut}(\mathbb{S}^1)$ such that $\chi(\xi) = -\xi$ where $\xi = e^{2\pi i/4r}$.

4. $\mathbb{R}$ is a $\mathbb{Q}$ vector space. Let H be a basis for $\mathbb{R}$ as a $\mathbb{Q}$-vector space such that $1 \in H$.

$$\mathbb{R} \simeq \mathbb{Q} \oplus \left(\bigoplus_{\substack{h \in H \\ h \neq 1}} \mathbb{Q} \right).$$

So

$$\mathbb{R}/\mathbb{Z} \simeq \mathbb{Q}/\mathbb{Z} \oplus \left(\bigoplus_{\substack{h \in H \\ h \neq 1}} \mathbb{Q} \right).$$

Denoting by $\mathbb{Z}(p^\infty) = \left\{ \frac{a}{p^n}; a \in \mathbb{Z}, n \in \mathbb{Z}^+ \right\} / \mathbb{Z}$, we have

$$\mathbb{Q}/\mathbb{Z} = \bigoplus_{p \text{ prime}} \mathbb{Z}(p^\infty)$$

and hence

$$\begin{aligned}
\mathbb{Q}/\mathbb{Z} &= \left(\bigoplus_{\substack{p \text{ prime} \\ p \nmid 2r}} \mathbb{Z}(p^\infty) \right) \oplus \left(\bigoplus_{\substack{p \text{ prime} \\ p \mid 2r}} \mathbb{Z}(p^\infty) \right) \\
&= \left(\bigoplus_{\substack{p \text{ prime} \\ p \nmid 2r}} \mathbb{Z}(p^\infty) \right) \oplus \mathbb{Z}((2r)^\infty).
\end{aligned}$$

Define $\chi_f \in \operatorname{Aut}(\mathbb{R}/\mathbb{Z})$ by $\chi_f = [(\bigoplus \mathrm{id}) \oplus f] \oplus (\bigoplus \mathrm{id})$, where $f : \mathbb{Z}((2r)^\infty) \to \mathbb{Z}((2r)^\infty)$ is given by

$$f\left(\frac{a}{(2r)^n}\right) = \left(\frac{(2r+1)a}{(2r)^n}\right).$$

Then $\chi_f \in \operatorname{Aut}(\mathbb{S}^1)$ is such that $\chi_f(\xi) = -\xi$. It follows that υ extends to an automorphism φ_f of G.

5. Note that the self-adjoint unitary $u = I - 2p \in G$ has determinant $(-1)^{\mathrm{trace}(p)}$, and hence

$$\upsilon(I-2p) = [\det(I-2p)](I-2p) = -(I-2p) = I - 2(I-p).$$

It follows that the map θ_{φ_f} sends rank one projections to their complements.

6. Note that the map φ_f takes $u \in G$ to a scalar multiple of itself. Indeed, let $u = \mu k$ with $\mu \in \mathbb{S}^1$ and $k \in K$. Then

$$\begin{aligned} \varphi_f(u) &= \chi_f(\mu)\upsilon(k) \\ &= \chi_f(\mu)(\det k)k \\ &= [\chi_f(\mu)(\det k)\bar{\mu}][\mu k] \\ &= [\chi_f(\mu)(\det k)\bar{\mu}][u], \end{aligned}$$

so $\varphi_f(u)u^* = \chi_f(\mu)(\det k)\bar{\mu} \in \mathbb{S}^1$. Setting $\eta(u) = \varphi_f(u)u^*$ we see that η is a character of G and $\varphi_f(u) = \eta(u)u$ for all $u \in G$.

Appendix B

Extending [Dy, Theorem 2]

Here we show that it is not necessary to exclude the I_{2n} case from [Dy, Theorem 2], more precisely, the following is Theorem B.0.0.9.

TheoremLet n be an integer strictly greater than 1 and let $\varphi : \mathcal{U}(\mathbb{M}_{2n}) \to \mathcal{U}(\mathbb{M}_{2n})$ be a group automorphism. Then there is either an automorphism or a conjugate linear automorphism Φ of $\mathbb{M}_{2n}$ and a character χ of $\mathcal{U}(\mathbb{M}_{2n})$ such that $\varphi(u) = \chi(u)\Phi(u)$ for all $u \in \mathcal{U}(\mathbb{M}_{2n})$.

B.0.0.3 Let $\mathbb{M}_{2n}$ be equipped with the normalized trace τ, and let $\varphi : \mathcal{U}(\mathbb{M}_{2n}) \to \mathcal{U}(\mathbb{M}_{2n})$ be a group automorphism with corresponding projection map θ. If $p \in \mathcal{P}(\mathbb{M}_{2n})$ such that $\tau(p) = \tau(\theta(p))$ we say that p is **trace preserved** by θ, and that p is **trace flipped** by θ if $\tau(p) = 1 - \tau(\theta(p))$.

B.0.0.4 Lemma *Let $\varphi : \mathcal{U}(\mathbb{M}_{2n}) \to \mathcal{U}(\mathbb{M}_{2n})$ be a group automorphism with corresponding projection map θ. If θ is not an orthoisomorphism then even rank projections are trace preserved by θ, and odd rank projections are trace flipped.*

Proof: Let $\{p_i\}_{i=1}^{2n}$ be a family of pairwise orthogonal minimal projections in $\mathcal{P}(\mathbb{M}_{2n})$. Since minimal projections in $\mathbb{M}_{2n}$ are all unitarily equivalent, if θ preserves orthogonality of two minimal projections then it preserves orthogonality of any orthogonal minimal projections, and it follows that θ is an orthoisomorphism (and hence preserves dimension). So we assume that θ does not preserve orthogonality of minimal projections. By (1) of (3.2.1.3) we have that $c_{p_1} = \ldots = c_{p_{2n}}$. By (3.2.2.3), we have $c_{p_i+p_j} = \bar{c}_{p_1}$, and therefore $\theta(I-p_i)\theta(I-p_j) = 0$, for all $i \neq j$. We have

$$I = \theta\left(\sum_{i=1}^{2n} p_i\right) = \theta\left(\mathop{\triangle}_{i=1}^{2n} p_i\right) = \mathop{\triangle}_{i=1}^{2n} \theta(p_i).$$

Since $2n$ is even we have by repeatedly using (3.1.22) of (3.1.0.2) that

$$\mathop{\triangle}_{i=1}^{2n} \theta(p_i) = \mathop{\triangle}_{i=1}^{2n} \theta(I-p_i),$$

and since the $\theta(I-p_i)$ are pairwise orthogonal we get

$$I = \sum_{i=1}^{2n} \theta(I-p_i).$$

Hence

$$1 = \tau(I) = \sum_{i=1}^{2n} \tau(\theta(I-p_i)) = 2n\tau(\theta(I-p_1)).$$

So $\tau(\theta(I-p_1)) = 1/2n = \tau(p_1)$, and hence $\tau(p_i) = 1 - \tau(\theta(p_i))$ for all i. So if θ is not an orthoisomorphism then θ flips the trace of minimal projections and their complements.

Now let $p \in \mathcal{P}(\mathbb{M}_{2n})$ be a projection and write $p = \sum_{i=1}^{m} p_i$ as a sum of pairwise orthogonal minimal projections. If m is even then

$$\theta(p) = \mathop{\triangle}_{i=1}^{m} \theta(p_i) = \mathop{\triangle}_{i=1}^{m} (I - \theta(p_i)) = \sum_{i=1}^{m} (I - \theta(p_i)) = \sum_{i=1}^{m} \theta(I-p_i)$$

and

$$\tau(\theta(p)) = \sum_{i=1}^{m} \tau(\theta(I-p_i)) = m(\tau(\theta(I-p_1))) = m\tau(p_1) = m/2n = \tau(p),$$

so the trace of p is preserved by θ. If m is odd, then

$$\begin{aligned}\theta(p) &= \mathop{\Delta}_{i=1}^{m} \theta(p_i) \\ &= I - \mathop{\Delta}_{i=1}^{m} (I - \theta(p_i)) && \text{by (3.1.22) of (3.1.0.2)} \\ &= I - \sum_{i=1}^{m} (I - \theta(p_i)) && \text{since } (I-\theta(p_i))(I-\theta(p_j)) = 0 \text{ when } i \neq j \\ &= I - \sum_{i=1}^{m} \theta(I - p_i)\end{aligned}$$

and

$$\begin{aligned}\tau(\theta(p)) &= 1 - \sum_{i=1}^{m} \tau(\theta(I - p_i)) = 1 - m(\tau(\theta(I - p_1))) \\ &= 1 - m\tau(p_1) = 1 - m/2n = 1 - \tau(p),\end{aligned}$$

so the trace of p is flipped by θ. ∎

B.0.0.5 Lemma *Let $\varphi : \mathcal{U}(\mathbb{M}_{2n}) \to \mathcal{U}(\mathbb{M}_{2n})$ be a group automorphism with corresponding projection map θ_φ. Let $\varphi_f : \mathcal{U}(\mathbb{M}_{2n}) \to \mathcal{U}(\mathbb{M}_{2n})$ denote the group automorphism constructed in Appendix A. If θ_φ is not an orthoisomorphism, then $\theta = \theta_\varphi \circ \theta_{\varphi_f}$ is an orthoisomorphism.*

Proof: We have that θ_φ and θ_{φ_f} flip the trace of minimal projections, and hence θ is trace preserving. It follows that θ preserves dimension, and hence orthogonality of minimal projections. Since θ also preserves the symmetric difference of arbitrary projections, it is an orthoisomorphism. ∎

B.0.0.6 Recall (cf. Appendix A) that K is the subgroup of $\mathcal{U}(\mathbb{M}_{2n})$ generated by the self-adjoint unitaries, and that

$$K = \{u \in \mathcal{U}(\mathbb{M}_{2n}); \det u = \pm 1\}.$$

B.0.0.7 Lemma *Suppose that $\varphi : \mathcal{U}(\mathbb{M}_{2n}) \to \mathcal{U}(\mathbb{M}_{2n})$ and $\eta : \mathcal{U}(\mathbb{M}_{2n}) \to \mathcal{U}(\mathbb{M}_{2n})$ are group automorphisms such that $\theta_\varphi = \theta_\eta$. Then $\varphi(k) = \eta(k)$ for all $k \in K$.*

Proof: Let $p \in \mathcal{P}(\mathbb{M}_{2n})$ and let $u = I - 2p$. Then

$$\varphi(u) = \varphi(I-2p) = I - 2\theta_\varphi(p) = I - 2\theta_\eta(p) = \eta(I-2p) = \eta(u).$$

So $\varphi(u) = \eta(u)$ for every generator of K. The result follows. ■

B.0.0.8 Lemma *Let $\varphi : \mathcal{U}(\mathbb{M}_{2n}) \to \mathcal{U}(\mathbb{M}_{2n})$ and $\eta : \mathcal{U}(\mathbb{M}_{2n}) \to \mathcal{U}(\mathbb{M}_{2n})$ be group isomorphisms that agree on K. Then there is a character χ of $\mathcal{U}(\mathbb{M}_{2n})$ such that $\varphi = \chi\eta$.*

Proof: The map $\chi(u) = \varphi(u)\eta(u)^*$ is a function from $\mathcal{U}(\mathbb{M}_{2n})$ into $\mathbb{S}^1$ since $\varphi = \eta$ on K. Since every $u \in \mathcal{U}(\mathbb{M}_{2n})$ can be written $u = \mu n$ for appropriate choices of $\mu \in \mathbb{S}^1$ and $n \in K$, and since $\varphi = \eta$ on K, it follows from a straightforward calculation that χ is a character of $\mathcal{U}(\mathbb{M}_{2n})$. ■

B.0.0.9 Theorem *Let n be an integer strictly greater than 1 and let $\varphi : \mathcal{U}(\mathbb{M}_{2n}) \to \mathcal{U}(\mathbb{M}_{2n})$ be a group automorphism. Then there is either an automorphism or a conjugate linear automorphism Φ of $\mathbb{M}_{2n}$ and a character χ of $\mathcal{U}(\mathbb{M}_{2n})$ such that $\varphi(u) = \chi(u)\Phi(u)$ for all $u \in \mathcal{U}(\mathbb{M}_{2n})$.*

Proof: We have that $\theta_\varphi = \theta_f \circ \theta$ for some orthoisomorphism θ. We have from [Dy, Theorem 1] that there is a map Φ' which is either an automorphism or an antiautomorphism of $\mathbb{M}_{2n}$ such that Φ' extends θ. If Φ' is an antiautomorphism then composing it with the map $x \mapsto x^*$ gives a conjugate linear map Φ which also extends θ. So there is either an automorphism or a conjugate linear automorphism Φ of $\mathbb{M}_{2n}$ which extends θ. In either case Φ is an automorphism of $\mathcal{U}(\mathbb{M}_{2n})$ and $\theta = \theta_\Phi$. So we have by (B.0.0.5) that $\theta_\varphi = \theta_{\varphi_f} \circ \theta_\Phi = \theta_{\varphi_f \circ \Phi}$, and hence $\varphi = \varphi_f \circ \Phi$ on K by (B.0.0.7). It follows from (B.0.0.8) that there is a character λ of $\mathcal{U}(\mathbb{M}_{2n})$ such that $\varphi(u) = \lambda(\mu)\varphi_f \circ \Phi(u)$. Since $\varphi_f(u) = \eta_f(u)u$ for some character η_f of $\mathcal{U}(\mathbb{M}_{2n})$ and for all $u \in \mathcal{U}(\mathbb{M}_{2n})$ (see (Appendix A)), we have that there is a character χ of $\mathcal{U}(\mathbb{M}_{2n})$ such that $\varphi(u) = \chi(u)\Phi(u)$ for all $u \in \mathcal{U}(\mathbb{M}_{2n})$. ■

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List of Symbols

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