INFORMATION TO USERS

This manuscript has been reproduced from the microfilm master. UMI films the text directly from the original or copy submitted. Thus, some thesis and dissertation copies are in typewriter face, while others may be from any type of computer printer.

The quality of this reproduction is dependent upon the quality of the copy submitted. Broken or indistinct print, colored or poor quality illustrations and photographs, print bleedthrough, substandard margins, and improper alignment can adversely affect reproduction.

In the unlikely event that the author did not send UMI a complete manuscript and there are missing pages, these will be noted. Also, if unauthorized copyright material had to be removed, a note will indicate the deletion.

Oversize materials (e.g., maps, drawings, charts) are reproduced by sectioning the original, beginning at the upper left-hand corner and continuing from left to right in equal sections with small overlaps.

ProQuest Information and Learning
300 North Zeeb Road, Ann Arbor, MI 48106-1346 USA
800-521-0600

UMI®

NOTE TO USERS

This reproduction is the best copy available.

UMI®

8C

ANALYSIS OF STIFFENED AND UNSTIFFENED SLABS.

BY

D.S.B. BHALLA

Université d'Ottawa
BIBLIOTHÈQUES
LIBRARIES
University of Ottawa

ENGINEERING REPORT

Submitted in Partial Fulfillment of the Requirements for the Degree of Master of Engineering in the Graduate School of the University Of Ottawa.

OTTAWA, CANADA.

JUNE 1972.

UNIVERSITAS SCIENTIAE OTTAVIENSIS

© D.S.B. Bhalla, Ottawa 1972.

UMI Number: EC52339

INFORMATION TO USERS

The quality of this reproduction is dependent upon the quality of the copy submitted. Broken or indistinct print, colored or poor quality illustrations and photographs, print bleed-through, substandard margins, and improper alignment can adversely affect reproduction.

In the unlikely event that the author did not send a complete manuscript and there are missing pages, these will be noted. Also, if unauthorized copyright material had to be removed, a note will indicate the deletion.

UMI®

UMI Microform EC52339
Copyright 2007 by ProQuest LLC
All rights reserved. This microform edition is protected against unauthorized copying under Title 17, United States Code.

ProQuest LLC
789 East Eisenhower Parkway
P.O. Box 1346
Ann Arbor, MI 48106-1346

ACKNOWLEDGEMENTS.

Constant encouragement of Dr. S.F. Ng, under whose supervision this study was conducted is greatly appreciated. Thanks are due to Dr. G.M. Lindberg and Dr. G.R. Cowper with whom the author had very useful discussions.

The co-operation extended by the personnel of the Computing Centre, University of Ottawa is acknowledged. The author is appreciative of the financial Grant No: A-4357 of National Research Council, which was of assistance in making the Graduate study possible.

Abstract:

High Precision Finite elements have been used to determine displacements and internal forces of rectangular and skewed slabs. The displacement model is derived on the assumptions of small deflection theory and the principles of linear elasticity.

The high precision triangular plate bending element with six degrees of freedom per node, developed by Cowper, Lindberg, Olson & Kosko has been used to study the behaviour of slabs under concentrated loads. A computer program is developed for the generation of the stiffness maxtrix of the individual elements. These matrices are used to represent the structure force-displacement relationships.

To widen the scope of study, linear elastic members are incorporated and some problems of stiffened slabs are analysed. A compatible beam element is developed to represent the stiffeners. In the case of eccentric stiffeners in-plane forces are developed. To take the in-plane forces into account a complete cubic function is assumed for the tangential displacements.

Comparisons with the existing solutions are made to establish the accuracy achieved on using high precision elements.

TABLE OF CONTENTS

LIST OF FIGURES

LIST OF TABLES

CHAPTER 1

INTRODUCTION

1.1 Objective:

The objective here is to lay the foundation for the development of high precision finite elements application program for the analysis of bridge deck, slabs or plate problems. In one part of the report emphasis is laid on comparison of influence coefficients relative to the values given by other researchers. Simultaneously a study is done on centrally loaded slabs, the idea being to establish the accuracy and confidence in the approach and results. The computer run is relatively inexpensive and can be easily set up for a specific case study.

The second part of the research was to widen the scope of the program to stiffened structures. In-plane strain element for slab is added to the program. A compatible beam element is considered, to analyse stiffened slabs. All these modifications increase the capability of the computer program to handle the problems on plate bending, in-plane analysis of plates and plate with stiffeners, concentric or eccentric.

1.2 General:

Many researchers have done work on the analysis of slabs of various shapes and boundary conditions under different types of loads. Theoretical solutions to most of the problems do not exist and the analytic procedures presented are limited to handle a specific problem at a time. Furthermore, to increase the structural efficiency of a slab, with no increase in weight or reduction in critical buckling load; the slabs are stiffened either concentrically or eccentrically with reference to the middle plane of the slab. Such structures are frequently encountered in ships and aircrafts structures, waffle-floor slabs and girder bridges. Although stiffeners make the structure economical and aesthetically pleasant but, it tends to complicate the already difficult problem of slab analysis.

Nowadays slab structures are so intricate that many idealizations and simplifying assumptions as regards to material or geometric properties have to be resorted to in analysis. In the present study the structure is idealized geometrically by using a finite element mathematical model.

The current study has been developed in steps. The first stage was to analyse rectangular slabs under uniform or concentrated loading conditions. In the second stage skew or non-orthogonal (non-rectangular) edges were taken into account. The scope of the work was further extended by incorporating beam elements to analyse stiffened structures. The program developed is capable of analysing stiffened or unstiffened slabs. The inherent ability of finite element formulation to handle arbitrary boundary conditions - simple support, free or clamped edge, column support etc. with relative ease, enables the analytic procedure to be extremely flexible and increases the utility of the program.

1.3 Background:

Slab structures have been analysed by various investigators using different techniques. Many idealizations and simplifying assumptions are made to reduce the structure to a form for which an approximate solution can be obtained. The analytic approach to slab structures can be categorized mainly as -

1.3a Finite Difference Solution:

In this approach the slab is divided into a gridwork or network. Finite difference operators are used to reduce the governing differential equations into a set of linear algebraic equations. Vogt (1940) used this technique to analyse skew slabs. Jensen (1941, 1947) presented his own finite difference network to analyse skew slabs and skew slabs with curbs. Robinson (1959), Naginchali (1959), Ghali (1967) have independantly analysed various single span and continous slabs. The accuracy of the analysis depends on the fineness of gridwork used.

1.3b Experimental Model Tests:

No exact solutions are available to predict the behaviour of skew stiffened and unstiffened slabs, and as a result a number of researchers and designers have resorted to model tests. Results obtained from such model tests serve to provide checks and verify the analytical values presented by other researchers.

Rushton (1963), Yeginobali (1959), Rush and Hergenroder (1969) have done substantial experimental study on skew slabs. Jensen (1941) and Gossard et al (1950), Kennedy, J.B. (1964) are reported to have done model tests on skew slabs with edge stiffening beams. Gamble (1961) performed experiments on the model of a nine-pannel continuous two way slab. Mehrain (1967) substantiated his finite element results for skew composite plates by plexiglass model tests.

1.3c Orthotropic Plate Theory:

Orthotropic plate theory has been most commonly used for the analysis of composite slabs. There are two theories, one as presented by Huber (1923) and the second, a more refined approach, by Pfluger (1955). Chu and Krishnamoorthy (1962) presented a closed-form solution to Huber's equation to analyse composite beam bridges. However, eccentricity of stiffeners was not considered in the analysis.

Bridge decks with different material properties for slab and girders were analysed by Vitols, Clifton and Au (1963) using Pfluger's refined theory. Although the refined solution considered the eccentricity of stiffeners and accounted for the in-plane displacements but, it is limited to right bridges with two opposite edges supported and the other two free. The effect of stiffeners is averaged ('smeared') over the plate width in orthotropic plate theory. This presents difficulty in separating plate and stiffener moments. It is also difficult to evaluate the torsional rigidity constant for the equivalent orthotropic plate.

1.3d Other Approaches:

Ehasz (1945), Lardy (1949), Quinlan (1962), Kennedy & Huggins (1964), Iyengar et al (1967), have made use of series solutions to analyse skew slabs under different loading conditions for various edge restraints.

Aggarwala (1967) made use of mapping function to analyse simply supported rhombic plates.

Skew clamped plates have been analysed by Dorman (1953) and Kennedy & Ng (1965) using Raleigh-Ritz energy approach. Kennedy (1967) applied Galerkin's variational principle to obtain deflection and moments for skew plates with fixed edges. McElman et al (1966) analysed stiffened plates employing the principle of minimum potential energy.

The method of equivalent grillage was used by Lightfoot & Sawko (1959) and Hendry & Jaegar (1958) to analyse stiffened plates. In this method the structure was reduced to a gridwork of bars, whose stiffness approximates the stiffness of the equivalent plate and beam structure.

One of the best methods for predicting local behaviour of stiffened deck plates is the one presented by Newmark, Siess & Chen (1938). The same principle was later used by Jensen (1941) and Kennedy and Huggins (1964). In this approach shearing stresses between the plate and the stiffening beams are ignored and the solution is obtained by considering the slab continuous over flexible supports.

1.3e Finite Element Method:

A rapid convergence to the true value and the ease with which the Finite Element Method can handle physically and geometrically complex structures has made it a very attractive and practical method of analysis.

The approach has been shown equivalent to the Raleigh-Ritz variational principle, applied to each individual element. Both the Raleigh-Ritz and Finite Element methods are based on the principle of minimum potential energy. In the Raleigh-Ritz approach the displacement function is chosen uniquely for the structure depending on the deformation pattern and boundary conditions. In the F.E. method the displacement function is selected for the element and the boundary conditions are applied to the assemblage of elements which represent the structure.

Since the first paper published on FEM by Turner, Clough, Martin and Topp for the solution of Plane stress problem, various elements have been derived and suggested to analyse plane stress and bending problems in continuum mechanics.

The success of the FEM has been established and demonstrated by numerous independant investigations. As the present work is similar to the study done by Sawko & Cope, Mehrain, McBean and Gustafson, it would be appropriate to review the elements used by these researchers.

A rectangular element with deflection and its derivatives for slope as degrees of freedom at each node has been used by Swako and Cope to analyse single span and continuous skew slabs. On the skew edge they used triangular elements whose displacements function was defined by imposing constraints due to boundary restraints and continuity requirements. In this element compatibility of slope is violated between the edges of the elements.

Gustafson investigated the composite action of slab and beams using bending and plane stress finite elements. He used parallelogramic elements. A complete cubic polynomial represents the transverse deflection and the in-plane displacement functions were linear.

Later, Mehrain did an extensive study of various possible elements to analyse skew composite plates. He also made use of a parallelogramic element for plates with three degrees of freedom, i.e., vertical displacement w and rotational displacements w_x and w_y at each node. He assumed a compatible beam element. To account for in-plane stresses in composite action he introducted two more unknowns u and v at each node, thus giving five degrees of freedom per node. To allow for linear variation of stresses within each element he refined his analysis by introducing additional nodes at mid-side of the plate and beam elements. At these additional nodes the degrees of freedom is the in-plane displacement in the direction of the side.

Recently Mcbean used the finite element approach to provide a more accurate and practical approach to the analysis of stiffened plates. He investigated only rectangular configurations. Mcbean used the quadrilateral bending element developed by Fraijs de Veubeke. Four triangular elements are combined to develop a quadrilateral element with four corner nodes and one node at middle of each side. A complete cubic polynomial for vertical displacements is assumed for a triangular element. Similarly a quadrilateral plane stress element is developed by combination of four triangular elements. A quadratic function for u and v is assumed in order to have a linear variation of strain. This quadrilateral plane stress element also has four corner nodes and four mid-side nodes. Beam element with a mid-side node similar to the one developed by Mehrain was used to idealize stiffeners. The assembled structure has five degrees of freedom at each corner node, vertical deflection and rotations and in-plane displacements. Each mid-side node has three degrees of freedom in-plane displacements and a rotation thus giving a total of 32 degrees of freedom per element.

1.4 NOTATIONS

a, b, c	Dimensions of the Triangular Plate Element
a_i	Coefficients of the quintic polynomial for the triangular plate bending element
$\{A\}$	Column vector for undetermined constants a_i
A_s	Area of the stiffener
b_i	Coefficients of the cubic polynomial for the triangular in - plane plate element
$\{B\}$	Column vector for undetermined constants b_i
c_i	Coefficients of the polynomial for the beam element
$\{C\}$	Column vector for undetermined constants
C_c	Constraint equations for oblique edge
d_i	Coefficients for the cubic polynomial for beam torsion element
D	Flexural rigidity of the plate, $\frac{Et^3}{12(1-\nu^2)}$
$\{D_s\}$	Generalized displacements for the complete structure
$\{D_T\}$	Column vector for undetermined constants d_i
D_1	Unit diagonal matrix consisting of elements of the diagonal of lower band matrix L
E	Modulus of elasticity for plate material
E_s	Modulus of elasticity for beam material
G	Modulus of rigidity
g_i, f_i	Exponents of the i^{th} term of the polynomial for beam bending and in - plane element
F(m,n)	Modified Euler's Beta function
h_i	Exponents of the i^{th} term of the polynomial for beam torsion element
I_x	Second moment of area of the stiffener element, with reference to the mid-plane of plate
J	Torsion constant for the stiffener
$\|k\|$	Stiffness matrix as defined in eq. (13)
$\|K_1\|$	Stiffness matrix as defined in eq. (15)
$\|K_p\|$	Stiffness matrix in global system for plate bending triangular element defined in eq.(20)

$\|k_{ep}\|$	Stiffness matrix as defined in eq. (31)
$\|K_{1ep}\|$	Stiffness matrix as defined in eq. (34)
$\|K_{2ep}\|$	Stiffness matrix as defined in eq. (38)
$\|K_o\|$, $\|K_{oc}\|$, $\|K_c\|$	Partitioned matrices of K_{2ep}
$\|K_I\|$	Stiffness matrix in global system for triangular in-plane plate element
$\|k_b\|$	Stiffness matrix as defined in eq. (66)
$\|K_b\|$	Stiffness matrix for the beam element as defined in eq. (68.1)
$\|k_t\|$	Stiffness matrix as defined in eq. (79)
$\|K_T\|$	Stiffness matrix for the beam torsion element defined in eq.(81)
$\|K\|$	Stiffness matrix for the complete structure
ℓ	Dimensions of the beam element
ℓ_x, ℓ_y, ℓ_0	Dimensions to define the dimensions of the plate
L	Lower band of the banded symmetric matrix K
m_i, n_i	Exponents of the i^{th} term of the polynomial for the plate bending elements
M	Half - band - width of the banded symmetric matrix K
M_x, M_y, M_{xy}	Moments
N	Size of the NxN symmetric banded matrix
m_1, m_2, m_3	Coefficients for moments
p	Entries to the consistent load vector for transverse load on the plate
$\{P_g\}$	Column vector for the loading for the complete structure
P_t	Point load
q_i, p_i	Exponents for the i^{th} term in the polynomial for in-plane displacements in η direction
r_i, s_i	Exponents for the i^{th} term in the polynomial for in-plane displacements of the plate element in ξ direction
$\|R\|$	Rotation matrix as defined in equation (16) and given in appendix 22
$\|R_{ep}\|$	Rotation matrix as defined in eq. (35) and given in appendix 4
S_x	First moment of area of the beam element with respect to the reference plane of the plate

t	Thickness of the plate
$\|T\|$	Transformation matrix for plate bending element as defined in eq. (8) and given in appendix 1.
$\|T_{ep}\|$	Transformation matrix for plate in - plane element as defined in eq. (26) and given in appendix 3
$\|T_b\|$	Transformation matrix for beam bending element as defined in eq. (55) and given in appendix 5
$\|T_t\|$	Transformation matrix for beam torsion element as defined in eq.(74)
u	In-plane displacements of triangular plate element in ξ direction
u_b	In-plane displacement of the beam element
u_c, v_c	Centroidal displacement for the in-plane triangular plate element
U_b	Strain energy for beam element in bending
U_e	Strain energy for plate element in bending
U_{ep}	Strain energy for plate element due to in plane deformation
U_t	Strain energy for beam torsion element
v	In - plane displacement of triangular plate element in η direction
$\{V_1\}$	Column vector of generalized displacements $(u_1, u_{\xi 1}, u_{\eta 1}, v_1, v_{\xi 1}, v_{\eta 1}, \ldots)$ in local coordinates for the in plane plate element
$\{V_2\}$	Column vector for generalized displacements $(u_1, u_{x1}, u_{y1}, v_1, v_{x1}, v_{y1}, \ldots)$ in global coordinates for the in-plane plate element
$\{V\}$	Reduced column vector for generalized displacements in global coordinates for the in plane plate element
V_e	Virtual work done in bending by the plate element
$\{W_1\}, \{W\}$	Column vector for generalized displacements for the plate bending element
$\{W_3\}$	Column vector for generalized
(V, V_s, W, W_s, W_{ss})	Kinematic boundary conditions on oblique edge
X, Y	Global coordinate system
ξ, η	Local coordinate system for plate elements
λ	Langrangian multipliers
β	Skew - angle
θ	angle between the local and global coordinate system
Π	Potential energy of the system
σ_x	Stress for beam element in bending
ϕ	Rotation angle for the beam element

1.5 Outline of the Report:

In the present chapter the literature has been reviewed. In chapter 2 the derivation of the stiffness matrices based on small deflection theory and energy principles is outlined. The derivation for consistent load vector for plate bending element is also included. Normal tranverse loads are the only type considered in this report.

Assembly of elements and solution of symmetric banded matrix is discussed in chapter 3. The various methods to handle oblique edges are also reviewed.

Comparison of results for unstiffened single span slabs is made in chapter 4. Conclusions are summed in this chapter for slabs under concentrated loads.

Two examples for eccentric stiffeners and one for concentric stiffener are presented in chapter 5.

Recommendations are outlined for further study. A note on the computer program and the flow chart are also included.

CHAPTER 2

DERIVATION OF ELEMENT PROPERTIES

2.1 General:

The accuracy of solution by the finite element method is dependant on how well the element deformation characteristics represent the deflected shape of the actual structure. For a detailed and complete discussion for the selection of displacement function and Finite element approximation reference is made to the books by Zienkiewicz & Cheung (33) and Przemieniecki (22). However the essential criteria for the FEM solution to approach the true solution with increasing fineness of gridwork and the criteria of the FEM to provide a bound to the strain energy convergence is briefly mentioned here.

The displacement polynomial must include all the rigid body modes and the states of uniform strain. The polynomial should be complete and must contain the terms to include the highest order derivative occurring in the strain energy integral. Transverse displacements and normal slopes should be continuous at the nodal points and the edges of the elements. The degrees of freedom should match the number of terms in the polynomial.

Interelement continuity of stresses in the displacement model is extremely difficult to achieve even if all the conditions of compatibility are satisfied. The elements presented have a quadratic representation of stresses and the stress discontinuities between adjacent elements is much smaller than lower order elements.

DERIVATION OF THE ELEMENTS

2.2 Basic assumptions:

The results presented are for thin plates, hence it will be appropriate to list the assumptions underlying small deflection (Elastic analysis) theory of thin plates. The assumptions are:

a) The thickness of plate is small as compared to lateral dimensions.

b) The deflection w, in the z - direction is a function of x and y only.

c) Normal stresses to the middle plane do not affect the deflection.

d) Normal to the undeformed plate and stiffner structure remains straight and normal to the deformed structure.

e) Hooke's Law is valid for in-plane strains.

f) Energy due to change in length of the middle surface of plate may be neglected i.e., deflections/thickness ratio is small; for plate bending.

The three elements made use of in the analysis are:

i. Plate bending element

ii Plane stress element

iii Beam element

The slab is assumed to be homogeneous, elastic and isotropic. The stiffners are made of homogeneous, elastic and isotropic material. No relative movement occurs between the plate and stiffner element during deformation i.e., slab and stiffners are considered integral. The beam element is considered torsionally weak and hence the torsion is uncoupled.

2.3 Plate Bending Element:

A twenty one degree of freedom, high precision triangular element developed by Cowper et al has been used.

The assumed displacement field in local co-ordinates fig (i) is a

$$w(\xi,\eta) = a_1 + a_2\xi + a_3\eta + a_4\xi^2 + a_5\xi\eta + a_6\eta^2 + a_7\xi^2 + a_8\xi^2\eta$$
$$+ a_9\xi\eta^2 + a_{10}\eta^2 + a_{11}\xi^4 + a_{12}\xi^3\eta + a_{13}\xi^2\eta^2 + a_{14}\xi\eta^3 + a_{15}\eta^4 +$$
$$a_{16}\xi^5 + a_{17}\xi^3\eta^2 + a_{18}\xi^2\eta^3 + a_{19}\xi\eta^4 + a_{20}\eta^5 \qquad (1)$$

$$w(\xi,\eta) = \sum_{i=1}^{20} a_i \xi^{m_i}\eta^{n_i} \qquad (2)$$

The $\xi^4\eta$ term has been omitted so as to have normal slope along edge $\eta = 0$ be a cubic in ξ. The condition that normal slope be a cubic along the other two edges of the element, yield

$$5b^4ca_{16} + (3_b{}^2c^3 - 2b^4c)a_{17} + (2bc^4 - 3b^3c^2)a_{18} \pm (c^5 - 4b^2c^3)a_{19}$$

$$- 5bc^4a_{20} = 0 \qquad (3)^*$$

$$5a^4ca_{16} + (3a^2c^3 - 2a^4c)a_{17} + (-2ac^4 + 3a^3c^2)a_{18} + (c^5 - 4a^2c^3)$$

$$+ 5ac^4a_{20} = 0 \qquad (4)^*$$

The condition $\xi^4\eta = 0$ and equations (3) and (4) assure continuity of displacement and slopes along the common edge of two elements. The variation of curvature is quadratic.

The eighteen generalized displacements for the element, deflection and its first and second derivatives at each node, can be related to the coefficients a_i as

$$\{w_1\} = [T_1]\{A\} \qquad (5)$$

where $\{w_1\}^T = (w_1, w_{\xi 1}, w_{\eta 1}, w_{\xi\xi 1}, w_{\xi\eta 1}, w_{\eta\eta 1}, w_2, \ldots w_{\eta\eta 3}) \qquad (6)$

and $\{A\}^T = (a_1, a_2, a_3, \ldots\ldots, a_{20}) \qquad (7)$

Considering the constraints (3) and (4) in (5) we have

$$\left\{\begin{matrix} w_1 \\ 0 \\ 0 \end{matrix}\right\} = [T]\{A\} \qquad (8)$$

The matrix [T] is listed in Appendix 1 and is defined by the geometry of the element. The determinant of [T] is $-64(a+b)^{17}c^{20}.(a^2+c^2)(b^2+c^2)$ which is singular only when the area of the triangle approaches zero.

$$\{A\} = [T]^{-1} \left\{\begin{matrix} w_1 \\ 0 \\ 0 \end{matrix}\right\} \qquad (9)$$

$$\{A\} = [T_2]\{w_1\} \qquad (10)$$

where $[T_2]$ is 20x18 matrix having first eighteen columns of $[T_1]^{-1}$.

---

*For detailed derivation see reference LR514 (4).

The strain energy of an isotropic plate element is

$$U_e = \frac{1}{2} D\iint \{w_{\xi\xi}{}^2 + w_{\eta\eta}{}^2 + 2\nu w_{\xi\xi} w_{\eta\eta} + 2(1-\nu) w_{\xi\eta}{}^2\} d_\xi d_\eta \quad (11)$$

U_e can be written in matrix notation as

$$U_e = \frac{1}{2} D\{A\}^T [k]\{A\} \quad (12)$$

where $k_{ij} = m_i m_j (m_i-1)(m_j-1) F(m_i+m_j-4, n_i+n_j) + n_i n_j (n_i-1).(n_j-1) F$

$(m_i+m_j,\ n_i+n_j-4) + \{2(1-\nu) m_i m_j n_i n_j + \nu m_i n_j (m_i-1)(n_j-1) + \nu (m_j-1)(n_i-1)$

$m_j n_i\} F(m_i+m_j-2,\ n_i+n_j-2)$ (13)

F(m,n) is Euler Beta function. All elements of the matrix [k] can be evaluated automatically on the computer. Using (10) we can write relation (12) as

$$U_e = \frac{1}{2} D\{w_1\}^T [T_2]^T [k][T_2]\{w_1\} \quad (14)$$

$$U_e = \frac{1}{2} D\{w_1\}^T [K_1]\{w_1\} \quad (15)$$

Displacements $\{w_1\}$ can be transformed to displacements in global systems by the rotation matrix [R], such that

$$\{w_1\} = [R]\{w\} \quad (16)$$

where $\{w\}^T = (w_1,\ w_{x_1},\ w_{y_1},\ w_{xx_1},\ w_{xy_1},\ w_{yy_1},\ w_2, \ldots\ldots\ldots)$ (17)

and [R] is given in Appendix 2.

(15) can be written as

$$U_e = \frac{1}{2} D\{w\}^T [R]^T [K_1][R]\{w\} \quad (18)$$

$$U_e = \frac{1}{2} D\{w\}^T [K_p]\{w\} \quad (19)$$

where $[K_p]$ is the element stiffness matrix for plate bending in global co-ordinate system, such that

$$[K_p] = [R]^T [T_2]^T [k][T_2][R] \quad (20)$$

2.4 In-Plane Plate Element

The in-plane displacements u and v are assumed to be a complete cubic polynomial

$$u = b_1 + b_2\xi + b_3\eta + b_4\xi^2 + b_5\xi\eta + b_6\eta^2 + b_7\xi^3 + b_8\xi^2\eta + b_9\xi\eta^2 + b_{10}\eta^3 \quad (21)$$

$$v = b_{11} + b_{12}\xi + b_{13}\eta + b_{14}\xi^2 + b_{15}\xi\eta + b_{16}\eta^2 + b_{17}\xi^3 + b_{18}\xi^2\eta + b_{19}\xi\eta^2 + b_{20}\eta^3 \quad (22)$$

$$u = \sum_{i=1}^{10} b_i \xi^{r_i} \eta^{s_i} \quad (23)$$

$$v = \sum_{i=11}^{20} b_i \xi^{p_i} \eta^{q_i} \quad (24)$$

The twenty generalized displacements to match twenty coefficients are u and v and their first derivatives at each vertex and u_c, v_c, the displacements at the centroid.

The generalized displacement vector

$$\{V_1\}^T = (u, u_{\xi 1}, u_{\eta 1}, v_1, v_{\xi 1}, v_{\eta 1}, u_2, \ldots, v_{\eta 3}, u_c, v_c) \quad (25)$$

The generalized displacements can be related to coefficients b_i as

$$\{V_1\} = [T_{ep}]\{B\} \quad (26)$$

where $\{B\}^T = (b_1, b_2, b_3, \ldots, b_{20})$ (27)

and $[T_{ep}]$ the transformation matrix is defined in Appendix 3.

The transformation matrix $[T_{cp}]$ has non zero determinant, so

$$\{B\} = [T_{cp}]^{-1}\{V_1\} \quad (28)$$

The strain energy due to t in plane displacements of an element

$$U_{cp} = \frac{Et}{2(1-\nu^2)} \iint \{u_\xi^2 + v_\eta^2 + 2\nu u_\xi v_\eta + \frac{1-\nu}{2}(u_\xi + v_\xi)^2\}\, d_\xi d_\eta \quad (29)$$

In matrix notation

$$U_{ep} = \frac{Et}{2(1-\nu^2)} \{B\}^T [k_{ep}] \{B\} \tag{30}$$

where the elements of $[k_{ep}]$ are given as

$$\begin{aligned} k_{ep\ ij} = {} & r_i r_j \ F(r_i + r_j - 2,\ s_i + s_j) + q_i q_j \ F(p_i + p_j,\ q_i + q_j - 2) \\ & + \frac{1-\nu}{2} [s_i s_j \ F(r_i + r_j,\ s_i + s_j - 2) + p_i p_j \ F(p_i + p_j - 2,\ q_j + q_i)] \\ & + [\frac{1-\nu}{2} \ s_j p_i + \nu r_j q_i] \ F(r_j + p_i - 1,\ s_j + q_i - 1) + [\frac{1-\nu}{2} \ s_i p_j + \nu r_i q_i]. \\ & F(r_i + p_j - 1,\ s_i + q_j - 1) \end{aligned} \tag{31}$$

r_i and s_i are zero for $i > 10$

p_i and q_i are zero for $i < 11$.

From (26) and (30) we have

$$U_{ep} = \frac{Et}{2(1-\nu^2)} \{V_1\}^T \ [T_{ep}]^{-1} \ [k_{ep}][T_{ep}]^{-1} \ \{V_1\} \tag{32}$$

$$U_{ep} = \frac{Et}{2(1-\nu^2)} \ \{V_1\}^T \ [K_{1ep}]\{V_1\} \tag{33}$$

where $[k_{1ep}] = [T_{ep}^{-1}]^T \ [k_{ep}][T_{ep}^{-1}]$ (34)

The general displacements in local co-ordinates can be transformed into the global system by the rotation matrix $[R_{ep}]$

$$\{V_1\} = [R_{ep}]\{V_2\} \tag{35}$$

where $[R_{ep}]$ is defined in Appendix 4.

The strain energy for tangential displacements in the global system is

$$U_{ep} = \frac{Et}{2(1-\nu^2)} \ \{V_2\}[R_{ep}]^T \ [K_{1ep}][R_{ep}]\{V_2\} \tag{36}$$

$$U_{ep} = \frac{Et}{2(1-\nu^2)} \{V_2\}[K_{2ep}]\{V_2\} \quad (37)$$

where $[K_{2ep}] = [R_{ep}]^T[K_{1ep}][R_{ep}]$ (38)

The equilibrium equation for an element is

$$[K_{2ep}]\{V_2\} = \{P_{ep}\} \quad (39)$$

where, $\{P_{ep}\}$ is the 20 x 1 load vector.
Equation (39) can be partitioned as

$$\begin{bmatrix} K_o & K_{oc} \\ K_{oc}^T & K_c \end{bmatrix} \begin{Bmatrix} V \\ V_c \end{Bmatrix} = \begin{Bmatrix} P_o \\ P_c \end{Bmatrix} \quad (40)$$

where the subscript c denotes the contribution of the centroidal displacements;

$$\{V_c\} = \begin{Bmatrix} u_c \\ v_c \end{Bmatrix} \quad (41)$$

and $\{V\}^T = \{u_1, u_{x1}, u_{y1}, v_1, v_{x1}, v_{y1}, u_2, \ldots, v_{y3}\}$ (42)

Equation (40) can be expanded to the form

$$[K_o]\{V\} + [K_{oc}]\{V_c\} = \{P_o\} \quad (43)$$

$$[K_{oc}]^T\{V\} + [K_c]\{V_c\} = \{P_c\} \quad (44)$$

Solving for V_c, we have

$$\{V_c\} = [K_c^{-1}]\{P_c\} - [K_c^{-1}][K_{oc}]^T\{V\} \quad (45)$$

Equations (43) and (45) yield

$$[K_o]\{V\} + [K_{oc}][K_c^{-1}]\{P_c\} - [K_{oc}][K_c^{-1}][K_{oc}]^T\{V\} = \{P_o\} \quad (46)$$

$$[K_o] - [K_{oc}][K_c^{-1}][K_{oc}]^T \{V\} = \{P_o\} - [K_{oc}][K_c^{-1}]\{P_c\} \tag{47}$$

$$[K_I] = [K_o] - [K_{oc}][K_c^{-1}][K_{oc}]^T \tag{48}$$

and $$\{P_p\} = \{P_o\} - [K_{oc}][K_c^{-1}]\{P_c\} \tag{49}$$

The equilibrium equation (39) for an element can be condensed to

$$[K_I]\{V\} = \{P_p\} \tag{50}$$

2.5 Beam Element

To be consistent with the plate bending and in-plane elements the beam element is derived by energy principles. The axial deformation function is a cubic and the bending displacement function is a quintic polynomial*, to be compatible with plate in-plane and plate-bending elements.

$$u_b = c_1 + c_2x + c_3x^2 + c_4x^3 \tag{51}$$

$$w_b = c_5 + c_6x + c_7x^2 + c_8x^3 + c_9x^9 + c_{10}x^5 \tag{52}$$

$$u = \sum_{i=1}^{4} c_i x^{g_i} \qquad g_i = i-1 = 0 \ , \ i > 5 \tag{53}$$

$$w = \sum_{i=5}^{10} c_i x^{f_i} \qquad f_i = i-5 = 0 \ , \ i < 4 \tag{54}$$

The generalized displacements considered are axial displacement u, its first derivative, deflection w and its first and second derivatives.

$$\{\delta\} = [T_b]\{c\} \tag{55}$$

where $\{\delta\}^T = (u_1, u_{x_1}, w_1, w_{x_1}, w_{xx_1}, \ldots\ldots, w_{xx_2})$

and $[T_b]$ is the transformation matrix defined in Appendix 5. The determinent of $[T_b]$ is $124\ell^{13}$ and is zero when the beam element reduces to a point.

---

*For the beam element in x-direction. (Fig. 2).

$$\{c\} = [T_b]^{-1} \{\delta\} \tag{56}$$

The strain energy of a beam element in bending is

$$U_b = \frac{1}{2} \int_0^{\ell} \iint_A \frac{\sigma_x^2}{E} \, dx dA \tag{57}$$

For an eccentric stiffner

$$\sigma_x = E\left(\frac{\partial u}{\partial x} - Z \frac{\partial^2 w}{\partial x^2}\right) \tag{58}$$

Equations (58) and (57) yields

$$U_b = \frac{E}{2} \int_0^{\ell} \iint_A \left(\frac{\partial u}{\partial x} - Z \frac{\partial^2 w}{\partial x^2}\right)^2 dx dA \tag{59}$$

$$U_b = \frac{E}{2} \int_0^{\ell} \iint_{A_s} \left\{\left(\frac{\partial u}{\partial x}\right)^2 - 2Z \left(\frac{\partial u}{\partial x}\right)\left(\frac{\partial^2 w}{\partial x^2}\right) + Z^2 \left(\frac{\partial^2 w}{\partial x^2}\right)\right\} dx dA \tag{60}$$

$$U_b = \frac{E}{2} \int_0^{\ell} \left\{A_s \left(\frac{\partial u}{\partial x}\right)^2 - 2 \, Sx \left(\frac{\partial u}{\partial x}\right)\left(\frac{\partial^2 w}{\partial x^2}\right) + I \left(\frac{\partial^2 w}{\partial x^2}\right)\right\} dx \tag{61}$$

where A_s is the area of the stiffner; Sx, I are the first and the second moment of area, respectively, about the reference plane, i.e. mid-surface of the slab.

Differentiating equation (53) we have

$$\frac{\partial u}{\partial x} = \sum_{i=1}^{4} c_i g_i \, x^{g_i - 1} \tag{62}$$

$$\left(\frac{\partial u}{\partial x}\right)^2 = \sum_{i=1}^{4} \sum_{j=1}^{4} c_i c_j g_i g_j \, x^{g_i + g_j - 2} \tag{63}$$

$$\int_0^{\ell} \left(\frac{\partial u}{\partial x}\right)^2 = \sum_{i=1}^{4} \sum_{j=1}^{4} c_i c_j g_i g_j \frac{\ell^{g_i + g_j - 1}}{g_i + g_j - 1} \tag{64}$$

Similarly expanding other terms of the strain energy expression, integrating and in matrix notation

$$U_b = \frac{1}{2} E \{C\}^T [k_b]\{C\} \tag{65}$$

where the elements of $[k_b]$ are given as

$$k_{b\,ij} = A_s g_i g_j \frac{\ell^{g_i+g_j-1}}{g_i+g_j-1} - S_x\{g_i f_j (f_{j-1}) \frac{\ell^{g_i+f_j-2}}{g_i+f_j-2} + g_i f_i (f_{i-1}) \frac{\ell^{g_j+f_i-2}}{g_j+f_i-2}\} + I\{f_i f_j (f_{i-1})(f_{j-1}) \frac{\ell^{f_i+f_j-3}}{f_i+f_j-3}\} \tag{66}$$

The elements of $[k_b]$ are given in Appendix 6.

From relations (66) and (65)

$$U_b = \frac{1}{2} E \{\delta\}^T [T_b^{-1}]^T [k_b][T_b^{-1}]\{\delta\} \tag{67}$$

$$U_b = \frac{1}{2} E \{\delta\}^T [K_b]\{\delta\} \tag{68}$$

where $$[K_b] = [T_b^{-1}]^T [k_b][T_b^{-1}] \tag{68.1}$$

As the direction of the stiffner is coincident to the global co-ordinate system, no rotation is required.

2.6 Beam Torsion Element

The rotation angle ϕ for the beam element is assumed to be a cubic. The generalized displacements for this element are ϕ and its first derivative at each node. A cubic function is assumed for ϕ in order to be compatible with the plate bending element and also to satisfy the requirement of normal slope to be a cubic along an edge of an element.

$$\phi = d_1 + d_2 x + d_3 x^2 + d_4 x^3 \tag{69}$$

$$\phi = \sum_{i=1}^{4} d_i \, x^{h_i} \qquad h_i = i-1 \tag{70}$$

In matrix notation

$$\{W_3\} = [T_T]\{D_T\} \tag{71}$$

where
$$\{W_3\}^T = (\phi_1, \phi_{x1}, \phi_2, \phi_{x2}) \tag{72}$$

$$\{D_T\}^T = (d_1, d_2, d_3, d_4) \tag{73}$$

And $[T_T]$ is a 4x4 transformation matrix required as

$$[T_T] = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & \ell & \ell^2 & \ell^3 \\ 0 & 1 & 2\ell & 3\ell^2 \end{vmatrix} \tag{74}$$

Strain energy due to rotation of a beam element

$$U_T = \frac{GJ}{2} \int_0^{\ell} \phi_x^{\,2} \, dx \tag{75}$$

where J is a torsion constant and is approximately

$$J \approx \frac{bh^3}{3} \quad \text{for a rectangular section,}$$

$$\phi_x = \sum_{i=1}^{4} d_i \, h_i \, x^{h_i - 1} \tag{76}$$

$$\phi_x^{\,2} = \sum_{i=1}^{4} \sum_{j=1}^{4} d_i d_j h_i h_j \, x^{h_i + h_j - 2} \tag{77}$$

Then equation (75) yields, in matrix form

$$U_T = \frac{GJ}{2}\{w_3\}^T [T_T^{-1}]^T [k_T][T_T^{-1}]\{w_3\} \tag{78}$$

where the elements of $[k_T]$ are given as

$$k_{T\,ij} = (i-1)(j-1)\,\frac{2^{h_i+h_j-1}}{h_i+h_j-1} \tag{79}$$

$$U_T = \frac{GJ}{2}\{w_3\}^T\,[K_T]\{w_3\} \tag{80}$$

where $$[K_T] = [T_T^{-1}]^T[k_T][T_T^{-1}] \tag{81}$$

The stiffness matrices $[K_B]$ and $[K_T]$ for the beam element are derived for the beam elements parallel to the x-direction of the global system. However, beam elements can be considered in y-direction by just substituting x by y in the above derivations

2.7 Load Vector for Transverse Loading

The loading condition can be simulated either by determining the contribution for every degree of freedom from the principle of virtual work or by lumping the equivalent load at the nodal points.

The work done by a load of intensity q over an element is

$$Ve = \iint q w\, d\xi d\eta \tag{82}$$

By relation (2) and (82)

$$Ve = \sum ai \iint q\xi^{mi}\eta^{ni}\; d\xi d\eta \tag{83}$$

in matrix notation $Ve = \{A\}^T\{p\}$ (84)

where $$P_i = \iint q\xi^{mi}\eta^{ni}\; d\xi d\eta \tag{85}$$

By relation (10) and (85)

$$Ve = \{w_1\}^T[T_2]^T\{p\} \tag{86}$$

By relation (16) and (86)

$$Ve = \{u\}^T[R]^T[T_2]^T\{p\} \tag{87}$$

The consistent load vector is

$$\{P\} = [R]^T[T_2]\{p\} \tag{88}$$

If the load is of uniform intensity q_o over the element then p_i in relation to (88) is given by

$$p_i = q_o \iint \xi^{m_i}\eta^{n_i}\, d\xi d\eta \tag{89}$$

$$p_i = q_o\, F(m_i,\, n_i) \tag{90}$$

If the load is a concentrated load q_p applied at a point ξ_p , η_p within the triangle, then in that case the terms of p_i in relation to (88) are given by

$$p_i = q_p\, \xi_p^{m_i}\, \eta_p^{n_i} \tag{91}$$

The load vector obtained as above is called a consistent load vector because it is derived from the principle of virtual work to be consistent with the derivation of the stiffness properties of the element. In the case of the lumped load vector the load intensity is converted into an equivalent system of loads acting at nodal points. The bound to strain energy is provided only for the case of consistent load vector. For a moving concentrated load acting at a nodal point, if the relation (91) is used, the only significant contribution in the load vector will be to the degree of freedom corresponding to transverse deflection, the rest of the terms working out to be nearly zero. This is equivalent to a lumped load system.

CHAPTER 3

ASSEMBLAGE OF ELEMENTS AND SOLUTION OF EQUATIONS

3.1 Assembly of Elements:

In the present work three different problems are considered viz (i) Single span slabs (ii) Slabs with concentric stiffeners (iii) Slabs with eccentric stiffeners. For single slab, , only plate bending type finite element is required to define the deformation pattern of the structure under normal transverse loads. In the case of slabs stiffened with concentric stiffeners i.e., the centre of gravity of the stiffener lies on the mid-plane of the slab; The finite elements necessary to represent the structure are plate bending element and the beam bending and torsion element. Plate bending element, in-plane slab element, beam bending element, coupled axial deformation and beam torsion element are required to represent a loaded composite slab.

The following requirements must be simultaneously satisfied while assembling finite elements.

a) Equilibrium - the internal element forces and the external applied forces at each nodal point must be in equilibrium.

b) Compatibility - in the loaded condition the adjacent elements must have continuous deformation.

On assembling two different structural type elements such as plate element and a beam element fig. (3), additional conditions of compatibility must be satisfied. The two elements must have compatible deformation pattern and equivalent degrees of freedom. For the stiffener in x direction, the generalized displacements w, w_x, w_{xx}, u_x, u, w_y, w_{xy} are added to the corresponding degrees of freedom of the plate element at the common node. This beam element in x direction has zero contribution to generalized displacements u_y, v, v_x, v_y and w_{yy} of the plate element.

3.2 Boundary Conditions:

By virtue of the principle of minimum potential energy, only natural or kinematic boundary conditions need be applied. Force boundary conditions are questionable and normally lead to added constraints and hence a stiffer structure.

There are a number of ways in which the oblique edges in skewed structures can be handled. Automatic boundary conditions on a simply supported edge making an angle β with the x - axis fig. (7) are

i) Flexural restraints

$$W = W_s = W_{ss} = 0 \qquad (92)$$

where W, W_s and W_{ss} are deflection, its first and second derivation in the direction of the edge.

ii) In-plane deformation restraints

$$V' = V's = 0 \qquad (93)$$

where V' and $V's$ are the in-plane displacement and its slope in the direction of the oblique edge.

These can be expressed in terms of derivatives of the global system as

$$W = 0 \qquad (94)$$

$$W_s = 0 = W_x \cos\beta + W_y \sin\beta \qquad (95)$$

$$W_{ss} = 0 = W_{xx}\cos^2\beta + 2W_{xy} \sin\beta\cos\beta + W_{yy} \sin^2\beta \qquad (96)$$

$$V' = U \cos\beta + V \sin\beta \qquad (97)$$

$$V's = Ux \cos^2\beta + (Uy + Vx) \sin\beta\cos\beta + Vy \sin^2\beta \qquad (98)$$

One way to apply these constraint equations is, say in equation (95), to express the term W_x in terms of W_y and thus eliminating W_x.

e.g.,
$$Wx = - Wy \frac{\sin\beta}{\cos\beta} \qquad (99)$$

similarly for other conditions. This increases the computational difficulties. Another way is to consider the equations (95) to (98) as constraints and the generalized displacements are obtained from minimization of the potential energy.

The potential energy of the system is

$$\Pi = \tfrac{1}{2}\{D_s\}^T[K]\{D_s\} - \{D_s\}^T\{P_g\} \qquad (100)$$

and the constraint equations are

$$[C_c]\{D_s\} = 0 \qquad (101)$$

The use of langrangian multipliers ..... result in a system of equations

$$\begin{bmatrix} K & C_c^T \\ C_c & 0 \end{bmatrix} \begin{Bmatrix} D_s \\ \lambda \end{Bmatrix} = \begin{Bmatrix} P_g \\ 0 \end{Bmatrix} \qquad (102)$$

which can be very easily solved. This technique though simple has a distinct disadvantage of being extremely demanding on computer storage, since the structure stiffness matrix loses the banded nature and has a lot of zeros.

The boundary restraints can be directly applied by locally rotating the nodes lying on the boundary by an orthogonal system such that the generalized displacements become W, Ws, Wn, Wss, Wsn, Wnn. Mathematically it implies to premultiply the stiffness matrix by the transpose of the rotation matrix and then to post multiply by the rotation matrix. This method has an advantage in that the symmetric and the banded nature of the structure stiffness matrix is retained; which can be used for economic storage location and operation time of the computer.

3.3 Solution of Equations:

On assembling the structure stiffness matrix and application of the boundary conditions the problem reduces to the solution of the matrix equation

$$[K]\{D_s\} = \{P\} \qquad (92)$$

where $[K]$ is the structure stiffness matrix.

$\{Ds\}$ is the displacement victor.
and $\{P\}$ is the load victor.

The size of the matrices in equation (92) is a function of the degrees of freedom per node and the fineness of gridwork. It is imperative to have an efficient method of solving the equation (92) especially when the computer memory and operation time are additional restraints.

When Langrangian multipliers are used to satisfy the boundary conditions, the banded property of the stiffness matrix is lost. In such a case Gauss total elimination method may be used. This method can take zero values on the leading diagonal and the precision of results is very good. It has a distinct disadvantage of being time consuming and very demanding on computer memory. The technique is attractive for it being very simple and easy to use particularly when the number of equations is not very large.

The symmetric and banded characteristics of the structure stiffness matrix are retained when the generalized displacements of the nodes on the oblique edge are locally rotated. The edge condition can be applied directly by zeroing out the corresponding row and column and putting a unity on the main diagonal of the structure stiffness matrix. There are numerous algorithms available for the solution of symmetric banded matrices. In the present work the solution of the linear equation as given by Wong (30) is obtained by applying crout factorization scheme to the banded symmetric matrix $[K]$.

The matrix $[K]$ can be split into a product of a lower triangular matrix L and an upper triangular matrix U (with a unit diagonal) such that

$$K = LU \tag{93}$$

Since K is symmetric

$$U = D_1^{-1} L^T \tag{94}$$

where D_1 is the matrix consistency of the diagonal of L and L^T is the transpose of L.

$$K = LD_1^{-1} L^T \tag{95}$$

The displacement vector D_s in equation (92) is obtained by performing the forward substitution $LY = P$ followed by the backward substitution $UW = Y$. As U is expressed, completely, in terms of elements of L, hence the substitution and factorization algorithm operates only on the elements of L. The limitation of the above scheme is that the stability is assured only for a positive definite matrix K.

The computer memory and storage are used most economically by storing the elements of the lower band of the stiffness matrix K in a one-dimensional array. The elements of the lower band of the symmetric matrix K are stored in a LBR (lower - band - row by row) mode, such that, the elements of each row immediately following the preceeding row are stored in a one-dimensional array.

The location of an element k(i,j) in the one-dimensional array S(k) is determined as

$$k = \frac{i(i^2-1)}{2} + j \qquad \text{for } i < m+1 \qquad (96)$$

and

$$k = im + j - \frac{m\,(m+1)}{2} \qquad \text{for } i \geqslant m+1 \qquad (97)$$

where m is the half-band width of the banded symmetric matrix K.

The size of the one-dimensional array S(k) to store the lower band of the N by N banded (half band width M) symmetic matrix is

$$d \geqslant \frac{(M+1)\;(2N-M)}{2} \qquad (98)$$

Special care should be exercised while applying the boundary conditions in the above mentioned scheme of storage since, the location of the element has been shifted.

CHAPTER 4

UNSTIFFENED SINGLE SPAN SLABS

4.1 General:

The accuracy and convergence of the high precision plate bending element has been very well documented by Cowper et al (4). This element has been used to analyse skew slabs of various boundary conditions, under uniformly distributed load, by Desikan (6). The present work is an extension of the work done by these researchers.

Slabs with two sides simply supported and two sides free are analysed for uniform load and the moment coefficients at various points have been compared with those obtained by Yeginobali (31,32) in Table 1 to Table 3. The results are in excellent agreement, except at the edges. This is due to the fact that in Finite element method kinematic boundary conditions are applied rather than force boundary conditions; besides, stresses at corners are not zero as given by Yeginobali. The convergence for a square plate is tabulated in Table 4 and comparison is made with the exact solution and with that given by Gustafson (7). The deflection at the free edge is only 0.133% in error to the exact solution and the values for 4x4 grid are excellent and more close to the exact solution than the ones given by Gustafson for a 10x10 grid using a parallelogramic element with 12 degrees of freedom.

Table 5 shows the convergence of the values obtained for parallelogramic plates with two sides simply supported and the other two free. It is observed that the central deflection, longitudinal moment and strain energy decrease with increase in skew, whereas the transverse moment increases with the increase in skew. The central deflection for zero skew is 11.5 times the central deflection for 60° skew. The variation of central deflection and longitudinal moment with the angle of skew is quite significant while the variation of the transverse moment is not that marked.

4.2.1 Simply Supported Square Slab under a central load.

The convergence of central deflection is plotted in fig. (11) and tabulated in Table 6. The plot of moment at the centre with the fineness of grid is shown in fig. (10). The plot shows that as the finite element grid is refined, the slab approaches the true plate condition at $n = \infty$, the stress under the point load tend to infinity. This is in accordance to the classical plate theory. Fig. (12) shows that the moment variation along the line of symmetry is in excellent agreement with the exact solution given by Timoshenko (27).

4.2.2 Centrally loaded slabs, supported on two sides and free on the other two:

Central deflections are compared in Table 7 with the experimental results given by Robinson (23). Robinson tested mild-steel plates of 1/4" thickness and width of 12" and spanning 12". For zero skew the finite element approximation gives a stiffer structure and the deviation from experimental results is nearly 6%. But for parallelogramic slabs the results are higher than those obtained experimentally. This may partially be due to the restraints applied at edges, in model tests, to prevent the plate corners from lifting.

Principal bending moments at mid-span are compared with those given by Robinson (23) for slabs with 0°, 30° and 45° skew. Under the load the finite element values are higher. However at other points they are in good agreement; even for high skew.

Robinson (23) also presented results for skew slabs using finite difference operators. Table 8 and Table 9 and Table 10 show the comparison of his values with the one obtained by high precision element. A difference of about 4 to 5% is observed between the two approximations. It should be observed that the central deflections and the Principle Moments decrease with the increase in skew angle.

The convergence of deflections and strain energy for skew slabs with the fineness of network is tabulated in Table 11.

Figs. (13) to figs. (21) show plot of longitudunal and transverse moments along the mid-span and line of symmetry for centrally loaded slabs. It is observed that the values for crude grid are not in much error to the ones obtained by the refined network.

The distribution of moment along the mid span for plate with 60° skew as shown in fig. (20) and fig. (21) is not uniform. The cusping* is quite predominant for transverse moment. The distribution of longitudunal moment fig. (19) along the line of symmetry A-A′ is relatively smooth for the same case. The cusping phenomena is observed for all plots (13) to (21) but it is not so distinct for the cases of 0°, 30° and 45° skew. A skew of 60° is a case of severe skew, however, the line joining values at nodal points for various grids is quite uniform.

4.2.3 Influence Co-efficients:

Checks for relative accuracy are made of influence co-efficients for moments at the centre point of the slabs; first with the values given by Yeginobali (31, 33) using finite difference operators and then with the experimental values presented by Rush and Morgenroder (24).

Influence co-efficients for moment at centre point for plate type II - A, B and C fig. (6) are tabulated for an 8 x 8 grid in Table 12 to Table 14. The stress pattern obtained by finite element analysis is similar to the one given by Yeginobali (31, 32). Mostly the values are the same. A relative difference of about 5% for zero and 30° skew increasing to about 10% and higher for 45° skew is observed at some nodal points. In finite element method curvatures are the generalized displacements and are obtained directly from the stiffness matrix, and thus should be more reliable. The convergence and other plots as indicated in section 4.2.1 and 4.2.2 add confidence to the finite element approximation

* cusping is due to stress discontinuity between displacement type elements.

Rush and Hergenroder (24) have presented values as a result of experimental tests carried on models made of special slow setting gypsum. They measured strains by means of electrical resistance strain gauges applied in the form of rosette. Since only one direction of the rosette is coincident with the co-ordinate system for finite element analysis, comparison is therefore made for moments in that direction only.

The results for 30° skew fig. (22) are in excellent agreement except near the edges. The experimental values are lower, which may be due to the edge restraints. It is interesting to note that for 45° skew fig. (45) the values at many grid points have a discrepancy of less than 1% but at some points the relative error is 10% and higher. High skew of 60° fig. (24) yields appreciable deviation from the experimental values. The difference is about 10% at most of the grid points. It has been pointed out by Robinson (23) that the tendency of the acute corners to lift becomes appreciable for large skew.

4.3 Summary and Conclusions:

Calculus of differences yields deflections and inaccuracy is anticipated on differentiation to calculate moments and stresses. The accuracy achieved is a function of the fineness of network. There are no bounds to the solution so it is difficult to ascertain the deviation from the true value.

The mathematical solution is very laborious and impracticle for skew slabs. The computational difficulties increase in the case of point loads. The development of sensitive measuring devices lead to reliable model testing. The model tests are limited due to the expense incurred and at the same time are extremely laborious. Furthermore, the solution by model tests is dependant on accuracy in measuring the physical properties of the material and simulation of the model. It has been indicated by Rush and Hergenroder (24) that the thickness of the slab and the rate of loading play an important role in determining the moments. The poissons ratio, modulus of elasticity should be measured with extreme caution and the creep phenomena of the material for model may also effect the results.

Since the exact solution is unknown, it is rather difficult to assess the accuracy achieved by any approximate method and hence relative checks can only be made to establish the validity of results.

The high precision plate bending finite element provides a lower bound to strain energy and to deflection in general only. It is interesting to observe that for skewed slabs the deflections obtained experimentally are generally lower relative to the ones given by finite element approximation.

The solution presented by current analysis is in good agreement with those given by other researchers. The moments evaluated on using high precision triangular plate bending element should be more reliable as curvatures being the generalised displacements are obtained directly from the stiffness matrix. The computations are carried out in a digital computer and the program can be modified without difficulty to analyse special cases.

CHAPTER 5

STIFFENED SLABS

5.1 General:

The eccentricity between the mid-plane of the plate and the axis of stiffener induces in-plane deformation under normal transverse loading. This makes it necessary to incorporate plate plane-stress element. Quadratic variation of stress is assumed to be compatible with the plate bending element. A compatible beam stiffener element is developed. The beam element can be either concentric or eccentric to the middle plane of the slab. Two examples similar to the ones analysed by McBean (16) are worked out using the high precision element. The objective was to determine the accuracy of the results.

5.2 Simply Supported Slab with One Stiffener:

Fig. (8) shows a simply supported square plate with one stiffener. In this example Younge's Modules is taken as 1.7×10^7 and poissons ratio as 0.3. A unit uniform load is applied acting downward on the plate. Due to double symmetry only one quarter of the structure is analysed. The example is worked out first by considering the stiffener to be concentric and then it being eccentric.

McBean (16) worked out the same example using finite element approximation. As there was no exact solution available, McBean applied Raleigh-Ritz method to obtain an approximate solution. He also determined the exact solution, but with stiffener properties averaged over the width of the plate.

The comparison is made with the results obtained by McBean (16) and are tabulated in Table 15. The refined element shows much faster convergence and the results are in excellent agreement with the exact solution. Plot of strain energy and deflection with the fineness of grid is shown in fig. (25) and fig. (26). The ratio of central deflection of Plate with concentric stiffener to the Plate with no stiffener is 5.72

and the ratio of the central deflection of the plate with eccentric stiffener to the plate with no stiffener is 19.28.

5.3 Slab-Beam Canopy:

A 12 pannel two-way slab shown in fig. (9) is analysed. The slab is supported at points 3, 12 and 14. The Younge's modulus is taken as 4.5×10^8 psf and poissons ratio as 0.15 for both the slab and the stiffener. Two triangular element represent each pannel.

Fig. (27) shows the deflections for the structure due to a uniform transverse load of 1.0 psf. These values are in excellent agreement with those obtained by McBean (16). However the nodal point 13 is lifted up 9.66% more, relative to the value given by McBean.

The same slab-beam canopy is analysed by applying a load of 100 lbs. at point 1. Again the deflections are in excellent agreement with those given by McBean, except at node point 13 fig. (28). The deflection at this point in this case is 9.45% less than that given by McBean.

5.4 Summary and Conclusions:

The elements used by McBean (16) were the most refined from the ones used by Gustafson (4) and Mehrain (19) to study composite slabs. He has 32 degrees of freedom per element in contrast to 36 generalized displacements for the high precision triangular element. In the study carried out by McBean and Mehrain mid-side nodes had to be added to achieve better stress distributions.

The solution presented by high precision elements is well in agreement with those given by McBean (16). It has been pointed out by McBean it is questionable whether to have very precise elements or achieve the same accuracy by a crude element with a very fine network. The high precision elements have most accurate inter element compatibility relative to other elements and the monotonic convergence of strain energy adds confidence to the analysis.

The ease with which a triangular element can be oriented to consider any geometric configuration so the limitation of the parallelogramic elements to handle orthogonal stiffener system for a skew composite section is overcome.

Recommendations for Further Study:

In the current study emphasis was on checks to the results obtained by high precision elements. The stiffness properties of plate plane-stress, plate bending and a compatible stiffener element are generated to idealize a composite section. Each element can be used individually or in combination with another element or elements. This leads to an unlimited scope for further study.

Influence coefficients can be presented for single span slabs with different aspect ratios and skew angles. The ease with which various boundary conditions can be handled, results can be tabulated for plates with different edge restraints. The program can be very conveniently extended to continuous skew slabs.

It shall be interesting to study the stress distribution for stiffened plates. The study can be extended to preparation of optimization design charts for stiffened Plate structures; by varying the material properties and rigidity of the stiffeners and the plate.

Previous studies were limited to handle longitudunal stiffeners only, in the case of skew stiffened plates. This limitation can be overcome with the use of triangular elements which can be oriented to suit any geometric configuration. The finite element has a constraint that the stiffener element lie along the edge of the plate element. The present work considered torsionally weak stiffeners. There should not be much difficulty to study torsionally stiff sections.

Analysis of stiffened shell structures can be the next extension. This would increase the utility of the computer program immensely.

The study need not be limited to static analysis only.

A Note On Computer Program

The computer program for the analysis of stiffened plate structure is writen in FORTRAN IV and was executed on IBM 360/65 - OS Computer at H - Level. Since a large high speed in-core capacity is available on the operating system so the program was run in-core. If the size of the structure stiffness matrix becomes too large, then auxialiary storage would have to be used which of course can be dealt with no difficulty.

A subroutine 'ELSTIF' generates the stiffness matrix for plate bending and plate in-plane elements. At the present moment the two matrices are uncoupled. The program can be modified for analysis of Shell problems by adding curvature terms to generate the shallow shell element developed by Cowper et al. Economy to computer memory is exercised by minimizing the arrays by destroying the matrices which are no further required during the program execution. The element stiffness matrices are assembled to represent the structure. A special subroutine 'BUILD' stores the entries to the lower band of the structure stiffness matrix. BUILD sorts the entries to the master stiffness matrix and gives the elements the desired location such that only the lower band is stored row by row in a one dimensional array.

A subroutine 'BEAM' generates the beam stiffness matrix. Beam torsion contribution although present but is uncoupled from the beam bending and axial deformation properties. Either subroutines 'SETUP' and 'STUP' or 'SETUP1' and 'STUP1' are executed to add the contributions of the stiffeners to the master matrix; depending on the direction of the stiffener being parallel to the X - axis or Y - axis respectly. The stiffeners can be concentric or eccentric.

After the application of boundary conditions the displacement vector is determined by performing Crout's Factorization algoritherm. Subroutine 'SLVBSR' writen by Wong (30) is used to obtain the solution to the equations.

The Computer output gives the deflection, slopes and bending moments at the nodal points.

BIBLIOGRAPHY

1. Adini, A. and Clough, R., "Analysis of Plate Bending by the Finite Element Method", Report submitted to the Nat. Sc. Found, Grant G7337, 1960.

2. Chu, K., and Krishnamoorthy, G., "Use of Orthotropic Plate Theory in Bridge Design", Journal of the Structural Division, A.S.C.E., Vol. 88, No. ST3, Proc. paper 3157, 1962.

3. Clough, R.W., and Tocher, J.L., "Finite Element Stiffness Matrices for Analysis of Plate Bending", Conference on Matrix Methods in Structural Mechanics Wright-Patterson AFB, Ohio, AFFDL-TR-66-80, 1966.

4. Cowper, G.R., Kosko, E., Lindberg, G.M. and Olson, M.D., "A High Precision Triangular Plate Bending Element", Aeronautical Report LR-514, Dec., 1968.

5. Dawe, D.J. "Parallelogrammic Elements in the Solution of Rhombic Cantilever Plate Problems", Journal of Strain Analysis Vol. 1, No. 3, 1966.

6. Desikan, M.S., "Refined Finite Element Analysis of Skew Stiffened and Unstiffened Plates", M.A. Sc. Thesis, University of Ottawa, September 1970.

7. Gustafson, W.C., "Analysis of Eccentrically Stiffened Skewed Plate Structures", Ph.D. Thesis, Univ. of Illinois, 1966.

8. Hilderbrand, F.B., "Methods of Applied Mathematics", Prentice Hall, New Jersey, 1952.

9. Holand, I. and Bell, K., "Finite Element Methods in Stress Analysis".

10. Iyengar, K., Sunderra, R. and Srinivasan, R., "Clamped Skew Plate Under Uniform Normal Loading", Journal of the Roy. Aero. Soc., Vol. 71, Feb., 1967.

11. Jensen, V.P., "Analysis of Skew Slabs", Univ. of Illinois, Eng. Exp. Sta., Bul. 332.

12. Kennedy, J.B. and Huggins, M.W., "Series Solution of Skewed Stiffened Plates", Journal of Eng. Mech. Division, A.S.C.E. Vol. 90, No. EM1.

13. Kennedy, J.B. and Tamberg, K.G., "Problems of Skew in Concrete Bridge Design", Dept. of Highways, Ont., D.H.O. Report No. RR 144, March, 1969.

14. Lindberg, G.M., Cowper, G.R., and Hrudey, T.M., "An analysis of a Circular Cylinder with a Rectangular cut-out-Lockheed Sample Problem No. 2" NRC Internal Report No. ST-137. National Aeronautical Establishment Ottawa, Canada.

15. Lindberg, G.M., and Oslson, M.D., "Free vibrations and Random Response of an Integrally-stiffened Panel" NRC Aeronautical Report LR-544, NRC, Ottawa.

16. McBean, R.P., "Analysis of Stiffened Plates by the Finite Element Method", P.hd. Dissertation, Stanford University, June 1968.

17. Mikhaiel, S.N., "An investigation on the Statics of Twist" M. Sc. thesis, Cairo university, U.A.R.

18. Mansfield, E.H., "The bending and Stretching of Plates" A Pergamon Press Book, 1964.

19. Mehrain, M. "Finite Element Analysis of Skew Composite Plates", Ph.D. Thesis, University of California, Berkeley, 1967.

20. Morley, L.S.D., "Skew Plates and Structures", A Pergamon Press Book, 1963.

21. Newmark, N.M., and Seiss, C.P., "Moments in I-Beam Bridges", Univ. of Illinois, Eng. Exp. Sta., Bul. 336, 1942.

22. Prezemieniecki, J.S., "Theory of Matrix Structural Analysis" Published by McGraw-Hill, 1968.

23. Robinson, K.E., "The behaviour of Simply supported Skew Bridge Slabs under concentrated loads", C & CA Research Report No. 8 November, 1959.

24. Rush-Hergenroder, "Einflu Bfelder der Momente schiefw - inklinger Platten", Werner-Verlag - Dusseldorf, 3 Auflage, 1969.

25. Sawko, F.S. and Cope, R.J., "The Analysis of Skew Bridge Decks - a new finite element approach", Structural Engineer, June, 1969.

26. Siess, C.P. and Viest, J.M., "Tests of continuous Right I-Beam Bridges". Univ. of Illinois, Eng. Exp. Sta. Bul. No. 416, Oct, 1953.

27. Timoshenko, S. and Woinowski Krieger, S., "Theory of Plates and Shells", McGraw-Hill, N.Y., 1959.

28. Tong, P. and Pian, T.H.H., "The Convergence of the Finite Element Method in Solving Linear Elastic Proglems", Int. J. Solids and Structures, Vol. 3, 1967.

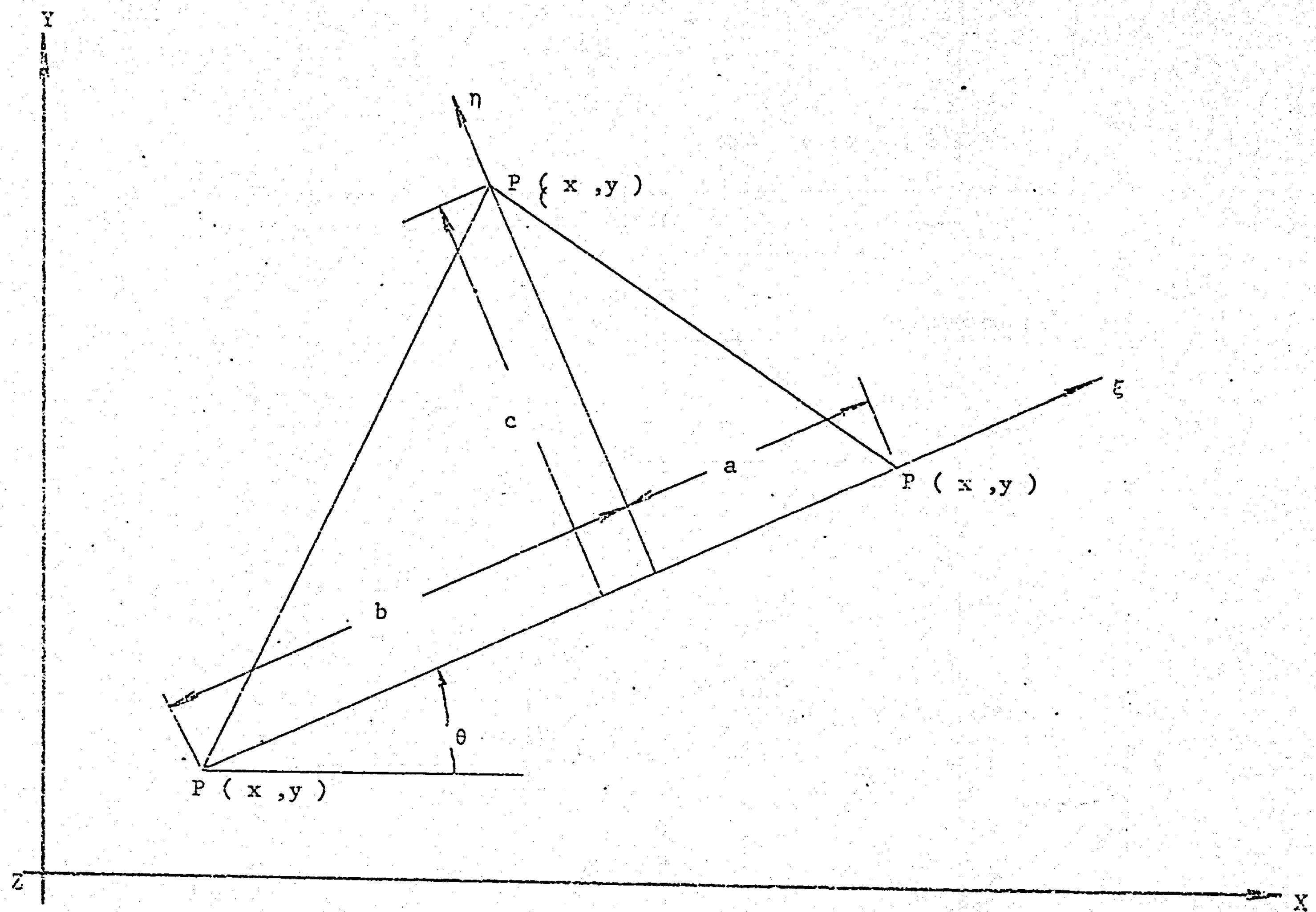
29. Vitols, V., Clifton, R.J., and Au, T., "Analysis of Composite Beam Bridges by orthotropic Plate Theory", Journal of the Structural Division, A.S.C.E. Vol. ST4, Aug. 1963.

30. Wong, K., "Solution of Banded, Symmetric System of Equations AX B", NRC TSS User,s Manual, NRC Ottawa Computations Centre, July 1971.

31 Yeginobali, A., "Continuous Skewed Slabs", Engineering Experiment Station Bulletin, The Ohio State Univ., Columbus, Nov. 1959.

32. Yeginobali, A., "Analysis of Continuous Skewed Slab Bridge Decks", Eng. Exp. Sta. Report No. EE S 170-2, The Ohio State University, Columbus, Ohio, Aug. 1962.

33. Zienkiewicz, O.C. "The Finite Element Method in Structural and Continuum Mechanics", Published by McGraw-Hill, 1963.

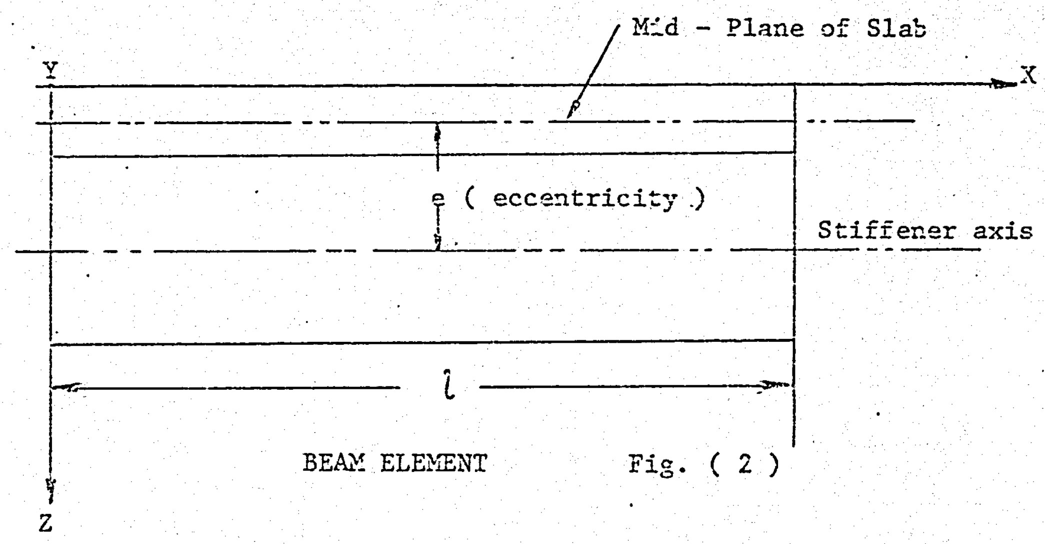


CO - ORDINATE SYSTEM

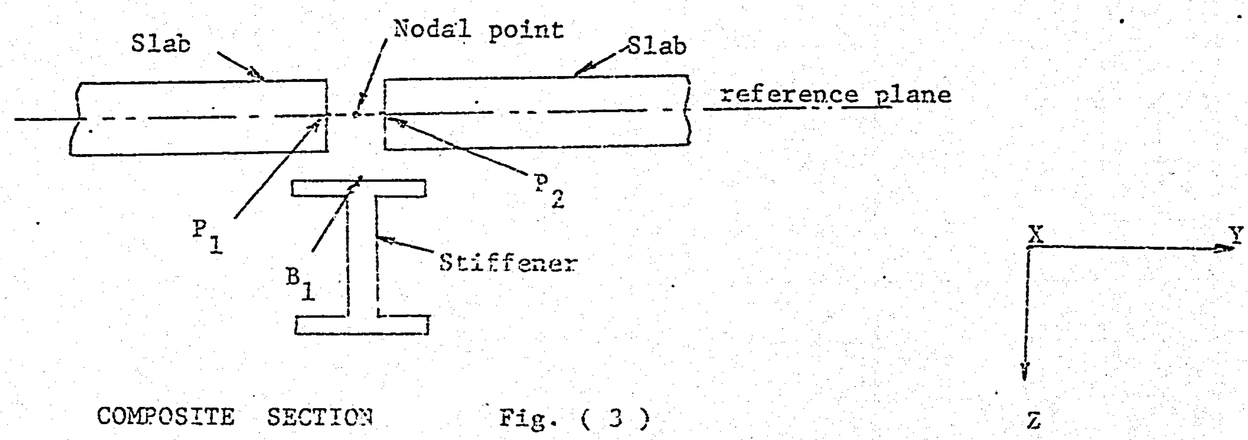
X - Y GLOBAL SYSTEM

ξ - η LOCAL SYSTEM

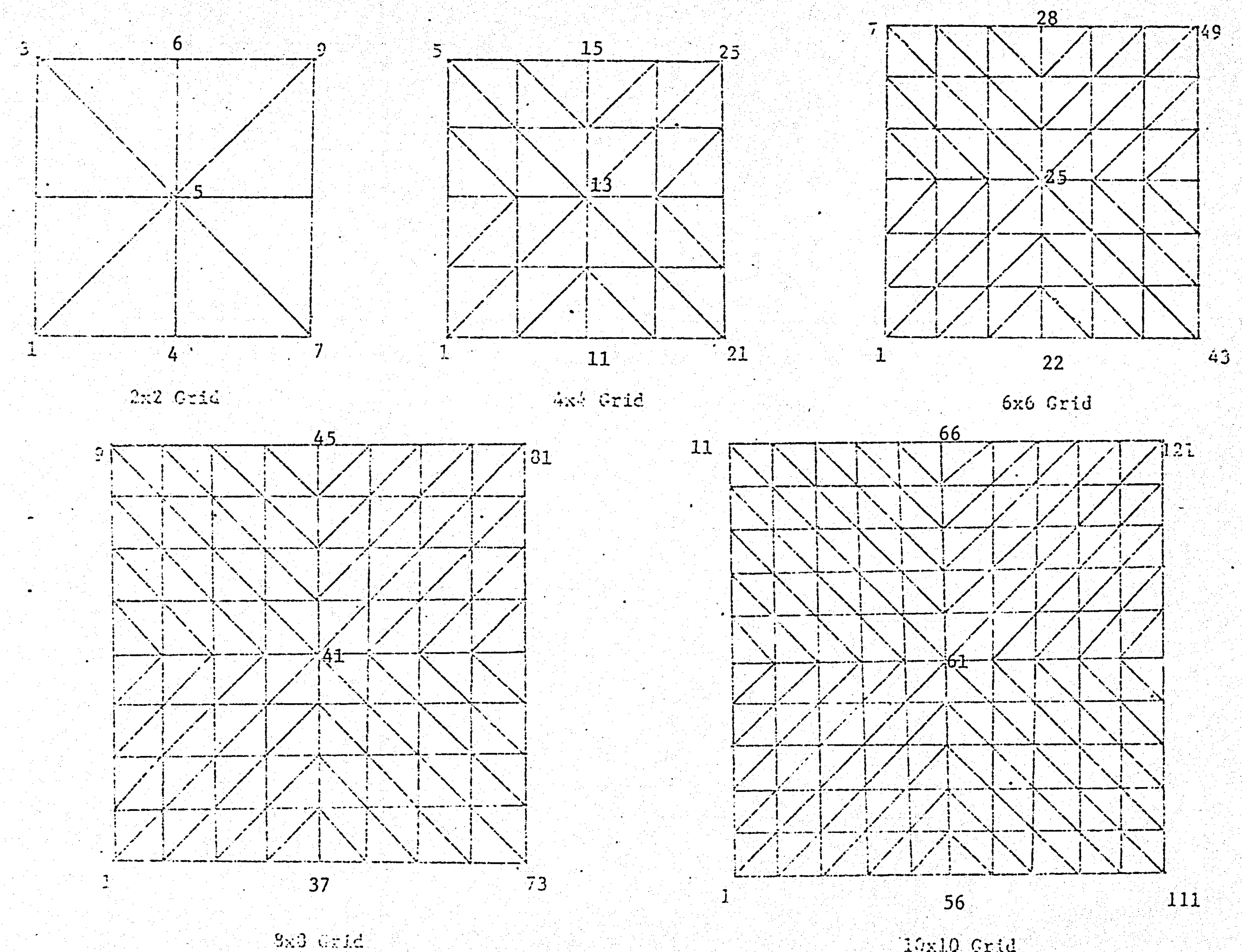
Fig. (1)



BEAM ELEMENT Fig. (2)



COMPOSITE SECTION Fig. (3)



FINITE ELEMENT NETWORK

Fig. (4)

PLATE TYPE I - (ASPECT RATIO = $\frac{l_x}{l_\phi}$ 1.0)

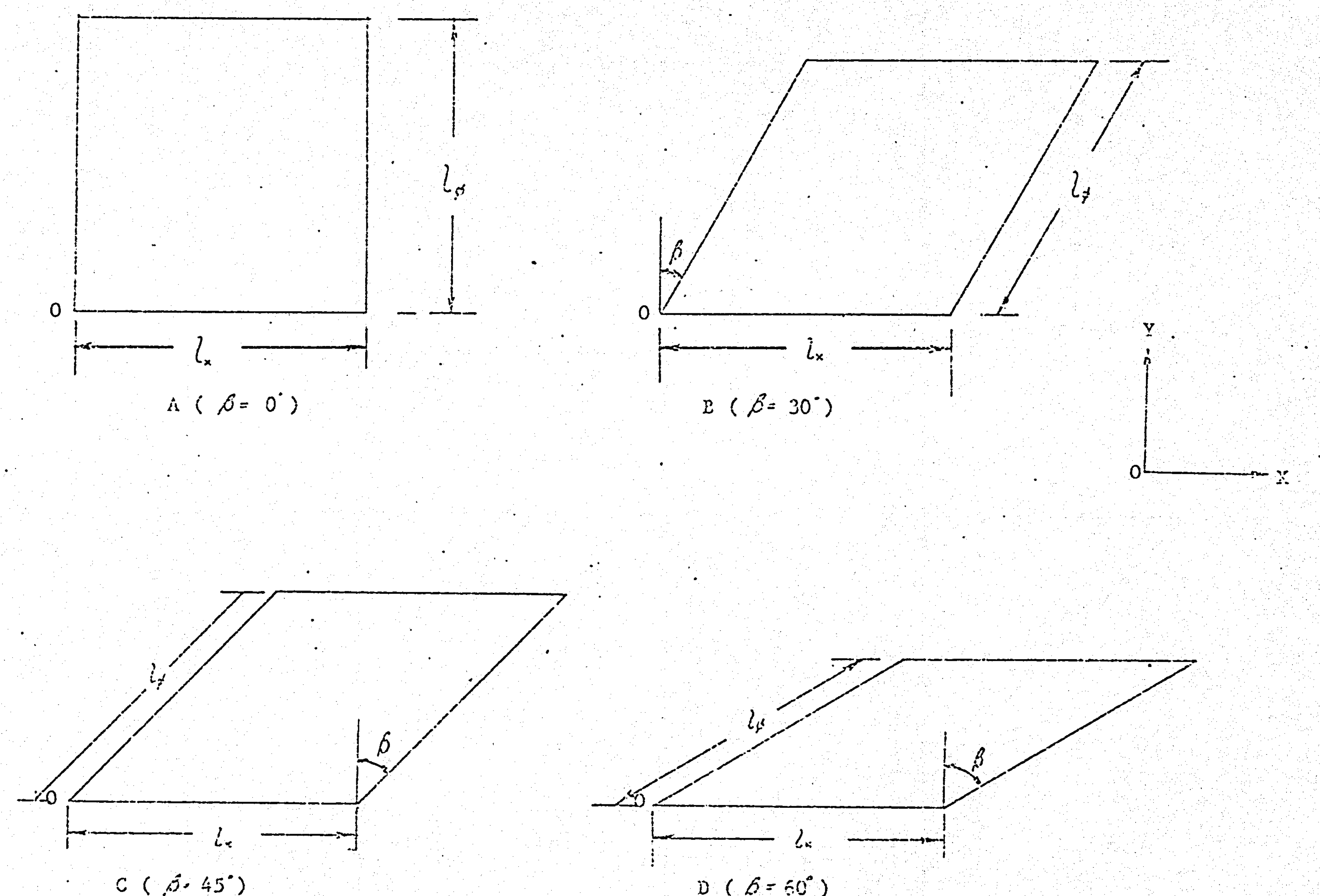


Fig. (5)

PLATE TYPE II - (ASPECT RATIO = $\frac{l_x}{l_y}$ = 1.0)

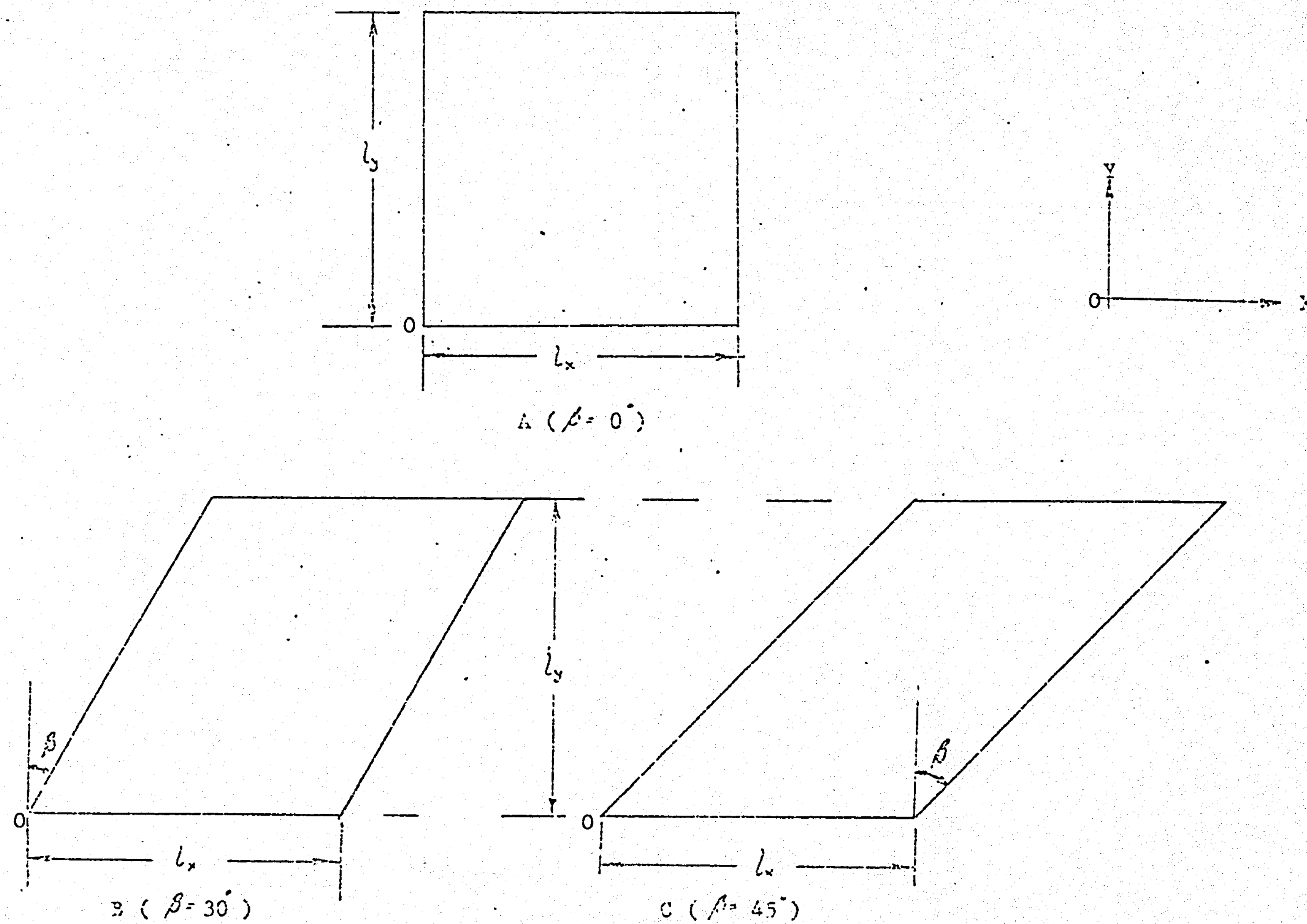
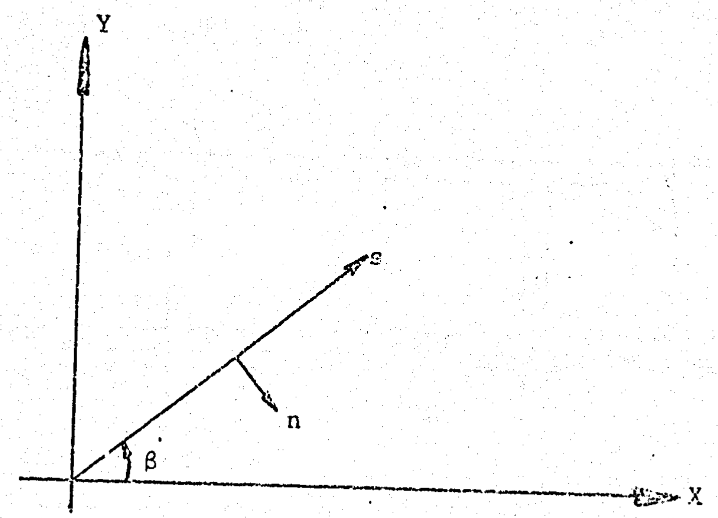
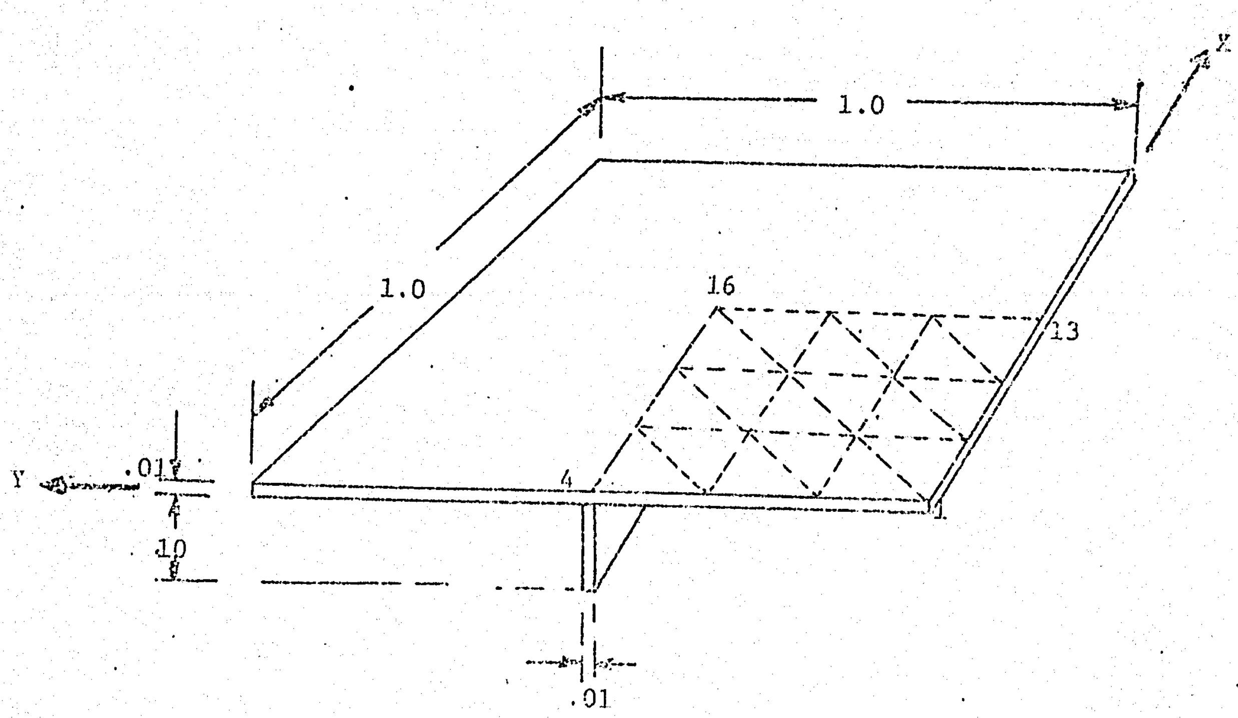


Fig. (6)

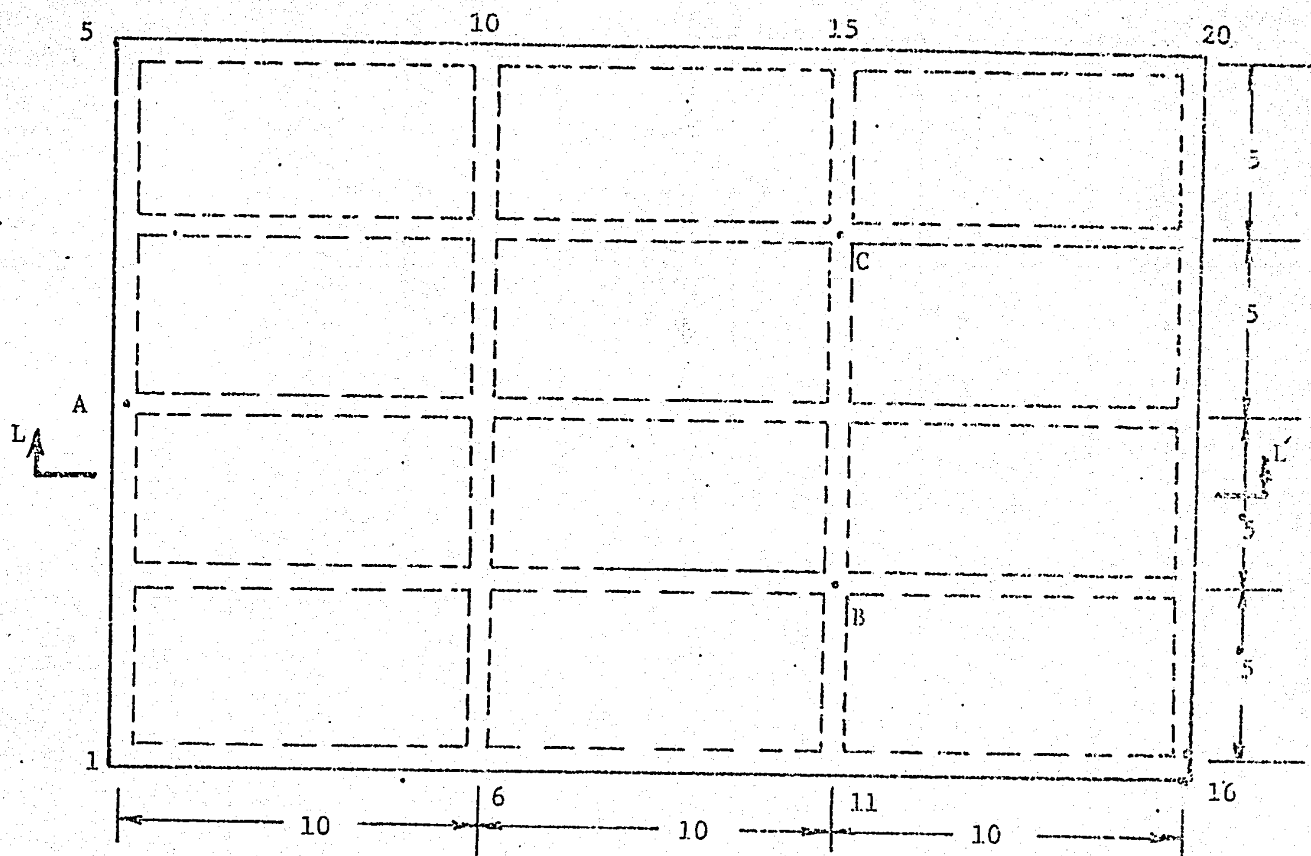


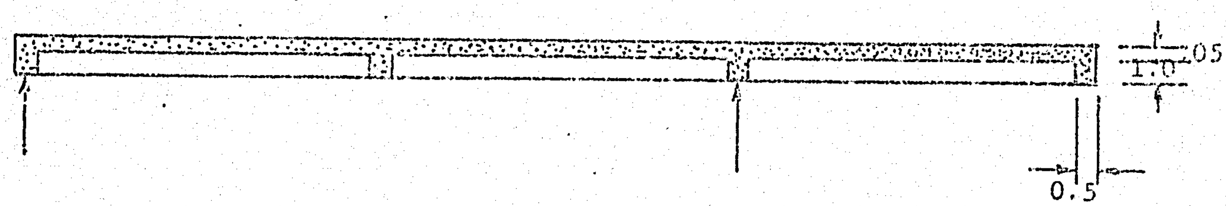
OBLIQUE BOUNDARY

Fig. (7)



SIMPLY SUPPORTED PLATE WITH ONE STIFFENER Fig. (8)





SECTION L - L'

SLAB BEAM CANOPY Fig. (9)

TABLE 1

MOMENT COEFFICIENTS FOR UNIFORMLY LOADED SQUARE SLAB, TWO SIDES SIMPLY-SUPPORTED AND THE OTHER TWO FREE

PLATE TYPE - II - A(p. 46)(8x8 Grid) POISSONS RATIO = 0.15

NODAL POINT	LONG. MOMENT $M_x = m_1 ql^2 \times 10^{-3}$		TRANS. MOMENT $M_y = m_2 ql^2 \times 10^{-3}$		TOR. MOMENT $M_{xy} = -m_3 ql^2 \times 10^{-3}$	
	FEM	Ref (32)	FEM	Ref (32)	FEM	Ref (32)
1	0.2324	0	0.03486	0	12.76	127.9
2	0.05142	0	0.00771	0	6.874	7.02
3	0.002986	0	0.00442	0	3.57	3.68
4	0.01926	0	0.00289	0	1.5211	1.57
5	0.00697	0	0.00091	0	0	0
10	56.504	56.30	-0.0208	0	11.3	11.40
11	55.05	55.05	3.5206	3.45	6.32	6.43
12	54.429	54.42	5.323	5.27	3.311	3.39
13	54.142	54.14	6.22	6.17	1.41	1.45
14	54.052	54.05	6.495	6.97	0	0
19	96.767	95.65	-0.0132	0	8.23	27.5
20	94.4896	94.44	5.589	5.51	4.771	4.93
21	93.3115	93.3	8.7779	8.67	2.536	2.59
22	92.765	92.76	10.4017	10.3	1.02	1.111
23	92.60	92.6	10.906	10.81	0	0
28	120.905	120.79	-0.0086	0	4.307	27.86
29	118.156	118.1	6.693	6.52	2.547	2.57
30	116.651	116.63	10.697	1.059	1.37	1.39
31	115.92	113.83	12.795	-1.28	0.589	0.6
32	115.7	115.71	13.446	13.32	0	0
37	128.694	128.84	-0.00231	0	0	0
38	126.047	125.99	7.047	6.97	0	0
39	124.43	124.41	11.319	11.20	0	0
40	123.641	121.37	13.574	-1.65	0	0
41	123.406	123.41	14.27	14.14	0	0

TABLE 2

MOMENT COEFFICIENTS FOR A UNIFORMLY LOADED 30° SLAB WITH TWO SIDES SIMPLY SUPPORTED AND OTHER TWO FREE.

PLATE TYPE - II - B (fig. 6 p. 46)

POISSONS RATIO=0.15

NODAL POINT	LONG MOMENT $M_x = m_1 q l^2 \times 10^{-3}$		TRANS. MOMENT $M_y = m_2 q l^2 \times 10^{-3}$		TOR. MOMENT $M_{xy} = -m_3 q l^2 \times 10^{-3}$	
	FEM	Ref (32)	FEM	Ref (32)	FEM	Ref (32)
1	-1.42	0	-0.374	0	-0.465	-4.23
2	0.846	1.43	-9.76	-1.43	-0.531	-0.82
3	1.44	1.77	-1.48	-1.77	-0.862	-1.03
4	1.486	1.71	-1.48	-1.71	-0.856	-0.99
5	0.878	1.05	-0.86	-1.05	-0.494	-0.91
6	-0.5	-0.2	0.543	0.3	0.317	0.17
7	-3.3	-2.91	3.372	2.91	1.95	1.66
8	-9.36	-8.94	11.534	8.93	6.93	5.16
9	-29.72	0	70.89	0	46.69	-546.05
10	18.23	18.48	0.0235	0	3.35	3.13
11	25.901	26.2	0.659	6.45	3.263	7.88
12	30.03	30.27	9.419	9.305	11.157	10.9
13	31.88	32.01	11.16	11.08	13.135	12.99
14	32.106	32.22	12.62	12.57	14.826	14.79
15	31.29	31.47	14.14	14.07	16.901	17.01
16	29.295	30.01	16.12	15.79	20.79	21.46
17	26.95	29.15	12.869	16.86	32.04	35.11
18	49.79	47.37	-7.31	0	42.62	43.95
19	44.23	44.28	0.0262	0	10.007	99.8
20	50.95	51.11	10.57	10.52	16.5	16.22
21	54.46	54.6	16.093	16.08	21.008	20.8
22	55.69	55.79	19.39	19.42	24.09	24.01
23	55.683	55.83	21.457	21.48	26.43	26.51
24	55.59	55.77	22.317	22.31	29.11	29.42
25	57.081	57.14	21.03	21.02	33.76	34.44
26	65.09	64.47	14.23	13.75	40.31	40.38
27	80.28	80.59	-0.72	0	41.44	41.13
28	67.69	67.69	0.0305	0	17.89	17.9
29	71.102	71.13	12.46	12.47	24.4	24.3
30	71.866	71.88	19.735	19.81	29.15	29.05
31	71.22	71.28	24.14	24.23	32.05	32.08
32	70.63	70.76	26.25	26.34	33.88	34.08
33	71.33	71.42	25.77	25.93	35.677	36.0
34	74.711	74.69	21.59	21.91	37.58	37.84
35	82.111	81.73	12.93	13.08	37.577	37.54
36	90.027	89.75	0.052	0	34.24	34.2
37	84.15	84.05	0	0	26.2	26.19
38	82.63	83.53	12.98	13.1	31.74	31.66
39	79.47	79.38	21.31	21.49	35.14	35.15
40	76.77	76.79	26.204	26.32	36.31	36.49
41	75.8	75.9	27.76	27.86	36.48	36.72

TABLE 5

MOMENT COEFFICIENTS FOR UNIFORMLY LOADED 45° SLAB , TWO SIDES SIMPLY SUPPORTED AND THE OTHER TWO FREE.

PLATE TYPE - 11 - C (fig. 6) (8x8 Grid) POISSONS RATIO = 0.15

NODAL POINT	LONG MOMENT $M_x = m_1 q l^2 \times 10^{-3}$		TRANS. MOMENT $M_y = m_2 q l^2 \times 10^{-3}$		TOR. MOMENT $M_{xy} = -m_3 q l^2 \times 10^{-3}$	
	FEM	Ref (32)	FEM	Ref (32)	FEM	Ref (32)
1	-1.06	0	-0.154	0	-0.449	-3.43
2	2.538	2.72	-2.704	-2.72	-0.061	0
3	5.195	5.2	-5.22	-5.2	-0.0108	0
4	5.89	5.8	-5.878	-5.8	0.006	0
5	4.516	4.49	-4.46	-4.49	0.0212	0
6	2.113	2.21	-1.94	-2.21	0.0605	0
7	-0.731	-0.62	1.7007	0.62	0.358	0
8	-6.78	-6.19	12.908	6.19	2.27	0
9	-46.09	0	82.75	0	13.548	-231.06
10	7.584	7.89	-0.118	0	1.466	1.35
11	15.63	15.71	3.914	3.999	5.999	5.58
12	19.916	19.79	5.99	6.12	9.371	8.75
13	20.43	20.32	8.435	8.58	11.514	10.73
14	18.59	18.68	11.424	11.49	12.37	11.57
15	16.245	16.3	14.209	14.32	12.66	11.85
16	13.06	13.0	17.07	17.52	13.919	12.85
17	2.392	4.25	22.298	25.27	22.48	21.27
18	25.636	21.15	-16.79	0	42.34	41.53
19	21.95	22.14	-0.125	0	6.2	6.24
20	29.99	29.9	8.723	8.79	13.402	13.1
21	32.55	32.34	14.489	14.65	18.67	18.01
22	31.04	31.02	19.464	19.62	21.326	20.52
23	28.29	28.53	23.44	23.55	21.778	21.08
24	26.084	26.21	26.09	26.21	22.235	21.53
25	23.406	23.36	27.645	28.35	25.605	25.17
26	26.379	27.68	21.55	20.71	36.29	36.83
27	51.83	49.49	0.994	0	39.379	42.72
28	37.28	37.27	12.066	0	21.55	13.15
29	41.98	41.67	-0.096	12.22	12.97	21.31
30	40.49	40.2	20.86	21.03	27.09	26.32
31	36.33	36.5	27.63	27.71	28.316	27.54
32	33.66	33.91	31.183	31.42	27.689	27.07
33	32.55	32.65	32.14	32.59	28.75	28.07
34	33.76	34.3	29.27	29.26	33.558	32.64
35	45.006	44.29	16.52	17.06	36.94	36.06
36	55.97	55.15	0.0568	0	30.296	30.42
37	49.83	49.46	-0.069	0	21.306	21.43
38	48.235	47.69	14.277	14.72	29.817	29.45
39	41.215	41.14	25.993	25.95	33.123	32.14
40	36.315	36.54	32.6	32.55	[illegible]	30.36
41	35.39	35.51	34.127	34.27	[illegible]	29.11

TABLE 4

Comparison of Deflections and Bending Moments of Uniformly Loaded Square Plate With Two Sides Simply Supported and the Other Two Free

Poissons Ratio = 0.3 Reference Point Location $(\frac{l_x}{2}, 0)$ Fig. (5)

GRID	$\frac{\text{Deflection} \times D}{ql^4}$	M_x/ql^2	M_y/ql^2	M_{xy}/ql^2
2 X 2	0.0150231	0.13175388	-0.00077344	0.0
4 X 4	0.01501156	0.13118169	-0.00006966	0.0
6 X 6	0.0150112	0.13111324	-0.00001303	0.0
8 X 8	0.0150112	0.13109861	-0.00000512	0.0
Gustafson (7) 10 X 10	0.01487	0.1311	0.0004	0.0006
Finite Difference	0.1511	0.1309	0	0
'Exact' (27)	0.01509	0.1318	0.00	0.00

Central Moments, Central Deflection and Strain Energy Convergence For Plate With

Two Sides Simply Supported and Two Sides Free

Plate Type I - A,B,C,D Poissons Ratio = 0.20

(fig. 5 p. 45)

	GRID	Deflection x D / $q \times l_x^4$	Moment $M_x / q \times l_x^2$	Moment $M_y / q \times l_x^2$	Strain Energy x D / $q^2 l^6$
0°	2 x 2	0.01294521	0.12107217	0.01879438	0.003150113
	4 x 4	0.01294675	0.12295347	0.01884053	0.003979227
	6 x 6	0.01294672	0.12302238	0.01871109	0.004248158
	8 x 8	0.01294673	0.12304248	0.01869527	0.004248161
30°	2 x 2	0.00768397	0.07566458	0.02240054	0.001698100
	4 x 4	0.00772381	0.07825446	0.02784774	0.002120368
	6 x 6	0.00772919	0.07803673	0.02746717	0.002254469
	8 x 8	0.00773123	0.07805010	0.02745986	0.002255665
45°	2 x 2	0.00367495	0.03732175	0.02823257	0.0007150769
	4 x 4	0.00375332	0.03940718	0.03016070	0.0009067494
	6 x 6	0.00376962	0.03940372	0.02950385	0.0009664605
	8 x 8	0.00377640	0.03942908	0.02934750	0.0009693078
60°	2 x 2	0.00100198	0.00751091	0.03002604	0.000168521
	4 x 4	0.00108589	0.00724786	0.02599722	0.0002374395
	6 x 6	0.00111003	0.00782027	0.02356369	0.0002585791
	8 x 8	0.00112207	0.00748845	0.02487722	0.0002620027

SIMPLY SUPPORTED SQUARE PLATE UNDER CENTRAL POINT LOAD P_t

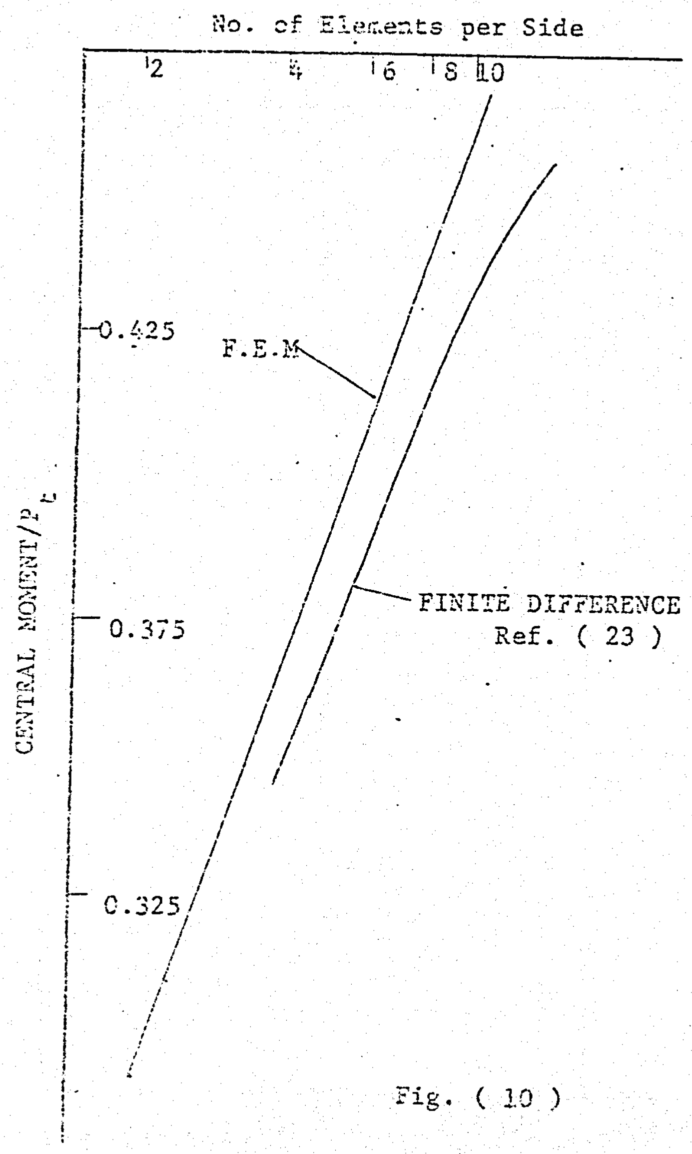


Fig. (10)

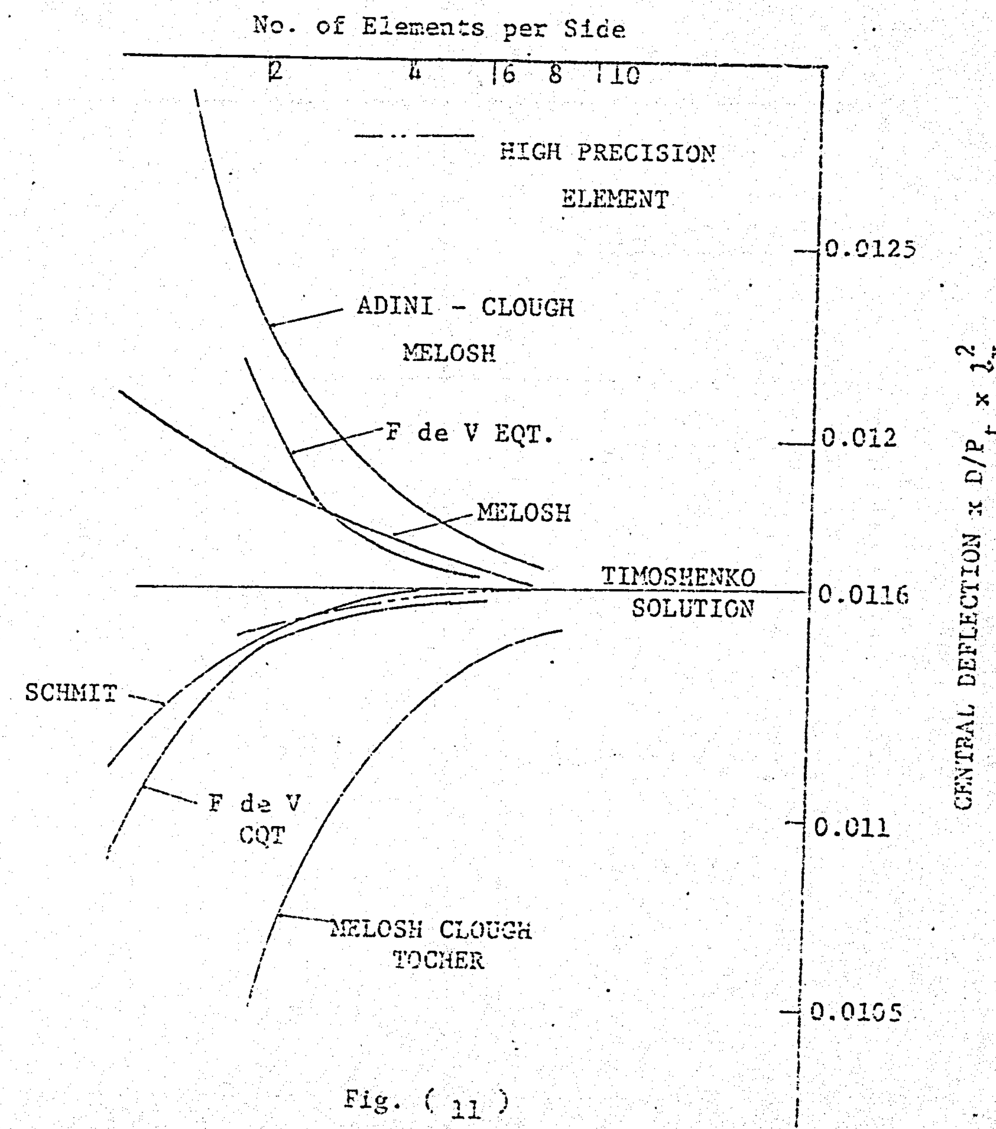


Fig. (11)

TABLE 6

Central Deflection and Moments Of Simply Supported Square Plate ν:0.3
Under Central Load P_t

GRID	$\frac{\text{DEFLECTION X D}}{P_t \times l^2}$	MOMENT (Mx)/P_t	MOMENT (My)/P_t	$\frac{\text{STRAIN ENERGY x D}}{P_t^2 \times l^4}$
2 x 2	0.01149279	0.29403802	0.28403802	0.005746395
4 x 4	0.01157422	0.36596004	0.36596004	0.005787112
6 x 6	0.01158896	0.40765662	0.40765662	0.00579448
8 x 8	0.01159419	0.43761156	0.43761156	0.00579095
10 x 10	0.01159659	0.46069435	0.46069435	

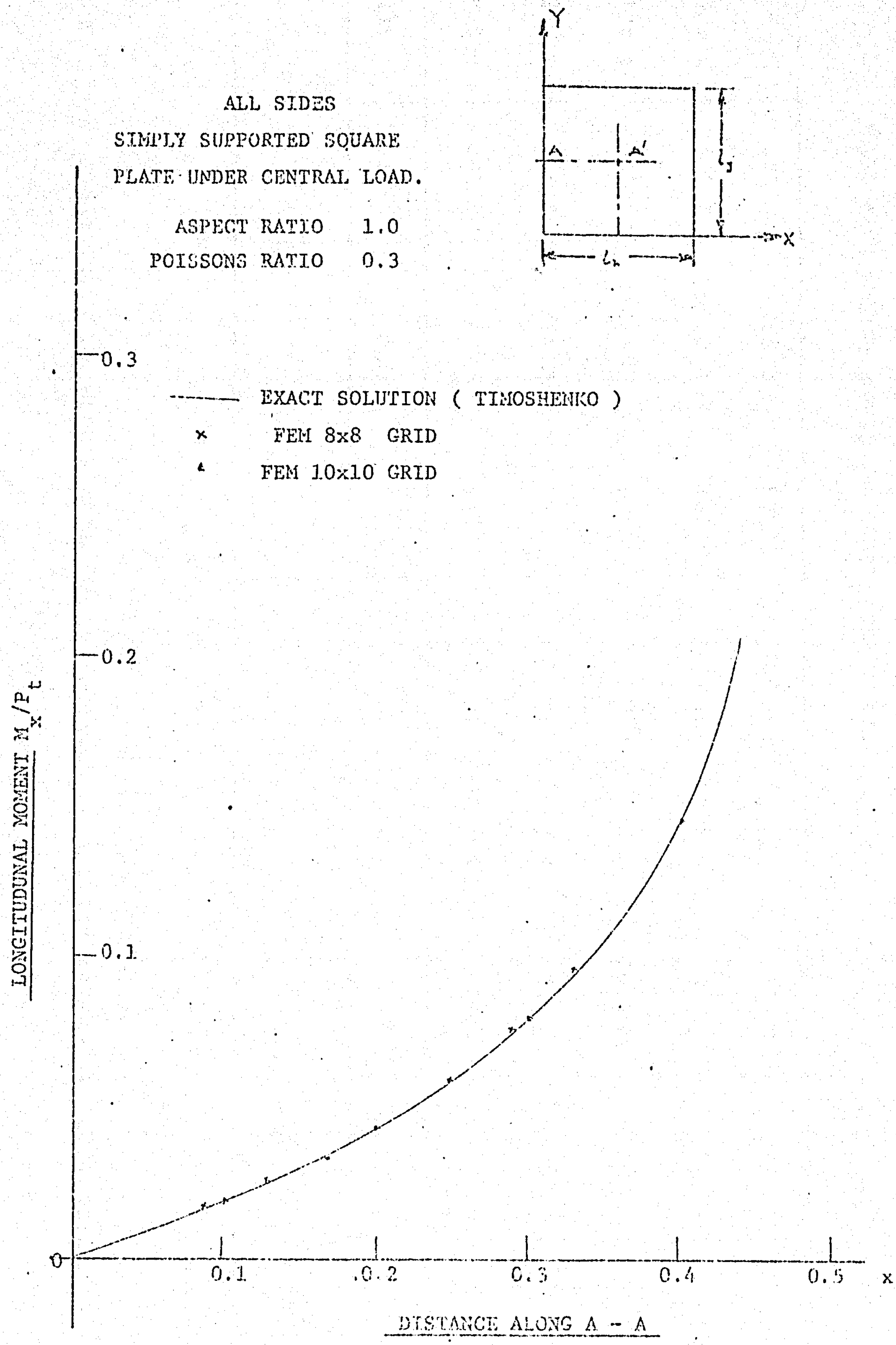

ALL SIDES
SIMPLY SUPPORTED SQUARE
PLATE UNDER CENTRAL LOAD.
ASPECT RATIO 1.0
POISSONS RATIO 0.3
Y
A
A'
X
EXACT SOLUTION (TIMOSHENKO)
FEM 8x8 GRID
FEM 10x10 GRID
LONGITUDUNAL MOMENT M_x/P_t
0.3
0.2
0.1
0
0.1
0.2
0.3
0.4
0.5
x
DISTANCE ALONG A - A

Fig. (12)

TABLE 7

Comparison of Central Deflection Coefficient d_1 Under A Central Point Load P_t For Plate Type I (8 x 8 Grid)

($d_1 \times 10^{-2}$) in/in

NODAL POINT	$\beta = 0^\circ$		$\beta = 30^\circ$		$\beta = 45^\circ$		$\beta = 60^\circ$	
	EXP *	FEM	EXP *	FEM	EXP *	FEM	EXP *	FEM
37	0.540	0.514	0.459	0.4803	0.397	0.429	0.322	0.3548
38	0.547	0.5266	0.473	0.4878	0.403	0.438	0.330	0.3653
39	0.592	0.5552	0.510	0.518	0.442	0.4725	0.376	0.401
40	0.634	0.5922	0.544	0.5658	0.491	0.5310	0.466	0.475
41	0.662	0.6217	0.587	0.601	0.533	0.596	0.549	0.551

Deflection = $d_1 \times P_t$ * Experimental Values As Given By Robinson (23)

TABLE 8

Comparison of Principal Bending Moments At Midspan For Plate Type I (8 x 8 Grid) Under Central Load

Maximum Principal Moment $/P_t$						Minimum Principal Moment $/P_t$						Direction of Principal Moment					
$\beta = 0°$		$\beta = 20°$		$\beta = 45°$		$\beta = 0°$		$\beta = 30°$		$\beta = 45°$		$\beta = 0°$		$\beta = 30°$		$\beta = 45°$	
EXP	FEM	EXP	FEM	EXP	FEM	EXP	FEM	EXP	FEM	EXP	FEM	EXP	FEM	EXP	FEM	EXP	FEM
0.158	0.17805	0.106	0.1255	0.049	0.07109	0	0.000016	0	-0.00111	0	-0.01407	0	0	0	16.49°	0	23.80°
0.172	0.18615	0.142	0.1500	0.091	0.10575	0.015	0.000363	0.005	0.000847	-0.004	-0.0049	0	0	21°	21.17°	32.5°	31.25°
0.195	0.21414	0.172	0.1872	0.142	0.15149	0.039	0.03618	0.027	0.03633	0.015	0.014389	0	0	24.5°	24.62°	37°	36.64°
0.259	0.2786	0.239	0.2576	0.214	0.2274	0.108	0.09834	0.089	0.09768	0.071	0.06816	0	0	27°	27.95°	40.5°	41.61°
0.382	0.5339	0.373	0.5110	0.354	0.4827	0.278	0.4106	0.284	0.3884	0.254	0.36012	0	0	28°	29.82°	46.5°	44.39°

Experimental Values As Given By Robinson (23)

TABLE 9

Comparison of Deflection and Moments Along Midspan Under A Central Load P_t of Plate With Two Sides Simply Supported and the Other Two Free

Plate Type I - C (8x8 Grid) Poissons Ratio = 0.30

NODAL POINT	Deflection x $\frac{D}{P_t \ell_x^2}$		Longitudinal Moment/P_t		Transverse Moment/P_t		Twisting Moment/P_t	
	Finite Diff.	FEM	Finite Diff.	FEM	Finite Diff.	FEM	Finite Diff.	FEM
37	0.008337	0.00801867	0.060	0.05721	0	-0.000191	0.032	0.03145
38	0.00851	0.00818631	0.080	0.7595	0.024	0.02489288	0.050	0.04908
39	0.0091875	0.0088306	0.106	0.10266	0.061	0.06322028	0.069	0.06565
40	0.0104	0.0099106	0.164	0.1571	0.131	0.13839	0.083	0.07907
41	0.0117	0.107977	0.298	0.422	0.329	0.42012	0.054	0.06128

Finite Difference Values As Given By Robinson (23)

TABLE 10

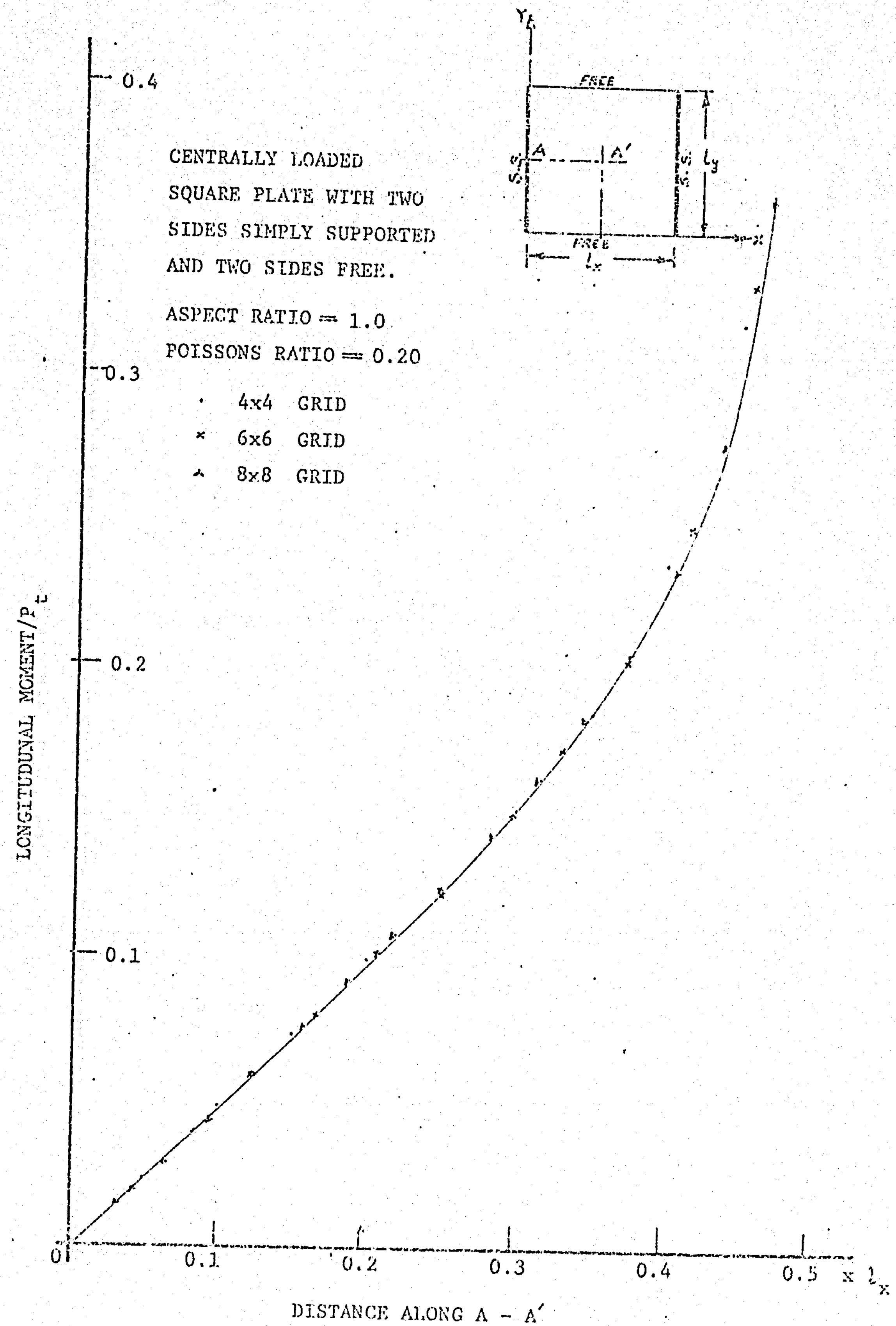
Comparison of Deflection Longitudinal and Transverse Moment Coefficients For a Centrally Loaded Slab With Two Sides Simply Supported and the Other Two Free

Plate Type I - A (8x8 Grid) Poissons Ratio = 0.3

Nodal Point	$\frac{\text{Deflection} \times D}{Pt \times 1x^2} \times 10^3 = d_1$		$\frac{\text{Moment (Mx)}}{Pt} \times 10^3 = m_1$		$\frac{\text{Moment (My)}}{Pt} \times 10^3 = m_2$	
	FEM	REF (23)	FEM	REF (23)	FEM	REF (23)
1	0	0	-0.127	0	-0.0383	0
2	0	0	-0.196	0	-0.059	0
3	0	0	0.116	0	0.0349	0
4	0	0	0.3606	0	0.108	0
5	0	0	0.137	0	0.041	0
10	7.367	7.4	63.025	62.0	-0.0759	0
11	7.407	7.45	62.696	63.0	5.466	6.0
12	7.667	7.725	63.03	63.0	16.043	15.0
13	7.973	8.075	62.29	62.0	28.92	29.0
14	8.109	8.25	61.03	62.0	35.55	39.0
19	13.661	13.725	120.06	118.0	0.0185	0
20	13.76	13.85	121.55	119.0	8.98	8.0
21	14.334	14.475	126.77	124.0	30.06	28.0
22	15.02	15.225	128.62	128.0	60.53	59.0
23	15.48	15.65	127.17	126.0	79.9	86.0
28	17.913	18.025	162.11	161.0	0.0524	0
29	18.11	18.275	167.58	167.0	9.49	9.0
30	18.987	19.225	185.09	183.0	36.30	37.0
31	20.158	20.475	204.3	205.0	87.03	87.0
32	20.793	21.325	213.4	216.0	152.73	161.0
37	19.419	19.55	178.05	178.0	0.0157	0

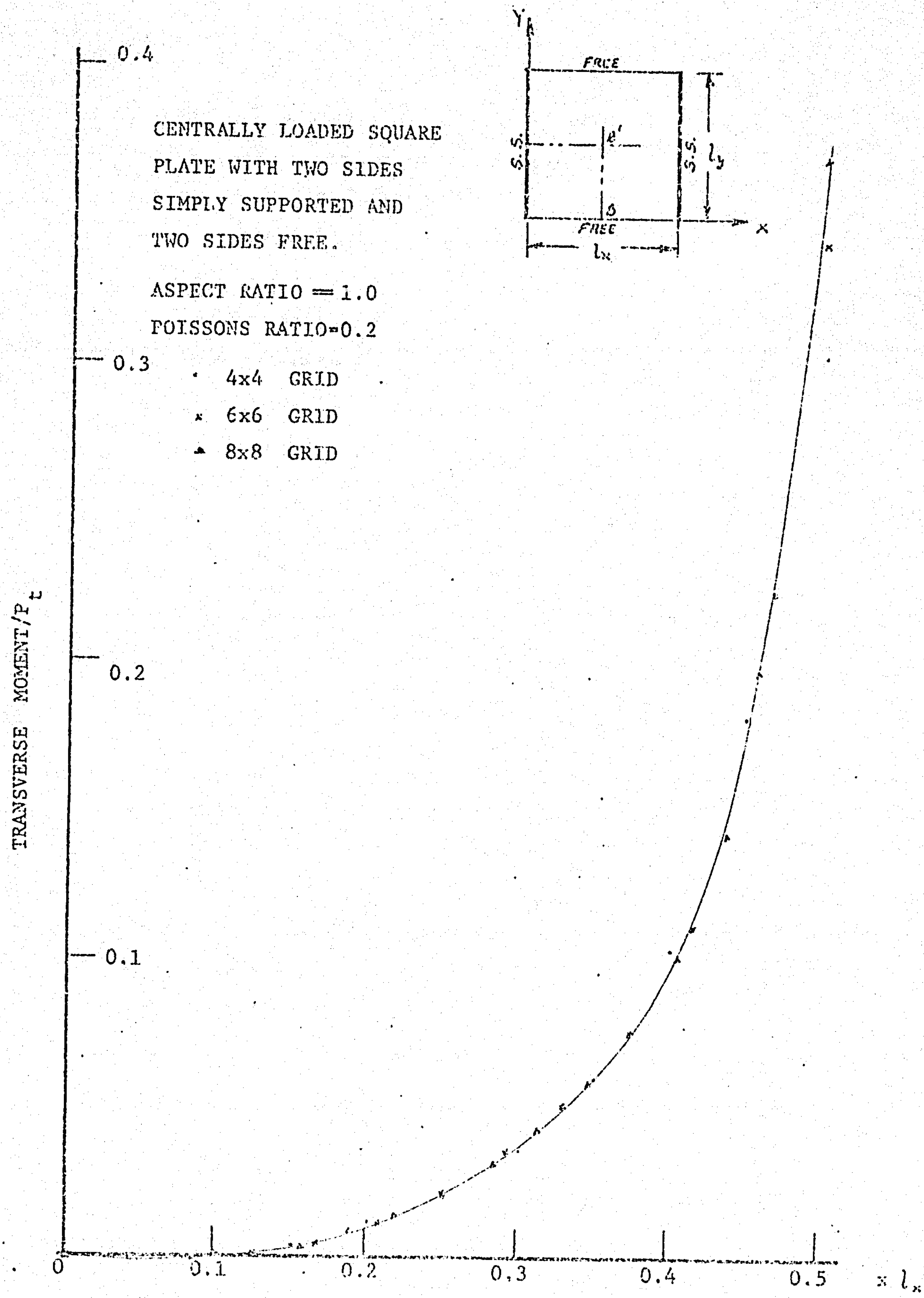
TABLE 10 CONT'D

odal oint	$\frac{\text{Deflection} \times D}{Pt \times 1x^2} \times 10^3 = d_1$		$\frac{\text{Moment } (Mx)}{Pt} \times 10^3 = m_1$		$\frac{\text{Moment } (My)}{Pt} \times 10^3 = m_2$	
	FEM	REF (23)	FEM	REF (23)	FEM	REF (23)
38	19.657	19.85	186.15	188.0	8.63	7.0
39	20.685	21.025	214.1	220.0	36.18	36.0
40	22.16	22.7	278.6	287.0	98.33	101.0
41	23.2	24.1	533.9	413.0	410.64	289.0



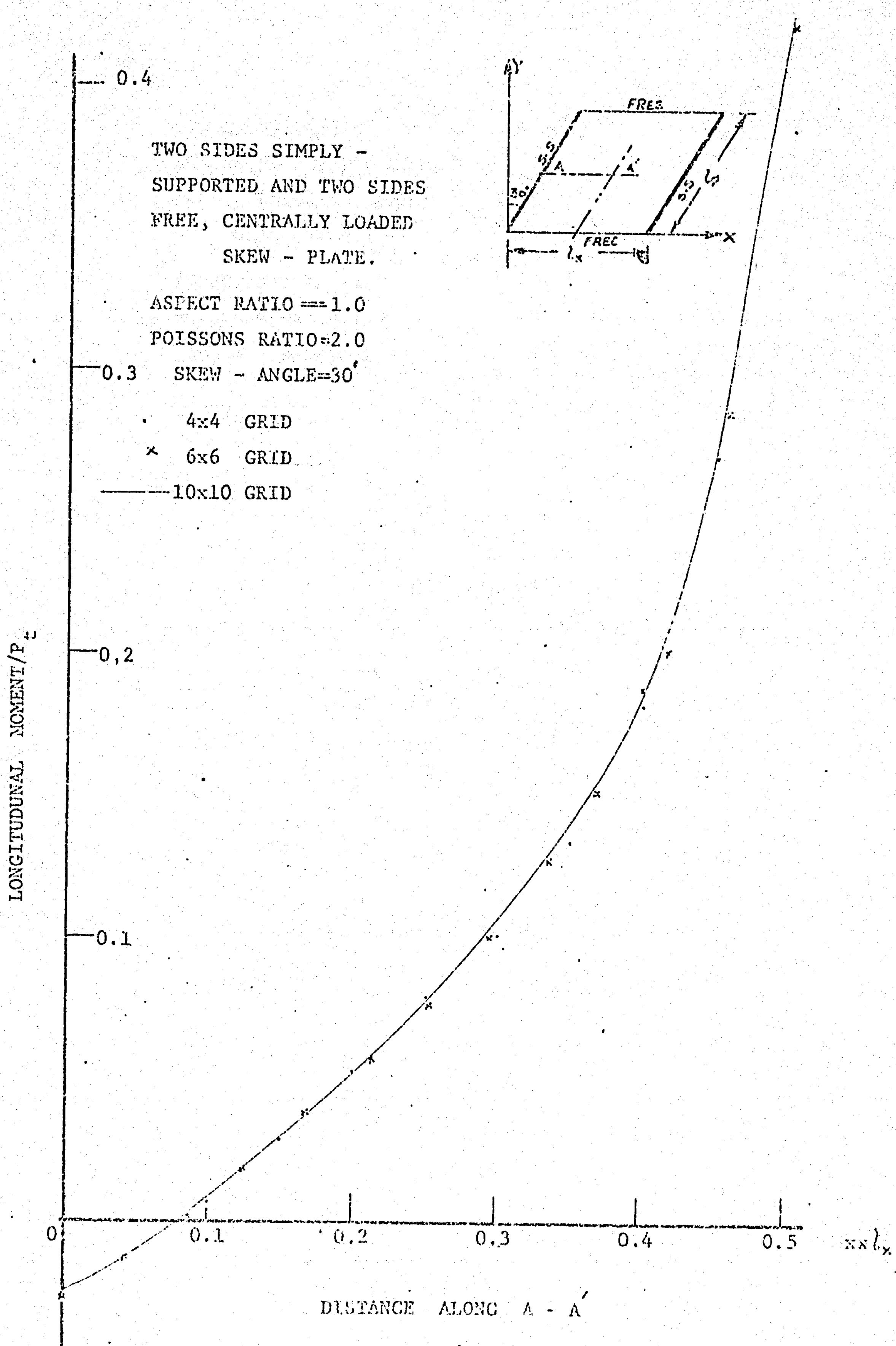
CENTRALLY LOADED
SQUARE PLATE WITH TWO
SIDES SIMPLY SUPPORTED
AND TWO SIDES FREE.
ASPECT RATIO = 1.0
POISSONS RATIO = 0.20
4x4 GRID
6x6 GRID
8x8 GRID
FREE
FREE
A
A′
LONGITUDINAL MOMENT/P
0.4
0.3
0.2
0.1
0
0.1
0.2
0.3
0.4
0.5
DISTANCE ALONG A - A′

Fig. (13)



DISTANCE ALONG B - B'

Fig. (14)



TWO SIDES SIMPLY - SUPPORTED AND TWO SIDES FREE, CENTRALLY LOADED SKEW - PLATE.
ASPECT RATIO = 1.0
POISSONS RATIO = 2.0
SKEW - ANGLE = 30°
4x4 GRID
6x6 GRID
10x10 GRID
FREE
S.S
30°
LONGITUDUNAL MOMENT/P
0.4
0.3
0,2
0.1
0
0.1
0.2
0.3
0.4
0.5
DISTANCE ALONG A - A'

Fig. (15)

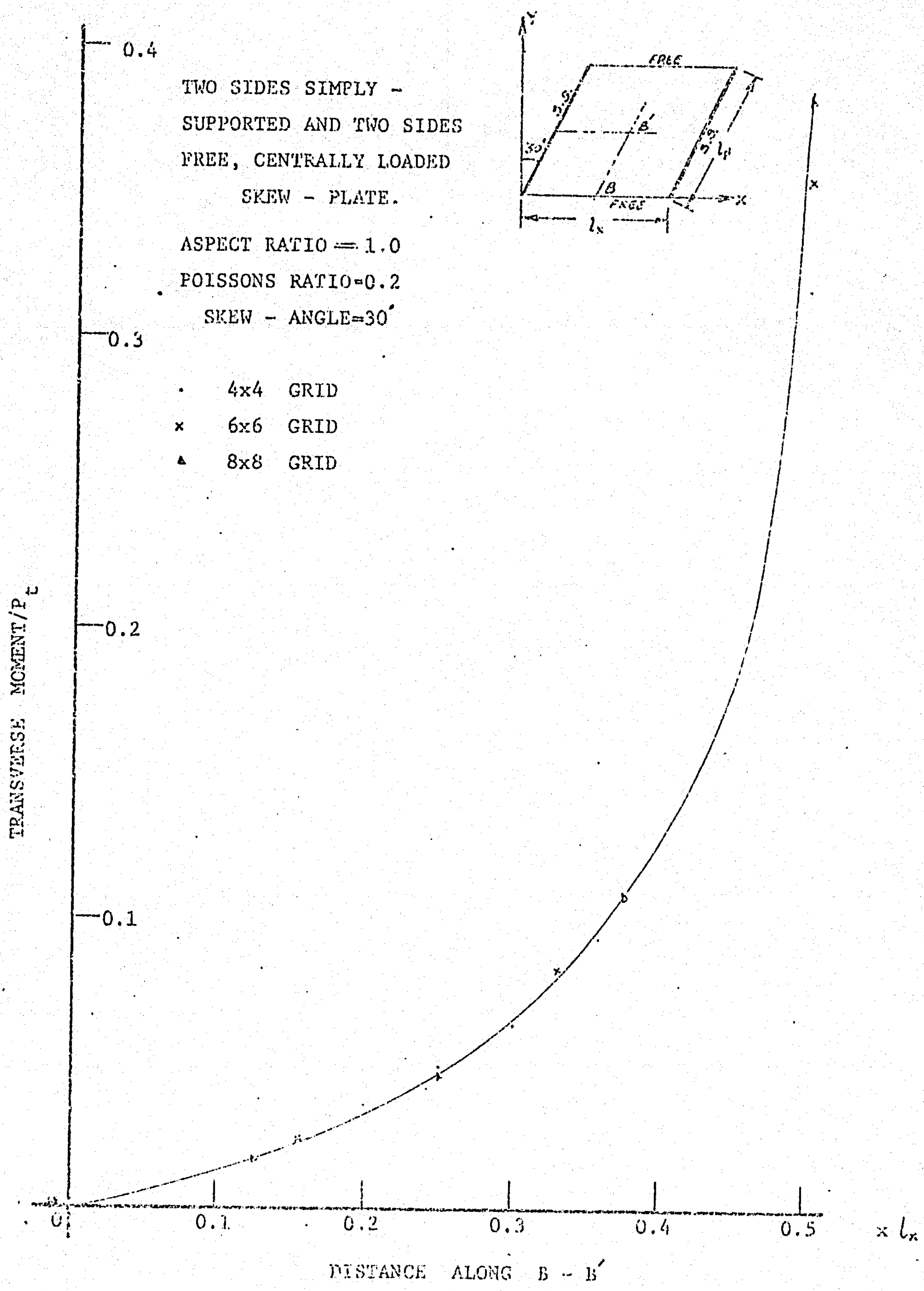

TWO SIDES SIMPLY - SUPPORTED AND TWO SIDES FREE, CENTRALLY LOADED SKEW - PLATE.
ASPECT RATIO = 1.0
POISSONS RATIO=0.2
SKEW - ANGLE=30°
4x4 GRID
6x6 GRID
8x8 GRID
FREE
B
B'
30°
l_x
l_y
X
Y
TRANSVERSE MOMENT/P_t
0.1
0.2
0.3
0.4
0
0.5
x l_x
DISTANCE ALONG B - B'

Fig. (16)

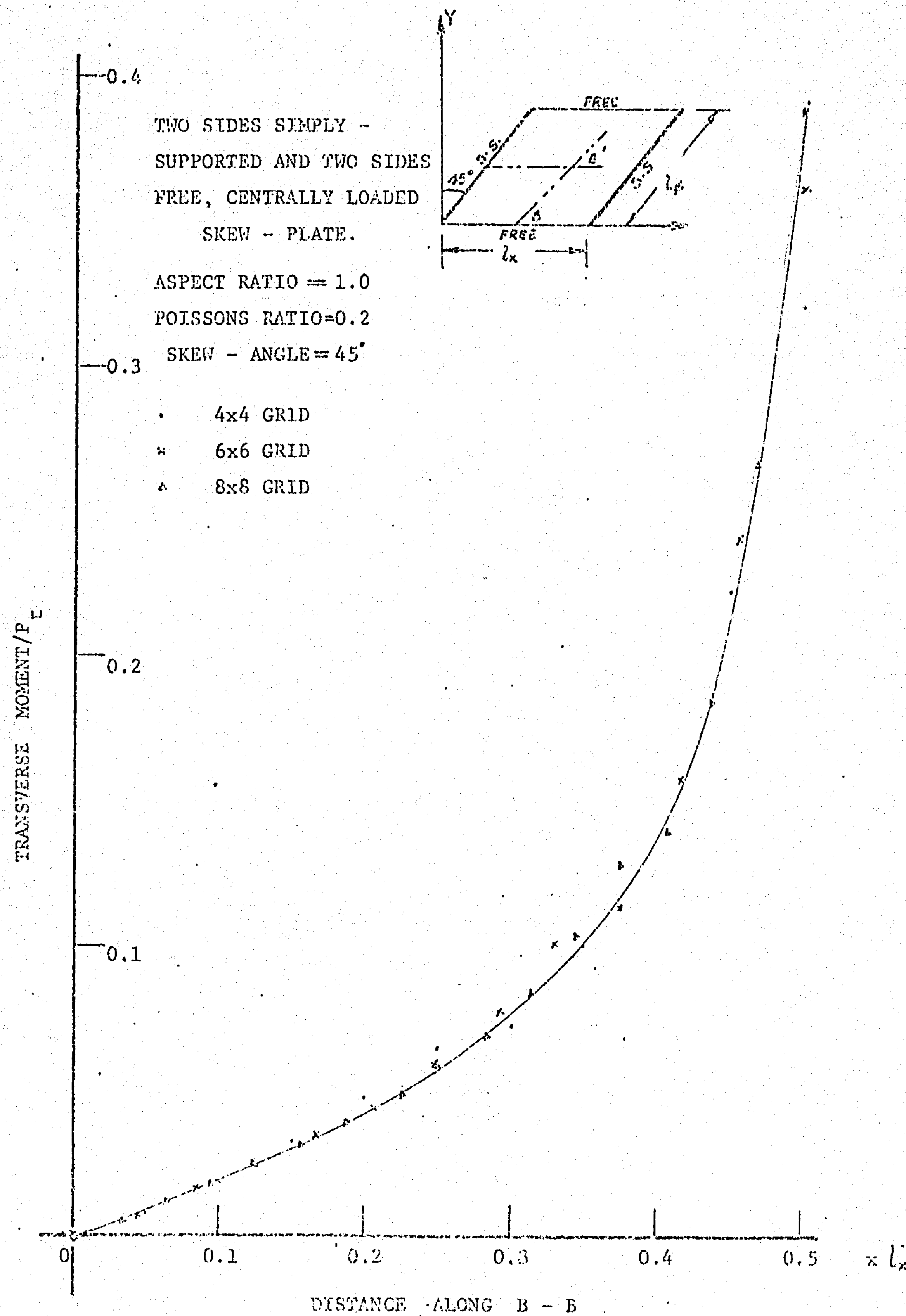

TWO SIDES SIMPLY - SUPPORTED AND TWO SIDES FREE, CENTRALLY LOADED SKEW - PLATE.
ASPECT RATIO = 1.0
POISSONS RATIO=0.2
SKEW - ANGLE = 45°
4x4 GRID
6x6 GRID
8x8 GRID
FREE
S.S
45°
B
FREE
Y
0.4
0.3
0.2
0.1
0
0.1
0.2
0.3
0.4
0.5
TRANSVERSE MOMENT/P
DISTANCE ALONG B - B

Fig. (17)

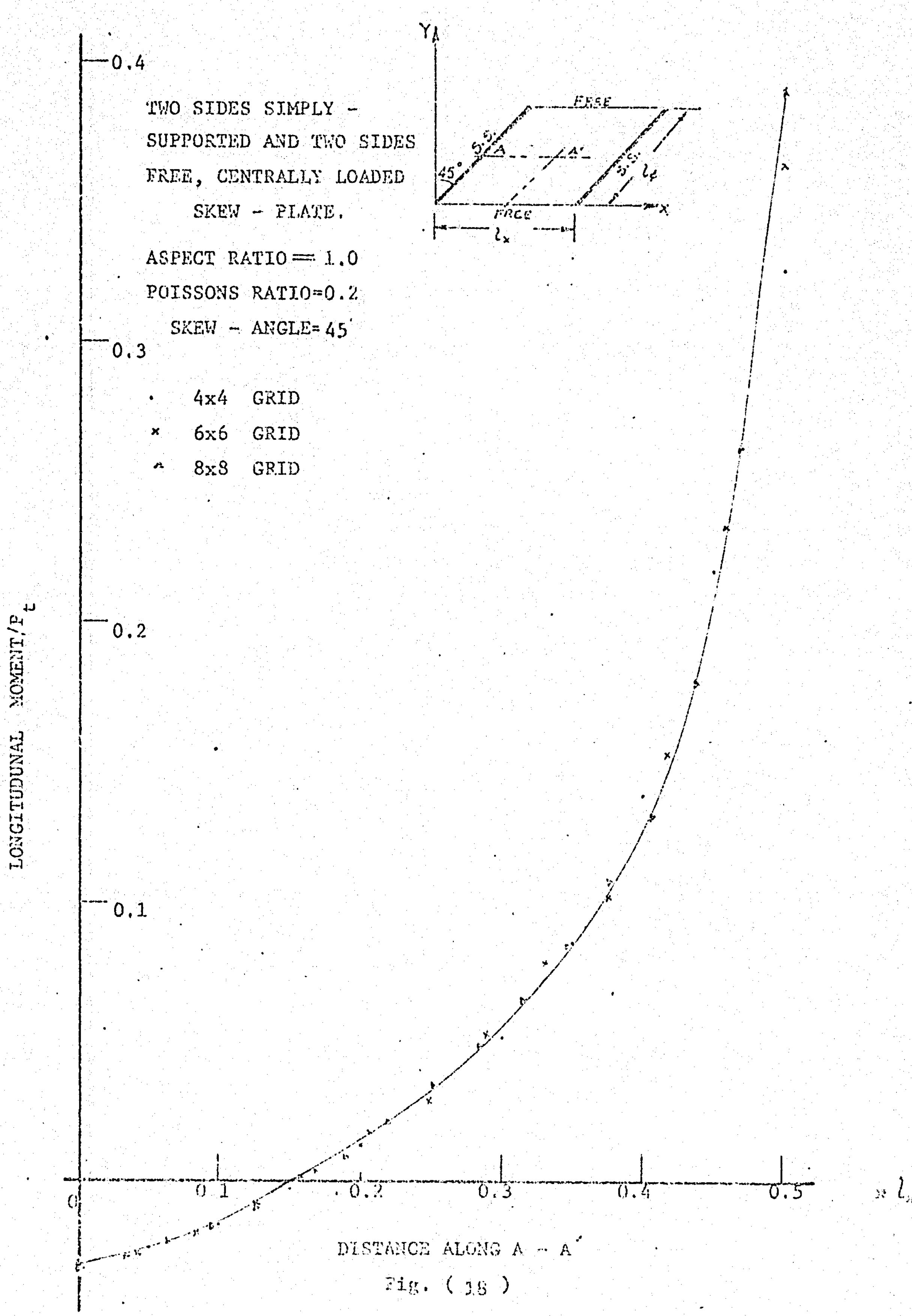

TWO SIDES SIMPLY - SUPPORTED AND TWO SIDES FREE, CENTRALLY LOADED SKEW - PLATE.
ASPECT RATIO = 1.0
POISSONS RATIO=0.2
SKEW - ANGLE= 45°
4x4 GRID
6x6 GRID
8x8 GRID
LONGITUDUNAL MOMENT/P_t
0.4
0.3
0.2
0.1
0
0.1
0.2
0.3
0.4
0.5
$\times l_x$
FREE
45°
l_x
l_y
Y
X
DISTANCE ALONG A - A'

Fig. (18)

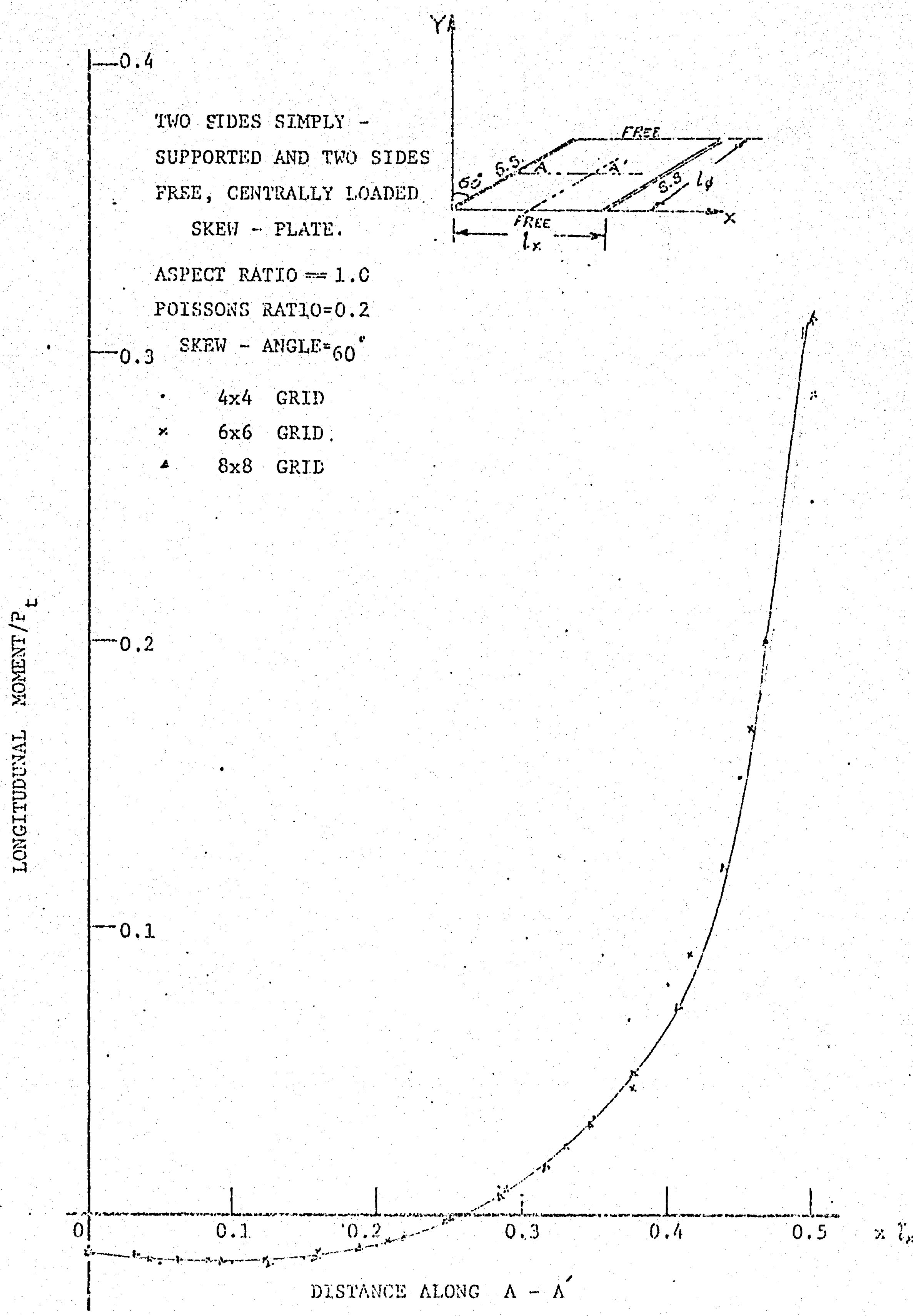

TWO SIDES SIMPLY -
SUPPORTED AND TWO SIDES
FREE, CENTRALLY LOADED
SKEW - PLATE.
ASPECT RATIO = 1.0
POISSONS RATIO=0.2
SKEW - ANGLE=60°
4x4 GRID
6x6 GRID
8x8 GRID
Y
X
FREE
FREE
60°
S.S
A
A'
l_y
l_x
LONGITUDINAL MOMENT/P_t
0.4
0.3
0.2
0.1
0
0.1
0.2
0.3
0.4
0.5
x l_x
DISTANCE ALONG A - A'

Fig. (19)

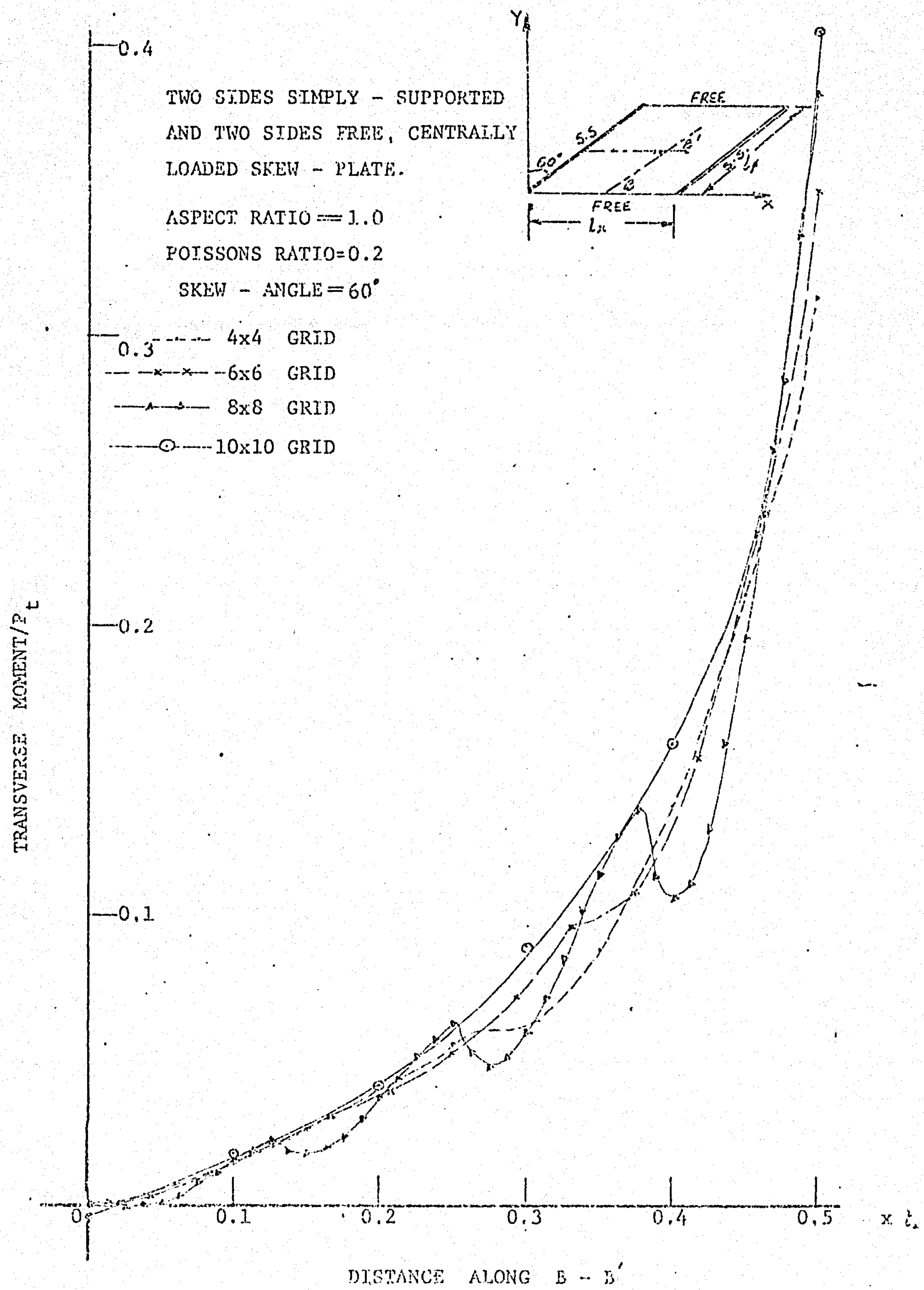

TWO SIDES SIMPLY - SUPPORTED
AND TWO SIDES FREE, CENTRALLY
LOADED SKEW - PLATE.
ASPECT RATIO = 1.0
POISSONS RATIO=0.2
SKEW - ANGLE = 60°
4x4 GRID
6x6 GRID
8x8 GRID
10x10 GRID
FREE
S.S
60°
FREE
0.4
0.3
0.2
0.1
0
0.1
0.2
0.3
0.4
0.5
TRANSVERSE MOMENT/P
DISTANCE ALONG B - B'

Fig. (20)

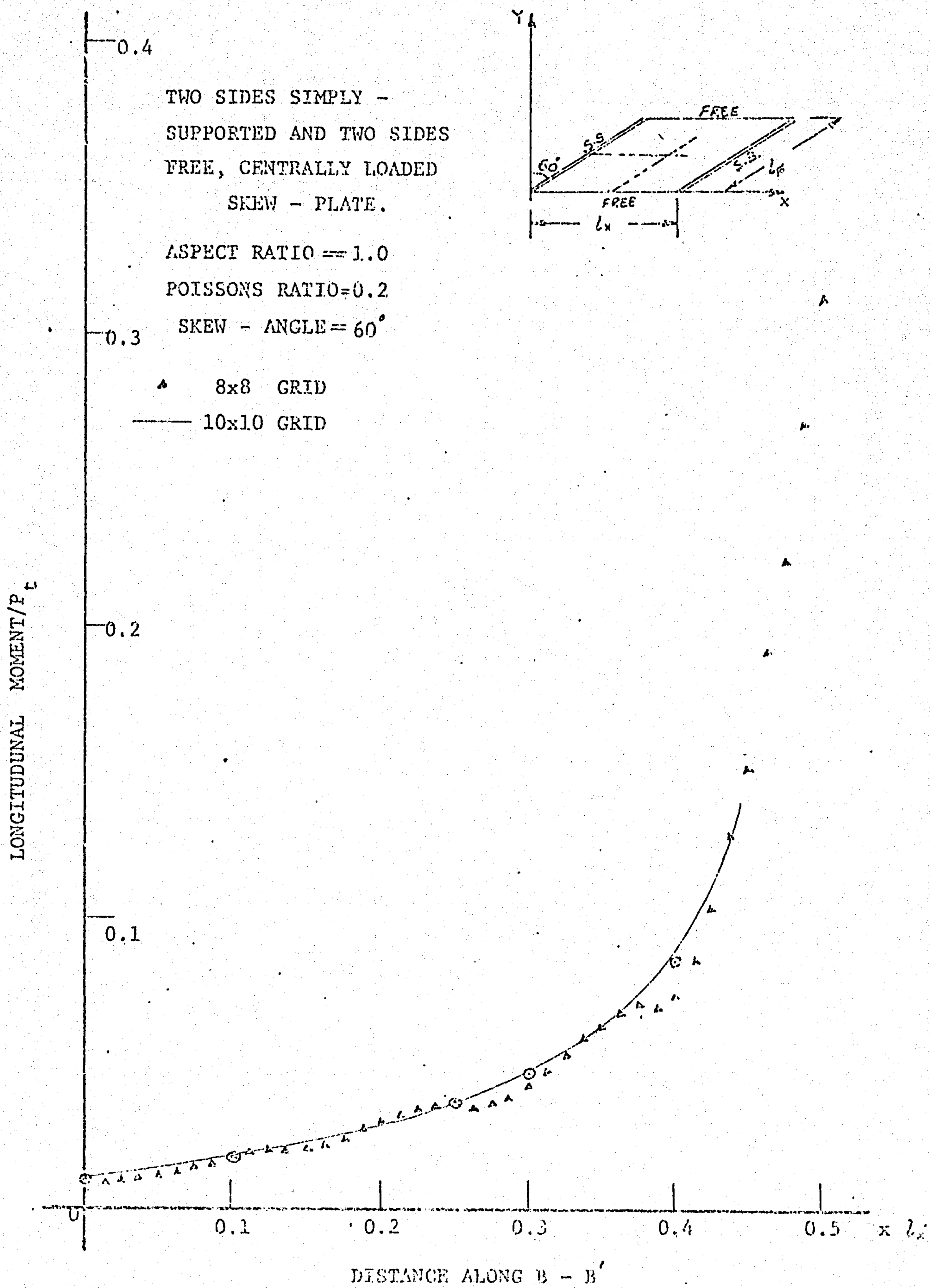

0.4
TWO SIDES SIMPLY - SUPPORTED AND TWO SIDES FREE, CENTRALLY LOADED SKEW - PLATE.
ASPECT RATIO = 1.0
POISSONS RATIO=0.2
SKEW - ANGLE = 60°
8x8 GRID
10x10 GRID
0.3
0.2
0.1
LONGITUDUNAL MOMENT/P
0
0.1
0.2
0.3
0.4
0.5
Y
FREE
S.S.
60°
S.S.
FREE
X
DISTANCE ALONG B - B'

Fig. (21)

TABLE 11

Central Deflections and Strain Energy For A Centrally Loaded Plate With Two Sides Simply Supported and the Other Two Free

Plate Type I - A, B, C $\nu = 0.20$

	GRID →	2 X 2	4 X 4	6 X 6	8 X 8	10 X 10
$\beta = 0°$	$\frac{\text{Deflection x D}}{P_t\ l_x^2}$	0.023136	0.02320953	0.02322420	0.02322942	0.02323181
	$\frac{\text{St.Energy x D}}{P_t \times l_x^4}$	0.01156813	0.01160477	0.0116121	0.01161471	0.01181591
$\beta = 30°$	$\frac{\text{Deflection x D}}{P_t \times l_x^2}$	0.01657257	0.01671238	0.01673593	0.01674435	0.01674825
	$\frac{\text{St.Energy x D}}{P_t \times l_x^4}$	0.008286285	0.00835618	0.008367967	0.008372173	0.008374124
$\beta = 45°$	$\frac{\text{Deflection x D}}{P_t \times l_x^4}$	0.01041895	0.01063950	0.01068067	0.01069640	0.01076436
	$\frac{\text{St.Energy x D}}{P_t \times l_x^4}$	0.005209477	0.005319751	0.0055340357	0.00534820	0.005382179

TABLE 12

Influence Coefficients For Moments At Point 41 (8x8 Grid)

Plate Type II - A Poissons Ration = 0.15

Slab With Two Sides Simply Supported and the Other Two Free

Nodal Point	Long Moment $Mx = m_1 Pt \times 10^{-3}$		Trans. Moment $My = m_2 P_t \times 10^{-3}$		Tors. Moment $M_{xy} = m_3 P_t \times 10^{-3}$	
	FEM	REF (32)	FEM	REF (32)	FEM	REF (32)
10	60.61	62.20	-20.07	-19.45	11.12	11.17
11	62.26	62.26	- 8.114	- 7.92	12.62	12.13
12	63.14	63.14	6.821	5.95	13.52	12.82
13	61.138	61.48	22.181	21.83	1.95	10.0
14	59.07	58.67	29.84	32.18	0	0
19	115.35	115.37	-39.22	-38.31	16.67	17.04
20	121.77	121.29	-17.69	-17.06	19.73	19.3
21	126.84	126.49	10.65	9.54	23.65	22.27
22	124.96	126.92	47.64	44.06	21.05	20.09
23	123.77	120.97	68.323	73.62	0	0
28	155.66	157.69	-5.449	-54.13	12.408	13.32
29	168.44	169.97	-27.05	-27.12	15.74	16.11
30	188.58	186.87	6.943	7.69	22.605	21.79
31	200.97	203.14	62.34	62.5	33.287	26.49
32	185.45	202.71	131.36	138.76	0	0
37	170.84	175.19	-60.49	-61.35	0	0
38	187.54	172.89	-32.19	-33.57	0	0
39	218.16	225.94	4.894	0.60	0	0
40	277.37	287.15	47.4	62.54	0	0
41	494.63	390.94	349.7	243.62	0	0

TABLE 13

Influence Coefficients For Moments at Point 41 (8x8 Grid)
Slab With Two Sides Simply Supported and the Other Two Free
Plate Type II - B Poissons Ratio = 0.15

NODAL POINT	LONG MOMENT		TRANS. MOMENT		TORS. MOMENT	
	FEM	REF (32)	FEM	REF (32)	FEM	REF(32)
10	18.01	18.37	-6.95	-7.65	23.82	24.31
11	20.593	20.97	-3.25	-3.60	24.63	25.12
12	23.960	24.70	4.529	4.01	25.68	26.21
13	28.189	28.24	16.69	16.92	25.91	26.03
14	32.32	31.37	32.46	34.50	20.75	20.41
15	42.18	42.19	38.706	38.19	8.4	9.12
16	53.89	54.89	22.63	21.87	4.93	4.95
17	55.066	55.45	4.09	4.30	13.52	12.67
18	49.014	49.84	-11.004	-10.21	23.73	23.46
19	37.260	37.70	-13.207	13.60	46.47	47.45
20	43.95	44.33	-3.403	-3.72	47.76	48.59
21	51.711	53.21	13.937	13.31	49.64	50.33
22	60.87	62.41	43.73	42.28	48.11	49.16
23	79.518	71.35	78.35	82.90	33.93	34.07
24	99.116	10.301	73.74	70.91	8.5	10.67
25	113.923	115.07	27.94	27.33	19.45	16.24
26	96.141	96.21	0.566	1.01	37.01	36.84
27	78.641	79.47	-20.24	-19.58	49.7	51.03
28	56.859	957.44	-17.42	-17.97	64.402	66.08
29	68.013	69.57	-1.729	-2.19	64.92	66.452
30	89.12	86.98	23.91	23.39	67.94	68.65
31	110.954	110.41	73.69	69.59	67.42	68.92

TABLE 13 CONT'D

NODAL POINT	LONG MOMENT		TRANS. MOMENT		TORS. MOMENT	
	FEM	REF (32)	FEM	REF(32)	FEM	REF(32)
32	117.357	135.9	147.63	156.85	43.19	41.59
33	190.434	194.01	79.33	81.53	17.68	13.28
34	144.81	143.93	24.48	27.04	51.02	50.97
35	107.12	107.51	-1.304	-1.25	60.56	62.47
36	89.82	85.67	-22.68	-22.69	68.32	70.81
37	74.313	75.30	-20.83	-21.25	73.13	75.48
38	93.34	93.77	-0.26	-1.34	70.916	73.11
39	122.81	124.49	31.622	27.92	70.35	74.35
40	181.9	184.61	74.63	82.6	82.16	80.68
41	414.964	342.92	360.85	266.59	54.25	54.25

TABLE 14

Influence Coefficients For Point 41 (8x8 Grid)

Plate Type II - C Poissons Ratio = 0.15

Slab With Two Sides Simply Supported and the Other Two Free

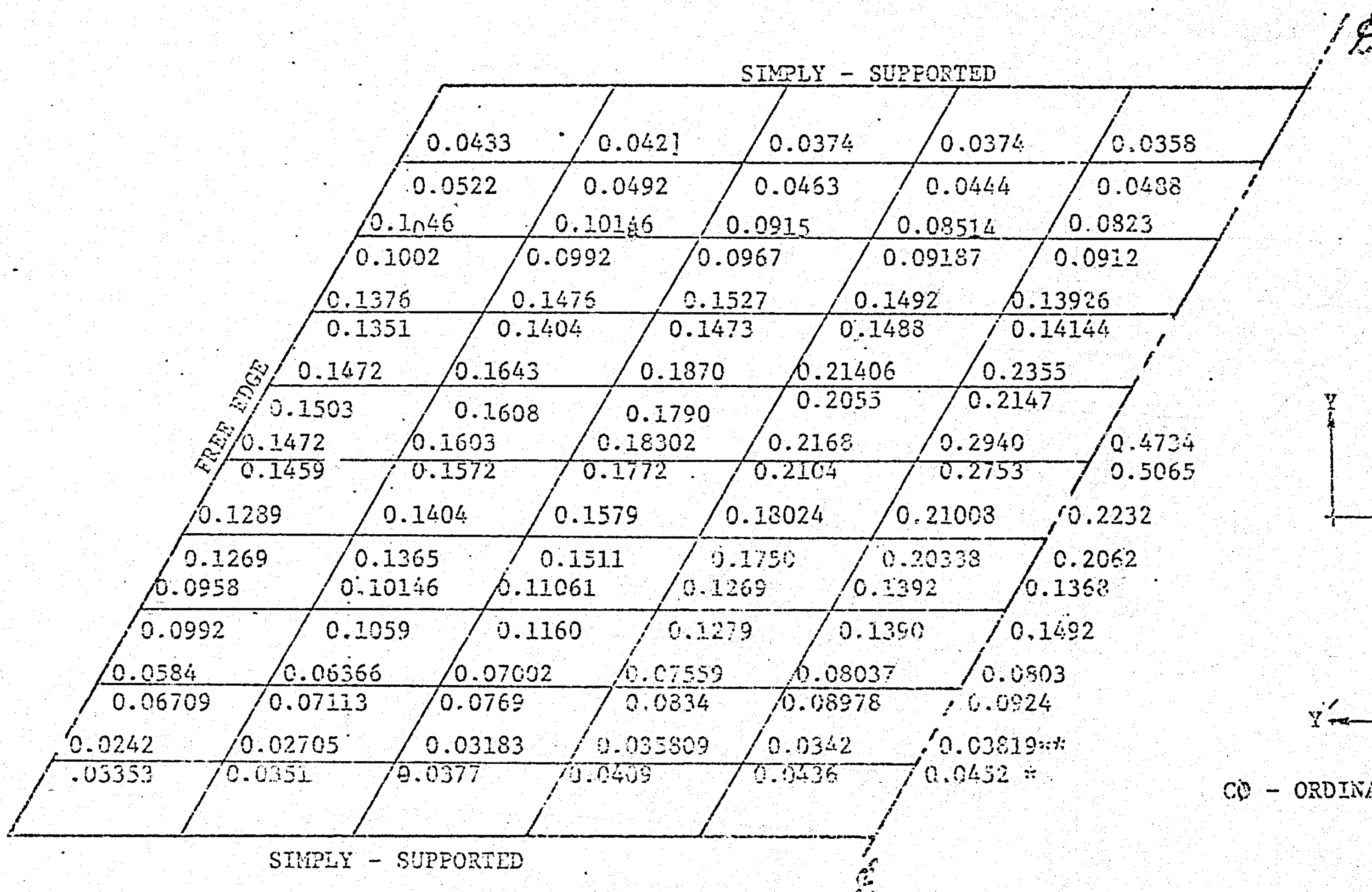
NODAL POINT	LONG MOMENT		TRANS. MOMENT		TORS. MOMENT	
	FEM	REF(32)	FEM	REF (32)	FEM	REF (32)
10		0.21		-0.09		11.65
11	1.003	1.13	1.52	1.14	12.82	12.59
12	2.849	3.47	5.409	46.59	14.72	14.89
13	5.71	7.02	13.58	12.96	18.11	18.51
14	15.80	10.74	30.815	32.69	18.54	18.26
15	21.933	26.39	43.97	43.05	5.73	6.76
16	45.5	44.08	24.64	25.65	6.977	6.98
17	32.23	34.44	10.30	10.13	20.15	19.58
18	14.80	15.63	3.94	4.65	25.11	26.27
19	0.762	0.91	0.77	0.40	24.23	23.55
20	3.47	4.02	4.88	4.37	26.56	26.07
21	8.153	10.261	14.51	13.89	31.56	31.27
22	21.48	19.50	39.69	35.92	35.87	37.38
23	37.52	33.83	76.23	83.95	30.21	30.86
24	83.85	80.53	84.8	78.95	2.64	8.34
25	82.27	86.20	33.11	32.52	34.27	31.69
26	39.37	39.51	17.51	18.07	40.79	41.46
27	16.09	15.72	6.668	6.77	42.58	43.48
28	1.98	2.68	2.37	1.88	35.74	35.09
29	8.39	9.47	10.03	9.78	39.89	39.09
30	26.24	22.03	29.41	26.74	48.18	46.6

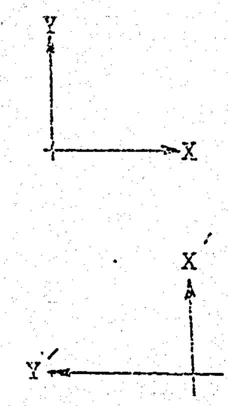
TABLE 14 CONT'D

NODAL POINT	LONG MOMENT		TRANS. MOMENT		TORS. MOMENT	
	FEM	REF(32)	FEM	REF(32)	FEM	REF(32)
31	48.01	43.96	71.32	64.57	51.52	54.11
32	62.62	90.07	150.75	158.63	36.804	37.45
33	159.34	160.89	83.77	88.10	39.06	37.67
34	66.76	67.67	38.92	42.01	56.66	55.43
35	30.92	29.72	20.68	19.85	50.38	51.01
32	11.26	10.84	6.81	6.31	48.72	68.58
37	5.30	5.99	4.53	4.18	45.119	44.48
38	19.25	18.17	17.60	15.97	49.077	48.77
39	42.30	41.23	41.37	38.57	56.139	56.85
40	86.84	90.96	82.19	88.50	73.549	63.83
41	33.42	26.947	348069	267.25	52.57	39.41

INFLUENCE COEFFICIENTS FOR TRANSVERSE MOMENT (m_2) AT POINT 61 (10x10) GRID

PLATE TYPE I-B POISSONS RATIO = 0.20





CO - ORDINATE SYSTEM

** VALUES (ON TOP) AS GIVEN BY RUSH & HERGENRODER (24) (X′ - Y′) SYSTEM

* VALUES (BELOW) AS OBTAINED BY FINITE ELEMENT METHOD (X - Y) SYSTEM

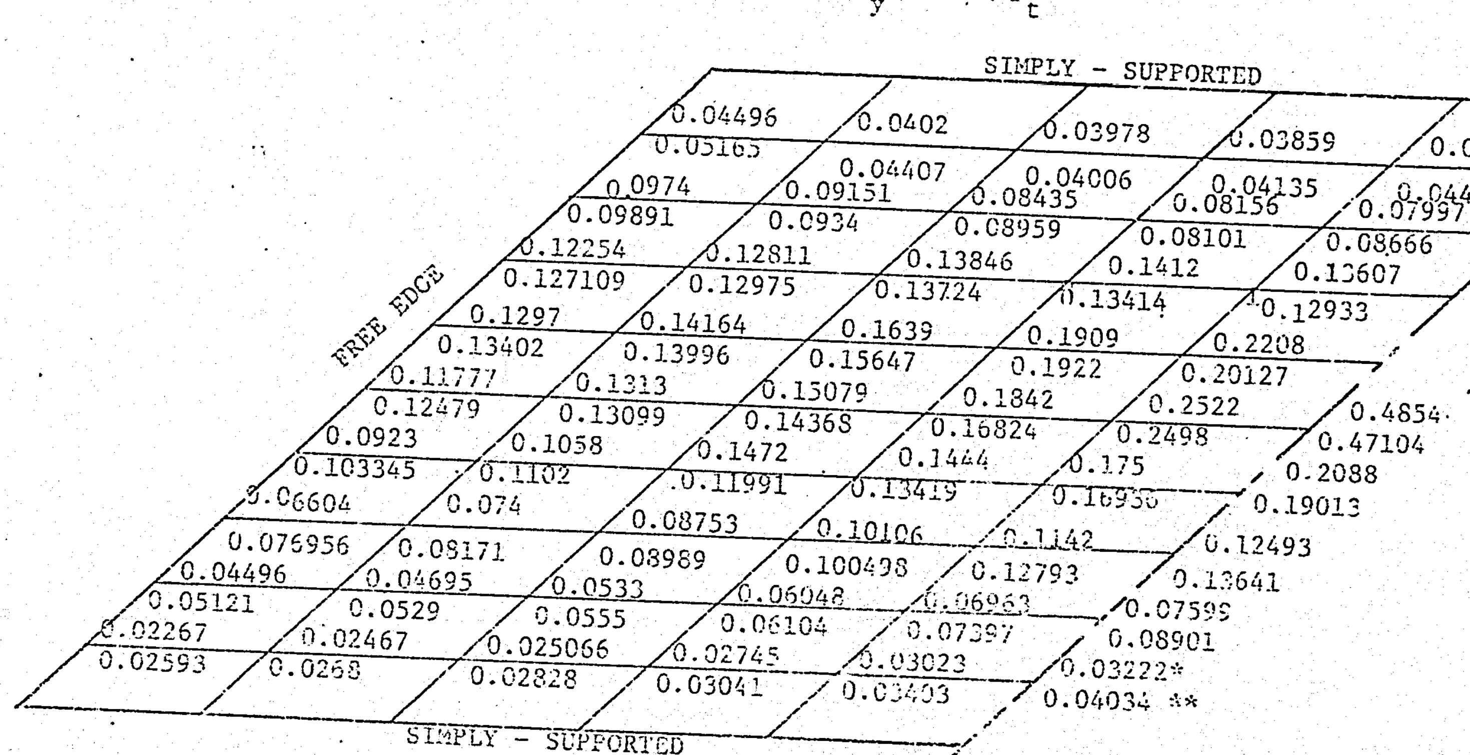
Fig. (22)

INFLUENCE COEFFICIENTS (m_z) FOR MOMENT M_y AT POINT 61 (10x10) GRID

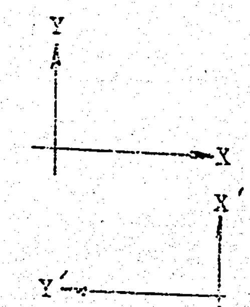
PLATE TYPE I - C

POISSONS RATIO = 0.20

MOMENT M_y m x P_t



* VALUES (ON TOP) AS GIVEN BY RUSH & HERGENRODER (24)
**VALUES (BELOW) AS OBTAINED BY FINITE ELEMENT METHOD (X - Y) SYSTEM



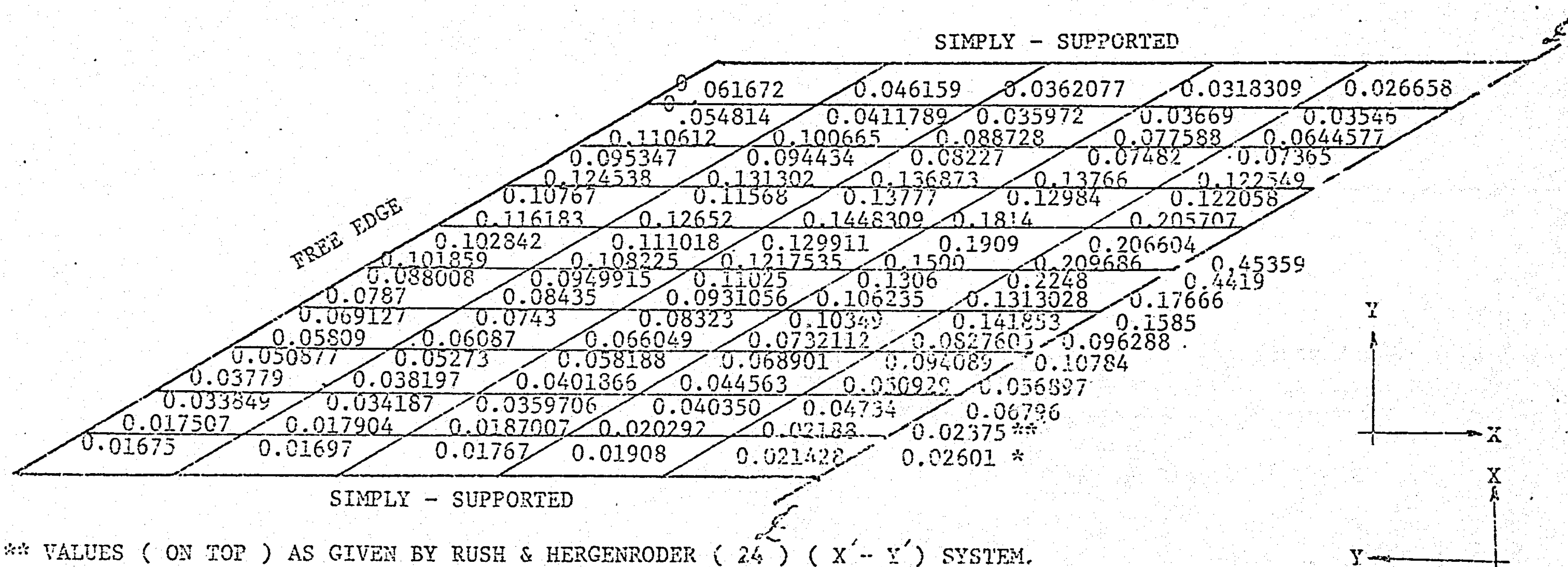
CO - ORDINATE SYSTEM

Fig. (23)

INFLUENCE COEFFICIENTS (m_2) FOR MOMENT M_y AT POINT 61 (10x10) GRID.

PLATE TYPE I - D. POISSONS RATIO = 0.20

MOMENT M_y m x P_t



** VALUES (ON TOP) AS GIVEN BY RUSH & HERGENRODER (24) (X′ - Y′) SYSTEM.

* VALUES (BELOW) AS OBTAINED BY FINITE ELEMENT METHOD (X - Y) SYSTEM

CO - ORDINATE SYSTEMS.

Fig. (24)

UNIFORMLY LOADED SIMPLY SUPPORTED PLATE WITH ONE STIFFENER Fig. (8)

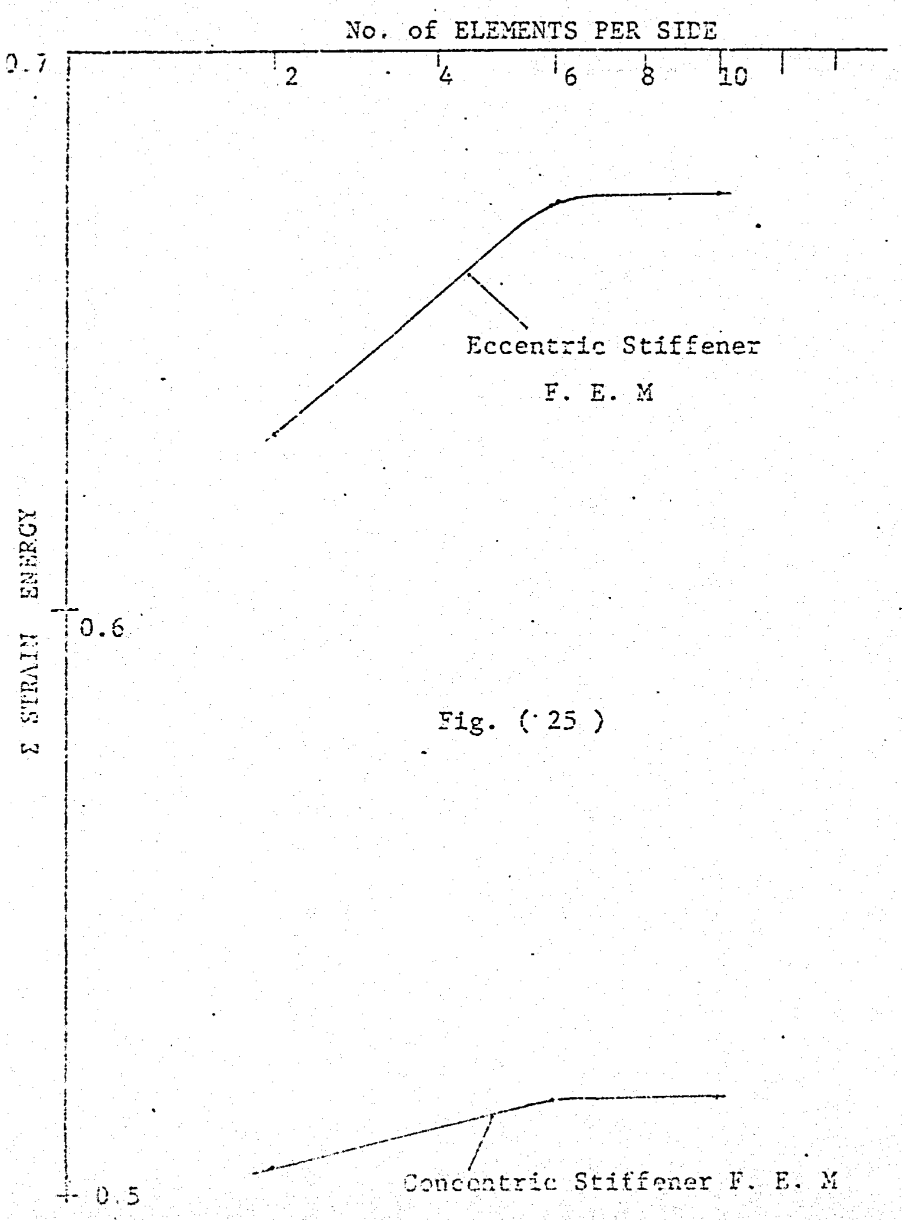


Fig. (25)

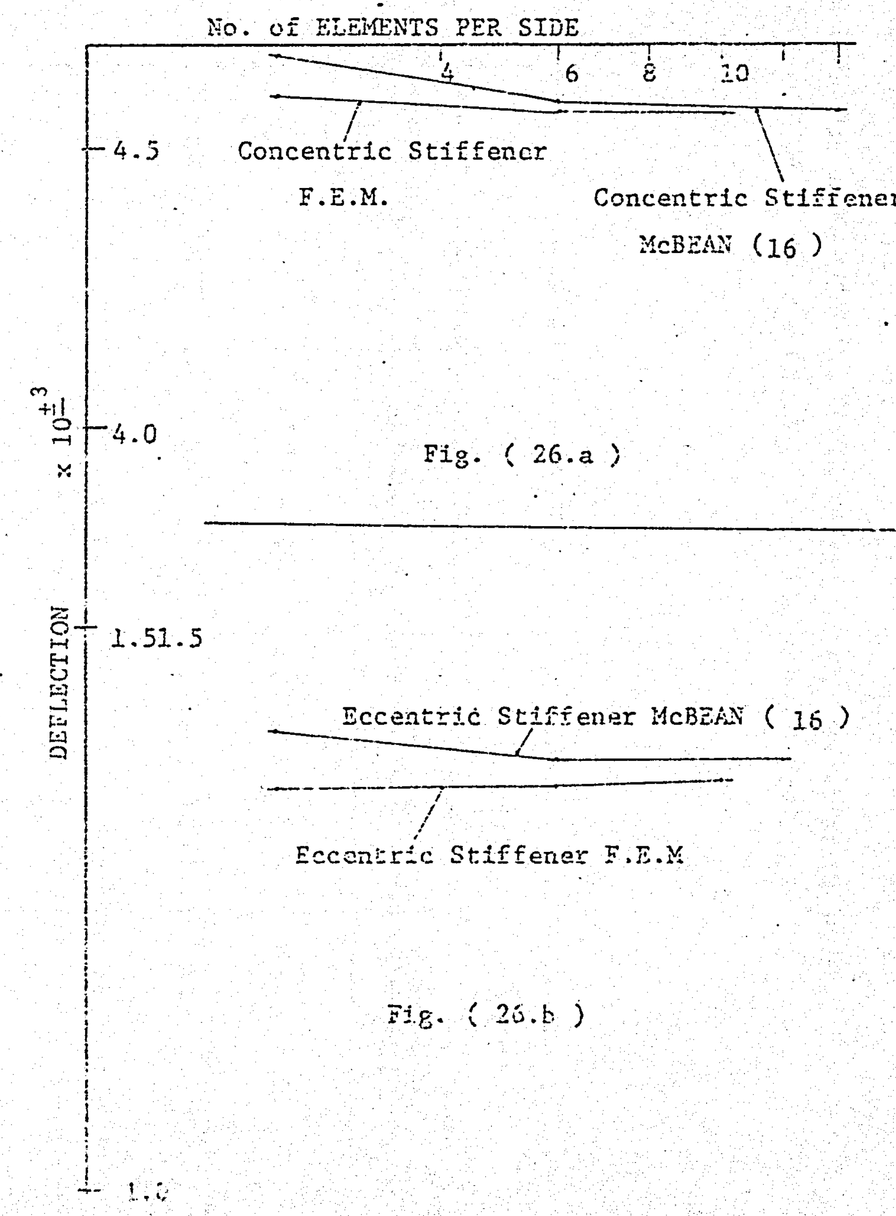


Fig. (26.a)

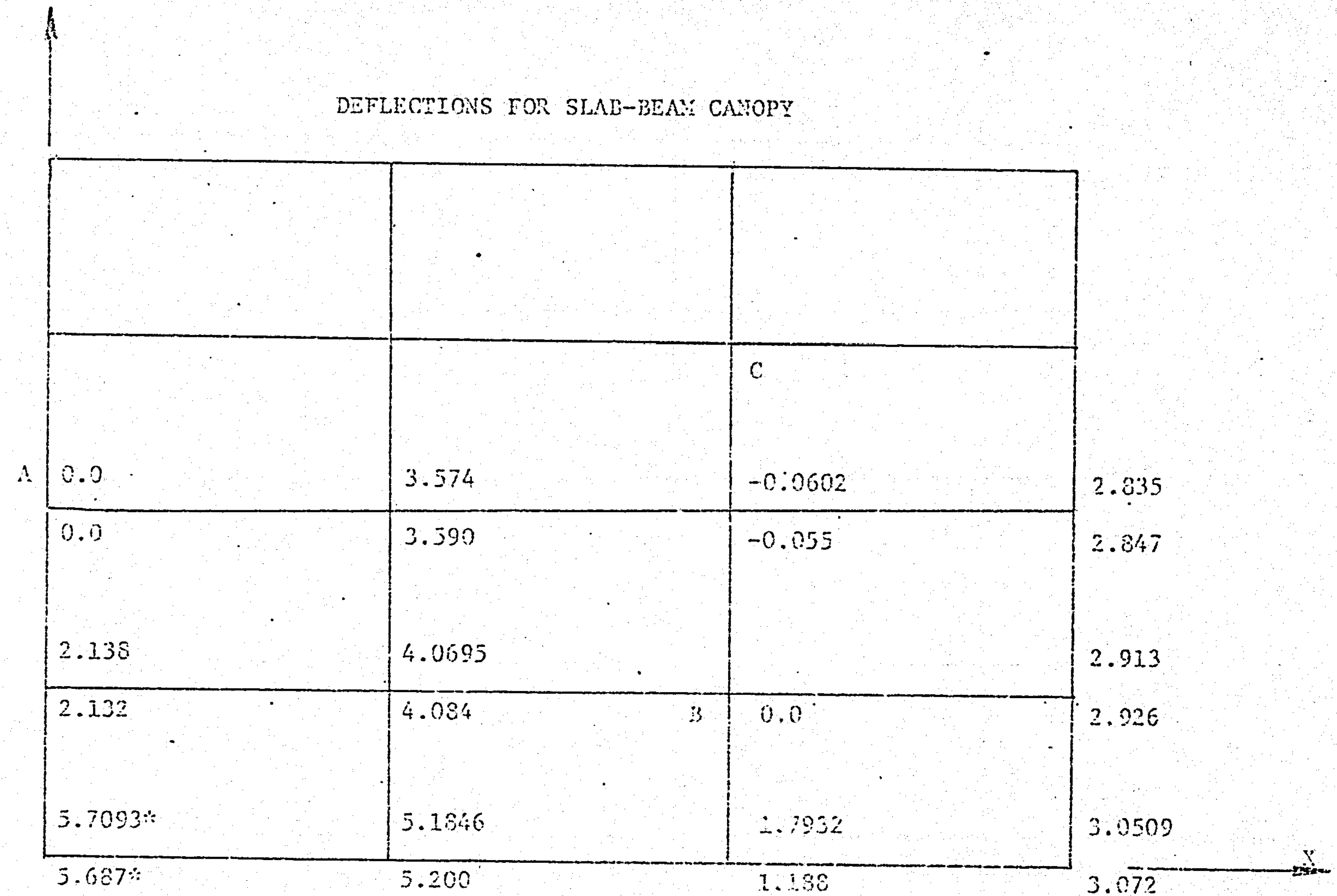
Fig. (26.b)

TABLE 15

COMPARISON OF DEFLECTIONS WITH THOSE GIVEN BY McBEAN FOR SIMPLY-SUPPORTED PLATE WITH ONE STIFFENER FIG.(8)

GRIDWORK OF QUARTER STRUCTURE	CONCENTRIC STIFFENER		ECCENTRIC STIFFENER	
	McBEAN (16)	REFINED ELEMENT	McBEAN'S (16)	REFINED ELEMENT
1 X 1	4.662×10^{-4}	4.5799×10^{-4}	1.398×10^{-4}	1.348×10^{-4}
2 X 2	4.568×10^{-4}	4.5598×10^{-4}	1.370×10^{-4}	1.349×10^{-4}
3 X 3		4.5595×10^{-4}		1.354×10^{-4}
4 X 4	4.559×10^{-4}		1.67×10^{-4}	
EXACT SOLUTION	4.561×10^{-4}		1.352×10^{-4}	
AVERAGED SOLUTION	4.523×10^{-4}		1.186×10^{-4}	

DEFLECTIONS FOR SLAB-BEAM CANOPY



All values to be multiplied by 10^{-5} ft.

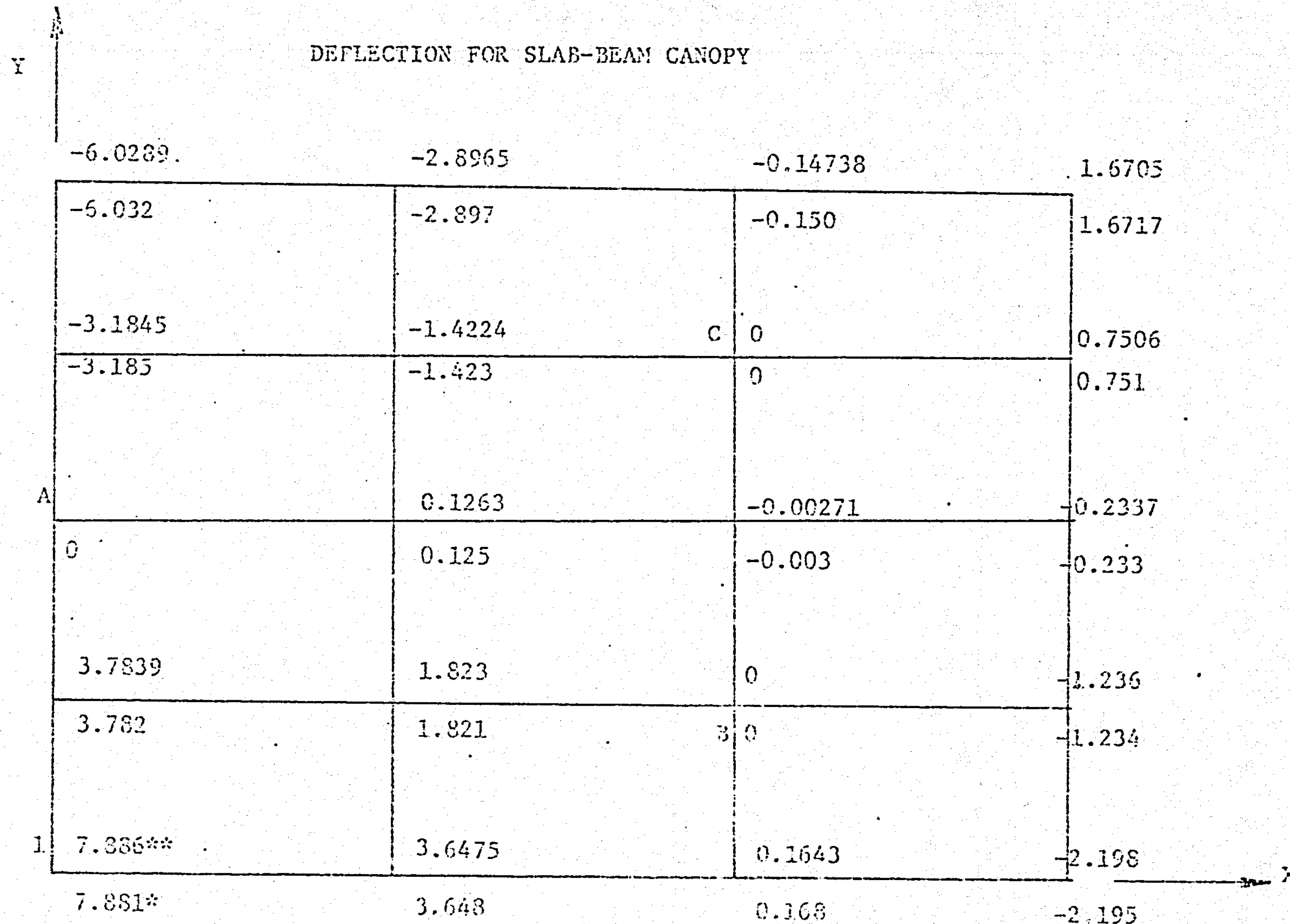
* As obtained by McBean

* As obtained by High Precision elements.

Boundary Condition: @ A W = V = 0 @ B & C W = U = 0

Fig. (27)

DEFLECTION FOR SLAB-BEAM CANOPY



All values to be multiplied by 10^{-4}
Load of 100 lb. at node 1.

* As obtained by McBean
** As obtained by high precision elements.

Boundary Conditions @ A: W = V = 0 @ B & C W = U = 0

Fig. (28)

APPENDIX 1
TRANSFORMATION MATRIX [T]

1	$-b$	0	b^2	0	0	$-b^3$	0	0	0	b^4	0	0	0	0	$-b^5$	0	0	0	0
0	1	0	$-2b$	0	0	$3b^2$	0	0	0	$-4b^3$	0	0	0	0	$5b^4$	0	0	0	0
0	0	1	0	$-b$	0	0	b^2	0	0	0	$-b^3$	0	0	0	0	0	0	0	0
0	0	0	2	0	0	$-6b$	0	0	0	$12b^2$	0	0	0	0	$-20b^3$	0	0	0	0
0	0	0	0	1	0	0	$-2b$	0	0	0	$3b^2$	0	0	0	0	0	0	0	0
0	0	0	0	0	2	0	0	$-2b$	0	0	0	$2b^2$	0	0	0	$-2b^3$	0	0	0
1	a	0	a^2	0	0	a^3	0	0	0	a^4	0	0	0	0	a^5	0	0	0	0
0	1	0	$2a$	0	0	$3a^2$	0	0	0	$4a^3$	0	0	0	0	$5a^4$	0	0	0	0
0	0	1	0	a	0	0	a^2	0	0	0	a^3	0	0	0	0	0	0	0	0
0	0	0	2	0	0	$6a$	0	0	0	$12a^2$	0	0	0	0	$20a^3$	0	0	0	0
0	0	0	0	1	0	0	$2a$	0	0	0	$3a^2$	0	0	0	0	0	0	0	0
0	0	0	0	0	2	0	0	$2a$	0	0	0	$2a^2$	0	0	0	$2a^3$	0	0	0
1	0	c	0	0	c^2	0	0	0	c^3	0	0	0	0	c^4	0	0	0	0	c^5
0	1	0	0	c	0	0	0	c^2	0	0	0	0	c^3	0	0	0	0	c^4	0
0	0	1	0	0	$2c$	0	0	0	$3c^2$	0	0	0	0	$4c^3$	0	0	0	0	$5c^4$
0	0	0	2	0	0	0	$2c$	0	0	0	0	$2c^2$	0	0	0	0	$2c^3$	0	0
0	0	0	0	1	0	0	0	$2c$	0	0	0	0	$3c^2$	0	0	0	0	$4c^3$	0
0	0	0	0	0	2	0	0	0	$6c$	0	0	0	0	$12c^2$	0	0	0	0	$20c^3$
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	$5a^4c$	$3a^2c^3-2a^4c$	$-2ac^4+3a^3c^2$	$c^5-4a^2c^3$	$5ac^4$
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	$5b^4c$	$3b^2c^3-2b^4c$	$2bc^4-3b^3c^2$	$c^5-4b^2c^3$	$-5bc^4$

APPENDIX 2

ROTATION MATRIX R

$$[R] = \begin{bmatrix} R_1 & 0 & 0 \\ 0 & R_1 & 0 \\ 0 & 0 & R_1 \end{bmatrix}$$

$$[R_1] = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \mathrm{Cos}\theta & \mathrm{Sin}\theta & 0 & 0 & 0 \\ 0 & -\mathrm{Sin}\theta & \mathrm{Cos}\theta & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathrm{Cos}^2\theta & 2\mathrm{Sin}\theta\mathrm{Cos}\theta & \mathrm{Sin}^2\theta \\ 0 & 0 & 0 & -\mathrm{Sin}\theta\mathrm{Cos}\theta & \mathrm{Cos}^2\theta-\mathrm{Sin}^2\theta & \mathrm{Sin}\theta\mathrm{Cos}\theta \\ 0 & 0 & 0 & \mathrm{Sin}^2\theta & 2\mathrm{Sin}\theta\mathrm{Cos}\theta & \mathrm{Cos}^2\theta \end{bmatrix}$$

APPENDIX 3

TRANSFORMATION MATRIX T_{ep}

$-b$	0	b^2	0	0	$-b^3$	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	0	$-2b$	0	0	$3b^2$	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	1	0	$-b$	0	0	b^2	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	1	$-b$	0	b^2	0	0	$-b^3$	0	0	0	0
0	0	0	0	0	0	0	0	0	0	1	0	$-2b$	0	0	$3b^2$	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	1	0	$-b$	0	0	b^2	0	0	0
a	0	a^2	0	0	a^3	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	0	$2a$	0	0	$3a^2$	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	1	0	a	0	0	a^2	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	1	a	0	a^2	0	0	a^3	0	0	0	0
0	0	0	0	0	0	0	0	0	0	1	0	$2a$	0	0	$3a^2$	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	1	0	a	0	0	a^2	0	0	0
0	c	0	0	c^2	0	0	0	c^3	0	0	0	0	0	0	0	0	0	0	0
1	0	0	c	0	0	0	c^2	0	0	0	0	0	0	0	0	0	0	0	0
0	1	0	0	$2c$	0	0	0	$3c^2$	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	1	0	c	0	0	c^2	0	0	0	c^3	0
0	0	0	0	0	0	0	0	0	0	1	0	0	c	0	0	0	c^2	0	0
0	0	0	0	0	0	0	0	0	0	0	1	0	0	$2c$	0	0	0	$3c^2$	0
$\frac{a-b}{3}$	$\frac{c}{3}$	$\frac{(a-b)^2}{9}$	$\frac{(a-b)c}{9}$	$\frac{c^2}{9}$	$\frac{(a-b)^3}{27}$	$\frac{(a-b)^2c}{27}$	$\frac{(a-b)c^2}{27}$	$\frac{c^3}{27}$	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	1	$\frac{a-b}{3}$	$\frac{c}{3}$	$\frac{(a-b)^2}{9}$	$\frac{(a-b)c}{9}$	$\frac{c^2}{9}$	$\frac{(a-b)^3}{27}$	$\frac{(a-b)^2c}{27}$	$\frac{(a-b)c^2}{27}$	$\frac{c^3}{27}$	0

APPENDIX 4

ROTATION R_{ep}

$$[R_{ep}] = \begin{bmatrix} R_2 & 0 & 0 & 0 \\ 0 & R_2 & 0 & 0 \\ 0 & 0 & R_2 & 0 \\ 0 & 0 & 0 & R_3 \end{bmatrix}$$

$$[R_2] = \begin{bmatrix} \cos\theta & 0 & 0 & \sin\theta & 0 & 0 \\ 0 & \cos^2\theta & \sin\theta\cos\theta & 0 & \sin\theta\cos\theta & \sin^2\theta \\ 0 & -\sin\theta\cos\theta & \cos^2\theta & 0 & -\sin^2\theta & \sin\theta\cos\theta \\ -\sin\theta & 0 & 0 & \cos\theta & 0 & 0 \\ 0 & -\sin\theta\cos\theta & -\sin^2\theta & 0 & \cos^2\theta & \sin\theta\cos\theta \\ 0 & \sin^2\theta & -\sin\theta\cos\theta & 0 & -\sin\theta\cos\theta & \cos^2\theta \end{bmatrix}$$

$$[R_3] = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

APPENDIX 5

TRANSFORMATION MATRIX T_b

$$[T_b] = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 \\ 1 & 1 & 1^2 & 1^3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 21 & 31^2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1^2 & 1^3 & 1^4 & 1^5 \\ 0 & 0 & 0 & 0 & 0 & 1 & 21 & 31^2 & 41^3 & 51^4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & 61 & 121^2 & 201^3 \end{bmatrix}$$

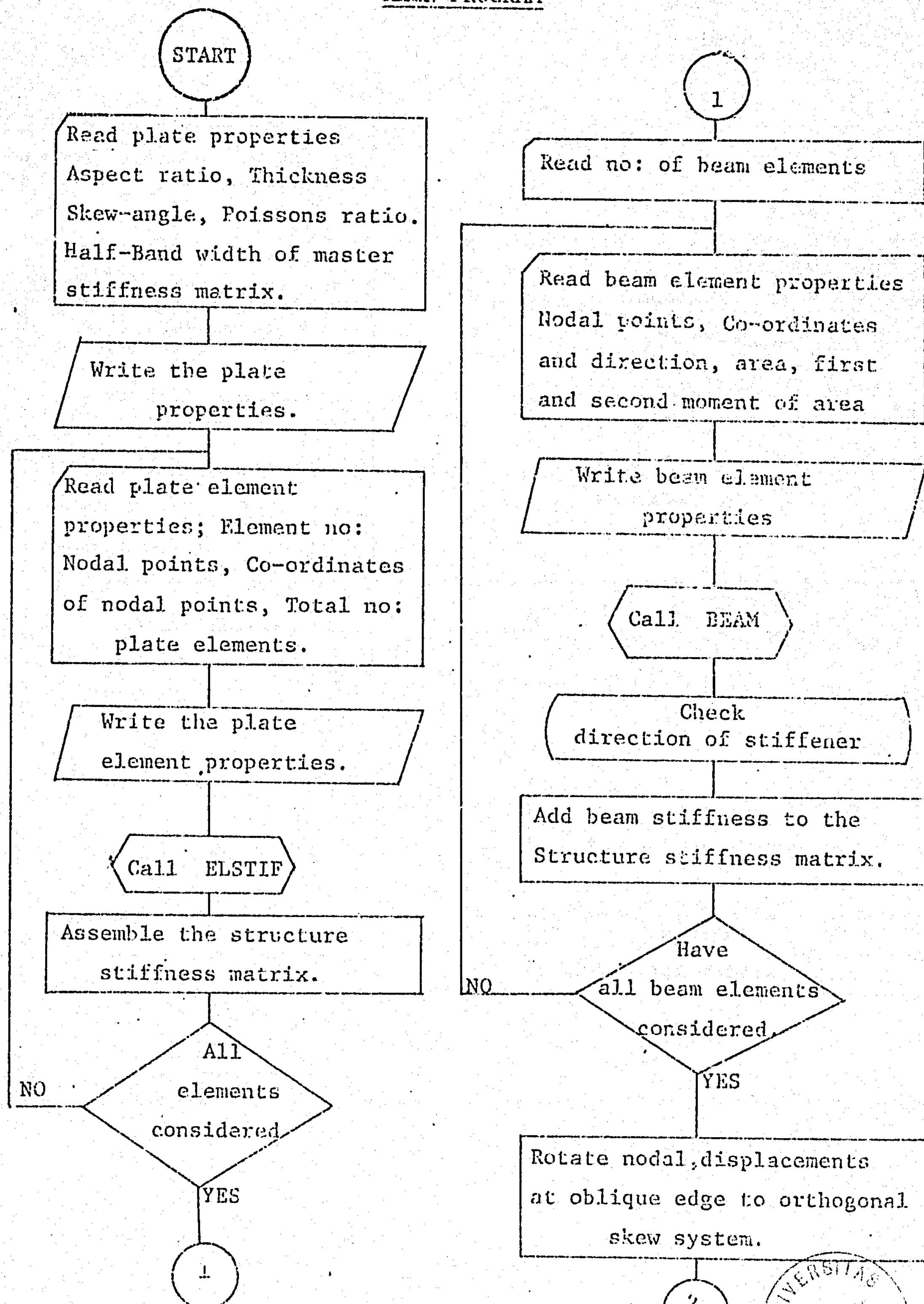
APPENDIX 6

MATRIX k_b

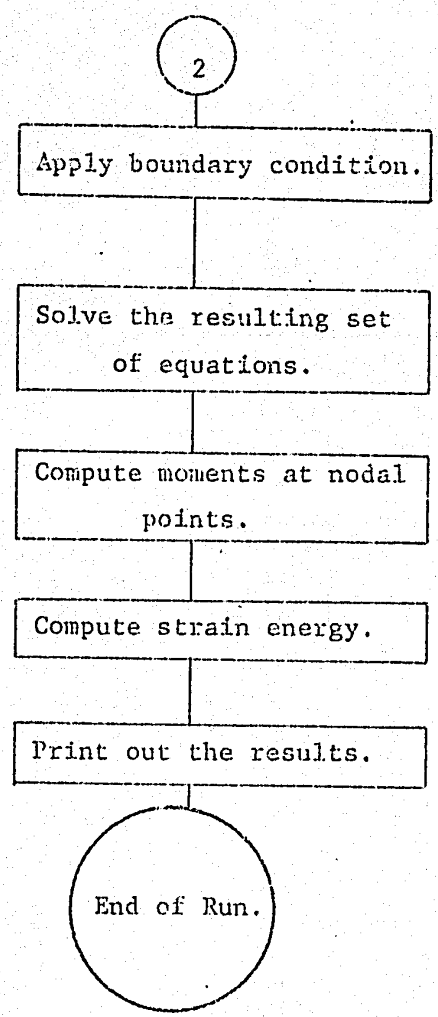
$$
\left[k_b\right] = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & A1 & A1^2 & A1^3 & 0 & 0 & -2S1 & -3S1^2 & -4S1^3 & -5S1^4 \\
0 & A1^2 & \frac{4}{3}A1^3 & \frac{3}{2}A1^4 & 0 & 0 & -2S1^2 & -4S1^3 & -6S1^4 & -8S1^5 \\
0 & A1^3 & \frac{3}{2}A1^4 & \frac{9}{5}A1^5 & 0 & 0 & -2S1^3 & -\frac{9}{2}S1^4 & -\frac{36}{3}S1^5 & -10S1^6 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -2S1 & -2S1^2 & -2S1^3 & 0 & 0 & 4I1 & 6I1^2 & 8I1^3 & 10I1^4 \\
0 & -3S1^2 & -4S1^3 & -\frac{9}{2}S1^4 & 0 & 0 & 6I1^2 & 12I1^3 & 18I1^4 & 24I1^5 \\
0 & -4S1^3 & -6S1^4 & -\frac{36}{3}S1^5 & 0 & 0 & 8I1^3 & 18I1^4 & \frac{144}{5}I1^5 & 40I1^6 \\
0 & -5S1^4 & -8S1^5 & -10S1^6 & 0 & 0 & 10I1^4 & 24I1^5 & 40I1^6 & \frac{400}{7}I1^7
\end{bmatrix}
$$

FLOW CHART

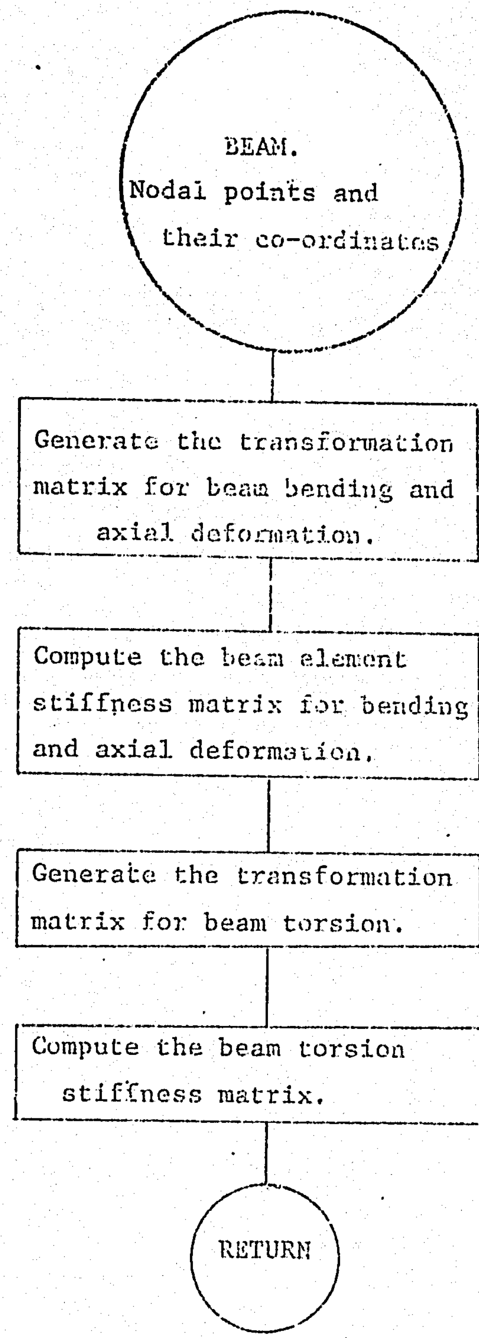
MAIN PROGRAM



Main Program
Con't

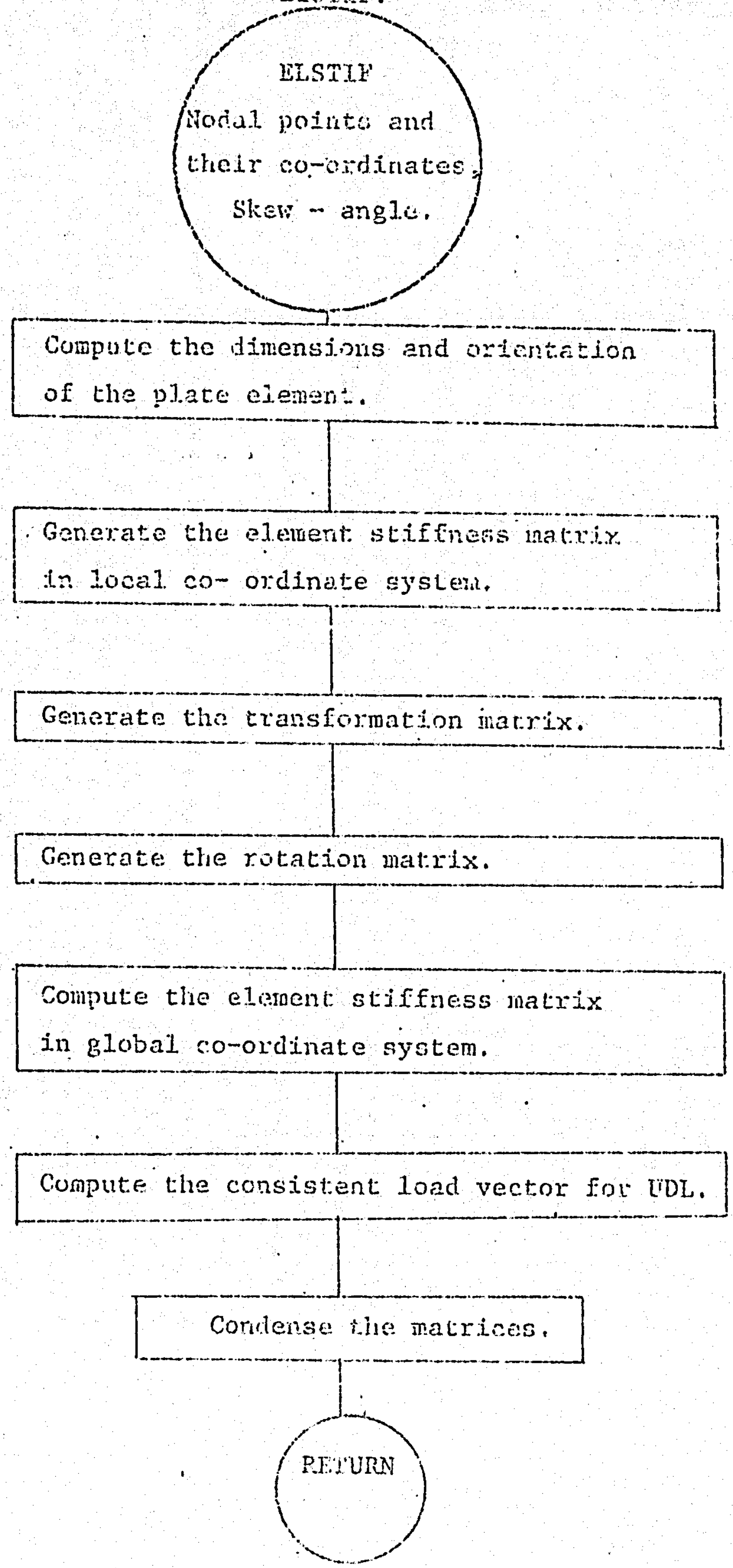


SUB - PROGRAM
BEAM.



SUB - PROGRAM

ELSTIF.



```
C*****************************************************************************
C     SUBROUTINE  FOR  THE  GENERATION  OF  THE  STIFFNESS  MATRIX  FOR
C   THE  SHALLOW  SHELL  ELEMENT  DEVELOPED  BY  COWPER ET AL.
C*****************************************************************************
      SUBROUTINE ELSTIF(X,Y,ES,VPL)                                     00001500
C*****************************************************************************
C     X -  AND  Y -  ARE THE CO - ORDINATES  OF  THE  PLATE  ELEMENT
C   ES   IS  THE  ELEMENT  STIFFNESS  MATRIX
C     VPL    IS    THE     LOAD   VECTOR ---  CONSTINT
C*****************************************************************************
      IMPLICIT REAL*8(A-H,O-Z)                                          00001510
      DIMENSION X(3),Y(3), M(20), N(20), MP(20), NQ(20), MR(20), NS(20),00001520
     .S(40,40), PT(40), T(40,40), RM(38,38), W(38,40), WS(38,40),PL(40),00001530
     . ES1(2,2), ES(36,36), VPL(36)                                     00001540
C*****************************************************************************
C   COMMON  BLOCK  FOR  THE  CASE  WHEN  THE  PLATE  IS  OF  UNIFORM
C   THICKNESS   AND  THE  MATERIAL  PROPERTIES  OF  THE  PLATE  AND  THE
C   STIFFENER  IS  THE  SAME.
C*****************************************************************************
      COMMON AR                                                         00001550
      COMMON ALFA,PR,TH                                                 00001560
C*****************************************************************************
C   ALFA  IS  THE  SKEW  ANGLE.
C*****************************************************************************
C*****************************************************************************
C     INITIALIZATION  OF  THE  MATRICES.
C*****************************************************************************
      DO 1000 I = 1,40                                                  00001570
      PT(I) = 0.0                                                       00001580
      PL(I) = 0.0                                                       00001590
      DO 1000 J = 1,40                                                  00001600
      S(I,J) = 0.0                                                      00001610
      T(I,J) = 0.0                                                      00001620
 1000  CONTINUE                                                         00001630
      DO 1001  I = 1,38                                                 00001640
      DO 1001  J = 1,40                                                 00001650
      W(I,J) = 0.0                                                      00001660
      WS(I,J) = 0.0                                                     00001670
 1001 CONTINUE                                                          00001680
      DO 1002  I = 1,38                                                 00001690
      DO 1002  J = 1,38                                                 00001700
      RM(I,J) = 0.0                                                     00001710
 1002 CONTINUE                                                          00001720
```

```
      DO 1003 I = 1,36
      VPL(I) = 0.0
      DO 1003 J = 1,36
      ES(I,J) = 0.0
 1003 CONTINUE
      DO 1004 I = 1,2
      DO 1004 J = 1,2
 1004 ES1(I,J) = 0.0
C***********************************************************************
C     CALCULATE   A    B    C
C   THE  DIMENSIONS  OF  THE  TRIANGLUAR  ELEMENT.
C***********************************************************************
      DATA M/0,1,0,2,1,0,3,2,1,0,0,0,0,0,0,0,0,0,0,0/
      DATA N/0,0,1,0,1,2,0,1,2,3,0,0,0,0,0,0,0,0,0,0/
      DATA NQ/0,0,0,0,0,0,0,0,0,0,0,1,0,1,2,0,1,2,3/
      DATA MP/0,0,0,0,0,0,0,0,0,0,1,0,2,1,0,3,2,1,0/
      DATA MR/0,1,0,2,1,0,3,2,1,0,4,3,2,1,0,5,3,2,1,0/
      DATA NS/0,0,1,0,1,2,0,1,2,3,0,1,2,3,4,0,2,3,4,5/
      X(1) = X(1) + Y(1)*DCOS(ALFA)
      Y(1) = Y(1)*DSIN(ALFA)
      X(2) = X(2) + Y(2)*DCOS(ALFA)
      Y(2) = Y(2)*DSIN(ALFA)
      X(3) = X(3) + Y(3)*DCOS(ALFA)
      Y(3) = Y(3)*DSIN(ALFA)
      XU = ((X(2) - X(1))**2)
      XV = ((Y(2) - Y(1))**2)
      XT = XU + XV
      RAD = DSQRT(XT)
      COST = (X(2) - X(1))/RAD
      SINT = (Y(2) - Y(1))/RAD
      A = ((X(2)-X(3))*(X(2)-X(1)) +(Y(2)-Y(3))*(Y(2)-Y(1)))/RAD
      B = ((X(3)-X(1))*(X(2)-X(1))+(Y(3)-Y(1))*(Y(2)-Y(1)))/RAD
      C = ((X(2)-X(1))*(Y(3)-Y(1)) - (X(3)-X(1))*(Y(2)-Y(1)))/RAD
C***********************************************************************
C     (K) MATRIX
C
C     PLANE STRESS CONTRIBUTION TO THE STIFFNESS MATRIX
C
C***********************************************************************
      IF(A.LT.0.01D-04) GO TO 81
      IF(B.LT.0.01D-04) GO TO 82
      GO TO 83
 81   A = 0.0D0
```

```
      GO TO 83                                                          00002140
   82 B = 0.0D0                                                         00002150
   83 PR1 = 0.5*(0.1D01 - PR)                                           00002160
      TH1 = TH**2/12.0                                                  00002190
      PR2 = 2.0*(0.1D01 - PR)                                           00002200
      DO 101 II = 1,20                                                  00002210
      I1 = II + 20                                                      00002220
      DO 101 JJ = 1,20                                                  00002230
      J1 = JJ + 20                                                      00002240
      PP1=M(II)*M(JJ)                                                   00002250
      IF (PP1) 102,103,102                                              00002260
  102 ICOM1=M(II)+M(JJ)-2                                               00002270
      IF (ICOM1) 103,104,104                                            00002280
  104 JCOM1=N(II)+N(JJ)                                                 00002290
      IF (JCOM1) 103,105,105                                            00002300
C***********************************************************************
C    SUBROUTINE   FACTOR   CALCULATES   THE   EULER'S  BETA  FUNCTION
C***********************************************************************
  105 CALL FACTOR (A,B,C,ICOM1,JCOM1,FUNC1)                             00002310
      P1=PP1*FUNC1                                                      00002320
      GO TO 106                                                         00002330
  103 P1=0.0                                                            00002340
  106 PP2=NQ(II)*NQ(JJ)                                                 00002350
      IF (PP2) 107,108,107                                              00002360
  107 ICOM2=MP(II)+MP(JJ)                                               00002370
      IF (ICOM2) 108,109,109                                            00002380
  109 JCOM2=NQ(II)+NQ(JJ)-2                                             00002390
      IF (JCOM2) 108,110,110                                            00002400
  110 CALL FACTOR (A,B,C,ICOM2,JCOM2,FUNC2)                             00002410
      P2=PP2*FUNC2                                                      00002420
      GO TO 111                                                         00002430
  108 P2=0.0                                                            00002440
  111 P3P = N(II)*N(JJ)                                                 00002450
      PP3 = PR1*P3P                                                     00002460
      IF (PP3) 112,113,112                                              00002470
  112 ICOM3=M(II)+M(JJ)                                                 00002480
      IF (ICOM3) 113,114,114                                            00002490
  114 JCOM3=N(II)+N(JJ)-2                                               00002500
      IF (JCOM3) 113,115,115                                            00002510
  115 CALL FACTOR (A,B,C,ICOM3,JCOM3,FUNC3)                             00002520
      P3=PP3*FUNC3                                                      00002530
      GO TO 116                                                         00002540
  113 P3=0.0                                                            00002550
```

```
116   P4P = MP(II)*MP(JJ)
      PP4 = PR1*P4P
      IF (PP4) 117,118,117
117   ICOM4 = MP(II) + MP(JJ) - 2
      IF (ICOM4) 118,119,119
  119 JCOM4=NQ(II)+NQ(JJ)
      IF (JCOM4) 118,120,120
  120 CALL FACTOR (A,B,C,ICOM4,JCOM4,FUNC4)
      P4=PP4*FUNC4
      GO TO 121
  118 P4=0.0
121   P5P1 = N(JJ)*MP(II)
      P5P2 = M(JJ)*NQ(II)
      PP5 = PR1*P5P1 + PR*P5P2
      IF (PP5) 122,123,122
  122 ICOM5=M(JJ)+MP(II)-1
      IF (ICOM5) 123,124,124
  124 JCOM5=N(JJ)+NQ(II)-1
      IF (JCOM5) 123,125,125
  125 CALL FACTOR (A,B,C,ICOM5,JCOM5,FUNC5)
      P5=PP5*FUNC5
      GO TO 126
  123 P5=0.0
126   P6P1 = N(II)*MP(JJ)
      P6P2 = M(II)*NQ(JJ)
      PP6 = PR1*P6P1 + PR*P6P2
      IF (PP6) 127,128,127
  127 ICOM6=M(II)+MP(JJ)-1
      IF (ICOM6) 128,129,129
  129 JCOM6=N(II)+NQ(JJ)-1
      IF (JCOM6) 128,130,130
  130 CALL FACTOR (A,B,C,ICOM6,JCOM6,FUNC6)
      P6=PP6*FUNC6
      GO TO 131
  128 P6=0.0
C*****************************************************************************
C
C     PLATE BENDING CONTRIBUTION TO THE STIFFNESS MATRIX
C
C*****************************************************************************
131   S(II,JJ) = P1 + P2 + P3 + P4 + P5 + P6
      P7P = MR(II)*MR(JJ)*(MR(II) - 1)*(MR(JJ) - 1)
      PP7 = TH1*P7P
```

```
      IF (PP7) 132,133,132
  132 ICOM7=MR(II)+MR(JJ)-4
      IF (ICOM7) 133,134,134
  134 JCOM7=NS(II)+NS(JJ)
      IF (JCOM7) 133,135,135
  135 CALL FACTOR (A,B,C,ICOM7,JCOM7,FUNC7)
      P7=PP7*FUNC7
      GO TO 136
  133 P7=0.0
136   P8P = NS(II)*NS(JJ)*(NS(II) - 1)*(NS(JJ) - 1)
      PP8 = TH1*P8P
      IF (PP8) 137,138,137
137   ICOM8 = MR(II) + MR(JJ)
      IF (ICOM8) 138,139,139
  139 JCOM8=NS(II)+NS(JJ)-4
      IF (JCOM8) 138,140,140
  140 CALL FACTOR (A,B,C,ICOM8,JCOM8,FUNC8)
      P8=PP8*FUNC8
      GO TO 141
  138 P8=0.0
141   P9P1 = MR(II)*MR(JJ)*NS(II)*NS(JJ)
      P9P2 = MR(II)*NS(JJ)*(MR(II) - 1)*(NS(JJ) - 1)
      P9P3 = MR(JJ)*NS(II)*(MR(JJ) - 1)*(NS(II) - 1)
      PP9 = TH1*(PR2*P9P1 + PR*P9P2 + PR*P9P3)
  142 ICOM9=MR(II)+MR(JJ)-2
       IF (ICOM9) 143,144,144
  144 JCOM9=NS(II)+NS(JJ)-2
      IF (JCOM9) 143,145,145
  145 CALL FACTOR (A,B,C,ICOM9,JCOM9,FUNC9)
      P9=PP9*FUNC9
      GO TO 101
  143 P9=0.0
C*****************************************************************************
C    CURVATURE TERMS CAN BE ADDED TO DEAL WITH THE SHELL PROBLEMS
C*****************************************************************************
101   S(I1,J1) = P7 + P8 + P9
C*************************************************************************
C      LOAD        VECTOR        CONTRIBUTION
C*************************************************************************
C*****************************************************************************
C   TRANSVERSE LOAD FOR PLATE ELEMENT.
C*****************************************************************************
      DO 786 IL = 1,20
```

```
      IL1 = IL + 20
      IVEC = MR(IL)
      JVEC = NS(IL)
      CALL FACTOR(A,B,C,IVEC,JVEC,FUNC)
      PT(IL1) = FUNC
  786  CONTINUE
C*************************************************************************
C  THE  LOAD  VECTOR  FOR  THE  IN - PLANE  LOAD FOR  THE  PLATE  ELEMENT  CAN
C  BE  ADDED  IF  NEED  BE.
C**********************************************************************
C     TRANSFORMATION MATRIX
C**********************************************************************
      I = 1
      Z = -B
   36 T(I,1)=1.0
      T(I,2) = Z
      T(I,4) = Z**2
      T(I,7) = Z**3
      I=I+1
      T(I,2)=1.0
      T(I,4) = 2.0*Z
      T(I,7) = 3.0*Z**2
      I=I+1
      T(I,3)=1.0
      T(I,5) = Z
      T(I,8) = Z**2
      I=I+10
      Z = A
      IF (I.EQ.13) GO TO 36
       T(25,1)=1.0
       T(25,3)=C
       T(25,6)=C**2
       T(25,10)=C**3
       T(26,2)=1.0
       T(26,5)=C
      T(26,9) = C**2
      T(27,3) = 1.0
       T(27,6)=2.0*C
       T(27,10)=3.0*C**2
       T(37,1)=1.0
      T(37,2)=(A-B)/3.0
      T(37,3)=C/3.0
      T(37,4)=(A-B)**2/9.0
```

```
      T(37,5)=(A-B)*C/9.0
      T(37,6)=C**2/9.0
      T(37,7)=(A-B)**3/27.0
      T(37,8)=(A-B)**2*C/27.0
      T(37,9)=(A-B)*C**2/27.0
      T(37,10)=C**3/27.0
      DO 202 II=1,3
      II1=II+3
      DO 202 JJ=1,10
      JJ1=JJ+10
      T(II1,JJ1)=T(II,JJ)
  202 CONTINUE
      DO 203 II = 13,15
      II1 = II + 3
      DO 203 JJ=1,10
      JJ1=JJ+10
      T(II1,JJ1)=T(II,JJ)
  203 CONTINUE
      DO 204 II=25,27
      II1=II+3
      DO 204 JJ=1,10
      JJ1=JJ+10
      T(II1,JJ1)=T(II,JJ)
  204 CONTINUE
      DO 205 JJ=1,10
      JJ1=JJ+10
      T(38,JJ1) = T(37,JJ)
  205 CONTINUE
      I=7
      Z = -B
   37 T(I,21)=1.0
      T(I,22) = Z
      T(I,24) = Z**2
      T(I,27) = Z**3
      T(I,31) = Z**4
      T(I,36) = Z**5
      I=I+1
      T(I,22)=1.0
      T(I,24) = 2.0*Z
      T(I,27) = 3.0*Z**2
      T(I,31) = 4.0*Z**3
      T(I,36) = 5.0*Z**4
      I=I+1
```

```
T(I,23)=1.0
T(I,25) = Z
T(I,28) = Z**2
T(I,32) = Z**3
I=I+1
T(I,24)=2.0
T(I,27) = 6.0*Z
T(I,31) = 12.0*Z**2
T(I,36) = 20.0*Z**3
I=I+1
T(I,25)=1.0
T(I,28) = 2.0*Z
T(I,32) = 3.0*Z**2
I=I+1
T(I,26)=2.0
T(I,29) = 2.0*Z
T(I,33) = 2.0*Z**2
T(I,37) = 2.0*Z**3
I=I+7
Z = A
IF(I.EQ.19) GO TO 37
T(31,21)=1.0
T(31,23)=C
T(31,26)=C**2
T(31,30)=C**3
T(31,35) = C**4
T(31,40)=C**5
T(32,22)=1.0
T(32,25)=C
T(32,29) = C**2
T(32,34) = C**3
T(32,39) = C**4
T(33,23)=1.0
T(33,26)=2.0*C
T(33,30)=3.0*C**2
T(33,35)=4.0*C**3
T(33,40)=5.0*C**4
T(34,24)=2.0
T(34,28)=2.0*C
T(34,33)=2.0*C**2
T(34,38)=2.0*C**3
T(35,25) = 1.0
T(35,29) = 2.0*C
```

```
      T(35,34)=3.0*C**2
      T(35,39) = 4.0*C**3
      T(36,26)=2.0
      T(36,30)=6.0*C
      T(36,35)=12.0*C**2
      T(36,40)=20.0*C**3
      T(39,36)=5.0*A**4*C
      T(39,37)=3.0*A**2*C**3-2.0*A**4*C
      T(39,38)=-2.0*A*C**4+3.0*A**3*C**2
      T(39,39)=C**5-4.0*A**2*C**3
      T(39,40)=5.0*A*C**4
      T(40,36) = 5.0*B**4*C
      T(40,37)=3.0*B**2*C**3-2.0*B**4*C
      T(40,38)=2.0*B*C**4-3.0*B**3*C**2
      T(40,39)=C**5-4.0*B**2*C**3
      T(40,40)=-5.0*B*C**4
      CALL MATINV(T,40)
C***********************************************************************
C     ROTATION MATRIX
C***********************************************************************
      CTH = COST
      STH = SINT
      CTH2 = COST*COST
      STH2 = SINT*SINT
      SCTH = COST*SINT
      RM(1,1)=CTH
      RM(1,4)=STH
      RM(2,2)=CTH2
      RM(2,3)=SCTH
      RM(2,5)=SCTH
      RM(2,6)=STH2
      RM(3,2) = -SCTH
      RM(3,3)=CTH2
      RM(3,5)=-STH2
      RM(3,6)=SCTH
      RM(4,1)=-STH
      RM(4,4)=CTH
      RM(5,2)=-SCTH
      RM(5,3)=-STH2
      RM(5,5)=CTH2
      RM(5,6)=SCTH
      RM(6,2)=STH2
      RM(6,3)=-SCTH
```

```
      RM(6,5)=-SCTH
      RM(6,6)=CTH2
      DO 302 II=1,6
      II1=II+12
      II2=II+24
      DO 302 JJ=1,6
      JJ1=JJ+12
      JJ2=JJ+24
      RM (II1,JJ1)=RM(II,JJ)
      RM(II2,JJ2) = RM(II,JJ)
  302 CONTINUE
      RM(7,7)=1.0
      RM(8,8)=CTH
      RM(8,9)=STH
      RM(9,8)=-STH
      RM(9,9)=CTH
      RM(10,10)=CTH2
      RM(10,11)=2.0*SCTH
      RM(10,12)=STH2
      RM(11,10)=-SCTH
      RM(11,11)=CTH2-STH2
      RM(11,12)=SCTH
      RM(12,10) = STH2
      RM(12,11)=-2.0*SCTH
      RM(12,12)=CTH2
      DO 303 II = 7,12
      II1=II+12
      II2=II+24
      DO 303 JJ=7,12
      JJ1=JJ+12
      JJ2=JJ+24
      RM(II1,JJ1)=RM(II,JJ)
      RM(II2,JJ2) = RM(II,JJ)
  303 CONTINUE
      RM(37,37)=CTH
      RM(37,38)=STH
      RM(38,37)=-STH
      RM(38,38)=CTH
      CALL MATMUT(RM,T,W,38,38,40)
      CALL MATMUL(W,S,WS,38,40,40)
      CALL MATMUX(W,PT,PL,38,40)
      CALL MATMUL(WS,T,W,38,40,38)
      CALL MATMUL(W,RM,WS,38,38,38)
```

```
                C*****************************************************************00005460
                C      CONDENSATION  OF   THE   MATRICES                            00005470
                C*****************************************************************00005480
0373                 ES1(1,1) = WS(37,37)                                           00005490
0374                 ES1(1,2) = WS(37,38)                                           00005500
0375                 ES1(2,1) = WS(38,37)                                           00005510
0376                 ES1(2,2) = WS(38,38)                                           00005520
0377                 CALL MATINV(ES1,2)                                             00005530
0378                 DO 501 I = 1,36                                                00005540
0379                 DO 501 J = 1,36                                                00005550
0380                 ES(I,J) = WS(I,J)-(WS(I,37)*ES1(1,1)*WS(37,J)+WS(I,38)*ES1(2,1)*  00005560
                    .WS(37,J)+WS(I,37)*ES1(1,2)*WS(38,J)+WS(I,38)*ES1(2,2)*WS(38,J))   00005570
0381            501  CONTINUE                                                       00005580
                C*****************************************************************
                C   YONE    IS    THE    YOUNGE'S    MODULUS
                C*****************************************************************
0382                 YONE = 0.450D00
0383                 YONE1 = YONE*TH/(1.0 - PR**2)                                  00005600
0384                 DO 505 I = 1,36                                                00005610
0385                 DO 505 J = 1,36                                                00005620
0386                 ES(I,J) = ES(I,J)*YONE1                                        00005630
0387            505  CONTINUE                                                       00005640
0388                 DO 502 I = 1,36                                                00005650
0389                 VPL(I) = PL(I)-(WS(I,37)*ES1(1,1)*PL(37)+WS(I,38)*ES1(2,1)*PL(37)  00005660
                    .+WS(I,37)*ES1(1,2)*PL(38)+WS(I,38)*ES1(2,2)*PL(38))            00005670
0390            502  CONTINUE                                                       00005680
0391                 RETURN                                                         00005690
0392                 END                                                            00005700
```

```
      C*****************************************************************************
      C   SUBROUTINE  FOR  THE   GENERATION  OF  THE  BEAM  ELEMENT  STIFFNESS
      C     MATRIX.
      C   BEAM   TORSION  IS  UNCOUPLED.
      C*****************************************************************************
0001           SUBROUTINE BEAM(NC,X1,X2,A,S,XI,BM1,M1,BT2,N1)                  00005710
      C*****************************************************************************
      C   X1 - AND  X2 -   ARE  THE  CO - ORDINATES  OF  THE  BEAM ELEMENT
      C*****************************************************************************
      C*****************************************************************************
      C   A  IS  THE  AREA  OF  THE  BEAM  ELEMENT.
      C   S   IS   THE  FIRST  MOMENT  OF  THE  AREA  WITH  RESPECT  TO  THE
      C    MID  PLANE   OF  THE  PLATE.
      C   XI  IS  THE  SECOND  MOMENT  OF  AREA  OF  THE  BEAM  ELEMENT  WITH
      C   RESPECT  TO  THE  MID - PLANE  OF  THE  PLATE.
      C*****************************************************************************
0002           IMPLICIT REAL*8(A-H,O-Z)                                         00005720
0003           DIMENSION T1(10,10), BM(10,10), SM(10,10), BM1(M1,M1), BT2(N1,N1) 00005730
0004           DIMENSION T3(4,4), BT(4,4), BT1(4,4)                             00005740
      C*****************************************************************************
      C   COMMON  BLOCK  FOR  THE  CASE  WHEN  THE  PLATE  IS  OF  UNIFORM
      C   THICKNESS   AND  THE  MATERIAL  PROPERTIES  OF  THE  PLATE  AND  THE
      C   STIFFENER  IS  THE  SAME.
      C*****************************************************************************
0005           COMMON AR                                                        00005750
0006           COMMON ALFA, PR,TH                                               00005760
      C*****************************************************************************
      C   INITIALIZATION   OF   THE   MATRICES
      C*****************************************************************************
0007           DO 1000 I = 1,10                                                 00005770
0008           DO 1000 J = 1,10                                                 00005780
0009           BM1(I,J) = 0.0                                                   00005790
0010           T1(I,J) = 0.0                                                    00005800
0011           BM(I,J) = 0.0                                                    00005810
0012           SM(I,J) = 0.0                                                    00005820
0013      1000 CONTINUE                                                         00005830
0014           DO 1001 I = 1,4                                                  00005840
0015           DO 1001 J = 1,4                                                  00005850
0016           T3(I,J) = 0.0                                                    00005860
0017           BT(I,J) = 0.0                                                    00005870
0018           BT2(I,J) = 0.0                                                   00005880
0019      1001 CONTINUE                                                         00005890
0020           BL = DABS(X2 - X1)*AR                                            00005900
```

```
      BL2 = BL*BL                                                       00005910
      BL3 = BL2*BL                                                      00005920
      BL4 = BL3*BL                                                      00005930
      BL5 = BL4*BL                                                      00005940
      BL6 = BL5*BL                                                      00005950
      BL7 = BL6*BL                                                      00005960
C*************************************************************************
C     TRANSFORMATION   MATRIX
C*************************************************************************
      T1(1,1) = 1.0                                                     00005970
      T1(2,2) = 1.0                                                     00005980
      T1(3,5) = 1.0                                                     00005990
      T1(4,6) = 1.0                                                     00006000
      T1(5,7) = 2.0                                                     00006010
      T1(6,1) = 1.0                                                     00006020
      T1(6,2) = BL                                                      00006030
      T1(6,3) = BL2                                                     00006040
      T1(6,4) = BL3                                                     00006050
      T1(7,2) = 1.0                                                     00006060
      T1(7,3) = 2.0*BL                                                  00006070
      T1(7,4) = 3.0*BL2                                                 00006080
      T1(8,5) = 1.0                                                     00006090
      T1(8,6) = BL                                                      00006100
      T1(8,7) = BL2                                                     00006110
      T1(8,8) = BL3                                                     00006120
      T1(8,9) = BL4                                                     00006130
      T1(8,10) = BL5                                                    00006140
      T1(9,6) = 1.0                                                     00006150
      T1(9,7) = 2.0*BL                                                  00006160
      T1(9,8) = 3.0*BL2                                                 00006170
      T1(9,9) = 4.0*BL3                                                 00006180
       T1(9,10) = 5.0*BL4                                               00006190
      T1(10,7) = 2.0                                                    00006200
      T1(10,8) = 6.0*BL                                                 00006210
      T1(10,9) = 12.0*BL2                                               00006220
      T1(10,10) = 20.0*BL3                                              00006230
C*************************************************************************
C     GENERATION   OF  THE  STIFFNESS  MATRIX  (  KB  )
C*************************************************************************
      BM(2,2) = A*BL                                                    00006240
      BM(2,3) = A*BL2                                                   00006250
      BM(2,4) = A*BL3                                                   00006260
      BM(3,3) = (4.0/3.0)*A*BL3                                         00006270
```

```
      BM(3,4) = (3.0/2.0)*A*BL4
      BM(4,4) = (9.0/5.0)*A*BL5
      BM(2,7) =-2.0*S*BL
      BM(2,8) =-3.0*S*BL2
      BM(2,9) =-4.0*S*BL3
      BM(2,10) =-5.0*S*BL4
      BM(3,7) =-2.0*S*BL2
      BM(3,8) =-4.0*S*BL3
      BM(3,9) =-6.0*S*BL4
      BM(3,10) =-8.0*S*BL5
      BM(4,7) =-2.0*S*BL3
      BM(4,8) =-(9.0/2.0)*S*BL4
      BM(4,9) =-(36.0/5.0)*S*BL5
      BM(4,10) =-10.0*S*BL6
      BM(7,7) = 4.0*XI*BL
      BM(7,8) = 6.0*XI*BL2
      BM(7,9) = 8.0*XI*BL3
      BM(7,10) = 10.0*XI*BL4
      BM(8,8) = 12.0*XI*BL3
      BM(8,9) = 18.0*XI*BL4
      BM(8,10) = 24.0*XI*BL5
      BM(9,9) = (144.0/5.0)*XI*BL5
      BM(9,10) = 40.0*XI*BL6
      BM(10,10) = (400.0/7.0)*XI*BL7
      DO 601 I = 1,10
      DO 601 J = 1,10
      BM(J,I) = BM(I,J)
  601 CONTINUE
      CALL MATINV(T1,10)
      CALL MATMIT(T1,BM,SM,10,10,10)
C***********************************************************************
C   YONE   IS   THE   YOUNGE'S   MODULUS
C***********************************************************************
      YONE = 0.450D00
      CALL MATMUL(SM,T1,BM1,10,10,10)
      DO 602 I = 1,10
      DO 602 J = 1,10
      BM1(I,J) = BM1(I,J)*YONE
 602  CONTINUE
C***********************************************************************
C   CONTRIBUTIONS   OF   THE   TORSIONAL   PROPERTIES   OF   THE   BEAM.
C***********************************************************************
C***********************************************************************
```

```
C     TRANSFORMATION    MATRIX
C*****************************************************************************
      T3(1,1) = 1.0
      T3(2,2) = 1.0
      T3(3,1) = 1.0
      T3(3,2) = BL
      T3(3,3) = BL2
      T3(3,4) = BL3
      T3(4,2) = 1.0
      T3(4,3) = 2.0*BL
      T3(4,4) = 3.0*BL2
      CALL MATINV(T3,4)
C*****************************************************************************
C     GENERATION   OF   THE   STIFFNESS   MATRIX
C*****************************************************************************
      DO 605 I = 1,4
      DO 605 J = 1,4
      AM = I - 1
      AN = J - 1
      XX = I + J - 3
      IF(XX) 5,8,5
5     BT(I,J) = AM*AN*(BL**XX)/XX
      GO TO 605
8     BT(I,J) = 0.0
605   CONTINUE
      CALL MATMIT(T3,BT,BT1,4,4,4)
      CALL MATMUL(BT1,T3,BT2,4,4,4)
C*****************************************************************************
C    YOG   IS   THE   MODULUS   OF   RIGIDITY
C   TOJ   IS   THE   TORSIONAL   CONSTANT
C*****************************************************************************
      TOJ = (0.1D01*0.5D0**3)/0.3D01
      YOG = YONE/(2.0*(1.0 + PR))
      DO 606 I = 1,4
      DO 606 J = 1,4
      BT2(I,J) = TOJ*YOG*BT2(I,J)
606   CONTINUE
      RETURN
      END
```

```
C*******************************************************************************
C   SUBROUTINE  FOR  STORING  THE  LOWER  BAND  OF  THE  MASTER  STIFFNESS
C   MATRIX  ROW  BY  ROW  IN A  ONE  DIMENSIONAL  ARRAY ( A ).
C*******************************************************************************
      SUBROUTINE BUILD(I,J,K,A,M1,D,MM)
C*******************************************************************************
C   D(36,36)  IS  THE STIFFFNESS MATRIX  FOR  THE  PLATE  ELEMENTFOR
C   PLATE  BENDING  AND  PLATE  IN - PLANE  STRESSES.
C   MM  IS  THE  HALF  BAND  WIDTH  OF  THE  STRUCTURE  STIFFNESSMATRIX
C    I,  J,  K  ARE  THE  NODAL  POINTS.
C*******************************************************************************
      IMPLICIT REAL*8(A-H,O-Z)
      DIMENSION A(M1), D(36,36)
      IR = MM*(MM + 1)/2
      DO 1 M = 1,12
      I1 = 12*I - 12 + M
      J1 = 12*J - 12 + M
      K1 = 12*K - 12 + M
      IR1 = (I1-1)*I1/2
      IR2 = (J1 - 1)*J1/2
      IR3 = (K1 - 1)*K1/2
      DO 1 N = 1,12
      I2 = 12*I - 12 + N
      J2 = 12*J - 12 + N
      K2 = 12*K - 12 + N
      IF(N.GT.M) GO TO 2
      IF(I1.GT.MM) GO TO 3
      KK = IR1 + I2
      GO TO 4
3     KK = I1*MM + I2 - IR
4     A(KK) = A(KK) + D(M,N)
      IF(J1.GT.MM) GO TO 5
      KK = IR2 + J2
      GO TO 6
5     KK = J1*MM + J2 - IR
6     A(KK) = A(KK) + D(M+12,N+12)
      IF(K1.GT.MM) GO TO 7
      KK = IR3 + K2
      GO TO 8
7     KK = K1*MM + K2 - IR
  8   A(KK) = A(KK) + D(M+24,N+24)
2     IF(I2.GT.J1) GO TO 10
      IF(J1.GT.MM) GO TO 21
```

```
      KK = IR2 + I2
      GO TO 22
21    KK = J1*MM + I2 - IR
22    A(KK) = A(KK) + D(M+12,N)
10    IF(I2.GT.K1) GO TO 11
      IF(K1.GT.MM) GO TO 31
      KK = IR3 + I2
      GO TO 32
31    KK = K1*MM + I2 - IR
32    A(KK) = A(KK) + D(M+24,N)
11    IF(J2.GT.I1) GO TO 12
      IF(I1.GT.MM) GO TO 41
      KK = IR1 + J2
      GO TO 42
41    KK = I1*MM + J2 - IR
42    A(KK) = A(KK) + D(M,N+12)
12    IF(J2.GT.K1) GO TO 13
      IF(K1.GT.MM) GO TO 51
      KK = IR3 + J2
      GO TO 52
51    KK = K1*MM + J2 - IR
52    A(KK) = A(KK) + D(M+24,N+12)
13    IF(K2.GT.I1) GO TO 14
      IF(I1.GT.MM) GO TO 61
      KK = IR1 + K2
      GO TO 62
61    KK = I1*MM + K2 - IR
62    A(KK) = A(KK) + D(M,N+24)
14    IF(K2.GT.J1) GO TO 1
      IF(J1.GT.MM) GO TO 71
      KK = IR2 + K2
      GO TO 72
71    KK = J1*MM + K2 - IR
72    A(KK) = A(KK) + D(M+12,N+24)
1     CONTINUE
      RETURN
      END
```

```
C*****************************************************************************
C    SUBROUTINE  FOR  ADDING  THE  CONTRIBUTION  OF THE BEAM  ELEMENT
C  (  BENDING  AND  AXIAL  DEFORMATION  )  IN  THE  X - DIRECTION.
C*****************************************************************************
      SUBROUTINE SETUP(I,J,A,M1,D,MM)
C*****************************************************************************
C   I,  J   ARE   THE   NODAL   POINTS   OF  THE  BEAM   ELEMENT
C      MM   IS   THE  HALF  BAND  WIDTH  OF  THE  STRUCTURE  STIFFNESS  MATRIX
C  D   IS   THE  BEAM  ELEMENT  STIFFNESS  MATRIX
C*****************************************************************************
      IMPLICIT REAL*8(A-H,O-Z)
      DIMENSION A(M1), D(10,10)
      IR = MM*(MM + 1)/2
      DO 1 M = 1,5
      MNM = M
      IF(M.EQ.3.OR.M.EQ.4) MNM = M + 4
      IF(M.EQ.5) MNM = M + 5
      I1 = 12*I - 12 + MNM
      J1 = 12*J - 12 + MNM
      IR1 = (I1-1)*I1/2
      IR2 = (J1 - 1)*J1/2
      DO 1 N = 1,5
      NMN = N
      IF(N.EQ.3.OR.N.EQ.4) NMN = N + 4
      IF(N.EQ.5) NMN = N + 5
      I2 = 12*I - 12 + NMN
      J2 = 12*J - 12 + NMN
      IF(N.GT.M) GO TO 2
      IF(I1.GT.MM) GO TO 3
      KK = IR1 + I2
      GO TO 4
    3 KK = I1*MM + I2 - IR
    4 A(KK) = A(KK) + D(M,N)
      IF(J1.GT.MM) GO TO 5
      KK = IR2 + J2
      GO TO 6
    5 KK = J1*MM + J2 - IR
    6 A(KK) = A(KK) + D(M+5,N+5)
    2 IF(I2.GT.J1) GO TO 10
      IF(J1.GT.MM) GO TO 21
      KK = IR2 + I2
      GO TO 22
   21 KK = J1*MM + I2 - IR
```

```
22    A(KK) = A(KK) + D(M+5,N)
10    IF(J2.GT.I1) GO TO 1
      IF(I1.GT.MM) GO TO 41
      KK = IR1 + J2
      GO TO 42
41    KK = I1*MM + J2 - IR
42    A(KK) = A(KK) + D(M,N+5)
1     CONTINUE
      RETURN
      END
```

```
          C**********************************************************************
          C   SUBROUTINE  FOR  ADDING  THE  CONTRIBUTION  OF THE BEAM  ELEMENT
          C  (  TORSION  )  IN THE  X - DIRECTION.
          C**********************************************************************
0001            SUBROUTINE  STUP(I,J,A,M1,D,MM)                              00009470
          C**********************************************************************
          C   I,  J   ARE   THE   NODAL   POINTS   OF  THE  BEAM   ELEMENT
          C     MM   IS   THE  HALF  BAND  WIDTH  OF  THE  STRUCTURE  STIFFNESS  MATRIX
          C  D   IS   THE  BEAM  ELEMENT  STIFFNESS  MATRIX
          C**********************************************************************
0002            IMPLICIT REAL*8(A-H,O-Z)                                     00009480
0003            DIMENSION A(M1),D(4,4)                                       00009490
0004            IR = MM*(MM + 1)/2                                           00009500
0005            DO 1 M = 1,2
0006            MNM = M                                                      00009520
0007            IF(M.EQ.1) MNM = M + 8                                       00009530
0008            IF(M.EQ.2) MNM = M + 9                                       00009540
0009            I1 = 12*I - 12 + MNM                                         00009550
0010            J1 = 12*J - 12 + MNM                                         00009560
0011            IR1 = (I1-1)*I1/2                                            00009570
0012            IR2 = (J1 - 1)*J1/2                                          00009580
0013            DO 1 N = 1,2
0014            NMN = N                                                      00009600
0015            IF(N.EQ.1) NMN = N + 8                                       00009610
0016            IF(N.EQ.2) NMN = N + 9                                       00009620
0017            I2 = 12*I - 12 + NMN                                         00009630
0018            J2 = 12*J - 12 + NMN                                         00009640
0019            IF(N.GT.M) GO TO 2                                           00009650
0020            IF(I1.GT.MM) GO TO 3                                         00009660
0021            KK = IR1 + I2                                                00009670
0022            GO TO 4                                                      00009680
0023      3     KK = I1*MM + I2 - IR                                         00009690
0024      4     A(KK) = A(KK) + D(M,N)                                       00009700
0025            IF(J1.GT.MM) GO TO 5                                         00009710
0026            KK = IR2 + J2                                                00009720
0027            GO TO 6                                                      00009730
0028      5     KK = J1*MM + J2 - IR                                         00009740
0029      6     A(KK) = A(KK) + D(M+2,N+2)                                   00009750
0030      2     IF(I2.GT.J1) GO TO 10                                        00009760
0031            IF(J1.GT.MM) GO TO 21                                        00009770
0032            KK = IR2 + I2                                                00009780
0033            GO TO 22                                                     00009790
0034      21    KK = J1*MM + I2 - IR                                         00009800
```

```
0035        22    A(KK) = A(KK) + D(M+2,N)                              00009810
0036        10    IF(J2.GT.I1) GO TO 1                                  00009820
0037              IF(I1.GT.MM) GO TO 41                                 00009830
0038              KK = IR1 + J2                                         00009840
0039              GO TO 42                                              00009850
0040        41    KK = I1*MM + J2 - IR                                  00009860
0041        42    A(KK) = A(KK) + D(M,N+2)                              00009870
0042        1     CONTINUE                                              00009880
0043              RETURN                                                00009890
0044              END                                                   00009900
```

```
              C***************************************************************************
              C   SUBROUTINE  FOR  ADDING  THE  CONTRIBUTION  OF THE BEAM  ELEMENT
              C  (  BENDING  AND  AXIAL  DEFORMATION  )  IN  THE  Y - DIRECTION.
              C***************************************************************************
 0001                SUBROUTINE SETUP1(I,J,A,M1,D,MM)
              C***************************************************************************
              C   I,  J   ARE  THE   NODAL   POINTS   OF  THE  BEAM   ELEMENT
              C     MM   IS   THE  HALF  BAND  WIDTH  OF  THE  STRUCTURE  STIFFNESS  MATRIX
              C  D   IS   THE  BEAM  ELEMENT  STIFFNESS  MATRIX
              C***************************************************************************
 0002                IMPLICIT REAL*8(A-H,O-Z)                                      00000020
 0003                DIMENSION A(M1), D(10,10)                                     00000030
 0004                IR = MM*(MM + 1)/2                                            00000040
 0005                DO 1 M = 1,5                                                  00000050
 0006                MNM = M                                                       00000060
 0007                IF(M.EQ.1) MNM = M + 3
 0008                IF(M.EQ.2.OR.M.EQ.3) MNM = M + 4
 0009                IF(M.EQ.4) MNM = M + 5
 0010                IF(M.EQ.5) MNM = M + 7
 0011                I1 = 12*I - 12 + MNM                                          00000090
 0012                J1 = 12*J - 12 + MNM                                          00000100
 0013                IR1 = (I1-1)*I1/2                                             00000110
 0014                IR2 = (J1 - 1)*J1/2                                           00000120
 0015                DO 1 N = 1,5                                                  00000130
 0016                NMN = N                                                       00000140
 0017                IF(N.EQ.1) NMN = N + 3
 0018                IF(N.EQ.2.OR.N.EQ.3) NMN = N + 4
 0019                IF(N.EQ.4) NMN = N + 5
 0020                IF(N.EQ.5) NMN = N + 7
 0021                I2 = 12*I - 12 + NMN                                          00000170
 0022                J2 = 12*J - 12 + NMN                                          00000180
 0023                IF(N.GT.M) GO TO 2                                            00000190
 0024                IF(I1.GT.MM) GO TO 3                                          00000200
 0025                KK = IR1 + I2                                                 00000210
 0026                GO TO 4                                                       00000220
 0027          3     KK = I1*MM + I2 - IR                                          00000230
 0028          4     A(KK) = A(KK) + D(M,N)                                        00000240
 0029                IF(J1.GT.MM) GO TO 5                                          00000250
 0030                KK = IR2 + J2                                                 00000260
 0031                GO TO 6                                                       00000270
 0032          5     KK = J1*MM + J2 - IR                                          00000280
 0033          6     A(KK) = A(KK) + D(M+5,N+5)                                    00000290
 0034          2     IF(I2.GT.J1) GO TO 10                                         00000300
```

```
      IF(J1.GT.MM) GO TO 21
      KK = IR2 + I2
      GO TO 22
21    KK = J1*MM + I2 - IR
22    A(KK) = A(KK) + D(M+5,N)
10    IF(J2.GT.I1) GO TO 1
      IF(I1.GT.MM) GO TO 41
      KK = IR1 + J2
      GO TO 42
41    KK = I1*MM + J2 - IR
42    A(KK) = A(KK) + D(M,N+5)
1     CONTINUE
      RETURN
      END
```

```
C*****************************************************************************
C   SUBROUTINE  FOR  ADDING  THE  CONTRIBUTION  OF THE BEAM  ELEMENT
C  (  TORSION  )  IN THE  Y - DIRECTION.
C*****************************************************************************
      SUBROUTINE STUP1(I,J,A,M1,D,MM)
C*****************************************************************************
C   I,  J   ARE   THE    NODAL   POINTS   OF  THE  BEAM   ELEMENT
C     MM   IS   THE  HALF  BAND  WIDTH  OF  THE  STRUCTURE  STIFFNESS  MATRIX
C*****************************************************************************
      IMPLICIT REAL*8(A-H,O-Z)                                          00000460
      DIMENSION A(M1),D(4,4)                                            00000470
      IR = MM*(MM + 1)/2                                                00000480
      DO 1 M = 1,2
      MNM = M                                                           00000500
      IF(M.EQ.1) MNM = M + 7
      IF(M.EQ.2) MNM = M + 9
      I1 = 12*I - 12 + MNM                                              00000530
      J1 = 12*J - 12 + MNM                                              00000540
      IR1 = (I1-1)*I1/2                                                 00000550
      IR2 = (J1 - 1)*J1/2                                               00000560
      DO 1 N = 1,2
      NMN = N                                                           00000580
      IF(N.EQ.1) NMN = N + 7
      IF(N.EQ.2) NMN = N + 9
      I2 = 12*I - 12 + NMN                                              00000610
      J2 = 12*J - 12 + NMN                                              00000620
      IF(N.GT.M) GO TO 2                                                00000630
      IF(I1.GT.MM) GO TO 3                                              00000640
      KK = IR1 + I2                                                     00000650
      GO TO 4                                                           00000660
    3 KK = I1*MM + I2 - IR                                              00000670
    4 A(KK) = A(KK) + D(M,N)                                            00000680
      IF(J1.GT.MM) GO TO 5                                              00000690
      KK = IR2 + J2                                                     00000700
      GO TO 6                                                           00000710
    5 KK = J1*MM + J2 - IR                                              00000720
    6 A(KK) = A(KK) + D(M+2,N+2)                                        00000730
    2 IF(I2.GT.J1) GO TO 10                                             00000740
      IF(J1.GT.MM) GO TO 21                                             00000750
      KK = IR2 + I2                                                     00000760
      GO TO 22                                                          00000770
   21 KK = J1*MM + I2 - IR                                              00000780
   22 A(KK) = A(KK) + D(M+2,N)                                          00000790
```

```
   10 IF(J2.GT.I1) GO TO 1
      IF(I1.GT.MM) GO TO 41
      KK = IR1 + J2
      GO TO 42
   41 KK = I1*MM + J2 - IR
   42 A(KK) = A(KK) + D(M,N+2)
    1 CONTINUE
      RETURN
      END
```

```
      C     SUBROUTINE FOR MATRIX INVERSION
      C                 .   IDENTIFICATION--SLVBSR
      C
      C                 SOLUTION OF BANDED,SYMETRIC SYSTEM OF EQUATION AX=B
      C        WRITTEN BY   WONG KAM    NRC  COMPUTATION CENTRE  JULY 71
      C
      C         METHOD-THE CROUT FACTORIZATION A=LU IS USED
      C                THE SYSTEM AX=B IS EQUIVALENT TO THE SYSTEMS LY=B AND U
      C        STORAGE-
      C               THE LOWER BAND OF THE ORIGINAL MATRIX A IS STORED ROWWIS
      C               IN AN ONE DIMENSIONAL ARRAY WO
      C        ENTRY-
      C             M  IS THE HALF-BANDWIDTH  (A(I,J)=0 FOR (ABS(I-J) GT M))
      C             N  IS THE NUMBER OF EQUATIONS
      C             PREC  SPECIFIES MINIMUM VALUE OF PIVOT
      C             IF PREC IS ZERO THE SUBROUTINE ASSUMES THE DEFAULT VALUE 0
      C             WO CONTAINS THE LOWER BAND OF THE MATRIX OF COEFFICIENTS
      C             DIMENSION OF WO  MUST BE GREATER THAN OR EQUAL
      C             TO (M+1)*(2*N-M)/2
      C             B CONTAINS THE RHS VECTOR WITH DIMENSION GREATER THAN OR E
      C             TO N
      C             TEMP IS A VECTOR OF DIMENSION GREATER THAN OR EQUAL TO M U
      C             IN SLVBSR FOR STORING SOME INTERMEDIATE CALCULATIONS
      C     RETURN- TEST EQUAL TO 0.DO INDICATES NORMAL COMPLETION
      C                B CONTAINS THE SOLUTION VECTOR
      C                WO CONTAINS THE LOWER TRIANGULAR FACTOR L OF THE
      C                CROUT FACTORIZATION
      C                DET CONTAINS THE DETERMINANT OF THE ORIGINAL MATRIX
      C             TEST NOT EQUAL TO ZERO INDICATES  SMALL PIVOT ENCOUNTERED
      C             DECOMPOSITION
      C                DET CONTAINS THE PRODUCT OF PIVOTS FOUND BEFORE THE SMA
      C                PIVOT WAS ENCOUNTERED
      C
0001        SUBROUTINE SLVBSR(WO,B,TEMP,N,M,PREC,DET,TEST)
0002        IMPLICIT REAL*8(A-H,O-Z)
0003        DIMENSION B(1),WO(1),TEMP(M)
0004        INTEGER  Q,SM,IMSM,U
      C
0005        ISW=1
0006        GO TO 1000
      C
      C                SLVBSE IS THE NAME OF AN ENTRY POINT FOR ENTERING THE
      C                SUBROUTINE WITH A NEW RIGHT HAND SIDE VECTOR C AT A POI
```

```
            C                BYPASSING THE FACTORIZATION(WHICH HAS BEEN PERFORMED)
            C                AND FINDING THE SOLUTION
            C
0007              ENTRY SLVBSE(B)
0008              ISW=-1
0009              GO TO 1
            C
0010         1000 M1= M+1
0011              MINLW=M1*(2*N-M)/2
            C
0012              PRES=DABS(PREC)
0013              IF(PRES .EQ. 0.D0) PRES=0.5D-14
            C
            C                TO GENERATE TEST USED FOR TESTING SMALL PIVOT
0014              TEST=0.D0
0015              DO 4 I=1,MINLW
0016            4 TEST=TEST+WO(I)**2
0017              TEST=DSQRT(TEST/MINLW)
0018              TEST=PRES*TEST
            C
            C       TO GENERATE ROW I OF THE TRIANGULAR FACTOR L WHICH THEN
            C       REPLACES THE ROW I OF THE ORIGINAL MATRIX
            C
0019              DET=1.D0
0020              SM=M*M1/2
0021            1 Q=1
0022              DO  100  I=1,N
0023              SUMF=0.D0
0024              IF(ISW.LT.0) GO TO 11
            C
            C          THE FIRST DIAGONAL ELEMENT OF THE MATRIX L IS IDENTICAL TO TH
            C          THE MATRIX A. PERFORM FORWARD SUBSTITUTION TO FIND Y(1) WHICH
            C          THEN STORED IN B(1)
0025              IF(I.LE.1) GO TO 177
            C
            C          M=0 IMPLIES THAT THE MATRIX A IS A DIAGONAL MATRIX. PERFORM
            C          SUBSTITUTION IMMEDIATELY
0026              SUMD=0.D0
0027              IF(M.EQ.0) GO TO 166
            C
0028              IM=I-M
0029              IF (IM.LE.0)  IM=1
0030              I1=I-1
```

```
             C
             C
             C          TO GENERATE OFF-DIAGONAL ELEMENTS OF L
 0031                IMSM=IM-SM
 0032                DO 21  J=IM,I1
 0033                SUM=0.D0
 0034                Q=Q+1
 0035                ITEMP=1
             C
             C          SET UP SUBSCRIPT U OF THE ELEMENTS OF THE UPPER ROW OF THE MA
             C          USED FOR GENERATING ELEMENTS OF THE LOWER ROW OF THE MATRIX L
 0036                J1=J-1
 0037                JM1=J-M1
 0038                IF(JM1.GT.0) GO TO 2
 0039                U=J1*J/2+IM
 0040                GO TO 3
 0041              2 U=J*M+IMSM
             C
             C          THE FIRST ELEMENT OF EACH ROW OF THE MATRIX L IS IDENTICAL TO
             C          THE MATRIX A
 0042              3 IF (J.LE.IM)  GO TO 20
             C
 0043                DO 10 K=IM,J1
             C          SUM IS USED FOR GENERATING OFF-DIAGONAL ELEMENTS
 0044                SUM=SUM+TEMP(ITEMP)*WO(U)
 0045                U=U+1
 0046             10 ITEMP=ITEMP+1
 0047                WO(Q)=WO(Q)-SUM
             C
             C          SUMF IS USED FOR FORWARD SUBSTITUTION(FOR ISW = 1)
 0048             20 SUMF=SUMF+WO(Q)*B(J)
             C
             C          TO STORE THE RATIOS L(I,J)/L(J,J)TEMPORARILY IN THE VECTOR TE
             C          USED FOR GENERATING OTHER ELEMENTS OF THE MATRIX L IN THE SAM
 0049                TEMP(ITEMP)=WO(Q)/WO(U)
             C
             C          SUMD IS USED FOR GENERATING DIAGONAL ELEMENTS
 0050                SUMD=SUMD+WO(Q)*TEMP(ITEMP)
 0051             21 CONTINUE
             C
             C          TO GENERATE DIAGONAL ELEMENTS OF L
 0052            166 Q=Q+1
 0053                WO(Q)=WO(Q)-SUMD
```

```
             C                                                                    00011330
             C          TEST FOR SMALL PIVOT                                      00011340
0054           177 IF (DABS(WO(Q)) .LE. TEST)  RETURN                             00011350
             C                                                                    00011360
             C          TO GENERATE THE DETERMINANT OF THE MATRIX OF COEFFICIENTS A 00011370
0055               DET=DET*WO(Q)                                                  00011380
0056               GO TO 17                                                       00011390
             C                                                                    00011400
             C                         FORWARD SUBSTITUTION(ISW = -1)             00011410
             C          FOR FINDING THE VECTOR Y WHICH IS THEN STORED IN VECTOR B 00011420
             C                                                                    00011430
0057            11 IF(I.LE.1) GO TO 17                                            00011440
0058               IF(M.EQ.0) GO TO 19                                            00011450
             C                                                                    00011460
0059               IM=I-M                                                         00011470
0060               IF (IM.LE.0)  IM=1                                             00011480
0061               I1=I-1                                                         00011490
0062               DO  18  K=IM,I1                                                00011500
0063               Q=Q+1                                                          00011510
0064            18 SUMF=SUMF+WO(Q)*B(K)                                           00011520
0065            19 Q=Q+1                                                          00011530
             C          FOR ISW = 1 OR -1                                         00011540
0066            17 B(I)=(B(I)-SUMF)/WO(Q)                                         00011550
0067           100 CONTINUE                                                       00011560
             C                                                                    00011570
             C          NORMAL COMPLETION OF FACTORIZATION AND FORWARD SUBSTITUTION 00011580
0068               IF (ISW .GT.0) TEST=0.D0                                       00011590
             C                                                                    00011600
             C                         BACKWARD SUBSTITUTION                      00011610
             C          FOR FINDING THE SOLUTION VECTOR X USING ELEMENTS OF WO CONSEC00011620
             C                                                                    00011630
0069               IF (M .EQ.0) RETURN                                            00011640
             C                                                                    00011650
             C          STORE TEMPORARILY THE PROUCTS L(N,I)*X(N),I=N-1           00011660
             C          TO N-M IN VECTOR TEMP  USED TO FIND X(N-1) TO X(N-M)      00011670
             C          NOTE THAT X(N) EQUALS Y(N)                                00011680
0070               DO 702 ITEMP=1,M                                               00011690
0071               Q=Q-1                                                          00011700
0072           702 TEMP(ITEMP)=WO(Q)*B(N)                                         00011710
             C                                                                    00011720
             C          TO FIND X(N-1) TO X(1)                                    00011730
0073               N1=N-1                                                         00011740
0074               DO 709 K=1,N1                                                  00011750
```

```
0075                I=N-K
0076                Q=Q-1
0077                B(I)=B(I)-TEMP(1)/WO(Q)
              C
0078                IF(I .EQ. 1)  RETURN
              C
0079                I1=M
0080                IF(I.LE.M) I1=I-1
0081                IF(I1.EQ.1) GO TO 708
0082                I11=I1-1
0083                DO 707 ITEMP=1,I11
0084                Q=Q-1
0085            707 TEMP(ITEMP)=TEMP(ITEMP+1)+WO(Q)*B(I)
0086            708 Q=Q-1
0087                TEMP(I1)=WO(Q)*B(I)
0088                IF(I1 .LT. M) TEMP(I1)=TEMP(I1)+TEMP(I1+1)
0089           709  CONTINUE
0090                RETURN
0091                END
```