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**ENDOGENOUS GROWTH: A SEARCH MODEL
FOR NEW TECHNOLOGY WITH APPLICATIONS
TO INTERNATIONAL TRADE**

By

MOHAMMAD HADI ZAHEDI VAFA

A thesis presented to the School of Graduate Studies and Research
of the University of Ottawa in partial fulfillment of the requirements
for the degree of Doctor of Philosophy in Economics

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In The Name Of God, The Most Compassionate, The Merciful

Dedicated to

HAZRAT FATEMEH ZAHRA (as)

Peace be upon you, Oh daughter of the Messenger of God! Peace be upon you, Oh daughter of the possessor of outstanding merit above all Prophets and messengers and angels! Peace be upon you, Oh Leader of the Women of the Worlds. Peace be upon you, Oh Wife of the Vicegerent of God! Peace be upon you, Oh mother of HASSAN and H'USSEIN, the Leaders of the Youth of Paradise!

Peace be upon you, Oh most truthful and martyr. Peace be upon you, who was pleased to resign to the will of God and God was pleased with you. Peace be upon you, Oh possessor of outstanding merit of above others, and the pure. Peace be upon you, who is holy and guarded herself against sins! Peace be upon you, who is knowledgeable and an authority on traditions of the Prophet.



*This Thesis was completed on Her Majesty's Martyrdom Anniversary
(Jamadi I 13th, 1422)*

ABSTRACT

A model in which firms carry out a sequential search to raise their technological level is formulated under the overlapping-generations framework. Technological improvements are random outcomes of a sequential search process, which is financed by the capital raised by firms. The amount of capital resources spent on searching for new technologies by a firm is endogenous and depends on the search undertaken by the firm and its capital. This model identifies some factors that affect the interactions between the search process and the production technology.

The basic model is extended to that of a two-sector economy that encompasses a high-technology sector and a more traditional sector. In the extended model, capital is sector-specific. Firms in the high-technology sector can raise their technological level by engaging in R&D, while those in the traditional sector have no more prospects for improving their technological capacity. The two sectors differ from each other in terms of technological level and factor intensity. Finally, the two-sector model is extended in to the world of international trade to analyze the impact of technological change on trade in goods and capital flow.

Keywords: Endogenous Growth, Search Theory, Innovation, Technology, Two-Sectors Growth, International Trade, And Dynamic Comparative Advantage.
JEL: O41, O31, D9, D83, F21, F43, L16.

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M. H. ZAHEDI VAFA

INTRODUCTION

Once one starts to think about [growth], it is hard to think about anything else. R. Lucas

Economists have long been interested in searching for the causes and effects of the growth of income and wealth of countries. And the development of economic growth theories has passed through many stages; periods of booms in different directions, and periods of neglect. The relevant literature gradually divided into two branches: the economics of growth, and the economics of development, with the former emphasizing the use of rigorous mathematical models describing the quantitative relationships between economic variables, and the latter paying more attention to economic structures and institutions.

In Chapter 1, we briefly discuss developments in the theory of economic growth over the past few decades. We will start from the neoclassical paradigm—identified mostly with the works of Solow (1956) and Swan (1956), and was later extended by Cass (1965) and Koopmans (1965). The basic assumptions underlying the neoclassical growth model are as follows: (i) the economy can be adequately characterized by a constant returns to scale production function with diminishing returns to capital and labor, (ii) firms are price-takers in a competitive market place, and (iii) technological change is entirely exogenous.

Since this model assumes diminishing returns to capital, there is a limit to how much capital accumulation can add to output per capita. Hence, the only way to increase output per worker in the long run is to have technological change. This is a major weakness of the neoclassical growth model, since long-run growth is exogenous.

With the second paradigm starting in the 1980s, a number of more sophisticated growth models have been developed. A key feature of them is that, unlike the neoclassical model, technological change is not assumed to be exogenous. In particular, the accumulation of knowledge plays a key role in driving growth.

There are essentially two strands in the endogenous growth literature. The first one starts with the works of Romer (1986) and Lucas (1988). In this variety of endogenous growth theories, the assumption of constant returns to scale is dropped. The second strand starts with the research of Romer (1990) and Aghion and Howitt (1992). In this stand the assumption of perfect competition is dropped.

The fundamental problem of endogenous growth models is that an increase in the size of the economy or in the size of the R&D sector, will increase the growth rate of the economy. Several efforts have been made to rescue endogenous growth theory from the scale effects. These efforts initiated the third paradigm for more sophisticated sustainable growth models. At the end of Chapter 1, we arrive at the frontier of current research on growth theories, and it is at this juncture that we clarify the position of the thesis in the literature.

In Chapter 2, we present the basic model of sustainable growth as a combination of capital accumulation and technological advance. The model is based on the aggregate growth model of Bental and Peled (1996), who drew on the research of Jovanovic and

MacDonald (1994). These last two researchers studied the evolution of a competitive industry with a homogenous product in which a fixed number of firms reduce costs by taking draws from a distribution of untried technologies. As the firm's technologies gradually improve, industry output expands. Bental and Peled, *op cit.*, built an aggregate model from the work of these two researchers and made the number of draws taken by a firm endogenous. The result is an endogenous growth model in which technological improvements are generated by a costly and uncertain search process, with search efforts financed by capital whose alternative use is in the production of consumption goods. According to this model, technological progress requires ever-growing investments in R&D and these growing investments can only be maintained when wealth is growing. The growth of wealth, in turn, is generated by successful R&D efforts through their effects on production technologies. The resulting growth path is characterized by invention cycles reflecting varying incentives to engage in R&D as technology improves.

Bental and Peled, *op cit.*, modeled R&D as a sequential search for better technologies among a population of technologies characterized by a Pareto distribution. The search is interpreted as a sampling process in which a fixed amount of capital must be expended each time to sample (test) a new technology. In any period, all the firms are assumed to have free access to the best technology employed by any firm in the previous period; that is, it takes one period for a new technology to diffuse completely. Thus, a profit-maximizing firm begins each period with a fallback technology, which was the best technology in the preceding period. If the firm chooses not to engage in a sequential search, the fallback technology is the one it will use in the production of the consumption good. On the other hand, if the firm chooses to engage in a sequential search, then every time it pays

to sample a new technology, it can either continue or stop the search. Continue the search means rejecting the newly discovered technology, which under their assumption of sampling with replacement, is assumed not to be available for use at the end of the sequential search of the period in question. If the firm chooses to stop the search, then it will use the latest technology found, providing that it is better than the fallback technology; otherwise, it will rely on the fallback technology in the production of the consumption good.

Chapter 2 of the thesis takes as the starting point the model of Bental and Peled, *op cit.*. It deepens some of their analyses, and presents some new results not found in their paper. More specifically, we drop the assumption of sampling with replacement. Although it might be reasonable, in the context of a job search model, to assume that one can not return to a possibility that has been rejected, it is difficult to justify the assumption that continuing the search means losing temporarily the latest found technology. Because R&D activities involve an accumulation of knowledge, and it is difficult to imagine without good reasons that knowledge once discovered can be lost, the assumption that the sequential search is “forgetful” is difficult to maintain. Once the assumption of sampling with replacement is dropped, the dynamic programming argument used to analyze the sequential search must be reworked. We have managed to find a closed-form expression for the equation of the boundary that separates the “continue the search” zone and the “stop the search” zone, and thus obtain an explicit solution to the sequential search problem. The explicit form for the equation of the boundary between “continue” and “stop” the sequential search allows us to conduct a detailed analysis of the transition dynamics, which is missing in Bental and Peled, *op cit.* We also offer some analytical results concerning the asymptotic behavior of the

system under sustained growth. Finally, a numerical example that illustrates the behavior of the economy through time is given.

In Chapter 3, the model will be extended to include a second sector in which a second good, say a more traditional good, is produced and in which there is no more prospect for technological progress. The good produced in the original sector will now be called the high-technology good, and the structure of this sector is exactly as described in the preceding chapter. Because there is no further technological progress in the traditional sector, competition in this sector is dictated by more static conditions, which is in sharp contrast with competition in the high-technology sector in which firms compete both through prices and through innovations.

As in Chapter 2, to operate in any period, a firm—whether in the high-technology sector or in the traditional sector—must be able to raise capital during the capital allocation stage of that period. We shall also assume that capital is sector specific and that the capital raised by a firm in the high-technology sector also becomes firm specific. The high-technology good is chosen as the numeraire. With the price of their product set equal to 1, the behavior of the firms in the high-technology sector is exactly as described in Chapter 2. At the sectoral level, the introduction of a second sector exerts an impact on the evolution of the high-technology sector in two fundamental ways. First, the high-technology sector must compete against the traditional sector in each period for a limited pool of workers. Second, with two types of capital to invest in, the portfolio choice made by the younger generation in each period, affects the size and composition of the economy's capital stock in the following period, which in turn will influence the R&D activities in the high-technology sector. Our objective in Chapter 3 is to analyse how the savings decision—namely the portfolio

choice—of the successive young generations, the innovating behavior of the firms in the high-technology sector, and the allocation of scarce labor resources between the two sectors, determine the endogenous growth of the economy.

In Chapter 4, we extend the two-sector model of Chapter 3 into the world of international trade. The world economy is made up of two countries whose economic structures are that of the two-sector model presented in Chapter 3. There are now two goods that can be traded: the high-technology good and the traditional good. Because each consumption good can also be used as sector-specific capital, trade in a consumption good is also trade in capital specific to this good. As in Chapter 3, capital is not mobile between sectors within an economy. However, now we allow sector-specific capital to be mobile between the same sectors of the two countries. Also, we assume that labor is immobile. Again, the high-technology good is chosen as the numeraire. Our objective in this chapter is to analyze the relations between trade and growth. In particular, we analyze how the savings decision—namely the portfolio choice—of the successive young generations, the innovating behavior of the firms in the high-technology sectors, and the allocation of scarce labor resources between the two sectors, determine the endogenous growth of each country in the global economy.

Chapter 4 departs from earlier works in several aspects. First, as we showed in the second chapter, sustainable growth is due to an optimal combination of capital accumulation and technological improvements. Hence, a complete analysis would need to account for the transitional effect of trade in goods and technology, in addition to the effects that persist in the steady state. Chapter 4 provides this analysis. Second, we study the impact of growth on dynamic comparative advantage. Third, our model is an *AK* model that embodies a process

innovation in which a rise in A (the technological level) represents a new and more efficient process. Also, the technology transfer happens independently of the trade in goods. This will allow us distinguish the impact of technological innovation from import-driven growth.

By its nature, accumulated knowledge, is partially excludable, and there exists a spillover of knowledge among the firms inside the industry that is called national spillover. Also, as a result of trade, there can be a scope for international spillover. International spillover can be complete or partial, depending on the barriers to knowledge diffusion and trade. To keep the model simple it is assumed that there is no spillover among the two industries in the benchmark model introduced in Chapter 4. However, this extension can be easily be accommodated. In the first part of Chapter 4, the basic set-up of the model is elaborated under a deterministic framework. In the second part of the chapter, the uncertain nature of R&D activities are explicitly considered to develop a stochastic version of the model with a macro perspective, in which the relationship between research and productivity is derived from the parameters underlying the search distribution.

Finally, the conclusion of the thesis follows Chapter 4. In this conclusion, we point out some avenues for future research.

Chapter 1

PARADIGMS OF GROWTH THEORY

1. INTRODUCTION

Economists have long been interested in searching for the causes and effects of the growth of income and wealth of countries. And the development of economic growth theories has passed many cycles; some periods of booms in different directions, and some periods of neglect. The relevant literature gradually divided into two branches: the economics of growth and the economics of development, with the former emphasizing the use of rigorous mathematical models describing the quantitative relationships between economic variables, and the latter paying more attention to economic structures and institutions.

In this chapter, we briefly review the growth literature. To obtain a complete perspective on this literature, we found the theory of Kuhn very useful in classifying these theories. Kuhn (1970), in his book *The Structure of Scientific Revolution*, describes the notion of paradigms and the emergence of new clusters of thought as reflections to the crisis in explanatory ability of theories. According to Kuhn, a scientific discipline can develop in

two qualitatively distinct ways: times when normal scientific knowledge grows cumulatively and times of revolutions, in which progress is non-cumulative.

Before a scientific discipline reaches a state of normal science for the first time, there is a period of pre-paradigmatic science. In this period, there are many different schools of thought. Because of the lack of a common body of belief to be taken for granted, every scientist has to build his field as new from its foundations. As a science becomes mature, however, this situation will change. Different schools of thought will disappear because of the triumph of one of them. This prominent school provides the first paradigm. This process can be explained as a conversion of the scientists to the triumphant school; those scientists who are not converted will not be taken seriously any more by the now-leading school, and their work is ignored by it.

Thus the dominant school starts to provide a paradigm for the discipline. Normal science is completely based upon this first paradigm. But how then are the scientists converted to a paradigm? In other words, how does a paradigm gain its status as a paradigm? First of all, because the accepted theory is more successful than its competitors in solving problems, it will generally be recognized as acute. However, this success is neither a complete success on one single problem nor a notable success dealing with a lot of problems. Whether a paradigm is successful to a large extent depends on its promise of success, i.e., the promise to solve some still unsolved problems.

Normal science, as Kuhn stated, consists in the fulfillment of that promise. In other words, if a paradigm dictates how we should look at the world, then during normal science scientists try to describe the world in that way. The paradigm restricts the vision of the discipline, and this restriction is an essential aspect of normal science. In focusing attention

on a small range of problems, some part of nature is investigated in detail, which would otherwise have been impossible.

According to Kuhn, normal science consists of three classes of problems, both empirical and theoretical: (i) determination of a significant fact, (ii) matching facts and theory, and (iii) articulation of the theory. These three classes exhaust the literature of normal science. Normal science does not aim to produce novelties, but unexpected results are of course inevitable. However, as unexpected results cannot satisfy any of the above classes, they remain mere facts, unrelated to any subsequent research.

But if normal science does not aim at producing novelties, and if unexpected results merely mean failure as a scientist, then the question will be why research is undertaken at all. The answer to this question is that the results of normal scientific research enlarge the scope and precision with which the paradigm can be applied. But this is just part of the answer. Also, the results may be uninteresting and/or anticipated, or the way to achieve them is not clear at all. In this respect, normal scientific research is much like puzzle solving. The paradigm also allows the existence of anomalies, something that would not be possible without the background of the paradigm. Because during normal science a paradigm determines expectations, the awareness of anomaly can arise when these expectations are violated.

So, again, only because the paradigm exists can the paradigm be violated. When a scientist is confronted with anomalous applications, then it becomes clear that somewhere the theory has gone wrong. If the problems cannot be solved, the situation becomes very uncomfortable. A crisis will result. This crisis can only be solved by switching to a new paradigm.

Novelties can only be discovered after the emergence of anomalies has shown that something must be wrong. According to Kuhn, the awareness of anomaly and scientific discovery go hand in hand and it is never easy or even possible to distinguish them in an unproblematic manner. The scientist who is confronted with anomalous facts will be prepared to test another theory, when one is at hand. This possibility is what Kuhn calls revolutionary science.

A scientist who rejects a paradigm simultaneously accepts another one. Otherwise, he would be rejecting the science itself. The transition from one paradigm to another one is a scientific revolution, according to Kuhn. This transition is far from cumulative. Because the revolution means a reconstruction of the fundamentals and rules of the discipline, there will be a large, but never complete, overlap between the problems that can be solved by both paradigms. Because the new paradigm is still in an early stage, and must be tried out on a lot of problems, it is still an incomplete paradigm. That brings us to a definition of a scientific revolution: revolutions are "those non-cumulative developmental episodes in which an older paradigm is replaced in whole or in part by an incomplete new one" (Kuhn 1970, page 91).

When a new theory emerges that can serve as an alternative to the existing paradigm it may be about three types of phenomena. First, it may be about problems that the existing paradigm has already solved. In this case the new theory will be ignored because there is no use for a shift to a new paradigm that is no better than the old one. Second, the new theory may be needed to explain details on topics that the old theory could explain but not in that detail. This is actually the same kind of development as the articulation of the old paradigm, and indeed the two theories are not mutually incompatible. Third, a scientific revolution may happen when the new theory can explain the problems that led the old theory into trouble,

i.e., the new theory can explain the anomalous facts. Since it can explain the facts that the old theory could not solve, the new theory must be incompatible with the old one. Again, in order to cause a revolutionary shift, the new theory must explain the anomalous facts of the old theory.

How, then, does a revolution resolve a scientific crisis, i.e., how are scientists converted to the new paradigm? In many cases they actually are not. The scientists who are unable to adapt to the new paradigm are ignored by the rest of the discipline. This is much like the scientists who are not persuaded by a first paradigm after the pre-paradigmatic stage. But there are some factors that may help scientists to be converted to the new paradigm. The first factor is the strongest one: the new paradigm has solved the problems that led the old paradigm into crisis. Secondly, the new theory may be neater, or simpler. This is usually the case when the new paradigm is still young and not so differentiated as the old theory. A third factor, finally, may be that the new paradigm has predicted phenomena which were unexpected by the old paradigm. This adds to the promise of the paradigm, which makes it look more favorable.

We found Kuhn's model very helpful for categorizing the development of economic growth theories as three paradigms, one starting from the twenties and thirties, known in the literature as neoclassical growth theory, another from the mid-eighties labeled as endogenous growth theories, and the third starting from the mid-nineties.

The purpose of this chapter is to show how our models relate to the current literature of growth theory, focusing on paradigms and trajectories of thoughts that seeded the successor paradigms and planted the third paradigm of economic growth theories in which our work stands. This literature has emphasized how the neoclassical growth theory has

been criticized, and what new ideas have been suggested for the second and third paradigms. This chapter, using some unified frameworks, can give the reader an overview and introduction to this growing literature. However, because the literature has become so voluminous, there are many issues and results that are not covered in this chapter. In the first paradigm of economic growth theory, namely neoclassical growth theories, we emphasize the core model that contributed to the literature, and its crisis that led to the emergence of the second paradigm to compensate for the theoretical and empirical shortcomings of these theories. It will be followed with the third paradigm of theories with lessons from two previous paradigms.

Section 2 begins with a summary of the basic features of the neoclassical growth theory. This will help the reader understand the recent criticism on this theory in Section 3. Section 4 explains the basic mechanics of the endogenous growth theory. Different models and how they endogenize the growth rates of economies are introduced and compared. We point out that some of the ideas that have been used and developed in several endogenous growth papers can be traced back to several papers in the sixties and seventies. In particular, we found some "old" studies in the sixties and seventies that developed endogenous growth models. These studies have apparently been forgotten and can be read together with more recent endogenous growth theories. In Section 5, we focus on the critiques of endogenous growth models. Before leaving the endogenous growth models, we summarize the lessons of this paradigm for market structure in Section 6. The sustainable growth responses to these critiques are presented in Section 7. Section 8 presents some concluding remarks of this chapter and explains the position of this thesis in the literature.

2. THE FIRST WAVE: THE NEOCLASSICAL PARADIGM

The history of the economic growth models goes back to the twenties and thirties, in particular the works of Ramsey (1928), Harrod (1939), and Domar (1946). While Ramsey was concerned about the maximization of inter-temporal utility, Harrod and Domar paid attention to the equilibrium path of an economy. The models of Harrod and Domar defined a “warranted” growth rate that depends on the saving rate and a “natural” growth rate, which should be equal in equilibrium. Their model suffers from the severe shortcoming that there was no reason to expect the two growth rates to be equal. Despite this shortcoming, their work has generated a series of papers on economic growth, and much attention was devoted to solving the Harrod-Domar problem. Since at that time the dynamic models that are the essential part of economic growth models had not been developed in economics, we observe little progress in growth theory for the next two decades. One suggestion, which relies on the possibilities of production substitution and is due mainly to the work of Solow (1956) and Swan (1956), has been called the neoclassical theory of growth that we explain in the first wave of theory.

2.1. *Static Neoclassical Theory of Growth*

In this section, we discuss a version of the models of Solow (1956) and Swan (1956). The Solow model is the starting point for the neoclassical models and can be used as a framework to analyze other models. Consider a one-sector, closed economy accommodating a large number of competitive firms. At time t , $t = 0, 1, \dots$, the firms produce a homogeneous good using a Cobb-Douglas production function:

$$(1) \quad Y = AK^\alpha L^{1-\alpha}, \quad 0 < \alpha < 1,$$

where Y is the output, A is a technological index, K the quantity of capital input, and L is the quantity of labor. For this closed, one-sector economy, Y is also its national income. Unless confusion arises, the time subscript is dropped for simplicity.

The assumption of linear homogeneity of the production function in (1) allows us to express the output-labor ratio (per capita output), y , in terms of the capital-labor ratio, k , in the following way:

$$(2) \quad y = f(k) = Ak^\alpha .$$

This function satisfies $f(0)=0$, $f'(k)>0$, and $f''(k)<0$. These properties imply that the marginal product of capital is positive, but it is declining as capital rises. In addition $f(k)$ satisfies the *Inada condition*: $\lim_{k \rightarrow 0} f'(k) = \infty$ and $\lim_{k \rightarrow \infty} f'(k) = 0$.

Part of national income will be consumed and rest will be saved. The saving rate, as a fraction of the economy's national income, is denoted by s . In equilibrium, saving is equal to investment $s = I/Y$. There are two ways to determine the optimal saving: (i) by some or all individuals in a decentralized economy and (ii) by a social planner who maximizes the per capita consumption. Also, the saving rate can be determined by maximization of the intertemporal utility of a representative consumer in a dynamic context. In the present context, we take s as a parameter. The saving of the economy is converted into investment, and the change of the capital stock over time is

$$(3) \quad \dot{K} = I - \delta K = sY - \delta K ,$$

where δ is the exogenous depreciation rate. Suppose that the population (and also labor

force) grows at an exogenous rate of n . In terms of the capital-labor ratio we can rewrite equation (3) as:

$$(4) \quad \dot{k} = sy - (\delta + n)k.$$

Equation (4) is the key equation in the Solow model. It states that the rate of change of capital per worker is the difference between sy and $(n + \delta)k$. If we differentiate the production function given by (2), we have the growth rate of the per capita output:

$$(5) \quad \hat{y} = \hat{A} + \alpha\hat{k},$$

where a variable with a “hat” denotes the rate of change of that variable; i.e., $\hat{k} = \dot{k}/k$. This equation means that the per capita output growth rate depends on that of the technology and that of the capital-labor ratio. If we take A as given, y can be expressed as a function of k :

$$y = f(k).$$

Now we solve for the steady state of the model. First, since the production function satisfies the *Inada conditions* $\lim_{k \rightarrow 0} f'(k) = \infty$ and $\lim_{k \rightarrow \infty} f'(k) = 0$, there exists a unique k^* so that $f'(k^*) = \delta$, and $f(k) < \delta k$ for all $k > k^*$. Note that $s < 1$, and (4) implies that $\dot{k} > 0$ for $k < k^*$. Hence k is bounded from above. Also k is bounded from below unless y approaches zero.

Now, in the present model, k (and thus the marginal product of any factor) must be stationary in a steady state. By (4), the equilibrium condition is

$$(6) \quad sf(k) = (n + \delta)k.$$

This condition is illustrated in Figure 1.

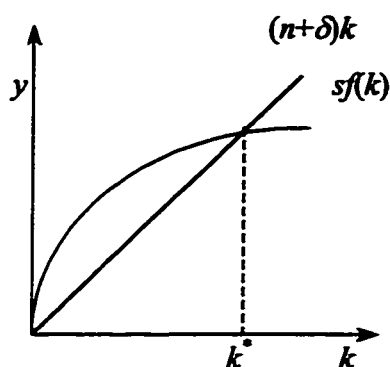


Figure 1. —The steady state level of capital

The schedule representing $sf(k)$ is strictly upward sloping, and the steady state occurs when this schedule has a value of $n + \delta$, giving a steady-state capital-labor ratio equal to k^* . Thus, in the present model the steady state exists and is unique. This result can also be derived for any linear homogeneous production function with the following properties: $f(k) > 0$, $f'(k) < 0$, $f'(0) = \infty$, and $f'(\infty) = 0$.

If we assume $\hat{A} = 0$, by (5), the per capita output is also stationary in the steady state. In other words, in the absence of any technological change both capital and output are growing at the same rate as the population.

This model features the substitutability of factors for each other. And in this way it does not suffer from the rigidity of the output-capital ratio that exists in the Harrod-Domar model. Also, this model implies that, regardless of the starting point, the economy converges to a stationary equilibrium—a situation where each variable of the model is constant through time. The parameter s has a critical role in the Solow model. The increase in s shifts the actual level of investment up, and so k^* rises. This is shown in Figure 2.

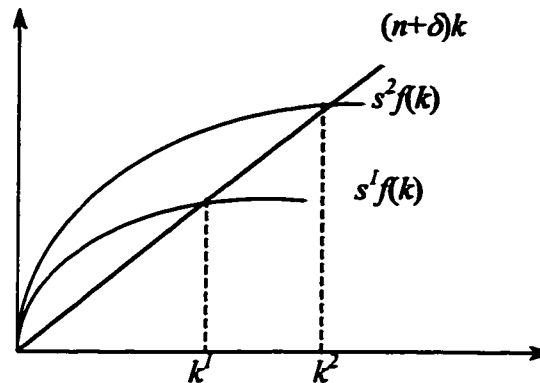


Figure 2. —The effect of an increase in the saving rate

When k reaches new value of k^* , growth stops. Thus a *permanent* increase in the saving rate produces only a *temporary* increase in the growth rate of output per worker. In other words, a change in the saving rate has a *level effect* but not a *growth effect*; it changes the stationary equilibrium point. One central question for every growth theory is to address the differences between countries in growth rates. The Solow model's answer to this question lies in differences in capital per worker (K/L). However, there are some problems with trying to account for large differences in incomes on the basis of differences in capital. First, the differences in capital per worker are far smaller than those needed to account for the differences in output per worker that we are trying to understand. Second, attributing differences in output to differences in capital, without differences in the effectiveness of labor, implies immense and implausible variations across countries in the rate of return on capital (Lucas 1990a). Thus differences in physical capital per worker cannot account for the differences in output per worker that we observe.

The Solow model does not include a dynamic optimizing analysis of households' saving behavior, and it simply took the fraction of income saved to be a given constant. In the following sub-section we look at another version of neoclassical growth theory developed by Ramsey (1928), Cass (1965), and Koopmans (1965), in which the dynamics of

economic aggregates are determined by decisions at the microeconomic level. As a result, the saving rate is no longer exogenous, and need not be constant.

2.2. Dynamic Neoclassical Growth Theory

Consider an economy populated by a large (but constant) number of separate households, each of which seeks to maximize the discounted value of a flow of utility:

$$(7) \quad W = \int_0^{\infty} e^{-\rho t} u(c(t)) dt,$$

where $c(t)$ is the per capita consumption of a typical household member, and $\rho > 0$ is the rate of time preference. This formulation assumes that the households' utility evaluated from time 0 into the indefinite future is the present value of all future flows of utility $u(c(t))$. The instantaneous utility function $u(c(t))$ is assumed to be well-behaved, i.e., it has the following properties: $u'(c(t)) > 0$, $u''(c(t)) < 0$, $u'(0) = \infty$, and $u'(\infty) = 0$. The analysis would not be appreciably altered if leisure time were included as a second argument, but to keep matters simple leisure will not be recognized in what follows. Instead, to simplify the analysis we abstract from population growth by assuming a constant labor force: $L = 1$.

Each household operates a technology with input-output possibilities described by a production function, say

$$(8) \quad Y_t = F(K_t, L_t),$$

where K_t and L_t are the household's quantities of labor and capital inputs and Y_t denotes output at time t . As before, the function F is assumed to be linear homogenous and we can express the per capita output as

$$(9) \quad Y = F(K,1) = f(k).$$

The competitive equilibrium, with perfect foresight and in the absence of any externalities, can be found by maximizing W subject to the following resource constraint

$$(10) \quad \dot{K} = F(K) - \delta K - c,$$

and subject to the transition rule of capital.

To maximize W , we write the current Hamiltonian:

$$(11) \quad H = u(c) + \lambda[F(K) - \delta K - c],$$

where λ is the shadow value of investment, evaluated in current utilities. The Hamiltonian represents net national product, i.e., consumption plus net investment, but in units of utilities. The following first order condition characterizes the optimal consumption at each instant.

$$(12) \quad u'(c) = \lambda.$$

The behavior of the shadow price of capital is governed by the following adjoint equation:

$$(13) \quad \rho\lambda = \lambda(F'(K) - \delta) + \dot{\lambda}.$$

The right hand side of (13) shows the increment of flow of income, including capital gains for incremental units in K . That is, the extra K will raise the flow of output by an amount equal to the marginal product minus depreciation, each of which has a utility value of λ plus the capital gain of $\dot{\lambda}$, and it should be equal to the “competitive rate of interest” ρ in utility terms.

The relevant transversality condition is

$$(14) \quad \lim_{t \rightarrow \infty} e^{-\rho t} \lambda K = 0.$$

The transversality condition rules out the possibility of accumulating capital forever without consuming it. In the stationary state, where both the capital stock, say K^* , and the shadow price λ are constant, we have

$$(15) \quad F'(K^*) = \rho + \delta.$$

Along an optimal growth path, capital increases whenever its marginal product $F'(K) - \delta$ is greater than the rate time preference ρ , and is decreasing otherwise. That is, the rate of time preference identifies the lower bound on the required rate of return on capital, the rate below which it is not optimal to invest in capital. This together with the Inada conditions, $\lim_{k \rightarrow 0} f'(k) = \infty$ and $\lim_{k \rightarrow \infty} f'(k) = 0$, imply that growth cannot be sustained indefinitely. It means, as in the Solow-Swan model with a fixed saving rate, that the capital stock will converge to a stationary state and growth will cease in long run. Thus, the central implication of the static model does not depend on its assumption of a fixed saving rate. Even if saving is endogenous, growth will halt in the long run if the engine of growth is limited to capital accumulation.

Here we would like to mention that another version of dynamic model was developed by Diamond (1965) in an overlapping-generation context, where in each period new households enter the picture while old individuals die off.

3. THE CRITIQUES OF THE NEOCLASSICAL GROWTH MODEL

The neoclassical model, however, has been under heavy criticism in recently. We

now present some of the more common criticisms, and explain how they are related to the recent contributions to the endogenous growth literature.

The first criticism leveled at the neoclassical growth model involves the *exogeneity of the its growth rate*. Equation (5) shows that the growth rate of the per capita output depends on those of the technology and the capital-labor ratio. In a steady state where k is constant, the per capita output then grows at the exogenous growth rate of technical progress, which receives no explanation from the neoclassical growth models. The neoclassical growth model thus leaves unexplained a major factor in the determination of per capita output growth. There are also many crucial implications of the model that need to be emphasized. First, in the absence of technological advance the economy will not grow. Second, according to condition (6), saving has only a *level effect* but no *growth effect* (Pitchford (1960), Lucas (1988)). This means that policies which subsidize saving or help to change the saving rate will only lead to a higher stationary level of the capital stock without inducing growth in the long run. Similarly, a once-and-for-all change in technology or population will not have any growth effect. Third, government policies that do not affect the growth rate of technology or that of population will not change the steady-state growth rate of the economy. For example, trade liberalization will not have any growth effect, as long as it does not affect the growth rate of technology (Lucas, 1988).

The second criticism involves *the disparities in international growth rates*. The neoclassical growth model in which the growth rate of the per capita output is equal to the sum of those of technology and the capital-labor ratio, suggests that any two countries with the same steady-state (or long-run) growth rate of technology should have the same steady-state growth rate of per capita income, regardless of their prevailing size or

technology level. What we see instead is that countries consistently have wide disparities of growth rates (Azariadis and Drazen (1990)).

The third criticism involves *the convergence of growth rates* that the neoclassical growth model predicts. Figure 1 suggests that two countries with the same technology will have the same steady state growth rate of per capita income. Consider two countries, which are identical except that country one has a higher initial capital-labor ratio, i.e., $k^1 > k^2$. Assuming that they are below the steady state level, we see that both k^1 and k^2 are increasing over time. The gap between the sy/k schedule and the line corresponding to $n + \delta$ gives the speed of adjustment of the capital-labor ratio (or the per capita output). Hence, the capital-labor ratio of country two will grow faster because its capital-labor ratio gives a bigger gap between sy/k and $n + \delta$. Since the sy/k schedule is strictly upward sloping for an economy with a Cobb-Douglas production function, the speed of adjustment of k or y decreases monotonically and k moves towards the steady state point. So country two is catching up until both countries have the same capital-labor ratio and the same growth rate.

Empirically, we lack evidence of convergence across countries (Romer (1986) and Lucas (1988)). The lack of convergence of the growth rates of different countries shows the inadequacy of the neoclassical growth theory in explaining the growth experience of countries. However, within a country, different regions give more satisfactory result for convergence prediction of neoclassical model. Barro and Sala-i-Martin (1992) do observe convergence among the 48 states of the United States in terms of the growth rates of their

per capita income and per capita gross state product. However, the convergence theory is much less satisfactory when cross-country data are used.¹

4. THE SECOND WAVE: THE ENDOGENOUS GROWTH PARADIGM

The interest of economists in economic growth was rekindled in the mid-eighties. First, there was the paper by Romer (1986) that initiated the second wave of economic growth theories by heavily criticizing the neoclassical theories. Romer also suggests a model that endogenizes the growth rate of economies. Then, Lucas (1988) provides some alternative, more appealing, ways of fixing some of the shortcomings of the neoclassical theory. In response to the various failures of the neoclassical model, there has been a flood of papers and books on endogenous growth. The common idea of this new wave is to develop models in which steady growth can be generated endogenously—i.e., can occur without any assumption on exogenous technical progress—at rates that may depend upon taste and technology parameters. There are two broad views: one with emphasis on knowledge and dynamics, with explicit models of knowledge accumulation; the other with a broader view of capital, including human capital. Inside each view there are numerous variants of the basic model, but we draw only the main trajectories of this paradigm in this section, paying particular attention to the attempts made by the papers along these trajectories to address the above three shortcomings of the neoclassical model.

4.1. *Growth with Knowledge Spillovers and Increasing Returns: The Romer AK Model*

¹ For a critic of tests of convergence hypothesis, see Barro and Sala-i-Martin (1995).

The main problem in modelling sustained growth is that the growth of technology depends on economic decisions. Various attempts were previously tried before the current wave of endogenous growth models, but the problem facing all these attempts was how to deal with increasing returns to scale in dynamic general equilibrium. More specifically, if attempts to improve A are supposed to be defined in an economic context, then these attempts should be rewarded like K and L . If F shows constant returns to scale in K and L , then the production function has increasing returns to scale in three factors and all factors can not be rewarded their marginal products.

Romer (1986) constructed a model based on the idea that growth can be sustained by increasing specialization of labor across an increasing variety of activities: as the economy grows, an increasing number of intermediates will be produced, which leads to higher productivity of labor and capital. His model has three main features. First, firms use knowledge as input. Second, knowledge can be aggregated so that we can talk about aggregate knowledge in the economy. Third, firms are competitive, taking prices and the aggregate knowledge as given. Knowledge and other factors are chosen optimally by firms, and knowledge is accumulated by sacrificing current consumption.

Assume the one-sector Cobb-Douglas production function, which can be written as

$$(16) \quad Y_i = AK^\beta K_i^\alpha L_i^{1-\alpha},$$

where $0 < \beta < 1$. The representative firm chooses a knowledge input of K_i . Let the aggregate knowledge be $K = \sum K_i$. The firm treats K as a parameter, while choosing K_i and L_i optimally. All firms have the same production function. The aggregate product function can be figured out by adding the individual firm's production function as

$$(17) \quad Y = AK^{\alpha+\beta} L^{1-\alpha}.$$

We assume $\beta = 1-\alpha$ and $L=1$. The production function in (17) reduces to an *AK* model asymptotically. And in terms of the per capita output the aggregate production function is

$$(18) \quad y = AK^{\beta} k^{\alpha},$$

while the growth rate of per capita output is:

$$(19) \quad \hat{y} = \hat{A} + \beta\hat{K} + \alpha\hat{k}.$$

In a steady state the capital-labor ratio, k , remains constant, and in the absence of technological progress, the growth of per capita output is proportional to that of knowledge capital. Romer (1986) assumes that knowledge capital displays strictly increasing marginal product, i.e., $\beta > (1-\alpha)$.

One way to escape from diminishing returns at the aggregate level is to model knowledge in a way that the improvement in technique can be shared in a normal manner by all producers. Models of this type were pioneered by Romer (1990) and Aghion and Howitt (1992). Romer (1990) extended his previous model by adding an entry fee for a new intermediate producer, whose outlay is compensated with monopoly rent. His model assumes a free-entry condition borrowed from the differentiated product literature. In this paper, Romer accommodates the concept of spillover based on the partially excludable nature of knowledge. In this way, two source of increasing returns are introduced: specialization or product differentiation, as in his previous paper, and knowledge spillover.

An important weakness of his approach is to rely on product variety for growth. Certainly the number of varieties can not grow unlimitedly. Furthermore, by introducing

some new varieties the old ones will go out of the market, a phenomenon called “obsolescence”. This weakness brought some researchers to think of a vertical model of quality improvement instead of a horizontal model of product variety.

4.2. Growth with Education: The Uzawa-Lucas Model

Among many researchers, Razin (1972a, 1972b), Manning (1975, 1976), and Findlay and Kierzkowski (1983) made efforts to explain the accumulation of human capital through education in a dynamic context. This trend in the research usually recognizes two types of labor, skilled workers and unskilled workers. However, most of these papers did not give an explanation of how to choose the skill level of the workers within the model. Rather they focus on a steady state economy with a fixed unskilled-skilled labor ratio.

Uzawa (1965) proposed to treat the skill level of workers as a variable that can accumulate over time, and he showed how sustained growth could be achieved at an endogenous rate in the neoclassical model. This researcher defined variable A as human capital that can be enhanced by the use of educational services. With a linear utility function, Uzawa drew the optimal path of accumulation for human and physical capital. Lucas (1988) extended this idea and allowed for external effects of human capital. We now present a simple version of their model.

Suppose that labor services are quantified in units of efficiency. For example, a worker with two efficiency units of labor is as productive as two workers working together, each having one efficiency unit of labor.

At any time there is a general knowledge available to all individuals that is denoted by h . The individuals who possess this general knowledge can acquire more knowledge by

getting education. Each individual is endowed with one unit of non-leisure time. The individual spends a fraction of his time endowment, denoted by τ , on getting education. And he spends the rest of his endowment, $1-\tau$, on work. The increase in human capital depends positively on the amount of time spent on education and on the existing human capital level. We assume that there is no depreciation of human capital and that human capital is accumulated according to the following differential equation:

$$(20) \quad \dot{h} = hg(\tau),$$

with $g(\tau)$ an increasing function of time spent on education. Equation (20) is called the human capital (education) production function. Note that h is both the average human capital stock and the human capital stock each individual acquires through education in the next period. An individual, taking the existing human and physical capital as given, chooses τ to maximize his utility subject to the budget constraint and condition (20). After an individual has accumulated human capital, his new level of knowledge is assumed to be immediately available to all other individuals.

Denote the labor force at any time by N , with an exogenous growth rate of n . Assume that all individuals are identical. Therefore the available efficiency units of labor is $L = (1-\tau)hN$. The per capita production function can be written as

$$(21) \quad y = (1-\tau)hAk^\alpha,$$

and the growth rate of the per capita output as

$$(22) \quad \hat{y} = -\hat{\tau} + \hat{h} + \hat{A} + \alpha\hat{k}.$$

Thus the growth of per capita output depends on those of τ , h , A , and k .

Because of diminishing returns to capital, in a steady state (balanced growth path) we have $\hat{k}=0$ and $\hat{z}=0$. This implies that the growth of human capital in a steady state is only a function of time spent on education so that

$$(23) \quad \hat{h} = g(\tau^*),$$

where τ^* is the steady-state value τ . Substituting these growth rates into (22), we have

$$\hat{y} = \hat{h} = g(\tau^*).$$

That is the per capita output and human capital grow at the same rates, and these rates depend on the steady-state value of τ^* , which is chosen endogenously by individuals.²

In this model, τ^* is the crucial parameter that determines the growth of an economy; any policy or economic factor that affects τ^* can thus change the economy's growth rate. This model gives a better explanation for international differences of countries in growth rates than the neoclassical growth model. For example, two countries which have the same technology may still grow at different rates in a steady state if individuals in different countries choose to spend different amounts of time on education (Azariadis and Drazen (1990)), or if they have different education policies. In particular, two countries may have two different steady states, and in general there is no reason to believe that their growth rates should converge.

This model has been extended in different directions. Stokey (1991) extends it to a model with a continuum of individuals with different human capital levels and a continuum

² See Caballe and Santos (1993) for other properties of the model

of products with different qualities. She assumes that the firms are competitive and hire individuals with higher levels of human capital to produce higher quality products. She also examines explicitly how human capital accumulation depends negatively on the rate of time preference but positively on the elasticity of inter-temporal substitution.

Grossman and Helpman (1991, Section 5.2) extended the model of Findlay and Kierzkowski (1983). They endogenized the choice of education in the presence of innovation, such that growth is driven not by education but by innovation. They found that an increase in the fraction of skilled workers does have a positive effect on the rate of innovation. Eicher (1996) paid more attention to the education sector. In his model this sector does the task of human capital accumulation. Also he introduced technological spillovers in his model. He showed that higher rates of technological progress and growth might be accompanied by a higher relative wage but a lower relative supply of skilled labor.

4.3 Growth with Learning by Doing

Learning by doing is another notion that introduces a channel through which human capital and knowledge of an individual or an economy accumulates. As Arrow (1962, p.155) described it, "Learning is the product of experience. Learning can only take place through the attempt to solve a problem and therefore only takes place during activity." The experience that a worker acquires through learning augments his productivity, which implies that for any given factor endowments, the production possibility set of the economy expands. This is similar to human capital accumulation through education, except that learning by doing requires very little, if any, resources: At least a worker does not have to (in fact shouldn't) stop working in order to learn.

However, an increase in workers' productivity may or may not lead to a perpetual growth of the economy. In formalizing the concept of learning by doing, Arrow, *op cit.*, postulated that the productivity of a given firm is an increasing function of cumulative investment in the industry. In his model, however, the growth rate of consumption converges to zero, because it is assumed that, for the economy as a whole, the marginal product of capital eventually falls to zero.

Lucas (1988) suggested an alternative formulation. He dropped the diminishing returns assumption made by Arrow and showed how the growth of an economy may depend positively on the rate of accumulation of human capital through learning by doing. Stokey (1988) and Young (1991, 1993) constructed other models of endogenous growth with learning by doing.

To be more precise, let us go back to the one-sector, Cobb-Douglas model described previously. During the process of learning, workers do not have to spend time, and the labor force in efficiency unit can be written as hN , where h is the average skill level. The production function in terms of per capita output is

$$(24) \quad y = hAk^\alpha,$$

or, in terms of growth rates,

$$(25) \quad \hat{y} = \hat{h} + \hat{A} + \alpha\hat{k}.$$

In a steady state $\hat{k} = 0$, and in the absence of technological progress, the growth rate of per capita output depends only on the human capital growth rate. In this model, learning is assumed to occur accidentally and individuals do not take it into account in making consumption and time-allocation decisions. However, Arrow, *op cit.*, suggested that the

experience a worker acquires through learning depends on the amount of activity he goes through. To catch his idea in this model, we assume that human capital is a positive function of a variable Z , which is an index of accumulated experience, i.e.,

$$(26) \quad h = g(Z),$$

where $g' > 0$. In terms of growth rates, (26) assumes the following form:

$$(27) \quad \hat{h} = \varepsilon \hat{Z},$$

where ε is the elasticity of the function $g(\cdot)$. To have perpetual growth, it is required that $\varepsilon \hat{Z}$ be endogenously determined and remain constant in a balanced growth path. To ensure the above perpetual growth condition, Lucas (1993) defined Z to be the cumulative human capital and

$$(28) \quad g(Z) = au \int_0^t h d\tau, \quad 0 < a, u < 1.$$

The variable u represents the fraction of time a representative worker spends on working, and is determined endogenously. The parameter a represents the efficiency unit of labor. According to equation (28), the growth rate of human capital is equal to au . Therefore, equation (28) implies that the growth rate of human capital is proportional to the amount of time individuals spend on producing the good: the more time they spend on producing the good, the faster human capital and per capita output will grow. In a steady state, u is a constant, and therefore the growth rates of human capital and per capita output in the absence of technological progress are both equal to au .

It is widely believed that the learning follows an S-shaped curve. In the sense that the learning experienced by a person in general rises rapidly initially, but then it slows down,

and eventually may become flat. Arrow, *op cit.*, was aware of this fact, and it is suggested that new goods continually appear while some old goods disappear. Lucas (1988) used the same argument, but he did not explicitly consider the continuing emergence of new goods. Stokey (1988) and Young (1991, 1993) assumed that there are diminishing returns in learning by doing with respect to any given product, but because of the emerging new products, growth can be sustained.

5. CRITIQUES OF ENDOGENOUS GROWTH MODELS

Do models of the type outlined in the preceding section make more sense than the neoclassical construct that they were designed to replace? Clearly they have the virtue of at least attempting to explain growth endogenously, but are these attempts logically satisfactory and empirically plausible? There are some highly attractive features of the models discussed above, including the possibility of knowledge externalities and the recognition that progress in terms of workforce skills relies in large part upon the allocation of resources to the production of such skills. But there are apparently some difficulties with these models that need to be considered before a conclusion can be drawn.

The first of these difficulties is that, in the Lucas model, never-ending growth requires a never-ending increase in human capital. But for such a variable, never-ending growth in human capital is implausible, because the skills in question are ones possessed by individual human beings and so are not automatically passed on to workers in succeeding generations. In this regard human capital is different from the stock of *knowledge*, which is possessed by society in general and is passed on from generation to generation, in the sense of being available to those who wish to draw upon it. Thus, it is some form of knowledge,

not human capital, that can plausibly provide the basis for never-ending growth (Grossman and Helpman (1994)).

Mankiw, Romer, and Weil (1992) and Mankiw (1995) argued that, although it has some weaknesses, the neoclassical model's empirical performance is much better than is suggested by the discussion of its endogenous growth critics. In particular, the neoclassical model is fairly successful in explaining cross-country differences in income levels, and is even more successful when the role of human capital is taken into account. This argument seems ingenious and their finding is interesting, but the suggestion that it serves to rescue the neoclassical model from its critics seems insufficient.

Young (1995b), in an influential paper, used growth accounting to show that much of East Asian growth can be attributed to capital accumulation, increased human capital, and increased labor force participation. Also Jorgenson (1995) concluded that the role of technological change in previous studies was overemphasized, and capital accumulation plays a better role in explaining growth.

A more fundamental problem in some of the endogenous growth models is that in these models an increase in the size of the economy or in the size of the R&D sector will increase the growth rate of the economy. This effect, which is called the *scale effect of R&D*, can be found in the growth equations of endogenous growth models of the previous section. Since the growth rate of the economy relies on the growth rate of the R&D sector, or of factor productivity, or of an increasing number of product varieties, the growth equations implies that an increase in the level of employment in the research sector will increase the growth rate of the economy.

The R&D models, however, came under criticism because the implication of the scale effects that lack empirical support. Jones (1995a) pointed out that the U. S. and other OECD growth rates exhibit no large persistent changes, and there is evidence of decreasing returns in the production of new knowledge because the more knowledge has already been accumulated, the harder it is to extend. Jones (1995b) further pointed out that while the number of scientists and engineers employed in R&D in the United States grew by more than five times from 1950 to 1988, the total factor productivity growth for the same period is constant or even negative. In other words, there is evidence of decreasing returns in the production of new innovations, which according to Jones contradicts the view that long-run growth relies on technological change.

Several efforts have been made to rescue endogenous-innovation growth theory. Jones (1995b) modified the R&D equation in a way that the increasing complexity of technology makes it necessary to raise R&D over time just to keep the innovation rate constant for each product. Segerstrom (1998) introduced human capital that grows through education and knowledge spillover. Young (1998) introduced an alternative formulation. His model accommodates both vertical innovation (quality improvement) and horizontal innovation (increased in the number of product varieties). He also assumed inter-temporal knowledge spillover in the vertical dimension but not in the horizontal dimension. A larger market will lead to an increase in the number of horizontal product varieties, thus affecting the *level* of utility, but not the growth rate.

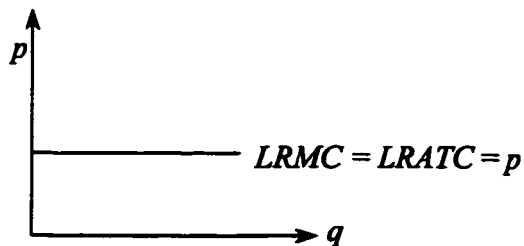
Despite their success in eliminating the scale effects, these papers do raise some questions. Because Jones (1995b) assumed a declining rate of innovation, the long-run growth rate would cease to be affected by the incentive to innovate as well as by the

incentive to accumulate capital, and would depend only on the rate of population growth. He described this model as “semi-endogenous”. Segerstrom’s model has endogenous growth, but endogeneity comes from education and human capital accumulation, not from R&D. In Young’s model, the absence of scale effect implies exogenous growth even though vertical and horizontal innovations are determined endogenously. Thus government R&D subsidies or trade policies have no growth effect, even though the number of product varieties and welfare may change.

Before leaving the endogenous growth models, we look at the lessons of this paradigm for market structure as we introduce technology as a productive input in production.

6. TECHNOLOGY AND MARKET STRUCTURE

Consider a firm using a constant returns-to-scale technology. Since the optimal bundle of inputs scales up in proportion to output with such a technology, given fixed input prices, we see that $c(q) = qc(1)$. Hence both long run average cost and marginal cost are constant (and equal). This means that the supply function of a competitive firm is also constant and is equal to this value.



The inverse demand curve can also be drawn on the above figure. Under the assumption of perfect competition, the firm faces a horizontal demand curve. Profit maximisation requires it sets output where $p=MC$ and in equilibrium $p=MC=ATC$, since any positive profit will encourage entrance of new firms that bring the price down to MC . The only equilibrium is where $p = MC$ is possible for the firm, and the firm can operate at any $q>0$ and the profit will be zero. In fact, any $q>0$ is optimal for the firm and the scale of production is indeterminate.

Let us assume the production function to be $Q = F(K, L)$. The economic assumption of constant returns to scale would amount to the mathematical assumption of linear homogeneity. Let us define capital per worker as $k = K/L$ so that we have

$$K \frac{\partial Q}{\partial K} + L \frac{\partial Q}{\partial L} = Kf'(k) + L[f(k) - kf'(k)] = Lf(k) = Q.$$

This equation is known as Euler's equation. Economically, this property means that under constant returns to scale, if each production factor is paid the amount of its marginal product, the total product will be exactly exhausted by the shares for all the input factors, or, equivalently, the pure economic profit will be zero. Although this situation describes the long run equilibrium under perfect competition, it should not be thought that only constant returns to scale technologies make sense in economics. The zero economic profit is brought about by the forces of competition through the entry and exit of firms, regardless of the specific nature of the production functions actually prevailing. Thus it is not mandatory to have a production function that ensures product exhaustion for any and all (K, L) pairs. Moreover, when imperfect competition exists in the factor markets, or in output market, the return to the factors will not be equal to the value of the marginal products, and,

consequently, Euler's theorem becomes irrelevant for the distribution of output among inputs.

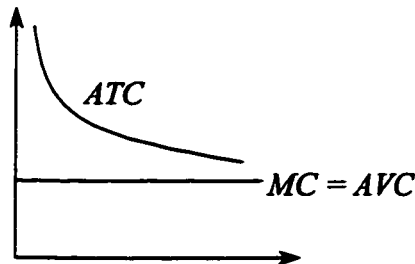
Now we look at the nature of technology and its implication for market structure. We introduce technology as a productive input in the production function. Let us assume the production function $Q = F(A, X)$, where A is technology input and X represents all rival inputs (like K and L). The technology has an important characteristic: the use of an item of knowledge in one application makes its use by someone else or in another application no more difficult. In other words, once the cost of creating a new technology has been incurred, the new technology can be used over and over again at no additional cost. Developing a new technology is equivalent to incurring a fixed cost. Shell (1973) and Romer (1990) emphasised this essential feature of all types of knowledge: they are *nonrival*. If the firm doubles X , leaving A constant, then Q doubles. If the firm doubles its conventional factors and doubles its stock of knowledge, then the firm's output must be more than doubled. An immediate implication of this fundamental property of knowledge is that if technical change is an argument of the production function, then constant returns to scale is not an attractive hypothesis. By introducing a nonrival input with productive value in the production function, the output cannot be a constant returns to scale function of all inputs taken together. The standard replication argument used to justify homogeneity of degree one does not apply because it is not necessary to replicate nonrival inputs. Hence the firm indeed faces increasing returns to scale, and it is obvious that the specification of market structure under increasing returns to scale needs more attention.

Formally, for production function $Q = F(A, X)$, where A is nonrival input and X represent all rival inputs, the replication argument implies constant returns to scale in X ,

$F(A, \lambda X) = \lambda F(A, \lambda X)$. If A is productive as well, it follows that F cannot be a concave production function because $F(\lambda A, \lambda X) > \lambda F(A, \lambda X)$. Because of the properties of homogenous functions (i.e., the Euler's theorem), it follows that the competitive model with free entry or costless adjustment of inputs cannot work.

In the case of increasing returns to scale, the average total cost decreases with output. This implies that marginal cost is below average total cost, and hence there is no finite, nonzero solution to the profit maximisation problem under the assumption of price-taking competitive behaviour. If $p = MC$, then, since $MC < ATC$, $p < ATC$. Hence, profit is negative and increasing returns to scale is therefore *inconsistent* with perfect competition.

In our case where we assume constant returns to scale in X , the AVC is the cost of the X factors. And by treating A as the fixed cost, AVC and MC are constant and horizontal.



If the price is to be equal to MC , the firm will only pay for the cost of variable inputs and so profits are negative. By Euler's theorem, if factors were rewarded their value marginal products, then payments to rival factors would exhaust output, with no revenue left over to pay for A . Formally, if $F(A, X) = X(\partial F / \partial X)(A, X)$, then the firm would suffer losses, or leave no room for payment of inventive activity because

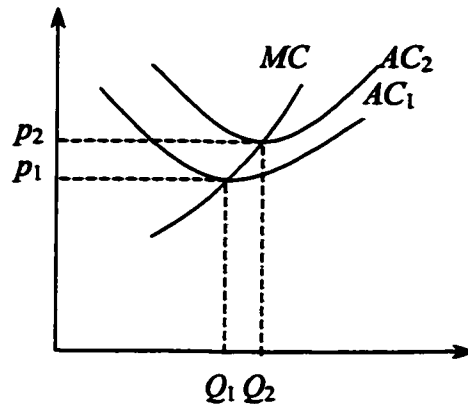
$$F(A, X) < A \frac{\partial F}{\partial A}(A, X) + X \frac{\partial F}{\partial X}(A, X).$$

When returns to scale are increasing and factors are paid their marginal product (in units of output), the product would be “overexhausted” in paying the factors; nothing would be left for paying technology. In terms of cost, a zero profit, profit-maximising, price-taking firm will not be found producing where there are decreasing (average) costs; i.e., where marginal costs is less than average cost. Therefore, the market structure must be one of imperfect competition.

How do we fix this problem? We must either drop the assumption of constant returns to scale in X ; or else we must drop the assumption of perfect competition, and introduce market power, either in the output or the input market.

Decreasing returns to scale: One way to solve the problem is assuming decreasing returns to scale in the rival input, X . If we assume that the production in rival factors is subject to decreasing returns to scale, then we will have an increasing MC and a U-shaped ATC . However, it can be argued, because of the replication argument, decreasing returns to scale is not a desirable assumption.

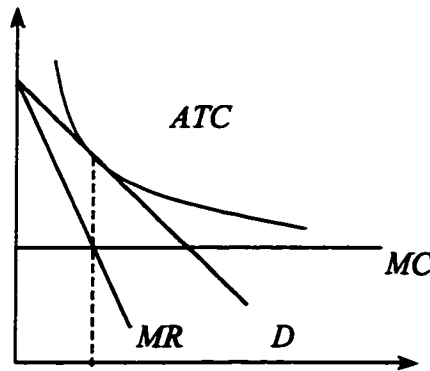
Fixed Capital: Shell (1973), in a partial equilibrium analysis of an industry, departed from perfect competition by assuming the number of operating firms is some finite and the level of technology is different across the firms. Although a finite number, n is large enough so that all firms consider themselves to be price taker. Assume there is one factor of production, say labor, that the firms can vary in the short run, but cannot vary K . The short run U-shape average cost curve for two firms with different level of technology can be illustrated as follows:



If the demand is less than Q_1 , then the supply is zero. If the demand is equal to Q_1 , then only the firm with higher technology level can operate. If the price level is equal, or greater than Q_2 , both firms will operate and the industry supply price of output is equal to the minimum average cost of most inefficient firm. When the industry demand is greater than Q_1 , the efficient firm is reaping positive quasi-rents on advanced technology. These quasi rents compensate for expenditure on research. The advantage of the interpretation that knowledge is compensated out of quasi rents is that it allows for intentional private investments in research and development and the firms are able to take a price higher than MC . Because the firm's capital stock is fixed (for one period), it can only increase output by hiring more labor, with the ensuing result of a diminishing marginal product—and therefore increasing marginal cost—it won't expand to cover the whole market.

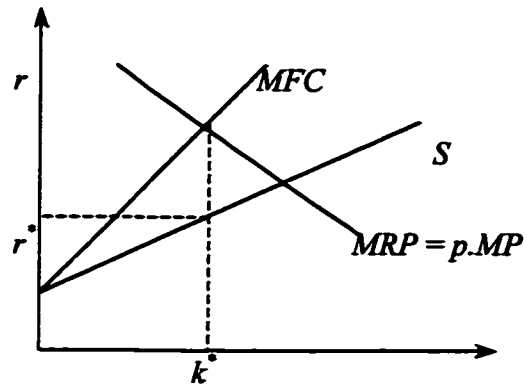
If we are interested in extending the argument of Shell for the long run in which capital also is variable, we cannot have constant returns to scale. Rather we can introduce an adjustment cost for capital. In this way long run marginal cost is increasing, and consequently, the long run average cost curve is a U-shaped function similar to the short run average cost curve.

Imperfect Competition in the Output Market: Shell (1973), Romer (1990), Grossman and Helpman (1991), and Aghion and Howitt (1992) developed models of private R&D activities. As suggested above, for R&D to be driven by economic incentives, knowledge must be at least partially excludable and the developer of a new idea must have some degree of market power for monopoly rent. In a monopolistic market structure with free entry, we come to the product differentiation monopolistic competition model at the aggregate level. Each firm faces its own downward-sloping demand curve, and the firm can charge a price higher than the marginal cost of production or the average total cost, i.e., $p > MR = MC$. It is now possible to have $ATC > MC$ (increasing returns to scale), $ATC = p$ (zero profit), and $MR = MC$ (profit-maximisation).



Imperfect Competition in the Input Market: Another solution is to depart from perfect competition in the input market. Remember the problem arises because of horizontal MR and MC curves, with $ATC > MC$. Now, if we have imperfect competition in the input market, then we can introduce an increasing MC together with a U-shaped ATC . Suppose the firm has monopsony power in the market for capital. As the firm demands more capital it faces an upward sloping supply curve of capital and must pay more for each unit of capital.

Under this situation, marginal revenue product is equal to marginal factor cost, $MRP = MFC$, and since this value is higher than rental rate of capital, r , and since $MRP = p \cdot MP > r$ we have $MP > r/p$.



If production takes place at time t , firms attract savings at time $t-1$ by issuing equities on time t profits. Each firm generates profits by choosing a rate of output along with R&D investment. Each firm makes decisions on behalf of its shareholders, whether to invest in production with known technology, or to invest in R&D with uncertain payoffs. The firm's profits are distributed to shareholders as returns on their savings.

Each young individual has a two-tier utility function. At the higher level of the utility function, the individual maximizes utility by allocating wages income between current consumption and savings for the next period. With the logarithmic utility function, the individual will save a constant fraction of his wage for to provide for future consumption. At the lower level of the utility function, the individual maximizes his portfolio by diversifying his investment in equities of n firms. Because risk-averse investors want to diversify across firms, an individual firm faces a rising cost of capital that reflects the risk premium demanded by investors (and hence an upward sloping MC curve).

The fixed costs of R&D under the present set-up give firms an incentive to increase its size since this pushes down their *ATC*. This creates a tendency towards one large firm (and hence monopoly). The desire by investors to spread their risks between many firms (since each firm is sampling independently in R&D) means that each individual firm will face a rising cost of capital, and hence an upward-sloping *MC* curve and a U-shaped *ATC* curve. That means the number of firms would be endogenous and all of them will have the same size in equilibrium, with a zero expected economic profit.

7. THE THIRD WAVE: THE SUSTAINABLE GROWTH PARADIGM

From the mid-nineties some authors, such as Jones (1995b), Kortum (1997), and Young (1998), among many, tried to construct endogenous growth models that use the analytical framework of the second wave but that could get around the scale effect problem. Indeed, the scale effect problem started a research discipline that looks deeper inside the black box of technological change. Generally, these new models are built in a way that R&D becomes progressively more difficult over time: the most obvious ideas are discovered first, making it harder to find new ideas subsequently.

However, the broader trajectory of research relies on the Schumpeterian tradition of creative destruction, in which the industry develops under pressure of technological innovation. In this tradition, an economic model emphasizes process and change as the key elements in the economic realms. This research trajectory that was backed in the early eighties by the works of Nelson and Winter (1982), which was overshadowed by the second wave. One problem with this tradition is a lack of proper mathematical tools available for describing and analyzing the properties of economic processes. Recent developments in

non-linear dynamic systems have paved the way for a more formal modeling along this trajectory.

The tradition tries to distance itself from the neoclassical interpretation of change that focuses on a converging processes, a process that uses the concept of steady state to study equilibrium but offers less explanation for transition dynamics. Obviously, for some researchers, this approach is not satisfactory for capturing the dynamic process because although some partial features may be characterized by a converging process, a convergence process is not sufficient to explain the dynamics fully.

In the Schumpeterian tradition, agents—including firms—are pictured as entities that *search for a best action and can learn*. Because the search process is inherently uncertain, this wave widely supports stochastic growth models, models that contain both random and systematic elements in the process of search. In these stochastic models, successful attempts will be selected and failed attempts rejected, and in the market only successful firms will survive. This view brings technological change into the core of competition, and necessarily looks for multiple agents in a micro approach for economic growth. Aghion and Howitt (1998) accept this tradition and pursue it in an open-ended way.

8. CONCLUSION

We saw in previous sections that the first paradigm puts capital accumulation at the heart of the growth process, and the second wave considers innovation and technological change as the engines of growth. Each approach has been advocated in many research articles, but the challenge faced by the authors considered in these waves has been to devise a theory of growth in which technical advance, as in endogenous growth theory, and capital

formation, as in neoclassical growth theory, together drive growth. And the theory must be capable of explaining the macroeconomic pattern, but on the basis of a micro-foundation. In this approach, the firms are the key actors—the entities that search, incubate, and carry knowledge and technologies.

The individual search process undertaken by a firm provides the source of its differential fitness; a firm whose R&D turns up better technologies will earn profits and grow relative to its competitors. But R&D also tends to bind firms together. Under this framework, a firm's R&D is partly a response to what its competitors are doing, and profitable innovations are, with a lag, imitated by other firms in the industry. New firms enter the industry and those firms that suffer losses exit. The profitability of any firm is determined by what it is doing, and what its competitors are doing, given the competition environment.

Considering the uncertainty that is an essential part of any search process, the logic of these models defines a dynamic stochastic system. A standard iteration that captures the evolution in time of an economic system can be described as follows. At any moment, a capital stock and a technological level at its disposal characterize each firm. Decision rules are defined by market conditions. Inputs employed, and outputs produced by firms are then determined. Then the market will determine the prices. Given the technology at its disposal, each firm's profitability is determined, and the investment rules determines how much each firm expands or contracts. A search strategy represents an aspect of the firm's behavior and capabilities, and (stochastically) comes up with proposed modifications, which may or may not be adopted. The system is now ready for the next period's iteration. Different information sets for economic agents can give different varieties of such models. For

example, if it is assumed that firms largely understand the details of the context in which they are operating and competing, then firms will be able to choose their best action in light of this knowledge as well and make a choice under rational expectations.

This thesis stays in the third paradigm by highlighting some of the above-mentioned aspects of this new discipline, in which a firm searches for a higher technological level that will affect their total factor productivity by using capital as an input to R&D activities. With attention to the work of Kortum (1997)—which showed that as the number of engineers and researchers increases, the number of patents is constant—we adopt the sequential search approach with diminishing returns in R&D investment.

The next chapter develops the search model in a one-sector economy with emphasis on the dynamics of the model. This structure can be used to analyse some important economic problems. In that chapter, we discuss different possible regimes that an economy may find itself in as a result of the interactions between capital accumulation and technological innovations. In Chapter 3, we extend the model formulated in Chapter 2 to a two-sector model. The international trade application of the two-sector model will be discussed in Chapter 4. The international trade model is made up of two countries—each with the economic structure of the two-sector model formulated and analysed in Chapter 3. It takes into account the possibility of knowledge spillover and capital flows across the borders. Chapter 5, the last chapter offers some concluding remarks and directions for possible future research.

Chapter 2

A SEARCH MODEL FOR NEW TECHNOLOGY

1. INTRODUCTION

Traditional growth theories often focus exclusively on the accumulation of physical capital as the main determinant of economic growth. Sometimes, labor is also used to explain the economic growth experienced by an economy although the population, and *a fortiori* the labor force, is always assumed to grow exponentially at an exogenous rate. Modern growth theories—known under the rubric of endogenous growth—on the other hand, relegate the accumulation of physical capital to a less prominent role and consider the accumulation of knowledge or the accumulation of human capital as the fundamental driving force behind economic growth.

In the last chapter, we surveyed this literature briefly. We mentioned that most previous endogenous growth models carry the implication of a scale effect. Since the scale effect is not supported by observed data, these models lose their theoretical underpinnings in the growth literature. Recent growth models have attempted to eliminate the scale effect in R&D. These new models are built in a way that R&D becomes progressively more difficult

over time. The challenge faced by economists is to devise a theory of growth in which technical advance, as in endogenous growth theory, and capital formation, and as in neo-classical growth theory, together drive growth.

Technological change occurs as firms search for a new technology. The history of search theory goes back to a seminal paper of Stigler (1961) that addressed the question of how large a fixed sample ought to be collected before making the choice of which alternative to accept. Later, Stigler (1970) and McCall (1970) used search theory in labor economics. Other researchers, such as Benhabib and Bull (1983), Morgan and Manning (1985) considered the choice of the set of opportunities to search over at any given time.

In a parallel attempt, Nelson (1961) used the idea of modeling R&D as a search process for a better technology. Evenson and Kislev (1975) developed his idea into a sequential search process. They built a search model in which applied research is a search in a given distribution, and basic research is a shift in the distribution searched. Tesler (1982) used the theory of optimal sequential search to study an industry where firms make homogenous products and incur the expense of research to obtain a new technology that may enable them to lower their costs. Muth (1986) completed the idea of random search for lower cost methods from a fixed population of technological possibilities.

Jovanovic and MacDonald (1994) studied the evolution of a competitive industry with a homogenous product in which a fixed number of firms reduce costs by taking draws from a distribution of untried technologies. As the firm's technologies gradually improve, industry output expands. Bental and Peled (1996) aggregate their model and make the number of draws taken by a firm endogenous. They formulated an endogenous growth model in which technological improvements are generated by a costly and uncertain search

process, with search efforts financed by capital whose alternative use is in the production of consumption goods. According to this model, technological progress requires ever-growing investments in R&D, and these growing investments can only be maintained when wealth is growing. The growth of wealth, in turn, is generated by successful R&D efforts through their effects on production technologies. The resulting growth path is characterized by invention cycles reflecting varying incentives to engage in R&D as technology improves.

In this chapter, we follow Bental and Peled, *op. cit.*, and model R&D as a sequential search for better technologies among a population of technologies characterized by a Pareto distribution. The search is interpreted as a sampling process in which a fixed amount of capital must be expended each time to sample (test) a new technology. In any period, all the firms are assumed to have free access to the best technology employed by any firm in the previous period; that is, it takes one period for a new technology to diffuse completely. Thus a profit-maximizing firm begins each period with a fallback technology, which was the best technology in the preceding period. If the firm chooses not to engage in a sequential search, the fallback technology is the one it will use in the production of the consumption good. On the other hand, if the firm chooses to engage in a sequential search, then every time it pays to sample a new technology, it can either continue or stop the search.

In the model to be developed here, we drop the assumption of sampling with replacement that was used by Bental and Peled, *op. cit.*. Although it might be reasonable, in the context of a job search model, to assume that one can not return to a possibility that has been rejected, it is difficult to justify the assumption that continuing the search means losing temporarily the latest found technology. Because R&D activities involve an accumulation of knowledge and because it is difficult to imagine without good reason that knowledge once

discovered can be lost, the assumption that the sequential search is “forgetful” is difficult to maintain. Once the assumption of sampling with replacement is dropped from the model, the dynamic programming argument used to analyze the sequential search must be reworked. We have managed to find a closed-form expression for the equation of the boundary that separates the “continue the search” zone and the “stop the search” zone, and thus obtain an explicit solution to the sequential search problem. The explicit form for the equation of the boundary between “continue” and “stop” the sequential search allows us to conduct a detailed analysis of the transition dynamics, which is missing in Bental and Peled’s paper. We also offer some analytical results concerning the asymptotic behavior of the system under sustained growth. Finally, a numerical example that illustrates the behavior of the economy is given.

The chapter is organized as follows. In Section 2, the one-sector model is presented. The production decisions in each period, after the sequential has been terminated, are analyzed in Section 3. In Section 4, the solution of the sequential search problem is presented. The technical arguments that are carried out to solve the sequential problem are relegated to Appendix 1. The expected lifetime utility maximization of a young individual is presented in Section 5. In Section 6, we classify the various regimes in which the economy might find itself and analyze the transition dynamics that the economy might undergo. Depending on the value of the parameter, the economy might experience sustained growth or not, and this question is analyzed in great detail in this section. In Section 7, we analyze the asymptotic behavior of the system when it experiences sustained growth. Section 8 presents a numerical example illustrating the working of the model. Some concluding

remarks are given in Section 9. To keep the economic logic more transparent, we have relegated most technical arguments to the appendices that appear at the end of the chapter.

2. THE MODEL

Time is discrete and denoted by t , $t = 0, 1, \dots$. Three classes of economic agents exist in each period: a young generation, an old generation, and a number of firms that produces a single consumption good.

There is no population growth in the economy. More specifically, in each period the young generation and the old generation are both assumed to have the cardinality of the continuum of measure one. An individual lives two periods and works when he is young. A young individual owns nothing except for one unit of labor that he supplies inelastically on the labor market. He divides his wage between current consumption and saving for his old age. Capital is the only real asset in the economy, and capital investments represent the only possible form of saving. The aggregate saving of the young generation in period $t-1$ thus constitutes the aggregate capital stock at the beginning of period t .

Firms produce the consumption good from labor and capital. The set of all potential firms is denoted by I , assumed to be a finite set. A firm $i \in I$ begins any period t with a certain level of technological competence—denoted by a_{it} —for producing the consumption good. We shall assume that all the firms begin period t with the same technological level, say $a_{it} = a_t, i \in I$. To operate in period t , the firm must attract savings from the generation of period $t-1$ by issuing equities on the profits that it generates in period t , and the profits will be distributed to share holders as returns on their savings. This process is assumed to take place at the beginning of each period during a stage called the capital allocation stage. Now

because all the firms are assumed to have the same technological level at the beginning of each period, with the same expected return, we have a symmetric equilibrium in which they will all be able to attract the same amount of capital in the capital allocation stage of each period from savers who will be fully and equally diversified between all firms. Thus for each period t , the amount of capital raised by firm i is $k_{it} = K_t / |I|, i \in I$. Here K_t is the aggregate capital stock of the economy at the beginning of period t and $|I|$ represents the number of firms.

Having raised k_{it} units of capital, firm $i, i \in I$, can either combine this amount of capital with hired labor to produce the consumption good, using the technology with which it begins the period, or engage in a sequential search for a new and better vintage of technology. A sequential search involves taking random draws—also called technology draws—from a population of untried technologies. The population of all possible technologies—found or yet to be found—is assumed to consist of all the Cobb-Douglas production functions of the following forms:

$$(1) \quad y(a, k, l) = ak^\alpha l^{1-\alpha}, \quad (a \geq 1).$$

In (1), $y(a, k, l)$ is the output of the consumption good that is produced from k units of capital and l units of labor; a is the technological level—also called the productivity—of the technology in question; and $\alpha, 0 < \alpha < 1$, is a parameter that characterizes the entire population in which the sequential search is conducted, while the technological level a identifies a specific technology within the population. The following distribution is imposed upon the population of possible technologies.

ASSUMPTION 1: *The probability of drawing a technology with productivity less than or equal to a is given by the following Pareto distribution:*

$$F(a | \lambda) = 1 - a^{-\lambda}, \quad (1 \leq a < \infty),$$

where λ is a positive parameter. Observe that for any given $a \geq 1$, $F(a | \lambda)$ is increasing in λ ; that is, the cumulative probability of finding a technology with productivity at least as high as a declines as λ increases. In other words, the Pareto distribution is stochastically decreasing in its parameter: a higher value for the parameter implies a less favorable population of potential technologies in which the sequential search is conducted.

In any period, a firm finances its sequential search from part of the capital it raised during the capital allocation stage. There is no fixed cost in the sequential search and the firm can take any number of technology draws at the cost of ε units of capital per draw. A technology draw completely reveals the productivity of the technology found. The maximum numbers of technology draws that firm i can take in period t is $\max\{0, \lceil k_{it} / \varepsilon \rceil\}$, where $\lceil x \rceil$ is defined to be the greatest integer less than or equal to x . The sequential nature of the search can be described as follows. Suppose that firm i starts the sequential search and finds a technology with productivity a^1 after the first draw. At this point in time, it has $k_{it} - \varepsilon$ units of capital left and the best technology under its command has a productivity equal to $\max\{a_{it}, a^1\}$. The firm might decide to stop or continue the search. If it continues the search, let a^2 be the productivity of the technology found in the second draw. After the second draw, the firm has $k_{it} - 2\varepsilon$ units of capital left and the best technology at its disposal has a productivity equal to $\max\{a_{it}, a^1, a^2\}$. At any instant during the sequential search, the

decision whether to stop or continue the search depends on how much capital remains and the best technology at its disposal.

The productivity of the technologies drawn in a sequential search are assumed to be independently and identically distributed. This assumption can be justified by noting that under the distribution F each productivity level $a \in [1, \infty)$ has probability zero. Furthermore, to rule out the possibility of strategic behavior by the firms during the search stage of any period, the technologies drawn by a firm in its sequential search are assumed to be observed only by the firm itself. Finally, to ensure that a firm will develop and use a technology it has found instead of selling it to another firm, and that it will engage in a sequential search—whenever profitable—the capital owned by a firm is assumed to become firm-specific.

Let \hat{a}_i and \hat{k}_i denote, respectively, the productivity of the best technology under the command of firm i and its remaining capital when this firm enters the production stage of period t . If the firm did not engage in a sequential search, then $\hat{a}_i = a_i$ and $\hat{k}_i = k_i$. On the other hand, if it engaged in a sequential search and took n technology draws, then $\hat{a}_i = \max\{a_i, a^1, \dots, a^n\}$ where a^1, \dots, a^n are the productivity of the technologies found in the search, and $\hat{k}_i = k_i - n\varepsilon$. The technology with which firm i enters the production function stage of period t is given by the following short-run production function:

$$(2) \quad y(\hat{a}_i, \hat{k}_i, l) = \hat{a}_i \hat{k}_i^\alpha l^{1-\alpha}.$$

Observe that in (2) capital is the fixed input while labor is the variable input. The fixity of \hat{k}_i follows from the assumption that the capital owned by a firm becomes firm

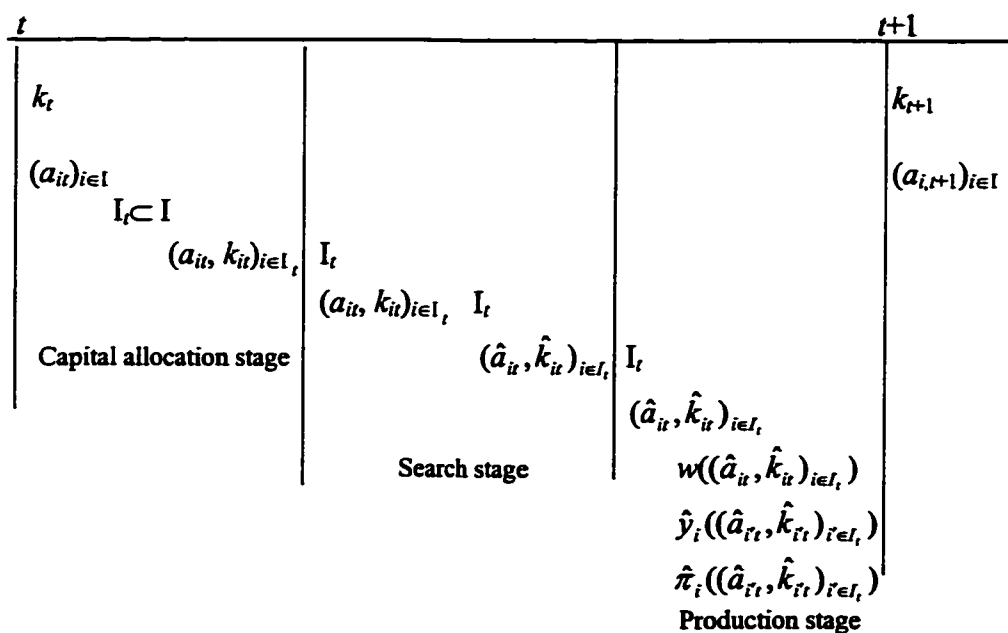
specific. We shall assume that the capital used in producing the consumption good depreciates completely at the end of the production process.

The stage is now set for describing the evolution of the economy through time. For each $t = 0, 1, \dots$, the state of the economy at the beginning of period t is represented by the vector (a_t, K_t) , where a_t is the common technological level of all the firms and K_t is the aggregate capital stock – all at the beginning of period t . The initial state of the economy, namely (a_0, K_0) is assumed to be known.

A firm might decide to engage in a sequential search to improve its productivity. Although such a firm begins the search stage with a known technological level a_{it} and a known capital stock k_{it} , the random nature of the search means that \hat{a}_{it} , the highest technological level, and \hat{k}_{it} , the remaining capital, under its command for producing the consumption good are only known when the search ends. Hence the profit generated by the firm is not known at the time it begins the search. However, once \hat{a}_{it} and \hat{k}_{it} are realized, the uncertainty about the production technology of firm i is completely resolved and is as given in (2).

Because the old generation in each period are the owners of the firms in that period, and because those individuals will not be alive in the following period, all that they care about is the current profit of the firms, not their future profitability. Furthermore, with full depreciation, the firms do nothing this period to affect its future and are essentially single period entities. Thus the firms will pursue the objective of maximizing profits in each period. Given this objective, and after \hat{a}_{it} and \hat{k}_{it} have been realized, firm i will combine \hat{k}_{it} with hired labor to maximize the short-run profit in the production of the consumption good.

The demand for labor by all the firms will then determine the realized equilibrium wage rate in period t . Once the equilibrium wage rate is known, a young individual in period t can decide on his current consumption and his saving. The aggregate saving of the young individuals of period t then constitute K_{t+1} , the aggregate capital stock at the beginning of period $t + 1$. We shall assume that it takes one period for the best technology at the end of the sequential search of period t to diffuse throughout the economy. Hence the common technological level with which all the firms begin the next period is given by $a_{t+1} = \max_{i \in I} \hat{a}_{it}$. The state of the economy at the beginning of period $t+1$ is then represented by the vector (K_{t+1}, a_{t+1}) , and the process just explained for period t repeats itself in period $t+1$, driving the system to a new state (K_{t+2}, a_{t+2}) in period $t+2$, and so forth.



The structure of the equilibrium in each period is that of an equilibrium extending over three successive stages—the capital allocation stage, the sequential search strategy, and the

production stage. The strategy for analyzing such equilibrium is by backward induction. We thus begin by analyzing the last stage.

The model thus describes the evolution of a market of a homogenous product. The supply side of the model comprises a fixed number of price-taking firms that maximize profits. To operate at time t , firms attract savings at period $t-1$ by issuing equities on time t profits. Each firm make decisions on behalf of shareholders, whether to invest in production with a known technology or to invest in R&D with uncertain payoffs. The firm's profits are distributed to shareholders as returns on their savings. There is no entry or exit. In a symmetric equilibrium, all firms are identical at the beginning of each period. This symmetry is motivated by the absence of any long-lasting impact of any action taken by the firms in any period. Consequently, the distribution of returns offered by all the firms in each period will be the same, and all risk-averse agents will be fully and equally diversified across firms. Provided we recognize that the firm's costs include the necessary risk premium, the desire by savers to spread their risks between many firms means that each individual firm will face a rising cost of capital. In this way the number of firms would be endogenous, all beginning their operations in any period with same amount of capital, pursuing the same search strategy. Since solving for capital market equilibrium and finding the endogenous number of firms are not our primary objectives in this thesis, we leave this important extension for future research.

3. THE PRODUCTION STAGE

In the production stage of period t , an operating firm i produces the consumption good according to the technology represented by the short-run production function (2). Let

w be the wage rate, and assume that w is taken as given by all the operating firms. Firm i solves the following profit maximization:

$$(3) \quad \max_i (\hat{a}_i \hat{k}_i^\alpha l^{1-\alpha} - wl).$$

The solution of (3) yields the following demand for labor by firm i :

$$(4) \quad l(\hat{a}_i, \hat{k}_i, w) = \left[\frac{(1-\alpha)\hat{a}_i}{w} \right]^{1/\alpha} \hat{k}_i.$$

Using (4) in (2), we obtain the following expression for the output of firm i :

$$(5) \quad y(\hat{a}_i, \hat{k}_i, l(\hat{a}_i, \hat{k}_i, w)) = \hat{a}_i^{1/\alpha} \hat{k}_i \left(\frac{1-\alpha}{w} \right)^{(1-\alpha)/\alpha}.$$

The profit (i.e. returns to capital) generated by firm i is:

$$(6) \quad \pi(\hat{a}_i, \hat{k}_i, w) = \alpha \hat{a}_i^{1/\alpha} \hat{k}_i \left(\frac{1-\alpha}{w} \right)^{(1-\alpha)/\alpha}.$$

Summing (4) over $i \in I$, then equating the result to 1, we obtain the following market-clearing condition for labor:

$$(7) \quad \left(\frac{1-\alpha}{w} \right)^{1/\alpha} \left(\sum_{i \in I} \hat{a}_i^{1/\alpha} \hat{k}_i \right) = 1.$$

The realized equilibrium wage rate is then given by:

$$(8) \quad \hat{w}((\hat{a}_i, \hat{k}_i)_{i \in I}) = (1-\alpha) \left(\sum_{i \in I} \hat{a}_i^{1/\alpha} \hat{k}_i \right)^\alpha.$$

Substituting the right side of (8) for w in (5), we obtain the following expression for the output of the firm i in terms of the data $(\hat{a}_{i'}, \hat{k}_{i'})_{i' \in I}$

$$(9) \quad \hat{y}_i((\hat{a}_{it}, \hat{k}_{it})_{i \in I}) = (\hat{a}_{it}^{1/\alpha}) \hat{k}_{it} \left(\sum_{i \in I} \hat{a}_{it}^{1/\alpha} \hat{k}_{it} \right)^{\alpha-1}.$$

Similarly, substituting the right side of (8) for w in (6), we obtain the following expression for the profit of the firm i in terms of the data $(\hat{a}_{it}, \hat{k}_{it})_{i \in I}$

$$(10) \quad \hat{\pi}_i((\hat{a}_{it}, \hat{k}_{it})_{i \in I}) = \alpha (\hat{a}_{it}^{1/\alpha}) \hat{k}_{it} \left(\sum_{i \in I} \hat{a}_{it}^{1/\alpha} \hat{k}_{it} \right)^{\alpha-1}.$$

Summing (9) over $i \in I$, we obtain the following expression for national income in terms of the data $(\hat{a}_{it}, \hat{k}_{it})_{i \in I}$:

$$(11) \quad \hat{Y}((\hat{a}_{it}, \hat{k}_{it})_{i \in I}) = \left(\sum_{i \in I} \hat{a}_{it}^{1/\alpha} \hat{k}_{it} \right)^{\alpha}.$$

Similarly, summing (11) over $i \in I$, we obtain the following expression for the gross income of the old generation of period t :

$$(12) \quad \sum_{i \in I} \hat{\pi}_i((\hat{a}_{it}, \hat{k}_{it})_{i \in I}) = \alpha \left(\sum_{i \in I} \hat{a}_{it}^{1/\alpha} \hat{k}_{it} \right)^{\alpha}.$$

The realized gross rate of return to the capital investment received by an old individual of period t is thus given by:

$$(13) \quad \hat{r}((\hat{a}_{it}, \hat{k}_{it})_{i \in I} | K_t) = \alpha \left(\sum_{i \in I} \hat{a}_{it}^{1/\alpha} \hat{k}_{it} \right)^{\alpha} / K_t.$$

4. THE SEARCH STAGE

Consider a firm, say i , that enters the search stage of period t with $k_{it} > 0$ as the amount of capital it raised from the young generation of period $t-1$. This firm has to decide whether to engage in a sequential search to find a new and better technology than the one at its disposal when the period begins.

Now the profit of firm i , as represented by (6), depends on the technological level and the remaining capital under its command when it enters the production stage as well as the wage rate that prevails in this stage. Furthermore, according to (8), the realized equilibrium wage rate depends on the technological levels and the remaining capital stocks of all the firms—including firm i —when the production stage begins. Thus strictly speaking, the search strategy chosen by a firm will have an impact on the realized equilibrium wage rate. A complete modeling of this impact however is quite involved.

A rational expectation equilibrium in the search stage consists of a distribution, say $G(w)$, of the wage rate and a search strategy for each operating firm. Two consistency conditions are imposed by such a rational expectations equilibrium. First, the search strategy pursued by each operating firm must be optimal, given that $G(w)$ represents its expectations about the equilibrium wage rate. Second, when these search strategies are carried out, the resulting equilibrium wage rate, as represented by (8), has $G(w)$ as its distribution.

Given the expectations that firm i holds about the equilibrium wage rate, we can find its optimal sequential search with the help of dynamic programming by carrying out an induction on the maximum numbers of technology draws that the firm can take. The following result is established in Appendix 1 of this chapter.

PROPOSITION 1: Let $\bar{k} : a \rightarrow \bar{k}(a), a \geq 1$, be the curve defined by

$$\bar{k}(a) = \varepsilon(1 + (\alpha\lambda - 1)a^\lambda)$$

and $\bar{a} : k \rightarrow \bar{a}(k), k \geq 0$, be the curve defined as follows:

$$\bar{a}(k) = 1 \text{ for } 0 \leq k \leq \varepsilon\alpha\lambda,$$

= the inverse of $\bar{k}(a)$ for $k > \varepsilon\alpha\lambda$, i.e., $\bar{a}(k)$ is the value of a defined implicitly

$$\text{by } \bar{k}(a) = \varepsilon(1 + (\alpha\lambda - 1)a^\lambda).$$

Next, let (a, k) be the current state of the sequential search of a firm. If $a < \bar{a}(k)$, then the firm should continue the search; otherwise, it is optimal for the firm to stop the search. As defined, $\bar{a}(k)$ represents the threshold technological level for the firm in terms of its remaining capital k such that if the best technology currently at the disposal of the firm falls short of this threshold level, then it is optimal for the firm to continue the search.

The following figure depicts the relationship between k and $\bar{a}(k)$.

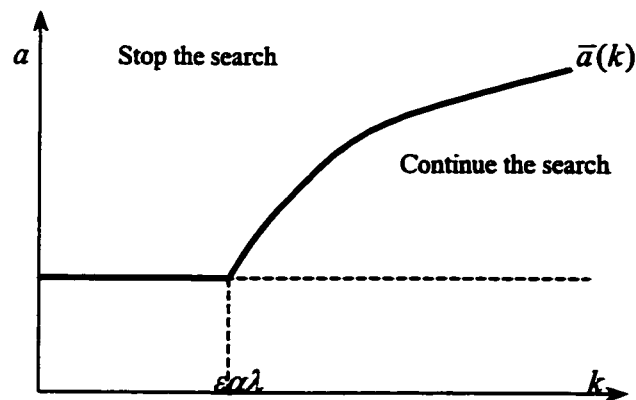


Figure 1.—The threshold technological level $\bar{a}(k)$

Given the initial condition (a_{it}, k_{it}) , the search strategy described in Proposition 1 completely determines the distribution of $(\hat{a}_{it}, \hat{k}_{it})$, and in particular the distribution of $\hat{\zeta}_{it} = \hat{a}_{it}^{1/\alpha} \hat{k}_{it}$, the random variable that embodies simultaneously the productivity and the remaining capital of firm i at the beginning of the production stage in period t . The derivation of these distributions is given in Appendix 2 of the chapter.

Figure 2 depicts a possible realization of the sequential search for the case $\varepsilon < k_{it} - \bar{k}(a_{it}) \leq 2\varepsilon$ under the event that the firm takes two technology draws. In this figure, S^0, S^1 , and S^2 denote, respectively, the state of the search after the setup cost is paid, after the first draw, and after the second draw.

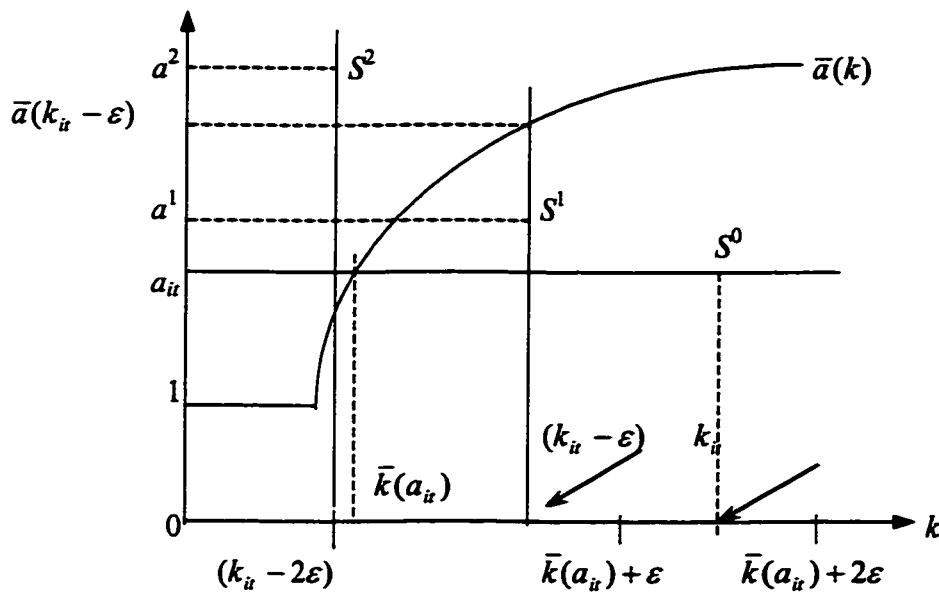


Figure 2.—A possible realization of the search process:

$$(a^1 < \bar{a}(k_{it} - \varepsilon), \varepsilon < k_{it} - \bar{k}(a_{it}) \leq 2\varepsilon)$$

5. LIFETIME UTILITY MAXIMIZATION

The preferences of an individual are assumed to satisfy the following assumption.

ASSUMPTION 2: The lifetime utility function of a young individual depends only on his current and future consumption, and is assumed to have the following form:

$$u(c^0, c^1) = \log c^0 + \delta \log c^1,$$

where $0 < \delta < 1$ is the discount factor; c^0 is the consumption when he is young; and c^1 is the consumption when he is old.

Let w be the wage rate received by a young individual. If c^0 is his current consumption, then his saving is $w - c^0$. This individual solves the following expected lifetime utility problem:

$$\max_{c^0} [\log c^0 + \delta E \log(\tilde{r}(w - c^0))],$$

where \tilde{r} is a random variable representing his anticipated gross rate of return to capital investment and E is the expectation operator with respect to the distribution of \tilde{r} . The solution of this expected lifetime utility problem yields the following saving function:

$$w - c^0 = \frac{\delta w}{1 + \delta},$$

which asserts that a young individual always saves a constant fraction of his wage. Thus if w_t is the wage rate in period t , then the aggregate capital stock at the beginning of the next period is

$$K_{t+1} = \frac{\delta w_t}{1 + \delta}.$$

6. CLASSIFICATION OF REGIMES AND THE TRANSITION DYNAMICS

Intuitively, we expect that the economy will experience sustained growth if R&D activities are highly productive. Now R&D activities are profitable if the share of capital in national income is high, and the population from which technologies are sampled are promising. In terms of the parameters of our model, these requirements translate into a high value of α and a low value of λ , or equivalently, a high value of $1/(1-\alpha)$ and a low value of λ .

In this section, we show that if $1/(1-\alpha) < \lambda$, then there is no sustained growth and the economy will converge to a stationary equilibrium in the long run. This result, which is a strengthened version of Proposition 4.4 in Bental and Peled, *op cit.*, is stated as our Proposition 2.

To analyze the dynamic behavior of the economy, we begin with its evolution through time when there is no R&D. To this end, let K_t be the aggregate capital stock at the beginning of period t . When there is no R&D, the technological level of the economy is constant, say $a_t = a$ for $t = 0, 1, \dots$. If we denote the aggregate output in period t by Y_t , then because the aggregate labor input is $L = 1$, we must have

$$(14) \quad Y_t = aK_t^\alpha,$$

and by using (4), we obtain the following expression for the aggregate capital stock of the next period

$$(15) \quad K_{t+1} = \frac{\delta}{1+\delta}(1-\alpha)aK_t^\alpha.$$

Equation (15) asserts that if the economy chooses never to conduct R&D, then as Solow's model the aggregate capital stock converges geometrically to the stationary level

$\left(\frac{\delta(1-\alpha)a}{1+\delta}\right)^{1/(1-\alpha)}$, as depicted in the following diagram.

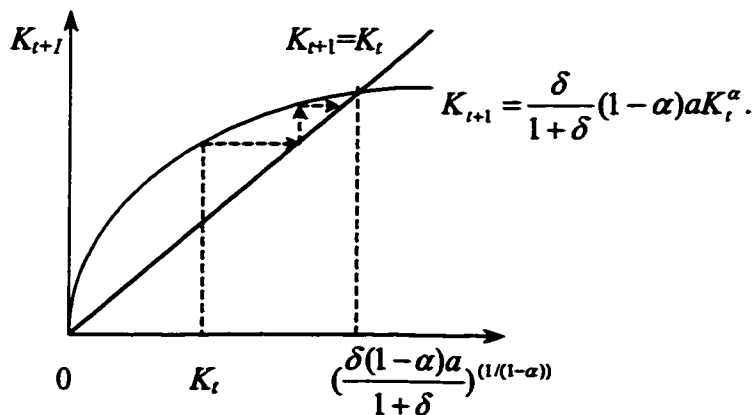


Figure 3.—The convergence of the aggregate capital stock to the stationary level

The following figure depicts the stationary equilibrium level of the aggregate capital stock as a function of the technological level a of the economy, given that no more R&D activities will be undertaken once a has been attained.

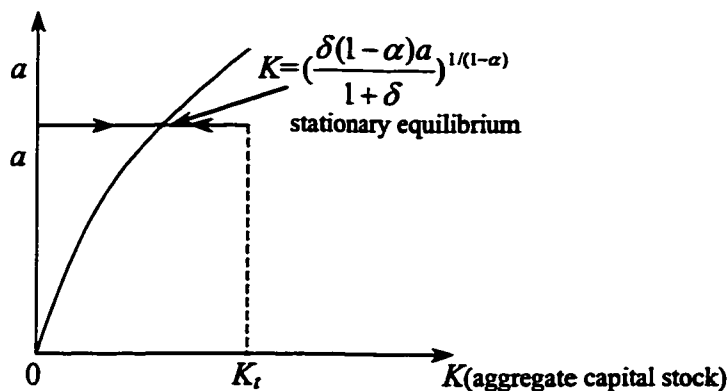


Figure 4.—Stationary equilibrium level of the aggregate capital stock

An alternative representation of the economy in its convergence to the stationary equilibrium depicts the equilibrium level of capital per firm as a function of the technological level a , given that no more R&D activity will be undertaken by the firms. The

curve $\overset{\circ}{k} : a \rightarrow \frac{1}{|I|} \left[\frac{\delta(1-\alpha)a}{1+\delta} \right]^{1/(1-\alpha)}$ represents the stationary equilibrium capital stock per firm

as a function of the economy's technological level a , given that no more R&D activities are undertaken. The dynamics of the economy is completely determined by the relative position

of the curves $\bar{k}(a) = \varepsilon(1 + (\alpha\lambda - 1)a^\lambda)$ and $\overset{\circ}{k}(a) = \frac{1}{|I|} \left[\frac{\delta(1-\alpha)a}{1+\delta} \right]^{1/(1-\alpha)}$. For each value of

a on the vertical axis, the horizontal distance of the curve $\bar{k} : a \rightarrow \varepsilon(1 + (\alpha\lambda - 1)a^\lambda)$ relative

to that of the curve $\overset{\circ}{k} : a \rightarrow \left[\frac{\delta(1-\alpha)a}{1+\delta} \right]^{1/(1-\alpha)} / |I|$, is given by the following ratio:

$$(16) \quad \frac{\varepsilon(1 + (\alpha\lambda - 1)a^\lambda)}{\left(\frac{\delta(1-\alpha)a}{1+\delta} \right)^{1/(1-\alpha)} / |I|} = \frac{\varepsilon |I|}{\left(\frac{\delta(1-\alpha)}{1+\delta} \right)^{1/(1-\alpha)}} \left(a^{-1/(1-\alpha)} + (\alpha\lambda - 1)a^{\lambda-1/(1-\alpha)} \right).$$

We shall call (16) the horizontal distance of \bar{k} relative to $\overset{\circ}{k}$. Observe that if $\lambda < 1/(1-\alpha)$,

then the right side of (16) declines monotonically from $\varepsilon\alpha\lambda |I| / \left(\frac{\delta(1-\alpha)}{1+\delta} \right)^{1/(1-\alpha)}$ to zero as

$a \rightarrow +\infty$. If $\lambda = 1/(1-\alpha)$, then it declines monotonically from $\varepsilon\alpha\lambda |I| / \left(\frac{\delta(1-\alpha)}{1+\delta} \right)^{1/(1-\alpha)}$ to

$\varepsilon\alpha\lambda |I| (\alpha\lambda - 1) / \left(\frac{\delta(1-\alpha)}{1+\delta} \right)^{1/(1-\alpha)}$ when $a \rightarrow +\infty$. When $\lambda > 1/(1-\alpha)$, the right side of (16)

tends to ∞ as $a \rightarrow +\infty$. Furthermore, depending on the value of α and λ , the horizontal

distance of \bar{k} relative to $\overset{\circ}{k}$ might rise monotonically, or it might first decline, reach a minimum, then rises monotonically to $+\infty$ as a rises from 1 (Appendix 3).

6.1. The case $1/(1-\alpha) < \lambda$

When $1/(1-\alpha) < \lambda$, the horizontal distance of \bar{k} relative to $\overset{\circ}{k}$ tends to 0 as $a \rightarrow +\infty$, i.e., $\overset{\circ}{k}$ will be to the left of \bar{k} when a is large. There are four possibilities to consider.

6.1.1. The case $1/(1-\alpha) < \lambda$ with $\overset{\circ}{k}$ completely on the left of \bar{k}

This case is depicted in the following figure. There are three distinct regions to consider:

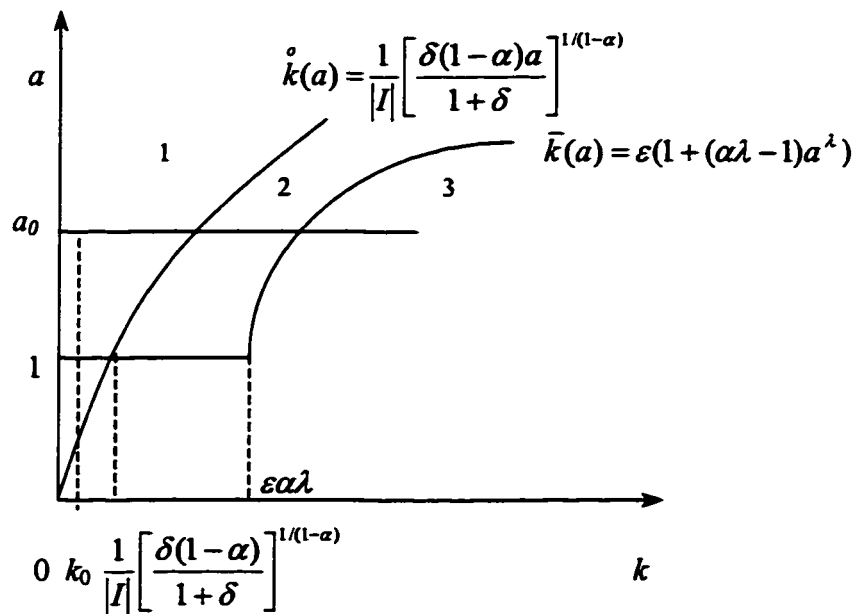


Figure 5.—No intersection between the two curves

$$\text{Region 1} = \left\{ (a, k) \mid a \geq 1, 0 < k \leq \overset{\circ}{k}(a) \right\},$$

$$\text{Region 2} = \left\{ (a, k) \mid a \geq 1, \hat{k}(a) < k \leq \bar{k}(a) \right\},$$

$$\text{Region 3} = \left\{ (a, k) \mid a \geq 1, \bar{k}(a) < k \right\}.$$

Let (a_0, k_0) be the initial state of the system. If the initial state of the system is in Region 1, no R&D activities will ever be undertaken in this period or any future period. With the same technological level a_0 in each period, the capital stock per firm rises monotonically to its stationary equilibrium level $\left[\frac{\delta(1-\alpha)a_0}{1+\delta} \right]^{1/(1-\alpha)} / |I|$. Similarly, if the initial state of (a_0, k_0) is located in Region 2, no R&D activity will be undertaken in period 0 or any future period. The capital stock per firm declines monotonically to the stationary equilibrium level $\left[\frac{\delta(1-\alpha)a_0}{1+\delta} \right]^{1/(1-\alpha)} / |I|$. Under these two possibilities, no long-run growth can be observed.

If (a_0, k_0) belongs to Region 3, such that the capital level is above the critical level of $\bar{k}(a_0)$, the state of the system at the beginning of period 1, will be in either Region 1 or Region 2.

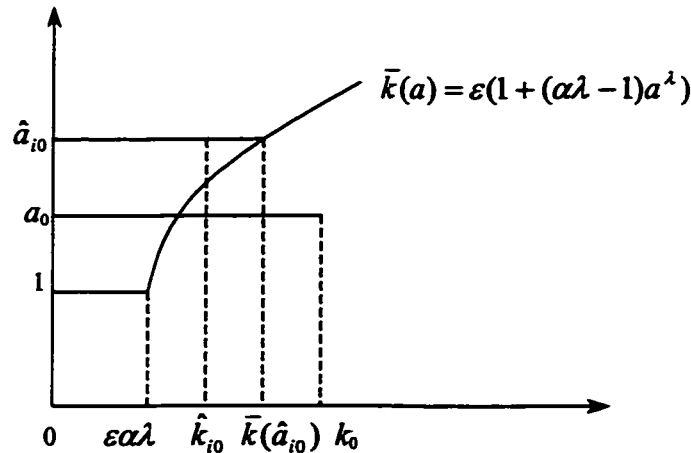


Figure 6.

More clearly, the firms will engage in a sequential search in period 0 to find a new and better technology. Let \hat{a}_{i0} and $\hat{k}_{i0}, i \in I$, be the technological level and the remaining capital stock, respectively, of firm i when this firm enters the production stage of period 0.

Now the technological level that the economy begins period 1 is given by $a_1 = \max_{i \in I} \hat{a}_{i0}$. And we have $\hat{k}_{i0} \leq \bar{k}(\hat{a}_{i0}) \leq \bar{k}(a_1), i \in I$. The output of the consumption good in period 0 is:

$$(17) \quad \left[\sum_{i \in I} (\hat{a}_{i0})^{1/\alpha} \hat{k}_{i0} \right]^{1/\alpha},$$

which is less or equal to

$$(18) \quad \left[\sum_{i \in I} a_1^{1/\alpha} \bar{k}(a_1) \right]^{1/\alpha} = a_1 (|I| \bar{k}(a_1))^\alpha.$$

Observe that the right side of (18) is the aggregate output of the consumption good if each of the firm's capital input is $\bar{k}(a_1)$ and each of them has the same technological level a_1 . In the following figure we can observe that if the right side of (18) were the actual output of the

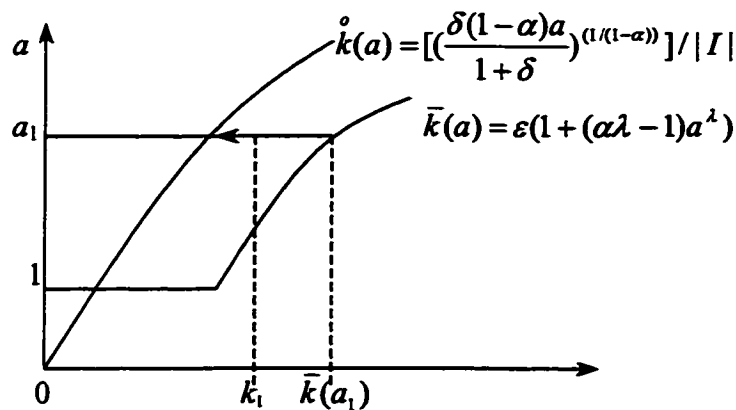


Figure 7.

consumption good in period 0, then because $\bar{k}(a_1) > \overset{\circ}{k}(a)$, the capital stock raised per firm in period 1 will be strictly less than $\bar{k}(a_1)$. Because the realized output of the consumption good, namely (17), can never exceed (18), the actual amount of capital raised per firm in period 1, namely k_1 , will be strictly less than $\bar{k}(a_1)$. Therefore, (a_1, k_1) , the state of the system at the beginning of period 1, will be in either Region 1 or Region 2. The analysis already carried out for Regions 1 and 2 can then be repeated verbatim for period 1. No more R&D activity will be undertaken, and the capital stock per firm will converge to its stationary equilibrium level $(\frac{\delta(1-\alpha)a_1}{1+\delta})^{1/(1-\alpha)} / |I|$. In its approach to the stationary equilibrium, the technological level of the economy remains at the level a_1 . We summarize the results just obtained in the following proposition.

6.1.2. *The case $1/(1-\alpha) < \lambda$, with $\overset{\circ}{k}$ on the left of and tangent to \bar{k}*

This case is depicted in Figure 8.

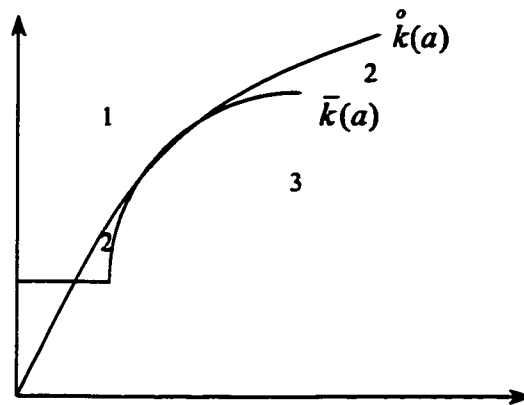


Figure 8.

Regions 1, 2, and 3 are as defined before, and the analysis is the same as in Section 6.1.1.

6.1.3. The case $1/(1-\alpha) < \lambda$, with $\overset{\circ}{k}$ crossing \bar{k} at two points

This case is depicted in the following figure.

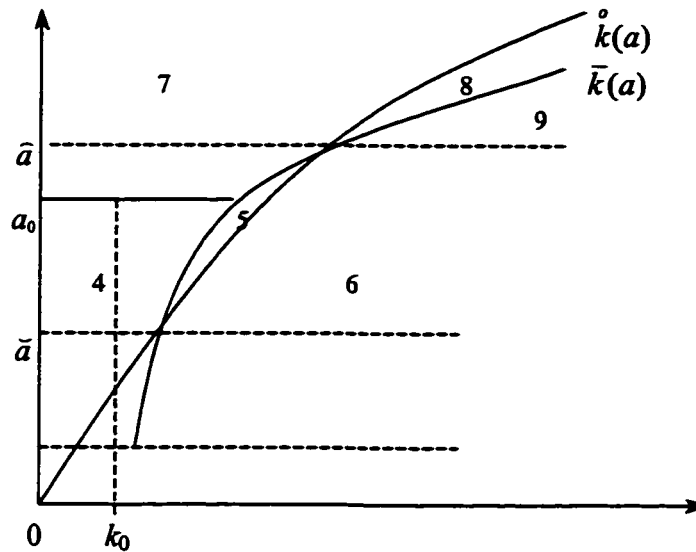


Figure 9.

In Region 7 and 8, firms do not carry out R&D; the economy converges to a stationary equilibrium. In Region 9 firms undertake R&D exactly in the first period, and the system then converges to stationary equilibrium after that. In Region 4, there is no R&D investment initially, and the economy accumulates capital while the technological level stays at a_0 until the capital raised per firm exceeds $[(\frac{\delta(1-\alpha)a_0}{1+\delta})^{1/(1-\alpha)}] / |I|$, at which time a sequential search is triggered.

In Regions 5 and 6 firms carry R&D in period 0. Let (a_1, k_1) be the state of the system at the beginning of period 1. There are two possibilities:

- (i) If (a_1, k_1) belongs to Regions 7 and 8, no more R&D will be carried out, and the economy converges to a stationary equilibrium.
- (ii) If (a_1, k_1) belongs to Region 4, R&D activities will be suspended temporarily until growth moves the economy into Region 5, at which time sequential search resumes.

Observe that for $\bar{a} < a < \hat{a}$, the curve $\overset{\circ}{k}(a), a \geq 1$, is horizontally further to the right (i.e., in the direction of increasing k) than the curve $\bar{k}(a), a \geq 1$. Hence (a_1, k_1) , the state of the system at the beginning of period 1, will never be in Region 6.

In Regions 1 and 2 no R&D will be taken by the firms and the economy converges to a stationary equilibrium. In Region 3, R&D will be carried out in period 0. If (a_1, k_1) belongs to Region 1 or Region 2, the economy will then converge to a stationary equilibrium. If (a_1, k_1) belongs to Regions 4, 7, or 8, the analysis already carried out for these regions can be repeated verbatim.

6.1.4. The case $1/(1-\alpha) < \lambda$, with $\overset{\circ}{k}$ crossing \bar{k} at exactly one point

This possibility is depicted in the following figure, in which the curve $\overset{\circ}{k}(a), a \geq 1$, is further to the right of the curve $\bar{k}(a), a \geq 1$, initially and crossing it from the below. In Regions of 4 and 5, no R&D will be undertaken and the economy converges to a stationary equilibrium. In Region 6, R&D will be carried out only in period 0, and after that the economy converges to stationary equilibrium from period 1 onward. In Regions 2 and 3, R&D will be carried out. In Region 1, the economy grows without undertaking any R&D activities until it enters Region 2 or Region 3. We thus see that \hat{a} plays the role of a critical

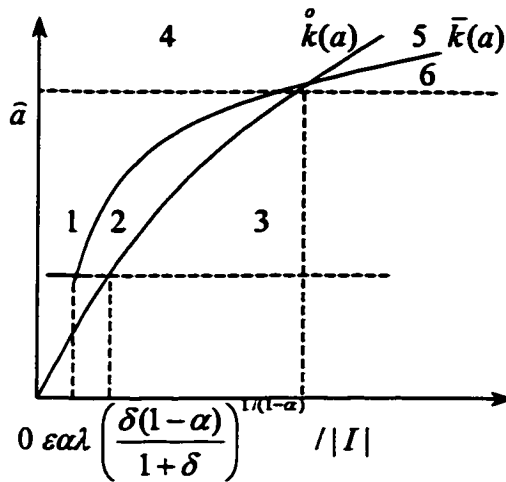


Figure 10.

technological level such that the sequential search will be terminated once the productivity of the economy rises above \hat{a} . Observe that as the number of firms $|I|$ increases the curve $\hat{k}(a)$ shifts to the left and this critical technological will shrink. In other words, a large number of firms limit the economic growth.

PROPOSITION 2: *If $1/(1-\alpha) < \lambda$, then depending on the configuration of the curve \hat{k} and the curve \bar{k} , the economy might engage in R&D for some periods. However, in the long run all growth stops and the economy converges to a stationary equilibrium.*

6.2. The case $\lambda = 1/(1-\alpha)$

In this case, (16) assumes the following form:

$$(19) \quad \frac{\varepsilon(1 + (\alpha\lambda - 1)\alpha^{\lambda})}{\left(\frac{\delta(1-\alpha)a}{1+\delta}\right)^{1/(1-\alpha)} / |I|} = \frac{\varepsilon |I|}{\left(\frac{\delta(1-\alpha)}{1+\delta}\right)^{1/(1-\alpha)}} (a^{-1/(1-\alpha)} + \alpha\lambda - 1),$$

which declines monotonically from $\frac{\varepsilon |I| \alpha \lambda}{\left(\frac{\delta(1-\alpha)}{1+\delta}\right)^{1/(1-\alpha)}}$ to $\frac{\varepsilon |I| (\alpha \lambda - 1)}{\left(\frac{\delta(1-\alpha)}{1+\delta}\right)^{1/(1-\alpha)}}$ as $\alpha \rightarrow +$. There

are three possibilities to consider. The first possibility involves $\frac{\varepsilon |I| (\alpha \lambda - 1)}{\left(\frac{\delta(1-\alpha)}{1+\delta}\right)^{1/(1-\alpha)}} \geq 1$, which

is depicted in the following figure:

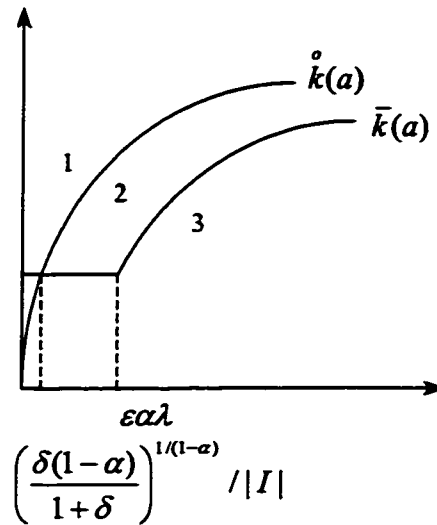


Figure 11.

This case has been dealt with in Proposition 2: R&D is carried out only in period 0 if the initial state of the economy is under the curve $\bar{a}(k)$. After that the system will converge monotonically to a stationary equilibrium without undertaking any R&D. The second possibility occurs when $\frac{\varepsilon |I| (\alpha \lambda - 1)}{\left(\frac{\delta(1-\alpha)}{1+\delta}\right)^{1/(1-\alpha)}} \leq 1$ and is depicted in the following figure.

Under this possibility, the economy will experience sustained long-run growth. More specifically, R&D activities are always carried out when the system is in Regions 2 and 3. If the system is in Region 1, then the economy will grow without undertaking R&D until the

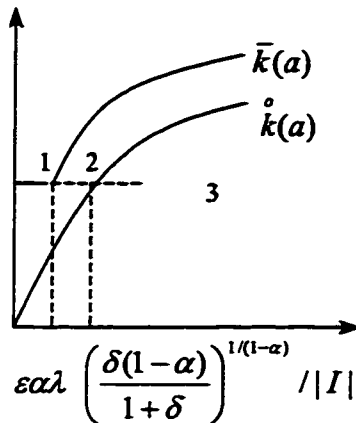


Figure 12.

system enters Region 2, at which time the sequential for a new technology will be launched. Because a random variable following a Pareto distribution can assume any arbitrarily large value with a positive probability, the technological level of the economy as well as the capital raised per firm will tend to infinity with positive probability in the long run: there is sustained growth.

The third possibility occurs if $\frac{\epsilon |I| (\alpha\lambda - 1)}{\left(\frac{\delta(1-\alpha)}{1+\delta}\right)^{1/(1-\alpha)}} < 1 < \frac{\epsilon |I| \alpha\lambda}{\left(\frac{\delta(1-\alpha)}{1+\delta}\right)^{1/(1-\alpha)}}$, i.e., if $\dot{k}(a)$ is

on the right side of $\bar{k}(a)$ but crosses the latter curve from under at exactly one point, as depicted in the following figure.

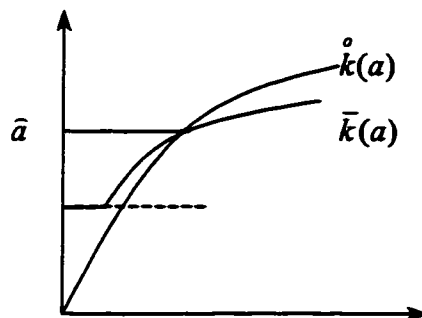


Figure 13.

The economy will carry out a sequential search until its technological level rises above the critical level \bar{a} , at which time all R&D activities will be terminated and the economy will then begin its long march to a stationary equilibrium without any improvement in its technological level.

PROPOSITION 3: *Suppose that $1/(1-\alpha) = \lambda$. There are three possibilities to consider*

(i) If $\frac{\varepsilon |I| (\alpha\lambda - 1)}{\left(\frac{\delta(1-\alpha)}{1+\delta}\right)^{1/(1-\alpha)}} \geq 1$, then the economy might engage in R&D at most in one

period. In the long run, the system converges to a stationary equilibrium.

(ii) If $\frac{\varepsilon |I| \alpha\lambda}{\left(\frac{\delta(1-\alpha)}{1+\delta}\right)^{1/(1-\alpha)}} < 1$, then the economy will experience sustained growth.

(iii) If $\frac{\varepsilon |I| (\alpha\lambda - 1)}{\left(\frac{\delta(1-\alpha)}{1+\delta}\right)^{1/(1-\alpha)}} < 1 < \frac{\varepsilon |I| \alpha\lambda}{\left(\frac{\delta(1-\alpha)}{1+\delta}\right)^{1/(1-\alpha)}}$, then the system might engage in R&D in

short run. However, in the long run growth stops and economy converges to a stationary equilibrium.

6.3. The case $1/(1-\alpha) > \lambda$

When $1/(1-\alpha) > \lambda$, the curve $\overset{\circ}{k}(a)$ will be to the right of the curve $\bar{k}(a)$ when a is large.

There are two possibilities to consider. First, we consider the possibility that the curve

$\overset{\circ}{k}(a)$ is on the left of the curve $\bar{k}(a)$ at the beginning, when a rises from 1 then cross the

latter curve from above. This case occurs when $\varepsilon\alpha\lambda > \left(\frac{\delta(1-\alpha)}{1+\delta}\right)^{1/(1-\alpha)} / |I|$ and is depicted in the following figure.

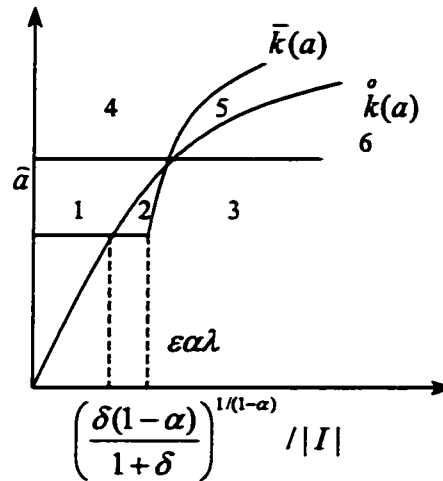


Figure 14.

If (a_0, k_0) is located in Region 4, the capital stock per firm grows and R&D is not undertaken until the system enters Region 5, at which time a sequential search is triggered. If (a_0, k_0) is in Region 5 or Region 6, R&D will take place in period 0. The state of the system at the beginning of period 1, namely (a_1, k_1) will be either in Region 4 or Region 5, but not in Region 6. If (a_1, k_1) belongs to Region 4, R&D activities will be temporarily suspended until the system returns to Region 5, at which time a sequential search will be resumed. If (a_1, k_1) belongs to region 5, R&D activities will be undertaken in period 1, and so forth. Obviously, if $a_0 > \hat{a}$, there will be sustained growth.

On the other hand, if the economy is located in Regions 1 and 2, the firms will not carry out R&D and the economy converges to stationary equilibrium. If (a_0, k_0) is located in region 3, R&D activity will be undertaken in period 0. The state of the system at the

beginning of period will not be in Region 3. If (a_1, k_1) is in Region 1 or Region 2, then the technological level of economy will remain forever at a_1 , and the system will converge to stationary equilibrium. And if $a_1 > \hat{a}$, the economy enters the region of sustained growth.

The argument in the preceding paragraph indicates that if a_0 and k_0 are low, there is no growth in the long run even though the economy has the potential to grow. If $a_0 < \hat{a}$ but for capital level of $k_0 > \varepsilon(1 + (\alpha\lambda - 1)a_0^{\lambda})$, R&D activities will be undertaken in period 0, and there is a positive probability that $a_1 > \hat{a}$, under which event the economy escapes the poverty trap. If $a_0 > \hat{a}$, there is sustained growth in the long run. Observe that an increase in the number of firms induces a rise in \hat{a} , i.e., getting out of the poverty trap becomes more difficult for the economy. We can conclude in this case that in the preliminary stage of growth, monopoly favors sustained growth.

Second, we consider the possibility $\varepsilon\alpha\lambda \leq \left(\frac{\delta(1-\alpha)}{1+\delta}\right)^{1/(1-\alpha)} |I|$. Under this

possibility, the curve $\overset{\circ}{k}(a)$ is completely on the right of the curve $\bar{k}(a)$, as depicted in the following figure.

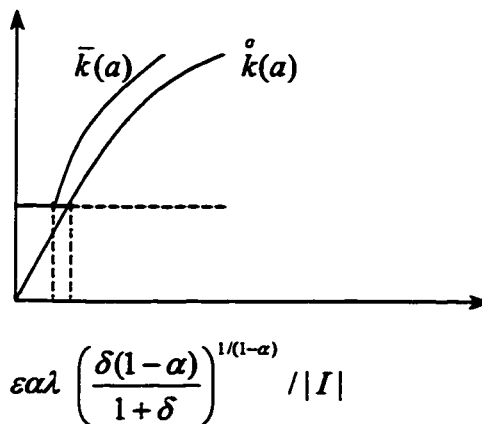


Figure 15.

There is sustained growth in the long run no matter what the initial condition (a_0, k_0) is. In this case the productivity of R&D is high or the cost of search is low.

PROPOSITION 4: *Suppose that $1/(1-\alpha) > \lambda$. There are two possibilities to consider*

- (i) *If $\varepsilon\alpha\lambda > \left(\frac{\delta(1-\alpha)}{1+\delta}\right)^{1/(1-\alpha)} / |I|$, then the configuration of $\overset{\circ}{k}(a)$ and $\bar{k}(a)$ as depicted in*

Figure 14 the system will experience sustained growth if initial stage is in Regions 4, 5, or 6. On the other hand, if the initial stage of system is in Regions 1 or 2, no R&D will carry out. Under this scenario the system will converge to stationary equilibrium also it has a potential to grow indefinitely. If the state of system in Region 3 the R&D activity will carry out.

- (ii) *If $\varepsilon\alpha\lambda \leq \left(\frac{\delta(1-\alpha)}{1+\delta}\right)^{1/(1-\alpha)} / |I|$. Then economy will experience sustained growth.*

7. THE ASYMPTOTIC BEHAVIOR OF THE SYSTEM UNDER SUSTAINED GROWTH ($1/(1-\alpha) > \lambda$)

Let \hat{a} be the technological level defined as follows. If $\varepsilon\alpha\lambda \leq \left(\frac{\delta(1-\alpha)}{1+\delta}\right)^{1/(1-\alpha)} / |I|$,

set $\hat{a}=1$; otherwise, let \hat{a} be the unique value of $a \geq 1$ that satisfies

$\left(\frac{\delta(1-\alpha)a}{1+\delta}\right)^{1/(1-\alpha)} / |I| = \varepsilon(1 + (\alpha\lambda - 1)a^\lambda)$. As defined, \hat{a} is the threshold technological

level such that the economy will enter the zone of sustained growth once its technological level rises above \hat{a} . We shall now study the behavior of the economy once it enters this zone.

For each $a \geq 1$, let $(k^n(a))_{n=0}^{\infty}$ be the sequence defined recursively as follows. For $n = 0$, set $k^n(a) = \bar{k}(a) = \varepsilon(1 + (\alpha\lambda - 1)a^{\lambda})$. For $n \geq 0$, $k^{n+1}(a)$ is the unique value of the k that satisfies the following relation:

$$\left(\frac{\delta(1-\alpha)}{1+\delta} \right) a (|I| k^{n+1})^{\alpha} / |I| = k^n(a), \quad (n = 0, 1, \dots),$$

i.e.,

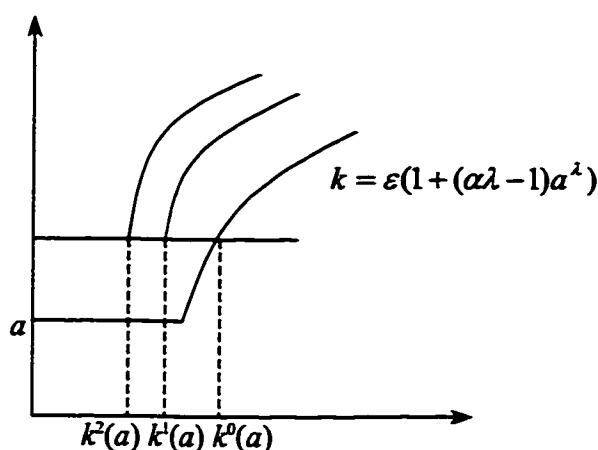


Figure 16.

$$[k^{n+1}(a)]^{\alpha} = \frac{|I|^{1-\alpha} k^n(a)}{\left(\frac{\delta(1-\alpha)a}{1+\delta} \right)}, \quad (n = 0, 1, \dots).$$

As defined, $(a, k^n(a))$, $n = 0, 1, \dots$, is the state of the economy in the next period, given that $(a, k^{n+1}(a))$ is the state in the current period. Now we follow an induction to compute $k^n(a)$.

We have

$$k^1(a) = \left[\frac{|I|^{1-\alpha} \varepsilon}{\left(\frac{\delta(1-\alpha)}{1+\delta} \right)} (1 + (\alpha\lambda - 1)a^{\lambda}) / a \right]^{1/\alpha}.$$

Because $\lambda > 1$, $k^1(a)$ will rise monotonically to $+\infty$ as $a \rightarrow +\infty$. Also, note that

$$\frac{k^1(a)}{k^0(a)} = \left[\frac{(\varepsilon |I|)^{1-\alpha} (1 + (\alpha\lambda - 1)a^\lambda)^{1-\alpha}}{\left(\frac{\delta(1-\alpha)}{1+\delta}\right) a} \right]^{1/\alpha}.$$

Now in the case we are considering $1/(1-\alpha) > \lambda$, which implies that $0 > \lambda(1-\alpha) - 1$. It

follows from this last inequality that

$$\lim_{a \rightarrow \infty} [(1 + (\alpha\lambda - 1)a^\lambda)^{1-\alpha} / a] = \lim_{a \rightarrow \infty} a^{\lambda(1-\alpha)-1} = 0.$$

Hence $\lim_{a \rightarrow \infty} \frac{k^1(a)}{k^0(a)} = 0$, i.e., the horizontal distance of $k^0(a) - k^1(a)$ tends to infinity when

$a \rightarrow +\infty$.

As for $k^2(a)$, we have;

$$\begin{aligned} [k^2(a)]^\alpha &= |I|^{1-\alpha} k^1(a) / \left(\frac{\delta(1-\alpha)a}{1+\delta}\right) \\ &= (|I|^{1-\alpha} / \left(\frac{\delta(1-\alpha)}{1+\delta}\right)a) [|I|^{1-\alpha} \varepsilon (1 + (\alpha\lambda - 1)a^\lambda) / \left(\frac{\delta(1-\alpha)}{1+\delta}\right)a]^{1/\alpha}. \end{aligned}$$

Thus

$$k^2(a) = [(\varepsilon |I|^{1-\alpha^2} / (\frac{\delta(1-\alpha)}{1+\delta} a)^{1+\alpha}) (1 + (\alpha\lambda - 1)a^\lambda)]^{1/\alpha^2}.$$

Now suppose that for any $n = 0, 1, \dots$, we have

$$k^n(a) = \left[\frac{\varepsilon |I|^{1-\alpha^n} (1 + (\alpha\lambda - 1)a^\lambda)}{\left(\frac{\delta(1-\alpha)}{1+\delta}\right)^{1+\alpha+\dots+\alpha^{n-1}} a^{1+\alpha+\dots+\alpha^{n-1}}} \right]^{1/\alpha^n}.$$

Using the definition of $k^n(a)$ and $k^{n+1}(a)$, we can write

$$[k^{n+1}(a)]^\alpha = \frac{|I|^{(1-\alpha)} k^n(a)}{\left(\frac{\delta(1-\alpha)}{1+\delta}\right)a} = \left[\frac{\varepsilon |I|^{(1-\alpha)\alpha^n + 1 - \alpha^n} (1 + (\alpha\lambda - 1)a^\lambda)}{\left(\frac{\delta(1-\alpha)}{1+\delta}\right)^{1+\alpha+\dots+\alpha^n} a^{1+\alpha+\dots+\alpha^n}} \right]^{1/\alpha^n}$$

Thus

$$k^{n+1}(a) = \left[\frac{\varepsilon |I|^{1-\alpha^{n+1}} (1 + (\alpha\lambda - 1)a^\lambda)}{\left(\frac{\delta(1-\alpha)}{1+\delta}\right)^{1+\alpha+\dots+\alpha^n} a^{1+\alpha+\dots+\alpha^n}} \right]^{1/\alpha^{n+1}}, \quad (n = 0, 1, \dots).$$

Now using the assumption $1/(1-\alpha) = 1 + \alpha + \alpha^2 + \dots > \lambda$, we can assert the existence of a positive integer, say $n(\alpha, \lambda)$, such that

$$(20) \quad \begin{aligned} 1 + \alpha + \alpha^2 + \dots + \alpha^n &< \lambda && \text{if } n \leq n(\alpha, \lambda), \\ 1 + \alpha + \alpha^2 + \dots + \alpha^n &> \lambda && \text{if } n > n(\alpha, \lambda). \end{aligned}$$

Using (20), we can then assert that when $n > n(\alpha, \lambda)$, we must have $\lim_{a \rightarrow 0} k^{n+1}(a) = 0$.

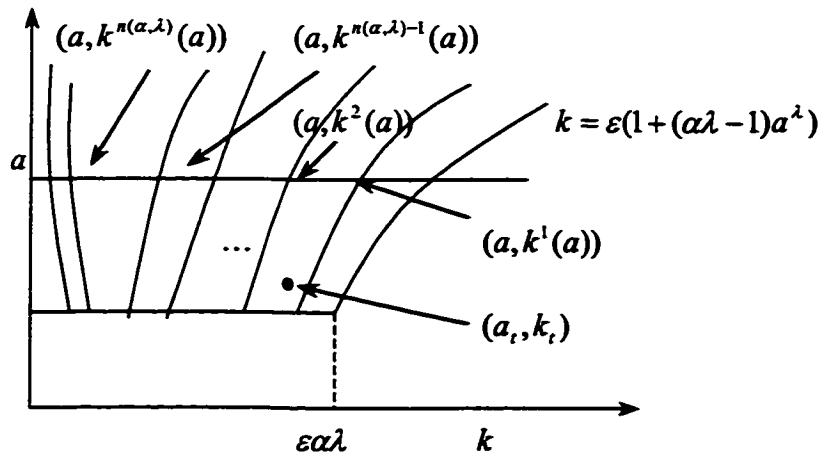


Figure 17.

If (a_t, k_t) , the state of economy at the beginning of period t , $t = 0, 1, \dots$, satisfies $k^n(a_t) < k_t \leq k^{n-1}(a_t)$, then R&D activities will not be undertaken in period $t, t+1, \dots, t+n-1$; the sequential search will only be resumed in period $t+n$.

Suppose that the economy is on a sustained growth path. For each period $t = 0, 1, \dots$, let $(\hat{a}_t, \hat{k}_t)_{i \in I}$ be the state of the economy at the end of the sequential search in period t . The state of the economy at the beginning of period $t+1$ is then given by

$$(a_{t+1}, Y_{t+1}) = (\max_{i \in I} \hat{a}_t, (\sum_{i \in I} \hat{a}_t^{1/\alpha} \hat{k}_t)^\alpha).$$

The state of a firm in period $t+1$ is thus

$$(a_{t+1}, k_{t+1}) = (\max_{i \in I} \hat{a}_t, \frac{\delta(1-\alpha)}{(1+\delta)|I|} (\sum_{i \in I} \hat{a}_t^{1/\alpha} \hat{k}_t)^\alpha).$$

Because the economy is on a sustained growth path, both a_t and k_t tend to infinity as t tends to infinity. Furthermore, because for any $n \geq n(\alpha, \lambda)$, $k^n(a)$ is a strictly decreasing to zero as a tends to infinity, the state of a firm at any point in time on the sustained growth path will remain on the right of the curve $k^n : a \rightarrow k^n(a)$, i.e., $k_t > k^n(a_t)$ for all t greater than a certain positive integer. Hence the number of periods during which the economy grows without engaging in R&D is bounded above by $n(\alpha, \lambda)$. Now for a given value of k_t , an increase in a_t will move the state (a_t, k_t) vertically upward, inducing a longer phase of growth without R&D. On the other hand, an increase in k_t , with a_t maintained at the same level, induces a shortening of such a phase. A successful search in period $t-1$ implies a rise in both a_t and k_t and it is not clear whether the search will be resumed sooner or later. We summarize the results just discussed in the following proposition.

PROPOSITION 5: *Suppose that $1/(1-\alpha) > \lambda$. Also, suppose that the economy is in the region of sustained growth path; that is, at a certain point in time, say t , the technological level a_t exceeds the critical value \bar{a} , defined at the beginning of Section 7. Then along the sustained growth path, the length of each phase during which the economy grows without engaging in R&D is bounded above by the positive integer defined by (20). It is not clear whether a more successful search will lengthen the no R&D phase that begins in the next period.*

Consider a technological level a and let c be the point $(a, \varepsilon(1 + (\alpha\lambda - 1)a^\lambda))$. If c is the state of the economy at the beginning of the current period, then the state of the economy at the beginning of the following period is the point

$$\begin{aligned} c' &= \left(a, \frac{\delta(1-\alpha)a}{|I|(1+\delta)} (|I|\varepsilon(1 + (\alpha\lambda - 1)a^\lambda))^\alpha \right) \\ &= \left(a, \frac{\delta(1-\alpha)a}{|I|^{1-\alpha}(1+\delta)} (\varepsilon(\alpha\lambda - 1)a^\lambda (1/(\alpha\lambda - 1)a^\lambda + 1))^\alpha \right) \\ &= \left(a, \frac{\delta(1-\alpha)}{|I|^{1-\alpha}(1+\delta)} (\varepsilon(\alpha\lambda - 1))^\alpha a^{\alpha\lambda+1} (1/(\alpha\lambda - 1)a^\lambda + 1)^\alpha \right). \end{aligned}$$

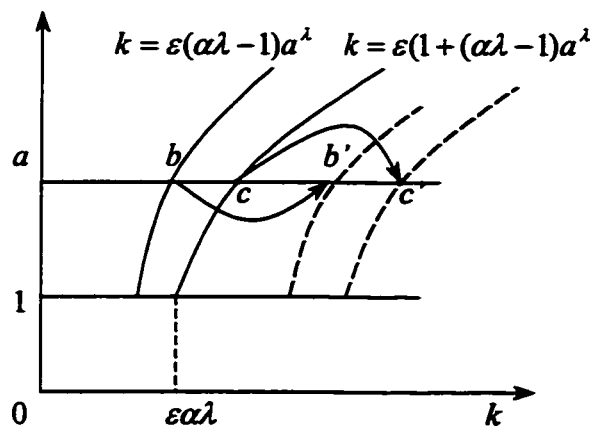


Figure 18.

If the point $b = (a, \varepsilon(\alpha\lambda - 1)a^\lambda)$ represents the state of the economy at the beginning of the current period, then the state of the economy at the beginning of the following period is represented by the point

$$\begin{aligned} b' &= \left(a, \frac{\delta(1-\alpha)a}{|I|(1+\delta)} (|I| \varepsilon(\alpha\lambda - 1)a^\lambda)^\alpha \right) \\ &= \left(a, \frac{\delta(1-\alpha)}{|I|^{1-\alpha}(1+\delta)} (\varepsilon(\alpha\lambda - 1)^\alpha a^{\alpha\lambda+1}) \right). \end{aligned}$$

The horizontal distance of point c relative to point b' is

$$\frac{\varepsilon(1 + (\alpha\lambda - 1)a^\lambda)}{\frac{\delta(1-\alpha)}{|I|^{1-\alpha}(1+\delta)} (\varepsilon(\alpha\lambda - 1))^\alpha a^{\alpha\lambda+1}}$$

In the present section, it is assumed that $1/(1-\alpha) > \lambda$, i.e., $\lambda - \alpha\lambda - 1 < 0$. Hence this ratio tends to 0 as a tends to infinity. The distance cb' —and a fortiori the distance bb' —thus tends to infinity as a tends to infinity. The horizontal distance between point b' and point c' is

$$\frac{\delta(1-\alpha)}{|I|^{1-\alpha}(1+\delta)} (\varepsilon(\alpha\lambda - 1))^\alpha a^{\alpha\lambda+1} ((1 + 1/(\alpha\lambda - 1)a^\lambda)^\alpha - 1),$$

which is shown in Appendix 4 to tend to infinity as a tends to infinity.

Now consider a period t in which R&D activities are carried out and suppose that the state of economy at the beginning of this period is (a_t, k_{it}) . If the R&D activities of the firms do not yield any technology better than a_t , then the state of a typical firm, say i , at the beginning of the production stage of period t is represented by the point $(\hat{a}_{it}, \hat{k}_{it}) = (a_t, k_{it})$, a point inside the interval $(b, c]$. The state of the economy at the beginning of period $t+1$ is (a_{t+1}, k_{t+1}) , which belongs to the interval $(b', c']$. If the R&D activities carried out by the

firms in period $t+1$ are a complete failure, the state of the system at the beginning of the production phase will again lie in the interval $(b, c]$, which

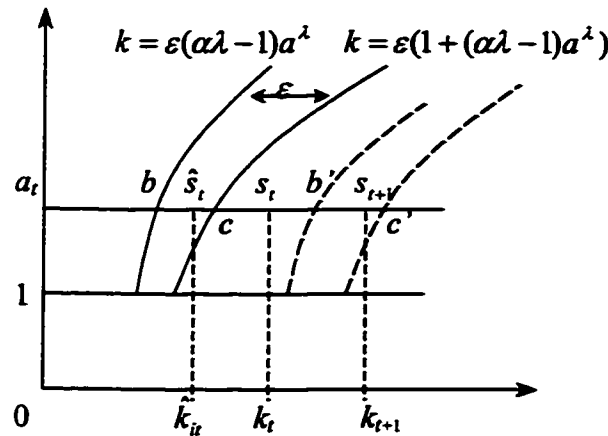


Figure 19.

in turn implies that the state of economy at the beginning of period $t+2$ will again belong to the interval $(b', c']$. This process might persist for a number of periods, and the economy's technological level remains at a_t during this phase, although the capital stock per firm during periods might vary between the horizontal distance of b' and the horizontal distance of c' . Bental and Peled, *op. cit.*, called the states that the economy assumes during such a phase *quasi-steady states*. During such a phase, the minimum number of technology draws taken by a firm is $\text{Ceiling}\left[\frac{d(c, b')}{\epsilon}\right]$, where $d(c, b')$ is the distance between point c and point b' , and $\text{Ceiling}[x]$ is the smallest integer greater than or equal to x . Also, the maximum number of technology draws taken by a firm is $\text{Ceiling}\left[\frac{d(c, c')}{\epsilon}\right]$, where $d(c, c')$ is the distance between points c and points c' . We already shown that $d(c, b') \rightarrow \infty$ and $d(b', c') \rightarrow \infty$ as $a \rightarrow \infty$. Hence for an advanced economy (i.e., an economy with a high value of a) more and

more resources will be devoted to R&D, actually without improving the production capacity of the economy. Under such a scenario, the probability of finding no better technology by a firm in a period is bounded above by

$$(21) \quad (1 - a^{-\lambda})^{\text{Ceiling}\left[\frac{d(c,b')}{\varepsilon}\right]},$$

and bounded below by

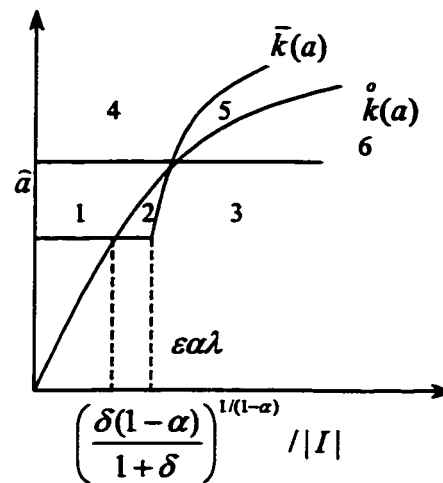
$$(22) \quad (1 - a^{-\lambda})^{\text{Ceiling}\left[\frac{d(c,c')}{\varepsilon}\right]}.$$

In Appendix 5, we show that (21) is strictly decreasing to zero when $a \rightarrow +\infty$. Thus as the economy becomes more advanced, the probability of failing to find a better technology tends to decrease. We summarize the results just discussed in the following proposition.

PROPOSITION 6: *Suppose that $1/(1 - \alpha) > \lambda$. Also, suppose that the economy is in the region of sustained growth. Then along a sustained growth path, more and more resources are devoted to R&D. Furthermore, the ever increase amount of resources devoted to R&D more than compensates for the diminishing return to R&D, and the end result is that the probability of success in discovering new and better technologies tends to one when the technological level $a \rightarrow +\infty$.*

8. A NUMERICAL EXAMPLE

The model was simulated for a 20-period time horizon. The following values for the parameters were used: $\alpha = 0.7$, $\lambda = 1.49$, $\varepsilon = 2.0$, and $\delta = 1$. Note that $1/(1 - \alpha) > \lambda$. Hence there is sustained growth.



The simulation started at a technological level above \bar{a} and in Region 5 of the above figure. The initial aggregate capital stock is 30 and there are three firms in the economy. The initial technological level is 20. The realized states of the economy, (a_t, K_t) —averaged over one hundred simulations—for the 20 successive periods are shown in the following table:

Period	Technological level	Aggregate capital
0	20	30
1	20.65	28.10
2	20.65	32.57
3	20.82	32.53
4	20.82	35.51
5	23.01	37.19
6	23.16	48.76
7	23.16	52.21
8	24.34	57.24
9	24.34	66.22
10	26.72	68.32
11	30.35	139.27
12	30.82	273.76
13	34.21	403.20
14	37.82	827.48
15	41.19	1235.25
16	62.61	4065.82
17	63.15	7965.76
18	93.35	17680.6
19	279.80	57413.5
20	465.15	575837.

9. CONCLUSION

In this chapter, we have presented and analyzed exhaustively a one-sector model of endogenous growth in which firms can improve their productivity by engaging in a sequential search for a new technology. The model we have formulated is based upon the work of Bental and Peled (1996). In the model, the sequential search, whose outcomes are random, is interpreted as a process of testing new and untried technologies. The testing, also called taking technology draws, require the expenditure of cumulable capital, say laboratory equipment. To operate in any period, a firm must raise capital from the young generation of the previous period. Part of the capital raised by a firm at the beginning of every period is spent in the sequential search; what is left of the capital it raised will be combined with labor according to the best technology at its disposal at the end of the search to produce the consumption good.

The new and untried technologies as well as those already found are assumed to come from a population of Cobb-Douglas technologies using labor and capital as inputs. The technologies in this population all have the same elasticity of output with respect to the capital input and what differentiates one member of the population from another is the difference in their technological levels, which are assumed to follow a Pareto distribution. The Pareto distribution that characterizes the technological levels of the technologies in the population is unbounded from above and thus the probability of finding a technology with a technological level higher than the one currently available is always positive. There is thus a possibility for sustained and unending growth. Another property of the Pareto distribution is that it exhibits diminishing returns: the higher is the current technological level, the harder it is to find a new and better technology using the same amount of cumulable capital as input

in the sequential search process. Such a distribution thus allows for the possibility of growth while not contradicting the empirical evidence of diminishing returns to R&D.

As the aggregate capital stock increases and the technological level of the economy remains the same, diminishing returns will set in. There comes a point when the marginal product of capital, given a constant labor force, is so low that it is better to use up part of the capital already accumulated to find a new and better technology. If the sequential search is successful and a new technology with a higher technological level has been found, the search is temporarily suspended. Production is now carried out using the latest technology, and more capital will be accumulated. Again, even with the new and better technology, there comes a point in time when diminishing returns reappear and a new sequential search is called for. The model thus exhibits cycles constituting of a phase during which R&D activities are undertaken to be followed by periods during which production model is carried out using the same technology and during which capital is accumulated.

Now the Pareto distribution is stochastically decreasing in its parameter, i.e., a distribution with a low parameter is more valuable than one with a higher parameter. Also, a higher elasticity of output with respect to the input capital enhances the contribution of this factor. The model of this chapter asserts that a low parameter for the Pareto distribution coupled with a high value for the elasticity of output with respect to the capital input could generate sustained growth. We have also shown that as the economy embarks on a sustained growth path, more and more resources are devoted to the sequential search. In its advanced state, the ever increasing amounts of capital used up in the search process more than compensate for the diminishing returns in R&D in the sense that the lower bound on the probability of failure tends to zero as the economy's technological level tends to infinity. In

the next chapter, we extend this model to a two-sector economy that is made up of a high-technology good sector similar to the one analyzed in this chapter and a traditional good sector without any prospect for technological advance.

APPENDIX 1 THE SEQUENTIAL SEARCH

Suppose that when period t begins the technological level of the high-technology good sector is a_t and its capital stock is K_t . Each firm i in the high-technology good sector thus begins the sequential stage of period t with $a_i = a_t$ as its technological level and $k_i = K_t / |I|$ as the amount of capital it has managed to raise in the capital allocation stage of this period. Let $G(w)$ be the distribution function of the wage rate that a firm i expects to prevail in the production stage of period t .

Let us situate ourselves at a particular instant during the sequential stage of period t and consider a particular firm in the high-technology good sector. Suppose that the highest technological level at the disposal of this firm is a and its remaining capital is k . The question we need to answer is whether the firm should continue or terminate its sequential search. To answer this question, we now proceed with a dynamic programming argument.

A1.1. *The case $0 < k < \varepsilon$*

When $0 < k < \varepsilon$, the optimal decision for the firm is to terminate the sequential search and invest the remaining capital in the technology with productivity a . Given the wage rate w , the profit associated with this decision is

$$\alpha \left(\frac{1-\alpha}{w} \right)^{\frac{1-\alpha}{\alpha}} a^{1/\alpha} k.$$

The expected profit of the firm, given its expectations about the wage rate prevailing in the production and consumption stage of period t is thus given by

$$\begin{aligned}
 v_1(a, k | G) &= \int \alpha \left(\frac{1-\alpha}{w} \right)^{\frac{1-\alpha}{\alpha}} a^{1/\alpha} k dG(w) \\
 &= \alpha a^{1/\alpha} k \int \left(\frac{1-\alpha}{w} \right)^{\frac{1-\alpha}{\alpha}} dG(w) \\
 &= v_1(a, k) \int \left(\frac{1-\alpha}{w} \right)^{\frac{1-\alpha}{\alpha}} dG(w), \quad 0 < k < \varepsilon.
 \end{aligned}$$

Where we have let

$$v_1(a, k) = \alpha a^{1/\alpha} k, \quad 0 < k < \varepsilon.$$

A1.2. The Case $\varepsilon < k < 2\varepsilon$

When $\varepsilon < k < 2\varepsilon$, the firm can take one more technology draw without reducing its capital stock to 0. If it terminates the sequential search, then its expected profit is given by

$$(A1.2.1) \quad \alpha a^{1/\alpha} k \int \left(\frac{1-\alpha}{w} \right)^{\frac{1-\alpha}{\alpha}} dG(w).$$

On the other hand, if it decides to take another technology draw, then its expected profit is given by

$$\begin{aligned}
 (A1.2.2) \quad & F(a)v_1(a, k - \varepsilon | G) + \int_a^\infty v_1(a', k - \varepsilon | G) dF(a') \\
 &= [F(a)v_1(a, k - \varepsilon) + \int_a^\infty v_1(a', k - \varepsilon) dF(a')] \int \left(\frac{1-\alpha}{w} \right)^{\frac{1-\alpha}{\alpha}} dG(w).
 \end{aligned}$$

In (A1.2.2), the first term on the left side captures the component of the expected

profit under the event that the technology draw fails to yield a better technology. This event occurs with probability $F(a)$ and conditioned on this event the expected profit of the firm is $v_1(a, k - \varepsilon | G)$. The second term on the left of (A1.2.2) captures the component of the expected profit under the event that the technology draw results in a new technology with productivity $a' > a$. Given $a' > a$, the expected profit is $v_1(a', k - \varepsilon | G)$.

The optimal expected profit of the firm is then

$$\begin{aligned}
& v_2(a, k - \varepsilon | G) \\
&= \max \left\{ [F(a)v_1(a, k - \varepsilon) + \int_a^\infty v_1(a', k - \varepsilon | G) dF(a')], \alpha a^{1/\alpha} k \right\} \int \left(\frac{1 - \alpha}{w} \right)^{\frac{1 - \alpha}{\alpha}} dG(w) \\
&= \max \left\{ [F(a)\alpha a^{1/\alpha} (k - \varepsilon) + \int_a^\infty \alpha (a')^{1/\alpha} (k - \varepsilon) dF(a')], \alpha a^{1/\alpha} k \right\} \int \left(\frac{1 - \alpha}{w} \right)^{\frac{1 - \alpha}{\alpha}} dG(w) \\
&= \alpha k \max \left\{ \left(1 - \frac{\varepsilon}{k}\right) [F(a)a^{1/\alpha} + \int_a^\infty (a')^{1/\alpha} dF(a')], a^{1/\alpha} \right\} \int \left(\frac{1 - \alpha}{w} \right)^{\frac{1 - \alpha}{\alpha}} dG(w) \\
&= \alpha k \max \{ \varphi_2(a, k), a^{1/\alpha} \} \int \left(\frac{1 - \alpha}{w} \right)^{\frac{1 - \alpha}{\alpha}} dG(w) \\
&= v_2(a, k) \int \left(\frac{1 - \alpha}{w} \right)^{\frac{1 - \alpha}{\alpha}} dG(w),
\end{aligned}$$

Where we have defined

$$(A1.2.3) \quad \varphi_2(a, k) = \left(1 - \frac{\varepsilon}{k}\right) \left[F(a)a^{1/\alpha} + \int_a^\infty (a')^{1/\alpha} dF(a') \right],$$

$$v_2(a, k) = \alpha k \max \{ \varphi_2(a, k), a^{1/\alpha} \},$$

Now let $m_2(a, k) = \varphi_2(a, k) - a^{1/\alpha}$,

Then

$$\begin{aligned}
 (A1.2.4) \quad \frac{\partial m_2(a, k)}{\partial a} &= \frac{\partial(\varphi_2(a, k) - a^{1/\alpha})}{\partial a} \\
 &= \left[\left(1 - \frac{\varepsilon}{k}\right) F(a) - 1 \right] \frac{a^{(1-\alpha)/\alpha}}{\alpha} < \frac{-\varepsilon a^{(1-\alpha)/\alpha}}{k \alpha} < \frac{-\varepsilon}{\alpha k} \leq \frac{-1}{2\alpha}
 \end{aligned}$$

Because the partial derivative $\frac{\partial m_2(a, k)}{\partial a}$ is negative and bounded above by $\frac{-1}{2\alpha}$, it is clear that (A1.2.2) will be negative—and thus the firm will not take another technology draw—if a is large. When $a = 1$, we have

$$(A1.2.5) \quad m_2(1, k) = \frac{1 - \alpha\varepsilon/k}{-1 + \alpha\lambda}.$$

To insure the convergence of the integral in (A1.2.3), we must and will assume that $-1 + \alpha\lambda > 0$. Furthermore, depending on the values of $\alpha, \lambda, \varepsilon$, and k , the right side of the last equality in (A1.2.5) might be positive, negative, or equal to 0. Note that if $-1 + \alpha\lambda > 0$ then $m_2(1, k) < 0$ when k is in a right neighborhood of ε . If $m_2(1, k) \geq 0$, there exists a unique value of $a \geq 1$ such that $m_2(a, k) = 0$. Such a value of a satisfies the following equation:

$$(A1.2.6) \quad \varphi_2(a, k) - a^{1/\alpha} = 0,$$

and will be denoted by $\bar{a}(k)$. When $m_2(1, k) < 0$ there is no value of a that satisfies (A1.2.1). In this case, we set $\bar{a}(k) = 1$. We can extend $\bar{a}(k)$ in the obvious way to $0 < k < 2\varepsilon$ by setting $\bar{a}(k) = 1$ for $0 < k < \varepsilon$. As defined, $\bar{a}(k)$, $0 < k < 2\varepsilon$ represent threshold technological level such that the sequential search will continue as long as the highest technological level under the firm's command falls short of $\bar{a}(k)$, i.e., as long as $a < \bar{a}(k)$, where k is the firm's remaining capital. Observe that the curve $a \rightarrow m_2(a, k)$, which has been shown to be strictly downward sloping, shifts upward when k increases. Hence the

curve $k \rightarrow \bar{a}(k), 0 < k < 2\varepsilon$, is increasing and will be strictly increasing once it has risen over 1. As already explained, $m_2(1, k) < 0$ when k is in a right neighborhood of ε . Hence $\bar{a}(k) = 1$ for k in a right neighborhood of ε . The following figure depicts the behavior of $k \rightarrow \bar{a}(k), 0 < k < 2\varepsilon$.

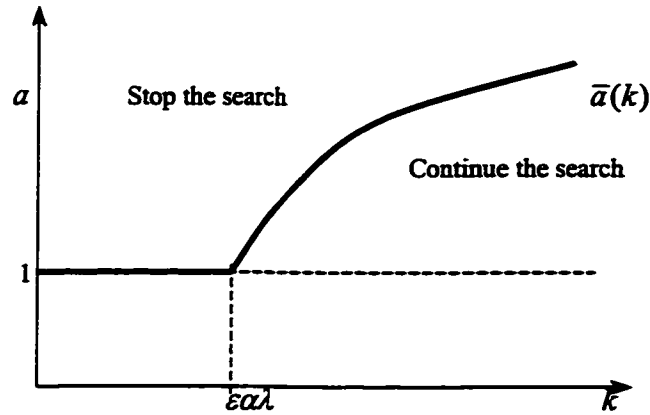


Figure 20.—The threshold technological level $\bar{a}(k)$

A1.3. The Case $2\varepsilon < k < 3\varepsilon$

When $2\varepsilon < k < 3\varepsilon$, the firm can take one more technology draw without reducing its capital stock to 0. If it terminates the sequential search, then its expected profit is given by

$$\alpha a^{1/\alpha} k \int \left(\frac{1-\alpha}{w} \right)^{(1-\alpha)/\alpha} dG(w).$$

On the other hand, if it decides to take another technology draw, then its expected profit is given by

$$F(a)v_2(a, k - \varepsilon | G) + \int_a^\infty v_2(a', k - \varepsilon | G) dF(a')$$

$$= [F(a)v_2(a, k - \varepsilon) + \int_a^{\infty} v_2(a', k - \varepsilon) dF(a')] \int \left(\frac{1 - \alpha}{w} \right)^{(1 - \alpha)/\alpha} dG(w)$$

Hence the firm's optimal expected profit is given by

$$\begin{aligned} v_3(a, k | G) &= \max \left\{ [F(a)v_2(a, k - \varepsilon) + \int_a^{\infty} v_2(a', k - \varepsilon) dF(a')], \alpha a^{1/\alpha} k \right\} \\ &\quad \int \left(\frac{1 - \alpha}{w} \right)^{(1 - \alpha)/\alpha} dG(w), \\ &= \max \left\{ \begin{aligned} &[F(a)\alpha(k - \varepsilon) \max\{\varphi_2(a, k - \varepsilon), a^{1/\alpha}\} + \\ &\int_a^{\infty} \alpha(k - \varepsilon) \max\{\varphi_2(a', k - \varepsilon), (a')^{1/\alpha}\} dF(a')], \alpha a^{1/\alpha} k \end{aligned} \right\} \\ &\quad \int \left(\frac{1 - \alpha}{w} \right)^{(1 - \alpha)/\alpha} dG(w), \\ &= \alpha k \max \left\{ \begin{aligned} &(1 - \frac{\varepsilon}{k}) [F(a) \max\{\varphi_2(a, k - \varepsilon), a^{1/\alpha}\} + \\ &\int_a^{\infty} \max\{\varphi_2(a', k - \varepsilon), (a')^{1/\alpha}\} dF(a')], a^{1/\alpha} \end{aligned} \right\} \\ &\quad \int \left(\frac{1 - \alpha}{w} \right)^{(1 - \alpha)/\alpha} dG(w), \\ &= \alpha k \max\{\varphi_3(a, k - \varepsilon), a^{1/\alpha}\} \int \left(\frac{1 - \alpha}{w} \right)^{(1 - \alpha)/\alpha} dG(w), \\ &= v_3(a, k) \int \left(\frac{1 - \alpha}{w} \right)^{(1 - \alpha)/\alpha} dG(w), \end{aligned}$$

where we have defined

$$\begin{aligned} \varphi_3(a, k - \varepsilon) &= (1 - \frac{\varepsilon}{k}) [F(a) \max\{\varphi_2(a, k - \varepsilon), a^{1/\alpha}\} \\ &\quad + \int_a^{\infty} \max\{\varphi_2(a', k - \varepsilon), (a')^{1/\alpha}\} dF(a')] \end{aligned}$$

$$v_3(a, k) = \max\{\varphi_3(a, k - \varepsilon), a^{1/\alpha}\}.$$

Observe that $\varphi_3(a, k - \varepsilon)$, $a \geq 1$, $2\varepsilon < k < 3\varepsilon$ is continuous in (a, k) . It is also strictly increasing in both a and k . Its partial derivative with respect to a exists at any point (a, k) with $a > 1$ except at the point $(\bar{a}(k), k)$ where $\bar{a}(k) > 1$.

Now define

$$m_3(a, k) = \varphi_3(a, k - \varepsilon) - a^{1/\alpha}$$

Next, observe that

$$\begin{aligned} \max\{\varphi_2(a, k - \varepsilon), a^{1/\alpha}\} &= \varphi_2(a, k - \varepsilon) \text{ if } a < \bar{a}(k - \varepsilon), \\ &= a^{1/\alpha}, \text{ otherwise.} \end{aligned}$$

Hence for $a < \bar{a}(k - \varepsilon)$, we have

$$\begin{aligned} \frac{\partial m_3(a, k)}{\partial a} &= (1 - \frac{\varepsilon}{k})(F'(a)\varphi_2(a, k - \varepsilon) + F(a)\frac{\partial \varphi_2(a, k - \varepsilon)}{\partial a}) - \varphi_2(a, k - \varepsilon) - \frac{a^{(1-\alpha)/\alpha}}{\alpha} \\ (A1.3.1) \quad &= [(1 - \frac{\varepsilon}{k})F(a)\frac{\partial \varphi_2(a, k - \varepsilon)}{\partial a} - \frac{a^{(1-\alpha)/\alpha}}{\alpha}] \\ &< \frac{\partial \varphi_2(a, k - \varepsilon)}{\partial a} - \frac{a^{(1-\alpha)/\alpha}}{\alpha} \\ &< \frac{-1}{2\alpha} \end{aligned}$$

where the last inequality in (A1.3.1) has been obtained with the help of (A1.2.4).

For $a > \bar{a}(k - \varepsilon)$, we have

$$\begin{aligned} \frac{\partial m_3(a, k)}{\partial a} &= \alpha k \left((1 - \frac{\varepsilon}{k})F(a)\frac{a^{(1-\alpha)/\alpha}}{\alpha} - \frac{a^{(1-\alpha)/\alpha}}{\alpha} \right) \\ (A1.3.2) \quad &= ka^{(1-\alpha)/\alpha} \left((1 - \frac{\varepsilon}{k})F(a) - 1 \right) < -\varepsilon a^{(1-\alpha)/\alpha} < -\varepsilon. \end{aligned}$$

According to (A1.3.1) and (A1.3.2) $\partial m_3(a, k) / \partial a$ is negative and uniformly bounded above for all $a \neq \bar{a}(k)$. Hence $a \rightarrow m_3(a, k)$ is strictly decreasing and tends to $-\infty$ when a tends to $+\infty$. Thus if $m_3(a, k) = 0$ for some a , then this value of a , say $\bar{a}(k)$ must be unique and satisfies the following relation:

$$(A1.3.3) \quad \varphi_3(a, k - \varepsilon) - a^{1/\alpha} = 0.$$

If no value of a satisfies (A1.3.3), then set $\bar{a}(k) = 1$. Observe that as a function of a , the left side of (A1.3.3) shifts upward as k increases. Hence $\bar{a}(k), 2\varepsilon < k \leq 3\varepsilon$ is increasing with k .

Now note that

$$(A1.3.4) \quad \lim_{k \rightarrow 2\varepsilon} [\varphi_3(a, k - \varepsilon) - a^{1/\alpha}] = \frac{1}{2} \left(F(a) \max\{\varphi_2(a, k - \varepsilon), a^{1/\alpha}\} + \int_a^{\infty} \max\{\varphi_2(a', k - \varepsilon), (a')^{1/\alpha}\} dF(a') \right) - a^{1/\alpha}$$

Furthermore, when $k = \varepsilon$, we have

$$(A1.3.5) \quad \varphi_2(a, k - \varepsilon) = (1 - \frac{\varepsilon}{k}) \left(F(a) a^{1/\alpha} + \int_a^{\infty} (a')^{1/\alpha} dF(a') \right) = 0.$$

Using (A1.3.5), we can rewrite (A1.3.4) as

$$\lim_{k \rightarrow 2\varepsilon} [\varphi_3(a, k - \varepsilon) - a^{1/\alpha}] = \frac{1}{2} \left(F(a) a^{1/\alpha} + \int_a^{\infty} (a')^{1/\alpha} dF(a') \right) - a^{1/\alpha},$$

which is nothing other than the form assumed by (A1.2.6) when $k = 2\varepsilon$. Therefore,

$$\lim_{k \rightarrow 2\varepsilon} \bar{a}(k) = \bar{a}(2\varepsilon).$$

Therefore, the curve $a \rightarrow \bar{a}(k), 0 < k < 3\varepsilon$, is increasing and continuous.

A1.4. The General Case $n\varepsilon < k < (n+1)\varepsilon, n > 0$

To find the solution to the general sequential search problem of the firm, let us now situate ourselves at a particular instant during the sequential search stage and suppose that (a, k) is the state of the search at this instant. Conditioned on this state and its expectations about the wage rate that will prevail when the search is over, let $v(a, k | G)$ be the optimal expected profit for the firm. Given the remaining capital k , the maximum number of technology draws that the firm can take is $\text{Ceiling}[k/\varepsilon]$, where $\text{Ceiling}[x]$ is the smallest integer greater than or equal to x . We have already solve the optimal search when $\text{Ceiling}[k/\varepsilon]$ is less than or equal to 3. Now let $n > 3$ be a given integer, and suppose that the optimal solution to the sequential search has been found for all the states (a, k) with $\text{Ceiling}[k/\varepsilon] < n$. Then proceeding as in previous case, we can assert that the optimal expected profit for the firm under a state (a, k) with $\text{Ceiling}[k/\varepsilon] = n$ is given by

$$\begin{aligned}
 v(a, k | G) &= v_n(a, k | G) = \max \left\{ [F(a)v_{n-1}(a, k - \varepsilon) + \int_a^\infty v_{n-1}(a', k - \varepsilon) dF(a')], \alpha a^{1/\alpha} k \right\} \\
 &\quad \int \left(\frac{1-\alpha}{w} \right)^{(1-\alpha)/\alpha} dG(w), \\
 &= \max \left\{ \begin{aligned} &[F(a)\alpha(k - \varepsilon) \max\{\varphi_{n-1}(a, k - \varepsilon), a^{1/\alpha}\} + \\ &\int_a^\infty \alpha(k - \varepsilon) \max\{\varphi_{n-1}(a', k - \varepsilon), (a')^{1/\alpha}\} dF(a')], \alpha a^{1/\alpha} k \end{aligned} \right\} \\
 &\quad \int \left(\frac{1-\alpha}{w} \right)^{(1-\alpha)/\alpha} dG(w),
 \end{aligned}$$

$$\begin{aligned}
&= \alpha k \max \left\{ \left(1 - \frac{\varepsilon}{k}\right) [F(a) \max\{\varphi_{n-1}(a, k - \varepsilon), a^{1/\alpha}\} + \right. \\
&\quad \left. \int_a^\infty \max\{\varphi_{n-1}(a', k - \varepsilon), (a')^{1/\alpha}\} dF(a')], a^{1/\alpha} \right\} \\
&\quad \int \left(\frac{1-\alpha}{w}\right)^{(1-\alpha)/\alpha} dG(w), \\
&= \alpha k \max\{\varphi_n(a, k - \varepsilon), a^{1/\alpha}\} \int \left(\frac{1-\alpha}{w}\right)^{(1-\alpha)/\alpha} dG(w), \\
&= v_n(a, k) \int \left(\frac{1-\alpha}{w}\right)^{(1-\alpha)/\alpha} dG(w),
\end{aligned}$$

where we have defined

$$\begin{aligned}
\varphi_n(a, k) &= \left(1 - \frac{\varepsilon}{k}\right) [F(a) \max\{\varphi_{n-1}(a, k - \varepsilon), a^{1/\alpha}\} \\
\text{(A1.4.1)} \quad &+ \int_a^\infty \max\{\varphi_{n-1}(a', k - \varepsilon), (a')^{1/\alpha}\} dF(a')],
\end{aligned}$$

$$v_n(a, k) = \alpha k \max\{\varphi_n(a, k), a^{1/\alpha}\}.$$

Now define

$$m_n(a, k) = \varphi_n(a, k) - a^{1/\alpha}.$$

be shown that $a \rightarrow m_n(a, k)$ is strictly decreasing and tends to $-\infty$ when a tends to $+\infty$.

Thus if $m_n(a, k) = 0$ for some a , then this value of a , say $\bar{a}(k)$ must be unique and satisfies the following relation:

$$\text{(A1.4.2)} \quad \varphi_n(a, k) - a^{1/\alpha} = 0$$

If no value of a satisfies (A1.4.2), then set $\bar{a}(k) = 1$. As defined, $\bar{a}(k), n\varepsilon < k < (n+1)\varepsilon, n > 0$ represent the threshold technological level such that the

sequential search will continue as long as the highest technological level under the firm's command falls short of $\bar{a}(k)$, i.e., as long as $a < \bar{a}(k)$ where k is the firm's remaining capital. Again, it can then be shown that the curve $a \rightarrow m_n(a, k)$, $0 < k < n\varepsilon$ is continuous and increasing.

Observe that the expression inside the square brackets on the right side of (A1.4.1) is strictly greater than $a^{1/\alpha}$. Thus for any given a , the left side of (A1.4.2) will be strictly positive when k is sufficiently large, i.e., $\bar{a}(k) > 1$ when k is large. This last result together with the fact that $\bar{a}(k)$ is strictly increasing in k once it rises above 1 then imply that $\bar{a}(k)$ rises monotonically to $+\infty$ as $k \rightarrow +\infty$. Using the functional forms of the production and search technologies, we can find the explicit form of the threshold technological level $\bar{a}(k)$ as follows.

Let k be a capital stock level such that $\bar{a}(k) > 1$. Then the firm will be indifferent between continuing and stopping the search when $(\bar{a}(k), k)$ is the state of its sequential search and the following equality must hold

$$(A1.4.3) \alpha(\bar{a}(k))^{1/\alpha} k \int \left(\frac{1-\alpha}{w} \right)^{(1-\alpha)/\alpha} dG(w) = F(\bar{a}(k))v(\bar{a}(k), k - \varepsilon | G) + \int_{\bar{a}(k)}^{\infty} v(a', k - \varepsilon | G) da'$$

Observe that the left side of (A1.4.3) represents the firm's expected profit if it stops the search, while the right side represents its expected profit if it chooses to take another technology draw. Because the threshold technological level is increasing in remaining capital and because $\bar{a}(k) > 1$, we must have $\bar{a}(k - \varepsilon) < \bar{a}(k)$. Hence the firm will enter the

region of no R&D after its remaining capital is reduced to $k - \varepsilon$. Using this fact to evaluate the right side of (A1.4.3) we can rewrite this expression as follows:

$$(A1.4.4) \quad (\bar{a}(k))^{1/\alpha} k = F(\bar{a}(k))\bar{a}(k)^{1/\alpha} (k - \varepsilon) + (k - \varepsilon) \int_{\bar{a}(k)}^{\infty} (a')^{1/\alpha} dF(a').$$

An alternative to (A1.4.4) as a representation of the technological threshold is given by the following curve:

$$k : a \rightarrow \bar{k}(a) = \varepsilon(1 + (\alpha\lambda - 1)a^\lambda), a > 1.$$

APPENDIX 2 THE DISTRIBUTION FUNCTION OF THE OUTCOMES OF A SEQUENTIAL SEARCH

Let $H_i(\hat{\zeta}_{it} | (a_{it}, k_{it}))$, $i \in I$, be the distribution function of $\hat{\zeta}_{it}$ and $H((\hat{\zeta}_{it})_{i \in I} | (a_{it}, k_{it})_{i \in I})$ be the distribution function of $(\hat{\zeta}_{it})_{i \in I}$. Because the outcomes of the search strategies of the firms in I are assumed to be independent, we have

$$(A2.1) \quad H((\hat{\zeta}_{it})_{i \in I} | (a_{it}, k_{it})_{i \in I}) = \prod_{i \in I} H_i(\hat{\zeta}_{it} | (a_{it}, k_{it})).$$

We shall now determine $H_i(\hat{\zeta}_{it} | (a_{it}, k_{it}))$ for each operating firm in period t .

If $k_{it} \leq k(\bar{a}_i)$, then firm i will not engage in a sequential search. In this case,

$(\hat{a}_{it}, \hat{k}_{it}) = (a_{it}, k_{it})$ with probability 1, and

$$(A2.2) \quad H_i(\hat{\zeta}_{it} | (a_{it}, k_{it})) = \begin{cases} 0 & \text{for } \hat{\zeta}_{it} < a_{it}^{1/\alpha} k_{it}, \\ 1 & \text{for } \hat{\zeta}_{it} \geq a_{it}^{1/\alpha} k_{it}, \end{cases} \quad (k_{it} \leq k(\bar{a}_i)).$$

If $k_{it} > k(\bar{a}_{it})$, then firm i will engage in a sequential search. In this case, the distribution of $\hat{\zeta}_{it}$ can be found by an induction on the maximum number of technology draws that firm i might take under σ_i^* .

First, let us consider the case $0 < k_{it} - \bar{k}(a_{it}) \leq \varepsilon$, where $\bar{k}(a)$ is the inverse function of $\bar{a}(k)$. In this case, firm i will take exactly one technology draw then stop the search. The distribution of $\hat{\zeta}_{it}$ is then given by

$$(A2.3) \quad H_i(\hat{\zeta}_{it} | (a_{it}, k_{it})) = H(\hat{\zeta}_{it} | (a_{it}, k_{it})), \quad (0 < k_{it} - \bar{k}(a_{it}) \leq \varepsilon),$$

where H is the distribution function defined as follows

$$(A2.4) \quad H(\hat{\zeta}_{it} | (a, k)) = \begin{cases} 0 & \text{for } \hat{\zeta}_{it} < a^{1/\alpha}(k - \varepsilon), \\ F((\hat{\zeta}_{it} / (k - \varepsilon))^\alpha) & \text{for } \hat{\zeta}_{it} \geq a^{1/\alpha}(k - \varepsilon), \end{cases} \quad (0 < k - \bar{k}(a) \leq \varepsilon).$$

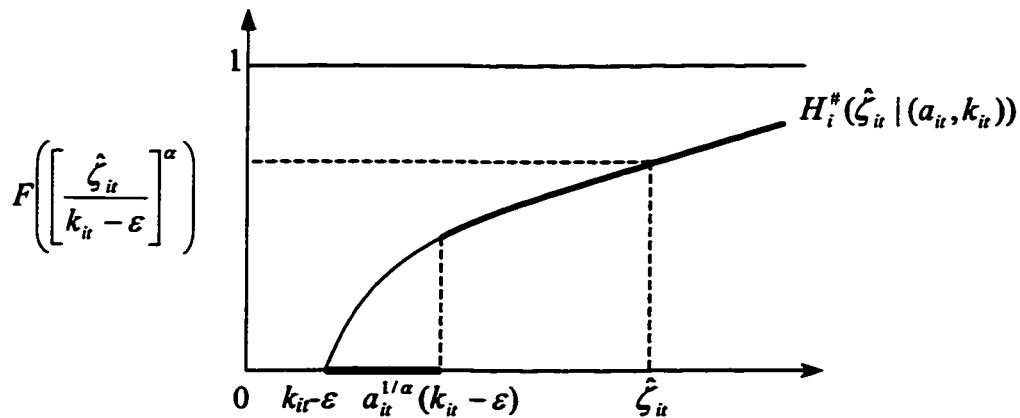


Figure 21.—The distribution of $\hat{\zeta}_{it}$, $0 < k_{it} - \bar{k}(a_{it}) \leq \varepsilon$.

Next, let us consider the case $\varepsilon < k_{it} - \bar{k}(a_{it}) \leq 2\varepsilon$. In this case, the firm will engage in a sequential search and might take up to two technology draws. Let a^1 be the productivity of the technology found in the first draw. If $a^1 \geq \bar{a}(k_{it} - \varepsilon)$, then the firm will stop the search

and invest the remaining capital $k_{it} - \varepsilon$ in the new technology to produce the consumption

good. In this case, we have $\hat{a}_{it} = a^1, \hat{k}_{it} = k_{it} - \varepsilon$, and

$$(A2.5) \quad \hat{\zeta}_{it} = (a^1)^{1/\alpha} (k_{it} - \varepsilon), \quad (a^1 \geq \bar{a}(k_{it} - \varepsilon), \varepsilon < k_{it} - \bar{k}(a_{it}) \leq 2\varepsilon),$$

and this event has probability $1 - F(\bar{a}(k_{it} - \varepsilon))$.

If $a^1 < \bar{a}(k_{it} - \varepsilon)$, then the firm will take another technology draw then stop the search. In

the case $a^1 \leq a_{it}$, an event with probability $F(a_{it})$, the distribution of $\hat{\zeta}_{it}$, conditional on a^1 , is given by

$$(A2.6) \quad H(\hat{\zeta}_{it} | (a_{it}, k_{it} - \varepsilon)), \quad (a^1 \leq a_{it}, \varepsilon < k_{it} - \bar{k}(a_{it}) \leq 2\varepsilon).$$

In the case $a_{it} < a^1 < \bar{a}(k_{it} - \varepsilon)$, the distribution of $\hat{\zeta}_{it}$, conditional on a^1 , is given by

$$(A2.7) \quad H(\hat{\zeta}_{it} | (a^1, k_{it} - \varepsilon)), \quad (a_{it} < a^1 < \bar{a}(k_{it} - \varepsilon), \varepsilon < k_{it} - \bar{k}(a_{it}) \leq 2\varepsilon).$$

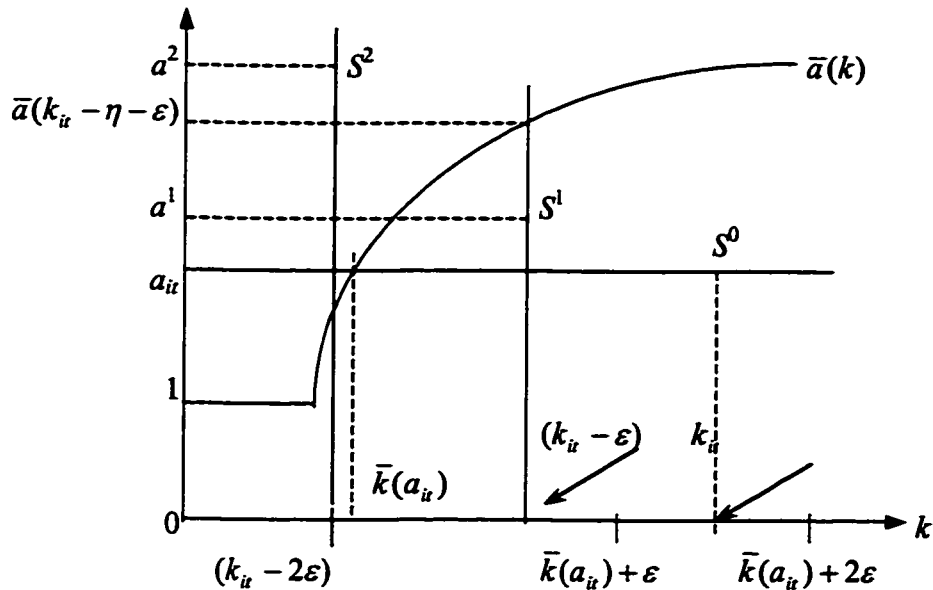


Figure 22.—A possible realization of the search process:

$$(a^1 < \bar{a}(k_{it} - \varepsilon), \varepsilon < k_{it} - \bar{k}(a_{it}) \leq 2\varepsilon)$$

This figure depicts a possible realization of the sequential search for the case $\varepsilon < k_{it} - \bar{k}(a_{it}) \leq 2\varepsilon$ under the event that the firm takes two technology draws. In this figure, S^0 , S^1 , and S^2 denote, respectively, the state of the search after the setup cost is paid, after the first draw, and after the second draw.

Using (A2.5), (A2.6), and (A2.7), we obtain the following distribution of $\hat{\zeta}_{it}$ for the case $\varepsilon < k_{it} - \bar{k}(a_{it}) \leq 2\varepsilon$.

$$\begin{aligned}
 (A2.8) \quad H_i(\hat{\zeta}_{it} | (a_{it}, k_{it})) &= F(a_{it})H(\hat{\zeta}_{it} | (a_{it}, k_{it} - \varepsilon)) \\
 &+ \int_{a_{it}}^{\bar{a}(k_{it} - \varepsilon)} H(\hat{\zeta}_{it} | (a^1, k_{it} - \varepsilon)) dF(a^1) \\
 &+ \chi_{E(k_{it} - \varepsilon)}(\hat{\zeta}_{it}) \left[F(\hat{\zeta}_{it} / (k_{it} - \varepsilon)^\alpha) - F(\bar{a}(k_{it} - \varepsilon)) \right], \\
 &\quad (\varepsilon < k_{it} - \bar{k}(a_{it}) \leq 2\varepsilon).
 \end{aligned}$$

In (A2.8), we have let $E(k)$ denote the interval $[(\bar{a}(k))^{1/\alpha} k, +\infty)$ and $\chi_{E(k)}(\zeta)$ denote the characteristic function of the set $E(k)$; that is

$$\chi_{E(k)}(\zeta) = \begin{cases} 1 & \text{if } \zeta \in E(k), \\ 0 & \text{otherwise.} \end{cases}$$

Now let $H(\hat{\zeta}_{it} | (a, k), k > \bar{k}(a))$, be the distribution of $\hat{\zeta}_{it}$, given (a, k) , the state of the search after it has begun. This distribution function has been defined for $0 < k - \bar{k}(a) \leq \varepsilon$ by (A2.4). For $\varepsilon < k - \bar{k}(a)$, the argument used to derive (A2.5) for the case $\varepsilon < k_{it} - \bar{k}(a_{it}) \leq 2\varepsilon$ can also be used to establish following recurrent relation:

$$\begin{aligned}
H(\hat{\zeta}_{it} | (a, k)) &= F(a)H(\hat{\zeta}_{it} | (a, k - \varepsilon)) \\
\text{(A2.9)} \quad &+ \int_a^{\bar{a}(k-\varepsilon)} H(\hat{\zeta}_{it} | (a^1, k - \varepsilon)) dF(a^1) \\
&+ \chi_{\varepsilon(k-\varepsilon)}(\hat{\zeta}_{it}) \left[F\left(\left(\frac{\hat{\zeta}_{it}}{k-\varepsilon}\right)^\alpha\right) - F(\bar{a}(k-\varepsilon)) \right], \quad (\varepsilon < k - \bar{k}(a)).
\end{aligned}$$

Using (A2.9), we obtain the following distribution of $\hat{\zeta}_{it}$ for the case $\varepsilon < k_{it} - \bar{k}(a_{it}) < \infty$,

$$\begin{aligned}
H_i(\hat{\zeta}_{it} | (a_{it}, k_{it})) &= F(a_{it})H(\hat{\zeta}_{it} | (a_{it}, k_{it} - \varepsilon)) \\
\text{(A2.10)} \quad &+ \int_a^{\bar{a}(k_{it}-\varepsilon)} H(\hat{\zeta}_{it} | (a^1, k_{it} - \varepsilon)) dF(a^1) \\
&+ \chi_{\varepsilon(k_{it}-\varepsilon)}(\hat{\zeta}_{it}) \left[F\left(\left(\frac{\hat{\zeta}_{it}}{k_{it}-\varepsilon}\right)^\alpha\right) - F(\bar{a}(k_{it}-\varepsilon)) \right], \\
&\quad (\varepsilon < k_{it} - \bar{a}(k_{it}) < \infty).
\end{aligned}$$

Two Technology Draws: $\bar{k}(a_{it}) + \varepsilon < k_{it} \leq \bar{k}(a_{it}) + 2\varepsilon$

If $\hat{\zeta} = a^{1/\alpha}(k - 2\varepsilon)$, then $H(a^{1/\alpha}(k - 2\varepsilon) | (a, k)) = (F(a))^2$.

If $a^{1/\alpha}(k - 2\varepsilon) \leq \hat{\zeta} \leq (a(k - \varepsilon))^{1/\alpha}(k - 2\varepsilon)$, then

$$\begin{aligned}
H(\hat{\zeta} | (a, k)) &= F(a)F\left(\left(\frac{\hat{\zeta}}{k-2\varepsilon}\right)^\alpha\right) + \int_a^{\left(\frac{\hat{\zeta}}{k-2\varepsilon}\right)^\alpha} F\left(\frac{\hat{\zeta}}{k-2\varepsilon}\right)^\alpha dF(a^1) \\
&= F(a)F\left(\left(\frac{\hat{\zeta}}{k-2\varepsilon}\right)^\alpha\right) + F\left(\left(\frac{\hat{\zeta}}{k-2\varepsilon}\right)^\alpha\right) \left(F\left(\left(\frac{\hat{\zeta}}{k-2\varepsilon}\right)^\alpha\right) - F(a) \right) \\
&= \left(F\left(\left(\frac{\hat{\zeta}}{k-2\varepsilon}\right)^\alpha\right) \right)^2.
\end{aligned}$$

If $(a(k - \varepsilon))^{1/\alpha}(k - 2\varepsilon) \leq \hat{\zeta} \leq (a(k - \varepsilon))^{1/\alpha}(k - \varepsilon)$, then

$$\begin{aligned}
 H(\hat{\zeta} | (a, k)) &= F(a)F\left(\left(\frac{\hat{\zeta}}{k-2\varepsilon}\right)^\alpha\right) + \int_a^{\bar{a}(k-\varepsilon)} F\left(\frac{\hat{\zeta}}{k-2\varepsilon}\right)^\alpha dF(a') \\
 &= F\left(\left(\frac{\hat{\zeta}}{k-2\varepsilon}\right)^\alpha\right) + F(\bar{a}(k-\varepsilon)).
 \end{aligned}$$

If $(a(k-\varepsilon))^{1/\alpha}(k-\varepsilon) < \hat{\zeta}$, then

$$H(\hat{\zeta} | (a, k)) = F(\bar{a}(k-\varepsilon))F\left(\left(\frac{\hat{\zeta}}{k-2\varepsilon}\right)^\alpha\right) + F\left(\left(\frac{\hat{\zeta}}{k-2\varepsilon}\right)^\alpha\right) - F(\bar{a}(k-\varepsilon)).$$

The General Case: n technology draws ($n > 2$)

The distribution of the outcomes of the sequential search in the general case can now be obtained by an induction argument on n .

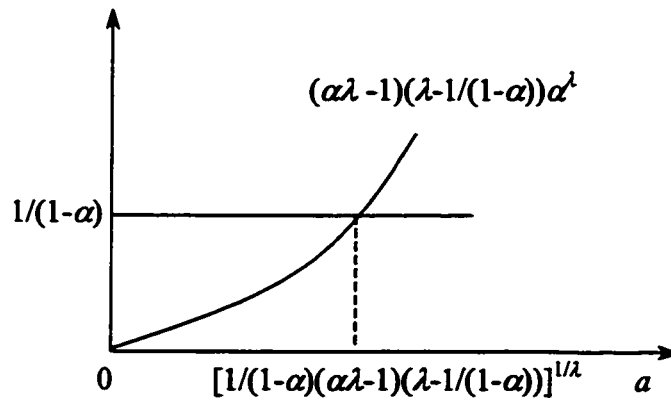
APPENDIX 3

THE HORIZONTAL DISTANCE OF \bar{k} RELATIVE TO \hat{k}

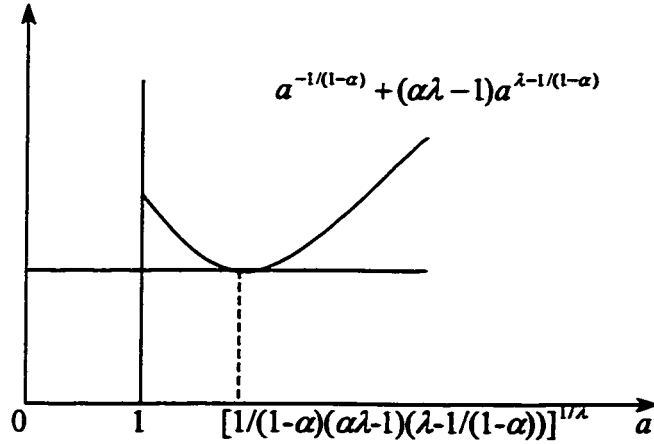
Here we analyse the behaviour of $a^{-1/(1-\alpha)} + (\alpha\lambda - 1)a^{\lambda-1/(1-\alpha)}$ as a rises from 1. Now

$$\text{(A3.1) } \frac{d[a^{-1/(1-\alpha)} + (\alpha\lambda - 1)a^{\lambda-1/(1-\alpha)}]}{da} = a^{(\alpha-2)/(1-\alpha)}[-1/(1-\alpha) + (\lambda-1/(1-\alpha))(\alpha\lambda - 1)a^\lambda],$$

($1 \leq a < \infty$).



If $[1/(1-\alpha)(\alpha\lambda-1)(\lambda-1/(1-\alpha))]^{1/\lambda} \leq 1$, then (A3.1) is positive for all $a > 1$, and (16) is strictly increasing in a . If $[1/(1-\alpha)(\alpha\lambda-1)(\lambda-1/(1-\alpha))]^{1/\lambda} > 1$, then (16) is strictly increasing as a rises from 1, reaches a minimum at $a = [1/(1-\alpha)(\alpha\lambda-1)(\lambda-1/(1-\alpha))]^{1/\lambda}$, then rises monotonically to $+\infty$ as $a \rightarrow +\infty$. The behavior of this function is shown in the following figure.



The behavior of distance of curves $\overset{\circ}{k}(a)$ and $\bar{k}(a)$

APPENDIX 4 THE HORIZONTAL DISTANCE OF TWO SUCCESSIVE STATES IN QUASI-STEADY STATE

Now

$$\lim_{a \rightarrow \infty} a^{\alpha\lambda+1} \left(\left(1 + \frac{1}{(\alpha\lambda-1)a^\lambda} \right)^\alpha - 1 \right) = \lim_{a \rightarrow \infty} \frac{\left(1 + \frac{1}{(\alpha\lambda-1)a^\lambda} \right)^\alpha - 1}{a^{-(\alpha\lambda+1)}}$$

Applying l'Hôpital rule we have:

$$\lim_{a \rightarrow \infty} \frac{-\alpha \left(1 + \frac{1}{(\alpha\lambda-1)a^\lambda} \right)^{\alpha-1} \left(\frac{1}{\alpha\lambda-1} \frac{\lambda a^{\lambda-1}}{a^{2\lambda}} \right)}{-(\alpha\lambda+1)a^{-(\alpha\lambda+2)}}$$

$$\begin{aligned}
&= \lim_{a \rightarrow \infty} \frac{\alpha \lambda \left(1 + \frac{1}{(\alpha \lambda - 1)a^\lambda}\right)^{\alpha-1}}{(\alpha \lambda - 1)(\alpha \lambda + 1)} a^{-\lambda + \alpha \lambda + 1} \\
&= \frac{\alpha \lambda}{(\alpha \lambda - 1)(\alpha \lambda + 1)} \lim_{a \rightarrow \infty} a^{-\lambda + \alpha \lambda + 1} = +\infty
\end{aligned}$$

because $\frac{1}{1-\alpha} > \lambda \Rightarrow 1 > \lambda - \alpha \lambda \Rightarrow -\lambda + \alpha \lambda + 1 > 0$.

APPENDIX 5

THE UPPER BOUND OF THE PROBABILITY OF STAYING IN QUASI-STEADY STATES (FINDING NO BETTER TECHNOLOGY)

We have

$$\begin{aligned}
d(c, b') &= \frac{\delta}{|I|^{1-\alpha}} (\varepsilon(\alpha \lambda - 1))^\alpha a^{\alpha \lambda + 1} - \varepsilon(1 + (\alpha \lambda - 1)a^\lambda) \\
&= \varepsilon a^{\alpha \lambda + 1} \left(\frac{\delta \varepsilon (\alpha \lambda - 1)^\alpha}{\varepsilon |I|^{1-\alpha}} - (a^{-\alpha \lambda - 1} + (\alpha \lambda - 1)a^{\lambda - \alpha \lambda - 1}) \right).
\end{aligned}$$

Let us approximate the minimum number of technology draws by

$$(A5.1) \quad \frac{d(c, b')}{\varepsilon} = a^{\alpha \lambda + 1} \left(\frac{\delta \varepsilon (\alpha \lambda - 1)^\alpha}{\varepsilon |I|^{1-\alpha}} - (a^{-\alpha \lambda - 1} + (\alpha \lambda - 1)a^{\lambda - \alpha \lambda - 1}) \right)$$

Using this equation in (21) we obtain the following lower bound for the probability of finding no better technology:

$$(A5.2) \quad (1 - a^{-\lambda})^{a^{\alpha \lambda + 1} \left(\frac{\delta \varepsilon (\alpha \lambda - 1)^\alpha}{\varepsilon |I|^{1-\alpha}} - (a^{-\alpha \lambda - 1} + (\alpha \lambda - 1)a^{\lambda - \alpha \lambda - 1}) \right)}$$

Taking the logarithm of (A5.2), we obtain:

$$(A5.3) \quad a^{\alpha \lambda + 1} \left(\frac{\delta \varepsilon (\alpha \lambda - 1)^\alpha}{\varepsilon |I|^{1-\alpha}} - (a^{-\alpha \lambda - 1} + (\alpha \lambda - 1)a^{\lambda - \alpha \lambda - 1}) \right) \log(1 - a^{-\lambda}).$$

Using the assumption that $\frac{1}{1-\alpha} > \lambda$, we can assert that the limit of (A5.3) when $a \rightarrow \infty$ is

given by

$$\frac{\delta\varepsilon(\alpha\lambda - 1)^\alpha}{\varepsilon |I|^{1-\alpha}} \lim_{a \rightarrow \infty} (a^{\alpha\lambda+1} \log(1 - a^{-\lambda})).$$

Now when a is large, $\log(1 - a^{-\lambda}) \approx -a^{-\lambda}$, which then implies that

$$a^{\alpha\lambda+1} \log(1 - a^{-\lambda}) \approx -a^{\alpha\lambda+1} a^{-\lambda} = -a^{\alpha\lambda+1-\lambda} \rightarrow -\infty$$

as $a \rightarrow +\infty$. This last result means that the probability of staying in quasi-steady states (finding no better technology) tends to 0 as a tends to infinity.

Chapter 3

THE TWO-SECTOR ECONOMY

1. INTRODUCTION

In the preceding chapter, we formulated and analyzed exhaustively a one-sector, one homogenous good model of endogenous growth in which firms can raise their productivity by engaging in a sequential search to find a new and better technology. That model will now be extended to include a second sector in which a second good, say a more traditional good, is produced and in which there is no more prospect for technological progress. The good produced in the original sector will now be called the high-technology good, and the structure of this sector is exactly as described in the preceding chapter. Because there is no further technological progress in the traditional sector, competition in this sector is dictated by more static conditions, which is in sharp contrast with competition in the high-technology sector in which firms compete both through prices and through innovations.

As in Chapter 2, to operate in any period, a firm—whether in the high-technology sector or in the traditional sector—must be able to raise capital during the capital allocation stage of that period. We shall also assume that capital is sector specific and that the capital raised by a firm in the high-technology sector also becomes firm specific. The price of the

high-technology good is chosen as the numeraire while the price of the traditional good and the wage rate are denoted by p and w , respectively. With the price of their product set equal to 1, the behavior of the firms in the high-technology sector is exactly as described in Chapter 2. At the sectoral level, the introduction of a second sector exerts an impact on the evolution of the high-technology sector in two fundamental ways. First, the high-technology sector must compete against the traditional sector in each period for a limited pool of workers. Second, with two types of capital to invest in, the portfolio choice made by the younger generation in each period affects the size and composition of the economy's capital stock in the following period, which in turn will influence the R&D activities in the high-technology sector. Our objective in this chapter is to analyze how the savings decision—namely the portfolio choice—of the successive young generations, the innovating behavior of the firms in the high-technology sector, and the allocation of scarce labor resources between the two sectors determine the endogenous growth of the economy.

The chapter is organized as follows. In Section 2, we explain the production side of economy. Section 3 discusses the lifetime utility maximization problem of a young individual. The definition of overlapping generation equilibrium comes in Section 4. Capital accumulation of two-sector economy will be covered in Section 5. The existence of overlapping generation equilibrium for the two-sector economy is provided in Section 6. The dynamic analysis appears in Section 7. Section 8 concludes our discussion in this chapter. More technical calculations have been relegated to the appendix at the end of the chapter.

2. THE PRODUCTION SIDE

The production side of the economy is made up of two sectors, one producing a high-technology good the other a traditional good. Each good is produced with the help of labor and its sector-specific capital. In each period, part of the production of a good is consumed; the remaining part is devoted to capital allocation, which constitutes the sector-specific capital stock of its sector in the next period.

2.1. The High-Technology Sector

As in Chapter two, let I be the set of firms in the high-technology sector. Again, we shall assume that I is finite but large enough so that the firms in the high-technology sector behave as price takers. Furthermore, technological diffusion is also assumed to be complete after one period. All the firms in the high-technology sector are thus identical at the beginning of each period.

Suppose that the economy begins period t with K_{Yt} as the capital stock of the high-technology sector and K_{Zt} as the capital stock of the traditional sector. Also, the technological level of the high-technology sector at the disposal of all the firms in this sector at the beginning of period t is denoted by a_t . Because all the firms in the high-technology sector are symmetric when the period begins, each of them will be able to raise the same amount of sector-specific capital given by $k_{it} = K_{Yt} / |I|$. The state of a firm, say i , in the high-technology sector at the end of the capital allocation stage of period t is thus (a_t, k_{it}) . A firm in the high-technology sector can engage in a sequential search to improve its productivity, paying ε units of capital out of the capital it raised in the capital allocation stage each time

it takes a technology draw. Let \hat{a}_i and \hat{k}_i denote, respectively, the highest technological level and the remaining firm-specific capital at the disposal of firm i at the end of the sequential search stage of period t . If firm i does not engage in a sequential search for a new and better technology, then $(\hat{a}_i, \hat{k}_i) = (a_i, k_i)$. The probability distribution function of (\hat{a}_i, \hat{k}_i) will be denoted by $H((\hat{a}_i, \hat{k}_i) | (a_i, k_i))$. Because the R&D activities of the firms in the high technology good sector are independent, the joint probability distribution of $(\hat{a}_i, \hat{k}_i)_{i \in I}$ is given by $\prod_{i \in I} H((\hat{a}_i, \hat{k}_i) | (a_i, k_i))$.

The short-run production function of firm i during the production stage of period t is $y = \hat{a}_i \hat{k}_i^\alpha l^{1-\alpha}$. In the production stage of period t , firm i , $i \in I$, takes the wage rate w as given and solves the following profit maximization problem:

$$\max_l [\hat{a}_i \hat{k}_i^\alpha l^{1-\alpha} - wl].$$

The solution of this profit maximization problem yields the following demand for labor by firm i :

$$l(\hat{a}_i, \hat{k}_i, w) = \left(\frac{1-\alpha}{w} \right)^{1/\alpha} \hat{a}_i^{1/\alpha} \hat{k}_i, \quad (i \in I).$$

Its output and its profit are given, respectively, by

$$y(\hat{a}_i, \hat{k}_i, l(\hat{a}_i, \hat{k}_i, w)) = \left(\frac{1-\alpha}{w} \right)^{(1-\alpha)/\alpha} \hat{a}_i^{1/\alpha} \hat{k}_i, \quad (i \in I),$$

and

$$\pi(\hat{a}_i, \hat{k}_i, w) = \alpha \left(\frac{1-\alpha}{w} \right)^{(1-\alpha)/\alpha} \hat{a}_i^{1/\alpha} \hat{k}_i, \quad (i \in I).$$

The aggregate demand for labor by the high-technology sector is:

$$L_Y((\hat{a}_i, \hat{k}_i)_{i \in I}, w) = \sum_{i \in I} l(\hat{a}_i, \hat{k}_i, w) = \left(\frac{1-\alpha}{w} \right)^{1/\alpha} \sum_{i \in I} \hat{a}_i^{1/\alpha} \hat{k}_i.$$

The aggregate output of the high-technology good is:

$$Y((\hat{a}_i, \hat{k}_i)_{i \in I}, w) = \left(\frac{1-\alpha}{w} \right)^{(1-\alpha)/\alpha} \sum_{i \in I} \hat{a}_i^{1/\alpha} \hat{k}_i.$$

The profit (or return to capital) generated by the high-technology sector can be calculated as:

$$\Pi((\hat{a}_i, \hat{k}_i)_{i \in I}, w) = \sum_{i \in I} \pi(\hat{a}_i, \hat{k}_i, w) = \alpha \left(\frac{1-\alpha}{w} \right)^{(1-\alpha)/\alpha} \sum_{i \in I} \hat{a}_i^{1/\alpha} \hat{k}_i.$$

The realized in kind gross rate of return to capital invested in the high-technology sector is:

$$r_Y((\hat{a}_i, \hat{k}_i)_{i \in I}, w | (a_i, k_i)_{i \in I}) = \frac{\alpha \left(\frac{1-\alpha}{w} \right)^{(1-\alpha)/\alpha} \sum_{i \in I} \hat{a}_i^{1/\alpha} \hat{k}_i}{\sum_{i \in I} k_i}.$$

2.2. The Traditional-Good Sector

We assume that the traditional-good sector is perfectly competitive and that it has no prospect for technological progress. More specifically, the traditional good is produced according to the following production function: $Z = bK_Z^\beta L_Z^{1-\beta}$, where Z is the aggregate output of the traditional good; K_Z is the sector-specific capital stock; L_Z is the labor input. Also, β , $0 < \beta < 1$, is a parameter representing the income share of the factor capital, and $b >$

0 is a constant representing the unchanging technological level of the traditional-good sector. The traditional sector solves the following profit maximization problem:

$$\max_{L_z} [pbK_z^\beta L_z^{1-\beta} - wL_z],$$

where p is the price of traditional good and w is the wage rate. The aggregate demand for labor by the traditional sector is:

$$L_z(K_z, w, p) = \left(\frac{1-\beta}{w}\right)^{1/\beta} (pb)^{1/\beta} K_z.$$

The aggregate output of the traditional good is

$$Z(K_z, w, p) = \left(\frac{(1-\beta)p}{w}\right)^{(1-\beta)/\beta} b^{1/\beta} K_z,$$

and the total revenue of this sector is:

$$pZ(K_z, w, p) = \left(\frac{1-\beta}{w}\right)^{(1-\beta)/\beta} (pb)^{1/\beta} K_z.$$

The profit, i.e., the return to capital investment, in the traditional-good sector is

$$\beta pZ(K_z, w, p) = \beta \left(\frac{(1-\beta)p}{w}\right)^{(1-\beta)/\beta} (pb)^{1/\beta} K_z.$$

The in kind gross rate of return to capital invested in the traditional sector will be:

$$r_z(K_z, w, p) = \beta \left(\frac{(1-\beta)p}{w}\right)^{(1-\beta)/\beta} b^{1/\beta}.$$

3. EXPECTED LIFETIME UTILITY MAXIMISATION

Suppose that the preferences over lifetime consumption of a young individual is represented by the following utility function:

$$u(c_Y^0, c_Z^0, c_Y^1, c_Z^1) = \sigma \log c_Y^0 + (1-\sigma) \log c_Z^0 + \delta [\sigma \log c_Y^1 + (1-\sigma) \log c_Z^1],$$

where c_Y^0 and c_Z^0 are the consumption of the high technology good and the consumption of traditional good when the individual is young, while c_Y^1 and c_Z^1 are the consumption of the high technology good and the consumption of traditional good when the individual is old. Also, σ , $0 < \sigma < 1$, is a parameter and δ , $0 < \delta < 1$, is the discount factor.

A young individual of period t solves the following lifetime utility maximization problem:

$$\max_{c_Y^0, c_Z^0, s_Y^0, s_Z^0} \left[\sigma \log c_Y^0 + (1-\sigma) \log c_Z^0 + \delta \mathcal{E} \left[\sigma \log [\sigma (\tilde{r}_{Y,t+1} s_Y^0 + \tilde{p}_{t+1} \tilde{r}_{Z,t+1} s_Z^0)] + (1-\sigma) \log \left[\frac{(1-\sigma) (\tilde{r}_{Y,t+1} s_Y^0 + \tilde{p}_{t+1} \tilde{r}_{Z,t+1} s_Z^0)}{\tilde{p}_{t+1}} \right] \right] \right]$$

subject to the budget constraint

$$w_t - c_Y^0 - s_Y^0 - p_t (c_Z^0 + s_Z^0) = 0.$$

Here w_t is the wage rate in period t ; p_t is the price of traditional good in period t ; and \tilde{p}_{t+1} is the price of traditional good in period $t+1$, not known in period t when the young individual makes his portfolio choice. Also, s_Y^0 represents the part of savings put under the form of sector-specific capital in the high technology sector and s_Z^0 represents the part of savings put under the form of sector-specific capital in the traditional sector. The random vector $(\tilde{r}_{Y,t+1}, \tilde{r}_{Z,t+1}, \tilde{p}_{t+1})$ represents the individual's expectations about the in kind gross rate of return to capital invested in high technology sector, the in kind gross rate of return to capital invested in traditional sector, and the price of the traditional good in period $t+1$, respectively.

Finally, \mathcal{E} is the expectation operator with respect to the distribution of the random vector $(\tilde{r}_{Y,t+1}, \tilde{r}_{Z,t+1}, \tilde{p}_{t+1})$.

With $\tilde{r}_{Y,t+1}s_{Yt}^0 + \tilde{p}_{t+1}\tilde{r}_{Z,t+1}s_{Zt}^0$ as the random old-age income of an individual born in period t , the consumption of the high-technology and traditional goods in old age of such an individual are given, respectively, by

$$\tilde{c}_{Y,t+1} = \sigma(\tilde{r}_{Y,t+1}s_{Yt}^0 + \tilde{p}_{t+1}\tilde{r}_{Z,t+1}s_{Zt}^0),$$

$$\tilde{c}_{Z,t+1} = (1-\sigma)(\tilde{r}_{Y,t+1}s_{Yt}^0 + \tilde{p}_{t+1}\tilde{r}_{Z,t+1}s_{Zt}^0) / \tilde{p}_{t+1}.$$

Given the vector of current consumption (c_{Yt}^0, c_{Zt}^0) and the investment portfolio (s_{Yt}^0, s_{Zt}^0) , the lifetime expected utility of young individual of period t is then given by:

$$(1) \quad \sigma \log c_{Yt}^0 + (1-\sigma) \log c_{Zt}^0 + \delta \mathcal{E}[\sigma \log \tilde{c}_{Y,t+1} + (1-\sigma) \log \tilde{c}_{Z,t+1}].$$

The problem of a young individual in period t is to choose a current consumption plan (c_{Yt}^0, c_{Zt}^0) and an investment portfolio (s_{Yt}^0, s_{Zt}^0) to maximize (1). Letting λ be the multiplier associated with the budget constraint, we can write the Lagrangian as follows:

$$\mathcal{L} = \sigma \log c_{Yt}^0 + (1-\sigma) \log c_{Zt}^0 + \delta \mathcal{E} \left[\begin{aligned} & \sigma \log[\sigma(\tilde{r}_{Y,t+1}k_{Y,t+1} + \tilde{p}_{t+1}\tilde{r}_{Z,t+1}k_{Z,t+1})] \\ & + (1-\sigma) \log \left[\frac{(1-\sigma)(\tilde{r}_{Y,t+1}k_{Y,t+1} + \tilde{p}_{t+1}\tilde{r}_{Z,t+1}\tilde{k}_{Z,t+1})}{\tilde{p}_{t+1}} \right] \end{aligned} \right] \\ + \lambda[w_t - c_{Yt}^0 - k_{Y,t+1} - p_t(c_{Zt}^0 + k_{Z,t+1})].$$

The first-order conditions for an interior solution to the problem of lifetime utility maximization problem are

$$\partial \mathcal{L} / \partial c_{Yt}^0 = \sigma / c_{Yt}^0 - \lambda = 0,$$

$$\partial \mathcal{L} / \partial c_{Z_t}^0 = (1 - \sigma) / c_{Z_t}^0 - \lambda p_t = 0,$$

$$\partial \mathcal{L} / \partial s_{Y_t}^0 = \delta \mathcal{E} \left[\frac{\tilde{r}_{Y,t+1}}{\tilde{r}_{Y,t+1} s_{Y_t}^0 + \tilde{p}_{t+1} \tilde{r}_{Z,t+1} s_{Z_t}^0} \right] - \lambda = 0,$$

$$\partial \mathcal{L} / \partial s_{Z_t}^0 = \delta \mathcal{E} \left[\frac{\tilde{p}_{t+1} \tilde{r}_{Z,t+1}}{\tilde{r}_{Y,t+1} s_{Y_t}^0 + \tilde{p}_{t+1} \tilde{r}_{Z,t+1} s_{Z_t}^0} \right] - \lambda p_t = 0,$$

$$\partial \mathcal{L} / \partial \lambda = w_t - c_{Y_t}^0 - s_{Y_t}^0 - p_t (c_{Z_t}^0 + s_{Z_t}^0) = 0.$$

It follows directly from the first-order conditions that

$$\begin{aligned} \lambda &= \frac{\sigma}{c_{Y_t}^0} = \frac{(1 - \sigma)}{p_t c_{Z_t}^0} = \delta \frac{\mathcal{E} \left[\frac{\tilde{r}_{Y,t+1} s_{Y_t}^0}{\tilde{r}_{Y,t+1} s_{Y_t}^0 + \tilde{p}_{t+1} \tilde{r}_{Z,t+1} s_{Z_t}^0} \right]}{s_{Y_t}^0} \\ &= \delta \frac{\mathcal{E} \left[\frac{\tilde{p}_{t+1} \tilde{r}_{Z,t+1} s_{Z_t}^0}{\tilde{r}_{Y,t+1} s_{Y_t}^0 + \tilde{p}_{t+1} \tilde{r}_{Z,t+1} s_{Z_t}^0} \right]}{p_t s_{Z_t}^0} = \frac{1 + \delta}{w_t}. \end{aligned}$$

Thus the optimal lifetime plan for a young individual of period t is given by the following four equations:

$$c_{Y_t}^0 = \sigma \frac{w_t}{1 + \delta}, \quad c_{Z_t}^0 = (1 - \sigma) \frac{w_t}{(1 + \delta) p_t},$$

$$\frac{\delta w_t}{1 + \delta} \mathcal{E} \left[\frac{\tilde{r}_{Y,t+1}}{\tilde{r}_{Y,t+1} s_{Y_t}^0 + \tilde{p}_{t+1} \tilde{r}_{Z,t+1} s_{Z_t}^0} \right] = \frac{\delta w_t}{(1 + \delta) p_t} \mathcal{E} \left[\frac{\tilde{p}_{t+1} \tilde{r}_{Z,t+1}}{\tilde{r}_{Y,t+1} s_{Y_t}^0 + \tilde{p}_{t+1} \tilde{r}_{Z,t+1} s_{Z_t}^0} \right] = 1.$$

Observe that a young individual saves a constant fraction of his wages w_t , regardless of his expectations about the rate of return on the capital invested in each sector and the price of the traditional good in his old age. The propensity to save is $\delta / (1 + \delta)$. This

property follows directly from the assumption that his preferences are of the Cobb-Douglas type.

4. THE DEFINITION OF OVERLAPPING-GENERATION EQUILIBRIUM

Suppose that the economy begins period t in state (a_t, K_{Yt}, K_{Zt}) . At the end of the capital allocation stage of this period, the state of a firm, say i , in the high-technology sector is (a_t, k_{it}) , where $k_{it} = K_{it}/|I|$. Let $(\hat{a}_{it}, \hat{k}_{it})$ be the state of this firm at the end of the sequential search of the period. If w and p denote, respectively, the wage rate and the price of the traditional good that prevails in the production stage of period t , then the market for labor will be in equilibrium if the following market-clearing condition hold:

$$(2) \quad \left(\frac{1-\alpha}{w}\right)^{1/\alpha} \sum_{i \in I} \hat{a}_{it}^{1/\alpha} \hat{k}_{it} + \left(\frac{(1-\beta)p}{w}\right)^{1/\beta} b^{1/\beta} K_{Zt} = 1$$

Observe that in order for the total demand for labor by the high-technology sector not to exceed the aggregate labor supply, the wage rate must be constrained to satisfy the following inequality:

$$(3) \quad w \geq (1-\alpha) \left(\sum_{i \in I} \hat{a}_{it}^{1/\alpha} \hat{k}_{it} \right)^\alpha.$$

For each wage rate that satisfies (3), let $p(w | \sum_{i \in I} \hat{a}_{it}^{1/\alpha} \hat{k}_{it}, K_{Zt})$ be the value of p that satisfies

(2). More precisely, we have

$$(4) \quad p(w | \sum_{i \in I} \hat{a}_i^{1/\alpha} \hat{k}_i, K_Z) = \frac{w}{(1-\beta)b} \left(\frac{1 - \left(\frac{1-\alpha}{w} \right)^{1/\alpha} \sum_{i \in I} \hat{a}_i^{1/\alpha} \hat{k}_i}{K_Z} \right)^\beta.$$

Equation (4) plays a fundamental role in the model of the present chapter and the model of international trade of Chapter 4; it represents the functional relationship between the wage rate and the price of the traditional good in the production stage of period t —once the outcomes of the R&D activities of the firms in the high-technology sector are known—that must be satisfied so that the labor market in this period is in equilibrium.

By a wage map we mean a continuous function

$$\Omega : ((\hat{a}_i, \hat{k}_i)_{i \in I}, K_Z) \rightarrow \Omega((\hat{a}_i, \hat{k}_i)_{i \in I}, K_Z)$$

such that for each array $((\hat{a}_i, \hat{k}_i)_{i \in I}, K_Z)$ of strictly positive numbers, $\Omega((\hat{a}_i, \hat{k}_i)_{i \in I}, K_Z) > 0$. We interpret $\Omega((\hat{a}_i, \hat{k}_i)_{i \in I}, K_Z)$ as the wage rate under Ω in the production stage of period t , given that the state of the economy at the beginning of this stage is $((\hat{a}_i, \hat{k}_i)_{i \in I}, K_Z)$.

Now consider a wage map Ω . For each array $((\hat{a}_i, \hat{k}_i)_{i \in I}, K_Z)$, let (\hat{w}_t, \hat{p}_t) be the price system defined by

$$(5) \quad \hat{w}_t = \Omega((\hat{a}_i, \hat{k}_i)_{i \in I}, K_Z), \quad \hat{p}_t = p(\hat{w}_t | (\hat{a}_i, \hat{k}_i)_{i \in I}, K_Z).$$

Under the price system (\hat{w}_t, \hat{p}_t) , the labor market during the production and consumption stage of period t is in equilibrium. Hence in searching for an overlapping generation equilibrium, it is sufficient to focus on the wage maps.

When the price system (\hat{w}_t, \hat{p}_t) prevails, the aggregate output of the high-technology good industry and its profit, are given, respectively, by

$$\hat{Y}_t = Y((\hat{a}_{it}, \hat{k}_{it})_{i \in I}, \hat{w}_t) = \left(\frac{1-\alpha}{\hat{w}_t} \right)^{(1-\alpha)/\alpha} \sum_{i \in I} \hat{a}_{it}^{1/\alpha} \hat{k}_{it},$$

and

$$\hat{\Pi}_t = \Pi((\hat{a}_{it}, \hat{k}_{it})_{i \in I}, \hat{w}_t) = \alpha \left(\frac{1-\alpha}{\hat{w}_t} \right)^{(1-\alpha)/\alpha} \sum_{i \in I} \hat{a}_{it}^{1/\alpha} \hat{k}_{it}.$$

As for the traditional-good industry, its output, its revenues, and its profits are given, respectively, by

$$(6) \quad \hat{Z}_t = Z(K_Z, \hat{w}_t, \hat{p}_t) = \left(\frac{(1-\beta)\hat{p}_t}{\hat{w}_t} \right)^{(1-\beta)/\beta} b^{1/\beta} K_Z,$$

$$\hat{p}_t \hat{Z}_t = \left(\frac{(1-\beta)}{\hat{w}_t} \right)^{(1-\beta)/\beta} (\hat{p}_t b)^{1/\beta} K_Z,$$

and

$$\beta \left(\frac{1-\beta}{\hat{w}_t} \right)^{(1-\beta)/\beta} (\hat{p}_t b)^{1/\beta} K_Z.$$

The income of an old individual is $\alpha \hat{Y}_t + \beta \hat{p}_t \hat{Z}_t$. Given his Cobb-Douglas preferences, the consumption vector of an old individual in period t is then

$$(\hat{c}_t^1, \hat{c}_t^2) = (\sigma(\alpha \hat{Y}_t + \beta \hat{p}_t \hat{Z}_t), \frac{(1-\sigma)(\alpha \hat{Y}_t + \beta \hat{p}_t \hat{Z}_t)}{\hat{p}_t}).$$

The wage income of a young individual is \hat{w}_t . His current consumption is given by the following consumption vector:

$$(\hat{c}_{Yt}^0, \hat{c}_{Zt}^0) = \left(\frac{\sigma \hat{w}_t}{1 + \delta}, \frac{(1 - \sigma) \hat{w}_t}{1 + \delta} \right).$$

Now let

$$a_{t+1} = \max_{i \in I} \hat{a}_i,$$

$$(7) \quad K_{Y,t+1} = \hat{Y}_t - \hat{c}_{Yt}^0 - \hat{c}_{Yt}^1,$$

$$(8) \quad K_{Z,t+1} = \hat{Z}_t - \hat{c}_{Zt}^0 - \hat{c}_{Zt}^1.$$

Because the economy being considered is a closed system, the aggregate output of each good must not fall short of the aggregate demand for this good for current consumption.

Hence only the wage maps that lead to $K_{Y,t+1} \geq 0$ and $K_{Z,t+1} \geq 0$ should be considered. The state of a firm, say i , in the high-technology sector at the beginning of the sequential search stage of period $t+1$ —as anticipated by a young individual of period t —is thus

$$(9) \quad (a_{i,t+1}, k_{i,t+1}) = \left(a_{t+1}, \frac{K_{Y,t+1}}{|I|} \right).$$

The initial condition (9), as demonstrated in Chapter 2, determines completely the behavior of this firm during the sequential search. Let $(\hat{a}_{i,t+1}, \hat{k}_{i,t+1})$ be its state at the end of the sequential search in period $t+1$. The price system that prevails in the production stage of this period under the wage map Ω is then given by

$$(10) \quad \hat{w}_{t+1} = \Omega((\hat{a}_{i,t+1}, \hat{k}_{i,t+1})_{i \in I}, K_{Z,t+1}) \quad \text{and} \quad \hat{p}_{t+1} = p(\hat{w}_{t+1} | (\hat{a}_{i,t+1}, \hat{k}_{i,t+1})_{i \in I}, K_{Z,t+1})$$

The realized in kind gross rates of return to capital investments in the high-technology good and the traditional-good sectors in period $t+1$ are then given, respectively, by

$$(11) \quad \hat{r}_{Y,t+1} = r_{Y,t+1}((\hat{a}_{i,t+1}, \hat{k}_{i,t+1})_{i \in I}, \hat{w}_{t+1} | (a_{i,t+1}, k_{i,t+1})_{i \in I}) = \frac{\alpha \left(\frac{1-\alpha}{\hat{w}_{t+1}} \right)^{(1-\alpha)/\alpha} \sum_{i \in I} \hat{a}_{i,t+1}^{1/\alpha} \hat{k}_{i,t+1}}{\sum_{i \in I} k_{i,t+1}},$$

$$(12) \quad \hat{r}_{Z,t+1} = r_{Z,t+1}(K_{Z,t+1}, \hat{p}_{t+1}, \hat{w}_{t+1}) = \beta \left(\frac{(1-\beta)\hat{p}_{t+1}}{\hat{w}_{t+1}} \right)^{(1-\beta)/\beta} b^{1/\beta}.$$

Recall that the joint distribution of the family of random variables $(\hat{a}_{i,t+1}, \hat{k}_{i,t+1})_{i \in I}$ is given by

$\prod_{i \in I} H((\hat{a}_{i,t+1}, \hat{k}_{i,t+1})_{i \in I} | (a_{i,t+1}, k_{i,t+1}))$, which can be used to determine the distribution of the price system (10) and the own gross rates of return to capital invested in the two sectors, as represented, respectively, by (11) and (12). For a young individual in period t , the amount of savings put into the high-technology sector, namely \hat{s}_{Yt}^0 , and the amount of savings put into the traditional-good sector, namely \hat{s}_{Zt}^0 , are the solution of the following system of equations:

$$(13) \quad \frac{\delta \hat{w}_t}{1 + \delta} \mathcal{E} \left[\frac{\hat{r}_{Y,t+1}}{\hat{r}_{Y,t+1} \hat{s}_{Yt}^0 + \hat{p}_{t+1} \hat{r}_{Z,t+1} \hat{s}_{Zt}^0} \right] = \frac{\delta \hat{w}_t}{(1 + \delta) \hat{p}_t} \mathcal{E} \left[\frac{\hat{p}_{t+1} \hat{r}_{Z,t+1}}{\hat{r}_{Y,t+1} \hat{s}_{Yt}^0 + \hat{p}_{t+1} \hat{r}_{Z,t+1} \hat{s}_{Zt}^0} \right] = 1.$$

Observe that in (13), \mathcal{E} is the expectation operator with respect to the joint distribution of $(\hat{p}_{t+1}, \hat{r}_{Y,t+1}, \hat{r}_{Z,t+1})$.

DEFINITION: A wage map $\Omega : ((\hat{a}_i, \hat{k}_i)_{i \in I}, K_Z) \rightarrow \Omega((\hat{a}_i, \hat{k}_i)_{i \in I}, K_Z)$ constitutes an overlapping-generation equilibrium if the following conditions are satisfied.

(i) For any array $((\hat{a}_i, \hat{k}_i)_{i \in I}, K_Z)$, the value of $K_{Y,t+1}$ and the value of $K_{Z,t+1}$, as defined by (7) and (8), respectively, are strictly positive.

(ii) The current market for the high-technology good clears, i.e.,

$$\hat{Y}_t = \hat{c}_{Y_t}^0 + \hat{s}_{Y_t}^0 + \hat{c}_{Y_t}^1.$$

(iii) The current market for the traditional good clears, i.e.,

$$\hat{Z}_t = \hat{c}_{Z_t}^0 + \hat{s}_{Z_t}^0 + \hat{c}_{Z_t}^1.$$

(iv) Expectations are rational, i.e.,

$$\hat{s}_{Y_t}^0 = K_{Y,t+1} \text{ and } \hat{s}_{Z_t}^0 = K_{Z,t+1}.$$

Observe that because the price systems during the production and consumption stage of periods t and $t+1$ are defined by (5) and (10), the labor market is always in equilibrium in these periods. Due to Walras' Law, when either of the two conditions (ii) and (iii) is satisfied, so is the other. Finally, (iv) asserts that the expectations about capital accumulation are correct in the sense that the capital investment decisions made by the young generation in period t —conditioned on their anticipations on the diffusion of new technologies—the accumulation of capital in the two sectors, as embodied in the wage map Ω , coincides exactly with their anticipations.

5. CAPITAL ACCUMULATION

Suppose that the state of the economy at the beginning of the production and consumption stage in period t is $(\hat{a}_i, \hat{k}_i)_{i \in I}, K_Z$. We want to study the production decisions of the firms in the two sectors as well as the consumption behavior of the old and young generations as the wage rate varies under the constraint that the price of the traditional good must adjust to the variation in the wage rate to maintain the equilibrium on the labor market.

5.1. Capital Accumulation in the High-technology sector

Let $w \geq (1-\alpha) \left(\sum_{i \in I} \hat{a}_i^{1/\alpha} \hat{k}_i \right)^\alpha$ be a given wage rate and consider the price system $(w, p(w | \sum_{i \in I} \hat{a}_i^{1/\alpha} \hat{k}_i, K_Z))$. Under this price system, the labor market is in equilibrium and the incomes of the young and old generations are given, respectively, by w and

$$\alpha Y(\sum_{i \in I} \hat{a}_i^{1/\alpha} \hat{k}_i, w) + \beta p(w | \sum_{i \in I} \hat{a}_i^{1/\alpha} \hat{k}_i, K_Z) Z(K_Z, w, p(w | \sum_{i \in I} \hat{a}_i^{1/\alpha} \hat{k}_i, K_Z)).$$

For the young generation, its total consumption of the high technology and its total consumption of the traditional good are given, respectively, by

$$C_Y^0(w | \sum_{i \in I} \hat{a}_i^{1/\alpha} \hat{k}_i, K_Z) = \frac{\sigma w}{1 + \delta}$$

and

$$C_Z^0(w | \sum_{i \in I} \hat{a}_i^{1/\alpha} \hat{k}_i, K_Z) = \frac{(1-\sigma)w}{(1+\delta)p(w | \sum_{i \in I} \hat{a}_i^{1/\alpha} \hat{k}_i, K_Z)}.$$

As for the old generation, its consumption of the high technology and traditional goods are given, respectively, by

$$C_Y^1(w | \sum_{i \in I} \hat{a}_i^{1/\alpha} \hat{k}_i, K_Z) =$$

$$\sigma[\alpha Y(\sum_{i \in I} \hat{a}_i^{1/\alpha} \hat{k}_i, w) + \beta p(w | \sum_{i \in I} \hat{a}_i^{1/\alpha} \hat{k}_i, K_Z) Z(K_Z, w, p(w | \sum_{i \in I} \hat{a}_i^{1/\alpha} \hat{k}_i, K_Z))],$$

and

$$C_Z^l(w | \sum_{i \in I} \hat{a}_i^{1/\alpha} \hat{k}_i, K_Z) =$$

$$(1-\sigma)[\alpha Y(\sum_{i \in I} \hat{a}_i^{1/\alpha} \hat{k}_i, w) / p(w | \sum_{i \in I} \hat{a}_i^{1/\alpha} \hat{k}_i, K_Z) + \beta Z(K_Z, w, p(w | \sum_{i \in I} \hat{a}_i^{1/\alpha} \hat{k}_i, K_Z))].$$

The excess of the aggregate supply of the high-technology good over the aggregate demand for this good for current consumption is given by¹

$$(14) \quad \begin{aligned} S_Y(w | \sum_{i \in I} \hat{a}_i^{1/\alpha} \hat{k}_i, K_Z) \\ &= Y(\sum_{i \in I} \hat{a}_i^{1/\alpha} \hat{k}_i, K_Y) - C_Y^0(w | \sum_{i \in I} \hat{a}_i^{1/\alpha} \hat{k}_i, K_Z) - C_Y^l(w | \sum_{i \in I} \hat{a}_i^{1/\alpha} \hat{k}_i, K_Z) \\ &= w \left(-\kappa_{Y0} + \kappa_{Y1} \left(\frac{1-\alpha}{w} \right)^{1/\alpha} \sum_{i \in I} \hat{a}_i^{1/\alpha} \hat{k}_i \right), \end{aligned}$$

where we let

$$\kappa_{Y0} = \frac{\beta\sigma\delta}{(1-\beta)(1+\delta)} + \frac{\sigma}{(1-\beta)(1+\delta)} > 0, \quad \text{and} \quad \kappa_{Y1} = \frac{\beta\sigma}{1-\beta} + \frac{(1-\alpha\sigma)}{1-\alpha} > 0.$$

Observe that K_Z does not appear in the right side of the second equation in (14). The following result is immediate.

LEMMA 1: *The curve $w \rightarrow S_Y(w | \sum_{i \in I} \hat{a}_i^{1/\alpha} \hat{k}_i, K_Z)$ is strictly convex and strictly decreasing from $\left(\frac{1-\sigma + \delta(1-\alpha\delta)}{(1-\alpha)(1+\delta)} \right) (1-\alpha) \left(\sum_{i \in I} \hat{a}_i^{1/\alpha} \hat{k}_i \right)^\alpha > 0$ to $-\infty$ as w rises from*

¹ This equation can be derived by brute force or rather effortlessly with the help of Mathematica.

$(1-\alpha)\left(\sum_{i \in I} \hat{a}_i^{1/\alpha} \hat{k}_i\right)^\alpha$ to $+\infty$. It assumes a value equal to 0 when $w = w^\#(\sum_{i \in I} \hat{a}_i^{1/\alpha} \hat{k}_i, K_Z)$,

where

$$w^\#(\sum_{i \in I} \hat{a}_i^{1/\alpha} \hat{k}_i, K_Z) = (1-\alpha)a[\kappa^\# \sum_{i \in I} \hat{a}_i^{1/\alpha} \hat{k}_i]^\alpha,$$

$$\text{and } \kappa^\# = \frac{\kappa_{Y1}}{\kappa_{Y0}}.$$

5.2. Capital Accumulation in the Traditional-Good Sector

The excess of the aggregate supply of the traditional good over the aggregate demand for this good for current consumption is given by²

$$\begin{aligned} (15) \quad S_Z(w | \sum_{i \in I} \hat{a}_i^{1/\alpha} \hat{k}_i, K_Z) \\ &= Z(\sum_{i \in I} \hat{a}_i^{1/\alpha} \hat{k}_i, K_Y) - C_Z^0(w | \sum_{i \in I} \hat{a}_i^{1/\alpha} \hat{k}_i, K_Z) - C_Z^1(w | \sum_{i \in I} \hat{a}_i^{1/\alpha} \hat{k}_i, K_Z) \\ &= b \left[\kappa_{Z0} - \kappa_{Z1} \left(\frac{1-\alpha}{w} \right)^{1/\alpha} \sum_{i \in I} \hat{a}_i^{1/\alpha} \hat{k}_i \right] \left[\frac{1 - \left(\frac{1-\alpha}{w} \right)^{1/\alpha} \sum_{i \in I} \hat{a}_i^{1/\alpha} \hat{k}_i}{K_Z} \right]^{-\beta}, \end{aligned}$$

where we have let

$$\kappa_{Z0} = \frac{\delta - \beta\delta + \sigma + \beta\delta\sigma}{1 + \delta} > 0, \text{ and } \kappa_{Z1} = \frac{(1 + \beta(-1 + \sigma) - \alpha\sigma)}{1 - \alpha} > 0.$$

It is simple to verify that $\kappa_{Z0} < \kappa_{Z1}$. Thus

$$(16) \quad \kappa^b = \frac{\kappa_{Z1}}{\kappa_{Z0}} > 1.$$

² This equation can be derived by brute force or rather effortlessly with the help of Mathematica.

The following result is immediate.

LEMMA 2: *The curve $w \rightarrow S_Z(w | \sum_{i \in I} \hat{a}_i^{1/\alpha} \hat{k}_i, K_Z)$ is strictly concave and strictly increasing from $-\infty$ to $\left(1 - \frac{(1-\sigma)(1+\beta\delta)}{1+\delta}\right) b K_Z^\beta$ as w rises from a right neighborhood of $(1-\alpha) \left(\sum_{i \in I} \hat{a}_i^{1/\alpha} \hat{k}_i\right)^\alpha$ to $+\infty$. It assumes a value equal to 0 when $w = w^b(\sum_{i \in I} \hat{a}_i^{1/\alpha} \hat{k}_i, K_Z)$, where*

$$w^b(\sum_{i \in I} \hat{a}_i^{1/\alpha} \hat{k}_i, K_Z) = (1-\alpha) a [\kappa^b \sum_{i \in I} \hat{a}_i^{1/\alpha} \hat{k}_i]^\alpha.$$

It can easily be verified that $\kappa^b < \kappa^\#$. Hence $w^b(\sum_{i \in I} \hat{a}_i^{1/\alpha} \hat{k}_i, K_Z) < w^\#(\sum_{i \in I} \hat{a}_i^{1/\alpha} \hat{k}_i, K_Z)$.

5.3. The Range of Admissible Wage Rates

Because our two-sector economy is a closed system, the aggregate demand for each good for current consumption cannot exceed its aggregate supply. Therefore, only the wage rates that satisfy the following constraints are admissible:

$$(17) \quad w^b(\sum_{i \in I} \hat{a}_i^{1/\alpha} \hat{k}_i, K_Z) \leq w \leq w^\#(\sum_{i \in I} \hat{a}_i^{1/\alpha} \hat{k}_i, K_Z).$$

Furthermore, in order for capital accumulation to take place through time, the inequalities in (17) must also be strict. Now for any wage rate w satisfying (17), the following national income identity must hold

$$(18) \quad Y(\sum_{i \in I} \hat{a}_i^{1/\alpha} \hat{k}_i, w) + p(w | \sum_{i \in I} \hat{a}_i^{1/\alpha} \hat{k}_i, K_Z) Z(K_Z, w, p(w | \sum_{i \in I} \hat{a}_i^{1/\alpha} \hat{k}_i, K_Z)) = \\ [C_Y^0(w | \sum_{i \in I} \hat{a}_i^{1/\alpha} \hat{k}_i, K_Z) + p(w | \sum_{i \in I} \hat{a}_i^{1/\alpha} \hat{k}_i, K_Z) C_Z^0(w | \sum_{i \in I} \hat{a}_i^{1/\alpha} \hat{k}_i, K_Z) + \frac{\delta w}{1+\delta}]$$

$$+ [C_Y^1(w | \sum_{i \in I} \hat{a}_i^{1/\alpha} \hat{k}_i, K_Z) + p(w | \sum_{i \in I} \hat{a}_i^{1/\alpha} \hat{k}_i, K_Z) C_Z^1(w | \sum_{i \in I} \hat{a}_i^{1/\alpha} \hat{k}_i, K_Z)].$$

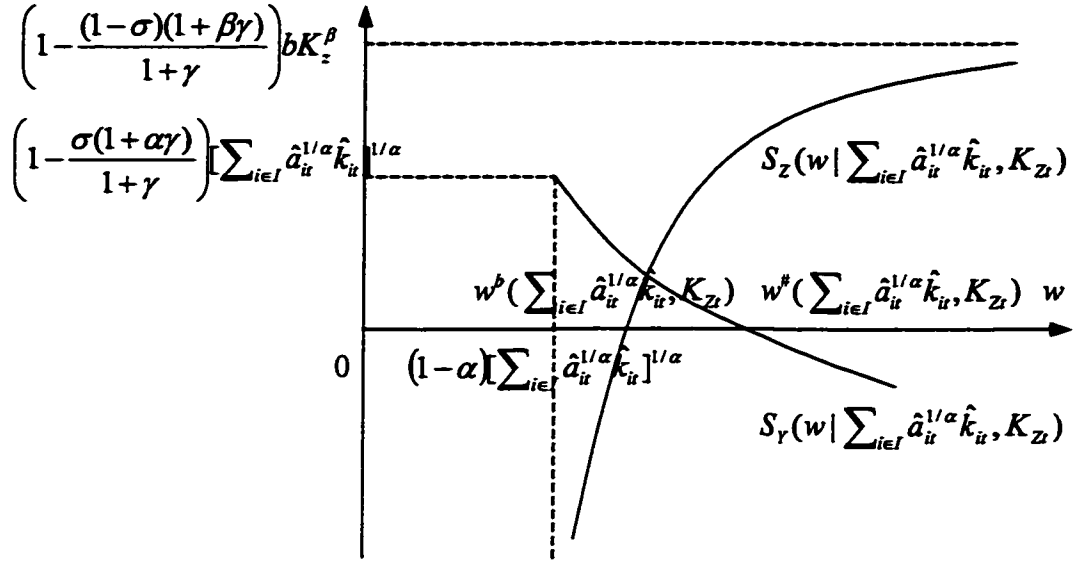
Observe that the right side of (18) is national income; the expression between the first pair of square brackets is equal to the labor income of the young generation; and the expression within the second pair of square brackets is equal to the capital income of the old generation.

A rearrangement of (18) yields the following identity:

$$\begin{aligned} (19) \quad & [Y(\sum_{i \in I} \hat{a}_i^{1/\alpha} \hat{k}_i, w) - C_Y^0(w | \sum_{i \in I} \hat{a}_i^{1/\alpha} \hat{k}_i, K_Z) - C_Y^1(w | \sum_{i \in I} \hat{a}_i^{1/\alpha} \hat{k}_i, K_Z)] + \\ & p(w | \sum_{i \in I} \hat{a}_i^{1/\alpha} \hat{k}_i, K_Z) \left(Z(K_Z, w, p(w | \sum_{i \in I} \hat{a}_i^{1/\alpha} \hat{k}_i, K_Z)) - \right. \\ & \left. C_Z^0(w | \sum_{i \in I} \hat{a}_i^{1/\alpha} \hat{k}_i, K_Z) - C_Z^1(w | \sum_{i \in I} \hat{a}_i^{1/\alpha} \hat{k}_i, K_Z) \right) \\ & = S_Y(w | \sum_{i \in I} \hat{a}_i^{1/\alpha} \hat{k}_i, K_Z) + p(w | \sum_{i \in I} \hat{a}_i^{1/\alpha} \hat{k}_i, K_Z) S_Z(w | \sum_{i \in I} \hat{a}_i^{1/\alpha} \hat{k}_i, K_Z) \\ & = \frac{\delta w}{1 + \delta}. \end{aligned}$$

The second equality in (19) asserts that as a function of the wage rate w , the sum of the value of the aggregate excess supply of the high-technology good over its aggregate demand for current consumption and the value of the aggregate excess supply of the traditional good over its aggregate demand for current consumption is always equal to the aggregate savings of the young generation.

The following figure depicts the savings functions $w \rightarrow S_Y(w | \sum_{i \in I} \hat{a}_i^{1/\alpha} \hat{k}_i, K_Z)$ and $w \rightarrow S_Z(w | \sum_{i \in I} \hat{a}_i^{1/\alpha} \hat{k}_i, K_Z)$ for $w > (1-a)[\sum_{i \in I} \hat{a}_i^{1/\alpha} \hat{k}_i]^\alpha$.



6. THE EXISTENCE OF OVERLAPPING-GENERATION EQUILIBRIUM

Let $(\Omega^n)_{n=0}^\infty$ be the sequence of wage maps defined recursively in the following manner. For $n = 0$, we have

$$\Omega^0 : (\hat{a}_i, \hat{k}_i)_{i \in I}, K_Z \rightarrow \Omega^0(\hat{a}_i, \hat{k}_i)_{i \in I}, K_Z = (1-\alpha)(\kappa^0)^\alpha \left(\sum_{i \in I} \hat{a}_i^{1/\alpha} \hat{k}_i \right)^\alpha$$

where κ^0 is equal to the constant κ^b as defined by (16). Observe that the wage rate under Ω^0 is the minimum admissible wage rate. To determine Ω^1 , we proceed as follows. Suppose that the system is in period t and that Ω^0 is the wage map that is anticipated—by the young individuals of the current period—to prevail in the next period. Now let $(\hat{a}_i, \hat{k}_i)_{i \in I}, K_Z$ be the state of the system at the beginning of the production and consumption stage of period t . To keep the notations from becoming too cumbersome, we shall let $\hat{\Xi}_t = \sum_{i \in I} \hat{a}_i^{1/\alpha} \hat{k}_i$. Given the anticipated wage map Ω^0 , we want to find the equilibrium wage rate for the current period. In the Appendix 1 of the end of the chapter, we establish the following result.

LEMMA 3: If Ω^0 is the wage map that is anticipated to prevail in the next period, then the equilibrium wage map for the current period is

$$\Omega^1 : (\hat{a}_i, \hat{k}_i)_{i \in I}, K_Z \rightarrow \Omega^1(\hat{a}_i, \hat{k}_i)_{i \in I}, K_Z = (1 - \alpha)(\kappa^1)^\alpha \left(\sum_{i \in I} \hat{a}_i^{1/\alpha} \hat{k}_i \right)^\alpha,$$

where $\kappa^1 = \zeta(\kappa^0)$ and ζ is the function defined by

$$\zeta : \kappa \rightarrow \zeta(\kappa) = \frac{(1 - \alpha)\beta\kappa_{Y1}[\kappa - 1] + \alpha\kappa_{Z1}}{(1 - \alpha)\beta\kappa_{Y0}[\kappa - 1] + \alpha\kappa_{Z0}}, \quad \kappa^b \leq \kappa \leq \kappa^\#.$$

The argument used to establish Lemma 3 can be repeated ad infinitum to obtain a sequence of wage maps $(\Omega^n)_{n=0}^\infty$ defined by

$$\Omega^n : (\hat{a}_i, \hat{k}_i)_{i \in I}, K_Z \rightarrow \Omega^n(\hat{a}_i, \hat{k}_i)_{i \in I}, K_Z = (1 - \alpha)(\kappa^n)^\alpha \left(\sum_{i \in I} \hat{a}_i^{1/\alpha} \hat{k}_i \right)^\alpha,$$

where

$$(20) \quad \kappa^n = \zeta(\kappa^{n-1}), \quad n > 0 \text{ and } \kappa^0 = \kappa^b.$$

It is clear that the sequence $(\kappa^n)_{n=0}^\infty$, as defined by (20), is strictly monotone increasing and bounded above by $\kappa^\#$. Hence it has a limit, say $\kappa^* = \lim_{n \rightarrow \infty} \kappa^n$, which is a fixed point of ζ . Furthermore, for any $n \geq 0$, if Ω^n is the wage map that is anticipated to prevail in the next period, then Ω^{n+1} is the equilibrium wage map for the current period. We have just established the following result.

PROPOSITION 1: *The wage map*

$$\Omega^* : (\hat{a}_i, \hat{k}_i)_{i \in I}, K_Z \rightarrow \Omega^*(\hat{a}_i, \hat{k}_i)_{i \in I}, K_Z = (1 - \alpha)(\kappa^*)^\alpha \left(\sum_{i \in I} \hat{a}_i^{1/\alpha} \hat{k}_i \right)^\alpha,$$

where κ^* is the fixed point of ζ , i.e.,

$$(21) \quad \kappa^* = \frac{(1-\alpha)\beta\kappa_{y1}[\kappa^* - 1] + \alpha\kappa_{z1}}{(1-\alpha)\beta\kappa_{y0}[\kappa^* - 1] + \alpha\kappa_{z0}}$$

is the unique overlapping-generation equilibrium.

The existence part of Proposition 1 has been established. The uniqueness argument is based on Lemma 3 and is given in Appendix 2. Note that the map $\zeta : \kappa \rightarrow \zeta(\kappa), \kappa^b \leq \kappa \leq \kappa^\#$, has only one fixed point. This assertion can be proved by noting that

- (i) ζ is a hyperbola
- (ii) it is continuous for $\kappa > 1$;
- (iii) $\zeta(1) = \kappa_{z1} / \kappa_{z0} = \kappa^b > 1$; and
- (iv) $\lim_{\kappa \rightarrow \infty} \zeta(\kappa) = \kappa_{y1} / \kappa_{y0} = \kappa^\# > \kappa^b$.

7. THE DYNAMICS OF OVERLAPPING-GENERATION EQUILIBRIUM

Suppose that the economy begins period $t=0$ in state (a_0, K_{y0}, K_{z0}) . Let

$((\hat{a}_{it}, \hat{k}_{it})_{i \in I}, \hat{K}_{Zt})_{t=0}^\infty$ be a realized path of the economy. For each $t = 0, 1, \dots$, let (\hat{w}_t, \hat{p}_t) be the realized price system in period t , where

$$(22) \quad \hat{w}_t = \Omega((\hat{a}_{it}, \hat{k}_{it})_{i \in I}, K_{Zt}) = (1-\alpha) \left(\kappa^* \sum_{i \in I} \hat{a}_{it}^{1/\alpha} \hat{k}_{it} \right)^\alpha$$

and

$$(23) \quad \hat{p}_t = p(\hat{w}_t | \sum_{i \in I} \hat{a}_{it} \hat{k}_{it}, K_{Zt})$$

$$\begin{aligned}
&= \frac{\hat{w}_t}{(1-\beta)b} \left(\frac{1 - \left(\frac{1-\alpha}{\hat{w}_t} \right)^{1/\alpha} \sum_{i \in I} \hat{a}_{it}^{1/\alpha} \hat{k}_{it}}{K_Z} \right)^\beta \\
&= \frac{\hat{w}_t}{(1-\beta)b} \left(\frac{1 - \frac{1}{\kappa^*}}{K_Z} \right)^\beta \\
&= \left(\frac{(1-\alpha)(\kappa^*)^{\alpha-\beta} (\kappa^* - 1)^\beta}{(1-\beta)b} \right) \left(\sum_{i \in I} \hat{a}_{it}^{1/\alpha} \hat{k}_{it} \right)^\alpha (K_Z)^{-\beta}.
\end{aligned}$$

The demand for labor by the high-technology sector is:

$$\hat{L}_{Yt} = L_Y((\hat{a}_{it}, \hat{k}_{it})_{i \in I}, \hat{w}_t) = \left(\frac{1-\alpha}{\hat{w}_t} \right)^{1/\alpha} \sum_{i \in I} \hat{a}_{it}^{1/\alpha} \hat{k}_{it} = \frac{1}{\kappa^*}.$$

The demand for labor by the traditional sector is:

$$\hat{L}_{Zt} = L_Z(K_Z, \hat{w}_t, \hat{p}_t) = 1 - \frac{1}{\kappa^*} = \frac{\kappa^* - 1}{\kappa^*}.$$

PROPOSITION 2: *Along a realized path for the two-sector economy that carries out R&D activities to raise its technological level, the demand for labor by the high-technology sector remains constant at the level $\frac{1}{\kappa^*}$ through time, while the demand for labor by the traditional sector remains constant at the level $\frac{\kappa^* - 1}{\kappa^*}$ from period to period.*

In any period t , the more successful the search (i.e., the higher the value of $\sum_{i \in I} \hat{a}_{it}^{1/\alpha} \hat{k}_{it}$) is, the higher will be the wage rate \hat{w}_t , and the higher will be the price of

traditional good, as can be seen from (22) and (23). Because the high-technology good industry uses the same amount of labor input, namely $\frac{1}{\kappa^*}$, in each period, a high value in the “effective capital input” $\sum_{i \in I} \hat{a}_i^{1/\alpha} \hat{k}_i$ raises the marginal product of labor in this sector, which in turn induces a rise in the wage rate \hat{w}_t . Because the traditional sector uses the same amount of labor input, namely $\frac{\kappa^* - 1}{\kappa^*}$, in each period, a rise in the price of the traditional good must accompany the rise in the wage rate—for any given value of the capital stock K_{Zt} —to keep the same numbers of workers in this sector.

Now when R&D is not carried out in period t , we have $\sum_{i \in I} \hat{a}_i^{1/\alpha} \hat{k}_i = a_t^{1/\alpha} K_{Yt}$. Using this result and (22) in (14), we obtain the following equation for the accumulation of capital in the high-technology sector:

$$(24) \quad K_{Y,t+1} = (1 - \alpha)(\kappa^*)^\alpha a_t K_{Yt}^\alpha \left(-\kappa_{Y0} + \frac{\kappa_{Y1}}{\kappa^*} \right) = \eta_Y a_t K_{Yt}^\alpha.$$

where

$$(25) \quad \eta_Y = (1 - \alpha)(\kappa^*)^{\alpha-1} (-\kappa_{Y0}\kappa^* + \kappa_{Y1}).$$

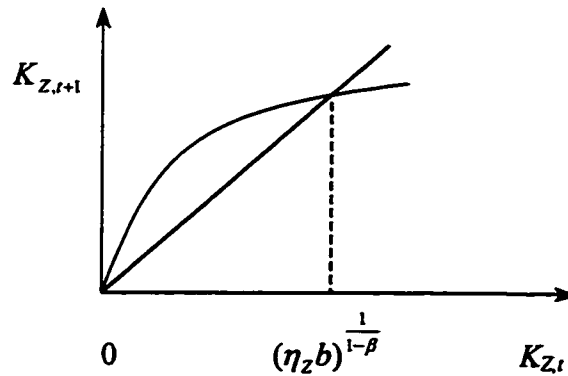
Thus if no R&D activities will ever be carried out, the technological level of the high-technology sector will remain constant through time, say $a_t = a = \text{constant}$. Under this scenario, the capital stock of the high-technology sector converges to the stationary equilibrium level $(\eta_Y a)^{\frac{1}{1-\alpha}}$.

As for the traditional sector, using (15) we obtain the following dynamic equation for its capital accumulation:

$$\begin{aligned}
 (26) \quad K_{Z,t+1} &= S_Z(\hat{w}_t | \sum_{i \in I} \hat{a}_{it}^{1/\alpha} \hat{k}_{it}, K_{Zt}) \\
 &= b \left(\kappa_{Z0} - \kappa_{Z1} \left(\frac{1-\alpha}{\hat{w}_t} \right)^{1/\alpha} \sum_{i \in I} \hat{a}_{it}^{1/\alpha} \hat{k}_{it} \right) \left(\frac{1 - \left(\frac{1-\alpha}{\hat{w}_t} \right)^{1/\alpha} \sum_{i \in I} \hat{a}_{it}^{1/\alpha} \hat{k}_{it}}{K_{Zt}} \right)^{-\beta} \\
 &= b \left(\kappa_{Z0} - \frac{\kappa_{Z1}}{\kappa^*} \right) \left(\frac{\kappa^* - 1}{\kappa^* K_{Zt}} \right)^{-\beta} \\
 &= \frac{b(\kappa_{Z0}\kappa^* - \kappa_{Z1})}{(\kappa^*)^{1-\beta} (\kappa^* - 1)^\beta} (K_{Zt})^\beta = \eta_Z b (K_{Zt})^\beta.
 \end{aligned}$$

where

$$(27) \quad \eta_Z = \frac{(\kappa_{Z0}\kappa^* - \kappa_{Z1})}{(\kappa^*)^{1-\beta} (\kappa^* - 1)^\beta}.$$



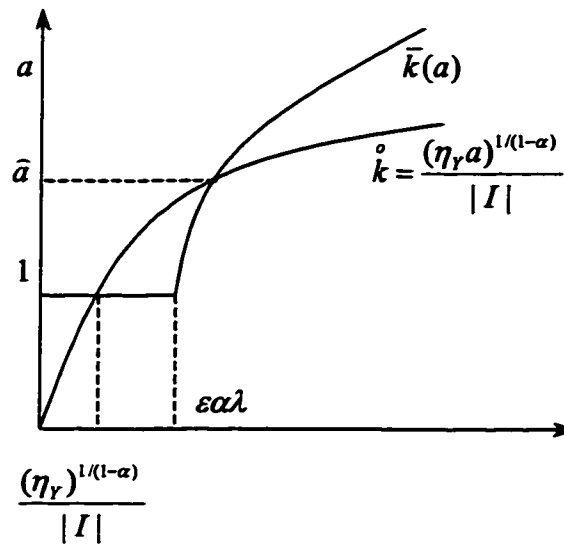
We have just established the following proposition:

PROPOSITION 3: *Although capital accumulation in the high-technology sector is random, capital accumulation in the traditional sector is completely deterministic. In the long run, the capital stock of the traditional sector converges to a stationary level, namely $(\eta_Z b)^{\frac{1}{1-\beta}}$, that depends on b , its unchanging technology level and a constant, namely η_Z , which depends on α, β, δ , and σ -the parameters of the model.*

At the most fundamental level, the real linkages between the two sectors exist through the interplay of the structural parameters α, β, δ , and σ of the model. The evolution of the high-technology sector impacts upon the traditional sector via the price of the traditional good: a successful R&D program in any period induces a higher price for the traditional good without affecting the value of the consumption of this good or the capital accumulation in this sector.

In Chapter 2, we explained that it is the values of the parameters on the supply side of the high-technology sector, namely α and λ , that determines the potential profits for R&D. Here, as in Chapter 2, the possibility of sustained growth depends on the relative position in the (a, k) -plane of the curve $\bar{k} : a \rightarrow \bar{k}(a)$ and the curve $\overset{\circ}{k} : a \rightarrow \overset{\circ}{k}(a) = \frac{(\eta_Y a)^{1/(1-\alpha)}}{|I|}$. In the following figure, we depict both $\bar{k} : a \rightarrow \bar{k}(a)$, which separates the region of sequential search and the region of no sequential search for a firm producing the high-technology good, and $\overset{\circ}{k} : a \rightarrow \overset{\circ}{k}(a) = \frac{(\eta_Y a)^{1/(1-\alpha)}}{|I|}$, which captures the relationship between the technological level of the high-technology industry and its capital stock in stationary equilibrium when more R&D activities are carried out. As in Chapter 2, if

$1/(1-\alpha) < \lambda$, there is no sustained growth in the long run. If $1/(1-\alpha) > \lambda$, the economy will grow without bound in the long run. For the borderline case $1/(1-\alpha) = \lambda$, there is no sustained growth unless the values of the parameters of the model place the curve $\dot{k}(a)$ above the curve $\bar{k}(a)$ for all $a \geq 1$.



The accumulation of capital in the high-technology sector in the periods that the firms engage in R&D activities is governed by the following dynamic equation:

$$\begin{aligned}
 K_{Y,t+1} &= S_Y(\hat{w}_t | \sum_{i \in I} \hat{a}_{it}^{1/\alpha} \hat{k}_{it}, K_Z) \\
 &= \hat{w}_t \left(-\kappa_{Y0} + \kappa_{Y1} \left(\frac{1-\alpha}{\hat{w}_t} \right)^{1/\alpha} \sum_{i \in I} \hat{a}_{it}^{1/\alpha} \hat{k}_{it} \right) \\
 &= (1-\alpha) \left(\kappa^* \sum_{i \in I} \hat{a}_{it}^{1/\alpha} \hat{k}_{it} \right)^\alpha \left(-\kappa_{Y0} + \frac{\kappa_{Y1}}{\kappa^*} \right) \\
 &= \eta_Y \left(\sum_{i \in I} \hat{a}_{it}^{1/\alpha} \hat{k}_{it} \right)^\alpha.
 \end{aligned}$$

During the periods that the firm engages in R&D activity, capital accumulation in the high-technology sector depends on $(\hat{a}_{it}, \hat{k}_{it})_{i \in I}$ and, therefore, is a random. We summarize the results just discussed in the following proposition:

PROPOSITION 4: *In a sustained growth path, the firm in the high-technology sector experiences both periods of no search and periods of search for new technologies. More specifically,*

- (i) *during periods of no search, the capital accumulation in this sector is driven by $K_{Y,t+1} = \eta_Y a K_{Yt}^\alpha$; and*
- (ii) *during periods of search for new technology, capital accumulation is governed by the dynamic equation $K_{Y,t+1} = \eta_Y \left(\sum_{i \in I} \hat{a}_{it}^{1/\alpha} \hat{k}_{it} \right)^\alpha$.*

In the long run, GDP can be calculated in the two-sector economy by adding the value of each sector. The contribution to GDP of the high-technology sector is

$$\hat{Y}_t = \left(\frac{1-\alpha}{\hat{w}_t} \right)^{(1-\alpha)/\alpha} \sum_{i \in I} \hat{a}_{it}^{1/\alpha} \hat{k}_{it} = \frac{\hat{w}_t}{(1-\alpha)\kappa^*}.$$

The contribution to GDP of the traditional sector, which can be obtained with the help of (6) and (23), is given by

$$\hat{p}_t \hat{Z}_t = \left(\frac{(1-\beta) \left(\frac{\kappa^* - 1}{\kappa^* K_Z} \right)^\beta}{(1-\beta)b} \right)^{(1-\beta)/\beta} b^{1/\beta} K_Z \frac{\hat{w}_t}{(1-\beta)b} \left(\frac{\kappa^* - 1}{\kappa^* K_Z} \right)^\beta = \frac{(\kappa^* - 1)\hat{w}_t}{(1-\beta)\kappa^*}.$$

Thus GDP is

$$\hat{Y}_t + \hat{p}_t \hat{Z}_t = \frac{\hat{w}_t}{\kappa^*} \left(\frac{1}{1-\alpha} + \frac{\kappa^* - 1}{1-\beta} \right).$$

The share of the production of traditional good in GDP is

$$\frac{\hat{p}_t \hat{Z}_t}{\hat{Y}_t + \hat{p}_t \hat{Z}_t} = \frac{\frac{\kappa^* - 1}{(1 - \beta)\kappa^*} \hat{w}_t}{\frac{\hat{w}_t}{\kappa^*} \left(\frac{1}{1 - \alpha} + \frac{\kappa^* - 1}{1 - \beta} \right)} = \frac{(1 - \alpha)(\kappa^* - 1)}{(1 - \beta) + (1 - \alpha)(\kappa^* - 1)} = \text{constant}.$$

In terms of value, the production of the traditional good—as a proportion of national income—is constant. In physical terms, the high-technology sector grows relative to the traditional sector. In value terms, the relative size remains constant through time.

8. CONCLUSION

In this chapter, we have extended the basic model of Chapter 2 to include a second sector, called the traditional sector, in which there is no prospect for technological progress. The extension serves two purposes. First, it adds some realism to the one-sector of Chapter 2. The economy now has two sectors, a high-technology sector—which nowadays is often identified with the information and computer industries—and a traditional sector that can be identified with the industries producing agricultural products. Second, the two-sector model of the present chapter will serve as a spring board for the model of international trade formulated in Chapter 4.

We have proved the existence of overlapping-generation equilibrium and established its uniqueness. The overlapping-generation equilibrium is represented by an equilibrium wage map that gives the equilibrium wage rate in each period in terms of the state of the high-technology sector at the end of the sequential stage of each period. The logarithmic structure of the two-sector economy allows us to find a closed form for the equilibrium wage map.

Using the equilibrium. We found that that the demand for labor in each sector remains constant. Furthermore, the evolution of the traditional sector is completely deterministic although the evolution of the high-technology sector is random due to the uncertain nature of the search process. In its deterministic path, the traditional sector converges to a stationary state. All the action takes place in the high-technology sector. A successful sequential search in any period raises the wage rate and the price of the traditional good. Although the high-technology sector grows in physical terms, its value as a percentage of national income remains constant through time along any realized path of the economy. The contribution of the traditional, as a fraction of national income, is also constant over time. Labor mobility between the two sectors guarantees that the wage rate in the traditional sector is the same as that in the high-technology sector. A high wage rate in the high-technology sector, triggered by a successful R&D program, thus induces a high wage rate in the traditional sector. The demand for the traditional good for current consumption and investment purposes then support a rise in the price of the traditional good.

APPENDIX 1

THE EXISTENCE OF OVERLAPPING-GENERATION EQUILIBRIUM

Let us pick an arbitrary wage rate w strictly between $w^b(\hat{\Xi}_t, K_z)$ and $w^\#(\hat{\Xi}_t, K_z)$. In order for the labor market in the current period to clear, the current price of the traditional good must be given by

$$(A1.1) \quad p(w | \hat{\Xi}_t, K_Z) = \frac{w}{(1-\beta)b} \left(\frac{1 - \left(\frac{1-\alpha}{w} \right)^{1/\alpha} \hat{\Xi}_t}{K_Z} \right)^\beta.$$

Suppose that under the price system $(w, p(w | \hat{\Xi}_t, K_Z))$ the current markets for the high-technology good and the traditional good also clear. Under this scenario, the capital stocks of the two sectors in the next period are given by

$$K_{Y,t+1} = S_Y(w | \hat{\Xi}_t, K_Z) = w \left(-\kappa_{Y0} + \kappa_{Y1} \left(\frac{1-\alpha}{w} \right)^{1/\alpha} \hat{\Xi}_t \right),$$

and

$$K_{Z,t+1} = S_Z(w | \hat{\Xi}_t, K_Z) = b \left[\kappa_{Z0} - \kappa_{Z1} \left(\frac{1-\alpha}{w} \right)^{1/\alpha} \hat{\Xi}_t \right] \left[\frac{1 - \left(\frac{1-\alpha}{w} \right)^{1/\alpha} \hat{\Xi}_t}{K_Z} \right]^{-\beta},$$

while the technological of the economy at the beginning of the next period is given by

$$a_{t+1} = \max_{i \in I} \hat{a}_{it}.$$

The economy thus begins period $t+1$ in state $(a_{t+1}, K_{Y,t+1}, K_{Z,t+1})$, and a firm in the high-technology sector begins the sequential stage of that period in state $(a_{i,t+1}, k_{i,t+1}) = (a_{t+1}, K_{Y,t+1} / I)$. Let $(\hat{a}_{i,t+1}, \hat{k}_{i,t+1})_{i \in I}$ be realized the state of the high-technology sector at the end of the sequential search of that period. The distribution of this array, we recall, is denoted by $\prod_{i \in I} H((\hat{a}_{i,t+1}, \hat{k}_{i,t+1}) | (a_{i,t+1}, k_{i,t+1}))$. Given that Ω^0 is the wage map that is anticipated to prevail in that period, the realized wage rate and the realized price of the traditional good will be given, respectively, by

$$\hat{w}_{t+1} = (1-\alpha)[\kappa^0 \hat{\Xi}_{t+1}]^\alpha$$

and

$$\begin{aligned} \hat{p}_{t+1} &= p(\hat{w}_{t+1} | (\hat{a}_{i,t+1}, \hat{k}_{i,t+1})_{i \in I}, K_{Z,t+1}) = \frac{\hat{w}_{t+1}}{(1-\beta)b} \left(\frac{1 - \left(\frac{1-\alpha}{\hat{w}_{t+1}} \right)^{1/\alpha} \hat{\Xi}_{t+1}}{K_{Z,t+1}} \right)^\beta \\ &= \frac{\hat{w}_{t+1}}{(1-\beta)b} \left(\frac{1 - \frac{1}{\kappa^0}}{K_{Z,t+1}} \right)^\beta \end{aligned}$$

where we have let $\hat{\Xi}_{t+1} = \sum_{i \in I} \hat{a}_{i,t+1}^{1/\alpha} \hat{k}_{i,t+1}$. The in-kind gross rate of return to the capital investment made by the young generation of period t is thus given by

$$\hat{r}_{Y,t+1} = r_{Y,t+1}((\hat{a}_{i,t+1}, \hat{k}_{i,t+1})_{i \in I}, \hat{w}_{t+1} | (a_{i,t+1}, k_{i,t+1})_{i \in I}) = \frac{\alpha \left(\frac{1-\alpha}{\hat{w}_{t+1}} \right)^{(1-\alpha)/\alpha} \hat{\Xi}_{t+1}}{K_{Y,t+1}},$$

and

$$\hat{r}_{Z,t+1} = r_{Z,t+1}(K_{Z,t+1}, \hat{p}_{t+1}, \hat{w}_{t+1}) = \beta \left(\frac{(1-\beta)\hat{p}_{t+1}}{\hat{w}_{t+1}} \right)^{(1-\beta)/\beta} b^{1/\beta} = \beta b \left(\frac{1 - \frac{1}{\kappa^0}}{K_{Z,t+1}} \right)^{1-\beta}.$$

Now in the present context, the optimal investment portfolio of a young individual of period t is characterized by the following first-order conditions:

$$(A1.2) \quad \frac{\delta w}{1+\delta} \mathcal{E} \left[\frac{\hat{r}_{Y,t+1}}{\hat{r}_{Y,t+1} K_{Y,t+1} + \hat{p}_{t+1} \hat{r}_{Z,t+1} K_{Z,t+1}} \right] = 1,$$

$$(A1.3) \quad \frac{\delta w}{(1+\delta)p(w|\hat{\Xi}_t, K_Z)} \mathcal{E} \left[\frac{\hat{p}_{t+1} \hat{r}_{Z,t+1}}{\hat{r}_{Y,t+1} K_{Y,t+1} + \hat{p}_{t+1} \hat{r}_{Z,t+1} K_{Z,t+1}} \right] = 1.$$

In (A1.2) and (A1.3), \mathcal{E} is the expectation operator with respect to the distribution of the random vector $(\hat{r}_{Y,t+1}, \hat{r}_{Z,t+1}, \hat{p}_{t+1})$. It follows from (A1.2) and (A1.3) that

$$(A1.4) \quad p(w|\hat{\Xi}_t, K_Z) \mathcal{E} \left[\frac{\hat{r}_{Y,t+1}}{\hat{r}_{Y,t+1} K_{Y,t+1} + \hat{p}_{t+1} \hat{r}_{Z,t+1} K_{Z,t+1}} \right] = \mathcal{E} \left[\frac{\hat{p}_{t+1} \hat{r}_{Z,t+1}}{\hat{r}_{Y,t+1} K_{Y,t+1} + \hat{p}_{t+1} \hat{r}_{Z,t+1} K_{Z,t+1}} \right].$$

Using the expressions for $\hat{r}_{Y,t+1}$ and $\hat{r}_{Z,t+1}$, we can rewrite (A1.4) as follows

$$p(w|\hat{\Xi}_t, K_Z) \mathcal{E} \left[\frac{\alpha \left(\frac{1-\alpha}{\hat{w}_{t+1}} \right)^{\frac{1-\alpha}{\alpha}} \hat{\Xi}_{t+1} / K_{Y,t+1}}{\alpha \left(\frac{1-\alpha}{\hat{w}_{t+1}} \right)^{\frac{1-\alpha}{\alpha}} \hat{\Xi}_{t+1} + \frac{\beta \hat{w}_{t+1}}{1-\beta} \left(1 - \frac{1}{\kappa^0} \right)} \right] = \mathcal{E} \left[\frac{\frac{\beta \hat{w}_{t+1}}{1-\beta} \left(\frac{1 - \frac{1}{\kappa^0}}{K_{Z,t+1}} \right)}{\alpha \left(\frac{1-\alpha}{\hat{w}_{t+1}} \right)^{\frac{1-\alpha}{\alpha}} \hat{\Xi}_{t+1} + \frac{\beta \hat{w}_{t+1}}{1-\beta} \left(1 - \frac{1}{\kappa^0} \right)} \right],$$

which with the help of (A1.1) becomes

$$(A1.5) \quad \frac{w}{(1-\beta)b} \left[\frac{1 - \left(\frac{1-\alpha}{w} \right)^{1/\alpha} \hat{\Xi}_t}{K_Z} \right]^\beta \mathcal{E} \left[\frac{\frac{\alpha}{(1-\alpha)K_{Y,t+1}} \left(\frac{1-\alpha}{\hat{w}_{t+1}} \right)^{\frac{1}{\alpha}} \hat{\Xi}_{t+1}}{\frac{\alpha}{1-\alpha} \left(\frac{1-\alpha}{\hat{w}_{t+1}} \right)^{\frac{1}{\alpha}} \hat{\Xi}_{t+1} + \frac{\beta}{1-\beta} \left(1 - \frac{1}{\kappa^0} \right)} \right] =$$

$$\mathcal{E} \left[\frac{\frac{\beta}{1-\beta} \left(\frac{1-\frac{1}{\kappa^0}}{K_{Z,t+1}} \right)}{\frac{\alpha}{1-\alpha} \left(\frac{1-\alpha}{\hat{w}_{t+1}} \right)^{\frac{1}{\alpha}} \hat{\Xi}_{t+1} + \frac{\beta}{1-\beta} \left(1 - \frac{1}{\kappa^0} \right)} \right]$$

Equation (A1.5) can next be rewritten as (A1.6), which is then transformed in to (A1.7)

(A1.6)

$$\frac{w}{(1-\beta)b} \left[\frac{1 - \left(\frac{1-\alpha}{w} \right)^{1/\alpha} \hat{\Xi}_t}{K_Z} \right]^\beta \mathcal{E} \left[\frac{\frac{\alpha}{(1-\alpha)K_{Y,t+1}} \frac{1}{\kappa^0}}{\frac{\alpha}{1-\alpha} \frac{1}{\kappa^0} + \frac{\beta}{1-\beta} \left(1 - \frac{1}{\kappa^0} \right)} \right] = \mathcal{E} \left[\frac{\frac{\beta}{1-\beta} \left(\frac{1-\frac{1}{\kappa^0}}{K_{Z,t+1}} \right)}{\frac{\alpha}{1-\alpha} \frac{1}{\kappa^0} + \frac{\beta}{1-\beta} \left(1 - \frac{1}{\kappa^0} \right)} \right]$$

$$(A1.7) \quad \frac{w}{(1-\beta)b} \left[\frac{1 - \left(\frac{1-\alpha}{w} \right)^{1/\alpha} \hat{\Xi}_t}{K_Z} \right]^\beta \left(\frac{\alpha}{(1-\alpha)K_{Y,t+1}} \frac{1}{\kappa^0} \right) = \frac{\beta}{1-\beta} \left(\frac{1-\frac{1}{\kappa^0}}{K_{Z,t+1}} \right)$$

Substituting the expressions for $K_{Y,t+1}$ and $K_{Z,t+1}$ in (A1.7), we obtain (A1.8)

$$\frac{w}{(1-\beta)b} \left[\frac{1 - \left(\frac{1-\alpha}{w} \right)^{1/\alpha} \hat{\Xi}_t}{K_Z} \right]^\beta \frac{\alpha}{(1-\alpha)w[-\kappa_{r0} + \kappa_{r1} \left(\frac{1-\alpha}{w} \right)^{1/\alpha} \hat{\Xi}_t]} \frac{1}{\kappa^0} = \frac{\beta}{1-\beta} \left(\frac{1-\frac{1}{\kappa^0}}{b[\kappa_{z0} - \kappa_{z1} \left(\frac{1-\alpha}{w} \right)^{1/\alpha} \hat{\Xi}_t]} \right) \left(\frac{1 - \left(\frac{1-\alpha}{w} \right)^{1/\alpha} \hat{\Xi}_t}{K_Z} \right)^\beta,$$

which can be rewritten successively as (A1.9), (A1.10), and (A1.11):

$$(A1.9) \quad \frac{\alpha}{(1-\alpha)[- \kappa_{y0} + \kappa_{y1} \left(\frac{1-\alpha}{w} \right)^{1/\alpha} \hat{\Xi}_r]} \frac{1}{\kappa^0} = \beta \left(\frac{1 - \frac{1}{\kappa^0}}{[\kappa_{z0} - \kappa_{z1} \left(\frac{1-\alpha}{w} \right)^{1/\alpha} \hat{\Xi}_r]} \right),$$

$$(A1.10) \quad \frac{\alpha}{(1-\alpha)[- \kappa_{y0} + \kappa_{y1} \hat{x}_r]} \frac{1}{\kappa^0} = \beta \left(\frac{1 - \frac{1}{\kappa^0}}{[\kappa_{z0} - \kappa_{z1} \hat{x}_r]} \right),$$

$$(A1.11) \quad \frac{\alpha}{(1-\alpha)[- \kappa_{y0} + \kappa_{y1} \hat{x}_r]} = \beta \left(\frac{\kappa^0 - 1}{[\kappa_{z0} - \kappa_{z1} \hat{x}_r]} \right).$$

Observe that in (A1.10) and (A1.11) we have let

$$(A1.12) \quad \hat{x}_r = \left(\frac{1-\alpha}{w} \right)^{1/\alpha} \hat{\Xi}_r.$$

Solving (A1.11) for \hat{x}_r , we obtain

$$(A1.13) \quad \hat{x}_r = \frac{(1-\alpha)\beta(\kappa^0 - 1)\kappa_{y0} + \alpha\kappa_{z0}}{(1-\alpha)\beta(\kappa^0 - 1)\kappa_{y1} + \alpha\kappa_{z1}}.$$

Using (A1.13) in (A1.12), we can solve for the equilibrium wage rate in the current period, namely

$$(A1.14) \quad w = (1-\alpha) \left[\frac{(1-\alpha)\beta(\kappa^0 - 1)\kappa_{y1} + \alpha\kappa_{z1}}{(1-\alpha)\beta(\kappa^0 - 1)\kappa_{y0} + \alpha\kappa_{z0}} \right]^\alpha \hat{\Xi}_r^\alpha \\ = (1-\alpha)[\kappa^1 \hat{\Xi}_r]^\alpha,$$

where

$$(A1.15) \quad \kappa^1 = \frac{(1-\alpha)\beta(\kappa^0 - 1)\kappa_{y1} + \alpha\kappa_{z1}}{(1-\alpha)\beta(\kappa^0 - 1)\kappa_{y0} + \alpha\kappa_{z0}}.$$

Lemma 3 is now proved.

APPENDIX 2

Let $\Omega : (\hat{a}_i, \hat{k}_i)_{i \in I}, K_Z) \rightarrow \Omega(\hat{a}_i, \hat{k}_i)_{i \in I}, K_Z)$ be an equilibrium wage map. Then we can write Ω as

$$\Omega : (\hat{a}_i, \hat{k}_i)_{i \in I}, K_Z) \rightarrow \Omega(\hat{a}_i, \hat{k}_i)_{i \in I}, K_Z) = (1 - \alpha) (\kappa (\sum_{i \in I} \hat{a}_i^{1/\alpha} \hat{k}_i, K_Z))^\alpha (\sum_{i \in I} \hat{a}_i^{1/\alpha} \hat{k}_i)^\beta,$$

where we have defined

$$\kappa (\sum_{i \in I} \hat{a}_i^{1/\alpha} \hat{k}_i, K_Z) = \left[\frac{\Omega(\hat{a}_i, \hat{k}_i)_{i \in I}, K_Z)}{1 - \alpha} \right]^{1/\beta} \sum_{i \in I} \hat{a}_i^{1/\alpha} \hat{k}_i.$$

Suppose that we are at the end of the sequential stage of period t and that the realized state of the two-sector economy is $(\hat{a}_i, \hat{k}_i)_{i \in I}, K_Z)$. The equilibrium wage rate in period t under Ω is thus

$$(A2.1) \quad \Omega(\hat{a}_i, \hat{k}_i)_{i \in I}, K_Z) = (1 - \alpha) (\kappa (\sum_{i \in I} \hat{a}_i^{1/\alpha} \hat{k}_i, K_Z))^\alpha (\sum_{i \in I} \hat{a}_i^{1/\alpha} \hat{k}_i)^\beta.$$

Next, let $(\hat{a}_{i,t+1}, \hat{k}_{i,t+1})_{i \in I}, K_{Z,t+1})$ be a possible realization of the state of the economy in period $t+1$. Under Ω , such a realization of the state of the world will lead to the following equilibrium wage rate:

$$(A2.2) \quad \Omega(\hat{a}_{i,t+1}, \hat{k}_{i,t+1})_{i \in I}, K_{Z,t+1}) = (1 - \alpha) (\kappa (\sum_{i \in I} \hat{a}_{i,t+1}^{1/\alpha} \hat{k}_{i,t+1}, K_{Z,t+1}))^\alpha (\sum_{i \in I} \hat{a}_{i,t+1}^{1/\alpha} \hat{k}_{i,t+1})^\beta.$$

If we now repeat the argument in Appendix 1, then we will obtain the following version of

$$(A1.15), \text{ with } \kappa (\sum_{i \in I} \hat{a}_{i,t+1}^{1/\alpha} \hat{k}_{i,t+1}, K_{Z,t+1}) \text{ replacing } \kappa^0 \text{ and } \kappa (\sum_{i \in I} \hat{a}_i^{1/\alpha} \hat{k}_i, K_Z) \text{ replacing } \kappa^1;$$

that is,

$$(A2.3) \quad \kappa (\sum_{i \in I} \hat{a}_i^{1/\alpha} \hat{k}_i, K_Z) = \zeta (\kappa (\sum_{i \in I} \hat{a}_{i,t+1}^{1/\alpha} \hat{k}_{i,t+1}, K_{Z,t+1})),$$

where we have defined

$$(A2.4) \quad \zeta(\kappa) = \frac{(1-\alpha)\beta(\kappa-1)\kappa_{Y1} + \alpha\kappa_{Z1}}{(1-\alpha)\beta(\kappa-1)\kappa_{Y0} + \alpha\kappa_{Z0}}.$$

Now the left side of (A2.3) is fixed because the uncertainty concerning R&D activities in period t has been resolved. Furthermore, as already mentioned after the statement of Proposition 1, ζ is strictly increasing for $\zeta > 1$. Also, because Ω is admissible, we must have $\kappa(\sum_{i \in I} \hat{a}_{i,t+1}^{1/\alpha} \hat{k}_{i,t+1}, K_{Z,t+1}) \geq \kappa^b > 1$. Hence $\kappa(\sum_{i \in I} \hat{a}_{i,t+1}^{1/\alpha} \hat{k}_{i,t+1}, K_{Z,t+1})$ must be equal to a constant, say $\bar{\kappa}$, regardless of the value of $(\hat{a}_{i,t+1}, \hat{k}_{i,t+1})_{i \in I}, K_{Z,t+1}$. Because Ω is an equilibrium wage map, we must also have $\kappa(\sum_{i \in I} \hat{a}_{it}^{1/\alpha} \hat{k}_{it}, K_{Zt}) = \bar{\kappa}$. Hence according to (A2.3), $\bar{\kappa}$ must be a fixed point of ζ . The uniqueness of overlapping-generation equilibrium is now immediate if we recall that κ^* is the only fixed point of ζ .

Chapter 4

INTERNATIONAL TRADE

1. INTRODUCTION

The recent literature on economies of scale, imperfect competition, and endogenous growth has been used to explain dynamic gains from trade. In this growing literature, one important question is the relation between trade and growth, and the dual question of that topic is the effect of trade and industrial policies on long-run rates of innovation and national welfare. This problem can be addressed from two perspectives. One perspective maintains that trade and industrial policies affect a country's growth rate through its effects on domestic resource allocation. According to this view, trade can be replaced with policies such as taxes and subsidies that alter the domestic resource allocation. The other perspective looks for a direct effect of international trade on productivity growth through technology transfer across the board. According to this view, total factor productivity is affected by the country's own R&D as well as by the R&D investments made by its trade partners. The literature also focuses on the impacts on

long run growth of both trade in intermediate goods as well and trade in final goods. In this strand of the literature, technology diffusion takes place through trade in intermediate inputs, which embody the know-how of the producers of these goods. The impact of importing new intermediate inputs might take various forms. First, there is a direct effect of employing a larger range of intermediate inputs in final production. Second, the import of specialized inputs might facilitate learning, imitation, and innovation of a competing product. Therefore, the countries that import to a larger extent from high-technology countries should import on average more and better differentiated input varieties than those countries that import largely from low-technology countries. Also, for a given composition of imports, technological diffusion is likely to be stronger, the greater is the overall import share of country.

In Chapter 3, we formulated and analyzed extensively a two-sector economy that consists of a high-technology sector and a traditional sector. In this chapter, we extend that model into a world of two countries that trade with each other. Each country has the economic structure of the two-sector economy described in the previous chapter. As in Chapter 3, we still assume that capital is sector-specific. However, capital—although sector-specific—is now allowed to move freely across the borders of the two countries. As for labor, it is completely mobile within each country but is not allowed to cross international borders. Again, the price of the high-technology good is chosen as the numeraire. As for the traditional good and the wage rate, their prices are still denoted by p and w , respectively. Under free trade, the prices of the high-technology good as well as the prices of the traditional good will be equalized across countries.

As in Chapters 2 and 3, technological diffusion is assumed to be complete after one period, both within national borders and across international frontiers. We have not been

particularly specific about how technological innovations are diffused across international frontiers. One can imagine that diffusion takes place through scientific channels, such as scientific journals or learned conferences where scientists and engineers present and exchange new ideas. Another possibility is through capital flow. Probably, the most effective channel for the transmission of technical know-how is through the importation of intermediate goods, which embody the most recent advanced technology. Under these channels, the discoveries made in one country can spill over into another country, helping the latter to grow.

Our main objective in this chapter is to analyze the relation of trade and growth. In particular, we want to know how the savings decision—namely the portfolio choice—of the successive young generations, the innovating behavior of the firms in the high-technology sector, and the allocation of scarce labor resources between the two sectors, determine the endogenous growth of each country in the global economy. The research presented in the present chapter departs from earlier works in several aspects. First, sustainable growth—as shown in Chapter 2—is due to an optimal combination of capital accumulation and technological improvements. Hence, a complete analysis must account for the transitional effects of technology improvements and trade in goods, in addition to the effects that persist in steady state. This chapter provides this analysis. Second, we look for the impact of growth on dynamic comparative advantage. Third, the technology transfers and the trade in intermediate inputs happen independently of the trade in goods. This allows us to distinguish the impact of technological innovation from import-driven growth.

The chapter is organized as follows. In Section 2, we discuss the lifetime utility maximization problem of a young individual. The capital accumulation in the world economy is

covered in Section 3. The definition of overlapping generation equilibrium comes in Section 4. The proof of the existence of an overlapping-generation equilibrium without R&D and its properties are given in Section 5. This existence of overlapping-generation equilibrium with R&D is asserted in Section 6. Because this existence proof is long and technical, it is presented in the appendix at the end of the chapter. The transition dynamics of the global economy is analyzed in Section 7. Some concluding remarks are given in Section 8.

2. LIFETIME UTILITY MAXIMIZATION

Suppose that the preference over lifetime consumption of a young individual is represented by the following utility function:

$$u(c_Y^0, c_Z^0, c_Y^1, c_Z^1) = \sigma \log c_Y^0 + (1 - \sigma) \log c_Z^0 + \delta [\sigma \log c_Y^1 + (1 - \sigma) \log c_Z^1].$$

Here c_Y^0 and c_Z^0 denote, respectively, the consumption of the high-technology good and the consumption of the traditional good when the individual is young, while c_Y^1 and c_Z^1 denote, respectively, the consumption of the high-technology good and the consumption of the traditional good when the individual is old. Also, σ , $0 < \sigma < 1$, is a parameter and δ , $0 < \delta < 1$, is the discount factor.

Let w_t be the wage rate and p_t be the price of the traditional good in period t . Let c_Y^0 and c_Z^0 be the consumption of the high-technology good and the consumption of the traditional good, respectively, of a young individual in period t . As for savings, let s_Y^0 and s_Z^0 denote, respectively, the capital investments in the high-technology sector and in the traditional-good

sector of such an individual. Then the following budget constraint must hold:

$$(1) \quad w_t^j - c_{Yt}^0 - s_{Yt}^0 - p_t(c_{Zt}^0 + s_{Zt}^0) = 0$$

Let $\tilde{r}_{Y,t+1}^j$ and $\tilde{r}_{Z,t+1}^j$ denote, respectively, the random in kind gross rate of return to capital invested in the high-technology sector, the in kind gross rate of return to capital invested in the traditional sector that a young individual in period t receives in his old age in country j . Also, let $(\tilde{p}_{t+1}, (w_t^j)_{j=1}^2)$ be the random price of the traditional good and wage rate in period $t+1$.

Because capital—although sector-specific—is perfectly mobile across the border at the beginning of each period, and because the firms in high-technology sectors in the two countries are identical at the beginning of each period, each of the firms receives the same amount of sector-specific capital at the beginning of each period. Thus a young individual in the current period will invest half of s_{Yt}^0 in each country and half of s_{Zt}^0 in each country. His future income is thus:

$$\sum_{j=1}^2 \left[\frac{1}{2} s_{Yt}^0 \tilde{r}_{Y,t+1}^j + \frac{1}{2} s_{Zt}^0 \tilde{r}_{Z,t+1}^j \tilde{p}_{t+1} \right] = s_{Yt}^0 \tilde{r}_{Y,t+1} + s_{Zt}^0 \tilde{r}_{Z,t+1} \tilde{p}_{t+1},$$

where \tilde{p}_{t+1} is the random price of the traditional good under free trade in period $t+1$ and

$$\tilde{r}_{Y,t+1} = \frac{1}{2} \sum_{j=1}^2 \tilde{r}_{Y,t+1}^j \quad \text{and} \quad \tilde{r}_{Z,t+1} = \frac{1}{2} \sum_{j=1}^2 \tilde{r}_{Z,t+1}^j.$$

The random consumption of the high-technology good and the random consumption of the traditional good of such an individual in his old age are then given, respectively, by

$$\tilde{c}_{Y,t+1}^1 = \sigma(\tilde{r}_{Y,t+1}s_{Yt}^0 + \tilde{p}_{t+1}\tilde{r}_{Z,t+1}s_{Zt}^0),$$

$$\tilde{c}_{Z,t+1}^1 = (1-\sigma)(\tilde{r}_{Y,t+1}s_{Yt}^0 + \tilde{p}_{t+1}\tilde{r}_{Z,t+1}s_{Zt}^0) / \tilde{p}_{t+1}.$$

The expected lifetime utility of a young individual of period t , as a function of his vector of current consumption (c_{Yt}^0, c_{Zt}^0) and future consumption $(\tilde{c}_{Y,t+1}^1, \tilde{c}_{Z,t+1}^1)$ is

$$\sigma \log c_{Yt}^0 + (1-\sigma) \log c_{Zt}^0 + \delta \mathcal{E}[\sigma \log \tilde{c}_{Y,t+1}^1 + (1-\sigma) \log \tilde{c}_{Z,t+1}^1].$$

Here \mathcal{E} denotes the expectation operator with respect to the joint distribution of $(\tilde{r}_{Y,t+1}, \tilde{r}_{Z,t+1}, \tilde{p}_{t+1})$ as anticipated by a young individual of period t . The problem faced by a young individual of period t is

$$\max_{(c_{Yt}^0, c_{Zt}^0, \tilde{c}_{Y,t+1}^1, \tilde{c}_{Z,t+1}^1)} [\sigma \log c_{Yt}^0 + (1-\sigma) \log c_{Zt}^0 + \delta \mathcal{E}[\sigma \log \tilde{c}_{Y,t+1}^1 + (1-\sigma) \log \tilde{c}_{Z,t+1}^1]]$$

subject to $c_{Yt}^0 \geq 0$, $c_{Zt}^0 \geq 0$, $\tilde{c}_{Y,t+1}^1 \geq 0$, $\tilde{c}_{Z,t+1}^1 \geq 0$ and (1).

The expected lifetime utility of a young individual of period t , as a function of his vector of current consumption (c_{Yt}^0, c_{Zt}^0) and his investment portfolio (s_{Yt}^0, s_{Zt}^0) is

$$\sigma \log c_{Yt}^0 + (1-\sigma) \log c_{Zt}^0 + \delta \mathcal{E} \left[\begin{aligned} &\sigma \log [\sigma (\tilde{r}_{Y,t+1}s_{Yt}^0 + \tilde{p}_{t+1}\tilde{r}_{Z,t+1}s_{Zt}^0)] + \\ &(1-\sigma) \log [(1-\sigma)(\tilde{r}_{Y,t+1}s_{Yt}^0 + \tilde{p}_{t+1}\tilde{r}_{Z,t+1}s_{Zt}^0) / \tilde{p}_{t+1}] \end{aligned} \right].$$

Here \mathcal{E} denotes the expectation operator with respect to the joint distribution of $(\tilde{r}_{Y,t+1}, \tilde{r}_{Z,t+1}, \tilde{p}_{t+1})$ as anticipated by a young individual of period t . The problem faced by a young individual of period t is

$$(2) \max_{(c_{Yt}^0, c_{Zt}^0, s_{Yt}^0, s_{Zt}^0)} \sigma \log c_{Yt}^0 + (1-\sigma) \log c_{Zt}^0 + \delta \mathcal{E} \left[\begin{aligned} & \sigma \log [\sigma (\tilde{r}_{Y,t+1} s_{Yt}^0 + \tilde{p}_{t+1} \tilde{r}_{Z,t+1} s_{Zt}^0)] + \\ & (1-\sigma) \log [(1-\sigma) (\tilde{r}_{Y,t+1} s_{Yt}^0 + \tilde{p}_{t+1} \tilde{r}_{Z,t+1} s_{Zt}^0) / \tilde{p}_{t+1}] \end{aligned} \right]$$

subject to $c_{Yt}^0 \geq 0$, $c_{Zt}^0 \geq 0$, $s_{Yt}^0 \geq 0$, $s_{Zt}^0 \geq 0$ and (1).

Letting λ be the multiplier associated with the budget constraint (1), we can write the Lagrangian for the lifetime utility maximization problem (2) as follows:

$$\begin{aligned} \mathcal{L} = & \sigma \log c_{Yt}^0 + (1-\sigma) \log c_{Zt}^0 + \delta \mathcal{E} \left[\begin{aligned} & \sigma \log [\sigma (\tilde{r}_{Y,t+1} s_{Yt}^0 + \tilde{p}_{t+1} \tilde{r}_{Z,t+1} s_{Zt}^0)] + \\ & (1-\sigma) \log [(1-\sigma) (\tilde{r}_{Y,t+1} s_{Yt}^0 + \tilde{p}_{t+1} \tilde{r}_{Z,t+1} s_{Zt}^0) / \tilde{p}_{t+1}] \end{aligned} \right] \\ & + \lambda [w_t - c_{Yt}^0 - s_{Yt}^0 - p_t (c_{Zt}^0 + s_{Zt}^0)] \end{aligned}$$

The following first-order conditions characterize the solution to the problem represented by

(2):

$$\frac{\partial \mathcal{L}}{\partial c_{Yt}^0} = \frac{\sigma}{c_{Yt}^0} - \lambda = 0,$$

$$\frac{\partial \mathcal{L}}{\partial c_{Zt}^0} = \frac{1-\sigma}{c_{Zt}^0} - \lambda p_t = 0,$$

$$\frac{\partial \mathcal{L}}{\partial s_{Yt}^0} = \delta \mathcal{E} \left[\frac{\tilde{r}_{Y,t+1}}{\tilde{r}_{Y,t+1} s_{Yt}^0 + \tilde{p}_{t+1} \tilde{r}_{Z,t+1} s_{Zt}^0} \right] - \lambda \leq 0,$$

with equality holding if $s_{Yt}^0 > 0$,

$$\frac{\partial \mathcal{L}}{\partial s_{Zt}^0} = \delta \mathcal{E} \left[\frac{\tilde{p}_{t+1} \tilde{r}_{Z,t+1}}{\tilde{r}_{Y,t+1} s_{Yt}^0 + \tilde{p}_{t+1} \tilde{r}_{Z,t+1} s_{Zt}^0} \right] - \lambda p_t \leq 0,$$

with equality holding if $s_Z^0 > 0$,

$$\frac{\partial \mathcal{L}}{\partial \lambda} = w_t^j - c_{Y_t}^0 - s_{Y_t}^0 - p_t(c_{Z_t}^0 + s_{Z_t}^0) = 0.$$

If $s_{Y_t}^0$ and $s_{Z_t}^0$ are both positive, then using above first-order conditions, we can write

$$\lambda = \frac{\sigma}{c_{Y_t}^0} = \frac{1-\sigma}{p_t c_{Z_t}^0} = \frac{\delta}{s_{Y_t}^0} \mathcal{E} \left[\frac{\tilde{r}_{Y,t+1} s_{Y_t}^0}{\tilde{r}_{Y,t+1} s_{Y_t}^0 + \tilde{p}_{t+1} \tilde{r}_{Z,t+1} s_{Z_t}^0} \right] = \frac{\delta}{p_t s_{Z_t}^0} \mathcal{E} \left[\frac{\tilde{p}_{t+1} \tilde{r}_{Z,t+1} s_{Z_t}^0}{\tilde{r}_{Y,t+1} s_{Y_t}^0 + \tilde{p}_{t+1} \tilde{r}_{Z,t+1} s_{Z_t}^0} \right] = \frac{1+\delta}{w_t^j}.$$

In this case, the optimal lifetime plan for a young individual of period t is given by

$$(3) \quad c_{Y_t}^0 = \frac{\sigma w_t}{1+\delta}, \quad c_{Z_t}^0 = \frac{(1-\sigma) w_t}{1+\delta},$$

$$(4) \quad \mathcal{E} \left[\frac{\tilde{r}_{Y,t+1}}{\tilde{r}_{Y,t+1} s_{Y_t}^0 + \tilde{p}_{t+1} \tilde{r}_{Z,t+1} s_{Z_t}^0} \right] = \frac{1}{p_t} \mathcal{E} \left[\frac{\tilde{p}_{t+1} \tilde{r}_{Z,t+1}}{\tilde{r}_{Y,t+1} s_{Y_t}^0 + \tilde{p}_{t+1} \tilde{r}_{Z,t+1} s_{Z_t}^0} \right] = \frac{1+\delta}{\delta w_t^j}.$$

3. GLOBAL SAVINGS IN THE HIGH-TECHNOLOGY SECTORS AND THE TRADITIONAL SECTORS

Suppose that the world economy begins period t in state $(a_t, (K_{Y_t}^j, K_{Z_t}^j)_{j=1}^2)$, where a_t is the common technological level of the two countries at the beginning of period t and $K_{Y_t}^j (K_{Z_t}^j)$ represents the capital stock of the high-technology sector (the capital stock of the traditional sector) in country $j, j=1,2$. Also, let v_{t-1}^j denote the share in the world capital stocks held by the old generation of period t in country j . Thus the amounts of capital in the high-technology sectors in country 1 and in country 2 held by the old generation of period t in country j are given by $v_{t-1}^j K_{Y_t}^1$ and $v_{t-1}^j K_{Y_t}^2$, respectively. Similarly, the amounts of capital in the traditional sectors in

the two countries held by the old generation of period t in country j are given by $v_{t-1}^j K_{Zt}^1$ and $v_{t-1}^j K_{Zt}^2$, respectively. In a more compact notation, the investment portfolio held by the old generation of period t in country j is represented by the four-dimensional vectors $v_{t-1}^j (K_{Yt}^1, K_{Zt}^1, K_{Yt}^2, K_{Zt}^2)$. Of course we have $0 \leq v_{t-1}^j \leq 1$ and $v_{t-1}^1 + v_{t-1}^2 = 1$.

In the high-technology sectors, each firm can finish the search stage with a different technological level and a different amount of remaining capital. We shall let $((\hat{a}_i^j, \hat{k}_i^j)_{i \in I}, K_{Zt}^j)_{j=1}^2$ be the state of the world economy at the end of the sequential search stage of period t . Here $(\hat{a}_i^j, \hat{k}_i^j)$ represents the state of firm i in country j at the end of the search stage of period t . We shall assume that there is full technology spillover at the beginning of the next period. Hence the state of the economy at the beginning of the next period is $(a_{t+1}, (K_{Y,t+1}^j, K_{Z,t+1}^j)_{j=1}^2)$. Because labor is only mobile within each country, the following equilibrium condition, namely equation (4) of Chapter 3, holds in the labor market of country $j, j=1,2$.

$$(5) \quad p(w | \hat{\Xi}_t^j, K_{Zt}^j) = \frac{w}{b(1-\beta)} \left[\frac{1 - \left(\frac{1-\alpha}{w} \right)^{1/\alpha} \hat{\Xi}_t^j}{K_{Zt}^j} \right]^\beta.$$

Here w is the wage rate in this country and p is the price of the high-technology good. Also, we have let $\hat{\Xi}_t^j = \sum_{i \in I} (\hat{a}_i^j)^{1/\alpha} \hat{k}_i^j$. In what follows, we shall let $w(p | \hat{\Xi}_t^j, K_{Zt}^j)$ denote the inverse of $p(w | \hat{\Xi}_t^j, K_{Zt}^j)$. As defined, $w(p | \hat{\Xi}_t^j, K_{Zt}^j)$ represents the equilibrium wage rate in country j , given

that (i) p is the price of the high-technology good under free trade and (ii) $(\hat{\Xi}_t^j, K_Z^j)$ is its state at the end of the search stage of t .

Under the price system $(p, w(p | \hat{\Xi}_t^j, K_Z^j))$, the output of the high-technology good and the output of the traditional good in country j are given, respectively, by

$$Y_t^j = \left(\frac{1-\alpha}{w(p | \hat{\Xi}_t^j, K_Z^j)} \right)^{(1-\alpha)/\alpha} \hat{\Xi}_t^j \quad \text{and} \quad Z_t^j = \left(\frac{(1-\beta)p}{w(p | \hat{\Xi}_t^j, K_Z^j)} \right)^{(1-\beta)/\beta} b^{1/\beta} K_Z^j.$$

The income of the old generation of period t in country j is $v_{t-1}^j \left(\alpha \sum_{j=1}^2 Y_t^j + \beta p \sum_{j=1}^2 Z_t^j \right)$ and its demand for the high-technology and the traditional goods are given, respectively, by

$$C_{Y_t}^{1j} = \sigma v_{t-1}^j \left(\alpha \sum_{j=1}^2 Y_t^j + \beta p \sum_{j=1}^2 Z_t^j \right) \quad \text{and} \quad C_{Z_t}^{1j} = \frac{1-\sigma}{p} v_{t-1}^j \left(\alpha \sum_{j=1}^2 Y_t^j + \beta p \sum_{j=1}^2 Z_t^j \right).$$

As for the young generation of period t in country j , its demand for the high-technology and traditional goods—for current consumption—are given, respectively, by

$$C_{Y_t}^{0j} = \frac{\sigma w(p | \hat{\Xi}_t^j, K_Z^j)}{1+\delta} \quad \text{and} \quad C_{Z_t}^{0j} = \frac{(1-\sigma)w(p | \hat{\Xi}_t^j, K_Z^j)}{p(1+\delta)}.$$

3.1. Global Savings in the High-Technology Sectors

Given $(\hat{\Xi}_t^j, K_Z^j)_{j=1}^2$, the excess global supply of the high-technology good over the global demand for this good for current consumption, as a function of p , is given by

$$\begin{aligned}
S_Y(p | (\hat{\Xi}_t^j, K_Z^j)_{j=1}^2) &= \sum_{j=1}^2 Y_t^j - \sum_{j=1}^2 C_{Y_t}^{0j} - \sum_{j=1}^2 C_{Y_t}^{1j} \\
&= \sum_{j=1}^2 \left[Y_t^j - \frac{\sigma w}{1+\delta} (p | \hat{\Xi}_t^j, K_Z^j) - \sigma(\alpha Y_t^j + \beta p Z_t^j) \right] \\
(6) \quad &= \sum_{j=1}^2 S_Y(w(p | \hat{\Xi}_t^j, K_Z^j) | \hat{\Xi}_t^j, K_Z^j) \\
&= \sum_{j=1}^2 w(p | \hat{\Xi}_t^j, K_Z^j) \left(-\kappa_{Y0} + \kappa_{Y1} \left(\frac{1-\alpha}{w(p | \hat{\Xi}_t^j, K_Z^j)} \right)^{1/\alpha} \hat{\Xi}_t^j \right)
\end{aligned}$$

In (6), the third and fourth equalities follows from the definition of $S_Y(w | \hat{\Xi}_t^j, K_Z^j)$, which—as defined by (14) in the two-sector model of the preceding chapter—represents the excess aggregate supply of the high-technology good over the aggregate demand for this good for current consumption for a closed economy.

Observe that $S_Y(w | \hat{\Xi}_t^j, K_Z^j)$ declines continuously from

$$\left(1 - \frac{\sigma(1+\alpha\delta)}{1+\delta} \right) \sum_{j=1}^2 (\hat{\Xi}_t^j)^\alpha > 0 \text{ to } -\infty \text{ when } p \text{ rises from zero to } +\infty. \text{ Hence there exists a}$$

unique value of p , say $p^*((\hat{\Xi}_t^j, K_Z^j)_{j=1}^2)$, with the following properties:

$$\begin{aligned}
S_Y(p | (\hat{\Xi}_t^j, K_Z^j)_{j=1}^2) &> 0 \text{ when } 0 \leq p < p^*((\hat{\Xi}_t^j, K_Z^j)_{j=1}^2), \\
&= 0 \text{ when } p = p^*((\hat{\Xi}_t^j, K_Z^j)_{j=1}^2), \\
&< 0 \text{ when } p > p^*((\hat{\Xi}_t^j, K_Z^j)_{j=1}^2).
\end{aligned}$$

3.2. Global Savings in the Traditional Sectors

Similarly, given $(\hat{\Xi}_t^j, K_Z^j)_{j=1}^2$, the excess global supply of the traditional good over the global demand for this good for current consumption—as a function of p —is

$$\begin{aligned}
\mathbf{S}_Z(p | (\hat{\Xi}_t^j, K_{Z_t}^j)_{j=1}^2) &= \sum_{j=1}^2 Z_t^j - \sum_{j=1}^2 C_{Z_t}^{0j} - \sum_{j=1}^2 C_{Z_t}^{1j} \\
&= \sum_{j=1}^2 \left[Z_t^j - \frac{(1-\sigma)w}{(1+\delta)p} (p | \hat{\Xi}_t^j, K_{Z_t}^j) - \frac{(1-\sigma)}{p} (\alpha Y_t^j + \beta p Z_t^j) \right] \\
(7) \quad &= \sum_{j=1}^2 S_Z(w(p | \hat{\Xi}_t^j, K_{Z_t}^j) | \hat{\Xi}_t^j, K_{Z_t}^j) \\
&= \frac{1}{(1-\beta)b} \sum_{j=1}^2 w(p | \hat{\Xi}_t^j, K_{Z_t}^j) \left(\kappa_{Z_0} - \kappa_{Z_1} \left(\frac{1-\alpha}{w(p | \hat{\Xi}_t^j, K_{Z_t}^j)} \right)^{1/\alpha} \hat{\Xi}_t^j \right).
\end{aligned}$$

Here note that the third and fourth equalities in (7) follow from the definition of

$S_Z(w(p | \hat{\Xi}_t^j, K_{Z_t}^j) | \hat{\Xi}_t^j, K_{Z_t}^j)$, as represented by (15) of the preceding chapter. Observe that

$p \rightarrow \mathbf{S}_Z(p | (\hat{\Xi}_t^j, K_{Z_t}^j)_{j=1}^2)$, is continuous and strictly increasing. Furthermore,

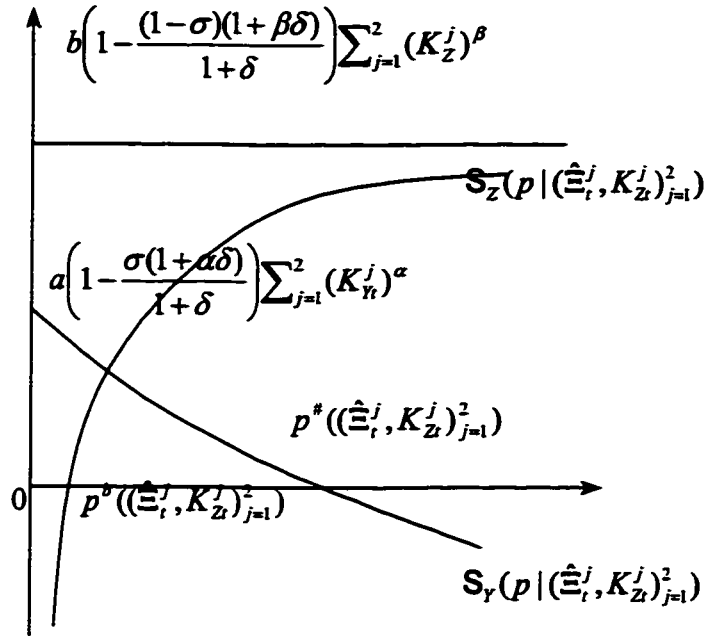
$$\lim_{p \downarrow 0} \mathbf{S}_Z(p | (\hat{\Xi}_t^j, K_{Z_t}^j)_{j=1}^2) = -\infty$$

and

$$\lim_{p \rightarrow \infty} \mathbf{S}_Z(p | (\hat{\Xi}_t^j, K_{Z_t}^j)_{j=1}^2) = b \left(1 - \frac{(1-\sigma)(1+\beta\delta)}{1+\delta} \right) \sum_{j=1}^2 (K_{Z_t}^j)^\beta > 0.$$

Hence there exists a unique value of p , say $p = p^b((\hat{\Xi}_t^j, K_{Z_t}^j)_{j=1}^2)$, with the following properties:

$$\begin{aligned}
\mathbf{S}_Z(p | (\hat{\Xi}_t^j, K_{Z_t}^j)_{j=1}^2) &> 0 \text{ when } p > p^b((\hat{\Xi}_t^j, K_{Z_t}^j)_{j=1}^2), \\
&= 0 \text{ when } p = p^b((\hat{\Xi}_t^j, K_{Z_t}^j)_{j=1}^2), \\
&< 0 \text{ when } p < p^b((\hat{\Xi}_t^j, K_{Z_t}^j)_{j=1}^2).
\end{aligned}$$



3.3 The Range of Admissible Prices for the Traditional Good under Free Trade

Let p be the current price of the traditional good under free trade. Then $w(p | \hat{\xi}_t^j, K_Z^j)$ will be the wage rate for country $j, j=1,2$, that clears the labor market in country j . The saving of a young individual in country j is thus $\frac{\delta w(p | \hat{\xi}_t^j, K_Z^j)}{1 + \delta}$ and we have

$$\frac{\delta w(p | \hat{\xi}_t^j, K_Z^j)}{1 + \delta} = S_Y(w(p | \hat{\xi}_t^j, K_Z^j) | \hat{\xi}_t^j, K_Z^j) + p S_Z(w(p | \hat{\xi}_t^j, K_Z^j) | \hat{\xi}_t^j, K_Z^j)$$

The value of global savings of all the young individuals in period t is then

$$(8) \quad \frac{\delta \sum_{j=1}^2 w(p | \hat{\xi}_t^j, K_Z^j)}{1 + \delta} = S_Y(p | (\hat{\xi}_t^j, K_Z^j)_{j=1}^2) + p S_Z(p | (\hat{\xi}_t^j, K_Z^j)_{j=1}^2).$$

Next, note that although each country is an open system in the world economy, the world economy, as a whole, is a closed system. Thus the world production of each good must be at

least as high as the world demand for that good for current consumption, and a price level p that leads to $S_Y(p | (\hat{\Xi}_t^j, K_{Z_t}^j)_{j=1}^2) < 0$ or $S_Z(p | (\hat{\Xi}_t^j, K_{Z_t}^j)_{j=1}^2) < 0$ is not possible. In order for a price level to be admissible, it must satisfy the following condition:

$$p^b((\hat{\Xi}_t^j, K_{Z_t}^j)_{j=1}^2) \leq p \leq p^*((\hat{\Xi}_t^j, K_{Z_t}^j)_{j=1}^2).$$

4. DEFINITION OF OVERLAPPING-GENERATION EQUILIBRIUM

By an admissible price map for the traditional good under free trade we mean a continuous function $\mathcal{P} : (\hat{\Xi}_t^j, K_{Z_t}^j)_{j=1}^2 \rightarrow \mathcal{P}((\hat{\Xi}_t^j, K_{Z_t}^j)_{j=1}^2)$ that satisfies the following condition:

$$p^b((\hat{\Xi}_t^j, K_{Z_t}^j)_{j=1}^2) < \mathcal{P}((\hat{\Xi}_t^j, K_{Z_t}^j)_{j=1}^2) < p^*((\hat{\Xi}_t^j, K_{Z_t}^j)_{j=1}^2).$$

Let \mathcal{P} be an admissible price map for the traditional good under free trade. By a world saving map that is consistent with \mathcal{P} we mean a continuous map

$$\begin{aligned} \mathcal{P} &: (\hat{\Xi}_t^j, K_{Z_t}^j)_{j=1}^2 \rightarrow \mathcal{P}((\hat{\Xi}_t^j, K_{Z_t}^j)_{j=1}^2) \\ &= (\mathcal{P}_Y^1((\hat{\Xi}_t^j, K_{Z_t}^j)_{j=1}^2), \mathcal{P}_Z^1((\hat{\Xi}_t^j, K_{Z_t}^j)_{j=1}^2), \mathcal{P}_Y^2((\hat{\Xi}_t^j, K_{Z_t}^j)_{j=1}^2), \mathcal{P}_Z^2((\hat{\Xi}_t^j, K_{Z_t}^j)_{j=1}^2)) \forall \in \mathbb{R}_+^4 \end{aligned}$$

with the following properties. For each $j=1,2$,

$$(9) \quad \begin{aligned} &\mathcal{P}_Y^j((\hat{\Xi}_t^j, K_{Z_t}^j)_{j=1}^2) + \mathcal{P}((\hat{\Xi}_t^j, K_{Z_t}^j)_{j=1}^2) \mathcal{P}_Z^j((\hat{\Xi}_t^j, K_{Z_t}^j)_{j=1}^2) \\ &= \frac{\delta}{1+\delta} w(\mathcal{P}((\hat{\Xi}_t^j, K_{Z_t}^j)_{j=1}^2) | \hat{\Xi}_t^j, K_{Z_t}^j). \end{aligned}$$

Here we interpret $\mathcal{P}_Y^j((\hat{\Xi}_t^j, K_{Z_t}^j)_{j=1}^2)$ and $\mathcal{P}_Z^j((\hat{\Xi}_t^j, K_{Z_t}^j)_{j=1}^2)$ as the saving in the high-technology good and the saving in the traditional good by the young generation of period t in country j . Condition (9) is nothing other than the saving budget constraint that must be respected by the

young generation of period t in country j .

Consider a pair $(\mathcal{P}, \mathcal{S})$, where \mathcal{P} is an admissible price map for the traditional good under free trade and \mathcal{S} is a world saving map that is consistent with \mathcal{P} . Suppose that $((\hat{a}_{it}^j, \hat{k}_{it}^j)_{i \in I}, K_{Zt}^j)$ is the state of the economy of country j at the end of the sequential search of period t . Let $\hat{p}_t = \mathcal{P}((\hat{\Xi}_t^j, K_{Zt}^j)_{j=1}^2)$ be the price of the traditional good under the price map \mathcal{P} . When the price \hat{p}_t prevails under free trade, the wage rate that clears the labor market in country j is given by $\hat{w}_t^j = w(\hat{p}_t | \hat{\Xi}_t^j, K_{Zt}^j)$. Under the price system $(\hat{p}_t, (\hat{w}_t^j)_{j=1}^2)$, the excess global supply of the high-technology good over the global demand for this good for current consumption is given by $\hat{S}_Y = S_Y(\hat{p}_t | (\hat{\Xi}_t^j, K_{Zt}^j)_{j=1}^2)$, while the excess global supply of the traditional good over the global demand for this good for current consumption is given by $\hat{S}_Z = S_Z(\hat{p}_t | (\hat{\Xi}_t^j, K_{Zt}^j)_{j=1}^2)$.

Let $a_{t+1} = \max_{i \in I, j=1,2} \hat{a}_{it}^j$, $K_{Y,t+1}^j = \frac{1}{2} \hat{S}_Y$, and $K_{Z,t+1}^j = \frac{1}{2} \hat{S}_Z$. Suppose now that a young individual of period t —in country 1 or 2—anticipates that the world economy begins period $t+1$ in state $(a_{t+1}, (K_{Y,t+1}^j, K_{Z,t+1}^j)_{j=1}^2)$ and that \mathcal{P} will be the price map for the traditional good under free trade, while \mathcal{S} will be the world saving map—both prevailing in period $t+1$. Let $((\hat{a}_{i,t+1}^j, \hat{k}_{i,t+1}^j)_{i \in I}, K_{Z,t+1}^j)$ be the realized state of the economy of country j at the end of the sequential search of period $t+1$. We recall from the preceding chapter that the distribution of the array $(\hat{a}_{i,t+1}^j, \hat{k}_{i,t+1}^j)_{i \in I}$ is given by

$$(10) \quad \prod_{i \in I} H(\hat{a}_{i,t+1}^j, \hat{k}_{i,t+1}^j | a_{i,t+1}^j, k_{i,t+1}^j)_{i \in I},$$

where $a_{i,t+1}^j = a_{t+1}$ and $k_{i,t+1}^j = K_{Y,t+1}^j / |I|$.

Using (10), we can then compute the distribution of $\hat{\Xi}_{t+1}^j = \sum_{i \in I} (\hat{a}_{i,t+1}^j)^{1/\alpha} \hat{k}_{i,t+1}^j$. Using the assumption that the R&D activities of the firms in high-technology sectors in the two countries are independent, we can then find the distribution of the array $(\hat{\Xi}_{t+1}^j, K_{Z,t+1}^j)_{j=1}^2$.

Let

$$(11) \quad \hat{p}_{t+1} = \mathcal{P}((\hat{\Xi}_{t+1}^j, K_{Z,t+1}^j)_{j=1}^2),$$

$$(12) \quad \hat{w}_{t+1}^j = w(\hat{p}_{t+1} | \hat{\Xi}_{t+1}^j, K_{Z,t+1}^j).$$

As defined, $(\hat{p}_{t+1}, (\hat{w}_{t+1}^j)_{j=1}^2)$ is the price system that prevails in period $t+1$ under \mathcal{P} , given that $((\hat{a}_{i,t+1}^j, \hat{k}_{i,t+1}^j)_{i \in I}, K_{Z,t+1}^j)$ is the realized state of the economy of country j at the end of the sequential search of period $t+1$. The realized in kind gross rate of return to capital invested in the high-technology sector of country j is

$$(13) \quad \hat{r}_{Y,t+1}^j = \alpha \left(\frac{1-\alpha}{\hat{w}_{Y,t+1}^j} \right)^{(1-\alpha)/\alpha} \frac{\hat{\Xi}_{t+1}^j}{K_{Y,t+1}^j}, \quad (j=1,2).$$

The realized in kind gross rate of return to capital invested in the traditional sector in country j is

$$(14) \quad \hat{r}_{Z,t+1}^j = \beta b^{1/\beta} \left(\frac{(1-\beta)\hat{p}_{t+1}}{\hat{w}_{Y,t+1}^j} \right)^{(1-\beta)/\beta}, \quad (j=1,2).$$

Observe that the joint distribution of $(\hat{p}_{t+1}, (\hat{r}_{Y,t+1}^j, \hat{r}_{Z,t+1}^j)_{j=1}^2)$ is completely determined by \mathcal{P} .

We are now ready to give a formal definition of overlapping-generation equilibrium for our trade model.

Let \mathcal{P} be an admissible price map for the traditional good under free trade and \mathcal{S} be a world saving map that is consistent with \mathcal{P} . For each array $((\hat{a}_i^j, \hat{k}_i^j)_{i \in I}, K_Z^j)_{j=1}^2$, which represents the state of the world economy at the end of the sequential search stage of period t , let

$$\hat{p}_t = \mathcal{P}((\hat{\Xi}_t^j, K_Z^j)_{j=1}^2),$$

$$\hat{w}_t^j = w(\hat{p}_t | \hat{\Xi}_t^j, K_Z^j), \quad (j=1,2),$$

where $\hat{\Xi}_t^j = \sum_{i \in I} (\hat{a}_i^j)^{1/\alpha} \hat{k}_i^j$. We say that the pair $(\mathcal{P}, \mathcal{S})$ is an overlapping-generation equilibrium if the following two conditions are satisfied.

First, $(\mathcal{P}_Y^j((\hat{\Xi}_t^j, K_Z^j)_{j=1}^2), \mathcal{P}_Z^j((\hat{\Xi}_t^j, K_Z^j)_{j=1}^2))$ is the optimal investment portfolio for a young individual of period t in country j , given that

- a) \hat{w}_t^j is his wage income,
- b) \hat{p}_t is the price of traditional good in period t ,
- c) \hat{p}_{t+1} , as represented by (11), is the random price of the traditional good in period $t+1$,
- d) $\hat{r}_{Y,t+1}^j$, as represented by (13), is the random in kind gross rate of return to capital invested in high-technology sector in country j ,
- e) $\hat{r}_{Z,t+1}^j$, as represented by (14), is the random in kind gross rate of return to capital invested in traditional sector in country j .

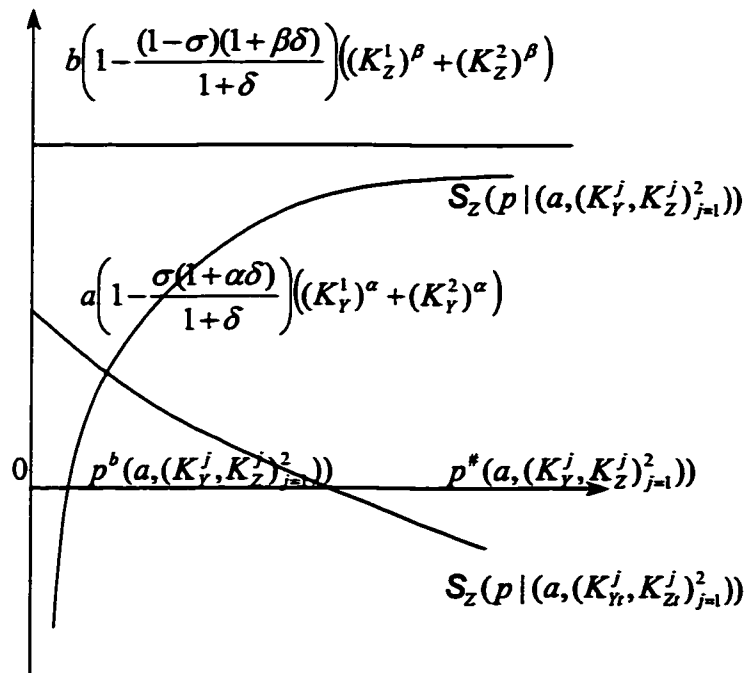
Second, expectations are realized, i.e.,

$$f) \sum_{j=1}^2 \mathcal{P}_Y^j((\hat{\Xi}_t^j, K_Z^j)_{j=1}^2) = \mathbf{S}_Y(\hat{p}_t | (\hat{\Xi}_t^j, K_Z^j)_{j=1}^2),$$

$$g) \sum_{j=1}^2 \mathcal{P}_Z^j((\hat{\Xi}_t^j, K_Z^j)_{j=1}^2) = \mathbf{S}_Z(\hat{p}_t | (\hat{\Xi}_t^j, K_Z^j)_{j=1}^2).$$

5. OVERLAPPING-GENERATION EQUILIBRIUM WITHOUT R&D

Suppose that the world economy begins the current period in state $(a, K_Y^1, K_Z^1, K_Y^2, K_Z^2)$, where a is the common technological level of all the firms in the high-technology sectors and (K_Y^j, K_Z^j) denotes the capital stock in the high-technology sector and the capital stock in the traditional sector in country $j, j = 1, 2$. Assume that no sequential search will ever be carried out from now on.



Let us pick p , $p^b(a, (K_Y^j, K_Z^j)_{j=1}^2) \leq p \leq p^*(a, (K_Y^j, K_Z^j)_{j=1}^2)$, and interpret p as the current price of the traditional good under free trade. Then the excess global supply of the high-technology good over the global demand for this good for current consumption is $S_Y(p | (a, (K_Y^j, K_Z^j)_{j=1}^2))$. Similarly, the excess global supply of the traditional good over the global demand for this good for current consumption is $S_Z(p | (a, (K_Y^j, K_Z^j)_{j=1}^2))$

Because capital in the high-technology sector of one country can move to the high-technology sector of another country at the beginning of each period, and also because of the symmetry of the high-technology sectors in the two countries, the excess global supply of the high-technology good over the global demand for this good for current consumption must fulfill the following identity:

$$S_Y(p | (a, (K_Y^j, K_Z^j)_{j=1}^2)) = 2K_{Y,+1}^j, \quad (j=1,2),$$

where $K_{Y,+1}^j$ is the capital stock of the high-technology sector of country j in the next period. Similarly, the excess global supply of the traditional good over the global demand for this good for current consumption satisfies the following relation:

$$S_Z(p | (a, (K_Y^j, K_Z^j)_{j=1}^2)) = 2K_{Z,+1}^j, \quad (j=1,2),$$

where $K_{Z,+1}^j$ is the capital stock of the traditional sector in country j at the beginning of the next period.

The capital structure of the world economy in the period next is then given by

$$(K_{Y,t+1}^1, K_{Z,t+1}^1, K_{Y,t+1}^2, K_{Z,t+1}^2) \\ = \left(\frac{1}{2} S_Y(p | (a, (K_Y^j, K_Z^j)_{j=1}^2)), \frac{1}{2} S_Z(p | (a, (K_Y^j, K_Z^j)_{j=1}^2)), \frac{1}{2} S_Y(p | (a, (K_Y^j, K_Z^j)_{j=1}^2)), \frac{1}{2} S_Z(p | (a, (K_Y^j, K_Z^j)_{j=1}^2)) \right) \gg 0.$$

Let

$$(15) \quad w_{+1}^j = (1-\alpha)a(\kappa^* K_{Y,+1}^j)^\alpha = (1-\alpha)a \left[\frac{\kappa^*}{2} S_Y(p | (a, (K_Y^j, K_Z^j)_{j=1}^2)) \right]^\alpha.$$

Here recall that κ^* is the fixed point of the map ζ , as defined by (21) of the preceding chapter.

Observe that $w_{+1}^1 = w_{+1}^2$. As defined, w_{+1}^j is the wage rate that will prevail in country j in the next period if this country is in autarky. Next, let

$$(16) \quad p_{+1} = \frac{1}{b(1-\beta)} w_{+1}^j \left(\frac{1 - \left(\frac{1-\alpha}{w_{+1}^j} \right)^{1/\alpha} a^{1/\alpha} K_{Y,+1}^j}{K_{Z,+1}^j} \right)^\beta \\ = \frac{1}{b(1-\beta)} w_{+1}^j \left(\frac{2(\kappa^* - 1)}{\kappa^* S_Z(p | (a, (K_Y^j, K_Z^j)_{j=1}^2))} \right).$$

Observe that if p_{+1} is the price of the traditional good that will prevail under free trade in the next period, then w_{+1}^j will be the wage rate that clears the labor market in country j in that period.

The gross rate of return to capital invested in the high-technology sector in country j in period $t+1$ is then

given by:

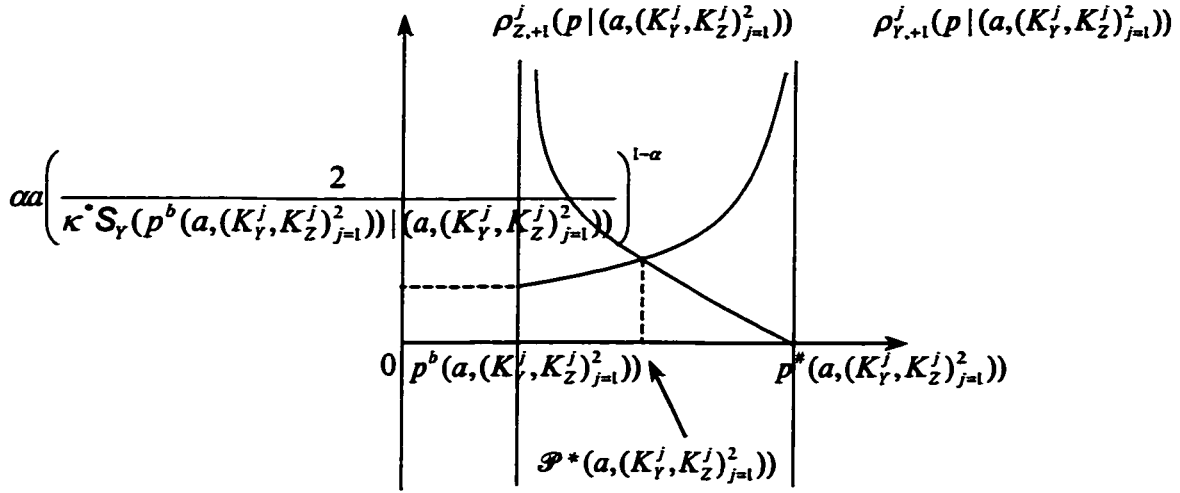
$$\begin{aligned} \rho_{Y,+1}^j(p | (a, (K_Y^j, K_Z^j)_{j=1}^2)) &= \alpha a^{1/\alpha} \left(\frac{1-\alpha}{w_{+1}^j} \right)^{\frac{1-\alpha}{\alpha}} \\ &= \alpha a^{1/\alpha} \left(\frac{1-\alpha}{(1-\alpha)a \left(\frac{1}{2} \kappa^* \mathcal{S}_Y(p | (a, (K_Y^j, K_Z^j)_{j=1}^2)) \right)^\alpha} \right)^{\frac{1-\alpha}{\alpha}} = \alpha a \left(\frac{2}{\kappa^* \mathcal{S}_Y(p | (a, (K_Y^j, K_Z^j)_{j=1}^2))} \right)^{1-\alpha} \end{aligned}$$

Note that the anticipated in kind rate of return to capital invested in the high-technology sector is strictly increasing in p . When investments and returns are expressed in terms of the numeraire, the anticipated rate of return to capital invested in the traditional-good sector in country j is

$$\begin{aligned} \rho_{Z,+1}^j(p | (a, (K_Y^j, K_Z^j)_{j=1}^2)) &= \beta b^{1/\beta} \left(\frac{(1-\beta)p_{+1}}{w_{+1}^j} \right)^{\frac{1-\beta}{\beta}} \frac{p_{+1}}{p} \\ &= \frac{\beta b^{1/\beta}}{p} \left(\frac{1}{b} \left(\frac{2(\kappa^* - 1)}{\kappa^* \mathcal{S}_Z(p | (a, (K_Y^j, K_Z^j)_{j=1}^2))} \right)^\beta \right)^{\frac{1-\beta}{\beta}} \frac{w_{+1}^j}{(1-\beta)b} \left(\frac{2(\kappa^* - 1)}{\kappa^* \mathcal{S}_Z(p | (a, (K_Y^j, K_Z^j)_{j=1}^2))} \right)^\beta \\ &= \frac{\beta w_{+1}^j}{(1-\beta)p} \frac{2(\kappa^* - 1)}{\kappa^* \mathcal{S}_Z(p | (a, (K_Y^j, K_Z^j)_{j=1}^2))} \\ &= \left(\frac{\beta}{1-\beta} \right) \left(\frac{2(\kappa^* - 1)}{\kappa^*} \right) \left(\frac{(1-\alpha)a \left(\frac{1}{2} \kappa^* \mathcal{S}_Y(p | (a, (K_Y^j, K_Z^j)_{j=1}^2)) \right)^\alpha}{p \mathcal{S}_Z(p | (a, (K_Y^j, K_Z^j)_{j=1}^2))} \right) \end{aligned}$$

In the preceding chain of equalities, the second equality is obtained with the help of (16) while the last equality is obtained with the help of (15). Since $\mathcal{S}_Y(p | (a, (K_Y^j, K_Z^j)_{j=1}^2))$ is strictly

decreasing while $S_Z(p|(a,(K_Y^j,K_Z^j)_{j=1}^2))$ is strictly increasing when p rises, $\rho_{Z,+1}^j(p|(a,(K_Y^j,K_Z^j)_{j=1}^2))$ must be strictly decreasing in p :



Observe that $\rho_{Y,+1}^j(p|(a,(K_Y^j,K_Z^j)_{j=1}^2))$ rises monotonically from

$$\alpha a \left(\frac{2}{\kappa \cdot S_Y(p^b(a,(K_Y^j,K_Z^j)_{j=1}^2)|a,(K_Y^j,K_Z^j)_{j=1}^2)} \right)^{1-\alpha}$$

to $+\infty$ as p rises from $p^b(a,(K_Y^j,K_Z^j)_{j=1}^2)$ to $p^*(a,(K_Y^j,K_Z^j)_{j=1}^2)$. Also, $\rho_{Z,+1}^j(p|(a,(K_Y^j,K_Z^j)_{j=1}^2))$ declines monotonically from $+\infty$ to zero as p rises from $p^b(a,(K_Y^j,K_Z^j)_{j=1}^2)$ to $p^*(a,(K_Y^j,K_Z^j)_{j=1}^2)$. Therefore, there exists a unique value of p , say $p = \mathcal{P}^*(a,(K_Y^j,K_Z^j)_{j=1}^2)$ such that

$$\rho_{Y,+1}^j(p|(a,(K_Y^j,K_Z^j)_{j=1}^2)) = \rho_{Z,+1}^j(p|(a,(K_Y^j,K_Z^j)_{j=1}^2)),$$

i.e.,
$$\alpha a \left(\frac{2}{\kappa \cdot S_Y(p|(a,(K_Y^j,K_Z^j)_{j=1}^2))} \right)^{1-\alpha} =$$

$$\left(\frac{\beta}{1-\beta} \right) \left(\frac{2(\kappa^* - 1)}{\kappa^*} \right) \left(\frac{(1-\alpha)a \left(\frac{1}{2} \kappa^* \mathcal{S}_Y(p | (a, (K_Y^j, K_Z^j)_{j=1}^2)) \right)^\alpha}{p \mathcal{S}_Z(p | (a, (K_Y^j, K_Z^j)_{j=1}^2))} \right)$$

After some simplification, the preceding equation becomes

$$(17) \quad \frac{(1-\alpha)\mathcal{S}_Y(p | (a, (K_Y^j, K_Z^j)_{j=1}^2))}{\alpha} = \frac{(1-\beta)p\mathcal{S}_Z(p | (a, (K_Y^j, K_Z^j)_{j=1}^2))}{\beta(\kappa^* - 1)}.$$

The preceding discussion leads to the following proposition, which asserts the existence of an overlapping-generation equilibrium.

PROPOSITION 1: *For any state of the world $(a_t, K_{Y,t}^1, K_{Z,t}^1, K_{Y,t}^2, K_{Z,t}^2)$ at the beginning of period t , $t=0,1,\dots$, let $p_t = \mathcal{P}^*(a_t, K_{Y,t}^1, K_{Z,t}^1, K_{Y,t}^2, K_{Z,t}^2)$ be the unique value of p that satisfies (17) for $(a, K_Y^1, K_Z^1, K_Y^2, K_Z^2) = (a_t, K_{Y,t}^1, K_{Z,t}^1, K_{Y,t}^2, K_{Z,t}^2)$, i.e.,*

$$(18) \quad \frac{(1-\alpha)\mathcal{S}_Y(p_t | (a, (K_{Y,t}^j, K_{Z,t}^j)_{j=1}^2))}{\alpha} = \frac{(1-\beta)p_t\mathcal{S}_Z(p_t | (a, (K_{Y,t}^j, K_{Z,t}^j)_{j=1}^2))}{\beta(\kappa^* - 1)}.$$

Then $\mathcal{P}^(a_t, K_{Y,t}^1, K_{Z,t}^1, K_{Y,t}^2, K_{Z,t}^2)$ is the equilibrium price for the traditional good under free trade in period t , given that no R&D activities will be carried out in the future.*

PROOF: Let $a = a_t = a_{t+1} = a_{t+2} = \dots$ be the unchanging technological level of the high-technology sector from time t on. Under the price map \mathcal{P}^* , the price of the traditional good under free trade in period t is given by $p_t = \mathcal{P}^*(a, K_{Y,t}^1, K_{Z,t}^1, K_{Y,t}^2, K_{Z,t}^2)$ while the wage rate that clears the labor market in country j is $w_t^j = w(p_t | a, K_{Y,t}^1, K_{Z,t}^1, K_{Y,t}^2, K_{Z,t}^2)$.

Now let $v_t^j = \frac{w_t^j}{w_t^1 + w_t^2}$ be the share in the global economy of the saving of the young

generation of period t in country j and suppose that

$$(19) \quad v_t^j (\mathbf{S}_Y(p_t | (a, (K_{Yt}^j, K_{Zt}^j)_{j=1}^2)), \mathbf{S}_Z(p_t | (a, (K_{Yt}^j, K_{Zt}^j)_{j=1}^2)))$$

is the investment portfolio of a young individual of period t in country j . Then the world capital structure in period $t+1$ will be

$$(20) \quad (K_{Y,t+1}^1, K_{Z,t+1}^1, K_{Y,t+1}^2, K_{Z,t+1}^2) =$$

$$\left(\frac{1}{2} \mathbf{S}_Y(p_t | (a, (K_{Yt}^j, K_{Zt}^j)_{j=1}^2)), \frac{1}{2} \mathbf{S}_Z(p_t | (a, (K_{Yt}^j, K_{Zt}^j)_{j=1}^2)), \frac{1}{2} \mathbf{S}_Y(p_t | (a, (K_{Yt}^j, K_{Zt}^j)_{j=1}^2)), \frac{1}{2} \mathbf{S}_Z(p_t | (a, (K_{Yt}^j, K_{Zt}^j)_{j=1}^2)) \right).$$

Under the price map \mathcal{P}^* , the price of the traditional good under free trade in period $t+1$ is

$$p_{t+1} = \mathcal{P}^*(a, K_{Y,t+1}^1, K_{Z,t+1}^1, K_{Y,t+1}^2, K_{Z,t+1}^2),$$

and according to definition of \mathcal{P}^* , we must have

$$(20) \quad \frac{1-\alpha}{\alpha} \mathbf{S}_Y(p_{t+1} | (a, (K_{Y,t+1}^j, K_{Z,t+1}^j)_{j=1}^2)) = \frac{1-\beta}{\beta(\kappa^* - 1)} p_{t+1} \mathbf{S}_Z(p_{t+1} | (a, (K_{Y,t+1}^j, K_{Z,t+1}^j)_{j=1}^2)).$$

Let w_{t+1}^j be the equilibrium wage rate in country j in period $t+1$ period when p_{t+1} is the prevailing price of the traditional good, i.e., $w_{t+1}^j = w(p_{t+1} | K_{Y,t+1}^j, K_{Z,t+1}^j)$. In terms of w_{t+1}^j , (20)

can be rewritten as follows

$$\begin{aligned} \frac{1-\alpha}{\alpha} \sum_{j=1}^2 w_{t+1}^j \left[-\kappa_{y0} + \kappa_{y1} \left(\frac{1-\alpha}{w_{t+1}^j} \right)^{1/\alpha} a^{1/\alpha} K_{y,t+1}^j \right] \\ = \frac{1-\beta}{\beta(\kappa^* - 1)} p_{t+1} \frac{1}{(1-\beta)p_{t+1}} \sum_{j=1}^2 w_{t+1}^j \left[-\kappa_{z0} + \kappa_{z1} \left(\frac{1-\alpha}{w_{t+1}^j} \right)^{1/\alpha} a^{1/\alpha} K_{y,t+1}^j \right], \end{aligned}$$

which can be simplified as

$$\begin{aligned} \sum_{j=1}^2 w_{t+1}^j \left[-\kappa_{y0} + \kappa_{y1} \left(\frac{1-\alpha}{w_{t+1}^j} \right)^{1/\alpha} a^{1/\alpha} K_{y,t+1}^j \right] \\ = \frac{\alpha}{\beta(1-\alpha)(\kappa^* - 1)} \sum_{j=1}^2 w_{t+1}^j \left[-\kappa_{z0} + \kappa_{z1} \left(\frac{1-\alpha}{w_{t+1}^j} \right)^{1/\alpha} a^{1/\alpha} K_{y,t+1}^j \right]. \end{aligned}$$

The preceding equation can be written as

$$\sum_{j=1}^2 w_{t+1}^j \left[-\left(\kappa_{y0} + \frac{\alpha}{\beta(1-\alpha)(\kappa^* - 1)} \kappa_{z0} \right) + \left(\kappa_{y1} + \frac{\alpha}{\beta(1-\alpha)(\kappa^* - 1)} \kappa_{z1} \right) \left(\frac{1-\alpha}{w_{t+1}^j} \right)^{1/\alpha} a^{1/\alpha} K_{y,t+1}^j \right].$$

Because $w_{t+1}^1 = w_{t+1}^2$ and $K_{y,t+1}^1 = K_{y,t+1}^2$, the preceding equation can be reduced to

$$2w_{t+1}^j \left[-\left(\kappa_{y0} + \frac{\alpha}{\beta(1-\alpha)(\kappa^* - 1)} \kappa_{z0} \right) + \left(\kappa_{y1} + \frac{\alpha}{\beta(1-\alpha)(\kappa^* - 1)} \kappa_{z1} \right) \left(\frac{1-\alpha}{w_{t+1}^j} \right)^{1/\alpha} a^{1/\alpha} K_{y,t+1}^j \right] = 0,$$

from which the following result follows immediately

$$\begin{aligned}
 (21) \quad \left(\frac{1-\alpha}{w_{t+1}^j} \right)^{1/\alpha} a^{1/\alpha} K_{Y,t+1}^j &= \frac{(\kappa_{Y0} + \frac{\alpha}{\beta(1-\alpha)(\kappa^* - 1)} \kappa_{Z0})}{(\kappa_{Y1} + \frac{\alpha}{\beta(1-\alpha)(\kappa^* - 1)} \kappa_{Z1})} \\
 &= \frac{\kappa_{Y0} \beta(1-\alpha)(\kappa^* - 1) + \alpha \kappa_{Z0}}{\kappa_{Y1} \beta(1-\alpha)(\kappa^* - 1) + \alpha \kappa_{Z1}} = \frac{1}{\kappa^*}.
 \end{aligned}$$

Observe that the last equality in (21) follows directly from the definition of κ^* , as given by (21) in Proposition 1 of the Chapter 3. It now follows immediately from the equality between the first and the last expressions in (21) that

$$(22) \quad w_{t+1}^j = (1-\alpha)a(\kappa^* K_{Y,t+1}^j)^\alpha, \quad (j = 1, 2).$$

Given (20) as the state of the world economy in period t , it follows from the definition of p_t that the rate of return to capital invested in the high-technology sectors and the rate of return to capital invested in the traditional sectors in the two countries are all the same. Hence it is immaterial for a young individual in period t in any country to invest his saving in any sector in any country. In particular, (19) is an optimal investment portfolio for a young individual in period t in any country, with half of the portfolio in each country. The proof that \mathcal{P}^* is an equilibrium price map for the traditional good under free trade is now complete. ■

Now according to (21), the equilibrium wage rate in country j in period $t+1$ under free trade is the same as the equilibrium wage rate that would prevail if it were in autarky. As a byproduct, the proof of Proposition 1 also shows that after one period the two sectors in each economy behave as if they were in autarky. In each of the future periods, the capital structures in the two countries are identical, with half of the capital stock in each sector owned by foreigners. Except for the cross ownership of capital, the two economies engage in an identical long march

to their stationary equilibrium, with one economy as a mirror image of the other. The dynamics of capital accumulation in each country is as described in Section 7 of Chapter 3. More specifically, the following versions of (24) and (25) in Section 7 of Chapter 3 describes the capital accumulation in the high-technology sector of each country:

$$(23) \quad K_{Y,s+1}^j = (1-\alpha)(\kappa^*)^\alpha a_t (K_{Ys}^j)^\alpha \left(-\kappa_{Y0} + \frac{\kappa_{Y1}}{\kappa^*} \right) = \eta_Y a_t (K_{Ys}^j)^\alpha, s > t + 1,$$

where

$$(24) \quad \eta_Y = (1-\alpha)(\kappa^*)^{\alpha-1} (-\kappa_{Y0}\kappa^* + \kappa_{Y1}).$$

Thus if no R&D activities will ever be carried out, the technological level of the high-technology sector will remain constant through time, say $a_t = a = \text{constant}$. Under this scenario, the capital

stock of the high-technology sector converges to the stationary equilibrium level $(\eta_Y a)^{\frac{1}{1-\alpha}}$.

As for the traditional sector in each country, the following versions of (26) and (27) in Section 7 of Chapter 3 govern its accumulation of capital

$$(25) \quad K_{Z,s+1}^j = \eta_Z b (K_{Zs}^j)^\beta, s > t + 1,$$

where

$$(26) \quad \eta_Z = \frac{(\kappa_{Z0}\kappa^* - \kappa_{Z1})}{(\kappa^*)^{1-\beta} (\kappa^* - 1)^\beta}.$$

The capital stock in the traditional sector of each country in stationary equilibrium is then

$$(\eta_Z b)^{\frac{1}{1-\beta}}.$$

6. THE EXISTENCE OF OVERLAPPING-GENERATION EQUILIBRIUM WITH R&D

Suppose that $((\hat{a}_i^j, \hat{k}_i^j)_{i \in I}, K_Z^j)_{j=1}^2$ is the state of the world economy at the end of sequential search stage of period t . Next, let $\hat{\Xi}_t^j = \sum_{j=1}^2 (\hat{a}_i^j)^{1/\alpha} \hat{k}_i^j$. Now let $\mathcal{P}^*((\hat{\Xi}_t^j, K_Z^j)_{j=1}^2)$ be the unique value of p that satisfies the following relation:

$$(27) \quad \frac{1-\alpha}{\alpha} S_Y(p | (\hat{\Xi}_t^j, K_Z^j)_{j=1}^2) = \frac{1-\beta}{\beta(\kappa^* - 1)} p S_Z(p | (\hat{\Xi}_t^j, K_Z^j)_{j=1}^2),$$

which is the generalization of (17) into the world of R&D. Here we interpret $\hat{p}_t = \mathcal{P}^*((\hat{\Xi}_t^j, K_Z^j)_{j=1}^2)$ as the free-trade price of the traditional good that prevails in period t under the price map $\mathcal{P}^* : (\hat{\Xi}_t^j, K_Z^j)_{j=1}^2 \rightarrow \mathcal{P}^*((\hat{\Xi}_t^j, K_Z^j)_{j=1}^2)$.

When \hat{p}_t prevails, the wage rate that clears the labor market in country j is given by $\hat{w}_t^j = w(\hat{p}_t | (\hat{\Xi}_t^j, K_Z^j), j=1,2$. The value of the saving by a young individual of period t in country j is then $\frac{\delta \hat{w}_t^j}{1+\delta}$. As a share in the value of global savings, the saving of the young generation of period t in country j is given by

$$(28) \quad \hat{v}_t^j = \frac{\delta \hat{w}_t^j / (1+\delta)}{\delta(\hat{w}_t^1 + \hat{w}_t^2) / (1+\delta)} = \frac{\hat{w}_t^j}{\hat{w}_t^1 + \hat{w}_t^2}.$$

Next, let

$$\mathcal{P}_Y^j((\hat{\Xi}_t^j, K_Z^j)_{j=1}^2) = \hat{v}_t^j S_Y(\hat{p}_t | (\hat{\Xi}_t^j, K_Z^j)_{j=1}^2) \text{ and } \mathcal{P}_Z^j((\hat{\Xi}_t^j, K_Z^j)_{j=1}^2) = \hat{v}_t^j S_Z(\hat{p}_t | (\hat{\Xi}_t^j, K_Z^j)_{j=1}^2).$$

We interpret $(\mathcal{P}_Y^j((\hat{\Xi}_t^j, K_Z^j)_{j=1}^2), \mathcal{P}_Z^j((\hat{\Xi}_t^j, K_Z^j)_{j=1}^2))$ as the investment portfolio of a young individual of period t in country j under the world saving map

$$(29) \quad \mathcal{P}^* : ((\hat{\Xi}_t^j, K_{Z_t}^j)^2_{j=1}) \rightarrow$$

$$(\mathcal{P}_Y^* ((\hat{\Xi}_t^j, K_{Z_t}^j)^2_{j=1}), \mathcal{P}_Z^* ((\hat{\Xi}_t^j, K_{Z_t}^j)^2_{j=1}), \mathcal{P}_Y^{2*} ((\hat{\Xi}_t^j, K_{Z_t}^j)^2_{j=1}), \mathcal{P}_Z^{2*} ((\hat{\Xi}_t^j, K_{Z_t}^j)^2_{j=1})).$$

In the appendix at the end of this chapter, we establish the following proposition:

PROPOSITION 2: *The pair $(\mathcal{P}^*, \mathcal{P}^*)$, as defined by (27) and (29) constitutes an overlapping-generation equilibrium with R&D.*

7. THE DYNAMICS OF THE GLOBAL ECONOMY WITH SEQUENTIAL SEARCH

In Chapter 3, we explained that the possibility of sustained growth depends on the relative position in the (a, k) -plane of the curve $\bar{k} : a \rightarrow \bar{k}(a)$ and the curve and the curve

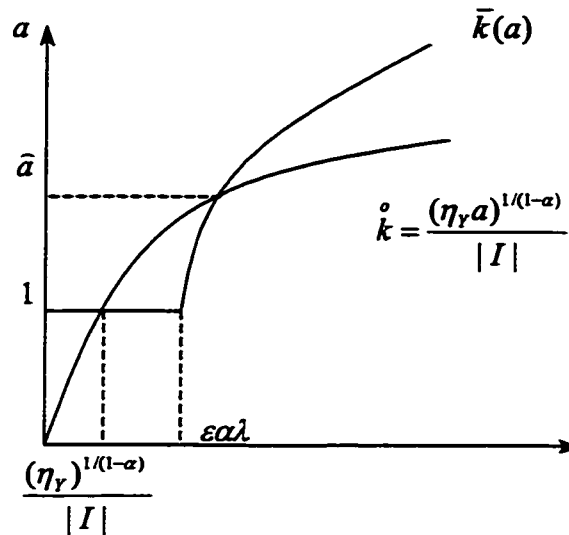
$\overset{\circ}{k} : a \rightarrow \overset{\circ}{k}(a) = \frac{(\eta_Y a)^{1/(1-\alpha)}}{|I|}$. The same discussion is also valid in the present context. If

$1/(1-\alpha) < \lambda$, there is no sustained growth in the long run. If $1/(1-\alpha) > \lambda$, each economy will grow without bound in the long run. For the borderline case $1/(1-\alpha) = \lambda$, there is no sustained

growth unless the values of the parameters of the model place the curve $\overset{\circ}{k}(a)$ is above the curve

$\bar{k}(a)$ for all $a \geq 1$. In this section, we shall concentrate only on the case $1/(1-\alpha) > \lambda$ in analyzing

the interactions between growth and trade.



In the absence of technological innovations, comparative advantage analysis is static because with an unchanging technological level, growth will stop in the long run as the factor endowments of the world economy converge to their stationary levels. When factor accumulation is driven by technological innovations and vice versa, we can examine trade in a dynamic context. In other words, extension of growth models to international trade opens a scope for dynamic comparative advantage¹.

Suppose then that the world economy begins period t in state $(a_t, (K_{Yt}^j, K_{Zt}^j)_{j=1}^2)$, where a_t is the common technological level of the two countries, $K_{Yt}^1 = K_{Yt}^2$, and $K_{Zt}^1 = K_{Zt}^2$. Let $((\hat{a}_it^j, \hat{k}_it^j)_{i \in I}, K_{Zt}^j)_{j=1}^2$ be the state of the world economy at the end of the sequential stage of this period.

Also, let $\hat{\Xi}_t^j = \sum_{i \in I} (\hat{a}_it^j)^{1-\alpha} \hat{k}_it^j$ be the “effective aggregate capital stock” in the high-technology

1. See Findlay (1984) and Smith (1984) for two surveys on trade in a dynamic context.

sector of country j at the end of the sequential search of period t . According to Proposition 2, the realized price of the traditional good, say \hat{p}_t , under free trade satisfies the following relation:

$$\frac{1-\alpha}{\alpha} S_Y(\hat{p}_t | (\hat{\Xi}_t^j, K_Z^j)_{j=1}^2) = \frac{1-\beta}{\beta(\kappa^* - 1)} \hat{p}_t S_Z(\hat{p}_t | (\hat{\Xi}_t^j, K_Z^j)_{j=1}^2).$$

The equilibrium wage rate in country j in period t , say \hat{w}_t^j , and \hat{p}_t are linked by the following market clearing condition on the labor market of that country:

$$\hat{p}_t = \frac{\hat{w}_t^j}{(1-\beta)b} \left(\frac{1 - \left(\frac{1-\alpha}{\hat{w}_t^j} \right)^{1/\alpha} \hat{\Xi}_t^j}{K_Z^j} \right)^\beta.$$

In country j , the output of the high-technology sector and the output of the traditional sector are given, respectively, by

$$(30) \quad Y_t^j = \left(\frac{1-\alpha}{\hat{w}_t^j} \right)^{(1-\alpha)/\alpha} \hat{\Xi}_t^j$$

and

$$(31) \quad Z_t^j = \left((1-\beta) \frac{\hat{p}_t}{\hat{w}_t^j} \right)^{(1-\beta)/\beta} b^{1/\beta} K_Z^j$$

$$= \left(\frac{1}{b} \left(\frac{1 - \left(\frac{1-\alpha}{\hat{w}_t^j} \right)^{1/\alpha} \hat{\Xi}_t^j}{K_Z^j} \right)^\beta \right)^{(1-\beta)/\beta} b^{1/\beta} K_Z^j$$

$$= b (K_Z^j)^\beta \left(1 - \left(\frac{1-\alpha}{\hat{w}_t^j} \right)^{1/\alpha} \hat{\Xi}_t^j \right)^{1-\beta}$$

$$= b(K_Z^j)^\beta \left(1 - \frac{(Y_t^j)^{1/(1-\alpha)} \hat{\Xi}_t^j}{(\hat{\Xi}_t^j)^{\alpha/(1-\alpha)} \hat{\Xi}_t^j} \right)^{1-\beta},$$

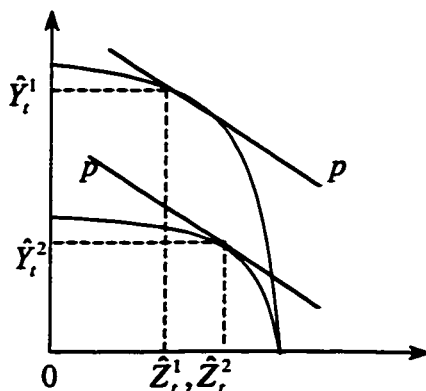
where the last expression in (31) has been obtained with the help of (30). Given K_Z^j and $\hat{\Xi}_t^j$, equation (31) expresses the relationship between the output of the high-technology good and the output of the traditional good in country j in period t . If we consider Y_t^j and Z_t^j as variables, then (31) represents the production possibility curve of country j at the end of the search stage of period t . Under this interpretation, we have

$$\frac{\partial Y_t^j}{\partial Z_t^j} = -\frac{1-\alpha}{1-\beta} \frac{(\hat{\Xi}_t^j)^{\alpha/(1-\alpha)}}{[b(K_Z^j)^\beta]^{1/(1-\beta)}} (Z_t^j)^{\beta/(1-\beta)}.$$

For a given Z_t^j , it is clear that $\left| \frac{\partial Y_t^j}{\partial Z_t^j} \right|$ rises with $\hat{\Xi}_t^j$. It is also clear that $\frac{\partial Y_t^j}{\partial Z_t^j} < 0$ and

$$\frac{\partial^2 Y_t^j}{\partial (Z_t^j)^2} < 0, \text{ i.e., the production possibility curve is concave.}$$

The difference in the technological levels of the high-technology sectors in the two countries leads to different production possibility frontier for two countries. We depict these production possibility frontiers in the following figure.



Intuitively, we expect that the country with a more successful sequential search will have a higher wage rate and, a higher output of the high-technology good, and a lower output of the traditional good. The following proposition confirms this intuition.

PROPOSITION 3: *Suppose that $\hat{\Xi}_t^1 > \hat{\Xi}_t^2$, then (i) $\hat{w}_t^1 > \hat{w}_t^2$, (ii) $\hat{Y}_t^1 > \hat{Y}_t^2$, and (iii) $\hat{Z}_t^1 < \hat{Z}_t^2$.*

PROOF: Pick any $Z \leq b(K_z^j)^\beta$. Then

$$\left. \frac{\partial Y_t^1}{\partial Z_t^1} \right|_{Z_t^1=Z} > \left. \frac{\partial Y_t^2}{\partial Z_t^2} \right|_{Z_t^2=Z}.$$

Furthermore, $(\hat{Y}_t^1, \hat{Z}_t^1)$ is the point on the production possibility frontier of country 1, where the slope is \hat{p}_t . Similarly, $(\hat{Y}_t^2, \hat{Z}_t^2)$ is the point on the production possibility curve of country 2, where the slope is \hat{p}_t . Therefore, $\hat{Y}_t^1 > \hat{Y}_t^2, \hat{Z}_t^1 < \hat{Z}_t^2$. Because the output of the traditional good in country 1 depends on $\frac{\hat{p}_t}{\hat{w}_t^1}$, while the output of traditional good in country 2 depends on $\frac{\hat{p}_t}{\hat{w}_t^2}$, we must also have $\frac{\hat{p}_t}{\hat{w}_t^1} < \frac{\hat{p}_t}{\hat{w}_t^2}$, i.e. $\hat{w}_t^1 > \hat{w}_t^2$. ■

The next proposition describes the world pattern of production in period t when the outcomes of R&D activities are dramatic.

PROPOSITION 4: *When $\hat{\Xi}_t^j \rightarrow \infty$, we have*

(i) $\hat{p}_t \rightarrow \infty$, and $\hat{w}_t^j \rightarrow \infty, j' = 1, 2$; that is, the price of the traditional good under

free trade and the wage rates in both countries all tend to infinity;

$$(ii) \quad \frac{(1-\sigma)}{(1+\delta)2b(K_z^j)^\beta} < \frac{\hat{p}_t}{\hat{w}_t^j} \leq \frac{1}{b(1-\beta)(K_z^j)^\beta}, \text{ i.e., the price-wage ratio in country } j \text{ is}$$

bounded from below and above;

- (iii) *in the limit, country j' , $j' \neq j$, will specialize in the production of the traditional good in the limit. Country j , on the other hand does not specialize in the production of the high technology good. Although in the limit country j produces an infinite amount of the high-technology good, its output of the traditional good remains bounded away from 0.*

PROOF: The labor demand by the high-technology sector of country j in period t satisfies

$$\text{the inequality } \left(\frac{1-\alpha}{\hat{w}_t^j} \right)^{1/\alpha} \hat{\Xi}_t^j \leq 1. \text{ Hence } \hat{w}_t^j \rightarrow \infty \text{ when } \hat{\Xi}_t^j \rightarrow \infty.$$

The demand for the traditional good by the young generation of period t in country j is

$(1-\sigma)\hat{w}_t^j/(1+\delta)\hat{p}_t$. Because the global output of the traditional good is bounded above by $2b(K_z^j)^\beta$, we must have $(1-\sigma)\hat{w}_t^j/(1+\delta)\hat{p}_t < 2b(K_z^j)^\beta$, which in turn implies

$$(32) \quad \hat{p}_t / \hat{w}_t^j > (1-\sigma)/(1+\delta)2b(K_z^j)^\beta > 0.$$

It follows from (32) and the result $\hat{w}_t^j \rightarrow \infty$ when $\hat{\Xi}_t^j \rightarrow \infty$, already proven, that \hat{p}_t also tends to $+\infty$ when $\hat{\Xi}_t^j \rightarrow \infty$.

The demand for labor by the traditional sector of country j' , $j' = 1, 2$, in period t satisfies

the following inequality $\left(b(1-\beta)\frac{\hat{p}_t}{\hat{w}_t^{j'}}, which implies that$

$$(33) \quad \frac{\hat{p}_t}{\hat{w}_t^{j'}} \leq \frac{1}{b(1-\beta)(K_Z^{j'})^\beta}.$$

Now a version of (32) also holds for country j' , which together with (33) allow us to write

$$(34) \quad \frac{(1-\sigma)}{(1+\delta)2b(K_Z^{j'})^\beta} < \frac{\hat{p}_t}{\hat{w}_t^{j'}} \leq \frac{1}{b(1-\beta)(K_Z^{j'})^\beta}.$$

Thus $\hat{w}_t^{j'}$ also tends to infinity when $\hat{\Xi}_t^j \rightarrow +\infty$.

Now when $\hat{w}_t^{j'} \rightarrow +\infty$ the demand for labor in period t by the high-technology sector of country j' , given $\hat{\Xi}_t^{j'}$, tends to 0, i.e., $\left(\frac{1-\alpha}{\hat{w}_t^{j'}}\right)^{1/\alpha} \hat{\Xi}_t^{j'} \rightarrow 0$. Therefore, when $\hat{\Xi}_t^j \rightarrow +\infty$, country j' will specialize in the production of the traditional good.

Finally, if country j specializes in the production of the high-technology good when

$\hat{\Xi}_t^j \rightarrow +\infty$, then we must have $\left(\frac{1-\alpha}{\hat{w}_t^j}\right)^{1/\alpha} \hat{\Xi}_t^j \rightarrow 1$, which together with

$$\hat{p}_t = \frac{\hat{w}_t^j}{b(1-\beta)} \left(\frac{1 - \left(\frac{1-\alpha}{\hat{w}_t^j}\right)^{1/\alpha} \hat{\Xi}_t^j}{K_Z^j} \right)^\beta$$

implies that

$$\frac{\hat{p}_t}{\hat{w}_t^j} = \frac{1}{b(1-\beta)} \left(\frac{1 - \left(\frac{1-\alpha}{\hat{w}_t^j}\right)^{1/\alpha} \hat{\Xi}_t^j}{K_Z^j} \right)^\beta \rightarrow 0.$$

However this result contradicts the version of (34) for country j . Hence in the limit, country j still produces both goods. ■

Recall that the income of the old generation of period t in country j is

$$\hat{v}_{t-1}^j \left(\sum_{j=1}^2 \alpha \hat{Y}_t^j + \sum_{j=1}^2 \beta \hat{p}_t \hat{Z}_t^j \right),$$

where $\hat{v}_{t-1}^j = \frac{\hat{w}_{t-1}^j}{\hat{w}_{t-1}^1 + \hat{w}_{t-1}^2}$ is the share in global capital ownership in period t of an old individual

of this period. The following national income accounting represents the export of the high-technology good in period t by country j :

$$(35) \quad \hat{E}_Y^j = \hat{Y}_t^j - \frac{\sigma \hat{w}_t^j}{1 + \delta} - \sigma \hat{v}_{t-1}^j \left(\sum_{j=1}^2 \alpha \hat{Y}_t^j + \sum_{j=1}^2 \beta \hat{p}_t \hat{Z}_t^j \right) - K_{Y,t+1}^j$$

$$= \hat{Y}_t^j - \frac{\sigma \hat{w}_t^j}{1 + \delta} - \sigma \hat{v}_{t-1}^j \left(\sum_{j=1}^2 \alpha \hat{Y}_t^j + \sum_{j=1}^2 \beta \hat{p}_t \hat{Z}_t^j \right) - \frac{1}{2} \hat{S}_Y.$$

In (35), the second, third, and last terms on the right side represent the demand for current consumption by the young generation, the demand for current consumption by the old generation, and the demand for investment by the high-technology sector.

Similarly, the export of traditional good in period t by country j is given by

$$(36) \quad \hat{E}_Z^j = \hat{Z}_t^j - \frac{(1-\sigma) \hat{w}_t^j}{\hat{p}_t (1+\delta)} - \frac{(1-\sigma)}{\hat{p}_t} \hat{v}_{t-1}^j \left(\sum_{j=1}^2 \alpha \hat{Y}_t^j + \sum_{j=1}^2 \beta \hat{p}_t \hat{Z}_t^j \right) - K_{Z,t+1}^j$$

$$= \hat{Z}_t^j - \frac{(1-\sigma) \hat{w}_t^j}{\hat{p}_t (1+\delta)} - \frac{(1-\sigma)}{\hat{p}_t} \hat{v}_{t-1}^j \left(\sum_{j=1}^2 \alpha \hat{Y}_t^j + \sum_{j=1}^2 \beta \hat{p}_t \hat{Z}_t^j \right) - \frac{1}{2} \hat{S}_Z.$$

Using (35) and (36), we can write the current account in period t of country j as

$$\begin{aligned}
(37) \quad CA_t^j &= \hat{E}_t^j + \hat{p}_t \hat{E}_t^j \\
&= \hat{Y}_t^j - \frac{\sigma \hat{w}_t^j}{1 + \delta} - \sigma \hat{v}_{t-1}^j \left(\sum_{j'=1}^2 \alpha \hat{Y}_t^{j'} + \sum_{j'=1}^2 \beta \hat{p}_t \hat{Z}_t^{j'} \right) - \frac{1}{2} \hat{S}_t^j \\
&\quad + \hat{p}_t \hat{Z}_t^j - \frac{(1 - \sigma) \hat{w}_t^j}{(1 + \delta)} - (1 - \sigma) \hat{v}_{t-1}^j \left(\sum_{j'=1}^2 \alpha \hat{Y}_t^{j'} + \sum_{j'=1}^2 \beta \hat{p}_t \hat{Z}_t^{j'} \right) - \frac{1}{2} \hat{S}_t^j.
\end{aligned}$$

Observe that a positive value of TB_t^j means a capital outflow for country j in period t .

Now, suppose that $\hat{\Xi}_t^1 = \hat{\Xi}_t^2$. Then the wage rates in period t in the two countries are identical and the two economies produce the same output of the high-technology good and the same output of the traditional good. Furthermore, if $\hat{v}_{t-1}^1 = \hat{v}_{t-1}^2$, i.e., the world capital stock in this period is split equally between the old generations in the two countries, then (35) and (36) are both equal to zero for each country: there is no trade.

Next, suppose that $\hat{\Xi}_t^1 = \hat{\Xi}_t^2$ still holds, but $\hat{v}_{t-1}^1 > \hat{v}_{t-1}^2$. Then (35) and (36) are both negative for country 1; it experiences a trade deficit in each good and these trade deficits are financed by drawing down the financial assets that the old generation of period t holds against the rest of the world.

To continue, suppose that $\hat{v}_{t-1}^1 > \hat{v}_{t-1}^2$ and that $\hat{\Xi}_t^1$ is slightly higher than $\hat{\Xi}_t^2$. By continuity, we can assert that (35) and (36) are still negative; that is, a comparative advantage in producing the high-technology good that is produced by a more successful sequential search does not necessarily mean that the country in question will export that good. The exact pattern of trade thus depends on the outcomes of current R&D activities and the current structure of the

ownership of the world capital stocks. However, when one country is much more successful than the other in its sequential search and when the shares in the world capital stocks of the old generations in the two countries are not too unequal, the pattern of trade is clear, as asserted by the following proposition:

PROPOSITION 5: *Suppose that v_{t-1}^1 and v_{t-1}^2 are not too different from each other. If $\hat{\Xi}_t^1$ is much greater than $\hat{\Xi}_t^2$, then country 1 will export the high-technology good and import the traditional good.*

PROOF: According to Proposition 4(iii), country 2 will specialize in the production of the traditional good when $\hat{\Xi}_t^1 \rightarrow \infty$. Hence Y_t^2 will be small when $\hat{\Xi}_t^1$ is large, which means that country 2 will have to import the high-technology good for current consumption. Furthermore, because (i) $Z_t^2 > Z_t^1$, (ii) $\hat{w}_t^1 > \hat{w}_t^2$, and (iii) $v_{t-1}^1 \approx v_{t-1}^2$, we must have $\hat{E}_{Z_t}^1 < \hat{E}_{Z_t}^2$, which then implies that $\hat{E}_{Z_t}^1 = -\hat{E}_{Z_t}^2 < 0$. ■

Because sector-specific capital is allowed to move freely across international borders at the beginning of each period, the capital structures of both economies are identical when each period begins. Furthermore, because the outcomes of the R&D activities of the two countries are assumed to be independent, there is no systematic bias through time for a country to enjoy a comparative advantage in the production of the high-technology good. Each country has the

same chance to obtain a better outcome in its search for productivity growth as the other. In this manner, comparative advantage has a dynamic nature, and over time there is a possibility of comparative advantage reversal. As for the current account, its behavior through time is more predictable. The current account of a country, one expect, should alternate between a surplus and a deficit. Indeed, if the outcome of the R&D activities of a country in the current period are more favorable, the current young generation of this country should own more capital than the current generation of the other country in the next period. The current young generation of the former country will thus have more income in the next period when it is old, and thus will demand more of both consumption goods, triggering a current account deficit in the future. Finally, observe that because of capital flow, the current account of a country does not always have to be equal to zero. That is, we might have $E_{Y_t}^j + \hat{p}_t E_{Z_t}^j > 0$, which does not preclude the possibility of both $E_{Y_t}^j > 0$ and $E_{Z_t}^j > 0$.

8. CONCLUSION

In this chapter, we have extended the two-sector model of Chapter 3 in to the world of international trade. There is now free trade in the high-technology good and the traditional good. Although capital is still sector-specific, we now allow sector-specific capital to move across international border. There is now an intertemporal dimension to trade. A country whose sequential search is a spectacular success in a period enjoys a comparative advantage in the production of the high-technology good. And this dynamic comparative advantage might change over time. Because of free capital flow, each country is on the same footing with the other at the beginning of every period. There is thus no systematic bias for one country to command a

dynamic comparative advantage over the other all the time.

We have provided a proof for the existence of overlapping-generation equilibrium then proceeded to investigate its properties. Unlike the two-sector model, the traditional sectors in the two countries do not evolve independently from the high-technology sectors in a deterministic manner. The uncertain nature of R&D activities now spills over into the traditional sectors and cause these sectors also to behave in a random manner. Furthermore, technological superiority does not always imply current account surplus. Whether country enjoys a current account surplus depends both on the outcomes of its R&D activities and the proportion of the world capital stocks that belongs to the old generation of this country. A persistent current account surplus rests on repeated successes in the sequential search, which is unlikely because the capital stocks of both countries are identical at the beginning of each period. The model predicts that the current account of a country will alternate between a surplus and a deficit over time.

APPENDIX

THE EXISTENCE OF OVERLAPPING-GENERATION EQUILIBRIUM WITH SEQUENTIAL SEARCH

Let us imagine that the price map \mathcal{P}^* is expected to prevail at the end of the sequential search of every period. Now suppose that the state of the world economy at the end of the sequential search of period t is $((\hat{\alpha}_i^j, \hat{k}_i^j)_{i \in I}, K_Z^j)_{j=1}^2$. Then $\hat{p}_t = \mathcal{P}^*((\hat{\Xi}_t^j, K_Z^j)_{j=1}^2)$ is the price of the traditional good that prevails under free trade in this period. The wage rate that prevails in country j in this period is $\hat{w}_t^j = w(\hat{p}_t | \hat{\Xi}_t^j, K_Z^j)$. The global savings in the high-technology good

and the global savings in the traditional good are then given, respectively, by $\hat{S}_Y = S_Y(\hat{p}_t | (\hat{\Xi}_t^j, K_Z^j)_{j=1}^2)$ and $\hat{S}_Z = S_Z(\hat{p}_t | (\hat{\Xi}_t^j, K_Z^j)_{j=1}^2)$. The investment portfolio of a young individual in country j in this period is thus $v_t^j(\hat{S}_Y, \hat{S}_Z)$, where $v_t^j = \hat{w}_t^j / (\hat{w}_t^1 + \hat{w}_t^2)$ is the share of the young generation of country j in total global savings in period t . Observe that this investment portfolio is divided equally between the two countries.

Next, suppose that $((\hat{a}_{i,t+1}^j, \hat{k}_{i,t+1}^j)_{i \in I}, K_{Z,t+1}^j)_{j=1}^2$ is the state of the world economy at the end of the sequential search of period $t+1$. In period $t+1$, the price of the traditional good under free trade is $\hat{p}_{t+1} = \mathcal{P}^*((\hat{\Xi}_{t+1}^j, K_{Z,t+1}^j)_{j=1}^2)$ and the wage rate in country j is $\hat{w}_{t+1}^j = w(\hat{p}_{t+1} | \hat{\Xi}_{t+1}^j, K_{Z,t+1}^j)$. The realized in kind rate of return to capital invested in the high-technology sectors is

$$\hat{r}_{Y,t+1} = \frac{\sum_{j=1}^2 \left[\alpha \left(\frac{1-\alpha}{\hat{w}_{t+1}^j} \right)^{\frac{1-\alpha}{\alpha}} \hat{\Xi}_{t+1}^j \right]}{\hat{S}_Y}.$$

The realized in kind rate of return to capital invested in the traditional sectors is

$$\hat{r}_{Z,t+1} = \frac{\beta b^{1/\beta}}{2} \sum_{j=1}^2 \left[(1-\beta) \frac{\hat{p}_{t+1}}{\hat{w}_{t+1}^j} \right]^{\frac{1-\beta}{\beta}}.$$

We shall now establish the following result:

$$\text{LEMMA: } \frac{\hat{r}_{Y,t+1}}{\hat{S}_Y \hat{r}_{Y,t+1} + \hat{p}_{t+1} \hat{S}_Z \hat{r}_{Z,t+1}} = \frac{1}{\hat{p}_t} \frac{\hat{p}_{t+1} \hat{r}_{Z,t+1}}{\hat{S}_Y \hat{r}_{Y,t+1} + \hat{p}_{t+1} \hat{S}_Z \hat{r}_{Z,t+1}}.$$

PROOF: Applying (6) to period $t+1$, we obtain

$$(A.1) \quad S_Y(\hat{p}_{t+1} | (\hat{\Xi}_{t+1}^j, K_{Z,t+1}^j)_{j=1}^2) = \sum_{j=1}^2 \hat{w}_{t+1}^j \left[-\kappa_{Y0} + \kappa_{Y1} \left(\frac{1-\alpha}{\hat{w}_{t+1}^j} \right)^{1/\alpha} \hat{\Xi}_{t+1}^j \right],$$

Applying (7) to period $t+1$ and taking advantage of the functional relationship

$$(A.2) \quad \hat{p}_{t+1} = \frac{\hat{w}_{t+1}^j \left(\frac{1 - \left(\frac{1-\alpha}{\hat{w}_{t+1}^j} \right)^{1/\alpha} \hat{\Xi}_{t+1}^j}{K_{Z,t+1}^j} \right)^\beta}{b(1-\beta)}$$

that captures the equilibrium on the labor market of country j in period $t+1$, we obtain

$$(A.3) \quad S_Z(\hat{p}_{t+1} | (\hat{\Xi}_{t+1}^j, K_{Z,t+1}^j)_{j=1}^2) = \frac{1}{(1-\beta)\hat{p}_{t+1}} \sum_{j=1}^2 \hat{w}_{t+1}^j \left[\kappa_{Z0} - \kappa_{Z1} \left(\frac{1-\alpha}{\hat{w}_{t+1}^j} \right)^{1/\alpha} \hat{\Xi}_{t+1}^j \right].$$

Applying (18) to $\hat{p}_{t+1} = \mathcal{P}^*((\hat{\Xi}_{t+1}^j, K_{Z,t+1}^j)_{j=1}^2)$ and using (A.1) as well as (A.3), we obtain

$$(A.4) \quad \frac{1-\alpha}{\alpha} \sum_{j=1}^2 \hat{w}_{t+1}^j [-\kappa_{Y0} + \kappa_{Y1} \hat{x}_{t+1}^j] = \frac{1}{\beta(\kappa^* - 1)} \sum_{j=1}^2 \hat{w}_{t+1}^j [\kappa_{Z0} - \kappa_{Z1} \hat{x}_{t+1}^j].$$

In (A.4), we have let $\hat{x}_{t+1}^j = \left(\frac{1-\alpha}{\hat{w}_{t+1}^j} \right)^{1/\alpha} \hat{\Xi}_{t+1}^j$. Equation (A.4) can be rewritten as

$$\sum_{j=1}^2 \hat{w}_{t+1}^j \left[-\left(\kappa_{Y0} + \frac{\alpha \kappa_{Z0}}{(1-\alpha)\beta(\kappa^* - 1)} \right) + \left(\kappa_{Y1} + \frac{\alpha \kappa_{Z1}}{(1-\alpha)\beta(\kappa^* - 1)} \right) \hat{x}_{t+1}^j \right] = 0$$

$$\text{or } \sum_{j=1}^2 \hat{w}_{t+1}^j \left[-1 + \frac{\left(\kappa_{Y1} + \frac{\alpha \kappa_{Z1}}{(1-\alpha)\beta(\kappa^* - 1)} \right)}{\left(\kappa_{Y0} + \frac{\alpha \kappa_{Z0}}{(1-\alpha)\beta(\kappa^* - 1)} \right)} \hat{x}_{t+1}^j \right] = 0.$$

Using the expression for κ^* from (21) in Chapter 3, we can simplify the preceding equation as

$$(A.5) \quad \sum_{j=1}^2 \hat{w}_{t+1}^j = \kappa^* \sum_{j=1}^2 \hat{w}_{t+1}^j \hat{x}_{t+1}^j.$$

Now we have

$$(A.6) \quad \frac{\hat{r}_{Y,t+1}}{\hat{S}_Y \hat{r}_{Y,t+1} + \hat{p}_{t+1} \hat{S}_Z \hat{r}_{Z,t+1}} = \frac{\sum_{j=1}^2 \left[\alpha \left(\frac{1-\alpha}{\hat{w}_{t+1}^j} \right)^{\frac{1-\alpha}{\alpha}} \hat{z}_{t+1}^j \right]}{\hat{S}_Y} = \frac{\sum_{j=1}^2 \left[\alpha \left(\frac{1-\alpha}{\hat{w}_{t+1}^j} \right)^{\frac{1-\alpha}{\alpha}} \hat{z}_{t+1}^j \right] + \hat{p}_{t+1} \hat{S}_Z \frac{\beta b^{1/\beta}}{2} \sum_{j=1}^2 \left[(1-\beta) \frac{\hat{p}_{t+1}}{\hat{w}_{t+1}^j} \right]^{\frac{1-\beta}{\beta}}}{\hat{S}_Y}.$$

Next, note that

$$(A.7) \quad \hat{p}_{t+1} \hat{S}_Z \beta b^{1/\beta} \sum_{j=1}^2 \left[(1-\beta) \frac{\hat{p}_{t+1}}{\hat{w}_{t+1}^j} \right]^{\frac{1-\beta}{\beta}}$$

$$\begin{aligned}
&= \hat{p}_{t+1} 2K_{Z,t+1}^j \beta b^{1/\beta} \sum_{j=1}^2 \left[\frac{1}{b} \left(\frac{1 - \left(\frac{1-\alpha}{\hat{w}_{t+1}^j} \right)^{\frac{1}{\alpha}} \hat{\underline{z}}_{t+1}^j}{K_{Z,t+1}^j} \right)^\beta \right]^{\frac{1-\beta}{\beta}} \\
&= 2\hat{p}_{t+1} K_{Z,t+1}^j \beta b \sum_{j=1}^2 \left[\frac{1 - \left(\frac{1-\alpha}{\hat{w}_{t+1}^j} \right)^{\frac{1}{\alpha}} \hat{\underline{z}}_{t+1}^j}{K_{Z,t+1}^j} \right]^{1-\beta} \\
&= 2\beta b \sum_{j=1}^2 \left[\left(1 - \left(\frac{1-\alpha}{\hat{w}_{t+1}^j} \right)^{\frac{1}{\alpha}} \hat{\underline{z}}_{t+1}^j \right) \hat{p}_{t+1} \left(\frac{1 - \left(\frac{1-\alpha}{\hat{w}_{t+1}^j} \right)^{\frac{1}{\alpha}} \hat{\underline{z}}_{t+1}^j}{K_{Z,t+1}^j} \right)^{-\beta} \right] \\
&= 2\beta b \sum_{j=1}^2 \frac{1}{(1-\beta)b} \hat{w}_{t+1}^j \left(1 - \left(\frac{1-\alpha}{\hat{w}_{t+1}^j} \right)^{\frac{1}{\alpha}} \hat{\underline{z}}_{t+1}^j \right), \text{ using} \\
&= \frac{2\beta}{1-\beta} \sum_{j=1}^2 \hat{w}_{t+1}^j \left(1 - \left(\frac{1-\alpha}{\hat{w}_{t+1}^j} \right)^{\frac{1}{\alpha}} \hat{\underline{z}}_{t+1}^j \right).
\end{aligned}$$

Using (A.7), we can rewrite (A.6) as follows:

$$(A.8) \quad \frac{\hat{r}_{Y,t+1}}{\hat{S}_Y \hat{r}_{Y,t+1} + \hat{p}_{t+1} \hat{S}_Z \hat{r}_{Z,t+1}}$$

$$\begin{aligned}
& \sum_{j=1}^2 \left[\alpha \left(\frac{1-\alpha}{\hat{w}_{t+1}^j} \right)^{\frac{1-\alpha}{\alpha}} \hat{\Xi}_{t+1}^j \right] \\
= & \frac{1}{\hat{S}_Y} \frac{\sum_{j=1}^2 \left[\alpha \left(\frac{1-\alpha}{\hat{w}_{t+1}^j} \right)^{\frac{1-\alpha}{\alpha}} \hat{\Xi}_{t+1}^j \right] + \frac{\beta}{1-\beta} \sum_{j=1}^2 \hat{w}_{t+1}^j (1-\hat{x}_{t+1}^j)}{\sum_{j=1}^2 \left[\alpha \left(\frac{1-\alpha}{\hat{w}_{t+1}^j} \right)^{\frac{1-\alpha}{\alpha}} \hat{\Xi}_{t+1}^j \right]} \\
= & \frac{1}{\hat{S}_Y} \frac{1}{1 + \frac{\beta}{1-\beta} \frac{\sum_{j=1}^2 \hat{w}_{t+1}^j (1-\hat{x}_{t+1}^j)}{\sum_{j=1}^2 \left[\alpha \left(\frac{1-\alpha}{\hat{w}_{t+1}^j} \right)^{\frac{1-\alpha}{\alpha}} \hat{\Xi}_{t+1}^j \right]}} \\
= & \frac{1}{\frac{\alpha(1-\beta)}{(1-\alpha)\beta(\kappa^*-1)} \hat{p}_t \hat{S}_Z \left[1 + \frac{\beta}{1-\beta} \frac{\sum_{j=1}^2 \hat{w}_{t+1}^j (1-\hat{x}_{t+1}^j)}{\frac{\alpha}{1-\alpha} \sum_{j=1}^2 \hat{w}_{t+1}^j \hat{x}_{t+1}^j} \right]} \\
= & \frac{1}{\hat{p}_t \hat{S}_Z \left[\frac{\alpha(1-\beta)}{(1-\alpha)\beta(\kappa^*-1)} + \frac{1}{(\kappa^*-1)} \frac{\sum_{j=1}^2 \hat{w}_{t+1}^j (1-\hat{x}_{t+1}^j)}{\sum_{j=1}^2 \hat{w}_{t+1}^j \hat{x}_{t+1}^j} \right]} \\
= & \frac{1}{\hat{p}_t \hat{S}_Z \left[\frac{\alpha(1-\beta)}{(1-\alpha)\beta(\kappa^*-1)} + \frac{1}{(\kappa^*-1)} \left[\frac{\sum_{j=1}^2 \hat{w}_{t+1}^j}{\sum_{j=1}^2 \hat{w}_{t+1}^j \hat{x}_{t+1}^j} - 1 \right] \right]} \\
= & \frac{1}{\hat{p}_t \hat{S}_Z \left[\frac{\alpha(1-\beta)}{(1-\alpha)\beta(\kappa^*-1)} + \frac{1}{(\kappa^*-1)} \left[\frac{\sum_{j=1}^2 \hat{w}_{t+1}^j}{\sum_{j=1}^2 \hat{w}_{t+1}^j \hat{x}_{t+1}^j} - 1 \right] \right]}, \text{ using (A.5),}
\end{aligned}$$

$$= \frac{1}{\hat{p}_t \hat{S}_z \left[\frac{\alpha(1-\beta)}{(1-\alpha)\beta(\kappa^*-1)} + 1 \right]}$$

Having evaluated the left side of the equality stated in the lemma, we next evaluate its right side. We have

$$\begin{aligned}
 \text{(A.9)} \quad & \frac{1}{\hat{p}_t} \frac{\hat{S}_N \hat{r}_{t,t+1}}{\hat{S}_N \hat{r}_{t,t+1} + \hat{p}_{t+1} \hat{S}_z \hat{r}_{z,t+1}} = \frac{1}{\hat{p}_t \hat{S}_z} \frac{1}{\frac{\hat{S}_N \hat{r}_{t,t+1}}{\hat{S}_z \hat{p}_{t+1} \hat{r}_{z,t+1}} + 1} \\
 & = \frac{1}{\hat{p}_t \hat{S}_z} \frac{1}{\frac{\sum_{j=1}^2 \left[\alpha \left(\frac{1-\alpha}{\hat{w}_{t+1}^j} \right)^{\frac{1-\alpha}{\alpha}} \hat{\pi}_{t+1}^j \right]}{\hat{p}_{t+1} \hat{S}_z \frac{\beta b^{1/\beta}}{2} \sum_{j=1}^2 \left[(1-\beta) \frac{\hat{p}_{t+1}}{\hat{w}_{t+1}^j} \right]^{\frac{1-\beta}{\beta}} + 1}} \\
 & = \frac{1}{\hat{p}_t \hat{S}_z} \frac{1}{(1-\beta) \sum_{j=1}^2 \left[\alpha \left(\frac{1-\alpha}{\hat{w}_{t+1}^j} \right)^{\frac{1-\alpha}{\alpha}} \hat{\pi}_{t+1}^j \right]} \text{, using (A.7),} \\
 & \quad \frac{1}{\beta \sum_{j=1}^2 \hat{w}_{t+1}^j \left[1 - \left(\frac{1-\alpha}{\hat{w}_{t+1}^j} \right)^{\frac{1}{\alpha}} \hat{\pi}_{t+1}^j \right]} + 1 \\
 & = \frac{1}{\hat{p}_t \hat{S}_z} \frac{1}{\frac{(1-\beta)\alpha \sum_{j=1}^2 [\hat{w}_{t+1}^j \hat{\pi}_{t+1}^j]}{\beta(1-\alpha) \sum_{j=1}^2 \hat{w}_{t+1}^j [1 - \hat{\pi}_{t+1}^j]} + 1} .
 \end{aligned}$$

$$= \frac{1}{\hat{p}_t \hat{S}_Z} \frac{1}{(1-\beta)\alpha \frac{1}{\beta(1-\alpha) \kappa^* - 1} + 1},$$

which is identical to (A.8). The lemma is now proved. ■

It follows directly from the lemma that

$$\frac{\hat{r}_{Y,t+1}}{\hat{v}_t^j \hat{S}_Y \hat{r}_{Y,t+1} + \hat{p}_{t+1} \hat{v}_t^j \hat{S}_Z \hat{r}_{Z,t+1}} = \frac{1}{\hat{p}_t} \frac{\hat{p}_{t+1} \hat{r}_{Z,t+1}}{\hat{v}_t^j \hat{S}_Y \hat{r}_{Y,t+1} + \hat{p}_{t+1} \hat{v}_t^j \hat{S}_Z \hat{r}_{Z,t+1}},$$

which implies that the investment portfolio $(s_{Y_t}^0, s_{Z_t}^0) = \hat{v}_t^j(\hat{S}_Y, \hat{S}_Z)$ satisfies the first-order condition (5) for lifetime utility maximization. Therefore, the price map for the traditional good

$$\mathcal{P}^* : (\hat{\Xi}_t^j, K_{Z_t}^j)_{j=1}^2 \rightarrow \mathcal{P}^* (\hat{\Xi}_t^j, K_{Z_t}^j)_{j=1}^2$$

represents an overlapping-generation equilibrium. ■

CONCLUDING REMARKS

We end the thesis here with some concluding remarks, which point out some avenues for future research.

First, the number of firms in the high-technology sector should be endogenized. One way of accomplishing this task is to let entry take place until the number of firms in the market yields the maximum expected utility for the old generation in the period. One can rationalize this approach by arguing that if the number of firms in the high-technology sector is too small, a larger number of firms in the industry might induce R&D activities that would result in an equilibrium in the production stage of the period yielding a higher expected utility for the old individuals of the period who must allocate their savings (capital) among these firms. On the other hand, too many firms in the market will result in too little capital for each firm, with the ensuing consequence that none of them will engage in a sequential search. The end result is that no R&D activities will be carried out. Without the possibility of a higher productivity, the old generation of the current might be worse-off, which will induce them to invest in fewer firms. The endogenous number of firms thus should lie somewhere between these two extremes.

Second, because the possibility of sustained growth require a lower parameter for the Pareto distribution and a higher elasticity of output with respect to the capital input, a policy such as R&D subsidies will have no impact on long-run growth. A more efficient policy is to carry out fundamental research to discover more promising populations in which new and more productive technologies can be sampled. In modelling basic research, one can incorporate learning by using the techniques of statistical decision theory. Also, the basic research can be financed by a lump sum tax, probably on the successive young generations because they might enjoy the fruit of this type of research. There are now equity issues involved across successive young generations.

Third and finally, one might adopt a more realistic approach in modelling technological diffusion. In the thesis, we have assumed that technological diffusion takes place within a country and across international borders and is complete after one period. A more appropriate approach might be to allow for diffusion to proceed at different speeds, one within national borders and one across international frontiers. Also, diffusion might not be complete after one period. Under these assumptions about technological diffusion, the two countries in the world economy are no longer symmetric at the beginning of each period and the proof of the existence of overlapping-generation equilibrium must be completely reworked.

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