

REPRESENTATIONS OF MULTIPLIERS ON A NORMED ALGEBRA

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by

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Abstract

We present here two aspects of representation theory of multipliers acting on a normed algebra.

We show that if a normed algebra A has a weak left (right) identity then the algebra of left (right) continuous linear multipliers on A can be embedded into the second conjugate space A^{**} of A . This kind of representation of multipliers was obtained by L. Máté. On the other hand, if a normed algebra A has a minimal left (right) approximate identity it is possible to embed A isometrically into a Banach algebra A^K such that every left (right) multiplier on A is given by the restriction of the left (right) multiplication operator determined by an element of A^K . This approach to multiplier theory was developed by K. McKennon. The rest of the thesis is concerned mainly with results obtained by B. J. Tomiuk and B. D. Malviya. They give a characterization of the dual B^* -algebra and the algebra of bounded linear operators on Hilbert space in terms of their multipliers.

Introduction

The purpose of this thesis is to give some aspects of representation theory of multipliers on a normed algebra.

Chapter 1 deals with the question of continuity of multipliers and the representation of continuous (right) multipliers on a Banach algebra A in terms of the elements of the second conjugate space A^{**} of A . In §1 we show that if a Banach algebra A has an approximate identity then every right multiplier on A is a continuous linear operator. In §2 we show that if the Banach algebra A has a weak right identity then the algebra of right continuous linear multipliers can be embedded isomorphically into the second conjugate space A^{**} considered as a Banach algebra under an Arens product.

In chapter 2 we are concerned with the representation of continuous linear multipliers on a normed algebra A with a minimal left approximate identity. We first consider a normed algebra which is left faithful and define a topology on A , called the K -topology, which is the weakest topology on A

for which the semi-norms $^a ||\cdot||: b \rightarrow ||ba||$, for all $a \in A$, are continuous on A . This topology is a Hausdorff locally convex topology on A , so that A admits a unique completion A^c .

Let $A_r = \{x \in A: ||x|| \leq r\}$, for $r > 0$, and let A_r^K be the closure of A_r in A^c . Let $A^K = \bigcup_{r>0} A_r^K$. We show that A^K

is a Banach algebra with norm $||\cdot||^K$. It follows that

$||x|| = ||x||^K$ for all $x \in A$ and the $||\cdot||^K$ -closure of A in

A^K is a left ideal of A^K . The K -topology and the algebra A^K are discussed in § 1. In § 2 we show that if a normed algebra A has a minimal left approximate identity then for every left continuous linear multiplier h on A there exists a unique $a \in A^K$ such that $h(x) = ax$ for all $x \in A$.

Chapter 3 is of a more specialized nature. Here we give a characterization of the dual B^* -algebra and the algebra of bounded linear operators on Hilbert space in terms of their multipliers. We show that a B^* -algebra A is dual if and only if the algebra $M(A)$ of right multipliers on A can be identified in a certain way with the second conjugate space A^{**} of A . Suppose A is a B^* -algebra containing minimal left ideals and let I be a minimal left ideal of A . Let M be the closed two-sided ideal of A generated by I . Then A is the algebra of all continuous linear operators on some Hilbert space if and only if A is $*$ -isomorphic to M^{**} , the second conjugate space of M with Arens product.

Chapter 1

Multipliers

§ 1. The continuity of multipliers on a Banach algebra A.

We shall assume for simplicity that all algebras and vector spaces in this section as well as in the rest of the thesis are over the complex field.

Definition (1.1.1). Let A be an associative algebra. A right (resp. left) multiplier on A is a map $T : A \rightarrow A$ such that

$$T(ab) = aT(b) \quad (\text{resp. } T(ab) = T(a)b)$$

for all $a, b \in A$

Example : For each $a \in A$ define the mapping $R_a : A \rightarrow A$ by $R_a(x) = xa, x \in A$. Then R_a is a right multiplier on A . Similarly $L_a : x \rightarrow ax, x \in A$ is a left multiplier on A .

This section is devoted to proving the following theorem.

Theorem (1.1.2). Let A be a Banach algebra with a bounded approximate identity. Then every right (left) multiplier on A is a continuous linear operator.

In order to prove Theorem (1.1.2) we shall need the concept of a left Banach A -module.

Definition. (1.1.3). Let A be a Banach algebra and V a Banach space over the complex field. V is called a left Banach A -module if it is a left A -module in the algebraic sense and if $\|av\| \leq \|a\| \|v\|$, for all $a \in A, v \in V$.

We shall now follow Rieffel [12] in proving Th. (1.1.2).

Lemma (1.1.4). Let A be a Banach algebra and let V be a left Banach A -module. Suppose that there is a constant, M , such that for every $a \in A$, $v \in V$ and $\epsilon > 0$ there exists an element $e \in A$ such that $\|e\| \leq M$, $\|a - ae\| < \epsilon$, and $\|v - ev\| < \epsilon$. Then for every $v \in V$ and $\epsilon > 0$ there exist elements $a \in A$ and $w \in V$ such that $v = aw$.

Proof. We may assume that $M = 1$. As Koosis [7] remarks, the proof for the general case is obtained by replacing each term of the form $(2\delta - e_n)$ below by $(\delta + p(\delta - e_n))$ for a sufficiently small positive number p . We let A_1 denote the Banach algebra obtained by adjoining an identity δ to A . Then every element in A_1 is of the form $\lambda\delta + x$, where $x \in A$ and λ is a complex number, and V becomes an A_1 -module in the obvious way.

Let n be any fixed positive integer. For every $a \in A$, $v \in V$ and $\epsilon > 0$, there exist elements $e_i \in A$, $i = 1, 2, \dots, n$, such that

$$\begin{aligned} \|e_i\| &\leq 1, \\ \|v - e_i v\| &< \epsilon 2^{-i}, \quad i = 1, 2, \dots, n. \end{aligned}$$

Let

$$\begin{aligned} a_n &= (2\delta - e_1)^{-1} (2\delta - e_2)^{-1} \dots (2\delta - e_n)^{-1}, \\ w_n &= (2\delta - e_n) (2\delta - e_{n-1}) \dots (2\delta - e_1) v, \end{aligned}$$

Then $a_n \in A_1$, $w_n \in V$ and $v = a_n w_n$. Clearly

$$(1) \quad a_n = 2^{-n} \delta + R_n$$

where $R_n \in A$. Choose e_{n+j} , $j = 1, 2, \dots$, so that

$$\begin{aligned} \|e_{n+j}\| &\leq 1, \\ \|R_n - R_n e_{n+j}\| &< 2^{-n}, \end{aligned}$$

$$||w_n - e_{n+j}w_n|| < \epsilon 2^{-(n+j)}.$$

We now show that both $\{a_n\}$ and $\{w_n\}$ are Cauchy sequences.

Consider $\{a_n\}$ first. Since

$$(2\delta - e_{n+1})^{-1} = 2^{-1}(\delta + 2^{-1}e_{n+1} + 2^{-2}e_{n+1}^2 + \dots),$$

$$|| (2\delta - e_{n+1})^{-1} || \leq 1.$$

Clearly

$$(2\delta - e_{n+1})^{-1} - \delta = (e_{n+1} - \delta)(2\delta - e_{n+1})^{-1}.$$

Now

$$a_{n+1} = a_n(2\delta - e_{n+1})^{-1},$$

and so

$$\begin{aligned} a_{n+1} - a_n &= a_n[(2\delta - e_{n+1})^{-1} - \delta] \\ &= 2^{-n}(e_{n+1} - \delta)(2\delta - e_{n+1})^{-1} + \\ &\quad R_n(e_{n+1} - \delta)(2\delta - e_{n+1})^{-1}. \end{aligned}$$

Then

$$||a_{n+1} - a_n|| \leq 2^{-(n-1)} + ||R_n e_{n+1} - R_n|| \leq 2^{-(n-2)}.$$

Thus $\{a_n\}$ converges to an element $a \in A_1$. Since $2^{-n}\delta$ converges to 0, from (1) it is clear that $a \in A$.

Next we consider $\{w_n\}$. Now

$$w_{n+1} = (2\delta - e_{n+1})w_n,$$

so that

$$w_{n+1} - w_n = w_n - e_{n+1}w_n,$$

and so

$$||w_{n+1} - w_n|| < \epsilon 2^{-(n+1)}.$$

Hence $\{w_n\}$ converges to an element $w \in V$. Since $v = a_n w_n$ for each positive integer n , we have $v = aw$.

Remark: The above proof does not appear in [12]. It was communicated to us by Rieffel.

Lemma (1.1.5). Let A be a Banach algebra. Suppose that there is a constant, M , such that for every $a \in A$, every finite collection v_1, v_2, \dots, v_k of elements of A , and every $\epsilon > 0$, there exists an element $e \in A$ such that $\|e\| \leq M$, $\|a - ae\| < \epsilon$ and $\|v_j - ev_j\| < \epsilon$ for $1 \leq j \leq k$. Then for every sequence $\{v_n\}$ of elements of A which converges to 0, there exists $a \in A$ and a sequence $\{w_n\}$ of A which converges to 0 such that $v_n = aw_n$, for all n .

Proof. Let $C_0(A)$ be the Banach space of all sequences of elements of A which converges to 0 with the supremum norm. Then $C_0(A)$ is an A -module with respect to the usual coordinate-wise operations. We show that $C_0(A)$ satisfies the hypothesis of Lemma (1.1.4). Let $\{v_n\} \in C_0(A)$. Then for any positive number ϵ , there exists a positive integer N such that

$$\|v_n\| < \frac{\epsilon}{2(1+M)} \quad (n \geq N).$$

By hypothesis, for every $a \in A$, and the finite collection v_1, \dots, v_N , there is an element $e \in A$ such that

$$\|e\| \leq M,$$

$$\|a - ae\| < \epsilon,$$

$$\|v_j - ev_j\| < \frac{\epsilon}{2} \quad (j = 1, 2, \dots, N).$$

For any $n \geq N$, we have

$$\begin{aligned} \|v_n - ev_n\| &\leq \|v_n\| + \|ev_n\| \leq (1 + \|e\|) \|v_n\| \\ &< (1 + \|e\|) \frac{\varepsilon}{2(1+M)} < \frac{\varepsilon}{2}. \end{aligned}$$

Hence for every $a \in A$, $v = \{v_n\} \in C_0(A)$ and $\varepsilon > 0$, there exists an element $e \in A$ such that

$$\|e\| \leq M,$$

$$\|a - ae\| < \varepsilon,$$

$$\|v - ev\| = \|\{v_n - ev_n\}\| = \sup_n \|v_n - ev_n\| \leq \frac{\varepsilon}{2} < \varepsilon.$$

Thus $C_0(A)$ satisfies the hypothesis of Lemma (1.1.4) and so for each $v = \{v_n\} \in C_0(A)$ there exist an element $a \in A$ and an element $w = \{w_n\} \in C_0(A)$ such that

$$v = aw, \quad \text{i.e.}$$

$$v_n = aw_n \quad (n=1,2, \dots).$$

Proof of Theorem (1.1.2). Let T be a right multiplier on A and let $\{e_\alpha : \alpha \in \Omega\}$ be a bounded approximate identity in A . Then $\|e_\alpha\| \leq M$ for all α and some constant M . For every $a \in A$, every finite collection v_1, v_2, \dots, v_k in A , and every $\varepsilon > 0$, there exists an $\alpha_0 \in \Omega$ such that

$$\|e_{\alpha_0}\| \leq M,$$

$$\|a - ae_{\alpha_0}\| < \varepsilon,$$

$$\|v_j - e_{\alpha_0} v_j\| < \varepsilon, \quad (j = 1, 2, \dots, k).$$

Let $\{a_n\}$ be a sequence of elements in A which converges to 0.

We can apply the right-hand version of Lemma (1.1.5) to

obtain an element $c \in A$ and a sequence $\{b_n\}$ in A with b_n converging to 0, such that $a_n = b_n c$ for all n . Then

$$T(a_n) = T(b_n c) = b_n T(c)$$

which converges to 0. This implies that T is continuous at 0.

For any elements a_1, a_2 in A , $\{a_1, a_2, 0, 0, \dots\}$ is a sequence in A which converges to 0. Then by Lemma (1.1.5) there exist elements b_1, b_2, c in A such that $a_1 = b_1 c$, $a_2 = b_2 c$. For any complex numbers λ_1, λ_2 ,

$$\begin{aligned} T(\lambda_1 a_1 + \lambda_2 a_2) &= T[(\lambda_1 b_1 + \lambda_2 b_2)c] = (\lambda_1 b_1 + \lambda_2 b_2)T(c) \\ &= \lambda_1 b_1 T(c) + \lambda_2 b_2 T(c) = \lambda_1 T(b_1 c) + \lambda_2 T(b_2 c) \\ &= \lambda_1 T(a_1) + \lambda_2 T(a_2). \end{aligned}$$

Hence T is linear and so continuous on all of A .

§ 2. Multipliers and the second conjugate space.

We shall show in this section that under certain conditions on the Banach algebra A every continuous right multiplier on A can be represented by an element of the second conjugate space A^{**} of A . In order to do this we shall need the concept of multiplication on A^{**} introduced by Arens. There are two such multiplications on A^{**} under which A^{**} is a Banach algebra.

We now sketch one of these operations which we shall use.

Let A^* be the conjugate space of A .

Definition (1.2.1.). Let $\phi, \psi \in A$; $\Phi \in A^*$; $F, G \in A^{**}$.

$(\Phi \circ \phi)\psi = \Phi(\phi\psi)$. This defines $\Phi \circ \phi$ as an element of A^* .

$(F \circ \Phi)\phi = F(\Phi \circ \phi)$. This defines $F\circ\Phi$ as an element of A^* .
 $(F \circ G)\phi = F(G \circ \phi)$. This defines $F\circ G$ as an element of A^{**} .
 We will call $F\circ G$ the Arens product.

Definition (1.2.2). We say that a Banach algebra A has a weak right identity if there exists a net $\{e_\alpha : \alpha \in \Omega\}$ in A and a constant $M > 0$ such that $\|e_\alpha\| \leq M, \alpha \in \Omega$ and $\lim_{\alpha} \phi(\phi e_\alpha - \phi) = 0$ for each $\phi \in A, \phi \in A^*$.

We shall show below that if A is a Banach algebra with weak right identity then the algebra $M(A)$ of all right multipliers on A may be identified (anti-isomorphically) as a subalgebra of A^{**} with Arens product. We need the following lemma which is due to Civin and Yood [2; p. 55, Lemma 3.8].

Lemma (1.2.3). A Banach algebra A has a weak right identity if and only if A^{**} has a right identity.

Proof. Suppose A has a weak right identity $\{e_\alpha : \alpha \in \Omega\}$ with $\|e_\alpha\| \leq M, \alpha \in \Omega$. Since the natural embedding π is an isometry of A into A^{**} , we have $\|\pi e_\alpha\| \leq M, \alpha \in \Omega$. Since the ball of radius M in A^{**} is compact with respect to the w^* -topology, there exists a subnet $\{e_\beta : \beta \in \Lambda\}$ such that $\pi e_\beta \xrightarrow{w^*} I \in A^{**}$. Then I is a right identity for A^{**} . If $\phi \in A^*, \phi \in A$, we have

$$\begin{aligned} (I \circ \Phi)\phi &= I(\Phi \circ \phi) = \lim_{\beta} (\pi e_\beta)(\Phi \circ \phi) = \\ &= \lim_{\beta} (\Phi \circ \phi)e_\beta = \lim_{\beta} \Phi(\phi e_\beta) = \Phi(\phi). \end{aligned}$$

Hence $I \circ \phi = \phi$, for all $\phi \in A^*$. Thus for all $F \in A^{**}$, we have

$$(F \circ I)\phi = F(I \circ \phi) = F(\phi), \quad (\phi \in A^*),$$

and so

$$F \circ I = F \quad (F \in A^{**}).$$

Suppose that A^{**} has a right identity I . By [3; p.425, Corollary 6], there is a net $\{\pi e_\alpha : \alpha \in \Omega\}$ such that $\|\pi e_\alpha\| \leq 1$ for all $\alpha \in \Omega$, and $w^* \text{-}\lim_{\alpha} \pi e_\alpha = I$. Since I is a right identity in A^{**} ,

$$F(\phi) = (F \circ I)\phi = F(I \circ \phi) \quad (F \in A^{**}, \phi \in A^*).$$

Hence

$$I \circ \phi = \phi \quad (\phi \in A^*).$$

Consequently for any $\phi \in A$, $\phi \in A^*$, we have

$$\begin{aligned} \phi(\phi) &= (I \circ \phi)\phi = I(\phi \circ \phi) = \lim_{\alpha} (\pi e_\alpha)(\phi \circ \phi) \\ &= \lim_{\alpha} (\phi \circ \phi)e_\alpha = \lim_{\alpha} \phi(\phi e_\alpha). \end{aligned}$$

This shows that $\{e_\alpha : \alpha \in \Omega\}$ is a weak right identity in A .

Corollary (1.2.4). If $I \in A^{**}$ is a right identity, then $I \circ \phi = \phi$ for all $\phi \in A^*$.

For each $F \in A^{**}$, the mapping $\phi \rightarrow F \circ \phi$ on A^* into itself is a continuous linear operator. We say that this mapping is the operator represented by F .

For a Banach algebra A with a weak right identity we have the following representation theorem for continuous right multipliers on A due to Máté [9; p.810, Theorem 1]:

Theorem (1.2.5). Let A be a Banach algebra with weak right identity. Then for every continuous right multiplier T on

A, there exists $F \in A^{**}$ such that T^* is the operator represented by F , i.e.

$$(2) \quad (F \circ \phi)\phi = \phi(T\phi) \quad (\phi \in A, \phi \in A^*).$$

Proof. If T is a continuous right multiplier on A , then for every $\phi \in A^*$; $\phi, \psi \in A$,

$$\begin{aligned} [(T^*\phi) \circ \phi]\psi &= (T^*\phi)\phi\psi = \phi(T(\phi\psi)) \quad \text{and} \\ [T^*(\phi \circ \phi)]\psi &= (\phi \circ \phi)(T\psi) = \phi(\phi(T\psi)). \end{aligned}$$

Since T is a right multiplier, we have

$$(3) \quad (T^*\phi) \circ \phi = T^*(\phi \circ \phi) \quad (\phi \in A^*, \phi \in A).$$

Similarly, for each $G \in A^{**}$, $\phi \in A^*$, $\phi \in A$,

$$\begin{aligned} [G \circ (T^*\phi)]\phi &= G[(T^*\phi) \circ \phi], \\ [(T^{**}G) \circ \phi]\phi &= (T^{**}G)(\phi \circ \phi) = G[T^*(\phi \circ \phi)]. \end{aligned}$$

By (3), we have

$$(4) \quad G \circ (T^*\phi) = (T^{**}G) \circ \phi \quad (G \in A^{**}, \phi \in A^*).$$

Since A has a weak right identity, by Lemma (1.2.3) A^{**} has a right identity I for the Arens product. Replacing G by I in (4), we have

$$I \circ (T^*\phi) = (T^{**}I) \circ \phi.$$

Hence by Corollary (1.2.4), we have

$$T^*\phi = (T^{**}I) \circ \phi.$$

If we take

$$(5) \quad F = T^{**}I,$$

then $F \in A^{**}$ and

$$(F \circ \phi)\phi = [(T^{**}I) \circ \phi]\phi = (T^*\phi)\phi = \phi(T\phi)$$

for all $\phi \in A$, $\phi \in A^*$.

Remark. In Theorem (1.2.5) the condition that weak right identity exists in A cannot be omitted. For example, if I is the identity operator on A , then I is a continuous right multiplier on A . Suppose there is $F \in A^{**}$ such that

$$(F \circ \Phi)\phi = \Phi(I\phi) = \Phi(\phi) \quad (\Phi \in A^*, \phi \in A).$$

Then we have

$$F \circ \Phi = \Phi \quad (\Phi \in A^*),$$

which implies that F is a right identity for A^{**} . However A^{**} has a right identity for the Arens product if and only if there exists a weak right identity in A .

In following the A -topology on A^* means the weak*-topology $\sigma(A^*, A)$.

Lemma (1.2.6). A bounded linear operator of A^* is the conjugate of a bounded linear operator of A if and only if it is continuous in the A -topology on A^* .

For the proof of this lemma see [16; Lemma 5.10].

Lemma (1.2.7). Let T be a continuous linear operator on A . If there is $F \in A^{**}$ such that $T^*\Phi = F \circ \Phi$ for all $\Phi \in A^*$, then T is a right multiplier on A .

Proof. For every $\Phi \in A^*$; ϕ, ψ and $\eta \in A$,

$$[(\Phi \circ \phi) \circ \psi]\eta = (\Phi \circ \phi)\psi\eta = \Phi(\phi\psi\eta) = (\Phi \circ \phi\psi)\eta.$$

We have

$$(6) \quad (\Phi \circ \phi) \circ \psi = \Phi \circ \phi\psi \quad (\Phi \in A^*; \phi, \psi \in A).$$

Furthermore,

$$\Phi(T(\phi\psi)) = (T^*\Phi)\phi\psi = (F \circ \Phi)\phi\psi = F(\Phi \circ \phi\psi)$$

and

$$\begin{aligned}\Phi(\phi(T\psi)) &= (\Phi\circ\phi)(T\psi) = (T^*(\Phi\circ\phi))\psi = (F\circ(\Phi\circ\phi))\psi \\ &= F((\Phi\circ\phi)\circ\psi).\end{aligned}$$

By (6), we obtain

$$\Phi(T(\phi\psi)) = \Phi(\phi(T\psi)) \quad (\Phi \in A^*; \phi, \psi \in A).$$

Hence $T(\phi\psi) = \phi T(\psi)$. That is, T is a right multiplier on A .

For the connection between the operator product and Arens product, we have the following theorem.

Theorem (1.2.8). Let T_1, T_2 be continuous linear operators on A . If the operators T_1^*, T_2^* are represented by F_1, F_2 in A^{**} respectively, then $T_1^*T_2^*$ is represented by $F_1\circ F_2$.

Proof. Since T_1^* and T_2^* are represented by F_1 and F_2 in A^{**} , by Lemma (1.2.7), T_1 and T_2 are right multipliers on A . From (4) in the proof of theorem (1.2.5), we obtain

$$G\circ(T^*\phi) = (T^{**}G)\circ\phi$$

for each continuous right multiplier T on A , $\phi \in A^*$ and $G \in A^{**}$. Then

$$\begin{aligned}[(F_1\circ F_2)\circ\phi]\phi &= (F_1\circ F_2)(\phi\circ\phi) = F_1(F_2\circ(\phi\circ\phi)) = \\ &F_1(T_2^*(\phi\circ\phi)) = (T_2^{**}F_1)(\phi\circ\phi) = ((T_2^{**}F_1)\circ\phi)\phi = \\ &(F_1\circ(T_2^*\phi))\phi = (T_1^*(T_2^*\phi))\phi = ((T_1^*T_2^*)\phi)\phi\end{aligned}$$

for all $\phi \in A$, and $\phi \in A^*$. It follows that

$$(F_1 \circ F_2) \circ \phi = (T_1^* T_2^*) \phi \quad (\phi \in A^*).$$

That is, $T_1^* T_2^*$ is represented by $F_1 \circ F_2$.

We shall now characterize those $F \in A^{**}$ which are, in the sense of (2) in Theorem (1.2.5), the conjugate operators of continuous right multipliers on A .

Theorem (1.2.9). If A is a Banach algebra, then the operator represented by $F \in A^{**}$ is the conjugate operator of a certain continuous right multiplier on A if and only if it is continuous in the A -topology on A^* .

Proof. Let K be the operator represented by $F \in A^{**}$. If $K = T^*$ for some continuous right multiplier on A , then by Lemma (1.2.6), K is continuous in the A -topology on A^* .

Conversely, if K is continuous in the A -topology on A^* , then by Lemma (1.2.6), $K = T^*$ for some continuous linear operator T on A . Hence

$$T^*(\phi) = K(\phi) = F \circ \phi \quad (\phi \in A^*).$$

By Lemma (1.2.7), T is a right multiplier on A .

Chapter 2

Multipliers and K-topology

In this Chapter we shall give McKennon's approach [10] to the representation of multipliers on a normed algebra with a minimal left approximate identity. Since he gives the theory in terms of left multipliers and since we want to follow his notation we shall thus assume that all multipliers in this chapter are continuous linear left multipliers.

§ 1. K-topology.

Let $(A, +, \cdot, \|\cdot\|)$ be a normed algebra which is left faithful, that is, for each non-zero element $a \in A$, there is some $b \in A$ such that $ab \neq 0$.

For each $a \in A$, let ${}^a\|\cdot\|$ be the semi-norm on A given by ${}^a\|b\| = \|ba\|$, for all $a \in A$. Then for every subset B of A , the family $\{{}^b\|\cdot\| : b \in B\}$ of semi-norms determines a locally convex topology which is a coarsest topology on A compatible with the algebraic structure in which each member of the family $\{{}^b\|\cdot\| : b \in B\}$ is continuous [13; p. 15, Theorem 3]. We denote this topology by $K(A, B, \|\cdot\|)$. Since A is left faithful, $K(A, A, \|\cdot\|)$ is Hausdorff. For any positive number r , let $A_r = \{x \in A : \|x\| \leq r\}$. It follows from [10; P. 5, Lemma 1.3] that $K(A, B, \|\cdot\|)$ and $K(A, \bar{B}, \|\cdot\|)$ coincide on A_r for every $r > 0$, where \bar{B} denotes the closure of B in A in the norm topology. When we consider A with the topology $K(A, B, \|\cdot\|)$, we will write it as $(A, K(A, B, \|\cdot\|))$.

Let X and Y be topological vector spaces over the same field. A map f from a subset $A \subset X$ into Y is said to be uniformly continuous if for every neighborhood (nhd.) W of 0 in Y there exists a nhd. V of 0 in X such that $x, y \in A$ and $x - y \in V$ implies $f(x) - f(y) \in W$. If f is uniformly continuous on A then it is continuous on A . A continuous linear map f on X into Y is uniformly continuous since

$$f(x) - f(y) = f(x - y).$$

Lemma (2.1.1). Suppose that B is a subalgebra of a normed algebra $(A, \|\cdot\|)$. Let r be any positive number and endow the Cartesian product $A_r \times A_r$ with the relative topology induced by $K(A, B, \|\cdot\|) \times K(A, B, \|\cdot\|)$. Then the multiplication is uniformly continuous on $A_r \times A_r$.

Proof. Let x_0, y_0 be any elements in A_r . Let $\varepsilon > 0, a \in B$ and let

$$U(x_0) = \{ x - x_0 : x \in A_r \text{ and } \|(x - x_0)a\| \leq \frac{\varepsilon}{6r} \},$$

$$V(y_0) = \{ y - y_0 : y \in A_r \text{ and } \|(y - y_0)a\| \leq \frac{\varepsilon}{6r} \}.$$

Then $U(x_0) + x_0$ and $V(y_0) + y_0$ are nhds. of x_0, y_0 , respectively.

Under the multiplication,

$$\begin{aligned} (U(x_0) + x_0, V(y_0) + y_0) &\rightarrow U(x_0)V(y_0) + U(x_0)y_0 + \\ &\quad + x_0V(y_0) + x_0y_0. \end{aligned}$$

For $U(x_0)V(y_0)$, we have

$$\begin{aligned} ||(x - x_0)(y - y_0)a|| &= ||xya - xy_0a - x_0ya + x_0y_0a|| \\ &\leq ||(y - y_0)a|| (||x|| + ||x_0||) \leq \frac{\epsilon}{6r} \cdot 2r = \frac{\epsilon}{3}. \end{aligned}$$

For $U(x_0)y_0$, we have

$$||x - x_0|| y_0 a || \leq \frac{\epsilon}{3}.$$

For $x_0V(y_0)$, we have

$$||x_0(y - y_0)a|| \leq \frac{\epsilon}{3}.$$

Hence

$$a ||(U(x_0)V(y_0) + U(x_0)y_0 + x_0V(y_0) + x_0y_0) - x_0y_0|| \leq \epsilon.$$

Thus the nhd. $(U(x_0) + x_0, V(y_0) + y_0)$ is mapped by the

product into the nhd. $\{w \in A: ||(w - x_0y_0)a|| \leq \epsilon\}$ about x_0y_0 .

This means that the product is continuous, for if $x \in A_r$ such

that $||x - x_0|| a || \leq \frac{\epsilon}{6r}$ and $y \in A_r$ such that $||y - y_0|| a || \leq \frac{\epsilon}{6r}$,

then $||xy - x_0y_0|| a || < \epsilon$. It is easy to see that the product

is also uniformly continuous on $A_r \times A_r$. In fact, let

$W = \{w \in A: ||wa|| \leq \epsilon\}$ be any nhd. of 0 in A. For the nhd.

$W_1 = \{w \in A: ||wa|| \leq \frac{\epsilon}{2}\}$ of 0, there exists $U_1 = \{x \in A_r:$

$||xa|| \leq \frac{\epsilon}{3r}\}$ such that (U_1, U_1) is mapped into W_1 by the product.

Let $(u,v), (x,y)$ be ordered pairs in $A_r \times A_r$ such that

$(u,v) - (x,y) \in (U_1, U_1)$. Then

$$\begin{aligned} ||(uv - xy)a|| &\leq ||uva|| + ||xya|| \leq \\ &||u|| ||va|| + ||x|| ||ya|| \leq r \frac{\epsilon}{3r} + r \frac{\epsilon}{3r} < \epsilon. \end{aligned}$$

Hence $uv - xy \in W$. Thus the product is uniformly continuous

on $A_r \times A_r$.

Lemma (2.1.2). Let X be a topological vector space, A a subset of X and Y a complete Hausdorff topological vector space over the same field as X . If f is a uniformly continuous map from A into Y , Then there exists a unique uniformly continuous map \bar{f} from \bar{A} , the closure of A , into Y such that $\bar{f}(x) = f(x)$ for all $x \in A$. Moreover, if A is a subspace of X and f is linear, then \bar{f} is linear.

For the proof of this lemma see [4].

Since the topological vector space $(A, K(A, A, ||\cdot||))$ is Hausdorff, there exists a complete Hausdorff topological vector space A^c containing A as a dense subspace and A^c is unique to within isomorphism [14; p. 17, Theorem 1.5]. For each positive number r , let A_r^K be the closure of A_r in A^c and let $A^K = \bigcup_{r>0} A_r^K$. It is easy to see that A^K is a vector space. By Lemmas (2.1.1) and (2.1.2), for each $r > 0$, the multiplication on $A_r \times A_r$ into the complete Hausdorff topological vector space A^c is uniformly continuous, and hence can be extended uniquely to a uniformly continuous function from $A_r^K \times A_r^K$ into A^c . It is easy to check that A^K is an algebra containing A as a subalgebra.

Since A is left faithful, each $a \in A$ may be viewed as an endomorphism of A by identifying it with $L_a : A \rightarrow A$, where $L_a(x) = ax$, $x \in A$. For convenience, in the rest of this chapter we shall assume that all algebras A are such that for each $a \in A$,

$||a|| = ||L_a||$, and hence that

$$(1) \quad ||a|| = \sup \{ ||ab|| : b \in A, ||b|| \leq 1 \}.$$

In particular if A contains an approximate identity $\{e_\alpha\}$, then $\|L_a\| = \|a\|$, for all $a \in A$.

Definition (2.1.3). For each $a \in A^K$, we define

$$\|a\|^K = \inf \{ r > 0 : a \in A_r^K \}.$$

Lemma (2.1.4). (i) For each $a \in A$, $\|a\| = \|a\|^K$.

(ii) $\|\cdot\|^K$ is a norm on A^K such that $\|xy\|^K \leq \|x\|^K \|y\|^K$, for all $x, y \in A^K$.

Proof. (i) Let $a \in A$ and let $\|a\| = r$. Then $a \in A_r \subseteq A_r^K$, and so $\|a\|^K \leq \|a\|$. Suppose that $\|a\|^K < \|a\|$. Then there exists a positive number p such that $\|a\|^K < p < \|a\|$. Then $a \in A_p^K$. Choose a net $\{a_\alpha\}$ in A_p such that $\lim_\alpha a_\alpha = a$ in $K(A, A, \|\cdot\|)$. Since $\|a\| = \sup \{ \|ab\| : b \in A, \|b\| \leq 1 \}$, we may choose an element $b \in A_1$ such that $\|ab\| > p$. we thus have

$$p < \|ab\| = \|a\| \|b\| = \lim_\alpha \|a_\alpha b\| \leq \overline{\lim}_\alpha \|a_\alpha\| \cdot \|b\| \leq p \cdot 1 = p,$$

a contradiction. Hence

$$\|a\|^K = \|a\|, \quad (a \in A).$$

(ii) (1) It is clear that $\|x\|^K \geq 0$ for each $x \in A^K$, and $\|x\|^K = 0$ if and only if $x = 0$.

(2) Let $x, y \in A^K$. Then $x \in A_{r_1}^K$ and $y \in A_{r_2}^K$ for some $r_1 > 0$ and $r_2 > 0$.

Let $\{x_\alpha\}$ and $\{y_\alpha\}$ be nets in A_{r_1} and A_{r_2} respectively such that $\lim_\alpha x_\alpha = x$, $\lim_\alpha y_\alpha = y$ in A^c . Since A^c is a topological vector space and $A^K \subset A^c$ we have

$$\lim_{\alpha} (x_{\alpha} + y_{\alpha}) = x + y.$$

Moreover since $x_{\alpha} + y_{\alpha} \in A_{r_1+r_2}^K$ and $A_{r_1+r_2}^K$ is closed in A^c ,

we have $x + y \in A_{r_1+r_2}^K$. Hence $||x + y||^K \leq r_1 + r_2$

for all $r_1 > 0, r_2 > 0$ such that $x \in A_{r_1}^K, y \in A_{r_2}^K$, and consequently

$$||x + y||^K \leq ||x||^K + ||y||^K.$$

Thus, for any $x, y \in A^K$, we have

$$||x + y||^K \leq ||x||^K + ||y||^K.$$

(3) Let λ be any complex number and $x \in A^K$. Then $x \in A_r^K$ for

some $r > 0$. Choose a net $\{x_{\alpha}\}$ in A_r such that $\lim_{\alpha} x_{\alpha} = x$

in A^c . By the continuity of scalar multiplication in A^c ,

$$\lim_{\alpha} \lambda x_{\alpha} = \lambda x.$$

Since $\lambda x_{\alpha} \in A_{|\lambda|r}$ for all α , we have $\lambda x \in A_{|\lambda|r}^K$. Hence

$||\lambda x||^K \leq |\lambda|r$ for all $r > 0$ with $x \in A_r^K$, and consequently

$$||\lambda x||^K \leq |\lambda| ||x||^K.$$

Replacing x by λx and λ by λ^{-1} , we obtain

$$||\lambda^{-1} \lambda x||^K \leq |\lambda^{-1}| ||\lambda x||^K, \text{ i.e.,}$$

$$|\lambda| ||x||^K \leq ||\lambda x||^K.$$

Hence $||\lambda x||^K = |\lambda| ||x||^K$ for all $x \in A^K$ and complex number λ .

(4) Let $x, y \in A^K$. Then there exist $r_1 > 0$ and $r_2 > 0$ such that

$x \in A_{r_1}^K$ and $y \in A_{r_2}^K$, respectively. Choose a net $\{x_{\alpha}\}$ in A_{r_1}

and a net $\{y_{\alpha}\}$ in A_{r_2} such that $\lim_{\alpha} x_{\alpha} = x, \lim_{\alpha} y_{\alpha} = y$ in A^c .

Thus (x, y) belongs to the closure of $A_r \times A_r$ in $A^c \times A^c$. Therefore

by the uniform continuity of multiplication on $A_r \times A_r$ and Lemma (2.1.2), we have $x_\alpha y_\alpha \rightarrow xy$ in the topology of A^c . Since $x_\alpha y_\alpha \in A_{r_1 r_2}^K$ and $A_{r_1 r_2}^K$ is closed in A^c , we have $xy \in A_{r_1 r_2}^K$.

Hence $\|xy\|^K \leq r_1 r_2$ for all $r_1 > 0, r_2 > 0$ such that $x \in A_{r_1}^K, y \in A_{r_2}^K$. Consequently

$$\|xy\|^K \leq \|x\|^K \|y\|^K \quad (x, y \in A^K).$$

(1), (2), (3), and (4) together completes (ii).

Lemma (2.1.5). For each positive number r , $A_r^K = (A^K)_r$, where $(A^K)_r = \{x \in A^K: \|x\|^K \leq r\}$.

Proof. Let $x \in A_r^K$. Then by the definition of $\|x\|^K$, $\|x\|^K \leq r$, and clearly $x \in A^K$. Hence $x \in (A^K)_r$. Conversely, suppose that $x \in (A^K)_r$. Then $\|x\|^K \leq r$. If $\|x\|^K < r$, then clearly $x \in A_r^K$. If $\|x\|^K = r$, then $\|\frac{1}{2}x\|^K = \frac{1}{2}r$. Hence $\frac{1}{2}x \in A_r^K$.

Since A_r^K is convex, $x = \frac{1}{2}x + \frac{1}{2}x \in \frac{1}{2}A_r^K + \frac{1}{2}A_r^K = A_r^K$, i.e. $x \in A_r^K$.

Thus $A_r^K = (A^K)_r$.

Lemma (2.1.6). The topologies $K(A, A, \|\cdot\|)$ and $K(A^K, A, \|\cdot\|^K)$ coincide on A_r , for each positive number r .

Proof. In order to show that the topologies coincide on A_r , by [20; p.146, Cor. 2], we have to show that they have the same convergent nets on A_r . Let $\{a_\alpha\}$ be any net in A_r which $K(A, A, \|\cdot\|)$ -converges to some element $a \in A_r$. For any $b \in A$, we have

$$\lim_\alpha \|b\| \|a_\alpha - a\|^K = \lim_\alpha \|(a_\alpha - a)b\|^K = \lim_\alpha \|(a_\alpha - a)b\| = 0.$$

It follows that $\lim_\alpha a_\alpha = a$ in $K(A^K, A, \|\cdot\|^K)$.

Hence $K(A, A, \|\cdot\|) \supset K(A^K, A, \|\cdot\|^K)$ on A_r . Similarly we have $K(A, A, \|\cdot\|) \subset K(A^K, A, \|\cdot\|^K)$ on A_r . Consequently

$$K(A, A, \|\cdot\|) = K(A^K, A, \|\cdot\|^K) \text{ on } A_r.$$

Corollary (2.1.7). For each $r > 0$, $A_r^K =$ the $K(A^K, A, \|\cdot\|^K)$ -closure of A_r in A^K .

A collection $\{e_\alpha : \alpha \in \Omega\}$ of elements of a normed algebra A , where the index Ω is a directed set, is called a minimal approximate identity for A if the following two conditions are satisfied : $\|e_\alpha\| \leq 1$, for each α , and $\lim_\alpha e_\alpha x = \lim_\alpha x e_\alpha = x$, for each $x \in A$.

Theorem (2.1.8) . The following statements hold :

- (i) for each $a \in A$, we have $\|a\| = \|a\|^K$;
- (ii) the quadruple $(A^K, +, \cdot, \|\cdot\|^K)$ constitutes a Banach algebra;
- (iii) for each $r > 0$, the topologies $K(A, A, \|\cdot\|)$ and $K(A^K, A, \|\cdot\|^K)$ coincide on A_r ;
- (iv) the $\|\cdot\|^K$ -closure \bar{A} of A in A^K is a left ideal in A^K ;
- (v) A^K is commutative if A is commutative;
- (vi) a necessary and sufficient condition for A^K to have a left identity element is for A to possess a minimal left approximate identity.

Proof. (i) This follows from the proof of Lemma (2.1.4).

(ii) We have shown that A^K is an associative algebra containing A as a subalgebra and $(A^K, \|\cdot\|^K)$ is a normed algebra. We now show that $(A^K, \|\cdot\|^K)$ is complete. Let τ_1 be the $\|\cdot\|^K$ -topology and τ_2 the $K(A^K, A, \|\cdot\|^K)$ -topology on A^K . Then τ_1

is finer than τ_2 since every semi-norm $a \|\cdot\|^K$ for each $a \in A$ is continuous in τ_1 . Both (A^K, τ_1) and (A^K, τ_2) are topological linear spaces. The family $\{A_{\frac{1}{n}}^K : n \text{ is a positive integer}\}$ is a 0-nhd. base in (A^K, τ_1) . Since $A_{\frac{1}{n}}^K =$ the τ_2 -closure of $A_{\frac{1}{n}}$, by Corollary (2.1.7), $A_{\frac{1}{n}}^K$ is complete in τ_2 , for each positive integer n . Hence, by [14; Chapter 1, 1.6], (A^K, τ_1) is τ_1 -complete, i.e., $(A^K, \|\cdot\|^K)$ is a Banach algebra.

(iii) This is proved in Lemma (2.1.6).

(iv) By [10; p. 5, Lemma 1.3] and Lemma (2.1.5) the topologies $K(A^K, \bar{A}, \|\cdot\|^K)$ and $K(A^K, A, \|\cdot\|^K)$ coincide on A_r^K for each $r > 0$.

Since $\bar{A} \subset A^K, \bar{A}_r \subset (A^K)_r$. By Lemma (2.1.5), we have

$$A_r \subset \bar{A}_r \subset A_r^K \quad (r > 0).$$

Since $K(A^K, \bar{A}, \|\cdot\|^K)$ and $K(A^K, A, \|\cdot\|^K)$ coincide on A_r^K , the $K(A^K, \bar{A}, \|\cdot\|^K)$ -closure of $\bar{A}_r =$ the $K(A^K, A, \|\cdot\|^K)$ -closure of \bar{A}_r . We denote them by \bar{A}_r^K . Since $A_r \subset \bar{A}_r$ and $K(A^K, A, \|\cdot\|^K)$ -closure of $A_r = A_r^K$, it follows that $\bar{A}_r^K = A_r^K$.

Hence

$$\bigcup_{r>0} \bar{A}_r^K = \bigcup_{r>0} A_r^K$$

and consequently

$$\bar{A}^K = A^K.$$

Thus we can assume in the rest of the proof that $(A, \|\cdot\|)$ is complete.

Let $x \in A^K$, then $x \in A_r^K$ for some $r > 0$. Choose a net $\{x_\alpha\}$ in A_r such that $\lim_\alpha x_\alpha = x$ in A^c . Then $\{x_\alpha\}$ is a $K(A, A, \|\cdot\|)$ -Cauchy net in A . For each $a \in A$, $\{x_\alpha a\}$ is a $\|\cdot\|$ -Cauchy net in A . Since $(A, \|\cdot\|)$ is complete, there exists an element $y \in A$ such that $\|x_\alpha a - y\| \rightarrow 0$. However, $0 = \lim_\alpha \|x_\alpha - x\|^K = \lim_\alpha \|x_\alpha a - xa\|^K$. Since $\|xa - y\|^K \leq \|xa - x_\alpha a\|^K + \|x_\alpha a - y\|^K$, $\|xa - y\|^K$ may be arbitrarily small. It follows that $\|xa - y\|^K = 0$. Therefore $xa = y$, i.e., $xa \in A$, for all $x \in A^K$ and $a \in A$.

Let $x \in A^K$, $b \in \bar{A}$. Choose a net $\{b_\alpha\}$ in A such that $\|b_\alpha - b\|^K \rightarrow 0$. By the continuity of multiplication, we have $\|xb_\alpha - xb\|^K \rightarrow 0$.

Since $xb_\alpha \in A$ for each α , we have that $xb \in \bar{A}$. Hence \bar{A} is a left ideal in A^K .

(v) Let $x, y \in A^K$. Then there exists $r > 0$ such that $x \in A_r^K$ and $y \in A_r^K$. We have $xy \in A_{r^2}^K$. Choose nets $\{x_\alpha\}$ and $\{y_\alpha\}$ in A_r such that $x_\alpha \rightarrow x$ and $y_\alpha \rightarrow y$ in $K(A^K, A, \|\cdot\|^K)$. Since the multiplication on $A_r \times A_r$ into $A_{r^2}^K$ is uniformly continuous,,

it is continuous on $A_r \times A_r$. Now $(x_\alpha, y_\alpha) \rightarrow (x, y)$ in $K(A^K, A, \|\cdot\|^K) \times K(A^K, A, \|\cdot\|^K)$. By Lemma (2.1.1), $x_\alpha y_\alpha \rightarrow xy$ in $K(A^K, A, \|\cdot\|^K)$. Similarly $y_\alpha x_\alpha \rightarrow yx$ in $K(A^K, A, \|\cdot\|^K)$.

Suppose that A is commutative, then $x_\alpha y_\alpha = y_\alpha x_\alpha$ for all α . Since the topology $K(A^K, A, \|\cdot\|^K)$ is Hausdorff, we have $xy = yx$ for all $x, y \in A^K$, i.e. A^K is commutative.

(vi) Suppose that A possesses a minimal left approximate identity $\{a_\alpha\}$. Then $\|a_\alpha x - x\| \rightarrow 0$ for all $x \in A$. It follows that $\{a_\alpha\}$ is a $K(A, A, \|\cdot\|)$ -Cauchy net in A . There exists an element $e \in A^c$ such that $\lim a_\alpha = e$ in A^c . Since $\|a_\alpha\| \leq 1$ for all α , it follows that $e \in A_1^K$. For each $b \in A$, we have

$$\begin{aligned} \|eb - b\|^K &= \overline{\lim}_\alpha \|eb - a_\alpha b + a_\alpha b - b\|^K \leq \\ &\overline{\lim}_\alpha \|b\| \|e - a_\alpha\|^K + \overline{\lim}_\alpha \|a_\alpha b - b\|^K = 0. \end{aligned}$$

Thus $eb = b$ for all $b \in A$. Let $b \in A^K$. Then $b \in A_r^K$ for some $r > 0$. Let $b_\alpha \rightarrow b$, $b_\alpha \in A_r^K$. Since $eb_\alpha = b_\alpha$ and multiplication is continuous in A^K , it follows that $eb_\alpha \rightarrow eb$ and so $eb = b$. This shows that e is a left identity in A^K .

On the other hand, suppose A^K contains a left identity e . Since $(A^K, \|\cdot\|^K)$ is a Banach algebra, $\|e\|^K = 1$ and so $e \in A_1^K$. There exists a net $\{a_\alpha\}$ in A_1 such that $\lim a_\alpha = e$ in $K(A^K, A, \|\cdot\|^K)$. Since $\|a_\alpha\| \leq 1$ for all α and $\lim a_\alpha x = ex = x$ for all $x \in A$, the net $\{a_\alpha\}$ is a minimal left approximate identity in A .

§ 2. The algebra A^K and multipliers on A .

Theorem (2.2.1). Suppose that the normed algebra A has a minimal

left approximate identity and let h be any continuous linear left multiplier on A . Then there exists an element $a \in A^K$ such that $h(x) = ax$ for all $x \in A$.

Proof. Let $\{x_\alpha\}$ be any net in A such that $x_\alpha \rightarrow x \in A$ in $K(A, A, \|\cdot\|)$. For each $y \in A$, we have

$$\begin{aligned} \overline{\lim}_\alpha^y \|h(x_\alpha - x)\| &= \overline{\lim}_\alpha \| (h(x_\alpha - x))y \| = \\ \overline{\lim}_\alpha \| h((x_\alpha - x)y) \| &\leq \overline{\lim}_\alpha \|h\| \| (x_\alpha - x)y \| = \\ \|h\| \cdot \overline{\lim}_\alpha^y \|x_\alpha - x\| &= 0. \end{aligned}$$

Hence $\overline{\lim}_\alpha^y \|h(x_\alpha - x)\| = 0$. Thus $\lim_\alpha^y \|h(x_\alpha - x)\| = 0$

and, since h is linear, we have $h(x_\alpha) \rightarrow h(x)$ which implies that h is $K(A, A, \|\cdot\|)$ -continuous on A . Since $\|a\| = \|a\|^K$ for each $a \in A$, $K(A, A, \|\cdot\|)$ -continuity of h on A implies the $K(A^K, A, \|\cdot\|^K)$ -continuity of h on A . Since $A_r \subset A$, $A_r^K \subset$ the $K(A^K, A, \|\cdot\|^K)$ -closure of A , and so $A^K \subset$ the $K(A^K, A, \|\cdot\|^K)$ -closure of A . Hence A is dense in A^K . By Lemma (2.1.2), there exists a unique $K(A^K, A, \|\cdot\|^K)$ -continuous linear extension \bar{h} to all of A^K .

To show that \bar{h} is a multiplier on A^K , let $x, y \in A^K$.

Then there exists a positive number r such that $x, y \in A_r^K$.

Choose nets $\{x_\alpha\}$ and $\{y_\alpha\}$ in A_r such that $x_\alpha \rightarrow x$ and $y_\alpha \rightarrow y$

in $K(A^K, A, \|\cdot\|^K)$. Then $(x_\alpha, y_\alpha) \rightarrow (x, y)$ in $K(A^K, A, \|\cdot\|^K) \times K(A^K, A, \|\cdot\|^K)$. By the uniform continuity of multiplication operator of $A_r \times A_r$ into $A_{r^2}^K$, we have

$$x_\alpha y_\alpha \rightarrow xy \quad \text{in } K(A^K, A, \|\cdot\|^K).$$

Hence

$$\bar{h}(x_\alpha y_\alpha) \rightarrow \bar{h}(xy) \quad \text{in } K(A^K, A, \|\cdot\|^K).$$

Since $\bar{h}(x_\alpha y_\alpha) = \bar{h}(x_\alpha) y_\alpha$, and $\bar{h}(x_\alpha) y_\alpha \rightarrow \bar{h}(x) y$ in $K(A^K, A, \|\cdot\|^K)$,

$$\bar{h}(x_\alpha y_\alpha) \rightarrow \bar{h}(x) y \quad \text{in } K(A^K, A, \|\cdot\|^K).$$

Thus clearly $\bar{h}(xy) = \bar{h}(x)y$ for all $x, y \in A^K$. This shows that \bar{h} is a left multiplier on A^K . Since A has a minimal left approximate identity, by Theorem (2.1.8) (vi), A^K possesses a left identity e . Let $a = \bar{h}(e)$. Then for each element $x \in A$, $h(x) = \bar{h}(x) = \bar{h}(ex) = \bar{h}(e)x = ax$. This completes the proof.

Chapter 3

Multipliers on B^* -algebras.

The purpose of this chapter is to give a characterization of the dual B^* -algebra and the algebra of bounded linear operators on Hilbert space in terms of their multipliers.

§ 1. Some definitions.

A Banach algebra A with involution $*$ is called a Banach $*$ -algebra. If the involution in A is continuous with respect to the given norm $||\cdot||$, it is easy to show that there exists an equivalent norm $||\cdot||'$ on A such that $||x^*||' = ||x||'$ [11; page 180]. A B^* -algebra A is a Banach $*$ -algebra in which the involution $*$ and the norm $||\cdot||$ satisfy the condition $||xx^*|| = ||x||^2$ for all $x \in A$. Thus in a B^* -algebra A , $||x^*|| = ||x||$ for all $x \in A$.

Let A be a Banach $*$ -algebra with continuous involution. For each $f \in A^*$, let f^* be defined by $f^*(x) = \overline{f(x^*)}$, $x \in A$; f^* is called the adjoint functional of f . It is easy to see that $f^* \in A^*$, $(\lambda f)^* = \overline{\lambda} f^*$ and $(f^*)^* = f$ for all complex number λ and $f \in A^*$.

For any Hilbert space H , $L(H)$ will denote the algebra of all continuous linear operators on H with the usual operator bound norm, $LC(H)$ the subalgebra of $L(H)$ consisting of all compact linear operators on H , and $\mathcal{K}(H)$ the

subalgebra of trace class operators on H . We shall denote the trace function on $\tau c(H)$ by $\text{tr}(\cdot)$ and the trace norm by $\tau(\cdot)$ which are defined as follows : Let $\{\phi_\alpha\}$ be a complete orthonormal set in H then $\text{tr}(T) = \sum_\alpha (T\phi_\alpha, \phi_\alpha)$ and $\tau(T) = \text{tr}((T^*T)^{\frac{1}{2}})$ for all $T \in \tau c(H)$. The value $\text{tr}(T)$ is independent of the set $\{\phi_\alpha\}$ [15; p.37, Lemma 11].

$L(H)$ and $LC(H)$ are B^* -algebras in which the involution is given by taking the adjoint T^* for every $T \in L(H)$. Under this involution $\tau c(H)$ is a Banach $*$ -algebra with $\tau(T^*) = \tau(T)$.

Let $\{A_\lambda : \lambda \in \Lambda\}$ be a family of Banach algebras, and let ΣA_λ be the set of functions on Λ such that $f(\lambda) \in A_\lambda$, for each λ , and such that $\|f\| = \sup_\lambda \|f(\lambda)\| < \infty$. Under the usual operations for functions and the norm $\|f\|$, ΣA_λ is a Banach algebra. It is called the normed full direct sum of the algebras A_λ [11; P. 77]. Let $(\Sigma A_\lambda)_0$ be the subset of ΣA_λ consisting of all f such that, for every $\epsilon > 0$, the set $\{\lambda : \|f(\lambda)\| \geq \epsilon\}$ is finite. Then $(\Sigma A_\lambda)_0$ is a closed subalgebra of ΣA_λ [11; p. 107].

For any set S in a Banach algebra A , let $\ell(S)$ and $r(S)$ be the left and right annihilators of S , respectively. A is called dual if $\ell(r(J)) = J$ and $r(\ell(R)) = R$ for every closed left ideal J and every closed right ideal R of A .

Let A be a Banach algebra with an approximate identity.

For every $a \in A$, let T_a be the right multiplication operator on A . Then T_a is a right multiplier and $I_A = \{ T_a : a \in A \}$ a closed left ideal of $M(A)$, the algebra of all bounded right multipliers on A . Hence the mapping $a \rightarrow T_a$, $a \in A$, is an isometric anti-isomorphism of A onto I_A .

§ 2. Multipliers on $LC(H)$.

As a Banach space $\tau c(H)$ can be identified with the conjugate space of $LC(H)$ in the following way: For each continuous linear functional f on $LC(H)$ there exists a unique T in $\tau c(H)$ such that $f(S) = \text{tr}(ST)$ for all $S \in LC(H)$ and $\|f\| = \tau(T)$ [15; p.46, Theorem 1]. Similarly the conjugate space of $\tau c(H)$ can be identified with $L(H)$ [15; p.47, Theorem 2]. Thus as a Banach space $L(H)$ can be identified with the second conjugate space of $LC(H)$. We now show that this identification is actually a $*$ -isomorphism between the B^* -algebra $L(H)$ and the second conjugate space of $LC(H)$ considered as a B^* -algebra with Arens product. That A^{**} is a B^* -algebra with Arens product is given in [2; p.869, Th. 7.1].

Lemma (3.2.1). Let $A = LC(H)$ and denote the elements of A^{**} by F, G etc. Considering A^* as the algebra $\tau c(H)$, let F^*, G^* denote the adjoint functionals of F, G over $\tau c(H)$. Then the isometric isomorphism which identifies A^{**} with $L(H)$ is a $*$ -isomorphism between the B^* -algebra A^{**} (with Arens product) and the B^* -algebra $L(H)$; that is, if the operators T, U in

$L(H)$ correspond through this isomorphism respectively to the linear functionals F, G in A^{**} , then $F \circ G$ corresponds to TU and F^* to T^* .

Proof. For each $f \in A^*$, let T_f be the corresponding operator in $\text{rc}(H)$. Then $F(S) = \text{tr}(ST) = \text{tr}(TS)$ and $F^*(S) = \overline{F(S^*)} = \overline{\text{tr}(S^*T)} = \text{tr}(S^*T\phi_\alpha, \phi_\alpha) = \sum_\alpha (S^*T\phi_\alpha, \phi_\alpha) = \sum_\alpha (T\phi_\alpha, S\phi_\alpha) = \sum_\alpha (S\phi_\alpha, T\phi_\alpha) = \sum_\alpha (T^*S\phi_\alpha, \phi_\alpha) = \text{tr}(T^*S) = \text{tr}(ST^*)$. Thus $F^*(S) = \text{tr}(ST^*)$ for all $S \in \text{rc}(H)$ and hence F^* corresponds to T^* . ([15; p. 45, Lemma 1]).

It remains to show that $F \circ G$ corresponds to TU . Now, for $Q, R \in A$ and $f \in A^*$, we have

$$(f \circ Q)R = f(QR) = \text{tr}(T_f QR)$$

which shows that $f \circ Q$ is given by the operator $T_f Q$. Hence

$$(G \circ f)Q = G(f \circ Q) = \text{tr}(T_f QU) = \text{tr}(UT_f Q).$$

Thus $G \circ f$ is given by the operator UT_f and consequently

$$(F \circ G)f = F(G \circ f) = \text{tr}(UT_f T) = \text{tr}(TUT_f)$$

for all $f \in A^*$. Hence $F \circ G$ corresponds to the product TU .

Theorem (3.2.2). To each right multiplier T on the algebra $LC(H)$ there corresponds a unique element a_T in $L(H)$ such that $T(s) = sa_T$ for all $s \in LC(H)$; $\|T\| = \|a_T\|$. Thus the mapping $T \rightarrow a_T$ is an isometric anti-isomorphism of $M(LC(H))$ onto $L(H)$.

Proof. Let $A = LC(H)$ and let $T \in M(A)$. Since the B^* -algebra

A has an approximate identity [11; p.245, Theorem (4.8.14)], by Theorem (1.2.5), there exists a unique element $F \in A^{**}$ such that

$$(1) \quad (Fof)s = f(T(s)) \quad (s \in A, f \in A^*).$$

For $s \in A$ and $f \in A^*$, let t_f and t_{fos} be the elements of $\tau c(H)$ such that $f(a) = \text{tr}(at_f)$ and $(fos)a = \text{tr}(at_{fos})$ for all $a \in A$. (See [15; p.46, Theorem 1]). Since

$$\text{tr}(t_fsa) = f(sa) = (fos)a = \text{tr}(t_{fos}a) \quad (a \in A),$$

[12; p.45, Lemma 1] shows that

$$(2) \quad t_{fos} = t_f s \quad (s \in A, f \in A^*).$$

Thus

$$(3) \quad (Fof)s = F(fos) = \text{tr}(t_{fos}t_F) = \text{tr}(t_fst_F)$$

($s \in A, f \in A^*$) where t_F is the unique element in $L(H)$ such that $F(f) = \text{tr}(t_f t_F)$ for all $f \in A^*$. (See [15; p.47, Theorem 2]). But $f(T(s)) = \text{tr}(t_f T(s))$. Hence from (1) and (3) it follows that

$$\text{tr}(t_f T(s)) = \text{tr}(t_fst_F) \quad (f \in A^*).$$

Recalling [15; p.45, Lemma 1], we see that $T(s) = st_F$ for all $s \in A$. Taking $a_T = t_F$, $T(s) = sa_T$ for all $s \in A$.

We now show that $\|T\| = \|a_T\|$. Since $F = T^{**}I$, where I is an identity in A^{**} , (see (5) in the proof of Theorem (1.2.5)), we have $\|F\| = \|T^{**}I\| \leq \|T^{**}\| \|I\| = \|T\|$. Thus $\|F\| \leq \|T\|$. On the other hand, for all $f \in A^*$, $s \in A$ with $\|f\| = 1, \|s\| = 1$, we have

$\|F\| \geq \|F(fos)\| = \|(Fof)s\| = \|f(T(s))\|$,
 and so, $\|F\| \geq \sup \{ \|f(T(s))\| : f \in A^* , s \in A, \|f\| = 1, \|s\| = 1 \}$. That is, $\|F\| \geq \|T\|$. Thus $\|F\| = \|T\|$.
 By [15; p.47, Theorem 2], $\|F\| = \|t_F\|$. Since $a_T = t_F$,
 we have $\|T\| = \|a_T\|$. This completes the proof.

Corollary (3.2.3). Let $A = LC(H)$. Then there exists an isometric anti-isomorphism ϕ of $M(A)$ onto A^{**} such that $\phi(I_A) = \pi(A)$, where $\pi(A)$ is the canonical image of A in A^{**} .

Proof. For each $a \in A$, the right multiplication operator T_a is a right multiplier on A . Hence by Theorem (3.2.2), there exists $b \in L(H)$ such that $T_a x = xb$ for all $x \in A$. But this means that $xa = xb$, for all $x \in A$, which clearly implies that $a = b$. Let ϕ_1 be the mapping $T \rightarrow a_T$ of $M(A)$ onto $L(H)$ given in Theorem (3.2.2), then $\phi_1(T_a) = a$ for all $a \in A$. Let ϕ_2 be the mapping $a \rightarrow F_a$ which identifies $L(H)$ with A^{**} ; by Lemma (3.2.1) ϕ_2 is an isometric *-isomorphism of $L(H)$ onto A^{**} . Let $\phi = \phi_2 \circ \phi_1$. Then ϕ is an isometric anti-isomorphism of $M(A)$ onto A^{**} . We show that $\phi(I_A) = \pi(A)$. Since $F_a(f) = \text{tr}(t_f a) = f(a) = (\pi(a))f$ for each $a \in A$ and $f \in A^*$, we have $F_a = \pi(a)$ for all $a \in A$. Hence

$$\phi(T_a) = \phi_2 \circ \phi_1(T_a) = \phi_2(\phi_1(T_a)) = \phi_2(a) = F_a = \pi(a)$$
 for each $a \in A$, and so $\phi(I_A) = \pi(A)$. This completes the proof.

§ 3. Some useful lemmas.

Lemma (3.3.1). Let $\{A_\lambda : \lambda \in \Lambda\}$ be a family of semi-simple Banach algebras and let $A = (\Sigma A_\lambda)_0$. For each $\lambda \in \Lambda$, let $I_\lambda = \{f \in A : f(\mu) = 0 \text{ if } \mu \neq \lambda\}$ and $B_\lambda = \{f \in A : f(\lambda) = 0\}$. Then

(i) $I_\lambda \cap B_\lambda = (0)$ and $I_\lambda + B_\lambda = A$.

(ii) $\ell(I_\lambda) = r(I_\lambda) = B_\lambda$ and $\ell(B_\lambda) = r(B_\lambda) = I_\lambda$.

Proof. (i) If $f \in I_\lambda \cap B_\lambda$, then $f(\mu) = 0$ for all $\mu \in \Lambda$. This shows that $f = 0$ and so $I_\lambda \cap B_\lambda = (0)$. We now show that $I_\lambda + B_\lambda = A$ for each $\lambda \in \Lambda$. It is clear that $I_\lambda + B_\lambda \subseteq A$. On the other hand, for any element $f \in A$, we define two functions on Λ such that

$$f_1(\mu) = \begin{cases} f(\mu) & \text{if } \mu = \lambda \\ 0 & \text{if } \mu \neq \lambda \end{cases}$$

$$f_2(\mu) = \begin{cases} 0 & \text{if } \mu = \lambda \\ f(\mu) & \text{if } \mu \neq \lambda \end{cases}$$

Then $f_1 \in I_\lambda$, $f_2 \in B_\lambda$, and $f = f_1 + f_2$. Hence $A \subseteq I_\lambda + B_\lambda$. Consequently $A = I_\lambda + B_\lambda$.

(ii) It is easy to see that the semi-simplicity for each A_λ implies semi-simplicity for $(\Sigma A_\lambda)_0 = A$, and so A is an annihilator algebra (11; p. 107). Since I_λ is a two-sided ideal in A , $I_\lambda \cap \ell(I_\lambda) = (0)$ and $\ell(I_\lambda) = r(I_\lambda)$ (11; p. 99; Lemma 2.8.10). Following (i), we have $\ell(I_\lambda) \subseteq B_\lambda$. Conversely, let $f \in B_\lambda$. For any $g \in I_\lambda$, $fg(\mu) = f(\mu)g(\mu) = 0$ for all $\mu \in \Lambda$. Hence $fg = 0$. Hence $f \in \ell(I_\lambda)$ and so $B_\lambda \subseteq \ell(I_\lambda)$.

Consequently $\ell(I_\lambda) = B_\lambda$. Similarly we can show that $r(I_\lambda) = B_\lambda$ and $\ell(B_\lambda) = r(B_\lambda) = I_\lambda$.

Lemma (3.3.2). Let A , I_λ and B_λ be as in Lemma (3.3.1).

Let $T \in M(A)$. Then

- (i) T leaves I_λ invariant, i.e. $T(I_\lambda) \subseteq I_\lambda$,
- (ii) if T_λ denotes the restriction of T to I_λ , then $\|T\| = \sup \|T_\lambda\|$.

Proof. (i) Let $x \in B_\lambda$ and $y \in I_\lambda$. Then $xy = 0$ and $0 = T(xy) = xT(y)$. This shows that $Ty \in r(B_\lambda) = I_\lambda$ by Lemma (3.3.1). Hence $T(I_\lambda) \subseteq I_\lambda$.

(ii) since T_λ is the restriction of T to I_λ , $\|T_\lambda\| \leq \|T\|$ for all λ . Let $\varepsilon > 0$ be given. Then there exists $f \in A$, $\|f\| = 1$, such that $\|T\| - \varepsilon \leq \|Tf\|$. Since $A = (\sum A_\lambda)_0$, there exists $\lambda_1, \lambda_2, \dots, \lambda_n$ in A such that $\|f(\lambda_i)\| \geq \frac{\varepsilon}{\|T\|}$ and $\|f(\lambda)\| < \frac{\varepsilon}{\|T\|}$ for $\lambda \neq \lambda_i$, $i = 1, 2, 3, \dots, n$. Let $g \in A$ be such that $g(\lambda_i) = f(\lambda_i)$ and $g(\lambda) = 0$ for $\lambda \neq \lambda_i$, $i = 1, 2, \dots, n$. Then clearly

$$\|f\| = \|g\| \text{ and } \|Tg\| = \sup_{\lambda} \|Tg(\lambda)\| = \sup_{\lambda} \|T_\lambda(g(\lambda))\| = \sup_{1 \leq i \leq n} \|T_{\lambda_i}(g(\lambda_i))\|.$$

We now show that $\|Tf\| = \|Tg\|$.

It is clear that $\|Tg\| \leq \|Tf\|$. On the other hand, since $Tf \in A$, there exist $\mu_1, \mu_2, \dots, \mu_k$ in A such that

$$\|Tf(\mu_j)\| \geq \varepsilon \text{ and } \|Tf(\mu)\| < \varepsilon \text{ for } \mu \neq \mu_j, j = 1, \dots, k.$$

Then for each $j = 1, 2, \dots, k$, $\mu_j = \lambda_i$ for some $i = 1, \dots, n$.

Suppose not. There is some j such that $\mu_j \neq \lambda_i$ for all $i = 1, 2, \dots, n$. Then $\epsilon \leq ||Tf(\mu_j)|| \leq ||T|| \cdot ||f(\mu_j)|| <$
 $< ||T|| \frac{\epsilon}{||T||} = \epsilon$, a contradiction. Hence $\{\mu_1, \dots, \mu_k\} \subseteq$

$\{\lambda_1, \dots, \lambda_n\}$ and so $||Tf|| = \sup_{\mu} ||Tf(\mu)|| = \sup_{1 \leq j \leq k} ||Tf(\mu_j)|| =$
 $\sup_{1 \leq j \leq k} ||T_{\mu_j}(f(\mu_j))|| \leq \sup_{1 \leq i \leq n} ||T_{\lambda_i}(f(\lambda_i))|| = \sup_{1 \leq i \leq n} ||T_{\lambda_i}(g(\lambda_i))||$
 $= ||Tg|| \leq ||Tf||$. Consequently $||Tf|| = ||Tg|| =$

$\sup_{1 \leq i \leq n} ||T_{\lambda_i}(g(\lambda_i))||$, so that $||Tf|| = ||T_{\lambda_{i_0}}(g(\lambda_{i_0}))||$ for

some i_0 , $1 \leq i_0 \leq n$. Since $||T_{\lambda_{i_0}}(g(\lambda_{i_0}))|| \leq ||T_{\lambda_{i_0}}||$,

we have $||T|| - \epsilon \leq ||T_{\lambda_{i_0}}||$. Hence $||T|| = \sup_{\lambda} ||T_{\lambda}||$

Lemma (3.3.3). Let $\{A_{\lambda} : \lambda \in \Lambda\}$ be a family of semi-simple Banach algebras and let $A = (\Sigma A_{\lambda})_0$. Then $M(A)$ is isometrically isomorphic to the normed full direct sum of the algebras $M(A_{\lambda})$.

Proof. For each $\lambda \in \Lambda$, let $I_{\lambda} = \{f \in A : f(\mu) = 0 \text{ if } \mu \neq \lambda\}$ and, for each $T \in M(A)$, let T_{λ} be the restriction of T to I_{λ} ; T_{λ} is a right multiplier on I_{λ} . Since A_{λ} is isometrically isomorphic to I_{λ} , each T_{λ} may be identified as an element of $M(A_{\lambda})$ with the same norm. This isomorphism ψ is given as follows: For each $x \in A_{\lambda}$, let $\psi(x) = (x_{\mu}) \in I_{\lambda}$ such that $x_{\mu} = 0$ for $\mu \neq \lambda$ and $x_{\mu} = x$ for $\mu = \lambda$. Now for $T \in M(I_{\lambda})$, define ξ_T on A_{λ} by the relation

$\xi_T(x) = \psi^{-1}(T(x_\mu))$, where $\psi(x) = (x_\mu)$. Then ξ_T is a right multiplier on A_λ and $T \rightarrow \xi_T$ is an isometric isomorphism into $M(A_\lambda)$. To see that it is onto, let $\xi \in M(A_\lambda)$ and define T on I_λ by the relation $T(x_\mu) = \psi(\xi(x))$. Then T is a right multiplier on I_λ and such that $\psi^{-1}(T(x_\mu)) = \xi(x)$. Thus $T \rightarrow \xi_T$ is onto.

For $T \in M(A)$, let η_T be the function on Λ such that $\eta_T(\lambda) = T_\lambda$. By Lemma (3.3.2), η_T is an element of the normed full direct sum $\Sigma M(A_\lambda)$ with $||\eta_T|| = ||T||$. For each element $T \in M(A)$, define the mapping $\phi: T \rightarrow \eta_T$ of $M(A)$ into $\Sigma M(A_\lambda)$. This mapping ϕ is an isometric isomorphism of $M(A)$ into $\Sigma M(A_\lambda)$. To show that this mapping ϕ is onto, let $\eta \in \Sigma M(A_\lambda)$ and let T be a mapping on A such that $(Tf)(\lambda) = \eta(\lambda)f(\lambda)$. Since $(T(fg))\lambda = \eta(\lambda)(fg(\lambda)) = f(\lambda)(\eta(\lambda)g(\lambda)) = f(\lambda)(Tg)(\lambda) = (f(Tg))\lambda$ for all $f, g \in A$, and $\lambda \in \Lambda$, and so $T \in M(A)$. Moreover, $\eta = \eta_T$ and $||T|| = ||\eta_T||$ imply that $||T|| = ||\eta||$. Thus the mapping ϕ is an isometric isomorphism of $M(A)$ onto $\Sigma M(A_\lambda)$ and this completes the proof.

§ 4. Multipliers and the duality in B^* -algebras.

Let $\{H_\lambda: \lambda \in \Lambda\}$ be a family of Hilbert spaces H_λ and

let $(\Sigma \tau c(H_\lambda))_1$ denote the family of all functions f defined on Λ such that $f(\lambda) \in \tau c(H_\lambda)$ for all λ and such that $\Sigma ||f(\lambda)|| < \infty$. It follows that $(\Sigma \tau c(H_\lambda))_1$ is a Banach algebra under the norm $||f|| = \Sigma ||f(\lambda)||$, and the usual operations for functions. It is clearly a *-subalgebra of $(\Sigma LC(H_\lambda))_0$.

Lemma (3.4.1). As a Banach space, $(\Sigma \tau c(H_\lambda))_1$ is isometrically isomorphic to the conjugate space of $(\Sigma LC(H_\lambda))_0$.

Proof. See [18; p. 45, Theorem (4.1.1)].

Lemma (3.4.2). Let $A = (\Sigma LC(H_\lambda))_0$. As a Banach space, the normed full direct sum $\Sigma L(H_\lambda)$ is isometrically isomorphic to the second conjugate space A^{**} of A .

Proof. See [18; p. 51, Theorem (4.2.1)].

Theorem (3.4.3). Let A be a B^* -algebra. Then A is a dual algebra if and only if there exists an isometric anti-isomorphism ϕ of $M(A)$ onto A^{**} such that $\phi(I_A) = \pi(A)$.

Proof. Suppose that A is dual. Then there exists a family of Hilbert spaces $\{H_\lambda : \lambda \in \Lambda\}$ such that A is *-isomorphic to $(\Sigma LC(H_\lambda))_0$ [6; p. 221, Lemma 2.3]. By Lemmas (3.4.1) and (3.4.2), A^{**} is isometrically isomorphic to $(\Sigma \tau c(H_\lambda))_1$ the L_1 -direct sum of the algebras $\tau c(H_\lambda)$ and that in turn

A^{**} is isometrically isomorphic to the normed full direct sum $\Sigma L(H_\lambda)$ of the algebras $L(H_\lambda)$. Letting $LC(H_\lambda) = A_\lambda$ and identifying A with $(\Sigma A_\lambda)_0$, Lemma (3.3.3) shows that $M(A)$ is isometrically isomorphic to the normed full direct sum of the algebras $M(A_\lambda)$. But, by Corollary (3.2.3), $M(A_\lambda)$ is isometrically anti-isomorphic to $L(H_\lambda)$ for each λ . Hence $M(A)$ is isometrically anti-isomorphic to $\Sigma L(H_\lambda)$. Since $\Sigma L(H_\lambda)$ is $*$ -isomorphic to A^{**} , it follows that $M(A)$ is isometrically anti-isomorphic to A^{**} . Let ϕ_1 be the isomorphism of $M(A)$ onto $\Sigma L(H_\lambda)$ and let $\phi_{1\lambda} = \phi_1|_{M(A_\lambda)}$, the restriction of ϕ_1 to $M(A_\lambda)$. Let $T \in M(A)$, then $T = (T_\lambda)$ where $T_\lambda \in M(A_\lambda)$ for each λ , and so $\phi_1(T) = \phi_1((T_\lambda)) = (\phi_{1\lambda} T_\lambda) = (a_{T_\lambda})$, where a_{T_λ} is an element in $L(H_\lambda)$ corresponding to T_λ through the isomorphism $\phi_{1\lambda}$ for each λ . Since $\sup_\lambda \|a_{T_\lambda}\| = \|T\|$, it follows that the element $a_T = (a_{T_\lambda})$ belongs to $\Sigma L(H_\lambda)$. It is clear that $\phi_1(T_a) = a$, for each $a \in A$. Let ϕ_2 be the mapping $a \rightarrow F_a$ which identifies $\Sigma L(H_\lambda)$ with A^{**} , where $a = (a_\lambda)$ with $a_\lambda \in L(H_\lambda)$. Then ϕ_2 is an isometric $*$ -isomorphism of $\Sigma L(H_\lambda)$ onto A^{**} . It is easy to see that $F_a \in A^{**}$ can be written in the form $F_a = (F_{a_\lambda})$ with $F_{a_\lambda} \in A_\lambda^{**}$. Let $\phi = \phi_2 \circ \phi_1$. Then, for each $a \in A$, $\phi(T_a) = \phi_2(\phi_1(T_a)) = \phi_2(a) = F_a$, i.e., $\phi(T_a) = F_a$. Let $f \in A^*$ and let f_λ be the restriction of f to $LC(H_\lambda)$ for each λ . (We identify $LC(H_\lambda)$ as a sub-

algebra of A). Then each $f_\lambda \in LC(H_\lambda)^*$ and $f = (f_\lambda)$. Hence

$$\begin{aligned} F_a(f) &= (F_{a_\lambda}(f_\lambda)) = (\text{tr}(t_{f_\lambda} a_\lambda)) = (f_\lambda(a_\lambda)) = \\ &(\pi(a_\lambda)f_\lambda) = \pi(a)f. \end{aligned}$$

Hence $F_a = \pi(a)$ for each $a \in A$. Consequently $\phi(T_a) = \pi(a)$ for each $a \in A$. This shows that $\phi(I_A) = \pi(A)$.

Conversely, suppose that there exists an isometric anti-isomorphism of $M(A)$ onto A^{**} such that $\phi(I_A) = \pi(A)$. Since I_A is a closed left ideal of $M(A)$, it follows that $\pi(A)$ is a closed right ideal of A^{**} . But $\pi(A)$ is a $*$ -subalgebra of A^{**} . Hence $\pi(A)$ is a closed two-sided ideal of A^{**} . Therefore, by [19; p.533, Theorem 5.1], A is dual. This completes the proof.

Corollary (3.4.4). A B^* -algebra A is dual if and only if every multiplier on $\pi(A)$ is given by the restriction to $\pi(A)$ of the right multiplication operator T_a , for some $a \in A^{**}$.

§ 5. Multipliers and the characterization of $L(H)$.

Theorem(3.5.1). Let A be a B^* -algebra containing minimal left ideals. Let I be a minimal left ideal of A , M is the closed two-sided ideal generated by I and M^{**} the second conjugate space of M . Then A is $*$ -isomorphic to $L(H)$, for some Hilbert space H , if and only if A is $*$ -isomorphic to M^{**} when M^{**} is considered as a B^* -algebra with Arens product.

Proof. Suppose that A is $*$ -isomorphic to $L(H)$. Identify A with $L(H)$. Since I is a minimal left ideal of $L(H)$, I consists of elements of rank one, and so $I \subseteq LC(H)$. Since M is the smallest closed two-sided ideal containing I , $M \subseteq LC(H)$. However, $LC(H)$ is a simple B^* -algebra [18; p. 11, Theorem(1.2.5)]. Consequently $M = LC(H)$. By Lemma (3.2.1), M^{**} is $*$ -isomorphic to $L(H)$.

Conversely, suppose that A is $*$ -isomorphic to M^{**} . Since A is a B^* -algebra containing minimal left ideals, the socle S_A of A is defined [11; p. 261, Theorem(4.10.2)]. Let B be the closure of S_A . Since every minimal left (right) ideal of A is of the form Ae (eA), where e is a self-adjoint minimal idempotent [11; p. 261, Theorem (4.10.1)], it follows that every minimal left (right) ideal of A is also a minimal left (right) ideal of B . Thus B is a B^* -algebra with dense socle and consequently is dual by [6; p. 222, Theorem 2.1]. Hence the closed two-sided ideal M generated by I is a minimal closed two-sided ideal of B [1; p. 158, Theorem 5]. As B is dual, M is $*$ -isomorphic to $LC(H)$ for some Hilbert space H [11; p. 269, Corollary (4.10.20)]. Hence M^{**} is $*$ -isomorphic to $L(H)$ by Lemma (3.2.1), and so A is $*$ -isomorphic to $L(H)$. This completes the proof.

We remark that a B^* -algebra A contains minimal left ideals if and only if it contains minimal right ideals. This easily follows from the continuity of the involution and [11; p. 261, Lemma (4.10.1)]. Thus Theorem(3.5.1) can

also be stated in terms of right ideals. We also observe that the Hilbert space H in Theorem (3.5.1) is essentially unique. For if $L(H_1)$ is $*$ -isomorphic to $L(H_2)$ then H_1 is isometrically isomorphic to H_2 . (See [21; p.538]).

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