NUMERICAL TREATMENT OF SOME LINEAR AND NON-LINEAR BOUNDARY VALUE PROBLEMS IN APPLIED MECHANICS

BY

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ABSTRACT

Error distribution principles have been widely used in the past for the solution of boundary value problems in applied mechanics. Of all the numerical schemes that are based on the principles of error distribution, Galerkin's method is generally the most rapidly converging method, while the collocation method is definitely the most simple numerical scheme. However, both methods have their drawbacks. Computationwise, the Galerkin method is usually very inefficient, as it involves the tedious and some times formidable task of definite integrations. The collocation method, though simple in theory and application, is not very reliable, since the solution can fluctuate greatly for arbitrary choices of collocation points.

In this study, means of refining the collocation method and simplifying the Galerkin method as applied to the solution of boundary value problems in applied mechanics is investigated. For the collocation method, two improved versions of the method are proposed. The first is a least square augmented collocation scheme, while the second is a combination of orthogonality and collocation. For the Galerkin method, a simplified form of the method, termed Vlasov's method, is studied.

To demonstrate the simplicity in application and good accuracy of the proposed methods, typical applied mechanics boundary value problems such as the torsion of bars and the bending of plates are formulated and used as illustrative examples. Such complex boundary value problems

as the linear and non-linear analyses of orthotropic plates and sandwich plates are solved with great ease. The results obtained are presented in tabular and graphical forms, and whenever possible, are compared with existing solutions based on more tedious and lengthier methods of analysis. The comparisons are generally very favorable.

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NOMENCLATURE

x, y, z	rectangular Cartesian coordinates
u, v, w	displacements in x,y, and z-directions
σ_x, σ_y	direct stresses
$\tau, \tau_{xy}, \tau_x, \tau_y$	shear stresses
$\varepsilon_x, \varepsilon_y$	direct strains
$\gamma, \gamma_{xy}, \gamma_x, \gamma_y$	shear strains
E_x, E_y, G_{xy}	moduli of elasticity and shear modulus of isotropic material
$E, G,$	modulus of elasticity and shear modulus of isotropic material
ν	Poisson's ratios for isotropic material
ν_x, ν_y	Poisson's ratios for orthotropic material
h	plate thickness
D	flexural rigidity of plates
D_x, D_y, D_{xy}	bending and twisting stiffnesses of orthotropic plates
k	modulus of elastic foundation
q, P	lateral load per unit area
a, b	plate dimensions in x and y-directions
λ	aspect ratio of plate, (a/b)
U, V, W	dimensionless displacements in x, y, and z-directions
S_x, S_y, S_{xy}	dimensionless direct and shear stresses
ξ, η	dimensionless parameters of x and y directional coordinates, ($\xi = x/a$, $\eta = y/b$)

K	dimensionless modulus of elastic foundation
Q	dimensionless parameter of load
W_o	perturbation parameter
w_i, u_i, v_i, m_i, n_i	dimensionless functions of ξ and η
q_i	constants in series expansion of lateral load
$A_i, B_i, C_i, A_{ij}, B_{ij}, \ldots$	adjustable constants
P_i', P_i'', P_i'''	orthogonal polynomial sets
$\alpha_i', \alpha_i'', \alpha_i'''$	constants
δ_{ij}	Kronecker delta
E_f, G_f	modulus of elasticity and shear modulus of face layers of sandwich plates.
E_c, G_c	modulus of elasticity and shear modulus of core layer of sandwich plates.
H	twisting moments in sandwich plates
$h-t$	thickness of core layer of sandwich plates
t	thickness of face layers of sandwich plates
S	shearing force parallel to the plane of the plate
ν_f	Poisson's ratio of face layers of sandwich plates
V_x, V_y	shearing forces perpendicular to the plane of the plate
M_x, M_y	bending moments in x and y-directions
N_x, N_y	normal forces in x and y-directions
μ	t/a
θ	h/a
e	$(w_+ - w_-)/h$

H	$(S_+ - S_-)h/2$
M_x	$(N_{x+} - N_{x-})h/2$
M_y	$(N_{y+} - N_{y-})h/2$
N_x	$N_{x+} + N_{x-}$
N_y	$N_{y+} + N_{y-}$
p	$p_+ + p_-$
q	$(p_+ - p_-)/2$
S	$S_+ + S_-$
u	$(u_+ + u_-)/2$
v	$(v_+ + v_-)/2$
w	$(w_+ + w_-)/2$
α	$(u_+ - u_-)/h$
β	$(v_+ - v_-)/h$
$\bar{\sigma}_z$	$(\sigma_{z+} + \sigma_{z-})/2$
λ_n	eigenvalue corresponding to the n^{th} mode of vibration
X_m	functions of x only
Y_n	functions of y only
ϕ_{mn}, ψ_{mn}	functions of x and y $[X_m(x).Y_n(y)]$
W_{mn}	adjustable constants
m,n,i,j,	integers

Note:

1) Partial differentiation is denoted by a comma in subscript.

2) Single and double primes appearing with the stress or strain components denote membrane and bending effects, respectively.

3) Subscripts (+) and (-) are quantities referring to the upper and lower membranes of sandwich plates where $z = \pm h/2$.

4) All symbols not given here are defined immediately following the formula in which they first appear.

CHAPTER I

INTRODUCTION

1.1 General:

The governing differential equations of boundary value problems in applied mechanics are usually rather complex, exact solutions to these differential equations can only be obtained for a few simple cases. In the majority of cases it is almost impossible to find a relatively simple function which will simultaneously satisfy both the governing differential equations and the boundary conditions. Some times, exact solutions are not possible to obtain even for relatively simple cases. Confronted with these problems, the researcher frequently has to resort to numerical methods to effect a solution. Numerical methods have frequently been used in the past when rigorous mathematical solutions have failed. The real impetus to their development, however, was given by the invention of digital computers which came into wide use in the last two decades.

1.2 Brief Discussion of Some Numerical Methods:

The following is a brief discussion of some of the more popular numerical techniques that have been applied successfully to various problems in applied mechanics by researchers in the past.

a) Ritz Method: The Ritz method is one of the energy methods. The method has found particular application in the analysis of very complicated problems. The Ritz method is based on the principle of minimizing the total potential energy of a system when the system is in stable equilibrium. In applying this method to a particular problem, an assumed function with undetermined adjustable parameters, satisfying various essential boundary conditions of the problem is chosen. The undetermined parameters of the assumed function can be evaluated from the minimizing condition of the system. In using this method, the governing differential equations are not involved and hence need not be known. Generally, this will save a considerable amount of mathematical work. The use of the Ritz method is recommended when computers are not readily available and the solution must be obtained by manual computation. However, some difficulty is always encountered if the problem is not symmetric. For example, the problem of bending of a plate with one side simply supported and all other sides clamped, the choice of a complete deflection function to meet the geometric boundary conditions is indeed difficult. Another drawback of this method is when the number of undetermined parameters in the assumed solution is increased, the amount of arithmetic work can be formidable.

b) Fourier Series Method: The Fourier series are very useful in the analytical treatment of many problems in the field of applied mechanics, such as the bending of plates. The extension of the Fourier series leads to Fourier integrals and Fourier transforms, the latter methods are considered to be powerful tools of higher analysis. Once the governing differential equation of a problem is determined, a rigorous solution to the problem would involve the adjusting of certain constants in order to satisfy the prescribed boundary conditions.

Fourier series has found application in the solution of many problems in applied mechanics because of its ability to represent discontinuous functions.

c) Perturbation Method: This method is often used to solve non-linear boundary value problems. In applying this technique, the solution of the problem is sought in the form of ascending powers of some arbitrary small perturbation parameter, then the original non-linear problem is reduced to a sequence of linear or perturbed equations. These perturbed equations can be solved by various approximate methods, such as Galerkin's method, the method of weighted least square, collocation and the power series method. The disadvantage of the perturbation method lies in the amount of lengthy arithmetic work required in solving the perturbed equations. Except for the collocation method, all the other methods mentioned above require lengthy manual computations.

d) Finite Difference Method: The finite difference method is one of the most general numerical methods in the field of applied mechanics. It can be effectively used to solve a wide variety of problems. Although the method has been known for a long time, it has gained considerable importance only after the invention of high speed digital computers. In this method, the governing differential equation (and the equations of the boundary conditions) are replaced by corresponding finite difference equations, which in turn yields a system of simultaneous algebraic equations. The advantages of the method are:

1. Simplicity in application.
2. Versatility.
3. The resulting numerical equations can be easily programmed using desk-top calculators or digital computers.
4. Acceptable accuracy for most technical purposes, provided that a relatively fine mesh is used.

Unfortunately, this method is characterized (beyond a certain mesh width) by slow convergence. Generally, a relatively fine mesh is required to obtain an acceptable accuracy. The accuracy deteriorates when the order of derivatives is increased. Consequently, the method is not recommended when higher than fourth-order derivatives are involved or when high accuracy in the solution is required [48].

e) The Finite Element Method: The recently developed finite element method has proved to be extremely powerful and versatile for the analysis of a wide variety of structural problems. The most critical, and simultaneously the most difficult, phase of the analysis is the evaluation of the element-stiffness coefficients. Fortunately, the stiffness properties of some of the more commonly used elements, which yield sufficiently accurate results, are readily available. Once the element-stiffness coefficients have been determined, the analysis of the structural system follows the familiar procedure of matrix methods used in structural mechanics for which standard computer programs are available.

The most important advantages of the finite element method are [48]:

1. The solution is obtained without the use of the governing differential equations, thus avoiding the mathematical analysis of the problem.
2. Arbitrary boundary and loading conditions can be handled with great ease.
3. It permits the complete automation of all procedures.
4. It permits the combination of various structural elements, such as plates, beams, and shells.
5. It can be extended to cover virtually all fields of continuum mechanics.

The major disadvantages are:

1. It requires the use of electronic digital computers of considerable speed and storage capacity, especially in the case of non-linear problems.
2. The preparation of data for each element can be time-consuming and is the most general source of human error in the solution.
3. Some problems may require sophisticated programming techniques and hence the aid of computer specialists.
4. When large structural systems are analyzed, it is difficult to ascertain the accuracy of the results.

f) <u>Error Distribution Methods:</u> In the treatment of boundary value problems, the problems are often solved by assuming an approximate solution to the differential equation; this approximate solution is usually in the form of an arbitrary linear combination of a set of independent functions and is dependent on a number of adjustable parameters such that for arbitrary values of the parameters,

(1) the differential equation is satisfied exactly, but not the boundary conditions ("boundary" method)

or (2) the boundary conditions are satisfied exactly, but not the differential equation ("interior" method),

or (3) the assumed solution satisfies neither the differential equation nor the boundary conditions ("mixed" method).

It is evident that, if by some numerical scheme, the undetermined parameters can be obtained such that the assumed solution satisfies

in case (1) (boundary method) the boundary conditions exactly,

in case (2) (interior method) the differential equation exactly,

in case (3) (mixed method) the boundary conditions and the differential equation exactly,

then no error would result if we substitute the assumed solution into the governing differential equation. Obviously, this is rarely possible.

A variety of approximate methods falling into the category of error distribution methods can be employed to distribute the error as uniformly as possible throughout the domain of the solution.

Among these methods are:

(i) Collocation Method.

(ii) Least Squares Method.

(iii) Least Squares with Weighting Functions.

(iv) Partition Method.

(v) Relaxation

(vi) Galerkin's Method.

The ultimate aim of these methods is to determine the undetermined parameters in such a manner that, throughout the entire domain of the solution, the assumed solution satisfies the differential equation, or the boundary conditions, or both the differential equation and the boundary conditions as accurately as possible, i.e., the resulting error be as close to zero as possible.

The methods mentioned above are sometimes referred to as methods of weighted residuals. The majority of these methods, with the exception of collocation and relaxation, involve the tedious process of definite integration over the region or boundary where the problem is defined. Hence, in terms of ease of computation, i.e., automated computation, these methods should be avoided whenever possible.

From previous experiences, Galerkin's method proved to be the most rapidly converging method. The collocation method, though simple in application, suffers the drawback of uncertainty of results due to the nature of the method.

1.3 Object and Scope:

The main objective of the thesis is to make a comprehensive investigation of possible means of improving the collocation method, and a study of an important variation of Galerkin's method for the approximate solution of difficult plate problems. The scope of this work covers the application of two modified forms of the collocation method to typical boundary value problems such as the torsion of prismatic bars, bending of plates, and the application of the modified Galerkin's method to plates with irregular boundary conditions. To pursue the investigation further, the problem of bending of sandwich plates and orthotropic plates is also studied.

1.4 Outline of the Thesis:

Since the majority of the problems studied in the thesis are related to the bending of plates, existing literature relating to the topic are briefly reviewed in Chapter 2. Chapters 3 and 4 are devoted to the improvement of collocation as an interior method. In Chapter 3, a modified form of the collocation method, termed the collocation least square method is formulated, and applied to the problem of torsion of prismatic bars of rectangular cross-section, the linear and non-linear analyses of uniformly loaded, clamped plates of rectangular, elliptical and circular planform, resting on an elastic foundation and the linear and non-linear analyses of uniformly loaded, clamped, orthotropic plates of rectangular planform. In Chapter 4, another modified form of the collocation method is formulated. Application of the method is demonstrated by applying the method to some of the problems considered in Chapter 3, viz. the torsion problem and the linear and non-linear analyses of the rectangular isotropic and orthotrophic plates. Also included in the demonstration is the complex problem of the linear and non-linear analyses of uniformly loaded, clamped rectangular sandwich plates.

Chapter 5 is devoted to the investigation of a variation of Galerkin's method termed Vlasov's method, this method is applied to the problem of bending of uniformly loaded rectangular plates with two opposite sides clamped and the other sides simply supported, and also to the problem of bending of uniformly loaded, simply supported rectangular sandwich plates.

In the final Chapter, the conclusions drawn are summarized.

Numerical and graphical results of all the analyses are presented. Whenever possible, such results are compared with solutions obtained by other investigators.

CHAPTER II

REVIEW OF LITERATURE

The governing differential equations of large deflections of thin isotropic plates were obtained by Von Karman [53] in 1910, and were consequently named after him. In 1940, Rostovtsev [43] modified the Von Karman equations and obtained the governing differential equations for the case of large deflection of orthotropic plates. In 1948, Reissner [42], based on several fundamental assumptions, derived the basic differential equations for the finite deflection of isotropic sandwich plates, his assumptions and differential equations were later verified by Gerard [18] and Alwan [2].

An attempt is made here to review some of the previous research works which employ numerical schemes described in Chapter I as methods of solutions.

a) Ritz method: This method was used by Way [53], for the large deflection problem of uniformly loaded clamped rectangular plates, by Weil and Newmark [60], for the solution of large deflections of uniformly loaded clamped elliptical plates, and by Ku [24], for the analysis of the small deflection problem of clamped skew plates on elastic foundations, subjected to uniformly distributed and concentrated loads. March [32] and Ericksen [15] used this method to solve the small deflection problem of clamped rectangular sandwich plates. Application of the Ritz method to the small or large deflection problem of plates results in the solution of a system of linear or non-linear algebraic equations.

b) Fourier Series Method: For rectangular plates with simply supported edges, the Fourier series method proves to be extremely powerful. In 1820, Navier presented a paper to the French Academy of Sciences on the solution of small deflection of simply supported rectangular plates by double Fourier series. Levy [30], using a similar method, solved the corresponding large deflection problem. Using this method, Yen et al. [62] solved the small deflection problem of a simply supported rectangular sandwich plate, while Alwan [2] solved the corresponding large deflection problem.

c) Perturbation Method and Series Solution: In applying the perturbation method, the Von Karman equations are reduced to a set of linear equations by the application of the perturbation procedure of Poincare. This method has been applied by many investigators to the large deflection problem of a variety of uniformly loaded clamped plates. The method was first employed by Chien [8] in analyzing the clamped circular plate subjected to uniform pressure. Subsequent works utilyzing this method are those by Chan [6], Kennedy and Ng [22], Nash [36], Ng [37,38,39] and Walter [54].

Stippes and Hausrath [46], solved the case of a simply supported circular plate using a nondimensionalized load as a perturbation parameters, while all the other investigators have used a dimensionless central deflection as their perturbation parameter.

Using this method, some solutions to the non-linear analysis of uniformly loaded clamped sandwich plates were obtained by a few investigators. e.g., Kan et al [20], who presented results for that of a square sandwich plate, and Ng [40], who obtained solutions to sandwich plates of circular and elliptical planform. Chia [7], also using this method, investigated the large deflections of clamped rectangular orthotropic plates.

d) Finite Difference Method: Wang [56,57], has used this method for simply supported rectangular plates of various aspect ratios. Szilard [48], solved a variety of plate problems using this method. He also has an extensive discussion of the method, and of means of refining the method as applied to plate bending.

e) Finite Element Method: In the last ten years, due to the improvement in computing facilities, this method has been widely used to solve plate problems. Haskell [18] and Melliere [33], among others, presented solutions for large deflections of rectangular plates with various boundary conditions. In the application of the finite element method to sandwich plates, Kwok [26] and Monfortan et al. [34], solved the small deflection problem of skew sandwich plates.

In using this method, quite a large number of finite elements are often required to obtain sufficiently accurate answers. In the case of large deflection problems, the solution often involves cycles of iteration. Consequently, the use of the finite element method has been found to be rather inadequate to the solution of large deflection problems of homogeneous and orthotropic plates.

f) Error Distribution Methods: The most widely used error distribution methods are those of Galerkin and collocation. Walter [54] treated the large deflection problem of a variety of plates by means of the collocation method. The Galerkin method was applied to the small deflections of clamped plates of various planforms on elastic foundations by Ng [37], to the large deflections of circular plates with various boundary conditions by Bolton [5], and to the large deflections of simply supported rectangular sandwich plates by Dundrova et al [14].

CHAPTER III

THE COLLOCATION LEAST SQUARE METHOD

3.1 General:

Of all the numerical methods discussed so far, the easiest but not exactly the most elegant method is the collocation method. This method was first systematically discussed in a report by Frazer et. al [16] in 1937.

The literal definition of the word "collocation" is the act of setting in a place or position; which is the fundamental idea of the method so named. There are three different types of the collocation method, viz., interior collocation, boundary collocation and mixed collocation. In this thesis, only the interior collocation method will be discussed in detail.

3.2 The Collocation Method:

To illustrate the method, consider the problem of determining a function W(x,y) which satisfies a linear partial differential equation:

$$L^R(x,y,W,W,_x,W,_y,\dots\dots\dots) = f \qquad (3.2.1)$$

and which satisfies the prescribed linear boundary condition:

$$L^S(x,y,W,W,_x,W,_y,\dots\dots\dots) = 0 \qquad (3.2.2)$$

where L is a differential operator,

R is the region where the differential equation is defined,

S is the boundary adjoining the region R and

f is a prescribed function known throughout R.

For an interior method, an approximate solution of Equation (3.2.1) can be assumed in the form:

$$W \approx \overline{W}(x,y,a_1,a_2, \ldots\ldots\ldots, a_n) \qquad (3.2.3)$$

where $\overline{W}$ represents an arbitrary linear combination of a set of independent functions, each one of which satisfies Equation (3.2.2), and $a_1,\ldots,a_n$ are undetermined adjustable parameters.

Substitution of Equation (3.2.3) into Equation (3.2.1) defines an error (or residual) function ε of the form:

$$\varepsilon(x,y,a_1,\ldots\ldots,a_n) = L^R(x,y,\overline{W},\overline{W}_{,x},\overline{W}_{,y},\ldots\ldots)-f \qquad (3.2.4)$$

Next, the parameters $a_1,\ldots\ldots,a_n$ in the assumed solution are determined by setting the error ε to zero at some n prior chosen points in the region R. This is equivalent to forcing the differential equation to be satisfied exactly at these n points. Such a procedure will lead to n linear equations for determining the n unknown parameters $a_1,\ldots\ldots,a_n$. i.e.,

$$\varepsilon_i\ (x_i,y_i,a_1,\ldots\ldots,a_n) = 0 \quad (i = 1,\ldots,n) \qquad (3.2.5)$$

In practice only a limited number of undetermined parameters can be taken in the assumed solution, hence, the error can only be set to zero at a limited number of points, the magnitude of the error at any other points besides the n chosen points remains unknown. Hopefully, it is small. Consequently, the approximate solution of a given boundary value problem depends, to a great extent, upon the choice of collocation points. Collatz [9], indicates that the choice of collocation points is a matter of some uncertainty and the effect of the distribution of collocation points on the results is unknown. Crandall [11] points out that the locations of the points are arbitrary, but are usually such that the region R is covered more or less uniformly in a simple pattern. For a limited number (six to nine) of undetermined parameters, depending on the type of boundary value problem considered, the results can differ by as much as 100% for arbitrary choices of collocation points.

3.3 The Collocation Least Square Method:

From the discussion above, it seems logical that if the error function ε is forced to be zero at m points instead of n points, where $m >> n$, and the undetermined parameters $a_1, \ldots, a_n$ are evaluated in such a manner that ε be zero or as close to zero as possible at these m points, the results obtained would certainly be improved, and such results will be somewhat less independent of the choice of collocation points.

However, by setting ε to zero at m points, an over determined system of linear simultaneous equations would result. For the sake of convenience, let these equations be expressed in matrix notations as:

$$\underset{m\times n}{[C]}\ \underset{n\times 1}{\{A\}} = \underset{m\times 1}{\{R\}} \tag{3.3.1}$$

where $[C]$ is the mxn coefficient matrix of the system of equations,

$\{A\}$ is the nx1 column vector of the undetermined parameters

$a_1,\ldots,a_n$,

and $\{R\}$ is the mx1 right hand side column vector.

Having generated m equations in n unknowns, the n unknowns, viz., $a_1,\ldots,a_n$ are then solved in a manner analogous to the fitting of a curve through a given set of data points. To effect this, the least squares procedure, which is often used in statistics to produce a so-called "best fitting curve" is applied to the equations.

Consider equation (3.3.1). For any particular column vector $\{A\}$, it is very unlikely that equation (3.3.1) will be satisfied identically. Let the errors associated with the equation be expressed by the mx1 column vector $\{E\}$. i.e.,

$$\{E\} = [C]\{A\} - \{R\} \tag{3.3.2}$$

Expanding the above matrix equation, we have:

$$\begin{aligned} e_1 &= c_{11}a_1 + c_{12}a_2 + \ldots + c_{1n}a_n - r_1 \\ e_2 &= c_{21}a_1 + c_{22}a_2 + \ldots + c_{2n}a_n - r_2 \\ &\vdots \\ e_m &= c_{m1}a_1 + c_{m2}a_2 + \ldots + c_{mn}a_n - r_n \end{aligned} \tag{3.3.3}$$

According to the least squares method, the criterion for choosing the undetermined parameters $a_1, \dots, a_n$ is such that the sum of the squares of the errors, i.e.,

$$e_1^2 + e_2^2 + \dots + e_n^2$$

be a minimum. Adopting the notation $< >$ as a symbol of summation so that, e.g., $<c_{i1}c_{i1}> = c_{11}^2 + c_{21}^2 + \dots + c_{m1}^2$, $<c_{i1}c_{i2}> = c_{11}c_{12} + c_{21}c_{22} + \dots + c_{m1}c_{m2}$, the sum S of the squares of the errors is then:

$$\begin{aligned} S = {} & <c_{i1}c_{i1}> a_1^2 + <c_{i2}c_{i2}> a_2^2 + <c_{i3}c_{i3}> a_3^2 + \dots + <c_{in}c_{in}> a_n^2 \\ & + 2<c_{i1}c_{i2}> a_1 a_2 + 2<c_{i1}c_{i3}> a_1 a_3 + \dots + 2<c_{i(n-1)}c_{in}> a_{n-1}a_n \\ & - 2<c_{i1}r_i> a_1 - 2<c_{i2}r_i> a_2 - \dots - 2<c_{in}r_i> a_n + <r_i r_i> \end{aligned} \qquad \dots\dots \quad (3.3.4)$$

In order that S be a minimum, its derivatives with respect to $a_1, a_2, \dots a_n$ must vanish. i.e., :

$$\begin{aligned} & <c_{i1}c_{i1}> a_1 + <c_{i1}c_{i2}> a_2 + <c_{i1}c_{i3}> a_3 + \dots + <c_{i1}c_{in}> a_n = <c_{i1}r_i> \\ & <c_{i1}c_{i2}> a_1 + <c_{i2}c_{i2}> a_2 + <c_{i2}c_{i3}> a_3 + \dots + <c_{i2}c_{in}> a_n = <c_{i2}r_i> \\ & \quad \vdots \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \vdots \qquad\qquad \vdots \\ & <c_{i1}c_{in}> a_1 + \dots\dots\dots\dots\dots\dots + <c_{in}c_{in}> a_n = <c_{in}r_n> \end{aligned}$$

$$\dots\dots \quad (3.3.5)$$

It can be seen that equation (3.3.5) is equivalent to premultiplying both sides of equation (3.3.1) by the transpose of the coefficient matrix $[c]$, i.e.,

$$[C]^T[C]\{A\} = [C]^T\{R\} \tag{3.3.6}$$

and consequently,

$$\{A\} = ([C]^T[C])^{-1}[C]^T[R] \tag{3.3.7}$$

Such operations as multiplications and inversions of matrices can be easily performed on a digital computer. Thus, with simple matrix operations of transposition, multiplication and inversion, results of the conventional simple but crude method of collocation can be greatly improved by the least square augmentation as proposed in this thesis.

In the following sections, the success of the collocation least square method will be illustrated by applying it to some boundary value problems in applied mechanics.

To facilitate a solution for the problems considered, a general programme coded in FORTRAN IV for an IBM 360/65 computer was developed. The programme is quite general in that it can more or less handle all kinds of boundary-value problems with very little modifications. A note on the programme along with a typical listing of the programme is given in Appendix C.

3.4 Torsion of Prismatic Bars of Rectangular Cross-Section:

Before considering the more complex problem of large deflections of plates, a well known problem in elasticity, viz., the torsion of prismatic bars of rectangular cross-section, is selected to serve as a trial problem using the collocation least square method.

For a thin rectangular bar with cross-section and coordinate system as shown in Figure 1, the governing differential equation for torsions of such bars is [50]:

$$\phi,_{xx} + \phi,_{yy} = -2G\theta \tag{3.4.1}$$

And the boundary conditions are:

$$\phi = 0 \quad \text{at } x = \pm a \text{ and at } y = \pm b \tag{3.4.2}$$

where ϕ is the stress function and θ is the angle of twist per unit length.

The torsional moment m_t can be calculated by:

$$m_t = 2\int_{-b}^{b}\int_{-a}^{a} \phi\,dx\,dy \tag{3.4.3}$$

and the maximum shearing stress is:

$$\tau_{max} = -\phi,_x \Big|_{x=a,\ y=0} \tag{3.4.4}$$

For convenience of calculation, the following dimensionless ratios are used:

$$\xi = x/a$$

$$\eta = y/b$$

$$\lambda = b/a$$

Substituting the dimensionless ratios into Equation (3.4.1) gives:

$$\lambda^2 \phi,_{\xi\xi} + \phi,_{\eta\eta} = -2G\theta b^2 \qquad (3.4.5)$$

As can be seen from Figure 1, the cross-section possesses mutually perpendicular axes of symmetry, resulting in quandrant symmetry. In view of this and the boundary conditions, a suitable approximate solution can be chosen in the form:

$$\phi = (1-\xi^2)\ (1-\eta^2)(A_o+A_1\xi^2 + A_2\eta^2 + A_3\xi^4 + A_4\xi^2\eta^2 + A_5\eta^4) \qquad (3.4.6)$$

Solutions are obtained by holding the half-breadth "a" constant, while the distance "b" is varied. The aspect ratio λ= b/a has values between 1.0 and 10.0. The manner in which the collocation points are distributed is as shown in Figure 3. The number of collocation points used varied from 150 to 200.

For comparison, coefficients C_1 and C_2 for maximum shear stress τ_{max} and maximum torsional moment m_t respectively are calculated, i.e.,

$$\tau_{max} = C_1(2G\theta a) \quad \text{and} \quad m_t = C_2 G\theta (2a)^3(2b)$$

Results are tabulated in Table 1. Comparisons are made with results obtained by Timoshenko [50], where much more laborious computations are used to analyse the same problem. As seen in Table 1, such comparisons are generally very favorable. The C_2 values are in excellent agreement with those obtained by Timoshenko for all cases considered, while the maximum discrepancy in the C_1 values is between 5 and 6 percent. Thus, it is reasonable to conclude that the simple augmented collocation method yields results with comparable accuracy as those obtained by Timoshenko.

Having succeeded in applying the collocation least square method to the problem of twisting of rectangular bars, the problem of large deflections of plates is now investigated using the proposed collocation least square technique.

3.5 Formulation of the Problem of Large Deflections of Plates:

I. Introductory Comments:

Solutions to plate problems that consider only the bending of a plate subjected to lateral loads are in the "small-deflection" category and are characterized by their linear load-deflection relations; solutions that consider bending as well as stretching of the middle surface of the plate belong to the "large-deflection" category and exhibits a non-linear load-deflection relationship. The significance of "large-deflection" can be seen by the fact that when this condition of "large-deflection" is realized, the plate is much stiffer than indicated by the classical linear theory. Hence, the design of structural members employing the linear theory can be over conservative, when the deflection of the plate is relatively large, i.e., in the order of one-half the thickness of the plate or more.

The large deflection theory has gained popularity in recent years through the utilization of materials with thin sections and high elastic strengths. To gain advantage of these materials, efficient design criteria based on the "large-deflection" theory has to be employed, provided that such deflections are not objectionable.

II. Basic Assumptions:

The classical theory of plates is based on the following well-known assumptions:

a) Strains in the middle surface produced by in-plane forces can usually be neglected compared with strains due to bending.

b) Straight lines initially normal to the middle plane of the plate remain straight lines and normal to the middle surface of the plate after bending.

c) Stresses normal to the middle surface of the plate are of a negligible order of magnitude compared with stresses in the plane of the plate.

d) The slopes of the deflected middle surface are small compared to unity.

These assumptions have been shown to be quite satisfactory when the maximum lateral displacement of a loaded plate does not exceed approximately one-half of the plate thickness. When the lateral displacement exceeds this limit, assumption (a) will be violated, i.e., the middle surface strains will have such magnitudes that it should no longer be neglected. For such problems, the use of the Von Karman theory which takes into account the stretching of the plate must be considered.

The Von Karman equations for the large deflections of plates takes the form [53]:

$$D\nabla^4 w = hL(w,F) + q \qquad (3.5.1)$$

$$2\nabla^4 F = -EL(w,w) \qquad (3.5.2)$$

where D: flexural rigidity of the plate

∇^4: biharmonic operator

w: out-of-plan displacement

h: constant plate thickness

F: membrane stress function

q: lateral load

E: modulus of elasticity

L(W,F) is defined by:

$$L(w,F) = w_{,xx}F_{,yy} + w_{,yy}F_{,xx} - 2w_{,xy}F_{,xy},$$

and L(w,w) can be obtained by replacing w for F in the above expression.

Equation (3.5.1) is a result of summation of forces in the direction normal to the plane of the plate, and Equation (3.5.2) is derived by considering compatibility after summing up forces in mutually perpendicular in-plane directions.

Although the Von Karman theory assumes that the deflections of the plate are larger than one-half the thickness of the plate, experiments seem to indicate that the theory is valid for plate deflection to thickness (w/h) ratio of perhaps not greater than two. In other words; deflections are large enough for the induced membrane forces to be important but small enough so that linearized formulae for curvatures are still applicable.

III Derivation of Differential Equations:

The usual procedures in the derivation of differential equations in applied mechanics are: 1) Formulate strain-displacement relationship of the problem. 2) Provide a relation between stresses and strains. In most cases, this relation will be a linear one - i.e., Hooke's law. 3) Consider the equilibrium of an element. For static analyses, this procedure consists of equating to zero all internal and external forces and moments in the coordinate directions. These steps are normally combined to eliminate components of stresses and strains. These procedures will be followed throughout this work.

Consider a thin elastic plate of an arbitrary planform, let the plate rest on a WINKLER type elastic foundation and possess rectilinear orthotropy. Adopting a rectangular Cartesian coordinate system with the origin located at some arbitrary point in the middle plane of the plate, let the axis of principal stiffness coincide with the x and y directions. Applying an arbitrary distributed load $q(x,y)$ acting normal to the plane of the plate will cause displacements in the x, y and z directions which are denoted by u, v and w respectively. Thus:

a) The strain - displacement relations are [49]:

For membrane strains:

$$\varepsilon'_x = u,_x + \frac{1}{2}(w,_x)^2 \qquad (3.5.3)$$

$$\varepsilon'_y = v,_y + \frac{1}{2}(w,_y)^2 \qquad (3.5.4)$$

$$\gamma'_{xy} = u,_x + v,_y + w,_x w,_y \qquad (3.5.5)$$

For bending strains:

$$\varepsilon''_x = -zw,_{xx} \qquad (3.5.6)$$

$$\varepsilon''_y = -zw,_{yy} \qquad (3.5.7)$$

$$\gamma''_{xy} = -2zw,_{xy} \qquad (3.5.8)$$

The non-linear terms on the right hand side of Equations (3.5.3), (3.5.4) and (3.5.5) are due to the stretching of the middle surface.

b) The stress-strain relations according to Hooke's law for orthotropic materials are [27]:

For membrane stresses:

$$\sigma'_x = E'_x\,\varepsilon'_x + E''\,\varepsilon'_y \qquad (3.5.9)$$

$$\sigma'_y = E'_y\,\varepsilon'_y + E''\,\varepsilon'_x \qquad (3.5.10)$$

$$\tau'_{xy} = G_{xy}\gamma'_{xy} \qquad (3.5.11)$$

For bending stresses:

$$\sigma''_x = E'_x \varepsilon''_x + E'' \varepsilon''_y \quad (3.5.12)$$

$$\sigma''_y = E'_y \varepsilon''_y + E'' \varepsilon''_x \quad (3.5.13)$$

$$\tau''_{xy} = G_{xy}\gamma''_{xy} \quad (3.5.14)$$

Where the material constants are defined by

$$E'_x = E_x/(1 - \nu_x \nu_y)$$

$$E'_y = E_y/(1 - \nu_x \nu_y)$$

$$E'' = \nu_x E'_y = \nu_y E'_x$$

and E_x = modulus of elasticity in the x - direction.

E_y = modulus of elasticity in the y - direction.

G_{xy} = shear modulus in the xy plane.

ν_x = ratio of strain in the y - direction to strain in the x - direction due to uniaxial stress in the x - direction.

ν_y = ratio of strain in the x - direction to strain in the y - direction due to uniaxial stress in the y - direction.

c) The equilibrium equations are:

$$\sigma'_{x,x} + \tau'_{xy,y} = 0 \quad (3.5.15)$$

$$\sigma'_{y,y} + \tau'_{xy,x} = 0 \quad (3.5.16)$$

$$D_x w_{,xxxx} + 2H w_{,xxyy} + D_y w_{,yyyy} + kw =$$

$$q + h[\sigma'_x w_{,xx} + \sigma'_y w_{,yy} + 2\tau'_{xy} w_{,xy}] \qquad (3.5.17)$$

where k is the foundation modulus, h is the thickness of the plate and,

$$D_x = E'_x h^3/12$$

$$D_y = E'_y h^3/12$$

$$H = D' + 2D_{xy}$$

$$D' = E'' h^3/12$$

$$D_{xy} = G_{xy} h^3/12$$

Equation (3.5.15) and (3.5.16) are obtained by the summation of forces in the x and y directions respectively, while Equation (3.5.17) is a result of summation of forces in the z-direction with appropriate substitution of the shearing forces by the derivatives of moments[49].

One further step is taken to express the equilibrium equations entirely in terms of displacements and derivatives of displacements. Substitution of the membrane stress-strain and strain-displacement relations into the equilibrium equations yields:

$$E'_x \; [u_{,xx} + w_{,x} w_{,xx} + \nu_y v_{,xy} + \nu_y w_{,y} w_{,xy}]$$

$$+ G_{xy} \; [u_{,yy} + v_{,xy} + w_{,x} w_{,yy} + w_{,y} w_{,xy}] = 0 \qquad (3.5.18)$$

$$E'_y\,[v,_{yy}+w,_y w,_{yy}+\nu_x u,_{xy}+\nu_x w,_x w,_{xy}]$$

$$+\ G_{xy}\,[v,_{xx}+u,_{xy}+w,_y w,_{xx}+w,_x w,_{xy}] = 0 \qquad (3.5.19)$$

$$D_x w,_{xxxx}+2Hw,_{xxyy}+D_y w,_{yyyy}+kw =$$

$$q+h\{E'_x\ w,_{xx}[u,_x+\tfrac{1}{2}(w,_x)^2+\nu_y(v,_y+\tfrac{1}{2}(w,_y)^2)]$$

$$+\ E'_y\ w,_{yy}[v,_y+\tfrac{1}{2}(w,_y)^2+\nu_x(u,_x\ \ \tfrac{1}{2}(w,_x)^2)]$$

$$+\ 2G_{xy}w,_{xy}[u,_y+v,_x+w,_x w,_y]\} \qquad (3.5.20)$$

These three equations, two of the second order and one of the fourth order, are the equilibrium equations in terms of displacements, and, with $k = 0$, they are the counterpart to the two fourth order equations derived by Rostovatev [43].

For ease of computation, it is convenient to render these equations dimensionless. This is done by considering "a" and "b" as two characteristic length of the plate and letting:

$$U = au/h^2 \qquad \xi = x/a$$

$$V = av/h^2 \qquad \eta = y/b$$

$$W = w/h \qquad \lambda = a/b$$

$$Q = qa^4/D_x h \qquad K = ka^4/D_x$$

Substitution of the above dimensionless ratios into Equations (3.5.18), (3.5.19) and (3.5.20) gives the following dimensionless equations:

$$[U,_{\xi\xi}+W,_{\xi}W,_{\xi\xi}+\nu_y\lambda V,_{\xi\eta}+\nu_y\lambda^2W,_{\eta}W,_{\xi\eta}]$$
$$+(D_{xy}\lambda/D_x)[\lambda U,_{\eta\eta}+V,_{\xi\eta}+\lambda W,_{\xi}W,_{\eta\eta}+\lambda W,_{\eta}W,_{\xi\eta}] = 0 \quad (3.5.21)$$

$$(D_y\lambda/D_x)[\lambda V,_{\eta\eta}+\lambda^2W,_{\eta}W,_{\eta\eta}+\nu_xU,_{\xi\eta}+\nu_xW,_{\xi}W,_{\xi\eta}]$$
$$+(D_{xy}/D_x)[V,_{\xi\xi}+\lambda U,_{\xi\eta}+\lambda W,_{\eta}W,_{\xi\xi}+\lambda W,_{\xi}W,_{\xi\eta}] = 0 \quad (3.5.22)$$

$$W,_{\xi\xi\xi\xi}+(2H\lambda^2/D_x)W,_{\xi\xi\eta\eta}+(D_y\lambda^4/D_x)W,_{\eta\eta\eta\eta}+KW = Q$$
$$+12W,_{\xi\xi}[U,_{\xi}+\tfrac{1}{2}(W,_{\xi})^2+\nu_y\lambda V,_{\eta}+\tfrac{1}{2}\nu_y\lambda^2(W,_{\eta})^2]$$
$$+12(D_y\lambda^2/D_x)W,_{\eta\eta}[\lambda V,_{\eta}+\tfrac{1}{2}\lambda^2(W,_{\eta})^2+\nu_xU,_{\xi}+\tfrac{1}{2}\nu_x(W,_{\xi})^2]$$
$$+(24D_{xy}\lambda/D_x)W,_{\xi\eta}[\lambda U,_{\eta}+V,_{\xi}+\lambda W,_{\xi}W,_{\eta}] \quad (3.5.23)$$

The above equations are next simplified by the application of the perturbation procedure of Poincare, thus reducing them to a sequence of linear or perturbed equations.

In applying the perturbation method, a perturbation parameter is first selected, solutions of Equations (3.5.21), (3.5.22) and (3.5.23) are then sought in the form of ascending powers of the perturbation parameter as mentioned earlier in Chapter I. Let Wo be the nondimensionalized perturbation parameter, representing the centre deflection of the plate. It must be noted here that, whenever possible, advantage should be taken of the symmetry of the plate by locating the origin of the Cartesian coordinate system at the intersection of the axes of symmetry, since in most cases, this would simplify the problem considerably.

The displacements and lateral load are expressed in terms of increasing powers of Wo, i.e.,

$$U \doteq u_2(\xi,\eta)W_o^{\,2} + u_4(\xi,\eta)W_o^{\,4} + \ldots\ldots \quad (3.5.24)$$

$$V = v_2(\xi,\eta)W_o^{\,2} + v_4(\xi,\eta)W_o^{\,4} + \ldots\ldots \quad (3.5.25)$$

$$W = w_1(\xi,\eta)W_o + w_3(\xi,\eta)W_o^{\,3} + w_5(\xi,\eta)W_o^{\,5} + \ldots \quad (3.5.26)$$

$$Q = q_1 p(\xi,\eta)W_o + q_3 p(\xi,\eta)W_o^{\,3} + q_5 p(\xi,\eta)W_o^{\,5} + \ldots \quad (3.5.27)$$

In the above equations, the u_i, v_i and w_i are unknown functions of ξ and η which satisfy the boundary conditions, and the q_i are constants which represent the stiffness of the plate. The function p is a load function determined from the load distribution, and is subjected to the condition that p(o,o) = 1. For a uniformly distributed load, p will have a constant value of one throughout the entire region of the plate.

The reason that only even powers of W_o are present in Equation (3.5.24) and (3.5.25) can be explained by the fact that a change in sign of W_o obtained by reversing the load, leaves U and V unaltered. Similarly, only odd powers of W_o are required in Equations (3.5.26) and (3.5.27), because a change in sign of W_o corresponds to a change in sign of W and Q. The functions $u_o(\xi,\eta)$ and $v_o(\xi,\eta)$ are absent from Equations (3.5.24) and (3.5.25) since they represent in-plane displacements when the lateral load is zero, i.e., these displacements are due to externally applied in-plane forces. Since the only in-plane displacements considered here are those induced by the membrane effect, consequently, $u_o = v_o = 0$.

From the series for W, Equation (3.5.26), it is obvious that, in order that the lateral displacement be W_o at the origin of the coordinate system as defined, it is necessary that the following conditions hold, i.e.,

$$w_1(0,0) = 1 \quad \text{and} \quad w_3(0,0) = w_5(0,0) = 0$$

Substitution of the power series for the displacement components and load into Equations (3.5.21), (3.5.22) and (3.5.23) results in a system of linear partial differential equations. By the equating terms of order W_o, the first order approximation - i.e. the usual small deflection equation is obtained,

$$w_{1,\xi\xi\xi\xi} + (2H\lambda^2/D_x)\, w_{1,\xi\xi\eta\eta} + (D_y\lambda^4/D_x) w_{1,\eta\eta\eta\eta} + Kw_1 = q_1 p \quad (3.5.28)$$

The constant q_1 and the function w_1 can be readily solved from the above equation.

Equating terms of order W_o^2, yields the second order approximation:

$$u_{2,\xi\xi} + [\nu_y + (D_{xy}/D_x)]\lambda v_{2,\xi\eta} + (D_{xy}/D_x)\lambda^2 u_{2,\eta\eta}$$

$$= -w_{1,\xi} w_{1,\xi\xi} - (D_{xy}/D_x)\lambda^2 w_{1,\xi} w_{1,\eta\eta} - [\nu_y + (D_{xy}/D_x)]\lambda^2 w_{1,\eta} w_{1,\xi\eta}$$

(3.5.29)

$$(D_{xy}/D_x)v_{2,\xi\xi}+[(\nu_x D_y+D_{xy})/D_x]\lambda u_{2,\xi\eta}+(D_y/D_x)\lambda^2 v_{2,\eta\eta}$$

$$= -(D_y/D_x)\lambda^3 w_{1,\eta}w_{1,\eta\eta} -(D_{xy}/D_x)\lambda w_{1,\eta}w_{1,\xi\xi}-[(\nu_x D_y+D_{xy})/D_x]\lambda w_{1,\xi}w_{1,\xi\eta}$$

(3.5.30)

Knowing the function w_1, the functions u_2 and v_2 could be evaluated from Equations (3.5.29) and (3.5.30).

Equating terms of order W_o^3, gives the third order approximation:

$$w_{3,\xi\xi\xi\xi} +(2H\lambda^2/D_x)w_{3,\xi\xi\eta\eta} +(D_y\lambda^4/D_x)w_{3,\eta\eta\eta\eta}+Kw_3 =$$

$$q_3 p +12w_{1,\xi\xi}[u_{2,\xi}+\tfrac{1}{2}(w_{1,\xi})^2+\nu_y\lambda v_{2,\eta}+\tfrac{1}{2}\nu_y\lambda^2(w_{1,\eta})^2]$$

$$+ 12(D_y\lambda^2/D_x)w_{1,\eta\eta}[\lambda v_{2,\eta}+\tfrac{1}{2}\lambda^2(w_{1,\eta})^2+\nu_x u_{2,\xi}+\tfrac{1}{2}\nu_x(w_{1,\xi})^2]$$

$$+ (24D_{xy}\lambda/D_x)w_{1,\xi\eta}[\lambda u_{2,\eta}+ v_{2,\xi}+ \lambda w_{1,\xi}w_{1,\eta}] \qquad (3.5.31)$$

Theoretically, further approximations could be obtained by equating higher order terms of W_o. However, as pointed out by previous investigators, such procedures are not called for since, the solution of the function W_3 and the constant q_3 from Equation (3.5.31) defines the displacements well into the large deflection regime.

To calculate the membrane and bending stresses, the generalized Hooke's law relating stresses and strains and ultimately stresses and displacements as given in Equations (3.5.9) to (3.5.14) are rewritten in a nondimensionalized form by adopting the following dimensionless stress ratios:

Membrane Stresses:

$$S'_x = \sigma'_x a^2/E'_x h^2, \quad S'_y = \sigma'_y a^2/E'_x h^2, \quad S'_{xy} = \tau'_{xy} a^2/G_{xy}h^2$$

Bending Stresses:

$$S''_x = \sigma''_x a^2/E'_x h^2, \quad S''_y = \sigma''_y a^2/E'_x h^2, \quad S'_{xy} = \tau''_{xy} a^2/G_{xy}h^2$$

Total Stresses:

$$S_x = \sigma_x a^2/E'_x h^2 = S'_x + S''_x, \quad S_y = \sigma_y a^2/E'_x h^2 = S'_y + S''_y, \quad S_{xy} = \tau_{xy} a^2/G_{xy}h^2 = S'_{xy} + S''_{xy}$$

The stress-displacement relations in their proper dimensionless forms are:

$$S'_x = (w_o/h)^2[u_{2},_{\xi} + \frac{1}{2}(w_{1},_{\xi})^2 + \nu_y v_{2},_{\eta} + \frac{1}{2}\nu_y \lambda^2 (w_{1},_{\eta})^2] \qquad (3.5.32)$$

$$S'_y = (w_o/h)^2[\frac{E_y}{E_x}\lambda v_{2},_{\eta} + \frac{E_y}{E_x}\lambda^2 \frac{1}{2}(w_{1},_{\eta})^2 + \nu_y u_{2},_{\xi} + \frac{1}{2}\nu_y (w_{1},_{\xi})^2] \qquad (3.5.33)$$

$$S'_{xy} = (w_o/h)^2 [\lambda u_{2},_{\eta} + v_{2},_{\xi} + \lambda w_{1},_{\xi} w_{1},_{\eta}] \qquad (3.5.34)$$

$$S''_x = \frac{1}{2}(w_o/h) [w_{1},_{\xi\xi} + \nu_y \lambda^2 w_{1},_{\eta\eta}] + \frac{1}{2}(w_o/h)^3[w_{3},_{\xi\xi} + \nu_y \lambda^2 w_{3},_{\eta\eta}] \qquad (3.5.35)$$

$$S''_y = \frac{1}{2}(w_o/h)[\frac{E_y}{E_x}\lambda^2 w_{1},_{\eta\eta} + \nu_y w_{1},_{\xi\xi}] + \frac{1}{2}(w_o/h)^3[\frac{E_y}{E_x}\lambda^2 w_{3},_{\eta\eta} + \nu_y w_{3},_{\xi\xi}] \qquad (3.5.36)$$

$$S''_{xy} = (w_o/h)\ \lambda w_{1},_{\xi\eta} + (w_o/h)^3\ \lambda w_{3},_{\xi\eta} \qquad (3.5.37)$$

Where the direct and shearing stresses S''_x, S''_y and S''_{xy} due to bending are evaluated at the extreme fibers $z = \pm h/2$.

For a plate of homogeneous isotropic material under a uniformly distributed load, Equations (3.5.28), (3.5.29), (3.5.30) and (3.5.31) can be slightly simplified. Since for homogeneous materials, $E_x = E_y = E$, $\nu_x = \nu_y = \nu$ and $G_{xy} = G = E/2(1+\nu)$, consequently $D_x = D_y = H = D = Eh^3/12(1-\nu^2)$. For a uniformly distributed load, the load function $p(\xi,\eta)$ takes on a constant value of one. Thus, the four perturbed equations, viz., (3.5.28), (3.5.29), (3.5.30) and (3.5.31) become:

$$w_{1,\xi\xi\xi\xi} + 2\lambda^2 w_{1,\xi\xi\eta\eta} + \lambda^4 w_{1,\eta\eta\eta\eta} + Kw_1 = q \tag{3.5.38}$$

$$\begin{aligned} &2u_{2,\xi\xi} + (1+\nu)\lambda v_{2,\xi\eta} + (1-\nu)\lambda^2 u_{2,\eta\eta} \\ &= -2w_{1,\xi}w_{1,\xi\xi} - (1-\nu)\lambda^2 w_{1,\xi}w_{1,\eta\eta} - (1+\nu)\lambda^2 w_{1,\eta}w_{1,\xi\eta} \end{aligned} \tag{3.5.39}$$

$$\begin{aligned} &(1-\nu)v_{2,\xi\xi} + (1+\nu)\lambda u_{2,\xi\eta} + 2\lambda^2 v_{2,\eta\eta} \\ &= -2\lambda^3 w_{1,\eta}w_{1,\eta\eta} - (1-\nu)\lambda w_{1,\eta}w_{1,\xi\xi} - (1+\nu)\lambda w_{1,\xi}w_{1,\xi\eta} \end{aligned} \tag{3.5.40}$$

$$\begin{aligned} &w_{3,\xi\xi\xi\xi} + 2\lambda^2 w_{3,\xi\xi\eta\eta} + \lambda^4 w_{3,\eta\eta\eta\eta} + Kw_3 = q_3 \\ &+ 12w_{1,\xi\xi}\left[u_{2,\xi} + \tfrac{1}{2}(w_{1,\xi})^2 + \nu\lambda v_{2,\eta} + \tfrac{1}{2}\nu\lambda^2(w_{1,\eta})^2\right] \\ &+ 12\lambda^2 w_{1,\eta\eta}\left[\lambda v_{2,\eta} + \tfrac{1}{2}\lambda^2(w_{1,\eta})^2 + \nu u_{2,\xi} + \tfrac{1}{2}\nu(w_{1,\xi})^2\right] \\ &+ 12(1-\nu)\lambda w_{1,\xi\eta}\left[\lambda u_{2,\eta} + v_{2,\xi} + \lambda w_{1,\xi}w_{1,\eta}\right] \end{aligned} \tag{3.5.41}$$

Similar changes of the material constants are to be made in the equations relating stresses to displacements; i.e., Equations (3.5.32) to (3.5.37).

3.6 Linear and Non-Linear Analyses of Uniformly Loaded Clamped Plates by the Collocation Least Square Method:

I. Rectangular Plates on Elastic Foundations:

(i) Solution of Problems:

For a clamped, homogeneous, isotropic rectangular plate with coordinate system as shown in Figure 1, the perturbed equations, Equations (3.5.38) to (3.5.41), are to be solved using the collocation least square method.

For the first order approximation, i.e., linear analysis, a solution of Equation (3.5.28) can be taken in the form of an algebraic polynomial [6]:

$$w_1 = (1-\xi^2)^2(1-\eta^2)^2 f_1(\xi,\eta) \tag{3.6.1}$$

Where the function f_1 is defined by:

$$f_1(\xi,\eta) = 1+C_1\xi^2+C_2\eta^2+ C_3\xi^4+ C_4\xi^2\eta^2+ C_5\eta^4+ C_6\xi^4\eta^2+ C_7\xi^2\eta^4+ C_8\xi^4\eta^4$$

and C_i are the undetermined coefficients. The associated boundary conditions for this first order approximation are:

$$w_{1,\xi} = w_1 = 0 \quad \text{at} \quad \xi = \pm 1 \tag{3.6.2}$$

$$w_{1,\eta} = w_1 = 0 \quad \text{at} \quad \eta = \pm 1 \tag{3.6.3}$$

It can be easily verified that Equation (3.6.1) satisfies the above boundary conditions and the condition that $w_1 = 1$ at $\xi = 0$ and $\eta = 0$.

The boundary conditions for the second order approximation are:

$$u_2 = v_2 = 0 \text{ at } \xi = \pm 1 \text{ and } \eta = \pm 1 \tag{3.6.4}$$

which can be satisfied if we assume [6]:

$$u_2 = \xi(1-\xi^2)(1-\eta^2)f_2(\xi,\eta) \tag{3.6.5}$$

$$v_2 = \eta\ (1-\xi^2)(1-\eta^2)f_3(\xi,\eta) \tag{3.6.6}$$

where the functions f_2 and f_3 are:

$$f_2(\xi,\eta) = D_o + D_1\xi^2 + D_2\eta^2 + D_3\xi^4 + D_4\xi^2\eta^2 + D_5\eta^4 + D_6\xi^4\eta^2 + D_7\xi^2\eta^4 + D_8\xi^4\eta^4$$

$$f_3(\xi,\eta) = E_o + E_1\xi^2 + E_2\eta^2 + E_3\xi^4 + E_4\xi^2\eta^4 + E_5\eta^4 + E_6\xi^4\eta^2 + E_7\xi^2\eta^4 + E_8\xi^4\eta^4$$

and the undetermined coefficients D_i and E_i are to be solved in this approximation.

In the third order approximation, the boundary condition to be met are:

$$w_{3,\xi} = w_3 = 0 \quad \text{at } \xi = \pm 1 \tag{3.6.7}$$

$$w_{3,\eta} = w_3 = 0 \quad \text{at } \eta = \pm 1 \tag{3.6.8}$$

To satisfy these boundary conditions and the condition that $w_3(0,0) = 0$, the assume solution for w_3 is taken as [6]:

$$w_3 = (1-\xi^2)^2(1-\eta^2)^2 f_4(\xi,\eta) \qquad (3.6.9)$$

where the function f_4 is

$$f_4(\xi,\eta) = F_1\xi^2+F_2\eta^2+F_3\xi^4+F_4\xi^2\eta^2+F_5\eta^4+F_6\xi^4\eta^2+F_7\xi^2\eta^4+F_8\xi^4\eta^4$$

Solution of the undetermined coefficients F_i from this approximation will completely define the large deflection problem.

Solutions are obtained for plates of aspect ratios ranging from 1/2 to 1 with the dimensionless foundation modulus varying from 0 to 200.

To investigate the variation of the results due to the number of collocation points used in solving each equation, the case of zero foundation modulus is solved using 25, 50 and 100 collocation points. The distribution of these collocation points is analogous to the pattern shown in Figure 3.

Results from this investigation indicates that this effect is very minor. From Table 2, it can be seen that for the linear analysis, results obtained by using 25 and 50 collocation points deviated less than 0.2% from those obtained by using 100 collocation points. While from Table 3, this deviation is shown to be no greater than 4% for the non-linear analysis. Consequently, the problem of rectangular plates on elastic foundations is solved by using 100 collocation points per equation in the recursive solution. These collocation points are distributed as shown in Figure 3.

(ii) Comparison and Discussion of Results:

For plates with zero foundation modulus, results of the linear analysis are tabulated in Table 2 along with values obtained by Timoshenko [49]. As can be observed, the comparison is excellent. Plots of load vs. deflection and load vs. total edge and center stress are shown in Figures 5 and 9 respectively. In both figures, the results shown are those obtained by employing 100 collocation points per equation and $\nu = 0.3$. The solutions due to Way [59] and Chan [6] are also shown for comparison. The deflections obtained here are in better agreement with Way's accurate results than with those obtained by Chan whose results tends to be slightly more conservative than the present solution.

In general, these results are reasonably consistent though slight deviations are seen for the case of $\lambda = 1/2$. This discrepancy may well be due to the inability of the assumed displacement functions to represent the actual deflected shape of a plate of small aspect ratio since, such a plate tends to take on the behavior of a beam.

For plates on elastic foundations, a poisson's ratio of $\nu = 1/3$ is used. Maximum center small deflections for various foundation moduli and aspect ratios are tabulated in Table 4 and values of the constants q_1 and q_3 are shown in Table 5. Figures 6 to 8 shows curves of load vs. deflection and Figures 10 to 12 shows curves of load vs. maximum total edge and center stress. Results obtained by Ng [38] are also shown. As expected, the figures show that in general, the present results are again slightly below those obtained by Ng [38], as was noted in the case of zero foundation modulus. This slight over-estimation of results might be due to the limited number of undetermined coefficients used by Ng [38] and Chan [6] in their solutions.

From this investigation, the following results were observed:

1) In analyzing the large deflections of rectangular plates, the collocation least square method provides results which are comparable to those obtained by much more laborious computational methods. Though the investigation is more or less carried out by using 100 collocation points per equation, it seems that the number of collocation points required to obtain a sufficiently accurate result is about two to three times the number of unknown parameters, provided that these collocation points are distributed in a fairly uniform manner.

2) For a given aspect ratio λ, the maximum center deflection of the plate decreases with increasing values of the foundation modulus. This should be expected since, the object of the elastic foundation is to reduce the lateral pressure.

3) The effectiveness of the elastic foundation in reducing the maximum center deflection of the plate is more pronounced for small aspect ratios than it is for aspect ratios approaching unity. For instance, with the foundation modulus increasing

from 0 to 200, the decrease in the maximum small deflection at the center of a plate of aspect ratio $\lambda = 1/2$ is 86.62%; while for a plate of aspect ratio $\lambda = 1$, the corresponding decrease is only 73.81%. This is so because the deflection of a plate of small aspect ratio is greater than a plate of aspect ratio approaching one, and since the foundation reaction is proportional to the deflection, hence, the reduction in deflections due to an increase in the dimensionless foundation modulus K, will obviously be more significant for long rectangular plates than for plates approaching a square planform.

4) The maximum total stress occuring at the mid-point of the longer side of the plate is much greater than the maximum total stress occurring at the center of the plate.

5) The effects of the elastic foundation on the non-linear analysis is less significant. For example, increasing the foundation modulus from 0 to 200, the q_1 value of a square plate increased 73.81%, while the q_3 value increased only 16.31%. Hence, deflections tend to become increasingly linear as the foundation modulus is increased.

6) The magnitude of the membrane stress at the edge of the plate is relatively small when compared with the bending stress. However, at the center of the plate, where the stretching is most severe, the membrane stress is of comparable magnitude with the bending stress.

7) Due to the presence of the elastic foundation, the stresses of the plate are reduced. This reduction is more pronounced at the center of the plate than at the edge and is most significant in the

bending stress. The effect of the elastic foundation on the non-linear stresses (i.e. membrane and non-linear bending stress) is negligible.

II. Elliptical and Circular Plates on Elastic Foundations:

(i) Solution of the Problem:

For the clamped, homogeneous, isotropic elliptical plate as shown in Figure 2, the governing differential equations for the displacements corresponding to each stage of the successive approximation process is identical to that of the rectangular plate, viz., Equations (3.5.38) to (3.5.41), and the boundary conditions associated with each approximation are:

$$w_1 = w_{1,\xi} = w_{1,\eta} = 0 \quad \text{at} \quad \xi^2 + \eta^2 = 1 \tag{3.6.10}$$

$$u_2 = v_2 = 0 \quad \text{at} \quad \xi^2 + \eta^2 = 1 \tag{3.6.11}$$

$$w_3 = w_{3,\xi} = w_{3,\eta} = 0 \quad \text{at} \quad \xi^2 + \eta^2 = 1 \tag{3.6.12}$$

In order to satisfy these boundary conditions, the solutions are taken in the form [39]:

$$w_1 = (1 - \xi^2 - \eta^2) f_1(\xi,\eta) \tag{3.6.13}$$

$$u_2 = \xi(1 - \xi^2 - \eta^2) f_2(\xi,\eta) \tag{3.6.14}$$

$$v_2 = \eta(1 - \xi^2 - \eta^2) f_3(\xi,\eta) \tag{3.6.15}$$

$$w_3 = (1 - \xi^2 - \eta^2)^2 f_4(\xi,\eta) \tag{3.6.16}$$

where the functions f_1, f_2, f_3 and f_4 are as defined in the previous problem. It is evident that the assumed solutions w_1 and w_3 also satisfy the condition $w_1(0,0) = 1$ and $w_3(0,0) = 0$.

For comparison of results, solutions are obtained for plates with aspect ratios between 2 and 1. For $\lambda = 1$, the elliptical plate becomes a circular plate. The range of the dimensionless foundation modulus varied from 0 to 200.

The effect of the number of collocation points on the solutions is again investigated by solving the case of $K = 0$ with 25, 50 and 100 collocation points. The collocation points are distributed in a manner similar to the pattern shown in Figure 4. For the linear analysis, the exact solution was obtained regardless of the number of collocation points and the number of undetermined parameters used. In all cases, the undetermined coefficients in the polynomial w_1 turned out to be identically zero. Table 6 shows results of the non-linear analysis. From the results shown, the maximum deviation between the results is about 0.2%. For the analysis of plates on elastic foundations, all the results are obtained by using 100 collocation points per equation. The locations of the collocation points are shown in Figure 4..

(ii) Comparison and Discussion of Results

The results of the linear analysis of elliptical plates on elastic foundations are shown in Table 7. Comparisons of these results are made with results obtained by Ng [39]. The agreement is excellent with the maximum error not exceeding 1.5%. Table 8 shows values of q_1 and q_3. For comparison of the non-linear analysis, plates of load vs. deflection and load vs. maximum total edge and center stress are shown in Figures 13 to 15 and Figures 16 to 18. All the results of the elliptical plates are compared with Chan [6], for the case of K = 0, the results are also compared with Weil and Newmark [60]. Results of circular plates with K = 0 are compared with Way [58], while those of K $\neq$ 0 are compared with Sinha [45].

As can be seen from the graphs presented, the present solution yields results which are in good agreement with results of previous investigators. The comparisons are exceptionally good in the case of a circular plate with or without the presence of the elastic foundations.

From the comparisons of results for rectangular and elliptical plates, it is observed that the agreement is slightly better in the case of elliptical plates than it is with rectangular plates. This is to be expected since, unlike the assumed solutions of rectangular plates, the assumed solutions for the circular or elliptical plates takes on the exact mathematical expression of a circular or elliptical boundary.

From the results for stresses for elliptical plates, it is observed that the maximum total stress occurs at the end of the minor axis. This stress is of greater magnitude than the positive total stress at the center of the plate.

The effect of the elastic support in reducing the deflections is seen to be more significant for plates of aspect ratios approaching one than it is for plates of greater aspect ratios. For example, for an aspect ratio of one by increasing K from 0 to 200, the decrease in the central deflection is 68.1 %; however, the corresponding decrease is only 22.5% for such plates with aspect ratios equal to two. This finding is somewhat contradictory to the results of the rectangular plates. But, recalling the dimensionless form adopted for the foundation modulus, i.e., $K = ka^4/D$, and the variation of the aspect ratio in this problem, i.e. $a/b = 1$ to $a/b = 2$, it can be seen that by increasing the aspect ratio, i.e. holding "b" constant and increasing "a", the actual foundation modulus k is decreased by a factor of a^4. Consequently, for a certain value of K, say K = 40, taking the semi-minor axis "b" as unity, when $a/b = 1$, k has a value of 40D, whereas when $a/b = 2$, k becomes 2.5D. Hence, it can be observed that for a given change in the plate aspect ratio, the increase in deflection as the plate approaches an infinite strip, is not enough to off set the decrease of the actual foundation modulus k. Apart from this, all the other findings in this problem are identical to those of the rectangular plates.

III Rectangular Orthotropic Plates:

(i) Solution of Problem:

To further demonstrate the validity of the collocation least square method, the large deflection of uniformly loaded, clamped, rectangular orthotropic plates are investigated here. For this problem, the perturbed governing differential equations, Equations (3.5.28), (3.5.29), (3.5.30) and (3.5.31) are slightly modified for uniformly distributed loads and absence of the elastic foundation by setting $K = 0$ and $p = 1$ in Equations (3.5.28) and (3.5.31).

The boundary conditions here are identical to those of the rectangular homogeneous plates. Thus, the assumed displacement functions of that problem can be taken for the solution of the present problem.

For comparison of results, the numerical values of the elastic constants used by Chia [7] are adopted. These values are:

Material	E_y/E_x	G_{xy}/E_x	ν_y
Glass-Epoxy	3.0	1/2	0.25
Boron-Epoxy	10.0	1/3	0.22
Graphite-Epoxy	40.0	0.6	0.25

The aspect ratios of the plate are varied from 0.5 to 1.0.

All results are obtained by using 100 collocation points for each equation. The distribution of these points is identical to that of the homogeneous rectangular plate.

(ii) Comparison and Discussion of Results:

Results of the analysis are shown in Table 9 and Figures 19 to 24. Results from Chia [7] are also shown for comparison. For the boron-epoxy plate, the load vs. deflection curve of a homogeneous square plate is also plotted.

From the comparisons made, again, the agreement is seen to be good. The load vs. deflection plot of the boron-epoxy plate shows that the curve of the square isotropic plate approaches the curve of the orthotropic plate of $\lambda = 3/4$. This is so because, the tensile modulus of the isotropic plate is equal to that of the orthotropic plate in the transverse direction and to one-third of that in the filament direction. For these orthotropic plates, due to the different tensile moduli, the stresses of a square plate at the mid-points are not equal to each other and the stress in the y - direction associated with the higher tensile modulus is greater than that in the x - direction.

3.7 Concluding Remarks on the Collocation Least Square Method.

From the various problems demonstrated in this Chapter, the collocation least square method, though simple in its mathematical concept, proves to be an extremely valuable tool for the solution of problems of applied mechanics involving complex differential equations. Such difficult problems as large deflection of plates on elastic foundations and plates of orthotropic materials were handled with great ease. The results of the variety of problems considered in this Chapter were obtained with acceptable accuracy, and if these results were in error at all, it is believed that the errors were generally on the safe side.

In short, the simplicity, versatility and ease of handling makes this method a very powerful means for the solution of many difficult boundary value problems in the field of applied mechanics.

CHAPTER IV

THE ORTHOGONAL COLLOCATION METHOD

4.1 General:

Presented in this chapter, is another method for the improvement of the conventional collocation method as applied to symmetrical boundary value problems. In this method, the error (or residual) is represented as an orthogonal polynomial over the region of the problem. The criteria for the selection of the collocation points are based on the orthogonality conditions. For some choices of weight functions in the orthogonality relations, the results obtained from this method are comparable to the Galerkin interior method, which will be briefly described next.

4.2 Galerkin's Method:

Galerkin's method can be successfully applied to a variety of boundary value problems in applied mechanics. Typical of which are such problems as vibration, stability and small and large deflection theories in plates and shells. Though the mathematical theory behind this method is very complex, its physical explanation is rather simple.

Consider a structural system in equilibrium ,the sum of all the external and internal forces is zero. The equilibrium condition of an infinitesimal element can be represented by the following differential equations:

$$L_1(u, v, w) - p_x = 0$$
$$L_2(u, v, w) - p_y = 0 \qquad (4.2.1)$$
$$L_3(u, v, w) - p_z = 0$$

which describe the equilibrium of all forces in the x, y and z directions, respectively. In the above equations, L_1, L_2 and L_3 are differential operators operating on the displacement functions, while p_x, p_y and p_z are external forces. The equilibrium of the structural system is obtained by integrating these differential equations over the entire structure.

Expressing the small arbitrary variations of the displacement functions by δu_i, δv_i and δw_i, and noting that although the displacement components are interrelated, their arbitrary varations are not interrelated, the virtual work of the external and internal forces,

$$\delta W_i + \delta W_e = \delta(W_i + W_e) = 0, \qquad (4.2.2)$$

can be obtained directly from the differential equations of equilibrium without determining the actual potential energy of the system. Thus,

$$\iiint_V [L_1(u, v, w) - p_x](\delta u)dV = 0$$

$$\iiint_V [L_2(u, v, w) - p_y](\delta v)dV = 0$$

$$\iiint_V [L_3(u, v, w) - p_z](\delta w)dV = 0 \tag{4.2.3}$$

Strictly speaking, these variational equations are valid only if the displacement functions u, v, and w are the exact solutions of the problem under consideration. However, these equations will not be greatly violated if proper approximate expressions for the displacement functions are chosen and the variations carried out accordingly. Replacing the exact solutions for the displacements by approximate expressions of the form:

$$u = \sum_{i=1}^{\ell} a_i \alpha_i(x, y, z),$$

$$v = \sum_{i=1}^{m} b_i \beta_i(x, y, z),$$

and

$$w = \sum_{i=1}^{n} c_i \gamma_i(x, y, z), \tag{4.2.4}$$

where $\alpha_i(x, y, z)$, $\beta_i(x, y, z)$ and $\gamma_i(x, y, z)$ are functions that satisfy all the prescribed boundary conditions, and a_i, b_i and c_i are undetermined constants, it is also required that the displacement functions (4.2.4) should have at least the same order derivatives as called for by the differential operators in Equation (4.2.3).

Expressing the small arbitrary variations of the displacements by:

$$\delta u = \sum_{i=1}^{\ell} \alpha_i(x, y, z)\delta a_i,$$

$$\delta v = \sum_{i=1}^{m} \beta_i(x, y, z)\delta b_i,$$

and

$$\delta w = \sum_{i=1}^{n} \gamma_i(x, y, z)\delta c_i, \tag{4.2.5}$$

where the variations are carried out term by term. Substituting Equation (4.2.5) into Equation (4.2.3) results in

$$\sum_{i=1}^{\ell} \delta a_i \iiint_V [L_1(u, v, w) - p_x]\alpha_i(x, y, z)dV = 0$$

$$\sum_{i=1}^{m} \delta b_i \iiint_V [L_2(u, v, w) - p_y]\beta_i(x, y, z)dV = 0$$

$$\sum_{i=1}^{n} \delta c_i \iiint_V [L_3(u, v, w) - p_z]\gamma_i(x, y, z)dV = 0$$

(4.2.6)

Since the variations of the expansion coefficients δa_i, δb_i and δc_i are arbitrary and not interrelated, the only way that the above equations can be identically zero is that

$$\iiint_V [L_1(u, v, w) - p_x]\ \alpha_i(x, y, z)dV = 0$$

$$\iiint_V [L_2(u, v, w) - p_y]\ \beta_i(x, y, z)dV = 0$$

$$\iiint_V [L_3(u, v, w) - p_z]\ \gamma_i(x, y, z)dV = 0$$

(4.2.7)

This provides $m + n + \ell$ equations for calculating the $m + n + \ell$ undetermined coefficients a_i, b_i and c_i.

It should be noted that the differential operators, L( ), act on the entire series expressions of the displacement components, which in turn are multiplied by the individual terms of the functions α_i, β_i and γ_i, resulting in simple analytic expressions. Integrating these expressions over the entire structural system, a set of coupled algebraic equations for determining the unknown coefficients a_i, b_i and c_i is obtained.

4.3 The Orthogonal Collocation Method:

To illustrate the orthogonal collocation method, consider a symmetrical second order boundary value problem in one independent variable, x, in the region $x^2 < 1$. The differential equation is:

$$L(y) = 0 \quad \text{for } x^2 < 1 \tag{4.3.1}$$

and the boundary conditions are:

$$y = y(1) \quad \text{at } x^2 = 1 \tag{4.3.2}$$

$$y_{,x} = 0 \quad \text{at } x = 0 \tag{4.3.3}$$

For interior collocation, the assumed solution is chosen such that the boundary conditions are satisfied. A suitable function is:

$$y = y(1) + (1 - x^2) \sum_{i=0}^{n-1} a_i P_i'(x^2) \tag{4.3.4}$$

where $P_i'(x^2)$ are polynomials of degree i in x^2, yet to be specified and the a_i are undetermined constants.

Once y has been adjusted to satisfy Equation (4.3.1) at n collocation points $x_1, \ldots\ldots x_n$, the residual function L(y) either vanishes everywhere or contains a polynomial factor $G_n(x^2)$ of degree n in x^2 whose zeroes are the collocation points. Then by analogy with Galerkin's method, which specifies that the residual be orthogonal to all the trial functions, the collocation points are selected by specifying that $G_n(x^2)$ be orthogonal to all the functions $(1 - x^2)P_i'(x^2)$ of Equation (4.3.4) over the region $x^2 < 1$. Such a specification is automatically satisfied by taking $G_n(x^2)$ and $P_n'(x^2)$ from the orthogonal polynomial set defined by

$$\int_0^1 (1 - x^2)P_i'(x^2)P_n'(x^2)\,dx = \alpha_i'\delta_{in} \qquad (4.3.5)$$

for all positive integers i and n, where α_i' is a constant and δ_{in} is the Kronecker delta.

The orthogonality relation in Equation (4.3.5) ensures that the zeroes of $P_n'(x^2)$ are real, distinct and located within the open interval 0, 1.

The key formula here is Equation (4.3.5) which provides both the trial functions and the collocation points.

The collocation method shown here is a discrete analogy of Galerkin's method. It is based on the orthogonality, not of the residual function, but of a polynomial which vanishes at the same points.

In the Galerkin interior method, the approximate solution of Equation (4.3.1) is obtained by setting the differential-equation residual L(y) orthogonal to all the trial functions. For the assumed solution, Equation (4.3.4), this orthogonality relation over the region $x^2 < 1$ becomes

$$\int_o^1 (1 - x^2)P_i'(x^2)\ [L(y)]\ dx = 0 \qquad (i = 0,\ldots\ldots.n - 1) \qquad (4.3.6)$$

The present collocation method, on the other hand, uses the orthogonality relation

$$\int_o^1 (1 - x^2)P_i'(x^2)\ [(x^2 - x_1^2)\ldots\ldots\ldots(x^2 - x_n^2)]\ dx = 0 \qquad (i = 0,\ldots\ldots n - 1) \qquad (4.3.7)$$

to define the collocation points, $x_1,\ldots\ldots x_n$ where the residual L(y) is to vanish. The two methods agree if L(y) is a polynomial of degree $d \leq n$ in x^2.

A weight least-squares method may be written for this symmetrical problem as

$$\{\int_o^1 W(x^2)\ [L(y)]^2\ dx\}_{,a_i} = 0 \qquad (i = 0,\ldots.,n - 1) \qquad (4.3.8)$$

where $W(x^2)$ is a weight function, positive for $x^2 < 1$. For comparison, Equation (4.3.7) can be combined to give

$$\left\{ \int_0^1 (1 - x^2)[(x^2 - x_1^2).....(x^2 - x_n^2)]^2 \, dx \right\}_{,x_i^2} = 0$$

$$(i = 1,.....,n) \qquad (4.3.9)$$

which shows the collocation points also satisfy a least-squares criterion. Hence, if one chooses a weight function $W(x^2) = 1 - x^2$ and if the residual is a polynomial of degree $d \leq n$ in x^2, then Equation (4.3.8) leads to the same result as the present collocation method.

Although the derivation here is based on a one-dimensional second order problem, the present method can be easily extended to two-dimensional problems and problems involving higher order derivatives. To demonstrate this, boundary value problems such as torsion of bars, linear and non-linear analysis of plates will be solved in the following sections using the present method. In order to meet the requirements posed by these problems, other orthogonal polynomial sets have to be formulated since, the weight function in the orthogonality relation, Equation (4.3.4), must be replaced by a suitable function to meet the various requirements of a particular boundary value problem. For the purpose of analysing the problems under investigation, orthogonal polynomials $P_i''(x^2)$ and $P_i'''(x^2)$ are defined by

$$\int_0^1 x(1 - x^2)P_i''(x^2)P_n''(x^2)dx = \alpha_i''\delta_{in} \qquad (4.3.10)$$

and

$$\int_0^1 (1 - x^2)^2 P_i'''(x^2)P_n'''(x^2)dx = \alpha_i'''\delta_{in} \qquad (4.3.11)$$

Although, in general, the residuals in these problems will no longer be a polynomial of degree $d \leq n$ in x^2, yet, the results obtained by employing this method is of comparable accuracy to those of other investigators using much more laborious techniques, as will be shown later.

Construction of these orthogonal polynomials can be easily achieved via a simple computer programme, and the roots of the polynomials, i.e., the required collocation points, obtained with very little effort through standard iteration schemes such as Bairstow's method. Table 10 shows the polynomials $P_i'(x^2)$, $P_i''(x^2)$ and $P_i'''(x^2)$; and their constants α_i', α_i'' and α_i''', while Table 11 shows the roots of these polynomials. Since all the calculations are carried out in double precision arithmetic, the values shown are accurate up to the 16th decimal digit.

4.4 Torsion of Rectangular Bars:

To verify the analogy between the orthogonal collocation method and the Galerkin method, and to prove its validity, the torsion problem considered in section 3.4 is used here to serve as an illustrative example.

For the present method, an admissible solution of Equation (3.4.5) can be taken as

$$\phi = (1-\xi^2)(1-\eta^2) \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} A_{ij} P_i'(\xi^2) P_j'(\eta^2) \quad (4.4.1)$$

and the collocation points where the residual is to be set equal to zero are combinations of the roots of $P_n'(\xi^2)$ and $P_n'(\eta^2)$.

For the particular case of a square bar and a one term solution, i.e., n = 1, substituting Equation (4.4.1) into Equation (3.4.5) results in

$$A_{oo} \left[(-2 + 2\eta^2) + (-2 + 2\xi^2)\right] + 2\ G\theta = 0 \quad (4.4.2)$$

Since there is only one unknown, viz., A_{oo}, by setting Equation (4.4.2) to zero at a single collocation point will yield the solution of this particular case. From Table 11, the root of $P_1'(x^2)$ is $1/\sqrt{5}$, hence the location of the collocation point is $(1/\sqrt{5}, 1/\sqrt{5})$.

Substituting the ξ and η coordinate of this collocation point into Equation (4.4.2) gives $A_{oo} = 0.625G\theta$

from which the constant for maximum shear stress $C_1 = 0.625$, and the constant for the torque $C_2 = 0.139$. These values agree very favourably with Timoshenko's value of $C_1 = 0.675$ and $C_2 = 0.141$, an extremely accurate result considering the crudeness of the one term solution used.

The results obtained here are also identical with the one term solution of the Galerkin method and the Ritz method ([55], p. 165 and p. 158). This is to be expected since, the differential-equation residual of this problem is always a polynomial of degree $d = n$ in ξ^2 and η^2, hence, the results obtained here should agree with those of Galerkin's or Ritz's method, as stated in the previous section.

To investigate the convergence of the present collocation method, results are obtained for a 4 term, 9 term and a 16 term solution, i.e., $n = 2$, $n = 3$, and $n = 4$. These results are shown in Tables 12 and 13.

As can be seen from the results presented, the convergence is very consistent, and the agreement of these results with those of the collocation least square method and Timoshenko [50] is excellent.

From this simple example, it can be said that the present collocation method provides results which are of comparable accuracy with other solutions based on more powerful but lengthier methods of analysis such as the Galerkin method and Ritz method.

4.5 Linear and Non-Linear Analyses of Isotropic and Orthotropic Rectangular Plates:

(i) Solution of the Problem:

To demonstrate the ability of the orthogonal collocation method in handling higher order complex differential equations, the plate problems considered in Chapter III are analysed here using the present collocation scheme.

For both the isotropic and orthotropic plates considered here, the effect of the elastic foundation will not be accounted for, though, this effect can be easily incorporated in the solution.

Since the boundary conditions posed by these two problems in each stage of the recursive solution are the same, their assumed solutions can be taken as polynomials with identical forms.

Hence, neglecting the terms involving the foundation modulus K and considering only uniformly distributed loads, the assumed solution for the first order approximation, i.e., Equation (3.5.38) or (3.5.28) is of the form:

$$w_1 = (1-\xi^2)^2(1-\eta^2)^2 \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} A_{ij} P_i''' (\xi^2) P_j''' (\eta^2) \qquad (4.5.1)$$

The collocation points here consist of combinations of the roots of $P_n'''(\xi^2)$ and $P_n'''(\eta^2)$. However, two problems arise as a consequence of the present choice for w_1. Firstly, it is obvious that the condition $w_1(0,0) = 1$ is not met by Equation (4.5.1), and secondly, for the present problem, such a choice of w_1 will invariably lead to $n^2 + 1$ unknowns, viz., the polynomial coefficients $A_{oo}, \ldots, A_{(n-1)(n-1)}$ and the constant q_1, while the number of equations available are n^2 equations generated from n^2 prior chosen collocation points, resulting in an under-determined system of equations. Hence, in order to eliminate these two obstacles, an additional equation is introduced. This equation takes the form

$$\sum_{i=o}^{n-1} \sum_{j=o}^{n-1} A_{ij} = 1 \tag{4.5.2}$$

It is evident that this equation will satisfy the condition $w_1(0,0) = 1$. Thus, Equation (4.5.2) along with the n^2 equation generated from the n^2 collocation points constitute a system of $n^2 + 1$ equations for the solution of the $n^2 + 1$ unknowns.

The second order approximation for both problems consists of two coupled perturbed equations in terms of the displacements in the ξ and η directions, where the first of the two equation is a result of summation of forces in the ξ direction, while the second equation is derived from summing up forces in the η direction.

In the discussion of Galerkin's method (Section 4.2), it was pointed out that though the displacements in the ξ and η directions are interrelated, their arbitrary variations are not interrelated. † Consequantly, for the two equations of the second order approximation, the first equation will only be subjected to an arbitrary variation of the displacement function u_2, while the second equation only to the displacement function v_2.

Thus, following this argument, the approximate solutions for Equations (3.5.39) and (3.5.40) or Equations (3.5.29) and (3.5.30) are taken as:

$$u_2 = \xi(1-\xi^2)(1-\eta^2) \sum_{i=o}^{n-1} \sum_{j=o}^{n-1} B_{ij} P_i''(\xi^2) P_j'(\eta^2) \qquad (4.5.3)$$

$$v_2 = \eta(1-\xi^2)(1-\eta^2) \sum_{i=o}^{n-1} \sum_{j=o}^{n-1} C_{ij} P_i'(\xi^2) P_j''(\eta^2) \qquad (4.5.4)$$

From Equation (4.5.3), a set of n^2 collocation points is obtained, in which the ξ coordinates and η coordinates of the points are roots of the polynomials $P_n''(\xi^2)$ and $P_n'(\eta^2)$ respectively, and by evaluating the residuals of Equation (3.5.39) or (3.5.29) at these n^2 collocation points, a set of n^2 equations is obtained. Similarly, from Equation (4.5.4), another set of n^2 collocation points is obtained, with the ξ and η coordinates of the points being roots of the polynomials $P_n'(\xi^2)$ and $P_n''(\eta^2)$ respectively, and another set of n^2 equations generated by the evaluation of the residuals of Equation (3.5.40) or (3.5.30) at these n^2 collocation points. Thus, providing $2n^2$ equations for solving the $2n^2$ undetermined coefficients associated with the two displacement functions u_2 and v_2.

For the third order approximation, an admissible assumed solution of Equation (3.5.41) or (3.5.31) is:

$$w_3 = (1-\xi^2)^2(1-\eta^2)^2 \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} D_{ij} P_i'''(\xi^2) P_j'''(\eta^2) \qquad (4.5.5)$$

The situtation here is almost identical to that of the first order approximation, except that the condition posed by the perturbation procedure at this stage of the recursive solution is $w_3(0,0)=0$, hence, the required additional equation is taken in the form

$$\sum_{i=0}^{n-1} \sum_{j=0}^{n-1} D_{ij} = 0 \qquad (4.5.6)$$

In order to investigate the variation of the results due to the number of terms adopted in each series, results are obtained for a 4 term, 9 term and a 16 term solution.

(ii) Comparison and Discussion of Results:

For the isotropic plates, values of the maximum small deflection are tabulated in Table 14 and comparison made with Timoshenko [49]. It can be seen that the agreement here is excellent. In spite of a crude 4 term solution, the results deviated no more than 2% from those of Timoshenko [49].

Values of the constants q_1 and q_3 are tabulated in Table 15, also shown are results from the collocation least square method and those due to Chan [6]. The results shown compare well with those obtained by using the collocation least square method or the more elaborate power series method. Though, for λ less than 1, the q_3 values obtained from a

4 term solution are not as accurate, however, by retaining additional number of terms in the solution, the results obtained converged towards those of the collocation least square method, which are somewhat more accurate than those obtained by the power series method, as was pointed out earlier in Chapter III.

For the orthotropic plates, the constants q_1 and q_3 are shown in Table 16 along with results from Chapter III. Again, it can be observed that even the results of a 4 term solution are comparable with those obtained from the collocation least square method. Although slight deviations are noted when the orthotropy of the plates become more pronounced, such deviations are rapidly reduced when the number of terms are increased as can be seen by the results of the 9 term solution, and for a 16 term solution, results from the two collocation schemes are more or less identical.

For verification of the fulfillment of conditions (4.5.2) and (4.5.6), typical numerical values of the resulting polynomial coefficients for the isotropic plates are listed in Tables 17 to 19.

From this investigation, the orthogonal collocation method is seen to be an extremely efficient and accurate scheme for the analysis of difficult symmetrical boundary value problems such as those considered in this section. However, the extent to which the present method is applicable does not cease at this point, as will be seen by its application to the very complex problem of linear and non-linear analyses of sandwich panels in the following section.

4.6 Linear and Non-Linear Analyses of Rectangular Sandwich Plates:

(i) Introductory Comments:

Sandwich material is defined as: a laminar construction; composed of a combination of alternating dissimilar simple or complex materials, assembled and intimately fixed in relation to each other, so as to use the properties of each to attain specific structural advantage for the whole assembly. This defination is necessarily general. Specifically, sandwich plates are a type of three layer construction, consisting of two very thin sheets of high strength material between which a thicker layer of comparatively soft and light material is sandwiched. The two thin sheets are termed face sheets or skins and the middle layer is called the core. Sandwich construction is an efficient way of obtaining a light-weight structure with comparatively high strength. Such structures have found particular application in the aerospace industry.

In actual engineering construction, the material for the facings are usually light metals with high elastic moduli such as aluminum alloy, while the core material is generally of honeycomb type and is neither homogeneous nor isotropic. In fact it is not even a continuum. However, when the dimensions of the structural element in which this material is used, are large compared to the individual cell dimensions, the mechanical properties of the honeycomb frequently are idealized as those of an isotropic or orthotropic homogeneous continuum. Of course, other light-weight materials such as expanded plastic or balsa wood have also been used as core materials, and in such cases, the aforementioned idealization is simply a fact.

Since the elastic modulus of the core layer in the plane of the plate is of a negligible magnitude in comparison to that of the facings, the normal stresses in the core are of little importance in resisting bending moments, even though the usual ratio of the thickness of the facings to that of the core is between one-tenth and one-hundredth. On the other hand, the core performs the task of transmitting shearing forces and undergoes considerable shearing deformations because of its low modulus of shear. As a consequence of such shearing deformations, the differential equations governing this problem are different from those of the general homogeneous structure.

As in the case of homogeneous plates, the sandwich plates can be analyzed by the general linear theory when the plate deflection to thickness (w/h) ratio is small. But for large values of this ratio, the elaborate non-linear theory must be employed for a more realistic solution.

The governing differential equations of the non-linear theory were first formulated by Reissner [42]. He obtained two coupled non-linear differential equations of the fourth order by considering the equilibrium and compatability of an infinitesimal element of the sandwich plate. The Reissner equations in rectangular Cartesian coordinates are:

$$\nabla^2\nabla^2 F = 2tE_f[(w,_{xy})^2 - (w,_{xx})(w,_{yy})] \qquad (4.6.1)$$

$$D\nabla^2\nabla^2 w = [1-(thE_f/2(1-\nu_f^{\;2})G_c)\nabla^2] \times [P + F,_{yy}w,_{xx} - 2F,_{xy}w,_{xy} + F,_{xx}w,_{yy}] \qquad (4.6.2)$$

where

∇^2 = Laplacian operator

F = membrane stress function

t = thickness of the facings

E_f = modulus of elasticity of facing material

w = out-of-plane displacement

D = flexural rigidity of the sandwich plate

$= E_f t h^2 / 2(1-\nu_f^{\,2})$

γ_f = Poisson's ratio of facing material

h = overall thickness of sandwich plate

G_c = shear modulus of core material

p = lateral load,

Using the principle of complementary energy, Alwan [2] arrived at an identical set of two fourth order differential equations.

It can be seen that these two equations are of a form similar to that of the Von Karman equations. Equation (4.6.1) is in fact identical to Equation (3.5.2), while from Equation (4.6.2), it can be observed that the transverse shear deformability of the core introduces a group of new terms on the right hand side, and for $G_c = \infty$, the equation reduces to the well known form of the homogeneous plate, Equation (3.5.1).

It is obvious that the Reissner equations are of a very complex nature and no exact solution can be possible. For the present study, a set of differential equations, which is of a form more suitable for an approximate solution of the problem is derived. The steps taken in deriving these equations are more or less identical to those taken by Reissner, and, it will be shown later that the resulting equations are of a form equivalent to the two Reissner equations.

(ii) <u>Derivation of the Governing Differential Equations</u>:

Consider a sandwich plate consisting of two face layers of thickness "t" and a core layer of thickness "h-t". Assuming that "t" is small compared with "h" and that the values of the elastic constants E_f, G_f for the face layers are large compared with the values of the elastic constants E_c, G_c for the core layer, such that the products tE_f, tG_f are large compared with the values hE_c and hG_c.

Based on the assumption that $t \ll h$, it can be assumed that the stresses in the faces parallel to their plane are distributed uniformly over the thickness of the face layers. From the assumption that $hE_c \ll tE_f$, the face parallel stresses in the core layer and their effect on the deformation of the composite plate may be neglected. Thus, the sandwich plate considered here is treated as a combination of two plates with no bending stiffness (the face layers), and of a third plate (the core layer) offering resistance only to transverse shear stresses and transverse normal stresses.

Although the following derivation will be restricted to face and core materials that are isotropic, this restriction can be easily removed from the derivation without causing major complications.

For a sandwich plate subjected to some arbitrary distributed load p, using the subscript '+' to indicate quantitites referring to the upper facing and the subscript '-' to indicate quantities referring to the lower facing, the strain-displacement relations for the face membrane are known to be of the following form [49]:

$$\varepsilon_{x\pm} = u_{\pm},_x + \frac{1}{2}(w_{\pm},_x)^2 \quad (4.6.3)$$

$$\varepsilon_{y\pm} = v_{\pm},_y + \frac{1}{2}(w_{\pm},_y)^2 \quad (4.6.4)$$

$$\gamma_{\pm} = u_{\pm},_y + v_{\pm},_x + w_{\pm},_x\, w_{\pm},_y \quad (4.6.5)$$

and those for the core layer may be written as:

$$\varepsilon_z = w,_z \quad (4.6.6)$$

$$\gamma_x = u,_z + w,_x \quad (4.6.7)$$

$$\gamma_y = v,_z + w,_y \quad (4.6.8)$$

from Hooke's Law, the stress-strain relations for the facings are:

$$\varepsilon_{x\pm} = \frac{1}{E_f t}(N_{x\pm} - \nu_f N_{y\pm}) \quad (4.6.9)$$

$$\varepsilon_{y\pm} = \frac{1}{E_f t}(N_{y\pm} - \nu_f N_{x\pm}) \quad (4.6.10)$$

$$\gamma_{\pm} = \frac{1}{G_f t} S_{\pm} \quad (4.6.11)$$

and those for the core layer are:

$$\varepsilon_z = \sigma_z / E_c \tag{4.6.12}$$

$$\gamma_x = \tau_x / G_c \tag{4.6.13}$$

$$\gamma_y = \tau_y / G_c \tag{4.6.14}$$

With the notation of Figure 25, the equilibrium differential equations for the face membrane are the following:

$$N_{x\pm,x} + S_{\pm,y} \mp \tau_{x\pm} = 0 \tag{4.6.15}$$

$$S_{\pm,x} + N_{y\pm,y} \mp \tau_{y\pm} = 0 \tag{4.6.16}$$

$$(N_{x\pm} w_{\pm,x})_{,x} + (S_{\pm} w_{\pm,x})_{,y} + (S_{\pm} w_{\pm,y})_{,x} + (N_{y\pm} w_{\pm,y})_{,y} + p_{\pm} \mp \sigma_{z\pm} \mp \tau_{x\pm} w_{\pm,x} \mp \tau_{y\pm} w_{\pm,y} = 0 \tag{4.6.17}$$

The equilibrium equations for the core layer are, under the assumption of negligible face parallel core stresses,

$$\tau_{x,z} = 0 \tag{4.6.18}$$

$$\tau_{y,z} = 0 \tag{4.6.19}$$

$$\tau_{x,x} + \tau_{y,y} + \sigma_{z,z} = 0 \tag{4.6.20}$$

Equations (4.6.6) to (4.6.8) and Equations (4.6.18) to (4.6.20) for the core layer must be integrated over the depth of the core, and the results of the integration combined with the remaining equations for the face layers in such a way that a system of differential equations for the composite plate is obtained.

Equations (4.6.18) and (4.6.19) indicate that τ_x and τ_y do not vary across the thickness of the core. Thus, the transverse shear stress resultants can be defined by means of the following equations:

$$V_x = h\tau_x \tag{4.6.21}$$

$$V_y = h\tau_y \tag{4.6.22}$$

Integration of Equation (4.6.20) gives

$$V_{x,x} + V_{y,y} + \sigma_{z+} - \sigma_{z-} = 0 \tag{4.6.23}$$

From Equations (4.6.6) and (4.6.12) and from the fact that σ_z varies linearly over the thickness, it then follows that

$$w_+ - w_- = h(\sigma_{z+}+\sigma_{z-})/2E_c \tag{4.6.24}$$

Equations (4.6.7), (4.6.8) and Equation (4.6.13), (4.6.14) imply the following relations

$$V_x/G_c = \left(\int_{-h/2}^{h/2} w\,dz\right)_{,x} + u_+ - u_- \tag{4.6.25}$$

$$V_y/G_c = \left(\int_{-h/2}^{h/2} w\,dz\right)_{,y} + v_+ - v_- \tag{4.6.26}$$

The term inside the brackets on the right hand side of the above equations may be further written as:

$$\int_{-h/2}^{h/2} w dz = wz \Big|_{-h/2}^{h/2} - \int_{-h/2}^{h/2} w_{,z} \, z dz$$

$$= \frac{h}{2} (w_+ + w_-) - \int_{-h/2}^{h/2} (\sigma_z/E_c) z dz \qquad (4.6.27)$$

Since σ_z is a linear function of z, it can be written as

$$\sigma_z = \frac{1}{2} (\sigma_{z+} + \sigma_{z-}) + \frac{z}{h} (\sigma_{z+} - \sigma_{z-})$$

and therewith

$$\int_{-h/2}^{h/2} w dz = \frac{h}{2}(w_+ + w_-) - (\frac{h^2}{12E_c}) (\sigma_{z+} - \sigma_{z-}) \qquad (4.6.28)$$

Substituting the last term of Equation (4.6.28) by the relation given in Equation (4.6.23), the following equations are established:

$$V_x/G_c = [\frac{h}{2}(w_+ + w_-) + \frac{h^2}{12E_c} (V_{x,x} + V_{y,y})]_{,x} + u_+ - u_- \qquad (4.6.29)$$

$$V_y/G_c = [\frac{h}{2}(w_+ + w_-) + \frac{h^2}{12E_c} (V_{x,x} + V_{y,y})]_{,y} + v_+ - v_- \qquad (4.6.30)$$

Equations (4.6.24), (4.6.29) and (4.6.30) are the stress, strain relations for the core layer in a form suitable for use in the derivation of the equations for the composite plate.

In order to derive the equations for the composite plate, the following appropriate variables are defined:

$$\alpha = (u_+ - u_-)/h, \quad \beta = (v_+ - v_-)/h \qquad (4.6.31)$$

representing the effective changes of slope of the normal to the middle surface;

$$w = (w_+ + w_-)/2 \qquad (4.6.32)$$

representing the effective transverse deflection of the middle surface;

$$u = (u_+ + u_-)/2, \qquad v = (v_+ + v_-)/2 \qquad (4.6.33)$$

representing the effective in-plane displacement components of the middle surface; and

$$e = (w_+ - w_-)/h \qquad (4.6.34)$$

representing the effective transverse normal strain for the composite plate.

In addition to the transverse shear stress resultants V_x and V_y defined by Equations (4.6.21) and (4.6.22), the stress resultants and couples for the composite plate are defined as follows:

$$N_x = N_{x+} + N_{x-}, \; N_y = N_{y+} + N_{y-}, \quad S = S_+ + S_- \tag{4.6.35}$$

$$M_x = (N_{x+} - N_{x-})h/2, \; M_y = (N_{y+} - N_{y-})h/2, \; H = (S_+ - S_-)h/2 \quad \ldots\ldots \tag{4.6.36}$$

Finally, defining the effective transverse normal stress in the core by

$$\bar{\sigma}_z = (\sigma_{z+} + \sigma_{z-})/2 \tag{4.6.37}$$

and the external load terms p and q by means of the following relations:

$$p = p_+ + p_-, \quad q = (p_+ - p_-)/2 \tag{4.6.38}$$

The differential equations of the composite plate are obtained by combining the six equations of equilibrium, Equations (4.6.15) to (4.6.17), by means of suitable additions and subtractions. From Equation (4.6.15),

$$N_{x,x} + S_{,y} = 0 \tag{4.6.39}$$

$$M_{x,x} + H_{,y} - V_x = 0 \tag{4.6.40}$$

From Equation (4.6.16),

$$S_{,x} + N_{y,y} = 0 \tag{4.6.41}$$

$$H_{,x} + M_{y,y} - V_y = 0 \tag{4.6.42}$$

From Equation (4.6.17), after some transformations, the following two relations are derived:

$$p + V_{x},_{x} + V_{y},_{y} + N_{x}w,_{xx} + 2Sw,_{xy} + N_{y}w,_{yy} + M_{x}e,_{xx} + 2He,_{xy}$$

$$+ M_{y}e,_{yy} - V_{x}e,_{x} - V_{y}e,_{y} = 0 \qquad (4.6.43)$$

$$q - \bar{\sigma}_{z} + \frac{h}{4}(N_{x}e,_{xx} + 2Se,_{xy} + N_{y}e,_{yy}) + \frac{1}{h}(M_{x}w,_{xx} + 2Hw,_{xy}$$

$$+ M_{y}w,_{yy}) = 0 \qquad (4.6.44)$$

The physical meaning of Equations (4.6.39) and (4.6.41) are the usual equations of horizontal force equilibrium. Equations (4.6.40) and (4.6.42) are the usual equations of moment equilibrium. Equation (4.6.43) is the condition of transverse force equilibrium, and contains terms that do not occur when homogeneous isotropic plates are considered.

The significance of Equation (4.6.44) is that it gives the local change of thickness of the plate caused directly by the external loads through the term q and resulting indirectly from the external loads by way of the non-linear terms having stress resultants and couples as factors.

For the purpose of practical application, the stress-strain relations of the composite plate must also be obtained. These relations can easily be found by combining Equations (4.6.3) to (4.6.5) and Equations (4.6.35) and (4.6.36). The resulting equations are as follows [42]:

$$u,_{x} + \frac{1}{2}[(w,_{x})^{2} + \frac{h}{4}^{2}(e,_{x})^{2}] = (N_{x} - \nu_{f}N_{y})/2tE_{f} \qquad (4.6.45)$$

$$v,_{y} + \frac{1}{2}[(w,_{y})^{2} + \frac{h}{4}^{2}(e,_{y})^{2}] = (N_{y} - \nu_{f}N_{x})/2tE_{f} \qquad (4.6.46)$$

$$u,_y + v,_x + w,_x w,_y + \frac{h^2}{4} e,_x e,_y = S/2tG_f \qquad (4.6.47)$$

$$\alpha,_x + w,_x e,_x = (M_x - \nu_f M_y)/(th^2 E_f/2) \qquad (4.6.48)$$

$$\beta,_y + w,_y e,_y = (M_y - \nu_f M_x)/(th^2 E_f/2) \qquad (4.6.49)$$

$$\alpha,_y + \beta,_x + w,_x e,_y + w,_y e,_x = H/(th^2 E_f/2) \qquad (4.6.50)$$

In addition to the above six stress-strain relations, Equations (4.6.29), (4.6.30) and (4.6.24) may be written as:

$$w,_x + \alpha = \frac{V_x}{hG_c} - \frac{h}{12E_c} (V_x,_x + V_y,_y),_x \qquad (4.6.51)$$

$$w,_y + \beta = \frac{V_y}{hG_c} - \frac{h}{12E_c} (V_y,_y + V_x,_x),_y \qquad (4.6.52)$$

$$e = \bar{\sigma}_z/E_c \qquad (4.6.53)$$

Furthermore, the following discussion will be restricted to cases corresponding to the relation:

$$e \simeq q/E_c \qquad (4.6.54)$$

This should be true in most cases of practical interest. Also, by assuming that $q \simeq p$, it can be seen that the terms involving e in the above equations are negligible, provided that:

$$p/E_c \ll 1 \qquad (4.6.55)$$

which relation again is true in most practical cases.

Since the effect of local change of thickness of the plate is negligibly small so long as the above assumptions are valid, it can be concluded that as far as the present investigation is concerned, Equation (4.6.44) will be of negligible influence in the solution of the problem, and hence, can be discarded. Similarly, for all the other equations, terms involving e may be neglected without introducing appreciable errors. Thus, reducing the problem to a set of five equations with five unknowns.

From the assumption that e is small compared to the total deformation, the quantity $w,_z$ may be set equal to zero and:

$$V_x = hG_c(w,_x + \alpha) \tag{4.6.56}$$

$$V_y = hG_c(w,_y + \beta) \tag{4.6.57}$$

from which

$$V_x,_x = hG_c(w,_{xx} + \alpha,_x) \tag{4.6.58}$$

$$V_y,_y = hG_c(w,_{yy} + \beta,_y) \tag{4.6.59}$$

It is seen that by dropping terms involving e in the stress strain relations for the composite plate, the resulting relations are identical to the ones obtained by Alwan [2] through a variational approach.

With Equations (4.6.56) through (4.6.59) and the stess-strain relations, the equations of equilibrium may be written in the following manner [40]:

$$2u,_{xx} + (1-\nu_f)u,_{yy} + (1+\nu_f)v,_{xy} + [(w,_x)^2 + \nu_f(w,_y)^2],_x$$

$$+ \quad (1-\nu_f)[w,_{xy}w,_y + w,_x w,_{yy}] = 0 \qquad (4.6.60)$$

$$2v,_{yy} + (1-\nu_f)v,_{xx} + (1+\nu_f)u,_{xy} + [(w,_y)^2 + \nu_f(w,_x)^2],_y$$

$$+ \quad (1-\nu_f)[w,_{xy}w,_y + w,_y w,_{xx}] = 0 \qquad (4.6.61)$$

$$\frac{thE_f}{1-\nu_f^2}[2\alpha,_{xx} + (1-\nu_f)\alpha,_{yy} + (1+\nu_f)\beta,_{xy}] - 4G_c(w,_x + \alpha) = 0 \qquad (4.6.62)$$

$$\frac{thE_f}{1-\nu_f^2}[2\beta,_{yy} + (1-\nu_f)\beta,_{xx} + (1+\nu_f)\alpha,_{xy}] - 4G_c(w,_y + \beta) = 0 \qquad (4.6.63)$$

$$p + hG_c[w,_{xx} + w,_{yy} + \alpha,_x + \beta,_y] + \frac{tE_f}{1-\nu_f^2}\{ w,_{xx}[2(u,_x + \nu_f v,_y) +$$

$$(w,_x)^2 + \nu_f(w,_y)^2] + w,_{yy}[2(v,_y + \nu_f u,_x) + (w,_y)^2 + \nu_f(w,_x)^2]$$

$$+ 2(1-\nu_f)w,_{xy}[u,_y + v,_x + w,_x w,_y]\} = 0 \qquad (4.6.64)$$

The above equations, Equations (4.6.60) to (4.6.64) are the governing differential equations for sandwich plates in terms of displacements.

From the equations, it can be seen that as in the case of the homogeneous plate, the fourth order differential equation of compatability, i.e., Equation (4.6.1) is replaced by two differential equations of the second order, viz., Equations (4.6.60) and (4.6.61). These two equations are in fact of an identical form as the corresponding two second order equations of the homogeneous plate.

The other fourth order differential equation, i.e., Equation (4.6.2), is seen to be substituted by three second order equations. Viz., Equations (4.6.62), (4.6.63) and (4.6.64). In Reissners' derivation of Equation (4.6.2), the equations of moment equilibrium, Equations (4.6.40) and (4.6.42), were introduced into Equation (4.6.43) to replace the derivatives of the transverse shear stress resultants V_x and V_y, resulting in:

$$M_{x},_{xx} + 2H,_{xy} + M_{y},_{yy} + p + N_x w,_{xx} + 2Sw,_{xy} + N_y w,_{yy} = 0 \qquad (4.6.65)$$

then by means of the stress-strain relations, Equations (4.6.48) to (4.6.50), the above equation is transformed into:

$$D\nabla^2(\alpha,_x + \beta,_y) + p + N_x w,_{xx} + 2Sw,_{xy} + N_y w,_{yy} = 0 \qquad (4.6.66)$$

and from the relations (4.6.56) and (4.6.58), it follows that

$$\alpha,_x + \beta,_y = -\nabla^2 w + \frac{1}{hG_c}(V_x,_x + V_y,_y) \qquad (4.6.67)$$

By introducing Equation (4.6.67) into Equation (4.6.66), and observing once more Equation (4.6.43), the unknown function α and β are eliminated implicitly, thus, resulting in a single equation in terms of the out-of-plane displacement w and the membrane stress function F, and together with Equation (4.6.1), constitute a system of two fourth order non-linear differential equations for the solution of the two unknowns w and F. Meanwhile, in the present derivation, substitution of the stress-strain relations into Equation (4.6.43) leads to an equation involving the unknown functions α, β, w, u and v. Instead of eliminating the unknowns α and β as Reissner had done, Equations (4.6.40) and (4.6.42) are used to supplement Equation (4.6.43), and along with Equations (4.6.39) and (4.6.41), make up a set of five equations, which are expressed explicitly in terms of the unknowns α, β, w, u and v. Thus, it can be concluded that the present system of five equations are actually Reissners' equations written in a modified form.

For ease of computation, the five governing differential equations, Equations (4.6.60) to (4.6.64) are converted to a non-dimensional form by adopting the following dimensionless ratios:

$$\lambda = a/b; \qquad \xi = x/a; \qquad \eta = y/b,$$

$$W = w/h; \qquad U = ua/h^2; \qquad V = va/h^2,$$

$$Q = pa^4/Dh = 2(1-\nu_f^2)pa^4/Eth^3; \qquad \mu = t/a; \qquad \theta = h/a$$

By substituting the above dimensionless quantities into the governing differential equations, the following equations are obtained:

$$2U,_{\xi\xi} + (1+\nu_f)\lambda V,_{\xi\eta} + (1-\nu_f)\lambda^2 U,_{\eta\eta} + 2W,_{\xi}W,_{\xi\xi} + (1-\nu_f)\lambda^2 W,_{\xi}W,_{\eta\eta} + (1+\nu_f)\lambda^2 W,_{\eta}W,_{\xi\eta} = 0 \qquad (4.6.68)$$

$$2\lambda^2 V,_{\eta\eta} + (1+\nu_f)\lambda U,_{\xi\eta} + (1-\nu_f)V,_{\xi\xi} + 2\lambda^3 W,_{\eta}W,_{\eta\eta} + (1-\nu_f)\lambda W,_{\eta}W,_{\xi\xi} + (1-\nu_f)\lambda W,_{\xi}W,_{\xi\eta} = 0 \qquad (4.6.69)$$

$$\frac{\mu\theta E_f}{(1-\nu_f^2)}\left[2\alpha,_{\xi\xi} + (1+\nu_f)\lambda\beta,_{\xi\eta} + (1-\nu_f)\lambda^2\alpha,_{\eta\eta}\right] - 4G_c\left[\theta W,_{\xi} + \alpha\right] = 0 \qquad \ldots\ldots\ldots\ldots(4.6.70)$$

$$\frac{\mu\theta E_f}{(1-\nu_f^2)}\left[2\lambda^2\beta,_{\eta\eta} + (1+\nu_f)\lambda\alpha,_{\xi\eta} + (1-\nu_f)\beta,_{\xi\xi}\right] - 4G_c\left[\theta\lambda W,_{\eta} + \beta\right] = 0 \qquad \ldots\ldots\ldots\ldots(4.6.71)$$

$$Q + \frac{2(1-\nu_f^2)G_c}{E_f\mu\theta^2}\left[\theta W,_{\xi\xi} + \lambda^2\theta W,_{\eta\eta} + \alpha,_{\xi} + \lambda\beta,_{\eta}\right] + 2W,_{\xi\xi}\left[2U,_{\xi} + 2\nu_f\lambda V,_{\eta} + (W,_{\xi})^2 + \nu_f\lambda^2(W,_{\eta})^2\right] + 2\lambda^2 W,_{\eta\eta}\left[2\lambda V,_{\eta} + 2\nu_f U,_{\xi} + \lambda^2(W,_{\eta})^2 + \nu_f(W,_{\xi})^2\right] + 4(1-\nu_f)\lambda W,_{\xi\eta}\left[\lambda U,_{\eta} + V,_{\xi} + \lambda W,_{\xi}W,_{\eta}\right] = 0 \qquad (4.6.72)$$

These equations are next simplified by the application of the perturbation method. Following procedures similar to those in Chapter III, the dimensionless center deflection W_o of the plate is chosen as a perturbation parameter and the lateral load and displacement functions expressed as ascending powers of the parameter by the following equations:

$$\alpha = m_1(\xi,\eta)W_o + m_3(\xi,\eta)\ W_o^3 + \ldots\ldots \tag{4.6.73}$$

$$\beta = n_1(\xi,\eta)W_o + n_3(\xi,\eta)\ W_o^3 + \ldots\ldots \tag{4.6.74}$$

$$W = w_1(\xi,\eta)W_o + w_3(\xi,\eta)\ W_o^3 + \ldots\ldots \tag{4.6.75}$$

$$U = u_2(\xi,\eta)W_o^2 + u_4(\xi,\eta)\ W_o^4 + \ldots\ldots \tag{4.6.76}$$

$$V = v_2(\xi,\eta)W_o^2 + v_4(\xi,\eta)W_o^4 + \ldots\ldots \tag{4.6.77}$$

$$Q = q_1W_o + q_3W_o^3 + \ldots\ldots\ldots \tag{4.6.78}$$

where Equation (4.6.78) represents a case of uniformly distributed load - the case of loading to be considered in the present study.

Substituting Equations (4.6.73) to (4.6.78) into Equations (4.6.68) to (4.6.72) and equating terms of order W_o, the first order approximation is obtained - i.e., the small deflection equations,

$$\frac{\mu\theta E_f}{(1-\nu_f^2)}\left[2m_{1,\xi\xi} + (1+\nu_f)\lambda\, n_{1,\xi\eta} + (1-\nu_f)\lambda^2 m_{1,\eta\eta}\right]$$

$$- 4G_c[\theta w_{1,\xi} + m_1] = 0 \tag{4.6.79}$$

$$\frac{\mu\theta E_f}{(1-\nu_f^2)}\left[2\lambda^2 n_{1,\eta\eta} + (1+\nu_f)\lambda\, m_{1,\xi\eta} + (1-\nu_f) n_{1,\xi\xi}\right] - 4G_c[\theta\lambda w_{1,\eta} + n_1] = 0 \tag{4.6.80}$$

$$q_1 + \frac{2(1-\nu_f^2)G_c}{E_f\mu\theta^2}\left[\theta w_{1,\xi\xi} + \lambda^2\theta w_{1,\eta\eta} + m_{1,\xi} + \lambda n_{1,\eta}\right] = 0 \tag{4.6.81}$$

Collecting terms of order W_o^2 yields the second order approximation:

$$2u_{2,\xi\xi} + (1+\nu_f)\lambda v_{2,\xi\eta} + (1-\nu_f)\lambda^2 u_{2,\eta\eta} + 2w_{1,\xi}w_{1,\xi\xi} + (1-\nu_f)\lambda^2 w_{1,\xi}w_{1,\eta\eta} + (1+\nu_f)\lambda^2 w_{1,\eta}w_{1,\xi\eta} = 0 \tag{4.6.82}$$

$$2\lambda^2 v_{2,\eta\eta} + (1+\nu_f)\lambda\, u_{2,\xi\eta} + (1-\nu_f)v_{2,\xi\xi} + 2\lambda^3 w_{1,\eta}w_{1,\eta\eta} + (1-\nu_f)\lambda w_{1,\eta}w_{1,\xi\xi} + (1+\nu_f)\lambda w_{1,\xi}w_{1,\xi\eta} = 0 \tag{4.6.83}$$

Finally, equating terms of order W_o^3, the third order approximation is obtained:

$$\frac{\mu\theta E_f}{(1-\nu_f^2)}\left[2m_{3,\xi\xi} + (1+\nu_f)\lambda n_{3,\xi\eta} + (1-\nu_f)\lambda^2 m_{3,\eta\eta}\right] - 4G_c[\theta w_{3,\xi} + m_3] = 0 \tag{4.6.84}$$

$$\frac{\mu\theta E_f}{(1-\nu_f^2)}\left[2\lambda^2 n_{3,\eta\eta}+(1+\nu_f)\lambda m_{3,\xi\eta}+(1-\nu_f)n_{3,\xi\xi}\right]$$

$$-4G_c\left[\theta\lambda w_{3,\eta}+n_3\right] = 0 \tag{4.6.85}$$

$$q_3+\frac{2(1-\nu_f^2)G_c}{E_f\mu\theta^2}\left[\theta w_{3,\xi\xi}+\lambda^2\theta w_{3,\eta\eta}+m_{3,\eta}+\lambda n_{3,\eta}\right]$$

$$+2w_{1,\xi\xi}\left[2u_{2,\xi}+2\nu_f\lambda v_{2,\eta}+(w_{1,\xi})^2+\nu_f\lambda^2(w_{1,\eta})^2\right]$$

$$+2\lambda^2 w_{1,\eta\eta}\left[2\lambda v_{2,\eta}+2\nu_f u_{2,\xi}+\lambda^2(w_{1,\eta})^2+\nu_f(w_{1,\xi})^2\right]$$

$$+4(1-\nu_f)\lambda w_{1,\xi\eta}\left[\lambda u_{2,\eta}+v_{2,\xi}+\lambda w_{1,\xi}w_{1,\eta}\right] = 0 \tag{4.6.86}$$

When Equations (4.6.79) to (4.6.81) are solved, the problem of the non-linear analysis of uniformly loaded sandwich plates is considered completed.

(iii) Solution of the problem:

For the linear and non-linear analyses of clamped sandwich plates of a rectangular geometry as shown in Figure 1, the differential equations (4.6.79) to (4.6.81) are solved by means of the orthogonal collocation method.

For the linear analysis, i.e., first order approximation, the boundary conditions to be met by the unknown functions m_1, n_1 and w_1 are:

$$w_1 = m_1 = n_1 = 0 \quad \text{at } \xi = \pm 1 \text{ and } \eta = \pm 1 \tag{4.6.87}$$

$$w_{1,\xi} = 0 \quad \text{at} \quad \xi = \pm 1 \tag{4.6.88}$$

$$w_{1,\eta} = 0 \quad \text{at} \quad \eta = \pm 1 \tag{4.6.89}$$

and the condition $w_1(0,0) = 1$ as required by the definition of Equation (4.6.75).

Following the principle that arbitrary small variations of the functions m_1, n_1 and w_1 are not interrelated and bearing in mind the requirements of the boundary conditions, the assumed solutions of the three coupled linear differential equations (4.6.79), (4.6.80) and (4.6.81) can be taken in the following forms:

$$m_1 = \xi(1-\xi^2)(1-\eta^2) \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} A_{ij} P_i''(\xi^2) P_j'(\eta^2) \tag{4.6.90}$$

$$n_1 = \eta(1-\xi^2)(1-\eta^2) \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} B_{ij} P_i'(\xi^2) P_j''(\eta^2) \tag{4.6.91}$$

$$w_1 = (1-\xi^2)(1-\eta^2)^2 \sum_{i=0}^{n-1} \sum_{n=0}^{n-1} C_{ij} P_i'''(\xi^2) P_j'''(\eta^2) \tag{4.6.92}$$

The undetermined coefficients in the series for m_1, n_1 and w_1 together with the constant q_1 presents $3n^2 + 1$ unknowns. Equation (4.6.90) provides n^2 collocation points with the ξ coordinates of the points being roots of the polynomial $P_n''(\xi^2)$ and the η coordinates roots of the polynomial $P_n'(\eta^2)$. The residual of Equation (4.6.79) is to be evaluated at these n^2 collocation points, thus creating a set of n^2 equations. Similarly, a second set of n^2 equations can be obtained by evaluating the residual of Equation (4.6.80) at the n^2 points provided by Equation (4.6.91), and a third set of n^2 equations generated via Equation (4.6.81) and the collocation points furnished by Equation (4.6.92). Finally, introducing the equation

$$\sum_{i=0}^{n-1} \sum_{j=0}^{n-1} C_{ij} = 1 \tag{4.6.93}$$

a system of $3n^2+1$ equations for the solution of the $3n^2+1$ unknowns is established.

The boundary conditions for the second order approximation are:

$$u_2 = v_2 = 0 \quad \text{at} \quad \xi = \pm 1 \quad \text{and} \quad \eta = \pm 1 \tag{4.6.94}$$

The solution of Equations (4.6.82) and (4.6.83) follows a procedure similar to the solution of the corresponding equations of a homogeneous plate (c.f. Sec. 4.5). The assumed solutions for the functions u_2 and v_2 are:

$$u_2 = \xi(1-\xi^2)(1-\eta^2) \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} D_{ij} P_i''(\xi^2) P_j'(\eta^2) \tag{4.6.95}$$

$$v_2 = \eta(1-\xi^2)(1-\eta^2) \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} E_{ij} P_i'(\xi^2) P_j''(\eta^2) \qquad (4.6.96)$$

For the final stage of the approximation, the boundary conditions are:

$$w_3 = m_3 = n_3 = 0 \quad \text{at} \quad \xi = \pm 1, \text{ and } \eta = \pm 1 \qquad (4.6.97)$$

$$w_{3,\xi} = 0 \quad \text{at} \quad \xi = \pm 1 \qquad (4.6.98)$$

$$w_{3,\eta} = 0 \quad \text{at} \quad \eta = \pm 1 \qquad (4.6.99)$$

To satisfy these boundary conditions, the assumed solutions for Equations (4.6.84), (4.6.85) and (4.6.86) are taken as

$$m^3 = \xi(1-\xi^2)(1-\eta^2) \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} F_{ij} P_i''(\xi^2) P_j'(\eta^2) \qquad (4.6.100)$$

$$n_3 = \eta(1-\xi^2)(1-\eta^2) \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} G_{ij} P_i'(\xi^2) P_j''(\eta^2) \qquad (4.6.101)$$

$$w_3 = (1-\xi^2)^2(1-\eta^2)^2 \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} H_{ij} P_i'''(\xi^2) P_j'''(\eta^2) \qquad (4.6.102)$$

then in the manner explained in the first order approximation, a set of $3n^2$ equations is generated, and with the additional equation,

$$\sum_{i=0}^{n-1} \sum_{j=0}^{n-1} H_{ij} = 0 \qquad (4.6.103)$$

which is required in order that $w_3(0,0) = 0$, a system of $3n^2 + 1$ equations for the solution of the $3n^2$ undetermined polynomial coefficients and the constant q_3 is obtained.

For the purpose of comparison of results, in the linear analysis, the following numerical examples are considered:

plate No. 1: $E_f = 10 \times 10^6$ psi, $G_c = 500$ psi

$\nu_f = 0.32$, $\mu = 0.00125$, $\theta = 0.05125$

plate No. 2: $E_f = 10 \times 10^6$ psi, $G_c = 100{,}000$ psi

$\nu_f = 0.32$, $\mu = 0.00125$, $\theta = 0.05125$

plate No. 3: $E_f = 10.5 \times 10^6$ psi, $G_c = 50{,}000$ psi

$\nu_f = 0.30$, $\mu = 0.0006$, $\theta = 0.04$

The first two problems were solved by Monforton et.al [34] for the particular case of a square using the finite element method, the third problem was solved by Kan et. al [20] for plate aspect ratios ranging from 0 to 1 using a perturbation and power series solution. Whenever possible, the present solution of these problems are also compared with those according to March [32].

In the non-linear analysis, the only source of comparison available to this writer was that by Kan et. al [20]. But unfortunately, it seems that in their derivation of the governing differential equations, the equations relating the local change of thickness to the overall deflection, i.e., Equation (4.6.44), was not totally deleted as was done here. The terms involving products of moments and derivatives of deflections in this equation were retained, resulting in three equations for the second order approximation, two of which are identical to Equations (4.6.82) and (4.6.83) and a third equation expressed in terms of the functions m_1, n_1 and w_1, which are all known quantities determined from the first order approximation. Logically speaking, a set of two equations is all that is required for the solution of the unknowns u_2 and v_2, thus, in the opinion of this writer, the third equation involving the known functions m_1, n_1 and w_1 seems unjustifiable, and as such, no comparisons will be made for the non-linear analysis.

In addition to the non-linear analysis of plate No. 3, a similar analysis is also performed on plate No. 2.

For all the problems considered, the variation of results due to the number of terms retained in each series is investigated by obtaining results from a 4 term, 9 term and a 16 term solution.

(iv) Comparison and Discussion of Results:

For the linear analysis, the results are tabulated in Tables 20 to 25. Comparisons of results are made with March [32], Monforton et. al [34] and Kan et. al [20]. The solution due to March [32] consists of two formulae, one for 0.7< b/a <1.4 and the other for large values of b/a. In a discussion by Planterna [41], the formula for large values of b/a was found to be inaccurate in estimating the additional deflections arising from the transverse shear deformability of the core, furthermore, the results become worse as the shear rigidity of the core is decreased. Therefore comparison with Ref. [32] is limited to plates of aspect ratios greater than 2/3, while deflections for large values of b/a as given by Ref. [32] are listed as a reference. It can be observed that the results of the present solution are generally comparable with those due to March [32].

For the two problems solved by Monforton et. al [34], the present results are in excellent agreement with those obtained by the accurate finite element method. The results for plate No. 3 are also in good agreement with the solutions of Kan et. al [20]. Furthermore, convergence of the results can be seen as the number of terms is increased.

For the non-linear analysis, results are presented in Tables 26 and 27. Load vs. deflection curves are shown in Figures 26 and 27. For reasons explained earlier, no comparison is made here. In general, it can be seen that the non-linear results, i.e., the q_3 values, follow a trend similar to the homogeneous plate, viz., the gradual but slight increase in q_3 for increasing plate aspect ratios up to the value of 3/4 and a very sharp rise when the aspect ratio is 1.

4.7 Concluding Remarks on the Orthogonal Collocation Method:

The collocation method developed here permits rapid solution of many types of difficult boundary value problems. The accuracy of the method is excellent as can be seen by the results presented herein.

Rather than distributing the collocation points at random as was done in the collocation least square method, the collocation points here are chosen in a well defined manner, such that very accurate results can be obtained even for a 4 term (4 collocation points) solution, a feat praotically impossible by the conventional collocation method, and with a nine term (9 collocation points) solution, the results are almost identical if not better than those obtained by the collocation least square method which uses a large number of points for a solution.

The orthogonal collocation method differs from other weighted residual methods in that the residual here is not directly orthogonalized, but is matched to an orthogonal function at its zeroes. The tedious task of integrating the residual is thereby avoided, and the calculations are correspondingly simplified.

The vital part of the solution lies in the construction of the orthogonal polynomial sets, once these polynomials are established and their zeroes obtained, the solution of a problem becomes very straight forward. Although the additional step taken in constructing the polynomials seems to make this method slightly more complicated than the collocation least square method, its efficiency in computation, compactness of results and excellent accuracy compensates for the additional step taken.

In conclusion, it can be said that in addition to its simplicity in application, the orthogonal collocation method has an accuracy comparable to other weighted residual methods, and as such, can be used as a convenient tool in the numerical treatment of very complex boundary value problems.

CHAPTER V

VLASOV'S METHOD

5.1 General:

Although Galerkin's method is the most rapidly converging error distribution method [16], the process of solving a boundary value problem by this method is usually a very tedious one, since it involves evaluation of the integrals of the residuals. However, by choosing functions with special mathematical properties for the assumed solution, and by utilizing these mathematical properties, Galerkin's method can be greatly simplified. Such a procedure was suggested by Vlasov [51] and can be efficiently applied to the solution of many linear and non-linear problems in mechanics.

5.2 Vlasov's Method:

To demonstrate the formulation of Vlasov's method as applied to the problem of plate bending, consider a rectangular plate of width 'a' and length 'b' with arbitrary boundary conditions, the well-known governing differential equation is:

$$D\nabla^2\nabla^2 w = q \tag{5.2.1}$$

The lateral deflection can be expressed by an infinite series of the form:

$$w(x,y) = \sum_m \sum_n W_{mn} \phi_{mn}(x,y) \tag{5.2.2}$$

Similarly, the lateral load can be expressed as:

$$q(x,y) = \sum_m \sum_n q_{mn} \psi_{mn}(x,y) \tag{5.2.3}$$

Where the functions $\phi_{mn}(x,y)$ and $\psi_{mn}(x,y)$ are the product of two functions, each of which depends on just a single argument, i.e.,

$$\phi_{mn}(x,y) = X_m(x) \cdot Y_n(y), \tag{5.2.4}$$

and

$$\psi_{mn}(x.y) = X_m(x) \cdot Y_n(y) \tag{5.2.5}$$

Thus, by separation of the variables, the variational problem is reduced to the selection of two linearly independent sets of functions $X_m(x)$ and $Y_n(y)$, which satisfy all the boundary conditions. For these functions, Vlasov used the eigenfunctions of vibrating beams, with identical boundary conditions as those of the plate.

The eigenfunctions and their derivatives satisfy certain important mathematical relations. Let $X_m(x)$ and $X_n(x)$ be any two eigenfunctions of a vibrating beam of length 'ℓ', corresponding to the circular frequencies ω_m and ω_n respectively. Then, for different modes

$(m \neq n)$, the following relations hold:

$$\int_o^\ell X_m(x) \cdot X_n(x)dx = 0 \quad \text{and} \quad \int_o^\ell X''_m(x) \cdot X''_n(x)dx = 0 \qquad (5.2.6)$$

i.e., the eigenfunctions and their second derivatives are said to be orthogonal. The same holds for their fourth derivatives, while the desirable property $\int_o^\ell X''_m(x) \cdot X_n(x)dx = 0$, which plays a role in the solution, is slightly violated. Strictly speaking, these functions are only quasi-orthogonal. The orthogonality conditions however, do not hold for free and guided, or elastically supported edges.

As the eigenfunctions are othogonal, another useful property common to all orthogonal functions can be utilized; i.e., by expanding the lateral load $q(x,y)$ in terms of the eigenfunctions, the constants q_{mn} in Equation (5.2.3) can be determined by multiplying both sides of the equation by $X_m(x) \cdot Y_n(y)$ and integrating the product; a procedure that yields:

$$q_{mn} = \frac{\int_o^a \int_o^b q(x,y)X_m(x) \cdot Y_n(y)dxdy}{\int_o^a \int_o^b X_m^2(x)Y_n^2(y)dxdy} \qquad (5.2.7)$$

Expressing the plate deflection by eigenfunctions in the form:

$$w(x,y) = \sum_m \sum_n W_{mn}\phi_{mn}(x,y) = \sum_m \sum_n W_{mn}X_m(x) \cdot Y_n(y) \qquad (5.2.8)$$

and using a similar expression for the lateral load q, the variational equation of the plate can then be written as

$$D \sum_m \sum_n W_{mn} \int_0^a \int_0^b \phi_{ik} \nabla^4 \phi_{mn} dxdy - \sum_m \sum_n q_{mn} \int_0^a \int_0^b \psi_{mn} \phi_{mn} dxdy = 0 \qquad \ldots\ldots\ldots \quad (5.2.9)$$

For this particular choice of expression for the lateral deflection w and the load q, i.e., $\phi_{mn} = \psi_{mn} = X_m(x) \cdot Y_n(y)$, the first integral term in Equation (5.2.9) can be written as

$$\int_0^a \int_0^b \phi_{ik} \nabla^4 \phi_{mn} dxdy = \int_0^a \int_0^b [X_m^{IV}(x) Y_n(y) X_i(x) Y_k(y) + 2X_m''(x) Y_n''(y) X_i(x) Y_k(y)$$

$$+ Y_n^{IV}(y) X_m(x) X_i(x) Y_k(y)] \, dxdy \qquad (5.2.10)$$

Neglecting the terms with nonidentical subscripts mi and nk, the error induced is zero or negligible. Thus, by introducing the following notations:

$$I_1 = \int_0^a X_m^{IV}(x) X_m(x) dx$$

$$I_2 = \int_0^a Y_n(y) Y_n(y) dy$$

$$I_3 = \int_0^a X_m''(x) X_m(x) dx$$

$$I_4 = \int_o^b Y_n''(y) Y_n(y)\, dy$$

$$I_5 = \int_o^b Y_n^{IV}(y) Y_n(y)\, dy$$

$$I_6 = \int_o^a X_m(x) X_m(x)\, dx \qquad (5.2.11)$$

Equation (5.2.10) can be written as

$$\int_o^a \int_o^b \phi_{ik} \nabla^4 \phi_{mn} dxdy = I_1 I_2 + 2I_3 I_4 + I_5 I_6 \qquad (5.2.12)$$

In a similar manner, the integrals of the second term in Equation (5.2.9) are expressed as

$$\int_o^a \int_o^b \psi_{mn} \phi_{ik} dxdy = \int_o^a \int_o^b X_m^2(x) Y_n^2(y)\, dxdy = I_7 I_8 \qquad (5.2.13)$$

where

$$I_7 = \int_o^a X_m^2(x)\, dx \quad \text{and} \quad I_8 = \int_o^b Y_n^2(y)\, dy \qquad (5.2.14)$$

For a particular set of m,n values, the variational equation of the plate problem is then reduced to

$$DW_{mn}(I_1 I_2 + 2I_3 I_4 + I_5 I_6) - q_{mn} I_7 I_8 = 0 \qquad (5.2.15)$$

Consequently, the undetermined expansion coefficients W_{mn} can be calculated from

$$W_{mn} = \frac{q_{mn} I_7 I_8}{(I_1 I_2 + 2I_3 I_4 + I_5 I_6) D} \qquad (5.2.16)$$

which, upon substitution of Equation (5.2.7) becomes

$$W_{mn} = \frac{\int_0^a \int_0^b q(x,y) X_m(x) Y_n(y)\, dxdy}{(I_1 I_2 + 2I_3 I_4 + I_5 I_6) D} \qquad (5.2.17)$$

Thus, the approximate solution of the problem of plate bending is reduced to the evaluation of simple definite integrals. Furthermore, the eigenfunctions reduce the required numerical work by a very great extent.

A similar approach can be taken with eigenfunctions of column buckling since, these functions are also quasi-orthogonal, i.e., for $m \neq n$ they satisfy

$$\int_0^{\ell} X_m^{IV} X_n dx = 0 \quad \text{and} \quad \int_0^{\ell} X_m'' X_n dx = 0 \qquad (5.2.18)$$

while

$$\int_0^{\ell} X_m X_n dx \neq 0 \quad \text{for } m \neq n \qquad (5.2.19)$$

a violation which is of minor importance.

Since the eigenfunctions of vibrating beams or column buckling are readily available, the dilemma of choosing an appropriate function which will satisfy all the prescribed boundary conditions of a plate problem is thereby avoided.

5.3 Rectangular Plates with Two Opposite Sides Simply Supported and the Other Two Sides Clamped:

As an illustrative example of the application of Vlasov's method, consider the bending of the rectangular plate shown in Figure 28. The boundary conditions are:

$$w = w,_y = 0 \quad \text{at } y = \pm b \tag{5.3.1}$$

$$w = w,_{xx} = 0 \text{ at } \quad x = 2a, 0 \tag{5.3.2}$$

In order to meet the above requirements, the assumed solution is taken as the product of the eigenfunctions of a clamped-clamped beam and a simply supported beam. i.e, the functions X_m and Y_n in Equation (5.2.8) take the form

$$X_m(x) = \sin \frac{m\pi x}{2a} \quad (m = 1, 3, 5, \ldots.) \tag{5.3.3}$$

$$Y_n(y) = \frac{\cosh\lambda_n \frac{y}{b}}{\cosh \lambda_n} - \frac{\cos \lambda_n \frac{y}{b}}{\cos \lambda_n} \quad (n = 1, 2, 3, ..) \tag{5.3.4}$$

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$$I_6 = \int_o^{2a} X_m X_m dx = a$$

$$I_7 = \int_o^{2a} X_m^2 dx = a$$

$$I_8 = \int_{-b}^{b} Y_n^2 dy = 2b \quad (5.3.5)$$

For a uniformly distributed load of intensity q_o, the numerator in expression (5.2.17) becomes:

$$q_o \int_o^{2a} \int_{-b}^{b} X_m Y_n dxdy = \frac{8abq_o}{m\pi\lambda_n} [\tanh\lambda_n - \tan\lambda_n] \quad (5.3.6)$$

For comparison with Timoshenko [49], the coefficients W_{mn} can be determined from the following dimensionless expressions:

$$W_{mn} = \frac{Q}{[\lambda^4 I_1 I_2 + 2\lambda^2 I_3 I_4 + I_5 I_6]} \quad (5.3.7)$$

where $\lambda = b/a$

and $$Q = \frac{8b^4 q_o}{m\pi\lambda_n} [\tanh\lambda_n - \tan\lambda_n] \quad (5.3.8)$$

for $b>a$,

$$W_{mn} = \frac{Q}{[I_1 I_2 + 2\lambda^2 I_3 I_4 + \lambda^4 I_5 I_6]} \quad (5.3.9)$$

where $\lambda_1 = 2.365020372$

$\lambda_2 = 5.497803919$

$\lambda_3 = 8.639379829$

$\lambda_4 = 11.78097245$

and for large values of n

$$\lambda_n = (4n - 1)\ \pi/4 \ .$$

Thus, the integrals $I_1, \ldots\ldots, I_8$ become :

$$I_1 = \int_o^{2a} X_m^{IV} X_m dx = \frac{m^4 \pi^4}{16a^4} \ a$$

$$I_2 = \int_{-b}^{b} Y_n Y_n dy = 2b$$

$$I_3 = \int_o^{2a} X_m'' X_m dx = - \frac{m^2 \pi^2}{4a^2} \ a$$

$$I_4 = \int_{-b}^{b} Y_n'' Y_n dy = \frac{\lambda_n}{b} \left[\frac{1}{\cosh^2 \lambda_n} \left(\frac{\sinh 2\lambda_n}{2} + \lambda_n \right) - \frac{1}{\cos^2 \lambda_n} \left(\frac{\sin 2\lambda_n}{2} + \lambda_n \right) \right]$$

$$I_5 = \int_{-b}^{b} Y_n^{IV} Y_n dy = \frac{\lambda_n^4}{b^4} \ 2b$$

where $\lambda = a/b$

and $$Q = \frac{8a^4 q_o}{m\pi\lambda_n} [\tanh \lambda_n - \tan \lambda_n] \qquad (5.3.10)$$

The maximum deflection which occurs at the center of the plate can be calculated from

$$w_{max} = \sum_m \sum_n W_{mn} (-1)^{(m-1)/2} \left(\frac{1}{\cosh \lambda_n} - \frac{1}{\cos \lambda_n}\right) \qquad (5.3.11)$$

Values of the maximum deflection obtained by using a 1 term, 4 term, 9 term and a 16 term solution are tabulated in Table 28 along with results reported by Timoshenko [49].

From the results shown, it can be said that the results obtained from the present solution are in excellent agreement with those of Timoshenko [49]. For a one term solution, the maximum deviation is no more than 5%, while the other results are more or less identical with those due to Timoshenko.

5.4 Simply Supported Rectangular Sandwich Plates:

As a second illustrative example, consider the linear analysis of a simply supported rectangular sandwich plate (Figure 1), subjected to a uniformly distributed load of intensity q_o. The governing differential equation can be obtained by dropping the non-linear terms in Reissner's equations, Equations (4.6.1) and (4.6.2). The resulting expression is:

$$\nabla^2\nabla^2 w = \frac{q_o}{D} - \frac{1}{S}\nabla^2 q_o \tag{5.4.1}$$

where $S = hG_c$

For simply supported edges, the boundary conditions are

$$w = w,_{xx} = 0 \quad \text{at } x = \pm a \tag{5.4.2}$$

and
$$w = w,_{yy} = 0 \quad \text{at } y = \pm b \tag{5.4.3}$$

These boundary conditions will be automatically satisfied if the assumed solution for w is taken as the product of the eigenfunctions of two simply supported beams. Thus,

$$X_m = \cos\frac{m\pi x}{2a} \quad (m = 1, 3, 5, \ldots) \tag{5.4.4}$$

$$Y_n = \cos\frac{n\pi y}{2b} \quad (n = 1, 3, 5, \ldots) \tag{5.4.5}$$

and the integrals $I_1, \ldots, I_8$ become:

$$I_1 = \int_{-a}^{a} X_m^{IV} X_m dx = \frac{m^4\pi^4}{16a^4} a$$

$$I_2 = \int_{-b}^{b} Y_n Y_n dy = b$$

$$I_3 = \int_{-a}^{a} X_m'' X_m dx = -\frac{m^2\pi^2}{4a^2} a$$

$$I_4 = \int_{-b}^{b} Y''_n Y_n dy = -\frac{n^2\pi^2}{4b^2} b$$

$$I_5 = \int_{-b}^{b} Y_n^{IV} Y_n dy = \frac{n^4\pi^4}{16b^4} b$$

$$I_6 = \int_{-a}^{a} X_m X_m dx = a$$

$$I_7 = \int_{-a}^{a} X_m^2 dx = a$$

$$I_8 = \int_{-b}^{b} Y_n^2 dy = b \qquad (5.4.6)$$

From the relation given in Equation (5.2.7), the constants q_{mn} are found to be:

$$q_{mn} = \frac{16q_o}{\pi^2 mn} (-1)^{(m+n-2)/2} \qquad (5.4.7)$$

The coefficients W_{mn} can be determined from the expression:

$$W_{mn} = \frac{q_{mn} I_7 I_8}{D[I_1 I_2 + 2I_3 I_4 + I_5 I_6]} - \frac{q_{mn}[I_2 I_3 + I_4 I_6]}{S[I_1 I_2 + 2I_3 I_4 + I_5 I_6]}$$

$$= W_{mn}^b + W_{mn}^S \qquad (5.4.8)$$

It can be observed that the first term on the right hand side of Equation (5.4.8) is the coefficient corresponding to the deflection w^b of an ordinary plate of bending stiffness D, while the second term is the coefficient for the deflection w^S arising from the transverse shear deformability of the core. The total deflection w of the sandwich plate is simply the sum of w^b and w^S.

Thus, upon substitution of Equations (5.4.6) and (5.4.7) into the expression for W_{mn}^b and W_{mn}^S, the deflections w^b and w^S become:

$$w^b = \sum_m \sum_n \frac{256q_o(-1)^{(m+n-2)/2}}{\pi^6 D[mn(\frac{m^2}{a^2} + \frac{n^2}{b^2})]} \cos\frac{m\pi x}{2a} \cos\frac{n\pi y}{2b} \qquad (5.4.9)$$

$$w^S = \sum_m \sum_n \frac{64q_o(-1)^{(m+n-2)/2}}{\pi^6 D[mn(\frac{m^2}{a^2} + \frac{n^2}{b^2})]} \cos\frac{m\pi x}{2a} \cos\frac{n\pi y}{2b} \qquad (5.4.10)$$

The maximum value of these deflections occurs at $x = 0$, $y = 0$. The product of the two cosines in Equations (5.4.9) and (5.4.10) then reduces to unity.

The results given here, i.e., expressions (5.4.9) and (5.4.10) are in complete agreement with those given by Planterma [41].

Numerical values of the nondimensional maximum deflections for various values of b/a are given in Table 29. These numerical values are identical to the ones given in Figure 1 of Ref. [15].

5.5 Concluding Remarks on Vlasov's Method:

From the two illustrative problems solved here, it can be seen that Vlasov's method provides an efficient and accurate solution for plate bending problems. The amount of arithmetic work required in this method is a mere fraction of that required in the conventional Galerkin method, in addition, the choice of a suitable expression to meet the prescribed boundary conditions is simply a matter of choosing the appropriate beam functions. Thus, Vlasovs' method is highly recommended for the solution of complex plate problems when computer facilities are not readily available.

CHAPTER VI

SUMMARY OF CONCLUSIONS

(i) The Collocation Least Square Method:

1) By the application of the least squares concept, the accuracy of the conventional collocation method can be greatly improved.

2) Although the problem of selecting "correct" locations for the collocation points is avoided by the use of a large number of collocation points, these collocation points must however be distributed in a sensibly uniform manner over the entire region of the problem under consideration. Accurate results cannot be expected if all the collocation points are unreasonably crowded into a particular area of the region. Furthermore, for an "interior method", the results would not be as accurate if some of the collocation points fall on the boundary.

3) The collocation least square technique proposed here is a simple yet powerful tool for the solution of complex boundary value problems in the field of applied mechanics.

4) This method seems to yield better results for problems with compact geometries, i.e., square or circular boundaries.

5) The collocation method represents a great saving in human and machine efforts as a result of its simple mathematical concept and the relatively small amount of computer time and storage space required for a solution.

6) In the majority of cases, results obtained by the use of this

method are more accurate than those obtained through the laborious power series solution. Although slight deviations are observed in some of the results, it is believed that such deviations are generally on the conservative side.

(ii) The Orthogonal Collocation Method:

1) The formulation of the orthogonal collocation method is based on the orthogonality, not of the residual function, but of a polynomial which vanishes at the same points. Such a condition is in effect a discrete analogy of Galerkin's method. Conceivably, results obtained from this method should be much more accurate than those obtained from the conventional collocation method which is based more on chance than a sound mathematical theory.

2) The most crucial phase of the solution lies in the construction of the orthogonal polynomial sets, a step that provides both the assumed solution as well as the collocation points. Once these orthogonal polynomial sets are formulated, the solution follows the simple procedures of the conventional collocation method.

3) The solutions obtained from this collocation method would agree completely with Galerkin's method if the differential equation residual is a polynomial of degree $d \leq n$ in x^2.

4) Although this collocation method seems to be slightly more complicated than the collocation least square method, it is more efficient computationwise, and the accuracy is as good if not better than the

collocation least square method.

5) By the application of this method, just as accurate results can be obtained without going through the formidable task of definite integrations required in most error distribution methods.

(iii) Vlasov's Method:

1) By choosing eigenfunctions of vibrating beams or column buckling, the amount of numerical work required in the Galerkin method can be greatly reduced as a result of the orthogonality conditions of these functions.

2) The choice of an appropriate expression for the assumed solution is simply a matter of choosing beam functions with identical boundary conditions as those of the plate.

3) The accuracy of this method is excellent in spite of the relatively little amount of calculations involved.

4) The solution becomes increasingly tedious as the number of terms is increased. However, from the results shown in the previous chapter, it seems that in general, not too many terms are required to produce a solution accurate enough for all practical purposes.

5) This method definitely has its merits in the manual solution of plate problems.

APPENDIX A FIGURES

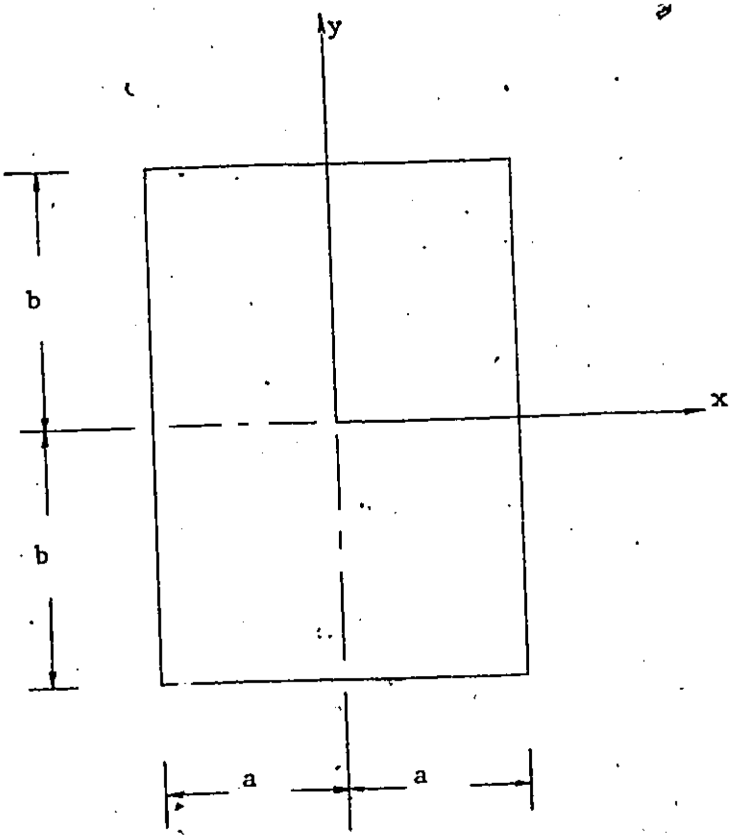


Figure 1 - Rectangular Geometry Defined by a Cartesian Coordinate System.

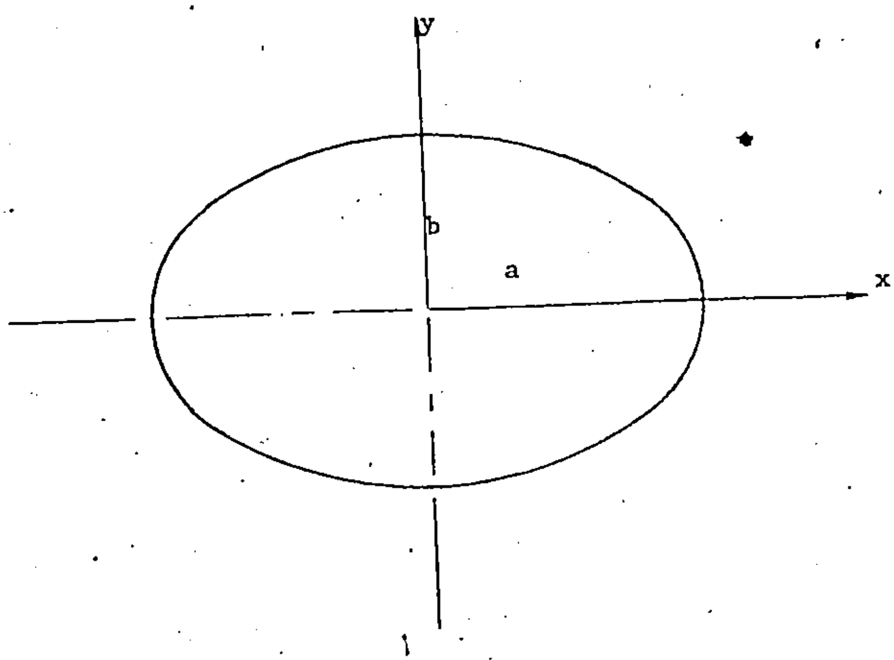


Figure 2 - Elliptical Geometry Defined by a Cartesian Coordinate System

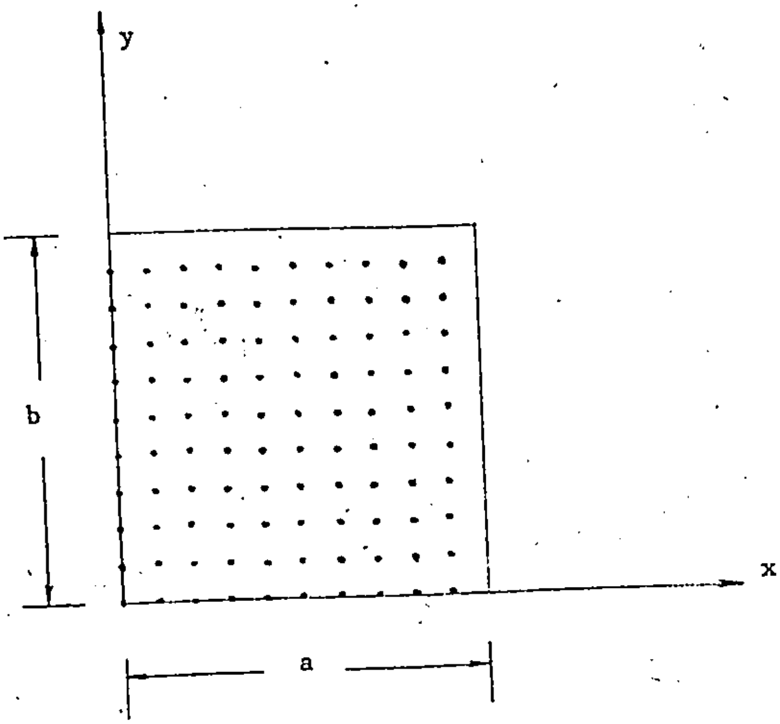


Figure 3 - Distribution of Collocation Points for Rectangular Geometries.

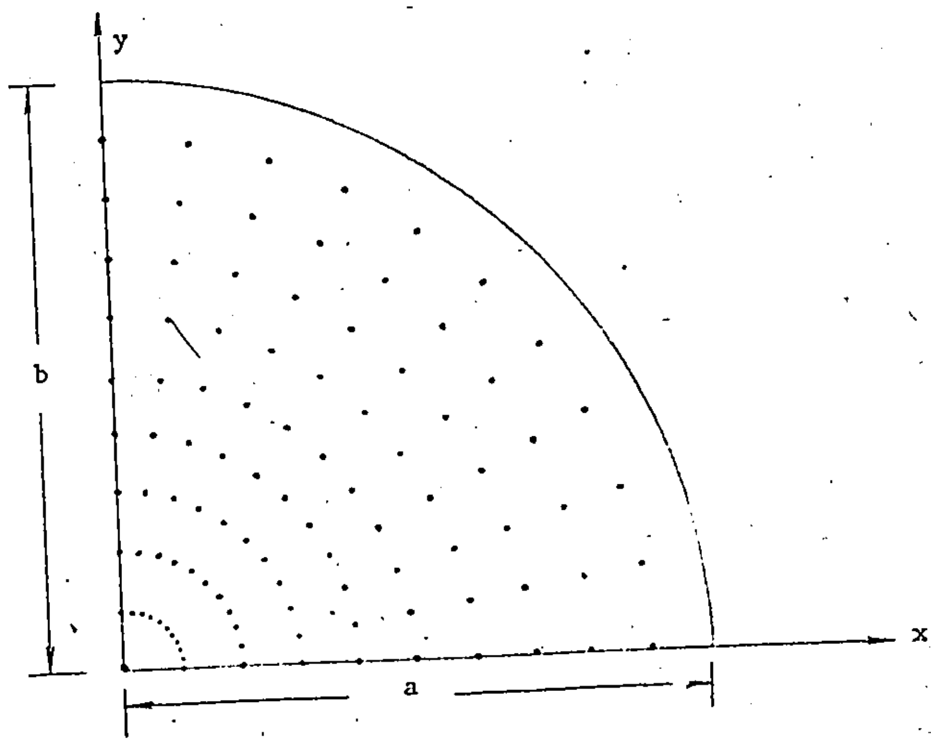


Figure 4 - Distribution of Collocation Points for Circular and Elliptical Geometries.

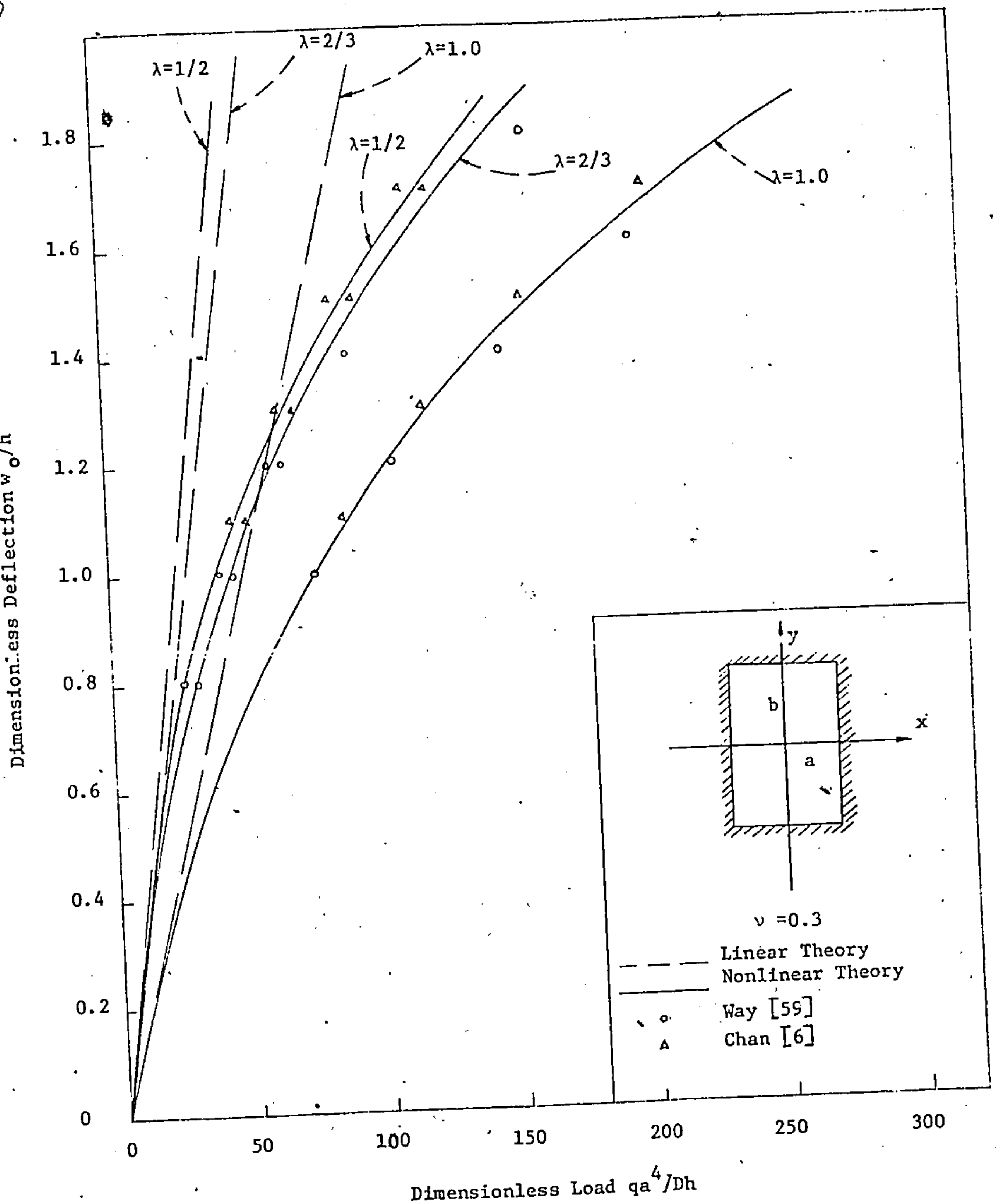


Figure 5 - Variation of Central Deflection with Lateral Pressure for Clamped Rectangular Plates.

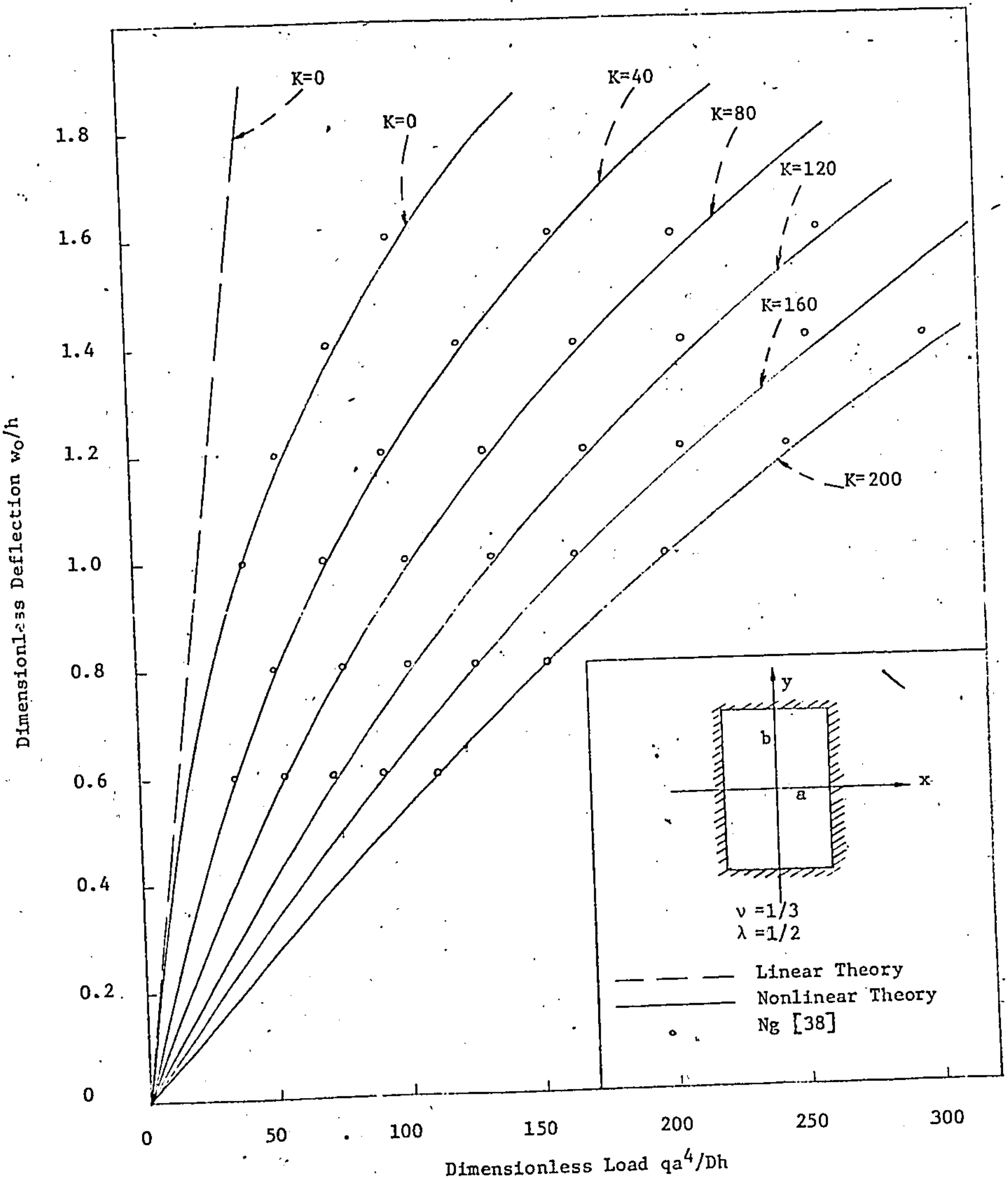


Figure 6 – Variation of Central Deflection with Lateral Pressure and Foundation Modulus for Clamped Rectangular Plates of $\lambda=\frac{1}{2}$

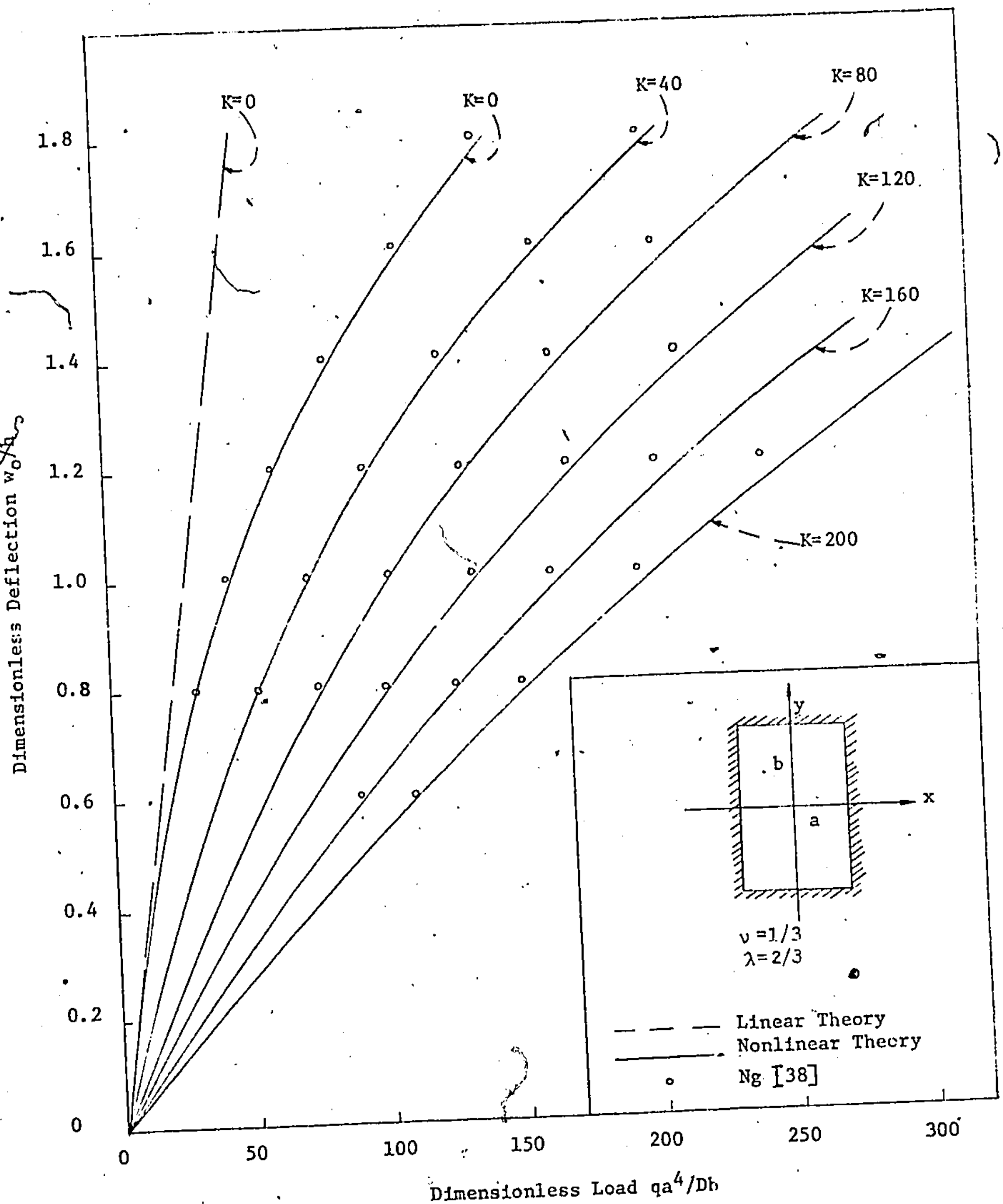


Figure 7 - Variation of Central Deflection with Lateral Pressure and Foundation Modulus for Clamped Rectangular Plates of $\lambda=2/3$

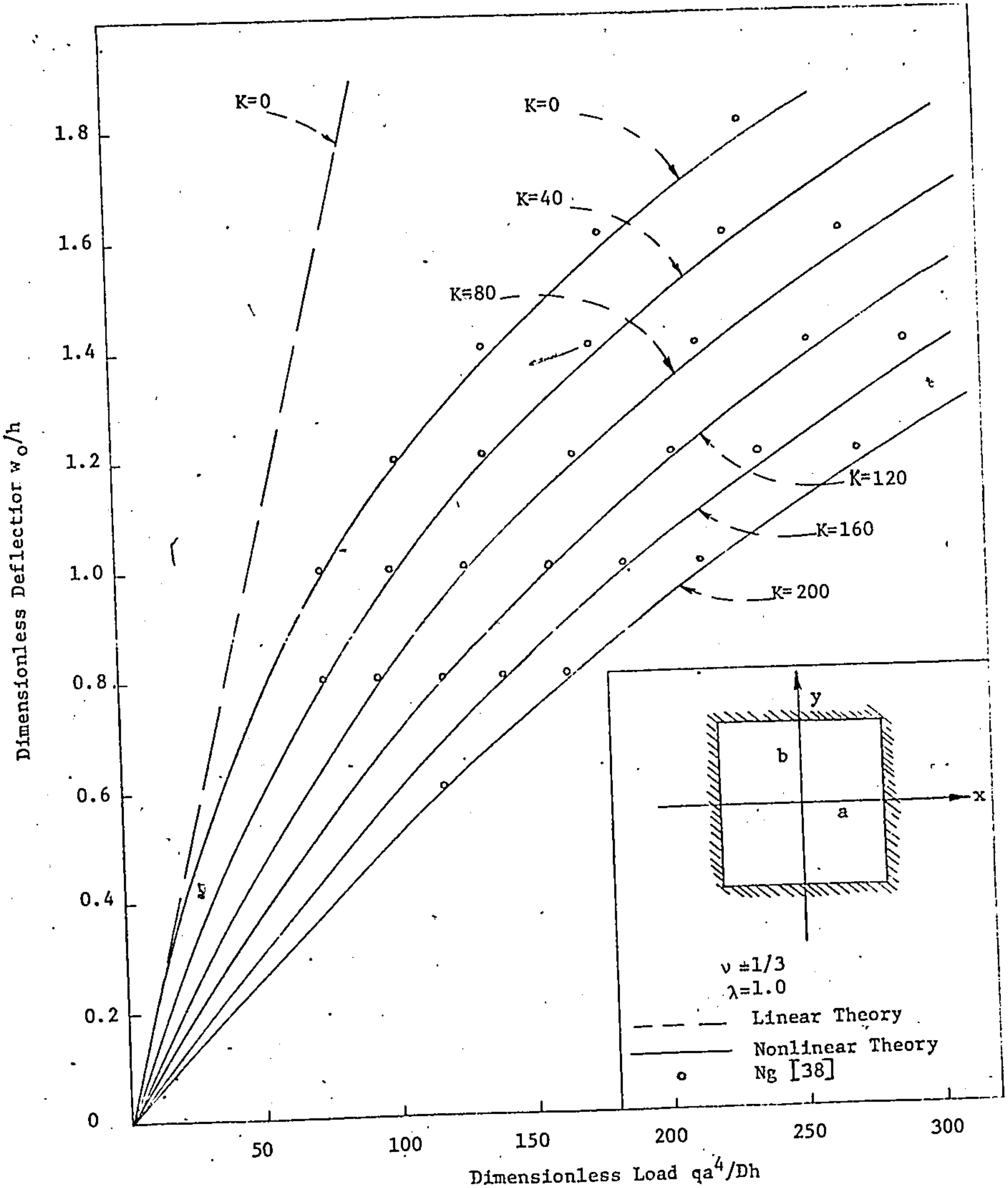


Figure 8 - Variation of Central Deflection with Lateral Pressure and Foundation Modulus for Clamped Rectangular Plates of λ=1.0

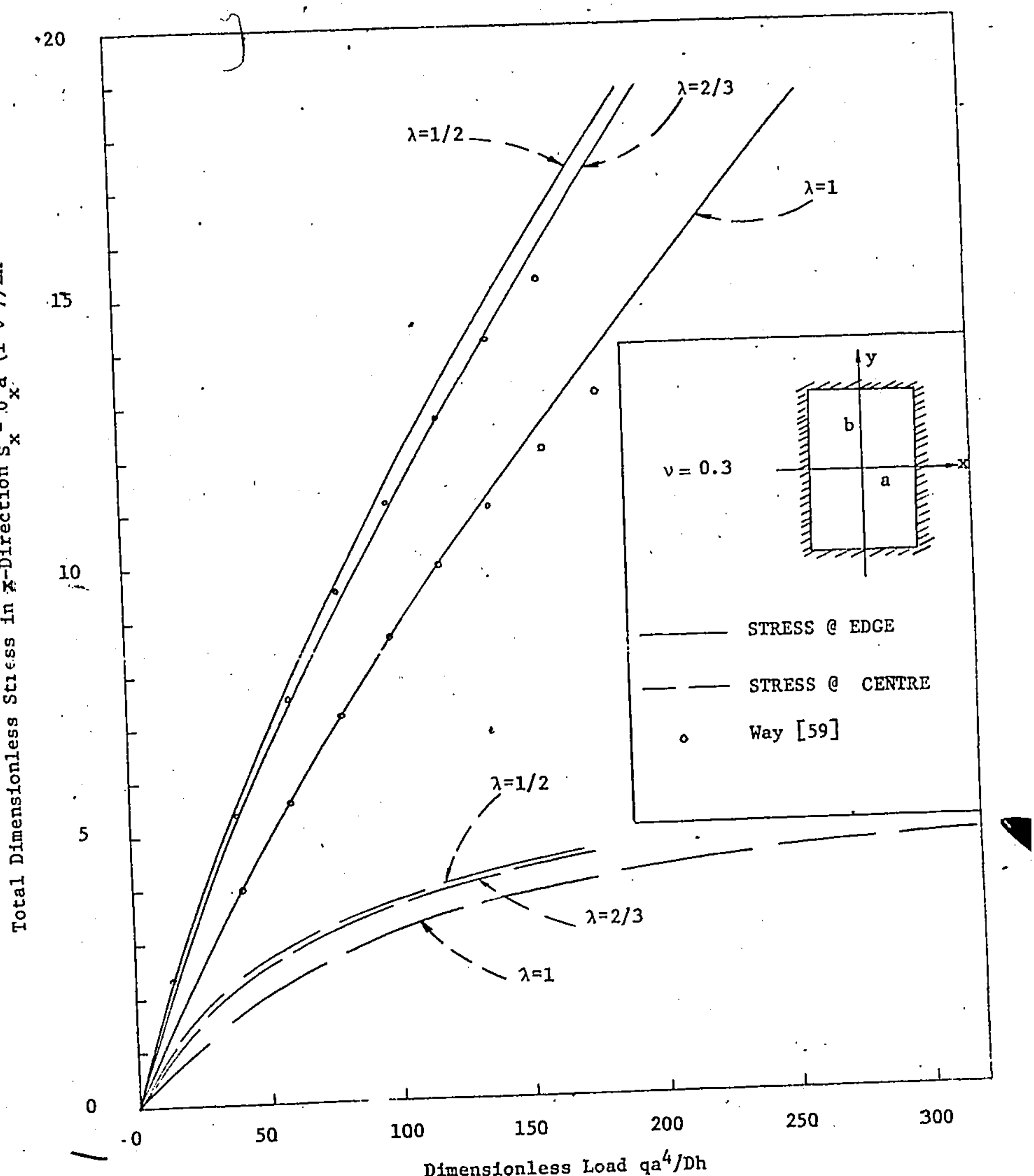


Figure 9 - Variation of Total Maximum Edge and Centre Stress with Lateral Pressure for Clamped Rectangular Plates.

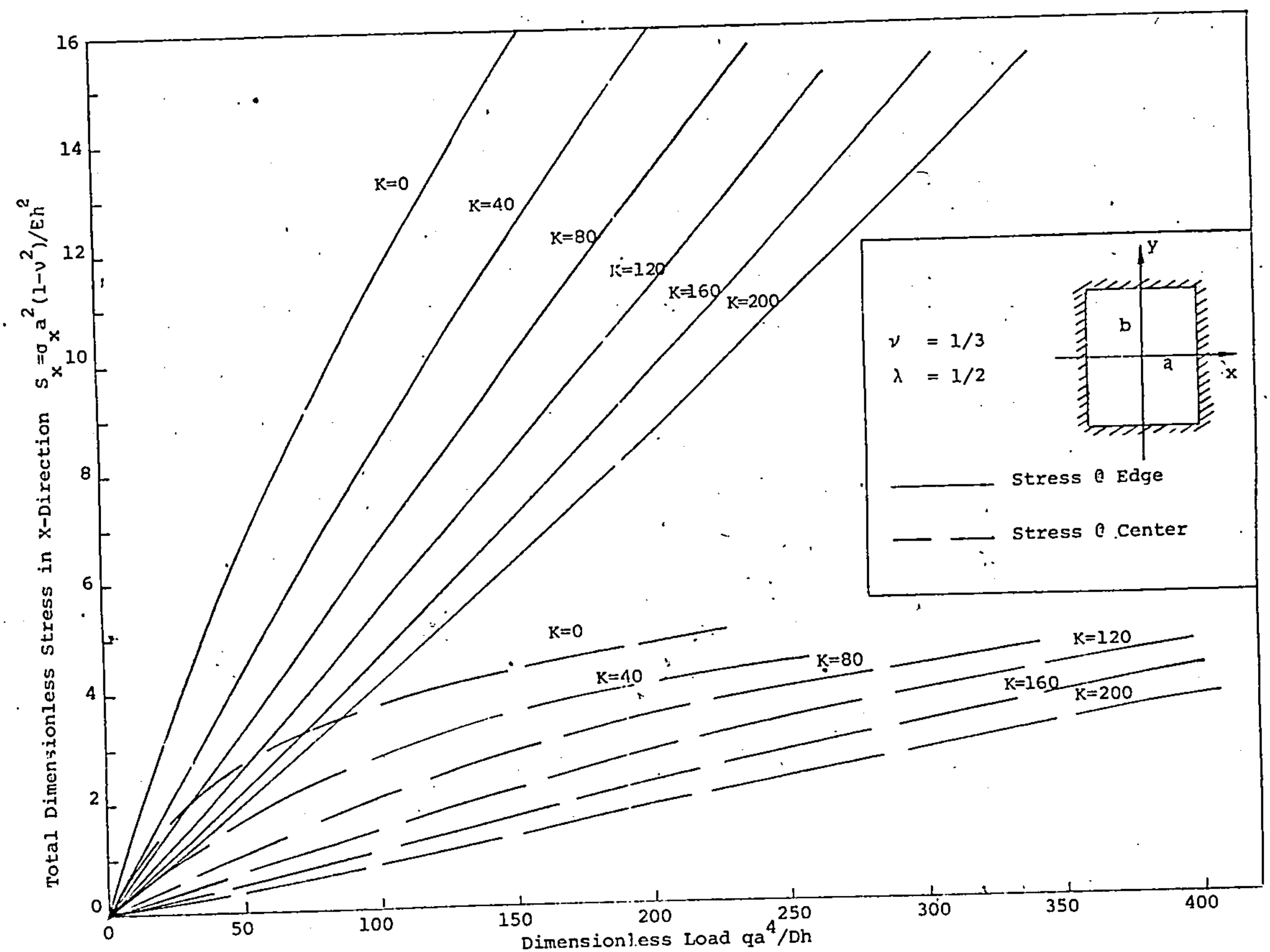


Figure 10 Variation of Total Maximum Edge and Centre Stress with Lateral Pressure and Foundation Modulus for Clamped Rectangular Plates of $\lambda = 1/2$.

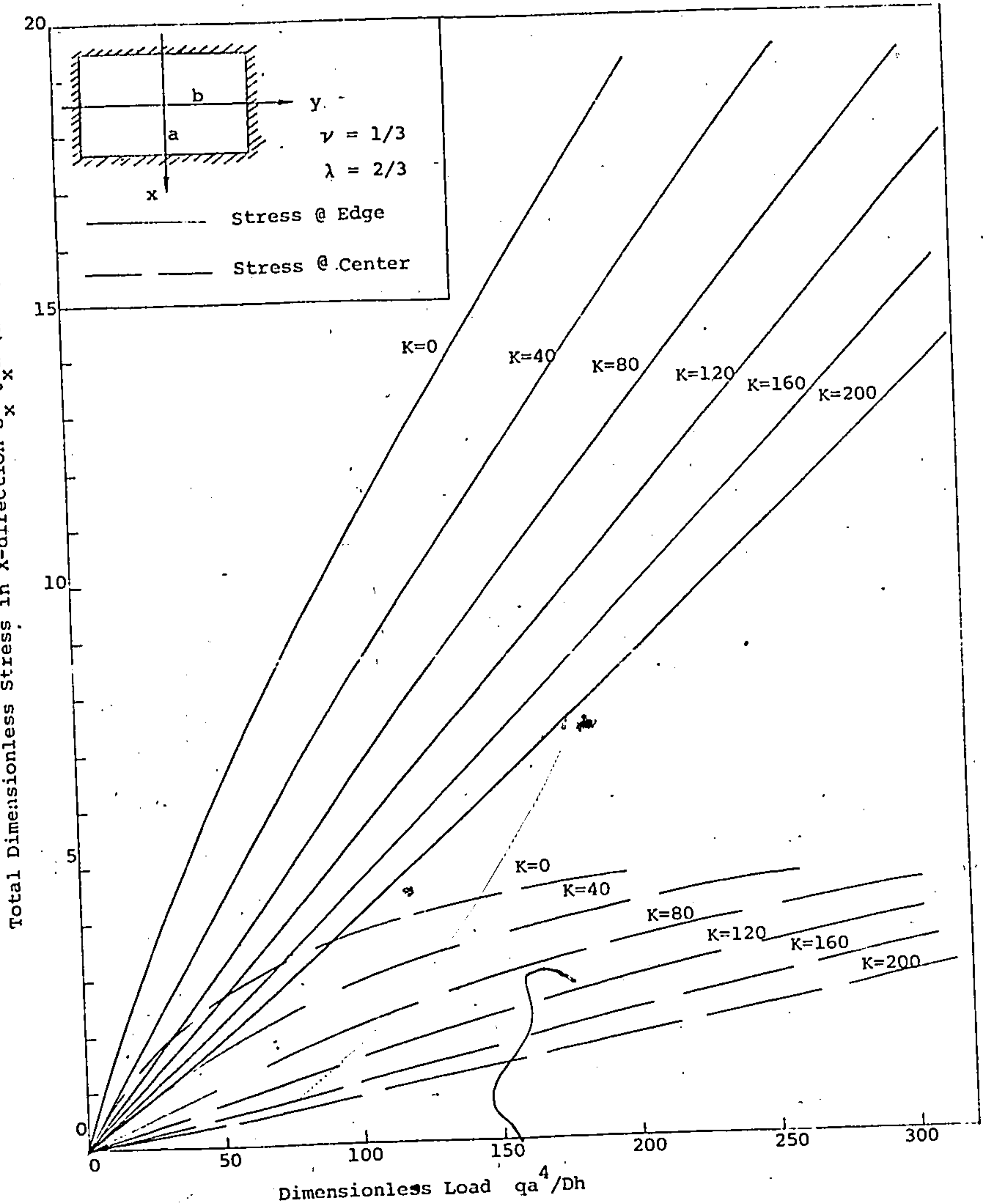


Figure 11 Variation of Total Maximum Edge and Centre Stress with with Lateral Pressure and Foundation Modulus for Clamped Rectangular Plates of λ = 2/3.

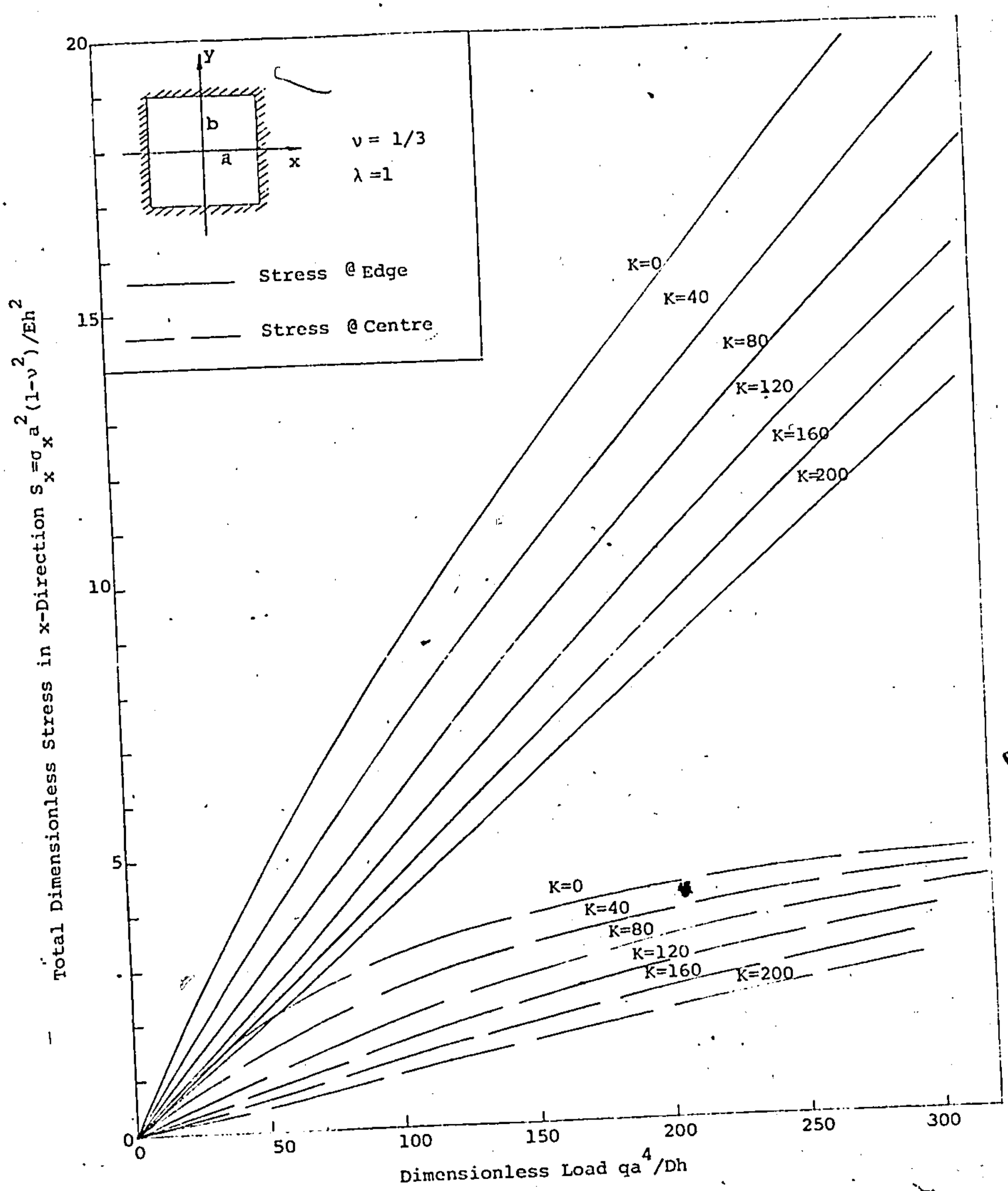


Figure 12 Variation of Total Maximum Edge and Centre Stress with Lateral Pressure and Foundation Modulus for Clamped Square Plates.

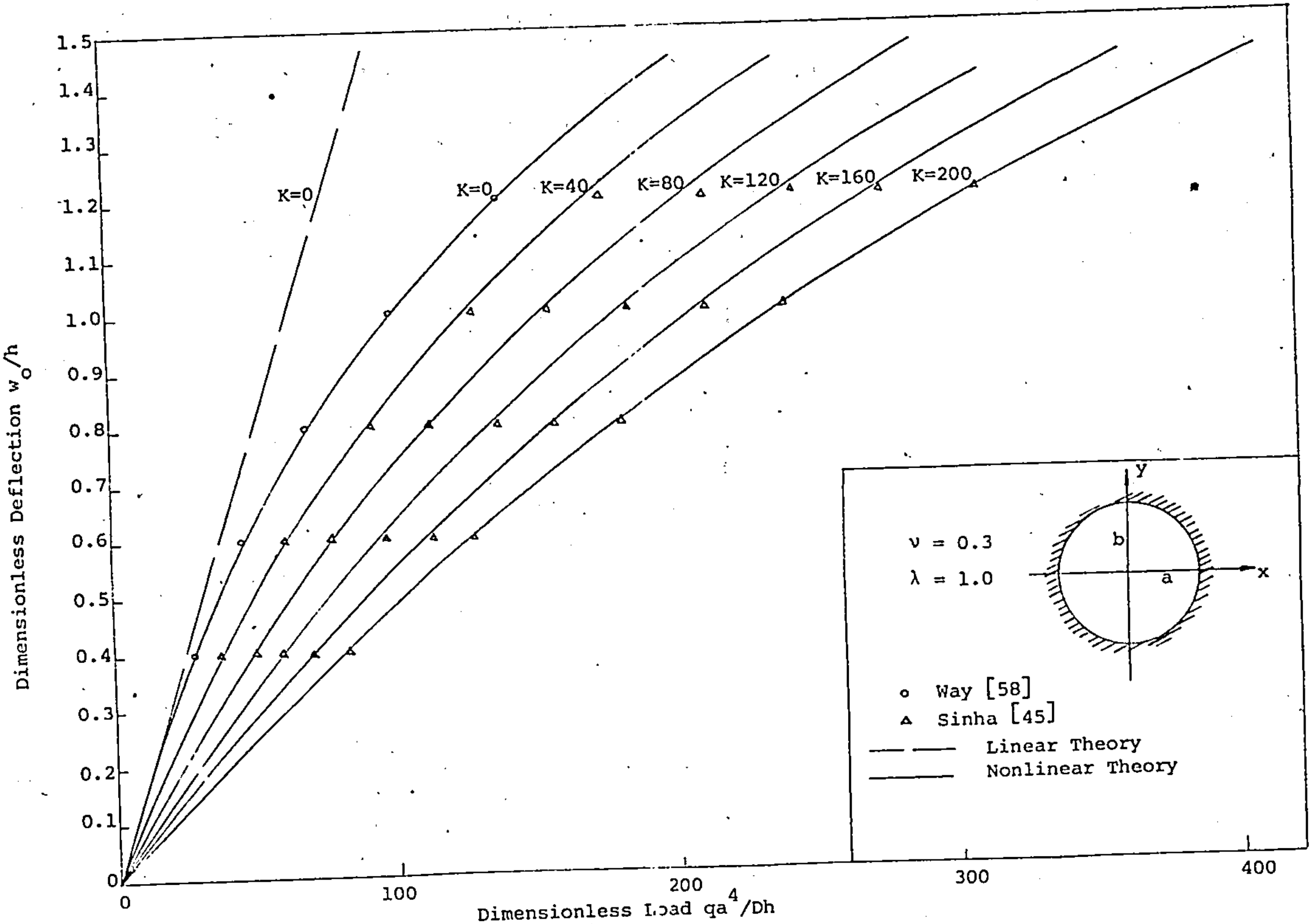


Figure 13 Variation of Central Deflection with Lateral Pressure and Foundation Modulus for Clamped Circular Plates.

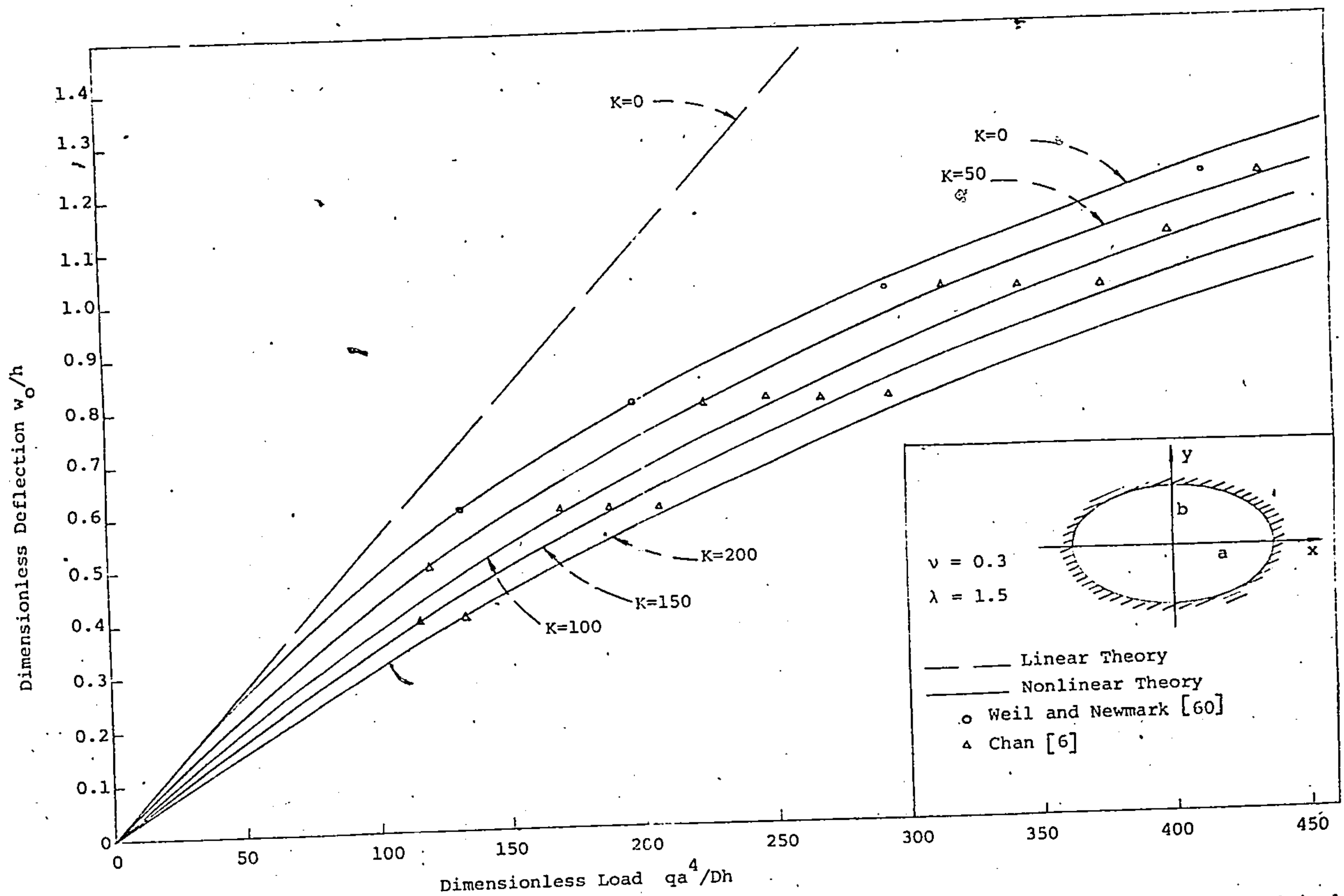


Figure 14 Variation of Central Deflection with Lateral Pressure and Foundation Modulus for Clamped Ellipitical Plates of λ = 1.5.

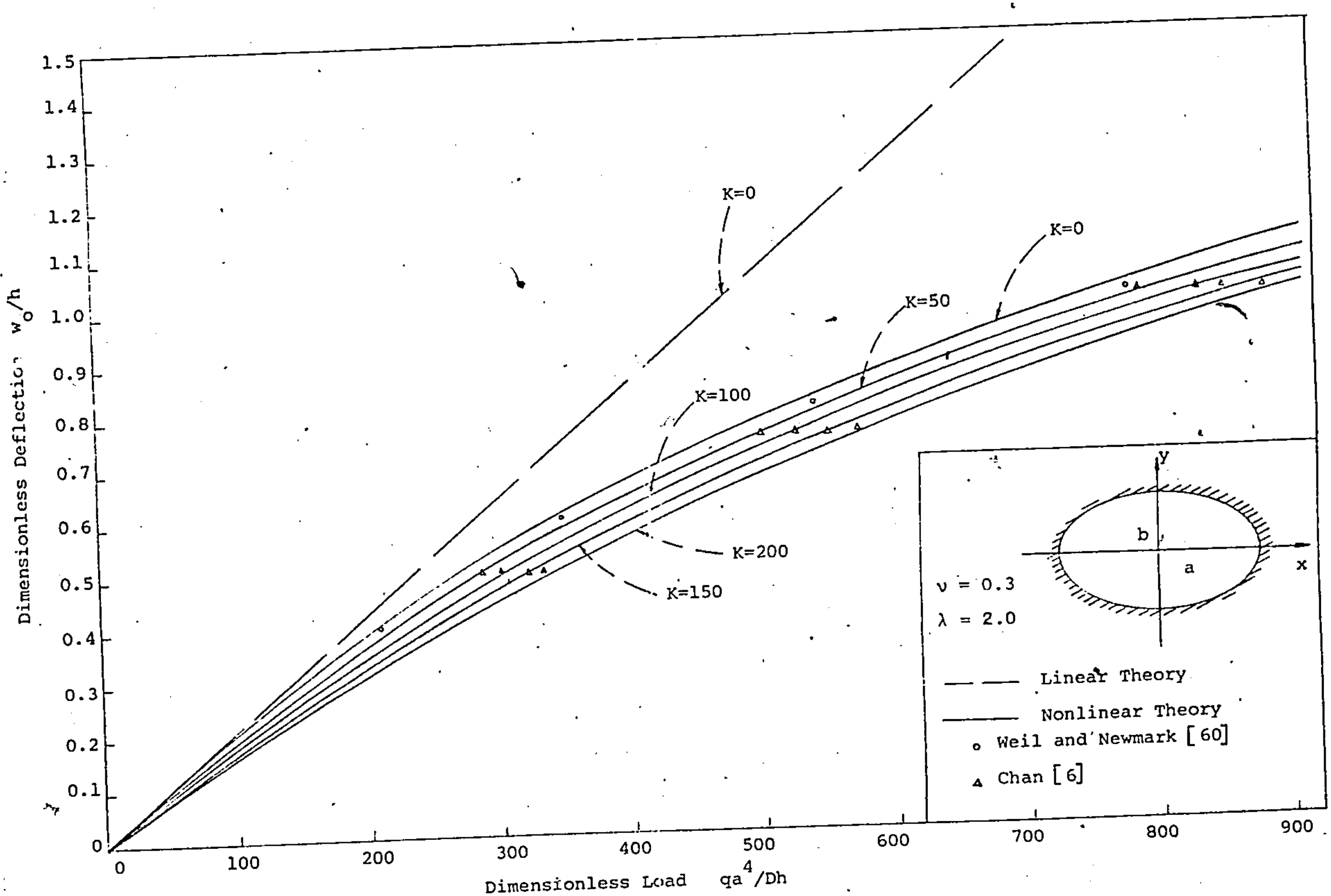


Figure 15 Variation of Central Deflection with Lateral Pressure and Foundation Modulus for Clamped Ellipitical plates of λ =2.0.

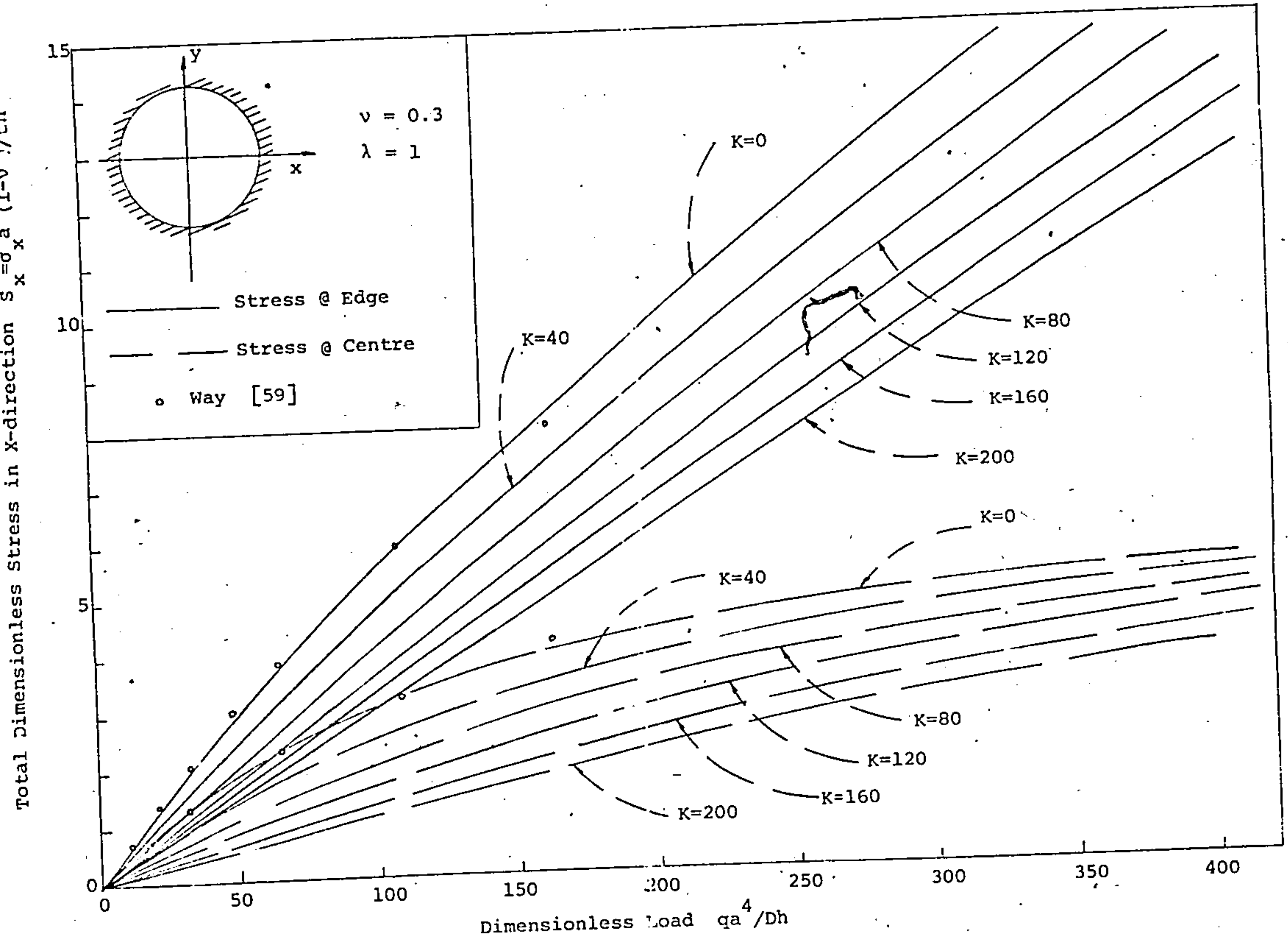


Figure 16 Variation of Total Maximum Edge and Centre Stress with Lateral Pressure and Foundation Modulus for Clamped Circular Plates.

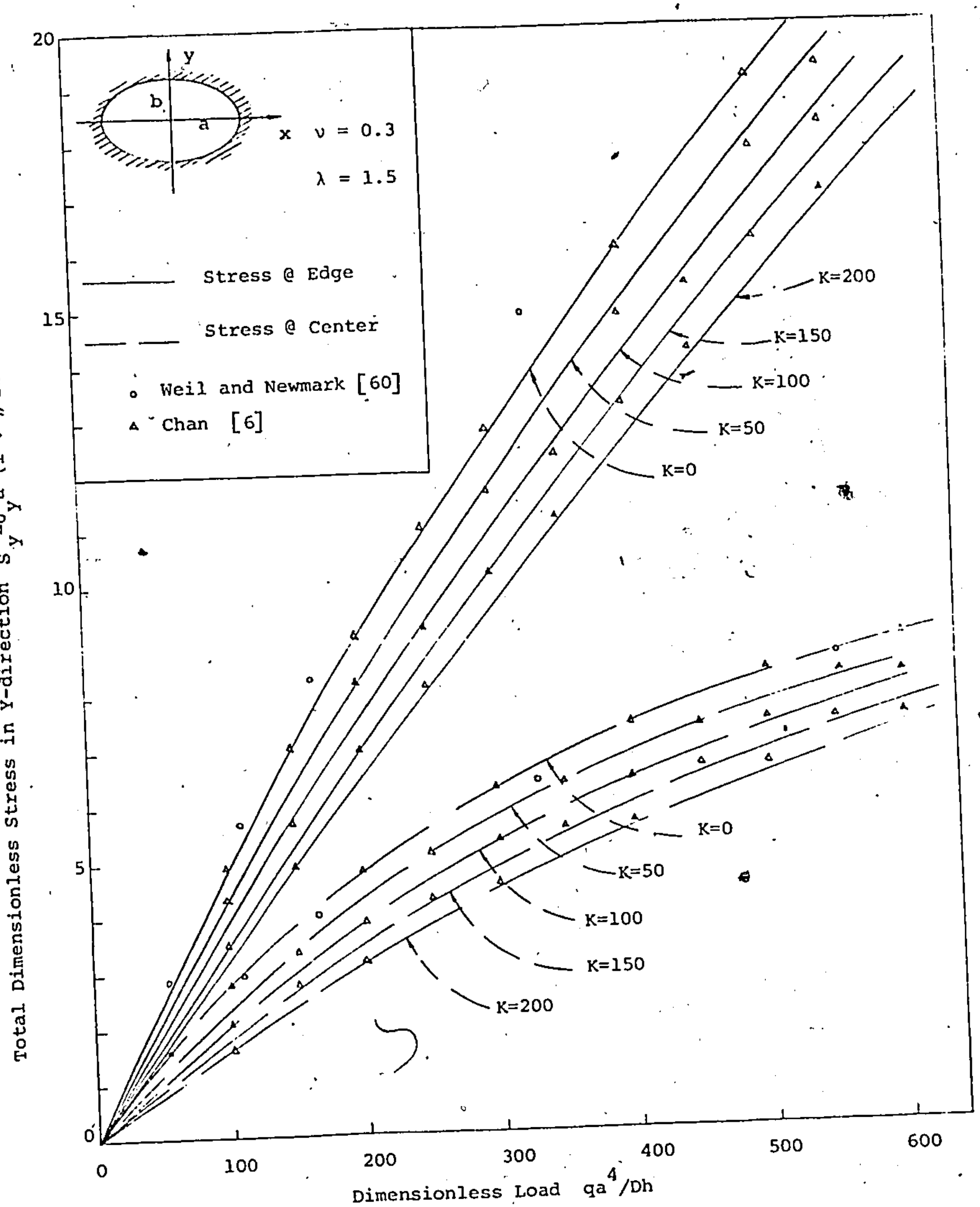


Figure 17 Variation of Total Maximum Edge and Centre Stress with Lateral Pressure and Foundation Modulus for Clamped Elliptical Plates of $\lambda = 1.5$.

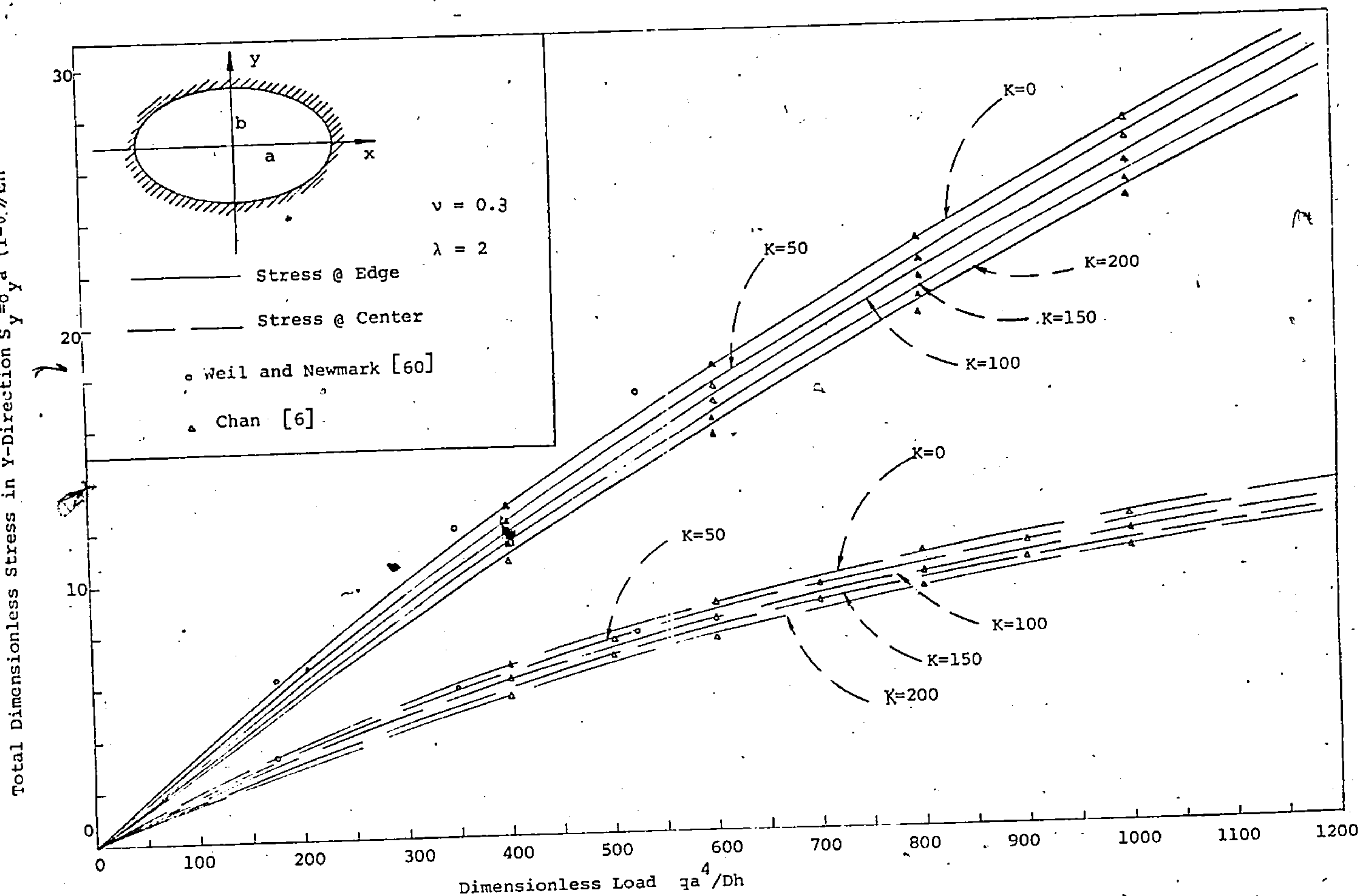


Figure 18 Variation of Total Maximum Edge and Centre Stress with Lateral Pressure and Foundation Modulus for Clamped Elliptical Plates of λ =2.0.

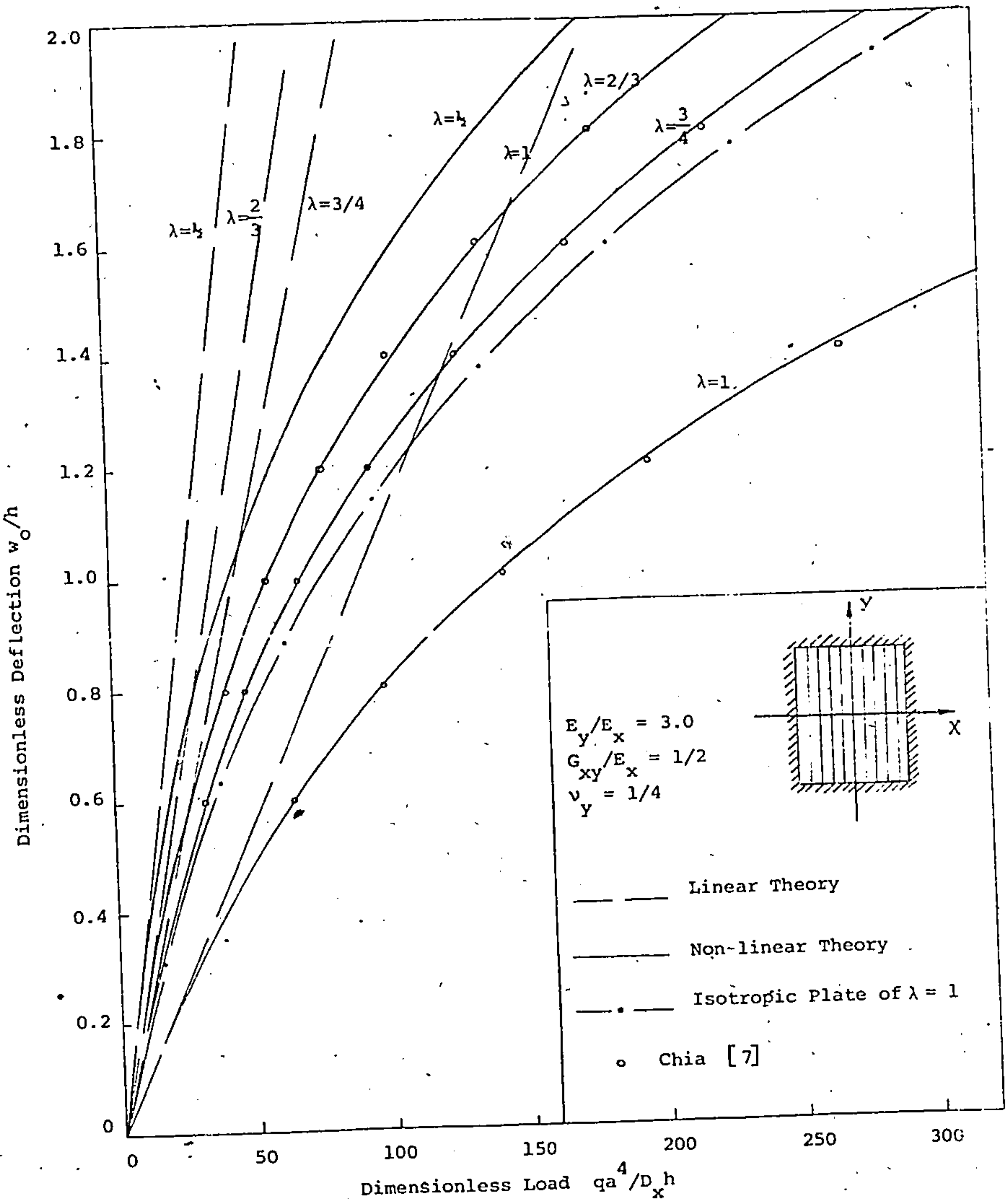


Figure 19 Variation of Central Deflection with Lateral Pressure for Clamped Gasll-Epoxy Rectangular Plates.

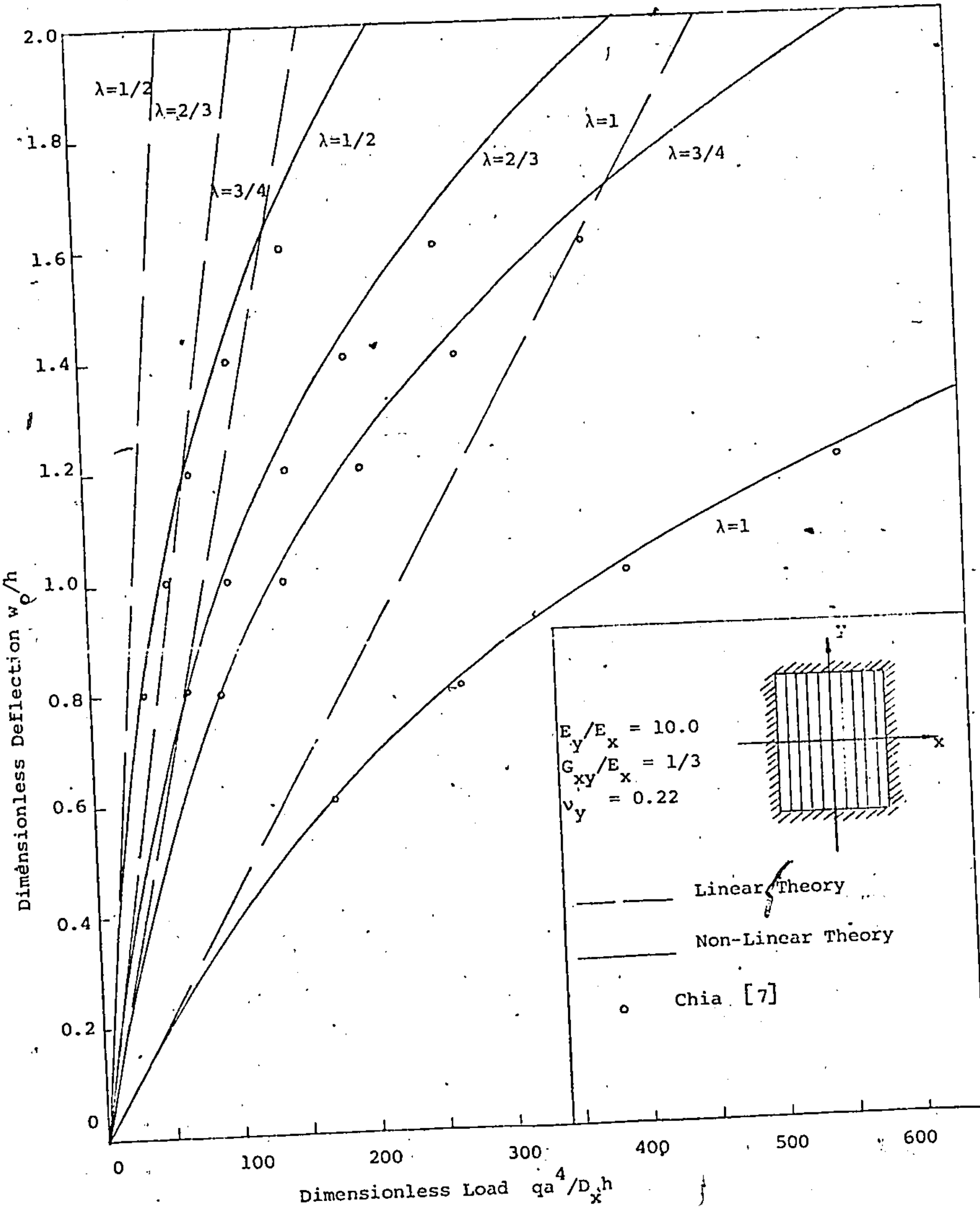


Figure 20 Variation of Central Deflection with Lateral Pressure for Clamped Boron-Epoxy Rectangular Plates.

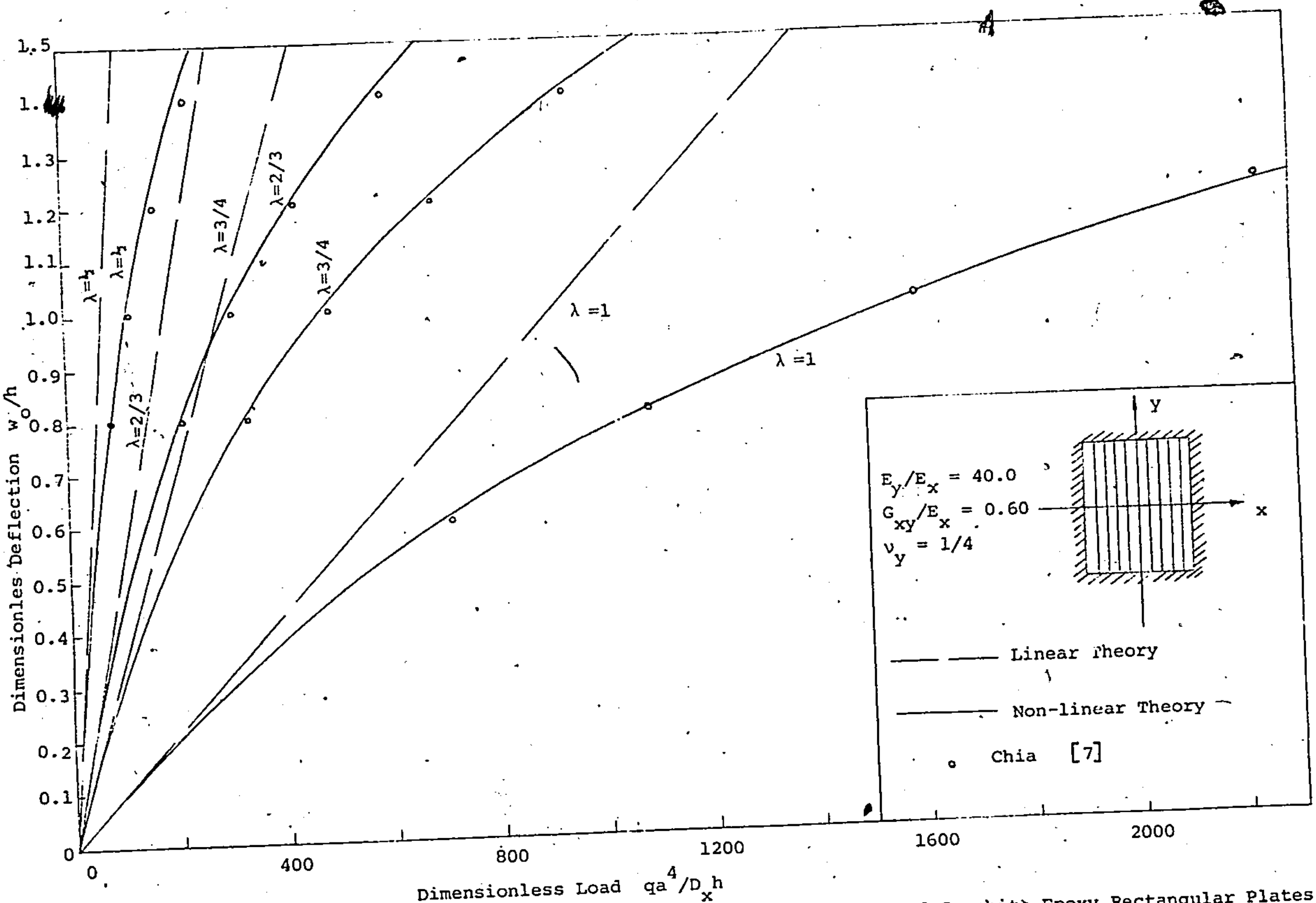


Figure 21 Variation of Central Deflection with Lateral Pressure for Clamped Graphite-Epoxy Rectangular Plates.

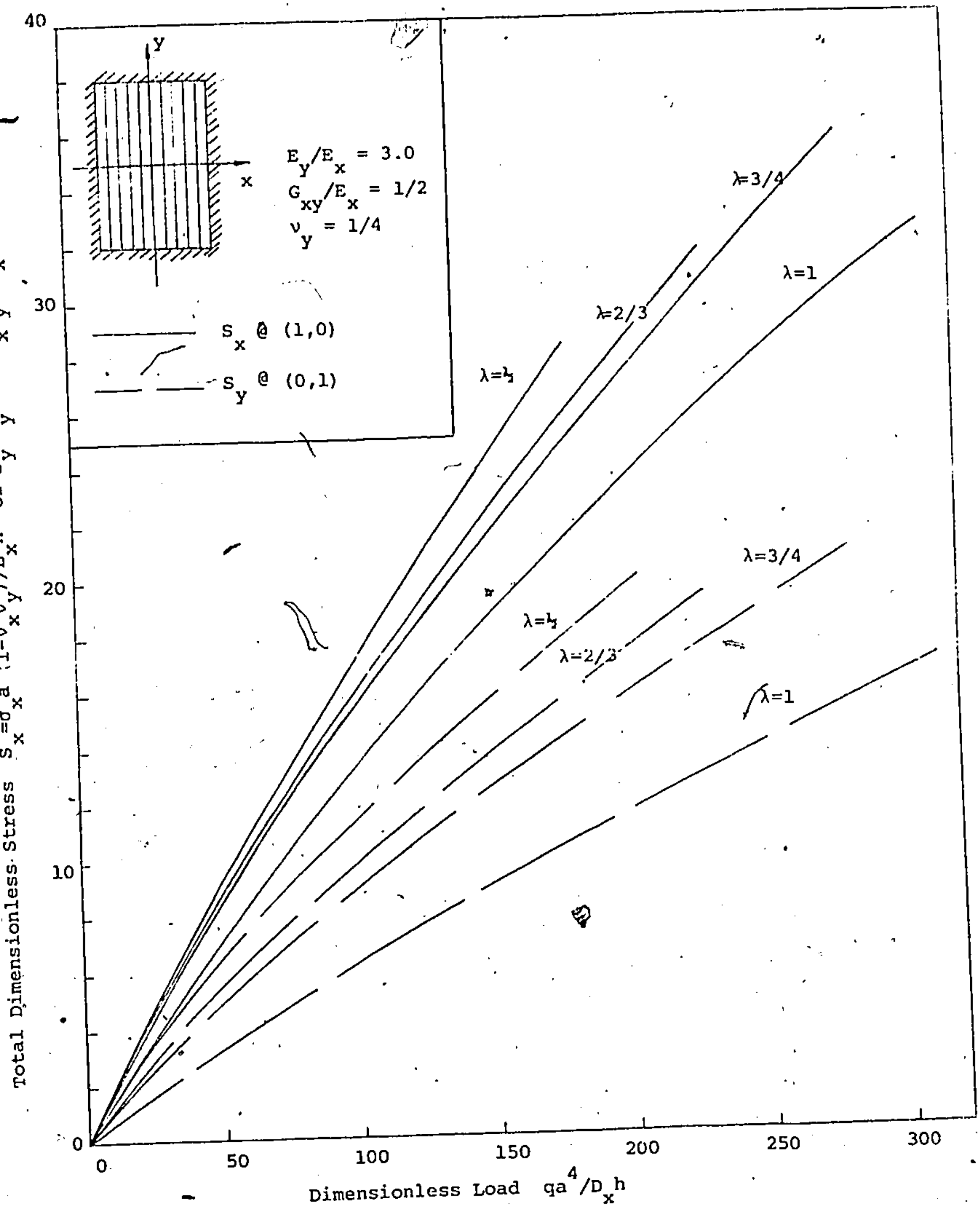


Figure 22 Variation of Total Maximum Edge Stress with Lateral Pressure for Clamped Glass-Epoxy Rectangular Plates.

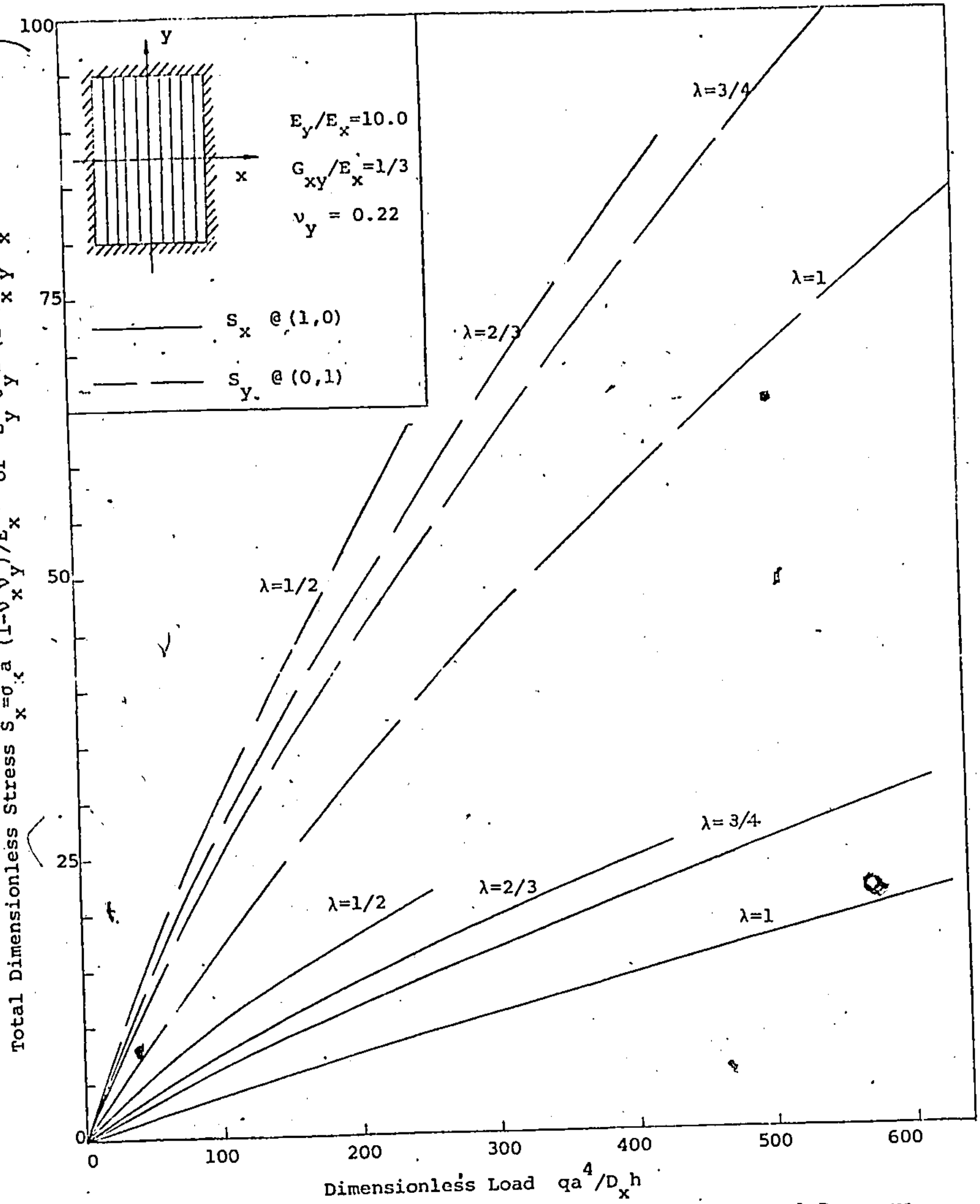


Figure 23 Variation of Total Maximum Edge Stress with Lateral Pressure for Clamped Boron-Epoxy Rectangular Plates.

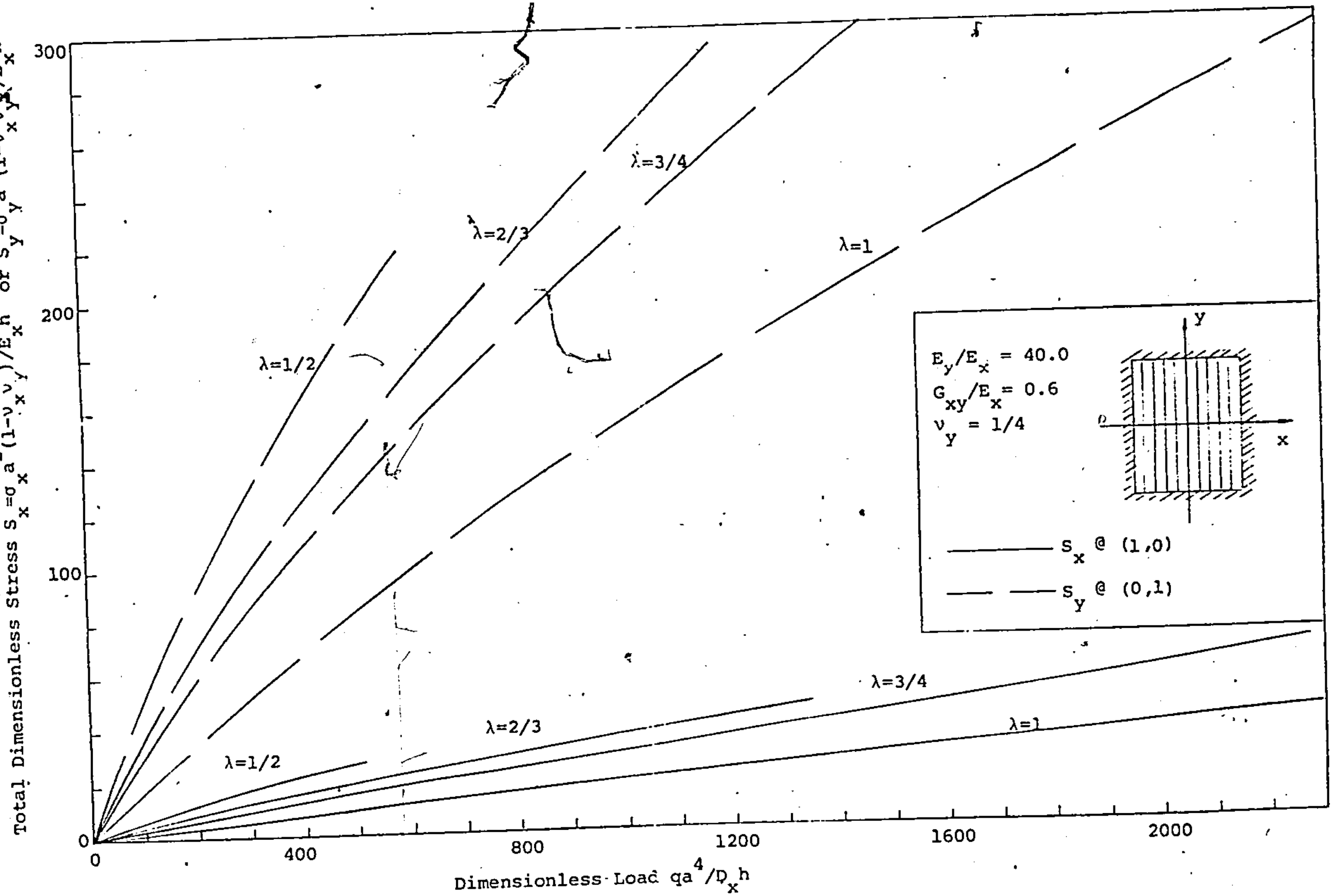
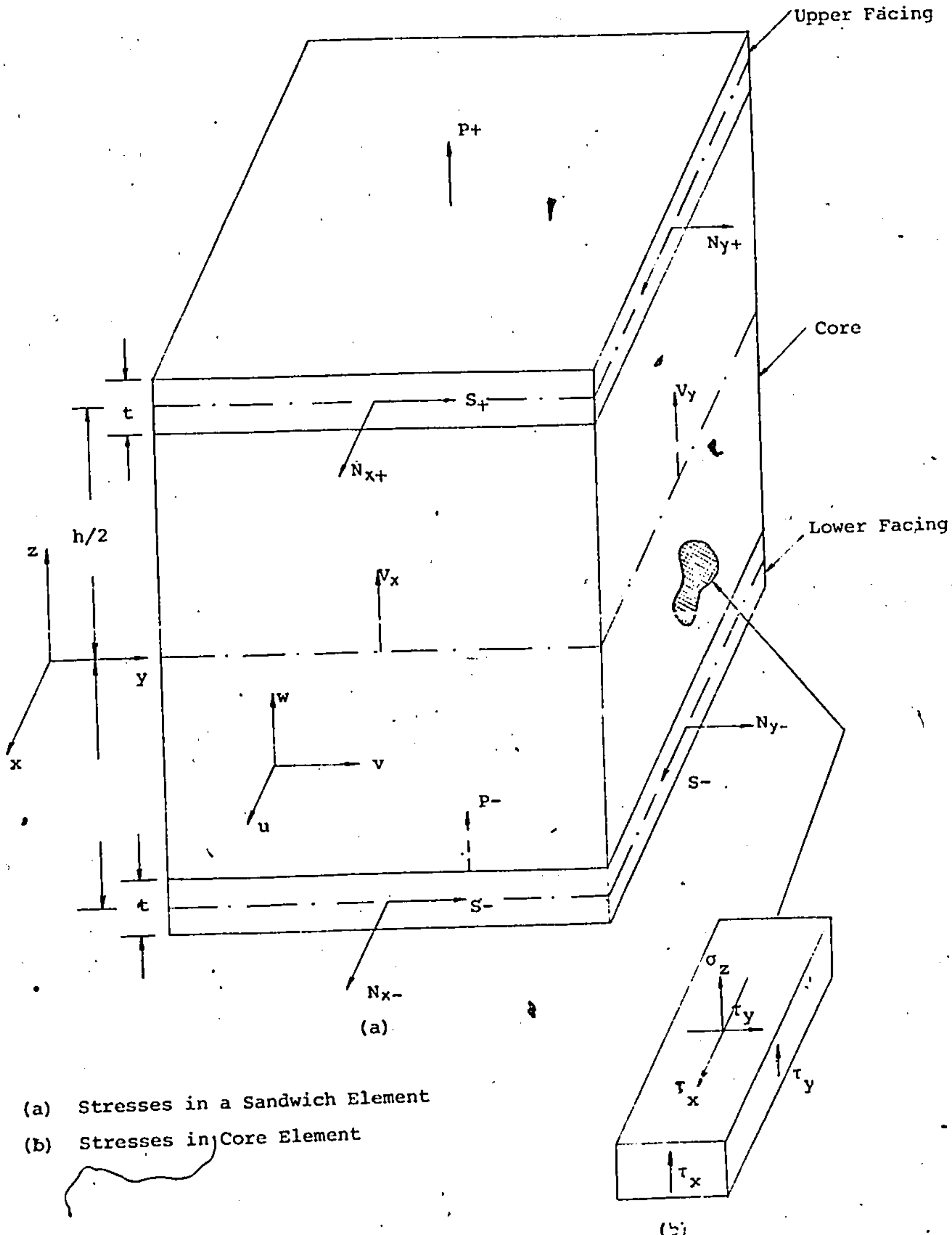


Figure 24 Variation of Total Maximum Edge Stress with Lateral Pressure for Clamped Graphite-Epoxy Rectangular Plates.



(a) Stresses in a Sandwich Element

(b) Stresses in Core Element

Figure 25 Stresses in an Element of a Sandwich Plate

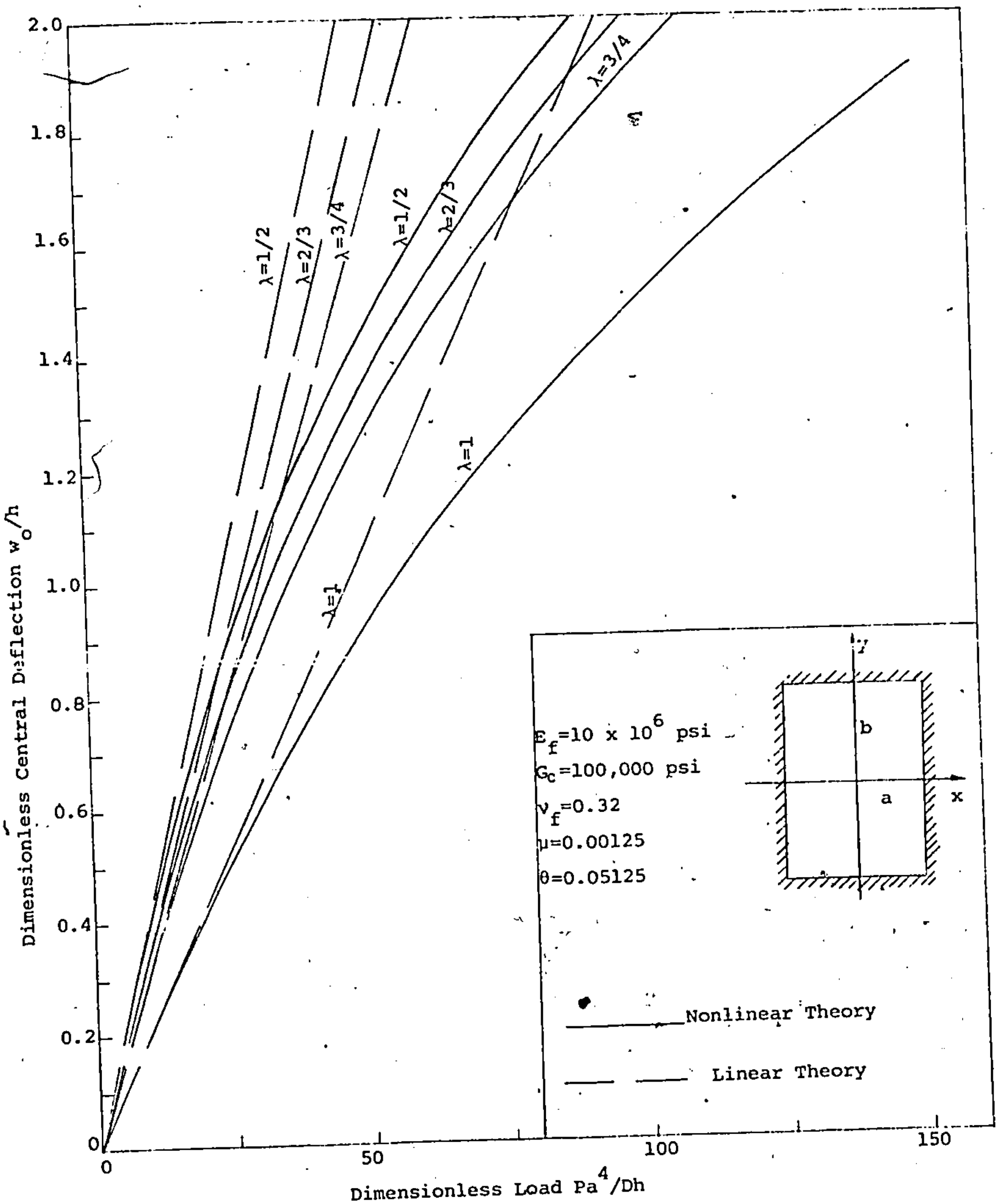


Figure 26 Variation of Central Deflection with Lateral Pressure for Sandwich Plate No. 2.

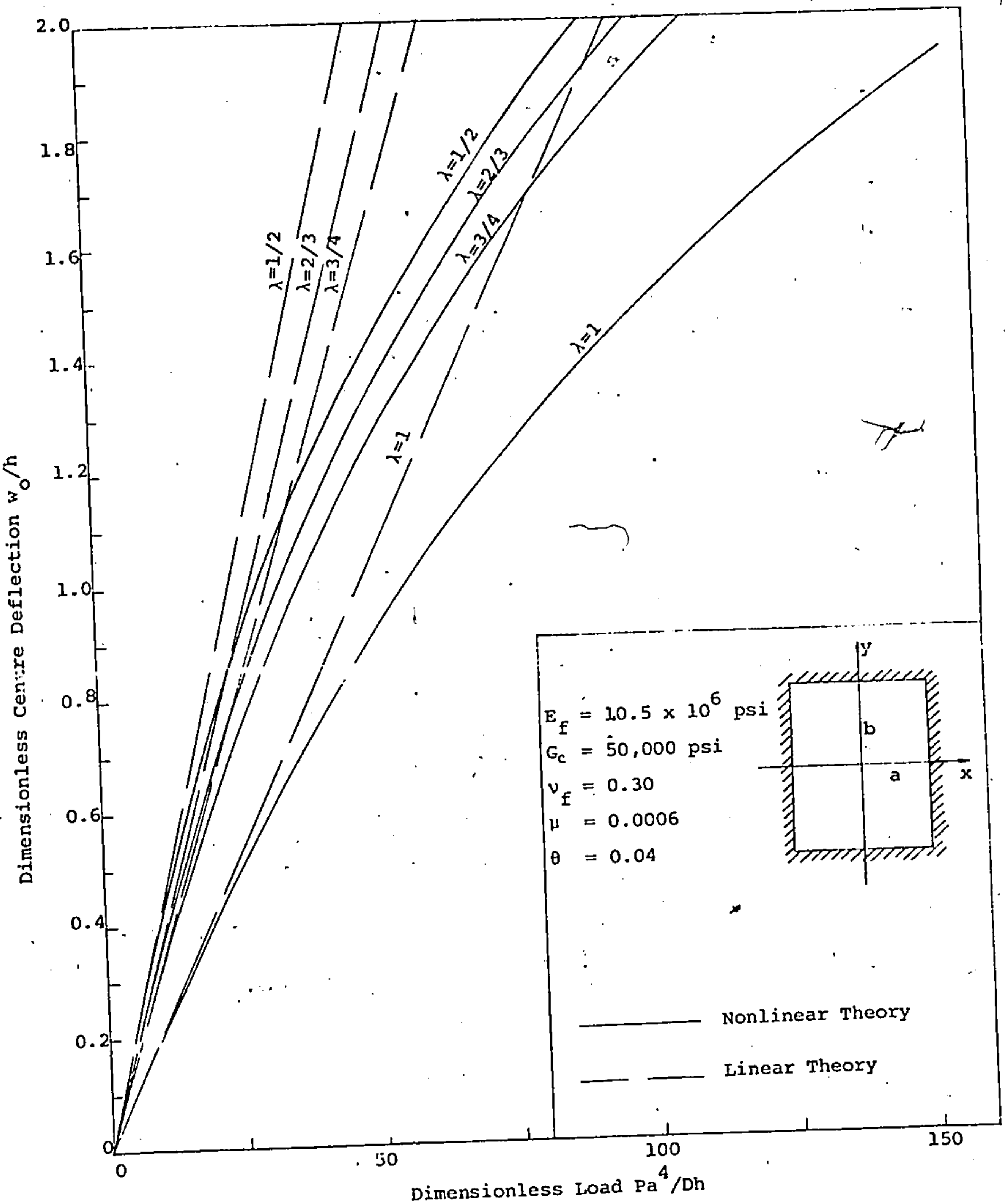


Figure 27 Variation of Central Deflection with Lateral Pressure for Sandwich Plate No. 3.

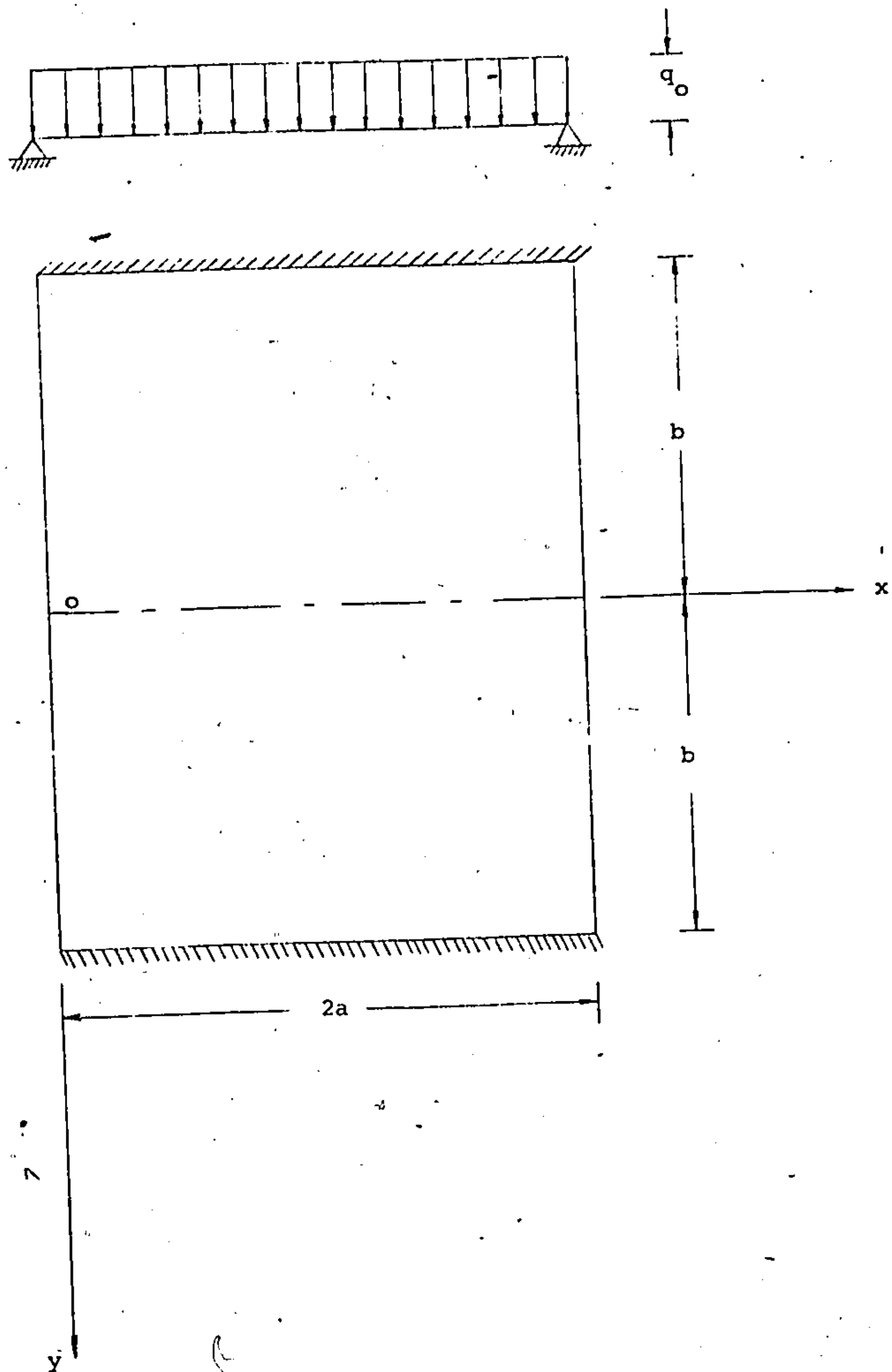


Figure 28 Uniformly Loaded Rectangular Plate with Two Opposite Edges Simply Supported and the Other Two Edges Clamped.

APPENDIX B TABLES

$\lambda = b/a$; $\tau_{max} = C_1(2G\,a)$; $M_t = C_2 G\theta (2a)^3 (2b)$

λ	COLLOCATION LEAST SQUARE METHOD						TIMOSHENKO REF. [50]	
	C_1			C_2			C_1	C_2
	*	**	***	*	**	***		
1.0	0.671	0.672	0.671	0.141	0.141	0.141	0.675	0.141
1.2	0.750	0.751	0.750	0.166	0.166	0.166	0.759	0.166
1.5	0.835	0.835	0.834	0.196	0.196	0.196	0.848	0.196
2.0	0.915	0.915	0.914	0.229	0.229	0.229	0.930	0.229
2.5	0.954	0.954	0.954	0.249	0.249	0.249	0.968	0.249
3.0	0.975	0.975	0.975	0.263	0.263	0.263	0.985	0.263
4.0	0.997	0.998	0.998	0.281	0.281	0.281	0.997	0.281
5.0	1.013	1.014	1.013	0.292	0.292	0.292	0.999	0.291
10.0	1.057	1.061	1.060	0.312	0.312	0.312	1.000	0.312

*, **, ***, indicates results obtained from solution using 150, 175 and 200 collocation points, respectively.

Table 1- Results of the Collocation Least Square Method Applied to Torsion of Rectangular Bars. Coefficients C_1 and C_2 for Maximum Shear Stress τ_{max} and Torque m_t respectively for various ratios of Sides b/a.

ASPECT RATIO $\lambda = a/b$	NO. OF COLLOCATION POINTS USED			TIMOSHENKO REF. [49]
	25	50	100	
1/2	0.040510	0.040510	0.040545	0.04064
2/3	0.035139	0.035090	0.035090	0.03520
3/4	0.031470	0.031422	0.031412	---
1	0.020243	0.020221	0.020201	0.02016

$$W_{max} = \alpha \frac{qa^4}{D}$$

Table 2 - Variation of the Maximum Small-Deflection Coefficient α with the Number of Collocation Points used in the Solution for Rectangular Plates with Built-in Edges.

NO. OF COLLOCATION POINTS USED	ASPECT RATIO $\lambda = a/b$					
	1/2		2/3		1	
	q_1	q_3	q_1	q_3	q_1	q_3
25	24.6855	15.4644	28.4583	15.7517	49.4000	25.4105
50	24.6855	15.4722	28.4983	15.9116	49.4543	25.8136
100	24.6640	15.5199	28.4982	16.0658	49.5018	26.2275

($\nu = 0.3$)

Table 3 - Variation of the Constants q_1 and q_3 with the Number of Collocation Points used in the Solution for the Large-Deflection of Rectangular Plates with Built-in Edges.

DIMENSIONLESS FOUNDATION MODULUS K	ASPECT RATIO λ = a/b			
	1/2	2/3	3/4	1
0	0.040545	0.035090	0.031412	0.020201
20	0.025525	0.023807	0.022201	0.016033
40	0.018411	0.017840	0.017036	0.013245
60	0.014307	0.014167	0.013741	0.011251
80	0.011656	0.011689	0.011463	0.009756
100	0.009812	0.009910	0.009797	0.008594
120	0.008459	0.008575	0.008530	0.007665
140	0.007427	0.007540	0.007534	0.006903
160	0.006616	0.006713	0.006734	0.006377
180	0.005963	0.006041	0.006076	0.005745
200	0.005425	0.005484	0.005528	0.005291

$$w_{max} = \alpha \frac{qa^4}{D}$$

Table 4 - Variation of the Maximum Small-Deflection Coefficient α of Clamped Rectangular Plates with the Dimensionless Foundation Modulus K.

DIMENSIONLESS FOUNDATION MODULUS K	ASPECT RATIO = a/b					
	1/2		2/3		1	
	q_1	q_3	q_1	q_3	q_1	q_3
0	24.664	15.555	28.498	16.147	49.502	26.423
40	54.314	17.922	56.053	18.040	75.499	27.722
80	85.790	18.798	85.554	19.433	102.502	28.881
120	118.216	18.813	116.620	20.393	130.456	29.906
160	151.145	18.830	148.959	20.983	159.310	30.801
200	184.325	18.850	182.342	21.254	189.013	31.571

$(\nu = 1/3)$

Table 5 - Variation of the Constants q_1 and q_3 for the Large-Deflection of Clamped Rectangular Plates with the Dimensionless Foundation Modulus K.

NO. OF COLLOCATION POINTS USED	ASPECT RATIO $\lambda = a/b$					
	1		1.5		2.0	
	q_1	q_3	q_1	q_3	q_1	q_3
25	64.0000	35.3674	181.5000	102.6214	472.0000	279.0387
50	64.0000	35.28317	181.5000	102.4089	472.0000	278.5391
100	64.0000	35.2973	181.5000	102.4199	472.0000	278.4600

(ν = 0.3)

Table 6 - Variation of the Constants q_1 and q_3 with the Number of Collocation Points used in the Solution for the Large-Deflection of Elliptical and Circular Plates with Built-in Edges.

K	$\lambda = 1.0$		$\lambda = 1.25$		$\lambda = 1.5$		$\lambda = 1.75$		$\lambda = 2.0$	
	PRESENT SOLUTION	NG REF. [39]	PRESENT SOLUTION	NG REF. [39]	PRESENT SOLUTION	NG REF. [39]	PRESENT SOLUTION	NG REF. [39]	PRESENT SOLUTION	NG REF. [39]
0	1.5625	1.5625	0.9294	0.9294	0.5510	0.5510	0.3355	0.3355	0.2119	0.2118
20	1.3009	1.3017	0.8298	0.8300	0.5139	0.5140	0.3211	0.3212	0.2060	0.2060
40	1.1120	1.1143	0.7488	0.7495	0.4814	0.4816	0.3079	0.3080	0.2004	0.2004
60	0.9694	0.9730	0.6817	0.6998	0.4526	0.4529	0.2957	0.2958	0.1950	0.1951
80	0.8579	0.8627	0.6252	0.6268	0.4268	0.4273	0.2844	0.2846	0.1900	0.1900
100	0.7684	0.7741	0.5769	0.5791	0.4038	0.4044	0.2739	0.2741	0.1852	0.1852
120	0.6950	0.7013	0.5353	0.5378	0.3829	0.3838	0.2641	0.2643	0.1806	0.1807
140	0.6337	0.6405	0.4990	0.5019	0.3640	0.3651	0.2549	0.2552	0.1762	0.1763
160	0.5818	0.5889	0.4671	0.4703	0.3469	0.3480	0.2463	0.2467	0.1720	0.1721
180	0.5373	0.5446	0.4388	0.4422	0.3311	0.3324	0.2383	0.2387	0.1680	0.1682
200	0.4987	0.5060	0.4135	0.4172	0.3167	0.3181	0.2307	0.2312	0.1642	0.1644

$\lambda = a/b$; $K = ka^4/D$; $w_{max} = \alpha(qa^4/D)(10^{-2})$

Table 7 - Variation of the Maximum Small-Deflection Coefficient α of Clamped Elliptical and Circular Plates with the Dimensionless Foundation Modulus K.

DIMENSIONLESS FOUNDATION MODULUS K	ASPECT RATIO λ = a/b					
	1		1.5		2.0	
	q_1	q_3	q_1	q_3	q_1	q_3
0	64,0000	35.2973	181.5000	102.3199	472.0000	278.4510
40	89.9221	36.3548	207.7331	---	499.0968	---
50	96.5154	36.6084	214.3414	104.3114	505.8959	281.2752
80	116.5595	37.3435	234.2845	---	526.3518	---
100	130.1382	37.8123	247.6768	106.1224	540.0376	284.0116
120	143.8860	38.2642	261.1456	---	553.7616	---
150	164.8179	38.9105	281.4896	107.8544	574.4185	286.6703
160	171.8763	39.1175	288.3080	---	581.3229	---
200	200.5057	39.9037	315.7639	109.5085	609.0323	289.2524

(ν = 0.3)

Table 8 - Variation of the Constants q_1 and q_3 for the Large-Deflection of Clamped Elliptical and Circular Plates with the Dimensionless Foundation Modulus K.

PLATE MATERIALS	ASPECT RATIO λ = a/b							
	1/2		2/3		3/4		1	
	q_1	q_3	q_1	q_3	q_1	q_3	q_1	q_3
GLASS-EPOXY	26.744	15.337	36.571	19.669	44.913	23.992	90.978	50.093
BORON-EPOXY	33.210	18.905	60.811	34.803	85.237	49.777	230.487	148.822
GRAPHITE-EPOXY	70.566	40.702	183.245	115.395	286.663	190.946	917.882	695.226

Table 9- Values of the Constants q_1 and q_3 for the Large-Deflection of Clamped Rectangular Orthotropic Plates.

$i = 0$	
$p'_i(x^2) = 1$	$\alpha'_i = 2/3$
$p''_i(x^2) = 1$	$\alpha''_i = 1/4$
$p'''_i(x^2) = 1$	$\alpha'''_i = 8/15$
$i = 1$	
$p'_i(x^2) = 1-5x^2$	$\alpha'_i = 16/21$
$p''_i(x^2) = 1-3x^2$	$\alpha''_i = 1/8$
$p'''_i(x^2) = 1-7x^2$	$\alpha'''_i = 224/315$
$i = 2$	
$p'_i(x^2) = 1-14x^2 + 21x^4$	$\alpha'_i = 128/165$
$p''_i(x^2) = 1-8x^2 + 10x^4$	$\alpha''_i = 1/12$
$p'''_i(x^2) = 1-18x^2 + 33x^4$	$\alpha'''_i = 3072/4096$
$i = 3$	
$p'_i(x^2) = 1- 27x^2 + 99x^4 - 85.8x^6$	$\alpha'_i = 2048/2625$
$p''_i(x^2) = 1- 15x^2 + 45x^4 - 35x^6$	$\alpha''_i = 1/16$
$p'''_i(x^2) = 1- 33x^2 + 143x^4 - 143x^6$	$\alpha'''_i = 12288/16065$
$i = 4$	
$p'_i(x^2) = 1-44x^2 +286x^4 -572x^6 +347.2857142859954x^8$	$\alpha'_i = 0.7821458407942714$
$p''_i(x^2) = 1-24x^2 +126x^4 -224x^6 +126x^8$	$\alpha''_i \simeq 0.05$
$p'''_i(x^2) = 1-52x^2 +390x^4 -884x^6 +599.8571428570319x^8$	$\alpha'''_i = 0.7719881026195806$

Table 10 - The Polynomials $p'_i(x^2)$, $p''_i(x^2)$, $p'''_i(x^2)$ and their Constants α'_i, α''_i, α'''_i for $i \leq 4$.

i	$p_i'(x^2)$	$p_i''(x^2)$	$p_i'''(x^2)$
	x_i	x_i	x_i
1	$\sqrt{1/5}$	$\sqrt{1/3}$	$\sqrt{1/7}$
2	0.2852315164806451	0.3937651910995717	0.2505628070857316
	0.7650553239294647	0.8030871523554077	0.6947465906068657
3	0.2092992094130445	0.2976373041716384	0.1886774224907859
	0.5917004235824805	0.6398960591198522	0.5406046373873587
	0.8717397758045273	0.8875016722031682	0.8198459954634870
4	0.1652789576663599	0.2389648428421374	0.1516316642932670
	0.4779249498103827	0.5261587342601463	0.4414329761085338
	0.7387738651054443	0.7639309081120975	0.6920606182568650
	0.9195339081664319	0.9274913129815385	0.8814085756174183

Table 11 - Roots of the Polynomials $p_i'(x^2)$, $p_i''(x^2)$ and $p_i'''(x^2)$ for $i \leq 4$

ASPECT RATIO b/a	COEFFICIENT C_1 FOR MAXIMUM SHEAR STRESS				
	ORTHOGONAL COLLOCATION METHOD			COLLOCATION LEAST SQUARE 200 POINTS	TIMOSHENKO REF. [50]
	4 TERMS	9 TERMS	16 TERMS		
1.0	0.672	0.678	0.674	0.671	0.675
1.2	0.752	0.762	0.757	0.750	0.759
1.5	0.836	0.852	0.845	0.834	0.848
2.0	0.910	0.937	0.927	0.914	0.930
2.5	0.940	0.978	0.964	0.954	0.968
3.0	0.949	0.997	0.980	0.975	0.985
4.0	0.943	1.014	0.990	0.998	0.997
5.0	0.931	1.022	0.991	1.013	0.999
10.0	0.895	1.049	0.998	1.060	1.000

$$\tau_{max} = C_1(2G\theta a)$$

Table 12 - Results of Torsion Problem: Coefficient C_1 for Maximum Shear Stress τ_{max} for Various Ratios of Sides of Rectangular Bars.

ASPECT RATIO b/a	COEFFICIENT C_2 FOR TORQUE				
	ORTHOGONAL COLLOCATION METHOD			COLLOCATION LEAST SQUARE 200 POINTS	TIMOSHENKO REF. [50]
	4 TERMS	9 TERMS	16 TERMS		
1.0	0.141	0.141	0.141	0.141	0.141
1.2	0.166	0.166	0.166	0.166	0.166
1.5	0.196	0.196	0.196	0.196	0.196
2.0	0.229	0.229	0.229	0.229	0.229
2.5	0.249	0.249	0.249	0.249	0.249
3.0	0.263	0.263	0.263	0.263	0.263
4.0	0.280	0.281	0.281	0.281	0.281
5.0	0.289	0.291	0.291	0.292	0.291
10.0	0.305	0.311	0.312	0.312	0.312

$$m_t = C_1 G\theta(2a)^3(2b)$$

Table 13 - Results of Torsion Problem: Coefficient C_2 for Torque m_t for Various Ratios of Sides of Rectangular Bars.

ASPECT RATIO b/a	COEFFICIENT α FOR MAXIMUM SMALL DEFLECTION			
	ORTHOGONAL COLLOCATION METHOD			TIMOSHENKO REF. [49]
	4 TERMS	9 TERMS	16 TERMS	
1.0	0.02022	0.02024	0.02024	0.02016
1.1	0.02411	0.02413	0.02412	0.02400
1.2	0.02757	0.02760	0.02759	0.02752
1.3	0.03053	0.03058	0.03058	0.03056
1.4	0.03301	0.03309	0.03309	0.03312
1.5	0.03501	0.03514	0.03514	0.03520
1.6	0.03661	0.03680	0.03680	0.03680
1.7	0.03782	0.03811	0.03811	0.03808
1.8	0.03875	0.03914	0.03914	0.03920
1.9	0.03942	0.03994	0.03993	0.03984
2.0	0.03987	0.04054	0.04053	0.04064

$$W_{max} = \alpha q a^4 / D$$

Table 14 - Variation of the Maximum Small Deflection Coefficient α with the Number of Terms used in the Solution of Clamped Isotropic Homogeneous Rectangular Plates.

NUMBER OF TERMS USED	ASPECT RATIO λ = a/b							
	λ = 1/2		λ = 2/3		λ = 3/4		λ = 1	
	q_1	q_3	q_1	q_3	q_1	q_3	q_1	q_3
4	25.081	20.139	28.563	17.701	31.837	18.198	49.429	25.784
9	24.666	15.593	28.455	16.025	31.774	17.260	49.397	25.808
16	24.674	15.621	28.454	16.078	31.773	17.356	49.395	26:037
COLLOCATION LEAST SQUARE	24.664	15.520	28.498	16.066	31.835	---	49.502	26.227
CHAN [6]	25.261	14.242	29.144	14.748	---	---	50.3813	23.940

Table 15 - Variation of the Constants q_1 and q_3 with the Number of Terms used in the Solution of Large Deflection of Clamped Isotropic Homogeneous Rectangular Plates.

NUMBER OF TERMS USED	ASPECT RATIO $\lambda = a/b$							
	$\lambda = 1/2$		$\lambda = 2/3$		$\lambda = 3/4$		$\lambda = 1$	
	q_1	q_3	q_1	q_3	q_1	q_3	q_1	q_3
4	26.789	17.048	36.516	19.779	44.838	23.826	90.921	52.439
9	26.720	15.293	36.515	19.337	44.844	23.540	90.856	49.561
16	26.717	15.327	36.514	19.510	44.843	23.773	90.851	49.860
COLLOCATION LEAST SQUARE	26.744	15.337	36.571	19.669	44.913	23.992	90.978	50.093

Table 16a - Variation of the Constants q_1 and q_3 with the Number of Terms used in the Solution of Large Deflection of Clamped Glass-Epoxy Rectangular Plates.

NUMBER OF TERMS USED	ASPECT RATIO $\lambda = a/b$							
	$\lambda = 1/2$		$\lambda = 2/3$		$\lambda = 3/4$		$\lambda = 1$	
	q_1	q_3	q_1	q_3	q_1	q_3	q_1	q_3
4	33.209	19.595	60.824	36.338	85.327	53.282	231.397	177.722
9	33.250	18.574	60.895	34.265	85.405	49.319	231.489	150.193
16	33.250	18.771	60.896	34.596	85.401	49.643	231.363	149.534
COLLOCATION LEAST SQUARE	33.210	18.905	60.811	34.803	85.237	49.777	230.487	148.822

Table 16b - Variation of the Constants q_1 and q_3 with the Number of Terms used in the Solution of Large Deflection of Clamped Boron-Epoxy Rectangular Plates.

NUMBER OF TERMS USED	ASPECT RATIO $\lambda = a/b$							
	$\lambda = 1/2$		$\lambda = 2/3$		$\lambda = 3/4$		$\lambda = 1$	
	q_1	q_3	q_1	q_3	q_1	q_3'	q_1	q_3
4	70.636	43.166	184.380	134.127	290.217	235.797	968.592	1096.327
9	70.725	40.203	184.090	116.210	288.418	193.483	928.431	701.271
16	70.724	40.538	184.015	115.946	288.188	192.095	926.983	694.100
COLLOCATION LEAST SQUARE	70.566	40.702	183.245	115.395	286.663	190.946	917.882	695.226

Table 16c - Variation of the Constants q_1 and q_3 with the Number of Terms used in the Solution of Large Deflection of Clamped Graphite-Epoxy Rectangular Plates.

** POLYNOMIAL COEFFICIENTS **
(NO. OF TERMS USED : 4)
** ASPECT RATIO A/B = 0.50000 **

```
A00 =  0.1237775602447786D 01        B00 =  0.9663153502009610D-01
A01 = -0.2367741561268116D 00        B01 =  0.1590808787034938D 00
A10 = -0.1277032289300899D-01        B10 =  0.3371571621120644D 00
A11 =  0.1176887657203465D-01        B11 =  0.1607144811945216D 00

C00 = -0.5995940035689397D-01        D00 =  0.2495419580753286D 00
C01 =  0.9697361812085350D-01        D01 = -0.2240507049636154D 00
C10 =  0.9682951870705913D-02        D10 = -0.3974170085350140D-01
C11 = -0.3489078711720005D-02        D11 =  0.1425044774178820D-01
        Q1 =   25.0807551            Q3 =   20.1394215
```

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** POLYNOMIAL COEFFICIENTS **
(NO. OF TERMS USED : 4)
** ASPECT RATIO A/B = 0.66667 **

```
A00 =  0.1129831272726804D 01        B00 =  0.8093389635142640D-01
A01 = -0.1196664426748667D 00        B01 =  0.1353954062810732D 00
A10 = -0.1941654603853140D-01        B10 =  0.2916222754384408D 00
A11 =  0.9251715986594029D-02        B11 =  0.1522406490150457D 00

C00 = -0.1759168701414425D-01        D00 =  0.1220872053004800D 00
C01 =  0.1561626431704560D 00        D01 = -0.9728011746029370D-01
C10 =  0.4361250868259099D-01        D10 = -0.3488949167894514D-01
C11 =  0.6181688432333212D-01        D11 =  0.1008240383875884D-01
        Q1 =   28.5634985            Q3 =   17.7012163
```

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Table 17a - Polynomial Coefficients for Isotropic Rectangular Plates (4 term Solution, λ = 1/2, 2/3)

** POLYNOMIAL COEFFICIENTS **
(NO. OF TERMS USED : 4)
** ASPECT RATIO A/B = 0.75000 **

A00 =	0.1105232273582505D 01	B00 =	0.7294484623704030D-01
A01 =	-0.8942439161787850D-01	B01 =	0.1259960514413666D 00
A10 =	-0.2403352166448757D-01	B10 =	0.2799077660403510D 00
A11 =	0.8225639699860760D-02	B11 =	0.1479970534185074D 00
C00 =	0.2998509224873191D-03	D00 =	0.9785881295352360D-01
C01 =	0.1823544308831085D 00	D01 =	-0.7197380774239800D-01
C10 =	0.5939129671562474D-01	D10 =	-0.3491020177717086D-01
C11 =	0.8399645546876590D-01	D11 =	0.9025196566045188D-02
Q1 =	31.8368926	Q3 =	18.1976698

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** POLYNOMIAL COEFFICIENTS **
(NO. OF TERMS USED : 4)
** ASPECT RATIO A/B = 1.00000 **

A00 =	0.1082039938622297D 01	B00 =	0.4572292205611690D-01
A01 =	-0.4451346313603723D-01	B01 =	0.1045218239994643D 00
A10 =	-0.4451346313603723D-01	B10 =	0.2598864006107893D 00
A11 =	0.6986987649777540D-02	B11 =	0.1359694180012270D 00
C00 =	0.4572292205611687D-01	D00 =	0.7691157784457360D-01
C01 =	0.2598864006107893D 00	D01 =	-0.4244178358357982D-01
C10 =	0.1045218239994646D 00	D10 =	-0.4244178358357983D-01
C11 =	0.1359694180012270D 00	D11 =	0.7971989322585954D-02
Q1 =	49.4287547	Q3 =	25.7837730

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Table 17b - Polynomial Coefficients for Isotropic Rectangular Plates (4 Term Solution, $\lambda = 3/4, 1$)

** POLYNOMIAL COEFFICIENTS **
(NO. OF TERMS USED : 9)
** ASPECT RATIO A/B = 0.50000 **

A00 =	0.1216127466892445D 01	B00 =	0.7866089442544360D-01
A01 =	-0.2323133092779775D 00	B01 =	0.4647619263176394D-01
A02 =	0.1987865632366683D-01	B02 =	-0.1793394160730061D-01
A10 =	-0.1236888313093866D-01	B10 =	0.2903538141182986D 00
A11 =	0.1156311741448599D-01	B11 =	0.8998557921252700D-01
A12 =	-0.2780392667260582D-02	B12 =	-0.1028449817085791D-02
A20 =	0.2607247573630427D-04	B20 =	0.7828687023825960D-01
A21 =	-0.1873409691477127D-03	B21 =	0.2984538650997091D-01
A22 =	0.5461293899065890D-04	B22 =	0.3511748548717110D-02
C00 =	0.5193064961354158D-01	D00 =	0.1697222841943500D 00
C01 =	0.9905494856246170D-01	D01 =	-0.1524888451159910D 00
C02 =	-0.6655793818267430D-01	D02 =	0.1322254185222062D-01
C10 =	0.1101853066608522D 00	D10 =	-0.3657969332763002D-01
C11 =	0.4296845903455736D-01	D11 =	0.9806107195468810D-02
C12 =	-0.4373414643419311D-01	D12 =	-0.3978748489248343D-02
C20 =	0.1893583994093091D-01	D20 =	0.8860365669858639D-04
C21 =	0.8858140720872368D-02	D21 =	-0.5902398327806733D-04
C22 =	-0.1501746677428636D-01	D22 =	0.2667740174094186D-03
Q1 =	24.6662919	Q3 =	15.5929584

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** POLYNOMIAL COEFFICIENTS **
(NO. OF TERMS USED : 9)
** ASPECT RATIO A/B = 0.66667 **

A00 =	0.1125103953772320D 01	B00 =	0.5951036208534078D-01
A01 =	-0.1183384264851432D 00	B01 =	0.7144746053688980D-01
A02 =	0.4546773633654894D-02	B02 =	0.4649056990586062D-02
A10 =	-0.1916905955126036D-01	B10 =	0.2490665815881308D 00
A11 =	0.8657386628438256D-02	B11 =	0.1016522937657391D 00
A12 =	-0.5770802754850073D-03	B12 =	0.2102043980446328D-01
A20 =	0.5957465733121130D-04	B20 =	0.6030356570985024D-01
A21 =	-0.6099281252784874D-04	B21 =	0.2909365798538602D-01
A22 =	-0.2221295673279403D-03	B22 =	0.7686423088094682D-02
C00 =	0.3908543278403011D-01	D00 =	0.1128622611922697D 00
C01 =	0.1249910388650977D 00	D01 =	-0.8522735960355370D-01
C02 =	-0.2851597666633262D-01	D02 =	0.1525260966219597D-02
C10 =	0.1220060202644624D 00	D10 =	-0.3527385121019623D-01
C11 =	0.7241679162737640D-01	D11 =	0.8021246397764540D-02
C12 =	-0.1431213992543541D-01	D12 =	-0.2054940091865337D-02
C20 =	0.2125864285417942D-01	D20 =	0.2683441135652276D-04
C21 =	0.1673418148045936D-01	D21 =	-0.1624216419191295D-03
C22 =	-0.5310866678622344D-02	D22 =	0.2829695799239653D-03
Q1 =	28.4547992	Q3 =	16.0250864

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Table 18a - Polynomial Coefficients for Isotropic Rectangular Plates (9 Term Solution, λ = 1/2, 2/3)

** POLYNOMIAL COEFFICIENTS **
(NO. OF TERMS USED : 9)
** ASPECT RATIO A/B = 0.75000 **

A00 =	0.1102904325556312D 01	B00 =	0.5299302115065335D-01
A01 =	-0.8869376582369110D-01	B01 =	0.8488181603066890D-01
A02 =	0.2341948971874597D-02	B02 =	0.1023053233151379D-01
A10 =	-0.2387213152645302D-01	B10 =	0.2363107433520928D 00
A11 =	0.7667562390217916D-02	B11 =	0.1049958283073911D 00
A12 =	-0.1341306058210715D-03	B12 =	0.2519715864879696D-01
A20 =	0.8943599043661342D-04	B20 =	0.6250350759927449D-01
A21 =	0.2268721407316423D-04	B21 =	0.2744541783277266D-01
A22 =	-0.3259321669490890D-03	B22 =	0.7735969959864538D-02

C00 =	0.3632660392981513D-01	D00 =	0.9768371571843610D-01
C01 =	0.1441317277550552D 00	D01 =	-0.6799440115144000D-01
C02 =	-0.1200325069698640D-01	D02 =	0.4085537280769838D-03
C10 =	0.1242610731078170D 00	D10 =	-0.3627049687258647D-01
C11 =	0.8408138011652010D-01	D11 =	0.7700957551041822D-02
C12 =	-0.3274996500061406D-02	D12 =	-0.1540713152607167D-02
C20 =	0.2163391897721404D-01	D20 =	-0.1771473734599370D-04
C21 =	0.2050164110953770D-01	D21 =	-0.2545976424684240D-03
C22 =	-0.1800861057822013D-02	D22 =	0.2846965588929584D-03

Q1 = 31.7741716 Q3 = 17.2604712

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** POLYNOMIAL COEFFICIENTS **
(NO. OF TERMS USED : 9)
** ASPECT RATIO A/B = 1.00000 **

A00 =	0.1081474378750858D 01	B00 =	0.4273400135031896D-01
A01 =	-0.4438803758946433D-01	B01 =	0.1231447746956725D 00
A02 =	0.4144143413215402D-03	B02 =	0.2045570487439418D-01
A10 =	-0.4438803758946435D-01	B10 =	0.2112468719245633D 00
A11 =	0.6546477368636738D-02	B11 =	0.1113121489453727D 00
A12 =	0.1881187348203018D-03	B12 =	0.2916455483981175D-01
A20 =	0.4144143413215352D-03	B20 =	0.2757181651176281D-01
A21 =	0.1881187348203006D-03	B21 =	0.1864559291766313D-01
A22 =	-0.4498470928494845D-03	B22 =	0.5130660466172084D-02

C00 =	0.4273400135031874D-01	D00 =	0.8299962568521000D-01
C01 =	0.2112468719245634D 00	D01 =	-0.4454168817941328D-01
C02 =	0.2757181651176281D-01	D02 =	-0.1251170010261822D-03
C10 =	0.1231447746956726D 00	D10 =	-0.4454168817941328D-01
C11 =	0.1113121489453726D 00	D11 =	0.7435083934158363D-02
C12 =	0.1864559291766317D-01	D12 =	-0.6930168294911661D-03
C20 =	0.2045570487439404D-01	D20 =	-0.1251170010261850D-03
C21 =	0.2916455483981194D-01	D21 =	-0.6930168294911634D-03
C22 =	0.5130660466172049D-02	D22 =	0.2849344004927763D-03

Q1 = 49.3972723 Q3 = 25.8078468

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Table 18b - Polynomial Coefficients for Isotropic Rectangular Plates (9 Term Solution, λ=3/4, 1).

** POLYNOMIAL COEFFICIENTS **
(NO. OF TERMS USED : 16)
** ASPECT RATIO A/B = 0.50000 **

A00 =	0.1216684862150708D 01	B00 =	0.7634987718391710D-01
A01 =	-0.2324426708341491D 00	B01 =	0.5535495327838217D-01
A02 =	0.1954170285403932D-01	B02 =	-0.1769797294449241D-01
A03 =	-0.2445179932432800D-03	B03 =	-0.9048903883935453D-02
A10 =	-0.1251548014294401D-01	B10 =	0.2901109485600147D 00
A11 =	0.1151872734611214D-01	B11 =	0.9472895490239160D-01
A12 =	-0.2272467609095137D-02	B12 =	-0.3869258918026969D-02
A13 =	-0.2441001195913308D-03	B13 =	-0.1215002226350156D-01
A20 =	0.3927892025465922D-04	B20 =	0.7916305227507980D-01
A21 =	-0.7611818547891424D-04	B21 =	0.3041781258939932D-01
A22 =	-0.2462889156950256D-03	B22 =	0.9430325982291715D-03
A23 =	0.2801184581733999D-03	B23 =	-0.2353775086792321D-02
A30 =	0.1284971247462398D-04	B30 =	-0.3587634491194560D-02
A31 =	-0.4558854187399023D-04	B31 =	0.3116464142615909D-03
A32 =	0.7952479243525704D-04	B32 =	0.9662041354527826D-03
A33 =	-0.6983189212655330D-04	B33 =	0.2337494550152855D-03

C00 =	0.5480126582560637D-01	D00 =	0.1679068191739718D 00
C01 =	0.8156366138857510D-01	D01 =	-0.1524455793481487D 00
C02 =	-0.5124602983490754D-01	D02 =	0.1450800094703634D-01
C03 =	-0.2265424118456668D-03	D03 =	0.5189668907110434D-04
C10 =	0.1098255573387593D 00	D10 =	-0.3693457686518534D-01
C11 =	0.3738990232800684D-01	D11 =	0.9964076420997774D-02
C12 =	-0.2694046143213545D-01	D12 =	-0.3819397156690347D-02
C13 =	-0.2024423885070523D-02	D13 =	0.5790997385419775D-03
C20 =	0.1880084577447787D-01	D20 =	0.5549933921731740D-04
C21 =	0.1078020388626821D-01	D21 =	-0.1166358862480849D-04
C22 =	-0.6980534979710266D-02	D22 =	0.2212722660051510D-03
C23 =	-0.4071073359305733D-03	D23 =	-0.7097597522745528D-04
C30 =	0.1894162547123840D-02	D30 =	-0.9403973906478514D-05
C31 =	0.1632629427335447D-02	D31 =	0.4541474578496832D-06
C32 =	-0.2800890889948718D-03	D32 =	0.1828239524047851D-04
C33 =	-0.8715416096552206D-03	D33 =	-0.1380420975666503D-04

Q1 = 24.6745483 Q3 = 15.6212556

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Table 19a–Polynomial Coefficients for Isotropic Rectangular Plates (16 Term Solution, $\lambda = 1/2$)

** POLYNOMIAL COEFFICIENTS **
(NO. OF TERMS USED : 16)
** ASPECT RATIO A/B = 0.66667 **

A00	=	0.1125270843112378D 01
A01	=	-0.1185340768491140D 00
A02	=	0.4478304972273478D-02
A03	=	0.1609971554091162D-03
A10	=	-0.1936498883808036D-01
A11	=	0.8789428720891575D-02
A12	=	-0.3041736422014241D-03
A13	=	-0.4234213726916358D-03
A20	=	0.7571036288866083D-04
A21	=	0.3878661884300808D-04
A22	=	-0.5193554786632069D-03
A23	=	0.3636801142400903D-03
A30	=	0.2856931942712002D-04
A31	=	-0.9873199990001559D-04
A32	=	0.1609532499281385D-03
A33	=	-0.1225254456281460D-03

B00	=	0.5933073791762997D-01
B01	=	0.7799377420638250D-01
B02	=	-0.5070876363932679D-02
B03	=	-0.5769483937634939D-02
B10	=	0.2499534066658018D 00
B11	=	0.1053921714010728D 00
B12	=	0.1445723233698575D-01
B13	=	-0.4803290607352151D-02
B20	=	0.6208523299116778D-01
B21	=	0.2987305735888736D-01
B22	=	0.4806637220467721D-02
B23	=	0.4030287321805500D-03
B30	=	-0.5719699017713233D-02
B31	=	-0.1313373665020690D-02
B32	=	0.2874136388911119D-03
B33	=	0.2408782937018362D-03

C00	=	0.5315440455617479D-01
C01	=	0.1192056300974546D 00
C02	=	-0.2407417559450654D-01
C03	=	-0.1258419077958617D-01
C10	=	0.1223395613714879D 00
C11	=	0.7606869060921600D-01
C12	=	-0.3118562911669492D-02
C13	=	-0.7370798682380531D-02
C20	=	0.1947299143977559D-01
C21	=	0.1878956268583770D-01
C22	=	-0.2298221873897318D-02
C23	=	-0.1724037246388520D-02
C30	=	0.1710319653633942D-02
C31	=	0.2328294343649579D-02
C32	=	0.7937460053325652D-03
C33	=	-0.6638957746451577D-03

D00	=	0.1140297404429143D 00
D01	=	-0.8599191116870980D-01
D02	=	0.1870642645152278D-02
D03	=	-0.2720684912832600D-03
D10	=	-0.3560769716807970D-01
D11	=	0.7843391215938405D-02
D12	=	-0.2019344489745988D-02
D13	=	0.9631186735918648D-04
D20	=	-0.4992995245084978D-05
D21	=	-0.1342520542221806D-03
D22	=	0.2408484973704223D-03
D23	=	-0.2557657871159024D-05
D30	=	-0.3654462646272224D-04
D31	=	-0.1409302090530517D-04
D32	=	0.2589149670843914D-04
D33	=	-0.2336449291789950D-04

Q1'= 28.4542527   Q3 = 16.0776102

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Table 19b- Polynomial Coefficients for Isotropic Rectangular Plates (16 Term solution, λ = 2/3)

** POLYNOMIAL COEFFICIENTS **
(NO. OF TERMS USED : 16)
** ASPECT RATIO A/B = 0.75000 **

A00 =	0.1103045312748268D 01	B00 =	0.5367407924721094D-01
A01 =	-0.8889162700378050D-01	B01 =	0.8975103745467600D-01
A02 =	0.2313978390403419D-02	B02 =	0.4872579477278476D-03
A03 =	0.1468122940229356D-03	B03 =	-0.4063665507248977D-02
A10 =	-0.2406815296171246D-01	B10 =	0.2377949131306273D 00
A11 =	0.7829761228122927D-02	B11 =	0.1082905325926513D 00
A12 =	0.6403159767285462D-04	B12 =	0.1937135080078680D-01
A13 =	-0.3862003320895253D-03	B13 =	-0.2470378676365817D-02
A20 =	0.1050980128145283D-03	B20 =	0.5493757104493534D-01
A21 =	0.1156056667460299D-03	B21 =	0.2859987607364371D-01
A22 =	-0.6057562876450815D-03	B22 =	0.5166513008360814D-02
A23 =	0.3627552643866698D-03	B23 =	0.1068362561366031D-02
A30 =	0.3880333355509459D-04	B30 =	-0.7087604498186350D-02
A31 =	-0.1306630113875681D-03	B31 =	-0.2168477350921438D-02
A32 =	0.2009078933036359D-03	B32 =	-0.5723884932347189D-04
A33 =	-0.1406668326804214D-03	B33 =	0.1266057997582791D-03

C00 =	0.5020373150424754D-01	D00 =	0.9963444527379750D-01
C01 =	0.1409480343885809D 00	D01 =	-0.6899972984868120D-01
C02 =	-0.9458739103450237D-02	D02 =	0.5245566584390797D-03
C03 =	-0.1352191397365345D-01	D03 =	-0.2709764802047448D-03
C10 =	0.1250221684447000D 00	D10 =	-0.3672307798118816D-01
C11 =	0.8831618029320910D-01	D11 =	0.7437167358404210D-02
C12 =	0.5202237043029572D-02	D12 =	-0.1518538547079440D-02
C13 =	-0.7134438052798035D-02	D13 =	0.3169717818801481D-04
C20 =	0.1879503202224431D-01	D20 =	-0.6007568217470066D-04
C21 =	0.2150011910686613D-01	D21 =	-0.2362032484201188D-03
C22 =	-0.4349042802460066D-03	D22 =	0.2483796021860657D-03
C23 =	-0.1590864910083741D-02	D23 =	0.1069826267220990D-04
C30 =	0.1391007683540965D-02	D30 =	-0.6035076728323319D-04
C31 =	0.2368330711498784D-02	D31 =	-0.1888791811188688D-04
C32 =	0.1205255284986293D-02	D32 =	0.2688656174664592D-04
C33 =	-0.5365829160192240D-03	D33 =	-0.2599042229031326D-04

Q1 = 31.7730601 Q3 = 17.3561056

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Table 19c - Polynomial Coefficients for Isotropic Rectangular Plates (16 Term Solution, $\lambda = 3/4$)

** POLYNOMIAL COEFFICIENTS **
(NO. OF TERMS USED : 16)
** ASPECT RATIO A/B = 1.00000 **

A00 =	0.1081587776714482D 01	B00 =	0.4508373134312412D-01
A01 =	-0.4457667839924374D-01	B01 =	0.1253451314696647D 00
A02 =	0.4230656102886279D-03	B02 =	0.1351725235982372D-01
A03 =	0.8044373617585084D-04	B03 =	-0.5511518599511400D-03
A10 =	-0.4457667839924377D-01	B10 =	0.2140940609124376D 00
A11 =	0.6727467431001331D-02	B11 =	0.1150159733119062D 00
A12 =	0.2908227106249807D-03	B12 =	0.2651450828509703D-01
A13 =	-0.2434393579139719D-03	B13 =	0.1311695421883253D-02
A20 =	0.4230656102886550D-03	B20 =	0.3301160739464564D-01
A21 =	0.2908227106249523D-03	B21 =	0.2242124541119625D-01
A22 =	-0.7032349705622529D-03	B22 =	0.3759066631822914D-02
A23 =	0.3005732547413564D-03	B23 =	0.1837465199268001D-02
A30 =	0.8044373617584264D-04	B30 =	-0.1201914635534633D-01
A31 =	-0.2434393579139540D-03	B31 =	-0.5102564485420862D-02
A32 =	0.3005732547413459D-03	B32 =	-0.9705936465451062D-03
A33 =	-0.1615842842672323D-03	B33 =	-0.2394210204637438D-03

C00 =	0.5236318369221289D-01	D00 =	0.8554411461331790D-01
C01 =	0.2111822799728024D 00	D01 =	-0.4541692087285742D-01
C02 =	0.2864393598519270D-01	D02 =	-0.1863681278423894D-03
C03 =	-0.1201914635534629D-01	D03 =	-0.1578176865426172D-03
C10 =	0.1253451314696654D 00	D10 =	-0.4541692087285744D-01
C11 =	0.1150159733119063D 00	D11 =	0.7104072965516414D-02
C12 =	0.2242124541119631D-01	D12 =	-0.6810413752427309D-03
C13 =	-0.5102564485420787D-02	D13 =	-0.1998785703320125D-04
C20 =	0.1351725235982380D-01	D20 =	-0.1863681278423792D-03
C21 =	0.2651450828509703D-01	D21 =	-0.6810413752427353D-03
C22 =	0.3759066631822895D-02	D22 =	0.255304146988243D-03
C23 =	-0.9705936465451456D-03	D23 =	0.2485982673171136D-04
C30 =	-0.5511518599511562D-03	D30 =	-0.1578176865426169D-03
C31 =	0.1311695421883264D-02	D31 =	-0.1998785703319872D-04
C32 =	0.1837465199267970D-02	D32 =	0.2485982673171023D-04
C33 =	-0.2394210204637665D-03	D33 =	-0.2893954025009945D-04

Q1 = 49.3950074   Q3 = 26.0366766

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Table 19d - Polynomial Coefficients for Isotropic Rectangular Plates (16 Term Solution, λ = 1)

SANDWICH PLATE # 1

$E_f = 10\times10^6$ psi $\quad G_c = 500$ psi

$\nu_f = 0.32 \quad \mu = 0.00125 \quad \theta = 0.05125$

NUMBER OF TERMS USED	COEFFICIENT α FOR MAXIMUM SMALL DEFLECTION			
	ASPECT RATIO $\lambda = a/b$			
	$\lambda = 1/2$	$\lambda = 2/3$	$\lambda = 3/4$	$\lambda = 1$
4	0.310799	0.276026	0.255451	0.196127
9	0.370218	0.323472	0.298340	0.228553
16	0.352876	0.310520	0.286846	0.220047

$W_{max} = \alpha p a^4/D, \quad D = t h^2 E_f / 2(1-\nu_f^2)$

Table 20 - Variation of the Maximum Small Deflection Coefficient α with the Number of Terms used in the Solution of the Linear Analysis of Sandwich Plate No. 1

NUMBER OF TERMS USED	MAXIMUM CENTER SMALL DEFLECTION IN INCHES			
	PLATE DIMENSIONS 2b x 2a IN INCHES			
	80 x 40	60 x 40	53-1/3 x 40	40 x 40
4	0.339879	0.301853	0.279353	0.214478
9	0.404858	0.353738	0.326254	0.249938
16	0.385893	0.339574	0.313685	0.240636
MARCH [32]	0.353457*	0.353457*	0.298742	0.228190
MONFORTON et. al [34]	---	---	---	0.2483

p = 1 psi

* Deflections corresponding to large values of b/a. Ref. [32].

Table 21 - Comparison of the Maximum Center Small Deflection of Plate No. 1

SANDWICH PLATE # 2

$E_f = 10 \times 10^6$ psi $G_c = 100{,}000$ psi

$\nu_f = 0.32$ $\mu = 0.00125$, $\theta = 0.05125$

NUMBER OF TERMS USED	COEFFICIENT α FOR MAXIMUM SMALL DEFLECTION			
	ASPECT RATIO $\lambda = a/b$			
	$\lambda = 1/2$	$\lambda = 2/3$	$\lambda = 3/4$	$\lambda = 1$
4	0.045890	0.037674	0.033235	0.021078
9	0.042724	0.037424	0.033593	0.021826
16	0.041791	0.036367	0.032643	0.021216

$$W_{max} = \alpha p a^4/D, \quad D = t h^2 E_f/2(1-\nu_f^2)$$

Table 22 - Variation of the Maximum Small Deflection Coefficient α with the Number of Terms used in the Solution of the Linear Analysis of Sandwich Plate No. 2

NUMBER OF TERMS USED	MAXIMUM CENTER SMALL DEFLECTION IN INCHES			
	PLATE DIMENSIONS 2b X 2a IN INCHES			
	80 X 40	60 X 40	53-1/3 X 40	40 X 40
4	0.050184	0.041199	0.036345	0.023050
9	0.046722	0.040925	0.036736	0.023868
16	0.045701	0.039769	0.035697	0.023201
MARCH [32]	0.046485*	0.046485*	0.036716	0.023482
MONFORTON et. al [34]	---	---	---	0.023480

p = 1 psi

* Deflection corresponding to large values of b/a Ref. [32]

Table 23 - Comparison of the Maximum Center Small Deflection of Plate No. 2.

SANDWICH PLATE # 3

$E_f = 10.5 \times 10^6$ psi. $G_c = 50{,}000$ psi

$\nu_f = 0.30$ $\mu = 0.0006$ $\theta = 0.04$

NUMBER OF TERMS USED	COEFFICIENT α FOR MAXIMUM SMALL DEFLECTION			
	ASPECT RATIO $\lambda = a/b$			
	$\lambda = 1/2$	$\lambda = 2/3$	$\lambda = 3/4$	$\lambda = 1$
4	0.046291	0.037931	0.033406	0.021081
9	0.042204	0.037034	0.033242	0.021559
16	0.041430	0.036024	0.032321	0.020962
KAN et. al [20]	0.041746	0.036249	----	0.021039

$W_{max} = \alpha p a^4/D, \quad D = th^2 E_f/2(1-\nu_f^2)$

Table 24 - Variation of the Maximum Small Deflection Coefficient α with the Number of Terms used in the Solution of the Linear Analysis of Sandwich Plate No. 3

NUMBER OF TERMS USED	MAXIMUM CENTER SMALL DEFLECTION IN INCHES			
	PLATE DIMENSIONS 2b X 2a IN INCHES			
	40 X 20	30 X 20	26-2/3 X 20	20 X 20
4	0.083581	0.068487	0.060316	0.038063
9	0.076202	0.066867	0.060020	0.038926
16	0.074804	0.065433	0.058357	0.037848
MARCH [32]	0.076070*	0.076070*	0.060150	0.038402
KAN et. al [20]	0.075375	0.065496	---	0.037987

p = 1 psi

* Deflections corresponding to large values of b/a. Ref. [32]

Table 25 - Comparison of the Maximum Center Small Deflection of Plate No. 3.

SANDWICH PLATE # 2

$E_f = 10 \times 10^6$ psi, $G_c = 100{,}000$ psi, $\nu_f = 0.32$, $\mu = 0.00125$, $\theta = 0.05125$

NUMBER OF TERMS USED	ASPECT RATIO $\lambda = a/b$							
	$\lambda = 1/2$		$\lambda = 2/3$		$\lambda = 3/4$		$\lambda = 1$	
	q_1	q_3	q_1	q_3	q_1	q_3	q_1	q_3
4	21.7911	4.5825	26.5433	5.0787	30.0890	5.5105	47.4439	8.2237
9	23.4062	5.1436	26.7207	5.1808	29.7678	5.5857	45.8175	8.3686
16	23.9285	5.2555	27.4977	5.3995	30.6339	5.8251	47.1347	8.7160

Table 26 - Results of the Non-Linear Analysis of Plate No. 2: Variation of the Constants q_1 and q_3 with the Number of Terms used.

SANDWICH PLATE # 3

$E_f = 10.5 \times 10^6$ pxi, $G_c = 50{,}000$ psi, $\nu_f = 0.30$, $\mu = 0.0006$, $\theta = 0.04$

NUMBER OF TERMS USED	ASPECT RATIO $\lambda = a/b$							
	$\lambda = 1/2$		$\lambda = 2/3$		$\lambda = 3/4$		$\lambda = 1$	
	q_1	q_3	q_1	q_3	q_1	q_3	q_1	q_3
4	21.6023	4.4005	26.3635	4.9415	29.9345	5.3923	47.4372	8.1147
9	23.6944	5.1800	27.0023	5.1894	30.0826	5.5753	46.3844	8.3409
16	24.1368	5.2591	27.7594	5.3972	30.9393	5.8192	47.7056	8.7032
KAN et.al [20]	23.9543	---	27.5873	---	---	---	47.5310	---

Table 27 - Results of the Non-Linear Analysis of Plate No. 3: Variation of the Constants q_1 and q_3 with the Number of Terms used.

b < a

a/b	$W_{max} = \alpha q_o b^4/D$				
	NO. OF TERMS USED				Timoshenko Ref. [49]
	1	4	9	16	
∞	0.04218	0.04162	0.04168	0.04166	0.04160
2	0.04738	0.04105	0.04197	0.04177	0.04160
1.5	0.04282	0.03934	0.03971	0.03963	0.03952
1.4	0.04137	0.03839	0.03869	0.03863	0.03840
1.3	0.03963	0.03713	0.03737	0.03732	0.03744
1.2	0.03753	0.03546	0.03565	0.03561	0.03568
1.1	0.03501	0.03331	0.03346	0.03343	0.03344

b > a

b/a	$W_{max} = \alpha q_o a^4/D$				
	NO. OF TERMS USED				Timoshenko Ref. [49]
	1	4	9	16	
1	0.03197	0.03058	0.03071	0.03068	0.03072
1.1	0.04200	0.04033	0.04048	0.04044	0.04015
1.2	0.05297	0.05095	0.05113	0.05109	0.05014
1.3	0.06456	0.06212	0.06235	0.06230	0.06208
1.4	0.07645	0.07351	0.07380	0.07374	0.07360
1.5	0.08836	0.08484	0.08520	0.08512	0.08496
1.6	0.10006	0.09588	0.09632	0.09622	0.09648
1.7	0.11137	0.10643	0.10697	0.10685	0.10688
1.8	0.12216	0.11638	0.11703	0.11689	0.11712
1.9	0.13236	0.12564	0.12643	0.12626	0.12640
2.0	0.14192	0.13418	0.13512	0.13491	0.13504
3.0	0.20587	0.18406	0.18773	0.18681	0.18688
∞	0.20914	0.20828	0.20834	0.20833	0.20832

Table 28 - Variation of the Coefficient α with the Number of Terms used in the Solution of Rectangular Plates with Two Opposite Edges Simply Supported and the Other Two Edges Clamped

$\frac{b}{a}$	$\frac{D}{q_o a^4} (W^b_{max})$	$\frac{S}{q_o a^2} (W^S_{max})$
1.0	0.06496	0.29480
1.2	0.09024	0.34720
1.4	0.11280	0.38680
1.6	0.13280	0.41680
1.8	0.14896	0.43920
2.0	0.16208	0.45560
3.0	0.19568	0.49080
4.0	0.20512	0.49800
5.0	0.20752	0.49960
∞	0.20832	0.50000

Table 29 - Nondimensional Deflections for Simply Supported Rectangular Sandwich Plates Subjected to a Uniformly Distributed Load q_o.

APPENDIX C COMPUTER PROGRAMMES

A NOTE ON THE COMPUTER PROGRAMMES

For any collocation scheme as long as the assumed solution is taken in the form of an algebraic series, the computer programmes shown in the following pages can be slightly modified and applied to the solution of boundary value problems.

As the assumed solutions adopted for the problems in this thesis are generally algebraic functions of the form

$$p(x,y) = g(x,y) \cdot f(x,y),$$

where the functions g(x,y) contain say m terms and the functions f(x,y) n terms, the functions p(x,y) can thus be expanded into polynomials with mXn terms. These polynomials can be conveniently stored in arrays of dimension (3, m, n) with the first, second and third mXn array containing the constants, powers of x and powers of y respectively of each term.

The partial differentiation of these polynomials is performed term by term and is simply a matter of subtracting the powers of x or y and multiplying the constants by the original values of the powers of x or y.

To evaluate these polynomials or their derivatives at a particular collocation point, each term is evaluated in turn and the result added to that of the previous terms. The evaluation process can be briefly described as follows: The x and y of a term are evaluated separately by taking on the values of the x and y of the collocation point and self-multiplying as many times as indicated by the x and y powers of that term which are used as do-loop indices. The evaluated values of the x and y are then multiplied together with the constant associated with that term and the final result added to that of the previous terms. Such a procedure is continued until all the terms in the polynomial are exhausted.

The input informations required for these programmes are the number of collocation points to be used, the number of terms in the functions g and f, and the constant, powers of x and y associated with each term in these two functions. These input data can be precisely prepared with a minimum of human effort, and thus, a solution can be obtained with the least of human errors. Whereas if performed manually, such operations as expansion, differentiation and evaluation of the polynomials are generally very time consuming and error prone.

To guard against round-off errors, double precision arithmetic is employed for all the operations in these programmes. The following is a brief description of the function of the subroutines:

EXPAND : Expand the polynomials prior to differentiation.

DIFF : Perform partial differentiation on the polynomials.

POINTS : Set up the collocation points.

ZERO : Zero out the arrays prior to any calculations.

SETUPL : Set up the left hand side coefficient matrix.

SETUPR : Set up the right hand side column vector.

LSTSQR : Perform the least square operations.

ADD : Add two matrix equations together. (For the second order approximation, instead of simultaneously generating 200 equations for the two differential equations, the collocation least square procedure can be applied separately to the two differential equations and the resulting equations added together prior to determining the polynomial coefficients, a procedure which yields identical results but would only require an array of dimension 100 X 18 to store the numbers as compared to an array of dimension 200 X 18).

MATINV : Matrix inversion routine.

MATMPY : Matrix multiplication routine.

```
C     *******************************************************************
C     *                                                                 *
C     *              LINEAR AND NON-LINEAR ANALYSES OF                  *
C     *      CLAMPED RECTANGULAR PLATES ON ELASTIC FOUNDATION           *
C     *         USING THE COLLOCATION LEAST SQUARE METHOD               *
C     *    NO. OF COLLOCATION POINTS USED IN THIS PROGRAMME : 100       *
C     *         MATHEW Y. T. CHAN,   DEPT. OF CIVIL ENG.,               *
C     *                  UNIVERSITY OF OTTAWA                           *
C     *                                                                 *
C     *                                                                 *
C     *******************************************************************
C
      IMPLICIT REAL*8(A-H, O-Z)
      DIMENSION AC(9),AX(9),AY(9),BC(9),BX(9),BY(9),W(3,9,9)
      DIMENSION D1WX(3,9,9),D1WY(3,9,9),D2WX(3,9,9),D2WXY(3,9,9),
     *D2WY(3,9,9),D4WX(3,9,9),D4WXY(3,9,9),D4WY(3,9,9)
      DIMENSION D1UX(3,9,4),D1UY(3,9,4),D2UX(3,9,4),D2UXY(3,9,4),
     *D2UY(3,9,4)
      DIMENSION D1VX(3,9,4),D1VY(3,9,4),D2VX(3,9,4),D2VXY(3,9,4),
     *D2VY(3,9,4)
      DIMENSION B(18,18),B1(18,18),QQ1(18),QQ2(18)
      DIMENSION C(18),CW1(9),CU(9),CV(9),CW3(9)
      DIMENSION A(100,18),X(100),Y(100),Q(100),CO(100)
      DIMENSION XX(2),YY(2),W1X2(2),W1Y2(2),W1XSQ(2),W1YSQ(2),W3X2(2),
     *W3Y2(2),UX(2),VY(2),SXM(2),SXBL(2),SXBNL(2)
      READ(5,10)NWA,NWB,NUVA,NUVB,NPTS,NTERMS
   10 FORMAT(6I5)
C
C      *******************************************************************
C      *                                                                 *
C      *        OBTAIN DERIVATIVES OF THE DISPLACEMENT FUNCTIONS U, V & W *
C      *                                                                 *
C      *******************************************************************
C
      READ(5,20)(AC(I),AX(I),AY(I), I=1,NUVA)
      READ(5,20)(BC(I),BX(I),BY(I), I=1,NUVB)
   20 FORMAT(3F10.5)
      CALL EXPAND(AC,AX,AY,NWA,BC,BX,BY,NWB,W,NUVB,NUVA)
      CALL DIFF(W,NWB,NWA,D1UX,NUVB,NUVA,1)
      CALL DIFF(W,NWB,NWA,D1UY,NUVB,NUVA,2)
      CALL DIFF(W,NWB,NWA,D2UX,NUVB,NUVA,3)
      CALL DIFF(W,NWB,NWA,D2UXY,NUVB,NUVA,4)
      CALL DIFF(W,NWB,NWA,D2UY,NUVB,NUVA,5)
      READ(5,20)(AC(I),AX(I),AY(I), I=1,NUVA)
      CALL EXPAND(AC,AX,AY,NWA,BC,BX,BY,NWB,W,NUVB,NUVA)
      CALL DIFF(W,NWB,NWA,D1VX,NUVB,NUVA,1)
      CALL DIFF(W,NWB,NWA,D1VY,NUVB,NUVA,2)
      CALL DIFF(W,NWB,NWA,D2VX,NUVB,NUVA,3)
      CALL DIFF(W,NWB,NWA,D2VXY,NUVB,NUVA,4)
      CALL DIFF(W,NWB,NWA,D2VY,NUVB,NUVA,5)
      READ(5,20)(AC(I),AX(I),AY(I), I=1,NWA)
      CALL EXPAND(AC,AX,AY,NWA,BC,BX,BY,NWB,W,NWB,NWA)
      CALL DIFF(W,NWB,NWA,D1WX,NWB,NWA,1)
      CALL DIFF(W,NWB,NWA,D1WY,NWB,NWA,2)
      CALL DIFF(W,NWB,NWA,D2WX,NWB,NWA,3)
      CALL DIFF(W,NWB,NWA,D2WXY,NWB,NWA,4)
      CALL DIFF(W,NWB,NWA,D2WY,NWB,NWA,5)
```

```
      CALL DIFF(W,NWB,NWA,D4WX,NWB,NWA,6)
      CALL DIFF(W,NWB,NWA,D4WXY,NWB,NWA,7)
      CALL DIFF(W,NWB,NWA,D4WY,NWB,NWA,8)
C
C     ******************************************************************
C     *                                                                *
C     *                SET UP THE COLLOCATION POINTS                   *
C     *                                                                *
C     ******************************************************************
C
      CALL POINTS(X,Y,100,1.0D 00,10.0D 00,1.0D 00,10.0D 00,10,90)
      XX(1)=1.0D 00
      YY(1)=0.0D 00
      XX(2)=0.0D 00
      YY(2)=0.0D 00
      V=1.0D 00/3.0D 00
    1 READ(5,30)R
   30 FORMAT(D30.16)
      R2=R*R
      R3=R2*R
      R4=R2*R2
      IF(R)999,40,40
C
C     ******************************************************************
C     *                                                                *
C     *                FIRST ORDER APPROXIMATION                       *
C     *                                                                *
C     ******************************************************************
C
   40 FM=-40.0D 00
      DO 888 III=1,6
      FM=FM+40.0D 00
      CALL ZERO(A,1,NPTS,NTERMS)
      CALL ZERO(Q,1,1,NPTS)
      CALL ZERO(CO,1,1,NPTS)
      CALL ZERO(CW1,1,1,NWB)
      COEFF=1.0D 00
      CALL SETUPL(A,NPTS,NTERMS,X,Y,NPTS,D4WX,NWB,NWA,2,-1,COEFF,1)
      COEFF=2.0D 00*R2
      CALL SETUPL(A,NPTS,NTERMS,X,Y,NPTS,D4WXY,NWB,NWA,2,-1,COEFF,1)
      COEFF=R4
      CALL SETUPL(A,NPTS,NTERMS,X,Y,NPTS,D4WY,NWB,NWA,2,-1,COEFF,1)
      COEFF=FM
      CALL SETUPL(A,NPTS,NTERMS,X,Y,NPTS,W,NWB,NWA,2,-1,COEFF,1)
      COEFF=1.0D 00
      CALL SETUPR(Q,CO,NPTS,X,Y,NPTS,D4WX,CW1,NWB,NWA,COEFF,1,0)
      COEFF=2.0D 00*R2
      CALL SETUPR(Q,CO,NPTS,X,Y,NPTS,D4WXY,CW1,NWB,NWA,COEFF,1,0)
      COEFF=R4
      CALL SETUPR(Q,CO,NPTS,X,Y,NPTS,D4WY,CW1,NWB,NWA,COEFF,1,0)
      COEFF=FM
      CALL SETUPR(Q,CO,NPTS,X,Y,NPTS,W,CW1,NWB,NWA,COEFF,1,0)
      DO 50 I=1,NPTS
      A(I,NWB)=-1.0D 00
      Q(I)=-Q(I)
   50 CONTINUE
      CALL LSTSQR(A,NPTS,NTERMS,B,Q,QQ1,NPTS,NWB)
      CALL MATINV(B,NTERMS,NWB)
```

```
0079          CALL MATMPY(B,NTERMS,QQ1,C,NWB)
0080          CW1(1)=1.0D 00
0081          DO 60 I=2,NWB
0082          J=I-1
0083          CW1(I)=C(J)
0084       60 CONTINUE
0085          Q1=C(NWB)
      C
      C       ****************************************************************
      C       *                                                              *
      C       *                SECOND ORDER APPROXIMATION                    *
      C       *                                                              *
      C       ****************************************************************
      C
0086          CALL ZERO(A,1,NPTS,NTERMS)
0087          CALL ZERO(Q,1,1,NPTS)
0088          CALL ZERO(C0,1,1,NPTS)
      C
      C       ****************************************************************
      C       *                                                              *
      C       *                     FIRST EQUATION                           *
      C       *                                                              *
      C       ****************************************************************
      C
0089          COEFF=2.0D 00
0090          CALL SETUPL(A,NPTS,NTERMS,X,Y,NPTS,D2UX,NUVB,NUVA,1,0,COEFF,1)
0091          COEFF=(1.0-V)*R2
0092          CALL SETUPL(A,NPTS,NTERMS,X,Y,NPTS,D2UY,NUVB,NUVA,1,0,COEFF,1)
0093          COEFF=(1.0+V)*R
0094          CALL SETUPL(A,NPTS,NTERMS,X,Y,NPTS,D2VXY,NUVB,NUVA,1,NUVB,COEFF,1)
0095          CALL SETUPR(Q,C0,NPTS,X,Y,NPTS,D1WX,CW1,NWB,NWA,1.0D 00,1,2)
0096          COEFF=-2.0D 00
0097          CALL SETUPR(Q,C0,NPTS,X,Y,NPTS,D2WX,CW1,NWB,NWA,COEFF,1,1)
0098          COEFF=-(1.0-V)*R2
0099          CALL SETUPR(Q,C0,NPTS,X,Y,NPTS,D2WY,CW1,NWB,NWA,COEFF,1,1)
0100          CALL ZERO(C0,1,1,NPTS)
0101          CALL SETUPR(Q,C0,NPTS,X,Y,NPTS,D1WY,CW1,NWB,NWA,1.0D 00,1,2)
0102          COEFF=-(1.0+V)*R2
0103          CALL SETUPR(Q,C0,NPTS,X,Y,NPTS,D2WXY,CW1,NWB,NWA,COEFF,1,1)
0104          CALL LSTSQR(A,NPTS,NTERMS,B,Q,QQ1,NPTS,NTERMS)
      C
      C       ****************************************************************
      C       *                                                              *
      C       *                     SECOND EQUATION                          *
      C       *                                                              *
      C       ****************************************************************
      C
0105          CALL ZERO(A,1,NPTS,NTERMS)
0106          CALL ZERO(Q,1,1,NPTS)
0107          COEFF=(1.0+V)*R
0108          CALL SETUPL(A,NPTS,NTERMS,X,Y,NPTS,D2UXY,NUVB,NUVA,1,0,COEFF,1)
0109          COEFF=(1.0-V)
0110          CALL SETUPL(A,NPTS,NTERMS,X,Y,NPTS,D2VX,NUVB,NUVA,1,NUVB,COEFF,1)
0111          COEFF=2.0*R2
0112          CALL SETUPL(A,NPTS,NTERMS,X,Y,NPTS,D2VY,NUVB,NUVA,1,NUVB,COEFF,1)
0113          COEFF=-2.0D 00*R3
0114          CALL SETUPR(Q,C0,NPTS,X,Y,NPTS,D2WY,CW1,NWB,NWA,COEFF,1,1)
0115          COEFF=-(1.0-V)*R
```

```
      CALL  SETUPR(Q,CO,NPTS,X,Y,NPTS,D2WX,CW1,NWB,NWA,COEFF,1,1)
      CALL  ZERO(CO,1,1,NPTS)
      CALL  SETUPR(Q,CO,NPTS,X,Y,NPTS,D1WX,CW1,NWB,NWA,1.0D 00,1,2)
      COEFF=-(1.0+V)*R
      CALL  SETUPR(Q,CO,NPTS,X,Y,NPTS,D2WXY,CW1,NWB,NWA,COEFF,1,1)
      CALL  LSTSQR(A,NPTS,NTERMS,B1,Q,QQ2,NPTS,NTERMS)
      CALL  ADD(B,B1,QQ1,QQ2,NTERMS)
      CALL  MATINV(B,NTERMS,NTERMS)
      CALL  MATMPY(B,NTERMS,QQ1,C,NTERMS)
      DO 70 I=1,NUVB
      J=I+NUVB
      CU(I)=C(I)
      CV(I)=C(J)
   70 CONTINUE
C
C     ***********************************************************************
C     *                                                                     *
C     *                    THIRD ORDER APPROXIMATION                        *
C     *                                                                     *
C     ***********************************************************************
C
      CALL  ZERO(A,1,NPTS,NTERMS)
      CALL  ZERO(Q,1,1,NPTS)
      CALL  ZERO(CO,1,1,NPTS)
      COEFF=1.0D 00
      CALL  SETUPL(A,NPTS,NTERMS,X,Y,NPTS,D4WX,NWB,NWA,2,-1,COEFF,1)
      COEFF=2.0D 00*R2
      CALL  SETUPL(A,NPTS,NTERMS,X,Y,NPTS,D4WXY,NWB,NWA,2,-1,COEFF,1)
      COEFF=R4
      CALL  SETUPL(A,NPTS,NTERMS,X,Y,NPTS,D4WY,NWB,NWA,2,-1,COEFF,1)
      COEFF=FM
      CALL  SETUPL(A,NPTS,NTERMS,X,Y,NPTS,W,NWB,NWA,2,-1,COEFF,1)
      CALL  SETUPR(Q,CO,NPTS,X,Y,NPTS,D2WX,CW1,NWB,NWA,1.0D 00,1,2)
      COEFF=12.0D 00
      CALL  SETUPR(Q,CO,NPTS,X,Y,NPTS,D1UX,CU,NUVB,NUVA,COEFF,1,1)
      COEFF=12.0D 00*V*R
      CALL  SETUPR(Q,CO,NPTS,X,Y,NPTS,D1VY,CV,NUVB,NUVA,COEFF,1,1)
      CALL  ZERO(CO,1,1,NPTS)
      CALL  SETUPR(Q,CO,NPTS,X,Y,NPTS,D2WY,CW1,NWB,NWA,1.0D 00,1,2)
      COEFF=12.0*R3
      CALL  SETUPR(Q,CO,NPTS,X,Y,NPTS,D1VY,CV,NUVB,NUVA,COEFF,1,1)
      COEFF=12.0D 00*V*R2
      CALL  SETUPR(Q,CO,NPTS,X,Y,NPTS,D1UX,CU,NUVB,NUVA,COEFF,1,1)
      CALL  ZERO(CO,1,1,NPTS)
      CALL  SETUPR(Q,CO,NPTS,X,Y,NPTS,D1WX,CW1,NWB,NWA,1.0D 00,1,3)
      COEFF=6.0D 00
      CALL  SETUPR(Q,CO,NPTS,X,Y,NPTS,D2WX,CW1,NWB,NWA,COEFF,1,1)
      COEFF=6.0D 00*V*R2
      CALL  SETUPR(Q,CO,NPTS,X,Y,NPTS,D2WY,CW1,NWB,NWA,COEFF,1,1)
      CALL  ZERO(CO,1,1,NPTS)
      CALL  SETUPR(Q,CO,NPTS,X,Y,NPTS,D1WY,CW1,NWB,NWA,1.0D 00,1,3)
      COEFF=6.0D 00*R4
      CALL  SETUPR(Q,CO,NPTS,X,Y,NPTS,D2WY,CW1,NWB,NWA,COEFF,1,1)
      COEFF=6.0D 00*V*R2
      CALL  SETUPR(Q,CO,NPTS,X,Y,NPTS,D2WX,CW1,NWB,NWA,COEFF,1,1)
      CALL  ZERO(CO,1,1,NPTS)
      CALL  SETUPR(Q,CO,NPTS,X,Y,NPTS,D2WXY,CW1,NWB,NWA,1.0D 00,1,2)
      COEFF=12.0D 00*(1.0-V)*R2
```

```
0167          CALL  SETUPR(Q,C0,NPTS,X,Y,NPTS,D1UY,CU,NUVB,NUVA,COEFF,1,1)
0168          COEFF=12.0D 00*(1.0-V)*R
0169          CALL  SETUPR(Q,C0,NPTS,X,Y,NPTS,D1VX,CV,NUVB,NUVA,COEFF,1,1)
0170          CALL  SETUPR(Q,C0,NPTS,X,Y,NPTS,D1WX,CW1,NWB,NWA,1.0D 00,1,4)
0171          COEFF=12.0D 00*(1.0-V)*R2
0172          CALL  SETUPR(Q,C0,NPTS,X,Y,NPTS,D1WY,CW1,NWB,NWA,COEFF,1,1)
0173          DO 80  I=1,NPTS
0174          A(I,NWB)=-1.0D 00
0175       80 CONTINUE
0176          CALL  LSTSQR(A,NPTS,NTERMS,B,Q,QQ1,NPTS,NWB)
0177          CALL  MATINV(B,NTERMS,NWB)
0178          CALL  MATMPY(B,NTERMS,QQ1,C,NWB)
0179          CW3(1)=0.0D 00
0180          DO 90  I=2,NWB
0181          J=I-1
0182          CW3(I)=C(J)
0183       90 CONTINUE
0184          Q3=C(NWB)
0185          WRITE(6,100)R,FM,Q1,Q3
0186      100 FORMAT(///,18X,'ASPECT RATIO =',F10.5,10X,'K =',F10.5,10X,'Q1 =',
             *F10.5,10X,'Q3 =',F10.5/)
0187          WRITE(6,110)
0188      110 FORMAT('0',42X,'******* POLYNOMIAL COEFFICIENTS *******'//)
0189          WRITE(6,120)(CW1(I),CU(I),CV(I),CW3(I), I=1,NWB)
0190      120 FORMAT(' ',4D33.16)
0191          WRITE(6,130)
0192      130 FORMAT(' ',//,37X,'**** THE FOLLOWING ARE THE LOAD-DEFLECTION RELA
             *TIONS ****'/)
      C
      C       ****************************************************************
      C       *                                                              *
      C       *         CALCULATION OF DEFLECTIONS AND STRESSES              *
      C       *                                                              *
      C       ****************************************************************
      C
0193          RR=0.0D 00
0194          DO 140  I=1,20
0195          RR=RR+0.10D 00
0196          W1=Q1*RR
0197          W3=W1+(Q3*(RR**3))
0198          WRITE(6,150)RR,W1,W3
0199      150 FORMAT(' ',31X,3F20.8)
0200      140 CONTINUE
0201          CALL  SETUPR(C,UX,2,XX,YY,2,D1UX,CU,NUVB,NUVA,1.0,1,2)
0202          CALL  SETUPR(C,VY,2,XX,YY,2,D1VY,CV,NUVB,NUVA,1.0,1,2)
0203          CALL  SETUPR(C,W1X2,2,XX,YY,2,D2WX,CW1,NWB,NWA,1.0,1,2)
0204          CALL  SETUPR(C,W1Y2,2,XX,YY,2,D2WY,CW1,NWB,NWA,1.0,1,2)
0205          CALL  SETUPR(C,W1XSQ,2,XX,YY,2,D1WX,CW1,NWB,NWA,1.0,1,3)
0206          CALL  SETUPR(C,W1YSQ,2,XX,YY,2,D1WY,CW1,NWB,NWA,1.0,1,3)
0207          CALL  SETUPR(C,W3X2,2,XX,YY,2,D2WX,CW3,NWB,NWA,1.0,1,2)
0208          CALL  SETUPR(C,W3Y2,2,XX,YY,2,D2WY,CW3,NWB,NWA,1.0,1,2)
0209          DO 160  I=1,2
0210          SXM(I)=UX(I)+(0.5*W1XSQ(I))+(R*V*VY(I))+(0.5*V*R2*W1YSQ(I))
0211          SXBL(I)=0.5*(W1X2(I)+(V*R2*W1Y2(I)))
0212          SXBNL(I)=0.5*(W3X2(I)+(V*R2*W3Y2(I)))
0213      160 CONTINUE
0214          WRITE(6,170)
0215      170 FORMAT(' ',//,36X,'**** THE FOLLOWING ARE THE STRESS-DEFLECTION RE
```

```
         *LATIONS ****')
          SS=1.0D 00
          DO 180 I=1,2
          IF(I .EQ. 2)SS=-1.0D 00
          WRITE(6,190)XX(I),YY(I)
      190 FORMAT('0',50X,'**** STRESS @ X =',F3.0,1X,'Y =',F3.0,' ****'//)
          RR=0.0D 00
          DO 200 J=1,20
          RR=RR+0.10D 00
          STB=RR*SXBL(I)
          STM=(RR**2)*SXM(I)
          STBNL=(RR**3)*SXBNL(I)
          STOTAL=(STB*SS)+STM+(STBNL*SS)
          WRITE(6,210)RR,STB,STM,STBNL,STOTAL
      210 FORMAT(' ',5F24.8)
      200 CONTINUE
      180 CONTINUE
      888 CONTINUE
          GO TO 1
      999 STOP
          END
```

```
0001          SUBROUTINE EXPAND(CA,PXA,PYA,ND,CB,PXB,PYB,MD,ABC,M,N)
0002          IMPLICIT REAL*8(A-H, O-Z)
0003          DIMENSION CA(ND),PXA(ND),PYA(ND),CB(MD),PXB(MD),PYB(MD),
             *ABC(3,MD,ND)
0004          DO 10 I=1,M
0005          DO 10 J=1,N
0006          ABC(1,I,J)=CB(I)*CA(J)
0007          ABC(2,I,J)=PXB(I)+PXA(J)
0008          ABC(3,I,J)=PYB(I)+PYA(J)
0009       10 CONTINUE
0010          RETURN
0011          END
```

```
      SUBROUTINE DIFF(A1,MO,NO,A2,M,N,NNN)
      IMPLICIT REAL*8(A-H, O-Z)
      DIMENSION A1(3,MO,NO),A2(3,M,N)
      GO TO(10,20,30,40,50,60,70,80),NNN
   10 DO 1 I=1,M
      DO 1 J=1,N
      A2(1,I,J)=A1(1,I,J)*A1(2,I,J)
      A2(2,I,J)=A1(2,I,J)-1.0
      A2(3,I,J)=A1(3,I,J)
    1 CONTINUE
      GO TO 999
   20 DO 2 I=1,M
      DO 2 J=1,N
      A2(1,I,J)=A1(1,I,J)*A1(3,I,J)
      A2(2,I,J)=A1(2,I,J)
      A2(3,I,J)=A1(3,I,J)-1.0
    2 CONTINUE
      GO TO 999
   30 DO 3 I=1,M
      DO 3 J=1,N
      A2(1,I,J)=A1(1,I,J)*A1(2,I,J)*(A1(2,I,J)-1.0)
      A2(2,I,J)=A1(2,I,J)-2.0
      A2(3,I,J)=A1(3,I,J)
    3 CONTINUE
      GO TO 999
   40 DO 4 I=1,M
      DO 4 J=1,N
      A2(1,I,J)=A1(1,I,J)*A1(2,I,J)*A1(3,I,J)
      A2(2,I,J)=A1(2,I,J)-1.0
      A2(3,I,J)=A1(3,I,J)-1.0
    4 CONTINUE
      GO TO 999
   50 DO 5 I=1,M
      DO 5 J=1,N
      A2(1,I,J)=A1(1,I,J)*A1(3,I,J)*(A1(3,I,J)-1.0)
      A2(2,I,J)=A1(2,I,J)
      A2(3,I,J)=A1(3,I,J)-2.0
    5 CONTINUE
      GO TO 999
   60 DO 6 I=1,M
      DO 6 J=1,N
      A2(1,I,J)=A1(1,I,J)*A1(2,I,J)*(A1(2,I,J)-1.0)*(A1(2,I,J)-2.0)*(A1(
     *2,I,J)-3.0)
      A2(2,I,J)=A1(2,I,J)-4.0
      A2(3,I,J)=A1(3,I,J)
    6 CONTINUE
      GO TO 999
   70 DO 7 I=1,M
      DO 7 J=1,N
      A2(1,I,J)=A1(1,I,J)*A1(2,I,J)*(A1(2,I,J)-1.0)*A1(3,I,J)*(A1(3,I,J
     *)-1.0)
      A2(2,I,J)=A1(2,I,J)-2.0
      A2(3,I,J)=A1(3,I,J)-2.0
    7 CONTINUE
      GO TO 999
   80 DO 8 I=1,M
      DO 8 J=1,N
      A2(1,I,J)=A1(1,I,J)*A1(3,I,J)*(A1(3,I,J)-1.0)*(A1(3,I,J)-2.0)*
```

```
                    *(A1(3,I,J)-3.0)
0057                 A2(2,I,J)=A1(2,I,J)
0058                 A2(3,I,J)=A1(3,I,J)-4.0
0059               8 CONTINUE
0060             999 RETURN
0061                 END
```

FORTRAN IV G LEVEL 21 POINTS DATE = 75225 17/33/27

```
      SUBROUTINE POINTS(X,Y,M,A,B,C,D,NP1,NP2)
      IMPLICIT REAL*8(A-H, O-Z)
      DIMENSION X(M), Y(M)
      X(1)=0.0
      Y(1)=0.0
      X1=0.0
      DO 10 I=2,NP1
      X1=X1+(A/B)
      Y(I)=X1
      X(I)=0.0
   10 CONTINUE
      X1=0.0
      DO 20 J=NP1,NP2,NP1
      X1=X1+(C/D)
      DO 20 L=1,NP1
      K=J+L
      Y(K)=Y(L)
      X(K)=X1
   20 CONTINUE
      RETURN
      END
```

```
      SUBROUTINE ZERO(A,L,M,N)
      IMPLICIT REAL*8(A-H, O-Z)
      DIMENSION A(L,M,N)
      DO 1 I=1,L
      DO 1 J=1,M
      DO 1 K=1,N
      A(I,J,K)=0.0
    1 CONTINUE
      RETURN
      END
```

FORTRAN IV G LEVEL 21 SETUPL DATE = 75225 17/33/27

```
      SUBROUTINE SETUPL(A,MO,NO,X,Y,M,B,M1,N1,MM,M3,COEFF,KO)
      IMPLICIT REAL*8(A-H, O-Z)
      DIMENSION A(MO,NO),X(M),Y(M),B(3,M1,N1)
      IF(KO .EQ. 1)KK=0
      IF(KO .EQ. 2)KK=M
      IF(KO .EQ. 3)KK=2*M
      DO 80 I=1,M
      NN=KK+I
      X1=X(I)
      Y1=Y(I)
      DO 70 L1=MM,M1
      L=L1+M3
      TERM1=0.0
      DO 60 K=1,N1
      IF(B(1,L1,K) .EQ. 0.0)GO TO 60
      LL=B(2,L1,K)
      IF(LL .EQ. 0)X3=1.0
      IF(LL .EQ. 1)X3=X1
      IF(LL .GT. 1)GO TO 45
      GO TO 21
   45 ML=LL-1
      X3=X1
      DO 46 JL=1,ML
      X3=X3*X1
   46 CONTINUE
   21 JJ=B(3,L1,K)
      IF(JJ .EQ. 0)Y3=1.0
      IF(JJ .EQ. 1)Y3=Y1
      IF(JJ .GT. 1)GO TO 47
      GO TO 24
   47 NL=JJ-1
      Y3=Y1
      DO 48 IL=1,NL
      Y3=Y3*Y1
   48 CONTINUE
   24 TERM1=TERM1+(COEFF*B(1,L1,K)*X3*Y3)
   60 CONTINUE
      A(NN,L)=A(NN,L)+TERM1
   70 CONTINUE
   80 CONTINUE
      RETURN
      END
```

```
      SUBROUTINE SETUPR(V,CO,MO,X,Y,M,B,C,M1,N1,COEFF1,KO,NNN)
      IMPLICIT REAL*8(A-H, O-Z)
      DIMENSION V(MO),X(M),Y(M),B(3,M1,N1),C(M1),CO(MO)
      IF(KO .EQ. 1)KK=0
      IF(KO .EQ. 2)KK=M
      IF(KO .EQ. 3)KK=2*M
      IF(NNN .EQ. 0)GO TO 10
      GO TO 20
   10 DO 80 I=1,M
      NN=I+KK
      X1=X(I)
      Y1=Y(I)
      TERM2=0.0
      DO 70 I1=1,N1
      IF(B(1,1,I1) .EQ. 0.0)GO TO 70
      LL=B(2,1,I1)
      IF(LL .EQ. 0)X3=1.0
      IF(LL .EQ. 1)X3=X1
      IF(LL .GT. 1)GO TO 45
      GO TO 21
   45 ML=LL-1
      X3=X1
      DO 46 JL=1,ML
      X3=X3*X1
   46 CONTINUE
   21 JJ=B(3,1,I1)
      IF(JJ .EQ. 0)Y3=1.0
      IF(JJ .EQ. 1)Y3=Y1
      IF(JJ .GT. 1)GO TO 47
      GO TO 24
   47 NL=JJ-1
      Y3=Y1
      DO 48 IL=1,NL
      Y3=Y3*Y1
   48 CONTINUE
   24 TERM2=TERM2+B(1,1,I1)*X3*Y3
   70 CONTINUE
      V(NN)=V(NN)+TERM2*COEFF1
   80 CONTINUE
      GO TO 999
   20 DO 30 I=1,M
      NN=KK+I
      X1=X(I)
      Y1=Y(I)
      TERM2=0.0
      DO 50 L1=1,M1
      TERM3=0.0D 00
      DO 40 K=1,N1
      IF(B(1,L1,K) .EQ. 0.0)GO TO 40
      LL=B(2,L1,K)
      IF(LL .EQ. 0)X3=1.0
      IF(LL .EQ. 1)X3=X1
      IF(LL .GT. 1)GO TO 31
      GO TO 32
   31 ML=LL-1
      X3=X1
      DO 33 JL=1,ML
      X3=X3*X1
```

```
   33 CONTINUE
   32 JJ=B(3,L1,K)
      IF(JJ .EQ. 0)Y3=1.0
      IF(JJ .EQ. 1)Y3=Y1
      IF(JJ .GT. 1)GO TO 35
      GO TO 36
   35 NL=JJ-1
      Y3=Y1
      DO 37 IL=1,NL
      Y3=Y3*Y1
   37 CONTINUE
   36 XX=B(1,L1,K)*X3*Y3
      TERM3=TERM3+XX
   40 CONTINUE
      TERM5=TERM3*C(L1)
      TERM2=TERM2+TERM5
   50 CONTINUE
      GO TO(51,52,53,54),NNN
   51 V(NN)=V(NN)+(CO(NN)*TERM2*COEFF1)
      GO TO 30
   52 CO(NN)=TERM2
      GO TO 30
   53 CO(NN)=TERM2*TERM2
      GO TO 30
   54 CO(NN)=TERM2*CO(NN)
   30 CONTINUE
  999 RETURN
      END
```

```
      SUBROUTINE LSTSQR(P,MD,ND,Q,R,S,M,N)
      IMPLICIT REAL*8(A-H, O-Z)
      DIMENSION P(MD,ND), Q(ND,ND), R(MD), S(ND)
      DO 1 I=1,N
      DO 1 J=1,N
      Q(I,J)=0.0
      DO 1 K=1,M
    1 Q(I,J)=Q(I,J)+P(K,I)*P(K,J)
      DO 2 L=1,N
      S(L)=0.0
      DO 2 L1=1,M
    2 S(L)=S(L)+P(L1,L)*R(L1)
      RETURN
      END
```

```
FORTRAN IV G LEVEL  21                    ADD                   DATE = 75225        17/33/27

            SUBROUTINE ADD(B,B1,Q,Q1,N)
            IMPLICIT REAL*8(A-H, O-Z)
            DIMENSION B(N,N), B1(N,N), Q(N), Q1(N)
            DO 10 I=1,N
            Q(I)=Q(I)+Q1(I)
            DO 10 J=1,N
            B(I,J)=B(I,J)+B1(I,J)
      10    CONTINUE
            RETURN
            END
```

```
      SUBROUTINE MATINV(A,ND,N)
      IMPLICIT REAL*8(A-H, O-Z)
      DIMENSION A(ND,ND),INDEX(18,2)
      DO 108 I=1,N
  108 INDEX(I,1)=0
      II=0
  109 AMAX=-1.0D 00
      DO 110 I=1,N
      IF(INDEX(I,1) .NE. 0)GO TO 110
      DO 112 J=1,N
      IF(INDEX(J,1) .NE. 0)GO TO 112
      TEMP=DABS(A(I,J))
      IF(TEMP .LE. AMAX)GO TO 112
      IROW=I
      ICOL=J
      AMAX=TEMP
  112 CONTINUE
  110 CONTINUE
      IF(AMAX)225,115,116
  116 INDEX(ICOL,1)=IROW
      IF(IROW .EQ. ICOL)GO TO 118
      DO 120 J=1,N
      TEMP=A(IROW,J)
      A(IROW,J)=A(ICOL,J)
  120 A(ICOL,J)=TEMP
      II=II+1
      INDEX(II,2)=ICOL
  118 PIVOT=A(ICOL,ICOL)
      A(ICOL,ICOL)=1.0
      PIVOT=1.0D 00/PIVOT
      DO 121 J=1,N
  121 A(ICOL,J)=A(ICOL,J)*PIVOT
      DO 122 I=1,N
      IF(I .EQ. ICOL)GO TO 122
      TEMP=A(I,ICOL)
      A(I,ICOL)=0.0
      DO 124 J=1,N
      A(I,J)=A(I,J)-A(ICOL,J)*TEMP
      IF(DABS(A(I,J)) .LE. 0.10D-14)A(I,J)=0.0D 00
  124 CONTINUE
  122 CONTINUE
      GO TO 109
  125 ICOL=INDEX(II,2)
      IROW=INDEX(ICOL,1)
      DO 126 I=1,N
      TEMP=A(I,IROW)
      A(I,IROW)=A(I,ICOL)
  126 A(I,ICOL)=TEMP
      II=II-1
  225 IF(II)125,127,125
  100 FORMAT('0', 1X, 'ZERO PIVOT', 2I3, D15.7)
  115 WRITE(3,100)ICOL,IROW,AMAX
  127 CONTINUE
      RETURN
      END
```

FORTRAN IV G LEVEL 21 MATMPY DATE = 75225 17/33/27

```
      SUBROUTINE MATMPY(T,ND,U,D,N)
      IMPLICIT REAL*8(A-H, O-Z)
      DIMENSION T(ND,ND),U(ND),D(ND)
      DO 3 I=1,N
      D(I)=0.0
      DO 3 J=1,N
      D(I)=D(I)+T(I,J)*U(J)
    3 CONTINUE
      RETURN
      END
```

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