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Numerical Modelling of Shallow Water Flows over Mobile Beds

by

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Abstract

This Ph.D. thesis aims to develop numerical models for two-dimensional and three-dimensional shallow water systems over mobile beds. In order to accomplish the goal of this dissertation, the following sub-projects are defined and completed.

1: The first sub-project consists in developing a robust two-dimensional coupled numerical model based on an unstructured mesh, which can simulate rapidly varying flows over an erodible bed involving wet-dry fronts that is a complex yet practically important problem. In this task, the central-upwind scheme is extended to simulation of bed erosion and sediment transport, a modified shallow water system is adopted to improve the model, a wetting and drying scheme is proposed for tracking wet-dry interfaces and stably predict the bed erosion near wet-dry area. The shallow water, sediment transport and bed evolution equations are coupled in the governing system. The proposed model can efficiently track wetting and drying interfaces while preserving stability in simulating the bed erosion near the wet-dry fronts. The additional terms in shallow water equations can improve the accuracy of the simulation when intense sediment-exchange exists; the central-upwind method adopted in the current study shows great accuracy and efficiency compared with other popular solvers; the developed model is robust, efficient and accurate in dealing with various challenging cases.

2: The second sub-project consists in developing a novel numerical scheme for a coupled two-dimensional hyperbolic system consisting of the shallow water equations with friction terms coupled with the equations modeling the sediment transport and bed evolution. The resulting 5×5 hyperbolic system of balance laws is numerically solved using a Godunov-type central-upwind scheme on a triangular grid. A spatially second-order and temporally third-order central-upwind scheme has been derived to discretize the conservative hyperbolic sub-system. However, such schemes need a correct evaluation of local wave speeds to avoid instabilities. To address such an issue, a mathematical result by the Lagrange theorem is used in the proposed scheme. Consequently, a computationally expensive process of finding all of the eigenvalues of the Jacobian matrices is avoided: The upper/lower bounds on the largest/smallest local speeds of propagation are estimated using the Lagrange theorem. In addition, a special discretization of the bed-slope term is proposed to guarantee the well-balanced property of the designed scheme.

3: The third sub-project consists in designing a novel scheme to estimate bed-load fluxes which can produce more accurate results than the previously reported coupled model. Using a pair of local wave speeds different from those used for the flow, a novel wave estimator in conjunction with the central upwind method is proposed and successfully applied to the coupled water-sediment system involving a rapid bed-erosion process. It was demonstrated that, in comparison with the decoupled model, applying the proposed novel scheme to approximate the bed-load fluxes can successfully avoid the numerical oscillations caused by simple and less stable schemes, e.g. simple upwind methods; in comparison with the coupled model using same flux-estimator for both hydrodynamic and morphological systems, the proposed numerical scheme successfully prevents excessive numerical diffusion for prediction of bed evolution. Consequently, the proposed scheme has advantages in terms of accuracy which are shown in several numerical tests. In addition, analytical expressions

have been provided for calculating the eigenvalues of the coupled shallow-water-Exner system, which greatly enhances the efficiency of the proposed method.

4: The fourth sub-project consists in developing a three-dimensional numerical model for the simulation of unsteady non-hydrostatic shallow water flows on unstructured grids using the finite volume method. The free surface variations are modeled by a characteristics-based scheme which simulates sub- and super-critical flows. Three-dimensional velocity components are considered in a collocated arrangement with a σ -coordinate system. A special treatment of the pressure term is developed to avoid the water surface oscillations. Convective and diffusive terms are approximated explicitly, and an implicit discretization is used for the pressure term. The unstructured grid in the horizontal direction and the σ coordinate in the vertical direction facilitate the use of the model in complicated geometries.

5: The fifth sub-project consists in developing a well-balanced three-dimensional shallow water model which is able to simulate shock waves over dry bed. Due to the hydrostatic simplification of the vertical momentum equation, the governing system of equations is not hyperbolic and can not be solved using standard hyperbolic solvers. That is, one can not use a high-order Godunov-type scheme to compute all fluxes through cell-interfaces. This may cause the model to fail in simulations of some unsteady-flows with discontinuities, e.g., dam-break flows and floods. To overcome this difficulty, a novel numerical scheme for the three-dimensional shallow water equations is proposed using a relaxation approach in order to convert the system to a hyperbolic one. Thus, a high-order Godunov-type central-upwind scheme based on the finite volume method can be applied to approximate the numerical fluxes. The proposed model can also preserve the “lake at rest” state and positivity of water depth over irregular bottom topographies based on special reconstruction of the corresponding parameters.

6: The sixth sub-project consists in extending the result of the fifth sub-project to development of a three-dimensional numerical model for shallow water flows over mobile beds, which is able to simulate morphological evolutions under shock waves, e.g. dam-break flows. The hydrodynamic model solves the three-dimensional shallow water equations using a finite volume method on prismatic cells in σ -coordinates based on the scheme proposed in sub-project 5. The morphodynamic model solves an Exner equation consisting of bed-load sediment transportation. The performance of the proposed model has been demonstrated by several laboratory experiments of dam-break flows over mobile beds.

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Chapter 1

Introduction

1.1 Background Knowledge

In this section, several key theories and terminologies which are pivotal in the current research project are briefly introduced and explained.

1.1.1 2-D and 3-D shallow water equations

The 2-D shallow water equations are a set of hyperbolic partial differential equations (if viscous shear is ignored) that describe a thin layer of fluid of constant density under hydrostatic assumption, bounded by the bottom topography and a free water surface. The system is also called Saint-Venant system in its unidimensional form. The 2-D shallow water equations can be derived from Navier-Stokes equations by depth-averaged integration under the assumption of constant density, Reynolds-Averaging, Newtonian-Fluid and hydrostatic status. A typical two-dimensional shallow water system can be written as:

$$\frac{\partial h}{\partial t} + \frac{\partial}{\partial x}(uh) + \frac{\partial}{\partial y}(vh) = 0 \quad (1.1)$$

$$\frac{\partial}{\partial t}(uh) + \frac{\partial}{\partial x}\left(u^2h + \frac{1}{2}gh^2\right) + \frac{\partial}{\partial y}(uvh) = -gh\frac{\partial B}{\partial x} \quad (1.2)$$

$$\frac{\partial}{\partial t}(vh) + \frac{\partial}{\partial x}(uvh) + \frac{\partial}{\partial y}\left(v^2h + \frac{1}{2}gh^2\right) = -gh\frac{\partial B}{\partial y} \quad (1.3)$$

in which t = time, h = the water depth, x and y are horizontal coordinates, u and v are the depth-averaged velocity-components in the x and y directions, respectively, B is the bed level, and g is the gravitational acceleration. Figure 1.1 shows a typical water column in 2-D SW system in the papers presented in this thesis.

Equation 1.1 is commonly called continuity equation, and equations 1.2 and 1.3 are commonly called momentum equations in x and y directions, respectively.

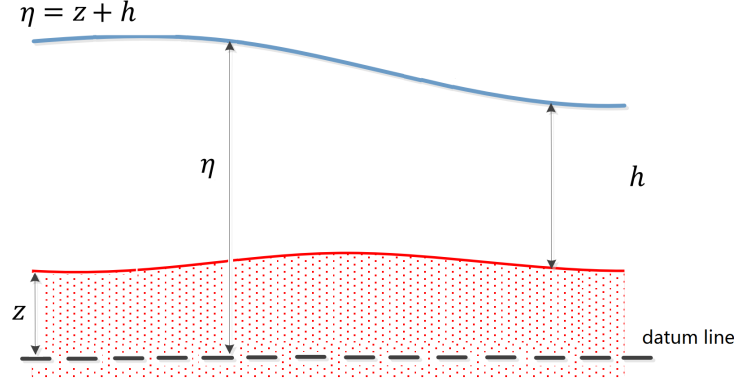


Figure 1.1: A typical water column in 2-D shallow water system

The 3-D shallow water equations can also be derived from the Navier-Stokes equations under the assumption of constant density, Reynolds-averaging and Newtonian-fluid. Note that the derivation of 3-D SWEs does not include the step of depth-averaged integration. It can thus calculate the vertical structure of shallow water flows. If the simplification of 3-D SWEs is under the assumption of hydrostatic pressure, the typical hydrostatic 3-D shallow water system can be written as:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \quad (1.4)$$

$$\frac{\partial u}{\partial t} + \frac{\partial uu}{\partial x} + \frac{\partial uv}{\partial y} + \frac{\partial uw}{\partial z} = -g \frac{\partial \eta}{\partial x} \quad (1.5)$$

$$\frac{\partial v}{\partial t} + \frac{\partial vu}{\partial x} + \frac{\partial vv}{\partial y} + \frac{\partial vw}{\partial z} = -g \frac{\partial \eta}{\partial y} \quad (1.6)$$

where z is the vertical coordinate, w is the velocity in the z direction and η is the elevation of the water surface.

If the hydrodynamic pressure is also considered, the governing equations of non-hydrostatic 3-D shallow water system can be written as:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \quad (1.7)$$

$$\frac{\partial u}{\partial t} + \frac{\partial uu}{\partial x} + \frac{\partial uv}{\partial y} + \frac{\partial uw}{\partial z} = -g \frac{\partial \eta}{\partial x} - \frac{1}{\rho} \frac{\partial p}{\partial x} \quad (1.8)$$

$$\frac{\partial v}{\partial t} + \frac{\partial vu}{\partial x} + \frac{\partial vv}{\partial y} + \frac{\partial vw}{\partial z} = -g \frac{\partial \eta}{\partial y} - \frac{1}{\rho} \frac{\partial p}{\partial y} \quad (1.9)$$

$$\frac{\partial w}{\partial t} + \frac{\partial wu}{\partial x} + \frac{\partial wv}{\partial y} + \frac{\partial ww}{\partial z} = -\frac{1}{\rho} \frac{\partial p}{\partial z} \quad (1.10)$$

1.1.2 Hyperbolic systems

Quantities in most differential equations of physics depend on both space and time, and partial derivatives and partial differential equations (PDE) are involved inevitably.

What is a hyperbolic partial differential equation?

In mathematics, partial differential equations can be generally classified into three types, namely hyperbolic, parabolic and elliptic. Generally, the flow equations are non-linear. In order to classify the non-linear equations, currently, the conventional method is linearizing the equations first, and then analyzing the linearized system. For example, a first order system of quasi-linear equations with two independent variables is

$$\mathbf{u}_t + \mathbf{A}\mathbf{u}_x = 0 \quad (1.11)$$

where $\mathbf{u}$ is a vector function of x and t and $\mathbf{A}$ is a matrix function of $\mathbf{u}$ as well as x and t . The determinant ($\det|\mathbf{A} - \lambda\mathbf{I}|$) (where $\mathbf{I}$ is identity matrix) can be written as

$$a\lambda^2 + b\lambda + c = 0 \quad (1.12)$$

If $b^2 - 4ac > 0$, $\mathbf{A}$ is a diagonalizable matrix and there are two real and distinct eigenvalues. The system in this case is hyperbolic.

If $b^2 - 4ac = 0$, there is only one real eigenvalue and the system is parabolic.

If $b^2 - 4ac < 0$, there are two imaginary eigenvalues and the system is elliptic.

Classification of higher order PDE systems is more complicated and is rarely involved in water related problems.

What is the significance of classification of PDEs?

Generally, different types of PDEs describe different flow characteristics and physical meanings, moreover, the mathematical properties and numerical methods for them are different. One has to choose different algorithms to solve different types of PDEs.

(1) For example, to solve the elliptical equation

$$u_{xx} + u_{yy} = f(x, y), \quad (x, y) \in D \quad (1.13)$$

where D is a bounded domain, one only needs the boundary conditions. However, to solve the hyperbolic equation

$$u_{tt} = u_{xx} \quad (1.14)$$

or parabolic equation

$$u_t + u_x = u_{xx} \quad (1.15)$$

, one has to know both boundary conditions and initial conditions.

(2) The zones affected by solutions of hyperbolic and parabolic equations are bounded, and the upwind methods are better choices for hyperbolic and parabolic equations because they use characteristic information similar to the analytical solution. The zone affected by solutions of elliptical equations is the whole domain, and the central-differencing method is more appropriate because it preserves symmetry of the system.

1.1.3 Shock wave and shock capturing schemes

A shock wave is a type of propagating disturbance. A shock wave is generated when a wave moves faster than the speed of sound in the propagation fluid, and it is characterized by an abrupt and nearly discontinuous change in temperature, density and pressure of the medium through which the shock wave propagates.

In computational fluid dynamics, numerical schemes that detect and compute a solution with a shock discontinuity automatically, without requiring any special treatment for the shock, are known as shock capturing schemes[124].

The classical central or symmetric schemes do not take any information about the wave propagation into account in the discretization. Comparing with the classical central or symmetric schemes, modern shock-capturing methods are generally upwind based. When using the upwind-type differencing schemes and considering the wave propagation, the hyperbolic PDEs are discretized by differencing biased in the direction determined by the characteristic speeds.

In order to avoid unphysical numerical oscillations, a stable simulation of a shock requires a certain amount of numerical dissipation no matter what type of shock-capturing scheme is used. When applying classical shock-capturing methods, numerical dissipation is generally linear and unadjustable. This leads to the result that the classical shock-capturing methods only show accurate results for smooth solutions, but unstable and oscillatory results for shock waves. On the contrary, a non-linear numerical dissipation is introduced in modern shock-capturing methods, and the amount of dissipation in any cell can automatically be adjusted according to the gradients of the computed unknowns. This leads to the result that modern shock-capturing methods can stably and accurately solve the solutions across discontinuities.

Examples of classical shock-capturing methods include the BeamWarming method [12], LaxWendroff method [98], and MacCormack method [115]. Some of the modern shock-capturing schemes include the flux-corrected transport scheme proposed by Boris and Book [19], Monotonic Upstream-centered Schemes for Conservation Laws (MUSCL) introduced by Van Leer [166] based on the Godunov method, higher-order Total Variation Diminishing (TVD) schemes first introduced by Harten [70], and Essentially Non-Oscillatory schemes (ENO) proposed by Harten et al. [71]. The central-upwind method used in the current thesis is a Godunov-type modern shock-capturing method [93]. Approximate Riemann solvers present an important class of modern shock-capturing methods, examples include the approximate Riemann solvers developed by Roe [133], Shu and Osher [142], and Harten et al. [72].

1.1.4 Central upwind method

Kurganov and Tadmor [93] introduced a new class of semi-discrete central schemes, which have been proven to be efficient and accurate as a shock-capturing method. This is achieved by utilizing local speeds of propagation to achieve a more precise estimate of the width of the Riemann fans, but still without any attempt to (approximately) solve Riemann

problems. The central-upwind schemes are Godunov-type central scheme because the evolution step employs integration over Riemann fans and does not require a Riemann solver and a characteristic decomposition. They also have an upwind nature, since one-sided informations are used to estimate the width of the Riemann fans. The central-upwind method for the Saint-Venant system was proposed in [86]. A well-balanced positivity preserving central-upwind scheme on triangular grids for the Saint-Venant system was proposed in [23].

The general form of the central-upwind method can be written as:

$$\mathbf{F}_{ik} = \frac{[S_R \mathbf{F}(\mathbf{U}_L)_{ik} + S_L \mathbf{F}(\mathbf{U}_R)_{ik}] \cdot \mathbf{n} - S_R S_L [(\mathbf{U}_R)_{ik} - (\mathbf{U}_L)_{ik}]}{(S_R + S_L)} \quad (1.16)$$

in which $\mathbf{F}(\mathbf{U}_L)_{ik}$ and $\mathbf{F}(\mathbf{U}_R)_{ik}$ are normal fluxes on the left and right sides of the k th interface, respectively. S_R and S_L are one-sided local speeds of propagation on right and left sides of k th interface, respectively, which are

$$\begin{aligned} S_R &= \max\{\lambda_N(\mathbf{U}_L)_{ik}, \lambda_N(\mathbf{U}_R)_{ik}, 0\} \\ S_L &= -\min\{\lambda_1(\mathbf{U}_L)_{ik}, \lambda_1(\mathbf{U}_R)_{ik}, 0\} \end{aligned} \quad (1.17)$$

with $\lambda_1 \leq \dots \leq \lambda_N$ being the N eigenvalues of the Jacobian matrix $\partial F_n(\mathbf{U})/\partial \mathbf{U}$ using reconstructed variables $(\mathbf{U}_L)_{ik}$ or $(\mathbf{U}_R)_{ik}$ at the center of the k th interface of the i th cell.

When a cell is dry or nearly dry, “ $S_R + S_L \approx 0$ ” may exist in the denominator of equation (1.16) and may cause numerical instability. In order to handle the situation that both S_R and S_L are zero or very close to zero, following the suggestion in [23], (1.16) reduces to

$$\mathbf{F}_{ik} = 0.5[\mathbf{F}(\mathbf{U}_L)_{ik} + \mathbf{F}(\mathbf{U}_R)_{ik}], \quad \text{if } S_L + S_R < 10^{-8} \quad (1.18)$$

1.1.5 Well-balanced property

A convective system can be written as

$$\mathbf{U}_t + \mathbf{F}(\mathbf{U}) = \mathbf{S}(\mathbf{U}) \quad (1.19)$$

Different numerical methods are used to approximate all three terms in 1.19. The well-balanced property of the scheme is guaranteed if, for the steady states, the discretized cell average of the source term, $\mathbf{S}_i$, exactly balances the numerical fluxes $\mathbf{F}(\mathbf{U})$ so that the right-hand side (RHS) of equation 1.19 vanishes.

In the current research, specifically, the balance between the numerical discretization of flux and bed slope terms in the SWEs is referred to as the well-balanced property which guarantees that “lake at rest” solutions that satisfy

$$\eta \equiv \eta_c = \text{Const}, \quad u \equiv v \equiv 0, \quad (1.20)$$

are exactly preserved by the applied numerical scheme.

Using the 2-D SWEs 1.1–1.3 as an example, under the “lake at rest” steady states 1.20, equations 1.1–1.3 become

$$\frac{\partial h}{\partial t} = 0 \quad (1.21)$$

$$\frac{\partial}{\partial t}(uh) = -\frac{\partial}{\partial x}\left(\frac{1}{2}gh^2\right) - gh\frac{\partial B}{\partial x} \quad (1.22)$$

$$\frac{\partial}{\partial t}(vh) = -\frac{\partial}{\partial y}\left(\frac{1}{2}gh^2\right) - gh\frac{\partial B}{\partial y} \quad (1.23)$$

In order to guarantee that the steady solutions $\frac{\partial}{\partial t}(uh) = 0$ and $\frac{\partial}{\partial t}(vh) = 0$ are preserved, the right hand sides (RHS) of the equations 1.22 and 1.23 have to be exactly zero using the proposed numerical methods. In the current project, the first term on the RHS of 1.22 or 1.23 is part of the flux term and is generally computed by a specific shock-capturing scheme. In order to exactly cancel the first term on the RHS, special numerical methods have to be proposed to discretize the second term (bed-slope source term) on the RHS to preserve the steady state. A scheme that succeeds in doing this, is considered to be able to preserve the well-balanced property.

1.1.6 Positivity-preserving property

A positivity-preserving scheme is a numerical scheme that guarantees positivity of the predicted water depth when the system of ODEs is discretized in time, using the forward Euler method or a higher-order strong-stability-preserving ODE solver.

1.2 Objectives of the project

The main objective of this research project was developing numerical models for rapid bed erosion processes and shallow water flows, which could be divided into the following sub-objectives:

1. Developing a two dimensional coupled model for rapid bed-erosion process under shallow water flows.
2. Developing a 3-D hydrodynamic model using 3-D shallow water equations; and extending this 3-D hydrodynamic model to the study of bed erosion problems.

1.3 Significances and novelties of the project

The subject of this research project was the development of numerical models for rapid bed erosion process under shallow water flows. Six components of the study in the scope of this project are presented in chapter 2 – chapter 7, respectively. A brief description of the significance and novelty of each component is provided in the following.

1.3.1 Novelties and contributions of the study in [chapter 2](#)

- (1) A robust model of water-sediment system is developed. It can deal with the following extreme cases: large hydraulic gradients, steep bed slopes, very small water depths, wetting and drying, shock wave propagation, intense sediment exchange and rapid bed erosion. The proposed model is also very efficient, because it uses the central-upwind method.
- (2) A wetting-drying scheme for erodible bed is proposed. It can stably track the wetting-drying fronts in the presence of bed erosion.
- (3) The effect of additional terms on the right hand side of modified SWEs is investigated. These components can also decelerate the wave propagation due to the inertia of sediment. It can also be understood as a fact that the sediment absorbs the momentum from water.
- (4) The central-upwind method is extended to the study of the water-sediment system, and it shows good efficiency and accuracy.

1.3.2 Novelties and contributions of the study in [chapter 3](#)

- (1) A novel scheme combining “Lagrange” theorem with central-upwind scheme when deriving finite volume method for hyperbolic system is proposed. This is a substantial contribution because it provides the readers with a new tool for solving the coupled water-sediment-Exner system and other systems coupling multiple physical processes.
- (2) A novel well-balanced discretization proposed in this chapter is designed for discontinuous bed level Z , approximated using a discontinuous piecewise polynomial approximation of Z , which is now evolved in the framework of a finite-volume method.

1.3.3 Novelties and contributions of the study in [chapter 4](#)

- (1) A novel wave estimator in conjunction with central-upwind method is proposed. Most reported coupled methods of water-erosion system use the same hyperbolic solver (estimated from hydrodynamic model) for the Exner equation. This leads to excessive numerical diffusion. This defect prevents the application of the coupled method to real world cases. In the proposed novel scheme, the bed-load transport is estimated using a pair of local wave speeds different from those used for the flow. This method successfully avoids the excessive diffusion on the premise that the stability is guaranteed for rapid bed erosion processes. The proposed method shows good accuracy and stability in real world applications.

(2) An analytical expression of Jacobian matrix of the coupled system is given by the author, which allows for a direct application of the results in different cases.

1.3.4 Novelties and contributions of the study in [chapter 5](#)

(1) This study uses upwind methods on unstructured grids for 3-D non-hydrostatic free surface flows in σ -coordinates.

(2) A numerical model is proposed to employ upwind methods on unstructured grids for 3-D non-hydrostatic free surface shallow water equations.

1.3.5 Novelties and contributions of the study in [chapter 6](#)

(1) A novel numerical model of 3-D shallow water system is proposed using a relaxation approach. The existing schemes for 3-D SWEs cannot deal with shocks and discontinuities because no shock-capturing method has been applied to the 3-D SWEs due to the lack of hyperbolicity. In order to solve this problem, a method is proposed that uses the relaxation approach to build a new hyperbolic system, thus shock capturing schemes can be used in calculation of horizontal discharges. As a result, the proposed model can successfully capture the discontinuities in flows. Cases like dam break and flood can thus be simulated by the proposed method.

(2) Well-balanced and positivity-preserving properties are preserved by the proposed model. Therefore, the model can deal with “lake at rest” status, very small water depth, and wetting and drying processes.

1.3.6 Novelties and contributions of the study in [chapter 7](#)

(1) Extending the proposed scheme in [chapter 6](#) to the current study, a novel 3-D numerical model for dam-break flows over mobile bed is developed, which is capable of simulating morphological evolution under shock waves, e.g. dam-break flows.

(2) Two interesting aspects are investigated: one interesting aspect of the proposed 3-D model is about the way it performs for dam-break flows over mobile bed compared with 2-D depth-averaged models; another interesting aspect of the proposed 3-D model is the comparison of the use of depth-averaged velocity with near bed speed to estimate bed-load transport.

Chapter 2

A robust coupled 2-D model for rapidly varying flows over erodible bed using central-upwind method with wetting and drying

2.1 Abstract

This paper aims to develop a robust two-dimensional coupled numerical model based on an unstructured mesh, which can simulate rapidly varying flows over an erodible bed involving wet-dry fronts that is a complex yet practically important problem. Using a modified spatial reconstruction based on the finite volume method, the well-balanced property is preserved, which is important for accurate and efficient simulation of morphological problems. In the present study, the central-upwind scheme is extended to simulation of bed erosion and sediment transport for the first time. It is demonstrated that the proposed scheme shows good accuracy and high efficiency. A modified shallow water system is adopted to improve the model. The shallow water equations, sediment transport equation and bed evolution equation are coupled in the governing system. Multiple test cases are employed to demonstrate the robustness, accuracy, and efficiency of the current model. Furthermore, with a field scale dam-break test case, the efficiency and accuracy of the central-upwind method is verified in comparison with other popular Riemann solvers. The effects of the additional source terms in the adopted modified shallow water equations are also investigated by comparing the numerical results with a laboratory study available in the literature. The proposed scheme can efficiently track wetting and drying interfaces while preserving stability in simulating the bed erosion near the wet-dry fronts. The added terms in shallow water equations can improve the accuracy of the simulation when intense sediment-exchange exists; the central-upwind method adopted in the current study shows great accuracy and efficiency compared with other popular solvers; the developed model is robust, efficient and accurate to deal with various challenging cases.

Keywords: Dam-break flows, Central-upwind method, Wetting and drying, Bed ero-

sion, Sediment transport, Shallow water equations, Finite volume method.

2.2 Introduction

Rapidly varying flows over a mobile bed are commonly observed in engineering applications although limited studies have been dedicated to their modeling, which is a challenging task in terms of numerical aspects. Such flows may experience large hydraulic gradients, steep bed slopes, very small water depths, wetting and drying, shock wave propagation, intense sediment exchange and rapid bed erosion. Thus, developing a robust model that can deal with these items is challenging but necessary. Some numerical models that simulate rapid shallow water flows over an erodible bed have been developed in the literature [180, 182, 178, 120, 126]. However, in order to build a robust model, the schemes for wetting and drying, very small water depth, positivity of water depth and well-balanced property need to be included in the study.

Central-upwind method on general triangular grids for solving two-dimensional (2-D) systems of conservation laws were developed by Kurganov and Petrova [90], and enjoy the main advantages of the Godunov-type central schemes, i.e., simplicity, universality and robustness and can be applied to problems with complicated geometries. This relatively new method has not been adapted to the study of bed erosion so far. In order to build a robust numerical model, a coupling approach, second order temporal and spatial reconstruction, well-balanced discretization of bed-slope term, non-negative water depth reconstruction, special treatment of friction term and wet-dry tracking technique are necessary to be included in the model. The computational cost of such a robust and complete model could be very expensive. Thus, it inevitably requires a solver, e.g., the central-upwind method which is less expensive than other popular Riemann solvers while still capable of simulating complex flows.

Flow over an initially dry bed is very common in many engineering problems, and it involves complicated boundary conditions. For irregular topography, both positive and negative bed slopes generally exist, which may lead to cell drying and wetting with moving fronts, which may not be easily modeled using simple treatment methods typically used for boundary conditions. Some techniques have been developed using the finite volume method (FVM) for shallow water equations to track wet-dry fronts. Zhao et al. [189] and Sleigh et al. [145] introduced two similar schemes to track the wetting and drying fronts, in which cells are divided into wet, dry and partially dry types according to two tolerances. Brufau et al. [22] proposed a technique using unsteady wetting and drying conditions in flows and claimed that the method gave zero mass error, which is valid for a FVM with only first-order accuracy. Falconer et al. [52] developed a wetting and drying method for regular grid finite difference model, which is recently refined for triangular grids by Xia et al. [180]. However, the above-mentioned schemes are only developed and applied for fixed bed conditions. For cases involving erodible bed, the water depth is often present in the denominator of erosion related terms and it can cause numerical oscillations and instabilities if the water depth is very small. In this paper, in order to simulate the bed erosion and wave propagation in areas exposed to wetting and drying condition, which

cause instability issues with most available schemes, we develop a scheme for tracking wet-dry interfaces over erodible beds.

In this paper, a coupled system consisting of a hydrodynamic model for rapidly evolving shallow water and sediment transport as well as bed evolution is developed. The modified 2-D shallow water and sediment transport equations are adopted as suggested by Cao et al. [29] and Yue et al. [182], in which several terms are added by considering the flowing fluid as a water-sediment mixture instead of clear water. The impact of those added terms are verified in this paper using numerical experiments. The central-upwind method is adopted in order to efficiently and precisely approximate numerical fluxes through cell interfaces. A wetting and drying treatment technique for mobile bed is proposed in order to track wet-dry fronts and predict the bed erosion in these areas while ensuring the stability of the numerical method. The bed evolution equation is reformulated as suggested by Fagherazzi and Sun [51] in order to couple all the governing equations and build a system of conservation laws. In addition, the proposed method offers multiple desirable features such as second-order accuracy in space and time, non-negative water depth reconstruction, well-balanced discretization of the bed slope terms and point-implicit treatment of the friction term, which makes it an attractive and robust model for a wide range of practical applications in river and coastal engineering.

2.3 Governing Equations

2.3.1 Description of Dam-break Hydraulic and Sediment Transport System

As suggested in [182], the following governing equations are adopted:

$$\frac{\partial h}{\partial t} + \frac{\partial}{\partial x}(uh) + \frac{\partial}{\partial y}(vh) = \frac{E - D}{1 - p} \quad (2.1)$$

$$\begin{aligned} \frac{\partial}{\partial t}(uh) + \frac{\partial}{\partial x}\left(u^2h + \frac{1}{2}gh^2\right) + \frac{\partial}{\partial y}(uvh) = & -gh\left(\frac{\partial Z}{\partial x} + S_f^x\right) - gh^2\frac{(\rho_s - \rho_f)}{2\rho_m}\frac{\partial c}{\partial x} \\ & - u\frac{\rho_0 - \rho_m}{\rho_m}\frac{E - D}{1 - p} \end{aligned} \quad (2.2)$$

$$\begin{aligned} \frac{\partial}{\partial t}(vh) + \frac{\partial}{\partial x}(uvh) + \frac{\partial}{\partial y}\left(v^2h + \frac{1}{2}gh^2\right) = & -gh\left(\frac{\partial Z}{\partial y} + S_f^y\right) - gh^2\frac{(\rho_s - \rho_f)}{2\rho_m}\frac{\partial c}{\partial y} \\ & - v\frac{\rho_0 - \rho_m}{\rho_m}\frac{E - D}{1 - p} \end{aligned} \quad (2.3)$$

$$\frac{\partial}{\partial t}(hc) + \frac{\partial}{\partial x}(uhc) + \frac{\partial}{\partial y}(vhc) = E - D \quad (2.4)$$

in which t =time, h =the water depth, x and y are horizontal coordinates, u and v are the depth-averaged velocity components in the x and y directions, respectively, Z is

the bed level, g is the gravitational acceleration, p is the bed porosity, ρ_s is the density of sediment particles, ρ_f is the density of clear water, ρ_o is the density of saturated bed given by $\rho_o = \rho_f p + \rho_s(1 - p)$, ρ_m is the density of water and sediment mixture obtained from $\rho_m = \rho_f(1 - c) + \rho_s c$, in which c is volumetric sediment concentration and is assumed to be uniform on the depth. S_f^x and S_f^y are friction slope terms in the x , y directions, respectively, which can be determined by conventional formulas involving Manning roughness coefficient n_b , $S_f^x = n_b^2 h^{-4/3} u \sqrt{u^2 + v^2}$, $S_f^y = n_b^2 h^{-4/3} v \sqrt{u^2 + v^2}$. Furthermore, in this work, water surface level $\eta = h + Z$ is used in the second order spatial reconstruction and the non-negative water depth reconstruction. The E and D terms with units of m/s are sediment entrainment and deposition terms which are defined in the [subsection 2.3.3](#).

According to Cao et al. [29], the term on the right hand side of continuity equation (2.1) represents the mass transfer from the erodible bed; the last two terms in the momentum equations (2.2) and (2.3) represent the effects of spatial variations of sediment concentration and momentum exchange between sediment and water, respectively. These two terms are only significant when rapid erosion and hyper concentration occurs in flows such as the dam-break case. The impacts of these terms will be investigated in the Dam break flow over movable bed test. In addition, in the current study, an activation condition is added for these two terms to avoid unphysical perturbation under "lake at rest" condition when sediment deposition exists, which will be explained in the Numerical scheme section.

2.3.2 Morphological Model

As suggested in [51], the morphological equations can be transformed to:

$$\frac{\partial \psi}{\partial t} + \frac{\partial}{\partial x} (uhc) + \frac{\partial}{\partial y} (vhc) = 0 \quad (2.5)$$

where ψ can be treated as a variable and can be calculated in a coupled system. The bed-level Z can be updated using the relation $\psi = (1 - p)Z + hc$ at the end of each time-step.

2.3.3 Sediment Deposition and Entrainment Fluxes

Cao et al. [29] introduced a set of scaling relations and approximate formulas, and recast the formulas to be used in shallow water system to determine the entrainment flux E and deposition flux D based on assumptions of non-cohesive sediment. The term D can be calculated as $D = \omega_o(1 - C_a)^m C_a$, where ω_o is settling velocity of a single particle in tranquil water and $\omega_o = \sqrt{(13.95\nu/d)^2 + 1.09sgd} - 13.95\nu/d$ [186], in which ν is kinematic viscosity of water, d is mean diameter of sediment particles, s is the submerged specific gravity of sediment given by $s = \frac{\rho_s}{\rho_f} - 1$, m is an exponent indicating the effect of hindered settling due to high sediment concentration, C_a is local near-bed sediment concentration in volume, which can be determined with $C_a = \alpha c$, where coefficient α is usually larger than unity and c must not exceed $(1 - p)$. Cao et al. [29] suggested $\alpha = \min[2, (1 - p)/c]$.

An empirical entrainment formulation suggested in [29] is chosen.

$$E = \begin{cases} \varphi(\theta - \theta_c)\sqrt{u^2 + v^2}d^{-0.2}h^{-1} & \text{if } \theta \geq \theta_c \\ 0 & \text{otherwise} \end{cases} \quad (2.6)$$

where φ is a coefficient to control the erosion force and $\varphi = 0.015(m^{1.2})$, θ is dimensionless shear stress which can be calculated as $\theta = hS'/(sd)$, in which $S' = S_f = \sqrt{S_f^{x2} + S_f^{y2}}$. θ_c is critical bed shear stress which is calculated using Van Rijn's method [168]:

2.4 Numerical Scheme

Equations (2.1), (2.2), (2.3), (2.4) and (2.5) constitute a hyperbolic system that can be presented in the following vector form:

$$\frac{\partial \mathbf{U}}{\partial t} + \frac{\partial \mathbf{F}}{\partial x} + \frac{\partial \mathbf{G}}{\partial y} = \mathbf{S} \quad (2.7)$$

where $\mathbf{U}$ is the vector of conserved variables; $\mathbf{F}$ and $\mathbf{G}$ are the convective fluxes in x and y directions, respectively; $\mathbf{S}$ is the source term. These vectors are defined as:

$$\mathbf{U} = \begin{pmatrix} h \\ uh \\ vh \\ ch \\ \psi \end{pmatrix}, \quad \mathbf{F} = \begin{pmatrix} uh \\ u^2h + \frac{gh^2}{2} \\ vuh \\ uch \\ uch \end{pmatrix}, \quad \mathbf{G} = \begin{pmatrix} vh \\ uvh \\ v^2h + \frac{gh^2}{2} \\ vch \\ vch \end{pmatrix}, \quad (2.8)$$

$$\mathbf{S} = \begin{pmatrix} \frac{E-D}{1-p} \\ -gh \left(\frac{\partial Z}{\partial x} + S_f^x \right) - gh^2 \frac{(\rho_s - \rho_f)}{2\rho_m} \frac{\partial c}{\partial x} - u \frac{\rho_0 - \rho_m}{\rho_m} \frac{E-D}{1-p} \\ -gh \left(\frac{\partial Z}{\partial y} + S_f^y \right) - gh^2 \frac{(\rho_s - \rho_f)}{2\rho_m} \frac{\partial c}{\partial y} - v \frac{\rho_0 - \rho_m}{\rho_m} \frac{E-D}{1-p} \\ E - D \\ 0 \end{pmatrix} \quad (2.9)$$

The Jacobian matrix $\partial F_n(\mathbf{U})/\partial \mathbf{U} = \frac{\partial \mathbf{F}}{\partial \mathbf{U}}n_x + \frac{\partial \mathbf{G}}{\partial \mathbf{U}}n_y$ of the coupled system has five eigenvalues:

$$\lambda_1, \lambda_3 = (un_x + vn_y) \pm a, \quad \lambda_2 = \lambda_4 = un_x + vn_y, \quad \lambda_5 = 0 \quad (2.10)$$

where λ are eigenvalues, $a = \sqrt{gh}$ is local dynamic wave velocity, n_x and n_y are components of unit normal vector in the x and y directions, respectively.

2.4.1 Discretization of the Governing Equations

In the present study, the coupled system including five equations is discretized on an unstructured triangular grid by finite volume method. The line-integration can be discretized as follows by summing fluxes on each boundary of the triangular control volume:

$$\frac{\partial \mathbf{U}_i}{\partial t} = -\frac{1}{A_i} \sum_{k=1}^3 \mathbf{F}_{ik} l_{ik} + \mathbf{S}_i \quad (2.11)$$

in which l_{ik} is the length of the k th edge of control volume i , $\mathbf{F}_{ik}$ is numerical convective flux across l_{ik} and can be calculated as $\mathbf{F}_{ik} = \mathbf{F}^*[(\mathbf{U}_L)_{ik}, (\mathbf{U}_R)_{ik}]$ in which $\mathbf{F}^*$ is an Godunov-type solver for the coupled system, $(\mathbf{U}_L)_{ik}$ and $(\mathbf{U}_R)_{ik}$ are the reconstructions of $\mathbf{U}$ on the left and right sides of the k th interface of cell i .

2.4.2 Central-upwind Method for the Interface Fluxes

Applying the central-upwind scheme, one obtains:

$$\mathbf{F}_{ik} = \frac{[S_R \mathbf{F}(\mathbf{U}_L)_{ik} + S_L \mathbf{F}(\mathbf{U}_R)_{ik}] \cdot \mathbf{n} - S_R S_L [(\mathbf{U}_R)_{ik} - (\mathbf{U}_L)_{ik}]}{(S_R + S_L)} \quad (2.12)$$

in which $\mathbf{F}(\mathbf{U}_L)_{ik}$ and $\mathbf{F}(\mathbf{U}_R)_{ik}$ are normal fluxes on the left and right sides of the k th edge, respectively. S_R and S_L are one-sided local speeds of propagation on right and left sides of k th edge, respectively. $S_R = \max\{\lambda_N(\mathbf{U}_L)_{ik}, \lambda_N(\mathbf{U}_R)_{ik}, 0\}$ and $S_L = -\min\{\lambda_1(\mathbf{U}_L)_{ik}, \lambda_1(\mathbf{U}_R)_{ik}, 0\}$ with $\lambda_1 \leq \dots \leq \lambda_N$ being the N eigenvalues of the Jacobian matrix $\partial F_n(\mathbf{U})/\partial \mathbf{U}$ using reconstructed variables $(\mathbf{U}_L)_{ik}$ or $(\mathbf{U}_R)_{ik}$.

When a cell is dry or nearly dry, " $S_R + S_L \approx 0$ " may exist in the denominator of equation (2.12) and may cause numerical instability. In order to handle the situation that both S_R and S_L are zero or very close to zero, following the suggestion in [23], (2.12) reduces to $\mathbf{F}_{ik} = 0.5[\mathbf{F}(\mathbf{U}_L)_{ik} + \mathbf{F}(\mathbf{U}_R)_{ik}]$, if $S_L + S_R < 10^{-8}$.

2.4.3 Spatial linear Reconstruction

In order to use a MUSCL-type piecewise linear-reconstruction proposed in [166] to obtain a higher order spatial accuracy, the reconstructed values $\mathring{\mathbf{U}}_L^M$ and $\mathring{\mathbf{U}}_R^M$ on left and right side of midpoint M of k th edge can be interpolated as:

$$\mathring{\mathbf{U}}_L^M = \mathbf{U}_i + 0.5\varphi(\mathbf{r}_i) [(\mathbf{U}_i - \mathbf{U}_m)\mathbf{r}_i], \quad \mathring{\mathbf{U}}_R^M = \mathbf{U}_j - 0.5\varphi(\mathbf{r}_j) [(\mathbf{U}_s - \mathbf{U}_j)\mathbf{r}_j] \quad (2.13)$$

in which a minmod-type limiter $\varphi(r) = \max[0, \min(r, 1)]$ proposed by Roe and Baines [135] is used, j represents a neighboring cell on k th edge, the variable values $\mathbf{U}_m$ and $\mathbf{U}_s$

at nodes m and s can be determined by the inverse-power interpolation scheme proposed by Scott and Barber [138], conserved variables r_i and r_j are calculated as

$$r_i = \frac{2a}{a+b} \frac{U_j - U_i}{U_i - U_m + \varepsilon_p}, \quad r_j = \frac{2b}{a+b} \frac{U_j - U_i}{U_s - U_j + \varepsilon_p} \quad (2.14)$$

where a and b are the distances between the centroid of cells i and j and the middle point M of their interface. ε_p is a very small number ($\varepsilon_p = 10^{-6}$) to avoid the division by zero.

In order to minimize the oscillations and enhance the model efficiency and stability, the appearance of local extrema is prevented by using a limited directional correction of the form of a maximum principle constraint [24, 79]:

$$\min(U_i, U_j) \leq \mathring{U}_L^M \leq \max(U_i, U_j), \quad \min(U_i, U_j) \leq \mathring{U}_R^M \leq \max(U_i, U_j) \quad (2.15)$$

With this constraint, the new local extrema caused by the reconstruction (2.13) is successfully prevented in an efficient way.

In order to preserve the well-balanced property for second order accurate schemes, as suggested in [8], water surface levels $\mathring{\eta}_L^M, \mathring{\eta}_R^M$, water depths $\mathring{h}_L^M, \mathring{h}_R^M$, flow discharges $(\mathring{hu})_L^M, (\mathring{hu})_R^M, (\mathring{hv})_L^M, (\mathring{hv})_R^M$ and sediment discharges $(\mathring{hc})_L^M, (\mathring{hc})_R^M$ are reconstructed at the midpoint M of the considered edges. The reconstructed bed elevations at M are given by $\mathring{Z}_L^M = \mathring{\eta}_L^M - \mathring{h}_L^M$ and $\mathring{Z}_R^M = \mathring{\eta}_R^M - \mathring{h}_R^M$.

A threshold ε will be introduced later in the Time integration section for defining a dry cell. The second order reconstructions are only applicable to the wet cells. In a dry cell, the variables on both sides of the midpoint of interfaces are calculated using first order scheme to prevent numerical instabilities [103, 176].

2.4.4 Non-negative water depth reconstruction

The water depth values at left or right sides of interfaces could become negative after linear reconstruction, especially at wet-dry wave fronts. Hence, a proper modification for the variables developed by Liang [103] is applied in current study at these interfaces to assure non-negative water depth and the well-balanced property.

Firstly, the bed level at the midpoint of an edge is defined as $Z^M = \min(\mathring{Z}_L^M, \mathring{Z}_R^M, \mathring{\eta}_L^M)$, then the non-negative water depth values on both sides are reconstructed as:

$$h_R^M = \max(0, \mathring{\eta}_R^M - Z_b^M) - \max(0, \mathring{Z}_{bR}^M - Z_b^M), \quad h_L^M = \mathring{\eta}_L^M - Z^M \quad (2.16)$$

the discharges at M are in turn reconstructed as

$$\begin{aligned} (hu)_L^M &= h_L^M \cdot \mathring{u}_L^M, & (hu)_R^M &= h_R^M \cdot \mathring{u}_R^M, & (hv)_L^M &= h_L^M \cdot \mathring{v}_L^M, \\ (hv)_R^M &= h_R^M \cdot \mathring{v}_R^M, & (hc)_L^M &= h_L^M \cdot \mathring{c}_L^M, & (hc)_R^M &= h_R^M \cdot \mathring{c}_R^M. \end{aligned} \quad (2.17)$$

2.4.5 Well-balanced treatment of the bed-slope source term

Hou et al.[74] developed a novel slope source term treatment to transform the slope source of a cell into a flux form which can preserve the well-balanced property. However, it could be found that non-uniform sediment concentration over irregular topography in quiescent water may lead to the perturbation of the "lake at rest" status due to the effect of the last two terms added in the momentum equations (2.2) and (2.3). To avoid this problem, we use the criterion for (7.33) (i.e. $\theta \geq \theta_c$) as activation condition for those two terms in order to preserve the well-balanced property. The justification of the well-balanced scheme especially with the sediment is described as follows.

The "lake at rest" status is represented by $\eta = \text{constant}$, $u = 0$, $v = 0$ everywhere. It can be easily shown that, under the "lake at rest" condition, $\mathbf{r}_i$ and $\mathbf{r}_j$ in (2.14) will be zero. Under this quiescent water condition, one can get $h_L^M = h_R^M$ by applying the reconstruction (2.16), $(hu)_L^M = (hu)_R^M = 0$, $(hv)_L^M = (hv)_R^M = 0$, $S_f^x = S_f^y = 0$, $q_b = 0$ and $E = 0$ for k th interface of cell i . Accordingly, the integral of slope source $S_b(\mathbf{U})$ over cell i can be rewritten as:

$$\int_{A_i} S_b(\mathbf{U}) dA = \oint_{\Gamma} F_{nb}(\mathbf{U}) d\Gamma = \frac{1}{A_i} \sum_{k=1}^3 F_{nb}(\mathbf{U})_{ik} l_{ik} \quad (2.18)$$

According to [74], the slope source terms can be formulated as fluxes by assuming linear variations of bed elevation and water level within a cell, the vector of source term $F_{nb}(\mathbf{U})_{ik}$ at the considered faces k in cell i becomes

$$F_{nb}(\mathbf{U})_{ik} = \begin{bmatrix} 0 \\ -n_x g(h_L^M + h_i)(Z^M - Z_i)/2 \\ -n_y g(h_L^M + h_i)(Z^M - Z_i)/2 \\ 0 \\ 0 \end{bmatrix} \quad (2.19)$$

In this case, equation (2.11) can be rearranged into

$$\frac{\partial \mathbf{U}_i}{\partial t} = -\frac{1}{A_i} \sum_{k=1}^3 [\mathbf{F}_{ik} - F_{nb}(\mathbf{U})_{ik}] l_{ik} + \mathbf{S}_s \quad (2.20)$$

where $\mathbf{S}_s$ are other non-zero terms under "lake at rest" condition with sediment and can be formulated as:

$$\mathbf{S}_s = \left[\frac{-D}{1-p}, 0, 0, -D, 0 \right]^T \quad (2.21)$$

Applying central-upwind scheme under "lake at rest" condition, the numerical flux through cell interface becomes

$$\mathbf{F}_{ik} = [0, n_x g (h_L^{M2}/2), n_y g (h_L^{M2}/2), 0, 0]^T \quad (2.22)$$

Since $Z^M - Z_i = h_i - h_L^M$ under quiescent steady state, the slope flux vector $F_{nb}(\mathbf{U})_{ik}$ at the same interface is

$$F_{nb}(\mathbf{U})_{ik} = [0, n_x g (h_L^{M2} - h_i^2)/2, n_y g (h_L^{M2} - h_i^2)/2, 0, 0]^T \quad (2.23)$$

Substituting the flux vector (2.22) and (2.23) into equation (2.20), one can get

$$\frac{\partial \mathbf{U}_i}{\partial t} = -\frac{1}{A_i} \sum_{k=1}^3 \begin{bmatrix} 0 \\ n_x g (h_i^2/2) \\ n_y g (h_i^2/2) \\ 0 \\ 0 \end{bmatrix} l_{ik} + \mathbf{S}_s = -\frac{1}{A_i} \frac{g h_i^2}{2} \sum_{k=1}^3 \begin{bmatrix} 0 \\ n_x l_{ik} \\ n_y l_{ik} \\ 0 \\ 0 \end{bmatrix} + \mathbf{S}_s \quad (2.24)$$

it is easy to prove that

$$\sum_{k=1}^3 n_x l_{ik} = \sum_{k=1}^3 n_y l_{ik} = 0 \quad (2.25)$$

thus equation (2.24) becomes

$$\frac{\partial \mathbf{U}_i}{\partial t} = \mathbf{S}_s = \begin{bmatrix} -D/(1-p) \\ 0 \\ 0 \\ -D \\ 0 \end{bmatrix} \quad (2.26)$$

Under "lake at rest" condition, $\frac{\partial \psi}{\partial t} = 0$ and $\frac{\partial h c}{\partial t} = -D$ which lead to $\frac{\partial Z}{\partial t} = D/(1-p) = -\frac{\partial h}{\partial t}$. Thus, the time derivative of water level η is calculated as:

$$\frac{\partial \eta}{\partial t} = \frac{\partial h}{\partial t} + \frac{\partial Z}{\partial t} = 0 \quad (2.27)$$

Equations (2.26) and (2.27) represent the satisfaction of the well-balanced property under "lake at rest" condition with sediment.

2.4.6 Friction source term treatment

When considering a case involving very shallow water or wet-dry interfaces, since the water depth is present in the denominator of the friction term, it actually dominates the stability of a numerical scheme. When the water depth becomes very small, the Manning

formulation of friction may lead to an exaggerated force that can even reverse the flow direction, which is physically incorrect [104].

Therefore, a splitting point-implicit method derived by Liang and Marche [104] is adopted to evaluate the friction source terms on unstructured grids in the current study.

2.4.7 Time integration

In order to obtain the second-order accuracy in time and retain the stability of current model, the two-stage explicit TVD Runge-Kutta method is applied.

Since the numerical model is explicit, its stability can be controlled by the Courant-Friedrichs-Lewy (CFL)-type condition. For a triangular grid system, a CFL-type condition adopted in [74] is modified by adding a small number ($\varepsilon_p = 1 \times 10^{-8}m$) to avoid a denominator equal to 0 under a dry bed condition to applied to estimate an appropriate time step in this paper. It can be expressed as

$$\Delta t = C_r \cdot \min \left(\frac{R_{ik}}{\sqrt{u_i^2 + v_i^2} + \sqrt{gh_i} + \varepsilon_p} \right) \quad i = 1, 2, \dots, n \quad (2.28)$$

in which Δt is the time step which is automatically adjusted according to the minimum grid size, C_r is the Courant number specified in the range $0 < C_r \leq 1$, n is the total number of cells. R_i is the minimum distance from the centroid to the k th edges of the i th triangle cell.

2.4.8 Treatment of wetting and drying fronts

In the present study, a wetting and drying scheme is proposed for tracking wet-dry interfaces and stably predict the bed erosion near wet-dry area. The scheme can be summarized as:

1. A tolerance water depth ε is introduced. In present model, $\varepsilon = 1 \times 10^{-6}m$ is generally used. If the water depth in a cell i is smaller than ε or the reconstructed water depths h_L^M or h_R^M is negative, first order reconstruction method will be applied instead of a high order linear reconstruction.

2. For a dry cell with $h_i^n \leq \varepsilon$, only the continuity equation (2.1) will be solved to deal with potential wetting.

3. To deal with the cell-drying, if there is a cell with $h_i^{n+1} \leq 0$, it is treated as a completely dry cell in the following time-step and all hydraulic parameters are set to zero in this cell.

4. For a dry interface, sided local wave-speeds S_L and S_R for k th edge are zero, the method description in the Numerical scheme section will be implemented to maintain numerical stability.

5. As water depth is present in the denominator of the shear stress parameter in the bed-load transport equation S_f and sediment entrainment E , a very small water depth will cause a nonphysically large erosion quantity. To maintain the stability of current model, the bed-load transport and sediment entrainment are only activated when the water depth is larger than $1 \times 10^{-3}m$.

All of the above steps are employed for each time-step to handle the wetting and drying action of flow over the erodible bed.

2.5 Numerical Tests

In this section, several numerical tests are used to test the performance of proposed scheme. The first test case deals with well balanced property and wet-dry interface in presence of sediment. It shows that the deposition of sediment will not affect the preserving of "lake at rest" status in current model, which is the well-balanced property. The second test is a real dam-break case (actual field scale) with wetting and drying, in which the efficiency and the accuracy of the current model with the central-upwind method is shown in comparison with Roe and HLLC solvers. The third test is a comparison between our numerical predictions and laboratory data, in which the improvements obtained by the added terms in the modified shallow water equations are investigated. The fourth test case is a very challenging test after adding initial dry bed condition. It shows the robustness of the proposed numerical model which can stably simulate the bed-erosion near the wet-dry fronts without oscillations. The fifth test is a benchmark test that demonstrates the dam-break flow over a dry and erodible bed, in which good agreements between simulated results and measured data are reached.

2.5.1 Quiescent water around a hump with sediment deposition

This test case is designed to demonstrate the well-balanced property of the proposed method in the case of non-zero sediment concentration which the focus of the present paper. Another purpose of this test case is to evaluate the performance of the proposed scheme in preserving a stationary boundary between wet and dry zones. To this end, initial suspended load is added to a quiescent lake around an island. To conduct this test, the elevation of the island is defined as

$$Z(x, y) = \max \{0m, 2m - 0.32m^{-1}[(x - 4m)^2 + (y - 4m)^2]\} \quad (2.29)$$

The island is located at the centre of a $8m \times 8m$ computational domain, the height of the hump is $2m$. A quiescent lake around the hump is defined with an initial water surface elevation of $1m$, so that the test can involve the wet-dry interfaces. The initial sediment concentration in the water is set to 0.3, the mean diameter of the sediment is set to $5mm$ and the porosity of the saturated bed is 0.4. The domain is divided into 4362 triangular cells. The model runs 50s with a time step $\Delta t = 0.1s$ so that all the sediment can be

deposited on the bed. Figure 2.1 shows the 3-D views of the simulated bed profile and still water surface at $t = 0s$ and $t = 50s$, respectively. An undisturbed water surface is observed throughout the entire simulation. The exact bed level after complete sediment

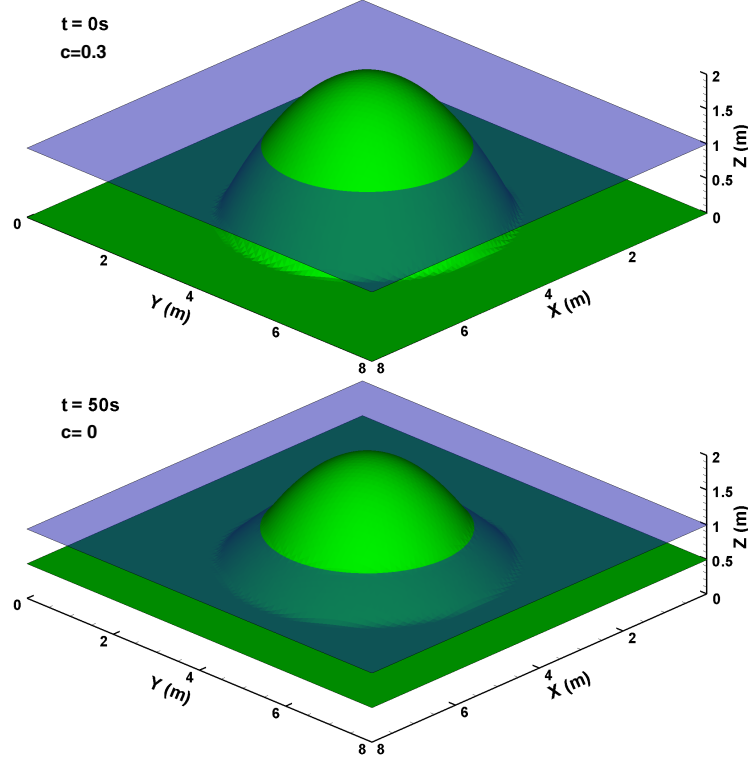


Figure 2.1: Quiescent lake around a hump with initial sediment concentration and wet-dry boundaries at $t=0s$ and $50s$.

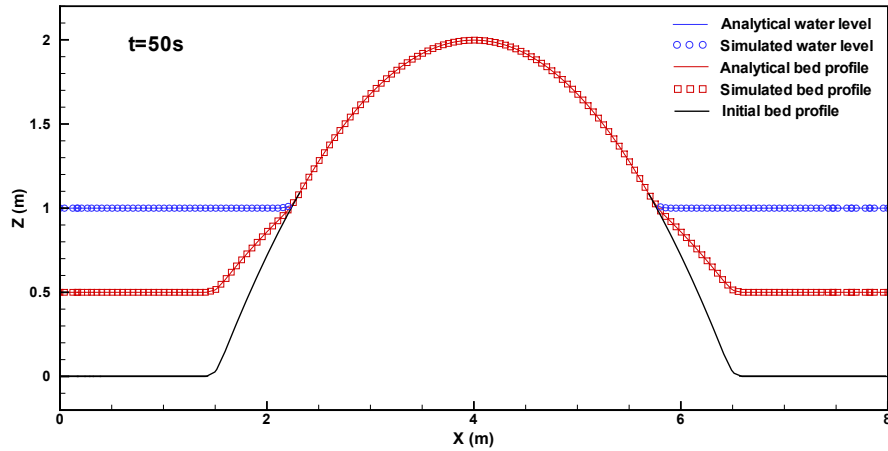


Figure 2.2: Bed profile and undisturbed quiescent water surface at $t=50s$.

deposition in cell i can be analytically estimated by $Z_i = 1 + c_i (\eta_i - Z_i) / (1 - p)$. Figure 2.2 presents both the simulated and analytical water levels and bed profiles at $t = 50s$ on cross-section $y = 4m$. As shown in Figure 2.2, the predicted water level and bed profile exactly

agree with the analytical solution. Moreover, the sediment deposition does not disturb the quiescent water surface, thus, the well-balanced property is verified.

2.5.2 1-D bed erosion with exact solution

In this test, the accuracy of the proposed model is investigated by a test of dam-break flow over erodible bed in which the bottom shear stress has been neglected in the momentum equation and only the bed-load transport is considered. The exact solution was worked out by a process of subsequent wave-field construction demonstrated in [139]. In this test, the variable Ψ is updated by

$$\frac{\partial \psi}{\partial t} = - \left(\frac{\partial q_{bx}}{\partial x} + \frac{\partial q_{by}}{\partial y} \right) \quad (2.30)$$

after the coupled system (2.1)-(2.5) is solved. The q_{bx} and q_{by} terms are given by $q_{bx} = A_g u(u^2 + v^2)$ and $q_{by} = A_g v(u^2 + v^2)$ assuming a material characterized by $A_g = 0.01$ and $p = 0.4$. The term $\frac{E-D}{1-p}$ in added parts of governing system is replaced by $\frac{\partial z}{\partial t}$ following the suggestion by Xia et al. [180] in order to consider the mass and momentum exchange between water and sediment. The initial conditions are given by $h_L = 2.25m$, $u_L = 0.2m/s$ and $z_L = 5.0m$, for $x < 0$, $h_R = 1.188686m$, $u_R = 2.4321238m/s$ and $z_R = 5.124685m$ for $x > 0$. A $30m$ long domain is discretized by cells with uniform size of $0.01m$, and the simulated time is $2s$.

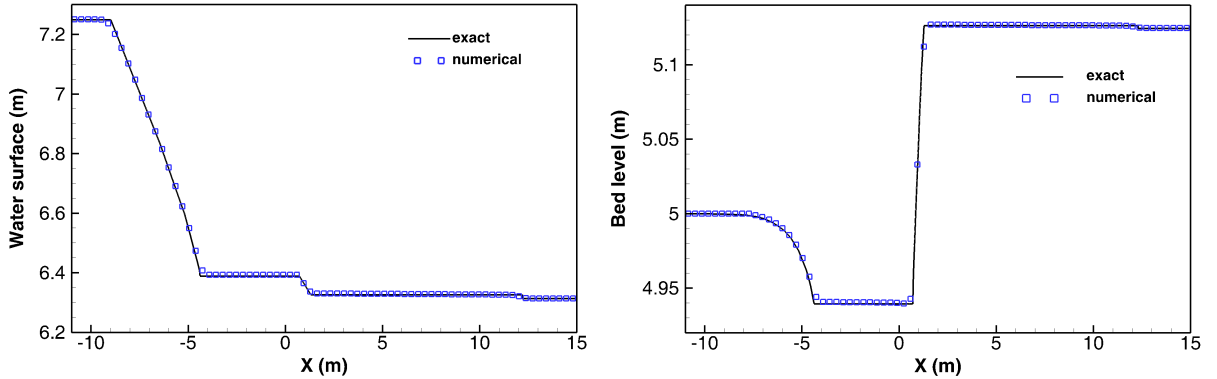


Figure 2.3: The exact solution and the simulated solution of water surface and bed level.

Figure 2.3 shows the comparisons between exact and numerical solutions for the water surface and bed level. Although some small discrepancies are found, the simulated results show very good agreement with exact solution. The results indicate the numerical solution of proposed model converges to the exact solution and high accuracy is achieved. As observed in Figure 2.3, the positions of the shock wave and rarefaction wave are accurately calculated.

2.5.3 The field dam-break case of Malpasset dam

This field test with a large number of grid points and long simulation time is used to evaluate the efficiency and accuracy of the central-upwind method compared with popular solvers, e.g., the Roe method and the HLLC scheme. The stable and accurate results of this test which involves large hydraulic gradient and initial dry bed condition show the capacity of developed model.

The Malpasset dam-break happened in 1959 due to the intense rainfall. After the dam failure, the maximum levels reached by the water at 17 different survey points along both sides of the Reyran River Valley were recorded by the French police. The shutdown times of three electric transformers were recorded as the wave arrival times at corresponding locations. In addition, a non-distorted 1:400 scale model was built in laboratory in 1964 [64]. The maximum water levels and the arrival times of front wave were measured by 9 gauges along the valley. The locations of electrical transformers, survey points and experimental gauges are indicated in Figure 2.4. This case has been used as a benchmark test to investigate the model capacity for dam-break flow by many researchers [184, 64, 165, 59, 144].

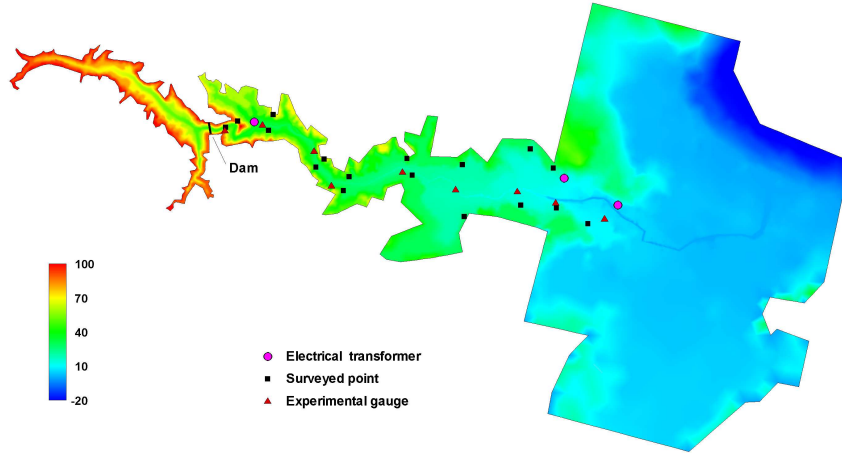


Figure 2.4: Initial topography of Reyran Valley and measuring points

The topography of the valley is shown in Figure 2.4. It ranges from -20m below the sea level to 100m above the sea level. Figure 2.5 shows the computational grids generated in [64], which consists of 13,541 nodes and 26,000 triangular cells. Figure 2.5 also shows close-up view of grid points around the dam. As suggested by Goutal [64], the real arch dam is approximated as a straight line in the current simulation between the points of coordinates $(4701.18\text{m}, 4143.41\text{m})$ and $(4655.50\text{m}, 4392.10\text{m})$, and the remnants of the dam after failure are not taken into account. The dam is assumed to be suddenly removed at $t=0\text{s}$ as also suggested by Goutal [64]. The water level in the reservoir is set to be 100m , other part of the topography except the sea is assumed initially dry. All the boundaries are treated as solid walls. As suggested in [184, 64, 165, 59, 144], the Manning's coefficient is presumed to be $0.033 \text{ s/m}^{1/3}$ for the field case and $0.023 \text{ s/m}^{1/3}$ for the laboratory case.

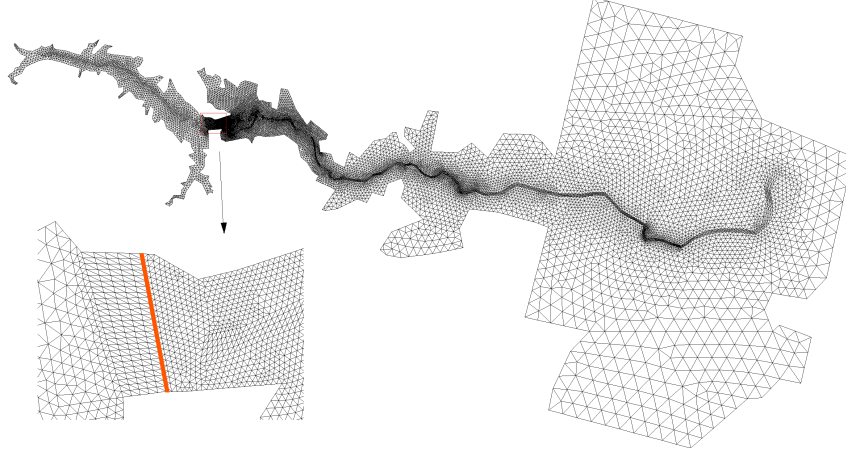


Figure 2.5: Simulating grids and a close view of the grids around dam

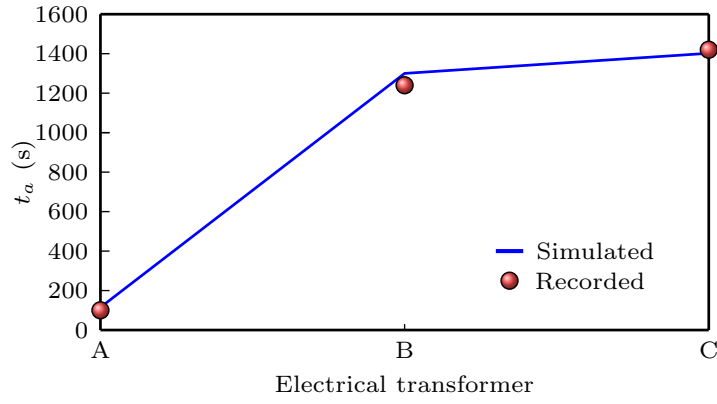


Figure 2.6: Recorded and simulated wave arrived time at electrical transformers

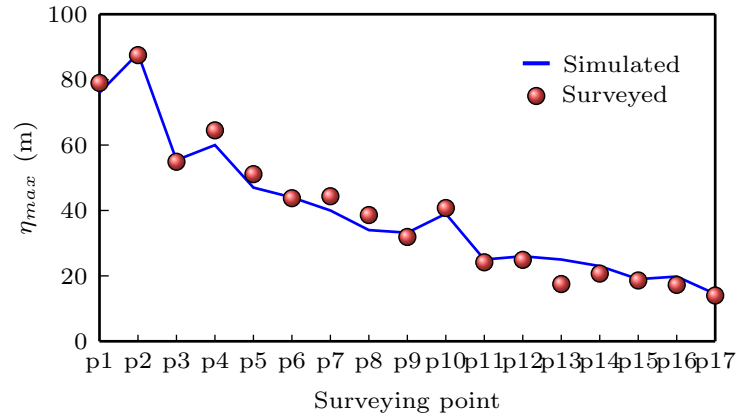


Figure 2.7: Surveyed and simulated maximum water level at survey points

Figure 2.6 shows the arrival times of the front wave at different electrical transformers. Good agreement is observed in the figure between the simulated and measured arrival times. The simulated wave arrival at transformer B is somewhat late compared with the

recorded value. Figure 2.7 presents the comparison between the simulated and measured maximum water levels at different survey points. The simulated level generally agrees well with the measured values. Discrepancies are observed at some points probably due to the topography erosion during the flood and other uncertain events in surveying.

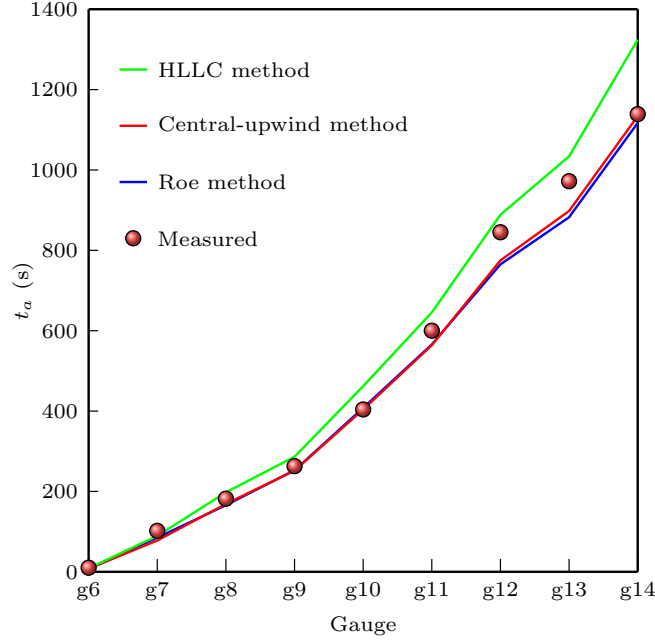


Figure 2.8: Measured and simulated wave arrived time at experimental gauges

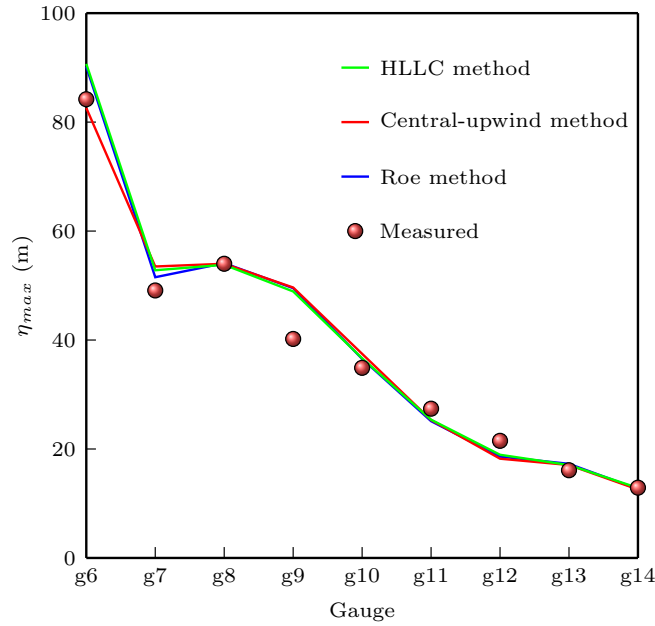


Figure 2.9: Measured and simulated maximum water level at experimental gauges

Figure 2.8 compares the simulated and measured wave arrival times and Figure 2.9 presents a comparison of the simulated and measured values of the maximum water levels, at different gauges in the laboratory-scale model. The proposed method is compared with two widely used approximate Riemann solvers, and their numerical results are presented. An overall good agreement is observed between the simulated and the measured values in Figure 2.9. Table 2.1 shows the comparison of average absolute errors and relative run time among the three different solvers. The average absolute errors are calculated by $[\sum_1^n |error|]/n$, where n is the total number of the gauges. It can be seen that the central-upwind method requires the minimum computational cost and yields better accuracy compared with Roe method and HLLC method in this test.

Table 2.1: Average absolute errors of arrived time and maximum water level, and relative run time of different solvers for laboratory-scale model

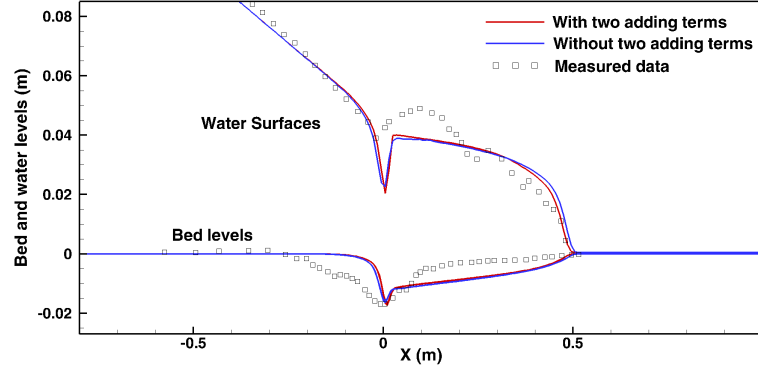
Solver used in the model	Average absolute error of arrived time	Average absolute error of maximum water level	Relative run time
Central-upwind	26.2 s	2.74 m	1.00
Roe	30.7 s	2.91 m	1.21
HLLC	49.8 s	2.94 m	1.19

2.5.4 Dam break flow over movable bed

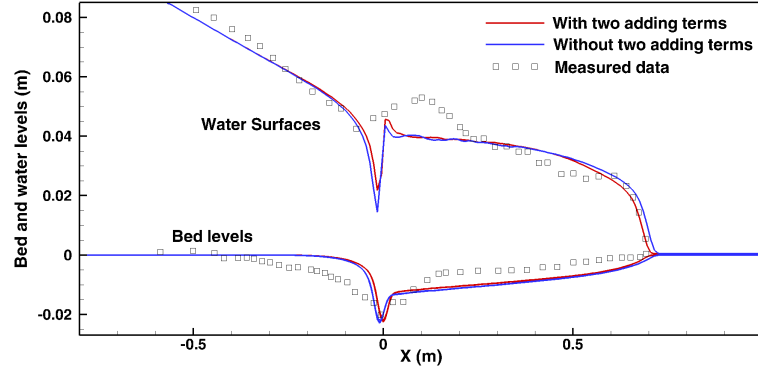
In this test, an experiment on dam-break flow over erodible bed reported by Fraccarollo and Capart [56] is used to demonstrate how the two added terms in the momentum equations affect the results. Experimental setup concerns small-scale laboratory dam-break waves of initial water depth $h_0 = 10$ cm released over erodible bed in channels. Sediment is modeled by cylindrical PVC pellets which have an equivalent spherical diameter of 3.5 mm; density = 1540 kg/m^3 , fall velocity $\omega \approx 18 \text{ cm/s}$. The testing reach was of length=2.5 m with an initial flat bed. The Manning roughness coefficient is set to $0.025 \text{ s/m}^{1/3}$. At the beginning, 10 cm initial water depth is raised upstream. The dam-break wave is then released by rapidly lifting the sluice gate.

In the numerical simulation, the domain is discretized by 7269 triangular grids. An initial dry bed is set in the downstream of the dam. The total running time is $10t_0$, in which the hydrodynamic time scale $t_0 = \sqrt{h_0/g} \approx 0.101 \text{ s}$.

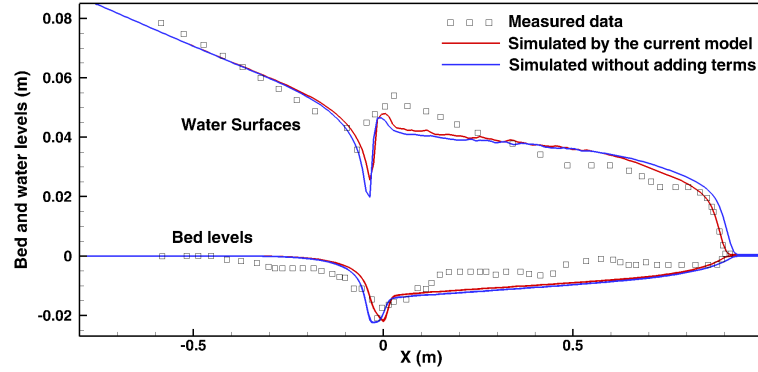
Figure 2.10 shows good agreement for water surface and bed profile between simulated and measured results at different times, which are very close to the results reported in the study by [178]. It can be observed that the current model with modified shallow water equations shows better agreement with measured data compared with the classical shallow water equations without added terms related to sediment. It can be observed that these terms clearly retard the wave propagation due to the fact that the intense sediment entrainment at front waves will "absorb" momentum from the fluid. The existence of



(a) $t=5t_0$



(b) $t=7.5t_0$



(c) $t=10t_0$

Figure 2.10: Simulated and measured water levels and bed profiles at different times

these terms improve the model because of considering the effect of stream-wise variable concentration and momentum transfer between sediment and flow.

2.5.5 2-D Dam break flow on initially dry and erodible bed

This test case was first introduced by Fennema and Chaudhry [53] in a numerical method study, which has since been used in other studies [5, 6]. The original case is mostly applied over an initially wet and fixed bed. It is modified in the present study for the purpose of testing the capacity of current model to simulate the front wave propagation over an initial dry bed with wet-dry interface tracking, with a particular attention to the 2-D aspects of the flow motion; and the model performance in prediction of fast erosion under highly energetic flow in large scale. The modeling domain area is a $200\text{m} \times 200\text{m}$ basin over a flat and dry bed, which is discretized by 9866 triangular cells. A 10m-thick dam splits the basin into two equal-sized regions. The clear water depth are 10m and 0m on the left and right sides of the dam, respectively. At $t = 0\text{s}$, a 75m wide breach centered at $y = 125\text{m}$ is assumed to form instantaneously. The duration of the simulation is 14s. The initial velocity of the whole modeling area is 0m/s and the outlet boundary at $x = 200\text{m}$ is specified with a free out-flow boundary condition, meanwhile all other boundaries are set to be standard wall condition.

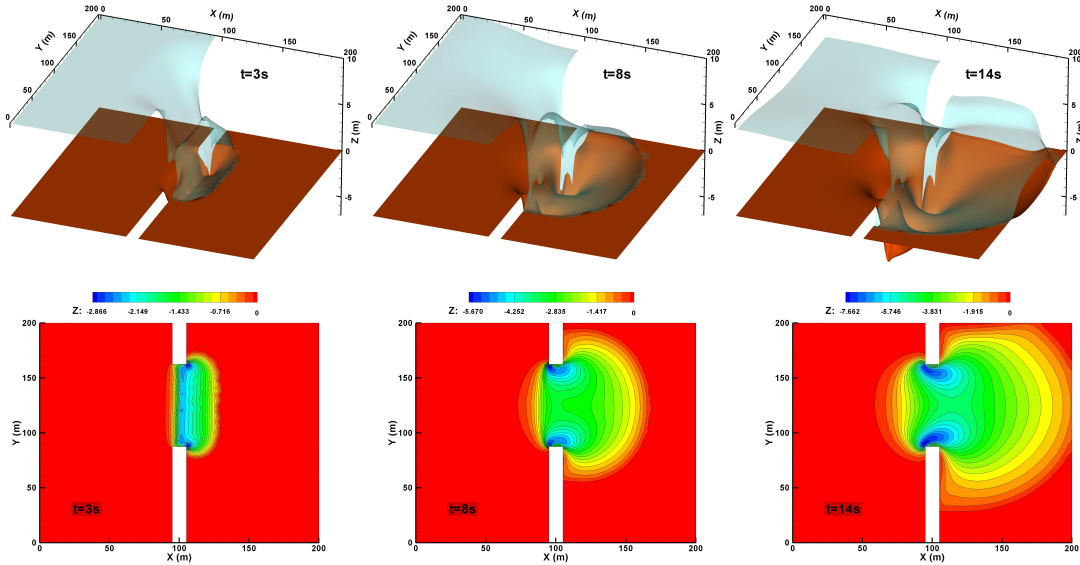


Figure 2.11: Wave front propagation of dam break flow over erodible bed (first row) and contours of erosion (second row) at different times

Figure 2.11 shows 3-D views of the water front-wave propagation and predicted bed erosion at different times (first row). It can be seen that a shock wave forms and propagates downstream and a negative wave travels upstream during the simulated period. No non-physically high velocities and oscillations are generated at the front-wave locations where wetting and drying interfaces exist. An intense bed evolution process is observed due to the

dam break flow. The second row of [Figure 2.11](#) presents the contours of erosion magnitude at different times. The proposed methods exhibits a very stable behavior even with 10 m initial surface gradient at the initial location of the dam. No oscillation is observed during the process and no nonphysically high velocity is produced at the interface of wet-dry front attributed to the proposed treatment in current model.

2.5.6 2-D Dam break flow over mobile bed with sudden enlargement

This test of a 2-D dam-break flow over initially dry bed with sudden enlargement is based on the experiment carried out in the Hydraulic Laboratory of the Université catholique de Louvain (Belgium) which is reported by Palumbo et al. [\[126\]](#). To simulate this test, only bed-load sediment transport is considered. Thus, after the coupled system is solved, the variable ψ is then updated by equation (2.30), in which, the Meyer-Peter and Müller formulas, originally developed in [\[116\]](#), is employed

$$q_{bx} = 8\sqrt{sgd^3} \cdot (\theta - \theta_c)^{1.5} \frac{u}{\sqrt{u^2 + v^2}}; \quad q_{by} = 8\sqrt{sgd^3} \cdot (\theta - \theta_c)^{1.5} \frac{v}{\sqrt{u^2 + v^2}} \quad (2.31)$$

where ξ is the included angle between θ and x-direction.

The experiment was performed in a 6 m long flume, presenting a non-symmetrical sudden widening from 0.25 m to 0.5 m width, located 1 m downstream of the gate which is located at $x=3$ m. The channel width is 0.25m before $x=4$ m and 0.5m after $x=4$ m (as shown in [Figure 2.12](#)). The flow depth are recorded at P1(3.75 m,0.125 m), P2(4.2m,0.125m), P3(4.45m,0.125m), P4(4.95 m,0.125 m), P5(4.2m,0.375m), P6(4.95m,0.375m), respectively. In this test, two locations with measured cross-section profiles of topography are chosen to verify the simulation, which are at $x=4.1$ m and $x=4.4$ m, respectively. The configuration of the test is shown in [Figure 2.12](#).

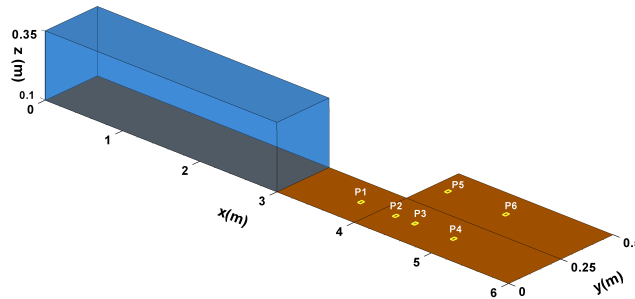


Figure 2.12: Sketch of experiential setup of dam break flow over erodible bed at $t=0$ s

The initial clear-water depth is 0.25m upstream of the gate. The initial conditions consist of a 0.1 m high horizontal layer of fully saturated and compacted sand over the whole flume. The sediment used is a uniform coarse sand with a median diameter $d_{50}=1.72$ mm, and relative density of 2.63, deposited with a volume fraction of 0.3922 [\[13\]](#).

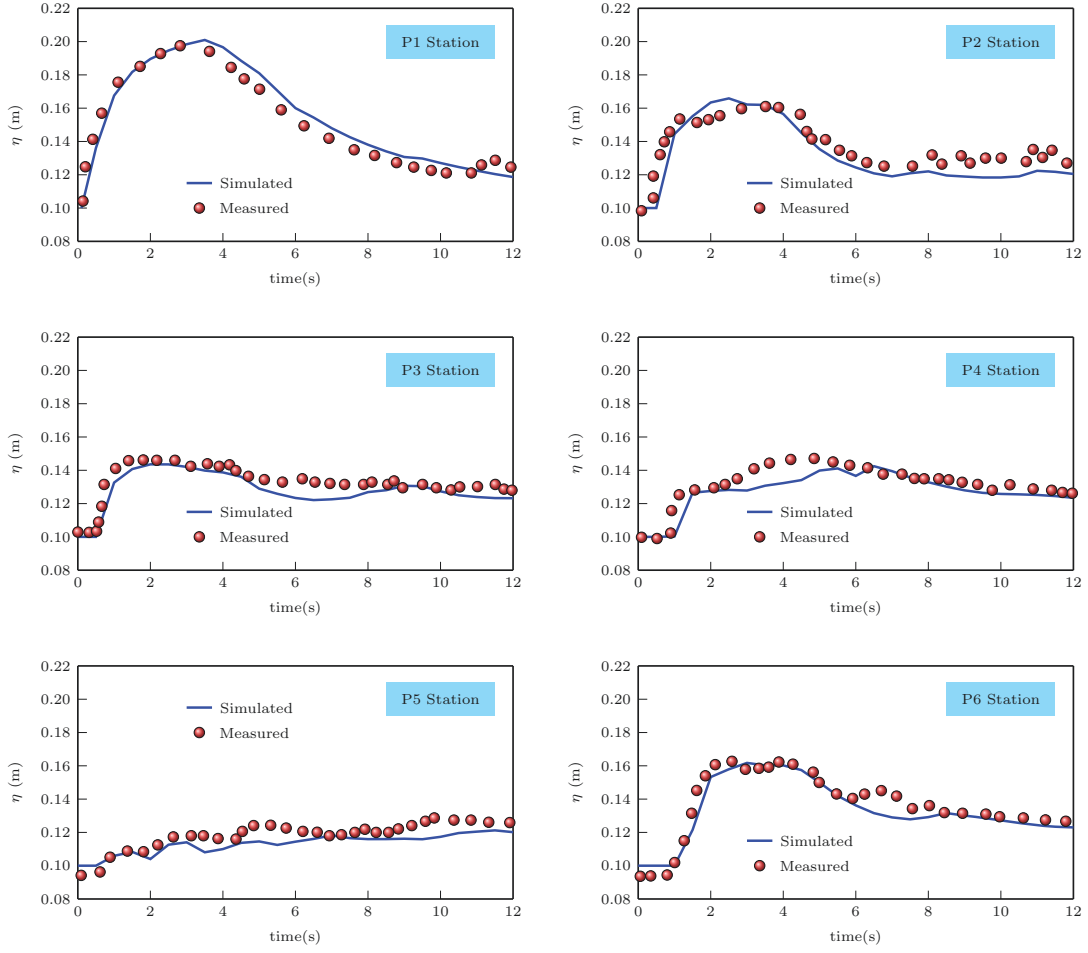


Figure 2.13: Comparison between measured and simulated water level evolution at different stations

The domain is discretized into 9168 triangular cells which are refined at the section of enlargement. A standard closed boundary condition is applied to all the side-walls, and a free-outlet condition is applied at the downstream outlet. The value of the Manning coefficient is set to be $0.018 \text{ s/m}^{1/3}$ in our test which is smaller than other researchers' reports [180, 184] due to the different choice of bed-evolution formula.

Figure 2.13 shows a comparison between predicted water level and experimentally measured values at the 6 aforementioned monitoring stations over the test period. The simulated results show very good agreement with the measured water surface at all 6 selected monitoring stations, from which it may be concluded that the proposed model shows a desirable accuracy and capacity of prediction of dam-break flow when it is affected by morphological evolution.

Figure 2.14 shows the comparisons between the simulated and measured bed level profiles at two measured cross-sections at $x=4.1\text{m}$ and $x=4.4\text{m}$, respectively. At cross-sections $x=4.1\text{m}$, the simulated deepest scouring elevation agrees well with the measured

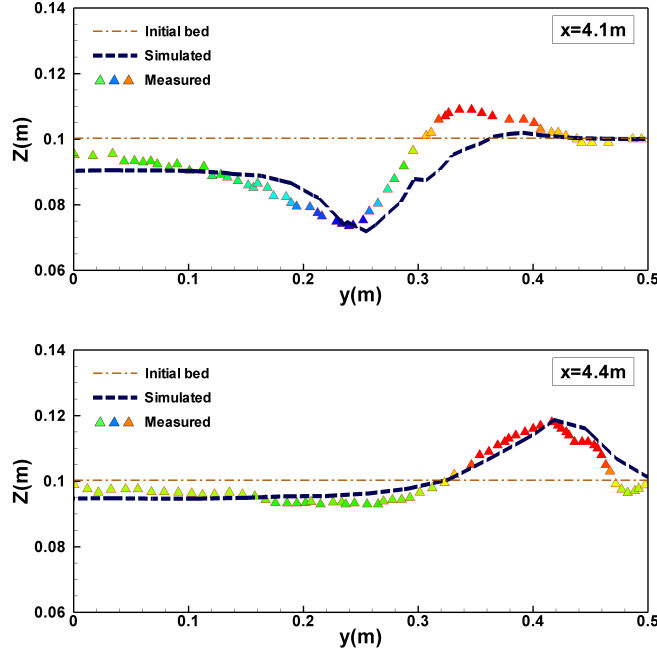


Figure 2.14: Simulated and measured bed profile at different cross sections

value. The simulation did not show the same deposition volume near the upper wall as compared with the measured data, the simulated position of the scouring hole is slightly shifted to the right along the cross-section comparing with experimental data. At cross-sections at $x=4.4\text{m}$, the lateral bed profile is predicted correctly, the simulated position and peak value for deposition near side-wall shows great agreement with the measured data. The results show that the proposed model has a very good capacity and accuracy to predict the rapid morphological evolution due to dam-break flow.

2.6 Concluding remarks

A robust 2-D model for predicting shallow water flows over erodible bed using the central-upwind method with wetting and drying on an unstructured mesh is successfully developed in the current study. While maintaining second order spatial accuracy, the method ensures non-negative water depth. Moreover, the well-balanced discretization of the bed slope term is a significant property that is offered by the proposed model. Furthermore, the splitting approach allows for point-implicit treatment of the friction term, which ensures desirable stability properties of the scheme particularly close to wet-dry fronts that are typically very challenging even without the movable bed.

The Godunov-type central-upwind methods do not require characteristic decompositions, and they are not tied up to the specific eigenstructure. This leads to significantly less computational cost compared with other popular Riemann solvers, i.e., Roe method and HLLC method. This superiority makes the proposed scheme much more efficient for

the coupled modeling of hydrodynamics, sediment transport and bed-evolution. For the central-upwind method, the "upwind" nature makes it capable to precisely simulate sharp gradients and even discontinuities and the "central" nature makes it enjoy the advantages of simplicity and high-resolution. In the comparisons in the Field dam-break case of Malpasset dam, the central-upwind method shows good accuracy as Roe method and HLLC method with less computational cost. The developed wetting-drying treatment method is verified to be able to stably track the wet-dry fronts where erosion also occurs. No oscillations or instabilities are observed. The proposed method successfully prevents the instability caused by the presence of small values of h in the denominator of some erosion-related terms.

It should be pointed out that the treatment of adding a threshold as an activating condition may cause a slight loss of accuracy of erosion in the very shallow water areas ($h < 1 \times 10^{-3}m$). However, since the value of the threshold is very small, these very shallow areas will be filled or completely drained very quickly under dam-break flows. Thus, the loss of erosion volume in these very shallow areas can be neglected considering the very short time of the duration of wetting and drying.

The effects of the adding components related to sediment introduced by Cao et al. [29], are investigated in the present study and show improvements in simulations. It can be seen that in the case of erosion, in which $E > D$, those terms will decelerate the wave propagations. The modified shallow water equations proposed by [29] consider the fluid as a sediment-water mixture instead of clear water. Thus a more complete conservation law is built. This is very important for cases like dam-break flow over erodible bed in which very intense transport and hyper concentration exist. Therefore, the added components improve the performance of the proposed model.

To summarize, the main contributions of the present study are the following:

(1) The subject of tracking the wetting and drying in presence of sediment transport and bed erosion is challenging and rarely reported. The difficulty is basically due to the existence of very small water depth in the denominator of some erosion-related terms. This can cause numerical oscillations and instabilities in the regions subjected to wetting and drying. In this paper, we proposed a wetting and drying scheme; while being highly stable, the proposed scheme can efficiently track wetting and drying interfaces and stably simulate the bed erosion near the wet-dry fronts.

(2) Any numerical model that deals with bed-evolutions, requires integration of many numerical components, and if the employed numerical methods are not efficient, the resulting model can be computationally too expensive. In order to avoid this problem, in this paper, the central-upwind scheme is employed, which is proven to be highly efficient as it involves less computations. This is the first time (up to the best of our knowledge) that the central-upwind scheme is adapted in the study of bed erosion and sediment transport in conjunction with shallow water flows. The numerical tests in this paper show that the central-upwind method can precisely predict the bed erosion and shallow water flows. They also demonstrate that this scheme leads to good accuracy comparable to Roe and HLLC methods with significantly less computational cost, as demonstrated in the the Field dam-

break case of Malpasset dam test. These superiorities make the central-upwind method an ideal choice for a robust model.

(3) This paper also investigated and discussed how the added terms related to sediment in the shallow water equations improve the model. They lead to very interesting features as described in the following. The added components related to sediment in the momentum equation account for the inertia effect of sediment onto the flow, which are important especially when the hyper concentration and intense sediment transport exist. These components can also decelerate the wave propagation due to the inertia of sediment, as demonstrated in the Dam break flow over movable bed test. It can also be understood as a fact that the sediment "absorbs" the momentum from the water. The modified SW equations with new components in this paper model the system as a single fluid which is a water-sediment mixture instead of clear water system only (classical SW equations), thus a more complete conservation law can be built. As a consequence, a better agreement with measured data was achieved by adopting the modified SW equations, as demonstrated in the Dam break flow over movable bed test.

(4) To verify the robustness of the current model, we increased the difficulties of several challenging benchmark-tests in this study to include the sediment or dry-bed condition. For example, we have added initial sediment concentration and wet-dry interfaces in the Quiescent water around a hump with sediment deposition test. Moreover, the 2-D Dam break flow on initially dry and erodible bed test is already a very challenging test originally designed on an initially wet bed; we made it even more difficult by adding the initial dry bed condition and bed erosion. We have also included a field test case (Malpasset dam-break), to emphasize on the applicability of the scheme in engineering application, not only on academic benchmark tests.

The above-mentioned features makes the proposed scheme an attractive tool for numerous practical applications related to sediment transport and bed evolution in river and coastal engineering problems. The extension of the proposed scheme to three-dimensional hydrostatic as well as non-hydrostatic cases is currently in progress by the authors, which are the subjects of subsequent papers.

Chapter 3

Well-balanced central-upwind scheme for a fully coupled shallow water system modeling flows over erodible bed

3.1 Abstract

Intense sediment transport and rapid bed evolution are frequently observed under highly-energetic flows, and bed erosion sometimes is of the same magnitude as the flow itself. Simultaneous simulation of multiple physical processes requires a fully coupled system to achieve an accurate hydraulic and morphodynamical prediction. In this paper, we develop a high-order well-balanced finite-volume method for a new fully coupled two-dimensional hyperbolic system consisting of the shallow water equations with friction terms coupled with the equations modeling the sediment transport and bed evolution.

The nonequilibrium sediment transport equation is used to predict the sediment concentration variation. Since bed-load, sediment entrainment and deposition have significant effects on the bed evolution, an Exner-based equation is adopted together with the Grass bed-load formula and sediment entrainment and deposition models to calculate the morphological process. The resulting 5×5 hyperbolic system of balance laws is numerically solved using a Godunov-type central-upwind scheme on a triangular grid. A computationally expensive process of finding all of the eigenvalues of the Jacobian matrices is avoided: The upper/lower bounds on the largest/smallest local speeds of propagation are estimated using the Lagrange theorem. A special discretization of the bed-slope term is proposed to guarantee the well-balanced property of the designed scheme. The proposed fully coupled model is verified on a number of numerical experiments.

Keywords: Fully coupled shallow water system, sediment transport, dam-break flow, finite-volume method, well-balanced central-upwind scheme, Lagrange theorem.

3.2 Introduction

Shallow water systems are widely used in predicting hydrodynamics of surface flows such as water flows in rivers, channels, flood plains and coastal regions. It is well-known that the shallow water systems can accurately predict the hydraulic parameters under conditions of slow erosion and low sediment concentration. However, in cases of highly energetic flows like dam-break or flood flows, the effects of intense sediment exchange and rapid bed evolution cannot be neglected.

In recent years, a number of dam-break hydraulics models have been developed to predict the fluid flow and sediment transport [102, 148, 162, 165, 192, 193]. However, the majority of the current models are based on the decoupled equations of flow and sediment transport and thus they are restricted to erosion rates which are considerably weaker than the flow, see, e.g., [102, 187]. However, dam-break flows may induce intense sediment transport and rapid morphological changes which are not independent processes but have mutually significant interactions with the fluid flow. In many dam-break cases and in strong fluvial processes in rivers, the bed evolution is almost of the same order of magnitude as the free-surface level changes [186]. Hence, developing a dam-break model that takes into account coupling hydraulic model with the sediment transport and bed evolution equations is required in such cases.

Many dam-break models based on fixed-bed conditions have been developed, see, e.g., [16, 21, 23, 148, 165, 192, 193]. In order to evaluate morphological changes during the high-energy dam-break flows, some earlier numerical models use uncoupled strategy to first solve the hydrodynamic model, and then to solve the sediment transport and bed deformation equations separately, see, e.g., [54, 55, 131, 162]. More recently, noting that the flow, sediment transport and bed evolution can be strongly coupled to each other while the rate of bed deformation being considerable compared to that of the flow evolution, several coupled mathematical modeling strategies for simulating dam-break flows over mobile beds have been developed. Fagherazzi and Sun [51] presented a 1-D coupled system consisting of a simplified shallow water system, sediment concentration and bed level calculation. Cao *et al.* [29] proposed a 1-D coupled model and the sediment entrainment and deposition are considered to update the bed-levels. Wu and Wang presented a similar 1-D coupled model for dam-break flows over erodible beds in [178] including the bed-load transport. Capart and Young [30] developed a one-dimensional (1-D) coupled dam-break model based on an explicit finite-difference algorithm.

Several two-dimensional (2-D) coupled models for dam-break flows over mobile beds have been recently introduced. Simpson and Castelltort [143] presented a 2-D model for the free surface flow, sediment transport and bed level change based on the 1-D model from [29]. Yue *et al.* [182] extended the 1-D model from [29] to 2-D for modeling alluvial processes with intense sediment transport and rapid bed evolution. Similar 2-D coupled mathematical models have been also studied in [77, 180]. Li and Duffy [101] proposed a fully coupled 2-D system including the modified shallow water system, sediment transport and bed evolution. In their study, the bed evolution formula is rewritten to couple the sediment entrainment and deposition effects by substituting the sediment transport equation into it. Meanwhile, the sediment exchange terms are treated as source terms which will not affect

the solution of the hyperbolic system. In [15], Benkhaldoun *et al.* presented a coupled 2-D model, which consists of both bed-load and sediment exchange in the morphological equation, and the bed-load terms are treated as a flux term coupled in the hyperbolic system. Hudson and Sweby [80] investigated the accuracy and determined the validity of both the steady approach and unsteady approach of five different types of governing formulations coupling the morphology continuity equation on rectangular meshes. They concluded that one conservative reformulation form based on the unsteady approach, was overall the best. However, the drawback of this form is its dependence of the Jacobian matrix on z , which was shown and discussed in [140]. Castro Díaz *et al.* [32] studied the numerical approximation of bed-load sediment transport due to shallow layer flows on unstructured meshes with a second-order MUSCL-type reconstruction. Murillo and Garcia-Navarro [121] studied an Exner-based coupled model for 2-D transient flows over erodible beds on triangular unstructured meshes and developed a general “Grass formula” format expressing several commonly used empirical bed-load formulas. A similar scheme is adopted in the current study using the Meyer-Peter and Müller formula. Soares-Frazão and Zech [149] built a coupled system using the HLLC method and proposed an approximate analytical expression for the wave celerities, which was valid for any Froude number.

In this paper, we propose a new fully coupled hyperbolic system consisting of hydrodynamic model, sediment transport and morphological evolution. The obtained hyperbolic system of balance laws consists of five coupled equations, for which no analytical expressions for the eigenstructure are available. This makes it difficult to develop a robust and efficient upwind scheme for the proposed model. We develop a Riemann-problem-solver-free finite volume method for the studied fully coupled system. Our method is based on the efficient, robust and highly accurate semi-discrete central-upwind scheme, originally introduced in [87, 88, 90, 93] for general multidimensional hyperbolic systems of conservation laws. The central-upwind scheme was first extended to the Saint-Venant system of shallow water equations in [86]. A more robust, well-balanced and positivity preserving central-upwind scheme for the shallow water equations was developed on both Cartesian [92] and triangular [23] grids. These schemes are capable of exactly preserving “lake at rest” steady-state solutions and preserving positivity of the computed water depth. In this paper, we extend the second-order semi-discrete central-upwind scheme to the fully coupled shallow water system modeling flows over erodible bed. Since, according to the central-upwind methodology, no (generalized) Riemann problems are to be (approximately) solved, the numerical fluxes are obtained in a straightforward manner. The one-sided local speeds needed to evaluate numerical fluxes are estimated using the Lagrange theorem [95, 117], which significantly reduces the numerical cost (the same approach was used in [42, 89] in the context of the two- and three-layer shallow water equations). We complete the development of the well-balanced (in the sense that it is capable of exactly preserving “lake at rest” states) central-upwind scheme by designing a positivity preserving piecewise linear reconstruction and a well-balanced quadrature for the bed-slope terms.

This paper is organized as follows. In §3.3, we present the governing equations, for which in §3.4 we develop the central-upwind scheme. Several numerical experiments are then presented in §3.5. Some concluding remarks complete the paper in §3.6.

3.3 Governing Equations

In this section, we will present the model equations (§3.3.1) and write down the studied model in the vector form (§3.3.2).

3.3.1 Fully Coupled Shallow Water System Modeling Flows over Erodeable Bed

We follow the approach in [29] and modify the continuity equation to account for the sediment effect. As in the 1-D model from [178], we consider both bed-load and suspended load effects in the momentum equations. The resulting 2-D modified shallow water equations read:

$$h_t + (hu)_x + (hv)_y = -Z_t, \quad (3.1)$$

$$(hu)_t + \left(hu^2 + \frac{g}{2}h^2\right)_x + (huv)_y = -gh(Z_x + S_f^x), \quad (3.2)$$

$$(hv)_t + (huv)_x + \left(hv^2 + \frac{g}{2}h^2\right)_y = -gh(Z_y + S_f^y). \quad (3.3)$$

The sediment transportation and morphological evolution are modeled based on the mass and momentum conservation of sediment and water-sediment mixture. We model the bed evolution using an Exner-based equation proposed in [67], which we modify to comprise both bed-load and suspended-load. The 2-D sediment transport equation for suspended load and the morphological evolution equation are given by

$$(hc)_t + (huc)_x + (hvc)_y = E - D \quad (3.4)$$

and

$$Z_t + \frac{\mu}{1-p} [u(u^2 + v^2)]_x + \frac{\mu}{1-p} [v(u^2 + v^2)]_y = -\frac{E - D}{1-p}, \quad (3.5)$$

respectively.

In equations (3.1)–(3.5), t is time, x and y are horizontal coordinates. The unknown functions are the water depth $h = h(x, y, t)$, the x - and y -components of the depth-averaged velocity, $u = u(x, y, t)$ and $v = v(x, y, t)$, respectively, the volumetric sediment concentration $c = c(x, y, t)$, and the bed elevation $Z = Z(x, y, t)$. The following quantities are parameters of the model: g is the gravitational acceleration; p is the bed porosity; μ is a coefficient usually obtained experimentally by taking into account the grain diameter and the kinematic viscosity of the sediment mixture; E and D are sediment entrainment and deposition fluxes across the bottom boundary of flow, which represent the sediment exchange between the bed and water column; S_f^x and S_f^y are friction slope terms in the x - and y -directions, respectively, which are computed using Manning's roughness coefficient n_b as follows: $S_f^x = n_b^2 u \sqrt{u^2 + v^2} / h^{4/3}$, $S_f^y = n_b^2 v \sqrt{u^2 + v^2} / h^{4/3}$. In all of our numerical experiments, we set $n_b = 0.02$.

Bed sediment entrainment due to turbulence, E , and sediment deposition due to gravity, D , are two distinct mechanisms involved in the sediment exchange and bed evolution processes. And they are considered for source of suspended sediment in the current study. Several empirical formulations derived on various premises were reviewed in [27, 29]. We adopted the formulas used in [29], where the coefficients in E and D were determined based on non-cohesive sediment assumptions. First,

$$D := \omega_0 (1 - C_a)^m C_a, \quad (3.6)$$

where ω_0 is the settling velocity of a single particle in tranquil water, which can be calculated as:

$$\omega_0 = \sqrt{(13.95\nu/d)^2 + 1.09sgd} - 13.95\nu/d. \quad (3.7)$$

In formulas (3.6), (3.7), ν is the kinematic viscosity of water ($\nu = 1.2 \times 10^{-6}$), d is the average diameter of sediment particles, s is the submerged specific gravity of sediment given by $s = \frac{\rho_s}{\rho_f} - 1$, where ρ_s is the density of sediment particles and ρ_f is the density of clear water; m is an exponent indicating the effects of hindered settling due to high sediment concentration, C_a is the local near-bed sediment concentration in volume, which can be determined by $C_a = \alpha c$, where the coefficient α is usually larger than 1 and must not exceed $(1 - p)$. α is empirically specified by $\alpha = \min[2, (1 - p)/c]$. And E is calculated as:

$$E := \begin{cases} \frac{\varphi (\theta - \theta_c) \sqrt{u^2 + v^2}}{h d^{1/5}}, & \text{if } \theta \geq \theta_c, \\ 0, & \text{otherwise.} \end{cases}$$

Here, φ is a coefficient to control the erosion force (in our numerical examples, we take $\varphi = 0.015$), θ is the dimensionless shear stress which can be calculated as $\theta = hS_f/(sd)$, and θ_c is the critical Shields parameter for initiation of sediment movement (the commonly used value is experimentally determined as $\theta_c = 0.047$).

3.3.2 Vector Form of the Fully Coupled Shallow Water System

In order to make it easier to design a well-balanced scheme for the studied fully coupled shallow water system, we introduce the water surface variable $\eta := h + Z$ and rewrite the system (3.1)–(3.5) in terms of the equilibrium variables $\mathbf{U} := (\eta, hu, hv, hc, Z)^T$ in the following vector form:

$$\mathbf{U}_t + \mathbf{F}_x + \mathbf{G}_y = \mathbf{S} + \mathbf{K}. \quad (3.8)$$

Here, $\mathbf{F}$ and $\mathbf{G}$ are:

$$\mathbf{F} = \begin{pmatrix} hu \\ \frac{(hu)^2}{\eta - Z} + \frac{g}{2}\eta^2 - g\eta Z \\ \frac{(hu)(hv)}{\eta - Z} \\ \frac{(hu)(hc)}{\eta - Z} \\ \frac{\mu}{1-p} \frac{(hu)[(hu)^2 + (hv)^2]}{(\eta - Z)^3} \end{pmatrix}, \quad \mathbf{G} = \begin{pmatrix} hv \\ \frac{(hu)(hv)}{\eta - Z} \\ \frac{(hv)^2}{\eta - Z} + \frac{g}{2}\eta^2 - g\eta Z \\ \frac{(hv)(hc)}{\eta - Z} \\ \frac{\mu}{1-p} \frac{(hv)[(hu)^2 + (hv)^2]}{(\eta - Z)^3} \end{pmatrix}, \quad (3.9)$$

and $\mathbf{S}$ and $\mathbf{K}$ are:

$$\mathbf{S} = \begin{pmatrix} 0 \\ -g\eta Z_x \\ -g\eta Z_y \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{K} = \begin{pmatrix} 0 \\ -g(\eta - Z)S_f^x \\ -g(\eta - Z)S_f^y \\ E - D \\ \frac{E - D}{1-p} \end{pmatrix}. \quad (3.10)$$

In order to define a quasi-linear hyperbolic system, the geometric source term $\mathbf{S} = P(\mathbf{U})\mathbf{U}_x + Q(\mathbf{U})\mathbf{U}_y$ is incorporated into the Jacobian matrix following the approach in [121, 140]. Notice that, for smooth solutions, the system (3.8)–(3.10) can be rewritten in the following form:

$$\mathbf{U} + [A(\mathbf{U}) - P(\mathbf{U})]\mathbf{U}_x + [B(\mathbf{U}) - Q(\mathbf{U})]\mathbf{U}_y = \mathbf{K},$$

where

$$A(\mathbf{U}) = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ -\frac{(hu)^2}{(\eta - Z)^2} + g(\eta - Z) & \frac{2hu}{\eta - Z} & 0 & 0 & \frac{(hu)^2}{(\eta - Z)^2} - g\eta \\ -\frac{(hu)(hv)}{(\eta - Z)^2} & \frac{hv}{\eta - Z} & \frac{hu}{\eta - Z} & 0 & \frac{(hu)(hv)}{(\eta - Z)^2} \\ -\frac{(hu)(hc)}{(\eta - Z)^2} & \frac{hc}{\eta - Z} & 0 & \frac{hu}{\eta - Z} & \frac{(hu)(hc)}{(\eta - Z)^2} \\ -3\frac{\mu}{1-p} \frac{(hu)[(hu)^2 + (hv)^2]}{(\eta - Z)^4} & \frac{\mu}{1-p} \frac{3(hu)^2 + (hv)^2}{(\eta - Z)^3} & \frac{\mu}{1-p} \frac{2(hu)(hv)}{(\eta - Z)^3} & 0 & 3\frac{\mu}{1-p} \frac{(hu)[(hu)^2 + (hv)^2]}{(\eta - Z)^4} \end{pmatrix}, \quad (3.11)$$

$$B(\mathbf{U}) = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ -\frac{(hu)(hv)}{(\eta - Z)^2} & \frac{hv}{\eta - Z} & \frac{hu}{\eta - Z} & 0 & \frac{(hu)(hv)}{(\eta - Z)^2} \\ -\frac{(hv)^2}{(\eta - Z)^2} + g(\eta - Z) & 0 & \frac{2hv}{\eta - Z} & 0 & \frac{(hv)^2}{(\eta - Z)^2} - g\eta \\ -\frac{(hv)(hc)}{(\eta - Z)^2} & 0 & \frac{hc}{\eta - Z} & \frac{hv}{\eta - Z} & \frac{(hv)(hc)}{(\eta - Z)^2} \\ -3\frac{\mu}{1-p} \frac{(hv)[(hu)^2 + (hv)^2]}{(\eta - Z)^4} & \frac{\mu}{1-p} \frac{2(hu)(hv)}{(\eta - Z)^3} & \frac{\mu}{1-p} \frac{3(hv)^2 + (hu)^2}{(\eta - Z)^3} & 0 & 3\frac{\mu}{1-p} \frac{(hv)[(hu)^2 + (hv)^2]}{(\eta - Z)^4} \end{pmatrix}, \quad (3.12)$$

and

$$P(\mathbf{U}) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -g\eta \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad Q(\mathbf{U}) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -g\eta \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \quad (3.13)$$

3.4 Semi-Discrete Central-Upwind Scheme

We now turn to the derivation of the semi-discrete central-upwind scheme for the system (3.8)–(3.10).

We assume that an unstructured triangulation $\mathcal{T} := \bigcup_i T_i$ of the computational domain, consisting of triangular cells T_i of size $|T_i|$, is given. We denote by $\mathbf{n}_{ik} := (\cos(\theta_{ik}), \sin(\theta_{ik}))^T$ the outer unit normals to the corresponding sides of T_i of length ℓ_{ik} , $k = 1, 2, 3$, see Figure 3.1. Let (x_i, y_i) be the coordinates of the centroid of T_i and $M_{ik} = (x_{ik}, y_{ik})$ be the midpoint of the k -th side of the triangle T_i , $k = 1, 2, 3$. We denote by T_{i1} , T_{i2} and T_{i3} the neighboring triangles that share a common side with T_i .

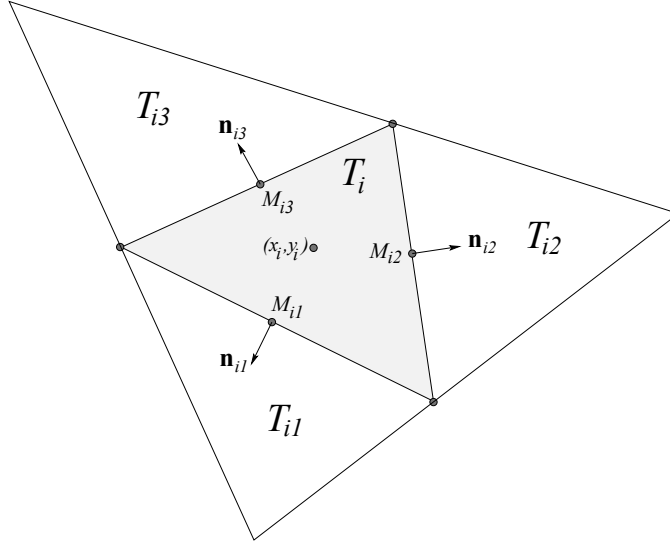


Figure 3.1: A typical triangular cell with three neighbors.

A semi-discrete scheme for (3.8)–(3.10) is a system of ODEs for the approximations of the cell averages of the solution:

$$\bar{U}_i(t) \approx \frac{1}{|T_i|} \int_{T_i} U(x, y, t) dx dy.$$

We refer the reader to [90], where a general form of a “triangular” central-upwind scheme for systems of hyperbolic conservation laws is derived. We follow [23, 90] and obtain the

semi-discrete second-order central-upwind scheme, which reads:

$$\begin{aligned}
\frac{d\bar{\mathbf{U}}_i}{dt} = & -\frac{1}{|T_i|} \sum_{k=1}^3 \frac{\ell_{ik} \cos(\theta_{ik})}{a_{ik}^{\text{in}} + a_{ik}^{\text{out}}} [a_{ik}^{\text{in}} \mathbf{F}(\mathbf{U}_{ik}(M_{ik})) + a_{ik}^{\text{out}} \mathbf{F}(\mathbf{U}_i(M_{ik}))] \\
& -\frac{1}{|T_i|} \sum_{k=1}^3 \frac{\ell_{ik} \sin(\theta_{ik})}{a_{ik}^{\text{in}} + a_{ik}^{\text{out}}} [a_{ik}^{\text{in}} \mathbf{G}(\mathbf{U}_{ik}(M_{ik})) + a_{ik}^{\text{out}} \mathbf{G}(\mathbf{U}_i(M_{ik}))] \\
& +\frac{1}{|T_i|} \sum_{k=1}^3 \ell_{ik} \frac{a_{ik}^{\text{in}} a_{ik}^{\text{out}}}{a_{ik}^{\text{in}} + a_{ik}^{\text{out}}} [\mathbf{U}_{ik}(M_{ik}) - \mathbf{U}_i(M_{ik})] + \bar{\mathbf{S}}_i + \bar{\mathbf{K}}_i.
\end{aligned} \tag{3.14}$$

Here, $\mathbf{U}_i(M_{ik})$ and $\mathbf{U}_{ik}(M_{ik})$ are the corresponding values at M_{ik} of the piecewise linear reconstruction

$$\mathring{\mathbf{U}}(x, y) := \bar{\mathbf{U}}_i + (\hat{\mathbf{U}}_x)_i(x - x_i) + (\hat{\mathbf{U}}_y)_i(y - y_i), \quad (x, y) \in T_i \tag{3.15}$$

of $\mathbf{U}$ at time t , that is:

$$\mathbf{U}_i(M_{ik}) := \lim_{(x,y) \rightarrow M_{ik}; (x,y) \in T_i} \mathring{\mathbf{U}}(x, y), \quad \mathbf{U}_{ik}(M_{ik}) := \lim_{(x,y) \rightarrow M_{ik}; (x,y) \in T_{ik}} \mathring{\mathbf{U}}(x, y), \tag{3.16}$$

where the numerical derivatives $(\hat{\mathbf{U}}_x)_i$ and $(\hat{\mathbf{U}}_y)_i$ are (at least) first-order, componentwise approximations of $\mathbf{U}_x(x_i, y_i, t)$ and $\mathbf{U}_y(x_i, y_i, t)$, respectively, computed via a nonlinear limiter used to minimize the oscillations of the reconstruction (3.15), see §3.4.1.

The directional local speeds a_{ik}^{in} and a_{ik}^{out} in (3.14) are defined by

$$\begin{aligned}
a_{ik}^{\text{in}}(M_{ik}) &= -\min\{\lambda_1[V_{ik}(\mathbf{U}_i(M_{ik}))], \lambda_1[V_{ik}(\mathbf{U}_{ik}(M_{ik}))], 0\}, \\
a_{ik}^{\text{out}}(M_{ik}) &= \max\{\lambda_5[V_{ik}(\mathbf{U}_i(M_{ik}))], \lambda_5[V_{ik}(\mathbf{U}_{ik}(M_{ik}))], 0\},
\end{aligned} \tag{3.17}$$

where $\lambda_1[V_{ik}] \leq \dots \leq \lambda_5[V_{ik}]$ are the eigenvalues of the matrices

$$\begin{aligned}
V_{ik}(\mathbf{U}_i(M_{ik})) &:= \cos(\theta_{ik}) [A(\mathbf{U}_i(M_{ik})) + P(\mathbf{U}_i(M_{ik}))] \\
&\quad + \sin(\theta_{ik}) [B(\mathbf{U}_i(M_{ik})) + Q(\mathbf{U}_i(M_{ik}))],
\end{aligned} \tag{3.18}$$

and

$$\begin{aligned}
V_{ik}(\mathbf{U}_{ik}(M_{ik})) &:= \cos(\theta_{ik}) [A(\mathbf{U}_{ik}(M_{ik})) + P(\mathbf{U}_{ik}(M_{ik}))] \\
&\quad + \sin(\theta_{ik}) [B(\mathbf{U}_{ik}(M_{ik})) + Q(\mathbf{U}_{ik}(M_{ik}))],
\end{aligned} \tag{3.19}$$

where A , B , P and Q are given by (3.11)–(3.13). Unfortunately, no analytical expressions for $\lambda_1[V_{ik}]$ and $\lambda_5[V_{ik}]$ are available. We therefore follow the approach in [42, 89] and establish a lower bound for $\lambda_1[V_{ik}]$ and an upper bound for $\lambda_5[V_{ik}]$ using the Lagrange theorem [95, 117], see §3.4.2.

Finally, $\bar{\mathbf{S}}_i$ and $\bar{\mathbf{K}}_i$ in (3.14) are the cell averages of the bed-slope term $\mathbf{S}$ and the source term $\mathbf{K}$, respectively, over the cell T_i .

3.4.1 Non-Oscillatory Piecewise Linear Reconstruction

In this section, we describe a nonlinear limiter used to evaluate the numerical derivatives $(\hat{\mathbf{U}}_x)_i$ and $(\hat{\mathbf{U}}_y)_i$ in (3.15) in a non-oscillatory manner. A variety of limiters on triangular grids can be found in, for example, [1, 23, 43, 49, 83, 90, 127, 177]. In the present study, the limiting approach proposed by Jawahar and Kamath in [83] is adopted.

First, we compute the nonlimited gradients $(\nabla \mathbf{U})_j$ in every cell T_j by taking the x - and y -derivatives of the planes passing through the points $(x_{j1}, y_{j1}, \mathbf{U}_{j1})$, $(x_{j2}, y_{j2}, \mathbf{U}_{j2})$ and $(x_{j3}, y_{j3}, \mathbf{U}_{j3})$ outlined in Figure 3.2:

$$\begin{aligned} (\mathbf{U}_x)_j &= \frac{(y_{j3} - y_{j1})(\mathbf{U}_{j2} - \mathbf{U}_{j1}) - (y_{j2} - y_{j1})(\mathbf{U}_{j3} - \mathbf{U}_{j1})}{(y_{j3} - y_{j1})(x_{j2} - x_{j1}) - (y_{j2} - y_{j1})(x_{j3} - x_{j1})}, \\ (\mathbf{U}_y)_j &= \frac{(x_{j2} - x_{j1})(\mathbf{U}_{j3} - \mathbf{U}_{j1}) - (x_{j3} - x_{j1})(\mathbf{U}_{j2} - \mathbf{U}_{j1})}{(x_{j2} - x_{j1})(y_{j3} - y_{j1}) - (x_{j3} - x_{j1})(y_{j2} - y_{j1})}. \end{aligned} \quad (3.20)$$

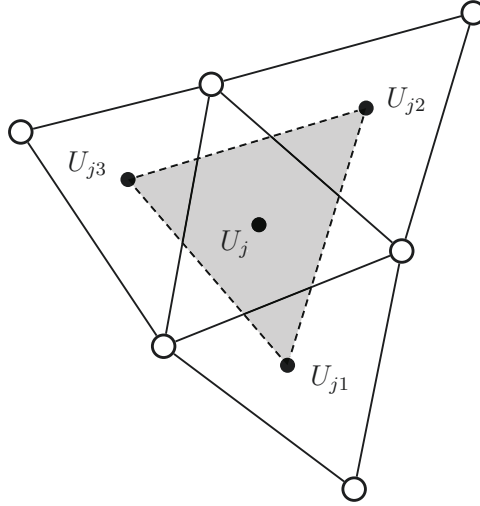


Figure 3.2: Stencil for calculating nonlimited derivatives.

We then follow the approach in [127] and obtain the limited numerical gradient $(\nabla \hat{\mathbf{U}})_i$ by taking the weighted average of three nonlimited gradients:

$$(\nabla \hat{\mathbf{U}})_i = \Lambda_{i1}(\nabla \mathbf{U})_{i1} + \Lambda_{i2}(\nabla \mathbf{U})_{i2} + \Lambda_{i3}(\nabla \mathbf{U})_{i3}, \quad (3.21)$$

where Λ_{i1} , Λ_{i2} and Λ_{i3} are weights and $(\nabla \mathbf{U})_{i1}$, $(\nabla \mathbf{U})_{i2}$ and $(\nabla \mathbf{U})_{i3}$ are the three nonlimited gradients computed using (3.20) for $j = i1, i2$ and $i3$, respectively.

The weights are calculated as follows:

$$\begin{aligned}\Lambda_{i1} &= \frac{\|(\nabla \mathbf{U})_{i2}\|_2^2 \|(\nabla \mathbf{U})_{i3}\|_2^2 + \varepsilon}{\|(\nabla \mathbf{U})_{i1}\|_2^4 + \|(\nabla \mathbf{U})_{i2}\|_2^4 + \|(\nabla \mathbf{U})_{i3}\|_2^4 + 3\varepsilon}, \\ \Lambda_{i2} &= \frac{\|(\nabla \mathbf{U})_{i1}\|_2^2 \|(\nabla \mathbf{U})_{i3}\|_2^2 + \varepsilon}{\|(\nabla \mathbf{U})_{i1}\|_2^4 + \|(\nabla \mathbf{U})_{i2}\|_2^4 + \|(\nabla \mathbf{U})_{i3}\|_2^4 + 3\varepsilon}, \\ \Lambda_{i3} &= \frac{\|(\nabla \mathbf{U})_{i1}\|_2^2 \|(\nabla \mathbf{U})_{i2}\|_2^2 + \varepsilon}{\|(\nabla \mathbf{U})_{i1}\|_2^4 + \|(\nabla \mathbf{U})_{i2}\|_2^4 + \|(\nabla \mathbf{U})_{i3}\|_2^4 + 3\varepsilon},\end{aligned}$$

where the parameter ε is a small positive number introduced to prevent division by zero. In all of the numerical examples presented in §3.5, we have taken $\varepsilon = 10^{-14}$.

Equipped with the numerical gradient (3.21), we obtain the piecewise linear approximations (3.15) for each of the conservative variables. These approximants $\mathring{\mathbf{U}}(x, y)$ are to be used to evaluate the point values of $\mathbf{U}$ at the midpoints of the cell edges, M_{ik} . To prevent appearance of any negative water depth values at these points, we replace the corresponding point values of the bottom elevation with

$$\mathring{U}^{(5)}(M_{ik}) := \min(\mathring{U}^{(5)}(M_{ik}), \mathring{U}^{(1)}(M_{ik})).$$

3.4.2 Estimating One-Sided Local Speeds of Propagation

We now turn to establishing an upper bound on the largest eigenvalue and a lower bound on the smallest eigenvalue of the matrices $V_{ik}(\mathbf{U}_i(M_{ik}))$ and $V_{ik}(\mathbf{U}_{ik}(M_{ik}))$ given by (3.18) and (3.19), respectively.

To this end, we employ the following theorem originally introduced by Lagrange in [95], see also [117] for the proof of the theorem and precise description on the way to obtain an upper bound on the largest positive root of a polynomial. The Lagrange theorem can be formulated as follows:

Let $P_n(\lambda) = \lambda^n + b_1\lambda^{n-1} + \dots + b_{n-1}\lambda + b_n$ be a nonconstant polynomial of degree n . Then, the largest nonnegative root of P_n is smaller than the sum of the largest and the second largest numbers in the set $\{|b_j|^{1/j} : b_j \in J_{\max}\}$, where $J_{\max}$ is the set of the negative coefficients of P_n .

For the matrices $V_{ik}(\mathbf{U}_i(M_{ik}))$ and $V_{ik}(\mathbf{U}_{ik}(M_{ik}))$, the characteristic polynomials read

$$[\lambda^4 + b_1\lambda^3 + b_2\lambda^2 + b_3\lambda + b_4] \left(\frac{hu}{\eta - Z} \cos(\theta_{ik}) + \frac{hv}{\eta - Z} \sin(\theta_{ik}) - \lambda \right), \quad (3.22)$$

where the coefficients b_1 , b_2 , b_3 and b_4 can be written explicitly using the definitions of the matrices A and B given by (3.11) and (3.12), respectively (we do not provide the expressions of these coefficients for the sake of brevity).

Then, $a_{ik}^{\text{out}}(M_{ik})$ is obtained using the upper bound of the largest nonnegative root of the fourth-order polynomial in the square brackets in equation (3.22), which is obtained

using the Lagrange theorem. Similarly, $a_{ik}^{\text{in}}(M_{ik})$ is obtained using a lower bound on the smallest nonpositive root of the fourth-order polynomial in the square bracket in equation (3.22) (according to the Lagrange theorem, this root is larger than the sum of the smallest and second smallest numbers in the set $\{-|d_j|^{1/j} : b_j \in J_{\min}\}$, where $J_{\min}$ is the set of the negative coefficients of the polynomial $\lambda^4 + d_1\lambda^3 + d_2\lambda^2 + d_3\lambda + b_4$ with $d_j = (-1)^j b_j, \forall j$).

3.4.3 Well-Balanced Discretization of the Bed-slope Terms

In this section, we describe a well-balanced discretization of the bed-slope terms, which guarantees that “lake at rest” solutions that satisfy

$$\eta \equiv \eta_c = \text{Const}, \quad u \equiv v \equiv 0, \quad (3.23)$$

are exactly preserved by the resulting central-upwind scheme. Notice that unlike the case of the classical Saint-Venant system, the “lake at rest” state in the current study is not a steady-state solution since both hc and Z variables may change in time while the identities (3.23) are satisfied. However, the “lake at rest” state is still physically significant and the approach of designing a special well-balanced quadrature for the geometric source term proposed in [23] can be extended to the studied system (3.8)–(3.10) so that the identities (3.23) are satisfied by the computed solution provided they are satisfied by the initial data.

First, we note that the second and third components of the source term K vanish at “lake at rest” states and therefore, the well-balanced property of the scheme will be guaranteed if the discretized cell average of the geometric term $\bar{\mathbf{S}}_i$, exactly balances the rest of numerical fluxes so that the second and third components on the right-hand side (RHS) of (3.14) vanish for the data satisfying (3.23).

After a substitution of a “lake at rest” state into (3.14), we conclude that a well-balanced quadrature of S should satisfy the following two conditions:

$$-\frac{g}{|T_i|} \sum_{k=1}^3 \ell_{ik} \cos(\theta_{ik}) \left[\frac{\eta_c^2}{2} - \eta_c \frac{a_{ik}^{\text{in}} Z_{ik}(M_{ik}) + a_{ik}^{\text{out}} Z_i(M_{ik})}{a_{ik}^{\text{in}} + a_{ik}^{\text{out}}} \right] + \bar{S}_i^{(2)} = 0 \quad (3.24)$$

and

$$-\frac{g}{|T_i|} \sum_{k=1}^3 \ell_{ik} \sin(\theta_{ik}) \left[\frac{\eta_c^2}{2} - \eta_c \frac{a_{ik}^{\text{in}} Z_{ik}(M_{ik}) + a_{ik}^{\text{out}} Z_i(M_{ik})}{a_{ik}^{\text{in}} + a_{ik}^{\text{out}}} \right] + \bar{S}_i^{(3)} = 0, \quad (3.25)$$

where

$$\bar{S}_i^{(2)} \approx -\frac{g}{|T_i|} \int_{T_i} \eta Z_x dx dy, \quad \bar{S}_i^{(3)} \approx -\frac{g}{|T_i|} \int_{T_i} \eta Z_y dx dy.$$

To derive the desired well-balanced quadrature, we first apply Green’s formula,

$$\int_{T_i} \text{div } \mathbf{g} dx dy = \int_{\partial T_i} \mathbf{g} \cdot \mathbf{n} ds,$$

to the vector field $\mathbf{g} = (\frac{1}{2}\eta^2 - \eta Z, 0)$ and obtain:

$$-\int_{T_i} \eta Z_x dx dy = \sum_{k=1}^3 \int_{(\partial T_i)_k} \left(\frac{\eta^2}{2} - \eta Z \right) \cos(\theta_{ik}) ds - \int_{T_i} (\eta - Z) \eta_x dx dy, \quad (3.26)$$

where $(\partial T_i)_k$ is the k -th side of the triangle T_i , $k = 1, 2, 3$. We then apply the midpoint rule to the integrals on the RHS of (3.26) and obtain

$$\begin{aligned} \bar{S}_i^{(2)} &= -g(\bar{\eta}_i - \bar{Z}_i) \eta_x(x_i, y_i) \\ &+ \frac{g}{|T_i|} \sum_{k=1}^3 \ell_{ik} \cos(\theta_{ik}) \frac{a_{ik}^{\text{in}} \left(\frac{\eta_{ik}(M_{ik})}{2} - Z_{ik}(M_{ik}) \right) + a_{ik}^{\text{out}} \left(\frac{\eta_i(M_{ik})}{2} - Z_i(M_{ik}) \right)}{a_{ik}^{\text{in}} + a_{ik}^{\text{out}}} \eta_i(M_{ik}). \end{aligned} \quad (3.27)$$

Finally, we apply the divergence theorem to the first term on the RHS of (3.27) and arrive at the following quadrature for the cell average $\bar{S}_i^{(2)}$:

$$\begin{aligned} \bar{S}_i^{(2)} &= \frac{g}{|T_i|} \sum_{k=1}^3 \ell_{ik} \cos(\theta_{ik}) \left[\frac{a_{ik}^{\text{in}} \left(\frac{\eta_{ik}(M_{ik})}{2} - (M_{ik}) Z_{ik}(M_{ik}) \right) + a_{ik}^{\text{out}} \left(\frac{\eta_i(M_{ik})}{2} - Z_i(M_{ik}) \right)}{a_{ik}^{\text{in}} + a_{ik}^{\text{out}}} \eta_i(M_{ik}) \right. \\ &\quad \left. - (\bar{\eta}_i - \bar{Z}_i) \frac{\eta_i(M_{ik}) + \eta_{ik}(M_{ik})}{2} \right]. \end{aligned} \quad (3.28)$$

Likewise, we obtain the quadrature for the cell average $\bar{S}_i^{(3)}$:

$$\begin{aligned} \bar{S}_i^{(3)} &= \frac{g}{|T_i|} \sum_{k=1}^3 \ell_{ik} \sin(\theta_{ik}) \left[\frac{a_{ik}^{\text{in}} \left(\frac{\eta_{ik}(M_{ik})}{2} - Z_{ik}(M_{ik}) \right) + a_{ik}^{\text{out}} \left(\frac{\eta_i(M_{ik})}{2} - Z_i(M_{ik}) \right)}{a_{ik}^{\text{in}} + a_{ik}^{\text{out}}} \eta_{ik}(M_{ik}) \right. \\ &\quad \left. - (\bar{\eta}_i - \bar{Z}_i) \frac{\eta_i(M_{ik}) + \eta_{ik}(M_{ik})}{2} \right]. \end{aligned} \quad (3.29)$$

Notice that since at “lake at rest” states $\eta_x \equiv \eta_y \equiv 0$, the quadratures (3.28), (3.29) satisfy (3.24), (3.25) as long as $\eta \equiv \eta_c, u \equiv v \equiv 0$.

Since the source term $\mathbf{K}$ in equation (3.8), does not affect the well-balanced property of the developed central-upwind scheme, we discretize its cell averages, $\{\bar{\mathbf{K}}_i\}$ in a straightforward manner by using the midpoint rule:

$$\bar{\mathbf{K}}_i = \mathbf{K}(x_i, y_i).$$

3.4.4 Time Integration

The semi-discretization (3.14) is a system of time-dependent ODEs, which should be integrated using a stable and sufficiently accurate ODE solver. In our numerical experiments,

we have used the third-order strong stability preserving (SSP) Runge-Kutta solver, see, e.g., [61, 62].

The time step restriction is determined by the CFL condition (for details, see [90]) and can be expressed as

$$\Delta t = C_r \min_{i,k} \left[\frac{R_i}{\max(a_{ik}^{\text{in}}, a_{ik}^{\text{out}})} \right],$$

where C_r is the Courant number typically taken in the range $0 < C_r \leq 1$ and R_i is the minimum distance from the centroid of the cell T_i to its edges.

3.5 Numerical Experiments

In this section, we demonstrate the performance of the proposed central-upwind scheme on five test problems.

Example 1—Small Perturbation of a “Lake at Rest” State

This numerical example is a modification of the benchmark proposed in [99]. It is designed to test the ability of the designed scheme to accurately capture small perturbations of “lake at rest” states.

In this test, the computational domain is $[0, 2] \times [0, 1]$ and it is divided into 20658 triangular cells. The initial conditions are

$$\begin{aligned} \eta(x, y, 0) &= \begin{cases} 1.01, & \text{if } 0.05 < x < 0.15, \\ 1, & \text{otherwise,} \end{cases} & u(x, y, 0) \equiv v(x, y, 0) \equiv 0, \\ c(x, y, 0) &\equiv 0.05, & Z(x, y, 0) = 0.8 \exp[-5(x - 0.9)^2 - 50(y - 0.5)^2]. \end{aligned}$$

We have used zero-order extrapolation at the right and left boundaries, while the solid wall boundary conditions have been implemented in the y -direction.

The gravity acceleration is set to $g = 1$, the bed porosity is taken to be $p = 0.3$, and the coefficient μ in equation (3.5) set to $\mu = 0.001$. In this example, we do not use equation (3.7) to determine the settling velocity of particles, but instead we set it to $\omega_0 = 1$.

Figure 3.3 (left) shows the contour lines of water surface for the right-going disturbance as it propagates past the hump at times $t = 0.6, 0.9, 1.2, 1.5$ and 1.8 , respectively. As one can see, complex small features of the flow are resolved in an oscillation-free manner and the results are in good agreement with the ones reported in [99]. To demonstrate the necessity and importance of the well-balanced property, we also apply a non-well-balanced central-upwind scheme to the same problem. The non-well-balanced scheme is obtained by replacing the bed-slope terms $\bar{S}_i^{(2)}$ and $\bar{S}_i^{(3)}$ in (3.28) and (3.29) by a straightforward

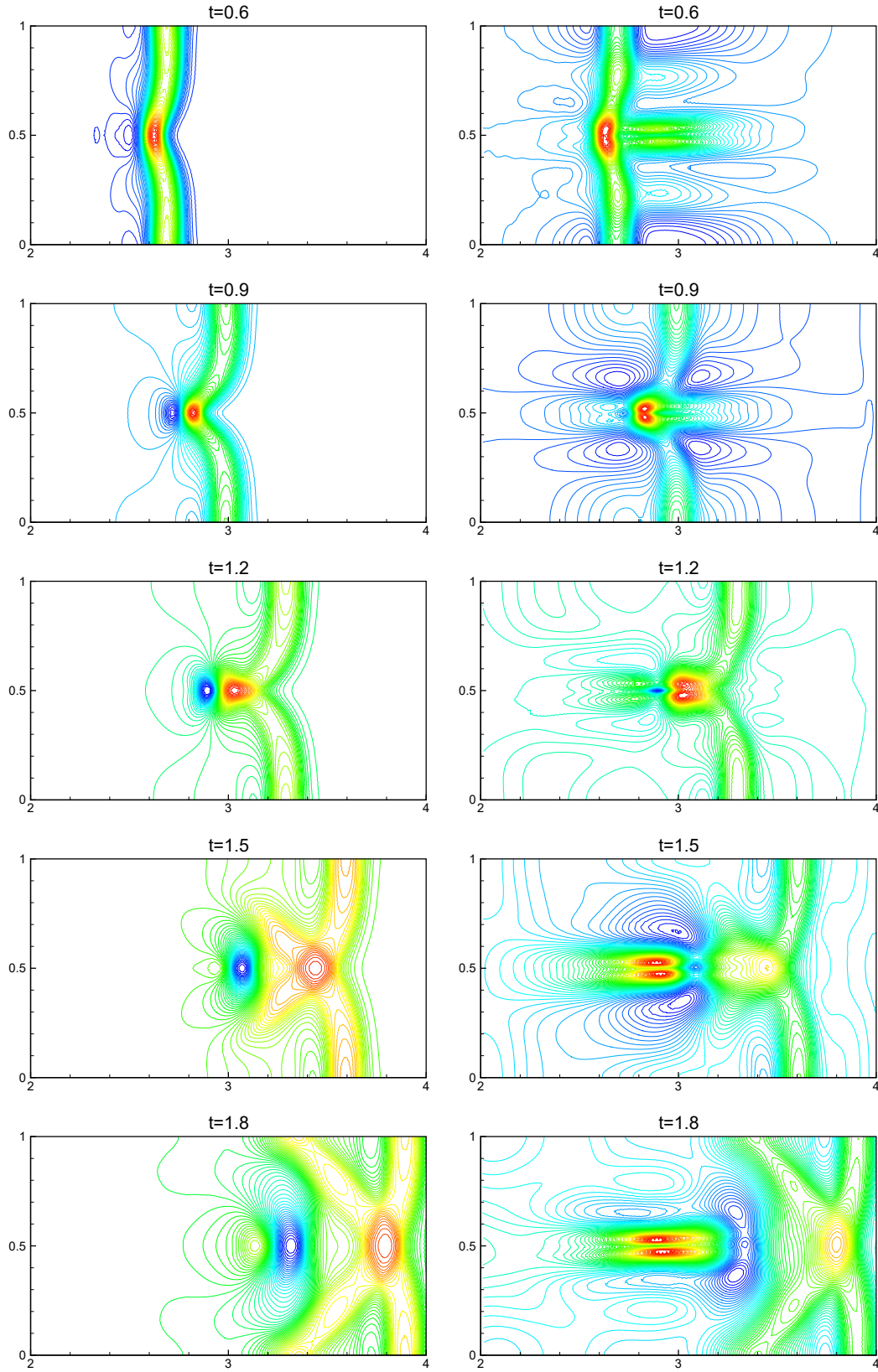


Figure 3.3: Example 1: Water surface η computed by well-balanced (left) and non-well-balanced (right) schemes.

midpoint rule discretization:

$$\bar{S}_i^{(2)} = -\eta(x_i, y_i) \frac{g}{|T_i|} \sum_{k=1}^3 \ell_{ik} \cos(\theta_{ik}) \frac{Z_i(M_{ik}) + Z_{ik}(M_{ik})}{2},$$

$$\bar{S}_i^{(3)} = -\eta(x_i, y_i) \frac{g}{|T_i|} \sum_{k=1}^3 \ell_{ik} \sin(\theta_{ik}) \frac{Z_i(M_{ik}) + Z_{ik}(M_{ik})}{2}.$$

The non-well-balanced solution computed on the same grid is presented in Figure 3.3 (right). It can be clearly observed that spurious waves are generated by the non-well-balanced scheme at the plateau of the hump. Moreover, the spurious waves are of approximately the same magnitude as the physical waves, which suggests that the non-well-balanced scheme should be used on much finer grid only, which would make it impractical.

Figure 3.4 shows the 1-D slice along the line $y = 0.5$ of both the water surface and bed evolutions (the latter occurs due to the sediment deposition) at the same times $t = 0.6, 0.9, 1.2, 1.5$ and 1.8 . It should be observed that even in the presence of sediment concentration, the well-balanced property of the proposed scheme has prevented numerical oscillations that are typically observed when the computation is conducted using a non-well-balanced scheme.

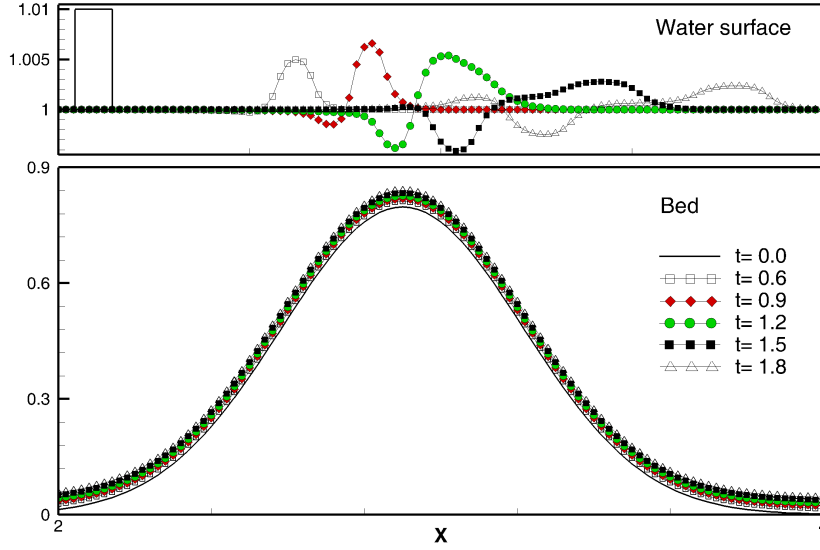


Figure 3.4: Example 1: 1-D slices of the water surface η and bed Z along the line $y = 0.5$.

Example 2—Sand Deposition in Quiescent Water

This test is designed to verify that the added source terms will not affect the well-balanced property in quiescent water with uniform sediment deposition.

We take the computational domain $[0, 2] \times [0, 1]$, which is divided into 3628 triangular cells, set the following initial conditions:

$$\begin{aligned}\eta(x, y, 0) &\equiv 1, & u(x, y, 0) &\equiv v(x, y, 0) \equiv Z(x, y, 0) \equiv 0, \\ c(x, y, 0) &= 0.7 \exp \left[-5(x - 0.9)^2 - 50(y - 0.5)^2 \right],\end{aligned}$$

and use zero-order extrapolation at all of the boundaries.

We also use the following values of the parameters: the gravity acceleration is $g = 9.8$, the bed porosity is $p = 0.28$, the sediment diameter is $d = 0.01$, the density of sediment particles is $\rho_s = 2400$, the density of water is $\rho_f = 1000$, and the coefficient μ in equation (3.5) is $\mu = 0$.

In Figure 3.5, we show the quiescent water surface and bed profile at a large time $t = 100$. As one can see, no oscillations are developed at the quiescent water surface, so that the sediment deposition does not affect the numerical “lake at rest” steady state. The 1-D slice along the line $y = 0.5$ of the obtained results is shown in Figure 3.6.

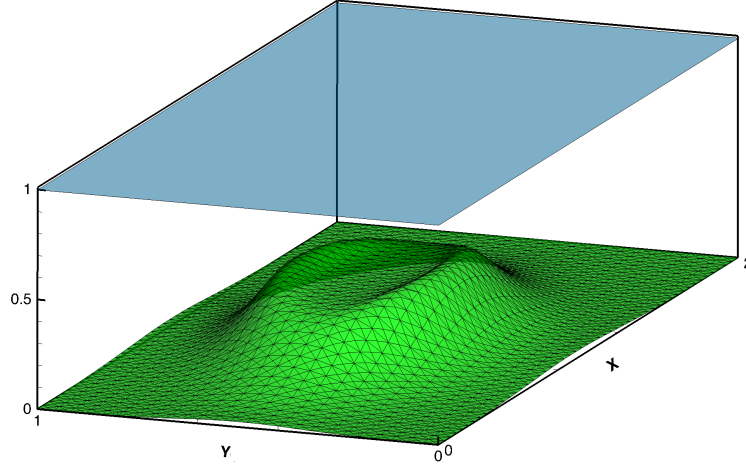


Figure 3.5: Example 2: Bed Z and quiescent water surface η at $t = 100$.

Example 3—Dam-Break Flow in a Long Channel over Erodible Bed

This test is performed to investigate the advantage of the proposed central-upwind scheme applied to the fully coupled system (3.1)–(3.5) over a simpler partially coupled approach, in which the central-upwind scheme is applied to the first four equations (3.1)–(3.4) only, while the equation for the bed evolution (3.5) is solved using an upwind method as it is described below.

For simplicity, the sediment entrainment effects are deactivated so that we take $E \equiv D \equiv 0$.

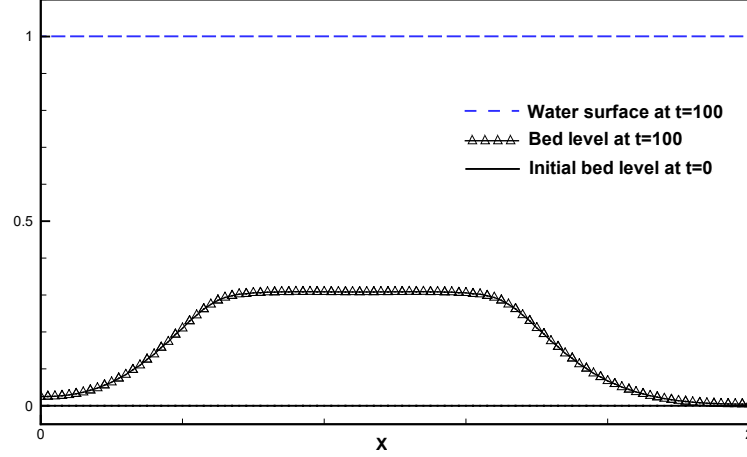


Figure 3.6: Example 2: 1-D slice along the line $y = 0.5$ of the results reported in Figure 3.5.

In the partially coupled method, the first four conservative quantities, η , hu , hv and hc , are evolved in time by solving the same ODE system (3.14), but with different one-sided speeds of propagation, which are now obtained using the largest and smallest eigenvalues of the matrix

$$\tilde{V}_{ik} := \cos(\theta_{ik})\tilde{A}(\mathbf{U}(M_{ik})) + \sin(\theta_{ik})\tilde{B}(\mathbf{U}(M_{ik})),$$

where

$$\tilde{A}(\mathbf{U}) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -\frac{(hu)^2}{(\eta - Z)^2} + g(\eta - Z) & \frac{2hu}{\eta - Z} & 0 & 0 \\ -\frac{(hu)(hv)}{(\eta - Z)^2} & \frac{hv}{\eta - Z} & \frac{hu}{\eta - Z} & 0 \\ -\frac{(hu)(hc)}{(\eta - Z)^2} & \frac{hc}{\eta - Z} & 0 & \frac{hu}{\eta - Z} \end{pmatrix},$$

and

$$\tilde{B}(\mathbf{U}) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ -\frac{(hu)(hv)}{(\eta - Z)^2} & \frac{hv}{\eta - Z} & \frac{hu}{\eta - Z} & 0 \\ -\frac{(hv)^2}{(\eta - Z)^2} + g(\eta - Z) & 0 & \frac{2hv}{\eta - Z} & 0 \\ -\frac{(hv)(hc)}{(\eta - Z)^2} & 0 & \frac{hc}{\eta - Z} & \frac{hv}{\eta - Z} \end{pmatrix}.$$

Unlike the case of a fully coupled model, the eigenvalues of $\tilde{V}_{ik}(\mathbf{U}_i(M_{ik}))$ can be computed

analytically and they are

$$\begin{aligned}\lambda_{1,4} &= \frac{hu}{\eta - Z} \cos(\theta_{ik}) + \frac{hv}{\eta - Z} \sin(\theta_{ik}) \pm \sqrt{g(\eta - Z)}, \\ \lambda_2 &= \lambda_3 = \frac{hu}{\eta - Z} \cos(\theta_{ik}) + \frac{hv}{\eta - Z} \sin(\theta_{ik}).\end{aligned}$$

The bottom topography function is evolved in time by applying the semi-discrete upwind discretization to equation (3.5):

$$\begin{aligned}\frac{d\bar{Z}_i}{dt} &= - \frac{\mu}{|T_i|(1-p)} \sum_{k=1}^3 \ell_{ik} \cos(\theta_{ik}) \left[\beta_{ik} u_{ik}(M_{ik}) (u_{ik}^2(M_{ik}) + v_{ik}^2(M_{ik})) \right. \\ &\quad \left. + (1 - \beta_{ik}) u_i(M_{ik}) (u_i^2(M_{ik}) + v_i^2(M_{ik})) \right] \\ &\quad - \frac{\mu}{|T_i|(1-p)} \sum_{k=1}^3 \ell_{ik} \sin(\theta_{ik}) \left[\beta_{ik} v_{ik}(M_{ik}) (u_{ik}^2(M_{ik}) + v_{ik}^2(M_{ik})) \right. \\ &\quad \left. + (1 - \beta_{ik}) v_i(M_{ik}) (u_i^2(M_{ik}) + v_i^2(M_{ik})) \right],\end{aligned}$$

where

$$\beta_{ik} = \begin{cases} 0, & \frac{u_{ik}(M_{ik}) + u_i(M_{ik})}{2} \cos(\theta_{ik}) + \frac{v_{ik}(M_{ik}) + v_i(M_{ik})}{2} \sin(\theta_{ik}) \geq 0, \\ 1, & \text{otherwise.} \end{cases}$$

In this numerical test, we simulate a dam-break flow over erodible bed. The computational domain $[0, 40] \times [0, 0.5]$ is divided into 2560 triangular cells. The initial conditions are

$$\eta(x, y, 0) = \begin{cases} 3.1, & x \leq 20, \\ 0.2, & x > 20, \end{cases} \quad u(x, y, 0) \equiv v(x, y, 0) \equiv c(x, y, 0) \equiv Z(x, y, 0) \equiv 0,$$

and zero-order extrapolation is used at all of the boundaries. The parameter values are $g = 9.8$, $p = 0.28$ and $\mu = 0.001$.

In Figure 3.7, we show the bed profile computed at $t = 1$ using the central-upwind scheme applied to both fully coupled and partially coupled models. As one can clearly see, even under a high-energetic flow considered in this example, the fully coupled model can predict a stable bed erosion process and leads to a smooth and physically expectable bed profile. On the contrary, the use of the partially coupled model results in an unphysical and oscillatory bed profile.

Example 4—1-D Anti-Dune Evolution

In this numerical test, we simulate a case with a configuration which allows the occurrence of the anti-dune phenomenon, see [9]. The computational domain in $[0, 24]$ and it is

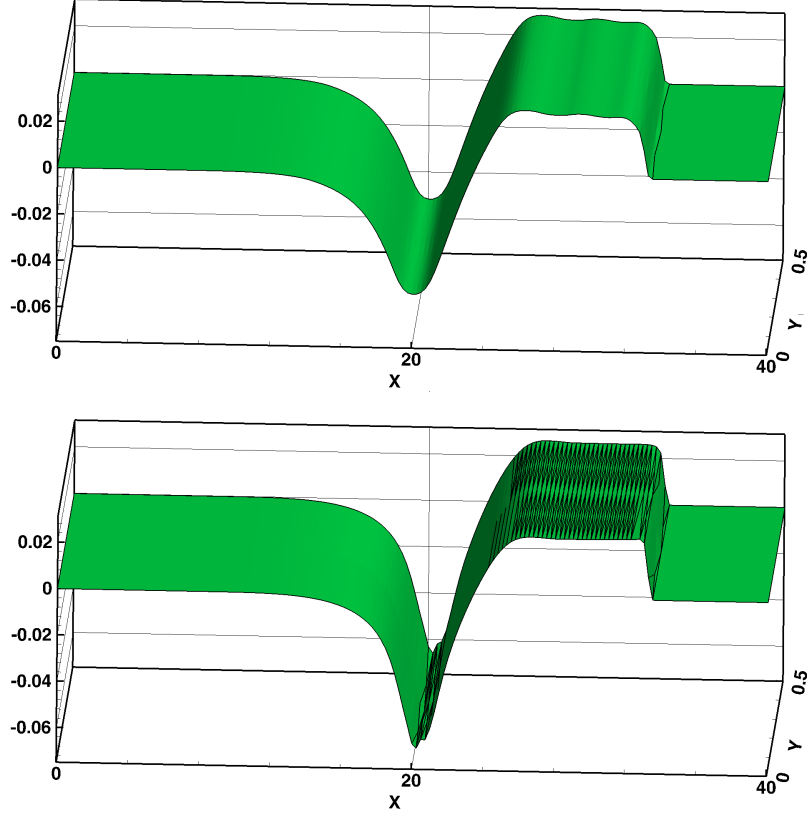


Figure 3.7: Example 3: Bed profile $Z(x, y, 1)$ computed using the central-upwind scheme applied to the fully coupled (top) and partially coupled (bottom) models.

discretized on the mesh with an average size $\Delta x = 0.1$. The initial topography contains a bump and is given by

$$Z(x, 0) = \begin{cases} 0.2 - 0.05(x - 10)^2, & \text{if } 8 \leq x \leq 12, \\ 0, & \text{otherwise.} \end{cases}$$

The inflow discharge is uniform and set to be $q_0 = 1.7$. The water height of the inflow is the stationary supercritical profile determined by Bernoulli's law

$$\begin{cases} \frac{q_0^2}{2gh^2} + h + Z = H_0, \\ q(x, t) = q_0, \end{cases}$$

where $H_0 = q_0^2/(2gh_0^2) + h_0 + Z(0, 0)$ is the total hydraulic head at inflow, and the water height at inflow is $h_0 = 0.5$. In this test, we set $A_g = \mu/(1 - p) = 0.001$, $g=9.8$ and $E \equiv D \equiv 0$.

In Figure 3.8, we show the computed bed evolution and water level at times $t = 0, 15$ and 30. As one can observe, the dune migrates counter to the flow direction because the sediment is deposited on the upstream side and eroded from the downstream side. It shows that the proposed model has the ability to accurately simulate the anti-dune phenomenon which occurs in supercritical regimes.

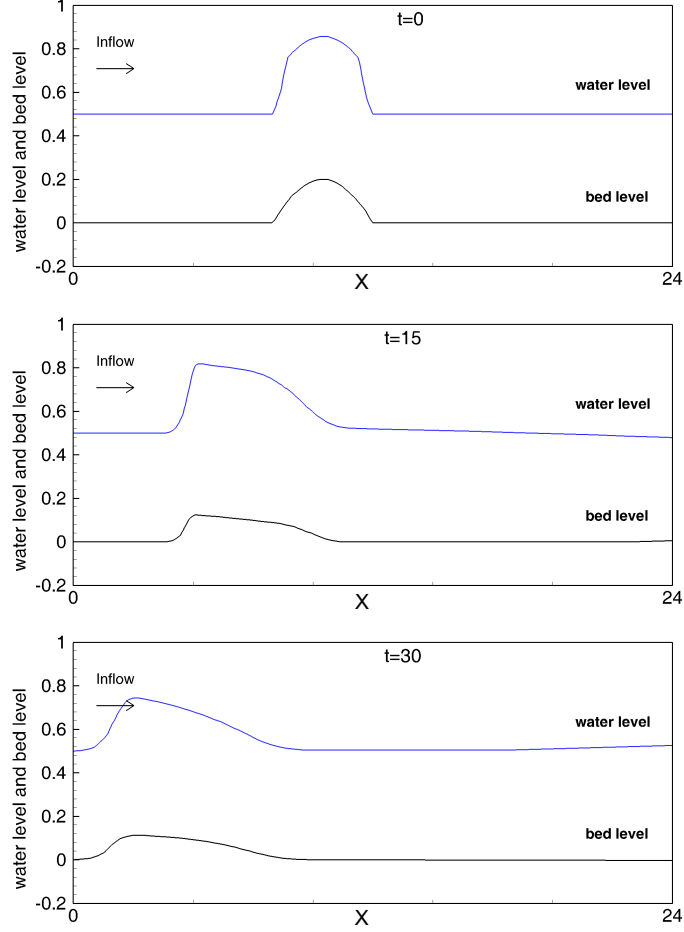


Figure 3.8: Example 4: Water level (η) and bed profile (Z) computed by the proposed central-upwind scheme at times $t = 0, 15$ and 30 .

Example 5—Partial Dam-Break Flow over Mobile Bed

In the last numerical example, we demonstrate the performance of the proposed central-upwind scheme on a 2-D dam break problem with rapidly varying unsteady flow and rapid bed evolution. This test problem is based on the numerical test that was first introduced in [53] and then widely used as a benchmark, see, e.g., [5, 6, 175, 180, 189]. The original case is mostly applied on an initially wet and fixed bed, which has been modified in the current study by adding bed evolution.

We consider the domain $[0, 200] \times [0, 200]$ with the rectangular dam located in $[95, 105] \times [0, 200]$. The initial conditions are

$$\eta(x, y, 0) = \begin{cases} 6.8, & x \leq 95, \\ 1, & x \geq 105, \end{cases} \quad u(x, y, 0) \equiv v(x, y, 0) \equiv c(x, y, 0) \equiv Z(x, y, 0) \equiv 0.$$

At time $t = 0$, a breach in the dam at $[95, 105] \times [85, 160]$ is assumed to form instantaneously and the water starts flowing through the breach. The computational domain

covered with 10508 triangular cells is outlined in Figure 3.9. Notice that at the breach region where the gradients of the water depth are large and intense sediment transport occurs, the mesh is refined to obtain high resolution. The outlet boundary condition is set at $x = 200$, while the solid wall boundary conditions are used at all other boundaries. In this example, we use the following parameter values: $g = 9.8$, $p = 0.28$, $d = 0.01$,

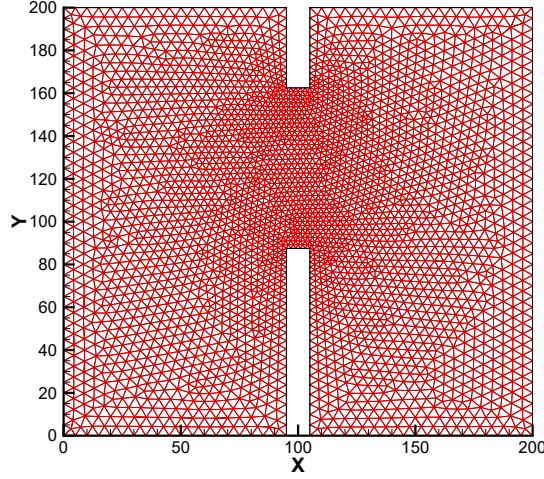


Figure 3.9: Example 5: Computational domain and its unstructured triangulation.

$\rho_s = 2400$, $\rho_f = 1000$, $\mu = 0.001$, and compute the solution up to time $t = 8$. During the modeling process, a shock wave forms and propagates downstream and a depression wave spreads upstream.

We first solve the shallow water system (3.1)–(3.3) assuming that the bed is fixed, that is, $Z_t \equiv Z_x \equiv Z_y \equiv 0$. The predicted water level and velocity distribution are shown in Figure 3.10: The obtained results are in good agreement with those reported in [25, 174, 180, 188].

We then consider a general case of an erodible bed and solve the system (3.1)–(3.5) using both the fully and partially coupled models. For both models the computed water surfaces (not shown here for the sake of brevity) contain no spurious oscillations and the obtained flow patterns are quite similar.

However, the bed elevations predicted by the fully and partially coupled models are very different, as one can see in Figure 3.11, where we show the bottom topography snapshots at times $t = 4, 6$ and 8 . It can be clearly observed that at the corners of both sides of the breach where intense scouring and sediment entrainment are induced by strong vortices, the partially coupled model produces very sharp unphysical scouring holes and slight oscillations (see Figure 3.11 (right)), which may develop spurious oscillations when a flow with higher energy is applied. In contrast with the results obtained using the partially coupled model, the bed elevations predicted using the fully coupled model are more accurate and smoother at the corners of the breach and no unphysical scouring and oscillations can be observed in Figure 3.11 (left).

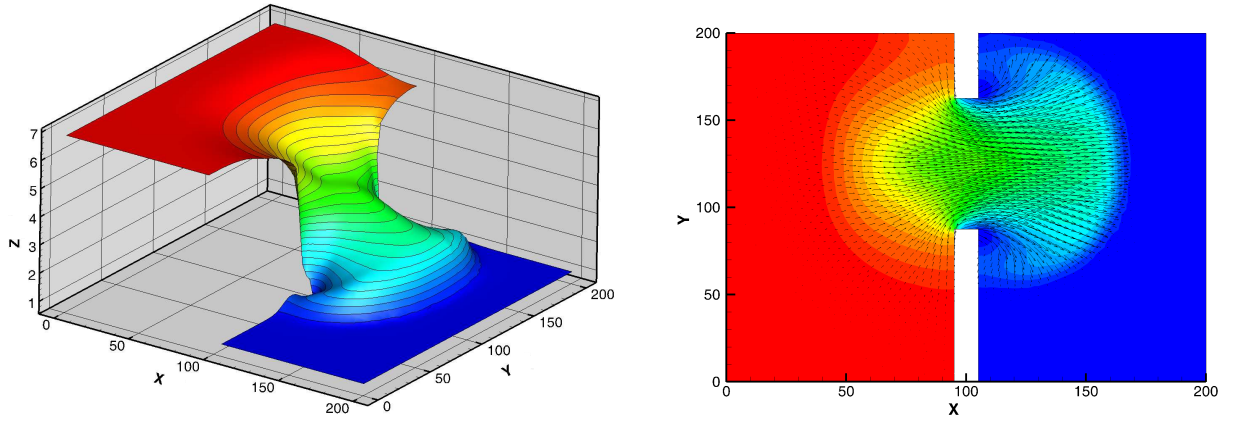


Figure 3.10: Example 5 with fixed bed: Water level $\eta(x, y, 8)$; surface plot (left) and contour plot with the velocity field (right).

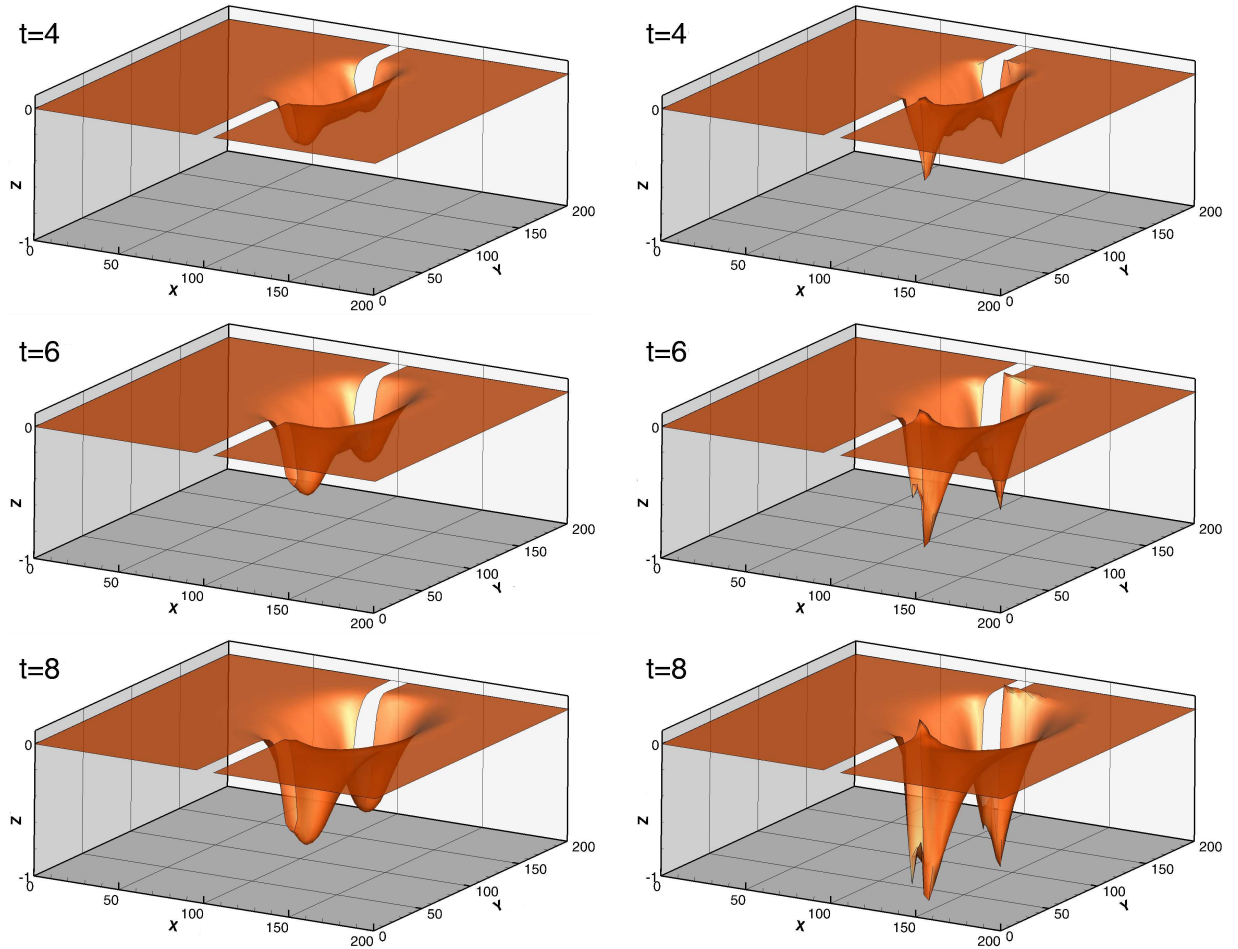


Figure 3.11: Example 5: Bed evolution computed using the proposed fully coupled model (left) and partially coupled model (right).

3.6 Conclusion

In this paper, we have proposed and studied a fully coupled model that contains the shallow water system, sediment transport equation and bed evolution equation. We have developed a well-balanced, second-order, semi-discrete central-upwind scheme on unstructured triangular grids. The scheme has been tested on a number of numerical examples, which have demonstrated that the developed central-upwind scheme is an efficient and stable tool for solving the proposed fully coupled hyperbolic system that models shallow water flows with intense sediment exchange and bed erosion process.

Some of the major conclusions have been drawn:

1. Applying the Lagrange theorem efficiently reduces the cost of finding the smallest and largest eigenvalues of the complicated 5×5 Jacobian. By comparing the results of the fully and partially coupled models, one may claim that the upper/lower bounds on the largest/smallest eigenvalues obtained using the Lagrange theorem are quite sharp;
2. The model proposed in this paper couples the bed evolution equation into the hyperbolic system, which leads to more stable and accurate results. The bed evolution process is less oscillatory compared with a partially coupled model, especially when intense scouring and sediment exchange occur. This implies that the proposed fully coupled model is advantageous for highly energetic flows like the dam-break one and fluvial flood;
3. The well-balanced discretization proposed in the current paper is designed for discontinuous Z , approximated using a (generically discontinuous) piecewise polynomial, which is evolved in the framework of a finite-volume methods. This approach proves to be very robust and accurate.

Chapter 4

A coupled two-dimensional numerical model for rapidly varying flow, sediment transport and bed morphology

4.1 Abstract

This paper presents a coupled two-dimensional model that can produce a more stable numerical simulation of rapid bed evolution than the conventional decoupled model. To solve the coupled bed-load sediment transport terms using a Godunov-type central-upwind method, a novel scheme to estimate the bed-load fluxes which can produce more accurate results than the previously reported coupled model is proposed using a pair of local wave speeds different from those used for the flow. The two-dimensional shallow water equations are used to solve the flow velocities and water depth. The bed level is solved by an Exner-based equation containing bed-load sediment transport as numerical flux terms and sediment entrainment and deposition as source terms. Analytical formulas to compute the eigenvalues of the Jacobian matrix are developed. The linear reconstruction of variables with a multi-dimensional slope limiter, and the second-order Runge-Kutta scheme are employed to achieve higher accuracy in space and time. For the case of rapid bed-erosion, the accuracy and stability of the proposed coupled model are verified by several numerical tests.

Keywords: Bed-load transport, coupled model, direct numerical simulation methods, finite volume method, shallow flows, two-dimensional models.

4.2 Introduction

Conventional decoupled models for simulating shallow water equations (SWEs) and bed erosion have normally been developed assuming that the rate of bed morphological evo-

lution is of a much lower order of magnitude than the flow changes. In such models, the hydrodynamic module is assumed to be identical to that of a fixed-bed flow, and the bed evolution equation is treated separately from the hydrodynamic model. Such approaches are accurate and stable when dealing with slow water flows with a very slow bed erosion process over a long period of time. However, when considering faster flows with a relatively fast bed evolution process, decoupled approaches might not be suitable any more, as the interactions between water and sediment variables become stronger.

Two fundamentally different directions are considered to account for strong interactions between water flow and bed erosion [28]. The first direction is recognizing that the momentum equation for flow mixture over an erodible bed differs from that for fixed-bed clear water flow in the sense that the former contains terms related to the density of the mixture and bed mobility. It requires a different momentum equation to be built for an intense sediment transport layer to account for inertial effects when large amounts of sediments are transported by the flow. Several morphological models were successfully developed considering this direction. A two-layer model was developed by Fraccarollo and Capart ([56]) assuming that the water-sediment mixture is in the thick bed-load layer under clear water, in which the conservation laws are built for both layers. This model was later improved by considering velocity decoupling (Capart & Young, [31]), granular phase dilatation (Spinewine & Zech, [156]; Zech, Soares-Frazão, Spinewine, Savary, & Goutiere, [183]), equations coupling (Spinewine, Guinot, Soares-Frazão, & Zech, [155]; Swartenbroekx, Soares-Frazão, Spinewine, Guinot, & Zech, [159]), sediment inertia effect on flows (Castro-Díaz, Fernández-Nieto, González-Vida, & Parés-Madroñal, [33]), large shear stress between two layers (Castro-Díaz et al., [33]), hyperbolicity of the system (Castro-Díaz et al., [33]; Savary & Zech, [137]; Spinewine et al., [155]; Swartenbroekx et al., [159]) and turbulence (Ferreira, Franca, Leal, & Cardoso, [54]). The two-phase models have gained popularity in modelling intense sediment transport under sheet-flow conditions, in which the sediment phase can be written in either Eulerian or Lagrangian form. The “Euler-Lagrange” two-phase models (Drake & Calantoni [48]; Gotoh & Sakai [60]; Yeganeh-Bakhtiary, Shabani, Gotoh, & Wang, [181]) can better account for particle-particle interactions in the hyper-concentrated layer. The “Euler-Euler two-phase models (Bakhtyar, Yeganeh-Bakhtiary, Barry, & Gha-heri, [10]; Greco, Iervolino, Leopardi, & Vacca, [68]; Hsu, Chang, & Hsieh, [75]; Liu & Sato, [107]) consider sediment phase as a continuum that follows different constitutive laws from those for the clear water, and the momentum transfer between two phases is taken into account. The above models lead to relatively expensive computational costs for building the mass and momentum conservation laws separately for two layers or two phases. To solve the general cases of rapid bed erosion under shallow water flows, the conventional single-layer models were kept having been widely studied. Cao, Pender, Wallis and Carling ([29]) modified the classic SWEs by adding components which account for mass and momentum exchange between erodible beds and water-mixture flows. The induced entraining sediment was considered as a depth-averaged suspended load, and a non-capacity erosion model was developed in their study. Yue, Cao, Li and Che ([182]) extended the model developed by Cao et al. ([29]) to two-dimensional (2-D) cases. This modified system was applied by Wu and Wang ([178]) and Xia, Lin, Falconer and Wang ([180]) with different bed-load and suspended-load formulas.

The second direction is coupling the Saint-Venant-Exner equations and solving them in a synchronous procedure. The strategy of coupling the Exner-based equation with the governing hydrodynamic system and approximating the bed-load fluxes using a Riemann solver has attracted increasing attention from researchers. Cao et al. ([29]) coupled the SWEs with a suspended sediment transport equation in their 1-D study. Similar coupled models have been used in (Simpson & Castelltort, [143]) and (Hu & Cao, [76]). Fagherazzi and Sun ([51]) reformulated the bed-evolution equation by introducing new conservative variables in order to build a fully coupled system and correctly capture discontinuous solutions. Li and Duffy ([101]) extended this approach to 2-D cases. Soulis ([151]) and Liu, Landry and Garcia ([109]) solved the coupled system of hydrodynamic and morphodynamic equations by a finite volume method, and the bed-load transport was considered in their coupled system. However, the bed-load transport was not considered in the above-mentioned models, and the bed-evolution equation is not coupled with the governing system. Hudson and Swebby ([80]) investigated the accuracy and determined the validity of both steady and unsteady approaches for 5 different types of governing formulations, coupling the morphology continuity equation with hydrodynamic equations on rectangular meshes. They concluded that one of the conservative reformulated forms based on the unsteady approach, was the best overall. However, a drawback of this form is its dependence on z in the Jacobian matrix, which was shown and discussed in (Murillo & García-Navarro, [121]) and (Serrano-Pacheco, Murillo, & Garcia-Navarro, [140]). Castro-Díaz, Fernández-Nieto and Ferreiro ([32]) studied the numerical approximation of bed-load sediment transport caused by shallow layer flows on unstructured meshes with a second-order MUSCL-type reconstruction. Murillo and García-Navarro ([121]) studied an Exner-based coupled model for 2-D transient flows over erodible beds on triangular unstructured meshes, and developed a general "Grass formula" format expressing several commonly used empirical bed-load formulas. However, in the study by Murillo and García-Navarro ([121]), the same eigenvalues and eigenvectors were used to estimate the fluxes for both the hydrodynamic and morphological models, which may lead to excessive diffusion for the bed-evolution process. In addition, although considering the bed-load coefficient as a constant in the Jacobian matrix can significantly reduce the computational cost and difficulties for solving the system, ignoring the variability of the bed-load coefficient may lead to inaccurate estimations of the eigen-structure. Liu, Mohammadian, Kurganov and Infante Sedano ([111]) proposed a coupled strategy which using the Lagrange theorem to estimate the bounds of the eigenvalues in order to avoid difficulty in calculating those eigenvalues. They focused on very rapid erosion processes, which need a higher diffusion level. Their approach may need to be adjusted for real-world applications. Soares-Frazão and Zech ([149]) built a coupled system using the HLLC method and chose different pairs of eigenvalues to estimate the numerical fluxes. However, based on the fact that they only used two specific eigenvalues during the simulation process, excessive diffusion might still be introduced in simulating the bed profile.

For most real-world applications, even for rapid bed-load transport cases, the rates of morphological evolution caused by bed-load transport are still considerably smaller than the rate of the flow changes. Excessive numerical diffusion may be introduced when approximating bed-load transport fluxes using the same directional local speeds from the

hydrodynamic system in the flux-solver. Accordingly, in Godunov-type flux approximations, it is necessary to consider smaller directional local speeds in the solver in order to accurately approximate the bed-load sediment flux terms.

In this paper, we focus on the second direction described, and we propose a numerical scheme which uses different pairs of eigenvalues of the Jacobian matrix to estimate the numerical fluxes of flow and bed-load transport. A central-upwind method with novel estimators of local wave-speeds is developed based on the analytical expressions of the eigenvalues of the Jacobian matrix. A single-layer model is adopted and it is demonstrated that with a suitable numerical scheme, the single-layer model could lead to an efficient and accurate scheme for a wide range of applications. Using the single-layer model, a coupling strategy is proposed, and a coupled model for rapid bed erosion is developed. The sediment concentrations are also updated at every time-step in the system, and sediment entrainment and deposition are included in the source terms, as they are effective in various practical tests. Multiple test cases are presented in order to demonstrate the advantages of the proposed scheme in terms of numerical performance such as better accuracy and non-oscillatory behaviour.

4.3 Governing equations

4.3.1 Description of hydraulic and sediment transport system

In the current study, the governing equations, consisting of the shallow water system, sediment transport, and Exner-based equation, are given by

$$\frac{\partial h}{\partial t} + \frac{\partial}{\partial x}(uh) + \frac{\partial}{\partial y}(vh) = 0 \quad (4.1)$$

$$\frac{\partial}{\partial t}(uh) + \frac{\partial}{\partial x}\left(u^2h + \frac{1}{2}gh^2\right) + \frac{\partial}{\partial y}(uvh) = -gh\frac{\partial z}{\partial x} - ghS_{fx} \quad (4.2)$$

$$\frac{\partial}{\partial t}(vh) + \frac{\partial}{\partial x}(uvh) + \frac{\partial}{\partial y}\left(v^2h + \frac{1}{2}gh^2\right) = -gh\frac{\partial z}{\partial y} - ghS_{fy} \quad (4.3)$$

$$\frac{\partial}{\partial t}(hc) + \frac{\partial}{\partial x}(uhc) + \frac{\partial}{\partial y}(vhc) = E - D \quad (4.4)$$

$$\frac{\partial z}{\partial t} + \xi \frac{\partial q_{bx}}{\partial x} + \xi \frac{\partial q_{by}}{\partial y} = \frac{D - E}{1 - p} \quad (4.5)$$

in which t is time, h is the water depth, x and y are horizontal coordinates, u and v are the depth-averaged velocity-components in the x and y directions, respectively, z is the bed level, g is the gravitational acceleration, p is the bed porosity, ξ is a coefficient equal to $(1 - p)^{-1}$, and c is the volumetric sediment concentration. In addition, S_{fx} and S_{fy} are the friction slope terms in x and y directions, respectively, which can be determined by conventional formulas involving the Manning roughness coefficient n_b , i.e. $S_{fx} = n_b^2 h^{-4/3} u \sqrt{u^2 + v^2}$ and $S_{fy} = n_b^2 h^{-4/3} v \sqrt{u^2 + v^2}$, q_{bx} and q_{by} are components of bed-load discharge q_b in x and y directions, respectively.

4.3.2 Bed-load formulas

In the current study, the bed-load sediment transport rate is estimated by the Meyer-Peter and Müller (MPM) formula (Meyer-Peter & Müller, [116]). The formula is as follows

$$q_b = 8\sqrt{(s-1)gd^3} \cdot (\theta - \theta_c)^{\frac{3}{2}} \quad (4.6)$$

in which s is the specific gravity of sediment given by $s = \rho_s/\rho_f$, where ρ_s is the density of sediment particles, ρ_f is the density of water, d is the mean particle diameter, θ_c is the critical bed shear stress and $\theta_c = 0.047$, and θ is the dimensionless shear stress which can be calculated as $\theta = hS'/[(s-1)d]$. Using $S' = S_f = \sqrt{S_{fx}^2 + S_{fy}^2}$, we arrive at

$$\theta = \frac{n_b^2}{(s-1)dh^{1/3}}|u|^2 \quad (4.7)$$

in which $|u| = \sqrt{u^2 + v^2}$.

4.3.3 Sediment deposition and entrainment fluxes

Sediment entrainment E , due to turbulence, and sediment deposition D , due to gravitational action, are two distinct mechanisms involved in sediment exchange between the fluid and the erodible bed. The relationships in (Cao et al., [29]) are used in this study

$$\begin{aligned} E &= \begin{cases} \varphi(\theta - \theta_c)|u|d^{-0.2}h^{-1} & \text{if } \theta \geq \theta_c \\ 0 & \text{otherwise} \end{cases} , \\ D &= \omega_o(1 - C_a)^m C_a \end{aligned} \quad (4.8)$$

where ω_o is the settling velocity of a single particle in tranquil water, which can be calculated as $\omega_o = \sqrt{(13.95\nu/d)^2 + 1.09sgd} - 13.95\nu/d$ (Zhang & Xie, [186]), where ν is the kinematic viscosity of water, m is an exponent representing the effect of hindered settling due to sediment concentration, and C_a is the local near-bed sediment volumetric concentration, which can be determined by $C_a = \alpha c$, in which the coefficient α is usually larger than unity and c must not exceed $(1-p)$. Cao et al. ([29]) suggested using $\alpha = \min[2, (1-p)/c]$, and φ as a coefficient to control the erosion force, which may be obtained using empirical formulas for intense and strong turbulent bursting entrainment processes. Parameter φ is calibrated for a relatively weak sediment entrainment process in this paper, which is required to extend this formula for situations of bed erosion with a less turbulent bursting effect.

4.4 Numerical scheme

4.4.1 Coupled governing system

Equations (4.1)–(4.5) constitute the governing system which can be presented in the following vector form

$$\frac{\partial \mathbf{U}}{\partial t} + \frac{\partial \mathbf{F}}{\partial x} + \frac{\partial \mathbf{G}}{\partial y} - \mathbf{S}_b = \mathbf{S} \quad (4.9)$$

where $\mathbf{U}$ is the vector of conserved variables, $\mathbf{F}$ and $\mathbf{G}$ are the convective fluxes in the x and y directions respectively, $\mathbf{S}_b$ is the bed-slope term and $\mathbf{S}$ is the source term. These vectors are defined as

$$\mathbf{U} = \begin{pmatrix} h \\ uh \\ vh \\ ch \\ z \end{pmatrix}, \quad \mathbf{F} = \begin{pmatrix} uh \\ u^2h + 0.5gh^2 \\ vuh \\ uch \\ \xi q_{bx} \end{pmatrix}, \quad \mathbf{G} = \begin{pmatrix} vh \\ uvh \\ v^2h + 0.5gh^2 \\ vch \\ \xi q_{by} \end{pmatrix} \quad (4.10)$$

$$\mathbf{S}_b = \begin{pmatrix} 0 \\ -gh \frac{\partial z}{\partial x} \\ -gh \frac{\partial z}{\partial y} \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{S} = \begin{pmatrix} 0 \\ -gh S_{fx} \\ -gh S_{fy} \\ E - D \\ \frac{D-E}{1-p} \end{pmatrix} \quad (4.11)$$

In this study, following the coupled approach developed in (Murillo & García-Navarro, [121]), a new flux term associated to the bed slope $\mathbf{S}_b$, normal to a direction given by the unit vector $\mathbf{n}$ is defined by

$$\mathbf{T} \cdot \mathbf{n} = (0, -gh \delta z n_x, -gh \delta z n_y, 0, 0)^T \quad (4.12)$$

Accordingly, one can get:

$$\mathbf{J}_n \delta \mathbf{U} = \delta F_n(\mathbf{U}) - \mathbf{S}_b = \delta(\mathbf{F} n_x + \mathbf{G} n_y) \cdot \mathbf{n} - \mathbf{T} \cdot \mathbf{n} \quad (4.13)$$

where $\mathbf{n}$ is the outward unit vector normal to the cell interfaces, n_x and n_y are components of unit normal vector $\mathbf{n}$ in the x and y directions, respectively. The analytical form of the Jacobian matrix $\mathbf{J}_n$ is written as

$$\mathbf{J}_n = \begin{pmatrix} 0 & n_x & n_y & 0 & 0 \\ -(uh)q^\perp/(h^2) + ghn_x & q^\perp/h + \frac{uh}{h}n_x & (uh)n_y/h & 0 & ghn_x \\ -(vh)q^\perp/(h^2) + ghn_y & (vh)n_x/h & q^\perp/h + (vh)n_y/h & 0 & ghn_y \\ -(ch)q^\perp/(h^2) & (ch)n_x/h & (ch)n_y/h & q^\perp/h & 0 \\ \tilde{A} & \tilde{B} & \tilde{C} & 0 & 0 \end{pmatrix} \quad (4.14)$$

in which $q^\perp = (uh)n_x + (vh)n_y$. $\tilde{A}$, $\tilde{B}$ and $\tilde{C}$ are

$$\begin{aligned}\tilde{A} &= -\frac{28\xi n_b^2 \sqrt{gd}}{\sqrt{(s-1)} |q|} \frac{q^\perp}{|q|} \left[\frac{n_b^2}{(s-1)d} \frac{|q|^2}{h^{7/3}} - \theta_c \right]^{1/2} \frac{|q|^2}{h^{10/3}} \\ \tilde{B} &= \frac{24\xi n_b^2 \sqrt{gd}}{h^{7/3} \sqrt{(s-1)}} (uh) \left[\frac{n_b^2}{(s-1)d} \frac{|q|^2}{h^{7/3}} - \theta_c \right]^{1/2} \frac{q^\perp}{|q|} - 8\xi \sqrt{g(s-1)d^3} \\ &\quad \left[\frac{n_b^2}{(s-1)d} \frac{|q|^2}{h^{7/3}} - \theta_c \right]^{3/2} \left[\frac{n_x}{|q|} - (uh) \frac{q^\perp}{|q|^3} \right] \\ \tilde{C} &= \frac{24\xi n_b^2 \sqrt{gd}}{h^{7/3} \sqrt{(s-1)}} (vh) \left[\frac{n_b^2}{(s-1)d} \frac{|q|^2}{h^{7/3}} - \theta_c \right]^{1/2} \frac{q^\perp}{|q|} - 8\xi \sqrt{g(s-1)d^3} \\ &\quad \left[\frac{n_b^2}{(s-1)d} \frac{|q|^2}{h^{7/3}} - \theta_c \right]^{3/2} \left[\frac{n_x}{|q|} - (vh) \frac{q^\perp}{|q|^3} \right]\end{aligned}$$

in which $|q| = \sqrt{(uh)^2 + (vh)^2}$.

The eigenvalues of $\mathbf{J}_n$ are the roots of the polynomial $f(\lambda, \mathbf{U})$ of the determinant $\left| \frac{\partial F_n(\mathbf{U})}{\partial \mathbf{U}} - \lambda \mathbf{I} \right|$, and the $f(\lambda, \mathbf{U})$ is calculated as:

$$f(\lambda, \mathbf{U}) = -\left(\lambda - \frac{q^\perp}{h}\right)^2 \left\{ \lambda^3 - \frac{2q^\perp}{h} \lambda^2 + \left[\left(\frac{q^\perp}{h}\right)^2 - g(h + h\tilde{D}) \right] \lambda + gh \left(\frac{q^\perp}{h} \tilde{D} - \tilde{E} \right) \right\} \quad (4.15)$$

where $\tilde{D} = \tilde{B}n_x + \tilde{C}n_y$ and $\tilde{E} = [(uh)\tilde{B} + (vh)\tilde{C}]/(6h)$. Two of the eigenvalues of $\mathbf{J}_n$ are $\lambda_1 = \lambda_2 = q^\perp/h$ from Eq. (4.15), the other three eigenvalues can be obtained by solving the following cubic:

$$P(\lambda) = \lambda^3 + a_1 \lambda^2 + a_2 \lambda + a_3 \quad (4.16)$$

in which $a_1 = -2q^\perp/h$, $a_2 = [(q^\perp/h)^2 - g(h + h\tilde{D})]$ and $a_3 = gh(q^\perp \tilde{D}/h - \tilde{E})$. We define

$$Q = \frac{1}{9} (3a_2 - a_1^2), \quad R = \frac{1}{54} (9a_1 a_2 - 27a_3 - 2a_1^3) \quad (4.17)$$

The governing system is strictly hyperbolic when condition $|q^\perp/h| < 6\sqrt{gh}$ is held (Cordier, Le, & De Luna, [45]), all three roots are real and unequal, and can be determined using the following definitions (Spiegel, [154])

$$\begin{aligned}\lambda_3 &= 2\sqrt{-Q} \cos\left(\frac{\gamma}{3}\right) - \frac{a_1}{3}, \\ \lambda_4 &= 2\sqrt{-Q} \cos\left(\frac{\gamma + 2\pi}{3}\right) - \frac{a_1}{3}, \\ \lambda_5 &= 2\sqrt{-Q} \cos\left(\frac{\gamma + 4\pi}{3}\right) - \frac{a_1}{3}\end{aligned} \quad (4.18)$$

in which $\cos \gamma = R/\sqrt{-Q^3}$.

4.4.2 Discretization of governing equations

In the present study, the coupled system comprising five governing equations is discretized on unstructured triangular grids using the finite volume method. Integrating the system over the control volume, one obtains

$$\int_{A_i} \frac{\partial \mathbf{U}}{\partial t} dA + \oint_{\Gamma} [\mathbf{F}_n(\mathbf{U}) - \mathbf{T}_n(\mathbf{U})] d\Gamma = \int_{A_i} \mathbf{S}(\mathbf{U}) dA \quad (4.19)$$

in which Γ denotes the boundary of control volume A_i , $\mathbf{F}_n(\mathbf{U}) = \mathbf{F}n_x + \mathbf{G}n_y$ and $\mathbf{T}_n(\mathbf{U}) = \mathbf{T} \cdot \mathbf{n}$. The line-integral terms in Eq. (4.19) can be discretized using the divergence theorem

$$\frac{\partial \mathbf{U}_i}{\partial t} = -\frac{1}{A_i} \sum_{k=1}^3 \mathbf{F}_{ik} l_{ik} + \mathbf{S}_i \quad (4.20)$$

in which l_{ik} is the length of the k th edge of A_i , and $\mathbf{F}_{ik}$ is numerical flux across l_{ik} :

$$\mathbf{F}_{ik} = \mathbf{F}^A[(\mathbf{U}_L)_{ik}, (\mathbf{U}_R)_{ik}] \quad (4.21)$$

in which $\mathbf{F}^A$ is a hyperbolic solver estimating the numerical fluxes through the cell interfaces, and $(\mathbf{U}_L)_{ik}$ and $(\mathbf{U}_R)_{ik}$ are the reconstructed values of $\mathbf{U}$ on the left and right sides of l_{ik} respectively.

4.4.3 Central-upwind method

Central-upwind schemes on general triangular grids for solving two-dimensional systems of conservation laws were developed in Kurganov & Petrova, ([90]); they offer the main advantages of the Godunov-type central schemes while keeping simplicity, universality, and robustness, and can be applied to problems with complicated geometries. Applying the central-upwind method, Eq. (4.21) can be written by

$$\mathbf{F}_{ik} = \frac{[S_r \vec{\mathbf{F}}(\mathbf{U}_L)_{ik} + S_l \vec{\mathbf{F}}(\mathbf{U}_R)_{ik}] \cdot \mathbf{n} - S_r S_l [(\mathbf{U}_R)_{ik} - (\mathbf{U}_L)_{ik}]}{(S_r + S_l)} \quad (4.22)$$

in which $\vec{\mathbf{F}}(\mathbf{U}_L)_{ik}$ and $\vec{\mathbf{F}}(\mathbf{U}_R)_{ik}$ are normal fluxes on the left and right sides of l_{ik} , respectively, and S_r and S_l are the one-sided local speeds of propagation on the right and left sides of l_{ik} respectively.

Wave speed estimator for the hydrodynamic model

For the hydrodynamic model, the conventional central-upwind method is used. The one-sided local speeds S_r and S_l are determined by seeking the absolute values of both the maximum non-negative eigenvalue and the maximum non-positive eigenvalue.

$$\begin{aligned} S_r &= \max\{\lambda_N [\mathbf{J}_n(\mathbf{U}_L)]_{ik}, \lambda_N [\mathbf{J}_n(\mathbf{U}_R)]_{ik}, 0\}, \\ S_l &= -\min\{\lambda_1 [\mathbf{J}_n(\mathbf{U}_L)]_{ik}, \lambda_1 [\mathbf{J}_n(\mathbf{U}_R)]_{ik}, 0\} \end{aligned} \quad (4.23)$$

where $\lambda_1 [\mathbf{J}_n(\mathbf{U})]_{ik} \leq \dots \leq \lambda_N [\mathbf{J}_n(\mathbf{U})]_{ik}$ being the N eigenvalues of the Jacobian matrix $\mathbf{J}_n(\mathbf{U})$ using the reconstructed variables $(\mathbf{U}_L)_{ik}$ or $(\mathbf{U}_R)_{ik}$.

Wave speed estimator for bed-load sediment transport

The conventional decoupled model using the straightforward central-difference method or the simple upwind method to approximate the bed-load fluxes, will introduce the numerical oscillations and instabilities. Thus, it requires a coupling strategy and a modern upwind scheme to stably approximate the bed-load fluxes, instead of the conventional decoupled method. However, in the real-world cases of fast-propagating flows, the wave speeds of the bed-load sediment transport are much smaller than the wave speeds of the hydrodynamic flows. Therefore, a lower diffusion-level than hydrodynamic solver is essentially required to apply the modern upwind method to the Exner-based equation. Using the same one-sided local speeds S_r of the hydrodynamic model, which are calculated in Eq. (4.23), will cause additional bed deformation deviating from the measured values, which are typically referred to as numerical diffusion.

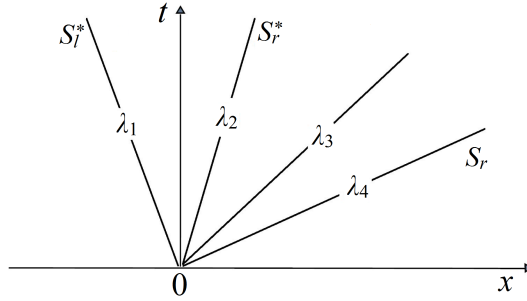


Figure 4.1: Typical characteristic waves of the system

A typical wave structure is illustrated in Fig. 4.1. To estimate the local wave-speeds of the bed-load sediment transport, in this paper we propose a novel wave-estimator using a different characteristic wave with a smaller shock wave speed to approximate the numerical fluxes of bed-load transport terms using the central-upwind method. The corresponding one-sided local speeds for bed-load sediment transport S_r^* and S_l^* are determined by

$$\begin{aligned} S_r^* &= \max\{\lambda_K [\mathbf{J}_n(\mathbf{U}_L)]_{ik}, \lambda_K [\mathbf{J}_n(\mathbf{U}_R)]_{ik}, 0\}, \\ S_l^* &= -\min\{\lambda_1 [\mathbf{J}_n(\mathbf{U}_L)]_{ik}, \lambda_1 [\mathbf{J}_n(\mathbf{U}_R)]_{ik}, 0\} \end{aligned} \quad (4.24)$$

with $\lambda_1 [\mathbf{J}_n(\mathbf{U})]_{ik} \leq \dots \leq 0 \leq \lambda_K [\mathbf{J}_n(\mathbf{U})]_{ik} \leq \dots \leq \lambda_N [\mathbf{J}_n(\mathbf{U})]_{ik}$ being the N eigenvalues of the Jacobian matrix $\mathbf{J}_n(\mathbf{U})$ using the reconstructed variables $(\mathbf{U}_L)_{ik}$ or $(\mathbf{U}_R)_{ik}$. Applying the central-upwind method with this novel local wave-estimator, the fluxes of bed-load sediment transport in the Exner-based equation (4.5) can be estimated by

$$\mathbf{F}_{ik} = \frac{[S_r^* \vec{\mathbf{F}}(\mathbf{U}_L)_{ik} + S_l^* \vec{\mathbf{F}}(\mathbf{U}_R)_{ik}] \cdot \mathbf{n} - S_r^* S_l^* [(\mathbf{U}_R)_{ik} - (\mathbf{U}_L)_{ik}]}{(S_r^* + S_l^*)} \quad (4.25)$$

In order to handle the situation where both S_r^* and S_l^* are zero or very close to zero, in this paper the flux computed in Eq. (4.25) is replaced by $\mathbf{F}_{ik} = 0.5[\vec{\mathbf{F}}(\mathbf{U}_L)_{ik} + \vec{\mathbf{F}}(\mathbf{U}_R)_{ik}]$, if $S_l^* + S_r^* < 10^{-6}$.

4.4.4 Linear reconstruction and multi-dimensional slope limiter

In the current study, the following reconstruction of unknowns is employed to acquire second-order spatial accuracy

$$\mathbf{U}(x, y) = \mathbf{U}_i + (\check{\mathbf{U}}_x)_i(x - x_i) + (\check{\mathbf{U}}_y)_i(y - y_i), \quad (x, y) \in A_i \quad (4.26)$$

where $\mathbf{U}(x, y)$ is the reconstructed value of variables at point (x, y) inside of cell i , x_i and y_i are the coordinates of the centre of cell i , $(\check{\mathbf{U}}_x)_i$ and $(\check{\mathbf{U}}_y)_i$ are component-wise approximations of numerical gradient $\nabla \check{\mathbf{U}}_i$ which are computed via the multidimensional limiter (Jawahar & Kamath, [83]), which is used to minimize the oscillations of the reconstructions:

$$\nabla \check{\mathbf{U}}_i = \Lambda_a \nabla \mathbf{U}_a + \Lambda_b \nabla \mathbf{U}_b + \Lambda_c \nabla \mathbf{U}_c \quad (4.27)$$

where $\nabla \mathbf{U}_a, \nabla \mathbf{U}_b$ and $\nabla \mathbf{U}_c$ are the three unlimited gradients chosen following (Pan & Cheng, [127]), (the calculation of these unlimited gradients is given in Section 4.4.5); the weights Λ_a, Λ_b and Λ_c are calculated by

$$\begin{aligned} \Lambda_a(g_a, g_b, g_c) &= \frac{(g_b g_c + \epsilon^2)}{(g_a^2 + g_b^2 + g_c^2 + 3\epsilon^2)} \\ \Lambda_b(g_a, g_b, g_c) &= \frac{(g_a g_c + \epsilon^2)}{(g_a^2 + g_b^2 + g_c^2 + 3\epsilon^2)} \\ \Lambda_c(g_a, g_b, g_c) &= \frac{(g_a g_b + \epsilon^2)}{(g_a^2 + g_b^2 + g_c^2 + 3\epsilon^2)} \end{aligned} \quad (4.28)$$

where g_a, g_b , and g_c are given by $g_a = \|\nabla \mathbf{U}_a\|_2^2$, $g_b = \|\nabla \mathbf{U}_b\|_2^2$, and $g_c = \|\nabla \mathbf{U}_c\|_2^2$. The parameter ϵ is a small number which is introduced to prevent indeterminacy caused by the vanishing of the three gradients, in regions of uniform flow.

4.4.5 Calculation of derivatives

In order to calculate the derivative in a cell i , e.g. the derivatives of U_i , a triangular zone formed by the centres of three adjacent cells is considered (the shaded zone in Fig. 4.2 with surface area A'_i). Using the divergence theorem,

$$\frac{\partial U_i}{\partial x} = \frac{1}{A'_i} \int_{A'} \frac{\partial U_i}{\partial x} dA'_i \approx \frac{(U_{i3} + U_{i2}) \Delta y_1 + (U_{i1} + U_{i3}) \Delta y_2 + (U_{i2} + U_{i1}) \Delta y_3}{2A'_i} \quad (4.29)$$

$$\frac{\partial U_i}{\partial y} = \frac{1}{A'_i} \int_{A'} \frac{\partial U_i}{\partial y} dA'_i \approx -\frac{(U_{i3} + U_{i2}) \Delta x_1 + (U_{i1} + U_{i3}) \Delta x_2 + (U_{i2} + U_{i1}) \Delta x_3}{2A'_i} \quad (4.30)$$

where

$$\begin{aligned} \Delta x_1 &= (x_{i3} - x_{i2}), \Delta x_2 = (x_{i1} - x_{i3}), \Delta x_3 = (x_{i2} - x_{i1}) \\ \Delta y_1 &= (y_{i3} - y_{i2}), \Delta y_2 = (y_{i1} - y_{i3}), \Delta y_3 = (y_{i2} - y_{i1}) \end{aligned}$$

This approach is used to calculate the unlimited gradients of variables.

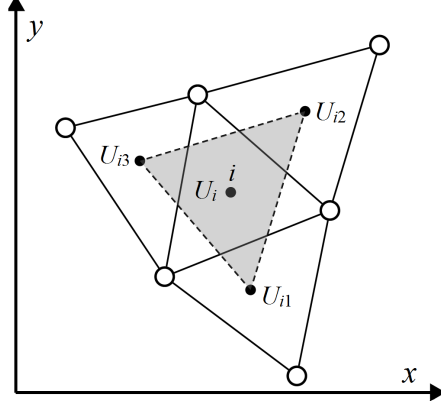


Figure 4.2: A sketch of the derivatives calculation

4.4.6 Time integration

In order to obtain a second-order accuracy in time, the two-stage explicit TVD Runge-Kutta method adopted in (Song, Zhou, Guo, Zou, & Liu, [150]) is applied in the current study. Since the numerical model is explicit, its stability can be controlled by a Courant-Friedrichs-Lewy (CFL)-type condition. For a triangular grid system, a CFL-type condition is applied to estimate an appropriate time step. It can be expressed as

$$\Delta t = C_r \min \left(\frac{R_i}{S_r} \right)_{ik} \quad i = 1, 2, \dots, N \quad (4.31)$$

where C_r is the Courant number specified in the range $0 < C_r \leq 1$, N is the total number of cells, R_i is the minimum distance from the centre to the edges of the i th triangular cell, and S_r is the right-sided local speed of propagation for the k th edge of cell i , which is estimated in Eq. (4.23).

4.5 Numerical tests

In this section, several test cases are performed, to evaluate the performance of the proposed coupled model. Test cases 4.5.1 and 4.5.4 compare the proposed method with decoupled models in which the bed-load fluxes in the Exner-based equation are solved separately by a simple upwind method, i.e.

$$\mathbf{F}_{ik} = \begin{cases} \vec{\mathbf{F}}(\mathbf{U}_L)_{ik} & \text{if } p^\perp \geq 0 \\ \vec{\mathbf{F}}(\mathbf{U}_R)_{ik} & \text{if } p^\perp < 0 \end{cases} \quad (4.32)$$

4.5.1 Dam failure due to over-topping flow

This test is chosen to verify the accuracy of the proposed model in predicting the erosion caused by a rapid flow on an initially irregular bed. The corresponding laboratory experiment was carried out and reported in (Tingsanchali & Chinnarasri, [162]). The tests were performed in a rectangular flume with a 35 m length, 1.0 m depth, and 1.0 m width. The height and crest-width of the dam were fixed at 0.80 m and 0.30 m, respectively. The upstream slope was fixed at 1V:3H, while the downstream slope (DIS) was varied. Sand, described as Sand I in (Tingsanchali & Chinnarasri, [162]), is used as the bed sediment in the current study. The mean grain size of Sand I is $d=1.13$ mm. In this test, DIS=1V:2.5H is chosen for the simulation. The reservoir inflow discharge is set at 1.23 l s^{-1} .

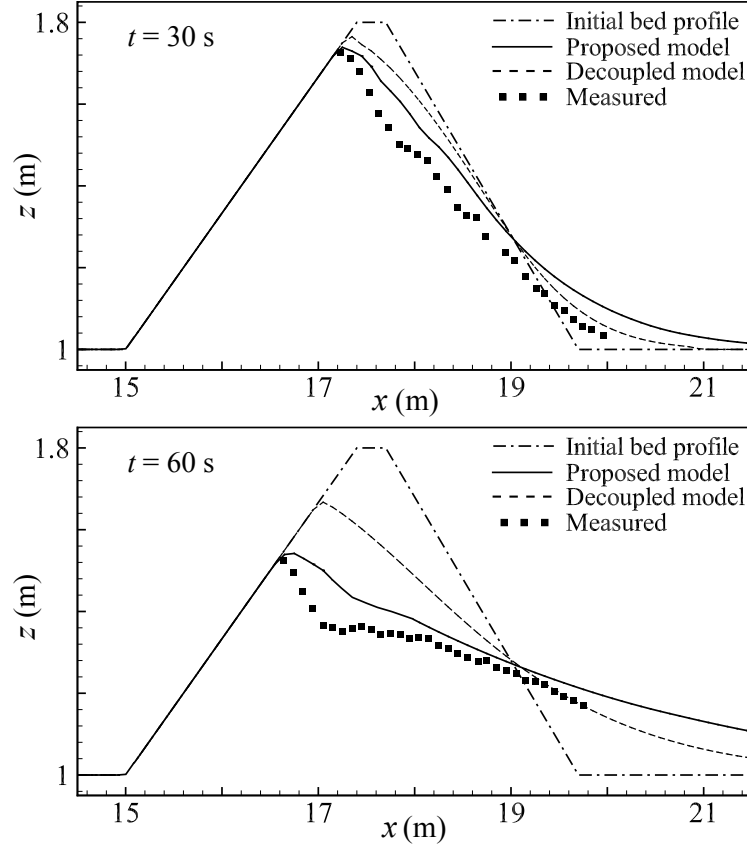


Figure 4.3: Measured and simulated bed profiles at $t=30$ s and 60 s

Only the bed-load transport is considered in this test. The computational domain is discretized by a grid of uniform cell size of 0.05 m. Figure 4.3 shows the simulated bed profiles from the proposed coupled model and a conventional decoupled model at 30 s and 60 s, respectively. These are also compared with the experimental measurements. It can be seen that the proposed model is able to more accurately predict the bed profile under a rapid varying flow than the conventional decoupled model which does not yield sufficient erosion under the fast over-topping flow.

4.5.2 Dam failure due to over-topping flow

In this test, an experiment with a dam-break flow over an erodible bed, as reported by Fraccarollo and Capart ([56]), is used. The experimental set concerns small-scale laboratory dam-break waves of initial water depth $h_0 = 10$ cm released over an erodible channel bed. The sediment consists of cylindrical PVC pellets with an average spherical diameter of 3.5 mm, a density of 1540 kg m^{-3} and a fall velocity $\omega_0 \approx 18 \text{ cm s}^{-1}$. The testing reach had a length of 2.5 m with an initial flat bed. The Manning roughness coefficient is set to $0.025 \text{ s m}^{-1/3}$. At the beginning, a 10 cm initial water depth is set upstream. The dam-break wave is then released by rapidly lifting the sluice gate. In the numerical simulation, the computational domain is discretized by grids of size 0.005 m. The total simulated time is $10t_0$, in which the hydrodynamic time scale $t_0 = \sqrt{h_0/g} \approx 0.101 \text{ s}$.

Figure 4.4 shows good agreement of the water surface and bed profile between the simulated and measured results. Compared with the numerical results from (Fraccarollo, Capart, & Zech, [57]), the current model shows a better agreement with the data measured upstream of the dam; the current model can precisely simulate the scouring hole at the position of the collapsed dam, with a better agreement for the front waves. The current model predicts excessive erosion in the downstream area, however, whereas the results from (Fraccarollo et al., [57]) showed a better agreement for bed level than the current model.

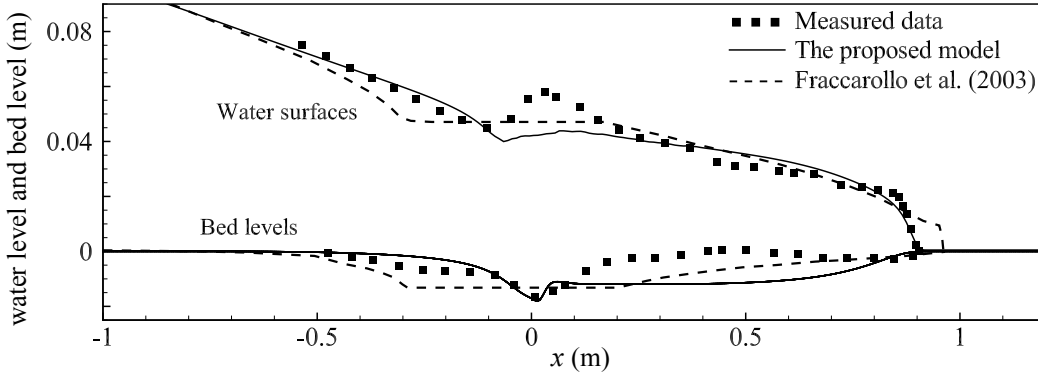


Figure 4.4: Simulated and measured water levels and bed profiles at $t=10t_0$

4.5.3 2-D dam-break flows over mobile bed with sudden enlargement

This test on a two-dimensional dam-break flow over an erodible bed with sudden enlargement followed the experiment carried out at the Hydraulic Laboratory of the Université catholique de Louvain (Belgium) that was reported by Palumbo, Soares-Frazão, Goutière, Pianese and Zech ([126]) and Goutiere, Soares-Frazão and Zech ([66]). The experiment was performed in a 6 m-long flume presenting a non-symmetrical sudden widening from 0.25 m to 0.5 m width, located 1 m downstream of the gate. The channel width is 0.25 m before $x=4$ m and 0.5 m after $x=4$ m.

The data at six locations, which are P1 (3.75 m, 0.125 m), P2 (4.2m, 0.125m), P3 (4.45 m, 0.125 m), P4 (4.95 m, 0.125 m), P5 (4.2m, 0.375m), and P6 (4.95m, 0.375m), are chosen in this test to be compared with the simulated results. In this test, measured cross-section profiles at three locations, which are at $x=4.2$ m, $x=4.3$ m and $x=4.5$ m, are chosen to verify the simulation.

An initial clear-water depth of 0.25 m is set upstream of the gate at $x = 3$ m. The initial conditions consist of a 0.1 m-thick horizontal layer of fully saturated sand along the entire length of the flume. The sediment used is uniform coarse sand with a median diameter of $d_{50} = 1.72$ mm and relative density of 2.63, deposited with a porosity of 0.3922.

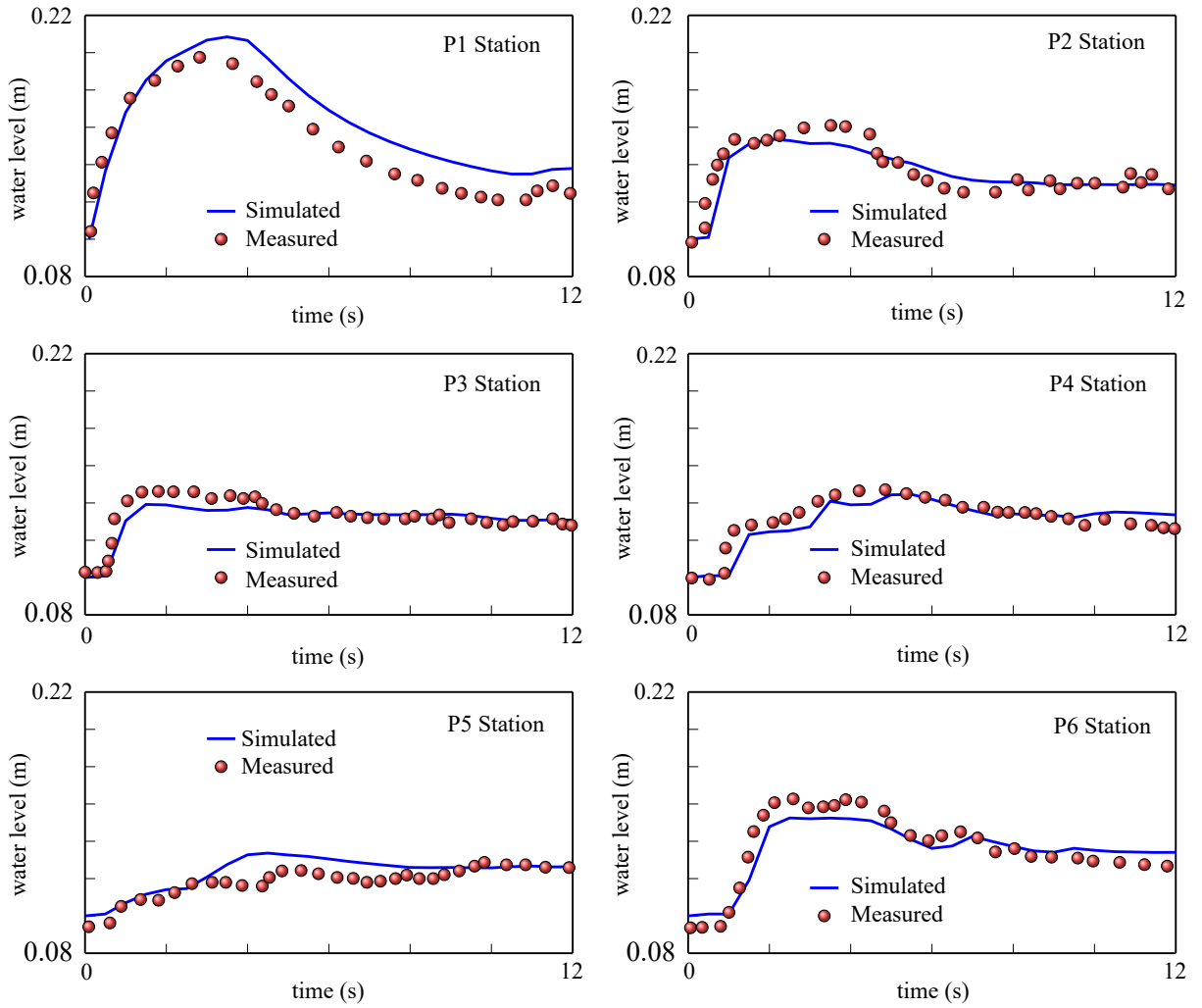


Figure 4.5: Comparisons between measured and simulated water-level evolutions at different stations

The domain is discretized into 9168 triangular cells with an average area of 2.32×10^{-4} m², which are refined at the section of enlargement. A standard closed boundary condition is applied to all side-walls, and a free-outflow boundary condition is applied at the downstream outlet. The Manning coefficient is set to be $0.018 \text{ s m}^{-1/3}$ in our test. In order to

compare the proposed method with the method used by Soares-Frazão and Zech ([149]), a first-order spatial reconstruction is implemented, the suspended sediment is ignored, and only bed-load transport is considered.

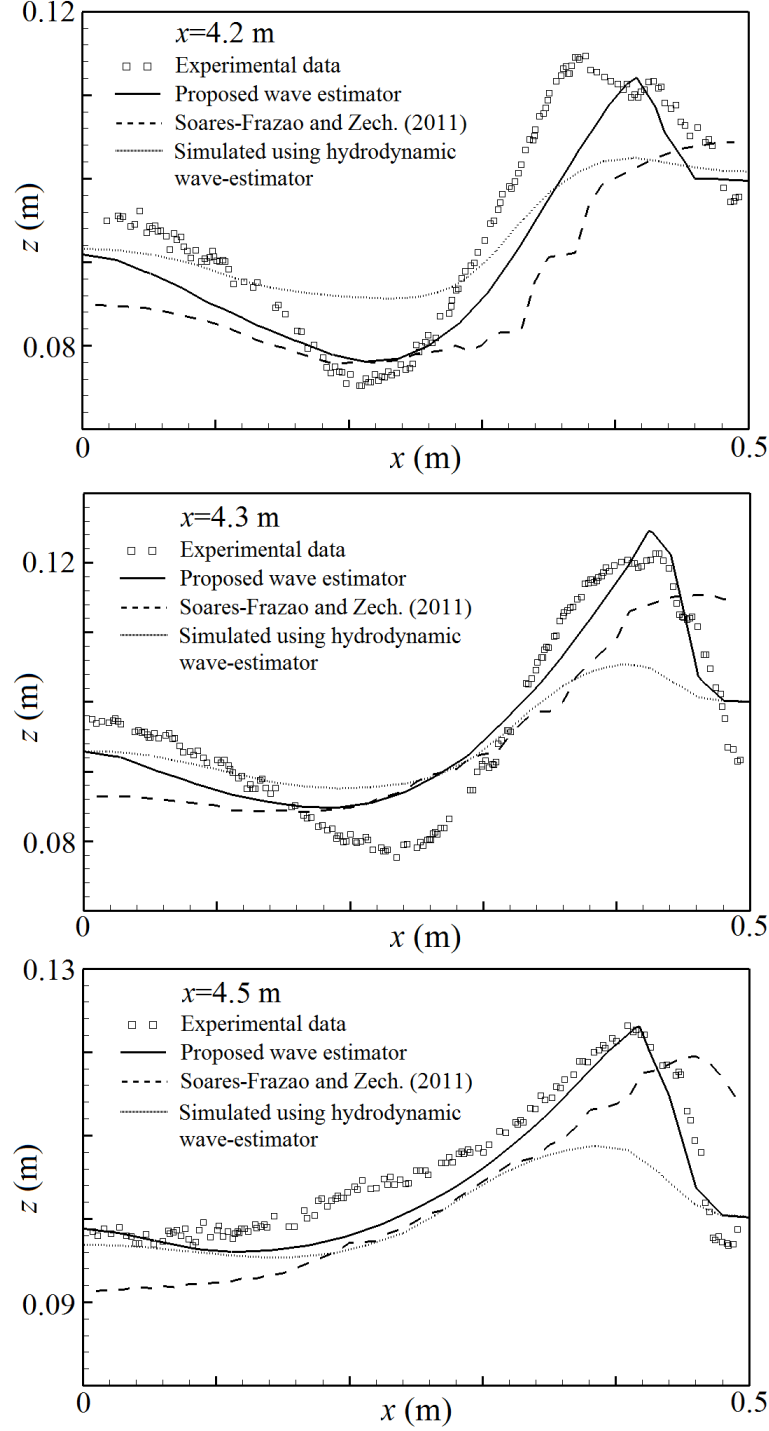


Figure 4.6: Simulated and measured bed profiles at different cross-sections ($x= 4.2$ m, 4.3 m, and 4.4 m) compared with the results from (Soares-Frazão & Zech, [149])

Figure 4.5 shows the comparison between predicted water levels and experimentally measured values at the aforementioned monitoring stations over the test period. The proposed model shows very good agreement with the measured water surface at monitoring stations, from which it can be concluded that the proposed model shows a desirable accuracy and is able to predict the water surface profile of a rapidly varying flow affected by bed evolution.

In Fig. 4.6, the results of simulated bed profiles from the proposed model with novel wave speed estimator and a coupled model using hydrodynamic wave estimator, in which the maximum shock celerity S_r is used for bed-load sediment fluxes, are shown at measured cross-sections at $x=4.2$ m, 4.3 m and 4.5 m, respectively.

It can be seen that, the coupled model using S_r (computed from Eq. (4.23) for hydrodynamic system) to compute bed-load fluxes introduces excessive diffusion and results in a flattened bed profile. The results from (Soares-Frazão and Zech, [149]), are also compared with our proposed model and the measured data. It can be seen that, comparing with the simulated results by Soares-Frazão and Zech ([149]), our proposed method can yield more accurate prediction of bed erosion; the positions of trenches and dunes predicted by the proposed method are more accurate. Moreover, it can be observed that the simulated bed-profile from the proposed model is accurate and sharp, while the results from (Soares-Frazão & Zech, [149]) seem more diffusive. The reason may be explained as follows. In the proposed sediment wave speed estimator, the smallest positive characteristic wave from the five characteristic waves is chosen. Therefore, when bed-evolution in the x -direction is dominant, minimum diffusion is introduced in the y -direction. However, in the wave-speed estimator from Soares-Frazão and Zech ([149]), the width of the Riemann fan is larger than what is needed, and thus it introduces excessive diffusion in practical cases.

4.5.4 2-D dam-break flow over an erodible bed

To further examine the numerical performance of the present scheme, a benchmark test of a 2-D dam-break problem with a rapidly varying unsteady flow is chosen in this paper. This test case was first introduced by Fennema and Chaudhry ([53]) in a numerical method study, which has been widely used by many researchers (Anastasiou & Chan, [6]; Jawahar & Kamath, [83]; Wang & Liu, [175]). The original case was mostly applied on an initially wet and fixed bed. It is modified in the present study with the aim of testing the ability of the current model to simulate front wave propagation over an initially dry bed with wet-dry interface tracking, with particular attention to the 2-D aspects of the flow motion; and the model's performance in predicting rapid erosion under a highly energetic flow on a large scale. The modelling domain area is a 200 m×200 m basin with a flat, dry bed. A 10 m-thick dam splits the basin into two equally-sized regions. The clear water depths are 7 m and 0.5 m on the left and right sides of the dam, respectively. At $t=0$ s, a 75 m-wide breach centered at $y=125$ m is assumed to form instantaneously. The duration of the simulation is 10 s. The initial velocity of the whole modelling area is 0m/s and the outlet boundary at $x=200$ m is specified with a free outflow boundary condition; meanwhile, all other boundaries are set to a standard wall condition. The computational domain is discretized into 5134 triangles with an average area of 7.53 m².

Figure 4.7 shows the meshes generated for this test. Figure 4.8 shows the 3-D views of the simulated water surface on a fixed bed at $t=10$ s. In Fig. 4.8, it can be seen that a shock wave forms and propagates downstream and a depression wave spreads upstream during the simulation process. The predicted water profile with a fixed bed from the proposed model closely agrees with other researchers' results (Anastasiou & Chan, [6]; Fennema & Chaudhry, [53]; Jawahar & Kamath, [83]; Wang & Liu, [175]).

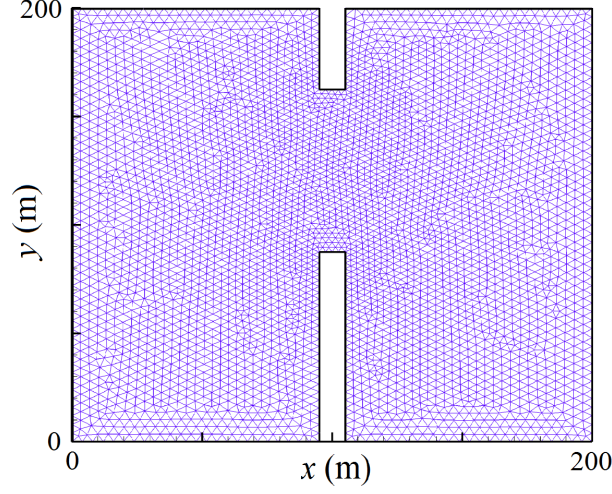


Figure 4.7: Discretized meshes

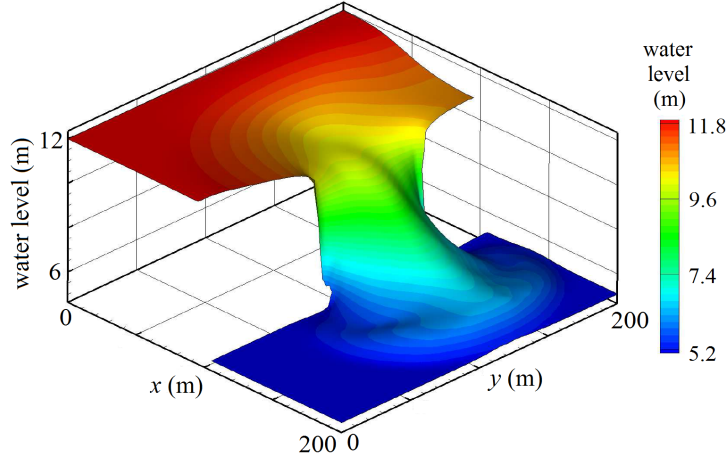


Figure 4.8: Simulated water level on a fixed bed at $t=10$ s using the proposed model

Figure 4.9 shows the simulated bed profile at $t=10$ s for both the proposed model and a conventional decoupled model. It can be observed that the erosion depths of both models basically agree with each other, since the sediment entrainment E and deposition D terms in Eq. (4.5) dominate the erosion depth. However, since bed-load transport exists as a numerical flux, it actually control the oscillation of the simulated bed-profile. The bed erosion from the proposed model in Fig. 4.9a shows very good non-oscillatory form during the simulation process while the conventional decoupled model shows clear oscillations at the corners of the broken dam (Fig. 4.9b), which is incorrect.

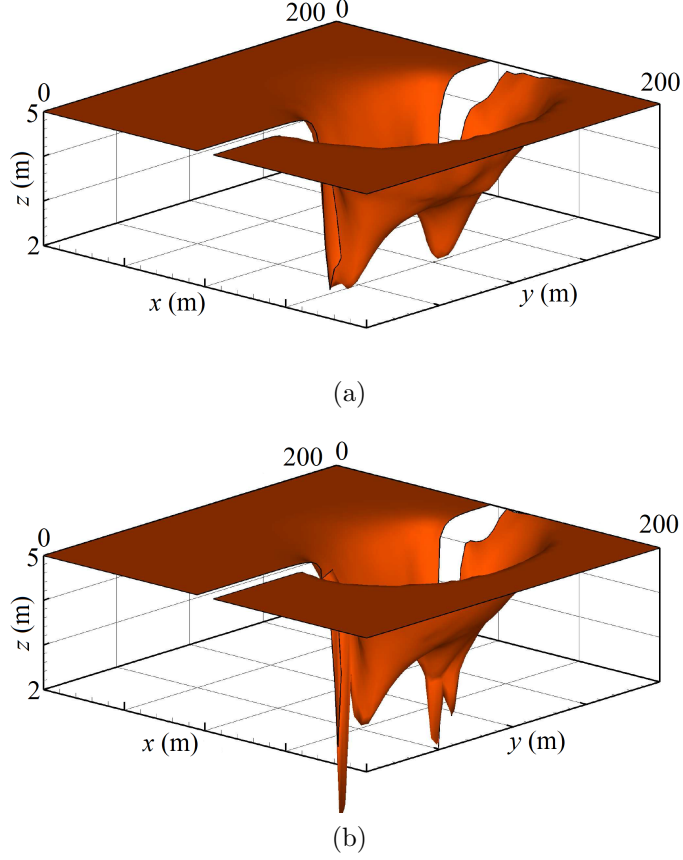


Figure 4.9: Simulated bed profiles z at $t=10$ s from (a) the proposed model and (b) the decoupled model

4.6 Conclusion

In this paper, an accurate and stable two-dimensional coupled numerical model involving the bed-erosion process is proposed to simultaneously predict rapidly varying flow, sediment transport and rapid bed evolution using the finite volume method based on unstructured triangular grids. To account for the relatively strong interaction effects between the water and the bed-evolution under a rapidly varying flow, a coupled strategy is adopted, and the bed-load sediment transport terms are solved as fluxes by the central-upwind method. In the numerical tests, the proposed model shows enhanced performance compared with the conventional decoupled model and previously reported coupled models.

A novel wave estimator in conjunction with the central-upwind method is proposed and successfully applied to the coupled water-sediment system involving a rapid bed-erosion process. It was demonstrated that, in comparison with the decoupled model, applying the central-upwind method to approximate the bed-load fluxes can successfully avoid the numerical oscillations caused by simple and less stable schemes, e.g., simple upwind methods (see test 4.5.4). This illustrates the advantage of a coupled approach for bed-load transport under rapid flows. However, it was also found that excessive bed deformation (shown in Fig. 4.6) could be introduced by using the same one-sided local wave speed S_r as in the

hydrodynamic model in the solver of bed-load fluxes (see test 4.5.3). This implies that the largest non-negative eigenvalue of the coupled system will cause excessive numerical diffusion to bed-evolution and can only be used for hydrodynamic model in the central-upwind method. Based on the fact that the bed-deformation speed is much smaller than the speed of the fluid, even for fast-propagating flows, we proposed a novel wave speed estimator seeking the smallest non-negative eigenvalue of coupled system as characteristic wave speed (S_r^*) in the central-upwind method to compute the bed-load fluxes. Based on the results of the numerical tests, the proposed scheme can predict accurate results for both the hydrodynamic system and the morphodynamic system, with better agreement with the experimental data than the decoupled and coupled models using S_r . Consequently, the proposed scheme has advantages in terms of accuracy and stability which are shown in the numerical tests. In addition, we have provided analytical expressions for calculating the eigenvalues of the coupled shallow-water-Exner system, which greatly enhances the efficiency of the proposed method.

The current study shows that the proposed scheme provides better accuracy and stability than conventional decoupled methods and can avoid excessive numerical diffusions and oscillations.

Chapter 5

Three-dimensional modeling of non-hydrostatic free-surface flows on unstructured grids

5.1 Abstract

A three-dimensional numerical model is presented for the simulation of unsteady non-hydrostatic shallow water flows on unstructured grids using the Finite volume method. The free surface variations are modeled by a characteristics-based scheme which simulates sub- and super-critical flows. Three-dimensional velocity components are considered in a collocated arrangement with a σ -coordinate system. A special treatment of the pressure term is developed to avoid the water surface oscillations. Convective and diffusive terms are approximated explicitly, and an implicit discretization is used for the pressure term to ensure exact mass conservation. The unstructured grid in the horizontal direction and the σ coordinate in the vertical direction facilitate the use of the model in complicated geometries. Solution of the non-hydrostatic equations enables the model to simulate short-period waves and vertically circulating flows.

Keywords: Three-dimensional, non-hydrostatic, free surface flow, unstructured grid, finite volume method.

5.2 Introduction

Shallow-water (SW) equations are usually used for the simulation of rivers, estuaries, lakes and other hydraulic situations where the vertical dimensions are much smaller than the horizontal ones and also where the flow is vertically mixed. Three-dimensional (3D) models are necessary when the vertical structure is important for other purposes like sediment or pollutant transport. There are also several other cases, like density currents, where vertical profiles are variable, and since the depth-averaged equations are not good enough to specify the flow conditions, a 3D model is needed [173].

Hydrostatic pressure is popularly assumed in most 3D SW models because of its simplicity and the substantial reduction of computational cost. Several 3D hydrostatic models have been presented in the literature, such as the Princeton Ocean model [18], which uses an explicit scheme with a σ -coordinate system. (See also [35], for a stable semi-implicit scheme). However, in some cases, like vertically circulating flows and short-period waves, hydrostatic pressure assumption fails, and non-hydrostatic models are necessary.

The fractional step method is used in most non-hydrostatic models, where the hydrostatic component is calculated first and the hydrodynamic pressure calculated in a subsequent step. Stansby and Zhou [157] developed a pressure corrective algorithm which uses a SIMPLE-type algorithm [128] for the calculation of hydrodynamic pressure. Pressure correction algorithms have also been used by Zhou [190] and Ouillon and Dartus [125] for the simulation of non-hydrostatic flows. Pressure correction algorithms generally have a low rate of convergence with a large aspect ratio, as in most hydrodynamic cases where the vertical scale is much smaller than the horizontal one. Namin et al. [122] developed an efficient implicit non-hydrostatic 2D vertical model in which water surface elevation, hydrodynamic pressure and velocity components are calculated simultaneously via a block tri-diagonal implicit system of equations. This scheme has a high stability because of implicit discretization of source terms, and also the method is free of iteration.

Due to increases in computing abilities, 3D non-hydrostatic models have become feasible, and a number of numerical models have been developed. Li and Yu [?] developed a quasi-3D model in which the solution procedure is split into convection, diffusion and propagation steps. Casulli and Stelling [36] developed a 3-D quasi-hydrostatic model where the pressure terms were decomposed into hydrostatic and non-hydrostatic components. Casulli [34] developed an accurate semi-implicit finite difference method in which a correction for the free-surface is performed such that the new free-surface elevation is implicitly coupled with the non-hydrostatic pressure and with the new velocity field. He showed that this scheme is unconditionally stable with respect to the gravity wave speed, wind stress, vertical viscosity and bottom friction. Kocyigit et al. [85] developed a fractional step algorithm in which convective terms are discretized explicitly and a Poisson equation is solved for the calculation of hydrodynamic pressure. Lin and Li [106] developed a σ -coordinate model with an operator-splitting approach and showed that energy is conserved with that approach. They derived the modified analytical form of the Poisson equation of pressure in a σ -coordinate system and used it for the calculation of hydrodynamic pressure. Chen [41] developed a two-step algorithm in which each stage consists of a two-step predictor-corrector scheme and used a Poisson equation for non-hydrostatic pressure. Ai et al. [3] developed a non-hydrostatic 3-D free surface flow model on a new grid arrangement under a horizontal Cartesian grid framework. However, it is limited to structured grids.

All of the above-mentioned models use a structured grid with a staggered arrangement of velocity components [69]. A problem with the structured grid is that local grid refinement is not straightforward. Also, a staggered grid in a curvilinear coordinate system is too expensive, and most staggered grids generally use the orthogonal curve-linear coordinate, which is not so flexible for general applications.

An unstructured grid is a helpful tool to overcome the above mentioned problems.

Combined with a σ -coordinate system in the vertical direction, it is an attractive mesh for practical applications. Several studies have been reported in the last decade. Casulli and Walters [37] developed an unstructured 3D model. Chen et al. [40] developed a 3D, primitive equations ocean model on unstructured grids. Boscheri et al. [20] developed a semi-implicit scheme for 3D free surface flows with a high-order velocity reconstruction on unstructured voronoi meshes. Above-mentioned studies are based on hydrostatic pressure assumption. Casulli and Zanolli [38] developed a semi-implicit scheme on unstructured grids for non-hydrostatic free-surface flows. Cea et al. [39] proposed a non-hydrostatic 3D free surface layer-structured finite volume model for short wave propagation, in which the flow equations are discretized in a collocated mesh, which is unstructured in the horizontal plane, but structured in the vertical direction. Ai and Jin [4] developed a 3D non-hydrostatic model for free-surface flows with unstructured grids. Fringer et al. [58] proposed a nonhydrostatic, parallel coastal ocean simulator using finite-volume method on unstructured grids.

The objective of this paper is to develop a non-hydrostatic 3D model with an unstructured grid in the horizontal direction and a sigma coordinate system in the vertical direction. With this type of grid, the model benefits from the properties of the unstructured grid in the horizontal direction, like modeling of complicated geometries and local grid refinement. Also, the σ -coordinate system enables the model to efficiently simulate flows with topographic changes, and temporal variations of the computational grid in the vertical direction are easily incorporated into the equations. A problem with an unstructured grid is that classical finite difference algorithms, which generally use efficient ADI algorithms, cannot be used. Special attention should be paid to the simulation of water surface elevations, which generally exhibits a hyperbolic behaviour, and classical schemes fail to simulate it in all conditions. In order to overcome this problem, in this paper a characteristic-based finite volume scheme is used to simulate the water surface variations. Another problem is that, in contrast to structured grids, a staggered arrangement of velocities leads to a very expensive model and is not feasible in finite volume methods. So in this paper, a collocated arrangement of variables is employed. A special treatment is used to avoid the well-known checkerboard problem, and an efficient diagonal system of equations is developed for calculating the hydrodynamic pressure.

The structure of this paper is as follows. In Section 2, the 3D shallow-water equations are presented. In Section 3, the numerical scheme for different parts of the model is introduced and the pressure equation is derived. In Section 4, the performance of the proposed numerical method is verified by different test cases in which the simulated results are compared with exact solutions.

5.3 The three-dimensional free-surface flow equations

Three-dimensional free-surface flows can be described by the Navier-Stokes equations, which express the physical principle of conservation of mass and momentum.

The mass conservation is expressed by the following incompressibility condition:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \quad (5.1)$$

where x , y and z are the Cartesian coordinates, and u , v and w are velocity components.

Integrating the continuity equation (5.1) over the depth and applying the kinematic boundary condition leads to the following free-surface (continuity) equation:

$$\frac{\partial \eta}{\partial t} + \frac{\partial}{\partial x} \int_{-h}^{\eta} u dz + \frac{\partial}{\partial y} \int_{-h}^{\eta} v dz = 0 \quad (5.2)$$

where t is the time, η is the water surface above the horizontal datum, and $h(x, y)$ is the elevation of the bed below the datum.

Total pressure P is the combination of hydrodynamic pressure p and hydrostatic pressure. Thus, the pressure gradient can be written as:

$$\frac{\partial P}{\partial x} = \frac{\partial p}{\partial x} + \rho g \frac{\partial \eta}{\partial x} \quad (5.3)$$

where p is the hydrodynamic pressure, g is the gravitational acceleration and ρ is the fluid density. Substituting (5.3) in the Navier-Stokes equations, the momentum conservation can be expressed as:

$$\frac{\partial u}{\partial t} + \frac{\partial(uu)}{\partial x} + \frac{\partial(uv)}{\partial y} + \frac{\partial(uw)}{\partial z} = f v - g \frac{\partial \eta}{\partial x} - \frac{1}{\rho} \frac{\partial p}{\partial x} + \frac{\partial}{\partial x} (\nu_h \frac{\partial u}{\partial x}) + \frac{\partial}{\partial y} (\nu_h \frac{\partial u}{\partial y}) + \frac{\partial}{\partial z} (\nu_v \frac{\partial u}{\partial z}) \quad (5.4)$$

$$\frac{\partial v}{\partial t} + \frac{\partial(vu)}{\partial x} + \frac{\partial(vv)}{\partial y} + \frac{\partial(vw)}{\partial z} = -f u - g \frac{\partial \eta}{\partial y} - \frac{1}{\rho} \frac{\partial p}{\partial y} + \frac{\partial}{\partial x} (\nu_h \frac{\partial v}{\partial x}) + \frac{\partial}{\partial y} (\nu_h \frac{\partial v}{\partial y}) + \frac{\partial}{\partial z} (\nu_v \frac{\partial v}{\partial z}) \quad (5.5)$$

$$\frac{\partial w}{\partial t} + \frac{\partial(wu)}{\partial x} + \frac{\partial(wv)}{\partial y} + \frac{\partial(ww)}{\partial z} = -\frac{1}{\rho} \frac{\partial p}{\partial z} + \frac{\partial}{\partial x} (\nu_h \frac{\partial w}{\partial x}) + \frac{\partial}{\partial y} (\nu_h \frac{\partial w}{\partial y}) + \frac{\partial}{\partial z} (\nu_v \frac{\partial w}{\partial z}) \quad (5.6)$$

where f is the Coriolis parameter and ν_h and ν_v are the kinematic eddy viscosity coefficients in the horizontal and vertical directions, respectively.

The transformation between the Cartesian coordinate system and the σ -coordinate system is expressed by the following relationship:

$$\sigma = \frac{z - h(x, y)}{D(x, y, t)} \quad (5.7)$$

where σ is the transformed vertical coordinate and $D(x, y, t)$ is the water depth calculated by:

$$D(x, y, t) = \eta(x, y, t) - h(x, y) \quad (5.8)$$

, $\eta(x, y, t)$ is the water surface level and $h(x, y)$ is the bed level.

The transformed momentum equations in the (x, y, σ) coordinates system can be written in the following matrix form:

$$\frac{\partial \mathbf{u}}{\partial t} + \nabla \mathbf{f} + \mathbf{p} = \mathbf{s} \quad (5.9)$$

where

$$\mathbf{u} = \begin{pmatrix} Du \\ Dv \\ Dw \end{pmatrix} \quad (5.10)$$

is the vector of variables, and $\mathbf{f}$ is the flux vector,

$$\mathbf{f} = \mathbf{c} + \mathbf{d} \quad (5.11)$$

where $\mathbf{c}$ and $\mathbf{d}$ are the convective and diffusive flux vectors respectively, as follows:

$$\mathbf{c} = \left[\begin{pmatrix} Duu \\ Dvu \\ Dwu \end{pmatrix}, \begin{pmatrix} Duv \\ Dvv \\ Dwv \end{pmatrix}, \begin{pmatrix} Du\omega \\ Dv\omega \\ Dw\omega \end{pmatrix} \right] \quad (5.12)$$

and

$$\mathbf{d} = \left[\begin{pmatrix} -D\nu_h \partial u / \partial x \\ -D\nu_h \partial v / \partial x \\ -D\nu_h \partial w / \partial x \end{pmatrix}, \begin{pmatrix} -D\nu_h \partial u / \partial y \\ -D\nu_h \partial v / \partial y \\ -D\nu_h \partial w / \partial y \end{pmatrix}, \begin{pmatrix} -(\nu_v/D) \partial u / \partial \sigma \\ -(\nu_v/D) \partial v / \partial \sigma \\ -(\nu_v/D) \partial w / \partial \sigma \end{pmatrix} \right]. \quad (5.13)$$

$\mathbf{s}$ is the vector of the source terms

$$\mathbf{s} = \begin{pmatrix} fDv - Dg\partial\eta/\partial x \\ -fDu - Dg\partial\eta/\partial y \\ 0 \end{pmatrix} \quad (5.14)$$

and $\mathbf{p}$ is the pressure gradients vector

$$\mathbf{p} = \begin{pmatrix} (D/\rho)\partial p/\partial x \\ (D/\rho)\partial p/\partial y \\ (1/\rho)\partial p/\partial \sigma \end{pmatrix}. \quad (5.15)$$

ω is the transformed vertical velocity in σ -coordinate system, which is defined as follows:

$$\omega = D \frac{d\sigma}{dt}. \quad (5.16)$$

ω and w are related via the following relationship:

$$\omega = w - u \left(\sigma \frac{\partial D}{\partial x} + \frac{\partial \eta}{\partial x} \right) - v \left(\sigma \frac{\partial D}{\partial y} + \frac{\partial \eta}{\partial y} \right) - \left(\sigma \frac{\partial D}{\partial t} + \frac{\partial \eta}{\partial t} \right) \quad (5.17)$$

The vertical eddy viscosity can be estimated by a two-layer mixing length model [132] as:

$$\nu_v = l_m^2 \left[\left(\frac{\partial u}{\partial z} \right)^2 + \left(\frac{\partial v}{\partial z} \right)^2 \right]^{1/2} \quad (5.18)$$

where l_m is the mixing length which can be approximated as follows:

$$\begin{aligned} l_m &= \kappa z & \text{for } \kappa z \leq 0.1H, \\ l_m &= 0.1H & \text{for } \kappa z > 0.1H. \end{aligned} \quad (5.19)$$

where $\kappa=0.4$ is the Von Karman constant.

5.4 Numerical solution procedure

The numerical solution procedure consists of solution of the depth-integrated continuity equation (5.2), calculation of the water surface elevation, calculation of the hydrostatic pressure gradient by a characteristics-based method, calculation of the hydrodynamic pressure from the incompressibility condition (5.1) and finally, solution of the momentum equations (5.9) and calculation of the velocity components, which shall be explained in detail in this section.

5.4.1 Computational grid and arrangement of variables

The computational grid consists of prismatic control volumes with triangular horizontal faces. All three dimensional variables are located at the center of each prism. The center of each prism is assumed to be the mid-point of the line connecting the centers of the top and bottom triangular faces of the prismatic control volume. A schematic view of a control volume is presented in Figure 5.1.

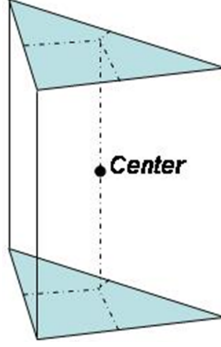


Figure 5.1: A schematic view of a control volume.

Since the equations are transformed to the sigma coordinate system, the heights of control volumes are variable in time. A vertical section of the grid is shown in Figure 5.2. The transformed vertical sizes of the control volumes are constant in time; however, the transformed vertical sizes of different levels are not necessarily equal and could be variable, as depicted in Figure 5.2.

5.4.2 The depth-integrated continuity equation

In order to calculate the water surface from the depth-integrated continuity equation, the domain is considered as one-layer. The depth-integrated continuity equation can be integrated once more in every control volume as:

$$\int_v \left(\frac{\partial \eta}{\partial t} + \nabla \mathbf{E} \right) dv = 0, \quad (5.20)$$

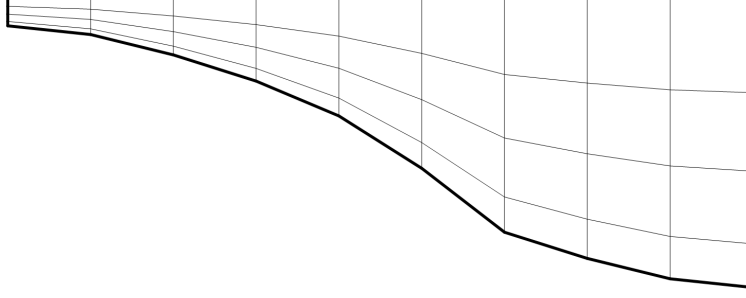


Figure 5.2: A vertical section of the grid and σ -coordinate system.

where $\mathbf{E} = \left(\int_{-h}^{\eta} u dz, \int_{-h}^{\eta} v dz \right)$ is the flux vector.

The volume integral can be transformed into a surface integral as:

$$\int_v \frac{\partial \eta}{\partial t} dv + \oint_s (\mathbf{E} \cdot \mathbf{n}) ds = 0, \quad (5.21)$$

where $\mathbf{n}$ is the normal vector of each edge. The surface integral is approximated by summation on the triangle edges as:

$$\oint_s (\mathbf{E} \cdot \mathbf{n}) ds = \sum_{k=1}^3 (E_k \cdot n_k) ds_k \quad (5.22)$$

The flux vector $\mathbf{E}$ on the cell interfaces (edges) can be calculated as:

$$\mathbf{E} \cdot \mathbf{n} = \frac{1}{l_{ik}} [\alpha_L \mathbf{E}_L + \alpha_R \mathbf{E}_R] \cdot \mathbf{n} \quad (5.23)$$

where $\mathbf{E}_L = \mathbf{E}(u_L, v_L)$, $\mathbf{E}_R = \mathbf{E}(u_R, v_R)$ are the left and right flux vectors. The subscripts R, L represent the evaluations on the right- and left-hand sides of the interface, respectively. l_{ik} is the distance between barycenter of two adjacent triangles, α_L and α_R are distance from the barycenter to common interface in two adjacent triangles, respectively.

Substituting (5.23) into (5.22) and discretizing the temporal derivative by the forward Euler explicit scheme, the following equation is achieved for calculation of the water surface:

$$\eta_i^{n+1} = \eta_i^n - 0.5 \frac{\Delta t}{\vartheta_i} \sum_{k=1}^3 \left[\frac{1}{l_{ik}} (\alpha_L E_L + \alpha_R E_R)_k n_k \right] ds_k, \quad (5.24)$$

where ϑ_i is the volume of each control volume.

5.4.3 The hydrostatic pressure gradients

The hydrostatic pressure gradients ($g \partial \eta / \partial x$ and $g \partial \eta / \partial y$) are calculated separately in each triangle to use in the 3D momentum equations. They can be approximated using the following equations:

$$g \frac{\partial \eta}{\partial x} = \frac{g}{A_i} \sum_{k=1}^3 \hat{\eta}_k (x_2^k - x_1^k), \quad g \frac{\partial \eta}{\partial y} = -\frac{g}{A_i} \sum_{k=1}^3 \hat{\eta}_k (y_2^k - y_1^k) \quad (5.25)$$

where A_i is the area of each triangle, $\hat{\eta}_k$ is the approximate water surface elevation on each edge, and x_1^k, y_1^k, x_2^k and y_2^k are geometrical coordinates of each edge k , as defined in Figure 5.3.

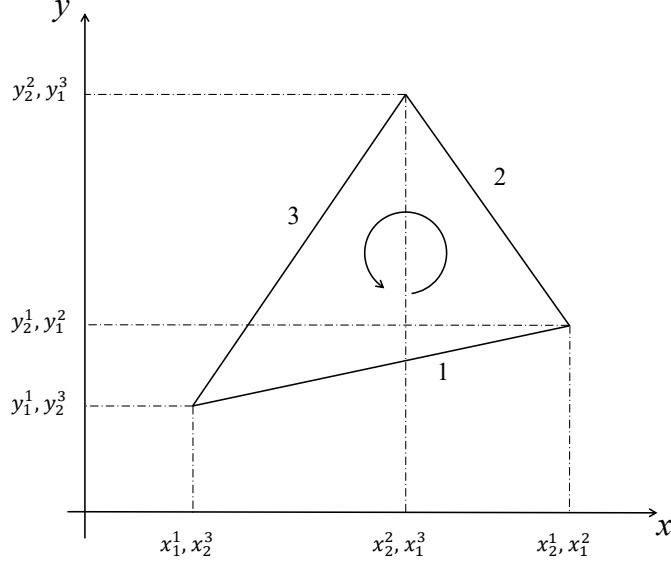


Figure 5.3: Nomination of geometrical coordinates of each edge.

The approximate water surface elevation $\hat{\eta}_k$ can be calculated by either an exact or approximate Riemann solver. For the sake of simplicity, in this paper an approximate Riemann solver is used [163], which has the following form:

$$D_* = \frac{1}{g} \left[\frac{1}{2} \left(\sqrt{gD_L} + \sqrt{gD_R} \right) + \frac{1}{4} (U_L + U_R) \right]^2, \quad (5.26)$$

where U is the normal velocity at the edge and D_* is the approximate value of the water depth on edge. Having the water depth and the bed elevation on each edge, the approximate water surface elevation $\hat{\eta}_k$ can be easily determined.

5.4.4 Numerical discretization of the momentum equations

Integrating (5.9) over a prismatic control volume, the integral form of the momentum equations is obtained:

$$\int_v \left(\frac{\partial \mathbf{u}}{\partial t} + \nabla \mathbf{f} + \mathbf{p} - \mathbf{s} \right) dv = 0 \quad (5.27)$$

Transforming the volume integral into a surface integral, (5.27) leads to the following form:

$$\frac{\partial \mathbf{u}}{\partial t} \vartheta + \oint_s (\mathbf{f} \cdot \mathbf{n}) ds + \mathbf{p} \vartheta - \mathbf{s} \vartheta = 0 \quad (5.28)$$

In this paper, the temporal derivative is discretized by the forward Euler explicit scheme, the hydrodynamic pressure is approximated implicitly and all other terms are approximated explicitly as follows:

$$\frac{\mathbf{u}^{n+1} - \mathbf{u}^n}{\Delta t} \vartheta + \left[\oint_s (\mathbf{f} \cdot \mathbf{n}) ds \right]^n + \mathbf{p}^{n+1} \vartheta - \mathbf{s}^n \vartheta = 0 \quad (5.29)$$

The surface integral on the control volume faces is approximated as follows:

$$\oint_s (\mathbf{f} \cdot \mathbf{n}) ds = \sum_{k=1}^5 (\mathbf{f} \cdot \mathbf{n})_k A_k \quad (5.30)$$

where A_k is the area of each face of the control volume.

Applying the upwind method, the convective momentum flux is calculated in the following form:

$$\mathbf{c} \cdot \mathbf{n} = 0.5 [Q \mathbf{u}'_L + Q \mathbf{u}'_R + |Q| (\mathbf{u}'_R - \mathbf{u}'_L)] \quad (5.31)$$

where Q is the flow flux through each face, which is calculated as:

$$Q = \mathbf{u} \cdot \mathbf{n} \quad (5.32)$$

where $\mathbf{u}'$ is the vector of variables in the sigma coordinate system and

$$\mathbf{u}' = \begin{pmatrix} u \\ v \\ \omega \end{pmatrix}, \quad (5.33)$$

The horizontal and vertical diffusion fluxes are calculated by the central differencing scheme:

$$\mathbf{d} \cdot \mathbf{n} = \frac{1}{l_{ik}} [\alpha_L \mathbf{d}_L + \alpha_R \mathbf{d}_R] \cdot \mathbf{n} \quad (5.34)$$

5.4.5 Increasing the order of spatial accuracy

The values of the variables at the left and right sides of the interface are needed to compute the numerical fluxes. Various methods have been proposed for developing accurate higher-order upwind models. Here, we use a MUSCL-type approach, proposed by van Leer [167], based on an upwind monotonic approximation of the variables. Those are located in two adjacent triangles, as shown in Figure 5.4.

For a scalar variable c , the interface values c_L and c_R (on the left and right sides of a given face, respectively) are calculated by the κ scheme [11]. For example, c_L is calculated as:

$$c_L = c_{b_L} + \frac{s}{4} [(1 - \kappa s) \Delta_- + (1 + \kappa s) \Delta_+], \quad (5.35)$$

$$\Delta_- = (c_{b_L} - c_p) \quad , \quad \Delta_+ = \left[2l_L / (l_L + l_R) \right] (c_{b_R} - c_{b_L}) . \quad (5.36)$$

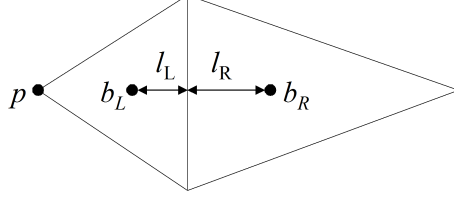


Figure 5.4: Two adjacent triangles with barycenters b_L and b_R , where l_L and l_R are the lengths between the midpoint node of the common face and b_L and b_R , respectively.

where $\kappa=-1$ leads to a completely upwind scheme, $\kappa=0$ to the Fromm scheme, $\kappa=1$ to a central scheme and $\kappa=1/3$ to a third-order scheme (in 1-D). We will use $\kappa=1/3$ in this study. Note that the spatial reconstruction in the current 3-D case study is separately implemented among two adjacent cell centers and midpoint of interface, which can be deemed as a 1-D case locally.

The slope limiter s is chosen according to [11]:

$$s = \frac{2\Delta_- \Delta_+ + \varepsilon}{\Delta_-^2 + \Delta_+^2 + \varepsilon}, \quad (5.37)$$

where ε is a very small number in order to avoid division by zero in the regions of mild slope. A similar approach is used to calculate the values of variables at the horizontal cell interfaces.

5.4.6 The incompressibility condition

The incompressibility condition (5.1), in the cartesian-coordinate system can be written as follows:

$$\nabla \mathbf{u} = 0 \quad (5.38)$$

Integrating (5.38) over a prismatic control volume and transforming the volume integral into a surface integral, the following form is obtained:

$$\oint_s (\mathbf{u} \cdot \mathbf{n}) ds = 0 \quad (5.39)$$

which can be approximated as

$$\sum_{k=1}^5 (\mathbf{u} \cdot \mathbf{n})_k A_k = 0 \quad (5.40)$$

where $\mathbf{u}$ is the vector of variables evaluated on the surfaces of the prismatic control volume.

In order to adjust the hydrodynamic pressure such that (5.39) is satisfied, we substitute $\mathbf{u}^{n+1}$ from (5.29) into (5.40). $\mathbf{u}^{n+1}$ from (5.29) can be written in the following form:

$$\mathbf{u}^{n+1} = -\Delta t \mathbf{p}^{n+1} + \mathbf{R} \quad (5.41)$$

where $\mathbf{R}$ contains all explicit terms evaluated at time step n , as follows:

$$\mathbf{R} = \mathbf{u}^n - \frac{\Delta t}{\vartheta} \left[\oint_s (\mathbf{f} \cdot \mathbf{n}) ds \right]^n + \Delta t \mathbf{s}^n \quad (5.42)$$

Substituting $\mathbf{u}^{n+1}$ from (5.41) into (5.40), we arrive at:

$$\sum_{k=1}^5 [\mathbf{R} \cdot \mathbf{n} - \Delta t \mathbf{p}^{n+1} \cdot \mathbf{n}]_k A_k = 0 \quad (5.43)$$

$\mathbf{R} \cdot \mathbf{n}$ in (5.43) can be approximated by:

$$\mathbf{R} \cdot \mathbf{n} = \frac{1}{2} [\mathbf{R}_L + \mathbf{R}_R] \cdot \mathbf{n} \quad (5.44)$$

Similarly, $\mathbf{p}^{n+1} \cdot \mathbf{n}$ in (5.43) can be approximated by:

$$\mathbf{p}^{n+1} \cdot \mathbf{n} = \frac{1}{2} [\mathbf{p}_L^{n+1} + \mathbf{p}_R^{n+1}] \cdot \mathbf{n} \quad (5.45)$$

where $\mathbf{n}$ is the normal direction of each face.

Substituting (5.44) and (5.45) into (5.43), the following Poisson-type equation for the hydrodynamic pressure is obtained:

$$\sum_{k=1}^5 \left\{ \frac{[\mathbf{R}_L + \mathbf{R}_R]}{2} \cdot \mathbf{n} - \frac{\Delta t}{2} [\mathbf{p}_L^{n+1} + \mathbf{p}_R^{n+1}] \cdot \mathbf{n} \right\}_k A_k = 0 \quad (5.46)$$

in which

$$\mathbf{p}_L^{n+1} = \begin{pmatrix} (D/\rho) \partial p^{n+1} / \partial x \\ (D/\rho) \partial p^{n+1} / \partial y \\ (1/\rho) \partial p^{n+1} / \partial \sigma \end{pmatrix}_L, \quad \mathbf{p}_R^{n+1} = \begin{pmatrix} (D/\rho) \partial p^{n+1} / \partial x \\ (D/\rho) \partial p^{n+1} / \partial y \\ (1/\rho) \partial p^{n+1} / \partial \sigma \end{pmatrix}_R \quad (5.47)$$

in which $\partial p^{n+1} / \partial x$, $\partial p^{n+1} / \partial y$ and $\partial p^{n+1} / \partial \sigma$ are discretized by the divergence theorem over control volumes.

Expanding (5.46) for each control volume produces a 14-diagonal system of equations, which can be solved by any direct or iterative solver. Once the hydrodynamic pressure is calculated, it is used in (5.41) to calculate the three-dimensional velocity components.

5.4.7 Stability

In the proposed model, the stability is limited by explicit schemes. Specifically, in the proposed model, an upwind method is applied to convective fluxes, see Equation (5.31); a central-differencing method is applied to diffusion terms, see Equation (5.34); an upwind method (approximate Riemann solver) is applied to the hydrostatic pressure gradients,

see Equation (5.26). The stability of explicit discretization using upwind method and central-differencing can be approximated by Courant–Friedrichs–Lewy (CFL) condition.

For convection terms in momentum equations, the time-step is limited by:

$$\left(\frac{|u_{ik}^\perp| \Delta t}{\alpha_L} + \frac{|\omega_i| \Delta t}{(\Delta \sigma)_{ik}} \right)^2 < 1 \quad (5.48)$$

where $u_{ik}^\perp$ is the horizontal velocity normal to interface s_k of triangle i , $(\Delta \sigma)_{ik}$ is the distance from the cell-center to top/bottom face in a prism in σ coordinates.

For horizontal diffusion terms and vertical diffusion terms, the time-step is limited by:

$$\left(\frac{2\nu_h \Delta t}{\alpha_L^2} \right) \leq 1, \quad k = 1, 2, 3 \quad \text{and} \quad \left(\frac{2\nu_v \Delta t}{(\Delta \sigma)_{ik}^2} \right) \leq 1, \quad k = 4, 5 \quad (5.49)$$

For hydrostatic pressure gradients, the time-step is limited by:

$$\left(\frac{(|\bar{u}_{ik}^\perp| + \sqrt{gD_i}) \Delta t}{\alpha_L} \right) \leq 1, \quad k = 1, 2, 3 \quad (5.50)$$

in which $\bar{u}_{ik}^\perp$ is depth averaged horizontal velocity normal to interface s_k of triangle i .

5.4.8 Solution algorithm

The algorithm of numerical scheme can be summarized as follows:

- 1- Calculation of the water surface from the discretized continuity equation (5.24)
- 2- Calculation of the hydrostatic pressure gradients from (5.25)
- 3- Calculation of the hydrodynamic pressure from the incompressibility condition (5.46)
- 4- Calculation of the velocity components from the discretized continuity equation (5.41)
- 5- Calculation of the transformed vertical velocity component from (5.17)

5.5 Numerical results

5.5.1 Test case 1: Coastal flow in an annular section

In this numerical experiment, a three-dimensional flow in an annular section is considered [114]. The flow is driven by tidal forcing at the open boundary. The angular frequency of the tidal flow is given by $\omega = 1.41 \times 10^{-4} \text{ s}^{-1}$, which corresponds to a period of 12.4 hr. The problem geometry is shown in Figure 5.5, where the computational grid is also shown and where 15 equal vertical layers were employed. The bottom boundary condition is no-slip. The water depth h varies quadratically with $1/r$ as $h = h_0 r^{-2}$, with $r_1 = 60960$

m, $r_2=152400$ m and $h_0=3.048$ m. A time step size of $\Delta t=190$ s was considered. The Coriolis parameter was assumed to be zero ($f=0$). The vertical turbulent eddy viscosity was assumed to be given by:

$$\omega h^2 / \nu_v = 10$$

as suggested by Lynch and Werner [114].

The boundary condition at r_2 is the tidal surface elevation given by:

$$\eta = 0.1 \cos(2\theta) \cos(\omega t)$$

and the other boundaries are free-slip walls.

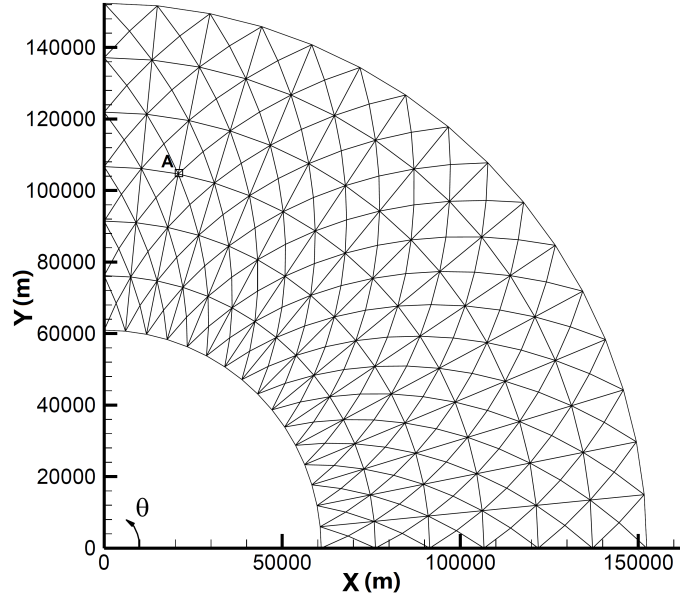


Figure 5.5: Problem geometry and computational grid for test case 1.

A linearized analytical solution can be obtained for this test case using the method proposed in [113]. Figure 5.6 shows the analytical and numerical velocity components at the surface at point A over one tidal cycle. Figures 5.7 and 5.8 respectively show the numerical results for velocity profiles u and v at mesh point A (shown in Figure 5.5) compared with the analytical solutions at four different times ($t = 0, 3.1, 6.2, 9.3$ hours). As can be seen in Figures 5.6, 5.7 and 5.8, the modeling results are in very good agreement with the exact solution, despite the coarse grid used.

5.5.2 Test case 2: Wind-induced flow in a closed basin

In this test case, we consider a steady wind-induced circulatory flow in a closed basin with a flat bed. An analytical solution was derived by Heaps [73] assuming convective acceleration, horizontal diffusivity and Coriolis forces are negligible compared with the vertical shear

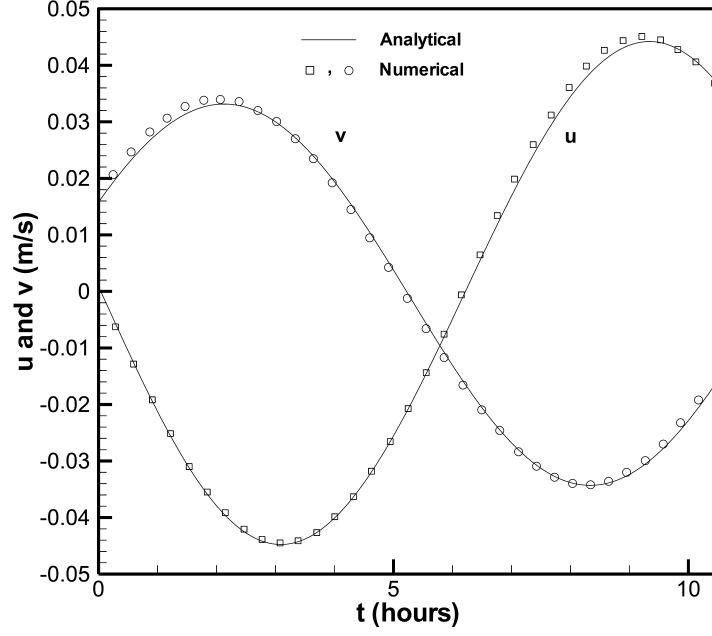


Figure 5.6: Analytical and numerical velocity components at the surface at point A over one tidal cycle.

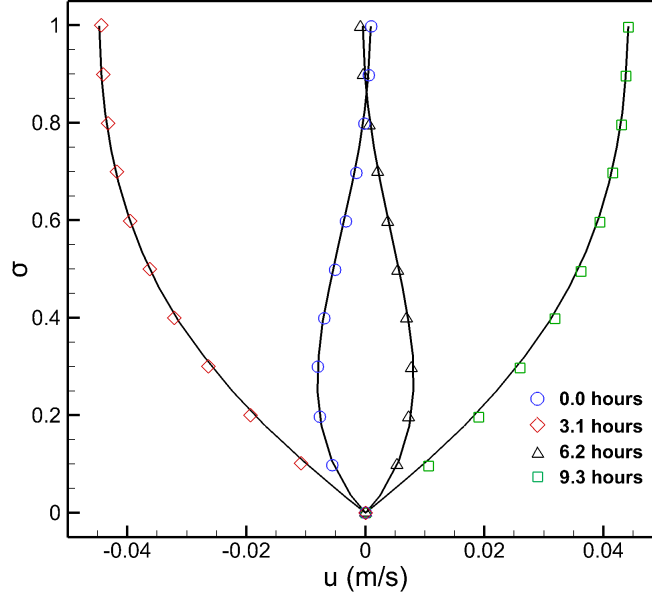


Figure 5.7: Numerical results (solid lines) for horizontal velocity profile u at mesh point A (shown in Figure 5.5) compared with the analytical solutions (symbols) at four different times.

gradient. For the no-slip bottom condition and constant vertical eddy viscosity, the solution is written as:

$$u(z) = \frac{\tau_s}{4\rho\nu h} z(3z - h)$$

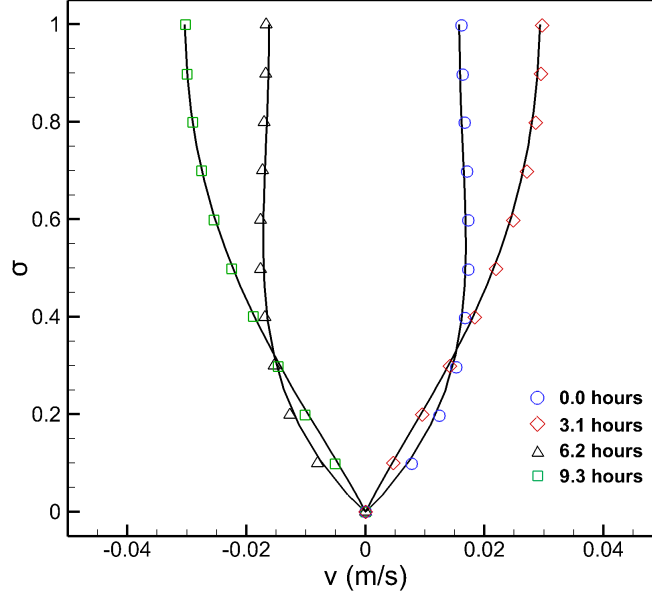


Figure 5.8: Numerical results (solid lines) for vertical velocity profile v at mesh point A (shown in Figure 5.5) compared with the analytical solutions (symbols) at four different times.

where $u(z)$ is velocity at any depth z , τ_s is the wind stress on the surface, h is the water depth, ν is vertical eddy viscosity and ρ is water density.

For this test case, a basin of length 30 km and width 4 km is considered. The computational grid is shown in Figure 5.9, and 14 equal vertical layers were employed. The wind stress was assumed to be $\tau_s = 0.2 \text{ N/m}^2$. The water depth was set to $h = 10 \text{ m}$, the density of water is 1000 kg/m^3 and the vertical eddy viscosity was set to $0.01 \text{ m}^2/\text{s}$. The numerical solution was begun with a zero velocity field and $\Delta t = 15 \text{ s}$ and continued until the steady state condition was obtained. A comparison of the analytical and numerical solutions is shown in Figure 5.10, which shows an excellent agreement between them.

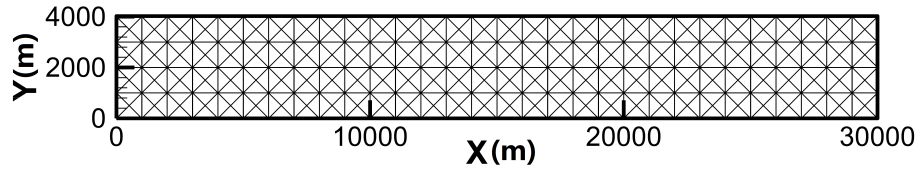


Figure 5.9: Problem geometry and computational grid for test case 2.

Table 5.1 shows the averaged absolute errors ε of simulated horizontal velocity and relative computational time using 7 and 14 layers, respectively. The averaged absolute errors ε are calculated by

$$\varepsilon = \frac{1}{n} \sum_{i=1}^n |true\ value - approximated\ value|$$

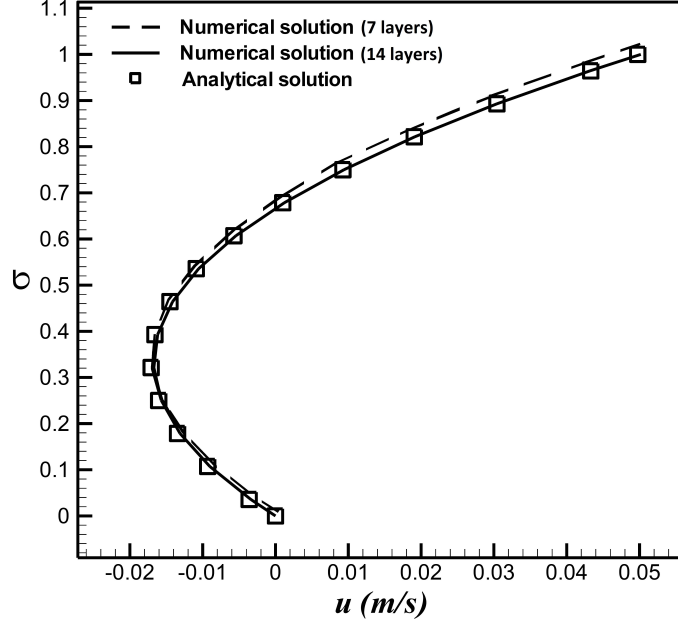


Figure 5.10: Analytical and numerical velocity profiles for the wind-driven flow test.

where n is the total number of the sampling points. In this error analysis, 11 sampling points are selected that equally distribute on σ axis.

Table 5.1: Averaged absolute errors ε of u -velocity by proposed model.

Number of vertical layers	Averaged absolute errors ε	Relative run time
7	0.00196 m/s	1.00
14	0.00085 m/s	1.71

5.5.3 Test case 3: Standing wave in a closed basin

This test is a benchmark for evaluating energy and mass conservation properties of the scheme as well as its accuracy for computing the wave dispersion problems. In this test case, we consider a standing wave in a closed basin of a constant water depth $h=10$ m. The initial velocity is assumed to be zero, and the analytical solution is given by

$$\eta(x, t) = a \cos(kx) \cos\left(\frac{2\pi}{T}t\right), \quad 0 \leq x \leq L \quad (5.51)$$

where a is the amplitude of the small wave, k is the wave number and $k = 2\pi/L$, L is the length of the basin. T is the wave period. The above analytical solution can be used to define boundary conditions by setting $t=0$ in (5.51). The details of the analytical solution can be found in [82].

A computational domain of length $L=10$ m is divided into 4 horizontal layers, with a grid of $\Delta x=L/40$. The simulated duration is $t=10T$, and a time step $\Delta t=T/100$ is used in this test case.

We consider a deep water case ($kh = 2\pi$) and a very small wave with $a=0.05$ m is employed. The viscous terms are ignored in this test.

Figure 5.11 shows the comparison between the analytical solutions and the simulated results for the water surface elevation at $x=5.3$ m. It shows that the proposed non-hydrostatic 3-D free surface model can accurately simulate the amplitude and the phase of infinitesimal wave with a low level of damping and consequently energy dissipation.

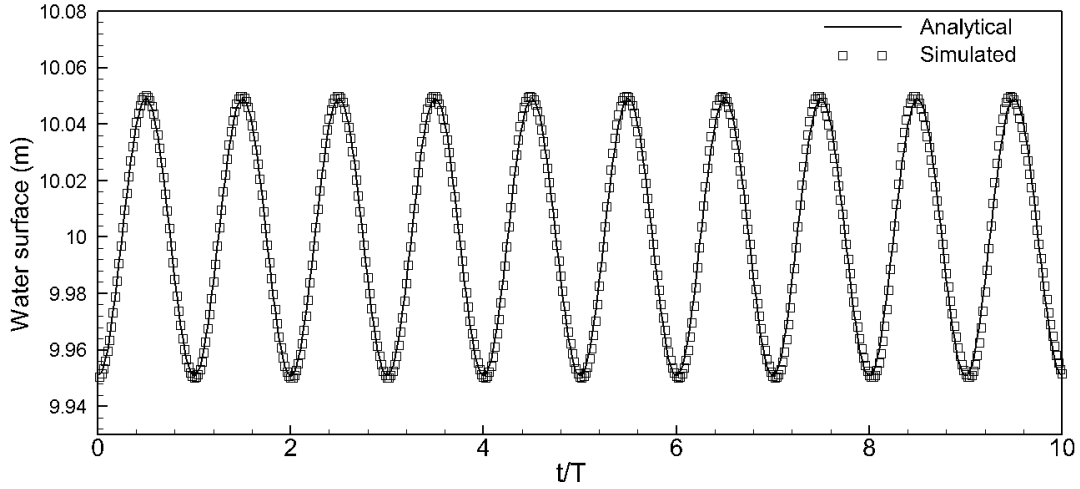


Figure 5.11: Comparison between analytical and simulated water surface at $x=5.3$ m.

5.5.4 Test case 4: Steady current flow over a trench

A test of steady flow over a trench with experimental data is used to test the performance of the proposed model. The detailed geometry of the trench and the locations of the measuring stations are shown in Figure 5.12. The velocity profile at inlet boundary condition is given by

$$u = \frac{V_*}{\kappa} \ln \frac{30.0(z - z_0)}{k_s}, \quad v = 0 \quad \text{and} \quad w = 0 \quad (5.52)$$

where k_s is the equivalent roughness height which is set to 0.002 m in this test, V_* is the shear velocity, z_0 is the elevation of topography. The shear velocity at inlet boundary is 0.033 ms^{-1} , the depth at outlet boundary is 0.2 m. Equation (5.52) is also used as bottom boundary condition. This particular trench was investigated experimentally at Delft Hydraulics Laboratory (see [169]). The computational domain is discretized into 1628 triangular cells in horizontal and 20 layers in vertical. The steady solutions shown in the figures are achieved at $t=80$ s.

Figure 5.13 shows the reasonable agreement between simulated u velocities under steady states and experimental measurements.

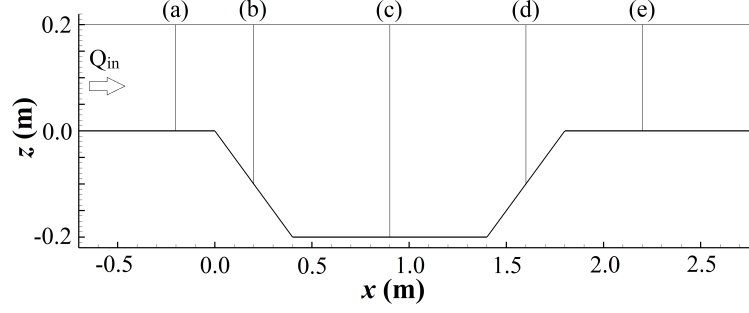


Figure 5.12: Trench dimensions and measuring stations.

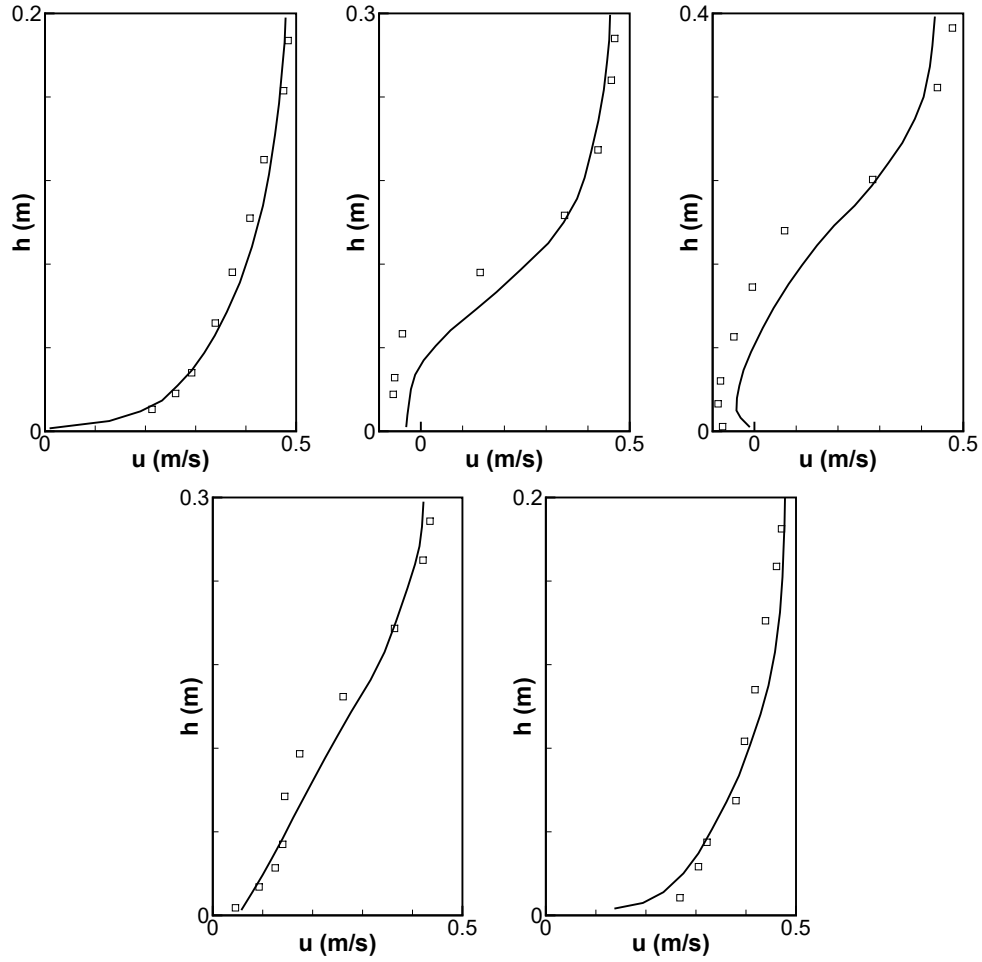


Figure 5.13: Comparison between experimental ($\square$) and simulated ($—$) velocities at different measuring stations.

5.5.5 Test case 5: Steady flow over a hump

A benchmark test developed by Goutal and Maurel [65] is used in the current study to verify the ability of simulating sub- and super-critical flows using proposed model. In this

test, a 25 m long frictionless channel is considered. The bottom topography is defined by

$$h(x, 0) = \begin{cases} 0.2 - 0.05(x - 10)^2, & \text{if } 8 \leq x \leq 12, \\ 0, & \text{otherwise} \end{cases} \quad (5.53)$$

The steady flow over the topography defined by (5.53) can be sub-, trans- or super-critical flow depending on the initial and boundary conditions. The trans-critical flow without a hydraulic jump is chosen to be simulated in this test. Corresponding analytical solutions can be found in [65]. The solution is considered to be convergent if following condition [191] is satisfied:

$$\sqrt{\sum_i \left(\frac{D_i^n - D_i^{n-1}}{D_i^n} \right)^2} < 1 \times 10^{-6} \quad (5.54)$$

In order to achieve a trans-critical flow, a constant discharge flowing through the channel at the upstream inflow-boundary is given by $1.53 \text{ m}^2/\text{s}$, and a fixed water depth of 0.66 m at downstream outflow-boundary is given. A test with same condition has been previously reported in [164] for depth-averaged shallow water flows.

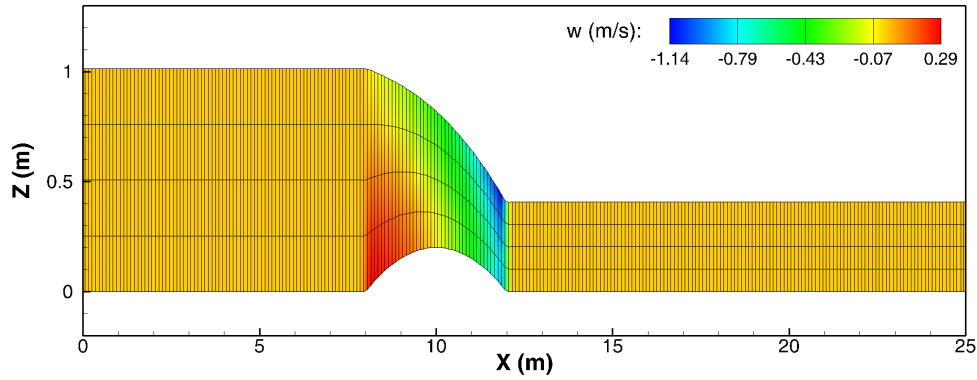


Figure 5.14: Vertical cross-section of grids and contour of vertical velocity w (top), and velocity vectors of steady trans-critical flow.

The domain is discretized by triangular cells with averaged area of 0.005 m^2 and 4 vertical layers. Figure 5.14 shows reasonable vertical velocity profile over the whole computational domain.

Figures 5.15, 5.16 and 5.17 show that a very good agreement between simulated result and analytical solution under steady status is achieved for water surface level, discharge and froude number, respectively. The results demonstrate that the proposed model is able to solve sub- and super-critical flows.

5.6 Conclusion

A numerical model was presented for the simulation of three-dimensional unsteady non-hydrostatic shallow water flows.

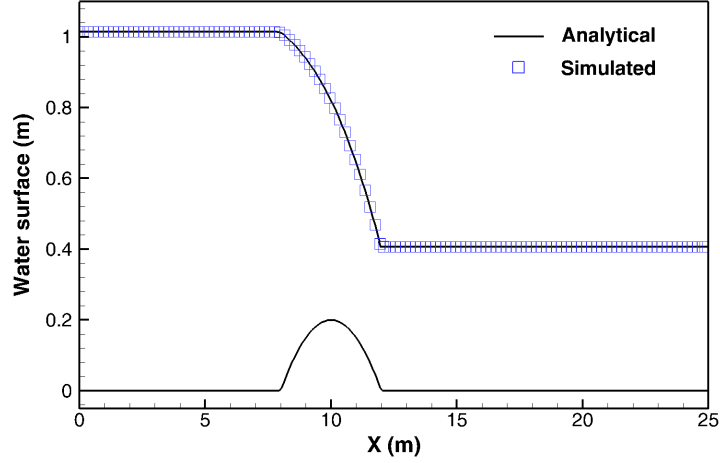


Figure 5.15: Comparison between analytical and simulated water surface at $x=5.3$ m.

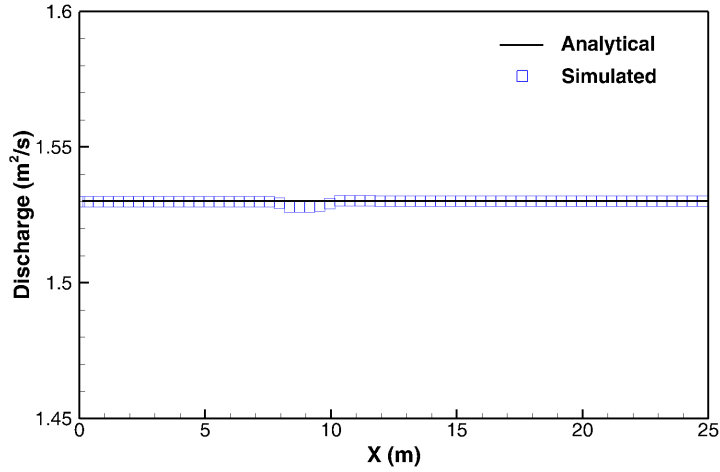


Figure 5.16: Comparison between analytical and simulated discharge at $x=5.3$ m.

A sigma coordinate system with a collocated arrangement of three-dimensional velocity components was used to simulate the variation of water depth during the time steps. A special treatment of the pressure term was developed to avoid the water surface oscillations.

By using an unstructured grid, the model benefits from the properties of the unstructured grid in the horizontal direction, like modeling of complicated geometries and grid refinement. Also, the σ -coordinate system enables the model to simulate flows with irregular topography.

Use of a Godunov-type scheme for free surface modeling enables the model to simulate sub- and super-critical flows. Non-hydrostatic pressure is calculated by a Poisson-type equation which ensures exact mass conservation.

The numerical results showed that the model is capable of producing non-oscillatory and highly accurate results. By solving the governing equations without the hydrostatic pressure assumption, the model is able to simulate short-period waves and vertically cir-

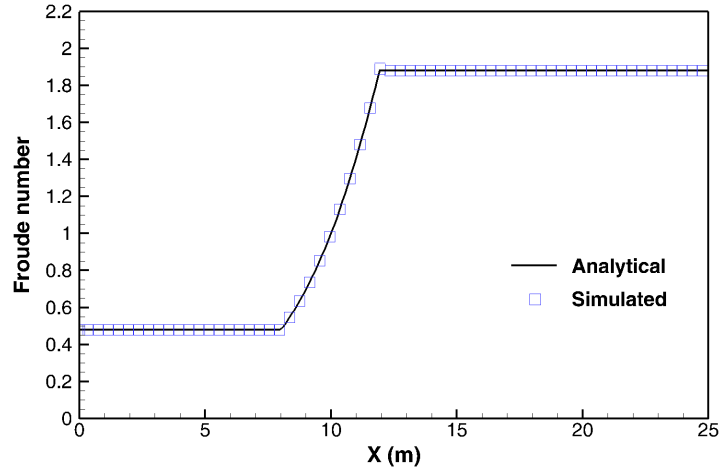


Figure 5.17: Comparison between analytical and simulated froude number at $x=5.3$ m.

culating flows, which is beyond the capability of hydrostatic models.

Chapter 6

Three-dimensional shallow water system: a relaxation approach

6.1 Abstract

We study a three-dimensional shallow water system, which is obtained from the three-dimensional Navier-Stokes equations after Reynolds averaging and under the simplifying hydrostatic pressure assumption. Since the three-dimensional shallow water system is generically not hyperbolic, it cannot be numerically solved using hyperbolic shock capturing schemes. At the same time, existing simple finite-difference and finite-volume methods may fail in simulations of unsteady flows with shock discontinuities, such as dam-break and flood flows. To overcome this difficulty, we propose a novel numerical method, which is based on a relaxation approach utilized to “hyperbolize” the three-dimensional shallow water system. The extended relaxation system is hyperbolic and we develop a second-order semi-discrete central-upwind scheme for it. The proposed numerical method can preserve “lake at rest” steady states and positivity of water depth over irregular bottom topography. The accuracy, stability and robustness of the developed numerical method is verified on five numerical experiments.

Keywords: Three-dimensional shallow water equations, central-upwind scheme, relaxation approach, well-balanced, positivity preserving, finite-volume method.

6.2 Introduction

Two-dimensional (2-D) depth-averaged shallow water equations (SWEs) have been widely used in studying various hydrodynamic phenomena including dam-break flows, flood waves, and tidal flows among others. The popularity of the 2-D SWEs hinges on the following facts. First, they are substantially simpler than the three-dimensional (3-D) Navier-Stokes equations and thus can be more efficiently solved numerically. At the same time, the 2-D SWEs are capable of describing the main features of the water flow in regimes when the

wave lengths are much larger than the water depth. Since the 2-D shallow water system is a hyperbolic system of balance laws, it can be accurately solved using the finite-volume methods, in particular, Godunov-type schemes, which are designed to capture both shocks and contact discontinuities. For applications of Godunov-type schemes to the 2-D SWEs, we refer the reader to, e.g., [5, 7, 23, 46, 74, 86, 92, 104, 118, 150, 179] and references therein.

A major drawback of the 2-D SWEs is that they are depth-averaged and thus do not take into account any vertical variation of the water flow. The 3-D shallow water system, proposed in [17, 18], is still substantially simpler than the 3-D Navier-Stokes equations, and unlike the 2-D SWEs it can predict the vertical distribution of primitive variables, information on which is essential in many practical cases. Therefore, the 3-D SWEs have recently attracted a lot of attention as a more accurate alternative to the 2-D SWEs; see, e.g., [3, 20, 37, 40, 47, 58, 84, 100, 105, 106, 130, 158].

After Reynolds averaging and under the simplifying hydrostatic pressure assumption, the 3-D SWEs with constant density and free surface are derived from the 3-D Navier-Stokes equations. The governing equations have the following form [17, 18]:

$$\begin{aligned} u_x + v_y + w_z &= 0, \\ u_t + (uu)_x + (uv)_y + (uw)_z &= -g\eta_x, \\ v_t + (vu)_x + (vv)_y + (vw)_z &= -g\eta_y, \end{aligned} \tag{6.1}$$

where t is the time; x , y and z are the spatial coordinates; $u(x, y, z, t)$, $v(x, y, z, t)$ and $w(x, y, z, t)$ are the velocity components in the x -, y - and z -directions, respectively; g is the gravitational acceleration; and $\eta(x, y, t)$ is the water surface elevation. The system (6.1) together with either kinematic or dynamic boundary conditions for η constitute a closed system of equations for u , v , w and η .

In order to avoid the difficulties with applying the boundary conditions on the free surface and representing the irregular bottom, we follow the approach in [17, 18, 129] and replace the vertical coordinate z with the σ coordinate, which normalizes the vertical dimension to unity by the following transformation:

$$\sigma = \frac{z - \eta(x, y, t)}{D(x, y, t)},$$

in which σ is the transformed vertical coordinate that varies between -1 and 0 , and $D(x, y, t)$ is the depth of the water column computed by

$$D(x, y, t) = \eta(x, y, t) - h(x, y),$$

where $h(x, y)$ is the bed level. Accordingly, the governing equations (6.1) can be written in the following conservative form:

$$\eta_t + (Du)_x + (Dv)_y + \Omega_\sigma = 0, \tag{6.2}$$

$$(Du)_t + \left(Du^2 + \frac{g}{2} D^2 \right)_x + (Dvu)_y + (\Omega u)_\sigma = -gDh_x, \tag{6.3}$$

$$(Dv)_t + (Dvu)_x + \left(Dv^2 + \frac{g}{2} D^2 \right)_y + (\Omega v)_\sigma = -gDh_y, \tag{6.4}$$

where Ω represents the vertical velocity in the σ -direction, which is related to w through

$$w = \Omega + u(\sigma D_x + \eta_x) + v(\sigma D_y + \eta_y) + (\sigma D_t + \eta_t).$$

To close the system (6.2)–(6.4), we apply the kinematic boundary conditions, namely, $\Omega(x, y, \sigma = -1, t) = 0$ and $\Omega(x, y, \sigma = 0, t) = 0$, and integrate equation (6.2) with respect to σ from -1 to 0 to obtain

$$\eta_t + \left[\int_{-1}^0 Du \, d\sigma \right]_x + \left[\int_{-1}^0 Dv \, d\sigma \right]_y = 0. \quad (6.5)$$

Note that using the same kinematic boundary conditions, one can rewrite equation (6.2) by integrating it with respect to the σ -coordinate as follows:

$$\Omega(x, y, \sigma, t) = - \int_{-1}^{\sigma} \left[\eta_t(x, y, t) + (Du)_x(x, y, \xi, t) + (Dv)_y(x, y, \xi, t) \right] d\xi. \quad (6.6)$$

The 3-D shallow water system (6.2)–(6.4), (6.5) has been intensively studied and several finite difference (see, e.g., [3, 36, 37, 47, 84, 100, 105, 106, 130, 158]) and finite-volume (see, e.g., [20, 40, 58]) methods for (6.2)–(6.4), (6.5) have been developed. However, these methods are only applicable to smooth solutions due to the fact that the 3-D shallow water system is not hyperbolic, which prevents one from applying shock-capturing schemes. Therefore, the existing methods may fail when unsteady flows with shocks and contact discontinuities such as dam-break and flood flows are simulated. In order to address this issue, we propose a relaxation approach inspired by [44, 136], where a hyperbolic model of compressible two-phase flow was developed using the pressure relaxation, and [2], where an unconditionally hyperbolic two-layer SWEs were obtained using the relaxation in two auxiliary layer depth variables. In §6.3, we present a relaxation model of the 3-D SWEs, which replaces the system (6.2)–(6.4) with the hyperbolic one that includes an additional equation on an auxiliary vertical velocity variable.

We then develop a second-order semi-discrete central-upwind scheme for the obtained relaxation system. Godunov-type central-upwind schemes were developed in [87, 88, 93, 94] for general multidimensional hyperbolic systems of conservation laws. The central-upwind scheme was extended to the one-dimensional (1-D) SWEs in [86]. A more robust, well-balanced and at the same time positivity preserving central-upwind scheme for the 1-D SWEs was developed in [92]. The central-upwind scheme for the 2-D SWEs was proposed using both Cartesian [86, 92], triangular [23], cell-vertex [14] and unstructured quadrilateral [141] grids. These schemes are capable of exactly preserving “lake at rest” steady states and ensuring the positivity of the computed water depth. In this paper, we propose a new central-upwind scheme on 3-D prismatic grids and apply it to the relaxation model of the 3-D SWEs. Our method, presented in §6.4, includes a well-balanced discretization of the bed-slope source term, continuous piecewise linear approximation of the bottom topography and positivity preserving linear reconstruction of the water surface: as we

show, this ensures both well-balanced and positivity preserving properties of the resulting scheme.

Finally, in §6.5, we present several numerical experiments, where the robustness and high accuracy of the proposed relaxation approach and new central-upwind scheme on prismatic grids are illustrated. Some concluding remarks complete the paper in §6.6.

6.3 A Relaxation Model

In this section, we present a relaxation model of the 3-D SWEs, which is hyperbolic and equivalent to the original model in the zero relaxation limit. In order to obtain the relaxation model, we introduce an auxiliary variables Ω' and add a PDE with a stiff source term to the original system, in which the variable Ω is replaced with Ω' :

$$\eta_t + (Du)_x + (Dv)_y + \Omega'_\sigma = 0, \quad (6.7)$$

$$(Du)_t + \left(Du^2 + \frac{g}{2}D^2\right)_x + (Duv)_y + (\Omega'u)_\sigma = -gDh_x, \quad (6.8)$$

$$(Dv)_t + (Duv)_x + \left(Dv^2 + \frac{g}{2}D^2\right)_y + (\Omega'v)_\sigma = -gDh_y, \quad (6.9)$$

$$\Omega'_t + W^*\Omega'_\sigma = -\frac{1}{\varepsilon}(\Omega' - \Omega). \quad (6.10)$$

Here, $\varepsilon > 0$ is a relaxation parameter and W^* is a propagation speed associated with the auxiliary variable Ω' (in all of our numerical experiments, we have taken $W^* = 1$).

The system (6.7)–(6.10) can be written in the following vector form:

$$\mathbf{U}_t + \mathbf{F}(\mathbf{U}, h)_x + \mathbf{G}(\mathbf{U}, h)_y + \mathbf{J}(\mathbf{U}, h)_\sigma = \mathbf{S}(\mathbf{U}, h) + \mathbf{R}(\mathbf{U}), \quad \mathbf{U} := (\eta, Du, Dv, \Omega')^T, \quad (6.11)$$

where $\mathbf{F}$, $\mathbf{G}$ and $\mathbf{J}$ are the fluxes:

$$\begin{aligned} \mathbf{F}(\mathbf{U}, h) &:= \left(Du, \frac{(Du)^2}{\eta - h} + \frac{g}{2}(\eta - h)^2, \frac{(Du)(Dv)}{\eta - h}, 0 \right)^T, \\ \mathbf{G}(\mathbf{U}, h) &:= \left(Dv, \frac{(Du)(Dv)}{\eta - h}, \frac{(Dv)^2}{\eta - h} + \frac{g}{2}(\eta - h)^2, 0 \right)^T, \\ \mathbf{J}(\mathbf{U}, h) &:= \left(\Omega', \frac{(Du)\Omega'}{\eta - h}, \frac{(Dv)\Omega'}{\eta - h}, W^*\Omega' \right)^T, \end{aligned} \quad (6.12)$$

and $\mathbf{S}$ and $\mathbf{R}$ are the source terms:

$$\mathbf{S}(\mathbf{U}, h) := (0, -(\eta - h)gh_x, -(\eta - h)gh_y, 0)^T, \quad \mathbf{R}(\mathbf{U}) := \left(0, 0, 0, -\frac{1}{\varepsilon}(\Omega' - \Omega) \right)^T. \quad (6.13)$$

The relaxation system (6.7)–(6.10) can be also written in the following quasi-linear form:

$$\mathbf{U}_t + A(\mathbf{U}, h)\mathbf{U}_x + B(\mathbf{U}, h)\mathbf{U}_y + C(\mathbf{U}, h)\mathbf{U}_\sigma = \mathbf{S}(\mathbf{U}, h) + \mathbf{R}(\mathbf{U}),$$

where the Jacobians $A := \frac{\partial \mathbf{F}}{\partial \mathbf{U}}$, $B := \frac{\partial \mathbf{G}}{\partial \mathbf{U}}$ and $C := \frac{\partial \mathbf{J}}{\partial \mathbf{U}}$ are

$$A(\mathbf{U}, h) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -\frac{(Du)^2}{(\eta-h)^2} + g(\eta-h) & \frac{2Du}{\eta-h} & 0 & 0 \\ -\frac{(Du)(Dv)}{(\eta-h)^2} & \frac{Dv}{\eta-h} & \frac{Du}{\eta-h} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$B(\mathbf{U}, h) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ -\frac{(Du)(Dv)}{(\eta-h)^2} & \frac{Dv}{\eta-h} & \frac{Du}{\eta-h} & 0 \\ -\frac{(Dv)^2}{(\eta-h)^2} + g(\eta-h) & 0 & \frac{2Dv}{\eta-h} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

and

$$C(\mathbf{U}, h) = \begin{pmatrix} 0 & 0 & 0 & 1 \\ -\frac{(Du)\Omega'}{(\eta-h)^2} & \frac{\Omega'}{\eta-h} & 0 & \frac{Du}{\eta-h} \\ -\frac{(Dv)\Omega'}{(\eta-h)^2} & 0 & \frac{\Omega'}{\eta-h} & \frac{Dv}{\eta-h} \\ 0 & 0 & 0 & W^* \end{pmatrix},$$

respectively. Their eigenvalues are given by

$$\text{spec}(A(\mathbf{U}, h)) = \left\{ \frac{Du}{\eta-h} - \sqrt{g(\eta-h)}, \frac{Du}{\eta-h}, \frac{Du}{\eta-h} + \sqrt{g(\eta-h)}, 0 \right\},$$

$$\text{spec}(B(\mathbf{U}, h)) = \left\{ \frac{Dv}{\eta-h} - \sqrt{g(\eta-h)}, \frac{Dv}{\eta-h}, \frac{Dv}{\eta-h} + \sqrt{g(\eta-h)}, 0 \right\},$$

and

$$\text{spec}(C(\mathbf{U}, h)) = \left\{ \frac{\Omega'}{\eta-h}, \frac{\Omega'}{\eta-h}, 0, W^* \right\}.$$

In the numerical method presented in the next section, we will solve the relaxation system (6.11)–(6.13) using an operator splitting approach: We first evolve the solution by solving the propagation system

$$\mathbf{U}_t + \mathbf{F}(\mathbf{U}, h)_x + \mathbf{G}(\mathbf{U}, h)_y + \mathbf{J}(\mathbf{U}, h)_\sigma = \mathbf{S}(\mathbf{U}, h), \quad (6.14)$$

and then update it according to the infinitely fast relaxation step $\mathbf{U}_t = \mathbf{R}(\mathbf{U})$ with $\varepsilon \rightarrow 0+$, that is, upon the completion of the evolution step, we set

$$\Omega' = \Omega$$

throughout the entire computational domain.

6.4 Numerical Method

In this section, we present a numerical method for the 3-D SWEs. Notice that even though the system (6.7)–(6.10) is hyperbolic, one cannot solve it directly since the water surface variable η is independent of σ . We therefore proceed in the following four steps:

Step 1 (§6.4.1). The discharge components Du and Dv are evolved by numerically solving equations (6.8) and (6.9) from a given time level t to the next time level $t + \Delta t$;

Step 2 (§6.4.2). The water surface elevation η is then updated by numerically solving the depth-averaged equation (6.5);

Step 3 (§6.4.3). The vertical velocity component Ω is computed by an appropriate quadrature applied to equation (6.6) using Du and Dv computed in Step 1 and η computed in Step 2;

Step 4. The new values of Ω' are then obtained from the relaxation step using the infinitely fast relaxation with $\varepsilon \rightarrow 0^+$ as described at the end of §6.3.

6.4.1 Central-Upwind Scheme for the Propagation Step (6.14)

In this section, we develop a semi-discrete second-order central-upwind scheme on prismatic grid for the system (6.14). In fact, as we mentioned in the beginning of §6.4, we will only use the developed central-upwind scheme to evolve the discharge components of the solution, Du and Dv , only.

Our choice of the prismatic grid is motivated by the fact that when working in the σ coordinates, the horizontal domain may be arbitrary, while in the vertical direction the domain is always $[-1, 0]$. We therefore use the 2-D triangulation coupled with the uniform grid in the σ -direction with the small scale $\Delta\sigma$. In Figure 6.1, we show a typical prismatic cell.

Consider a prismatic cell V_i of volume $|V_i|$ with the centroid (x_i, y_i, σ_i) . It has five faces of areas A_{ik} , $k = 1, \dots, 5$. We denote their corresponding outer unit normals by $\mathbf{n}_{ik} := (\cos(\theta_{ik}), \sin(\theta_{ik}), 0)^T$, $k = 1, 2, 3$, $\mathbf{n}_{i4} := (0, 0, 1)^T$ and $\mathbf{n}_{i5} := (0, 0, -1)^T$; see Figure 6.1. We also denote by $M_{ik} = (x_{ik}, y_{ik}, \sigma_{ik})$ the centroid of the k -th face of the cell V_i and by V_{ik} the corresponding neighboring cell, $k = 1, \dots, 5$.

A semi-discrete scheme for (6.14) is a system of ODEs for the approximations of the cell averages of the solution:

$$\bar{U}_i(t) \approx \frac{1}{|V_i|} \iiint_{V_i} U(x, y, \sigma, t) dx dy d\sigma.$$

For the prismatic cells, a direct extension of the triangular [23, 90] and Cartesian [86, 87,

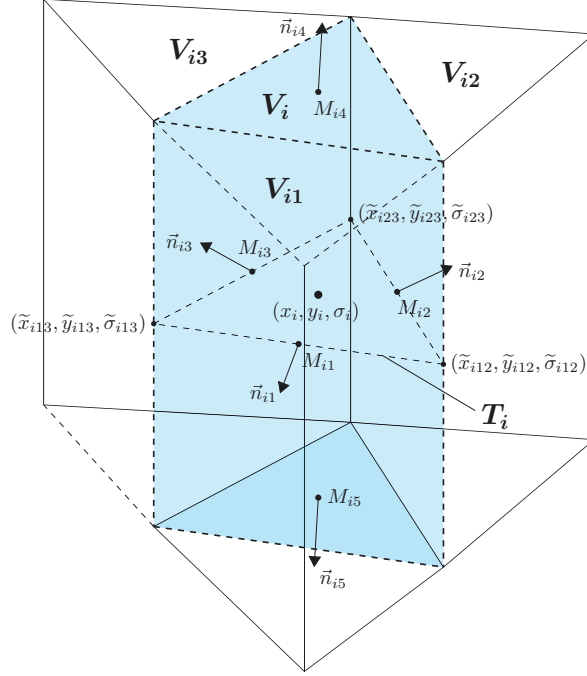


Figure 6.1: A typical prismatic cell with three neighbors.

88, 92, 94] second-order central-upwind schemes reads:

$$\begin{aligned}
\frac{d\bar{\mathbf{U}}_i}{dt} = & -\frac{1}{|V_i|} \sum_{k=1}^3 \frac{A_{ik} \cos(\theta_{ik})}{a_{ik}^{\text{in}} + a_{ik}^{\text{out}}} [a_{ik}^{\text{in}} \mathbf{F}(\mathbf{U}_{ik}(M_{ik})) + a_{ik}^{\text{out}} \mathbf{F}(\mathbf{U}_i(M_{ik}))] \\
& -\frac{1}{|V_i|} \sum_{k=1}^3 \frac{A_{ik} \sin(\theta_{ik})}{a_{ik}^{\text{in}} + a_{ik}^{\text{out}}} [a_{ik}^{\text{in}} \mathbf{G}(\mathbf{U}_{ik}(M_{ik})) + a_{ik}^{\text{out}} \mathbf{G}(\mathbf{U}_i(M_{ik}))] \\
& +\frac{1}{|V_i|} \sum_{k=1}^3 A_{ik} \frac{a_{ik}^{\text{in}} a_{ik}^{\text{out}}}{a_{ik}^{\text{in}} + a_{ik}^{\text{out}}} [\mathbf{U}_{ik}(M_{ik}) - \mathbf{U}_i(M_{ik})] \\
& -\frac{1}{\Delta\sigma} \sum_{k=4}^5 \frac{(-1)^k}{b_{ik}^{\text{in}} + b_{ik}^{\text{out}}} [b_{ik}^{\text{in}} \mathbf{J}(\mathbf{U}_{ik}(M_{ik})) + b_{ik}^{\text{out}} \mathbf{J}(\mathbf{U}_i(M_{ik}))] \\
& +\frac{1}{\Delta\sigma} \sum_{k=4}^5 \frac{b_{ik}^{\text{in}} b_{ik}^{\text{out}}}{b_{ik}^{\text{in}} + b_{ik}^{\text{out}}} [\mathbf{U}_{ik}(M_{ik}) - \mathbf{U}_i(M_{ik})] + \bar{\mathbf{S}}_i.
\end{aligned} \tag{6.15}$$

Here, $\mathbf{U}_i(M_{ik})$ and $\mathbf{U}_{ik}(M_{ik})$ are the reconstructed point values at M_{ik} , $k = 1, 2, 3, 4, 5$, which are obtained using a piecewise linear reconstruction of $\mathbf{U}$ at time t ,

$$\tilde{\mathbf{U}}(x, y, \sigma) := \bar{\mathbf{U}}_i + (\hat{\mathbf{U}}_x)_i(x - x_i) + (\hat{\mathbf{U}}_y)_i(y - y_i) + (\hat{\mathbf{U}}_\sigma)_i(\sigma - \sigma_i), \quad (x, y, \sigma) \in V_i, \tag{6.16}$$

namely,

$$\mathbf{U}_i(M_{ik}) := \lim_{\substack{(x, y, \sigma) \rightarrow M_{ik} \\ (x, y, \sigma) \in V_i}} \tilde{\mathbf{U}}(x, y, \sigma), \quad \mathbf{U}_{ik}(M_{ik}) := \lim_{\substack{(x, y, \sigma) \rightarrow M_{ik} \\ (x, y, \sigma) \in V_{ik}}} \tilde{\mathbf{U}}(x, y, \sigma). \tag{6.17}$$

To prevent oscillations, the slopes in (6.16) are to be computed using a nonlinear limiter. To evaluate the horizontal derivatives, $(\widehat{\mathbf{U}}_x)_i$ and $(\widehat{\mathbf{U}}_y)_i$, we use the limiter developed in [83] (see also [111]) with the small parameter needed to be used to avoid division by zero set to be 10^{-8} ; for details, see [111]. To obtain the vertical derivatives, $(\widehat{\mathbf{U}}_\sigma)_i$, we use the minmod limiter; see, e.g., [123, 134, 161, 166]. Both limiters are used in a componentwise manner.

The horizontal directional local speeds of propagation, a_{ik}^{in} and a_{ik}^{out} , are defined by

$$\begin{aligned} a_{ik}^{\text{in}} &:= -\min \left\{ \lambda_1[E_{ik}(\mathbf{U}_i(M_{ik}), h(M_{ik}))], \lambda_1[E_{ik}(\mathbf{U}_{ik}(M_{ik}), h(M_{ik}))], 0 \right\}, \\ a_{ik}^{\text{out}} &:= \max \left\{ \lambda_4[E_{ik}(\mathbf{U}_i(M_{ik}), h(M_{ik}))], \lambda_4[E_{ik}(\mathbf{U}_{ik}(M_{ik}), h(M_{ik}))], 0 \right\}, \end{aligned} \quad k = 1, 2, 3, \quad (6.18)$$

where $\lambda_1[E_{ik}] \leq \lambda_2[E_{ik}] \leq \lambda_3[E_{ik}] \leq \lambda_4[E_{ik}]$ are the eigenvalues of the matrix

$$E_{ik}(\mathbf{U}, h) = \cos(\theta_{ik})A(\mathbf{U}, h) + \sin(\theta_{ik})B(\mathbf{U}, h), \quad k = 1, 2, 3. \quad (6.19)$$

Similarly, the vertical directional local speeds of propagation, b_{ik}^{in} and b_{ik}^{out} , are defined by

$$\begin{aligned} b_{ik}^{\text{in}} &:= -\min \left\{ \lambda_1[C(\mathbf{U}_i(M_{ik}), h(M_{ik}))], \lambda_1[C(\mathbf{U}_{ik}(M_{ik}), h(M_{ik}))], 0 \right\}, \\ b_{ik}^{\text{out}} &:= \max \left\{ \lambda_4[C(\mathbf{U}_i(M_{ik}), h(M_{ik}))], \lambda_4[C(\mathbf{U}_{ik}(M_{ik}), h(M_{ik}))], 0 \right\}, \end{aligned} \quad k = 4, 5, \quad (6.20)$$

where $\lambda_1[C] \leq \lambda_2[C] \leq \lambda_3[C] \leq \lambda_4[C]$ are the eigenvalues of the matrix $C(\mathbf{U}, h)$.

Finally, the source terms $\bar{\mathbf{S}}_i$ in (6.15) are to be approximated using a certain quadrature:

$$\bar{\mathbf{S}}_i(t) \approx \frac{1}{|V_i|} \iiint_{V_i} \mathbf{S}(\mathbf{U}(x, y, \sigma, t), h(x, y)) dx dy d\sigma.$$

We use a well-balanced quadrature described in §6.4.1 below.

Remark. All of the evolved and reconstructed indexed quantities in (6.15)–(6.20) depend on t , but we suppress this dependence for the sake of brevity.

Remark. In order to avoid division by 0 (or by a number close to 0), in all the reported experiments, when $a_{ik}^{\text{in}} + a_{ik}^{\text{out}} < 10^{-6}$, the scheme (6.15) was replaced with

$$\begin{aligned} \frac{d\bar{\mathbf{U}}_i}{dt} &= -\frac{1}{|V_i|} \sum_{k=1}^3 \frac{A_{ik} \cos(\theta_{ik})}{2} [\mathbf{F}(\mathbf{U}_{ik}(M_{ik})) + \mathbf{F}(\mathbf{U}_i(M_{ik}))] \\ &\quad - \frac{1}{|V_i|} \sum_{k=1}^3 \frac{A_{ik} \sin(\theta_{ik})}{2} [\mathbf{G}(\mathbf{U}_{ik}(M_{ik})) + \mathbf{G}(\mathbf{U}_i(M_{ik}))] \\ &\quad - \frac{1}{\Delta\sigma} \sum_{k=4}^5 \frac{(-1)^k}{2} [\mathbf{J}(\mathbf{U}_{ik}(M_{ik})) + \mathbf{J}(\mathbf{U}_i(M_{ik}))] + \bar{\mathbf{S}}_i. \end{aligned}$$

A fully discrete scheme can be developed from (6.15) using a stable ODE solver. In the present study, a three-stage third-order strong stability preserving (SSP) Runge-Kutta solver [61, 62, 142] was used. The time-step Δt is limited by the CFL condition

$$\Delta t < \min_{i,k} \left[\min \left\{ \frac{r_{ik}}{3 \max(a_{ik}^{\text{in}}, a_{ik}^{\text{out}})}, \frac{\Delta\sigma}{2 \max(b_{ik}^{\text{in}}, b_{ik}^{\text{out}})} \right\} \right],$$

in which r_{ik} , $k = 1, 2, 3$ are the three corresponding altitudes of the triangular sides of the prismatic cell V_i .

Piecewise Linear Approximation of the Bottom

In this section, we conduct a piecewise linear approximation which is necessary for developing the well-balanced and positivity preserving schemes in the present study. The approach is applicable for both continuous and discontinuous topography function.

We denote by $h_{i\kappa}$ the values of the bottom topography function $H(\tilde{x}_{i\kappa}, \tilde{y}_{i\kappa})$ at the vertices $(\tilde{x}_{i\kappa}, \tilde{y}_{i\kappa}, \tilde{\sigma}_{i\kappa})$, $\kappa = 12, 23, 31$, of the horizontal cross-sections of the cell V_i ; see Figure 6.1.

First, we compute $h_{i\kappa}$ according to the following formula:

$$h_{i\kappa} := \frac{1}{2} \left[\min_{\varrho^2 + \zeta^2 = 1} \lim_{\delta_x, \delta_y \rightarrow 0} H(\tilde{x}_{i\kappa} + \delta_x \varrho, \tilde{y}_{i\kappa} + \delta_y \zeta) + \max_{\varrho^2 + \zeta^2 = 1} \lim_{\delta_x, \delta_y \rightarrow 0} H(\tilde{x}_{i\kappa} + \delta_x \varrho, \tilde{y}_{i\kappa} + \delta_y \zeta) \right],$$

which is simplified to

$$h_{i\kappa} := H(\tilde{x}_{i\kappa}, \tilde{y}_{i\kappa})$$

if the function H is continuous at $(\tilde{x}_{i\kappa}, \tilde{y}_{i\kappa})$.

The continuous piecewise linear approximation $\tilde{H}(x, y)$ is then given by

$$\begin{vmatrix} x - \tilde{x}_{i12} & y - \tilde{y}_{i12} & \tilde{H}(x, y) - h_{i12} \\ \tilde{x}_{i23} - \tilde{x}_{i12} & \tilde{y}_{i23} - \tilde{y}_{i12} & h_{i23} - h_{i12} \\ \tilde{x}_{i31} - \tilde{x}_{i12} & \tilde{y}_{i31} - \tilde{y}_{i12} & h_{i31} - h_{i12} \end{vmatrix} = 0, \quad (x, y, \sigma) \in V_i. \quad (6.21)$$

We denote the value of the reconstruction $\tilde{H}$ at the point M_{ik} by $h_{ik} := \tilde{H}(M_{ik})$, and at the center of mass (x_i, y_i, σ_i) of V_i by $h_i := \tilde{H}(x_i, y_i)$, which, as one can easily show, is equal to

$$h_i = \frac{1}{|V_i|} \iiint_{V_i} \tilde{H}(x, y) dx dy d\sigma = \frac{1}{3}(h_{i1} + h_{i2} + h_{i3}) = \frac{1}{3}(h_{i12} + h_{i23} + h_{i31}).$$

Well-balanced Discretization of the Source Term

In order to maintain the “lake at rest” steady states ($\eta \equiv \hat{\eta}$, $u \equiv v \equiv \Omega \equiv 0$, where $\hat{\eta}$ is a constant), an appropriate discretization of the cell average of source term $\bar{\mathbf{S}}_i$ has to be developed to exactly balance the numerical fluxes computed by the proposed semi-discrete central-upwind scheme (6.15). We extend the well-balanced treatment of the 2-D triangular cells in [23] to the studied 3-D model.

Note that at the “lake at rest” steady states, $\mathbf{U}_{ik}(M_{ik}) \equiv \mathbf{U}_i(M_{ik}) \equiv (\hat{\eta}, 0, 0, 0)^T$, $\forall i, k$. After a substitution of this “lake at rest” steady state into the right-hand side (RHS) of

(6.15) and taking into account the fact that $a_{ik}^{\text{in}} = a_{ik}^{\text{out}}$, we conclude that the “lake at rest” state will be preserved at the discrete level provided the source quadrature satisfies the following two conditions:

$$-\frac{g}{|V_i|} \sum_{k=1}^3 A_{ik} \cos(\theta_{ik}) \frac{(\hat{\eta} - h_{ik})^2}{2} + \bar{S}_i^{(2)} = 0, \quad (6.22)$$

and

$$-\frac{g}{|V_i|} \sum_{k=1}^3 A_{ik} \sin(\theta_{ik}) \frac{(\hat{\eta} - h_{ik})^2}{2} + \bar{S}_i^{(3)} = 0, \quad (6.23)$$

where $\bar{\mathbf{S}}_i = (0, \bar{S}_i^{(2)}, \bar{S}_i^{(3)}, 0)^T$ with

$$\begin{aligned} \bar{S}_i^{(2)} &\approx -\frac{g}{|V_i|} \iiint_{V_i} (\hat{\eta} - h(x, y)) h_x(x, y) dx dy d\sigma, \\ \bar{S}_i^{(3)} &\approx -\frac{g}{|V_i|} \iiint_{V_i} (\hat{\eta} - h(x, y)) h_y(x, y) dx dy d\sigma. \end{aligned}$$

In order to derive the desired quadratures, we first apply the divergence theorem

$$\iiint_{V_i} \operatorname{div} \mathbf{g} dV = \iint_{\partial V_i} \mathbf{g} \cdot \mathbf{n} dS,$$

to the vector field $\mathbf{g} = (\frac{1}{2}(\eta(x, y) - h(x, y))^2, 0, 0)$ and obtain

$$\begin{aligned} -\iiint_{V_i} (\eta(x, y) - h(x, y)) h_x(x, y) dx dy d\sigma &= \sum_{k=1}^3 \iint_{(\partial V_i)_k} \frac{(\eta(x, y) - h(x, y))^2}{2} \cos(\theta_{ik}) dS \\ &\quad - \iiint_{V_i} (\eta(x, y) - h(x, y)) \eta_x(x, y) dx dy d\sigma. \end{aligned} \quad (6.24)$$

Next, we apply the midpoint rule to the integrals on the RHS of (6.24) and arrive at the following quadrature for the cell average $\bar{S}_i^{(2)}$:

$$\bar{S}_i^{(2)} = \frac{g}{2|V_i|} \sum_{k=1}^3 A_{ik} \cos(\theta_{ik}) (\eta_i(M_{ik}) - h_{ik})^2 - g(\eta_x)_i (\bar{\eta}_i - h_i), \quad (6.25)$$

where $(\eta_x)_i$ is the x -slope of the i th piece of the piecewise linear reconstruction of η . Similarly, we obtain the quadrature for the cell average $\bar{S}_i^{(3)}$:

$$\bar{S}_i^{(3)} = \frac{g}{2|V_i|} \sum_{k=1}^3 A_{ik} \sin(\theta_{ik}) (\eta_i(M_{ik}) - h_{ik})^2 - g(\eta_y)_i (\bar{\eta}_i - h_i), \quad (6.26)$$

where $(\eta_y)_i$ is the y -slope of the i th piece of the piecewise linear reconstruction of η .

Notice that since at “lake at rest” states $(\eta_x)_i \equiv (\eta_y)_i \equiv 0, \forall i$ and $\eta_i(M_{ik}) \equiv \hat{\eta}, \forall i, k$, the quadratures (6.25) and (6.26) satisfy (6.22) and (6.23), respectively, as long as $\mathbf{U}_i \equiv (\hat{\eta}, 0, 0, 0)^T$.

Non-negative Linear Reconstruction for η

We note that the use of the limited piecewise linear reconstruction (6.16) for η and piecewise linear approximation (6.21) of the bottom topography cannot guarantee that the reconstructed values (6.17) of the water surface level $\eta_i(M_{ik})$ and $\eta_{ik}(M_{ik})$ are above the bed level h_{ik} . This may introduce unphysical negative water depths at M_{ik} and therefore, the piecewise linear reconstruction (6.16) for η in the problematic cells have to be corrected to ensure the positivity preserving property, that is, that $D_i(M_{ik}) := \eta_i(M_{ik}) - h_{ik} \geq 0$ and $D_{ik}(M_{ik}) := \eta_{ik}(M_{ik}) - h_{ik} \geq 0$.

In this section, we extend the 2-D correction algorithm from [23] to the studied 3-D SWEs. By modifying $\tilde{\eta}$, we will in fact guarantee that $\tilde{\eta}(x, y) \geq \tilde{H}(x, y)$ throughout the entire computational domain.

The correction is performed as follows. First, for the dry cells in which $\bar{\eta}_i = h_i$, we simply set $\tilde{\eta}(x, y) = \tilde{H}(x, y)$. For the wet cells in which $\bar{\eta}_i > h_i$, the following two problematic cases are possible.

- Case 1: There is only one vertex, say, with index $\kappa = 12$, for which $\eta_i(\tilde{x}_{i\kappa}, \tilde{y}_{i\kappa}) < h_{i\kappa}$. In this case, we first correct this physically irrelevant reconstructed value by setting $\eta_i(\tilde{x}_{i12}, \tilde{y}_{i12}) = h_{i12}$. Next, we determine $\eta_i(\tilde{x}_{i\kappa}, \tilde{y}_{i\kappa})$ at the remaining two vertices with $\kappa = 23$ and 31 by setting $\eta_i(\tilde{x}_{i23}, \tilde{y}_{i23}) - h_{i23} = \eta_i(\tilde{x}_{i31}, \tilde{y}_{i31}) - h_{i31}$. Then, the conservation requirement yields

$$\eta_i(x_{i23}, y_{i23}) - h_{i23} = \eta_i(x_{i31}, y_{i31}) - h_{i31} = \frac{3}{2}(\bar{\eta}_i - h_i), \quad (6.27)$$

and the original reconstruction (6.16) for η is replaced with a new one defined by

$$\begin{vmatrix} x - x_i & y - y_i & \tilde{\eta}(x, y) - \bar{\eta}_i \\ \tilde{x}_{i12} - x_i & \tilde{y}_{i12} - y_i & h_{i12} - \bar{\eta}_i \\ \tilde{x}_{i23} - x_i & \tilde{y}_{i23} - y_i & \frac{3}{2}(\bar{\eta}_i - h_i) + h_{i23} - \bar{\eta}_i \end{vmatrix} = 0, \quad (x, y) \in V_i. \quad (6.28)$$

- Case 2: There are two vertices, say, with indices $\kappa = 12$ and 23, for which $\eta_i(\tilde{x}_{i\kappa}, \tilde{y}_{i\kappa}) < h_{i\kappa}$. In this case, we first correct these two physically irrelevant reconstructed values by setting $\eta_i(\tilde{x}_{i12}, \tilde{y}_{i12}) = h_{i12}$ and $\eta_i(\tilde{x}_{i23}, \tilde{y}_{i23}) = h_{i23}$. Then, the conservation requirement yields

$$\eta_i(\tilde{x}_{i31}, \tilde{y}_{i31}) - h_{i31} = 3(\bar{\eta}_i - h_i), \quad (6.29)$$

and the original reconstruction (6.16) for η is replaced with a new one defined by

$$\begin{vmatrix} x - x_i & y - y_i & \tilde{\eta}(x, y) - \bar{\eta}_i \\ \tilde{x}_{i12} - x_i & \tilde{y}_{i12} - y_i & h_{i12} - \bar{\eta}_i \\ \tilde{x}_{i31} - x_i & \tilde{y}_{i31} - y_i & 3(\bar{\eta}_i - h_i) + h_{i31} - \bar{\eta}_i \end{vmatrix} = 0, \quad (x, y) \in V_i. \quad (6.30)$$

Equipped with the positivity preserving reconstruction $\tilde{\eta}$, the point values of the velocities u and v needed in the computation of the one-sided local speeds in (6.18) and (6.20) are recalculated using the same desingularization procedure as in [23, 92]; namely (the i, k indexes are omitted):

$$u = \frac{\sqrt{2} D(Du)}{\sqrt{D^4 + \max(D^4, \varphi)}}, \quad v = \frac{\sqrt{2} D(Dv)}{\sqrt{D^4 + \max(D^4, \varphi)}},$$

where φ is a prescribed tolerance (in all of the numerical experiments reported below we have used $\varphi = 10^{-10}$). After obtaining the desingularized point values of u and v , we recompute the x - and y -discharges and fluxes accordingly, that is, we set

$$\begin{aligned} (Du) &:= D \cdot u, & (Dv) &:= D \cdot v, \\ \mathbf{F} &= \left((Du), (Du) \cdot u + \frac{g}{2} D^2, (Du) \cdot v, 0 \right)^T, \\ \mathbf{G} &= \left((Dv), (Dv) \cdot u, (Dv) \cdot v + \frac{g}{2} D^2, 0 \right)^T, \\ \mathbf{J} &= (\Omega', u \cdot \Omega', v \cdot \Omega', W^* \Omega')^T, \end{aligned}$$

at the points where these quantities are to be calculated.

6.4.2 Central-Upwind Scheme for the Depth-Averaged Equation (6.5)

We evolve the water surface elevation η by applying the “triangular” central-upwind scheme from [23, 91] to the vertically integrated free surface equation (6.5):

$$\begin{aligned} \frac{d\bar{\eta}_i}{dt} &= - \frac{1}{|T_i|} \sum_{k=1}^3 \frac{\ell_{ik} \cos(\theta_{ik})}{a_{ik}^{\text{in}} + a_{ik}^{\text{out}}} \left[a_{ik}^{\text{in}} ((\mathring{D}u)_{ik}(M_{ik})) + a_{ik}^{\text{out}} ((\mathring{D}u)_i(M_{ik})) \right] \\ &\quad - \frac{1}{|T_i|} \sum_{k=1}^3 \frac{\ell_{ik} \sin(\theta_{ik})}{a_{ik}^{\text{in}} + a_{ik}^{\text{out}}} \left[a_{ik}^{\text{in}} ((\mathring{D}v)_{ik}(M_{ik})) + a_{ik}^{\text{out}} ((\mathring{D}v)_i(M_{ik})) \right] \\ &\quad + \frac{1}{|T_i|} \sum_{k=1}^3 \ell_{ik} \frac{a_{ik}^{\text{in}} a_{ik}^{\text{out}}}{a_{ik}^{\text{in}} + a_{ik}^{\text{out}}} \left[\eta_{ik}(M_{ik}) - \eta_i(M_{ik}) \right]. \end{aligned} \quad (6.31)$$

Here, T_i is the triangular horizontal cross-section of the prism V_i ; ℓ_{ik} , $k = 1, 2, 3$ are the lengths of the corresponding sides of the triangle T_i ; $\mathring{D}u$ and $\mathring{D}v$ are vertically integrated fluxes calculated as follows. We first compute the vertically averaged quantities

$$\overline{(\mathring{D}u)} = \sum_{m: (x_m, y_m) = (x_i, y_i)} (\overline{Du})_m \Delta\sigma, \quad \overline{(\mathring{D}v)} = \sum_{m: (x_m, y_m) = (x_i, y_i)} (\overline{Dv})_m \Delta\sigma,$$

which are then used to obtain the required point values $(\mathring{D}u)_i(M_{ik})$ and $(\mathring{D}v)_i(M_{ik})$ for $k = 1, 2, 3$ using piecewise linear reconstruction (6.16), (6.17) with $\sigma = \sigma_i$ and the limiter

developed in [83]. Finally, the one-sided local speeds a_{ik}^{in} and a_{ik}^{out} are calculated using equations (6.18) and (6.19) with Du and Dv replaced with $\mathring{D}u$ and $\mathring{D}v$, respectively.

Remark. As before, we avoid division by 0 (or by a number close to 0), by replacing the scheme (6.31) with

$$\begin{aligned} \frac{d\bar{\eta}_i}{dt} = & -\frac{1}{|T_i|} \sum_{k=1}^3 \frac{\ell_{ik} \cos(\theta_{ik})}{2} \left[((\mathring{D}u)_{ik}(M_{ik})) + ((\mathring{D}u)_i(M_{ik})) \right] \\ & - \frac{1}{|T_i|} \sum_{k=1}^3 \frac{\ell_{ik} \sin(\theta_{ik})}{2} \left[((\mathring{D}v)_{ik}(M_{ik})) + ((\mathring{D}v)_i(M_{ik})) \right], \end{aligned}$$

whenever $a_{ik}^{\text{in}} + a_{ik}^{\text{out}} < 10^{-6}$.

Remark. The use of the continuous piecewise linear approximation of the bottom (see §6.4.1) and a nonnegative piecewise linear reconstruction of the water surface (see §6.4.1) guarantees the positivity preserving property of the presented scheme. Following the proof in [?], one can establish the following positivity preserving result.

Theorem 6.4.1 *Consider equation (6.5) and the semi-discrete central-upwind scheme (6.31) with the nonnegative piecewise linear reconstruction of the water surface presented in §6.4.1. Assume that the system of ODEs (6.31) is solved by either the forward Euler method or a SSP Runge-Kutta solver and that for all i , $\bar{\eta}_i^n - h_i \geq 0$ at time $t = t^n$. Then, for all i , $\bar{\eta}_i^{n+1} - h_i \geq 0$ at time $t = t^{n+1} = t^n + \Delta t$, provided that $\Delta t \leq \frac{1}{6a} \min_{i,k} \{r_{ik}\}$, where r_{ik} , $k = 1, 2, 3$ are the three corresponding altitudes of triangle T_i and $a := \max_{i,k} \{\max(a_{ik}^{\text{out}}, a_{ik}^{\text{in}})\}$.*

6.4.3 Numerical Scheme for Updating the Vertical Velocity Ω

After the water surface η and discharges Du and Dv have been evolved from time level $t = t^n$ to the new time level $t = t^{n+1} = t^n + \Delta t$, we use the obtained values to update the vertical velocity Ω as it is described in this section.

Let us assume that the $[-1, 0]$ interval is split into L uniform subintervals so that $\sigma_m := -1 + \Delta\sigma/L$, $m = 0, \dots, L$. We consider a prismatic cell V_i with the horizontal surfaces $\sigma = \sigma_m$ (top) and $\sigma = \sigma_{m-1}$ (bottom) for some $m \in \{1, \dots, L\}$. Then, sampling equation (6.6) at time $t = t^{n+1}$ and at $\sigma = \sigma_m$ and $\sigma = \sigma_{m-1}$ and subtracting one from the other results in

$$\begin{aligned} & \frac{1}{|T_i|} \int_{T_i} [\Omega(x, y, \sigma_m, t^{n+1}) - \Omega(x, y, \sigma_{m-1}, t^{n+1})] dx dy \\ &= -\frac{1}{|T_i|} \int_{T_i} \left[\eta_t(x, y, t^{n+1}) + \int_{\sigma_{m-1}}^{\sigma_m} \{ (Du)_x(x, y, \sigma, t^{n+1}) + (Dv)_y(x, y, \sigma, t^{n+1}) \} d\sigma \right] dx dy. \end{aligned} \tag{6.32}$$

Introducing the horizontal cell averages

$$\bar{\Omega}_i^m(t) := \frac{1}{|T_i|} \int_{T_i} \Omega(x, y, \sigma_m, t) dx dy,$$

using the second-order backward difference approximation of η_t , the central-upwind approximation of the $(Du)_x$ and $(Dv)_y$ terms, and the midpoint quadrature, equation (6.32) yields

$$\begin{aligned} \bar{\Omega}_i^m(t^{n+1}) &= \bar{\Omega}_i^{m-1}(t^{n+1}) - \frac{\Delta\sigma}{2\Delta t} [3\bar{\eta}_i(t^{n+1}) - 4\bar{\eta}_i(t^n) + \bar{\eta}_i(t^{n-1})] \\ &\quad - \frac{1}{|V_i|} \sum_{k=1}^3 \frac{A_{ik} \cos(\theta_{ik})}{a_{ik}^{\text{in}} + a_{ik}^{\text{out}}} \left[a_{ik}^{\text{in}}((Du)_{ik}(M_{ik})) + a_{ik}^{\text{out}}((Du)_i(M_{ik})) \right] \\ &\quad - \frac{1}{|V_i|} \sum_{k=1}^3 \frac{A_{ik} \sin(\theta_{ik})}{a_{ik}^{\text{in}} + a_{ik}^{\text{out}}} \left[a_{ik}^{\text{in}}((Dv)_{ik}(M_{ik})) + a_{ik}^{\text{out}}((Dv)_i(M_{ik})) \right] \\ &\quad + \frac{1}{|V_i|} \sum_{k=1}^3 A_{ik} \frac{a_{ik}^{\text{in}} a_{ik}^{\text{out}}}{a_{ik}^{\text{in}} + a_{ik}^{\text{out}}} \left[\eta_{ik}(M_{ik}) - \eta_i(M_{ik}) \right], \end{aligned} \tag{6.33}$$

where all of the quantities in the last three sums on the RHS of (6.33) are computed according to the algorithm described in §6.4.1, but using the data at the new time level $t = t^{n+1}$.

We note that the boundary condition $\Omega(x, y, \sigma = 0, t) = 0$ implies that for all cells V_i located at the bottom of the computational domain, $\bar{\Omega}_i^0(t^{n+1}) = 0$, and thus all of the values of $\bar{\Omega}_i^m(t^{n+1})$ can be recursively obtained from (6.33). Finally, we use the trapezoidal rule in the σ -direction to conclude with

$$\bar{\Omega}_i(t^{n+1}) = \frac{1}{2} \left[\bar{\Omega}_i^m(t^{n+1}) + \bar{\Omega}_i^{m-1}(t^{n+1}) \right],$$

where, as before,

$$\bar{\Omega}_i(t^{n+1}) \approx \frac{1}{|V_i|} \int_{V_i} \Omega(x, y, \sigma, t) dx dy d\sigma.$$

Remark. We note that at the first time-step, the second-order temporal discretization of η_t in (6.33) should be replaced with the first-order one, $\frac{\Delta\sigma}{\Delta t} [\bar{\eta}_i(t^{n+1}) - \bar{\eta}_i(t^n)]$.

6.5 Numerical Experiments

In this section, we demonstrate the performance of the proposed well-balanced and positivity preserving central-upwind scheme on five numerical examples.

6.5.1 Example 1—Transcritical Flow over a Hump

In this numerical example taken from [65, 172], we test the ability of the proposed numerical method to capture hydraulic jump discontinuities.

We numerically solve the 3-D SWEs with $g = 9.81$ and the bottom topography defined by

$$h(x, y) = \begin{cases} 0.2 - 0.05(x - 10)^2, & \text{if } 8 \leq x \leq 12, \\ 0, & \text{otherwise,} \end{cases}$$

on the computational domain $[0, 25] \times [0, 0.2] \times [-1, 0]$ discretized using prismatic cells of average volume 0.00125 with 4 vertical layers (Figure 6.2 shows the horizontal and vertical arrangement of the mesh).

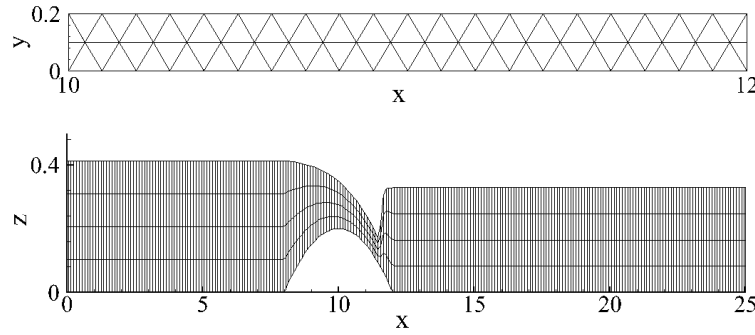


Figure 6.2: The horizontal (top) and vertical (bottom) mesh at the steady state.

We study the convergence of the computed solution in time towards steady flow. The solution is considered to be convergent if the following condition [191] is satisfied:

$$\sum_i \left(\frac{\bar{D}_i^n - \bar{D}_i^{n-1}}{\bar{D}_i^n} \right)^2 < 10^{-12}.$$

We consider the following set of initial and boundary data:

$$\begin{aligned} \eta(x, y, 0) &= 0.33, \quad (Du)(x, y, \sigma, 0) = 0, \quad (Dv)(x, y, \sigma, 0) = 0, \quad \Omega(x, y, \sigma, 0) = 0, \\ (Du)(0, y, \sigma, t) &= 0.18, \quad (Dv)(0, y, \sigma, t) = 0, \quad \eta(25, y, t) = 0.33, \end{aligned}$$

which corresponds to the transcritical steady flow with a shock. The analytical steady-state solutions can be found in [65].

Figure 6.3 (top) demonstrates a very good agreement between the computed and analytical values of the water surface. Figure 6.3 (bottom) shows that when the computed and analytical values of the discharge are compared, they also agree quite well though some oscillations can be observed near the turning point of water surface.

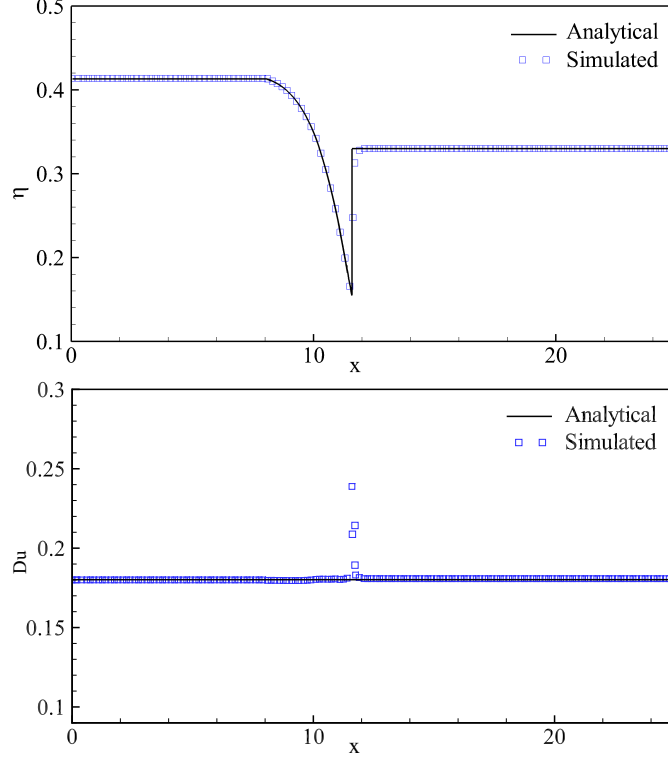


Figure 6.3: Comparison between computed and analytical steady states for water surface level η (top) and discharge Du (bottom).

6.5.2 Example 2—Wind-Induced Water Circulation

In this example, we test the accuracy of the proposed numerical method in predicting the vertical distribution of velocity. We simulate the wind-induced water circulation in an rectangular closed basin.

In this numerical experiment, we simplify the 3-D SWEs by neglecting the horizontal advection terms. At the same time, we include the vertical diffusion terms so that the relaxation system (6.7)–(6.10) becomes the following 1-D system:

$$\begin{aligned} (Du)_t + (\Omega' u)_\sigma &= \frac{\partial}{\partial \sigma} \left(\frac{\nu_v}{D} \frac{\partial u}{\partial \sigma} \right), \\ (Dv)_t + (\Omega' v)_\sigma &= \frac{\partial}{\partial \sigma} \left(\frac{\nu_v}{D} \frac{\partial v}{\partial \sigma} \right), \\ \Omega'_t + W^* \Omega'_\sigma &= -\frac{1}{\varepsilon} (\Omega' - \Omega), \end{aligned}$$

where ν_v is the vertical eddy viscosity. The diffusion terms are approximated in the flux form using centered differences. At the bottom, the centered-difference approximations are replaced with the one-sided ones. At the top surface, we replace the vertical gradients $\frac{\partial u}{\partial \sigma}$ and $\frac{\partial v}{\partial \sigma}$ with τ_x/ρ and τ_y/ρ , where τ_x and τ_y are the x - and y -components of the wind shear stress, respectively, and ρ is the water density. Note that in the studied setting, both the bottom topography and water surface are flat so that $D = \text{Const}$.

We take the following parameter values:

$$\nu_v = 0.01, \quad \rho = 1000, \quad \tau_x = 1.5, \quad \tau_y = 0, \quad D = 10,$$

for which the analytical steady-state solution is (see [73]):

$$u(\sigma) = \frac{D\tau_x(\sigma+1)(3\sigma+1)}{4\rho\nu_v}, \quad v(\sigma) \equiv 0.$$

We run the simulations using the zero initial conditions ($u(\sigma, 0) \equiv v(\sigma, 0) \equiv \Omega(\sigma, 0) \equiv 0$) and compute the solution until the final time $t = 200$, by which the solution is very close to its steady state. The simulation results obtained using evenly distributed 4, 7 and 10 layers are compared with the analytical solution in Figure 6.4. As one can see, the computed solutions are in good agreement with the analytical one and the accuracy is improved with higher resolution in the vertical direction.

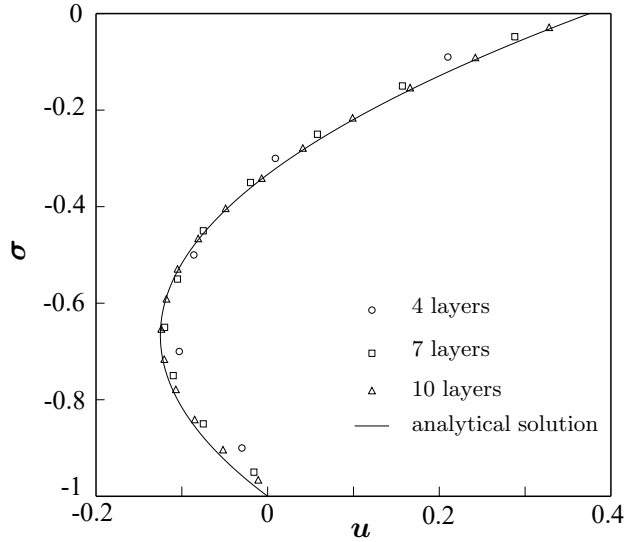


Figure 6.4: The numerical solution $u(\sigma, 200)$ computed using different number of vertical layers and the analytical (solid line) steady-state solution.

6.5.3 Example 3—Small Perturbation of a “Lake at Rest” Steady State

In this example, we demonstrate importance of the well-balanced property of the developed numerical method.

We consider the 3-D version of the initial value problem (IVP) studied in [23] (it is a modification of the benchmark originally proposed in [99]). We solve the 3-D SWEs with $g = 1$ on the computational domain is $[0, 2] \times [0, 1] \times [-1, 0]$. The bottom topography is an elliptical shaped hump defined by

$$h(x, y) = 0.8 \exp \left[-5(x - 0.9)^2 - 40(y - 0.5)^2 \right].$$

The initial conditions are

$$\eta(x, y, 0) = \begin{cases} 1 + \phi, & \text{if } 0.05 < x < 0.15, \\ 1, & \text{otherwise,} \end{cases} \quad u(x, y, \sigma, 0) \equiv v(x, y, \sigma, 0) \equiv \Omega(x, y, \sigma, 0) \equiv 0,$$

where ϕ is the height of a small perturbation of the water surface. Zero-order extrapolation is used at the right and left boundaries, while periodic boundary conditions are set in the y -direction.

We first take a very small $\phi = 10^{-14}$, which is comparable with the machine error, and numerically verify the well-balanced property of the proposed numerical method. In Figure 6.5, we plot $\max_{x,y} |\eta - 1|$ as a function of t , computed on a coarse mesh using 128 triangles in the horizontal cross section and 5 vertical layers. As one can clearly see, no instabilities are developed and the balance between the fluxes and source terms is numerically preserved.

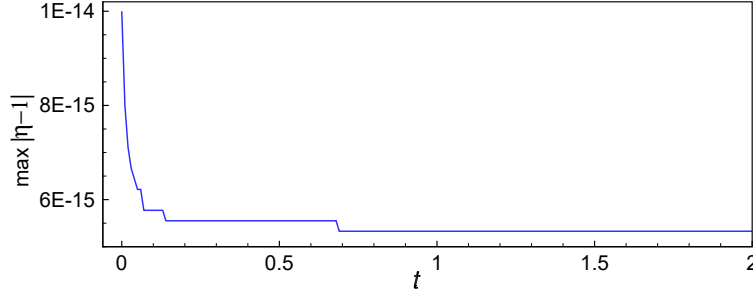


Figure 6.5: $\max_{x,y} |\eta - 1|$ computed on a coarse grid as a function of t .

We then take a larger, but still very small perturbation height $\phi = 10^{-4}$. Figure 6.6 (left column) displays the right-going disturbance as it propagates past the hump. The water surface, $\eta(x, y, t)$, computed on the mesh using 80000×10 prismatic cells, is presented at times $t = 0.6, 0.9, 1.2, 1.5$ and 1.8 . To demonstrate that the importance of the well-balanced property, we apply a non well-balanced method, obtained by replacing the well-balanced quadratures (6.25) and (6.26) by a straightforward midpoint rule discretization:

$$\bar{S}_i^{(2)} = -g(\bar{\eta}_i - h_i)(h_x)_i, \quad \bar{S}_i^{(3)} = -g(\bar{\eta}_i - h_i)(h_y)_i,$$

where $(h_x)_i$ and $(h_y)_i$ are the slopes of the piecewise linear approximation of the bottom topography given by (6.21). A “non well-balanced” solution, computed on the same 80000×10 grid, is shown in the right column of Figure 6.6. As one can clearly see there, the solution, computed by the non well-balanced scheme contains unphysical waves, which are of about the same magnitude as the waves generated by the small perturbation. We note that the obtained well-balanced results are similar to the well-balanced results reported in [23, Figure 5], but there is a certain difference attributed to the fact that we have solved the 3-D SWEs using 10 vertical layers, while the results in [?] were obtained for the single-layer 2-D SWEs.

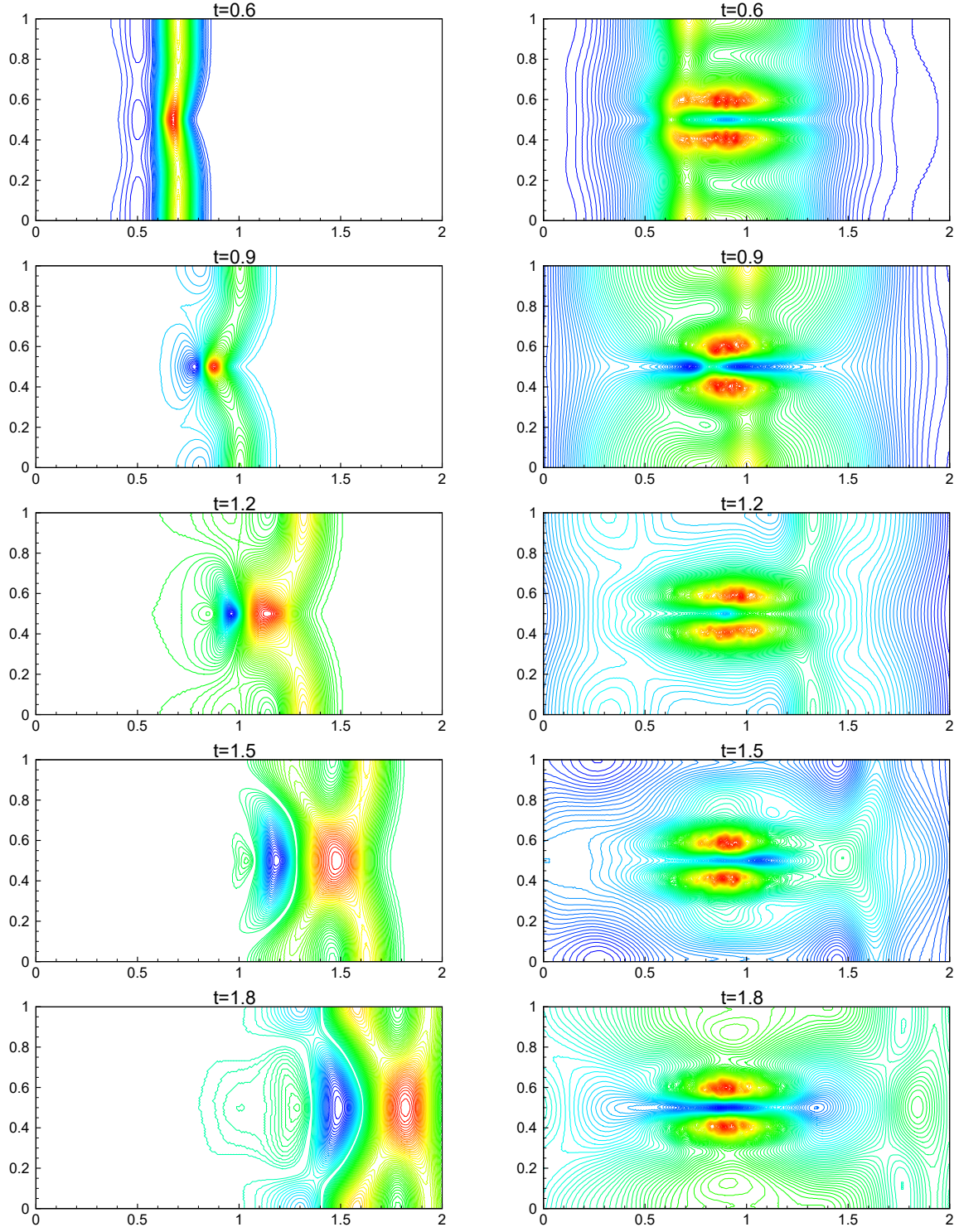


Figure 6.6: The η component of the solution computed by the well-balanced (left column) and non well-balanced (right column) methods.

6.5.4 Example 4—Wave Propagating over an Island

This test is designed to verify the positivity preserving property of the proposed numerical method by simulating a shallow overtopping wave.

The problem setting is a modification of the IVP studied in Example 3. The gravity acceleration is set to be $g = 2$ in this test. The bottom topography is

$$h(x, y) = 0.4 \exp \left[-5(x - 0.9)^2 - 10(y - 0.5)^2 \right].$$

The initial conditions are

$$\eta(x, y, 0) = \begin{cases} 0.5, & \text{if } 0.05 < x < 0.15, \\ 0.35, & \text{otherwise,} \end{cases} \quad u(x, y, \sigma, 0) \equiv v(x, y, \sigma, 0) \equiv \Omega(x, y, \sigma, 0) \equiv 0,$$

so that the right-going wave will travel past an elliptical shaped island. During this process, a very shallow overtopping flow will occur at the time when a small portion of the wave washes over the island and introduces a very small water depth on the initial dry island.

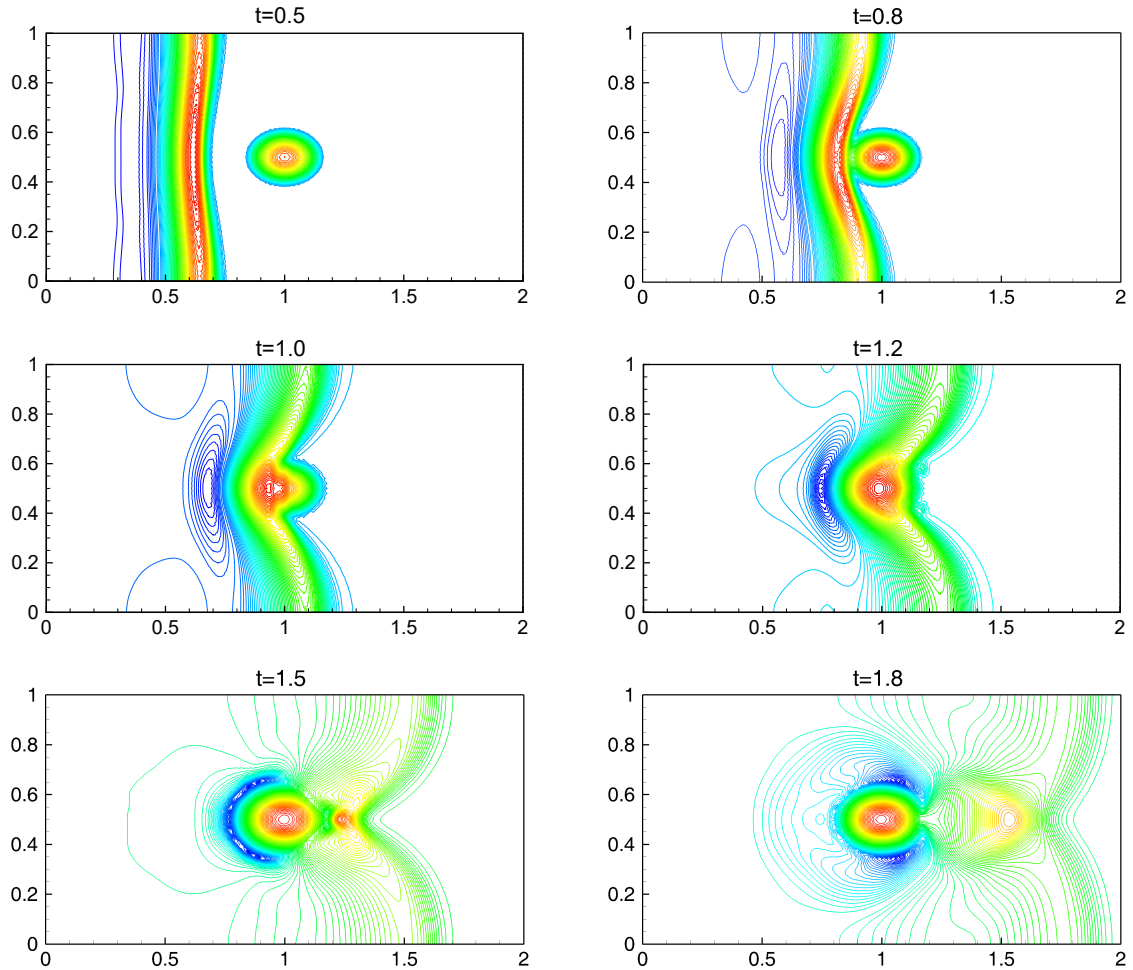


Figure 6.7: The η component of the computed solution.

Figure 6.7 shows the surface waves computed by the proposed numerical method at times $t = 0.5, 0.8, 1, 1.2, 1.5$ and 1.8 using 12800×8 prismatic cells. As one can see, no spurious waves are generated and no instabilities are developed when a very shallow wave washes over the initially dry island. Figure 6.8 shows the computed velocity field at the times $t = 0.8, 1$ and 1.2 when the wave travels past the island. As one can see, no oscillations or unphysical velocity values are observed.

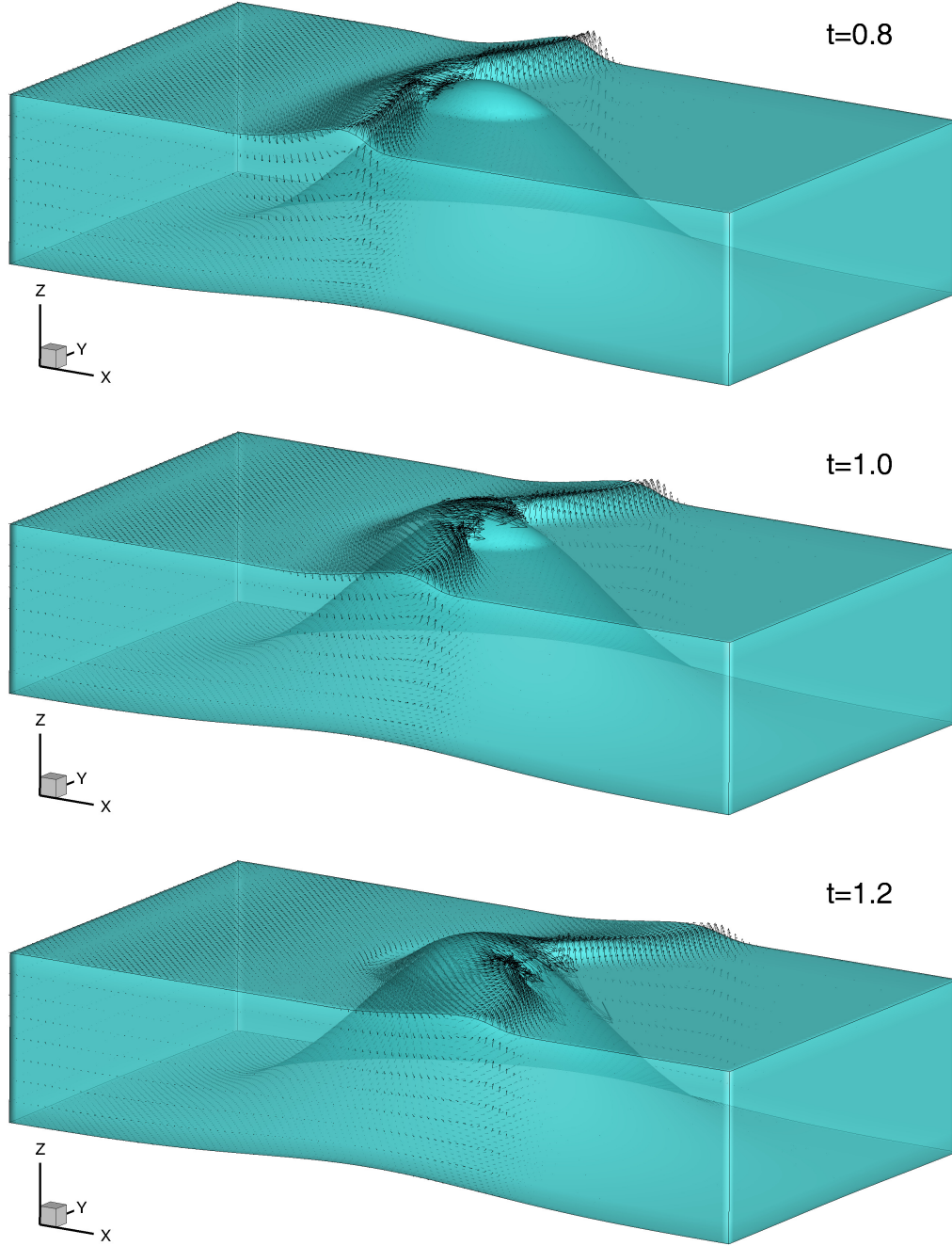


Figure 6.8: The velocity vector field on the surface of the computational domain.

6.5.5 Example 5—Laboratory Dam-Break Flow over a Triangular Hump

In the last numerical example, the accuracy and positivity preserving property of the proposed numerical method are tested by simulating a laboratory dam-break flow over a triangular hump in a long straight flume.

In this example, we modify the discharge equations (6.8) and (6.9) by including the vertical diffusion terms so that the modified discharge equations read as

$$\begin{aligned} (Du)_t + \left(Du^2 + \frac{g}{2}D^2\right)_x + (Duv)_y + (\Omega'u)_\sigma &= -gDh_x + \frac{\partial}{\partial\sigma} \left(\frac{\nu_v}{D} \frac{\partial u}{\partial\sigma}\right), \\ (Dv)_t + (Duv)_x + \left(Dv^2 + \frac{g}{2}D^2\right)_y + (\Omega'v)_\sigma &= -gDh_y + \frac{\partial}{\partial\sigma} \left(\frac{\nu_v}{D} \frac{\partial v}{\partial\sigma}\right), \end{aligned}$$

where ν_v is the vertical eddy viscosity. As in Example 2, we approximate the vertical diffusion terms in the flux form using centered differences. At the top of the computational domain, the centered-difference approximations are replaced with the one-sided ones, while at the bottom, we replace the vertical gradients $\frac{\partial u}{\partial\sigma}$ and $\frac{\partial v}{\partial\sigma}$ with τ_x/ρ and τ_y/ρ , where ρ is the water density and τ_x and τ_y represent the bottom stress, which is determined from a quadratic law as follows:

$$\tau_x = \rho C_f u_b (u_b^2 + v_b^2)^{\frac{1}{2}}, \quad \tau_y = \rho C_f v_b (u_b^2 + v_b^2)^{\frac{1}{2}}.$$

Here, u_b and v_b are velocities at the bottom and C_f is an empirical coefficient of the bottom friction.

In this numerical example, we take the following parameter values:

$$g = 9.81 \text{ m/s}^2, \quad \rho = 1000 \text{ kg/m}^3, \quad C_f = 0.012,$$

and estimate the vertical eddy viscosity by a two-layer mixing length model [132]:

$$\nu_v = \left(\frac{\ell_m}{D}\right)^2 \left[\left(\frac{\partial u}{\partial\sigma}\right)^2 + \left(\frac{\partial v}{\partial\sigma}\right)^2 \right]^{\frac{1}{2}},$$

in which the mixing length ℓ_m can be estimated by

$$\ell_m = \begin{cases} = 0.4(\sigma + 1)D, & \text{if } \sigma \leq -0.75, \\ 0.1D, & \text{otherwise.} \end{cases}$$

The experimental setup described in [119] is schematically shown in Figure 6.9. At time $t = 0$ s, the dam is located at $x = 15.5$ m in a 38 m long flume; the reservoir in the upstream is initially filled with water of 0.75 m deep; an initial dry bed is used in the downstream of the reservoir and a symmetric triangular hump of 0.4 m high is installed between $x = 25.5$ m and $x = 31.5$ m. The upstream end is a solid wall and the downstream end is a free outlet. The flow depth was measured by several gauges including Gauge 1,

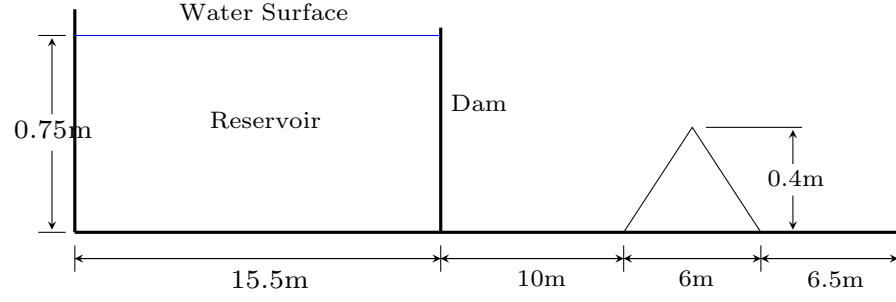


Figure 6.9: Experimental setup of the laboratory dam-break flow.

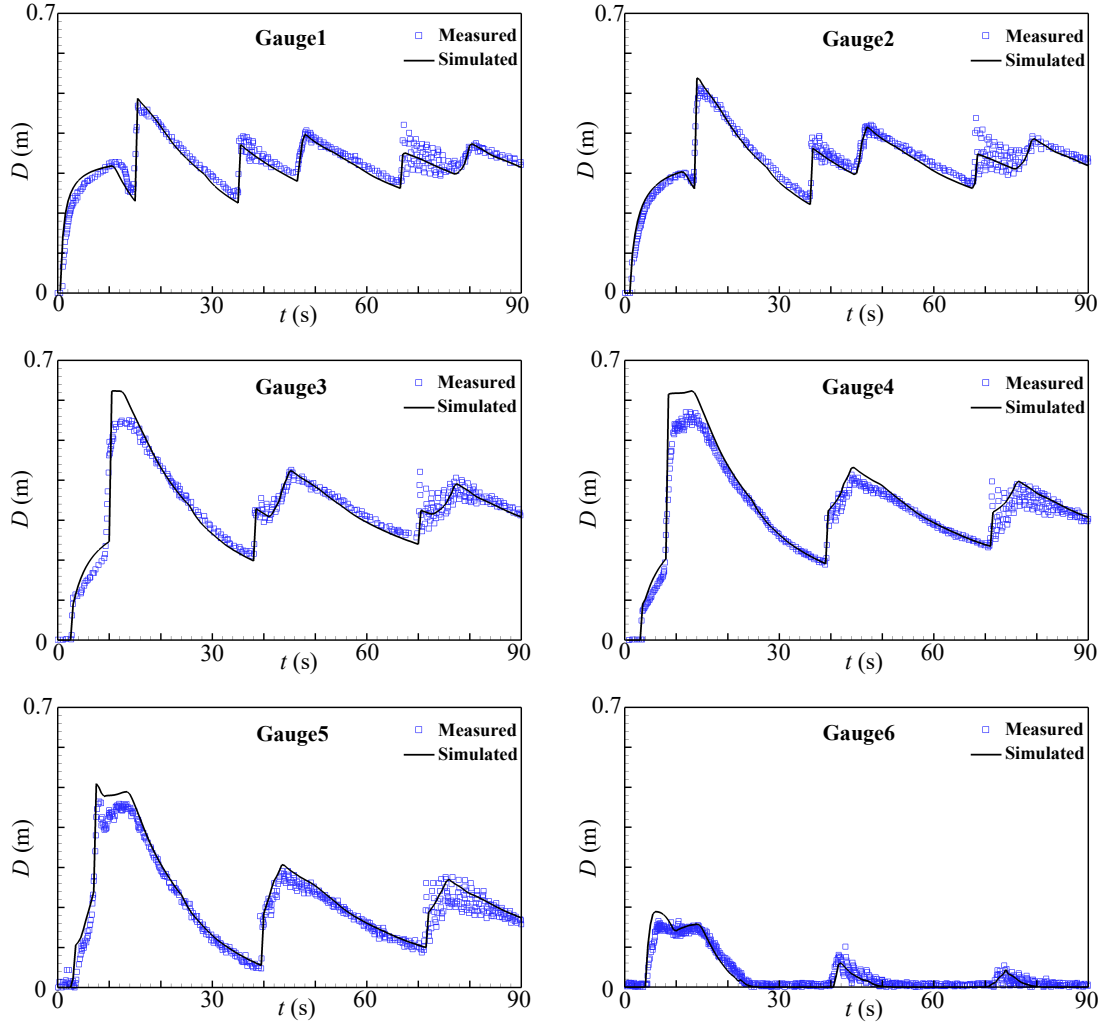


Figure 6.10: The comparison of computed and measured water depth versus time at different gauge points.

Gauge 2, Gauge 3, Gauge 4, Gauge 5 and Gauge 6, located at 2 m, 4 m, 8 m, 10 m, 11 m and 13 m, respectively, downstream from the dam.

We compute the solution until the final time $t = 90$ s on the computational domain $[0, 38] \times [0, 0.2] \times [-1, 0]$ using 11400×4 prismatic cells. Figure 6.10 shows the comparison between the computed and measured water depths over 90 s at the aforementioned 6 gauge points. It can be seen that the simulated results are in quite good agreement with the laboratory data at all of the gauge points.

In Figure 6.11, we take a close look at the wave front when it runs up the initially dry triangular hump at $t = 4$ s. As one can see, no unphysically large or spurious velocity is generated near the front. This confirms that the positivity preserving reconstruction successfully maintains stability in the areas with a small water depth. The effect of the bottom friction is also clearly demonstrated as one can see that the closer to the bottom, the smaller the velocity magnitude.

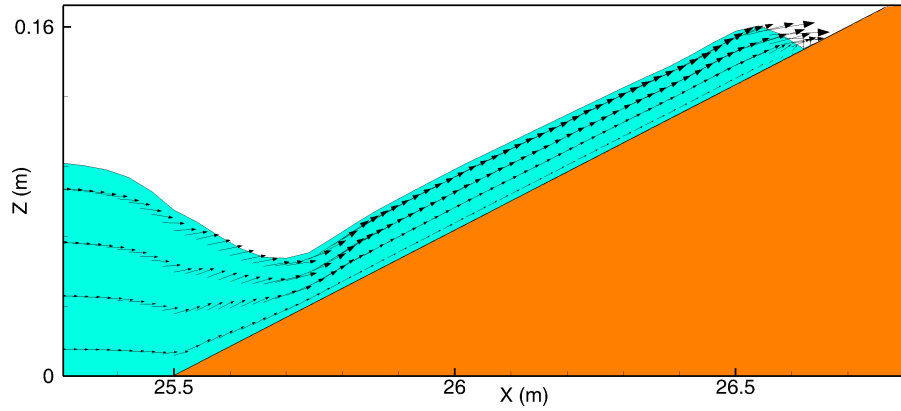


Figure 6.11: A cross-section at $y = 0$ of the water surface and velocity vector-field at time $t = 4$ s when a wave runs up the hump.

6.6 Conclusion

In this paper, we have developed a new well-balanced and positivity preserving shock capturing numerical method for the 3-D SWEs. The contributions and conclusions of the present study are discussed below.

The main contribution of this paper is that we have applied a relaxation approach to the 3-D shallow water system and “hyperbolized” it. We then developed a well-balanced and positivity preserving second-order semi-discrete central-upwind scheme for the extended relaxation system. The resulting shock capturing numerical method is applicable to cases with shock discontinuities arising in, for example, dam-break flows. This extends the applicability of the 3-D shallow water model.

We have ensured the well-balanced property of the proposed numerical method by extending the well-balanced discretization of the geometric source terms proposed in [23] for 2-D SWEs discretized on triangular grids, to the 3-D SWEs discretized on prismatic cells. Based on conducted numerical experiments, we draw the conclusion that the proposed well-balanced discretization can guarantee the balance between the numerical fluxes and bottom

topography source terms and the well-balanced property can prevent the appearance of unphysical waves in the simulations of quasi-steady states. In addition, in order to maintain the stability in cells with very small water depth and preserve the positivity of water depth throughout the entire computational domain, a continuous piecewise linear reconstruction of the bottom topography and a positivity preserving reconstruction of the water surface are applied to guarantee that the resulting numerical method is positivity preserving. The success in preserving the positivity of the water depth can also greatly extend the application range of the 3-D shallow water model.

The proposed relaxation based numerical method has been tested on five numerical experiments. The ability of our method to accurately capture shock discontinuities has been demonstrated in Examples 1 and 5. The accuracy of the proposed numerical method is verified by comparing the computed results with analytical solutions or laboratory data; see Example 1, 2 and 5. The importance of the well-balanced has been illustrated in Examples 3 and 4. The capability of the developed method to preserve the positive water depth have been demonstrated in Examples 4 and 5.

In summary, we have provided a novel framework for building 3-D shallow water systems, and thus their applicability range can be extended by adding other components, such as the Coriolis force, non-hydrostatic pressure and horizontal diffusion.

Chapter 7

A numerical model for three-dimensional shallow water flows with shocks over mobile topography

7.1 Abstract

This study aims to develop a novel three-dimensional (3-D) numerical model for shallow water flows over mobile topography, which is capable of simulating morphological evolution under shock waves, e.g. dam-break flows. The hydrodynamic model solves the three-dimensional shallow water equations (SWEs) using a finite volume method on prismatic cells in σ -coordinates. The morphodynamic model solves an Exner equation consisting of bed-load sediment transportation. Using a relaxation approach, a hyperbolic system is built for hydrodynamic system, which allows for using a Godunov-type central-upwind method to capture the shocks and approximate the numerical fluxes. Consequently, the 3D-SWEs-Exner model proposed in the present study can stably and accurately solve the dam-break flows over mobile beds. A spatially and temporally second-order “prismatic” central-upwind method is used for approximating the numerical fluxes through cell interfaces. The Exner equation is solved using an upwind method at the end of each time step. Using spatially linear reconstruction and explicit two-stage Runge-Kutta time discretization, second order accuracy is achieved in space and time. The proposed model is validated by several tests with experimental measurements, and is compared with the simulated results using reported two-dimensional (2-D) models.

Keywords: Three-dimensional shallow water, bed erosion, Dam-break, relaxation approach, σ -coordinates, finite volume method.

7.2 Introduction

In the past few decades, a number of 2-D numerical models for shallow water (SW) flows with shocks over mobile topography have been developed based on 2-D depth-averaged

SWEs. Vásquez [170] have developed a 2-D river morphology model referred as River2D-MOR which has been tested for dam-break flows over mobile bed in [171]. Zhang and Wu have developed a 2-D hydrodynamic and sediment transport model for dam break flows using finite volume method based on quad-tree grids [185]. Xia et al. [180] have developed a 2-D numerical model for dam-break flows over mobile bed using a coupled approach. Evangelista et al. [50] proposed a 2-D model to simulate dam-break waves on movable beds using a multi-stage centered scheme. Swartenbroekx et al. [160] have built a 2-D two-layer shallow water model for dam break flows with significant bed load transport. Liu et al. [112] developed a robust 2-D model for rapidly varying flows over erodible bed with wetting and drying. Canelas et al. [26] proposed a novel two-dimensional depth-averaged simulation tool for highly unsteady discontinuous flows over complex time-evolving geometries. Liu et al. [111] developed a well-balanced 2-D coupled model for dam-break flow over erodible bed. Wei et al. [78] have conducted a 2D hydrodynamic and sediment transport modeling of mega flood due to dam-break. Liu et al. [108] have proposed a two dimensional numerical model for rapidly varying flow over mobile bed, which is suitable for dam-break cases.

There are only a limited number of studies of 3-D shallow water models over erodible beds. Lesser et al. [97] discussed the operation of the DELFT3D module and presented the key features of the formulations used to model the fluid flow and sediment transport over mobile bed, in which the hydrodynamic module solves the 3-D shallow water equations [17, 18] based on assumption of hydrostatic status, using the finite difference method. However, the numerical model proposed in [97] cannot guarantee shock-capturing ability, therefore, it may failed in simulations of bed-erosion under unsteady-flows with shocks and discontinuities, e.g., dam-break flows. This prevents the extension of this model to the cases with shocks waves.

Inspired by the study [97] in which the 3-D SWEs and Exner formula have been used, the goal of this study is developing a 3-D numerical model for shallow water flows which is able to capture the shocks over mobile topography. Unlike the previously reported 3-D SW model over mobile bed [97] which may fail in dam-break cases, the proposed model can stably simulate the shock waves over mobile bed. This is accomplished by converting the 3-D SWEs into a hyperbolic form using a relaxation approach based on the work [2, 110], which allows for the use of shock capturing schemes, e.g. central upwind method. Consequently, the model is able to simulate dam-break flows without numerical instability. Furthermore, a second order “prismatic” central upwind method is used to estimate the fluxes through interfaces based on explicit finite volume method. The temporal evolution of topography is estimated by an Exner equation consisting of bed-load transport which is estimated by an upwind method. A stable total variation diminishing two-stage Runge-Kutta (TV-DRK2) solver is used in the proposed model to gain second order temporal accuracy. The model is validated by several laboratory tests with experimental data; the proposed 3-D model is compared with several reported 2-D models; and the simulated results using depth-averaged velocities are also compared with those using near bed velocities.

7.3 Governing equations

7.3.1 Hydrodynamic model

3-D SWEs with constant density are derived from the 3-D Navier-Stokes equations, after Reynolds-averaging and under the simplifying assumptions of hydrostatic pressure [17, 18]. Ignoring the horizontal viscous terms and coriolis effect, the convection-dominated 3-D SWEs have the following form:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \quad (7.1)$$

$$\frac{\partial u}{\partial t} + \frac{\partial uu}{\partial x} + \frac{\partial vu}{\partial y} + \frac{\partial wu}{\partial z} = \frac{\partial}{\partial z} \left(\nu_v \frac{\partial u}{\partial z} \right) - g \frac{\partial \eta}{\partial x} \quad (7.2)$$

$$\frac{\partial v}{\partial t} + \frac{\partial uv}{\partial x} + \frac{\partial vv}{\partial y} + \frac{\partial wv}{\partial z} = \frac{\partial}{\partial z} \left(\nu_v \frac{\partial v}{\partial z} \right) - g \frac{\partial \eta}{\partial y} \quad (7.3)$$

where t is the time; x , y and z are the Cartesian coordinates; $u(x, y, z, t)$ and $v(x, y, z, t)$ are the velocity components in the x - and y -directions, respectively; g is the gravitational acceleration; $\eta(x, y, t)$ is the water surface elevation; ρ is the density of fluid; and τ is the shear stress acting on the free surface or bottom surface. The system (7.1)–(7.3) together with either kinematic or dynamic boundary conditions for η constitute a closed system of equations for u , v , w and η . Figure 7.1 shows a typical water column in shallow water system in the current study.

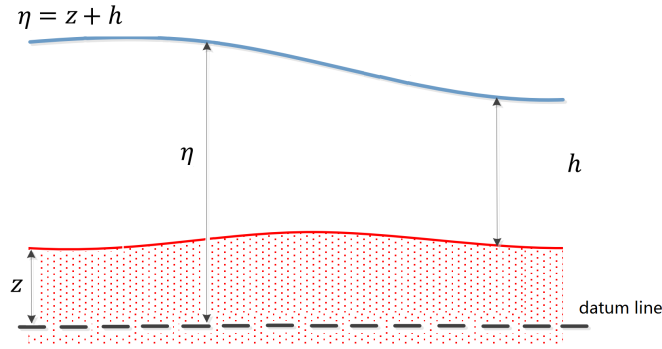


Figure 7.1: A typical water column in shallow water system in the current study.

In order to avoid the difficulties with representing the irregularly mobile topography in this study, the governing equations (7.1)–(7.3) are first transformed by replacing the vertical coordinate z with the σ coordinate using the following transformation [17, 18, 129]:

$$\sigma = \frac{z - \eta(x, y, t)}{D(x, y, t)}, \quad (7.4)$$

in which σ is the transformed vertical coordinate that varies between -1 and 0 , and $D(x, y, t)$ is the depth of the water column computed by

$$D(x, y, t) = \eta(x, y, t) - B(x, y), \quad (7.5)$$

where $B(x, y)$ is the bed level.

Accordingly, the 3-D SWEs can be written in the following conservative form in σ -coordinates

$$\frac{\partial \eta}{\partial t} + \frac{\partial(Du)}{\partial x} + \frac{\partial(Dv)}{\partial y} + \frac{\partial \omega}{\partial \sigma} = 0 \quad (7.6)$$

$$\frac{\partial(Du)}{\partial t} + \frac{\partial}{\partial x} \left(Du^2 + \frac{g}{2} D^2 \right) + \frac{\partial(Dvu)}{\partial y} + \frac{\partial(\omega u)}{\partial \sigma} = \frac{\partial}{\partial \sigma} \left(\frac{\nu_v}{D} \frac{\partial u}{\partial \sigma} \right) - Dg \frac{\partial B}{\partial x} \quad (7.7)$$

$$\frac{\partial(Dv)}{\partial t} + \frac{\partial(Duv)}{\partial x} + \frac{\partial}{\partial y} \left(Dv^2 + \frac{g}{2} D^2 \right) + \frac{\partial(\omega v)}{\partial \sigma} = \frac{\partial}{\partial \sigma} \left(\frac{\nu_v}{D} \frac{\partial v}{\partial \sigma} \right) - Dg \frac{\partial B}{\partial y} \quad (7.8)$$

where ω represents the vertical velocity in the σ -direction, which is related to w through

$$w = \omega + u \left(\sigma \frac{\partial D}{\partial x} + \frac{\partial \eta}{\partial x} \right) + v \left(\sigma \frac{\partial D}{\partial y} + \frac{\partial \eta}{\partial y} \right) + \left(\sigma \frac{\partial D}{\partial t} + \frac{\partial \eta}{\partial t} \right). \quad (7.9)$$

To close the system (7.6)-(7.8), we apply the kinematic boundary conditions, namely, $\omega(x, y, \sigma = -1, t) = 0$ and $\omega(x, y, \sigma = 0, t) = 0$, and integrate Eq. (7.6) with respect to σ from -1 to 0 to obtain

$$\eta_t + \left[\int_{-1}^0 Du \, d\sigma \right]_x + \left[\int_{-1}^0 Dv \, d\sigma \right]_y = 0. \quad (7.10)$$

The vertical eddy viscosity can be estimated by a two-layer mixing length model [132]:

$$\nu_v = l_m^2 \left[\left(\frac{\partial u}{\partial z} \right)^2 + \left(\frac{\partial v}{\partial z} \right)^2 \right]^{1/2} \quad (7.11)$$

where l_m is the mixing length which can be approximated as follows:

$$\begin{aligned} l_m &= \kappa z & \text{for } \kappa z \leq 0.1D, \\ l_m &= 0.1D & \text{for } \kappa z > 0.1D. \end{aligned} \quad (7.12)$$

where $\kappa=0.41$ is the Von Karman constant.

7.3.2 Morphodynamic model

In the current study, the topography is evaluated by the Exner equation,

$$\frac{\partial B}{\partial t} + \xi \frac{\partial q_{b,x}}{\partial x} + \xi \frac{\partial q_{b,y}}{\partial y} = 0 \quad (7.13)$$

in which ξ is a coefficient equal to $(1 - p)^{-1}$; p is the bed porosity; and $q_{b,x}$ and $q_{b,y}$ are components of bed-load discharge q_b in x and y -directions, respectively.

In this study, the bed-load transport rate is estimated by the Meyer-Peter and Müller (MPM) formula [116],

$$q_b = C_B \sqrt{(s-1)gd^3} \cdot (\theta - \theta_c)^{\frac{3}{2}} \quad (7.14)$$

in which $C_B=8.0$ is the bed-load coefficient; s is the specific gravity of sediment given by $s = \rho_s/\rho_f$, where ρ_s is the density of sediment particles and ρ_f is the density of water; d is the mean particle diameter; θ_c is the critical bed shear stress that can be estimated by the following formula [153]

$$\theta_c = \frac{0.3}{1 + 1.2d^*} + 0.055(1 - e^{-0.02d^*}) \quad (7.15)$$

where d^* is the dimensionless grain size calculated by

$$d^* = d \left[\frac{g(s-1)}{v_f^2} \right]^{1/3} \quad (7.16)$$

where v_f is the kinematic viscosity of fluid. Considering the slope effect, θ_c can be modified by [152]

$$\theta'_c = \theta_c \frac{\cos \psi \sin \beta + \sqrt{\cos^2 \beta \tan^2 \phi - \sin^2 \psi \sin^2 \beta}}{\tan \phi} \quad (7.17)$$

where $\phi = 30^\circ$ is the angle of repose of the bed material (sand) used in the current study, β is the angle of the bed and ψ is the angle between the upslope direction of the sloping bed and the direction of the flow.

The dimensionless bed shear stress, θ , can be calculated as

$$\theta = \frac{\tau_b}{[gd(\rho_s - \rho_f)]} \quad (7.18)$$

in which the bed shear stress τ_b normal to the cell interface is determined as

$$\begin{aligned} \tau_b &= \tau_{b,x}n_x + \tau_{b,y}n_y, & \tau_{b,x} &= \rho_f C_d u \sqrt{u^2 + v^2} \quad \text{and} \\ \tau_{b,y} &= \rho_f C_d v \sqrt{u^2 + v^2} \end{aligned} \quad (7.19)$$

where $\tau_{b,x}$ and $\tau_{b,y}$ are the horizontal components of τ_b in the x and y directions, respectively; u_0 and v_0 are the horizontal components of near-bed flow velocity in the x and y directions, respectively; and C_d is the drag coefficient that can be locally determined in terms of the Nikuradse roughness k_s [153],

$$C_d = \left[\frac{\kappa}{0.71 + \ln \left(\frac{k_s/30}{D} \right)} \right]^2 \quad (7.20)$$

where k_s is estimated by $k_s = C_N d$ in which C_N is the coefficient that is set to 3.5 in this study.

Substituting the Eqs. (7.18) and (7.19) into Eq. (7.14), Eq. (7.14) can be written as

$$\theta = \frac{C_d}{gd(s-1)(\eta - B)^2} q^\perp |q| \quad (7.21)$$

in which $q^\perp = (Du)n_x + (Dv)n_y$; and $|q| = \sqrt{(Du)^2 + (Dv)^2}$.

7.4 Numerical scheme

Since the governing system (7.6)–(7.8) is not in a hyperbolic form, one can not apply a shock capturing solver on the system. Consequently, the shock-capturing ability of the numerical method cannot be guaranteed. In order to stably and accurately simulate the sharp gradient and discontinuities in 3-D shallow water flows, in the present study, a relaxation model of the 3-D SWEs [110] is built, which is hyperbolic and equivalent to the original model in the zero relaxation limit ($\varepsilon \rightarrow 0^+$). The original work for transforming a system into a hyperbolic one, is presented in [2]. In order to obtain the relaxation model, an auxiliary variables ω' is introduced and a partial differential equation with a stiff source term is added to the original system, in which the variable ω is replaced with ω' as

$$\frac{\partial \eta}{\partial t} + \frac{\partial(Du)}{\partial x} + \frac{\partial(Dv)}{\partial y} + \frac{\partial \omega'}{\partial \sigma} = 0 \quad (7.22)$$

$$\frac{\partial(Du)}{\partial t} + \frac{\partial}{\partial x} \left(Du^2 + \frac{gD^2}{2} \right) + \frac{\partial(Dvu)}{\partial y} + \frac{\partial(\omega'u)}{\partial \sigma} = \frac{\partial}{\partial \sigma} \left(\frac{\nu_v}{D} \frac{\partial u}{\partial \sigma} \right) - Dg \frac{\partial B}{\partial x} \quad (7.23)$$

$$\frac{\partial(Dv)}{\partial t} + \frac{\partial(Duv)}{\partial x} + \frac{\partial}{\partial y} \left(Dv^2 + \frac{gD^2}{2} \right) + \frac{\partial(\omega'v)}{\partial \sigma} = \frac{\partial}{\partial \sigma} \left(\frac{\nu_v}{D} \frac{\partial v}{\partial \sigma} \right) - Dg \frac{\partial B}{\partial y} \quad (7.24)$$

$$\omega'_t + W^* \omega'_\sigma = -\frac{1}{\varepsilon} (\omega' - \omega) \quad (7.25)$$

where $\varepsilon > 0$ is a relaxation parameter and W^* is a propagation speed associated with the auxiliary variable ω' (in all of the numerical tests, $W^* = 1$).

The system (7.22)–(7.25) can be written in the following vector form:

$$\mathbf{U}_t + \mathbf{F}(\mathbf{U})_x + \mathbf{G}(\mathbf{U})_y + \mathbf{K}(\mathbf{U})_\sigma = \mathbf{S}(\mathbf{U}) + \mathbf{P}(\mathbf{U}) + \mathbf{R}(\mathbf{U}) \quad (7.26)$$

where $\mathbf{U} := (\eta, Du, Dv, \omega', B)^T$; $\mathbf{F}$, $\mathbf{G}$ and $\mathbf{J}$ are the fluxes:

$$\mathbf{F}(\mathbf{U}, h) := \left(Du, \frac{(Du)^2}{\eta - h} + \frac{g}{2} (\eta - h)^2, \frac{(Du)(Dv)}{\eta - h}, 0 \right)^T \quad (7.27)$$

$$\mathbf{G}(\mathbf{U}, h) := \left(Dv, \frac{(Du)(Dv)}{\eta - h}, \frac{(Dv)^2}{\eta - h} + \frac{g}{2} (\eta - h)^2, 0 \right)^T \quad (7.28)$$

$$\mathbf{K}(\mathbf{U}, h) := \left(\omega', \frac{(Du)\omega'}{\eta - h}, \frac{(Dv)\omega'}{\eta - h}, W^* \omega' \right)^T \quad (7.29)$$

and $\mathbf{S}$, $\mathbf{P}$ and $\mathbf{R}$ are the source terms:

$$\mathbf{S}(\mathbf{U}) := \left(0, -(\eta - B)g \frac{\partial B}{\partial x}, -(\eta - B)g \frac{\partial B}{\partial y}, 0 \right)^T \quad (7.30)$$

$$\mathbf{P}(\mathbf{U}) := \left(0, \frac{\partial}{\partial \sigma} \left(\frac{\nu_v}{D} \frac{\partial u}{\partial \sigma} \right), \frac{\partial}{\partial \sigma} \left(\frac{\nu_v}{D} \frac{\partial v}{\partial \sigma} \right), 0 \right)^T \quad (7.31)$$

$$\mathbf{R}(\mathbf{U}) := \left(0, 0, 0, -\frac{1}{\varepsilon} (\omega' - \omega) \right)^T \quad (7.32)$$

7.4.1 Eigenstructure of the relaxed hydrodynamic model

In the coupling model, the relaxation system (7.22)–(7.25) can be written in the following quasi-linear form:

$$\mathbf{U}_t + A_n(\mathbf{U})\mathbf{U}_x + B_n(\mathbf{U})\mathbf{U}_y + C_n(\mathbf{U})\mathbf{U}_\sigma = \mathbf{S}(\mathbf{U}) + \mathbf{P}(\mathbf{U}) + \mathbf{R}(\mathbf{U}),$$

where the matrices $A_n := \frac{\partial \mathbf{F}}{\partial \mathbf{U}}$, $B_n := \frac{\partial \mathbf{G}}{\partial \mathbf{U}}$ and $C_n := \frac{\partial \mathbf{K}}{\partial \mathbf{U}}$. The matrices $\mathbf{J}_n = A_n n_x + B_n n_y$ and C_n are

$$\mathbf{J}_n = \begin{pmatrix} 0 & n_x & n_y & 0 \\ -\frac{(Du)q^\perp}{(\eta-B)^2} + g(\eta-B)n_x & \frac{q^\perp}{\eta-B} + \frac{Du}{\eta-B}n_x & \frac{(Du)}{\eta-B}n_y & 0 \\ -\frac{(Dv)q^\perp}{(\eta-B)^2} + g(\eta-B)n_y & \frac{(Dv)}{\eta-B}n_x & \frac{q^\perp}{\eta-B} + \frac{Dv}{\eta-B}n_y & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad (7.33)$$

in which $q^\perp = (Du)n_x + (Dv)n_y$, n_x and n_y are components of unit normal vector $\mathbf{n}$ in the x and y directions, respectively.

C_n is the Jacobian matrix of the system in the vertical direction.

$$\mathbf{C}_n = \begin{pmatrix} 0 & 0 & 0 & 1 \\ -\frac{(Du)\omega'}{(\eta-h)^2} & \frac{\omega'}{\eta-h} & 0 & \frac{Du}{\eta-h} \\ -\frac{(Dv)\omega'}{(\eta-h)^2} & 0 & \frac{\omega'}{\eta-h} & \frac{Dv}{\eta-h} \\ 0 & 0 & 0 & W^* \end{pmatrix} \quad (7.34)$$

The eigenvalues of $\mathbf{J}_n$ are the roots of the characteristic polynomial $f(\lambda, \mathbf{U}) = \det(\mathbf{J}_n - \lambda \mathbf{I})$, which are

$$\lambda(\mathbf{J}_n) = \left\{ \frac{q^\perp}{(\eta-B)} + \sqrt{g(\eta-B)}, \frac{q^\perp}{(\eta-B)}, \frac{q^\perp}{(\eta-B)} - \sqrt{g(\eta-B)}, 0 \right\} \quad (7.35)$$

and the spectrum of $\mathbf{C}_n$ are

$$\lambda(\mathbf{C}_n) = \left\{ \frac{\omega'}{\eta-h}, \frac{\omega'}{\eta-h}, 0, W^* \right\} \quad (7.36)$$

7.4.2 Central upwind method for governing system

Using a split-step method, the governing system is solved in two steps, i.e., relaxation step and propagation step.

Relaxation step

In this step, one first implicitly solves the system

$$\mathbf{U}_t = \mathbf{R}(\mathbf{U}). \quad (7.37)$$

In the small relaxation limit ($\epsilon \rightarrow 0^+$), this step sets the local equilibrium condition and assigns the auxiliary variables to their relaxed values. At the first time step ($t = 0$) of the simulation, $\omega' = \omega$ is set.

Propagation step

In this step, a semi-discrete second-order central-upwind scheme on prismatic grid is used.

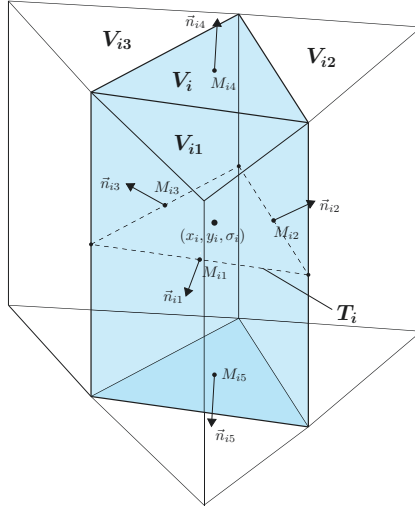


Figure 7.2: A typical prismatic cell in σ coordinates.

Figure 7.2 shows a typical triangular prismatic cell V_i in σ coordinates, with the centroid (x_i, y_i, σ_i) and of volume $|V_i|$. The prismatic cell has five faces of areas A_{ik} , $k = 1, \dots, 5$, whose corresponding outer unit normals are denoted by $\mathbf{n}_{ik} := (n_{x,ik}, n_{y,ik}, 0)^T$, $k = 1, 2, 3$, $\mathbf{n}_{i4} := (0, 0, 1)^T$ and $\mathbf{n}_{i5} := (0, 0, -1)^T$, see Figure 7.2. The corresponding neighboring cell shares the k -th face of the cell V_i is denoted by V_{ik} , $k = 1, \dots, 5$. The center of the k -th face of the cell V_i is denoted by $M_{ik} = (x_{ik}, y_{ik}, \sigma_{ik})$, $k = 1, \dots, 5$.

The second-order central-upwind schemes originally proposed for triangular grids [23, 91] and Cartesian grids [88, 86, 92] are extended to a prismatic cell in this paper as

$$\begin{aligned}
\frac{d\bar{\mathbf{U}}_i}{dt} = & -\frac{1}{|V_i|} \sum_{k=1}^3 \frac{A_{ik} n_{x,ik}}{a_{ik}^{\text{in}} + a_{ik}^{\text{out}}} [a_{ik}^{\text{in}} \mathbf{F}(\mathbf{U}_{ik}(M_{ik})) + a_{ik}^{\text{out}} \mathbf{F}(\mathbf{U}_i(M_{ik}))] \\
& -\frac{1}{|V_i|} \sum_{k=1}^3 \frac{A_{ik} n_{y,ik}}{a_{ik}^{\text{in}} + a_{ik}^{\text{out}}} [a_{ik}^{\text{in}} \mathbf{G}(\mathbf{U}_{ik}(M_{ik})) + a_{ik}^{\text{out}} \mathbf{G}(\mathbf{U}_i(M_{ik}))] \\
& +\frac{1}{|V_i|} \sum_{k=1}^3 A_{ik} \frac{a_{ik}^{\text{in}} a_{ik}^{\text{out}}}{a_{ik}^{\text{in}} + a_{ik}^{\text{out}}} [\mathbf{U}_{ik}(M_{ik}) - \mathbf{U}_i(M_{ik})] \\
& -\frac{1}{|V_i|} \sum_{k=4}^5 \frac{A_{ik} (-1)^k}{b_{ik}^{\text{in}} + b_{ik}^{\text{out}}} [b_{ik}^{\text{in}} \mathbf{K}(\mathbf{U}_{ik}(M_{ik})) + b_{ik}^{\text{out}} \mathbf{K}(\mathbf{U}_i(M_{ik}))] \\
& +\frac{1}{|V_i|} \sum_{k=4}^5 A_{ik} \frac{b_{ik}^{\text{in}} b_{ik}^{\text{out}}}{b_{ik}^{\text{in}} + b_{ik}^{\text{out}}} [\mathbf{U}_{ik}(M_{ik}) - \mathbf{U}_i(M_{ik})] + \bar{\mathbf{S}}_i + \bar{\mathbf{P}}_i
\end{aligned} \tag{7.38}$$

where $\mathbf{U}_i(M_{ik})$ and $\mathbf{U}_{ik}(M_{ik})$ are the reconstructed values at M_{ik} , $k = 1, 2, 3, 4, 5$, which are obtained using a piecewise linear reconstruction,

$$\tilde{\mathbf{U}}(x, y, \sigma) := \bar{\mathbf{U}}_i + (\hat{\mathbf{U}}_x)_i(x - x_i) + (\hat{\mathbf{U}}_y)_i(y - y_i) + (\hat{\mathbf{U}}_\sigma)_i(\sigma - \sigma_i), \quad (x, y, \sigma) \in V_i, \tag{7.39}$$

namely,

$$\mathbf{U}_i(M_{ik}) := \lim_{\substack{(x, y, \sigma) \rightarrow M_{ik} \\ (x, y, \sigma) \in V_i}} \tilde{\mathbf{U}}(x, y, \sigma), \quad \mathbf{U}_{ik}(M_{ik}) := \lim_{\substack{(x, y, \sigma) \rightarrow M_{ik} \\ (x, y, \sigma) \in V_{ik}}} \tilde{\mathbf{U}}(x, y, \sigma). \tag{7.40}$$

The slopes in linear reconstruction Eq. (7.39) have to be limited to suppress potential nonphysical oscillations near discontinuities. In the current study, the horizontal limited slopes $\hat{\mathbf{U}}_x$ and $\hat{\mathbf{U}}_y$ are computed using a multidimensional slope limiter proposed in [83]; the vertical limited slope $\hat{\mathbf{U}}_\sigma$ is computed using the minmod limiter (see, e.g., [123, 134, 166]).

The horizontal directional local speeds of propagation, a_{ik}^{in} and a_{ik}^{out} , are defined by

$$\begin{aligned}
a_{ik}^{\text{in}} &:= -\min \{ \lambda_1[J_n(\mathbf{U}_i(M_{ik}))], \lambda_1[J_n(\mathbf{U}_{ik}(M_{ik}))], 0 \} \\
a_{ik}^{\text{out}} &:= \max \{ \lambda_n[J_n(\mathbf{U}_i(M_{ik}))], \lambda_n[J_n(\mathbf{U}_{ik}(M_{ik}))], 0 \}
\end{aligned} \quad k = 1, 2, 3, \tag{7.41}$$

where $\lambda_1[J_n] \leq \lambda_2[J_n] \leq \dots \leq \lambda_{n-1}[J_n] \leq \lambda_n[J_n]$ are the eigenvalues of the Jacobian matrix J_n .

Similarly, the vertical directional local speeds of propagation, b_{ik}^{in} and b_{ik}^{out} , are defined by

$$\begin{aligned}
b_{ik}^{\text{in}} &:= -\min \{ \lambda_1[C_n(\mathbf{U}_i(M_{ik}))], \lambda_1[C_n(\mathbf{U}_{ik}(M_{ik}))], 0 \}, \\
b_{ik}^{\text{out}} &:= \max \{ \lambda_4[C_n(\mathbf{U}_i(M_{ik}))], \lambda_4[C_n(\mathbf{U}_{ik}(M_{ik}))], 0 \},
\end{aligned} \quad k = 4, 5, \tag{7.42}$$

where $\lambda_1[C_n] \leq \lambda_2[C_n] \leq \dots \leq \lambda_{n-1}[C_n] \leq \lambda_n[C_n]$ are the eigenvalues of the Jacobian matrix C_n .

The terms $\bar{\mathbf{S}}_i$ and $\bar{\mathbf{P}}_i$ in Eq. (7.38) are treated as source terms which are discretized using the divergence theorem.

In order to avoid division by 0 (or close to 0), in all the reported experiments, when $a_{ik}^{\text{in}} + a_{ik}^{\text{out}} < 10^{-6}$, the scheme (7.38) is simplified to

$$\begin{aligned} \frac{d\bar{\mathbf{U}}_i}{dt} = & -\frac{1}{|V_i|} \sum_{k=1}^3 \frac{A_{ik}n_{x,ik}}{2} [\mathbf{F}(\mathbf{U}_{ik}(M_{ik})) + \mathbf{F}(\mathbf{U}_i(M_{ik}))] \\ & -\frac{1}{|V_i|} \sum_{k=1}^3 \frac{A_{ik}n_{y,ik}}{2} [\mathbf{G}(\mathbf{U}_{ik}(M_{ik})) + \mathbf{G}(\mathbf{U}_i(M_{ik}))] \\ & -\frac{1}{|V_i|} \sum_{k=4}^5 \frac{A_{ik}(-1)^k}{2} [\mathbf{J}(\mathbf{U}_{ik}(M_{ik})) + \mathbf{J}(\mathbf{U}_i(M_{ik}))] + \bar{\mathbf{S}}_i + \bar{\mathbf{P}}_i \end{aligned} \quad (7.43)$$

7.4.3 Updating the water surface level η

After solving the horizontal discharges in section 7.4.2, the water surface is estimated by the continuity equation (7.10) which can be written as

$$\frac{\partial \eta}{\partial t} + \frac{\partial}{\partial x}(\bar{D}u) + \frac{\partial}{\partial y}(\bar{D}v) = 0. \quad (7.44)$$

where $\bar{D}u$ and $\bar{D}v$ are depth-averaged fluxes calculated using the results from section 7.4.2. In order to avoid the oscillations caused by discontinuities, typical 2D depth-averaged SWEs can be built in this step so that a shock-capturing solver can be applied, e.g., the "triangular" central upwind method for 2D SWEs in [23].

7.4.4 Updating the vertical velocity ω

In this step, the vertical velocity ω is updated in order to satisfy the continuity equation (7.6) using the η updated by Eq. (7.44). Applying the central-upwind method to terms $\partial(Du)/\partial x$ and $\partial(Dv)/\partial y$ in Eq. (7.6), and applying the second order backward differencing temporal discretization to term $\partial\eta/\partial t$ in Eq. (7.6), the equation (7.6) is solved as:

$$\begin{aligned} \omega_{it}(t + \Delta t) = & \omega_{ib}(t + \Delta t) - L \frac{3\eta_i(t + \Delta t) - 4\eta_i(t) + \eta_i(t - \Delta t)}{2\Delta t} \\ & - \frac{L}{|V_i|} \sum_{k=1}^5 \frac{A_{ik}n_{x,ik}}{a_{ik}^{\text{in}} + a_{ik}^{\text{out}}} [a_{ik}^{\text{in}}((Du)_{ik}(M_{ik})) + a_{ik}^{\text{out}}((Du)_i(M_{ik}))] \\ & - \frac{L}{|V_i|} \sum_{k=1}^5 \frac{A_{ik}n_{y,ik}}{a_{ik}^{\text{in}} + a_{ik}^{\text{out}}} [a_{ik}^{\text{in}}((Dv)_{ik}(M_{ik})) + a_{ik}^{\text{out}}((Dv)_i(M_{ik}))] \\ & + \frac{L}{|V_i|} \sum_{k=1}^5 A_{ik} \frac{a_{ik}^{\text{in}} a_{ik}^{\text{out}}}{a_{ik}^{\text{in}} + a_{ik}^{\text{out}}} [\eta_{ik}(M_{ik}) - \eta_i(M_{jk})]. \end{aligned} \quad (7.45)$$

where ω_{it} and ω_{ib} are vertical velocities at the top and bottom surfaces of the cell i , respectively. Note that, in the equation (7.45), $Du = Du(t + \Delta t)$, $Dv = Dv(t + \Delta t)$ and $\eta = \eta(t + \Delta t)$ are computed in sections 7.4.2 and 7.4.3. At the cell center, $\omega_i(t + \Delta t)$ is calculated by

$$\omega_i(t + \Delta t) = [\omega_{it}(t + \Delta t) + \omega_{ib}(t + \Delta t)] / 2 \quad (7.46)$$

7.4.5 Updating the bed profile

The bed level in the current model is updated by applying a semi-discrete upwind discretization to the Exner equation (7.13):

$$\frac{d\bar{B}_i}{dt} = - \frac{1}{|E_i|} \sum_{k=1}^3 l_{ik} n_{x,ik} \left[\beta_{ik} q_{b,x}(\mathbf{U}_i(M_{ik})) + (1 - \beta_{ik}) q_{b,x}(\mathbf{U}_{ik}(M_{ik})) \right] \quad (7.47)$$

$$- \frac{1}{|E_i|} \sum_{k=1}^3 l_{ik} n_{y,ik} \left[\beta_{ik} q_{b,y}(\mathbf{U}_i(M_{ik})) + (1 - \beta_{ik}) q_{b,y}(\mathbf{U}_{ik}(M_{ik})) \right] \quad (7.48)$$

where $|E_i|$ is the area of cross-sectional triangle of each prismatic control volume, l_{ik} is the length of each edge of E_i , β_{ik} is an upwind coefficient on l_{ik} , which can be estimated by

$$\beta_{ik} = \begin{cases} 0, & \text{if } \frac{1}{2} [u_i(M_{ik}) + u_{ik}(M_{ik})] n_{x,ik} + \frac{1}{2} [v_i(M_{ik}) + v_{ik}(M_{ik})] n_{y,ik} \geq 0, \\ 1, & \text{otherwise.} \end{cases} \quad (7.49)$$

7.4.6 Stability Criterion

A fully discrete scheme can be developed from (7.38), (7.43) and (7.47) using a stable ODE solver. In the present study, a total variation diminishing two-stage Runge-Kutta (TVDRK2) solver [63] is used in order to achieve second order accuracy in time. The time-step size Δt is limited by the CFL condition; see [91]:

$$\Delta t < \frac{1}{3} \min_{i,k} \left[\frac{r_{ik}}{\max(a_{ik}^{in}, a_{ik}^{out})}, \frac{e_{ik}}{\max(b_{ik}^{in}, b_{ik}^{out})} \right] \quad (7.50)$$

in which r_{ik} , $k = 1, 2, 3$ are the 3 corresponding altitudes of the prism V_i in the horizontal direction; and e_{ik} , $k = 4, 5$ are the 2 corresponding altitudes of the prism V_i in the vertical direction.

7.5 Model validation

7.5.1 Dam-break flow over mobile bed

A series of dam-break experiments were conducted by Leal et al. [96] in a 19.2 m long horizontal flume with a width of 0.5 m and a height of 0.7 m. A vertical gate that was originally located in the center of the flume was suddenly removed at $t=0$ s in order to simulate the dam-break. The wave-front travels to the downstream end of the flume in 6 s. Three groups of tests were carried out with respect to the bed types which are fixed bed, sand bed and pumice bed. In the current simulation, a test (Ts.28 in [96]) with sand bed is considered. The depth-averaged velocities are used to estimate the bed shear stress.

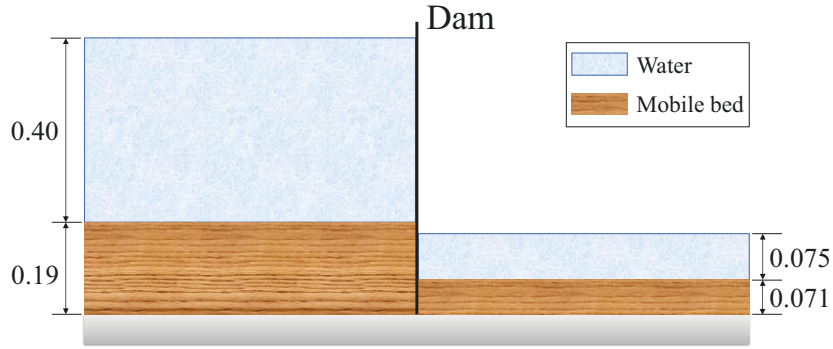


Figure 7.3: Initial condition of the dam-break over mobile bed.

At $t=0$ s, a 0.4 m thick flat sand bed is built under a 0.4 m deep quiescent water on the upstream side of the gate, and a 0.071 m thick flat sand bed is built under a 0.075 m deep quiescent water on the downstream side of the gate. The upstream side boundary is defined as a wall boundary condition, and the downstream side boundary is defined as a free outflow boundary condition with a 0.075 m fixed water level. The initial condition is illustrated in Figure 7.3.

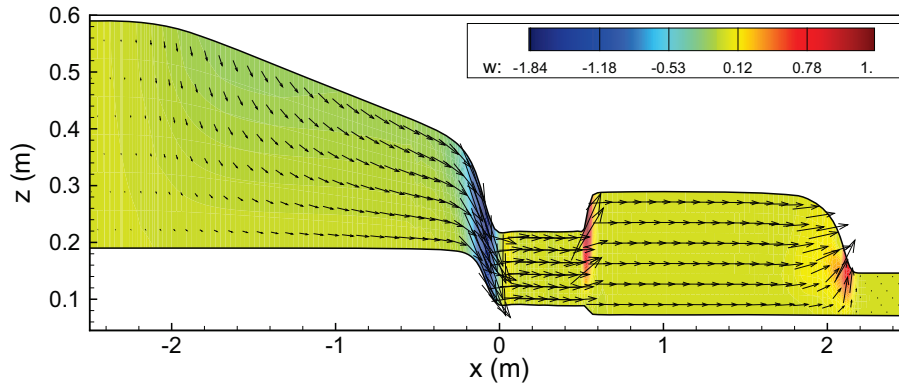


Figure 7.4: Velocity vectors of the dam-break flow at $t=1$ s.

In the current test, the computational domain is a $20 \text{ m} \times 0.5 \text{ m}$ area which is discretized

by 8628 unstructured triangular cells in the horizontal direction and 6 layers in the vertical direction. The horizontal meshes are locally refined in the area near the dam.

Figure 7.4 shows the velocity vectors of the dam-break flow at $t=1$ s. Relative high vertical velocity can be observed at the dam-break location ($x=0$ m). An obvious gradient of horizontal velocity at the downstream side of the dam can be seen at around $x=0.6$ m location caused by trans-critical flows, and a hydraulic jump is also observed. This test case emphasizes one of the advantages of the proposed model which is its ability to capture shock waves over erodible bed.

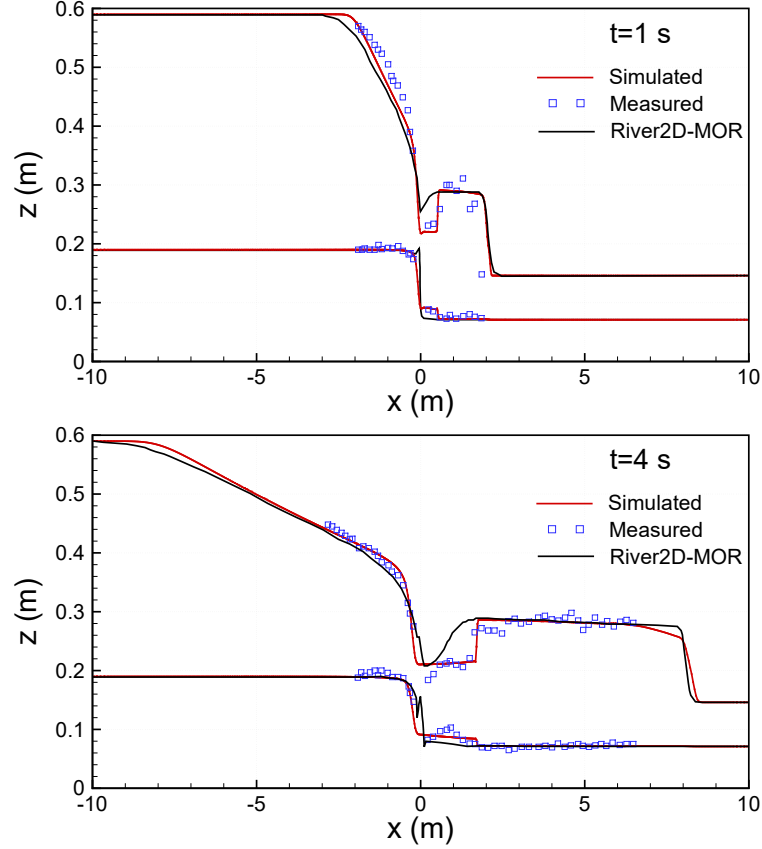


Figure 7.5: Simulated and measured results for dam-break flow over mobile bed at $t=1$ s and 4 s.

Figure 7.5 shows the comparisons between numerical and experimental results of the current test. It can be observed that the simulated results by the proposed 3-D model show generally good agreement with the experimental measurements at both $t=1$ s and 4 s. The simulated results by the proposed 3-D model are also compared with a published 2-D numerical model (River2D-MOR) reported in [171]. It can be seen that the proposed 3-D model yields better results than the 2-D model on the downstream side near the location of the initial dam. This may be attributed to the fact that, in locations where the vertical velocity cannot be neglected, e.g. the flow at the initial dam location ($x=0$ m) in this test, see Figure 7.4, 3-D SW model can yield more accurate prediction of horizontal velocities

than a 2-D model. This may lead to more accurate prediction of bed-load transport which is basically driven by horizontal velocities.

7.5.2 Dam-break flow over mobile bed with a sudden enlargement

This test case is chosen to verify the stability and accuracy of the current model simulating dam-break flows over erodible beds. The laboratory experiment was conducted in the Université catholique de Louvain, Belgium. The experimental results were reported in [126, 66, 81].

The experiments were conducted in a 6 m long flume, presenting a non-symmetrical sudden widening from 0.25 m to 0.5 m width, located 1 m downstream of the gate which is located at $x=3\text{m}$. The channel width is 0.25m on the upstream side of $x=4\text{m}$ and 0.5m on the downstream side of $x=4\text{m}$. Ultrasonic gauges and digital imagery equipments are placed and focused on different locations to take the measurements of the water surface and bed profile. The water surface levels are recorded at P1(3.75 m,0.125 m), P2(4.2m,0.125m), P3(4.45m,0.125m), P4(4.95 m,0.125 m), P5(4.2m,0.375m), P6(4.95m,0.375m), respectively. In this test, two locations with measured cross-section profiles of bed are chosen to verify the numerical results, which are at $x=4.1\text{m}$ and $x=4.4\text{m}$, respectively. The configuration of the current test is shown in Figure 7.6.

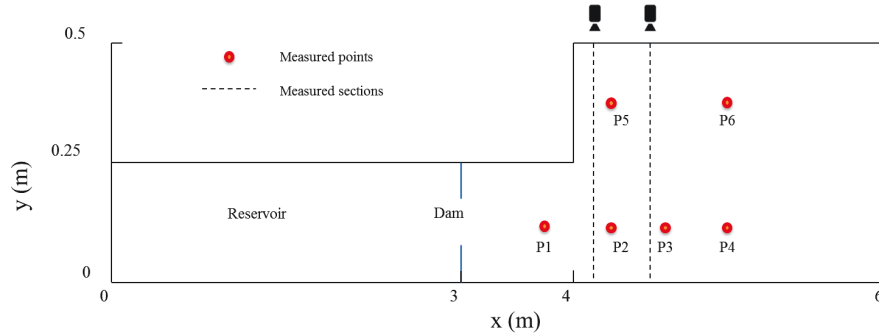


Figure 7.6: The configuration of measured points and measured sections of the current test.

A coarse and almost uniform sand was used as bed material with a median diameter of $d_{50}=1.72\text{ mm}$; specific density of 2.63 and a porosity of 0.39 [13]. At $t=0\text{ s}$, a 0.1 m thick layer of fully saturated sand over the whole flume is setup, and stationary water with a 0.25 m depth is stored in the reservoir on the upstream side of the gate. The gate is suddenly removed at $t=0\text{ s}$, and a dam-break flow with induced bed-load sediment transport travels towards downstream side. A free outflow boundary condition is used at the downstream outlet at $x=6\text{ m}$, and a free-slip boundary condition is applied at all side walls.

The computational domain is discretized by 6389 unstructured triangular cells in the horizontal direction and 5 layers in the vertical direction. Figure 7.7 shows the computational grid arrangement at $t=0\text{ s}$.

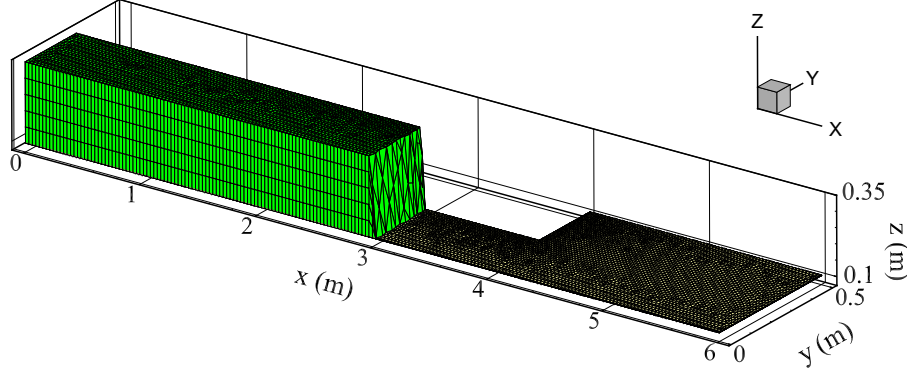


Figure 7.7: Arrangement of the 3-D computational cells.

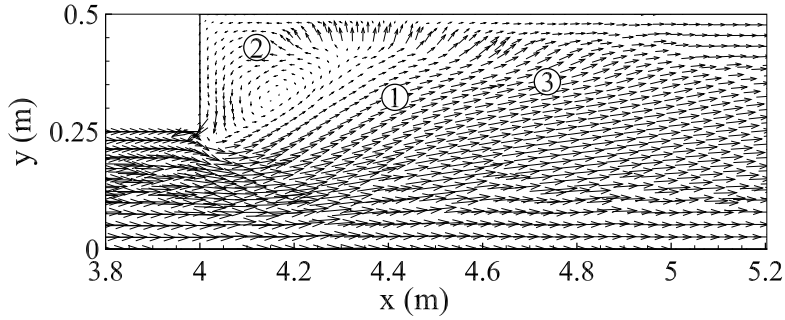


Figure 7.8: A comparison between simulated and measured horizontal velocity vectors.

Figure 7.8 shows the simulated horizontal velocity vectors on the water surface. As can be observed, the length of circulating zone is approximately 0.45 m which is close to the measured value in [66]. The simulated flow field is close to the measured one in [66], and different components of the flow structure can be found at three indicated regions: in region ①, deviated velocity orientations in the wider reach is observed; in region ②, a recirculation area at the corner forms; and in region ③, an oblique hydraulic jump caused by reflection against the sidewall is observed. Similar features are reported in [66].

Figure 7.9 shows the comparisons between the simulated and experimentally measured water levels at 6 measured points. An overall good agreement between simulated results and experimental measurements is observed. In Figure 7.9, NBV denotes the simulated results by the proposed 3-D model using near bed velocity to calculate bed evolution; and DAV denotes the simulated results by the proposed 3-D model using depth-averaged velocity to calculate bed evolution. Only very small differences are observed between NBV and DAV results in Figure 7.9.

Figure 7.10 shows the comparisons between the simulated and experimentally measured bed profiles at $t=12$ s at 2 measured cross-sections which are $x=4.1$ m and 4.4 m, respectively. It can be observed that, at $x=4.1$ m, the maximum scour depth predicted using depth-averaged velocity (DAV) is larger than that using near bed velocity (NBV). This may be due to the fact that near bed velocity is generally smaller than the depth-averaged velocity in 3-D model due to the bottom friction, which leads to different erosion depths.

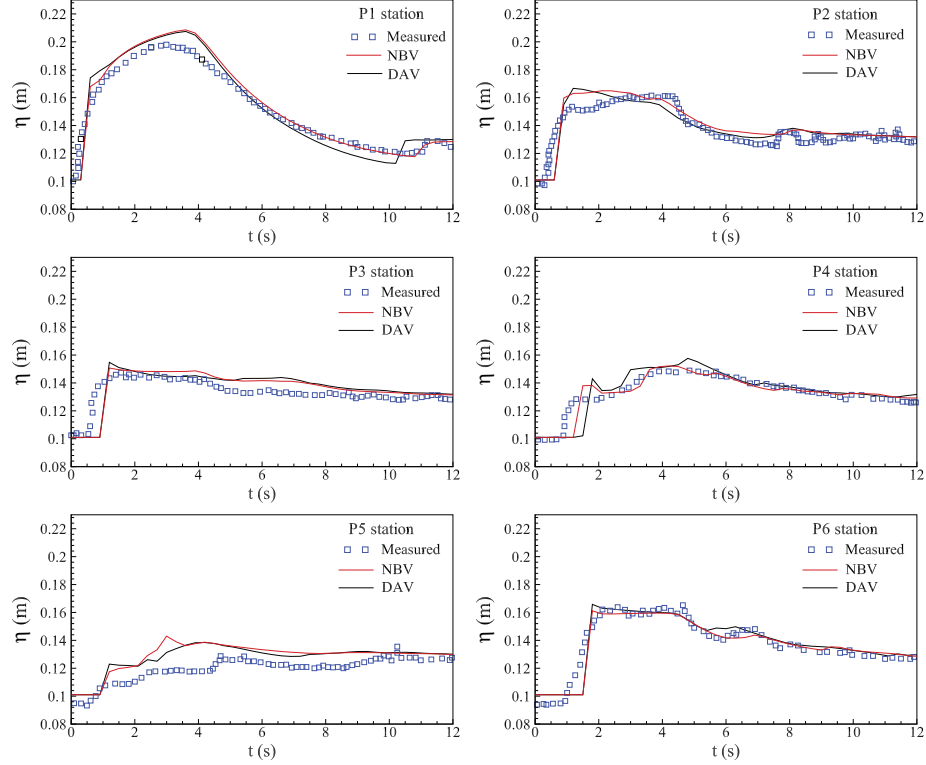


Figure 7.9: Simulated and measured water level η at 6 different measured points.

A similar problem is also observed at the $x=4.4$ m location, where the altitude of the dune simulated using DAV is larger than using NBV. At both measured locations, the trends of the topography variation are very close to the measured bed profile. Overall, one can conclude that both DAV and NBV methods can generally simulate the bed profile accurately. Small differences are shown between these two methods in the current test; however, the results by DAV method show generally better agreement with the measured data than NBV method. Therefore, the use of DAV is recommended for shallow water cases based on the proposed 3-D model.

7.5.3 Dam-break flow over mobile bed

The experiments were conducted at the Hydraulics Unit of the Mechanical and Civil Engineering Laboratory in Université catholique de Louvain, Belgium and the experimental results were published in [147]. The experiment was conducted in a 3.6 m wide flume with a useful length of 27 m. The breached dam was represented by two impervious blocks at about 12 m from upstream end and a 1 m wide gate located between the blocks. The origin of the axes is taken at the gate center. The configuration of experimental setup is shown in Figure 7.11. More details of the flume dimensions can be found in [147].

The grey area in Figure 7.11 was covered by an 85 mm thick sand layer, which extended over 1.5 m upstream of it and over 9 m downstream of the gate. A flat sand bed without

evolutions were recorded by 8 ultrasonic gauges which were located at: g1(0.64 m, -0.5 m), g2(0.64 m, -0.165 m), g3(0.64 m, 0.165 m), g4(0.64 m, 0.5 m), g5(1.94 m, -0.99 m), g6(1.94 m, -0.33 m), g7(1.94 m, 0.33 m) and g8(1.94 m, 0.99 m). At $t=0$ s, the gate is suddenly removed in order to simulate the dam-break. The experiment was considered to last for 20 s.

The computational domain is discretized by 15717 triangular cells in horizontal (see Figure 7.12) and 6 layers in vertical. The boundary at $x=15.5$ m is set to be free outlet boundary condition, and other boundaries are set to be free-slip wall condition.

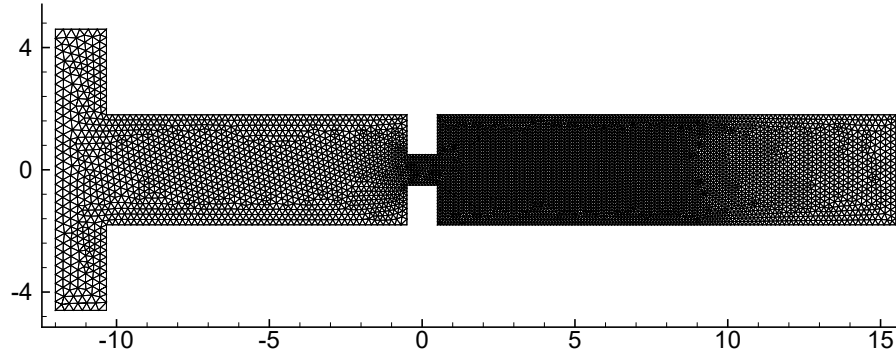


Figure 7.12: Horizontal mesh for the computational domain.

Figure 7.13 shows the temporal evolution of topography under dam-break flow.

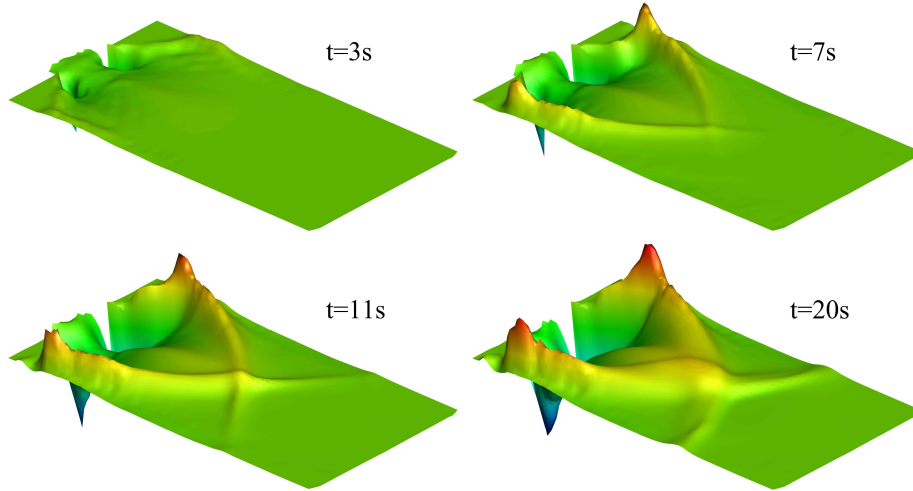


Figure 7.13: Temporal evolution of topography.

Based on the comparisons between DAV and NBV methods in Figure 7.14, one can observe that the DAV method yields more accurate results than NBV method. This may be attributed to the fact that the depth-averaged velocity is generally higher than the near bed velocity in the shallow water cases, which leads to larger magnitude of bed erosion.

The numerical results estimated by 2D-UCL1 model were reported in [146, 147], and are used in comparisons with the proposed 3-D model in this study, see Figure 7.14. The 2D-UCL1 model uses two-dimensional depth-averaged shallow water equations with MPM formula. It shows that the proposed 3-D model with DAV method yields more accurate results than the reported 2-D model (2D-UCL1).

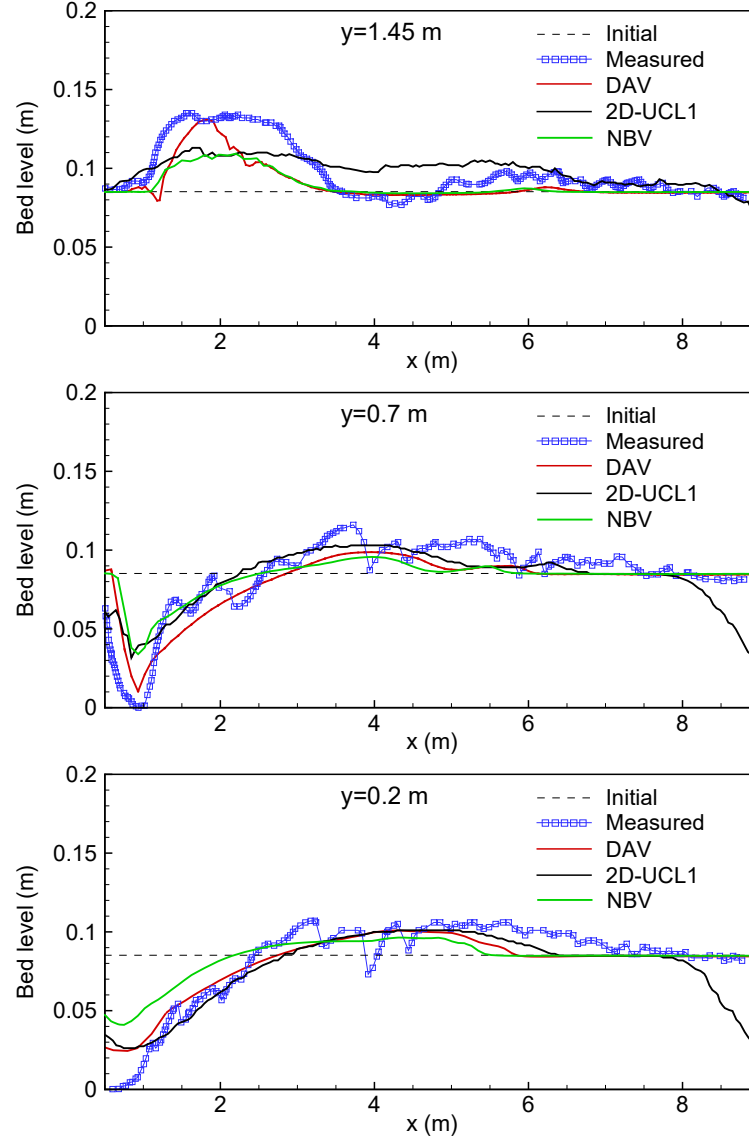


Figure 7.14: Comparisons among the measured bed profiles and simulated results by DAV method, NBV method and 2D-UCL1 model at $t=20$ s.

Figure 7.15 shows the comparisons between simulated and measured water levels at gauges 1 and 6 throughout the dam-break process. Good agreements are observed, and the accuracy of the proposed 3-D model is verified.

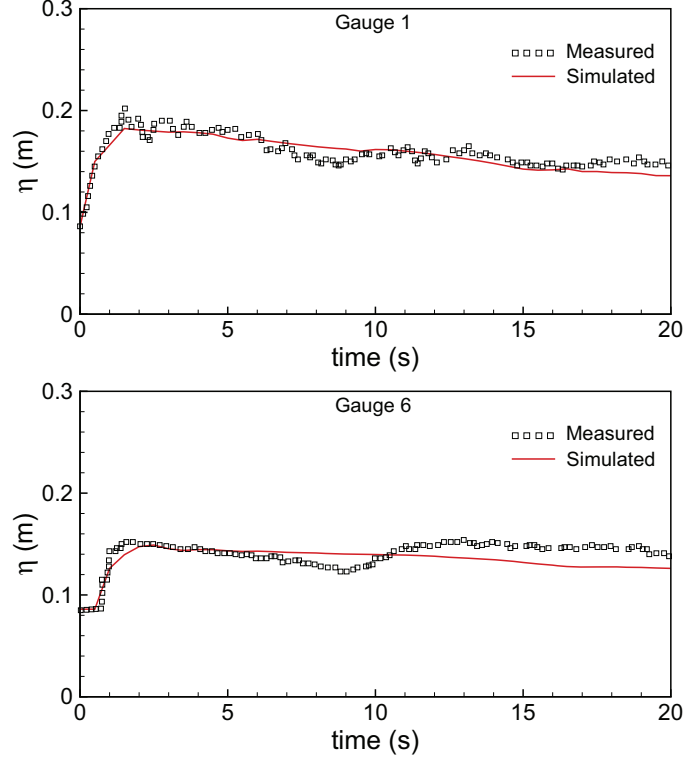


Figure 7.15: Simulated and measured water levels at gauge 1 and 6.

7.6 Concluding remarks

In the present paper, a novel 3-D numerical model for shallow water flows with shocks over mobile bed is successfully developed. In hydrodynamic model, the 3-D shallow water equations in sigma coordinate are converted to a hyperbolic form using a relaxation approach, which allows for the use of shock capturing schemes, e.g. central upwind method. Consequently, the model is able to simulate dam-break flows without numerical instability. A second order “prismatic” central upwind method is used to estimate the fluxes through interfaces based on explicit finite volume method. The temporal evolution of topography is estimated by an Exner equation consisting of MPM bed-load transport formula which has been widely used in practice. Water and bed levels are updated in a depth-averaged form based on the continuity equation. In order to capture the sharp gradient of water surface, a hyperbolic depth-averaged 2-D shallow water system is built, which allows for an approximation of fluxes by the central-upwind method. The bed-load fluxes in Exner equation are estimated by an upwind method on triangular cells in order to maintain numerical stability. The proposed model has been tested by several laboratory experiments of dam-break flows over mobile beds. Two main concluding remarks can be drawn as follows:

(1) One interesting aspect of 3-D model is about the way it performs for dam-break flows over mobile bed compared with 2-D depth-averaged models. In the test 7.5.1 and 7.5.3, the proposed 3-D model is compared with reported 2-D depth-averaged models; it shows that both models exhibit generally good agreements with experimental measurements.

However, the proposed 3-D model can yield better results near the dam-break location due to more accurate prediction of velocity vectors where three-dimensional features are dominant.

(2) Another interesting problem of 3-D model is the comparison of the use of depth-averaged velocity with near bed velocity to estimate bed-load transport. The model demonstrates which one gives more accurate results for shallow water cases. In the test [7.5.2](#) and [7.5.3](#), the numerical results predicted by the proposed 3-D model with NBV and DAV, are compared with experimental measurements respectively. It is shown that, both NBV and DAV methods can correctly predict the trends and locations of scour hole and deposition dune. However, the NBV method underestimates the magnitude of erosion, which can be attributed to the fact that the near bed velocity is smaller than depth-averaged velocity due to the bed friction, and that the bed-load transport is governed by horizontal velocities. After being verified by experimental data, one can conclude that the results by DAV method show generally better agreement with measured data than the results by NBV method in shallow water cases. Therefore, using depth-averaged velocity to estimate bed-load fluxes is generally recommended for shallow water cases using the proposed 3-D model.

Chapter 8

Conclusion and Recommendation

The objective of this project has been to develop and evaluate novel numerical models for shallow water flows and bed erosion. Both 2-D and 3-D shallow water systems are developed and have been extended to the cases with bed-erosion.

In order to solve 2-D shallow water flows with a rapid bed-erosion process, a two-dimensional coupled numerical model involving bed erosion process is proposed to simultaneously simulate rapidly varying flow, sediment transport and rapid bed evolution using the Finite Volume Method based on unstructured triangular grids. In addition to the studies of 2-D SWEs, novel numerical models for 3-D shallow water system are also developed and investigated in the current project. A novel model for 3-D SWEs is developed for the cases that involve discontinuities and shocks, e.g. dam break flows, based on the σ -coordinates. The proposed 3-D shallow water system is also extended to simulation of bed-erosion under dam-break flows.

In [chapter 2](#), a wetting and drying method is proposed which, while being highly stable, can efficiently track wet-dry fronts and stably simulate the bed erosion near the wet-dry fronts. In this paper, the central-upwind scheme is employed, which is proven to be highly efficient as it involves less computations. The numerical tests in this part show that the central-upwind method can precisely predict the bed erosion and shallow water flows. These advantages make the central-upwind method an ideal choice for a robust water-sediment-erosion model. The difficulties of several challenging benchmark-tests in this study are increased by including the sediment or dry-bed condition. Accordingly, the robustness of the current model is also verified.

In [chapter 3](#), a fully coupled model of water-sediment-bed system is developed. One can conclude that, applying the Lagrange theorem efficiently reduces the cost of finding the smallest and largest eigenvalues of the complicated 5×5 Jacobian. By comparing the results of the fully and partially coupled models, one may claim that the upper/lower bounds on the largest/smallest eigenvalues obtained using the Lagrange theorem are quite sharp; the bed evolution process is less oscillatory compared with a partially coupled model, especially when intense scouring and sediment exchange occur. The well-balanced discretization proposed in the current paper is designed for discontinuous bed level Z , approximated

using a (discontinuous) piecewise polynomial, which is evolved in the framework of finite-volume methods. This approach proves to be very robust and accurate.

In [chapter 4](#), a novel wave speed estimator seeking the smallest non-negative eigenvalue of coupled system as characteristic wave speed (S_r) in the central-upwind method, is proposed to compute the bed-load fluxes. Based on the results of the numerical tests, the proposed scheme can predict accurate results for both hydrodynamic and morphodynamic systems, with better agreement with the experimental data than the decoupled and coupled models using S_r . Consequently, the proposed scheme has advantages in terms of accuracy and stability which are shown in the numerical tests. In addition, we have provided analytical expressions for calculating the eigenvalues of the coupled shallow-water-Exner system, which greatly enhances the efficiency of the proposed method. The study in this chapter shows that the proposed scheme provides better accuracy and stability than conventional decoupled methods for rapid bed evolution under shallow water, and can avoid excessive numerical diffusions and oscillations.

In [chapter 5](#), a numerical model was developed for the simulation of 3-D unsteady non-hydrostatic shallow water flows. A sigma coordinate system with a collocated arrangement of three-dimensional velocity components was used to simulate the variation of water depth during the time steps. A special treatment of the pressure term was developed to avoid the water surface oscillations. By using an unstructured grid, the model benefits from the properties of the unstructured grid in the horizontal direction, like modeling of complicated geometries and grid refinement. Also, the σ -coordinate system enables the model to simulate flows with irregular topography. Use of a Godunov-type scheme for free surface modeling enables the model to simulate sub- and super-critical flows. Non-hydrostatic pressure is calculated by a Poisson-type equation which ensures exact mass conservation. The numerical results showed that the model is capable of producing non-oscillatory and highly accurate results. By solving the governing equations without the hydrostatic pressure assumption, the model is able to simulate short-period waves and vertically circulating flows, which is beyond the capability of hydrostatic models.

In [chapter 6](#), a three-dimensional shallow water model in which shocks and discontinuities can be solved by applying the central-upwind method is proposed, and the “well-balanced” and “positivity-preserving” properties are guaranteed. The contributions and conclusions of this study can be summarized as follows. In order to apply a Godunov-type central-upwind method for computing the horizontal velocities, a hyperbolic system is required. The main contribution of the study in this chapter is that a relaxation approach has been applied on the governing system, and thus one can build a hyperbolic system which allows the use of the central-upwind method. One main conclusion that can be drawn is that the proposed relaxation model in conjunction with the central-upwind method can stably and accurately simulate cases with shocks and discontinuities e.g. dam-break flows. This allows for the use of the 3-D shallow water model and its extension to a wider range. Unlike most existing 3-D shallow water models, in order to apply the central-upwind method, the governing equations are formulated in a conservative form in which one needs to calculate the gradient of bed level in the source term rather than the gradient of water surface. Thus, a special treatment of the bed-slope source term has to be performed in order to preserve the “well-balanced” property. We extend the “well-balanced” discretization in [\[23\]](#) to 3-D

case in prismatic cells. Based on the numerical tests, one can draw the conclusion that the “well-balanced” discretization can guarantee the balance between the numerical flux and bed-slope source term under the “lake at rest” state, and the “well-balanced” property can diminish the appearance of unphysical waves under quasi-steady states. Furthermore, in order to maintain the stability in cells of very small water depth and to preserve the positivity of water depth throughout the whole computational domain, a continuous reconstruction of topography and a proposed reconstruction of water surface are cooperatively applied to guarantee that the proposed 3-D model is “positivity-preserving”. The success in preserving the positivity of the water depth can greatly extend the application range of the 3-D shallow water model. The advantageous properties and accuracy of the proposed 3-D relaxation model are tested through several numerical experiments. One important superiority of the proposed model is the ability to capture the shocks and discontinuities. The accuracy of the model is also verified by comparing the computed results with analytical solutions or laboratory data. The advantages of the “well-balanced” property and the capacity to preserve the positive water depth are also demonstrated. In summary, a 3-D shallow water model was successfully built by using a relaxation approach and integrating the “well-balanced” and “positivity-preserving” properties. The model is verified to be accurate and robust. Note that in this paper, we provide a novel framework for building a 3-D shallow water system, and its application field can be extended by allowing the addition of other components, e.g., Coriolis term, non-hydrostatic pressure and horizontal diffusion.

In [chapter 7](#), the model proposed in [chapter 6](#) is extended to the simulations of bed-erosion, and a 3-D numerical model for dam-break flows over mobile bed is successfully developed. In the hydrodynamic model, the 3-D shallow water equations in σ coordinates are converted to a hyperbolic form using a relaxation approach, which allows for the use of shock capturing schemes, e.g. central upwind method. Consequently, the model is able to simulate dam-break flows without numerical instability. The bed-load fluxes in the Exner equation are estimated by an upwind method on triangular cells. The proposed model has been tested by several laboratory experiments of dam-break flows over mobile beds. Two main concluding remarks can be drawn as follows. (i) One interesting aspect of the 3-D model is about the way it performs for dam-break flows over mobile bed compared with 2-D depth-averaged models. In the comparison of the proposed 3-D model with reported 2-D depth-averaged models, it is shown that both models exhibit generally good agreements with experimental measurements. However, the proposed 3-D model can yield better results near the dam-break location due to more accurate prediction of velocity vectors where three-dimensional features are dominant. (ii) Another interesting feature of the 3-D model is the comparison of the use of depth-averaged velocity with near bed velocity to estimate bed-load transport. The model demonstrates which one gives more accurate results for shallow water cases. Based on various test cases, the numerical results predicted by the proposed 3-D model with NBV and DAV, are compared with experimental measurements respectively. It is shown that, both NBV and DAV methods can correctly predict the trends and locations of scour hole and deposition dune. However, the NBV method underestimates the magnitude of erosion, which can be attributed to the fact that the near bed velocity is smaller than depth-averaged velocity due to the bed friction, and

that the bed-load transport is governed by horizontal velocities. After verification with experimental data, one can conclude that the results by the DAV method show generally better agreement with measured data than the results by the NBV method in shallow water cases. Therefore, the use of depth-averaged velocity to estimate bed-load fluxes is more recommended for shallow water cases using the proposed 3-D model.

Based on the current project, the following recommendations can be suggested.

The aim of the current project is developing numerical methods. Although the proposed methods had been validated by experimental measurements and numerical tests with analytical solutions, it is always interesting to see how the proposed models perform when they are applied to real-world cases. Due to the complexity and uniqueness of such cases, several assumptions and empirical formulas used in the current project have to be modified and replaced in order to achieve more accurate and reasonable simulations. For examples, the MPM bed-load formula may need to be modified or replaced by other empirical formulas for different bed-materials; the classification of bed-material size and the sorting of multi-sized particles can be considered; the mechanism of sediment entrainment at the dam-break location needs to be further studied and applied to the proposed model; other classes of turbulence model may be adopted instead of the ones used in the current project. By including the non-hydrostatic pressure in the proposed 3-D SW model with relaxation approach, the model can be applied to a wider range of real-world cases. Finally, suspended load transport can be added to the proposed 3-D SW-erosion model in order to cover a wider range of applications.

References

- [1] R. Abgrall. On essentially non-oscillatory schemes on unstructured meshes: analysis and implementation. *Journal of Computational Physics*, 114:45–58, 1994.
- [2] R. Abgrall and S. Karni. Two-layer shallow water system: a relaxation approach. *SIAM Journal on Scientific Computing*.
- [3] C. Ai, S. Jin, and B. Lv. A new fully non-hydrostatic 3d free surface flow model for water wave motions. *International Journal for Numerical Methods in Fluids*, 66(11):1354–1370, 2011.
- [4] C. Ai and J. Sheng. Three-dimensional non-hydrostatic model for free-surface flows with unstructured grid. *Journal of Hydrodynamics, Ser. B*, 20(1):108–116, 2008.
- [5] F. Alcrudo and P. Garcia-Navarro. A high-resolution Godunov-type scheme in finite volumes for the 2D shallow-water equations. *International Journal for Numerical Methods in Fluids*, 16(6):489–505, 1993.
- [6] K Anastasiou and C.T. Chan. Solution of the 2D shallow water equations using the finite volume method on unstructured triangular meshes. *International Journal for Numerical Methods in Fluids*, 24(11):1225–1245, 1997.
- [7] E. Audusse, F. Bouchut, M.-O. Bristeau, R. Klein, and B. Perthame. A fast and stable well-balanced scheme with hydrostatic reconstruction for shallow water flows. *SIAM Journal on Scientific Computing*, 25(6):2050–2065, 2004.
- [8] E. Audusse and M. O. Bristeau. A well-balanced positivity preserving second-order scheme for shallow water flows on unstructured meshes. *Journal of Computational Physics*, 206(1):311–333, 2005.
- [9] E. Audusse, O. Delestre, M. H. Le, M. Masson-Fauchier, P. Navaro, and R. Serra. Parallelization of a relaxation scheme modelling the bedload transport of sediments in shallow water flow. In *ESAIM: Proceedings*, volume 43, pages 80–94. EDP Sciences, 2013.
- [10] R. Bakhtyar, A. Yeganeh Bakhtiary, D. A. Barry, and A. Ghaheri. Two-phase hydrodynamic and sediment transport modeling of wave-generated sheet flow. *Advances in Water Resources*, 32(8):1267–1283, 2009.

- [11] J. T. Batina. Implicit flux-split Euler schemes for unsteady aerodynamic analysis involving unstructured dynamic meshes. *AIAA journal*, 29(11):1836–1843, 1991.
- [12] R. M. Beam and R. F. Warming. An implicit finite-difference algorithm for hyperbolic systems in conservation-law form. *Journal of computational physics*, 22(1):87–110, 1976.
- [13] A. Becker and D Kaiser. Ecoulement consécutif à une rupture de barrage dans un canal présentant un élargissement brusque : mesures de la surface libre et de la topographie finale. M.S. thesis, Université catholique de Louvain, Louvain-la-Neuve, Belgium, 2008.
- [14] A Beljadid, A Mohammadian, and A Kurganov. Well-balanced positivity preserving cell-vertex central-upwind scheme for shallow water flows. *Applied Mathematical Modelling, Preprint, submitted for publication*.
- [15] F. Benkhaldoun, I. Elmahi, S. Sari, and M. Seaïd. A two-dimensional finite volume solution of dam-break hydraulics over erodible sediment beds. In *Finite volumes for complex applications. VI. Problems & perspectives. Volume 1, 2*.
- [16] C. Biscarini, S. Di Francesco, and P. Manciola. CFD modelling approach for dam break flow studies. *Hydrology and Earth System Sciences*, 14(4):705–718, 2010.
- [17] A. F. Blumberg and G. L. Mellor. A coastal ocean numerical model. In *Mathematical modelling of estuarine physics*, pages 203–219. Springer, 1980.
- [18] A. F. Blumberg and G. L. Mellor. A description of a three-dimensional coastal ocean circulation model. *Three-dimensional coastal ocean models*, 208:1–16, 1987.
- [19] J. P. Boris and D. L. Book. Flux-corrected transport. iii. minimal-error fct algorithms. *Journal of Computational Physics*, 20(4):397–431, 1976.
- [20] W. Boscheri, M. Dumbser, and M. Righetti. A semi-implicit scheme for 3d free surface flows with high-order velocity reconstruction on unstructured voronoi meshes. *International Journal for Numerical Methods in Fluids*, 72(6):607–631, 2013.
- [21] P. Brufau and P. Garcia Navarro. Two-dimensional dam break flow simulation. *International Journal for Numerical Methods in Fluids*, 33(1):35–57, 2000.
- [22] P. Brufau, P. Garcia-Navarro, and M.E. Vazquez-Cendon. Zero mass error using unsteady wetting-drying conditions in shallow flows over dry irregular topography. *International Journal for Numerical Methods in Fluids*, 45(10):1047–1082, 2004.
- [23] S. Bryson, Y. Epshteyn, A. Kurganov, and G. Petrova. Well-balanced positivity preserving central-upwind scheme on triangular grids for the Saint-Venant system. *ESAIM: Mathematical Modelling and Numerical Analysis*, 45(03):423–446, 2011.
- [24] T. Buffard and S. Clain. Monoslope and multislope muscl methods for unstructured meshes. *Journal of Computational Physics*, 229(10):3745–3776, 2010.

- [25] V. Caleffi, A. Valiani, and A. Zanni. Finite volume method for simulating extreme flood events in natural channels. *Journal of Hydraulic Research*, 41(2):167–177, 2003.
- [26] Ricardo Canelas, Javier Murillo, and Rui ML Ferreira. Two-dimensional depth-averaged modelling of dam-break flows over mobile beds. *Journal of Hydraulic Research*, 51(4):392–407, 2013.
- [27] Z. Cao and P. A. Carling. Mathematical modelling of alluvial rivers: reality and myth. Part 2: Special issues. In *Proceedings of the Institution of Civil Engineers-Water Maritime and Engineering*, volume 154, pages 297–308, 2002.
- [28] Z. Cao, R. Day, and S. Egashira. Coupled and decoupled numerical modelling of flow and morphological evolution in alluvial rivers. *Journal of Hydraulic Engineering*, 128(3):306–321, 2002.
- [29] Z. Cao, G. Pender, S. Wallis, and P. Carling. Computational dam-break hydraulics over erodible sediment bed. *Journal of Hydraulic Engineering*, 128(5):460–472, 2002.
- [30] H. Capart and D. L. Young. Formation of jump by the dam-break wave over a granular bed. *Journal of Fluid Mechanics*, 372:165–187, 1998.
- [31] H. Capart and D. L. Young. Two-layer shallow water computations of torrential geomorphic flows. In *Proceedings of River Flow*, pages 1003–1012. Balkema: Louvain-la-Neuve, 2002.
- [32] M. J. Castro-Díaz, E. D. Fernández-Nieto, and A. M. Ferreiro. Sediment transport models in shallow water equations and numerical approach by high order finite volume methods. *Computers & Fluids*, 37(3):299–316, 2008.
- [33] M. J. Castro-Díaz, E. D. Fernández Nieto, J. M. González-Vida, and C. Parés Madronal. Numerical treatment of the loss of hyperbolicity of the two-layer shallow-water system. *Journal of Scientific Computing*, 48(1-3):16–40, 2011.
- [34] V. Casulli. A semi-implicit finite difference method for non-hydrostatic, free-surface flows. *International Journal for Numerical Methods in Fluids*, 30(4):425–440, 1999.
- [35] V. Casulli and R. Cheng. Semi-implicit finite difference methods for three-dimensional shallow water flow. *International Journal for numerical methods in fluids*, 15(6):629–648, 1992.
- [36] V. Casulli and G. S. Stelling. Numerical simulation of 3D quasi-hydrostatic, free-surface flows. *Journal of Hydraulic Engineering*, 124(7):678–686, 1998.
- [37] V. Casulli and R. A. Walters. An unstructured grid, three-dimensional model based on the shallow water equations. *International journal for numerical methods in fluids*, 32(3):331–348, 2000.
- [38] V. Casulli and P. Zanolli. Semi-implicit numerical modeling of nonhydrostatic free-surface flows for environmental problems. *Mathematical and computer modelling*, 36(9):1131–1149, 2002.

- [39] L. Cea, G. Stelling, and M. Zijlema. Non-hydrostatic 3d free surface layer-structured finite volume model for short wave propagation. *International Journal for Numerical Methods in Fluids*, 61(4):382–410, 2009.
- [40] C. Chen, H. Liu, and R. C. Beardsley. An unstructured grid, finite-volume, three-dimensional, primitive equations ocean model: application to coastal ocean and estuaries. *Journal of atmospheric and oceanic technology*, 20(1):159–186, 2003.
- [41] X. Chen. A fully hydrodynamic model for three-dimensional, free-surface flows. *International Journal for Numerical Methods in Fluids*, 42(9):929–952, 2003.
- [42] A. Chertock, A. Kurganov, Z. Qu, and T. Wu. Three-layer approximation of two-layer shallow water equations. *Mathematical Modelling and Analysis*.
- [43] I. Christov and B. Popov. New nonoscillatory central schemes on unstructured triangulations for hyperbolic systems of conservation laws. *Journal of Computational Physics*, 227:5736–5757, 2008.
- [44] F. Coquel, K. El Amine, E. Godlewski, B. Perthame, and P. Rascle. A numerical method using upwind schemes for the resolution of two-phase flows. *J. Comput. Phys.*, 136:272–288, 1997.
- [45] S. Cordier, M. H. Le, and T. M. De Luna. Bedload transport in shallow water models: Why splitting (may) fail, how hyperbolicity (can) help. *Advances in Water Resources*, 34(8):980–989, 2011.
- [46] A. I. Delis and T. Katsaounis. Numerical solution of the two-dimensional shallow water equations by the application of relaxation methods. *Applied Mathematical Modelling*, 29(8):754–783, 2005.
- [47] M. I. Drago and L. Iovenitti. σ -Coordinates hydrodynamic numerical model for coastal and ocean three-dimensional circulation. *Ocean Engineering*, 27(10):1065–1085, 2000.
- [48] T. G. Drake and J. Calantoni. Discrete particle model for sheet flow sediment transport in the nearshore. *Journal of Geophysical Research: Oceans (1978–2012)*, 106(C9):19859–19868, 2001.
- [49] L. J. Durlofsky, B. Engquist, and S. Osher. Triangle based adaptive stencils for the solution of hyperbolic conservation laws. *Journal of Computational Physics*, 98:64–73, 1992.
- [50] S. Evangelista, M. S. Altinakar, C. Di Cristo, and A. Leopardi. Simulation of dam-break waves on movable beds using a multi-stage centered scheme. *International Journal of Sediment Research*, 28(3):269–284, 2013.
- [51] S. Fagherazzi and T. Sun. Numerical simulations of transportational cyclic steps. *Computers & Geosciences*, 29(9):1143–1154, 2003.

- [52] R. A. Falconer, B. Lin, Y. Wu, and E. Harries. Divast model: reference manual. Environmental Water Management Research Centre, School of Engrg, Cardiff Univ, 1999.
- [53] R. J. Fennema and M. H. Chaudhry. Explicit methods for 2-d transient free surface flows. *Journal of Hydraulic Engineering*, 116(8):1013–1034, 1990.
- [54] R. M. L. Ferreira, M. J. Franca, Joao G. A. B. Leal, and A. H. Cardoso. Mathematical modelling of shallow flows: Closure models drawn from grain-scale mechanics of sediment transport and flow hydrodynamics. *Canadian Journal of Civil Engineering*, 36(10):1605–1621, 2009.
- [55] L. Fraccarollo and A. Armanini. A semi-analytical solution for the dam-break problem over a movable bed. *Proceedings, European Concerted Action on Dam-Break Modeling*, pages 145–152, 1998.
- [56] L. Fraccarollo and H. Capart. Riemann wave description of erosional dam-break flows. *Journal of Fluid Mechanics*, 461:183–228, 2002.
- [57] L. Fraccarollo, H. Capart, and Y. Zech. A godunov method for the computation of erosional shallow water transients. *International Journal for Numerical Methods in Fluids*, 41(9):951–976, 2003.
- [58] O. B. Fringer, M. Gerritsen, and R. L. Street. An unstructured-grid, finite-volume, nonhydrostatic, parallel coastal ocean simulator. *Ocean Modelling*, 14(3):139–173, 2006.
- [59] D. L. George. Adaptive finite volume methods with well-balanced riemann solvers for modeling floods in rugged terrain: Application to the malpasset dam-break flood (France, 1959). *International Journal for Numerical Methods in Fluids*, 66(8):1000–1018, 2011.
- [60] H. Gotoh and T. Sakai. Numerical simulation of sheetflow as granular material. *Journal of waterway, port, coastal, and ocean engineering*, 123(6):329–336, 1997.
- [61] S. Gottlieb, D. Ketcheson, and C. Shu. *Strong stability preserving Runge-Kutta and multistep time discretizations*. World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2011.
- [62] S. Gottlieb, C. Shu, and E. Tadmor. Strong stability-preserving high-order time discretization methods. *SIAM Review*, 43:89–112, 2001.
- [63] S. Gottlieb and C. W. Shu. Total variation diminishing runge-kutta schemes. *Mathematics of computation of the American Mathematical Society*, 67(221):73–85, 1998.
- [64] N. Goutal. The malpasset dam failure. an overview and test case definition. Spain: Univ. of Zaragoza, 4th CADAM meeting, 1999.

- [65] N. Goutal and F. Maurel. *Proceedings of the 2nd workshop on dam-break wave simulation*. Electricité de France. Direction des études et recherches, 1997.
- [66] L. Goutiere, S. Soares Frazao, and Y. Zech. Dam-break flow on mobile bed in abruptly widening channel: experimental data. *Journal of Hydraulic Research*, 49(3):367–371, 2011.
- [67] A.J. Grass. Sediment transport by waves and currents. *SERC London Centre for Marine Technology, Report No. FL29*, 1981.
- [68] M. Greco, M. Iervolino, A. Leopardi, and A. Vacca. A two-phase model for fast geomorphic shallow flows. *International Journal of Sediment Research*, 27(4):409–425, 2012.
- [69] F. H. Harlow and J. E. Welch. Numerical calculation of time-dependent viscous incompressible flow of fluid with free surface. *Physics of fluids*, 8(12):2182, 1965.
- [70] A. Harten. High resolution schemes for hyperbolic conservation laws. *Journal of computational physics*, 49(3):357–393, 1983.
- [71] A. Harten, B. Engquist, S. Osher, and S. R. Chakravarthy. Uniformly high order accurate essentially non-oscillatory schemes, iii. *Journal of computational physics*, 71(2):231–303, 1987.
- [72] Amiram Harten, Peter D Lax, and Bram van Leer. On upstream differencing and godunov-type schemes for hyperbolic conservation laws. *SIAM review*, 25(1):35–61, 1983.
- [73] N. S. Heaps. Vertical structure of current in homogeneous and stratified waters. In *Hydrodynamics of lakes*, pages 153–207. Springer, 1984.
- [74] J. Hou, F. Simons, M. Mahgoub, and R. Hinkelmann. A robust well-balanced model on unstructured grids for shallow water flows with wetting and drying over complex topography. *Computer Methods in Applied Mechanics and Engineering*, 257:126–149, 2013.
- [75] T. W. Hsu, H. K. Chang, and C. M. Hsieh. A two-phase flow model of wave-induced sheet flow. *Journal of Hydraulic Research*, 41(3):299–310, 2003.
- [76] P. Hu and Z. Cao. Fully coupled mathematical modeling of turbidity currents over erodible bed. *Advances in Water Resources*, 32(1):1–15, 2009.
- [77] P. Hu, Z. Cao, G. Pender, and G. Tan. Numerical modelling of turbidity currents in the Xiaolangdi reservoir, Yellow River, China. *Journal of Hydrology*, 464:41–53, 2012.
- [78] W. Huang, Z. X. Cao, P. Carling, and G. Pender. Coupled 2d hydrodynamic and sediment transport modeling of megaflood due to glacier dam-break in altai mountains, southern siberia. *Journal of Mountain Science*, 11(6):1442–1453, 2014.

- [79] M. E. Hubbard. Multidimensional slope limiters for muscl-type finite volume schemes on unstructured grids. *Journal of Computational Physics*, 155(1):54–74, 1999.
- [80] J. Hudson and P. K. Sweby. Formulations for numerically approximating hyperbolic systems governing sediment transport. *Journal of Scientific Computing*, 19(1-3):225–252, 2003.
- [81] M. Iervolino, A. Leopardi, S. Soares-Frazão, C. Swartenbroekx, Y. Zech, et al. 2d-h numerical simulation of dam-break flow on mobile bed with sudden enlargement. In *River Flow 2010 Conference*, 2010.
- [82] J. A. Jankowski. *A non-hydrostatic model for free surface flows*. PhD thesis, Inst. für Strömungsmechanik und Elektronisches Rechnen im Bauwesen, 1999.
- [83] P. Jawahar and H. Kamath. A high-resolution procedure for euler and navier–stokes computations on unstructured grids. *Journal of Computational Physics*, 164(1):165–203, 2000.
- [84] M. B. Koçyigit and R. A. Falconer. Three-dimensional numerical modelling of wind-driven circulation in a homogeneous lake. *Advances in Water Resources*, 27(12):1167–1178, 2004.
- [85] M. B. Koçyigit, R. A. Falconer, and B. Lin. Three-dimensional numerical modelling of free surface flows with non-hydrostatic pressure. *International Journal for Numerical Methods in Fluids*, 40(9):1145–1162, 2002.
- [86] A. Kurganov and D. Levy. Central-upwind schemes for the saint-venant system. *ESAIM: Mathematical Modelling and Numerical Analysis*, 36(03):397–425, 2002.
- [87] A. Kurganov and C. Lin. On the reduction of numerical dissipation in central-upwind schemes. *Communications in Computational Physics*, 2:141–163, 2007.
- [88] A. Kurganov, S. Noelle, and G. Petrova. Semi-discrete central-upwind scheme for hyperbolic conservation laws and Hamilton-Jacobi equations. *SIAM Journal on Scientific Computing*, 23:707–740, 2001.
- [89] A. Kurganov and G. Petrova.
- [90] A. Kurganov and G. Petrova. Central-upwind schemes on triangular grids for hyperbolic systems of conservation laws. *Numerical Methods for Partial Differential Equations*, 21(3):536–552, 2005.
- [91] A. Kurganov and G. Petrova. Central-upwind schemes on triangular grids for hyperbolic systems of conservation laws. *Numer. Methods Partial Diff. Equ.*, 21(3):536–552, 2005.
- [92] A. Kurganov and G. Petrova. A second-order well-balanced positivity preserving central-upwind scheme for the Saint-Venant system. *Communications in Mathematical Sciences*, 5:133–160, 2007.

- [93] A. Kurganov and E. Tadmor. New high-resolution central schemes for nonlinear conservation laws and convection–diffusion equations. *Journal of Computational Physics*, 160(1):241–282, 2000.
- [94] A. Kurganov and E. Tadmor. Solution of two-dimensional riemann problems for gas dynamics without riemann problem solvers. *Numer. Methods Partial Differential Equations*, 18:584–608, 2002.
- [95] J. L. Lagrange. *Traité de la résolution des équations numériques*, paris (1798). *Reprinted in Œuvres*, 8, 1879.
- [96] J. G. A. B. Leal, R. M. Ferreira, A. H. Cardoso, D. Bousmar, and Y. Zech. Dam-break waves on movable bed. In *River Flow*, pages 981–990, 2002.
- [97] G. R. Lesser, J. A. Roelvink, J. A. T. M. Van Kester, and G. S. Stelling. Development and validation of a three-dimensional morphological model. *Coastal engineering*, 51(8):883–915, 2004.
- [98] R. J. LeVeque. *Numerical methods for conservation laws*, volume 132. Springer, 1992.
- [99] R. J. LeVeque. Balancing source terms and flux gradients in high-resolution godunov methods: the quasi-steady wave-propagation algorithm. *Journal of Computational Physics*, 146(1):346–365, 1998.
- [100] C. W. Li and B. Zhu. A sigma coordinate 3D k - ϵ model for turbulent free surface flow over a submerged structure. *Applied Mathematical Modelling*, 26(12):1139–1150, 2002.
- [101] S. Li and C. J. Duffy. Fully coupled approach to modeling shallow water flow, sediment transport, and bed evolution in rivers. *Water Resources Research*, 47(3), 2011.
- [102] L. Liang, J. Ni, A. G. L. Borthwick, and B. D. Rogers. Simulation of dike-break processes in the Yellow River. *Science in China Series E: Technological Science*, 45(6):606–619, 2002.
- [103] Q. Liang. Flood simulation using a well-balanced shallow flow model. *Journal of Hydraulic Engineering*, 136(9):669–675, 2010.
- [104] Q. Liang and F. Marche. Numerical resolution of well-balanced shallow water equations with complex source terms. *Advances in Water Resources*, 32(6):873–884, 2009.
- [105] P. Lin. A multiple-layer σ -coordinate model for simulation of wavestructure interaction. *Computers & Fluids*, 35(2):147–167, 2006.
- [106] P. Lin and C. W. Li. A σ -coordinate three-dimensional numerical model for surface wave propagation. *International Journal for Numerical Methods in Fluids*, 38(11):1045–1068, 2002.

- [107] H. Liu and S. Sato. A two-phase flow model for asymmetric sheetflow conditions. *Coastal Engineering*, 53(10):825–843, 2006.
- [108] X. Liu, J. Á. Infante Sedano, and A. Mohammadian. A coupled two-dimensional numerical model for rapidly varying flow, sediment transport and bed morphology. *Journal of Hydraulic Research*, pages 1–13, 2015.
- [109] X. Liu, B. J. Landry, and M. H. Garcia. Two-dimensional scour simulations based on coupled model of shallow water equations and sediment transport on unstructured meshes. *Coastal Engineering*, 55(10):800–810, 2008.
- [110] X. Liu, A. Mohammadian, J. Á. Infante Sedano, and A. Kurganov. Three-dimensional shallow water system: A relaxation approach. *Submitted*.
- [111] X. Liu, A. Mohammadian, A. Kurganov, and J. Á. Infante Sedano. Well-balanced central-upwind scheme for a fully coupled shallow water system modeling flows over erodible bed. *Journal of Computational Physics*, 300:202–218, 2015.
- [112] X. Liu, J. A. Sedano Infante, and A. Mohammadian. A robust coupled 2-d model for rapidly varying flows over erodible bed using central-upwind method with wetting and drying. *Canadian Journal of Civil Engineering*, 42(8):530–543, 2015.
- [113] D. R. Lynch and C. B. Officer. Analytic test cases for three-dimensional hydrodynamic models. *International journal for numerical methods in fluids*, 5(6):529–543, 1985.
- [114] D. R. Lynch and F. E. Werner. Three-dimensional hydrodynamics on finite elements. part ii: Non-linear time-stepping model. *International Journal for Numerical Methods in Fluids*, 12(6):507–533, 1991.
- [115] R. MacCormack. The effect of viscosity in hypervelocity impact cratering. *Journal of spacecraft and rockets*, 40(5):757–763, 2003.
- [116] E. Meyer-Peter and R. Müller. Formulas for bed-load transport. *Proceedings of the 2nd Meeting of the International Association of Hydraulic Structures Research, Delft, Netherland*, 1948.
- [117] M. Mignotte and D. Stefanescu. On an estimation of polynomial roots by lagrange. Technical Report 025/2002, pp. 1–17, IRMA Strasbourg, 2002.
- [118] A. Mohamadian, D. Y. Le Roux, M. Tajrishi, and K. Mazaheri. A mass conservative scheme for simulating shallow flows over variable topographies using unstructured grid. *Advances in Water Resources*, 28(5):523–539, 2005.
- [119] M. Morris. Cadam: concerted action on dam break modelling final report. *HR Wallingford, Wallingford*, 2000.
- [120] J. Murillo and P. Garca-Navarro. Improved riemann solvers for complex transport in two-dimensional unsteady shallow flow. *Journal of Computational Physics*, 230(19):7202–7239, 2011.

- [121] J. Murillo and P. García-Navarro. An exner-based coupled model for two-dimensional transient flow over erodible bed. *Journal of Computational Physics*, 229(23):8704–8732, 2010.
- [122] M. M. Namin, B. Lin, and R. A. Falconer. An implicit numerical algorithm for solving non-hydrostatic free-surface flow problems. *International Journal for Numerical Methods in Fluids*, 35(3):341–356, 2001.
- [123] H. Nessyahu and E. Tadmor.
- [124] Pradip Niyogi. *Introduction to computational fluid dynamics*. Pearson Education India, 2006.
- [125] S. Ouillon and D. Dartus. Three-dimensional computation of flow around groyne. *Journal of hydraulic Engineering*, 123(11):962–970, 1997.
- [126] A. Palumbo, S. Soares Frazao, L. Goutière, D. Pianese, Y. Zech, et al. Dam-break flow on mobile bed in a channel with a sudden enlargement. In *Proceedings of the River Flow 2008 Conference*, 2008.
- [127] D. Pan and J. Cheng. Upwind finite-volume Navier-Stokes computations on unstructured triangular meshes. *AIAA journal*, 31(9):1618–1625, 1993.
- [128] S. V. Patankar and D. B. Spalding. A calculation procedure for heat, mass and momentum transfer in three-dimensional parabolic flows. *International Journal of Heat and Mass Transfer*, 15(10):1787–1806, 1972.
- [129] N. A. Phillips. A coordinate system having some special advantages for numerical forecasting. *Journal of the Meteorological*, 14(2):184–185, 1957.
- [130] J. Pietrzak, J. B. Jakobson, H. Burchard, H. Jacob Vested, and O. Petersen. A three-dimensional hydrostatic model for coastal and ocean modelling using a generalised topography following co-ordinate system. *Ocean Modelling*, 4(2):173–205, 2002.
- [131] D. Pritchard and A. J. Hogg. On sediment transport under dam-break flow. *Journal of Fluid Mechanics*, 473:265–274, 2002.
- [132] Wolfgang Rodi. *Turbulence models and their application in hydraulics*. CRC Press, 1993.
- [133] P. L. Roe. Approximate riemann solvers, parameter vectors, and difference schemes. *Journal of computational physics*, 43(2):357–372, 1981.
- [134] P. L. Roe. Characteristic-based schemes for the euler equations. *Annual Review of Fluid Mechanics*, 18:337–365, 1986.
- [135] P. L. Roe and M. J. Baines. Algorithms for advection and shock problems. In *Proceedings Conference on Numerical Methods in Fluid Mechanics*., 1:281–290, 10 1982.

- [136] R. Saurel and R. Abgrall.
- [137] C. Savary, Y. Zech, M. Greco, M. Iervolino, and A. Vacca. Boundary conditions in a two-layer geomorphological model: Application to a hydraulic jump over a mobile bed. *Journal of Hydraulic Research*, 46(6):856–860, 2008.
- [138] L. J. Scott and R. W. Barber. A comparison of interpolation methods for processing randomly scattered bathymetric data. *Environmental Studies*, 1999.
- [139] A. Serrano-Pacheco, J. Murillo, and P. Garcia-Navarro. Exact solutions for unsteady 1D shallow water flow over mobile bed. *River, Coastal and Estuarine Morphodynamics*, 2009.
- [140] A. Serrano-Pacheco, J. Murillo, and P. Garcia-Navarro. Finite volumes for 2d shallow-water flow with bed-load transport on unstructured grids. *Journal of Hydraulic Research*, 50(2):154–163, 2012.
- [141] H. Shirkhani, A. Mohammadian, O. Seidou, and A. Kurganov. A well-balanced positivity-preserving central-upwind scheme for shallow water equations on unstructured quadrilateral grids. *Computers & Fluids*, 126:25–40, 2016.
- [142] C. Shu and S. Osher. Efficient implementation of essentially non-oscillatory shock-capturing schemes. *Journal of Computational Physics*, 77(2):439–471, 1988.
- [143] G. Simpson and S. Castelltort. Coupled model of surface water flow, sediment transport and morphological evolution. *Computers & Geosciences*, 32(10):1600–1614, 2006.
- [144] J. Singh, M. S. Altinakar, and Y. Ding. Two-dimensional numerical modeling of dam-break flows over natural terrain using a central explicit scheme. *Advances in Water Resources*, 34(10):1366–1375, 2011.
- [145] P. A. Sleigh, P. H. Gaskell, M. Berzins, and N. G. Wright. Finite-volume two-dimensional unsteady flow model for river basins. *Journal of Hydraulic Engineering*, 27(4):479–508, 1998.
- [146] S. Soares-Frazão. Dam-break flow on mobile bed benchmark. 2010. http://www.uclouvain.be/cps/ucl/doc/gce/documents/Benchmark_UCL.pdf.
- [147] S. Soares-Frazão, R. Canelas, Z. Cao, L. Cea, H. M. Chaudhry, A. Die Moran, K. El Kadi, R. Ferreira, I. F. Cadórniga, N. Gonzalez-Ramirez, et al. Dam-break flows over mobile beds: experiments and benchmark tests for numerical models. *Journal of Hydraulic Research*, 50(4):364–375, 2012.
- [148] S. Soares Frazão, D. Lories, S. Taminiau, and Y. Zech. Dam-break flow in a channel with a sudden enlargement. In *at the XXX IAHR Congress, Thessaloniki, Greece*, pages 25–31, 2003.

- [149] S. Soares-Frazão and Y. Zech. Hllc scheme with novel wave-speed estimators appropriate for two-dimensional shallow-water flow on erodible bed. *International Journal for Numerical Methods in Fluids*, 66(8):1019–1036, 2011.
- [150] L. Song, J. Zhou, J. Guo, Q. Zou, and Y. Liu. A robust well-balanced finite volume model for shallow water flows with wetting and drying over irregular terrain. *Advances in Water Resources*, 34(7):915–932, 2011.
- [151] J. V. Soulis. A fully coupled numerical technique for 2d bed morphology calculations. *International journal for numerical methods in fluids*, 38(1):71–98, 2002.
- [152] R. Soulsby. *Dynamics of marine sands: a manual for practical applications*. Thomas Telford, 1997.
- [153] R. L. Soulsby and R. J. S. Whitehouse. Threshold of sediment motion in coastal environments. In N.Z. Christchurch, editor, *Pacific Coasts and Ports '97: Proceedings of the 13th Australasian Coastal and Ocean Engineering Conference and the 6th Australasian Port and Harbour Conference*, volume 1, pages 145–150, 1997.
- [154] M. Spiegel. *Schaum's mathematical handbook of formulas and tables*. McGraw-Hill Osborne Media, 1999.
- [155] B. Spinewine, V. Guinot, S. Soares Frazão, and Y. Zech. Solution properties and approximate riemann solvers for two-layer shallow flow models. *Computers & Fluids*, 44(1):202–220, 2011.
- [156] B. Spinewine and Y. Zech. Dam-break on a movable bed in presence of an initial bed discontinuity: laboratory experiments and simulations with a multi-layer shallow water model. In *Proceedings of the 31st IAHR Congress*, pages 3422–3433, Seoul, Korea, 2005.
- [157] P. K. Stansby and J. Zhou. Shallow-water flow solver with non-hydrostatic pressure: 2d vertical plane problems. *International Journal for Numerical Methods in Fluids*, 28(3):541–563, 1998.
- [158] G. Stelling and M. Zijlema. An accurate and efficient finite-difference algorithm for non-hydrostatic free-surface flow with application to wave propagation. *International Journal for Numerical Methods in Fluids*, 43(1):1–23, 2003.
- [159] C. Swartenbroekx, S. Soares-Frazão, B. Spinewine, V. Guinot, and Y. Zech. Hyperbolicity preserving hll solver for two-layer shallow-water equations applied to dam-break flows. In *Proceedings of River Flow*, pages 1379–1387, Bundesanstalt für Wasserbau: Braunschweig, 2010.
- [160] C. Swartenbroekx, Y. Zech, and S. Soares-Frazão. Two-dimensional two-layer shallow water model for dam break flows with significant bed load transport. *International Journal for Numerical Methods in Fluids*, 73(5):477–508, 2013.
- [161] P. K. Sweby.

- [162] T. Tingsanchali and C. Chinnarasri. Numerical modelling of dam failure due to flow overtopping. *Hydrological sciences journal*, 46(1):113–130, 2001.
- [163] E. F. Toro. *Shock-capturing methods for free-surface shallow flows*. Wiley, 2001.
- [164] A. Valiani and L. Begnudelli. Divergence form for bed slope source term in shallow water equations. *Journal of Hydraulic Engineering*, 132(7):652–665, 2006.
- [165] A. Valiani, V. Caleffi, and A. Zanni. Case study: Malpasset dam-break simulation using a two-dimensional finite volume method. *Journal of Hydraulic Engineering*, 128(5):460–472, 2002.
- [166] B. Van Leer. Towards the ultimate conservative difference scheme. v. a second-order sequel to godunov’s method. *Journal of computational Physics*, 32(1):101–136, 1979.
- [167] B. Van Leer. Flux-vector splitting for the euler equations. In *Eighth international conference on numerical methods in fluid dynamics*, pages 507–512. Springer, 1982.
- [168] L. C. van Rijn. Sediment transport, Part ii: Suspended load transport. *Journal of Hydraulic Engineering*, 110(11):1613–1641, 1984.
- [169] L.C. Van Rijn. The computation of the flow and turbulence field in dredged trenches. *Report S*, pages 488–1, 1982.
- [170] J. A. Vásquez. *Two-dimensional finite element river morphology model*. PhD thesis, University of British Columbia, Vancouver, Canada, 2005.
- [171] J. A. Vásquez and J. G. Leal. Two-dimensional dambreak simulation over movable beds with an unstructured mesh. *Ferreira, Alves, Leal & Cardoso (ed.) River Flow*, 2:1483–1491, 2006.
- [172] M. E. Vazquez-Cendon. Improved treatment of source terms in upwind schemes for the shallow water equations in channels with irregualr geometry. *J. Comput. Phys.*, 148:497–526, 1999.
- [173] C. B. Vreugdenhil. *Numerical methods for shallow-water flow*, volume 13. Springer Science & Business Media, 2013.
- [174] J. Wang and R. Liu. A comparative study of finite volume methods on unstructured meshes for simulation of 2D shallow water wave problems. *Mathematics and Computers in Simulation*, 53(3):171–184, 2000.
- [175] J. Wang and R. Liu. The composite finite volume method on unstructured meshes for the two-dimensional shallow water equations. *International journal for numerical methods in fluids*, 37(8):933–949, 2001.
- [176] Y. Wang, Q. Liang, G. Kesserwani, and J. W. Hall. A 2d shallow flow model for practical dam-break simulations. *Journal of Hydraulic Research*, 49(3):307–316, 2011.

- [177] M. Wierse. A new theoretically motivated higher order upwind scheme on unstructured grids of simplices. *Advances in Computational Mathematics*, 7(3):303–335, 1997.
- [178] W. Wu and S. S. Wang. One-dimensional modeling of dam-break flow over movable beds. *Journal of hydraulic engineering*, 133(1):48–58, 2007.
- [179] J. Xia, R. A. Falconer, B. Lin, and G. Tan. Modelling flood routing on initially dry beds with the refined treatment of wetting and drying. *International Journal of River Basin Management*, 8(3-4):225–243, 2010.
- [180] J. Xia, B. Lin, R. A. Falconer, and G. Wang. Modelling dam-break flows over mobile beds using a 2d coupled approach. *Advances in Water Resources*, 33(2):171–183, 2010.
- [181] A. Yeganeh Bakhtiary, B. Shabani, H. Gotoh, and S. S. Wang. A three-dimensional distinct element model for bed-load transport. *Journal of Hydraulic Research*, 47(2):203–212, 2009.
- [182] Z. Yue, Z. Cao, X. Li, and T. Che. Two-dimensional coupled mathematical modeling of fluvial processes with intense sediment transport and rapid bed evolution. *Science in China Series G: Physics, Mechanics and Astronomy*, 51(9):1427–1438, 2008.
- [183] Y. Zech, S. Soares Frazão, Savary C. Spinewine, B., and L. Goutiere. Inertia effects in bed-load transport models. *Canadian Journal of Civil Engineering*, 36:1587–1597, 2009.
- [184] M. Zhang and W. M. Wu. A two dimensional hydrodynamic and sediment transport model for dam break based on finite volume method with quadtree grid. *Applied Ocean Research*, 33(4):297–308, 2011.
- [185] M. Zhang and W. M. Wu. A two dimensional hydrodynamic and sediment transport model for dam break based on finite volume method with quadtree grid. *Applied Ocean Research*, 33(4):297–308, 2011.
- [186] R. Zhang and J. Xie. *Sedimentation research in China: Systematic selections*. China water and Power Press, Beijing, China, 1993.
- [187] X. Zhang and G. Wang. Flow analysis and scour computation of the dike-break. *Journal of Sedimentary Research*, 1:18–24, 2002.
- [188] D. Zhao, H. Shen, J. Lai, and G. Q. Tabios III. Approximate riemann solvers in fvm for 2d hydraulic shock wave modeling. *Journal of Hydraulic Engineering*, 122(12):692–702, 1996.
- [189] D. Zhao, H. Shen, G. Q. Tabios III, J. Lai, and W. Tan. Finite-volume two-dimensional unsteady flow model for river basins. *Journal of Hydraulic Engineering*, 120(7):863–883, 1994.

- [190] J. Zhou. Velocity-depth coupling in shallow-water flows. *Journal of Hydraulic Engineering*, 121(10):717–724, 1995.
- [191] J. Zhou, D. M. Causon, C. G. Mingham, and D. M. Ingram. The surface gradient method for the treatment of source terms in the shallow-water equations. *Journal of Computational Physics*, 168(1):1–25, 2001.
- [192] J. Zhou, D. M. Causon, C. G. Mingham, and D. M. Ingram. Numerical prediction of dam-break flows in general geometries with complex bed topography. *Journal of Hydraulic Engineering*, 130(4):332–340, 2004.
- [193] C. Zoppou and S. Roberts. Numerical solution of the two-dimensional unsteady dam break. *Applied Mathematical Modelling*, 24(7):457–475, 2000.