Ratio set of boundary actions

Tianyi Zhou

Thesis submitted to the University of Ottawa
in partial Fulfillment of the requirements for the
Mathematics & Statistics - Master Degree

Mathematics & Statistics Program
Science Faculty
University of Ottawa

Abstract

Given an action of a countable group with a quasi-invariant measure, there exists a multiplicative group in $(0,\infty)$, called the **ratio set of the group action**, which in a sense describes the values of the Radon-Nikodym derivative. The main purpose of this thesis is to find the ratio set of the action of a finitely generated free group $\mathcal{F}$ on its topological boundary $\partial\mathcal{F}$ (the set of infinite words) for a certain natural class of quasi-invariant boundary measures.

In **Section 1**, we focus on the general ergodic theory of equivalence relations. We outline the set-up, borrow from **[1], [4]** the definitions of the central notions of the theory, including counting measures (**Proposition 1.8**), quasi-invariance (**Definition 1.6**), Radon-Nikodym cocycle (**Definition 1.15**) and raio set (**Definition 1.19**), and illustrate them on the example of the orbit equivalence relation of a Markov shift (**Definition 1.22**). We also introduce the principal object: the boundary action of a finitely generated free group (see **Section 1.2**).

In **Section 2**, we define the class of multiplicative Markov measures (**Definition 2.1**). These are the measures on a topological Markov chain entirely determined just by an initial (base) distribution and the admissibility matrix; the transition probabilities are then just the normalized restrictions of the base distribution onto the set of admissible transitions (see **[7]**). In the case of the free group, its boundary has a natural structure of a topological Markov chain (determined by the irreducibility condition from the definition of a free group: consecutive letters should not cancel each other), and in this case, we show that the multiplicative Markov measures are precisely the ones for which the Radon-Nikodym cocycle is a product cocycle (i.e. a cocycle whose potential only depends on the first letter of the input; see **Definition 2.8**). The final result of this section is an explicit description of the ratio set of the boundary action with respect to multiplicative Markov measures.

In **Section 3**, given a probability measure μ on the set of free generators and their inverses, the definition of the associated **nearest neighbor random walk** is given. According to **Furstenberg's Theorem** (proof provided in **Appendix**), in this random walk, sample paths converge almost surely to a random boundary point, and the resulting limit distribution on the boundary of the free group is called the **harmonic measure of the random walk** (see **Section 3.1**). We show that the harmonic measure is a multiplicative measure (**Theorem 3.3**), and therefore the results of **Section 2** allow us to describe the ratio set of the harmonic measure (**Theorem 3.5**). A significant role in these considerations is played by the passage probabilities of the random walk (given a group element, the probability that it is ever visited by a random walk). Since the harmonic measure is multiplicative, its potential only depends on the first letter, and this dependence actually amounts to taking the inverse of the corresponding passage probability (**Proposition 2.9, Remark 2.10**). Finally, we establish a one-to-one correspondence between three families of numbers indexed by the alphabet of the free group and subject to natural conditions; these are the step distributions of the random walk, the base of the harmonic measure (which is multiplicative Markov) and the family of passage probabilities (**Theorem 3.6**).

In **Section 4**, we discuss another method for finding the ratio set of the harmonic measure based on using Martin theory (see **[2]**).

In the **Appendix**, we prove **Furstenberg's Theorem**, a result used for defining the harmonic measure in **Section 3**. Actually, it is applicable not only for the nearest neighbor random walk (i.e. not only when the probability measure μ is supported on the alphabet set) but also the more general case where the support of the step distribution generates the free group. Moreover, in addition to the existence it also characterizes the harmonic measure as the unique μ-stationary measure on the boundary.

Acknowledgement

I would like to especially thank my supervisor, Dr. Vadim Kaimanovich, for his recognition, patience and support in the last two years. It is always one of my greatest pleasures to study with you. I would also like to thank Dr. Charles Starling and Dr. Thierry Giordano for their willingness to be the examiners, and advice to the final draft of this thesis.

Contents

1 Ratio set of a general Markov shift

1.1 Countable equivalence eelations

Definition 1.1. Let $(X, \mathcal{B})$ be a standard Borel space and suppose R is an equivalence relation defined on X such that for each $x \in X$, the equivalence class of x is countable and R as a subset of $X \times X$ belongs to $\mathcal{B} \times \mathcal{B}$. In this case, we call R a **countable measurable equivalence relation with respect to** $\mathcal{B}$. For each $x \in X$ we use $[x]$ to denote the **equivalence class** of x, and for each $B \in \mathcal{B}$, we use $R(B) = \bigcup_{b \in B}[b]$ to denote the R-**saturation** of the set B.

In particular, if a group G is acting a space X, then the associated **orbit equivalence relation** is defined as: given $x, y \in X$, $x \sim y$ iff $g \cdot x = y$ for some $g \in G$.

Proposition 1.2. *Given a standard Borel space $(X, \mathcal{B})$ and a countable group G acting on X by measurable transformations, the associated orbit equivalence (denoted by R_G) is a measurable equivalence relation in the sense of **Definition 1.1**.*

Proof. In $X \times X$, given $h \in G$, since the mapping $x \mapsto h \cdot x$ is measurable, the graph of this mapping:

$$\Gamma(h) = \{(x, h \cdot x) \mid x \in X\}$$

is then measurable with respect to $\mathcal{B} \times \mathcal{B}$. We then have:

$$R_G = \bigcup_{h \in G} \Gamma(h).$$

The right hand side is a countable union of $\mathcal{B} \times \mathcal{B}$-measurable sets, and hence is $\mathcal{B} \times \mathcal{B}$-measurable.

□

In a dynamical system $(X, \mathcal{B}, T)$ where X is a T_1 and $\mathcal{B}$ is the Borel σ-algebra of X, if T is not necessarily invertible, then the orbit equivalence relation will be defined as: $x \sim y$ iff there is $k, l \in \mathbb{N}$ such that $T^k x = T^l y$; if T is invertible, then the k, l can be picked from $\mathbb{Z}$. This equivalence relation will be denoted by R_T. To make R_T a discrete measurable equivalence relation, we want T to be continuous with respect to the given topology in X (therefore measurable with respect to $\mathcal{B}$) and $T^{-1}(\{x\})$ countable for each $x \in X$.

Proposition 1.3. *Given a Polish space X endowed with the Borel σ-algebra $\mathcal{B}$, and a continuous map $T : X \to X$ such that $T^{-1}\big(\{x\}\big)$ is countable for each $x \in X$, the orbit equivalence relation R_T is a countable measurable equivalence relation in the sense of **Definition 1.1***

Proof. Because X is a Polish space, the topology X is equipped with is equivalent to a complete separable metric topology. Therefore every singleton will be closed. Because T is continuous and $T^{-1}(\{x\})$ countable for each $x \in X$, we have that for each $n \in \mathbb{N} \cup \{0\}$, T^n is continuous and $(T^n)^{-1}(\{x\})$ countable for each $x \in X$. Hence, for each $x \in X$ and $n, m \in \mathbb{N} \cup \{0\}$, the equivalence class:

$$[x] = \bigcup_{n,m \in \mathbb{N} \cup \{0\}} \{y \in X \mid T^n(x) = T^m(y)\} = \bigcup_{n,m \in \mathbb{N} \cup \{0\}} (T^m)^{-1}\{T^n(x)\}$$

is a countable union of countable closed sets and hence countable and measurable. Now fix $n, m \in \mathbb{N}$, we have the following set:

$$\Big\{(x, y) \in X \times X \mid T^n(x) = T^m(y)\Big\}$$

as the pre-image of the diagonal in $X \times X$ under $T^n \times T^m$, is closed in the product topology (generated by the given topology in X). Therefore:

$$R = \bigcup_{n,m \in \mathbb{N} \cup \{0\}} \Big\{(x, y) \mid T^n(x) = T^m(y)\Big\}$$

is a countable union of closed sets and hence belongs to $\mathcal{B} \times \mathcal{B}$.

□

1.2 The topological Markov chain associated to the boundary of a free group

In this subsection, we shall illustrate the notions introduced in **Section 1.1** by looking at our principal object, which is the action of the free group on its topological boundary. Given a finite set of free generators and their inverses A, we will consider the dynamical system determined by the action of the free group $\mathcal{F}$ generated by A on the (topological) boundary $\partial\mathcal{F}$, then introduce the shift transformation on $\partial\mathcal{F}$, and our first example of **orbit equivalence** (see **Definition 1.4**).

Each $v \in \mathcal{F}$ can be written as, for some $m \in \mathbb{N}$:

$$v = v_1 v_2 v_3 \cdots v_m \qquad\qquad v_i \in A \quad \text{and} \quad v_i v_{i+1} \neq e \quad \forall\, i \in \{1, 2, \cdots, m-1\}.$$

where e denotes the identity of $\mathcal{F}$. Here m is the length of v, denoted by $|\,v\,|$ and the above property:

$$v_i v_{i+1} \neq e \quad \forall\, i \in \{1, 2, \cdots, m-1\}$$

is called **irreducibility**. Then we use $\partial\mathcal{F}$ to denote the set of infinite (to the right) words where each word has the form $v = v_1 v_2 \cdots$ subject to the same irreducibility condition. We call $\partial\mathcal{F}$ the **boundary of** $\mathcal{F}$, and define $\overline{\mathcal{F}} = \mathcal{F} \cup \partial\mathcal{F}$. Clearly with respect to the δ-metric, $\partial\mathcal{F}$ is the topological boundary of $\overline{\mathcal{F}}$. One can also notice that the (directed) Cayley graph with A the generating set is a tree. Further, given $v \in \overline{\mathcal{F}}$ and $k \in \mathbb{N}$, we define the k**-truncation** of an element v as:

$$[v]_k = \begin{cases} v_1 v_2 \cdots v_k, & k < |\,v\,| \\ v, & k \geqslant |\,v\,| \end{cases}$$

Given $u, v \in \overline{\mathcal{F}}$, let $u \wedge v$ denote the **confluent** (the common part of u and v starting from the left) of u and v and $|\,u \wedge v\,|$ the length of it. Then define:

$$\delta : \overline{\mathcal{F}} \times \overline{\mathcal{F}} \to [0, \infty), \qquad (u, v) \mapsto \begin{cases} 0, & u = v \\ \exp(-|\,u \wedge v\,|), & u \neq v \end{cases}$$

Hence δ is a metric defined on $\overline{\mathcal{F}}$. We will then show the metric space $(\overline{\mathcal{F}}, \delta)$ is compact by showing that it is complete and totally bounded.

Given a Cauchy sequence $\{u_n\}_{n \in \mathbb{N}}$, we have that $\limsup_{n,m} \delta(u_n, u_m) = 0$. Hence, for each $k \in \mathbb{N}$, we can find $n_k \in \mathbb{N}$ so that $\delta(u_n, u_m) \leqslant e^{-k}$ for all $n, m \geqslant n_k$, which implies $[u_n]_k = [u_m]_k$ for all $n, m \geqslant n_k$. If we let $v_k = [u_{n_k}]_k$, then $\{v_k\}_{k \in \mathbb{N}}$ will converge to an element v where $[v]_k = v_k$ for each $k \in \mathbb{N}$. Hence, for each $k \in \mathbb{N}, \delta(v, u_n) \leqslant e^{-k}$ for each $k \geqslant n_k$ and this proves that $(\overline{\mathcal{F}}, \delta)$ is complete.

Then we will show that $\overline{\mathcal{F}}$ is totally bounded (i.e. in a metric space, for each $\epsilon > 0$, the space can be covered by finitely many open balls with radius ϵ). Then we will fix $\epsilon > 0$ and find $k \in \mathbb{N}$ so that $e^{-k} < \epsilon$. Let $\mathcal{B}_k$ be the finite set of all k-length words. Hence, each element in $\overline{\mathcal{F}}$ will be in $B(u, e^{-k})$, the closed ball centred at u with radius e^{-k} for some $u \in \mathcal{B}_k$. For each $r > 0$ and $x \in \overline{\mathcal{F}}$, we will use $B(x, r)$ to denote the closed ball in $\overline{\mathcal{F}}$. Hence $(\overline{\mathcal{F}}, \delta)$ is compact and $(\partial\mathcal{F}, \delta)$, as a closed subspace, is therefore compact.

In $\partial\mathcal{F}$, given $g = g_1 \cdots g_n \in \mathcal{F}$, we define an **elementary cylinder set in** $\partial\mathcal{F}$ as the following:

$$C_g = \{v \in \partial\mathcal{F} \,|\, v_1 = g_1, \cdots, v_n = g_n\}$$

Also notice that for each $v \in C_g$, $B(v, e^{-n}) \cap \partial\mathcal{F} = C_g$. We use $\mathfrak{B}$ to denote the σ-algebra generated by elementary cylinder sets in $\partial\mathcal{F}$, which coincides with the Borel σ-algebra in $\partial\mathcal{F}$. The group action of $\mathcal{F}$ acting on $\partial\mathcal{F}$, namely the **boundary action**, is defined by the concatenation with subsequent cancellation (if necessary), and is continuous.

Next, we define the **shift transformation** T on $\partial\mathcal{F}$ as the following:

$$T : \partial\mathcal{F} \to \partial\mathcal{F}, \quad v \mapsto [v]_1^{-1} v$$

where $[v]_1$ is the first letter of the input element. T is continuous.

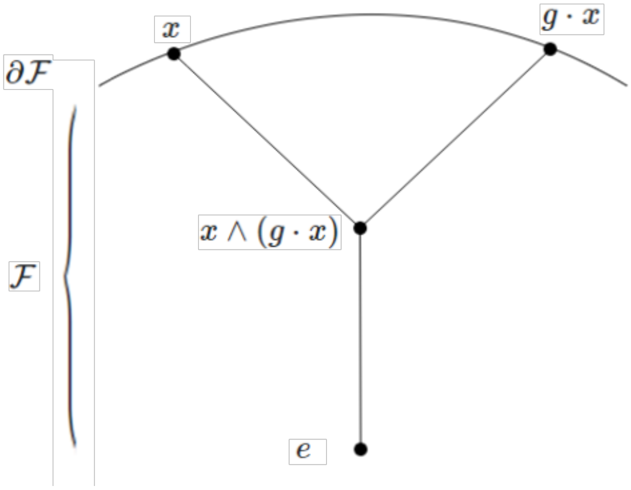
Definition 1.4. Given two measurable spaces, $(X, \mathcal{B}_X)$ and $(Y, \mathcal{B}_Y)$, each of which endowed with respective countable measurable equivalence relation R_X and R_Y, an **isomorphism** between these two relations is an invertible $\mathcal{B}_X - \mathcal{B}_Y$ measurable bijection which maps equivalence classes to equivalence classes. If R_X and R_Y are isomorphic relations of respective dynamical systems on X and Y, these two systems are said to be **orbit equivalent**.

Proposition 1.5. *The boundary action of the free group is orbit equivalent to the shift T.*

Proof. We denote by $R_{\mathcal{F}}$ the orbit equivalence relation of the boundary action and R_T the orbit equivalence of the shift T on $\partial\mathcal{F}$. We will first show that if x and y are $R_{\mathcal{F}}$-equivalent, then they are R_T-equivalent. In the sense of **Definition 1.1**, we want to show that, given an arbitrary $x \in \partial\mathcal{F}$ and $g \in \mathcal{F}$, x and $y = g \cdot x$ are R_T-equivalent. Recall that given $x, \neq y \in \partial\mathcal{F}$, we denote $x \wedge y$ by the (finite) common part of x and y, and $|\, x \wedge y\,|$ is the length of that. Set $u = x \wedge y$. Notice that g acting on x will remove the first $|\, u^{-1} \wedge x\,|$ letters of x and then add the first $|\,u\,| - |\,u^{-1} \wedge x\,|$ of u letters to what remains in x. Therefore:

$$T^{|\,u^{-1}\wedge x\,|}x = T^{|\,u\,|-|\,u^{-1}\wedge x\,|}y \quad \Longrightarrow \quad \forall\, x \in \partial\mathcal{F}\ \forall\, g \in \mathcal{F}, \quad x \overset{R_T}{\sim} y.$$

Visually, we have:



Now given $(x, y) \in R_T$, we can then find $n_1, n_2 \in \mathbb{N}$ such that $T^{n_1}x = T^{n_2}y = z$. Then by taking $g = x_1x_2\cdots x_{n_1} \in \mathcal{F}, h = y_1y_2\cdots y_{n_2} \in \mathcal{F}$, we have:

$$g^{-1} \cdot x = T^{n_1}x = T^{n_2}y = h^{-1} \cdot y \quad \Longrightarrow \quad y \overset{R_{\mathcal{F}}}{\sim} z.$$

□

1.3 Non-singularity and quasi-invariance

First we need the definitions of a **non-singular** (countable measurable) equivalence relation, the associated **flip transformation**, and of the **left** and **right coordinate projections**. Throughout this section, only finite measure states spaces are considered. **Definition 1.6, 1.7** and **Proposition 1.8** are from [4].

Definition 1.6. Given a Borel measure space $(X, \mathcal{B}, \mu)$ and a countable measurable equivalence relation R, we call μ **quasi-invariant for** R if:

$$\forall\, B \in \mathcal{B}, \quad \mu[R(B)] = 0 \iff \mu(B) = 0.$$

In this case, we call R **non-singular** with respect to μ.

Definition 1.7. Given a measurable space $(X, \mathcal{B})$, the **flip transformation** θ is defined as:

$$\theta : X \times X \to X \times X, \quad (x_1, x_2) \mapsto (x_2, x_1).$$

Observe that $\mathcal{B} \times \mathcal{B} = \{B_1 \times B_2 \mid B_1, B_2 \in \mathcal{B}\}$ is a σ-algebra in $X \times X$ and is generated by product of sets in $\mathcal{B}$. Hence, θ is measurable with respect to $\mathcal{B} \times \mathcal{B}$. Similarly, the **left projection**:

$$\pi_l : X \times X \to X \quad (x_1, x_2) \mapsto x_1$$

and the **right projection**:

$$\pi_r : X \times X \to X \quad (x_1, x_2) \mapsto x_2$$

are both measurable with respect to $\mathcal{B}_R = \{C \cap R \mid C \in \mathcal{B} \times \mathcal{B}\}$.

Proposition 1.8. *Given a standard Borel space $(X, \mathcal{B})$, a probability measure μ on it, and a countable measurable equivalence relation R,*

(i) for any $C \in \mathcal{B}_R$, the function:

$$X \to [0, \infty], \quad x \mapsto |\, \pi_l^{-1}(\{x\}) \cap C \,| \ \textbf{\textit{(the cardinality)}}$$

is measurable and:

$$\nu_l : \mathcal{B}_R \to [0, \infty], \quad C \mapsto \int_X |\, \pi_l^{-1}(\{x\}) \cap C \,|\, d\mu(x)$$

*is a σ-finite measure on $\mathcal{B}_R$. We call ν_l the **left counting measure** and in the same way we define the **right counting measure** as:*

$$\nu_r : \mathcal{B}_R \to [0, \infty], \quad C \mapsto \int_X |\, \pi_r^{-1}(\{x\}) \cap C \,|\, d\mu(x)$$

(ii) For each $C \in \mathcal{B}_R$, $\nu_l(C) = 0$ iff $\mu\big[\pi_l(C)\big] = 0$ and $\nu_r(C) = 0$ iff $\mu\big[\pi_r(C)\big] = 0$

For proving **Proposition 1.8**, we will use the following results from [**5.** §18]

Definition 1.9. Given two sets X, Y and $P \subseteq X \times Y$, a **uniformization of** P is a subset $P^* \subseteq P$ such that, given π_X the projection from $X \times Y$ onto X, we have $\pi_X(P) = \pi_X(P^*)$ and π_X restricted to P^* is bijective.

Theorem 1.10. *Let $(X, \mathcal{B}_X), (Y, \mathcal{B}_Y)$ be standard Borel spaces and let $P \subseteq X \times Y$ be $\mathcal{B}_X \times \mathcal{B}_Y$-measurable. If, for each $x \in X$, $P_x = \pi_l^{-1}(\{x\}) \cap P$ is countable, then P has a uniformation that is also $\mathcal{B}_X \times \mathcal{B}_Y$-measurable. Moreover, there exists a countable partition $\{E_n\}$ for P such that π_l is injective on each E_n.*

*Proof of **Proposition 1.8.*** According to **Theorem 1.10**, there exists a countable partition of R, denoted by $\{E_n\}$, such that π_l is injective on each E_n. Therefore, we will have $X = \pi_l(R) = \bigcup_{n \in \mathbb{N}} \pi_l(E_n)$. Hence we can define $f_n : \pi_l(E_n) \to E_n$ such that for each $x \in E_n$, $f_n(x)$ is the unique point in E_n such that $\pi_l\big[f_n(x)\big] = x$, and each f_n is then a $\mathcal{B} - (\mathcal{B} \times \mathcal{B})$ measurable bijection between $\pi_l(E_n)$ and E_n. Therefore, we have that for each $n \in \mathbb{N}$, f_n^{-1} is equal to π_l restricted to E_n. Hence, for each $C \in \mathcal{B}_R$, we have:

$$\pi_l(C) = \bigcup_{n \in \mathbb{N}} \pi_l(C \cap E_n) = \bigcup_{n \in \mathbb{N}} \pi_l\Big|_{E_n}(C \cap E_n) = \bigcup_{n \in \mathbb{N}} f_n^{-1}(C \cap E_n).$$

(i) Fix $C \in \mathcal{B}_R$ and notice that for each $x \in X$:

$$\pi_l^{-1}\big(\{x\}\big) \cap C \neq \varnothing \quad \Longleftrightarrow \quad x \in \pi_l(C). \tag{1}$$

Because $X = \bigcup_{n \in \mathbb{N}} \pi_l(E_n)$, with (1), we then have:

$$|\,\pi_l^{-1}\big(\{x\}\big) \cap C\,| = \sum_{n\in\mathbb{N}} \chi_C\big[f_n(x)\big]$$

and hence, according to **Dominance Convergence Theorem**:

$$\int_X |\,\pi_l^{-1}\big(\{x\}\big) \cap C\,|\, d\mu(x) = \int_X \sum_{n\in\mathbb{N}} \chi_C\big[f_n(x)\big]\, d\mu(x) = \sum_{n\in\mathbb{N}} \int_X \chi_C\big[f_n(x)\big]\, d\mu(x).$$

Since each f_n is bijective from $\pi_l(E_n)$ to E_n, again with (1), for each $n \in \mathbb{N}$, we then have:

$$\int_X \chi_C\big[f_n(x)\big]\, d\mu(x) = \int_{\pi_l(E_n \cap C)} 1\, d\mu(x) = \mu\big[\pi_l(C \cap E_n)\big]$$

which implies that:

$$\int_X |\,\pi_l^{-1}\big(\{x\}\big) \cap C\,|\, d\mu(x) = \sum_{n\in\mathbb{N}} \mu\big[\pi_l(C \cap E_n)\big]. \tag{2}$$

Since the cardinality function is sub-additive, we can now conclude that ν_l is a measure defined on $\mathcal{B}_R$, and that for each $n \in \mathbb{N}$, $\nu_l(E_n) = \mu\big[\pi_l(E_n)\big] < \infty$. Hence, ν_l is a σ-finite measure defined on $\mathcal{B}_R$. Replacing π_l by π_r and adjusting the definition of each f_n can also show that ν_r is a σ-finite measure defined on $\mathcal{B}_R$.

(ii) Given $C \in \mathcal{B}_R$, if $\mu\big[\pi_l(C)\big] = 0$, then for each $n \in \mathbb{N}$:

$$0 \leqslant \mu\big[\pi_l(C \cap E_n)\big] = \mu\big[\pi_l(C) \cap \pi_l(E_n)\big] \leqslant \mu\big[\pi_l(C)\big] = 0.$$

With (2), we then can conclude $\nu_l(C) = 0$. Conversely, if $\nu_l(C) = 0$, by definition of ν_l, we have that for μ-almost each $x \in X$, $|\,\pi_l^{-1}\big(\{x\}\big) \cap C\,| = 0$, or $\pi_l^{-1}\big(\{x\}\big) \cap C = \varnothing$. By (1), we then have $\pi_l(C)$ is μ-null set, or $\mu\big[\pi_l(C)\big] = 0$. Replacing π_l by π_r can prove that, for each $C \in \mathcal{B}_R$, $\nu_r(C) = 0$ iff $\mu\big[\pi_r(C)\big] = 0$.

□

Proposition 1.11. *Given a non-singular equivalence relation $(X, \mathcal{B}, \mu, R)$, the left and right counting measure (defined in **Proposition 1.8**) are equivalent.*

Proof. Assume that $\{E_n\}_{n\in\mathbb{N}}$ is a partition of R (given by **Theorem 1.10**) such that π_l is injective on each E_n, and that $\{F_m\}_{m\in\mathbb{N}}$ is a partition of R (also given by **Theorem 1.10**) such that π_r is injective on each F_m. Therefore, we have that $\{E_n \cap F_m\}_{n,m\in\mathbb{N}}$ is a partition of R where both π_l and π_r are injective on each $E_n \cap F_m$. Hence, without losing generality, we can assume that $\{E_n\}$ is a partition of R where both π_l and π_r are injective on each E_n. As a result, we have that, for each $C \in \mathcal{B}_R$:

$$\nu_l(C) = \sum_{n\in\mathbb{N}} \mu\big[\pi_l(C \cap E_n)\big] \qquad \nu_r(C) = \sum_{n\in\mathbb{N}} \mu\big[\pi_r(C \cap E_n)\big].$$

Observe that for each $C \in \mathcal{B}_R$, we have $R\big[\pi_l(C)\big] = R\big[\pi_r(C)\big]$. Because R is non-singular, we then can conclude:

$$\mu\big[\pi_l(C)\big] = 0 \quad\Longleftrightarrow\quad \mu\Big(R\big[\pi_l(C)\big]\Big) = 0 \quad\Longleftrightarrow\quad \mu\Big(R\big[\pi_r(C)\big]\Big) = 0 \quad\Longleftrightarrow\quad \mu\big[\pi_r(C)\big] = 0.$$

Combining equations above gives us:

$$\nu_l(C) = 0 \quad\Longleftrightarrow\quad \mu\big[\pi_l(C \cap E_n)\big]\,\forall\, n \in \mathbb{N} \quad\Longleftrightarrow\quad \mu\big[\pi_r(C \cap E_n\big]\,\forall\, n \in \mathbb{N} \quad\Longleftrightarrow\quad \nu_r(C) = 0.$$

□

Definition 1.12. A **partial isomorphism** Φ of a measurable equivalence relation $(X, \mathcal{B}, R)$ is a measurable bijection between $A, B \in \mathcal{B}$ such that the graph of Φ is contained in R.

Proposition 1.13. *Given a non-singular equivalence relation $(X, \mathcal{B}, \mu, R)$, if $\Phi : A \to B$ is a partial isomorphism with $\mu(A) > 0$, then $\Phi\mu_A$ and μ_A are equivalent, where μ_A denotes the restriction of the measure μ to A.*

Proof. By definition of a partial isomorphism, we have $\Gamma(\Phi) \subseteq R$. Then, for each measurable subset $C \subseteq B$, we will have:

$$\pi_l[\Gamma(\Phi) \cap \pi_r^{-1}(C)] \subseteq \Phi^{-1}(C)$$

Suppose $x \in \Phi^{-1}(C)$. Then, $\Phi(x) \in C$, and hence:

$$(x, \Phi(x)) \in \Gamma(\Phi) \cap \pi_r^{-1}(C) \qquad \Longrightarrow \qquad x \in \pi_l[\Gamma(\phi) \cap \pi_r^{-1}(C)]$$

which implies $\Phi^{-1}(C) = \pi_l[\Gamma(\Phi) \cap \pi_r^{-1}(C)]$. Similarly, we first have $\pi_r[\Gamma(\Phi) \cap \pi_r^{-1}(C)] \subseteq C$. Then given $y \in C$, we will have:

$$(\Phi^{-1}(y), y) \in \Gamma(\Phi) \cap \pi_r^{-1}(C \cap B)$$

which implies:

$$\pi_r(\Phi^{-1}(y), y) = y \in \pi_r[\Gamma(\Phi) \cap \pi_r^{-1}(C \cap B)]$$

Therefore $\pi_r[\Gamma(\phi) \cap \pi_r^{-1}(C)] = C$. From **Proposition 1.8**, we have that ν_l is equivalent to ν_r when R is non-singular with respect to μ. Therefore:

$$\begin{aligned} &\mu(C) = \mu\Big(\pi_r\big[\Gamma(\Phi) \cap \pi_r^{-1}(C)\big]\Big) = 0 \\ \Longleftrightarrow\ &\nu_r\big[\Gamma(\Phi) \cap \pi_r^{-1}(C)\big] = 0 \\ \Longleftrightarrow\ &\nu_l\,\big[\Gamma(\Phi) \cap \pi_r^{-1}(C)\big] = 0 \\ \Longleftrightarrow\ &\mu\Big(\pi_l\big[\Gamma(\Phi) \cap \pi_r^{-1}(C)\big]\Big) = \mu\big[\Phi^{-1}(C)\big] = 0 \end{aligned}$$

which implies that, for each measurable subset $C \subseteq B$, $\mu(C) = 0$ iff $\mu\big[\Phi^{-1}(C)\big] = 0$.

□

Theorem 1.14. *Given a measurable equivalence relation $(X, \mathcal{B}, R)$ and a measure μ defined on $\mathcal{B}$, the following properties are equivalent:*

(1) R is non-singular with respect to μ.

*(2) The left and right counting measure, ν_l and ν_r (defined in **Proposition 1.8**) are equivalent.*

(3) For each partial isomorphism $\Phi : A \to B$ with $\mu(A) > 0$, we have $\Phi\mu_A$ is equivalent to μ_B.

Proof. **Proposition 1.11** shows that (1) $\Longrightarrow$ (2) and **Proposition 1.13** shows that (1) $\Longrightarrow$ (3). We will prove the theorem by showing that (2) $\Longrightarrow$ (1) and (3) $\Longrightarrow$ (2).

(2) $\Longrightarrow$ (1): Let $\{E_n\}$ be a partition of R (given by **Theorem 1.10**) such that π_l is injective on each E_n. Hence, according to the proof of **Proposition 1.8**, we have that for each $C \in \mathcal{B}_R$:

$$\nu_l(C) = \sum_{n \in \mathbb{N}} \mu\big[\pi_l(C \cap E_n)\big].$$

To prove that R is non-singular with respect to μ, it suffices to show that $\mu(B) = 0$ implies $\mu\big[R(B)\big] = 0$. Suppose $B \in \mathcal{B}$ is a μ-null set. Hence, for each $n \in \mathbb{N}$, we have:

$$\mu\Big(\pi_l\big[\pi_l^{-1}(B) \cap E_n\big]\Big) \leqslant \mu(B) = 0 \qquad \Longrightarrow \qquad \nu_l\big[\pi_l^{-1}(B)\big] = \sum_{n \in \mathbb{N}} \mu\Big(\pi_l\big[\pi_l^{-1}(B) \cap E_n\big]\Big) = 0.$$

Because we assume that ν_l is equivalent to ν_r in $\mathcal{B}_R$, we then have $\nu_r\big[\pi_l^{-1}(B)\big] = 0$ and that:

$$\nu_r\big[\pi_l^{-1}(B)\big] = \sum_{n\in\mathbb{N}} \mu\Big(\pi_r\big[\pi_l^{-1}(B) \cap E_n\big]\Big) = 0 \quad\Longrightarrow\quad \forall\, n \in \mathbb{N},\ \mu\Big(\pi_r\big[\pi_l^{-1}(B) \cap E_n\big]\Big) = 0.$$

Notice that $\pi_r\big[\pi_l^{-1}(B)\big] = R(B)$. Since $\{E_n\}_{n\in\mathbb{N}}$ is a partition of R, we have $X = \bigcup_{n\in\mathbb{N}} \pi_r(E_n) = \pi_r(R)$. Hence, for each $n \in \mathbb{N}$, we have:

$$\mu\Big(\pi_r\big[\pi_l^{-1}(B) \cap E_n\big]\Big) = \mu\Big(\pi_r\big[\pi_l^{-1}(B)\big] \cap \pi_r(E_n)\Big) = \mu\big[R(B) \cap \pi_r(E_n)\big] = 0.$$

which implies:

$$0 \leqslant \mu\big[R(B)\big] = \mu\big[\bigcup_{n\in\mathbb{N}} R(B) \cap \pi_r(E_n)\big] \leqslant \sum_{n\in\mathbb{N}} \mu\big[R(B) \cap \pi_r(E_n)\big] = 0 \quad\Longrightarrow\quad \mu\big[R(B)\big] = 0.$$

(3) $\Longrightarrow$ (2): To show that ν_l is equivalent to ν_r, it suffices to show that, for each $C \in \mathcal{B}_R$, $\nu_l(C) = 0$ implies that $\nu_r(C) = 0$ and then the other direction follows by replacing ν_l by ν_r. Again let $\{E_n\}$ be the partition of R (provided by **Theorem 1.10**) such that π_l is injective on each E_n. As in the proof of **Proposition 1.8**, let $f_n : \pi_l(E_n) \to E_n$ be defined as the following: for each $x \in E_n$, $f_n(x)$ is the unique point in E_n such that $\pi_l\big[f_n(x)\big] = x$. Define $g_n : \pi_r(E_n) \to E_n$ similarly, and then define:

$$\Phi_n : \pi_l(E_n) \to \pi_r(E_n), \quad x \mapsto g_n^{-1} \circ f_n(x)$$

Because both f_n and g_n are measurable and bijective, Φ_n is also measurable and bijective. Also, because $E_n = \pi_l(E_n) \times \pi_r(E_n)$ and $E_n \subseteq R$, we then have the graph of Φ_n is in R, and hence ϕ_n is a partial isomorphism between $\pi_l(E_n)$ and $\pi_r(E_n)$. Now given $C \in \mathcal{B}_R$, if $\nu_l(C) = 0$, according to the proof in **Proposition 1.8**:

$$\nu_l(C) = \sum_{n\in\mathbb{N}} \mu\big[\pi_l(C \cap E_n)\big] = 0 \quad\Longrightarrow\quad \forall\, n \in \mathbb{N},\ \mu\big[\pi_l(C \cap E_n)\big] = 0.$$

By assumption, we then have, for each $n \in \mathbb{N}$:

$$\mu\big[\pi_l(C \cap E_n)\big] = 0 \quad\Longrightarrow\quad \mu\Big(\Phi_n\big[\pi_l(C \cap E_n)\big]\Big) = \mu\big[\pi_r(C \cap E_n)\big] = 0.$$

which implies that:

$$\nu_r(C) = \sum_{n\in\mathbb{N}} \mu\big[\pi_r(C \cap E_n)\big] = 0.$$

$\square$

1.4 Radon-Nikodym cocycle

Definition 1.15. Given a non-singular measurable equivalence relation $(X, \mathcal{B}, \mu, R)$, a **real-valued cocycle of** $R\,(\mathrm{mod}\ 0)$ is a measurable function $c(\cdot,\cdot) : R \to \mathbb{R}$ such that there exists a measurable subset $X_0 \subseteq X$ where $\mu(X_0) = \mu(X)$ and:

- for every $x \in X_0$, $c(x,x) = 1$,
- for every $x, y \in X_0$ with $(x,y) \in R$, $c(x,y)c(y,x) = 1$,
- for every $x, y, z \in X_0$ with $(x,y) \in R$ and $(y,z) \in R$, we have $c(x,y)c(y,z) = c(x,z)$.

Proposition 1.16. *If $\Phi : A \to B$ is a partial isomorphism of a non-singular equivalence relation $(X, \mathcal{B}, \mu, R)$ with $\mu(A), \mu(B) > 0$, then*

$$\frac{d\,\Phi\mu}{d\,\mu}(y) = \frac{d\,\nu_l}{d\,\nu_r}(\Phi^{-1}(y), y)$$

for almost each $y \in B$.

Proof. Again we use $\Gamma(\Phi)$ to denote the graph of Φ. Fix a measurable subset $C \subseteq B$. In **Proposition 1.13**, we observed that:

$$\pi_l\Big[\Gamma(\Phi) \cap \pi_l^{-1}(C)\Big] = \Phi^{-1}(C)$$

Therefore:

$$\Phi\mu(C) = \mu\big[\Phi^{-1}(C)\big] = \mu\Big(\pi_l\big[\Gamma(\Phi) \cap \pi_l^{-1}(C)\big]\Big) \tag{3}$$

Observe that:

$$\Gamma(\Phi) \cap \pi_l^{-1}(C) = \Big\{(\Phi^{-1}(y), y) \,|\, y \in C\Big\} \tag{4}$$

Because Φ is bijective, π_l is then injective on $\Gamma(\Phi) \cap \pi_l^{-1}(C)$ and hence:

$$\nu_l\big[\Gamma(\phi) \cap \pi_l^{-1}(C)\big] = \int_X \big|\,\pi_l^{-1}\{x\} \cap \Gamma(\Phi) \cap \pi_l^{-1}(C)\,\big|\, d\mu(x) = \mu\big[\Phi^{-1}(C)\big] \tag{5}$$

Combining $(3), (4)$ and (5) gives us:

$$\begin{aligned}
&\nu_l\big[\Gamma(\Phi) \cap \pi_l^{-1}(C)\big] \\
&= \int_{\Gamma(\Phi)\cap\pi_l^{-1}(C)} \frac{d\,\nu_l}{d\,\nu_r}(x,y)\, d\nu_r(x,y) \\
&= \int_C \frac{d\,\nu_l}{d\,\nu_r}\big[\Phi^{-1}(y), y\big]\, d\mu(y) \\
&= \mu\Big(\pi_l\big[\Gamma(\Phi) \cap \pi_l^{-1}(C)\big]\Big) = \mu\big[\Phi^{-1}(C)\big] \\
&= \int_C \frac{d\,\Phi\mu}{d\,\mu}(y)\, d\mu(y)
\end{aligned}$$

which implies:

$$\int_C \frac{d\,\nu_l}{d\,\nu_r}\big[\Phi^{-1}(y), y\big]\, d\mu(y) = \int_C \frac{d\,\Phi\mu}{d\,\mu}(y)\, d\mu(y).$$

Since C is arbitrarily picked, we then can conclude, for almost each $y \in B$:

$$\frac{d\,\Phi\mu}{d\,\mu}(y) = \frac{d\,\nu_l}{d\,\nu_r}\big[\Phi^{-1}(y), y\big].$$

□

Proposition 1.17. *Given two measurable spaces $(X, \mathcal{B}_X), (Y, \mathcal{B}_Y)$, two equivalent measures μ and σ defined on $\mathcal{B}_X$, and a measurable bijection $\Phi : X \to Y$, the image measures $\Phi\mu$ and $\Phi\sigma$ are equivalent and, for $\Phi\mu$-almost every $y \in Y$:*

$$\frac{d\,\Phi\mu}{d\,\Phi\sigma}(y) = \frac{d\,\mu}{d\,\sigma} \circ \Phi^{-1}(y).$$

Proof. Since, for each $B \in \mathcal{B}_Y$, $\Phi^{-1}(B) \in \mathcal{B}_X$ and $\mu \sim \sigma$ on the entire $\mathcal{B}_X$, we then have:

$$\Phi\mu(B) = 0 \quad\Longleftrightarrow\quad \mu\big[\Phi^{-1}(B)\big] = 0 \quad\Longleftrightarrow\quad \sigma\big[\Phi^{-1}(B)\big] = 0 \quad\Longleftrightarrow\quad \Phi\sigma(B) = 0$$

To establish the equality, it suffices to show that, for each $B \in \mathcal{B}_Y$

$$\int_B \frac{d\,\Phi\mu}{d\,\Phi\sigma}(y)\, d(\Phi\sigma) = \int_B \left[\frac{d\,\mu}{d\,\sigma} \circ \Phi^{-1}(y)\right] d(\Phi\sigma)$$

Fix $B \in \mathcal{B}_Y$. Since Φ^{-1} is well-defined on in Y, we then have:

$$\begin{aligned}
&\int_B \frac{d\,\Phi\mu}{d\,\Phi\sigma}(y)\,d(\Phi\sigma) = \int_B \chi_B(y) d(\Phi\mu)\\
&= \int_{\Phi^{-1}(B)} \chi_B \circ \Phi(x)\,d\mu\\
&= \int_{\Phi^{-1}(B)} \Big[\chi_B \circ \Phi(x)\Big] \cdot \frac{d\,\mu}{d\,\sigma}(x)\,d\sigma\\
&= \int_{\Phi^{-1}(B)} \Big[\chi_B \circ \Phi(x)\Big] \cdot \frac{d\,\mu}{d\,\sigma}(x)\,d\big[\Phi^{-1}(\Phi\sigma)\big]\\
&= \int_B \Big[\chi_B \circ \Phi \circ \Phi^{-1}(y)\Big] \cdot \Big[\frac{d\,\mu}{d\,\sigma} \circ \Phi^{-1}(y)\Big]\,d(\Phi\sigma)\\
&= \int_B \frac{d\,\mu}{d\,\sigma} \circ \Phi^{-1}(y)\,d(\Phi\sigma)
\end{aligned}$$

which implies:

$$\int_B \frac{d\,\Phi\mu}{d\,\Phi\sigma}(y)\,d(\Phi\sigma) = \int_B \Big[\frac{d\,\mu}{d\,\sigma} \circ \Phi^{-1}(y)\Big]\,d(\Phi\sigma)$$

Since B is an arbitrary measurable subset of Y, the desired equality holds for almost each $y \in Y$.

□

Theorem 1.18. *Given a non-singular equivalence relation $(X, \mathcal{B}, \mu, R)$, the Radon-Nikodym derivative of the left counting measure with respect to the right counting measure, $d\,\nu_l/d\,\nu_r$, is a real-valued cocycle of R (mod 0).*

Proof. Let $\{E_n\}$ be the partition of R (given by **Theorem 1.10**) where π_l is injective in each E_n. As we have seen from the proof of the implication (3) $\implies$ (2) from **Theorem 1.14**, if R is non-singular with respect to μ, for each $n \in \mathbb{N}$, there exists a partial isomorphism Φ_n from $\pi_l(E_n)$ to $\pi_r(E_n)$ such that, for each $(a, b) \in E_n$, $\Phi_n(a) = b$. Therefore, we have $X = \bigcup_{n\in\mathbb{N}} \pi_r(E_n)$. For each $n \in \mathbb{N}$, define:

$$R_n = \pi_r(E_n)\backslash \bigcup_{i<n} \pi_r(E_i).$$

Hence, $X = \bigcup_{n\in\mathbb{N}} R_n$ and the family $\{R_n\}$ are pair-wise disjoint. Without losing generality, we can assume each R_n has positive μ-measure (or exclude all those μ-null sets from $\{R_n\}$). From **Proposition 1.13**, we will have $\Phi_n\mu_{\pi_r(E_n)}$ is equivalent to $\mu_{\pi_l(E_n)}$ in the relative σ-algebra $\{C \subseteq \pi_r(E_n) \,|\, C \in \mathcal{B}\}$. For each $n \in \mathbb{N}$, without losing generality, suppose D_n is a representative that is equal to the Radon-Nikodym derivative of $\Phi_n\mu$ with respect to μ almost everywhere. Then, according to **Proposition 1.16** and **Proposition 1.17**, for each $n, m \in \mathbb{N}$, there exists $F_{n,m} \subseteq R_n$ with $\mu(F_{n,m}) = \mu(R_n)$ such that for (exactly) each $x \in F_{n,m}$, we have:

$$D_n(x) = \frac{d\,\nu_l}{d\,\nu_r}\big(\Phi_n^{-1}(x), x\big), \qquad \frac{d\,\Phi_m(\Phi_n\mu)}{d\,\Phi_n\mu}(x) = D_n \circ \Phi_m^{-1}(x). \tag{6}$$

For each $n \in \mathbb{N}$, set $G_n = \bigcap_{m\in\mathbb{N}} F_{n,m}$. Then we will have $\mu(G_n) = \mu(R_n)$ and that, for all $m \in \mathbb{N}$ and for (exactly) each x in G_n, (6) holds. Hence, we will have that:

$$\mu(X) = \sum_{n\in\mathbb{N}} \mu(R_n) = \sum_{n\in\mathbb{N}} \mu(G_n) = \mu\Big[\bigcup_{n\in\mathbb{N}} G_n\Big].$$

Set $X_0 = \bigcup_{n\in\mathbb{N}} G_n$. Suppose $x, y, z \in X_0$ and that $(x, y), (y, z) \in R$. Without losing generality, suppose $(x, y) \in E_n$ and $(y, z) \in E_m$, which implies that $y \in \pi_r(E_n)$ and hence $y \in G_n$, and that $z \in \pi_r(E_m)$ and hence $z \in G_m$. Since we have $\Phi_n(x) = y$, $\Phi_m(y) = z$ and $\Phi_m \circ \Phi_n(x) = z$, according to (6), we will have:

$$D_m(z) = \frac{d\,\nu_l}{d\,\nu_r}(\Phi_m^{-1}(z), z) = \frac{d\,\nu_l}{d\,\nu_r}(y, z), \qquad D_n(y) = \frac{d\,\nu_l}{d\,\nu_r}(\Phi_n^{-1}(y), y) = \frac{d\,\nu_l}{d\,\nu_r}(x, y), \tag{7}$$

and:

$$\frac{d\,(\Phi_m \circ \Phi_n)\mu}{d\,\mu}(z) = \frac{d\,\nu_l}{d\,\nu_r}\big[(\Phi_m \circ \Phi_n)^{-1}(z), z\big] = \frac{d\,\nu_l}{d\,\nu_r}(x, z). \tag{8}$$

Meanwhile, from **Proposition 1.17**, we have:

$$\frac{d\,\nu_l}{d\,\nu_r}(x,y) = \frac{d\,\nu_l}{d\,\nu_r}\Big(\Phi_n^{-1}\big[\Phi_m^{-1}(z)\big], \Phi_m^{-1}(z)\Big) = D_n \circ \Phi_m^{-1}(z) = \frac{d\,(\Phi_m \circ \Phi_n)\mu}{d\,\Phi_m \mu}(z) \tag{9}$$

Combining (7), (8) and (9) gives us:

$$\frac{d\,\nu_l}{d\,\nu_r}(x,y)\frac{d\,\nu_l}{d\,\nu_r}(y,z) = \frac{d\,\nu_l}{d\,\nu_r}(x,z).$$

Because $\mu(X_0) = \mu(X)$, we can now conclude that $d\,\nu_l/d\,\nu_r$ is a real-valued cocycle of R (mod 0).

□

1.5 Definition and properties of ratio set

Definition 1.19. Given a non-singular equivalence relation $(X, \mathcal{B}, \mu, R)$, the **ratio set** $r_\mu(R)$ is the set of all non-negative real numbers λ such that: for each $\epsilon > 0$ and for any $B \in \mathcal{B}$ with $\mu(B) > 0$, there exists two measurable subsets of B, say B_1, B_2, such that $\mu(B_1), \mu(B_2) > 0$ and, there exists a partial isomorphism $\Phi : B_1 \to B_2$ such that for μ-almost each $x \in B_2, |\,D_\mu(\Phi^{-1}(x), x) - \lambda\,| < \epsilon$ where D_μ is the Radon-Nikodym cocycle given by **Theorem 1.18**. In particular, according to **Theorem 1.14**, by removing μ-null sets from B_1, B_2 we can claim the following equality:

$$B_2 = \Phi(B_1) = \big\{x \in B_2 \,|\, D_\mu(\Phi^{-1}(x), x) \in (\lambda - \epsilon, \lambda + \epsilon)\big\}.$$

The following proof is inspired by **[1]**.

Proposition 1.20. *Given a non-singular countable equivalence relation* $(X, \mathcal{B}, \mu, R)$*, its ratio set is a non-empty closed subset of* $\mathbb{R}$ *in usual topology and* $r_\mu(R)\backslash\{0\}$ *is a multiplicative subgroup of* $(0, \infty)$*.*

Proof. Throughout this proof, we will use $\mathcal{B}^+$ to denote the set of non-μ-null measurable sets.

(i). Firstly, we will show that $1 \in r_\mu(R)$. By **Definition 1.17**, if we use id_X to denote the identity mapping in X, then $(id_X)_*\mu = \mu$ and, for each $x \in X$, $D_\mu(x, id_X^{-1}(x))$ is constantly equal to 1. Hence, for each $A \in \mathcal{B}^+$:

$$A = id_X(A) = \{x \in A : |\,D_\mu(x, id_X^{-1}(x)) - 1\,| < \epsilon\}$$

hence condition $(*)$ is satisfied and hence 1 is in the ratio set.

(ii). After showing that the ratio set is always non-empty, we then will show it is a closed set (in the usual topology) in the real line. Given a converging sequence $\{a_n\}_{n\in\mathbb{N}} \subseteq r_\mu(R)$, assume $a_n \to a \in [0, \infty)$. Fix $A \in \mathcal{B}^+, \epsilon > 0$ and suppose $|\,a_n - a\,| < \epsilon$ for all $n \geqslant k$. Then, we can find Φ a partial isomorphism with domain A such that:

$$A = \Phi(A) = \{x \in A \,:\, |\,D_\mu(x, \Phi^{-1}(x)) - a_k\,| \leqslant \epsilon\} \in \mathcal{B}^+$$

Then, since $|\,a_k - a\,| \leqslant \epsilon$, we then have:

$$A = \{x \in A \,:\, |\,D_\mu(x, \phi^{-1}(x)) - a_k\,| \leqslant \epsilon\} \subseteq \{x \in A \,:\, |\,D_\mu(x, \Phi^{-1}(x)) - a\,| \leqslant 2\epsilon\}$$

and hence:

$$\Big\{x \in X : |\,D_\mu(x, \Phi^{-1}(x)) - a\,| \leqslant 2\epsilon\Big\} \in \mathcal{B}^+$$

Thus $a \in r_\mu(R)$.

(iii). Next, we will show that the ratio set is a multiplicative subgroup of $(0, \infty)$. The following proof is inspired by [**1 Lemma** 9.7]. Suppose $p, q \in r_\mu(R)$. We will first show that $pq \in r_\mu(R)$. Fix $C \in \mathcal{B}^+$. Fix $\epsilon > 0$. Because multiplication is jointly continuous in $\mathbb{R}$, we can find $\delta > 0$ so that

$$\{xy : x \in (p - \epsilon, p + \epsilon), \quad y \in (q - \epsilon, q + \epsilon)\} \subseteq (pq - \delta, pq + \delta)$$

By definition, we can find $C_1, C_2 \subseteq C$, both of which have positive measure, and a partial isomorphism $\Phi_p : C_1 \to C_2$ for each $x \in C_2$, $D_\mu(x, \Phi_p^{-1}(x)) \in (p - \epsilon, p + \epsilon)$. Therefore, we have the following equality:

$$C_2 = \Phi_p(C_1) = \{x \in C_2 : |\, D_\mu(x, \phi_p^{-1}(x)) - p \,| \leqslant \epsilon\} \in \mathcal{B}^+$$

Given $\mu(C_1) > 0$, then we can find $D_1, D_2 \subseteq C_1$, both of which have positive measure and $\Phi_q : D_1 \to D_2$ such that, similar to the equation above for C_2, we have:

$$D_2 = \Phi_q(D_1) = \{x \in D_2 : |\, D_\mu(x, \Phi_q^{-1}(x)) - q \,| \leqslant \epsilon\} \in \mathcal{B}^+$$

By **Proposition 1.16**, we have that, for almost each $b \in C_2$, we have:

$$\frac{d\,(\Phi_p)_*\big[(\Phi_q)_*\mu\big]}{d\,\mu}(b) = \frac{d\,(\Phi_p)_*\big[(\Phi_q)_*\mu\big]}{d\,(\phi_p)_*\mu}(b) \cdot \frac{d\,(\Phi_p)_*\,\mu}{d\,\mu}(b) = \frac{d\,(\Phi_q)_*\mu}{d\,\mu} \circ \Phi_p^{-1}(b) \cdot \frac{d\,(\Phi_p)_*\mu}{d\,\mu}(b)$$

In terms of the Radon-Nikodym Cocycle, we have the following equivalent equation:

$$D_\mu(b, (\Phi_p \circ \Phi_q)^{-1}(b)) = D_\mu[\Phi_p^{-1}(b), \Phi_q^{-1} \circ \Phi_p^{-1}(b)] \cdot D(b, \Phi_p^{-1}(b))$$

Since $\Phi_p\mu$ is equivalent to μ in C_2, given $D_2 \subseteq C_1$ with $\mu(D_2) > 0$, we will have $\Phi_p(D_2)$ also has positive measure and $\Phi_p(D_2) \subseteq \Phi_p(C_1) = C_2$. Hence, for each $b \in \Phi_p(D_2)$:

$$\begin{aligned} D_\mu(b, (\Phi_p \circ \Phi_q)^{-1}(b)) &= D_\mu[\Phi_p^{-1}(b), \Phi_q^{-1} \circ \Phi_p^{-1}(b)] \cdot D(b, \Phi_p^{-1}(b)) \\ &\in \{x_1 x_2 : x_1 \in (q - \epsilon, q + \epsilon), \quad x_2 \in (p - \epsilon, p + \epsilon)\} \\ &\subseteq (pq - \delta, pq + \delta) \end{aligned}$$

Therefore, for each $b \in \Phi_p(D_2)$, we have:

$$\begin{aligned} b &\in C_2 \cap \Phi_p \circ \Phi_q(D_1) \cap \{x \in X : D_\mu(x, (\Phi_p \circ \Phi_q)^{-1}(x)) - pq \,| \leqslant \epsilon\} = \Phi_p \circ \Phi_q(D_1) \\ &\subseteq C_2 \cap \Phi_p \circ \Phi_q(C_1) \cap \{x \in X : D_\mu(x, (\Phi_p \circ \Phi_q)^{-1}(x)) - pq \,| \leqslant \epsilon\} \\ &\subseteq C \cap \Phi_p \circ \Phi_q(C) \cap \{x \in X : D_\mu(b, (\Phi_p \circ \Phi_q)^{-1}(b)) - pq \,| \leqslant \epsilon\} \end{aligned}$$

The set in the last line has positive measure for containing $\Phi_p \circ \Phi_q(D_1)$. Now we can conclude, given $C \in \mathcal{B}^+$, we can find a partial isomorphism $\Phi_p \circ \Phi_q$ between $D_1.\Phi_p \circ \Phi_q(D_1)$, both of which have positive measure, such that:

$$C \cap \Phi_p \circ \Phi_q(C) \cap \{x \in X : D_\mu(x, (\Phi_p \circ \Phi_q)^{-1}(x)) \in (pq - \epsilon, pq + \epsilon)\} \in \mathcal{B}^+$$

Hence $pq \in r_\mu(R)$.

(iv). At last we want to show, given $d \in r_\mu(R)$, we will have $d^{-1} \in r_\mu(R)$, by using continuity the map $x \mapsto x^{-1}$ in $(0, \infty)$. Fix $A \in \mathcal{B}^+, \epsilon > 0$, a partial isomorphism ψ from A_1 to A_2, both of which are contained in A and have positive measure such that:

$$A_2 = \psi(A_1) = \{x \in X \mid D_\mu(x, \psi^{-1}(x)) \in (d - \epsilon, d + \epsilon)\}$$

Again by **Proposition 1.17**, we have, for almost each $a \in A_2$:

$$\frac{d\,\mu}{d\,\psi * \mu}(a) = \frac{d\,\psi * \psi^{-1} * \mu}{d\,\psi * \mu}(a) = \frac{d\,\psi^{-1} * \mu}{d\,\mu} \circ \psi^{-1}(a)$$

In terms of Radon-Nikodym Cocycle, we have the following equation equivalent to the one above:

$$D_\mu(a, \psi^{-1}(a))^{-1} = D_\mu(\psi^{-1}(a), \psi \circ \psi^{-1}(a)) = D_\mu(\psi^{-1}(a), a)$$

Since we can find $\delta > 0$ such that $\{x^{-1} \mid x \in (d - \epsilon, d + \epsilon)\} \subseteq (d^{-1} - \delta, d^{-1} + \delta)$, then:

$$\begin{aligned} A_2 &= \{x \in A_2 \mid D_\mu(x, \psi^{-1}(x)) \in (d - \epsilon, d + \epsilon)\} \\ &= \{x \in A_2 \mid D_\mu(\psi^{-1}(x), x)^{-1} \in (d - \epsilon, d + \epsilon)\} \\ &\subseteq A_2 \cap \psi(A_1) \cap \{x \in X \mid D_\mu(\psi^{-1}(x), x) \in (d^{-1} - \delta, d^{-1} + \delta)\} \end{aligned}$$

Since ψ is a bijection from A_1 to A_2, we then have:

$$\psi^{-1}(A_2) = A_1 \subseteq \{\psi^{-1}(x), x \in A_2 \,|\, D_\mu(\psi^{-1}(x), x) \in (d^{-1} - \delta, d^{-1} + \delta)\}$$

Therefore:

$$A_1 = \psi^{-1}(A_2) = \{y \in A_1 \,|\, D_\mu(y, \psi(y)) \in (d^{-1} - \delta, d^{-1} + \delta)\} \in \mathcal{B}^+$$

where ψ^{-1} is a partial isomorphism from A_2 to A_1 and $\psi(y) = (\psi^{-1})^{-1}(y)$. Since δ approaches to zero when ϵ does, we then can conclude d^{-1} is also in $r_\mu(R)$

□

Proposition 1.21. *Given a non-singular equivalence relation $(X, \mathcal{B}, \mu, R)$ and another measure σ equivalent to μ, the ratio set with respect to μ is the same as the ratio set with respect to σ.*

Proof. It suffices to show that the ratio of one is included in the other. Fix r in the ratio set with respect to μ and an arbitrary $B \in \mathcal{B}$ with $\mu(B) > 0$, $\epsilon > 0$. According to **Definition 1.18**, there will be $B_1, B_2 \subseteq B$, both of which have positive μ-measure, and a partial isomorphism Φ from B_1 to B_2 such that the following set:

$$B_2 \cap \Phi(B_1) \cap \{x \in X \,|\, D_\mu(x, \Phi^{-1}(x)) \in (r - \epsilon, r + \epsilon)\}$$

is equal to B_2 by **Definition 1.18** and hence has positive μ-measure by assumption. Meanwhile, by $\mu \sim \sigma$, for each $A \in \mathcal{B}$, we have:

$$\mu[\Phi^{-1}(A)] = 0 \iff \sigma[\Phi^{-1}(A)] = 0.$$

Fix $\delta \in (0, 1)$ and set $U = (1 - \delta, 1 + \delta)$. If we use U^2 to denote the set of products of elements from U, U^{-1} the set of reciprocals of elements from U, then, without losing generality, we can assume the following set:

$$(U^2) \cdot r \cdot (U^{-1})^2 = \{r_1 r r_2 \,|\, r_1 \in U^2, r_2 \in (U^{-1})^2\}$$

is contained in $(r - \delta, r + \delta)$. If, for each $q \in \mathbb{Q}$, we set $Uq = \{r_1 q \,|\, r_1 \in U\}$. Then $\{Uq\}_{q \in \mathbb{Q}}$ is an open cover of $\mathbb{R}$ because $\mathbb{Q}$ is dense in $\mathbb{R}$. Because μ is equivalent to σ, the derivative $d\mu/d\sigma$ is positive almost everywhere in X. With $\mu(B) > 0$, $d\mu/d\sigma$ is positive almost everywhere in B. Suppose $\{q_n\}_{n \in \mathbb{N}}$ is a enumeration of $\mathbb{Q}$ and set $U_n = Uq_n$ for each $n \in \mathbb{N}$. Now define $O_1 = U_1$ and for $n > 1$, define:

$$O_n = U_n \backslash \bigcup_{1 \leqslant i < n} U_i.$$

Then we have:

$$B = B \cap \Big[\frac{d\mu}{d\sigma}(x)\Big]^{-1}(\mathbb{R}) = B \cap \bigcup_{n \in \mathbb{N}} \Big[\frac{d\mu}{d\sigma}(x)\Big]^{-1}(O_n) = \bigcup_{n \in \mathbb{N}} B \cap \Big[\frac{d\mu}{d\sigma}(x)\Big]^{-1}(O_n)$$

The equation above can be visually displayed as the following:

Since the family $\{O_n\}_{n \in \mathbb{N}}$ is pair-wise disjoint, the set on the very right hand side is a disjoint countable union. Since $\mu(B) > 0$, we can find $N \in \mathbb{N}$ such that the following set:

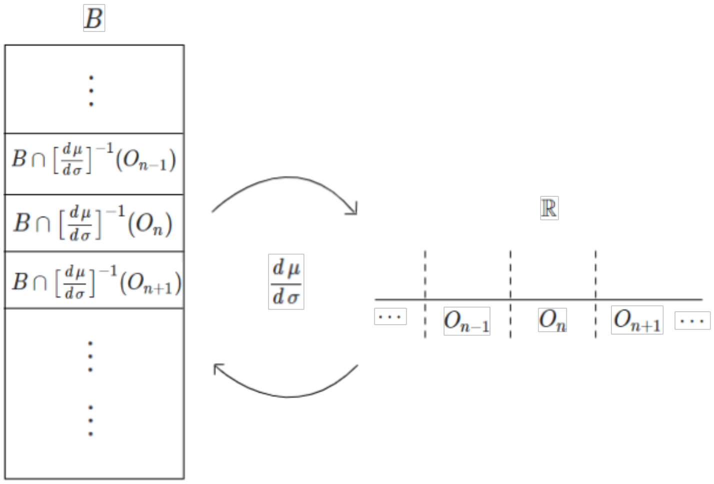
$$B \cap \Big[\frac{d\mu}{d\sigma}(x)\Big]^{-1}(O_N)$$

has positive μ-measure. Since $O_N \subseteq U_N = Uq_N$, the following set:

$$B^{'} = \Big\{x \in B \,|\, \frac{d\mu}{d\sigma}(x) \in Uq_N\Big\}$$

has positive μ-measure. Since $U \cdot r \cdot U^{-1}$ is an open neighborhood of r, we can find $\delta_1 > 0$ such that $(r - \delta_1, r + \delta_1) \subseteq U \cdot r \cdot U^{-1}$. Apply the definition of ratio set again with the previously fixed r, δ_1, but with B replaced by $B^{'}$. Then we can find $B_1^{'}, B_2^{'} \subseteq B^{'}$, both of which have positive measure, and a Φ, a desired partial isomorphism from $B_1^{'}$ to $B_2^{'}$ such that the following set:

$$B_1^{'} = \Phi(B_2^{'}) = \{x \in B_1^{'} \,|\, D_\mu(x, \Phi^{-1}(x)) \in (r - \delta_1, r + \delta_1)\}$$



has positive measure. Meanwhile, because, for each $b \in B'$, $\frac{d\mu}{d\sigma}(b) \in Uq_N$, and $\Phi^{-1}(B_2') = B_1' \subseteq B'$, according to **Proposition 1.17**, we will have:

$$\forall\, b \in B_2', \quad \frac{d\,\Phi\sigma}{d\,\Phi\mu}(b) = \frac{d\,\sigma}{d\,\mu} \circ \Phi^{-1}(b) = \Big(\frac{d\,\mu}{d\,\sigma} \circ \Phi^{-1}(b)\Big)^{-1} \in (Uq_N)^{-1} = q_N^{-1}U^{-1}.$$

Therefore, for each $b \in B_2'$, we have:

$$\begin{aligned}
D_\sigma(b, \Phi^{-1}(b)) &= \frac{d\,\Phi\sigma}{d\,\Phi\mu}(b) \cdot D_\mu(b, \Phi^{-1}(b)) \cdot \frac{d\,\mu}{d\,\sigma}(b) \\
&= \frac{d\,\sigma}{d\,\mu} \circ \Phi^{-1}(b) \cdot D_\mu(b, \Phi^{-1}(b)) \cdot \frac{d\,\mu}{d\,\sigma}(b) \\
&\in \big(q_N^{-1}U^{-1}\big) \cdot (r - \delta_1, r + \delta_1) \cdot (Uq_N) \\
&\subseteq \big(q_N^{-1}U^{-1}\big) \cdot \big(U \cdot r \cdot U^{-1}\big) \cdot (Uq_N) \subseteq (r - \epsilon, r + \epsilon)
\end{aligned}$$

Because $B_2' = \Phi(B_1') \in \mathcal{B}^+$, we will have:

$$\begin{aligned}
B_2' &= B_2' \cap \Phi(B_1') \cap \{x \in B_2' \mid D_\sigma(x, \Phi^{-1}(x)) \in (r - \epsilon, r + \epsilon)\} \\
&\subseteq B' \cap \Phi(B') \cap \{x \in X \mid D_\sigma(x, \Phi^{-1}(x)) \in (r - \epsilon, r + \epsilon)\} \\
&\subseteq B \cap \Phi(B) \cap \{x \in X \mid D_\sigma(x, \Phi^{-1}(x)) \in (r - \epsilon, r + \epsilon)\}
\end{aligned}$$

which implies:

$$B \cap \Phi(B) \cap \{x \in X \mid D_\sigma(x, \Phi^{-1}(x)) \in (r - \epsilon, r + \epsilon)\} \in \mathcal{B}^+$$

because it contains a B_2' with $\mu(B_2') > 0$. Since ϵ, B are arbitrary, we can conclude r is in the ratio set with respect to σ.

□

1.6 Ratio set of a Markov shift

Definition 1.22. Given a finite alphabet set A (considered as a discrete topological space), the product space $A^{\mathbb{Z}_+}$ is called the **full shift over** A where $\mathbb{Z}_+$ is the set of all non-negative integers. We use $\boldsymbol{a} = (a_n)_{n \in \mathbb{Z}_+}$ to denote elements of $A_M^{\mathbb{Z}_+}$. It is equipped with the product topology and endowed with the continuous shift T defined by $(T\boldsymbol{a})_n = a_{n+1}$. Throughout this section, when a full shift over a alphabet set is given, we will use $\mathcal{B}$ to denote the σ-algebra generated by finite cylinder sets (which is the same as the Borel σ-algebra).

An **admissibility matrix** M over a finite alphabet set A is an $A \times A$ matrix with entries 0 or 1. The associated **topological Markov chain** is the following subset of the full shift

$$A_M^{\mathbb{Z}_+} = \{\boldsymbol{a} \in A^{\mathbb{Z}_+} \mid M_{a_n, a_{n+1}} = 1 \, \forall \, n \in \mathbb{Z}^+\}.$$

A stochastic matrix over A is a $A \times A$ matrix, say $P = [P_{ij}]_{i,j \in A}$, that has entries from $[0, 1]$ and entries in each row sum up to 1. When a topological Markov chain $A_M^{\mathbb{N}}$ is given, and $P_{ij} > 0$ implies that $M_{ij} = 1$, we say the stochastic matrix P is **subordinate to** M. We call M **irreducible** if, for every two $i, j \in A$, we can always find $N \in \mathbb{N}$ such that $(M^N)_{ij} \neq 0$. Similar, we call a stochastic matrix P **irreducible** iff for every $i, j \in A$, there exists $N \in \mathbb{N}$ such that $(P^N)_{ij} \neq 0$. From now on both the admissibility matrix and the transition matrix will be assumed irreducible.

When a topological Markov chain $A_M^{\mathbb{Z}_+}$ is given, a **Markov measure P** defined on $\mathcal{B}$ is determined by a stochastic matrix P subordinated to M, and an **initial distribution** $\boldsymbol{\lambda} = (\lambda_i)_{i \in A}$. Given an **elementary cylinder set** (a cylinder set that has the following form)

$$C_{i_0, \cdots, i_n} = \{\boldsymbol{a} \in A_M^{\mathbb{Z}_+} \mid a_0 = i_0, \cdots, a_n = i_n\}$$

we have:

$$\mathbf{P}(C_{i_0, \cdots, i_n}) = \boldsymbol{\lambda}(i_0) \prod_{1 \leqslant i < n} P_{i_{n-1} i_n}. \tag{10}$$

Whenever $\boldsymbol{\lambda} P = \boldsymbol{\lambda}$, we call the initial distribution **stationary**. In general, for an arbitrary (real-valued) irreducible transition matrix P, according to **Perron-Frobenius theorem**, there exists a unique left eigenvector, which is the desired stationary initial distribution.

Below, we shall introduce the following result that will be used later in this chapter. For the set-up, for each $N \in \mathbb{N}$, use $\mathcal{B}_0^N$ to denote the coordinate σ-algebra in $A_M^{\mathbb{Z}_+}$ corresponding to the time interval $[0, N]$. Hence, $\{\mathcal{B}_0^N\}_{N \in \mathbb{N}}$ is an increasing sequence of σ-algebras that generate $\mathcal{B}$. By the **Martingale Convergence Theorem** (see [**3**]), for each $B \in \mathcal{B}$, we have that **P**-almost surely:

$$\mathbf{P}(B \mid \mathcal{B}_0^N) \xrightarrow{N \to \infty} \chi_B.$$

In other words, for **P**-almost each $\boldsymbol{a} \in B$:

$$\lim_N \frac{\mathbf{P}(B \cap C_{a_0, \cdots, a_N})}{\mathbf{P}(C_{a_0, \cdots, a_N})} = 1.$$

Lemma 1.23. *Given a topological Markov chain $A_M^{\mathbb{Z}_+}$, a full-support initial distribution $\boldsymbol{\lambda}$ (i.e. $\lambda_i > 0 \, \forall \, i \in A$), a subordinate transition matrix $P = [P_{ij}]_{i,j \in A}$, and the associated measure* **P**, *if we put $B_{i,j} = B \cap C_{i,j}$ for an admissible pair (i, j) and a measurable set B with positive measure, then:*

i) $T\Big|_{B_{i,j}} : B_{i,j} \to T(B_{i,j})$ is a partial isomorphism.

ii) The restriction $m_{i,j}^{'}$ of **P** *to $T(B_{i,j})$ is equivalent to the restriction $m_{i,j}$ of* **P** *to $B_{i,j}$.*

iii) For almost all $\boldsymbol{a} \in T(B_{i,j})$, the corresponding Radon-Nikodym derivative $dm_{i,j}/dm_{i,j}^{'}$ defined on $T(B_{i,j})$ satisfies:

$$\frac{d\,m_{i,j}}{d\,m_{i,j}^{'}}(\boldsymbol{a}) = \frac{\lambda_i P_{ij}}{\lambda_j}.$$

Proof. First of all, in the case where B is an elementary cylinder set. we define:

$$F : T(B_{i,j}) \to B_{i,j}, \quad \boldsymbol{a} \mapsto (F\boldsymbol{a})_n = \begin{cases} i, & n = 0 \\ a_{n-1}, & n > 0 \end{cases}$$

Given $\boldsymbol{x} \neq \boldsymbol{y} \in C_{i,j}$, there must exist $n > 1$ such that $x_n \neq y_n$, and hence $T(\boldsymbol{x}) \neq T(\boldsymbol{y})$. By definition of $B_{i,j}$, we can see that the inverse of $T\Big|_{B_{i,j}}$ is F, and that for each $\boldsymbol{x} \in B_{i,j}$, $(\boldsymbol{x}, T(\boldsymbol{x})) \in R_T$. Hence, $T\Big|_{B_{i,j}}$ is a partial isomorphism. When B is an elementary cylinder set, by (10), we have that:

$$m_{i,j}(B_{i,j}) = \frac{\lambda_i P_{ij}}{\lambda_j} m'_{i,j}\big[T(B_{i,j})\big], \tag{11}$$

In general, fix $j \in A$ and the elementary cylinder set C_j. We can define:

$$F_i : C_j \to C_{i,j}, \quad \boldsymbol{a} \mapsto (F_i \boldsymbol{a})_n = \begin{cases} i, & n = 0 \\ a_{n-1}, & n > 0 \end{cases}$$

From what we previously proved, F_i is a partial isomorphism and has inverse $T\Big|_{C_{i,j}}$. If $D \subseteq C_j$ is a cylinder set, we can write $D = \bigsqcup_{n \leqslant M} C^n$, a finite union of pair-wise disjoint elementary cylinder sets. Then, by (11), we will have:

$$m'_{i,j}(D) = \mathbf{P}\big[F_i(D)\big] = \sum_{n \leqslant M} \mathbf{P}\big[F_i(C^n)\big] = \sum_{n \leqslant M} \frac{\lambda_i P_{ij}}{\lambda_j} \mathbf{P}(C^n) = \frac{\lambda_i P_{ij}}{\lambda_j} \mathbf{P}(D) \tag{12}$$

Since, (12) holds for an arbitrary cylinder sets D, we then can conclude that the two measures $m'_{i,j}$ and $(\lambda_i P_{ij} \mathbf{P})/\lambda_j$ coincides in the relative σ-algebra of C_j and hence equivalent. This will imply, for almost each $\boldsymbol{a} \in C_j = T(C_{i,j})$:

$$\frac{d\,m_{i,j}}{d\,m'_{i,j}}(\boldsymbol{a}) = \frac{\lambda_i P_{ij}}{\lambda_j}$$

Because, for each $B \in \mathcal{B}$, and $i, j \in A$, $B_{i,j} \subseteq C_{i,j}$, whenever $\mathbf{P}\big[T(B_{i,j})\big] > 0$, the desired equality holds.

□

Proposition 1.24. *In the set-up of the previous lemma, an initial distribution $\boldsymbol{\lambda}$ is stationary iff the associated measure $\mathbf{P}$ is shift invariant, i.e. for each $B \in \mathcal{B}$:*

$$\mathbf{P}(B) = \mathbf{P}\big[T^{-1}(B)\big].$$

Proof. Fix $B \in \mathcal{B}$, $i, j \in A$. Define $B_{i,j}$ as in **Lemma 1.23** and $B_j = \{\boldsymbol{a} \in B \,|\, a_0 = j\}$. According to **Lemma 1.23**, we will have:

$$\mathbf{P}(B_{i,j}) = \frac{\lambda_i P_{ij}}{\lambda_j} \mathbf{P}\big[T(B_{i,j})\big].$$

Notice that B is a disjoint union of $B_{i,j}$ where i, j range over all letters from A, and that $T\big[T^{-1}(B)_{i,j}\big] = B_j$. We then have:

$$\mathbf{P}\big[T^{-1}(B)\big] = \sum_{i,j\ inA} \mathbf{P}\big[T^{-1}(B)_{i,j}\big] = \sum_{i,j \in A} \frac{\lambda_i P_{ij}}{\lambda_j} \mathbf{P}(B_j) = \sum_{j \in A} \frac{\mathbf{P}(B_j)}{\lambda_j} \sum_{i \in A} \lambda_i P_{ij}.$$

Therefore, $\mathbf{P}(B) = \mathbf{P}\big[T^{-1}(B)\big]$ precisely when, for each $j \in A$, $\lambda_j = \sum_{i \in A} \lambda_i P_{ij}$, which is equvialent to that $\boldsymbol{\lambda}$ being stationary.

□

Proposition 1.25. *Given a topological Markov chain $A_M^{\mathbb{Z}_+}$, a subordinate transition matrix $P = [P_{i,j}]_{i,j \in A}$ and its stationary distribution $\boldsymbol{\lambda}$, the associated measure $\mathbf{P}$ in the path space is quasi-invariant with respect to the orbit equivalence relation of the shift T.*

Proof. Since $A_M^{\mathbb{Z}_+}$ is a Polish space, the shift T is continuous, and, for each $\boldsymbol{a} \in A_M^{\mathbb{Z}_+}$, $T^{-1}\{\boldsymbol{a}\}$ is finite, by **Proposition 2.3**, its orbit equivalence relation R_T is a countable measurable equivalence relation. To show that R_T is non-singular with respect to $\mathbf{P}$, it suffices to show that, for each $B \in \mathcal{B}$, $\mathbf{P}(B) = 0$ implies $\mathbf{P}\big[R_T(B)\big] = 0$. Fix $B \in \mathcal{B}$ with $\mathbf{P}(B) = 0$. From **Proposition 2.3**, we have:

$$R_T(B) = \bigcup_{n,m \in \mathbb{Z}_+} (T^m)^{-1}\big[T^n(B)\big]. \tag{13}$$

Let us prove that $\mathbf{P}\big[T(B)\big] = 0$. Let $B_{i,j}$ and B_j be defined as in the proof of **Proposition 1.24**. Since, for each $i, j \in A$, $T(B_{i,j}) = T(B)_j$. By **Lemma 1.23**, for an arbitrarily fixed $i \in A$, we have:

$$\mathbf{P}\big[T(B)\big] = \sum_{j \in A} \mathbf{P}\big[T(B)_j\big] = \sum_{j \in A} \mathbf{P}\big[T(B_{i,j})\big] = \sum_{j \in A} \frac{\lambda_j}{\lambda_i P_{ij}} \mathbf{P}(B_{i,j}).$$

Because $\mathbf{P}(B) = 0$, $\mathbf{P}(B_{i,j}) = 0$ for each $i, j \in A$ and hence $\mathbf{P}\big[T(B)\big] = 0$. By replacing B by $T(B)$, we can then conclude, for each $n \in \mathbb{N}$, $\mathbf{P}\big[T^n(B)\big] = 0$. Along with **Proposition 1.24**, we can then conclude for each $m \in \mathbb{Z}$, $\mathbf{P}\big[(T^m)^{-1}(B)\big] = 0$ whenever $\mathbf{P}(B) = 0$. Hence, in (13), the right hand side is a countable union of $\mathbf{P}$-null sets, which implies $R_T(B)$ is also $\mathbf{P}$-null.

□

Corollary 1.26. *In the set-up of* ***Proposition 1.25****, the Radon-Nikodym cocycle of the equivalence relation* R_T *(which exists according to* ***Theorem 1.18****) has the property that, for almost each* $\boldsymbol{a} \in A_M^{\mathbb{Z}_+}$

$$D(T\boldsymbol{a}, \boldsymbol{a}) = \frac{\lambda_{a_0} P_{a_0 a_1}}{\lambda_{a_1}}.$$

Proof. Fix $\boldsymbol{a} \in A_M^{\mathbb{Z}_+}$ such that (a_0, a_1) is admissible. Without losing generality, suppose $a_0 = i$, $a_1 = j$ and take $C_{i,j} = \{\boldsymbol{a} \in A_M^{\mathbb{Z}_+} \mid a_0 = i, a_1 = j\}$. Then according to **Lemma 1.23**, $T\Big|_{C_{i,j}}$ is a partial isomorhpism between $C_{i,j}$ and its image, and, according to **Proposition 1.16** and **Theorem 1.18**, for almost each $\boldsymbol{a} \in T(C_{i,j}) = C_j$:

$$\frac{d\mathbf{P}}{d\,(T\mathbf{P})}(\boldsymbol{a}) = D\Big[\boldsymbol{a}, \Big(T\big|_{C_{i,j}}\Big)^{-1}(\boldsymbol{a})\Big] = \frac{\lambda_i P_{ij}}{\lambda_j}$$

Hence, we can conclude that, for almost each $\boldsymbol{a} \in C_{i,j}$:

$$D(T\boldsymbol{a}, \boldsymbol{a}) = \frac{\lambda_i P_{ij}}{\lambda_j}, \tag{14}$$

Since $A_M^{\mathbb{Z}_+} = \bigcup_{i,j \in A} C_{i,j}$, the disjoint union of all $C_{i,j}$ where (i, j) are admissible, we are done.

□

Theorem 1.27. *Given an irreducible topological Markov chain* $(A_M^{\mathbb{Z}_+}, \mathcal{B}, T)$*, the measure* $\mathbf{P}$ *determined by the stationary initial distribution* $\boldsymbol{\lambda}$ *and a subordinate stochastic matrix* $P = [P_{ij}]_{i,j \in A}$*, the ratio set of the orbit equivalence relation of the shift transformation is the closure of the multiplicative group generated by the numbers of the form* $P_{i_1 i_2} \cdots P_{i_{n-1} i_n} P_{i_n i_1}$ *where* $P_{i_1 i_2}, \cdots, P_{i_n i_1} \neq 0$.

Proof. We shall first show that Γ, the multiplicative group generated by $\{P_{i,j}\}_{i,j \in A}$ is contained in the ratio set. Fix $\alpha = P_{i_1 i_2} \cdots P_{i_{n-1} i_n} P_{i_n i_1}$ where all $(i_1, i_2), \cdots, (i_{n-1}, i_n), (i_n, i_1)$ are admissible pairs. We first start from an elementary cylinder set, and then approximate an arbitrary measurable set using elementary cylinder sets. Fix an elementary cylinder set $C = C_{n_0, \cdots, n_N}$. Because $\mathbf{P}$ is irreducible, there exists a smallest $L \in \mathbb{N}$ such that $(P^L)_{n_N i_1} > 0$ (*i.e.*, the shortest possible path from n_N to i_1). If $n_N \to k_1 \to \cdots \to k_l \to i_1$ is a path from n_N to i_1 and of length L (when $L = 1$, we then have (n_N, i_1) is an admissible pair), then define:

$$\begin{aligned} C_a &= C_{n_0, \cdots, n_N, k_1, \cdots, k_l, i_1} \\ C_b &= C_{n_0, \cdots, n_N, k_1, \cdots, k_l, i_1, \cdots, i_n, i_1}. \end{aligned}$$

We then have $C_a \neq \varnothing$ and, by the assumption that $P_{i_1, i_2}, \cdots, P_{i_{n-1}, i_n}, P_{i_n, i_1} \neq 0$, $C_b \neq \varnothing$. Now define a mapping Φ from C_a to C_b:

$$\Phi : C_a \to C_b, \qquad \boldsymbol{a} = (a_i)_{i \in \mathbb{Z}_+} \mapsto \begin{cases} [\Phi(v)]_k = \boldsymbol{a}_k, & 0 \leqslant k < N + l + 1 \\ [\Phi(v)]_k = i_{k-N-l}, & N + l + 1 \leqslant k \leqslant N + l + n \\ [\Phi(v)]_k = i_1, & k = N + l + n + 1 \\ [\Phi(v)]_k = \boldsymbol{a}_{k-n}, & k > N + l + n + 1 \end{cases}$$

Given $u, v \in C_a$, if $u \neq v$. then they will differ at a coordinate after the $(N+l+1)$-th one and hence $\Phi(u) \neq \Phi(v)$. Given $\mathbf{b} \in C_b$, we can find $\mathbf{a} \in C_a$ such that:

$$\forall\, k \in \mathbb{N}, \quad \boldsymbol{a}_{N+l+k} = \boldsymbol{b}_{N+l+n+k}$$

which implies $\Phi(\boldsymbol{a}) = \boldsymbol{b}$. Hence, for each $\boldsymbol{a} \in C_a$, we will have:

$$T^{N+l}(\boldsymbol{a}) = T^{N+l+n}\big[\Phi(\boldsymbol{a})\big]$$

Hence, for each $\boldsymbol{a} \in C_a$, we have $(\boldsymbol{a}, \Phi(\boldsymbol{a})) \in R_T$, and now we can conclude Φ is a measurable partial isomorphism with respect to R, and so is Φ^{-1}. Therefore, for each elementary cylinder set $C \subseteq C_a$, we will have:

$$\frac{\mathbf{P}[\Phi(C)]}{\mathbf{P}(C)} = \alpha \tag{15}$$

According to the proof of **Lemma 1.23**, the set C in (15) can be replaced by a cylinder set, and then by an arbitrary measurable subset in C_a because $\mathbf{P}$ is equivalent to $\Phi^{-1}\mathbf{P}$ in the relative σ-algebra in C_a. Hence, the following set will have the same measure as C_a:

$$\mathcal{C}_a = \Big\{x \in C_a \mid \frac{d\,(\Phi^{-1})\mathbf{P}}{d\,\mathbf{P}}(x) = \alpha\Big\}.$$

For an arbitrary measurable set B with $\mathbf{P}(B) > 0$, according to the **Martingale Convergence Theorem**, we have that for each $u \in B$:

$$\lim_{M} \frac{\mathbf{P}\big(B \cap C_{u_0,\cdots,u_M}\big)}{\mathbf{P}\big(C_{u_0,\cdots,u_M}\big)} = 1 \tag{16}$$

Fix $u \in B$. Let $U = \{u_n\}_{n\in\mathbb{Z}_+}$, the set of all the letters that appear in u, which will be finite. Then, for each letter $a \in U$, there exists a shortest possible path l_a that starts from a and ends at i_1. Suppose $l_a = a \to a_1 \to \cdots \to a_M \to i_1$, and define:

$$\lambda_{l_a} = P_{aa_1} \prod_{1\leqslant j<M} P_{a_j a_{j+1}} P_{a_M i_1}$$

Further, define:

$$\lambda = \min_{a\in U} \min_{l_a} \lambda_{l_a}$$

which is the minimum among all λ_{l_a} where each l_a is a shortest possible path from a to i_1 and among all $a \in U$. Since U is finite and, for each $a \in U$, the set of shortest possible paths from a to i_1 is also finite, λ will be positive. Back to (16). Suppose $N \in \mathbb{N}$ is large enough such that $\alpha^N < \lambda$ and M is large enough such that

$$\mathbf{P}(B \cap C_{u_0,\cdots,u_M}) > (1-\alpha^{N+1})\mathbf{P}(C_{u_0,\cdots,u_M})$$

Let $C_M = C_{u_0,\cdots,u_M}$. According to the previous part, we then can find two elementary cylinder sets $C_M^1, C_M^2 \subseteq C_M$, $C_M^2 \subseteq C_M^1$ such that there exists a partial isomorphism $\Phi_M : C_M^1 \to C_M^2$ and that the following set:

$$\mathcal{C}_M^1 = \Big\{x \in C_M^1 \mid \frac{d(\Phi_M^{-1})\mathbf{P}}{d\,\mathbf{P}}(x) = \alpha\Big\}$$

has the same measure as C_M^1. Meanwhile, according to how we find C_M^1, C_M^2, we will have $\mathbf{P}(C_M^1) \geqslant \lambda \mathbf{P}(C_M)$ and $\mathbf{P}(C_M^2) = \alpha \mathbf{P}(C_M^1)$. Hence, we will have:

$$\begin{aligned}
\mathbf{P}(B \cap C_M^1) &= \mathbf{P}(B \cap C_M) - \mathbf{P}\big[B \cap (C_M \backslash C_M^1)\big] \\
&> (1-\alpha^{N+1})\mathbf{P}(C_M) - \mathbf{P}(C_M \backslash C_M^1) \\
&= \mathbf{P}(C_M^1) - \alpha^{N+1}\mathbf{P}(C_M) \geqslant (\lambda - \alpha^{N+1})\mathbf{P}(C_M) > 0,
\end{aligned} \tag{17}$$

which implies that $\mathbf{P}(B \cap \mathcal{C}_M^1) > 0$. Similarly:

$$\begin{aligned}\mathbf{P}(B\cap C_M^2) &= \mathbf{P}(B\cap C_M) - \mathbf{P}\big[B\cap (C_M\backslash C_M^2)\big]\\ &> (1-\alpha^{N+1})\mathbf{P}(C_M) - \mathbf{P}(C_M\backslash C_M^2)\\ &= \mathbf{P}(C_M^2) - \alpha^{N+1}\mathbf{P}(C_M)\\ &= \alpha\mathbf{P}(C_M^1) - \alpha^{N+1}\mathbf{P}(C_M) = \alpha(\lambda-\alpha^N)\mathbf{P}(C_M) > 0\end{aligned}\tag{18}$$

Therefore, with (17), (20), we have:

$$\begin{gathered}\mathbf{P}\big[(B\cap C_M^2)\backslash\Phi_M(B\cap C_M^1)\big] = \mathbf{P}(B\cap C_M^2) - \mathbf{P}\big[\Phi_M(B\cap\mathcal{C}_M^1)\big]\\ = \mathbf{P}(B\cap C_M^2) - \alpha\mathbf{P}(B\cap C_M^1) < \mathbf{P}(B\cap C_M^2) - \alpha\big[\mathbf{P}(C_M^2) - \alpha^{N+1}\mathbf{P}(C_M)\big]\end{gathered}$$

and hence:

$$\begin{aligned}&\mathbf{P}\big[(B\cap C_M^2)\cap\Phi_M(B\cap C_M^1)\big] = \mathbf{P}(B\cap C_M^2) - \mathbf{P}\big[(B\cap C_M^2)\backslash\Phi_M(B\cap C_M^1)\big]\\ &> \alpha\big[\mathbf{P}(C_M^2) - \alpha^{N+1}\mathbf{P}(C_M)\big] = \alpha^2\big[\mathbf{P}(C_M^1) - \alpha^N\mathbf{P}(C_M)\big]\\ &\geqslant \alpha^2(\lambda-\alpha^N)\mathbf{P}(C_M) > 0.\end{aligned}$$

Hence both $B\cap C_M^1$ and $\Phi_M(B\cap C_M^1)\cap(B\cap C_M^2)$ are subsets of B, both have positive measure, and the restricted Φ_M is a partial isomorphism between them. Since B is an arbitrary measurable set, we then can conclude that α is in the ratio set. In view of **Proposition 1.20**, we then can conclude that Γ is contained in the ratio set.

Let us prove that the ratio set does not contain anything else. Assume by contradiction that, γ is a number in the ratio set that is not in Γ, and then there exists $\delta > 0$ such that γ is δ-away (in the usual metric in $\mathbb{R}$) from Γ. Fix $a\in A$ such that $\mathbf{P}(C_a) > 0$. Then, by definition of the ratio set **Definition 1.22**, for an $\epsilon\in(0,\delta)$, there exists $D_1, D_2\subseteq C_a$, both of which have positive $\mathbf{P}$-measure, and a partial isomorphism $\Psi : D_1\to D_2$ such that the following set:

$$\mathcal{D}_2 = \Big\{\boldsymbol{a}\in D_2 \mid \frac{d\,\Psi\mathbf{P}}{d\,\mathbf{P}}(\boldsymbol{a})\in(\gamma-\epsilon,\gamma+\epsilon)\Big\}$$

has the same measure as D_2. Fix $\boldsymbol{a}\in\mathcal{D}_2$ and set $\boldsymbol{b} = \Psi^{-1}(\boldsymbol{a})$. Since $\boldsymbol{a},\boldsymbol{b}\in C_a$, we have that $a_0 = b_0$. Since $(\boldsymbol{a},\boldsymbol{b})\in R_T$, there exists $n, m\in\mathbb{Z}^+$ such that $T^n(\boldsymbol{a}) = T^m(\boldsymbol{b})$, which implies that $a_n = b_m$ and, for each $j\in\mathbb{N}$, $T^{n+j}(\boldsymbol{a}) = T^{m+j}(\boldsymbol{b})$. Since, according to **Proposition 1.24**, the shift T is measure-preserving, hence by the **Poincaré recurrence Theorem**, the following set has the same measure as C_a:

$$\mathcal{C}_a = \{\boldsymbol{c}\in C_a \mid \forall\, n\in\mathbb{N}\,\exists\, N\in\mathbb{N}(N>n) \text{ such that } T^N(\boldsymbol{c})\in C_a\}.$$

Without losing generality (or by replacing D_2 by $D_2\cap\mathcal{C}_a$), we can assume the letter a appears for infinitely many times in $\boldsymbol{a}$. Hence, we can then find $N\in\mathbb{N}$ such that $a_{n+N} = a_0 = a = b_0 = b_{m+N}$. By **Definition 2.15**, **Proposition 1.16**, **Theorem 1.18**, and **Corollary 1.26**, we will have:

$$\begin{aligned}&\frac{d\,\Psi\mathbf{P}}{d\,\mathbf{P}}(\boldsymbol{a}) = D(\boldsymbol{b},\boldsymbol{a}) = D\big[\boldsymbol{b}, T^{m+N}(\boldsymbol{b})\big]\,D\big[T^{n+N}(\boldsymbol{a}),\boldsymbol{a}\big]\\ &= \Big(\prod_{0\leqslant i<m+N} D\big[T^{i+1}(\boldsymbol{b}), T^i(\boldsymbol{b})\big]\Big)^{-1}\Big(\prod_{0\leqslant i<n+N} D\big[T^{i+1}(\boldsymbol{a}), T^i(\boldsymbol{a})\big]\Big)\\ &= \Big(\prod_{0\leqslant i<m+N}\frac{\lambda_{b_i}P_{b_ib_{i+1}}}{\lambda_{b_{i+1}}}\Big)^{-1}\Big(\prod_{0\leqslant i<n+N}\frac{\lambda_{a_i}P_{a_ia_{i+1}}}{\lambda_{a_{i+1}}}\Big)\\ &= \frac{\lambda_{b_{m+N}}}{\lambda_{b_0}}\Big(\prod_{0\leqslant i<m+N}P_{b_ib_{i+1}}\Big)^{-1}\frac{\lambda_{a_0}}{\lambda_{a_{n+N}}}\Big(\prod_{0\leqslant i<n+N}P_{a_ia_{i+1}}\Big)\\ &= \Big(\prod_{0\leqslant i<m+N}P_{b_ib_{i+1}}\Big)^{-1}\Big(\prod_{0\leqslant i<n+N}P_{a_ia_{i+1}}\Big)\in(\gamma-\epsilon,\gamma+\epsilon)\end{aligned}\tag{19}$$

Since $a = a_0, a_1, \cdots, a_{n+N-1}, a_{n+N} = a$ and $a = b_0, b_1, \cdots, b_{m+N-1}, b_{m+N}$ are the cycles described in the formulation of **Theorem 1.27**, the number $\Big(\prod_{0\leqslant i<m+N}P_{b_ib_{i+1}}\Big)^{-1}\Big(\prod_{0\leqslant i<n+N}P_{a_ia_{i+1}}\Big)$ is contained in Γ

and hence should be at least δ away from γ, which contradicts (19). Therefore, we then can conclude that Γ is the entire ratio set.

$\square$

Remark 1.28. *In the set-up of **Theorem 1.27**, if we replace* $\boldsymbol{\lambda}$*, the given stationary fully-supported initial distribution, by another fully-supported initial distribution* $\boldsymbol{\lambda}^{'}$*, and keep the same transition matrix, then the new measure* $\mathbf{P}^{'}$ *associated to* $\boldsymbol{\lambda}^{'}$ *and the same transition matrix will be equivalent to the original* $\mathbf{P}$ *according to formula* (10)*, which implies that these two Markov measures have the same ratio set by **Proposition 1.21**.*

2 Boundary measures

2.1 Multiplicative Markov chain associated to $\partial\mathcal{F}$

Definition 2.1. A Markov measure $\boldsymbol{\nu}$ on a topological Markov chain $A_M^{\mathbb{N}}$ (with M the admissibility matrix) is called **multiplicative** if its transition probabilities are determined by its initial distribution ν in such a way that, for each admissible pair $a, b \in A$:

$$P_{a,b} = \frac{\nu(b)}{\sum\limits_{\substack{x \in A \\ M_{ax}=1}} \nu(x)}$$

From now on, we only consider fully supported initial distributions.

In particular, when A is a finite set of free generators and their inverses of a free group $\mathcal{F}$, the admissibility matrix is $M = [M_{a,b}]_{a,b \in A}$ where $M_{a,b} = 0$ precisely when $ab = ba = e$, and $M_{a,b} = 1$ otherwise. The matrix M is irreducible, and the topological Markov chain $A_M^{\mathbb{N}}$ can be identified with the boundary $\partial\mathcal{F}$ (which was defined in **Section 1** as the set of infinite irreducible words). In view of **Definition 2.1**, any distribution ν defined on A determines the associated transition matrix:

$$P_{ab} = \frac{\nu(b)}{1 - \nu(a^{-1})}$$

for each admissible pair (a, b). Then the associated Markov measure $\boldsymbol{\nu}$ will be multiplicative. By the general formula (10), the values of $\boldsymbol{\nu}$ on elementary cylinder sets are:

$$\boldsymbol{\nu}(C_{a_1,\cdots,a_n}) = \nu(a_1) \prod_{1 \leqslant j < n} P_{a_j,a_{j+1}} = \nu(a_1) \prod_{1 \leqslant j < n} \frac{\nu(a_{j+1})}{1 - \nu(a_j^{-1})}. \tag{20}$$

where $a_1, \cdots, a_n$ is an irreducible sequence, i.e., $a_1 a_2 \cdots a_n$ is an element of the free group. Further, for each $a \in A$, define:

$$\pi_a = \frac{\nu(a)}{1 - \nu(a^{-1})}.$$

and hence formula (20) can be written as:

$$\boldsymbol{\nu}(C_{a_1,\cdots,a_n}) = \nu(a_n) \prod_{1 \leqslant j < n} \pi_j. \tag{21}$$

We will again use $\mathcal{B}$ to denote the σ-algebra generated by elementary cylinder sets.

Proposition 2.2. *For any full support initial distribution ν on A the associated multiplicative Markov measure $\boldsymbol{\nu}$ is quasi-invariant with respect to the group action.*

Proof. According to **Proposition 1.5**, it suffices to show that $\boldsymbol{\nu}$ is quasi-invariant for the orbit equivalence relation of the shift transformation T. The claim then immediately follows from **Proposition 1.25**.

□

2.2 Definition of an action cocycle

The framework of this and following subsections is inspired by **[7]**.

Definition 2.3. Given a commutative group of values $(V, +)$ and an action of a countable group G on a space X, a function:

$$c : G \times X \to V$$

is called a (V-valued) **cocycle** if for each $x \in X$ and $g_1, g_2 \in V$:

$$c(g_1 g_2, x) = c(g_1, x) + c(g_2, g_1^{-1} x)$$

Therefore, if we use e to denote the identity of G, then for each $x \in X$, we will have $c(e,x) = c(e,x) + c(e,x)$, which implies that $c(e,x)$ is the identity of V for every $x \in X$.

Remark 2.4. *If one has an action $G \curvearrowright X$ of a group G on a set X, then there are two associated groupoids:*

*i) The **action groupoid of** $G \curvearrowright X$ with the set of objects X, the set of morphisms:*

$$\{(x, g, gx) \mid x \in X, g \in G\}$$

the source map $(x, g, gx) \mapsto x$, and the target map $(x, g, gx) \mapsto gx$. Two morphisms $(x, g, gx), (y, h, hy)$ are composable if $y = gx$ and in this case the composition is:

$$(x, g, gx)(y, h, hy) = (x, hg, hgx)$$

The set of units is $\{(x, e, x) \mid x \in X\}$, and the inverse of a morphism (x, g, gx) is (gx, g^{-1}, x)

ii) In general, given R an equivalence relation defined on X, the associated groupoid has the set of objects X and the set of morphisms R (viewed as a subset of $X \times X$). Two morphisms (x_1, y_1) and (x_2, y_2) are composable if $y_1 = x_2$ and in this case the composition is:

$$(x_1, y_1)(x_2, y_2) = (x_1, y_2)$$

The set of units is the diagonal $\{(x, x) \mid x \in X\}$ and $(x, y)^{-1} = (y, x)$ for each morphism (x, y). In particular, a group action gives rise to the groupoid determined by its orbit equivalence relation.

*Given two groupoids G_1, G_2, a **groupoid homomorphism** $\Phi : G_1 \to G_2$ is a mapping that maps the objects of G_1 to the objects of G_2, and the morphisms of G_1 to the morphisms of G_2 and respects the structure maps. In other words, a groupoid homomorphism is a functor between two categories G_1 and G_2. In our situation there is a homomorphism Φ from the action groupoid in (i) to the orbit equivalence relation groupoid in (ii) determined:*

$$\Phi : (x, g, gx) \mapsto (x, gx)$$

The groupoid homomorphism Φ is an isomorphism if and only if the group action is free (i.e. all point stabilizers are trivial).

*Given a commutative group of values $(V, +)$, a V-valued cocycle of a groupoid is a homomorphism to V. From this point of view, cocycles of equivalence relations (see **Definition 1.15**) and cocycles of group actions are precisely cocycles of the associated groupoids. Further, a cocycle of the orbit equivalence relation of an action can be lifted to an action cocycle. If an action is free, then two groupoids coincide, and therefore in this case there is no difference between the action cocycles and orbit equivalence relation cocycles.*

Definition 2.5. Recall that a measure m defined on a measurable space $(X, \mathcal{B})$ is called **quasi-invariant** with respect to a countable group G acting on X if, for each $g \in G$, the image measure gm with respect to the mapping $x \mapsto g \cdot x$, is equivalent to m. The **Radon-Nikodym cocycle** of a measure m that is quasi-invariant with respect to a countable group G acting on $(X, \mathcal{B})$ is given by the family of Radon-Nikodym derivatives:

$$\Delta_m : G \times X \to \mathbb{R}_+^*, \quad (g, x) \mapsto \frac{d\,gm}{d\,m}(x)$$

By an argument analogous to **Proposition 1.16** and **Proposition 1.17**, Δ_m is a cocycle of the group action.

Definition 2.6. Given a commutative group V, the **potential** of a V-valued cocycle c of the boundary action $\mathcal{F} \curvearrowright \partial\mathcal{F}$ is defined as:

$$\xi : \partial\mathcal{F} \to V, \quad \gamma \mapsto c(\gamma_1, \gamma)$$

where γ_1 denotes the first letter of γ.

Proposition 2.7. *Any function $\xi : \partial\mathcal{F} \to V$ can be uniquely extended to a V-valued cocycle c of the boundary action $\mathcal{F} \curvearrowright \partial\mathcal{F}$ such that the potential of c is ξ.*

Proof. Given a function $\xi : \partial\mathcal{F} \to V$, for each a in A (the set of free generators and their inverses), first define a function $c : A \times \partial\mathcal{F} \to V$ as the following:

$$c(a,\gamma) = \begin{cases} \xi(\gamma), & \gamma \in C_a \\ -\xi(a^{-1}\gamma), & \gamma \notin C_a \end{cases} . \tag{22}$$

Then, for each $a, b \in A$ and $\gamma \in \partial\mathcal{F}$, define $c(ab,\gamma) = c(a,\gamma) + c(b, a^{-1}\gamma)$, and, by induction, we then can have $c(g,\gamma)$ defined for each $g \in \mathcal{F}$ and $\gamma \in \partial\mathcal{F}$. For each $a \in A$ and $g \in \mathcal{F}$, according to the definition of c, we will have $c(ag,\gamma) = c(a,\gamma) + c(g, a^{-1}\gamma)$ for each $\gamma \in \partial\mathcal{F}$. Again by induction, we then have for each $h, g \in \mathcal{F}$ and $\gamma \in \partial\mathcal{F}$, $c(hg,\gamma) = c(h,\gamma) + c(g, h^{-1}\gamma)$, and hence c is a cocycle.

Fix an arbitrary $a \in A$, if $\gamma \in C_a$, then $c(a,\gamma) = \xi(\gamma)$ by definition; otherwise, we will have $a^{-1}\gamma \in C_{a^{-1}}$ and hence $c(a^{-1}, a^{-1}\gamma) = -c(a,\gamma) = \xi(a^{-1}\gamma)$, or $c(a,\gamma) = -\xi(a^{-1}\gamma)$. Therefore, the potential of the c defined according to ξ is indeed ξ.

□

Definition 2.8. Given an assignment $\pi = \{\pi_a\}_{a \in A}$ of positive weights to the set of generators and their inverses, the associated $\mathbb{R}_+^*$-valued **product cocycle** $r = r_\pi$ of the boundary action $\mathcal{F} \curvearrowright \partial\mathcal{F}$ is the cocycle of the boundary action determined by the potential:

$$\rho : \partial\mathcal{F} \to \mathbb{R}_+^*, \quad \gamma \mapsto \frac{1}{\pi_{\gamma_1}}$$

where γ_1 is the first letter of the input γ. Namely, by **Proposition 2.7**, for each $a \in A$:

$$r(a,\gamma) = \begin{cases} (\pi_a)^{-1}, & \gamma \in C_a \\ \pi_{a^{-1}}, & \gamma \notin C_a \end{cases}$$

Proposition 2.9. *In the set-up of **Proposition 2.2**, the potential of the Radon-Nikodym cocycle of the measure $\boldsymbol{\nu}$ with respect to the boundary action of the free group is:*

$$D_{\boldsymbol{\nu}}(\boldsymbol{a}) = \frac{1 - \nu(\boldsymbol{a}_1^{-1})}{\nu(\boldsymbol{a}_1)} \tag{23}$$

Proof. For each $a \in A$, we use $a\boldsymbol{\nu}$ to denote the associated translate of the measure $\boldsymbol{\nu}$. Fix an elementary cylinder set $C_{i_1,\cdots,i_n}$. According to (20), we will have:

$$(a^{-1}\boldsymbol{\nu})(C_{i_1,\cdots,i_n}) = \begin{cases} \nu(a) P_{a,i_1} \prod_{1 \leqslant j < n} P_{i_j,i_{j+1}}, & i_1 \neq a^{-1} \\ \nu(i_2) \prod_{2 \leqslant j < n} P_{i_j,i_{j+1}}, & i_1 = a^{-1} \end{cases}$$

which implies that:

$$\frac{(a^{-1}\boldsymbol{\nu})(C_{i_1,\cdots,i_n})}{\boldsymbol{\nu}(C_{i_1\cdots,i_n})} = \begin{cases} \dfrac{\nu(a)}{1 - \nu(a^{-1})}, & i_1 \neq a^{-1} \\ \\ \dfrac{1 - \nu(a)}{\nu(a^{-1})}, & i_1 = a^{-1} \end{cases} \tag{24}$$

Fix a cylinder set C. Since a cylinder set is a disjoint union of elementary cylinder sets, from (24), we then have:

$$\frac{(a^{-1}\boldsymbol{\nu})(C)}{\boldsymbol{\nu}(C)} = \begin{cases} \dfrac{\nu(a)}{1 - \nu(a^{-1})}, & C \cap C_{a^{-1}} = \varnothing \\ \\ \dfrac{1 - \nu(a)}{\nu(a^{-1})}, & C \subseteq C_{a^{-1}} \end{cases}$$

and hence the equation above holds when C (in both the equation and conditions) is replaced by an arbitrary measurable subset. Therefore, by **Proposition 1.16**, for each $a \in A$ there exists a full measure set $X_a \subseteq A_M^{\mathbb{N}}$ such that the following equation holds:

$$\frac{d\,(a^{-1}\boldsymbol{\nu})}{d\,\boldsymbol{\nu}}(\boldsymbol{a}) = D_{\boldsymbol{\nu}}(\boldsymbol{a}) = \frac{1-\nu(\boldsymbol{a}_1^{-1})}{\nu(\boldsymbol{a}_1)} \tag{25}$$

and hence for each element in $\bigcap_{a \in A} X_a$, (23) holds.

$\square$

Remark 2.10. *By* ***Proposition 2.9****, the Radon-Nikodym cocycle of a multiplicative measure (with the ν the initial distribution) $\boldsymbol{\nu}$ is the product cocycle determined by the family of weights $\{\pi_a\}_{a \in A}$ where for each $a \in A$:*

$$\pi_a = \frac{\nu(a)}{1-\nu(a^{-1})}. \tag{26}$$

More specifically, the potential of such Radon-Nikodym cocycle, and hence the ratio set, is determined by the weights.

2.3 Radon-Nikodym problem

The **Radon-Nikodym problem** consists of the following: given a $\mathbb{R}_+^*$-valued cocycle c (with respect to a countable group G acting on a measurable space), finding a measure m on X quasi-invariant with respect to the group action such that the RN-cocycle Δ_m coincides with c almost everywhere. If such a measure exists, one can further ask about its uniqueness.

Proposition 2.9 shows that a multiplicative measure associated to $\partial\mathcal{F}$ determines a product cocycle. It turns out that the converse is also true under certain circumstances.

Theorem 2.11. *Given a collection of positive weights $\pi = \{\pi_a\}_{a \in A}$ on the alphabet A, the associated product cocycle $r = r_\pi$ (by* ***Definition 2.7****) of the boundary action $\mathcal{F} \curvearrowright \partial\mathcal{F}$ can be realized as the RN-cocycle Δ_ν of a quasi-invariant probability measure $\boldsymbol{\nu}$ on $\partial\mathcal{F}$ iff $\{\pi_a\}_{a \in A} \subseteq (0,1)$ and:*

$$\sum_{a \in A} \frac{\pi_a(1-\pi_{a^{-1}})}{1-\pi_a\pi_{a^{-1}}} = 1 \tag{27}$$

Under these conditions $\boldsymbol{\nu}$ is unique and is the multiplicative Markov measure determined by the initial distribution $[\nu_a]_{a \in A}$ where:

$$\nu_a = \boldsymbol{\nu}(C_a) = \frac{\pi_a(1-\pi_{a^{-1}})}{1-\pi_a\pi_{a^{-1}}} \tag{28}$$

Proof. Suppose first that $\boldsymbol{\nu}$ is a quasi-invariant probability measure on $\partial\mathcal{F}$ with the RN-cocycle $\Delta_{\boldsymbol{\nu}} = r_\pi$. Since the boundary action is minimal, $\boldsymbol{\nu}$ will be fully supported and hence $\nu_a = \boldsymbol{\nu}(C_a) \in (0,1)$ for each $a \in A$, and obviously:

$$\sum_{a \in A} \nu_a = 1$$

According to **Definition 2.7** and the proof of **Proposition 2.9**, we will have:

$$\begin{aligned} \frac{(a\boldsymbol{\nu})(C_a)}{\boldsymbol{\nu}(C_a)} = \frac{1}{\pi_a} \qquad &\Longrightarrow \qquad \pi_a\boldsymbol{\nu}(a^{-1}C_a) = \boldsymbol{\nu}(C_a) \\ \Longrightarrow \ \pi_a\boldsymbol{\nu}(\partial\mathcal{F}\backslash C_{a^{-1}}) &= \pi_a\big[1-\boldsymbol{\nu}(C_{a^{-1}})\big] = \boldsymbol{\nu}(C_a) \\ \Longrightarrow \ \pi_a &= \frac{\nu(a)}{1-\nu(a^{-1})} \end{aligned}$$

which implies that $\pi_a \in (0,1)$ for each $a \in A$. Further, for each $a \in A$, we have:

$$\begin{cases} \pi_a\big[1-\nu_{a^{-1}}\big] = \nu_a \\ \pi_{a^{-1}}\big[1-\nu_a\big] = \nu_{a^{-1}} \end{cases} \quad \Longrightarrow \quad \nu_a = \frac{\pi_a(1-\pi_a^{-1})}{1-\pi_a\pi_{a^{-1}}}$$

and hence:

$$\sum_{a\in A}\nu_a = \sum_{a\in A}\frac{\pi_a(1-\pi_a^{-1})}{1-\pi_a\pi_{a^{-1}}} = 1 \tag{29}$$

Conversely, given the assignment of positive $\{\pi_a\}_{a\in A}$ that satisfies conditions in the formulation, for each $a\in A$, define:

$$\nu_a = \frac{\pi_a(1-\pi_{a^{-1}})}{1-\pi_a\pi_{a^{-1}}}$$

and for each $a,b\in A$, define:

$$P_{a,b} = \begin{cases} \dfrac{\nu_b}{1-\nu_{a^{-1}}}, & b\neq a^{-1} \\ \\ 0, & b = a^{-1} \end{cases}$$

Then the multiplicative measure ν generated by the initial distribution $[\nu_a]_{a\in A}$ and the transition matrix $P = [P_{a,b}]_{a,b\in A}$ will be quasi-invariant by **Proposition 2.2** and its RN-cocycle $\Delta_{\boldsymbol{\nu}}$ will coincide with the product cocycle r_π by **Proposition 2.9** and **Definition 2.7**. Obviously such $\boldsymbol{\nu}$ is unique to the fixed assignment of weights π.

□

Corollary 2.12. *The Radon-Nikodym problem for a product cocycle of the boundary action of a free group is solvable if and only if condition (27) is satisfied, and in this situation the solution is unique.*

Corollary 2.13. *In the set-up of **Proposition 2.11**, the ratio set of $\boldsymbol{\nu}$ is the closure of the multiplicative group generated by the set $\{\pi_a\}_{a\in A}$.*

Proof. According to **Theorem 1.27** and **Remark 1.28**, since the initial distribution ν is fully supported, we then can conclude that the ratio set of $\boldsymbol{\nu}$ is the closure of the multiplicative group generated by numbers of the following form:

$$P_{i_1,i_2}P_{i_2,i_3}\cdots P_{i_n,i_1} = \Big(\prod_{1\leqslant j<n}\frac{\nu(i_{j+1})}{1-\nu(i_j^{-1})}\Big)\frac{\nu(i_1)}{1-\nu(i_n^{-1})} = \prod_{1\leqslant j\leqslant n}\pi_{i_j}$$

which is the same as the closure of the multiplicative group generated by $\{\pi_a\}_{a\in A}$.

□

Remark 2.14. *Formulas (26), (28) establish a one-to-one correspondence between two collections of numbers in the interval $(0,1)$ indexed by A, the alphabet of the free group: the base distribution of a multiplicative Markov measure $\{\nu_a\}_{a\in A}$, subject to the condition that $\{\nu_a\}_{a\in A}$ sums up to 1, and the weights $\{\pi_a\}_{a\in A}$ of the potential of the associated product RN-cocycle, subject to (29).*

2.4 Perron-Frobenius interpretation

There is yet another interpretation of condition (27). Let M be the admissibility matrix introduced in **Definition 2.1** and $D_\pi = \operatorname{diag}(\pi_a)_{a\in A}$. Since M is irreducible, so is MD_π or $D_\pi M$, and hence by **Perron-Frobenius Theorem**, there exists a unique largest eigenvalue for MD_π, which will be denoted by $\rho(MD_\pi)$ (and similar for $\rho(D_\pi M)$).

Theorem 2.15 (**Perron-Frobenius interpretation**). *Let $\pi = \{\pi_a\}_{a\in A}$ be a collection of positive weights on A and M, D_π be defined as above. Then $\rho(D_\pi M) = 1$ iff $\{\pi_a\}_{a\in A}\subseteq(0,1)$ and equation (27) holds.*

Proof. Assume first that $\{\pi_a\}_{a\in A} \subseteq (0,1)$ and that the equation (27) holds. Let ν_a be defined as in the formulation of **Theorem 2.11** and define:

$$P_{a,b} = \begin{cases} \dfrac{\nu_b}{1-\nu_{a^{-1}}}, & b \neq a^{-1} \\ \\ 0, & b = a^{-1} \end{cases}$$

If define $D_\nu = \operatorname{diag}(\nu_a)_{a\in A}$, then for the stochastic matrix $P = [P_{a,b}]_{a,b\in A}$, one can observe that $P = D_\nu^{-1} D_\pi M D_\nu$ and $\rho(P) = 1$. Since eigenvalues do not depend on the basis, as D_ν is invertible, we then have the spectrum of P is the same as $D_\pi M$, which implies that $\rho(P) = \rho(D_\pi M) = 1$.

Conversely, assume $\rho(D_\pi M) = 1$ and by contradiction that $\pi_a \geqslant 1$ for some $a \in A$. We first observe that:

$$\begin{bmatrix} \pi_a & \pi_a \\ \pi_b & \pi_b \end{bmatrix} \begin{bmatrix} \pi_a \\ \pi_b \end{bmatrix} = (\pi_a + \pi_b) \begin{bmatrix} \pi_a \\ \pi_b \end{bmatrix}$$

where $\begin{bmatrix} \pi_a & \pi_a \\ \pi_b & \pi_b \end{bmatrix}$ is a submatrix of $D_\pi M$ and, by normalizing the given vector, we then have:

$$\rho\Big(\begin{bmatrix} \pi_a & \pi_a \\ \pi_b & \pi_b \end{bmatrix} \Big) \geqslant \pi_a + \pi_b > 1.$$

Since $D_\pi M$ is irreducible and positive, then for any principal submatrix B of $D_\pi M$, we will have $\rho(B) < \rho(D_\pi M)$. Then, by induction, we will have:

$$\rho(D_\pi M) > \rho\Big(\begin{bmatrix} \pi_a & \pi_a \\ \pi_b & \pi_b \end{bmatrix} \Big) > 1$$

when $\pi_a \geqslant 1$, which contradicts our assumption. Hence, when $\rho(D_\pi M) = 1$, we will have $\{\pi_a\}_{a\in A} \subseteq (0,1)$. Next, if we use $\sigma(D_\pi M)$ to denote the spectrum of $D_\pi M$, we then will have $\sigma(D_\pi M) \cup \{0\} = \sigma(M D_\pi) \cup \{0\}$ and hence,by **Perron-Frobenius Theorem**, there exists a unique row vector $\boldsymbol{\phi} = [\phi_a]_{a\in A}$ such that $\boldsymbol{\phi} D_\pi M = \boldsymbol{\phi}$. Namely, for each $b \in A$:

$$\phi_b = \sum_{a\in A\setminus\{b^{-1}\}} \pi_a \phi_a = S - \pi_{b^{-1}} \phi_{b^{-1}}$$

where

$$S = \sum_{a\in A} \pi_a \phi_a.$$

Hence, for each $b \in A$, we will have:

$$\begin{cases} \phi_b = \displaystyle\sum_{a\in A\setminus\{b^{-1}\}} \pi_a \phi_a = S - \pi_{b^{-1}} \phi_{b^{-1}} \\ \\ \phi_{b^{-1}} = \displaystyle\sum_{a\in A\setminus\{b\}} \pi_a \phi_a = S - \pi_b \phi_b \end{cases} \implies \phi_b = \frac{1-\pi_{b^{-1}}}{1-\pi_b \pi_{b^{-1}}} S$$

and hence:

$$\sum_{a\in A} \frac{\pi_a(1-\pi_{a^{-1}})}{1-\pi_a \pi_{a^{-1}}} = \frac{1}{S} \sum_{a\in A} \pi_a \phi_a = 1$$

□

3 The harmonic measure on $\partial\mathcal{F}$

3.1 Sample path space

Given A, a set of free generators and their inverses of the free group $\mathcal{F}$, and a probability measure μ on $\mathcal{F}$ with $\operatorname{supp}(\mu) = A$, the product measure μ^{∞} is defined on $A^{\mathbb{N}}$ as follows: given an **elementary cylinder set in** $A_{\mathbb{N}}$:

$$D_{a_1,\cdots,a_n} = \{\boldsymbol{v} \in A^{\mathbb{N}} \mid v_1 = a_1, v_2 = a_2, \cdots, v_n = a_n\}$$

where $a_1, \cdots, a_n \in A$. Let:

$$\mu^{\infty}(D_{a_1,\cdots,a_n}) = \prod_{1\leqslant j\leqslant n} \mu(a_{i_j}).$$

Consider the following mapping:

$$S : A^{\mathbb{N}} \to \mathcal{F}^{\mathbb{Z}_+}, \qquad \boldsymbol{v} = (v_i)_{i\in\mathbb{N}} \mapsto (e, v_1, v_1v_2, \cdots, v_1v_2\cdots v_n, \cdots),$$

and we use $\mathbf{P}_\mu$ to denote the image of μ^{∞} under the mapping S. Given $h_1, \cdots, h_n \in \mathcal{F}$, an **elementary cylinder set in** $\mathcal{F}^{\mathbb{Z}_+}$ is defined as:

$$C_{h_1,\cdots,h_n} = \{\boldsymbol{g} \in \mathcal{F}^{\mathbb{Z}_+} \mid \boldsymbol{g}_0 = e, \boldsymbol{g}_1 = h_1, \cdots, \boldsymbol{g}_n = h_n\}$$

and the $\mathbf{P}_\mu$-value of $C_{h_1,\cdots,h_n}$ is:

$$\mathbf{P}_\mu(C_{h_1,\cdots,h_n}) = \mu(h_1) \prod_{1\leqslant j<n} \mu(h_j^{-1}h_{j+1}).$$

For each $n \in \mathbb{N}$ and $h \in \mathcal{F}$, we define:

$$C(n,h) = \{\boldsymbol{g} \in \mathcal{F}^{\mathbb{Z}_+} \mid \boldsymbol{g}_n = h\}.$$

and the $\mathbf{P}_\mu$-value of $C(n,h)$ is:

$$\mathbf{P}_\mu\big[C(n,h)\big] = \prod_{\substack{a_1,\cdots,a_n\in A\\ a_1a_2\cdots a_n=h}} \prod_{1\leqslant j\leqslant n} \mu(a_j).$$

Therefore, the one-dimensional distribution of $\mathbf{P}_\mu$ at time n, (i.e., the distribution of the n-th position of $\boldsymbol{g}$ in $\mathcal{F}^{\mathbb{Z}_+}$), by the definition of $\mathbf{P}_\mu$ is the n-fold convolution of the step distribution μ:

$$\mu^{*n}(h) = \mathbf{P}_\mu\big[C(n,h)\big].$$

Equivalently, the measure $\mathbf{P}_\mu$ is the measure on the space of sample paths of the Markov chain with the transition probabilities:

$$P_{h_1,h_2} = \mu(h_1^{-1}h_2)$$

issued from the group identity at time 0. We use $(\mathcal{F}, \mu)$ to denote this random walk. Next, we need the following theorem (whose proof is in **Appendix**) for defining the harmonic measure $\boldsymbol{\nu}$ in $\partial\mathcal{F}$.

Theorem 3.1 (Furstenberg). *For $\boldsymbol{P}_\mu$-almost every $\boldsymbol{g} \in \mathcal{F}^{\mathbb{Z}_+}$, $\{\boldsymbol{g}_i\}_{i\in\mathbb{N}}$ converges to a point in $\partial\mathcal{F}$ with respect to δ-metric. Hence the limit map:*

$$L : \mathcal{F}^{\mathbb{Z}_+} \to \partial\mathcal{F}, \qquad \boldsymbol{g} = (g_i)_{i\in\mathbb{Z}_+} \mapsto \lim_i g_i$$

is well-defined $\boldsymbol{P}_\mu$-almost everywhere.

For each $n \in \mathbb{N}$, define:

$$P_n : \mathcal{F}^{\mathbb{Z}_+} \to \overline{\mathcal{F}}, \qquad \boldsymbol{g} \mapsto g_n.$$

Since L is the point-wise limit of $\{P_n\}_{n\in\mathbb{N}}$, a sequence of measurable maps from $\mathbf{P}_\mu$ to $\overline{\mathcal{F}}$, then L is also measurable. In the measurable space $(\partial\mathcal{F}, \mathfrak{B})$ where $\mathfrak{B}$ is the σ-algebra generated by elementary cylinder sets, the **harmonic measure** $\boldsymbol{\nu}$ is defined as the image of $\mathbf{P}_\mu$ on $\mathcal{F}^{\mathbb{Z}_+}$ under the map L (i.e., for each $A \in \mathcal{B}$, $\boldsymbol{\nu}(A) = \mathbf{P}_\mu\big[L^{-1}(A)\big]$).

3.2 Properties of the harmonic measure

Recall that μ is a probability measure defined on $\mathcal{F}$ with support on A, the finite set of free generators and their inverses. Let L be the limit map given by **Theorem 3.1**. For simplicity, for each $a \in A$, define $\mu_a = \mu(a)$. Recall that for each $h \in \mathcal{F}$ and $l \in \mathbb{N}$, $C(l,h) = \{\boldsymbol{g} \in \mathcal{F}^{\mathbb{Z}_+} \mid g_l = h\}$. For each $a \in A$, define:

$$\pi_a = \mathbf{P}_\mu\Big[\bigcup_{l\in\mathbb{N}} C(l,a)\Big].$$

which is the probability of ever visiting a by the random walk. Therefore:

$$\pi_a = \mu_a + \pi_a \sum_{b\in A\setminus\{a\}} \mu_b \pi_{b^{-1}} \tag{30}$$

Since $\mu_a \in (0,1)$ for each $a \in A$, we then have $\pi_a \in (0,1)$. If, for each $a \in A$, we define:

$$y = 1 - \sum_{a\in A} \mu_a \pi_{a^{-1}} \qquad x_a = 1 - \sum_{b\in A\setminus\{a\}} \mu_b \pi_{b^{-1}}$$

then (30) can be rewritten as:

$$\pi_a x_a = \mu_a = \pi_a(y + \mu_a \pi_{a^{-1}}) = \pi_a y + \mu_a \pi_a \pi_{a^{-1}} \tag{31}$$

Now back to the random walk $(\mathcal{F},\mu)$. For each $a \in A$, define ν_a to be the probability of the following event in $\mathcal{F}^{\mathbb{Z}_+}$:

$$\nu_a = \mathbf{P}_\mu[\boldsymbol{g} \in \mathcal{F}^{\mathbb{Z}_+} : L(\boldsymbol{g})_1 = a].$$

By definition of the harmonic measure $\boldsymbol{\nu}$ (given in **Section 3.1**), we then have:

$$\nu_a = \boldsymbol{\nu}\Big(\{x \in \partial\mathcal{F} \mid x_1 = a\}\Big)$$

and $\boldsymbol{\nu}$ has full support because the boundary action is minimal. By definition of π_a, given $a, b \in A$ with $ab \neq e$, we have:

$$\begin{aligned} &\boldsymbol{\nu}(C_{a,b}) = \mathbf{P}_\mu\big[\boldsymbol{g} \in \mathcal{F}^{\mathbb{Z}_+} : L(\boldsymbol{g})_1 = a, L(\boldsymbol{g})_2 = b\big] = \pi_a \mathbf{P}_\mu\big[\boldsymbol{g} \in \mathcal{F}^{\mathbb{Z}_+} : L(\boldsymbol{g})_1 = b\big] \\ \Longrightarrow\ &\boldsymbol{\nu}(C_{a,b}) = \pi_a \cdot (a\boldsymbol{\nu})\big(C_{a,b}\big) = \pi_a \cdot \boldsymbol{\nu}(C_b) = \pi_a \nu_b \end{aligned} \tag{32}$$

Therefore, for each $a \in A$:

$$\begin{aligned} &\nu_a = \mathbf{P}_\mu\big[g \in \mathcal{F}^{\mathbb{Z}_+} : L(\boldsymbol{g})_1 = a\big] = \sum_{b\in A\setminus\{a^{-1}\}} \mathbf{P}_\mu\big[\boldsymbol{g} \in \mathcal{F}^{\mathbb{Z}_+} : L(\boldsymbol{g})_1 = a, L(\boldsymbol{g})_2 = b\big] \\ \Longrightarrow\ &\nu_a = \sum_{b\in A\setminus\{a^{-1}\}} \pi_a \nu_b = \pi_a(1 - \nu_{a^{-1}}) \\ \Longrightarrow\ &\begin{cases} \pi_a(1-\nu_{a^{-1}}) = \nu_a \\ \pi_{a^{-1}}(1-\nu_a) = \nu_{a^{-1}} \end{cases} \quad \Longrightarrow\ \nu_a = \frac{\pi_a(1-\pi_{a^{-1}})}{1-\pi_a\pi_{a^{-1}}} \end{aligned}$$

Since each $\pi_a \in (0,1)$, we must have $\nu_a \in (0,1)$ and the family $\{\nu_a\}_{a\in A}$ sums up to 1 since $\mathbf{P}_\mu$ is a probability measure.

Theorem 3.2. *The harmonic measure $\boldsymbol{\nu}$ is a multiplicative Markov measure.*

Proof. By inductively iterating the formula (32), we obtain that for each $g = g_1 g_2 \cdots g_n \in \mathcal{F}$:

$$\boldsymbol{\nu}(C_{g_1,\cdots,g_n}) = \nu_{g_n} \prod_{1\leqslant i<n} \pi_{g_i}$$

which is precisely the formula (21).

□

Remark 3.3. *In view of* ***Remark 2.14****, the passage probabilities* π_a *can now be interpreted as the weights of the potential of the product RN-cocycle of the harmonic measure.*

Corollary 3.4. *The Radon-Nikodym cocycle of the harmonic measure is the product cocycle determined by* π_a *(given in* ***Remark 3.2****).*

Theorem 3.5. *The ratio set of the harmonic measure defined on the boundary action is the multiplicative group generated by* $\{\pi_a\}_{a\in A}$ *(given in* ***Remark 3.2****)*

Proof. Immediate according to **Corollary 2.12**.

□

We recall that, according to **Remark 2.14**, there is a one-to-one correspondence between the families of weights $\{\nu_a\}_{a\in A}$ and $\{\pi_a\}_{a\in A}$ subject to the respective conditions. It turns out that one can extend it to a bijection with yet another family of weights $\{\mu_a\}_{a\in A}$ (the values of the step distribution) subject to the natural condition that the family $\{\mu_a\}_{a\in A}$ sums up to 1.

Theorem 3.6. *Equation* (30) *establishes a bijective correspondence between a fully supported probability measure* $\mu = \{\mu_a\}_{a\in A}$ *on* A *and the associated passage probabilities* $\{\pi_a\}_{a\in A}$*. Namely, given a probability measure* $\mu = \{\mu_a\}_{a\in A}$ *on* A*, the resulting passage probabilities all belong to the interval* $(0,1)$ *and satisfy condition (27), and, conversely, any collection of weights* $\{\pi_a\}_{a\in A}$ *with the above property uniquely determines a probability measure* μ *on* A*, such that* $\{\pi_a\}_{a\in A}$ *are the passage probabilities of the random walk* $(\mathcal{F},\mu)$*.*

Proof. Given μ a probability measure defined and fully supported on A, the new collection of weights $\{\pi_a\}_{a\in A}$ given by **Remark 3.2** is unique to μ and satisfies equation (27) and $\{\pi_a\}_{a\in A} \subseteq (0,1)$. Conversely, suppose $\{\pi_a\}_{a\in A} \subseteq (0,1)$ and satisfies (27). Then we need to find the solution set $\{\mu_a\}_{a\in A}$ to the linear system of equations in (27) and establish its uniqueness. From now on, assume $\{\mu_a\}_{a\in A}$ is a set of variables. By (31), if we set:

$$S = \sum_{a\in A} \mu_a \pi_{a^{-1}}$$

we then will have, for each $a \in A$:

$$\mu_a(1-\pi_a\pi_{a^{-1}}) = \pi_a(1-S).$$

Further, define:

$$Z = \sum_{a\in A} \frac{\pi_a\pi_{a^{-1}}}{1-\pi_a\pi_{a^{-1}}}$$

which will give us:

$$S = \sum_{a\in A} \mu_a\pi_{a^{-1}} = (1-S)\sum_{a\in A} \frac{\pi_a\pi_{a^{-1}}}{1-\pi_a\pi_{a^{-1}}} = (1-S)Z \quad \Longrightarrow \quad S = 1 - \frac{1}{1+Z}$$

Therefore, for each $a \in A$:

$$\mu_a = \frac{1}{1+Z}\frac{\pi_a}{1-\pi_a\pi_{a^{-1}}}.$$

Since $\{\pi_a\}_{a\in A}$ satisfies equation (27), we then have:

$$1 = \sum_{a\in A} \frac{\pi_a(1-\pi_{a^{-1}})}{1-\pi_a\pi_{a^{-1}}} = \sum_{a\in A} \frac{\pi_a}{1-\pi_a\pi_{a^{-1}}} - Z \quad \Longrightarrow \quad \sum_{a\in A} \frac{\pi_a}{1-\pi_a\pi_{a^{-1}}} = 1 + Z$$

which implies that $p = \{\mu_a\}_{a\in A}$ is a probability measure defined and fully supported on A.

□

4 Finding the ratio set by Martin Kernel

The method to find the ratio set of a system generated by a random walk defined on a hyperbolic group Γ is originally from **[2]**. Here we show how it applies to our case of a nearest neighbor random walk on a free group, which provides an alternative approach to finding the ratio set of the harmonic measure.

4.1 Preliminary

i) The random walk (Γ, p) is assumed **transient**: i.e. the **Green Function** defined below converges for each $x, y \in \Gamma$:

$$G(x,y) = \sum_{n \geqslant 0} p^{(n)}(x,y) \qquad \text{where} \quad P^n = [p^{(n)}(x,y)]_{x,y \in \Gamma}$$

where $p^{(n)}$ is the n-fold convolution. For each $s \in S$, set $q_s = G(e,s)$. Given $x, y \in \Gamma$, if $|\, x^{-1}y \,| = k$, then, for each $n \in \mathbb{N}(n \geqslant k)$, we have:

$$p^{(k)}(x,y) = p^{(k)}(e, x^{-1}y)$$

where $[x^{-1}y]_i$ denotes the i-th letter of $x^{-1}y$. Therefore, $G(x,y) = G(e, x^{-1}y)$. Recall the definition of P_μ, the measure associated with μ defined on the space of sample paths. By **Theorem 3.1**, we have that P_μ-almost every sample path converges to an element in $\partial\mathcal{F}$ and we continue to use L to denote the limit map as in **Section 3.1**.

ii) The **Martin Kernel** is defined by:

$$K(\cdot,\cdot) : \mathcal{F} \times \mathcal{F} \to \mathbb{R}, \quad (x,y) \mapsto \frac{G(x,y)}{G(e,y)}.$$

The **Martin compacification** $\mathcal{F}_1$ is the smallest compact (topological) space that the metric space $(\mathcal{F}, \delta)$ (see **Section 1.2**) as a discrete subspace, and where, for each $x \in \mathcal{F}$, $K(x, \cdot)$ is continuous in $\mathcal{F}_1$. Below we will show that the topology in $\mathcal{F}_1$ coincides with the δ-metric topology in $(\overline{\mathcal{F}}, \delta)$ (i.e., $\mathcal{F}_1 = \overline{\mathcal{F}}$).

Proposition 4.1. *The **Martin Compacification** of the metric space $(\mathcal{F}, \delta)$ is the space $(\overline{\mathcal{F}}, \delta)$. Therefore, for each $x \in \mathcal{F}, K(x, \cdot)$ is also continuous on $\partial\mathcal{F}$ and, for each $x = x_1 \cdots x_n \in \mathcal{F}$ and $\xi \in \partial\mathcal{F}$, if we have $m = |\, x \wedge \xi \,|$, then:*

$$K(x,\xi) = \Big[\prod_{1 \leqslant j \leqslant n-m} q_{x^{-1}_{m+j}} \Big] \Big[\prod_{1 \leqslant j \leqslant m} q_{\xi_j} \Big]^{-1}$$

Proof. Fix $x = x_1 \cdots x_m \in \mathcal{F}$. For each $g = g_1 g_2 \cdots g_n \in \mathcal{F}$, if we set $k = |\, x \wedge g \,|$, we then have:

$$K(x,g) = \frac{G(e, x^{-1}g)}{G(e,g)} = \Big[\prod_{1 \leqslant j \leqslant m-k} q_{x^{-1}_{k+j}} \prod_{1 \leqslant j \leqslant n-k} q_{g_{k+j}} \Big] \Big[\prod_{1 \leqslant j \leqslant n} q_{g_j} \Big]^{-1} = \Big[\prod_{1 \leqslant j \leqslant m-k} q_{x^{-1}_{k+j}} \Big] \Big[\prod_{1 \leqslant j \leqslant k} q_{g_j} \Big]^{-1}$$

We proved that $(\overline{\mathcal{F}}, \delta)$ is compact in **Section 1**. Given $\{g_n\}_{n \in \mathbb{N}} \subseteq \mathcal{F}$ and $g_n \to \xi \in \partial\mathcal{F}$ in δ-metric topology, because $x \in \mathcal{F}$,$|\, x \wedge g_n \,|$ will eventually be constant so that $K(x, g_n)$ will eventually be constant. Hence, for each $x \in \mathcal{F}$, $K(x, \cdot)$ can be continuously extended to $\partial\mathcal{F}$, which implies that the δ-metric topology is stronger.

Meanwhile, given $\{x_n\}_{n \in \mathbb{N}} \subseteq \mathcal{F}$ and $x_n \to \xi$ in $\mathcal{F}_1$, we then have for each $x \in \mathcal{F}$, $K(x, x_n) \to K(x, \xi)$, which implies that $|\, x \wedge x_n \,| \to |\, x \wedge \xi \,|$ for each $x \in \mathcal{F}$. Hence, for each $N \in \mathbb{N}$ (for definition of $[\xi]_N$ see **Section 1.2**):

$$
\begin{aligned}
&\big|\,[\xi]_N \wedge x_n\,\big| \overset{n\to\infty}{\longrightarrow} \big|\,[\xi_N] \wedge \xi\,\big| \\
\implies\ & [\xi]_N \wedge x_n \overset{n\to\infty}{\longrightarrow} [\xi]_N
\end{aligned}
$$

which can then imply $\delta(x_n, \xi) \to 0$, or $x_n \to \xi$ in δ-metric topology.

□

4.2 Finding the generators of the ratio set by Martin Kernel

Proposition 4.2. *For each $g \in \mathcal{F}$, set $g^+ = \lim_{n\to\infty} g^n$. Then, for each $g \in \mathcal{F}, \lambda_g = K(g, g^+)$ is in the ratio set.*

Proof. Since the metric δ defined on $\overline{\mathcal{F}}$ is given by $\delta(x,y) = \exp(-|\,x \wedge y\,|)$ is an ultrametric, each open ball will be both closed and open. Since $(\overline{\mathcal{F}}, \delta)$ is a compact metric space, it will be also separable and we can assume the set $F = \{f_n\}_{n\in\mathbb{N}}$ is dense in $\overline{\mathcal{F}}$. Because δ is an ultrametric, we can further assume that $F \subseteq \mathcal{F}$. For each $n \in \mathbb{N}$, we can find a finite set $F_n \subseteq F$ such that:

$$\overline{\mathcal{F}} = \bigcup_{f\in F_n} B(f, e^{-n})$$

We can further assume $F_n \subseteq F_{n+1}$ for each $n \in \mathbb{N}$ (by replacing F_n by the union of the first n sets). Then, for each $n \in \mathbb{N}$, we use $\mathcal{B}_n$ to denote the σ-algebra generated by $\{B(f, e^{-n})\}_{f\in F_n}$. For each $n \in \mathbb{N}$, once $\mathcal{B}_n$ is defined, we define $\mathcal{B}_{n+1}$ to be the σ-algebra generated by the following two sets:

$$\mathcal{B}_n, \quad \{B(f, e^{-n+1})\}_{f\in F_{n+1}}$$

Now we have a increasing sequence of σ-algebras $\{\mathcal{B}_n\}_{n\in\mathbb{N}}$. Notice that the Borel σ-algebra $\mathcal{B}$ in $(\overline{\mathcal{F}}, \delta)$ is generated by open balls. For each open ball B in $\overline{\mathcal{F}}$, we can find $f_N \in F$ and N such that $B(f_N, e^{-N}) \subseteq B$ is contained in the open ball we picked. Then, we can further find $f \in F_m$ for some $m \in \mathbb{N}$ such that f is closed enough to f_N and $B(f, e^{-m}) \subseteq B$. This shows that $\mathcal{B}$ is contained in the union of all $\mathcal{B}_n$ and hence $\{\mathcal{B}_n\}_{n\in\mathbb{N}}$ is a filtration that generates $\mathcal{B}$. Recall $\boldsymbol{\nu}$ is the harmonic measure defined in **Section 3**. By **Martingale Convergence Theorem for Conditional Probabilities**, for each measurable subset $A \subseteq \partial\mathcal{F}$, we have that for $\boldsymbol{\nu}$-almost each $x \in \partial\mathcal{F}$:

$$\boldsymbol{\nu}(A\,|\,\mathcal{B}_n)(x) \to \chi_A(x)$$

Let $\mathcal{B}_1$ denote the set of all Borel measurable sets in $\partial\mathcal{F}$ that have positive $\boldsymbol{\nu}$-measure. If we fix $A \in \mathcal{B}_1$, then the convergence above implies that for $\boldsymbol{\nu}$-almost each $x \in A$:

$$\lim_n \frac{\boldsymbol{\nu}[A \cap B(x, e^{-n})]}{\boldsymbol{\nu}[B(x, e^{-n})]} = 1$$

Notice that $B(x, e^{-n})$ can be replaced by $B([x]_n, e^{-n})$ (see **Section 1.2**). Fix $x \in A$. Now back to the proof of this proposition. Fix $A \in \mathcal{B}_1$ and $g \in \mathcal{F}$. Then we can find $k \in \mathbb{N}$ such that $k > |\,g\,|$ and $A \cap B([x]_k, e^{-k}) \in \mathcal{B}_1$. By the convergence above, we can further assume the integer k we picked satisfies:

$$1 - \epsilon < \frac{\boldsymbol{\nu}[A \cap B([x]_k, e^{-k})]}{\boldsymbol{\nu}[B([x]_k, e^{-k})]} < 1 + \epsilon$$

Without losing generality (or consider a proper subsequence), we can further assume the inequality above holds for all large enough k such that:

$$\boldsymbol{\nu}[A \cap B([x]_k, e^{-k})] > \boldsymbol{\nu}[B([x]_k, e^{-k})](1-\epsilon) \quad \implies \quad \boldsymbol{\nu}[B([x]_k, e^{-k})\backslash A] < \epsilon\, B([x]_k, e^{-k}) \tag{33}$$

By **Proposition 4.1**, $K(g, \cdot)$ is continuous on $\partial\mathcal{F}$ with respect to the δ-metric topology. Hence, we can find $N \in \mathbb{N}$ such that:

$$B(g^+, e^{-N}) \subseteq K(g, \cdot)^{-1}(\lambda_g - \epsilon, \lambda_g + \epsilon)$$

Since such N is independent to the choice of K that satisfies (33), we can then assume $k > N$ from now on. Fix $M \in \mathbb{N}$ such that $|\, g^M \,| > N$. Then we will have $g^M \in B(g^+, e^{-N})$ and, since δ is an ultrametric, we then have $B(g^M, e^{-N}) = B(g^+, e^{-N})$. Since, for each $h \in \mathcal{F}$:

$$\lim_k [x]_k = \lim_k [x]_k h = x \quad \Longrightarrow \quad \lim_k \delta([x]_k h, [x]_k) = 0$$

we can then assume k is large enough (and still greater than N) such that:

$$\begin{aligned} & \delta([x]_k g^M, [x]_k) = \delta([x]_k g^{M-1}, [x]_k) = e^{-k} \quad \Longrightarrow \quad [x]_k g^M \in B([x]_k, e^{-k}) \\ \Longrightarrow\ & B([x]_k g^M, e^{-k}) = B([x]_k, e^{-k}) \\ \Longrightarrow\ & [x]_k^{-1}[B([x]_k g^M, e^{-k})] = B(g^M, e^{-N}) \subseteq B(g^+, e^{-N}). \end{aligned}$$

Since, for each $x \in \mathcal{F}, K(x, \cdot) = d(x\boldsymbol{\nu})/d\boldsymbol{\nu}$, by **Proposition 1.16** and **Proposition 1.17**, for each $\xi \in \partial\mathcal{F}$ and $h \in \mathcal{F}$, we have:

$$\begin{aligned} K(h, (hgh^{-1})^{-1}\xi) &= \frac{d\,h\boldsymbol{\nu}}{d\boldsymbol{\nu}} \circ (hgh^{-1})^{-1}(\xi) = \frac{d\,(hgh^{-1})\,h\boldsymbol{\nu}}{d\,(hgh^{-1})\boldsymbol{\nu}}(\xi) = \frac{d\,(hg)\boldsymbol{\nu}}{d\,(hgh^{-1})\boldsymbol{\nu}}(\xi) \\ K(g, h^{-1}\xi) &= \frac{d\,g\boldsymbol{\nu}}{d\boldsymbol{\nu}} \circ (h^{-1}\xi) = \frac{d\,hg\boldsymbol{\nu}}{d\,h\boldsymbol{\nu}}(\xi) \end{aligned}$$

Therefore, for each $\xi \in \partial\mathcal{F}$:

$$K(hgh^{-1}, \xi) = \frac{d\,hgh^{-1}\boldsymbol{\nu}}{d\boldsymbol{\nu}}(\xi) = \Big[\frac{d\,hgh^{-1}\boldsymbol{\nu}}{d\,hg\boldsymbol{\nu}}(\xi)\Big] \cdot \Big[\frac{d\,hg\boldsymbol{\nu}}{d\,h\boldsymbol{\nu}}(\xi)\Big] \cdot \Big[\frac{d\,h\boldsymbol{\nu}}{d\boldsymbol{\nu}}(\xi)\Big].$$

Together with **Proposition 1.16** and **Proposition 1.17**, we will have:

$$K(hgh^{-1}, \xi) = K(g, h^{-1}\xi)\frac{K(h, \xi)}{K(h, (hgh^{-1})^{-1}\xi)} = K(g, h^{-1}\xi)\frac{K(h, \xi)}{K(h, hg^{-1}h^{-1}\xi)}. \tag{34}$$

According to the general formula of the Martin Kernel proved in **Proposition 4.1**. For an arbitrary $h \in \mathcal{F}, |\, h \,| = N$ and $\xi \in \partial\mathcal{F}$, if we let $n = |\, h \wedge \xi \,|$, we then have:

$$K(h, \xi) = \Big[\prod_{1 \leqslant i \leqslant N-n} q_{h^{-1}_{n+i}}\Big]\Big[\prod_{1 \leqslant j \leqslant n} q_{\xi_j}\Big]^{-1}.$$

Hence, for each $\xi_0 \in \partial\mathcal{F}$ with $\delta(\xi, \xi_0) = e^{-n}$, we have $K(h, \xi_0) = K(h, \xi)$. Thus:

$$\forall\, \xi \in B(h, e^{-N})\, \forall\, \xi_0 \in B(\xi, e^{-N}), \qquad K(h, \xi_0) = K(h, \xi)$$

which implies that, for each $h \in \mathcal{F}$, $K(h, \cdot)$ is locally constant. According to our choice of k and N, since $\delta([x]_k g^M, [x]_k) = \delta([x]_k g^{M-1}, [x]_k) = e^{-k}$, we then have:

$$\begin{aligned} & B([x]_k, e^{-k}) = B([x]_k g^M, e^{-k}) = B([x]_k g^{M-1}, e^{-k}) \\ \Longrightarrow\ & \forall\, y \in B([x]_k, e^{-k}), \quad K([x]_k, y) = K([x]_k g^M, y) = K([x]_k g^{M-1}, y). \end{aligned}$$

From now on, set $h = [x]_k$ and fix an arbitrary y from the following set:

$$hg^M h^{-1}\big[A \cap B(h, e^{-k})\big] = hg^M h^{-1}(A) \cap B(hg^M, e^{-k}).$$

Therefore, $hg^{-1}h^{-1}y \in B(hg^{M-1}, e^{-k})$. Combining results above with equations (33), (34) gives us:

$$\frac{d\,hgh^{-1}\boldsymbol{\nu}}{d\boldsymbol{\nu}}(y) = K(hgh^{-1}, y) = K(g, h^{-1}y) \in (\lambda_g - \epsilon, \lambda_g + \epsilon).$$

Since y is arbitrarily picked, we then can conclude:

$$hg^M h^{-1}\big[A \cap B(h, e^{-k})\big] \subseteq K(hgh^{-1}, \cdot)^{-1}(\lambda_g - \epsilon, \lambda_g + \epsilon). \tag{35}$$

Set $r = \boldsymbol{\nu}[A \cap B(h, e^{-k})]$ and $X_M = hg^M h^{-1}\big[A \cap B(h, e^{-k})\big]$. By (33), $r > 0$ and by (35):

$$\boldsymbol{\nu}\big[hgh^{-1}(X_M)\big] = \int_{X_M} K(hgh^{-1}, \xi)\, d\boldsymbol{\nu}(\xi) \geqslant \boldsymbol{\nu}(X_M)(\lambda_g - \epsilon).$$

According to **Proposition 2.2**, the boundary action is non-singular with respect to $\boldsymbol{\nu}$ and T_h is a partial isomorphism for each $h \in \mathcal{F}$. Since $r > 0$, there exists $\epsilon(M)$ such that $\epsilon(M)$ approaches to zero as M approaches to infinity and the following implication holds:

$$\begin{aligned} & 0 < \boldsymbol{\nu}\big[B(h, e^{-k})\backslash A\big] < \epsilon\, \boldsymbol{\nu}\big[B(h, e^{-k})\big] \\ \Longrightarrow\ \ & 0 < \boldsymbol{\nu}\Big(hg^M h^{-1}\big[B(h, e^{-k})\backslash A\big]\Big) < \epsilon(M). \end{aligned} \tag{36}$$

As $k \to \infty$, we will have the difference between $\boldsymbol{\nu}\big[A \cap B(h, e^{-k})\big]$ and $\boldsymbol{\nu}\big[B(h, e^{-k})\big]$ is approaching zero. Since k is assumed to be always greater than M, we can further assume that k is large enough such that:

$$\boldsymbol{\nu}(X_M)(\lambda_g - \epsilon) > \epsilon(M). \tag{37}$$

Notice that:

$$\begin{aligned} hg^{M+1}h^{-1}(A) \cap B(hg^{M+1}, e^{-k}) &= \Big(\big[hg^M h^{-1}(A) \cap B(hg^{M+1}, e^{-k})\big] \cap hgh^{-1}(X_M)\Big) \\ &\quad \bigcup \Big(\big[hg^M h^{-1}(A) \cap B(hg^{M+1}, e^{-k})\big]\backslash hgh^{-1}(X_M)\Big) \\ &= \big[hg^M h^{-1}(A) \cap hgh^{-1}(X_M)\big] \\ &\quad \bigcup \Big(\big[hg^M h^{-1}(A) \cap B(hg^{M+1}, e^{-k})\big]\backslash hgh^{-1}(X_M)\Big). \end{aligned}$$

Indeed $X_M \subseteq B(hg^M, e^{-k})$ implies that $hgh^{-1}(X_M) \subseteq B(hg^{M+1}, e^{-k})$. Hence, we have:

$$\begin{aligned} & \boldsymbol{\nu}\big[hg^M h^{-1}(A) \cap hgh^{-1}(X_M)\big] \\ &= \boldsymbol{\nu}\big[hg^M h^{-1}(A) \cap B(hg^{M+1}, e^{-k})\big] - \boldsymbol{\nu}\Big(\big[hg^M h^{-1}(A) \cap B(hg^{M+1}, e^{-k})\big]\backslash hgh^{-1}(X_M)\Big). \end{aligned} \tag{38}$$

Also, we have:

$$\begin{aligned} & hg^M h^{-1}(A) \cap B(hg^{M+1}, e^{-k}) \subseteq B(hg^{M+1}, e^{-k}) \\ \Longrightarrow\ \ & \nu\Big(\big[hg^M h^{-1}(A) \cap B(hg^{M+1}, e^{-k})\big]\backslash hgh^{-1}(X_M)\Big) \leqslant \nu\Big(B(hg^{M+1}, e^{-k})\backslash hgh^{-1}(X_M)\Big). \end{aligned} \tag{39}$$

Notice that:

$$B(hg^{M+1}, e^{-k})\backslash hgh^{-1}(X_M) = hg^{M+1}h^{-1}\Big(B(h, e^{-k})\backslash\big[A \cap B(h, e^{-k})\big]\Big) = hg^{M+1}h^{-1}\big[B(h, e^{-k})\backslash A\big].$$

Similarly, we have:

$$\begin{aligned} X_M &= hg^M h^{-1}(A)\ \cap\ B(hg^M, e^{-k}) \\ &= \Big(hg^M h^{-1}(A) \cap B(hg^{M+1}, e^{-k})\Big)\ \cup\ \Big(\big[hg^M h^{-1}(A) \cap B(hg^M, e^{-k})\big]\backslash B(hg^{M+1}, e^{-k})\Big) \\ &\subseteq \Big(hg^M h^{-1}(A) \cap B(hg^{M+1}, e^{-k})\Big)\ \cup\ \Big(B(hg^M, e^{-k})\backslash B(hg^{M+1}, e^{-k}\Big) \\ \Longrightarrow\ \ & \nu\Big(hg^M h^{-1}(A) \cap B(hg^{M+1}, e^{-k})\Big) \geqslant \nu(X_M) - \nu\big[B(hg^M, e^{-k})\backslash B(hg^{M+1}, e^{-k})\big] \end{aligned} \tag{40}$$

and:

$$\begin{aligned} & B(hg^M, e^{-k})\backslash hgh^{-1}(X_M) \\ &= \Big(\big[B(hg^M, e^{-k}\backslash hgh^{-1}(X_M)\big] \cap B(hg^{M+1}, e^{-k})\Big) \cup \Big(\big[B(hg^M, e^{-k}\backslash hgh^{-1}(X_M)\big]\backslash B(hg^{M+1}, e^{-k})\Big) \\ &= \Big(B(hg^{M+1}, e^{-k})\backslash hgh^{-1}(X_M)\Big) \cup \Big(B(hg^M, e^{-k})\backslash B(hg^{M+1}, e^{-k})\Big) \end{aligned} \tag{41}$$

because $hgh^{-1}(X_M) \subseteq B(hg^{M+1}, e^{-k})$. Combining (31) $\sim$ (36) gives us:

$$\begin{aligned}
&\boldsymbol{\nu}\big[hg^M h^{-1}(A) \cap hgh^{-1}(X_M)\big] \\
&\geqslant \boldsymbol{\nu}\big[hg^M h^{-1}(A) \cap B(hg^{M+1}, e^{-k})\big] - \boldsymbol{\nu}\Big(B(hg^{M+1}, e^{-k}) \backslash hgh^{-1}(X_M)\Big) \\
&\geqslant \boldsymbol{\nu}(X_M) - \boldsymbol{\nu}[B(hg^M, e^{-k}) \backslash B(hg^{M+1}, e^{-k})] - \boldsymbol{\nu}\Big(B(hg^{M+1}, e^{-k}) \backslash hgh^{-1}(X_M)\Big) \\
&= \boldsymbol{\nu}(X_M) - \boldsymbol{\nu}\Big(B(hg^M, e^{-k}) \backslash hgh^{-1}(X_M)\Big) \\
&= \boldsymbol{\nu}(X_M) + \boldsymbol{\nu}\big[hgh^{-1}(X_M)\big] - \boldsymbol{\nu}\big[B(hg^M, e^{-k})\big] \\
&= \boldsymbol{\nu}\big[hgh^{-1}(X_M)\big] - \boldsymbol{\nu}\Big(hg^M h^{-1}\big[B(h, e^{-k}) \backslash A\big]\Big) \geqslant \boldsymbol{\nu}(X_M)(\lambda_g - \epsilon) - \epsilon(M) > 0
\end{aligned}$$

and hence:

$$\begin{aligned}
&hg^M h^{-1}(A) \cap hgh^{-1}(X_M) = hg^M h^{-1}(A) \cap hg^{M+1}h^{-1}\big[A \cap B(h, e^{-k}\big] \\
&= hg^M h^{-1}(A) \cap hg^{M+1}h^{-1}\big[A \cap B(h, e^{-k})\big] \cap \big\{hg^{M+1}h^{-1}(\xi) \,|\, \xi \in \partial\mathcal{F},\ K(hgh^{-1}, hg^{M+1}h^{-1}\xi) \in (\lambda_g - \epsilon, \lambda_g + \epsilon)\big\} \\
&= hg^M h^{-1}\Big(\underbrace{A \cap hgh^{-1}\big[A \cap B(h, e^{-k}\big] \cap \big\{\xi \in \partial\mathcal{F} \,|\, K(hgh^{-1}, \xi) \in (\lambda_g - \epsilon, \lambda_g + \epsilon)\big\}}_{A_0}\Big).
\end{aligned}$$

Again by **Theorem 1.14**, since $hg^M h^{-1}(A_0)$ has positive $\boldsymbol{\nu}$-measure, A_0 also has positive $\boldsymbol{\nu}$-measure. Then we can conclude the following set, which contains A_0, has positive $\boldsymbol{\nu}$-measure:

$$A \cap hgh^{-1}(A) \cap \big\{\xi \in \partial\mathcal{F} \,|\, K(hgh^{-1}, \xi) \in (\lambda_g - \epsilon, \lambda_g + \epsilon)\big\}$$

Since $hgh^{-1} \in \mathcal{F}$ and ϵ, A are arbitrarily picked, we then can conclude λ_g is in the ratio set.

$\square$

Lemma 4.3. *Apply the terminology from **Proposition 3.2**. We will have that for each $g \in \mathcal{F}$ and $m \in \mathbb{Z}$, $K(g^m, g^+) = \lambda_g^m = K(g, g^+)^m$.*

Proof. Recall that for each $\xi \in \partial\mathcal{F}$, we have that for $\boldsymbol{\nu}$-almost every $\xi \in \partial\mathcal{F}$:

$$K(g, \xi) = \frac{d\, g\boldsymbol{\nu}}{d\boldsymbol{\nu}}(\xi).$$

Then, by **Proposition 1.16** and **Proposition 1.17**, we have that for almost each $\xi \in \partial\mathcal{F}$

$$K(g^2, \xi) = \frac{d\, g^2\, \boldsymbol{\nu}}{d\boldsymbol{\nu}}(\xi) = \Big(\frac{d\, g\, \boldsymbol{\nu}}{d\boldsymbol{\nu}}(\xi)\Big) \cdot \Big(\frac{d\, g^2\, \boldsymbol{\nu}}{d\, g\boldsymbol{\nu}}\Big) = \Big(\frac{d\, g\, \boldsymbol{\nu}}{d\boldsymbol{\nu}}(\xi)\Big) \cdot \Big(\frac{d\, g\, \boldsymbol{\nu}}{d\boldsymbol{\nu}} \circ g^{-1}(\xi)\Big) = K(g, \xi)K(g, g^{-1}\xi).$$

Since $g^+ = \lim_{n\to\infty} g^n$, we then have:

$$K(g^2, g^+) = K(g, g^+)K(g, g^{-1}g^+) = K(g, g^+)^2 = \lambda_g^2$$

By induction, we then have for each $m \in \mathbb{N}$, $K(g^m, g^+) = \lambda_g^m$. Since $K(e, \cdot)$ is constantly equal to 1 on $\partial\mathcal{F}$, again by **Proposition 1.16** and **Proposition 1.17**, we will have:

$$K(e, g^+) = K(g, g^+)K(g^{-1}, g^{-1}g^+) = 1 \quad \Longrightarrow \quad K(g^{-1}, g^+) = \lambda_g^{-1}$$

and then:

$$K(g^{-2}, g^+) = K(g^{-1}, g^+)K(g^{-1}, gg^+) = K(g^{-1}, g^+)^2 = \lambda_g^{-2}.$$

Our desired conclusion follows by induction.

$\square$

Theorem 4.4. *Recall that we use $r_{\boldsymbol{\nu}}(\mathcal{F})$ to denote the ratio set of the system $(\partial\mathcal{F}, \Omega, \boldsymbol{\nu}, \mathcal{F})$. Then $r_{\nu}(\mathcal{F})\backslash\{0\}$ is a multiplicative subgroup of $(0, \infty)$ and is generated by the set $\{\lambda_g\}_{g\in\mathcal{F}}$.*

Proof. Recall that, for each $g \in \mathcal{F}$, the constant λ_g is defined by:

$$\lambda_g = \lim_n K(g, g^n)$$

From **Proposition 1.20**, we have that $r_\nu(\mathcal{F})\backslash\{0\}$ contains the multiplicative group generated by the set $\{\lambda_g\}_{g\in\mathcal{F}}$. If $\{\lambda_g\}_{g\in\mathcal{F}}$ contains more than one element, then $\{\lambda_g\}_{g\in\mathcal{F}}$ generates the entire $(0, \infty)$. Therefore, because $r_\nu(\mathcal{F})\backslash\{0\}$ is contained in $(0, \infty)$, $r_\nu(\mathcal{F}) = (0, \infty)$, namely the multiplicative group genernated by $\{\lambda_g\}_{g\in\mathcal{F}}$. If $\{\lambda_g\}_{g\in\mathcal{F}}$ only contains one element, suppose for each $g \in \mathcal{F}$, $\lambda_g = \lambda$ for some $\lambda \in (0, 1)$. What remains unproved is that there are no other elements in $r_{\boldsymbol{\nu}}(\mathcal{F})$. Suppose $r \in r_{\boldsymbol{\nu}}(\mathcal{F})$. Recall that, in **Proposition 4.2**, for each $g \in \mathcal{F}$, we define:

$$g^+ = \lim_n g^n.$$

Fix $g \in \mathcal{F}$ and $\epsilon \in (0, r)$. For each $m \in \mathbb{N}$, we can find $N(m) \in \mathbb{N}$ such that $N(m) > |\, g^m \,|$ and $N(m) \bmod |\, g^m \,| < |\, g \,|$. Since r is in the ratio set, we then can find $h(m) \in \mathcal{F}$ such that:

$$\mathcal{A}_r = B(g^+, e^{-N(m)}) \cap h(m)^{-1}\big[B(g^+, e^{-N(m)})\big] \cap \big\{\xi \in \partial\mathcal{F} \,|\, K(h(m), \xi) \in (r - \epsilon, r + \epsilon)\big\}$$

has positive $\boldsymbol{\nu}$-measure. Now fix an arbitrary $\xi \in \mathcal{A}_r$. From $N(m) > |\, g^m \,|$, we then can find $x, y \in \partial\mathcal{F}$ such that:

$$\xi = g^m x = h(m)^{-1} g^m y \quad \Longrightarrow \quad h(m) g^m x = g^m y.$$

In general, there will be only two possibilities for $h(m)$: either $h(m) = g^k$ for some $k \in \mathbb{Z}$, or $h(m) = g^k l_m$ for some $k \in \mathbb{Z}$ and $l_m \in \mathcal{F}$ such that l_m does not contain any power of g in the front (which is equivalent to $|\, l_m \wedge g \,| < |\, g \,|$ and $|\, l_m \wedge g^{-1} \,| < |\, g \,|$). If the second case is true, we then have:

$$h(m) g^m x = g^k l_m g^m x = g^m \tag{42}$$

Below we discuss possibly values for the integer k:

(i) When $k \geqslant 0$, we must have $k \geqslant m$ because $|\, l_m \wedge g \,| < |\, g \,|$ by assumption. We then have:

$$g^m x = h(m)^{-1} g^m y = l_m^{-1} g^{-k} g^m y = l_m^{-1} g^{-(m-k)} y.$$

Because we already assumed that $|\, l_m \wedge g^{-1} \,| < |\, g^{-1} \,|$, which implies $|\, l_m^{-1} \wedge g \,| < |\, g \,|$, we then obtain a contradiction.

(ii) When $k < 0$, according to (37), l_m must contain at least one g in the front, which again contradicts our assumption.

Therefore, now we can conclude that $h(m)$ is a power of g and we assume $h(m) = g^{K(m)}$ for some $K(m) \in \mathbb{Z}$. According to **Lemma 4.3**, we have:

$$K(h(m), g^+) = \lambda_g^{K(m)} \in (r - \epsilon, r + \epsilon)$$

(because multiplying g^+ by any power of g still returns g^+). Again, we discuss possibilities for the sequence $\{K(m)\}_{m\in\mathbb{N}}$:

(1) If there exists a subsequence, say $\{k(m)\}_{m\in\mathbb{N}}$, of $\{K(m)\}_{m\in\mathbb{N}}$ such that $\lim_m k(m) = \infty$, then we will have that for each arbitrarily small ϵ:

$$r < \lim_m \lambda_g^{k(m)} + \epsilon = \epsilon$$

because $\lambda_g \in (0, 1)$, which implies that $r = 0$, contradicting our assumption that $r \in r_{\boldsymbol{\nu}}(\mathcal{F})\backslash\{0\}$.

(2) If there exists a subsequence, say $\{k(m)\}_{m\in\mathbb{N}}$, of $\{K(m)\}_{m\in\mathbb{N}}$ such that $\lim_m k(m) = -\infty$, then we will have that for each arbitrarily small ϵ:

$$r > \lim_m \lambda_g^{k(m)} - \epsilon = \lim_m (\frac{1}{\lambda_g})^{-k(m)} - \epsilon = \infty$$

which is absurd because $\lambda_g \in (0, 1)$.

Therefore, we can now conclude that the sequence $\{K(m)\}_{m\in\mathbb{N}}$ is bounded both above and below. Suppose $\{K(m)\}_{m\in\mathbb{N}} \subseteq \{-K, -K+1, \cdots, K-1, K\}$ for some $K \in \mathbb{N}$. We then have $r \in \{\lambda_g^{-K}, \lambda_g^{-K+1}, \cdots, \lambda_g^{K}\}$. This shows that r is in the cyclic group generated by λ_g, and this proves that $r_{\boldsymbol{\nu}}(\mathcal{F})$ is the cyclic group generated by λ_g whenever $\{\lambda_g\}_{g\in\mathcal{F}}$ has only one element.

$\square$

A Proof of Furstenberg's Theorem

Definition A.1. A discrete time **martingale** is a discrete time real-valued stochastic process such that:

$$\forall\, n \in \mathbb{N},\ \ E[X_n] < \infty \qquad \text{and} \qquad E[X_{n+1} \,|\, X_1, X_2, \cdots, X_n] = X_n.$$

Remark A.2. *Recall that in **Section 3.1**, $\mathcal{F}^{\mathbb{Z}_+}$ denotes the sample path space. We use $\mathcal{A}$ to denote the σ-algebra generated by elementary cylinder sets in $\mathcal{F}^{\mathbb{Z}_+}$, $\mathcal{A}_n (n \in \mathbb{N})$ to denote the σ-algebra generated by cylinder set in the first n coordinates, and $\mathbf{P}_\mu$ the probability measure in $\mathcal{P}$, which arises from the step distribution in the random walk $(\mathcal{F}, \mu)$ (where the support of μ is only in the alphabet set). Now suppose f is a μ-harmonic function defined on $\mathcal{F}$. Then for an arbitrary $m \in \mathbb{N}$, define:*

$$\Phi_m : \mathcal{F}^{\mathbb{Z}_+} \to \mathbb{R}, \quad \boldsymbol{g} \mapsto f(\boldsymbol{g}_m).$$

*By definition of the **conditional expectation with respect to a σ-algebra**, we have that, for an arbitrary $A_n \in \mathcal{A}_n$, given $m \in \mathbb{N}$ with $m \geqslant n$:*

$$E[\Phi_m \,|\, \mathcal{A}_n]\mathbf{P}_\mu(A_n) = \int_{A_n} E[\Phi_m \,|\, \mathcal{A}_n]\, d\mathbf{P}_\mu = \int_{A_n} \Phi_m \, d\mathbf{P}_\mu = \int_{A_n} f(\boldsymbol{g}_m)\, d\mathbf{P}_\mu(\boldsymbol{g}).$$

Since f is μ-harmonic, with the fixed $n \in \mathbb{N}$ and a fixed $\boldsymbol{g} \in \mathcal{F}^{\mathbb{Z}_+}$, we have:

$$\int_{A_n} f(\boldsymbol{g}_{n+1})\, d\mathbf{P}_\mu(\boldsymbol{g}) = \int_{A_n} \Big(\sum_{h \in \mathcal{F}} f(\boldsymbol{g}_n h)\mu(h) \Big)\, d\mathbf{P}_\mu(\boldsymbol{g}) = \int_{A_n} f(\boldsymbol{g}_n)\, d\mathbf{P}_\mu(\boldsymbol{g})$$

Indeed, since $A_n \in \mathcal{A}_n$, for each $\boldsymbol{g} \in A_n$, there will be no restrictions on $\boldsymbol{g}_{n+1}$. Hence, for each $m \in \mathbb{N}$ with $m \geqslant n$, we have:

$$\begin{aligned} E\big[\Phi_m \,|\, \mathcal{A}_n\big]\mathbf{P}_\mu(A_n) &= \int_{A_n} f(\boldsymbol{g}_m)\, d\mathbf{P}_\mu(\boldsymbol{g}) = \int_{A_n} f(\boldsymbol{g}_n)\, d\mathbf{P}_\mu(\boldsymbol{g}) \\ &= \int_{A_n} \Phi_n \, d\mathbf{P}_\mu(\boldsymbol{g}) = \Phi_n \mathbf{P}_\mu(A_n) \\ \Longrightarrow \quad E\big[\Phi_m \,|\, \mathcal{A}_n\big]\mathbf{P}_\mu(A_n) &= \Phi_n \mathbf{P}_\mu(A_n) \end{aligned}$$

Since A_n is arbitrarily picked from $\mathcal{A}_n$, we have that $E[\Phi_m \,|\, \mathcal{A}_n] = \Phi_n$. Now we can conclude $\{\Phi_m\}_{m \in \mathbb{N}}$ is a discrete time martingale.

Proposition A.3. *The boundary action of $\mathcal{F}$ has no finite invariant sets.*

Proof. Since the boundary action is minimal (i.e. any orbit is dense), there are no closed invariant sets, and, in particular, no finite invariant sets.

□

Definition A.4. In a finite measure space (X, Ω, m), given $A \in \Omega$ with $m(A) > 0$, a **ϵ-partition of A** is a (at most countable) partition $\{A_i\}_{i \leqslant N}$ of A such that $m(A_i) \in [0, \epsilon)$ for each $i \in \mathbb{N}$. In this case, whenever an ϵ-partition of A exists, we say m **admits an ϵ-partition of** A.

Lemma A.5. *In a finite measure space (X, Ω, θ), if θ is purely non-atomic, then for each $\epsilon \in (0, 1)$, θ admits an ϵ-partition of X.*

Proof. Without losing generality, assume $\theta(X) = 1$. Define:

$$l_1 = \inf \big\{ \epsilon \in (0, 1) \,|\, \theta \text{ admits an } \epsilon\text{-partition of } X \big\}$$

Equivalently, we need to prove that $l_1 = 0$. Assume by contradiction that $l_1 > 0$. Therefore, for each $\epsilon \in (l_1, 1)$, if θ admits an ϵ-partition of X, then there is a set in that ϵ-partition with θ-measure greater than or equal to l_1. Since θ is purely non-atomic, we have $l_1 < \theta(X) = 1$. Next, set:

$$n_1 = \min \Big\{ n \in \mathbb{N} \,\Big|\, \theta \text{ admits an } (l_1 + \frac{1}{n})\text{-partition of } X \text{ and } l_1 + \frac{1}{n} < 1 \Big\}.$$

Without losing generality (or restrict to a subsequence of $\mathbb{N}$), assume for each $n \geqslant n_1$, θ admits an $(l_1 + \frac{1}{n})$-partition. For each $n \geqslant n_1$, by taking intersection between sets from a $(l_1 + \frac{1}{n})$-partition and a $(l_1 + \frac{1}{n+1})$-partition, we can further assume that the fixed $(l_1 + \frac{1}{n+1})$-partition is a refinement of the fixed $(l_1 + \frac{1}{n})$-partition. For each $n \geqslant n_1$, fix a $(l_1 + \frac{1}{n})$-partition of X and let $C_{n,1}$ be the set in the fixed $(l + \frac{1}{n})$-partition such that $\theta(C_{n,1}) \geqslant l$. Then by assumption we have:

$$\forall\, n \geqslant n_1,\ \ l_1 \leqslant \theta(C_{n,1}) < l_1 + \frac{1}{n}$$

Again by assumption, we have the sequence $\{\theta(C_{n,1})\}_{n \geqslant n_1}$ is non-increasing. Hence:

$$\forall\, n \geqslant n_1, \quad l \leqslant \theta\Big(\bigcap_{n \geqslant n_1} C_{n,1}\Big) < l_1 + \frac{1}{n} \quad \implies \quad \theta\Big(\bigcap_{n \geqslant n_1} C_{n,1}\Big) = l_1$$

Set $A_1 = \bigcap_{n \geqslant n_1} C_n$ and then we have $\theta(A_1) = l_1$. Next, we turn to $X \backslash A_1$ and define:

$$l_2 = \inf \big\{\epsilon \in (0,1) \,|\, \theta \text{ admits an } \epsilon\text{-partition of } X \backslash A_1\big\}$$

If $l_2 = 0$, then there exists $\alpha \in (0, l_1)$ such that θ admits an α-partition of $X \backslash A_1$. Since θ is purely non-atomic, we can find $B_1 \subsetneq A_1$ such that $\theta(B_1) \in \big(0, \theta(A_1)\big)$. If we define:

$$\beta = \max \big\{\alpha, \theta(B_1), \theta(A_1 \backslash B_1)\big\}$$

then the α-partition of $X \backslash A_1$ along with B_1 and $A_1 \backslash B_1$ consist of an β-partition of X and $\beta < l_1$, contradicting the definition of l_1. Therefore, $l_2 > 0$.

By the same approach, we then can find $A_2 \subseteq X \backslash A_1$ such that $\theta(A_2) = l_2$. Since, for each $\epsilon \in (l_1, 1)$ and each measurable subset $A \subseteq X$, an ϵ-partition of X induces an ϵ-partition of A, we then necessarily have $l_2 \leqslant l_1$, or $\theta(A_2) \leqslant \theta(A_1)$. By induction, for each $n \in \mathbb{N}$, we have that $\{A_n\}_{n \in \mathbb{N}}$ is a partition of X and $\{\theta(A_n)\}_{n \in \mathbb{N}}$ is a non-increasing sequence. Since $\theta(X) = 1$, there exists $N \in \mathbb{N}$ such that $\theta(A_{N+1}) < \theta(A_N) \leqslant l_1$. Since θ is purely non-atomic, for each $i \in \mathbb{N}$ and $i \leqslant N$, there exists $B_i \subsetneq A_i$ such that $0 < \theta(B_i) < \theta(A_i) = l_1$. Set:

$$l_0 = \max \big\{\theta(B_1), \cdots, \theta(B_N), \theta(A \backslash B_1), \cdots, \theta(A \backslash B_N), \theta(A_{N+1})\big\}$$

Therefore, the following partition:

$$\big\{A_n\big\}_{n > N} \cup \big\{B_i\big\}_{i \leqslant N} \cup \big\{A_i \backslash B_i\big\}_{i \leqslant N}$$

is an l_0-partition with $l_0 < l_1$, which contradicts the definition of l. Hence, we have $l_1 = 0$

□

Proposition A.6. *If θ is a purely non-atomic (regular) probability measure defined on the metric space $(\partial\mathcal{F}, \delta)$, then for each $\epsilon > 0$, there exists $N_\epsilon \in \mathbb{N}$ such that for each $g \in \mathcal{F}$ with $|\, g \,| \geqslant N_\epsilon$, $\theta(C_g) < \epsilon$.*

Proof. Since $(\partial\mathcal{F}, \delta)$ is compact (hence totally bounded) and δ is an ultra-metric, each open set can be written as a finite disjoint union of elementary cylinder sets. In particular, given an open subset $O \subseteq \partial\mathcal{F}$ and $\epsilon \in (0,1)$, we can find $\{g_i\}_{i \leqslant M} \subseteq \mathcal{F}$ such that:

$$O = \dot{\bigcup}_{i \leqslant M} C_{g_i}$$

and $\exp\big(-\,|\, g_i \,|\big) < \epsilon$. Fix $\epsilon \in (0,1)$. By **Lemma A.5**, compactness of the space and (outer) regularity of θ, there exists a finite ϵ-partition of $\partial\mathcal{F}$, say $\{O_i\}_{i \leqslant K}$, where each O_i. Then by our previous observation, we can find $\{g_j\}_{j \leqslant N} \subseteq \mathcal{F}$ such that the open cover $\{C_{g_j}\}_{j \leqslant N}$ is a refinement of the open cover $\{O_i\}_{i \leqslant K}$. Next, set:

$$N_\epsilon = \max_{j \leqslant N} |\, g_j \,|$$

Notice that, for each $\xi \in \partial\mathcal{F}$, if $\xi \in C_{g_j}$ for some $j \leqslant N$, we then have $B\big(\xi, \exp(-N)\big) \subseteq C_{g_j}$ for all $N \geqslant N_\epsilon$. Therefore, for each $g \in \mathcal{F}$ with $|\, g \,| \geqslant N_\epsilon$, there exists g_j for some $j \leqslant N$ such that:

$$C_g \subseteq C_{g_j} \quad \Longrightarrow \quad \theta(C_g) \leqslant \theta(C_{g_j}) \leqslant \max_{i\leqslant K}\theta(O_i) < \epsilon$$

□

Theorem A.7. *In the set-up of **Section 3.1**, let $\mathcal{F}^{\mathbb{Z}_+}$ denote the space of all sample paths of the random walk $(\mathcal{F},\mu)$, and $\mathbf{P}_\mu$ denote the measure in $\mathcal{F}^{\mathbb{Z}_+}$ arising from μ. If the group generated by the support of μ is the whole group, then:*

i) For $\mathbf{P}_\mu$-almost each sample path $\boldsymbol{g}$ of the random walk in $\mathcal{F}^{\mathbb{Z}_+}$, $\{\boldsymbol{g}_m\}_{m\in\mathbb{N}}$ converges to an element in $\partial\mathcal{F}$ (in δ-metric topology) and hence the following mapping:

$$L : \mathcal{F}^{\mathbb{Z}_+} \to \partial\mathcal{F}, \qquad \boldsymbol{g} \mapsto \lim_m \boldsymbol{g}_m$$

*is well-defined $\mathbf{P}_\mu$-almost everywhere. The **hitting distribution** on $\partial\mathcal{F}$ is defined as the image of $\mathbf{P}_\mu$ on $\mathcal{F}^{\mathbb{Z}_+}$ under L.*

ii) The following limit of convolution powers of μ:

$$\lim_{n\to\infty}\mu^{*n}$$

exists and defines a Borel probability measure $\boldsymbol{\nu}$ on $\partial\mathcal{F}$. This limit coincides with the hitting distribution above, is purely non-atomic, and the unique μ-stationary probability measure on $\partial\mathcal{F}$, i.e., the unique probability measure on $\partial\mathcal{F}$ such that $\mu * \boldsymbol{\nu} = \boldsymbol{\nu}$.

Proof.

i) Suppose θ is a purely non-atomic (regular) probability measure on $\partial\mathcal{F}$. Thus any singelton will be θ-null and hence so is any countable set. Suppose $\{g_n\}_{n\in\mathbb{N}}$ is a sequence in $\mathcal{F}$ such that $\lim_n g_n = g_\infty$ for some $g_\infty \in \partial\mathcal{F}$. Then we will show that the sequence of measures $\{g_n\theta\}_{n\in\mathbb{N}}$ converges weakly to δ_{g_∞} (the Dirac measure defined on g_∞) in the following sense:

$$\forall\, F \in C(\partial\mathcal{F}), \quad \lim_n \int_{\partial\mathcal{F}} F(\xi)\, d(g_n\theta)(\xi) = \lim_n \int_{\partial\mathcal{F}} F(g_n\xi)\, d\theta(\xi) = \int_{\partial\mathcal{F}} F(\xi)\, d\delta_{g_\infty}(\xi) = F(g_\infty) \tag{43}$$

By **Portemanteau Theorem** (**Theorem 13.16** in [**6**]), since for each $n \in \mathbb{N}$, $g_n\theta(\partial\mathcal{F}) = \theta(\partial\mathcal{F}) = 1$, it suffices to show that for all closed subsets $F \subseteq \partial\mathcal{F}$, $\limsup_n g_n\theta(F) \leqslant \delta_{g_\infty}(F)$. As we showed in the proof of **Proposition A.6**, for each open subset $O \subseteq \partial\mathcal{F}$ and $\epsilon(0,1)$, we have find $\{g_i\}_{i\leqslant K} \subseteq \mathcal{F}$ such that:

$$O = \dot{\bigcup}_{i\leqslant K} C_{g_i}$$

where $\exp\big(-\,|\,g_i\,|\big) < \epsilon$ for all $i \leqslant K$. Hence:

$$O^c = \bigcap_{i\leqslant K} (C_{g_j})^c = \bigcap_{i\leqslant K}\left(\dot{\bigcup_{\substack{g\in\mathcal{F}\setminus\{g_i\}\\ |\,g\,|=|\,g_i\,|}}} C_g\right)$$

and we can now conclude that every closed set can be written as a finite disjoint union of elementary cylinder sets. Hence, it suffices to show that for each $h \in \mathcal{F}$, $\limsup_n g_n\theta(C_h) \leqslant \delta_{g_\infty}(C_h)$. If $g_\infty \in C_h$, then $\delta_{g_\infty}(C_h) = 1$ and the inequality immediately holds since θ is a probability measure. Otherwise, without losing generality (or start from a large enough n), suppose $|\,g_n\,| > |\,h\,|$ for all $n \in \mathbb{N}$. Therefore, $g_n^{-1}C_h = C_{g_n^{-1}h}$ and $|\,g_n^{-1}h\,| \xrightarrow{n\to\infty} \infty$. By **Proposition A.6**, we then have:

$$\limsup_n (g_n\theta)(C_h) = \limsup_n \theta(C_{g_n^{-1}h}) = 0 = \delta_{g_\infty}(C_h)$$

Now we can conclude that $g_n\theta$ converges to δ_{g_∞} weakly and hence (43) holds.

ii) Let θ be an arbitrary probability measure and hence the following set:

$$\Big\{\frac{1}{n}\sum_{0\leqslant k<n}(\mu^{*\,k})\,*\,\theta\Big\}_{n\in\mathbb{N}}$$

is a sequence of probability measures defined on $\partial\mathcal{F}$. According to the weak compactness of the closed unit ball of $[C(\partial\mathcal{F})]^*$, this sequence has a sub-sequence converging to some θ_0 in weak-$*$ topology. Suppose:

$$\frac{1}{n_i}\sum_{0\leqslant k<n_i}(\mu^{*k})\,*\,\theta\to\theta_0.$$

Then

$$\begin{aligned}
&\mu\,*\,\Big[\frac{1}{n_i}\sum_{0\leqslant k<n_i}(\mu^{*k})\,*\,\theta\Big]\\
=&\,\frac{1}{n_i}\sum_{1\leqslant k\leqslant n_i}(\mu^{*k})\,*\,\theta\\
=&\,\Big[\frac{1}{n_i}\sum_{0\leqslant k<n_i}(\mu^{*k})\,*\,\theta\Big]-\frac{1}{n_i}\theta+\frac{1}{n_i}(\mu^{*n_i}\,*\,\theta)\overset{n\to\infty}{\to}\theta_0.
\end{aligned}$$

Indeed, for each $n\in\mathbb{N},\mu^{*n}*\theta$ is a probability measure on $\partial\mathcal{F}$ and hence $\frac{1}{n}\mu^{*n}\,*\,\theta$, as a bounded linear functional, will converge to zero in norm as n becomes arbitrarily large. We can then conclude θ_0 is μ-stationary (namely $\mu\,*\,\theta_0=\theta_0$) and hence there always exists a μ-stationary probability measure in $\partial\mathcal{F}$.

iii) Given θ_0 a μ-stationary probability measure, suppose it has atoms and we use $\mathcal{A}$ to denote the set of θ_0-atoms. Set:

$$\lambda=\sup_{x\in\mathcal{A}}\theta_0(\{x\}).$$

Since θ_0 is a probability measure, we will have $\lambda\in[0,1]$. If θ_0 can not attain λ for some $x\in\mathcal{A}$, then for a fixed $\epsilon\in(0,\lambda)$, we can find an infinite set $\{x_n\}_{n\in\mathbb{N}}\subseteq\mathcal{A}$ such that $\theta(\{x_n\})\geqslant\lambda-\epsilon$ for all large n. As a result, $\theta_0(\partial\mathcal{F})=\infty$, which is absurd. Hence, we can find $x_0\in\mathcal{A}$ such that $\theta_0(\{x_0\})=\lambda$. Because θ_0 is μ-stationary, we then have:

$$\theta_0(\{x_0\})=(\mu*\theta_0)(\{x_0\})=\sum_{h\in\mathcal{F}}\theta_0(\{h^{-1}x_0\})\mu(h)\tag{44}$$

Because $\mu(h)\in[0,1]$ for each $h\in\mathcal{F}$, for the equality above to hold, we must have $\theta_0(\{h^{-1}x_0\})=\theta_0(\{x_0\})$ precisely when $\mu(h)>0$. Set $\mathcal{S}$ to be the support of μ and $\mathcal{L}=\{s^{-1}x_0\,|\,s\in\mathcal{S}\}$. Since θ_0 is a probability measure, $\mathcal{L}$ must be finite and has the same size of $\mathcal{S}$. Hence, according to (44), for each $s\in\mathcal{S}$, we have:

$$\begin{aligned}
&\theta_0(\{x_0\})=\sum_{h\in\mathcal{F}}\theta_0\big[\{(h^{-1}s)\,(s^{-1}x_0)\big]\mu(h)=\sum_{g\in\mathcal{F}}\theta_0\big[\{g^{-1}\,(s^{-1}x_0)\}\big]\mu(sg)\\
\Longrightarrow\;&\theta_0(\{x_0\})=\frac{1}{|\,\mathcal{S}\,|}\sum_{s\in\mathcal{S}}\sum_{g\in\mathcal{F}}\theta_0\big[\{g^{-1}\,(s^{-1}x_0)\}\big]\mu(sg)=\frac{1}{|\,\mathcal{S}\,|}\sum_{g\in\mathcal{F}}\sum_{s\in\mathcal{S}}\theta_0\big[\{g^{-1}\,(s^{-1}x_0)\}\big]\mu(sg)
\end{aligned}$$

which implies:

$$\forall\,g\in\mathcal{F}\,\forall\,s\in\mathcal{S},\quad\theta_0\big[g^{-1}\,(s^{-1}x_0)\big]=\theta_0(\{x_0\})\iff\mu(sg)>0.$$

Suppose there exists $g\in\mathcal{F}$ and $l\in\mathcal{L}$ such that $g^{-1}l\notin\mathcal{L}$. This implies $\mathcal{S}\cap(\mathcal{S}\cdot g)\neq\mathcal{S}$ where $\mathcal{S}\cdot g=\{sg\,|\,s\in\mathcal{S}\}$. Therefore:

$$\begin{aligned}
&\theta_0(\{x_0\})\\
&= \sum_{h\in\mathcal{S}} \theta_0(\{h^{-1}x_0\})\mu(h) = \sum_{h\in\mathcal{F}} \theta_0(\{h^{-1}x_0\})\mu(h)\\
&= \sum_{h\in\mathcal{F}} \theta_0\big[\{(gh^{-1})\,x_0\}\big]\mu(hg^{-1}) = \sum_{f\in\mathcal{S}\cdot g} \theta_0(\{f^{-1}x_0\})\mu(f)\\
&= \sum_{f\in\mathcal{S}\cap\mathcal{S}\cdot g} \theta_0(\{f^{-1}x\})\mu(f) < \sum_{h\in\mathcal{S}} \theta_0(\{h^{-1}x_0\})\mu(h) = \theta_0(\{x_0\})
\end{aligned}$$

which is absurd. Hence we have $g^{-1}\cdot\mathcal{L} \subseteq \mathcal{L}$ for each $g\in\mathcal{F}$, or $\mathcal{L}$ is a $\mathcal{F}$-invariant finite subset in $\partial\mathcal{F}$, contradicting **Proposition A.3**. Hence, a μ-stationary probability measure is purely non-atomic.

iv) Fix a μ-stationary probability measure λ. Then, for each $\hat{f}\in C[\partial\mathcal{F}]$, define a function on $\mathcal{F}$ by:

$$f(g) = \int_{\partial\mathcal{F}} \hat{f}(\xi)\, d\,[g\lambda(\xi)]$$

By definition the $\mathcal{F}$-action on the space of probability measures, we have have:

$$f(g) = \int_{\partial\mathcal{F}} \hat{f}(g\xi)\, d[\lambda(\xi)]. \tag{45}$$

For each $\hat{f}\in C[\partial\mathcal{F}]$, we claim that the function f defined above is μ-harmonic: namely, for each $g\in\mathcal{F}$, $f(g) = \sum_{h\in\mathcal{F}} f(gh)\mu(h)$. Then, by definition of $\mu*\lambda$:

$$\begin{aligned}
\sum_{h\in\mathcal{F}} f(gh)\mu(h) &= \sum_{h\in\mathcal{F}} \Big(\int_{\partial\mathcal{F}} \hat{f}(\xi)\, d\,[gh\lambda(\xi)]\Big)\mu(h) = \sum_{h\in\mathcal{F}} \Big(\int_{\partial\mathcal{F}} \hat{f}(g\xi)\, d\,[h\lambda(\xi)]\Big)\mu(h)\\
&= \int_{\partial\mathcal{F}} \hat{f}(g\xi)\, d\Big(\sum_{h\in\mathcal{F}} [h\lambda(\xi)\mu(h)]\Big) = \int_{\partial\mathcal{F}} \hat{f}(g\xi)\, d\Big(\sum_{h\in\mathcal{F}} [\lambda(h^{-1}\xi)\mu(h)]\Big)\\
&= \int_{\partial\mathcal{F}} \hat{f}(g\xi)\, d\Big(\mu*\lambda(\xi)\Big) = \int_{\partial\mathcal{F}} \hat{f}(g\xi)\, d\lambda(\xi)\\
&= f(g)
\end{aligned}$$

which proves that each f defined in this way is μ-harmonic function on $\mathcal{F}$.

v) In the set-up of **Remark A.2** and the previous section, let $\mathcal{F}^{\mathbb{Z}_+}$ be the space of sample paths and λ be a μ-stationary probability measure on $\partial\mathcal{F}$. Recall that for each $h\in\mathcal{F}$, χ_h denotes the indicator function of C_h. Since the family of continuous function $\{\chi_h\}_{h\in\mathcal{F}}$ separates points, by **Stone-Weierstrass Theorem**, the unital algebra generated by $\{\chi_h\}_{h\in\mathcal{F}}$ is dense in $C(\partial\mathcal{F})$ with respect to the sup-norm. In this case, the unital algebra generaed by $\{\chi_h\}_{h\in\mathcal{F}}$ is equal to the span of $\{\chi_h\}_{h\in\mathcal{F}}$. For each $h\in\mathcal{F}$, as in (45), define:

$$f_h(g) = \int_{\partial\mathcal{F}} \chi_h(g\xi)\, d\lambda(\xi)$$

For each $h\in\mathcal{F}$ and $i\in\mathbb{N}$, define:

$$N_m^h : \mathcal{F}^{\mathbb{Z}_+} \to \mathbb{R}, \qquad \boldsymbol{g} \mapsto f_h(\boldsymbol{g}_m).$$

where f_h is μ-harmonic by (**iv**). By **Remark A.2**, for each $h\in\mathbb{N}$, $\{N_m^h\}_{m\in\mathbb{N}}$ is a discrete time martingale. Hence, for a fixed $h\in\mathcal{F}$, and for $\mathbf{P}_\mu$-almost each $\boldsymbol{g}\in\mathcal{F}^{\mathbb{Z}_+}$, according to the **Doob's First Martingale Convergence Theorem** (see [**3**]), the following limit exists for $\mathbf{P}_\mu$-almost each $\boldsymbol{g}\in\mathcal{F}^{\mathbb{Z}_+}$:

$$\lim_{m\to\infty} N_m^h(\boldsymbol{g}) = \lim_{m\to\infty} f_h(\boldsymbol{g}_m) = \lim_{m\to\infty} \int_{\partial\mathcal{F}} \chi_h(\xi)\, d\Big[\boldsymbol{g}_m\lambda(\xi)\Big] = \lim_{m\to\infty} \boldsymbol{g}_m\lambda(C_h). \tag{46}$$

Since $\mathcal{F}$ is countable, we then can conclude that for $\mathbf{P}_\mu$-almost each $\boldsymbol{g} \in \mathcal{F}^{\mathbb{Z}_+}$, $\lim_m N_m^h(\boldsymbol{g})$ exists. By approximating coefficients of elements in $\mathbb{C} - \mathrm{Span}(\chi_h\}_{h\in\mathcal{F}}$ using complex numbers with rational parts, we then can conclude that for $\mathbf{P}_\mu$-almost each $\boldsymbol{g} \in \mathcal{F}^{\mathbb{Z}_+}$, for each $F \in C(\partial\mathcal{F})$, the following limit exists:

$$\lim_{m\to\infty} \int_{\partial\mathcal{F}} F(\xi)\, d\big[\boldsymbol{g}_m\lambda(\xi)\big] = \lim_{m\to\infty} \int_{\partial\mathcal{F}} F(\boldsymbol{g}_m\xi)\, d\lambda(\xi). \tag{47}$$

Now fix an $\boldsymbol{g} \in \mathcal{F}^{\mathbb{Z}_+}$ so that the limit in (47) exists for each $F \in C(\partial\mathcal{F})$. Since $\{\boldsymbol{g}_m\}$ is in $(\overline{F}, \delta)$, by (sequentially-)compactness, we can assume $\{\boldsymbol{g}_{m_i}\}_{i\in\mathbb{N}} \subseteq \{\boldsymbol{g}_m\}_{m\in\mathbb{N}}$ is a convergent subsequence (and there could be more) which converges to $\boldsymbol{g}_\infty \in \partial\mathcal{F}$. Fix $h \in \mathcal{F}$. By (46), we have:

$$\begin{aligned}
&\lim_m \int_{\partial\mathcal{F}} \chi_h(\boldsymbol{g}_m\xi)\, d\lambda(\xi) = \lim_m \boldsymbol{g}_m\lambda(C_h) = \liminf_m \boldsymbol{g}_m\lambda(C_h) \\
&\leqslant \liminf_i \boldsymbol{g}_{m_i}\lambda(C_h) \leqslant \limsup_i \boldsymbol{g}_{m_i}\lambda(C_h) \leqslant \limsup_m \boldsymbol{g}_m\lambda(C_h) = \lim_m \boldsymbol{g}_m\lambda(C_h) \\
&\implies \lim_i \boldsymbol{g}_{m_i}\lambda(C_h) = \lim_m \boldsymbol{g}_m\lambda(C_h)
\end{aligned}$$

In the same way we deduce (47), we then have for each $F \in C(\partial\mathcal{F})$:

$$\lim_i \int_{\partial\mathcal{F}} F(\xi)\, d\big[\boldsymbol{g}_{m_i}\lambda(\xi)\big] = \lim_m \int_{\partial\mathcal{F}} F(\xi)\, d\big[\boldsymbol{g}_m\lambda(\xi)\big]$$

By **i), iii)**, we then can conclude for each $F \in C(\partial\mathcal{F})$:

$$F(\boldsymbol{g}_\infty) = \lim_i \int_{\partial\mathcal{F}} F(\xi)\, d\big[\boldsymbol{g}_{m_i}\lambda(\xi)\big] = \lim_m \int_{\partial\mathcal{F}} F(\xi)\, d\big[\boldsymbol{g}_m\lambda(\xi)\big] \tag{48}$$

By **Portemanteau Theorem**, we then can conclude that $\boldsymbol{g}_m\lambda$ converges weakly to $\delta_{\boldsymbol{g}_\infty}$. Further, for the $\boldsymbol{g}$ we fix, the set $\{\boldsymbol{g}_m\}_{m\in\mathbb{N}}$ can be replaced by $\{\boldsymbol{g}_{m_i}\}_{i\in\mathbb{N}}$ without ambiguity. Therefore, since $\boldsymbol{g}$ is arbitrarily picked, the following map can be defined $\mathbf{P}_\mu$-almost everywhere (if necessary, also by **Axiom of Choice**):

$$\phi : \mathcal{F}^{\mathbb{Z}_+} \to \partial\mathcal{F}, \quad \boldsymbol{g} \mapsto \boldsymbol{g}_\infty$$

where for each $\boldsymbol{g}$ in the domain, $\boldsymbol{g}_\infty \in \partial\mathcal{F}$ is picked from the set of clustered points of $\{\boldsymbol{g}_m\}$. For each $m \in \mathbb{N}$, define:

$$\phi_m : \mathcal{F}^{\mathbb{Z}_+} \to \mathcal{F}, \quad \boldsymbol{g} \mapsto \boldsymbol{g}_m$$

and each ϕ_m is measurable. Hence ϕ as a pointwise limit of $\{\phi_m\}$ is also measurable. Since, for each Borel subset $B \subseteq \mathcal{F}$, we have:

$$\mu^{*m}(B) = \mathbf{P}_\mu\big[\phi_m^{-1}(B)\big]$$

Then, given a Borel subset $C \subseteq \partial\mathcal{F}$, we then can define the **hitting distribution** based on ϕ:

$$\Big(\lim_{m\to\infty} \mu^{*m}\Big)(C) = \mathbf{P}_\mu\big[\phi^{-1}(C)\big]$$

By definition, $\lim_{m\to\infty} \mu^{*m}$ is μ-stationary. For now we will use μ_ϕ^∞ to denote $\lim_{m\to\infty} \mu^{*m}$ and in the next section, based on what we previously showed, prove that the existence of $\lim_{m\to\infty} \mu^{*m}$ does not depend on the definition of ϕ (or the choice of clustered points). Then we will show μ^∞ is the unique μ-stationary probability measure defined on $\partial\mathcal{F}$, and hence purely non-atomic by **iii)**.

vi) Let λ be a μ-stationary probability measure. Then for each $F \in C[\partial\mathcal{F}]$, we have:

$$\begin{aligned}
&\lambda(F) = (\mu * \lambda)(F)\\
&= \int_{\partial\mathcal{F}} F(\xi)\, d(\mu * \lambda)(\xi)\\
&= \int_{\partial\mathcal{F}} F(\xi)\, d\Big[\sum_{h\in\mathcal{F}} \lambda(h^{-1}\xi)\mu(h)\Big] = \int_{\partial\mathcal{F}} F(\xi)\, d\Big[\sum_{h\in\mathcal{F}} \lambda(h^{-1}\xi)\mu(h)\Big]\\
&= \int_{\partial\mathcal{F}} \Big(\sum_{h\in\mathcal{F}} F(h\xi)\mu(h)\Big)\, d\lambda(\xi) = \sum_{h\in\mathcal{F}} \mu(h) \int_{\partial\mathcal{F}} F(h\xi)\, d\lambda(\xi)\\
&= \int_{\mathcal{F}^{\mathbb{Z}_+}} \Big(\int_{\partial\mathcal{F}} F(\boldsymbol{g}_1\xi)\, d\lambda(\xi)\Big)\, d\mathbf{P}_\mu(\boldsymbol{g}) = \int_{\mathcal{F}^{\mathbb{Z}_+}} \int_{\partial\mathcal{F}} F(\xi)\, d\Big(\boldsymbol{g}_1\lambda(\xi)\Big)\, d\mathbf{P}_\mu(\boldsymbol{g})\\
&= \int_{\mathcal{F}^{\mathbb{Z}_+}} \boldsymbol{g}_1\lambda(F)\, d\mathbf{P}_\mu(\boldsymbol{g}).
\end{aligned}$$

Hence, for each $n \in \mathbb{N}$ and for each $F \in C[\partial\mathcal{F}]$:

$$\lambda(F) = (\mu^{*\,n}) * \lambda(F) = \underbrace{(\mu * \mu * \cdots * \mu)}_{\times\, n} * \lambda(F) = \int_{\mathcal{F}^{\mathbb{Z}_+}} \boldsymbol{g}_n\lambda(F)\, d\,\mathbf{P}_\mu(\boldsymbol{g}).$$

Hence, for each $n \in \mathbb{N}$ and $F \in C(\partial\mathcal{F})$, the mapping $\boldsymbol{g} \mapsto \boldsymbol{g}_n\lambda(F)$ is a measurable and $\mathbf{P}_\mu$-integrable function defined on $\mathcal{F}^{\mathbb{Z}_+}$. Since, from **v)**, for $\mathbf{P}_\mu$-almost each $\boldsymbol{g} \in \mathcal{F}^{\mathbb{Z}_+}$, $\lim_m \boldsymbol{g}_m\lambda(F)$ exists for each $F \in C(\partial\mathcal{F})$, according to (48) and **Dominance Convergence Theorem**, we have:

$$\lambda(F) = \lim_n \int_{\mathcal{F}^{\mathbb{Z}_+}} (\boldsymbol{g}_n\lambda)(F)\, d\,\mathbf{P}_\mu(\boldsymbol{g}) = \int_{\mathcal{F}^{\mathbb{Z}_+}} \Big(\lim_n \boldsymbol{g}_n\lambda(F)\Big)\, d\,\mathbf{P}_\mu(\boldsymbol{g}). \tag{49}$$

Since, in **v)**, we showed $\lim_m \boldsymbol{g}_m\lambda(F) = F\big[\phi(\boldsymbol{g})\big]$, (48) can be written as:

$$\lambda(F) = \int_{\mathcal{F}^{\mathbb{Z}_+}} \Big(\lim_n \boldsymbol{g}_n\lambda(F)\Big)\, d\,\mathbf{P}_\mu(\boldsymbol{g}) = \int_{\mathcal{F}^{\mathbb{Z}_+}} F \circ \phi(\boldsymbol{g}) d\,\mathbf{P}_\mu(\boldsymbol{g}) = \int_{\partial\mathcal{F}} F(\xi) d\Big(\mathbf{P}_\mu\big[\phi^{-1}(\xi)\big]\Big) = \int_{\partial\mathcal{F}} F(\xi) d\,\mu_\phi^\infty(\xi).$$

Since in **v)**, we showed that $\lim_n \boldsymbol{g}_n\lambda(F)$ is independent to the choice of the clustered point $\boldsymbol{g}_\infty$ of $\{\boldsymbol{g}_n\}_n$, therefore, the existence of $\mu^\infty = \lim_m \mu^{*m}$ (as a bounded linear functional in $C(\partial\mathcal{F})$) is independent to the definition of ϕ and hence the **hitting distribution** is well-defined. Moreover, for each $F \in C(\partial\mathcal{F})$:

$$\lambda(F) = \int_{\partial\mathcal{F}} F(\xi)\, d\lambda(\xi) = \int_{\partial\mathcal{F}} F(\xi)\, d\mu^\infty(\xi).$$

which implies μ^∞ coincides with λ as a probability measure. Since λ is an arbitrary μ-stationary probability measure, μ^∞ is the unique μ-stationary probability measure.

□

References

[1] J. Hawkins, *Ergodic dynamics: from basic theory to applications*, Springer. 2021

[2] M. Izumi, S. Neshveyev, and R. Okayasu., *The ratio set of the harmonic measure of a random walk on a hyperbolic group*. Israel. J. Math. 163 (2008), 285-316

[3] J. L. Doob, *Measure theory*, Springer. 1994

[4] J. Feldman, C. C. Moore, *Ergodic equivalence relations, cohomology, and von Neumann algebras. I*, Trans. AMS (1977), 289-324

[5] A. Kechris, *Classical descriptive set theory*. Springer. 1995

[6] A. Klenke, *Probability theory: a comprehensive course*. Springer. 2014

[7] V. Kaimanovich, *Multiplicative markov measures, Radon-Nikodym problem, and random walks: the free group case*, preprint (2023)