ON THE SURFACE GEOMETRY OF ORDERED SETS

By
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September 1996

A Ph.D. Thesis
submitted to the School of Graduate Studies and Research
in partial fulfillment of the requirements
for the degree of
Doctor of Philosophy in Mathematics[1]

[1] The Ph.D. Program is a joint program with Carleton University, administered by the Ottawa-Carleton Institute of Mathematics and Statistics

ISBN 0-612-19970-3

Abstract

This dissertation has two aims. The first is to study upward drawings of ordered sets on two-dimensional surfaces and secondly to study the geometry of the surfaces on which ordered sets can be drawn without crossing edges. *Critical points*, in particular, *saddle points* of ordered sets will play a decisive role. The *Discrete Index Theorem*, too, is fundamental.

We present a characterization, in terms of critical points, of *spherical ordered sets* — ordered sets which have upward drawings without crossing edges on the sphere.

A class of ordered sets, each member of which is called a *spiral*, plays an important role in the complexity of upward drawings without crossing edges on a specific surface. For a subclass of spirals we characterize those surfaces on which members of this subclass have upward drawings, and we apply spirals, as "gadgets", in the reduction to prove that this decision-making problem, whether an ordered set is spherical, is *NP-complete*. In the process, we derive a new and simpler proof for NP-completeness of upward planarity testing.

From a different point of view, we explore the interrelationship between the upward drawing of ordered sets and the geometry of surfaces. For each smooth, two-dimensional surface S embedded in $\mathbf{R}^3$, we shall construct an ordered set P to *fit* S, that is, (i) P has an upward drawing, without crossing edges on S, (ii) P contains the ordered set $critical(S)$ of critical points of S and (iii) if S' is any two-dimensional surface of *genus* g on which P has an upward drawing, without crossing edges, then $critical(S) \subseteq critical(S')$.

A leading unsolved problem is whether an ordered set has a *realizable* upward drawing on a surface of genus of its covering graph. We consider what seems to be a likely candidate for a counterexample — the upward drawing of the lattice of subspaces of the projective plane of order two.

Acknowledgements

It is my pleasure to acknowledge the many people who offered their help and advice throughout the course of this research.

First of all, I would like to express my most sincere gratitude to my supervisor, Dr. Ivan Rival, for his unfailing assistance, guidance and encouragement. I know that it has been his kindness above all that has given me the many constructive opportunities I needed to complete this work.

I wish to extend my thanks also to the members of my committee, Dr. Barry Jessup, Dr. Bruce Richter and Dr. Jorge Urrutia for their useful comments and kind encouragement.

Finally, I must acknowledge with gratitude the substantial financial support provided by my supervisor, Dr. Ivan Rival, and the Department of Mathematics. I also express my sincere thanks to the Algorithms Research Group of the Department of Computer Science for its financial and material support.

Dedication

This Work is lovingly dedicated to my wife, Miryam, who put her whole heart into helping me accomplish this work. For the priceless gift of her love, I thank her.

To my children, Hossein and Hosna, I extend my deepest love and thanks for their patience and tolerance.

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Overview

A *(partially) ordered set*, $P(X, <)$, consists of a set X and a binary relation $<$, less than, defined on $X \times X$ such that

$<$ is *transitive*, that is, $a < b$ and $b < c$ imply $a < c$, and

$<$ is *asymmetric*, that is, $a < b$ implies $b \not< a$.

Ordered sets are widely used for representing hierarchical structures. They occur in computation, in scheduling, in sorting, in social sciences and even in geography.

For many years much research work has focussed on graphical data structures. In particular, geometric representation and upward drawings play a decisive role in most of the applications, especially, in problems in which decisions must be made from among alternatives ranked according to precedence or preference relations.

Our main focus is surface geometry of ordered sets. We study the problem of representing ordered sets on planes and two-dimensional surfaces and investigate the interrelationship of upward drawings and geometric properties of surfaces. Many of the techniques come together to prove our central result — the NP-completeness of sphericity testing.

The Details

Chapter 1, "Drawing", describes upward drawings of ordered sets and directed graphs (shortened to "digraph") on planes. We present the evolution of and motivation for the planarity of upward drawings and complexity of upward planarity testing. Our main result in this chapter is a new and more transparent proof of the NP-completeness of planarity testing (cf. Theorem 1.5).

Chapters 2 through 7 are devoted to the concept of upward drawings on surfaces. Collectively, these chapters describe upward drawings on surfaces and study some topological and geometrical properties of ordered sets.

Embeddings and the genus of ordered sets, a lifting procedure, the concept of critical points and the Discrete Index Theorem will be introduced in Chapter 2.

Chapter 3, "Spherical Ordered Sets", describes upward drawings on the sphere. As the main results we present two characterizations of spherical ordered sets (cf. Theorems 3.1 and 3.5). The first characterization echoes planarity characterization for digraphs (cf. Theorem 1.1), and the second is based on critical points, in particular the saddle points. We shall also introduce another approach to drawings on the plane called "Circular Drawing". These may provide an auxiliary embedding of the covering graph of a spherical ordered set on the plane which is transferable to an upward drawing on the sphere.

Chapter 4, entitled "Spirals and Upward Drawings", is motivated by the gadgets that have been applied for the proof of the NP-completeness of planarity testing. In this chapter we develop a class of ordered sets of genus zero which are called spirals. These ordered sets lighten the idea of critical points of ordered sets, specifically the saddle points and their role in upward drawings. We shall develop a theorem based on which, for a large class of ordered sets, we can determine all saddle points. Accordingly, upward drawings of these ordered sets, on appropriate surfaces of genus zero, are fully understood (cf. Theorems 4.1 and 4.2). It is also our intention to show that for a large class of ordered sets, including spirals, planarity and sphericity are equivalent, that is, each such ordered set has upward drawing on the sphere without crossing edges if and only if it is planar.

We shall discuss the complexity of upward sphericity testing of ordered sets and digraphs in Chapter 5. We shall generalize the gadgets which have been used in the proof of the NP-completeness of upward planarity testing and develop some techniques to prove that upward sphericity testing is also NP-complete (cf. Theorem 5.1). For further research in this direction, a set of open problems will be presented at the end of this chapter.

Chapter 6, "Upward Drawing to fit surfaces", promotes a different point of view on the problem of upward drawings on surfaces. In this chapter the starting point is a two-dimensional surface with a set of prescribed conditions. Then the objective is to construct, based on some topological and geometrical property of the surface, an ordered set whose upward drawing "fits" the surface (cf. Theorem 6.1). We shall also present a classification of all ordered sets whose upward drawings fit two-dimensional surfaces (cf. Theorem 6.6).

Our final chapter is Chapter 7, "Realizable Surfaces", which is mainly devoted to the embedding of ordered sets on realizable surfaces and open problems in this area.

Although we have tried to keep this work self-contained, some definitions and basic results of graph theory, topology and the theory of ordered sets (at the level of [**Bondy** and **Murty** (1976)], [**Massey** (1991)] and [**Rival** (1994)], respectively) are assumed.

Chapter 1

Drawing

1.1 Introduction

To visualize an ordered set, we usually draw a graph on the plane with vertices corresponding to the elements of the ordered set, with edges drawn monotonically with respect to a fixed direction in order to represent the order relation between elements. This, in general, is called a *drawing* of an ordered set.

In this chapter we first review *upward drawing*, which is chief among graphical data structures for ordered sets, and introduce the concept of *planarity*. Then we review some of the previous works on *upward planarity testing* and the complexity of this problem, which was a long-standing problem for many years. In 1994, [**Garg** and **Tamassia** (1994)] made a major breakthrough by showing that upward planarity testing is NP-complete. We present another proof which is simpler and the idea of the proof can be extended to prove a similar result for upward drawing on other surfaces of genus zero, in particular, the sphere (see Chapter 5)[**Hashemi, Kisielewicz** and **Rival** (1995)].

1.2 The Upward Drawing

It is customary to identify an ordered set with its geometric representation, called its *upward drawing*. According to this convention, the elements of the ordered set P are drawn on a surface, traditionally a plane, as disjoint small circles, arranged in such a way that for $a, b \in P$, the circle corresponding to a is higher than the circle corresponding to b whenever $a > b$ and an arc, monotonic with respect to a fixed direction, usually south to north, is drawn to join them only if a *covers* b (that is, if each $x \in P$, $a > x \geq b$ implies $x = b$). We say that a is an *upper cover* of b or b is a *lower cover* of a, and write $a \succ b$ or $b \prec a$. These arcs are drawn, of course, to avoid the incidence of any other circle on it, to avoid unwanted comparabilities and when possible, to avoid intersections, except where two arcs meet at a circle.

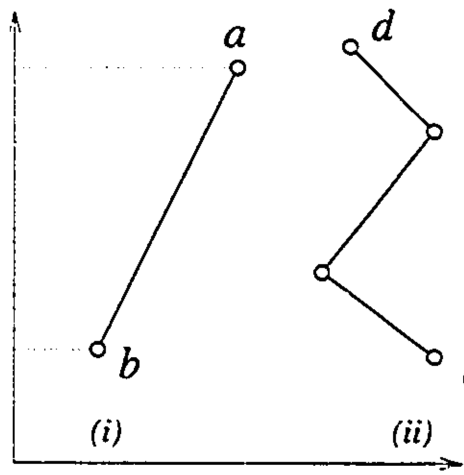


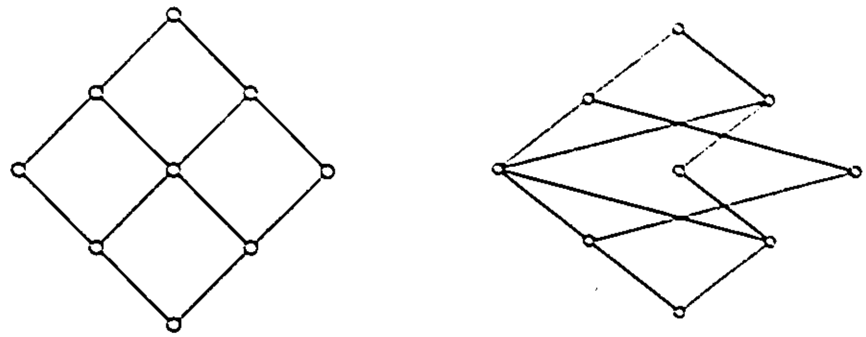
Figure 1: Upward drawings of ordered sets, In (*i*) a covers b, in (*ii*) $d > c$ but d does not cover c

1.2.1 Planar Ordered Sets

An ordered set is *planar* if it has an upward drawing on the plane in which no arcs cross, although they may meet at a vertex with which they are incident. Usually, by a planar upward drawing of an ordered set, we mean a drawing in which all edges are *straight lines*. It is a fundamental fact that every planar ordered set has a *straight line* planar representation [Kelly (1987)]. Therefore, without loss of generality, we can use monotonic curved lines.

Planarity is a property of the order and a planar ordered set may, indeed, have upward drawings which are not planar (see Figure 2).

The *covering graph* of an ordered set P, denoted *cover*(P), is the graph whose vertices are the elements of P, and with edges $x \sim y$, if either $x \succ y$ or $x \prec y$ in P. A necessary condition for an ordered set to be planar, is that its covering graph be planar. The condition



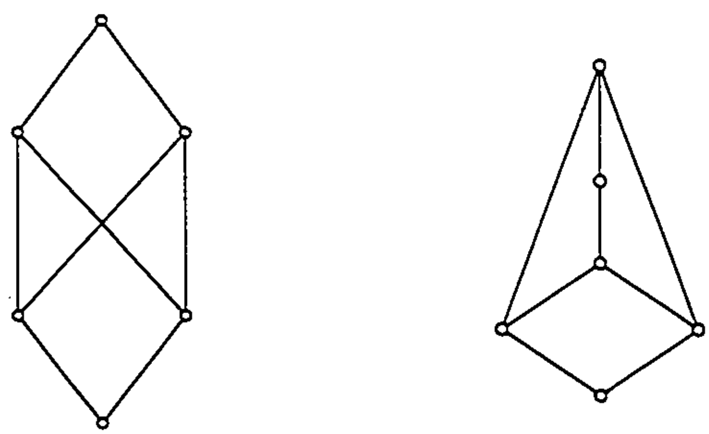
(a) (b)

Figure 2: (a) A planar ordered set, (b) An upward drawing of the same ordered set with crossing edges

is not sufficient. Thus, the six element ordered set

$$P = \{x_0 < x_1 < x_3 < x_5,\ x_0 < x_2 < x_4 < x_5,\ x_1 < x_4, x_2 < x_3\}$$

has a planar covering graph, while as an ordered set it is not planar (cf. Figure 3).



(a) (b)

Figure 3: (a) A nonplanar ordered set , (b) Its covering graph is planar

The edges of the covering graph of an ordered set can be directed with respect to the order, that is, if $a \sim b$ is a covering edge then we define a direction from a to b provided that $a \prec b$. To any covering edge $a \sim b$, corresponding to $a \prec b$, we usually associate two values, "−" to that end of the covering edge outgoing from a and "+" to that end of it incoming to b.

1.3 Digraphs and Upward Drawing

Although our interest is in the upward drawing of ordered sets, the concept of upward drawing can be extended to directed graphs, as well.

In general, a directed graph, which is abbreviated to *digraph*, has an *upward drawing* if it has a drawing in which all edges are monotonic arcs, and all point upward. Of course, such a digraph must be *acyclic* — it contains no directed cycles.

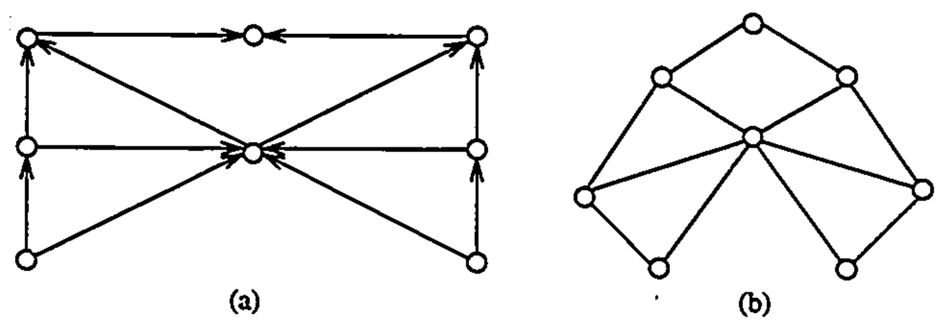


Figure 4: (a) An acyclic directed graph (b) Its upward drawing

All digraphs we consider here are finite, without loops and all are biconnected that is, they have no vertex such that its removal disconnects the underlying graph.

A vertex v in a digraph is called *maximal* (or a *sink*) if all the edges incident with it are directed toward v, and it is called *minimal* (or a *source*) if all the incident edges are directed away from it. A vertex is *extremal* if it is either minimal or maximal.

A *plane digraph* P is a digraph drawn in the Euclidean plane such that all edges are arcs and no two edges cross. A *planar* digraph D is a digraph isomorphic to a plane digraph P. In such a case we say also that P is a *planar representation* of D or P is an *embedding* (in the plane) of D. If we delete the vertices and edges of a plane digraph P from its representation in the plane, the remainder falls into connected components, called its *faces*. Each plane digraph has exactly one unbounded face called its *exterior* face. The remaining faces, if any, are called *interior*. The boundary of the exterior face is referred to simply as the *boundary* of the given plane digraph.

A *planar upward drawing* of a digraph D is a plane digraph U such that all arcs are straight-line segments directed upwards. By virtue of [**Kelly** (1987)] straight-line segments here may be replaced by monotonic arcs. If a digraph has a planar upward drawing then it is necessarily acyclic and planar. A planar upward drawing U (a plane digraph U) is *similar* to a plane digraph P if U and P are isomorphic as digraphs and have the same structure of faces, that is, the same face boundaries and the same exterior face boundary. Similar planar representations are considered as identical.

Note that, a *planar upward drawing* of an ordered set is a planar upward drawing of its directed covering graph. An ordered set is called *planar*, if it has a planar upward drawing.

Next, we provide an overview of some of the known combinatorial characterizations of upward planarity, in particular planar st-graphs and a characterization for biconnected digraphs.

1.4 Characterization

The search for an efficient upward planarity testing algorithm for ordered sets[1] is a long-standing problem, much sought after by theoreticians of graphical data structures. It has always been a mystery how upward planarity testing for orders could be so difficult if its undirected companion, planarity-testing for graphs is so easy. In spite of well-known linear time algorithms for graph planarity testing (e.g. [**Hopcroft** and **Tarjan** (1974)] and [**Lempel, Even** and **Cederbaum** (1967)]), it is not even self-evident that there is a finite algorithm for upward planarity testing.

For many years progress has been slow.

- *Planar lattices are dismantlable* [**Baker, Fishburn** and **Roberts** (1971)].
- *Algorithmics of planar lattices* [**Kelly** and **Rival** (1975)].
- *Planarity-testing for lattices using graph planarity-testing* [**Platt** (1976)].
- *Straight lines for planar upward drawings* [**Kelly** (1987)].
- *A digraph G is upward planar if and only if it is a spanning subgraph of a planar st-digraph* [**Di Batttista** and **Tamassia** (1988)].

[1]In fact, by adjoining subdivision points as needed we may just as well consider *directed acyclic graphs*.

- *Bipartite planar upward drawings* [**Di Battista, Liu** and **Rival** (1990)].
- *Planar ordered sets of width two* [**Czyzowicz, Pelc** and **Rival** (1990)].
- *Planar ordered sets with bottom* [**Hutton** and **Lubiw** (1991)].
- *Planar triangle-free graphs have planar orientations* [**Kisielewicz** and **Rival** (1993)].
- *An embedded digraph is upward planar if and only if it is acyclic, bimodal and admits a consistent assignment of minimals and maximals to faces* [**Bertolazzi, Di Battista, Liotta** and **Mannino** (1994)].

1.4.1 *st*-Digraphs

A *planar st-digraph*, by definition, is a planar acyclic dircted graph which has an embedding in the plane (a planar representation) with exactly one minimal element (a vertex with all incident edges are outgoing edges) which is called *bottom*, and exactly one maximal element (a vertex with all incident edges are incoming edges) which is called *top*, and a directed edge connecting the bottom to the top. In some articles (*e.g.* [**Di Battista** and **Tamassia** (1988)]) the bottom element is called *source* denoted by s, and the top element is called *sink* denoted by t. For instance the directed graph depicted in Figure 5:(b) is an *st*-digraph. [**Kelly** (1987)] and [**Di Battista** and **Tamassia** (1988)], independently

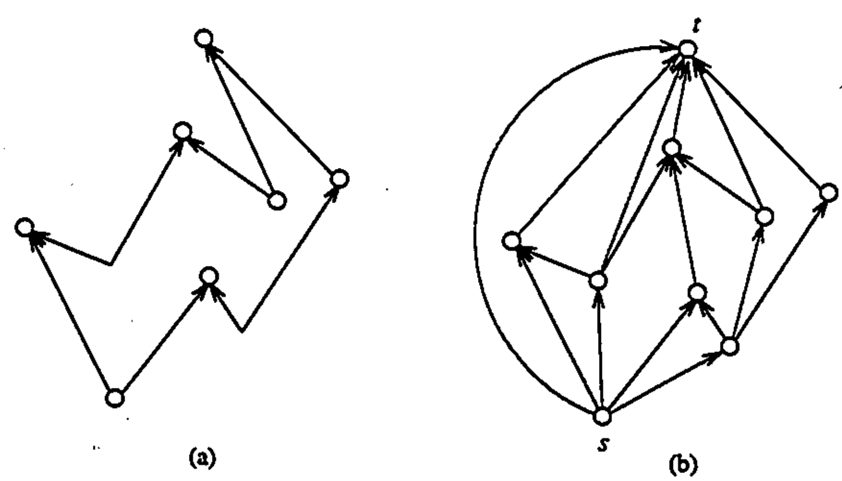


Figure 5: (a) An acyclic digraph (b) A planar *st*-digraph

characterized upward planarity in terms of st-digraphs.

Theorem 1.1 *A digraph $\vec{G}$ has a planar upward drawing if and only if it is a spanning subgraph of a planar st-digraph (i.e., edges can be added to $\vec{G}$ such that the resulting digraph $\vec{H}$ is a planar st-digraph).*

Example 1.1 In Figure 5 the acyclic digraph depicted in (a) has a planar upward drawing because by adding new edges, as shown in (b), it can be extended to a planar st-digraph.

Thus according to this theorem, one algorithmic way to test whether a directed graph has an upward drawing is as follows.

First test the planarity of the underlying graph, this can be done in linear time using the [**Hopcroft** and **Tarjan** (1974)] algorithm. Next, the acyclicity of the digraph must be tested by using, for instance, a depth-first-search algorithm (cf. [**Corman, Leiserson** and **Rivest** (1990)] which is also a linear time algorithm. Finally, one must add all possible subsets of edges and test whether the resulting digraph is an st-digraph. This yields an exponential time upward planarity testing algorithm (cf. [**Rival** (1993)]). Thus,

Theorem 1.2 *Upward planarity testing belongs to NP.*

1.4.2 Assignment of Large Angles

Let P be an embedded acyclic (connected) digraph (for example, a directed covering graph of an ordered set). For every vertex $v \in P$ the directed edges incident upon v can be partitioned into intervals of incoming edges and outgoing edges so that all the edges of each interval occur, consecutively, with respect to a fixed rotation on a neighbourhood about v (which contains no other vertex of P). If there is only one such interval then we say that the *alternation* about v is zero, *alternation*$(v) = 0$. If there are only two such intervals then the alternation about v is one, *alternation*$(v) = 1$. Otherwise, the alternation about v is greater than one (see Figure 6).

Observation 1.4.1 *Assume that P has a similar planar upward drawing, that is, a planar upward drawing with the same structure of faces. Then for every vertex $v \in P$, alternation$(v) < 2$.*

According to this observation, if P has a similar planar upward drawing, the directed edges incident upon each vertex v can be partitioned into two intervals (with possibly one empty),

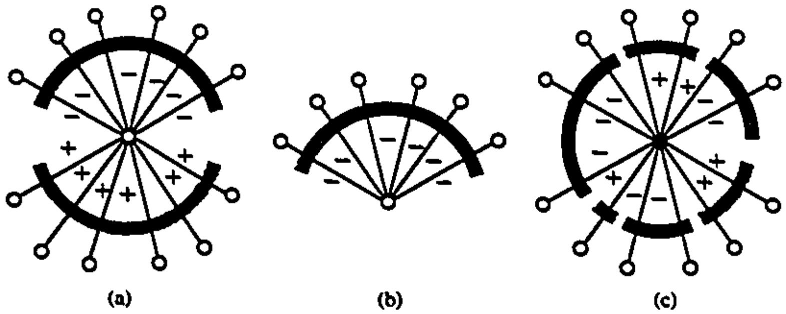


Figure 6: (a) A vertex with alternation one (b) A vertex with alternation zero (c) A vertex with alternation three

of incoming edges and outgoing edges. In fact this is a necessary condition for an ordered set (digraph) to be upward planar. In some articles (*e.g.* [Garg and Tamassia (1994)]) a digraph with this property is called *bimodal*.

Fix a straight line planar upward drawing of an ordered set. Consider the angles formed by adjacent pairs of incoming or adjacent pairs of outgoing edges of this upward drawing. An angle is called *large* if its measure is strictly greater than π, and it is called *small* if its measure is strictly less than π. One can observe that if a vertex is an internal vertex (is not extremal) then it has no large angles and if it is an extremal vertex it has exactly one large angle. Thus, if $L(v)$ stands for the number of large angles of vertex v, then

$$L(v) = \begin{cases} 0 & \text{if } v \text{ is an internal vertex} \\ 1 & \text{if } v \text{ is an extremal vertex} \end{cases}$$

This unique large angle of the extremal vertex v is in the corner of some face f. Thus, given an upward drawing, if the large angle of v is in the face f, then we say that v is *assigned* to f. As we will see later on, a consistent assignment of large angles to the faces can determine an upward drawing.

Let f be a face of a planar upward drawing of an ordered set. Consider all angles, inside the face, formed by pairs of incoming or outgoing edges of the boundary of this face. Let $L(f)$ stand for the number of large angles of f and $S(f)$ for the number of small angles of f. Assume that f is a *bounded* interior face that is, it is an interior face whose boundary consists of two maximal chains sharing the same top and the same bottom. Then it has only two small angles and no large angles at all. If f is a bounded exterior face, then it has two large angles and no small angles. Thus we have

$$L(f) - S(f) = \begin{cases} -2 & \text{if } f \text{ is an interior face} \\ +2 & \text{if } f \text{ is an exterior face} \end{cases}$$

One can apply a simple induction to show that the same relationship holds between large angles and small angles for any face which is not necessarily a bounded face.

If one walks around the boundary of a face in a fixed direction, then going from a *local minimum*, a vertex from which both incident edges in the cycle are outgoing, to the next local minimum, one must pass through a *local maximum*, a vertex into which both incident edges in the cycle are incoming. It follows that the number of angles (small or large) in the face f formed by pairs of incoming edges equals to the number of angles (small or large) in the face, formed by pairs of outgoing edges. This number is determined by the embedding and does not depend on the upward drawing. Thus, if $d(f)$ stands for the number of all maximal chains (directed paths) in the boundary of face f, then clearly $d(f)$ is an even number which indeed, is the number of local maxima and local minima of the boundary of this face.

$$d(f) = L(f) + S(f) \quad \text{for every face } f$$

We define the *switching index* $\sigma(f)$ as follows.

$$(\star) \qquad \sigma(f) = \begin{cases} \frac{1}{2}d(f) - 1 & \text{if } f \text{ is an interior face} \\ \frac{1}{2}d(f) + 1 & \text{if } f \text{ is an exterior face} \end{cases}$$

In fact a combination of the previous formulas shows that $\sigma(f)$ is the actual number of large angles in face f. On the other hand large angles can be formed only by extremal vertices. Thus, the existence of a similar planar upward drawing for an embedded digraph P depends on the existence of an assignment of extremal vertices to the faces so that it complies with $(\star)$.

Given an embedded digraph P, let ϕ be an assignment of extremal vertices of P to the faces. Then ϕ is an *upward assignment* of P, if for every face f the number $\alpha(f) = |\phi^{-1}(f)|$ of extremal vertices of P assigned to the face f equals $\sigma(f)$, i.e.,

$$\alpha(f) - \sigma(f) = 0 \quad \text{for every face } f$$

Theorem 1.3 *An embedded ordered set (acyclic digraph) P has a similar planar upward drawing if and only if all vertices of P have alternation < 2 and there exists an upward* *assignment ϕ of P. In this case, P has an upward drawing such that for every face f and every extremal vertex v of P, an angle in f, incident to v, has measure greater than π if and only if $\phi(v) = f$.*

This result (using slightly different terminology) is proved in [**Bertolazzi, Di Battista, Liotta and Mannino (1994)**] (an independent proof is given also in [**Kisielewicz (1994)**]). It is shown in [**Bertolazzi, Di Battista, Liotta and Mannino (1994)**] that checking whether P has an upward assignment (and finding one) may be done in polynomial time using the Ford-Fulkerson algorithm solving the max-flow problem on a related network. A direct effective procedure is given in [**Kisielewicz (1994)**].

Given an upward assignment of P, one can construct the upward drawing using a procedure of inserting edges described in [**Bertolazzi, Di Battista, Liotta and Mannino (1994)**]. For small digraphs, the fact that every assignment of an extremal vertex to a face corresponds to a non-convex angle, is often enough by itself to make a suitable drawing. (See Figure 7: for every face the value of $\sigma(f)$ is shown and for every extremal vertex the face assigned to it is labelled by an arrow.) According to the remarks above, testing whether

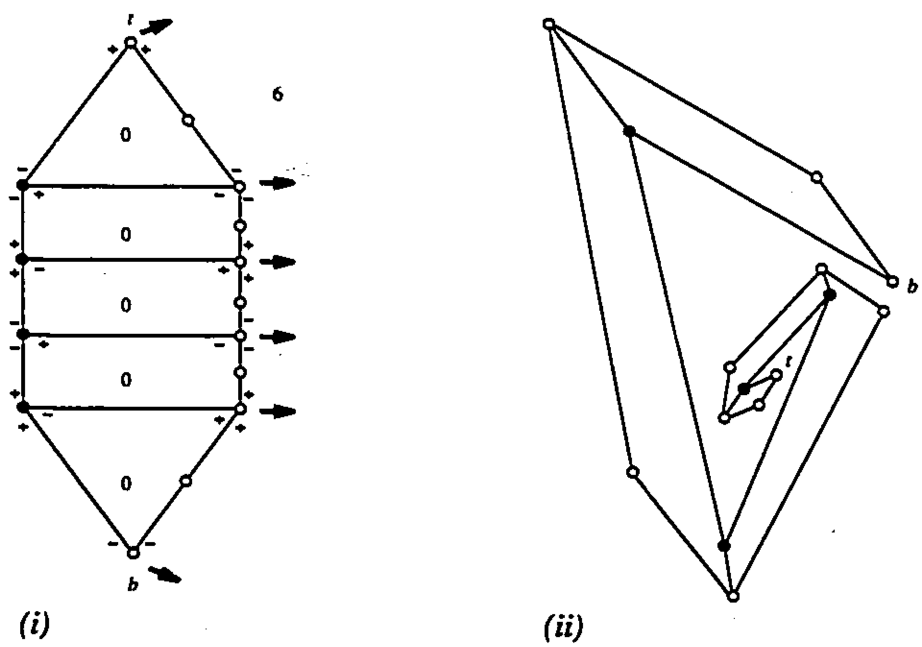


Figure 7: A plane digraph with an upward assignment and the corresponding planar upward drawing

a plane digraph P has a similar planar upward drawing may be done in polynomial time. This also yields a polynomial-time procedure to test whether a triconnected digraph, that is, a digraph with no two-element subset of vertices whose removal disconnects the underlying graph, has a planar upward drawing (since the number of distinct planar representations of a triconnected digraph is "small").

In fact, the argument used in [**Bertolazzi, Di Battista, Liotta** and **Mannino** (1994)] may be extended for a broader class, e.g., for all subdivisions of triconnected digraphs, as well.

Since testing upward planarity for digraphs can be easily reduced in polynomial time to testing upward planarity of their biconnected components, in looking for a digraph that is hard to test we must concentrate on planar digraphs whose connectivity is just two (cf. [**Kisielewicz** (1994)]).

1.5 NP-completeness of Upward Planarity Testing

Recently, there has been a major breakthrough: By transforming the *NOT-ALL-EQUAL-3-SAT*[2] decision problem into an auxiliary flow decision problem with integer coordinates, and then providing yet another transformation to upward planarity, it has been proved that upward planarity-testing is NP-complete [**Garg** and **Tamassia** (1994)].

We present a proof of this result using a direct transformation from *EXACT COVER BY 3-SETS*. It seems simpler and more transparent.

Roughly speaking, our idea is the following.

We start with an embedded biconnected st-digraph D and its planar upward drawing P. Such a digraph always has a planar upward drawing P, with edge $s \to t$ on the boundary of the exterior face (cf. [**Di Battista** and **Tamassia** (1988)]).

Note that all faces are bounded so that the boundary for each face consists of two maximal chains, that is, $d(f) = 2$ for every face f. Therefore, $\sigma(f) = 0$ for every interior face, and $\sigma(f) = 2$ for the exterior face. According to this, no large angle can be assigned to the interior faces. There are only two extremal vertices, namely, s and t. If ϕ is an assignment of the extremal vertices s and t to the exterior face it is the unique upward assignment.

[2] Given a set of clauses with three literals each, is there a truth assignment such that each clause has at least one true literal and one false literal?

Let S be a plane biconnected digraph with vertices x and y on its boundary, such that there is no other planar representation of S with x and y on the boundary.

Suppose, we have chosen n edges $x_i \rightarrow y_i$, $i = 1, 2, \ldots, n$ in P. Replace each edge $x_i \rightarrow y_i$ by a copy of S, identifying x and y with x_i and y_i, respectively. Denote the resulting plane digraph by P'.

Then there are at least 2^n distinct planar representations of P' obtained just by flipping subsets of the copies of S in P (see Figure 11:(b) for an example with $n = 9$).

Obviously, for some of these representations there may be similar upward drawings, while for others there may not. To check that P' has no upward drawing at all (using Theorem 1.3) we need to consider each of 2^n planar representations, which done in a natural way requires exponential time.

Generally, it may happen that P' has still more than 2^n planar representations. To exclude this possibility (and thus to keep full control over possible planar upward drawings) our initial choice of P will be such that P has precisely one planar upward drawing with vertices s, t and w on the boundary. To exclude possible upward drawings with other exterior faces, we add five new vertices and nine new edges in the exterior of P, as in Figure 8. We will refer to this by saying that a *frame* is added to P on vertices s and t.

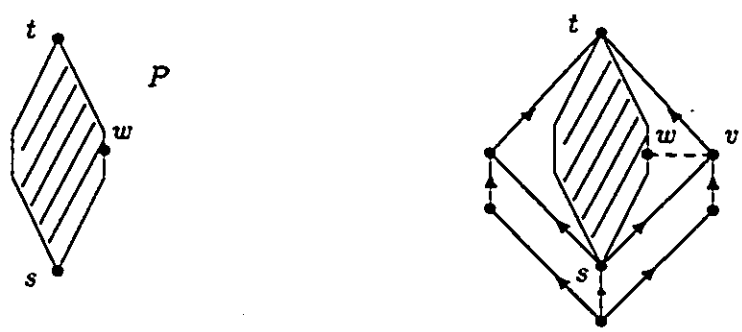


Figure 8: Adding a frame to a digraph P on vertices s and t

Such a frame has a unique planar upward drawing. This can be seen directly as is illustrated in Figure 8, or by using Theorem 1.3. The following fact is therefore immediate.

Lemma 1.4 *If a digraph P' has an edge from s to t, and P'' is obtained from P' by adding a frame on s and t, then P'' has a planar upward drawing if and only if P' has a planar upward drawing with s and t on the boundary.* □

To fix this embedding, we shall also add a suitable path joining vertex w in P' with a vertex v on the frame as in Figure 8.

Our ultimate aim is to construct a digraph P' such that flipping S-components corresponds to an instance of the following, well-known NP-complete problem [**Garey and Johnson** (1979)]:

EXACT COVER BY 3-SETS (X3C).

INSTANCE: Set X with $|X| = 3q$ and a collection C of 3-element subsets of X.

QUESTION: Does C contain an exact cover for X?

Our NP-completeness proof is based on standard techniques of "component design" as in [**Garey and Johnson**(1979)].

First we define *spiral sets* S_m, which will be our basic flipping components. Using these we define a planar ordered set Q, which has three spiral sets intended to represent elements of the set X, such that Q has a desired planar upward drawing if and only if either all the components are flipped or none. For every member c of C there will be a copy of Q in P'; flipping the components corresponds to choosing the member c to be an element of the exact cover we are looking for.

Next, we define planar ordered sets $P(d)$ with d spiral sets intended to represent members of C, such that $P(d)$ has a desired planar upward drawing if and only if exactly one of the spiral sets is flipped. For every element x in X there is a copy of a suitable $P(d)$ in P'; flipping a component corresponds to choosing a member of C in order to cover the element x.

There will also be "communication" edges in P' guaranteeing that choices in Q-components and $P(d)$-components agree. In fact, we join the pair of basic components S_m by a suitable pair of edges. Yet, because many connections are needed, the communication edges may cross in many places; these places are replaced by a specially designed "crossover" R.

The resulting digraph P' is constructed in polynomial time and (with a frame added) has an upward drawing if and only if the corresponding instance of X3C has an affirmative solution. This will yield a new proof (one of the principal results of this thesis) for the Theorem due originally to Garg and Tamassia:

Theorem 1.5 *The decision problem whether an ordered set (or a directed digraph) has a planar upward drawing is NP-complete.*

1.6 Gadgets for the Reduction

In this section we construct the basic gadgets that will be used for the reduction procedure in the proof of Theorem 1.5.

1.6.1 Spiral Sets

Let S_m be an ordered set with $6m+3$ elements denoted by $v_1, \ldots, v_{2m}$, $u_1, \ldots, u_{2m}$, b, t, and $z_1, \ldots, z_{2m+1}$, with the following comparabilities:

$$v_{2j} \geq v_{2j-1}, v_{2j+1}$$

$$u_{2j} \geq v_{2j}, z_{2j}, z_{2j+1}$$

$$u_{2j-1} \leq v_{2j-1}, z_{2j-1}, z_{2j}$$

where, $j = 1, 2, \ldots, m$ and, in addition,

$$t \geq z_1, v_1, \quad \text{and} \quad b \leq z_{2m+1}, v_{2m}$$

These comparabilities describe, in fact, the covering relations of S_m; the covering digraph is given in Figure 9:(a). We call this ordered set an *m-spiral* or simply a *spiral*, if there is no confusion about the size of this ordered set. The m-spirals are our basic flipping components.

In Figure 7:(b) a planar upward drawing of S_2 is shown. Note that vertices $z_1, z_2, \ldots, z_{2m+1}$ have the role of subdivision points to replace nonessential edges with a chain so that the digraph becomes a covering graph of an ordered set. To prove the NP-completeness of upward planarity testing for arbitrary digraphs, these vertices can be adjoined.

The vertices t and b will be called the *top* and the *bottom* of S_m, respectively.

Note that S_m is biconnected and it has precisely one planar representation with t and b on the boundary. There is also precisely one similar upward drawing of S_m.

Indeed, looking for an upward assignment we first observe that for every interior face of S_m the switching index $\sigma(f) = 0$. It follows that all extremal vertices must be assigned to the exterior face. The extremal vertices of S_m are $u_1, \ldots, u_{2m}$ and the top and the bottom of S_m.

Below, we will always use S_m as a subgraph of a plane digraph P whose vertices have alternation < 2, in such a way that $\{t, b\}$ forms a separation pair in P. If, in addition, there

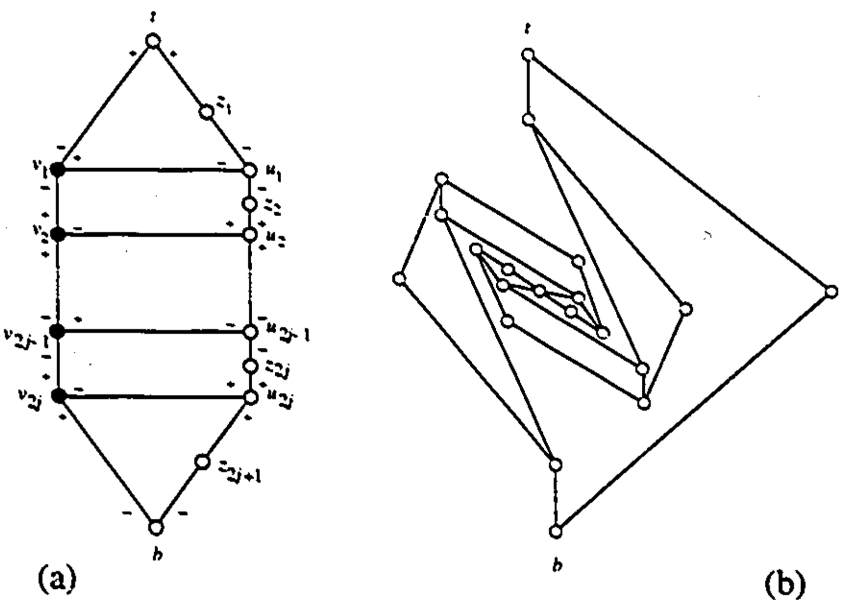


Figure 9: (a) Covering digraph of S_m (b) A planar upward drawing of $P(2)$

is an edge in P outgoing from t and there is an edge in P incoming to b, then we will say that a copy of S_m *is amalgamated* (on t and b) into P. (See, for example, Figure 9:(b) for two amalgamated copies of S_1).

If S_m is amalgamated into P, then $u_1, \ldots, u_{2m}$ are, of course, extremal vertices of P, while t and b are not. The (undirected) path $t, v_1, \ldots, v_{2m}, b$, which is called the *left*-path of S_m, is a part of a boundary of one face in P and the remaining part of the boundary of S_m, which is called the *right*-path of S_m, is a part of a boundary of another face in P. Denote the faces by F_1 and F_2, respectively, and suppose that ϕ is an upward assignment of P. Then, from our discussion so far, it follows that all the extremal vertices $u_1, \ldots, u_{2m}$ on the *right*-path of S_m must be assigned to F_2. Thus, using the notation preceding Theorem 1.3, we see that the contribution of S_m to $\alpha(F_2)$ is $2m$. On the other hand, the contribution of S_m to $\sigma(F_2)$ is m (since $d(F)$ is equal to the number of extremal vertices on the undirected cycle bounding F). Therefore, the total contribution of S_m to the difference $\alpha(F_2) - \sigma(F_2)$ equals m. Similarly, the contribution of S_m to the difference $\alpha(F_1) - \sigma(F_1)$ equals $-m$.

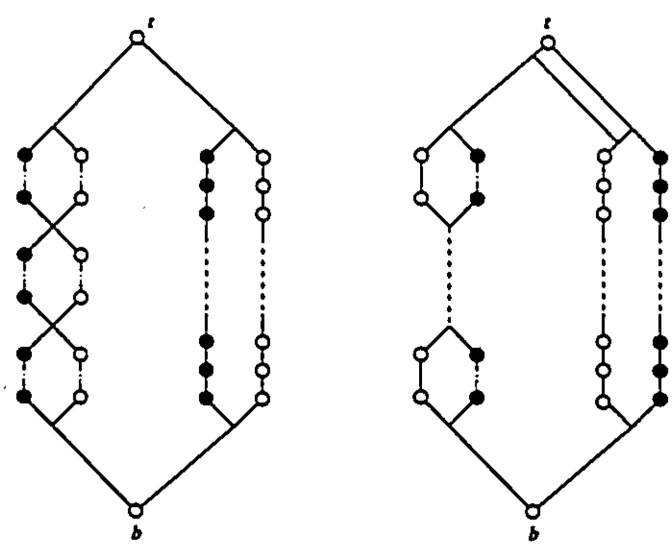
We sum up our discussion in the following.

Lemma 1.6 *Suppose that S is a copy of S_m amalgamated into a plane digraph P and F is* *a face of P sharing boundary with S. Suppose that o is an upward assignment of P. Then the contribution of S to the difference* $\alpha(F) - \sigma(F)$ *equals m if F shares the right-path of S, or* $-m$*, otherwise.* □

In Figure 9, the vertices in the *left*-path are denoted with filled vertices. For the exterior face F, the total contribution of two amalgamated copies of S_1 in Figure 9:(b) to the difference $\alpha(F) - \sigma(F)$ is zero, which in view of Theorem 1.3 is a necessary condition for the existence of a planar upward drawing.

1.7 Components

Let D be a plane digraph consisting of two directed paths $b \to x \to t$ and $b \to y \to t$. Replace vertex y by a copy of S_3, and vertex x by a chain of three copies of S_1 in such a way that all these copies are amalgamated into the resulting plane digraph Q. The *right*-path of the copy of S_3 and the *left*-paths of the copies of S_1 contribute to the exterior face of Q (see Figure 10:(a); every copy of S_m is labeled by m and its *left*-path is indicated with filled vertices). Similarly, we define a plane digraph $P(d)$ ($d > 1$) as one obtained from a



(a) (b)

Figure 10: Schematic diagrams of Q and $P(d)$

plane digraph D consisting of two directed paths $b \to x \to z \to t$ and $b \to y \to t$, where y is replaced by an amalgamated copy of S_{d-2}, and x is replaced by a chain of d amalgamated copies of S_1 (by S_0 we mean just a single vertex, so for $d = 2$ the vertex y is not replaced at all). In contrast to Q, the *right*-paths of the copies of S_1 and the *left*-path of the copy of S_{d-2} are assumed to contribute to the exterior face. Moreover, for $d > 2$ we assume that $P(d)$ has an additional edge from z_1 in the copy of S_{d-2} to z (which is intended to protect flipping this copy) (see Figure 10:(b)). Note that for $d = 2$, the vertex z is not needed.

The vertices t and b are again called the *top* and the *bottom* of Q and $P(d)$, respectively. The part of the boundary of Q (of $P(d)$) corresponding to the directed path $b \to y \to t$ in D will be called the *right*-path of Q (of $P(d)$), and the remaining part — the *left*-path.

A copy of Q (or $P(d)$) being a subgraph of a plane digraph P is said to be ***amalgamated*** into P if

- $\{t, b\}$ is a separation pair in P, and
- there are edges outgoing from t and incoming to b in P.

Lemma 1.7 *A plane digraph Q' obtained from Q by flipping its spiral set components has a similar planar upward drawing if and only if either all the four components are flipped, or none. A plane digraph P' obtained from $P(d)$ by flipping its S_1-components has a similar plane upward drawing if and only if exactly one of these components is flipped.* □

The proof is an application of Theorem 1.3. One just observes that by virtue of Lemma 1.6, in each case, for every face F, the total contribution of all spiral sets to the difference $\alpha(F) - \sigma(F)$ must be zero. Figure 9:(b) shows a planar upward drawing of $P(2)$ after flipping one of the S_1-components.

1.8 Communication and Crossovers

Given two copies S' and S'' of S_1 on the plane, the pair of non-crossing edges directed from the bottom and the top of S' to the bottom and the top of S'' will be called the *pair of communication edges from S' to S''* (see Figure 11:(a)). We allow joining the top of S' with the bottom of S'' (rather than with the top of S''). The following definition does not depend on this choice.

Let D be a plane digraph obtained from two copies S' and S'' of S_1 joined with the pair of communication edges, and let F be an exterior face of D. We say that S' and S'' are

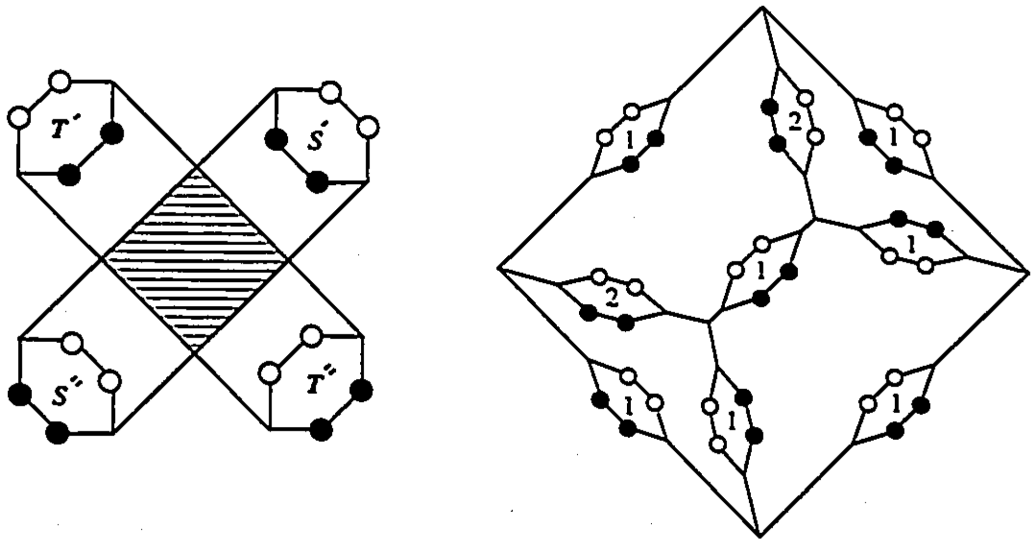


Figure 11: (a) Two pairs of communication edges crossing (b) Crossover R, to replace the shadowed area

flipped in the same way in D, if *right*-paths of S' and S'' contribute one of them, to the exterior face of D, and the other one, to an interior face of D (as indicated with non-filled vertices in Figure 11:(a)).

Again, by Theorem 1.3, we have immediately

Lemma 1.8 *A plane digraph D obtained from two copies S' and S'' of S_1 joined with the pair of communication edges has a similar planar upward drawing if and only if both the copies are flipped in the same way.* □

This allows us to place in a plane digraph two copies of S_1 such that in every planar upward drawing they must be flipped in the same way. The last problem to resolve is possible crossing of communication edges. We avoid it in a classical manner by designing a component called a *crossover*. To this end we may use the digraph R given in Figure 11:(b) in which every component being a copy of S_1 or S_2 is shown schematically as a shaped box labeled with 1 or 2, respectively, and is assumed to be amalgamated into R. We show that the pairs of components (S', S'') and (T', T'') must be flipped in the same way to allow an upward drawing of R.

Lemma 1.9 *Let R be the plane digraph in Figure 11:(b) and F its exterior face. Then R has a similar planar upward drawing if and only if the pairs (S', S'') and (T', T'') are flipped* *in the same way, that is, both the total contribution of (S', S'') to the difference $\alpha(F) - \sigma(F)$ and the total contribution of (T', T'') to the difference $\alpha(F) - \sigma(F)$ are equal to zero.* □

Proof First note that all vertices in R have *alternation* < 2. Suppose that S' contributes with the *left*-path to the exterior face (that is, with -1 to the difference $\alpha(F) - \sigma(F)$). In Figure 11:(b), we denote this path with the filled vertices on the left hand side of S' and with non-filled vertices on the right hand side. Then, applying Theorem 1.3 to the face F of R, on the right hand side of S' we see that its unique S_2-component must be flipped just as shown in Figure 11:(b) in order to make the difference $\alpha(F) - \sigma(F)$ equal zero. Also, further flipping is a consequence of flipping S', which proves the part of the theorem for the pair (S', S''). A similar argument applies independently to T' and T''.

On the other hand, we see that if all the components are flipped in the suggested way, then the conditions of the theorem are satisfied, and therefore R has a similar upward drawing. □

To fix attention, in the construction below, we assume that components S' and T' in every copy R' of R are flipped initially so that their *right*-paths contribute to the exterior face of R', and the remaining are flipped so that R' has a similar planar upward drawing.

1.9 Construction

Now suppose we are given an instance of X3C, Exact cover by 3-set, with $X = \{x_1, x_2, \ldots, x_n\}$ where $n = 3q$, and $C = \{C_1, C_2, \ldots, C_m\}$. Without loss of generality we may assume that each x_j is a member of at least two sets in C; otherwise the possible choice is partially determined. From this we construct a plane digraph G' as follows (we use elements of X and C to label some vertices of G'):

(*i*) Draw (without crossing) a cycle consisting of the directed paths: $p_1 = b \rightarrow C_1 \rightarrow C_2 \rightarrow \ldots \rightarrow C_m \rightarrow w$, $p_2 = w \rightarrow x_1 \rightarrow x_2 \rightarrow \ldots \rightarrow x_n \rightarrow t$, and the edge $b \rightarrow t$.

(*ii*) Draw a directed edge $C_i \rightarrow x_j$ inside the cycle whenever $x_j \in C_i$; allow edges to cross, but two edges cross in at most one point. (see Figure 12:(a)).

Replacing parts of G' by suitable components we construct digraph G as follows:

(*iii*) Replace every vertex C_i by a copy of Q in such a way that all of them are amalgamated into G and the *right*-path of each contributes to the exterior face.

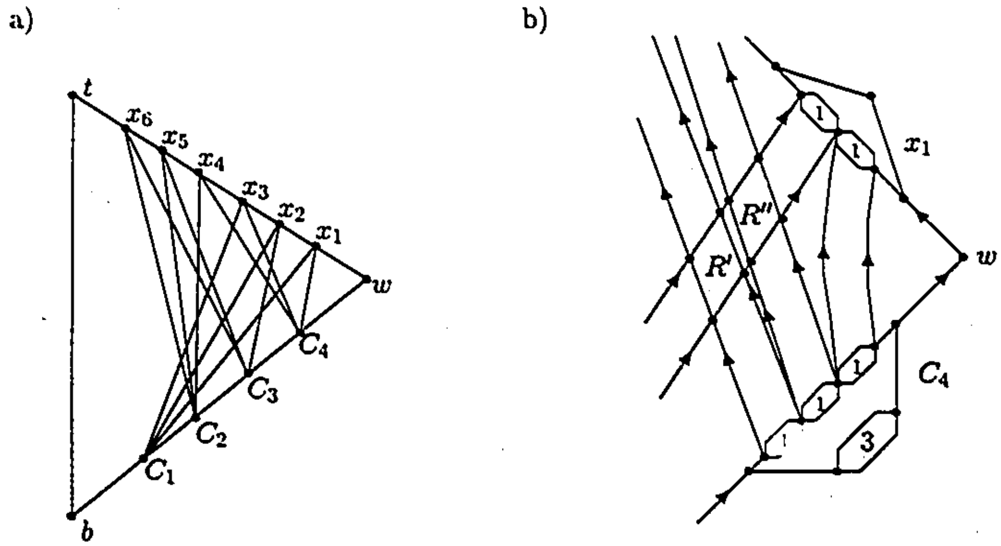


Figure 12: Digraph G' and a part of G containing components C_4 and x_1

(*iv*) Replace every vertex x_j by a copy of $P(d)$, where d is the degree of x_j, in such a way that all are amalgamated into G and the *right*-path of each contributes to the exterior face.

(*v*) Replace every edge $C_i \rightarrow x_j$ by a pair of communication edges joining (arbitrarily chosen) S_1-components of C_i and x_j, respectively (not allowing crossing areas to overlap) (see Figure 12:(a)).

(*vi*) Replace every crossing area produced by two pairs of communication edges, by a copy of the crossover R as in Figure 11:(b), as illustrated in Figure 12.

(*vii*) Subdivide nonessential edges by adding vertices of degree two, if necessary. Note that to prove the NP-completeness of upward planarity testing for arbitrary digraphs this step can be omitted (see the remark following the description of S_m in Section 1.6.1).

(*viii*) Add a frame on vertices t and b.

(*ix*) Add edges from w to a new vertex x and from x to the vertex v on the frame as in Figure 8, and replace x by an amalgamated copy of S_p with $p = |2n - 3m|$ (see Figure 13).

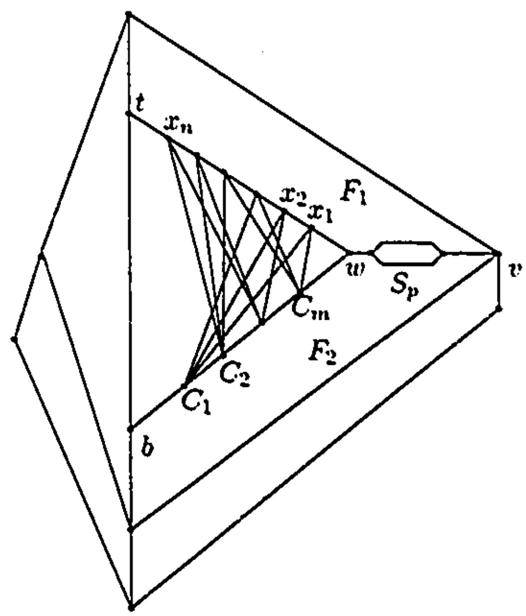


Figure 13: Schematic illustration of G

Clearly, G can be constructed in $O(m+n)$ time. It remains to show that G has a planar upward drawing if and only if C contains an exact cover of X.

Suppose, that $C' \subseteq C$ is an exact cover of X. First, note that G is constructed so that all vertices have alternation < 2. We flip some spiral set components in G to obtain another planar representation G^* of G.

At first, we flip all spiral sets in components Q corresponding to $C_i \in C'$, and then those copies of S_1 in components $P(d)$ that correspond by communication edges to flipped copies of S_1 in components Q. Next, all spiral sets in involved crossovers are flipped suitably. (By involved crossovers we mean those lying on communication edges joining flipped components.)

Observe that the extremal vertices of G^* are precisely the vertices u_i on *right*-paths of spiral set components and, in addition, two vertices on the boundary. If every spiral set component is replaced by a vertex, then the resulting digraph is an *st*-digraph; in particular, for every interior face F, the switching index $\sigma(F) = 0$. It follows, by Theorem 1.3, that G^* has an upward assignment if and only if for every face F of G^* the total contribution of spiral set components to the difference $\alpha(F) - \sigma(F)$ equals zero.

We have chosen flipping some components in G so that by Lemma 1.6, this condition holds for every face of G^* other than faces F_1 and F_2 in Figure 13. We proceed to show that this condition holds for F_1 and F_2, as well.

For each $d_1, d_2, \ldots, d_n$, $P(d_i)$ corresponds to the element $x_i \in X, i = 1, 2, \ldots, n$. Then the total contribution of all spiral set components to the face F_1 is

$$I = -\sum_{i=1}^{n}(d_i - 2) = 2n - \sum_{i=1}^{n} d_i = 2n - 3m. \tag{1}$$

On the other hand, since there are precisely q components Q with their spiral sets flipped (where $n = 3q$), the total contribution of spiral set components to F_2 is

$$3(m - q) - 3q = 3m - 2n \tag{2}$$

whence, assuming that the component S_p with $p = |2n - 3m|$ is flipped suitably, we see that the condition holds for F_1 and F_2, as well, thus proving that G has a planar upward drawing similar to G^*.

Conversely, from the construction it follows also that if G has a planar upward drawing, then it must be similar to a plane digraph G^* obtained from G just by flipping some of its spiral set components. It follows also, by Theorem 1.3, that the flipping must be such that for every face F the total contribution of such components to the difference $\alpha(F) - \sigma(F)$ equals zero. In particular, by Lemma 1.7, in every component $P(d)$ there is precisely one copy of S_1 flipped, and by properties of the construction, the components Q with their spiral set components flipped, yield an exact cover of the set X. This completes the proof of Theorem 1.5. □

Chapter 2

Upward Drawing on Surfaces

2.1 Introduction

If we wish to have upward drawings without crossing edges, we are led ineluctably to study upward drawings on surfaces other than planes in $\mathbf{R}^3$. A common artifice, especially contrived to avoid edge crossing, is to draw the diagram on closed, compact, orientable surfaces, that is, on surfaces homeomorphic to the surface of a sphere with attached handles, in $\mathbf{R}^3$.

In this chapter, after presenting some examples of nonplanar ordered sets (some of them, even, with planar covering graphs), we formally define the concept of upward drawings without crossing edges on surfaces, that is, *embedding* of ordered sets on surfaces. In Section 2.3 a *lifting procedure*, which is fundamental in upward drawings on surfaces, will be introduced. *Critical points*, in particular *saddle points*, which play a decisive role in upward drawings, and the *Discrete Index Theorem* will be presented in Section 2.4.

2.1.1 Examples of Nonplanar Ordered Sets

Although a planar ordered set must, of course, have a planar covering graph, an ordered set with a planar covering graph need not be a planar ordered set. The following three examples describe nonplanar ordered sets with planar covering graphs.

Example 2.1 Consider the six-element ordered set defined in Section 1.2.1. One can verify, for instance, by examining the maximal chains that this ordered set has no planar upward drawing, while its covering graph is planar.

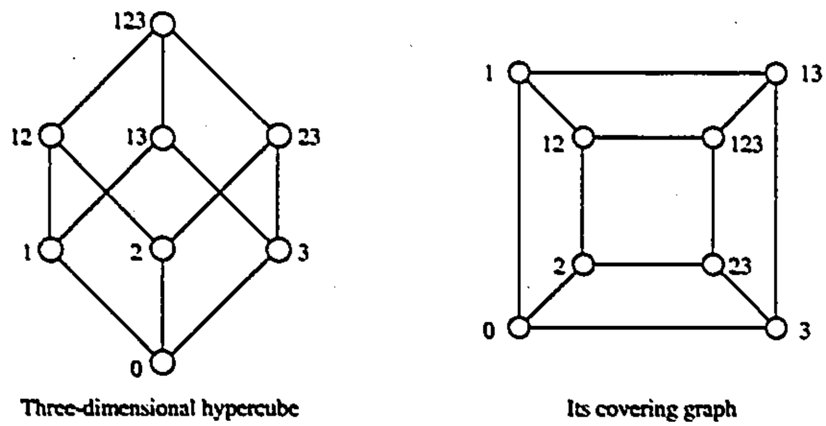


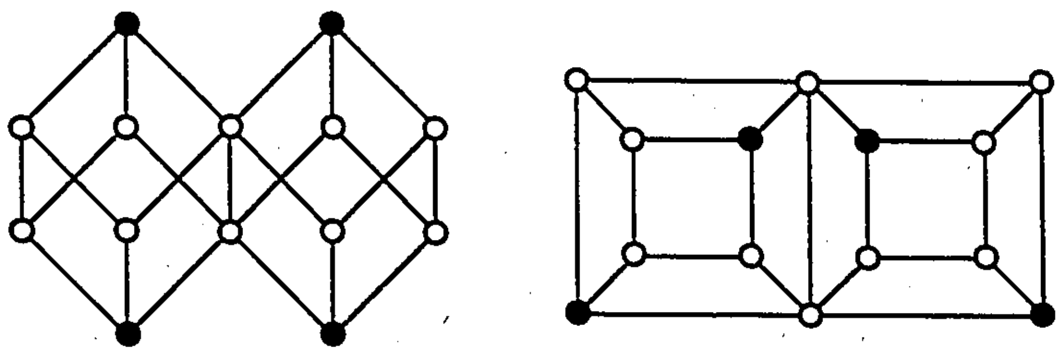
Figure 14: Hypercube and its covering graph

Example 2.2 The ordered set of all subsets of a three-element set with respect to inclusion is called the *three-dimensional hypercube* denoted by $\mathbf{Q}^3$. Figure 14 shows that the covering graph of this ordered set is planar, while the ordered set itself is nonplanar. The covering graph of $\mathbf{Q}^3$ has a unique embedding so that its top and bottom are not in the same face. On the other hand all faces are bounded faces, with switching index $\sigma(f) = 0$ if f is an interior face and $\sigma(f) = 2$ if f is an exterior face (cf. Section 1.4.2 for definition of switching index). Thus, by virtue of Theorem 1.3, the extremal vertices top and bottom must be assigned to the exterior face. Therefore, it admits no upward assignment.

The three-dimensional hypercube can be drawn, for instance, on the surface of a simple sphere $S = \{(x, y, z) : x^2 + y^2 + z^2 = r^2\}$ in three-dimensional space $\mathbf{R}^3$ in such a way that the arcs corresponding to the edges of the ordered set all lie on the surface and they are monotonically increasing with respect to the positive direction of the z-axis. We say that $\mathbf{Q}^3$ is embedded on the sphere (see Figure 16).

Example 2.3 If we paste two copies of $\mathbf{Q}^3$ along an edge not incident with an extremal vertex, the result is called the *doublecube*. An upward drawing of this ordered set is depicted in Figure 15. The covering graph of the doublecube is planar, but obviously it is not upward planar since even the three-dimensional hypercube is not planar. On the other hand, unlike $\mathbf{Q}^3$, it has no upward drawing without crossing of edges on a simple sphere.

In fact, an upward drawing of this ordered set requires only a surface topologically homeomorphic to the sphere. Geometrically, with respect to a height function, it must



The doublecube Its planar covering graph

Figure 15: The doublecube

contain however at least one saddle point [**Ewacha, Li, and Rival (1991)**], [**Reuter and Rival (1991)**], [**Musin, Rival and Tarasov (1993)**]. Figure 17, shows an upward drawing of this ordered set on a surface of genus zero.

Note that, if the covering graph of an ordered set is nonplanar, then obviously it has no upward drawing on a surface of genus zero.

Example 2.4 In Example 2.1, if we add another chain of length> 2, connecting its bottom to its top, then the covering graph contains a graph homeomorphic to the complete bipartite graph $K_{3,3}$. In effect the covering graph is a nonplanar graph[**Kuratowski (1963)**]. Figure 18, provides an upward drawing of this ordered set on the torus, which is topologically homeomorphic to the sphere with a handle is attached — a surface of genus one.

2.2 Embedding of Ordered Sets

The study of upward drawings on surfaces has been motivated by graph embedding, and topological graph theory, whose literature is extensive (cf. for example, [**Gross and Tucker (1987)**]). Thus, although it is customary and convenient to draw a diagram of an ordered set on the plane, whether or not edges cross, we may also wish to draw them on other surfaces, especially if this avoids the crossing of edges.

According to this scheme, small circles corresponding to the elements of the ordered set are located on the surface in such a way that for a and b in the ordered set P, the circle

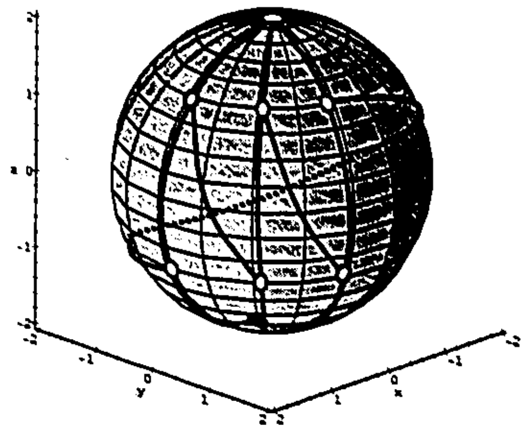


Figure 16: An upward drawing without crossing edges of the hypercube on the sphere

corresponding to a is farther north (has greater z-value) than the circle corresponding to b, whenever $a > b$, and a monotonic curve is drawn to join the circles just if a covers b. See Figure 16 for an upward drawing of hypercube $\mathbf{Q}^3$ on the sphere

There are major differences between graph embedding and order embedding. A surface's shape and curvature, and especially its *critical points* play an important role for order embedding although they play no role at all for graph embedding. In fact, for instance, all genus zero surfaces are topologically homeomorphic with a simple sphere, which in turn, from the point of view of graph embedding, is equivalent to a plane. That is, all genus zero graphs are planar. But as we have seen in the previous examples, the critical points of these surfaces, are important in that an ordered set with planar covering graph may have an embedding in a genus zero surface homeomorphic to the sphere, but may fail to have embedding onto the sphere itself (see Example 2.3 and Figure 17).

To be more precise, we consider the case of a two-dimensional orientable closed compact manifold, which is called a *two-dimensional surface* or simply a *surface*. It is well known (cf. [Massey (1991)]) that any such manifold is homeomorphic to a sphere with a certain number g, its *genus*, of handles attached and thus, could be embedded into $\mathbf{R}^3$. Thus an upward drawing of an ordered set can be defined as follows.

A directed covering graph of an ordered set P, denoted $\overrightarrow{cover}(P)$, is the covering graph of P where the edges are equipped with directions identical with the order. That is, if $a \prec b$ then the edge $a \sim b$ is directed from a to b.

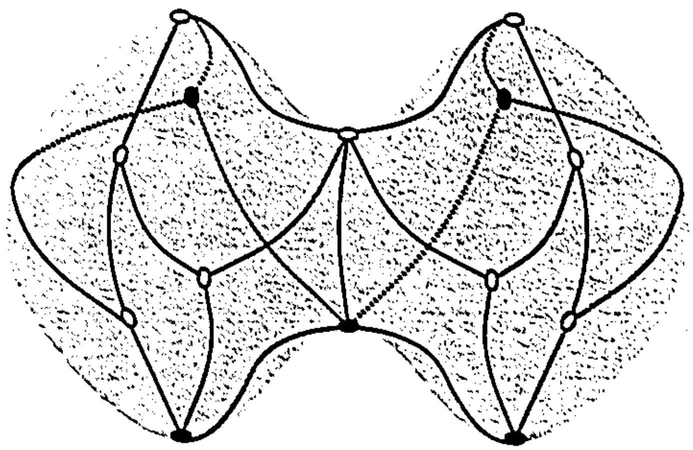


Figure 17: An upward drawing without crossing edges of doublecube on a surface of genus zero

An *upward drawing* of an ordered set P on a surface S is a pair of embeddings

$$\varphi : \overrightarrow{cover}(P) \longrightarrow S \quad \text{and} \quad \psi : S \longrightarrow \mathbf{R}^3$$

such that the images of the directed edges of $\overrightarrow{cover}(P)$, are monotonic with respect to a height function in $\mathbf{R}^3$.

This definition implies that such upward drawings on surfaces have no crossing of edges. We usually refer to such an upward drawing as an *embedding* of the ordered set (or *order embedding*) onto the surface.

The *order genus* of an ordered set P, denoted $genus(P)$, is the minimal genus of the surface on which an upward drawing of P exists.

An embedding is called *cellular*, if all faces are simply connected, that is, they are homeomorphic to a disk.

2.3 Lifting Procedure

The upward drawing of an ordered set is far from unique. In fact, there may be many associated cell complexes on the surface on which the ordered set is (cellularly) embedded. Nonetheless, there is one fairly obvious upward drawing associated with a *lifting* procedure

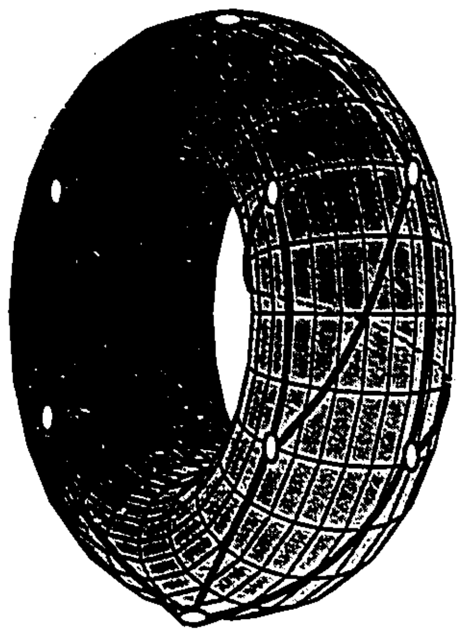


Figure 18: An upward drawing on a surface of genus one

(cf. [Ewacha, Li, and Rival (1991)]).

Start with *cover*(P), the covering graph of an ordered set P, which, as a graph, has genus, say g. Embed *cover*(P), without crossing edges on a two-dimensional surface of genus g. It is convenient to represent this embedding inside a polygon in the plane $\{(x,y,z) : z = 0\}$ with $4g$ sides (if $g > 0$), in which *cover*(P) is drawn planar, possibly with repeated edges and vertices, and in which $2g$ pairs of sides are tagged for identification. Figure 19 illustrates an embedding of the covering graph of the ordered set introduced in Example 2.4 on a 4-gon corresponding to the torus.

A *height function* of an ordered set is any strict isotone, real-valued function h of P, that is, $h(x) < h(y)$ whenever $x < y$ in P. A common example is

$$h(x) = \max\{|C| : C \text{ is a chain in } P \text{ with } top(C) = x\}$$

Next, define any (real-valued) height function h on the vertices and continue it by linearity onto the polygon, using a triangulation of the embedding on the polygon, thus

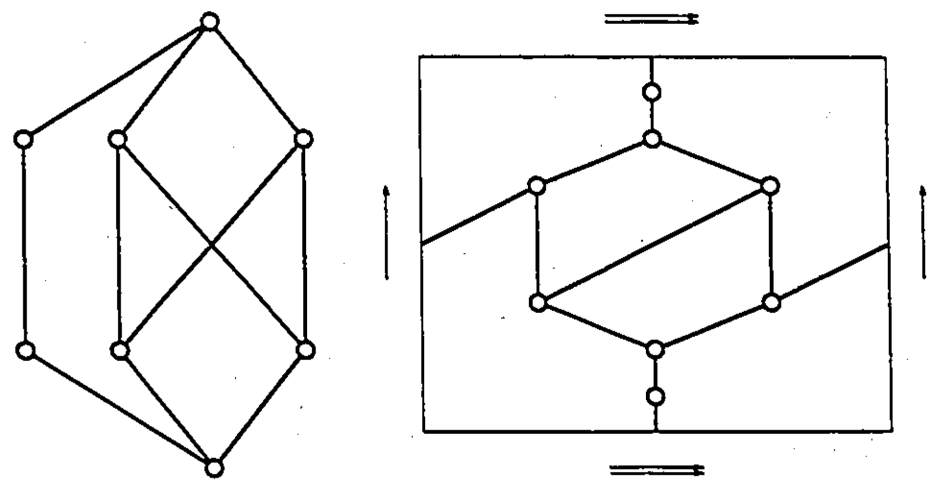


Figure 19: Embedding of an ordered set in a 4-gon corresponding to torus

producing a piecewise-linear homotopic image which can then be piecewise linearly reconstructed, monotonically, with respect to h in $\mathbf{R}^3$, by gluing pairs of sides of the polygon (already tagged for identification).

Unlike the planar case, a triangulation for graphs embedded on two-dimensional surfaces with *genus* > 0, may be impossible — without additional points (cf. Section 6.3.3 for details).

In summary, by the lifting and subsequent gluing of polygon sides, we can manufacture a piecewise linear two-dimensional surface in $\mathbf{R}^3$. It may happen that in higher genus ordered sets ($g > 0$) such a surface is not *realizable* in $\mathbf{R}^3$, that is, we may have self-intersections — an *immersion*[1] — despite the fact that the surface is topologically homeomorphic to one without intersections, that is, a sphere with attached handles.

Thus, if we relax the condition for realizability in $\mathbf{R}^3$ of the surface on which the ordered set is drawn without crossing edges, then the lifting procedure proves the following fundamental result about upward drawings on surfaces.

Theorem 2.1 [Ewacha, Li, and Rival (1991)] *The order genus of an upward drawing equals the graph genus of its covering graph.*

[1]Let X and Y be manifolds and $f : X \longrightarrow Y$ a differentiable map. We say that f is an *immersion* at x if $T_x f : T_x X \longrightarrow T_{f(x)} Y$ is injective. [Berger and Gostiaux (1988)].

According to Theorem 2.1, we can check whether an ordered set has an upward drawing without crossing edges on a surface of genus zero (a surface topologically homeomorphic to a sphere) in linear time. In fact, it is sufficient to check for the planarity of its covering graph which can be done in linear time using for instance the [**Hopcroft** and **Tarjan** (1979)] algorithm.

An *orientation* of a graph is an order (or upward drawing) with this graph as covering graph. The fundamental problem in the theory of orientations is to construct nontrivial *invariants* with respect to all orientations of a covering graph. The number of elements, edges, etc. are examples of trivial invariants. Thus, every orientation of a three element ordered set has three elements, while all of the orientations need not have same length (two or three). In fact, none of the familiar parameters *height*, *width*, or *dimension*, is such an invariant (cf. [**Rival** (1993)]).

The lifting procedure, on the other hand, provides the first nontrivial orientation invariant. As the order genus, by Theorem 2.1, depends only on the covering graph, every orientation of a fixed (covering) graph has the same order genus.

Theorem 2.2 [**Ewacha, Li,** and **Rival** (1991)] *Order genus is an orientation invariant.*

The lifting procedure suggests a *good* algorithm for upward drawing of ordered sets of genus zero, that is, those with planar covering graphs. Thus we are always able to construct an upward drawing on a surface of genus zero, realizable in $\mathbf{R}^3$. On the other hand, unlike (covering) graphs, it is not yet known whether an ordered set (of *order genus* $g > 0$) has an upward drawing on a two-dimensional surface of genus g, which itself has *no* self-intersections — an *embedding*. We have the following result.

Theorem 2.3 [**Ewacha, Li,** and **Rival** (1991)] *The decision problem whether the upward drawing of an ordered set can be drawn on a two-dimensional surface with fixed order genus and with a prescribed list of critical points, belongs to NP* (cf. [**Musin, Rival** and **Tarasov** (1993)]).

2.4 Critical Points

For a fixed surface and a fixed ordered set which constraints must be considered to guarantee an embedding of this ordered set on this surface?

Of course, the genus of an ordered set which is, by Theorem 2.1, equal to the genus of its covering graph, must be known. Calculating the order genus is an "NP-hard" problem, since the problem for covering graphs is NP-hard [**Thomassen** (1990)]. On the other hand for a fixed genus g we can check in $O(n^{O(g)})$ time whether the covering graph of an ordered set has genus g (cf. [**Filotti, Miller and Reif** (1979)]).

Thus, if we assume the order genus of an ordered set, the next natural problem, as the examples at the beginning of this chapter show, is to study the critical points of the surface on which the order has an upward drawing without crossing edges. Some of the vertices of the ordered set correspond to these points on the surface, which are called *critical points* of the ordered set.

In the next section, we first review the critical points of surfaces and the Poincaré Index Theorem. Then the critical points for ordered sets and a discrete analog of the Poincaré Index Theorem will be studied.

2.4.1 The Poincaré Index Theorem

The Critical Point Theorem for height functions on an embedded surface in $\mathbf{R}^3$ is one of the most important theorems of *geometry*. This theorem, in fact, relates a geometric property of the embedded surface namely, the sum of a set of geometrically defined indices of critical points, to a topological property of the surface, the Euler-Poincare characteristic $\chi(S)$. Although, the smoothness of surfaces is assumed, the theorem has analogs for polyhedral surfaces embedded on $\mathbf{R}^3$ [**Banchoff** (1967)] and ordered sets embedded on surfaces [**Glass** (1973)]. We shall now briefly review the Critical Point Theorem for surfaces and then, their discrete relatives for ordered sets will be discussed.

Consider a smooth surface S embedded in $\mathbf{R}^3$ and consider a linear function h on $\mathbf{R}^3$ which projects all of $\mathbf{R}^3$ to the line determined by a unit vector $\vec{h}$. A point p of S is said to be a *critical point* for h if the tangent plane to S at p is perpendicular to h. All other points of S are called *ordinary points* for h. Usually, one can consider the standard unit vector $\vec{k} = (0,0,1)$ and the height of each point is its z-value in the Cartesian coordinate system. Thus, for instance, the sphere has two critical points, one maximum and one minimum, and torus has four critical points one maximum, one minimum and two (nondegenerate) saddle points.

The *Critical Point Theorem* for height functions states that if h has a finite number of critical points on S of the three types maximum, minimum and saddle points which all are *nondegenerate*[2], then

$$|\text{ local maxima }| + |\text{ local minima }| - |\text{ saddle points }| = \chi(S) \quad (3)$$

where $\chi(S)$ is the *Euler-Poincaré characteristic* of S. The *Euler-Poincaré characteristic* $\chi(S)$ is the alternating sum of the number of cells of all dimension in any cell partition of S. Thus for a *cell complex*, that is a family of cells whose sum is S, such that the intersection of any two is either empty or the union of other cells

$$\sum_{i\geq 0}(-1)^i(\text{number of cells of dimension } i) = \chi(S)$$

The Euler-Poincaré characteristic is a topological invariant, that is, it remains fixed under any continuous deformation of the surface. Thus, for a cartographic map, say, of countries (simply connected regions) drawn on the surface of a sphere, the alternating sum of vertices, edges and faces is the constant integer two. In other words, the Euler-Poincaré characteristic of the sphere is two. More generally, the Euler-Poincaré characteristic of any orientable two-dimensional manifold equals $2 - 2g$, where g is the genus of the manifold.

There are several ways that one can define an *index* for each critical point.

In Morse Theory [Milnor (1963)] the *index* of a critical point is given by considering the sign of the determinant of the matrix of second derivatives at that point.

If we consider a smooth *vector field* on a surface, that is smoothly assign to each point on the surface, a vector taken from its tangential hyperplane, then a point is called *critical* (*singular*) if the corresponding vector at that point is zero. Let p be such a critical point and suppose it is *isolated*, that is, there is a neighborhood containing it in which it is the only critical point. Take a counterclockwise circular path inside this neighborhood. The *index* of p is the number of revolutions of the vector field along this path.

[2]Let $f : \mathbf{R}^k \longrightarrow \mathbf{R}$ be a smooth function and Suppose f has a critical point at x. If the *Hessian matrix*

$$H_x f = \left(\frac{\partial^2 f}{\partial x_i \partial x_j}\right)_x$$

is nonsingular at the critical point x, then x is said to be a *nondegenerate* critical point of f. The concept of nondegeneracy makes sense on manifolds via local parametrizations. Suppose that $f : X \longrightarrow \mathbf{R}$ has a critical point at x and that ϕ is a local parametrization carrying the origin to x. Then $\mathbf{0}$ is a critical point for the function $f \circ \phi$. x is said to be *nondegenerate* for f if $\mathbf{0}$ is nondegenerate for $f \circ \phi$ [Guillemin and Pollack (1974)] . A function whose critical points are all nondegenerate is called a *Morse function*. The case of nondegenerate critical points is the common situation. In fact it can be deduced from Sard's Theorem (cf. [Guillemin and Pollack (1974)]), that the vast majority of functions are actually Morse functions.

Here is still another definition for index of a critical point. For a height function h, we define $index(p, h) = 1$ if p is a local maximum or minimum and $index(p, h) = -1$ if p is a (nondegenerate) saddle point. The theorem then states

$$\sum_{p \text{ critical}} index(p, h) = \chi(S) \quad (4)$$

which is called Poincaré Index Theorem.

2.4.2 Critical Points of Ordered Sets

Suppose that an ordered set (directed graph) P is embedded without crossing edges on a closed compact surface S. Assume, moreover, that the embedding is *cellular*, that is, it partitions the surface into simply connected (null-homotopic) faces F, bounded by edges E, connecting vertices V. Accordingly,

$$|V| - |E| + |F| = \chi(S)$$

Now, a discrete analog of the Poincaré index theorem can be obtained for this upward drawing ([Glass (1973), cf. [Musin, Rival and Tarasov (1993)]) as follows.

To any covering edge $a \succ b$ in P we associate two values, "+" to that end of the covering edge outgoing from a and "−" to that end of it incoming to b. In this way, every covering edge acquires two values and, for every element $a \in P$, every incident covering edge associates, in this way, a sign (+ or −), to a. A minimal element of P will acquire all − values and a maximal element all + values. In general, to every element there is associated a circular sequence of +'s and −'s corresponding to a clockwise orientation in a neighborhood about it. What is of importance is the number of alternations of +'s and −'s. Thus, we call an element *ordinary* just if this sequence consists of an interval of +'s and and interval of −'s that is one *alternation*. An *extremal* element, that is, a maximal or a minimal element has no alternations at all. If an element's circular sequence has two or more alternations then it must be a *saddle point* — the surface on which the ordered set is drawn cannot be spherical (see Figure 6).

2.4.3 Discrete Index Theorem

The above assignment of +'s and −'s to the edges can be summarized by formulating the index of a critical point in a covering graph.

For each vertex, the incident edges "fan out" forming a circle. Mark such an incident edge + if its other endpoint u satisfies $u \succ v$ and mark it − if $u \prec v$, with respect to the order. Let n_+ stand for the number of intervals of +'s, and let n_- stand for the number of intervals of −'s in this fan. There are in effect, three cases: if

$$n_+ + n_- = 2 \quad \text{then } index(v) = 0$$

and the corresponding vertex is ordinary; if

$$n_+ = 0 \text{ or } n_- = 0 \text{ then } index(v) = 1$$

and the corresponding vertex is critical, an *extremum* — either a *local maximum* or a *local minimum*; finally,

$$n_+ + n_- > 2 \text{ and then } index(v) = 1 - n_-$$

in which case v is a *saddle point*.

To compute the index of a face we consider its local minima. Thus, call a vertex v of the face C a *local minimum* if, in the order, v is less than its (two) neighbors in the boundary cycle of face C. And, call the face C *ordinary* if it has only one (local) minimum, else call it *nonordinary* and , in any case, set

$$index(C) = 1 - |\text{ local minima }|$$

Here is the discrete analog of the Poincaré Index Theorem.

Theorem 2.4 Discrete Index Theorem [Glass (1973)] *For any directed graph embedded on a compact surface S*

$$\sum_{v \in V} index(v) + \sum_{C \in F} index(C) = \chi(S) \tag{5}$$

A directed graph is *acyclic* if it has no directed cycle. If an acyclic directed graph contains a bounded face C, in the sense that its boundary cycle can be partitioned into two maximal chains which share the same top and the same bottom, then $index(C) = 0$ since it has only one local minimum. The smallest examples of such faces are triangles.

Consider an acyclic directed graph, for instance, the directed comparability graph of an ordered set, so that all faces are bounded faces. As any face C is a bounded face with precisely one local minimum, then $index(C) = 0$ and it suffices to sum the index just over all vertices and Formula (5) reduces to (4), its continuous forebear. In particular, in a triangulated direct graph one can simply calculate all critical points.

We close this section by applying the index theorem in proving the fact that any upward drawing of the doublecube on a surface of genus zero requires at least a saddle point (cf. Example 2.3).

The proof goes as follows. The covering graph of doublecube is planar so that it has two *three*-connected components, namely the copies of the cube, so it has few embeddings in the plane. An embedding of this covering graph is illustrated in Figure 15. All interior faces are bounded faces with index zero. The exterior face is the only nonordinary face due to the two minima, the bottoms of the copies of the cube, thus its index is -1.

The doublecube has four extremals (top and bottom of each cube) whose contribution to the summation of indices is four. Note that since the covering graph is planar its Euler-Poincaré characteristic is at most two. Thus,

$$\begin{aligned} 2 \geq \chi(S) &= \sum_{v \in V} index(v) + \sum_{c \in F} index(C) \\ &= 4 + \sum_{v \neq extremal} index(v) - 1 \\ &= 3 + \sum_{v \neq extremal} index(v) \end{aligned}$$

Therefore,

$$\sum_{v \neq extremal} index(v) \leq -1$$

This proves the existence of at least one saddle point.□

Chapter 3

Spherical Ordered Sets

3.1 Introduction

In this chapter we focus on the upward drawings on the round sphere

$$S = \{(x, y, z) : x^2 + y^2 + z^2 = 1\}$$

On the one hand, it is a closed, compact, orientable surface in $\mathbf{R}^3$, and, therefore, has a simple structure. On the other hand, from the upward drawing point of view, it provides more freedom than the plane in that there are nonplanar ordered sets whose edges can wrap around the sphere preserving monotonicity and avoiding crossing edges. It is known that the sphere and the plane are equivalent for (undirected) graph embeddings: that is, an undirected graph has an embedding on the sphere if and only if it has a similar embedding on the plane. But surprisingly, Upward drawings of ordered sets (or directed graphs) are a different story. Indeed there are many ordered sets, all nonplanar, which have upward drawings without crossing edges on the sphere — they are all *spherical*.

Thus, as the next natural candidate for upward drawings without crossing edges, we shall study upward drawings on this surface.

In Section 3.2 we shall characterize all spherical ordered sets in terms of critical points. In Section 3.3 we define the concept of a circular drawing on the plane which is of independent interest and we shall show that it is equivalent to upward drawing on the sphere, in the sense that an ordered set is spherical if and only if it has a circular drawing on the plane without crossing edges.

3.2 Characterization

An ordered set is *spherical* if it has an upward drawing on the round sphere

$$S = \{(x, y, z) : x^2 + y^2 + z^2 = 1\}$$

such that all edges are monotonic paths with respect to a fixed direction, say the positive direction of the z-axis, the northerly direction, and no two edges cross.

Sphericity for acyclic directed graphs is defined in a similar manner. An acyclic directed graph is *spherical*, if it has a drawing on the sphere so that all monotonic paths corresponding to the directed edges point upward. Although our primary interest, in this work, bears on ordered sets, all the results can easily be extended to acyclic directed graphs.

[**Foldes, Rival and Urrutia** (1992)] showed that an ordered set is spherical if it has a *top*, a *bottom* and its covering graph is *planar*. Our definition of spherical ordered sets includes a wide range of ordered sets.

In the following we show that in fact, by adding new edges, if necessary, any spherical ordered set can be extended to an acyclic directed graph such that it has a top, a bottom and its covering graph is planar.

Theorem 3.1 *An ordered set is spherical if and only if by adding new edges (if necessary) it can be extended to an acyclic embedded digraph which has exactly one maximal vertex (all incident edges have sign '+') and exactly one minimal vertex (all incident edges have sign '−').*

Proof Let P be an ordered set such that by adding some new edges, if necessary, it can be extended to an acyclic directed graph P^*, where P^* has a planar underlying graph with exactly one maximal and one minimal. Then, by [**Foldes, Rival and Urrutia** (1992)], P^* has an upward drawing without crossing edges on the sphere. Now, removing all edges in $P^* \setminus P$ provides an upward drawing of P on the sphere, proving that P is spherical.

Conversely[1], suppose that we have an upward drawing without crossing edges of P, on the sphere, such that all edges are monotonically increasing in the positive direction of the z-axis. We also assume that the sphere, itself, has a fixed position in three-dimensional space $\mathbf{R}^3$.

[1]This part of the proof closely follows [**Di Battista and Tamassia** (1988)] (for the planar case).

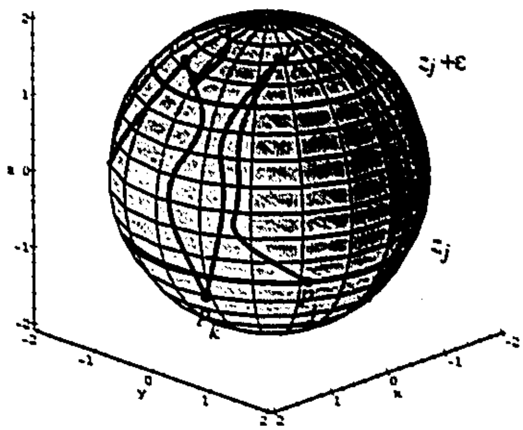


Figure 20:

We denote the vertices of P by $p_1, p_2, \ldots, p_n$, sorted from bottom to top, and with $z_1, z_2, \ldots, z_n$ their heights (their z-ordinates), respectively. Without loss of generality we can assume that p_1 is located at the south pole and p_n at the north pole. Otherwise, we may modify the upward drawing to provide an equivalent drawing on the sphere with this property.

Let p_j be a maximal element other than p_1 and p_n (with height different from both z_1 and z_n) and consider that portion of the sphere between the two level curves obtained by cutting the sphere with parallel planes $Z = z_j$ and $Z = z_j + \varepsilon$ where $\varepsilon > 0$ is small enough so that this portion contains no vertices p_i $(1 \leq i \leq n)$ in its interior. This portion is subdivided into connected regions bounded by the monotonic curves corresponding to the edges of P (see Figure 20).

Next, let $p_k \prec p_l$ be a covering edge on the boundary of any region that contains p_j. We draw a monotonic path starting at p_j going closely along the path corresponding to $p_k \prec p_l$. This means that we can add an edge $p_j \prec p_l$ to P with sign'+' at p_l and sign '−' at p_j. Thus p_j is no longer a maximal.

Now, repeating this procedure, we can eliminate all maximals except the one at z_n and similarly we can eliminate all minimals except the one at z_1.

To complete the construction, we shall now add an edge from z_1 to all remaining minimals and from all remaining maximals to z_n. Evidently, the resulting directed graph P^* is acyclic with exactly one maximal and exactly one minimal and the underlying graph is planar.□

A *similar planar embedding* of a planar graph, is an embedding in the plane which preserves the face structure of the original graph.

In view of the properties of the digraph P^* described in Theorem 3.1 and the Index Theorem (cf. Theorem 2.4), we can immediately deduce the following lemmas.

Lemma 3.2 *In a similar planar embedding of P^* all vertices have alternation at most one.*

Proof The maximal and minimal vertices of P^* contribute $+2$ to the summation of the indices. Thus, if there is a vertex with *alternation* > 1 , then it contradicts the fact that the Euler characteristic of the sphere is 2 .□

Therefore, the edges incident to each vertex of P^* can be partitioned into two intervals (with the possibility that one is empty), so that all edges in any interval have the same sign and they occur around the vertex consecutively. In other words, it has no *saddle points.*

Lemma 3.3 *All faces of P^* are ordinary faces with index zero.*

Proof We note that each nonordinary face contributes a negative integer to the summation of indices, thus similar to the proof of the previous lemma, P^* contains no nonordinary face.□

By definition a bounded face is a face with a top and a bottom element. The boundary of a bounded face can be partitioned into two maximal chains sharing the same top and bottom. In fact, using Lemma 3.3, we can partition the boundary of every face of directed graph P^* described in Theorem 3.1, into two maximal chains C_1 and C_2 such that $C = C_1 \cup C_2$, where C is the boundary of the face.

By *triangulation*, we mean an acyclic triangulation with signed edges. We have the following result.

Lemma 3.4 *A bounded face can be triangulated without increasing the alternation about any vertex.*

Proof Let C be a bounded face consisting of maximal chains C_1 and C_2. If the face is not a triangle, then by adding any *chordal* edge, that is, a directed edge which connects the tail of a directed path to its head, we obtain two smaller bounded faces where each of them can be triangulated with the property that it does not increase the alternation of any vertex.□

Note that any acyclic triangle as a face is a bounded face with index zero, so that for an acyclic triangulated graph, the index formula depends only on the indices of the vertices. A combination of the preceding results proves the following theorem.

Theorem 3.5 *Fix an embedding of the directed covering graph of an ordered set (or an embedding of an acyclic directed graph). The ordered set has an upward drawing on the sphere if and only if it has an acyclic triangulation with no saddle point.*

Corollary 3.6 *If every acyclic triangulation of an embedded covering graph of an ordered set has a vertex which is a saddle point then this embedded covering graph has no similar upward drawing without crossing edges on the sphere.*

Every planar ordered set is a spanning subgraph of a *st*-digraph whose faces are bounded so that they can be triangulated without creating any saddle point. Thus, we have the following corollary.

Corollary 3.7 *Every planar ordered set has a triangulation with no saddle point.*

3.3 Circular Drawing

For the presentation of ordered data, the order among the elements must, of course, be readily apparent. Thus, for elements a and b represented by vertices on the plane with different y-ordinates the fact that a is comparable to b is depicted by a monotonic path from the vertex a to the vertex b. A vertical path may be the easiest to discern.

It is customary to use upward planar drawings to display and read the order relation.

There are several other ways to represent an ordered set or a directed graph in the plane, each of special interest, with different advantages or applications (cf. [**Rival (1992)**]). Here are some examples.

- Upward drawings which minimize the number of different slopes used in drawing the covering edges, (cf. [**Czyzowicz, Pelc** and **Rival** (1990)], [**Czyzowicz, Pelc, Rival** and **Urrutia** (1990)] and [**Czyzowicz** (1991)]).

- Vertical drawing (k-vertical drawing), to draw the disjoint chains of a given chain decomposition of an ordered set, vertically (with non-vertical edges between these chains) and a k-channel drawing (possibly with bends and stretching) (cf. [**Nowakowski** and **Rival** (1989)]).

- Directed visibility representation (for a directed graph) consists of representing the vertices of a digraph by means of horizontal segments and the edges by means of vertical segments so that the vertical segment associated with an edge $v \rightarrow w$ has its endpoints on the segments representing the vertices. Moreover, every segment representing an edge is directed from the lower to the higher endpoint (cf. [**Di Battista** and **Tamassia** (1988)]).

- A monotonic grid drawing is an upward drawing such that the vertices are placed at grid points, and the edges are drawn as polygonal lines that bend only at grid points. Minimizing the area occupied and the number of bends along the edges are the criteria for this drawing (cf. **Jourdan, Rival** and **Zaguia** (1994)]).

In this section, we introduce *circular drawings* on the plane in which, loosely speaking, the vertices are placed on concentric circles and, for a covering edge $a \prec b$, a path is drawn so that a is its initial point and b its terminal point, and it is transversal to all circles with increasing radius. This particular drawing can be used as a tool for studying spherical ordered sets. We shall show that an ordered set is spherical if and only if it has a *circular embedding*, that is, a circular drawing in which no paths corresponding to the edges of the ordered set cross. We begin by the following example.

Example 3.1 Consider the six-element ordered set represented by its diagram in Figure 21(i). It is well known that this ordered set has no planar upward drawing, while it does have an upward drawing without crossing edges on the sphere.

We shall now give an embedding of this ordered set in the plane with its order relation displayed. To this end we first define a function

$$\mathcal{H} : V(P) \longrightarrow \mathbf{Z}^+$$

from the vertices of P to the set of nonnegative integers such that $\mathcal{H}$ is order preserving, that is, $\mathcal{H}(x) \leq \mathcal{H}(y)$ if and only if $x \leq y$. In Figure 21(i), the label of each vertex is the value of $\mathcal{H}$ at that vertex.

Now consider a set of concentric circles with origin O and radius $\mathcal{H}(x)$ for each $x \in V(P)$. We locate vertex x on the circle $C(r_{\mathcal{H}(x)}, O)$ and if $x \sim y$ is an edge in the ordered set then we draw a simple path, starting at the vertex x and ending at the vertex y such that it is transversal to all circles $C(r, O)$ where $\mathcal{H}(x) \leq r \leq \mathcal{H}(y)$. Moreover, the path meets the circles with increasing radii, transversally (see Figure 21(ii)).

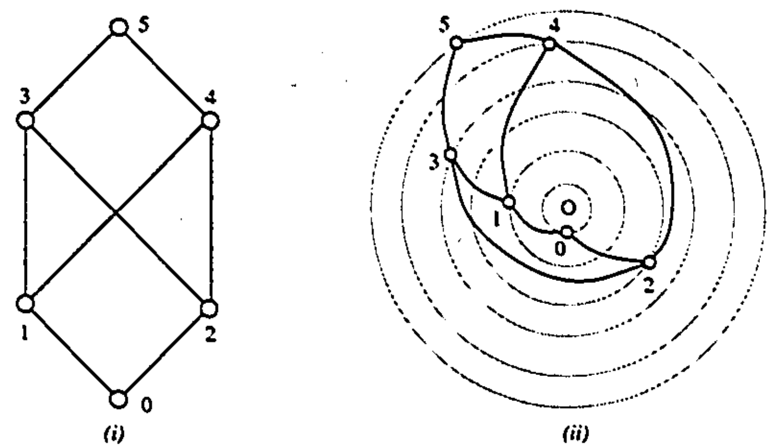


Figure 21: (i) A nonplanar ordered set (ii) Its circular embedding

One can note that although the ordered set has no planar upward drawing, this drawing gives an embedding which, at the same time, shows the order relation, namely, if $x \leq y$ then the circle on which x is located has radius less than the circle on which y is located and there is a sequence of continuous paths $f_1, f_2, \ldots, f_n$ with the aforementioned properties such that

$$f_1(0) = x,\ f_1(1) = z_1, \ldots,\ f_n(0) = z_{n-1} \text{ and } f_n(1) = y$$

where $x \prec z_1 \prec z_2 \prec \ldots \prec z_{n-1} \prec y$.

We shall now give the formal definition of circular drawing of an ordered set and examine its properties. Of course, both the definition and the result can be extended to any acyclic directed graph, even if we do not mention this explicitly.

Fix a horizontal plane XY in the three-dimensional Euclidean space $\mathbf{R}^3$ with the origin O. Fix this point as a center for all *concentric circles* with center O and radius $r \geq 0$, where r is a real nonnegative number.

A circular drawing of an ordered set or, in general, of an acyclic directed graph P, is a graph Γ with a set of vertices corresponding to the vertices of P, and edges so that if $x \prec y$ is a covering edge in the ordered set, then the corresponding vertices $\overline{x}$ and $\overline{y}$ in Γ are placed on two distinguished circles, say $C(r_x, O)$ and $C(r_y, O)$, with $r_x <_{\neq} r_y$ and a continuous path

$$f : [0,1] \longrightarrow XY \ , \ f(0) = \overline{x}, \ f(1) = \overline{y}$$

which meets the concentric circles transversally, that is, nowhere is tangent to a circles $C(r, O)$. Moreover, if $t_1, t_2 \in [0,1]$ with $t_1 < t_2$ then the path f meets circles $C(r_{t_1}, O)$ and $C(r_{t_2}, O)$ with $r_{t_1} < r_{t_2}$.
In other words, corresponding to each edge $x \rightarrow y$ in P there is a path from $\overline{x}$ to $\overline{y}$ which meets circles, transversally.
A circular drawing is planar if no two paths meet, except possibly at their initial or terminal points. In fact, a *planar circular drawing* is a special embedding of acyclic directed graphs into the plane with the above property. In this section we consider only planar circular drawings which we call *circular embeddings*.

Observation 3.3.1 If a directed graph has a planar upward drawing in the plane then it has a circular embedding.

To see this, consider the half upper plane $\{(x, y) : y \geq 0\}$ with all semicircles with centre O the origin. Now by modifying the original planar upward drawing, we can provide an embedding which satisfies the criteria for a circular embedding (see Figure 22).

We note that the converse, in general, does not hold. Figure 21(ii) shows a circular embedding of a six-element ordered set in Example 3.1 which has no planar upward drawing.

The advantage of the circular embedding, as we shall see in the following result, is to provide a tool for studying embedding of acyclic digraphs on the sphere.

Theorem 3.8 *An ordered set (directed graph) has a circular embedding if and only if it is spherical.*

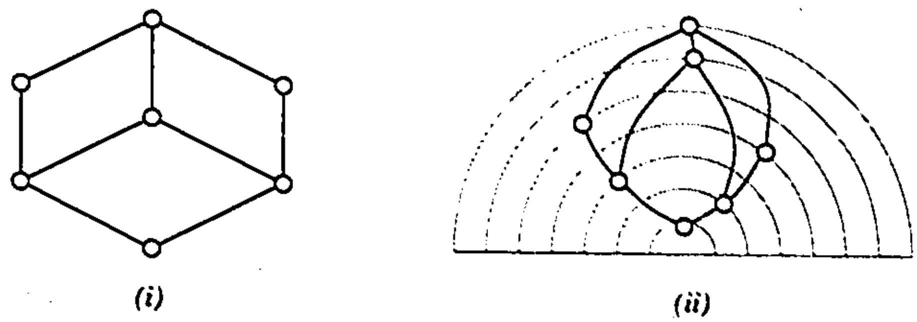


Figure 22: (i) A planar ordered set (ii) Its circular embedding

Proof Fix a circular embedding of an ordered set. Locate a sphere of arbitrary radius on the plane such that the south pole of the sphere is placed at the origin (at the center of concentric circles) and using the inverse of the usual *stereographic* projection, one can obtain an upward drawing of the ordered set on the sphere. That the edges are upward, is a consequence of the property of paths in the circular embedding.

Conversely, given an upward drawing of an ordered set on the sphere. We fix a position of the sphere which presents this upward drawing such that the south pole is located at the origin of the plane. If the north pole is a vertex of the upward drawing, it is not difficult to see that by perturbing the vertex on the north pole to a point in a neighborhood of it, the upward drawing remains unchanged. Now, since all edges are strictly monotonic in the direction of the north pole, by using a stereographic projection, we obtain a circular embedding of the ordered set in the plane.□

Chapter 4

Spirals and Upward Drawing

4.1 Introduction

The purpose of this chapter is to study the surface geometry of a certain class of ordered sets, each member of which is constructed from a set of "m-spirals" of type S_m (cf. Section 1.6.1). Each member of the class may be modelled by a set of positive integers $A = \{i_1, i_2, \ldots, i_n\}$ designated by $S(i_1, i_2, \ldots, i_n)$.

There are at least two advantages in studying the members of this class;

1. We shall find a better understanding of critical points, in particular, saddle points of ordered sets.

2. It gives a new direction to understand the complexity of *sphericity testing*, that is, to test whether an ordered set has an upward drawing without crossing edges on the sphere.

In the first part we show that one can calculate all saddle points of each member of the class; therefore, their upward drawings on surfaces of genus zero with the same number of saddle points are well-understood.

For the second part, we shall show that for this class of ordered sets, sphericity and planarity are equivalent, in the sense that each member of the class has a planar upward drawing if and only if it is spherical. This leads to the conjecture of NP-completeness of sphericity testing which will be discussed in Chapter 5.

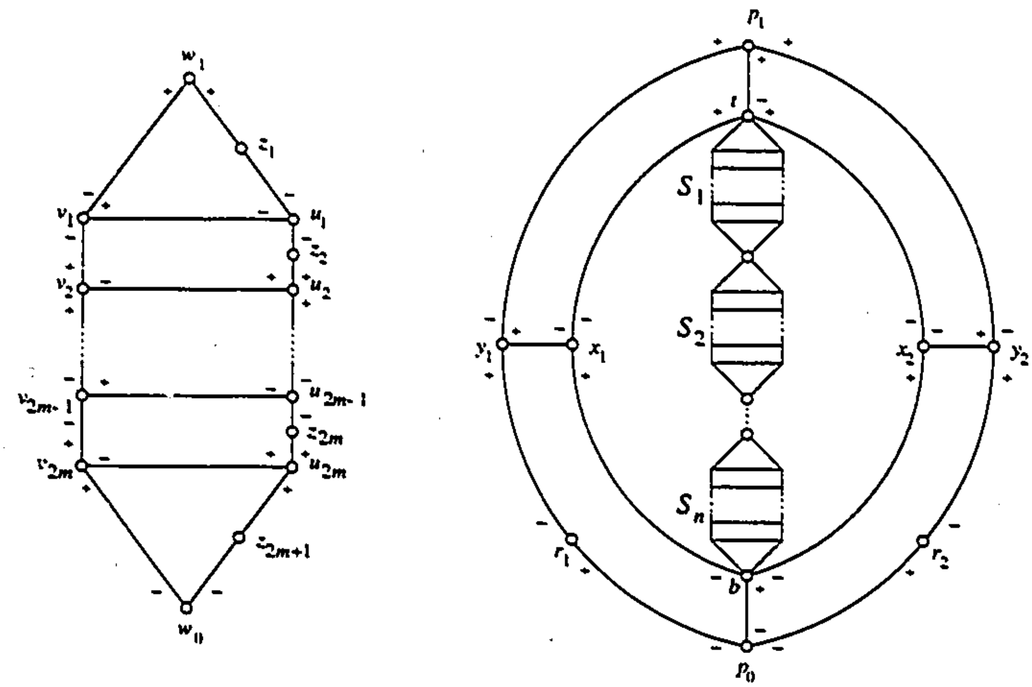


Figure 23: (a) The covering graph of an m-spiral S_m (b) The covering graph of the spiral of type $S(i_1, i_2, \ldots, i_n)$

4.2 Spirals of Type $S(i_1, i_2, \ldots, i_n)$

A spiral of type $S(i_1, i_2, \ldots, i_n)$ may be modelled by a set of positive integers $A = \{i_1, i_2, \ldots, i_n\}$ as follows.

For each integer i_j let S_{i_j} be a spiral (as defined in Section 1.6.1) which is an ordered set consisting of $6i_j + 3$ elements denoted by $v_1, \ldots, v_{2i_j}$, $u_1, \ldots, u_{2i_j}, w_0, w_1$ and $z_1, \ldots, z_{2i_j+1}$, with the following comparabilities:

$$v_{2k} \geq v_{2k-1}, v_{2k+1}, \quad \text{where } 1 \leq k \leq i_j \tag{6}$$

$$u_{2k} \geq v_{2k}, z_{2k}, z_{2k+1}, \quad \text{where } 1 \leq k \leq i_j \tag{7}$$

$$u_{2k-1} \leq v_{2k-1}, z_{2k-1}, z_{2k}, \quad \text{where } 1 \leq k \leq i_j \tag{8}$$

for $j = 1, 2, \ldots n$ and, in addition,

$$w_1 \geq z_1, v_1 \quad \text{and} \quad w_0 \leq z_{2i_j+1}, v_{2i_j} \tag{9}$$

The covering graph is illustrated in Figure 23:(a).

We shall now "amalgamate" these spirals so that we identify elements w_0 of S_{i_j} and w_1 of $S_{i_{j+1}}$ for every $j = 1, 2 \ldots, n-1$. Moreover, we define another ordered set over the ten-element set $\{x_1, x_2, y_1, y_2, p_0, p_1, r_1, r_2, b, t\}$ with the following comparabilities.

$$p_1 \geq y_k \geq p_0 \tag{10}$$

$$t \geq x_k \geq b \tag{11}$$

$$y_k \geq x_k \tag{12}$$

$$p_1 \geq t \text{ and } b \geq p_0 \tag{13}$$

$$p_0 \leq r_k \leq y_k \tag{14}$$

where, $k = 1, 2$. We refer to this ordered set as the *frame*.

Next, we amalgamate these two ordered sets by identifying vertex w_1 of spiral S_{i_1} with vertex t of the frame and vertex w_0 of spiral S_{i_n} with vertex b of the frame. The resulting covering graph of this ordered set is schematically depicted in Figure 23. We call this ordered set a *spiral* (over n positive integers $i_1, i_2, \ldots, i_n$) and it is designated by $S(i_1, i_2, \ldots, i_n)$.

Example 4.1 Figure 24 illustrates an upward drawing of a spiral over one positive integer $\{1\}$ which is a nonplanar ordered set. It has an upward drawing on a surface of genus zero with at least one saddle point.

Example 4.2 In Figure 25 the covering graph and the upward drawing of the ordered set $S(1, 1)$, spiral over two positive integers $i_1 = 1$ and $i_2 = 1$, is depicted. This ordered set has a planar upward drawing.

4.2.1 Structure

It can be verified that the frame suborder of $S(i_1, i_2, \ldots, i_n)$, described above, has a three-connected covering graph which therefore has a unique embedding in the plane (up to exterior face). Thus, any embedding of $S(i_1, i_2, \ldots, i_n)$ depends on the embedding of the spirals S_{i_j}, for $j = 1, \ldots, n$.

Each m-spiral S_m $(m = i_1, i_2, \ldots, i_n)$ has a *two*-connected covering graph with a unique embedding subject to the condition that the top and the bottom of spiral (the vertices w_0 and w_1) are in its exterior face (cf. Figure 23:(a)).

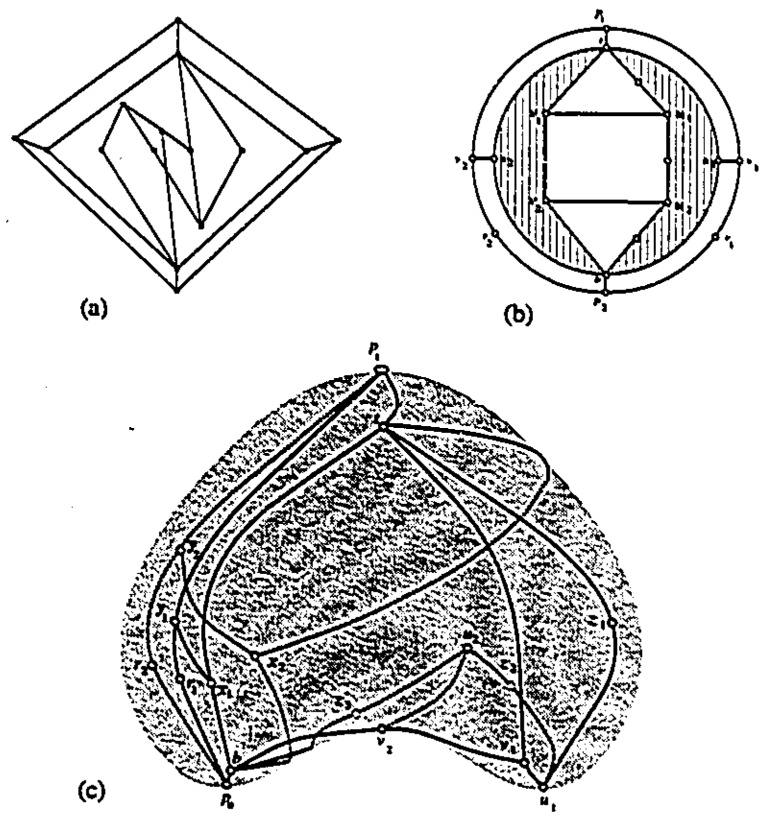


Figure 24: (a) The Spiral $S(1)$ (b) The planar covering graph of $S(1)$ (c) An upward drawing of $S(1)$ on a surface with one saddle point

Accordingly, in the boundary of the m-spiral S_m we distinguish two paths. The path $w_1 \leftarrow v_1 \rightarrow \ldots \rightarrow v_{2m} \leftarrow w_0$, is called the *left*-path, and the path $w_1 \leftarrow z_1 \leftarrow u_1 \rightarrow \ldots \rightarrow u_{2m} \leftarrow z_{2m+1} \leftarrow w_0$, is called the *right*-path of the spiral.

Every m-spiral S_m has an upward drawing (cf. Section 1.4.2 and Section 1.6.1 for details) "spiral-like", with both boundary paths following one another closely. An upward drawing of the 2-spiral S_2 is depicted in Figure 7:(ii) in which the filled vertices represent vertices of the *left*-path.

Thus, for an embedding of the i_j-spiral of $S(i_1, i_2, \ldots, i_n)$ we have two options:

1. Embedding the i_j-spiral S_{i_j} in such a way that, the *left*-path remains on the left hand side of the starting vertex w_1.

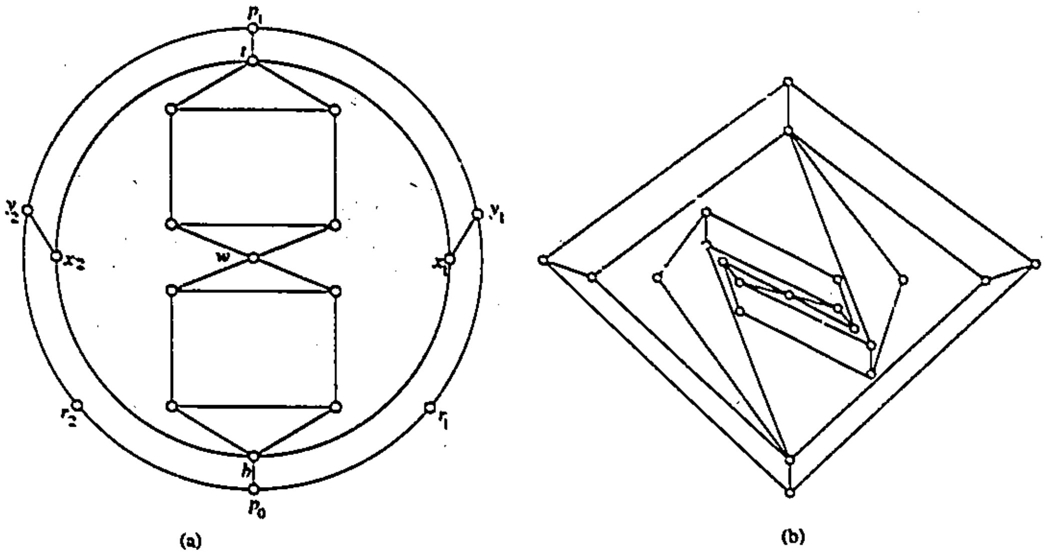


Figure 25: (a) The planar covering graph of $S(1,1)$ (b) A planar upward drawing of $S(1,1)$

2. Embedding the i_j-spiral S_{i_j} with *reflection*, according to which, we *flip* S_m so that the *left*-path is on the right hand side of the starting vertex w_1.

Therefore, the covering graph of $S(i_1, i_2, \ldots, i_n)$ has several embeddings in the plane which depend on the flipping of the spirals S_{i_j}. We refer to the elements $v_1, v_2, \ldots, v_{2m}$, on the *left*-path of S_m, as the "v-elements", and $u_1, u_2, \ldots, u_{2m}$ on the *right* as the "u-elements".

We shall now apply index theory (cf. Section 2.4) to partition faces of the covering graph of $S(i_1, i_2, \ldots, i_n)$ according to their indices. One can note that there are two types of faces:

- Faces containing u-elements or v-elements, and
- faces with no such elements.

If a face contains no u-element or v-element it has only one local minimum, therefore its index is zero, that is, an *ordinary* face. Examples of such faces are among the faces of the frame and exterior face (cf. Figure 23:(b)).

Faces containing u-elements or v-elements are of two types, too.

- Interior faces of the i_j-spirals S_{i_j}, and
- faces which contain a *right*-path or a *left*-path of an i_j-spiral S_{i_j}, for some $j = 1, 2, \ldots, n$.

In the first type of face, only u-elements with odd subscript are local minima. Each face of this type contains only one such element, thus they are ordinary faces with index zero.

There are only two faces of the last type, both nonordinary faces with *index* < 0, and they have nearly the same structure. The u-elements with odd subscripts are minimal elements, and those with even subscripts are maximal elements in the order, and therefore in the faces containing these elements. The v-elements are not maximal or minimal in the order, but inside these two nonordinary faces, the v-elements with odd subscript are minimal elements, and those with even subscript are maximal elements.

Thus, both u-elements and v-elements have the same role, in terms of maximality and minimality, inside the nonordinary faces. Figure 26 shows the covering graph of the spiral $S(i, j)$. The nonordinary faces are illustrated by the shaded regions.

Next, the *right*-path of each spiral S_{i_j} is in one of these nonordinary faces and the *left*-path is in the other one. Note that the u-elements and the v-elements of each i_j-spiral inside these faces are alternatively maximal or minimal with size $2i_j$. Therefore, each i_j-spiral contributes exactly i_j local minima in each face. For both nonordinary faces, the element b is also a local minimum. Thus, according to the index formula for a face (cf. Section 2.4.3), we have

$$\begin{aligned} \textit{index}\ (\text{nonordinary face}) &= 1 - (i_1 + i_2 + \cdots + i_n + 1) \\ &= -\,(i_1 + i_2 + \cdots + i_n) \end{aligned}$$

4.3 Saddle Points of $S(i_1, i_2, \ldots, i_n)$

The aim of this section is to determine the saddle points, if any, of the spiral $S(i_1, i_2, \ldots, i_n)$.. This enables us to describe an appropriate surface of genus zero, on which $S(i_1, i_2, \ldots, i_n)$ has an upward drawing without crossing edges. In particular we restrict our attention to the spirals modeled on a set of only two positive integers, that is, spirals of type $S(i, j)$. For this class of ordered sets the saddle points can be fully described and this helps in the construction of their upward drawings, in the sense that we can always present upward drawings of these ordered sets on appropriate surfaces in which there are no crossing edges.

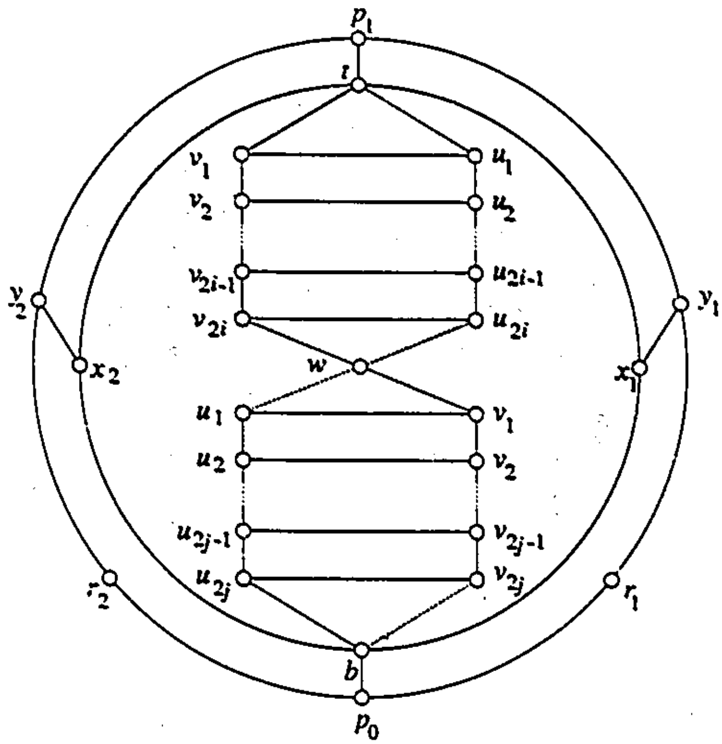


Figure 26: The covering graph of ordered set $S(i, j)$, the shaded regions show the nonordinary faces

Here are the main results of this section.

In general, for an arbitrary set of positive integers $A = \{i_1, i_2, \ldots, i_n\}$, we have an upper bound for the number of saddle points of the spiral $S(i_1, i_2, \ldots, i_n)$.

Theorem 4.1 *The spiral $S(i_1, i_2, \ldots, i_n)$ has an upward drawing on a surface of genus zero with at most $\sum_{j=1}^{n} i_j$, saddle points.*

Theorem 4.2 *The spiral $S(i, j)$ has at least $|i - j|$ saddle points and it has an upward drawing, without crossing of edges, on a two-dimensional surface of genus zero with $|i - j|$ saddle points.*

Note that this lower bound is tight, in the sense that we can present an acyclic triangulation of *cover*$(S(i, j))$ with exactly $|i - j|$ saddle points.

The interesting feature of this class of ordered sets is the fact that, if an ordered set of

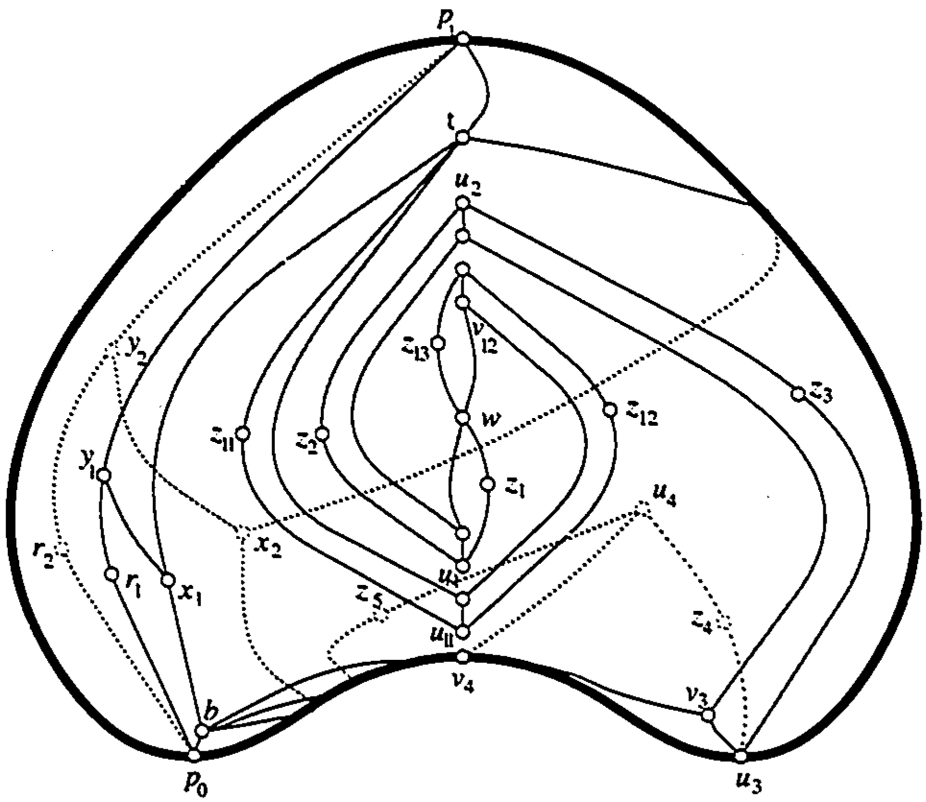


Figure 27: An embedding of spiral ordered set $S(1,2)$, on a surface of genus zero, with one saddle point

this class has no saddle point, then, according to Theorem 3.5, it is a spherical ordered set. We shall show that it is also planar. Thus, in fact:

Theorem 4.3 *For all spirals of type $S(i,j)$, sphericity and planarity are equivalent, that is, each such ordered set is spherical if and only if it is planar.*

This theorem raises the question of the NP-completeness of sphericity testing (which will be discussed in Chapter 5).

4.4 Strategy of the Proof

We recall the characterization of planar ordered sets in Section 1.4.2 and spherical ordered sets in Section 3.2. We consider two cases. If $i = j$, we shall show that $S(i,j)$ has a planar upward drawing and, consequently, it is spherical and has no saddle point. If $i \neq j$, we shall prove that any triangulation of the faces of $S(i,j)$ contains at least $|i-j|$ saddle points, provided that the *alternation* > 2 is not allowed. To this end, we need to prove a technical result which is of independent interest.

First we introduce some terminology.

Chain Chordal Regions Given a face f, start with any vertex and walk around the boundary of f. Replace any directed path $v_0 \rightarrow v_1 \rightarrow \cdots \rightarrow v_n$ with a directed edge $v_0 \rightarrow v_n$ which is called a *chordal edge*. Repeat this operation until every directed path has such a chordal edge. Apparently, the process terminates with a triangulated face or with a region whose boundary has no directed path of *length* > 1. We call it a *chain chordal region* (see Figure 28).

Observation 4.4.1 *Adding chordal edges does not increase the alternation about each vertex.* □

Observation 4.4.2 *The number of vertices on the boundary of a chain chordal region is always even.*□

Let P be an ordered set and $\overrightarrow{cover}(P)$ its directed covering graph. Suppose that $\mathcal{F}$ is the boundary of a chain chordal region which is a subgraph of $\overrightarrow{cover}(P)$. If we delete all edges and vertices of $\overrightarrow{cover}(P)\backslash\mathcal{F}$, then the result is an even cycle with only maximal and minimal vertices, that is, for each vertex of this cycle both incident edges have sign $+$ or both have sign $-$.

We now partition the vertices of $\mathcal{F}$ into two classes, called *extremal* and *interior* vertices.

- A vertex is called an *extremal* vertex if it is extremal in the covering graph, that is, all incident edges have the same sign.
- A vertex is called an *interior* vertex with respect to sign $-$ $(+)$ if it is maximal (minimal) with respect to $\mathcal{F}$ and there is an incident edge (not on the boundary of $\mathcal{F}$) with sign $-$ $(+)$ (see Figure 28).

Theorem 4.4 *Fix a chain chordal region whose boundary consists of two strings, one of m extremal vertices and the other of n interior vertices. If $n - m > 2$ then any acyclic triangulation of this region has at least one saddle point.*

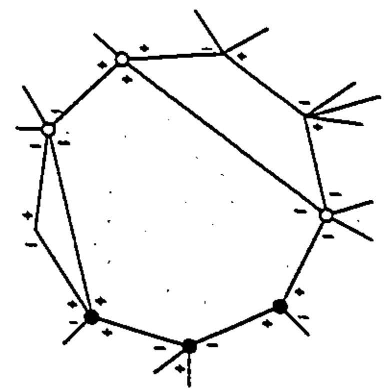


Figure 28: A face, the shaded region is the corresponding chain chordal region. The filled vertices are interior and non-filled vertices are extremal vertices.

Remark 4.1 *If a chain chordal region consists only of extremal vertices, then there are several ways to construct an acyclic triangulation of it so that it contains no saddle points at all. The following procedure is one example.*

Procedure Fix a direction on the boundary and label the vertices of the region by $u_1, u_2, \ldots, u_{2n-1}, u_{2n}$, so that those with even subscripts are maximals (both incident edges have signs + at these endpoints), and those with odd subscripts are minimals (both incident edges have signs − at these endpoints).

Next, we add the set of edges $\{u_k \sim u_{2n-(k-1)} : 2 \leq k \leq n-1\}$ whose directions are determined according to the rule: if u_i is minimal and u_j is maximal, then the edge $u_i \sim u_j$ has direction $u_i \rightarrow u_j$, where the edge has sign − at the endpoint u_i and sign + at the endpoint u_j.

To complete the triangulation, we add the set of edges $\{u_j \sim u_{2n-j} : 1 \leq j \leq n-1\}$. Each edge of this set connects either two maximals or two minimals. Thus, each such edge at one endpoint has sign similar to the signs of other incident edges, and at the other endpoint its sign is opposite to the signs of the other incident edges at that endpoint. We note that even in this case the alternation of the corresponding vertex is less than two and no saddle point is created (see Figure 29).

In upward drawings we are usually interested in surfaces with nondegenerate saddle points (and these have *alternation* ≤ 2). Thus, if we restrict our attention to the acyclic triangulations with the extra condition that no vertex has *alternation* > 2, then in a special case of the Theorem 4.4, where all vertices making up the boundary of the region are interior vertices, we can obtain a better bound for the number of saddle points created by any acyclic triangulation of the region.

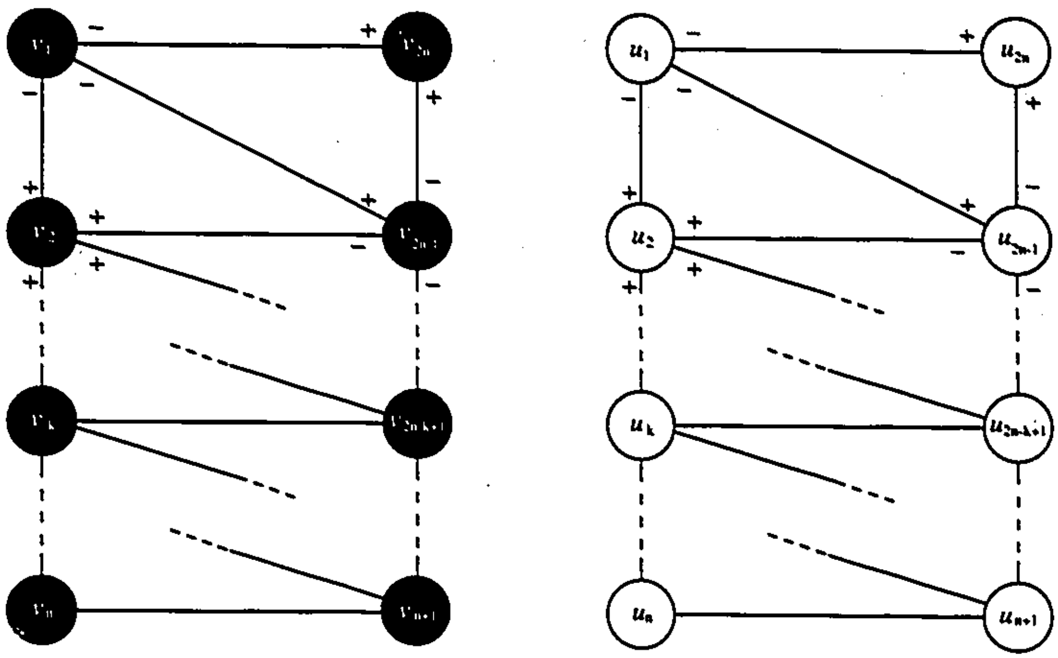


Figure 29:

Proposition 4.5 *Suppose that a chain chordal region consists only of n interior vertices. If no vertex has alternation > 2, then any acyclic triangulation of this region produces at least $(n-2)/2$ saddle points. Moreover, this bound is tight in the sense that such a triangulation exists.*

The proof of this result leads to an improvement in the lower bound of our result in Theorem 4.4. In fact, any attempt to triangulate a region $\mathcal{F}$ which satisfies the hypotheses of Theorem 4.4, with the least number of saddle points, ends up with a region which satisfies the hypotheses of Proposition 4.5. Hence, we end up with a region whose boundary consists only of interior vertices of size

$$interior(\mathcal{F}) - extremal(\mathcal{F})$$

where $interior(\mathcal{F})$ stands for the number of interior vertices and $extremal(\mathcal{F})$ stands for the number of extremal vertices of $\mathcal{F}$. Thus, according to the Proposition 4.5, any acyclic triangulation of such a region has at least

$$\frac{interior(\mathcal{F}) - extremal(\mathcal{F}) - 2}{2}$$

saddle points.

In Proposition 4.6 we shall show that, in fact, this is the best lower bound for the number of saddle points. Moreover, in the following, we present a procedure for triangulation of such a region so that it has exactly

$$\frac{interior(\mathcal{F}) - extremal(\mathcal{F}) - 2}{2}$$

saddle points. This confirms the tightness of this bound. The procedure is implicit in derives from the proof of Proposition 4.6.

Triangulation Procedure Assume that the interior and extremal vertices are arranged in two consecutive strings, and $interior(\mathcal{F}) = n$ and $extremal(\mathcal{F}) = m$.

We fix a direction on the boundary of this region, and label the string of interior vertices by $v_1, v_2, \ldots, v_n$, and the string of extremal vertices by $u_1, u_2, \ldots, u_m$, in which v_n is adjacent to u_1, and u_m is adjacent to v_1.

To construct the triangulation, we add the set of edges $\{u_k \sim v_{n-(k-1)} : 2 \leq k \leq m-1\}$ in which the direction of edges are determined with the rule: the sign of the edge at each endpoint is similar to the signs of the two other edges in the boundary incident with that endpoint. Thus, so far we have not changed the alternation of the vertices.

Next, we add the set of edges $\{u_k \sim v_{n-k} : 1 \leq k \leq m-1\}$ in which the direction of the edges are chosen so that at the endpoints incident with v_k's, they are identical with the signs of incident boundary edges to v_k, and in contrast, the sign of the other ends of these edges have opposite signs with respect to the signs of boundary edges incident with this endpoint. But we note that these endpoints are extremal vertices and, in effect, their alternation does not increase. Hence, up to this step we have not created any saddle point in triangulated part, and eventually, we are left with a region consisting of $(n-m)+1$ interior vertices and only one extremal vertex u_m.

Now, to triangulate the remainder of this region, we connect the extremal vertex u_m to the interior vertex v_{n-m} and v_{n-m} to v_1 (or u_m to v_2 and v_2 to v_{n-m+1}, if v_{n-m} and v_1 are interior vertices with respect to the same signs), without creating any saddle point. This process produces a region consisting only of $n-m$ interior vertices. Therefore, by Proposition 4.5, one can triangulate this region with exactly $(n-m-2)/2$ saddle points.

Remark 4.2 *If $\mathcal{F}$ is a chain chordal region such that*

$$interior(\mathcal{F}) - extremal(\mathcal{F}) \leq 2$$

then a similar procedure produces a region consisting only of

$$extremal(\mathcal{F}) - interior(\mathcal{F})$$

extremal vertices, for which the procedure in Remark 4.1 produces a triangulation without saddle points.

Let $saddle(\mathcal{F})$ stand for the least number of saddle points produced by any acyclic triangulation of the region $\mathcal{F}$.

Proposition 4.6 *Let $\mathcal{F}$ be a chain chordal region whose boundary consists of two strings, one of m extremal vertices and the other one of n interior vertices, then*

$$saddle(\mathcal{F}) = \begin{cases} \frac{n-m-2}{2} & \text{if } n-m > 2 \\ 0 & \text{if } n-m \leq 2 \end{cases}$$

4.4.1 Proof of the Results

Proof of Theorem 4.4 The proof is by induction. Let $\mathcal{F}$ be a chain chordal region whose boundary consists of a string of m extremal vertices $\{u_1, u_2, \ldots, u_m\}$ and a string of n interior vertices $\{v_1, v_2, \ldots, v_n\}$, where $n-m > 2$.

For the initial step of the induction we consider a chain chordal region consisting of $n = 4$ interior vertices with no extremal vertices ($m = 0$) which is the smallest region satisfying the theorem. One can easily verify that any acyclic triangulation of this region contains either an edge connecting two interior vertices with respect to sign $+$, or those with respect to sign $-$. In both cases this creates a saddle point.

Thus, assume that the result holds for any chain chordal region whose boundary consists of a string of i extremal vertices and a string of j interior vertices, where $i+j < m+n$ and $j-i > 2$. We prove that the assertion holds for a region consisting of n interior vertices and m extremal vertices where $n-m > 2$.

To this end, we shall consider all cases in which an edge of the triangulation may connect two vertices to each other and show that in all such cases, the triangulation contains at least one saddle point.

For convenience, we assign a sign + (−) to a vertex if both edges of the boundary incident with this vertex have signs + (−).

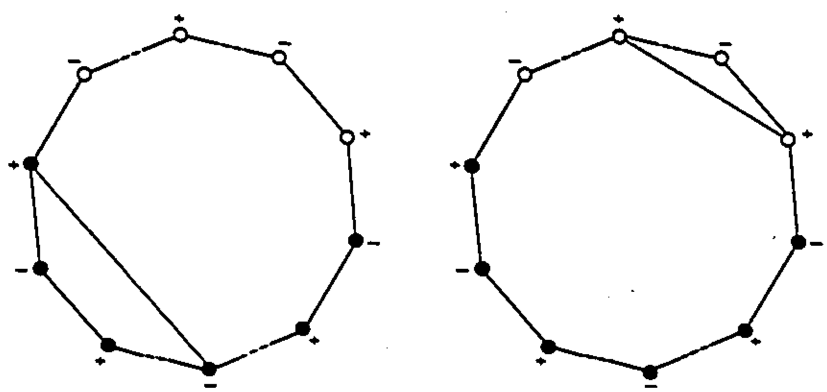


Figure 30: (a) The shaded face satisfies the induction hypothesis in part (i). (b) The shaded face satisfies the induction hypothesis in part (ii).

(i) Suppose that in a triangulation of region $\mathcal{F}$, there is an edge connecting two nonadjacent interior vertices. Then, depending on the signs of these vertices, either one of them becomes a saddle point or this edge provides a chain chordal region made up of at least four interior vertices and no extremal vertices. As this satisfies the induction hypothesis (see Figure 30:(a)) such a triangulation creates at least one saddle point.

(ii) Suppose that in a triangulation of region $\mathcal{F}$, there is an edge connecting two nonadjacent extremal vertices where the signs of the edge agree with the signs of two other incident edges at these vertices. Then we obtain two smaller regions, where one of them contains all interior vertices and the number of its extremal vertices is less than m. This region satisfies the induction hypothesis and any triangulation of it creates a saddle point.

Note that, if the signs of the end points of the edges do not agree with the extremal vertices, then by adding chordal edges, if necessary, we have a similar situation (see Figure 30:(b)).

Next, we consider all triangulations in which there are edges connecting interior vertices to the extremal vertices.

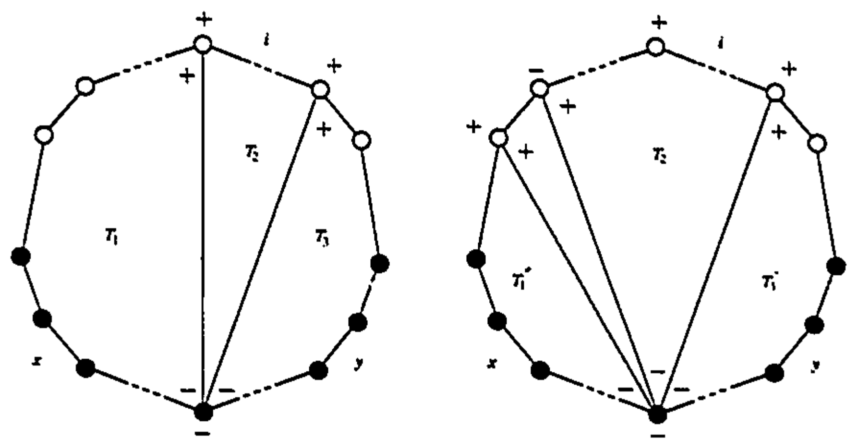


Figure 31:

(*iii*) Suppose that in a triangulation of the region $\mathcal{F}$, the interior vertex v_j is connected to the extremal vertices u_{r_1} and u_{r_i}, where $i > 1$ and the signs of the edges agree with the extremal vertices. These edges divide the region $\mathcal{F}$ into three smaller regions T_1, T_2 and T_3 (see Figure 31). Region T_1 consists of R interior vertices and $i_1 + 1$ extremal vertices. Region T_2 consists of i extremal vertices $u_{r_1}, u_{r_2}, \ldots, u_{r_i}$ and one interior vertex v_j. Region T_3 consists of S interior vertices and $i_2 + 1$ extremal vertices. We have

$$R + S = n + 1 \quad \text{and} \quad i_1 + i_2 = m - i \tag{15}$$

We claim that at least one of the regions T_1 or T_3 must satisfy the induction hypothesis. Clearly, these regions are smaller chain chordal regions and their boundaries consist of two strings of interior and extremal vertices. To prove the claim, we have to show that for at least one of these two regions, the number of the interior vertices exceeds the number of the extremal vertices at least by two:

$$interior(T_k) - extremal(T_k) > 2 \qquad k = 1 \text{ or } 3 \tag{16}$$

To this end, by way of contradiction, we assume that in both of the regions T_1 and T_3 the inequality (16) fails, so we have the following inequalities.

$$R - (i_1 + 1) \leq 2$$

$$S - (i_2 + 1) \leq 2$$

Adding both sides of these inequalities, we have

$$R + S - (i_1 + i_2) - 2 \leq 4$$

Now, by using (15) we have

$$n - m \leq 5 - i \quad \text{where} \quad i > 1, \quad \text{which implies} \quad n - m \leq 3$$

On the other hand $n + m$ is even, so either n and m are both odd numbers, or both even numbers, $n - m \neq 3$. Therefore $n - m \leq 2$, which contradicts the fact that $n - m > 2$. This proves the claim. Therefore, at least one of the regions T_1 or T_3 must satisfy the induction hypothesis. Thus, any triangulation of this region generates at least one saddle point.

Note that, if one of the signs of the end points of the edges does not agree with the corresponding extremal vertex, then by adding chordal edges, if necessary, we obtain chain chordal regions T_1' and T_3'. We can similarly show that at least one of them satisfies the induction hypothesis.

(***iv***) Suppose that an extremal vertex u_i is connected to interior vertices v_{r_1} and v_{r_j} where $j > 1$ (see Figure 32:(a)). These edges divide the region $\mathcal{F}$ into three smaller regions T_1, T_2 and T_3. We show that at least one of these regions satisfies the induction hypothesis. To this end, first assume that the signs of the edges agree with their endpoints. Therefore, all of these smaller regions are chain chordal. We distinguish several cases which depend on the value of j.

If $j > 3$, then the region T_2 consists of a string of j interior vertices and a string of only one extremal vertex u_i, so that $j - 1 > 3 - 1 = 2$. Therefore, the region T_2 satisfies the induction hypothesis.

If $j = 3$, then the region T_2 consists of a string of interior vertices v_{r_1}, v_{r_2} and v_{r_3} where v_{r_1} and v_{r_3} are of the same sign. If there is an edge connecting v_{r_1} to v_{r_3}, then according to what we have seen in part (i), we have at least one saddle point. Otherwise, to triangulate the region T_2, we have to add the edge $u_i \rightarrow v_{r_2}$ (see Figure 32:(b)). In this case u_i becomes an interior vertex for regions T_1 and T_3.

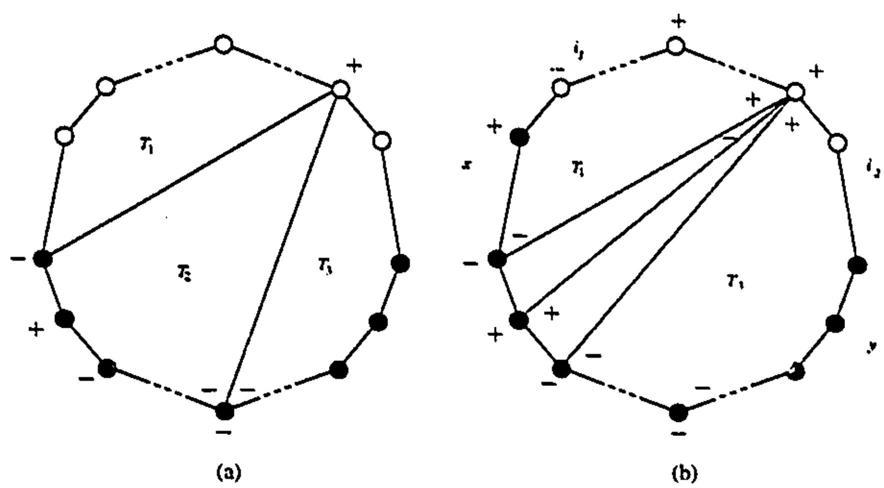


Figure 32:

Region T_1 is a smaller chain chordal region consisting of a string of R old interior vertices, the new interior vertex u_i and a string of i_1 extremal vertices. Similarly, T_3 is a smaller chain chordal region consisting of a string of S old interior vertices, the new interior vertex u_i and a string of i_2 extremal vertices, so that

$$R + S = n - 1 \quad \text{and} \quad i_1 + i_2 = m - 1 \tag{17}$$

We claim that at least one of these two regions satisfies the induction hypothesis from which we have

$$interior(T_k) - extremal(T_k) > 2 \quad k = 1 \text{ or } 3$$

To prove this claim, assume on the contrary that this condition fails for both regions. Thus, we have

$$R + 1 - i_1 \leq 2$$

$$S + 1 - i_2 \leq 2$$

Adding both sides of these inequalities we have

$$R + S - (i_1 + i_2) \leq 2$$

By using (17), we have

$$n - m \leq 2$$

This contradicts the fact that $n - m > 2$, proving the claim.

If $j = 2$, that is, u_i is connected to two adjacent interior vertices v_{r_1} and v_{r_2}, then the sign of one of the edges does not agree with u_i. Therefore, we have to add chordal edges in the corresponding region. The result is similar to the case that $j = 3$ or it can be considered as in part (iii). Thus, in any case, at least one of the regions satisfies the induction hypothesis. Therefore, any triangulation of that region creates at least one saddle point.

(v) In parts (iii) and (iv), we excluded the cases where $i = 1$ or $j = 1$. In both cases we have to consider a triangulation in which there is an edge, connecting an interior vertex v_j to an extremal vertex u_i. Considering parts (i) and (ii), there must be an edge from the interior vertex v_j to an extremal vertex adjacent to u_i or an edge from the extremal vertex u_i to an interior vertex adjacent to v_j. Therefore we have a case similar to (iii) or (iv).

Now, since in any acyclic triangulation of region $\mathcal{F}$, one of the cases (i)-(iv) occurs, then any acyclic triangulation of this region contains at least one saddle point. This completes the proof.□

Proof of Proposition 4.5 The proof is by induction. As we have seen in the proof of the last theorem, the smallest such region is a chain chordal region where the boundary consists of $n = 4$ interior vertices. Any acyclic triangulation of this face leaves at least $(4-2)/2 = 1$ saddle points. Moreover, there is one triangulation with that many saddle points.

Thus, we assume that the statement of the proposition holds for any chain chordal region consisting only of interior vertices of size $< n$. We prove the proposition for a chain chordal region $\mathcal{F}$ so that its boundary consists only of n interior vertices.

In any triangulation of this region, there is either an edge connecting two interior vertices with respect to the same signs, or an edge connecting two interior vertices with respect to different signs. We treat both cases to complete the induction.

Suppose that v_r is an interior vertex with respect to sign +, v_s is the other interior vertex with respect to sign −, and $v_r \rightarrow v_s$ is a directed edge of a triangulation of this region. The $v_r \rightarrow v_s$ divides the region into two smaller ones of sizes R and S.

Now, in this case, both smaller regions are made up only of interior vertices of sizes $R < n$ and $S < n$. Thus, by the induction hypothesis any triangulation of these regions produces at least $(R-2)/2$ saddle points in one of them and $(S-2)/2$ in the other one. Therefore, in the whole region the number of saddle points is

$$\frac{R-2}{2} + \frac{S-2}{2} = \frac{R+S-4}{2} \tag{18}$$

But $R + S = n + 2$, therefore any triangulation of the region has at least $(n-2)/2$ saddle points, as required.

If $v_r \rightarrow v_s$ is an edge of some triangulation so that at least one of its endpoints is already a saddle point, then by adding new chordal edges as it is necessary, we obtain smaller chain chordal regions which satisfy the induction hypothesis and a simple calculation similar to that in the last case completes the induction. This completes the proof of the Proposition.□

Proof of Proposition 4.6 If for a chain chordal region $\mathcal{F}$

$$interior(\mathcal{F}) - extremal(\mathcal{F}) \leq 2$$

then by virtue of Remark 4.2, $saddle(\mathcal{F}) = 0$

Next, let $interior(\mathcal{F}) = n$ and $extremal(\mathcal{F}) = m$ and $n - m > 2$. The proof is by induction on the size of the strings of extremal and interior vertices, and it follows the proof of Theorem 4.4 and Propositions 4.5, closely.

The smallest region which satisfies the hypotheses of the Proposition is the one with a string of four interior vertices and no extremal vertices. Therefore, by Proposition 4.5 it has at least $(4-2)/2 = 1$ saddle points.

Thus, we assume that the conclusion on the Proposition holds for any chain chordal region made up of two strings, one consisting of i extremal vertices, and the other one consisting of j interior vertices, so that $i+j < m+n$ and $j-i > 2$. Let $\mathcal{F}$ be a chain chordal region whose boundary consists of a string of m extremal vertices and a string of n interior vertices such that $n - m > 2$. We shall now consider all possible edges which an acyclic triangulation may contain, and we shall show that in any case, $saddle(\mathcal{F}) \geq (n-m)/2$.

(*i*) If a triangulation has an edge connecting two nonadjacent extremal vertices, it divides the region into two smaller regions T_1 and T_2. T_1 consists of i_1 xtremal vertices and no interior vertices. T_2 consists of i_2 extremal vertices and n interior vertices. Hence, T_1 can be triangulated without creating saddle points, that is, $saddle(T_1) = 0$. The smaller region T_2 satisfies the induction hypothesis, therefore, $saddle(T_2) = (n - i_2 - 2)/2$. Since $i_2 \leq m$ we have

$$\begin{aligned} saddle(\mathcal{F}) &\geq saddle(T_1) + saddle(T_2) \\ &= 0 + (n - i_2 - 2)/2 \\ &\geq \frac{n - m - 2}{2} \end{aligned}$$

(*ii*) If a triangulation has an edge connecting two nonadjacent interior vertices v_{j_1} and v_{j_R}, it divides the region $\mathcal{F}$ into smaller chain chordal regions T_1 and T_2 (by adding chordal edges where necessary) so that the boundary of T_1 consists of R interior vertices and no extremal vertices, while the boundary of T_2 is made up of S interior vertices and m extremal vertices, where, $R + S = n + 2$ (see Figure 30). Thus,

$$\begin{aligned} saddle(\mathcal{F}) &\geq saddle(T_1) + saddle(T_2) \\ &= \frac{R-2}{2} + \frac{S-m-2}{2} \\ &= \frac{n-m-2}{2} \end{aligned}$$

provided $S - m > 2$. Note that, even if $saddle(T_2) = 0$, having such an edge creates a triangulation with at least $(n - m - 2)/2$ saddle points.

(*iii*) If, in any triangulation, an interior vertex is to be connected to two extremal vertices u_{r_1} and u_{r_i}, for $i \geq 3$ (that is, an interior vertex is connected to two nonadjacent extremal vertices) the region splits into three smaller chain chordal regions (by adding chordal edges if necessary) T_1, T_2 and T_3. T_1 consists of i_1 extremal vertices and S interior vertices, T_2 consists of i ($i \geq 3$) extremal vertices and one interior vertex, and T_3 consists of i_2 extremal vertices and R interior vertices. Counting the vertices, we have that $i_1 + i_2 + i = m + 2$ and $R + S = n + 1$. Since $i \geq 3$ therefore, $saddle(T_2) = 0$ and

$$\begin{aligned} saddle(\mathcal{F}) &\geq saddle(T_1) + saddle(T_3) \\ &= \frac{S - i_1 - 2}{2} + \frac{R - i_2 - 2}{2} \end{aligned}$$

$$
\begin{aligned}
&= \frac{R+S-(i_1+i_2)-4}{2}\\
&= \frac{n-m+(i-3)-2}{2}\\
&\geq \frac{n-m-2}{2}
\end{aligned}
$$

(*iv*) If, in a triangulation there is an extremal vertex which is connected to two nonadjacent interior vertices, then the region is divided into three smaller regions T_1, consisting of i_1 extremal vertices and S_1 interior vertices, T_2, consisting of one extremal vertex and R ($R \geq 3$) interior vertices, and T_3, consisting of i_2 extremal vertices and S_2 interior vertices, where

$$S_1 + S_2 + R = n+2 \qquad \text{and} \qquad i_1 + i_2 = m+1$$

See Figure 32:(a).

We first note that in triangulation of the region T_2, if we have an edge connecting two nonadjacent interior vertices, then we have a situation similar to part (*ii*) and we are done. Therefore, we consider a triangulation, in which all edges connect the extremal vertex u to all consecutive interior vertices $v_{j_1}, v_{j_2}, \ldots, v_{j_R}$.

Now, one can verify that an extremal vertex u can be connected to at most three consecutive interior vertices without being a saddle point. But, in this case, since it has an edge with opposite sign at the end point u, then u is changed to an interior vertex for the regions T_1 and T_2 (see Figure 32:(b)). Of course, it is possible to keep this vertex as the extremal vertex, by connecting only the edges whose signs at the endpoint u are similar to the sign of u, but this creates at least one more saddle point for the region T_2, because in this case, alternate interior vertices are saddle points. We shall now study these cases separately.

Thus, first assume that in a triangulation of T_2, u becomes an interior vertex for the regions T_1 and T_2. All three smaller regions satisfy the induction hypothesis.

- If $saddle(T_1) > 0$ and $saddle(T_2) > 0$, then

$$
\begin{aligned}
saddle(\mathcal{F}) &\geq saddle(T_1) + saddle(T_2) + saddle(T_3)\\
&= \frac{(S_1+1)-(i_1-1)-2}{2} + \frac{R-1-2}{2} + \frac{(S_2+1)-(i_2-1)-2)}{2}\\
&= \frac{(S_1+R+S_2)-(i_1+i_2)-3}{2}\\
&= \frac{n+2-(m+1)-3}{2}
\end{aligned}
$$

$$= \frac{n-m-2}{2}$$

• If for one of the regions T_1 or T_3, say T_3, $saddle(T_3) = 0$, that is, $(S_2+1)-(i_2-1) \leq 2$, the Proposition holds, since $S_1 + R = n - S_2 + 2$, and then

$$\begin{aligned} saddle(\mathcal{F}) &\geq saddle(T_1) + saddle(T_2) \\ &= \frac{(S_1+1)-(i_1-1)-2)}{2} + \frac{R-1-2}{2} \\ &= \frac{(S_1+R+-i_1-3}{2} \\ &= \frac{n-S_2+2-i_1-3}{2} \\ &= \frac{n-(i_1+i_2)-1}{2} \\ &= \frac{n-m-2}{2} \end{aligned}$$

• If simultaneously, $saddle(T_1) = 0$, $saddle(T_2) = 0$ and $saddle(T_3) = 0$, that is,

$$\begin{aligned} (S_1+1)-(i_1-1) &\leq 2 \\ R-1 &\leq 2 \\ (S_2+1)-(i_2-1) &\leq 2 \end{aligned}$$

then by adding both sides of the above inequalities we have $n - m \leq 2$ which contradicts our assumption that $n - m > 2$.

• If $R > 3$ then it may happen that $saddle(T_1) = 0$ and $saddle(T_3) = 0$, that is,

$$\begin{aligned} (S_1+1)-(i_1-1) &\leq 2 \\ (S_2+1)-(i_2-1) &\leq 2 \end{aligned}$$

or equivalently, $S_1 + S_2 \leq i_1 + i_2$. We recall that $R = n + 2 - (S_1 + S_2)$, hence,

$$\begin{aligned} saddle(\mathcal{F}) &\geq \frac{R-1-2}{2} \\ &= \frac{n+2-(S_1+S_2)-3}{2} \\ &\geq \frac{n+2-(i_1+i_2)-3}{2} \\ &= \frac{n-m-2}{2} \end{aligned}$$

This proves the validity of the Proposition for all possible triangulations (for the first case).

Next, we shall analyze the case of a triangulation for the region $\mathcal{F}$ in which we keep the vertex u as an extremal vertex, in the sense that in the process of triangulation, we never connect an edge whose endpoint at the vertex u has a sign opposite to the sign of the other incident edges at this vertex.

Now, in triangulating the region T_2, if we have an edge connecting two interior vertices, then we face a situation similar to that in (ii) and we are done. Thus, we assume that all edges of the triangulation connect the extremal vertex u to all interior consecutive vertices. If we keep u as extremal vertex in this triangulation, this produces one more saddle point than the least number of saddle points which can triangulate this face, that is, $saddle(T_2) = (R-1-2)/2+1$.

We now consider the triangulation of the regions T_1 and T_3 which depends on the number of their interior and extremal vertices. We must either connect two extremal or two interior vertices to each other for which we face a situation similar to that in (i) and (ii), or at least in one of these two regions we create one more saddle point than its least number of saddle points. Thus

$$\begin{aligned} saddle(\mathcal{F}) &\geq saddle(T_1) + saddle(T_2) + saddle(T_3) \\ &= \frac{S_1 - i_1 - 2}{2} + \frac{R-1-2}{2} + 1 + \frac{S_2 - i_2 - 2}{2} + 1 \\ &= \frac{n-m-2}{2} \end{aligned}$$

This completes the proof of the proposition.□

We shall now apply the previous results of the last section to prove the main theorems.

Proof of the Theorem 4.1 Fix an embedding of the covering graph of $S(i_1, i_2, \dots, i_n)$ so that all i_j-spirals contribute their *left*-paths into the one nonordinary face and their *right*-paths into the other nonordinary face. We note that the vertices b and t are counted as interior vertices for the nonordinary faces. Thus, if we do not flip any of the i_j-spirals then one of the nonordinary faces has $(\sum_{i_j \in A} 2i_j) + 2$ interior vertices (we add cordal edges where needed) and no extremal vertex and the other one has $\sum_{i_j \in A} 2i_j$ extremal vertices, and only two interior vertices, namely b and t.

Now, applying Proposition 4.5, We see that any triangulation of the nonordinary face which contains only interior vertices will have at least

$$\frac{(\sum_{i_j \in A} 2i_j) + 2 - 2}{2} = \sum_{i_j \in A} i_j$$

saddle points.

For an upward drawing of this ordered set, we fix in $\mathbf{R}^3$ a surface of genus zero with only one maximum, $\sum_{i_j \in A} i_j$ saddle points, and $(\sum_{i_j \in A} i_j) + 1$ minima. To construct an upward drawing on this surface we first draw the frame by placing the top vertex p_1 at the maximum point of the surface and next draw the two faces $p_1 \sim t \sim x_1 \sim y_1 \sim p_1$ and $x_1 \sim y_1 \sim r_1 \sim p_0 \sim b \sim x_1$ at the front of the surface so that the bottom of the frame (the minimal vertex p_0) is placed at the first minimum point (from left to right) of the surface. We shall now draw the two other faces of the frame on the back of the surface by wrapping the edge $t \sim x_2$ around the surface (see Figure 33).

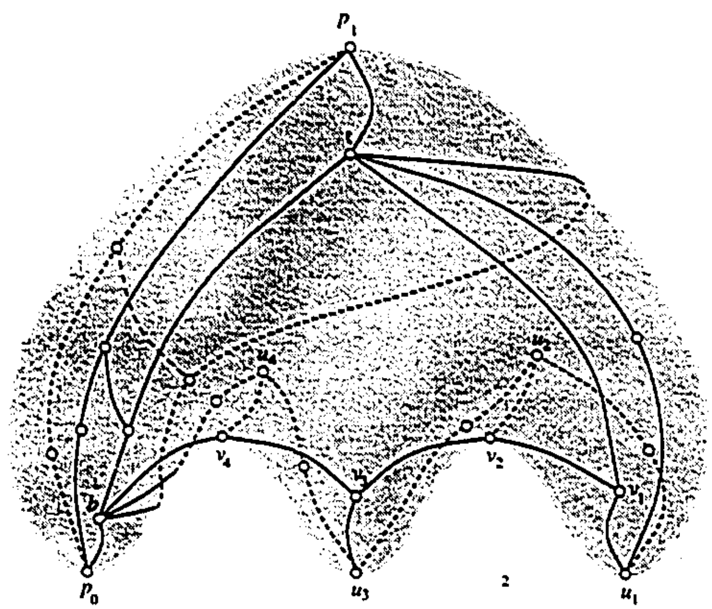


Figure 33:

Next, place the first extremal vertex u_1 of the i_1-spiral at the last minimal point of the surface, and the interior vertex v_2 at its adjacent saddle point and continue the drawing by placing the consecutive extremal vertices with odd subscripts at the minimum points of the surface and the consecutive interior vertices with even subscripts at their adjacent saddle points, respectively, until ultimately we reach the minimal vertex b. See Figure 33 for an upward drawing of $S(1,1)$ on a surface of genus zero with two saddle points. Note that, in this drawing, the faces of the i_j-spirals are drawn so that all interior vertices with odd subscripts are placed at the front of the surface and all extremal vertices with even subscripts are placed at the back of the surface[1].□

Proof of Theorem 4.2 Given a spiral of type $S(i, j)$, fix an embedding of its covering graph so that only one of the spirals S_i or S_j is flipped (cf. Page 55). Thus S_i and S_j do not contribute their *left*-paths to the same nonordinary face.

Now considering the structure of spiral $S(i, j)$, it follows that vertices b and t are both interior vertices for the nonordinary faces. Therefore, the boundary of one of the nonordinary faces consists of a string of $(2i + 2)$ interior and j extremal vertices and the boundary of the other one consists of a string of $(2j + 2)$ interior and i extremal vertices.

Now depending on whether $i \geq j$ or $j \geq i$, one of the nonordinary faces (by adding chordal edges where necessary) satisfies the hypotheses of Proposition 4.6. Therefore, any triangulation of this face produces at least $|i - j|$ saddle points.

Next we present an upward drawing of this ordered set on a surface of genus zero with $|i - j|$ saddle points.

If $i = j$, the order is planar. To obtain its upward drawing, we can use the procedure of assigning extremal vertices to the faces (cf. Section 1.4.2) which in turn, they are drawn, with large angles, in the corresponding faces.

The frame suborder has a unique upward drawing so that vertices p_0 and p_1 are assigned to the exterior face. The i-spiral S_i and j-spiral S_j must be drawn inside the face $t \sim x_1 \sim b \sim x_2 \sim t$, since the top of S_i and the bottom of S_j must be identified with vertices t and b, respectively. On the other hand, the top and the bottom of each component must be in the same face, requiring that b and t belong to the same face, and $t \sim x_1 \sim b \sim x_2 \sim t$ is the only face with this property. Each interior face of S_i and S_j in this drawing is called a *branch of the spiral* or simply a *branch*. Here is our proposed upward drawing.

First identify vertex w_1 of S_i with vertex t of the frame. Loosely speaking, it will be drawn in a spiral-like way with both boundary paths following closely along. It has $2i + 1$ branches, which spiral inward close to a center, until the total number of branches is $2i + 1$.

Next, S_j (which we assume has been flipped) has $2j + 1$ branches which are drawn spirally outward from the center, until the total number of branches is $2j + 1$. Since $i = j$,

[1]Note that, in view of the number of saddle points and upward drawings, the spirals $S(i_1, i_2, \ldots, i_n)$ and $S(i_1 + i_2 + \cdots + i_n)$ are the same, because the directed paths produced by the top and the bottom of the i_j-spirals have no important role in upward drawing and saddle points. Thus, they have the same number of saddle points and, therefore, similar upward drawings.

both components have the same number of branches. Therefore, the last branch of S_j can be drawn long enough, such that vertex w_0 of this component has the least y-coordinate. Then it can be identified with vertex b (see Figure 25:(b)).

This gives a planar upward drawing on the plane which clearly it provides an upward drawing on the sphere (which is a surface with zero saddle points) as well.

If $i \neq j$ then $S(i,j)$ has $|i-j|$ saddle points. We design an upward drawing which is an amalgamation of the planar upward drawing (the case $i = j$) and the upward drawing on the surface in the proof of the Theorem 4.1[2]. To see this intuitively, we first make an auxiliary pattern of upward drawing on the plane (with crossing edges) which is helpful in presenting an upward drawing (without crossing edges) on an appropriate surface.

To this end, suppose $i < j$. We first draw the i-spiral S_i without flipping, so that its branches are drawn spirally inward to be close to a center. The j-spiral S_j is flipped and drawn in such a way that the first $(2i+1)$ branches spiral outward from the centre to cancel the first $(2i+1)$ branches of the i-spiral S_i. The $(2i+1)$st branch is drawn long enough, such that the edge $u_{2i+1} \to v_{2i+1}$ is placed below the branches we have already drawn. We call this part of the drawing the *planar portion* of spiral $S(i,j)$. See Figure 34.

Next, the remainder of the $2(j-i)$ branches of the j-spiral S_j are drawn as follows.

- The $(2i+2)$nd branch is drawn upward.
- The $(2i+3)$rd branch is flipped to the right and pulled downward, this producing a crossing of edges.
- The $(2i+4)$th branch is pulled upward.
- The $(2i+5)$th branch is flipped to the left and pulled downward, this making a crossing of edges as well.

We continue this procedure for every group of four consecutive branches (see Figure 34 for the spiral $S(1,3)$).

The last branch (the $(2i+2(j-i)+1)$st branch), is pulled downward, in such a way that vertex w_0 of this component has the minimum y-coordinate. Thus, it can be identified with the vertex b.

[2]It is very similar to the amalgamation of the upward drawings of spiral $S(i,i)$ (which has a planar upward drawing) and $S(j-i)$, for $j > i$ (which by Theorem 4.1 it requires a surface of genus zero with $|i-j|$ saddlle points using the same frame).

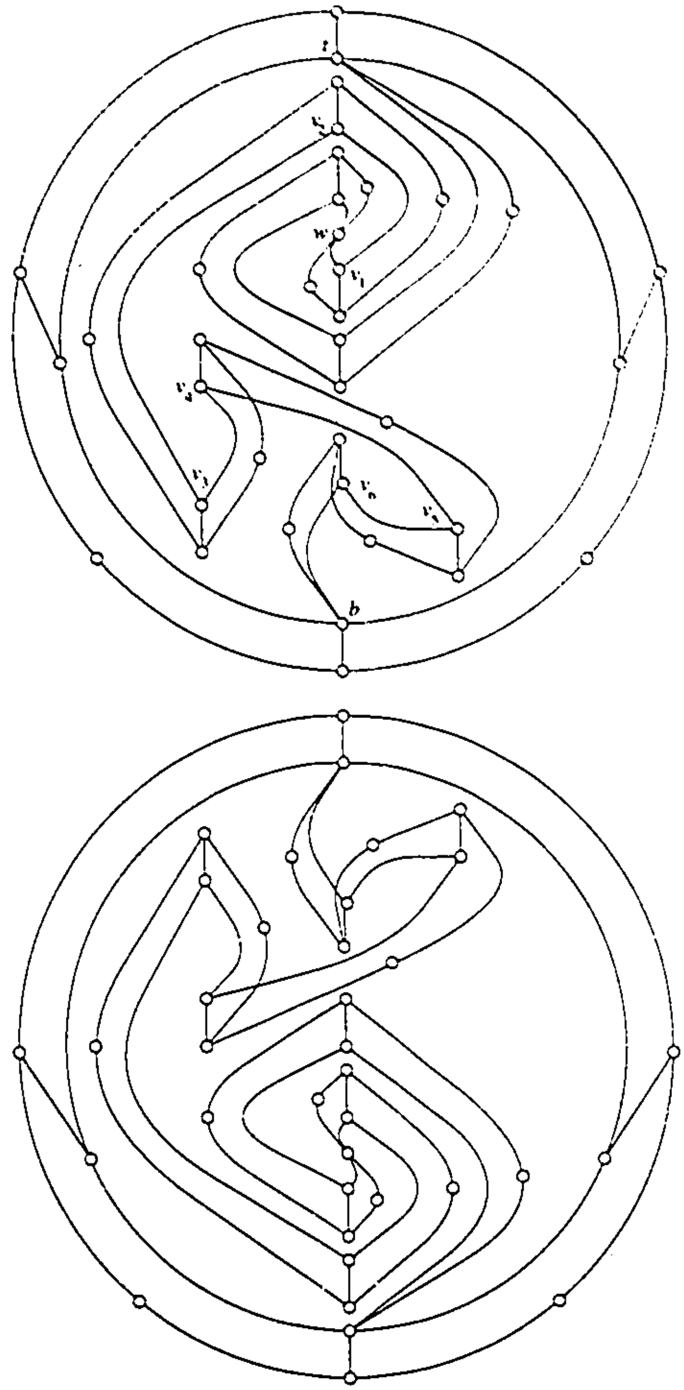


Figure 34: An upward drawing of spiral ordered sets $S(1,3)$ and $S(3,1)$, with two crossing of edges

Now, fix a surface of genus zero with a top (a maximal point) and $(j-i)$ saddle points, and $(j-i)+1$ minimum points.

For the actual upward drawing on the surface, we first draw the frame, exactly as we did in the upward drawing of spiral $S(i_1, i_2, \ldots, i_n)$, as explained in the proof of Theorem 4.1.

Now we can locate the planar portion of $S(i,j)$ in the front of the surface as in the pattern[3]. Finally, locate the minimal element u_{2i+1} at the last minimum point and v_{2i+2} at the adjacent saddle point to this minimum point and continue in this way by placing all v_{2i+2k} $(k = 1, 2, \ldots, j-i)$, at consecutive saddle points, respectively[4] (cf. Figures 27 for an upward drawing of $S(1,2)$).

If $i > j$, the procedure for upward drawing is similar. In this case, we may start with the nonplanar part on a surface which has one minimum and $(i-j)$ saddle points and $(i-j)+1$ maxima, and complete the upward drawing by adding the planar part. This upward drawing and the position of the surface looks like dual of the last case. This completes the construction of upward drawing. □

[3]The procedure for this upward drawing is similar to that for the case that $S(i,j)$ is planar.

[4]The upward drawing of this part is similar to the upward drawing of the spiral $S(j-i)$ which we have seen in the Theorem 4.1.

Chapter 5

Sphericity Testing

5.1 Introduction

In Chapter 4 we have seen that for a large class of ordered sets called spirals, planarity and sphericity are equivalent. That is, each member of this class is planar if and only if it has an upward drawing on the sphere without crossing of edges.

The basic components for the members of this class are the basic flipping components of the gadgets for the reduction in the proof of NP-completeness of planarity testing (cf. Chapter 1). This, in fact, provides the whole idea of the "intractability" of *upward sphericity testing*:

Upward Sphericity Testing

INSTANCE Given an ordered set P.

QUESTION Does P have an upward drawing on the round sphere

$$\{(x, y, z) : x^2 + y^2 + z^2 = 1\}$$

without crossing of edges?

Like *upward planarity testing* the complexity of *upward sphericity testing* seems far from obvious. Indeed, according to [**Ewacha, Li and Rival (1991)**] — and unlike *upward planarity testing* — the decision problem as to whether or not an ordered set has an *upward drawing* on a surface of genus zero (that is, a *topological* sphere) is itself polynomial. The reason is that its planar covering graph can be "lifted" to an upward drawing on a surface of genus zero. On the other hand, it is our main result that *upward sphericity testing* is still *NP-complete.*

Theorem 5.1 *Upward sphericity testing is NP-complete.*

The *NP-completeness* reduction is based on techniques of "component design" (cf. [**Garey and Johnson (1979)**]).

As in the proof of NP-completeness of planarity testing, our aim is to construct an acyclic digraph G such that "flipping" its components corresponds to an instance of **Exact Cover By 3-Sets** a well-known NP-complete problem:

INSTANCE Given a set X with $|X| = 3q$ and a collection C of three-element subsets of X.

QUESTION Does C contain an exact cover for X, that is, a subcollection $C' \subseteq C$ such that every element of X occurs in exactly one member of C'?

5.1.1 Spherical Ordered Sets at a Glance

What is the difference between the sphere $\{(x, y, z) : x^2 + y^2 + z^2 = 1\}$ and an arbitrary homeomorph of it, that is, any compact surface of genus zero in $\mathbf{R}^3$?

In a word, ***saddles.***

We recall the characterization of spherical ordered sets in Chapter 3, an algorithmic analysis of a graph on a surface ultimately entails a triangulation of it. Any face which itself (as an ordered subset) contains a *top* and *bottom* (a bounded face) can be triangulated without increasing the alternation about any of its vertices (cf. Lemma 3.4). The essential conclusion to which all of these observations lead the following.

Theorem [3.5] *An ordered set has an upward drawing on a sphere if and only if its covering graph has a triangulation with no saddle points at all.*□

5.1.2 Strategy of the Proof

Our intention in the proof of NP-completeness of sphericity testing is similar to that in the proof of NP-completeness of planarity testing (cf. Section 1.5). To this end, we first generalize all gadgets which have been used for the reduction problem, and we shall show that for every generalized form of the gadgets, *planarity and sphericity are equivalent.*

Next, we shall form a directed graph G for the reduction, using these generalized components. We shall also prove that for this directed graph planarity and sphericity are equivalent. Then, the proof of NP-completeness for sphericity testing is completed by observing that this generalized directed graph, constructed with generalized gadgets, can also be applied for the reduction problem in planarity testing.

5.2 Proof

5.2.1 Generalized Components

Let D be a plane digraph consisting of two directed paths $b \to x \to t$ and $b \to y \to t$. Replace vertex y by a copy of spiral S_{3k}, and vertex x by a chain of three copies of spiral S_k, where k is a positive integer, in such a way that all these copies are amalgamated into the resulting plane digraph Q_k. The *right*-path of the copy of S_{3k} and the *left*-paths of the copies of S_k contribute to the exterior face of Q_k (see Figure 35:(b); every copy of S_m is labelled by m and its *left*-path is indicated with filled vertices).

Similarly, we define the plane digraph $P_k(d)$ ($d > 1$ is a positive integer) as the one obtained from a plane digraph D consisting of two directed paths $b \to x \to z \to t$ and $b \to y \to t$, where y is replaced by an amalgamated copy of $S_{(d-2)k}$, and x is replaced by a chain of d amalgamated copies of S_k (by S_0 we mean just a single vertex, so for $d = 2$ the vertex y is not replaced at all). In contrast to Q_k, the *right*-paths of the copies of S_k and the *left*-path of the copy of $S_{(d-2)k}$ are assumed to contribute to the exterior face. Moreover, for $d > 2$, we assume that $P_k(d)$ has an additional edge from z_1 in the copy of $S_{(d-2)k}$ to z (which is intended to prevent flipping this copy) (see Figure 35:(c)). Note that for $d = 2$, the vertex z is not needed.

The vertices t and b are again called the *top* and the *bottom* of Q_k and $P_k(d)$, respectively. The part of the boundary of Q_k (of $P_k(d)$) corresponding to the directed path $b \to y \to t$ will be called the *right*-path of Q_k (of $P_k(d)$), and the remaining part, the *left*-path.

A copy of Q_k (or $P_k(d)$), being a subgraph of a plane digraph P, is said to be ***amalgamated*** in P if

- $\{t, b\}$ is a separation pair in P, and
- there are edges outgoing from t and incoming to b in P.

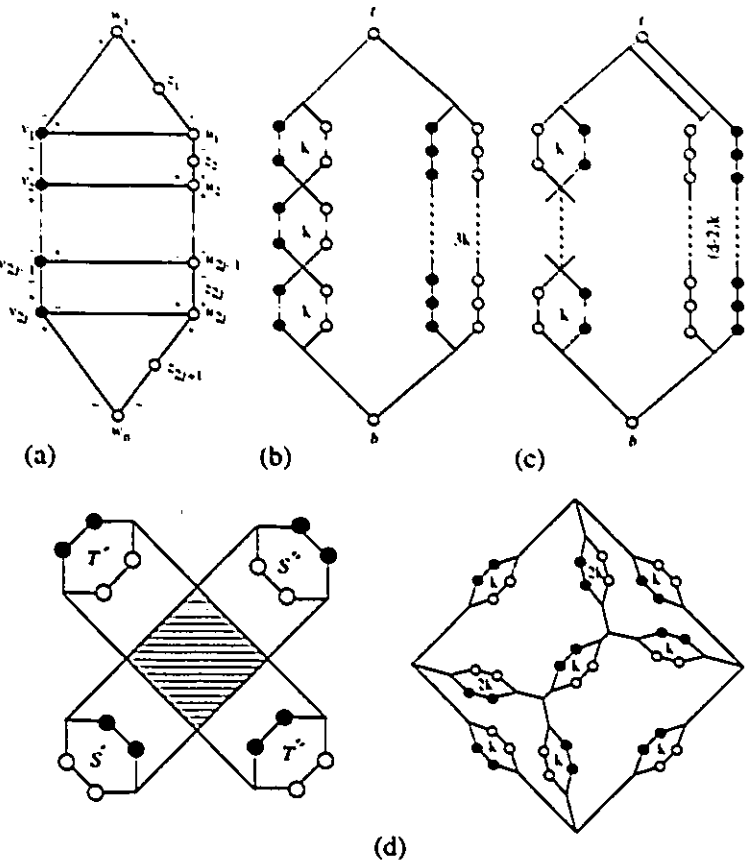


Figure 35: Schematic diagrams of (a) S_m (b)Q_k and (c) $P_k(d)$ (d) R_k

Lemma 5.2 *A plane digraph Q'_k obtained from Q_k by flipping its spiral set components has a similar planar upward drawing if and only if either all the four components are flipped, or none of the components is flipped. A plane digraph P' obtained from $P_k(d)$ by flipping its S_k-components has a similar plane upward drawing if and only if exactly one of these components is flipped.*□

The proof is similar to the proof of Lemma 1.7. If $k > 2$ then we have the following parallel result for sphericity of these components.

Lemma 5.3 *A plane digraph Q'_k obtained from Q_k by flipping its spiral set components is spherical if and only if either all the four components are flipped, or none is flipped. A* *plane digraph P' obtained from $P_k(d)$ by flipping its S_k-components is spherical if only if exactly one of these components is flipped.*

5.2.2 Generalized Communication and Crossovers.

Given two copies S' and S'' of S_k on the plane, the pair of non-crossing edges directed from the bottom and the top of S' to the bottom and the top of S'' will be called the *pair of communication edges from S' to S''* (see Figure 35:(d)).

We allow the joining of the top of S' with the bottom of S'' (rather than with the top of S''). The following definition does not depend on this choice.

Let D be a plane digraph obtained from two copies S' and S'' of S_k joined with the pair of communication edges, and let F be an exterior face of D. We say that S' and S'' are *flipped in the same way* in D if the *right*-path of S' contributes to the exterior face of D and the *right*-path of S'' contributes to an interior face of D (as indicated with non-filled vertices in Figure 35:(d)).

Again, by Theorem 1.3, we have immediately

Lemma 5.4 *A plane digraph D obtained from two copies S' and S'' of S_k joined with the pair of communication edges has a similar planar upward drawing if and only if both the copies are flipped in the same way.* □

This allows us to place two copies of S_k in a plane digraph such that in every planar upward drawing they must be flipped in the same way. The last problem to resolve is the possible crossing of communication edges. We avoid it in a classical manner by designing a component called a *crossover*. To this end we may use the digraph R_k given in Figure 35:(d): every component is a copy of S_k or S_{2k} (is shown schematically as a shaped box labelled with k or $2k$, respectively) and is assumed to be amalgamated into R_k. We show that the pairs of components (S', S'') and (T', T'') must be flipped in the same way to allow an upward drawing of R_k.

Lemma 5.5 *Let R_k be the plane digraph in Figure 35:(d) and F its exterior face. Then R_k has a similar planar upward drawing if and only if the pairs (S', S'') and (T', T'') are flipped in the same way, that is both the total contribution of (S', S'') to the difference $\alpha(F) - \sigma(F)$ and the total contribution of (T', T'') to the difference $\alpha(F) - \sigma(F)$ are equal to zero.*□

The proof is similar to the proof of Lemma 1.9. If $k > 2$ then we have the following parallel result for the sphericity of these ordered sets.

Lemma 5.6 *A plane digraph D obtained from two copies S' and S'' of S_k joined with the pair of communication edges is spherical if and only if both the copies are flipped in the same way.*

Lemma 5.7 *Let R_k be the plane digraph in Figure 35:(d) and F its exterior face. Then R_k is spherical if and only if the pairs (S', S'') and (T', T'') are flipped in the same way.*

5.2.3 Proof of the Lemmas

The idea of of the proof of these lemmas is based on the characterization of spherical ordered sets in terms of critical points (cf. Section 3.5). To this end, we need to triangulate all faces of each component or crossover.

All these ordered sets described in the above lemmas are obtained by amalgamation of m-spirals, where m is a positive integer.

Basically, all these ordered sets have two types of faces.

- The interior faces of m-spirals,
- Faces whose boundaries share a *right*-path or a *left*-path of the m-spirals.

All faces in the first type are bounded faces which by virtue of Lemma 3.4 can be triangulated without increasing the alternation around vertices. Thus, the question of sphericity of these ordered sets depends on the triangulation of the faces in the second type. All these faces are in fact nonordinary faces (cf. Section 2.4.2).

Next we shall show that if flipping of the basic components (m-spirals for an ordered set) fails to comply with the requirements of the corresponding lemma, then any acyclic triangulation of one of its faces has at least one saddle point. Therefore, in view of the Theorem 3.5, the ordered set is not spherical.

The proof of this fact is based on a technical result which generalizes Theorem 4.4 for the chain chordal regions consisting of more than two strings of interior and extremal vertices (cf. Section 4.3). As before, let *interior*$(\mathcal{F})$ stand for the number of interior vertices of a chain chordal region $\mathcal{F}$ and *extremal*$(\mathcal{F})$ stand for the number of its extremal vertices.

Theorem 5.8 *Let $\mathcal{F}$ be a chain chordal region consisting of $2m$ strings of interior and extremal vertices so that the total number of these vertices satisfies the following condition*

$$interior(\mathcal{F}) - extremal(\mathcal{F}) > 2m \tag{19}$$

Then, any triangulation of this region creates at latest one saddle point.

Proof The proof is by induction on m, the number of alternating strings of interior and extremal vertices.

For the initial step of the induction, that is, when $m = 1$, the assertion is exactly the Theorem 4.4, which has already been proved. Thus, we assume that for any chain chordal region consisting of $2k$ strings of interior and extremal vertices which satisfies the condition

$$interior(\mathcal{F}) - extremal(\mathcal{F}) > 2k$$

where $1 < k < m$, any triangulation contains at least one saddle point. We show that the conclusion is also true for a chain chordal region consisting of $2m$ strings of interior and extremal vertices satisfying condition (19).

To simplify referring to the strings of extremal and interior vertices, we label the strings from 1 to m, so that with respect to a fixed direction on the boundary cycle of the region, the consecutive strings of interior and extremal vertices have the same labels (see Figure 36).

Claim 1 *No saddle-free triangulation of region $\mathcal{F}$ can have an edge connecting two interior vertices.*

Proof If the interior vertices are in the same string, then as we have seen in the Proposition 4.5, either one of the endpoints becomes a saddle point, or the region consisting only of interior vertices has the property that any triangulation of this region contains a saddle point.

If the interior vertices are in distinct strings, then connecting them by such an edge, the region $\mathcal{F}$ is divided into two smaller regions F_1 and F_2, each of whose numbers of strings is less than m. We shall now show that at least one of these regions must satisfy the induction hypothesis. To this end, assume that we connect an interior vertex in the ith string to an interior vertex in the $(i + k)$th string. The endpoints are common for both smaller regions (see Figure 36). By contradiction, we assume that none of these regions satisfies the induction hypothesis, that is

$$interior(F_1) - extremal(F_1) \leq 2k$$

and

$$interior(F_2) - extremal(F_2) \leq 2(m-k)$$

Adding up these inequalities, we have

$$interior(\mathcal{F}) - extremal(\mathcal{F}) \leq 2m - 2$$

which contradicts the condition (19). This proves the Claim 1.

Claim 2 *No saddle-free triangulation can have an edge connecting two extremal vertices.*

Proof Suppose that there is an edge connecting two extremal vertices. Without loss of generality, we can assume that the edge connects two extremal vertices in distinct strings. Otherwise, we consider the triangulation of the region which has the same number of strings and fewer extremal vertices, which satisfies the condition of the theorem.

Thus, assume that the edge connects a vertex of the ith string of extremal vertices to the $(i+k)$th string of the extremal vertices, so that it divides the region into two smaller regions F_1 and F_2. One of them has $2k$ strings and the other one has $2(m-k)$ strings. We show that at least one of them satisfies the induction hypothesis. To this end, we proceed by contradiction and suppose that the condition of the theorem fails for both of them. Then

$$interior(F_1) - extremal(F_1) \leq 2k \tag{20}$$

and

$$interior(F_2) - extremal(F_2) \leq 2(m-k) \tag{21}$$

Now, without loss of generality, we can assume that at least one of the inequalities in (20) and (21) is strict[1]. Thus, adding up both sides of these inequalities, and noting that the

[1] If, for both of these regions F_1 and F_2 in (20) and (21) we have equality then in any triangulation of one of these regions, say F_1, to each extremal vertex there is an incident edge which has an opposite sign, with respect to the sign of two other edges in the boundary of this region, at this endpoint. Consequently, the two common extremal vertices of the regions have the role of interior vertices for the region F_2. Therefore, for region F_2, we have to count two more interior vertices and two fewer extremal vertices. Also, for this region the number of strings may increase by at most two. Thus, in this case (21) would be changed as follows.

$$interior(F_2) + 2 - (extremal(F_2) - 2) \leq 2(m-k) + 2$$

difference cannot be an odd number, we have

$$\begin{aligned} interior(\mathcal{F}) - (extremal(\mathcal{F}) + 2) &< 2m \\ interior(\mathcal{F}) - extremal(\mathcal{F}) &\leq 2m \end{aligned}$$

which contradicts (19). This proves Claim 2.

Hence, by the last two claims, if there is a saddle-free triangulation of the region, then each edge must connect an interior vertex to an extremal vertex. Next, we shall show that in a saddle-free triangulation, we have some constraints for connecting these edges. In fact, the following claim states that such a triangulation should not contain an edge connecting an extremal vertex of a string to an interior vertex in a nonadjacent string.

Claim 3 *No saddle-free triangulation can contain an edge connecting an interior vertex in the jth string to an extremal vertex in the ith string, where $j \neq i$.* (That is, there is no edge connecting an interior vertex of some string of interior vertices to an extremal vertex in a string of extremal vertices which is not itself adjacent to the former string.)

Proof Assume that in a triangulation of the region $\mathcal{F}$, there is an edge connecting an interior vertex of the jth string to an extremal vertex of the ith string so that $j \neq i$ (they are not adjacent). Label all vertices in the string of extremal vertices by $i_1, i_2, \ldots, i_r$ and vertices in the string of interior vertices by $j_1, j_2, \ldots, j_s$ (see Figure 36).

We first show that, in a saddle-free triangulation, j_1 cannot be connected to any extremal vertex $i_1, i_2, \ldots, i_r$. So suppose on the contrary that j_1 is connected to i_r. (Figure 36 illustrates this case.) This edge divides the region $\mathcal{F}$ into two smaller regions, each of which must be triangulated.

Now consider the triangulation of the shaded region, depicted in Figure 36. In any triangulation of this region, either the interior vertex adjacent to i_r must be connected to the interior vertex j_1, or the extremal vertex adjacent to j_1 must be connected to the extremal vertex i_r. But, in either case, one of the Claims 1 or 2 implies that any triangulation of the region $\mathcal{F}$ contains a saddle point, which contradicts the assumption that $\mathcal{F}$ is saddle-free.

---

or

$$interior(F_2) - extremal(F_2) \leq 2(m-k) - 2$$

This fact is immediate, from the proof of the Theorem 4.4, in the case that $m = 1$. By induction, we can assume that it is true for a region consisting of $2k$ strings, where $k < m$. This property for a region consisting of $2m$ strings will occur again later in the proof.

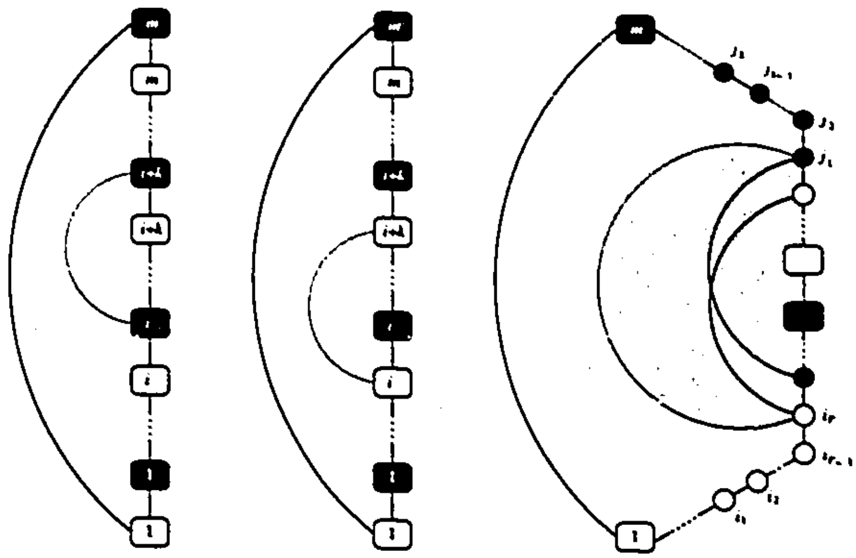


Figure 36:

Next, the interior vertex j_1 cannot be connected to i_{r-1}. Because in this case any triangulation of the shaded region must contain either the edge $j_1 \sim i_r$ or an edge which connects the extremal vertex i_{r-1} to the extremal vertex adjacent to the vertex j_1. The first case is the situation that has been considered above, and the second case satisfies the hypothesis of Claim 2. Therefore any triangulation of the region $\mathcal{F}$ contains a saddle point which contradicts the assumption.

Continuing in this way, we can recursively prove that no saddle-free triangulation can have an edge connecting the interior vertex j_1 to any extremal vertex $i_1, i_2, \ldots, i_r$. With a similar argument, we can also show that no saddle-free triangulation can have an edge connecting any extremal vertex $i_1, i_2, \ldots, i_r$ of the ith string to any interior vertex $j_1, j_2, \ldots, j_s$ in the jth string. This proves Claim 3.

Thus, in a saddle-free triangulation of the region (if one exists) the vertices in a string of interior vertices must be connected to the vertices of a string of extremal vertices which is itself adjacent to the former. Now, this reduces the problem to a situation similar the one we have considered in Theorem 4.4[2]. We distinguish two cases.

[2]This means that, in this situation, we have to begin the triangulation with a region consisting only of two strings, one of the consecutive interior vertices and the other one of consecutive extremal vertices.

If the number of interior vertices in the string exceeds the number of extremal vertices in the adjacent string, then triangulation creates at least one saddle point or we have to connect one interior vertex to an extremal vertex in a nonadjacent string (which by Claim 3 creates a saddle point) and we are done.

Otherwise, we continue the triangulation by connecting the interior vertices to the extremal vertices in the adjacent string. We note that, as we have seen in the proof of the Theorem 4.4, no interior vertex (in a saddle-free triangulation) can be connected to more than two consecutive extremal vertices. In other words, each interior vertex corresponds to only one extremal vertex. Thus, this case will ultimately yield a region which has fewer strings and the difference of the total number of its interior and extremal vertices satisfies the induction hypothesis. This completes the proof of the theorem. □

Proof of the Lemmas The idea of the proof for all of the above lemmas is similar. If we flip the m-spirals as stated in the corresponding lemma, then the ordered set is planar and consequently it is spherical. If we flip (the basic components) m-spirals in a different way to that stated in the lemma, then one of the nonordinary faces has more interior vertices than extremal vertices. If we choose k large enough, then flipping one of the basic components introduces at most two strings, while it contributes $2k$ interior vertices and at the same time it decreases the number of extremal vertices by $2k$. Then the total number of vertices, in the corresponding nonordinary face satisfies the hypothesis of Theorem 5.8, so that any acyclic triangulation has at least one saddle point.□

The parallel lemmas for sphericity of the components and crossover suggest a similar procedure for the proof of the NP-completeness of upward sphericity testing. Thus, in order to be able to use the same directed graph for the reduction, we need two results.

Lemma 5.9 *The proof of NP-completeness of planarity testing remains valid if we replace all m-spirals with mk-spirals, where k is a positive integer.*□

The proof is a recalculation of the respective contributions of the *left*-paths and *right*-paths to the faces (cf. Sections 2.4.2) together with the observation that the factor k always is canceled from both sides of the equations.□

Now, given an instance of the Exact Cover by 3-Sets, we form a directed graph G (or, by subdivision of some edges, if necessary, we form an ordered set) corresponding to this instance. The directed graph G is constructed in $O(m+n)$ time (where $m = |C|$ and $n = |X|$) as we constructed in Section 1.9 using the generalized form of the components Q_k and $P_k(d)$, and crossovers R_k. (see Figures 12 and 13). We shall show that

Lemma 5.10 *If k is large enough, then the directed graph G is spherical if and only if it is planar.*

Proof Choose k large enough so that it is greater than the largest number of strings of interior and extremal vertices that may occur on the boundary of any face. According to the structure of the directed graph G, the maximum number of strings can occur in one of the faces F_1 or F_2 (if $\max\{|X|, |C|\} > 3$). Note that the boundaries of these faces consist of strings of interior and extremal vertices corresponding to the *left*-paths and the *right*-paths of m-spirals of components $P_k(d)$ and Q_k. Thus, we can choose, for instance,

$$k > \max\{|X|, |C|\} + 3. \tag{22}$$

Now, if directed graph G is planar, it is clearly spherical. For the converse, we shall show that if G is not planar then it is not spherical. To this end, we observe that if G is not planar then there is a face of G in which the contribution of the *right*-paths and the *left*-paths of the flipping components are not equal. But if this happens, for one of the faces the number of interior vertices exceeds the number of extremal vertices by at least $2k$, which is greater than the total number of strings in that corresponding face (see (22)). An application of Theorem 5.8 shows that any triangulation of this face has at least one saddle point and the directed graph G is not spherical. This completes the proof of the lemma.□

Proof of Theorem 5.1 The proof follows from Lemmas 5.9 and 5.10.□

5.3 Open Problems

Although upward planarity testing is difficult, it may be that, if it is known that an ordered set already has an upward drawing on a genus zero surface then there is an efficient procedure to decide whether it has an upward drawing on a plane, too.

Upward Planarity Testing of Spherical Ordered Sets

INSTANCE Given an ordered set P with upward drawing on a sphere.
QUESTION Does P have an upward drawing on a plane?

Problem 5.1 *Is* **Upward Planarity Testing of Spherical Ordered Sets** *polynomial?*

Our analysis here has made no distinction between one or more saddle points.

m-Saddle Point Surfaces

INSTANCE Given an ordered set P with planar covering graph and a positive integer s.
QUESTION Does P have an upward drawing on a smooth surface of genus zero with at most s saddle points?

Problem 5.2 *Is the s-saddle point problem NP-complete?*

The bridge between the *NP-completeness* of *upward planarity testing* and *upward sphericity testing* is built over *spirals*. In effect, a *spiral* is spherical if and only if it is planar.

Problem 5.3 *Characterize those ordered sets for which sphericity implies planarity.*

Chapter 6

Upward Drawing to Fit Surfaces

6.1 Introduction

Much like the approximation of an arbitrary smooth function by an interpolating polynomial, we propose to approximate smooth two-dimensional surfaces by polyhedral surfaces, that is, piecewise linear two-dimensional surfaces, themselves modelled by "ordered sets" and their *upward drawings*. The analogy is fairly accurate for, much like numerical interpolation techniques which use the function's values and derivatives at certain points, our starting point is the set of critical points of the surface. In this respect it is natural, for our purposes, to assume that a *(smooth) surface* S is a

- *closed, compact, two-dimensional orientable manifold,*
- *embedded in* $\mathbf{R}^3$ *above the horizontal plane* $z = 0$,
- *smooth enough to have a continuously turning tangent plane,*
- *such that every point with tangent plane parallel to* $z = 0$ *has a neighborhood in which it is the only critical point.*

Of course, the study of these surfaces and their critical points is not new (cf. [**Milnor** (1963)]), although many new ideas are still launched (cf. [**de Rezende and Franzosa** (1993)]). Indeed, the common ground between flows and topology dates at least to Poincaré, whose well-known Index Theorem, relating the Euler characteristic of the surface to the sum of the indices of its critical points (with respect to some gradient flow), has surprising discrete analogues (cf. Chapter 2). Thus, for a cartographic map, say of countries drawn (as simply connected regions) on the surface of a sphere, the alternating sum of the number of vertices, number of edges, and the number of faces, is a constant (two) — its Euler characteristic. What is new, however, is the study of order types and their upward drawings, whose "critical point characteristics" are unlabelled, in order to interpolate a surface.

The application metaphors of this subject are diverse, too. Computational graphics, for instance, is useful in cartography as well as in dynamical systems. The graphic search of topographic maps for peaks, pits, and valleys aids in the identification, for example, of local watersheds. A qualitative description of its critical points will also determine the phase portrait of a system of differential equations. A common feature of both applications is a surface in $\mathbf{R}^3$ — perhaps a sphere or a torus, or any other two-dimensional surface equipped with a smooth vector field. And, if we fix such a surface on which every critical point is isolated, its critical points can be ordered: $x < y$ if, with respect to the z-axis (fixed direction), there is a strictly monotonic path on this surface from x to y.

We shall suppose that every critical point on a two-dimensional surface is isolated, that is, each critical point is contained in a neighborhood in which it is the only critical point. Fix an embedding of a surface satisfying the criteria stated on page 93 and let *critical*(S) stand for the set of all its critical points. Our objective is the following.

Theorem 6.1 *For any smooth two-dimensional embedded surface S of genus g there is an ordered set P such that*

(i) P has an upward drawing, without crossing edges, on S,

(ii) P contains the ordered set critical(S) of critical points of S,

(iii) if S' is any two-dimensional surface of genus g on which P has an upward drawing, without crossing edges, then critical$(S) \subseteq$ critical(S').

We say that the upward drawing of P ***fits*** the two-dimensional surface S, and that P *interpolates* S.

6.2 Surface Topology

In this section we briefly review some terminology and definitions of surface topology, which may be found in any standard text of combinatorial or algebraic topology (see e.g. [**Henle** (1979)] and [**Massey** (1991)]).

A *surface* is a topological space in which every point has a neighborhood that is topologically equivalent to an open disk, that is, there is a one-to-one continuous map from this neighborhood onto the open disk so that its inverse is continuous. In words, surfaces are locally similar to planes.

A surface is a *triangulable space* if it can be obtained from a set of triangles by identification of edges and vertices subject to the restriction that any two triangles are identified either along a single edge or at a single vertex, or are completely disjoint. In addition

- each edge is identified with exactly one other edge and
- the triangles identified at each vertex can always be arranged in a cycle $T_1, T_2, \cdots, T_n, T_1$ so that adjacent triangles are identified along an edge.

A topological space X is compact if and only if every *open cover* of X — a collection $\psi = \{c_\alpha : \alpha \in \Lambda\}$ of open sets in X whose union is X — has a finite subcover of X.

A combinatorial interpretation of compactness for surfaces is the following.

Theorem 6.2 *A surface is compact if and only if every triangulation uses only a finite number of triangles* (cf. [**Henle (1979)**]). □

A topological space X is *connected* if and only if no separation of X exists, that is, there exists no pair (U, V) of nonempty open subsets of X, such that $X = U \cup V$ and $U \cap V = \emptyset$.

The most important theorem in the topological theory of surfaces is the following classification theorem.

Theorem 6.3 [**Dehn and Heergaad (1907)**] *Every compact, connected surface is topologically equivalent to a sphere, or a connected sum of tori, or a connected sum of projective planes* (cf. [**Massey (1991)**]). □

6.2.1 Critical Points

The geometry of surfaces plays an important role in upward drawings. Thus we restrict our attention to those orientable surfaces embedded in three-dimensional space $\mathbf{R}^3$ with a fixed "position" in this space.

We define a *position* of a surface to be a fixed embedding of the surface in three-dimensional space $\mathbf{R}^3$ such that with respect to the height function

$$h(x, y, z) = z \ , \ \text{where} \ \ (x, y, z) \in S$$

it has only nondegenerate critical points, that is the height function is a *Morse function*.

Let $h : S \longrightarrow$ be a height function, and let $\mathcal{V} : S \longrightarrow TS$ (TS stands for the tangent space of S) be the *gradient vector field* of h. A gradient vector field $\mathcal{V}$ on S is a continuous vector field such that at each point p, the vector $\mathcal{V}(p)$ lies in the tangent plane to S at p and points in the direction of greatest increase of h, that is, along the paths of steepest ascent on S. The conditions we assumed for surfaces guarantee the existence of a gradient vector field.

A point p of S is said to be a *critical point* for h if the tangent plane to S at p is parallel to the plane $z = 0$, (or equivalently $\mathcal{V}(p) = 0$). We denote the set of critical points of S by *critical*(S). All other points of S are called *ordinary points* for h.

If p is an ordinary point, then $\mathcal{V}(p)$ is not zero and since $\mathcal{V}$ is continuous, it is non-zero in a neighborhood of p. If p is a critical point, then it may still be possible to choose a neighborhood so that $\mathcal{V}$ vanishes only at p in the neighborhood. Then p is called an *isolated critical point*.

Remark 6.1 *Through out this chapter we assume that (with respect to a height function) all critical points are* nondegenerate *(see Page 36). In such circumstances, all critical points are isolated and the compactness of the surface implies that the set of critical points is finite.*

We can define an order on the set of critical points of a surface in a natural way as follows.

For each x and y in *critical*(S), define $x \leq y$ just if there is a strictly monotonic path from x to y. A monotonic path, by definition, is a path $f : [0,1] \longrightarrow S$ from a unit interval $[0,1]$ to the surface S so that for $x < y$ (in *critical*(S)) , $f(0) = x, f(1) = y$ and for every t_1 and t_2 in $[0,1]$, $t_1 < t_2$ implies that

$$h(f(t_1)) \leq h(f(t_2)),$$

where h is a height function. Clearly (*critical*(S) , $\leq$) is an ordered set which we call the *ordered set associated* with the critical points of S and designate by *critical*(S).

Example 6.1 Consider the heart-shaped surface of genus zero illustrated in Figure 37. Apparently, various positions of this surface have different numbers and types of critical points (see also Figure 48). Here we consider a position of this surface in $\mathbf{R}^3$ which has two maxima, one minimal and one saddle point.

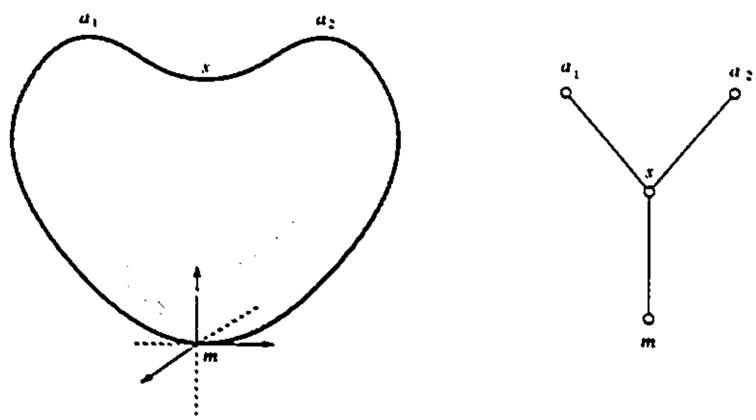


Figure 37: A surface of genus zero and the ordered set associated to its critical points

Since there are monotonic paths from the saddle point s to maximal points a_1 and a_2 and from the minimal point m to saddle point s, the ordered set associated with the critical points S is the four-element ordered set

$$critical(S) = \{m < s,\ s < a_1,\ s < a_2\}$$

Example 6.2 A torus is a surface of genus one. An embedding of the torus T in $\mathbf{R}^3$ is depicted in Figure 38. It has four critical points consisting of one minimal, one maximal and two saddle points. The ordered set associated to its critical points is a four-element chain

$$critical(T) = \{m < s_1 < s_2 < a\}$$

6.3 Construction Idea

The idea in the construction of an ordered set P whose upward drawing fits a surface S lies in forming a direct covering graph $\overrightarrow{cover}(P)$ such that

- its genus equals the genus of the surface S, and
- it has a set of critical points corresponding to the critical points of S so that they are all *essential* that is, in any embedding of $\overrightarrow{cover}(P)$ they are critical points with invariant type.

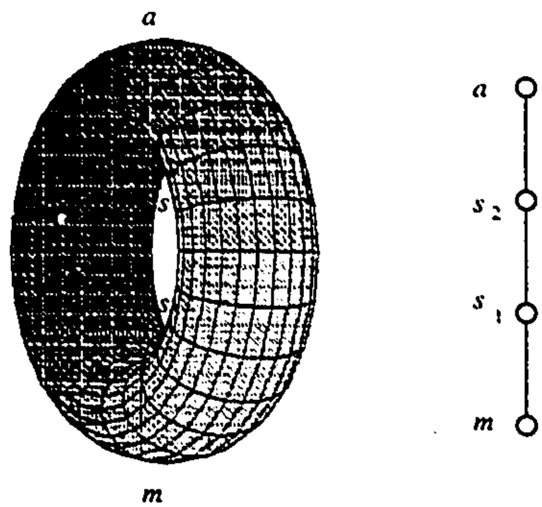


Figure 38: A surface of genus one and the ordered set associated to its critical points

- It includes the ordered set *critical*(S).[1]

Of course, such an ordered set must have an upward drawing without crossing edges on the surface so that a height function of the order can be extended linearly to the surface. Such an ordered set might be considered as a piecewise linear approximation of the surface.

A two-dimensional surface is always triangulable and a triangulation is the standard way to study a two-dimensional surface. According to the definition, two triangles in a triangulation of a surface have either two, one, or no vertices in common and two distinct triangles have distinct sets of vertices. It follows that the surface can be described by labeling vertices and listing the triangles by their vertex sets. This gives a combinatorial description of the surfaces. Also, a height function on vertices of a triangulation can be extended linearly to the interior points of triangles and therefore to the whole surface. Furthermore, it is well-known that a graph consisting of vertices and edges of a triangulation of a given surface has genus equal to the genus of the surface [Youngs (1963)].

[1]There is an injective order preserving map

$$f : critical(S) \longrightarrow critical(P)$$

from the set of critical points of S to the set of critical points of P, such that it preserves the type of the critical points. For convenience, we designate the image of f by *critical*(S).

A triangulation of a covering graph fixes the "rotation system" about vertices, in particular, in an acyclic directed triangulation (a triangulation in which all edges are directed edges) the alternation about each vertex under any other embedding remains unchanged. It follows that the critical points of the triangulated covering graph are all essential.

On the other hand, according to the index theory of ordered sets in Section 2.4 and the lifting procedure in Section 2.3, having a triangulation of the covering graph of an ordered set describes a surface topologically homeomorphic to the actual surface on which the ordered set has an upward drawing. Thus, the crucial fact in the study of an ordered set to fit a prescribed surface is a triangulation which meets certain conditions.

6.3.1 Ordered Set Associated to the Critical Points on the Surface

Fix a closed compact surface S embedded in $\mathbf{R}^3$ (with a fixed position in this space). We assume that it meets all necessary conditions stated in Section 6.1 and Theorem 6.1, and it is equipped with a gradient vector field $\mathcal{V}$ associated with a height function h.

We draw the ordered set $critical(S)$ on the surface by appointing vertices for the critical points and drawing monotonic paths corresponding to the covering edges in the ordered set.

We also assume that paths cross transversally and two paths intersect in at most one point (see the following observation).

Lemma 6.4 *If two monotonic paths corresponding to the covering edges of the ordered set associated to the critical points of a surface intersect in a finite set of points, then we can always find two other alternative monotonic paths with the same endpoints so that they intersect, at most, at one point.*

Suppose that $a \succ b$ and $c \succ d$ in $critical(S)$, and P_1 and P_2 are monotonic paths from a to b and from c to d, respectively. Let $P_1 \cap P_2 = \{x_1, x_2, \ldots, x_n\}$, and $x_1 > x_2 >, \ldots, > x_n$.

Now, let q_{11} be the portion of path P_1 which connects a to x_1 and q_{12} be the portion of path P_2 which connects x_1 to $x_2, \ldots,$ and alternatively we choose the portions of paths P_1 and P_2 which connect x_i to x_{i+1}, where $i = 1, \ldots, n$. The final portion $q_{1(n+1)}$ will be chosen from the path P_1 to connect x_n to b. Next, define

$$Q_1 = \bigcup_{i=1}^{n+1} q_{1i}$$

Clearly Q_1 is a monotonic path from a to b. Similarly, we can construct a monotonic path Q_2 from c to d such that these two paths have the same intersection set as P_1 and P_2.

Now, by a small perturbation we can separate these two monotonic paths Q_1 and Q_2 such that they have only a single point x_n in common (see Figure 39). □

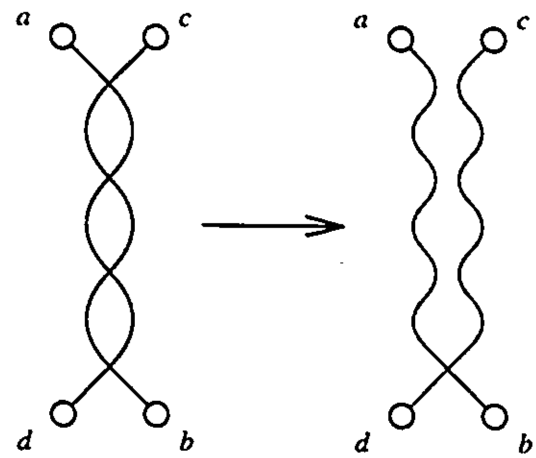


Figure 39:

For the purpose of upward drawing, we need to have an embedding of *critical*(S) on the surface S. Thus, if two paths intersect at an ordinary point which cannot be avoided by other drawings, then we introduce a new vertex at the intersection and it is marked as an ordinary vertex for the ordered set.

These additional vertices do not change the order relation and do not create any new comparabilities. To see this, let $a, b, c, d \in$ *critical*(S) and $b \prec a$ and $d \prec c$. Let f_1 be a path corresponding to the covering relation $b \prec a$ and f_2 be a path corresponding to the covering relation $d \prec c$ such that f_1 and f_2 intersect at a single point x where $x \notin$ *critical*(S) (x is an ordinary point).

The additional vertex x produces comparabilities $d \leq a$ and $b \leq c$. We shall now show that these are the actual comparabilities among the critical points of the surface S. To this end, let f_{11} be the portion of path f_1 which connects critical point b to the additional vertex x and f_{22} the portion of path f_2 which connects additional vertex x to the critical point c. Similarly, let f_{21} and f_{12} be the portions of paths f_2 and f_1 which connect critical point d to additional vertex x and the additional vertex x to the critical point a, respectively. Now it is easy to see that the paths $f_1' = f_{11} \cup f_{22}$ and $f_2' = f_{12} \cup f_{21}$ are monotonic paths on the surface from b to c and from d to a, respectively. Therefore, according to the definition (see page 96) $d \leq a$ and $b \leq c$ are the actual comparabilities in $critical(S)$. For convenience, if there is no danger of confusion, we designate this enhanced ordered set of critical points by $critical(S)$ as well.

6.3.2 How to Design Ordered Sets to Fit Surfaces

In this section we sketch the construction of an ordered set whose upward drawing fits a surface S.

Thus, let S be a surface of genus g and $critical(S)$ be the ordered set associated to its critical points which is embedded on the surface[2].

According to our assumption for the surface, since it is equipped with a gradient vector field $\mathcal{V} : S \longrightarrow TS$ of the height function h, there is an assignment $\mathcal{A} : critical(S) \longrightarrow \{0, 1, 2\}$ such that

$$\mathcal{A}(v) = \begin{cases} 0 & \text{if } v \text{ is a maximum or a minimum} \\ 1 & \text{if } v \text{ is an ordinary vertex} \\ 2 & \text{if } v \text{ is a saddle point} \end{cases}$$

This shall determine the type and the alternation about vertices. It is convenient to embed this ordered set into a polygon model of the surface.

Next, we direct all edges of the embedded ordered set by assigning labels $-$ and $+$, with respect to the order relation of their endpoints. Thus, if the edge e on this polygon model corresponds to $a \prec b$, then we assign $-$ to that end of it outgoing from a and $+$ to that end of it incoming to b.

For each vertex we consider a circular neighborhood such that it contains no other vertex of the ordered set. A fixed direction, say counterclockwise, is also considered for all boundaries of these neighborhoods (This is possible because we are only considering orientable surfaces). Next, with respect to this fixed direction, every vertex v must satisfy (perhaps by introducing additional vertices and comparabilities where necessary)

$$alternation(v) = \mathcal{A}(v) \tag{23}$$

In particular, if a vertex x corresponds to a saddle point of the surface, it usually requires

[2] If there are some unavoidable crossing points then we consider the enhanced ordered set associated to the critical points (cf. Section 6.3.1).

additional vertices a and b with covering relations $a \prec x$ or $b \succ x$, such that they create necessary alternation about vertex x.

Note that vertices added in any step must be marked as ordinary vertices, that is, if x is an additional vertex, then

$$alternation(x) = 1$$

Next, we triangulate the polygon model in such a way that among the vertices of this triangulation we include all critical points and additional vertices, and among the edges we include segments making up the embedded covering graph. Moreover, the triangulation must be acyclic and does not increase the alternation about vertices and all vertices must have different heights. Of course, the ordered set must be triangle-free, but this can be fulfilled by subdividing edges. This completes the construction of the directed graph on the polygon model.

Example 6.3 Consider the heart-shaped surface S described in Example 6.1. We follow the procedure discussed in Section 6.3.2 to construct an ordered set P whose upward drawing fits the surface. Figure 40:(a) presents an embedding of $critical(S)$ on a 2-gon model of surface S with assignment:

$$\mathcal{A}(a_1) = \mathcal{A}(a_2) = \mathcal{A}(m) = 0 \text{ and } \mathcal{A}(s) = 2$$

In Figure 40:(b), all edges are directed with respect to the order by assigning + and − to the edges and also, in order to provide the necessary alternation about vertex s ($alternation(s) = \mathcal{A}(s)$), we introduce vertex b with $b \prec s$.

Figure 40:(c) presents a directed triangulation of the surface which contains all vertices and edges of the directed graph constructed in the last step. In order to avoid transitive edges, all nonessential edges are subdivided in Figure 40:(d) which produces the complete construction of $\overrightarrow{cover}(P)$.

Finally in Figure 41 an upward drawing of the constructed ordered set on the plane and an embedding of this ordered set on the surface are depicted.

Example 6.4 In this example we construct an ordered set such that its upward drawing fits a more complex surface of genus one. We consider a position of this surface which has three maxima, four saddles and one minimum. The surface and the ordered set associated to its critical points are illustrated in Figure 42[3]. The corresponding assignment for the

[3]Figure 42(on the left) has been reproduced from [Moffat and Mottram (1979)].

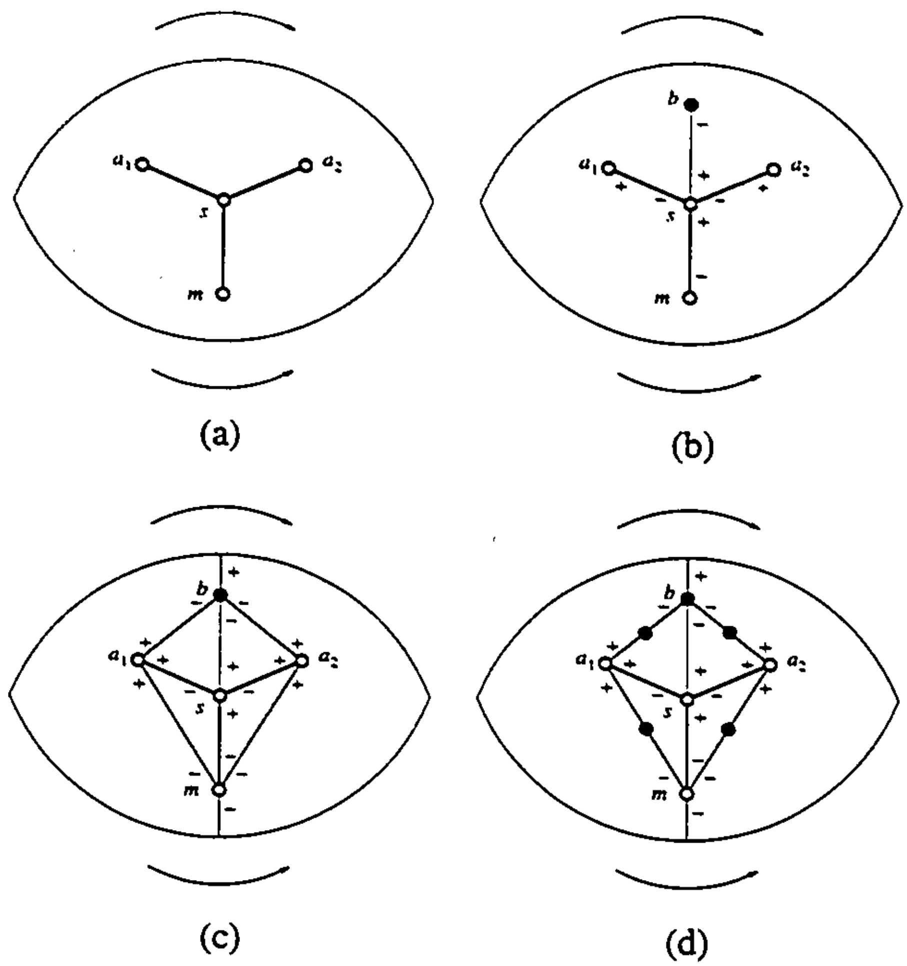


Figure 40:

critical points with respect to this position is

$$\mathcal{A}(0) = \mathcal{A}(5) = \mathcal{A}(6) = \mathcal{A}(7) = 0$$

$$\mathcal{A}(1) = \mathcal{A}(2) = \mathcal{A}(3) = \mathcal{A}(4) = 2$$

We embed this ordered set into the 4-gon model of the surface. Figure 43(a) illustrates one way of embedding this ordered set on the polygon model in which the heavy edges indicate the covering edges of the order. We then introduce additional vertices (which have been shown by the filled vertices in Figure 43(b)) in order to provide the required alternation about saddle points. Next we complete a triangulation of the surface which includes all

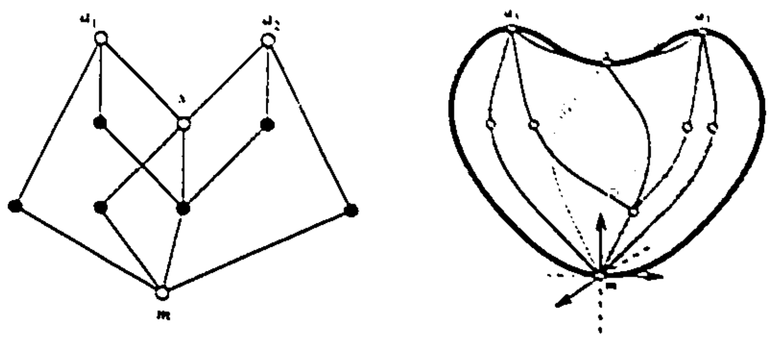


Figure 41:

vertices and edges of the associated ordered set so that it satisfies the necessary conditions. Figure 43(c) illustrates this step. Construction of the covering graph of the ordered set is completed by subdividing all nonessential (transitive) edges. An upward drawing of the constructed ordered set on the plane is depicted in Figure 44. We finally place the critical points of this ordered set on the corresponding critical points of the surface and complete an upward drawing without crossing edges on the original surface. Figure 45 illustrates this stage.

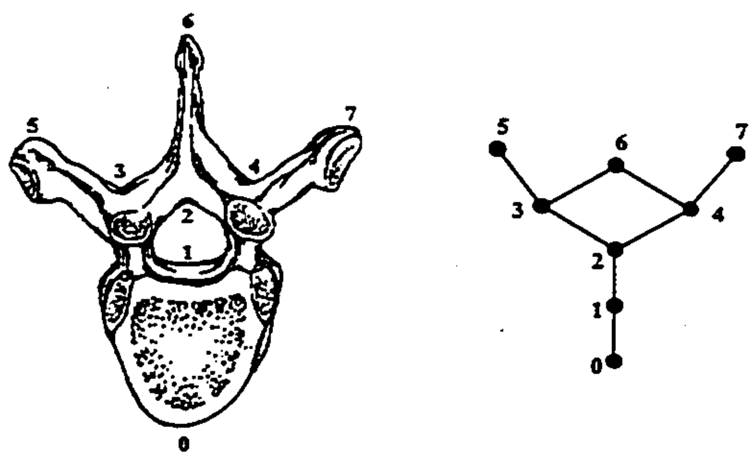
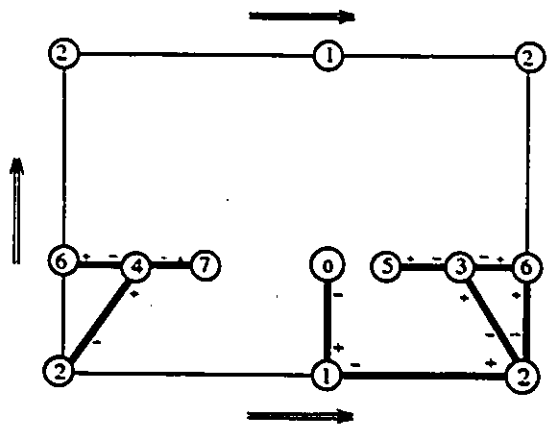
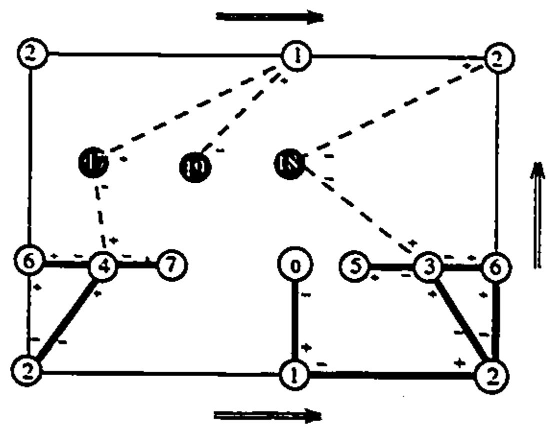


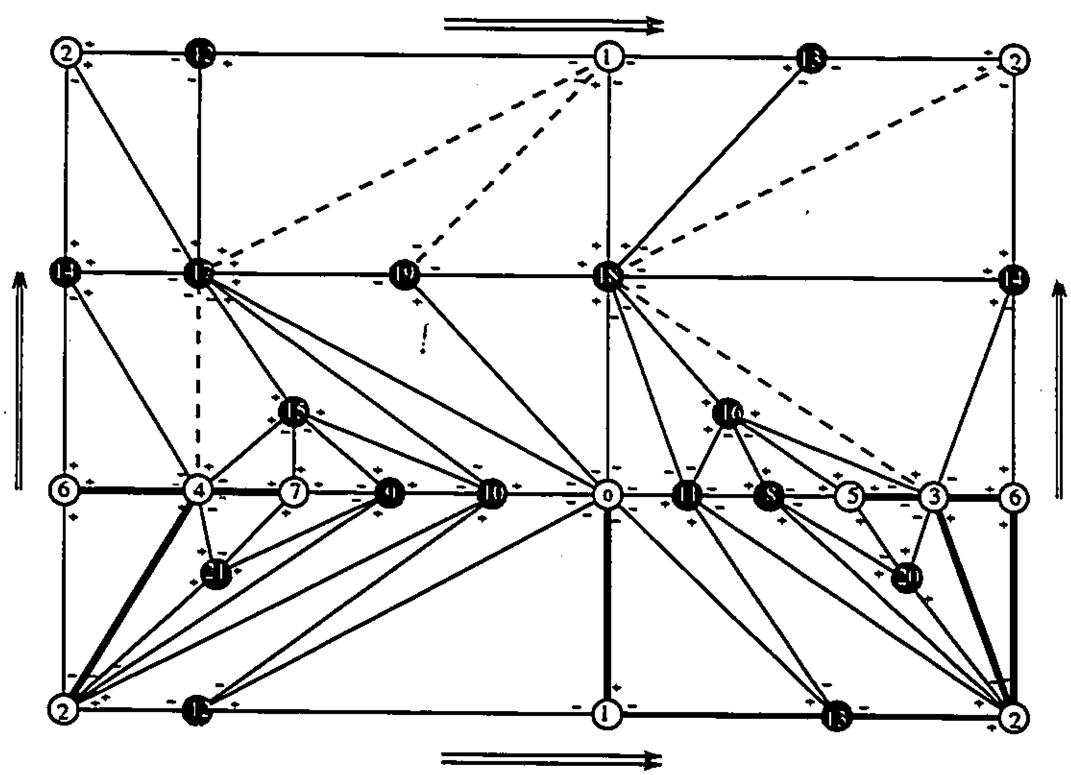
Figure 42:



(a)



(b)



(c)

Figure 43:

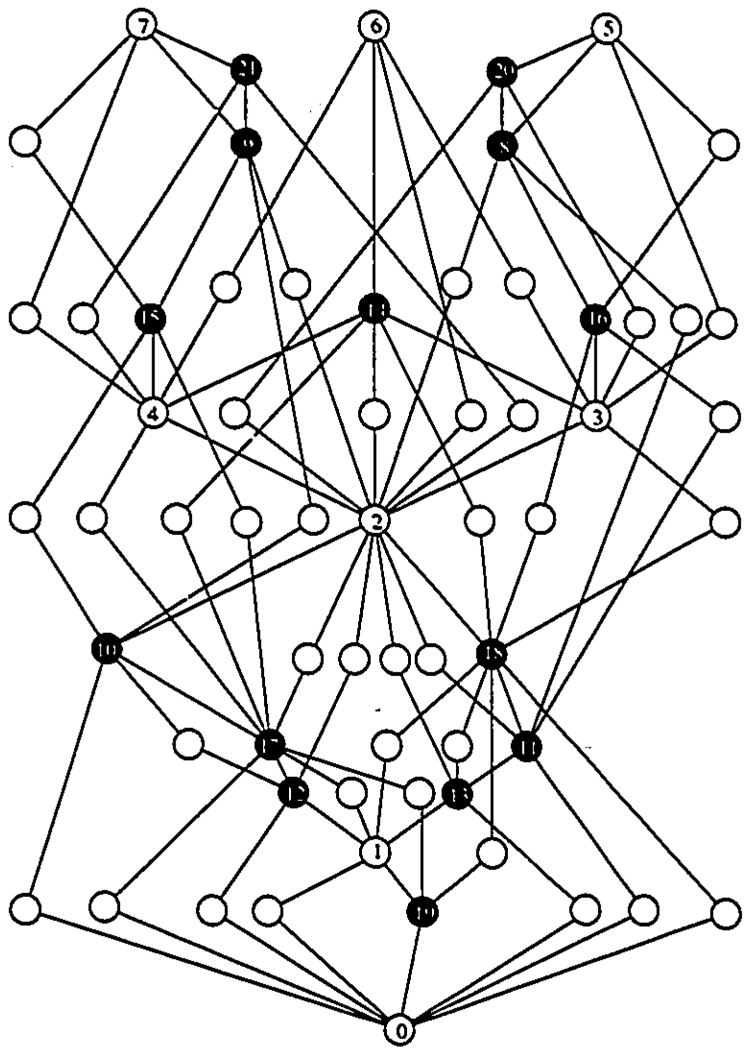


Figure 44:

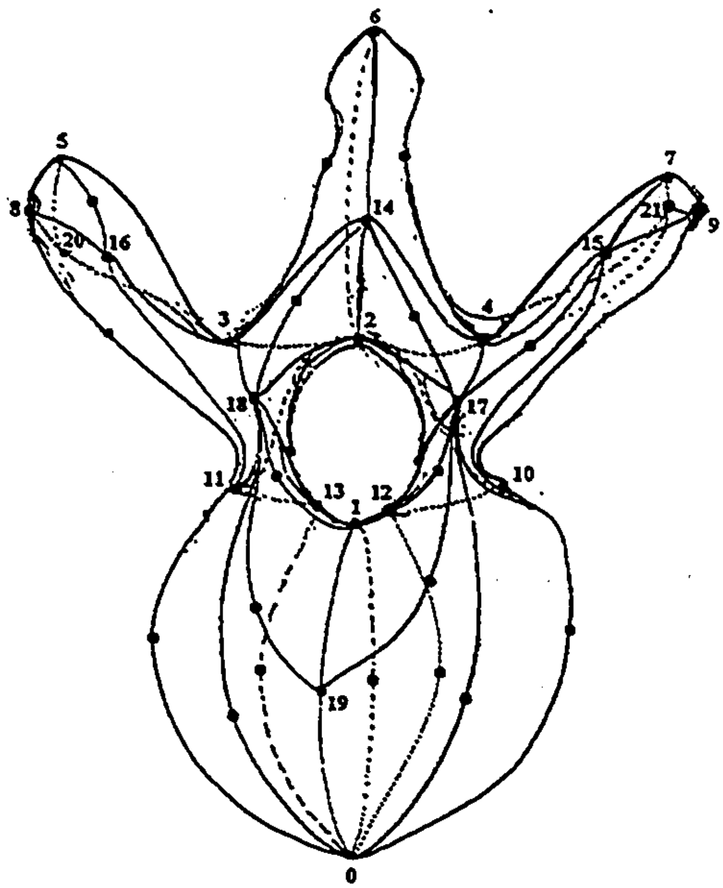


Figure 45:

6.3.3 Triangulation

We shall now investigate a triangulation of a given digraph on a surface, say for instance, a directed covering graph of an ordered set embedded on a surface of genus g. This triangulation must be a *proper* triangulation of the surface on which the directed graph is embedded. That is,

- the triangulation must include all edges and vertices of the directed graph
- any two triangles are identified either along a single edge or at a single vertex, or triangles are completely disjoint.

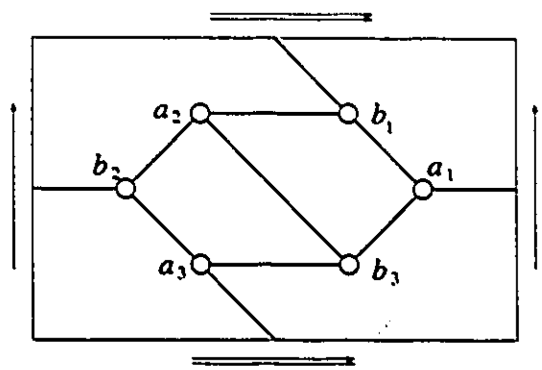


Figure 46: Embedding of $K_{3,3}$ on polygon model of torus.

- each edge is identified with exactly one other edge and the triangles identified at each vertex can always be arranged in a cycle $T_1, T_2, \cdots, T_n, T_1$ so that adjacent triangles are identified along an edge.

Unlike the planar covering graph, it may be impossible to triangulate a covering graph of higher genus without introducing additional vertices. To see this, consider a triangulation of the complete bipartite graph $K_{3,3}$ with bipartition $\mathcal{A} = \{a_1, a_2, a_3\}$ and $\mathcal{B} = \{b_1, b_2, b_3\}$. It is well-known that $K_{3,3}$ has essentially two distinct embeddings on the torus [**Gross and Tucker** (1987)]. Figure 46 illustrates an embedding of $K_{3,3}$ on the polygon model of torus in which the arrows indicate the pairs of edges to be identified.

Although some of the faces can be easily triangulated with necessary conditions, difficulty may arise in triangulating the face:

$$a_1 \sim b_2 \sim a_3 \sim b_1 \sim a_2 \sim b_2 \sim a_1 \sim b_1 \sim a_3 \sim b_3 \sim a_1$$

Since the graph is a complete bipartite graph, each vertex in $\mathcal{A}$ is already connected to all vertices in $\mathcal{B}$. Thus, to avoid parallel edges (edges with the same endpoints), in any triangulation we can only have edges with both endpoints in the same partition. But it can be checked that any triangulation with even this constraint requires parallel edges.

On the other hand, by adding at least one vertex to this face it has a proper triangulation which meets the necessary conditions.

A triangulation need not be unique at all. We now propose an algorithmic way for inserting additional vertices and triangulating an embedded covering graph which satisfies the necessary conditions. The additional vertices might be more than what are really needed, but the procedure works for all covering graphs.

Start with *cover*(P) which is embedded in a surface of genus g. It is convenient to represent this embedding inside a polygon on the plane $z = 0$ in $\mathbf{R}^3$ with $4g$ sides ($g \geq 1$), in which *cover*(P) is drawn planar, possibly with repeated edges and vertices, and in which $2g$ pairs of sides are to be identified.

Of course the embedding contains no loop, an edge with both end points identified, and without loss of generality, we can assume that it has no cut edge, an edge whose removal disconnects the graph. If there is such an edge, then by introducing a new vertex and connecting this additional vertex to the endpoints of the cut edge, we can avoid such edges.

Forming Triangles

First of all, subdivide all edges by introducing new vertices s such that $a \sim s \sim b$, where $a \sim b$ is an edge of the covering graph. In the resulting graph, triangulate all faces by joining successive subdivision points on the boundary of each face. Thus, suppose that $a_1 \sim a_2 \sim a_3 \sim ... \sim a_n \sim a_1$ is the boundary of a face of *cover*(P). Let $a_i \sim s_i \sim a_{i+1}$, $i = 1, 2, ..., n-1$, $a_n \sim s_n \sim a_1$ stand for the successive subdivision points. In the first step of the triangulation, introduce new edges, $s_n \sim s_1, s_1 \sim s_2, \ldots, s_{n-1} \sim s_n$. Each such edge creates one triangle whose vertices consist of two subdivision points and one of the vertices of the face. Thus it creates n triangles, where n is the number of edges on the boundary of the face.

At the next iteration, introduce edges $s_n \sim s_2, s_2 \sim s_4, \ldots, s_{2(k-1)} \sim s_{2k}$, where $2k \leq n$. This creates $[n/2]$ triangles.

At the j^{th} iteration, introduce edges $s_n \sim s_{2^j}, s_{2^j} \sim s_{2(2^j)}, \ldots, s_{2^j(k-1)} \sim s_{2^j k}$ where $2^j k \leq n$, which creates $[n/2^j]$ triangles.

This process will end in the r^{th} iteration if r is least so that $2^{r+1} > n$.

It is clear that in this process by adding each edge we create one triangle and ultimately this procedure triangulates all faces (although we may have some parallel edges). Thus, if φ is a face of degree $d\varphi$ (the number of edges on the boundary of the face), the total number of triangles created in this way is

$$t\varphi = d\varphi + [\frac{d\varphi}{2}] + \cdots + [\frac{d\varphi}{2^j}] + \cdots + [\frac{d\varphi}{2^{[\log_2(d\varphi)]}}] + \alpha$$

where α, with respect to $d\varphi$, is a relatively small constant. In fact the exact number of

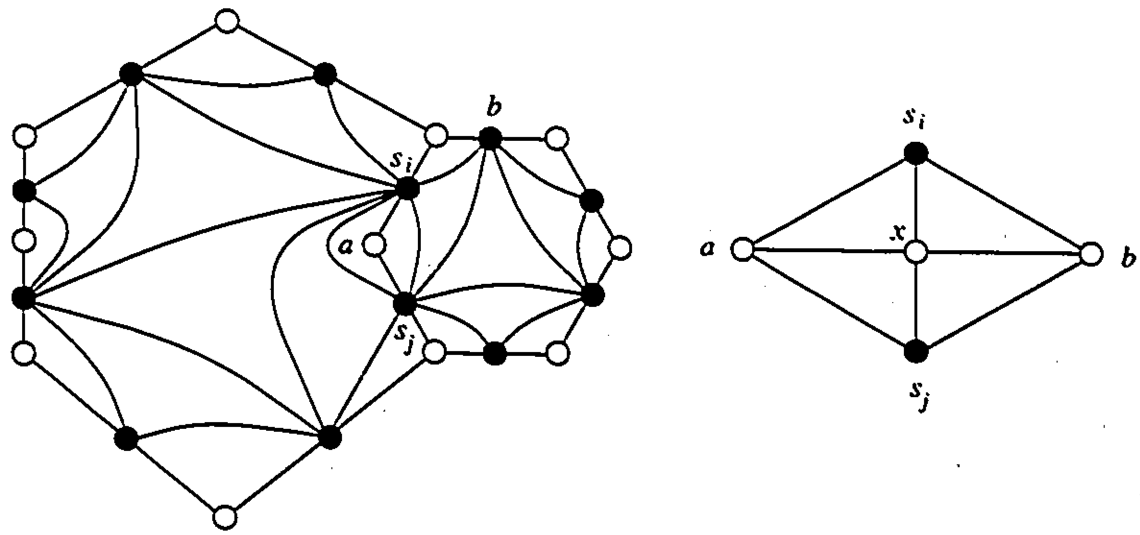


Figure 47:

triangles made at this stage is

$$\sum_{\varphi \in F} t\varphi \quad \text{where} \quad t\varphi = 2d\varphi - 2 \quad \text{is the number of triangles of face } \varphi$$

Correction

If two faces share more than one edge in their boundary, the procedure in the last stage produces some parallel edges. The parallel edges can be marked by recording all edges in an adjacency matrix or using any standard search algorithm.

Correction then starts with replacing any parallel edge $s_i \sim s_j$, in order to modify the triangulation to a proper one. If there exists a parallel edge, then locally we have the situation illustrated in Figure 47. The parallel edge $s_i \sim s_j$ separates two triangles $s_i \sim s_j \sim a \sim s_i$ and $s_i \sim s_j \sim b \sim s_i$. To avoid this edge, we introduce an additional vertex x to subdivide it and two edges $x \sim a$ and $x \sim b$, which connect x to nonadjacent vertices a and b in the separated triangles.

This process does not create any new parallel edge, so, for each parallel edge the correction iterates only once. Therefore, the correction step can be done in time $O(e)$, where e stands for the number of edges.

This can be summarized in the following lemma.

Lemma 6.5 *Fix an embedding of the covering graph of an ordered set on the polygon model of a surface on which it is embedded. We can construct a proper triangulation of this embedding, in time* $O(e)$, *where* e *is the number of edges.*

6.3.4 Existence

In this section we shall show that for each two-dimensional surface S satisfying the conditions stated in Theorem 6.1 we can always construct an ordered set whose upward drawing fits the surface. Thus, assume that we have an embedding of the ordered set *critical*(S) (or its enhanced ordered set) described in Section 6.3.1. We triangulate the surface in such a way that among the vertices of this triangulation, we include all critical points and additional vertices, and among the edges we include segments making up the embedded covering graph. It is always possible to modify the triangulation[4] (by a slight distortion of the surface if necessary) so that no triangle contains two vertices of the same height and all edges are monotonic.

Note that, by the prescribed triangulation, if we direct all edges with respect to their height, all critical vertices remain essential, no other critical point is produced (because all additional vertices are ordinary points), and this triangulation is acyclic.

Now, subdividing edges provides an ordered set whose upward drawing fits the surface. Eventually, Theorem 6.1(i) follows from the construction on the surface which has already an upward drawing of this ordered set on the surface. Theorem 6.1(ii) follows from the particular triangulation constructed above, since it contains all critical points of the surface and preserves their type. Since all other vertices of this triangulation are ordinary points, it therefore provides a one-to-one correspondence between the critical points of the surface and the critical points of the ordered set, that is, *critical*$(S) \subseteq$ *critical*(P).

Next, since all critical points of the constructed ordered set P are essential, an upward drawing of this ordered set on any surface S' with $genus(S') = genus(S)$ requires that S' contains a set of critical points corresponding to the set of critical points of the ordered set P. This implies Theorem 6.1(iii).

Note that, by applying a homeomorphism $\Gamma : S \longrightarrow \bar{S}$ where $\bar{S}$ is the polygon model of S, we can transfer all of the triangles on S to $\bar{S}$ under this homeomorphism, and prescribe a height value for each vertex v with $h'(v) = h \circ \Gamma^{-1}(v)$ and assign signs to the incident edges of vertex v in the direction of edges incident with $\Gamma^{-1}(v)$. This confirms that the covering

[4] A similar idea has been used by [Banchoff (1970)] for the proof of the index theorem.

graph of the desired ordered set can always be constructed on the polygon model and the proof is complete. □

6.4 Equivalent Surfaces and Ordereds Set to Fit Them

The structure of an ordered set whose upward drawing fits a two dimensional surface depends on the genus of the surface and its embedding in $\mathbf{R}^3$.

If a surface S has genus g and (with respect to a fixed height function) it has position $\wp$ (see Page 95 for definition), then we denote it by $S_{g,\wp}$[5]. Figure 48 illustrates some positions of surfaces and the ordered sets associated with the critical points of each position are also depicted in this Figure.

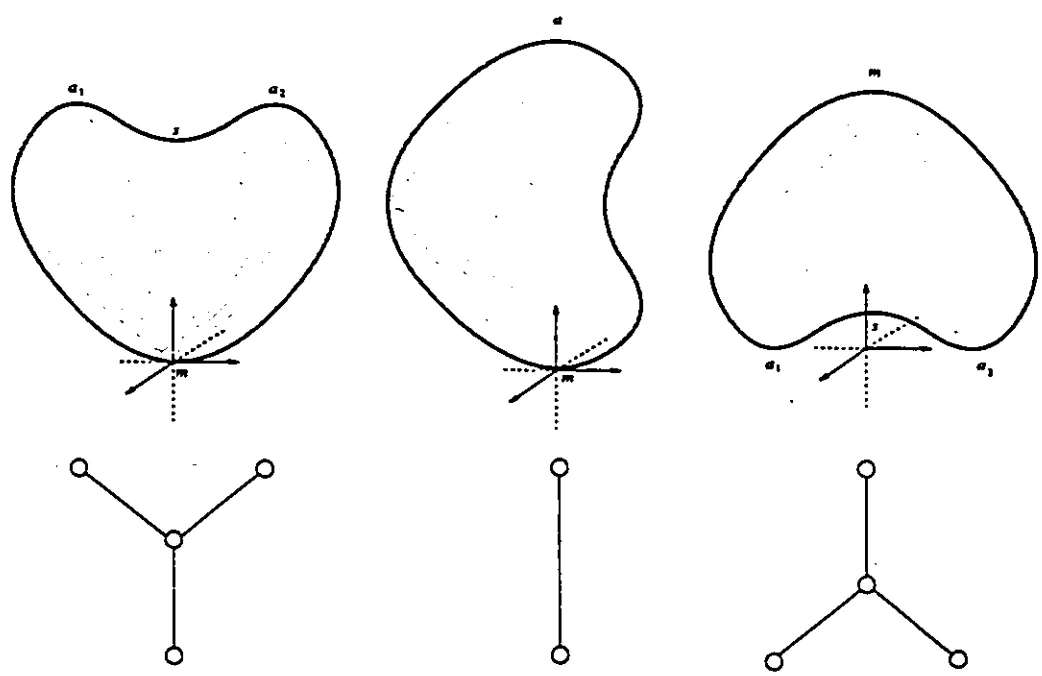


Figure 48:

[5]In this section we fix the height function h to be the projection of the surface on the z-axis.

Classification

The aim here is to classify all positions of surfaces and all ordered sets whose upward drawings fit them.

Fix a height function. Let $S_{g,\mathfrak{p}}$ and $S'_{g',\mathfrak{p}'}$ be positions of surfaces. We can define a relation on all positions of surfaces in three-dimensional space $\mathbf{R}^3$ as follows:

$$S_{g,\mathfrak{p}} \;\mathcal{R}\; S'_{g',\mathfrak{p}'}$$

if and only if $g = g'$ and there is a bijective order preserving map

$$\phi : critical(S_{g,\mathfrak{p}}) \longrightarrow critical(S'_{g',\mathfrak{p}'})$$

which also preserves the type of critical points. Then $S_{g,\mathfrak{p}}$ and $S'_{g',\mathfrak{p}'}$ are said to be *order equivalent*

Observation 6.4.1 *Relation "$\mathcal{R}$" is an equivalence relation on the set of all positions of surfaces.*

Proof Suppose that $S_{g,\mathfrak{p}}$ stands for some position of surface S, if we let $\phi = id$ (the identity map) then we have reflexive property, that is, $S_{g,\mathfrak{p}} \;\mathcal{R}\; S_{g,\mathfrak{p}}$. To show this relation is symmetric, assume that $S_{g,\mathfrak{p}} \;\mathcal{R}\; S'_{g,\mathfrak{p}'}$ then there is a bijective order preserving map $\phi : critical(S_{g,\mathfrak{p}}) \longrightarrow critical(S'_{g,\mathfrak{p}'})$ which preserve the type of elements. Now,

$$\phi^{-1} : critical(S'_{g,\mathfrak{p}'}) \longrightarrow critical(S_{g,\mathfrak{p}})$$

is a bijective order preserving map which preserves the type of elements, thus $S'_{g,\mathfrak{p}'} \;\mathcal{R}\; S_{g,\mathfrak{p}}$.

Finally, we suppose that $S_{g,\mathfrak{p}} \;\mathcal{R}\; S'_{g,\mathfrak{p}'}$ and $S'_{g,\mathfrak{p}'} \;\mathcal{R}\; S''_{g,\mathfrak{p}''}$, then there are bijective order preserving maps

$$\phi : critical(S_{g,\mathfrak{p}}) \longrightarrow critical(S'_{g,\mathfrak{p}'}) \text{ and } \psi : critical(S'_{g,\mathfrak{p}'}) \longrightarrow critical(S''_{g,\mathfrak{p}''})$$

which preserve the type of the elements. Now it is straightforward to see that

$$\psi \circ \phi : critical(S_{g,\mathfrak{p}}) \longrightarrow critical(S''_{g,\mathfrak{p}''})$$

is an order preserving map which preserves the type of elements and as a result it shows the transitivity of relation $\mathcal{R}$.□

We denote the *equivalence class* of $S_{g,\mathfrak{p}}$ by $[S_{g,\mathfrak{p}}]$.

Note that various positions of a specific surface may or may not belong to the same equivalence class with respect to the relation $\mathcal{R}$.

Example 6.5 Any position of the round sphere in three-dimensional space $\mathbf{R}^3$ has two critical points corresponding to the north pole and to the south pole which are maximum and minimum, respectively. The ordered sets associated with the critical points of all positions of the sphere are two-element chains. Therefore, all positions of the sphere belong to the same equivalence class.

Example 6.6 Figure 48 describes positions of surfaces which belong to three different classes.

Example 6.7 The torus is a surface of genus one which belongs to only one equivalence class. Each position of the torus has four critical points, one maximal, one minimal, and two saddle points such that the ordered set $critical(T_{1,\mathfrak{p}})$ is a four-element chain (see Figure 38).

Observation 6.4.2 *If an upward drawing of an ordered set fits one element of an equivalence class* $[S_{g,\mathfrak{p}}]$*, then it fits all elements of this class.*□

This is immediate from the construction of fit ordered sets.□

An ordered set whose upward drawing fits a position of a surface is not unique. Regarding the above observation, we note that upward drawing of all ordered sets which fit $S_{g,\mathfrak{p}}$, they fit all members of the equivalence class $[S_{g,\mathfrak{p}}]$. Thus, let $\mathcal{P}$ stand for the set of all ordered sets whose upward drawing fit positions of surfaces, and let $P_{S_{g,\mathfrak{p}}}$ stand for the ordered set whose upward drawing fits the position $S_{g,\mathfrak{p}}$. We can define a relation on $\mathcal{P}$ as follows:

$$P_{S_{g,\mathfrak{p}}} \ \mathcal{Z} \ Q_{S'_{g',\mathfrak{p}'}} \quad \text{if and only if} \quad S_{g,\mathfrak{p}} \ \mathcal{R} \ S'_{g',\mathfrak{p}'} \tag{24}$$

where, $P_{S_{g,\mathfrak{p}}}, Q_{S'_{g',\mathfrak{p}'}} \in \mathcal{P}$

Observation 6.4.3 *Relation "$\mathcal{Z}$" in (24), is an equivalence relation on $\mathcal{P}$.*

Proof Let $P_{S_{g,\mathfrak{p}}}$ be any element of $\mathcal{P}$. Since by Observation 6.4.1 $S_{g,\mathfrak{p}} \ \mathcal{R} \ S_{g,\mathfrak{p}}$, then by definition $P_{S_{g,\mathfrak{p}}} \ \mathcal{Z} \ P_{S_{g,\mathfrak{p}}}$. Thus, relation $\mathcal{Z}$ is reflexive. To show that $\mathcal{Z}$ is symmetric, assume that $P_{S_{g,\mathfrak{p}}} \ \mathcal{Z} \ Q_{S'_{g',\mathfrak{p}'}}$ then by definition $S_{g,\mathfrak{p}} \ \mathcal{R} \ S'_{g',\mathfrak{p}'}$. Since $\mathcal{R}$ is a symmetric relation then $S'_{g',\mathfrak{p}'} \ \mathcal{R} \ S_{g,\mathfrak{p}}$ and therefore by definition $Q_{S'_{g',\mathfrak{p}'}} \ \mathcal{Z} \ P_{S_{g,\mathfrak{p}}}$. Similarly, we can prove the transitivity of $\mathcal{Z}$ by using the transitivity of $\mathcal{R}$.□

We denote the equivalence class of $P_{S_{g,p}}$ with respect to Ξ by $[P_{S_{g,p}}]$. Clearly, the upward drawing of each element of class $[P_{S_{g,p}}]$ fits each element of class $[S_{g,p}]$. The following result, which follows from our observations so far, will prove this significant fact that for each class $[S_{g,p}]$ of positions of surfaces there is a unique class $[P_{S_{g,p}}]$ of ordered sets whose upward drawings fit the elements of the former class.

Theorem 6.6 *There is a one-to-one correspondence between the equivalence classes of all positions of surfaces and the equivalence classes of ordered sets whose upward drawings fit surfaces.*

Proof Let $\sum$ stand for the set of all equivalence classes with respect to '$\mathcal{R}$' and $\prod$ stand for the set of all equivalence classes with respect to 'Ξ'. We define the map

$$\Gamma : \sum \longrightarrow \prod \qquad \text{where } \Gamma([P_{S_{g,p}}]) = [S_{g,p}] \tag{25}$$

Γ is a well-defined map. For, if $[P_{S_{g,p}}] = [Q_{S'_{g',p'}}]$ then $P_{S_{g,p}} \; \Xi \; Q_{S'_{g',p'}}$ and therefore by definition $S_{g,p} \; \mathcal{R} \; S'_{g,p'}$, thus $[S_{g,p}] = [S'_{g',p'}]$. Next assume that $\Gamma([P_{S_{g,p}}]) = \Gamma([Q_{S'_{g',p'}}])$. Then by (25), $[S_{g,p}] = [S'_{g',p'}]$ and therefore $S_{g,p} \; \mathcal{R} \; S'_{g',p'}$. Thus, by definition $P_{S_{g,p}} \; \Xi \; Q_{S'_{g',p'}}$ therefore $[P_{S_{g,p}}] = [Q_{S'_{g',p'}}]$, that is, Γ is one-to-one.□

Chapter 7

Realizable Surfaces

7.1 Introduction

The lifting procedure (cf. Section 2.3) constructs an upward drawing of an ordered set P on a surface S with

$$genus(S) = genus(cover(P))$$

Nevertheless, if $genus(P) > 1$, this lifting procedure does not guarantee that the constructed surface itself is embeddable in $\mathbf{R}^3$. That is, for a given ordered set whose covering graph has genus greater than one, the *existence* of upward drawings on surfaces "realizable" in $\mathbf{R}^3$ is an *unsolved* problem. In other words, if we restrict our attention to the embedding of ordered sets on surfaces realizable in $\mathbf{R}^3$, the problem to determine the *genus* of an ordered set is still *unsolved*.

In this last chapter we discuss some of the open problems about surface embeddings of ordered sets and highlight some new directions for further research. We conjecture that the ordered set of closed subspaces of the projective plane of order two, denoted $PP(2)$, has no upward drawing without crossing edges on a realizable surface whose genus equals the genus of its covering graph.

In this chapter we actually construct[1] an embedding of the covering graph of this ordered set on the polygon model of a surface of genus three.

[1] We are grateful to D. Archdeacon, B. Richter and J. Siran for pointing out that the covering graph of this ordered set must have genus three.

7.1.1 Existence of Realizable embeddings

Fix an upward drawing of an ordered set P on the surface of a sphere, perhaps with crossing edges. Suppose that the edge e_1 crosses the edge e_2. On both sides of e_2 close to the crossing point remove two small disks (each small enough that it touches no edge of P). Attach a handle to the removed disks and re-route the edge e_1 on this handle such that it traverses the handle instead of crossing the edge e_2 while it preserves the monotonicity [**Reuter** and **Rival** (1991)].

With this procedure one can remove all crossings, one at a time, and obtain a surface S realizable in $\mathbf{R}^3$, on which the ordered set P is embedded (not though necessarily cellularly). Thus, if $u(P)$ is any upward drawing of P on the sphere (which is not necessarily an embedding), and $cross(u(P))$ stands for the number of crossing edges of $u(P)$ then

$$genus(P) \leq cross(u(P))$$

Thus, if

$$cross(P) = \min\{\ cross(u(P)) \mid u(P) \text{ is an upward drawing of } P \text{ on the sphere}\}$$

then

$$genus(P) \leq cross(P)$$

Note that although this procedure gives some intuitive idea of the existence of a realizable embedding of an ordered set, there are many examples in which the genus is far from the crossing number. Indeed, we can construct ordered sets of genus zero with a large number of crossing edges. To this end, recall the spirals of type $S(i,j)$ (cf. Section 4.3), for a given positive integer K, where i and j are positive integers such that $K = |i-j|$. The ordered set $S(i,j)$ has at least $|i-j|$ saddle points and corresponding to each saddle point there is at least one crossing of edges, that is, $cross(S(i,j)) \geq K$. On the other hand, this ordered set has an upward drawing without crossing edges on a surface of genus zero with K saddle points.

Recall that the crossing number for the covering graph of an ordered set is determined by the genus of the surface on which it is embedded. But the crossing number for the upward drawing of the ordered sets may vary from one surface to the other one — even if the surface has genus zero. Thus, for instance, the ordered set $\mathbf{Q}^3$ of the three-dimensional cube has crossing number ≥ 1 on the plane while its crossing number on the sphere is zero.

In conclusion, the crossing number for an ordered set depends on the surface on which the upward drawing occurs, and thus it has to be defined with respect to a fixed embedded surface.

Problem 7.1 *Fix an ordered set. Fix a surface embedded in three-dimensional space* $\mathbf{R}^3$. *Find (or estimate) the minimum number of crossing edges required for an upward drawing of this ordered set on this surface.*

Of special interest are the plane and the round sphere. If we have a saddle point in a planar embedding of the covering graph of an ordered set then clearly there is no similar planar upward drawing of this ordered set on the plane (cf. Section 1.4), thus

$$saddle(P) \leq cross(P)$$

Of course, since upward planarity testing is NP-complete, the crossing number, too, must be an intractable problem, otherwise planarity can be determined through testing for crossing number (an ordered set is planar if and only if its crossing number on the plane is zero). On the other hand, for ordered sets whose upward drawings on a specific surface contain some crossing of edges, reducing the number of crossing edges is of special interest.

7.1.2 Examples of Embeddings

For some classes of ordered sets there are interesting results according to which the embedding of each member of the class is known. For instance, the embedding of each member of the class of spirals of type $S(i,j)$ on surfaces of genus zero is fully determined (cf. Theorem 4.2).

As another example, K. Reuter and I. Rival have studied the genus of a class of lattices each called a *spider*. They showed that the genus of the k-spider is an increasing function of k.

Theorem 7.1 [Reuter and Rival (1991)] *The order genus of the k-spider is* $\lceil \frac{k-2}{4} \rceil$.

Problem 7.2 *Characterize a class of ordered sets whose genus (and/or their embeddings) can be calculated using "some" parameters of the ordered set.*

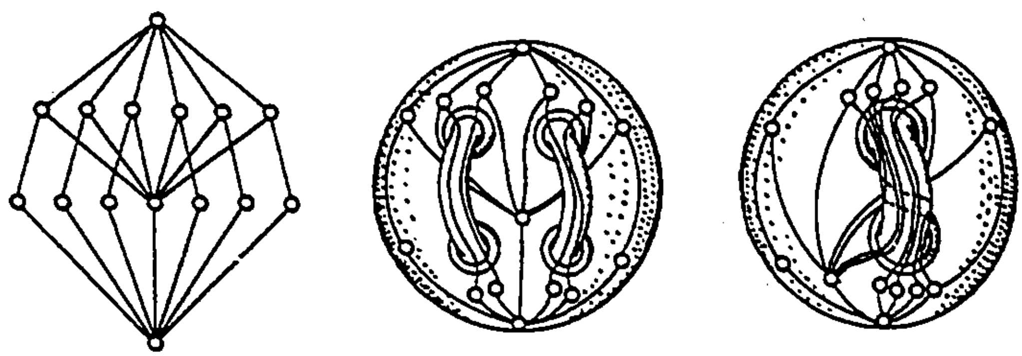


Figure 49:

Apart from the fact that, for lattices, planarity is equivalent to dimension[2] at most two, there is no further apparent connection between order genus and order dimension. For instance, the k-spider has order dimension at most three [Rival (1993)]. In contrast, Theorem 7.1 shows that, the order genus of the k-spider grows with k.

Problem 7.3 *Find nontrivial parameters of the ordered sets (lattices) which grow monotonically with the genus of the ordered sets (lattices).*

7.2 Projective Planes

In this section we study the ordered set of closed subspaces of the projective plane of order two. We denote this ordered set by $PP(2)$. For this ordered set there are two approaches.

1. Present an upward drawing of $PP(2)$ on a surface S realizable in $\mathbf{R}^3$ with $genus(S) = genus(cover(PP(2)))$.

2. Prove that there is no upward drawing of $PP(2)$ on a surface S realizable in $\mathbf{R}^3$ with $genus(S) = genus(cover(PP(2)))$.

Certainly, any attack on this problem requires some tools which are useful in the study of upward drawings on surfaces and the genus of ordered sets.

[2]The *dimension* of an ordered set $P = (X, \mathcal{R})$, denoted $dim(X, \mathcal{R})$, is the least positive integer t for which there exists a family $\{L_1, L_2, \ldots L_t\}$ of linear extensions of $\mathcal{R}$ so that $\mathcal{R} = \bigcap_{j=1}^{t} L_j$.

7.2.1 Structure of $PP(2)$

Consider a collection $\mathcal{P}$ of seven elements called points, and a set $\mathcal{L}$ of 3-element subsets of $\mathcal{P}$ called lines, having the property that any two points lie on (are incident with) exactly one line.

To fix the notation, we assume that the members of $\mathcal{P}$ are labelled by the elements of the finite group $\mathbf{Z}_7$ of integers modulo seven, that is, $\mathcal{P} = \{0,1,2,3,4,5,6\}$ and members of $\mathcal{L}$ are denoted by strings of three digits corresponding to their elements. Thus, if $\ell = \{i,j,k\} \in \mathcal{L}$ is a line then it is denoted by $\ell = ijk$. Note that by this definition

1. $\mathcal{L}$ has also seven elements which can be generated, for instance, by $\ell_i = i(i+1)(i+3)$ where $i \in \mathbf{Z}_7$ and $+$ is the addition operation in $\mathbf{Z}_7$,
2. any point is incident with three lines and
3. two lines meet in a unique point.

We shall now define an ordered set $PP(2) = (\mathcal{P} \cup \mathcal{L} \cup \{t,b\}, \leq)$ by b is its bottom, t is its top, and every point p is below every line with which it is incident. Thus all covering relations are as follows.

$$b \prec i \quad \text{where} \quad i \in \mathcal{P}$$

$$t \succ \ell \quad \text{where} \quad \ell \in \mathcal{L}$$

$$\ell = ijk \succ i,j,k$$

An upward drawing of ordered set $PP(2)$ on the plane is depicted in Figure 50.

7.2.2 Embedding of $cover(PP(2))$

To study the upward drawing of $PP(2)$, in particular, to have a lower bound for the genus of this ordered set, it is necessary to obtain an embedding of its covering graph. This does not solve the order embedding but it could be interesting in its own right and may lead to some insights and new directions in the order embedding. We need some terminology.

A *rotation* at a vertex v of a graph is an ordered list, unique up to a cyclic permutation, of the edges incident on v.

A *(pure) rotation system* is an assignment of a rotation to each vertex for a graph in which every edge is "orientation-preserving" (cf. [**Gross and Tucker** (1987)]). The following result shows that an embedding is determined by a rotation system for the graph.

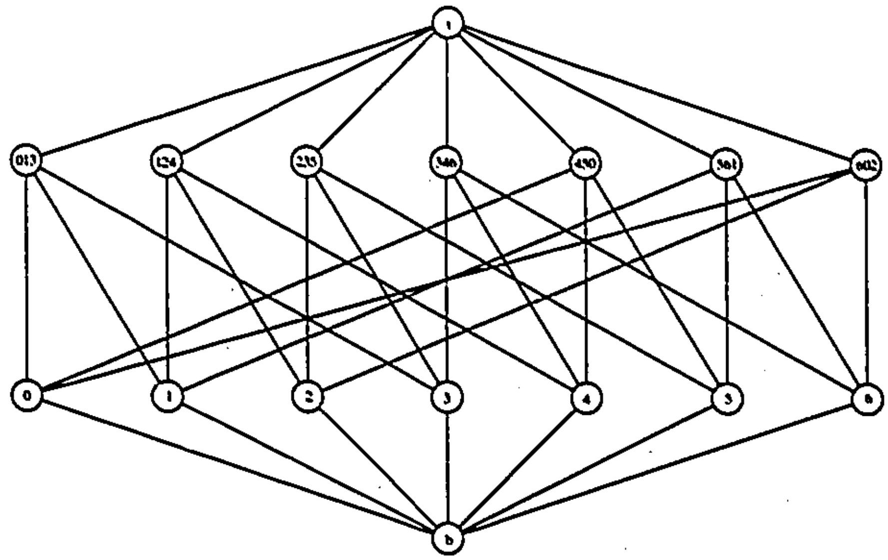


Figure 50: An upward drawing of $PP(2)$ on the plane

Theorem 7.2 **[Heffter (1891)] [Edmonds (1960)]** *Every pure rotation system for a graph G induces (up to orientation-preserving equivalence) a unique embedding of G into an orientable surface. Conversely, every embedding of a graph G into an orientable surface induces a unique pure rotation system for G.* □

Since we want to obtain a minimal embedding of the covering graph (which is a cellular embedding in an orientable surface with least genus) then according to the Euler formula

$$genus(cover(PP(2))) = 1-\frac{1}{2}[\mid V(cover(PP(2))) \mid - \mid E(cover(PP(2))) \mid + \mid F(cover(PP(2))) \mid] \quad (26)$$

we must construct an embedding with the most number of faces. We proceed by defining a rotation about each vertex of $cover(PP(2))$ as follows.

Each vertex corresponding to a line in $\mathcal{L}$ is connected to the top element t and three vertices in $\mathcal{P}$. Let $rotation(\ell)$ stand for a counterclockwise rotation about vertex ℓ. We define a rotation about each such vertex in $\mathcal{L}$ according to the following rule.

$$rotation(ijk): \quad t \quad i \quad j \quad k$$

The following table lists the rotation about all vertices in $\mathcal{L}$.

$rotation(013):$	t	0	1	3
$rotation(124):$	t	1	2	4
$rotation(235):$	t	2	3	5
$rotation(346):$	t	3	4	6
$rotation(450):$	t	4	5	0
$rotation(561):$	t	5	6	1
$rotation(602):$	t	6	0	2

If j is a vertex corresponding to a point in $\mathcal{P}$, it is connected to the bottom element b and three vertices of the set $\mathcal{L}$. We define a rotation at each vertex in $\mathcal{P}$ according to the following rule.

$$rotation(j): \quad b \quad j\alpha\beta \quad \alpha'\beta'j \quad \alpha''j\beta''$$

The following table lists the rotations about all vertices corresponding to the points in $\mathcal{P}$.

$rotation(0):$	b	013	450	602
$rotation(1):$	b	124	561	013
$rotation(2):$	b	235	602	124
$rotation(3):$	b	346	013	235
$rotation(4):$	b	450	124	346
$rotation(5):$	b	561	235	450
$rotation(6):$	b	602	346	561

We finally define the rotation about the bottom b and the top t as follows.

$rotation(b):$	6	5	4	3	2	1	0
$rotation(t):$	346	013	450	124	561	235	602

This completes a rotation system on the $cover(PP(2))$. The embedding is determined by this rotation system (using, for instance, the so-called *face tracing* algorithm) which consists of seven faces each of degree four sharing the vertex b, seven faces each of degree four sharing the vertex t, and one face of degree fourteen whose boundary cycle is a Hamiltonian cycle in the subgraph $cover(PP(2))\backslash\{t,b\}$.

All faces of degree four which have the bottom vertex b as their corner are listed in the following table (see also Figure 51).

$$f_0:\quad b \sim 0 \sim 013 \sim 1 \sim b$$
$$f_1:\quad b \sim 1 \sim 124 \sim 2 \sim b$$
$$f_2:\quad b \sim 2 \sim 235 \sim 3 \sim b$$
$$f_3:\quad b \sim 3 \sim 346 \sim 4 \sim b$$
$$f_4:\quad b \sim 4 \sim 450 \sim 5 \sim b$$
$$f_5:\quad b \sim 5 \sim 561 \sim 6 \sim b$$
$$f_6:\quad b \sim 6 \sim 602 \sim 0 \sim b$$

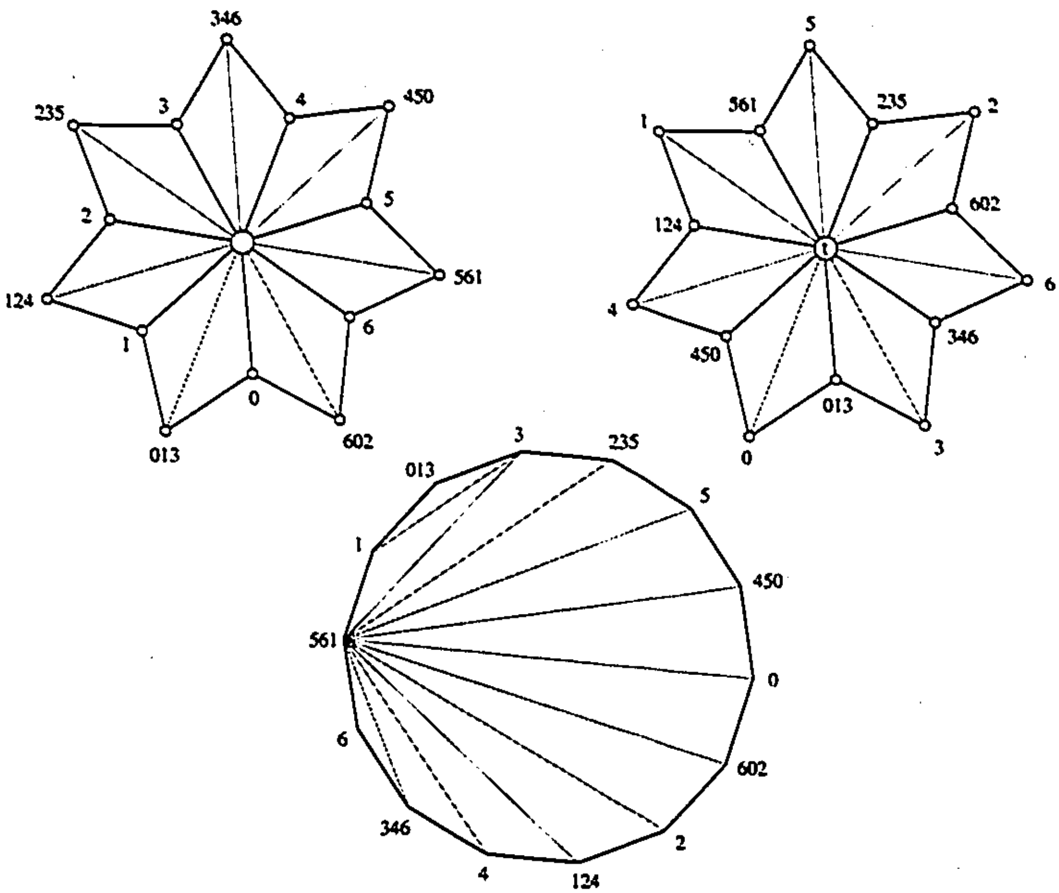


Figure 51: All triangulated faces of the projective plane

All faces of degree four containing vertex t as their corner are listed as follows (see also Figure 51).

$$g_0: \quad t \sim 013 \sim 0 \sim 450 \sim t$$
$$g_1: \quad t \sim 124 \sim 1 \sim 561 \sim t$$
$$g_2: \quad t \sim 235 \sim 2 \sim 602 \sim t$$
$$g_3: \quad t \sim 346 \sim 3 \sim 013 \sim t$$
$$g_4: \quad t \sim 450 \sim 4 \sim 124 \sim t$$
$$g_5: \quad t \sim 561 \sim 5 \sim 235 \sim t$$
$$g_6: \quad t \sim 602 \sim 6 \sim 346 \sim t$$

The single face of degree fourteen has all vertices corresponding to $\mathcal{P} \cup \mathcal{L}$ as its corner.

$$h: \quad 013 \sim 1 \sim 561 \sim 6 \sim 346 \sim 4 \sim 124 \sim 2 \sim 602 \sim 0 \sim 450 \sim 5 \sim 235 \sim 3 \sim 013$$

Now using the Euler Formula (26), since for this embedding we have

$$\mid V(cover(PP(2))) \mid = 16, \mid E(cover(PP(2))) \mid = 35 \text{ and } \mid F(cover(PP(2))) \mid = 15,$$

it requires a surface of genus three. We shall now show that in fact

$$genus(cover(PP(2))) > 2$$

proving that this embedding is a minimal embedding of $cover(PP(2))$.

First of all we note that the covering graph of $PP(2)$ is a bipartite graph with bipartition $\mathcal{A} = \mathcal{P} \cup \{t\}$ and $\mathcal{B} = \mathcal{L} \cup \{b\}$. Thus, it contains no odd cycle. It follows that in any embedding for each face f

$$degree(f) \geq 4 \tag{27}$$

Obviously, the graph is nonplanar. If there is an embedding of $cover(PP(2))$ on a surface of genus one then using (26) implies $\mid F(cover(PP(2))) \mid = 19$. We recall that

$$\sum_{i=1}^{\mid F(cover(PP(2))) \mid} degree(f_i) = 2\mid E(cover(PP(2))) \mid \quad \text{where} \quad f_i \in F(cover(PP(2))) \tag{28}$$

Now by substituting (27) in (28) we have

$$4\mid F(cover(PP(2))) \mid \leq 2\mid E(cover(PP(2))) \mid \quad \text{or} \quad 4(19) \leq 2(35)$$

which is a contradiction. Thus $genus(cover(PP(2))) \geq 2$.

Next if f is any face whose boundary cycle does not contain vertices b and t then by the structure of $PP(2)$, $degree(f) \geq 6$. For, if $f = x_1 \sim x_2 \sim x_3 \sim x_4 \sim x_1$ is such a face with $degree(f) = 4$, and, for instance, $x_2, x_4 \in P$ then the pair x_2 and x_4 are incident with two lines x_1 and x_3 which is not possible. On the other hand, since $degree(t) = 7$ in any embedding there are at most seven faces which have vertex t as their corner. Similarly, there are at most seven faces which have vertex b as their corner.

Next if there is an embedding of $cover(PP(2))$ on a surface of genus two then using (26) implies $| F(cover(PP(2))) | = 17$. Now a similar argument, using (28) implies that

$$2| E(cover(PP(2))) | \geq 4| F_1 | + 6| F_2 |$$

where for every $f \in F_1$, $degree(f) \geq 4$ and for every $f \in F_2$, $degree(f) \geq 6$. Thus

$$2| E(cover(PP(2))) | \geq 4(14) + 6(3)$$

which is a contradiction. It follows that $genus(cover(PP(2))) > 2$.

To embed $cover(PP(2))$ on the polygon model of a surface of genus three, we follow the procedure of the proof of the classification Theorem for surfaces (cf. [Massey (1991)]). To adapt this with the *cut and paste* procedure in the proof of this theorem, we paste all triangulated faces along their common edges in order to obtain a polygon on the plane with possibly repeated edges on the boundary of this polygon (see Figure 52 where the heavy edges represent the edges of $cover(PP(2))$). Next, we follow the procedure of reducing the resultant polygon to a standard polygon model so that in all steps we preserve the edges of the original graph (which is not necessary for the proof of the classification theorem). In this way, at the final stage, when we construct the standard polygon model of the surface, we ultimately come up with an embedding of the $cover(PP(2))$ on this polygon model. Figure (53) illustrates the final embedding of the triangulated covering graph of $PP(2)$ on the standard polygon model of an orientable two-dimensional surface of genus three. Note that we have labelled the boundary edges with $\gamma, \delta, \xi, \rho, \psi$ and τ so that pairs of edges with the same label are target edges to be identified. The boundary cycle of this polygon model (with respect to a fixed direction) is

$$\gamma\, \delta\, \gamma^{-1}\, \delta^{-1}\, \xi\, \rho\, \xi^{-1}\, \rho^{-1}\, \psi\, \tau\, \psi^{-1}\, \tau^{-1}$$

which is homeomorphic to a connected sum of three tori. In Figure 54 all edges for triangulation are deleted and the actual embedding of $cover(PP(2))$ is presented.

In general for an embedding of an ordered set P on a surface realizable in $\mathbf{R}^3$

$$genus(P) \geq genus(cover(P))$$

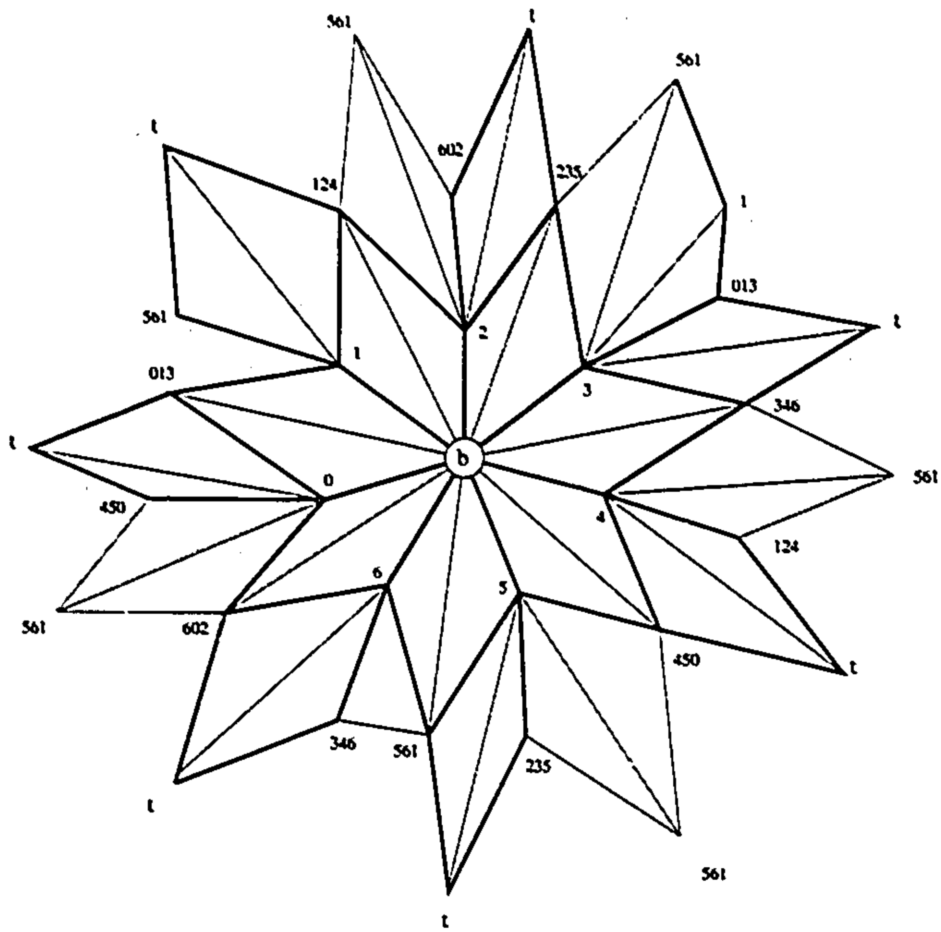


Figure 52:

For many examples, including $PP(2)$, it is possible to obtain an upward drawing of an ordered set P on a surface S with

$$genus(S) = genus(cover(P)) + 1$$

Problem 7.4 *Determine the smallest positive integer α such that any ordered set P can be drawn upward on a surface S realizable in $\mathbf{R}^3$ and*

$$genus(S) = genus(cover(P)) + \alpha$$

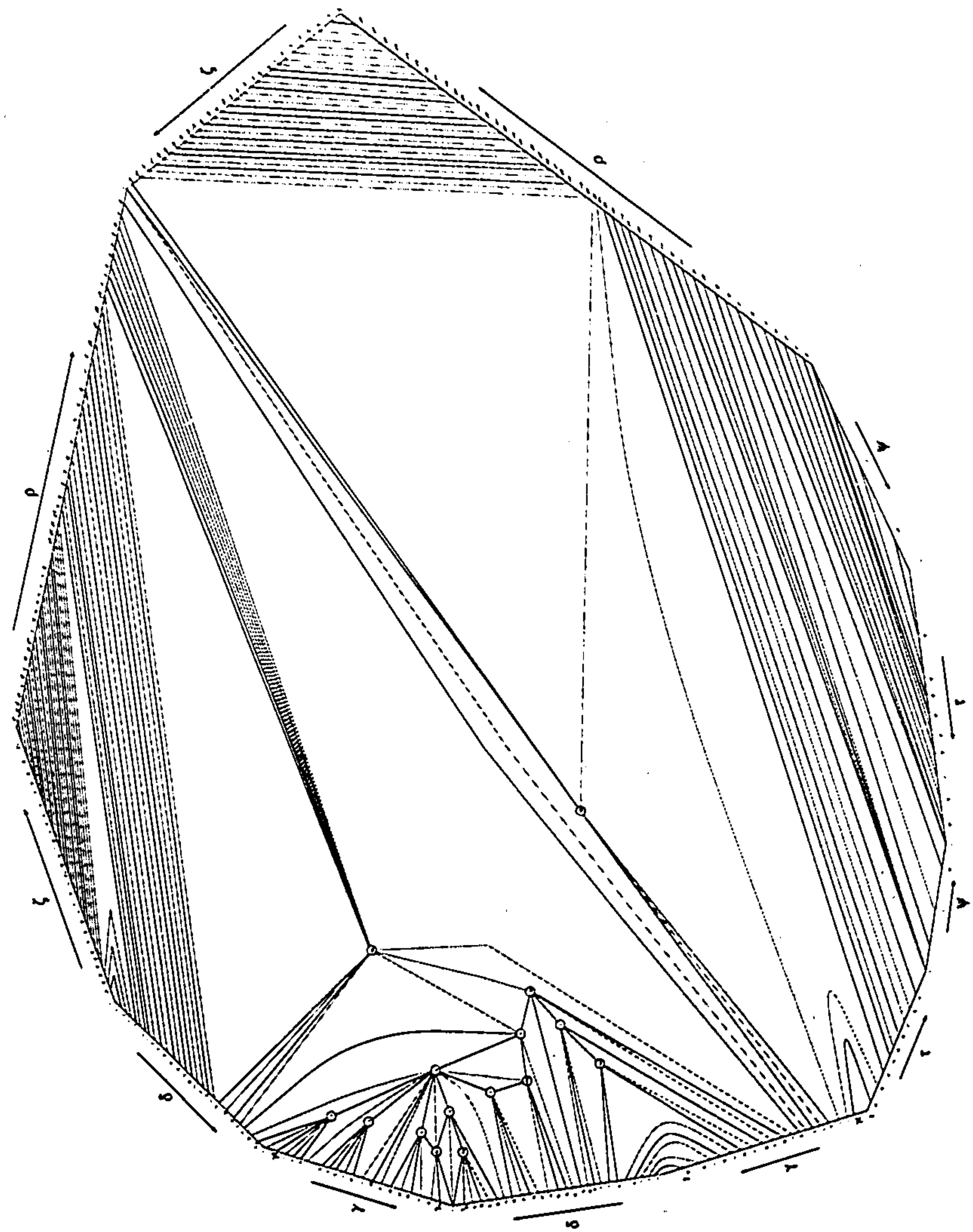


Figure 53: An embedding of the triangulated covering graph of $PP(2)$

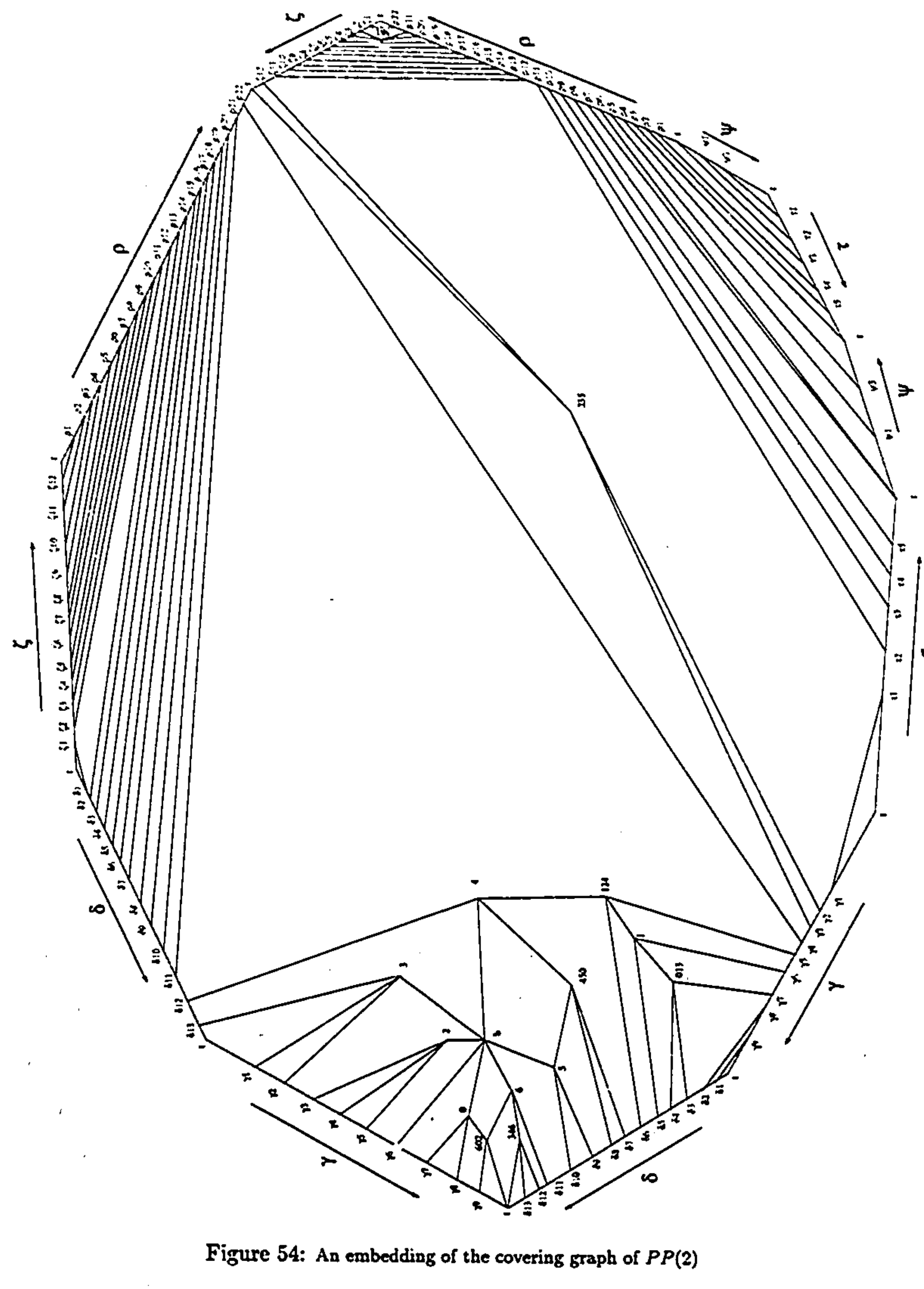


Figure 54: An embedding of the covering graph of $PP(2)$

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