Spherical functions on supersymmetric spaces and Calogero-Moser-Sutherland operators

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Abstract

In this thesis, we formulate the notion of a spherical function on a supersymmetric space, based on the canonical Hopf superalgebra structure of enveloping superalgebras. We present a comprehensive framework for spherical functions within the context of symmetric pairs $(\mathfrak{g}, \mathfrak{k})$ of Lie superalgebras. Our analysis begins by constructing an algebraic radial component map for the spherical functions. Subsequently, we focus on the supersymmetric pair $(\mathfrak{gl}(1|2), \mathfrak{osp}(1|2))$. Despite the simplicity of this pair, we uncover an interesting connection to Euler zigzag numbers. Finally, we thoroughly investigate three families of supersymmetric pairs whose restricted root systems are of type $A(m, n)$. For each of these cases, we derive a differential equation for spherical functions depending on a parameter k, which takes on the values 1, $\frac{1}{2}$, or $\frac{m-1-2n}{2}$. In mathematical physics literature, this differential equation is known as the Calogero-Moser-Sutherland operator (CMS), and it occurs as the Hamiltonian of the quantum n-body problem in one dimension. Our main result in this context, which extends Sergeev's earlier work, is that for the three families of supersymmetric spaces described above, the CMS operator is obtained from the action of the Casimir operator.

Dedications

To my parents

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Contents

Chapter 1

Introduction

A spherical function on a reductive Lie group is a continuous complex-valued function that is bi-invariant with respect to the maximal compact subgroup of G [14]. Spherical functions play a major role in Lie theory, and their theory lies at the intersection of analysis, algebraic combinatorics, and differential geometry. A readable reference for the abstract functional analytic theory of spherical functions is the expository paper of Roger Godement [10]. One remarkable feature of spherical functions is that they lead to well-known families of symmetric functions, such as Schur and more generally Jack polynomials [39].

This thesis aims to extend the theory of spherical functions to the realm of Lie supergroups. Lie supergroups (and their associated Lie superalgebras) are $\mathbb{Z}_2$-graded analogues of Lie groups (and Lie algebras) [8, 19]. Lie supergroups and Lie superalgebras emerged as critical structures in mathematical physics during the 1970s, particularly in the study of supersymmetry and supergravity. The concept of supersymmetry introduced a new type of symmetry that extends the Poincaré symmetry of spacetime by incorporating transformations between bosons and fermions. This led to the development of Lie superalgebras, which generalize classical Lie algebras by incorporating a $\mathbb{Z}_2$-grading. The seminal works of physicists and mathematicians, including Wess, Zumino, and Kac, laid the foundation for the formal theory of Lie superalgebras and their representations [42, 18].

Victor Kac's classification of simple Lie superalgebras provided a comprehensive framework for understanding the structure and representation theory of these algebraic objects [18]. Subsequent research has focused on exploring the rich algebraic and geometric properties of Lie supergroups and their homogeneous spaces. These studies revealed deep connections with various branches of mathematics, including algebraic geometry, representation theory, and combinatorics.

One of the primary motivations for studying Lie supergroups is their application in theoretical physics, particularly in the formulation of supersymmetric field theories. The representation theory of Lie superalgebras plays a crucial role in understanding the symmetries of these physical systems. Moreover, the theory of Lie supergroups and their homogeneous spaces has significant implications for the study of integrable systems, special functions, and invariant theory [36, 35, 17].

However, the geometry of homogeneous superspaces is usually substantially more technical than their non-super analogues (e.g., Riemannian symmetric spaces). This creates significant challenges in extending the classical theory of spherical functions to the super setting. To circumvent the above technical difficulty, we pursue an indirect, purely algebraic approach to study spherical functions on supersymmetric spaces in a systematic way. Our strategy begins with formulating the notion of a spherical function based on the canonical Hopf superalgebra structure of enveloping algebras. Then, the Hopf superalgebra structure of these enveloping superalgebras allows us to equip their dual with an associative algebraic structure. This approach was first introduced by Sergeev [38, 34] for two families of symmetric pairs $(\mathfrak{gl}(m|n) \oplus \mathfrak{gl}(m|n), \mathfrak{gl}(m|n))$ and $(\mathfrak{gl}(m|2n), \mathfrak{osp}(m|2n))$. Our work will go beyond Sergeev's work. We will formulate a general theory in terms of abstract root data of Lie superalgebras.

Given a finite dimensional Lie superalgebra $\mathfrak{g}$, an irreducible representation (π, V) of $\mathfrak{g}$, and any Lie subalgebra $\mathfrak{k} \subseteq \mathfrak{g}$, we can construct the elements $\theta_{v,v^*}(x) := \langle v^*, \pi(x)v\rangle$ of $U(\mathfrak{g})^\star$, where $v \in V$ and $v^* \in V^*$ are $\mathfrak{k}$-fixed [34]. We refer to the θ_{v,v^*} as algebraic spherical functions. Also, given any Lie subalgebra $\mathfrak{k} \subseteq \mathfrak{g}$, we define the $\mathfrak{k}$-biinvariant functionals as functionals on $U(\mathfrak{g})$ which vanish on $\mathfrak{k}U(\mathfrak{g})+U(\mathfrak{g})\mathfrak{k}$. The space of $\mathfrak{k}$-biinvariant functionals will be denoted by $\mathcal{A}(\mathfrak{g}, \mathfrak{k})$. These $\mathfrak{k}$-biinvariant functionals exhibit several remarkable properties, particularly those highlighted in Proposition 2.3.9, which states that every $\lambda \in \mathcal{A}(\mathfrak{g}, \mathfrak{k})^\circ$, i.e., every element of $\mathcal{A}(\mathfrak{g}, \mathfrak{k})$ that satisfies a natural finiteness property, can be expressed as a matrix coefficient of spherical modules. These properties enable the functionals $\lambda \in \mathcal{A}(\mathfrak{g}, \mathfrak{k})$ to serve as spherical functions for the supersymmetric pair $(\mathfrak{g}, \mathfrak{k})$, where $\mathfrak{k}$ is the fixed point subalgebra of an involution θ of $\mathfrak{g}$ and $\mathfrak{p}$ is the -1 eigenspace of θ.

In the following, we describe the original results of this thesis. The first major goal of the thesis is to find a suitable notion of the radial part of spherical functions in the superized setting. Specifically, for a given supersymmetric pair $(\mathfrak{g}, \mathfrak{k})$ with the property that the maximal abelian subalgebra $\mathfrak{a} \subset \mathfrak{p}$ is self-centralizing, we are able to find a natural injection of $\mathcal{A}(\mathfrak{g}, \mathfrak{k})^\circ$ into the dual of $S(\mathfrak{a})$. This injection is given in Corollary 3.2.7 and serves as the radial part map, a key concept for the subsequent chapters of this thesis.

For the supersymmetric pair $(\mathfrak{gl}(1|2), \mathfrak{osp}(1|2))$ which is actually Case II of Table 4.1, we are able to find a basis of $\mathcal{I} = \mathfrak{k}U(\mathfrak{g}) + U(\mathfrak{g})\mathfrak{k}$ in Theorem 3.3.7. Remarkably, these basis elements have connection to Euler Zigzag numbers. The argument involves tedious computations with a suitable PBW basis of $U(\mathfrak{g})$.

Starting from Chapter 4, we focus our attention on three cases I, II, and III of Table 4.1 whose restricted root systems are of type $A(l, l')$. For any $\mu \in \mathcal{A}(\mathfrak{g}, \mathfrak{k})$, we associate a generating series Φ_μ and proceed to prove the (super)symmetry of Φ_μ. The key idea in this proof involves finding interwoven pairs of $\mathfrak{gl}_2$ subalgebras within $\mathfrak{g}$, a technique that applies consistently across the three cases. Furthermore, we verify the symmetry (the Weyl group invariance) of spherical functionals. Symmetry follows from invariance under the generators of the Weyl group of the restricted root system, and we prove this invariance property in Proposition 4.4.3 by utilizing certain rank one symmetric pairs $(\mathfrak{s}_\alpha, \mathfrak{s}_\alpha \cap \mathfrak{k})$, defined in Definition 4.4.1, that embed in $(\mathfrak{g}, \mathfrak{k})$.

The spherical functional λ can in principle be recovered from the formal power series

Φ_λ. Thus, the generating function Φ_λ plays the role of a spherical function on the associated supersymmetric space, in an algebraic way. We explore the concept of supersymmetry in distinct Cases I and II in Table 4.1. In Case III, supersymmetry is not expected because the restricted root system is purely even. The condition of supersymmetry will be identified by considering the differential equation obtained in Chapter 6 for each case. Supersymmetry is expected to take the form

$$\left(\frac{\partial}{\partial x_i} + k\frac{\partial}{\partial y_j}\right)\Phi_\lambda = 0,$$

for each hyperplane $x_i - y_j = 0$, where $i = 1, \ldots, n$, $j = 1, \ldots, m$, and k is a parameter extracted from the restricted root system according to [37, Sec. 4]. Specifically, $k = 1$ in Case I and $k = 1/2$ in Case II.

Finally, the spherical power series Φ_λ satisfy the (in a suitable sense) differential equations that are analogous to those obtained in the classical work of Sekiguchi and Debiard for the non-super setting [7, 32]. Physicists are particularly interested in the integrable model corresponding to the quantum n-body problem in one dimension. The Schrödinger operator of this model is known as the CMS operator. A general form of this operator in the context of root systems of Lie superalgebras was given by Chalykh, Feigin, and Veselov [5]. Sergeev [37] proved that for Cases I and II, this operator arises from the action of the Casimir of $\mathfrak{g}$, which is either $\mathfrak{gl}(m|n) \oplus \mathfrak{gl}(m|n)$ or $\mathfrak{gl}(m|2n)$. In this thesis, we verify that this also holds in Case III: the CMS operator arises from the Casimir of the Lie superalgebra $\mathfrak{gosp}(m|2n)$. By utilizing the action of the Casimir operator, which is in the center of the enveloping algebra, one can describe spherical functions as eigenfunctions of suitable differential operators, which are stated in (6.0.1) as follows:

$$\begin{aligned}\mathcal{D}_k = k &\sum_{\substack{1 \leqslant i,j \leqslant l+1 \\ i \neq j}} \frac{x_i}{x_i - x_j}\left(x_i\frac{\partial}{\partial x_i} - x_j\frac{\partial}{\partial x_j}\right) - \sum_{\substack{1 \leqslant i,j \leqslant l'+1 \\ i \neq j}} \frac{y_{\bar{i}}}{y_{\bar{i}} - y_{\bar{j}}}\left(y_{\bar{i}}\frac{\partial}{\partial y_{\bar{i}}} - y_{\bar{j}}\frac{\partial}{\partial y_{\bar{j}}}\right) \\ &- \sum_{\substack{1 \leqslant i \leqslant l+1 \\ 1 \leqslant j \leqslant l'+1}} \frac{x_i + y_{\bar{j}}}{x_i - y_{\bar{j}}}\left(x_i\frac{\partial}{\partial x_i} + ky_{\bar{j}}\frac{\partial}{\partial y_{\bar{j}}}\right) + \sum_{1 \leqslant i \leqslant l+1}\left(x_i\frac{\partial}{\partial x_i}\right)^2 - k\sum_{1 \leqslant i \leqslant l'+1}\left(y_{\bar{i}}\frac{\partial}{\partial y_{\bar{i}}}\right)^2,\end{aligned}$$

where the parameter $k = 1, \frac{1}{2}, \frac{m-1}{2} - n$ corresponds to Cases I, II, and III, respectively, in Table 4.1. These parameter values correspond to those associated to the occurring deformed root systems [31, Table 1].

Chapter 2

Generalities on Lie superalgebras and enveloping algebras

In this chapter, we first review some basic definitions such as Lie superalgebras and the enveloping algebras of Lie superalgebras. Then, we use the Hopf superalgebra structure of these enveloping superalgebras, to equip their dual with an associative algebraic structure. Furthermore, we define algebraic spherical functions in the setting of superized homogeneous spaces as functionals in the restricted dual of enveloping superalgebras, with certain vanishing properties that will be introduced in Section 2.3.
Throughout this thesis, the base field will be $\mathbb{C}$.

2.1 Spherical Functions on Lie Groups

Let G be a reductive Lie group and K a maximal compact subgroup of G. The space of all complex-valued continuous functions of compact support on G, which are bi-invariant with respect to K, is denoted by $L(G,K)$. Specifically, a function $f \in L(G,K)$ satisfies

$$f(k_1 x k_2) = f(x), \quad \text{for all } x \in G \text{ and } k_1, k_2 \in K.$$

The convolution of two functions $f, g \in L(G,K)$ is given by

$$(f * g)(x) = \int_G f(xy^{-1}) g(y)\, dy.$$

Definition 2.1.1. A spherical function on G respect to K is a complex-valued continuous function ϕ on G satisfying the following properties:
(i) $\phi(k_1 x k_2) = \phi(x), \quad$ for all $x \in G$ and $k_1, k_2 \in K$,
(ii) $\phi * f = \lambda_f \phi, \quad$ for all $f \in L(G,K)$, where $\lambda_f \in \mathbb{C}$,
(iii) $\phi(1) = 1$.

The first condition ensures bi-K-invariance, the second establishes the eigenfunction property with respect to convolution, and the third normalizes the spherical function at the identity element of G.

Example 2.1.2 (Spherical Functions on the Sphere)**.** Consider the homogeneous space $S^{n-1} = O(n)/O(n-1)$, where $O(n)$ is the orthogonal group and $O(n-1)$ is a maximal compact subgroup. The space S^{n-1} represents the $(n-1)$-dimensional unit sphere.

In this case, a spherical function on S^{n-1} is a continuous function $\phi : O(n) \to \mathbb{C}$ satisfying:

1. Invariance under $O(n-1)$-action: For any $k_1, k_2 \in O(n-1)$ and $x \in S^{n-1}$,
$$\phi(k_1 x k_2) = \phi(x).$$
2. Eigenfunction property under convolution: For any bi-$O(n-1)$-invariant function f on $O(n)$,
$$\phi * f = \lambda_f \phi, \quad \lambda_f \in \mathbb{C}.$$
3. Normalization: $\phi(1) = 1$, where 1 is the identity element of $O(n)$.

A classical example of a spherical function on S^{n-1} is the Legendre polynomial $P_k(\cos\theta)$, where θ is the geodesic distance on the sphere and k is a non-negative integer. These functions arise naturally in harmonic analysis on the sphere and are eigenfunctions of the Laplace-Beltrami operator, satisfying the symmetry imposed by the $O(n-1)$-action.

2.2 Enveloping superalgebra

Superalgebra deals with performing algebra in the category of supervector spaces. The objects of this category are $\mathbb{Z}_2$-graded vector spaces, and the morphisms are grading-preserving linear maps. However, the difference between this category and the category of graded vector spaces is that the symmetry isomorphism $V \otimes W \to W \otimes V$ is different. Specifically, it is defined as
$$v \otimes w \mapsto (-1)^{|v|\cdot|w|} w \otimes v,$$
where $|v|$ and $|w|$ denote the *parity* of v and w, respectively, taking values in $\mathbb{Z}_2 = \{\bar{0}, \bar{1}\}$. Here, $|v| = \bar{0}$ if v is even and $|v| = \bar{1}$ if v is odd. For generalities on super linear algebra, we refer the reader to [4]. We adopt the standard notation of representing a supervector space by $\mathbb{C}^{m|n}$, where m and n stand for the dimensions of the even and odd parts, respectively. That is, $\mathbb{C}^{m|n}_{\bar{0}} = \mathbb{C}^m$ and $\mathbb{C}^{m|n}_{\bar{1}} = \mathbb{C}^n$.

A superalgebra $\mathcal{A}$ is a $\mathbb{Z}_2$-graded algebra $\mathcal{A} = \mathcal{A}_{\bar{0}} \oplus \mathcal{A}_{\bar{1}}$, that is, $\mathcal{A}_i\mathcal{A}_j \subseteq \mathcal{A}_{i+j}$ for $i, j \in \mathbb{Z}_2$ and $\mathcal{A}_{\bar{i}} + \mathcal{A}_{\bar{i}} \subseteq \mathcal{A}_{\bar{i}}$ for $i \in \{\bar{0}, \bar{1}\}$. Note that in this thesis we consider both associative and nonassociative superalgebras.

Definition 2.2.1. [6, Ch.1] A Lie superalgebra is a $\mathbb{Z}_2$-graded algebra $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ with a bilinear binary operation $[.,.]$ satisfying the following two axioms: for homogeneous elements $a, b, c \in \mathfrak{g}$,

(i) Skew-supersymmetry: $[a,b] := -(-1)^{|a||b|}[b,a]$.

(ii) Super Jacobi identity: $[a,[b,c]] = [[a,b],c] + (-1)^{|a||b|}[b,[a,c]]$.

The formulas in the above definition, and more generally all formulas in super linear algebra, are stated for homogeneous elements and then extended by linearity.

Definition 2.2.2. Let $V = V_{\bar{0}} \oplus V_{\bar{1}}$ be a vector superspace. A bilinear form

$$B(\cdot,\cdot) : V \times V \to V$$

is called even (respectively, odd) if $B(V_i, V_j) = 0$ unless $i + j = \bar{0}$ (respectively, $i + j = \bar{1}$). An even bilinear form B is said to be supersymmetric if $B|_{V_{\bar{0}}\times V_{\bar{0}}}$ is symmetric and $B|_{V_{\bar{1}}\times V_{\bar{1}}}$ is skew-symmetric, and it is called skew-supersymmetric if $B|_{V_{\bar{0}}\times V_{\bar{0}}}$ is skew-symmetric and $B|_{V_{\bar{1}}\times V_{\bar{1}}}$ is symmetric.

Example 2.2.3. (I) Let $\mathcal{A}$ be an associative superalgebra. We can equip $\mathcal{A}$ with a Lie superalgebra structure by defining a super commutator as follows:

$$[a,b] := ab - (-1)^{|a||b|}ba.$$

(II) Let V be a vector superspace. Then $\mathrm{End}(V)$ is an associative superalgebra. The Lie superalgebra associated to $\mathrm{End}(V)$ is denoted by $\mathfrak{gl}(V)$. Also, $\mathfrak{gl}(V)_{\bar{0}}$ is the subspace of linear maps that preserve grading and $\mathfrak{gl}(V)_{\bar{1}}$ is the subspace of linear maps that interchange grading. Therefore,

$$\mathfrak{gl}(V)_{\bar{0}} = \mathfrak{gl}(V_{\bar{0}}) \oplus \mathfrak{gl}(V_{\bar{1}}) \cong V_{\bar{0}} \otimes V_{\bar{0}}^* \oplus V_{\bar{1}} \otimes V_{\bar{1}}^*, \quad \mathfrak{gl}(V)_{\bar{1}} = V_{\bar{0}} \otimes V_{\bar{1}}^* \oplus V_{\bar{1}} \otimes V_{\bar{0}}^*.$$

Example 2.2.4. Let m and n be positive integers. Set $r := \lfloor \frac{m}{2} \rfloor$. Let J^+ be the $m \times m$ matrix defined by

$$J^+ := \begin{bmatrix} 1 & 0_{1\times r} & 0_{1\times r} \\ 0_{r\times 1} & 0_{r\times r} & I_{r\times r} \\ 0_{r\times 1} & I_{r\times r} & 0_{r\times r} \end{bmatrix} \text{ if } \quad m = 2r+1, \quad \text{and} \quad J^+ := \begin{bmatrix} 0_{r\times r} & I_{r\times r} \\ I_{r\times r} & 0_{r\times r} \end{bmatrix} \text{ if } \quad m = 2r,$$

and let

$$J^- := \begin{bmatrix} 0_{n\times n} & I_{n\times n} \\ -I_{n\times n} & 0_{n\times n} \end{bmatrix}.$$

Consider the even supersymmetric bilinear form $B : \mathbb{C}^{m|2n} \times \mathbb{C}^{m|2n} \to \mathbb{C}$ defined by

$$B(e_i, e_j) = J^+_{i,j}, \quad B(e'_i, e'_j) = J^-_{i,j}, \quad \text{and} \quad B(e_i, e'_j) = 0.$$

where $\{e_i\}_{i=1}^m \cup \{e'_j\}_{j=1}^{2n}$ is the standard homogeneous basis of $\mathbb{C}^{m|2n}$. Then the Lie superalgebra $\mathfrak{osp}(m|2n)$ can be realized as the subalgebra of $\mathfrak{gl}(m|2n)$ that leaves the bilinear form B invariant. Thus, the Lie superalgebra $\mathfrak{g} := \mathfrak{osp}(2r+1|2n)$ is realized as follows:

$$\mathfrak{g}_{\bar{0}} := \begin{bmatrix} \mathfrak{o}(2r+1) & 0 \\ 0 & \mathfrak{sp}(2n) \end{bmatrix},$$

and

$$\mathfrak{g}_{\bar{1}} := \begin{bmatrix} 0 & 0 & 0 & -y^t & x^t \\ 0 & 0 & 0 & -D^t & B^t \\ 0 & 0 & 0 & -C^t & A^t \\ x & A & B & 0 & 0 \\ y & C & D & 0 & 0 \end{bmatrix}.$$

where $x, y \in \mathrm{Mat}_{n\times 1}$ and $A, B, C, D \in \mathrm{Mat}_{n\times r}$. The structure of the Lie superalgebra $\mathfrak{osp}(2r|2n)$ is similar. The only difference is that the first row and column of the matrix realization of $\mathfrak{osp}(2r+1|2n)$ should be removed.

By adding a one-dimensional center to $\mathfrak{osp}(m|2n)$, we obtain a Lie superalgebra $\mathfrak{gosp}(m|2n)$, which can be realized naturally as a subalgebra of $\mathfrak{gl}(m|2n)$. Indeed,

$$\mathfrak{gosp}(m|2n) = \mathfrak{osp}(m|2n) \oplus \mathbb{C}I.$$

Here I is the identity matrix of size $(m+2n)\times(m+2n)$.

Definition 2.2.5. Let V be a vector superspace over a field $\mathbb{C}$. For any non-negative integer r, we define the r-th tensor power of V to be:

$$\mathcal{T}^r V = V^{\otimes r} = V \otimes \cdots \otimes V, \quad \mathcal{T}^0 V = \mathbb{C}.$$

Then, the tensor algebra of the vector superspace V is defined in the usual way:

$$\mathcal{T}(V) = \bigoplus_{r\geqslant 0} \mathcal{T}^r V.$$

Definition 2.2.6. Let $\mathfrak{g}$ be any Lie superalgebra, $\mathcal{T}(\mathfrak{g})$ denote the tensor algebra of $\mathfrak{g}$, and J be the two sided ideal in $\mathcal{T}(\mathfrak{g})$ generated by

$$x \otimes y - (-1)^{|x||y|} y \otimes x - [x, y], \quad x, y \in \mathfrak{g} = \mathcal{T}^1(\mathfrak{g}).$$

The universal enveloping algebra of $\mathfrak{g}$ is the quotient algebra: $U(\mathfrak{g}) = \mathcal{T}(\mathfrak{g})/J$.

Let $\mathfrak{g}$ be any Lie superalgebra. Then the universal enveloping algebra $U(\mathfrak{g})$ is a Hopf superalgebra, with co-product $\Delta : U(\mathfrak{g}) \to U(\mathfrak{g}) \otimes U(\mathfrak{g})$ defined by $\Delta(x) := 1 \otimes x + x \otimes 1$ for $x \in \mathfrak{g}$, antipode defined by $S(x) = -x$ for $x \in \mathfrak{g}$, and counit defined by $\varepsilon(1) = 1$ and $\varepsilon(x) = 0$ for $x \in \mathfrak{g}$. Using the comultiplication of $U(\mathfrak{g})$, we can equip the dual $U(\mathfrak{g})^*$ with an associative algebra structure. The multiplication of $U(\mathfrak{g})^*$ is given by

$$(\phi\psi)(x) := \sum (-1)^{|x_1||\psi|} \phi(x_1)\psi(x_2) \text{ for } x \in U(\mathfrak{g}), \text{ where } \Delta(x) = \sum x_1 \otimes x_2.$$

For later use, we define left and right translation representations of $U(\mathfrak{g})$ on itself as follows:

$$L_x : y \mapsto xy \quad \text{and} \quad R_x : y \mapsto (-1)^{|x|.|y|} yx.$$

Furthermore, given any $U(\mathfrak{g})$-module V, we equip $V^\star$ with the canonical contragredient module structure defined by

$$x \cdot \lambda(y) := -(-1)^{|x|.|\lambda|} \lambda(x \cdot y), \quad \text{for } x \in U(\mathfrak{g}), \text{ and } \lambda \in V^\star.$$

2.3 Restricted dual

Let (π, V) be a finite dimensional representation of $\mathfrak{g}$. Associated to this representation, we can construct elements of $U(\mathfrak{g})^*$ as follows. First, note that by the universal property of $U(\mathfrak{g})$, the representation yields a homomorphism of associative algebras

$$\pi : U(\mathfrak{g}) \to \mathrm{End}(V). \tag{2.3.1}$$

We choose any $v \in V$, $v^* \in V^*$ and we set

$$\theta_{v,v^*}(x) := \langle v^*, \pi(x)v\rangle \text{ for all } x \in U(\mathfrak{g}).$$

Remark 2.3.1. The linear functional $\theta_{v,v^*} \in U(\mathfrak{g})^*$ satisfies the following important property: $\ker(\theta_{v,v^*})$ contains a finite co-dimensional two-sided ideal of $U(\mathfrak{g})$. This is because if $\mathrm{Ann}(V)$ denotes the annihilator of the module V, then $U(\mathfrak{g})/\mathrm{Ann}(V)$ embeds in $\mathrm{End}_{\mathbb{C}}(V)$, which is finite dimensional. Furthermore, $\mathrm{Ann}(V) \subseteq \ker(\theta_{v,v^*})$.

The following lemma is well known.

Lemma 2.3.2. Let $\varphi, \varphi_1, \ldots, \varphi_n$ be linear functionals on a vector space E. If

$$\bigcap_{i=1}^{n} \ker(\varphi_i) \subseteq \ker(\varphi),$$

then φ is expressible as a linear combination of the φ_i for $1 \leqslant i \leqslant n$.

Proposition 2.3.3. Let $\phi \in U(\mathfrak{g})^*$ be such that $\ker(\phi)$ contains a finite co-dimensional two-sided ideal of $U(\mathfrak{g})$. Then, ϕ can be expressed as a linear combination of finitely many linear functionals of the form θ_{v_i,v_j^*}.

Proof. Let I be a finite co-dimensional two-sided ideal of $U(\mathfrak{g})$ contained in $\ker(\phi)$. Set $V = U(\mathfrak{g})/I$. Then, we have a finite dimensional representation of $U(\mathfrak{g})$ as follows:

$$\begin{aligned} \rho : U(\mathfrak{g}) &\to \mathrm{End}(V) \\ u &\mapsto T_u, \quad T_u(v+I) = uv + I. \end{aligned}$$

Note that ρ is a quotient of the left translation module $x \mapsto L_x$. Now we can see that

$$\begin{aligned} \mathrm{Ann}(V) &= \{u \in U(\mathfrak{g}) : uV = 0\} \\ &= \{u \in U(\mathfrak{g}) : u(v+I) = 0 \quad \text{for all} \quad v \in U(\mathfrak{g})\} \\ &= \{u \in U(\mathfrak{g}) : uv \in I \quad \text{for all} \quad v \in U(\mathfrak{g})\} = I. \end{aligned}$$

Suppose that $\{v_1, \cdots, v_n\}$ is a basis for V and $\{v_1^*, \cdots, v_n^*\}$ is a basis for V^*. Thus, by Lemma 2.3.2, it suffices to show that I contains the intersection of $\ker(\theta_{v_i,v_j^*})$ for $i, j = 1, \cdots, n$. To verify this, suppose that u is an arbitrary element of the intersection of $\ker(\theta_{v_i,v_j^*})$ for $1 \leqslant i, j \leqslant n$, so

$$\theta_{v_i,v_j^*}(u) := \langle v_j^*, \rho(u)v_i\rangle = 0 \text{ for } 1 \leqslant i, j \leqslant n.$$

Thus, for all $v \in V$, $\rho(u)v = 0$. But, $\mathrm{Ann}(V) = I$, so $u \in I$ which implies that

$$\bigcap_{i,j=1}^{n} \ker(\theta_{v_i, v_j^*}) \subseteq I.$$

Therefore, by Lemma 2.3.2 ϕ is the linear combination of finitely many linear functionals of the form

$$\{\theta_{v_i, v_j^*} \mid 1 \leqslant i, j \leqslant n\}.$$

∎

Definition 2.3.4. An element $\phi \in U(\mathfrak{g})^*$ is said to be of *finite type* if its kernel contains a two-sided ideal of finite co-dimension.

Now let $\mathfrak{k} \subseteq \mathfrak{g}$ be any Lie subalgebra, that is, a subspace that is a superalgebra[1]

Definition 2.3.5. An element $\lambda \in U(\mathfrak{g})^*$ is called $\mathfrak{k}$-biinvariant if

$$\lambda\big(\mathfrak{k}U(\mathfrak{g}) + U(\mathfrak{g})\mathfrak{k}\big) = 0.$$

We will denote the subspace of $U(\mathfrak{g})^*$ consisting of bi-invariant elements by $\mathcal{A}(\mathfrak{g}, \mathfrak{k})$.

Lemma 2.3.6. The superspace $\mathcal{A}(\mathfrak{g}, \mathfrak{k})$ is a sub-superalgebra of $U(\mathfrak{g})^*$.

Proof. Obviously, the sum and scalar multiplication of $\mathfrak{k}$-biinvariant functionals in $U(\mathfrak{g})^*$ are biinvariant functionals. Now let $\lambda, \mu \in \mathcal{A}(\mathfrak{g}, \mathfrak{k})$. We want to prove that $\lambda\mu \in \mathcal{A}(\mathfrak{g}, \mathfrak{k})$ under the induced product of $U(\mathfrak{g})^*$. Then for $k \in \mathfrak{k}$ and $u \in U(\mathfrak{g})$ we have

$$\begin{aligned}
\lambda\mu(ku) &= \lambda \otimes \mu\left((k \otimes 1 + 1 \otimes k)\,\Delta(u)\right) \\
&= \lambda \otimes \mu\Big((k \otimes 1 + 1 \otimes k) \sum u_1 \otimes u_2\Big) \\
&= \lambda \otimes \mu\Big(\sum \big(ku_1 \otimes u_2 + (-1)^{|k|\cdot|u_1|} u_1 \otimes ku_2\big)\Big) \\
&= \sum \big(\lambda\,(ku_1) \otimes \mu\,(u_2) + (-1)^{|k|\cdot|u_1|} \lambda\,(u_1) \otimes \mu\,(ku_2)\big) = 0
\end{aligned}$$

where Δ is the co-product of the Hopf superalgebra $U(\mathfrak{g})$, that is a homomorphism of algebras. Similarly, we can prove that $\lambda\mu$ is right $\mathfrak{k}$-invariant. Therefore, $\lambda\mu \in \mathcal{A}(\mathfrak{g}, \mathfrak{k})$ and hence $\mathcal{A}(\mathfrak{g}, \mathfrak{k})$ is a subalgebra of $U(\mathfrak{g})^*$.

Now, let $\lambda \in \mathcal{A}(\mathfrak{g}, \mathfrak{k})$ be written as

$$\lambda = \lambda_{\bar{0}} + \lambda_{\bar{1}}$$

such that

1. $\lambda_{\bar{0}} \in U(\mathfrak{g})^*_{\bar{0}}$, and $\lambda_{\bar{0}}|_{U(\mathfrak{g})_{\bar{0}}} = \lambda_{U(\mathfrak{g})_{\bar{0}}}$,
2. $\lambda_{\bar{1}} \in U(\mathfrak{g})^*_{\bar{1}}$, and $\lambda_{\bar{1}}|_{U(\mathfrak{g})_{\bar{1}}} = \lambda_{U(\mathfrak{g})_{\bar{1}}}$.

[1] By a Lie subalgebra we don't mean that $\mathfrak{k}$ is purely even. Some references use the term Lie sub-superalgebra, but we prefer to abbrivariate it to Lie subalgebra.

Set $E = \mathfrak{k}U(\mathfrak{g}) + U(\mathfrak{g})\mathfrak{k}$. Since $\mathfrak{k}$ is $\mathbb{Z}_2$-graded,

$$E_{\bar{0}} = E \cap U(\mathfrak{g})_{\bar{0}}, \quad E_{\bar{1}} = E \cap U(\mathfrak{g})_{\bar{1}}.$$

Since $\lambda_{\bar{0}} \in U(\mathfrak{g})^*_{\bar{0}}$, the restriction of $\lambda_{\bar{0}}$ to $U(\mathfrak{g})_{\bar{1}}$ and hence to $E_{\bar{1}}$ is zero. On the other hand, $\lambda_{\bar{0}}|_{U(\mathfrak{g})_{\bar{0}}} = \lambda|_{U(\mathfrak{g})_{\bar{0}}}$ and $\lambda|_{E_{\bar{0}}} = 0$ which implies that $\lambda_{\bar{0}}|_{E_{\bar{0}}} = 0$ and hence $\lambda_{\bar{0}}|_E = 0$. Similarly, we can verify that $\lambda_{\bar{1}}|_E = 0$. Thus, $\mathcal{A}(\mathfrak{g}, \mathfrak{k})$ is a sub-superalgebra of $U(\mathfrak{g})^*$. ∎

Remark 2.3.7. It is well known [40, Sec. 6.2] that linear functionals of finite type in any Hopf algebra also form a Hopf algebra. We will denote this latter Hopf superalgebra as $U(\mathfrak{g})^\circ$. Furthermore, we set

$$\mathcal{A}(\mathfrak{g}, \mathfrak{k})^\circ := \mathcal{A}(\mathfrak{g}, \mathfrak{k}) \cap U(\mathfrak{g})^\circ.$$

Lemma 2.3.8. Let E be a vector space and let $W \subseteq E^*$ be finite dimensional. Then the map $\mathrm{ev} : E \to W^*$, given by $\mathrm{ev}_e(\lambda) = \lambda(e)$, is a surjection.

Proof. Choose a basis $\lambda_1, \dots, \lambda_n$ for W. For each $1 \le i \le n$, by Lemma 2.3.2 there exists v_j in E such that $\lambda_i(v_j) = \delta_{i,j}$. The linear functionals ev_{v_i} form a dual basis for W^*. Hence, the map ev is a surjection. ∎

The following proposition is a refinement of Proposition 2.3.3.

Proposition 2.3.9. Let $\lambda \in \mathcal{A}(\mathfrak{g}, \mathfrak{k})^\circ$. Then $\lambda(x) = \langle v^*, \pi(x)v\rangle$ for $v \in V$ and $v^* \in V^*$, where V is a finite dimensional $\mathfrak{g}$-module, V^* is the dual of V, and both v and v^* are $\mathfrak{k}$-fixed.

Proof. Let $\lambda \in \mathcal{A}(\mathfrak{g}, \mathfrak{k})^\circ$. Then, $\lambda(\mathfrak{k}U(\mathfrak{g}) + U(\mathfrak{g})\mathfrak{k}) = 0$ and $\lambda(I) = 0$ for a two-sided ideal I of finite codimension. For the remainder of this proof, we assume $U := U(\mathfrak{g})$ and we will show that there exists a finite-dimensional representation (ρ, W) of $U(\mathfrak{g})$ and $\mathfrak{k}$-fixed vectors $w \in W$ and $w^* \in W^*$ such that $\lambda(x) = \langle w^*, \rho(x)w\rangle$.Since the sum of matrix coefficients on two $\mathfrak{g}$-modules V_1 and V_2 is a matrix coefficient on $V_1 \oplus V_2$, we can assume that λ is homogeneous.

For $x \in U$ recall that $L_x : U \to U$ is the left translation representation:

$$L_x(y) := xy.$$

We define a dual U-action on U^* as follows:

$$L^*_x\mu(y) := (-1)^{|x||\mu|}\mu(L_{S(x)}y) = (-1)^{|x||\mu|}\mu(S(x)y), \quad \text{because } S^2(x) = x, \tag{2.3.2}$$

where $S : U \to U$ is the antipode of U. Now set $W := \mathrm{Span}\{L^*_x(\lambda \circ S) : x \in U\} \subseteq U^*$. For homogeneous elements $u \in I$ and $x \in U$ we have

$$L^*_u(\lambda \circ S)(x) = (-1)^{|u||x|+|u||\lambda|}\lambda(S(x)u) = 0,$$

because $S(x)u \in I$. Therefore the kernel of the linear map $U(\mathfrak{g}) \to W$, $x \mapsto L^*_x(\lambda \circ S)$, contains I. Hence, $\dim W \le \dim U/I < \infty$. Set

$$W^\perp = \{x \in U : \mu(x) = 0 \quad \text{for all} \quad \mu \in W\}.$$

Now, consider the evaluation map

$$\mathrm{ev} : U(\mathfrak{g}) \to W^*, \quad u \mapsto \mathrm{ev}_u,$$

where $\mathrm{ev}_u(\mu) := \mu(u)$. Since W is finite dimensional, by Lemma 2.3.8 the above map is a surjection onto W^* and it follows that $U/W^\perp \cong W^*$. We can now obtain λ as a matrix coefficient for the representation (ρ, W) where $\rho(x) = (-1)^{|x|}L_x^*$. Note that if π is a representation of $\mathfrak{g}$ then $x \mapsto (-1)^{|x|}\pi(x)$ is also a representation of $\mathfrak{g}$ because the map $x \mapsto (-1)^{|x|}x$ is an automorphism of $\mathfrak{g}$. By (2.3.2) above, we have

$$\lambda(x) = (-1)^{|x||\lambda|}\mathrm{ev}_1(L_x^*(\lambda \circ S)).$$

The vector $\lambda \circ S$ is $\mathfrak{k}$-invariant because for any $k \in \mathfrak{k}$ we have:

$$L_k^*(\lambda \circ S)(y) = (-1)^{|k||\lambda|}\lambda(S(y)k) = 0.$$

The dual vector $1 = \mathrm{ev}_1$ is also $\mathfrak{k}$-invariant because from $S(\mathfrak{k}) \subseteq \mathfrak{k}$ it follows that

$$k \cdot \mathrm{ev}_1(L_x^*(\lambda \circ S)) = (-1)^{|k||\lambda|}L_x^*(\lambda \circ S(k)) = (\lambda \circ S)(S(x)k) = (-1)^{|k||x|}\lambda(S(k)x) = 0. \quad \blacksquare$$

Remark 2.3.10. From the perspective of the Peter-Weyl theorem, Proposition 2.3.9 highlights the decomposition of $\mathcal{A}(\mathfrak{g}, \mathfrak{k})^\circ$ in terms of matrix coefficients of finite-dimensional $\mathfrak{g}$-modules. The pairing

$$\lambda(x) = \langle v^*, \pi(x)v \rangle$$

expresses λ as a matrix coefficient, where the $\mathfrak{k}$-fixed elements $v \in V$ and $v^* \in V^*$ ensure compatibility with $\mathfrak{k}$-invariance. This mirrors the decomposition of bi-K-invariant functions on G into irreducible components under the symmetry imposed by the Weyl group. Such an interpretation aligns with the role of invariant subspaces in the Peter-Weyl theorem, emphasizing the harmonic structure underlying the representation theory of G and K.

Chapter 3

Structure of supersymmetric pairs

In this chapter, our goal is to define the radial part of spherical functions on a supersymmetric space in an algebraic way. Recall from the previous chapter that bi-invariant functions are replaced by bi-invariant functionals on the enveloping superalgbera. We consider symmetic pairs of the form $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ where $\mathfrak{k}$ is the fixed point of an involution θ of $\mathfrak{g}$. We consider a maximally split and θ-invariant Cartan subalgebra of $\mathfrak{g}$ of the form $\mathfrak{h} = \mathfrak{t} \oplus \mathfrak{a}$. Our main goal is to give an embedding of $\mathcal{A}(\mathfrak{g}, \mathfrak{k})$ into the dual of $S(\mathfrak{a})$, the symmetric algebra of the split part of the Cartan subalgebra. Our assumptions on $(\mathfrak{g}, \mathfrak{k})$ are quite general and therefore our results are applicable to a broad setting. Our strategy is to use a refinement of a lemma originally due to Lepowsky [23, Lemma 3.7].

3.1 The algebra of functions on the symmetric pair $(\mathfrak{g}, \mathfrak{k})$

For convenience we assume that $\mathfrak{g}$ is a Lie superalgebra of either $\mathfrak{gl}$ or $\mathfrak{gosp}$ type defined in Example 2.2.3 and Example 2.2.4. Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$. Then, the root decomposition of $\mathfrak{g}$ with respect to $\mathfrak{h}$ is given by

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta} \mathfrak{g}_\alpha,$$

where $\Delta \subseteq \mathfrak{h}^* \backslash \{0\}$ and

$$\mathfrak{g}_\alpha = \{x \in \mathfrak{g} \,\big|\, \forall h \in \mathfrak{h}, \mathrm{ad}_h(x) = \alpha(h)x\}.$$

Example 3.1.1. In this example, we review the unrestricted root system of the Lie superalgebra $\mathfrak{gl}(m|n)$. The notation that is laid out in this example will be used in the rest of the thesis. We choose the Cartan subalgebra of $\mathfrak{gl}(m|n)$ to be as follows:

$$\mathfrak{h} = \mathrm{Span}\left\{E_{i,i}, E_{\bar{j},\bar{j}} \,|\, 1 \leq i \leq m, \bar{1} \leq \bar{j} \leq \bar{n}\right\},$$

where $\bar{j} = m+j$. Consider the dual basis of the Cartan subalgebra to be $\{\boldsymbol{\varepsilon}_1, ..., \boldsymbol{\varepsilon}_m, \boldsymbol{\delta}_{\bar{1}}, ..., \boldsymbol{\delta}_{\bar{n}}\}$ such that

$$\boldsymbol{\varepsilon}_i(E_{j,j}) = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}, \quad \boldsymbol{\delta}_{\bar{i}}(E_{\bar{j},\bar{j}}) = \begin{cases} 1 & \bar{i} = \bar{j} \\ 0 & \bar{i} \neq \bar{j} \end{cases}.$$

Note that we are using bold Greek letters for characters of the Cartan subalgebra $\mathfrak{h}$. The absolute root system consists of the roots $\boldsymbol{\varepsilon}_i - \boldsymbol{\varepsilon}_j$, $\pm(\boldsymbol{\varepsilon}_i - \boldsymbol{\delta}_{\bar{i}})$, $\boldsymbol{\delta}_{\bar{i}} - \boldsymbol{\delta}_{\bar{j}}$ for $1 \leq i \neq j \leq m$ and $\bar{1} \leq \bar{i} \neq \bar{j} \leq \bar{n}$. We choose the positive roots to be $\boldsymbol{\varepsilon}_i - \boldsymbol{\varepsilon}_j$, $\pm(\boldsymbol{\varepsilon}_i - \boldsymbol{\delta}_{\bar{i}})$, $\boldsymbol{\delta}_{\bar{i}} - \boldsymbol{\delta}_{\bar{j}}$ for $1 \leq i < j \leq m$ and $\bar{1} \leq \bar{i} < \bar{j} \leq \bar{n}$. Then the fundamental system $\boldsymbol{\Phi}$ corresponding to the choice of positive roots is

$$\boldsymbol{\Phi} = \{\boldsymbol{\varepsilon}_i - \boldsymbol{\varepsilon}_{i+1}\}_{i=1}^{m} \cup \left\{\boldsymbol{\delta}_{\bar{i}} - \boldsymbol{\delta}_{\overline{i+1}}\right\}_{\bar{i}=\bar{1}}^{\bar{n}} \cup \{\boldsymbol{\varepsilon}_m - \boldsymbol{\delta}_{\bar{1}}\}.$$

The absolute root system of $\mathfrak{gl}(m|n)$ lies in $\mathfrak{h}^*$. The vectors $E_{i,i}, E_{\bar{i},\bar{i}} \in \mathfrak{h}$ form a natural dual basis of $\boldsymbol{\varepsilon}_i$ and $\boldsymbol{\delta}_{\bar{i}}$ for $\mathfrak{h}$.

Let $\mathfrak{g}$ be a Lie superalgebra. A *Cartan decomposition* of $\mathfrak{g}$ is a decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, where $\mathfrak{k}$ is a subalgebra of $\mathfrak{g}$, $\mathfrak{p}$ is a subspace of $\mathfrak{g}$, and the equivalent following conditions hold:
(i) $[\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{k}, \quad [\mathfrak{k}, \mathfrak{p}] \subset \mathfrak{p}, \quad [\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{k}$,
(ii) There exists an involutive automorphism $\theta : \mathfrak{g} \to \mathfrak{g}$ such that $\theta|_{\mathfrak{k}} = +1$ and $\theta|_{\mathfrak{p}} = -1$.
In the case of Lie superalgebras, the subspaces $\mathfrak{k}$ and $\mathfrak{p}$ are typically considered with respect to the $\mathbb{Z}_2$-grading of $\mathfrak{g}$, ensuring that the decomposition respects the superalgebraic structure. This decomposition corresponds to the unique automorphism θ, and we assume that the center $\mathfrak{z}$ of $\mathfrak{g}$ satisfies $\theta|_{\mathfrak{z}} = -1$, so $\mathfrak{z} \subset \mathfrak{p}$.

Associated to this Cartan involution, we consider $\mathfrak{h} = \mathfrak{t} \oplus \mathfrak{a}$ to be a Cartan subalgebra of $\mathfrak{g}$ such that $\mathfrak{t} \subseteq \mathfrak{k}$ and $\mathfrak{a}$ is a maximal abelian and self-centralizing subspace of $\mathfrak{p}$. That is, $[\mathfrak{a}, \mathfrak{a}] = 0$, and furthermore, for every subspace $\mathfrak{a} \subsetneq \mathfrak{a}' \subsetneq \mathfrak{p}$ we have $[\mathfrak{a}, \mathfrak{a}'] \neq 0$. The existence of such a Cartan subalgebra for our examples of interest will be established in Chapter 4. Note that the elements of $\mathfrak{a}$ are ad-diagonalizable on $\mathfrak{g}$.

Lemma 3.1.2. For $x \in \mathfrak{g}_\alpha$ we have $\theta(x) \in \mathfrak{g}_{\theta(\alpha)}$ where $\theta(\alpha) := \alpha \circ \theta$.

Proof. Since $\theta : \mathfrak{g} \to \mathfrak{g}$ is an involutive automorphism and $\theta(\mathfrak{h}) = \mathfrak{h}$, for $h \in \mathfrak{h}$ and $x \in \mathfrak{g}_\alpha$ we have

$$[h, \theta(x)] = [\theta(\theta(h)), \theta(x)] = \theta(\alpha(\theta(h))(x)) = \alpha(\theta(h))\theta(x) = (\theta(\alpha)(h))\theta(x),$$

where $\theta(\alpha) = \alpha \circ \theta$. ∎

Definition 3.1.3. [19, Ch.1] Let V be a vector superspace and J be the two-sided ideal of $\mathcal{T}(V)$ generated by $x \otimes y - (-1)^{|x||y|} y \otimes x$ for $x, y \in V$. Then the super symmetric algebra $S(V)$ is the quotient $\mathcal{T}(V)/J$.

We also have a super-symmetrization map $\mathsf{s} : S(\mathfrak{p}) \to U(\mathfrak{g})$ given by

$$x_1 \cdots x_r \in S^r(\mathfrak{p}) \mapsto \frac{1}{r!} \sum_{\pi \in S_r} (-1)^{|\pi|} x_{\pi(1)} \cdots x_{\pi(r)},$$

where $(-1)^{|\pi|}$ is defined as follows: if $x_{i_1}, x_{i_2}, \ldots, x_{i_s}$ are the odd vectors among $x_1, \ldots, x_r$, then π induces a permutation on the indices $i_1, \ldots, i_s$, and $(-1)^{|\pi|}$ denotes the sign of the latter permutation. Specifically, $|\pi|$ is the number of pairwise swaps of odd elements required to sort the indices $i_1, \ldots, i_s$ in ascending order.

Example 3.1.4. Let $x_1 \in \mathfrak{p}_{\bar{0}}$ and $x_2, x_3 \in \mathfrak{p}_{\bar{1}}$, then,

$$\mathsf{s}(x_1x_2x_3) = \frac{1}{6}(x_1x_2x_3 - x_1x_3x_2 + x_2x_1x_3 + x_2x_3x_1 - x_3x_2x_1 - x_3x_1x_2).$$

We set $\mathcal{I}_L := U(\mathfrak{g})\mathfrak{k}$ and $\mathcal{I} := U(\mathfrak{g})\mathfrak{k} + \mathfrak{k}U(\mathfrak{g})$.

Proposition 3.1.5. The natural map $\bar{\mathsf{s}} : S(\mathfrak{p}) \to U(\mathfrak{g})/\mathcal{I}_L$ that is induced by $\mathsf{s} : S(\mathfrak{p}) \to U(\mathfrak{g})$ is an isomorphism of vector spaces.

Proof. We prove that $\bar{\mathsf{s}}$ is surjective and injective in three steps:
Step 1. First, choose an ordered homogeneous basis $\{p_1, \dots, p_m\}$ of $\mathfrak{p}$ and an ordered homogeneous basis of $\mathfrak{k}$. This yields an ordered basis of $\mathfrak{g}$. Then by the PBW theorem we have:

$$U(\mathfrak{g}) = \mathcal{I}_L + \sum_{r_1,\dots,r_m \geq 0} \mathbb{C}p_1^{r_1} \cdots p_m^{r_m}.$$

Equivalently

$$U(\mathfrak{g}) = \mathcal{I}_L + \sum_{d \geqslant 1} V_d,$$

where

$$V_d = \operatorname{Span}\{\tilde{p}_1 \cdots \tilde{p}_r | \tilde{p}_i \in \mathfrak{p}, r \leqslant d, 1 \leqslant i \leqslant r\}.$$

Step 2. We prove by induction on d that for any monomial of degree d, the image of $\bar{\mathsf{s}}$ contains $\mathcal{I}_L + \tilde{p}_1 \cdots \tilde{p}_d$. For $\tilde{p}_1\tilde{p}_2 \cdots \tilde{p}_d \in S(\mathfrak{p})$, we have

$$\bar{\mathsf{s}}(\tilde{p}_1\tilde{p}_2 \cdots \tilde{p}_d) = \frac{1}{r!}\sum_{\pi \in S_d} (-1)^{|\pi|}\tilde{p}_{\pi(1)}\tilde{p}_{\pi(2)} \cdots \tilde{p}_{\pi(d)} + \mathcal{I}_L.$$

Next, note that

$$\begin{aligned}\tilde{p}_{\pi(1)} \cdots \tilde{p}_{\pi(j)}\tilde{p}_{\pi(j+1)} \cdots \tilde{p}_{\pi(d)} &= \pm\tilde{p}_{\pi(1)} \cdots \tilde{p}_{\pi(j+1)}\tilde{p}_{\pi(j)} \cdots \tilde{p}_{\pi(d)} \\ &\quad + \tilde{p}_{\pi(1)} \cdots [\tilde{p}_{\pi(j)}, \tilde{p}_{\pi(j+1)}] \cdots \tilde{p}_{\pi(d)},\end{aligned} \tag{3.1.1}$$

where the sign of the first term on the right hand side of (3.1.1) is $(-1)^{|\tilde{p}_{\pi(j)}||\tilde{p}_{\pi(j+1)}|}$. Set $q := [\tilde{p}_{\pi(j)}, \tilde{p}_{\pi(j+1)}]$. Then $q \in \mathfrak{k}$ and for $2 \leqslant r \leqslant d - j$,

$$\tilde{p}_{\pi(1)} \cdots q\tilde{p}_{\pi(j+r)} \cdots \tilde{p}_{\pi(d)} = \pm\tilde{p}_{\pi(1)} \cdots \tilde{p}_{\pi(j+r)}q \cdots \tilde{p}_{\pi(d)} + \tilde{p}_{\pi(1)} \cdots [q, \tilde{p}_{\pi(j+r)}] \cdots \tilde{p}_{\pi(d)}, \tag{3.1.2}$$

where $p' := [q, \tilde{p}_{\pi(j+r)}] \in \mathfrak{p}$ and the second term on the right hand side of (3.1.2) is a product of $d - 1$ elements of $\mathfrak{p}$.
For the first term on the right hand side of (3.1.2), we can repeat the process and eventually we end up with $\tilde{p}_{\pi(1)} \cdots \tilde{p}_{\pi(j)}\tilde{p}_{\pi(j+r)} \cdots \tilde{p}_{\pi(l)}q$, which is in $\mathcal{I}_L$. The extra terms that will be obtained by this process are products of $d - 1$ elements of $\mathfrak{p}$ and by induction hypothesis they lie in $\bar{\mathsf{s}}(S(\mathfrak{p}))$ modulo $\mathcal{I}_L$. Therefore, we can rearrange the order of $\tilde{p}_{\pi(j)}$'s until we reach at the monomial $\tilde{p}_1\tilde{p}_2 \dots \tilde{p}_d \in S(\mathfrak{p})$ plus some terms with lower degrees which are products of elements of $\mathfrak{p}$ plus some terms in $\mathcal{I}_L$;

$$\bar{\mathsf{s}}(\tilde{p}_1\tilde{p}_2 \cdots \tilde{p}_d) = \tilde{p}_1\tilde{p}_2 \cdots \tilde{p}_d + \text{lower degree terms } + \mathcal{I}_L. \tag{3.1.3}$$

Since the lower degree terms are in the image of $\bar{\mathsf{s}}$ (modulo $\mathcal{I}_L$), it follows that $\mathcal{I}_L + \tilde{p}_1 \cdots \tilde{p}_d$ is also in the image of $\bar{\mathsf{s}}$.
Step 3. Recall that $\{p_1, \cdots, p_m\}$ is a basis of $\mathfrak{p}$, then for $r_i \geqslant 0$, the monomials $p_1^{r_1} \cdots p_m^{r_m}$ form a basis of $S(\mathfrak{p})$. Therefore, by (3.1.3) we have:

$$\bar{\mathsf{s}}(p_1^{r_1} p_2^{r_2} \cdots p_m^{r_m}) = p_1^{r_1} p_2^{r_2} \cdots p_m^{r_m} + V(r_1, \cdots, r_m) + \mathcal{I}_L, \tag{3.1.4}$$

where $V(r_1, \cdots, r_m) \in V_{N-1}$ and $N = \sum_{i=1}^{m} r_i$. Then $V(r_1, \cdots, r_m)$ is linear combination of monomials of the form $\tilde{p}_1 \cdots \tilde{p}_r$ for $r \leqslant N-1$. But we can express $\tilde{p}_1 \cdots \tilde{p}_r$ as a linear combination of monomials of the form $p_1^{s_1} \cdots p_m^{s_m}$ where $\sum_{i=1}^{m} s_i \leqslant r$.
Recall that the monomials $p_1^{r_1} \cdots p_m^{r_m}$ represent a basis for $U(\mathfrak{g})/\mathcal{I}_L$. By the above dicussion, if we order this basis according to the basis then the transition matrix between the $\bar{\mathsf{s}}(p_1^{r_1} p_2^{r_2} \cdots p_m^{r_m})$ and this basis is a triangular matrix with 1's on the diagonal. Hence, the transition matrix is invertible, which implies that $\{\bar{\mathsf{s}}(p_1^{r_1} p_2^{r_2} \cdots p_m^{r_m}) | r_i \geqslant 0\}$ is a basis of $U(\mathfrak{g})/\mathcal{I}_L$. Therefore $\bar{\mathsf{s}}$ maps a basis to a basis and hence $\bar{\mathsf{s}}$ is injective. ∎

Corollary 3.1.6. The map $\bar{\mathsf{s}} : S(\mathfrak{p}) \to U(\mathfrak{g})/\mathcal{I}_L$ dualizes to an isomorphism of vector superspaces

$$S(\mathfrak{p})^* \cong (U(\mathfrak{g})/\mathcal{I}_L)^*.$$

3.2 Spherical functionals on a symmetric superspace

Recall that $\mathcal{I}_L := U(\mathfrak{g})\mathfrak{k}$ and $\mathcal{I} := U(\mathfrak{g})\mathfrak{k} + \mathfrak{k}U(\mathfrak{g})$. The Lie superalgebra $\mathfrak{k}$ acts on $S(\mathfrak{p})$ by the adjoint action.

Lemma 3.2.1. The image of $\big(\mathrm{ad}_{\mathfrak{k}}(S(\mathfrak{p}))\big)$ under $\bar{\mathsf{s}}$ lies in $\mathcal{I}$.

Proof. For $a \in \mathfrak{k}$ and $x_1, \ldots, x_r \in \mathfrak{p}$ we have

$$\begin{aligned}
\bar{\mathsf{s}}\big(\mathrm{ad}_a(x_1 \cdots x_r)\big) &= \frac{1}{r!} \sum_{\pi \in S_r} \sum_{i=1}^{r} (-1)^{\pi} x_{\pi(1)} \cdots x_{\pi(i-1)} [a, x_{\pi(i)}] x_{\pi(i+1)} \cdots x_{\pi(r)} + \mathcal{I}_L \\
&= \frac{1}{r!} \sum_{\pi \in S_r} (-1)^{\pi} [a, x_{\pi(1)} \cdots x_{\pi(r)}] + \mathcal{I}_L \\
&= \frac{1}{r!} \sum_{\pi \in S_r} (-1)^{\pi} a x_{\pi(1)} \cdots x_{\pi(r)} + \mathcal{I}_L.
\end{aligned}$$

The above argument proves that $\bar{s}(\mathrm{ad}_{\mathfrak{k}}(S(\mathfrak{p}))) \subseteq \mathfrak{k}U(\mathfrak{g}) + \mathcal{I}_L$. This completes the proof. ∎

Lemma 3.2.2. Let V be a finite dimensional complex vector space and let $S \subseteq V$ be Zariski dense. Then $S^n(V)$ is spanned by $\{a^n : a \in S\}$.

Proof. We prove this lemma by the way of contradiction. Suppose the statement is false; then there exists a nonzero functional $f : S^n(V) \to \mathbb{C}$ such that $f(a^n) = 0$ for all $a \in S$. Now, choose a basis $e_1, \ldots, e_d$ for V. This gives us a basis of $S^n(V)$ of the form $e_1^{m_1} \cdots e_d^{m_d}$, where the $m_i \geq 0$ and $\sum_i m_i = n$. Set $c_{m_1,\ldots,m_d} := f(e_1^{m_1} \cdots e_d^{m_d})$. Then for $a \in S$ of the form $a = a_1e_1 + \cdots + a_de_d$ with $a_i \in \mathbb{C}$, we have

$$\begin{aligned} 0 = f(a^n) &= f\left(\sum_{m_1+\cdots+m_d=n} \binom{m_1, \ldots, m_d}{n} a_1^{m_1} \cdots a_d^{m_d} e_1^{m_1} \cdots e_d^{m_d} \right) \\ &= \sum_{m_1+\cdots+m_d=n} \binom{m_1, \ldots, m_d}{n} c_{m_1,\ldots,m_d} a_1^{m_1} \cdots a_d^{m_d}. \end{aligned}$$

The right hand side is a polynomial in $a_1, \ldots, a_d$, and since S is Zariski dense it follows that $c_{m_1,\ldots,m_d} = 0$. Hence $f = 0$, which is a contradiction. ∎

Lemma 3.2.3. Let $x \in \mathfrak{g}_\alpha$, where $\alpha \in \Delta$, and let $a \in \mathfrak{a}$. Then, for every positive integer i, we have

$$\mathrm{ad}_{x+\theta(x)}(a^i) = -i\alpha(a)(x - \theta(x))a^{i-1}.$$

Proof. We use induction on i. For $i = 1$, from $(\theta(\alpha))(a) = -\alpha(a)$ it follows that

$$\mathrm{ad}_{x+\theta(x)}(a) = \mathrm{ad}_x(a) + \mathrm{ad}_{\theta(x)}(a) = -\alpha(a)x + \alpha(a)\theta(x) = -\alpha(a)(x - \theta(x)).$$

Assuming the assertion holds for $i - 1$ and using the fact that $S(\mathfrak{p})$ is commutative and $x - \theta(x) \in \mathfrak{p}$, we have

$$\begin{aligned} \mathrm{ad}_{(x+\theta(x))}(a^i) &= (\mathrm{ad}_{(x+\theta(x))}(a))a^{i-1} + a(\mathrm{ad}_{(x+\theta(x))}(a^{i-1})) \\ &= (-\alpha(a)(x - \theta(x))a^{i-1}) + a(-(i-1)\alpha(a)(x - \theta(x))a^{i-2}). \end{aligned}$$

From the above calculation it follows that

$$\mathrm{ad}_{(x+\theta(x))}(a^i) = -i\alpha(a)(x - \theta(x))a^{i-1}.$$ ∎

For $\alpha \in \Delta$ we set $\mathfrak{e}_\alpha := \mathfrak{g}_\alpha + \mathfrak{g}_{\theta(\alpha)}$. Then we have the following lemma.

Lemma 3.2.4. If $\mathfrak{e}_\alpha \cap \mathfrak{p} \neq \{0\}$, then $\alpha|_\mathfrak{a} \neq 0$.

Proof. Suppose that $\alpha|_\mathfrak{a} = 0$. It implies that $\alpha \circ \theta|_\mathfrak{a} = -\alpha|_\mathfrak{a} = 0$. Therefore, $\mathfrak{e}_\alpha := \mathfrak{g}_\alpha + \mathfrak{g}_{\theta(\alpha)}$ commutes with $\mathfrak{a}$ and hence $\mathfrak{e}_\alpha \cap \mathfrak{p}$ commutes with $\mathfrak{a}$. But $C_\mathfrak{p}(\mathfrak{a}) = \mathfrak{a}$ which implies that $\mathfrak{e}_\alpha \cap \mathfrak{p} = \{0\}$ which is a contradiction. ∎

We need the following proposition, which is originally due to Lepowsky [23, Lemma 3.7] in the case of Lie algebras.

Proposition 3.2.5. The space $S(\mathfrak{p})$ can be expressed as

$$S(\mathfrak{p}) = S(\mathfrak{a}) + \mathrm{ad}_\mathfrak{k}(S(\mathfrak{p})).$$

Proof. We prove a stronger statement: if W is the smallest $\mathrm{ad}_{\mathfrak{k}}$-invariant subspace of $S^r(\mathfrak{p})$ that contains $S^r(\mathfrak{a})$, then $W = S^r(\mathfrak{p})$. Note that

$$W = S^r(\mathfrak{a}) + \sum_{x\in\mathfrak{k}} \mathrm{ad}_x S^r(\mathfrak{a}) + \sum_{x,x'\in\mathfrak{k}} \mathrm{ad}_x \mathrm{ad}_{x'} S^r(\mathfrak{a}) + \cdots \subseteq S^r(\mathfrak{a}) + \mathrm{ad}_{\mathfrak{k}} S^r(\mathfrak{p}).$$

Thus, if we prove that $W = S^r(\mathfrak{p})$, then the proposition follows. Recall that $\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha\in\Delta} \mathfrak{g}_\alpha$ where $\Delta \subseteq \mathfrak{h}^*\backslash\{0\}$. By Lemma 3.1.2 for $x \in \mathfrak{g}_\alpha$ we have $\theta(x) \in \mathfrak{g}_{\theta(\alpha)}$ where $\theta(\alpha) := \alpha \circ \theta$. For every $\alpha \in \mathfrak{h}^*$, the subspace $\mathfrak{e}_\alpha := \mathfrak{g}_\alpha + \mathfrak{g}_{\theta(\alpha)}$ is θ-invariant because $\theta(\mathfrak{g}_\alpha) \subseteq \mathfrak{g}_{\theta(\alpha)}$ and $\mathfrak{g}_{\theta(\alpha)} \subseteq \mathfrak{g}_\alpha$. It follows that $\mathfrak{e}_\alpha = (\mathfrak{e}_\alpha \cap \mathfrak{k}) \oplus (\mathfrak{e}_\alpha \cap \mathfrak{p})$. In addition,

$$\mathfrak{e}_\alpha \cap \mathfrak{k} = \{x + \theta(x) \,:\, x \in \mathfrak{g}_\alpha\} \quad \text{and} \quad \mathfrak{e}_\alpha \cap \mathfrak{p} = \{x - \theta(x) \,:\, x \in \mathfrak{g}_\alpha\}.$$

It follows that $\mathfrak{p} = \mathfrak{a} \oplus \mathfrak{q}$ where $\mathfrak{q} := \bigoplus_{\alpha\in\Delta}(\mathfrak{e}_\alpha \cap \mathfrak{p})$. Next note that

$$S^r(\mathfrak{p}) = \bigoplus_{i=0}^{r} S^{r-i}(\mathfrak{q}) S^i(\mathfrak{a}).$$

We prove that $S^{r-i}(\mathfrak{q})S^i(\mathfrak{a}) \subseteq W$ for $0 \leq i \leq r$ by a reverse induction on i. The statement is trivial for $i = r$. Next assume that it holds for all $i' \geq i$. Since

$$S^{r-(i-1)}(\mathfrak{q}) = S^{r-i}(\mathfrak{q}) S^1(\mathfrak{q}) = S^{r-i}(\mathfrak{q})\mathfrak{q},$$

it suffices to prove that $qea^{i-1} \in W$ for every $q \in S^{r-i}(\mathfrak{q})$, $a \in \mathfrak{a}$, and $e \in \mathfrak{q}$. Since elements of $\mathfrak{q}$ are linear combinations of elements of $\mathfrak{e}_\alpha \cap \mathfrak{p}$, it suffices to assume that $e \in \mathfrak{e}_\alpha \cap \mathfrak{p}$ for some α. Thus, we can assume that $e = x - \theta(x)$ for some $x \in \mathfrak{g}_\alpha$. By choosing e to be homogeneous with respect to the $\mathbb{Z}_2$-grading, we can assume x is also homogeneous. Note that we can also assume that q is homogeneous.
Since $e \in \mathfrak{e}_\alpha \cap \mathfrak{p}$ and we can assume $e \neq 0$, from Lemma 3.2.4 it follows that $\alpha|_{\mathfrak{a}} \neq 0$. Recall that $e := x - \theta(x)$ where $x \in \mathfrak{g}_\alpha$. In particular, we have $x + \theta(x) \in \mathfrak{k}$, so that by Lemma 3.2.3 for $a \in \mathfrak{a}$ and $q \in S^{r-i}(\mathfrak{q})$ we have

$$\begin{aligned}\mathrm{ad}_{x+\theta(x)}(qa^i) &= (\mathrm{ad}_{x+\theta(x)} q)a^i + (-1)^{|q||x|} q\,\mathrm{ad}_{(x+\theta(x))}(a^i)\\ &= (\mathrm{ad}_{x+\theta(x)} q)a^i - (-1)^{|q||x|} i\alpha(a) q(x - \theta(x))a^{i-1}.\end{aligned}$$

By the induction hypothesis, the first term on the right hand side belongs to W. Also, the left hand side belongs to W (by $\mathrm{ad}_{\mathfrak{k}}$-invariance of W and the induction hypothesis). It follows that

$$\alpha(a)q(x - \theta(x))a^{i-1} \in W.$$

Since by Lemma 3.2.4 $\alpha\big|_{\mathfrak{a}} \neq 0$, for a Zariski open (hence dense) set $S \subseteq \mathfrak{a}$ we have if $a \in S$ then $\alpha(a) \neq 0$. Thus for $a \in S$,

$$q(x - \theta(x))a^{i-1} \in W.$$

Since the choices of $e = x - \theta(x)$ form a spanning set of $\mathfrak{q}$, from the above arguments it follows that

$$(S^{r-i}(\mathfrak{q})\mathfrak{q})a^{i-1} \subseteq W \text{ if } a \in S.$$

Now by Lemma 3.2.2 we have $S^{i-1}(\mathfrak{a})$ is spanned by elements of the form a^{i-1}, where $a \in S$. ∎

Since $\mathcal{I}_L \subseteq \mathcal{I}$, we have a natural quotient map Let $q : U(\mathfrak{g})/\mathcal{I}_L \to U(\mathfrak{g})/\mathcal{I}_L$.

Corollary 3.2.6. The universal enveloping algebra $U(\mathfrak{g})$ can be expressed as $U(\mathfrak{g}) = \mathcal{I} + \bar{\mathsf{s}}(S(\mathfrak{a}))$. In particular the natural map $q \circ \bar{\mathsf{s}} : S(\mathfrak{a}) \to U(\mathfrak{g})/\mathcal{I}$ is a surjection.

Proof. By Proposition 3.1.5 we have:

$$\bar{\mathsf{s}}(S(\mathfrak{p})) + \mathcal{I}_L = U(\mathfrak{g}).$$

Therefore, by Proposition 3.2.5, $\bar{\mathsf{s}}(S(\mathfrak{a})) + \bar{\mathsf{s}}(\mathrm{ad}_{\mathfrak{k}} S(\mathfrak{p})) + \mathcal{I}_L = U(\mathfrak{g})$ and by Lemma 3.2.1, $\bar{\mathsf{s}}(\mathrm{ad}_{\mathfrak{k}} S(\mathfrak{p})) \subseteq \mathcal{I}$ which implies that $\bar{\mathsf{s}}(S(\mathfrak{a})) + \mathcal{I} = U(\mathfrak{g})$. Hence, the natural map $\bar{\mathsf{s}} : S(\mathfrak{a}) \to U(\mathfrak{g})/\mathcal{I}$ is a surjection. ∎

Corollary 3.2.6 implies the following assertion. Recall that $\mathcal{A}(\mathfrak{g}, \mathfrak{k})$ is the dual of $U(\mathfrak{g})/\mathcal{I}$.

Corollary 3.2.7. The map $(q \circ \bar{\mathsf{s}})^* : \mathcal{A}(\mathfrak{g}, \mathfrak{k}) \to S(\mathfrak{a})^*$ that is dual to $S(\mathfrak{a}) \to U(\mathfrak{g})/\mathcal{I}$ is an injection.

Proof. The injectivity of $(q \circ \bar{\mathsf{s}})^*$ follows from Corollary 3.2.6. Since $\mathfrak{a}$ is abelian, the map $\mathsf{s} : S(\mathfrak{a}) \to U(\mathfrak{g})$ is the canonical embedding of Hopf algebras. The coalgebra structures of $U(\mathfrak{g})$ and $S(\mathfrak{a})$ induce the algebra structures of $\mathcal{A}(\mathfrak{g}, \mathfrak{k})$ and $S(\mathfrak{a})^*$, and $(q \circ \bar{\mathsf{s}})^*$ is compatible with these structures.
Finally, let $\lambda \in \mathcal{A}(\mathfrak{g}, \mathfrak{k})^\circ$. Then:

$$\dim(S(\mathfrak{a})/(\ker(\lambda) \cap S(\mathfrak{a}))) \leq \dim(U(\mathfrak{g})/(U(\mathfrak{g}) \cap \ker(\lambda))) < \infty,$$

hence $(q \circ \bar{\mathsf{s}})^*(\lambda) \in S(\mathfrak{a})^*$. ∎

Remark 3.2.8. Let G be a real reductive Lie group with a Cartan decomposition $G = KAK$, where K is the maximal compact subgroup of G. Then the restriction of a bi-K-invariant function on G to A is invariant under the action of the small Weyl group associated to A. It is natural to expect that, analogously, the image of the map $(q \circ s)^*$ of Corollary 3.2.7 can be characterized by symmetry conditions imposed by the Weyl groupoid of the restricted root system of $(\mathfrak{g}, \mathfrak{k})$.

3.3 Explicit computation of $\mathcal{I}$ for the pair $(\mathfrak{gl}(1|2), \mathfrak{osp}(1|2))$

In this section, we compute an explicit basis for $\mathcal{I} = \mathfrak{k}U(\mathfrak{g}) + U(\mathfrak{g})\mathfrak{k}$, in the context of the Lie superalgebra quotient $\mathfrak{gl}(1|2)/\mathfrak{osp}(1|2)$. A particularly intriguing aspect of this computation is the emergence of Euler Zigzag numbers within the structure of the basis elements. This phenomenon suggests a deeper interplay between the representation theory of Lie superalgebras and combinatorics, potentially paving the way for broader exploration in more general settings.

An alternating permutation or zigzag permutation of the set $\{1, 2, 3, \ldots, n\}$ is a permutation where each entry alternates between being greater than and less than the preceding entry. This type of permutation was first studied by Désiré André in the 19th century [25]. The first few values of A_n, which represent the number of alternating permutations of the set $\{1, 2, 3, \ldots, n\}$, are:

$$1, 1, 1, 2, 5, 16, 61, 272, 1385, 7936, 50521, \cdots$$

Determining A_n is known as André's problem. These numbers A_n are referred to as Euler zigzag numbers, or up/down numbers. Specifically, when n is even, A_n is known as a secant number, and when n is odd, it is called a tangent number. These names originate from the generating function associated with the sequence, expressed as the following well-known result, known as André's theorem [2, Ch. 2.2]:

$$\tan(x) + \sec(x) = \sum_{n \geqslant 0} \frac{A_n}{n!} x^n.$$

A geometric interpretation of this result can be given using a generalization of a theorem by Johann Bernoulli [15]. Furthermore, the Maclaurin series for $\tan(x)$ and $\sec(x)$ are:

$$\begin{aligned} \tan(x) &= x + \frac{x^3}{3} + \frac{2x^5}{15} + \frac{17x^7}{315} + \frac{62x^9}{2835} + \cdots \\ &= \sum_{n=1}^{\infty} (-1)^{n-1} \frac{2^{2n}(2^{2n}-1)B_{2n}}{(2n)!} x^{2n-1}, \end{aligned}$$

and

$$\begin{aligned} \sec(x) &= 1 + \frac{x^2}{2} + \frac{5x^4}{24} + \frac{61x^6}{720} + \frac{277x^8}{8064} + \cdots \\ &= \sum_{n=0}^{\infty} (-1)^n \frac{E_{2n}}{(2n)!} x^{2n}, \end{aligned}$$

where the numbers E_k in the expansion of $\sec(x)$ are Euler numbers [12, Ch. 6.2] and the numbers B_k appearing in the series for $\tan(x)$ are the Bernoulli numbers [12, Ch. 6.5].

Now, let $(\mathfrak{g}, \mathfrak{k}) = (\mathfrak{gl}(1|2), \mathfrak{osp}(1|2))$. Our goal is to find a basis of $\mathcal{I}$. Let $I_{\bar{0}} = \{1\}$ and $I_{\bar{1}} = \{\bar{1}, \bar{2}\}$ and let $\{E_{ij} \mid i, j \in I = I_{\bar{0}} \sqcup I_{\bar{1}}\}$ be a basis for $\mathfrak{gl}(1|2)$ consisting of standard

matrix units. Recall that $|j| = 0$ if $j \in I_{\bar{0}}$, and $|j| = 1$ if $j \in I_{\bar{1}}$.
Consider the Cartan involution $\theta(X) = -PX^{st}P^{-1}$ where $X^{st} = \sum_{i,j\in I}(-1)^{|j|(|i|+|j|)}x_{ij}E_{ji}$ and

$$P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix},$$

Then,

$$\mathfrak{k} = \{X \in \mathfrak{gl}(1|2) | \theta(X) = X\} = \left\{ \begin{bmatrix} 0 & b & c \\ -c & e & f \\ b & h & -e \end{bmatrix} | \, b, c, e, f, h \in \mathbb{C} \right\}.$$

Therefore, $\mathfrak{k}_{\bar{0}} \cong \mathfrak{sl}(2, \mathbb{C})$. Also,

$$\mathfrak{p} = \{X \in \mathfrak{gl}(1|2) | \theta(X) = -X\} = \left\{ \begin{bmatrix} a & b & c \\ c & e & 0 \\ -b & 0 & e \end{bmatrix} | \, a, b, c, e \in \mathbb{C} \right\}.$$

Thus

$$\mathfrak{p}_{\bar{0}} = \left\{ \begin{bmatrix} a & 0 & 0 \\ 0 & e & 0 \\ 0 & 0 & e \end{bmatrix} | \, a, e \in \mathbb{C} \right\}, \quad \mathfrak{p}_{\bar{1}} = \left\{ \begin{bmatrix} 0 & b & c \\ c & 0 & 0 \\ -b & 0 & 0 \end{bmatrix} | \, b, c \in \mathbb{C} \right\}.$$

Hence, the maximal abelian subspace of $\mathfrak{p}$ will be as follows:

$$\mathfrak{a} = \operatorname{Span} \left\{ h_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, h_{\bar{1}} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right\}.$$

Our next goal is to choose a basis for $\mathfrak{gl}(1|2)$ with convenient commutator relations. To this end, we make the following choices:

$$p = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \in \mathfrak{p}_{\bar{0}}, \quad e = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} \in \mathfrak{p}_{\bar{1}}, \quad f = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \in \mathfrak{p}_{\bar{1}}.$$

$$k = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \in \mathfrak{k}_{\bar{0}}, \quad e' = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \in \mathfrak{k}_{\bar{1}}, \quad f' = \begin{bmatrix} 0 & 0 & 1 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \in \mathfrak{k}_{\bar{1}}.$$

$$k_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \in \mathfrak{k}_{\bar{0}}, \quad k_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \in \mathfrak{k}_{\bar{0}}.$$

Then $\{k, k_1, k_2, e', f', p, e, f\}$ forms a basis of $\mathfrak{g}$. Consider this order on the basis elements of $\mathfrak{g}$. Therefore by PBW theorem

$$k^{r_1}k_1^{r_2}k_2^{r_3}(e')^{r_4}(f')^{r_5}p^{s_1}e^{s_2}f^{s_3}$$

form a basis of $U(\mathfrak{g})$. Set

$$v_{\mathfrak{p}} = \begin{bmatrix} 0 & a & b \\ b & 0 & 0 \\ -a & 0 & 0 \end{bmatrix} \in \mathfrak{p}_{\bar{1}}, \quad v_{\mathfrak{k}} = \begin{bmatrix} 0 & a & b \\ -b & 0 & 0 \\ a & 0 & 0 \end{bmatrix} \in \mathfrak{k}_{\bar{1}},$$

where $a, b \in \mathbb{C}$. Note that $v_{\mathfrak{k}}$ and $v_{\mathfrak{p}}$ depend on the vector $v = [a, b] \in \mathbb{C}^2$. We can see that $[v_{\mathfrak{p}}, p] = -v_{\mathfrak{k}}$ and $[v_{\mathfrak{k}}, p] = -v_{\mathfrak{p}}$.

Lemma 3.3.1. Let n be a positive integer. Then

$$p^n v_{\mathfrak{k}} = v_{\mathfrak{k}} \sum_{\substack{0 \leqslant i \leqslant n \\ i \text{ is even}}} \binom{n}{i} p^{n-i} + v_{\mathfrak{p}} \sum_{\substack{0 \leqslant i \leqslant n \\ i \text{ is odd}}} \binom{n}{i} p^{n-i},$$

and $v_{\mathfrak{p}} p^m \in \mathcal{I}$ for all non-negative integers $m \leqslant n$.

Proof. We have $[v_{\mathfrak{p}}, p] = -v_{\mathfrak{k}}$ and $[v_{\mathfrak{k}}, p] = -v_{\mathfrak{p}}$. Therefore,

$$p v_{\mathfrak{k}} = v_{\mathfrak{k}} p - [v_{\mathfrak{k}}, p] = v_{\mathfrak{k}} p + v_{\mathfrak{p}},$$

which implies that $v_{\mathfrak{p}} \in \mathcal{I}$ and therefore, $\mathfrak{p}_{\bar{1}} \subseteq \mathcal{I}$. Now suppose the assertion holds for $n-1$. First, we assume that n is odd (the argument for even n is analogous). Then we have,

$$p^{n-1} v_{\mathfrak{k}} = v_{\mathfrak{k}} \sum_{\substack{0 \leqslant i \leqslant n-1 \\ i \text{ is even}}} \binom{n-1}{i} p^{n-1-i} + v_{\mathfrak{p}} \sum_{\substack{0 \leqslant i \leqslant n-1 \\ i \text{ is odd}}} \binom{n-1}{i} p^{n-1-i},$$

and $v_{\mathfrak{p}} p^m \in \mathcal{I}$ for all positive integers $m \leqslant n-1$. Therefore,

$$\begin{aligned} p^n v_{\mathfrak{k}} &= p v_{\mathfrak{k}} \sum_{\substack{0 \leqslant i \leqslant n-1 \\ i \text{ is even}}} \binom{n-1}{i} p^{n-1-i} + p v_{\mathfrak{p}} \sum_{\substack{0 \leqslant i \leqslant n-1 \\ i \text{ is odd}}} \binom{n-1}{i} p^{n-1-i} \\ &= (v_{\mathfrak{k}} p + v_{\mathfrak{p}}) \sum_{\substack{0 \leqslant i \leqslant n-1 \\ i \text{ is even}}} \binom{n-1}{i} p^{n-1-i} + (v_{\mathfrak{p}} p + v_{\mathfrak{k}}) \sum_{\substack{0 \leqslant i \leqslant n-1 \\ i \text{ is odd}}} \binom{n-1}{i} p^{n-1-i} \\ &= v_{\mathfrak{k}} \left(\sum_{\substack{0 \leqslant i \leqslant n-1 \\ i \text{ is even}}} \binom{n-1}{i} p^{n-i} + \sum_{\substack{0 \leqslant i \leqslant n-1 \\ i \text{ is odd}}} \binom{n-1}{i} p^{n-1-i} \right) \\ &+ v_{\mathfrak{p}} \left(\sum_{\substack{0 \leqslant i \leqslant n-1 \\ i \text{ is even}}} \binom{n-1}{i} p^{n-1-i} + \sum_{\substack{0 \leqslant i \leqslant n-1 \\ i \text{ is odd}}} \binom{n-1}{i} p^{n-i} \right). \end{aligned}$$

But

$$\sum_{\substack{0 \leqslant i \leqslant n-1 \\ i \text{ is even}}} \binom{n-1}{i} p^{n-i} + \sum_{\substack{0 \leqslant i \leqslant n-1 \\ i \text{ is odd}}} \binom{n-1}{i} p^{n-1-i}$$

$$= p^n + \sum_{\substack{2 \leqslant i \leqslant n-1 \\ i \text{ is even}}} \left(\binom{n-1}{i} + \binom{n-1}{i-1} \right) p^{n-i}$$
$$= p^n + \sum_{\substack{2 \leqslant i \leqslant n-1 \\ i \text{ is even}}} \binom{n}{i} p^{n-i} = \sum_{\substack{0 \leqslant i \leqslant n-1 \\ i \text{ is even}}} \binom{n}{i} p^{n-i}.$$

By a similar argument we can see that

$$\left(\sum_{\substack{0 \leqslant i \leqslant n-1 \\ i \text{ is odd}}} \binom{n-1}{i} p^{n-1-i} + \sum_{\substack{0 \leqslant i \leqslant n-1 \\ i \text{ is even}}} \binom{n-1}{i} p^{n-i} \right) = \sum_{\substack{0 \leqslant i \leqslant n \\ i \text{ is odd}}} \binom{n}{i} p^{n-i}.$$

Hence,

$$p^n v_{\mathfrak{k}} = v_{\mathfrak{k}} \sum_{\substack{0 \leqslant i \leqslant n \\ i \text{ is even}}} \binom{n}{i} p^{n-i} + v_{\mathfrak{p}} \sum_{\substack{0 \leqslant i \leqslant n \\ i \text{ is odd}}} \binom{n}{i} p^{n-i}.$$

Since the left hand side and the first summation on the right hand side of the above equation are in $\mathcal{I}$, and by the hypothesis of induction, $v_{\mathfrak{p}} p^m \in \mathcal{I}$ for all positive integers $m \leqslant n-1$, we also obtain $v_{\mathfrak{p}} p^n \in \mathcal{I}$. ∎

The commutator relations between v_p, v_k, and p imply that for every $n \in \mathbb{Z}^{\geqslant 0}$ there exist unique polynomials $\alpha_n(x)$ and $\beta_n(x)$, independent of the choice of v, such that

$$p^n v_{\mathfrak{k}} = v_{\mathfrak{k}} \alpha_n(p) + \beta_{n-1}(p) v_{\mathfrak{p}}.$$

Lemma 3.3.2. Let $\alpha_n(x)$ and $\beta_n(x)$ be two polynomials such that $p^n v_{\mathfrak{k}} = v_{\mathfrak{k}} \alpha_n(p) + \beta_{n-1}(p) v_{\mathfrak{p}}$. Then, the following assertions hold:
(i) $\deg(\alpha_n) = \deg(\beta_n) = n$.
(ii) $v_{\mathfrak{p}} p^n = \alpha_n(p) v_{\mathfrak{p}} - v_{\mathfrak{k}} \beta_{n-1}(p)$.
(iii) $\alpha_{n+1}(x) = x\alpha_n(x) - \sum_{i=1}^n a_i \beta_{i-1}(x)$ and $\beta_n(x) = x\beta_{n-1}(x) + \sum_{i=0}^n a_i \alpha_i(x)$,
where $\alpha_n(x) = a_0 + a_1 x + \cdots + a_n x^n$.

Proof. We prove this lemma by induction on n. For $n = 1$ the lemma is trivial ($\alpha_1(x) = x$ and $\beta_0(x) = 1$). Now, assuming that the assertion holds for n, we have

$$\begin{aligned}
v_{\mathfrak{p}} p^{n+1} &= \alpha_n(p) v_{\mathfrak{p}} p - v_{\mathfrak{k}} \beta_{n-1}(p) p \\
&= \alpha_n(p)(p v_{\mathfrak{p}} - v_{\mathfrak{k}}) - v_{\mathfrak{k}} \beta_{n-1}(p) p \\
&= \alpha_n(p) p v_{\mathfrak{p}} - \alpha_n(p) v_{\mathfrak{k}} - v_{\mathfrak{k}} \beta_{n-1}(p) p \\
&= \alpha_n(p) p v_{\mathfrak{p}} - \sum_{i=0}^n a_i p^i v_{\mathfrak{k}} - v_{\mathfrak{k}} \beta_{n-1}(p) p \\
&= \alpha_n(p) p v_{\mathfrak{p}} - \sum_{i=0}^n a_i (v_{\mathfrak{k}} \alpha_i(p) + \beta_{i-1}(p) v_{\mathfrak{p}}) - v_{\mathfrak{k}} \beta_{n-1}(p) p
\end{aligned}$$

$$= \left(p\alpha_n(p) - \sum_{i=1}^{n} a_i\beta_{i-1}\right) v_{\mathfrak{p}} - v_{\mathfrak{k}} \left(p\beta_{n-1}(p) + \sum_{i=0}^{n} a_i\alpha_i\right).$$

But by the hypothesis of induction and straightforward computation we have

$$\begin{aligned} v_{\mathfrak{k}}\alpha_{n+1} + \beta_n v_{\mathfrak{p}} = p^{n+1}v_{\mathfrak{k}} &= pv_{\mathfrak{k}}\alpha_n(p) + p\beta_{n-1}(p)v_{\mathfrak{p}} \\ &= (v_{\mathfrak{k}}p + v_{\mathfrak{p}})\alpha_n(p) + p\beta_{n-1}(p)v_{\mathfrak{p}} \\ &= v_{\mathfrak{k}}p\alpha_n(p) + v_{\mathfrak{p}}\alpha_n(p) + p\beta_{n-1}(p)v_{\mathfrak{p}} \\ &= v_{\mathfrak{k}}p\alpha_n(p) + \sum_{i=0}^{n} a_i\alpha_i v_{\mathfrak{p}} - v_{\mathfrak{k}}\sum_{i=1}^{n} a_i\beta_{i-1} + p\beta_{n-1}(p)v_{\mathfrak{p}} \\ &= v_{\mathfrak{k}}\left(p\alpha_n(p) - \sum_{i=1}^{n} a_i\beta_{i-1}\right) + \left(p\beta_{n-1}(p) + \sum_{i=0}^{n} a_i\alpha_i\right)v_{\mathfrak{p}}. \end{aligned}$$

Comparing the result of the latter calculation with the assumption
$p^{n+1}v_{\mathfrak{k}} = v_{\mathfrak{k}}\alpha_{n+1}(p) + \beta_n(p)v_{\mathfrak{p}}$, and using the PBW theorem, we obtain the recursions

$$\alpha_{n+1}(x) = x\alpha_n(x) - \sum_{i=1}^{n} a_i\beta_{i-1}(x)$$

and

$$\beta_n(x) = x\beta_{n-1}(x) + \sum_{i=0}^{n} a_i\alpha_i(x).$$

Furthermore, from the right-hand side of the previous calculation of $v_{\mathfrak{p}}p^{n+1}$, we obtain $v_{\mathfrak{p}}p^{n+1} = \alpha_{n+1}(p)v_{\mathfrak{p}} - v_{\mathfrak{k}}\beta_n(p)$. The statements about the degree of α_n and β_n follow from the recursion as well. ∎

Lemma 3.3.3. Let $p^n v_{\mathfrak{k}} = v_{\mathfrak{k}}\alpha_n(p) + \beta_{n-1}(p)v_{\mathfrak{p}}$. Then

$$\alpha_n(x) = \sum_{k=0}^{n} E_{2k}\binom{n}{2k}x^{n-2k},$$

where E_i are Euler numbers.

Proof. Assume that for some numbers $\bar{E}_i$,

$$\alpha_n(x) = \sum_{k=0}^{n} \bar{E}_{2k}\binom{n}{2k}x^{n-2k}.$$

Our goal is to prove that $\bar{E}_i = E_i$ for all i. By Lemmas 3.3.1 and 3.3.2 we have,

$$p^n v_{\mathfrak{k}} = v_{\mathfrak{k}} \sum_{\substack{0 \leqslant i \leqslant n \\ i \text{ is even}}} \binom{n}{i} p^{n-i} + v_{\mathfrak{p}} \sum_{\substack{0 \leqslant i \leqslant n \\ i \text{ is odd}}} \binom{n}{i} p^{n-i}$$

$$
\begin{aligned}
&= v_{\mathfrak{k}} \sum_{\substack{0\leqslant i\leqslant n\\ i \text{ is even}}} \binom{n}{i} p^{n-i} + \sum_{\substack{0\leqslant i\leqslant n\\ i \text{ is odd}}} \binom{n}{i} v_{\mathfrak{p}} p^{n-i}\\
&= v_{\mathfrak{k}} \sum_{\substack{0\leqslant i\leqslant n\\ i \text{ is even}}} \binom{n}{i} p^{n-i} + \sum_{\substack{0\leqslant i\leqslant n\\ i \text{ is odd}}} \binom{n}{i} (\alpha_{n-i}(p)v_{\mathfrak{p}} - v_{\mathfrak{k}}\beta_{n-i-1}(p))\\
&= v_{\mathfrak{k}} \left(\sum_{\substack{0\leqslant i\leqslant n\\ i \text{ is even}}} \binom{n}{i} p^{n-i} - \sum_{\substack{0\leqslant i\leqslant n\\ i \text{ is odd}}} \binom{n}{i} \beta_{n-i-1}(p)) \right) + \sum_{\substack{0\leqslant i\leqslant n\\ i \text{ is odd}}} \binom{n}{i} \alpha_{n-i}(p)v_{\mathfrak{p}}.
\end{aligned}
$$

Comparing the right hand side of the above calculation and the assumption $p^n v_{\mathfrak{k}} = v_{\mathfrak{k}}\alpha_n(p) + \beta_{n-1}(p)v_{\mathfrak{p}}$ and also by using PBW theorem we obtain

$$
\beta_{n-1}(x) = \sum_{\substack{0\leqslant i\leqslant n\\ i \text{ is odd}}} \binom{n}{i} \alpha_{n-i}(x),
$$

and

$$
\alpha_n(x) = \sum_{\substack{0\leqslant i\leqslant n\\ i \text{ is even}}} \binom{n}{i} x^{n-i} - \sum_{\substack{0\leqslant i,j\leqslant n\\ i,j \text{ are odd}}} \binom{n}{i}\binom{n-i}{j} \alpha_{n-i-j}(x).
$$

Therefore, we have

$$
\sum_{k=0}^{n} \bar{E}_{2k} \binom{n}{2k} x^{n-2k} = \sum_{\substack{0\leqslant i\leqslant n\\ i \text{ is even}}} \binom{n}{i} x^{n-i} - \sum_{\substack{0\leqslant i,j\leqslant n\\ i,j \text{ are odd}}} \binom{n}{i}\binom{n-i}{j} \alpha_{n-i-j}(x).
$$

Then by comparing the coefficient of x^{n-2k} on both sides we obtain:

$$
\bar{E}_{2k}\binom{n}{2k} = \binom{n}{2k} - \sum_{\substack{0\leqslant i,j\leqslant n\\ i,j \text{ are odd}}} \binom{n}{i,j,n-2k,2k-i-j} \bar{E}_{2k-i-j}.
$$

We adopt the convention that $E_k = 0$ for $k < 0$. Thus,

$$
\bar{E}_{2k} = 1 - \sum_{\substack{0\leqslant i,j\\ i,j \text{ are odd}}} \binom{2k}{i,j,2k-i-j} \bar{E}_{2k-i-j}.
$$

Note that by the above convention we can also remove the constraint $i, j \leqslant n$. Dividing both sides by $(2k)!$, we obtain:

$$
\frac{\bar{E}_{2k}}{(2k)!} = \frac{1}{(2k)!} - \sum_{\substack{0\leqslant i,j\\ i,j \text{ are odd}}} \frac{1}{i!j!}\frac{\bar{E}_{2k-i-j}}{(2k-i-j)!}.
$$

Then, by substituting $i = 2A+1$ and $j = 2B+1$, we reach the following expression.

$$\frac{\bar{E}_{2k}}{(2k)!} = \frac{1}{(2k)!} - \sum_{0 \leqslant A, B} \frac{1}{(2A+1)!(2B+1)!} \frac{\bar{E}_{2k-2A-2B-2}}{(2k-2A-2B-2)!}.$$

Thus,

$$\sum_{k \geqslant 0} \frac{\bar{E}_{2k}}{(2k)!} x^{2k} = \sum_{k \geq 0} \frac{x^{2k}}{(2k)!} - \left(\sum_{A \geq 0} \frac{x^{2A+1}}{(2A+1)!} \right) \left(\sum_{B \geq 0} \frac{x^{2B+1}}{(2B+1)!} \right) \sum_{r \geqslant 0} \frac{\bar{E}_{2r}}{(2r)!} x^{2r}.$$

But we have

$$\cosh(x) = \sum_{k \geq 0} \frac{x^{2k}}{(2k)!}, \quad \sinh(x) = \sum_{k \geq 0} \frac{x^{2k+1}}{(2k+1)!}.$$

Therefore,

$$\sum_{k \geqslant 0} \frac{\bar{E}_{2k}}{(2k)!} x^{2k} = \left(\frac{e^x + e^{-x}}{2} \right) - \left(\frac{e^x - e^{-x}}{2i} \right)^2 \sum_{r \geqslant 0} \frac{\bar{E}_{2r}}{(2r)!} x^{2r},$$

which implies that

$$\sum_{k \geqslant 0} \frac{\bar{E}_{2k}}{(2k)!} x^{2k} = \frac{2}{e^x + e^{-x}}.$$

The right hand side of the above equation is $\operatorname{sech}(x)$. Since $\operatorname{sech}(x) = \sec(ix)$, the Taylor series of $\operatorname{sech}(x)$ is

$$\operatorname{sech}(x) = \sum_{k \geqslant 0} \frac{E_{2k}}{(2k)!} x^{2k},$$

where E_{2k} are Euler numbers. It follows that $\bar{E}_{2k} = E_{2k}$. ∎

Lemma 3.3.4. Let $p^n v_{\mathfrak{k}} = v_{\mathfrak{k}} \alpha_n(p) + \beta_{n-1}(p) v_{\mathfrak{p}}$. Then

$$\beta_{n-1}(x) = \sum_{k=0}^{n} \frac{2^{2k+2}(2^{2k+2}-1)B_{2k+2}}{2k+2} \binom{n}{n-1-2k} x^{n-1-2k},$$

where B_i are Bernoulli numbers.

Proof. According to the proof of Lemma 3.3.3,

$$\beta_{n-1}(x) = \sum_{\substack{0 \leqslant j \leqslant n \\ j \text{ is odd}}} \binom{n}{j} \alpha_{n-j}(x).$$

Assume that for some numbers $\bar{B}_i$,

$$\beta_{n-1}(x) = \sum_{k=0}^{n} \frac{2^{2k+2}(2^{2k+2}-1)\bar{B}_{2k+2}}{2k+2} \binom{n}{n-1-2k} x^{n-1-2k}.$$

Then by Lemma 3.3.3 we obtain

$$\sum_{k=0}^{n} \frac{2^{2k+2}(2^{2k+2}-1)\bar{B}_{2k+2}}{2k+2}\binom{n}{n-1-2k}x^{n-1-2k} = \sum_{\substack{0\leqslant j\leqslant n\\ j \text{ is odd}}}\binom{n}{j}\alpha_{n-j}(x)$$

$$= \sum_{\substack{0\leqslant j\leqslant n\\ j \text{ is odd}}}\sum_{r=0}^{n-j}\binom{n}{j}\binom{n-j}{2r}E_{2r}x^{n-j-2r}$$

where the E_i are Euler numbers. Therefore, by comparing the coefficient of x^{n-1-2k} on both sides we obtain

$$\frac{2^{2k+2}(2^{2k+2}-1)\bar{B}_{2k+2}}{2k+2}\binom{n}{n-1-2k}$$

$$= \sum_{\substack{0\leqslant j\leqslant n\\ j \text{ is odd}}}\sum_{k=j-1}^{n-j}\binom{n}{j, 2k-j+1, n-2k-1}E_{2k+1-j}.$$

Hence

$$\frac{2^{2k+2}(2^{2k+2}-1)\bar{B}_{2k+2}}{2k+2} = \sum_{\substack{0\leqslant j\leqslant n\\ j \text{ is odd}}}\sum_{k=j-1}^{n-j}\frac{(2k+1)!}{j!(2k+1-j)!}E_{2k+1-j}.$$

Dividing both sides by $(2k+1)!$, we obtain

$$\frac{2^{2k+2}(2^{2k+2}-1)\bar{B}_{2k+2}}{(2k+2)!} = \sum_{\substack{0\leqslant j\leqslant n\\ j \text{ is odd}}}\sum_{k=j-1}^{n-j}\frac{E_{2k+1-j}}{j!(2k+1-j)!}.$$

Then, by the substitiution $j = 2A+1$ we obtain

$$\sum_{k\geqslant 0}\frac{2^{2k+2}(2^{2k+2}-1)\bar{B}_{2k+2}}{(2k+2)!}x^{2k+1} = \sum_{A\geqslant 0}\frac{x^{2A+1}}{(2A+1)!}\sum_{r\geqslant 0}\frac{E_{2r}}{(2r)!}x^{2r}.$$

But we have

$$\operatorname{sech}(x) = \sum_{r\geqslant 0}\frac{E_{2r}}{(2r)!}x^{2r}, \quad \sinh(x) = \sum_{k\geq 0}\frac{x^{2k+1}}{(2k+1)!}.$$

Therefore,

$$\sum_{k\geqslant 0}\frac{2^{2k+2}(2^{2k+2}-1)\bar{B}_{2k+2}}{(2k+2)!}x^{2k+1} = \frac{e^x-e^{-x}}{2}\,\frac{2}{e^x+e^{-x}} = \frac{e^x-e^{-x}}{e^x+e^{-x}}.$$

The right hand side of the above equation is the hyperbolic tangent function $\tanh(x)$. The Taylor series of $\tanh(x)$ is

$$\tanh(x) = \sum_{k\geqslant 0}\frac{2^{2k+2}(2^{2k+2}-1)B_{2k+2}}{(2k+2)!}x^{2k+1},$$

where B_{2k+2} are the Bernoulli numbers. Therefore, $\bar{B}_{2k+2} = B_{2k+2}$. ∎

Lemma 3.3.5. For every

$$k_0 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \alpha & \beta \\ 0 & \gamma & -\alpha \end{bmatrix} \in \mathfrak{k}_{\bar{0}}$$

we have

$$\begin{aligned}
(i) \quad & p^n k_0 = k_0 p^n \in \mathfrak{k}U(\mathfrak{g}), \\
(ii) \quad & p^n e k_0 = k_0 p^n e + p^n(\alpha e + \beta f), \\
(iii) \quad & p^n f k_0 = k_0 p^n f + p^n(\gamma e - \alpha f), \\
(iv) \quad & p^n e f k_0 = k_0 p^n e f + \beta k_1 p^n - \gamma k_2 p^n.
\end{aligned}$$

Proof. By straightforward calculations one can see that

$$\begin{aligned}
& p^n k_0 = k_0 p^n \in \mathfrak{k}U(\mathfrak{g}), \\
& p^n e k_0 = p^n k_0 e + p^n(\alpha e + \beta f) = k_0 p^n e + p^n(\alpha e + \beta f), \\
& p^n f k_0 = p^n k_0 f + p^n(\gamma e - \alpha f) = k_0 p^n f + p^n(\gamma e - \alpha f).
\end{aligned}$$

and

$$\begin{aligned}
p^n e f k_0 &= p^n e(k_0 f + \gamma e - \alpha f) = p^n e k_0 f + \gamma p^n e^2 - \alpha p^n e f \\
&= p^n(k_0 e + (\alpha e + \beta f))f + \gamma p^n e^2 - \alpha p^n e f \\
&= p^n k_0 e f + \alpha p^n e f + \beta p^n f^2 + \gamma p^n e^2 - \alpha p^n e f \\
&= k_0 p^n e f + \frac{1}{2}\beta p^n [f, f] + \frac{1}{2}\gamma p^n [e, e] \\
&= k_0 p^n e f + \beta p^n k_1 - \gamma p^n k_2 = k_0 p^n e f + \beta k_1 p^n - \gamma k_2 p^n.
\end{aligned}$$

∎

Lemma 3.3.6. Let $k_1, k_2, e', f', p, e, f$ and v_k, v_p be as defined in the begining of this section. Then,

$$\begin{aligned}
(i) \quad & p^n e v_{\mathfrak{k}} = -v_{\mathfrak{k}} \alpha_n(p) e + a k_2 \beta_{n-1}(p) - b k \beta_{n-1}(p) + b \beta_{n-1}(p) e f - b p^{n+1}, \\
(ii) \quad & p^n f v_{\mathfrak{k}} = -v_{\mathfrak{k}} \alpha_n(p) f - a \beta_{n-1}(p) e f - b k_1 \beta_{n-1}(p) + a p^{n+1}, \\
(iii) \quad & p^n e f v_{\mathfrak{k}} = E - b k_1 \beta_{n-1}(p) e - b \beta_{n-1}(p) f + p^{n+1} v_{\mathfrak{p}} - a e' \alpha_n(p) - \beta_{n-1}(p) e.
\end{aligned}$$

where $E := v_{\mathfrak{k}} \alpha_n(p) e f - a k_2 \beta_{n-1}(p) + b k \beta_{n-1}(p) f$.

Proof. By straightforward calculations we have

$$\begin{aligned}
& [e, v_{\mathfrak{k}}] = -bp, \quad [f, v_{\mathfrak{k}}] = ap, \quad [e, e] = -2k_2, \quad [f, f] = 2k_1, \quad [e, p] = -e', \quad [e', p] = -e, \\
& [e', e'] = 2k_2, \quad [f', f'] = -2k_1, \quad [p, k_1] = [p, k_2] = [p, k] = 0, \quad [e, k_1] = f.
\end{aligned}$$

Therefore, by Lemmas 3.3.3 and 3.3.4 we obtain

$$\begin{aligned}
p^n e v_{\mathfrak{k}} &= p^n(-v_{\mathfrak{k}} e - bp) = -p^n v_{\mathfrak{k}} e - b p^{n+1} = -v_{\mathfrak{k}} \alpha_n(p) e - \beta_{n-1}(p) v_{\mathfrak{p}} e - b p^{n+1} \\
&= -v_{\mathfrak{k}} \alpha_n(p) e - \beta_{n-1}(p)(ae + bf)e - b p^{n+1}
\end{aligned}$$

$$
\begin{aligned}
&= -v_{\mathfrak{k}}\alpha_n(p)e - a\beta_{n-1}(p)e^2 - b\beta_{n-1}(p)(k-ef) - bp^{n+1},\\
&= -v_{\mathfrak{k}}\alpha_n(p)e + a\beta_{n-1}(p)k_2 - b\beta_{n-1}(p)k + b\beta_{n-1}(p)ef - bp^{n+1}\\
&= -v_{\mathfrak{k}}\alpha_n(p)e + ak_2\beta_{n-1}(p) - bk\beta_{n-1}(p) + b\beta_{n-1}(p)ef - bp^{n+1}\\
p^n f v_{\mathfrak{k}} &= p^n(-v_{\mathfrak{k}}f + ap) = -p^n v_{\mathfrak{k}} f + ap^{n+1} = -v_{\mathfrak{k}}\alpha_n(p)f - \beta_{n-1}(p)v_{\mathfrak{p}}f + ap^{n+1}\\
&= -v_{\mathfrak{k}}\alpha_n(p)f - \beta_{n-1}(p)(ae+bf)f + ap^{n+1}\\
&= -v_{\mathfrak{k}}\alpha_n(p)f - a\beta_{n-1}(p)ef - b\beta_{n-1}(p)f^2 + ap^{n+1}\\
&= -v_{\mathfrak{k}}\alpha_n(p)f - a\beta_{n-1}(p)ef - b\beta_{n-1}(p)k_1 + ap^{n+1}\\
&= -v_{\mathfrak{k}}\alpha_n(p)f - a\beta_{n-1}(p)ef - bk_1\beta_{n-1}(p) + ap^{n+1},
\end{aligned}
$$

and

$$
\begin{aligned}
p^n ef v_{\mathfrak{k}} &= p^n\left(v_{\mathfrak{k}}ef + e[v_{\mathfrak{k}}, f] - [v_{\mathfrak{k}}, e]f\right) = p^n v_{\mathfrak{k}} ef + p^n e(ap) + bp^{n+1}f\\
&= p^n v_{\mathfrak{k}} ef + ap^n(pe - e') + bp^{n+1}f = p^n v_{\mathfrak{k}} ef + ap^{n+1}e - ap^n e' + bp^{n+1}f\\
&= p^n v_{\mathfrak{k}} ef + p^{n+1}v_{\mathfrak{p}} - ap^n e'.
\end{aligned}
$$

From the definitions of α_n and β_{n-1} it follows that

$$ap^n e' = ae'\alpha_n(p) + \beta_{n-1}(p)e.$$

This implies that

$$
\begin{aligned}
p^n ef v_{\mathfrak{k}} &= v_{\mathfrak{k}}\alpha_n(p)ef + \beta_{n-1}(p)v_{\mathfrak{p}}ef + p^{n+1}v_{\mathfrak{p}} - ae'\alpha_n(p) - \beta_{n-1}(p)e\\
&= v_{\mathfrak{k}}\alpha_n(p)ef + \beta_{n-1}(p)(ae+bf)ef + p^{n+1}v_{\mathfrak{p}} - ae'\alpha_n(p) - \beta_{n-1}(p)e\\
&= v_{\mathfrak{k}}\alpha_n(p)ef + a\beta_{n-1}(p)e^2 + b\beta_{n-1}(p)fef + p^{n+1}v_{\mathfrak{p}} - ae'\alpha_n(p) - \beta_{n-1}(p)e\\
&= v_{\mathfrak{k}}\alpha_n(p)ef - a\beta_{n-1}(p)k_2 + b\beta_{n-1}(p)(k-ef)f + p^{n+1}v_{\mathfrak{p}} - ae'\alpha_n(p) - \beta_{n-1}(p)e\\
&= E - b\beta_{n-1}(p)ef^2 + p^{n+1}v_{\mathfrak{p}} - ae'\alpha_n(p) - \beta_{n-1}(p)e\\
&= E - b\beta_{n-1}(p)ek_1 + p^{n+1}v_{\mathfrak{p}} - ae'\alpha_n(p) - \beta_{n-1}(p)e\\
&= E - b\beta_{n-1}(p)(k_1e + f) + p^{n+1}v_{\mathfrak{p}} - ae'\alpha_n(p) - \beta_{n-1}(p)e\\
&= E - bk_1\beta_{n-1}(p)e - b\beta_{n-1}(p)f + p^{n+1}v_{\mathfrak{p}} - ae'\alpha_n(p) - \beta_{n-1}(p)e,
\end{aligned}
$$

where $E := v_{\mathfrak{k}}\alpha_n(p)ef - ak_2\beta_{n-1}(p) + bk\beta_{n-1}(p)f$. ∎

For the next proposition recall that $\mathfrak{g} = \mathfrak{gl}(1|2)$ and $\mathfrak{k} = \mathfrak{osp}(1|2)$.

Theorem 3.3.7. Let $\mathcal{I} = \mathfrak{k}U(\mathfrak{g}) + U(\mathfrak{g})\mathfrak{k}$ be the subspace of $U(\mathfrak{g})$. Let $\{k, k_1, k_2, e', f', p, e, f\}$ be chosen as an ordered basis of $\mathfrak{g}$. Then the following vectors form a basis of $\mathcal{I}$:
(i) $p^n e$ and $p^n f$ where $n \geqslant 0$,
(ii) $\beta_{n-1}(p)ef - p^{n+1}$ where $n \geqslant 1$,
(iii) $k^{r_1}k_1^{r_2}k_2^{r_3}(e')^{r_4}(f')^{r_5}p^{s_1}e^{s_2}f^{s_3}$ where at least one of $r_1, \cdots, r_5 > 0$.

Proof. Let $\mathcal{B}_i$, $\mathcal{B}_{ii}$, and $\mathcal{B}_{iii}$ denote the sets of vectors defined in $(i)-(iii)$ above respectively. Recall that by PBW theorem

$$k^{r_1}k_1^{r_2}k_2^{r_3}(e')^{r_4}(f')^{r_5}p^{s_1}e^{s_2}f^{s_3}$$

are basis elements of $U(\mathfrak{g})$ for non-negative integers $r_1, r_2, r_3, r_5, s_1, s_2, s_3$. Since

$$e^2 = \frac{1}{2}[e,e] = -k_2, \quad f^2 = \frac{1}{2}[f,f] = k_1, \quad e'^2 = \frac{1}{2}[e',e'] = k_2, \quad f'^2 = \frac{1}{2}[f',f'] = -k_1,$$

the integers r_4, r_5, s_2, s_3 are 0 or 1. Therefore, For finding a basis for $U(\mathfrak{g})\mathfrak{k}$ we need to check eight cases which have been computed in Lemmas 3.3.5 and 3.3.6 and it follows that the vectors $\mathcal{B}_i$ and $\mathcal{B}_{ii}$ generate $U(\mathfrak{g})\mathfrak{k}$. Also, by considering the basis of $U(\mathfrak{g})$, the monomials $k^{r_1}k_1^{r_2}k_2^{r_3}(e')^{r_4}(f')^{r_5}p^{s_1}e^{s_2}f^{s_3}$ when at least one of the exponents $r_1, \ldots, r_5$ is nonzero, generate $\mathfrak{k}U(\mathfrak{g})$. Hence, $\mathcal{B}_i \cup \mathcal{B}_{ii} \cup \mathcal{B}_{iii}$ generate $\mathcal{I}$. Now we aim to prove that the vectors in $\mathcal{B}_i \cup \mathcal{B}_{ii} \cup \mathcal{B}_{iii}$ are linearly independent. Consider any linear dependence relation between the elements of $\mathcal{B}_i \cup \mathcal{B}_{ii} \cup \mathcal{B}_{iii}$. Let c_n denote the coefficient of $\beta_{n-1}(p)ef - p^{n+1}$ for $n \geqslant 1$, and set $N := \max\{n : c_n \neq 0\}$. Recall that $\deg(\beta_{N-1}) = N - 1$.

Now, keeping the PBW basis of $U(\mathfrak{g})$ in mind, note that the summand $c_N p^{N-1}ef$ cannot be canceled in the linear dependence relation. This leads to a contradiction. ∎

Chapter 4

Weyl group invariance of spherical functionals

The main purpose of this chapter is to verify the Weyl group invariance of spherical functionals in Theorem 4.4.3. To this end, we reduce the invariance property to a rank one symmetric pair $(\mathfrak{s}_\alpha, \mathfrak{s}_\alpha \cap \mathfrak{k})$ that is embedded in $(\mathfrak{g}, \mathfrak{k})$, , where α is an even root, either $\varepsilon_i - \varepsilon_j$ or $\delta_i - \delta_j$ in the restricted root system. In Cases I, II the reduction will be to the symmetric pairs $(\mathfrak{gl}_2 \oplus \mathfrak{gl}_2, \mathfrak{gl}_2)$, $(\mathfrak{gl}_2, \mathfrak{so}_2)$, and $(\mathfrak{gl}_4, \mathfrak{sp}_4)$. Case III is a rank one symmetric pair and therefore basically $\mathfrak{s}_\alpha = \mathfrak{g}$. Additionally, for each case, we verify that $C_{\mathfrak{p}}(\mathfrak{a}) = \mathfrak{a}$, which is a necessary condition for proving Lemma 3.2.4.

	$\mathfrak{g}$	$\mathfrak{k}$	$\mathfrak{s}_{\pm(\varepsilon_i - \varepsilon_j)}$	$\mathfrak{s}_{\pm(\delta_i - \delta_j)}$	l	l'
I	$\mathfrak{gl}(m\|n) \oplus \mathfrak{gl}(m\|n)$	$\mathfrak{gl}(m\|n)$	$\mathfrak{gl}_2 \oplus \mathfrak{gl}_2$	$\mathfrak{gl}_2 \oplus \mathfrak{gl}_2$	$m-1$	$n-1$
II	$\mathfrak{gl}(m\|2n)$	$\mathfrak{osp}(m\|2n)$	$\mathfrak{gl}_2$	$\mathfrak{gl}_4$	$m-1$	$n-1$
III	$\mathfrak{gosp}(m+1\|2n)$	$\mathfrak{osp}(m\|2n)$	$\mathfrak{gosp}(m+1\|2n)$	$-$	1	-1

Table 4.1: The pair $(\mathfrak{g}, \mathfrak{k})$, $\mathfrak{s}_\alpha$, and the restricted root system of type $A(l, l')$

4.1 Restricted root systems and Weyl groups

From now on, we assume that $\mathfrak{h}$ is the Cartan subalgebra of $\mathfrak{g}$ chosen in Subsection 3.1. The idea of a restricted root system is to replace the Cartan subalgebra $\mathfrak{h}$ with a subalgebra $\mathfrak{a}$ that is also contained in $\mathfrak{p}$. Then, for the standard root system associated to the root decomposition of $\mathfrak{g}$ with respect to $\mathfrak{a}$, we can define the Weyl group associated to this restricted root system.

The roots of $\mathfrak{g}$ with respect to a Cartan subalgebra $\mathfrak{h}$ are restricted to $\mathfrak{a}$, yielding the decomposition:

$$\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{a} \oplus \bigoplus_{\gamma \in \Delta'} \mathfrak{g}_\gamma,$$

where:

- $\Delta' \subset \mathfrak{a}^* \setminus \{0\}$ is the set of restricted roots, obtained as the nonzero restrictions of the roots $\alpha \in \Delta$ of $\mathfrak{g}$ (relative to $\mathfrak{h}$) to $\mathfrak{a}$,
- $\mathfrak{g}_\gamma = \{X \in \mathfrak{g} : [H, X] = \gamma(H)X \ \forall H \in \mathfrak{a}\}$ is the restricted root space associated with γ,
- $\mathfrak{m} = \bigoplus_{\alpha\in\Delta, \alpha|_{\mathfrak{a}}=0} \mathfrak{g}_\alpha$ is the centralizer of $\mathfrak{a}$ in $\mathfrak{k}$.

Definition 4.1.1. The set Δ' is called the restricted root system of $\mathfrak{g}$.

We consider $\mathfrak{g}$ as a Lie subalgebra of the matrix algebra of endomorphisms of the *natural* representation of $\mathfrak{g}$ (recall from Chapter 2 that this matrix algebra is indeed an associative superalgebra). Now we can define a bilinear form $\kappa : \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$ as follows. We set

$$\kappa(x, y) := \mathrm{str}(xy), \quad \text{for} \quad x, y \in \mathfrak{g},$$

where $\mathrm{str}(\cdot)$ denotes the supertrace of the matrix.

Lemma 4.1.2. The supersymmetric bilinear form $\kappa(\cdot, \cdot)$ is invariant, i.e., for all $x, y, z \in \mathfrak{g}$,

$$\kappa([x, y], z) = \kappa(x, [y, z]).$$

Proof. The supersymmetry of κ follows from the supertrace identity $\mathrm{str}(xy) = (-1)^{|x||y|}\mathrm{str}(yx)$. Also, by using this property we have:

$$\begin{aligned}
\kappa([x, y], z) &= \mathrm{str}((xy - (-1)^{|x||y|}yx)z) = \mathrm{str}(xyz - (-1)^{|x||y|}yxz) = \mathrm{str}(xyz) - (-1)^{|x||y|}\mathrm{str}(yxz) \\
&= \mathrm{str}(xyz) - (-1)^{|x||y|}(-1)^{|y|(|x|+|z|)}\mathrm{str}(xzy) = \mathrm{str}(xyz) - (-1)^{|z||y|}str(xzy) \\
&= \mathrm{str}(x(yz - (-1)^{|z||y|}zy)) = \kappa(x, [y, z]).
\end{aligned}$$

∎

According to [31, Section 1], κ is non-degenerate. One can check by a direct calculation that the restriction of κ to $\mathfrak{a} \times \mathfrak{a}$ is a symmetric non-degenerate bilinear form as well.

Remark 4.1.3. Since the bilinear form $\kappa\big|_{\mathfrak{a}\times\mathfrak{a}}$ is non-degenerate, we have $\mathfrak{a} \cong \mathfrak{a}^*$ via the map $x \mapsto \kappa(x, .) = \alpha_x$ and in this way we can define a nondegenerate symmetric form on $\mathfrak{a}^*$ as follows:

$$\kappa(\alpha_x, \alpha_y) := \kappa(x, y).$$

For $\alpha \in \Delta'$, let $s_\alpha : \mathfrak{a}^* \to \mathfrak{a}^*$ be the reflection defined by:

$$s_\alpha(\beta) = \beta - 2\frac{\kappa(\beta, \alpha)}{\kappa(\alpha, \alpha)}\alpha \quad \text{for} \quad \beta \in \mathfrak{a}^*. \tag{4.1.1}$$

Lemma 4.1.4. For $\alpha \in \Delta'$, s_α leaves κ invariant.

Proof. Let β, $\gamma \in \mathfrak{a}^*$. We want to prove that $\kappa(s_\alpha(\beta), s_\alpha(\gamma)) = \kappa(\beta, \gamma)$. We have

$$\begin{aligned}
\kappa(s_\alpha(\beta), s_\alpha(\gamma)) &= \kappa\left(\beta - 2\frac{\kappa(\beta, \alpha)}{\kappa(\alpha, \alpha)}\alpha, \gamma - 2\frac{\kappa(\gamma, \alpha)}{\kappa(\alpha, \alpha)}\alpha\right) \\
&= \kappa(\beta, \gamma) - 2\frac{\kappa(\beta, \alpha)}{\kappa(\alpha, \alpha)}\kappa(\alpha, \gamma) - 2\frac{\kappa(\gamma, \alpha)}{\kappa(\alpha, \alpha)}\kappa(\alpha, \beta) + 4\frac{\kappa(\beta, \alpha)}{\kappa(\alpha, \alpha)}\frac{\kappa(\gamma, \alpha)}{\kappa(\alpha, \alpha)}\kappa(\alpha, \alpha) \\
&= \kappa(\beta, \gamma).
\end{aligned}$$

∎

Definition 4.1.5. Let $\kappa : \mathfrak{a}^* \times \mathfrak{a}^* \to \mathbb{C}$ be defined as in Remark 4.1.3. Then the reflections s_α of (4.1.1) generate a subgroup of $\mathrm{GL}(\mathfrak{a}^*)$ that preserves κ and this subgroup will be called the restricted Weyl group of $(\mathfrak{g}, \mathfrak{k})$.

Example 4.1.6. The restricted Weyl group of the pairs $(\mathfrak{gl}_m, \mathfrak{so}_m)$ and $(\mathfrak{gl}_{2n}, \mathfrak{sp}_{2n})$ are S_m and S_n, respectively.

Let $\Delta' = \Delta'_{\bar{0}} \sqcup \Delta'_{\bar{1}}$ where $\Delta'_{\bar{0}}$ is the even part of the restricted root system containing the even roots $\{\pm(\varepsilon_i - \varepsilon_{i'}), \pm(\delta_{\bar{j}} - \delta_{\bar{j}'}),\, 1 \leqslant i \neq i' \leqslant l,\, 1 \leqslant \bar{j} \neq \bar{j}' \leqslant l'\}$. Throughout this chapter, the Weyl group refers to the group generated by the reflections that correspond to the even restricted root system $\Delta'_{\bar{0}}$.
Now, recall that κ canonically induces the isomorphism:

$$\begin{aligned} \phi :&\mathfrak{a} \to \mathfrak{a}^* \\ &a \mapsto \phi_a, \quad \phi_a(b) = \kappa(b, a) \quad \text{for} \quad b \in \mathfrak{a}. \end{aligned}$$

In this fashion, the action of the Weyl group on $\mathfrak{a}^*$ can be carried over to $\mathfrak{a}$ such that

$$g \cdot \phi_a(b) = \phi_a(g^{-1} \cdot b) = \kappa(g^{-1} \cdot b, a) = \kappa(b, g \cdot a) = \phi_{g \cdot a}(b).$$

By another canonical extension, the action of the Weyl group can be carried over to the symmetric algebra $S(\mathfrak{a})$ and subsequently to $S(\mathfrak{a})^*$.

4.2 Interwoven $\mathfrak{gl}_2$ bases and Weyl group symmetry

Let $\mathfrak{g}$, θ, $\mathfrak{k}$, $\mathfrak{p}$, and $\mathfrak{a}$ be as in Section 4.1.

Definition 4.2.1. By an interwoven pair of $\mathfrak{gl}_2$ bases of $\mathfrak{g}$ we mean a choice of elements $k, x, y, h_1, h_2, e, f \in \mathfrak{g}$ satisfying the following relations:
(i) $\{k, x, y, z\}$ forms a basis for a $\mathfrak{gl}_2$ subalgebra, i.e.,

$$[k, x] = 2x\ ,\ [k, y] = -2y\ ,\ [x, y] = k\ ,\ [z, x] = [z, y] = [z, k] = 0.$$

Moreover, $k \in \mathfrak{k}, x, y \in \mathfrak{p}$.
(ii) $\{h_1, h_2, e, f\}$ is a standard $\mathfrak{gl}_2$-basis, i.e.,

$$[h_1, e] = e, \quad [h_1, f] = -f \quad [h_2, e] = -e, \quad [h_2, f] = f, \quad [e, f] = h_1 - h_2.$$

(iii) $k = i(e - f)$ where $i = \sqrt{-1}$ and $x + y = -h_1 + h_2$.

The interwoven $\mathfrak{gl}_2$ bases of Cases I, II, and III are given in Section 4.8.

Lemma 4.2.2. Let $\{k, x, y, h_1, h_2, e, f\}$ be an interwoven pair of $\mathfrak{gl}_2$ subalgebras chosen as in Definition 4.2.1. Then the following relations hold in $U(\mathfrak{g})$;

$$[k, x^a] = 2ax^a, \quad [k, y^a] = -2ay^a, \quad \text{for} \quad a \in \mathbb{N},\ \text{and}\ x^a, y^a \in U(\mathfrak{g}).$$

Proof. We use induction on a. For $a = 1$, the relations follow from Definition 4.2.1. Now, suppose that the above relations hold for $a-1$. Then,

$$[k, x^a] = [k, x]x^{a-1} + x[k, x^{a-1}].$$

But, by the hypothesis of induction, $[k, x^{a-1}] = 2(a-1)x^{a-1}$ and hence:

$$[k, x^a] = 2x^a + 2(a-1)x^a = 2ax^a.$$

Similarly, we can prove that $[k, y^a] = -2ay^a$. ∎

Recall that $\mathcal{I} = \mathfrak{k}U(\mathfrak{g}) + U(\mathfrak{g})\mathfrak{k}$.

Lemma 4.2.3. Let $\{k, x, y, h_1, h_2, e, f\}$ be chosen as in Definition 4.2.1. Then the following elements are in $\mathcal{I}$.
(i) $k^a x^b y^c z^r$ where $a > 0$.
(ii) $x^b y^c z^r$ where $b \neq c$.

Proof. Recall that we consider products of the form $k^a x^b y^c z^r$ as elements of $U(\mathfrak{g})$. If $a > 0$ then $k^a x^b y^c z^r \in \mathfrak{k}U(\mathfrak{g}) \subseteq \mathcal{I}$. Furthermore, any linear combination of monomials of the form $k^a x^b y^c z^r k$ is in $U(\mathfrak{g})\mathfrak{k}$, and by Lemma 4.2.2, we have:

$$\begin{aligned} k^a x^b y^c z^r k &= k^a x^b y^c k z^r \\ &= k^a x^b (ky^c + 2cy^c) z^r \\ &= k^a x^b k y^c z^r + 2ck^a x^b y^c z^r \\ &= k^a (kx^b - 2bx^b) y^c z^r + 2ck^a x^b y^c z^r \\ &= k^a (k - 2b + 2c) x^b y^c z^r. \end{aligned}$$

Taking the difference with the monomial $k^{a+1} x^b y^c z^r$ (which is also in $\mathcal{I}$), which implies that $\mathcal{I}$ contains the elements $2(b-c)x^b y^c z^r$. When $b \neq c$, it follows that $x^b y^c z^r$ is also in $\mathcal{I}$. ∎

Lemma 4.2.4. Keeping the notation of Lemma 4.2.3 for $k, x, y, z, h_1, h_2 \in \mathfrak{g}$, we have the following identity in $U(\mathfrak{g})$:

$$h_1^a h_2^b - h_2^a h_1^b = \frac{1}{2^{a+b}} \left(\sum_{n=0}^{a} \sum_{m=0}^{b} \binom{a}{n} \binom{b}{m} z^{n+m} (x+y)^{a-n} (x+y)^{b-m} C_{m,n}^{a,b} \right). \tag{4.2.1}$$

where $C_{m,n}^{a,b} = ((-1)^{b-m} - (-1)^{a-n})$.

Proof. Observe that

$$h_1 = \frac{1}{2}(z - x - y) \quad \text{and} \quad h_2 = \frac{1}{2}((h_1 + h_2) + (-h_1 + h_2)) = \frac{1}{2}(z + x + y).$$

Thus

$$h_1^a h_2^b - h_2^a h_1^b = \frac{1}{2^{a+b}} \left((z - x - y)^a (z + x + y)^b - (z + x + y)^a (z - x - y)^b \right)$$

$$
\begin{aligned}
&= \frac{1}{2^{a+b}}(\sum_{n=0}^{a}\binom{a}{n}z^n(x+y)^{a-n})(\sum_{m=0}^{b}\binom{b}{m}z^m(-(x+y))^{b-m})\\
&- \frac{1}{2^{a+b}}(\sum_{n=0}^{a}\binom{a}{n}z^n(-(x+y))^{a-n})(\sum_{m=0}^{b}\binom{b}{m}z^m(x+y)^{b-m})\\
&= \frac{1}{2^{a+b}}\sum_{n=0}^{a}\sum_{m=0}^{b}\binom{a}{n}\binom{b}{m}z^nz^m(x+y)^{a-n}(x+y)^{b-m}(-1)^{b-m}\\
&- \frac{1}{2^{a+b}}\sum_{n=0}^{a}\sum_{m=0}^{b}\binom{a}{n}\binom{b}{m}z^nz^m(x+y)^{a-n}(x+y)^{b-m}(-1)^{a-n}\\
&= \frac{1}{2^{a+b}}\left(\sum_{n=0}^{a}\sum_{m=0}^{b}\binom{a}{n}\binom{b}{m}z^nz^m(x+y)^{a-n}(x+y)^{b-m}((-1)^{b-m}-(-1)^{a-n})\right).
\end{aligned}
$$

This completes the proof of the assertion. ∎

Recall that $\Delta' = \Delta'_{\bar{0}} \sqcup \Delta'_{\bar{1}}$ is the restricted root system of $\mathfrak{g}$, where $\Delta'_{\bar{0}}$ denotes the even part of the restricted root system from Section 4.1. Then, the following proposition establishes the symmetry of spherical functionals with respect to a reflection in the Weyl group corresponding to $\Delta'_{\bar{0}}$.

Proposition 4.2.5. We adopt the notation of Lemma 4.2.3. Then for every $\lambda \in \mathcal{A}(\mathfrak{g}, \mathfrak{k})$ we have

$$\lambda(h_1^{a_1}h_2^{a_2}) = \lambda(h_1^{a_2}h_2^{a_1}), \quad \text{for all non-negative integers} \quad a_1, a_2.$$

Proof. First, we analyze the right hand side of (4.2.1). We note $(-1)^{b-m} - (-1)^{a-n} = 0$ unless $(b-m) \not\equiv (a-n) \bmod 2$ and hence the only terms that remain from the right hand side of (4.2.1) are

$$\binom{a}{r}\binom{b}{s}(x+y)^r(x+y)^s z^{a-r+b-s} \qquad \text{where } r \not\equiv s \bmod 2.$$

Thus, it will be sufficient to prove that $(x+y)^{2k+1} \in \mathcal{I}$ for every integer $k \geq 0$. This can be verified as follows: if we expand $(x+y)^{2k+1}$, it will be a linear combination of terms of the form

$$x^{a_1}y^{b_1}x^{a_2}y^{b_2}\cdots x^{a_r}y^{b_r}, \tag{4.2.2}$$

where $\sum_i a_i + \sum_i b_i = 2k+1$. By Lemma 4.2.2 the k-weight of x^a (i.e., the eigenvalue of the action of ad_k on x^a) is $2a$ and the "k-weight" of y^b is $-2b$. Therefore,

$$[k, x^ay^b] = [k, x^a]y^b + x^a[k, y^b] = 2ax^ay^b - 2bx^ay^b = (2a-2b)x^ay^b.$$

Hence, the k-weight of $x^{a_1}y^{b_1}x^{a_2}y^{b_2}\cdots x^{a_r}y^{b_r}$ is equal to $2(\sum a_i - \sum b_i) = 2(2\sum a_i - (2k+1))$. The last number is of the form $4\ell + 2$ for some integer ℓ.

We aim to express a monomial in x, y, k as a linear combination of ordered monomials of the form $k^ax^by^c$. This can be achieved by using the commutation relations between x, y, and k to successively move all occurrences of k to the left, followed by all occurrences of x. Importantly, the monomials that arise during this process retain the same k-weight as the original monomial, as the difference between the number of x's and y's remains invariant throughout the reordering. Specifically, when applying the operation

$$(\cdots xk\cdots) = (\cdots kx\cdots) + (\cdots(-2x)\cdots),$$

the number of x's and y's does not change. Similarly, under the operation

$$(\cdots yx\cdots) = (\cdots xy\cdots) + (\cdots(-k)\cdots),$$

the counts of x's and y's both decrease by one. These observations imply that the k-weight of any monomial is preserved under reordering. Furthermore, the k-weight of a monomial of the form $k^a x^b y^c z^r$ is $4b$, which is necessarily divisible by 4. Consequently, any monomial of this form cannot appear in the expansion of (4.2.2) as a linear combination of reordered monomials. Thus, (4.2.2) is expressible as a linear combination of monomials belonging to $\mathcal{I}$. ∎

4.3 The restricted root systems: Type A cases

Let $\mathfrak{h} = \mathfrak{t} \oplus \mathfrak{a}$ be the Cartan subalgebra of $\mathfrak{g}$ as described in Section 3.1 which is consistent with the Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. Thus $\mathfrak{a}$ is a maximal abelian and self-centralizing subspace of $\mathfrak{p}$. Then, in each case I, II, and III there is a choice of a basis for $\mathfrak{a}$, denoted

$$\{h_1, ..., h_l, h_{\bar{1}}, ..., h_{\bar{l}'}\}.$$

These vectors h_i's and $h_{\bar{j}}$'s are explicitly described for Cases II and III in Sections 4.6 and 4.7 (and in Section 4.3 for Case I).

Analogous to Example 3.1.1, we will reserve Greek letters ε and δ in unemphasized font for natural characters of the split part of the Cartan subalgebra, namely for $\mathfrak{a}$. More specifically, we define ε_i's and δ_j's to be the dual of the h_i's and $h_{\bar{j}}$'s selected above such that

$$\varepsilon_i(h_j) = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}, \quad \delta_{\bar{i}}(h_{\bar{j}}) = \begin{cases} 1 & \bar{i} = \bar{j} \\ 0 & \bar{i} \neq \bar{j} \end{cases}.$$

One can observe that the restricted root systems we consider are of type $A(l-1, l'-1)$ and lie on $l + l'$-dimensional space that is spanned by the ε_i's and δ_j's. The ε_i's and δ_j's form a basis of a^*. The numbers l and l' for each Case I, II, and III are given in the following proposition.

Proposition 4.3.1. The restricted root system of $\mathfrak{g}$ with respect to $\mathfrak{a}$ is of type $A(l-1, l'-1)$ where $l + l' = \dim(\mathfrak{a}^*) - 1$. The values of l and l' in Cases I and II are m and n respectively. Also, in Case III, $l = 2$ and $l' = 0$.

Proof. This is verified by direct calculation. More precisely, one can see that for all Lie superalgebras in Table 4.1 we can choose linear functionals $\varepsilon_i, \delta_{\bar{j}} \in \mathfrak{a}^*$ such that we obtain a realization of the restricted root system in the following form:

$$\Delta' = \{\pm(\varepsilon_i - \varepsilon_{i'}), \pm(\delta_{\bar{j}} - \delta_{\bar{j}'}), \pm(\varepsilon_i - \delta_{\bar{j}}),\ 1 \leqslant i \neq i' \leqslant l,\ 1 \leqslant \bar{j} \neq \bar{j}' \leqslant l'\}.$$

In Case I,

$$\mathfrak{a} = \operatorname{Span}\left\{h_i, h_{\bar{j}} \,\middle|\, 1 \leqslant i \leqslant m,\ 1 \leqslant j \leqslant n\right\}$$

where according to Section 4.5 $h_i = (E_{i,i}, -E_{i,i})$ and $h_{\bar{j}} = (E_{\bar{j},\bar{j}}, -E_{\bar{j},\bar{j}})$. Then

$$\varepsilon_i = \boldsymbol{\varepsilon}_{i,L}|_a = -\boldsymbol{\varepsilon}_{i,R}|_a,$$

where $\boldsymbol{\varepsilon}_{i,L}$ and $\boldsymbol{\varepsilon}_{i,R}$ are the choices of the bases for the Cartan subalgebra as in Section 4.1 for the left and the right $\mathfrak{gl}(m|n)$ factors. The δ_i are defined analogously. The restricted root system is of type $A(m-1|n-1)$. For Cases II and III analogous descriptions are given in Sections 4.6 and 4.7. ∎

4.4 Weyl group invariance for the pair $(\mathfrak{g},\mathfrak{k})$

We assume that $(\mathfrak{g},\mathfrak{k})$ and $\mathfrak{a}$ are as in Section 4.1. Furthermore, we assume that the restricted root system of $\mathfrak{g}$ (with respect to $\mathfrak{a}$) is of type $A(m,n)$, realized in the usual way by a choice of ε_i's and δ_j's in $\mathfrak{a}^*$. (The explicit choices of these functionals will be explained in each of the cases that we will be considering.) In what follows we will need the subalgebras $\mathfrak{s}_\alpha$ defined in the next definition:

Definition 4.4.1. Let $\alpha = \varepsilon_i - \varepsilon_{i+1}$ be a simple even root of the restricted root system Δ'. We define

$$\mathfrak{s}_\alpha = \mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha} \oplus [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] \oplus \mathfrak{a}_\alpha,$$

where

$$\mathfrak{a}_\alpha = \bigcap_{\alpha = \tilde{\alpha}|_{\mathfrak{a}}} \ker(\tilde{\alpha}) \cap \bigcap_{\substack{1 \leqslant j \leqslant n \\ j \neq i, i+1}} \ker(\varepsilon_j).$$

Lemma 4.4.2. $\mathfrak{s}_\alpha$ is a subalgebra of $\mathfrak{g}$.

Proof. Obviously, $[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] \subseteq \mathfrak{s}_\alpha$. By using the relation $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subseteq \mathfrak{g}_{\alpha+\beta}$ we have

$$[\mathfrak{g}_\alpha, [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]] \subseteq \mathfrak{g}_\alpha, \quad \text{and} \quad [\mathfrak{g}_{-\alpha}, [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]] \subseteq \mathfrak{g}_{-\alpha}.$$

Since 2α and -2α are never roots, we have $[\mathfrak{g}_\alpha, \mathfrak{g}_\alpha] = 0 = [\mathfrak{g}_{-\alpha}, \mathfrak{g}_{-\alpha}]$. On the other hand because $\mathfrak{a}_\alpha \subseteq \mathfrak{h}$ and $\mathfrak{s}_\alpha$ is a direct sum of $\mathfrak{a}_\alpha$ and restricted root spaces, it follows that $[\mathfrak{a}_\alpha, \mathfrak{s}_\alpha]$ is a subspace of $\mathfrak{s}_\alpha$. ∎

Now, recall that by Proposition 2.3.9, $\lambda \in \mathcal{A}(\mathfrak{g},\mathfrak{k})$ plays the role of spherical functionals and by Corollary 3.2.7 we obtain an injection $\mathcal{A}(\mathfrak{g},\mathfrak{k}) \to S(\mathfrak{a})^*$. Our next goal is to verify that the image of $\mathcal{A}(\mathfrak{g},\mathfrak{k})$ in $S(\mathfrak{a})^*$ lies in the subalgebra of Weyl group invariants. This is the content of the next proposition.

Theorem 4.4.3. Let $\lambda \in \mathcal{A}(\mathfrak{g},\mathfrak{k})$ be a $\mathfrak{k}$-biinvariant functional of the pair $(\mathfrak{g},\mathfrak{k})$. Assume the vector space $\mathfrak{a}^*$ has a natural basis $\varepsilon_1,...,\varepsilon_m,\delta_{\bar{1}},...\delta_{\bar{n}}$. Then, by keeping the same notation $h_1,....,h_m,h_{\bar{1}},....,h_{\bar{n}}$ for the dual basis elements as in Section 4.3 and for every permutation $\sigma \in S_l$ and $\delta \in S_{l'}$, we have

$$\lambda(h_1^{a_1}\cdots h_m^{a_l}h_{\bar{1}}^{b_1}\cdots h_{\bar{l'}}^{b_n}) = \lambda(h_1^{\sigma(a_1)}\cdots h_l^{\sigma(a_l)}h_{\bar{1}}^{\delta(b_1)}\cdots h_{\bar{l'}}^{\delta(b_{l'})}).$$

Proof. For each $\varepsilon_i \in \mathfrak{a}^*$ corresponding to $h_i \in \mathfrak{a}$ we define

$$\varepsilon_i(x) = \langle h_i, x\rangle \quad \text{for all}\, x \in \mathfrak{a},$$
$$\delta_{\bar{i}}(x) = \langle h_{\bar{i}}, x\rangle \quad \text{for all}\, x \in \mathfrak{a}.$$

Then by straightforward computation we have

$$s_{\varepsilon_i-\varepsilon_{i+1}}(h_i) = h_{i+1}, \quad s_{\varepsilon_i-\varepsilon_{i+1}}(h_{i+1}) = h_i, \quad i \in I_{\bar{0}},$$
$$s_{\delta_{\bar{i}}-\delta_{\bar{i}+1}}(h_{\bar{i}}) = h_{\bar{i}+1}, \quad s_{\delta_{\bar{i}}-\delta_{\bar{i}+1}}(h_{i+1}) = h_{\bar{i}}, \quad \bar{i} \in I_{\bar{1}}.$$

Set $\mathfrak{k}_\alpha = \mathfrak{s}_\alpha \cap \mathfrak{k}$. Then $\theta\Big|_{\mathfrak{k}_\alpha} = +1$ where θ is the involution associated to $(\mathfrak{g},\mathfrak{k})$.
Assume that $\lambda \in \mathcal{A}(\mathfrak{g},\mathfrak{k})$. Set

$$\lambda' = \lambda\Big|_{U(\mathfrak{s}_{\varepsilon_i-\varepsilon_{i+1}})^*},$$
$$\lambda'' = \lambda\Big|_{U(\mathfrak{s}_{\delta_{\bar{i}}-\delta_{\bar{i}+1}})^*}.$$

The Weyl group invariance for the pair $(\mathfrak{s}_\alpha,\mathfrak{k}_\alpha)$ has been verified in Proposition 4.2.5 which implies that the restriction $\lambda\Big|_{U(\mathfrak{s}_\alpha)^*}$ is a $\mathfrak{k}_\alpha$-biinvariant functional and hence,

$$\lambda'(h_i^a h_{i+1}^b) = \lambda'(h_i^b h_{i+1}^a), \quad \lambda''(h_{\bar{i}}^a h_{\bar{i+1}}^b) = \lambda''(h_{\bar{i}}^b h_{\bar{i}+1}^a),$$

or equivalently,

$$(h_i^{a_i}h_{i+1}^{a_{i+1}} - h_i^{a_{i+1}}h_{i+1}^{a_i}) \in \mathfrak{k}_\alpha U(\mathfrak{s}_\alpha) + U(\mathfrak{s}_\alpha)\mathfrak{k}_\alpha, \quad \text{for} \quad \alpha := \varepsilon_i - \varepsilon_{i+1},$$

and

$$(h_{\bar{i}}^{a_i}h_{\bar{i}+1}^{a_{i+1}} - h_{\bar{i}}^{a_{i+1}}h_{\bar{i}+1}^{a_i}) \in \mathfrak{k}_\alpha U(\mathfrak{s}_\alpha) + U(\mathfrak{s}_\alpha)\mathfrak{k}_\alpha, \quad \text{for} \quad \alpha := \delta_{\bar{i}} - \delta_{\bar{i}+1}.$$

First, assume $\alpha := \varepsilon_i - \varepsilon_{i+1}$. Then for $j \neq i, i+1$ we have $[h_j,\mathfrak{s}_\alpha] = 0$ and for $\bar{j} \in I_{\bar{1}}$ we have $[h_{\bar{j}},\mathfrak{s}_\alpha] = 0$. Therefore, for corresponding values of j we have $[h_j,\mathfrak{k}_\alpha] = 0$ and $[h_{\bar{j}},\mathfrak{k}_\alpha] = 0$. It follows that

$$h_1^{a_1}\cdots h_{i-1}^{a_{i-1}}(h_i^{a_i}h_{i+1}^{a_{i+1}} - h_i^{a_{i+1}}h_{i+1}^{a_i})h_{i+2}^{a_{i+2}}\cdots h_l^{a_l}h_{\bar{1}}^{b_1}\cdots h_{\bar{l'}}^{b_{l'}} \in \mathfrak{k}_\alpha U(\mathfrak{g}) + U(\mathfrak{g})\mathfrak{k}_\alpha \subseteq \mathfrak{k}U(\mathfrak{g}) + U(\mathfrak{g})\mathfrak{k} = \mathcal{I}.$$

Now consider $\alpha := \delta_{\bar{i}} - \delta_{\bar{i}+1}$. Then, similarly, for $\bar{j} \neq \bar{i}, \bar{i}+1$ we have $[h_{\bar{j}},\mathfrak{s}_\alpha] = 0$ and for $j \in I_{\bar{0}}$ we have $[h_j,\mathfrak{s}_\alpha] = 0$. Thus, $[h_j,\mathfrak{k}_\alpha] = 0$ and $[h_{\bar{j}},\mathfrak{k}_\alpha] = 0$ for mentioned $j,\bar{j}$. It follows that

$$h_1^{a_1}\cdots h_l^{a_l}h_{\bar{1}}^{b_1}\cdots(h_{\bar{i}}^{b_i}h_{\bar{i}+1}^{b_{i+1}} - h_{\bar{i}}^{b_{i+1}}h_{\bar{i}+1}^{b_i})\cdots h_{\bar{l'}}^{b_{l'}} \in \mathfrak{k}_\alpha U(\mathfrak{g}) + U(\mathfrak{g})\mathfrak{k}_\alpha \subseteq \mathfrak{k}U(\mathfrak{g}) + U(\mathfrak{g})\mathfrak{k} = \mathcal{I}.$$

Thus,

$$\lambda(h_1^{a_1}\cdots h_i^{a_i}h_{i+1}^{a_{i+1}}\cdots h_l^{a_l}h_{\bar{1}}^{b_1}\cdots h_{\bar{i}}^{b_i}h_{\bar{i}+1}^{b_{i+1}}\cdots h_{\bar{n}}^{b_n}) = \lambda(h_1^{a_1}\cdots h_i^{a_{i+1}}h_{i+1}^{a_i}\cdots h_m^{a_m}h_{\bar{1}}^{b_1}\cdots h_{\bar{i}}^{b_{i+1}}h_{\bar{i}+1}^{b_i}\cdots h_{\bar{l'}}^{b_{l'}}),$$

and hence, for all permutations $\sigma \in S_l$ and $\delta \in S_{l'}$,

$$\lambda(h_1^{a_1}\cdots h_l^{a_l}h_{\bar{1}}^{b_1}\cdots h_{\bar{l'}}^{b_{l'}}) = \lambda(h_1^{\sigma(a_1)}\cdots h_l^{\sigma(a_l)}h_{\bar{1}}^{\delta(b_1)}\cdots h_{\bar{l'}}^{\delta(b_{l'})}).$$ ∎

4.5 The h_i and the $\mathfrak{s}_\alpha$ for Case I : $\mathfrak{g} = \mathfrak{gl}(m|n) \oplus \mathfrak{gl}(m|n)$ and $\mathfrak{k} = \mathfrak{gl}(m|n)$

Let $\mathfrak{g} = \mathfrak{gl}(m|n) \oplus \mathfrak{gl}(m|n)$ and suppose that the involutive automorphism θ on $\mathfrak{g}$ is as follows:

$$\theta(X, Y) = (Y, X), \quad \text{for} \quad X, Y \in \mathfrak{gl}(m|n).$$

Therefore,

$$\mathfrak{k} = \{(X, Y) \quad | \quad \theta(X, Y) = (X, Y), \quad X, Y \in \mathfrak{gl}(m|n)\} = \{(X, X), X \in \mathfrak{gl}(m|n)\} \cong \mathfrak{gl}(m|n),$$

and

$$\mathfrak{p} = \{(X, Y) \quad | \quad \theta(X, Y) = (-X, -Y), \quad X, Y \in \mathfrak{gl}(m|n)\} = \{(X, -X), X \in \mathfrak{gl}(m|n)\}.$$

We choose the Cartan subalgebra $\mathfrak{h}$ consistent with Cartan decomposition to be:

$$\mathfrak{h} = \{\operatorname{diag}(s, t) \oplus \operatorname{diag}(s', t') : s, s' \in \mathbb{C}^m \quad \text{and} \quad t, t' \in \mathbb{C}^n\}.$$

Thus,

$$\mathfrak{a} = \{\operatorname{diag}(s, t) \oplus \operatorname{diag}(-s, -t) : s = (s_1, \cdots, s_m) \in \mathbb{C}^m \quad \text{and} \quad t = (t_1, \cdots, t_n) \in \mathbb{C}^n\}.$$

Lemma 4.5.1. For $\mathfrak{g} = \mathfrak{gl}(m|n) \oplus \mathfrak{gl}(m|n)$ and $\mathfrak{k} = \mathfrak{gl}(m|n)$, $C_{\mathfrak{p}}(\mathfrak{a}) = \mathfrak{a}$.

Proof. Consider $P = (X, -X) \in \mathfrak{p}$ where

$$X = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \mathfrak{gl}(m|n),$$

and then for $a \in \mathfrak{a}$ we have

$$[a, P] = ([\operatorname{diag}(s, t), X], [\operatorname{diag}(-s, -t), -X]).$$

But

$$[\operatorname{diag}(s, t), X] = [\operatorname{diag}(-s, -t), -X] = \begin{bmatrix} \operatorname{diag}(s)A - A\operatorname{diag}(s) & \operatorname{diag}(s)B - B\operatorname{diag}(t) \\ \operatorname{diag}(t)C - C\operatorname{diag}(s) & \operatorname{diag}(t)D - D\operatorname{diag}(t) \end{bmatrix}.$$

If a non-diagonal entry of P is nonzero, then a suitable choice of s and t yields a nonzero commutator $[a, P]$, hence P is not in $C_{\mathfrak{p}}(\mathfrak{a})$. Thus, for every arbitrary $a \in \mathfrak{a}$, we have $[a, P] = 0$ if and only if $P \in \mathfrak{a}$. ∎

Let $I_{\bar{0}} = \{1, \cdots, m\}$ and $I_{\bar{1}} = \{\bar{1}, \bar{2}, \cdots, \bar{n}\}$. Let $\{E_{ij} \,|\, i, j \in I = I_{\bar{0}} \sqcup I_{\bar{1}}\}$ be the basis for $\mathfrak{g}$ consisting of matrix units. Note that $|j| = 0$ if $j \in I_{\bar{0}}$, and $|j| = 1$ if $j \in I_{\bar{1}}$. Then equivalently,

$$\mathfrak{a} = \operatorname{Span}\left\{h_i, h_{\bar{j}} \,\middle|\, 1 \leqslant i \leqslant m,\, 1 \leqslant j \leqslant n\right\}$$

where $h_i = (E_{i,i}, -E_{i,i})$ and $h_{\bar{j}} = (E_{\bar{j},\bar{j}}, -E_{\bar{j},\bar{j}})$. Also,

$$\varepsilon_i = \boldsymbol{\varepsilon}_{i,L}|_{\mathfrak{a}} = -\boldsymbol{\varepsilon}_{i,R}|_{\mathfrak{a}},$$

where $\boldsymbol{\varepsilon}_{i,L}$ and $\boldsymbol{\varepsilon}_{i,R}$ are the choices of the bases for the Cartan subalgebra as in Section 4.1 for the left and the right $\mathfrak{gl}(m|n)$ factors. The δ_i are defined analogously. The restricted root system is of type $A(m-1|n-1)$ and for each simple even root $\alpha \in \left\{\varepsilon_i - \varepsilon_{i+1}, \delta_{\bar{j}} - \delta_{\overline{j+1}}, i \in I_{\bar{0}}, \bar{j} \in I_{\bar{1}}\right\}$ we have:

$$\mathfrak{g}_\alpha = \begin{cases} \operatorname{Span}\left\{(E_{i,i+1}, 0), (0, E_{i+1,i})\right\}, & \text{if} \quad \alpha = \varepsilon_i - \varepsilon_{i+1} \\ \operatorname{Span}\left\{(E_{\bar{j},\overline{j+1}}, 0), (0, E_{\overline{j+1},\bar{j}})\right\}, & \text{if} \quad \alpha = \delta_{\bar{j}} - \delta_{\overline{j+1}} \end{cases},$$

$$\mathfrak{g}_{-\alpha} = \begin{cases} \operatorname{Span}\left\{(E_{i+1,i}, 0), (0, E_{i,i+1})\right\}, & \text{if} \quad \alpha = \varepsilon_i - \varepsilon_{i+1} \\ \operatorname{Span}\left\{(E_{\overline{j+1},\bar{j}}, 0), (0, E_{\bar{j},\overline{j+1}})\right\}, & \text{if} \quad \alpha = \delta_{\bar{j}} - \delta_{\overline{j+1}} \end{cases}.$$

Therefore,

$$[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] = \begin{cases} \operatorname{Span}\left\{(E_{i,i} - E_{i+1,i+1}, 0), (0, E_{i+1,i+1} - E_{i,i})\right\}, & \text{if} \quad \alpha = \varepsilon_i - \varepsilon_{i+1} \\ \operatorname{Span}\left\{(E_{\bar{j},\bar{j}} - E_{\overline{j+1},\overline{j+1}}, 0), (0, E_{\overline{j+1},\overline{j+1}} - E_{\bar{j},\bar{j}})\right\}, & \text{if} \quad \alpha = \delta_{\bar{j}} - \delta_{\overline{j+1}} \end{cases},$$

and

$$\mathfrak{a}_\alpha = \begin{cases} \operatorname{Span}\left\{(E_{i,i}, E_{i,i}), (E_{i+1,i+1}, E_{i+1,i+1})\right\}, & \text{if} \quad \alpha = \varepsilon_i - \varepsilon_{i+1} \\ \operatorname{Span}\left\{(E_{\bar{j},\bar{j}}, E_{\bar{j},\bar{j}}), (E_{\overline{j+1},\overline{j+1}}, E_{\overline{j+1},\overline{j+1}})\right\}, & \text{if} \quad \alpha = \delta_{\bar{j}} - \delta_{\overline{j+1}} \end{cases}.$$

Thus,

$$\begin{aligned} \mathfrak{s}_{\varepsilon_i - \varepsilon_{i+1}} &= \operatorname{Span}\left\{ \begin{array}{l} (E_{i,i+1}, 0), (0, E_{i+1,i}), (E_{i+1,i}, 0), (0, E_{i,i+1}), (E_{i,i} - E_{i+1,i+1}, 0), \\ (0, E_{i+1,i+1} - E_{i,i}), (E_{i,i}, E_{i,i}), (E_{i+1,i+1}, E_{i+1,i+1}) \end{array} \right\}, \\ \mathfrak{s}_{\delta_{\bar{j}} - \delta_{\overline{j+1}}} &= \operatorname{Span}\left\{ \begin{array}{l} (E_{\bar{j},\overline{j+1}}, 0), (0, E_{\overline{j+1},\bar{j}}), (E_{\overline{j+1},\bar{j}}, 0), (0, E_{\bar{j},\overline{j+1}}), (E_{\bar{j},\bar{j}} - E_{\overline{j+1},\overline{j+1}}, 0), \\ (0, E_{\overline{j+1},\overline{j+1}} - E_{\bar{j},\bar{j}}), (E_{\bar{j},\bar{j}}, E_{\bar{j},\bar{j}}), (E_{\overline{j+1},\overline{j+1}}, E_{\overline{j+1},\overline{j+1}}) \end{array} \right\}. \end{aligned}$$

It follows that for simple even roots $\alpha \in \left\{\varepsilon_i - \varepsilon_{i+1}, \delta_{\bar{j}} - \delta_{\overline{j+1}}, i \in I_{\bar{0}}, \bar{j} \in I_{\bar{1}}\right\}$ we have

$$\mathfrak{s}_\alpha \cong \mathfrak{gl}_2(\mathbb{C}) \oplus \mathfrak{gl}_2(\mathbb{C}).$$

4.6 The h_i and the $\mathfrak{s}_\alpha$ for Case II : $\mathfrak{g} = \mathfrak{gl}(m|2n)$ and $\mathfrak{k} = \mathfrak{osp}(m|2n)$

Let $\mathfrak{g} = \mathfrak{gl}(m|2n)$ and let $I_{\bar{0}} = \{1, \cdots, m\}$ and $I_{\bar{1}} = \{\bar{1}, \bar{2}, \cdots, \overline{2n}\}$; let $\{E_{ij} \mid i, j \in I = I_{\bar{0}} \sqcup I_{\bar{1}}\}$ be the basis for $\mathfrak{gl}(m|2n)$ consisting of matrix units. Recall that $|j| = 0$ if $j \in I_{\bar{0}}$, and $|j| = 1$ if $j \in I_{\bar{1}}$.
Consider $\theta(X) = -PX^{st}P^{-1}$ where $X^{st} = \sum_{i,j \in I}(-1)^{|j|(|i|+|j|)} x_{ij} E_{ji}$ is the supertranspose of X and

$$P = \begin{bmatrix} I_m & 0 & 0 \\ 0 & 0 & I_n \\ 0 & -I_n & 0 \end{bmatrix}.$$

Then,

$$\begin{aligned}\mathfrak{k} &= \{X \in \mathfrak{gl}(m|2n) \,|\, \theta(X) = X\} \\ &= \left\{ \begin{bmatrix} A_{m\times m} & B_{m\times 2n} \\ C_{2n\times m} & D_{2n\times 2n} \end{bmatrix} \in \mathfrak{gl}(m|2n) \,|\, A = -A^t, D = JD^tJ, C = -JB^t \right\},\end{aligned}$$

where

$$J = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix}. \tag{4.6.1}$$

Moreover,

$$\begin{aligned}\mathfrak{p} &= \{X \in \mathfrak{gl}(m|2n) \,|\, \theta(X) = -X\} \\ &= \left\{ \begin{bmatrix} A_{m\times m} & B_{m\times 2n} \\ C_{2n\times m} & D_{2n\times 2n} \end{bmatrix} \in \mathfrak{gl}(m|2n) \,|\, A = A^t, D = -JD^tJ, C = JB^t \right\}.\end{aligned}$$

Thus,

$$\mathfrak{a} = \operatorname{Span}\left\{h_i, h_{\bar{j}} \,|\, 1 \le i \le m, 1 \le j \le n\right\},$$

where

$$h_i := E_{i,i}, \quad h_{\bar{j}} := E_{\bar{j},\bar{j}} + E_{\bar{j}+n,\bar{j}+n},$$

and

$$\varepsilon_i = \boldsymbol{\varepsilon}_i|_{\mathfrak{a}}, \quad \text{and} \quad \delta_j = \boldsymbol{\delta}_{\bar{j}}|_{\mathfrak{a}} = \boldsymbol{\delta}_{\bar{j}+n}|_{\mathfrak{a}} \quad \text{for} \quad 1 \le i \le m, 1 \le j \le n.$$

By considering $\{\varepsilon_i\}_{i=1}^m \cup \{\delta_{\bar{i}}\}_{\bar{i}=1}^n$ as a basis of $\mathfrak{a}^*$, the restricted root system is of the type $A(m-1|n-1)$ and for each simple even root we have:

$$\mathfrak{g}_\alpha = \begin{cases} \operatorname{Span}\{E_{i,i+1}\}, & \text{if} \quad \alpha = \varepsilon_i - \varepsilon_{i+1} \\ \operatorname{Span}\{E_{\bar{i},\overline{i+1}}, E_{\bar{i},\overline{i+1+n}}, E_{\overline{i+n},\overline{i+1}}, E_{\overline{i+n},\overline{i+1+n}}\}, & \text{if} \quad \alpha = \delta_{\bar{i}} - \delta_{\overline{i+1}} \end{cases},$$

$$\mathfrak{g}_{-\alpha} = \begin{cases} \operatorname{Span}\{E_{i+1,i}\}, & \text{if} \quad \alpha = \varepsilon_i - \varepsilon_{i+1} \\ \operatorname{Span}\{E_{\overline{i+1},\bar{i}}, E_{\overline{i+1+n},\bar{i}}, E_{\overline{i+1},\overline{i+n}}, E_{\overline{i+1+n},\overline{i+n}}\}, & \text{if} \quad \alpha = \delta_{\bar{i}} - \delta_{\overline{i+1}} \end{cases},$$

$$[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] = \begin{cases} \operatorname{Span}\{E_{i,i} - E_{i+1,i+1}\}, & \text{if} \quad \alpha = \varepsilon_i - \varepsilon_{i+1} \\ \operatorname{Span}(L \cup L'), & \text{if} \quad \alpha = \delta_{\bar{i}} - \delta_{\overline{i+1}} \end{cases},$$

where

$$\begin{aligned} L &= \{E_{\bar{i},\bar{i}} - E_{\overline{i+1},\overline{i+1}}, E_{\bar{i},\bar{i}} - E_{\overline{i+1+n},\overline{i+1+n}}, E_{\overline{i+n},\overline{i+n}} - E_{\overline{i+1},\overline{i+1}}\}, \\ L' &= \{E_{\overline{i+n},\overline{i+n}} - E_{\overline{i+1+n},\overline{i+1+n}}, E_{\bar{i},\overline{i+n}}, E_{\overline{i+n},\bar{i}}\}. \end{aligned}$$

Also,

$$\mathfrak{a}_\alpha = \begin{cases} \operatorname{Span}\{E_{i,i} + E_{i+1,i+1}\}, & \text{if} \quad \alpha = \varepsilon_i - \varepsilon_{i+1} \\ \operatorname{Span}(L''), & \text{if} \quad \alpha = \delta_{\bar{i}} - \delta_{\overline{i+1}} \end{cases},$$

where

$$L'' = \{E_{\bar{i},\overline{i+1}} + E_{\overline{i+1},\bar{i}}, E_{\bar{i},\overline{i+1+n}} + E_{\overline{i+1+n},\bar{i}}, E_{\overline{i+n},\overline{i+1}} + E_{\overline{i+1},\overline{i+n}}, E_{\overline{i+n},\overline{i+1+n}} + E_{\overline{i+1+n},\overline{i+n}}\}.$$

Thus,

$$\mathfrak{s}_\alpha = \begin{cases} \text{Span}\,\{E_{i,i}, E_{i+1,i+1}, E_{i,i+1}, E_{i+1,i}\}\,, & \text{if} \quad \alpha = \varepsilon_i - \varepsilon_{i+1} \\ \text{Span}\,(L_1 \cup L_2 \cup L_3)\,, & \text{if} \quad \alpha = \delta_{\bar{i}} - \delta_{\overline{i+1}} \end{cases},$$

where

$$\begin{aligned} L_1 &= \{E_{\bar{i},\bar{i}}, E_{\overline{i+1},\overline{i+1}}, E_{\overline{i+n},\overline{i+n}}, E_{\overline{i+1+n},\overline{i+1+n}}\}, \\ L_2 &= \{E_{\bar{i},\overline{i+1}}, E_{\bar{i},\overline{i+n}}, E_{\bar{i},\overline{i+1+n}}, E_{\overline{i+1},\bar{i}}, E_{\overline{i+1},\overline{i+n}}, E_{\overline{i+1},\overline{i+1+n}}\}, \\ L_3 &= \{E_{\overline{i+n},\bar{i}}, E_{\overline{i+n},\overline{i+1}}, E_{\overline{i+n},\overline{i+1+n}}, E_{\overline{i+1+n},\bar{i}}, E_{\overline{i+1+n},\overline{i+1}}, E_{\overline{i+1+n},\overline{i+n}}\}. \end{aligned}$$

Now by setting $E := E_{i,i+1}, F := E_{i+1,i}, h_1 := E_{i,i}, h_2 := E_{i+1,i+1}$ for $1 \le i \le m$ we obtain an isomorphism

$$\mathfrak{s}_\alpha \to \mathfrak{gl}_2(\mathbb{C}) \quad \text{for} \quad \alpha = \varepsilon_i - \varepsilon_{i+1}.$$

Also, we obtain an isomorphism of Lie algebras

$$\mathfrak{s}_\alpha \to \mathfrak{gl}_4(\mathbb{C}), \text{ for } \alpha = \delta_{\bar{i}} - \delta_{\overline{i+1}},$$

by mapping each $E_{a,b}$ to $E_{\eta(a),\eta(b)}$ where

$$\eta(\bar{i}) = 1,\ \eta(\overline{i+1}) = 2,\ \eta(\overline{i+n}) = 3,\ \eta(\overline{i+n+1}) = 4.$$

Hence $\mathfrak{s}_\alpha \cong \mathfrak{gl}_4(\mathbb{C})$.

Lemma 4.6.1. Let $\mathfrak{g} = \mathfrak{gl}(m|2n)$ and $\mathfrak{k} = \mathfrak{osp}(m|2n)$. Then $C_{\mathfrak{p}}(\mathfrak{a}) = \mathfrak{a}$.

Proof. For any element

$$P = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \mathfrak{p},$$

and $a = \text{diag}(s,t) \in \mathfrak{a}$ where $s = (s_1, \cdots, s_m) \in \mathbb{C}^m$ and $t = (t_1, \cdots, t_n, -t_1, \cdots, -t_n) \in \mathbb{C}^{2n}$ we have

$$[a, P] = \begin{bmatrix} \text{diag}(s)A - A\text{diag}(s) & \text{diag}(s)B - B\text{diag}(t) \\ \text{diag}(t)C - C\text{diag}(s) & \text{diag}(t)D - D\text{diag}(t) \end{bmatrix}.$$

Obviously, $\mathfrak{a} \subseteq C_{\mathfrak{p}}(\mathfrak{a})$. On the other hand, the entries of $\text{diag}(s)A - A\text{diag}(s)$ are of the form $a_{ij}(s_i - s_j)$ and $\text{diag}(t) - D\text{diag}(t)$ has entries of the form $d_{ij}(t_i - t_j)$. Then if $a_{ij} \neq 0$, for distinct s_i and s_j we obtain a nonzero entry of $[a, P]$. Similarly, if $d_{ij} \neq 0$, for distinct t_i and t_j we obtain a nonzero entry of $[a, P]$. Also, $\text{diag}(s)B - B\text{diag}(t)$ and $\text{diag}(t)C - C\text{diag}(s)$ respectively have entries of the form $s_i b_{ij} - b_{ij} t_j$ and $t_i c_{ij} - c_{ij} s_j$ which for nonzero b_{ij}, c_{ij} and distinct s_i, t_j, we have $[a, P] \neq 0$. Thus $C_{\mathfrak{p}}(\mathfrak{a}) \subseteq \mathfrak{a}$ as well. ∎

4.7 The h_i and the $\mathfrak{s}_\alpha$ for Case III : $\mathfrak{g} = \mathfrak{gosp}(m+1|2n)$ and $\mathfrak{k} = \mathfrak{osp}(m|2n)$

In this section we describe the absolute and restricted root systems of the Lie superalgebra $\mathfrak{gosp}(m+1|2n)$ with respect to the Cartan space $\mathfrak{a}$. The Cartan space $\mathfrak{a}$ is associated with a subalgebra $\mathfrak{k}$, where $\mathfrak{k} = \mathfrak{osp}(m|2n)$. We divide the analysis into the two cases $m+1 = 2r$ and $m+1 = 2r+1$ for $m \geqslant 0$.

4.7.1 The case $m+1 = 2r$

Recall that the Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ of diagonal matrices has a basis given by

$$H_i := E_{i,i} - E_{i+r,i+r}, \quad 1 \leqslant i \leqslant r,$$
$$H_{\bar{j}} := E_{\bar{j},\bar{j}} - E_{\bar{j}+n,\bar{j}+n}, \quad 1 \leqslant j \leqslant n.$$

Let the dual basis of the Cartan subalgebra be $\{\boldsymbol{\varepsilon}_1, ..., \boldsymbol{\varepsilon}_r, \boldsymbol{\delta}_{\bar{1}}, ..., \boldsymbol{\delta}_{\bar{n}}\}$ such that

$$\boldsymbol{\varepsilon}_i(H_j) = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}, \quad \boldsymbol{\delta}_{\bar{i}}(H_{\bar{j}}) = \begin{cases} 1 & \bar{i} = \bar{j} \\ 0 & \bar{i} \neq \bar{j} \end{cases}.$$

Then the root system $\boldsymbol{\Phi} = \boldsymbol{\Phi}_{\bar{0}} \cup \boldsymbol{\Phi}_{\bar{1}}$ with respect to $\mathfrak{h}$ is

$$\{\pm\boldsymbol{\varepsilon}_i \pm \boldsymbol{\varepsilon}_j, \pm\boldsymbol{\delta}_{\bar{k}} \pm \boldsymbol{\delta}_{\bar{l}}, \pm 2\delta_{\bar{p}}\} \cup \{\pm\boldsymbol{\varepsilon}_q \pm \boldsymbol{\delta}_{\bar{p}}\},$$

where $1 \leqslant i < j \leqslant r$, $1 \leqslant k < l \leqslant n$, $1 \leqslant p \leqslant r$, and $1 \leqslant q \leqslant n$.
In this case the involutive automorphism θ on $\mathfrak{gosp}(m+1, 2n)$ will be

$$\theta(x) = \begin{cases} wxw^{-1}, & \text{if} \quad x \in \mathfrak{osp}(2r|2n) \\ -x, & \text{if} \quad x = I_{2r+2n} \end{cases}$$

where

$$w = \begin{bmatrix} W_1 & 0 \\ 0 & W_2 \end{bmatrix}, \quad W_1 = \begin{array}{c|cccc} & 1 & r-1 & 1 & r-1 \\ \hline 1 & 0 & 0 & 1 & 0 \\ r-1 & 0 & I_{r-1} & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ r-1 & 0 & 0 & 0 & I_{r-1} \end{array}, \quad \text{and } W_2 = I_{2n}.$$

The small numbers above and on the left side of the matrix W_1 respectively represent the number of columns and rows in a batch. By a straightforward calculation $w = w^{-1}$. Consider any $x = x_{\bar{0}} + x_{\bar{1}} \in \mathfrak{g}$. Then $x_{\bar{0}}$ can be expressed as the following matrix:

$$x_{\bar{0}} = \left[\begin{array}{cc|cc|c} \boldsymbol{a} & X & 0 & U & \\ Y & A & -U^t & B & \\ \hline 0 & V & -\boldsymbol{a} & -Y^t & \\ -V^t & C & -X^t & -A^t & \\ & & & & D \end{array}\right] + \boldsymbol{c}I_{2r+2n}, \tag{4.7.1}$$

where $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c} \in \mathbb{C}$, $A, B, C \in \mathrm{Mat}_{(r-1)\times(r-1)}$ satisfy the relations $B + B^T = C + C^T = 0$, $X, U, V, Y^t \in \mathrm{Mat}_{1\times(r-1)}$, and $D \in \mathrm{MAT}_{2n\times 2n}$ satisfies the relation $D = JD^tJ$ where

$$J = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix}. \tag{4.7.2}$$

Remark 4.7.1. Throughout this section, in order to avoid confusion between Lie algebra elements and scalars that appear in matrix entries we use bold symbols such as $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$ for scalar entries.

Also $x_{\bar{1}}$ can be expressed as the following matrix:

$$x_{\bar{1}} = \left[\begin{array}{cc|cc|cc|cc} & & & & -\boldsymbol{d} & -M^t & \boldsymbol{b} & V^t \\ & & & & -L^t & -D^t & U^t & B^t \\ \hline & & & & -\boldsymbol{c} & -F^t & \boldsymbol{a} & Y^t \\ & & & & -E^t & -C^t & X^t & A^t \\ \boldsymbol{a} & X & \boldsymbol{b} & U & & & & \\ Y & A & V & B & & & & \\ \hline \boldsymbol{c} & E & \boldsymbol{d} & L & & & & \\ F & C & M & D & & & & \end{array}\right], \tag{4.7.3}$$

where $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c} \in \mathbb{C}$, $X, U, E, L \in \mathrm{Mat}_{1\times(r-1)}$, $Y, V, F, M \in \mathrm{Mat}_{(n-1)\times 1}$, and $A, B, C, D \in \mathrm{Mat}_{(n-1)\times(r-1)}$. Then

$$\theta(x_{\bar{0}}) = \left[\begin{array}{cc|cc|c} -\boldsymbol{a} & V & 0 & -Y^t & \\ -U^t & A & Y & B & \\ \hline 0 & X & \boldsymbol{a} & U & \\ -X^t & C & -V^t & -A^t & \\ & & & & D \end{array}\right] - \boldsymbol{c} I_{2r+2n},$$

and

$$\theta(x_{\bar{1}}) = \left[\begin{array}{cc|cc|cc|cc} & & & & -\boldsymbol{c} & -F^t & \boldsymbol{a} & Y^t \\ & & & & -L^t & -D^t & U^t & B^t \\ \hline & & & & -\boldsymbol{d} & -M^t & \boldsymbol{b} & V^t \\ & & & & -E^t & -C^t & X^t & A^t \\ \boldsymbol{b} & X & \boldsymbol{a} & U & & & & \\ V & A & Y & B & & & & \\ \hline \boldsymbol{d} & E & \boldsymbol{c} & L & & & & \\ M & C & F & D & & & & \end{array}\right].$$

Hence,

$$\mathfrak{k}_{\bar{0}} = \left\{ \left[\begin{array}{cc|cc|c} 0 & X & 0 & U & \\ Y & A & -U^t & B & \\ \hline 0 & V & 0 & -Y^t & \\ -V^t & C & -X^t & -A^t & \\ & & & & D \end{array}\right] \in \mathfrak{g}_{\bar{0}} \;\middle|\; V = X,\, U = -Y^t,\, D = JD^tJ \right\},$$

and for the odd part we have

$$\mathfrak{k}_{\bar{1}} = \left\{ \left[\begin{array}{cc:cc|cc:cc} & & & & -\boldsymbol{c} & -F^t & \boldsymbol{a} & Y^t \\ & & & & -L^t & -D^t & U^t & B^t \\ \cdashline{5-8} & & & & -\boldsymbol{d} & -M^t & \boldsymbol{b} & V^t \\ & & & & -E^t & -C^t & X^t & A^t \\ \hline \boldsymbol{b} & X & \boldsymbol{a} & U & & & & \\ V & A & Y & B & & & & \\ \cdashline{1-4} \boldsymbol{d} & E & \boldsymbol{c} & L & & & & \\ M & C & F & D & & & & \end{array} \right] \in \mathfrak{g}_{\bar{1}} \,\middle|\, \boldsymbol{a} = \boldsymbol{b}, \boldsymbol{c} = \boldsymbol{d}, M = F, V = Y \right\}.$$

Lemma 4.7.2. Let $\{e_i\}_{i=1}^{2r+2n}$ be the standard basis of $\mathbb{C}^{2r|2n}$. We consider $\mathbb{C}^{2r+2n} \cong \mathbb{C}^{2r|2n}$ as the standard module of $\mathfrak{osp}(2r|2n)$. Then the stabilizer of $e_1 - e_{r+1}$ in $\mathfrak{gosp}(2r|2n)$ is equal to $\mathfrak{k}$. In particular, $\mathfrak{k} \cong \mathfrak{osp}(2r-1|2n)$.

Proof. Note that $e_1 - e_{r+1}$ is not an isotropic vector and for $x = x_{\bar{0}} + x_{\bar{1}} \in \mathfrak{gosp}(2r|2n)$ where $x_{\bar{0}}$ and $x_{\bar{1}}$ are the matrices given in (4.7.1) and (4.7.3), by direct computations one can see that the stabilizer of $e_1 - e_{r+1}$ in $\mathfrak{gosp}_{\bar{0}}(2r|2n)$ is

$$\{x_{\bar{0}} \in \mathfrak{g}_{\bar{0}} \,|\, x_{\bar{0}} \cdot (e_1 - e_{r+1}) = 0\} = \left\{x_{\bar{0}} \in \mathfrak{g}_{\bar{0}} \,|\, \boldsymbol{a} = 0,\ V = X,\ U = -Y^t,\ D = JD^tJ\right\} = \mathfrak{k}_{\bar{0}},$$

and the stabilizer of $e_1 - e_{r+1}$ in $\mathfrak{gosp}_{\bar{1}}(2r|2n)$ is

$$\{x_{\bar{1}} \in \mathfrak{g}_{\bar{1}} \,|\, x_{\bar{1}} \cdot (e_1 - e_{r+1}) = 0\} = \{x_{\bar{1}} \in \mathfrak{g}_{\bar{1}} \,|\, \boldsymbol{a} = \boldsymbol{b}, \boldsymbol{c} = \boldsymbol{d}, M = F, V = Y\} = \mathfrak{k}_{\bar{1}}.$$

Now, set

$$R = \frac{1}{2} \left[\begin{array}{cc:cc|c} 1 & 0 & 1 & 0 & \\ 0 & \sqrt{2} & 0 & 0 & \\ \cdashline{1-4} -1 & 0 & 1 & 0 & \\ 0 & 0 & 0 & \sqrt{2} & \\ \hline & & & & I \end{array} \right] \in \mathrm{GOSp}(2r|2n)_{\bar{0}}.$$

Then for every $x_{\bar{0}} \in \mathfrak{k}_{\bar{0}}$ we have

$$Rx_{\bar{0}}R^t = \frac{1}{2} \left[\begin{array}{cc:cc|c} 0 & \sqrt{2}X & 0 & -\sqrt{2}Y^t & \\ \sqrt{2}Y & A & 0 & B & \\ \cdashline{1-4} 0 & 0 & 0 & 0 & \\ -\sqrt{2}X^t & C & 0 & -A^t & \\ \hline & & & & D \end{array} \right],$$

where $A, B, C \in \mathrm{Mat}_{(r-1)\times(r-1)}$ satisfy the relations $B+B^T = C+C^T = 0$, $D = JD^tJ$ where J is the matrix given in (4.7.2) and $X, Y^t \in \mathrm{Mat}_{1\times(r-1)}$. Also, for every $x_{\bar{1}} \in \mathfrak{k}_{\bar{1}}$ we have

$$Rx_{\bar{1}}R^t = \frac{1}{4}\begin{bmatrix} & & & & -2\boldsymbol{c} & -2F^t & 2\boldsymbol{a} & 2Y^t \\ & & & & -\sqrt{2}L^t & -\sqrt{2}D^t & \sqrt{2}U^t & \sqrt{2}B^t \\ & & & & 0 & 0 & 0 & 0 \\ & & & & -\sqrt{2}E^t & -\sqrt{2}C^t & \sqrt{2}X^t & \sqrt{2}A^t \\ 2\boldsymbol{a} & \sqrt{2}X & 0 & \sqrt{2}U & & & & \\ 2Y & \sqrt{2}A & 0 & \sqrt{2}B & & & & \\ 2\boldsymbol{c} & \sqrt{2}E & 0 & \sqrt{2}L & & & & \\ 2F & \sqrt{2}C & 0 & \sqrt{2}D & & & & \end{bmatrix},$$

where $\boldsymbol{a}, \boldsymbol{c} \in \mathbb{C}$, $X, U, E, L \in \mathrm{Mat}_{1\times(r-1)}$, $Y, F \in \mathrm{Mat}_{(n-1)\times 1}$, and $A, B, C, D \in \mathrm{Mat}_{(n-1)\times(r-1)}$. Then by Example 2.2.4 one can see that this is exactly the structure of $\mathfrak{gosp}(2r-1|2n)$. Hence, $\mathfrak{k} = \mathfrak{k}_{\bar{0}} \oplus \mathfrak{k}_{\bar{1}} \cong \mathfrak{gosp}(2r-1|2n)$. ∎

Since $\mathfrak{p}$ is the -1 eigenspace of θ, by analogous computations we obtain

$$\mathfrak{p}_{\bar{0}} = \left\{ \begin{bmatrix} \boldsymbol{a} & X & 0 & U & \\ Y & 0 & -U^t & 0 & \\ 0 & V & -\boldsymbol{a} & -Y^t & \\ -V^t & 0 & -X^t & 0 & \\ & & & & 0 \end{bmatrix} + \boldsymbol{c}I_{2r+2n} \in \mathfrak{g}_{\bar{0}} \;\middle|\; V = -X, U = Y^t \right\},$$

where $X, Y^t, U, V \in \mathrm{Mat}_{1\times(r-1)}$ and $\boldsymbol{a}, \boldsymbol{c} \in \mathbb{C}$. Also,

$$\mathfrak{p}_{\bar{1}} = \left\{ \begin{bmatrix} & & & & -\boldsymbol{c} & -F^t & \boldsymbol{a} & Y^t \\ & & & & 0 & 0 & 0 & 0 \\ & & & & -\boldsymbol{d} & -M^t & \boldsymbol{b} & V^t \\ & & & & 0 & 0 & 0 & 0 \\ \boldsymbol{b} & 0 & \boldsymbol{a} & 0 & & & & \\ V & 0 & Y & 0 & & & & \\ \boldsymbol{d} & 0 & \boldsymbol{c} & 0 & & & & \\ M & 0 & F & 0 & & & & \end{bmatrix} \in \mathfrak{g}_{\bar{1}} \;\middle|\; \boldsymbol{a} = -\boldsymbol{b}, \boldsymbol{c} = -\boldsymbol{d}, M = -F, V = -Y \right\},$$

where $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d} \in \mathbb{C}$, and $Y, F, V, M \in \mathrm{Mat}_{(n-1)\times 1}$.
We choose the Cartan subalgebra $\mathfrak{h}$ that is consistent with the Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ to be:

$$\mathfrak{h} = \{\mathrm{diag}(s, -s, t, -t) \,|\, s := (s_1, \cdots, s_m) \in \mathbb{C}^m \quad \text{and} \quad t := (t_1, \cdots, t_n) \in \mathbb{C}^n\}.$$

Thus,

$$\mathfrak{a} = \operatorname{Span}\{a, a'\},$$

where

$$a' = I_{2r+2n} \quad \text{and} \quad a = \operatorname{diag}(\bar{s}, -\bar{s}, 0_{2n}) \quad \text{for} \quad \bar{s} := (2, 0, \cdots, 0) \in \mathbb{C}^r.$$

We define

$$h_1 = \frac{1}{2}(a + a'), \quad \text{and} \quad h_2 = \frac{1}{2}(a' - a).$$

For the matrix realization of elements of this algebra see (4.7.1) and (4.7.3). Therefore, the restricted root system has only two roots $\alpha = \pm(\varepsilon_1 - \varepsilon_2)$. Hence, $\dim(\mathfrak{a}^*) = 2$ and the restricted root system is of the type $A(1) = A(1, -1)$. Note that α is the restriction of $\boldsymbol{\varepsilon}_1$. The choices of ε_1 and ε_2 in $\mathfrak{a}^*$ such that $\varepsilon_1 - \varepsilon_2 = \boldsymbol{\varepsilon}_1|_{\mathfrak{a}}$ do not matter.

Lemma 4.7.3. Let $\mathfrak{g} = \mathfrak{gosp}(2r|2n)$. The centralizer of $\mathfrak{a}$ in $\mathfrak{p}$ is $\mathfrak{a}$.

Proof. For every element $P_0 \in \mathfrak{p}_{\bar{0}}$ of the form

$$P_0 = \left[\begin{array}{cc:cc c} \boldsymbol{a}+1 & X & 0 & Y^t & \\ Y & 1 & -Y & 0 & \\ \hdashline 0 & -X & -\boldsymbol{a}+1 & -Y^t & \\ X^t & 0 & -X^t & 1 & \\ & & & & I_{2n} \end{array}\right],$$

where $X, Y^t \in \operatorname{Mat}_{1\times(r-1)}$, we have

$$[a, P_0] = \left[\begin{array}{cc:cc c} 0 & X & 0 & Y^t & \\ -Y & 0 & -Y & 0 & \\ \hdashline 0 & X & 0 & Y^t & \\ -X^t & 0 & -X^t & 0 & \\ & & & & 0 \end{array}\right],$$

Thus, $C_{\mathfrak{p}_{\bar{0}}}(\mathfrak{a}) = \mathfrak{a}$. Also, for

$$P_1 = \left[\begin{array}{cccc cc:cc} & & & & -\boldsymbol{c} & -F^t & \boldsymbol{a} & Y^t \\ & & & & 0 & 0 & 0 & 0 \\ \hdashline & & & & \boldsymbol{c} & F^t & -\boldsymbol{a} & -Y^t \\ & & & & 0 & 0 & 0 & 0 \\ -\boldsymbol{a} & 0 & \boldsymbol{a} & 0 & & & & \\ -Y & 0 & Y & 0 & & & & \\ \hdashline -\boldsymbol{c} & 0 & \boldsymbol{c} & 0 & & & & \\ -F & 0 & F & 0 & & & & \end{array}\right] \in \mathfrak{p}_{\bar{1}},$$

we have

$$[a, P_1] = \left[\begin{array}{cc|cc|cc|cc} & & & & -\boldsymbol{c} & -F^t & \boldsymbol{a} & Y^t \\ & & & & 0 & 0 & 0 & 0 \\ \hline & & & & -\boldsymbol{c} & -F^t & \boldsymbol{a} & Y^t \\ & & & & 0 & 0 & 0 & 0 \\ \boldsymbol{a} & 0 & \boldsymbol{a} & 0 & & & & \\ Y & 0 & Y & 0 & & & & \\ \hline \boldsymbol{c} & 0 & \boldsymbol{c} & 0 & & & & \\ F & 0 & F & 0 & & & & \end{array}\right]$$

which implies that $[a, P_1] = 0$ if and only if $P_1 = 0$. Thus, $C_{\mathfrak{p}_{\bar{1}}}(\mathfrak{a}) = \mathfrak{a}$ and hence $C_{\mathfrak{p}}(\mathfrak{a}) = \mathfrak{a}$. ∎

4.7.2 The case $m+1 = 2r+1$

Recall that the Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ of diagonal matrices has a basis given by

$$\begin{aligned} H_i &:= E_{i+1,i+1} - E_{i+r+1,i+r+1}, \quad 1 \leqslant i \leqslant r, \\ H_{\bar{j}} &:= E_{\bar{j},\bar{j}} - E_{\bar{j}+n,\bar{j}+n}, \quad 1 \leqslant j \leqslant n. \end{aligned}$$

Let the dual basis of the Cartan subalgebra be $\{\boldsymbol{\varepsilon}_1, ..., \boldsymbol{\varepsilon}_r, \boldsymbol{\delta}_{\bar{1}}, ..., \boldsymbol{\delta}_{\bar{n}}\}$ such that

$$\boldsymbol{\varepsilon}_i(H_j) = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}, \quad \boldsymbol{\delta}_{\bar{i}}(H_{\bar{j}}) = \begin{cases} 1 & \bar{i} = \bar{j} \\ 0 & \bar{i} \neq \bar{j} \end{cases}.$$

Then the root system $\boldsymbol{\Phi} = \boldsymbol{\Phi}_{\bar{0}} \cup \boldsymbol{\Phi}_{\bar{1}}$ with respect to $\mathfrak{h}$ is

$$\{\pm\boldsymbol{\varepsilon}_i \pm \boldsymbol{\varepsilon}_j, \pm\boldsymbol{\delta}_{\bar{k}} \pm \boldsymbol{\delta}_{\bar{l}}, \pm 2\boldsymbol{\delta}_{\bar{p}}, \pm\boldsymbol{\varepsilon}_q\} \cup \{\pm\boldsymbol{\varepsilon}_q \pm \boldsymbol{\delta}_{\bar{p}}, \pm\boldsymbol{\delta}_{\bar{p}}\},$$

where $1 \leqslant i < j \leqslant r$, $1 \leqslant k < l \leqslant n$, $1 \leqslant p \leqslant r$, and $1 \leqslant q \leqslant n$.
In this case the involutive automorphism θ on $\mathfrak{gosp}(m+1, 2n)$ will be as follows:

$$\theta(x) = \begin{cases} wxw^{-1}, & \text{if} \quad x \in \mathfrak{osp}(2r+1|2n) \\ -x, & \qquad\quad \text{if} \quad x = I_{2r+2n} \end{cases}$$

where

$$w = \begin{bmatrix} W_1 & 0 \\ 0 & W_2 \end{bmatrix}$$

and

$$W_1 = \begin{array}{c} \\ 1 \\ 1 \\ r-1 \\ 1 \\ r-1 \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & 1 & r-1 & 1 & r-1 \end{array} \\ \left[\begin{array}{c|cc|cc} 1 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & I_{r-1} & 0 & 0 \\ \hline 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & I_{r-1} \end{array}\right] \end{array}, \quad W_2 = I_{2n}.$$

Again, small numbers above and on the left side of the matrix W_1 respectively represent the number of columns and rows. By a straightforward calculation $w = w^{-1}$. Consider any $x = x_{\bar{0}} + x_{\bar{1}} \in \mathfrak{g}$. Then, $x_{\bar{0}}$ can be expressed as the following matrix:

$$x_{\bar{0}} = \left[\begin{array}{c:cc:cc c}
0 & -\boldsymbol{c} & -M^t & -\boldsymbol{b} & -L^t & \\
\hdashline
\boldsymbol{b} & \boldsymbol{a} & X & 0 & U & \\
L & Y & A & -U^t & B & \\
\hdashline
\boldsymbol{c} & 0 & V & -\boldsymbol{a} & -Y^t & \\
M & -V^t & C & -X^t & -A^t & \\
 & & & & & D
\end{array}\right] + \boldsymbol{z} I_{2r+1+2n} \tag{4.7.4}$$

where $\boldsymbol{a}', \boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{z} \in \mathbb{C}$, $A, B, C \in \mathrm{Mat}_{(r-1)\times(r-1)}$ satisfy the relation $B + B^T = C + C^T = 0$, $D = JD^tJ$ where J is the matrix given in (4.7.2), and $X, U, V, Y^t.M^t, L^t \in \mathrm{Mat}_{1\times(r-1)}$. Also, $x_{\bar{1}}$ can be expressed as the following matrix:

$$x_{\bar{1}} = \left[\begin{array}{ccccc cc:cc}
 & & & & & -\boldsymbol{g} & -G^t & \boldsymbol{j} & J^t \\
\cdashline{6-9}
 & & & & & -\boldsymbol{d} & -M^t & \boldsymbol{b} & V^t \\
 & & & & & -L^t & -D^t & U^t & B^t \\
\cdashline{6-9}
 & & & & & -\boldsymbol{c} & -F^t & \boldsymbol{a} & Y^t \\
 & & & & & -E^t & -C^t & X^t & A^t \\
\boldsymbol{j} & \boldsymbol{a} & X & \boldsymbol{b} & U & & & & \\
\boldsymbol{J} & Y & A & V & B & & & & \\
\cdashline{1-5}
\boldsymbol{g} & \boldsymbol{c} & E & \boldsymbol{d} & L & & & & \\
G & F & C & M & D & & & &
\end{array}\right] \tag{4.7.5}$$

where $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d}, \boldsymbol{g}, \boldsymbol{j} \in \mathbb{C}$, $X, U, E, L \in \mathrm{Mat}_{1\times(r-1)}$, $Y, V, F, M, J, G \in \mathrm{Mat}_{(n-1)\times 1}$, and $A, B, C, D \in \mathrm{Mat}_{(n-1)\times(r-1)}$. Then,

$$\theta(x_{\bar{0}}) = \left[\begin{array}{c:cc:cc c}
0 & -\boldsymbol{b} & -M^t & -\boldsymbol{c} & -L^t & \\
\hdashline
\boldsymbol{c} & -\boldsymbol{a} & V & 0 & -Y^t & \\
L & -U^t & A & Y & B & \\
\hdashline
\boldsymbol{b} & 0 & X & \boldsymbol{a} & U & \\
M & -X^t & C & -V^t & -A^t & \\
 & & & & & D
\end{array}\right] - \boldsymbol{z} I_{2r+1+2n}$$

and

$$\theta(x_{\bar 1}) = \left[\begin{array}{ccccc|cccc}
 & & & & & \boldsymbol{j} & J^t & \boldsymbol{g} & G^t \\
 & & & & & -\boldsymbol{c} & -F^t & \boldsymbol{a} & Y^t \\
 & & & & & -L^t & -D^t & U^t & B^t \\
 & & & & & -\boldsymbol{d} & -M^t & \boldsymbol{b} & V^t \\
 & & & & & -E^t & -C^t & X^t & A^t \\
\hline
\boldsymbol{g} & \boldsymbol{b} & X & \boldsymbol{a} & U & & & & \\
G & V & A & Y & B & & & & \\
\boldsymbol{j} & \boldsymbol{d} & E & \boldsymbol{c} & L & & & & \\
J & M & C & F & D & & & &
\end{array}\right].$$

It follows that

$$\mathfrak{k}_{\bar 0} = \left\{ \left[\begin{array}{ccccc|c}
0 & -\boldsymbol{c} & -M^t & -\boldsymbol{b} & -L^t & \\
\boldsymbol{b} & 0 & X & 0 & U & \\
L & Y & A & -U^t & B & \\
\boldsymbol{c} & 0 & V & 0 & -Y^t & \\
M & -V^t & C & -X^t & -A^t & \\
\hline
 & & & & & D
\end{array}\right] \in \mathfrak{g}_{\bar 0} \;\middle|\; \boldsymbol{b}=\boldsymbol{c}, V=X, U=-Y^t, \right\}$$

and

$$\mathfrak{k}_{\bar 1} = \left\{ \left[\begin{array}{ccccc|cccc}
 & & & & & \boldsymbol{j} & J^t & \boldsymbol{g} & G^t \\
 & & & & & -\boldsymbol{c} & -F^t & \boldsymbol{a} & Y^t \\
 & & & & & -L^t & -D^t & U^t & B^t \\
 & & & & & -\boldsymbol{d} & -M^t & \boldsymbol{b} & V^t \\
 & & & & & -E^t & -C^t & X^t & A^t \\
\hline
\boldsymbol{g} & \boldsymbol{b} & X & \boldsymbol{a} & U & & & & \\
G & V & A & Y & B & & & & \\
\boldsymbol{j} & \boldsymbol{d} & E & \boldsymbol{c} & L & & & & \\
\boldsymbol{J} & M & C & F & D & & & &
\end{array}\right] \in \mathfrak{g}_{\bar 1} \;\middle|\; \boldsymbol{a}=\boldsymbol{b}, \boldsymbol{c}=\boldsymbol{d}, F=M, V=Y \right\}$$

Lemma 4.7.4. Let $\{e_i\}_{i=1}^{2r+1+2n}$ be the standard basis of $\mathbb{C}^{2r+1|2n}$. We consider $\mathbb{C}^{2r+1|2n}$ as the standard module of $\mathfrak{osp}(2r+1|2n)$. Then the stabilizer of $e_2 - e_{r+2}$ in $\mathfrak{gosp}(2r+1|2n)$ is equal to $\mathfrak{k}$. In particular, $\mathfrak{k} \cong \mathfrak{osp}(2r|2n)$.

Proof. Note that $e_2 - e_{r+2}$ is not an isotropic vector. By direct computations one can see that the stabilizer of $e_2 - e_{r+2}$ in $\mathfrak{gosp}(2r+1|2n)$ is $\mathfrak{k}$. Similar to Lemma 4.7.2 one can see that the stabilizer of $e_2 - e_{r+2}$ also stabilizes its orthongal complement $\langle e_2 - e_{r+2}\rangle^\perp$, which is a nondegenerate orthosymplectic space of dimention $(2r|2n)$ and this completes the proof. ∎

Furtheremore,

$$\mathfrak{p}_{\bar{0}} = \left\{ \left[\begin{array}{c|cc|cc|c} 0 & -\boldsymbol{c} & 0 & -\boldsymbol{b} & 0 & \\ \hline \boldsymbol{b} & \boldsymbol{a} & X & 0 & U & \\ 0 & Y & 0 & -U^t & 0 & \\ \hline \boldsymbol{c} & 0 & V & -\boldsymbol{a} & -Y^t & \\ 0 & -V^t & 0 & -X^t & 0 & \\ & & & & & 0 \end{array} \right] + zI_{m+1+2n} \in \mathfrak{g}_{\bar{0}} \;\middle|\; \boldsymbol{c} = -\boldsymbol{b}, V = -X, U = Y^t \right\},$$

and

$$\mathfrak{p}_{\bar{1}} = \left\{ \left[\begin{array}{ccccccccc} & & & & & 0 & 0 & 0 & 0 \\ & & & & & -\boldsymbol{d} & -M^t & \boldsymbol{b} & V^t \\ & & & & & 0 & 0 & 0 & 0 \\ & & & & & -\boldsymbol{c} & -F^t & \boldsymbol{a} & Y^t \\ & & & & & 0 & 0 & 0 & 0 \\ 0 & a & 0 & b & 0 & & & & \\ 0 & Y & 0 & V & 0 & & & & \\ 0 & \boldsymbol{c} & 0 & \boldsymbol{d} & 0 & & & & \\ 0 & F & 0 & M & 0 & & & & \end{array} \right] \in \mathfrak{g}_{\bar{1}} \;\middle|\; \boldsymbol{b} = -\boldsymbol{a}, \boldsymbol{d} = -\boldsymbol{c}, M = -F, V = -Y \right\}.$$

Thus,

$$\mathfrak{a} = \operatorname{Span}\{a, a'\}$$

where

$$a' = I_{2r+1+2n},\ a = \operatorname{diag}(0, \bar{s}, -\bar{s}, 0_{2n}) \quad \text{for} \quad \bar{s} = (2, \cdots, 0) \in \mathbb{C}^r.$$

We define the h_i as follows

$$h_1 = \frac{1}{2}(a + a'), \quad \text{and} \quad h_2 = \frac{1}{2}(a' - a).$$

Recall that the realizations of even and odd parts of the Lie superalgebra $\mathfrak{gosp}(m+1, 2n)$ are defined in matrix forms (4.7.4) and (4.7.5). Therefore, the restricted root system has only two roots $\pm(\varepsilon_1 - \varepsilon_2)$. Hence, $\dim(\mathfrak{a}^*) = 2$ and the restricted root system is of the type $A(1) = A(1, -1)$.

Proposition 4.7.5. For the restricted root $\alpha = \varepsilon_1 - \varepsilon_2$ we have

$$\mathfrak{s}_\alpha = \mathfrak{gosp}(m+1|2n).$$

Proof. Recall that the root decomposition of $\mathfrak{g}$ with respect to $\mathfrak{h}$ is

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\beta \in \boldsymbol{\Phi}} \mathfrak{g}_\beta.$$

Now consider the decomposition $\mathfrak{h} = \mathfrak{t} \oplus \mathfrak{a}$ where $\mathfrak{t} = \mathfrak{h} \cap \mathfrak{k}$ and $\mathfrak{a} = \mathfrak{h} \cap \mathfrak{p}$. Then

$$\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{t} \oplus \bigoplus_{\beta \in \boldsymbol{\Phi}} \mathfrak{g}_\beta = \mathfrak{a} \oplus \mathfrak{t} \oplus \bigoplus_{\beta|_\mathfrak{a} = 0} \mathfrak{g}_\beta \oplus \bigoplus_{\beta|_\mathfrak{a} = \alpha} \mathfrak{g}_\beta \oplus \bigoplus_{\beta|_\mathfrak{a} = -\alpha} \mathfrak{g}_\beta = \mathfrak{a} \oplus \mathfrak{t} \oplus \bigoplus_{\beta|_\mathfrak{a} = 0} \mathfrak{g}_\beta \oplus \underline{\mathfrak{g}}_{\pm\alpha}$$

where $\underline{\mathfrak{g}}_{\pm\alpha} = \bigoplus \mathfrak{g}_\beta$ with $\beta|_\mathfrak{a} = \pm\alpha$ and on the other hand, the restricted root decomposition of $\mathfrak{g}$ is

$$\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{a} \oplus \bigoplus_{\gamma \in \Delta'} \mathfrak{g}_\gamma$$

where $\mathfrak{m} = \mathfrak{m}_{\bar{0}} \oplus \mathfrak{m}_{\bar{1}} = C_\mathfrak{k}(\mathfrak{a})$. For $m+1 = 2r$, where r is a positive integer, $\mathfrak{m}_{\bar{0}}$ and $\mathfrak{m}_{\bar{1}}$ have the following matrix forms

$$\mathfrak{m}_{\bar{0}} = \left\{ \left[\begin{array}{cc:cc c} 0 & 0 & 0 & 0 & \\ 0 & A & 0 & B & \\ \hdashline 0 & 0 & 0 & 0 & \\ 0 & C & 0 & -A^t & \\ & & & & D \end{array} \right] \in \mathfrak{gosp}(2r|2n) \,\Big|\, B = -B^T, C = -C^T \right\}$$

and

$$\mathfrak{m}_{\bar{1}} = \left\{ \left[\begin{array}{cccc cc:cc} & & & & 0 & 0 & 0 & 0 \\ & & & & -L^t & -D^t & U^t & B^t \\ \hdashline & & & & 0 & 0 & 0 & 0 \\ & & & & -E^t & -C^t & X^t & A^t \\ 0 & X & 0 & U & & & & \\ 0 & A & 0 & B & & & & \\ 0 & E & 0 & L & & & & \\ 0 & C & 0 & D & & & & \end{array} \right] \right\}.$$

and when $m+1 = 2r+1$, for a positive integer r, $\mathfrak{m}_{\bar{0}}$ and $\mathfrak{m}_{\bar{1}}$ have the following matrix forms

$$\mathfrak{m}_{\bar{0}} = \left\{ \left[\begin{array}{cccccc} 0 & 0 & -M^t & 0 & -L^t & \\ 0 & a & 0 & 0 & 0 & \\ L & 0 & A & 0 & B & \\ 0 & 0 & 0 & -a & 0 & \\ M & 0 & C & 0 & -A^t & \\ & & & & & D_{2n\times 2n} \end{array} \right] \in \mathfrak{gosp}(2r|2n) \Big| B = -B^T, C = -C^T \right\}$$

and

$$\mathfrak{m}_{\bar{1}} = \left\{ \begin{bmatrix} & & & & & j & J^t & j & J^t \\ & & & & & 0 & 0 & 0 & 0 \\ & & & & & -L^t & -D^t & U^t & B^t \\ & & & & & 0 & 0 & 0 & 0 \\ & & & & & -E^t & -C^t & X^t & A^t \\ j & 0 & X & 0 & U & & & & \\ J & 0 & A & 0 & B & & & & \\ -j & 0 & E & 0 & L & & & & \\ -J & 0 & C & 0 & D & & & & \end{bmatrix} \right\}.$$

Our goal is to show that $\mathfrak{m} \subseteq [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]$. To do this, it suffices to prove that $\mathfrak{m} \subseteq \sum[\mathfrak{g}_\gamma, \mathfrak{g}_{\gamma'}]$ where γ and γ' are two absolute roots with nonzero restriction on $\mathfrak{a}$ [16, Sec. III.10]. In order to prove this claim, let $x_\beta \in \mathfrak{g}_\beta$ where $\beta|_{\mathfrak{a}} = 0$. then for $m+1 = 2r$ explicitly we have

$$\begin{cases} \text{for } \boldsymbol{\varepsilon}_i - \boldsymbol{\varepsilon}_j, & \gamma := \boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_j \quad \gamma' := \boldsymbol{\varepsilon}_i - \boldsymbol{\varepsilon}_1 \\ \text{for } \boldsymbol{\varepsilon}_i + \boldsymbol{\varepsilon}_j, & \gamma := \boldsymbol{\varepsilon}_i - \boldsymbol{\varepsilon}_1 \quad \gamma' := \boldsymbol{\varepsilon}_j + \boldsymbol{\varepsilon}_1 \\ \text{for } \boldsymbol{\delta}_{\bar{k}} - \boldsymbol{\delta}_{\bar{l}}, & \gamma := \boldsymbol{\delta}_{\bar{k}} - \boldsymbol{\varepsilon}_1 \quad \gamma' := \boldsymbol{\varepsilon}_1 - \boldsymbol{\delta}_{\bar{l}} \\ \text{for } \boldsymbol{\delta}\bar{k} + \boldsymbol{\delta}_{\bar{l}}, & \gamma := \boldsymbol{\delta}_{\bar{k}} - \boldsymbol{\varepsilon}_1 \quad \gamma' := \boldsymbol{\varepsilon}_1 + \boldsymbol{\delta}_{\bar{l}} \\ \text{for } 2\boldsymbol{\delta}_{\bar{p}}, & \gamma := \boldsymbol{\varepsilon}_1 + \boldsymbol{\delta}_{\bar{p}} \quad \gamma' := \boldsymbol{\delta}_{\bar{p}} - \boldsymbol{\varepsilon}_1 \\ \text{for } \boldsymbol{\varepsilon}_{\bar{q}} - \boldsymbol{\delta}_{\bar{j}}, & \gamma := \boldsymbol{\varepsilon}_1 + \boldsymbol{\varepsilon}_q \quad \gamma' := -\boldsymbol{\varepsilon}_1 - \boldsymbol{\delta}_{\bar{p}} \end{cases},$$

where $2 \leqslant i < j \leqslant r$, $1 \leqslant k < l \leqslant n$, $1 \leqslant p \leqslant r$, and $1 \leqslant q \leqslant n$. Also, when $m+1 = 2r+1$ we have

$$\begin{cases} \text{for } \boldsymbol{\varepsilon}_i - \boldsymbol{\varepsilon}_j, & \gamma := \boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_j \quad \gamma' := \boldsymbol{\varepsilon}_i - \boldsymbol{\varepsilon}_1 \\ \text{for } \boldsymbol{\varepsilon}_i + \boldsymbol{\varepsilon}_j, & \gamma := \boldsymbol{\varepsilon}_i - \boldsymbol{\varepsilon}_1 \quad \gamma' := \boldsymbol{\varepsilon}_j + \boldsymbol{\varepsilon}_1 \\ \text{for } \boldsymbol{\delta}_{\bar{k}} - \boldsymbol{\delta}_{\bar{l}}, & \gamma := \boldsymbol{\delta}_{\bar{k}} - \boldsymbol{\varepsilon}_1 \quad \gamma' := \boldsymbol{\varepsilon}_1 - \boldsymbol{\delta}_{\bar{l}} \\ \text{for } \boldsymbol{\delta}\bar{k} + \boldsymbol{\delta}_{\bar{l}}, & \gamma := \boldsymbol{\delta}_{\bar{k}} - \boldsymbol{\varepsilon}_1 \quad \gamma' := \boldsymbol{\varepsilon}_1 + \boldsymbol{\delta}_{\bar{l}} \\ \text{for } 2\boldsymbol{\delta}_{\bar{p}}, & \gamma := \boldsymbol{\varepsilon}_1 + \boldsymbol{\delta}_{\bar{p}} \quad \gamma' := \boldsymbol{\delta}_{\bar{p}} - \boldsymbol{\varepsilon}_1 \\ \text{for } \boldsymbol{\varepsilon}_{\bar{q}} - \boldsymbol{\delta}_{\bar{j}}, & \gamma := \boldsymbol{\varepsilon}_1 + \boldsymbol{\varepsilon}_q \quad \gamma' := -\boldsymbol{\varepsilon}_1 - \boldsymbol{\delta}_{\bar{p}} \\ \text{for } \boldsymbol{\varepsilon}_{\bar{q}}, & \gamma := \boldsymbol{\varepsilon}_1 + \boldsymbol{\varepsilon}_q \quad \gamma' := -\boldsymbol{\varepsilon}_1 + \boldsymbol{\varepsilon}_q \\ \text{for } \boldsymbol{\delta}_{\bar{p}}, & \gamma := \boldsymbol{\varepsilon}_1 + \boldsymbol{\delta}_{\bar{p}} \quad \gamma' := -\boldsymbol{\varepsilon}_1 \end{cases},$$

where $3 \leqslant i < j \leqslant r$, $1 \leqslant k < l \leqslant n$, $1 \leqslant p \leqslant r$, and $1 \leqslant q \leqslant n$. One can see that every absolute root with zero restriction on $\mathfrak{a}$ can be written as $\gamma + \gamma'$ where

$$\gamma|_{\mathfrak{a}} = \alpha, \quad \gamma'|_{\mathfrak{a}} = -\alpha.$$

Thus, $x_\beta \in [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]$ which implies that $\mathfrak{m} \subseteq [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]$. ∎

Proposition 4.7.6. Let $\mathfrak{g} = \mathfrak{gosp}(2r+1, 2n)$. The centralizer of $\mathfrak{a}$ in $\mathfrak{p}$ is $\mathfrak{a}$.

Proof. For any arbitrary element

$$P_0 = \left[\begin{array}{c|cc|cc c}
0 & b & 0 & -b & 0 & \\
\hline
b & a & X & 0 & Y^t & \\
0 & Y & 0 & -Y & 0 & \\
\hline
-b & 0 & -X & -a & -Y^t & \\
0 & X^t & 0 & -X^t & 0 & \\
 & & & & & 0
\end{array}\right] \in \mathfrak{p}_{\bar{0}},$$

we have,

$$[a, P_0] = \left[\begin{array}{c|cc|cc c}
0 & -b & 0 & -b & 0 & \\
\hline
b & 0 & X & 0 & Y^t & \\
0 & -Y & 0 & -Y & 0 & \\
\hline
b & 0 & X & 0 & Y^t & \\
0 & -X^t & 0 & -X^t & 0 & \\
 & & & & & 0
\end{array}\right].$$

Thus $[a, P_0] = 0$ if and only if $b = c = 0$ and $X = U = V = Y^t = 0$. Also, for any arbitrary element

$$P_1 = \left[\begin{array}{c|cc|cc cc|cc}
 & & & & & 0 & 0 & 0 & 0 \\
\cline{6-9}
 & & & & & c & F^t & -a & -Y^t \\
 & & & & & 0 & 0 & 0 & 0 \\
\cline{6-9}
 & & & & & -c & -F^t & a & Y^t \\
 & & & & & 0 & 0 & 0 & 0 \\
0 & a & 0 & -a & 0 & & & & \\
0 & Y & 0 & -Y & 0 & & & & \\
\cline{1-5}
0 & c & 0 & -c & 0 & & & & \\
0 & F & 0 & -F & 0 & & & &
\end{array}\right] \in \mathfrak{p}_{\bar{1}},$$

we have

$$[a, P_1] = \left[\begin{array}{c|cc|cc cc|cc}
 & & & & & 0 & 0 & 0 & 0 \\
\cline{6-9}
 & & & & & c & F^t & -a & -Y^t \\
 & & & & & 0 & 0 & 0 & 0 \\
\cline{6-9}
 & & & & & c & F^t & -a & -Y^t \\
 & & & & & 0 & 0 & 0 & 0 \\
0 & -a & 0 & -a & 0 & & & & \\
0 & -Y & 0 & -Y & 0 & & & & \\
\cline{1-5}
0 & -c & 0 & -c & 0 & & & & \\
0 & -F & 0 & -F & 0 & & & &
\end{array}\right].$$

Thus $[a, P_1] = 0$ if and only if $b = c = 0$ and $F = M = Y^t = V^t = 0$. Hence, the centralizer of $\mathfrak{a}$ in $\mathfrak{p}$ is $\mathfrak{a}$. ∎

4.8 Examples of the interwoven pairs of $\mathfrak{gl}_2$ subalgebras in symmetric pair $(\mathfrak{s}_\alpha, \mathfrak{k}_\alpha)$

In Section 4.1, we introduced an interwoven pair of $\mathfrak{gl}_2$ subalgebras of $\mathfrak{g}$ as defined in Definition 4.2.1. In Sections 4.5, 4.6, and 4.7, we explicitly determined the $\mathfrak{s}_\alpha$ for Cases I, II, and III. In this section, we proceed to explore an example of interwoven $\mathfrak{gl}_2$ subalgebras for each pair $(\mathfrak{s}_\alpha, \mathfrak{k}_\alpha)$.

4.8.1 Examples of the interwoven pairs of $\mathfrak{gl}_2$ subalgebras in symmetric pairs $(\mathfrak{gl}_2, \mathfrak{so}_2)$, $(\mathfrak{gl}_4, \mathfrak{sp}_4)$, and $(\mathfrak{gl}_2 \oplus \mathfrak{gl}_2, \mathfrak{gl}_2)$

In this subsection we aim to specify elements k, x, y, h_1, h_2, e, f as the begining of this section for pairs $(\mathfrak{gl}_2, \mathfrak{so}_2)$, $(\mathfrak{gl}_4, \mathfrak{sp}_4)$, $(\mathfrak{gl}_2 \oplus \mathfrak{gl}_2, \mathfrak{gl}_2)$. First we concrete describe the $\mathfrak{gl}_2$ subalgebra spanned by h_1, h_2, e, f.

(i) In the case of $\mathfrak{g} = \mathfrak{gl}_2 = \mathfrak{k} \oplus \mathfrak{p}$ with respect to $\theta(x) := -x^T$, we consider the following standard $\mathfrak{gl}_2-$basis of $\mathfrak{gl}_2$,

$$e = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad f = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad h_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad h_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

Then

$$[h_1, e] = e, \ [h_1, f] = -f, \ [h_2, e] = -e, \ [h_2, f] = f, \ [e, f] = h_1 - h_2.$$

Thus

$$\mathfrak{k} = \{X \in \mathfrak{gl}_2 \,|\, \theta(X) = X\} = \mathrm{Span}\{e - f\} \cong \mathfrak{so}_2$$

and $\mathfrak{a} = \mathfrak{h} = \mathrm{Span}\{h_1, h_2\}$.
(ii) For the symmetric pair $(\mathfrak{g} = \mathfrak{gl}_4,\ \mathfrak{k} = \mathfrak{sp}_4)$ with respect to $\theta(x) := J_2 X^t J_2$, where

$$J_2 = \begin{bmatrix} 0 & I_2 \\ -I_2 & 0 \end{bmatrix}.$$

We use the usual basis of $\mathfrak{gl}_4$ which is the set of E_{ij} for $1 \leqslant i, j \leqslant 4$. In this case

$$\mathfrak{k} = \{X \in \mathfrak{gl}_4 | X = J_2 X^t J_2\} \cong \mathfrak{sp}_4$$

and if we write X as 4 blocks of size 2, then we have

$$\begin{aligned} \mathfrak{k} &= \left\{ \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \mathfrak{gl}_4 \;\middle|\; \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} -D^t & B^t \\ C^t & -A^t \end{bmatrix} \right\} \\ &= \left\{ \begin{bmatrix} A & B \\ C & -A^t \end{bmatrix} \in \mathfrak{gl}_4 \;\middle|\; B = B^t, C = C^t \right\}. \end{aligned}$$

Also,

$$\begin{aligned}\mathfrak{p} &= \left\{ \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \mathfrak{gl}_4 \;\middle|\; \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} D^t & -B^t \\ -C^t & A^t \end{bmatrix} \right\} \\ &= \left\{ \begin{bmatrix} A & B \\ C & A^t \end{bmatrix} \in \mathfrak{gl}_4 \;\middle|\; B + B^t = C + C^t = 0 \right\}.\end{aligned}$$

As in Example 3.1.1, we consider the Cartan subalgebra of $\mathfrak{g}$ to be

$$\mathfrak{h} = \{E_{ii}, i = 1, \cdots, 4\}.$$

According to 3.1, $\mathfrak{h} = \mathfrak{t} \oplus \mathfrak{a}$ such that $\mathfrak{t} = \mathfrak{h} \cap \mathfrak{k}$ and $\mathfrak{a} = \mathfrak{h} \cap \mathfrak{p}$ is a maximal abelian and self-centralizing subspace of $\mathfrak{p}$. Recall that this means that $[\mathfrak{a}, \mathfrak{a}] = 0$, and furthermore, for every subspace $\mathfrak{a} \subsetneq \mathfrak{a}' \subsetneq \mathfrak{p}$ we have $[\mathfrak{a}, \mathfrak{a}'] \neq 0$. Indeed $\mathfrak{a} = \text{span}\{h_1, h_2\}$ where

$$h_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad h_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

If we consider e, f to be

$$e = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad f = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

then we can see that e, f are two root vectors of $\mathfrak{a}$ and

$$\begin{aligned} [h_1, e] &= e, \quad [h_1, f] = -f, \\ [h_2, e] &= -e, \quad [h_2, f] = f. \end{aligned}$$

(iii) For the pair $(\mathfrak{gl}_2 \oplus \mathfrak{gl}_2, \mathfrak{gl}_2)$ with involution $\theta(x, y) = (y, x)$ consider the following standard $\mathfrak{gl}_2-$basis,

$$\begin{aligned} e &= \left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix} \right), \quad f = \left(\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix} \right), \\ h_1 &= \left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix} \right), \quad h_2 = \left(\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix} \right). \end{aligned}$$

Then

$$[h_1, e] = e, \; [h_1, f] = -f, \; [h_2, e] = -e, \; [h_2, f] = f, \; [e, f] = h_1 - h_2.$$

Also, one can see that

$$\mathfrak{k} = \left\{ (x, y) \in \mathfrak{gl}_2 \oplus \mathfrak{gl}_2 \,\middle|\, \theta(x, y) = (x, y) \right\} = \left\{ (x, y) \in \mathfrak{gl}_2 \oplus \mathfrak{gl}_2 \,\middle|\, (y, x) = (x, y) \right\} \cong \mathfrak{gl}_2$$

and

$$\begin{aligned}\mathfrak{p} &= \left\{ (x, y) \in \mathfrak{gl}_2 \oplus \mathfrak{gl}_2 \,\middle|\, \theta(x, y) = -(x, y) \right\} = \left\{ (x, y) \in \mathfrak{gl}_2 \oplus \mathfrak{gl}_2 \,\middle|\, (y, x) = (-x, -y) \right\} \\ &= \left\{ (x, -x) \,\middle|\, x \in \mathfrak{gl}_2 \right\}.\end{aligned}$$

Hence

$$\mathfrak{a} = \text{Span}\{\text{diag}(s) \oplus \text{diag}(-s), \; \middle| \; s \in \mathbb{C}^2\}.$$

Lemma 4.8.1. Set $k = i(e-f)$, $x = \mathrm{Ad}_g e$, and $y = \mathrm{Ad}_g f$, where Ad_g is the adjoint action of G on $\mathfrak{g}$, defined as the conjugation of matrices: $\mathrm{Ad}_g X = gXg^{-1}$ for $g \in G$ and $X \in \mathfrak{g}$. Let

$$g = M = \begin{bmatrix} i & -i \\ 1 & 1 \end{bmatrix} \in GL(2) \quad \text{if} \quad \mathfrak{g} = \mathfrak{gl}_2, \quad \text{and} \quad g = \begin{bmatrix} M & 0 \\ 0 & M \end{bmatrix} \in GL(4) \quad \text{if} \quad \mathfrak{g} = \mathfrak{gl}_4,$$
$$g = (M, M) \in GL(2) \times GL(2) \quad \text{if} \quad \mathfrak{g} = \mathfrak{gl}_2 \oplus \mathfrak{gl}_2.$$

Then $\{x, y, k, z\}$ is a standard $\mathfrak{gl}_2$-quadruple, i.e.,

$$[k, x] = 2x \ , \ [k, y] = -2y \ , \ [x, y] = k \ , \ [z, x] = [z, y] = [z, k] = 0.$$

Moreover, $k \in \mathfrak{k}$, $x, y \in \mathfrak{p}$ and $x + y = -h_1 + h_2$.

Proof. All of these statements can be checked directly as in Appendix A. For the reader's convenience we give explicit forms of the matrices for k, x, y, z.
For $\mathfrak{g} = \mathfrak{gl}(2)$:

$$k = \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix}, \quad x = \frac{1}{2}\begin{bmatrix} -1 & i \\ i & 1 \end{bmatrix}, \quad y = \frac{1}{2}\begin{bmatrix} -1 & -i \\ -i & 1 \end{bmatrix}, \quad z = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_{2\times 2}$$

and for $\mathfrak{g} = \mathfrak{gl}(4)$,

$$k = \begin{bmatrix} 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \end{bmatrix}, \quad x = \frac{1}{2}\begin{bmatrix} -1 & i & 0 & 0 \\ i & 1 & 0 & 0 \\ 0 & 0 & -1 & i \\ 0 & 0 & i & 1 \end{bmatrix}, \quad y = \frac{-1}{2}\begin{bmatrix} -1 & -i & 0 & 0 \\ -i & 1 & 0 & 0 \\ 0 & 0 & -1 & -i \\ 0 & 0 & -i & 1 \end{bmatrix}, \quad z = I_{4\times 4}.$$

For $\mathfrak{g} = \mathfrak{gl}_2 \oplus \mathfrak{gl}_2$,

$$k = \left(\begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix}, \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix}\right), \quad x = \left(\frac{1}{2}\begin{bmatrix} -1 & i \\ i & 1 \end{bmatrix}, \frac{-1}{2}\begin{bmatrix} -1 & i \\ i & 1 \end{bmatrix}\right)$$
$$y = \left(\frac{1}{2}\begin{bmatrix} -1 & -i \\ -i & 1 \end{bmatrix}, \frac{-1}{2}\begin{bmatrix} -1 & -i \\ -i & 1 \end{bmatrix}\right), \quad \text{and} \quad z = (I_{2\times 2}, I_{2\times 2}).$$

∎

4.8.2 An interwoven pairs of $\mathfrak{gl}_2$ subalgebras in symmetric pair $(\mathfrak{gosp}(m+1, 2n), \mathfrak{osp}(m, 2n))$

Let $m = 2r$ for an integer r. Recall that

$$\mathfrak{a} = \mathrm{Span}\,\{a, a'\}$$

where

$$a' = I_{2r+2n} \quad \text{and} \quad a = \mathrm{diag}(\bar{s}, -\bar{s}, 0_{2n} \quad \text{for} \quad \text{where} \quad \bar{s} := (2, 0, \cdots, 0) \in \mathbb{C}^r,$$

and

$$h_1 = \frac{1}{2}(a + a'), \quad \text{and} \quad h_2 = \frac{1}{2}(a' - a).$$

Set

$$e = \left[\begin{array}{cc:cc} 0 & X & 0 & U \\ 0 & 0 & -U^t & 0 \\ \hdashline 0 & 0 & 0 & 0 \\ 0 & 0 & -X^t & 0 \\ & & & & 0 \end{array}\right], \quad f = \left[\begin{array}{cc:cc} 0 & 0 & 0 & 0 \\ -Y^t & 0 & 0 & 0 \\ \hdashline 0 & V & 0 & Y \\ -V^t & 0 & 0 & 0 \\ & & & & 0 \end{array}\right],$$

where $X, U, Y, V \in \mathrm{Mat}_{1\times(r-1)}$. Then e, f, and a satisfy the following relations:

$$[a, e] = 2e, \quad [a, f] = -2f.$$

Suppose that

$$X = U = \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix}, Y = V = -X,$$

then

$$[e, f] = a.$$

Now, let

$$k := i(e - f) = \left[\begin{array}{cc:cc} 0 & iX & 0 & iX \\ -iX^t & 0 & -iX^t & 0 \\ \hdashline 0 & iX & 0 & iX \\ -iX^t & 0 & -iX^t & 0 \\ & & & & 0 \end{array}\right].$$

Then $k \in \mathfrak{k}$. Set matrices x, y, to be:

$$x := \frac{1}{2}(ie + if - a) = \frac{1}{2}\left[\begin{array}{cc:cc} -2 & iX & 0 & iX \\ iX^t & 0 & -iX^t & 0 \\ \hdashline 0 & -iX & 2 & -iX \\ iX^t & 0 & -iX^t & 0 \\ & & & & 0 \end{array}\right],$$

and

$$y := \frac{1}{2}(-ie - if - a) = \frac{1}{2}\left[\begin{array}{cc:cc} -2 & -iX & 0 & -iX \\ -iX^t & 0 & iX^t & 0 \\ \hdashline 0 & iX & 2 & iX \\ -iX^t & 0 & iX^t & 0 \\ & & & & 0 \end{array}\right].$$

Therefore,

$$[k, x] = 2x, \quad [k, y] = -2y, \quad [x, y] = k.$$

Thus, $\{e, f, h_1, h_2, k, x, y, z = a'\}$ is a $\mathfrak{gl}_2$-interwoven and hence by Proposition 4.2.5, for every $\lambda \in \mathcal{A}(\mathfrak{g}, \mathfrak{k})$,

$$\lambda(h_1^r(h_2)^s) = \lambda(h_1^s(h_2)^r) \text{ for } r, s \geq 0$$

where $h_1 + h_2 = a'$ and $h_1 - h_2 = a$.

Now, let $m + 1 = 2r + 1$ for a nonzero positive integer r. Recall that

$$a' = I_{2r+2n} \quad \text{and} \quad a = \operatorname{diag}(0, \bar{s}, -\bar{s}, 0_{2n} \quad \text{for} \quad \bar{s} := (2, 0, \cdots, 0) \in \mathbb{C}^r.$$

Root vectors of $\mathfrak{a}$ are $e, f \in \mathfrak{g}_{\bar{0}}$;

$$e = \begin{bmatrix} \begin{array}{c|c|c|cc} 0 & 0 & 0 & -b & 0 \\ \hline b & 0 & X & 0 & U \\ 0 & 0 & 0 & -U^t & 0 \\ \hline 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -X^t & 0 \end{array} & \\ & D \end{bmatrix}, \quad f = \begin{bmatrix} \begin{array}{c|cc|c|c} 0 & -c & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 \\ 0 & -Y^t & 0 & 0 & 0 \\ \hline c & 0 & V & 0 & Y \\ 0 & -V^t & 0 & 0 & 0 \end{array} & \\ & D \end{bmatrix}.$$

Assume,

$$X = U = \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix}, \quad Y = V = -X.$$

Thus,

$$[a, e] = 2e, \quad [a, f] = -2f, \quad [e, f] = a.$$

Set

$$k := i(e - f) = \begin{bmatrix} \begin{array}{c|cc|cc} 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & iX & 0 & iX \\ 0 & -iX^t & 0 & -iX^t & 0 \\ \hline 0 & 0 & iX & 0 & iX \\ 0 & -iX^t & 0 & -iX^t & 0 \end{array} & \\ & 0 \end{bmatrix}.$$

Then $k \in \mathfrak{k}$ and consider new matrices:

$$x := \frac{1}{2}(ie + if - a) = \frac{1}{2} \begin{bmatrix} \begin{array}{c|cc|cc} 0 & 0 & 0 & 0 & 0 \\ \hline 0 & -2 & iX & 0 & iX \\ 0 & iX^t & 0 & -iX^t & 0 \\ \hline 0 & 0 & -iX & 2 & -iX \\ 0 & iX^t & 0 & -iX^t & 0 \end{array} & \\ & 0 \end{bmatrix},$$

and

$$y := \frac{1}{2}(-ie - if - a) = \frac{1}{2}\left[\begin{array}{c:cc:cc} 0 & 0 & 0 & 0 & 0 & \\ \hdashline 0 & -2 & -iX & 0 & -iX & \\ 0 & -iX^t & 0 & iX^t & 0 & \\ \hdashline 0 & 0 & iX & 2 & iX & \\ 0 & -iX^t & 0 & iX^t & 0 & \\ & & & & & 0 \end{array}\right].$$

Therefore, simple computation shows that

$$[k, x] = 2x, \quad [k, y] = -2y, \quad [x, y] = k,$$

and $x + y = -a$. Thus, $\{e, f, h_1, h_2, k, x, y, z = a'\}$ is a $\mathfrak{gl}_2$-interwoven and hence by Proposition 4.2.5, for every $\lambda \in \mathcal{A}(\mathfrak{g}, \mathfrak{k})$,

$$\lambda(h_1^r h_2^s) = \lambda(h_1^s h_2^r) \text{ for } r, s \geq 0.$$

Chapter 5

Supersymmetry

The primary object of study in this chapter and the next chapter is a generating function Φ_λ associated to any $\lambda \in \mathcal{A}(\mathfrak{g}, \mathfrak{k})$ which will play the role of a spherical function on the associated supersymmetric space, in an algebraic way. Henceforth we denote any product of $x_i^{a_i}$ for $1 \leqslant i \leqslant m$ by $\prod_{i=1}^{m} x_i^{a_i}$. Let $h_1, \ldots, h_m, h_{\bar{1}}, \ldots, h_{\bar{n}}$ be the associated elements of $\mathfrak{a}$ defined in Sections 4.5, 4.6, and 4.7. For any $\lambda \in S(\mathfrak{a})^*$ we consider the formal power series

$$\Phi_\lambda := \sum_{\vec{a} \in (\mathbb{Z}^+)^{(m+n)}} \frac{\lambda[\vec{a}]}{\vec{a}!} \prod_{i \in I_{\bar{0}}} x_i^{a_i} \prod_{\bar{j} \in I_{\bar{1}}} y_{\bar{j}}^{a_{\bar{j}}},$$

where $\vec{a} := (a_1, \cdots, a_m, a_{\bar{1}}, \cdots, a_{\bar{n}})$, $\lambda[\vec{a}] := \lambda(h_1^{a_1} \cdots h_m^{a_n} h_{\bar{1}}^{a_{\bar{1}}} \cdots h_{\bar{n}}^{a_{\bar{n}}})$, and $\vec{a}! := \prod_{i \in I_{\bar{0}}} a_i! \prod_{\bar{j} \in I_{\bar{1}}} a_{\bar{j}}!$. Note that $\mathbb{Z}^+$ is the set of non-negative integers and the formal power series Φ_λ is symmetric in $x_1, \ldots, x_n$ and in $y_1, \ldots, y_m$. In this chapter, we explore the concept of supersymmetry in distinct Cases I and II in Table 4.1. In Case III, supersymmetry is not expected because both variables are even. The condition of supersymmetry will be identified by considering the differential equation obtained in the next chapter for each case. Supersymmetry is expected to take the form

$$\left(\frac{\partial}{\partial x_i} + k \frac{\partial}{\partial y_j} \right) \Phi_\lambda = 0,$$

for each hyperplane $x_i - y_j = 0$, where $i = 1, \ldots, n$, $j = 1, \ldots, m$. Here, k is a parameter derived from the restricted root system, as described in [37, Sec. 4]. Specifically, $k = 1$ in Case I, and in this case, supersymmetry implies that Φ_λ vanishes when substituting $x_i = y_j = t$. Furthermore, $k = \frac{1}{2}$ in Case II. Notably, this parameter value aligns with the one appearing in the differential equation discussed in Chapter 6.

5.1 Supersymmetry for Case I of Table 4.1

Let $I_{\bar{0}} = \{1, \cdots, m\}$ and $I_{\bar{1}} = \{\bar{1}, \bar{2}, \cdots, \bar{n}\}$. Let $\{E_{ij} \mid i, j \in I = I_{\bar{0}} \sqcup I_{\bar{1}}\}$ be the basis for $\mathfrak{g} = \mathfrak{gl}(m|n) \oplus \mathfrak{gl}(m|n)$ consisting of matrix units. According to Section 4.5 we have

$$\mathfrak{a} = \operatorname{Span}\left\{ h_i, h_{\bar{j}} \,\middle|\, 1 \leqslant i \leqslant m,\ 1 \leqslant j \leqslant n \right\}$$

where $h_i = (E_{i,i}, -E_{i,i})$ and $h_{\bar{j}} = (E_{\bar{j},\bar{j}}, -E_{\bar{j},\bar{j}})$.
To verify the supersymmetry in this case we need to prove

$$\left(\frac{\partial}{\partial x_i} + \frac{\partial}{\partial y_j}\right)\Phi_\lambda = 0.$$

But before that we need the following lemma which proves the supersymmetry for this case when $m = n = 1$. In this case,

$$\mathfrak{a} = \operatorname{Span}\{h_1 := (E_{11}, -E_{11}), h_{\bar{1}} := (E_{\bar{1}\bar{1}}, -E_{\bar{1}\bar{1}})\}.$$

Lemma 5.1.1. Let $\lambda \in \mathcal{A}(\mathfrak{gl}(1|1) \oplus \mathfrak{gl}(1|1), \mathfrak{gl}(1|1))$. Then,

$$\lambda\left((h_1 + h_{\bar{1}})^n\right) = 0 \quad \text{for} \quad n \geqslant 1.$$

Proof. Set

$$\begin{aligned} e_{1\bar{1}} &= (E_{1\bar{1}}, E_{\bar{1}1}), \quad e_{\bar{1}1} = (E_{\bar{1}1}, E_{1\bar{1}}), \quad \tilde{h}_1 = (E_{11}, E_{11}), \\ \tilde{e}_{1\bar{1}} &= (E_{1\bar{1}}, -E_{\bar{1}1}), \quad \tilde{e}_{\bar{1}1} = (E_{\bar{1}1}, -E_{1\bar{1}}), \quad \tilde{h}_{\bar{1}} = (E_{\bar{1}\bar{1}}, E_{\bar{1}\bar{1}}). \end{aligned}$$

Then $\{e_{1\bar{1}}, e_{\bar{1}1}, h_1, h_{\bar{1}}, \tilde{e}_{1\bar{1}}, \tilde{e}_{1\bar{1}}, \tilde{h}_1, \tilde{h}_{\bar{1}}\}$ is a basis for $\mathfrak{gl}(1|1) \oplus \mathfrak{gl}(1|1)$. We have,

$$(h_1 + h_{\bar{1}})^n = \sum_{i=1}^{n} \binom{n}{i} z^i (z')^{n-i},$$

where $z = (E_{11} + E_{\bar{1}\bar{1}}, 0)$ and $z' = (0, E_{11} + E_{\bar{1}\bar{1}})$. Therefore, it suffices to prove $\lambda(z^r(z')^s) = 0$ for integers $r, s \geqslant 1$. Consider

$$x = \frac{1}{2}(e_{1\bar{1}} + e_{\bar{1}1} + \tilde{e}_{1\bar{1}} - \tilde{e}_{\bar{1}1}) \in \mathfrak{k}, \quad y = \frac{1}{2}(e_{\bar{1}1} + \tilde{e}_{\bar{1}1}) \in U(\mathfrak{g}), \quad y' = \frac{1}{2}(e_{1\bar{1}} - \tilde{e}_{1\bar{1}}) \in U(\mathfrak{g}).$$

By straightforward computation we have

$$[x, y] = z, \quad [x, y'] = z'$$

which implies that $z, z' \in \mathcal{I}$. On the other hand z and z' are in the center of $U(\mathfrak{g})$. Therefore, for some $k \in \mathfrak{k}$ and $u \in U(\mathfrak{g})$ we have

$$\begin{aligned} z^r(z')^s &= z\left(z^{r-1}(z')^s\right) = \sum (ku + uk)\left(z^{r-1}(z')^s\right) = \sum \left(kuz^{r-1}(z')^s + ukz^{r-1}(z')^s\right) \\ &= \sum \left(kuz^{r-1}(z')^s + uz^{r-1}(z')^s k\right) \in \mathfrak{k}U(\mathfrak{g}) + U(\mathfrak{g})\mathfrak{k} = \mathcal{I}. \end{aligned}$$

∎

Proposition 5.1.2. Let $\lambda \in \mathcal{A}(\mathfrak{g}, \mathfrak{k})$ and $I = I_{\bar{0}} \cup I_{\bar{1}}$. Then,

$$\left(\frac{\partial}{\partial x_i} + \frac{\partial}{\partial y_{\bar{j}}}\right)\Phi_\lambda = \sum_{p \geqslant 1} \sum_{k=0}^{p} \frac{1}{A} \lambda\left(\prod_{\substack{l \,\in\, I,\, a_l \,\in\, \mathbb{Z}^+ \\ l \neq i, \bar{j}}} h_l^{a_l} (h_i + h_{\bar{j}})^p\right) x_i^k y_{\bar{j}}^{p-k} \prod_{\substack{1 \leqslant r \neq i \leqslant m \\ a_r \,\in\, \mathbb{Z}^+}} x_r^{a_r} \prod_{\substack{1 \leqslant s \neq j \leqslant n \\ a_{\bar{s}} \,\in\, \mathbb{Z}^+}} y_{\bar{s}}^{a_{\bar{s}}},$$

where

$$A = \prod_{\substack{1 \leqslant r \neq i \leqslant m \\ a_r \,\in\, \mathbb{Z}^+}} a_r! \prod_{\substack{1 \leqslant s \neq j \leqslant n \\ a_{\bar{s}} \,\in\, \mathbb{Z}^+}} a_{\bar{s}}!.$$

Proof. By straightforward computation we have

$$\left(\frac{\partial}{\partial x_i}+\frac{\partial}{\partial y_{\bar{j}}}\right)\Phi_\lambda=\sum_{\vec{a}\in(\mathbb{Z}^+)^{(m+n)}}\frac{\lambda[\vec{a}]}{\vec{a}!}\left((a_i x^{a_i-1}y^{a_{\bar{j}}}+a_{\bar{j}}x^{a_i}y^{a_{\bar{j}}-1})\prod_{\substack{1\leqslant r\neq i\leqslant m\\ a_r\in\mathbb{Z}^+}}x_r^{a_r}\prod_{\substack{1\leqslant s\neq j\leqslant n\\ a_{\bar{s}}\in\mathbb{Z}^+}}y_{\bar{s}}^{a_{\bar{s}}}\right)$$

$$=\sum_{p\geqslant 0}\sum_{k=0}^{p}\frac{1}{A}\lambda\left(\prod_{\substack{l\in I,\,a_l\in\mathbb{Z}^+\\ l\neq i,\bar{j}}}h_l^{a_l}h_i^{k+1}h_{\bar{j}}^{p-k}\right)\frac{x_i^k}{k!}\frac{y_{\bar{j}}^{p-k}}{(p-k)!}\prod_{\substack{1\leqslant r\neq i\leqslant m\\ a_r\in\mathbb{Z}^+}}x_r^{a_r}\prod_{\substack{1\leqslant s\neq j\leqslant n\\ a_{\bar{s}}\in\mathbb{Z}^+}}y_{\bar{s}}^{a_{\bar{s}}}$$

$$+\sum_{p\geqslant 0}\sum_{k=0}^{p}\frac{1}{A}\lambda\left(\prod_{\substack{l\in I,\,a_l\in\mathbb{Z}^+\\ l\neq i,\bar{j}}}h_l^{a_l}h_i^{k}h_{\bar{j}}^{p-k+1}\right)\frac{x_i^k}{k!}\frac{y_{\bar{j}}^{p-k}}{(p-k)!}\prod_{\substack{1\leqslant r\neq i\leqslant m\\ a_r\in\mathbb{Z}^+}}x_r^{a_r}\prod_{\substack{1\leqslant s\neq j\leqslant n\\ a_{\bar{s}}\in\mathbb{Z}^+}}y_{\bar{s}}^{a_{\bar{s}}}$$

$$=\sum_{p\geqslant 0}\sum_{k=0}^{p}\frac{1}{A}\lambda\left(\prod_{\substack{l\in I,\,a_l\in\mathbb{Z}^+\\ l\neq i,\bar{j}}}h_l^{a_l}(h_i^{k+1}h_{\bar{j}}^{n-k}+h_i^{k}h_{\bar{j}}^{p-k+1})\right)\binom{p}{k}x_i^k y_{\bar{j}}^{p-k}\prod_{\substack{1\leqslant r\neq i\leqslant m\\ a_r\in\mathbb{Z}^+}}x_r^{a_r}\prod_{\substack{1\leqslant s\neq j\leqslant p\\ a_{\bar{s}}\in\mathbb{Z}^+}}y_{\bar{s}}^{a_{\bar{s}}}$$

$$=\sum_{p\geqslant 0}\sum_{k=0}^{p}\frac{1}{A}\lambda\left(\prod_{\substack{l\in I,\,a_l\in\mathbb{Z}^+\\ l\neq i,\bar{j}}}h_l^{a_l}(h_i+h_{\bar{j}})(h_i+h_{\bar{j}})^p\right)x_i^k y_{\bar{j}}^{p-k}\prod_{\substack{1\leqslant r\neq i\leqslant m\\ a_r\in\mathbb{Z}^+}}x_r^{a_r}\prod_{\substack{1\leqslant s\neq j\leqslant p\\ a_{\bar{s}}\in\mathbb{Z}^+}}y_{\bar{s}}^{a_{\bar{s}}}$$

$$=\sum_{p\geqslant 1}\sum_{k=0}^{p}\frac{1}{A}\lambda\left(\prod_{\substack{l\in I,\,a_l\in\mathbb{Z}^+\\ l\neq i,\bar{j}}}h_l^{a_l}(h_i+h_{\bar{j}})^p\right)x_i^k y_{\bar{j}}^{p-k}\prod_{\substack{1\leqslant r\neq i\leqslant m\\ a_r\in\mathbb{Z}^+}}x_r^{a_r}\prod_{\substack{1\leqslant s\neq j\leqslant n\\ a_{\bar{s}}\in\mathbb{Z}^+}}y_{\bar{s}}^{a_{\bar{s}}}.$$ ∎

Therefore, to verify the supersymmetry in this case, it suffices to prove that

$$\lambda\left(\prod_{\substack{l\in I,\,a_l\in\mathbb{Z}^+\\ l\neq i,\bar{j}}}h_l^{a_l}(h_i+h_{\bar{j}})^p\right)=0,\quad\text{for}\quad p\geqslant 1.$$

The following proposition will accomplish this goal.

Proposition 5.1.3. Let $\lambda\in\mathcal{A}(\mathfrak{g},\mathfrak{k})$. Then

$$\lambda\left(\prod_{\substack{l\in I,\,a_l\in\mathbb{Z}^+\\ l\neq i,\bar{j}}}h_l^{a_l}(h_i+h_{\bar{j}})^p\right)=0,\quad\text{for}\quad p\geqslant 1.$$

Proof. Set

$$e_{i\bar{j}}=(E_{i\bar{j}},E_{\bar{j}i}),\quad e_{\bar{j}i}=(E_{\bar{j}i},E_{i\bar{j}}),\quad \tilde{h}_i=(E_{ii},E_{ii}),$$

$$\tilde{e}_{i\bar{j}} = (E_{i\bar{j}}, -E_{\bar{j}i}), \quad \tilde{e}_{\bar{j}i} = (E_{\bar{j}i}, -E_{i\bar{j}}), \quad \tilde{h}_{\bar{j}} = (E_{\bar{j}\bar{j}}, E_{\bar{j}\bar{j}}),$$

and let $\mathfrak{g}'$ be the $\mathfrak{gl}(1|1) \oplus \mathfrak{gl}(1|1)$-subalgebra of $\mathfrak{g}$ generated by $\{e_{i\bar{j}}, e_{\bar{j}i}, h_i, h_{\bar{j}}, \tilde{e}_{i\bar{j}}, \tilde{e}_{i\bar{j}}, \tilde{h}_i, \tilde{h}_{\bar{j}}\}$ and $\mathfrak{k}' = \mathfrak{g}' \cap \mathfrak{k}$. Assume that $\lambda \in \mathcal{A}(\mathfrak{g}, \mathfrak{k})$. Set

$$\lambda' = \lambda\Big|_{U(\mathfrak{g}')}.$$

Then $\lambda' \in \mathcal{A}(\mathfrak{g}', \mathfrak{k}')$. Identify h_i and $h_{\bar{j}}$ respectively to be h_1 and $h_{\bar{1}}$. Since the h_ls for $l \neq i, \bar{j}$ are in the centralizer of $\mathfrak{g}'$, to verify this assertion, it suffices to prove that $\lambda\left((h_i + h_{\bar{j}})^p\right) = 0$ or equivalently $(h_i + h_{\bar{j}})^p \in \mathcal{I}$. But by Lemma 5.1.1, we have

$$\lambda'\left((h_i + h_{\bar{j}})^p\right) = 0.$$

Therefore,

$$\left(h_i + h_{\bar{j}}\right)^p \in \mathfrak{k}'U(\mathfrak{g}') + U(\mathfrak{g}')\mathfrak{k}' \subseteq \mathfrak{k}U(\mathfrak{g}) + U(\mathfrak{g})\mathfrak{k} = \mathcal{I}.$$

This completes the proof. ∎

5.2 Supersymmetry for Case II of Table 4.1

Let $I_{\bar{0}} = \{1, \cdots, m\}$ and $I_{\bar{1}} = \{\bar{1}, \bar{2}, \cdots, \bar{n}\}$. Let

$$\{E_{ij}, E_{i,\bar{j}}, E_{\bar{i},j}, E_{i,\bar{j}+n}, E_{\bar{i}+n,j}, E_{\bar{i},\bar{j}}, E_{\bar{i},\bar{j}+n}, E_{\bar{i}+n,\bar{j}}, E_{\bar{i}+n,\bar{j}+n} \,|\, i, j \in I_{\bar{0}}, \quad \bar{i}, \bar{j} \in I_{\bar{1}}\}$$

be the basis for $\mathfrak{g} = \mathfrak{gl}(m|2n) = \mathfrak{k} \oplus \mathfrak{p}$ consisting of matrix units. According to Section 4.6 we have

$$\mathfrak{k} = \left\{ \begin{bmatrix} A_{m\times m} & B_{m\times 2n} \\ C_{2n\times m} & D_{2n\times 2n} \end{bmatrix} \in \mathfrak{gl}(m|2n) \,|\, A = -A^t, D = JD^tJ, C = -JB^t \right\}$$

where J is given in (4.7.2) and

$$\mathfrak{p} = \left\{ \begin{bmatrix} A_{m\times m} & B_{m\times 2n} \\ C_{2n\times m} & D_{2n\times 2n} \end{bmatrix} \in \mathfrak{gl}(m|2n) \,|\, A = A^t, D = -JD^tJ, C = JB^t \right\}.$$

Thus,

$$\mathfrak{a} = \mathrm{Span}\left\{h_i, h_{\bar{j}} \,|\, 1 \leq i \leq m, 1 \leq j \leq n\right\}$$

where

$$h_i := E_{i,i}, \quad h_{\bar{j}} := E_{\bar{j},\bar{j}} + E_{\bar{j}+n,\bar{j}+n}.$$

To verify the supersymmetry in this case we need to prove

$$\left(\frac{\partial}{\partial x_j} + \frac{1}{2}\frac{\partial}{\partial y_j}\right)\Phi_\lambda = 0.$$

But before that we need the following lemma which proves the supersymmetry for this case when $m = n = 1$. In this case,

$$\mathfrak{a} = \mathrm{Span}\left\{ h_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, h_{\bar{1}} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right\}.$$

Lemma 5.2.1. Let $\lambda \in \mathcal{A}(\mathfrak{gl}(1|2), \mathfrak{osp}(1|2))$. Then,

$$\lambda(2h_1 + h_{\bar{1}}) = 0.$$

Proof. Consider two arbitrary elements $v_k \in \mathfrak{k}_{\bar{1}}$ and $v_p \in \mathfrak{p}_{\bar{1}}$ as follows

$$v_k = \begin{bmatrix} 0 & b & c \\ -c & 0 & 0 \\ b & 0 & 0 \end{bmatrix}, \quad v_p = \begin{bmatrix} 0 & b' & c' \\ c' & 0 & 0 \\ -b' & 0 & 0 \end{bmatrix}$$

where $b, c, b', c' \in \mathbb{C}$. Then we have

$$[v_k, v_p] = (bc' - b'c) \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = (bc' - b'c)(2h_1 + h_{\bar{1}}).$$

Consider selecting $b, c, b', c' \in \mathbb{C}$ such that $bc' - b'c \neq 0$. This condition ensures that

$$2h_1 + h_{\bar{1}} \in \mathcal{I}.$$ ∎

Proposition 5.2.2. Let $\lambda \in \mathcal{A}(\mathfrak{g}, \mathfrak{k})$ and $I = I_{\bar{0}} \cap I_{\bar{1}}$. Then,

$$\left(\frac{\partial}{\partial x_i} + \frac{1}{2}\frac{\partial}{\partial y_{\bar{j}}}\right)\Phi_\lambda$$
$$= \sum_{p\geqslant 0}\sum_{k=0}^{p} \frac{1}{A}\lambda\left(\prod_{\substack{l\,\in\, I,\, a_l\,\in\,\mathbb{Z}^+\\ l\,\neq\, i,\,\bar{j}}} h_r^{a_r}(h_i + \frac{1}{2}h_{\bar{j}})(h_i + h_{\bar{j}})^p\right) x_i^k y_{\bar{j}}^{p-k} \prod_{\substack{1\,\leqslant\, r\,\neq\, i\,\leqslant\, m\\ a_r\,\in\,\mathbb{Z}^+}} x_r^{a_r} \prod_{\substack{1\,\leqslant\, s\,\neq\, j\,\leqslant\, n\\ a_{\bar{s}}\,\in\,\mathbb{Z}^+}} y_{\bar{s}}^{a_{\bar{s}}},$$

where

$$A = \prod_{\substack{1\,\leqslant\, r\,\neq\, i\,\leqslant\, m\\ a_r\,\in\,\mathbb{Z}^+}} a_r! \prod_{\substack{1\,\leqslant\, s\,\neq\, j\,\leqslant\, n\\ a_{\bar{s}}\,\in\,\mathbb{Z}^+}} a_{\bar{s}}!.$$

Proof. By straightforward computation we have

$$\left(\frac{\partial}{\partial x_i} + \frac{1}{2}\frac{\partial}{\partial y_{\bar{j}}}\right)\Phi_\lambda = \sum_{\vec{a}\in(\mathbb{Z}^+)^{(m+n)}} \frac{\lambda[\vec{a}]}{\vec{a}!}\left((a_i x^{a_i-1} y^{a_{\bar{j}}} + \frac{1}{2}a_{\bar{j}} x^{a_i} y^{a_{\bar{j}}-1}) \prod_{\substack{1\,\leqslant\, r\,\neq\, i\,\leqslant\, m\\ a_r\,\in\,\mathbb{Z}^+}} x_r^{a_r} \prod_{\substack{1\,\leqslant\, s\,\neq\, j\,\leqslant\, n\\ a_{\bar{s}}\,\in\,\mathbb{Z}^+}} y_{\bar{s}}^{a_{\bar{s}}}\right)$$
$$= \sum_{p\geqslant 0}\sum_{k=0}^{p} \frac{1}{A}\lambda\left(\prod_{\substack{l\,\in\, I,\, a_l\,\in\,\mathbb{Z}^+\\ l\,\neq\, i,\,\bar{j}}} h_l^{a_l} h_i^{k+1} h_{\bar{j}}^{p-k}\right) \frac{x_i^k}{k!}\frac{y_{\bar{j}}^{p-k}}{(p-k)!} \prod_{1\leqslant r\neq i\leqslant m} x_r^{a_r} \prod_{1\leqslant s\neq j\leqslant n} y_{\bar{s}}^{a_{\bar{s}}}$$
$$+ \sum_{p\geqslant 0}\sum_{k=0}^{p} \frac{1}{2A}\lambda\left(\prod_{\substack{l\,\in\, I,\, a_l\,\in\,\mathbb{Z}^+\\ l\,\neq\, i,\,\bar{j}}} h_l^{a_l} h_i^{k} h_{\bar{j}}^{p-k+1}\right) \frac{x_i^k}{k!}\frac{y_{\bar{j}}^{p-k}}{(p-k)!} \prod_{\substack{1\,\leqslant\, r\,\neq\, i\,\leqslant\, m\\ a_r\,\in\,\mathbb{Z}^+}} x_r^{a_r} \prod_{\substack{1\,\leqslant\, s\,\neq\, j\,\leqslant\, n\\ a_{\bar{s}}\,\in\,\mathbb{Z}^+}} y_{\bar{s}}^{a_{\bar{s}}}$$

$$= \sum_{p\geqslant 0}\sum_{k=0}^{p}\frac{1}{A}\lambda\left(\prod_{\substack{l\,\in\, I,\, a_l\,\in\,\mathbb{Z}^+\\ l\,\neq\, i,\bar{j}}} h_l^{a_l}(h_i^{k+1}h_{\bar{j}}^{p-k}+\frac{1}{2}h_i^k h_{\bar{j}}^{p-k+1})\right)\binom{n}{k}x_i^k y_{\bar{j}}^{p-k}\prod_{\substack{1\,\leqslant\, r\,\neq\, i\,\leqslant\, m\\ a_r\,\in\,\mathbb{Z}^+}} x_r^{a_r}\prod_{\substack{1\,\leqslant\, s\,\neq\, j\,\leqslant\, n\\ a_{\bar{s}}\,\in\,\mathbb{Z}^+}} y_{\bar{s}}^{a_{\bar{s}}}$$

$$= \sum_{p\geqslant 0}\sum_{k=0}^{p}\frac{1}{A}\lambda\left(\prod_{\substack{l\,\in\, I,\, a_l\,\in\,\mathbb{Z}^+\\ l\,\neq\, i,\bar{j}}} h_l^{a_l}(h_i+\frac{1}{2}h_{\bar{j}})(h_i+h_{\bar{j}})^p\right)x_i^k y_{\bar{j}}^{p-k}\prod_{\substack{1\,\leqslant\, r\,\neq\, i\,\leqslant\, m\\ a_r\,\in\,\mathbb{Z}^+}} x_r^{a_r}\prod_{\substack{1\,\leqslant\, s\,\neq\, j\,\leqslant\, n\\ a_{\bar{s}}\,\in\,\mathbb{Z}^+}} y_{\bar{s}}^{a_{\bar{s}}}. \qquad \blacksquare$$

Therefore, to verify the supersymmetry in this case, it suffices to prove that

$$\lambda\left(\prod_{\substack{l\,\in\, I,\, a_l\,\in\,\mathbb{Z}^+\\ l\,\neq\, i,\bar{j}}} h_l^{a_l}(h_i+\frac{1}{2}h_{\bar{j}})(h_i+h_{\bar{j}})^p\right)=0, \quad \text{for} \quad p\geqslant 0,$$

or equivalently

$$\prod_{\substack{l\,\in\, I,\, a_l\,\in\,\mathbb{Z}^+\\ l\,\neq\, i,\bar{j}}} h_l^{a_l}(h_i+\frac{1}{2}h_{\bar{j}})(h_i+h_{\bar{j}})^p\in\mathcal{I}, \quad \text{for} \quad p\geqslant 0,$$

where $\mathcal{I}=\mathfrak{k}U(\mathfrak{g})+U(\mathfrak{g})\mathfrak{k}$. It will be accomplished by the following proposition.

Proposition 5.2.3. Set $p_{i\bar{j}}=2h_i+h_{\bar{j}}, \quad z_{i\bar{j}}=h_i+h_{\bar{j}}$. Then,

$$\forall\, r\geqslant 0, \quad p_{i\bar{j}}z_{i\bar{j}}^r\prod_{\substack{r\,\in\, I\\ r\,\neq\, i,\bar{j}}} h_r^{a_r}\in\mathcal{I}.$$

Proof. Set

$$\begin{gathered} e=E_{i,\bar{j}}-E_{\bar{j}+n,i}, \quad f=E_{i,\bar{j}+n}+E_{\bar{j},i}, \quad k=E_{\bar{j},\bar{j}}-E_{\bar{j}+n,\bar{j}+n},\\ k_1=E_{\bar{j},\bar{j}+n}, \qquad k_2=E_{\bar{j}+n,\bar{j}} \qquad e'=E_{i,\bar{j}}+E_{\bar{j}+n,i}, \qquad f=E_{i,\bar{j}+n}-E_{\bar{j},i}, \end{gathered}$$

and let $\mathfrak{g}'$ be the $\mathfrak{gl}(1|2)$-subalgebra of $\mathfrak{g}$ generated by $\{p_{i\bar{j}}, z_{i\bar{j}}, e, f, k, k_1, k_2, e', f'\}$ and $\mathfrak{k}'=\mathfrak{g}'\cap\mathfrak{k}$. Assume that $\lambda\in\mathcal{A}(\mathfrak{g},\mathfrak{k})$. Set

$$\lambda'=\lambda\Big|_{U(\mathfrak{g}')}.$$

Then $\lambda'\in\mathcal{A}(\mathfrak{g}',\mathfrak{k}')$. Identify h_i and $h_{\bar{j}}$ respectively with h_1 and $h_{\bar{1}}$. Therefore, by Lemma 5.2.1, we have

$$\lambda'(p_{i\bar{j}})=0,$$

and since $z_{i\bar{j}}=h_i+h_{\bar{j}}$ is in the center of $\mathfrak{g}'$ and hence in the center of $U(\mathfrak{g}')$, it commutes with every element in $U(\mathfrak{g}')$ and therefore,

$$\lambda'(p_{i\bar{j}}z_{i\bar{j}}^r)=0.$$

or equivalentely,

$$p_{i\bar{j}} z^r_{i\bar{j}} \in \mathfrak{k}' U(\mathfrak{g}') + U(\mathfrak{g}')\mathfrak{k}'.$$

On the other hand, for any $s \in I - \{i, \bar{j}\}$ we have $[h_s, \mathfrak{g}'] = 0$. Hence $[h_s, \mathfrak{k}'] = 0$ for such s. It follows that

$$p_{i\bar{j}} z^r_{i\bar{j}} \prod_{\substack{l \in I,\, a_l \in \mathbb{Z}^+ \\ l \neq i, \bar{j}}} h_l^{a_l} \in \mathfrak{k}' U(\mathfrak{g}') + U(\mathfrak{g}')\mathfrak{k}' \subseteq \mathfrak{k} U(\mathfrak{g}) + U(\mathfrak{g})\mathfrak{k} = \mathcal{I}. \qquad \blacksquare$$

Chapter 6

The action of Casimir on Spherical functionals

In sections 6.2-6.4, we show that if λ is an eigenfunction of the Casimir operator of $\mathfrak{g}$, then Φ_λ satisfies the differential equation $\mathcal{D}_k$ (6.0.1). This equation is identical to the one satisfied by Sergeev-Veselov's super Jack polynomials, up to an exponential coordinate change [34, eq. 1.2.8]. In particular if λ is the spherical matrix coefficient of an irreducible $\mathfrak{g}$-module, then λ is an eigenfunction of the Casimir operator and it satisfies the aforementioned differential equation. We point out that the idea of forming a differential equation is due to Sergeev but it is only worked out by ad hoc methods. In this chapter we develop the theory more systematically. The advantage is that it can be applied to more general symmetric pairs (such as case III of Table 4.1).
In the following theorem, we demonstrate that the Calogero operator of Chalykh, Feigin, and Veselov [5, eq. 1] is always derived from the Casimir operator in the three symmetric spaces corresponding to Cases I, II, and III in Table 4.1. While Sergeev established this result for Cases I and II [34, eq. 1.2.8], we extend the analysis by incorporating Case III into this framework.

For Cases I, II, and III in Table 4.1 with restricted root system of type $A(l, l')$, set $\bar{\Phi}_\mu := \Phi_\mu(\log x_1, \ldots, \log x_{l+1}, \log y_1, \ldots, \log y_{l'+1})$.

Theorem 6.0.1. Let $\mu \in \mathcal{A}(\mathfrak{g}, \mathfrak{k})$. Then $\bar{\Phi}_\mu$ satisfies the following differential equation

$$\begin{aligned}\mathcal{D}_k = k &\sum_{\substack{1\leqslant i,j\leqslant l+1\\ i\neq j}} \frac{x_i}{x_i - x_j}\left(x_i\frac{\partial}{\partial x_i} - x_j\frac{\partial}{\partial x_j}\right)\\ &- \sum_{\substack{1\leqslant i,j\leqslant l'+1\\ i\neq j}} \frac{y_{\bar{i}}}{y_{\bar{i}} - y_{\bar{j}}}\left(y_{\bar{i}}\frac{\partial}{\partial y_{\bar{i}}} - y_{\bar{j}}\frac{\partial}{\partial y_{\bar{j}}}\right)\\ &- \sum_{\substack{1\leqslant i\leqslant l+1\\ 1\leqslant j\leqslant l'+1}} \frac{x_i + y_{\bar{j}}}{x_i - y_{\bar{j}}}\left(x_i\frac{\partial}{\partial x_i} + k y_{\bar{j}}\frac{\partial}{\partial y_{\bar{j}}}\right)\\ &+ \sum_{1\leqslant i\leqslant l+1}\left(x_i\frac{\partial}{\partial x_i}\right)^2 - k\sum_{1\leqslant i\leqslant l'+1}\left(y_{\bar{i}}\frac{\partial}{\partial y_{\bar{i}}}\right)^2,\end{aligned} \tag{6.0.1}$$

where the parameter $k = 1, \frac{1}{2}, \frac{m-1}{2} - n$ corresponds to Cases I, II, and III, respectively, in Table 4.1.

Proof. The proof of each case will be explicitly provided in Sections 6.2, 6.3, and 6.4. ∎

6.1 Some general properties of Φ_λ

Throughout this subsection we assume that $(\mathfrak{g}, \mathfrak{k})$ is one of the symmetric pairs in Cases I, II and III of Table 4.1. By utilizing lemmas of this section in subsequent sections, we can effectively determine the action of each term of Casimir operator on $\lambda \in \mathcal{A}(\mathfrak{g}, \mathfrak{k})$.

From this point forward, we will reserve the dot symbol to represent the structure of the right translation module structure on $U(\mathfrak{g})^*$ according to Section 2.2. In other words,

$$v \cdot \lambda(u) := R_v(\lambda)(u) = (-1)^{(|\lambda||v|+|u||v|)}\lambda(uv) \quad \text{for} \quad \lambda \in U(\mathfrak{g})^* \quad \text{and} \quad u, v \in U(\mathfrak{g}).$$

Remark 6.1.1. Throughout this chapter, we use $\mu[\vec{a} + \vec{e}_i]$ and $\mu[\vec{a}, i, j, \alpha, \beta]$ to denote the following:
(i) $\mu[\vec{a} + \vec{e}_i] := \mu(h_1^{a_1} \cdots h_i^{a_i+1} \cdots h_m^{a_m} h_{\bar{1}}^{a_{\bar{1}}} \cdots h_{\bar{n}}^{a_{\bar{n}}})$,
(ii) $\mu[\vec{a}, i, j, \alpha, \beta] := \mu(h_1^{a_1} \cdots (h_i + \alpha)^{a_i} \cdots (h_j + \beta)^{a_j} \cdots h_m^{a_m} h_{\bar{1}}^{a_{\bar{1}}} \cdots h_{\bar{n}}^{a_{\bar{n}}})$.

Lemma 6.1.2. Let Φ_μ be the power series defined at the beginning of Chapter 5, and let $\mu \in \mathcal{A}(\mathfrak{g}, \mathfrak{k})$. Then,

$$\Phi_{h_i \cdot \mu} = \frac{\partial}{\partial x_i}\Phi_\mu, \quad \Phi_{h_{\bar{i}} \cdot \mu} = \frac{\partial}{\partial y_{\bar{i}}}\Phi_\mu.$$

Proof. Since h_i and $h_{\bar{j}}$ are even $h_i \cdot \mu(u) = \mu(uh_i)$ and $h_{\bar{j}} \cdot \mu(u) = \mu(uh_{\bar{j}})$. Thus,

$$\Phi_{h_i \cdot \mu} = \sum_{\vec{a} \in (\mathbb{Z}^+)^{(m+n)}} \frac{\mu[\vec{a} + \vec{e}_i]}{\vec{a}!} \prod_{i=1}^m x_i^{a_i} \prod_{j=1}^n y_{\bar{j}}^{a_{\bar{j}}} = \frac{\partial}{\partial x_i}\Phi_\mu,$$

where $\mu[\vec{a}+\vec{e}_i] := \mu(h_1^{a_1}\cdots h_i^{a_i+1}\cdots h_m^{a_m}h_{\bar{1}}^{a_{\bar{1}}}\cdots h_{\bar{n}}^{a_{\bar{n}}})$. By a similar argument the second equation can be proved. ∎

Lemma 6.1.3. Let Φ_μ be the power series defined in Chapter 5, and let $\mu \in \mathcal{A}(\mathfrak{g},\mathfrak{k})$. Then the following hold:

$$(i)\quad e^{\alpha x_i+\beta x_j}\Phi_\mu = \sum_{\vec{a}\in(\mathbb{Z}^+)^{(m+n)}} \frac{\mu[\vec{a},i,j,\alpha,\beta]}{\vec{a}!}\prod_{r\in I_{\bar{0}}} x_r^{a_r}\prod_{\bar{s}\in I_{\bar{1}}} y_{\bar{s}}^{a_{\bar{s}}},\quad \text{for}\quad i,j\in I_{\bar{0}},\ i\neq j$$

where $\mu[\vec{a},i,j,\alpha,\beta] := \mu(h_1^{a_1}\cdots(h_i+\alpha)^{a_i}\cdots(h_j+\beta)^{a_j}\cdots h_m^{a_m}h_{\bar{1}}^{a_{\bar{1}}}\cdots h_{\bar{n}}^{a_{\bar{n}}})$.

$$(ii)\quad e^{\alpha y_{\bar{i}}+\beta y_{\bar{j}}}\Phi_\mu = \sum_{\vec{a}\in(\mathbb{Z}^+)^{(m+n)}} \frac{\mu[\vec{a},\bar{i},\bar{j},\alpha,\beta]}{\vec{a}!}\prod_{r\in I_{\bar{0}}} x_r^{a_r}\prod_{\bar{s}\in I_{\bar{1}}} y_{\bar{s}}^{a_{\bar{s}}},\quad \text{for}\quad \bar{i},\bar{j}\in I_{\bar{1}},\ \bar{i}\neq\bar{j}$$

where $\mu[\vec{a},\bar{i},\bar{j},\alpha,\beta] := \mu(h_1^{a_1}\cdots h_m^{a_m}h_{\bar{1}}^{a_{\bar{1}}}\cdots(h_{\bar{i}}+\alpha)^{a_{\bar{i}}}\cdots(h_{\bar{j}}+\beta)^{a_{\bar{j}}}\cdots h_{\bar{n}}^{a_{\bar{n}}})$.

$$(iii)\quad e^{\alpha x_i+\beta y_{\bar{j}}}\Phi_\mu = \sum_{\vec{a}\in(\mathbb{Z}^+)^{(m+n)}} \frac{\mu[\vec{a},i,\bar{j},\alpha,\beta]}{\vec{a}!}\prod_{r\in I_{\bar{0}}} x_r^{a_r}\prod_{\bar{s}\in I_{\bar{1}}} y_{\bar{s}}^{a_{\bar{s}}},\quad \text{for}\quad i\in I_{\bar{0}}\quad\text{and}\quad \bar{j}\in I_{\bar{1}},$$

where $\mu[\vec{a},i,\bar{j},\alpha,\beta] := \mu(h_1^{a_1}\cdots(h_i+\alpha)^{a_i}\cdots h_m^{a_m}h_{\bar{1}}^{a_{\bar{1}}}\cdots(h_{\bar{j}}+\beta)^{a_{\bar{j}}}\cdots h_{\bar{n}}^{a_{\bar{n}}})$.

Proof. As the proofs of the three parts are similar, we will only prove (i). By straightforward calculations we have

$$\begin{aligned}
e^{\alpha x_i+\beta x_j}\Phi_\mu &= e^{\alpha x_i+\beta x_j}\sum_{\vec{a}\in(\mathbb{Z}^+)^{(m+n)}}\frac{\mu[\vec{a}]}{\vec{a}!}\prod_{r\in I_{\bar{0}}} x_r^{a_r}\prod_{\bar{s}\in I_{\bar{1}}} y_{\bar{s}}^{a_{\bar{s}}}\\
&= \sum_{\substack{\vec{a}\,\in\,(\mathbb{Z}^+)^{(m+n)}\\ k\,\geqslant\,0}}\frac{\mu[\vec{a}]}{\vec{a}!}\frac{(\alpha x_i+\beta x_j)^k}{k!}\prod_{r\in I_{\bar{0}}} x_r^{a_r}\prod_{\bar{s}\in I_{\bar{1}}} y_{\bar{s}}^{a_{\bar{s}}}\\
&= \sum_{\substack{\vec{a}\,\in\,(\mathbb{Z}^+)^{(m+n)}\\ k\,\geqslant\,0,\ 0\,\leqslant\,t\,\leqslant\,k}}\binom{k}{t}\frac{\alpha^t\beta^{k-t}\mu[\vec{a}]}{k!\vec{a}!}\prod_{\substack{r\,\in\,I_{\bar{0}},\\ r\,\neq\,i,j}} x_r^{a_r}x_i^{a_i+t}x_j^{a_j+k-t}\prod_{\bar{s}\in I_{\bar{1}}} y_{\bar{s}}^{a_{\bar{s}}}\\
&= \sum_{\substack{\vec{a}\,\in\,(\mathbb{Z}^+)^{(m+n)}\\ k\,\geqslant\,0,\ 0\,\leqslant\,t\,\leqslant\,k}}\frac{\alpha^t\beta^{k-t}\mu[\vec{a}]}{(k-t)!t!\vec{a}!}\prod_{\substack{r\,\in\,I_{\bar{0}},\\ r\,\neq\,i,j}} x_l^{a_l}x_i^{a_i+t}x_j^{a_j+k-t}\prod_{j\in I_{\bar{1}}} y_j^{a_{\bar{j}}}.
\end{aligned}$$

Therefore, by shifting the exponents we can see that

$$\begin{aligned}
e^{\alpha x_i+\beta x_j}\Phi_\mu &= \sum_{\substack{\vec{a}\,\in\,(\mathbb{Z}^+)^{(m+n)}\\ k\,\geqslant\,0,\ 0\,\leqslant\,t\,\leqslant\,k}}\frac{\alpha^t\beta^{k-t}\mu[\vec{a}-t\vec{e}_i-(k-t)\vec{e}_j]}{(k-t)!t!(\vec{a}-t\vec{e}_i-(k-t)\vec{e}_j)!}\prod_{i=1}^m x_i^{a_i}\prod_{j=1}^n y_j^{a_{\bar{j}}}\\
&= \sum_{\vec{a}\in(\mathbb{Z}^+)^{(m+n)}}\frac{\mu[\vec{a},i,j,\alpha,\beta]}{\vec{a}!}\prod_{i=1}^m x_i^{a_i}\prod_{j=1}^n y_j^{a_{\bar{j}}}.
\end{aligned}$$

∎

Lemma 6.1.4. Let $(\mathfrak{g},\mathfrak{k})$ be a symmetric pair. Let $\{e,f,h,z\}$ be a $\mathfrak{gl}_2-$basis in $\mathfrak{g}$ as in Section 4.8.1. Assume that either $e-f\in\mathfrak{k}$ or $e+f\in\mathfrak{k}$. Then

$$\mu((p(h-2)-p(h+2))ef)=-\mu(p(h+2)h),\quad \text{for}\quad \mu\in\mathcal{A}(\mathfrak{g},\mathfrak{k}),\quad p(x)\in\mathbb{C}[x].$$

Proof. We have

$$\begin{aligned}&(p(h-2)-p(h+2))ef+p(h+2)h=p(h+2)(h-ef)+p(h-2)ef\\ &=-p(h+2)fe+p(h-2)ef=-fp(h)e+ep(h)f\\ &=-fp(h)(e\mp f)+(e\mp f)p(h)f\in\mathcal{I}.\end{aligned}$$

Then from the above relation it follows that $\mu((p(h-2)-p(h+2))ef)=-\mu(p(h+2)h)$ for $\mu\in\mathcal{A}(\mathfrak{g},\mathfrak{k})$. ∎

6.2 The action of Casimir operator on $\mathcal{A}(\mathfrak{g},\mathfrak{k})$ in Case I of Table 4.1

In this section $(\mathfrak{g},\mathfrak{k}):=(\mathfrak{gl}(m|n)\oplus\mathfrak{gl}(m|n),\mathfrak{gl}(m|n))$. For $\lambda\in\mathcal{A}(\mathfrak{g},\mathfrak{k})$ we let Φ_λ be as in Chapter 5, where in this case from Section 4.5 we know that $h_i=(E_{i,i},-E_{i,i})$ for $1\leqslant i\leqslant m$ and $h_{\bar{j}}=(E_{\bar{j},\bar{j}},-E_{\bar{j},\bar{j}})$ for $1\leqslant j\leqslant n$.

Let $\{(E_{i,j},0),(0,E_{i,j})\,|\,i,j\in I=I_{\bar{0}}\sqcup I_{\bar{1}}\}$ be the standard basis for $\mathfrak{g}$ consisting of matrix units where $I_{\bar{0}}=\{1,\cdots,m\}$ and $I_{\bar{1}}=\{\bar{1},\bar{2},\cdots,\bar{n}\}$. Then the Casimir operator of $U(\mathfrak{g})$ is

$$\Omega=\sum_{i,j\in I}(-1)^{|i|}(E_{i,j},0)(E_{j,i},0)+(-1)^{|i|}(0,E_{i,j})(0,E_{j,i}).$$

Set

$$\begin{aligned}e_{i,j}&=(E_{i,j},-(-1)^{(|i|+|j|)}E_{j,i}),\quad h_i=(E_{i,i},-E_{i,i}),\\ \tilde{e}_{i,j}&=(E_{i,j},(-1)^{(|i|+|j|)}E_{j,i}),\quad \tilde{h}_i=(E_{i,i},E_{i,i}).\end{aligned}$$

Then, $\{e_{i,j},h_i,\tilde{e}_{i,j},\tilde{h}_i\,|\,i,j\in I=I_{\bar{0}}\sqcup I_{\bar{1}}\}$ is a basis of $\mathfrak{g}=\mathfrak{gl}(m|n)\oplus\mathfrak{gl}(m|n)$.

Lemma 6.2.1. Let $\mu\in\mathcal{A}(\mathfrak{g},\mathfrak{k})$. Set

$$\mu_1:=(e_{i,j}e_{j,i})\cdot\mu,\quad\text{and}\quad \mu_2=(\tilde{e}_{i,j}\tilde{e}_{j,i})\cdot\mu,\quad\text{for}\quad i,j\in I.$$

Then

$$(i)\quad \Phi_{\mu_1}=\Phi_{\mu_2}=\frac{-e^{x_i-x_j}}{e^{-x_i+x_j}-e^{x_i-x_j}}(\frac{\partial}{\partial x_i}-\frac{\partial}{\partial x_j})\Phi_\mu,\quad\text{where}\quad i,j\in I_{\bar{0}}\quad\text{and}\quad i\neq j,$$

$$(ii)\quad \Phi_{\mu_1}=\Phi_{\mu_2}=\frac{-e^{y_{\bar{i}}-y_{\bar{j}}}}{e^{-y_{\bar{i}}+y_{\bar{j}}}-e^{y_{\bar{i}}-y_{\bar{j}}}}(\frac{\partial}{\partial y_{\bar{i}}}-\frac{\partial}{\partial y_{\bar{j}}})\Phi_\mu,\quad\text{where}\quad \bar{i},\bar{j}\in I_{\bar{1}}\quad\text{and}\quad \bar{i}\neq\bar{j}.$$

Proof. By Lemma 6.1.3 we have

$$(1)\quad e^{x_i-x_j}\Phi_{\mu_1}=\sum_{\vec{a}\in(\mathbb{Z}^+)^{(m+n)}}\mu_1[\vec{a},i,j,1,-1]\frac{\prod_{i=1}^m x_i^{a_i}\prod_{j=1}^n y_j^{a_{\bar{j}}}}{\vec{a}!},$$

and

$$(2)\quad e^{-x_i+x_j}\Phi_{\mu_1}=\sum_{\vec{a}\in(\mathbb{Z}^+)^{(m+n)}}\mu_1[\vec{a},i,j,-1,1]\frac{\prod_{i=1}^m x_i^{a_i}\prod_{j=1}^n y_j^{a_{\bar{j}}}}{\vec{a}!}.$$

By straightforward computations we have

$$\begin{aligned}\mu_1[\vec{a},i,j,1,-1]&:=\mu_1(h_1^{a_1}\cdots(h_i+1)^{a_i}(h_j-1)^{a_j}\cdots h_m^{a_m}h_{\bar{1}}^{a_{\bar{1}}}\cdots h_{\bar{n}}^{a_{\bar{n}}})\\&=\frac{1}{2^{a_i+a_j}}\mu_1(h_1^{a_1}\cdots((z+(h+2))^{a_i}(z-(h+2))^{a_j})\cdots h_m^{a_m}h_{\bar{1}}^{a_{\bar{1}}}\cdots h_{\bar{n}}^{a_{\bar{n}}}),\end{aligned}$$

where $h=h_i-h_j$ and $z=h_i+h_j$. Also,

$$\begin{aligned}\mu_1[\vec{a},i,j,-1,1]&:=\mu_1(h_1^{a_1}\cdots(h_i+1)^{a_i}(h_j-1)^{a_j}\cdots h_m^{a_m}h_{\bar{1}}^{a_{\bar{1}}}\cdots h_{\bar{n}}^{a_{\bar{n}}})\\&=\frac{1}{2^{a_i+a_j}}\mu_1(h_1^{a_1}\cdots((z+(h-2))^{a_i}(z-(h-2))^{a_j})\cdots h_m^{a_m}h_{\bar{1}}^{a_{\bar{1}}}\cdots h_{\bar{n}}^{a_{\bar{n}}}).\end{aligned}$$

Set $p_{k,l}(h)=(-1)^{a_j-1}h^{a_i-k+a_j-l}$. Therefore,

$$\begin{aligned}&((z+(h-2))^{a_i}(z-(h-2))^{a_j})-((z+(h+2))^{a_i}(z-(h+2))^{a_j})\\&=\sum_{\substack{0\leqslant k\leqslant a\\0\leqslant l\leqslant b}}\binom{a_i}{k}\binom{a_j}{l}z^{k+l}(p_{k,l}(h-2)-p_{k,l}(h+2)).\end{aligned}$$

Since $\{e_{i,j},e_{j,i},h:=h_i-h_j\}$ is an $\mathfrak{sl}_2$-triple and $e_{i,j}-e_{j,i}\in\mathfrak{k}$, for the polynomials $p_{k,l}(h)$ we can apply Lemma 6.1.4 which implies that

$$\mu_1((p_{k,l}(h-2)-p_{k,l}(h+2)))=\mu((p_{k,l}(h-2)-p_{k,l}(h+2))e_{i,j}e_{j,i})=-\mu(p_{k,l}(h+2)h).$$

Thus,

$$\begin{aligned}&\mu_1[\vec{a},i,j,-1,1]-\mu_1[\vec{a},i,j,1,-1]\\&=\frac{1}{2^{a_i+a_j}}\sum_{\substack{0\leqslant k\leqslant a\\0\leqslant l\leqslant b}}\binom{a_i}{k}\binom{a_j}{l}\mu_1(h_1^{a_1}\cdots\left(z^{k+l}(p_{k,l}(h-2)-p_{k,l}(h+2))\right)\cdots h_m^{a_m}h_{\bar{1}}^{a_{\bar{1}}}\cdots h_{\bar{n}}^{a_{\bar{n}}})\\&=-\frac{1}{2^{a_i+a_j}}\sum_{\substack{0\leqslant k\leqslant a\\0\leqslant l\leqslant b}}\binom{a_i}{k}\binom{a_j}{l}\mu(h_1^{a_1}\cdots\left(z^{k+l}p_{k,l}(h+2)h\right)\cdots h_m^{a_m}h_{\bar{1}}^{a_{\bar{1}}}\cdots h_{\bar{n}}^{a_{\bar{n}}})\\&=-\mu\left(h_1^{a_1}\cdots((z+(h+2))/2)^{a_i}\,((z-(h+2))/2)^{a_j}\,h\cdots h_m^{a_m}h_{\bar{1}}^{a_{\bar{1}}}\cdots h_{\bar{n}}^{a_{\bar{n}}}\right)\\&=-\mu\left(h_1^{a_1}\cdots(h_i+1)^{a_i}\,(h_j-1)^{a_j}\,h\cdots h_m^{a_m}h_{\bar{1}}^{a_{\bar{1}}}\cdots h_{\bar{n}}^{a_{\bar{n}}}\right).\end{aligned}$$

From (1) and (2) and summing over both sides of the above multi-line calculation after multiplication with suitable monomials we obtain that the left hand side yields

$$(e^{-x_i+x_j}-e^{x_i-x_j})\Phi_{\mu_1},$$

and the right hand side yields

$$-e^{x_i-x_j}(\frac{\partial}{\partial x_i}-\frac{\partial}{\partial x_j})\Phi_{\mu}.$$

Also, since $\{\tilde{e}_{i,j},\tilde{e}_{j,i},h:=h_i-h_j\}$ is an $\mathfrak{sl}_2$-triple and $\tilde{e}_{i,j}+\tilde{e}_{j,i}\in\mathfrak{k}$ we can apply Lemma 6.1.4 which implies that

$$\mu_2(p_{k,l}(h-2)-p_{k,l}(h+2))=\mu((p_{k,l}(h-2)-p_{k,l}(h+2))\tilde{e}_{i,j}\tilde{e}_{j,i})=-\mu(p_{k,l}(h+2)h).$$

Thus, similarly it can be proved that

$$\Phi_{\mu_2}=\frac{-e^{x_i-x_j}}{e^{-x_i+x_j}-e^{x_i-x_j}}(\frac{\partial}{\partial x_i}-\frac{\partial}{\partial x_j})\Phi_{\mu}.$$ ∎

Lemma 6.2.2. Set

$$\mu'=((E_{i,j},0)(E_{j,i},0)+(0,E_{i,j})(0,E_{j,i}))\cdot\mu,\quad \text{for}\quad i,j\in I_{\bar{0}}\quad\text{and}\quad i\neq j,$$

and

$$\mu''=\big(-(E_{\bar{i},\bar{j}},0)(E_{\bar{j},\bar{i}},0)-(0,E_{\bar{i},\bar{j}})(0,E_{\bar{j},\bar{i}})\big)\cdot\mu,\quad \text{for}\quad \bar{i},\bar{j}\in I_{\bar{1}}\quad\text{and}\quad \bar{i}\neq\bar{j}.$$

Then

$$(i)\quad \Phi_{\mu'}=\frac{-e^{x_i-x_j}}{e^{-x_i+x_j}-e^{x_i-x_j}}(\frac{\partial}{\partial x_i}-\frac{\partial}{\partial x_j})\Phi_{\mu}-\frac{1}{2}(\frac{\partial}{\partial x_i}-\frac{\partial}{\partial x_j})\Phi_{\mu},$$

$$(ii)\quad \Phi_{\mu''}=\frac{e^{y_{\bar{i}}-y_{\bar{j}}}}{e^{-y_{\bar{i}}+y_{\bar{j}}}-e^{y_{\bar{i}}-y_{\bar{j}}}}(\frac{\partial}{\partial y_{\bar{i}}}-\frac{\partial}{\partial y_{\bar{j}}})\Phi_{\mu}+\frac{1}{2}(\frac{\partial}{\partial y_{\bar{i}}}-\frac{\partial}{\partial y_{\bar{j}}})\Phi_{\mu}.$$

Proof. First one can see that when $|i|=|j|$ we have

$$\{(E_{i,j},0)=\frac{1}{2}\left(e_{i,j}+\tilde{e}_{i,j}\right),\quad (E_{j,i},0)=\frac{1}{2}\left(e_{j,i}+\tilde{e}_{j,i}\right),\quad (E_{i,i}-E_{j,j},0)=h_i-h_j+\tilde{h}_i-\tilde{h}_j\},$$

$$\{(0,E_{i,j})=\frac{1}{2}\left(-e_{j,i}+\tilde{e}_{j,i}\right),\quad (0,E_{j,i})=\frac{1}{2}\left(-e_{i,j}+\tilde{e}_{i,j}\right),\quad (0,E_{i,i}-E_{j,j})=-h_i+h_j+\tilde{h}_i-\tilde{h}_j\},$$

which are $\mathfrak{sl}_2$-triples. Therefore, for $i,j\in I_{\bar{0}}$,

$$\begin{aligned}&(E_{i,j},0)(E_{j,i},0)+(0,E_{i,j})(0,E_{j,i})\\&=\frac{1}{4}\left(\left(e_{i,j}+\tilde{e}_{i,j}\right)\left(e_{j,i}+\tilde{e}_{j,i}\right)+\left(-e_{j,i}+\tilde{e}_{j,i}\right)\left(-e_{i,j}+\tilde{e}_{i,j}\right)\right)\\&=\frac{1}{4}\left(e_{i,j}e_{j,i}+e_{i,j}\tilde{e}_{j,i}+\tilde{e}_{i,j}e_{j,i}+\tilde{e}_{i,j}\tilde{e}_{j,i}+e_{j,i}e_{i,j}-e_{j,i}\tilde{e}_{i,j}-\tilde{e}_{j,i}e_{i,j}+\tilde{e}_{j,i}\tilde{e}_{i,j}\right).\end{aligned}$$

But $[e_{i,j},\tilde{e}_{j,i}],[\tilde{e}_{i,j},e_{j,i}]\in\mathfrak{k}$. Therefore,

$$\mu'[\vec{a}]=\frac{1}{2}\left(e_{i,j}e_{j,i}+\tilde{e}_{i,j}\tilde{e}_{j,i}-h\right)\cdot\mu[\vec{a}].$$

By Lemmas 6.1.2 and 6.2.1, the proof of the first equation follows and the second equation can be proved by a similar argument. ∎

Lemma 6.2.3. Let $(\mathfrak{g},\mathfrak{k})$ be a symmetric pair. Suppose that $h_1, h_2 \in \mathfrak{a}$ and $e, f, \tilde{e}, \tilde{f} \in \mathfrak{g}_{\bar{1}}$ are chosen such that $(f-\tilde{e}) \in \mathfrak{k}$, $(e-\tilde{f}) \in \mathfrak{k}$, and

$$[h,e]=2e, \quad [h,f]=-2f, \quad [e,\tilde{e}]=0, \quad [e,f]=z,$$
$$[h,\tilde{e}]=2\tilde{e}, \quad [h,\tilde{f}]=-2\tilde{f}, \quad [f,\tilde{f}]=0, \quad [\tilde{e},\tilde{f}]=z,$$

where $h=h_1-h_2$ and $z=h_1+h_2$. Then given any $\mu \in \mathcal{A}(\mathfrak{g},\mathfrak{k})$, for $p(x) \in \mathbb{C}[x]$ we have

$$\mu((p(h-2)-p(h+2))\,ef) = \mu((p(h-2)-p(h+2))\,\tilde{e}\tilde{f}) = -\mu(p(h+2)z).$$

Proof. For $p(x) \in \mathbb{C}$, since $(f-\tilde{e}) \in \mathfrak{k}$ we have

$$\begin{aligned}
(p(h-2) &- p(h+2)ef) + p(h+2)z \\
&= p(h+2)(z-ef) + p(h-2)ef \\
&= p(h+2)fe + p(h-2)ef \\
&= fp(h)e + ep(h)f \\
&= (f-\tilde{e}+\tilde{e})p(h)e + ep(h)(f-\tilde{e}+\tilde{e}) \\
&= (f-\tilde{e})p(h)e + ep(h)(f-\tilde{e}) + \tilde{e}p(h)e + ep(h)\tilde{e} \\
&= (f-\tilde{e})p(h)e + ep(h)(f-\tilde{e}) \in \mathcal{I}.
\end{aligned}$$

The last equality of the above equation is due to the following relations

$$[e,\tilde{e}]=0, \quad [h,\tilde{e}]=2\tilde{e}, \quad [h,e]=2e.$$

Similarly, we can prove that

$$\mu((p(h-2)-p(h+2))\,\tilde{e}\tilde{f}) = -\mu(p(h+2)z).$$

∎

Lemma 6.2.4. Let $\mu \in \mathcal{A}(\mathfrak{g},\mathfrak{k})$. Set

$$\mu_1 := \left(e_{i,\bar{j}}\tilde{e}_{\bar{j},i}\right)\cdot\mu, \quad \text{and} \quad \mu_2 = \left(\tilde{e}_{i,\bar{j}}e_{\bar{j},i}\right)\cdot\mu, \quad \text{for} \quad i,j \in I_{\bar{0}} \quad \text{and} \quad \bar{i},\bar{j} \in I_{\bar{1}}.$$

Then

$$(i) \quad \Phi_{\mu_1} = \Phi_{\mu_2} = \frac{-e^{x_i-y_{\bar{j}}}}{e^{-x_i+y_{\bar{j}}}-e^{x_i-y_{\bar{j}}}}\left(\frac{\partial}{\partial x_i}+\frac{\partial}{\partial y_{\bar{j}}}\right)\Phi_\mu, \quad \text{when} \quad i \in I_{\bar{0}} \quad \text{and} \quad \bar{j} \in I_{\bar{1}},$$

and

$$(ii) \quad \Phi_{\mu_1} = \Phi_{\mu_2} = \frac{-e^{y_{\bar{i}}-x_j}}{e^{-y_{\bar{i}}+x_j}-e^{y_{\bar{i}}-x_j}}\left(\frac{\partial}{\partial y_{\bar{i}}}+\frac{\partial}{\partial x_j}\right)\Phi_\mu, \quad \text{when} \quad \bar{i} \in I_{\bar{1}} \quad \text{and} \quad j \in I_{\bar{0}}.$$

Proof. By Lemma 6.1.3 we have

$$(1) \quad e^{x_i-y_{\bar{j}}}\Phi_{\mu_1} = \sum_{\vec{a}\in(\mathbb{Z}^+)^{(m+n)}} \mu_1[\vec{a},i,\bar{j},1,-1]\frac{\prod_{i=1}^m x_i^{a_i}\prod_{j=1}^n y_{\bar{j}}^{a_{\bar{j}}}}{\vec{a}!},$$

and

$$(2)\quad e^{-x_i+y_{\bar{j}}}\Phi_{\mu_1} = \sum_{\vec{a}\in(\mathbb{Z}^+)^{(m+n)}} \mu_1[\vec{a},i,\bar{j},-1,1]\frac{\prod_{i=1}^m x_i^{a_i}\prod_{j=1}^n y_{\bar{j}}^{a_{\bar{j}}}}{\vec{a}!}.$$

By straightforward computations we have

$$\begin{aligned}\mu_1[\vec{a},i,\bar{j},1,-1] &:= \mu_1(h_1^{a_1}\cdots h_m^{a_m}(h_i+1)^{a_i}(h_{\bar{j}}-1)^{a_{\bar{j}}}h_{\bar{1}}^{a_{\bar{1}}}\cdots h_{\bar{n}}^{a_{\bar{n}}})\\ &= \frac{1}{2^{a_i+a_{\bar{j}}}}\mu_1(h_1^{a_1}\cdots h_m^{a_m}\left((z+(h+2))^{a_i}(z-(h+2))^{a_{\bar{j}}}\right)h_{\bar{1}}^{a_{\bar{1}}}\cdots h_{\bar{n}}^{a_{\bar{n}}}),\end{aligned}$$

where $h=h_i-h_{\bar{j}}$ and $z=h_i+h_{\bar{j}}$. Also,

$$\begin{aligned}\mu_1[\vec{a},i,\bar{j},-1,1] &:= \mu_1(h_1^{a_1}\cdots h_m^{a_m}(h_i+1)^{a_i}(h_{\bar{j}}-1)^{a_{\bar{j}}}h_{\bar{1}}^{a_{\bar{1}}}\cdots h_{\bar{n}}^{a_{\bar{n}}})\\ &= \frac{1}{2^{a_i+a_{\bar{j}}}}\mu_1(h_1^{a_1}\cdots h_m^{a_m}\left((z+(h-2))^{a_i}(z-(h-2))^{a_{\bar{j}}}\right)h_{\bar{1}}^{a_{\bar{1}}}\cdots h_{\bar{n}}^{a_{\bar{n}}}).\end{aligned}$$

We have

$$\begin{aligned}&((z+(h-2))^{a_i}(z-(h-2))^{a_{\bar{j}}})-((z+(h+2))^{a_i}(z-(h+2))^{a_{\bar{j}}})\\ &=\sum_{\substack{0\leqslant k\leqslant a\\ 0\leqslant l\leqslant b}}\binom{a_i}{k}\binom{a_{\bar{j}}}{l}z^{k+l}(p_{k,l}(h-2)-p_{k,l}(h+2)),\end{aligned}$$

where $p_{k,l}(h)=(-1)^{a_{\bar{j}}-1}h^{a_i-k+a_{\bar{j}}-l}$. Set

$$e:=e_{i,\bar{j}},\quad f:=\tilde{e}_{\bar{j},i},\quad \tilde{e}:=\tilde{e}_{i,\bar{j}},\quad \tilde{f}:=e_{\bar{j},i},\quad h=h_i-h_{\bar{j}},\quad z:=h_i+h_{\bar{j}}.$$

Since $f-\tilde{e}\in\mathfrak{k}$, for the polynomials $p_{k,l}(h)$ we can apply Lemma 6.2.3 which implies that

$$\mu_1((p_{k,l}(h-2)-p_{k,l}(h+2)))=\mu((p_{k,l}(h-2)-p_{k,l}(h+2))e_{i,\bar{j}}\tilde{e}_{\bar{j},i})=-\mu(p_{k,l}(h+2)z).$$

Therefore, for $\vec{a}\in(\mathbb{Z}^+)^{(m+n)}$ we have

$$\begin{aligned}&\mu_1[\vec{a},i,j,-1,1]-\mu_1[\vec{a},i,j,1,-1]\\ &=\frac{1}{2^{a_i+a_j}}\sum_{\substack{0\leqslant k\leqslant a\\ 0\leqslant l\leqslant b}}\binom{a_i}{k}\binom{a_j}{l}\mu_1(h_1^{a_1}\cdots\left(z^{k+l}(p_{k,l}(h-2)-p_{k,l}(h+2))\right)\cdots h_m^{a_m}h_{\bar{1}}^{a_{\bar{1}}}\cdots h_{\bar{n}}^{a_{\bar{n}}})\\ &=-\frac{1}{2^{a_i+a_j}}\sum_{\substack{0\leqslant k\leqslant a\\ 0\leqslant l\leqslant b}}\binom{a_i}{k}\binom{a_j}{l}\mu(h_1^{a_1}\cdots\left(z^{k+l}p_{k,l}(h+2)z\right)\cdots h_m^{a_m}h_{\bar{1}}^{a_{\bar{1}}}\cdots h_{\bar{n}}^{a_{\bar{n}}})\\ &=-\mu\left(h_1^{a_1}\cdots((z+(h+2))/2)^{a_i}\,((z-(h+2))/2)^{a_j}\,z\cdots h_m^{a_m}h_{\bar{1}}^{a_{\bar{1}}}\cdots h_{\bar{n}}^{a_{\bar{n}}}\right)\\ &=-\mu\left(h_1^{a_1}\cdots(h_i+1)^{a_i}\left(h_{\bar{j}}-1\right)^{a_{\bar{j}}}z\cdots h_m^{a_m}h_{\bar{1}}^{a_{\bar{1}}}\cdots h_{\bar{n}}^{a_{\bar{n}}}\right).\end{aligned}$$

From (1) and (2) and summing over both sides of the above multi-line calculation after multiplication with suitable monomials we obtain that the left hand side yields

$$(e^{-x_i+y_{\bar{j}}}-e^{x_i-y_{\bar{j}}})\Phi_{\mu_1},$$

and the right hand side yields

$$-e^{x_i-y_{\bar{j}}}(\frac{\partial}{\partial x_i}+\frac{\partial}{\partial y_{\bar{j}}})\Phi_\mu.$$

Similarly it can be proved that

$$\Phi_{\mu_1}=\Phi_{\mu_2}=\frac{-e^{x_i-y_{\bar{j}}}}{e^{-x_i+y_{\bar{j}}}-e^{x_i-y_{\bar{j}}}}(\frac{\partial}{\partial x_i}+\frac{\partial}{\partial y_{\bar{j}}})\Phi_\mu.$$

∎

Lemma 6.2.5. Let $\mu\in\mathcal{A}(\mathfrak{g},\mathfrak{k})$. Set

$$\mu'=\big((E_{i,\bar{j}},0)(E_{\bar{j},i},0)+(0,E_{i,\bar{j}})(0,E_{\bar{j},i})\big)\cdot\mu,\quad\text{for}\quad i\in I_{\bar{0}}\quad\text{and}\quad\bar{j}\in I_{\bar{1}},$$

and

$$\mu''=\big(-(E_{\bar{i},j},0)(E_{j,\bar{i}},0)-(0,E_{\bar{i},j})(0,E_{j,\bar{i}})\big)\cdot\mu,\quad\text{for}\quad\bar{i}\in I_{\bar{1}}\quad\text{and}\quad j\in I_{\bar{0}}.$$

Then

$$(i)\quad\Phi_{\mu'}=\frac{-e^{x_i-y_{\bar{j}}}}{e^{-x_i+y_{\bar{j}}}-e^{x_i-y_{\bar{j}}}}(\frac{\partial}{\partial x_i}+\frac{\partial}{\partial y_{\bar{j}}})\Phi_\mu-\frac{1}{2}(\frac{\partial}{\partial x_i}+\frac{\partial}{\partial y_{\bar{j}}})\Phi_\mu,$$

$$(ii)\quad\Phi_{\mu''}=\frac{e^{y_{\bar{i}}-x_j}}{e^{-y_{\bar{i}}+x_j}-e^{y_{\bar{i}}-x_j}}(\frac{\partial}{\partial y_{\bar{i}}}+\frac{\partial}{\partial x_j})\Phi_\mu+\frac{1}{2}(\frac{\partial}{\partial y_{\bar{i}}}+\frac{\partial}{\partial x_j})\Phi_\mu.$$

Proof. By straightforward calculations for $i\in I_{\bar{0}}$ and $\bar{j}\in I_{\bar{1}}$ we have

$$(E_{i,\bar{j}},0)(E_{\bar{j},i},0)+(0,E_{i,\bar{j}})(0,E_{\bar{j},i})=\frac{1}{2}\left((\tilde{h}_i+\tilde{h}_{\bar{j}})+e_{i,\bar{j}}\tilde{e}_{\bar{j},i}+\tilde{e}_{i,\bar{j}}e_{\bar{j},i}-z\right).$$

But $(\tilde{h}_i+\tilde{h}_{\bar{j}})\in\mathfrak{k}$ and by Lemma 6.2.4 we have

$$\Phi_{e_{i,\bar{j}}\tilde{e}_{\bar{j},i}\cdot\mu}=\Phi_{\tilde{e}_{i,\bar{j}}e_{\bar{j},i}\cdot\mu}=\frac{-e^{x_i-y_{\bar{j}}}}{e^{-x_i+y_{\bar{j}}}-e^{x_i-y_{\bar{j}}}}(\frac{\partial}{\partial x_i}+\frac{\partial}{\partial y_{\bar{j}}})\Phi_\mu,$$

and by Lemmas 6.1.2:

$$\Phi_{z\cdot\mu}=(\frac{\partial}{\partial x_i}+\frac{\partial}{\partial y_{\bar{j}}})\Phi_\mu.$$

Therefore, by Lemmas 6.1.2 and 6.2.4 we have

$$\Phi_{\mu'}=\frac{-e^{x_i-y_{\bar{j}}}}{e^{-x_i+y_{\bar{j}}}-e^{x_i-y_{\bar{j}}}}(\frac{\partial}{\partial x_i}+\frac{\partial}{\partial y_{\bar{j}}})\Phi_\mu-\frac{1}{2}(\frac{\partial}{\partial x_i}+\frac{\partial}{\partial y_{\bar{j}}})\Phi_\mu.$$

The second part can be proved similarly. ∎

Proposition 6.2.6. Let $(\mathfrak{g},\mathfrak{k})=(\mathfrak{gl}(m|n)\oplus\mathfrak{gl}(m|n),\mathfrak{gl}(m|n))$ and $\mu\in\mathcal{A}(\mathfrak{g},\mathfrak{k})$. Then

$$\Phi_{\Omega\cdot\mu}=\sum_{\substack{i,j\in I_{\bar{0}}\\ i\neq j}}\frac{-e^{x_i-x_j}}{e^{-x_i+x_j}-e^{x_i-x_j}}(\frac{\partial}{\partial x_i}-\frac{\partial}{\partial x_j})\Phi_\mu$$

$$+ \sum_{\substack{\bar{i},\bar{j} \in I_{\bar{1}} \\ \bar{i} \neq \bar{j}}} \frac{e^{y_{\bar{i}}-y_{\bar{j}}}}{e^{-y_{\bar{i}}+y_{\bar{j}}} - e^{y_{\bar{i}}-y_{\bar{j}}}} (\frac{\partial}{\partial y_{\bar{i}}} - \frac{\partial}{\partial y_{\bar{j}}})\Phi_\mu$$
$$+ \sum_{\substack{i \in I_{\bar{0}} \\ \bar{j} \in I_{\bar{1}}}} \frac{-e^{x_i-y_{\bar{j}}}}{e^{-x_i+y_{\bar{j}}} - e^{x_i-y_{\bar{j}}}} (\frac{\partial}{\partial x_i} + \frac{\partial}{\partial y_{\bar{j}}})\Phi_\mu$$
$$+ \sum_{\substack{i \in I_{\bar{1}} \\ j \in I_{\bar{0}}}} \frac{e^{y_{\bar{i}}-x_j}}{e^{-y_{\bar{i}}+x_j} - e^{y_{\bar{i}}-x_j}} (\frac{\partial}{\partial y_{\bar{i}}} + \frac{\partial}{\partial x_j})\Phi_\mu$$
$$+ \frac{1}{2}\sum_{i\in I_{\bar{0}}} \frac{\partial^2}{\partial x_i^2}\Phi_\mu - \frac{1}{2}\sum_{\bar{i}\in I_{\bar{1}}} \frac{\partial^2}{\partial y_{\bar{i}}^2}\Phi_\mu.$$

Proof. Recall that

$$\begin{aligned}
\Omega &= \sum_{i,j\in I} (-1)^{|i|}(E_{i,j},0)(E_{j,i},0) + (-1)^{|i|}(0,E_{i,j})(0,E_{j,i}) \\
&= \sum_{\substack{i,j \in I_{\bar{0}} \\ i \neq j}} (E_{i,j},0)(E_{j,i},0) + (0,E_{i,j})(0,E_{j,i}) \\
&+ \sum_{\substack{\bar{i},\bar{j} \in I_{\bar{1}} \\ \bar{i} \neq \bar{j}}} -(E_{\bar{i},\bar{j}},0)(E_{\bar{j},\bar{i}},0) - (0,E_{\bar{i},\bar{j}})(0,E_{\bar{j},\bar{i}}) \\
&+ \sum_{\substack{i \in I_{\bar{0}} \\ \bar{j} \in I_{\bar{1}}}} (E_{i,\bar{j}},0)(E_{\bar{j},i},0) + (0,E_{i,\bar{j}})(0,E_{\bar{j},i}) \\
&+ \sum_{\substack{\bar{i} \in I_{\bar{1}} \\ j \in I_{\bar{0}}}} -(E_{\bar{i},j},0)(E_{j,\bar{i}},0) - (0,E_{\bar{i},j})(0,E_{j,\bar{i}}) \\
&+ \sum_{i\in I_{\bar{0}}} \left((E_{i,i},0)^2 + (0,E_{i,i})^2\right) + \sum_{\bar{i}\in I_{\bar{1}}} \left(-(E_{\bar{i},\bar{i}},0)^2 - (0,E_{\bar{i},\bar{i}})^2\right).
\end{aligned}$$

For $i \in I_{\bar{0}}$ we have

$$(E_{i,i},0)^2 + (0,E_{i,i})^2 = \frac{1}{4}\left((h_i+\tilde{h}_i)^2 + (\tilde{h}_i - h_i)^2\right) = \frac{1}{2}\left(h_i^2 + \tilde{h}_i^2\right).$$

But $\tilde{h}_i^2 \in \mathcal{I}$ and hence by Lemma 6.1.2 we have

$$\Phi_{h_i^2\cdot\mu} = \frac{\partial^2}{\partial x_i^2}.$$

Similarly for $\bar{i} \in I_{\bar{1}}$ we have

$$\Phi_{h_{\bar{i}}^2\cdot\mu} = \frac{\partial^2}{\partial y_{\bar{i}}^2}.$$

The latter relations together with Lemmas 6.2.2 and 6.2.5, yield:

$$\Phi_{\Omega\cdot\mu} = \sum_{\substack{i,j \in I_{\bar{0}} \\ i \neq j}} \frac{-e^{x_i-x_j}}{e^{-x_i+x_j} - e^{x_i-x_j}} (\frac{\partial}{\partial x_i} - \frac{\partial}{\partial x_j})\Phi_\mu - \frac{1}{2}\sum_{\substack{i,j \in I_{\bar{0}} \\ i \neq j}} (\frac{\partial}{\partial x_i} - \frac{\partial}{\partial x_j})\Phi_\mu$$

$$
\begin{aligned}
&+ \sum_{\substack{\bar{i},\bar{j} \in I_{\bar{1}} \\ \bar{i} \neq \bar{j}}} \frac{e^{y_{\bar{i}}-y_{\bar{j}}}}{e^{-y_{\bar{i}}+y_{\bar{j}}} - e^{y_{\bar{i}}-y_{\bar{j}}}} (\frac{\partial}{\partial y_{\bar{i}}} - \frac{\partial}{\partial y_{\bar{j}}})\Phi_\mu + \frac{1}{2} \sum_{\substack{\bar{i},\bar{j} \in I_{\bar{1}} \\ \bar{i} \neq \bar{j}}} (\frac{\partial}{\partial y_{\bar{i}}} - \frac{\partial}{\partial y_{\bar{j}}})\Phi_\mu \\
&+ \sum_{\substack{i \in I_{\bar{0}} \\ \bar{j} \in I_{\bar{1}}}} \frac{-e^{x_i-y_{\bar{j}}}}{e^{-x_i+y_{\bar{j}}} - e^{x_i-y_{\bar{j}}}} (\frac{\partial}{\partial x_i} + \frac{\partial}{\partial y_{\bar{j}}})\Phi_\mu - \frac{1}{2} \sum_{\substack{i \in I_{\bar{0}} \\ \bar{j} \in I_{\bar{1}}}} (\frac{\partial}{\partial x_i} + \frac{\partial}{\partial y_{\bar{j}}})\Phi_\mu \\
&+ \sum_{\substack{i \in I_{\bar{1}} \\ j \in I_{\bar{0}}}} \frac{e^{y_{\bar{i}}-x_j}}{e^{-y_{\bar{i}}+x_j} - e^{y_{\bar{i}}-x_j}} (\frac{\partial}{\partial y_{\bar{i}}} + \frac{\partial}{\partial x_j})\Phi_\mu + \frac{1}{2} \sum_{\substack{\bar{i} \in I_{\bar{1}} \\ j \in I_{\bar{0}}}} (\frac{\partial}{\partial y_{\bar{i}}} + \frac{\partial}{\partial x_j})\Phi_\mu \\
&+ \frac{1}{2}\sum_{i\in I_{\bar{0}}} \frac{\partial^2}{\partial x_i^2}\Phi_\mu - \frac{1}{2}\sum_{\bar{i}\in I_{\bar{1}}} \frac{\partial^2}{\partial y_{\bar{i}}^2}\Phi_\mu.
\end{aligned}
$$

But we have

$$
\begin{gathered}
\sum_{\substack{i,j \in I_{\bar{0}} \\ i \neq j}} \left(\frac{\partial}{\partial x_i} - \frac{\partial}{\partial x_j}\right) = 0, \qquad \sum_{\substack{\bar{i},\bar{j} \in I_{\bar{1}} \\ \bar{i} \neq \bar{j}}} \left(\frac{\partial}{\partial y_{\bar{i}}} - \frac{\partial}{\partial y_{\bar{j}}}\right) = 0, \\
-\frac{1}{2} \sum_{\substack{i \in I_{\bar{0}} \\ \bar{j} \in I_{\bar{1}}}} \left(\frac{\partial}{\partial x_i} + \frac{\partial}{\partial y_{\bar{j}}}\right) + \frac{1}{2} \sum_{\substack{\bar{i} \in I_{\bar{1}} \\ j \in I_{\bar{0}}}} \left(\frac{\partial}{\partial y_{\bar{i}}} + \frac{\partial}{\partial x_j}\right) = 0.
\end{gathered}
$$

This concludes the proof. ∎

Corollary 6.2.7. Let $\mu \in \mathcal{A}(\mathfrak{g},\mathfrak{k})$, and assume that Φ_μ (or equivalently μ) is an eigenfunction of the Casimir operator. Set $\bar{\Phi}_\mu := \Phi_\mu(\log x_1, \ldots, \log x_m, \log y_1, \ldots, \log y_n)$. Then, for some $c \in \mathbb{C}$ we have

$$
\begin{aligned}
c\bar{\Phi}_\mu = &\sum_{\substack{i,j\in I_{\bar{0}} \\ i\neq j}} \frac{x_i}{x_i - x_j} \left(x_i \frac{\partial}{\partial x_i} - x_j \frac{\partial}{\partial x_j}\right) \Phi_\mu \\
&- \sum_{\substack{\bar{i},\bar{j}\in I_{\bar{1}} \\ \bar{i}\neq\bar{j}}} \frac{y_{\bar{i}}}{y_{\bar{i}} - y_{\bar{j}}} \left(y_{\bar{i}} \frac{\partial}{\partial y_{\bar{i}}} - y_{\bar{j}} \frac{\partial}{\partial y_{\bar{j}}}\right) \Phi_\mu \\
&- \sum_{\substack{i\in I_{\bar{0}} \\ j\in I_{\bar{1}}}} \frac{x_i + y_{\bar{j}}}{x_i - y_{\bar{j}}} \left(x_i \frac{\partial}{\partial x_i} + y_{\bar{j}} \frac{\partial}{\partial y_{\bar{j}}}\right) \Phi_\mu \\
&+ \sum_{i\in I_{\bar{0}}} \left(x_i \frac{\partial}{\partial x_i}\right)^2 \Phi_\mu - \sum_{\bar{i}\in I_{\bar{1}}} \left(y_{\bar{i}} \frac{\partial}{\partial y_{\bar{i}}}\right)^2 \Phi_\mu.
\end{aligned}
$$

Proof. By Proposition 6.2.6 we have

$$
\Phi_{\Omega\cdot\mu} = \sum_{\substack{i,j\in I_{\bar{0}} \\ i\neq j}} \frac{-e^{x_i-x_j}}{e^{-x_i+x_j} - e^{x_i-x_j}} \left(\frac{\partial}{\partial x_i} - \frac{\partial}{\partial x_j}\right) \Phi_\mu
$$

$$
\begin{aligned}
&+\sum_{\substack{\bar{i},\bar{j}\in I_{\bar{1}}\\ \bar{i}\neq\bar{j}}}\frac{e^{y_{\bar{i}}-y_{\bar{j}}}}{e^{-y_{\bar{i}}+y_{\bar{j}}}-e^{y_{\bar{i}}-y_{\bar{j}}}}\left(\frac{\partial}{\partial y_{\bar{i}}}-\frac{\partial}{\partial y_{\bar{j}}}\right)\Phi_\mu\\
&+\sum_{\substack{i\in I_{\bar{0}}\\ \bar{j}\in I_{\bar{1}}}}\frac{-e^{x_i-y_{\bar{j}}}}{e^{-x_i+y_{\bar{j}}}-e^{x_i-y_{\bar{j}}}}\left(\frac{\partial}{\partial x_i}+\frac{\partial}{\partial y_{\bar{j}}}\right)\Phi_\mu\\
&+\sum_{\substack{\bar{i}\in I_{\bar{1}}\\ j\in I_{\bar{0}}}}\frac{e^{y_{\bar{i}}-x_j}}{e^{-y_{\bar{i}}+x_j}-e^{y_{\bar{i}}-x_j}}\left(\frac{\partial}{\partial y_{\bar{i}}}+\frac{\partial}{\partial x_j}\right)\Phi_\mu\\
&+\frac{1}{2}\sum_{i\in I_{\bar{0}}}\frac{\partial^2}{\partial x_i^2}\Phi_\mu-\frac{1}{2}\sum_{\bar{i}\in I_{\bar{1}}}\frac{\partial^2}{\partial y_{\bar{i}}^2}\Phi_\mu.
\end{aligned}
$$

Therefore, by applying the substitutions $x_i\to e^{2x_i}$ for $i=1,\cdots,m$ and $y_{\bar{i}}\to e^{2y_{\bar{i}}}$ for $i=1,\cdots,n$, we transform the variables to exponential coordinates. Thus, we obtain

$$
\begin{aligned}
\bar{\Phi}_{\Omega\cdot\mu}=\;&2\sum_{\substack{i,j\in I_{\bar{0}}\\ i\neq j}}\frac{x_i}{x_i-x_j}\left(x_i\frac{\partial}{\partial x_i}-x_j\frac{\partial}{\partial x_j}\right)\Phi_\mu\\
&-2\sum_{\substack{\bar{i},\bar{j}\in I_{\bar{1}}\\ \bar{i}\neq\bar{j}}}\frac{y_{\bar{i}}}{y_{\bar{i}}-y_{\bar{j}}}\left(y_{\bar{i}}\frac{\partial}{\partial y_{\bar{i}}}-y_{\bar{j}}\frac{\partial}{\partial y_{\bar{j}}}\right)\Phi_\mu\\
&-2\sum_{\substack{i\in I_{\bar{0}}\\ j\in I_{\bar{1}}}}\frac{x_i+y_{\bar{j}}}{x_i-y_{\bar{j}}}\left(x_i\frac{\partial}{\partial x_i}+y_{\bar{j}}\frac{\partial}{\partial y_{\bar{j}}}\right)\Phi_\mu\\
&+2\sum_{i\in I_{\bar{0}}}\left(x_i\frac{\partial}{\partial x_i}+x_i^2\frac{\partial^2}{\partial x_i^2}\right)\Phi_\mu\\
&-2\sum_{\bar{i}\in I_{\bar{1}}}\left(y_{\bar{i}}\frac{\partial}{\partial y_{\bar{i}}}+y_{\bar{i}}^2\frac{\partial^2}{\partial y_{\bar{i}}^2}\right)\Phi_\mu.
\end{aligned}
$$

Since the Casimir element is in the center of $U(\mathfrak{g})$, it acts on $\mu\in\mathcal{A}(\mathfrak{g},\mathfrak{k})$ by a scalar $c_\Omega\in\mathbb{C}$. Therefore,

$$
\begin{aligned}
\frac{1}{2}c_\Omega\bar{\Phi}_\mu=\;&\sum_{\substack{i,j\in I_{\bar{0}}\\ i\neq j}}\frac{x_i}{x_i-x_j}\left(x_i\frac{\partial}{\partial x_i}-x_j\frac{\partial}{\partial x_j}\right)\Phi_\mu\\
&-\sum_{\substack{\bar{i},\bar{j}\in I_{\bar{1}}\\ \bar{i}\neq\bar{j}}}\frac{y_{\bar{i}}}{y_{\bar{i}}-y_{\bar{j}}}\left(y_{\bar{i}}\frac{\partial}{\partial y_{\bar{i}}}-y_{\bar{j}}\frac{\partial}{\partial y_{\bar{j}}}\right)\Phi_\mu\\
&-\sum_{\substack{i\in I_{\bar{0}}\\ j\in I_{\bar{1}}}}\frac{x_i+y_{\bar{j}}}{x_i-y_{\bar{j}}}\left(x_i\frac{\partial}{\partial x_i}+y_{\bar{j}}\frac{\partial}{\partial y_{\bar{j}}}\right)\Phi_\mu
\end{aligned}
$$

$$+\sum_{i\in I_{\bar{0}}}\left(x_i\frac{\partial}{\partial x_i}\right)^2\Phi_\mu-\sum_{\bar{i}\in I_{\bar{1}}}\left(y_{\bar{i}}\frac{\partial}{\partial y_{\bar{i}}}\right)^2\Phi_\mu.$$ ∎

6.3 The action of Casimir operator on $\mathcal{A}(\mathfrak{g},\mathfrak{k})$ in Case II of Table 4.1

Let $(\mathfrak{g},\mathfrak{k})=(\mathfrak{gl}(m|2n),\mathfrak{osp}(m|2n))$. For $\lambda\in\mathcal{A}(\mathfrak{g},\mathfrak{k})$ we let Φ_λ be as in Chapter 5, where in this case from Section 4.6 we know that $h_i=E_{i,i}$ for $1\leqslant i\leqslant m$ and $h_{\bar{j}}=E_{\bar{j},\bar{j}}+E_{\bar{j}+n,\bar{j}+n}$ for $1\leqslant j\leqslant n$. Let $\{E_{ij}\,|\,i,j\in I=I_{\bar{0}}\sqcup I_{\bar{1}}\}$ be the basis for $\mathfrak{gl}(m|2n)$ consisting of matrix units where $I_{\bar{0}}=\{1,\cdots,m\}$ and $I_{\bar{1}}=\{\bar{1},\bar{2},\cdots,\overline{2n}\}$. Then the Casimir operator of $U(\mathfrak{g})$ is

$$\Omega:=\sum_{i,j\in I}(-1)^{|i|}E_{ij}E_{ji}.$$

Lemma 6.3.1. Every polynomial $p(x)\in\mathbb{C}[x]$ can be expressed as $q(x+2)-q(x-2)$ for some $q(x)\in\mathbb{C}[x]$.

Proof. We need to show that for a given polynomial $p(x)\in\mathbb{C}[x]$ of degree n, there exists a polynomial $q(x)\in\mathbb{C}[x]$ such that:

$$p(x)=q(x+2)-q(x-2).$$

Consider a substitution$x=2y+2$, leading to:

$$p(2y+2)=q(2y+4)-q(2y).$$

By defining new polynomials $p^*(y)=p(2y+2)$ and $q^*(y)=q(2y)$ we can rewrite the equation as:

$$p^*(y)=q^*(y+2)-q^*(y).$$

To proceed, we define $p^{**}(y)=p^*(2y)$ and $q^{**}(y)=q^*(2y)$, yielding:

$$p^{**}(\frac{y}{2})=q^{**}(\frac{y}{2}+1)-q^{**}(\frac{y}{2}),$$

and by substitution $w=y/2$ we have

$$p^{**}(w)=q^{**}(w+1)-q^{**}(w).$$

Hence, it suffices to prove that there exists a polynomial $q^{**}(x)\in\mathbb{C}[x]$ which satisfies the above equation. But we can consider a possible choice for $q^{**}(x)$ based on the falling factorial identity: [13, Section 2.1]

$$(x+1)^{\bar{n}}-x^{\bar{n}}=nx^{\overline{n-1}}.$$

where $x^{\bar{n}}=x(x-1)\dots(x-n+1)$ is a falling factorial, and the right-hand side is a polynomial of degree $n-1$. This suggests a construction for $q^{**}(x)$ that satisfies the equation. Therefore, such a polynomial $q(x)$ exists. ∎

Lemma 6.3.2. Let $(\mathfrak{g},\mathfrak{k}) = (\mathfrak{gl}(1|2), \mathfrak{osp}(1|2))$. Set

$$h_1 := E_{11}, \quad h_{\bar{1}} := E_{\bar{1}\bar{1}} + E_{\bar{2}\bar{2}}, \quad e_1 := E_{1\bar{1}}, \quad e_2 := E_{1\bar{2}}, \quad f_1 := E_{\bar{1}1}, \quad f_2 := E_{\bar{2}1}.$$

Then for every polynomial $p(x) \in \mathbb{C}[x]$ we have

$$p(h)e_1 \in \mathcal{I}, \quad p(h)e_2 \in \mathcal{I}, \quad p(h)f_1 \in \mathcal{I}, \quad p(h)f_2 \in \mathcal{I}.$$

Proof. One can see that $(e_1 + f_2), (e_2 - f_1) \in \mathfrak{k}$ and

$$[h, e_1] = 2e_1, \quad [h, e_2] = 2e_2, \quad [h, f_1] = -2f_1, \quad [h, f_2] = -2f_2,$$

where $h = h_1 - h_{\bar{1}}$ and $z = h_1 + h_{\bar{1}}$. Therefore, For $q(x) \in \mathbb{C}[x]$ we have

$$\begin{aligned}(e_1 + f_2)q(h) &= q(h-2)e_1 + q(h+2)f_2 \\ &= \frac{1}{2}\left(q(h-2) - q(h+2)\right)(e_1 - f_2) \\ &+ \frac{1}{2}\left(q(h-2) + q(h+2)\right)(e_1 + f_2).\end{aligned}$$

Since $e_1 + f_2 \in \mathfrak{k}$, from the above calculation we obtain

$$\frac{1}{2}\left(q(h-2) - q(h+2)\right)(e_1 - f_2) \in \mathcal{I}.$$

Also, for any $p(x) \in \mathbb{C}[x]$ we have $p(h)(e_1 + f_2) \in \mathcal{I}$. Therefore, from the last two relations and Lemma 6.3.1 we obtain $p(h)e_1, p(h)f_2 \in \mathcal{I}$. Furthermore,

$$\begin{aligned}(e_2 - f_1)q(h) &= q(h-2)e_2 + q(h+2)f_1 \\ &= \frac{1}{2}\left(q(h-2) - q(h+2)\right)(e_2 - f_1) \\ &+ \frac{1}{2}\left(q(h-2) + q(h+2)\right)(e_2 + f_1).\end{aligned}$$

Since $e_2 - f_1 \in \mathfrak{k}$, from the above calculation we obtain

$$\frac{1}{2}\left(q(h-2) - q(h+2)\right)(e_2 - f_1) \in \mathcal{I},$$

and for any $p(x) \in \mathbb{C}[x]$ we have $p(h)(e_2 - f_1) \in \mathcal{I}$. Hence, as in the previous case, for every polynomial $p(x) \in \mathbb{C}[x]$ we obtain $p(h)e_2, p(h)f_1 \in \mathbb{C}[x]$. ∎

Lemma 6.3.3. Let $p(x) \in \mathbb{C}[x]$. Then by keeping the same notation of Lemma 6.3.2 and for $\mu \in \mathcal{A}(\mathfrak{g},\mathfrak{k})$,

$$\mu((p(h-2) - p(h+2))\, e_1 f_1) = \mu((p(h-2) - p(h+2))\, e_2 f_2) = -\mu(p(h+2)(h_1 + \frac{1}{2}h_{\bar{1}})),$$

where $h = h_1 - h_{\bar{1}}$.

Proof. One can see that

$$[e_1, f_1] = z - \frac{1}{2}h_{\bar{1}} + \frac{1}{2}k_1$$
$$[e_2, f_2] = z - \frac{1}{2}h_{\bar{1}} + \frac{1}{2}k_2,$$

where $z = h_1 + h_{\bar{1}}$, $k_1 := E_{\bar{1}\bar{1}} - E_{\bar{2}\bar{2}} \in \mathfrak{k}$, and $k_2 := -E_{\bar{1}\bar{1}} + E_{\bar{2}\bar{2}} \in \mathfrak{k}$. Therefore,

$$\begin{aligned}
&(p(h-2) - p(h+2))\, e_1 f_1 + p(h+2)[e_1, f_1] \\
&= p(h-2)e_1 f_1 - p(h+2)e_1 f_1 + p(h+2)e_1 f_1 + p(h+2) f_1 e_1 \\
&= p(h-2)e_1 f_1 + p(h+2) f_1 e_1 \\
&= p(h-2)e_1 f_1 + p(h+2) f_1 e_1 + f_2 p(h) f_1 - f_2 p(h) f_1 \\
&= (e_1 + f_2)p(h) f_1 + f_1 p(h)(e_1 + f_2) \in \mathcal{I}.
\end{aligned}$$

The last equality is due to the relation $f_2 p(h) f_1 = -f_1 p(h) f_2$ because $f_1 f_2 = -f_2 f_1$. Thus, for $\mu \in \mathcal{A}(\mathfrak{g},\mathfrak{k})$,

$$\mu((p(h-2) - p(h+2))\, e_1 f_1) = -\mu(p(h+2)(z - \frac{1}{2}h_{\bar{1}} + \frac{1}{2}k_1)).$$

Also,

$$\begin{aligned}
&(p(h-2) - p(h+2))\, e_2 f_2 + p(h+2)[e_2, f_2] \\
&= p(h-2)e_2 f_2 - p(h+2)e_2 f_2 + p(h+2)e_2 f_2 + p(h+2) f_2 e_2 \\
&= p(h-2)e_2 f_2 + p(h+2) f_2 e_2 \\
&= p(h-2)e_2 f_2 + p(h+2) f_2 e_2 - f_1 p(h) f_2 + f_1 p(h) f_2 \\
&= (e_2 - f_1)p(h) f_2 + f_2 p(h)(e_2 - f_1) \in \mathcal{I}.
\end{aligned}$$

Again the last equality is due to the relation $f_2 p(h) f_1 = -f_1 p(h) f_2$. Thus, for $\mu \in \mathcal{A}(\mathfrak{g},\mathfrak{k})$,

$$\mu((p(h-2) - p(h+2))\, e_2 f_2) = -\mu(p(h+2)(z - \frac{1}{2}h_{\bar{1}} + \frac{1}{2}k_2)).$$

But $k_1, k_2 \in \mathfrak{k}$. Hence, the assertion follows. ∎

Lemma 6.3.4. Let $\mu \in \mathcal{A}(\mathfrak{g},\mathfrak{k})$. Set

$$\mu' = E_{ij}E_{ji} \cdot \mu \quad \text{for} \quad i, j \in I_{\bar{0}} \quad \text{and} \quad i \neq j.$$

Then,

$$\Phi_{\mu'} = \frac{-e^{x_i - x_j}}{e^{-x_i + x_j} - e^{x_i - x_j}}\left(\frac{\partial}{\partial x_i} - \frac{\partial}{\partial x_j}\right)\Phi_\mu.$$

Proof. By Lemma 6.1.3 we have

$$(1) \quad e^{x_i - x_j}\Phi_{\mu'} = \sum \mu'[\vec{a}, i, j, 1, -1] \frac{\prod_{r=1}^m x_r^{a_r} \prod_{s=1}^n y_s^{a_{\bar{s}}}}{\vec{a}!},$$

and

$$(2) \quad e^{-x_i+x_j}\Phi_{\mu'} = \sum \mu'[\vec{a}, i, j, -1, 1]\frac{\prod_{i=1}^m x_i^{a_i}\prod_{j=1}^n y_j^{a_{\bar{j}}}}{\vec{a}!}.$$

By straightforward computations we have

$$\begin{aligned}\mu'[\vec{a}, i, j, 1, -1] &:= \mu'(h_1^{a_1}\cdots(h_i+1)^{a_i}(h_j-1)^{a_j}\cdots h_m^{a_m}h_{\bar{1}}^{a_{\bar{1}}}\cdots h_{\bar{n}}^{a_{\bar{n}}}) \\ &= \frac{1}{2^{a_i+a_j}}\mu'(h_1^{a_1}\cdots((z+(h+2))^{a_i}(z-(h+2))^{a_j})\cdots h_m^{a_m}h_{\bar{1}}^{a_{\bar{1}}}\cdots h_{\bar{n}}^{a_{\bar{n}}}),\end{aligned}$$

where $h = h_i - h_j$ and $z = h_i + h_j$. Also,

$$\begin{aligned}\mu'[\vec{a}, i, j, -1, 1] &:= \mu'(h_1^{a_1}\cdots(h_i+1)^{a_i}(h_j-1)^{a_j}\cdots h_m^{a_m}h_{\bar{1}}^{a_{\bar{1}}}\cdots h_{\bar{n}}^{a_{\bar{n}}}) \\ &= \frac{1}{2^{a_i+a_j}}\mu'(h_1^{a_1}\cdots((z+(h-2))^{a_i}(z-(h-2))^{a_j})\cdots h_m^{a_m}h_{\bar{1}}^{a_{\bar{1}}}\cdots h_{\bar{n}}^{a_{\bar{n}}}).\end{aligned}$$

We have

$$\begin{aligned}&((z+(h-2))^{a_i}(z-(h-2))^{a_j}) - ((z+(h+2))^{a_i}(z-(h+2))^{a_j}) \\ &= \sum_{\substack{0 \leqslant k \leqslant a \\ 0 \leqslant l \leqslant b}} \binom{a_i}{k}\binom{a_j}{l} z^{k+l}(p_{k,l}(h-2) - p_{k,l}(h+2)),\end{aligned}$$

where $p_{k,l}(h) = (-1)^{a_j-1}h^{a_i-k+a_j-l}$. For $1 \le i < j \le m$, $h := h_i - h_j, z := h_i + h_j, E_{ij}, E_{ji}$ is a $\mathfrak{gl}_2$ basis and $E_{ij} - E_{ji} \in \mathfrak{k}$. By identifying h_{ij}, E_{ij}, E_{ji} to be respectively h, e, f for $1 \le i < j \le m$ we can utilize Lemma 6.1.4. It follows that

$$\mu'((p_{k,l}(h-2) - p_{k,l}(h+2))) = \mu((p_{k,l}(h-2) - p_{k,l}(h+2))E_{i,j}E_{j,i}) = -\mu(p_{k,l}(h+2)h).$$

Therefore,

$$\begin{aligned}&\mu'[\vec{a}, i, j, -1, 1] - \mu'[\vec{a}, i, j, 1, -1] \\ &= \frac{1}{2^{a_i+a_j}} \sum_{\substack{0 \leqslant k \leqslant a \\ 0 \leqslant l \leqslant b}} \binom{a_i}{k}\binom{a_j}{l}\mu'(h_1^{a_1}\cdots\left(z^{k+l}(p_{k,l}(h-2) - p_{k,l}(h+2))\right)\cdots h_m^{a_m}h_{\bar{1}}^{a_{\bar{1}}}\cdots h_{\bar{n}}^{a_{\bar{n}}}) \\ &= -\frac{1}{2^{a_i+a_j}} \sum_{\substack{0 \leqslant k \leqslant a \\ 0 \leqslant l \leqslant b}} \binom{a_i}{k}\binom{a_j}{l}\mu(h_1^{a_1}\cdots\left(z^{k+l}p_{k,l}(h+2)h\right)\cdots h_m^{a_m}h_{\bar{1}}^{a_{\bar{1}}}\cdots h_{\bar{n}}^{a_{\bar{n}}}) \\ &= -\mu\left(h_1^{a_1}\cdots((z+(h+2))/2)^{a_i}\left((z-(h+2))/2\right)^{a_j} h\cdots h_m^{a_m}h_{\bar{1}}^{a_{\bar{1}}}\cdots h_{\bar{n}}^{a_{\bar{n}}}\right) \\ &= -\mu\left(h_1^{a_1}\cdots(h_i+1)^{a_i}\left(h_j-1\right)^{a_j} h\cdots h_m^{a_m}h_{\bar{1}}^{a_{\bar{1}}}\cdots h_{\bar{n}}^{a_{\bar{n}}}\right).\end{aligned}$$

From (1) and (2) and summing over both sides of the above multi-line calculation after multiplication with suitable monomials we obtain that the left hand side yields

$$(e^{-x_i+x_j} - e^{x_i-x_j})\Phi_{\mu'},$$

and the right hand side yields

$$-e^{x_i-x_j}(\frac{\partial}{\partial x_i}-\frac{\partial}{\partial x_j})\Phi_\mu.$$

This completes the proof of the assertion. ∎

Lemma 6.3.5. For $1\leqslant i\neq j\leqslant n$ set

$$\begin{aligned}\mu_1&=(E_{\bar{i},\bar{j}}+E_{\bar{i}+n,\bar{j}+n})(E_{\bar{j}\bar{i}}+E_{\bar{j}+n,\bar{i}+n})\cdot\mu,\\ \mu_2&=(E_{\bar{i},\bar{j}+n}+E_{\bar{i}+n,\bar{j}})(E_{\bar{j}+n,\bar{i}}+E_{\bar{j},\bar{i}+n})\cdot\mu,\\ \mu_3&=(E_{\bar{i},\bar{j}}-E_{\bar{i}+n,\bar{j}+n})(E_{\bar{j},\bar{i}}-E_{\bar{j}+n,\bar{i}+n})\cdot\mu,\\ \mu_4&=(E_{\bar{i},\bar{j}+n}-E_{\bar{i}+n,\bar{j}})(E_{\bar{j}+n,\bar{i}}-E_{\bar{j},\bar{i}+n})\cdot\mu.\end{aligned}$$

Then,

$$\Phi_{\mu_1}=\Phi_{\mu_2}=\Phi_{\mu_3}=\Phi_{\mu_4}=\frac{-e^{y_{\bar{i}}-y_{\bar{j}}}}{e^{-y_{\bar{i}}+y_{\bar{j}}}-e^{y_{\bar{i}}-y_{\bar{j}}}}(\frac{\partial}{\partial y_{\bar{i}}}-\frac{\partial}{\partial y_{\bar{j}}})\Phi_\mu.$$

Proof. For $\bar{i}\neq\bar{j}$ one can observe that the sets

$$\begin{aligned}&\left\{(E_{\bar{i},\bar{j}}+E_{\bar{i}+n,\bar{j}+n}),(E_{\bar{j}\bar{i}}+E_{\bar{j}+n,\bar{i}+n}),h_{\bar{i}\bar{j}}:=h_{\bar{i}}-h_{\bar{j}},z_{\bar{i}\bar{j}}:=h_{\bar{i}}+h_{\bar{j}}\right\},\\ &\left\{(E_{\bar{i},\bar{j}+n}+E_{\bar{i}+n,\bar{j}}),(E_{\bar{j}+n,\bar{i}}+E_{\bar{j},\bar{i}+n}),h_{\bar{i}\bar{j}}:=h_{\bar{i}}-h_{\bar{j}},z_{\bar{i}\bar{j}}:=h_{\bar{i}}+h_{\bar{j}}\right\},\\ &\left\{(E_{\bar{i},\bar{j}}-E_{\bar{i}+n,\bar{j}+n}),(E_{\bar{j},\bar{i}}-E_{\bar{j}+n,\bar{i}+n}),h_{\bar{i}\bar{j}}:=h_{\bar{i}}-h_{\bar{j}},z_{\bar{i}\bar{j}}:=h_{\bar{i}}+h_{\bar{j}}\right\},\\ &\left\{(E_{\bar{i},\bar{j}+n}-E_{\bar{i}+n,\bar{j}}),(E_{\bar{j}+n,\bar{i}}-E_{\bar{j},\bar{i}+n}),h_{\bar{i}\bar{j}}:=h_{\bar{i}}-h_{\bar{j}},z_{\bar{i}\bar{j}}:=h_{\bar{i}}+h_{\bar{j}}\right\},\end{aligned}$$

are $\mathfrak{gl}_2$-bases. Hence, if we identify elements of every above $\mathfrak{gl}_2$-basis to be respectively e,f,h,z for $1\leq i\neq j\leq n$ we can apply Lemma 6.1.4 and then similar to the proof of Lemma 6.3.4 we can complete the proof. ∎

Lemma 6.3.6. Let $\mu\in\mathcal{A}(\mathfrak{g},\mathfrak{k})$. Then

$$\Phi_{(\sum_{\bar{i},\bar{j}\in I_{\mathrm{I}}}E_{\bar{i},\bar{j}}E_{\bar{j},\bar{i}})\cdot\mu}=2\sum_{1\leqslant i\neq j\leqslant n}\frac{-e^{y_{\bar{i}}-y_{\bar{j}}}}{e^{-y_{\bar{i}}+y_{\bar{j}}}-e^{y_{\bar{i}}-y_{\bar{j}}}}(\frac{\partial}{\partial y_{\bar{i}}}-\frac{\partial}{\partial y_{\bar{j}}})\Phi_\mu+\frac{1}{2}\sum_{1\leqslant i\leqslant n}\frac{\partial^2}{\partial y_{\bar{i}}^2}\Phi_\mu.$$

Proof. By straightforward computations we have

$$\begin{aligned}\sum_{\bar{i},\bar{j}\in I_{\bar{1}}}E_{\bar{i}\bar{j}}E_{\bar{j}\bar{i}}=&\frac{1}{2}\sum_{1\leq i\neq j\leq n}\big((E_{\bar{i}\bar{j}}+E_{\bar{i}+n,\bar{j}+n})(E_{\bar{j}\bar{i}}+E_{\bar{j}+n,\bar{i}+n})+(E_{\bar{i}\bar{j}}-E_{\bar{i}+n,\bar{j}+n})(E_{\bar{j}\bar{i}}-E_{\bar{j}+n,\bar{i}+n})\big)\\ &+\frac{1}{2}\sum_{1\leq i\neq j\leq n}\big((E_{\bar{i},\bar{j}+n}+E_{\bar{i}+n,\bar{j}})(E_{\bar{j}+n,\bar{i}}+E_{\bar{j},\bar{i}+n})+(E_{\bar{i},\bar{j}+n}-E_{\bar{i}+n,\bar{j}})(E_{\bar{j}+n,\bar{i}}-E_{\bar{j},\bar{i}+n})\big)\\ &+\frac{1}{2}\sum_{1\leq i\neq j\leq n}\big((E_{\bar{j}\bar{i}}+E_{\bar{j}+n,\bar{i}+n})(E_{\bar{i}\bar{j}}+E_{\bar{i}+n,\bar{j}+n})+(E_{\bar{j}\bar{i}}-E_{\bar{j}+n,\bar{i}+n})(E_{\bar{i}\bar{j}}-E_{\bar{i}+n,\bar{j}+n})\big)\\ &+\frac{1}{2}\sum_{1\leq i\neq j\leq n}\big((E_{\bar{j}+n,\bar{i}}+E_{\bar{j},\bar{i}+n})(E_{\bar{i},\bar{j}+n}+E_{\bar{i}+n,\bar{j}})+(E_{\bar{j}+n,\bar{i}}-E_{\bar{j},\bar{i}+n})(E_{\bar{i},\bar{j}+n}-E_{\bar{i}+n,\bar{j}})\big)\end{aligned}$$

$$+\frac{1}{2}\sum_{i=1}^{n}\big((E_{\bar{i},\bar{i}}+E_{\bar{i}+n,\bar{i}+n})(E_{\bar{i},\bar{i}}+E_{\bar{i}+n,\bar{i}+n})+(E_{\bar{i},\bar{i}}-E_{\bar{i}+n,\bar{i}+n})(E_{\bar{i},\bar{i}}-E_{\bar{i}+n,\bar{i}+n})\big)$$
$$+\sum_{i=1}^{n}(E_{\bar{i},\bar{i}+n}E_{\bar{i}+n,\bar{i}}+E_{\bar{i}+n,\bar{i}}E_{\bar{i},\bar{i}+n}).$$

But

$$[(E_{\bar{i}\bar{j}}+E_{\bar{i}+n,\bar{j}+n}),(E_{\bar{j}\bar{i}}+E_{\bar{j}+n,\bar{i}+n})]=[(E_{\bar{i}\bar{j}}-E_{\bar{i}+n,\bar{j}+n}),(E_{\bar{j}\bar{i}}-E_{\bar{j}+n,\bar{i}+n})]=h_{\bar{i}}-h_{\bar{j}},$$
$$[(E_{\bar{i},\bar{j}+n}+E_{\bar{i}+n,\bar{j}}),(E_{\bar{j}+n,\bar{i}}+E_{\bar{j},\bar{i}+n})]=[(E_{\bar{i},\bar{j}+n}-E_{\bar{i}+n,\bar{j}}),(E_{\bar{j}+n,\bar{i}}-E_{\bar{j},\bar{i}+n})]=h_{\bar{i}}-h_{\bar{j}},$$

and

$$(E_{\bar{i},\bar{i}}+E_{\bar{i}+n,\bar{i}+n})(E_{\bar{i},\bar{i}}+E_{\bar{i}+n,\bar{i}+n})=h_{\bar{i}}^2$$
$$(E_{\bar{i},\bar{i}}-E_{\bar{i}+n,\bar{i}+n})(E_{\bar{i},\bar{i}}-E_{\bar{i}+n,\bar{i}+n})=\tilde{h}_{\bar{i}}^2.$$

Therefore,

$$\sum_{\bar{i},\bar{j}\in I_{\bar{1}}}E_{\bar{i}\bar{j}}E_{\bar{j}\bar{i}}=\sum_{1\le i\neq j\le n}(E_{\bar{i}\bar{j}}+E_{\bar{i}+n,\bar{j}+n})(E_{\bar{j}\bar{i}}+E_{\bar{j}+n,\bar{i}+n})+\sum_{1\le i\neq j\le n}(E_{\bar{i}\bar{j}}-E_{\bar{i}+n,\bar{j}+n})(E_{\bar{j}\bar{i}}-E_{\bar{j}+n,\bar{i}+n})$$
$$+\sum_{1\le i\neq j\le n}(E_{\bar{i},\bar{j}+n}+E_{\bar{i}+n,\bar{j}})(E_{\bar{j}+n,\bar{i}}+E_{\bar{j},\bar{i}+n})+\sum_{1\le i\neq j\le n}(E_{\bar{i},\bar{j}+n}-E_{\bar{i}+n,\bar{j}})(E_{\bar{j}+n,\bar{i}}-E_{\bar{j},\bar{i}+n})$$
$$+\frac{1}{2}\sum_{i=1}^{n}h_{\bar{i}}^2+\frac{1}{2}\sum_{i=1}^{n}\tilde{h}_{\bar{i}}^2+\sum_{i=1}^{n}(E_{\bar{i},\bar{i}+n}E_{\bar{i}+n,\bar{i}}+E_{\bar{i}+n,\bar{i}}E_{\bar{i},\bar{i}+n})-2\sum_{1\le i\neq j\le n}h_{\bar{i}}-h_{\bar{j}}.$$

Since $\tilde{h}_{\bar{i}}\in\mathfrak{k}$ and $E_{\bar{i},\bar{i}+n},E_{\bar{i}+n,\bar{i}}$ are in the centralizer of $\mathfrak{a}$ in $\mathfrak{k}$, for $\mu\in\mathcal{A}(\mathfrak{g},\mathfrak{k})$ we have

$$\left(\frac{1}{2}\sum_{i=1}^{n}\tilde{h}_{\bar{i}}^2+\sum_{i=1}^{n}(E_{\bar{i},\bar{i}+n}E_{\bar{i}+n,\bar{i}}+E_{\bar{i}+n,\bar{i}}E_{\bar{i},\bar{i}+n})\right)\cdot\mu=0.$$

Thus, by Lemmas 6.1.2, and 6.3.5 we have

$$\Phi_{(\sum_{\bar{i},\bar{j}\in I_{\bar{1}}}E_{\bar{i},\bar{j}}E_{\bar{j},\bar{i}})\cdot\mu}=4\sum_{1\leqslant i\neq j\leqslant n}\frac{-e^{y_{\bar{i}}-y_{\bar{j}}}}{e^{-y_{\bar{i}}+y_{\bar{j}}}-e^{y_{\bar{i}}-y_{\bar{j}}}}(\frac{\partial}{\partial y_{\bar{i}}}-\frac{\partial}{\partial y_{\bar{j}}})\Phi_\mu$$
$$+\frac{1}{2}\sum_{1\leqslant i\leqslant n}\frac{\partial^2}{\partial y_{\bar{i}}^2}\Phi_\mu-2\sum_{1\leqslant i\neq\leqslant n}\left(\frac{\partial}{\partial y_{\bar{i}}}-\frac{\partial}{\partial y_{\bar{j}}}\right)\Phi_\mu$$
$$=2\sum_{1\leqslant i\neq j\leqslant n}\frac{e^{2y_{\bar{i}}}+e^{2y_{\bar{j}}}}{e^{2y_{\bar{i}}}-e^{2y_{\bar{j}}}}(\frac{\partial}{\partial y_{\bar{i}}}-\frac{\partial}{\partial y_{\bar{j}}})\Phi_\mu+\frac{1}{2}\sum_{1\leqslant i\leqslant n}\frac{\partial^2}{\partial y_{\bar{i}}^2}\Phi_\mu.\qquad\blacksquare$$

Lemma 6.3.7. Let $\mu\in\mathcal{A}(\mathfrak{g},\mathfrak{k})$. Set

$$\mu'=(E_{i,\bar{j}}E_{\bar{j},i}).\mu,\quad \mu''=(E_{i,\bar{j}+n}E_{\bar{j}+n,i}).\mu,\quad\text{for}\quad 1\leqslant i\leqslant m,\quad\text{and}\quad 1\leqslant j\leqslant n.$$

Then,

$$\Phi_{\mu'}=\Phi_{\mu''}=\frac{-e^{x_i-y_{\bar{j}}}}{e^{-x_i+y_{\bar{j}}}-e^{x_i-y_{\bar{j}}}}(\frac{\partial}{\partial x_i}+\frac{1}{2}\frac{\partial}{\partial y_{\bar{j}}})\Phi_\mu,\quad\text{for}\quad 1\leqslant i\leqslant m,\quad\text{and}\quad 1\leqslant j\leqslant n.$$

Proof. By Lemma 6.1.3 we have

$$(1) \quad e^{x_i - y_{\bar{j}}}\Phi_{\mu'} = \sum \mu'[\vec{a}, i, \bar{j}, 1, -1]\frac{\prod_{r=1}^m x_r^{a_r} \prod_{s=1}^n y_{\bar{s}}^{a_{\bar{s}}}}{\vec{a}!},$$

and

$$(2) \quad e^{-x_i + y_{\bar{j}}}\Phi_{\mu'} = \sum \mu'[\vec{a}, i, \bar{j}, -1, 1]\frac{\prod_{r=1}^m x_r^{a_r} \prod_{s=1}^n y_{\bar{s}}^{a_{\bar{s}}}}{\vec{a}!}.$$

By straightforward computations we have

$$\begin{aligned}\mu'[\vec{a}, i, \bar{j}, 1, -1] &:= \mu'(h_1^{a_1}\cdots h_m^{a_m}(h_i+1)^{a_i}(h_{\bar{j}}-1)^{a_{\bar{j}}}h_{\bar{1}}^{a_{\bar{1}}}\cdots h_{\bar{n}}^{a_{\bar{n}}})\\ &= \frac{1}{2^{a_i+a_{\bar{j}}}}\mu'(h_1^{a_1}\cdots h_m^{a_m}\left((z+(h+2))^{a_i}(z-(h+2))^{a_{\bar{j}}}\right)h_{\bar{1}}^{a_{\bar{1}}}\cdots h_{\bar{n}}^{a_{\bar{n}}}),\end{aligned}$$

where $h = h_i - h_{\bar{j}}$ and $z = h_i + h_{\bar{j}}$. Also,

$$\begin{aligned}\mu'[\vec{a}, i, \bar{j}, -1, 1] &:= \mu'(h_1^{a_1}\cdots h_m^{a_m}(h_i+1)^{a_i}(h_{\bar{j}}-1)^{a_{\bar{j}}}h_{\bar{1}}^{a_{\bar{1}}}\cdots h_{\bar{n}}^{a_{\bar{n}}})\\ &= \frac{1}{2^{a_i+a_{\bar{j}}}}\mu'(h_1^{a_1}\cdots h_m^{a_m}\left((z+(h-2))^{a_i}(z-(h-2))^{a_{\bar{j}}}\right)h_{\bar{1}}^{a_{\bar{1}}}\cdots h_{\bar{n}}^{a_{\bar{n}}}).\end{aligned}$$

We have

$$\begin{aligned}&((z+(h-2))^{a_i}(z-(h-2))^{a_{\bar{j}}}) - ((z+(h+2))^{a_i}(z-(h+2))^{a_{\bar{j}}})\\ &= \sum_{\substack{0\leqslant k\leqslant a\\ 0\leqslant l\leqslant b}} \binom{a_i}{k}\binom{a_{\bar{j}}}{l} z^{k+l}(p_{k,l}(h-2) - p_{k,l}(h+2)),\end{aligned}$$

where $p_{k,l}(h) = (-1)^{a_{\bar{j}}-1}h^{a_i-k+a_{\bar{j}}-l}$. Identify $E_{i,\bar{j}}, E_{\bar{j},i}, h_i, h_{\bar{j}}, h_{i\bar{j}}$ respectively to be $e_1, f_1, h_1, h_{\bar{1}}, h$. Then for the polynomials $p_{k,l}(h)$ we can apply Lemma 6.3.3 which implies that

$$\mu'((p_{k,l}(h-2) - p_{k,l}(h+2)) = -\mu(p_{k,l}(h+2)(h_i + \frac{1}{2}h_{\bar{j}})).$$

Therefore,

$$\begin{aligned}&\mu'[\vec{a}, i, j, -1, 1] - \mu'[\vec{a}, i, j, 1, -1]\\ &= \frac{1}{2^{a_i+a_j}} \sum_{\substack{0\leqslant k\leqslant a\\ 0\leqslant l\leqslant b}} \binom{a_i}{k}\binom{a_j}{l}\mu'(h_1^{a_1}\cdots\left(z^{k+l}(p_{k,l}(h-2)-p_{k,l}(h+2))\right)\cdots h_m^{a_m}h_{\bar{1}}^{a_{\bar{1}}}\cdots h_{\bar{n}}^{a_{\bar{n}}})\\ &= -\frac{1}{2^{a_i+a_j}} \sum_{\substack{0\leqslant k\leqslant a\\ 0\leqslant l\leqslant b}} \binom{a_i}{k}\binom{a_j}{l}\mu(h_1^{a_1}\cdots\left(z^{k+l}p_{k,l}(h+2)(h_1+\frac{1}{2}h_{\bar{1}})\right)\cdots h_m^{a_m}h_{\bar{1}}^{a_{\bar{1}}}\cdots h_{\bar{n}}^{a_{\bar{n}}})\\ &= -\mu\left(h_1^{a_1}\cdots((z+(h+2))/2)^{a_i}((z-(h+2))/2)^{a_j} z\cdots h_m^{a_m}h_{\bar{1}}^{a_{\bar{1}}}\cdots h_{\bar{n}}^{a_{\bar{n}}}\right)\\ &= -\mu\left(h_1^{a_1}\cdots(h_i+1)^{a_i}\left(h_{\bar{j}}-1\right)^{a_{\bar{j}}}(h_1+\frac{1}{2}h_{\bar{1}})\cdots h_m^{a_m}h_{\bar{1}}^{a_{\bar{1}}}\cdots h_{\bar{n}}^{a_{\bar{n}}}\right).\end{aligned}$$

From (1) and (2) and summing over both sides of the above multi-line calculation after multiplication with suitable monomials we obtain that the left hand side yields

$$(e^{-x_i+y_{\bar{j}}}-e^{x_i-y_{\bar{j}}})\Phi_{\mu'},$$

and the right hand side yields

$$-e^{x_i-y_{\bar{j}}}(\frac{\partial}{\partial x_i}+\frac{1}{2}\frac{\partial}{\partial y_{\bar{j}}})\Phi_{\mu}.$$

Hence

$$\Phi_{\mu'}=\frac{-e^{x_i-y_j}}{e^{-x_i+y_j}-e^{x_i-y_j}}(\frac{\partial}{\partial x_i}+\frac{1}{2}\frac{\partial}{\partial y_j})\Phi_{\mu}.$$

Also we can identify $E_{i,\bar{j}+n},E_{\bar{j}+n,i},h_i,h_{\bar{j}+n},h_{i,\bar{j}+n}$ respectively to be $e_2,f_2,h_1,h_{\bar{1}},h$. Again, by Lemma 6.3.3 we have

$$\mu''((p_{k,l}(h-2)-p_{k,l}(h+2))=-\mu(p_{k,l}(h+2)(h_i+\frac{1}{2}h_{\bar{j}})).$$

Hence, by a similar argument we obtain

$$\Phi_{\mu''}=\frac{-e^{x_i-y_{\bar{j}}}}{e^{-x_i+y_{\bar{j}}}-e^{x_i-y_{\bar{j}}}}(\frac{\partial}{\partial x_i}+\frac{1}{2}\frac{\partial}{\partial y_{\bar{j}}})\Phi_{\mu}.$$ ∎

Lemma 6.3.8. Let $\mu\in\mathcal{A}(\mathfrak{g},\mathfrak{k})$. Set

$$\mu'=(E_{\bar{j},i}E_{i,\bar{j}}).\mu,\quad \mu''=(E_{\bar{j}+n,i}E_{i,\bar{j}+n}).\mu,\quad \text{for}\quad 1\leqslant i\leqslant m,\quad \text{and}\quad 1\leqslant j\leqslant n.$$

Then,

$$\Phi_{\mu'}=\Phi_{\mu''}=\frac{-e^{-x_i+y_{\bar{j}}}}{e^{x_i-y_{\bar{j}}}-e^{-x_i+y_{\bar{j}}}}\left(\frac{\partial}{\partial x_i}+\frac{1}{2}\frac{\partial}{\partial y_{\bar{j}}}\right)\Phi_{\mu}.$$

Proof. By straightforward computation we have

$$[E_{i,\bar{j}},E_{\bar{j},i}]=E_{ii}+E_{\bar{j},\bar{j}}=h_i+\frac{1}{2}h_{\bar{j}}+\frac{1}{2}(E_{\bar{j},\bar{j}}-E_{\bar{j}+n,\bar{j}+n}),$$

and

$$[E_{\bar{j}+n,i},E_{i,\bar{j}+n}]=E_{\bar{j}+n,\bar{j}+n}+E_{i,i}=h_i+\frac{1}{2}h_{\bar{j}}+\frac{1}{2}(-E_{\bar{j},\bar{j}}+E_{\bar{j}+n,\bar{j}+n}).$$

Therefore,

$$E_{\bar{j},i}E_{i,\bar{j}}=h_i+\frac{1}{2}h_{\bar{j}}+\frac{1}{2}(E_{\bar{j},\bar{j}}-E_{\bar{j}+n,\bar{j}+n})-E_{i,\bar{j}}E_{\bar{j},i}$$

and

$$E_{\bar{j}+n,i}E_{i,\bar{j}+n}=h_i+\frac{1}{2}h_{\bar{j}}+\frac{1}{2}(-E_{\bar{j},\bar{j}}+E_{\bar{j}+n,\bar{j}+n})-E_{i,\bar{j}+n}E_{\bar{j}+n,i}.$$

But $(E_{\bar{j},\bar{j}} - E_{\bar{j}+n,\bar{j}+n}), (-E_{\bar{j},\bar{j}} + E_{\bar{j}+n,\bar{j}+n}) \in \mathfrak{k}$. Thus,

$$\left(E_{\bar{j},i}E_{i,\bar{j}}\right)\cdot\mu = \left(h_i + \frac{1}{2}h_{\bar{j}}\right)\cdot\mu - \left(E_{i,\bar{j}}E_{\bar{j},i}\right)\cdot\mu.$$

Also, we have

$$\left(E_{\bar{j}+n,i}E_{i,\bar{j}+n}\right)\cdot\mu = \left(h_i + \frac{1}{2}h_{\bar{j}}\right)\cdot\mu - \left(E_{i,\bar{j}+n}E_{\bar{j}+n,i}\right)\cdot\mu.$$

Hence, by Lemmas 6.1.2 and 6.3.7 we have

$$\begin{aligned}\Phi_{\mu'} = \Phi_{\mu''} &= \left(\frac{\partial}{\partial x_i} + \frac{1}{2}\frac{\partial}{\partial y_{\bar{j}}}\right)\Phi_\mu - \frac{-e^{x_i - y_{\bar{j}}}}{e^{-x_j+y_{\bar{j}}} - e^{x_i - y_{\bar{j}}}}\left(\frac{\partial}{\partial x_i} + \frac{1}{2}\frac{\partial}{\partial y_{\bar{j}}}\right)\Phi_\mu \\ &= \frac{-e^{-x_i+y_{\bar{j}}}}{e^{x_i-y_{\bar{j}}} - e^{-x_i+y_{\bar{j}}}}\left(\frac{\partial}{\partial x_i} + \frac{1}{2}\frac{\partial}{\partial y_{\bar{j}}}\right)\Phi_\mu.\end{aligned}$$

∎

Proposition 6.3.9. Let $\mu \in \mathcal{A}(\mathfrak{g},\mathfrak{k})$. Set $\bar{\Phi}_\mu := \Phi_\mu(\log x_1,\ldots,\log x_m,\log y_1,\ldots,\log y_n)$. Then, for some scalar $c \in \mathbb{C}$ we have

$$\begin{aligned}\bar{\Phi}_{\mu\cdot\Omega} = \frac{1}{2}&\sum_{\substack{i,j\,\in\, I_{\bar{0}}\\ i\neq j}} \frac{x_i}{x_i - x_j}(x_i\frac{\partial}{\partial x_i} - x_j\frac{\partial}{\partial x_j})\Phi_\mu + \sum_{i\in I_{\bar{0}}}\left(x_i\frac{\partial}{\partial x_i} + x_i^2\frac{\partial^2}{\partial x_i^2}\right)\Phi_\mu \\ &- \sum_{\substack{\bar{i},\bar{j}\,\in\, I_{\bar{1}}\\ \bar{i}\neq\bar{j}}} \frac{y_{\bar{i}}}{y_{\bar{i}} - y_{\bar{j}}}(y_{\bar{i}}\frac{\partial}{\partial y_{\bar{i}}} - y_{\bar{j}}\frac{\partial}{\partial y_{\bar{j}}})\Phi_\mu - \frac{1}{4}\sum_{\bar{i}\in I_{\bar{1}}}\left(y_{\bar{i}}\frac{\partial}{\partial y_{\bar{i}}} + y_{\bar{i}}^2\frac{\partial^2}{\partial y_{\bar{i}}^2}\right)\Phi_\mu \\ &+ \sum_{\substack{i\,\in\, I_{\bar{0}}\\ j\,\in\, I_{\bar{1}}}} \frac{x_i + y_{\bar{j}}}{x_i - y_{\bar{j}}}(x_i\frac{\partial}{\partial x_i} + \frac{1}{2}y_{\bar{j}}\frac{\partial}{\partial y_{\bar{j}}})\Phi_\mu.\end{aligned}$$

Proof. Recall that

$$\Omega := \sum_{i,j\in I}(-1)^{|i|}E_{ij}E_{ji} = \sum_{\substack{i,j\,\in\, I_{\bar{0}}\\ i\neq j}} E_{ij}E_{ji} + \sum_{i,j\in I_{\bar{0}}} E_{ii}^2 - \sum_{\bar{i},\bar{j}\in I_{\bar{1}}} E_{\bar{i}\bar{j}}E_{\bar{j}\bar{i}} - \sum_{\bar{i}\in I_{\bar{1}}} E_{\bar{i}\bar{i}}^2 + \sum_{\substack{i\,\in\, I_{\bar{0}},\\ \bar{j}\,\in\, I_{\bar{1}}}} E_{i\bar{j}}E_{\bar{j}i} - E_{\bar{j}i}E_{i\bar{j}}.$$

Thus, by Lemmas 6.3.4, 6.3.6, 6.3.7, and 6.3.8 we have

$$\begin{aligned}\Phi_{\Omega\cdot\mu} = &\sum_{1\leqslant i\neq j\leqslant m} \frac{-e^{x_i - x_j}}{e^{-x_i+x_j} - e^{x_i-x_j}}\left(\frac{\partial}{\partial x_i} - \frac{\partial}{\partial x_j}\right)\Phi_\mu + \sum_{i=1}^{m}\left(\frac{\partial^2}{\partial x_i^2}\right)\Phi_\mu \\ &- 2\sum_{1\leqslant i\neq j\leqslant n} \frac{e^{2y_{\bar{i}}} + e^{2y_{\bar{j}}}}{e^{2y_{\bar{i}}} - e^{2y_{\bar{j}}}}(\frac{\partial}{\partial y_{\bar{i}}} - \frac{\partial}{\partial y_{\bar{j}}})\Phi_\mu - \frac{1}{2}\sum_{1\leqslant i\leqslant n}\frac{\partial^2}{\partial y_{\bar{i}}^2}\Phi_\mu \\ &+ 2\sum_{\substack{1\,\leqslant\, i\,\leqslant\, m,\\ 1\,\leqslant\, j\,\leqslant\, n}} \frac{-e^{x_i - y_{\bar{j}}}}{e^{-x_i+y_{\bar{j}}} - e^{x_i-y_{\bar{j}}}}(\frac{\partial}{\partial x_i} + \frac{1}{2}\frac{\partial}{\partial y_{\bar{j}}})\Phi_\mu \\ &- 2\sum_{\substack{1\,\leqslant\, i\,\leqslant\, m,\\ 1\,\leqslant\, j\,\leqslant\, n}} \frac{-e^{-x_i + y_{\bar{j}}}}{e^{x_i-y_{\bar{j}}} - e^{-x_i+y_{\bar{j}}}}\left(\frac{\partial}{\partial x_i} + \frac{1}{2}\frac{\partial}{\partial y_{\bar{j}}}\right)\Phi_\mu,\end{aligned}$$

and also by using the substitutions $x_i \to e^{2x_i}$ for $i = 1, \cdots, m$ and $y_i \to e^{2y_i}$ for $i = 1, \cdots, n$ we obtain

$$\begin{aligned}
\bar{\Phi}_{\Omega\cdot\mu} = c_\Omega \Phi_\mu = {} & 2 \sum_{\substack{i,\, j \,\in\, I_{\bar{0}} \\ i \neq j}} \frac{x_i}{x_i - x_j}\Big(x_i \frac{\partial}{\partial x_i} - x_j \frac{\partial}{\partial x_j}\Big)\Phi_\mu + 4 \sum_{i \in I_{\bar{0}}} \left(x_i \frac{\partial}{\partial x_i} + x_i^2 \frac{\partial^2}{\partial x_i^2} \right) \Phi_\mu \\
& - 4 \sum_{\substack{\bar{i},\, \bar{j} \,\in\, I_{\bar{1}} \\ \bar{i} \neq \bar{j}}} \frac{y_{\bar{i}}}{y_{\bar{i}} - y_{\bar{j}}}\Big(y_{\bar{i}} \frac{\partial}{\partial y_{\bar{i}}} - y_{\bar{j}} \frac{\partial}{\partial y_{\bar{j}}}\Big)\Phi_\mu - 2 \sum_{\bar{i} \in I_{\bar{1}}} \left(y_{\bar{i}} \frac{\partial}{\partial y_{\bar{i}}} + y_{\bar{i}}^2 \frac{\partial^2}{\partial y_{\bar{i}}^2} \right) \Phi_\mu \\
& + 4 \sum_{\substack{i \,\in\, I_{\bar{0}} \\ j \,\in\, I_{\bar{1}}}} \frac{x_i + y_{\bar{j}}}{x_i - y_{\bar{j}}}\Big(x_i \frac{\partial}{\partial x_i} + \frac{1}{2} y_{\bar{j}} \frac{\partial}{\partial y_{\bar{j}}}\Big)\Phi_\mu. \qquad \blacksquare
\end{aligned}$$

Corollary 6.3.10. Let $\mu \in \mathcal{A}(\mathfrak{g},\mathfrak{k})$, and assume that Φ_μ (or equivalently μ) is an eigenfunction of the Casimir operator. Since the Casimir operator is in the center of $U(\mathfrak{g})$, it acts on $\mu \in \mathcal{A}(\mathfrak{g},\mathfrak{k})$ as a scalar which implies that for some scalar $c_\Omega \in \mathbb{C}$ we obtain

$$\begin{aligned}
c_\Omega \bar{\Phi}_\mu = {} & \frac{1}{2} \sum_{\substack{i,\, j \,\in\, I_{\bar{0}} \\ i \neq j}} \frac{x_i}{x_i - x_j}\Big(x_i \frac{\partial}{\partial x_i} - x_j \frac{\partial}{\partial x_j}\Big)\Phi_\mu + \sum_{i \in I_{\bar{0}}} \left(x_i \frac{\partial}{\partial x_i} + x_i^2 \frac{\partial^2}{\partial x_i^2} \right) \Phi_\mu \\
& - \sum_{\substack{\bar{i},\, \bar{j} \,\in\, I_{\bar{1}} \\ \bar{i} \neq \bar{j}}} \frac{y_{\bar{i}}}{y_{\bar{i}} - y_{\bar{j}}}\Big(y_{\bar{i}} \frac{\partial}{\partial y_{\bar{i}}} - y_{\bar{j}} \frac{\partial}{\partial y_{\bar{j}}}\Big)\Phi_\mu - \frac{1}{2} \sum_{\bar{i} \in I_{\bar{1}}} \left(y_{\bar{i}} \frac{\partial}{\partial y_{\bar{i}}} + y_{\bar{i}}^2 \frac{\partial^2}{\partial y_{\bar{i}}^2} \right) \Phi_\mu \\
& + \sum_{\substack{i \,\in\, I_{\bar{0}} \\ j \,\in\, I_{\bar{1}}}} \frac{x_i + y_{\bar{j}}}{x_i - y_{\bar{j}}}\Big(x_i \frac{\partial}{\partial x_i} + \frac{1}{2} y_{\bar{j}} \frac{\partial}{\partial y_{\bar{j}}}\Big)\Phi_\mu.
\end{aligned}$$

6.4 The action of Casimir operator on $\mathcal{A}(\mathfrak{g},\mathfrak{k})$ in Case III of Table 4.1

Let $(\mathfrak{g},\mathfrak{k}) = (\mathfrak{gosp}(m+1, 2n), \mathfrak{osp}(m, 2n))$. Recal that $I = I_{\bar{0}} \sqcup I_{\bar{1}}$ where $I_{\bar{0}} = \{1, \cdots, m+1\}$ and $I_{\bar{1}} = \{\bar{1}, \cdots, \bar{2n}\}$. According to Section 4.7 we have

$$\mathfrak{a} = \operatorname{Span}\left\{ h_1 := \frac{1}{2}(a + a'), h_2 := \frac{1}{2}(a' - a) \right\}$$

where $a' = I_{m+1+2n}$ and

$$a = \begin{cases} \operatorname{diag}(\bar{s}, -\bar{s}, 0_{2n}) & \text{when} \quad m+1 = 2r \\ \operatorname{diag}(0, \bar{s}, -\bar{s}, 0_{2n}) & \text{when} \quad m+1 = 2r+1. \end{cases}, \qquad \text{for} \quad \bar{s} := (2, 0, \cdots, 0) \in \mathbb{C}^r.$$

As in Chapter 5, the formal power series in this case is

$$\Phi_\mu := \sum_{a_1, a_2 \in \mathbb{Z}^+} \frac{\lambda(h_1^{a_1} h_2^{a_2})}{a_1! a_2!} x_1^{a_1} x_2^{a_2}.$$

In this case the realization of the Casimir operator will be as follows. Let

$$J := \begin{bmatrix} J' & 0 \\ 0 & J_+ \end{bmatrix} \quad \text{when} \quad m+1 = 2r, \quad J := \begin{bmatrix} J'' & 0 \\ 0 & J_+ \end{bmatrix} \quad \text{when} \quad m+1 = 2r+1,$$

such that

$$J' = \begin{bmatrix} 0 & I_r \\ I_r & 0 \end{bmatrix}, \quad J'' = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & I_r \\ 0 & I_r & 0 \end{bmatrix}, \quad J_+ = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix}.$$

Consider

$$\mathrm{Skew}(\mathbb{C}^{m+1+2n}) := \left\{ \begin{bmatrix} A & B \\ -B^T & C \end{bmatrix} \mid A = -A^T, C = C^T \right\}.$$

Thus the matrices $X \in \mathrm{Skew}(\mathbb{C}^{m+1+2n})$ satisfy $x_{ij} = -(-1)^{|i||j|} x_{ji}$. Set

$$\tilde{\mathrm{Skew}}(\mathbb{C}^{m+1+2n}) = \mathrm{Skew}(\mathbb{C}^{m+1+2n}) \oplus \mathbb{C}J.$$

Consider the map

$$\phi : \tilde{\mathrm{Skew}}(\mathbb{C}^{m+1+2n}) \to \mathrm{Mat}_{m+1+2n}(\mathbb{C}), \quad \tilde{X} \mapsto J^{-1}\tilde{X}.$$

We denote the entries of J by J_{ij}. The image of ϕ is the Lie superalgebra $\mathfrak{gosp}(m+1|2n)$ which preserves the supersymmetric bilinear form defined by

$$(e_i, e_j) = e_j^T J e_i = J_{ji}, \quad \text{for} \quad i, j \in I.$$

We equip $\mathrm{Mat}_{m+1+2n}(\mathbb{C})$ with the supertrace bilinear form. The pullback to $\tilde{\mathrm{Skew}}(\mathbb{C}^{m+1+2n})$ of this nondegenerate supersymmetric bilinear form is

$$\begin{cases} \langle X, Y \rangle := \mathrm{str}(J^{-1}\tilde{X}J^{-1}\tilde{Y}) & \text{for} \quad X, Y \in \mathrm{Skew}(\mathbb{C}^{m+1+2n}) \\ \langle J, J \rangle = \alpha & \alpha \neq 0 \\ \langle J, X \rangle = 0 & \text{for} \quad X \in \mathrm{Skew}(\mathbb{C}^{m+1+2n}) \end{cases}. \tag{6.4.1}$$

Remark 6.4.1. It will be seen that $\alpha = 8$, but for better readability we keep the parameter α until the end.

Lemma 6.4.2. The bilinear form (6.4.1) on $\tilde{\mathrm{Skew}}(\mathbb{C}^{m+1+2n})$ is nondegenerate.

Proof. Assume that $\tilde{X} = X + cJ$ where $X \in \mathrm{Skew}(\mathbb{C}^{m+1+2n})$. Let for every $Y \in \mathrm{Skew}(\mathbb{C}^{m+1+2n})$ we have

$$\langle \tilde{X}, Y + c'J \rangle = \mathrm{str}(J^{-1}\tilde{X}J^{-1}Y) = 0,$$

where $c' \in \mathbb{C}$ is an arbitrary scalar. Therefore, $\langle \tilde{X}, Y + c'J \rangle = \langle \tilde{X}, Y \rangle + c'\langle \tilde{X}, J \rangle = 0$ for every $c' \in \mathbb{C}$. Thus for $c' = 0, 1$ we have

$$\begin{cases} \langle \tilde{X}, Y \rangle = 0, \\ \langle \tilde{X}, Y + J \rangle = 0. \end{cases}$$

It follows that $\langle \tilde{X}, J\rangle = 0$. Thus,

$$\langle X + cJ, J\rangle = \langle X, J\rangle + c\langle J, J\rangle = 0.$$

Since $\langle X, J\rangle = \mathrm{str}(J^{-1}X) = 0$ and $\langle J, J\rangle = \alpha \neq 0$, it implies that $c = 0$. Hence $X \in \mathrm{Skew}(\mathbb{C}^{m+1+2n})$ and because the bilinear form is nondegenerate on $\mathrm{Skew}(\mathbb{C}^{m+1+2n})$, we can conclude that the bilinear form is nondegenerate on $\widetilde{\mathrm{Skew}}(\mathbb{C}^{m+1+2n})$ as well. ∎

Then, we divide the analysis into the two cases $m+1 = 2r$ and $m+1 = 2r+1$ for $m \geqslant 0$.

6.4.1 The case $m+1 = 2r$

Let

$$\{S_{i,j} := E_{ij} - (-1)^{|i||j|}E_{ji}, w = 2S_{1,1+r}, w' = J \,|\, i,j \in I,\, i \leqslant j,\, (i,j) \neq (1,1+r)\},$$

be a basis of $\widetilde{\mathrm{Skew}}(\mathbb{C}^{m+1+2n})$. Note that $\phi(w) = a$ and $\phi(w') = a'$. We set

$$\{S^*_{i,j} := JS_{ij}J^T, w^* = -w, (w')^* = -w' \,|\, i,j \in I,\, i \leqslant j,\, (i,j) \neq (1,1+r)\},$$

to be a dual basis of $\widetilde{\mathrm{Skew}}(\mathbb{C}^{m+1+2n})$ up to the scalars with respect to the above bilinear form. Then, we have

$$\left\langle -\frac{1}{8}w^*, w\right\rangle = 1, \quad \left\langle -\frac{1}{\alpha}(w')^*, w'\right\rangle = 1, \quad \langle w^*, w'\rangle = \langle (w')^*, w\rangle = 0.$$

By replacing w and w' with $w + w'$ and $w - w'$ respectively, one can obtain the following basis and dual basis of $\widetilde{\mathrm{Skew}}(\mathbb{C}^{m+1+2n})$ up to the scalars.

$$\begin{aligned}
&\{S_{i,j} := E_{ij} - (-1)^{|i||j|}E_{ji}, (w+w'), (w-w') \,|\, i,j \in I,\, i \leqslant j,\, (i,j) \neq (1,1+r)\},\\
&\{S^*_{i,j} := JS_{ij}J^T, \frac{1}{2}(w^* + (w')^*), \frac{1}{2}(w^* - (w')^*) \,|\, i,j \in I,\, i \leqslant j,\, (i,j) \neq (1,1+r)\}.
\end{aligned}$$

Thus we have

$$S^*_{ij} = \sum_{a,b\in I} J_{ai}J_{bj}S_{ab}.$$

First assume $1 \leqslant i < j \leqslant m+1$, $1 \leqslant k < l \leqslant m+1$, and $(i,j) \neq (1,1+r)$. Then

$$\begin{aligned}
\langle S_{ij}, S^*_{kl}\rangle = \mathrm{str}(J^{-1}S_{ij}S_{kl}J^T) &= \mathrm{tr}((J')^{-1}(E_{ij} - E_{ji})(E_{kl} - E_{lk})(j')^T)\\
&= \mathrm{tr}((E_{ij} - E_{ji})(E_{kl} - E_{lk})(j')^T(J')^{-1})\\
&= \mathrm{tr}((E_{ij} - E_{ji})(E_{kl} - E_{lk})) = \begin{cases} 0 & \text{if } (i,j) \neq (k,l),\\ -2 & \text{if } (i,j) = (k,l).\end{cases}
\end{aligned}$$

Next assume that $i,j\in I_{\bar 1}$ and $k,l\in I_{\bar 1}$. Then by similar calculation and using $J_+^T=-J_+$ we have

$$\langle S_{ij},S_{kl}^*\rangle=\mathrm{tr}((E_{ij}-E_{ji})(E_{kl}-E_{lk}))=\begin{cases}0 & \text{if } (i,j)\neq(k,l),\\ 2 & \text{if } (i,j)=(k,l) \text{ and } i\neq j,\\ 4 & \text{if } (i,j)=(k,l) \text{ and } i=j.\end{cases}$$

Next, assume $i\in I_{\bar 0}$ and $j\in I_{\bar 1}$. Then by a direct calculation we obtain

$$\langle S_{ij},S_{kl}^*\rangle=-2\mathrm{tr}(E_{ij}E_{lk}))=\begin{cases}0 & \text{if } (i,j)\neq(k,l),\\ -2 & \text{if } (i,j)=(k,l).\end{cases}$$

Finally, by a straightforward calculation we have

$$\langle\frac{1}{2}(w^*+(w')^*),(w+w')\rangle=\langle\frac{1}{2}(w^*-(w')^*),(w-w')\rangle=-\frac{1}{2}(\alpha+8).$$

From the above relations it follows that the Casimir operator of $\mathfrak{gosp}(m+1,2n)$ is up to a scalar equal to

$$\begin{aligned}\Omega:=&-\frac{1}{2}\sum_{\substack{1\leqslant i<j\leqslant m+1\\(i,j)\neq(1,1+r)}}\phi(S_{ij}^*)\phi(S_{ij})+\frac{1}{2}\sum_{\bar 1\leqslant i<j\leqslant 2\bar n}\phi(S_{ij}^*)\phi(S_{ij})\\&+\frac{1}{4}\sum_{i\in I_{\bar 1}}\phi(S_{ii}^*)\phi(S_{ii})-\frac{1}{2}\sum_{\substack{i\in I_{\bar 0}\\ j\in I_{\bar 1}}}\phi(S_{ij}^*)\phi(S_{ij})\\&-\frac{2}{\alpha+8}\left(\frac{1}{2}\phi(w^*+(w')^*)\phi(w+w')+\frac{1}{2}\phi(w^*-(w')^*)\phi(w-w')\right).\end{aligned}$$

But by a direct computation we have

$$\begin{aligned}\frac{1}{2}\phi(w^*+(w')^*)\phi(w+w')+\frac{1}{2}\phi(w^*-(w')^*)\phi(w-w')&=-a^2-(a')^2\\&=-(h_1-h_2)^2-(h_1+h_2)^2\\&=-2(h_1^2+h_2^2).\end{aligned}$$

Thus

$$\begin{aligned}\Phi_{\Omega.\mu}:=&-\frac{1}{2}\sum_{\substack{i,j\in I_{\bar 0},i<j\\(i,j)\neq(1,1+r)\\a_1,a_2\in\mathbb{Z}^+}}\frac{\lambda(h_1^{a_1}h_2^{a_2}\phi(S_{ij}^*)\phi(S_{ij}))}{a_1!a_2!}x_1^{a_1}x_2^{a_2}-\frac{1}{2}\sum_{\substack{i\in I_{\bar 0}\\ j\in I_{\bar 1}\\a_1,a_2\in\mathbb{Z}^+}}\frac{\lambda(h_1^{a_1}h_2^{a_2}\phi(S_{ij}^*)\phi(S_{ij}))}{a_1!a_2!}x_1^{a_1}x_2^{a_2}\\&+\frac{1}{4}\sum_{\substack{i\in I_{\bar 1}\\a_1,a_2\in\mathbb{Z}^+}}\frac{\lambda(h_1^{a_1}h_2^{a_2}\phi(S_{ii}^*)\phi(S_{ii}))}{a_1!a_2!}x_1^{a_1}x_2^{a_2}+\frac{1}{2}\sum_{\substack{i,j\in I_{\bar 1},i<j\\a_1,a_2\in\mathbb{Z}^+}}\frac{\lambda(h_1^{a_1}h_2^{a_2}\phi(S_{ij}^*)\phi(S_{ij}))}{a_1!a_2!}x_1^{a_1}x_2^{a_2}\\&+\frac{4}{\alpha+8}\sum_{a_1,a_2\in\mathbb{Z}^+}\frac{\lambda(h_1^{a_1}h_2^{a_2}(h_1^2+h_2^2))}{a_1!a_2!}x_1^{a_1}x_2^{a_2}.\end{aligned}$$

Lemma 6.4.3. Let $\mu \in \mathcal{A}(\mathfrak{g},\mathfrak{k})$. Then

$$\big(\phi(S_{ij}^*)\phi(S_{ij})\big)\cdot\mu = 0,$$

except when one of the following occurs:
(i) $i = 1, 1+r$ and $1 \leqslant j \leqslant 2r$ or $j = 1, 1+r$ and $1 \leqslant i \leqslant 2r$.
(ii) $i = 1, 1+r$ and $j \in I_{\bar{1}}$ or $j = 1, 1+r$ and $i \in I_{\bar{1}}$.

Proof. According to the proof of Proposition 4.7.5 we have $C_{\mathfrak{k}}(\mathfrak{a}) = \mathfrak{m} = \mathfrak{m}_{\bar{0}} \oplus \mathfrak{m}_{\bar{1}}$ where $\mathfrak{m}_{\bar{0}}$ and $\mathfrak{m}_{\bar{1}}$ have the following matrix forms

$$\mathfrak{m}_{\bar{0}} = \left\{ \left[\begin{array}{cc:cc c} 0 & 0 & 0 & 0 & \\ 0 & A & 0 & B & \\ \hdashline 0 & 0 & 0 & 0 & \\ 0 & C & 0 & -A^t & \\ & & & & D \end{array}\right] \in \mathfrak{gosp}(2r|2n) \;\Big|\; B = -B^T, C = -C^T \right\}$$

and

$$\mathfrak{m}_{\bar{1}} = \left\{ \left[\begin{array}{cccc cc:cc} & & & & 0 & 0 & 0 & 0 \\ & & & & -L^t & -D^t & U^t & B^t \\ \cdashline{5-8} & & & & 0 & 0 & 0 & 0 \\ & & & & -E^t & -C^t & X^t & A^t \\ 0 & X & 0 & U & & & & \\ 0 & A & 0 & B & & & & \\ \cdashline{1-4} 0 & E & 0 & L & & & & \\ 0 & C & 0 & D & & & & \end{array}\right] \right\}.$$

Note that for every $\phi(S_{ij}) \in \mathfrak{m}$, the action of $\phi(S_{ij}^*)\phi(S_{ij})$ on every $\mu \in \mathcal{A}(\mathfrak{g},\mathfrak{k})$ will be zero. This concludes the proof. ∎

Lemma 6.4.4. Let $\mu \in \mathcal{A}(\mathfrak{g},\mathfrak{k})$. Then,

$$\sum_{\substack{1\leqslant i<j\leqslant m+1\\(i,j)\neq(1,1+r)}} \phi(S_{ij}^*)\phi(S_{ij})\cdot\mu = \sum_{j=2}^{r}\big(2\phi(S_{1j}^*)\phi(S_{1j}) + 2\phi(S_{1,j+r}^*)\phi(S_{1,j+r})\big)\cdot\mu$$
$$+\,2(r-1)(E_{11} - E_{1+r,1+r})\cdot\mu.$$

Proof. By Lemma 6.4.3 we have

$$\sum_{\substack{1\leqslant i<j\leqslant m+1\\(i,j)\neq(1,1+r)}} \phi(S_{ij}^*)\phi(S_{ij})\cdot\mu = \sum_{j=2}^{r}\phi(S_{1j}^*)\phi(S_{1j})\cdot\mu + \sum_{j=2}^{r}\phi(S_{1+r,j+r}^*)\phi(S_{1+r,j+r})\cdot\mu$$
$$+\sum_{j=2}^{r}\phi(S_{1,j+r}^*)\phi(S_{1,j+r})\cdot\mu + \sum_{j=2}^{r}\phi(S_{j,1+r}^*)\phi(S_{j,1+r})\cdot\mu.$$

By straightforward computation one can see that

$$\begin{aligned} \phi(S_{1j}) &= E_{1+r,j} - E_{j+r,1}, & \phi(S_{1j}^*) &= E_{1,j+r} - E_{j,1+r}, \\ \phi(S_{1,j+r}) &= E_{1+r,j+r} - E_{j,1}, & \phi(S_{1,j+r}^*) &= E_{1,j} - E_{j+r,1+r}, \\ \phi(S_{j,1+r}) &= E_{j+r,1+r} - E_{1,j}, & \phi(S_{j,1+r}^*) &= E_{j,1} - E_{1+r,j+r}, \\ \phi(S_{1+r,j+r}) &= E_{1,j+r} - E_{j,1+r}, & \phi(S_{1+r,j+r}^*) &= E_{1+r,j} - E_{j+r,1}. \end{aligned}$$

Therefore, since

$$\phi(S_{1j}) = \phi(S_{1+r,j+r}^*), \quad \phi(S_{1j}^*) = \phi(S_{1+r,j+r}), \quad \phi(S_{1,j+r}) = -\phi(S_{j,1+r}^*), \quad \phi(S_{1,j+r}^*) = -\phi(S_{j,1+r}),$$

we obtain

$$\begin{aligned} \sum_{\substack{1\leqslant i<j\leqslant m+1 \\ (i,j)\neq(1,1+r)}} \phi(S_{ij}^*)\phi(S_{ij}) \cdot \mu &= \sum_{j=2}^{r} \big(\phi(S_{1j}^*)\phi(S_{1j}) + \phi(S_{1j})\phi(S_{1j}^*)\big) \cdot \mu \\ &\quad + \sum_{j=2}^{r} \big(\phi(S_{1,j+r}^*)\phi(S_{1,j+r}) + \phi(S_{1,j+r})\phi(S_{1,j+r}^*)\big) \cdot \mu. \end{aligned}$$

But, we have

$$\begin{aligned} [\phi(S_{1j}^*), \phi(S_{1j})] &= -(E_{11} - E_{1+r,1+r}) - (E_{jj} - E_{j+r,j+r}), \\ [\phi(S_{1,j+r}^*), \phi(S_{1,j+r}] &= -(E_{11} - E_{1+r,1+r}) + (E_{jj} - E_{j+r,j+r}). \end{aligned}$$

Therefore,

$$\begin{aligned} \sum_{\substack{1\leqslant i<j\leqslant m+1 \\ (i,j)\neq(1,1+r)}} \phi(S_{ij}^*)\phi(S_{ij}) \cdot \mu &= \sum_{j=2}^{r} \big(2\phi(S_{1j}^*)\phi(S_{1j}) + (E_{11} - E_{1+r,1+r}) + (E_{jj} - E_{j+r,j+r})\big) \cdot \mu \\ &\quad + \sum_{j=2}^{r} \big(2\phi(S_{1,j+r}^*)\phi(S_{1,j+r}) + (E_{11} - E_{1+r,1+r}) - (E_{jj} - E_{j+r,j+r})\big) \cdot \mu \\ &= \sum_{j=2}^{r} \big(2\phi(S_{1j}^*)\phi(S_{1j}) + 2\phi(S_{1,j+r}^*)\phi(S_{1,j+r})\big) \cdot \mu \\ &\quad + 2(r-1)(E_{11} - E_{1+r,1+r}) \cdot \mu. \end{aligned}$$

∎

Lemma 6.4.5. Let $\mu \in \mathcal{A}(\mathfrak{g}, \mathfrak{k})$. Then, for $2 \leqslant j \leqslant r$ we have

$$\begin{aligned} \mu((p(a-2) - p(a+2))\phi(S_{1,j}^* + S_{1,j+r}^*)\phi(S_{1,j} + S_{1,j+r})) &= \mu(p(a+2)a), \\ \mu((p(a-2) - p(a+2))\phi(S_{1,j}^* - S_{1,j+r}^*)\phi(S_{1,j} - S_{1,j+r})) &= \mu(p(a+2)a). \end{aligned}$$

Proof. Set

$$e := \phi(S_{1,j}^* + S_{1,j+r}^*) = E_{1,j+r} - E_{j,1+r} + E_{1,j} - E_{j+r,1+r},$$

$$f := \phi(S_{1,j} + S_{1,j+r}) = E_{1+r,j} - E_{j+r,1} + E_{1+r,j+r} - E_{j,1}.$$

One can see that $e + f \in \mathfrak{k}$ and

$$[a,e] = 2e, \quad [a,f] = -2f, \quad [e,f] = -a.$$

Therefore,

$$\begin{aligned}(p(a-2) - p(a+2))ef - p(a+2)a &= p(a-2)ef - p(a+2)(ef+a)\\ &= p(a-2)ef - p(a+2)fe\\ &= ep(a)f - fp(a)e + fp(a)f - fp(a)f\\ &= (e+f)p(a)f - fp(a)(e+f) \in \mathcal{I}.\end{aligned}$$

Then from the above relation it follows that $\mu((p(a-2) - p(a+2))ef) = \mu(p(a+2)a)$ for $\mu \in \mathcal{A}(\mathfrak{g},\mathfrak{k})$. Also, set

$$\begin{aligned}e' &:= \phi(S^*_{1,j} + S^*_{1,j+r}) = E_{1,j+r} - E_{j,1+r} - E_{1,j} + E_{j+r,1+r},\\ f' &:= \phi(S_{1,j} + S_{1,j+r}) = E_{1+r,j} - E_{j+r,1} - E_{1+r,j+r} + E_{j,1}.\end{aligned}$$

Since $e' - f' \in \mathfrak{k}$ and

$$[a,e'] = 2e, \quad [a,f'] = -2f, \quad [e',f'] = -a.$$

Therefore,

$$\begin{aligned}(p(a-2) - p(a+2))e'f' - p(a+2)a &= p(a-2)e'f' - p(a+2)(e'f'+a)\\ &= p(a-2)e'f' - p(a+2)f'e'\\ &= ep(a)f' - f'p(a)e' + f'p(a)f' - f'p(a)f'\\ &= (e'-f')p(a)f' - f'p(a)(e'-f') \in \mathcal{I}.\end{aligned}$$

Then from the above relation it follows that $\mu((p(a-2) - p(a+2))e'f') = \mu(p(a+2)a)$ for $\mu \in \mathcal{A}(\mathfrak{g},\mathfrak{k})$. ∎

Lemma 6.4.6. Let $\mu \in \mathcal{A}(\mathfrak{g},\mathfrak{k})$. For $2 \leqslant j \leqslant r$ set

$$\begin{aligned}\mu_1 &:= \phi(S^*_{1,j} + S^*_{1,j+r})\phi(S_{1,j} + S_{1,j+r}) \cdot \mu,\\ \mu_2 &:= \phi(S^*_{1,j} - S^*_{1,j+r})\phi(S_{1,j} - S_{1,j+r}) \cdot \mu,\\ \mu_3 &:= \big(\phi(S^*_{1,j})\phi(S_{1,j}) + \phi(S^*_{1,j+r})\phi(S_{1,j+r})\big) \cdot \mu.\end{aligned}$$

Then

$$\Phi_{\mu_1} = \Phi_{\mu_2} = \Phi_{\mu_3} = \frac{e^{x_1-x_2}}{e^{-x_1+x_2} - e^{x_1-x_2}}(\frac{\partial}{\partial x_1} - \frac{\partial}{\partial x_2})\Phi_\mu.$$

Proof. By Lemma 6.1.3 we have

$$e^{x_1-x_2}\Phi_{\mu_1} = \sum_{a_1,a_2\in\mathbb{Z}^+} \mu_1((h_1+1)^{a_1}(h_2-1)^{a_2})\frac{x_1^{a_1}x_2^{a_2}}{a_1!a_2!}$$

and

$$e^{-x_1+x_2}\Phi_{\mu_1} = \sum \mu_1((h_1-1)^{a_1}(h_2+1)^{a_2})\frac{x_1^{a_1}x_2^{a_2}}{a_1!a_2!}.$$

Therefore,

$$(e^{-x_1+x_2}-e^{x_1-x_2})\Phi_{\mu_1} = \sum_{a_1,a_2\in\mathbb{Z}^+} \mu_1(((h_1-1)^{a_1}(h_2+1)^{a_2}-(h_1+1)^{a_1}(h_2-1)^{a_2}))\frac{x_1^{a_1}x_2^{a_2}}{a_1!a_2!}.$$

Recall that

$$h_1 = \frac{1}{2}(a+a'), \quad h_2 = \frac{1}{2}(a'-a).$$

Set $p_{k,l}(a) = (-1)^{a_2-l}a^{a_1-k+a_2-l}$. Thus, through a straightforward computation and by applying Lemma 6.4.5 we derive

$$\begin{aligned}
&\mu_1(((h_1-1)^{a_1}(h_2+1)^{a_2}-(h_1+1)^{a_1}(h_2-1)^{a_2}))\\
&= \frac{1}{2^{a_1+a_2}}\mu_1\left((a'+(a-2))^{a_1}(a'-(a-2))^{a_2}-(a'+(a+2))^{a_1}(a'-(a+2))^{a_2}\right)\\
&= \frac{1}{2^{a_1+a_2}}\mu_1\left(\sum_{k=1}^{a_1}\sum_{l=1}^{a_2}\binom{a_1}{k}\binom{a_2}{l}(a')^{k+l}(p_{k,l}(a-2)-p_{k,l}(a+2))\right)\\
&= \frac{1}{2^{a_1+a_2}}\mu\left(\sum_{k=1}^{a_1}\sum_{l=1}^{a_2}\binom{a_1}{k}\binom{a_2}{l}(-1)^{a_2-l}(a')^{k+l}p_{k,l}(a+2)a\right)\\
&= \mu((\frac{a'+(a+2)}{2})^{a_1}(\frac{a'-(a+2)}{2})^{a_2}a) = \mu((h_1+1)^{a_1}(h_2-1)^{a_2}a).
\end{aligned}$$

Therefore,

$$(e^{-x_1+x_2}-e^{x_1-x_2})\Phi_{\mu_1} = \sum_{a_1,a_2\in\mathbb{Z}^+}\mu((h_1+1)^{a_1}(h_2-1)^{a_2}a)\frac{x_1^{a_1}x_2^{a_2}}{a_1!a_2!}.$$

Thus, by Lemmas 6.1.2, and 6.1.3 we have

$$\sum_{a_1,a_2\in\mathbb{Z}^+}\mu((h_1+1)^{a_1}(h_2-1)^{a_2}a)\frac{x_1^{a_1}x_2^{a_2}}{a_1!a_2!} = e^{x_1-x_2}(\frac{\partial}{\partial x_1}-\frac{\partial}{\partial x_2})\Phi_\mu.$$

Hence,

$$(e^{-x_1+x_2}-e^{x_1-x_2})\Phi_{\mu_1} = e^{x_1-x_2}(\frac{\partial}{\partial x_1}-\frac{\partial}{\partial x_2})\Phi_\mu$$

which implies that

$$\Phi_{\mu_1} = \frac{e^{x_1-x_2}}{e^{-x_1+x_2}-e^{x_1-x_2}}(\frac{\partial}{\partial x_1}-\frac{\partial}{\partial x_2})\Phi_\mu.$$

Similarly, one can verify that

$$\Phi_{\mu_2} = \frac{e^{x_1-x_2}}{e^{-x_1+x_2}-e^{x_1-x_2}}(\frac{\partial}{\partial x_1}-\frac{\partial}{\partial x_2})\Phi_\mu.$$

Moreover, since $\mu_3 = \frac{1}{2}(\mu_1+\mu_2)$, we obtain

$$\Phi_{\mu_1} = \Phi_{\mu_2} = \Phi_{\mu_3} = \frac{e^{x_1-x_2}}{e^{-x_1+x_2}-e^{x_1-x_2}}\left(\frac{\partial}{\partial x_1}-\frac{\partial}{\partial x_2}\right)\Phi_\mu. \qquad \blacksquare$$

Remark 6.4.7. For $j \in \{\bar{1},\ldots,\bar{n}\}$, the notation $j+n$ refers to the element in $\{\overline{n+1},\ldots,\overline{2n}\}$ obtained as follows: first, remove the bar from j, add n to the resulting integer, and then reapply the bar. For example, if $j=\bar{3}$ and $n=5$, then $j+n=\bar{8}$. This convention is used consistently throughout this section.

Lemma 6.4.8. Let $\mu \in \mathcal{A}(\mathfrak{g},\mathfrak{k})$. For $\bar{1} \leqslant j \leqslant \bar{n}$ set

$$\begin{aligned} e_1 &:= \phi(S^*_{1,j}), \quad f_1 := \phi(S_{1,j}) \\ e_2 &:= \phi(S^*_{1,j+n}), \quad f_2 := \phi(S_{1,j+n}). \end{aligned}$$

Then for $p(x) \in \mathbb{C}[x]$ we have

$$\mu\left((p(a-2)-p(a+2))(e_1f_1+e_2f_2)\right) = -\mu\left(p(a+2)a\right).$$

Proof. We have

$$\begin{aligned} \phi(S_{1,j}) &= E_{1+r,j}-E_{j+n,1}, \quad \phi(S^*_{1,j}) = -E_{1,j+n}-E_{j,1+r}, \\ \phi(S_{1,j+n}) &= E_{1+r,j+n}+E_{j,1}, \quad \phi(S^*_{1,j+n}) = E_{1,j}-E_{j+n,1+r}. \end{aligned}$$

Thus,

$$\begin{aligned} [a,e_1] &= 2e_1, \quad [a,f_1] = -2f_1, \quad [e_1,f_1] = \frac{1}{2}a+k, \\ [a,e_2] &= 2e_2, \quad [a,f_2] = -2f_2, \quad [e_2,f_2] = \frac{1}{2}a-k, \end{aligned}$$

where $k = E_{j+n,j+n}-E_{jj} \in \mathfrak{k}$. Since $e_1-f_2 \in \mathfrak{k}$ and $[f_1,f_2]=0$, we have

$$\begin{aligned} &(p(a-2)-p(a+2))e_1f_1 + p(a+2)(\frac{1}{2}a+k) = p(a+2)(\frac{1}{2}a+k-e_1f_1)+p(a-2)e_1f_1 \\ &= p(a+2)f_1e_1+p(a-2)e_1f_1 = f_1p(a)e_1+e_1p(a)f_1 \\ &= f_1p(a)e_1+e_1p(a)f_1-f_2p(a)f_1-f_1p(a)f_2 \\ &= f_1p(a)(e_1-f_2)+(e_1-f_2)p(a)f_1 \in \mathcal{I}. \end{aligned}$$

Therefore,

$$\mu\left((p(a-2)-p(a+2))(e_1f_1)\right) = -\mu\left(p(a+2)(\frac{1}{2}a+k)\right).$$

Furthermore, since $e_2+f_1 \in \mathfrak{k}$ and $[f_1,f_2]=0$, we have

$$\begin{aligned} &(p(a-2)-p(a+2))e_2f_2 + p(a+2)(\frac{1}{2}a-k) = p(a+2)(\frac{1}{2}a-k-e_2f_2)+p(a-2)e_2f_2 \\ &= p(a+2)f_2e_2+p(a-2)e_2f_2 = f_2p(a)e_2+e_2p(a)f_2 \end{aligned}$$

$$= f_2p(a)e_2 + e_2p(a)f_2 + f_2p(a)f_1 + f_1p(a)f_2$$
$$= f_2p(a)(e_2 + f_1) + (e_2 + f_1)p(a)f_2 \in \mathcal{I}.$$

Therefore,

$$\mu\left((p(a-2)-p(a+2))(e_2f_2)\right) = -\mu\left(p(a+2)(\frac{1}{2}a-k)\right).$$

Finally, from the above relations we obtain

$$\mu\left((p(a-2)-p(a+2))(e_1f_1+e_2f_2)\right) = -\mu\left(p(a+2)a\right).$$ ∎

Lemma 6.4.9. Let $\mu \in \mathcal{A}(\mathfrak{g},\mathfrak{k})$. Keeping the same notation as in Lemma 6.4.8, for $\bar{1} \leqslant j \leqslant \bar{n}$ set

$$\mu_1 := (e_1f_1 + e_2f_2)\cdot\mu.$$

Then

$$\Phi_{\mu_1} = \frac{-e^{x_1-x_2}}{e^{-x_1+x_2}-e^{x_1-x_2}}(\frac{\partial}{\partial x_1}-\frac{\partial}{\partial x_2})\Phi_\mu.$$

Proof. By Lemma 6.1.3 we have

$$e^{x_1-x_2}\Phi_{\mu_1} = \sum_{a_1,a_2\in\mathbb{Z}^+}\mu_1((h_1+1)^{a_1}(h_2-1)^{a_2})\frac{x_1^{a_1}x_2^{a_2}}{a_1!a_2!}$$

and

$$e^{-x_1+x_2}\Phi_{\mu_1} = \sum\mu_1((h_1-1)^{a_1}(h_2+1)^{a_2})\frac{x_1^{a_1}x_2^{a_2}}{a_1!a_2!}.$$

Therefore,

$$(e^{-x_1+x_2}-e^{x_1-x_2})\Phi_{\mu_1} = \sum_{a_1,a_2\in\mathbb{Z}^+}\mu_1(((h_1-1)^{a_1}(h_2+1)^{a_2}-(h_1+1)^{a_1}(h_2-1)^{a_2}))\frac{x_1^{a_1}x_2^{a_2}}{a_1!a_2!}.$$

Recall that

$$h_1 = \frac{1}{2}(a+a'), \quad h_2 = \frac{1}{2}(a'-a).$$

Set $p_{k,l}(a) = (-1)^{a_2-l}a^{a_1-k+a_2-l}$. Thus, through a straightforward computation and by applying Lemma 6.4.8 we derive

$$\begin{aligned}
&\mu_1(((h_1-1)^{a_1}(h_2+1)^{a_2}-(h_1+1)^{a_1}(h_2-1)^{a_2}))\\
&= \frac{1}{2^{a_1+a_2}}\mu_1\left((a'+(a-2))^{a_1}(a'-(a-2))^{a_2}-(a'+(a+2))^{a_1}(a'-(a+2))^{a_2}\right)\\
&= \frac{1}{2^{a_1+a_2}}\mu_1\left(\sum_{k=1}^{a_1}\sum_{l=1}^{a_2}\binom{a_1}{k}\binom{a_2}{l}(a')^{k+l}(p_{k,l}(a-2)-p_{k,l}(a+2))\right)\\
&= -\frac{1}{2^{a_1+a_2}}\mu\left(\sum_{k=1}^{a_1}\sum_{l=1}^{a_2}\binom{a_1}{k}\binom{a_2}{l}(-1)^{a_2-l}(a')^{k+l}p_{k,l}(a+2)a\right)
\end{aligned}$$

$$= -\mu((\frac{a'+(a+2)}{2})^{a_1}(\frac{a'-(a+2)}{2})^{a_2}a) = -\mu((h_1+1)^{a_1}(h_2-1)^{a_2}a).$$

Therefore,

$$(e^{-x_1+x_2}-e^{x_1-x_2})\Phi_{\mu_1} = \sum_{a_1,a_2\in\mathbb{Z}^+}\mu((h_1+1)^{a_1}(h_2-1)^{a_2}a)\frac{x_1^{a_1}x_2^{a_2}}{a_1!a_2!}.$$

But, if we identify a with h, by Lemmas 6.1.2, and 6.1.3 we have

$$\sum_{a_1,a_2\in\mathbb{Z}^+}\mu((h_1+1)^{a_1}(h_2-1)^{a_2}a)\frac{x_1^{a_1}x_2^{a_2}}{a_1!a_2!} = e^{x_1-x_2}(\frac{\partial}{\partial x_1}-\frac{\partial}{\partial x_2})\Phi_\mu.$$

Hence,

$$(e^{-x_1+x_2}-e^{x_1-x_2})\Phi_{\mu_1} = -e^{x_1-x_2}(\frac{\partial}{\partial x_1}-\frac{\partial}{\partial x_2})\Phi_\mu$$

which implies that

$$\Phi_{\mu_1} = \frac{-e^{x_1-x_2}}{e^{-x_1+x_2}-e^{x_1-x_2}}(\frac{\partial}{\partial x_1}-\frac{\partial}{\partial x_2})\Phi_\mu.$$ ∎

Lemma 6.4.10. Let $\mu\in\mathcal{A}(\mathfrak{g},\mathfrak{k})$. Then

$$\sum_{\substack{i\,\in\, I_{\bar{0}}\\ j\,\in\, I_{\bar{1}}}}\phi(S^*_{ij})\phi(S_{ij})\cdot\mu = \sum_{\bar{1}\leqslant j\leqslant\bar{n}}\big(2\phi(S^*_{1,j})\phi(S_{1,j})+2\phi(S^*_{1,j+n})\phi(S_{1,j+n})-a\big)\cdot\mu.$$

Proof. By Lemma 6.4.3 we have

$$\sum_{\substack{i\,\in\, I_{\bar{0}}\\ j\,\in\, I_{\bar{1}}}}\phi(S^*_{ij})\phi(S_{ij})\cdot\mu = \sum_{\bar{1}\leqslant j\leqslant\bar{n}}\phi(S^*_{1,j})\phi(S_{1,j})\cdot\mu + \sum_{\bar{1}\leqslant j\leqslant\bar{n}}\phi(S^*_{1,j+n})\phi(S_{1,j+n})\cdot\mu$$
$$+\sum_{\bar{1}\leqslant j\leqslant\bar{n}}\phi(S^*_{1+r,j})\phi(S_{1+r,j})\cdot\mu + \sum_{\bar{1}\leqslant j\leqslant\bar{n}}\phi(S^*_{1+r,j+n})\phi(S_{1+r,j+n})\cdot\mu.$$

We have

$$\begin{aligned}\phi(S_{1,j}) &= E_{1+r,j}-E_{j+n,1}, & \phi(S^*_{1,j}) &= -E_{1,j+n}-E_{j,1+r},\\ \phi(S_{1,j+n}) &= E_{1+r,j+n}+E_{j,1}, & \phi(S^*_{1,j+n}) &= E_{1,j}-E_{j+n,1+r},\\ \phi(S_{1+r,j}) &= E_{1,j}-E_{j+n,1+r}, & \phi(S^*_{1+r,j}) &= -E_{1+r,j+n}-E_{j,1},\\ \phi(S_{1+r,j+n}) &= E_{1,j+n}+E_{j,1+r}, & \phi(S^*_{1+r,j+n}) &= E_{1+r,j}-E_{j+n,1}.\end{aligned}$$

However, since

$$\begin{aligned}\phi(S_{1,j}) &= \phi(S^*_{1+r,j+n}), & \phi(S^*_{1,j}) &= -\phi(S_{1+r,j+n}),\\ \phi(S_{1,j+n}) &= -\phi(S^*_{1+r,j}), & \phi(S^*_{1,j+n}) &= \phi(S_{1+r,j}),\end{aligned}$$

we obtain

$$\phi(S^*_{1,j})\phi(S_{1,j})+\phi(S^*_{1,j+n})\phi(S_{1,j+n})+\phi(S^*_{1+r,j})\phi(S_{1+r,j})+\phi(S^*_{1+r,j+n})\phi(S_{1+r,j+n})$$

$$
\begin{aligned}
&= \phi(S^*_{1,j})\phi(S_{1,j}) + \phi(S^*_{1,j+n})\phi(S_{1,j+n}) - \phi(S_{1,j})\phi(S^*_{1,j}) - \phi(S_{1,j+n})\phi(S^*_{1,j+n}) \\
&= 2\phi(S^*_{1,j})\phi(S_{1,j}) - [\phi(S^*_{1,j}), \phi(S_{1,j})] + 2\phi(S^*_{1,j+n})\phi(S_{1,j+n}) - [\phi(S^*_{1,j+n}), \phi(S_{1,j+n})].
\end{aligned}
$$

But from the proof of Lemma 6.4.8, it follows that

$$[\phi(S^*_{1,j}), \phi(S_{1,j})] + [\phi(S^*_{1,j+n}), \phi(S_{1,j+n})] = a.$$

Therefore,

$$
\begin{aligned}
&\phi(S^*_{1,j})\phi(S_{1,j}) + \phi(S^*_{1,j+n})\phi(S_{1,j+n}) + \phi(S^*_{1+r,j})\phi(S_{1+r,j}) + \phi(S^*_{1+r,j+n})\phi(S_{1+r,j+n}) \\
&= 2\phi(S^*_{1,j})\phi(S_{1,j}) + 2\phi(S^*_{1,j+n})\phi(S_{1,j+n}) - a.
\end{aligned}
$$

From the above relation we have

$$\sum_{\substack{i\,\in\, I_{\bar{0}} \\ j\,\in\, I_{\bar{1}}}} \phi(S^*_{ij})\phi(S_{ij}) \cdot \mu = \sum_{\bar{1}\leqslant j\leqslant \bar{n}} \big(2\phi(S^*_{1,j})\phi(S_{1,j}) + 2\phi(S^*_{1,j+n})\phi(S_{1,j+n}) - a\big) \cdot \mu. \qquad \blacksquare$$

Proposition 6.4.11. Let $\mu \in \mathcal{A}(\mathfrak{g},\mathfrak{k})$. Set $\bar{\Phi}_\mu := \Phi_\mu(\log x_1, \log x_2)$. Then

$$
\begin{aligned}
\bar{\Phi}_{\Omega\cdot\mu} &= (r-1-n)\frac{x_1+x_2}{x_1-x_2}(x_1\frac{\partial}{\partial x_1} - x_2\frac{\partial}{\partial x_2})\Phi_\mu \\
&\quad + \frac{4}{\alpha+8}\left(x_1\frac{\partial}{\partial x_1}\right)^2\Phi_\mu + \frac{4}{\alpha+8}\left(x_2\frac{\partial}{\partial x_2}\right)^2\Phi_\mu.
\end{aligned}
$$

Proof. Recall that

$$
\begin{aligned}
\Omega\cdot\mu := &-\frac{1}{2}\sum_{\substack{1\leqslant i<j\leqslant m+1 \\ (i,j)\neq(1,1+r)}} \phi(S^*_{ij})\phi(S_{ij})\cdot\mu + \frac{1}{2}\sum_{\bar{1}\leqslant i<j\leqslant \bar{2n}} \phi(S^*_{ij})\phi(S_{ij})\cdot\mu \\
&+ \frac{1}{4}\sum_{\bar{1}\leqslant i\leqslant \bar{2n}} \phi(S^*_{ii})\phi(S_{ii})\cdot\mu - \frac{1}{2}\sum_{\substack{i\,\in\, I_{\bar{0}} \\ j\,\in\, I_{\bar{1}}}} \phi(S^*_{ij})\phi(S_{ij})\cdot\mu + \frac{4}{\alpha+8}(h_1^2+h_2^2)\cdot\mu.
\end{aligned}
$$

By Lemma 6.4.4 we have

$$
\begin{aligned}
\sum_{\substack{1\leqslant i<j\leqslant m+1 \\ (i,j)\neq(1,1+r)}} \phi(S^*_{ij})\phi(S_{ij})\cdot\mu &= \sum_{j=2}^{r}\big(2\phi(S^*_{1j})\phi(S_{1j}) + 2\phi(S^*_{1,j+r})\phi(S_{1,j+r})\big)\cdot\mu \\
&\quad + 2(r-1)(E_{11} - E_{1+r,1+r})\cdot\mu \\
&= (r-1)a\cdot\mu + \sum_{j=2}^{r}\big(2\phi(S^*_{1j})\phi(S_{1j}) + 2\phi(S^*_{1,j+r})\phi(S_{1,j+r})\big)\cdot\mu.
\end{aligned}
$$

Also, by Lemma 6.4.3 we have

$$\frac{1}{2}\sum_{\bar{1}\leqslant i<j\leqslant \bar{2n}} \phi(S^*_{ij})\phi(S_{ij})\cdot\mu + \frac{1}{4}\sum_{\bar{1}\leqslant i\leqslant \bar{2n}} \phi(S^*_{ii})\phi(S_{ii})\cdot\mu = 0,$$

and by Lemma 6.4.10 we have

$$\sum_{\substack{i\,\in\, I_{\bar 0}\\ j\,\in\, I_{\bar 1}}} \phi(S^*_{ij})\phi(S_{ij})\cdot\mu = \sum_{\bar 1\leqslant j\leqslant \bar n}\big(2\phi(S^*_{1,j})\phi(S_{1,j}) + 2\phi(S^*_{1,j+n})\phi(S_{1,j+n}) - a\big)\cdot\mu.$$

Therefore, by Lemmas 6.1.2, 6.4.6, 6.4.9 we obtain

$$\begin{aligned}\Phi_{\Omega\cdot\mu} &= -\frac{1}{2}(r-1-n)\frac{e^{x_1-x_2}+e^{-x_1-x_2}}{e^{-x_1+x_2}-e^{x_1-x_2}}\Big(\frac{\partial}{\partial x_1}-\frac{\partial}{\partial x_2}\Big)\Phi_\mu\\ &\quad+\frac{4}{\alpha+8}\left(\frac{\partial^2}{\partial x_1^2}\right)\Phi_\mu+\frac{4}{\alpha+8}\left(\frac{\partial^2}{\partial x_2^2}\right)\Phi_\mu.\end{aligned}$$

Then by using the substitutions $x_1\to e^{2x_1}$ and $x_2\to e^{2x_2}$ we obtain:

$$\begin{aligned}\bar\Phi_{\Omega\cdot\mu} &= (r-1-n)\frac{x_1+x_2}{x_1-x_2}\Big(x_1\frac{\partial}{\partial x_1}-x_2\frac{\partial}{\partial x_2}\Big)\Phi_\mu\\ &\quad+\frac{16}{\alpha+8}\left(x_1\frac{\partial}{\partial x_1}\right)^2\Phi_\mu+\frac{16}{\alpha+8}\left(x_2\frac{\partial}{\partial x_2}\right)^2\Phi_\mu.\end{aligned}$$

Hence, for $\alpha=8$ we have

$$\begin{aligned}\bar\Phi_{\Omega\cdot\mu} &= (r-1-n)\frac{x_1+x_2}{x_1-x_2}\Big(x_1\frac{\partial}{\partial x_1}-x_2\frac{\partial}{\partial x_2}\Big)\Phi_\mu\\ &\quad+\left(x_1\frac{\partial}{\partial x_1}\right)^2\Phi_\mu+\left(x_2\frac{\partial}{\partial x_2}\right)^2\Phi_\mu.\end{aligned}$$

∎

Corollary 6.4.12. Let $\mu\in\mathcal{A}(\mathfrak{g},\mathfrak{k})$, and assume that Φ_μ (or equivalently μ) is an eigenfunction of the Casimir operator. Then for some scalar $c\in\mathbb{C}$,

$$\begin{aligned}c\bar\Phi_{\mu} &= (r-1-n)\frac{x_1+x_2}{x_1-x_2}\Big(x_1\frac{\partial}{\partial x_1}-x_2\frac{\partial}{\partial x_2}\Big)\Phi_\mu\\ &\quad+\left(x_1\frac{\partial}{\partial x_1}\right)^2\Phi_\mu+\left(x_2\frac{\partial}{\partial x_2}\right)^2\Phi_\mu.\end{aligned}$$

Proof. By Proposition 6.4.11 we have

$$\begin{aligned}\bar\Phi_{\Omega\cdot\mu} &= (r-1-n)\frac{x_1+x_2}{x_1-x_2}\Big(x_1\frac{\partial}{\partial x_1}-x_2\frac{\partial}{\partial x_2}\Big)\Phi_\mu\\ &\quad+\left(x_1\frac{\partial}{\partial x_1}\right)^2\Phi_\mu+\left(x_2\frac{\partial}{\partial x_2}\right)^2\Phi_\mu.\end{aligned}$$

Since the Casimir operator is in the center of $U(\mathfrak{g})$, it acts on Φ_μ as a scalar $c\in\mathbb{C}$. Then by using the substitutions $x\to e^{2x}$ and $y\to e^{2y}$ we obtain:

$$\begin{aligned}c\bar\Phi_{\mu} &= (r-1-n)\frac{x_1+x_2}{x_1-x_2}\Big(x_1\frac{\partial}{\partial x_1}-x_2\frac{\partial}{\partial x_2}\Big)\Phi_\mu\\ &\quad+\left(x_1\frac{\partial}{\partial x_1}\right)^2\Phi_\mu+\left(x_2\frac{\partial}{\partial x_2}\right)^2\Phi_\mu.\end{aligned}$$

for some $c\in\mathbb{C}$. ∎

6.4.2 The case $m+1=2r+1$

In this subsection, the argument is nearly identical to the case $m+1=2r$. Here, we work out the details of the minor modifications required for the current case where $m+1=2r+1$. Let

$$\{S_{i,j} := E_{ij} - (-1)^{|i||j|}E_{ji}, w = 2S_{2,2+r}, w' = J \,|\, i,j \in I,\, i \leqslant j,\, (i,j) \neq (2,2+r)\},$$

be a basis of $\tilde{\mathrm{Skew}}(\mathbb{C}^{m+1+2n})$. Again we have $\phi(w)=a$ and $\phi(w')=a'$. We set

$$\{S^*_{i,j} := JS_{ij}J^T, w^* = -w, (w')^* = -w' \,|\, i,j \in I,\, i \leqslant j,\, (i,j) \neq (2,2+r)\},$$

to be a dual basis of $\tilde{\mathrm{Skew}}(\mathbb{C}^{m+1+2n})$ up to the scalars with respect to the above form. Then, we define

$$\langle -\frac{1}{8}w^*, w\rangle = 1, \quad \langle -\frac{1}{\alpha}(w')^*, w'\rangle = 1, \quad \langle w^*, w'\rangle = \langle (w')^*, w\rangle = 0.$$

As in Section 6.4.1, it will eventually turn out that $\alpha = 8$. By replacing w and w' with $w+w'$ and $w-w'$ respectively, one can obtain the following basis and dual basis of $\tilde{\mathrm{Skew}}(\mathbb{C}^{m+1+2n})$ up to the scalars.

$$\{S_{i,j} := E_{ij} - (-1)^{|i||j|}E_{ji}, (w+w'), (w-w') \,|\, i,j \in I,\, i \leqslant j,\, (i,j) \neq (2,2+r)\},$$
$$\{S^*_{i,j} := JS_{ij}J^T, \frac{1}{2}(w^* + (w')^*), \frac{1}{2}(w^* - (w')^*) \,|\, i,j \in I,\, i \leqslant j,\, (i,j) \neq (2,2+r)\}.$$

Thus we have

$$S^*_{ij} = \sum_{a,b\in I} J_{ai}J_{bj}S_{ab}.$$

First assume $1 \leqslant i < j \leqslant m+1$, $1 \leqslant k < l \leqslant m+1$, and $(i,j) \neq (2,2+r)$. Then

$$\begin{aligned}\langle S_{ij}, S^*_{kl}\rangle = \mathrm{str}(J^{-1}S_{ij}S_{kl}J^T) &= \mathrm{tr}((J')^{-1}(E_{ij}-E_{ji})(E_{kl}-E_{lk})(j')^T) \\ &= \mathrm{tr}((E_{ij}-E_{ji})(E_{kl}-E_{lk})(j')^T(J')^{-1}) \\ &= \mathrm{tr}((E_{ij}-E_{ji})(E_{kl}-E_{lk})) = \begin{cases} 0 & \text{if } (i,j)\neq(k,l), \\ -2 & \text{if } (i,j)=(k,l). \end{cases}\end{aligned}$$

Next assume that $i,j \in I_{\bar{1}}$ and $k,l \in I_{\bar{1}}$. Then by similar calculation and using $J_+^T = -J_+$ we have

$$\langle S_{ij}, S^*_{kl}\rangle = \mathrm{tr}((E_{ij}-E_{ji})(E_{kl}-E_{lk})) = \begin{cases} 0 & \text{if } (i,j)\neq(k,l), \\ 2 & \text{if } (i,j)=(k,l) \text{ and } i\neq j, \\ 4 & \text{if } (i,j)=(k,l) \text{ and } i=j. \end{cases}$$

Next, assume $i \in I_{\bar{0}}$ and $j \in I_{\bar{1}}$. Then by a direct calculation we obtain

$$\langle S_{ij}, S^*_{kl}\rangle = -2\mathrm{tr}(E_{ij}E_{lk})) = \begin{cases} 0 & \text{if } (i,j)\neq(k,l), \\ -2 & \text{if } (i,j)=(k,l). \end{cases}$$

Finally, by a straightforward calculation we have

$$\langle\frac{1}{2}(w^*+(w')^*),(w+w')\rangle=\langle\frac{1}{2}(w^*-(w')^*),(w-w')\rangle=-\frac{1}{2}(\alpha+8).$$

From the above relations it follows that the Casimir operator of $\mathfrak{gosp}(m+1,2n)$ is up to a scalar equal to

$$\begin{aligned}\Omega:=&-\frac{1}{2}\sum_{\substack{1\leqslant i<j\leqslant m+1\\(i,j)\neq(2,2+r)}}\phi(S_{ij}^*)\phi(S_{ij})+\frac{1}{2}\sum_{\bar{1}\leqslant i<j\leqslant\bar{2n}}\phi(S_{ij}^*)\phi(S_{ij})\\&+\frac{1}{4}\sum_{i\in I_{\bar{1}}}\phi(S_{ii}^*)\phi(S_{ii})-\frac{1}{2}\sum_{\substack{i\in I_{\bar{0}}\\j\in I_{\bar{1}}}}\phi(S_{ij}^*)\phi(S_{ij})\\&-\frac{2}{\alpha+8}\left(\frac{1}{2}\phi(w^*+(w')^*)\phi(w+w')+\frac{1}{2}\phi(w^*-(w')^*)\phi(w-w')\right).\end{aligned}$$

But by a direct computation we have

$$\begin{aligned}\frac{1}{2}\phi(w^*+(w')^*)\phi(w+w')+\frac{1}{2}\phi(w^*-(w')^*)\phi(w-w')&=-a^2-(a')^2\\&=-(h_1-h_2)^2-(h_1+h_2)^2\\&=-2(h_1^2+h_2^2).\end{aligned}$$

Thus

$$\begin{aligned}\Phi_{\Omega.\mu}:=&-\frac{1}{2}\sum_{\substack{i,j\in I_{\bar{0}},i<j\\(i,j)\neq(2,2+r)\\a_1,a_2\in\mathbb{Z}^+}}\frac{\lambda(h_1^{a_1}h_2^{a_2}\phi(S_{ij}^*)\phi(S_{ij}))}{a_1!a_2!}x_1^{a_1}x_2^{a_2}-\frac{1}{2}\sum_{\substack{i\in I_{\bar{0}}\\j\in I_{\bar{1}}\\a_1,a_2\in\mathbb{Z}^+}}\frac{\lambda(h_1^{a_1}h_2^{a_2}\phi(S_{ij}^*)\phi(S_{ij}))}{a_1!a_2!}x_1^{a_1}x_2^{a_2}\\&+\frac{1}{4}\sum_{\substack{i\in I_{\bar{1}}\\a_1,a_2\in\mathbb{Z}^+}}\frac{\lambda(h_1^{a_1}h_2^{a_2}\phi(S_{ii}^*)\phi(S_{ii}))}{a_1!a_2!}x_1^{a_1}x_2^{a_2}+\frac{1}{2}\sum_{\substack{i,j\in I_{\bar{1}},i<j\\a_1,a_2\in\mathbb{Z}^+}}\frac{\lambda(h_1^{a_1}h_2^{a_2}\phi(S_{ij}^*)\phi(S_{ij}))}{a_1!a_2!}x_1^{a_1}x_2^{a_2}\\&+\frac{4}{\alpha+8}\sum_{a_1,a_2\in\mathbb{Z}^+}\frac{\lambda(h_1^{a_1}h_2^{a_2}(h_1^2+h_2^2))}{a_1!a_2!}x_1^{a_1}x_2^{a_2}.\end{aligned}$$

Lemma 6.4.13. Let $\mu\in\mathcal{A}(\mathfrak{g},\mathfrak{k})$. Then

$$\left(\phi(S_{ij}^*)\phi(S_{ij})\right)\cdot\mu=0,$$

except when one of the following occurs:
(i) $i=2,2+r$ and $j\in I_{\bar{0}}$ or $j=2,2+r$ and $i\in I_{\bar{0}}$.
(ii) $i=2,2+r$ and $j\in I_{\bar{1}}$ or $j=2,2+r$ and $i\in I_{\bar{1}}$.

Proof. According to the proof of Proposition 4.7.5 we have $C_{\mathfrak{k}}(\mathfrak{a})=\mathfrak{m}=\mathfrak{m}_{\bar{0}}\oplus\mathfrak{m}_{\bar{1}}$. For

$m+1=2r+1$, where $\mathfrak{m}_{\bar 0}$ and $\mathfrak{m}_{\bar 1}$ have the following matrix forms

$$\mathfrak{m}_{\bar 0}=\left\{\begin{bmatrix} 0 & 0 & -M^t & 0 & -L^t & \\ 0 & 0 & 0 & 0 & 0 & \\ L & 0 & A & 0 & B & \\ 0 & 0 & 0 & 0 & 0 & \\ M & 0 & C & 0 & -A^t & \\ & & & & & D_{2n\times 2n} \end{bmatrix} \in \mathfrak{gosp}(2r|2n) \Big| B=-B^T, C=-C^T \right\}$$

and

$$\mathfrak{m}_{\bar 1}=\left\{\begin{bmatrix} & & & & & j & J^t & j & J^t \\ & & & & & 0 & 0 & 0 & 0 \\ & & & & & -L^t & -D^t & U^t & B^t \\ & & & & & 0 & 0 & 0 & 0 \\ & & & & & -E^t & -C^t & X^t & A^t \\ j & 0 & X & 0 & U & & & & \\ J & 0 & A & 0 & B & & & & \\ -j & 0 & E & 0 & L & & & & \\ -J & 0 & C & 0 & D & & & & \end{bmatrix}\right\}$$

where $A,B,C,D\in Mat(n-1\times r-1)$.
Note that for every $\phi(S_{ij})\in\mathfrak{m}$, the action of $\phi(S_{ij}^*)\phi(S_{ij})$ on every $\mu\in\mathcal{A}(\mathfrak{g},\mathfrak{k})$ will be zero. This completes the proof. ∎

Lemma 6.4.14. Let $\mu\in\mathcal{A}(\mathfrak{g},\mathfrak{k})$. Then,

$$\sum_{\substack{1\leqslant i<j\leqslant m+1\\(i,j)\neq(2,2+r)}}\phi(S_{ij}^*)\phi(S_{ij})\cdot\mu=(2r-1)(E_{22}-E_{2+r,2+r})\cdot\mu$$
$$+\sum_{j=2}^{r+1}\big(2\phi(S_{2j}^*)\phi(S_{2j})+2\phi(S_{2,j+r}^*)\phi(S_{2,j+r})\big)\cdot\mu.$$

Proof. By Lemma 6.4.13 we have

$$\sum_{\substack{1\leqslant i<j\leqslant m+1\\(i,j)\neq(2,2+r)}}\phi(S_{ij}^*)\phi(S_{ij})\cdot\mu=\sum_{j=2}^{r+1}\phi(S_{2j}^*)\phi(S_{2j})\cdot\mu+\sum_{j=2}^{r+1}\phi(S_{2+r,j+r}^*)\phi(S_{2+r,j+r})\cdot\mu$$
$$+\sum_{j=2}^{r+1}\phi(S_{2,j+r}^*)\phi(S_{2,j+r})\cdot\mu+\sum_{j=2}^{r+1}\phi(S_{j,2+r}^*)\phi(S_{j,2+r})\cdot\mu$$
$$+\sum_{J=2,2+r}^{m+1}\phi(S_{1j}^*)\phi(S_{1j})\cdot\mu.$$

By straightforward computation one can see that

$$\sum_{J=2,2+r}^{m+1}\phi(S_{1j}^*)\phi(S_{1j})=\phi(S_{12}^*)\phi(S_{12})+\phi(S_{1,2+r}^*)\phi(S_{1,2+r})$$

$$\phi(S^*_{12})\phi(S_{12})+\phi(S_{12})\phi(S^*_{12})=[\phi(S^*_{12}),\phi(S_{12})]=a.$$

Also, for $2\leqslant j\leqslant r+1$ we have

$$\begin{aligned}
\phi(S_{2j})&=E_{2+r,j}-E_{j+r,2}, \quad \phi(S^*_{2j})=E_{2,j+r}-E_{j,2+r},\\
\phi(S_{2,j+r})&=E_{2+r,j+r}-E_{j,2}, \quad \phi(S^*_{2,j+r})=E_{2,j}-E_{j+r,2+r},\\
\phi(S_{2+r,j+r})&=E_{2,j+r}-E_{j,2+r}, \quad \phi(S^*_{2+r,j+r})=E_{2+r,j}-E_{j+r,2},\\
\phi(S_{j,2+r})&=E_{j+r,2+r}-E_{2,j}, \quad \phi(S^*_{j,2+r})=E_{j,2}-E_{2+r,j+r}.
\end{aligned}$$

Therefore,

$$\phi(S_{2j})=\phi(S^*_{2+r,j+r}),\quad \phi(S^*_{2j})=\phi(S_{2+r,j+r}),\quad \phi(S_{2,j+r})=-\phi(S^*_{j,2+r}),\quad \phi(S^*_{2,j+r})=-\phi(S_{j,2+r}).$$

Thus, we have

$$\begin{aligned}
\sum_{\substack{1\leqslant i<j\leqslant m+1\\(i,j)\neq(2,2+r)}}\phi(S^*_{ij})\phi(S_{ij})\cdot\mu&=(E_{22}-E_{2+r,2+r})\cdot\mu+\sum_{j=2}^{r+1}\big(\phi(S^*_{2j})\phi(S_{2j})+\phi(S_{2j})\phi(S^*_{2j})\big)\cdot\mu\\
&+\sum_{j=2}^{r+1}\big(\phi(S^*_{2,j+r})\phi(S_{2,j+r})+\phi(S_{2,j+r})\phi(S^*_{2,j+r})\big)\cdot\mu.
\end{aligned}$$

But, we have

$$\begin{aligned}
[\phi(S^*_{2j}),\phi(S_{2j})]&=-(E_{22}-E_{2+r,2+r})-(E_{jj}-E_{j+r,j+r}),\\
[\phi(S^*_{2,j+r}),\phi(S_{2,j+r}]&=-(E_{22}-E_{2+r,2+r})+(E_{jj}-E_{j+r,j+r}).
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\sum_{\substack{1\leqslant i<j\leqslant m+1\\(i,j)\neq(2,2+r)}}\phi(S^*_{ij})\phi(S_{ij})\cdot\mu=\sum_{j=2}^{r+1}\big(2\phi(S^*_{2j})\phi(S_{2j})+(E_{22}-E_{2+r,2+r})+(E_{jj}-E_{j+r,j+r})\big)\cdot\mu\\
&+\sum_{j=2}^{r+1}\big(2\phi(S^*_{2,j+r})\phi(S_{2,j+r})+(E_{22}-E_{2+r,2+r})-(E_{jj}-E_{j+r,j+r})\big)\cdot\mu+(E_{22}-E_{2+r,2+r})\cdot\mu\\
&=(2r-1)(E_{22}-E_{2+r,2+r})\cdot\mu+\sum_{j=2}^{r+1}\big(2\phi(S^*_{2j})\phi(S_{2j})+2\phi(S^*_{2,j+r})\phi(S_{2,j+r})\big)\cdot\mu. \qquad\blacksquare
\end{aligned}$$

Lemma 6.4.15. Let $\mu\in\mathcal{A}(\mathfrak{g},\mathfrak{k})$. Then, for $2\leqslant j\leqslant r+1$ we have

$$\begin{aligned}
\mu((p(a-2)-p(a+2))\phi(S^*_{2,j}+S^*_{2,j+r})\phi(S_{2,j}+S_{2,j+r}))&=\mu(p(a+2)a),\\
\mu((p(a-2)-p(a+2))\phi(S^*_{2,j}-S^*_{2,j+r})\phi(S_{2,j}-S_{2,j+r}))&=\mu(p(a+2)a).
\end{aligned}$$

Proof. Set

$$e:=\phi(S^*_{2,j}+S^*_{2,j+r})=E_{2,j+r}-E_{j,2+r}+E_{2,j}-E_{j+r,2+r},$$

$$f := \phi(S_{2,j} + S_{2,j+r}) = E_{2+r,j} - E_{j+r,2} + E_{2+r,j+r} - E_{j,2}.$$

One can see that $e + f \in \mathfrak{k}$ and

$$[a, e] = 2e, \quad [a, f] = -2f, \quad [e, f] = -a.$$

Therefore,

$$\begin{aligned}(p(a-2) - p(a+2))ef - p(a+2)a &= p(a-2)ef - p(a+2)(ef + a)\\ &= p(a-2)ef - p(a+2)fe\\ &= ep(a)f - fp(a)e + fp(a)f - fp(a)f\\ &= (e+f)p(a)f - fp(a)(e+f) \in \mathcal{I}.\end{aligned}$$

Then from the above relation it follows that $\mu((p(a-2) - p(a+2))ef) = \mu(p(a+2)a)$ for $\mu \in \mathcal{A}(\mathfrak{g},\mathfrak{k})$. Also, set

$$\begin{aligned}e' &:= \phi(S^*_{2,j} + S^*_{2,j+r}) = E_{2,j+r} - E_{j,2+r} - E_{2,j} + E_{j+r,2+r},\\ f' &:= \phi(S_{2,j} + S_{2,j+r}) = E_{2+r,j} - E_{j+r,2} - E_{2+r,j+r} + E_{j,2}.\end{aligned}$$

Since $e' - f' \in \mathfrak{k}$ and

$$[a, e'] = 2e, \quad [a, f'] = -2f, \quad [e', f'] = -a.$$

Therefore,

$$\begin{aligned}(p(a-2) - p(a+2))e'f' - p(a+2)a &= p(a-2)e'f' - p(a+2)(e'f' + a)\\ &= p(a-2)e'f' - p(a+2)f'e'\\ &= ep(a)f' - f'p(a)e' + f'p(a)f' - f'p(a)f'\\ &= (e'-f')p(a)f' - f'p(a)(e'-f') \in \mathcal{I}.\end{aligned}$$

Then from the above relation it follows that $\mu((p(a-2) - p(a+2))e'f') = \mu(p(a+2)a)$ for $\mu \in \mathcal{A}(\mathfrak{g},\mathfrak{k})$. ∎

Lemma 6.4.16. Let $\mu \in \mathcal{A}(\mathfrak{g},\mathfrak{k})$. For $2 \leqslant j \leqslant r+1$ set

$$\begin{aligned}\mu_1 &:= \phi(S^*_{2,j} + S^*_{2,j+r})\phi(S_{2,j} + S_{2,j+r}) \cdot \mu,\\ \mu_2 &:= \phi(S^*_{2,j} - S^*_{2,j+r})\phi(S_{2,j} - S_{2,j+r}) \cdot \mu,\\ \mu_3 &:= \big(\phi(S^*_{2,j})\phi(S_{2,j}) + \phi(S^*_{2,j+r})\phi(S_{2,j+r})\big) \cdot \mu.\end{aligned}$$

Then

$$\Phi_{\mu_1} = \Phi_{\mu_2} = \Phi_{\mu_3} = \frac{e^{x_1-x_2}}{e^{-x_1+x_2} - e^{x_1-x_2}}(\frac{\partial}{\partial x_1} - \frac{\partial}{\partial x_2})\Phi_\mu.$$

Proof. As in the proof of Lemma 6.4.6, we can derive

$$(e^{-x_1+x_2} - e^{x_1-x_2})\Phi_{\mu_1} = \sum_{a_1,a_2\in\mathbb{Z}^+} \mu_1(((h_1-1)^{a_1}(h_2+1)^{a_2} - (h_1+1)^{a_1}(h_2-1)^{a_2}))\frac{x_1^{a_1}x_2^{a_2}}{a_1!a_2!}.$$

Set $p_{k,l}(a) = (-1)^{a_2-l}a^{a_1-k+a_2-l}$. Then, through a straightforward computation and by applying 6.4.15 we derive

$$\begin{aligned}
&\mu_1(((h_1-1)^{a_1}(h_2+1)^{a_2}-(h_1+1)^{a_1}(h_2-1)^{a_2}))\\
&=\frac{1}{2^{a_1+a_2}}\mu_1\left((a'+(a-2))^{a_1}(a'-(a-2))^{a_2}-(a'+(a+2))^{a_1}(a'-(a+2))^{a_2}\right)\\
&=\frac{1}{2^{a_1+a_2}}\mu_1\left(\sum_{k=1}^{a_1}\sum_{l=1}^{a_2}\binom{a_1}{k}\binom{a_2}{l}(a')^{k+l}(p_{k,l}(a-2)-p_{k,l}(a+2))\right)\\
&=\frac{1}{2^{a_1+a_2}}\mu\left(\sum_{k=1}^{a_1}\sum_{l=1}^{a_2}\binom{a_1}{k}\binom{a_2}{l}(-1)^{a_2-l}(a')^{k+l}p_{k,l}(a+2)a\right)\\
&=\mu((\frac{a'+(a+2)}{2})^{a_1}(\frac{a'-(a+2)}{2})^{a_2}a)=\mu((h_1+1)^{a_1}(h_2-1)^{a_2}a).
\end{aligned}$$

Therefore,

$$(e^{-x_1+x_2}-e^{x_1-x_2})\Phi_{\mu_1}=\sum_{a_1,a_2\in\mathbb{Z}^+}\mu((h_1+1)^{a_1}(h_2-1)^{a_2}a)\frac{x_1^{a_1}x_2^{a_2}}{a_1!a_2!}.$$

Thus, by Lemmas 6.1.2, and 6.1.3 we have

$$\sum_{a_1,a_2\in\mathbb{Z}^+}\mu((h_1+1)^{a_1}(h_2-1)^{a_2}a)\frac{x_1^{a_1}x_2^{a_2}}{a_1!a_2!}=e^{x_1-x_2}(\frac{\partial}{\partial x_1}-\frac{\partial}{\partial x_2})\Phi_\mu.$$

Hence,

$$(e^{-x_1+x_2}-e^{x_1-x_2})\Phi_{\mu_1}=e^{x_1-x_2}(\frac{\partial}{\partial x_1}-\frac{\partial}{\partial x_2})\Phi_\mu$$

which implies that

$$\Phi_{\mu_1}=\frac{e^{x_1-x_2}}{e^{-x_1+x_2}-e^{x_1-x_2}}(\frac{\partial}{\partial x_1}-\frac{\partial}{\partial x_2})\Phi_\mu.$$

Similarly, one can verify that

$$\Phi_{\mu_2}=\frac{e^{x_1-x_2}}{e^{-x_1+x_2}-e^{x_1-x_2}}(\frac{\partial}{\partial x_1}-\frac{\partial}{\partial x_2})\Phi_\mu.$$

Moreover, since $\mu_3=\frac{1}{2}(\mu_1+\mu_2)$, we obtain

$$\Phi_{\mu_1}=\Phi_{\mu_2}=\Phi_{\mu_3}=\frac{e^{x_1-x_2}}{e^{-x_1+x_2}-e^{x_1-x_2}}(\frac{\partial}{\partial x_1}-\frac{\partial}{\partial x_2})\Phi_\mu.$$

∎

Lemma 6.4.17. For $\bar{1}\leqslant j\leqslant\bar{n}$ set

$$\begin{aligned}
e_1 &:=\phi(S_{2,j}^*),\quad f_1:=\phi(S_{2,j}),\\
e_2 &:=\phi(S_{2,j+n}^*),\quad f_2:=\phi(S_{2,j+n}).
\end{aligned}$$

Then for $p(x)\in\mathbb{C}[x]$ we have

$$\mu\left((p(a-2)-p(a+2))(e_1f_1+e_2f_2)\right)=-\mu\left(p(a+2)a\right).$$

Proof. We have

$$\phi(S_{2,j}) = E_{2+r,j} - E_{j+n,2}, \quad \phi(S^*_{2,j}) = -E_{2,j+n} - E_{j,2+r},$$
$$\phi(S_{2,j+n}) = E_{2+r,j+n} + E_{j,2}, \quad \phi(S^*_{2,j+n}) = E_{2,j} - E_{j+n,2+r}.$$

Thus,

$$[a,e_1] = 2e_1, \quad [a,f_1] = -2f_1, \quad [e_1,f_1] = \frac{1}{2}a + k,$$
$$[a,e_2] = 2e_2, \quad [a,f_2] = -2f_2, \quad [e_2,f_2] = \frac{1}{2}a - k,$$

where $k = E_{j+n,j+n} - E_{jj} \in \mathfrak{k}$. Since $e_1 - f_2 \in \mathfrak{k}$ and $[f_1,f_2]=0$, we have

$$\begin{aligned}
&(p(a-2)-p(a+2))e_1f_1 + p(a+2)(\frac{1}{2}a+k) = p(a+2)(\frac{1}{2}a+k-e_1f_1) + p(a-2)e_1f_1\\
&= p(a+2)f_1e_1 + p(a-2)e_1f_1 = f_1p(a)e_1 + e_1p(a)f_1\\
&= f_1p(a)e_1 + e_1p(a)f_1 - f_2p(a)f_1 - f_1p(a)f_2\\
&= f_1p(a)(e_1-f_2) + (e_1-f_2)p(a)f_1 \in \mathcal{I}.
\end{aligned}$$

Therefore,

$$\mu\left((p(a-2)-p(a+2))(e_1f_1)\right) = -\mu\left(p(a+2)(\frac{1}{2}a+k)\right).$$

Furthermore, since $e_2 + f_1 \in \mathfrak{k}$ and $[f_1,f_2]=0$, we have

$$\begin{aligned}
&(p(a-2)-p(a+2))e_2f_2 + p(a+2)(\frac{1}{2}a-k) = p(a+2)(\frac{1}{2}a-k-e_2f_2) + p(a-2)e_2f_2\\
&= p(a+2)f_2e_2 + p(a-2)e_2f_2 = f_2p(a)e_2 + e_2p(a)f_2\\
&= f_2p(a)e_2 + e_2p(a)f_2 + f_2p(a)f_1 + f_1p(a)f_2\\
&= f_2p(a)(e_2+f_1) + (e_2+f_1)p(a)f_2 \in \mathcal{I}.
\end{aligned}$$

Therefore,

$$\mu\left((p(a-2)-p(a+2))(e_2f_2)\right) = -\mu\left(p(a+2)(\frac{1}{2}a-k)\right).$$

Hence, from the above relations we obtain

$$\mu\left((p(a-2)-p(a+2))(e_1f_1+e_2f_2)\right) = -\mu\left(p(a+2)a\right). \qquad \blacksquare$$

Lemma 6.4.18. Let $\mu \in \mathcal{A}(\mathfrak{g},\mathfrak{k})$. Keeping the same notation as in Lemma 6.4.17, for $\bar{1} \leqslant j \leqslant \bar{n}$ set

$$\mu_1 := (e_1f_1 + e_2f_2)\cdot\mu.$$

Then

$$\Phi_{\mu_1} = \frac{-e^{x_1-x_2}}{e^{-x_1+x_2} - e^{x_1-x_2}}(\frac{\partial}{\partial x_1} - \frac{\partial}{\partial x_2})\Phi_\mu.$$

Proof. Identical to the proof of 6.4.9 one can observe that

$$(e^{-x_1+x_2}-e^{x_1-x_2})\Phi_{\mu_1}=\sum_{a_1,a_2\in\mathbb{Z}^+}\mu_1(((h_1-1)^{a_1}(h_2+1)^{a_2}-(h_1+1)^{a_1}(h_2-1)^{a_2}))\frac{x_1^{a_1}x_2^{a_2}}{a_1!a_2!}.$$

Thus, through a straightforward computation and by applying 6.4.17 we derive

$$\begin{aligned}
&\mu_1(((h_1-1)^{a_1}(h_2+1)^{a_2}-(h_1+1)^{a_1}(h_2-1)^{a_2}))\\
&=\frac{1}{2^{a_1+a_2}}\mu_1\left((a'+(a-2))^{a_1}(a'-(a-2))^{a_2}-(a'+(a+2))^{a_1}(a'-(a+2))^{a_2}\right)\\
&=\frac{1}{2^{a_1+a_2}}\mu_1\left(\sum_{k=1}^{a_1}\sum_{l=1}^{a_2}\binom{a_1}{k}\binom{a_2}{l}(a')^{k+l}(p_{k,l}(a-2)-p_{k,l}(a+2))\right)\\
&=-\frac{1}{2^{a_1+a_2}}\mu\left(\sum_{k=1}^{a_1}\sum_{l=1}^{a_2}\binom{a_1}{k}\binom{a_2}{l}(-1)^{a_2-l}(a')^{k+l}p_{k,l}(a+2)a\right)\\
&=-\mu((\frac{a'+(a+2)}{2})^{a_1}(\frac{a'-(a+2)}{2})^{a_2}a)=-\mu((h_1+1)^{a_1}(h_2-1)^{a_2}a),
\end{aligned}$$

where $p_{k,l}=(-1)^{a_2-l}a^{a_1-k+a_2-l}$. Therefore,

$$(e^{-x_1+x_2}-e^{x_1-x_2})\Phi_{\mu_1}=\sum_{a_1,a_2\in\mathbb{Z}^+}\mu((h_1+1)^{a_1}(h_2-1)^{a_2}a)\frac{x_1^{a_1}x_2^{a_2}}{a_1!a_2!}.$$

But, if we identify a with h, by Lemmas 6.1.2, and 6.1.3 we have

$$\sum_{a_1,a_2\in\mathbb{Z}^+}\mu((h_1+1)^{a_1}(h_2-1)^{a_2}a)\frac{x_1^{a_1}x_2^{a_2}}{a_1!a_2!}=e^{x_1-x_2}(\frac{\partial}{\partial x_1}-\frac{\partial}{\partial x_2})\Phi_\mu.$$

Hence,

$$(e^{-x_1+x_2}-e^{x_1-x_2})\Phi_{\mu_1}=-e^{x_1-x_2}(\frac{\partial}{\partial x_1}-\frac{\partial}{\partial x_2})\Phi_\mu$$

which implies that

$$\Phi_{\mu_1}=\frac{-e^{x_1-x_2}}{e^{-x_1+x_2}-e^{x_1-x_2}}(\frac{\partial}{\partial x_1}-\frac{\partial}{\partial x_2})\Phi_\mu.$$

∎

Lemma 6.4.19. Let $\mu\in\mathcal{A}(\mathfrak{g},\mathfrak{k})$. Then

$$\sum_{\substack{i\,\in\, I_{\bar 0}\\ j\,\in\, I_{\bar 1}}}\phi(S_{ij}^*)\phi(S_{ij})\cdot\mu=\sum_{\bar 1\leqslant j\leqslant\bar n}\left(2\phi(S_{1,j}^*)\phi(S_{1,j})+2\phi(S_{1,j+n}^*)\phi(S_{1,j+n})-a\right)\cdot\mu.$$

Proof. By Lemma 6.4.13 we have

$$\sum_{\substack{i\,\in\, I_{\bar 0}\\ j\,\in\, I_{\bar 1}}}\phi(S_{ij}^*)\phi(S_{ij})\cdot\mu=\sum_{\bar 1\leqslant j\leqslant\bar n}\phi(S_{2,j}^*)\phi(S_{2,j})\cdot\mu+\sum_{\bar 1\leqslant j\leqslant\bar n}\phi(S_{2,j+n}^*)\phi(S_{2,j+n})\cdot\mu$$

$$+\sum_{\bar{1}\leqslant j\leqslant\bar{n}}\phi(S^*_{2+r,j})\phi(S_{2+r,j})\cdot\mu+\sum_{\bar{1}\leqslant j\leqslant\bar{n}}\phi(S^*_{2+r,j+n})\phi(S_{2+r,j+n})\cdot\mu.$$

By direct calculations for $2\leqslant j\leqslant r+1$ we have

$$\begin{aligned}
\phi(S_{2j})&=E_{2+r,j}-E_{j+r,2}, & \phi(S^*_{2j})&=E_{2,j+r}-E_{j,2+r},\\
\phi(S_{2,j+r})&=E_{2+r,j+r}-E_{j,2}, & \phi(S^*_{2,j+r})&=E_{2,j}-E_{j+r,2+r},\\
\phi(S_{2+r,j+r})&=E_{2,j+r}-E_{j,2+r}, & \phi(S^*_{2+r,j+r})&=E_{2+r,j}-E_{j+r,2},\\
\phi(S_{j,2+r})&=E_{j+r,2+r}-E_{2,j}, & \phi(S^*_{j,2+r})&=E_{j,2}-E_{2+r,j+r}.
\end{aligned}$$

However, since

$$\begin{aligned}
\phi(S_{2,j})&=\phi(S^*_{2+r,j+n}), & \phi(S^*_{2,j})&=-\phi(S_{2+r,j+n}),\\
\phi(S_{2,j+n})&=-\phi(S^*_{2+r,j}), & \phi(S^*_{2,j+n})&=\phi(S_{2+r,j}),
\end{aligned}$$

we obtain

$$\begin{aligned}
&\phi(S^*_{2,j})\phi(S_{2,j})+\phi(S^*_{2,j+n})\phi(S_{2,j+n})+\phi(S^*_{2+r,j})\phi(S_{2+r,j})+\phi(S^*_{2+r,j+n})\phi(S_{2+r,j+n})\\
&=\phi(S^*_{2,j})\phi(S_{2,j})+\phi(S^*_{2,j+n})\phi(S_{2,j+n})-\phi(S_{2,j})\phi(S^*_{2,j})-\phi(S_{2,j+n})\phi(S^*_{2,j+n})\\
&=2\phi(S^*_{2,j})\phi(S_{2,j})-[\phi(S^*_{2,j}),\phi(S_{2,j})]+2\phi(S^*_{2,j+n})\phi(S_{2,j+n})-[\phi(S^*_{2,j+n}),\phi(S_{2,j+n})].
\end{aligned}$$

But

$$[\phi(S^*_{2,j}),\phi(S_{2,j})]+[\phi(S^*_{2,j+n}),\phi(S_{2,j+n})]=a.$$

Therefore,

$$\begin{aligned}
&\phi(S^*_{2,j})\phi(S_{2,j})+\phi(S^*_{2,j+n})\phi(S_{2,j+n})+\phi(S^*_{2+r,j})\phi(S_{2+r,j})+\phi(S^*_{2+r,j+n})\phi(S_{2+r,j+n})\\
&=2\phi(S^*_{2,j})\phi(S_{2,j})+2\phi(S^*_{2,j+n})\phi(S_{2,j+n})-a.
\end{aligned}$$

From the above relation we have

$$\sum_{\substack{i\,\in\, I_{\bar{0}}\\ j\,\in\, I_{\bar{1}}}}\phi(S^*_{ij})\phi(S_{ij})\cdot\mu=\sum_{\bar{1}\leqslant j\leqslant\bar{n}}\big(2\phi(S^*_{2,j})\phi(S_{2,j})+2\phi(S^*_{2,j+n})\phi(S_{2,j+n})-a\big)\cdot\mu.$$ ∎

Proposition 6.4.20. Let $\mu\in\mathcal{A}(\mathfrak{g},\mathfrak{k})$. Set $\bar{\Phi}_\mu:=\Phi_\mu(\log x_1,\log x_2)$. Then

$$\begin{aligned}
\bar{\Phi}_{\Omega\cdot\mu}&=\left(\frac{2r-1}{2}-n\right)\frac{x_1+x_2}{x_1-x_2}(x_1\frac{\partial}{\partial x_1}-x_2\frac{\partial}{\partial x_2})\Phi_\mu\\
&\quad+\left(x_1\frac{\partial}{\partial x_1}\right)^2\Phi_\mu+\left(x_2\frac{\partial}{\partial x_2}\right)^2\Phi_\mu.
\end{aligned}$$

Proof. Recall that

$$\Omega\cdot\mu:=-\frac{1}{2}\sum_{\substack{1\,\leqslant\, i\,<\,j\,\leqslant\, m+1\\ (i,j)\,\neq\,(2,2+r)}}\phi(S^*_{ij})\phi(S_{ij})\cdot\mu+\frac{1}{2}\sum_{\bar{1}\leqslant i<j\leqslant\bar{2n}}\phi(S^*_{ij})\phi(S_{ij})\cdot\mu$$

$$+\frac{1}{4}\sum_{\bar{1}\leqslant i\leqslant 2\bar{n}}\phi(S_{ii}^*)\phi(S_{ii})\cdot\mu-\frac{1}{2}\sum_{\substack{i\,\in\, I_{\bar{0}}\\ j\,\in\, I_{\bar{1}}}}\phi(S_{ij}^*)\phi(S_{ij})\cdot\mu+\frac{4}{\alpha+8}(h_1^2+h_2^2)\cdot\mu.$$

By Lemma 6.4.14 we have

$$\sum_{\substack{1\,\leqslant\, i\,<\,j\,\leqslant\, m+1\\ (i,j)\,\neq\,(2,2+r)}}\phi(S_{ij}^*)\phi(S_{ij})\cdot\mu=(2r-1)(E_{22}-E_{2+r,2+r})\cdot\mu$$
$$+\sum_{j=2}^{r+1}\big(2\phi(S_{2j}^*)\phi(S_{2j})+2\phi(S_{2,j+r}^*)\phi(S_{2,j+r})\big)\cdot\mu.$$

Also, by Lemma 6.4.13 we have

$$\frac{1}{2}\sum_{\bar{1}\leqslant i<j\leqslant 2\bar{n}}\phi(S_{ij}^*)\phi(S_{ij})\cdot\mu+\frac{1}{4}\sum_{\bar{1}\leqslant i\leqslant 2\bar{n}}\phi(S_{ii}^*)\phi(S_{ii})\cdot\mu=0,$$

and by Lemma 6.4.19 we have

$$\sum_{\substack{i\,\in\, I_{\bar{0}}\\ j\,\in\, I_{\bar{1}}}}\phi(S_{ij}^*)\phi(S_{ij})\cdot\mu=\sum_{\bar{1}\leqslant j\leqslant\bar{n}}\big(2\phi(S_{1,j}^*)\phi(S_{1,j})+2\phi(S_{1,j+n}^*)\phi(S_{1,j+n})-a\big)\cdot\mu.$$

Therefore, by Lemmas 6.1.2, 6.4.16, and 6.4.18 we obtain

$$\Phi_{\Omega\cdot\mu}=\frac{1}{2}\left(\frac{2r-1}{2}-n\right)\frac{e^{x_1-x_2}+e^{-x_1-x_2}}{e^{x_1-x_2}-e^{-x_1+x_2}}\left(\frac{\partial}{\partial x_1}-\frac{\partial}{\partial x_2}\right)\Phi_\mu$$
$$+\frac{4}{\alpha+8}\left(\frac{\partial^2}{\partial x_1^2}\right)\Phi_\mu+\frac{4}{\alpha+8}\left(\frac{\partial^2}{\partial x_2^2}\right)\Phi_\mu.$$

Then by using the substitutions $x_1\to e^{2x_1}$ and $x_2\to e^{2x_2}$ we obtain:

$$\bar{\Phi}_{\Omega\cdot\mu}=\left(\frac{2r-1}{2}-n\right)\frac{x_1+x_2}{x_1-x_2}\left(x_1\frac{\partial}{\partial x_1}-x_2\frac{\partial}{\partial x_2}\right)\Phi_\mu$$
$$+\frac{16}{\alpha+8}\left(x_1\frac{\partial}{\partial x_1}\right)^2\Phi_\mu+\frac{16}{\alpha+8}\left(x_2\frac{\partial}{\partial x_2}\right)^2\Phi_\mu.$$

Then for $\alpha=8$, the above relation becomes:

$$\bar{\Phi}_{\Omega\cdot\mu}=\left(\frac{2r-1}{2}-n\right)\frac{x_1+x_2}{x_1-x_2}\left(x_1\frac{\partial}{\partial x_1}-x_2\frac{\partial}{\partial x_2}\right)\Phi_\mu$$
$$+\left(x_1\frac{\partial}{\partial x_1}\right)^2\Phi_\mu+\left(x_2\frac{\partial}{\partial x_2}\right)^2\Phi_\mu.$$

∎

Corollary 6.4.21. Let $\mu\in\mathcal{A}(\mathfrak{g},\mathfrak{k})$, and assume that Φ_μ (or equivalently μ) is an eigenfunction of the Casimir operator. Then for some scalar $c\in\mathbb{C}$,

$$c\bar{\Phi}_\mu=\left(\frac{2r-1}{2}-n\right)\frac{x_1+x_2}{x_1-x_2}\left(x_1\frac{\partial}{\partial x_1}-x_2\frac{\partial}{\partial x_2}\right)\Phi_\mu$$
$$+\left(x_1\frac{\partial}{\partial x_1}\right)^2\Phi_\mu+\left(x_2\frac{\partial}{\partial x_2}\right)^2\Phi_\mu.$$

Proof. By Proposition 6.4.20 we have

$$\begin{aligned}\bar{\Phi}_{\Omega\cdot\mu} &= \left(\frac{2r-1}{2}-n\right)\frac{x_1+x_2}{x_1-x_2}(x_1\frac{\partial}{\partial x_1}-x_2\frac{\partial}{\partial x_2})\Phi_\mu \\ &+\left(x_1\frac{\partial}{\partial x_1}\right)^2\Phi_\mu+\left(x_2\frac{\partial}{\partial x_2}\right)^2\Phi_\mu.\end{aligned}$$

Since the Casimir operator is in the center of $U(\mathfrak{g})$, it acts on Φ_μ as an scalar $c\in\mathbb{C}$. Then by using the substitutions $x\to e^{2x}$ and $y\to e^{2y}$ we obtain:

$$\begin{aligned}c\bar{\Phi}_{\mu} &= \left(\frac{2r-1}{2}-n\right)\frac{x_1+x_2}{x_1-x_2}(x_1\frac{\partial}{\partial x_1}-x_2\frac{\partial}{\partial x_2})\Phi_\mu \\ &+\left(x_1\frac{\partial}{\partial x_1}\right)^2\Phi_\mu+\left(x_2\frac{\partial}{\partial x_2}\right)^2\Phi_\mu,\end{aligned}$$

for some $c\in\mathbb{C}$. ∎

Appendix A

Explicit examples of interwoven pairs

Lemma A.0.1. Set $k = i(e - f)$, $x = \mathrm{Ad}_g e$, and $y = \mathrm{Ad}_g f$ where

$$g = M = \begin{bmatrix} i & -i \\ 1 & 1 \end{bmatrix} \in GL(2) \quad \text{if} \quad \mathfrak{g} = \mathfrak{gl}_2, \quad \text{and} \quad g = \begin{bmatrix} M & 0 \\ 0 & M \end{bmatrix} \in GL(4) \quad \text{if} \quad \mathfrak{g} = \mathfrak{gl}_4,$$
$$g = (M, M) \in GL(2) \times GL(2) \quad \text{if} \quad \mathfrak{g} = \mathfrak{gl}_2 \oplus \mathfrak{gl}_2.$$

Then $\{x, y, k\}$ is a standard $\mathfrak{gl}_2$-quadruple, i.e.,

$$[k, x] = 2x \ , \ [k, y] = -2y \ , \ [x, y] = k, \ , \ [z, x] = [z, y] = [z, k] = 0. \tag{A.0.1}$$

Moreover, $k \in \mathfrak{k}, x, y \in \mathfrak{p}$ and $x + y = -h_1 + h_2$.

Proof. For the pair $(\mathfrak{gl}_2, \mathfrak{so}_2)$, we construct a new basis for $\mathfrak{gl}_2$ in the following:
Assume that the Cartan subalgebra $\mathfrak{h} = \mathrm{Span}\{h_1 - h_2, h_1 + h_2\}$, then

$$[h_1 - h_2, e] = 2e, \quad [h_1 - h_2, f] = -2f, \quad [h_1 + h_2, e] = 0, \quad [h_1 + h_2, f] = 0.$$

Set

$$k := i(e - f) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad z := h_1 + h_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

and set the Cartan subalgebra $\mathfrak{h}' := \mathrm{Span}\{k, z\}$. Then, one can observe that

$$\mathrm{Ad}_g(h_1 - h_2) = g(h_1 - h_2)g^{-1} = k,$$

and

$$\mathrm{Ad}_g(h_1 + h_2) = g(h_1 + h_2)g^{-1} = z,$$

where $g = M$. Hence, the new basis elements of $\mathfrak{gl}_2$ are:

$$k := k' = i(e-f) = \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix}, \ x := x' = \mathrm{Ad}_g e = \frac{1}{2} \begin{bmatrix} -1 & i \\ i & 1 \end{bmatrix}, \ y := y' = \mathrm{Ad}_g f = \frac{1}{2} \begin{bmatrix} -1 & -i \\ -i & 1 \end{bmatrix}.$$

and $z := I_2$. Then $\{k, x, y, z\}$ is a standard $\mathfrak{gl}_2$-quadratic.

$$[k, x] = 2x \ , \ [k, y] = -2y \ , \ [x, y] = k, \ , \ [z, x] = [z, y] = [z, k] = 0.$$

Similarly, for the pair $(\mathfrak{gl}_4, \mathfrak{sp}_4)$, set;

$$k := i(e-f) = \begin{bmatrix} k' & 0 \\ 0 & k' \end{bmatrix}, \quad z := h_1 + h_2 = \begin{bmatrix} I_2 & 0 \\ 0 & I_2 \end{bmatrix}.$$

Next, by direct calculation we have:

$$\mathrm{Ad}_g(h_1 - h_2) = g(h_1 - h_2)g^{-1} = k,$$

and

$$\mathrm{Ad}_g(h_1 + h_2) = g(h_1 + h_2)g^{-1} = z,$$

where

$$g = \begin{bmatrix} M & 0 \\ 0 & M \end{bmatrix} \in GL(4).$$

Hence, the new basis elements of $\mathfrak{gl}(4)$ are:

$$k = \begin{bmatrix} k' & 0 \\ 0 & k' \end{bmatrix}, \quad z = \begin{bmatrix} I_2 & 0 \\ 0 & I_2 \end{bmatrix}, \quad x = \frac{1}{2}\begin{bmatrix} x' & 0 \\ 0 & x' \end{bmatrix}, \quad y = \frac{-1}{2}\begin{bmatrix} y' & 0 \\ 0 & y' \end{bmatrix}.$$

Hence, $\{k, x, y, z\}$ is a standard $\mathfrak{sl}_2$-quadratic. Similar to the above cases, the $\mathfrak{sl}_2$-quadratic of the pair $(\mathfrak{gl}_2 \oplus \mathfrak{gl}_2, \mathfrak{gl}_2)$ can be obtained. ∎

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