Random walks on free products of cyclic groups

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Abstract

In this thesis, we investigate examples of random walks on free products of cyclic groups. Free products are groups that contain words constructed by concatenation with possible simplifications[20]. Mairesse in [17] proved that the harmonic measure on the boundary of these random walks has a Markovian Multiplicative structure (this is a class of Markov measures which requires fewer parameters than the usual Markov measures for its description), and also showed how in the case of the harmonic measure these parameters can be found from Traffic Equations. Then Mairesse and Mathéus in [20] continued investigation of these random walks and the associated Traffic Equations. They introduced the Stationary Traffic Equations for the situation when the measure is shift-invariant in addition to being μ-invariant. In this thesis, we review these developments as well as explicitly describe several concrete examples of random walks on free products, some of which are new.

Dedications

First and foremost, I give my endless thanks to God (Allah) for giving me life, health, family and thousands of blessings " And if you should count the favors of Allah, you could not enumerate them. Indeed, Allah is Forgiving and Merciful " [Quran;16:18].

Then, I would like to dedicate this work to the great ancient Muslim scientists who had a massive role in most sciences. Some of their achivements in mathematics is the contribution towards, and invention of the arithmetical decimal system and the fundamental operations. In addition, they also introduced the zero concept to the world. Some of the famous mathematicians of Islam are Al-Khowarizmi, Al-Kindi and Al-Battani.

Also, I would like to thank my beloved family for all the help and unconditional love. I owe profound gratitude to my Mother, Maryam, whose love for me knew no bounds and whose words of encouragement and push for tenacity ring in my ears. From the bottom of my heart, I thank my father, Mohammed who is my role model and who gave me everything he could. I pray for him that he is happy now in heaven. Then, I would like to express my wholehearted thanks to my husband, Yousef, for the support and encouragement he gave me, in willingly accompanying me on a journey of exploration that we knew would be immensely challenging and painful in parts. I am tremendously appreciative to my siblings Maymonah, Moath, Anfal, Amjad, Abdullah and Joud who have always been supportive and have never left my side. Moreover, warm thanks and best wishes go to my children, Suhayb and Awss, who are the light of my life and made it so meaningful.

In my long journey, I met a lot of great friends and classmates who put joy in some tough days and lessen some difficulties that I had. To name a few friends, Thank you Nawal and Miss. Fatimah, your smiles are my inspiration. Majedah and Najah, you both taught me how friends could be like sisters. Njwd, you supported me in the most difficult of times. Rasha and Hessah, we shared similar concerns, hope and jokes. Shatha, now I know that great things come at the end. Nema and Miss Zainab, I consider both of you as a teacher rather than a friend, and my aunt Maha who knows me better than I do. Lastly, for all my kind-sweetheart relatives and friends who knowingly and unknowingly inspire me and make my life enjoyable.

Acknowledgement

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Also, I would like to offer my gratitude and appreciation to my supervisor, Vadim Kaimanovich, for believing in me, encouraging and supporting me. He made the Masters journey much easier and I couldnt think of a supervisor better than him.

Last but surely not least, I would like to thank Dr. Mohammed and Dr. Johannes for their time and effort to make this work feasible.

Contents

Chapter 1

Introduction

In this thesis, we are dealing with free products of cyclic groups. Free products are groups that contain words constructed by concatenation with possible simplifications [20]. More precisely, these words take elements alternately from the original groups. Changes may occur only at the contact points. If at a contact point we have elements from different groups, then we just write them as they are. On the other hand, if at a contact point we have elements from the same group then we have to multiply them. They may cancel each other if they are inverses, or we may have a new element, which is the product of these two elements. For these two later cases, we continue the procedure for the new elements we got. In the same way, we proceed for all contact points.

A Markov chain is a "memoryless" stochastic process with discrete time, meaning that its behaviour in the future only depends on its position at present rather than on the whole history of the process. If the state space of the Markov chain is a graph, it is usually called a random walk.

One basic example of the random walk is the simple random walk on a graph. The transition probabilities in the simple random walk are equidistributed on the neighbours of each vertex, which means the random walker chooses the next step from the neighbours with equal probability.

A Markovian Multiplicative measure is a special Markov chain meaning that, unlike a regular Markov chain, it requires only one probability vector on the state space S to describe its behaviour [20, 17]. We call this vector the base of the Markovian Multiplicative measure. This base serves as an initial distribution as well as for defining the transition probabilities by using another piece of data, which is the set of admissible transitions $T \subset S \times S$ between the points of the state space (alternatively, it can be done in terms of the set of forbidden transitions $F = S \times S \setminus T$) [20, 17]. Then the transition probability from a given point a is obtained by restricting the base probability vector to the set of points accessible from a and then normalizing it.

Example 1.0.1. *Consider a Markovian Multiplicative measure given by the state space $S = \{a, b, c\}$, a Markovian Multiplicative base $\theta = (\theta(a), \theta(b), \theta(c))$ and the set of forbidden transitions*

$$F = \{(a, a), (b, b), (c, c)\} .$$

Then, the initial distribution is

$$\lambda = (\theta(a), \theta(b), \theta(c)) .$$

Also, the transition probabilities for each $\alpha, \beta \in S$ can be found using

$$p(\alpha, \beta) = \frac{\theta(\beta)}{\sum_{\beta':(\alpha,\beta')\notin F} \theta(\beta')}$$

Thus we have

$$p(a, b) = \frac{\theta(b)}{\theta(b) + \theta(c)} ,$$

and

$$p(a, c) = \frac{\theta(c)}{\theta(b) + \theta(c)}$$

whereas $p(a, a) = 0$ as the transition (a, a) is forbidden.

In our setting, we consider random walks on free products of cyclic groups that start at the identity element 1 and proceed according to a step distribution μ concentrated on the generating set S. These random walks happen to be transient, and therefore will eventually converge to the boundary [20, 11, 5].

Mairesse in [17] found that the harmonic measure of the random walk, which is the hitting measure on the boundary of the free product, is a Markovian Multiplicative measure. To find its base, Mairesse has used the Traffic Equations in [17], and Mairesse and Mathéus have introduced Stationary Traffic Equations in [20] in the case where the harmonic measure is shift-invariant in addition to being μ-invariant.

The *Traffic equations* associated with nearest neighbour random walks on free products (G, μ) were used by Mairesse in [17, Equation:17] $\forall a \in S$,

$$r(a) = \mu(a) \sum_{u\in S\smallsetminus S_a} r(u) + \sum_{u*v=a} \mu(u)r(v) + \sum_{u\in S\smallsetminus S_a} \mu(u^{-1}) \frac{r(u)}{\sum_{v\in S\smallsetminus S_u} r(v)} r(a) .$$

In [20, Equation:12], Mairesse and Mathéus defined the *Stationary Traffic Equations* associated with (G, μ) as $\forall a \in S$,

$$r(a) = \mu(a)\frac{|I| - 1}{|I|} + \sum_{u*v=a} \mu(u)r(v) + r(a)\frac{|I|}{|I| - 1} \sum_{u\in S\smallsetminus S_a} \mu(u^{-1})r(u) .$$

Our aim is to investigate some concrete examples of random walks on free products of cyclic groups. The examples of free products of two cyclic groups are based on Mairess and Mathéus papers [20] and [19]. We discuss the example of the free product $\mathbb{Z}/2\mathbb{Z} * Z/3\mathbb{Z}$ and two cases of symmetric measures on the free product $Z/3\mathbb{Z} * Z/3\mathbb{Z}$.

Our new examples are free products of three cyclic groups. The examples we consider are $\mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z}$ and $\mathbb{Z}/3\mathbb{Z} * \mathbb{Z}/3\mathbb{Z} * \mathbb{Z}/3\mathbb{Z}$. In each one of these examples, we solve the Traffic Equations. We use the solutions we obtain in finding the drift (the speed of the random walk), the asymptotic entropy of the random walk and the Hausdorff dimension of the harmonic measure on the boundary.

The drift measures the speed of the random walk in which it escapes to infinity and is defined as

$$\gamma = \lim_{n\to\infty} \frac{1}{n}|x_n|_S \ ,$$

where $|.|$ denotes the word length associated with a finite symmetric set of generators of G [17, Equation:5][18, Equation:3].

In a nearest neighbour random walk, the drift is bounded between $0 \leq \gamma \leq 1$. In our setting of Markovian Multiplicative measure, according to [20, Equation:19] the drift can be calculated as follows:

$$\gamma = \sum_{a\in S} \mu(a) \left[-r(a^{-1}) + \sum_{b\in S\setminus S_a} r(b) \right]$$

where μ is the probability measure concentrated on the generating set and the vectors $r(a)$ are solutions to the Traffic Equations.

Another invariant we are interested in is the asymptotic entropy of the random walk. The asymptotic entropy h is the limit of Shannon's entropies for the n-fold convolution measure $H(\mu^{*n})$ divided by time n. The Shannon entropy measures the randomness of a random variable. It increases if the predictability of the outcomes is low and decreases if we have a high predictability of the outcomes. The uniform measure has the highest entropy.

The asymptotic entropy measures how much information about any single increment of the random walk is retained in their product. According to [18, Equation:16], the asymptotic entropy can be found by using the harmonic measure:

Set $q(s) = \frac{r(s)}{r(S\setminus S_s)}$

$$h = -\sum_{s\in S} \mu(s) \left[\log\left[\frac{1}{q(s^{-1})}\right] r(s^{-1}) + \sum_{g\in S_s\setminus s^{-1}} \log\left[\frac{q(sg)}{q(g)}\right] r(g) + \log[q(s)] \sum_{g\in S\setminus S_s} r(g) \right]$$

After that, we want to calculate the Hausdorff dimension of the harmonic measure. The Hausdorff dimension, unlike the usual one, can take noninteger values and "shows the degree of singularity" or "fractalness" [12]. To understand the Hausdorff dimension let us consider a metric space and a probability measure defined on that space. We take a point from the space and look at balls centered at this point, see[23]. We take logarithms of the measures of these balls and divide them by the radius of each ball. If the limit exists almost everywhere and it is the same, it is called dimension of the measure.

To find Hausdorff dimension in our setting, we need to normalize the asymptotic entropy h by the rate of escape γ

$$\mathrm{HD}_\nu = \frac{h}{\gamma} ,$$

which is first appeared in [12] with specific conditions. Then, it was generalized in [15].

I want to acknowledge that, in the background section, the group theory part is, mostly, based on [8] and the graph theory part is, mostly, based on [16, 1]. Also, the Markov process part is, mostly, based on [22, 13, 25]. The entropy part in the numerical asymptotic invariants part is, mostly, based on [10, 27, 3] and the Hausdorff dimension part is, mostly, based on [12, 28, 15, 23] . Moreover, the drift part as well as the random walk on free products of cyclic groups section is, mostly, based on [20, 17, 18]. The examples of free products of two cyclic groups in the last section is based on [20, 19]. However, the examples of free products of three cyclic groups are new.

Chapter 2

Background

2.1 Introduction

In this section, we will recall definitions and important notions from group theory, graph theory, Markov chains and some numerical asymptotic invariants. In group theory, a group is a nonempty set with a binary operation called multiplication. A group should contain identity element and every element in the group should have an inverse. A cyclic group is a special kind of group that can be generated by a single element. In this thesis, we are dealing with free products of cyclic groups.

Groups are important easy structures which can be used as building blocks to construct more general structures such as direct products and free products.

Below we shall be dealing with Cayley graphs for free products of cyclic groups. Thus, we need to review some basic notions about graphs, digraphs and Cayley diagrams. A *graph* is a pair of sets (V, E), where V is the set of *vertices* and E is the set of *edges*, formed by pairs of vertices. A *digraph* Γ is a triple consisting of a vertex set $V(\Gamma)$, and edge set $E(\Gamma)$, and a function assigning to each edge an orded pair of vertices. These vertices are called *endpoints*. Let G be a group with generating set S. The *Cayley graph* of G with respect to S, denoted $X = X(G; S)$, is the graph with $V(X) = G$, and an edge between vertices $g, h \in G$ if $g^{-1}h \in S \times S^{-1}$.

After that we discuss one kind of stochastic processes namely, random walks. A random walk is a special case of Markov chains, which is a memoryless process, where you only need to know the last position in order to know the next one. In other words the whole history of the chain is irrelevant in predicting where is the next step.

We take some examples of random walks. We consider Pòlya's walk on $\mathbb{Z}$ and $\mathbb{Z}^2$ as well as random walks on Cayley graphs of groups which are related to the main topic of this thesis (The random walk on free products of cyclic groups).

Finally, we review some numerical asymptotic invariants namely, the asymptotic entropy, the drift and the Hausdorff dimension. Entropy is a term that used in different sciences to mean different quantities. It was initiated in physics as a measure of randomness. Then, it was adopted in the communication theory as a measure of missing information or unpredictability and is called Shannon entropy. Another invariant is the drift, which measures the speed in which the random walk escapes to infinity. Finally, we discuss Hausdorff dimension of random walk and express that it is the infimum of the dimensions of the sets of full measures on the boundary. It can also be found using

$$H(D) = \frac{h}{\gamma} \ .$$

See [12] and [15] for more details.

2.2 Group theory

We need to recall some basic definitions from group theory regarding groups, cyclic groups, free products and free groups. These definitions can be found in most abstract algebra books, see for instance [8, 6].

Definition 2.2.1. *Monoid*

A nonempty set G equipped with a binary operation $\circ : G \times G \to G$ is called a monoid if the following conditions are satisfied:

- *The operation $\circ$ is associative (i.e. $(a \circ b) \circ c = a \circ (b \circ c)$).*
- *There is an identity e such that $e \circ a = a \circ e = a$ for all $a \in G$.*

.

Definition 2.2.2. *Group*

A group is a monoid satisfying an extra condition:

- *For every element $a \in G$ there is an inverse $a^{-1} \in G$, such that $a * a^{-1} = a^{-1} * a = e$.*

.

Definition 2.2.3. *Cyclic group*

A cyclic group is a group that can be generated by one single element called a generator.

Denote by g the generator of the group. Recall that the order of a group element is the minimal n such that $g^n = e$.

There are two cases: if g has an infinite order (i.e., no power of g is the group identity), then the group G consists of all powers of g and is isomorphic to the group of integers $\mathbb{Z}$. If g has a finite order n, then the group G is isomorphic to the group Z_n of integers modulo n.

Definition 2.2.4. ***Free monoid***

The set S^* that contains all finite strings of elements (including the empty string) from a set S is called the free monoid on S. The multiplication in S^* is defined as the string concatenation and the unit element is the empty string.

The following definition and discussion about the free group are based on [8].

Definition 2.2.5. *Let S be a set. The group $F[S]$ is called the free group on the subset of free generators S (not necessarily finite) if it satisfies the following two conditions:*

- *$F[S]$ is generated by S, and*
- *For any mapping ϕ from S into a group G, we can uniquely extend ϕ to a group homomorphism from $F[S]$ to G.*

Now, set

$$S^{-1} = \{a^{-1} | a \in S\} \ .$$

Following are some facts about the free group $F[S]$ which is based on [8].

- Elements from S and S^{-1} are called the *letters* of the free group $F[S]$. Elements from $F[S]$ are *finite words* respectively, *infinite words* which are finite respectively, infinite strings of letters from S or S^{-1} written in juxtaposition.

  Example 2.2.6. *Consider the generating set $S = \{a, b, c\}$ Some words in the free group $F[S]$ are*

  $$abcc, abb^{-1}a, c^{-1}cbbc, ba^{-1}a^{-1}ca_{-1}a$$

  However, usually, a word is denoted by its reduced word which is the word that does not have a letter and its inverse as consecutive letters and we replace them (a letter followed by its inverse) by the unit element 1. In a reduced word, also, consecutive copies of the same letter are replaced by a power to that letter. For example, the previous words often denoted by the simpler forms:

  $$abc^2, a^2, b^2c, ba^{-2}c \ ,$$

  respectively.

- For two words L_1 and L_2 in $F[S]$, $L_1.L_2$ is the word obtained by concatenation L_1L_2 of the two words L_1 and L_2.

  Example 2.2.7. *Let $L_1 = ab^2c^{-1}b$ and $L_2 = b^{-1}cbc$. The concatenation $L_1.L_2 = ab^2c^{-1}bb^{-1}cbc$. After concatenation, the resulting word may be written in a reduced form*

  $$L_1.L_2 = ab^3c \ .$$

- The identity of $F[S]$ is the *empty word* which is the word that has no letters. The empty word is usually denoted by 1. If $S = \Phi$, then $F[S] = \{1\}$.
- Consider a word $L = ab^{-2}cb^2c^{-1}$ in $F[S]$, the inverse word of L is

$$L^{-1} = cb^{-2}c^{-1}b^2a^{-1} .$$

Definition 2.2.8. ***Free product*** *This definition is based on [20]. Let $(G_i)_{i\in I}$ be a finite family of finite groups, with $|I| \geq 2$. Let 1_{G_i} be the identity of G_i. Set $S_i = G_i \setminus \{1_{G_i}\}$ and set $S = \sqcup_i S_i$. Let $\iota : S \to I$ be defined by $\iota(u) = j$ if $u \in S_j$. It is also convenient to set $S_a = S_{\iota(a)}$ for all $a \in S$. The* free product *$G = \star_{i\in I} G_i$ is the group with set of elements L, unit element 1, and group law $*$ defined by*

$$u_1 \cdots u_k * v_1 \cdots v_l = \begin{cases} u_1 \cdots (u_{k-1})(u_k)(v_1)(v_2) \cdots v_l & \text{if } \iota(u_k) \neq \iota(v_1) \\ u_1 \cdots (u_{k-1})(u_k * v_1)(v_2) \cdots v_l & \text{if } \iota(u_k) = \iota(v_1),\ u_k \neq v_1^{-1} \\ u_1 \cdots u_{k-1} * v_2 \cdots v_l & \text{if } u_k = v_1^{-1} \end{cases} ,$$

where in the second case, $(u_k * v_1)$ is the product in $G_{\iota(u_k)}$ of u_k and v_1. Roughly, the law of G is the concatenation with possible simplifications at the contact point to reach a normal form word. See 3.2 for more details.

2.3 Metric spaces

We need to recall the following definition which is based on [21].

Definition 2.3.1. *A metric space consists of a set X and a distance function $d : X \times X \to \mathbb{R}$ such that, for any x, y, and $z \in X$*

- $d(x, y) \geq 0$,
- $d(x, y) = 0$ *if and only if* $x = y$,
- $d(x, y) = d(y, x)$,
- $d(x, y) + d(y, z) \geq d(x, z)$.

2.4 Graph theory

In this thesis, we deal with graphs and Cayley graphs. Therefore, we would like to begin with a brief introduction about graphs, digraphs and Cayley graphs and give some examples. For more details see [16], [1].

Definition 2.4.1. ***Graph***
A graph $\Gamma(V, E)$ is a pair of sets (V, E), where V is the set of vertices and E is the set of edges, formed by pairs of vertices. The vertices $u, v \in V$ are said to be joined or connected by the edge$\{u, v\}$ if $\{u, v\} \in E$.

Some terminologies

- *For the edge (u, v) we call u and v end vertices, u and v are called adjacent or neighbours.*
- *A loop is an edge of the form (u, u).*
- *A sequence of vertices $u_0 \sim u_1 \sim ... \sim u_n, n \geqslant 3$ is a cycle if there is no repetition except $u_n = u_0$.*
- *Edges are called parallel if they have the same end vertices.*
- *A simple graph is a graph that has no parallel edges or loops.*
- *Edges that share a common end vertex are adjacent.*
- *A path is a sequence of vertices in which each consecutive pair of vertices is an edge in the graph.*

Definition 2.4.2. ***Directed graph***
A directed graph or digraph $\Gamma(V, E)$ is a graph defined by three components: a vertex set $V(\Gamma)$, an edge set $E(\Gamma)$ that consists of ordered pairs of vertices and a function that connects each edge to its corresponding ordered pair of vertices. The first vertex of the ordered pair is called the tail of the edge and the second is called the head.

Definition 2.4.3. ***Trees***
A connected graph T that does not have any loops or cycles is a tree.

Example 2.4.4. *The homogeneous tree T_M is the tree where all vertices have the degree M [31], see figure 2.7 where we apply a general random walk on the homogeneous tree T_3.*

Example 2.4.5. *See figure 2.1 for an example of a inhomogeneous tree.*

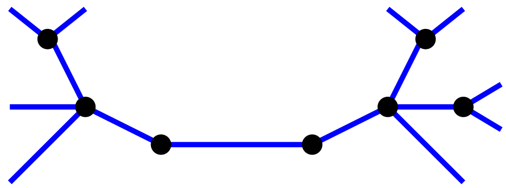

Figure 2.1: An example of an inhomogeneous tree

2.4.1 Cayley graphs

Definition 2.4.6. ***Cayley graph***

Let G be a group with generating set S. The Cayley graph of G with respect to S, denoted $X = X(G;S)$, is the graph with $V(X) = G$, and an edge between vertices $g, h \in G$ if and only if $g^{-1}h \in S$ [20].

The edges are labelled with generators; in general, the Cayley graph is directed; if the generating set is symmetric then it becomes a non-directed graph. For examples of Cayley graphs see figures 2.2,2.3,2.4,2.5 and 2.6.

Example 2.4.7. ***Examples of Cayley graphs***

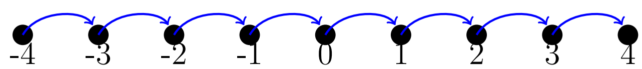

Figure 2.2: Cayley graph on $\mathbb{Z}$ with one generator $\{1\}$

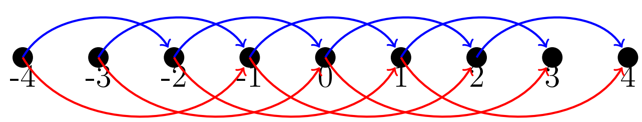

Figure 2.3: Cayley graph on $\mathbb{Z}$ with two generators $\{2, 3\}$

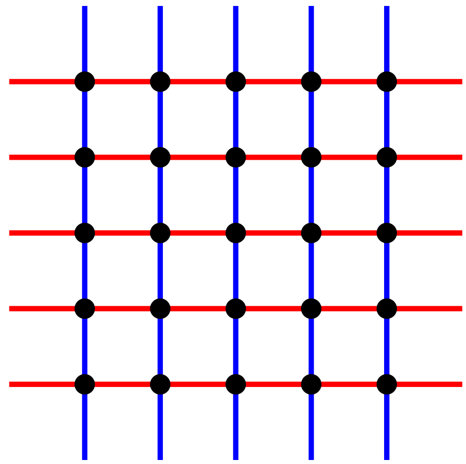

Figure 2.4: Cayley graphs on $\mathbb{Z}^2$ with respect to the generating set $\{(1,0)(red),(0,1)(blue)\}$

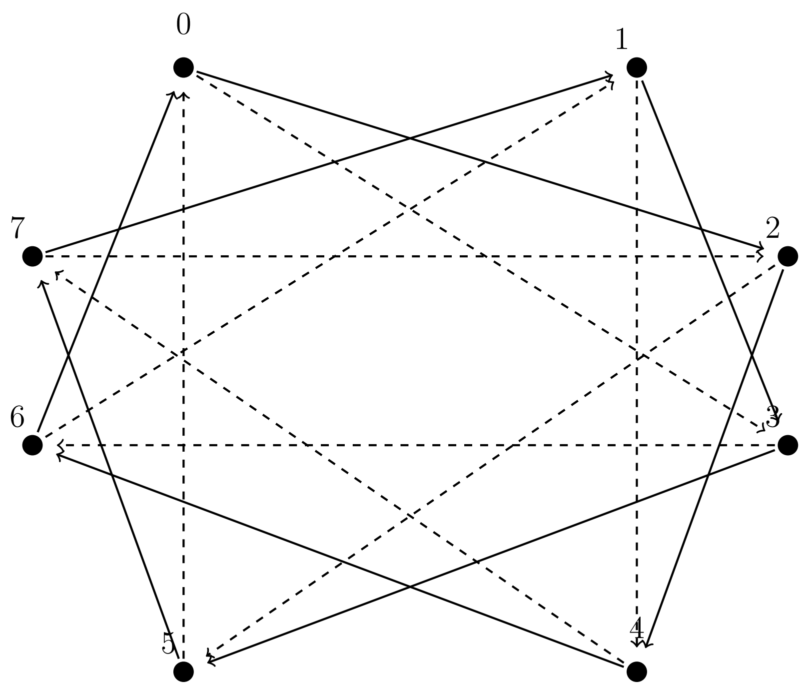

Figure 2.5: Cayley graphs on $\mathbb{Z}_8$ with the generators set: $\{2(\text{the solid line}), 3(\text{the dashed line})\}$

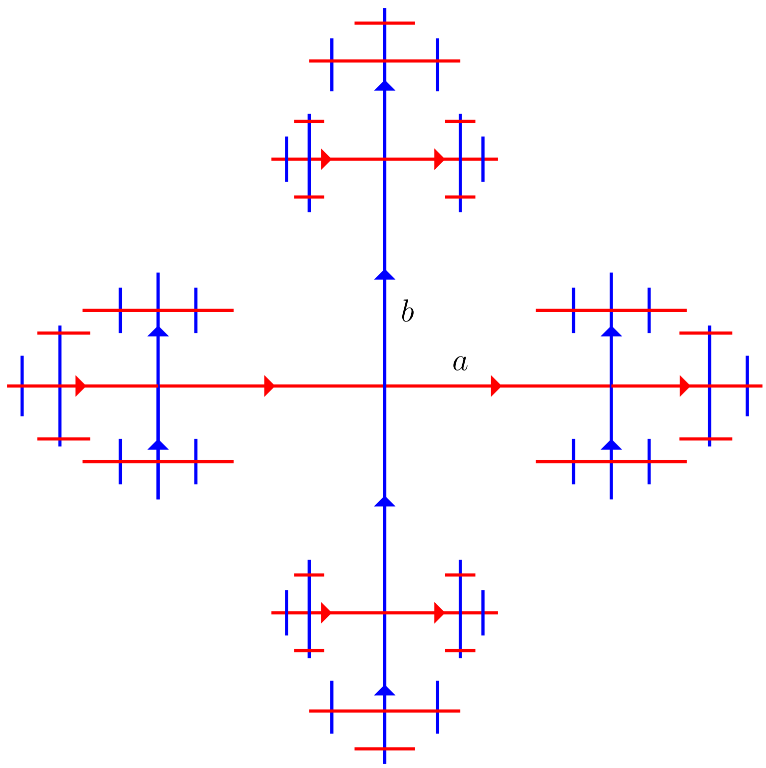

Figure 2.6: Cayley graph of the free group F_2 with generators set $\{a(red), b(blue)\}$

2.5 Markov chains

The main topic of this thesis is random walks on free products of cyclic groups. A random walk is a special case of Markov chains. A Markov chain is a stochastic process where future depends only on the present while the rest of the history of the process is irrelevant. Our examples are of homogeneous discrete Markov chains. We give some examples of simple random walks on integer lattices in dimensions one and two, random walks on Cayley graphs and random walks on groups. Let I be a countable set. We shall call its elements $i \in I$ *states*, and the whole set I will be called the *state space* [22]. Below we shall be talking about *measures* on the state space I. However, since we are only considering the case when the state space is countable, the "measure theory" on countable spaces becomes much easier. Namely, a measure on I is just a non-negative real function on I. The *total mass* of a measure m is

$$\|m\| = \sum_{i \in I} m(i) \ .$$

If $\|m\| = 1$, then m is called a *probability measure* or a *distribution*.

2.5.1 Transition probabilities

A Markov chain is determined by an initial distribution and a set of conditional probabilities called transition probabilities [13], defined as:

$$P_{i_1,i_2}(n) = \text{the probability that the process moves from state } i_1 \text{ to state } \ i_2$$

$$\text{when the time changes from } n \text{ to } n+1 \ ,$$

We emphasize that in our setting, we only use time-homogeneous Markov chains. Thus, we have

$$P_{i_1,i_2}(n) = P_{i_1,i_2}(m) \text{for all nonnegative integers } n \text{ and } m \ ,$$

and we will simply write P_{i_1,i_2} . These transition probabilities define the *stochastic* transition matrix

$$P = \begin{pmatrix} P_{1,1} & \cdots & P_{1,n} \\ \vdots & & \vdots \\ P_{n,1} & \cdots & P_{n,n} \end{pmatrix}$$

We remind that a matrix is called *stochastic* if all its entries are non-negative, and the sum of the entries in each row is equal to 1 [13]. For a square stochastic matrix whose rows and columns are parameterized by a countable state space I (possibly infinite) its rows can be considered as distributions on I. Therefore, in this situation a stochastic matrix Π is the same as a collection $\{\pi_i\}_{i \in I}$ of probability measures on I, parameterized by states $i \in I$. The following definition is based on [22].

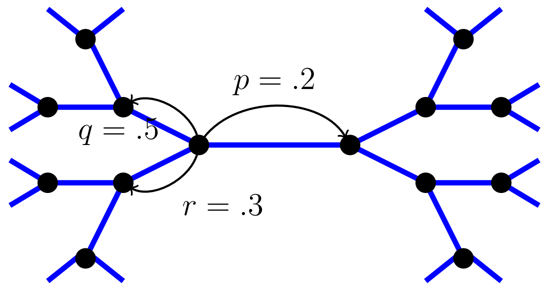

Figure 2.7: A random walk on the homogeneous tree T_3

Definition 2.5.1. ***Markov chain*** *We say that $(X_n)_{n\geq 0}$ is a Markov chain with initial distribution λ and transition matrix P if for all $n \geq 0$ and $i_0, ..., i_{n+1} \in I$,*

- *(i) $P(X_0 = i_0) = \lambda_{i_0}$;*
- *(ii) $P(X_{n+1} = i_{n+1}|X_0 = i_0, ..., X_n = i_n) = P(X_{n+1} = i_{n+1}|X_n = i_n) = p_{i_n i_{n+1}}$.*

For short, we say $(X_n)_{n\geq 0}$ is Markov(λ, P).

2.5.2 Some definitions

These definitions are based on [25].

Definition 2.5.2. ***The n-step transition probability*** *The n-step transition probability is defined as*

$$p^{(n)}(x, y) = \mathbb{P}_x[X_n = y] \ .$$

This gives the probability to get from x to y in n steps.

Definition 2.5.3. ***Irreducible Markov chain*** *We say a Markov chain is irreducible if for every $i_1, i_2 \in X$ there is some $n \in N$ such that*

$$p^{(n)}(i_1, i_2) < 0 \ .$$

2.5.3 Examples of random walks

If the state space of a Markov chain is a graph it is usually called random walk.

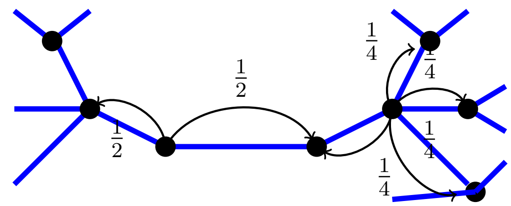

Figure 2.8: The simple random walk on a inhomogeneous tree

Random walks on graphs

One basic example of the random walk is the simple random walk on a graph. The transition probabilities in the simple random walk are equidistributed on the neighbours of each vertex which mean the random walker chooses the next step from the neighbours with equal probability. The following definition is based on [31].

Definition 2.5.4. ***The simple random walk*** *The simple random walk (SRW) on a locally finite graph G is the Markov chain whose state space is G and the transition probabilities are given by*

$$p(x, y) = \begin{cases} \frac{1}{deg(x)}, \textit{if } \ y \sim x \\ 0, \textit{otherwise.} \end{cases}$$

Note that a locally finite graph is a graph with a finite degree and a degree $deg(x)$ of a vertex x is the number of its neighbours. The symbol $y \sim x$ means there is an edge between y and x. See an example in figure 2.8.

Definition 2.5.5. ***Pòlya's walk*** *Pòlya's walk is the simple random walk on the d-dimensional grid, denoted by $\mathbb{Z}^d$, which is the graph whose vertices are integer points in d dimensions, and where two points are linked by an edge if they are at a distance 1. For more details about Polia's walk see [31].*

The simple random walk is a generalization of Pòlya's walk. The simple random walk may have different degrees for different vertices while in Pòlya's walk the number of neighbours in all vertices is fixed and the graphs are Cayley graphs of Z^d.

Example 2.5.6. *The simple random walk on $\mathbb{Z}$ is the Markov chain with state space $G = \mathbb{Z}$ and transition probabilities*

$$p(x, x + 1) = p(x, x - 1) = \frac{1}{2}\ , \quad \textit{for all} \ \ x \in \mathbb{Z}$$

See figure 2.9.

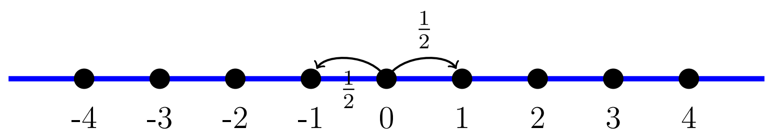

Figure 2.9: Simple random walk on $\mathbb{Z}$

The transition matrix for the simple random walk on $\mathbb{Z}$

$$\begin{pmatrix}
\dots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\
\dots & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\
\dots & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & \dots \\
\dots & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & \dots \\
\dots & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 & \dots \\
\dots & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & \dots \\
\dots & 0 & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & \dots \\
\dots & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & \dots \\
\dots & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & \dots \\
\dots & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & \dots \\
\dots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots
\end{pmatrix}$$

Random walks on groups:

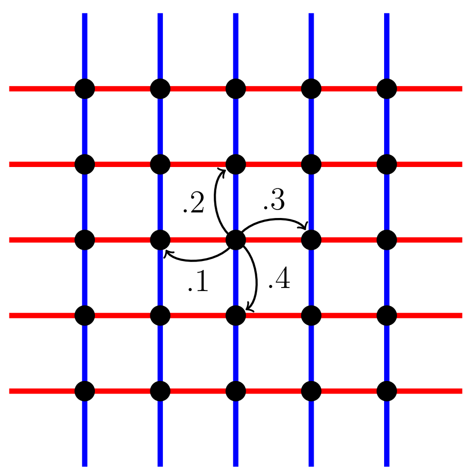

Figure 2.10: A random walk on $\mathbb{Z}^2$

Definition 2.5.7. ***Random walk on a group.*** *The right random walk (G, μ) on a group G determined by a probability measure μ is the time-homogeneous Markov* *chain with the state space G and with the transition probabilities (see [11])*

$$p_{ij} = \mu(i^{-1}j), i, j \in G,$$

which are invariant under the canonical left action of G on itself. See figure 2.10 for an example of a random walk on $\mathbb{Z}^2$.

2.6 Numerical asymptotic invariants

2.6.1 Introduction

This introduction is based on [4].

Information is mostly thought of as news, data or laws. However, in information theory, this term is more general than these limited meanings. Information theory is related to various sciences such as molecular biology, human communication and evolution of language. Information could include texts, pictures, conversations, radio signals and genes on a molecule of DNA.

In fact, chemical and physical laws alone are not sufficient to reveal and understand nature. "Nature must be interpreted as matter, energy and information" [4].

Two of the first and most important papers in information theory were published by Claude Shannon in the Bell System Technical Journal in 1948. The papers contain theorems aimed to find a way to send messages efficiently with the least time and cost. Fortunately, he was able to generalize information and establish laws that hold for all kinds of information. The papers offer new ways to look at world processes instead of classical ideas. Moreover, they "dwelt" with important aspects such as "order and disorder, error and the control of error, possibilities and the actualizing of possibilities, uncertainty and the limits to uncertainty" [4].

Most important applications for information theory were in the signals transmission in colour television, the ability to send and receive clear messages from spacecrafts and the design of early-warning radar systems.

2.6.2 The idea of entropy

Entropy first appeared in thermodynamics by Clausius as a measure of disorder in 19th century [24]. The Second Law of thermodynamics states that the entropy of an isolated physical system never decreases. For example, if you put a drop of ink in water after a while you see that they are completely mixed and it is difficult to reorder the water and the ink. In other words, systems without external influence tend to reach the maximum entropy or disorder.

After that, Gibbs work on statistical mechanics connects the entropy with probability. Later, entropy gains more popularity and significance after Shannon published his papers in 1948, to emphasize the importance of entropy as a measure of randomness (unpredictability) of the information content in a random variable. If you obtain a lot of information in each time you do the trial; then the entropy is high. In the reverse case, if after doing the trial a few times you have a high predictability about the next outcomes and you are not obtaining a lot of information; then the entropy is low. This means the entropy is maximized by the uniform distribution.

Shanon, also, discuss the relative entropy to measure the redundancy in languages. He noticed that certain combinations of letters are more likely to occur and that approach led to many important applications [24].

Shannon Entropy

We will start with Shannon entropy and the following exposition is based on [10]. Entropy means the amount of information we gain from a random variable. It also measures the amount of uncertainty we have before we do the experiment. It is obvious that if we have one probable outcome and the others are unlikely to happen, there is no uncertainty in the trail. In the probabilistic language if $p(x) = 1$, then $H(\mu) = 0$. Also, we can see that the entropy $H(\mu)$ is increased as the number of the state space increased. For instance, if we have two probable outcomes, then the entropy is less than if we have a hundred states. Thus, if the number of states in this experiment equals k we know that the entropy must be a function $f(k)$.

To discover other characteristics of this function let us consider two independent experiments α and β [10]. Assume that they are equidisributed on their state space and denote their cardinalities by k and l, respectively. If we want to study their joint experiment which equals to occuring of α and β simulantesly, it is clear that the number of probable outcomes is equal to the multiplication of the two numbers kl. It is natural to predict that the amount of uncertainty is increased and it is equal to the sum of the two entropies. That means the entropy function should satisfy

$$f(\alpha\beta) = f(\alpha) + f(\beta) \ .$$

From this relation, we can see that the appropriate function for entropy is $\log(k)$ which, also, satisfies the previous conditions that in the case of $p(x) = 1$ it gives $H(X) = 0$ and it is increased when k is increased . We conclude that $H(\mu) = \log(k)$ in the case of equidistributed measure. Thus, we can see that for each outcome x_1 with $p(x_1) = \frac{1}{k}$ we have $H(x_1) = \frac{1}{k} \log k = -\frac{1}{k} \log \frac{1}{k}$. Therefore, we can generalize this to any measure.

Let us assume that we have k probable outcomes with different probabilities given by $p(x_1) = p_1, p(x_2) = p_2, \cdots, p(x_k) = p_k$. We find that $H(\mu) = -p_1 \log(p_1) - p_2 \log(p_2) - \cdots - p_k \log(p_k)$.

Shannon in his paper [27] explained how he constructed the entropy. He explained that we need a measure of information or randomness. This measure intuitively should satisfy some properties.

- H should be continuous for all p_i.

- H is maximized by the uniform measure and it should be an increasing function of n.

- If we break an option to two successive choices, the original H sholud be the weighted sum of the individual values of H, see figure 2.11. At the left, we have three possibilities $p_1 = \frac{1}{2}\ , p_2 = \frac{1}{3}\ , p_3 = \frac{1}{6}$. On the right we first choose between two probabilities each with probability $\frac{1}{2}$. If the second choice happened we have to choose from the options with probabilities $\frac{2}{3}\ , \frac{1}{3}$. The final results have the same probabilities as before. In this case, it is required that

$$H(\frac{1}{2}\ , \frac{1}{3}\ , \frac{1}{6}) = H(\frac{1}{2}, \frac{1}{2}) + \frac{1}{2}H(\frac{2}{3}\ , \frac{1}{3})\ .$$

  We put $\frac{1}{2}$ before the second summand in the right-hand side because the second case happen in half the time.

Then Shannon expressed that the only such a function satisfying the above assumptions is of the form

$$H = -k \sum_{i=1} p_i \log p_i\ ,$$

where k is a positive constant. The following definition is based on [20]

Definition 2.6.1. *The entropy of a probability measure μ with finite support S is defined by*

$$H(\mu) = -\sum_{g \in S} \mu(x) \log[\mu(g)]\ . \tag{2.6.1}$$

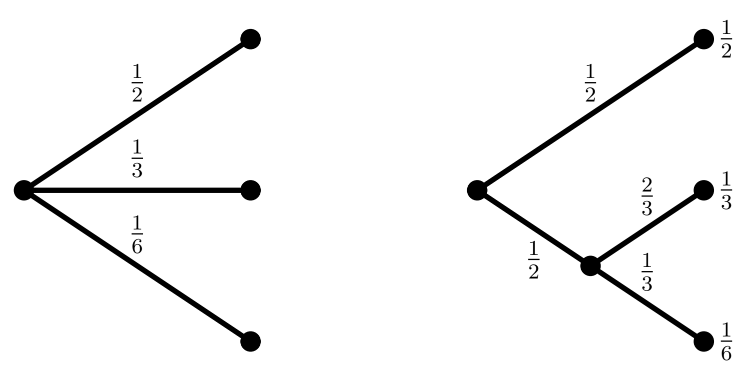

Figure 2.11: Decomposition of choices

Entropy properties

For more details about entropy properties see [10, 3].

- $$H(\mu) \leq \log |\text{supp}\mu| \ .$$

- The entropy is never negative. The probabilities are between $0 \leq p(x) \leq 1$ and the logarithms of these values are negative. After multiplying them by (-) we get
$$H(\mu) \geq 0 \ .$$

- If you have a map that takes some atoms from the first distribution to atoms from the second $\mu \to (\mu\prime)$, then the entropy does not increase.
$$H(\mu\prime) \leq H(\mu) \ .$$
This follows from convexity of the logarithmic function.

- If you have two independent distributions then the entropy of product distribution is equal to the sum of the entropies.
$$H(\mu \times \mu\prime) = H(\mu) + H(\mu\prime) \ .$$

In this paper, we are interested in investigating random walks which are sequences of random variables. As known, in a random walk each step X_n is dependent on the preceding one X_{n-1} which means we expect the unpredictibility$H(X_n)$ to be increased as n increases. Therefore, the entropy of a random walk measures how much faster $H(X_n)$ is growing. The two last properties imply existence of this entropy which called the asymptotic entropy.

Asymptotic entropy

Avez [2] was the first to discuss the asymptotic entropy (as cited in [17]), and the following definition is based on [17].

Definition 2.6.2. ***Asymptotic entropy***
Let G be a discrete countable group, μ a probability measure on G with finite entropy $H(\mu)$.

$$h = h(G, \mu) = \lim_{n\to\infty} \frac{H(\mu^{*n})}{n} = \lim_{n} -\frac{1}{n} \log \mu^{*n}(X_n) \ , \tag{2.6.2}$$

a.s. and in L^p, for all $1 \leq p < \infty$. This limit is called the asymptotic entropy of the pair(G, μ).

The existence of the limits as well as their equality follow from Kingman's Subadditive Ergodic Theorem ([2] and [7], as cited in [17]). The measure μ^{*n} is the time n position of the random walk (n fold convolution of μ). Thus, dividing the entropy at time n by n gives the asymptotic entropy h.

2.6.3 Drift

One main question regarding a random walk is whether it is transient or recurrent. If the answer is transient, the next natural question is how fast does the walk increase its distance from the identity? [17]

Let $G = \cup_{i\in S} G_i$ be finitely generated groups by symmetric generators sets S_i, respectively. This means if $a \in S_i$ that implies $a^{-1} \in S_i$. Define the length of a group element (number of letters in a word) (see [20, 18]) u in L with respect to the generator set S by

$$|u|_S = \min\{k | k = a_1 * a_2 * \cdots * a_k, a_i \in S\} \ .$$

We will replace $|u|_S$ by $|u|$ for simplicity.

Guivarc'h [9] (as cited in [20]) noticed that because of $|u * v| \leq |u| + |v|$, he can imply from Kingman's subadditive ergodic theorem the existence of a constant $\gamma \in R_+$ such that almost surely and in L^P, for all $1 \leq p \leq \infty$,

$$\lim_{n\to\infty} \frac{|X_n|_S}{n} = \gamma \ .$$

If $p(P) \leq 1$, that means for a typical random walk that it drifts to ∞ at linear speed at least [31].

In a nearest neighbour random walk in each time unit we go for most 1 unit. Therefore, the drift should be $0 \leq \gamma \leq 1$. To find the drift, we use the formula

$$\gamma = \lim_{n\to\infty} \frac{1}{n} \sum_{x\in G} |x| \mu^{*n}(x) \ ,$$

where $|.|$ denotes the word norm associated with a finite symmetric set of generators of G.

2.6.4 Hausdorff dimension

Hausdorff dimension and measure generalize Lebesgue measure and unlike the latter, they can take non integer values. Hausdorff dimension "shows the degree of singularity" or "fractalness" [12]. To understand the Hausdorff dimension let us consider a metric space and a probability measure defined on that space. We take a point from the space and look at balls centered at this point. We take logarithms of the measures of these balls and divide them by the radius of each ball. If the limit exists almost everywhere and it is the same, it is called dimension of the measure. For more details about Hausdorff dimension see [23].

Let (X, d) be a metric space, and ν be a Borel measure on X. The Hausdorff dimension of the measure ν is the smallest Hausdorff dimension of sets of full measure ν [28]. Hausdorff dimension is defined for a metric space HDX.

$$HD(\mu) = \inf\{HD(A) : \mu(A) = 1\}$$

If

$$d = \lim_{\epsilon \to 0} \frac{\log \mu(B(x, \epsilon)}{\log \epsilon}$$

exixts and is the same almost every where

$$d = HD(\mu)$$

Let us consider the metric d defined on the free product G as the distance between infinite words,

$$d(Y_1, Y_1) = 0 \quad \text{and,} \quad d(Y_1, Y_2) = e^{-(Y_1 \wedge Y_2)} \text{ for } \; Y_1 \neq Y_2 \; ,$$

where $(Y_1 \wedge Y_2)$ is the length of the common prefix of the normal reduced words of Y_1 and Y_2 [14]. The following exposition is based on [15], the metric d can be extended to the completion $\bar{F} = F \cup \partial F$ of F with respect to the metric d. The boundary ∂F is a compact space and can be represented as the space of infinite reduced words L^∞
.

Chapter 3

Random walks on free products of cyclic groups

3.1 Introduction

This chapter is based on Mairesse and Mathéus (2008) [20] unless otherwise stated. It has been proved that the random walk on an infinite graph is not only transient but it is also, convergent to the boundary. The hitting distribution of the random walk is called harmonic measure. An important result is that the harmonic measure of a transient random walk on a finitely generated free group is Markovian (see[26, Section:5] and [30, Section:6] as cited in [20]).

Let μ be a probability measure over a generating set S and denote its support by supp μ that is,

$$\text{supp}\mu = \{g \in G : \mu(g) > 0\} \ .$$

Assume that $\bigcup_{n\in\mathbb{N}^*} \text{supp } \mu^{*n} = \star_{i\in I} G_i$, where μ^{*n} is the n-fold convolution of μ. Let $(X_n)_n$ be a realization of the random walk $(\star_{i\in I} G_i, \mu)$ as defined above. The sequence $(X_n)_n$ can be viewed as a Markov chain on L. If the support of μ is included in S we say that this random walk is of the nearest neighbour random walk type (with respect to the generators S).

We want to consider nearest neighbour random walks on free products of cyclic groups with finite sets of generators, the same setting discussed by Mairesse and Mathéus (2008) in [20]. This means the harmonic measure has the Markovian property. In [17] Mairesse proves that it has a special structure that he called Markovian Multiplicative. He explains that this Markovian Multiplicative measure is entirely determined by its base r, where r is the unique solution of a finite set of polynomial equations that Mairesse called the Traffic Equations.

A Markovian Multiplicative measure is a special Markov chain meaning that it, unlike regular Markov chain, needs one probability vector to describe its behaviour, (see also [17]). We call this vector the base of the Markovian Multiplicative measure. This base serves as an initial distribution as well as transition probabilities after normalizing. To get transition probabilities we need to divide each transition probability $p(a,b)$ by the sum of the probabilities of the corresponding Σ_a that contains all admissible transitions from the element a.

Notations

Let N be the set of non-negative integers and N^* the set of positive integers. Let S be a symmetric generators set meaning that if $a \in S$ that implies $a^{-1} \in S$. If μ is a probability measure on a group $(G,*)$ defined on the generating set S. Then, μ^{*n} is the n-fold convolution product of μ, that is the image of the product measure $\mu^{\otimes n}$ by the product map

$$G \times \cdots \times G \to G,\ (g_1,\ldots,g_k) \mapsto g_1 * g_2 * \cdots * g_k \ .$$

Denote the support of μ by suppμ and assume that it generates the whole group, that is,

$$\bigcup_{n\in\mathbb{N}^*} supp(\mu^{*n}) = G \ .$$

The symbol $\sqcup$ is used for the disjoint union of sets. Given a finite set S_1, a vector $x \in R^S$, and $S_1 \subset S$, set

$$x(S_1) = \sum_{u\in S_1} x(u) \ .$$

3.2 Random walk on free products

Let $(G_i)_{i\in I}$ be a finite family of finite groups, with $|I| \geq 2$. Let 1_{G_i} be the unit of G_i. Set $S_i = G_i \setminus \{1_{G_i}\}$ and set $S = \sqcup_i S_i$. Define the map $\iota : S \to I$ by $\iota(u) = j$ if $u \in S_j$. It is also convenient to set $S_a = S_{\iota(a)}$ for all $a \in S$.

Let S^* be the free monoid over the alphabet S and denote its unit, the empty word, by 1. Define the set of *normal form words* $L \subset S^*$ by

$$L = \{u_1 \cdots u_k \in S^*,\ \forall i \in \{1,\cdots,k-1\}, \iota(u_i) \neq \iota(u_{i+1})\} \ . \qquad (3.2.1)$$

Hence, L consists of all words over the alphabet S whose consecutive letters come from different subalphabets S_i. Observe that $1 \in L$.

The *free product* $G = \star_{i \in I} G_i$ is the group with set of elements L, unit element 1, and group law $*$ defined by:

$$u_1 \cdots u_k * v_1 \cdots v_l = \begin{cases} u_1 \cdots (u_{k-1})(u_k)(v_1)(v_2) \cdots v_l & \text{if } \iota(u_k) \neq \iota(v_1) \ , \\ u_1 \cdots (u_{k-1})(u_k * v_1)(v_2) \cdots v_l & \text{if } \iota(u_k) = \iota(v_1), \ u_k \neq v_1^{-1} \ , \\ u_1 \cdots u_{k-1} * v_2 \cdots v_l & \text{if } u_k = v_1^{-1} \ , \end{cases} ,$$

where in the second case, $(u_k * v_1)$ is the product in $G_{\iota(u_k)}$ of u_k and v_1. Roughly, the law of G is the concatenation with possible simplifications at the contact point to reach a normal form word.

3.3 Main theorem

This thesis is based on the following results:

- The random walk on a free product of cyclic groups converges to the boundary (infinite words) [11, 5], so that the harmonic measure is well-defined:

$$Y_n \to Y_\infty \in \partial G \ , \quad n \to \infty \ .$$

- The harmonic measure on the boundary of a nearest neighbour random walk on a free product of cyclic groups is Markovian Multiplicative with the coefficients determined by the solutions of the Traffic Equations [[20, 17, 18]].

3.4 Geometry of free product

A Cayley graph of an infinite group has a boundary which is the space of infinite words. A sequence of words converges to an infinite word if their initial segments converge. Set $[g_n]_k$ as the initial segments which consist of the first k letters.

$$[g_n]_k \to [g_\infty]_k \ , \ k \to \infty \ .$$

The random walks on graphs that we are considering in this thesis are convergent to this boundary. See Figures 3.1 and 3.2.

Definition 3.4.1. *The* length *of an element u of $(\star_{i \in I} G_i)$ is the length (i.e. number of letters) of the word u in L. We denote it by $|u|$. Observe that*

$$|u| = \min\{k \mid u_1 * \cdots * u_k = u, \ u_i \in S\} = |u|_S \ .$$

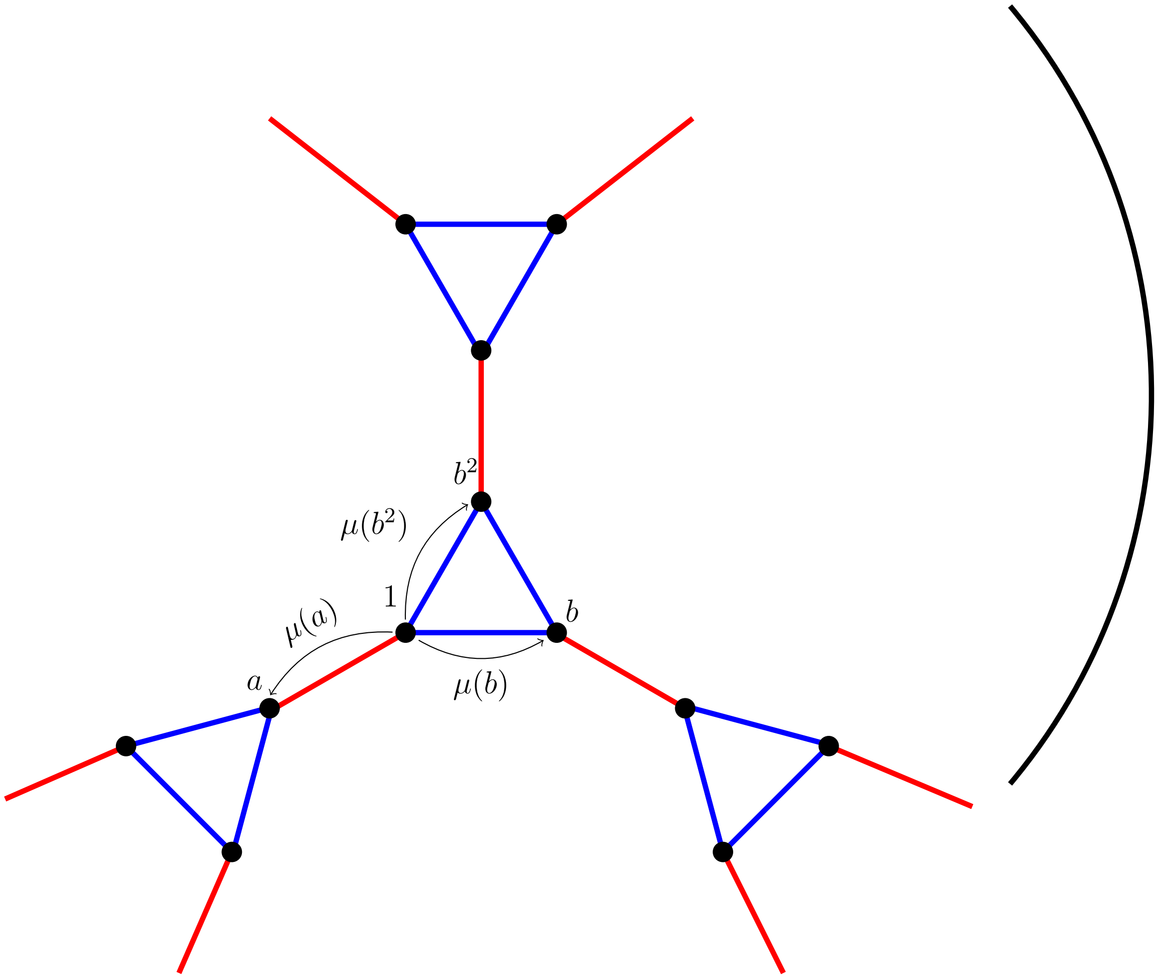

Figure 3.1: The boundary of a nearest neighbour random walk on $Z/2Z * Z/3Z$.

The harmonic measure is the law of $X_\infty = \lim_n X_n$. In the case where X is the Cayley graph of G, the support of the harmonic measure is the boundary ∂G[14].

Definition 3.4.2. *Let $L(G, S)$ be defined as before (3.2.1). Consider the set S^N equipped with the product topology. Denote by $(u_1 \cdots u_n S^N)$ the order-n cylinder in S^N defined by $u_1 \cdots u_n$. Define the set of right infinite normal form words $L^\infty \subset S^N$ by*

$$L^\infty = u_0 u_1 ... u_k ... \in S^N, \forall i \in N, \iota(u_i) \neq \iota(u_{i+1}) \ .$$

A word belongs to L^∞ iff all its finite prefixes belong to $L(G, S)$.

Consider the map

$$S \times L^\infty \to L^\infty, (a, \xi) \mapsto a \cdot \xi$$

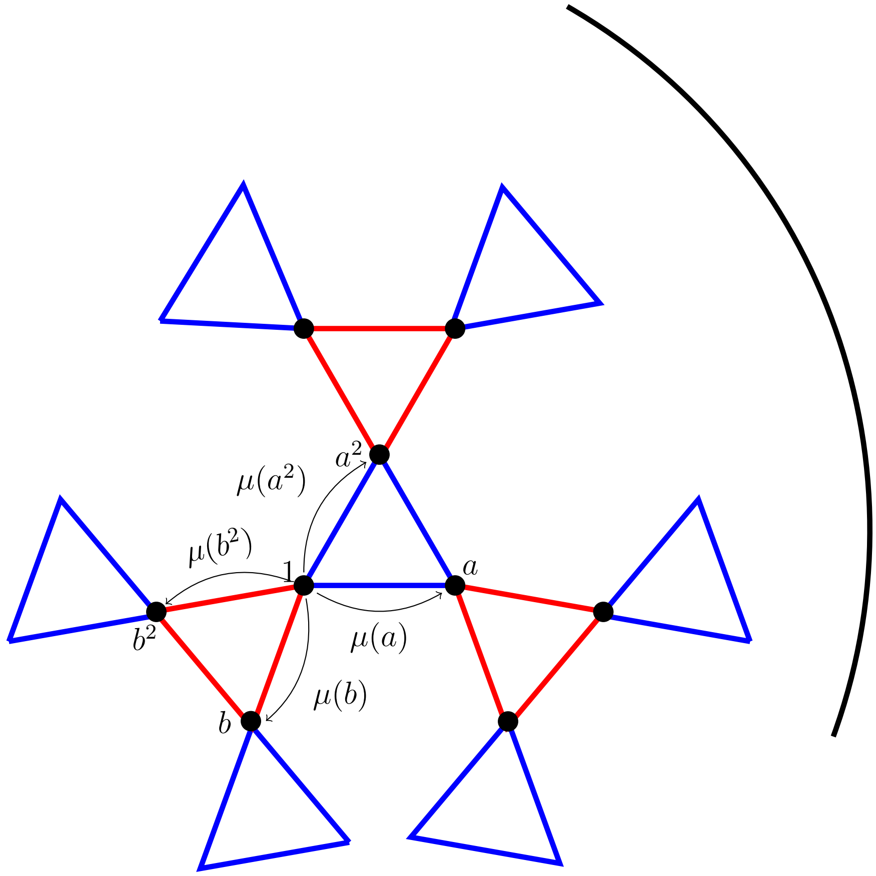

Figure 3.2: The boundary of a nearest neighbour random walk on $\mathbb{Z}/3\mathbb{Z} * \mathbb{Z}/3\mathbb{Z}$.

defined by

$$\begin{cases} a \cdot \xi = a\xi_0\xi_1 \cdots & if \iota(a) \neq \iota(\xi_0), \\ a \cdot \xi = (a * \xi_0)\xi_1 \cdots & \text{if} \iota(a) = \iota(\xi_0), a \neq \xi_0^{-1}, \\ a \cdot \xi = \xi_1\xi_2 \cdots & \text{if} a = \xi_0^{-1}. \end{cases}$$

Equip $S^{\mathbb{N}}$ with the Borel σ-algebra associated with the product topology. This induces a σ-algebra on L^∞.

Definition 3.4.3. *Given a measure ν^∞ on L^∞ and $a \in S$, define the measure $a\nu^\infty$ by: $\int f(\xi) d(a\nu^\infty)(\xi) = \int f(a \cdot \xi) d\nu^\infty(\xi)$. A probability measure ν^∞ on L^∞ is* μ-invariant *if*

$$\nu^\infty(\cdot) = \sum_{a \in S} \mu(a)[a\nu^\infty](\cdot) \,. \tag{3.4.1}$$

Proposition 3.4.4. *[20, Proposition 1] There exists a r.v. X_∞ valued in L^∞ such that a.s.*

$$\lim_{n\to\infty} X_n = X_\infty \ ,$$

meaning that the length of the common prefix between X_n and X_∞ goes to infinity a.s.

Let μ^∞ be the distribution of X^∞. The probability μ^∞ is μ-invariant and is the only μ-invariant probability on L^∞. We call it the *harmonic measure* of (G, μ).

Mairesse and Mathéus in [20] discussed the cases where the harmonic measure is shift-invariant (on top of being μ-invariant). To that purpose, they associate two sets of equations with the random walk: the *Traffic Equations* as in [17] and the *Stationary Traffic Equations*.

3.5 Markovian Multiplicative measure

A Markovian Multiplicative measure is a special Markov chain meaning that it, unlike the regular Markov chain, needs one probability vector to describe its behaviour. We call this vector the base of the Markovian Multiplicative measure. This base serves as initial distribution as well as transition probabilities after normalizing. To get transition probabilities we need to divide each transition probability $p(a, b)$ by the sum of the corresponding Σ_a that contains all admissible transitions probabilities from the element a.

Example 3.5.1. *Consider a Markovian Multiplicative measure defined on the generating set*

$$S = \{a, a^{-1}, b, b^{-1}\}$$

with a Markovian Multiplicative base

$$\theta = (\theta(a), \theta(a^{-1}), \theta(b), \theta(b^{-1})) \ .$$

The admissable transitions are:

$$T = \{(a, a), (a, b), (a, b^{-1}), (b, b), (b, a), (b, a^{-1}),$$
$$(a^{-1}, a^{-1}), (a^{-1}, b), (a^{-1}, b^{-1}), (b^{-1}, b^{-1}), (b^{-1}, a), (b^{-1}, a^{-1})\}$$

Then, the initial distribution is:

$$\lambda = (\theta(a), \theta(a^{-1}), \theta(b), \theta(b^{-1}))$$

Also, the transition probabilities for each $\alpha, \beta \in S$ can be found using:

$$p(\alpha, \beta) = \frac{\theta(\beta)}{\sum_{\beta' : (\alpha, \beta') \in T} \theta(\beta')}$$

Note that $a = b$ is possible. Thus we have:

$$p(a,a) = \frac{\theta(a)}{\theta(a) + \theta(b) + \theta(b^{-1})} \; .$$

Also,

$$p(a,b) = \frac{\theta(b)}{\theta(a) + \theta(b) + \theta(b^{-1})}$$

In the same way,

$$p(a,b^{-1}) = \frac{\theta(b^{-1})}{\theta(a) + \theta(b) + \theta(b^{-1})}$$

We can continue the procedure to get the remaining transition probabilities.

Example 3.5.2. *Consider a Markovian Multiplicative measure defined on the generating set*

$$S = \{a, b, c\}$$

with Markovian Multiplicative base

$$\theta = (\theta(a), \theta(b), \theta(c)) \; .$$

The forbidden transitions are:

$$F = \{(a,a), (b,b), (c,c)\}$$

Then, the initial distribution is:

$$\lambda = (\theta(a), \theta(b), \theta(c))$$

Also, the transition probabilities for each $\alpha, \beta \in S$ can be found using:

$$p(\alpha,\beta) = \frac{\theta(\beta)}{\sum_{\beta' : (\alpha,\beta') \notin F} \theta(\beta')}$$

Thus we have:

$$p(a,b) = \frac{\theta(b)}{\theta(b) + \theta(c)}$$

Also,

$$p(a,c) = \frac{\theta(c)}{\theta(b) + \theta(c)}$$

Definition 3.5.3. *Define* $\mathring{B} = \{x \in \mathbb{R}^{\mathbb{S}} \mid \forall u,\ x(u) > 0,\ \sum_u x(u) = 1\}$. *Consider* $r \in \mathring{B}$. *Define the matrix* P *of dimension* $S \times S$ *by*

$$P_{u,v} = \begin{cases} r(v)/r(S \setminus S_u) & \text{if } \iota(v) \neq \iota(u) \\ 0 & \text{otherwise} \end{cases} . \tag{3.5.1}$$

It is the transition matrix of an irreducible Markov Chain on the state space S. *Set* $p = (r(a)r(S \setminus S_a), a \in S)$ *and* $\pi = p/p(S)$. *Observe that* $\pi P = \pi$. *In words,* π *is the stationary distribution of the Markov chain defined by* P.

Let $(U_n)_n$ *be a realization of the Markov chain with transition matrix* P *and starting from* U_1 *such that* $P\{U_1 = x\} = r(x)$. *Set* $U^\infty = \lim_n U_1 \cdots U_n$, *and let* ν^∞ *be the distribution of* U^∞. *Clearly, the support of* ν^∞ *is included in* L^∞. *For* $u_1 \cdots u_k \in L$, *we have*

$$\begin{aligned} \nu^\infty(u_1 \cdots u_k S^N) &= r(u_1) P_{u_1,u_2} \cdots P_{u_{k-1},u_k} \\ &= r(u_1) \frac{r(u_2)}{r(S \setminus S_{u_1})} \cdots \frac{r(u_k)}{r(S \setminus S_{u_{k-1}})} \\ &= \frac{r(u_1)}{r(S \setminus S_{u_1})} \cdots \frac{r(u_{k-1})}{r(S \setminus S_{u_{k-1}})} r(u_k) . \end{aligned}$$

We call ν^∞ *the* Markovian Multiplicative *probability measure associated with* r.

3.6 Traffic Equations

The harmonic measure of a nearest neighbour random walk is described by two sets of equations of order two that have at most one solution in $\mathring{B}$. The first set are called general Traffic Equations and they have unique solutions. The second set is called Stationary Traffic Equations. These equations have solutions if and only if the harmonic measure is shift-invariant.

Definition 3.6.1 (Traffic Equations)**.** *[17, Equation:17]*

The Traffic Equations associated with (G, μ) *are defined by* $\forall a \in S$,

$$r(a) = \mu(a) \sum_{u \in S \setminus S_a} r(u) + \sum_{u * v = a} \mu(u) r(v) + \sum_{u \in S \setminus S_a} \mu(u^{-1}) \frac{r(u)}{\sum_{v \in S \setminus S_u} r(v)} r(a) . \tag{3.6.1}$$

Lemma 3.6.2. *[20, Lemma 3.6.1] If the harmonic measure* μ^∞ *is the Markovian Multiplicative measure associated with* $r \in B$, *then* r *is a solution to the Traffic Equations 3.6.1. Conversely, if the Traffic Equations admit a solution* $r \in B$, *then the harmonic measure* μ^∞ *is the Markovian Multiplicative measure associated with* r.

Theorem 3.6.3. *[20, Theorem: 3.5] Let $G = *_{i\in I} G_i$ be the free product of a finite family of finite groups, with $|I| \geqslant 2$, and for all $i, |G| > 1$. Let μ be a probability measure on $S = \sqcup_i G_i \setminus \{1_{G_i}\}$. Assume that $\bigcup_n \in N^* = G$ and that the random walk (G, μ) is transient. Then, the Traffic Equations have a unique solution $r \in B$. The harmonic measure of the random walk is the Markovian Multiplicative measure associated with r.*

Definition 3.6.4 (Stationary Traffic Equations). *[20, Equation:12]*
The Traffic Equations associated with (G, μ) are defined by $\forall a \in S$,

$$x(a) = \mu(a)\frac{|I|-1}{|I|} + \sum_{u*v=a} \mu(u)x(v) + x(a)\frac{|I|}{|I|-1} \sum_{u\in S\setminus S_a} \mu(u^{-1})x(u) \,. \tag{3.6.2}$$

Lemma 3.6.5. *[20, Lemma: 3.4] The harmonic measure μ^∞ is Markovian Multiplicative associated with r* and *ergodic if and only if the Stationary Traffic equations admit a solution r in $\mathring{\mathbb{B}}$.*

It follows from Lemma 3.6.5 and Theorem 3.6.3 that the harmonic measure is a shift-invariant if and only if the Stationary Traffic Equations have a solution.

3.7 Asymptotic invariants and harmonic measure

The solutions we obtain from solving the Traffic Equations can be used as a base to find the asymptotic numerical invariants as described below. In particular, if the probabilities $\mu(a)$ are algebraic numbers, then the drift and the entropy are algebraic numbers (see also [18]).

3.7.1 The drift of the random walk on free products

The drift γ gives the speed of the random walk. In the context of random walk on free products of cyclic groups, it is the expected change of length of an infinite normal form word distributed according to μ^∞, when we multiply it from left by an element distributed according to μ. According to ([20], Equation:19) it can be calculated as follows:

$$\gamma = \sum_{a\in S} \mu(a) \left[-r(a^{-1}) + \sum_{b\in S\setminus S_a} r(b) \right] , \tag{3.7.1}$$

where $r(a)$ is the solution tho the Traffic Equations.

3.7.2 The asymptotic entropy of the random walk on free products

The asymptotic entropy h of a random walk measures the amount of information we take from μ^{*n} divided by n and can be found using [18, Equation:16]

Set $q(s) = \frac{r(s)}{r(S \setminus S_s)}$, then

$$h = -\sum_{s \in S} \mu(s) \left[\log\left[\frac{1}{q(s^{-1})}\right] r(s^{-1}) + \sum_{g \in S_s \setminus s^{-1}} \log\left[\frac{q(sg)}{q(g)}\right] r(g) + \log[q(s)] \sum_{g \in S \setminus S_s} r(g) \right] \tag{3.7.2}$$

such that $r(s)$ is the solution for the Traffic Equations.

3.7.3 The Hausdorff dimension of the harmonic measure

Hausdorff dimension of the harmonic measure "shows the degree of singularity or fractalness" (Kaimanovich, 1997, p1)[12]. We can define a metric as a distance between infinite words see [20]. In mathematical language, for each two infinite words ω and $\omega\prime$

$$d(\omega, \omega\prime) = e^{-n}$$

such that n is the"confluent" (the length of the initial common part of ω and $\omega\prime$. To find the Hausdorff dimension of the harmonic measure in our setting we can apply the formula

$$\mathrm{HD}_\nu = \frac{h}{\gamma} \ , \tag{3.7.3}$$

which is first appeared in [12] with specific conditions. Then, it has been generalized in [15].

Volum of a group and the fundamental inequality

The *volume* of the group G with respect to the finite set of generators S is

$$v = v(S) = \lim_n \frac{1}{n} \log \#\{g \in G, |g| = n\} \ ,$$

where $|g|$ are words have at most n letters. The limit exists by subadditivity. The following fundamental inequality was proved and highlighted in [29] (as cited in [20])

$$h/\gamma \le v \ ,$$

where we recall that γ is the drift of the random walk. The interpretation is that the proportion of a typical element visited by the walk is less than or equal to the total number of elements.

Note that, in our setting of free products of finite groups, h/γ and v can be interpreted respectively as the Hausdorff dimension of the harmonic measure μ^∞ and the Hausdorff dimension of its support [20].

The following exposition is based on [14]. In the case where X is the Cayley graph of the hyperbolic group G equipped with the word metric (w.r.t. a certain system of generators S), the dimension of the boundary ∂G is equal to the growth $v(G, S)$ of G (w.r.t. the same S). Moreover, the Hausdorff measure is then finite and nonzero.

Chapter 4

Examples of random walks on the free product of cyclic groups

4.1 Introduction

As we have explained, Mairesse in [17] found that the harmonic measure on the boundary of random walks on free products of cyclic groups is Markovian Multiplicative measure. This means that this measure needs one probability vector to describe its behaviour. This base of the Markovian Multiplicative measure serves as an initial distribution as well as transition probabilities after normalizing.

Let $(G_i)_{i\in I}$ be a finite family of finite groups, with $|I| \geq 2$. Set $S_i = G_i \setminus \{1_{G_i}\}$ the set of generators of the respective group. Set $S = \cup_{i\in I} A_i$. Let 1_{G_i} be the unit of G_i.

Mairesse uses the Traffic Equation 3.6.1 in [17] to find this vector.

$$r(a) = \mu(a) \sum_{u\in S\setminus S_a} r(u) + \sum_{u*v=a} \mu(u)r(v) + \sum_{u\in S\setminus S_a} \mu(u^{-1}) \frac{r(u)}{\sum_{v\in S\setminus S_u} r(v)} r(a) \, .$$

In [20], Mairesse and Mathéus have discovered a special case when the measure is a shift-invariant on top of being μ-invariant. In this case, the measure satisfies the so-called Stationary Traffic Equations.

$$x(a) = \mu(a) \frac{|I|-1}{|I|} + \sum_{u*v=a} \mu(u)x(v) + x(a) \frac{|I|}{|I|-1} \sum_{u\in S\setminus S_a} \mu(u^{-1})x(u) \, .$$

In this section, we investigate some examples of random walks on free products of cyclic groups. The examples of free products of two cyclic groups are based on Mairess and Mathéus papers[20] and [19] with some added explanations. We discuss the example of the free product $\mathbb{Z}/2\mathbb{Z} * Z/3\mathbb{Z}$ and two cases of symmetric measures on the free product $Z/3\mathbb{Z} * Z/3\mathbb{Z}$.

Our new examples are free products of three cyclic groups. The examples we consider are $\mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z}$ and $Z/3\mathbb{Z} * Z/3\mathbb{Z} * Z/3\mathbb{Z}$.

In each one of these examples, we consider a nearest neighbour random walk on the free product of the cyclic groups. These random walks start at the identity element 1 and proceeds according to the probability measure μ. In each case, we show the Cayley graph of the free product and solve either the set of the general Traffic Equation or the special case, namely, which is the Stationary Traffic Equations. We use the solutions we obtain which are probabilty vectors in finding the drift (or the speed of the random walk) and draw a figure of the drift as a function of p. After that, we compute the asymptotic entropy and the Haudsdorff dimension of the probability measure.

4.2 Examples of random walks on the free product of two cyclic groups

4.2.1 Random walks on $\mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/3\mathbb{Z}$

We want to study a nearest neighbour random walk on

$$\mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/3\mathbb{Z} = \langle a, b | a^2 = 1, b^3 = 1 \rangle .$$

This example is based on [20] and [19] with some added explanations.

Note that the group $\mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/3\mathbb{Z}$ is isomorphic to the modular group $PSL(2, \mathbb{Z})$, i.e. the group of 2×2 matrices with integer entries and determinant 1, quotiented by $\pm Id$. Let a and b be the respective generators of $\mathbb{Z}/2\mathbb{Z}$ and $\mathbb{Z}/3\mathbb{Z}$.

A possible faithful representation of the group is (see [19])

$$a = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, b = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}.$$

Let $(G_i)_{i \in I}$ be a finite family of finite groups, with $|I| \geq 2$. Set $S_i = G_i \setminus \{1_{G_i}\}$ the set of generators of the respective group. Let 1_{G_i} be the unit of G_i.

Consider a nearest neighbour random walk on the free product of the cyclic groups $\mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/3\mathbb{Z}$. This random walk starts at the identity element 1 and proceeds according to the probability measure μ depending on one parameter p only, see figure 4.1. The probability distribution is given by:

$$\mu(a) = 1 - 2p, \mu(b) = \mu(b^2) = p, p \in (0, 1/2) .$$

Solving the Traffic Equations (3.6.1) for this example:

$$r(a) = \mu(a) \sum_{u \in S \setminus S_a} r(u) + \sum_{u * v = a} \mu(u) r(v) + \sum_{u \in S \setminus S_a} \mu(u^{-1}) \frac{r(u)}{\sum_{v \in S \setminus S_u} r(v)} r(a) .$$

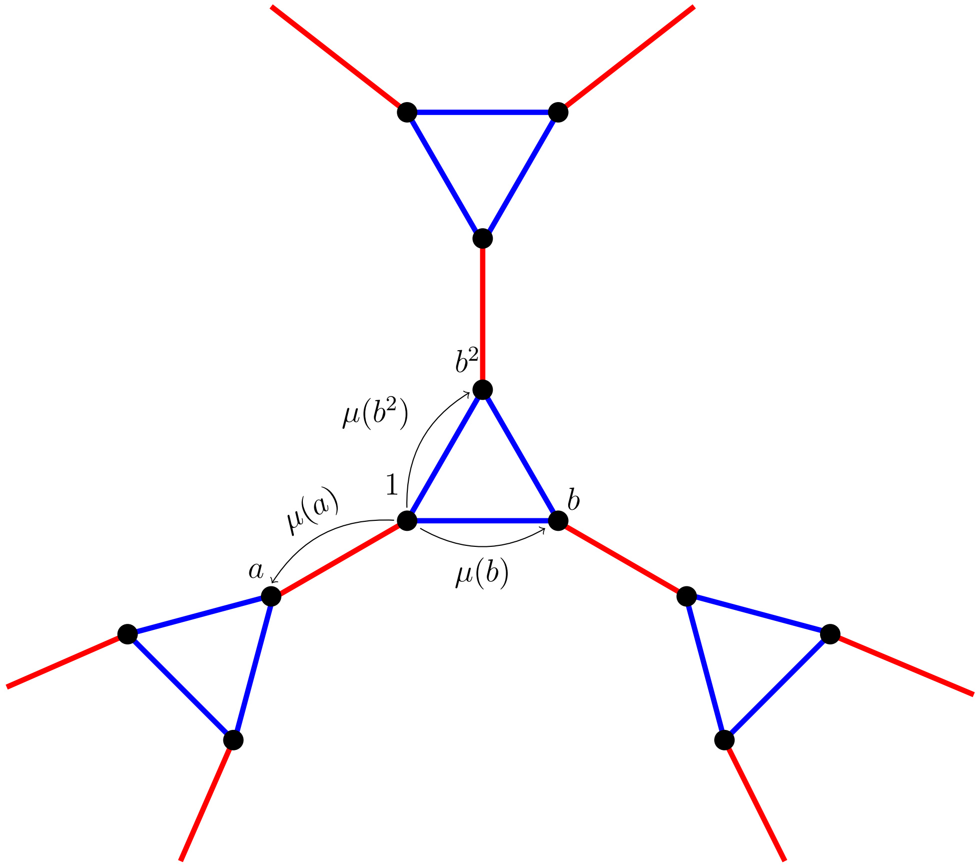

Figure 4.1: A nearest neighbour random walk on $Z/2Z * Z/3Z$

$$r(a) = (1-p)(r(b) + r(b^2)) \; . \tag{4.2.1}$$

$$r(b) = pr(a) + pr(b^2) + (1-2p)\left(\frac{r(a)}{r(b) + r(b^2)}\right) r(b) \; . \tag{4.2.2}$$

$$r(b^2) = pr(a) + pr(b) + (1-2p)\left(\frac{r(a)}{r(b) + r(b^2}\right) r(b^2) \; . \tag{4.2.3}$$

Because of the symmetry in the probabilities, we have

$$r(b) = r(b^2)$$

which simplifies the Traffic equations to:

$$r(a) = (1-p)(2r(b)) \; . \tag{4.2.4}$$

From the definition of the Markovian Multiplicative Measure, we have

$$\sum_{j \in I} r(A_j) = 1 \ ,$$

see([20],3.9). This gives us

$$r(a) + 2r(b) = 1 \ .$$

After plugging the formula of $r(a)$ from 4.2.4, we have

$$(1-p)(2r(b)) + 2r(b) = 1 \ . \tag{4.2.5}$$

This means:

$$r(b) = \frac{1}{2-2p+2} \ .$$

After simplifying, we obtain

$$r(b) = r(b^2) = \frac{1}{2(2-p)} \ .$$

And from 4.2.4, we have

$$r(a) = \frac{1-p}{2-p} \ .$$

Here, in this free product, we can go from a to either b or b^2 with the same probability $\frac{1}{2}$ because they are from a group different than the group of a. However, from b and b^2 we just have one accepted next step which is to a. Thus, the transition matrix defining the Markovian Multiplicative harmonic measure with respect to the order $\{a, b, b^2\}$ is:

$$P = \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

As we can see, this harmonic measure is not stationary. Now we want to compute the drift.

We used the harmonic measure to compute the drift and came to the same conclusion as in ([19],2).

Following([20],19), the drift can be computed as follows

$$\gamma = \sum_{a \in S} \mu(a) \left[-r(a^{-1}) + \sum_{b \in S \setminus S_a} r(b) \right] .$$

Plugging the formulas for $r(a)$, $r(b)$ and $r(b^2)$, we obtain

$$\gamma = (1-2p)\left[-\left(\frac{1-p}{2-p}\right) + 2\left(\frac{1}{2(2-p)}\right)\right] + 2p\left[-\left(\frac{1}{2(2-p)}\right) + \left(\frac{1-p}{2-p}\right)\right] \ .$$

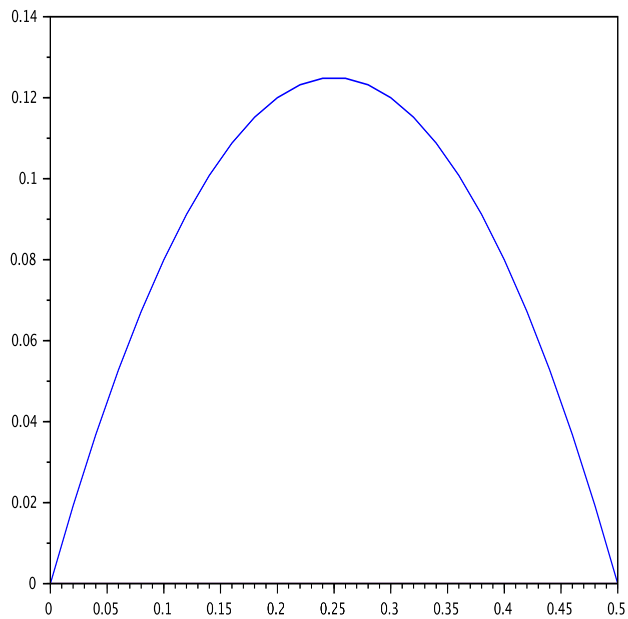

Figure 4.2: The drift of the free product of the cyclic groups $\mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/3Z$, the X axis represents p and the Y axis represents the entropy.

$$\gamma = (1-2p)\left[\frac{p}{2-p}\right] + \left[\left(\frac{-p}{2-p}\right) + \left(\frac{2p(1-p)}{2-p}\right)\right] .$$

Upon opening brackets, we have

$$\gamma = \frac{p - 2p^2 + 2p - 2p^2 - p}{2-p} .$$

Grouping similar terms together, we get

$$\gamma = \frac{2p(1-2p)}{2-p} ,$$

see figure 4.2.

Now, we want to derive the asymptotic entropy. According to 3.7.2 the asymptotic entropy can be calculated as follows

$$-h = (1-2p)\left[log\left[\frac{1}{1-p}\right]\left(\frac{1-p}{2-p}\right) + log[1-p]\left(\frac{1}{2-p}\right)\right]$$

$$+2p\left[log[2-2p]\left(\frac{1}{2(2-p)}\right)+log\left[\frac{1}{2-2p}\right]\left(\frac{1-p}{2-p}\right)\right]$$

Collecting similar terms together, we get

$$-h=(1-2p)\left[log[1-p]\left(\frac{1}{2-p}-\frac{1-p}{2-p}\right)\right]+2p\left[log[2-2p]\left(\frac{1}{2(2-p)}-\frac{2-2p}{2(2-p)}\right)\right]$$

Simplifying, we obtain

$$-h=log[1-p](1-2p)\left(\frac{2p}{2(2-p)}\right)+log[2-2p](2p)\left(\frac{2p-1}{2(2-p)}\right)$$

Collecting similar terms together gives us

$$-h=\frac{[log[1-p]]\left(2p-4p^2\right)+[log[2-2p]]\left(4p^2-2p\right)}{2(2-p)}$$

Taking the common factor

$$-h=\frac{2p-4p^2}{2(2-p)}\left[log[1-p]-log[2-2p]\right]$$

Using logarithm rules:

$$-h=\left(\frac{p-2p^2}{2-p}\right)\left[log\left[\frac{1-p}{2-2p}\right]\right]$$

$$h=\left(\frac{p-2p^2}{2-p}\right)\left[log\left[\frac{1}{2}(p-1)\right]\right]$$

At this point, we can calculate the Hausdorff dimension. According to 3.7.3, it can be calculated as follows:

$$\mathrm{HD}_\nu=\frac{h}{\gamma}$$

Plugging the formulas that we have, we obtain

$$\mathrm{HD}=\frac{\left[p-2p^2\left[log\left[\frac{1}{2}(p-1)\right]\right]\right]}{[2p(1-2p)]}$$

4.2.2 Random walks on $\mathbb{Z}/3\mathbb{Z}*\mathbb{Z}/3\mathbb{Z}$

This example is based on [20] with some added details and explanations. Consider a nearest neighbor random walk on the free product of the cyclic groups $\mathbb{Z}/3\mathbb{Z}*\mathbb{Z}/3\mathbb{Z}$. This random walk starts at the identity element 1 and proceeds according to the probability measure μ, see figure 4.3.

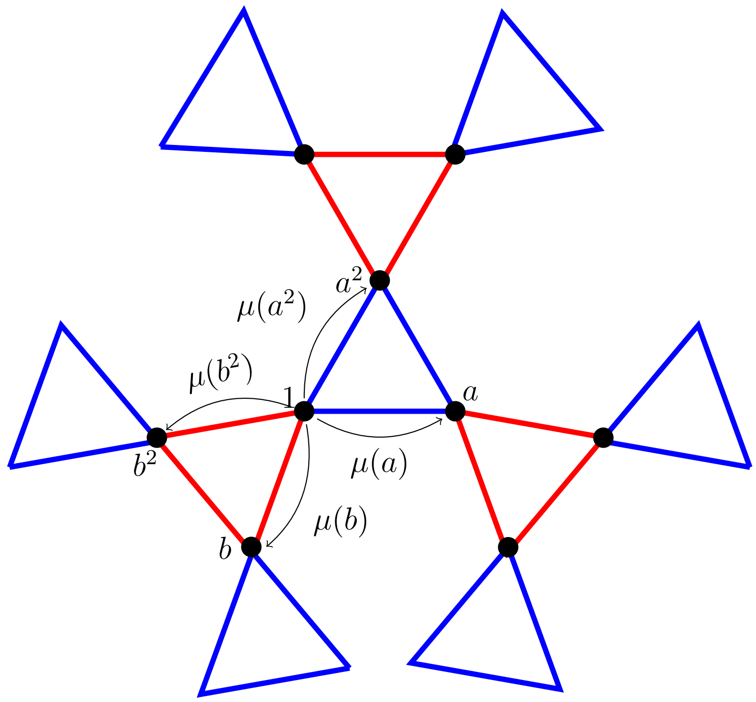

Figure 4.3: A nearest neighbour random walk on $\mathbb{Z}/3\mathbb{Z} * \mathbb{Z}/3\mathbb{Z}$

First case

First, consider a probability measure μ given by

$$\mu(a) = \mu(b) = p, \mu(a^2) = \mu(b^2) = q = 1/2 - p \text{ where } p \in (0, \frac{1}{2}) .$$

In this example, we have two groups which means $| I |= 2$.

We plug the values from this example in the Stationary Traffic Equations(3.6.2)

$$x(a) = \mu(a)\frac{|I|-1}{|I|} + \sum_{u*v=a} \mu(u)x(v) + x(a)\frac{|I|}{|I|-1} \sum_{u \in S \setminus S_a} \mu(u^{-1})x(u) .$$

Because of symmetry in the probabilities we obtain the following:

$$A = r(a) = r(b) = \frac{p}{2} + \left(\frac{1}{2} - p\right) r(a^2) + 2r(a)\{r(a)\left(\frac{1}{2} - p\right) + r(a^2)p\} . \quad (4.2.6)$$

$$B = r(a^2) = r(b^2) = \left(\frac{1}{2} - p\right)\frac{1}{2} + pr(a) + 2r(a^2)\{r(a)\left(\frac{1}{2} - p\right) + r(a^2)p\} . \quad (4.2.7)$$

As $r(a)$ for all $a \in S$ are probability vectors, this means

$$\sum_{a \in S} r(a) = 1 .$$

Thus we can use $A = \frac{1-2B}{2}$ and moving everything to the right-hand side:

$$0 = \left(\frac{2B-1}{2}\right) + \left(\frac{p}{2}\right) + \left(\frac{1}{2} - p\right) B + (1-2B) \left[pB + \left(\frac{1}{2} - p\right) \left(\frac{1-2B}{2}\right)\right] \quad (4.2.8)$$

Simplifying the fourth summand in the right-hand side:

$$\begin{aligned} &(1-2B) \left[pB + \left(\frac{1}{2} - p\right) \left(\frac{1-2B}{2}\right)\right] \\ &= (1-2B) \left[pB + \frac{1}{4} - \frac{p}{2} - \frac{B}{2} + pB\right] \\ &= (1-2B) \left[B \left(2p - \frac{1}{2}\right) + \frac{1-2p}{4}\right] \end{aligned}$$

Opening brackets:

$$\begin{aligned} &= B \left(2p - \frac{1}{2}\right) + \frac{1-2p}{4} - 2 \left[B^2 \left(2p - \frac{1}{2}\right) + B \left(\frac{1-2p}{4}\right)\right] \\ &= \frac{1-2p}{4} + B(3p-1) - 2B^2 \left(2p - \frac{1}{2}\right) \end{aligned}$$

Plug into (4.2.8)

$$0 = \frac{2B-1}{2} + \frac{p}{2} + \left(\frac{1}{2} - p\right) B + \frac{1-2p}{4} + B(3p-1) - 2B^2 \left(2p - \frac{1}{2}\right)$$

Collecting similar terms together, we have

$$0 = \frac{-1}{4} + B \left(2p + \frac{1}{2}\right) - 2B^2 \left(2p - \frac{1}{2}\right)$$

It is a second order equation, which can be solved using the quadratic formula, here we have:

$$a = -2 \left(2p - \frac{1}{2}\right) \ , b = \left(2p + \frac{1}{2}\right) \ \text{ and } \ c = \frac{-1}{4}$$

$$B = \frac{-2p - \frac{1}{2} \pm \sqrt{\left(2p + \frac{1}{2}\right)^2 - 2\left(2p - \frac{1}{2}\right)}}{-4\left(2p - \frac{1}{2}\right)}$$

$$B = \frac{-2p - \frac{1}{2} \pm \sqrt{4p^2 - 2p + \frac{5}{4}}}{-4\left(2p - \frac{1}{2}\right)}$$

Multiplying the numerator and denominator by -2:

$$B = \frac{4p + 1 \pm \sqrt{16p^2 - 8p + 5}}{4(4p - 1)}$$

We should choose the solution with(-) to get a positive answer.

Using $A = \frac{1-2B}{2}$ we find

$$A = \frac{1}{2} - \left[\frac{4p + 1 - \sqrt{16p^2 - 8p + 5}}{4(4p - 1)}\right]$$

$$A = \frac{(8p - 2) - 4p - 1 + \sqrt{16p^2 - 8p + 5}}{4(4p - 1)}$$

Upon Simplifying, we obtain

$$A = \frac{4p - 3 + \sqrt{16p^2 - 8p + 5}}{4(4p - 1)}$$

Hence, according to Lemma 3.6.5 and the following note that this measure is a shift-invariant on top of being μ-invariant.

Now we want to derive the drift. According to ([20],19) it can be calculated as follows:

$$\gamma = \sum_{a \in S} \mu(a) \left[-r(a^{-1}) + \sum_{b \in S \setminus S_a} r(b)\right]$$

$$\gamma = p[-B+A+B] + \left(\frac{1}{2} - p\right)[-A+A+B] + p[-B+A+B] + \left(\frac{1}{2} - p\right)[-A+A+B]$$

Collecting similar terms together:

$$\gamma = p(2A) + \left(\frac{1}{2} - p\right)(2B)$$

$$\gamma = 2p\left[\frac{4p - 3 + \sqrt{16p^2 - 8p + 5}}{4(4p - 1)}\right] + (1 - 2p)\left[\frac{4p + 1 - \sqrt{16p^2 - 8p + 5}}{4(4p - 1)}\right]$$

Multiplying:

$$\gamma = \left[\frac{8p^2 - 6p + 2p\sqrt{16p^2 - 8p + 5}}{4(4p - 1)}\right] + \left[\frac{4p + 1 - \sqrt{16p^2 - 8p + 5}}{4(4p - 1)}\right]$$

$$- \left[\frac{8p^2 + 2p - 2p\sqrt{16p^2 - 8p + 5}}{4(4p - 1)}\right]$$

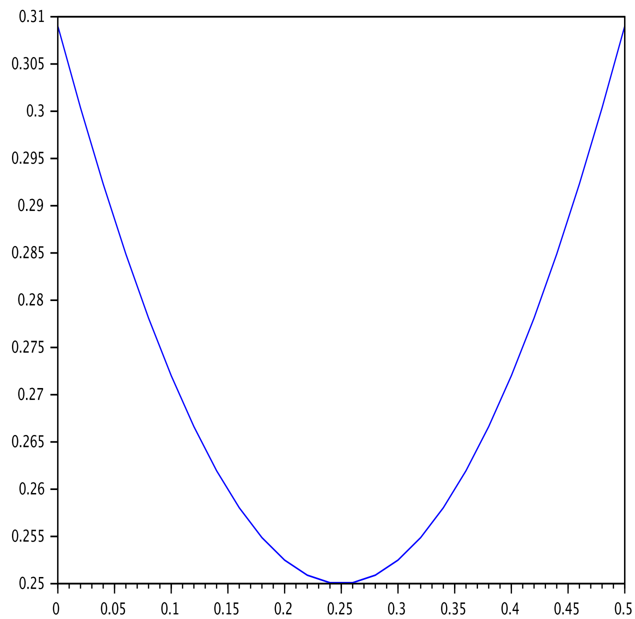

Figure 4.4: The drift of the free product of the cyclic groups $\mathbb{Z}/3\mathbb{Z} * \mathbb{Z}/3\mathbb{Z}$ (the first case)

Collecting similar terms together:

$$\gamma = \frac{-4p + 1 + (4p-1)\sqrt{16p^2 - 8p + 5}}{4(4p-1)}$$

Simplifying we obtain:

$$\gamma = \frac{-1}{4} + \frac{1}{4}\sqrt{16p^2 - 8p + 5}\,,$$

see figure 4.4.

The entropy can be found using (3.7.2): Set $q(s) = \frac{r(s)}{r(S \setminus S_s)}$

$$h = -\sum_{s \in S} \mu(s) \left[\log\left[\frac{1}{q(s^{-1})}\right] r(s^{-1}) + \sum_{g \in S_s \setminus s^{-1}} \log\left[\frac{q(sg)}{q(g)}\right] r(g) + \log[q(s)] \sum_{g \in S \setminus S_s} r(g) \right]$$

Recall that

$$\mu(a) = \mu(b) = p, \mu(a^2) = \mu(b^2) = q = 1/2 - p \;\text{ where }\; p \in (0, \frac{1}{2})\,.$$

We find

$$q(a) = q(b) = \frac{r(a)}{r(S \setminus S_a)} = \frac{A}{A+B}$$

Also,

$$q(a^{-1}) = q(a^2) = q(b^2) = \frac{B}{A+B}$$

In case $s = a = b$ We find $g \in S_s \setminus s^{-1} = \{a\}$ This leads to

$$\log \frac{q(gs)}{q(s)} = \log \frac{q(a^2)}{q(a)} = \log \left[\frac{B}{A+B} \times \frac{A+B}{A} \right] = \log \frac{B}{A}$$

In case $s = a^2 = b^2$ We find $g \in S_s \setminus s^{-1} = \{a^2\}$ This leads to

$$\log \frac{q(a)}{q(a^2)} = \log \left[\frac{A}{A+B} \times \frac{A+B}{B} \right] = \log = \frac{A}{B}$$

Plugging these formulas into the equation (3.7.2)

$$-h = 2p \left[\log \left[\frac{A+B}{B} \right] B + \log \left[\frac{B}{A} \right] A + \log \left[\frac{A}{A+B} \right] A + B \right]$$

$$+(1-2p) \left[\log \left[\frac{A+B}{A} \right] A + \log \left[\frac{A}{B} \right] B + \log \left[\frac{B}{A+B} \right] (A+B) \right]$$

Opening brackets

$$-h = \left[2pB \log \left[\frac{A+B}{B} \right] + 2pA \log \left[\frac{B}{A} \right] + (2pA + 2pB) \log \left[\frac{A}{A+B} \right] \right]$$

$$\left[(A-2pA) \log \left[\frac{A+B}{A} \right] + (B-2pB) \log \left[\frac{A}{B} \right] + (A+B-2pA-2pB) \log \left[\frac{B}{A+B} \right] \right]$$

Collecting similar logarithms

$$-h = \log[A+B][2pB - 2pA - 2pB + A - 2pA - A - B + 2pA + 2pB]$$

$$+ \log[A][-2pA + 2pA + 2pB - A + 2pA + B - 2pB]$$

$$+ \log[B][-2pB + 2pA - B + 2pB + A + B - 2pA - 2pB]$$

Adding similar terms

$$-h = \log[A+B][-2pA - B + 2pB] + \log[A][-A + 2pA + B] + \log[B][A - 2pB]$$

Recall that $A+B=\frac{1}{2}$ and taking common factors

$h=(0.693147\ldots)[-2pA+B(-1+2p)]-\log[A][A(-1+2p)+B]-\log[B][A-2pB]$

To find the Hausdorff measure we use the formula

$$\mathrm{HD}=\frac{h}{\gamma}\,.$$

Plug the formulas that we have

$$\mathrm{HD}=\frac{(0.693147\ldots)[-2pA+B(-1+2p)]-\log[A][A(-1+2p)+B]-\log[B][A-2pB]}{\frac{-1}{4}+\frac{1}{4}\sqrt{16p^2-8p+5}}$$

Second case

Now we want to consider this neighbour random walk with another symmetry. Namely, we have the probability measure μ defined by

$$\mu(a)=\mu(a^2)=p\ ,\mu(b)=\mu(b^2)=\frac{1}{2}-p \text{ where } \ p\in(0,\frac{1}{2}).$$

Solving the Stationary Traffic Equations 3.6.2

$$\sum_{u*v=a}\mu(u)x(v)+x(a)\frac{|I|}{|I|-1}\sum_{u\in S\smallsetminus S_a}\mu(u^{-1})x(u)\,.$$

Because of symmetry we have:

$$A=r(a)=r(a^2)=p(2B)+(pA)+2(\frac{1}{2}-p)\left[\frac{B}{2A}A\right]\quad , \tag{4.2.9}$$

$$B=r(b)=r(b^2)=\left(\frac{1}{2}-p\right)(2A)+\left(\frac{1}{2}-p\right)B+2p\frac{A}{2B}B$$

From 4.2.9 and knowing that:

$$B=\frac{1-2A}{2} \tag{4.2.10}$$

we obtain:

$$0=2p\left(\frac{1-2A}{2}\right)+pA+\left(\frac{1}{2}-p\right)\left(\frac{1-2A}{2}\right)-A$$

Opening brackets:

$$0 = p - 2pA + pA + \left[\frac{1}{4} - \frac{1}{2}A - \frac{p}{2} + pA\right] - A$$

Collecting similar terms together:

$$0 = \frac{p}{2} + \frac{1}{4} + A\left(\frac{-3}{2}\right)$$

Thus:

$$A = \frac{2p+1}{6}$$

From 4.2.10 we find:

$$B = \frac{1}{2} - \frac{2p+1}{6} = \frac{1-p}{3}$$

Hence, according to Lemma 3.6.5 and the following note that this measure is a shift-invariant on top of being μ-invariant. Now, we want to calculate the drift:

$$\gamma = 2p[-A + 2B] + [1 - 2p][-B + 2A]$$

Opening brackets:

$$\gamma = -2pA + 4pB - B + 2A + 2pB - 4pA$$

Collecting similar terms together:

$$\gamma = A[-2p + 2 - 4p] + B[4p - 1 + 2p]$$

Simplifying

$$\gamma = A[-6p + 2] + B[6p - 1]$$

Putting the values of A and B, we got:

$$\gamma = \left(\frac{2p+1}{6}\right)(-6p+2) + \left(\frac{1-p}{3}\right)(6p-1)$$

Opening brackets:

$$\gamma = \frac{-12p^2 + 4p - 6p + 2}{6} + \frac{6p - 1 - 6p^2 + p}{3}$$

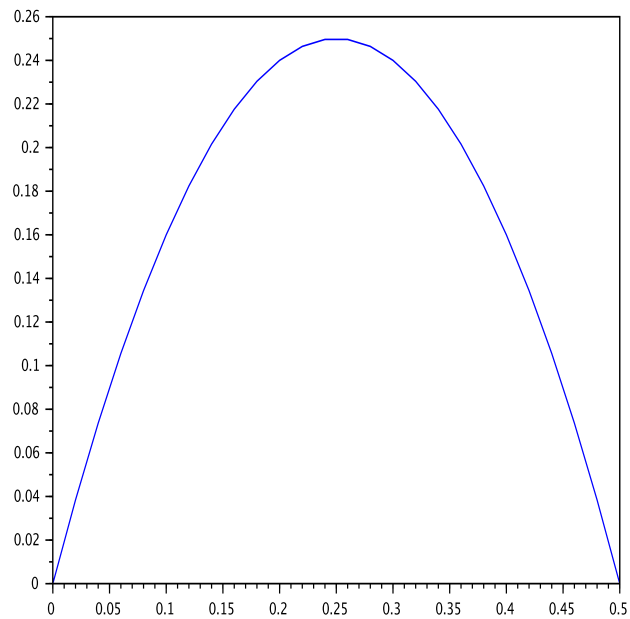

Figure 4.5: The drift of the free product of the cyclic groups $\mathbb{Z}/3\mathbb{Z} * \mathbb{Z}/3\mathbb{Z}$ (the second case)

, the X axis represents p and the Y axis represents the entropy.

Simplifying:

$$\gamma = \frac{-24p^2 + 12p}{6}$$

Thus we have:

$$\gamma = -4p^2 + 2p \; ,$$

see figure 4.5.

In the subcase:$p = \frac{1}{4}$

$$A = B = \gamma = \frac{1}{4}$$

The entropy can be found using (3.7.2): Set $q(s) = \frac{r(s)}{r(S \setminus S_s)}$

$$h = -\sum_{s \in S} \mu(s) \left[\log\left[\frac{1}{q(s^{-1})}\right] r(s^{-1}) + \sum_{g \in S_s \setminus s^{-1}} \log\left[\frac{q(sg)}{q(g)}\right] r(g) + \log[q(s)] \sum_{g \in S \setminus S_s} r(g) \right]$$

Recall that

$$\mu(a) = \mu(a^2) = p \ , \mu(b) = \mu(b^2) = \frac{1}{2} - p \text{ where } \ p \in (0, \frac{1}{2}).$$

We can find

$$q(a) = q(a^2) = \frac{A}{2B}$$

Also,

$$q(b) = q(b^2) = \frac{B}{2A}$$

The second summent in the right-hand side equals 0 for all alphabets in this example because:

$$g \in S_s \setminus s^{-1} = s$$

This leads to

$$\log \frac{q(gs)}{q(g)} = \log \frac{q(s^{-1})}{q(s)}$$

For example:

In case $s = a$ this gives us

$$\log \left[\frac{A}{2B} \times \frac{2B}{A} \right] = 0$$

The entropy then can be found by plugging the previous findings:

$$-h = 2p \left[\log \left[\frac{2B}{A} \right] A + \log[A] 2B \right]$$

$$+(1-2p) \left[\log \left[\frac{2A}{B} \right] B + \log \left[\frac{B}{2A} \right] [2A] \right]$$

Opening brackets

$$-h = \log \left[\frac{2B}{A} \right] [2pA] + \log[A][4pB] + \log \left[\frac{2A}{B} \right] [B - 2pB] + \log \left[\frac{B}{2A} \right] [2A - 4pA]$$

Collecting similar logarithms together

$$-h = \log[A][-2pA + 4pB] + \log[2A][B - 2pB - 2A + 4pA]$$

$$+ \log[B][2A - 4pA - B + 2pB] + \log[2B][2pA]$$

Simplifying

$$-h = \log[A][4pB - 2pA] + \log[2A] \left[A(-2 + 4p) + B(1 - 2p) \right]$$

$$+\log[B]\,[A(2-4p)+B(2p-1)]+\log[2B][2pA]$$

Collecting similar terms together

$$-h = A\,[-2p\log[A]+(-2+4p)\log[2A]+(2-4p)\log[B]+2p\log[2B]]$$

$$+B\,[4p\log[A]+(1-2p)\log[2A]+(-1+2p)\log[B]]$$

Taking common factors

$$-h = 2A\,[p(-\log[A]+\log[2B]+(-1+2p)\,(\log[2A]-\log[B])]$$

$$+B\,[4p\log[A]+(1-2p)((\log[2A]-log[B])]$$

To find the Hausdorff measure we use the formula

$$\text{HD} = \frac{h}{\gamma}\ .$$

Plug the formulas that we have

$$\text{HD} = \frac{A\,[p(-\log[A]+\log[2B]+(-1+2p)\,(\log[2A]-\log[B])]}{2p^2-p}$$

$$+\frac{B\,[4p\log[A]+(1-2p)((\log[2A]-log[B])]}{4p^2-2p}$$

4.3 Examples of random walks on the free product of three cyclic groups

4.3.1 Random walks on $\mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z}$

Consider a nearest neighbour random walk on the free product of three cyclic groups

$$G = \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z} .$$

Let us denote these groups by G_1, G_2, G_3, respectively, and let a, b, c be the generators of the respective groups G_1, G_2, G_3, see figure 4.6. We shall consider a probability distribution μ on this generating set defined by

$$\mu(a) = \mu(b) = p \quad \text{and} \quad \mu(c) = 1 - 2p \text{ where } \ p \in (0, \frac{1}{2}) .$$

Recall the Stationary Traffic Equations (3.6.2)

$$r(a) = \mu(a)\frac{|I|-1}{|I|} + \sum_{u*v=a} \mu(u)x(v) + r(a)\frac{|I|}{|I|-1} \sum_{u \in S \setminus S_a} \mu(u^{-1})x(u) .$$

Set $A = r(a) = r(b) \quad \text{and} \quad B = r(c)$.

Solving the stationary equation give us:

$$A = p\left(\frac{2}{3}\right) + A\left(\frac{3}{2}\right)[pA + (1-2p)B] .$$

$$B = (1-2p)\left(\frac{2}{3}\right) + B\left(\frac{3}{2}\right)[2pA] \tag{4.3.1}$$

Recall that $\sum_{a \in S} r(a) = 1$, this means

$$A = \frac{1-B}{2} \tag{4.3.2}$$

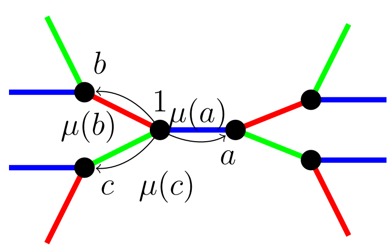

Figure 4.6: Random walk on $\mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z}$

Using (4.3.2) in (4.3.1) we find

$$0 = (1-2p)\left(\frac{2}{3}\right) + 3Bp\left[\frac{1-B}{2}\right] - B$$

Opening brackets

$$0 = (1-2p)\left(\frac{2}{3}\right) + \left(\frac{3}{2}Bp\right) - \left(\frac{3}{2}B^2p\right) - B$$

Collecting similar terms together

$$0 = (1-2p)\left(\frac{2}{3}\right) + B\left[\frac{3}{2}p - 1\right] - \frac{3}{2}B^2p$$

It is a second-order equation

$$a = \frac{-3}{2}p \ \ , b = \frac{3}{2}p - 1 \ \text{ and } c = \frac{2}{3}(1-2p) \ \ .$$

The solutions are

$$B = \frac{\left(\left(\frac{-3}{2}\right)p + 1\right) \pm \sqrt{\left(\frac{3}{2}p - 1\right)^2 + 2(3p)\frac{2}{3}(1-2p)}}{-3p}$$

$$B = \frac{\left(\left(\frac{-3}{2}\right)p + 1\right) \pm \sqrt{\frac{9}{4}p^2 + 1 - 3p + 4p(1-2p)}}{-3p}$$

Collecting similar terms together

$$B = \frac{\left(\left(\frac{-3}{2}\right)p + 1\right) - \sqrt{\frac{-23}{4}p^2 + 1 + p}}{-3p}$$

Note that, we choose the solution with (-) sign because the solution with (+) sign gives us a negative value. To find A we use the formula $A = \frac{1-B}{2}$

$$A = \frac{1}{2} + \frac{\left(\left(\frac{-3}{2}\right)p + 1\right) - \sqrt{\frac{-23}{4}p^2 + 1 + p}}{6p}$$

Adding the two fractions

$$A = \frac{\left(3p - \left(\frac{3}{2}\right)p + 1\right) - \sqrt{\frac{-23}{4}p^2 + 1 + p}}{6p}$$

Collecting similar terms

$$A = \frac{\left(\left(\frac{3}{2}\right)p + 1\right) - \sqrt{\frac{-23}{4}p^2 + 1 + p}}{6p}$$

Hence, according to Lemma 3.6.5 and the following note that this measure is a shift-invariant on top of being μ-invariant.

To find the drift, we use the formula 3.7.1

$$\gamma = 2p[-A + B] + (1 - 2p)[-B + 2A]$$

Opening brackets

$$\gamma = -2pA + 2pB - B + 2A + 2pB - 4pA$$

Collecting similar terms together

$$\gamma = A[2 - 6p] + B[4p - 1]$$

Plug the formulas for A and B

$$\gamma = (2-6p)\left[\frac{\left(\left(\frac{3}{2}\right)p + 1\right) - \sqrt{\frac{-23}{4}p^2 + 1 + p}}{6p}\right] + (4p-1)\left[\frac{\left(\left(\frac{-3}{2}\right)p + 1\right) - \sqrt{\frac{-23}{4}p^2 + 1 + p}}{-3p}\right]$$

Opening brackets and multiplying the first summand in the right-hand side by 2 to equalize the denominators

$$\gamma = \left[\frac{(6p + 4 - 18p^2 - 12p) - (4 - 12p)\sqrt{\frac{-23}{4}p^2 + 1 + p}}{3p}\right]$$

$$+ \left[\frac{\left(6p^2 - 4p - \left(\frac{3}{2}p\right) + 1\right) + (4p - 1)\sqrt{\frac{-23}{4}p^2 + 1 + p}}{3p}\right]$$

Collecting similar terms together

$$\gamma = \frac{\left(-12p^2 - \left(\frac{23}{2}p\right) + 5\right) + (16p - 5)\sqrt{\frac{-23}{4}p^2 + 1 + p}}{3p}$$

Simplifying, we obtain

$$\gamma = \left[-4p - \frac{23}{6} + \frac{5}{3p}\right] + \left[\frac{16}{3} - \frac{5}{3p}\right]\sqrt{\frac{-23}{4}p^2 + 1 + p}\,,$$

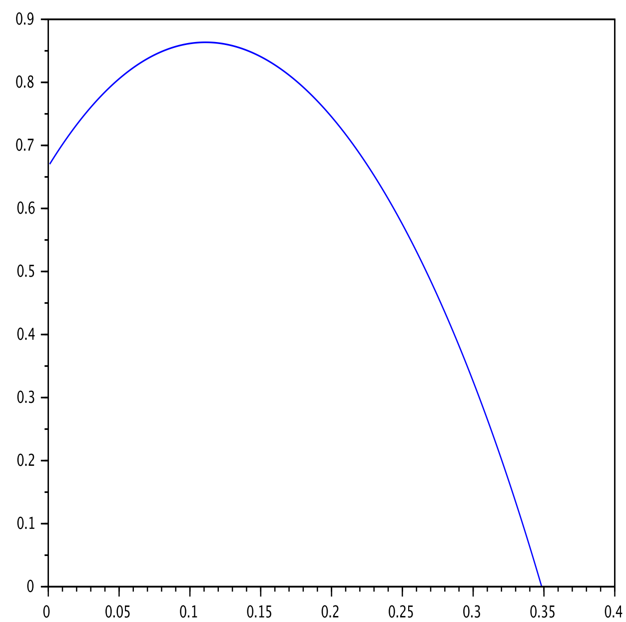

Figure 4.7: The drift of the free product of the cyclic groups $\mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2Z * \mathbb{Z}/2Z$, the X axis represents p and the Y axis represents the entropy.

see figure 4.7.

To calculate the asymptotic entropy for the probability measure μ, we use the formula (3.7.2) Recall that

$$\mu(a) = \mu(b) = p \quad \text{and} \quad \mu(c) = 1 - 2p \ , \text{where} \ p \in (0, \frac{1}{2}) \ .$$

We can find that

$$q(a) = q(b) = \frac{A}{A+B} \quad \text{and} \ \ q(c) = \frac{B}{2A}$$

Then, the entropy is calculated as follows

$$-h = 2p \left[\log \left[\frac{A+B}{A} \right] A + \log \left[\frac{A}{A+B} \right] [A+B] \right]$$
$$+(1-2p) \left[\log \left[\frac{2A}{B} \right] B + \log \left[\frac{B}{2A} \right] [2A] \right]$$

Using logarithms ruls and collecting similar terms together, we have

$$-h = 2p[\log[A+B][A-A-B]+\log[A][A+B-A]]+(1-2p)[\log[2A][B-2A]+\log[B][-B+2A]]$$

Opening brackets

$$-h = [\log[A+B][-2pB]+\log[A][2pB]]+[\log[2A][B-2A-2pB+4pA]+\log[B][-B+2A+2pB-4pA]]$$

Collecting similar terms together, we obtain

$$-h = A \left[\log[2A](-2+4p) + \log[B](2-4p) \right]$$
$$+B \left[\log[A+B](-2p) + \log[A](2p) + \log[2A](1-2p) + \log[B](-1+2p) \right]$$

Simplifying, we get

$$h = A(2-4p) \left[\log[2A] - \log[B] \right]$$
$$+B \left[(2p) \left[\log[A+B] - \log[A] \right] + (-1+2p) \left[\log[2A] - \log[B] \right] \right]$$

To find the Hausdorff measure we use the formula 3.7.3

$$\mathrm{HD} = \frac{h}{\gamma} \ .$$

Plug the formulas that we have

$$\mathrm{HD} = \frac{A(2-4p) \left[\log[2A] - \log[B] \right] + B \left[(2p) \left[\log[A+B] - \log[A] \right] + (-1+2p) \left[\log[2A] - \log[B] \right] \right]}{\left[-4p - \frac{23}{6} + \frac{5}{3p} \right] + \left[\frac{16}{3} - \frac{5}{3p} \right] \sqrt{\frac{-23}{4} p^2 + 1 + p}}$$

4.3.2 Random walks on $\mathbb{Z}/3\mathbb{Z} * \mathbb{Z}/3\mathbb{Z} * \mathbb{Z}/3\mathbb{Z}$

Consider a nearest neighbour random walk on the free product of three cyclic groups

$$G = \mathbb{Z}/3\mathbb{Z} * \mathbb{Z}/3\mathbb{Z} * \mathbb{Z}/3\mathbb{Z} .$$

Let us denote these groups by G_1, G_2, G_3, respectively, and let a, b, c be the generators of the respective groups G_1, G_2, G_3, see figure 4.8. We shall consider a probability distribution μ on this generating set defined by

$$\mu(a) = \mu(b) = \mu(c) = p \quad \text{and} \quad \mu(a^2) = \mu(b^2) = \mu(c^2) = \frac{1-3p}{3} \text{ where } p \in (0, \frac{1}{3}) .$$

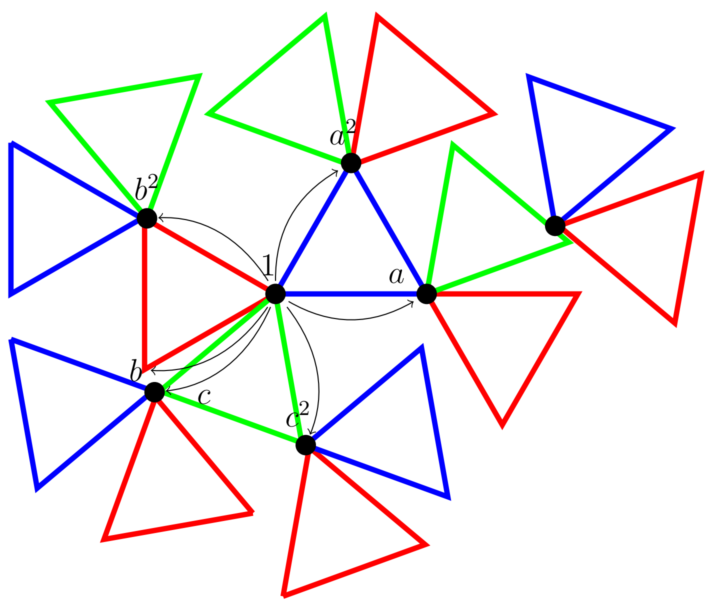

Figure 4.8: A nearest neighbour random walk on $\mathbb{Z}/3\mathbb{Z} * \mathbb{Z}/3\mathbb{Z} * \mathbb{Z}/3\mathbb{Z}$

Recall the Stationary Traffic Equations (3.6.2)

$$x(a) = \mu(a)\frac{|I|-1}{|I|} + \sum_{u*v=a} \mu(u)x(v) + x(a)\frac{|I|}{|I|-1} \sum_{u\in S\setminus S_a} \mu(u^{-1})x(u) .$$

Solving this equation give us:

$$A = r(a) = r(b) = r(c) = \frac{2}{3}p + \left[\frac{1-3p}{3}B\right] + 2\binom{3}{2} A \left[A\frac{1-3p}{3} + pB\right] .$$

$$A = \frac{2}{3}p + \left[\frac{1-3p}{3}B\right] + A^2(1-3p) + 3pAB . \tag{4.3.3}$$

$$B = \left[\left(\frac{1-3p}{3}\right)\left(\frac{2}{3}\right)\right] + pA + 2\left(\frac{3}{2}\right)B\left[A\left(\frac{1-3p}{3}\right) + pB\right] .$$

Since:

$$3A + 3B = 1 ,$$

This means:

$$B = \frac{1-3A}{3} \tag{4.3.4}$$

Using(4.3.4) in (4.3.3) we get:

$$A = \frac{2}{3}p + \left[\left(\frac{1-3p}{3}\right)\left(\frac{1-3A}{3}\right)\right] + A^2(1-3p) + 3pA\left[\frac{1-3A}{A}\right] .$$

Opening brackets and multiplying we obtain:

$$A = \frac{2}{3}p + \left[\frac{1}{9} - \frac{p}{3} - \frac{A}{3} + pA\right] + A^2(1-3p) + pA - 3pA^2 .$$

Moving everything to the right-hand side and grouping similar terms together:

$$0 = \left[\frac{1}{3}p + \frac{1}{9}\right] + A\left[2p - \frac{4}{3}\right] + A^2(1-6p) .$$

It is a second order equation which can be solved using the quadratic formula. Here we have:

$$a = 1-6p \quad b = \left[2p - \frac{4}{3}\right] \text{and} c = \left[\frac{p}{3} + \frac{1}{9}\right] .$$

The solutions are:

$$A = \frac{\frac{4}{3} - 2p \pm \sqrt{\left[2p - \frac{4}{3}\right]^2 - 4\left[(1-6p)\left(\frac{p}{3} + \frac{1}{9}\right)\right]}}{2-12p} .$$

Opening brackets:

$$A = \frac{\frac{4}{3} - 2p \pm \sqrt{\left[4p^2 + \frac{16}{9} - \frac{16}{3}p\right] + \left[\frac{-4}{3}p + 8p^2 - \frac{4}{9} + \frac{24}{9}p\right]}}{2-12p} .$$

Grouping similar terms together:

$$A = \frac{\frac{4}{3} - 2p \pm \sqrt{12p^2 + \frac{4}{3} - 4p}}{2-12p} .$$

Dividing both the numerator and denominator by 2 we obtain:

$$A = \frac{\frac{2}{3} - p \pm \sqrt{3p^2 + \frac{1}{3} - p}}{1 - 6p} \,. \tag{4.3.5}$$

We should choose the solution with the (-) sign, because the solution with (+) gives $A > 0$ for $0 < p < \frac{1}{6}$ and gives $A < 0$ for $\frac{1}{6} < p < \frac{1}{3}$. Using(4.3.4) in (4.3.5) we get:

$$B = \frac{1}{3} - \left[\frac{\frac{2}{3} - p - \sqrt{3p^2 + \frac{1}{3} - p}}{1 - 6p} \right] \,.$$

$$B = \frac{1 - 6p - 2 + 3p + 3\sqrt{3p^2 - p + \frac{1}{3}}}{3 - 18p} \,.$$

Grouping similar terms together:

$$B = \frac{-1 - 3p + 3\sqrt{3p^2 - p + \frac{1}{3}}}{3 - 18p} \,.$$

Hence, according to Lemma 3.6.5 and its following note, this measure is a shift-invariant on top of being μ-invariant. The drift can be found using (3.7.1):

$$\gamma = \sum_{a \in S} \mu(a) \left[-r(a^{-1}) + \sum_{b \in S \setminus S_a} r(b) \right]$$

After plugging the formulas, we obtain

$$\gamma = 3p[-B + 2A + 2B] + (1 - 3p)[-A + 2A + 2B]$$

Simplifying, we have

$$\gamma = 3p[2A + B] + (1 - 3p)[2B + A]$$

Upon grouping similar terms together, we get

$$\gamma = A[6p + 1 - 3p] + B[3p + 2 - 6p]$$

$$\gamma = A(3p + 1) + B(-3p + 2)$$

$$\gamma = (3p + 1) \left[\frac{\frac{2}{3} - p \pm \sqrt{3p^2 + \frac{1}{3} - p}}{1 - 6p} \right] + (-3p + 2) \left[\frac{-1 - 3p + 3\sqrt{3p^2 - p + \frac{1}{3}}}{3 - 18p} \right]$$

After opening brackets, we obtain

$$\gamma = \left[\frac{\frac{2}{3} - p + 2p - 3p^2 - (3p+1)\sqrt{3p^2 + \frac{1}{3} - p}}{1-6p}\right] + \left[\frac{3p + 9p^2 - 2 - 6p + (6-9p)\sqrt{3p^2 - p + \frac{1}{3}}}{3-18p}\right]$$

Equalizing denominators:

$$\gamma = \left[\frac{-9p^2 + 2 + 3p - (9p+3)\sqrt{3p^2 + \frac{1}{3} - p}}{1-6p}\right] + \left[\frac{9p^2 - 2 - 3p + (6-9p)\sqrt{3p^2 - p + \frac{1}{3}}}{3-18p}\right]$$

Collecting similar terms together:

$$\gamma = \frac{(-18p+3)\sqrt{3p^2 + \frac{1}{3} - p}}{3-18p}$$

$$\gamma = \sqrt{3p^2 + \frac{1}{3} - p}\ ,$$

see figure 4.9. The entropy can be found using (3.7.2): Set $q(a) = \frac{r(a)}{r(S \smallsetminus S_a)}$

$$h = -\sum_{a\in S} \mu(a) \left[\log\left[\frac{1}{q(a^{-1})}\right] r(a^{-1}) + \sum_{b \in S \smallsetminus a^{-1}} \log\left[\frac{q(ab)}{q(b)}\right] r(b) + \log[q(a)] \sum_{b\in S\smallsetminus S_a} r(b)\right]$$

Recall that

$$\mu(a) = \mu(b) = \mu(c) = p \quad \text{and} \quad \mu(a^2) = \mu(b^2) = \mu(c^2) = \frac{1-3p}{3}\ .$$

To find $q(a^{-1})$

$$q(a^{-1}) = \frac{r(a^2)}{2A+2B} = \frac{B}{2A+2B} = q(b^{-1}) = q(c^{-1})$$

$$q\left((a^2)^{-1}\right) = q(a) = \frac{r(a)}{r(S \smallsetminus S_a)} = q\left((b^2)^{-1}\right) = q\left((c^2)^{-1}\right)$$

Note that in our example:

$$b \in (S_a \smallsetminus a^{-1}) = a, \ \text{ for all } \ a \in S\ ,$$

because we just have the letter and its inverse in each generating set S_a.

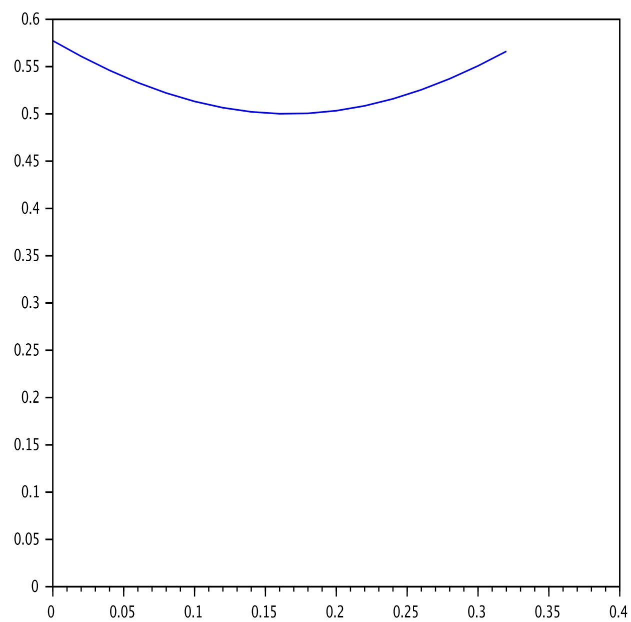

Figure 4.9: The drift of the free product of the cyclic groups $\mathbb{Z}/3\mathbb{Z} * \mathbb{Z}/3Z * \mathbb{Z}/3Z$, the X axis represents p and the Y axis represents the entropy.

For a, b, c

$$\frac{q(ab)}{q(b)} = \frac{q(a^2)}{q(a)} = \frac{B}{2A+2B} \times \frac{2A+2B}{A} = \frac{B}{A}\ .$$

In the same way, for a^2, b^2, c^2

$$\frac{q(a^2b)}{q(b)} = \frac{A}{2A+2B} \times \frac{2A+2B}{B} = \frac{A}{B}$$

We plug these formulas into the equation (3.7.2):

$$-h = 3p\left[\left[\log\frac{2A+2B}{B}\right]B + \log\left[\frac{B}{A}\right]A + \log\left[\frac{A}{2A+2B}\right](2A+2B)\right]$$

$$+(1-3p)\left[\log\left[\frac{2A+2B}{A}\right]A + \log\left[\frac{A}{B}\right]B + log\left[\frac{B}{2A+2B}\right](2A+2B)\right]\ .$$

Collecting similar logarithms:

$$-h = 3p\left[\log[2A+2B](B-2A-2B) + \log[A](2A+2B-A) + \log[B](A-B)\right]$$

$$+(1-3p)\left[\log[2A+2B](A-2A-2B) + \log[A](B-A) + \log[B](2A+2B-B)\right]$$

Opening brackets:

$$-h = \log[2A+2B](-6pA-3pB) + \log[A](3pA+6pB) + \log[B](3pA-3pB)$$

$$+\log[2A+2B](-A+3pA-2B+6pB) + \log[A](B-3pB-A+3pA] + \log[B](2A-6pA+B-3pB)$$

Collecting similar logarithms:

$$-h = \log[2A+2B](-A+3pA-2B+6pB-6pA-3pB)$$

$$+\log[A](3pA+6pB+B-3pB-A+3pA)$$

$$+\log[B](3pA-3pB+2A-6pA+B-3pB)$$

After simplification:

$$-h = \log[2A+2B](-A-3pA-2B+3pB) + \log[A](6pA+3pB-A+B) + \log[B](-3pA-6pB+B+2A)$$

Recall that:$2A+2B = \frac{2}{3}$;

Taking common factors, we obtain

$$-h = [-.41](A(-1-3p)+B(3p-2)) + \log[A](A(6p-1)+B(3p+1)) + \log[B](A(2-3p)+B(1-6p))$$

$$h = [.41](A(-1-3p)+B(3p-2)) - \log[A](A(6p-1)+B(3p+1)) - \log[B](A(2-3p)+B(1-6p))$$

Collecting similar terms together

$$h = A\,[[.41](-1-3p) - (6p-1)\log[A] - \log[B](2-3p)]$$
$$+B\,[[.41](3p-2) - \log[A](3p+1) - \log[B](1-6p)]$$

To find the Hausdorff measure we use the formula

$$\mathrm{HD} = \frac{h}{\gamma}\ .$$

$$\mathrm{HD} = \frac{A\,[[.41](-1-3p) - (6p-1)\log[A] - \log[B](2-3p)]}{\sqrt{3p^2+\frac{1}{3}-p}}$$
$$+\frac{B\,[[.41](3p-2) - \log[A](3p+1) - \log[B](1-6p)]}{\sqrt{3p^2+\frac{1}{3}-p}}$$

Appendix A

List of symbols

N	The set of non-negative integers.
N^*	The set of positive integers.
G	A group.
g	A generator of a cyclic group.
(V, E)	A graph.
V	A set of vertices.
E	A set of edges (u, v).
Γ	A diagraph.
X	A Cayley graph.
S^*	A free monoid.
$*$	The group multiplication.
$F[S]$	A free group.
Φ	A function.
$\iota : S \to I$	A map from a letter to a group number.
L	A word.
$\circ$	A binary operation.
$*$	A group law (concatenation).
d	A distance function.
T_M	A homogeneus tree of order M.
$\|\| m \|\|$	A total mass function.
$p^{(n)}$	n-step transition probbility.

deg	The degree of a vertex means the number of its neighbours.
$y \sim x$	There is an edge between x and y.
$supp$	The support of the measure.
μ	A probability measure.
S	A generators set.
$p(a, b)$	The transition probability from a to b
$p^{(n)}$	The n-step transition probability.
1	The identity element of a group G.
(G, μ)	A random walk.
$\|\| m \|\|$	A total mass function.
$T \subset S * S$	The admissible transitions.
$F \subset S * S$	The forbidden transitions.
θ	A Markovian Multiplicative base.
$r(a)$	A solution to a Traffic Equation.
λ	An initial distribution of a Markov chain.
I	The number of groups.
γ	The drift.
h	The asymptotic entropy.
H	Shannon entropy.
HD_μ	The Hausdorff dimension of the probability measure μ.
$\| . \|$	The word length.

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