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# TIME-DOMAIN CHARACTERIZATION OF LINEAR SYSTEMS WITH ARBITRARY INPUTS

by

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1965.

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## ABSTRACT

The subject of this thesis is the characterization of time-invariant or separable time-variant linear systems. Specifically, the problem considered is "The determination of an approximate linear operator which maps a arbitrary given set of inputs into a corresponding set of outputs, on the time domain."

In this thesis, the application of a numerical procedure toward the determination of a linear operator, ~~for real time computation~~ is mechanized ~~proposed~~ and applied. The conditions for the existence of the approximate solution are given and a bound on the errors of the process is established. Also, an algebra for manipulations of weighting functions of linear systems is given.

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I agree that this is a simple form of linear time-variant systems but they are of special interest, because of their simplicity and usefulness in constructing more complex systems.

## ACKNOWLEDGMENTS

The author is pleased to express his sincere thanks and gratitude to Professor G. S. Glinski for suggesting the problem and for his guidance and encouragement throughout the course of this work.

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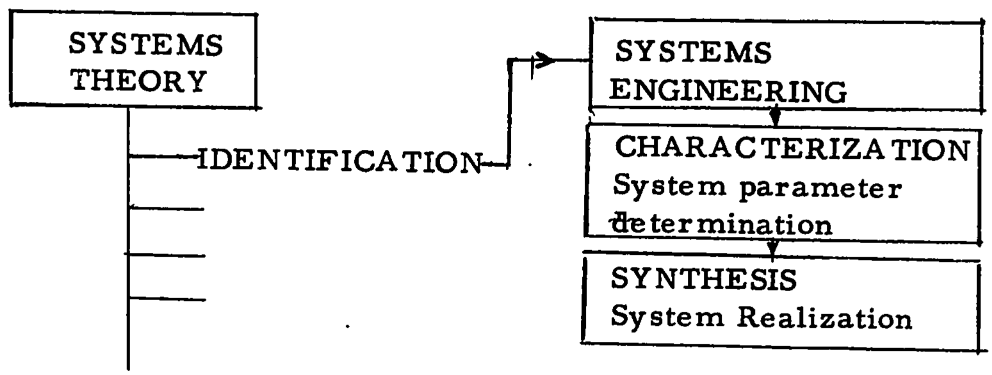
1. Introduction.

The problem of characterizing any technological system belongs to interacting hierarchies as shown below:




In general, the methods of systems theory entail minimal constraints, whereas the methods of systems engineering involve increasing constraints as progress is made toward physical realization. The problem dealt with in this thesis is properly classified as one of characterization which belongs to Systems Engineering [1].

In particular we are considering the characterization of a class of linear systems which have been identified as time-invariant or separable time-variant and having an arbitrary set of inputs $\{u\}$ and a corresponding set of outputs $\{y\}$. The specific problem which is attacked is this: determine an approximate linear operator which maps the set of inputs into the set of outputs, on the time-domain.

Mathematically, the problem is formidable since there may not exist a linear operator which gives an exact correspondence. Moreover, even if a solution exists, it may not be unique. However since our principal interest here is to find a solution to the characterization problem, we are not directly concerned with the question of uniqueness.

The case where the input is a unit impulse function has been extensively dealt with in the litrature [2,3,4]. For more general inputs, the problem has been solved in the frequency domain by transform methods[5,6,7]. However, the task of finding transforms of functions which are not given as simple analytical functions is often formidable.

In this thesis, a numerical procedure for real time computation is proposed and applied toward the solution of the above problem. The conditions for the existence of the solution are given and a bound on the errors of the process is established. It is shown that at a fixed value of the inependent variable, the errors in the solution tend to zero. Also, an algebra for manipulations of weighting functions of linear systems is given.

# 2. GENERAL BACKGROUND

## 2.1 Definitions

### a) System

In this thesis, the word system is used for any process that produces a response (output) when an excitation (input) is applied (the term transmittance is also used). A system may be depicted as a block diagram shown in Fig.(2.1), where W represents the system, u(t) represents the input, y(t) the output, and t is the independent variable.

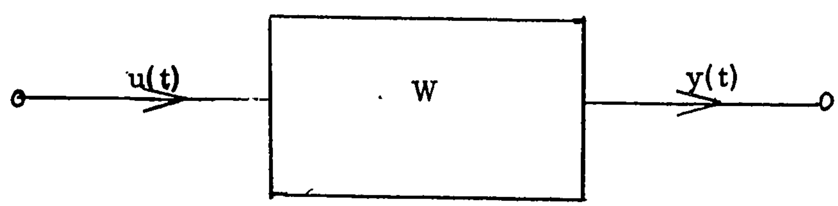



Fig. 2.1. A General System.

### b) Linear System

Suppose, with reference to a given system, that an input $u_1(t)$ produces an output $y_1(t)$, and $u_2(t)$ produces an output $y_2(t)$. The system is said to be linear if an input $c_1u_1(t) + c_2u_2(t)$ produces the output $c_1y_1(t) + c_2y_2(t)$, $c_1$ and $c_2$ being arbitrary constants.

Further, if an input u(t) produces an output y(t) and the input u(t-τ) produces an output y(t-τ), then the system is said to be time-invariant.

On the other hand, if an input u(t) produces an output y(t) and the input $u(t-\tau)$ produces an output different from $y(t-\tau)$, then the system is said to be time-variant.

## 2.2. Class of Systems to be Considered.

We shall be concerned only with real, causal (non-anticipative) continuous time, finite dimensional, linear systems. However we will be limited to the investigation of a class of linear systems that can be described by a set of ordinary differential equations defined on $(-\infty, \infty)$ and of the form[8,9].

$$\frac{dx(t)}{dt} = P(t)\,x(t) + q(t)\,u(t)$$

$$y(t) = s(t)\,x(t) \qquad 2.1$$

where

$u(t)$ = input (scalar)

$y(t)$ = output (scalar)

$x(t)$ = n-vector (the state vector)

$P(t)$ = n x n matrix

$q(t)$ = n-vector

$s(t)$ = n-vector

where $P(\cdot)$, $q(\cdot)$, $s(\cdot)$ are suitably differentiable functions of time.

The dimension of the state space is assumed to be the minimum consistent with known input-output information, in this case represented by the input-output relation.

Also[10,11]

$$\sum_{i=0}^{n} a_i(t) \frac{d^i y(t)}{dt^i} = \sum_{i=0}^{m} b_i(t) \frac{d^i u(t)}{dt^i}, \quad n \geqslant m, \quad a_n(t) \neq 0 \tag{2.2}$$

and

$$\left.\frac{d^i y(t)}{dt^i}\right|_{t=\tau} = 0, \quad i = 0, 1, 2, \ldots, n-1, \quad \tau \leqslant t < \infty$$

and the values of n and m are minimal.

In (2.2), the independent variable $t$ is the time and $\tau$ is the time of application of the input, the $a_i(t)$ and $b_i(t)$ are continuous and deterministic functions of time, $y(t)$ is the output of the system, and $u(t)$ is the input. Without loss of generality, $a_n(t)$ can be assumed to be unity and any or all of the $b_i(t)$'s may be zero in a particular case. The fact that $n \geqslant m$ implies that in a physical system there are no net differentiations between the input and output, which is a valid constraint from a physical standpoint[12] (for a causal system).

k is real time

The energy storage capabilities in lumped-parameter physical systems are associated with the mathematical process of integration in real time. ~~and~~ Moreover, most control systems possess a finite number of significant energy storages, or degrees of freedom, which is in conformity with the constraint imposed by our mathematical model (2.2).

The equations (2.1) and (2.2) give the time domain representation of the system to be discussed here. Although much of what is done will be in the time domain representation, it will be convenient on occasion to consider system from a slightly more abstract point of view than that suggested by (2.1) and (2.2). By way of explanation, we first note that with respect to the input-output relations, one can simply consider a system W as a mapping which takes an abstract space (the input space) into another space (the output space). The members of these abstract spaces may be concretely defined in terms of the time functions or functions related to these time functions by transformations of various kinds. In the latter case, there will be corresponding changes in the concrete representation of mapping between the two spaces. As far as we are concerned the words "input" and "output" will convey either time functions or specified linear transformations on time function.

## 2.3 Weighting Function Representation.

Another method of defining the types of systems investigated here is by means of their weighting functions[8,13] (unit impulse response function). The form of the weighting function can be established as follows:

For a given initial condition $x(t_o) = x_o$, the solution for the output $y(t)$ of (2.1)[8,12] is

$$y(t) = s(t)\phi(t,t_o)\,x_o + \int_{t_o}^{t} s(t)\phi(t,\tau)\,q(\tau)u(\tau)\,d\tau \qquad 2.3$$

where $\phi$ is a fundamental matrix solution of (2.1) satisfying the following equations

$$\frac{d\phi}{dt} = P(t)\phi\,, \qquad 2.4$$

and $\phi(t,t) = I$

Where I is the identity matrix.

We now define the weighting function (or unit impulse response function) as follows:

$$y(t) - s(t)\phi(t,t_o)\,x_o = \int_{t_o}^{t} W(t,\tau)\,u(\tau)\,d\tau \qquad 2.5$$

such that

$$W(t,\tau) = s(t)\phi(t,\tau)\,q(\tau) \qquad 2.6$$

and is defined for all t and $\tau$ such that $t \geqslant \tau$.

~~Further~~, if all initial conditions are zero, the weighting function then completely determines the input-output relation which takes the form

$$y(t) = \int_{t_0}^{t} W(t, \tau)\, u(\tau)\, d\tau \qquad 2.7$$

The weighting function $W(t,\tau)$ is assumed to satisfy the following conditions[14]:

(i) that it be measurable function of $t$ and $\tau$ for whatsoever? $t_0 \leq \tau \leq t$, and

(ii) that (2.7) be well defined for all finite $t \geq t_0$ and all bounded functions $u(t)$, i.e., for all $u(t)$ satisfying

$$|u(t)| < L, \qquad t \geq t_0$$

where $L$ is a finite positive constant.

<u>Definition:</u> A linear system satisfying equation (2.7) and conditions ( (i), (ii) ), is said to be stable[14-17] if bounded inputs give rise to bounded outputs. In other words, $|u(t)| < L$, $t \geq t_0$, implies $|y(t)| < N$, $t \geq t_0$ where $L$ and $N$ are two finite positive constants.

Now we shall give three important theorems concerning weighting functions.

<u>Theorem 1</u> : A linear system satisfying equation (2.7) and conditions ( (i), (ii) ), is stable[14-17] in the sense of the above definition if and only if there exists a finite positive constant $K$ such that

$$\int_{t_0}^{t} \left| W(t, \tau) \right| d\tau < K \qquad 2.8$$

for all finite $t \geq t_0$.

<u>Theorem 2</u>: A function W of two variables t and $\tau$ is a weighting function for a linear differential system (2.1 and 2.2) if and only if it is a separable function of the variables t and $\tau$ of the form

$$W(t, \tau) = \sum_{k=1}^{n} \psi_k(t)\phi_k(\tau) \qquad 2.9(a)$$

<u>Proof</u> See Appendix I.

<u>Theorem 3:</u> A necessary and sufficient condition that a weighting function of a stable system be realizable as a differential system is that

$$\begin{aligned} W(t, \tau) &= \sum_{k=1}^{n} \psi_k(t)\phi_k(\tau) \text{ for } t \geqslant \tau \\ &= 0 \qquad\qquad \text{for } t < \tau \qquad 2.9(b) \end{aligned}$$

If a given weighting function $W(t,\tau)$ is stable and satisfies the realizability condition (2.9b) then it may be realized in the following steps[11, 12]:

(1) A set of linearly independent functions $\{\phi_k(\tau)\}$ is formed.

(2) From the set $\{\phi_k(\tau)\}$ a complete set of orthonormal functions $\{\theta_k(\tau)\}$ is formed.

(3) An approximate expansion of $W(t,\tau)$ in terms of $\theta_k(\tau)$'s is formed with a finite number (n) of terms.

$$W(t, \tau) = \sum_{k=1}^{n} \psi_k(t)\, \theta_k(\tau) \qquad 2.10$$

where the coefficients are given by

$$\psi_k(t) = \int_a^b W(t, \tau)\, \theta_k(\tau)\, d\tau \qquad 2.11$$

Once the coefficients $\psi_k(t)$ are determined from equation (2. 11), the system may be realized as shown below Fig. (2. 2).

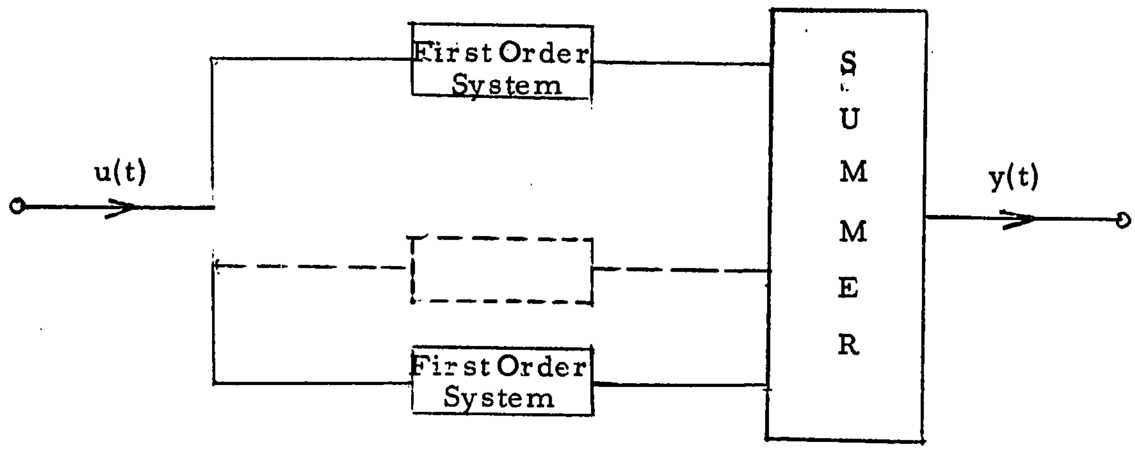



Fig. 2.2

## 2.4 Evaluation of the Weighting Function Representation.

To evaluate the weighting function representation, we ask the following three important questions.

(i). How difficult it is to arrive at this representation?

(ii) How much information about the system response is directly available from this representation?

(iii) How difficult are the operations of combining and manipulating systems using this representation?

The main value of the weighting function representation lies in the fact that it allows us to calculate the response of a system for any input by means of the convolution integral (2.7). Thus from the standpoint of criterion (ii), the weighting function is an excellent method of representation.

To examine question (iii), we shall consider the four main operations of combining the systems.

1. Cascade combination of systems.
2. Parallel combination of systems.
3. Finding an inverse system.
4. A system with feedback.

A cascade combination of systems is depicted in Figure (2. 3. a, b).

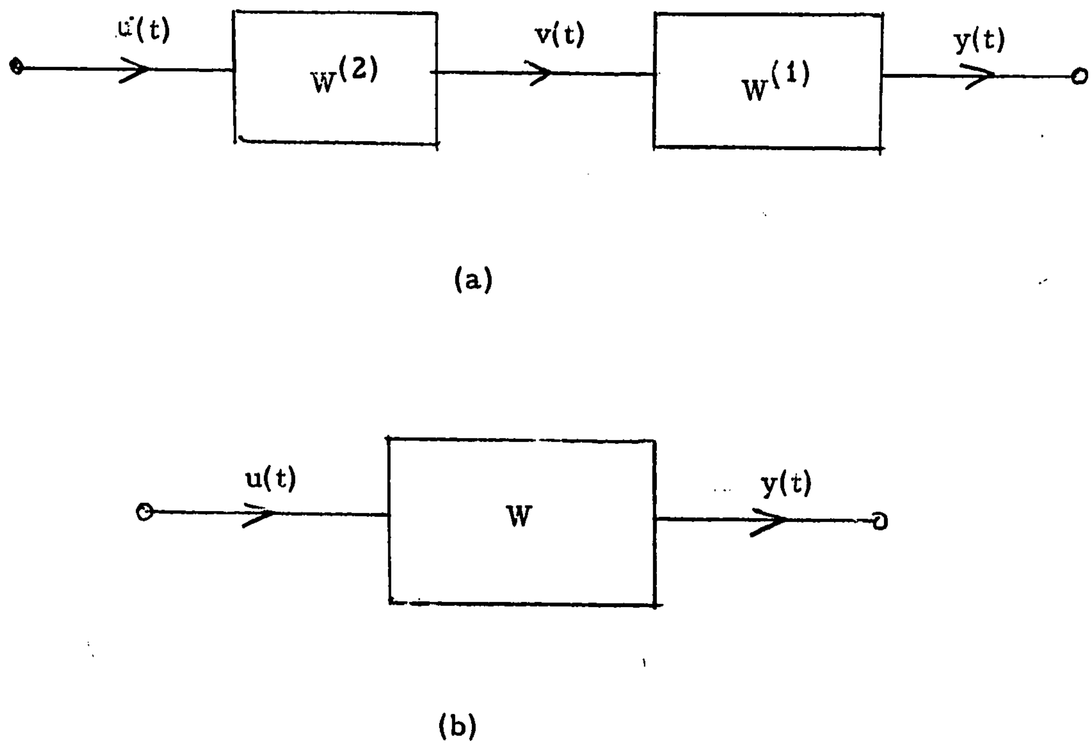



Fig. 2. 3 Cascade Combination of Systems.

Now, let the system $W^{(1)}$ have the weighting function $W^{(1)}(t, \tau)$, and $W^{(2)}$ have the weighting function $W^{(2)}(t, \tau)$.

Then

$$y(t) = \int_{t_o}^{t} W^{(1)}(t,\xi)\, v(\xi)\, d\xi,$$

$$v(t) = \int_{t_o}^{t} W^{(2)}(t, \tau)\, u(\tau)\, d\tau \qquad 2.12$$

Combining these equations, we get

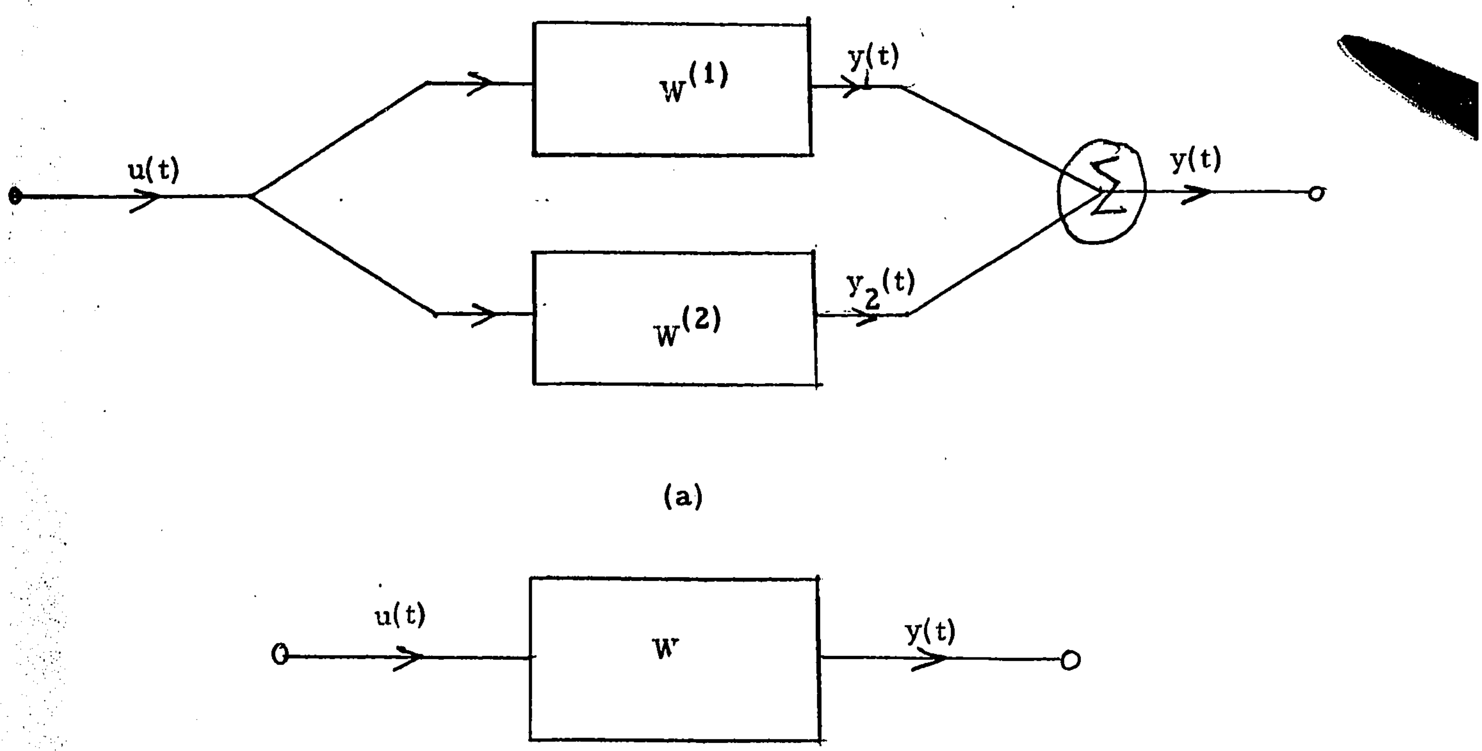
$$y(t) = \int_{t_0}^{t} d\xi\, W^{(1)}(t,\xi) \int_{\xi_0}^{\xi} W^{(2)}(\xi,\tau)\, u(\tau)\, d\tau$$

$$= \int_{t_0}^{t} d\tau\; u(\tau) \int_{\tau}^{t} W^{(1)}(t,\xi)\, W^{(2)}(\xi,\tau)\, d\xi . \qquad 2.13$$

The equivalent system W shown in Fig.(2.3.b) then has the weighting function

$$W(t,\tau) = \int_{\tau}^{t} W^{(1)}(t,\xi)\, W^{(2)}(\xi,\tau)\, d\xi \qquad 2.14$$

A parallel combination of systems is depicted in Figure (2.4. a,b)




(a)

(b)

Fig. 2.4. Parallel Combination of Systems.

Again, let the system $W^{(1)}$ have the weighting function $W^{(1)}(t, \tau)$; then

$$y_1(t) = \int_{t_o}^{t} W^{(1)}(t, \tau)\, u(\tau)\, d\tau \,, \qquad \text{(i)}$$

$$y_2(t) = \int_{t_o}^{t} W^{(2)}(t, \tau)\, u(\tau)\, d\tau \qquad \text{(ii)} \qquad 2.15$$

Adding (i) and (ii) in (2.15), we have

$$y(t) = y_1(t) + y_2(t) = \int_{t_o}^{t} \{ W^{(1)}(t, \tau) + W^{(2)}(t,\tau) \}\, u(\tau) d\tau \qquad 2.16$$

then the equivalent system shown in Fig. (2.4, b) has the weighting function $W(t,\tau)$ defined by

$$W(t,\tau) = W^{(1)}(t,\tau) + W^{(2)}(t,\tau) \qquad 2.17$$

Further, if $y(t) = y_1(t) - y_2(t)$ then the equivalent system $W(t,\tau)$ is

$$W(t,\tau) = W^{(1)}(t,\tau) - W^{(2)}(t,\tau) \qquad 2.18$$

The inverse operation is depicted in Figure(2.5).

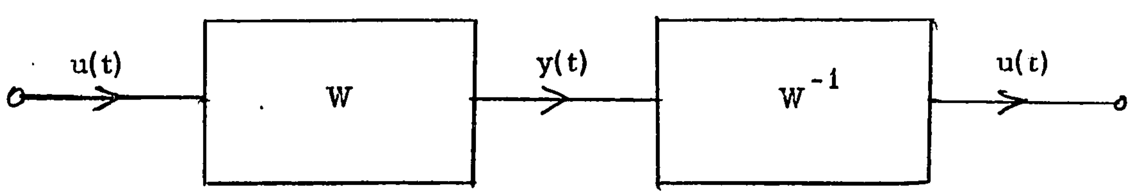



Fig. 2.5 Inverse System.

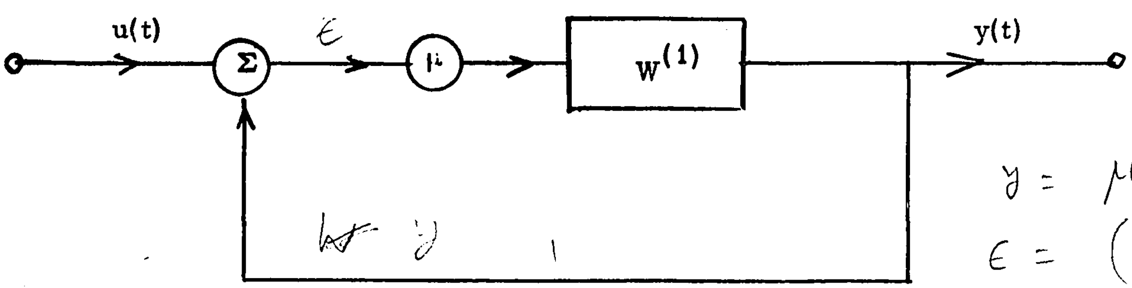
**Definition:** Two systems are inverse to each other if, when u(t) is applied as an input to the cascade combination of these systems, the output is also u(t). The system equivalent to this cascade combination has as its weighting function a dirac delta function, which is apparent from the convolution integral.

$$y(t) = \int_{t_0}^{t} \delta(t-\tau)\, u(\tau)\, d\tau = u(t) \qquad 2.19$$

Then, using (2.14), the weighting function of the inverse system $W^{-1}(t, \tau)$, is the solution of the Volterra integral equation

$$\delta(t-\tau) = \int_{\tau}^{t} W^{-1}(t,\xi)\, W(\xi,\tau)\, d\xi \qquad 2.20$$

where $W(t,\tau)$ represents the weighting function of the system W. Lastly, Figure (2.6. a, b) shows a system with feedback.



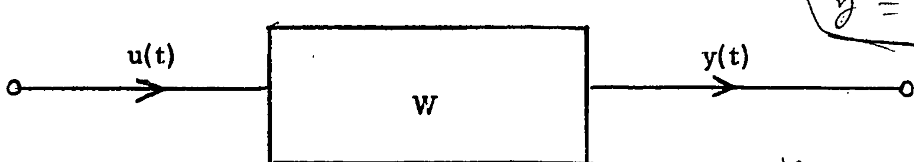

$y = \mu W \epsilon$

$\epsilon = (u + Wy)$

~~$y = \mu u + \mu W$~~

$y = (\mu W_1 u) + \mu W_1 y$

(a)




$W = (I - \mu W_1)^{-1} \mu W_1$

$y =$ ~~$W u$~~ $= W u.$

$y =$ ~~$\mu W_1 u$~~

$= \left(I + \sum \mu^{n-1} W_1^{(n)}\right) \mu W_1 u,$

(b)

Fig. 2.6 A System with Feedback.

$= \left(\mu W_1 + \sum_{n=1}^{\infty} \mu^n W_1^{n+1}\right) u$

$= \mu W_1 u +$

$y =$ ~~$\mu W_1 u$~~

$= \boxed{\left((I - \mu W_1)^{-1} \mu W_1\right) u}$

$y(t) = \int W(t, \tau)\, u(\tau)\, d\tau$

Let the system $W^{(1)}$ have the weighting function $W^{(1)}(t,\tau)$ and the equivalent system $W$ have the weighting function $W(t,\tau)$, ~~which is a solution of a Volterra integral equation of second kind~~[18,19]. what equation ?

Although such an equation cannot be solved explicitly in a closed form, it is possible to develop an infinite series solution for the weighting function $W(t,\tau)$. We note that the output of the system (2.6) may be found from the linear super-position of an infinite sequence of signals, each derived from the input signal u(t). The first signal is then obtained by applying u(t) directly to the weighting function $\mu\, W^{(1)}(t,\tau)$ ( $\mu$ is the gain introduced in the open loop system). feed forward
The second signal is obtained by applying the first signal as an input to the weighting function $\mu\, W^{(1)}(t,\tau)$ through the feedback loop. Similarly, we obtain the third signal and so on. Hence, a recurrence relationship for the infinite sequence of signals is given by (2.21)

$$y_o(t) = \mu \int_{t_o}^{t} W^{(1)}(t,\tau)\; u(\tau)\; d\tau \qquad 2.21$$

$$y_i(t) = \mu \int_{t_o}^{t} W^{(1)}(t,\tau)\; y_{i-1}\; u(\tau)\; d\tau$$

$$i = 1, 2, \ldots \infty$$

Therefore, $y_i(t)$ can be derived directly from u(t) by applying u(t) to a cascade of (i+1) identical weighting functions $\mu\, W^{(1)}(t,\tau)$.

The block diagram illustrating this operation is shown in Figure (2.7)

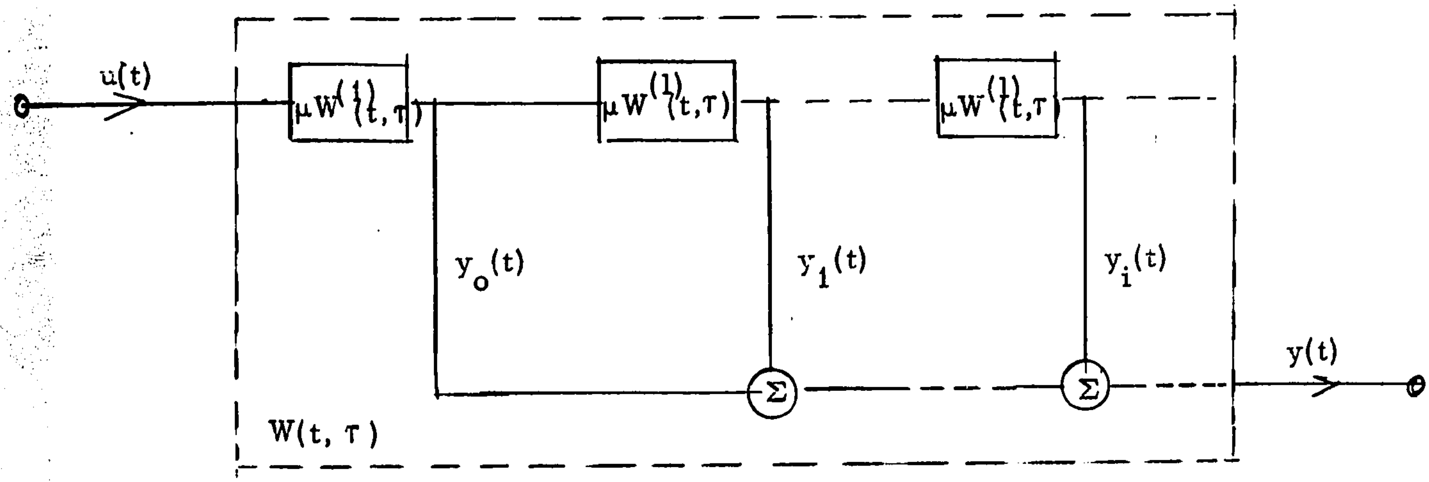



Fig. 2.7 Computation of Input Response Characteristic of a Closed Loop System.

The system (2.7) can be reduced to a single equivalent weighting function $W(t,\tau)$ by repeated application of (2.14). The configuration shown in Figure (2.7) indicates that the gain $\mu$ has a great influence on the composite weighting function $W(t, \tau)$.

Thus we see that the first two operations are simple and straightforward, but the third-the finding of an inverse, and the fourth-reduction of a system with feedback are generally difficult. Moreover, the solution may be in the form of an infinite series, and the solution must therefore be approximated usually by truncating the series.

The disadvantages of the weighting function are that it is very difficult to determine for a given general linear time-variable differential system, and that it is usually not expressible in a closed form. Actually, the amount of difficulty associated with determining a weighting function depends upon the form of the differential system. The problem of determining the weighting function is called the characterization problem.

## 3. METHOD OF CHARACTERIZATION FOR ARBITRARY INPUTS.

In this section, a method for determining weighting functions for a particular class of linear systems is presented. The method is first applied to linear time-invariant systems and further extended to linear time-variant systems whose outputs can be approximated by separable functions. A numerical procedure for real time computation will be proposed and applied. The existence of the solution and a bound on the errors of the process will be considered, proofs being given that at a fixed value of the independent variable, the errors in the solution tend to zero as the step length tends to zero.

### 3.1 Existence of the Solution.

The input-output relation for a linear system if all initial conditions are zero, is given by equation (2.7) and is of the form

$$y(t) = \int_{t_o}^{t} W(t, \tau)\, u(\tau)\, d\tau \qquad 2.7$$

which is depicted in Figure (3.1),

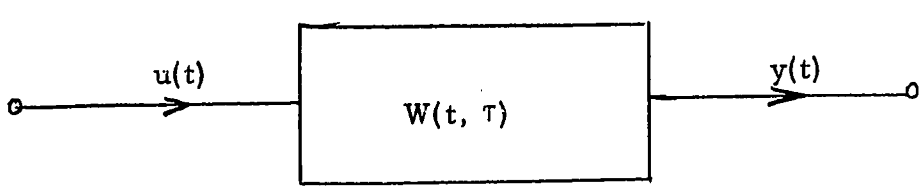



Fig. 3.1. A General System.

where u(t) represents the input, y(t) the output, $\tau$ the time of application of the input, and t is the time of observation. Let A be a linear operator which maps the input space U into the output space Y so the equation in (2.7) takes the form

$$y = Au. \tag{3.1}$$

In order to determine A, we postmultiply both sides of (3.1) by $\tilde{u}$ ($\tilde{u}$ is the transpose of u and is a row vector, taking u to be a column vector) Equation (3.1) takes the form

$$y\tilde{u} = Au\tilde{u} \tag{3.2}$$

Now $u\tilde{u}$ is ~~a~~ an nxn matrix. If its inverse exists, again post-multiplying both sides of equation (3.2) by $(u\tilde{u})^{-1}$ we get

$$y\tilde{u}(u\tilde{u})^{-1} = A \tag{3.3}$$

is not lin[illegible] the function

Hence, the solution of equation (3.1) for A exists if the matrix $u\tilde{u}$ is positive definite. Note that the matrix $u\tilde{u}$ is real, symmetric and has the principal diagonal terms greater than or equal to zero ($\geqslant 0$).

<u>Definition:</u> A real symmetric matrix A is positive definite[20] if and only if its determinant and principal co-factors are all positive. It is positive semidefinite if and only if its determinant and principal co-factors are all non-negative and at least one is zero (In fact the determinant of a positive semi-definite matrix is necessarily zero).

## 3.2 Linear-Time-Invariant Systems.

Consider the situation depicted in Fig. (3.2), where the

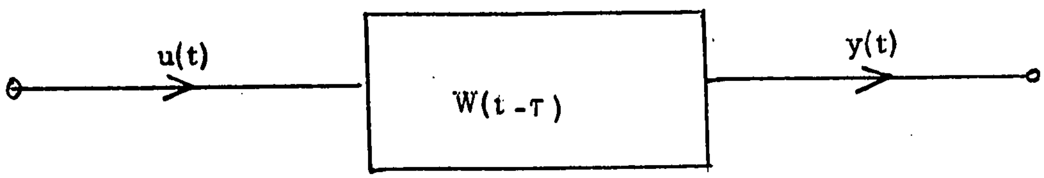



Fig. 3.2 A Time-Invariant System.

excitation u(t) and response y(t) are considered to be (single-valued) real functions vanishing for $t < 0$ and square integrable over the interval $0 < t < \infty$ i.e., $u(t), y(t) \in L_2(o,\infty)$. The function $W(t-\tau)$ represents the weighting function (or the unit impulse response) of the linear time-invariant system shown in Fig. (3.2). Thus,

$$y(t) = \int_0^t W(t-\tau)\, u(\tau)\, d\tau$$

$$= W * u \qquad 3.4$$

where the system is considered to be quiscent for $t < 0$. Our problem is basically to determine, if possible, a realizable function $W(t-\tau)$ from u(t) and y(t); however, the problem generally must be solved by an approximation technique, since $W(\tau)$ must represent a finite system in the practical case. Since u(t) and $W(t-\tau)$ may be interchanged in the convolution $W * u = u * W$ of equation (3.4), we may regard the physical problem as the determination of a system input from the corresponding output and impulse response functions given in the time domain. Therefore, equation (3.4) takes the form:

$$\int_0^t u(t-\tau)\, W(\tau)\, d\tau = y(t) \qquad 3.5$$

$$t - \tau \geqslant 0$$

### 3.2.1 Solution of the Characterization Problem for Linear Time-Invariant System.

Differentiating (3.5) with respect to t, we get

$$u(o)\, W(t) + \int_0^t u_t^{(1)}(t-\tau)\; W(\tau)\, d\tau = y^{(1)}(t) \qquad 3.6$$

Where $\frac{\partial u(t-\tau)}{\partial t} \triangleq u_t^{(1)}(t-\tau)$

and $\frac{dy}{dt} \triangleq y^{(1)}(t)$

If now $u(o) \neq 0$, the step by step process is started by means of the equation

$$u(o)\; W(o) = y^{(1)}(o) \qquad 3.7$$

If $u(o) = o$, further differentiation (n+1) times is required until we reach a derivative $u^{(n)}(o) \neq 0$.

Now, equal intervals of length $a$ are taken along the t-axis.

For the n-th step we have,

$$\int_0^{na} u(na-\tau)\; W(\tau)\, d\tau = y(na) \qquad 3.8$$

Choosing the value of W in any interval $(k-1)a \leqslant t < ka$, as the value in the middle and approximating the integral by the rectangular formula[21], and denoting the approximate value by an asterisk, we have

$$a\left\{\frac{1}{2} u(na)\; W(o) + \sum_{k=1}^{n-1} u\,\{(n-k)a\}\, W^*(ka) + \frac{1}{2}\, u(o)\, W^*(na)\right\} = y(na) \qquad 3.9$$

and we use the convention throughout that $\sum_{k=1}^{o} \equiv 0$.

Rearranging (3.9) we get

$$-\frac{a}{2}u(o)W^{*}(na) = -y\,(na) + a\,\{\frac{1}{2}u(na)W(o) + \sum_{k=1}^{n-1} u[(n-k)a]\,W^{*}(ka)\} \qquad 3.10$$

Unless u(o) = o, we can successively evaluate

$$W^{*}(a),\ W^{*}(2a),\ W^{*}(3a),\ldots\ldots$$

If u(o) = o, then the equivalent equation (3.6) or, if necessary an equation obtained by further differentiation is solved by the method described above.

Further, the exact equation corresponding to (3.10) is

$$-\frac{a}{2}u(o)W(na) = -y(na) + a\{\frac{1}{2}u(na)\,W(o) + \sum_{k=1}^{n-1} u[(n-k)a]\,W(ka)\} + \sum_{k=1}^{n} e_{n,k} \qquad 3.11$$

where the error for any interval $(k-1)\,a \leqslant \tau < ka$ is given by (3.12)

$$e_{n,k} = \int_{(k-1)a}^{ka} u(na-\tau)\,W(\tau)\,d\tau - \frac{a}{2}[u(na-\overline{k-1}a)W(\overline{k-1}a) + u(na-ka)\,W(ka)] \qquad 3.12$$

Practical ways of checking the accuracy of a numerical computation of this type are either to repeat the solution using a different step length (a), or to approximate the integrals in the equations by using more accurate quadrature formula[23,24] than one used here.

### 3.2.2. Sources of Error in the Solution.

There are two different types of errors in such a numerical computation

(i) Rounding errors

(ii) Truncation errors.

In going from n-th step to (n+1)-th step, truncation errors occur from two sources,

(a) due to replacing the integral by a quadrature formula.

(b) due to the fact that the values of the solution at the first n steps, all of which go into the quadrature formula for the evaluation of the solution at the (n+1)-th step, have corresponding errors in them.

Since it is shown in Appendix II (Compare equation (II. 8) and (II. 9) ) that the truncation errors are in general of opposite sign, a smoothing process may be adopted as follows:

Let us denote the sequence of values

$$W(o),\ W^*(2a),\ W^*(4a), \ldots\ldots$$

by $$W_f(o),\ W_f(2a),\ W_f(4a), \ldots\ldots$$

and complete the sequence

$$W_f(o),\ W_f(a),\ W_f(2a), \ldots\ldots$$

by suitable interpolation.

Similarly, from the sequence

$$W^*(a),\ W^*(3a) \ldots\ldots\ldots$$

the sequence

$$W_b(a),\ W_b(2a),\ W_b(3a), \ldots\ldots$$

is formed by suitable interpolation.

The two sequences then give approximate bounds to the solution, and smoothed solution is given by

$$W_s(na) = \frac{1}{2} [ W_f(na) + W_b(na) ] \qquad 3.13$$

Now, we will illustrate this procedure by the following example.

### 3.2.3 Example,

$$\int_0^t W(t-\tau)\, u(\tau)\, d\tau = y(t)$$

Where $u(\tau) = 1$, and $y(t) = \sin t$.

Obviously $W(t) = \cos t$.

The functions are given at intervals of 0.1 in $t$ up to 5 significant figures. To formulate the sequences $W_f(n\alpha)$, $W_b(n\alpha)$ polynomial interpolation is used. The results are tabulated in Table 1.

Table I

Forward Interpolation

| t | u(t)=1 | y(t) = sint | $W^*(t)$ Direct Solution | $W_f(t)$ ~~Forward Interpre-tation~~ | $W_b(t)$ ~~Backward Interpre-tation~~ | $W_s(t) = \frac{1}{2}[W_f+W_b]$ | Exact Value W(t) |
|---|---|---|---|---|---|---|---|
| 0 | 1.0000 | 0.00000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| 0.1 | 1.0000 | 0.09983 | 0.997 | 0.995 | 0.997 | 0.996 | 0.995 |
| 0.2 | 1.0000 | 0.19867 | 0.980 | 0.980 | 0.980 | 0.980 | 0.980 |
| 0.3 | 1.0000 | 0.29552 | 0.957 | 0.955 | 0.957 | 0.956 | 0.955 |
| 0.4 | 1.0000 | 0.38942 | 0.921 | 0.921 | 0.921 | 0.921 | 0.921 |
| 0.5 | 1.0000 | 0.47943 | 0.879 | 0.877 | 0.879 | 0.878 | 0.878 |
| 0.6 | 1.0000 | 0.56464 | 0.825 | 0.825 | 0.825 | 0.825 | 0.825 |
| 0.7 | 1.0000 | 0.64422 | 0.766 | 0.765 | 0.766 | 0.766 | 0.765 |
| 0.8 | 1.0000 | 0.71736 | 0.696 | 0.696 | 0.698 | 0.697 | 0.697 |
| 0.9 | 1.0000 | 0.78333 | 0.623 | 0.622 | 0.623 | 0.622 | 0.622 |
| 1.0 | 1.0000 | 0.84147 | 0.540 | | | | 0.540 |

3.2.4 Multivariable Systems. The same procedure can be extended to systems of simultaneous equations of the form

$$\int_0^t W(t-\tau)\, u(\tau)\, d\tau = y(t). \qquad 3.14$$

Read this or give [illegible]

where

$$W(\xi) = \begin{bmatrix} W_{11}(\xi) \cdots\cdots W_{1n}(\xi) \\ \vdots \qquad\qquad \vdots \\ W_{n1}(\xi) \cdots\cdots W_{nn}(\xi) \end{bmatrix} u(t) = \begin{bmatrix} u_1(t) \\ \vdots \\ u_n(t) \end{bmatrix} y(t) = \begin{bmatrix} y_1(t) \\ \vdots \\ y_n(t) \end{bmatrix}$$

## 3.3. Separable Linear Time-Variant Systems.

The analysis of linear differential time-varying systems[25,26] is, in general, made quite difficult by the lack of a general method for obtaining the "basis" functions of an n-th order operator. Fortunately, the use of digital computers somewhat simplifies the analysis problem. However, situations may arise in which it is not desirable to implement a computer solution to every input-output problem for a particular system. That is, in some cases, one might wish to have some analytical idea of the characteristics of the system, perhaps by making use of some initial computer calculations, so that one could approximately calculate the response to a given input without further recourse to the computer. One way of doing this is to represent the given system approximately by a separable (degenerate or algebraic[21]) weighting function. The input output relation for a linear system if all the initial conditions are zero, is given by the equation (2.7),

$$y(t) = \int_{t_0}^{t} W(t,\tau)\, u(\tau)\, d\tau \qquad 3.7$$

where $u(\tau)$ represents the excitation at time $\tau$ , $y(t)$ is the response and $w(t,\tau)$ is the impulse response measured at time $t$ due to an impulse applied at time $\tau$. For a physical system the non-zero data exists in the region which projects onto the horizontal plane shaded in Fig. (3.2).

We may note that for a time-invariant system

$$\frac{\partial W(t,\tau)}{\partial t} = 0. \qquad 3.15$$

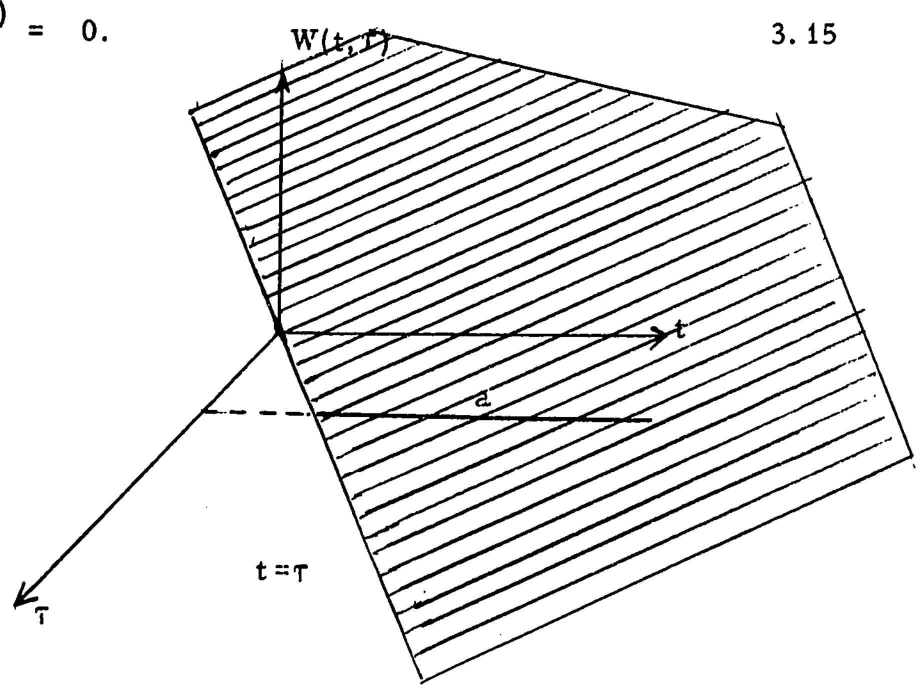



Fig. 3.2. Area of Non-Zero Data for Weighting Functions of Physical Systems.

The conditions of realizability can be expressed mathematically as follows:

$$\lim_{t \to \infty} W(t,\tau) = 0 \qquad 3.16$$

and $$\lim_{\tau \to -\infty} W(t,\tau) = 0$$ for all finite t

&

$t \to -\infty$

The possibility of conveniently solving equations with a separable system function (degenerate kernel) naturally leads to the thought that in solving an equation with an arbitrary system function we might replace this system function approximately by a separable one, and solve the resulting equation rather than the original one.

A suitable separable system function close to the given one can be found in many ways. In particular, as such a system function, one can adopt a segment of the Taylor series or a segment of the Fourier series, of the form

$$W(t,\tau) = \sum_{k=1}^{n} \psi_k(t)\phi_k(\tau), \qquad t \geqslant \tau \qquad 3.17$$

$$= 0 \qquad t < \tau$$

which converges to $W(t,\tau)$ in some sense.

This is illustrated by the following two examples.

<u>Example 1</u> Solve the equation

$$\int_0^{\frac{1}{2}} W(t,\tau)\, u(\tau)\, d\tau = y(t) \qquad 3.18$$

for $W(t,\tau)$, where $W(t,\tau)$ is taken as a segment of the Taylor series., i.e. $W(t,\tau) = \sum_{i,j=0}^{n} C_{ij}\, t^i \tau^j$ 3.19

and $u(\tau) = 1,\ y(t) = \frac{1}{t}(1 - \cos\frac{t}{2})$

$$= \frac{t}{8} - \frac{t^3}{384} + \frac{t^5}{46080} - \ldots\ldots$$

Substituting (3.19) in (3.18) we get

$$\int_0^{\frac{1}{2}} \sum_{i,j=o}^{n} C_{ij} t^i \tau^j \, u(\tau)\, d\tau = y(t)$$

Integrating we get,

$$\sum_{i,j=o}^{n} C_{ij}\, t^i \left| \frac{\tau^{j+1}}{j+1} \right|_o^{\frac{1}{2}} = y(t)$$

or

$$\sum_{i,j=o}^{n} C_{ij} t^i \frac{1}{(j+1)2^{j+1}} = \frac{t}{8} - \frac{t^3}{384} + \frac{t^5}{46080} ----- \qquad 3.20$$

Now, equating the coefficients of like powers of t, in (3.20) we get,

$$C_{oo} = o; \qquad C_{11} = 1; \; C_{22} = C_{21} = C_{12} = 0; \; C_{31} = C_{32} = C_{13} = C_{23} = 0$$

$$C_{33} = -\frac{1}{6}, \; C_{44} = C_{41} = C_{42} = C_{43} = C_{14} = C_{24} = C_{34} = 0$$

$$C_{55} = \frac{1}{120}, \; C_{51} = C_{52} = C_{53} = C_{54} = C_{15} = C_{25} = C_{35} = C_{45} = 0.$$

$$\therefore \quad W(t,\tau) = t\tau - \frac{t^3\tau^3}{6} + \frac{t^5\tau^5}{120} - \cdots\cdots\cdots\cdots$$

which is the Taylor's expansion of sint$\tau$ .

Hence $\qquad W(t,\tau) = \text{Sin}\; t\tau$ .

<u>Example 2</u>

Solve the equation

$$\int_0^1 W(t,\tau)\, u(\tau)\, d\tau = y(t) \qquad 3.21$$

for $W(t, \tau)$, where $W(t,\tau) = \sum_{i,j=o}^{n} C_{ij} t^i \tau^j$

and $u(\tau) = 1$ ; $y(t) = \frac{1}{t}(e^t - 1)$

Substituting $W(t, \tau) = \sum_{i,j=o}^{n} C_{ij} t^i \tau^j$ in (3.21) and integrating, we get

$$\sum_{i,j=o}^{n} C_{ij} t^i \left| \frac{\tau^{j+1}}{j+1} \right|_o^1 = y(t)$$

$$\sum_{i,j=o}^{n} C_{ij} \frac{t^i}{j+1} = 1 + \frac{t}{2} + \frac{t^2}{6} + \frac{t^3}{24} \; \ldots\ldots \qquad 3.22$$

Now, equating like powers of t in (3.22) we get

$$C_{oo} = 1, \; C_{11} = 1, \; C_{22} = \frac{1}{2} \; ; \; C_{21} = C_{12} = 0 \; ,$$

$$C_{33} = \frac{1}{6}, \; C_{31} = C_{32} = C_{13} = C_{23} = 0$$

$$\therefore \; W(t, \tau) = 1 + t\tau + \frac{t^2\tau^2}{2} + \frac{t^3\tau^3}{6} + \ldots\ldots$$

Hence $W(t,\tau) = e^{t\tau}$

Once the desired series has been found, the problem is reduced to synthesizing[27-31] a multi-output fixed linear system and a multi-input time-varying gain amplifier. The basic structure is shown in Figure (3.3). In some cases it is convenient and desirable to use single-input single-output subsystems as shown in Figure (3.4).

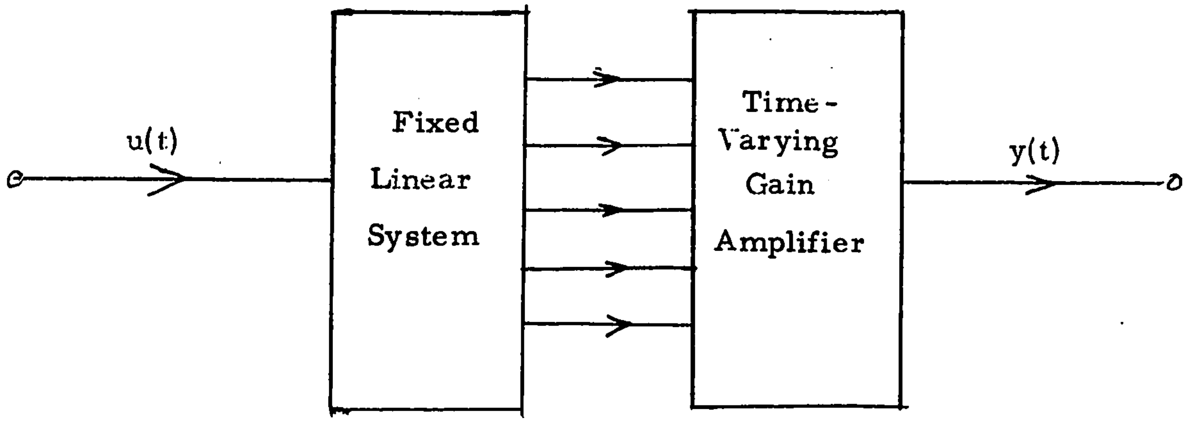



Fig. 3.3. Structural Decomposition of a Separable System.

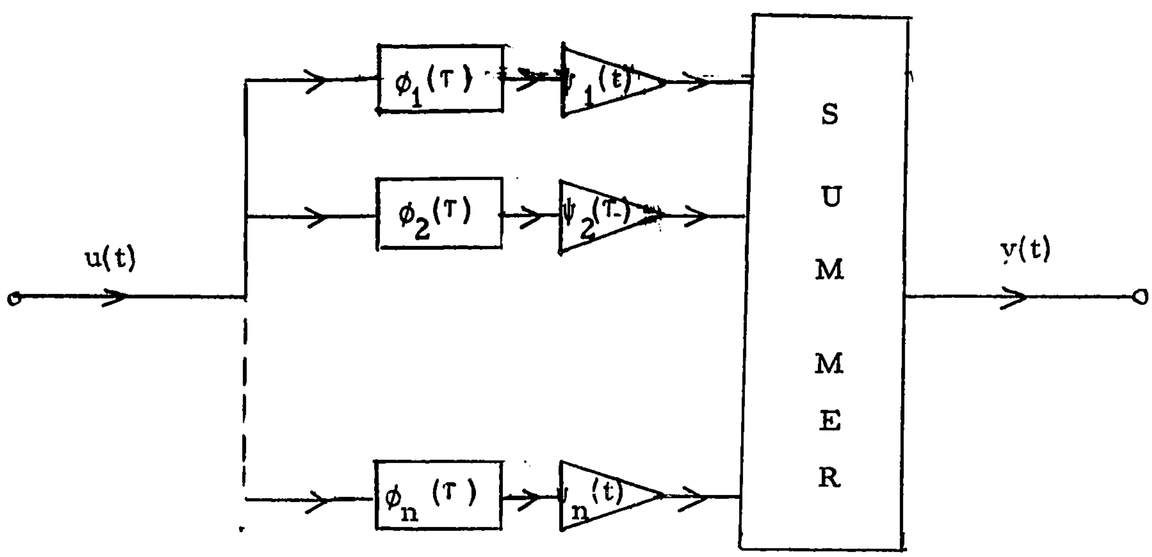



Fig. 3.4. A Special Case of the Structure in Figure 3.3.

Another structure which is a special case of the one in Figure (3.3) is depicted in Figure (3.5). Lees' Laguerre network[32], for example, leads to a structure such as that shown in Figure (3.5). A generalized impulse train approximation[27] to the weighting function $W(t, \tau)$ leads to a tapped delay line realization, which is also of the same form as depicted in Figure (3.5).

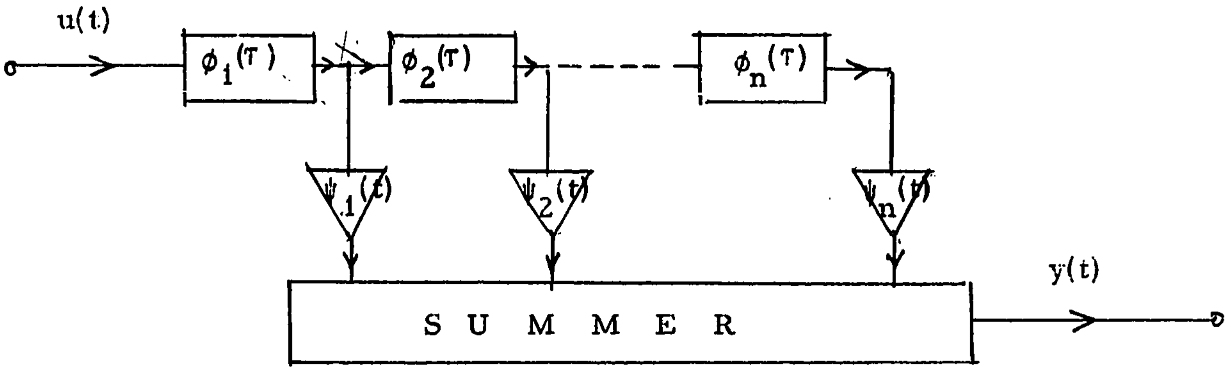
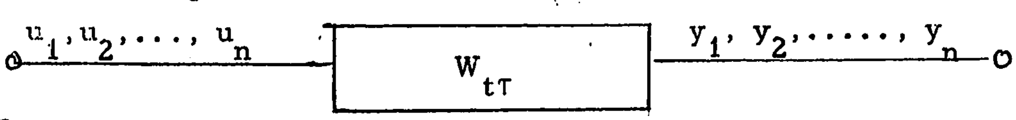



Fig. 3.5. Another Special Case of the Structure Depicted in Fig. 3.3.

3.3.1 Solution of the Characterization Problem for Separable Linear Time-Variant Systems.




$$y(t) = \int_{t_0}^{t} W(t, \tau)\, u(\tau)\, d\tau$$

$$y_{t\tau} = \sum_{1}^{n} W_{t\tau}\, u_{\tau}\ , \quad t = 1, \ldots\ldots, n.$$

$$y = W\, u \text{ (matrices)}$$

$$W_{t\tau} = 0,\ \tau > t \quad (\therefore \text{ Matrix } W \text{ is triangular}).$$

Fig. 3.6. Linear System with Discrete Input and Output Signals.

Fig. 3.6: Depicts a Linear Time-Variable System with Discrete Input and Output Signals.

The input is a sequence of n discrete variables (or samples) $u_1$, $u_2$, $\ldots u_n$, and the output is a sequence of n discrete variables (or samples) $y_1, y_2, \ldots, y_n$. Both sequences are ordered, by association with the n discrete times (or sampling times) $t_1, t_2, \ldots t_n$. Integration is now replaced by summation, and the weighting function W is an nxn matrix. For any element $W_{t\tau}$ in the matrix W, the row index t corresponds to the response time, and the column index $\tau$ corresponds to the excitation time of the continuous signals. As for any physical system, there is no response before excitation, therefore, all the elements in the matrix W which lie above the principal diagonal are zero. Consequently, the matrix W takes the form of a triangular matrix for any physical system.

Before going further, let us stipulate the conditions under which we are working.

We assume that the output y(t) and hence the system function $W(t, \tau)$ are piecewise differentiable with respect to t. ~~Further,~~ Also, ~~that~~ the system function is of bounded variation over the interval considered.

Defintion: Functions of bounded variation[35].

Let f(t) be bounded in (a,b) and let $a = t_o, t_1, t_2, \ldots t_{n-1}, t_n = b$, be a mode of division of this interval let $y_o, y_1, \ldots, y_{n-1}, y_n$ be the values of f(t) at these points.

Then

$$\sum_{o}^{n-1} (y_{r+1} - y_r) = f(b) - f(a) = p - n, \qquad 3.23$$

where p is the sum of the positive differences, and $-n$ the sum of the negative differences. The sum $\sum_{o}^{n-1} \left| y_{r+1} - y_r \right|$ is denoted by Z and we have

Cal

$$Z = \sum_{o}^{n-1} \left| y_{r+1} - y_r \right| \qquad o = p+n \tag{3.24}$$

To every mode of division of (a, b) into such partial intervals, there correspond sums Z, p and n.

When the sums Z corresponding to all possible modes of division of (a, b), are bounded and their upper bound is B, we say that B is the total variation of f(t) in (a, b), and that f(t) is of bounded variation in this interval.

Further we have assumed that the system function $W(t, \tau)$ is separable and of the form [34, 35]

$$\begin{aligned} W(t, \tau) &= f(t-\tau) \cdot g(\tau) \qquad t \geqslant \tau \\ &= 0 \qquad t < \tau \,. \end{aligned} \tag{3.25}$$

The input-output relation takes the form:

$$\int_{t_o}^{t} f(t-\tau)\, g(\tau)\, u(\tau)\, d\tau = y(t) \tag{3.26}$$

Differentiating (3.26) with respect to t, we have

$$f(t-t)\, g(t)\, u(t) + \int_{t_o}^{t} f^{(1)}(t-\tau)\, g(\tau)\, u(\tau)\, d\tau = y^{(1)}(t) \tag{3.27}$$

where $f^{(1)}_{t}(t-\tau) \triangleq \dfrac{\partial f(t-\tau)}{\partial t}$

and $y^{(1)}(t) \triangleq \dfrac{dy}{dt}$

Now, taking equal intervals of length $a$ along the $t$ axis as well as the $\tau$ -axis, for the i-th step, the equations (3.26) and (3.27) take the following form:

$$\int_{t_o}^{t_i} f(t_i-\tau)\, g(\tau)\, u(\tau)\, d\tau = y(t_i) \qquad 3.28$$

and

$$f(t_i-t_i)g(t_i)u(t_i) + \int_{t_o}^{t_i} f^{(1)}(t_i-\tau)g(\tau)u(\tau)\, d\tau = y^{(1)}(t_i) \qquad 3.29$$

$$\text{where } t_o + i\,a \leqslant t_i < t_o + (i+1)a$$

Now replacing the integration in (3.28) and (3.29) by a sum, using the rectangular formula for approximating an integral, we get

$$\sum_{j\leqslant i} f(t_i-t_j)\, g(t_j)u(t_j)a = y(t_i)$$

$$i = 0, \ldots, n \qquad 3.30$$

and

$$f(t_i-t_i)g(t_i)u(t_i) + \sum_{j\leqslant i} f^{(1)}(t_i-t_j)g(t_j)u(t_j)a = y^{(1)}(t_i) \qquad 3.31$$

$$i = 0, \ldots, n$$

For any two adjacent values $f(t_i-t_j)$, $f(t_i+1-t_j)$ on any line $a$ in Figure (3.2), when $a$ is sufficiently small, we assume a relationship of the form

$$f(t_i+1-t_j) = f(t_i-t_j) + a f^{(1)}(t_i-t_j), \; j\leqslant i, i=0, \ldots, n-1.$$

To simplify the notation we will rewrite (3.30), (3.31) and (3.32) in the following form

$$\sum_{j\leqslant i} f_{i-j}\, y_j\, u_j\, a = y_i \qquad 3.33$$

and

$$f_{i-i}\, g_i u_i + \sum_{j\leqslant i} f^{(1)}_{i-j}\, g_j u_j a = y_i^{(1)} \qquad 3.34$$

$$i = 0, \ldots, n$$

and

$$f_{i+1-j} = f_{i-j} + f^{(1)}_{i-j}\,a \qquad 3.35$$

$$j \leq i$$

$$i = 0, \ldots, n-1.$$

Writing (3.33) for $i = k$, and $i = k-1$, we get,

$$f_k g_o u_o a + f_{k-1} g_1 u_1 a + f_{k-2} g_2 u_2 a + \ldots$$

$$\ldots + f_2 g_{k-2} u_{k-2} a + f_1 g_{k-1} u_{k-1} a + f_o g_k u_k a = y_k \qquad 3.36$$

and

$$f_{k-1} g_o u_o a + f_{k-2} g_1 u_1 a + \ldots + f_2 g_{k-3} u_{k-3} a + f_1 g_{k-2} u_{k-2} a + f_o g_{k-1} u_{k-1} a$$

$$= y_{k-1} \qquad 3.37$$

Subtracting (3.37) from (3.36) and using (3.35) we get

$$(f^{(1)}_{k-1} g_o u_o a + f^{(1)}_{k-2} g_1 u_1 a + \ldots\ldots$$

$$+ f^{(1)}_1 g_{k-2} u_{k-2} a + f^{(1)}_o g_{k-1} u_{k-1} a)\, a + f_o g_k u_k a = y_k - y_{k-1} \qquad 3.38$$

Using (3.34) for $i = k-1$ in equation (3.38) we get

$$(y^{(1)}_{k-1} - f_o g_{k-1} u_{k-1} + f_o g_k u_k)\, a = y_k - y_{k-1} \qquad 3.39$$

Writing (3.33) for $i = 0$, we get

$$f_o\, g_o\, u_o\, a = y_o \qquad 3.40$$

If now $u(t_o) \neq 0$, and normalizing $g_o$ (i.e. taking $g_o = 1$) the step by step process is started by means of the equation (3.40). If $u(t_o) = o$, start from $t = t_o + \epsilon$ where $u(t_o + \epsilon) \neq 0$, and $\epsilon \ll a$.

Thus solving equation (3. 39) for $i=1, \ldots n$; we get $g_1, \ldots, g_n$.

Now solving (3. 33) for $i=1, \ldots n$; and using $f_o, g_o, g_1, \cdots g_n$; we get $f_1, f_2, \ldots, f_n$.

Now we can form the system matrix W as follows.

$$\begin{bmatrix} g_o \cdot f_o & & & & \\ g_o \cdot f_1 & g_1 \cdot f_o & & & \\ g_o \cdot f_2 & g_1 \cdot f_1 & g_2 \cdot f_o & & \\ \vdots & & & \ddots & \\ g_o \cdot f_n & g_1 \cdot f_{n-1} & g_2 \cdot f_{n-2} & \cdots & g_n \cdot f_o \end{bmatrix} \qquad 3.41$$

The matrix (W) represented by (3. 36) is called system matrix, because the matrix (W) consists of collection of numbers which are characteristics of the system and are independent of the input and output signals. Therefore, the matrix (W) constitutes a formal numerical representation of the system.

The above representation is ~~available~~ amenable to all the rules of interconnecting subsystems and has almost all the properties of matrix algebra. This is shown in detail in Appendix III.

Since the system matrix (W) is a triangular matrix, it offers us many remarkable advantages. Because all the elements above the diagonal are zero, the numerical operations such as multiplication and inversion are greatly simplified. Moreover, one can tell at a glance if the inverse matrix $W^{-1}$ exists by simply noting that none of the diagonal elements are zero. Any element above the diagonal being non-zero indicates the instability of the system, and that there exists at least one pole in the right half of the s-plane.

Since the product and sum of two or more triangular matrices is again a triangular matrix, it follows that the cascade and parallel combination of stable, physically realizable systems, are stable and physically realizable also.

3.3.2 Example.

The algorithm for calculating the system matrix, developed in section (3.3.1), was tried for the following example,

$$W(\tau) = 1. \quad y(t) = t\,e^{-t}; \quad 0 \leq t < \infty$$

The output was calculated using the convolution integral and taking the weighting function $W(t,\tau)$ to be $e^{-(t-\tau)} \cdot e^{-\tau}$, assuming all the initial conditions to be zero.

Preliminary calculations were made for intervals of length 0.01 sec, 0.005 sec, 0.0025 sec, and the final values for interval of length 0.001 sec, along with the exact values, are tabulated in Appendix IV. Examining table II, we conclude that our calculated values agree fairly-well with the exact values within the assumptions made.

A few words about the preliminary choice of the time-interval are appropriate here :

(i) According to Shannon's well known sampling theorem[36] any signal limited to the bandwidth W c.p.s. can be determined completely by specifying its ordinates at a series of points $1/2W$ second apart.

(ii) We also know that time constant is inversely proportional to the bandwidth.[37] Therefore (i) and (ii) enable us to make a reasonable preliminary choice of the time interval.

The following comments can be made regarding the realization of the system from the system matrix.

The finite number of equidistant values calculated above, corresponding to the time-invaria~~ble~~nt part of the system (i.e., $f_o\, f_1, \ldots, f_n$), are used to construct the function ~~f(t),~~ (38) whose Laplace transform F(s), is physically realizable as the transfer function of a passive network with lumped parameters. Then the system can be realized as shown in Figure (3.7).

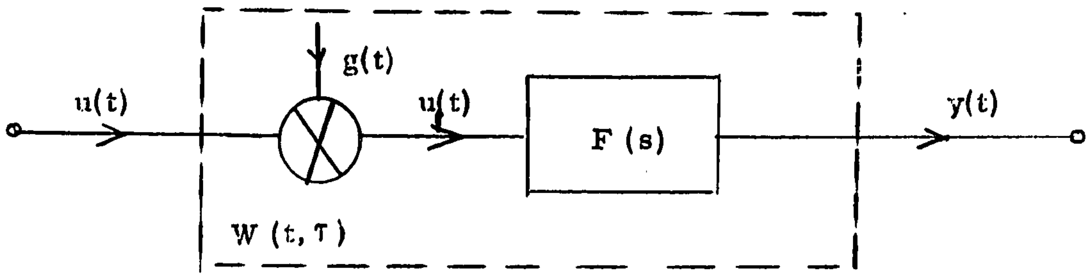



Fig. 3.7 Realization of Separable Linear Time-Variable System.

where F(s) represents the time-invaria~~ble~~nt portion of the system, and u(t) is the modified driving function which "includes" the time-variable portion.

## 3.4 Finding a Representation for Weighting Function from the System Matrix.

Once we have available to us the system matrix (W) in the form of regularly spaced points of data on the t, τ plane, the problem is to obtain a representation for weighting function W(t,τ) as a summation of products of some basis functions in t with basis functions in τ such that the resulting function fits the data points exactly. First of all, we have to assume that the system is asymptotically stable so that we can truncate the data in the direction of increasing t and $\tau$. The procedure for separable systems can be, in general, outlined as follows.

Firstly, we find a set of basis functions $[\theta'_i]$ which fit the data along the t-axis for any value of the discrete variable $\tau$. Then we find a set of basis functions $[\theta''_j]$ which fit the data along the $\tau$-axis for any value of the discrete variable t. Using the set of bases $[\theta'_i]$ and $[\theta''_j]$, we find a representation for the weighting function of the form

$$W(t,\tau) = \sum_{i,j}^{n} D_{ij} \; \theta'_i \; \theta''_j$$

Now, we shall give the procedure to represent the weighting function $W(t,\tau)$ on a set of exponential basis using Weiss - McDonough[39,40] version of the Prony's[41] classical method.

We have available to us 2n sampled values at the sampled interval $a$, which may be written as:

$$W^{(s)}(t) = W(t)\, i\,(t) \qquad 3.42$$

where $i\,(t)$ represents a series of unit impulses equally spaced in time

$$i(t) = \sum_{m=o}^{2n-1} \delta(t-ma) \qquad 3.43$$

We want to determine 2n parameters $[\beta_i, s_i]$ in the function

$$W_a(t) = \sum_{i=1}^{n} \beta_i e^{s_i t} \qquad 3.44$$

Such that $W_a(t)$ passes through 2n points given by the equation (3.42).

The frequency domain representation of (3.43) is,

$$I(s) = 1 + e^{-sa} + e^{-2sa} + \ldots\ldots$$

$$= \frac{1}{1 - e^{-sa}} \qquad 3.45$$

Let us represent the sampled value of the signal W(t) at the m'th sampling instant as $W_m$, then we have

$$W^{(s)}(t) = \sum_{m=o}^{2n-1} W_m \, \delta\,(t-ma) \qquad 3.46$$

and frequency domain representation of (3.46) is

$$W^{(s)}(s) = \sum_{m=o}^{2n-1} W_m \, e^{-msa} \qquad 3.47$$

$$\text{Let } z = e^{sa} \qquad 3.48$$

$$W^{(s)}(z) = W^{s}(s)$$

$$= \sum_{m=o}^{2n-1} W_m \, z^{-m}$$

$$= W_o + W_1 z^{-1} + W_2 z^{-2} \cdots + W_{2n-1} z^{-(2n-1)} \qquad 3.49$$

and we wish to have a rational function approximation $W_a^{(s)}(z)$ of the form

$$W_a^{(s)}(z) = \frac{b_n z^n + b_{n-1} z^{n-1} + \cdots + b_1 z}{z^n + \beta_{n-1} z^{n-1} + \cdots + \beta_1 z + \beta_o} \qquad 3.50$$

The time domain expansion of $W_a^{(s)}(z)$ has first 2n terms which match 2n terms of $W^{(s)}(t)$ given by (3.46).

Thus by equating (3.49) and (3.50) and multiplying both sides by the denominator of (3.50) we have

$$b_n z^n + b_{n-1} z^{n-1} + \cdots + b_2 z^2 + b_1 z$$

$$\simeq (z^n + \beta_{n-1} z^{n-1} + \cdots + \beta_1 z + \beta_o)$$

$$(W_o + W_1 z^{-1} + \cdots + W_{2n-1} z^{-(2n-1)}) \qquad 3.51$$

Equating the coefficient of like powers of $z$ on both sides of (3.51) we get,

$$\begin{aligned}
W_o\beta_o + W_1\beta_1 + \cdots + W_{n-1}\beta_{n-1} &= -W_n \\
W_1\beta_o + W_2\beta_1 + \cdots + W_n\beta_{n-1} &= -W_{n+1} \\
W_2\beta_o + W_3\beta_1 + \cdots + W_{n+1}\beta_{n-1} &= -W_{n+2} \\
\vdots \qquad\qquad & \qquad \vdots \\
W_{n-1}\beta_o + W_n\beta_1 + \cdots + W_{2n-2}\beta_{n-1} &= -W_{2n-1}
\end{aligned} \tag{3.52(a)}$$

and can also be written as (3.52(b))

$$(\beta_o, \beta_1, \ldots, \beta_{n-1}) \begin{bmatrix} W_o W_1 \cdots W_{n-1} \\ W_1 W_2 \cdots W_n \\ \vdots \\ W_{n-1} W_n \cdots W_{2n-2} \end{bmatrix} = (-W_n, -W_{n+1}, \ldots - W_{2n-1}). \tag{3.52(b)}$$

and

$$\begin{aligned}
W_o &= b_n \\
W_o\beta_{n-1} + W_1 &= b_{n-1} \\
&\vdots \\
W_o\beta_1 + W_1\beta_2 + \cdots + W_{n-2}\beta_{n-1} + W_{n-1} &= b_1
\end{aligned} \tag{3.53}$$

Solving (3.52) we get,

$$\beta_o, \beta_1, \ldots\ldots, \beta_{n-1} \qquad 3.54$$

And finding the roots of the polynomial

$$z^n + \beta_{n-1} z^{n-1} + \ldots\ldots + \beta_1 z + \beta_o = 0$$

and using $s_i = \frac{1}{T} \log z_i$, we get the exponents $[s_i]$.

Now computing $[b_i]$ from equation (3.53) and performing a partial function expansion of $\frac{W_a^{(s)}(z)}{z}$ we get

$$\frac{1}{z} W_a^{(s)}(z) = \sum_{i=1}^{n} \frac{\beta_i}{z - z_i} \qquad (3.55)$$

which gives us $[\beta_i]$.

Firstly, we find a set of exponentials ~~basis~~ $[s_i]$ which fit the data along the t axis for any value of the discrete variable $\tau$, say $\tau_{2n-1}$, and we obtain an expression of the form

$$W(t,\tau_{2n-1}) = \sum_{i=1}^{n} B_i(\tau_{2n-1}) e^{s'_i t} \qquad 3.56$$

Then we find a set of exponentials ~~bases~~ $[s''_j]$ which fit the data along the $\tau$-axis for any value of the discrete variable t, say $t_{2n-1}$, and we obtain an expression of the form

$$W(\tau, t_{2n-1}) = \sum_{j=1}^{n} C_j(t_{2n-1}) e^{s''_j \tau} \qquad 3.57$$

Using the sets of bases $[s'_i]$ and $[s''_i]$ we form the equation

$$W(t,\tau) = \sum_{i,j=1}^{n} D_{ij} e^{s'_i t} e^{s''_j \tau} \qquad 3.58$$

Rewriting (3.58) in the matrix form, we have

$$\begin{bmatrix} W_{00} & & & & \\ W_{10} & W_{11} & & & \\ W_{20} & W_{21} & W_{22} & & \\ \cdot & & & & \\ \cdot & & & & \\ W_{2n-1,0} & W_{2n-2,1} & W_{2n-3,2}, & \ldots, & W_{2n-1,2n-1} \end{bmatrix} =$$

$$\begin{bmatrix} e^{s'_1 t_o} & e^{s'_2 t_o} & \cdots\cdots & e^{s'_n t_o} \\ e^{s'_1 t_1} & e^{s'_2 t_1} & \cdots\cdots & e^{s'_n t_1} \\ e^{s'_1 t_2} & e^{s'_2 t_2} & \cdots\cdots & e^{s'_n t_2} \\ \cdot & & & \\ \cdot & & & \\ \cdot & & & \\ \cdot & & & \\ e^{s'_1 t_{2n-1}} & e^{s'_2 t_{2n-1}} & \cdots & e^{s'_n t_{2n-1}} \end{bmatrix} \begin{bmatrix} \\ D_{ij} \\ n \times n \\ \end{bmatrix} \begin{bmatrix} e^{s''_1 \tau_o} & e^{s''_1 \tau_1} & e^{s''_1 \tau_2} & \cdots\cdots & e^{s''_1 \tau_{2n-1}} \\ e^{s''_2 \tau_o} & e^{s''_2 \tau_1} & e^{s''_2 \tau_2} & \cdots\cdots & e^{s''_2 \tau_{2n-1}} \\ \cdot & \cdot & & & \\ \cdot & \cdot & & & \\ \cdot & \cdot & & & \\ \cdot & \cdot & & & \\ e^{s''_n \tau_o} & e^{s''_n \tau_1} & e^{s''_n \tau_2} & \cdots\cdots & e^{s''_n \tau_{2n-1}} \end{bmatrix} \qquad 3.59$$

Writing (3.59) for the first $n^2$-equations, we have

$$\begin{bmatrix} W \\ n \times n \end{bmatrix} = \begin{bmatrix} S' \\ n \times n \end{bmatrix} \begin{bmatrix} D \\ n \times n \end{bmatrix} \begin{bmatrix} S'' \\ n \times n \end{bmatrix} \qquad 3.60$$

From (3.60) we get

$$(S'^{-1})\,(W)\,(S'')^{-1} = (D) \qquad 3.61$$

provided $(S')^{-1}$ and $(S'')^{-1}$ exist.

Hence we get

$$W(t, \tau) = \sum_{i,j=1}^{n} D_{ij}\, e^{s'_i t}\, e^{s''_j \tau} \qquad 3.62$$

As a caution, we must remark that the exponents $[s]$ may occur with positive real parts, which interferes with the requirement of the passivity of the time invariant part of this system. However, it is sometimes possible to avoid this difficulty either by changing the sampling interval, or by enlarging the interval over which samples are taken, or both, but no general theory exists at present which gives us the conditions under which the above mentioned modification will have the desired effect.

## 4. Concluding Remarks.

To summarize the discussion of this thesis, we have presented a solution of the characterization problem for a class of linear systems which have been identified as time-invariant or separable time-variant, and having an arbitrary set of inputs $\{u\}$ and a corresponding set of outputs $\{y\}$. In particular, the systems discussed are (3.4), and (3.26) where the weighting function $W(t, \tau)$ is considered as the unknown to be determined on the time domain, the inputs being arbitrary.

The method of determination of weighting function $W(t, \tau)$, represented in this thesis~~,~~ is numerical~~,~~ and can be easily implemented on a digital computer. An algorithm in regard to the use of a digital computer is given and a bound on the errors of the process has been established. The word system matrix has been used for the matrix consisting of collection of numbers which are characteristics of the system. A procedure for finding a representation for weighting function $W(t, \tau)$ from the system matrix is developed. Also, an algebra for manipulations of weighting functions of linear systems is given. While the characterization of separable time-variant systems has been investigated in this thesis, it may be mentioned that the characterization of nonseparable time-variant systems is still an unsolved problem.

# APPENDICES

I. Proof of Theorem 2 on Weighting Functions.

II. A Bound on the Errors of Approximation Process.

III. An Algebra for Weighting Functions. of Linear Systems.

IV. Results of Digital Computation (Table II).

Notation

$$\frac{d^i y(t)}{dt^i} \triangleq y^{(i)}(t)$$

$$\frac{\partial^i W(t, \tau)}{\partial t^i} \triangleq W_t^{(i)}(t, \tau)$$

## APPENDIX I

Theorem 2 A function $W$ of two variables $t$ and $\tau$ is a weighting function for a linear differential system if and only if it is a separable function of the variables $t$ and $\tau$ of the form

$$W(t, \tau) = \sum_{k=1}^{n} \psi_k(t)\phi_k(\tau) \qquad \text{for } t \geq \tau$$

$$= 0 \qquad \text{for } t < \tau . \tag{I-1}$$

where $\psi_k(t)$'s are linearly independent functions of $t$, and $\phi_k(\tau)$'s are linearly independent functions of $\tau$ .

Definition 1 The weighting function of a physically realizable system can be defined as a function $W(t,\tau)$ such that if an input $u(t)$ is applied to the system, the output $y(t)$ is given by the convolution integral (I-2), assuming all the initial conditions to be zero,

$$y(t) = \int_{t_o}^{t} W(t, \tau)\, u(\tau)\, d\tau \tag{I-2}$$

The upper limit of the integral must be $t$, since the physical realizability of the system requires that

$$W(t,\tau) = 0 \qquad \text{for } \tau > t \tag{II-3}$$

Definition 2 If the system for which $W(t,\tau)$ is a weighting function can be described exactly by an n-th-order linear differential equation, ~~such systems are called linear differential systems~~ of the form

$$\sum_{i=0}^{n} a_i(t)\, y^{(i)}(t) = \sum_{i=0}^{m} b_i(t)\, u^{(i)}(t)\,, \quad n \geq m,\ a_n(t)=1$$

(I-4)

and $$y^{(i)}(t)\Big|_{t=\tau} = 0\,;\ i = 0,\ 1, 2, \dots, n-1$$

$$\tau \leq t < \infty$$

where $\dfrac{d^i y(t)}{dt} \triangleq y^{(i)}(t).$

such systems are called linear differential systems

Proof: (i) Necessity.

Assume now that a physical system of the form (I-4) has been guaranteed and it is desired to determine the weighting function which corresponds to (I-4).

In order to express explicitly the output of the system (I-4) in terms of its input, the knowledge of the complete set of homogeneous solutions of (I-4) is necessary. We shall not go into the details of determination of the complete set of homogeneous solutions of (I-4), but we assume that this particular set - also called the fundamental set of homogeneous solutions of (I-4) is known [10, 11]. Let us represent a complete set of homogeneous solutions of (I-4) by $\{\psi_k\}_{k=1,\dots,n}$.

In order to determine the forced solution of (I-4), we employ the method of variation-of-parameters expounded by Lagrange [13].

Let $\psi_1(t), \dots, \psi_n(t)$

be any fundamental set of homogeneous solutions of (I-4). Then a solution of (I-4) can be obtained in the form

$$y = \sum_{k=1}^{n} C_k(t)\, \psi_k(t) \tag{I-5}$$

where the functions $C^{(1)}_1(t), \; \ldots, \; C^{(1)}_n(t)$ are obtained as solutions of the system of algebraic equations. This system of equations can be written in the matrix notation as:

$$\begin{bmatrix} \psi_1(t) & \psi_2(t) \cdots\cdots\cdots & \psi_n(t) \\ \psi_1^{(1)}(t) & \psi_2^{(1)}(t) \cdots\cdots\cdots & \psi_n^{(1)}(t) \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \psi_1^{(n-1)} & \psi_2^{(n-1)}(t) \cdots\cdots & \psi_n^{(n-1)}(t) \end{bmatrix} \begin{bmatrix} C_1^{(1)}(t) \\ C_2^{(1)}(t) \\ \cdot \\ \cdot \\ C_n^{(1)}(t) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \cdot \\ \cdot \\ \sum\limits_{i=o}^{m} b_i(t) u^{(i)}(t) \end{bmatrix} \qquad \text{(I-6)}$$

The coefficient matrix of the functions $[\, C_k^{(1)}(t)]$ is recognized to be the Wronskian matrix of the homogeneous solutions of (I-5). Further, the fact that $[\psi_k(t)]$ is a complete set of homogeneous solutions of (I-5) guarantees the nonsingularity of this matrix.

Now, each of the functions $C_k(t)$ is a function of the input as follows:

$\boxed{C_k(t_o) = 0}$

$$C_k(t) = \int_{t_o}^{t} \phi_k(\tau) \left[ \sum_{i=o}^{m} b_i(\tau)\, u^{(i)}(\tau) \right] d\tau \qquad \text{(I-7)}$$

The set of functions $\{\phi_k(t)\}_{k=1,\ldots,n}$ are determined uniquely by the set of homongeneous solutions $[\psi_k(t)]_{k=1,\ldots,n}$ and are obtained by solving a system of algebraic equations, which are modified versions of those given by (I-6) and can be written in the matrix form as follows:

$$\begin{bmatrix} \psi_1(t) & \psi_2(t) \dots\dots & \psi_n(t) \\ \psi_1^{(1)}(t) & \psi_2^{(1)}(t) \dots\dots & \psi_n^{(1)}(t) \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \psi_1^{(n-1)}(t) & \psi_2^{(n-1)}(t) \dots\dots & \psi_n^{(n-1)}(t) \end{bmatrix} \begin{bmatrix} \phi_1(t) \\ \phi_2(t) \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \phi_n(t) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ 1 \end{bmatrix} \qquad \text{(I-8)}$$

Hence the input-output relationship of the system (I-5) takes the form:

$$y(t) = \sum_{k=1}^{n} [\psi_k(t) \int_{t_o}^{t} \phi_k(\tau) ( \sum_{i=o}^{m} b_i(\tau) u^{(i)}(\tau) ) \, d\tau ] \qquad \text{(I-9)}$$

Note that the functions $[\psi_k(t)]$ and $[\phi_k(t)]$ do not depend on the input in any way. The block diagram of the system given by equation (I-9) is shown in Figure (I-1)

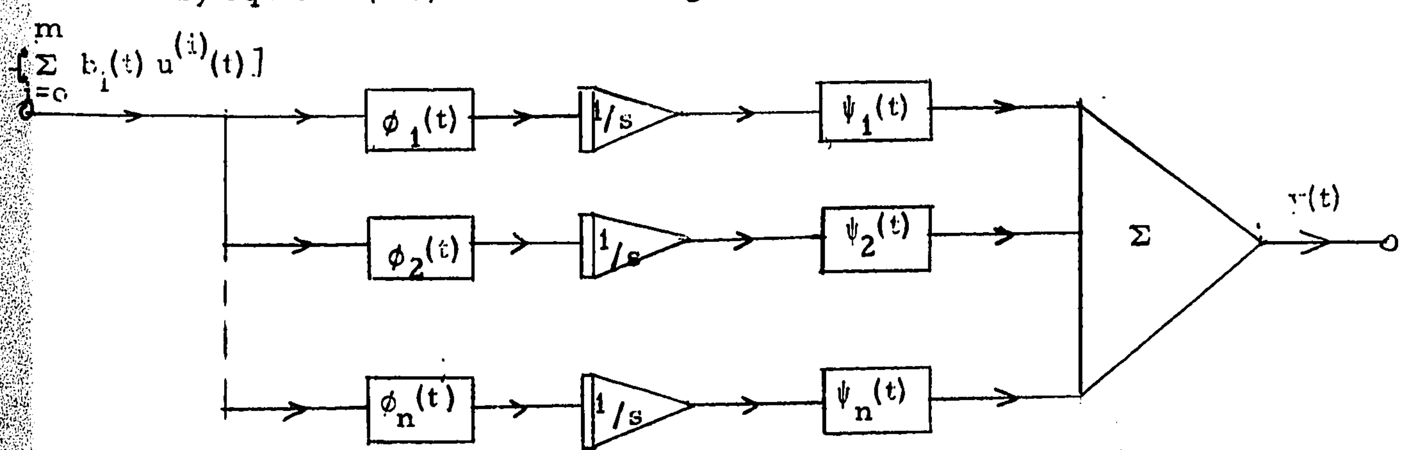



Fig. (I-1). Block Diagram Corresponding to Equation (I-9).

The mathematical model of the system shown in Figure (I-1) is denoted by the weighting function $W(t,\tau)$, defined mathematically as follows:

$$W(t,\tau) = \sum_{k=1}^{n} \psi_k(t)\,\phi_k(\tau), \qquad t \geq \tau$$
$$= 0, \qquad t < \tau \qquad \text{(I-10)}$$

To distinguish between the nonmnemonic operations occurring before and after the integrators shown in Figure (I-1), the weighting function is written as a function of two variables $t$ and $\tau$. The equation (I-9) in terms of $W(t,\tau)$ takes the form

$$y(t) = \int_{t_o}^{t} W(t,\tau)\,\left[\sum_{i=o}^{m} b_i(\tau)\,u^{(i)}(\tau)\right] d\tau \qquad \text{(I-11)}$$

The equation (I-11) tells us that a complete and explicit description of the input-output characteristics of the system (I-4) is provided by the weighting function $W(t,\tau)$, a separable function of two variables of the form (I-10), which proves the necessity.

(ii) Sufficiency

Assume now that a weighting function of the form (I-1) has been guaranteed and it is desired to determine the differential equation of the physical system which corresponds to (I-1). Since $[\psi_k(t)]$ is a set of $n$ independent solutions in $W(t,\tau)$, the order of the differential equation corresponding to (I-1) must be $n$, and is of the form

$$\sum_{i=o}^{n} \alpha_i(t)\,y^{(i)}(t) = \sum_{i=o}^{m} \beta_i(t)\,u^{(i)}(t), \qquad \text{(I-12)}$$
$$n \geq m, \quad \alpha_n(t) = 1$$

where $\alpha_i(t)$ and $\beta_i(t)$ are to be determined. Moreover, $[\psi_k(t)]$ is a set of homogeneous solutions of (I-12), therefore, we have

$$\sum_{i=0}^{n} \alpha_i(t)\psi_k^{(i)}(t) = 0, \quad k = 1, \ldots, n. \tag{I-13}$$

This is a set of n simultaneous equations which give us the n-unknowns $\alpha_i(t)$'s.

To determine $\beta_i(t)$'s proceed as follows:

Let us assume that the weighting function corresponding to the homogeneous part of (I-12), that is, the solution to the following problem, is known

$$\sum_{i=0}^{n} \alpha_i(t)\, k_t^{(i)}(t, \tau) = 0$$

$$\text{and} \quad k_t^{(i)}(t, \tau)\Big|_{t=\tau} = 0, \quad i = 0, 1, \ldots n-2, \tag{I-14}$$

$$k_t^{(n-1)}(t, \tau) = 1.$$

where $\dfrac{\partial^i k(t, \tau)}{\partial t^i} \triangleq k_t^{(i)}(t, \tau)$

If $k(t, \tau)$ is known, then the input-output relationship of the system (I-12), using (I-11), is of the form

$$y(t) = \int_{t_o}^{t} k(t, \tau)\left[\sum_{i=0}^{m} \beta_i(\tau)\, u^{(i)}(\tau)\right] d\tau \tag{I-15}$$

Using (I-2), we have

$$\int_{t_o}^{t} k(t, \tau)\left[\sum_{i=0}^{m} \beta_i(\tau)\, u^{(i)}(\tau)\right] d\tau = \int_{t_o}^{t} W(t, \tau)\, u(\tau)\, d\tau \tag{I-16}$$

Now operating on both sides of (I-16) by

$\left[ \sum_{l=o}^{m} a_l(t) \dfrac{d^l}{dt^l} \right]$ and using the fact that $k_t^{(n-1)}(t,\tau) = 1$ ,

we get (I-17)

$$\sum_{i=o}^{m} \beta_i(t)\, u^{(i)}(t) = \sum_{l=o}^{m} a_l(t) \frac{d^l}{dt^l} \left[ \int_{t_o}^{t} W(t,\tau)\, u(\tau)\, d\tau \right] \qquad \text{(I-17)}$$

The right-hand side of equation (I-17), using Leibnitz formula, can be expanded as

$$\sum_{l=o}^{m} a_l(t) \int_{t_o}^{t} W_t^{(l)}(t,\tau)\, u(\tau)\, d\tau + \sum_{l=o}^{m} a_l(t) \sum_{r=o}^{l-1} \frac{d^r}{dt^r} \left[ W_t^{(l-r-1)}(t,\tau)\, u(\tau) \Big|_{\tau=t-o} \right] \qquad \text{(I-18)}$$

and we use the convention throughout that $\sum_{r=o}^{-1} \equiv 0$.

Now by virtue of (I-14) and using Leibnitz formula, (I-18) reduces to

$$\sum_{l=o}^{m} a_l(t) \sum_{r=o}^{l-1} \sum_{i=o}^{r} \binom{r}{i} \frac{d^{r-i}}{dt^{r-i}} \left[ W_t^{(l-r-1)}(t,\tau) \Big|_{\tau=t-o} \right] u^{(i)}(t) \qquad \text{(I-19)}$$

Hence (I-17), becomes

$$\sum_{i=o}^{m} \beta_i(t)\, u^{(i)}(t) = \sum_{l=o}^{m} \sum_{r=o}^{l-1} \sum_{i=o}^{r} \binom{r}{i} a_l(t) \frac{d^{r-i}}{dt^{r-i}} \left[ W_t^{(l-n-1)}(t,\tau) \Big|_{\tau=t-o} \right] u^{(i)}(t) \qquad \text{(I-20)}$$

?

Rearranging the righthand side of equation (I-20), we get

$$\beta_i(t) = \sum_{l=i}^{m} \sum_{r=i}^{l-1} \binom{r}{i} a_l(t) \frac{d^{r-i}}{dt^{r-i}} \left[ \left| W_t^{(l-r-1)}(t, \tau) \right|_{t = \tau - 0} \right] \tag{I-21}$$

This is a set of (n+1) simultaneous equations which give us the (n+1) unknowns $\beta_i(t)$'s. Since the equation (I-17) is completely known, it establishes the sufficiency.

Hence our theorem.

## APPENDIX II

A Bound on the Errors of Approximate Integration.

The equation used to calculate

$$W^*(a)\ ,\ W^*(2a),\ W^*(3a),\ldots\ldots$$

is:

$$-\frac{a}{2}\,u(o)\,W^*(na) = -\,y(na) + a\left\{\frac{1}{2}u(na)\,W(o) + \sum_{k=1}^{n-1} u[(n-k)a]\,W^*(na)\right\}$$

$$\text{where}\quad \sum_{k=1}^{o} \equiv 0 \tag{II-1}$$

The exact equation corresponding to (II-1) is

$$-\frac{a}{2}\,u(o)\,W(na) = -\,y(na) + a\left\{\frac{1}{2}u(na)\,W(o). + \sum_{k=1}^{n-1} u[(n-k)a]\,W(ka)\right\}$$
$$+\sum_{k=1}^{n} e_{n,k} \tag{II-2}$$

where the error for any interval $(k-1)a \leq t < ka$ is given by (II-3)

$$e_{n,k} = \int_{(k-1)a}^{ka} u(na - \tau)\,W(\tau)\,d\tau$$
$$-\frac{a}{2}\left[u(na - \overline{k-1}\,a)\,W(\overline{k-1}\,a)\right.$$
$$\left. + \,u(na-ka)\,W(ka)\right] \tag{II-3}$$

Representing $E_n$, the error after n steps, as:

$$E_n = W^*(na) - W(na) \tag{II-4}$$

and substracting (II-2) from (II-1) we get:

$$\left[ -\frac{a}{2} u(o) \right] E_n = a \sum_{k=1}^{n-1} u(\overline{n-k}\, a)\, E_k - \sum_{k=1}^{n} e_{n,k}. \tag{II-5}$$

Replacing n by n+1 in (II-5) we have

$$\left[ -\frac{a}{2} u(o) \right] E_{n+1} = a \sum_{k=1}^{n} u(\overline{n+1-k}\, a)\, E_k - \sum_{k=1}^{n+1} e_{n+1,k} \tag{II-6}$$

Subtracting (II-5) from (II-6) and denoting

$$I_n = a \sum_{k=1}^{n-1} \left[ u(\overline{n+1-k}\, a) - u(\overline{n-k}\, a) \right] E_k$$

$$- e_{n+1,n+1} - \sum_{k=1}^{n} (e_{n+1,k} - e_{n,k}) \tag{II-7}$$

we get

$$-\frac{a}{2} u(o) \left[ E_{n+1} - E_n \right] = a\, u(a)\, E_n + I_n \tag{II-8}$$

subtracting $[\, a\, u(o) E_n ]$ from both sides of (II-8)

we have,

$$-\frac{a}{2} u(o) \left[ E_{n+1} + E_n \right] = a \left[ u(a) - u(o) \right] E_n + I_n \tag{II-9}$$

Comparing (II-9) and (II-8), we conclude that $[\, E_{n+1} + E_n ] < [\, E_{n+1} - E_n ]$,

i.e., $E_{n+1}$ and $E_n$ are in general of opposite sign, therefore, $E_n$ forms an oscillating sequence.

Rewriting (II-9) by replacing (n+1) for n, we get

$$-\frac{a}{2} u(o) \left[ E_{n+2} + E_{n+1} \right] = a \left[ u(a) - u(o) \right] E_{n+1} + I_{n+1} \tag{II-10}$$

Subtracting (II-9) from (II-10), we get

$$-\frac{a}{2}u(o)[E_{n+2}-E_n] = a(u(a)-u(o))[E_{n+1}-E_n] + I_{n+1}-I_n \tag{II-11}$$

We also know[23] that

$$e_{n,k} = -\frac{1}{12}a^2\left\{\frac{d}{d\tau}[u(na-\tau)W(\tau)]\tau = ka\right.$$

$$\left.-\frac{d}{d\tau}[u(na-\tau)W(\tau)]\tau = (k-1)a\right\} + o(a^5) \tag{II-12}$$

Assuming that W and u have bounded derivatives up to fourth order, it follows that the first and second differences of u are respectively of orders $a$ and $a^2$ and that the first and second differences of $e_{n,k}$ are respectively of orders $a^4$ and $a^5$. It can be easily proved by induction that a positive constant L can be found such that

$$|E_n| \leq f_n. \tag{II-13}$$

where

$$f_{n+2}-f_n = L[a^2\sum_{k=1}^{n-1} f_k + a(f_{n+1}+f_n) + na^4 + a^3] \tag{II-14}$$

where $f_1$ and $f_2$ are chosen so that (II-13) holds for $n=1$ and 2.

From (II-5)

$$f_1 = 0(a^2),\quad f_2 = 0(a^2) \tag{II-15}$$

and

$$f_1 < f_2 \tag{II-16}$$

Thus

$$f_{2n-1} < f_{2n} \quad \text{for all } n \tag{II-17}$$

and hence it may be shown by induction that if

$$F_{2n+2} - F_{2n} = L[\,a^2(2\sum_{k=1}^{n-1} F_{2k} + F_{2n}) + a(F_{2n} + F_{2n+2}) + 2na^4 + a^3\,] \quad \text{(II-18)}$$

and

$$F_2 = f_2 \quad \text{(II-19)}$$

Then, $f_{2n} < F_{2n}$ for all $n > 1$ (II-20)

From (II-18), we get the difference equation.

$$F_{2n+4}(1 - La) - F_{2n+2}(2 + La^2) + F_{2n}(1 - La^2 + 2a) = 2La^4 \quad \text{(II-21)}$$

Solving (II-21) we get [41]

$$F_{2n} = -a^2 + \omega_1 r_1^n + \omega_2 r_2^n \quad \text{(II-22)}$$

where

$$r_1, r_2 = \frac{1}{1-La}\left[\,1 + \frac{La^2}{2} \pm \left\{ L(2+L)a^2 - L^2a^3 + \frac{L^2a^4}{4} \right\}^{1/2} \right] \quad \text{(II-23)}$$

and the constants $\omega_1$ and $\omega_2$ are given by the initial conditions.

$$r_1\omega_1 + r_2\omega_2 = F_2 + a^2 \quad \text{(II-24)}$$

$$r_1^2\omega_1 + r_2^2\omega_2 = F_4 + a^2$$

From (II-15) and (II-19), $F_2 = 0(a^2)$ and from (II-18),

$F_4 = F_2(1 + 0(a))$. Also from (3.32).

$r_1 = 1 + 0(a)$ ; $r_2 = 1 + 0(a)$, $r_2 - r_1 = 0(a)$.

$\omega_1 = O(\alpha^2)$, $\omega_2 = O(\alpha)^2$

$F_{2n} = O(\alpha)^2$ or $E_n = O(\alpha)^2$

Thus from (II-24),

$$\omega_1 = \frac{r_2(F_2 + a^2) - (F_4 + a^2)}{r_1 (r_2 - r_1)}$$

or $$\omega_1 = \frac{0 (a^3)}{0 (a)} = 0 (a^2) \qquad \text{(II-25)}$$

Similarly $$\omega_2 = 0 (a^2) \qquad \text{(II-26)}$$

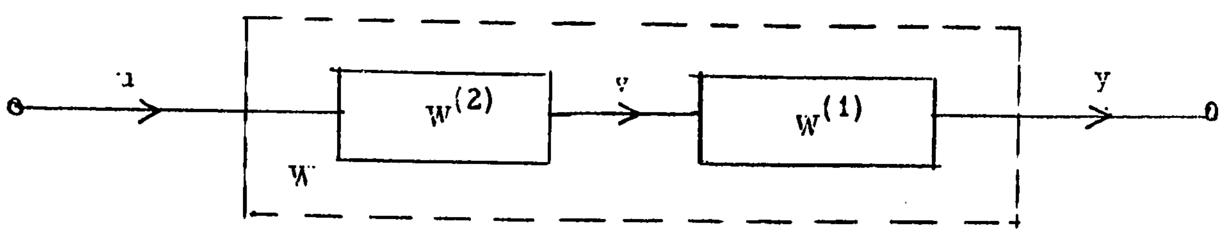
At a fixed value of $t$ we consider a sequence of values of $a$ such that $2na = t$, thus $n \to \infty$ as $a \to 0$.

Thus for small $a$, $r_1$ and $r_2$ in (II-22) are bounded, and $\omega_1$, $\omega_2$ are $0 (a^2)$. So, for fixed t, $F_{2n}$ is $0(a^2)$ and from (II-19), (II-17) and (II-19) it follows that $E_n = 0 (a^2)$.

## APPENDIX III

When linear systems are combined, their weighting functions may be combined by an algebra based on sums and convolutions[42]. The usual definition of convolution may be generalized so that the algebra applies to linear systems, time-invariant as well as time-variant. It then has almost all the properties of matrix algebra.

The cascade combination of two linear time-variant systems is depicted in Figure II-4, where $W^{(1)}$ and $W^{(2)}$ represent the weighting functions as well as the systems themselves.




a) Continuous:

Time-invariant $W = W^{(1)} * W^{(2)}$ (stationary)

Time-variant $W = W^{(1)} * W^{(2)}$ (nonstationary)

$$W(t, \tau) = \int_{\tau}^{t} W^{(1)}(t, \xi)\, W^{(2)}(\xi, \tau)\, d\tau \; .$$

b) Discrete:

$$W = W^{(1)} W^{(2)}$$

$$W_{t\tau} = \sum_{\xi} W^{(1)}_{t\xi} \; W^{(2)}_{\xi\tau}$$

Fig. III-1. Cascade-Connection of Linear Systems.

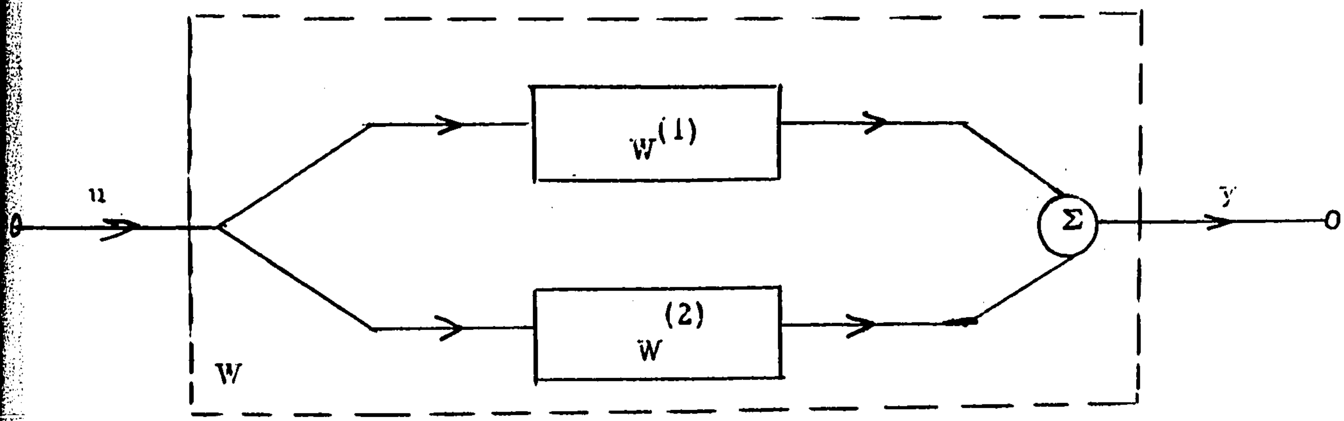
Now, considering the system from slightly abstract point of view, one can simply consider a system $W^{(2)}$ as a mapping which takes an abstract space $U$ (the input space) into another (the output space) space V. Similarly, $W^{(1)}$ is a map which takes an abstract space V (the output space of $W^{(2)}$) into another space Y. Let $u_1 \in U$, then its image $W^{(2)}(u_1)$ is in V which is the domain of $W^{(1)}$. Accordingly, we can find the image of $W^{(2)}(u_1)$ under the mapping $W^{(1)}$, that is, we can find $W^{(1)}(W^{(2)}(u_1))$. Thus we have a rule which assign to each element $u_1 \in U$ a corresponding element $W^{(1)}(W^{(2)}(u_1)) \in Y$. In other words, we have a map of $U$ into Y. This new map is called the product map or composition map[40] of $W^{(2)}$ and $W^{(1)}$ and is denoted by $W^{(1)}\, W^{(2)}$. When the systems are time-invariant the convolution expressed by $W = W^{(1)} * W^{(2)}$. is an ordinary convolution, that is, this relation is simply $G_W(s) = G_{W(1)}(s) + G_{W(2)}(s)$ in terms of the corresponding frequency domain representation. Further, the commutative law holds, that is, $W^{(1)} * W^{(2)} = W^{(2)} * W^{(1)} = W$. For time varying systems, the commutative law, in general, does not hold. But for systems representable by symmetric system functions (i.e. $W(t, \tau) = W(\tau, t)$), the commutative law does hold.

When the signals are discrete, W, $W^{(1)}$, and $W^{(2)}$ are matrices and W is simply the matrix product $W^{(1)} \cdot W^{(2)}$, expressed by the equation (III-1).

$$W_{t\tau} = \sum_{\xi} W^{(1)}_{t\xi}\, W^{(2)}_{\xi\tau} \qquad \text{(III-1)}$$

$$W^{(1)}\, W^{(2)} \neq W^{(2)}\, W^{(1)} \qquad \text{(III-2)}$$

The parallel combination of two linear time-varying systems is diagramatically shown in Fig. III-2.




4) Continuous

Time-invariant $W = W^{(1)} + W^{(2)}$ (stationary)

Time-variant $W = W^{(1)} + W^{(2)}$ (nonstationary)

$$W(t, \tau) = W^{(1)}(t, \tau) + W^{(2)}(t, \tau)$$

6) Discrete

$$W = W^{(1)} + W^{(2)}$$

$$W_{t\tau} = W^{(1)}_{t\tau} + W^{(2)}_{t\tau}$$

Fig. III-2. Parallel Combination of Linear Systems.

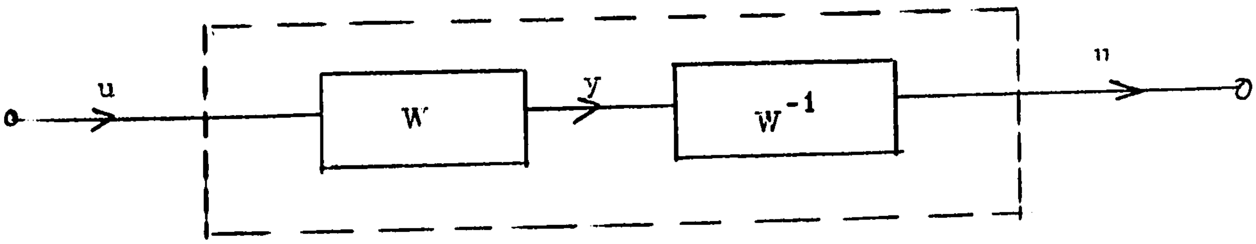
Thus we see that for continuous signal the equivalent system function is given by the equation (III-3)

$$W(t,\tau) = W^{(1)}(t, \tau) + W^{(2)}(t,\tau) \qquad \text{(III-3)}$$

Again, when the signals are discrete, $W$, $W^{(1)}$ and $W^{(2)}$ are matrices and $W$ is simply the matrix sum $W^{(1)} + W^{(2)}$. expressed by the equation

$$W_{t\tau} = W^{(1)}_{t\tau} + W^{(2)}_{t\tau} \qquad \text{(III-4)}$$

Lastly, the inverse operation is shown in Fig. III-3.




c) Continuous

$$W^{-1} * W = \delta$$

$$(W^{(1)} * W^{(2)})^{-1} = (W^{(2)})^{-1} = (W^{(1)})^{-1}$$

$$\delta(t-\tau) = \int_{\tau}^{t} W^{-1}(t,\xi)\, W(\xi,\tau)\, d\xi$$

d) Discrete

$$W^{-1}\, W = 1$$

$$(W^{(1)} W^{(2)})^{-1} = (W^{(2)})^{-1} (W^{(1)})^{-1}$$

Fig. III-3. Inverse System.

For continuous signals, the inverse system is given by the equation (III-5)

$$\delta(t-\tau) = \int_{\tau}^{t} W^{-1}(t,\xi)\, W(\xi,\tau)\, d\xi \qquad \text{(III-5)}$$

and for discrete signals, we have

$$W^{-1} W = I \tag{III-6}$$

Thus we have almost all the properties of finite order matrix algebra. For the most part, the relations require that orders of integration may be interchanged, and that inverses of weighting functions do exist.

# APPENDIX IV

## RESULTS OF DIGITAL COMPUTATION

```
C     NARESH GUPTA PROGRAM FOR SYSTEM METRIX DETERMINATION
      DIMENSION G(500), F(500)
      Y(TX)=TX*EXPF(-TX)
      U(TX)=1.
      DY(TX)=(1.-TX)*EXPF(-TX)
  213 READ 100,T1,N,ALFA,GO
  100 FORMAT(F5.3,I5,F5.3,F10.8)
      PRINT 103,T1,N,ALFA
  103 FORMAT (4HOT1=,F7.4,4X,2HN=,I5,4X,5HALFA=,F7.4)
      GU=GO*U(T1)
      FO=Y(T1)/(GU*ALFA)
      F(1)=FO
      G(1)=GO
      T=T1
      NG=1
      I=1
      PRINT 102
      GO TO 210
  211 NG=2
      DO 200 I=2,N
      IM=I-1
      TM=T
      T=T+ALFA
      G(I)=((Y(T)-Y(TM))/ALFA-DY(TM)+FO*G(IM)*U(TM))/(FO*U(T))
      F(I)=Y(T)/ALFA
      TK=T1
      DO 201 J=2,I
      TK=TK+ALFA
      K=I-J+1
  201 F(I)=F(I)-F(K)*G(J)*U(TK)
      F(I)=F(I)/GU
  210 EX=EXPF(-T)
      PRINT 101,T,F(I),G(I),EX
      GO TO (211,200),NG
  200 CONTINUE
      CALL EXIT
  102 FORMAT (1HO,3X,1HT,9X,1HF,11X,1HG,10X,3HEXP)
  101 FORMAT (1HO,F7.4,2X,3(F8.5,4X))
      END
```

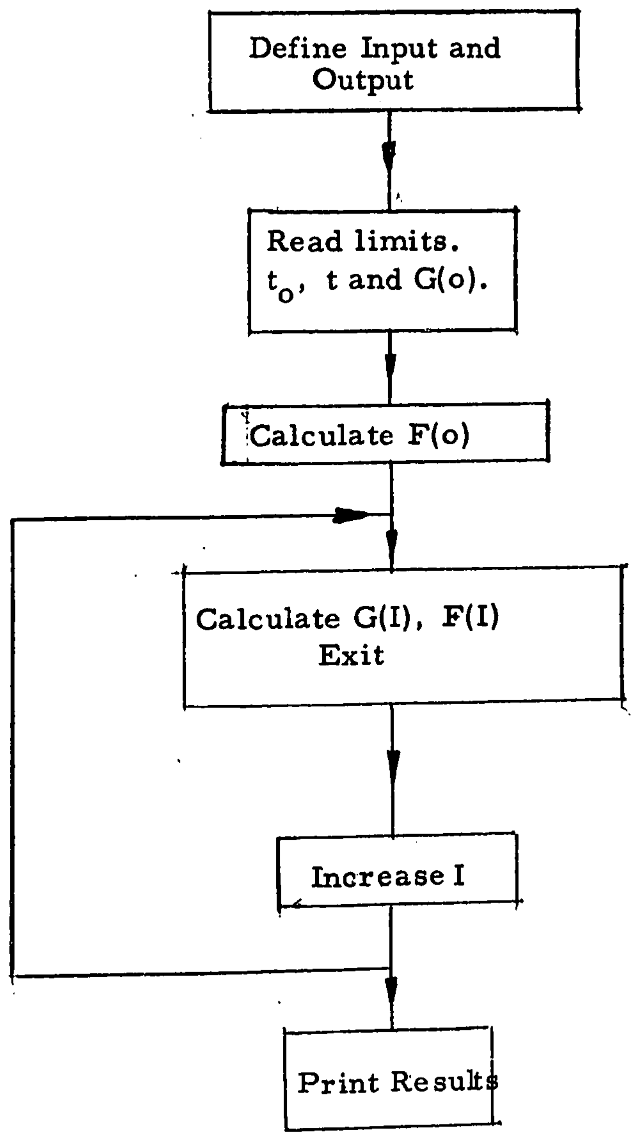



Fig. IV-1 Algorithm for Calculating System Matrix.

TABLE II

T1= .0010 N= 500 ALFA= .0010

| T | F (Cal.) | G (Cal.) | Exact Value |
|---|---|---|---|
| .0010 | .99900 | 1.00000 | .99900 |
| .0020 | .99800 | .99900 | .99800 |
| .0030 | .99700 | .99800 | .99700 |
| .0040 | .99600 | .99700 | .99600 |
| .0050 | .99500 | .99601 | .99501 |
| .0060 | .99401 | .99501 | .99401 |
| .0070 | .99301 | .99402 | .99302 |
| .0080 | .99201 | .99303 | .99203 |
| .0090 | .99102 | .99204 | .99104 |
| .0100 | .99002 | .99106 | .99004 |
| .0110 | .98903 | .99007 | .98906 |
| .0120 | .98804 | .98909 | .98807 |
| .0130 | .98704 | .98810 | .98708 |
| .0140 | .98605 | .98712 | .98609 |
| .0150 | .98506 | .98614 | .98511 |
| .0160 | .98406 | .98517 | .98412 |
| .0170 | .98307 | .98419 | .98314 |
| .0180 | .98208 | .98321 | .98216 |
| .0190 | .98109 | .98224 | .98117 |
| .0200 | .98010 | .98127 | .98019 |
| .0210 | .97911 | .98030 | .97921 |
| .0220 | .97812 | .97933 | .97824 |
| .0230 | .97713 | .97836 | .97726 |
| .0240 | .97615 | .97739 | .97628 |
| .0250 | .97516 | .97643 | .97530 |
| .0260 | .97417 | .97546 | .97433 |

| | | | |
|---|---|---|---|
| .0270 | .97318 | .97450 | .97336 |
| .0280 | .97220 | .97354 | .97238 |
| .0290 | .97121 | .97258 | .97141 |
| .0300 | .97023 | .97162 | .97044 |
| .0310 | .96925 | .97067 | .96947 |
| .0320 | .96826 | .96971 | .96850 |
| .0330 | .96727 | .96876 | .96753 |
| .0340 | .96630 | .96781 | .96657 |
| .0350 | .96531 | .96686 | .96560 |
| .0360 | .96433 | .96591 | .96464 |
| .0370 | .96334 | .96496 | .96367 |
| .0380 | .96237 | .96401 | .96271 |
| .0390 | .96139 | .96307 | .96175 |
| .0400 | .96041 | .96213 | .96078 |
| .0410 | .95942 | .96118 | .95982 |
| .0420 | .95845 | .96024 | .95886 |
| .0430 | .95747 | .95930 | .95791 |
| .0440 | .95649 | .95837 | .95695 |
| .0450 | .95552 | .95743 | .95599 |
| .0460 | .95454 | .95649 | .95504 |
| .0470 | .95356 | .95556 | .95408 |
| .0480 | .95258 | .95463 | .95313 |
| .0490 | .95161 | .95370 | .95218 |
| .0500 | .95063 | .95277 | .95122 |
| .0510 | .94966 | .95184 | .95027 |
| .0520 | .94869 | .95091 | .94932 |
| .0530 | .94771 | .94999 | .94838 |
| .0540 | .94674 | .94907 | .94743 |

| | | | |
|---|---|---|---|
| .0550 | .94576 | .94814 | .94646 |
| .0560 | .94480 | .94722 | .94553 |
| .0570 | .94382 | .94630 | .94459 |
| .0580 | .94285 | .94538 | .94364 |
| .0590 | .94188 | .94447 | .94270 |
| .0600 | .94091 | .94355 | .94176 |
| .0610 | .93994 | .94264 | .94082 |
| .0620 | .93897 | .94173 | .93988 |
| .0630 | .93801 | .94081 | .93894 |
| .0640 | .93703 | .93991 | .93800 |
| .0650 | .93607 | .93900 | .93706 |
| .0660 | .93510 | .93809 | .93613 |
| .0670 | .93414 | .93718 | .93519 |
| .0680 | .93317 | .93628 | .93426 |
| .0690 | .93221 | .93538 | .93332 |
| .0700 | .93123 | .93448 | .93239 |
| .0710 | .93028 | .93358 | .93146 |
| .0720 | .92930 | .93268 | .93053 |
| .0730 | .92834 | .93178 | .92960 |
| .0740 | .92739 | .93088 | .92867 |
| .0750 | .92641 | .92999 | .92774 |
| .0760 | .92546 | .92909 | .92681 |
| .0770 | .92450 | .92820 | .92588 |
| .0780 | .92354 | .92731 | .92496 |
| .0790 | .92258 | .92642 | .92403 |
| .0800 | .92161 | .92553 | .92311 |
| .0810 | .92066 | .92465 | .92219 |
| .0820 | .91969 | .92376 | .92127 |

| | | | |
|---|---|---|---|
| .0830 | .91874 | .92286 | .92035 |
| .0840 | .91778 | .92200 | .91943 |
| .0850 | .91682 | .92111 | .91851 |
| .0860 | .91586 | .92023 | .91759 |
| .0870 | .91491 | .91936 | .91667 |
| .0880 | .91396 | .91848 | .91576 |
| .0890 | .91299 | .91760 | .91484 |
| .0900 | .91204 | .91673 | .91393 |
| .0910 | .91109 | .91586 | .91301 |
| .0920 | .91013 | .91498 | .91210 |
| .0930 | .90918 | .91411 | .91119 |
| .0940 | .90823 | .91324 | .91028 |
| .0950 | .90728 | .91238 | .90937 |
| .0960 | .90631 | .91151 | .90846 |
| .0970 | .90537 | .91064 | .90755 |
| .0980 | .90442 | .90978 | .90664 |
| .0990 | .90347 | .90892 | .90574 |
| .1000 | .90253 | .90806 | .90483 |
| .1010 | .90157 | .90720 | .90393 |
| .1020 | .90062 | .90634 | .90302 |
| .1030 | .89967 | .90548 | .90212 |
| .1040 | .89873 | .90462 | .90122 |
| .1050 | .89777 | .90377 | .90032 |
| .1060 | .89683 | .90291 | .89942 |
| .1070 | .89588 | .90206 | .89852 |
| .1080 | .89494 | .90121 | .89762 |
| .1090 | .89399 | .90036 | .89673 |
| .1100 | .89305 | .89951 | [illegible] |

| | | | |
|---|---|---|---|
| .1110 | .89210 | .89867 | .89493 |
| .1120 | .89116 | .89782 | .89404 |
| .1130 | .89022 | .89697 | .89315 |
| .1140 | .88928 | .89613 | .89225 |
| .1150 | .88834 | .89529 | .89136 |
| .1160 | .88739 | .89445 | .89047 |
| .1170 | .88645 | .89361 | .88958 |
| .1180 | .88553 | .89277 | .88869 |
| .1190 | .88456 | .89193 | .88780 |
| .1200 | .88363 | .89110 | .88692 |
| .1210 | .88268 | .89026 | .88603 |
| .1220 | .88178 | .88943 | .88514 |
| .1230 | .88082 | .88860 | .88426 |
| .1240 | .87986 | .88777 | .88337 |
| .1250 | .87893 | .88694 | .88249 |
| .1260 | .87802 | .88611 | .88161 |
| .1270 | .87706 | .88528 | .88073 |
| .1280 | .87614 | .88446 | .87985 |
| .1290 | .87518 | .88364 | .87897 |
| .1300 | .87426 | .88281 | .87809 |
| .1310 | .87332 | .88199 | .87721 |
| .1320 | .87239 | .88118 | .87634 |
| .1330 | .87145 | .88035 | .87546 |
| .1340 | .87052 | .87953 | .87459 |
| .1350 | .86958 | .87872 | .87371 |
| .1360 | .86870 | .87790 | .87284 |
| .1370 | .86771 | .87709 | .87197 |
| .1380 | .86681 | .87627 | .87109 |

| | | | |
|---|---|---|---|
| .1390 | .86580 | .87547 | .87022 |
| .1400 | .86491 | .87466 | .86935 |
| .1410 | .86401 | .87385 | .86848 |
| .1420 | .86310 | .87304 | .86762 |
| .1430 | .86211 | .87224 | .86675 |
| .1440 | .86120 | .87143 | .86588 |
| .1450 | .86028 | .87063 | .86502 |
| .1460 | .85938 | .86982 | .86415 |
| .1470 | .85848 | .86902 | .86329 |
| .1480 | .85744 | .86822 | .86243 |
| .1490 | .85660 | .86742 | .86156 |
| .1500 | .85568 | .86662 | .86070 |
| .1510 | .85472 | .86583 | .85984 |
| .1520 | .85380 | .86503 | .85898 |
| .1530 | .85290 | .86423 | .85812 |
| .1540 | .85196 | .86345 | .85727 |
| .1550 | .85105 | .86266 | .85641 |
| .1560 | .85006 | .86186 | .85555 |
| .1570 | .84926 | .86107 | .85470 |
| .1580 | .84826 | .86028 | .85384 |
| .1590 | .84734 | .85949 | .85299 |
| .1600 | .84647 | .85871 | .85214 |
| .1610 | .84547 | .85793 | .85129 |
| .1620 | .84465 | .85715 | .85044 |
| .1630 | .84366 | .85636 | .84959 |
| .1640 | .84270 | .85559 | .84874 |
| .1650 | .84189 | .85481 | .84789 |
| .1660 | .84087 | .85402 | .84704 |

| | | | |
|---|---|---|---|
| .1670 | .84004 | .85325 | .84619 |
| .1680 | .83905 | .85247 | .84535 |
| .1690 | .83822 | .85170 | .84450 |
| .1700 | .83722 | .85092 | .84366 |
| .1710 | .83640 | .85015 | .84282 |
| .1720 | .83540 | .84938 | .84197 |
| .1730 | .83453 | .84861 | .84113 |
| .1740 | .83362 | .84784 | .84029 |
| .1750 | .83268 | .84707 | .83945 |
| .1760 | .83180 | .84631 | .83861 |
| .1770 | .83088 | .84555 | .83777 |
| .1780 | .82992 | .84478 | .83694 |
| .1790 | .82908 | .84401 | .83610 |
| .1800 | .82807 | .84325 | .83527 |
| .1810 | .82724 | .84249 | .83443 |
| .1820 | .82635 | .84174 | .83360 |
| .1830 | .82540 | .84098 | .83276 |
| .1840 | .82450 | .84022 | .83193 |
| .1850 | .82360 | .83947 | .83110 |
| .1860 | .82267 | .83870 | .83027 |
| .1870 | .82178 | .83795 | .82944 |
| .1880 | .82088 | .83720 | .82861 |
| .1890 | .81997 | .83645 | .82778 |
| .1900 | .81908 | .83570 | .82695 |
| .1910 | .81817 | .83495 | .82613 |
| .1920 | .81725 | .83420 | .82530 |
| .1930 | .8162[illegible] | .83346 | .8244[illegible] |

| | | | |
|---|---|---|---|
| .1940 | .81546 | .83272 | .82365 |
| .1950 | .81457 | .83197 | .82283 |
| .1960 | .81363 | .83122 | .82201 |
| .1970 | .81273 | .83049 | .82119 |
| .1980 | .81189 | .82975 | .82036 |
| .1990 | .81090 | .82900 | .81954 |
| .2000 | .80999 | .82827 | .81873 |
| .2010 | .80914 | .82753 | .81791 |
| .2020 | .80824 | .82679 | .81709 |
| .2030 | .80736 | .82606 | .81627 |
| .2040 | .80644 | .82533 | .81546 |
| .2050 | .80553 | .82459 | .81464 |
| .2060 | .80460 | .82386 | .81383 |
| .2070 | .80378 | .82313 | .81301 |
| .2080 | .80285 | .82240 | .81220 |
| .2090 | .80192 | .82167 | .81139 |
| .2100 | .80110 | .82095 | .81058 |
| .2110 | .80010 | .82022 | .80977 |
| .2120 | .79932 | .81949 | .80896 |
| .2130 | .79841 | .81877 | .80815 |
| .2140 | .79739 | .81805 | .80734 |
| .2150 | .79666 | .81732 | .80654 |
| .2160 | .79562 | .81661 | .80573 |
| .2170 | .79486 | .81589 | .80492 |
| .2180 | .79387 | .81517 | .80412 |
| .2190 | .79308 | .81445 | .80332 |
| .2200 | .79209 | .81374 | .80251 |
| .2210 | .79127 | .81302 | .80171 |
| .2220 | .79029 | .81230 | .80091 |

| | | | |
|---|---|---|---|
| .2230 | .78950 | .81159 | .80011 |
| .2240 | .78857 | .81088 | .79931 |
| .2250 | .78762 | .81017 | .79851 |
| .2260 | .78688 | .80947 | .79771 |
| .2270 | .78585 | .80876 | .79692 |
| .2280 | .78509 | .80806 | .79612 |
| .2290 | .78411 | .80735 | .79532 |
| .2300 | .78324 | .80664 | .79453 |
| .2310 | .78239 | .80594 | .79373 |
| .2320 | .78145 | .80523 | .79294 |
| .2330 | .78058 | .80454 | .79215 |
| .2340 | .77979 | .80383 | .79136 |
| .2350 | .77875 | .80313 | .79057 |
| .2360 | .77804 | .80243 | .78978 |
| .2370 | .77705 | .80174 | .78899 |
| .2380 | .77610 | .80105 | .78820 |
| .2390 | .77539 | .80035 | .78741 |
| .2400 | .77435 | .79965 | .78662 |
| .2410 | .77354 | .79897 | .78584 |
| .2420 | .77271 | .79827 | .78505 |
| .2430 | .77173 | .79758 | .78427 |
| .2440 | .77096 | .79690 | .78348 |
| .2450 | .77010 | .79620 | .78270 |
| .2460 | .76910 | .79552 | .78192 |
| .2470 | .76823 | .79484 | .78114 |
| .2480 | .76744 | .79415 | .78035 |
| .2490 | .76652 | .79346 | .77957 |
| .2500 | .76566 | .79276 | .77880 |

| | | | |
|---|---|---|---|
| .2510 | .76479 | .79210 | .77802 |
| .2520 | .76390 | .79142 | .77724 |
| .2530 | .76293 | .79074 | .77646 |
| .2540 | .76217 | .79006 | .77569 |
| .2550 | .76131 | .78938 | .77491 |
| .2560 | .76035 | .78871 | .77414 |
| .2570 | .75958 | .78802 | .77336 |
| .2580 | .75864 | .78735 | .77259 |
| .2590 | .75783 | .78668 | .77182 |
| .2600 | .75692 | .78601 | .77105 |
| .2610 | .75605 | .78533 | .77028 |
| .2620 | .75512 | .78467 | .76951 |
| .2630 | .75430 | .78400 | .76874 |
| .2640 | .75349 | .78333 | .76797 |
| .2650 | .75264 | .78266 | .76720 |
| .2660 | .75158 | .78200 | .76643 |
| .2670 | .75089 | .78133 | .76567 |
| .2680 | .74996 | .78067 | .76490 |
| .2690 | .74910 | .78001 | .76414 |
| .2700 | .74822 | .77935 | .76337 |
| .2710 | .74735 | .77868 | .76261 |
| .2720 | .74655 | .77802 | .76185 |
| .2730 | .74561 | .77736 | .76109 |
| .2740 | .74481 | .77671 | .76033 |
| .2750 | .74388 | .77605 | .75957 |
| .2760 | .74305 | .77539 | .75881 |
| .2770 | .74220 | .77474 | .75805 |
| .27[illegible] | .74152 | .77409 | .7[illegible]72[illegible] |

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|---|---|---|---|
| .2790 | .74050 | .77344 | .75653 |
| .2800 | .73960 | .77278 | .75578 |
| .2810 | .73874 | .77214 | .75502 |
| .2820 | .73789 | .77148 | .75427 |
| .2830 | .73702 | .77083 | .75351 |
| .2840 | .73615 | .77019 | .75276 |
| .2850 | .73529 | .76955 | .75201 |
| .2860 | .73446 | .76890 | .75126 |
| .2870 | .73359 | .76825 | .75051 |
| .2880 | .73277 | .76761 | .74976 |
| .2890 | .73182 | .76697 | .74901 |
| .2900 | .73107 | .76633 | .74826 |
| .2910 | .73020 | .76568 | .74751 |
| .2920 | .72928 | .76505 | .74676 |
| .2930 | .72837 | .76441 | .74602 |
| .2940 | .72766 | .76377 | .74527 |
| .2950 | .72671 | .76314 | .74453 |
| .2960 | .72590 | .76250 | .74378 |
| .2970 | .72510 | .76167 | .74304 |
| .2980 | .72414 | .76123 | .74230 |
| .2990 | .72341 | .76061 | .74155 |
| .3000 | .72242 | .75997 | .74081 |
| .3010 | .72173 | .75934 | .74007 |
| .3020 | .72076 | .75871 | .73933 |
| .3030 | .71994 | .75809 | .73859 |
| .3040 | .71913 | .75746 | .73786 |
| .3050 | .71814 | .75683 | .73712 |
| .3060 | .71746 | .75621 | .73638 |

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|---|---|---|---|
| .3070 | .71654 | .75559 | .73565 |
| .3080 | .71561 | .75496 | .73491 |
| .3090 | .71492 | .75434 | .73418 |
| .3100 | .71402 | .75372 | .73344 |
| .3110 | .71309 | .75310 | .73271 |
| .3120 | .71231 | .75248 | .73198 |
| .3130 | .71138 | .75186 | .73124 |
| .3140 | .71071 | .75124 | .73051 |
| .3150 | .70983 | .75063 | .72978 |
| .3160 | .70890 | .75001 | .72905 |
| .3170 | .70805 | .74939 | .72833 |
| .3180 | .70731 | .74878 | .72760 |
| .3190 | .70647 | .74817 | .72687 |
| .3200 | .70551 | .74756 | .72614 |
| .3210 | .70474 | .74695 | .72542 |
| .3220 | .70383 | .74634 | .72469 |
| .3230 | .70308 | .74573 | .72397 |
| .3240 | .70221 | .74512 | .72325 |
| .3250 | .70135 | .74452 | .72252 |
| .3260 | .70049 | .74391 | .72180 |
| .3270 | .69972 | .74331 | .72108 |
| .3280 | .69894 | .74271 | .72036 |
| .3290 | .69800 | .74211 | .71964 |
| .3300 | .69714 | .74151 | .71892 |
| .3310 | .69632 | .74090 | .71820 |
| .3320 | .69556 | .74031 | .71748 |
| .3330 | .69472 | .73971 | .71677 |

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