Reliability and Availability Analysis of Three-State Device Systems

Thesis submitted to the University of Ottawa
in partial fulfillments of the requirement for the degree
of Master of Science in Mechanical Engineering

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ISBN 0-612-20011-6

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Acknowledgments

I would like to express my sincere gratitude to my Thesis supervisor Prof. B.S. Dhillon for his continues advice and supervision throughout this work. I also express my gratitude to my colleagues Dr. Yang, N.F., Dr. Anude, O.C. and Fashandi, A for their useful discussion and help.

I wish to thank my parents and my sister for their moral and financial support.

Finally I wish to thank my wife, Yan, for her support and encouragement, without which this work would not have been possible.

Abstract

This study presents the reliability and availability analysis of three-state device systems with common-cause failures. The effect of common-cause failures on different systems is discussed. Different configurations are compared in order to obtain higher standby system reliability. Mathematical expressions are developed for the system reliability, the mean time to failure, the steady state availability and the time dependent availability of cold standby system, warm standby system, series system and parallel system. For system with constant common-cause failure rates, the *Markov State Space* approach is used to perform reliability analysis. The *Method of Stages* approach is used to investigate systems with non-constant common cause failure rates. Selected plots demonstrate the effect of the varying common cause failure rate on system reliability, mean time to failure, steady state availability and time dependent availability. This study clearly shows that the occurrence of common-cause failure has a negative effect on standby system reliability parameters. However, the system reliability, the mean time to failure, the steady state availability and the time dependent availability increase as the number of standby devices in the three-state device standby system increases.

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Chapter 1

Introduction

Often, engineering systems are composed of devices having only two states, i.e., normal functioning state and failure state. There are other engineering systems however, which are made up of three-state devices. A device is called a three-state device if it operates in its normal mode but can fail in either of two mutually exclusive modes: open mode failure and closed (shorted) mode failure. Typical examples of such devices are electrical relays, electrical switches, fluid flow valves and sensors. For instance, if a fluid flow valve remains in a closed position (i.e., it is stuck-closed and can not be opened), this type of failure is called open mode failure. However, if it remains in open position (i.e., it is stuck-open and can not be closed) this type of failure is called closed (or shorted) mode failure. One special characteristic of a three-state device is that it can not fail simultaneously in both the open and shorted (closed) mode and thus, the failures are mutually exclusive.

One application of reliability engineering is to increase system reliability given the poor component reliability. To achieve this, in many cases component redundancy is employed. There are several types of redundancies and one of those is known as the *standby redundancy* where one unit operates and the rest are kept as standby. The standby redundancy can be divided into two categories: *cold standby* and *warm standby*. In the case of *cold standby*, the standby units are inactive, whereas in *warm standby case*, the standby units are active but not operating. Figure 2.1 in Chapter 2 is a typical block diagram of a standby configuration.

In a redundant system, the assumption of each redundant unit failing independently can lead to an optimistic prediction in system reliability. But, this assumption may not be valid because of the occurrence of common-cause failures. A *common cause failure* is defined as any instance where multiple units or components fail due to a single underlying effect. Some of the reasons for the occurrence of a common-cause failures are design deficiencies, operation and maintenance error, extreme operating condition such as abnormally high (or low) temperature and humidity, and external catastrophe such as earthquakes, fires, floods, etc. This study considers the three-state device system with common-cause failures.

1.1 The Specific Property of Three-State Device System

Three-state devices were first investigated by Moore and Shannon [64] in 1956. They initially considered obtaining reliable circuits by using unreliable relays that can assume two kinds of failures. The first kind of failure was the failure of a relay contact to close which was defined as a closed mode failure. The second kind of failure was the failure of contact to open, called an open mode failure. In the same year Creveling [18] independently considered redundant circuits by analyzing the reliability of components made up of diodes. He argued that there are three states for a diode: normal operation, shorted-circuit and open-circuit. Creveling used a different method than that of Moore and Shannon.

Redundancy usually is a dominant factor in increasing a system reliability for a two-state device system without increasing the individual device reliability. However, for a system containing three-state devices, the redundancy may either increase or decrease the system reliability. This depends on the device's dominant mode of failure, the number of components in the system and the system configuration.

A three-state device system could be in various redundant configurations: series, parallel or standby and mixed arrangement such as series-parallel, parallel-series etc. As the complexity of configuration increases, the analysis become more difficult to perform.

For instance, a parallel system made up of two-state devices definitely has better reliability than individual devices which form the system. This however, could not be true for a three-state device system. In a parallel system containing three-state devices, one device is required to keep the system functioning if all the other devices fail due to an open mode failure. But if any of the devices fails due to a closed mode failure, the whole system fails. In a series system, the system made up of two-state devices fails if any of such devices fail. However, in a three-state device series system, the system fails only if all the devices fail due to a closed or shorted mode failure or if any of the devices fails due to an open mode failure.

1.2 Literature Review

In 1981, Dhillon [26] published a book containing a chapter on three state device systems and listing published literature prior to 1980. In 1992, Lesanovsky [60] published a review paper covering the same subject up to 1991. Thus, this review mainly covers works cited in both these publications with emphasis on the period after 1981. In addition, this review also covers published literature about multistate analysis. In real life a system and its devices are capable of being in a range of states, varying from a perfect functioning state through various levels of performance to complete failure. However, the system or devices must be in one of those states, all of which are mutually exclusive.

As stated earlier, the reliability of systems comprised of three-state devices was first studied by Moore and Shannon [64] and Creveling [18] in 1956. However, over the last four decades, there has been a significant growth in published literature on the subject that may be grouped into three categories:

(1) Multistate system definition and performance discussion - in this category, multistate system definition and performance were discussed by extending the concept of binary coherent systems to multistate systems.

(2) Reliability evaluation and optimization of the three-state device configurations - in this category, the block diagram method was used to evaluate and optimize the three-state system configurations.

(3) Analysis of three-state device system models - in this category, stochastic processes theory was used to analyze system reliability by developing mathematical models.

(1) Multistate system definition and the performance discussion.

Barlow and Proschan [6], in 1975, used the theory of coherent system to deal with the multistate system of multistate component. In the same year, Murchland [58], also discussed multistate system.

Block and Savits [9] obtained a decomposition for multistate structure functions and discussed the concepts of multistate importance and coherence. Natvig [67] introduced two suggestions of how to define a multistate coherent system. Nadar [66] introduced two types of a multistate coherent system and discussed relationship of different definitions of three-state device systems.

Hudson and Kapur [44] and [45] illustrated the analysis approach by a simple example, in which the system generally has various levels of operational performance and hence the total system effectiveness measure should reflect all of these performance levels and their reliability. They began by defining those functions and variables necessary to formally establish a system model. The development was centered on the structure function that given the system state as a function of component states and the equivalence classes of component state vector generated by structure function. They also discussed a variety of bounds on measure of system performance.

To clarify some of the confusion in multistate extensions to the concept of coherent systems, Bendell and Ansell [8] discussed the importance of relevancy and considered the generalization of properties of binary coherent systems to the various types of multistate systems. Funnemark and Natvig [32] obtained a series of bounds for the availability and unavailability in the fix time interval for a multistate system.

Xue [96] introduced the discrete function theory, which extended switching function theory and multiple valued logic function theory into multistate system analysis. Two approaches to compute the state probability of a multistate system were presented. One is based on inclusion-exclusion and the other is based on enumeration.

Aven [5] presented two efficient algorithms for reliability evaluation of monotone multistate systems with independent multistate components. The algorithms are based on the Donlliez and Hamonlle decomposition method. Algorithm one requires the minimal paths to be known. Algorithm two requires the minimal cut sets to be known (the state of the system need not be specified for each vector of component states).

Wood [95] applied binary algorithms to multistate components and systems. This paper showed how to model multistate components using binary variables. This modeling technique allowed binary algorithms for block diagram and fault tree to be applied to multistate systems. In this paper, several multistate examples are presented, and some cases in which computational efficiency can be enhanced are discussed.

Abouammoh and Al-kadi [2] reviewed the existing relevancy conditions and structure functions, investigated their interrelationship, and introduced a unified relevancy condition.

(2) Reliability evaluation and optimization of the three-state device configurations.

Proctor and Dhillon [78] illustrated a graphical technique to evaluate the reliability of any combination of series and parallel configurations (for identical units only) up to a maximum of fifteen elements per configuration. They also compared the numerical results obtained from the derived formulas and graphical techniques.

Dhillon [21] presented the delta-star transformation method. This method is very simple and powerful. It can be used to evaluate complex configurations such as the double bridge networks and a single-bridge network containing more than five elements. With delta-star transformation method, the network is simply reduced to a series and parallel combination. Dhillon and Rayapati [28] further developed the network reduction technique to evaluate reliability of three-state component complex systems. This method can be used in systems composed of series, parallel, series-parallel, parallel-series and bridge configurations in any combination.

Page and Perry [71] considered three state networks having two nodes designated as source and sink. Such a network is itself subject to each of the two failure modes. They showed that the network reliability problem for a three-state device can easily be reduced to a pair of problems involving a network made up of two-state devices. Consequently, such problems can be solved using algorithms based on the factoring theorem.

Jenney and Sherwin [51] developed a method to find the most reliable physical arrangement of 2, 3 and 4 units given the unit failure probability. They emphasized a method for finding the best arrangement of a limited number of components. The number can be limited by cost, space or weight considerations. Algorithms were developed for finding the most reliable array design for a given limit on the number of items to be used and minimum number of valid paths required. The problem of designing the most reliable configuration of a given number of identical three-state units is treated by Page and Perry [72]. They presented two simple algorithms for designing on optimal configuration and by performing analysis of 6, 8 and 20 unit systems, and they illustrated the extent to which other configurations can be more reliable than "series-parallel" or "parallel-series" arrays. Also, they illustrated an alternative algorithm called "keep 3" for selecting near optimized configurations for large systems. Malon [63] corrected an oversight in [51]. Malon pointed out that complicated structure such as those studied in [51] can allow system to fail in both modes simultaneously even though individual component can not, thus he introduced a formula which take this into account. Page and Perry [73] subsequently showed that the failure mode described by Malon does not affect the result in [72].

Sah and Stiglitz [82] derived several properties of the optimal K-out-of-N system which is subject to two kinds of failures. The characterization of the optimal K which maximizes the mean system-profit was obtained (a special case of this optimization criterion is the maximization of system reliability). They demonstrated how one can predict, based directly on the parameters of the system, whether the optimal K is smaller or larger than 1/2 N. The directions of change in the optimal K resulting from change in system parameter are also ascertained. Sah [84] derived and analyzed an explicit closed-form formula for the optimal K in K-out-of-N systems consisting of identical component systems which can be in one of two possible modes with pre-specified probability. This type component exhibits two failure modes. The cost of the two kinds of system failure is generally not identical. Since the formula is explicit, it permits a calculation of the optimal K directly in terms of the parameters of the system. In addition, it yields many results concerning both the bounds of the optimal K and the effects of a change in the parameters on the optimal K and on the optimized value of the systems expected profit.

Sharma, Panigrahi and Misra [88] considered multistate modelling of the network elements. The states which permit a flow less than the maximum assigned capacity are assumed as parallel and mutually exclusive in the network. A method is illustrated for the reliability evaluation of a communication network element. In their paper, the important criterion is taken care of by considering multistate modelling of branches as well as node. Each branch or node may have different capacity and therefore different probabilities are assigned for different capacities. Patra and Misra [76] present a simple method for evaluating the reliability of a multistate flow network by using the state enumeration technique. All successful states are obtained from the knowledge of the max-flow min-cuts of system graph. The proposed method considers cutsets of the network and examines their corresponding capacity vector for required flow capacity from the source node to sink node. System reliability evaluated through enumeration of selective states from enumerated capacity vectors.

Satoh, Sasaki, Yuge and Yanagi [85] considered three-state redundant systems in which the *system* can fail in both modes (open mode and closed mode) simultaneously, even though individual devices can not fail in this way. They defined system open mode failure as if any of devices in the system fails due to an open mode failure and on the other hand the system closed mode failure if any of the devices in the system fails due to closed mode failure. They developed two formulas for calculating the reliability of parallel-series or series-parallel systems which are formed from three-state devices and which may require multiple paths to function. They also presented the effect of the dominant failure-modes on the reliability of series and parallel systems and the reliability of parallel-series and series-parallel systems. Yuge, Sasaki and Yanagi [97] presented two approaches: a combinatorial approach and a simulation approach for computing the reliability of complex networks subject to two kinds of failure (open failure & shorted failure). They dealt mainly with the networks subject to flow quantity constraints such as the one which is required to control j or more separate paths.

Li and Chi [61] advanced a new algorithm to compute three-state device networks reliability by using the minimal cut--minimal path method. With this method, both a shorted mode failure probability and an open mode failure probability can be obtained simultaneously.

(3) Analysis of three-state device system models.

Proctor and Dhillon [78] presented a Markov model in which a device has one good state and two mutually exclusive failure modes, e.g., open and short modes. The failure and repair rate are assumed constant. This model was used to obtain the system availability.

Dhillon [20] discussed a three-state system containing two types of components (i.e., with open and short mode failures) when repair follows a general distribution. Expression for state probabilities and the availability function were obtained. The same author in Ref [22], discussed a four unit redundant system with common cause failure in which the units are subject to open mode failure and shorted mode failure. He developed the Laplace transforms of the state probabilities, the mean time to failure of system and the system reliability. Following year Dhillon [25] extended the model to the K-out-of-N three-state device system.

Chung [15] and [16], based on Dhillon's work [22] [25], developed the steady state probability and system availability equations. Further, the frequency of encountering a state and the average duration of residence in a state were also explored. He assumed [17] failed system repair times are arbitrarily distributed. Chung derived the Laplace transforms of the state probabilities and the availability of system, in addition to obtaining the steady-state availability.

Mahmond, Mokhles and Saleh [62] used the same model as Chung [15] but a different repair policy. They obtained the Laplace transforms of the state probabilities, the availability, the mean up times and down times as well as the steady-state probabilities and the mean time to system failure (MTSF).

Dhillon [21] presented a mathematical model representing a duplex unit system with three-state devices. A Markov model is developed to represent (M+1) identical duplex unit system. One of these duplex systems is operational and there are M on cold standbys. The operational duplex system may fail due to common-cause failures. Laplace transforms of state probability equations were developed.

Vaurio [94] derived the fundamental relations for the reliability, the availability and the other failure and repair characteristics of components with multiple failed states. These equations allow general failure and repair time distributions. Solution techniques were introduced for the time dependent and the asymptotic state probabilities and the failure and repair intensities.

Kumanoto, Ohrauk and Inoue [58] presented a formula to compute the figure of merit that included expected number of normal trips, spurious trips and hazards for design of prospective systems. They illustrated the 1-out-of-1:G, 1-out-of-2:G and 2-out-of-2:G configurations of protective systems with scheduled and unscheduled maintenance.

Cafaro, Francesco. C. and Francesco. V [13] made use of the Kronecker sum of the transition rate matrix of the component to analyze the structural properties of the Markov model system. As a result an efficient use of the Markov system (method) is possible by mapping the transition rates of the single components into the elements of the system transition rate matrix. They explained how to use the Markov approach for evaluating the availability and reliability of a system when the transition rates of each component depend on the state of the system.

Gupta with Agarwal [36] [37], Kumar [38], Sharma [39] [40], Singhal and Srivastava [41], and Sharma S. S. [42] developed a series of models based on Proctor and Dhillon work [78]. They presented the cost analysis of a three-state repairable system in which the units suffered partial and catastrophic failures. They discussed a one unit system, a two units in parallel redundancy system, a two units (one cold standby redundant) system and a system consisting of two subsystems connected in warm standby redundancy which have non-identical units in series. In these models, the failure and repair times for the systems follow exponential and general distributions respectively. The Laplace transforms of various state probabilities of the systems have been derived and steady state behavior of the system has also been examined. The time dependent probabilities which determine the expected profit as well as the availability of the system at any time were obtained for some models. Numerical examples were discussed.

Kececioglu and Jiang [52] analyzed a repairable standby system with imperfect sensing and switching units. They derived explicitly the system reliability expression and illustrated numerical examples. In addition, the effect of the sensing and switching unit's failure rates on the system reliabilities are discussed.

Tuteja, Arora and Taneja [92] presented three models for a two unit cold standby system under the assumption that each unit attains partial and complete failure. Three kinds of repair policy were used in the three models. In model one the repair man is always with the system. In model two the repair man comes at the moment a unit fails. In model three the repair man takes some random time in reaching the system. Formulas for the mean time to failure and steady-state availability of the system were derived. Profit is calculated for each model and a comparison of these profits was presented.

Guo and Cao [35] analyzed a multistate repairable system under a changing environment subject to a Markov process with two states. They assumed that the system has M types of failure under one environment and N types under another. By assuming constant failure rates and general repair rates distribution, they obtained the system availability, the frequency and the reliability functions.

Deng and Song [19] considered a system that operates in a randomly changing environment. The change of environment level was described by a Markov process with a finite state and the system consisted of several units and one repair facility. The repair time of a failed unit had an arbitrary distribution. They obtained the system reliability function, the availability, and the failure frequencies.

1.3 Objective

The main objective of this study is to analyze the effects of common-cause failures on a system made up of three-state devices. This study basically investigates the reliability of standby systems: cold and warm. Markov state space techniques were used in both standby cases. In addition, the reliability and the mean time to failure for parallel, series, and standby systems are compared with the presence of common-cause failures. The performance expressions are obtained by considering constant and non-constant common-cause failure rates. To analyze the case of non-constant common-cause failure rate, the *Method of Stages* [89] and [90] was used. To improve the system reliability, the system mean time to failure and the system availability, a new type of standby configuration, in which there are two units in parallel with one unit on standby, was studied.

The main difference between this study and those proposed by previous researchers are as follows:

- The standby system models proposed by previous researchers is for systems comprising of two-state devices only. In this study three-state device system are investigated.
- The *Method of Stages* is used to investigate three-state device systems with non-constant common-cause failure rates. Only the Markov state space method and the supplementary variable method are used in the three-state device systems studied by the previous researchers.

1.4 Organization of Study

This study is divided into six chapters:

Chapter 1 presents an introduction to three-state device systems and a literature review.

Chapter 2 presents a study of a cold standby system with two kinds of repair policies. The system reliability, the mean time to failure (MTTF), the steady state availability and the time dependent availability expressions are developed.

Chapter 3 presents a study of special cases for warm standby systems. The comparison is made with special cases of Chapter 2 for the same values of system parameters.

Chapter 4 presents a two-unit system reliability comparison for different configurations: parallel, series, and standby. To improve system reliability, a new type of standby configuration, in which there are two units in parallel with one unit on standby, is also presented.

Chapter 5 deals with the non-constant common cause failure rate. A comparison is performed for different configurations: parallel, series, and standby.

Chapter 6 discusses the results of the study and gives conclusions and recommendations.

Chapter 2

Three-State Device Cold Standby System

This chapter presents mathematical models to perform reliability analysis of various types of cold standby systems with common-cause failures. Three different situations of a cold standby system are considered: without Repair, with Type I Repair and with Type II Repair. The Markov processs and the Laplace transform method are used to obtain expressions for the system reliability, the mean time to failure, the steady state availability and the time dependent availability.

2.1 Three-state device cold standby system description

The block diagram of a cold standby system is shown in Figure 2.1. This type of redundancy represents the system in which one unit is operating and the remaining n units are on standby. When the operating unit fails, either due to an open mode failure or a short (closed) mode failure, one of the standby units switches in to replace the failed unit. All the units incorporated in this system are assumed to be identical and any functional unit is capable of keeping the system operating successfully for a period.

Therefore, there are three kinds of system failures. The first is open mode failure which means that the last unit in the system fails due to an open mode failure. The second is short (closed) mode failure which means that the last unit in the system fails due to a

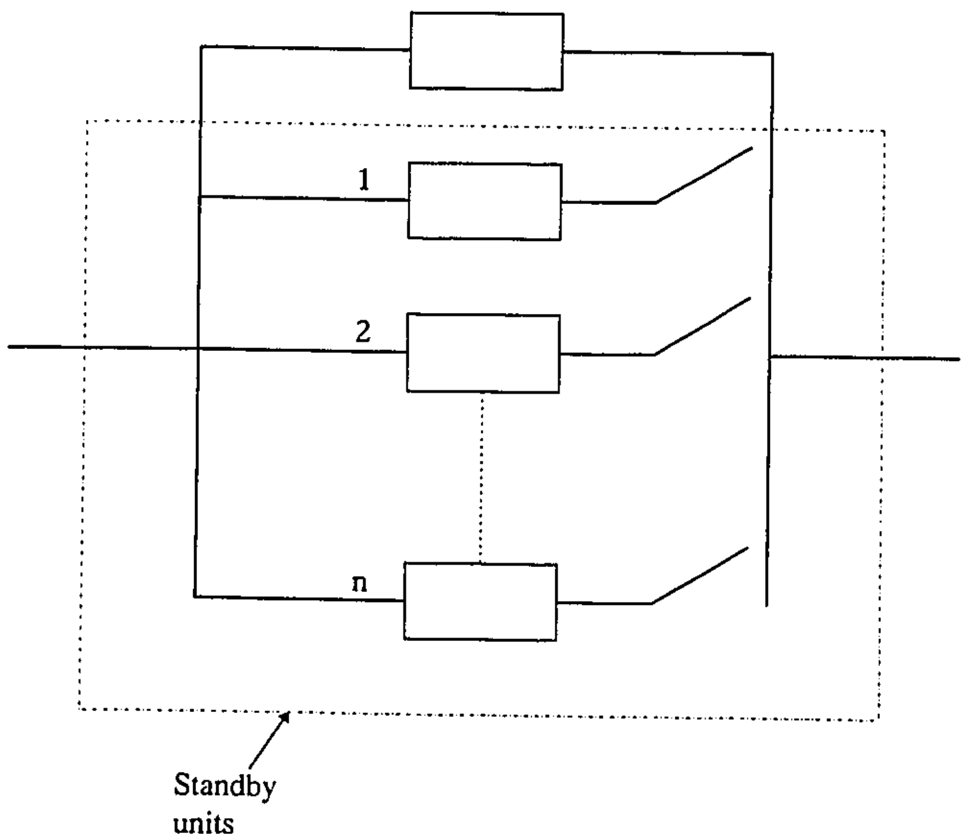


Figure 2.1 Block diagram of a standby system

short mode failure. Also, the system can fail due to a common-cause failure from any operable states as long as there are at least two operable units in the system.

Description of the System State

Referring to Figure 2.2, the first state (at time t=0) of the system considered in this chapter is defined as state 1, in which one unit is operating and the remaining *n* operable units are on standby. When the first operating unit fails, one of the *n* standby units starts operating and that leaves (*n-1*) units on standby in the system. In this case, there could be two different possibilities: i. If the operating unit results in an open mode failure, the state of system is defined as state O1. ii. If the operating unit results in a short mode failure, the state of system is defined as state S1. If two units have failed, there will be four possible states of system. The Table 2.1 shows the relationship of the number of possible states and the number of failed units as well as the number of units on standby.

Table 2.1 The relationship of the number of possible system states with the number of failed units and standby units

No. of failed units	0	1	2	3	...	n-2	n-1	n
No. of standby units	n	n-1	n-2	n-3	...	2	1	0
No. of system states	1	2	2^2	2^3	...	$2^{(n-2)}$	$2^{(n-1)}$	2^n

Table 2.1 indicates that the total number of possible operating states in a three-state unit standby system is $2^n + 2^{(n-1)} + ... + 4 + 2 + 1 = 2^{(n+1)} - 1$. To distinguish the different states in this system, the expression of system state is made up of the number of

failed units and failure types of the failed units. For example, the state OSO3 represents a system with three failed units or n-3 units on standby at time t (the last number represents the number of failed units in the system). The letter O represents a unit which failed due to an open mode failure and S indicates a short mode failure. These letters make up character sets which represent a sequence resulting in a failure. The first letter O means that the first unit failed due to an open mode failure. The second letter S indicates that the second unit is in a short mode failure. The third letter O means the third unit is in an open mode failure. In addition, in situations where there are 3 failed units or n-3 units on standby, the system could be in one of seven other states OOO3, OOS3, OSS3, SOO3, SOS3, SOS3 and SSS3.

Figure 2.2 is a state space diagram of a standby system without repair. In the first state and the states where there is only one operable unit, there are two possible failures: open mode failure and short mode failure. In the other operating state, there are three kinds of possible failures: open mode failure, short mode failure, and common-cause failure.

In this standby system two types of repair policies are considered.

Type I Repair: This repair policy deals with the repair of a system only when the system fails completely, either due to a common cause failure or to other types of failures (such as an open mode or a short mode failure), and the failed system is restored back to its initial state at time t=0.

Type II Repair: Unlike Type I Repair, Type II Repair only takes a partially failed system into consideration. The repair is conducted if any of the system units fail and the system is restored to its previous state. For instance, if any two units fail and the second standby unit is operating, once repaired, the first standby unit is operable and n-1 units are on standby.

Figure 2.3 is the state space diagram of a standby system with Type I Repair. The system is repaired from its failure states to its initial state. Figure 2.4 represents the state space diagram of a standby system with Type II Repair. The repair, when performed, returns a system from its current operable state to its previous state operable state.

Assumptions

The following assumptions are associated with cold standby system models:

1. The system contains $n+1$ independent and identical units. Specifically, one operating and n on standby.
2. A common cause failure can occur and trigger a whole system failure from any operable state if there are at least two operable units in the system.
3. A common cause failure and other failures are statistically independent.
4. A common cause failure and other failure rates are constant.
5. The repair rates are constant.
6. The switching arrangement of the standby system is perfect.
7. The repaired unit and system are as good as new.
8. In the Type I Repair case, both the repair rate due to an open mode failure and due to a short mode failure are the same.
9. In the Type II Repair case, all the repair rates are the same.

Notation

The following symbols are associated with this standby configuration:

n number of standby units

t time.

s Laplace transform variable.

λ_o Constant open-mode failure rate of an operating unit or device.

λ_s Constant short-mode failure rate of an operating unit or device.

λ_c Constant common-cause failure rate of the system.

μ Constant repair rate of a unit or the system.

μ_c Constant repair rate of the failed system from common-cause failure state to initial state 1.

$P_1(t)$ Probability that the system is in state 1.

$P_o(t)$ Probability that the system failed due to an open mode failure.

$P_s(t)$ Probability that the system failed due to a short mode failure.

$P_c(t)$ Probability that the system failed due to a common cause failure.

$R(t)$ System reliability.

$AV(t)$ System time dependent availability.

$MTTF$ System mean time to failure.

AV_{ss} System steady state availability.

2.2 Three-state device standby system without Repair

In this section, the general model and its three special cases are discussed. The system considered in the special cases consists of one to three standby units. The system reliability and mean time to failure are compared for all these models.

2.2.1 Standby system general model without Repair

Figure 2.1 shows a block diagram of the general model. The system of differential equations associated with Figure 2.2 (state space diagram) is

$$\frac{dP_1(t)}{dt} = -(\lambda_o + \lambda_s + \lambda_c)P_1(t) \tag{2.1}$$

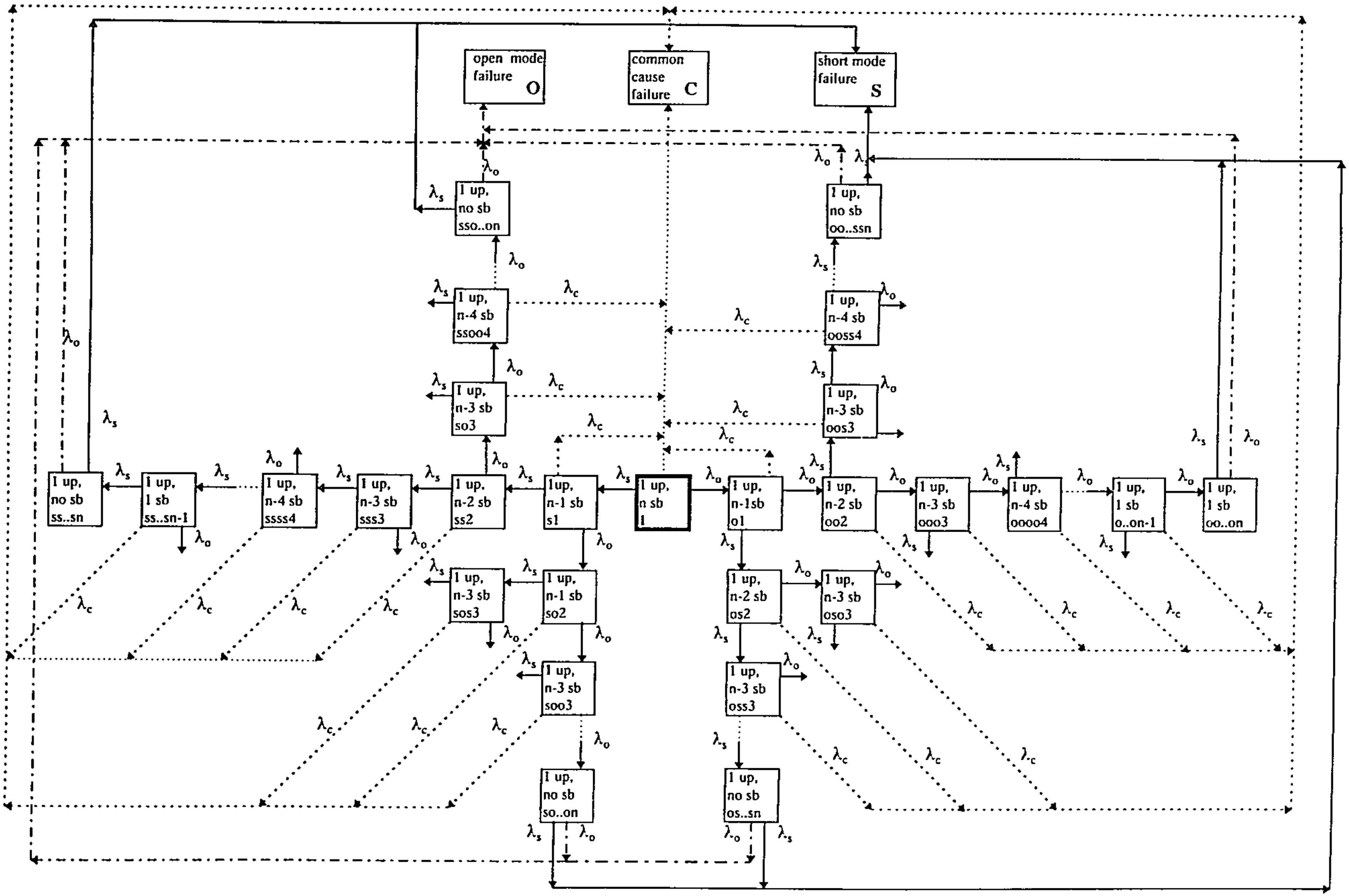


Figure 2.2 State space diagram of general model without repair

sb=standby

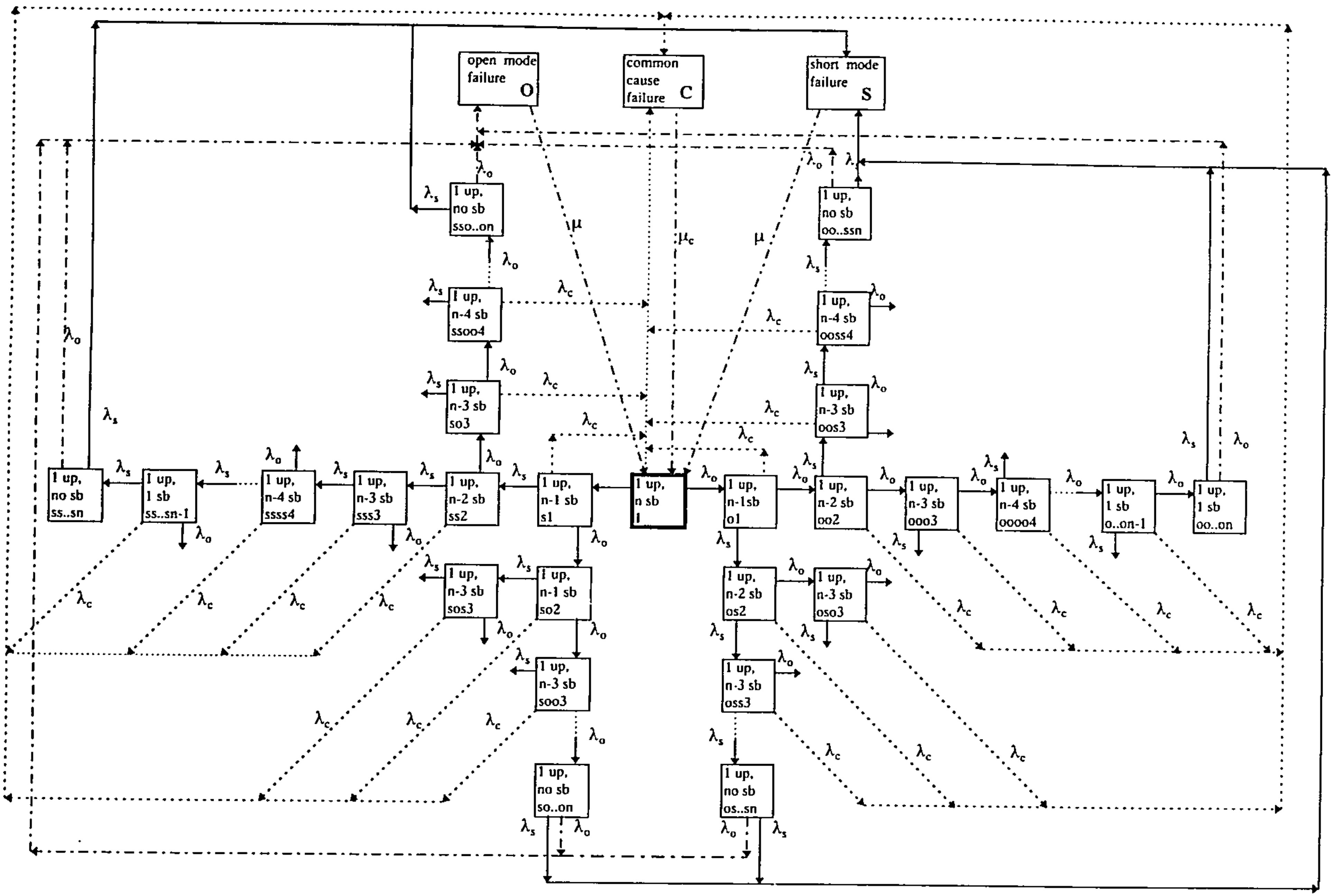


Figure 2.3 State space diagram of general model with Type I Repair

sb=standby

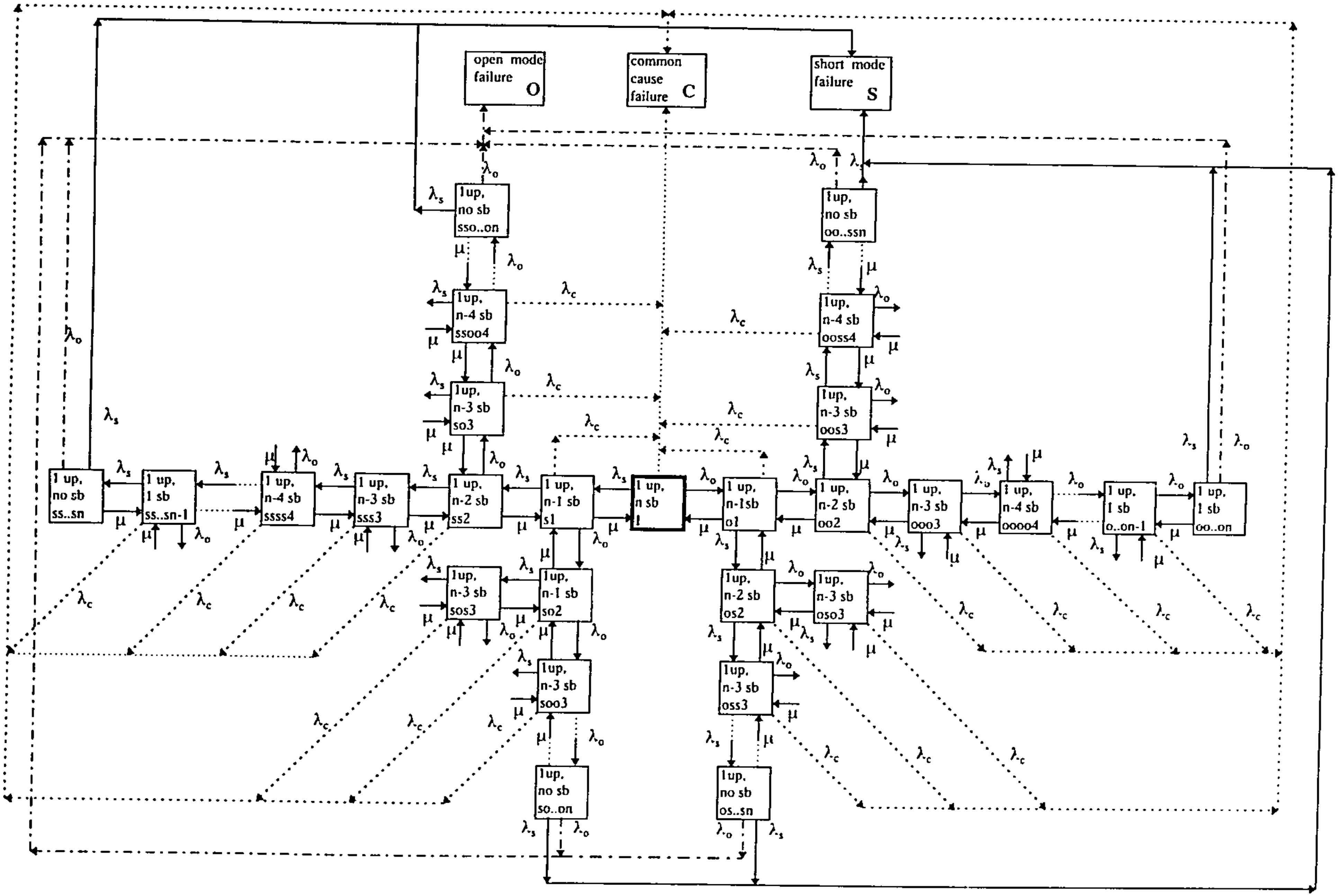


Figure 2.4 State space diagram of general model with Type II Repair

sb=standby

$$\frac{dP_{(o,s)ok}(t)}{dt} = -(\lambda_o + \lambda_s + \lambda_c)P_{(o,s)ok}(t) + \lambda_o P_{(o,s)k-1}(t) \tag{2.2}$$

$$\frac{dP_{(o,s)sk}(t)}{dt} = -(\lambda_o + \lambda_s + \lambda_c)P_{(o,s)sk}(t) + \lambda_s P_{(o,s)k-1}(t) \tag{2.3}$$

where $(o,s)ok$ in Equation (2.2) represents the system states in which there are k units failed and the last unit failed due to an open mode failure. $(o,s)sk$ in Equation (2.3) represents the system states in which the last unit failed due to a short mode failure. k=1,2,....,(n-1). (o,s) in Equations (2.2) and (2.3) is the character set indicating that there are (k-1) characters O and S as well as $2^{(k-1)}$ possible combinations of open mode failure and short mode failure. $P_{(o,s)k-1}(t) = P_1(t)$ when k=1.

$$\frac{dP_{(o,s)on}(t)}{dt} = -(\lambda_o + \lambda_s)P_{(o,s)on}(t) + \lambda_o P_{(o,s)n-1}(t) \tag{2.4}$$

$$\frac{dP_{(o,s)sn}(t)}{dt} = -(\lambda_o + \lambda_s)P_{(o,s)sn}(t) + \lambda_s P_{(o,s)n-1}(t) \tag{2.5}$$

where $(o,s)on$ in Equation (2.4) represents the system states that have no units on standby and only one unit operating as well as the last unit has failed due to an open mode failure. $(o,s)on$ in Equation (2.5) represents the system states that have no unit on standby and only one unit is operating as well as the last unit has failed due to a short mode failure. (o,s) in Equations (2.4) and (2.5) represents the character set where there are (n-1) characters of O and S as well as $2^{(n-1)}$ possible combination of open mode failure and short mode failure. $P_{(o,s)n-1}(t) = P_1(t)$ when n=1.

$$\frac{dP_o(t)}{dt} = \lambda_o \sum P_{(o,s)n}(t) \tag{2.6}$$

$$\frac{dP_s(t)}{dt} = \lambda_s \sum P_{(o,s)n}(t) \tag{2.7}$$

where $\sum P_{(o,s)n}(t)$ in Equations (2.6) and (2.7) represents the sum of the probabilities of all possible states in which n units have failed and only one unit is working.

$$\frac{dP_c(t)}{dt} = \lambda_c [P_1(t) + \sum_k P_{(o,s)k}(t)] \qquad \text{for k=1,2,3,...,(n-1).} \tag{2.8}$$

where $\sum_k P_{(o,s)k}(t)$ in Equation (2.8) represents the sum of the probabilities of all the possible states in which there is at least one unit on standby.

At time $t = 0$, $P_1(0) = 1$, and all the other initial condition probabilities are equal to zero.

Taking Laplace transforms of Equations (2.1) - (2.8) and solving the resulting equations, the Laplace transforms of the state probabilities are

$$P_1(s) = \frac{1}{\lambda_o + \lambda_s + \lambda_c + s} \tag{2.9}$$

$$P_{(o,s)ok}(s) = \frac{\lambda_o}{\lambda_o + \lambda_s + \lambda_c + s} P_{(o,s)k-1}(s) \qquad \text{for k=1,2,3,...(n-1)} \tag{2.10}$$

$$P_{(o,s)sk}(s) = \frac{\lambda_s}{\lambda_o + \lambda_s + \lambda_c + s} P_{(o,s)k-1}(s) \qquad \text{for k=1,2,3,...(n-1)} \tag{2.11}$$

$$P_{(o,s)on}(s) = \frac{\lambda_o}{\lambda_o + \lambda_s + s} P_{(o,s)n-1}(s) \tag{2.12}$$

$$P_{(o,s)sn}(s) = \frac{\lambda_s}{\lambda_o + \lambda_s + s} P_{(o,s)n-1}(s) \tag{2.13}$$

$$P_o(s) = \lambda_o \sum P_{(o,s)n}(s) \tag{2.14}$$

$$P_s(s) = \lambda_s \sum P_{(o,s)n}(s) \tag{2.15}$$

$$P_c(s) = \lambda_c [P_1(t) + \sum_k P_{(o,s)k}(s)] \qquad \text{for k=1,2,...(n-1)} \tag{2.16}$$

By taking inverse Laplace transforms, the time dependent state probabilities can be obtained. Thus, the system reliability is then given by

$$R(t) = P_1(t) + \sum_k P_{(o,s)k}(t) \qquad \text{for k=1,2,3,...,n} \tag{2.17}$$

The system mean time to failure (MTTF) is

$$MTTF = \lim_{s \to 0} [P_1(s) + \sum_k P_{(o,s)k}(s)] \qquad \text{for k=1,2,3,...,n} \tag{2.18}$$

2.2.2 Standby system without Repair: one unit operating, the other on standby

The system state space diagram is shown in Figure 2.5. By setting n=1 in Equations (2.1)-(2.8), we obtain the following differential equations:

$$\frac{dP_1(t)}{dt} = -(\lambda_o + \lambda_s + \lambda_c)P_1(t) \tag{2.19}$$

$$\frac{dP_{o1}(t)}{dt} = -(\lambda_o + \lambda_s)P_{o1}(t) + \lambda_o P_1(t) \tag{2.20}$$

$$\frac{dP_{s1}(t)}{dt} = -(\lambda_o + \lambda_s)P_{s1}(t) + \lambda_s P_1(t) \tag{2.21}$$

$$\frac{dP_s(t)}{dt} = \lambda_s[P_2(t) + P_3(t)] \tag{2.22}$$

$$\frac{dP_o(t)}{dt} = \lambda_o[P_2(t) + P_3(t)] \tag{2.23}$$

$$\frac{dP_c(t)}{dt} = \lambda_c P_1(t) \tag{2.24}$$

At time $t = 0$, $P_1(0) = 1$, and all the other initial condition probabilities are equal to zero.

By taking Laplace transforms of Equations (2.19) - (2.24) and solving the resulting equations, the Laplace transforms of the state probabilities are as follow:

$$P_1(s) = \frac{1}{\lambda_o + \lambda_s + \lambda_c + s} \tag{2.25}$$

$$P_{o1}(s) = \frac{\lambda_o}{(\lambda_o + \lambda_s + s)(\lambda_o + \lambda_s + \lambda_c + s)} \tag{2.26}$$

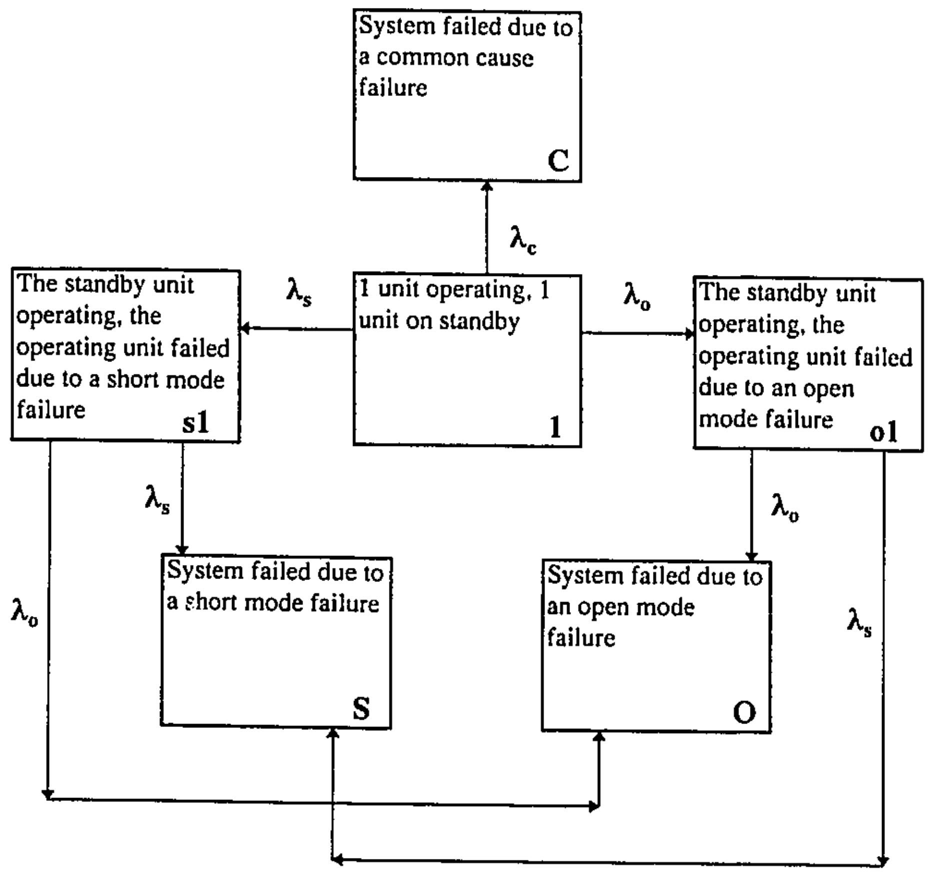


Figure 2.5 System state space diagram without repair: one unit operating, the other on standby

$$P_s(s) = \frac{\lambda_s(\lambda_o + \lambda_s)}{s(\lambda_o + \lambda_s + s)(\lambda_o + \lambda_s + \lambda_c + s)} \tag{2.28}$$

$$P_o(s) = \frac{\lambda_o(\lambda_o + \lambda_s)}{s(\lambda_o + \lambda_s + s)(\lambda_o + \lambda_s + \lambda_c + s)} \tag{2.29}$$

$$P_c(s) = \frac{\lambda_c}{s(\lambda_o + \lambda_s + \lambda_c + s)} \tag{2.30}$$

By taking inverse Laplace transforms of Equations (2.25) - (2.30) the following time dependent state probabilities result:

$$P_1(t) = \frac{1}{e^{(\lambda_o+\lambda_s+\lambda_c)t}} \tag{2.31}$$

$$P_{o1}(t) = \frac{\lambda_o}{\lambda_c}\left[\frac{1}{e^{(\lambda_o+\lambda_s)t}} - \frac{1}{e^{(\lambda_o+\lambda_s+\lambda_c)t}}\right] \tag{2.32}$$

$$P_{s1}(t) = \frac{\lambda_s}{\lambda_c}\left[\frac{1}{e^{(\lambda_o+\lambda_s)t}} - \frac{1}{e^{(\lambda_o+\lambda_s+\lambda_c)t}}\right] \tag{2.33}$$

$$P_s(t) = \frac{\lambda_s}{(\lambda_o + \lambda_s + \lambda_c)} - \frac{\lambda_s}{e^{(\lambda_o+\lambda_s)t}\lambda_c} + \frac{\lambda_s(\lambda_o + \lambda_s)}{e^{(\lambda_o+\lambda_s+\lambda_c)t}\lambda_c(\lambda_o + \lambda_s + \lambda_c)} \tag{2.34}$$

$$P_o(t) = \frac{\lambda_o}{(\lambda_o + \lambda_s + \lambda_c)} - \frac{\lambda_o}{e^{(\lambda_o+\lambda_s)t}\lambda_c} + \frac{\lambda_o(\lambda_o + \lambda_s)}{e^{(\lambda_o+\lambda_s+\lambda_c)t}\lambda_c(\lambda_o + \lambda_s + \lambda_c)} \tag{2.35}$$

$$P_c(t) = \frac{\lambda_c}{(\lambda_o + \lambda_s + \lambda_c)}\left[1 - \frac{1}{e^{(\lambda_o+\lambda_s+\lambda_c)t}}\right] \tag{2.36}$$

The system reliability is given by

$$\begin{aligned} R(t) &= P_1(t) + P_{o1}(t) + P_{s1}(t) \\ &= \frac{(\lambda_o + \lambda_s)}{e^{(\lambda_o+\lambda_s)t}\lambda_c} - \frac{\lambda_o + \lambda_s - \lambda_o}{e^{(\lambda_o+\lambda_s+\lambda_c)t}\lambda_c} \end{aligned} \tag{2.37}$$

The system mean time to failure (MTTF) is expressed by

$$MTTF = \lim_{s \to 0} R(s)$$
$$= \lim_{s \to 0} [P_1(s) + P_{o1}(s) + P_{s1}(s)]$$
$$= \frac{2}{(\lambda_o + \lambda_s + \lambda_c)} \quad (2.38)$$

where $R(s)$ is the Laplace transform of the system reliability.

The system reliability given by Equation (2.37) is plotted in Figure 2.6 for specified values of model parameters. The plots of Equations (2.31)-(2.36) are shown in Figure 2.7. Figure 2.8 and 2.9 present the plots of Equation (2.38). These plots as well as those of the other special cases are discussed at the end of Section 2.2.

2.2.3 Standby system without Repair: one unit operating, two units on standby

The system state space diagram is shown in Figure 2.10. By substituting n=2 in Equations (2.1) - (2.8), we obtain the following differential equations:

$$\frac{dP_1(t)}{dt} = -(\lambda_o + \lambda_s + \lambda_c) P_1(t) \quad (2.39)$$

$$\frac{dP_{o1}(t)}{dt} = -(\lambda_o + \lambda_s + \lambda_c) P_{o1}(t) + \lambda_o P_1(t) \quad (2.40)$$

$$\frac{dP_{s1}(t)}{dt} = -(\lambda_o + \lambda_s + \lambda_c) P_{s1}(t) + \lambda_s P_1(t) \quad (2.41)$$

$$\frac{dP_{oo2}(t)}{dt} = -(\lambda_o + \lambda_s) P_{oo2}(t) + \lambda_o P_{o1}(t) \quad (2.42)$$

$$\frac{dP_{ss2}(t)}{dt} = -(\lambda_o + \lambda_s) P_{ss2}(t) + \lambda_s P_{s1}(t) \quad (2.43)$$

$$\frac{dP_{os2}(t)}{dt} = -(\lambda_o + \lambda_s) P_{os2}(t) + \lambda_s P_{o1}(t) \quad (2.44)$$

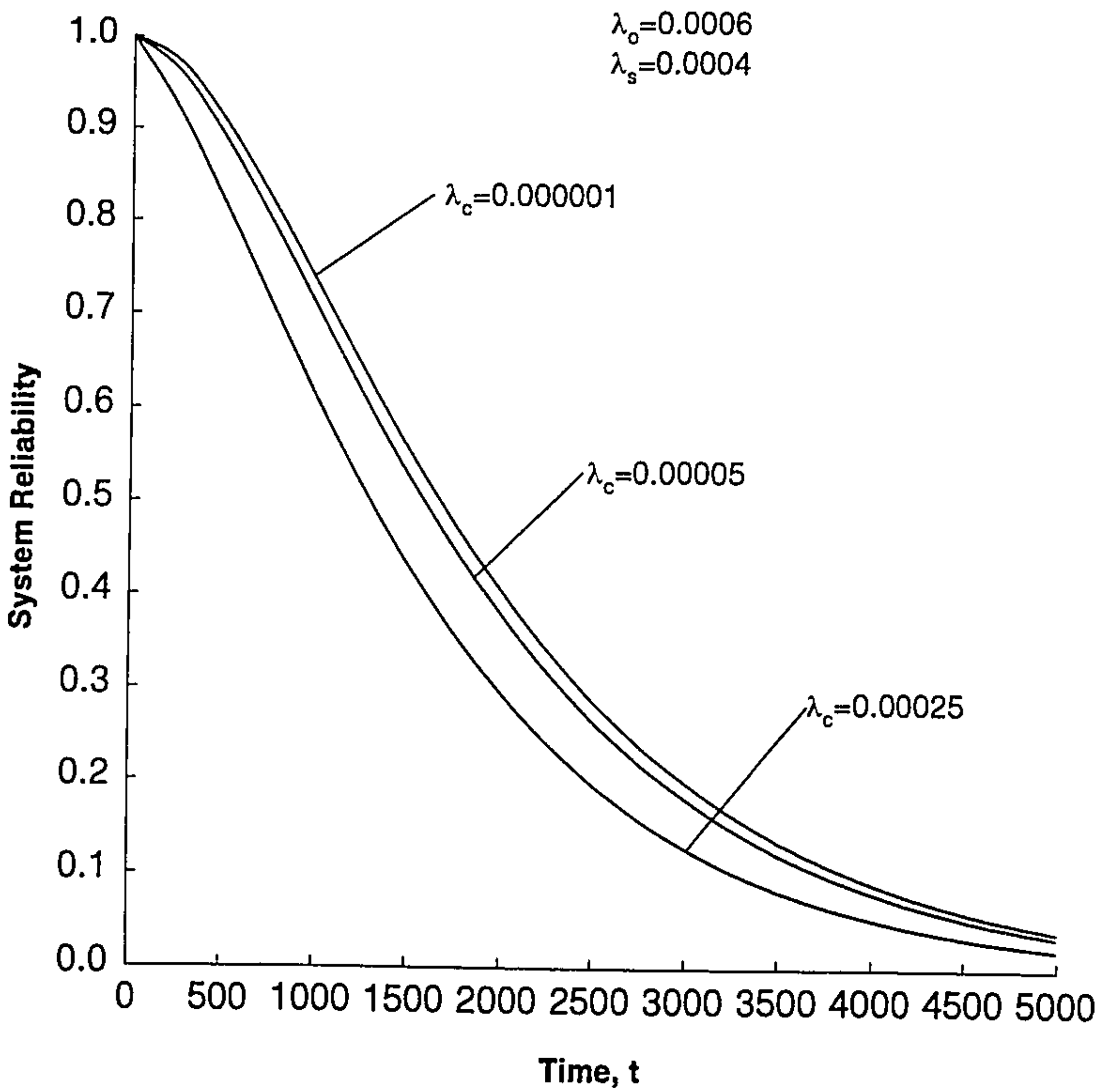


Figure 2.6 The standby system without repair reliability plots: one unit operating, the other on standby

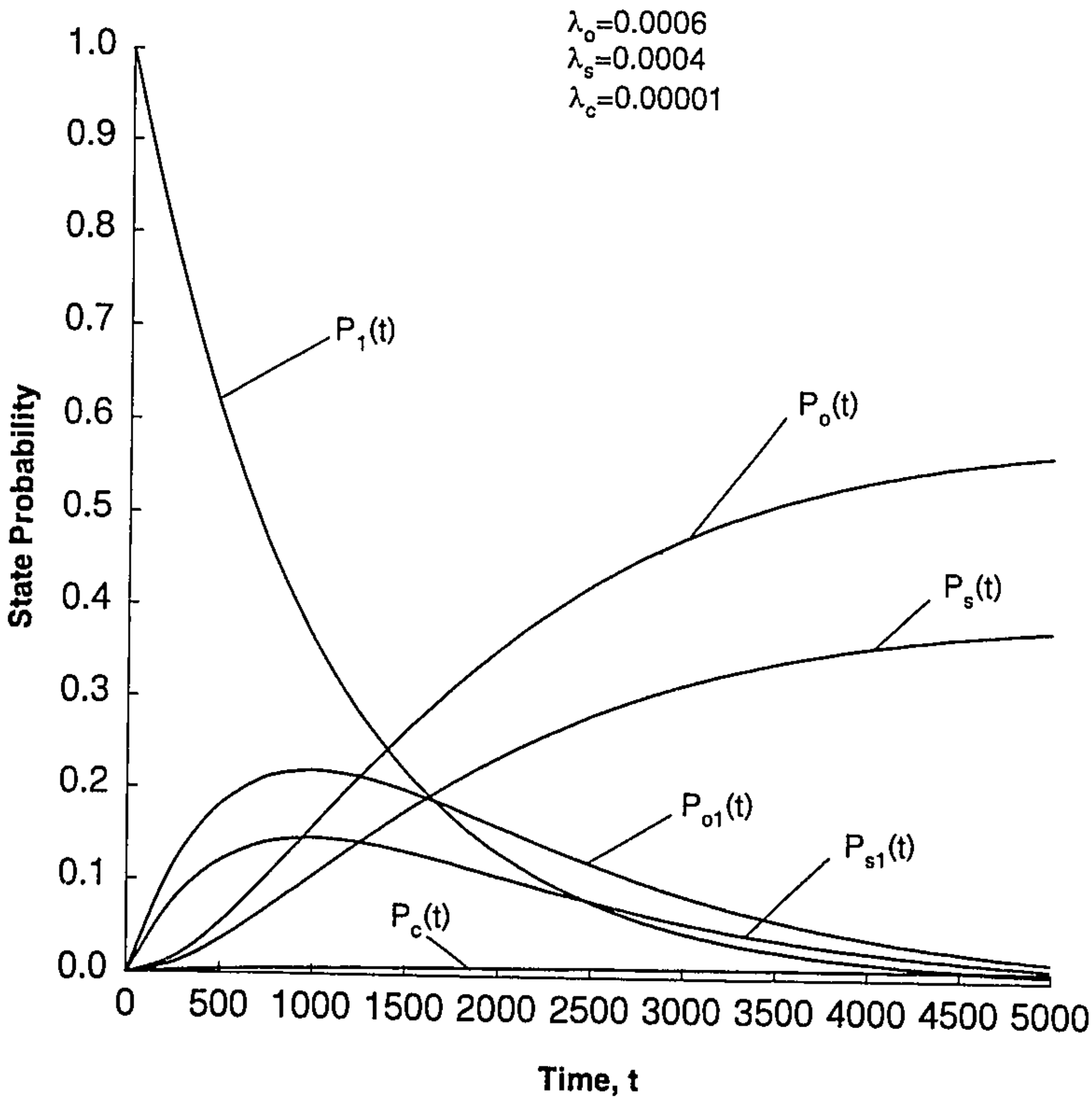


Figure 2.7 The standby system without repair state probability plot: one unit operating, the other on standby

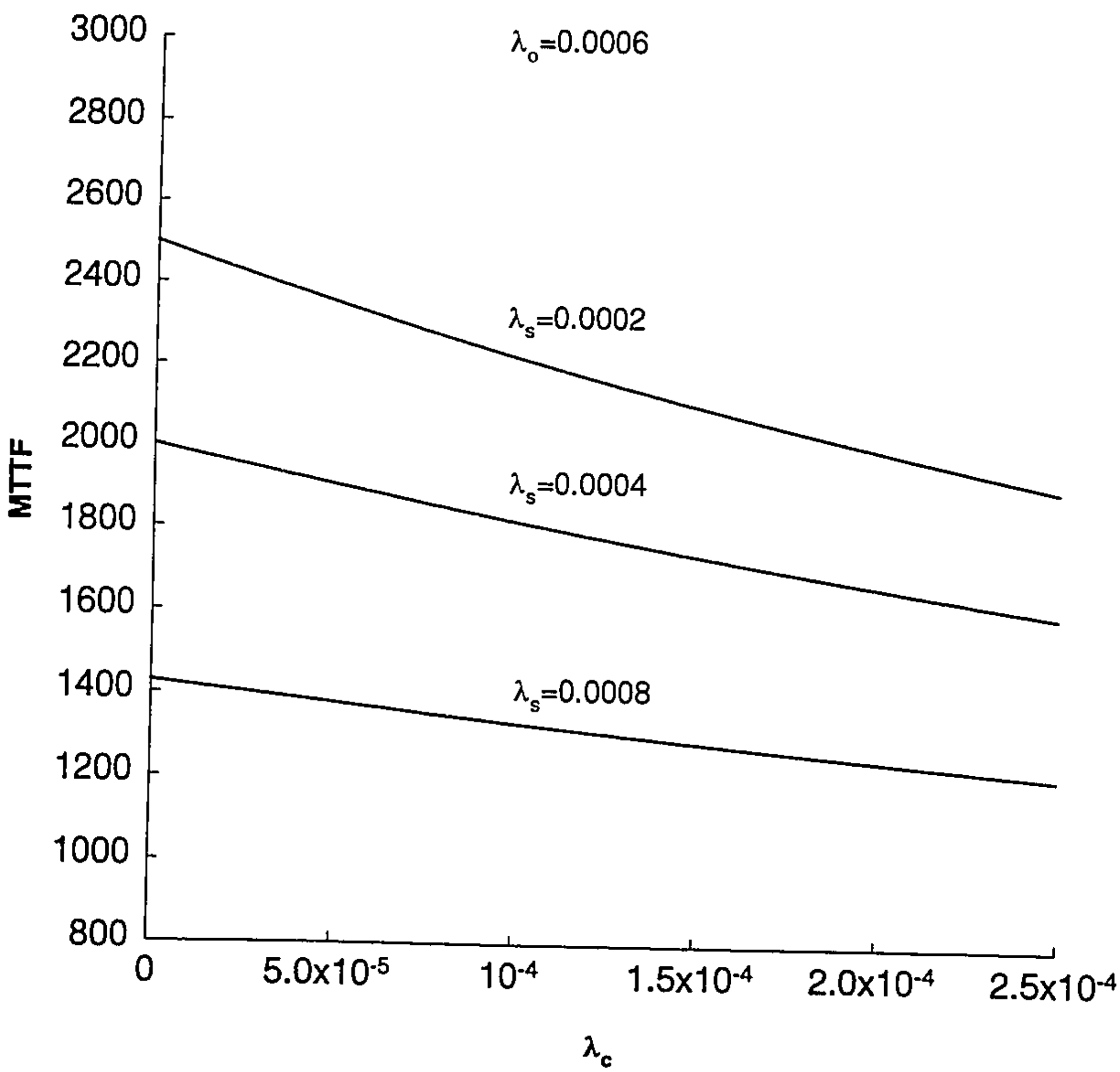


Figure 2.8 MTTF vs common cause failure rate, λ_c for the standby system (one unit operating, the other on standby) without repair

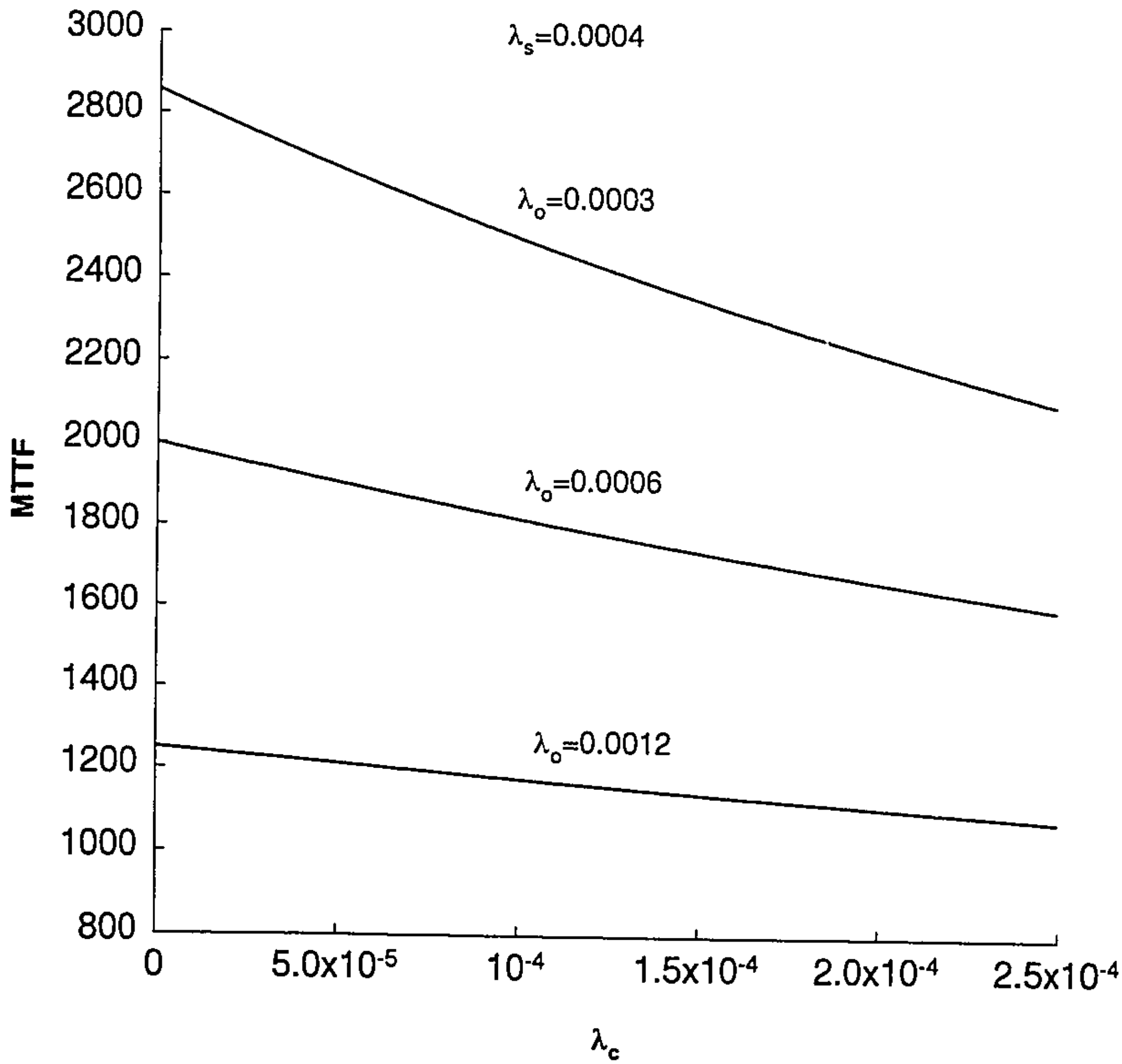


Figure 2.9 MTTF vs common cause failure rate, λ_c for the standby system (one unit operating, the other on standby) without repair

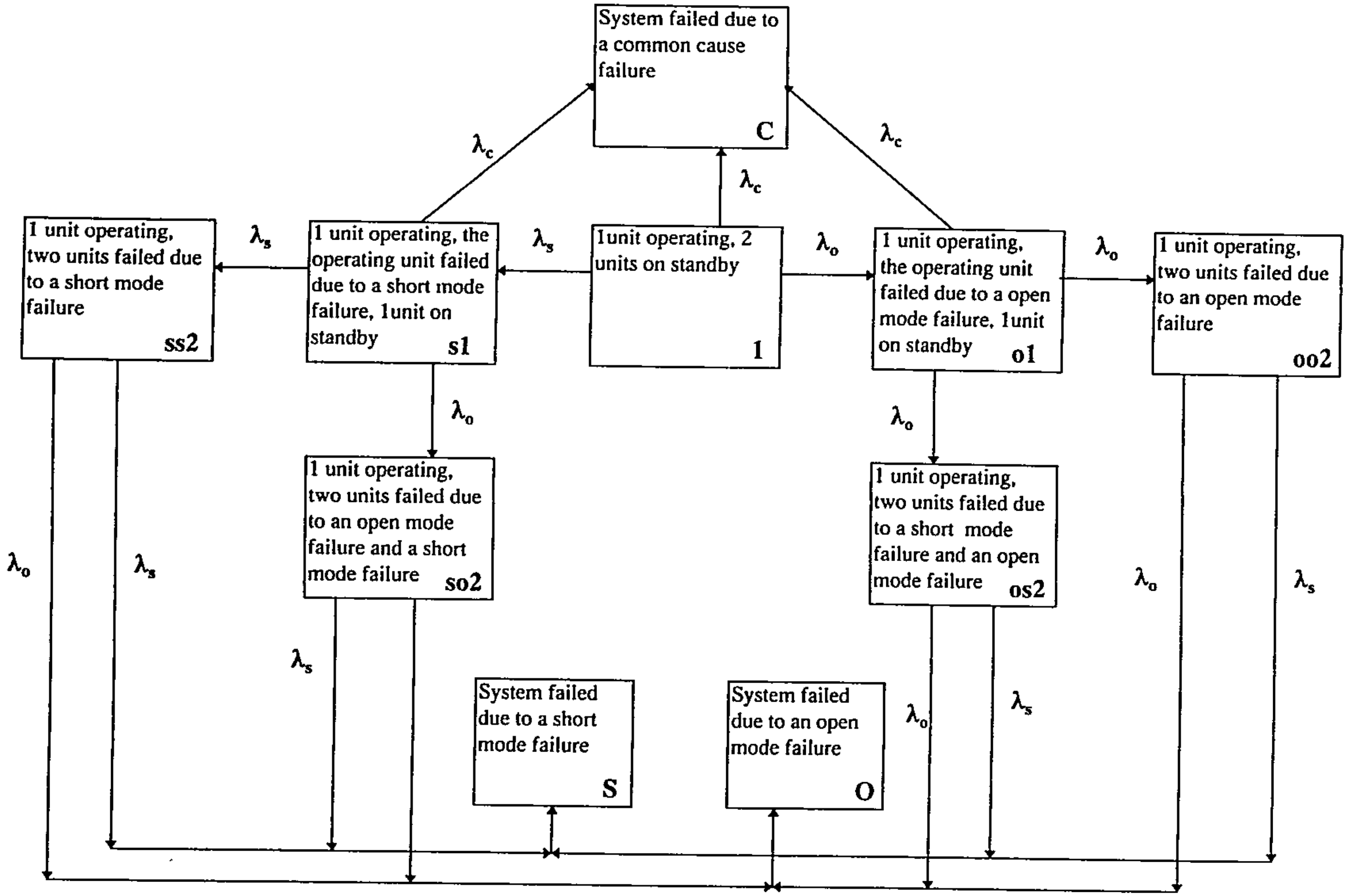


Figure 2.10 System state space diagram without repair: one unit operating, two units on standby

$$\frac{dP_{so2}(t)}{dt} = -(\lambda_o + \lambda_s) P_{so2}(t) + \lambda_o P_{s1}(t) \tag{2.45}$$

$$\frac{dP_s(t)}{dt} = \lambda_s [P_{oo2}(t) + P_{os2}(t) + P_{so2}(t) + P_{ss2}(t)] \tag{2.46}$$

$$\frac{dP_o(t)}{dt} = \lambda_o [P_{oo2}(t) + P_{os2}(t) + P_{so2}(t) + P_{ss2}(t)] \tag{2.47}$$

$$\frac{dP_c(t)}{dt} = \lambda_c [P_1(t) + P_{o1}(t) + P_{s1}(t)] \tag{2.48}$$

At time $t = 0$, $P_1(0) = 1$, and all the other initial condition probabilities are equal to zero.

By taking Laplace transforms of Equations (2.39) - (2.48) and solving the resulting equations, the Laplace transforms of the state probabilities are as follows:

$$P_1(s) = \frac{1}{\lambda_o + \lambda_s + \lambda_c + s} \tag{2.49}$$

$$P_{o1}(s) = \frac{\lambda_o}{(\lambda_o + \lambda_s + \lambda_c + s)^2} \tag{2.50}$$

$$P_{s1}(s) = \frac{\lambda_s}{(\lambda_o + \lambda_s + \lambda_c + s)^2} \tag{2.51}$$

$$P_{oo2}(s) = \frac{\lambda_o^{\ 2}}{(\lambda_o + \lambda_s + s)(\lambda_o + \lambda_s + \lambda_c + s)^2} \tag{2.52}$$

$$P_{ss2}(s) = \frac{\lambda_s^{\ 2}}{(\lambda_o + \lambda_s + s)(\lambda_o + \lambda_s + \lambda_c + s)^2} \tag{2.53}$$

$$P_{os2}(s) = \frac{\lambda_o \lambda_s}{(\lambda_o + \lambda_s + s)(\lambda_o + \lambda_s + \lambda_c + s)^2} \tag{2.54}$$

$$P_{so2}(s) = \frac{\lambda_o \lambda_s}{(\lambda_o + \lambda_s + s)(\lambda_o + \lambda_s + \lambda_c + s)^2} \tag{2.55}$$

$$P_s(s) = \frac{\lambda_s (\lambda_o + \lambda_s)^2}{s(\lambda_o + \lambda_s + s)(\lambda_o + \lambda_s + \lambda_c + s)^2} \tag{2.56}$$

$$P_o(s) = \frac{\lambda_o (\lambda_o + \lambda_s)^2}{s(\lambda_o + \lambda_s + s)(\lambda_o + \lambda_s + \lambda_c + s)^2} \tag{2.57}$$

$$P_c(s) = \frac{\lambda_c(2\lambda_o + 2\lambda_s + \lambda_c + s)}{s(\lambda_o + \lambda_s + s)(\lambda_o + \lambda_s + \lambda_c + s)^2} \tag{2.58}$$

By taking inverse Laplace transforms of Equations (2.49) - (2.58) the following time dependent state probabilities result:

$$P_1(t) = \frac{1}{e^{(\lambda_o+\lambda_s+\lambda_c)t}} \tag{2.59}$$

$$P_{o1}(t) = \frac{\lambda_o t}{e^{(\lambda_o+\lambda_s+\lambda_c)t}} \tag{2.60}$$

$$P_{s1}(t) = \frac{\lambda_s t}{e^{(\lambda_o+\lambda_s+\lambda_c)t}} \tag{2.61}$$

$$P_{oo2}(t) = \frac{\lambda_o^{\,2}}{e^{(\lambda_o+\lambda_s)t}\lambda_c^{\,2}} - \frac{\lambda_o^{\,2}}{e^{(\lambda_o+\lambda_s+\lambda_c)t}\lambda_c^{\,2}} - \frac{\lambda_o^{\,2} t}{e^{(\lambda_o+\lambda_s+\lambda_c)t}\lambda_c} \tag{2.62}$$

$$P_{ss2}(t) = \frac{\lambda_s^{\,2}}{e^{(\lambda_o+\lambda_s)t}\lambda_c^{\,2}} - \frac{\lambda_s^{\,2}}{e^{(\lambda_o+\lambda_s+\lambda_c)t}\lambda_c^{\,2}} - \frac{\lambda_s^{\,2} t}{e^{(\lambda_o+\lambda_s+\lambda_c)t}\lambda_c} \tag{2.63}$$

$$P_{os2}(t) = \frac{\lambda_o\lambda_s}{e^{(\lambda_o+\lambda_s)t}\lambda_c^{\,2}} - \frac{\lambda_o\lambda_s}{e^{(\lambda_o+\lambda_s+\lambda_c)t}\lambda_c^{\,2}} - \frac{\lambda_o\lambda_s t}{e^{(\lambda_o+\lambda_s+\lambda_c)t}\lambda_o} \tag{2.64}$$

$$P_{so2}(t) = \frac{\lambda_o\lambda_s}{e^{(\lambda_o+\lambda_s)t}\lambda_c^{\,2}} - \frac{\lambda_o\lambda_s}{e^{(\lambda_o+\lambda_s+\lambda_c)t}\lambda_c^{\,2}} - \frac{\lambda_o\lambda_s t}{e^{(\lambda_o+\lambda_s+\lambda_c)t}\lambda_o} \tag{2.65}$$

$$P_s(t) = -\frac{\lambda_s(\lambda_o+\lambda_s)}{e^{(\lambda_o+\lambda_s)t}\lambda_c^{\,2}} + \frac{\lambda_s(\lambda_o+\lambda_s)}{(\lambda_o+\lambda_s+\lambda_c)^2} + \frac{\lambda_s(\lambda_o+\lambda_s)^2(\lambda_o+\lambda_s+2\lambda_c)}{e^{(\lambda_o+\lambda_s+\lambda_c)t}\lambda_c^{\,2}(\lambda_o+\lambda_s+\lambda_c)^2} + \frac{\lambda_s(\lambda_o+\lambda_s)^2 t}{e^{(\lambda_o+\lambda_s+\lambda_c)t}\lambda_c(\lambda_o+\lambda_s+\lambda_c)} \tag{2.66}$$

$$P_o(t) = -\frac{\lambda_o(\lambda_o+\lambda_s)}{e^{(\lambda_o+\lambda_s)t}\lambda_c^{\,2}} + \frac{\lambda_o(\lambda_o+\lambda_s)}{(\lambda_o+\lambda_s+\lambda_c)^2} + \frac{\lambda_o(\lambda_o+\lambda_s)^2(\lambda_o+\lambda_s+2\lambda_c)}{e^{(\lambda_o+\lambda_s+\lambda_c)t}\lambda_c^{\,2}(\lambda_o+\lambda_s+\lambda_c)^2} + \frac{\lambda_o(\lambda_o+\lambda_s)^2 t}{e^{(\lambda_o+\lambda_s+\lambda_c)t}\lambda_c(\lambda_o+\lambda_s+\lambda_c)} \tag{2.67}$$

$$P_c(t) = \frac{\lambda_c(2\lambda_o+2\lambda_s+\lambda_c)}{(\lambda_o+\lambda_s+\lambda_c)^2} - \frac{\lambda_c(2\lambda_o+2\lambda_s+\lambda_c)}{e^{(\lambda_o+\lambda_s+\lambda_c)t}(\lambda_o+\lambda_s+\lambda_c)^2} - \frac{\lambda_c(\lambda_o+\lambda_s)t}{e^{(\lambda_o+\lambda_s+\lambda_c)t}(\lambda_o+\lambda_s+\lambda_c)} \tag{2.68}$$

The system reliability is given by

$$R(t) = P_1(t) + P_{o1}(t) + P_{s1}(t) + P_{oo2}(t) + P_{ss2}(t) + P_{os2}(t) + P_{so2}(t)$$
$$= \frac{(\lambda_o + \lambda_s)^2}{e^{(\lambda_o+\lambda_s)t}\lambda_c^{\ 2}} - \frac{(\lambda_o + \lambda_s)^2 - \lambda_c^{\ 2}}{e^{(\lambda_o+\lambda_s+\lambda_c)t}\lambda_c^{\ 2}} - \frac{(\lambda_o + \lambda_s)(\lambda_o + \lambda_s - \lambda_c)t}{e^{(\lambda_o+\lambda_s+\lambda_c)t}\lambda_c} \qquad (2.69)$$

The system mean time to failure (MTTF) is

$$MTTF = \lim_{s \to 0} R(s)$$
$$= \lim_{s \to 0} [P_1(s) + P_{o1}(s) + P_{s1}(s) + P_{oo2}(s) + P_{ss2}(s) + P_{os2}(s) + P_{so2}(s)]$$
$$= \frac{3\lambda_o + 3\lambda_s + \lambda_c}{(\lambda_o + \lambda_s + \lambda_c)^2} \qquad (2.70)$$

In order to make a comparison with the one unit on standby system, the same values of parameters (i.e. as for Equations (2.31)-(2.38)) are chosen for the reliability plots, the state probability plots, and the mean time to failure plots. The system reliability given by Equation (2.69) is plotted in Figure 2.11. The time dependent state probabilities plots are shown in Figure 2.12. The plots of Equation (2.70) are shown in Figure 2.13 and Figure 2.14.

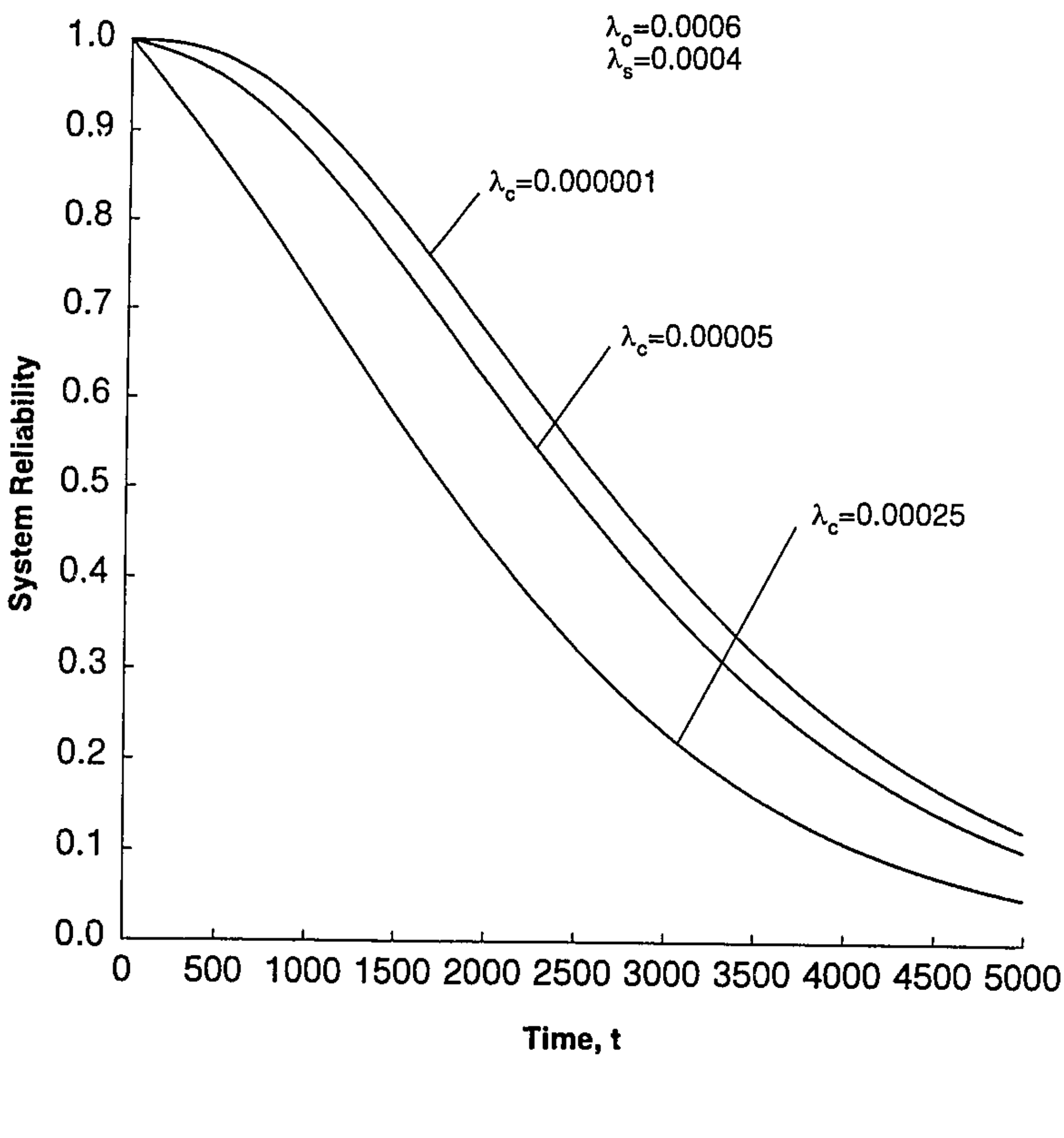


Figure 2.11 The standby system without repair reliability plots: one unit operating, two units on standby

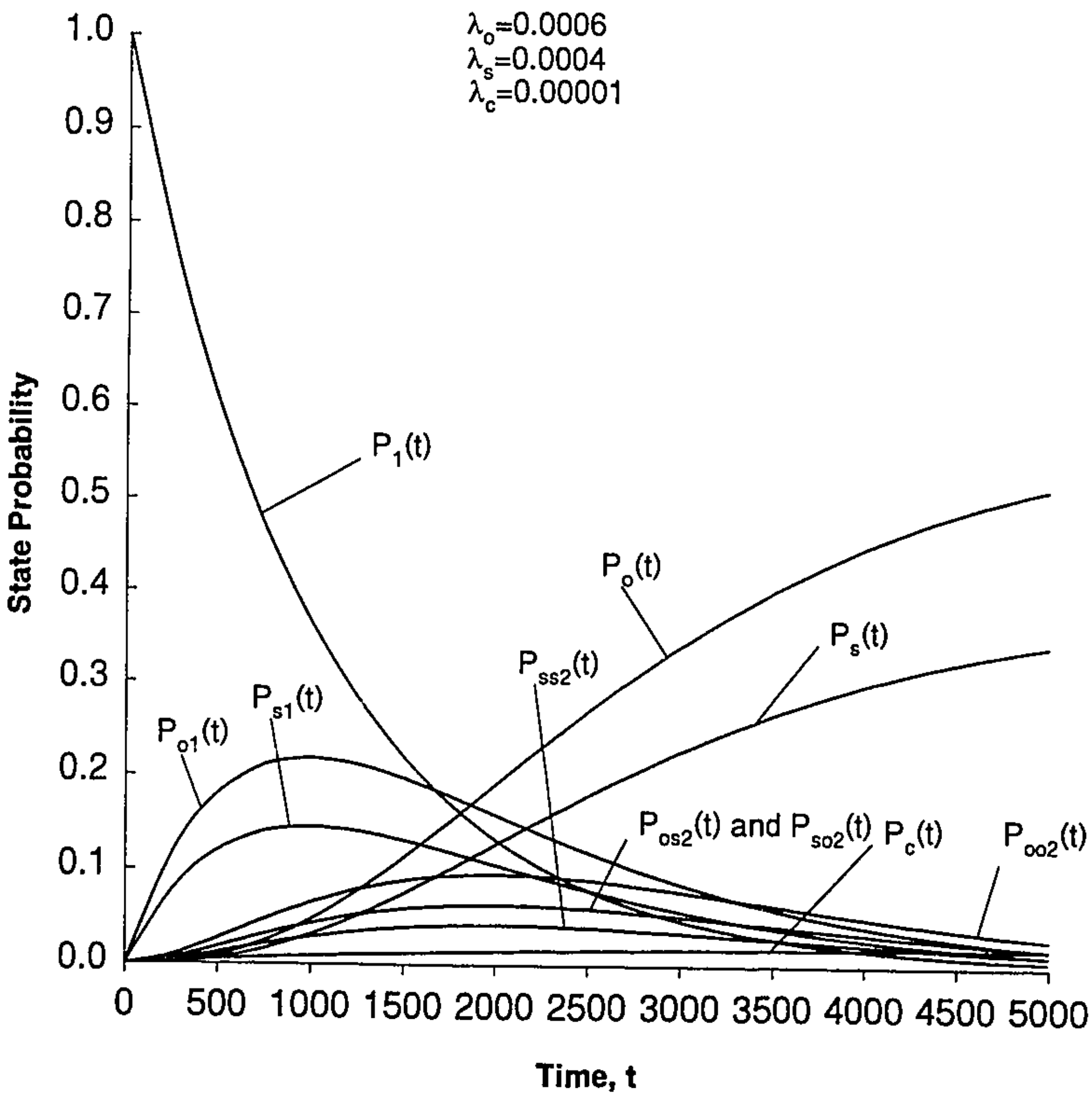


Figure 2.12 The standby system without repair state probability plots: one unit operating, two units on standby

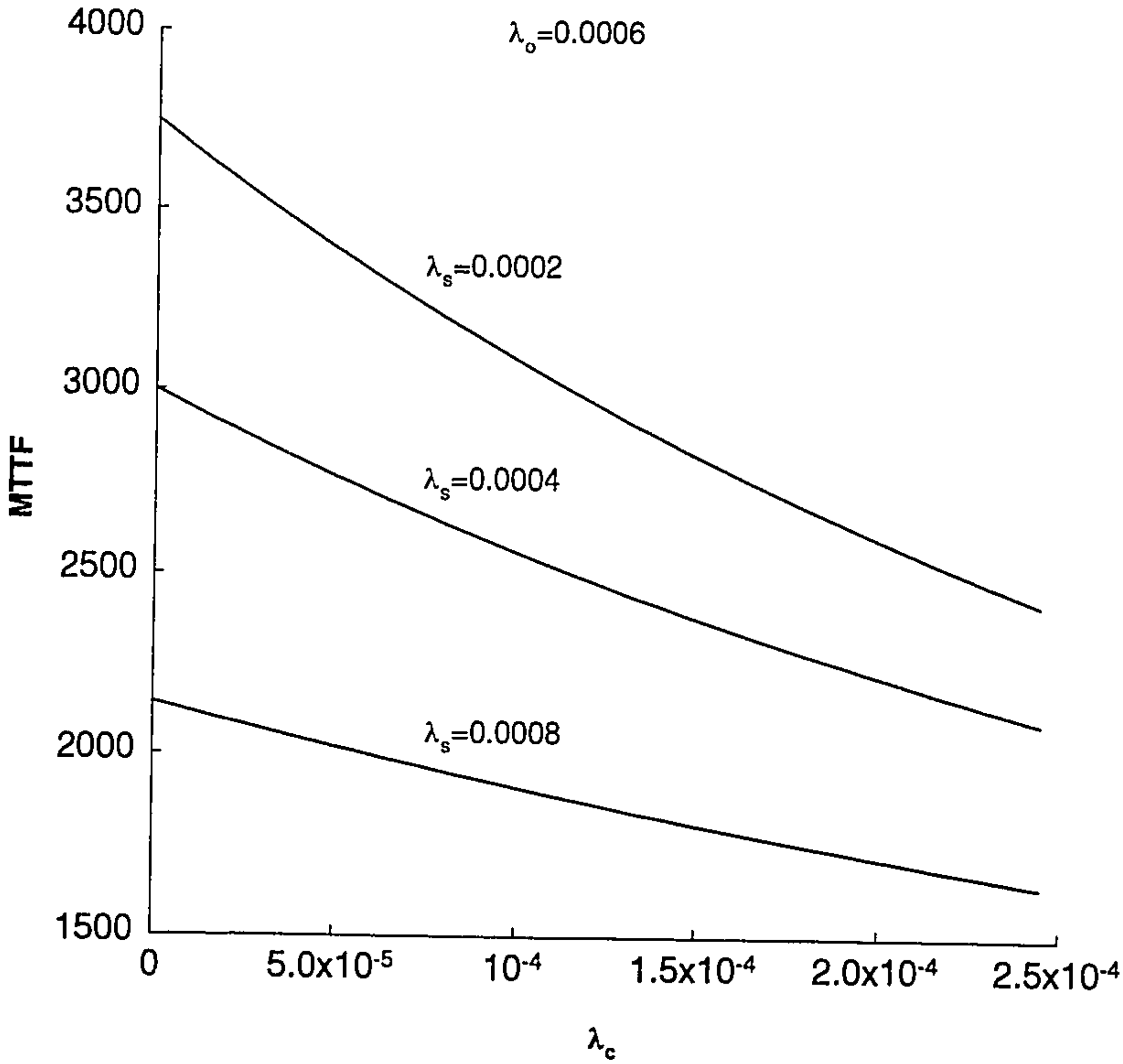


Figure 2.13 MTTF vs common cause failure rate, λ_c for the standby system (one unit operating, two units on standby) without repair

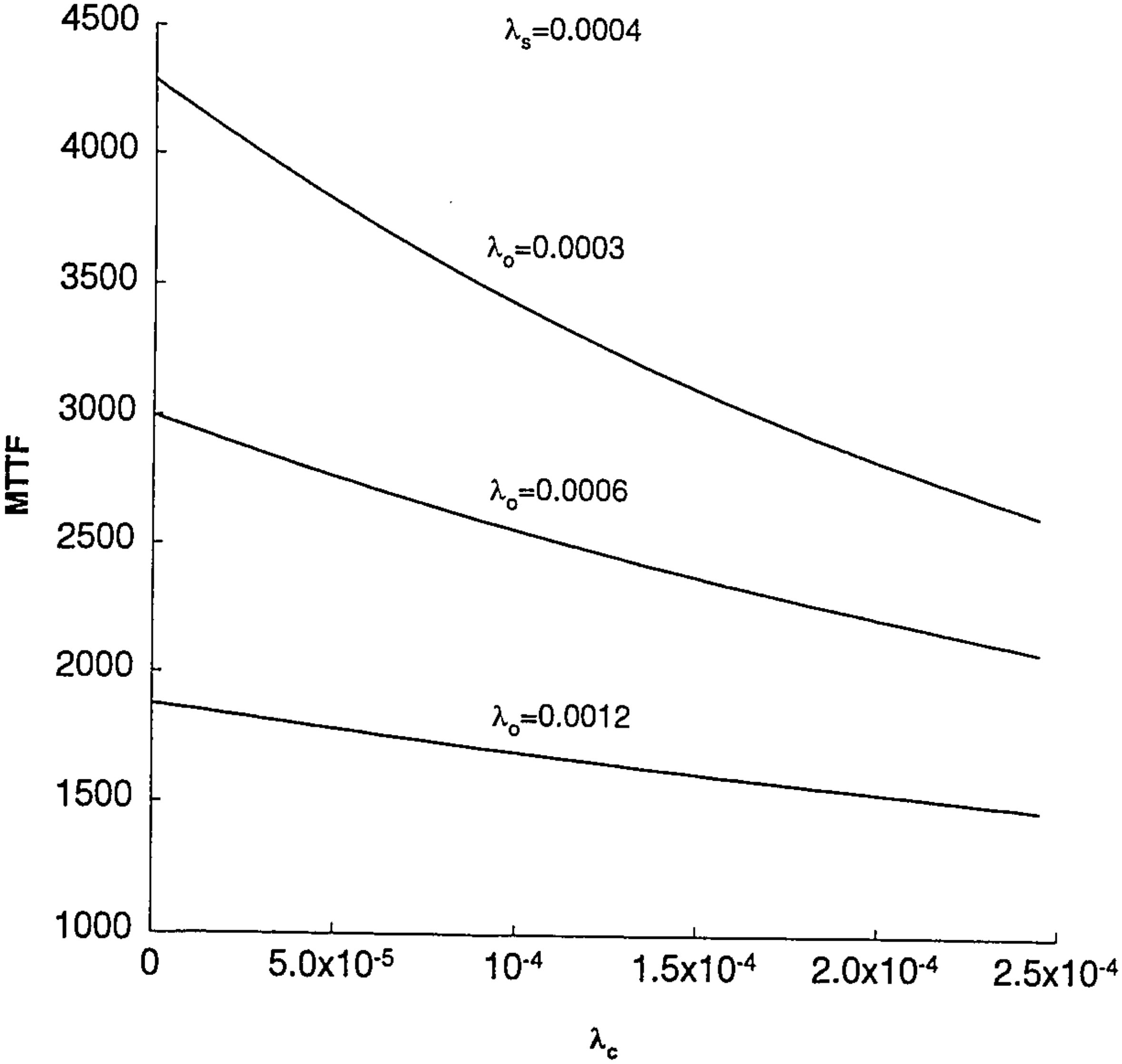


Figure 2.14 MTTF vs common cause failure rate, λ_c for the standby system (one unit operating, two units on standby) without repair

SB=standby

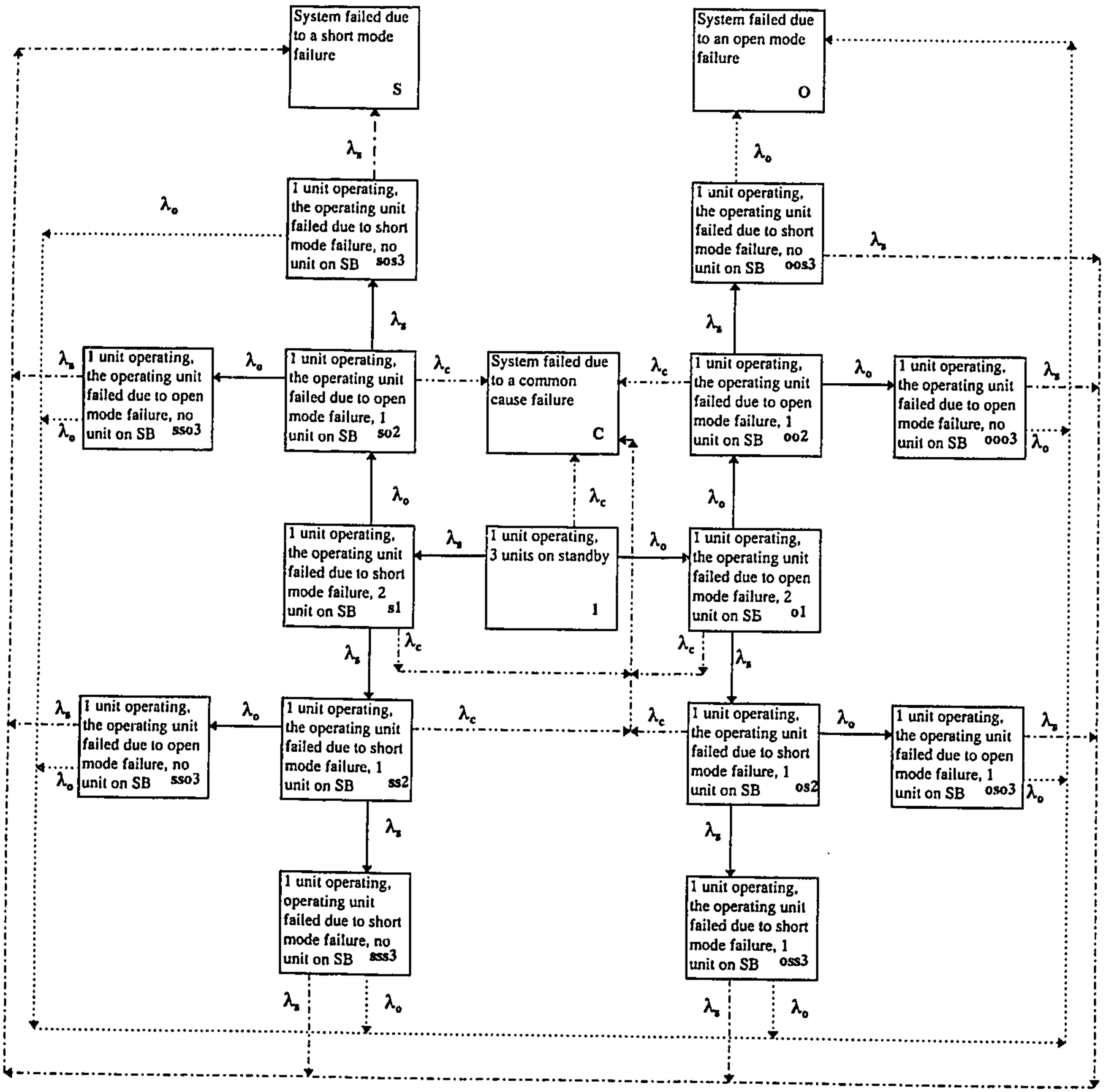


Figure 2.15 System state space diagram without repair: one unit operating, three units on standby

2.2.4 Standby system without Repair: one unit operating, three units on standby

The system state space diagram is shown in Figure 2.15. By setting n=3 in Equations (2.1) - (2.8), we obtain the following differential equations:

$$\frac{dP_1(t)}{dt} = -(\lambda_o + \lambda_s + \lambda_c) P_1(t) \tag{2.71}$$

$$\frac{dP_{o1}(t)}{dt} = -(\lambda_o + \lambda_s + \lambda_c) P_{o1}(t) + \lambda_o P_1(t) \tag{2.72}$$

$$\frac{dP_{s1}(t)}{dt} = -(\lambda_o + \lambda_s + \lambda_c) P_{s1}(t) + \lambda_s P_1(t) \tag{2.73}$$

$$\frac{dP_{oo2}(t)}{dt} = -(\lambda_o + \lambda_s + \lambda_c) P_{oo2}(t) + \lambda_o P_{o1}(t) \tag{2.74}$$

$$\frac{dP_{os2}(t)}{dt} = -(\lambda_o + \lambda_s + \lambda_c) P_{os2}(t) + \lambda_s P_{o1}(t) \tag{2.75}$$

$$\frac{dP_{ss2}(t)}{dt} = -(\lambda_o + \lambda_s + \lambda_c) P_{ss2}(t) + \lambda_s P_{s1}(t) \tag{2.76}$$

$$\frac{dP_{so2}(t)}{dt} = -(\lambda_o + \lambda_s + \lambda_c) P_{so2}(t) + \lambda_o P_{s1}(t) \tag{2.77}$$

$$\frac{dP_{ooo3}(t)}{dt} = -(\lambda_o + \lambda_s) P_{ooo3}(t) + \lambda_o P_{oo2}(t) \tag{2.78}$$

$$\frac{dP_{oos3}(t)}{dt} = -(\lambda_o + \lambda_s) P_{oos3}(t) + \lambda_s P_{oo2}(t) \tag{2.79}$$

$$\frac{dP_{oso3}(t)}{dt} = -(\lambda_o + \lambda_s) P_{oso3}(t) + \lambda_o P_{os2}(t) \tag{2.80}$$

$$\frac{dP_{oss3}(t)}{dt} = -(\lambda_o + \lambda_s) P_{oss3}(t) + \lambda_s P_{os2}(t) \tag{2.81}$$

$$\frac{dP_{sss3}(t)}{dt} = -(\lambda_o + \lambda_s) P_{sss3}(t) + \lambda_s P_{ss2}(t) \tag{2.82}$$

$$\frac{dP_{sso3}(t)}{dt} = -(\lambda_o + \lambda_s) P_{sso3}(t) + \lambda_o P_{ss2}(t) \tag{2.83}$$

$$\frac{dP_{sos3}(t)}{dt} = -(\lambda_o + \lambda_s)P_{sos3}(t) + \lambda_s P_{so2}(t) \tag{2.84}$$

$$\frac{dP_{soo3}(t)}{dt} = -(\lambda_o + \lambda_s)P_{soo3}(t) + \lambda_o P_{so2}(t) \tag{2.85}$$

$$\frac{dP_s(t)}{dt} = \lambda_s[P_{ooo3}(t) + P_{oos3}(t) + P_{oso3}(t) + P_{osss}(t) + P_{sss3}(t) + P_{sso3}(t) + P_{sos3}(t) + P_{soo3}(t)] \tag{2.86}$$

$$\frac{dP_o(t)}{dt} = \lambda_o[P_{ooo3}(t) + P_{oos3}(t) + P_{oso3}(t) + P_{osss}(t) + P_{sss3}(t) + P_{sso3}(t) + P_{sos3}(t) + P_{soo3}(t)] \tag{2.87}$$

$$\frac{dP_c(t)}{dt} = \lambda_c[P_1(t) + P_{o1}(t) + P_{s1}(t) + P_{oo2}(t) + P_{os2}(t) + P_{so2}(t) + P_{ss2}(t)] \tag{2.88}$$

At time $t = 0$, $P_1(0) = 1$, and all the other initial condition probabilities are equal to zero.

By taking Laplace transforms of Equations (2.71) - (2.88) and solving the resulting equations, the Laplace transforms of the state probabilities are as follow:

$$P_1(s) = \frac{1}{\lambda_o + \lambda_s + \lambda_c + s} \tag{2.89}$$

$$P_{o1}(s) = \frac{\lambda_o}{(\lambda_o + \lambda_s + \lambda_c + s)^2} \tag{2.90}$$

$$P_{s1}(s) = \frac{\lambda_s}{(\lambda_o + \lambda_s + \lambda_c + s)^2} \tag{2.91}$$

$$P_{oo2}(s) = \frac{\lambda_o^2}{(\lambda_o + \lambda_s + \lambda_c + s)^3} \tag{2.92}$$

$$P_{os2}(s) = P_{so2}(s) = \frac{\lambda_o\lambda_s}{(\lambda_o + \lambda_s + \lambda_c + s)^3} \tag{2.93}$$

$$P_{ss2}(s) = \frac{\lambda_s^2}{(\lambda_o + \lambda_s + \lambda_c + s)^3} \tag{2.94}$$

$$P_{ooo3}(s) = \frac{\lambda_o{}^3}{(\lambda_o + \lambda_s + s)(\lambda_o + \lambda_s + \lambda_c + s)^3} \tag{2.95}$$

$$P_{oos3}(s) = P_{oso3}(s) = P_{soo3}(s) = \frac{\lambda_o{}^2\lambda_s}{(\lambda_o+\lambda_s+s)(\lambda_o+\lambda_s+\lambda_c+s)^3} \tag{2.96}$$

$$P_{oss2}(s) = P_{sso3}(s) = P_{sos3}(s) = \frac{\lambda_o\lambda_s^2}{(\lambda_o+\lambda_s+s)(\lambda_o+\lambda_s+\lambda_c+s)^3} \tag{2.97}$$

$$P_{sss3}(s) = \frac{\lambda_s{}^3}{(\lambda_o+\lambda_s+s)(\lambda_o+\lambda_s+\lambda_c+s)^3} \tag{2.98}$$

$$P_s(s) = \frac{\lambda_s(\lambda_o+\lambda_s)^3}{s(\lambda_o+\lambda_s+s)(\lambda_o+\lambda_s+\lambda_c+s)^3} \tag{2.99}$$

$$P_o(s) = \frac{\lambda_o(\lambda_o+\lambda_s)^3}{s(\lambda_o+\lambda_s+s)(\lambda_o+\lambda_s+\lambda_c+s)^3} \tag{2.100}$$

$$P_c(s) = \frac{\lambda_c[3\lambda_c(\lambda_o+\lambda_s)^2+3\lambda_c(\lambda_o+\lambda_s)+3s(\lambda_o+\lambda_s)+(\lambda_c+s)^2]}{s(\lambda_o+\lambda_s+\lambda_c+s)^3} \tag{2.101}$$

By taking inverse Laplace transforms of Equations (2.89) - (2.101) the following time dependent state probabilities result:

$$P_1(t) = \frac{1}{e^{(\lambda_o+\lambda_s+\lambda_c)t}} \tag{2.102}$$

$$P_{o1}(t) = \frac{\lambda_o t}{e^{(\lambda_o+\lambda_s+\lambda_c)t}} \tag{2.103}$$

$$P_{s1}(t) = \frac{\lambda_s t}{e^{(\lambda_o+\lambda_s+\lambda_c)t}} \tag{2.104}$$

$$P_{oo2}(t) = \frac{\lambda_o^2 t^2}{2e^{(\lambda_o+\lambda_s+\lambda_c)t}} \tag{2.105}$$

$$P_{os2}(t) = P_{so2}(t) = \frac{\lambda_o\lambda_s t^2}{2e^{(\lambda_o+\lambda_s+\lambda_c)t}} \tag{2.106}$$

$$P_{ss2}(t) = \frac{\lambda_s^2 t^2}{2e^{(\lambda_o+\lambda_s+\lambda_c)t}} \tag{2.107}$$

$$P_{ooo3}(t) = \frac{\lambda_o^3}{e^{(\lambda_o+\lambda_s)t}\lambda_o^3} - \frac{\lambda_o^3}{e^{(\lambda_o+\lambda_s+\lambda_c)t}\lambda_o^3} - \frac{\lambda_o^3 t}{e^{(\lambda_o+\lambda_s+\lambda_c)t}\lambda_c^3} - \frac{\lambda_o^3 t^2}{2e^{(\lambda_o+\lambda_s+\lambda_c)t}\lambda_c} \tag{2.108}$$

$$P_{oos3}(t)=P_{oso3}(t)=P_{soo3}(t)=\frac{\lambda_o^2\lambda_s}{e^{(\lambda_o+\lambda_s)t}\lambda_c^3}-\frac{\lambda_o^2\lambda_s}{e^{(\lambda_o+\lambda_s+\lambda_c)t}\lambda_c^3}-\frac{\lambda_o^2\lambda_s t}{e^{(\lambda_o+\lambda_s+\lambda_c)t}\lambda_c^3}-\frac{\lambda_o^2\lambda_s t^2}{2e^{(\lambda_o+\lambda_s+\lambda_c)t}\lambda_c} \tag{2.109}$$

$$P_{oss3}(t)=P_{sso3}(t)=P_{sos3}(t)=\frac{\lambda_o\lambda_s^2}{e^{(\lambda_o+\lambda_s)t}\lambda_c^3}-\frac{\lambda_o\lambda_s^2}{e^{(\lambda_o+\lambda_s+\lambda_c)t}\lambda_c^3}-\frac{\lambda_o\lambda_s^2 t}{e^{(\lambda_o+\lambda_s+\lambda_c)t}\lambda_c^3}-\frac{\lambda_o\lambda_s^2 t^2}{2e^{(\lambda_o+\lambda_s+\lambda_c)t}\lambda_c} \tag{2.110}$$

$$P_{sss3}(t)=\frac{\lambda_s^3}{e^{(\lambda_o+\lambda_s)t}\lambda_c^3}-\frac{\lambda_s^3}{e^{(\lambda_o+\lambda_s+\lambda_c)t}\lambda_c^3}-\frac{\lambda_s^3 t}{e^{(\lambda_o+\lambda_s+\lambda_c)t}\lambda_c^3}-\frac{\lambda_s^3 t^2}{2e^{(\lambda_o+\lambda_s+\lambda_c)t}\lambda_c} \tag{2.111}$$

$$P_s(t)=-\frac{\lambda_s(\lambda_o+\lambda_s)^2}{e^{(\lambda_o+\lambda_s)t}\lambda_c^3}+\frac{\lambda_s(\lambda_o+\lambda_s)^3[(\lambda_o+\lambda_s)^2+3\lambda_c(\lambda_o+\lambda_s)+3\lambda_c^2)]}{e^{(\lambda_o+\lambda_s+\lambda_c)t}\lambda_c^3(\lambda_o+\lambda_s+\lambda_c)^3}+\frac{\lambda_s(\lambda_o+\lambda_s)^2}{(\lambda_o+\lambda_s+\lambda_c)^2}+\frac{\lambda_s(\lambda_o+\lambda_s)^3(\lambda_o+\lambda_s+2\lambda_c)t}{e^{(\lambda_o+\lambda_s+\lambda_c)t}\lambda_c^2(\lambda_o+\lambda_s+\lambda_c)^2}+\frac{\lambda_s(\lambda_o+\lambda_s)^3 t^2}{2e^{(\lambda_o+\lambda_s+\lambda_c)t}\lambda_c(\lambda_o+\lambda_s+\lambda_c)} \tag{2.112}$$

$$P_o(t)=-\frac{\lambda_o(\lambda_o+\lambda_s)^2}{e^{(\lambda_o+\lambda_s)t}\lambda_c^3}+\frac{\lambda_o(\lambda_o+\lambda_s)^3[(\lambda_o+\lambda_s)^2+3\lambda_c(\lambda_o+\lambda_s)+3\lambda_c^2)]}{e^{(\lambda_o+\lambda_s+\lambda_c)t}\lambda_c^3(\lambda_o+\lambda_s+\lambda_c)^3}+\frac{\lambda_o(\lambda_o+\lambda_s)^2}{(\lambda_o+\lambda_s+\lambda_c)^2}+\frac{\lambda_o(\lambda_o+\lambda_s)^3(\lambda_o+\lambda_s+2\lambda_c)t}{e^{(\lambda_o+\lambda_s+\lambda_c)t}\lambda_c^2(\lambda_o+\lambda_s+\lambda_c)^2}+\frac{\lambda_s(\lambda_o+\lambda_s)^3 t^2}{2e^{(\lambda_o+\lambda_s+\lambda_c)t}\lambda_c(\lambda_o+\lambda_s+\lambda_c)} \tag{2.113}$$

$$P_c(t)=\frac{\lambda_c[3(\lambda_o+\lambda_s)^2+3\lambda_c(\lambda_o+\lambda_s)+\lambda_c^2)}{(\lambda_o+\lambda_s+\lambda_c)^2}[1-\frac{1}{e^{(\lambda_o+\lambda_s+\lambda_c)t}}]-\frac{(\lambda_o+\lambda_s)\lambda_c(2\lambda_o+2\lambda_s+\lambda_c)t}{e^{(\lambda_o+\lambda_s+\lambda_c)t}(\lambda_o+\lambda_s+\lambda_c)^2}-\frac{\lambda_c(\lambda_o+\lambda_s)^2 t^2}{2e^{(\lambda_o+\lambda_s+\lambda_c)t}(\lambda_o+\lambda_s+\lambda_c)} \tag{2.114}$$

The system reliability is given by

$$R(t)=P_1(t)+P_{o1}(t)+P_{s1}(t)+P_{oo2}(t)+P_{os2}(t)+P_{ss2}(t)+P_{so2}(t)+P_{ooo3}(t)+P_{oos3}(t)+P_{oso3}(t)+P_{oss3}(t)+P_{soo3}(t)+P_{sos3}(t)+P_{sso3}(t)+P_{sss3}(t)$$

$$=\frac{(\lambda_o+\lambda_s)^3}{e^{(\lambda_o+\lambda_s)t}\lambda_c^3}-\frac{(\lambda_o+\lambda_s)^3-\lambda_c^3}{e^{(\lambda_o+\lambda_s+\lambda_c)t}\lambda_c^3}-\frac{(\lambda_o+\lambda_s)[(\lambda_o+\lambda_s)^2-\lambda_c^2]t}{e^{(\lambda_o+\lambda_s+\lambda_c)t}\lambda_c^3}-$$

$$\frac{(\lambda_o+\lambda_s)^2(\lambda_o+\lambda_s-\lambda_c)t^2}{2e^{(\lambda_o+\lambda_s+\lambda_c)t}\lambda_c} \tag{2.115}$$

The system mean time to failure (MTTF) is

$$\begin{aligned} MTTF &= \lim_{s\to 0} R(s) \\ &= \lim_{s\to 0}[P_1(s)+P_{o1}(s)+P_{s1}(s)+P_{oo2}(s)+P_{ss2}(s)+P_{os2}(s)+P_{so2}(s)+ \\ &\quad P_{ooo3}(s)+P_{oos3}(s)+P_{oso3}(s)+P_{oss3}(s)+P_{soo3}(s)+P_{sos3}(s)+P_{sso3}(s)+P_{sss3}(s)] \\ &= \frac{4(\lambda_o+\lambda_s)^2+3\lambda_c(\lambda_o+\lambda_s)+\lambda_c^2}{(\lambda_o+\lambda_s+\lambda_c)^2} \end{aligned} \tag{2.116}$$

In order to make a comparison with the one unit on standby system, the same values of parameters (i.e. as for Equations (2.31)-(2.38)) are chosen for the reliability plots, the state probability plots, and the mean time to failure plots. The system reliability given by Equation (2.115) is plotted in Figure 2.16. The time dependent state probability plots are shown in Figure 2.17. The plots of Equation (2.116) are shown in Figure 2.18 and Figure 2.19.

The results of the analysis indicate that the reliability of the three-state device standby system increases with the increasing number of standby units. The reliability plots show that the system reliability decreases with increasing common cause failure rate. Also, the system mean time to failure (MTTF) increases as the increasing number of standby units but decreases as the common-cause failure rate increases.

From the time dependent state probability plots, it is evident that at any time t, the sum of all state probabilities is equal to unity. It is also observed that the probability of a system open mode failure state P_o and a short mode failure state P_s decreases as the number of standby units increases and the probability of a common cause failure state P_c increases with the increasing number of standby units.

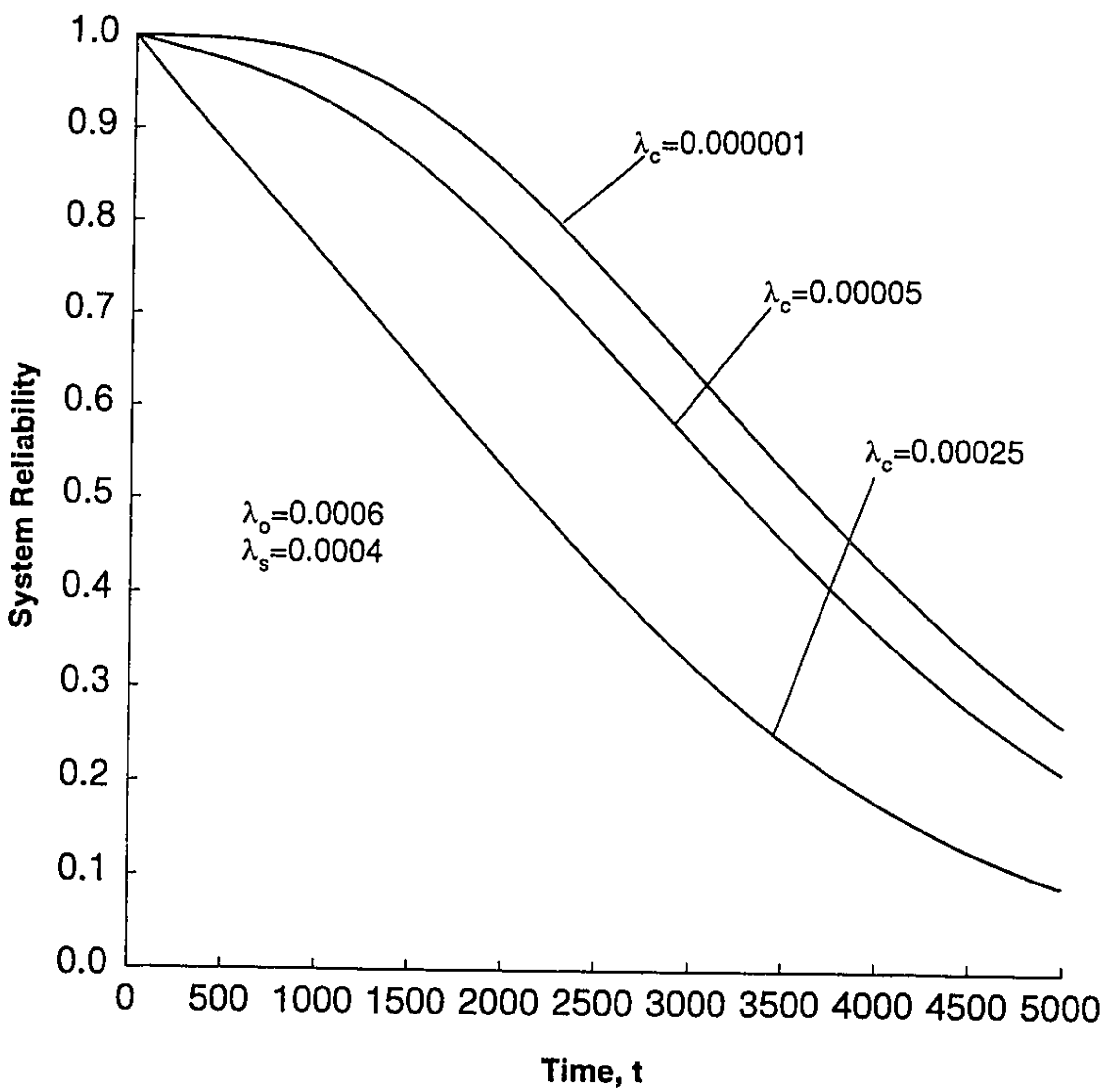


Figure 2.16 The standby system without repair reliability plots: one unit operating, three units on standby

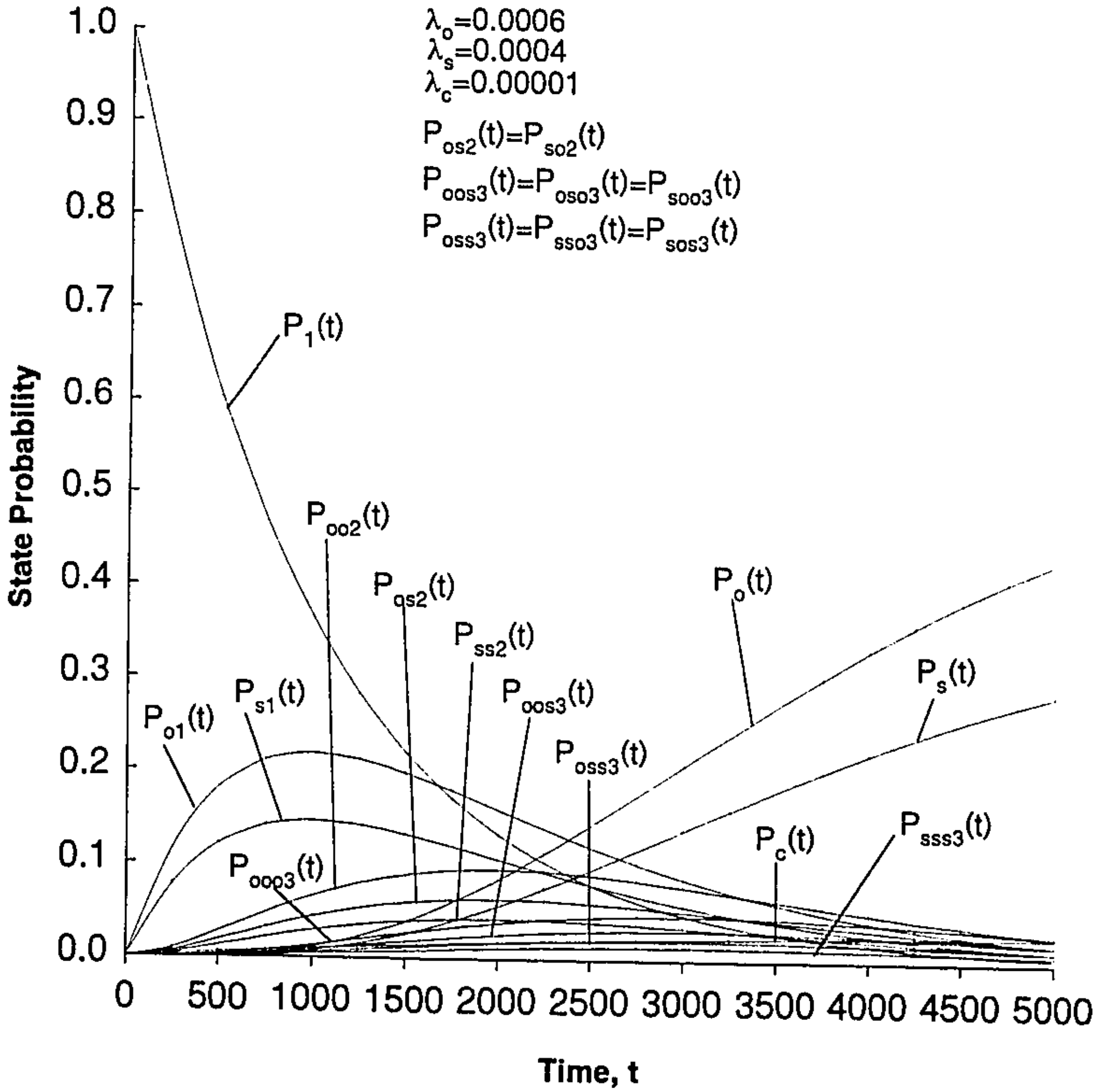


Figure 2.17 The standby system without repair state probability plot: one unit operating, three units on standby

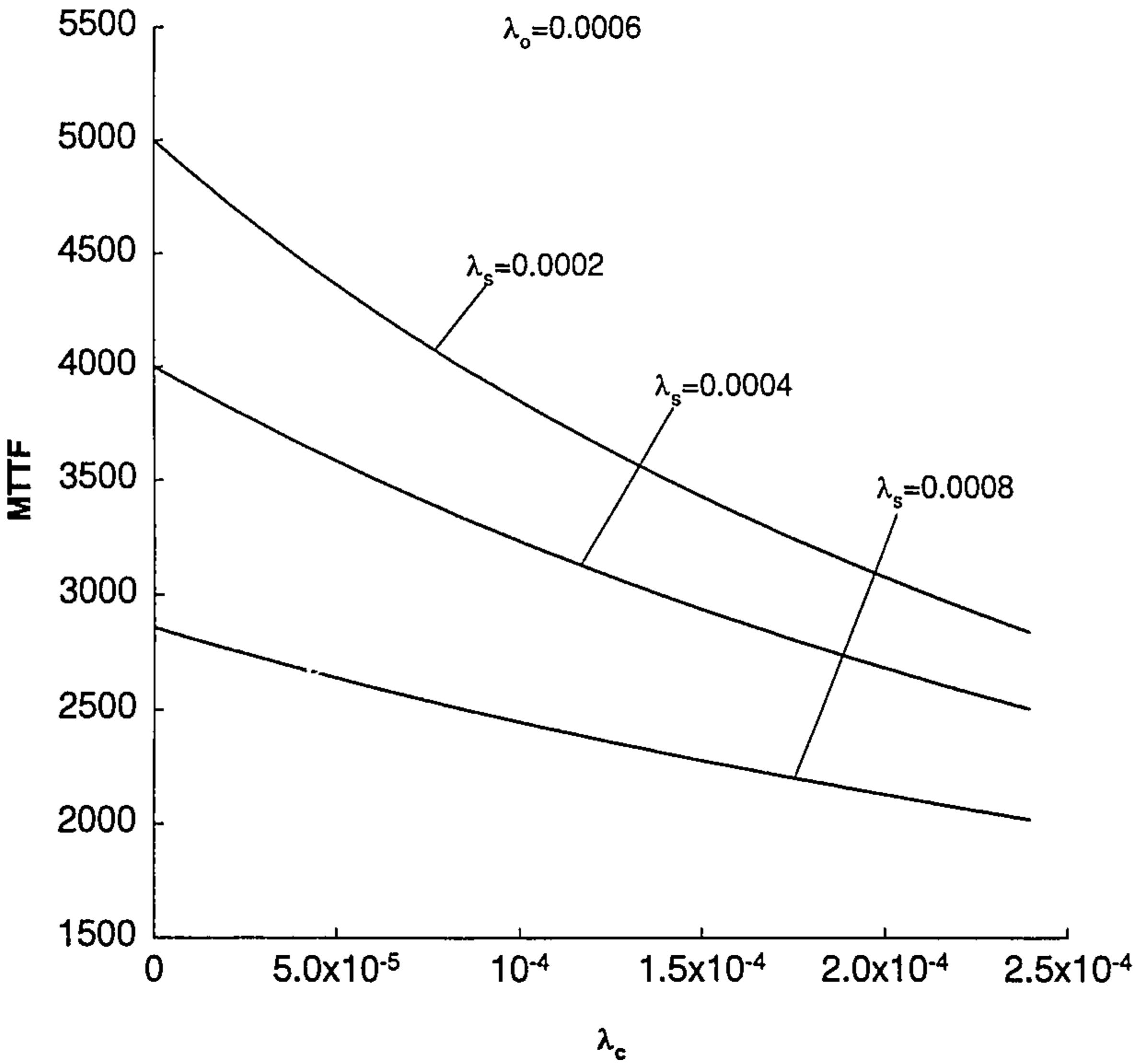


Figure 2.18 MTTF vs common cause failure rate, λ_c for the standby system (one unit operating, three units on standby) without repair

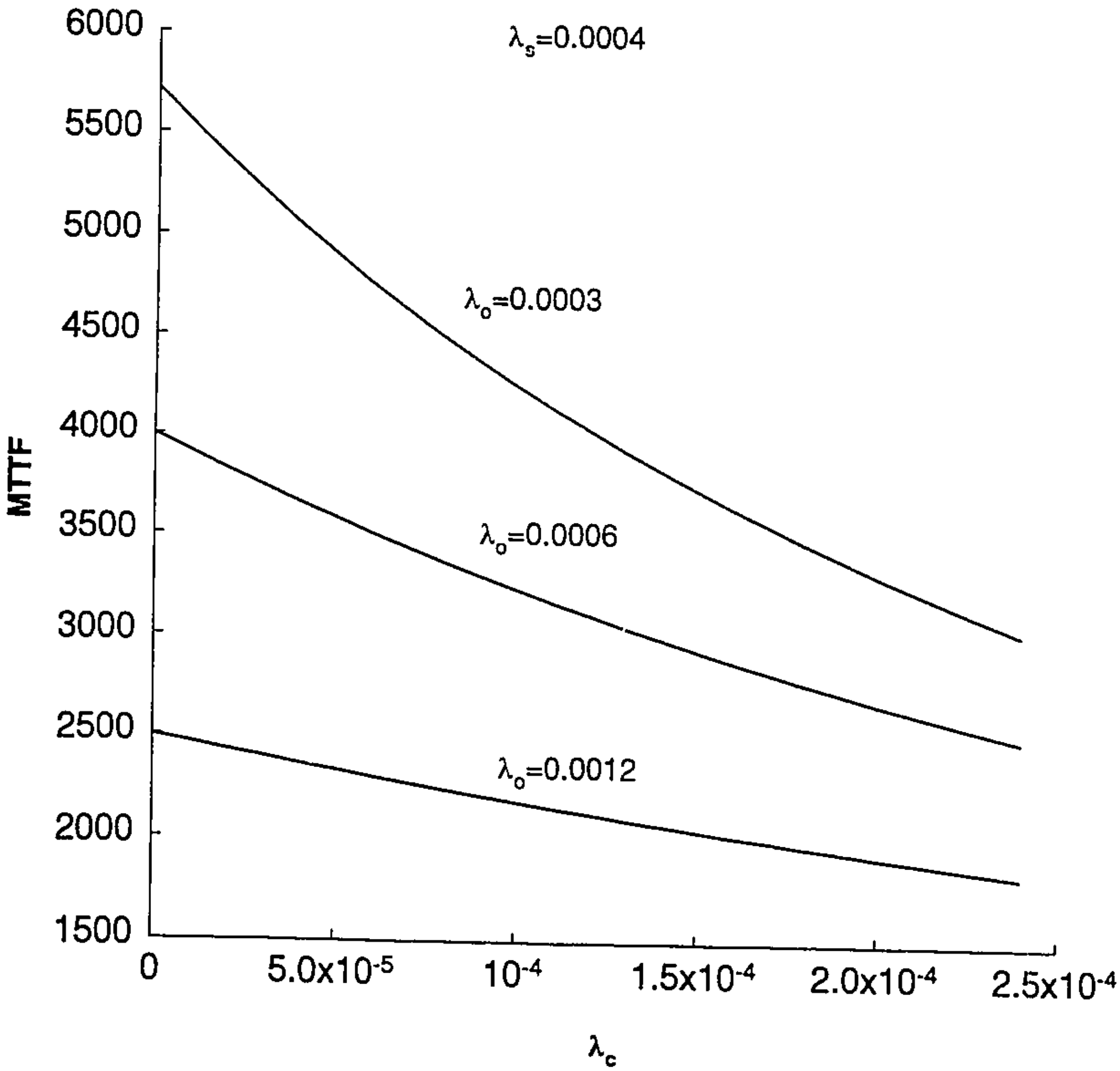


Figure 2.19 MTTF vs common cause failure rate, λ_c for the standby system (one unit operating, three units on standby) without repair

2.3 Three-state device standby system with Type I Repair

This repair policy is concerned with the repair of a total system failure due to common-cause failure or other failures such as an open/short mode failures. The partially failed system is not repaired. This is depicted in the state space diagram of general model in Figure 2.3. The total failed system is restored back to its fully operating state.

2.3.1 Standby system general model with Type I Repair

The system of differential equations associated with Figure 2.3 is

$$\frac{dP_1(t)}{dt} = -(\lambda_o + \lambda_s + \lambda_c)P_1(t) + \mu[P_o(t) + P_s(t)] + \mu_c P_c(t) \tag{2.117}$$

$$\frac{dP_{(o,s)ok}(t)}{dt} = -(\lambda_o + \lambda_s + \lambda_c)P_{(o,s)ok}(t) + \lambda_o P_{(o,s)k-1}(t) \tag{2.118}$$

$$\frac{dP_{(o,s)sk}(t)}{dt} = -(\lambda_o + \lambda_s + \lambda_c)P_{(o,s)sk}(t) + \lambda_s P_{(o,s)k-1}(t) \tag{2.119}$$

where $(o,s)ok$ and $(o,s)sk$ in Equations (2.118) and (2.119) have the same meaning as in Equations (2.2) and (2.3); k=1,2,3,....,(n-1). This is also applicable to (o,s) and k=1 $P_{(o,s)k-1}(t) = P_1(t)$.

$$\frac{dP_{(o,s)on}(t)}{dt} = -(\lambda_o + \lambda_s)P_{(o,s)on}(t) + \lambda_o P_{(o,s)n-1}(t) \tag{2.120}$$

$$\frac{dP_{(o,s)sn}(t)}{dt} = -(\lambda_o + \lambda_s)P_{(o,s)sn}(t) + \lambda_s P_{(o,s)n-1}(t) \tag{2.121}$$

where $(o,s)on$ and $(o,s)sn$ in Equation (2.120) and (2.121) have the same meaning as in Equation (2.4) and (2.5). This is also applicable to (o,s) and n=1 $P_{(o,s)n-1}(t) = P_1(t)$.

$$\frac{dP_o(t)}{dt} = -\mu P_o(t) + \lambda_o \sum P_{(o,s)n}(t) \tag{2.122}$$

$$\frac{dP_s(t)}{dt} = -\mu P_s(t) + \lambda_s \sum P_{(o,s)n}(t) \qquad (2.123)$$

where $\sum P_{(o,s)n}(t)$ in Equations (2.122) and (2.123) represents the sum of the probabilities of all the possible states in which n units have failed and only one unit is working.

$$\frac{dP_c(t)}{dt} = -\mu_c P_c(t) + \lambda_c [P_1(t) + \sum_k P_{(o,s)k}(t)] \qquad \text{for k=1,2,3,...,(n-1)} \qquad (2.124)$$

where $\sum_k P_{(o,s)k}(t)$ in Equation (2.124) represents the sum of the probabilities of all the possible states in which there is at least one unit on standby.

At time $t = 0$, $P_1(0) = 1$, and all the other initial condition probabilities are equal to zero.

By setting the derivatives of Equations (2.117) - (2.124) equal to zero and by making use of the relationship $\sum P_j = 1$, where j denotes jth system state, the steady state probability of being in each system state can be obtained.

The system steady state availability is

$$AV_{ss} = \sum_i P_i = \sum_i \lim_{s \to 0} s \bullet P_i(s) \qquad (2.125)$$

where i denotes all of the operating states.

By taking Laplace transforms of Equations (2.117) - (2.124) and solving the resulting equations, and then taking inverse Laplace transforms, the time dependent system availability can be obtained as expressed below.

$$AV(t) = L^{-1}[\sum_i P_i(s)] \qquad (2.126)$$

where i denotes all of the operating states.

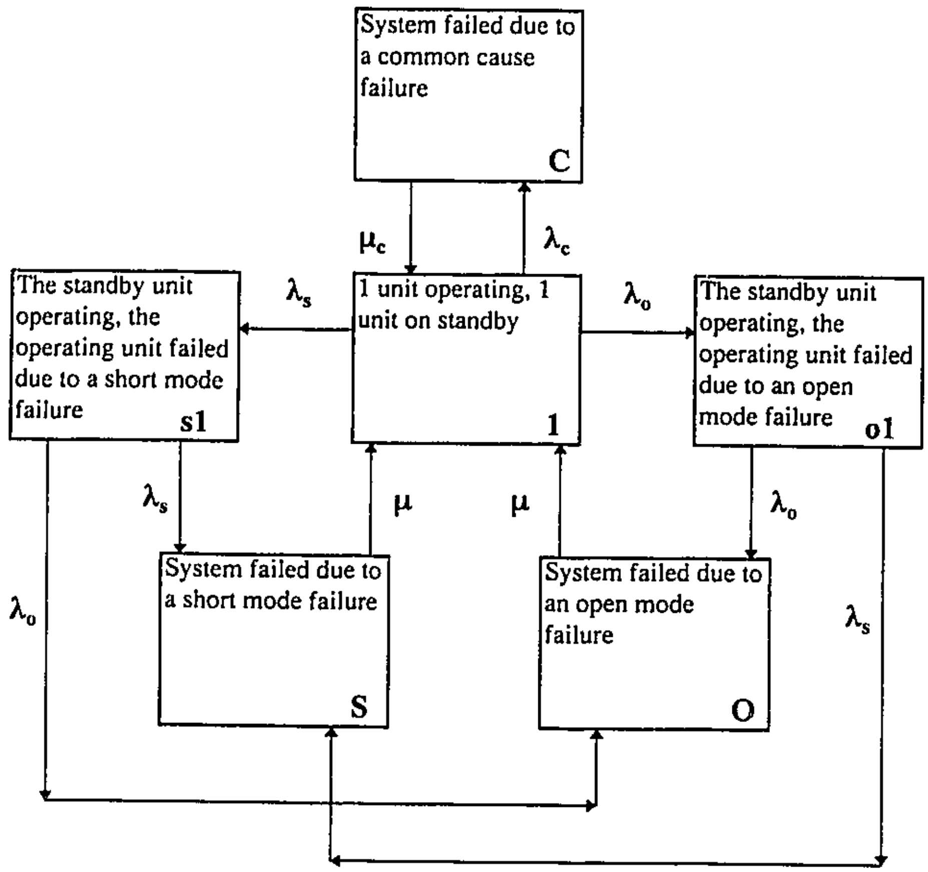


Figure 2.20 System state space diagram with Type I Repair: one unit operating, the other on standby

2.3.2 Standby system with Type I Repair: one unit operating, the other on standby

The system state space diagram is shown in Figure 2.20. By setting n=1 in Equations (2.117)-(2.124), we obtain the following differential equations:

$$\frac{dP_1(t)}{dt} = -(\lambda_o + \lambda_s + \lambda_c)P_1(t) + \mu[P_o(t) + P_s(t)] + \mu_c P_c(t) \tag{2.127}$$

$$\frac{dP_{o1}(t)}{dt} = -(\lambda_o + \lambda_s)P_{o1}(t) + \lambda_o P_1(t) \tag{2.128}$$

$$\frac{dP_{s1}(t)}{dt} = -(\lambda_o + \lambda_s)P_{s1}(t) + \lambda_s P_1(t) \tag{2.129}$$

$$\frac{dP_s(t)}{dt} = -\mu P_s(t) + \lambda_s[P_2(t) + P_3(t)] \tag{2.130}$$

$$\frac{dP_o(t)}{dt} = -\mu P_o(t) + \lambda_o[P_2(t) + P_3(t)] \tag{2.131}$$

$$\frac{dP_c(t)}{dt} = -\mu_c P_c(t) + \lambda_c P_1(t) \tag{2.132}$$

At time $t = 0$, $P_1(0) = 1$, and all the other initial condition probabilities are equal to zero.

The steady state probabilities can be obtained by setting the derivatives of above Equations (2.127) - (2.132) equal to zero and utilizing the relationship $\sum P_j = 1$, j represents all of the states in the system.

$$P_1 = \frac{\mu\mu_c}{(\lambda_o + \lambda_s)\mu_c + 2\mu\mu_c + \mu\lambda_c} \tag{2.133}$$

$$P_{o1} = \frac{\lambda_o \mu\mu_c}{(\lambda_o + \lambda_s)^2 \mu_c + 2\mu\mu_c(\lambda_o + \lambda_s) + \mu\lambda_c(\lambda_o + \lambda_s)} \tag{2.134}$$

$$P_{o1} = \frac{\lambda_s \mu\mu_c}{(\lambda_o + \lambda_s)^2 \mu_c + 2\mu\mu_c(\lambda_o + \lambda_s) + \mu\lambda_c(\lambda_o + \lambda_s)} \tag{2.135}$$

$$P_s = \frac{\lambda_s(\lambda_o+\lambda_s)\mu_c}{(\lambda_o+\lambda_s)^2\mu_c + 2\mu\mu_c(\lambda_o+\lambda_s) + \mu\lambda_c(\lambda_o+\lambda_s)} \tag{2.136}$$

$$P_o = \frac{\lambda_o(\lambda_o+\lambda_s)\mu_c}{(\lambda_o+\lambda_s)^2\mu_c + 2\mu\mu_c(\lambda_o+\lambda_s) + \mu\lambda_c(\lambda_o+\lambda_s)} \tag{2.137}$$

$$P_c = \frac{\lambda_c(\lambda_o+\lambda_s)\mu}{(\lambda_o+\lambda_s)^2\mu_c + 2\mu\mu_c(\lambda_o+\lambda_s) + \mu\lambda_c(\lambda_o+\lambda_s)} \tag{2.138}$$

The system steady state availability is given by

$$\begin{aligned} AV_{ss} &= P_1 + P_{o1} + P_{s1} \\ &= \frac{2\mu\mu_c}{(\lambda_o+\lambda_s)\mu_c + 2\mu\mu_c + \lambda_c\mu} \end{aligned} \tag{2.139}$$

By taking Laplace transforms of Equations (2.127) - (2.132) and solving the resulting equations, the Laplace transforms of the state probabilities are as follows:

$$P_1(s) = \frac{(\lambda_o+\lambda_s+s)(\mu+s)(\mu_c+s)}{A(2,1)} \tag{2.140}$$

$$P_{o1}(s) = \frac{\lambda_o(\mu+s)(\mu_c+s)}{A(2,1)} \tag{2.141}$$

$$P_{s1}(s) = \frac{\lambda_s(\mu+s)(\mu_c+s)}{A(2,1)} \tag{2.142}$$

$$P_s(s) = \frac{\lambda_s(\lambda_o+\lambda_s)(\mu_c+s)}{A(2,1)} \tag{2.143}$$

$$P_o(s) = \frac{\lambda_o(\lambda_o+\lambda_s)(\mu_c+s)}{A(2,1)} \tag{2.144}$$

$$P_c(s) = \frac{\lambda_c(\lambda_o+\lambda_s+s)(\mu+s)}{A(2,1)} \tag{2.145}$$

where A(2,1) is defined in Appendix 2.1.

The Laplace transform of the system availability is given by

$$AV(s) = P_1(s) + P_{o1}(s) + P_{s1}(s)$$
$$= \frac{(2\lambda_o + 2\lambda_s + s)(\mu + s)(\mu_c + s)}{A(2,1)} \tag{2.146}$$

The time dependent system availability and the state probabilities can be obtained by taking inverse Laplace transforms of Equation (2.146) and Equations (2.140)-(2.145).

The time dependent system availabilities are plotted in Figure 2.21 for specified values of model parameters. Figure 2.22 presents the time dependent state probabilities. The system steady state availability given by Equation (2.139) is plotted in Figure 2.23. These plots as well as those of the other special cases are discussed at the end of Section 2.3.

2.3.3 Standby system with Type I Repair: one unit operating, two units on standby

The system state space diagram is shown in Figure 2.24. By setting n=2 in Equations (2.117)-(2.124), we obtain the following differential equations:

$$\frac{dP_1(t)}{dt} = -(\lambda_o + \lambda_s + \lambda_c)P_1(t) + \mu[P_o(t) + P_s(t)] + \mu_c P_c(t) \tag{2.147}$$

$$\frac{dP_{o1}(t)}{dt} = -(\lambda_o + \lambda_s + \lambda_c)P_{o1}(t) + \lambda_o P_1(t) \tag{2.148}$$

$$\frac{dP_{s1}(t)}{dt} = -(\lambda_o + \lambda_s + \lambda_c)P_{s1}(t) + \lambda_s P_1(t) \tag{2.149}$$

$$\frac{dP_{oo2}(t)}{dt} = -(\lambda_o + \lambda_s)P_{oo2}(t) + \lambda_o P_{o1}(t) \tag{2.150}$$

$$\frac{dP_{ss2}(t)}{dt} = -(\lambda_o + \lambda_s)P_{ss2}(t) + \lambda_s P_{s1}(t) \tag{2.151}$$

$$\frac{dP_{os2}(t)}{dt} = -(\lambda_o + \lambda_s)P_{os2}(t) + \lambda_s P_{o1}(t) \tag{2.152}$$

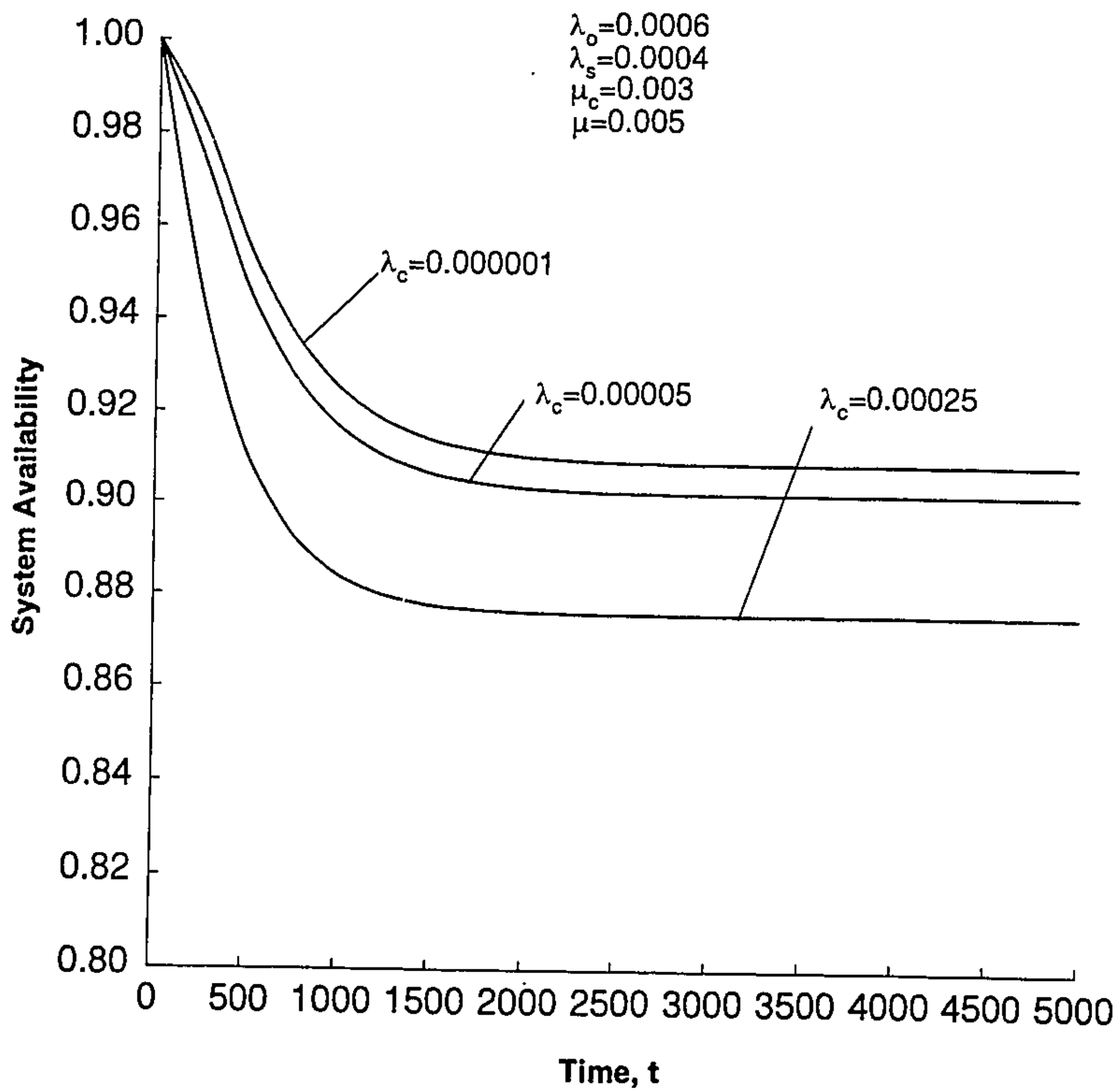


Figure 2.21 The standby system with type I repair availability plots: one unit operating, the other on standby

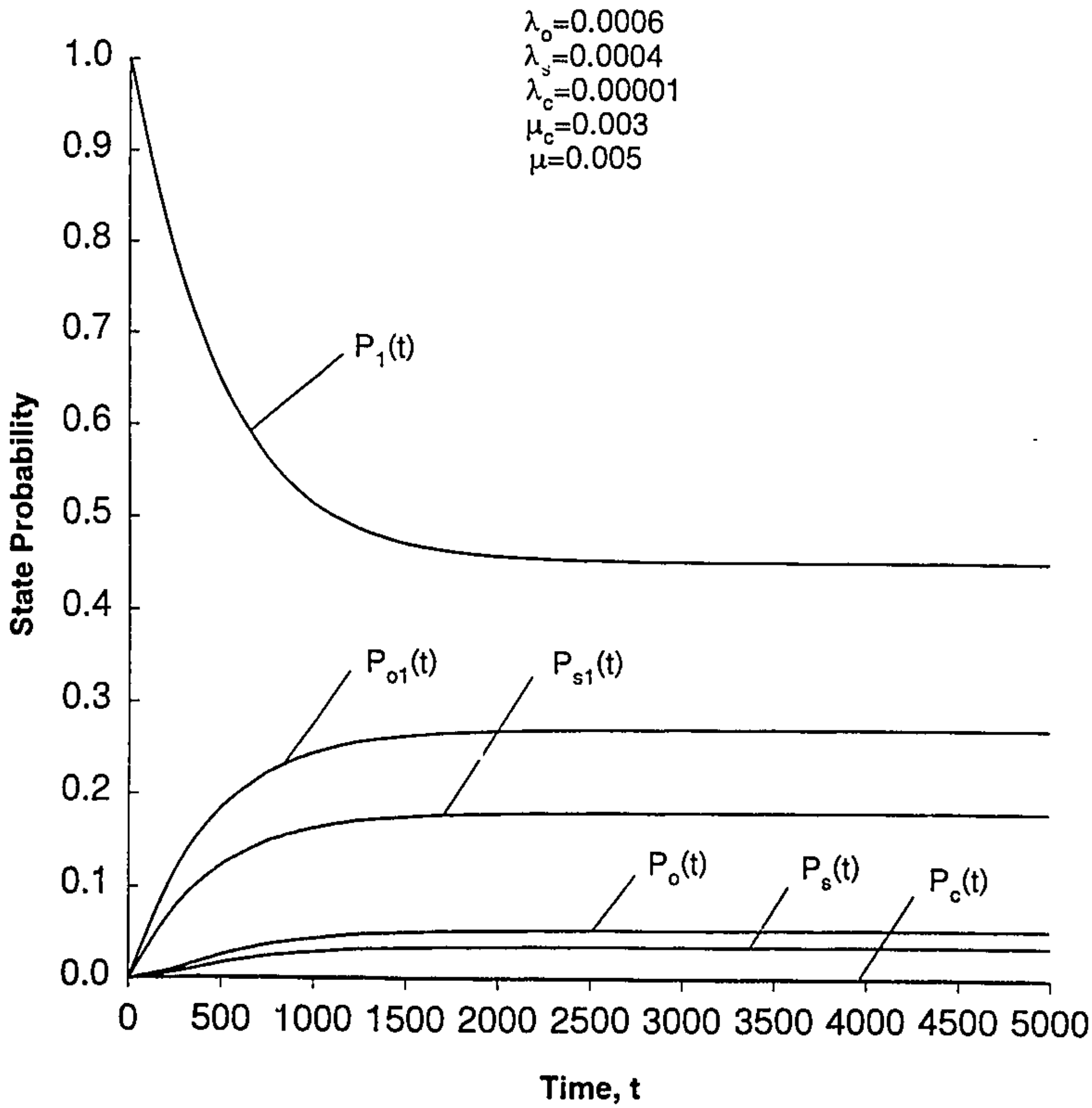


Figure 2.22 The standby system with type I repair state probability plots: one unit operating, the other on standby

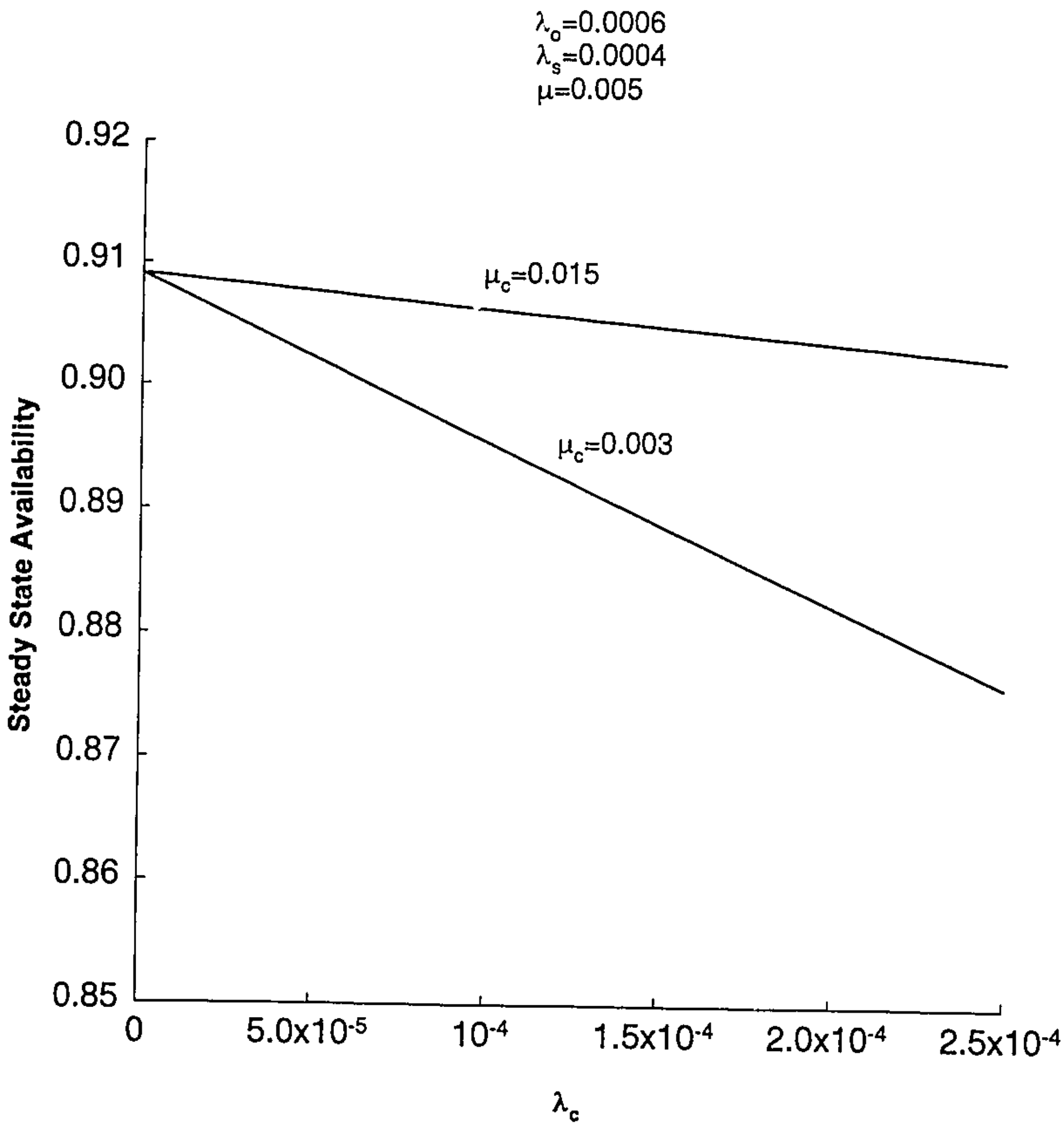


Figure 2.23 The standby system with type I repair steady state availability vs cause failure rate, λ_c plots (one unit operating, the other on standby)

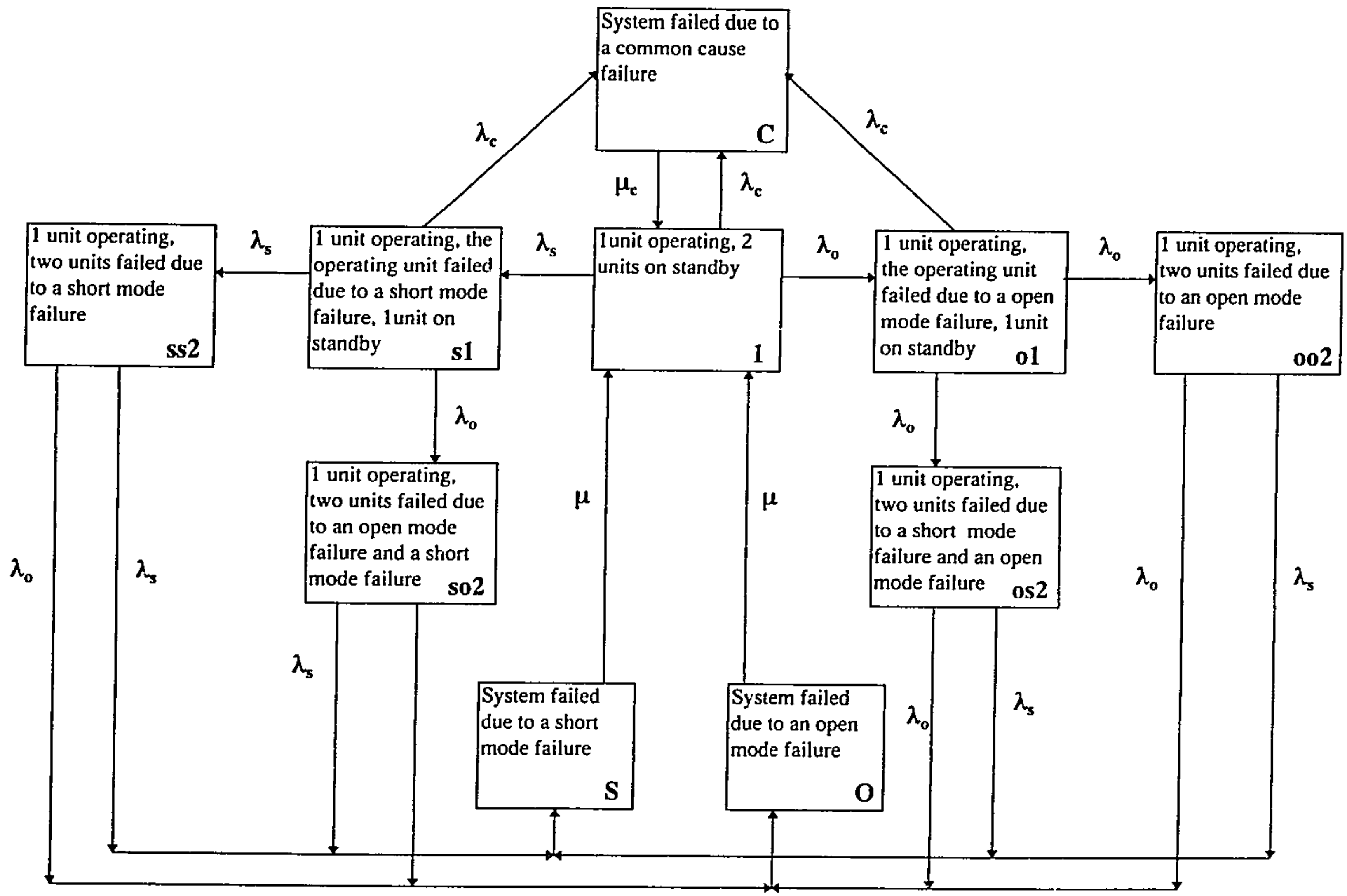


Figure 2.24 System state space diagram with Type I Repair:
one unit operating, two units on standby

$$\frac{dP_{so2}(t)}{dt} = -(\lambda_o + \lambda_s) P_{so2}(t) + \lambda_o P_{s1}(t) \tag{2.153}$$

$$\frac{dP_s(t)}{dt} = -\mu P_s(t) + \lambda_s [P_{oo2}(t) + P_{os2}(t) + P_{so2}(t) + P_{ss2}(t)] \tag{2.154}$$

$$\frac{dP_o(t)}{dt} = -\mu P_o(t) + \lambda_o [P_{oo2}(t) + P_{os2}(t) + P_{so2}(t) + P_{ss2}(t)] \tag{2.155}$$

$$\frac{dP_c(t)}{dt} = -\mu_c P_c(t) + \lambda_c [P_1(t) + P_{o1}(t) + P_{s1}(t)] \tag{2.156}$$

At time $t = 0$, $P_1(0) = 1$, and all the other initial condition probabilities are equal to zero.

The steady state probabilities can be obtained by setting the derivatives of above Equations (2.147)-(2.156) equal to zero and utilizing the relationship $\sum P_j = 1$, j represents all of the states in the system.

$$P_1 = \frac{(\lambda_o + \lambda_s + \lambda_c)\, \mu\mu_c}{(\lambda_o + \lambda_s)^2 \mu_c + 2(\lambda_o + \lambda_s)\, \mu\lambda_c + (3\lambda_o + 3\lambda_s + \lambda_c)\, \mu\mu_c + \lambda_c^2 \mu} \tag{2.157}$$

$$P_{o1} = \frac{\lambda_o\, \mu\mu_c}{(\lambda_o + \lambda_s)^2 \mu_c + 2(\lambda_o + \lambda_s)\, \mu\lambda_c + (3\lambda_o + 3\lambda_s + \lambda_c)\, \mu\mu_c + \lambda_c^2 \mu} \tag{2.158}$$

$$P_{s1} = \frac{\lambda_s\, \mu\mu_c}{(\lambda_o + \lambda_s)^2 \mu_c + 2(\lambda_o + \lambda_s)\, \mu\lambda_c + (3\lambda_o + 3\lambda_s + \lambda_c)\, \mu\mu_c + \lambda_c^2 \mu} \tag{2.159}$$

$$P_{oo2} = \frac{\lambda_{o,s}^2\, \mu\mu_c}{(\lambda_o + \lambda_s)[(\lambda_o + \lambda_s)^2 \mu_c + (\lambda_o + \lambda_s)(2\mu\lambda_c + 3\mu\mu_c) + \lambda_c \mu (\lambda_c + \mu_c)]} \tag{2.160}$$

$$P_{ss2} = \frac{\lambda_{o,s}^2\, \mu\mu_c}{(\lambda_o + \lambda_s)[(\lambda_o + \lambda_s)^2 \mu_c + (\lambda_o + \lambda_s)(2\mu\lambda_c + 3\mu\mu_c) + \lambda_c \mu (\lambda_c + \mu_c)]} \tag{2.161}$$

$$P_{os2} = P_{so2}$$

$$= \frac{\lambda_{o,s}^2\, \mu\mu_c}{(\lambda_o + \lambda_s)[(\lambda_o + \lambda_s)^2 \mu_c + (\lambda_o + \lambda_s)(2\mu\lambda_c + 3\mu\mu_c) + \lambda_c \mu (\lambda_c + \mu_c)]} \tag{2.162}$$

$$P_s = \frac{(\lambda_o + \lambda_s)\lambda_s\, \mu_c}{(\lambda_o + \lambda_s)^2 \mu_c + 2(\lambda_o + \lambda_s)\, \mu\lambda_c + (3\lambda_o + 3\lambda_s + \lambda_c)\, \mu\mu_c + \lambda_c^2 \mu} \tag{2.163}$$

$$P_o = \frac{(\lambda_o + \lambda_s)\lambda_o\, \mu_c}{(\lambda_o + \lambda_s)^2 \mu_c + 2(\lambda_o + \lambda_s)\, \mu\lambda_c + (3\lambda_o + 3\lambda_s + \lambda_c)\, \mu\mu_c + \lambda_c^2 \mu} \tag{2.164}$$

$$P_c = \frac{(2\lambda_o + 2\lambda_s + \lambda_c)\,\mu\lambda_c}{(\lambda_o + \lambda_s)^2\,\mu_c + 2(\lambda_o + \lambda_s)\,\mu\lambda_c + (3\lambda_o + 3\lambda_s + \lambda_c)\,\mu\mu_c + \lambda_c^2\,\mu} \tag{2.165}$$

The system steady state availability is given by

$$\begin{aligned} AV_{ss} &= P_1 + P_{o1} + P_{s1} + P_{oo2} + P_{ss2} + P_{os2} + P_{so2} \\ &= \frac{(3\lambda_o + 3\lambda_s + \lambda_c)\,\mu\lambda_c}{(\lambda_o + \lambda_s)^2\,\mu_c + 2(\lambda_o + \lambda_s)\,\mu\lambda_c + (3\lambda_o + 3\lambda_s + \lambda_c)\,\mu\mu_c + \lambda_c^2\,\mu} \end{aligned} \tag{2.166}$$

By taking Laplace transforms of Equations (2.147) - (2.156) and solving the resulting equations, the Laplace transforms of the state probabilities are as follows:

$$P_1(s) = \frac{(\lambda_o + \lambda_s + s)(\lambda_o + \lambda_s + \lambda_c + s)(\mu + s)(\mu_c + s)}{A(2,2)} \tag{2.167}$$

$$P_{o1}(s) = \frac{\lambda_o(\lambda_o + \lambda_s + s)(\mu + s)(\mu_c + s)}{A(2,2)} \tag{2.168}$$

$$P_{o1}(s) = \frac{\lambda_o(\lambda_o + \lambda_s + s)(\mu + s)(\mu_c + s)}{A(2,2)} \tag{2.169}$$

$$P_{oo2}(s) = \frac{\lambda_o^2(\mu + s)(\mu_c + s)}{A(2,2)} \tag{2.170}$$

$$P_{ss2}(s) = \frac{\lambda_s^2(\mu + s)(\mu_c + s)}{A(2,2)} \tag{2.171}$$

$$P_{os2}(s) = P_{so2}(s) = \frac{\lambda_o\lambda_s(\mu + s)(\mu_c + s)}{A(2,2)} \tag{2.172}$$

$$P_s(s) = \frac{\lambda_s(\lambda_o + \lambda_s)^2(\mu + s)(\mu_c + s)}{A(2,2)} \tag{2.173}$$

$$P_o(s) = \frac{\lambda_o(\lambda_o + \lambda_s)^2(\mu + s)(\mu_c + s)}{A(2,2)} \tag{2.174}$$

$$P_c(s) = \frac{\lambda_c(\lambda_o + \lambda_s + s)(2\lambda_o + 2\lambda_s + \lambda_c + s)(\mu + s)}{A(2,2)} \tag{2.175}$$

The Laplace transform of system availability is given by

$$R(s) = P_1(s) + P_{o1}(s) + P_{s1}(s) + P_{oo2}(s) + P_{ss2}(s) + P_{os2}(s) + P_{so2}(s)$$
$$= \frac{A(2,3)}{A(2,2)} \tag{2.176}$$

where A(2,2) and A(2,3) are defined in Appendix 2.1

The time dependent system availability and state probabilities can be developed by taking inverse Laplace transforms of Equation (2.176) and Equations (2.167)- (2.175).

In order to make a comparison with the system in which one unit is operating and the other is on standby, the same values of parameters (i.e. as for Equations (2.139)-(2.146)) are chosen for the steady state availability plots, the state probabilities plots, and the time dependent availability plots. The time dependent system availabilities are plotted in Figure 2.25. Figure 2.22 is the time dependent state probabilities. The system steady state availability given by Equation (2.166) is plotted in Figure 2.26.

2.3.4 Standby system with Type I Repair: one unit operating, three units on standby

The system state space diagram is shown in Figure 2.28. By setting n=3 in Equations (2.117)-(2.124), we obtain the following differential equations:

$$\frac{dP_1(t)}{dt} = -(\lambda_o + \lambda_s + \lambda_c)P_1(t) + \mu[P_o(t) + P_s(t)] + \mu_c P_c(t) \tag{2.177}$$

$$\frac{dP_{o1}(t)}{dt} = -(\lambda_o + \lambda_s + \lambda_c)P_{o1}(t) + \lambda_o P_1(t) \tag{2.178}$$

$$\frac{dP_{s1}(t)}{dt} = -(\lambda_o + \lambda_s + \lambda_c)P_{s1}(t) + \lambda_s P_1(t) \tag{2.179}$$

$$\frac{dP_{oo2}(t)}{dt} = -(\lambda_o + \lambda_s + \lambda_c)P_{oo2}(t) + \lambda_o P_{o1}(t) \tag{2.180}$$

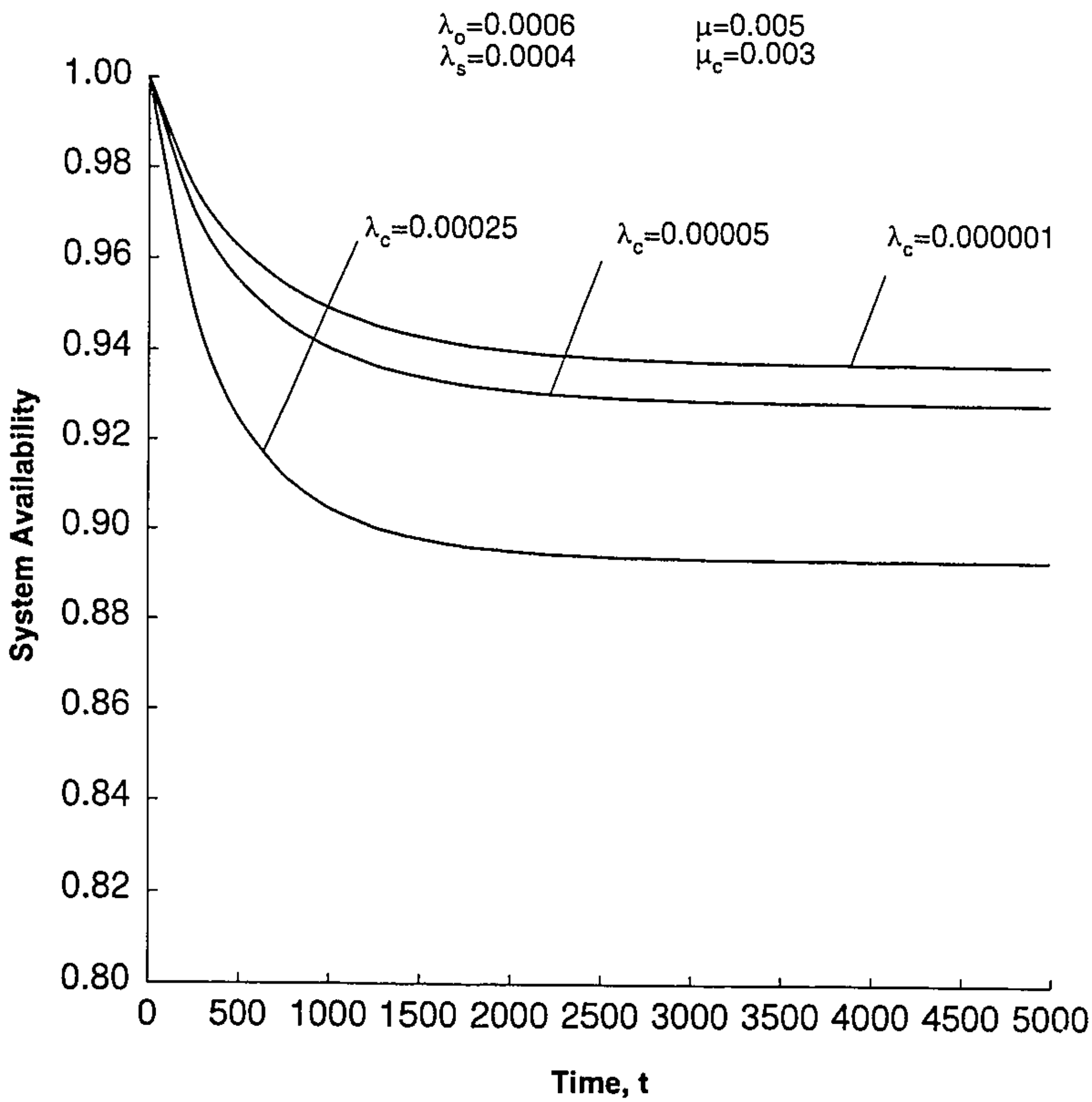


Figure 2.25 The standby system with type I repair availability plots: one unit operating, two units on standby

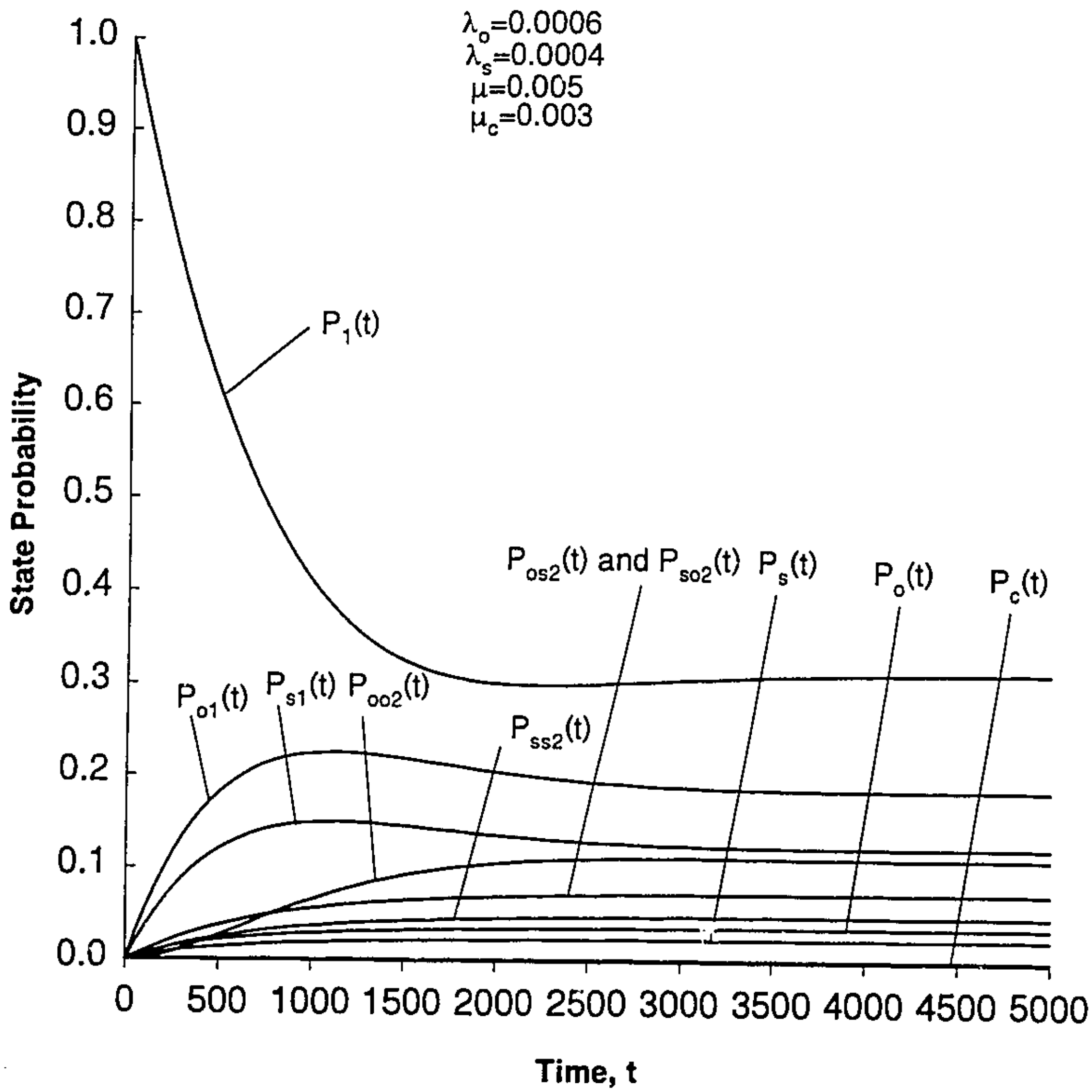


Figure 2.26 The standby system with type I repair state probability plots: one unit operating, two units on standby

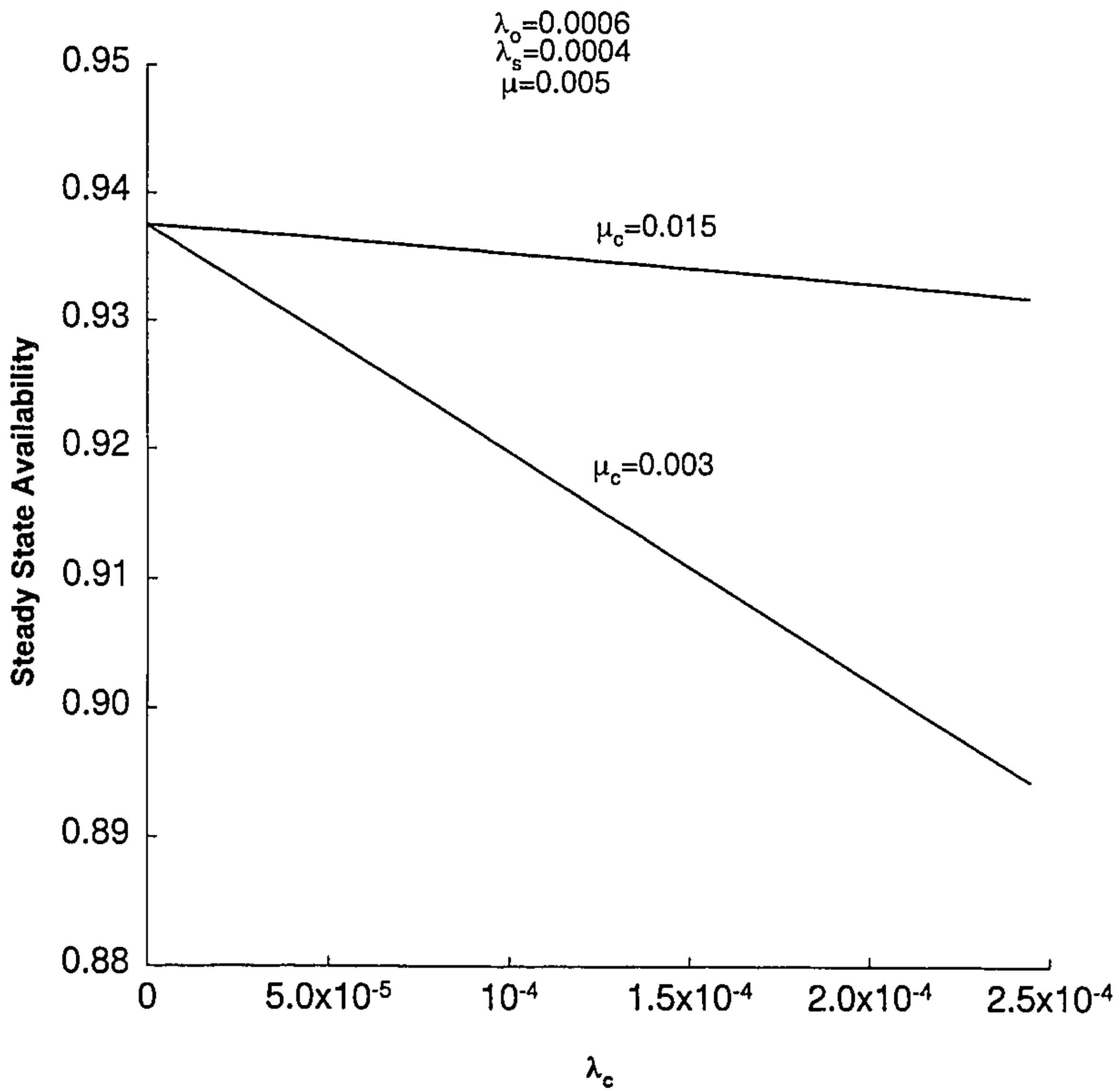


Figure 2.27 The standby system with type I repair steady state availability vs cause failure rate, λ_c plots (one unit operating, two units on standby)

SB=standby

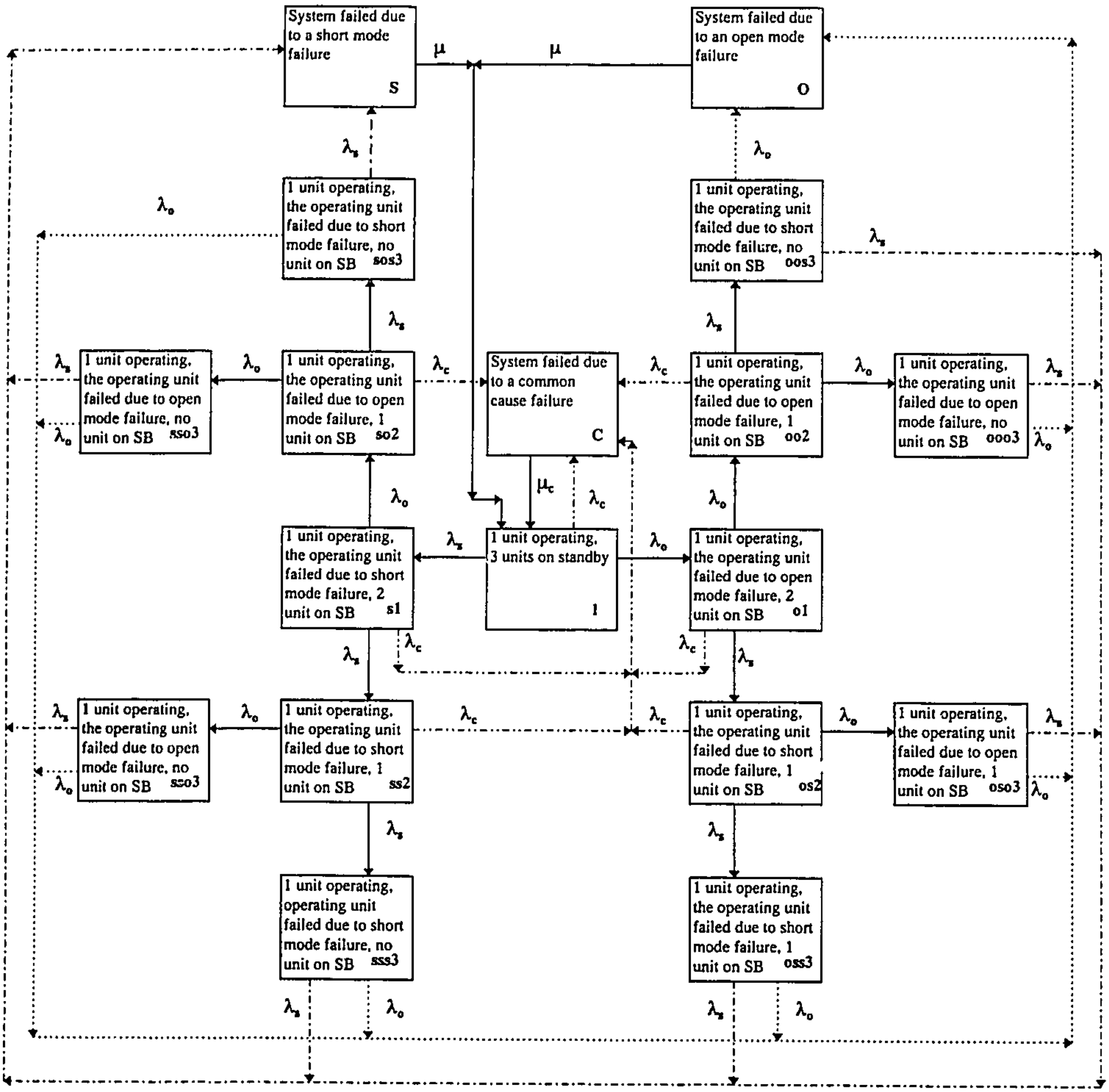


Figure 2.28 System state space diagram with Type I Repair: one unit operating, three units on standby

$$\frac{dP_{os2}(t)}{dt} = -(\lambda_o + \lambda_s + \lambda_c) P_{os2}(t) + \lambda_s P_{o1}(t) \tag{2.181}$$

$$\frac{dP_{ss2}(t)}{dt} = -(\lambda_o + \lambda_s + \lambda_c) P_{ss2}(t) + \lambda_s P_{s1}(t) \tag{2.182}$$

$$\frac{dP_{so2}(t)}{dt} = -(\lambda_o + \lambda_s + \lambda_c) P_{so2}(t) + \lambda_o P_{s1}(t) \tag{2.183}$$

$$\frac{dP_{ooo3}(t)}{dt} = -(\lambda_o + \lambda_s) P_{ooo3}(t) + \lambda_o P_{oo2}(t) \tag{2.184}$$

$$\frac{dP_{oos3}(t)}{dt} = -(\lambda_o + \lambda_s) P_{oos3}(t) + \lambda_s P_{oo2}(t) \tag{2.185}$$

$$\frac{dP_{oso3}(t)}{dt} = -(\lambda_o + \lambda_s) P_{oso3}(t) + \lambda_o P_{os2}(t) \tag{2.186}$$

$$\frac{dP_{oss3}(t)}{dt} = -(\lambda_o + \lambda_s) P_{oss3}(t) + \lambda_s P_{os2}(t) \tag{2.187}$$

$$\frac{dP_{sss3}(t)}{dt} = -(\lambda_o + \lambda_s) P_{sss3}(t) + \lambda_s P_{ss2}(t) \tag{2.188}$$

$$\frac{dP_{sso3}(t)}{dt} = -(\lambda_o + \lambda_s) P_{sso3}(t) + \lambda_o P_{ss2}(t) \tag{2.189}$$

$$\frac{dP_{sos3}(t)}{dt} = -(\lambda_o + \lambda_s) P_{sos3}(t) + \lambda_s P_{so2}(t) \tag{2.190}$$

$$\frac{dP_{soo3}(t)}{dt} = -(\lambda_o + \lambda_s) P_{soo3}(t) + \lambda_o P_{so2}(t) \tag{2.191}$$

$$\frac{dP_s(t)}{dt} = -\mu P_s(t) + \lambda_s [P_{ooo3}(t) + P_{oos3}(t) + P_{oso3}(t) + P_{osss}(t) + P_{sss3}(t) + P_{sso3}(t) + P_{sos3}(t) + P_{soo3}(t)] \tag{2.192}$$

$$\frac{dP_o(t)}{dt} = -\mu P_o(t) + \lambda_b [P_{ooo3}(t) + P_{oos3}(t) + P_{oso3}(t) + P_{osss}(t) + P_{sss3}(t) + P_{sso3}(t) + P_{sos3}(t) + P_{soo3}(t)] \tag{2.193}$$

$$\frac{dP_c(t)}{dt} = -\mu_c P_c(t) + \lambda_t [P_1(t) + P_{o1}(t) + P_{s1}(t) + P_{oo2}(t) + P_{os2}(t) + P_{so2}(t) + P_{ss2}(t)] \tag{2.194}$$

At time $t = 0$, $P_1(0) = 1$, and all the other initial condition probabilities are equal to zero.

The steady state probabilities can be obtained by setting the derivatives of above Equations (2.177) - (2.194) equal to zero and utilizing the relationship $\sum P_j = 1$, j represents all of the states in the system.

$$P_1 = \frac{(\lambda_o + \lambda_s + \lambda_c)^2 (\lambda_o + \lambda_s)\,\mu\mu_c}{A(2,4)} \tag{2.195}$$

$$P_{o1} = \frac{(\lambda_o + \lambda_s + \lambda_c)(\lambda_o + \lambda_s)\lambda_o\,\mu\mu_c}{A(2,4)} \tag{2.196}$$

$$P_{s1} = \frac{(\lambda_o + \lambda_s + \lambda_c)(\lambda_o + \lambda_s)\lambda_s\,\mu\mu_c}{A(2,4)} \tag{2.197}$$

$$P_{oo2} = \frac{\lambda_o^2 (\lambda_o + \lambda_s)\,\mu\mu_c}{A(2,4)} \tag{2.198}$$

$$P_{ss2} = \frac{\lambda_s^2 (\lambda_o + \lambda_s)\,\mu\mu_c}{A(2,4)} \tag{2.199}$$

$$P_{os2} = P_{so2} = \frac{\lambda_o \lambda_s (\lambda_o + \lambda_s)\,\mu\mu_c}{A(2,4)} \tag{2.200}$$

$$P_{ooo3} = \frac{\lambda_o^3\,\mu\mu_c}{A(2,4)} \tag{2.201}$$

$$P_{oos3} = P_{oso3} = P_{soo3} = \frac{\lambda_o^2 \lambda_s\,\mu\mu_c}{A(2,4)} \tag{2.202}$$

$$P_{oss2} = P_{sso3} = P_{sos3} = \frac{\lambda_o \lambda_s^2\,\mu\mu_c}{A(2,4)} \tag{2.203}$$

$$P_{sss3} = \frac{\lambda_s^3\,\mu\mu_c}{A(2,4)} \tag{2.204}$$

$$P_s = \frac{\lambda_s (\lambda_o + \lambda_s)^3\,\mu_c}{A(2,4)} \tag{2.205}$$

$$P_o = \frac{\lambda_o (\lambda_o + \lambda_s)^3\,\mu_c}{A(2,4)} \tag{2.206}$$

$$P_c = \frac{(\lambda_o + \lambda_s)[3(\lambda_o + \lambda_s)^2 + 3\lambda_c (\lambda_o + \lambda_s) + \lambda_c]\lambda_c\,\mu}{A(2,4)} \tag{2.207}$$

where A(2,4) is defined in Appendix 2.1

The system steady state availability is given by

$$AV_{ss} = P_1 + P_{o1} + P_{s1} + P_{oo2} + P_{ss2} + P_{os2} + P_{so2} + P_{ooo3} + P_{oos3} + P_{oso3} + P_{oss3} + P_{sso3} + P_{soo3} + P_{sos3} + P_{sss3}$$

$$= \frac{(\lambda_b + \lambda_s)[4(\lambda_b + \lambda_s)^2 + 3\lambda_c(\lambda_b + \lambda_s) + \lambda_c]\mu\mu_c}{A(2,4)} \tag{2.208}$$

By taking Laplace transforms of Equations (2.177) - (2.195) and solving the resulting equations, the Laplace transforms of the state probabilities are as follows:

$$P_1(s) = \frac{A(2,6)}{A(2,5)} \tag{2.209}$$

$$P_{o1}(s) = \frac{A(2,7)}{A(2,5)} \tag{2.210}$$

$$P_{s1}(s) = \frac{A(2,8)}{A(2,5)} \tag{2.211}$$

$$P_{oo2}(s) = \frac{A(2,9)}{A(2,5)} \tag{2.212}$$

$$P_{os2}(s) = P_{so2}(s) = \frac{A(2,10)}{A(2,5)} \tag{2.213}$$

$$P_{ss2}(s) = \frac{A(2,11)}{A(2,5)} \tag{2.214}$$

$$P_{ooo3}(s) = \frac{A(2,12)}{A(2,5)} \tag{2.215}$$

$$P_{oos3}(s) = P_{oso3}(s) = P_{soo3}(s) = \frac{A(2,13)}{A(2,5)} \tag{2.216}$$

$$P_{oss3}(s) = P_{sso3}(s) = P_{sos3}(s) = \frac{A(2,14)}{A(2,5)} \tag{2.217}$$

$$P_{sss3}(s) = \frac{A(2,15)}{A(2,5)} \tag{2.218}$$

$$P_s(s) = \frac{A(2,16)}{A(2,5)} \tag{2.219}$$

$$P_o(t) = \frac{A(2,17)}{A(2,5)} \tag{2.220}$$

$$P_o(t) = \frac{A(2,18)}{A(2,5)} \tag{2.221}$$

where A(2,5), A(2,6), A(2,7), A(2,8), A(2,9), A(2,10), A(2,11), A(2,12), A(2,13), A(2,14), A(2,15), A(2,16), A(2,17) and A(2,18) are defined in Appendix 2.1.

The Laplace transform of system availability is given by

$$\begin{aligned} AV(s) &= P_1(s) + P_{o1}(s) + P_{s1}(s) + P_{oo2}(s) + P_{os2}(s) + P_{ss2}(s) + P_{so2}(s) + P_{ooo3}(s) + P_{oos3}(s) + \\ &\quad P_{oso3}(s) + P_{oss3}(s) + P_{soo3}(s) + P_{sos3}(s) + P_{sso3}(s) + P_{sss3}(s) \\ &= \frac{A(2,19)}{A(2,5)} \end{aligned} \tag{2.222}$$

where A(2,19) is defined in Appendix 2.1.

The time dependent system availability and state probabilities can be developed by taking inverse Laplace transforms of Equation (2.222) and Equations (2.209)-(2.221).

In order to make a comparison with the system in which one unit is operating and the other is on standby, the same values of parameters (i.e. as for Equations (2.139)-(2.146)) are chosen for the steady state availability plots, the state probabilities plots, and the time dependent availability plots. The time dependent system availabilities are plotted in Figure 2.29. Figure 2.30 is for the time dependent state probabilities. The system steady state availability given by Equation (2.208) is plotted in Figure 2.31.

The result of analysis indicate that the steady state availability of standby systems increases with the increasing number of the standby units. Also, the steady state

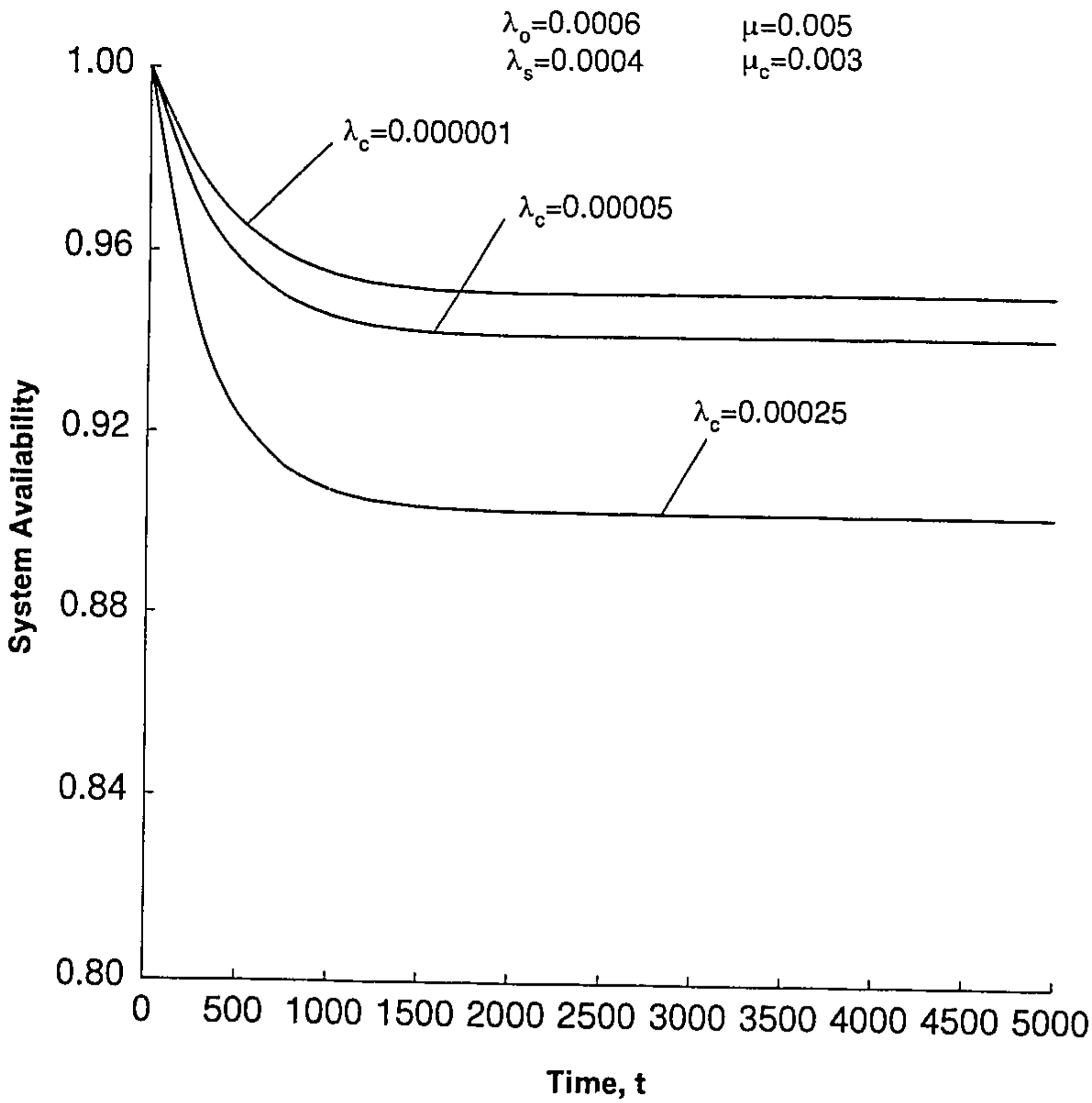


Figure 2.29 The standby system with type I repair availability plots: one unit operating, three units on standby

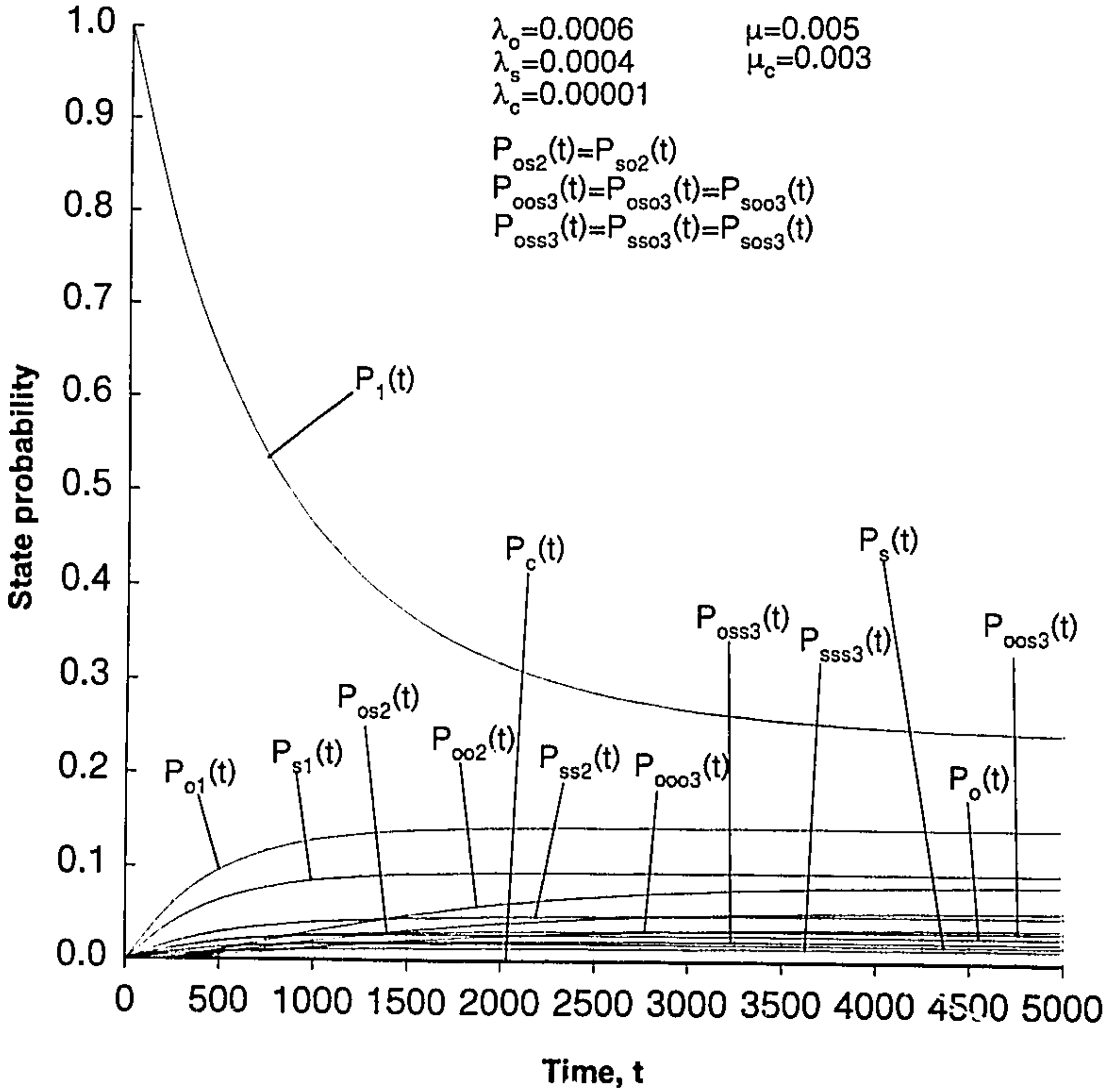


Figure 2.30 The standby system with type I repair state probability plots: one unit operating, three units on standby

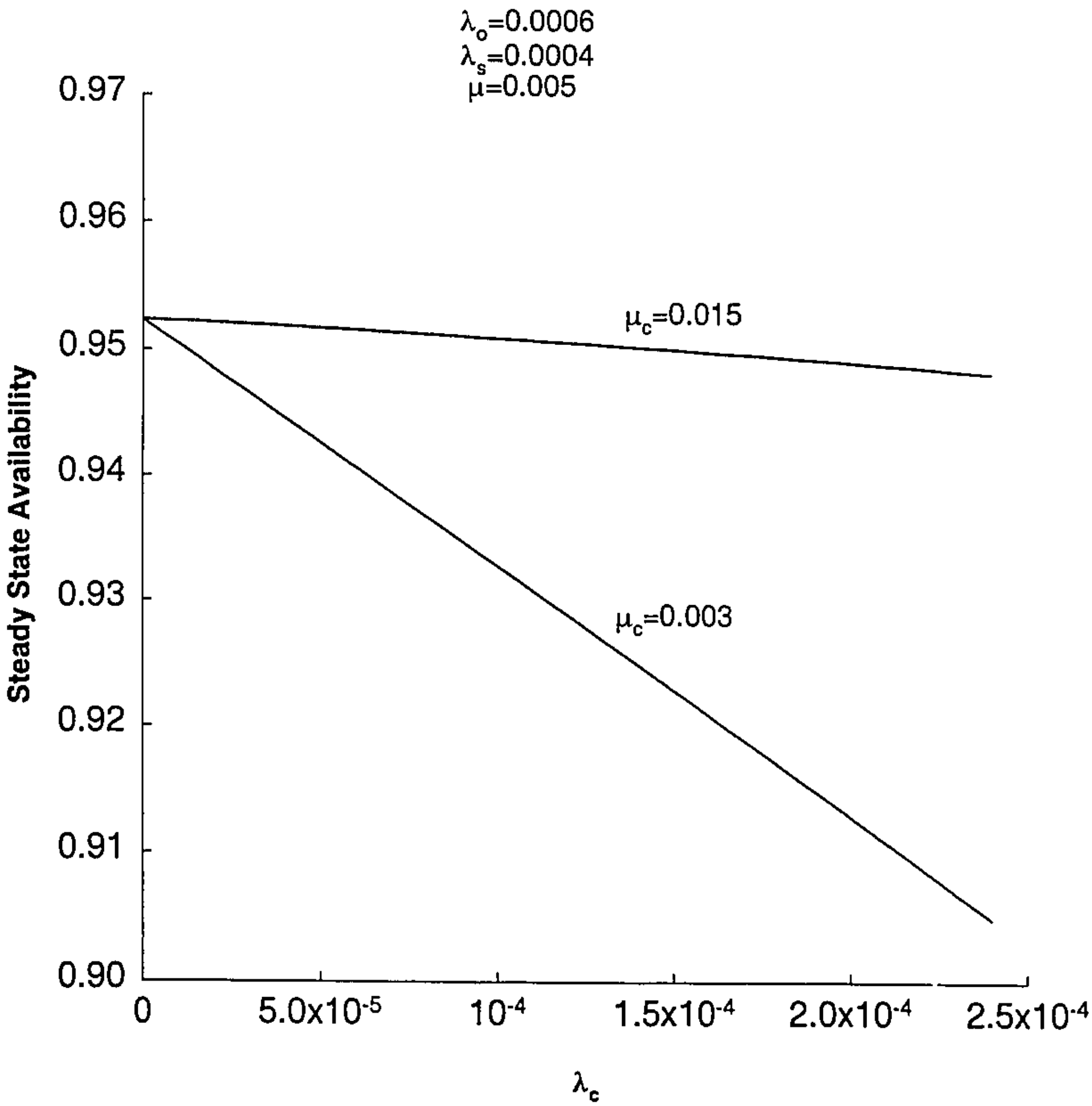


Figure 2.31 The standby system with type I repair steady state availability vs cause failure rate, λ_c plots (one unit operating, three units on standby)

availability plots indicates that the system steady state availability decreases with increasing common-cause failure rate.

For all of the special cases, the time dependent availability approaches to a constant value as time become large. In fact, that constant value is the system steady state availability. From time dependent probability plots, it is evident that at any time t, the sum of all state probabilities equals unity. As time increases, all of the state probabilities approach to steady state.

2.4 Three-state device standby system with Type II Repair

Under this repair policy, the entire failed system is not considered repairable but only the partially failed system. Therefore, for example, if the system failed due to the occurrence of common-cause failure, there would be no repair action because it would result in total system failure.

2.4.1 Standby system general model with Type II Repair

The system of differential equations associated with Figure 2.4 (state space diagram) is

$$\frac{dP_1(t)}{dt} = -(\lambda_o + \lambda_s + \lambda_c)P_1(t) + \mu[P_{o1}(t) + P_{s1}(t)] \tag{2.223}$$

$$\frac{dP_{(o,s)ok}(t)}{dt} = -(\lambda_o + \lambda_s + \lambda_c + \mu)P_{(o,s)ok}(t) + \lambda_o P_{(o,s)k-1}(t) + \mu[P_{(o,s)oo(k+1)}(t) + P_{(o,s)os(k+1)}(t)] \tag{2.224}$$

$$\frac{dP_{(o,s)sk}(t)}{dt} = -(\lambda_o + \lambda_s + \lambda_c + \mu)P_{(o,s)sk}(t) + \lambda_s P_{(o,s)k-1}(t) + \mu[P_{(o,s)oo(k+1)}(t) + P_{(o,s)os(k+1)}(t)] \tag{2.225}$$

where $(o,s)oo(k+1)$ in Equations (2.224) and (2.225) denotes the system states in which there are (k+1) failed units and the last unit failed due to an open mode failure. $(o,s)os(k+1)$ in Equations (2.224) and (2.225) denotes the system states in which there are (k+1) failed units and the last unit failed due to a short mode failure; k=1,2,3,....,(n-1). The remaining symbols denote the same meaning as in Equations (2.2) and (2.3).

$$\frac{dP_{(o,s)on}(t)}{dt} = -(\lambda_o + \lambda_s + \mu)P_{(o,s)on}(t) + \lambda_o P_{(o,s)n-1}(t) \tag{2.226}$$

$$\frac{dP_{(o,s)sn}(t)}{dt} = -(\lambda_o + \lambda_s + \mu)P_{(o,s)sn}(t) + \lambda_s P_{(o,s)n-1}(t) \qquad (2.227)$$

where $(o,s)on$ and $(o,s)sn$ in Equations (2.226) and (2.227) have the same meaning as in Equations (2.4) and (2.5). This is also applicable to (o,s) and at n=1 $P_{(o,s)n-1}(t) = P_1(t)$.

$$\frac{dP_o(t)}{dt} = \lambda_o \sum P_{(o,s)n}(t) \qquad (2.228)$$

$$\frac{dP_s(t)}{dt} = \lambda_s \sum P_{(os,)n}(t) \qquad (2.229)$$

where $\sum P_{(o,s)n}(t)$ in Equations (2.228) and (2.229) represents the sum of the probabilities of all the possible states in which n units have failed, and only one unit is working.

$$\frac{dP_c(t)}{dt} = \lambda_c[P_1(t) + \sum_k P_{(o,s)k}(t)] \qquad \text{for } k=1,2,3,\ldots,(n-1) \qquad (2.230)$$

where $\sum_k P_{(o,s)k}(t)$ in Equation (2.230) represents the sum of the probabilities of all the possible states in which there is at least one unit on standby.

At time $t = 0$, $P_1(0) = 1$, and all the other initial condition probabilities are equal to zero.

By taking Laplace transforms of Equations (2.223) - (2.230) and solving the resulting equations, and then taking inverse Laplace transforms, the following expression for system reliability can be obtained.

$$R(t) = L^{-1}[\sum P_i(s)] \qquad (2.231)$$

where i denotes all operating states.

The system mean time to failure (MTTF) is given by

$$MTTF = \lim_{s \to 0} R(s) = \sum_i \lim_{s \to 0} P_i(s) \qquad (2.232)$$

where i denotes all operating states.

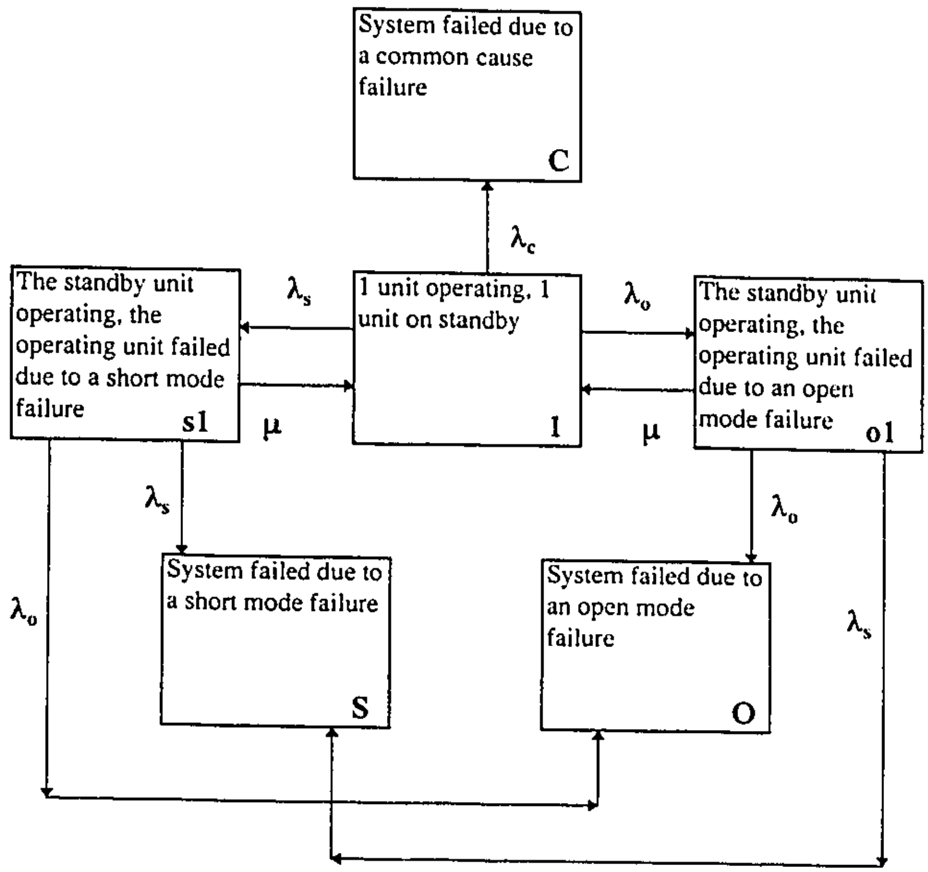


Figure 2.32 System state space diagram with type II repair: one unit operating, the other on standby

2.4.2 Standby system with Type II Repair: one unit operating, the other on standby

The system state space diagram is shown in Figure 2.32. By setting n=1 in Equations (2.223)-(2.231), we obtain the following differential equations:

$$\frac{dP_1(t)}{dt} = -(\lambda_o + \lambda_s + \lambda_c)P_1(t) + \mu[P_{o1}(t) + P_{s1}(t)] \tag{2.333}$$

$$\frac{dP_{o1}(t)}{dt} = -(\lambda_o + \lambda_s + \mu)P_{o1}(t) + \lambda_o P_1(t) \tag{2.334}$$

$$\frac{dP_{s1}(t)}{dt} = -(\lambda_o + \lambda_s + \mu)P_{s1}(t) + \lambda_s P_1(t) \tag{2.335}$$

$$\frac{dP_s(t)}{dt} = \lambda_s[P_2(t) + P_3(t)] \tag{2.336}$$

$$\frac{dP_o(t)}{dt} = \lambda_o[P_2(t) + P_3(t)] \tag{2.337}$$

$$\frac{dP_c(t)}{dt} = \lambda_c P_1(t) \tag{2.338}$$

At time $t = 0$, $P_1(0) = 1$, and all the other initial condition probabilities are equal to zero.

By taking Laplace transforms of Equations (2.333) - (2.338) and solving the resulting equations, the Laplace transforms of the state probabilities are as follows:

$$P_1(s) = \frac{\lambda_o + \lambda_s + \mu + s}{(\lambda_o + \lambda_s)^2 + (\lambda_o + \lambda_s)(\lambda_c + 2s) + (\mu + s)(\lambda_c + s)} \tag{2.339}$$

$$P_{o1}(s) = \frac{\lambda_o}{(\lambda_o + \lambda_s)^2 + (\lambda_o + \lambda_s)(\lambda_c + 2s) + (\mu + s)(\lambda_c + s)} \tag{2.340}$$

$$P_{s1}(s) = \frac{\lambda_s}{(\lambda_o + \lambda_s)^2 + (\lambda_o + \lambda_s)(\lambda_c + 2s) + (\mu + s)(\lambda_c + s)} \tag{2.341}$$

$$P_s(s) = \frac{\lambda_s(\lambda_o + \lambda_s)}{s[(\lambda_o + \lambda_s)^2 + (\lambda_o + \lambda_s)(\lambda_c + 2s) + (\mu + s)(\lambda_c + s)]} \tag{2.342}$$

$$P_o(s) = \frac{\lambda_o(\lambda_o + \lambda_s)}{s[(\lambda_o + \lambda_s)^2 + (\lambda_o + \lambda_s)(\lambda_c + 2s) + (\mu + s)(\lambda_c + s)]} \tag{2.343}$$

$$P_c(s) = \frac{\lambda_c(\lambda_o + \lambda_s + \mu + s)}{s[(\lambda_o + \lambda_s)^2 + (\lambda_o + \lambda_s)(\lambda_c + 2s) + (\mu + s)(\lambda_c + s)]} \tag{2.344}$$

The Laplace transform of system reliability is given by

$$\begin{aligned} R(s) &= \sum_{up} P_j(s) \\ &= P_1(s) + P_{o1}(s) + P_{s1}(s) \\ &= \frac{2\lambda_o + 2\lambda_s + \mu + s}{(\lambda_o + \lambda_s)^2 + (\lambda_o + \lambda_s)(\lambda_c + 2s) + (\mu + s)(\lambda_c + s)} \end{aligned} \tag{2.345}$$

The system mean time to failure (MTTF) is

$$\begin{aligned} MTTF &= \lim_{s \to 0} R(s) \\ &= \lim_{s \to 0} [P_1(s) + P_{o1}(s) + P_{s1}(s)] \\ &= \frac{2\lambda_o + 2\lambda_s + \mu}{(\lambda_o + \lambda_s)^2 + (\lambda_o + \lambda_s)\lambda_c + \mu\lambda_c} \end{aligned} \tag{2.346}$$

By taking inverse Laplace transforms of Equations (2.339) - (2.344) the state time dependent probabilities can be obtained. Similarly, the time dependent system reliability can be obtained by taking inverse Laplace transform of Equation (2.345).

The time dependent system reliabilities are plotted in Figure 2.33 for specified values of model parameters. Figure 2.34 is the time dependent state probabilities. The system mean time to failure given by Equation (2.346) is plotted in Figure 2.35. All these plots as well as those of special cases are discussed at the end of Section 2.4.

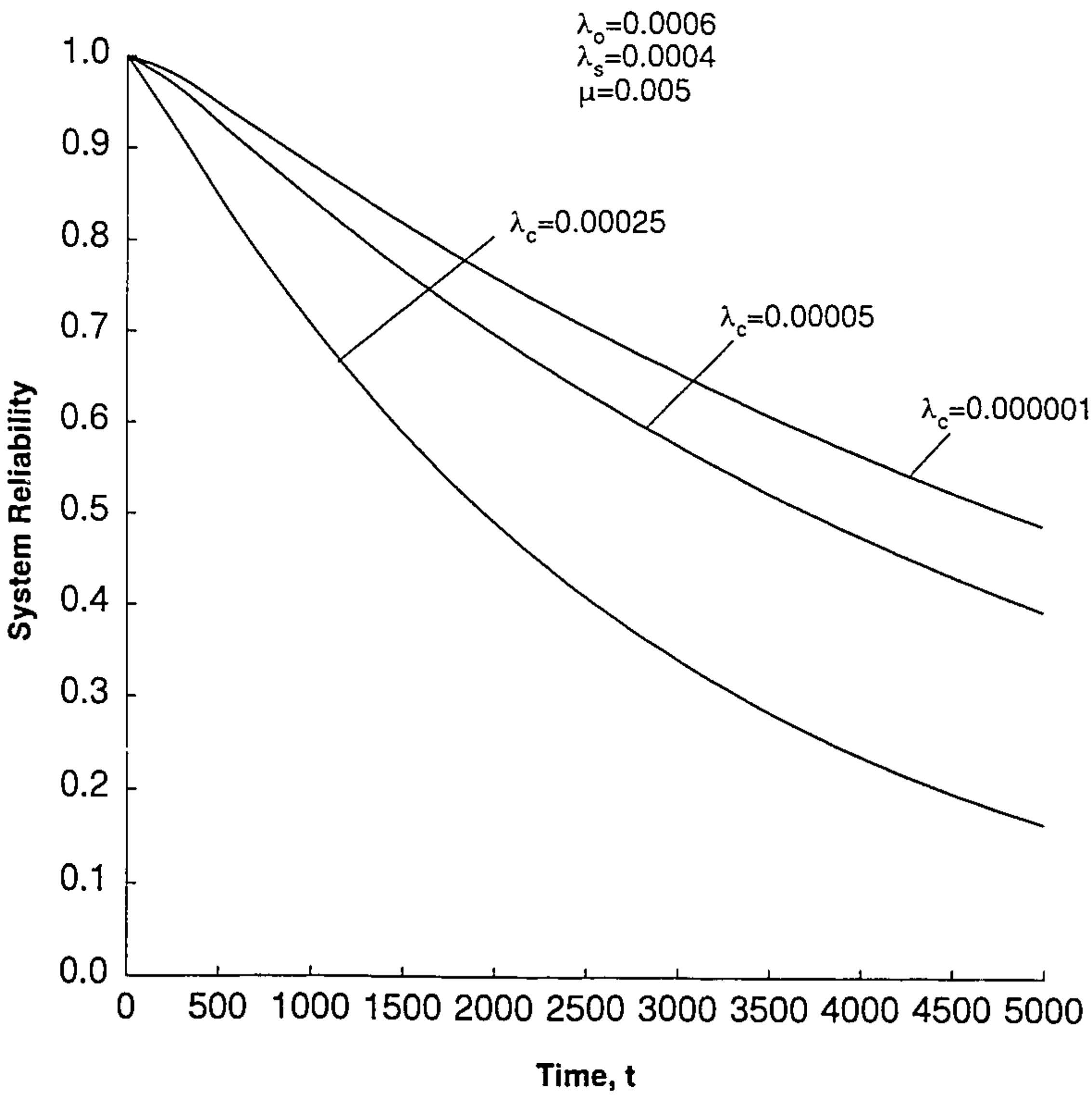


Figure 2.33 The standby system with type II repair reliability plots: one unit operating, the other on standby

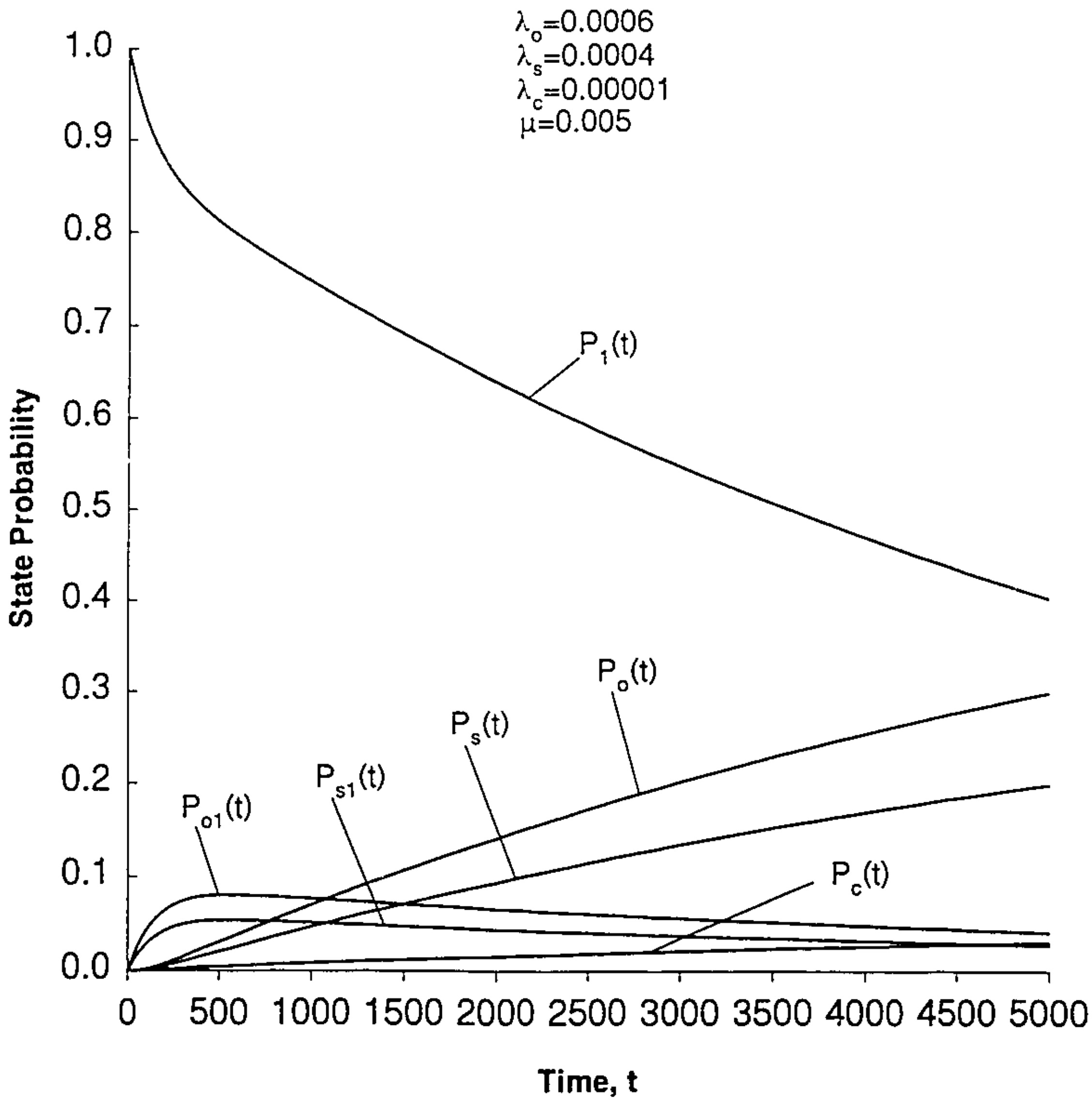


Figure 2.34 The standby system with type II repair state probability plots: one unit operating, the other on standby

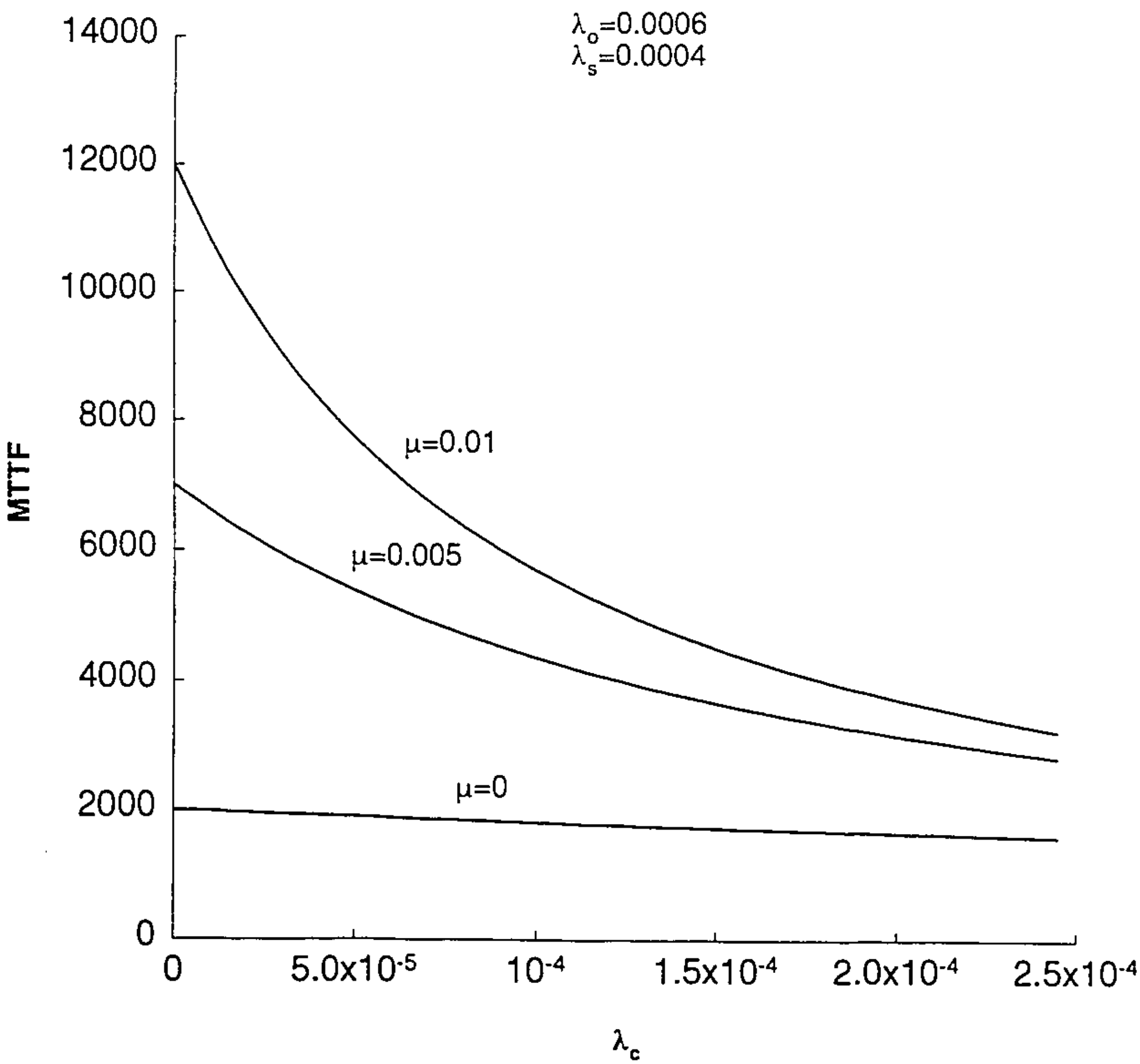


Figure 2.35 MTTF vs common cause failure rate, λ_c for standby system (one unit operating, the other on standby) with type II repair

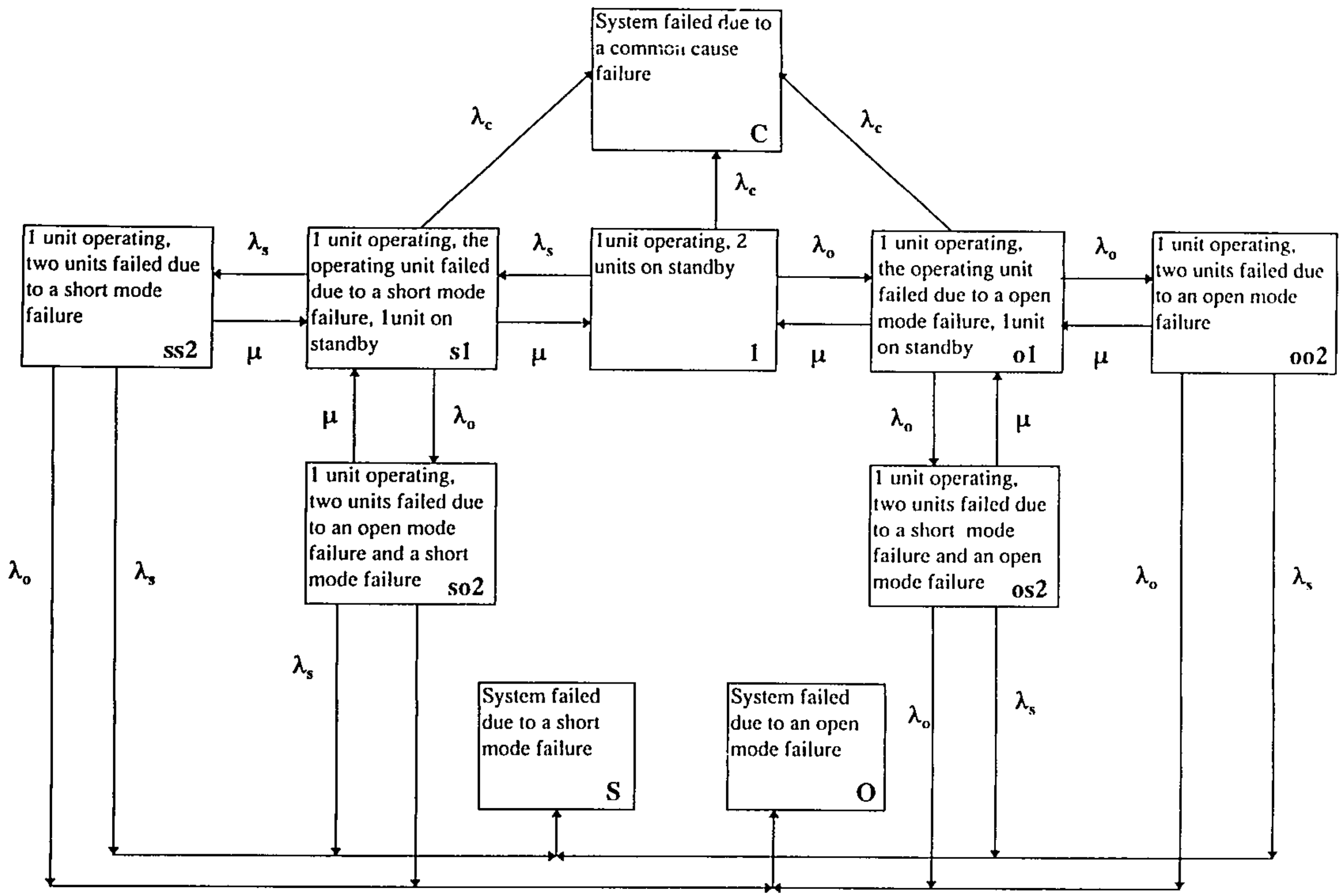


Figure 2.36 System state space diagram with type II repair: one unit operating, and two units on standby

2.4.3 Standby system with Type II Repair: one unit operating, two units on standby

The system state space diagram is shown in Figure 2.36. By setting n=2 in Equations (2.223)-(2.231), we obtain the following differential equations:

$$\frac{dP_1(t)}{dt} = -(\lambda_o + \lambda_s + \lambda_c)P_1(t) + \mu[P_{o1}(t) + P_{s1}(t)] \tag{2.347}$$

$$\frac{dP_{o1}(t)}{dt} = -(\lambda_o + \lambda_s + \lambda_c + \mu)P_{o1}(t) + \lambda_o P_1(t) + \mu[P_{oo2}(t) + P_{os2}(t)] \tag{2.348}$$

$$\frac{dP_{s1}(t)}{dt} = -(\lambda_o + \lambda_s + \lambda_c + \mu)P_{s1}(t) + \lambda_s P_1(t) + \mu[P_{so2}(t) + P_{ss2}(t)] \tag{2.349}$$

$$\frac{dP_{oo2}(t)}{dt} = -(\lambda_o + \lambda_s + \mu)P_{oo2}(t) + \lambda_o P_{o1}(t) \tag{2.350}$$

$$\frac{dP_{ss2}(t)}{dt} = -(\lambda_o + \lambda_s + \mu)P_{ss2}(t) + \lambda_s P_{s1}(t) \tag{2.351}$$

$$\frac{dP_{os2}(t)}{dt} = -(\lambda_o + \lambda_s + \mu)P_{os2}(t) + \lambda_s P_{o1}(t) \tag{2.352}$$

$$\frac{dP_{so2}(t)}{dt} = -(\lambda_o + \lambda_s + \mu)P_{so2}(t) + \lambda_o P_{s1}(t) \tag{2.353}$$

$$\frac{dP_s(t)}{dt} = \lambda_s[P_{oo2}(t) + P_{os2}(t) + P_{so2}(t) + P_{ss2}(t)] \tag{2.354}$$

$$\frac{dP_o(t)}{dt} = \lambda_o[P_{oo2}(t) + P_{os2}(t) + P_{so2}(t) + P_{ss2}(t)] \tag{2.355}$$

$$\frac{dP_c(t)}{dt} = \lambda_c[P_1(t) + P_{o1}(t) + P_{s1}(t)] \tag{2.356}$$

At time $t = 0$, $P_1(0) = 1$, and all the other initial condition probabilities are equal to zero.

By taking Laplace transforms of Equations (2.347) - (2.356) and solving the resulting equations, the Laplace transforms of the state probabilities are as follows:

$$P_1(s) = \frac{A(2,21)}{A(2,20)} \tag{2.357}$$

$$P_{o1}(s) = \frac{\lambda_o(\lambda_o + \lambda_s + \mu + s)}{A(2,20)} \tag{2.358}$$

$$P_{s1}(s) = \frac{\lambda_s(\lambda_o + \lambda_s + \lambda\mu + s)}{A(2,20)} \tag{2.359}$$

$$P_{oo2}(s) = \frac{\lambda_o^{\,2}}{A(2,20)} \tag{2.360}$$

$$P_{ss2}(s) = \frac{\lambda_s^{\,2}}{A(2,20)} \tag{2.361}$$

$$P_{os2}(s) = P_{so2}(s) = \frac{\lambda_o\lambda_s}{A(2,20)} \tag{2.362}$$

$$P_s(s) = \frac{\lambda_s(\lambda_o + \lambda_s)^2}{sA(2,20)} \tag{2.363}$$

$$P_o(s) = \frac{\lambda_o(\lambda_o + \lambda_s)^2}{sA(2,20)} \tag{2.364}$$

$$P_c(s) = \frac{A(2,22)}{sA(2,20)} \tag{2.365}$$

The Laplace transform of system reliability is given by

$$\begin{aligned} R(s) &= \sum_{u,r} P_j(s) \\ &= P_1(s) + P_{o1}(s) + P_{s1}(s) + P_{oo2}(s) + P_{ss2}(s) + P_{os2}(s) + P_{so2}(s) \\ &= \frac{A(2,23)}{A(2,20)} \end{aligned} \tag{2.366}$$

where A(2,20), A(2,21), A(2,22) and A(2,23) are defined in Appendix 2.1.

The system mean time to failure (MTTF) is

$$MTTF = \lim_{s\to 0} R(s)$$

$$= \lim_{s \to 0}[P_1(s) + P_{o1}(s) + P_{s1}(s) + P_{oo2}(s) + P_{ss2}(s) + P_{os2}(s) + P_{so2}(s)]$$

$$= \frac{3(\lambda_o + \lambda_s)^2 + (\lambda_o + \lambda_s)(\lambda_c + 2\mu) + \mu(\lambda_c + \mu)}{(\lambda_o + \lambda_s)^3 + 2\lambda_c(\lambda_o + \lambda_s)^2 + \lambda_c(\lambda_o + \lambda_s)(\lambda_c + 2\mu) + \lambda_c\mu(\lambda_c + \mu)} \qquad (2.367)$$

By taking inverse Laplace transforms of Equations (2.357) - (2.365) the state time dependent probabilities can be obtained. Also, an expression for the system reliability can be obtained by taking inverse Laplace transform of Equation (2.366).

In order to make a comparison with the system in which one unit is operating and the other is on standby, the same values of parameters (i.e. as for Equations (2.339)-(2.346)) are chosen for the reliability plots, the state probability plots, and the mean time to failure plots. The time dependent system reliabilities are plotted in Figure 2.37. Figure 2.38 is for the time dependent state probabilities. The system mean time to failure given by Equation (2.367) is plotted in Figure 2.39.

2.4.4 Standby system with Type II Repair: one unit operating, three units on standby

The system state space diagram is shown in Figure 2.40. By setting n=3 in Equations (2.224)-(2.231), we obtain the following differential equations:

$$\frac{dP_1(t)}{dt} = -(\lambda_o + \lambda_s + \lambda_c)P_1(t) + \mu[P_{o1}(t) + P_{s1}(t)] \qquad (2.368)$$

$$\frac{dP_{o1}(t)}{dt} = -(\lambda_o + \lambda_s + \lambda_c + \mu)P_{o1}(t) + \lambda_o P_1(t) + \mu[P_{oo2}(t) + P_{os2}(t)] \qquad (2.369)$$

$$\frac{dP_{s1}(t)}{dt} = -(\lambda_o + \lambda_s + \lambda_c + \mu)P_{s1}(t) + \lambda_s P_1(t) + \mu[P_{so2}(t) + P_{ss2}(t)] \qquad (2.370)$$

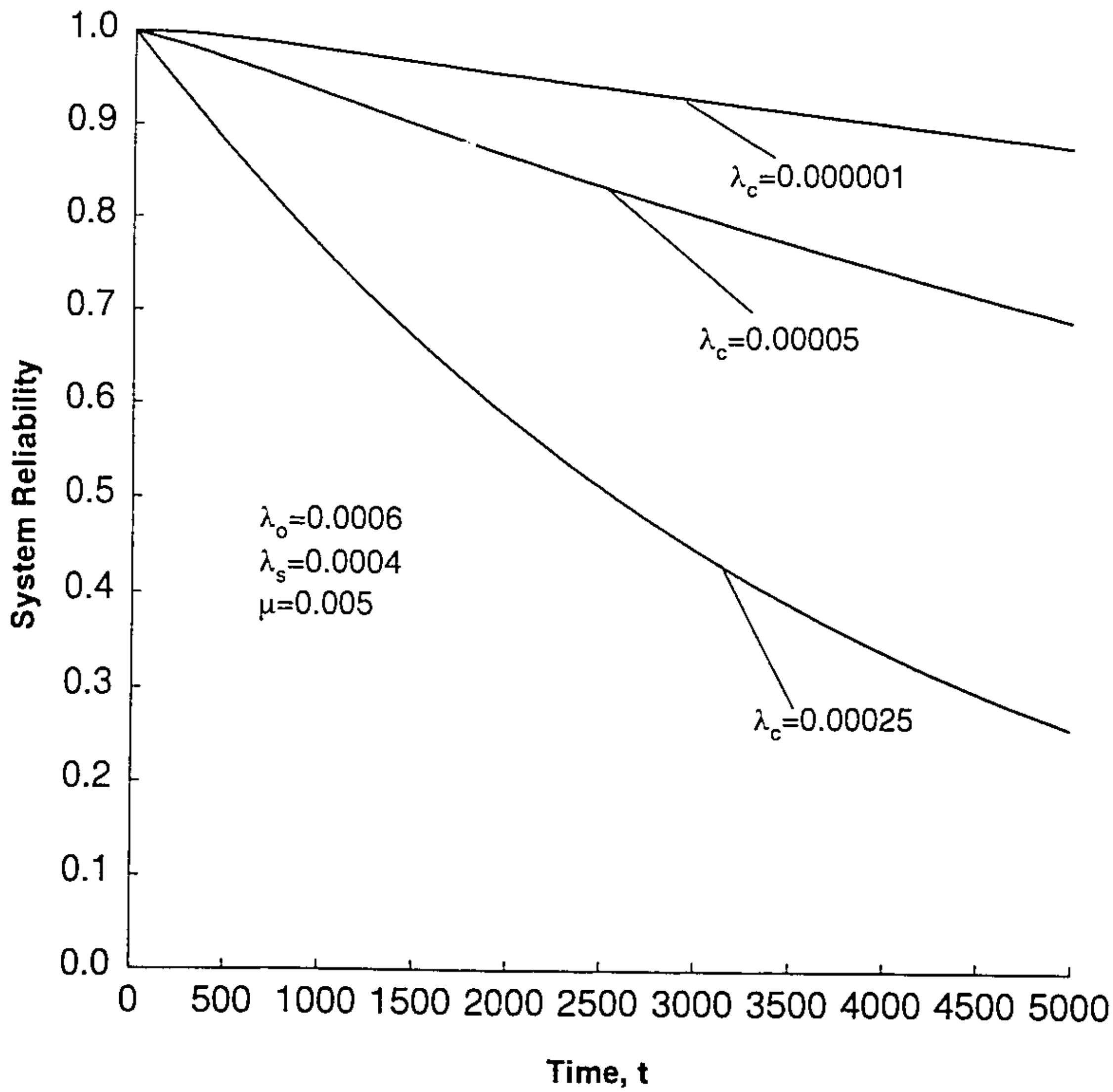


Figure 2.37 The standby system with type II repair reliability plots: one unit operating, two units on standby

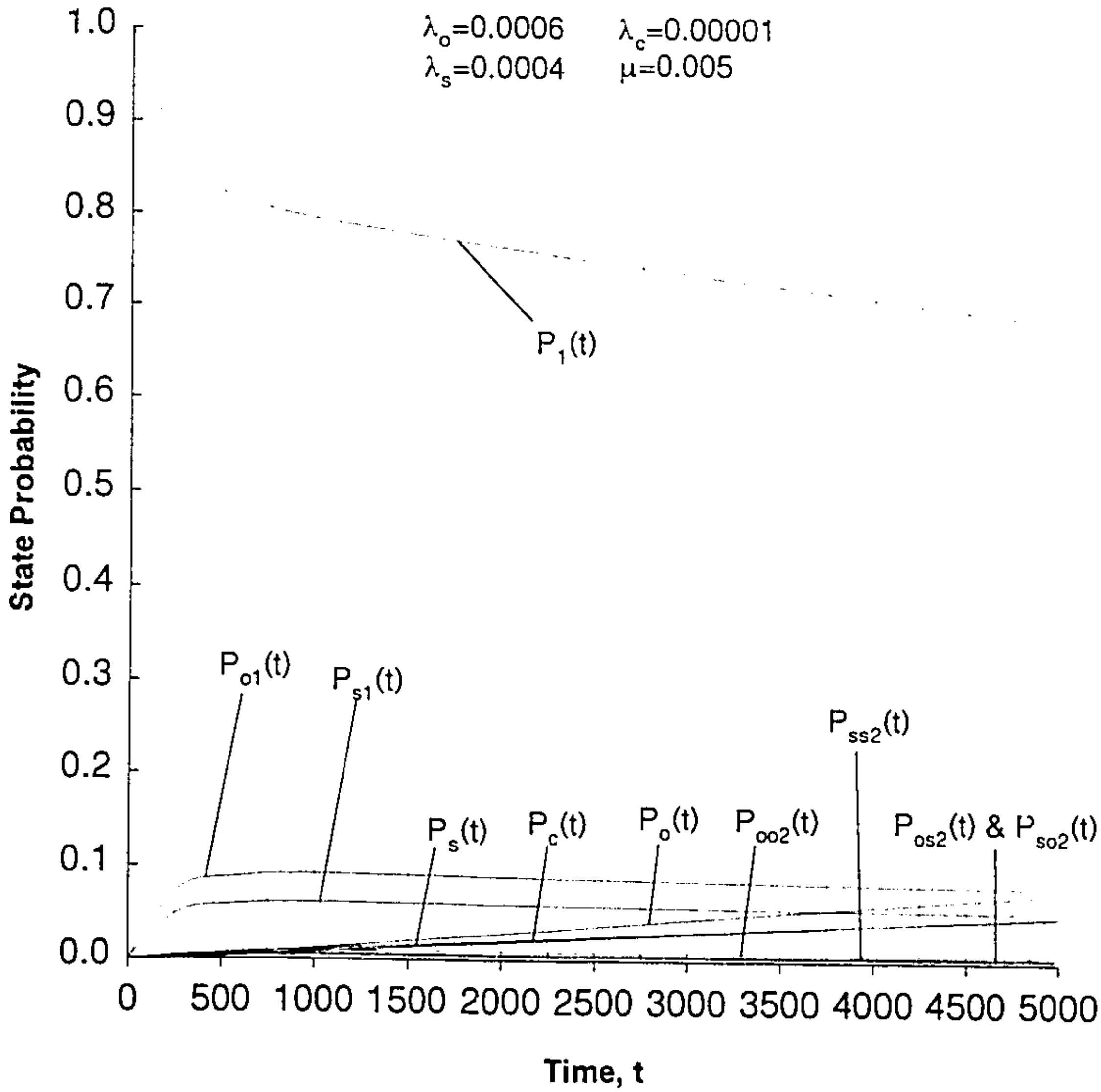


Figure 2.38 The standby system with type II repair state probability plots: one unit operating, two units on standby

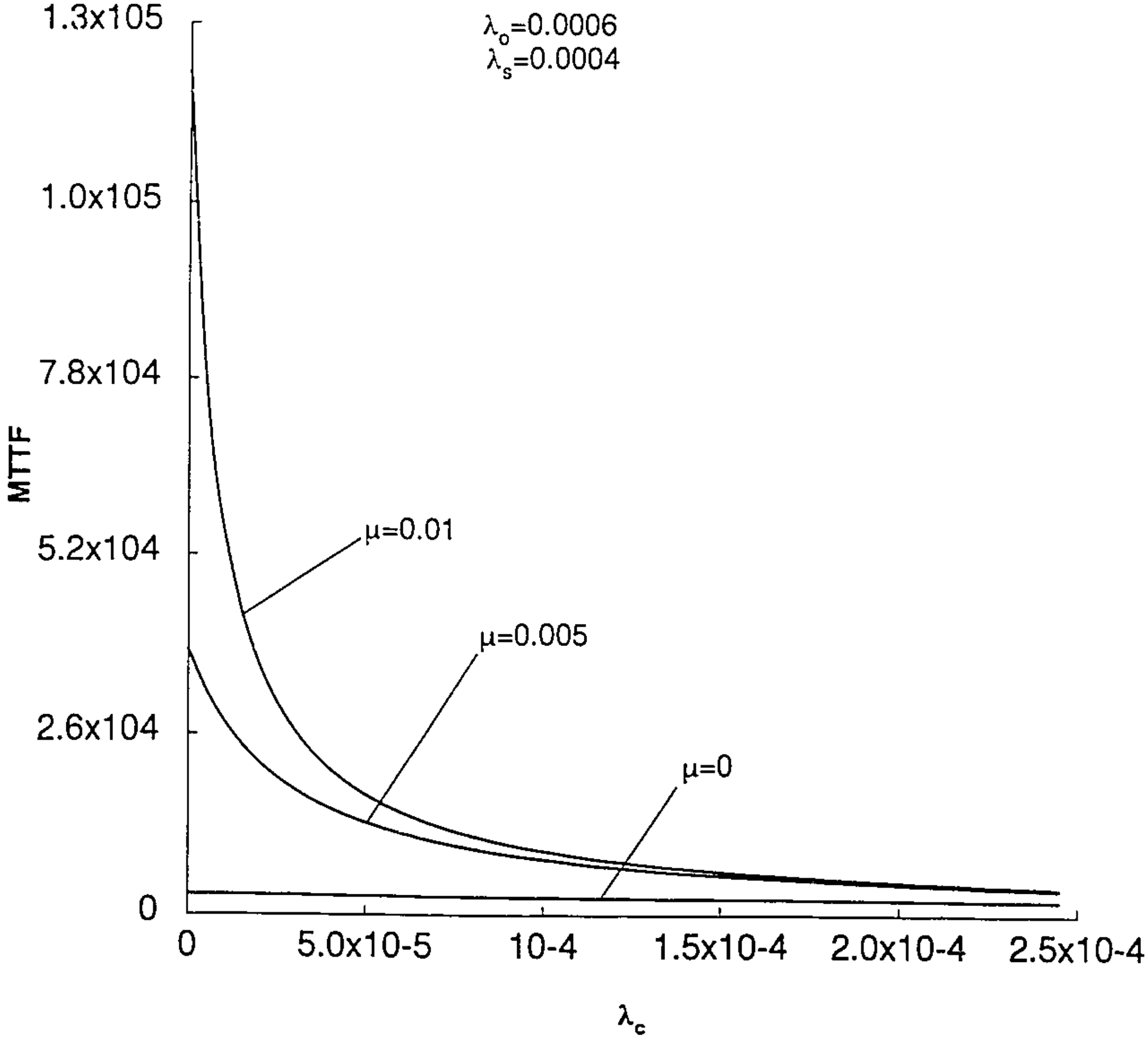


Figure 2.39 MTTF vs common cause failure rate, λ_c for standby system (one unit operating, two units on standby) with type II repair

SB=Standby

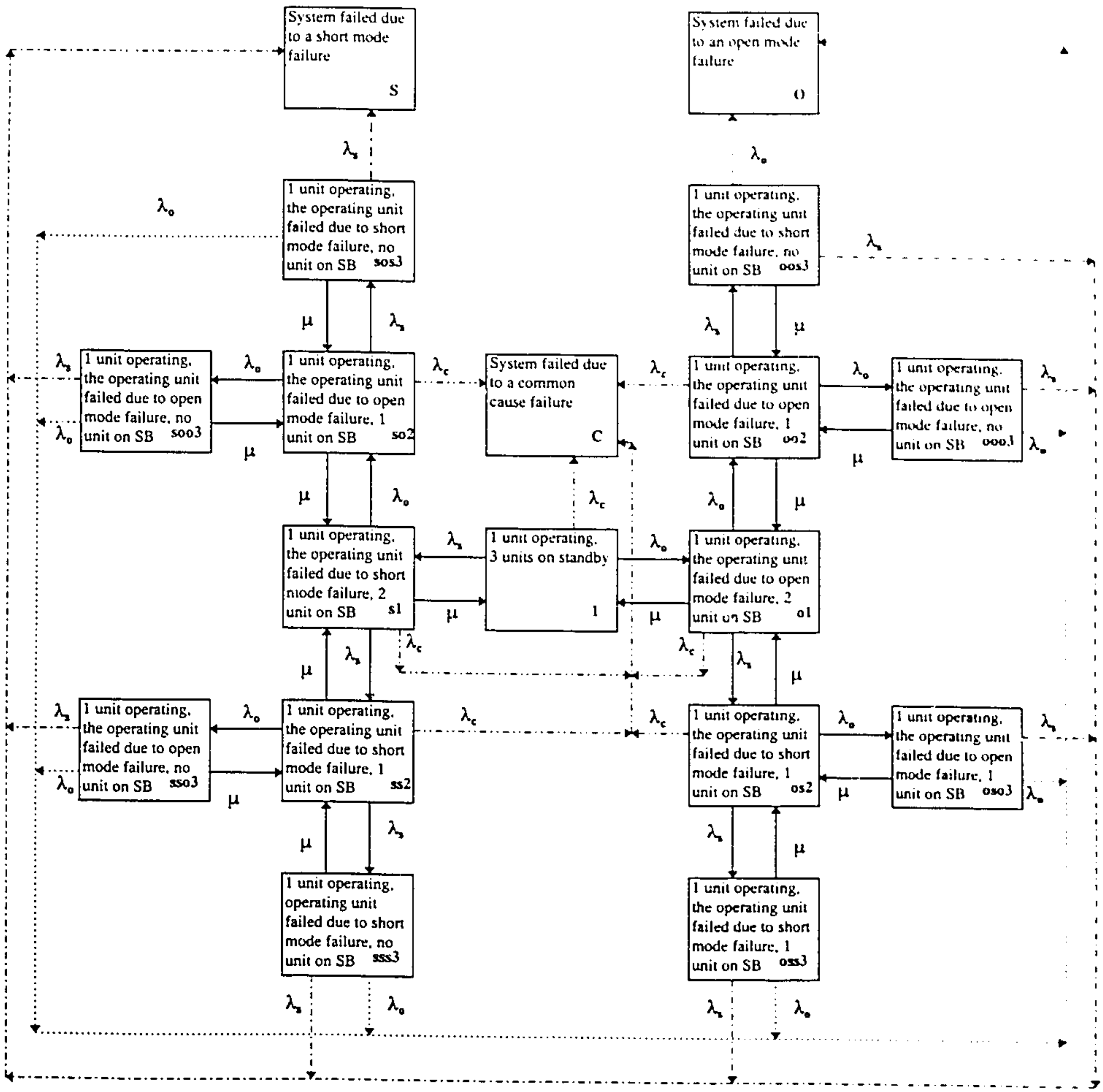


Figure 2.40 System state space diagram with type II repair: one unit operating, three units on standby

$$\frac{dP_{oo2}(t)}{dt} = -(\lambda_o + \lambda_s + \lambda_c + \mu)P_{oo2}(t) + \lambda_o P_{o1}(t) + \mu[P_{ooo3}(t) + P_{oss3}(t)] \tag{2.371}$$

$$\frac{dP_{os2}(t)}{dt} = -(\lambda_o + \lambda_s + \lambda_c + \mu)P_{os2}(t) + \lambda_s P_{o1}(t) + \mu[P_{oso3}(t) + P_{oss3}(t)] \tag{2.372}$$

$$\frac{dP_{ss2}(t)}{dt} = -(\lambda_o + \lambda_s + \lambda_c + \mu)P_{ss2}(t) + \lambda_s P_{s1}(t) + \mu[P_{sss3}(t) + P_{sso3}(t)] \tag{2.373}$$

$$\frac{dP_{so2}(t)}{dt} = -(\lambda_o + \lambda_s + \lambda_c + \mu)P_{so2}(t) + \lambda_o P_{s1}(t) + \mu[P_{soo3}(t) + P_{sos3}(t)] \tag{2.374}$$

$$\frac{dP_{ooo3}(t)}{dt} = -(\lambda_o + \lambda_s + \mu)P_{ooo3}(t) + \lambda_o P_{oo2}(t) \tag{2.375}$$

$$\frac{dP_{oos3}(t)}{dt} = -(\lambda_o + \lambda_s + \mu)P_{oos3}(t) + \lambda_s P_{oo2}(t) \tag{2.376}$$

$$\frac{dP_{oso3}(t)}{dt} = -(\lambda_o + \lambda_s + \mu)P_{oso3}(t) + \lambda_o P_{os2}(t) \tag{2.377}$$

$$\frac{dP_{oss3}(t)}{dt} = -(\lambda_o + \lambda_s + \mu)P_{oss3}(t) + \lambda_s P_{os2}(t) \tag{2.378}$$

$$\frac{dP_{sss3}(t)}{dt} = -(\lambda_o + \lambda_s + \mu)P_{sss3}(t) + \lambda_s P_{ss2}(t) \tag{2.379}$$

$$\frac{dP_{sso3}(t)}{dt} = -(\lambda_o + \lambda_s + \mu)P_{sso3}(t) + \lambda_o P_{ss2}(t) \tag{2.380}$$

$$\frac{dP_{sos3}(t)}{dt} = -(\lambda_o + \lambda_s + \mu)P_{sos3}(t) + \lambda_s P_{so2}(t) \tag{2.381}$$

$$\frac{dP_{soo3}(t)}{dt} = -(\lambda_o + \lambda_s + \mu)P_{soo3}(t) + \lambda_o P_{so2}(t) \tag{2.382}$$

$$\frac{dP_s(t)}{dt} = \lambda_s[P_{ooo3}(t) + P_{oos3}(t) + P_{oso3}(t) + P_{osss}(t) + P_{sss3}(t) + P_{sso3}(t) + P_{sos3}(t) + P_{soo3}(t)] \tag{2.383}$$

$$\frac{dP_o(t)}{dt} = \lambda_o[P_{ooo3}(t) + P_{oos3}(t) + P_{oso3}(t) + P_{osss}(t) + P_{sss3}(t) + P_{sso3}(t) + P_{sos3}(t) + P_{soo3}(t)] \tag{2.384}$$

$$\frac{dP_c(t)}{dt} = \lambda_c[P_1(t) + P_{o1}(t) + P_{s1}(t) + P_{oo2}(t) + P_{os2}(t) + P_{so2}(t) + P_{ss2}(t)] \tag{2.385}$$

At time $t = 0$, $P_1(0) = 1$, and all the other initial condition probabilities are equal to zero.

By taking Laplace transforms of Equations (2.368) - (2.385) and solving the resulting equations, the Laplace transforms of the state probabilities are as follows:

$$P_1(s) = \frac{A(2,25)}{A(2,24)} \tag{2.386}$$

$$P_{o1}(s) = \frac{\lambda_b A(2,26)}{A(2,24)} \tag{2.387}$$

$$P_{s1}(s) = \frac{\lambda_s A(2,26)}{A(2,24)} \tag{2.388}$$

$$P_{oo2}(s) = \frac{\lambda_b^2(\lambda_b + \lambda_s + \mu + s)}{A(2,24)} \tag{2.389}$$

$$P_{os2}(s) = P_{so2}(s) = \frac{\lambda_b\lambda_s(\lambda_b + \lambda_s + \mu + s)}{A(2,24)} \tag{2.390}$$

$$P_{ss2}(s) = \frac{\lambda_s^2(\lambda_b + \lambda_s + \mu + s)}{A(2,24)} \tag{2.391}$$

$$P_{ooo3}(s) = \frac{\lambda_b^{\ 3}}{A(2,24)} \tag{2.392}$$

$$P_{oos3}(s) = P_{oso3}(s) = P_{soo3}(s) = \frac{\lambda_b^{\ 2}\lambda_s}{A(2,24)} \tag{2.393}$$

$$P_{oss2}(s) = P_{sso3}(s) = P_{sos3}(s) = \frac{\lambda_b\lambda_s^2}{A(2,24)} \tag{2.394}$$

$$P_{sss3}(s) = \frac{\lambda_s^{\ 3}}{A(2,24)} \tag{2.395}$$

$$P_s(s) = \frac{\lambda_s(\lambda_b + \lambda_s)^3}{sA(2,24)} \tag{2.396}$$

$$P_o(s) = \frac{\lambda_b(\lambda_b + \lambda_s)^3}{sA(2,24)} \tag{2.397}$$

$$P_c(s) = \frac{A(2,27)}{sA(2,25)} \tag{2.398}$$

The Laplace transform of system reliability is given by

$$R(s) = \sum_{up} P_j(s)$$

$$= P_1(s) + P_{o1}(s) + P_{s1}(s) + P_{oo2}(s) + P_{os2}(s) + P_{ss2}(s) + P_{so2}(s) + P_{ooo3}(s) + P_{oos3}(s) +$$

$$P_{oso3}(s) + P_{oss3}(s) + P_{soo3}(s) + P_{sos3}(s) + P_{sso3}(s) + P_{sss3}(s)$$

$$= \frac{A(2,28)}{A(2,24)} \tag{2.399}$$

The system mean time to failure (MTTF) is

$$MTTF = \lim_{s \to 0} R(s)$$

$$= \lim_{s \to 0} [P_1(s) + P_{o1}(s) + P_{s1}(s) + P_{oo2}(s) + P_{ss2}(s) + P_{os2}(s) + P_{so2}(s) +$$

$$P_{ooo3}(s) + P_{oos3}(s) + P_{oso3}(s) + P_{oss3}(s) + P_{soo3}(s) + P_{sos3}(s) + P_{sso3}(s) + P_{sss3}(s)]$$

$$= \frac{A(2,30)}{A(2,29)} \tag{2.400}$$

By taking inverse Laplace transforms of Equations (2.386) - (2.398) the state time dependent probabilities can be obtained. Also, an expression for the system reliability can be obtained by taking inverse Laplace transforms of Equation (2.399).

In order to make a comparison with the system in which one unit is operating and the other is on standby, the same values of parameters (i.e. as for Equations (2.339)-(2.346)) are chosen for the system reliability plots, the state probability plots, and the mean time to failure plots. The time dependent system reliabilities are plotted in Figure 2.41. Figure 2.42 is for the time dependent state probabilities. The system mean time to failure given by Equation (2.400) is plotted in Figure 2.43.

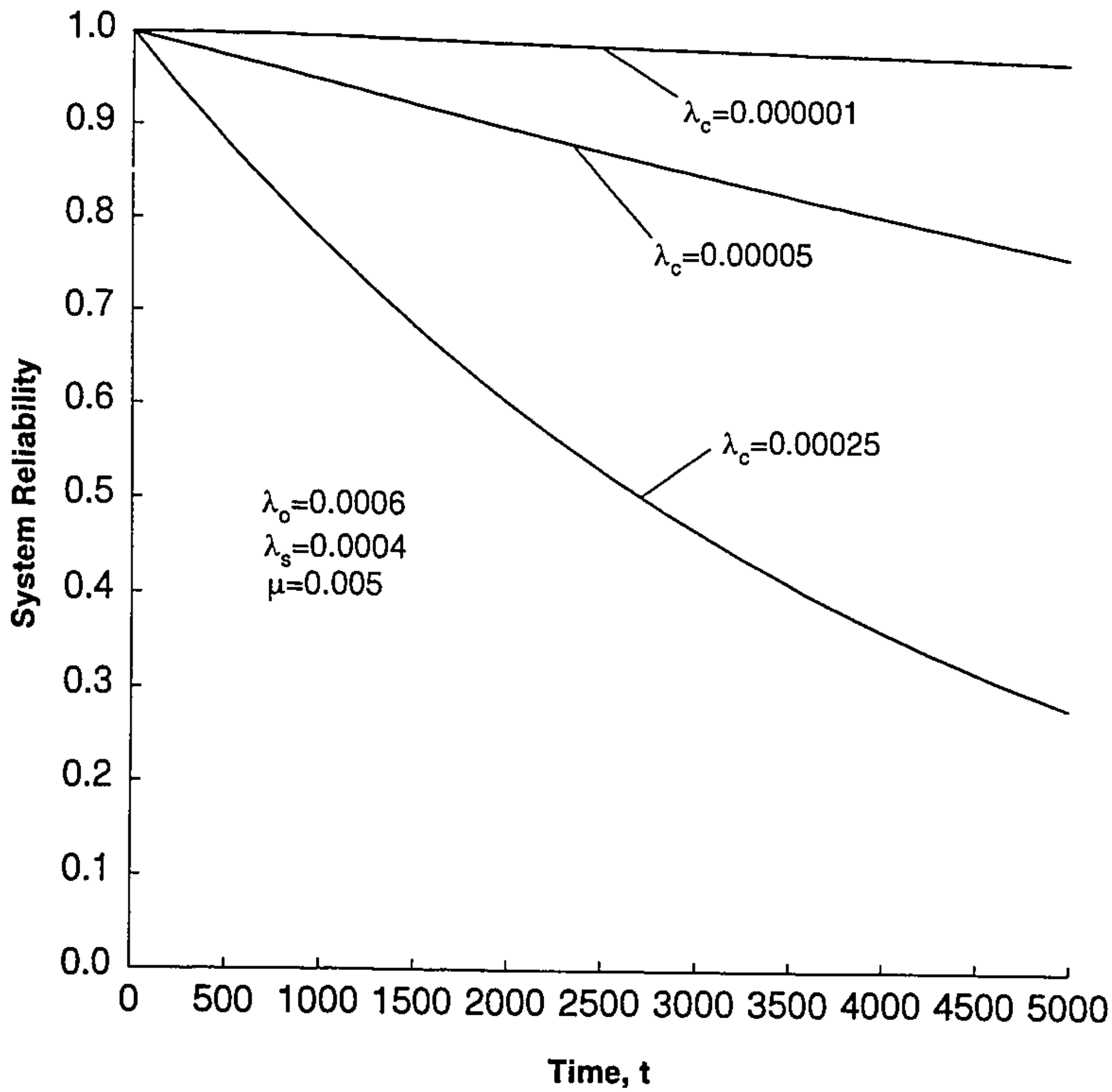


Figure 2.41 The standby system with type II repair reliability plots: one unit operating, three units on standby

* $P_{oos3}(t)$, $P_{oss3}(t)$ and P_{sss3} are too small, thus they do not appear.

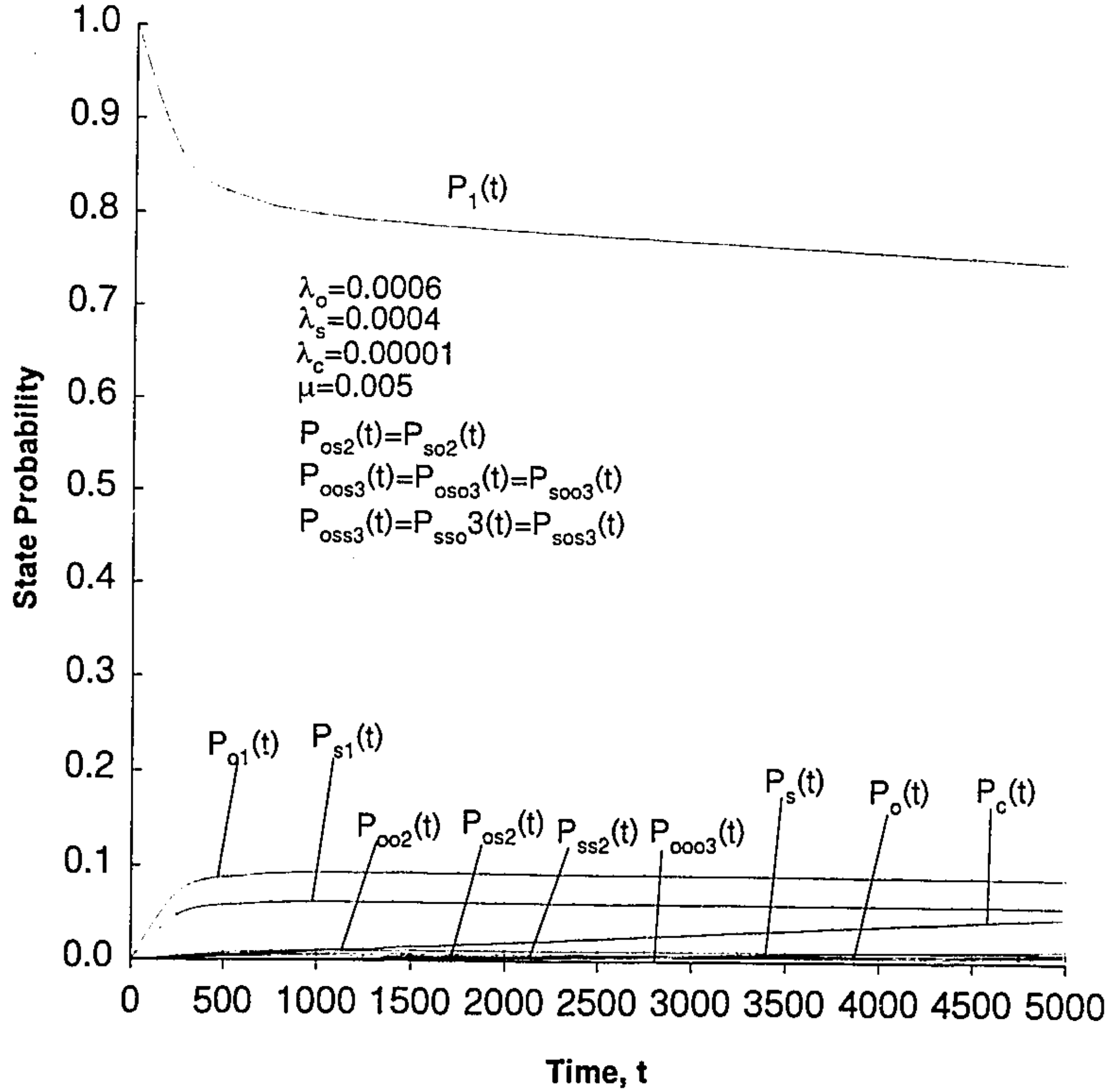


Figure 2.42 The standby system with type II repair state probability plots: one unit operating, three units on standby

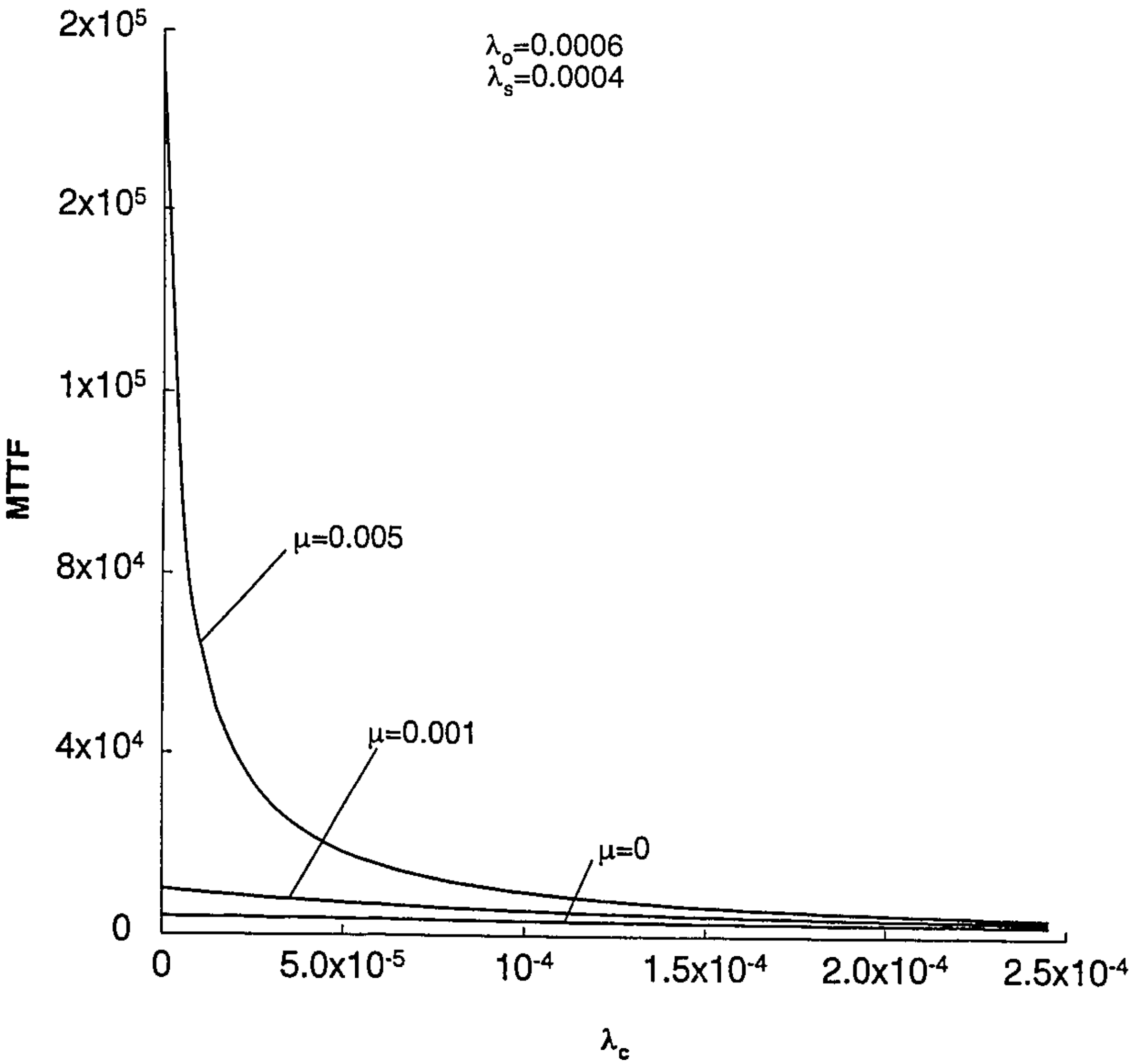


Figure 2.43 MTTF vs common cause failure rate, λ_c for standby system (one unit operating, three units on standby) with type II repair

The analysis performed in this section demonstrates basically the same result as for the standby system without repair: system reliability increases with increasing number of the standby units and decreases with an increase in value of common-cause failure rate. In addition, the repair action increases the system reliability dramatically.

From the system mean time to failure (MTTF) plots, it is noted that common-cause failure has significant affect on MTTF when Type II Repair is performed. The system mean time to failure (MTTF) increases as the number of standby units increase and decreases as the common-cause failure rate increases.

From the time dependent probability plots, it is evident that the probability of $P_1(t)$ decreases slower than without repair action. It is also observed that the probability P_o and P_s of system open mode failure state and short mode failure state, respectively, decrease with an increase in the number of standby units but the probability P_c of failure due to the occurrence of common-cause failure increase as the number of standby units increase.

Chapter 3

Three-State device Warm Standby System: Special Cases

In the previous Chapter we discussed the *cold standby* system, in which the standby unit is not active. However, this is not generally case in practice since *warm standby* systems are more common. This Chapter discusses the *warm standby* system with common cause failure. Because the warm standby system has more system states than the cold standby system, the calculations are more cumbersome. Thus, we have only considered special cases. The configurations of the system are the same as the configurations in the Chapter 2. The system reliability, the mean time to failure, the steady state availability and the time dependent availability are obtained. In order to make comparisons with the cold standby systems, the same values of the parameters as in Chapter 2 are chosen. The assumptions associated with the warm standby system are the same as with the cold standby system.

3.1 Warm standby system: one unit operating, the other one on standby

By setting n=1 in Figure 2.1, the configuration of a one unit on warm standby system can be obtained. In this system three cases are considerded: Without Repair, With Type I Repair and With Type II Repair (repair polcies are the same as described previously).

Notation

The following symbols are associated with this model:

t Time.

s Laplace transform variable.

λ_o Constant open-mode failure rate of an operating unit or device.

λ_s Constant short-mode failure rate of an operating unit or device.

λ_{so} Constant open-mode failure rate of a standby unit or device.

λ_{ss} Constant short-mode failure rate of a standby unit or device.

λ_c Constant common-cause failure rate of the system.

μ Constant repair rate of a unit or the system.

μ_c Constant repair rate of the failed system from common-cause failure state to initial state 1.

$P_1(t)$ Probability that the system is in state 1.

$P_2(t)$ Probability of the state in which one operating unit failed due to an open mode failure, the standby unit starts working.

$P_3(t)$ Probability of the state in which one operating unit failed due to a short mode failure, the standby unit starts working.

$P_4(t)$ Probability of the state in which one unit is operating and the standby unit failed due to an open mode failure.

$P_5(t)$ Probability of the state in which one unit is operating and the standby unit failed due to a short mode failure.

$P_s(t)$ Probability of the state that the system failed due to a short mode failure.

$P_o(t)$ Probability of the state that the system failed due to an open mode failure.

$P_c(t)$ Probability of the state that the system failed due to a common-cause failure.

$R(t)$ Reliability of system in [0,t].

$MTTF$ System mean time to failure.

AV_{ss} Steady state system availability.

$AV(t)$ Time dependent system availability.

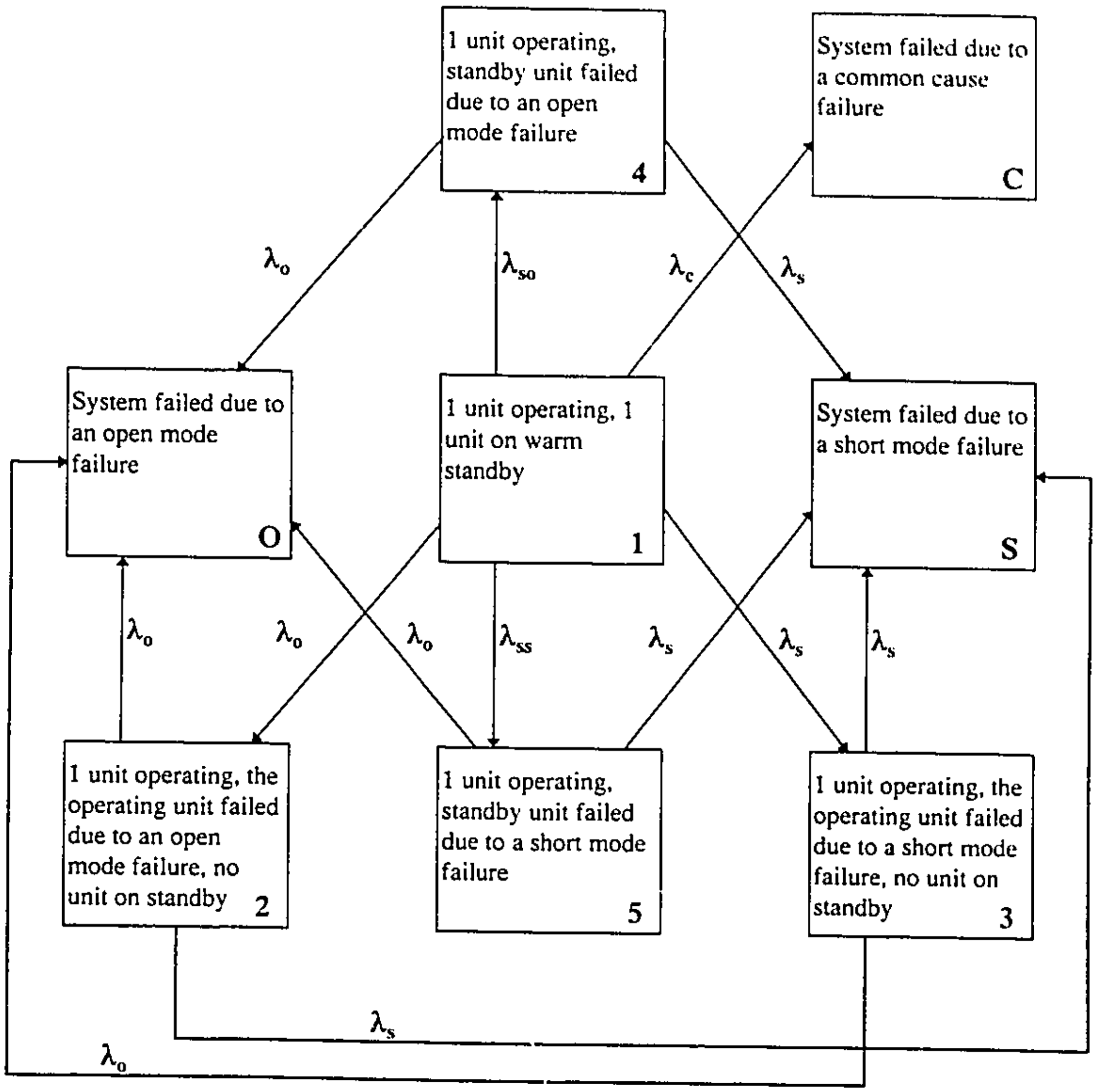


Figure 3.1 System state space diagram without repair: one unit operating, the other on warm standby

3.1.1 Warm standby system without Repair

The system state space diagram is shown in Figure 3.1. The following differential equations are associated with this system.

$$\frac{dP_1(t)}{dt} = -(\lambda_b + \lambda_{so} + \lambda_s + \lambda_{ss} + \lambda_c)P_1(t) \tag{3.1}$$

$$\frac{dP_2(t)}{dt} = -(\lambda_b + \lambda_s + \lambda_c)P_2(t) + \lambda_b P_1(t) \tag{3.2}$$

$$\frac{dP_3(t)}{dt} = -(\lambda_b + \lambda_s + \lambda_c)P_3(t) + \lambda_s P_1(t) \tag{3.3}$$

$$\frac{dP_4(t)}{dt} = -(\lambda_b + \lambda_s + \lambda_c)P_4(t) + \lambda_{so} P_1(t) \tag{3.4}$$

$$\frac{dP_5(t)}{dt} = -(\lambda_b + \lambda_s + \lambda_c)P_5(t) + \lambda_{ss} P_1(t) \tag{3.5}$$

$$\frac{dP_s(t)}{dt} = \lambda_s[P_2(t) + P_3(t) + P_4(t) + P_5(t)] \tag{3.6}$$

$$\frac{dP_o(t)}{dt} = \lambda_b[P_2(t) + P_3(t) + P_4(t) + P_5(t)] \tag{3.7}$$

$$\frac{dP_c(t)}{dt} = \lambda_c P_1(t) \tag{3.8}$$

At time $t = 0$, $P_1(0) = 1$, and all the other initial condition probabilities are equal to zero.

By taking Laplace transforms of Equations (3.1) - (3.8) and solving the resulting equations, the Laplace transforms of the state probabilities are as follows:

$$P_1(s) = \frac{1}{\lambda_b + \lambda_s + \lambda_{so} + \lambda_{ss} + \lambda_c + s} \tag{3.9}$$

$$P_2(s) = \frac{\lambda_b}{(\lambda_b + \lambda_s + s)(\lambda_b + \lambda_s + \lambda_{so} + \lambda_{ss} + \lambda_c + s)} \tag{3.10}$$

$$P_3(s) = \frac{\lambda_s}{(\lambda_b + \lambda_s + s)(\lambda_b + \lambda_s + \lambda_{so} + \lambda_{ss} + \lambda_c + s)} \tag{3.11}$$

$$P_4(s)=\frac{\lambda_{so}}{(\lambda_b+\lambda_s+s)(\lambda_b+\lambda_s+\lambda_{so}+\lambda_{ss}+\lambda_c+s)} \tag{3.12}$$

$$P_5(s)=\frac{\lambda_{ss}}{(\lambda_b+\lambda_s+s)(\lambda_b+\lambda_s+\lambda_{so}+\lambda_{ss}+\lambda_c+s)} \tag{3.13}$$

$$P_s(s)=\frac{\lambda_s(\lambda_b+\lambda_s+\lambda_{so}+\lambda_{ss})}{s(\lambda_b+\lambda_s+s)(\lambda_b+\lambda_s+\lambda_{so}+\lambda_{ss}+\lambda_c+s)} \tag{3.14}$$

$$P_o(s)=\frac{\lambda_b(\lambda_b+\lambda_s+\lambda_{so}+\lambda_{ss})}{s(\lambda_b+\lambda_s+s)(\lambda_b+\lambda_s+\lambda_{so}+\lambda_{ss}+\lambda_c+s)} \tag{3.15}$$

$$P_c(s)=\frac{\lambda_c}{s(\lambda_b+\lambda_s+\lambda_{so}+\lambda_{ss}+\lambda_c+s)} \tag{3.16}$$

The Laplace transform of the system reliability is given by

$$\begin{aligned}R(s)&=P_1(s)+P_2(s)+P_3(s)+P_4(s)+P_5(s)\\&=\frac{2\lambda_b+2\lambda_s+\lambda_{so}+\lambda_{ss}+s}{(\lambda_b+\lambda_s+s)(\lambda_b+\lambda_s+\lambda_{so}+\lambda_{ss}+\lambda_c+s)}\end{aligned} \tag{3.17}$$

By taking inverse Laplace transforms of Equations (3.9) - (3.17), the time dependent state probabilities are as follows:

$$P_1(t)=\frac{1}{e^{(\lambda_b+\lambda_s+\lambda_{so}+\lambda_{ss}+\lambda_c)t}} \tag{3.18}$$

$$P_2(t)=\frac{\lambda_b}{\lambda_{so}+\lambda_{ss}+\lambda_c}\left[\frac{1}{e^{(\lambda_b+\lambda_s)t}}-\frac{1}{e^{(\lambda_b+\lambda_s+\lambda_{so}+\lambda_{ss}+\lambda_c)t}}\right] \tag{3.19}$$

$$P_3(t)=\frac{\lambda_s}{\lambda_{so}+\lambda_{ss}+\lambda_c}\left[\frac{1}{e^{(\lambda_b+\lambda_s)t}}-\frac{1}{e^{(\lambda_b+\lambda_s+\lambda_{so}+\lambda_{ss}+\lambda_c)t}}\right] \tag{3.20}$$

$$P_4(t)=\frac{\lambda_{so}}{\lambda_{so}+\lambda_{ss}+\lambda_c}\left[\frac{1}{e^{(\lambda_b+\lambda_s)t}}-\frac{1}{e^{(\lambda_b+\lambda_s+\lambda_{so}+\lambda_{ss}+\lambda_c)t}}\right] \tag{3.21}$$

$$P_5(t)=\frac{\lambda_{ss}}{\lambda_{so}+\lambda_{ss}+\lambda_c}\left[\frac{1}{e^{(\lambda_b+\lambda_s)t}}-\frac{1}{e^{(\lambda_b+\lambda_s+\lambda_{so}+\lambda_{ss}+\lambda_c)t}}\right] \tag{3.22}$$

$$P_s(t)=\frac{\lambda_s(\lambda_b+\lambda_s+\lambda_{so}+\lambda_{ss})}{(\lambda_b+\lambda_s)(\lambda_{so}+\lambda_{ss}+\lambda_{so}+\lambda_{ss}+\lambda_c)}+$$

$$\frac{\lambda_s(\lambda_b+\lambda_s+\lambda_{so}+\lambda_{ss})}{\lambda_{so}+\lambda_{ss}+\lambda_c}[\frac{1}{e^{(\lambda_b+\lambda_s)t}(\lambda_b+\lambda_s)}-\frac{1}{e^{(\lambda_b+\lambda_s+\lambda_{so}+\lambda_{ss}+\lambda_c)t}(\lambda_b+\lambda_s+\lambda_{so}+\lambda_{ss}+\lambda_c)}]$$

(3.23)

$$P_5(t)=\frac{\lambda_b(\lambda_b+\lambda_s+\lambda_{so}+\lambda_{ss})}{(\lambda_b+\lambda_s)(\lambda_{so}+\lambda_{ss}+\lambda_{so}+\lambda_{ss}+\lambda_c)}+$$

$$\frac{\lambda_b(\lambda_b+\lambda_s+\lambda_{so}+\lambda_{ss})}{\lambda_{so}+\lambda_{ss}+\lambda_c}[\frac{1}{e^{(\lambda_b+\lambda_s)t}(\lambda_b+\lambda_s)}-\frac{1}{e^{(\lambda_b+\lambda_s+\lambda_{so}+\lambda_{ss}+\lambda_c)t}(\lambda_b+\lambda_s+\lambda_{so}+\lambda_{ss}+\lambda_c)}]$$

(3.24)

$$P_c(t)=\frac{\lambda_c}{(\lambda_b+\lambda_s+\lambda_{so}+\lambda_{ss}+\lambda_c)}[1-\frac{1}{e^{(\lambda_b+\lambda_s+\lambda_{so}+\lambda_{ss}+\lambda_c)t}}] \tag{3.25}$$

The system reliability is given by

$$\begin{aligned} R(t) &= \sum P(t) \\ &= P_1(t)+P_2(t)+P_3(t)+P_4(t)+P_5(t) \\ &= \frac{1}{\lambda_{so}+\lambda_{ss}+\lambda_c}[\frac{(\lambda_b+\lambda_s+\lambda_{so}+\lambda_{ss})}{e^{(\lambda_b+\lambda_s)t}}+\frac{(-\lambda_b-\lambda_s+\lambda_c)}{e^{(\lambda_b+\lambda_s+\lambda_{so}+\lambda_{ss}+\lambda_c)t}}] \end{aligned} \tag{3.26}$$

The system mean time to failure (MTTF) is

$$\begin{aligned} MTTF &= \lim_{s\to 0} R(s) \\ &= \lim_{s\to 0}[P_1(s)+P_2(s)+P_3(s)+P_4(s)+P_5(s)] \\ &= \frac{2\lambda_b+2\lambda_s+\lambda_{so}+\lambda_{ss}}{(\lambda_b+\lambda_s)(\lambda_b+\lambda_s+\lambda_{so}+\lambda_{ss}+\lambda_c)} \end{aligned} \tag{3.27}$$

The same values of model parameters corresponding to the cold standby system without repair (one unit operating, the other one on standby) as well as the failure rate of the standby unit are used in following figures. The system reliability given by Equation (3.26) is plotted in Figure 3.2. It is evident that the reliability of the warm standby system is lower than that of the cold standby system. The plots of Equation (3.18) - (3.25) are

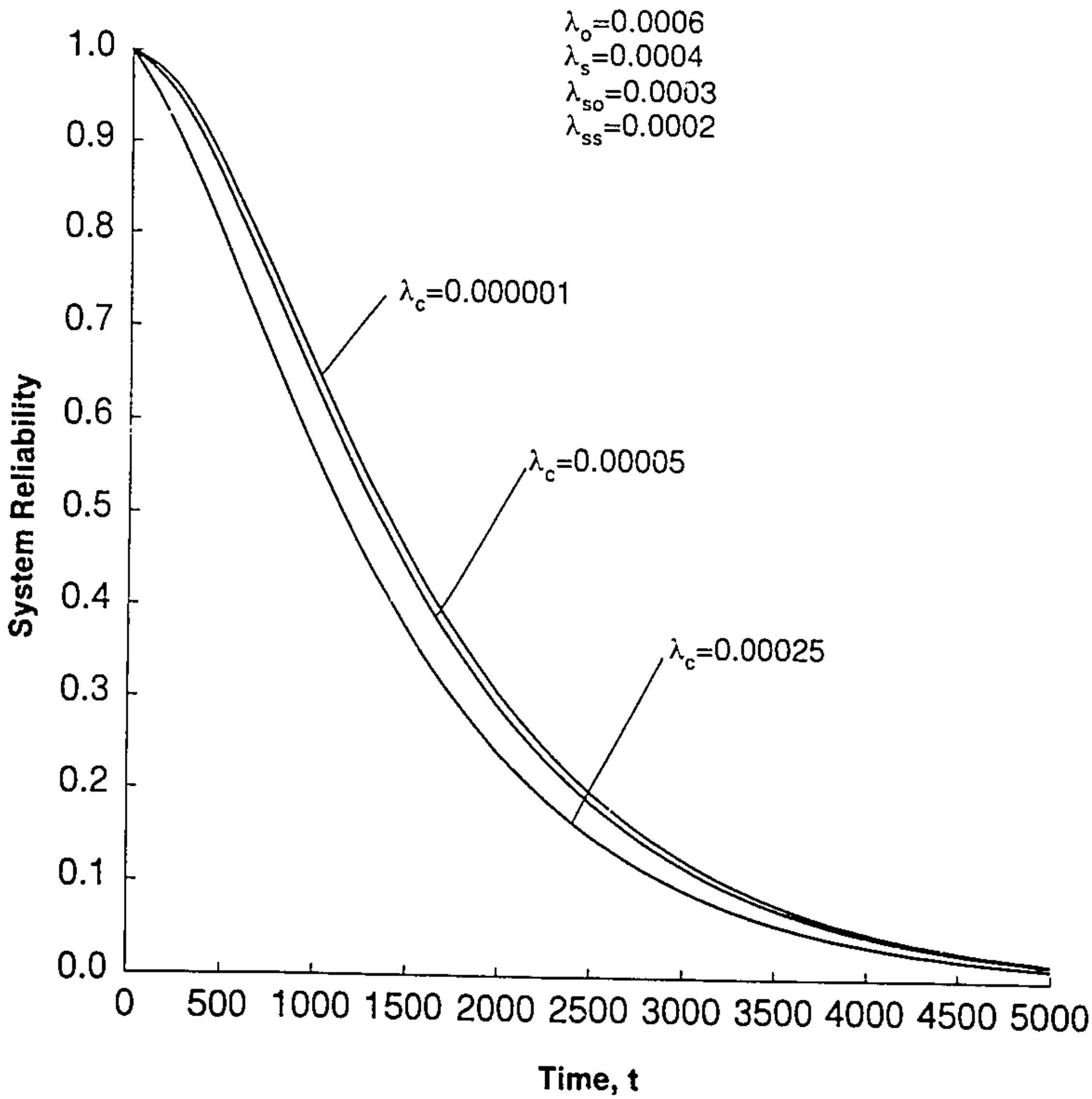


Figure 3.2 The warm standby system without repair reliability plots: one unit operating, the other on warm standby

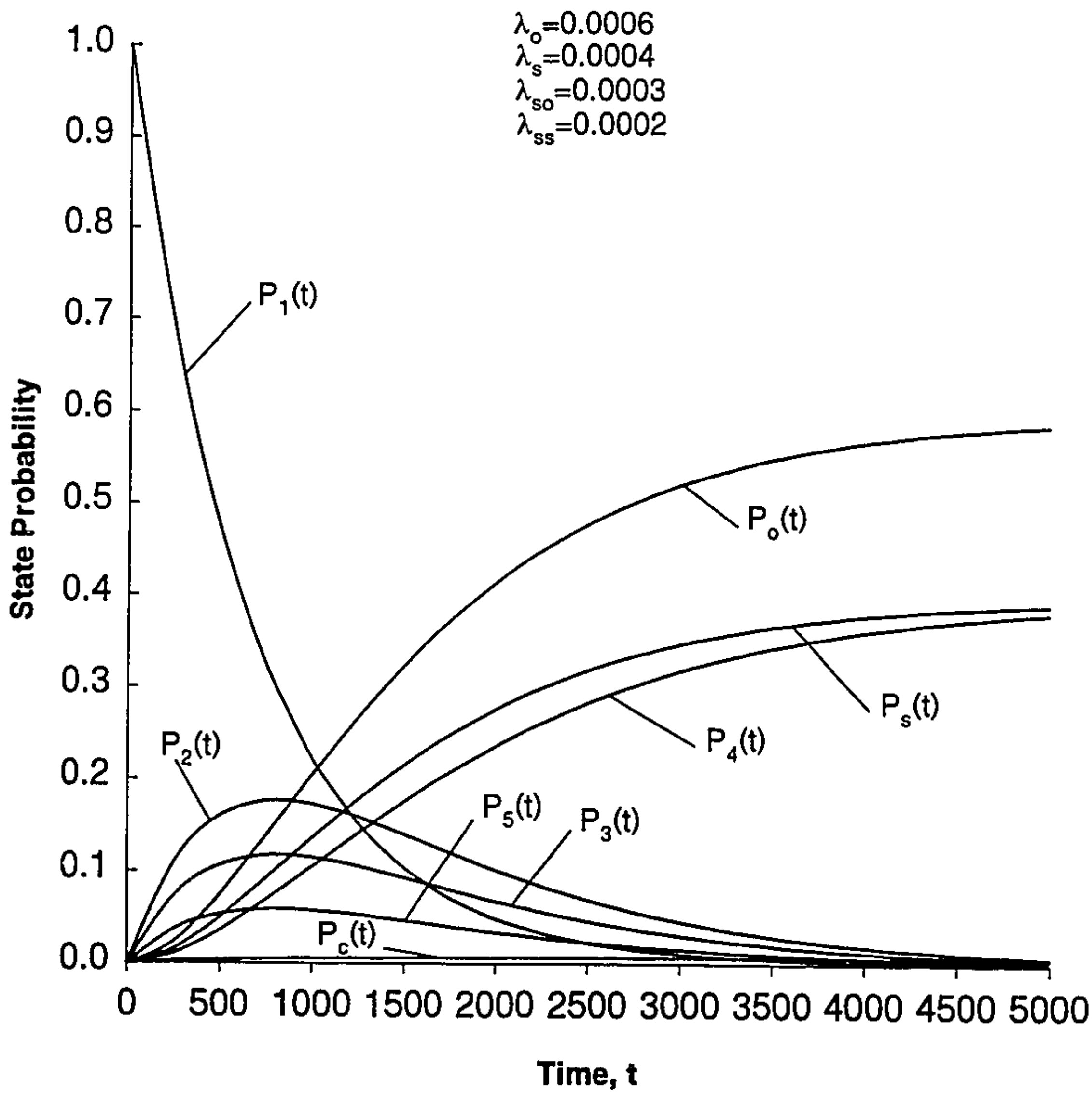


Figure 3.3 The warm standby system without repair state probability plots: one unit operating, the other on warm standby

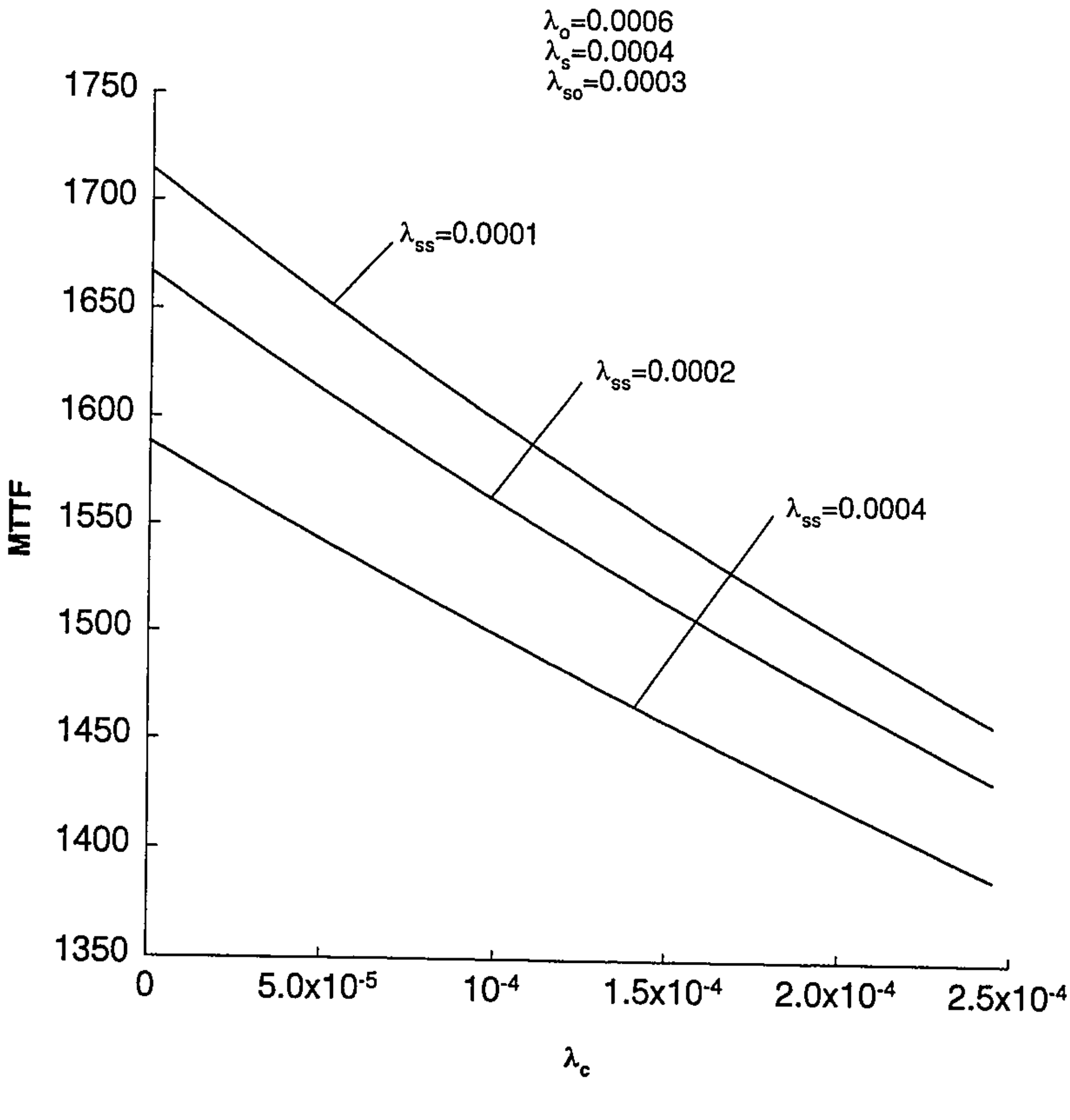


Figure 3.4 MTTF vs common cause failure rate, λ_c for warm standby system (one unit operating, the other on warm standby) without repair

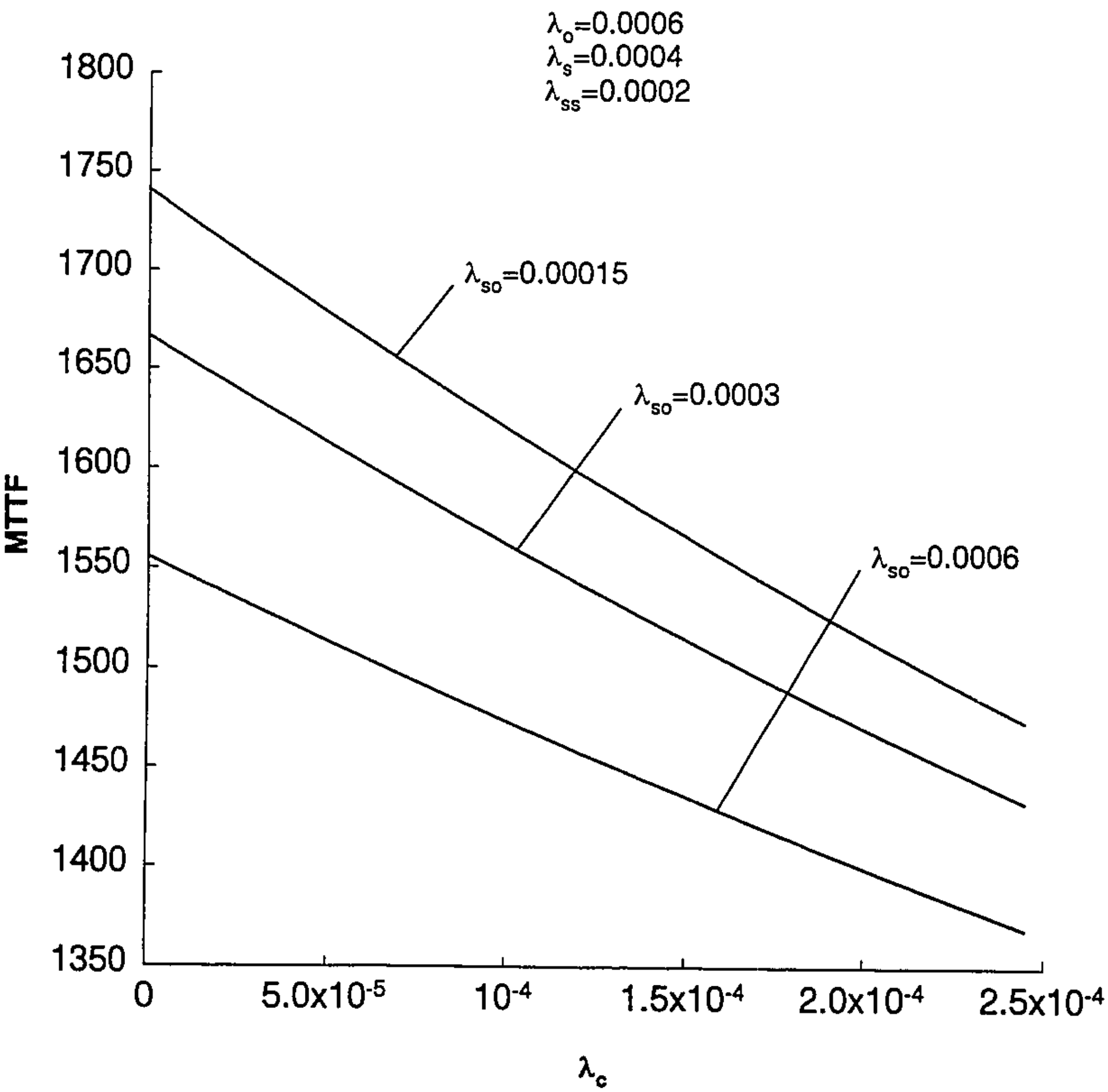


Figure 3.5 MTTF vs common cause failure rate, λ_c for warm standby system (one unit operating, the other on warm standby) without repair

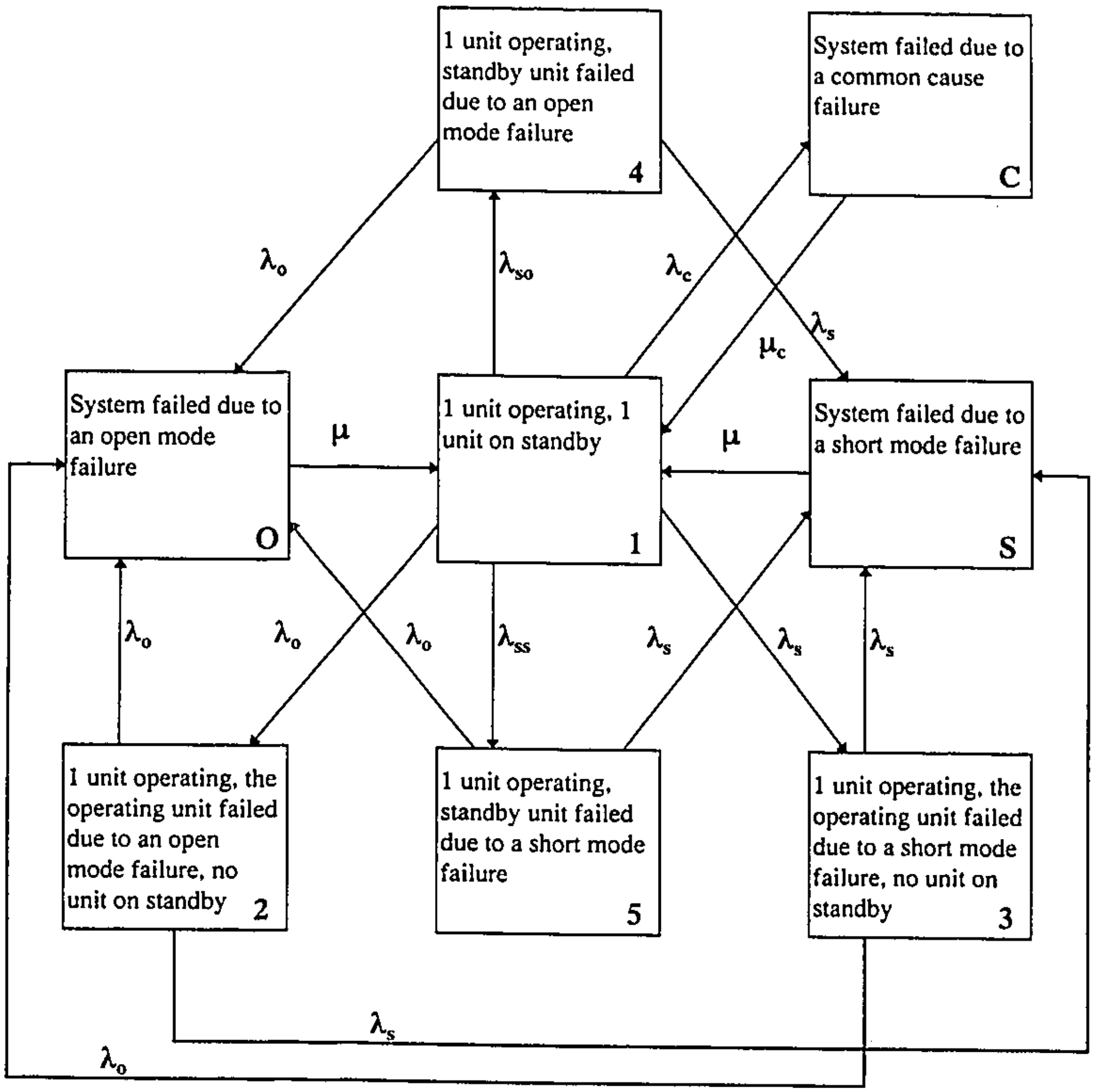


Figure 3.6 System state space diagram with Type I Repair: one unit operating, the other on warm standby

shown in Figure 3.3. The failure state probabilities $P_s(t)$ and $P_o(t)$ in Figure 3.3 are higher than $P_s(t)$ and $P_o(t)$ in Figure 2.6. It shows that the failure states of warm standby system contribute more probability than those of the cold standby system. Figure 3.4 and Figure 3.5. present the plots of Equation (3.27).

3.1.2 Warm standby system with Type I Repair

The system state space diagram is shown in Figure 3.6. The following differential equations are associated with this system.

$$\frac{dP_1(t)}{dt} = -(\lambda_b + \lambda_{so} + \lambda_s + \lambda_{ss} + \lambda_c)P_1(t) + \mu[P_o(t) + P_s(t)] + \mu_c P_c(t) \tag{3.28}$$

$$\frac{dP_2(t)}{dt} = -(\lambda_b + \lambda_s + \lambda_t)P_2(t) + \lambda_b P_1(t) \tag{3.29}$$

$$\frac{dP_3(t)}{dt} = -(\lambda_b + \lambda_s + \lambda_t)P_3(t) + \lambda_s P_1(t) \tag{3.30}$$

$$\frac{dP_4(t)}{dt} = -(\lambda_b + \lambda_s + \lambda_t)P_4(t) + \lambda_{so} P_1(t) \tag{3.31}$$

$$\frac{dP_5(t)}{dt} = -(\lambda_b + \lambda_s + \lambda_t)P_5(t) + \lambda_{ss} P_1(t) \tag{3.32}$$

$$\frac{dP_s(t)}{dt} = -\mu P_s(t) + \lambda_s[P_2(t) + P_3(t) + P_4(t) + P_5(t)] \tag{3.33}$$

$$\frac{dP_o(t)}{dt} = -\mu P_o(t) + \lambda_b[P_2(t) + P_3(t) + P_4(t) + P_5(t)] \tag{3.34}$$

$$\frac{dP_c(t)}{dt} = -\mu_c P_c(t) + \lambda_c P_1(t) \tag{3.35}$$

At time $t = 0$, $P_1(0) = 1$, and all the other initial condition probabilities are equal to zero.

The steady state probabilities can be obtained by setting derivatives of above Equations (3.28) - (3.35) equal to zero and utilizing the relationship $\sum P_j = 1$, j represents all of the states in the system.

$$P_1 = \frac{(\lambda_o + \lambda_s)\,\mu\mu_c}{A(3,1)} \tag{3.36}$$

$$P_2 = \frac{\lambda_o\,\mu\mu_c}{A(3,1)} \tag{3.37}$$

$$P_3 = \frac{\lambda_s\,\mu\mu_c}{A(3,1)} \tag{3.38}$$

$$P_4 = \frac{\lambda_{so}\,\mu\mu_c}{A(3,1)} \tag{3.39}$$

$$P_5 = \frac{\lambda_{ss}\,\mu\mu_c}{A(3,1)} \tag{3.40}$$

$$P_s = \frac{\lambda_s\,(\lambda_o + \lambda_s + \lambda_{so} + \lambda_{ss})\,\mu_c}{A(3,1)} \tag{3.41}$$

$$P_o = \frac{\lambda_o\,(\lambda_o + \lambda_s + \lambda_{so} + \lambda_{ss})\,\mu_c}{A(3,1)} \tag{3.42}$$

$$P_c = \frac{\lambda_c\,(\lambda_o + \lambda_s + s)\,\mu}{A(3,1)} \tag{3.43}$$

The system steady state availability is given by

$$AV_{ss} = \frac{(2\lambda_o + 2\lambda_s + \lambda_{so} + \lambda_{ss})\,\mu\mu_c}{A(3,1)} \tag{3.44}$$

where A(3,1) is defined in Appendix 3.1

By taking the Laplace transforms of Equations (3.28) - (3.35) and solving the resulting equations, the Laplace transforms of the state probabilities are as follows:

$$P_1(s) = \frac{(\lambda_o + \lambda_s + s)(\mu + s)(\mu_c + s)}{A(3,2)} \tag{3.45}$$

$$P_2(s) = \frac{\lambda_b(\mu+s)(\mu_c+s)}{A(3,2)} \quad (3.46)$$

$$P_3(s) = \frac{\lambda_s(\mu+s)(\mu_c+s)}{A(3,2)} \quad (3.47)$$

$$P_4(s) = \frac{\lambda_{so}(\mu+s)(\mu_c+s)}{A(3,2)} \quad (3.48)$$

$$P_5(s) = \frac{\lambda_{ss}(\mu+s)(\mu_c+s)}{A(3,2)} \quad (3.49)$$

$$P_s(s) = \frac{\lambda_s(\lambda_b+\lambda_s+\lambda_{so}+\lambda_{ss})(\mu_c+s)}{A(3,2)} \quad (3.50)$$

$$P_o(s) = \frac{\lambda_b(\lambda_b+\lambda_s+\lambda_{so}+\lambda_{ss})(\mu_c+s)}{A(3,2)} \quad (3.51)$$

$$P_c(s) = \frac{\lambda_c(\lambda_b+\lambda_s+s)(\mu+s)}{A(3,2)} \quad (3.52)$$

The Laplace transforms of system availability is given by

$$AV(s) = \frac{(2\lambda_b+2\lambda_s+\lambda_{so}+\lambda_{ss}+s)(\mu+s)(\mu_c+s)}{A(3,2)} \quad (3.53)$$

where A(3,2) is defined in Appendix 3.1

The time dependent system availability and state probabilities can be obtained by taking inverse of the Laplace transforms of Equation (3.53) and Equations (3.45)-(3.52).

The same values of model parameters corresponding to the cold standby system with type I repair (one unit operating, the other one on standby) as well as the failure rate of the standby unit are used in the following figure. The time dependent system availabilities are plotted in Figure 3.7. Comparing with Figure 2.21, it can be observed that the system availability of the warm standby system at any time is lower than that of the cold standby system. The steady state availability given by Equation (3.44) is plotted

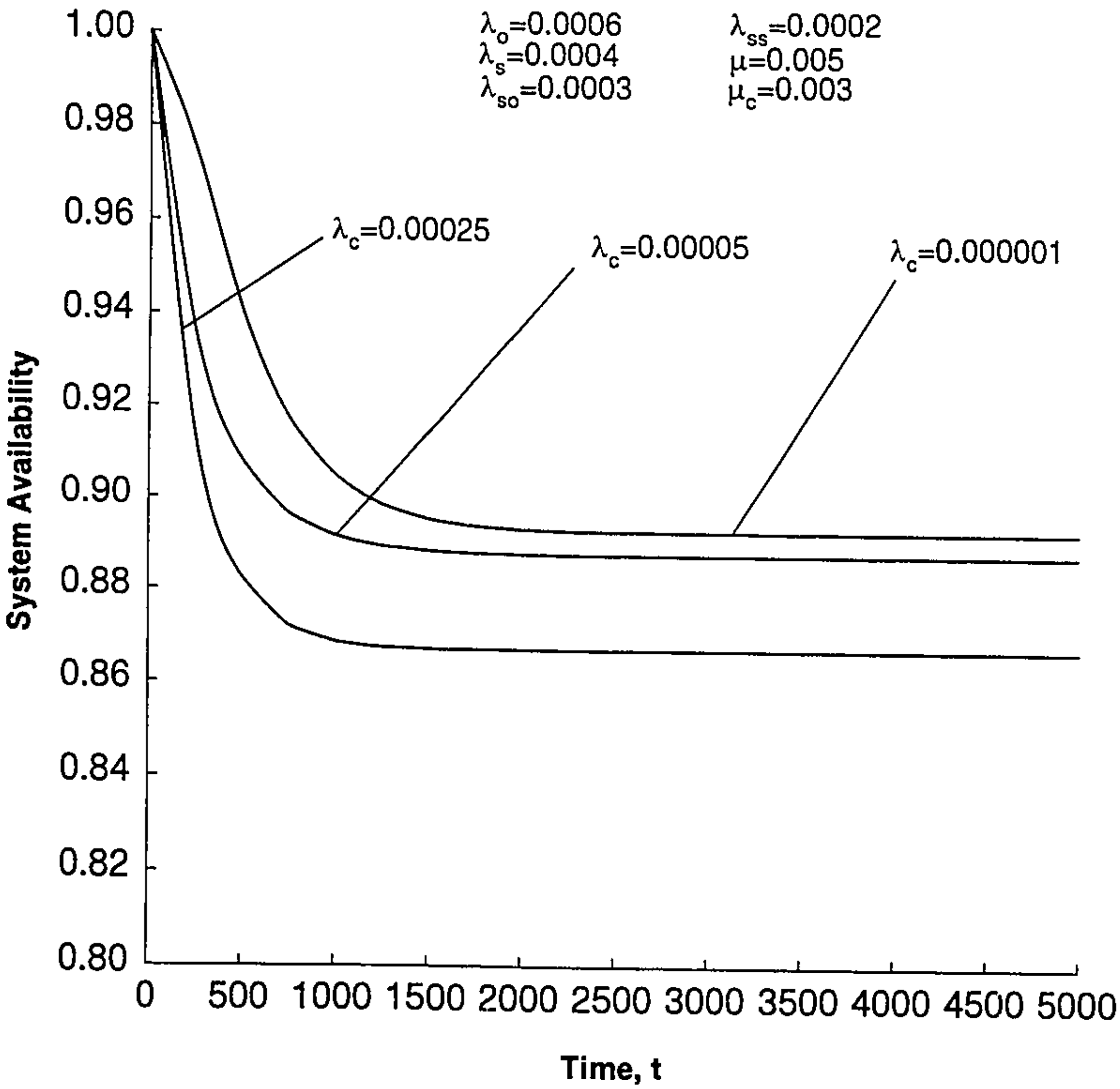


Figure 3.7 The warm standby system with type I repair availability plots: one unit operating, the other on warm standby

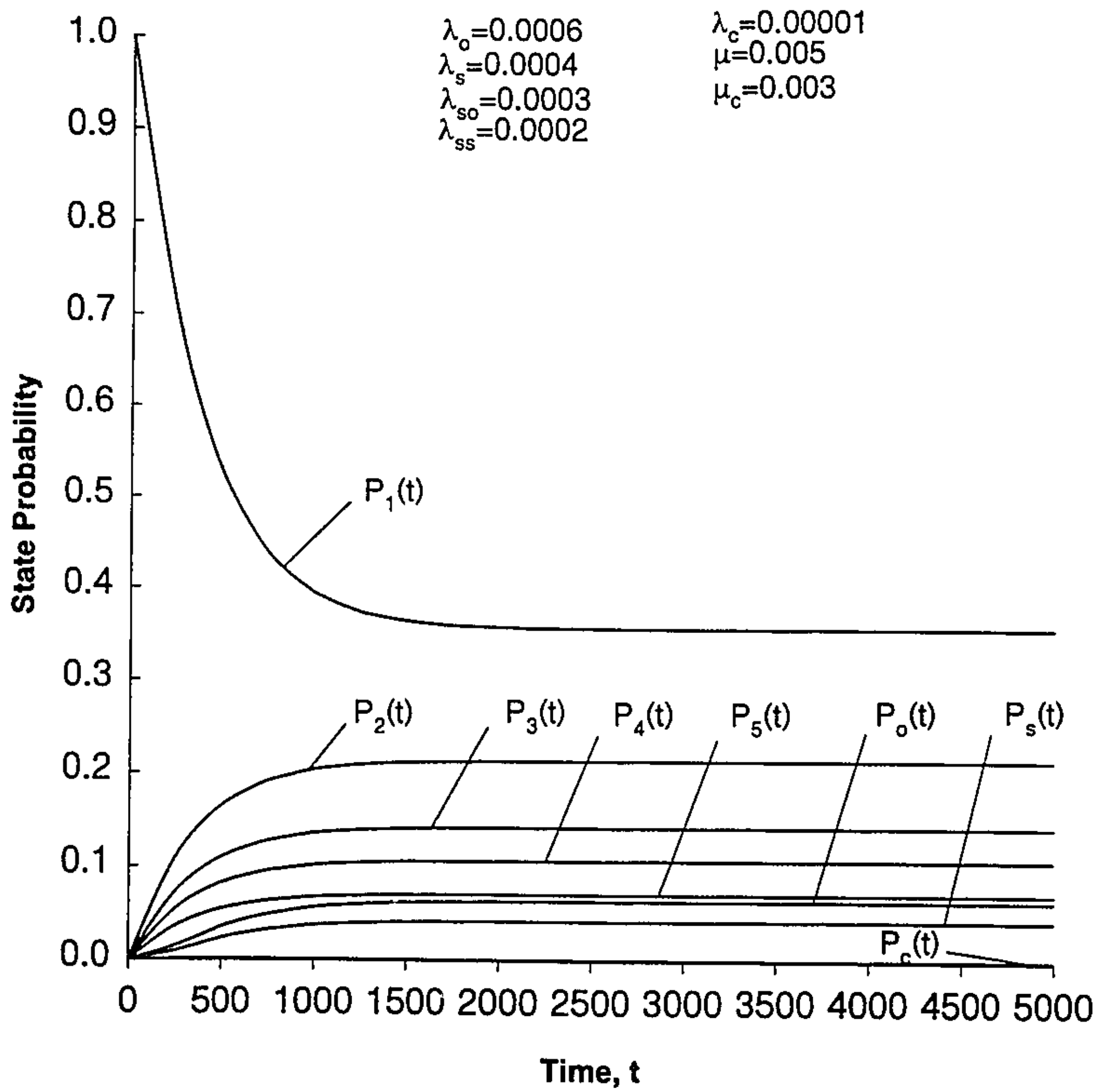


Figure 3.8 The warm standby system with type I repair state probability plots: one unit operating, the other on warm standby

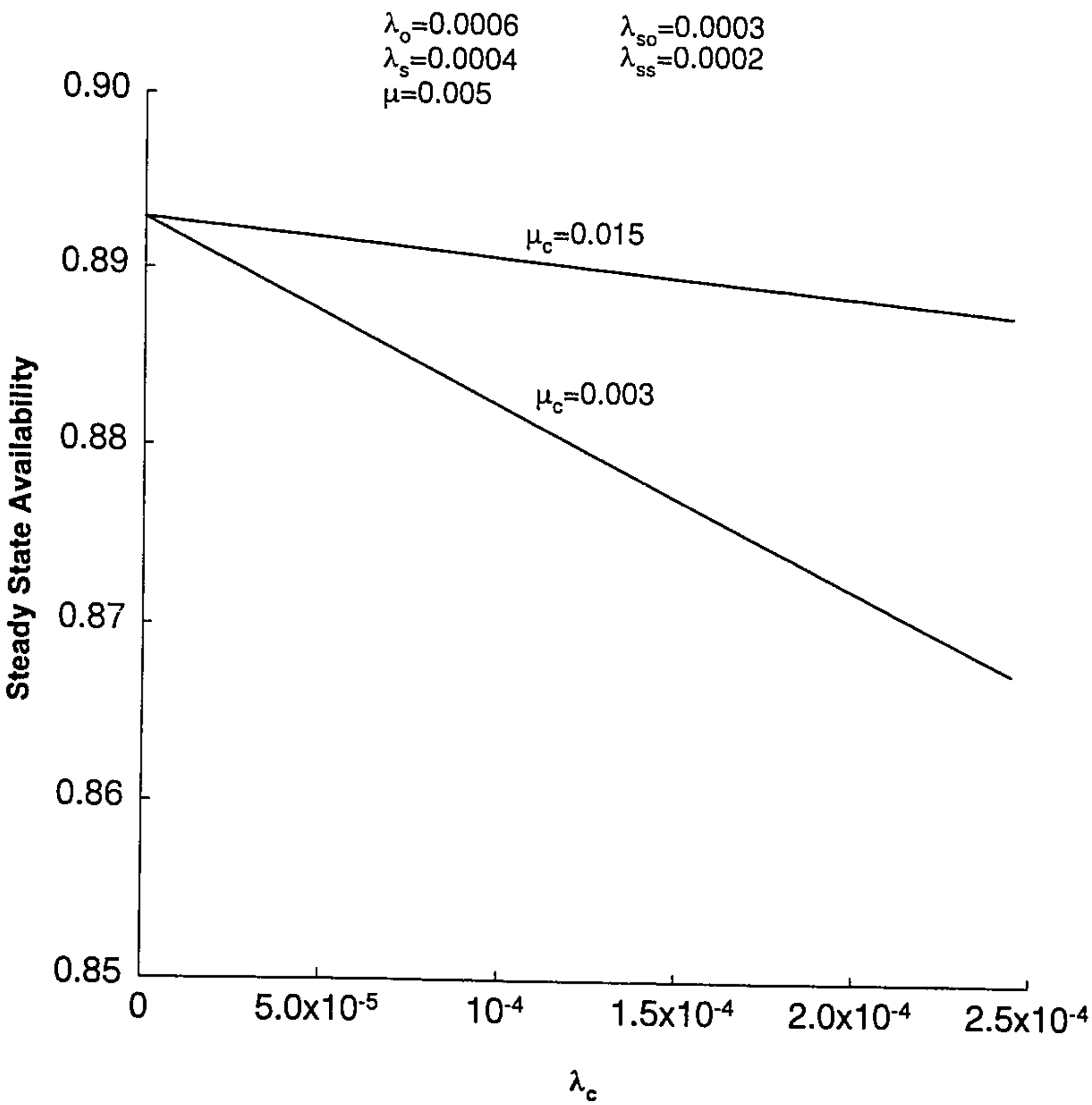


Figure 3.9 The warm standby system with type I repair steady state availability vs cause failure rate, λ_c plots (one unit operating, the other on warm standby)

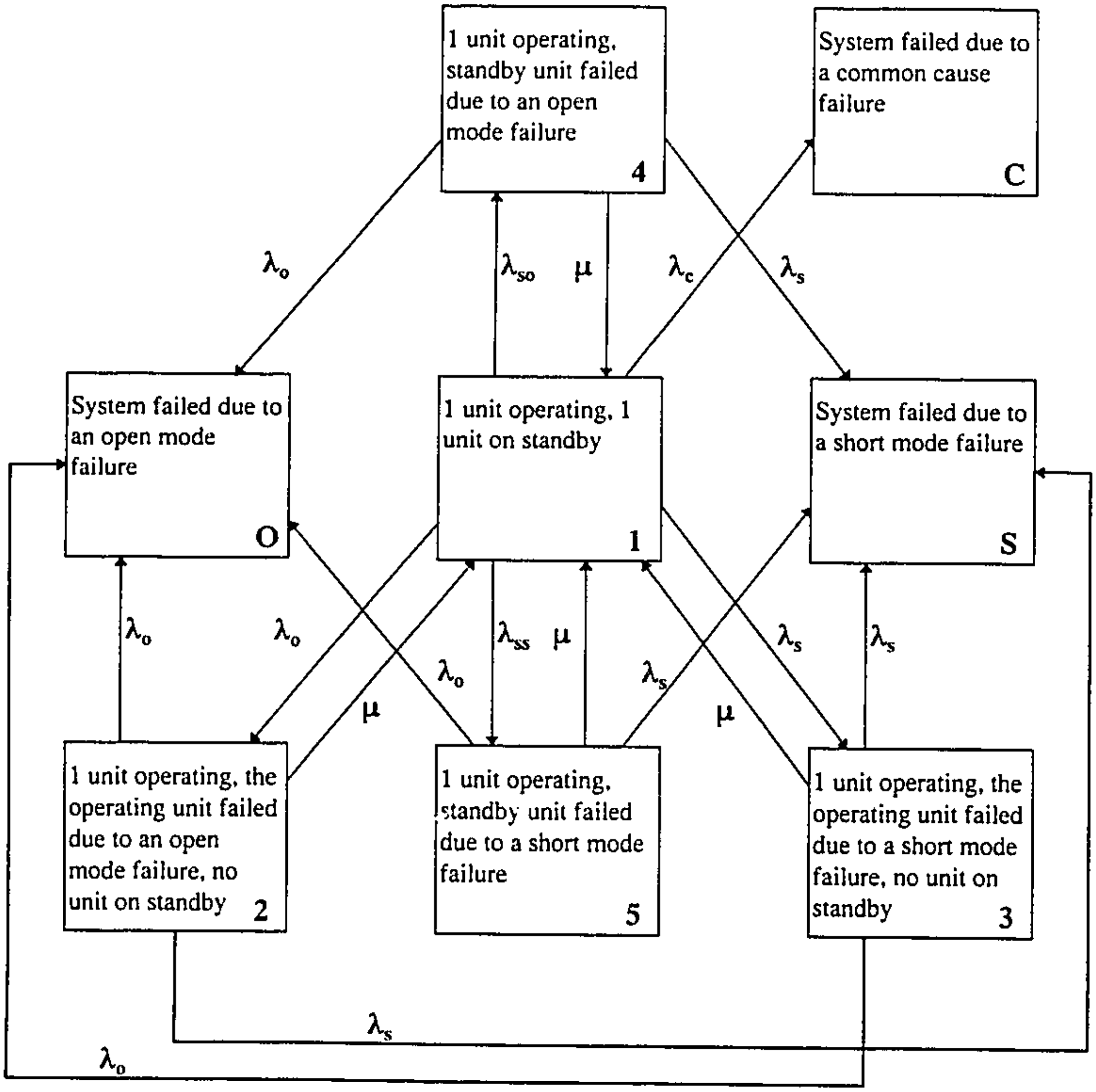


Figure 3.10 System state space diagram with Type II Repair:
one unit operating, the other on warm standby

in Figure 3.9. The same result is observed for both the time dependent availability and for the steady state availability. The time dependent state probabilities are presented in Figure 3.8.

3.1.3 Warm standby system with Type II Repair

The system state space diagram is shown in Figure 3.10. The following differential equations are associated with this system.

$$\frac{dP_1(t)}{dt} = -(\lambda_o + \lambda_{so} + \lambda_s + \lambda_{ss} + \lambda_c)P_1(t) + \mu[P_2(t) + P_3(t) + P_4(t) + P_5(t)] \quad (3.54)$$

$$\frac{dP_2(t)}{dt} = -(\lambda_o + \lambda_s + \lambda_c + \mu)P_2(t) + \lambda_o P_1(t) \quad (3.55)$$

$$\frac{dP_3(t)}{dt} = -(\lambda_o + \lambda_s + \lambda_c + \mu)P_3(t) + \lambda_s P_1(t) \quad (3.56)$$

$$\frac{dP_4(t)}{dt} = -(\lambda_o + \lambda_s + \lambda_c + \mu)P_4(t) + \lambda_{so} P_1(t) \quad (3.57)$$

$$\frac{dP_5(t)}{dt} = -(\lambda_o + \lambda_s + \lambda_c + \mu)P_5(t) + \lambda_{ss} P_1(t) \quad (3.58)$$

$$\frac{dP_s(t)}{dt} = \lambda_s[P_2(t) + P_3(t) + P_4(t) + P_5(t)] \quad (3.59)$$

$$\frac{dP_o(t)}{dt} = \lambda_o[P_2(t) + P_3(t) + P_4(t) + P_5(t)] \quad (3.60)$$

$$\frac{dP_c(t)}{dt} = \lambda_c P_1(t) \quad (3.61)$$

At time $t = 0$, $P_1(0) = 1$, and all the other initial condition probabilities are equal to zero.

By taking the Laplace transforms of Equations (3.54) - (3.61) and solving the resulting equations, the Laplace transforms of the state probabilities are as follows:

$$P_1(s) = \frac{(\lambda_o + \lambda_s + \mu + s)}{A(3,3)} \quad (3.62)$$

$$P_2(s) = \frac{\lambda_b}{A(3,3)} \tag{3.63}$$

$$P_3(s) = \frac{\lambda_s}{A(3,3)} \tag{3.64}$$

$$P_4(s) = \frac{\lambda_{so}}{A(3,3)} \tag{3.65}$$

$$P_5(s) = \frac{\lambda_{ss}}{A(3,3)} \tag{3.66}$$

$$P_s(s) = \frac{\lambda_s(\lambda_b + \lambda_s + \lambda_b + \lambda_s)}{sA(3,3)} \tag{3.67}$$

$$P_o(s) = \frac{\lambda_b(\lambda_b + \lambda_s + \lambda_b + \lambda_s)}{sA(3,3)} \tag{3.68}$$

$$P_c(s) = \frac{\lambda_c(\lambda_b + \lambda_s + \mu + s)}{sA(3,3)} \tag{3.69}$$

where A(3,3) is defined in Appendix 3.1

The Laplace transform of system reliability is given by

$$\begin{aligned} R(s) &= P_1(s) + P_2(s) + P_3(s) + P_4(s) + P_5(s) \\ &= \frac{(2\lambda_b + 2\lambda_s + \lambda_b + \lambda_s + \mu + s)}{A(3,3)} \end{aligned} \tag{3.70}$$

The system mean time to failure (MTTF) is

$$\begin{aligned} MTTF &= \lim_{s \to 0} R(s) \\ &= \frac{2\lambda_b + 2\lambda_s + \lambda_{so} + \lambda_{ss} + \mu}{(\lambda_b + \lambda_s)^2 + (\lambda_b + \lambda_s)(\lambda_c + \lambda_{so} + \lambda_{ss}) + \lambda_c \mu} \end{aligned} \tag{3.71}$$

The system reliability and time dependent state probabilities can be obtained by taking the inverse Laplace transforms of Equation (3.70) and Equations (3.62)-(3.69).

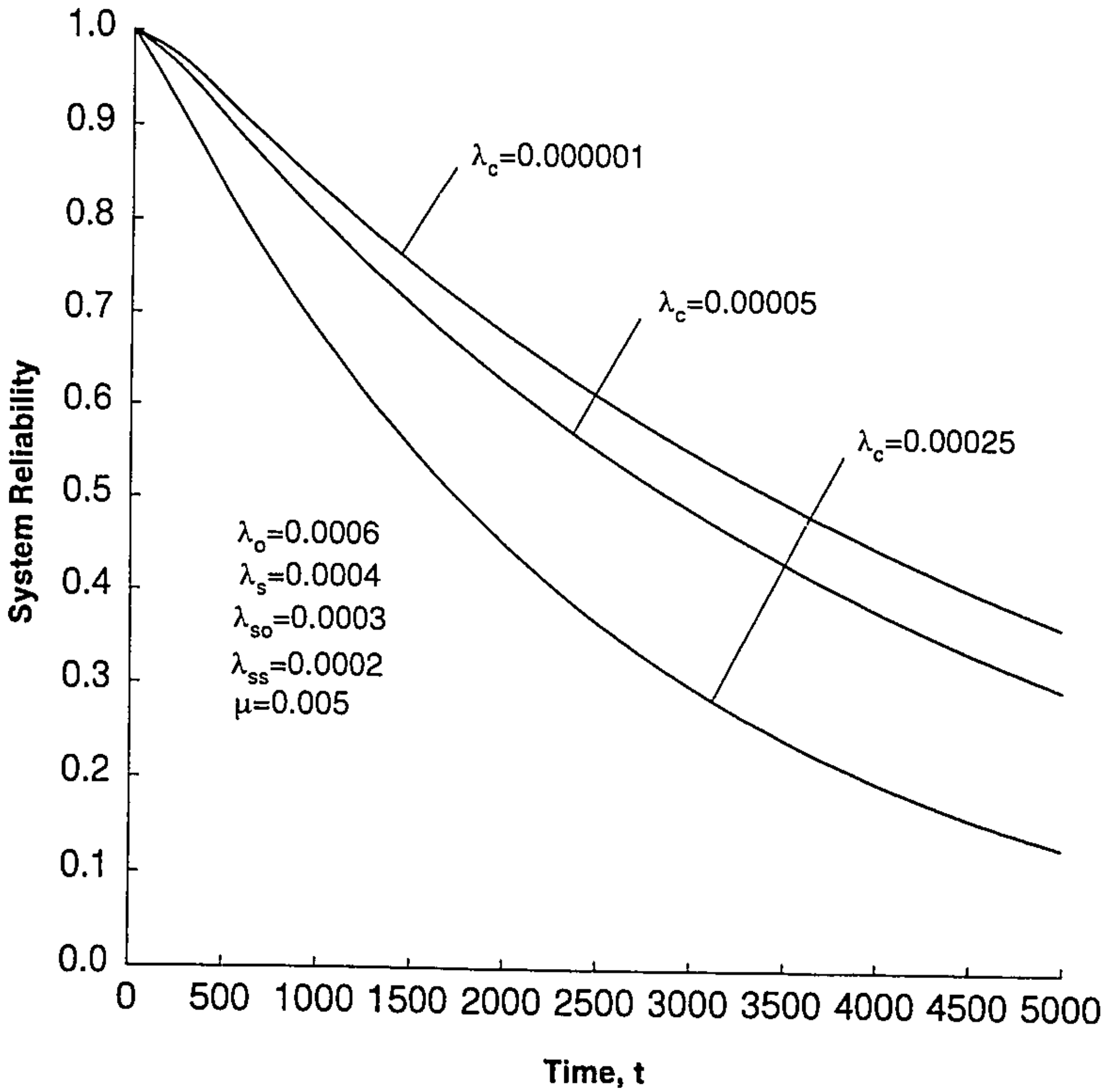


Figure 3.11 The warm standby system with type II repair reliability plots: one unit operating, the other on warm standby

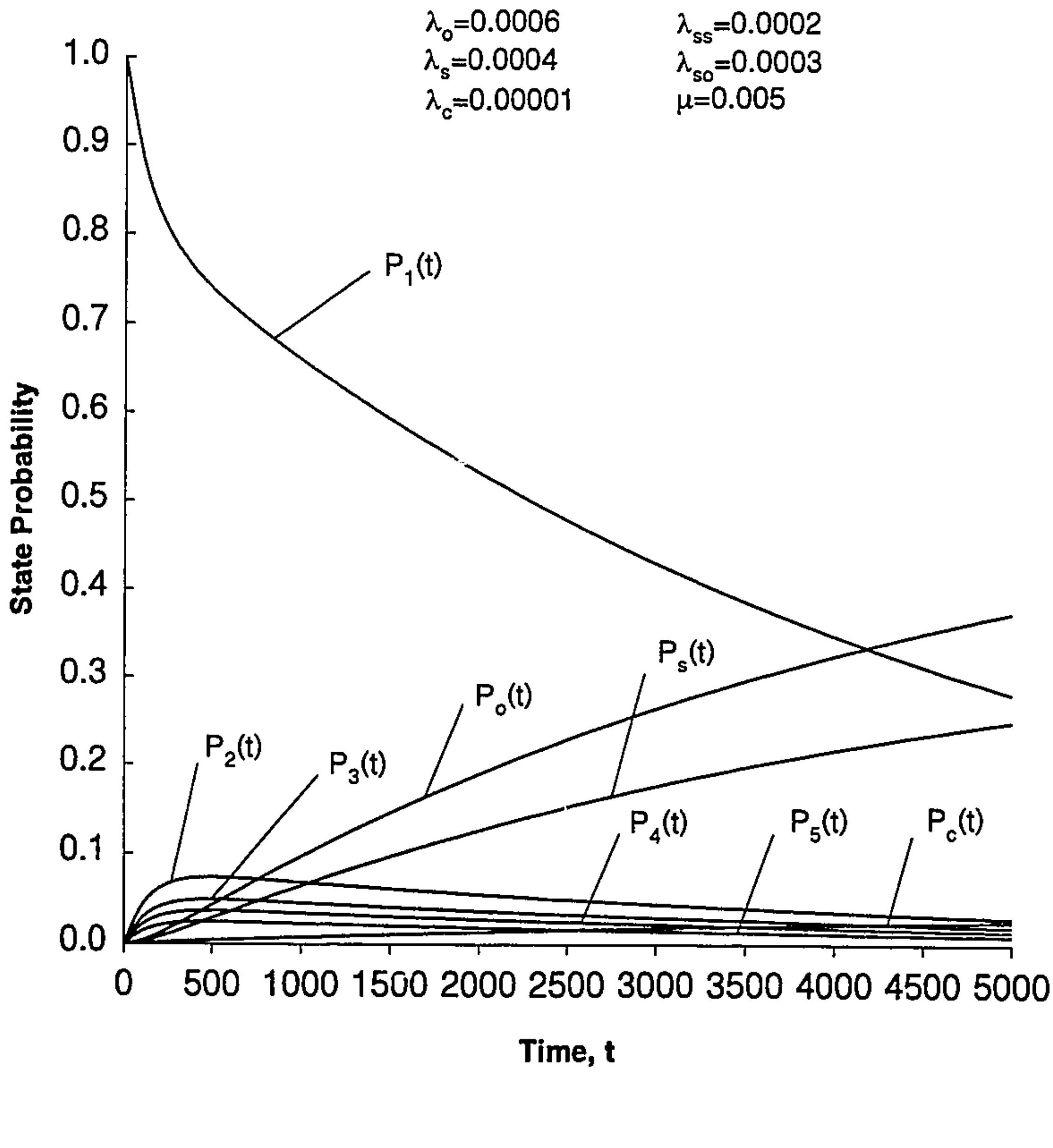


Figure 3.12 The warm standby system with type II repair state probability plots: one unit operating, the other on warm standby

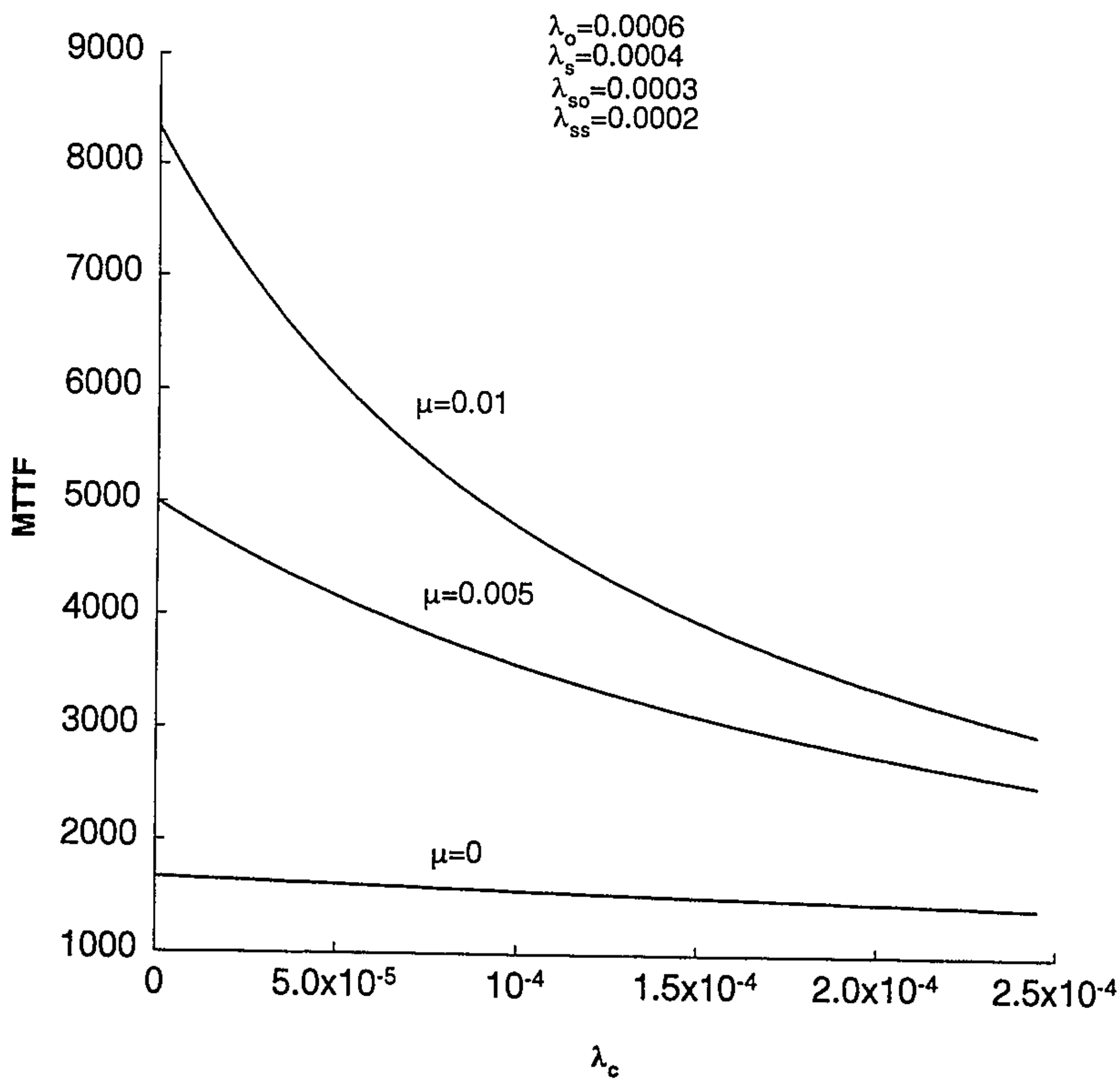


Figure 3.13 MTTF vs common cause failure rate, λ_c for warm standby system (one unit operating, the other on warm standby) with type II repair

The same values of model parameters corresponding to the cold standby system with type II repair (one unit operating, the other one on standby) as well as the failure rate of the standby unit are used in the following figures. The system reliabilities are plotted in Figure 3.11. Comparing with Figure 2.33, it is evident that the reliability of the warm standby system is lower than that of the cold standby system. Figure 3.12 is the time dependent state probability plots. The system mean time to failure (MTTF) plots given by Equation (3.71) are plotted in Figure 3.13. Comparing with Figure 2.35, it is noted that MTTF of the warm standby system is less than that of the cold standby system.

3.2 Warm standby system: one unit operating, two units on standby

By setting n=2 in Figure 2.1, the configuration consist of two units on warm standby mode can be obtained. In this system three cases are considerded: Without Repair, With Type I Repair and With Type II Repair (repair policies are the same as described previously).

Notation

The following symbols are associated with this model:

t time.

s Laplace transform variable.

λ_o Constant open-mode failure rate of an operating unit or device.

λ_s Constant short-mode failure rate of an operating unit or device.

λ_{so} Constant open-mode failure rate of a standby unit or device.

λ_{ss} Constant short-mode failure rate of a standby unit or device.

λ_c Constant common-cause failure rate of system or device.

μ Constant repair rate of a unit or the system.

μ_c Constant repair rate of the failed system from common-cause failure state to initial state 1.

$P_1(t)$ Probability that the system is in state 1.

$P_2(t)$ Probability of the system state in which one operating unit failed due to an open mode failure, one of the standby units starts working. The other one is still on warm standby.

$P_3(t)$ Probability of the system state in which one operating unit failed due to a short mode failure, one of the standby units starts working. The other one is still on warm standby.

$P_4(t)$ Probability of the system state in which one unit is operating and the one of standby units failed due to an open mode failure. The other one is still on warm standby.

$P_5(t)$ Probability of the system state in which one unit is operating and the one of standby units failed due to a short mode failure. The other one is still on warm standby.

$P_{21}(t)$ Probability of the system state in which two units failed due to an open mode failure. Only one unit is operating.

$P_{22}(t)$ Probability of the system state in which the first unit failed due to an open mode failure and the second one failed due to a short mode failure. Only one unit is operating.

$P_{23}(t)$ Probability of the system state in which the first operating unit failed due to an open mode failure and one of the standby units failed due to an open mode failure. Only one unit is operating.

$P_{24}(t)$ Probability of the system state in which the first operating unit failed due to an open mode failure and one of the standby units failed due to a short mode failure. Only one unit is operating.

$P_{31}(t)$ Probability of the system state in which the first operating unit failed due to a short mode failure and the second one failed due to an open mode failure. Only one is operating.

$P_{32}(t)$ Probability of system state in which two units failed due to a short mode failure. Only one unit is operating.

$P_{33}(t)$ Probability of system state in which the first operating unit failed due to a short mode failure and one of the standby units failed due to an open mode failure. Only one unit is operating.

$P_{34}(t)$ Probability of system state in which the first operating unit failed due to a short mode failure and one of the standby units failed due to a short mode failure. Only one is operating.

$P_{41}(t)$ Probability of the system state in which one of the standby units failed due to an open mode failure and the first operating unit failed due to an open mode failure. Only one unit is operating.

$P_{42}(t)$ Probability of the system state in which one of the standby units failed due to an open mode failure and the first operating unit failed due to a short mode failure. Only one unit is operating.

$P_{43}(t)$ Probability of the system state in which two standby units failed due to an open mode failure. Only one unit is operating.

$P_{44}(t)$ Probability of the system state in which one standby unit failed due to an open mode failure and the other standby unit failed due to a short mode failure. Only one unit is operating.

$P_{51}(t)$ Probability of the system state in which one standby unit failed due to a short mode failure and the first operating unit failed due to an open mode failure. Only one unit is operating.

$P_{52}(t)$ Probability of the system state in which one standby unit failed due to a short mode failure and the first operating unit failed due to a short mode failure. Only one unit is operating.

$P_{53}(t)$ Probability of the system state in which one standby unit failed due to a short mode failure and the other standby unit failed due to an open mode failure. Only one unit is operating.

$P_{54}(t)$ Probability of the system state in which one standby unit failed due to a short mode failure and the other standby unit failed due to a short mode failure. Only one unit is operating.

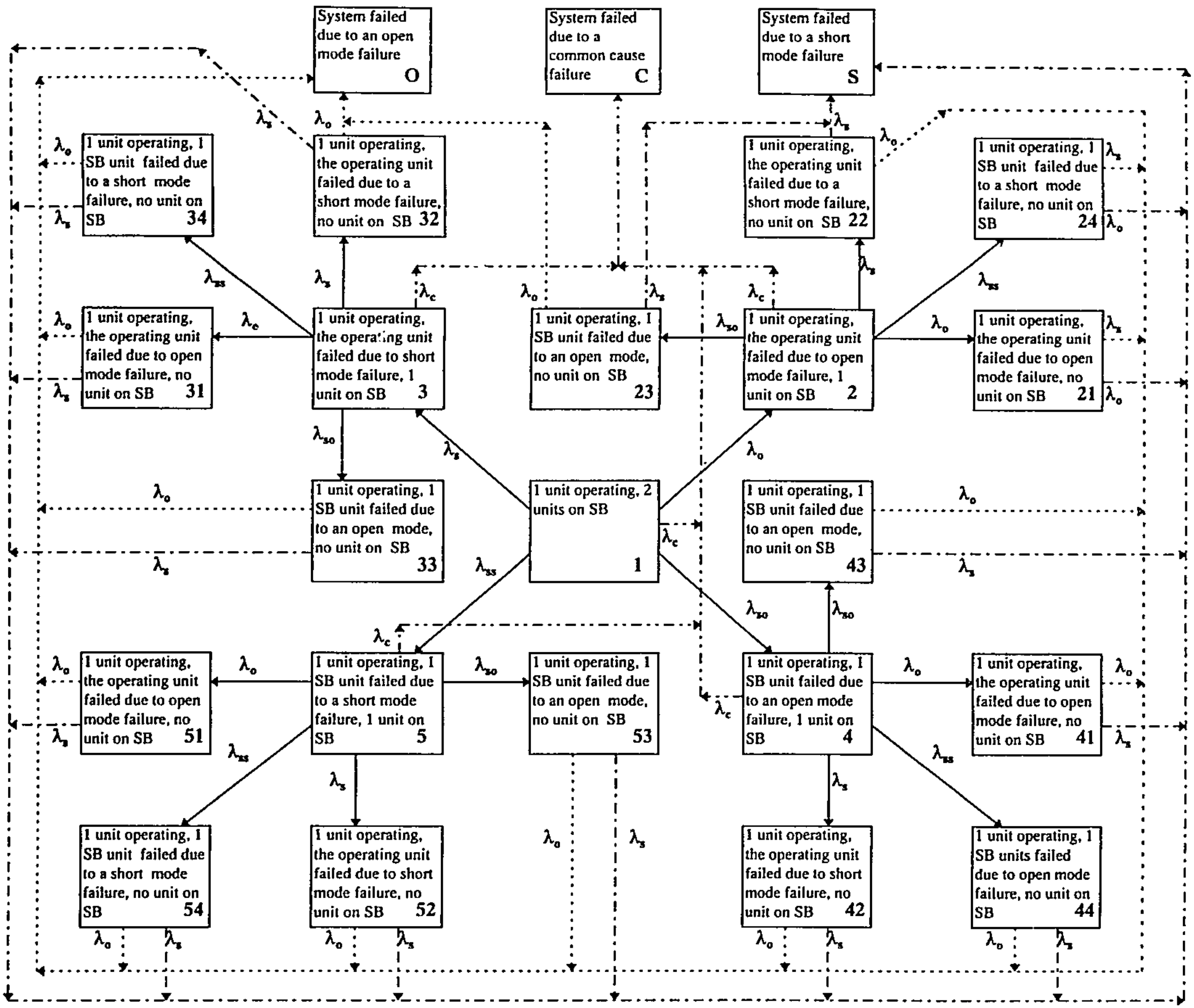
$P_s(t)$ Probability of state in which the system failed due to a short mode failure.

$P_o(t)$ Probability of state in which the system failed due to an open mode failure.

$P_c(t)$ Probability of state in which the system failed due to a common-cause failure.

$R(t)$ Reliability of the system in [0,t].

Figure 3.14 System state space diagram without Repair: one unit operating, two units on warm standby



SB=Standby

MTTF System mean time to failure.

AV_{ss} Steady state system availability.

$AV(t)$ Time dependent system availability.

3.2.1 Warm standby system without Repair

The system state space diagram is shown in Figure 3.14. The following differential equations are associated with this system.

$$\frac{dP_1(t)}{dt} = -(\lambda_b + \lambda_s + \lambda_{so} + \lambda_{ss} + \lambda_c)P_1(t) \quad (3.72)$$

$$\frac{dP_2(t)}{dt} = -(\lambda_b + \lambda_s + \lambda_{so} + \lambda_{ss} + \lambda_c)P_2(t) + \lambda_b P_1(t) \quad (3.73)$$

$$\frac{dP_3(t)}{dt} = -(\lambda_b + \lambda_s + \lambda_{so} + \lambda_{ss} + \lambda_c)P_3(t) + \lambda_s P_1(t) \quad (3.74)$$

$$\frac{dP_4(t)}{dt} = -(\lambda_b + \lambda_s + \lambda_{so} + \lambda_{ss} + \lambda_c)P_4(t) + \lambda_{so} P_1(t) \quad (3.75)$$

$$\frac{dP_5(t)}{dt} = -(\lambda_b + \lambda_s + \lambda_{so} + \lambda_{ss} + \lambda_c)P_5(t) + \lambda_{ss} P_1(t) \quad (3.76)$$

$$\frac{dP_{21}(t)}{dt} = -(\lambda_b + \lambda_s)P_{21}(t) + \lambda_b P_2(t) \quad (3.77)$$

$$\frac{dP_{22}(t)}{dt} = -(\lambda_b + \lambda_s)P_{22}(t) + \lambda_s P_2(t) \quad (3.78)$$

$$\frac{dP_{23}(t)}{dt} = -(\lambda_b + \lambda_s)P_{23}(t) + \lambda_{so} P_2(t) \quad (3.79)$$

$$\frac{dP_{24}(t)}{dt} = -(\lambda_b + \lambda_s)P_{24}(t) + \lambda_{ss} P_2(t) \quad (3.80)$$

$$\frac{dP_{31}(t)}{dt} = -(\lambda_b + \lambda_s)P_{31}(t) + \lambda_b P_3(t) \quad (3.81)$$

$$\frac{dP_{32}(t)}{dt} = -(\lambda_b + \lambda_s)P_{32}(t) + \lambda_s P_3(t) \quad (3.82)$$

$$\frac{dP_{33}(t)}{dt} = -(\lambda_b + \lambda_s)P_{33}(t) + \lambda_{so} P_3(t) \quad (3.83)$$

$$\frac{dP_{34}(t)}{dt} = -(\lambda_b + \lambda_s)P_{34}(t) + \lambda_{ss}P_3(t) \tag{3.84}$$

$$\frac{dP_{41}(t)}{dt} = -(\lambda_b + \lambda_s)P_{41}(t) + \lambda_o P_4(t) \tag{3.85}$$

$$\frac{dP_{42}(t)}{dt} = -(\lambda_b + \lambda_s)P_{42}(t) + \lambda_s P_4(t) \tag{3.86}$$

$$\frac{dP_{43}(t)}{dt} = -(\lambda_b + \lambda_s)P_{43}(t) + \lambda_{so}P_4(t) \tag{3.87}$$

$$\frac{dP_{44}(t)}{dt} = -(\lambda_b + \lambda_s)P_{44}(t) + \lambda_{ss}P_4(t) \tag{3.88}$$

$$\frac{dP_{51}(t)}{dt} = -(\lambda_b + \lambda_s)P_{51}(t) + \lambda_o P_5(t) \tag{3.89}$$

$$\frac{dP_{52}(t)}{dt} = -(\lambda_b + \lambda_s)P_{52}(t) + \lambda_s P_5(t) \tag{3.90}$$

$$\frac{dP_{53}(t)}{dt} = -(\lambda_b + \lambda_s)P_{53}(t) + \lambda_{so}P_5(t) \tag{3.91 }$$

$$\frac{dP_{54}(t)}{dt} = -(\lambda_b + \lambda_s)P_{54}(t) + \lambda_{ss}P_5(t) \tag{3.92}$$

$$\frac{dP_s(t)}{dt} = \lambda_s[P_{21}(t) + P_{22}(t) + P_{23}(t) + P_{24}(t) + P_{31}(t) + P_{32}(t) + P_{33}(t) + P_{34}(t) + P_{41}(t) + P_{42}(t) + P_{43}(t) + P_{44}(t) + P_{51}(t) + P_{52}(t) + P_{53}(t) + P_{54}(t)] \tag{3.93}$$

$$\frac{dP_o(t)}{dt} = \lambda_b[P_{21}(t) + P_{22}(t) + P_{23}(t) + P_{24}(t) + P_{31}(t) + P_{32}(t) + P_{33}(t) + P_{34}(t) + P_{41}(t) + P_{42}(t) + P_{43}(t) + P_{44}(t) + P_{51}(t) + P_{52}(t) + P_{53}(t) + P_{54}(t)] \tag{3.94}$$

$$\frac{dP_c(t)}{dt} = \lambda_c[P_1(t) + P_2(t) + P_3(t) + P_4(t) + P_5(t)] \tag{3.95}$$

At time $t = 0$, $P_1(0) = 1$, and all the other initial condition probabilities are equal to zero.

By taking the Laplace transforms of Equations (3.72) - (3.95) and solving the resulting equations, and then taking inverse Laplace transforms, the state probabilities can be obtained.

$$P_1(t) = \frac{1}{e^{(\lambda_b + \lambda_s + \lambda_{so} + \lambda_{ss} + \lambda_c)t}} \tag{3.96}$$

$$P_2(t) = \frac{\lambda_b t}{e^{(\lambda_b+\lambda_s+\lambda_{so}+\lambda_{ss}+\lambda_c)t}} \tag{3.97}$$

$$P_3(t) = \frac{\lambda_s t}{e^{(\lambda_b+\lambda_s+\lambda_{so}+\lambda_{ss}+\lambda_c)t}} \tag{3.98}$$

$$P_4(t) = \frac{\lambda_{so} t}{e^{(\lambda_b+\lambda_s+\lambda_{so}+\lambda_{ss}+\lambda_c)t}} \tag{3.99}$$

$$P_5(t) = \frac{\lambda_{ss} t}{e^{(\lambda_b+\lambda_s+\lambda_{so}+\lambda_{ss}+\lambda_c)t}} \tag{3.100}$$

$$P_{21}(t) = \frac{\lambda_b^2}{e^{(\lambda_b+\lambda_s+\lambda_{so}+\lambda_{ss}+\lambda_c)t}(\lambda_c+\lambda_{so}+\lambda_{ss})^2}[e^{(\lambda_{so}+\lambda_{ss}+\lambda_c)t} - (\lambda_c+\lambda_{so}+\lambda_{ss})t - 1] \tag{3.101}$$

$$P_{22}(t) = P_{31}(t)$$

$$= \frac{\lambda_b \lambda_s}{e^{(\lambda_b+\lambda_s+\lambda_{so}+\lambda_{ss}+\lambda_c)t}(\lambda_c+\lambda_{so}+\lambda_{ss})^2}[e^{(\lambda_{so}+\lambda_{ss}+\lambda_c)t} - (\lambda_c+\lambda_{so}+\lambda_{ss})t - 1] \tag{3.102}$$

$$P_{23}(t) = P_{41}(t)$$

$$= \frac{\lambda_b \lambda_{so}}{e^{(\lambda_b+\lambda_s+\lambda_{so}+\lambda_{ss}+\lambda_c)t}(\lambda_c+\lambda_{so}+\lambda_{ss})^2}[e^{(\lambda_{so}+\lambda_{ss}+\lambda_c)t} - (\lambda_c+\lambda_{so}+\lambda_{ss})t - 1] \tag{3.103}$$

$$P_{24}(t) = P_{51}(t)$$

$$= \frac{\lambda_b \lambda_{ss}}{e^{(\lambda_b+\lambda_s+\lambda_{so}+\lambda_{ss}+\lambda_c)t}(\lambda_c+\lambda_{so}+\lambda_{ss})^2}[e^{(\lambda_{so}+\lambda_{ss}+\lambda_c)t} - (\lambda_c+\lambda_{so}+\lambda_{ss})t - 1] \tag{3.104}$$

$$P_{32}(t) = \frac{\lambda_s^2}{e^{(\lambda_b+\lambda_s+\lambda_{so}+\lambda_{ss}+\lambda_c)t}(\lambda_c+\lambda_{so}+\lambda_{ss})^2}[e^{(\lambda_{so}+\lambda_{ss}+\lambda_c)t} - (\lambda_c+\lambda_{so}+\lambda_{ss})t - 1] \tag{3.105}$$

$$P_{33}(t) = P_{42}(t)$$

$$= \frac{\lambda_s \lambda_{so}}{e^{(\lambda_b+\lambda_s+\lambda_{so}+\lambda_{ss}+\lambda_c)t}(\lambda_c+\lambda_{so}+\lambda_{ss})^2}[e^{(\lambda_{so}+\lambda_{ss}+\lambda_c)t} - (\lambda_c+\lambda_{so}+\lambda_{ss})t - 1] \tag{3.106}$$

$$P_{34}(t) = P_{52}(t)$$

$$= \frac{\lambda_s \lambda_{ss}}{e^{(\lambda_b+\lambda_s+\lambda_{so}+\lambda_{ss}+\lambda_c)t}(\lambda_c+\lambda_{so}+\lambda_{ss})^2}[e^{(\lambda_{so}+\lambda_{ss}+\lambda_c)t} - (\lambda_c+\lambda_{so}+\lambda_{ss})t - 1] \tag{3.107}$$

$$P_{43}(t) = \frac{\lambda_{so}^2}{e^{(\lambda_b+\lambda_s+\lambda_{so}+\lambda_{ss}+\lambda_c)t}(\lambda_c+\lambda_{so}+\lambda_{ss})^2}[e^{(\lambda_{so}+\lambda_{ss}+\lambda_c)t} - (\lambda_c+\lambda_{so}+\lambda_{ss})t - 1] \tag{3.108}$$

$$P_{44}(t) = P_{53}(t)$$

$$= \frac{\lambda_{so} \lambda_{ss}}{e^{(\lambda_b+\lambda_s+\lambda_{so}+\lambda_{ss}+\lambda_c)t}(\lambda_c+\lambda_{so}+\lambda_{ss})^2}[e^{(\lambda_{so}+\lambda_{ss}+\lambda_c)t} - (\lambda_c+\lambda_{so}+\lambda_{ss})t - 1] \tag{3.109}$$

$$P_{54}(t) = \frac{\lambda_{ss}^2}{e^{(\lambda_b+\lambda_s+\lambda_{so}+\lambda_{ss}+\lambda_c)t}(\lambda_c+\lambda_{so}+\lambda_{ss})^2}[e^{(\lambda_{so}+\lambda_{ss}+\lambda_c)t} - (\lambda_c+\lambda_{so}+\lambda_{ss})t - 1] \tag{3.110}$$

$$P_s(t) = \frac{\lambda_s(\lambda_b+\lambda_s+\lambda_{so}+\lambda_{ss})^2}{(\lambda_{so}+\lambda_{ss})}[\frac{1}{(\lambda_b+\lambda_s+\lambda_{so}+\lambda_{ss}+\lambda_c)^2} - \frac{1}{e^{(\lambda_{so}+\lambda_s)t}(\lambda_c+\lambda_{so}+\lambda_{ss})^2}] +$$

$$\frac{\lambda_s(\lambda_b+\lambda_s+\lambda_{so}+\lambda_{ss})^2(\lambda_b+\lambda_s+2\lambda_c+2\lambda_{so}+2\lambda_{ss})}{e^{(\lambda_b+\lambda_s+\lambda_{so}+\lambda_{ss}+\lambda_c)t}(\lambda_c+\lambda_{so}+\lambda_{ss})^2(\lambda_b+\lambda_s+\lambda_c+\lambda_{so}+\lambda_{ss})^2} +$$

$$\frac{\lambda_s(\lambda_b+\lambda_s+\lambda_{so}+\lambda_{ss})^2 t}{e^{(\lambda_b+\lambda_s+\lambda_{so}+\lambda_{ss}+\lambda_c)t}(\lambda_c+\lambda_{so}+\lambda_{ss})(\lambda_b+\lambda_s+\lambda_c+\lambda_{so}+\lambda_{ss})} \tag{3.111}$$

$$P_o(t) = \frac{\lambda_b(\lambda_b+\lambda_s+\lambda_{so}+\lambda_{ss})^2}{(\lambda_{so}+\lambda_{ss})}[\frac{1}{(\lambda_b+\lambda_s+\lambda_{so}+\lambda_{ss}+\lambda_c)^2} - \frac{1}{e^{(\lambda_{so}+\lambda_s)t}(\lambda_c+\lambda_{so}+\lambda_{ss})^2}] +$$

$$\frac{\lambda_b(\lambda_b+\lambda_s+\lambda_{so}+\lambda_{ss})^2(\lambda_o+\lambda_s+2\lambda_c+2\lambda_{so}+2\lambda_{ss})}{e^{(\lambda_b+\lambda_s+\lambda_{so}+\lambda_{ss}+\lambda_c)t}(\lambda_c+\lambda_{so}+\lambda_{ss})^2(\lambda_o+\lambda_s+\lambda_c+\lambda_{so}+\lambda_{ss})^2} +$$

$$\frac{\lambda_b(\lambda_b+\lambda_s+\lambda_{so}+\lambda_{ss})^2 t}{e^{(\lambda_b+\lambda_s+\lambda_{so}+\lambda_{ss}+\lambda_c)t}(\lambda_c+\lambda_{so}+\lambda_{ss})(\lambda_b+\lambda_s+\lambda_c+\lambda_{so}+\lambda_{ss})} \tag{3.112}$$

$$P_o(t) = \frac{\lambda_c(2\lambda_o+2\lambda_s+\lambda_c+2\lambda_{so}+2\lambda_{ss})}{(\lambda_o+\lambda_s+\lambda_{so}+\lambda_{ss}+\lambda_c)^2}[1 - \frac{1}{e^{(\lambda_b+\lambda_s+\lambda_{so}+\lambda_{ss}+\lambda_c)t}}] -$$

$$\frac{\lambda_c(\lambda_o+\lambda_s+\lambda_{so}+\lambda_{ss})t}{e^{(\lambda_b+\lambda_s+\lambda_{so}+\lambda_{ss}+\lambda_c)t}(\lambda_o+\lambda_s+\lambda_{so}+\lambda_{ss}+\lambda_c)} \tag{3.113}$$

The system reliability is given by

$$\begin{aligned} R(t) &= \sum P(t) \\ &= P_1(t)+P_2(t)+P_3(t)+P_4(t)+P_5(t)+P_{21}(t)+P_{22}(t)+P_{23}(t)+P_{24}(t)+P_{31}(t)+ \\ &\quad P_{32}(t)+P_{33}(t)+P_{34}(t)+P_{41}(t)+P_{42}(t)+P_{43}(t)+P_{44}(t)+P_{51}(t)+P_{52}(t)+P_{53}(t)+ \\ &\quad P_{54}(t) \\ &= \frac{(\lambda_b+\lambda_s+\lambda_{so}+\lambda_{ss})^2}{e^{(\lambda_b+\lambda_s)t}(\lambda_c+\lambda_{so}+\lambda_{ss})^2} + \frac{(-\lambda_b-\lambda_s+\lambda_c)(\lambda_o+\lambda_s+\lambda_c+2\lambda_{so}+2\lambda_{ss})}{e^{(\lambda_b+\lambda_s+\lambda_{so}+\lambda_{ss}+\lambda_c)t}(\lambda_c+\lambda_{so}+\lambda_{ss})^2} + \\ &\quad \frac{(-\lambda_b-\lambda_s+\lambda_c)(\lambda_b+\lambda_s+\lambda_{so}+\lambda_{ss})t}{e^{(\lambda_b+\lambda_s+\lambda_{so}+\lambda_{ss}+\lambda_c)t}(\lambda_c+\lambda_{so}+\lambda_{ss})} \end{aligned} \tag{3.114}$$

The system mean time to failure (MTTF) is

$$
\begin{aligned}
MTTF &= \lim_{s\to 0} R(s) \\
&= \lim_{s\to 0}[P_1(s) + P_2(s) + P_3(s) + P_4(s) + P_5(s) + P_{21}(t) + P_{22}(s) + P_{23}(s) + P_{24}(s) + P_{31}(s) + \\
&\quad P_{32}(s) + P_{33}(s) + P_{34}(s) + P_{41}(s) + P_{42}(s) + P_{43}(s) + P_{44}(s) + P_{51}(s) + P_{52}(s) + P_{53}(s) + \\
&\quad P_{54}(s)] \\
&= \frac{3(\lambda_b + \lambda_s)^2 + (\lambda_b + \lambda_s)(\lambda_c + 4\lambda_{so} + 4\lambda_{ss}) + (\lambda_{so} + \lambda_{ss})^2}{(\lambda_b + \lambda_s)(\lambda_b + \lambda_s + \lambda_{so} + \lambda_{ss} + \lambda_c)^2}
\end{aligned}
\tag{3.115}
$$

The same values of model parameters corresponding to the cold standby system without repair (one unit operating, two units on standby) as well as the failure rate of the standby unit are used in the following plots. The system reliability given by Equation (3.114) is shown in Figure 3.15. It can be observed that the reliability of warm standby system is lower than that of cold standby system. The plots of Equations (3.96)-(3.113) are shown in Figure 3.16. It is noted that the failure states $P_s(t)$ and $P_o(t)$ in Figure 3.16 are higher than $P_s(t)$ and $P_o(t)$ in Figure 2.12 (cold standby system). It represents that failure states of the warm standby system contribute more probability than those of the cold standby system. Figure 3.4 and Figure 3.5 present Equation (3.115) plots.

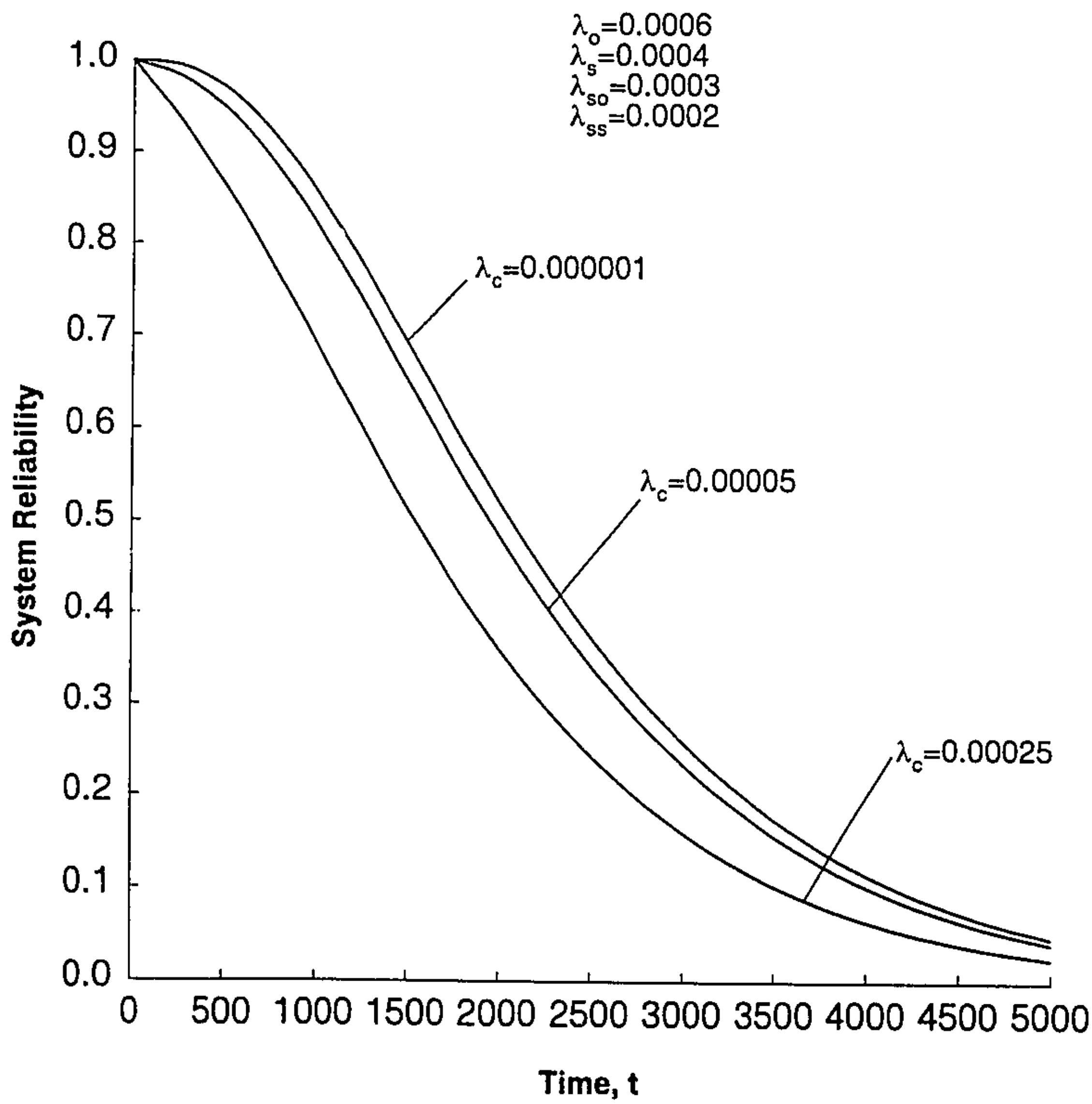


Figure 3.15 The warm standby system without repair reliability plots: one unit operating, two units on warm standby

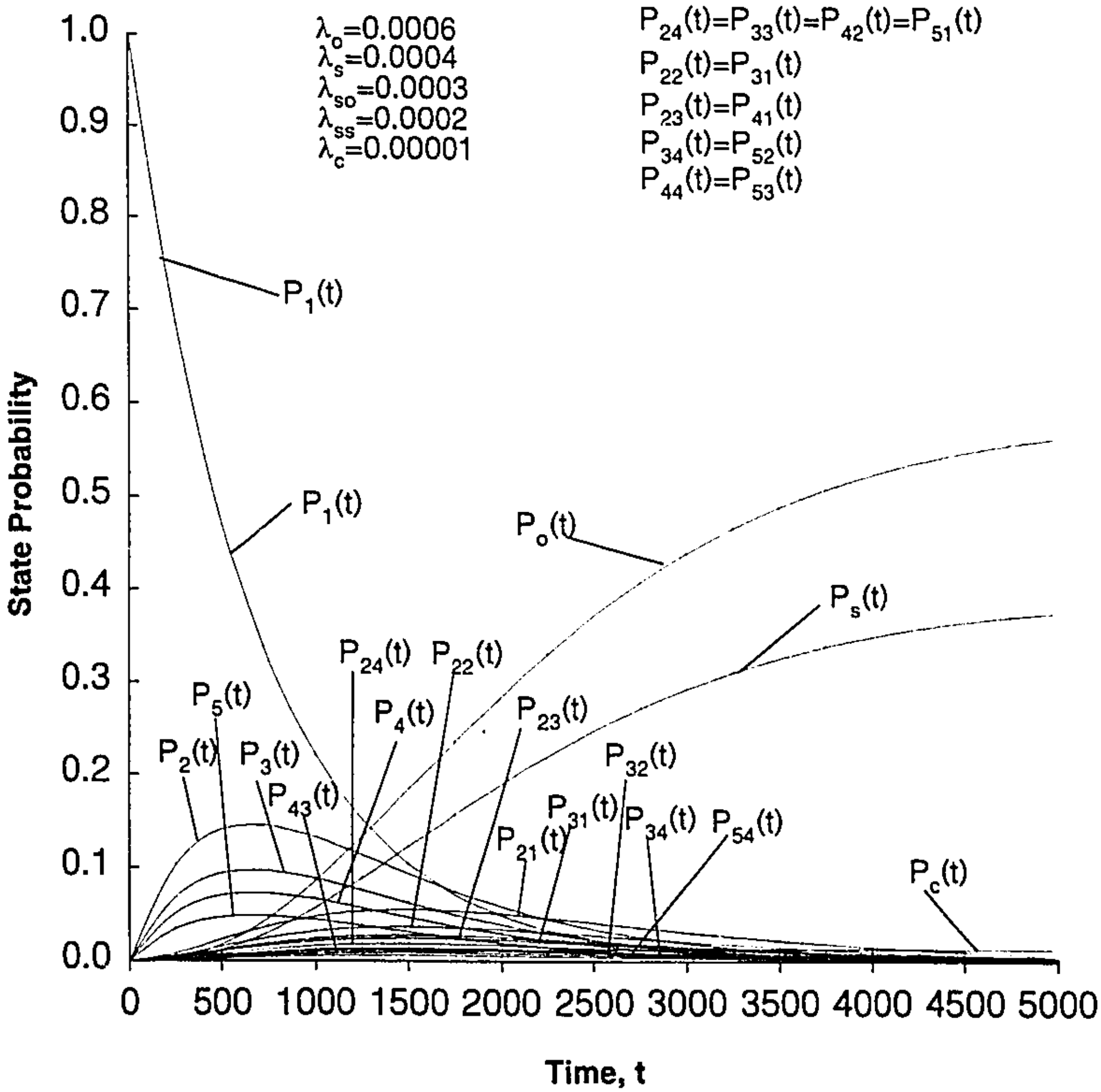


Figure 3.16 The warm standby system without repair state probability plots: one unit operating, two units on warm standby

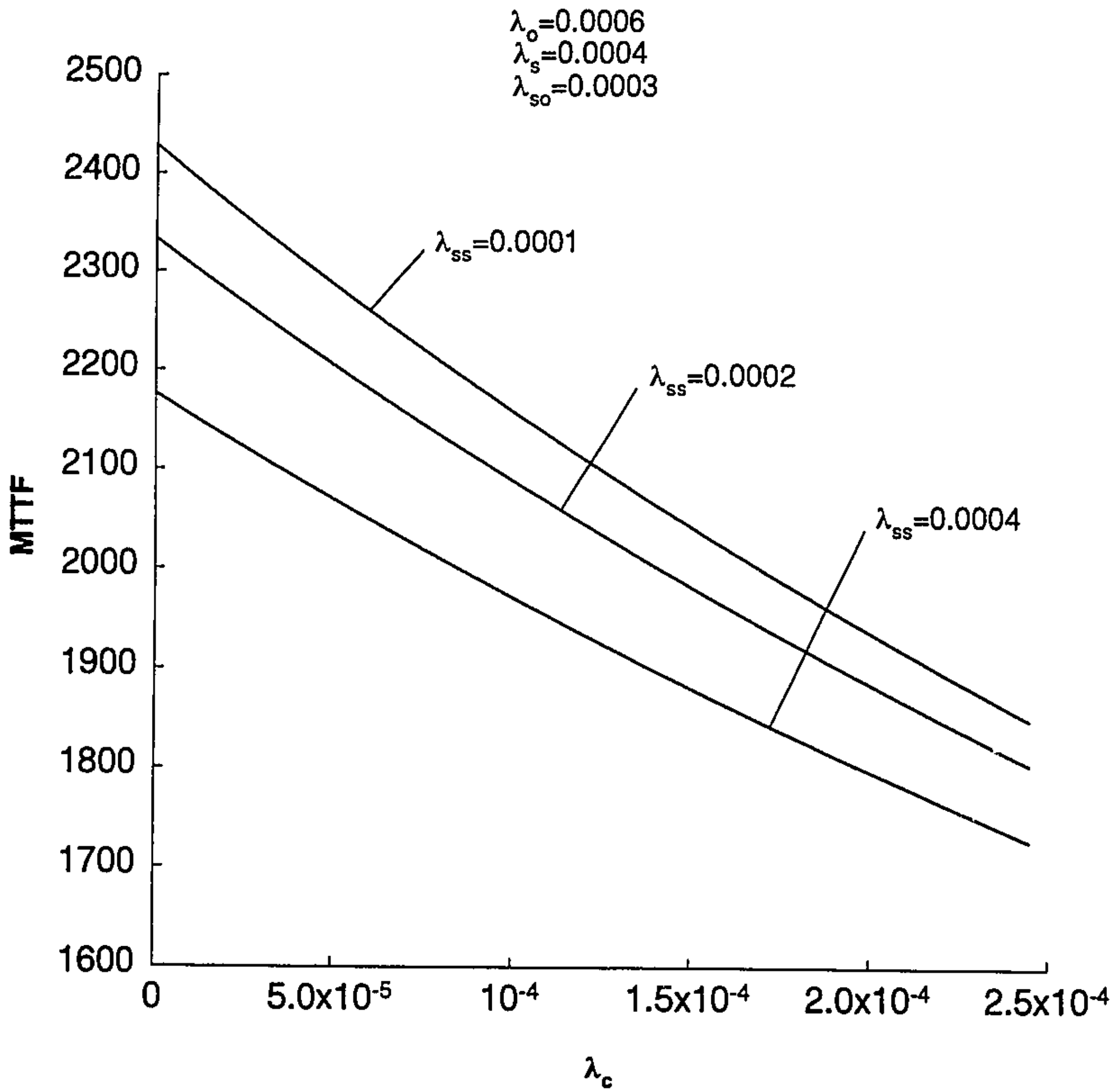


Figure 3.17 MTTF vs common cause failure rate, λ_c for warm standby system (one unit operating, two units on warm standby) without repair

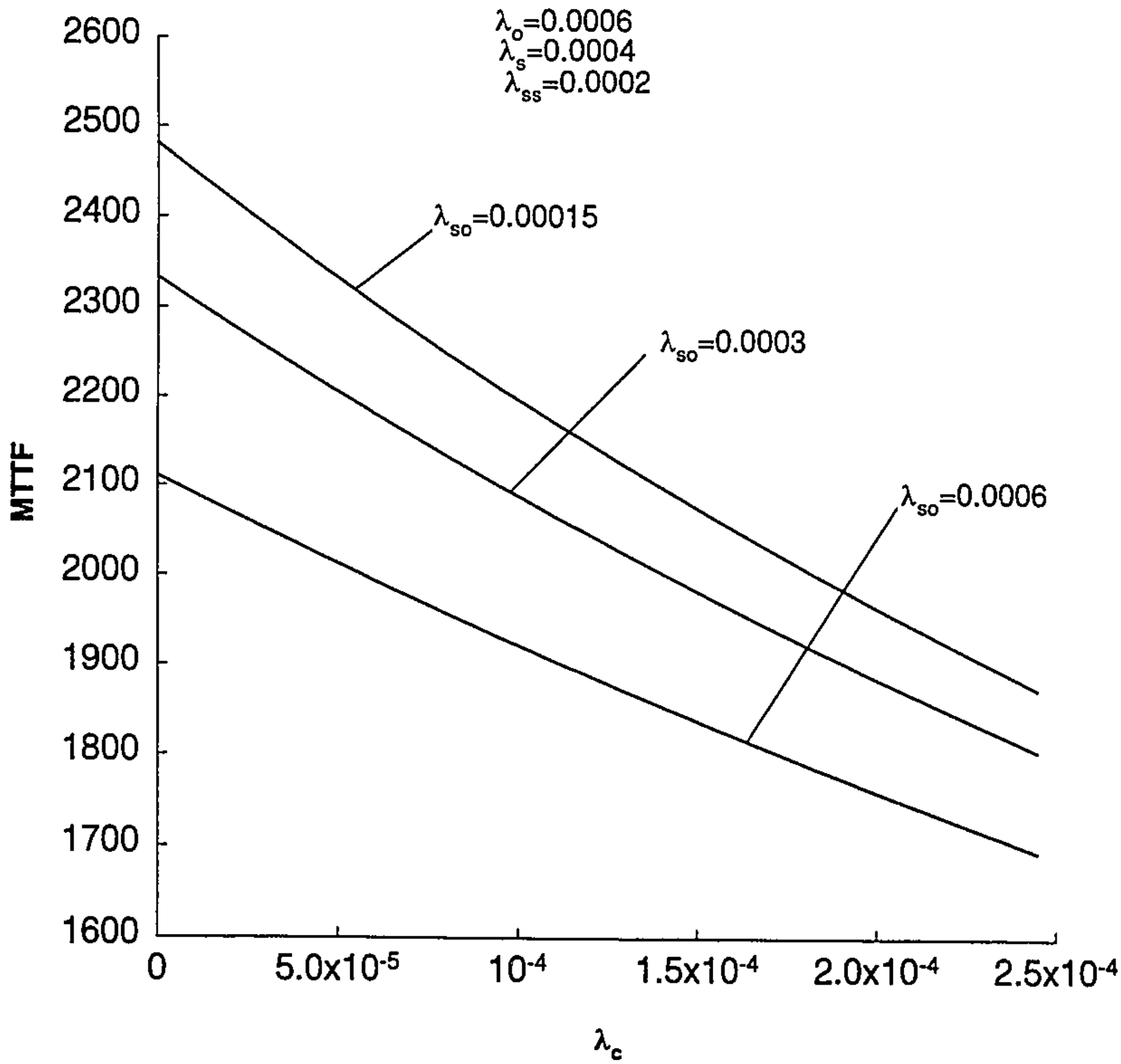
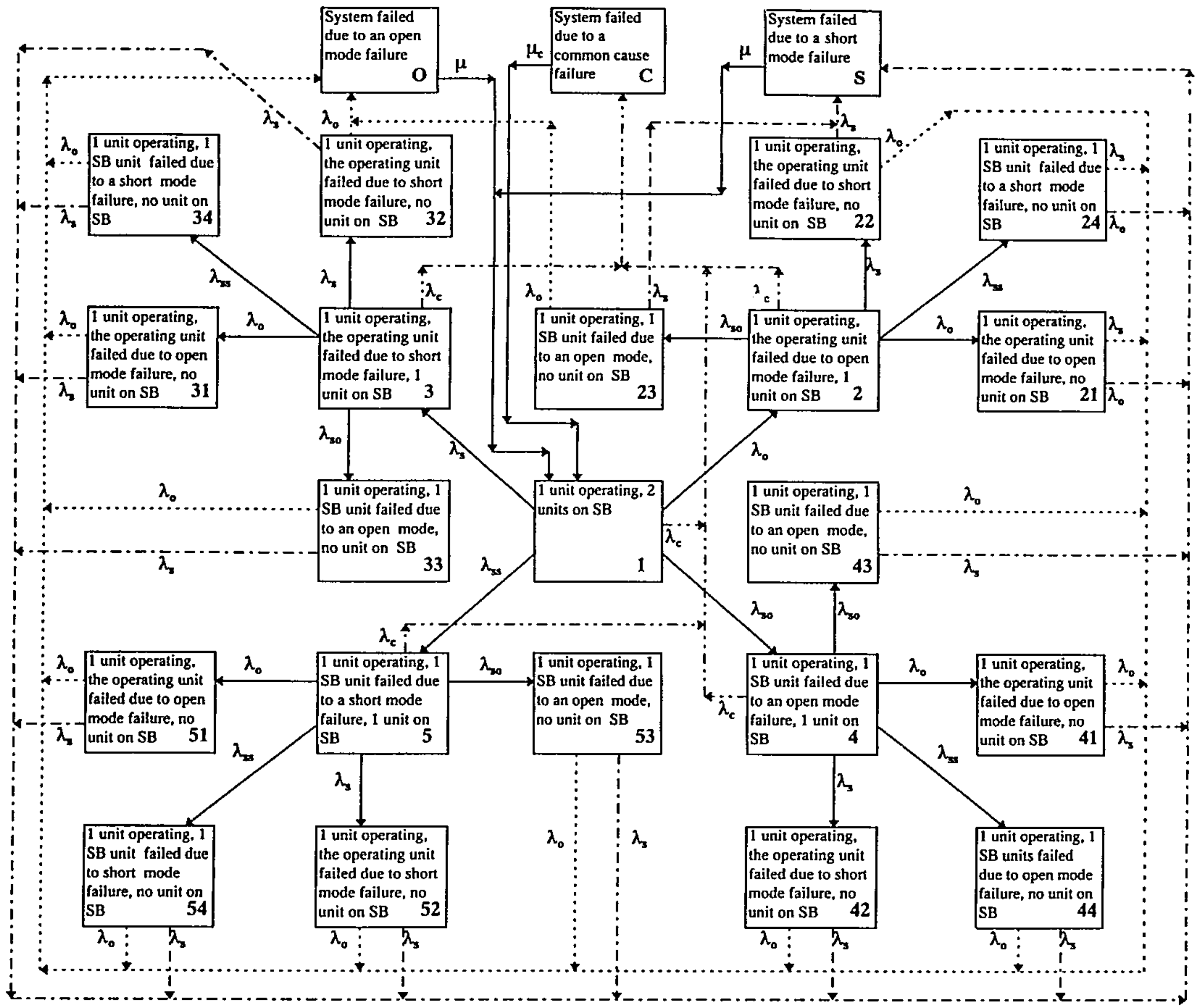


Figure 3.18 MTTF vs common cause failure rate, λ_c for warm standby system (one unit operating, two units on warm standby) without repair

Figure 3.19 System state space diagram with Type I Repair: one unit operating, two units on warm standby



SB=Standby

3.2.2 Warm standby system with Type I Repair

The system state space diagram is shown in Figure 3.19. The following differential equations are associated with this system.

$$\frac{dP_1(t)}{dt} = -(\lambda_o + \lambda_{so} + \lambda_s + \lambda_{ss} + \lambda_c)P_1(t) + \mu[P_s(t) + P_o(t)] + \mu_c P_c(t) \quad (3.116)$$

$$\frac{dP_2(t)}{dt} = -(\lambda_o + \lambda_s + \lambda_{so} + \lambda_{ss} + \lambda_c)P_2(t) + \lambda_o P_1(t) \quad (3.117)$$

$$\frac{dP_3(t)}{dt} = -(\lambda_o + \lambda_s + \lambda_{so} + \lambda_{ss} + \lambda_c)P_3(t) + \lambda_s P_1(t) \quad (3.118)$$

$$\frac{dP_4(t)}{dt} = -(\lambda_o + \lambda_s + \lambda_{so} + \lambda_{ss} + \lambda_c)P_4(t) + \lambda_{so} P_1(t) \quad (3.119)$$

$$\frac{dP_5(t)}{dt} = -(\lambda_o + \lambda_s + \lambda_{so} + \lambda_{ss} + \lambda_c)P_5(t) + \lambda_{ss} P_1(t) \quad (3.120)$$

$$\frac{dP_{21}(t)}{dt} = -(\lambda_o + \lambda_s)P_{21}(t) + \lambda_o P_2(t) \quad (3.121)$$

$$\frac{dP_{22}(t)}{dt} = -(\lambda_o + \lambda_s)P_{22}(t) + \lambda_s P_2(t) \quad (3.122)$$

$$\frac{dP_{23}(t)}{dt} = -(\lambda_o + \lambda_s)P_{23}(t) + \lambda_{so} P_2(t) \quad (3.123)$$

$$\frac{dP_{24}(t)}{dt} = -(\lambda_o + \lambda_s)P_{24}(t) + \lambda_{ss} P_2(t) \quad (3.124)$$

$$\frac{dP_{31}(t)}{dt} = -(\lambda_o + \lambda_s)P_{31}(t) + \lambda_o P_3(t) \quad (3.125)$$

$$\frac{dP_{32}(t)}{dt} = -(\lambda_o + \lambda_s)P_{32}(t) + \lambda_s P_3(t) \quad (3.126)$$

$$\frac{dP_{33}(t)}{dt} = -(\lambda_o + \lambda_s)P_{33}(t) + \lambda_{so} P_3(t) \quad (3.127)$$

$$\frac{dP_{34}(t)}{dt} = -(\lambda_o + \lambda_s)P_{34}(t) + \lambda_{ss} P_3(t) \quad (3.128)$$

$$\frac{dP_{41}(t)}{dt} = -(\lambda_o + \lambda_s)P_{41}(t) + \lambda_o P_4(t) \quad (3.129)$$

$$\frac{dP_{42}(t)}{dt} = -(\lambda_b + \lambda_s)P_{42}(t) + \lambda_s P_4(t) \tag{3.130}$$

$$\frac{dP_{43}(t)}{dt} = -(\lambda_b + \lambda_s)P_{43}(t) + \lambda_{so} P_4(t) \tag{3.131}$$

$$\frac{dP_{44}(t)}{dt} = -(\lambda_b + \lambda_s)P_{44}(t) + \lambda_{ss} P_4(t) \tag{3.132}$$

$$\frac{dP_{51}(t)}{dt} = -(\lambda_b + \lambda_s)P_{51}(t) + \lambda_o P_5(t) \tag{3.133}$$

$$\frac{dP_{52}(t)}{dt} = -(\lambda_b + \lambda_s)P_{52}(t) + \lambda_s P_5(t) \tag{3.134}$$

$$\frac{dP_{53}(t)}{dt} = -(\lambda_b + \lambda_s)P_{53}(t) + \lambda_{so} P_5(t) \tag{3.135}$$

$$\frac{dP_{54}(t)}{dt} = -(\lambda_b + \lambda_s)P_{54}(t) + \lambda_{ss} P_5(t) \tag{3.136}$$

$$\frac{dP_s(t)}{dt} = -\mu P_s(t) + \lambda_s[P_{21}(t) + P_{22}(t) + P_{23}(t) + P_{24}(t) + P_{31}(t) + P_{32}(t) + P_{33}(t) + P_{34}(t) + P_{41}(t) + P_{42}(t) + P_{43}(t) + P_{44}(t) + P_{51}(t) + P_{52}(t) + P_{53}(t) + P_{54}(t)] \tag{3.137}$$

$$\frac{dP_o(t)}{dt} = -\mu P_o(t) + \lambda_b[P_{21}(t) + P_{22}(t) + P_{23}(t) + P_{24}(t) + P_{31}(t) + P_{32}(t) + P_{33}(t) + P_{34}(t) + P_{41}(t) + P_{42}(t) + P_{43}(t) + P_{44}(t) + P_{51}(t) + P_{52}(t) + P_{53}(t) + P_{54}(t)] \tag{3.138}$$

$$\frac{dP_c(t)}{dt} = -\mu_c P_c(t) + \lambda_c[P_1(t) + P_2(t) + P_3(t) + P_4(t) + P_5(t)] \tag{3.139}$$

At time $t = 0$, $P_1(0) = 1$, and all the other initial condition probabilities are equal to zero.

The steady state probabilities can be obtained by setting the derivatives of above Equations (3.116) - (3.139) equal to zero and utilizing the relationship $\sum P_j = 1$, j represents all of the states in the system.

$$P_1 = \frac{(\lambda_b + \lambda_s)(\lambda_b + \lambda_s + \lambda_{so} + \lambda_{ss} + \lambda_c)\,\mu\mu_c}{A(3,4)} \tag{3.140}$$

$$P_2 = \frac{\lambda_b(\lambda_b + \lambda_s)\,\mu\mu_c}{A(3,4)} \tag{3.141}$$

$$P_3 = \frac{\lambda_s (\lambda_o + \lambda_s)\, \mu\mu_c}{A(3,4)} \tag{3.142}$$

$$P_4 = \frac{\lambda_{so} (\lambda_o + \lambda_s)\, \mu\mu_c}{A(3,4)} \tag{3.143}$$

$$P_5 = \frac{\lambda_{ss} (\lambda_o + \lambda_s)\, \mu\mu_c}{A(3,4)} \tag{3.144}$$

$$P_{21} = \frac{\lambda_o^2\, \mu\mu_c}{A(3,4)} \tag{3.145}$$

$$P_{22} = P_{31} = \frac{\lambda_o \lambda_s\, \mu\mu_c}{A(3,4)} \tag{3.146}$$

$$P_{23} = P_{41} = \frac{\lambda_o \lambda_{so}\, \mu\mu_c}{A(3,4)} \tag{3.147}$$

$$P_{24} = P_{51} = \frac{\lambda_o \lambda_{ss}\, \mu\mu_c}{A(3,4)} \tag{3.148}$$

$$P_{32} = \frac{\lambda_s^2\, \mu\mu_c}{A(3,4)} \tag{3.149}$$

$$P_{33} = P_{42} = \frac{\lambda_s \lambda_{so}\, \mu\mu_c}{A(3,4)} \tag{3.150}$$

$$P_{34} = P_{52} = \frac{\lambda_s \lambda_{ss}\, \mu\mu_c}{A(3,4)} \tag{3.151}$$

$$P_{43} = \frac{\lambda_{so}^2\, \mu\mu_c}{A(3,4)} \tag{3.152}$$

$$P_{44} = P_{53} = \frac{\lambda_{so} \lambda_{ss}\, \mu\mu_c}{A(3,4)} \tag{3.153}$$

$$P_{54} = \frac{\lambda_{ss}^2\, \mu\mu_c}{A(3,4)} \tag{3.154}$$

$$P_o = \frac{\lambda_o (\lambda_o + \lambda_s + \lambda_{so} + \lambda_{ss})^2\, \mu_c}{A(3,4)} \tag{3.155}$$

$$P_s = \frac{\lambda_s (\lambda_o + \lambda_s + \lambda_{so} + \lambda_{ss})^2\, \mu_c}{A(3,4)} \tag{3.156}$$

$$P_c = \frac{\lambda_c (\lambda_o + \lambda_s)(2\lambda_o + 2\lambda_s + 2\lambda_{so} + 2\lambda_{ss} + \lambda_c)\, \mu}{A(3,4)} \tag{3.157}$$

The system steady state availability is given by

$$AV_{ss} = \frac{A(3,5)}{A(3,4)} \tag{3.158}$$

where A(3,4) and A(3,5) are defined in Appendix 3.1.

By taking the Laplace transforms of Equations (3.116) - (3.139) and solving the resulting equations, the Laplace transforms of the state probabilities are as follows:

$$P_1(s) = \frac{(\lambda_b + \lambda_s + s)(\lambda_b + \lambda_s + \lambda_{so} + \lambda_{ss} + \lambda_t + s)(\mu + s)(\mu_c + s)}{A(3,6)} \tag{3.159}$$

$$P_2(s) = \frac{\lambda_b(\lambda_b + \lambda_s + s)(\mu + s)(\mu_c + s)}{A(3,6)} \tag{3.160}$$

$$P_3(s) = \frac{\lambda_s(\lambda_b + \lambda_s + s)(\mu + s)(\mu_c + s)}{A(3,6)} \tag{3.161}$$

$$P_4(s) = \frac{\lambda_{so}(\lambda_b + \lambda_s + s)(\mu + s)(\mu_c + s)}{A(3,6)} \tag{3.162}$$

$$P_5(s) = \frac{\lambda_{ss}(\lambda_b + \lambda_s + s)(\mu + s)(\mu_c + s)}{A(3,6)} \tag{3.163}$$

$$P_{21}(s) = \frac{\lambda_b^2(\mu + s)(\mu_c + s)}{A(3,6)} \tag{3.164}$$

$$P_{22}(s) = P_{31}(s) = \frac{\lambda_b\lambda_s(\mu + s)(\mu_c + s)}{A(3,6)} \tag{3.165}$$

$$P_{23}(s) = P_{41}(s) = \frac{\lambda_b\lambda_{so}(\mu + s)(\mu_c + s)}{A(3,6)} \tag{3.166}$$

$$P_{24}(s) = P_{51}(s) = \frac{\lambda_b\lambda_{ss}(\mu + s)(\mu_c + s)}{A(3,6)} \tag{3.167}$$

$$P_{32}(s) = \frac{\lambda_s^2(\mu + s)(\mu_c + s)}{A(3,6)} \tag{3.168}$$

$$P_{33}(s) = P_{42}(s) = \frac{\lambda_s\lambda_{so}(\mu + s)(\mu_c + s)}{A(3,6)} \tag{3.169}$$

$$P_{34}(s) = P_{52}(s) = \frac{\lambda_s \lambda_{ss}(\mu + s)(\mu_c + s)}{A(3,6)} \tag{3.170}$$

$$P_{43}(s) = \frac{\lambda_{so}^2(\mu + s)(\mu_c + s)}{A(3,6)} \tag{3.171}$$

$$P_{44}(s) = P_{53}(s) = \frac{\lambda_{so} \lambda_{ss}(\mu + s)(\mu_c + s)}{A(3,6)} \tag{3.172}$$

$$P_{54}(s) = \frac{\lambda_{ss}^2(\mu + s)(\mu_c + s)}{A(3,6)} \tag{3.173}$$

$$P_o(s) = \frac{\lambda_b(\lambda_b + \lambda_s + \lambda_{so} + \lambda_{ss})^2(\mu_c + s)}{A(3,6)} \tag{3.174}$$

$$P_s(s) = \frac{\lambda_s(\lambda_b + \lambda_s + \lambda_{so} + \lambda_{ss})^2(\mu_c + s)}{A(3,6)} \tag{3.175}$$

$$P_c(s) = \frac{\lambda_c(\lambda_b + \lambda_s + s)(2\lambda_b + 2\lambda_s + 2\lambda_{so} + 2\lambda_{ss} + \lambda_c + s)(\mu + s)}{A(3,6)} \tag{3.176}$$

The Laplace transform of system availability is given by

$$AV(s) = \frac{A(3,7)}{A(3,6)} \tag{3.177}$$

where A(3,6) and A(3,7) are defined in Appendix 3.1.

The time dependent system availability and state probabilities can be obtained by taking the inverse Laplace transform of Equation (3.177) and Equations (3.159)-(3.176).

The same values of model parameters corresponding to the cold standby system with Type I Repair (one unit operating, two units on standby) as well as the failure rate of standby unit are used in the following plots. The time dependent system availabilities are plotted in Figure 3.20. Comparing with Figure 2.25, it can be observed that system availability of warm standby system is lower than that of the cold standby system. The steady state availability given by Equation (3.158) is plotted in Figure 3.21. The same

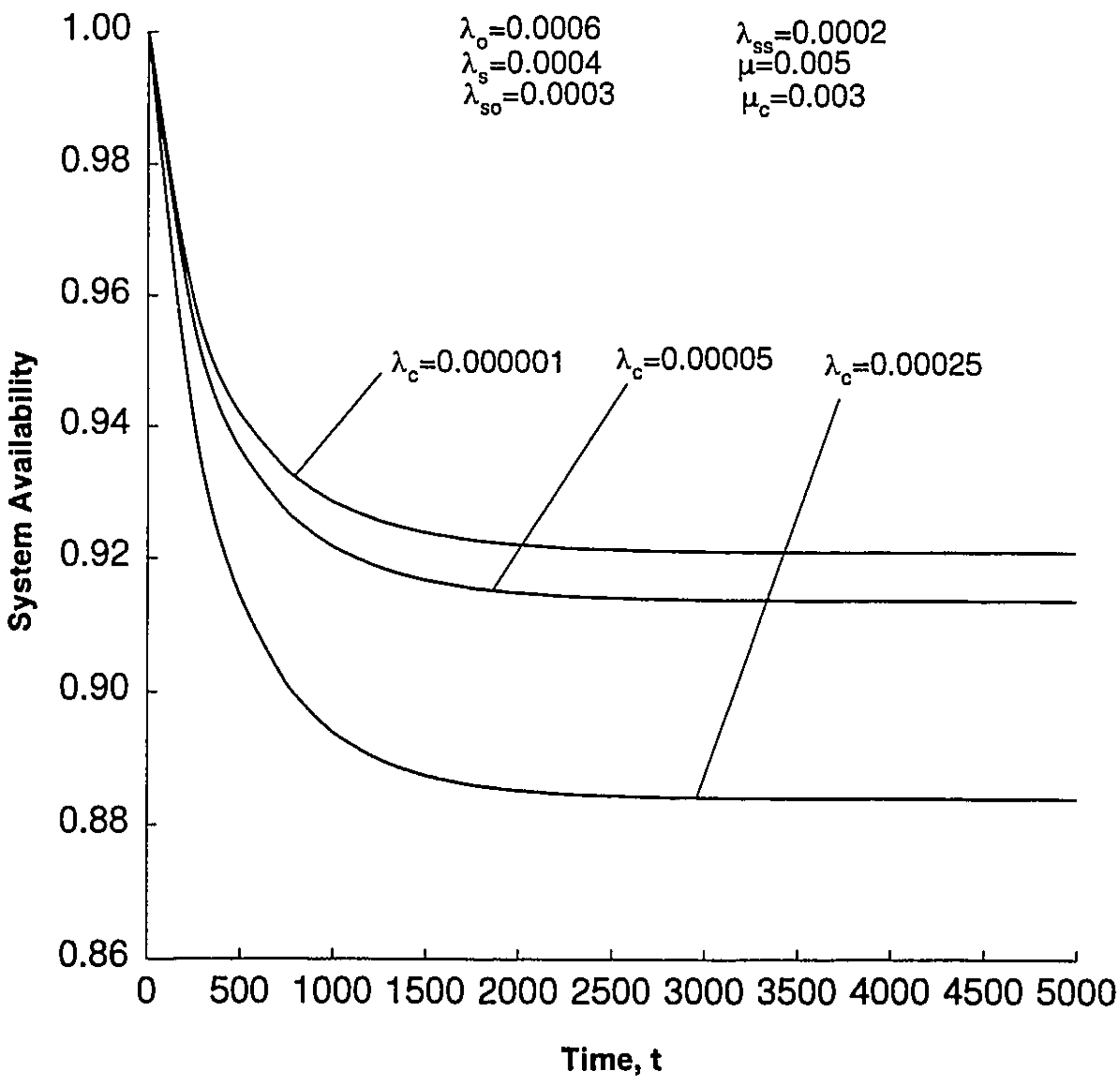


Figure 3.20 The warm standby system with type I repair availability plots: one unit operating, two units on warm standby

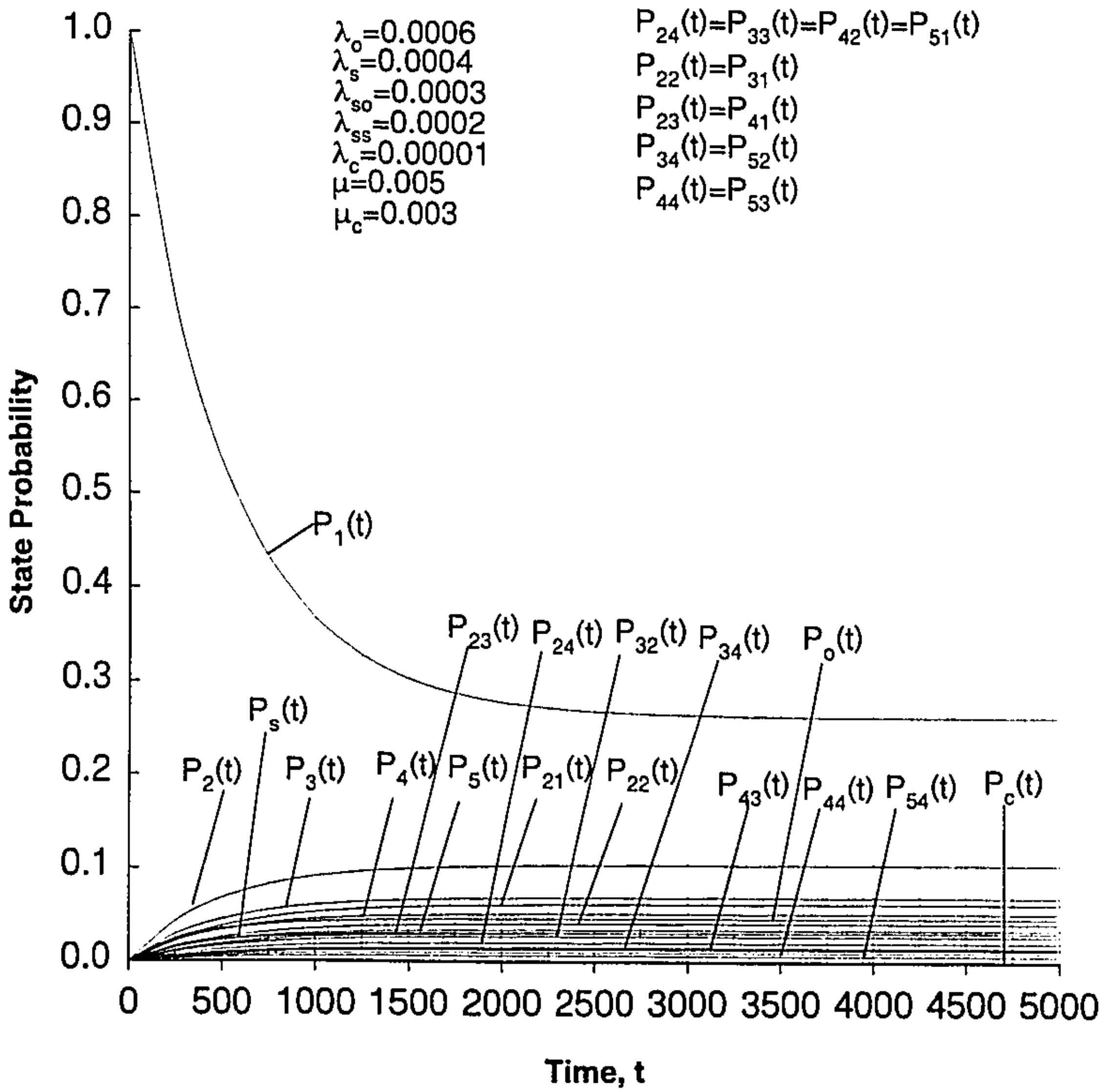


Figure 3.21 The warm standby system with type I repair state probability plots: one unit operating, the other one on warm standby

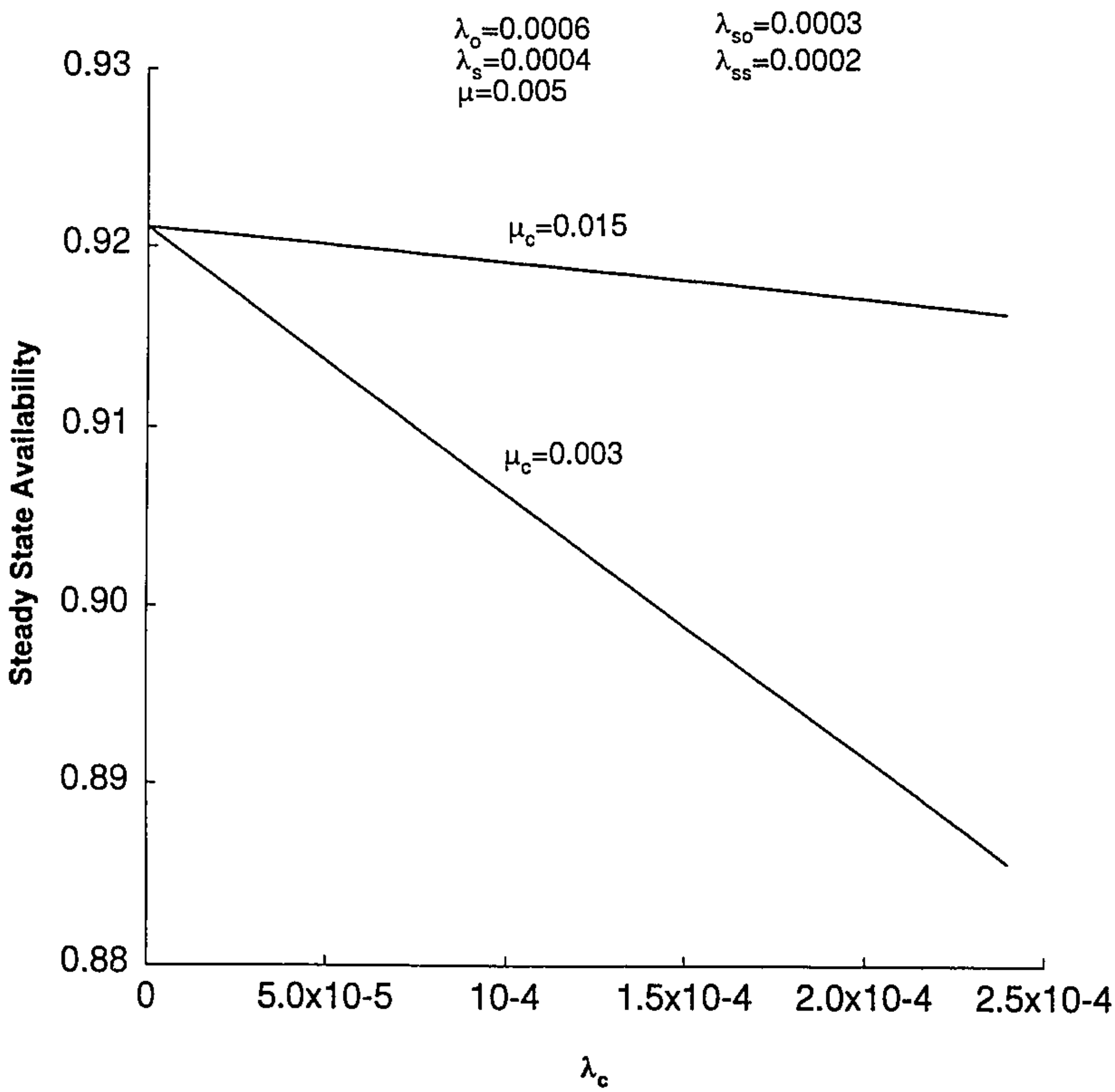
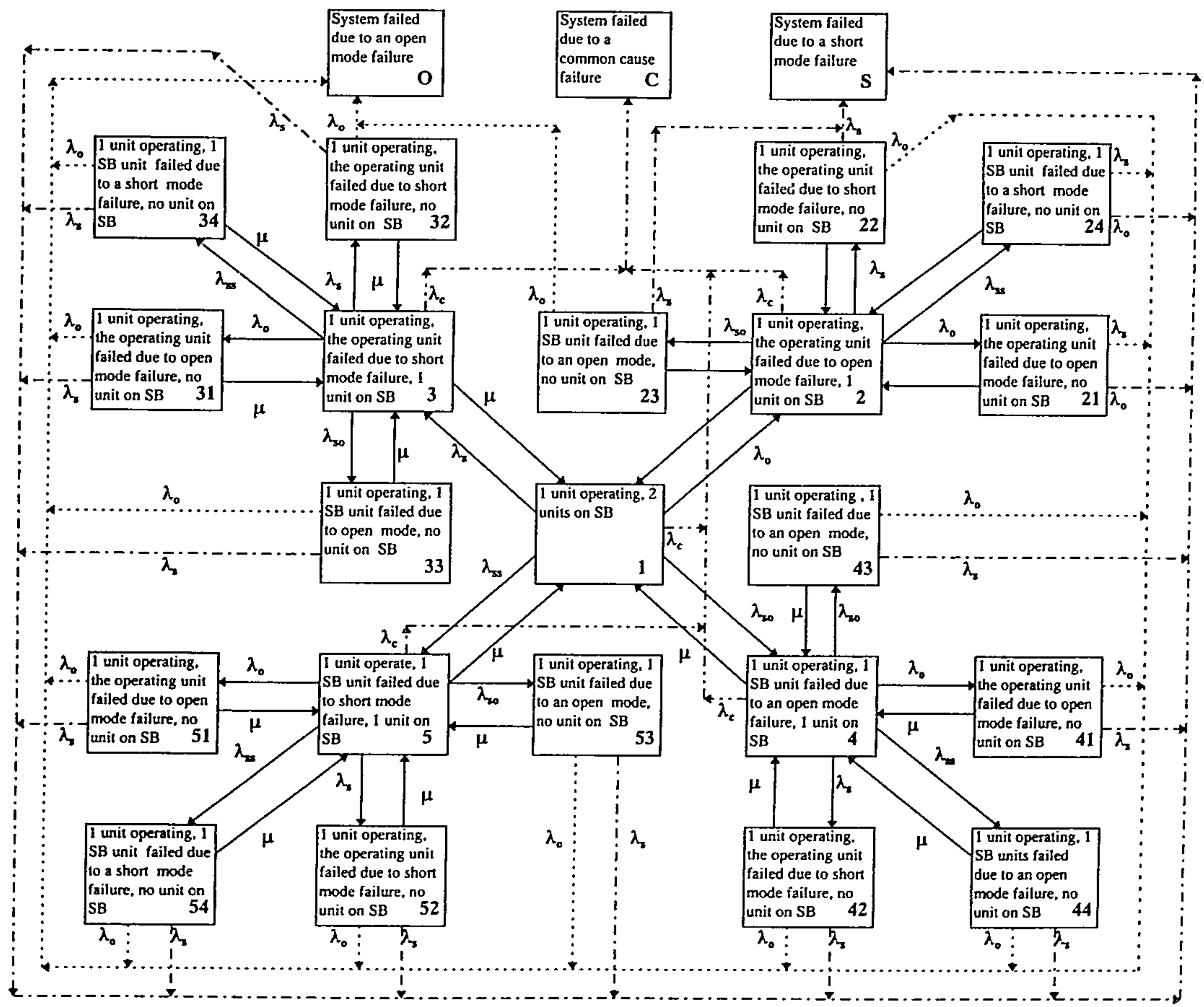


Figure 3.22 The warm standby system with type I repair steady state availability vs cause failure rate, λ_c plots (one unit operating, two units on warm standby)

Figure 3.23 System state space diagram with Type II Repair: one unit operating, two units on warm standby



SB=Standby

result is observed both for the time dependent availability and for the steady state availability. The time dependent state probabilities are presented in Figure 3.22.

3.2.3 Warm standby system with Type II Repair

The system state space diagram is shown in Figure 3.23. The following differential equations are associated with this system.

$$\frac{dP_1(t)}{dt} = -(\lambda_b + \lambda_{so} + \lambda_s + \lambda_{ss} + \lambda_c)P_1(t) + \mu[P_2(t) + P_3(t) + P_4(t) + P_5(t)] \quad (3.178)$$

$$\frac{dP_2(t)}{dt} = -(\lambda_b + \lambda_s + \lambda_{so} + \lambda_{ss} + \lambda_c + \mu)P_2(t) + \lambda_b P_1(t) + \mu[P_{21}(t) + P_{22}(t) + P_{23}(t) + P_{24}(t)] \quad (3.179)$$

$$\frac{dP_3(t)}{dt} = -(\lambda_b + \lambda_s + \lambda_{so} + \lambda_{ss} + \lambda_c + \mu)P_3(t) + \lambda_s P_1(t) + \mu[P_{31}(t) + P_{32}(t) + P_{33}(t) + P_{34}(t)] \quad (3.180)$$

$$\frac{dP_4(t)}{dt} = -(\lambda_b + \lambda_s + \lambda_{so} + \lambda_{ss} + \lambda_c + \mu)P_4(t) + \lambda_{so} P_1(t) + \mu[P_{41}(t) + P_{42}(t) + P_{43}(t) + P_{44}(t)] \quad (3.181)$$

$$\frac{dP_5(t)}{dt} = -(\lambda_b + \lambda_s + \lambda_{so} + \lambda_{ss} + \lambda_c + \mu)P_5(t) + \lambda_{ss} P_1(t) + \mu[P_{51}(t) + P_{52}(t) + P_{53}(t) + P_{54}(t)] \quad (3.182)$$

$$\frac{dP_{21}(t)}{dt} = -(\lambda_b + \lambda_s + \mu)P_{21}(t) + \lambda_b P_2(t) \quad (3.183)$$

$$\frac{dP_{22}(t)}{dt} = -(\lambda_b + \lambda_s + \mu)P_{22}(t) + \lambda_s P_2(t) \quad (3.184)$$

$$\frac{dP_{23}(t)}{dt} = -(\lambda_b + \lambda_s + \mu)P_{23}(t) + \lambda_{so} P_2(t) \quad (3.185)$$

$$\frac{dP_{24}(t)}{dt} = -(\lambda_b + \lambda_s + \mu)P_{24}(t) + \lambda_{ss} P_2(t) \quad (3.186)$$

$$\frac{dP_{31}(t)}{dt} = -(\lambda_b + \lambda_s + \mu)P_{31}(t) + \lambda_b P_3(t) \quad (3.187)$$

$$\frac{dP_{32}(t)}{dt} = -(\lambda_b + \lambda_s + \mu)P_{32}(t) + \lambda_s P_3(t) \tag{3.188}$$

$$\frac{dP_{33}(t)}{dt} = -(\lambda_b + \lambda_s + \mu)P_{33}(t) + \lambda_{so} P_3(t) \tag{3.189}$$

$$\frac{dP_{34}(t)}{dt} = -(\lambda_b + \lambda_s + \mu)P_{34}(t) + \lambda_{ss} P_3(t) \tag{3.190}$$

$$\frac{dP_{41}(t)}{dt} = -(\lambda_b + \lambda_s + \mu)P_{41}(t) + \lambda_b P_4(t) \tag{3.191}$$

$$\frac{dP_{42}(t)}{dt} = -(\lambda_b + \lambda_s + \mu)P_{42}(t) + \lambda_s P_4(t) \tag{3.192}$$

$$\frac{dP_{43}(t)}{dt} = -(\lambda_b + \lambda_s + \mu)P_{43}(t) + \lambda_{so} P_4(t) \tag{3.193}$$

$$\frac{dP_{44}(t)}{dt} = -(\lambda_b + \lambda_s + \mu)P_{44}(t) + \lambda_{ss} P_4(t) \tag{3.194}$$

$$\frac{dP_{51}(t)}{dt} = -(\lambda_b + \lambda_s + \mu)P_{51}(t) + \lambda_b P_5(t) \tag{3.195}$$

$$\frac{dP_{52}(t)}{dt} = -(\lambda_b + \lambda_s + \mu)P_{52}(t) + \lambda_s P_5(t) \tag{3.196}$$

$$\frac{dP_{53}(t)}{dt} = -(\lambda_b + \lambda_s + \mu)P_{53}(t) + \lambda_{so} P_5(t) \tag{3.197}$$

$$\frac{dP_{54}(t)}{dt} = -(\lambda_b + \lambda_s + \mu)P_{54}(t) + \lambda_{ss} P_5(t) \tag{3.198}$$

$$\frac{dP_s(t)}{dt} = \lambda_s[P_{21}(t) + P_{22}(t) + P_{23}(t) + P_{24}(t) + P_{31}(t) + P_{32}(t) + P_{33}(t) + P_{34}(t) + P_{41}(t) + P_{42}(t) + P_{43}(t) + P_{44}(t) + P_{51}(t) + P_{52}(t) + P_{53}(t) + P_{54}(t)] \tag{3.199}$$

$$\frac{dP_o(t)}{dt} = \lambda_b[P_{21}(t) + P_{22}(t) + P_{23}(t) + P_{24}(t) + P_{31}(t) + P_{32}(t) + P_{33}(t) + P_{34}(t) + P_{41}(t) + P_{42}(t) + P_{43}(t) + P_{44}(t) + P_{51}(t) + P_{52}(t) + P_{53}(t) + P_{54}(t)] \tag{3.200}$$

$$\frac{dP_c(t)}{dt} = \lambda_c[P_1(t) + P_2(t) + P_3(t) + P_4(t) + P_5(t)] \tag{3.201}$$

At time $t = 0$, $P_1(0) = 1$, and all the other initial condition probabilities are equal to zero.

By taking the Laplace transforms of Equations (3.178) - (3.201) and solving the resulting equations, the Laplace transforms of the state probabilities are as follows:

$$P_1(s) = \frac{A(3,9)}{A(3,8)} \tag{3.202}$$

$$P_2(s) = \frac{\lambda_o(\lambda_o + \lambda_s + \mu + s)}{A(3,8)} \tag{3.203}$$

$$P_3(s) = \frac{\lambda_s(\lambda_o + \lambda_s + \mu + s)}{A(3,8)} \tag{3.204}$$

$$P_4(s) = \frac{\lambda_{so}(\lambda_o + \lambda_s + \mu + s)}{A(3,8)} \tag{3.205}$$

$$P_5(s) = \frac{\lambda_{ss}(\lambda_o + \lambda_s + \mu + s)}{A(3,8)} \tag{3.206}$$

$$P_{21}(s) = \frac{\lambda_o^2}{A(3,8)} \tag{4.207}$$

$$P_{22}(s) = P_{31}(s) = \frac{\lambda_o\lambda_s}{A(3,8)} \tag{3.208}$$

$$P_{23}(s) = P_{41}(s) = \frac{\lambda_o\lambda_{so}}{A(3,8)} \tag{3.209}$$

$$P_{24}(s) = P_{51}(s) = \frac{\lambda_o\lambda_{ss}}{A(3,8)} \tag{3.210}$$

$$P_{32}(s) = \frac{\lambda_s^2}{A(3,8)} \tag{3.211}$$

$$P_{33}(s) = P_{42}(s) = \frac{\lambda_s\lambda_{so}}{A(3,8)} \tag{3.212}$$

$$P_{34}(s) = P_{52}(s) = \frac{\lambda_s\lambda_{ss}}{A(3,8)} \tag{3.213}$$

$$P_{43}(s) = \frac{\lambda_{so}^2}{A(3,8)} \tag{3.214}$$

$$P_{44}(s) = P_{53}(s) = \frac{\lambda_{so}\lambda_{ss}}{A(3,8)} \tag{3.215}$$

$$P_{54}(s) = \frac{\lambda_{ss}^2}{A(3,8)} \tag{3.216}$$

$$P_s(s) = \frac{\lambda_s(\lambda_o + \lambda_s + \lambda_{so} + \lambda_{ss})^2}{sA(3,8)} \tag{3.217}$$

$$P_o(s) = \frac{\lambda_b(\lambda_b + \lambda_s + \lambda_{so} + \lambda_{ss})^2}{sA(3,8)} \tag{3.218}$$

$$P_c(s) = \frac{A(3,10)}{sA(3,8)} \tag{3.219}$$

where A(3,8), A(3,9) and A(3,10) are defined in Appendix 3.1

The Laplace transform of system reliability is given by

$$R(s) = \frac{A(3,11)}{A(3,8)} \tag{3.220}$$

where A(3,11) is defined in Appendix 3.1.

The system mean time to failure (MTTF) is

$$\begin{aligned} MTTF &= \lim_{s \to 0} R(s) \\ &= \frac{A(3,12)}{A(3,13)} \end{aligned} \tag{3.221}$$

where A(3,12) and A(3,13) are defined in Appendix 3.1.

The system reliability and the time dependent state probabilities can be obtained by taking the inverse Laplace transforms of Equation (3.220) and Equations (3.202)-(3.219).

The same values of model parameters corresponding to the cold standby system without repair (one unit operating, two units on standby) as well as the failure rate of the standby units are used in the following figures. The system reliabilities are plotted in Figure 3.24. Comparing with Figure 2.37, it is evident that the reliability of warm standby system is lower than that of the cold standby system. Figure 3.25 is the time dependent state probabilities. The plots of system mean time to failure given by Equation (3.221) are shown in Figure 3.26. Comparing with Figure 2.39, it is noted that MTTF of warm standby system is less than that of the cold standby system.

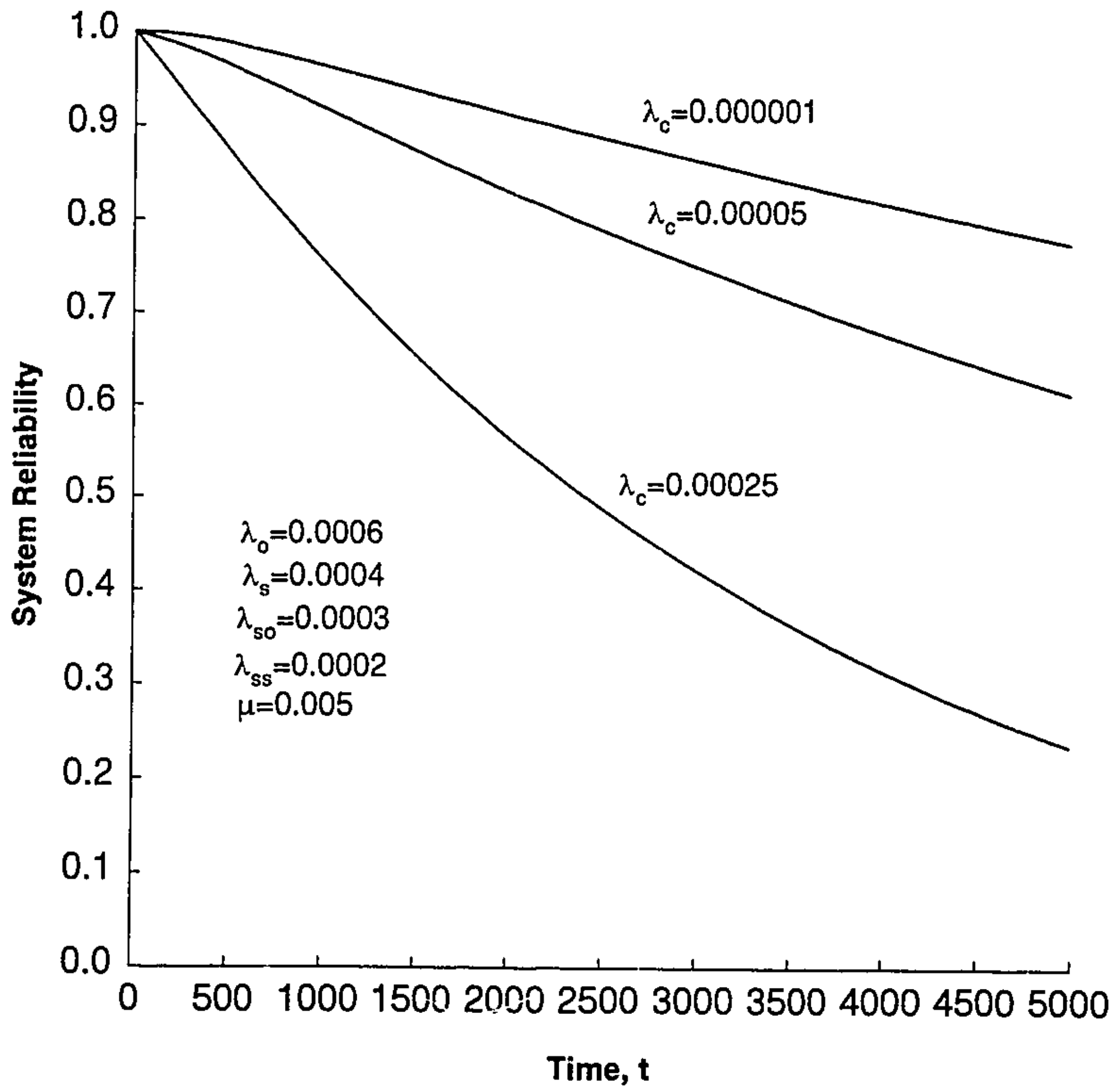


Figure 3.24 The warm standby system with type II repair reliability plots: one unit operating, two units on warm standby

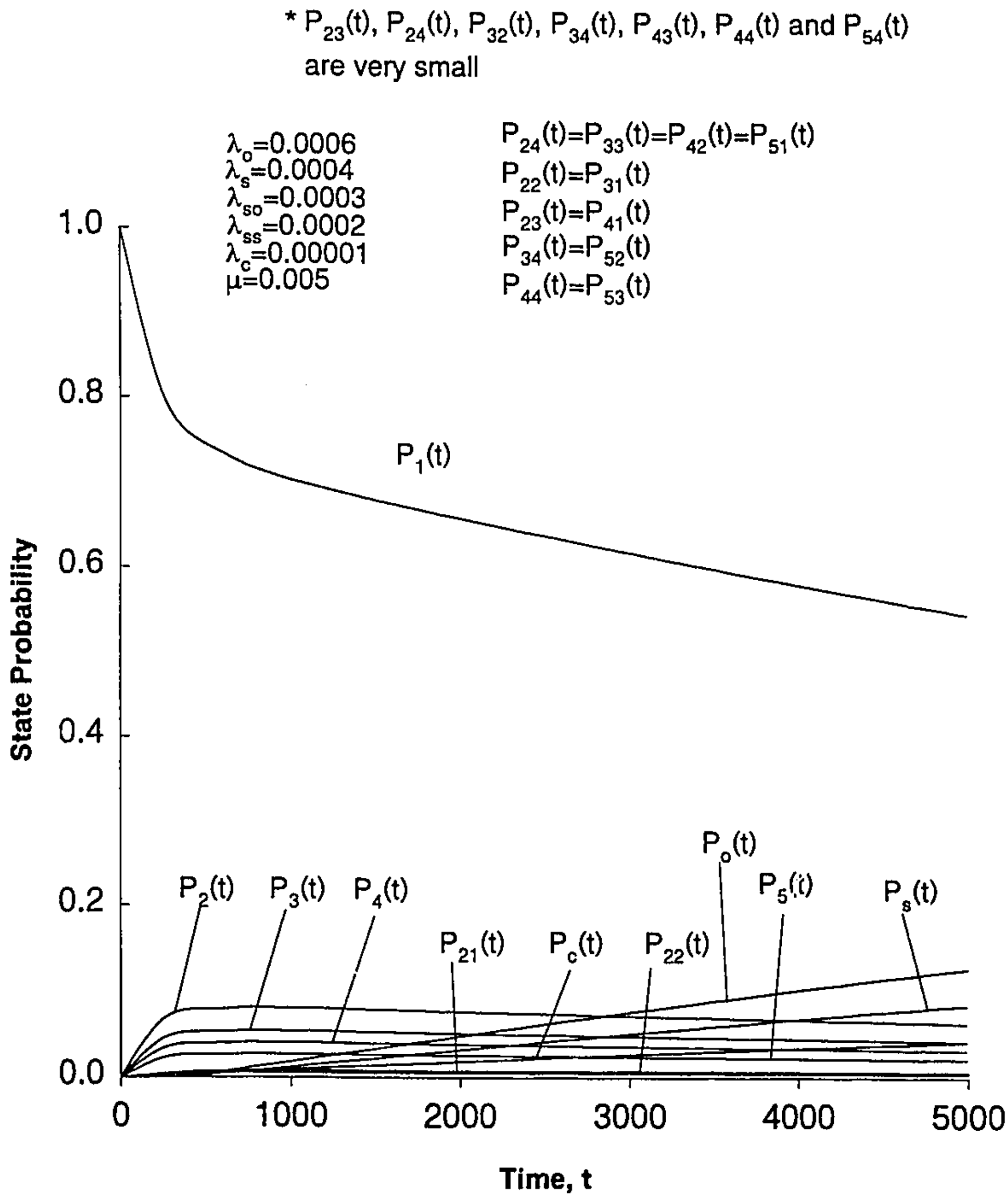


Figure 3.25 The warm standby system with type II repair state probability plots: one unit operating, two units on warm standby

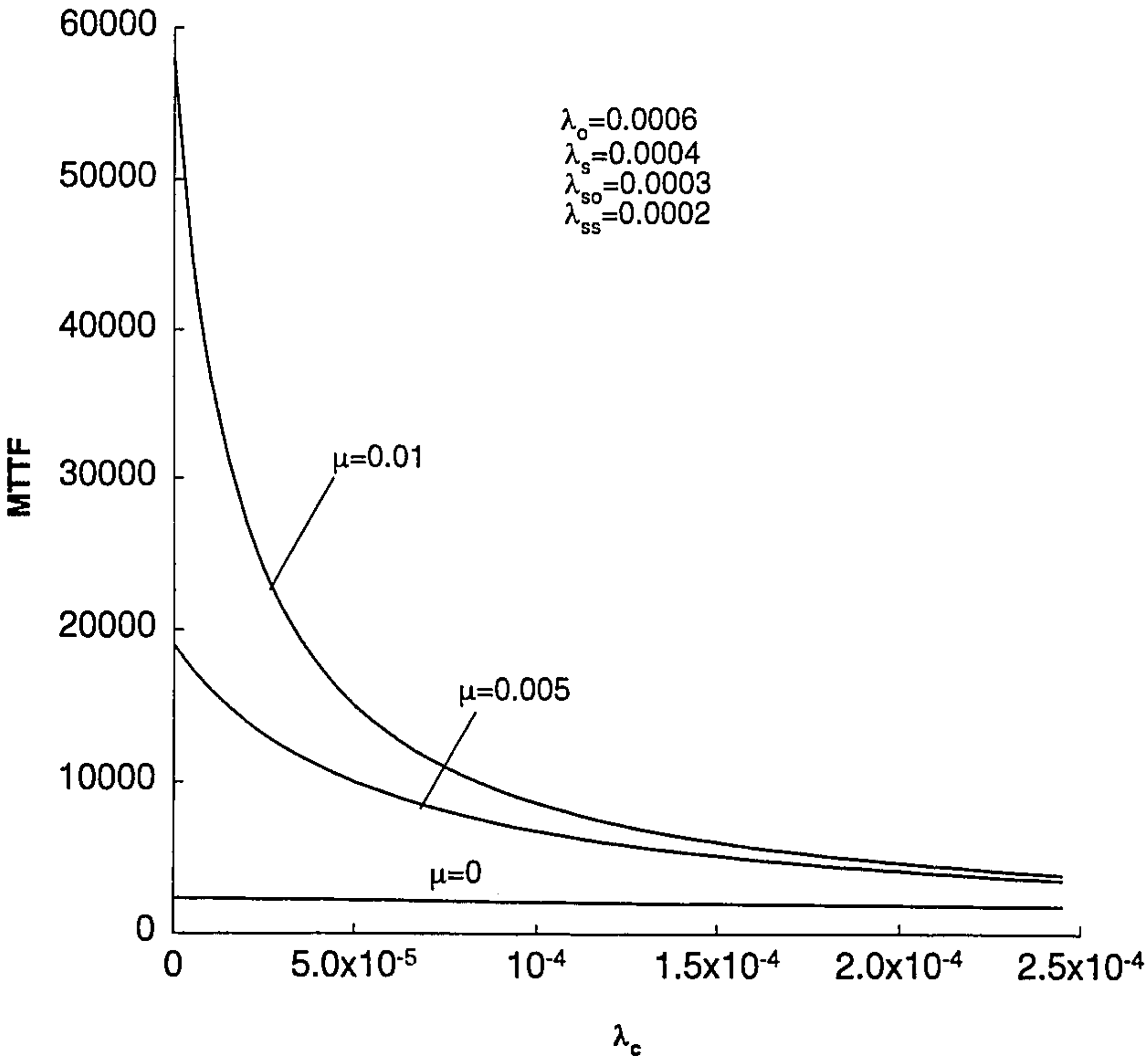


Figure 3.26 MTTF vs common cause failure rate, λ_c (with different repair rate) for three identical units warm standby system with type II repair

By comparing the cold standby system and warm standby system, the following conclusion can be made. When the same number of standby units and the same repair policy are considered, the system reliability, the mean time to failure, the steady state availability and the time dependent availability in the warm standby system are lower than these parameters in the cold standby system. Therefore, in real life, overlooking the possibility of standby unit failure can lead to overestimates of the system reliability and other parameters.

Chapter 4

System Reliability Comparison and Improvement

This chapter compares two identical three-state device systems with different configurations including parallel, series and standby. To improve the reliability of the standby system, a new kind of standby configuration, in which there are two units in parallel with one unit on standby, is introduced. The system reliability, the mean time to failure, the steady state availability and the time dependant availability for this configuration are obtained.

4.1 Comparison of the standby system, the parallel system, and the series system

There are three different configurations (Figure 4.1 for the standby, Figure 4.2 for the parallel and Figure 4.3 for the series) which are made up of two identical three-state devices. In this section, the reliability, the mean time to failure and the availability for three different configurations are discussed and compared.

Notation:

The following symbols are associated with these models:

t time.

λ_o Constant open-mode failure rate of an operating unit or device.

λ_s Constant short-mode failure rate of an operating unit or device.

λ_c Constant common-cause failure rate of the system.

μ Constant repair rate for failed system due to an open or a short mode failure.

μ_c Constant repair rate for failed system due to a common-causefailure rate.

$P_1(t)$ Probability that the system is in first state.

$P_s(t)$ Probability that the system is in the failed state due to a short mode failure.

$P_o(t)$ Probability that the system is in the failed state due to an open mode failure.

$P_c(t)$ Probability that the system is in the failed state due to a common-cause failure.

$R_p(t)$ Reliability of the parallel system in [0,t].

$R_s(t)$ Reliability of the series system in [0,t].

$R_{sb}(t)$ Reliability of the standby system in [0,t].

$MTTF_p$ Mean time to failure of the parallel system.

$MTTF_s$ Mean time to failure of the series system.

$MTTF_{sb}$ Mean time to failure of the standby system.

AV_p Steady state availability of parallel system.

AV_s Steady state availability of series system.

AV_{sb} Steady state availability of standby system.

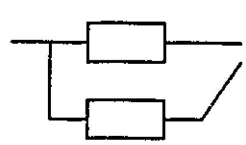


Figure 4.1 Standby system

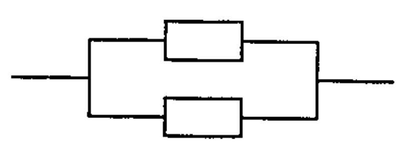


Figure 4.2 Parallel system

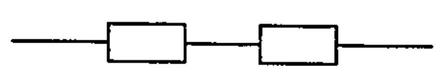


Figure 4.3 Series system

4.1.1 System Reliability and MTTF comparison without Repair

The state space diagram for the parallel and the series systems are shown in Figures 4.4 and 4.5 respectively.

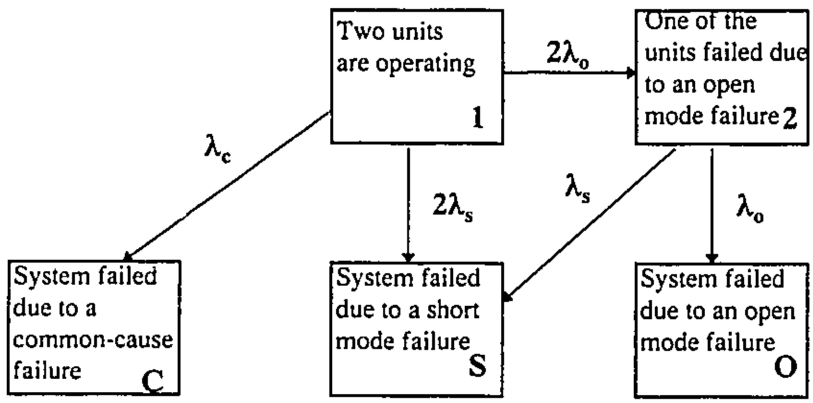


Figure 4.4 The state space diagram for a two unit in parallel system without repair

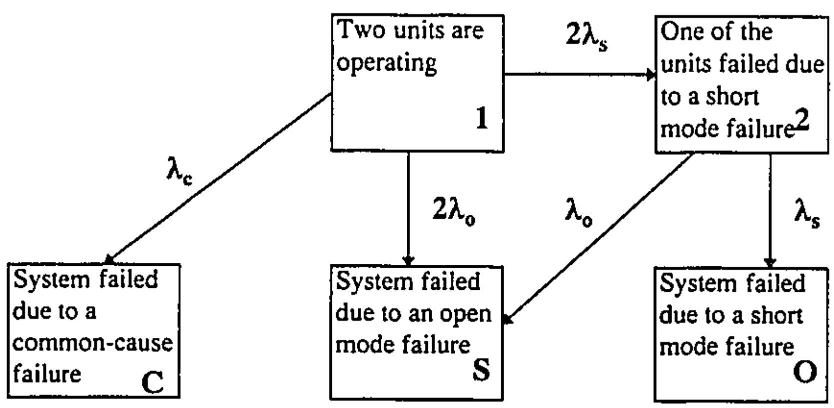


Figure 4.5 The state space diagram for a two unit in series system without repair

The system of differential equations associated with Figure 4.4 is as follows:

$$\frac{dP_1(t)}{dt} = -(2\lambda_o + 2\lambda_s + \lambda_c)P_1(t) \tag{4.1}$$

$$\frac{dP_2(t)}{dt} = -(\lambda_o + \lambda_s)P_2(t) + 2\lambda_o P_1(t) \tag{4.2}$$

$$\frac{dP_s(t)}{dt} = \lambda_s[2P_1(t) + P_2(t)] \tag{4.3}$$

$$\frac{dP_o(t)}{dt} = \lambda_o P_2(t) \tag{4.4}$$

$$\frac{dP_c(t)}{dt} = \lambda_c P_1(t) \tag{4.5}$$

At time $t = 0$, $P_1(0) = 1$, and all the other initial condition probabilities are equal to zero. $P_2(t)$ is the probability that one unit failed due to an open mode failure.

The system of differential equations associated with Figure 4.5 is as follows:

$$\frac{dP_1(t)}{dt} = -(2\lambda_o + 2\lambda_s + \lambda_c)P_1(t) \tag{4.6}$$

$$\frac{dP_2(t)}{dt} = -(\lambda_o + \lambda_s)P_2(t) + 2\lambda_s P_1(t) \tag{4.7}$$

$$\frac{dP_o(t)}{dt} = \lambda_o[2P_1(t) + P_2(t)] \tag{4.8}$$

$$\frac{dP_s(t)}{dt} = \lambda_s P_2(t) \tag{4.9}$$

$$\frac{dP_c(t)}{dt} = \lambda_c P_1(t) \tag{4.10}$$

At time $t = 0$, $P_1(0) = 1$, and all the other initial condition probabilities are equal to zero. $P_2(t)$ is the probability that one unit failed due to a short mode failure.

The state space diagram of standby system is shown in Figure 2.6. The differential equations associated with this model are given by (2.19) - (2.25). At time $t = 0$, $P_1(0) = 1$, and all the other initial condition probabilities are equal to zero.

Case I: Open/Short mode failure without common-cause failure

In this case common-cause failures are not considered. The reliability and mean time to failure (MTTF) for three different configurations are compared.

Setting λ_c=0 in Equations (4.1)-(4.5) and solving these equations with the same method used as in the previous chapters, the reliability and the mean time to failure (MTTF) of the parallel system are given by

$$R_p(t) = P_1(t) + P_2(t)$$
$$= \frac{1}{\lambda_o + \lambda_s}\left[\frac{2\lambda_o}{e^{(\lambda_o+\lambda_s)t}} + \frac{(-\lambda_o + \lambda_s)}{e^{(2\lambda_o+2\lambda_s)t}}\right] \qquad (4.11)$$

$$MTTF_p = \frac{3\lambda_o + \lambda_s}{2(\lambda_o + \lambda_s)^2} \qquad (4.12)$$

Setting λ_c=0 in Equations (4.6)-(4.10) and solving these equations with the same method used as in the previous chapters, the reliability and the mean time to failure (MTTF) of the series system are given by

$$R_s(t) = P_1(t) + P_2(t)$$
$$= \frac{1}{\lambda_o + \lambda_s}\left[\frac{2\lambda_s}{e^{(\lambda_o+\lambda_s)t}} + \frac{(\lambda_o - \lambda_s)}{e^{(2\lambda_o+2\lambda_s)t}}\right] \qquad (4.13)$$

$$MTTF_s = \frac{\lambda_o + 3\lambda_s}{2(\lambda_o + \lambda_s)^2} \qquad (4.14)$$

Setting λ_c=0 in Equations (2.19)-(2.25) and solving these equations with the same method used as in the previous chapters, the reliability and the mean time to failure (MTTF) of the standby system are

$$R_{sb}(t) = \frac{1+(\lambda_b+\lambda_s)t}{e^{(\lambda_b+\lambda_s)t}} \tag{4.15}$$

$$MTTF_{sb} = \frac{2}{\lambda_b+\lambda_s} \tag{4.16}$$

The reliability difference $\Delta R(t)$ between the standby system and the parallel system is

$$\Delta R(t) = R_{sb}(t) - R_p(t) = \frac{e^{(\lambda_b+\lambda_s)t}(\lambda_b+\lambda_s)^2 t + [e^{(\lambda_b+\lambda_s)t} - 1](\lambda_s - \lambda_b)}{e^{(2\lambda_b+2\lambda_s)t}(\lambda_b+\lambda_s)} \tag{4.17}$$

At time t=0, $\Delta R(t)$ in Equation (4.17) equals to zero, however, $\Delta R(t)$ will be greater than zero with the progression in time which indicates that the reliability of the standby system is higher than the reliability of the parallel system. This trend is proofed and discussed in detail in Appendix 4.1-(i).

The MTTF (mean time to failure) difference between the standby system and the parallel system is

$$\Delta MTTF = MTTF_{sb} - MTTF_p = \frac{\lambda_b+3\lambda_s}{2(\lambda_b+\lambda_s)^2} \tag{4.18}$$

Comparing the MTTF of the two system in Equation (4.18), it is evident that the standby system has higher MTTF than the parallel system.

Similarly, the reliability difference between the standby system and the series system $\Delta R(t)$ is

$$\Delta R(t) = R_{sb}(t) - R_s(t) = \frac{e^{(\lambda_b+\lambda_s)t}(\lambda_b+\lambda_s)^2 t + [e^{(\lambda_b+\lambda_s)t} - 1](\lambda_b - \lambda_s)}{e^{(2\lambda_b+2\lambda_s)t}(\lambda_b+\lambda_s)} \tag{4.19}$$

The MTTF (mean time to failure) difference between the standby system and the series system is

$$\Delta MTTF = MTTF_{sb} - MTTF_s = \frac{3\lambda_o + \lambda_s}{2(\lambda_o + \lambda_s)^2} \tag{4.20}$$

With the same analogy, it can be proved that $\Delta R(t)$ and $\Delta MTTF$ in Equations (4.19) and (4.20) are greater than zero, thus, the standby system has higher reliability and MTTF than the series system. The detailed proof is in Appendix 4.1-(ii).

Also, the reliability difference between the parallel system and the series system is

$$\Delta R(t) = R_p(t) - R_s(t) = \frac{(2\lambda_o - 2\lambda_s)}{\lambda_o + \lambda_s}\left[\frac{1}{e^{(\lambda_o+\lambda_s)t}} - \frac{1}{e^{(2\lambda_o+2\lambda_s)t}}\right] \tag{4.21}$$

The MTTF (mean time to failure) difference between the parallel system and the series system is

$$\Delta MTTF = MTTF_p - MTTF_s = \frac{\lambda_o - \lambda_s}{(\lambda_o + \lambda_s)^2} \tag{4.22}$$

Equations (4.21) and (4.22) indicate that if the open mode failure rate is greater than the short mode failure rate, the parallel system will have higher reliability and higher MTTF than the series system and vice versa.

Case II: Open/Short mode failure with common-cause failure

This case considers open/short mode failures with constant common-cause failure rate, and compares the reliability and the mean time to failure for three different configurations.

The reliability and the mean time to failure (MTTF) of the parallel system are given by solving Equations (4.1) - (4.5).

$$R_p(t) = P_1(t) + P_2(t)$$

$$= \frac{1}{\lambda_b + \lambda_s + \lambda_c}\left[\frac{2\lambda_b}{e^{(\lambda_b+\lambda_s)t}} + \frac{(-\lambda_b + \lambda_s + \lambda_c)}{e^{(2\lambda_b+2\lambda_s+\lambda_c)t}}\right] \tag{4.23}$$

$$MTTF_p = \frac{3\lambda_b + \lambda_s}{(\lambda_b + \lambda_s)(2\lambda_b + 2\lambda_s + \lambda_c)} \tag{4.24}$$

The reliability and the mean time to failure (MTTF) of the series system are given by solving Equations (4.6) - (4.10).

$$R_s(t) = P_1(t) + P_2(t)$$

$$= \frac{1}{\lambda_b + \lambda_s + \lambda_c}\left[\frac{2\lambda_s}{e^{(\lambda_b+\lambda_s)t}} + \frac{(\lambda_b - \lambda_s + \lambda_c)}{e^{(2\lambda_b+2\lambda_s+\lambda_c)t}}\right] \tag{4.25}$$

$$MTTF_s = \frac{\lambda_b + 3\lambda_s}{(\lambda_b + \lambda_s)(2\lambda_b + 2\lambda_s + \lambda_c)} \tag{4.26}$$

The reliability and the mean time to failure (MTTF) for the standby system are given by Equations (2.38) and (2.39). Thus, from Equations (2.38) and (4.23), the reliability difference $\Delta R(t)$ between the standby system and the parallel system is

$$\Delta R(t) = R_{sb}(t) - R_p(t)$$

$$= \frac{(\lambda_b + \lambda_s)^2[e^{(\lambda_b+\lambda_s+\lambda_c)t} - e^{(\lambda_b+\lambda_s)t}] + (\lambda_s - \lambda_b)\lambda_c[e^{(\lambda_b+\lambda_s+\lambda_c)t} - 1] + \lambda_c^2(e^{(\lambda_b+\lambda_s)t} - 1)}{e^{(2\lambda_b+2\lambda_s+\lambda_c)t}(\lambda_b + \lambda_s + \lambda_c)\lambda_c} \tag{4.27}$$

At time t=0, $\Delta R(t)$ in Equation (4.27) equals to zero, however, $\Delta R(t)$ will be greater than zero with progression in time which indicates that that the reliability of standby system is higher than that of the parallel system. This trend is proofed and discussed in detail in Appendix 4.1-(iii).

The MTTF difference between the standby system and the parallel system can be obtained by using Equations (2.39) and (4.24).

$$MTTF_{sb} - MTTF_{p} = \frac{\lambda_b(\lambda_b - \lambda_c) + 4\lambda_b\lambda_s + 3\lambda_s^2 + \lambda_s\lambda_c}{(\lambda_b + \lambda_s)(\lambda_b + \lambda_s + \lambda_c)(2\lambda_b + 2\lambda_s + \lambda_c)} \tag{4.28}$$

Since the open mode failure rate is greater than the common-cause failure rate, (usually common-cause failure is 1%-30% of component' failure rate) the solution of Equation (4.28) is greater than zero. Thus, the MTTF of the standby system is higher than that of the parallel system.

Similarly, from Equations (2.38) and (4.25) the reliability difference $\Delta R(t)$ between the standby system and the series sy stem is

$$\begin{aligned}\Delta R(t) &= R_{sb}(t) - R_s(t) \\ &= \frac{(\lambda_b + \lambda_s)^2[e^{(\lambda_b+\lambda_s+\lambda_c)t} - e^{(\lambda_b+\lambda_s)t}] + (\lambda_b - \lambda_s)\lambda_c[e^{(\lambda_b+\lambda_s+\lambda_c)t} + 1] + \lambda_c^2(e^{(\lambda_b+\lambda_s)t} - 1)}{e^{(2\lambda_b+2\lambda_s+\lambda_c)t}(\lambda_b + \lambda_s + \lambda_c)\lambda_c}\end{aligned} \tag{4.29}$$

At time t=0, $\Delta R(t)$ in Equation (4.29) equals to zero, and for $t > 0$, $\Delta R(t) > 0$. This indicates that the standby system has a better reliability than the parallel system. The detail proof and discussion are in Appendix 4.1-(iv).

The MTTF difference between the standby system and the series system can be obtained by using Equations (2.39) and (4.26).

$$MTTF_{sb} - MTTF_s = \frac{\lambda_o(\lambda_s - \lambda_c) + 4\lambda_o\lambda_s + 3\lambda_o^2 + \lambda_{os}\lambda_c}{(\lambda_o + \lambda_s)(\lambda_o + \lambda_s + \lambda_c)(2\lambda_o + 2\lambda_s + \lambda_c)} \tag{4.30}$$

Since the short mode failure rate is greater than the common-cause failure rate, (usually common-cause failure is 1%-30% of component' failure rate) the solution of Equation (4.30) is greater than zero. Thus, the MTTF of the standby system is higher than that of the series system.

From Equations (4.23) and (4.25), the reliability difference between the parallel and the series systems is

$$R_p(t) - R_s(t) = \frac{(2\lambda_o - 2\lambda_s)}{\lambda_o + \lambda_s + \lambda_c}\left[\frac{1}{e^{(\lambda_o+\lambda_s)t}} - \frac{1}{e^{(2\lambda_o+2\lambda_s+\lambda_c)t}}\right] \tag{4.31}$$

The MTTF difference between the standby system and the series system can be obtained by using Equations (4.24) and (4.26).

$$\Delta MTTF = MTTF_p - MTTF_s = \frac{2\lambda_o - 2\lambda_s}{(\lambda_o + \lambda_s)(2\lambda_o + 2\lambda_s + \lambda_c)} \tag{4.32}$$

Equations (4.31) and (4.32) indicate that if the open mode failure rate is greater than the short mode failure rate, then a parallel system will have higher reliability and higher MTTF than a series system and vice versa.

4.1.2 System availability comparison with Type I Repair

Figure 4.6 and Figure 4.7 are state space diagrams for the parallel and the series system with Type I Repair respectively.

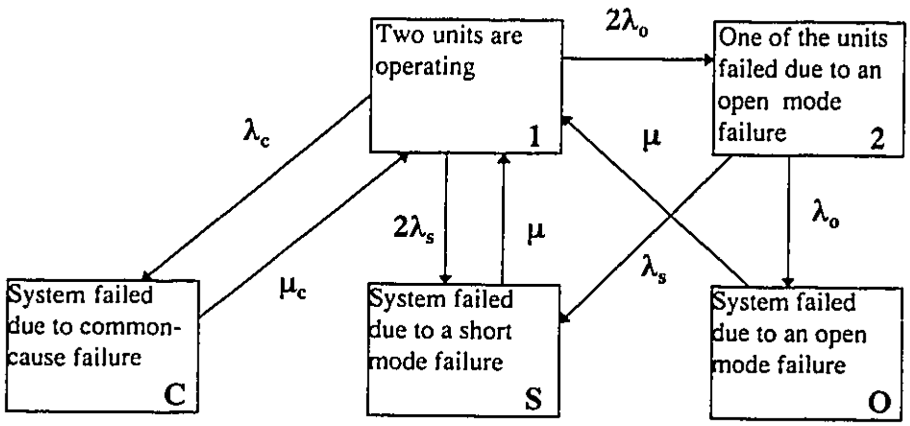


Figure 4.6 The state space diagram for a two unit in parallel system with Type I Repair

The following differential equations are associated with the model in Figure 4.6

$$\frac{dP_1(t)}{dt} = -(2\lambda_o + 2\lambda_s + \lambda_c)P_1(t) + \mu[P_o(t) + P_s(t)] + \mu_c P_c(t) \tag{4.33}$$

$$\frac{dP_2(t)}{dt} = -(\lambda_o + \lambda_s)P_2(t) + 2\lambda_o P_1(t) \tag{4.34}$$

$$\frac{dP_s(t)}{dt} = -\mu P_s(t) + \lambda_s[2P_1(t) + P_2(t)] \tag{4.35}$$

$$\frac{dP_o(t)}{dt} = -\mu P_o(t) + \lambda_o P_2(t) \tag{4.36}$$

$$\frac{dP_c(t)}{dt} = -\mu_c P_c(t) + \lambda_c P_1(t) \tag{4.37}$$

At time $t = 0$, $P_1(0) = 1$, and all the other initial condition probabilities are equal to zero.

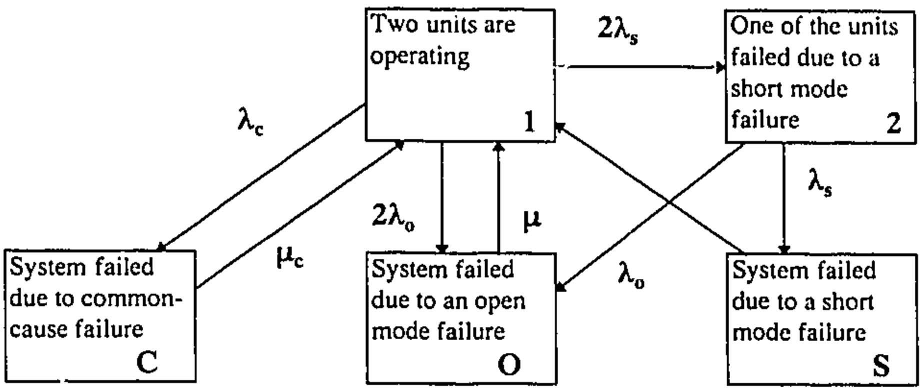


Figure 4.7 The state space diagram for a two unit in series system with Type I Repair

The following differential equations are associated with the model in Figure 4.7

$$\frac{dP_1(t)}{dt} = -(2\lambda_o + 2\lambda_s + \lambda_c)P_1(t) + \mu[P_o(t) + P_s(t)] + \mu_c P_c(t) \tag{4.38}$$

$$\frac{dP_2(t)}{dt} = -(\lambda_o + \lambda_s)P_2(t) + 2\lambda_s P_1(t) \tag{4.39}$$

$$\frac{dP_o(t)}{dt} = -\mu P_o(t) + \lambda_o[2P_1(t) + P_2(t)] \tag{4.40}$$

$$\frac{dP_s(t)}{dt} = -\mu P_s(t) + \lambda_s P_2(t) \tag{4.41}$$

$$\frac{dP_c(t)}{dt} = -\mu_c P_c(t) + \lambda_c P_1(t) \tag{4.42}$$

At time $t = 0$, $P_1(0) = 1$, and all the other initial condition probabilities are equal to zero.

The state space diagram of the standby system with Type I Repair is shown in Figure 2.20. The differential equations are given by (2.128) - (2.133). At time $t = 0$, $P_1(0) = 1$, and all other initial condition probabilities are equal to zero.

Case I: Open/Short mode failure without common-cause failure

In this case, common-cause failure is not considered. Both steady state availability and time dependent availability are compared.

Setting λ_c=0 and μ_c=0 in Equations (4.33) - (4.37) and by using the same method as in the previous chapter, the numerical examples of time dependent availability for the parallel system can be obtained. Also, the parallel system steady state availability can be obtained.

$$AV_p = \frac{(3\lambda_o + \lambda_s)\,\mu}{2\lambda_o^2 + 4\lambda_o\lambda_s + 2\lambda_s^2 + 3\lambda_o\,\mu + \lambda_s\,\mu} \tag{4.43}$$

Setting λ_c=0 and μ_c=0 in Equations (4.38) - (4.42) and by using the same method as in the previous chapter, the numerical examples of time dependent availability for the series system can be obtained. Also, the series system steady state availability can be obtained.

$$AV_s = \frac{(\lambda_o + 3\lambda_s)\,\mu}{2\lambda_o^2 + 4\lambda_o\lambda_s + 2\lambda_s^2 + \lambda_o\,\mu + 3\lambda_s\,\mu} \tag{4.44}$$

Setting λ_c=0 and μ_c=0 in Equations (2.128) - (2.133) and by using the same method as in the previous chapter, the numerical examples of time dependent availability for the standby system can be obtained. Also, the standby system steady state availability can be obtained.

$$AV_{sb} = \frac{2\mu}{\lambda_o + \lambda_s + 2\mu} \tag{4.45}$$

From Equations (4.45) and (4.43), the steady state availability difference between the standby system and the parallel system is given by

$$AV_{sb} - AV_p = \frac{(\lambda_o + 3\lambda_s)(\lambda_o + \lambda_s)\mu}{(\lambda_o + \lambda_s + 2\mu)(2\lambda_o^2 + 4\lambda_o\lambda_s + 2\lambda_s^2 + 3\lambda_o\mu + \lambda_s\mu)} \tag{4.46}$$

Similarly, from Equations (4.45) and (4.44), the steady state availability difference between the standby system and the series system is given by

$$AV_{sb} - AV_s = \frac{(3\lambda_o + \lambda_s)(\lambda_o + \lambda_s)\mu}{(\lambda_o + \lambda_s + 2\mu)(2\lambda_o^2 + 4\lambda_o\lambda_s + 2\lambda_s^2 + \lambda_o\mu + 3\lambda_s\mu)} \tag{4.47}$$

Also, from Equations (4.43) and (4.44), the steady state availability difference between the parallel system and the series system is given by

$$AV_p - AV_s = \frac{4(\lambda_o - \lambda_s)(\lambda_o + \lambda_s)^2\mu}{(2\lambda_o^2 + 4\lambda_o\lambda_s + 2\lambda_s^2 + 3\lambda_o\mu + \lambda_s\mu)(2\lambda_o^2 + 4\lambda_o\lambda_s + 2\lambda_s^2 + \lambda_o\mu + 3\lambda_s\mu)} \tag{4.48}$$

From Equations (4.46) and (4.47), it is evident that the standby system has higher steady state availability than the parallel and the series systems. Equation (4.48) indicates that if the open mode failure rate of unit is higher than the short mode failure rate, then a parallel system has better steady state availability than a series system.

The comparison plots of the time dependent availability for the parallel, series and standby systems are shown in Figure 4.8. It can be observed that at any time the standby system has higher availability than the parallel and the series systems.

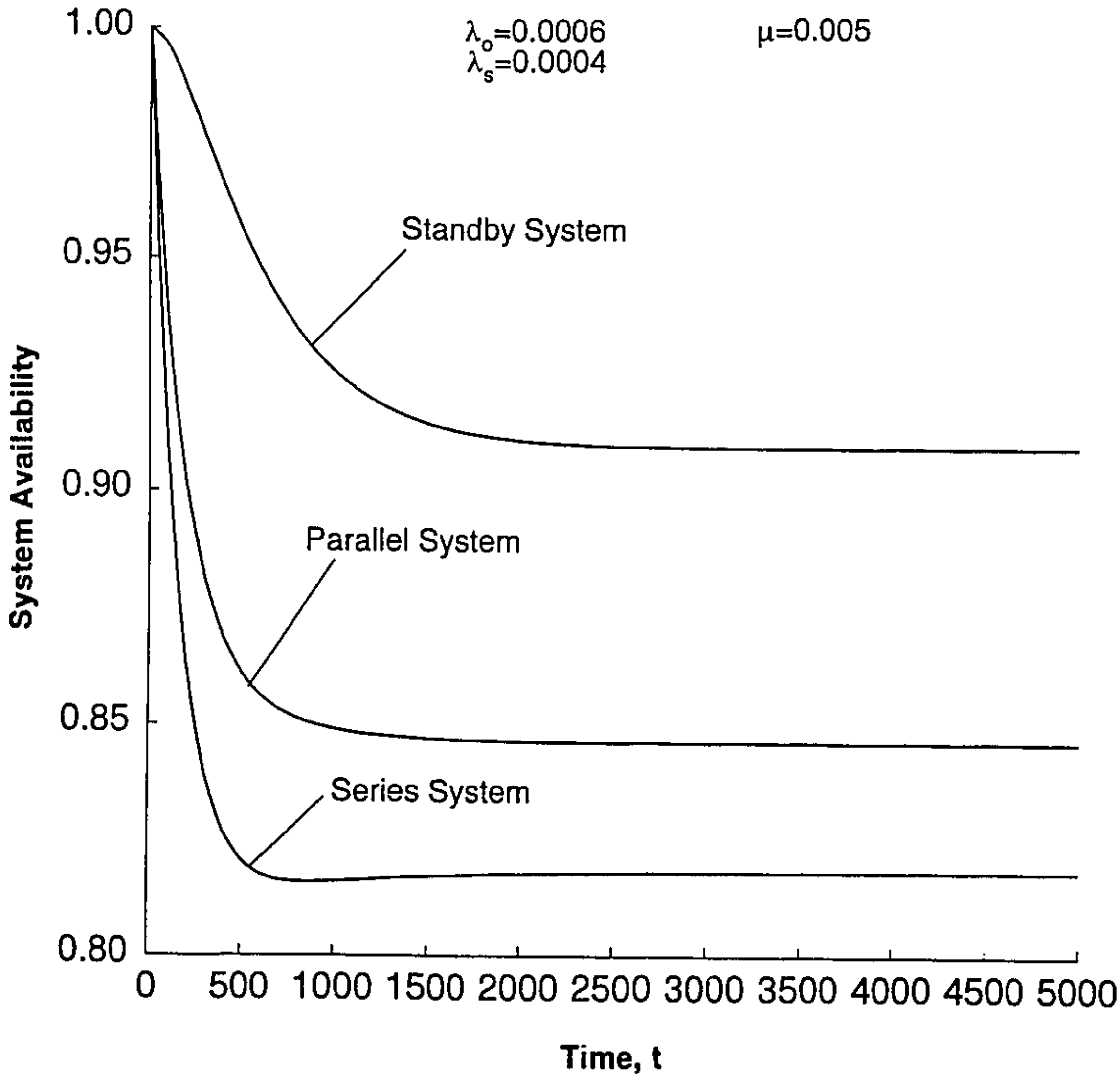


Figure 4.8 Avaliability comparison plots for the standby, parallel and series systems without common cause failure

Case II: Open/Short mode failure with common-cause failure

This case considers an open/short mode failure with common-cause failure and compares the steady state availability and the time dependent availability.

Solving Equations (4.33) - (4.37) by using the same method as in the previous chapter, the numerical examples of the time dependent availability for the parallel system can be obtained. The parallel system steady state availability is as follows:

$$AV_p = \frac{(3\lambda_o + \lambda_s)\,\mu\mu_c}{2(\lambda_o + \lambda_s)^2\,\mu\mu_c + (\lambda_o + \lambda_s)\lambda_c\,\mu + (3\lambda_o + \lambda_s)\,\mu\mu_c} \tag{4.49}$$

Solving Equations (4.38) - (4.42) by using the same method as in the previous chapter, the numerical examples of the time dependent availability for the series system can be obtained. The series system steady state availability is as follows:

$$AV_s = \frac{(\lambda_o + 3\lambda_s)\,\mu\mu_c}{2(\lambda_o + \lambda_s)^2\,\mu\mu_c + (\lambda_o + \lambda_s)\lambda_c\,\mu + (\lambda_o + 3\lambda_s)\,\mu\mu_c} \tag{4.50}$$

The standby system steady state availability is given by Equation (2.140). Using Equations (4.49) and (2.140), the steady state availability difference between the standby system and the parallel system is given by

$$AV_{sb} - AV_p = \frac{(\lambda_s\lambda_c\,\mu + \lambda_o^2\,\mu_c + 3\lambda_s^2\,\mu_c)\,\mu\mu_c + (4\lambda_s\,\mu_c - \lambda_c\,\mu)\lambda_o\,\mu\mu_c}{A(4,1)} \tag{4.51}$$

where

$$A(4,1) = [2(\lambda_o + \lambda_s)^2\,\mu\mu_c + (\lambda_o + \lambda_s)\lambda_c\,\mu + (3\lambda_o + \lambda_s)\,\mu\mu_c][(\lambda_o + \lambda_s)\,\mu_c + 2\,\mu\mu_c + \lambda_c\,\mu]$$

Similarly, using Equations (4.50) and (2.140), the steady state availability difference between the standby system and the series system is given by

$$A V_{sb} - A V_s = \frac{(\lambda_b \lambda_s \mu + 3\lambda_b^2 \mu_c + \lambda_s^2 \mu_c)\, \mu\mu_c + (4\lambda_b \mu_c - \lambda_c \mu)\lambda_s \mu\mu_c}{A(4,2)} \tag{4.52}$$

where

$$A(4,2) = [2(\lambda_b + \lambda_s)^2 \mu\mu_c + (\lambda_b + \lambda_s)\lambda_c \mu + (\lambda_b + 3\lambda_s)\, \mu\mu_c][(\lambda_b + \lambda_s)\, \mu_c + 2\mu\mu_c + \lambda_c \mu]$$

Since in practice, the value of the common-cause failure rate lies within the range of 1%-30% component's failure rate and the system repair rate for a common-cause failure is within the same range of system repair rates as for an open/short mode failure. Thus, in Equation (4.51), $4\lambda_s \mu_c - \lambda_c \mu \geq 0$ which indicates that the steady state availability of the standby system is higher than that of the parallel system. Using the same analogy, from Equation (4.52), it is evident that the steady state availability of the standby system is higher than that of the series system.

Also, comparing the Equations (4.49) and (4.50), the steady state availability difference between the parallel and the series systems is

$$A V_p - A V_s = \frac{2(\lambda_b^2 - \lambda_s^2)(\lambda_c \mu + 2\lambda_b \mu_c + 2\lambda_s \mu_c)\, \mu\mu_c}{A(4,3)} \tag{4.53}$$

where

$$A(4,3) = [2(\lambda_b + \lambda_s)^2 \mu\mu_c + (\lambda_b + \lambda_s)\lambda_c \mu + (\lambda_b + 3\lambda_s)\, \mu\mu_c] \bullet$$
$$2(\lambda_b + \lambda_s)^2 \mu\mu_c + (\lambda_b + \lambda_s)\lambda_c \mu + (3\lambda_b + \lambda_s)\, \mu\mu_c]$$

Equation (4.53) indicates that if the open mode failure rate of a unit is higher than the short mode failure rate, then the steady state availability of the parallel system is higher than that of series system and vice versa.

The comparison plots of the time dependent availability for the parallel, series and standby systems are shown in Figure 4.9. It can be observed that at any time the standby system has higher availability than the parallel or series systems.

The results of analysis in this section indicate that for a two three-state device system, standby configuration can attain the highest system reliability, MTTF and availability. If the open mode failure rate of a device is higher than a short mode failure rate, the parallel configuration has higher reliability, MTTF and availability than the series configuration and vice versa.

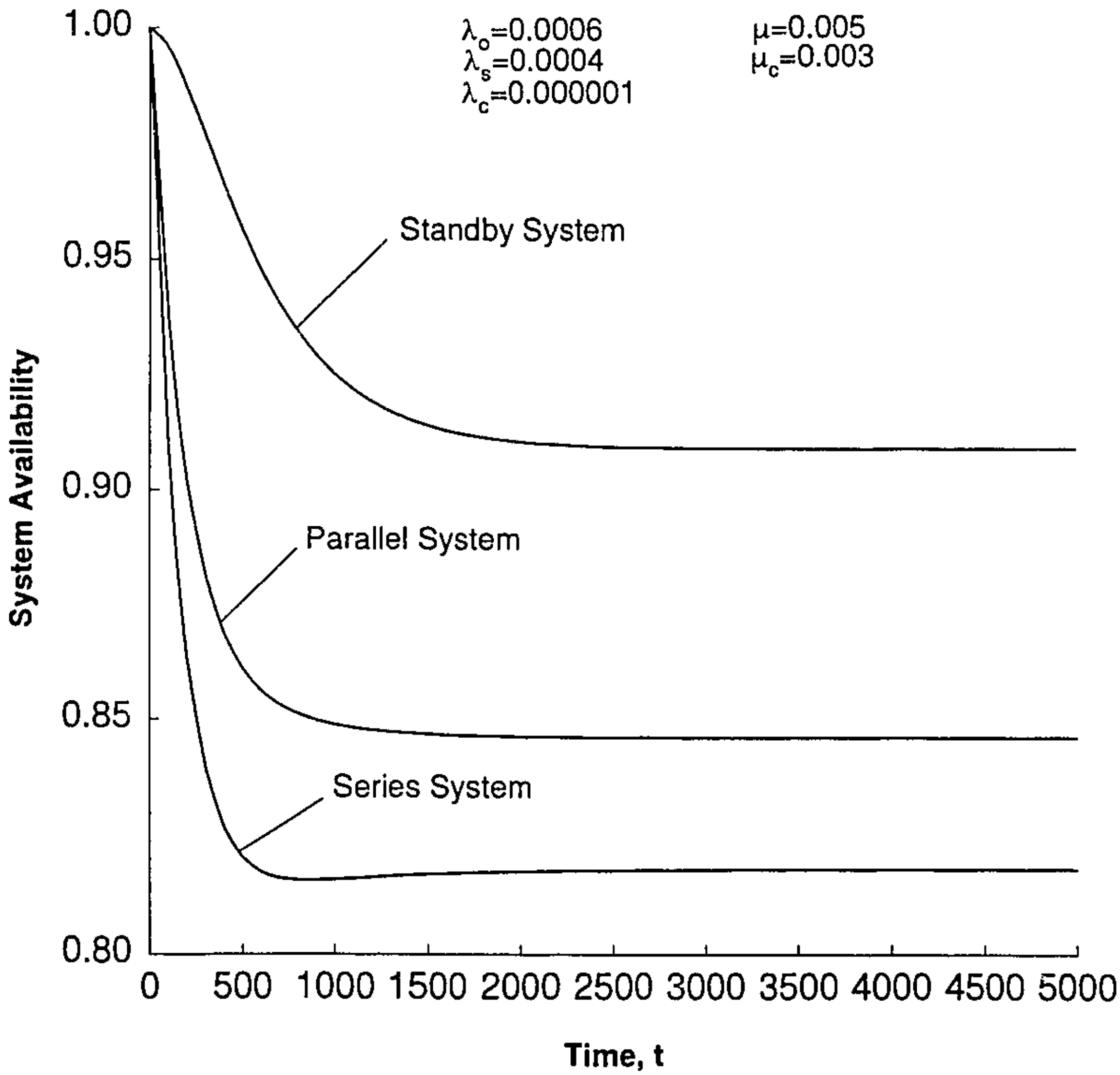


Figure 4.9 Avaliability comparison plots for the standby, parallel and series systems with common cause failure

4.2 Standby system reliability improvement

Under the conditions when the open mode failure rate of a unit is greater than its short mode failure rate, two units arranged in parallel would offer a higher reliability than a single unit (all units are identical). This is examined in Appendix 4.2. In order to improve the standby system reliability, the new configuration is introduced as shown in Figure 4.10. In Figure 4.10, there are two units in parallel with one unit on standby (all units are identical). As long as one of the two units fails due to a short mode failure or both units fail due to an open mode failure, the standby unit starts to operate.

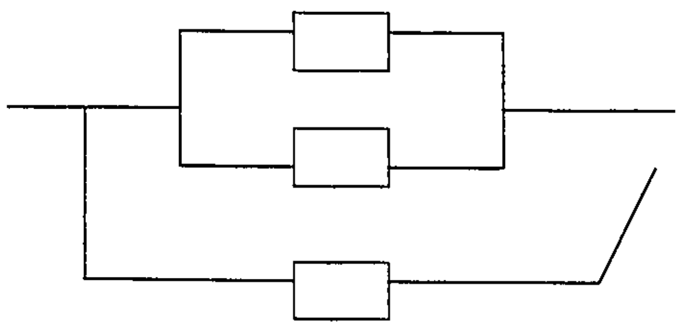


Figure 4.10 Block diagram of two units in parallel with one unit on standby system

Notation:

The following symbols are associated with the system of two units in parallel with one unit on standby.

t time.

s Laplace transform variable

λ_o Constant open-mode failure rate of an operating unit.

λ_s Constant short-mode failure rate of an operating unit.

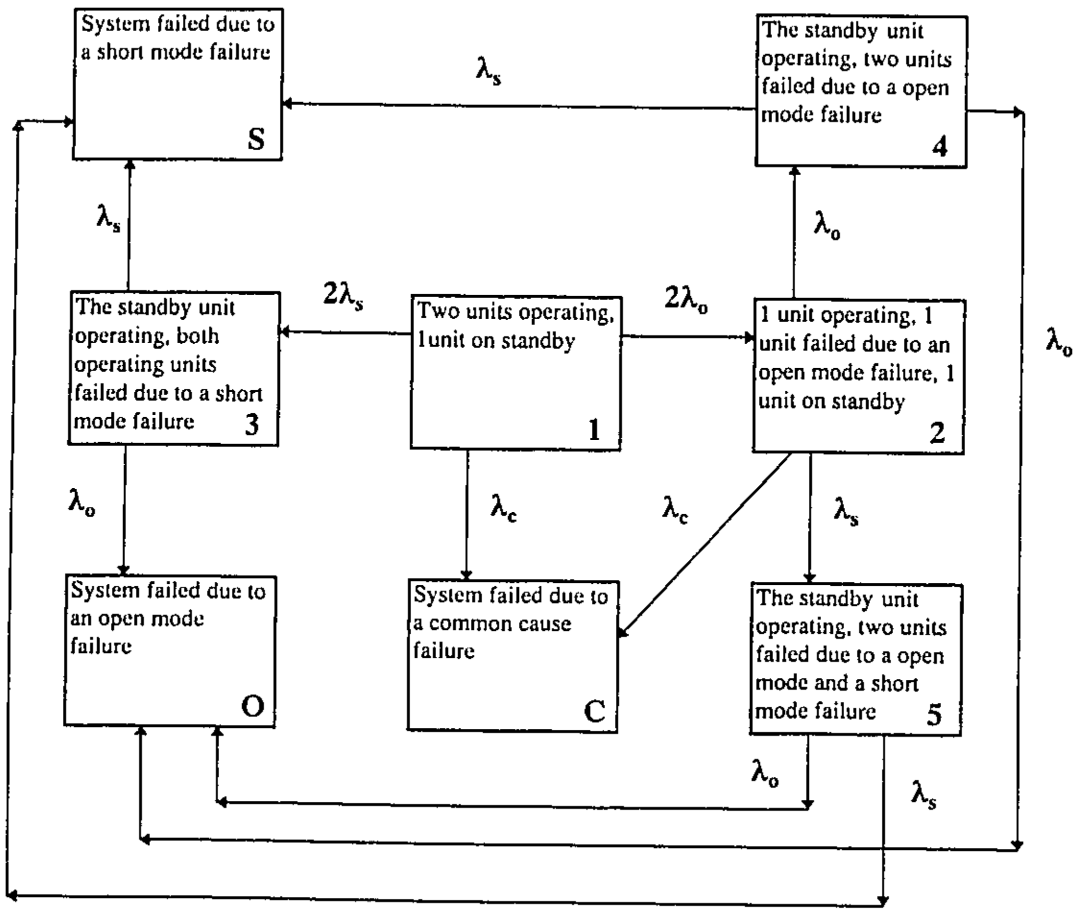


Figure 4.11 System state space diagram without Repair: two units in paralle! with one unit on standby system

λ_c Constant common-cause failure rate of system.

μ Constant repair rate for failed system due to an open or a short mode failure.

μ_c Constant repair rate for failed system due to a common-cause failure rate.

$P_1(t)$ Probability that the system is in the state 1.

$P_2(t)$ Probability of state that one of the operating units failed due to an open mode failure.

$P_3(t)$ Probability of state that the standby unit is operating and both operating units failed due to a short mode failure.

$P_4(t)$ Probability of state that the standby unit is operating and two operating units failed due to an open mode failure.

$P_5(t)$ Probability of state that standby unit is operating and two operating units failed due to an open mode failure and a short mode failure respectively.

$P_s(t)$ Probability that the system is in the failed state due to a short mode failure.

$P_o(t)$ Probability that the system is in the failed state due to an open mode failure.

$P_c(t)$ Probability that the system is in the failed state due to a common-cause failure.

$R(t)$ System reliability.

$AV(t)$ System availability.

$MTTF$ System mean time to failure.

AV_{ss} System steady state availability.

4.2.1 Two units in parallel with one unit on standby system without Repair

Figure 4.11 is state space diagram of the two units in parallel with one unit on standby system without repair. The system of differential equations associated with Figure 4.11 is as follows:

$$\frac{dP_1(t)}{dt} = -(2\lambda_o + 2\lambda_s + \lambda_c)P_1(t) \tag{4.54}$$

$$\frac{dP_2(t)}{dt} = -(\lambda_o + \lambda_s + \lambda_c)P_2(t) + 2\lambda_o P_1(t) \tag{4.55}$$

$$\frac{dP_3(t)}{dt} = -(\lambda_o + \lambda_s)P_3(t) + 2\lambda_s P_1(t) \tag{4.56}$$

$$\frac{dP_4(t)}{dt} = -(\lambda_o + \lambda_s)P_4(t) + \lambda_o P_2(t) \tag{4.57}$$

$$\frac{dP_5(t)}{dt} = -(\lambda_o + \lambda_s)P_5(t) + \lambda_s P_2(t) \tag{4.58}$$

$$\frac{dP_s(t)}{dt} = \lambda_s[P_3(t) + P_4(t) + P_5(t)] \tag{4.59}$$

$$\frac{dP_o(t)}{dt} = \lambda_o[P_3(t) + P_4(t) + P_5(t)] \tag{4.60}$$

$$\frac{dP_c(t)}{dt} = \lambda_c[P_1(t) + P_2(t)] \tag{4.61}$$

At time $t = 0$, $P_1(0) = 1$, and all the other initial condition probabilities are equal to zero.

By taking Laplace transforms of Equations (4.54)-(4.61) and solving the resulting equations, we can obtained the Laplace transforms of the state probabilities. Inverting these Laplace transforms of the state probabilities, the time dependent state probabilities result:

$$P_1(t) = \frac{1}{e^{(2\lambda_o + 2\lambda_s + \lambda_c)t}} \tag{4.62}$$

$$P_2(t) = \frac{2\lambda_o}{(\lambda_o + \lambda_s)}\left[\frac{1}{e^{(\lambda_o + \lambda_s + \lambda_c)t}} - \frac{1}{e^{(2\lambda_o + 2\lambda_s + \lambda_c)t}}\right] \tag{4.63}$$

$$P_3(t) = \frac{2\lambda_s}{(\lambda_o + \lambda_s + \lambda_c)}\left[\frac{1}{e^{(\lambda_o + \lambda_s)t}} - \frac{1}{e^{(2\lambda_o + 2\lambda_s + \lambda_c)t}}\right] \tag{4.64}$$

$$P_4(t) = \frac{-2\lambda_o^2}{e^{(\lambda_o + \lambda_s + \lambda_c)t}(\lambda_o + \lambda_s)\lambda_c} + \frac{2\lambda_o^2}{e^{(2\lambda_o + 2\lambda_s + \lambda_c)t}(\lambda_o + \lambda_s)(\lambda_o + \lambda_s + \lambda_c)} + \frac{2\lambda_o^2}{e^{(\lambda_o + \lambda_s)t}(\lambda_o + \lambda_s + \lambda_c)\lambda_c} \tag{4.65}$$

$$P_5(t)=\frac{-2\lambda_o\lambda_s}{e^{(\lambda_o+\lambda_s+\lambda_c)t}(\lambda_o+\lambda_s)\lambda_c}+\frac{2\lambda_o\lambda_s}{e^{(2\lambda_o+2\lambda_s+\lambda_c)t}(\lambda_o+\lambda_s)(\lambda_o+\lambda_s+\lambda_c)}+\frac{2\lambda_o\lambda_s}{e^{(\lambda_o+\lambda_s)t}(\lambda_o+\lambda_s+\lambda_c)\lambda_c}$$

(4.66)

$$P_s(t)=\frac{2\lambda_o\lambda_s}{e^{(\lambda_o+\lambda_s+\lambda_c)t}(\lambda_o+\lambda_s+\lambda_c)\lambda_c}+\frac{2\lambda_s(-\lambda_o+\lambda_s)}{e^{(2\lambda_o+2\lambda_s+\lambda_c)t}(\lambda_o+\lambda_s+\lambda_c)(2\lambda_o+2\lambda_s+\lambda_c)}+$$
$$\frac{-2\lambda_s(\lambda_o^2+\lambda_o\lambda_s+\lambda_s\lambda_c)}{e^{(\lambda_o+\lambda_s)t}(\lambda_o+\lambda_s+\lambda_c)(\lambda_o+\lambda_s)\lambda_c}+\frac{2\lambda_s(\lambda_o^2+2\lambda_o\lambda_s+\lambda_s^2+\lambda_s\lambda_c)}{(2\lambda_o+2\lambda_s+\lambda_c)(\lambda_o+\lambda_s+\lambda_c)(\lambda_o+\lambda_s)}$$

(4.67)

$$P_o(t)=\frac{2\lambda_o^2}{e^{(\lambda_o+\lambda_s+\lambda_c)t}(\lambda_o+\lambda_s+\lambda_c)\lambda_c}+\frac{2\lambda_o(-\lambda_o+\lambda_s)}{e^{(2\lambda_o+2\lambda_s+\lambda_c)t}(\lambda_o+\lambda_s+\lambda_c)(2\lambda_o+2\lambda_s+\lambda_c)}+$$
$$\frac{-2\lambda_o(\lambda_o^2+\lambda_o\lambda_s+\lambda_s\lambda_c)}{e^{(\lambda_o+\lambda_s)t}(\lambda_c+\lambda_s+\lambda_c)(\lambda_o+\lambda_s)\lambda_c}+\frac{2\lambda_o(\lambda_o^2+2\lambda_o\lambda_s+\lambda_s^2+\lambda_s\lambda_c)}{(2\lambda_o+2\lambda_s+\lambda_c)(\lambda_o+\lambda_s+\lambda_c)(\lambda_o+\lambda_s)}$$

(4.68)

$$P_c(t)=\frac{2\lambda_b\lambda_c}{e^{(\lambda_b+\lambda_s+\lambda_c)t}(\lambda_b+\lambda_s+\lambda_c)(\lambda_b+\lambda_s)}+\frac{\lambda_c(\lambda_b-\lambda_s)}{e^{(2\lambda_b+2\lambda_s+\lambda_c)t}(\lambda_b+\lambda_s)(2\lambda_b+2\lambda_s+\lambda_c)}+$$
$$\frac{\lambda_c(3\lambda_b+\lambda_s+\lambda_c)}{(2\lambda_b+2\lambda_s+\lambda_c)(\lambda_b+\lambda_s+\lambda_c)} \qquad (4.69)$$

The system reliability is given by

$$R(t)=P_1(t)+P_2(t)+P_3(t)+P_4(t)+P_5(t)$$
$$=\frac{2(\lambda_{b_o}^2+\lambda_o\lambda_s+\lambda_s\lambda_c)}{e^{(\lambda_o+\lambda_s)t}(\lambda_o+\lambda_s+\lambda_c)\lambda_c}+\frac{2\lambda_o(-\lambda_o-\lambda_s+\lambda_c)}{e^{(\lambda_o+\lambda_s+\lambda_c)t}(\lambda_o+\lambda_s)\lambda_c}+\frac{\lambda_o^2-\lambda_s^2-\lambda_o\lambda_s+\lambda_s\lambda_c}{e^{(2\lambda_o+2\lambda_s+\lambda_c)t}(\lambda_o+\lambda_s+\lambda_c)(\lambda_o+\lambda_s)}$$

(4.70)

The system mean time to failure (MTTF) is

$$MTTF=\lim_{s\to 0}R(s)$$
$$=P_1(s)+P_2(s)+P_3(s)+P_4(s)+P_5(s)$$
$$=\frac{(5\lambda_b^2+8\lambda_b\lambda_s+3\lambda_s^2+\lambda_b\lambda_c+3\lambda_s\lambda_c)}{(2\lambda_b+2\lambda_s+\lambda_c)(\lambda_b+\lambda_s+\lambda_c)(\lambda_b+\lambda_s)} \qquad (4.71)$$

Comparing with the system in which one unit operating and the other one on standby, the reliability difference between the system represented by Equations (4.70) and the system represented by Equation (2.37) is as follows:

$$\Delta R = \frac{2(\lambda_{o_o}^2 + \lambda_o\lambda_s + \lambda_s\lambda_c)}{e^{(\lambda_o+\lambda_s)t}(\lambda_o + \lambda_s + \lambda_c)\lambda_c} + \frac{2\lambda_o(-\lambda_o - \lambda_s + \lambda_c)}{e^{(\lambda_o+\lambda_s+\lambda_c)t}(\lambda_o + \lambda_s)\lambda_c} + \frac{\lambda_o^2 - \lambda_s^2 - \lambda_o\lambda_s + \lambda_s\lambda_c}{e^{(2\lambda_o+2\lambda_s+\lambda_c)t}(\lambda_o + \lambda_s + \lambda_c)(\lambda_o + \lambda_s)} -$$

$$\frac{(\lambda_o + \lambda_s)}{e^{(\lambda_o+\lambda_s)t}\lambda_c} - \frac{\lambda_o + \lambda_s - \lambda_o}{e^{(\lambda_o+\lambda_s+\lambda_c)t}\lambda_c}]$$

$$= \frac{(\lambda_o - \lambda_s)(\lambda_o + \lambda_s - \lambda_c)[e^{(\lambda_o+\lambda_s+\lambda_c)t}(\lambda_o + \lambda_s) - e^{(\lambda_o+\lambda_s)t}(\lambda_o + \lambda_s + \lambda_c) + \lambda_c]}{e^{(2\lambda_o+2\lambda_s+\lambda_c)t}(\lambda_o + \lambda_s + \lambda_c)(\lambda_o + \lambda_s)\lambda_c} \qquad (4.72)$$

Since $e^{(\lambda_o+\lambda_s+\lambda_c)t}(\lambda_o + \lambda_s) - e^{(\lambda_o+\lambda_s)t}(\lambda_o + \lambda_s + \lambda_c) + \lambda_c \geq 0$, the proof is shown in Appendix 4.1-(v), and $\lambda_o + \lambda_s \geq \lambda_c$, so if $\lambda_o \geq \lambda_s$, $\Delta R > 0$ which indicates the reliability of two units in parallel with one unit on standby system is higher than that of the one unit on standby system.

Further considering the mean time to failure (MTTF) of the two system, the difference in the MTTF between system represented by Equation (4.71) and system represented by Equation (2.38) is as follows:

$$\Delta MTTF = \frac{(5\lambda_o^2 + 8\lambda_o\lambda_s + 3\lambda_s^2 + \lambda_o\lambda_c + 3\lambda_s\lambda_c)}{(2\lambda_o + 2\lambda_s + \lambda_c)(\lambda_o + \lambda_s + \lambda_c)(\lambda_o + \lambda_s)} - \frac{2}{\lambda_o + \lambda_s + \lambda_c}$$

$$= \frac{(\lambda_o - \lambda_s)(\lambda_o + \lambda_s - \lambda_c)}{(2\lambda_o + 2\lambda_s + \lambda_c)(\lambda_o + \lambda_s + \lambda_c)(\lambda_o + \lambda_s)} \qquad (4.73)$$

Equation (4.73) indicates that if the open mode failure rate of a unit is greater than its short mode failure rate, $\Delta MTTF > 0$. Since the common-cause failure rate is much less than the open failure rate or the short mode failure rate. Thus, the MTTF of the two units in parallel with one unit on standby system is higher than the MTTF of the one unit on standby system.

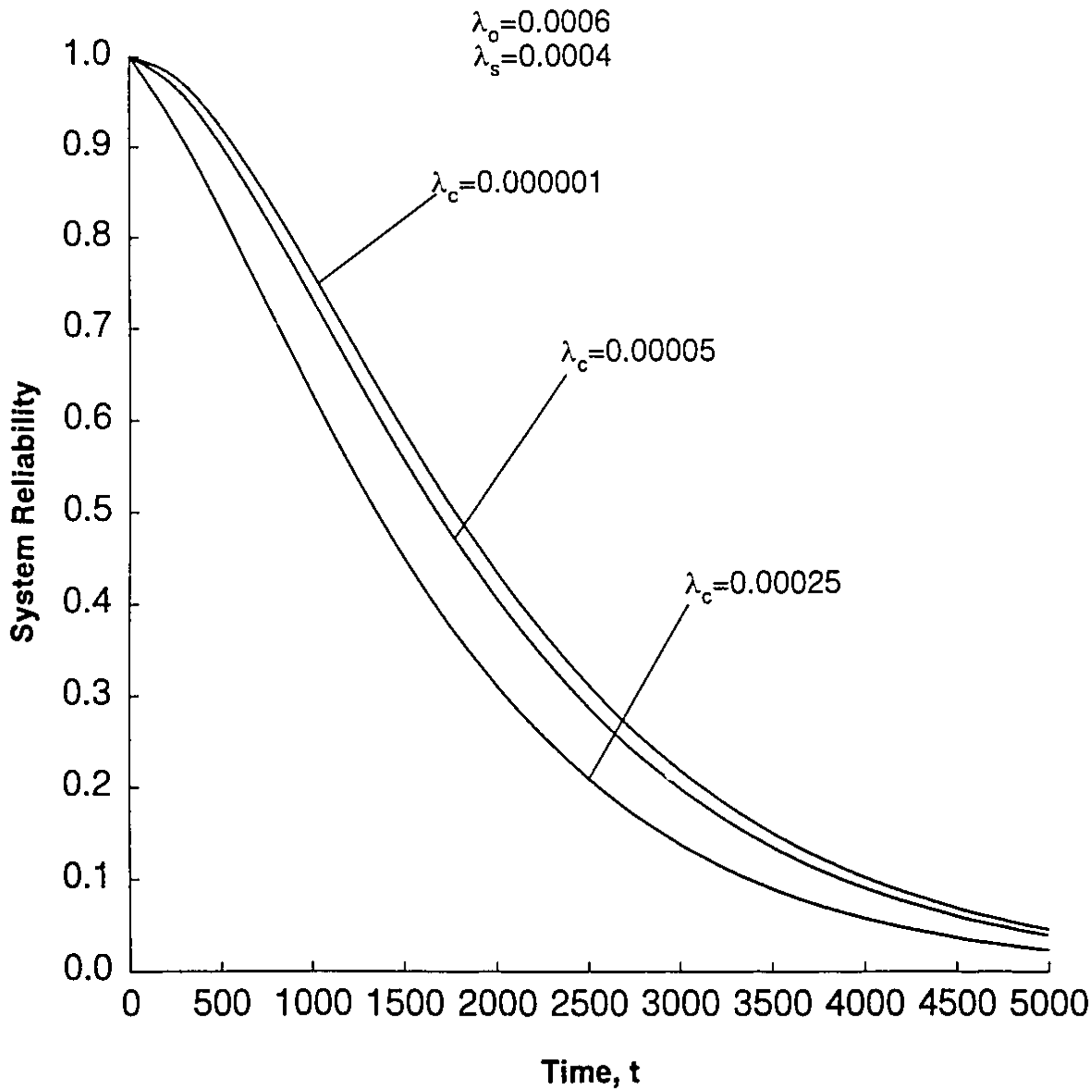


Figure 4.12 The standby system without repair reliability plots: two units in parallel with one unit on standby

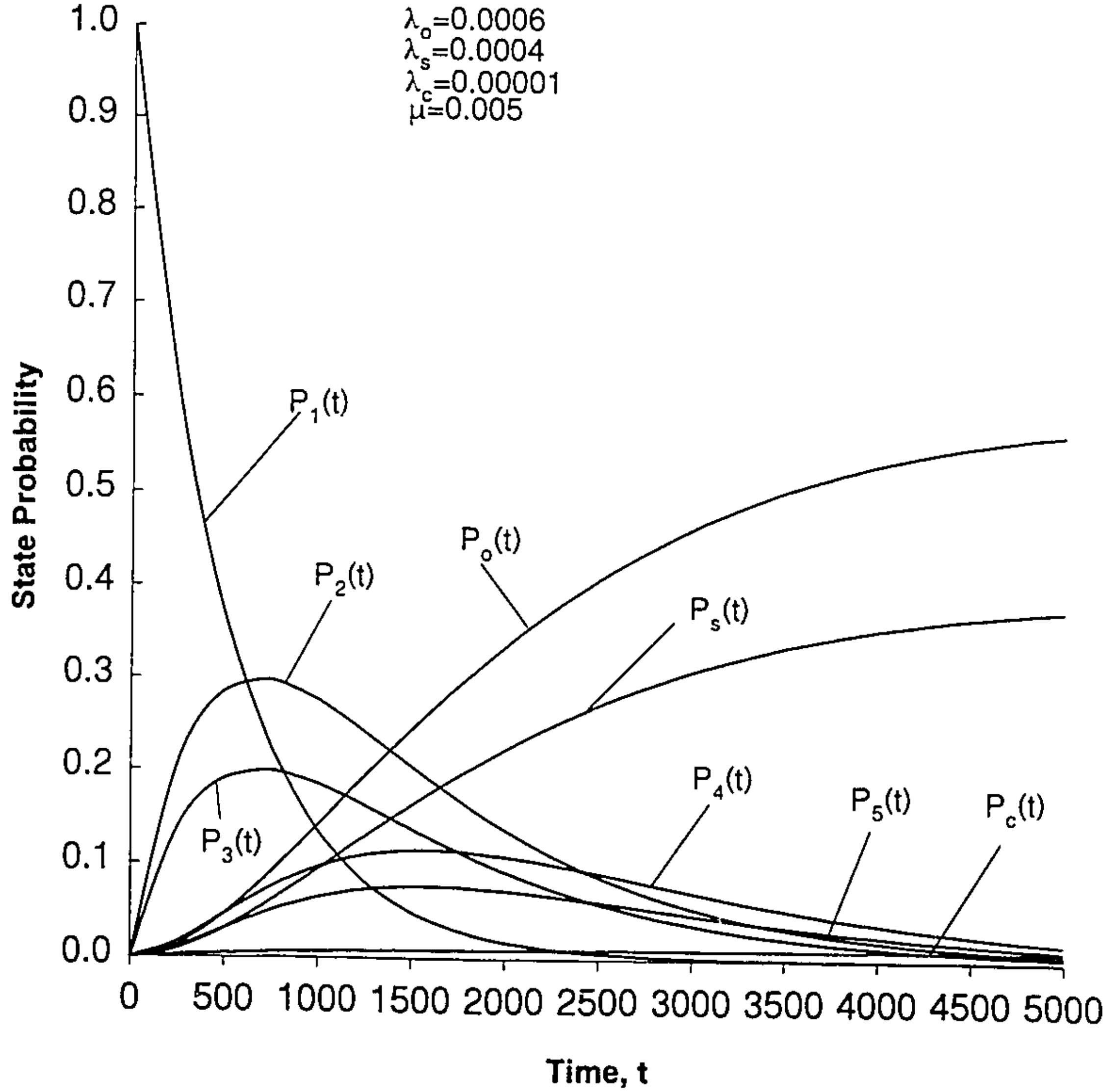


Figure 4.13 The standby system without repair state probability plots: two units in parallel with one unit on standby

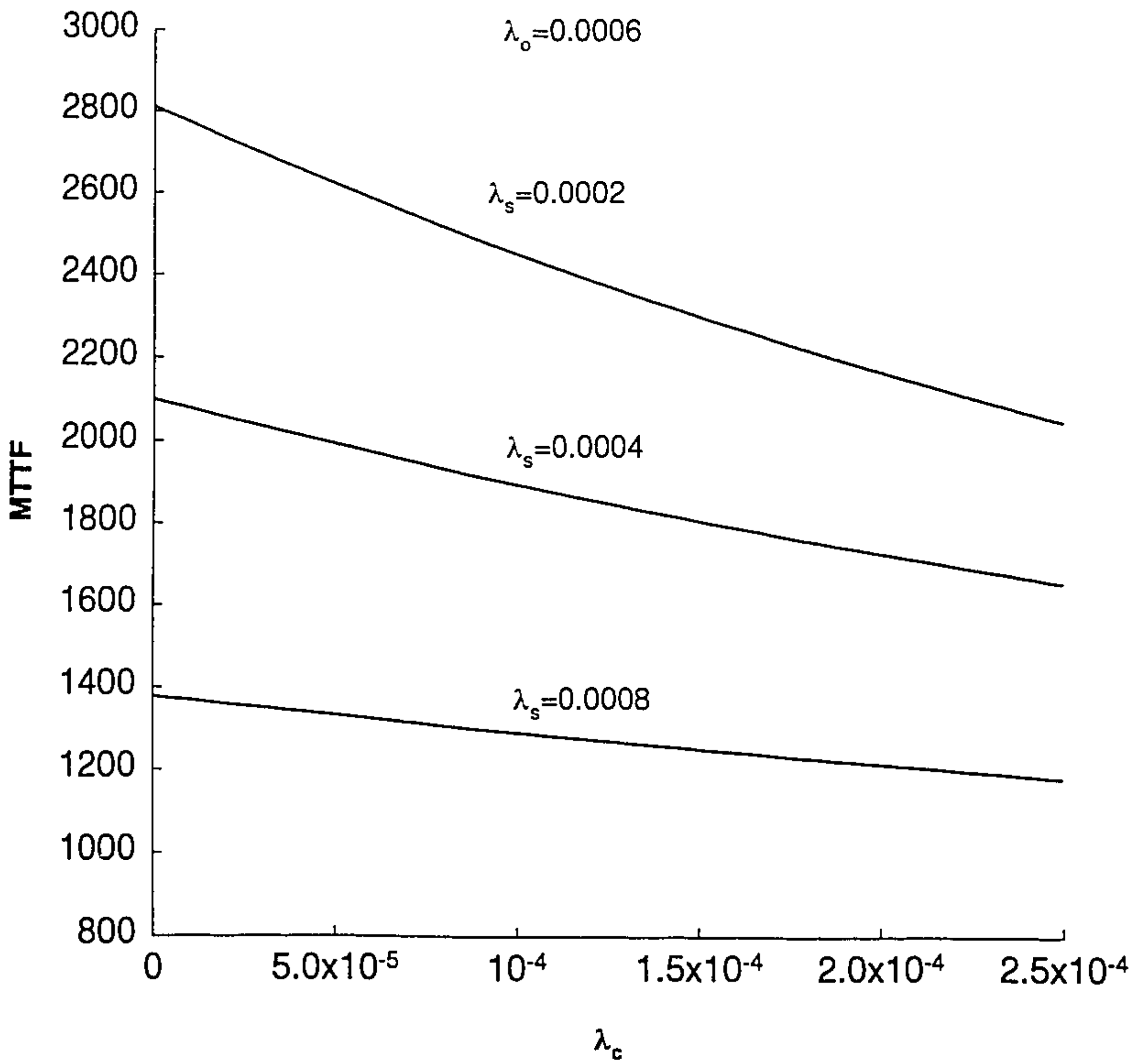


Figure 4.14 MTTF vs common cause failure rate, λ_c for the standby system (two units in parallel with one unit on standby) without repair

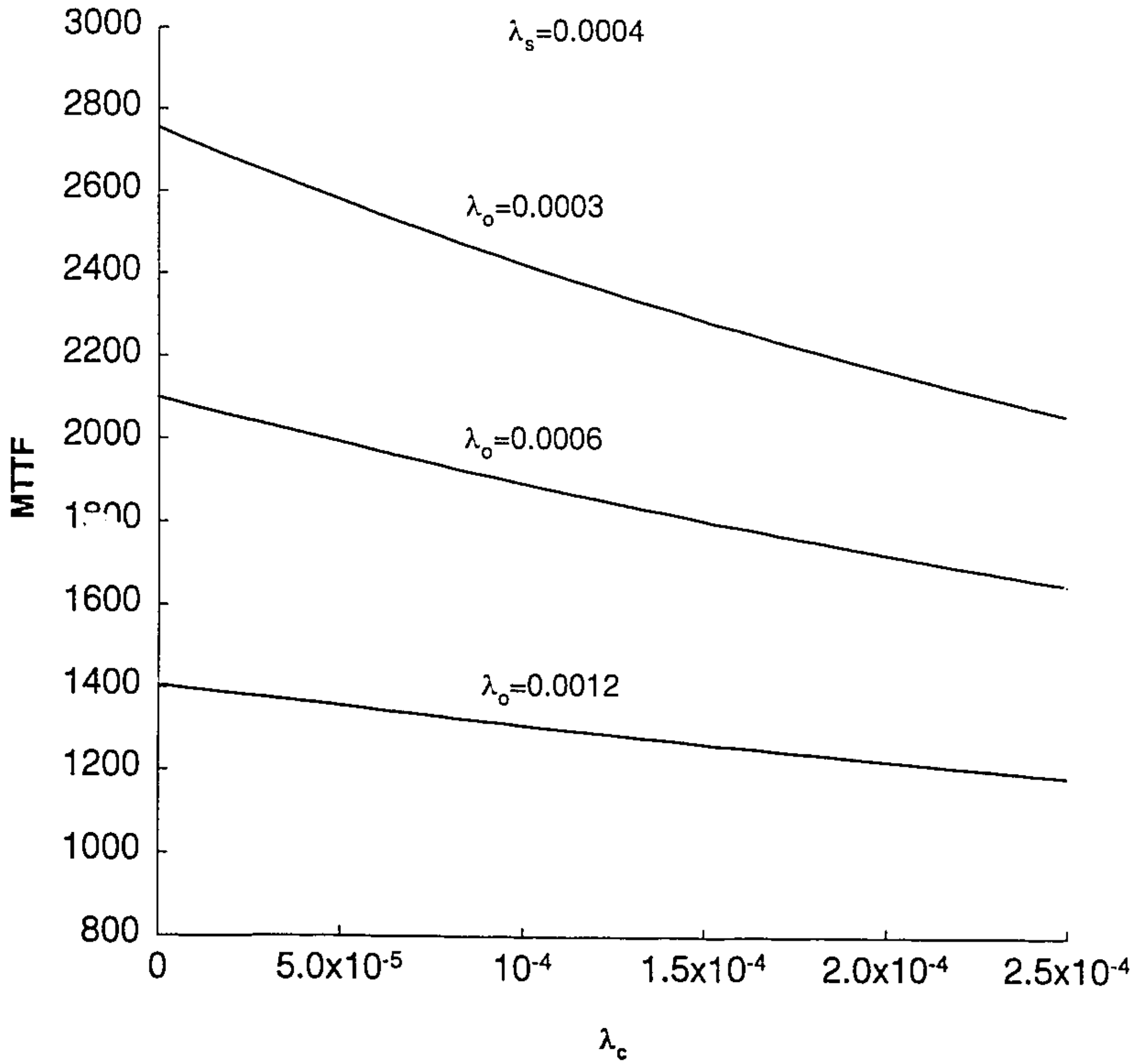


Figure 4.15 MTTF vs common cause failure rate, λ_c for the standby system (two units in parallel one unit on standby) without repair

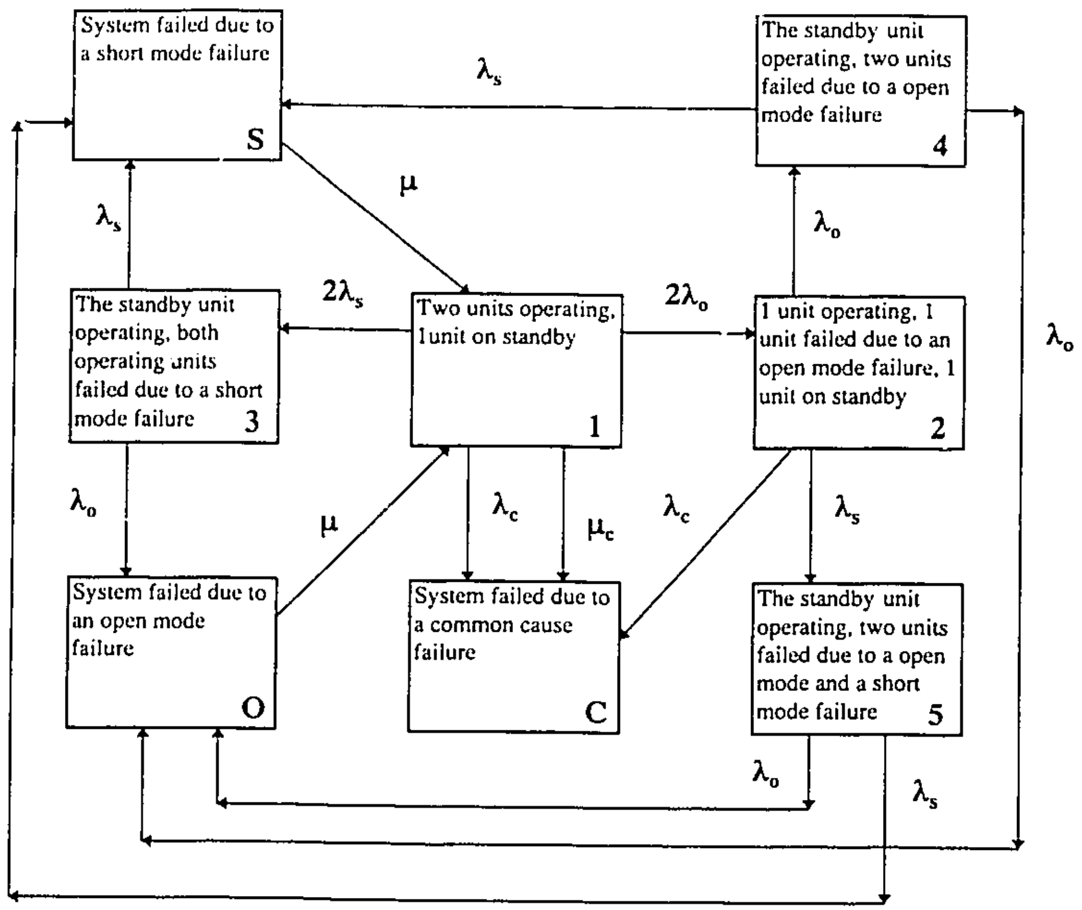


Figure 4.16. System state space diagram with Type I Repair: two units in parallel with one unit on standby system

In order to make a comparison with the one unit on standby system, the same values of parameters (i.e. as for Equations (2.31)-(2.38)) are chosen for the reliability plots, the state probability plots, and the mean time to failure plots. The system reliability given by Equation (4.70) is plotted in Figure 4.12. The time dependent state probabilities plots are shown in Figure 4.13. The plots of Equation (4.72) are shown in Figure 4.14 and Figure 4.15.

4.2.2 Two units in parallel with one unit on standby system with Type I Repair

Figure 4.16 is the state space diagram of the two units in parallel with one unit on standby system with Type I Repair. The system of differential equations associated with Figure 4.16 is as follows:

$$\frac{dP_1(t)}{dt} = -(2\lambda_o + 2\lambda_s + \lambda_c)P_1(t) + \mu[P_o(t) + P_s(t)] + \mu_c P_c(t) \quad (4.74)$$

$$\frac{dP_2(t)}{dt} = -(\lambda_o + \lambda_s + \lambda_c)P_2(t) + 2\lambda_o P_1(t) \quad (4.75)$$

$$\frac{dP_3(t)}{dt} = -(\lambda_o + \lambda_s)P_3(t) + 2\lambda_s P_1(t) \quad (4.76)$$

$$\frac{dP_4(t)}{dt} = -(\lambda_o + \lambda_s)P_4(t) + \lambda_o P_2(t) \quad (4.77)$$

$$\frac{dP_5(t)}{dt} = -(\lambda_o + \lambda_s)P_5(t) + \lambda_s P_2(t) \quad (4.78)$$

$$\frac{dP_s(t)}{dt} = -\mu P_s(t) + \lambda_s[P_3(t) + P_4(t) + P_5(t)] \quad (4.79)$$

$$\frac{dP_o(t)}{dt} = -\mu P_o(t) + \lambda_o[P_3(t) + P_4(t) + P_5(t)] \quad (4.80)$$

$$\frac{dP_c(t)}{dt} = -\mu_c P_c(t) + \lambda_t [P_1(t) + P_2(t)] \tag{4.81}$$

At time $t = 0$, $P_1(0) = 1$, and all the other initial condition probabilities are equal to zero.

The steady state probabilities can be obtained by setting the derivatives of above Equations (4.74)-(4.81) equal to zero and utilizing the relationship $\sum P_j = 1$, j represents all of the states in the system.

$$P_1 = \frac{(\lambda_b + \lambda_s)(\lambda_b + \lambda_s + \lambda_t)\,\mu\mu_c}{A(4,4)} \tag{4.82}$$

$$P_2 = \frac{2\lambda_b(\lambda_b + \lambda_s + \lambda_t)\,\mu\mu_c}{A(4,4)} \tag{4.83}$$

$$P_3 = \frac{2\lambda_s(\lambda_b + \lambda_s + \lambda_t)\,\mu\mu_c}{A(4,4)} \tag{4.84}$$

$$P_4 = \frac{2\lambda_b^2\,\mu\mu_c}{A(4,4)} \tag{4.85}$$

$$P_5 = \frac{2\lambda_b\lambda_s\,\mu\mu_c}{A(4,4)} \tag{4.86}$$

$$P_s = \frac{2\lambda_s\mu_c[(\lambda_b + \lambda_s)^2 + \lambda_s\lambda_t]}{A(4,4)} \tag{4.87}$$

$$P_o = \frac{2\lambda_b\mu_c[(\lambda_b + \lambda_s)^2 + \lambda_s\lambda_t]}{A(4,4)} \tag{4.88}$$

$$P_c = \frac{\lambda_t(\lambda_b + \lambda_s)(3\lambda_b + \lambda_s + \lambda_t)\,\mu}{A(4,4)} \tag{4.89}$$

The system steady state availability is then given by

$$\begin{aligned} AV_{ss} &= P_1 + P_2 + P_3 + P_4 + P_5 \\ &= \frac{(5\lambda_b^2 + 8\lambda_b\lambda_s + 3\lambda_s^2 + \lambda_o\lambda_c + 3\lambda_s\lambda_c)\,\mu\mu_c}{A(4,4)} \end{aligned} \tag{4,90}$$

where A(4,4) is defined in Appendix 4.3.

By taking Laplace transforms of Equations (2.127) - (2.132) and solving the resulting equations, the Laplace transforms of the state probabilities are as follows:

$$P_1(s) = \frac{(\lambda_b + \lambda_s + s)(\lambda_b + \lambda_s + \lambda_t + s)(\mu + s)(\mu_c + s)}{A(4,5)} \tag{4.91}$$

$$P_2(s) = \frac{2\lambda_b(\lambda_b + \lambda_s + \lambda_t + s)(\mu + s)(\mu_c + s)}{A(4,5)} \tag{4.92}$$

$$P_3(s) = \frac{2\lambda_s(\lambda_b + \lambda_s + \lambda_t + s)(\mu + s)(\mu_c + s)}{A(4,2)} \tag{4.93}$$

$$P_4(s) = \frac{2\lambda_b^2(\mu + s)(\mu_c + s)}{A(4,2)} \tag{4.94}$$

$$P_5(s) = \frac{2\lambda_b\lambda_s(\mu + s)(\mu_c + s)}{A(4,2)} \tag{4.95}$$

$$P_s(s) = \frac{2\lambda_s(\mu_c + s)[(\lambda_b + \lambda_s)^2 + \lambda_s(\lambda_t + s)]}{A(4,2)} \tag{4.96}$$

$$P_o(s) = \frac{2\lambda_b(\mu_c + s)[(\lambda_b + \lambda_s)^2 + \lambda_s(\lambda_c + s)]}{A(4,2)} \tag{4.97}$$

$$P_c(s) = \frac{\lambda_t(\lambda_b + \lambda_s + s)(3\lambda_b + \lambda_s + \lambda_t + s)(\mu + s)}{A(4,2)} \tag{4.98}$$

The Laplace transform of the system availability is given by

$$\begin{aligned} AV(s) &= P_1(s) + P_2(s) + P_3(s) + P_4(s) + P_5(s) \\ &= \frac{A(4,6)}{A(4,5)} \end{aligned} \tag{4.99}$$

where A(4,5) and A(4,6) are defined in Appendix 4.3.

The time dependent availability and the state probabilities can be obtained by taking the inverse Laplace transforms of Equations (4.99) and Equations (4.91)-(4.98).

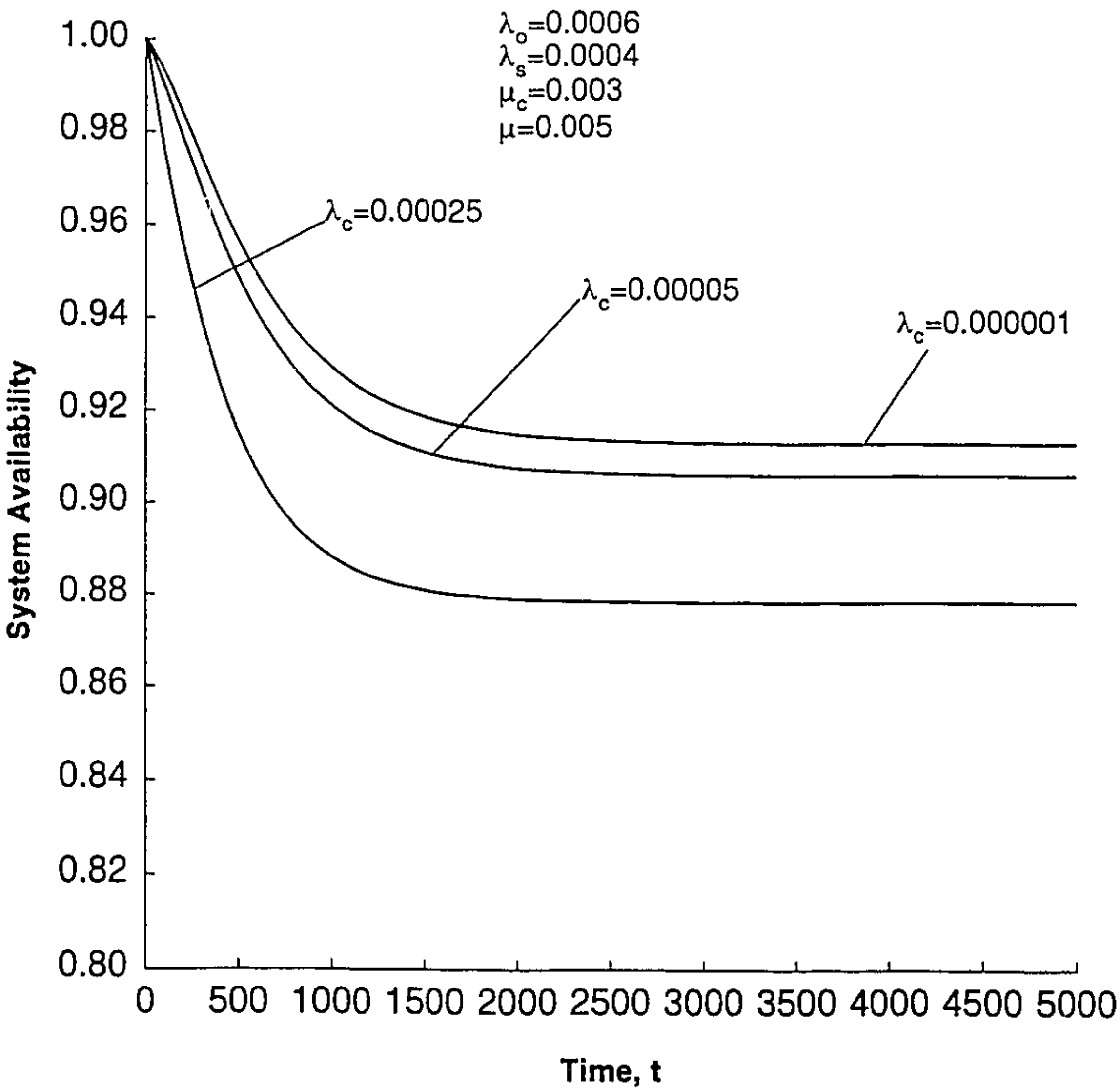


Figure 4.17 The standby system with type I repair availability plots: two units in parallel and one unit on standby

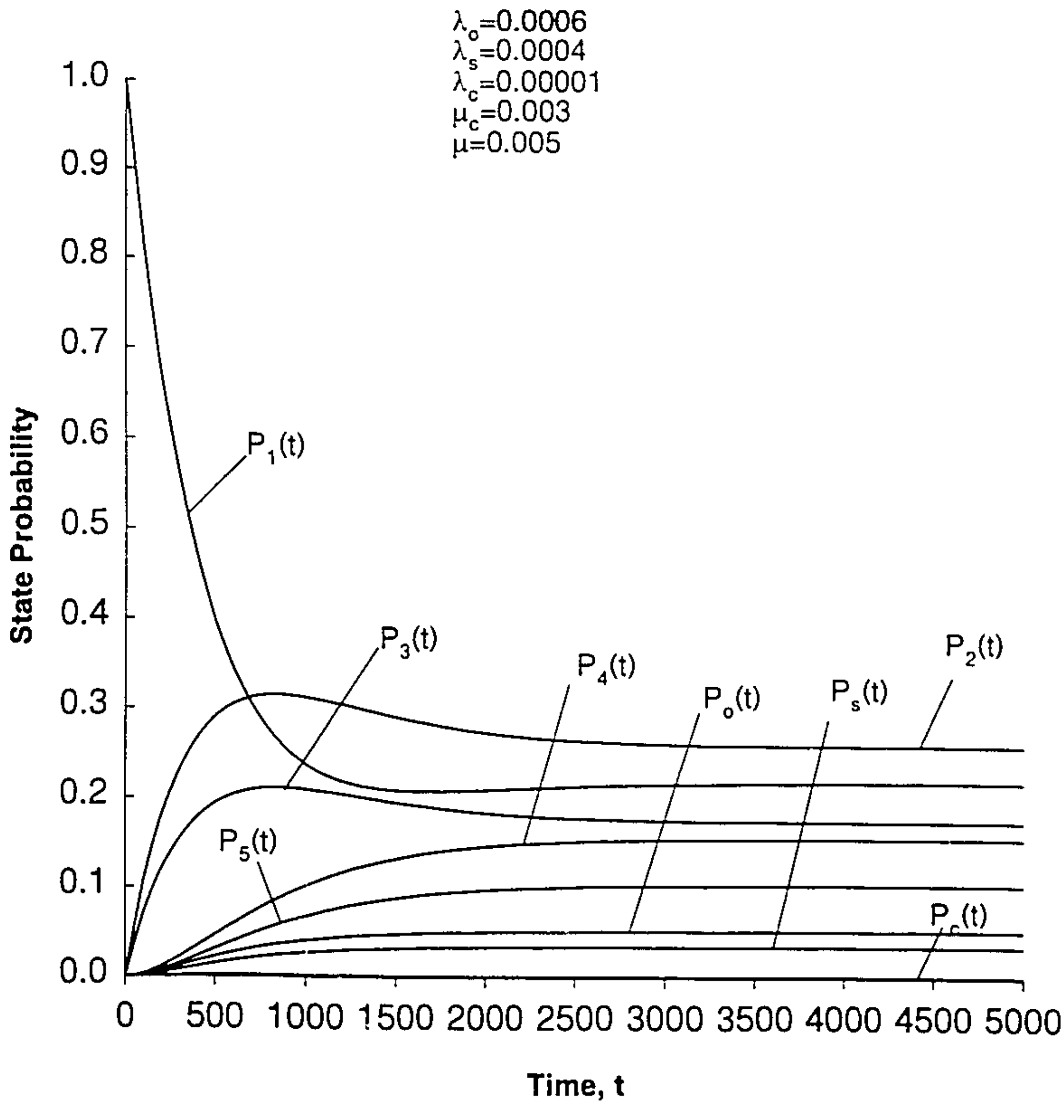


Figure 4.18 The standby system with type I repair state probability plots: two units in parallel with one unit on standby

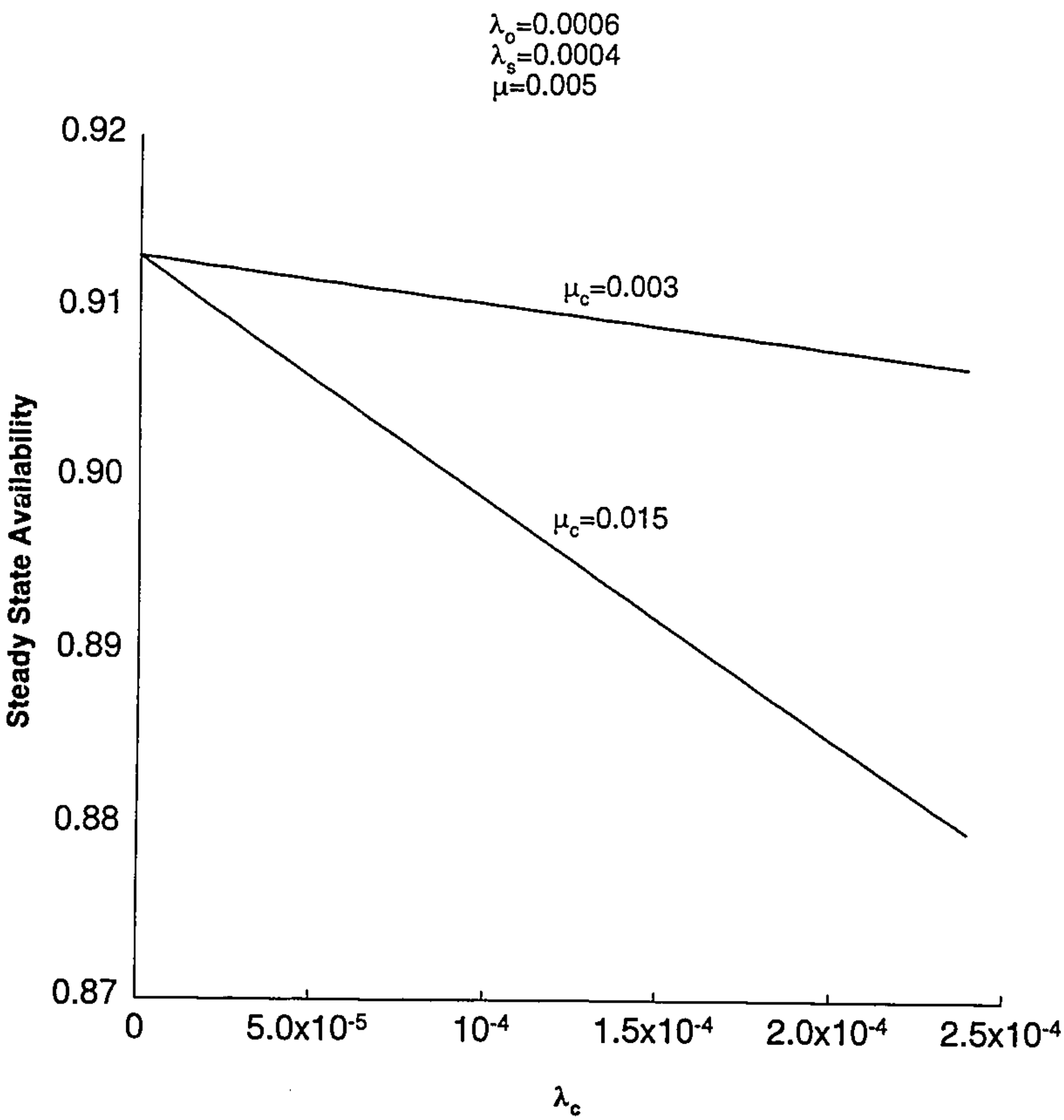


Figure 4.19 The standby system with type I repair steady state availability vs common cause failure rate, λ_c plots (two units in parallel with one unit on standby)

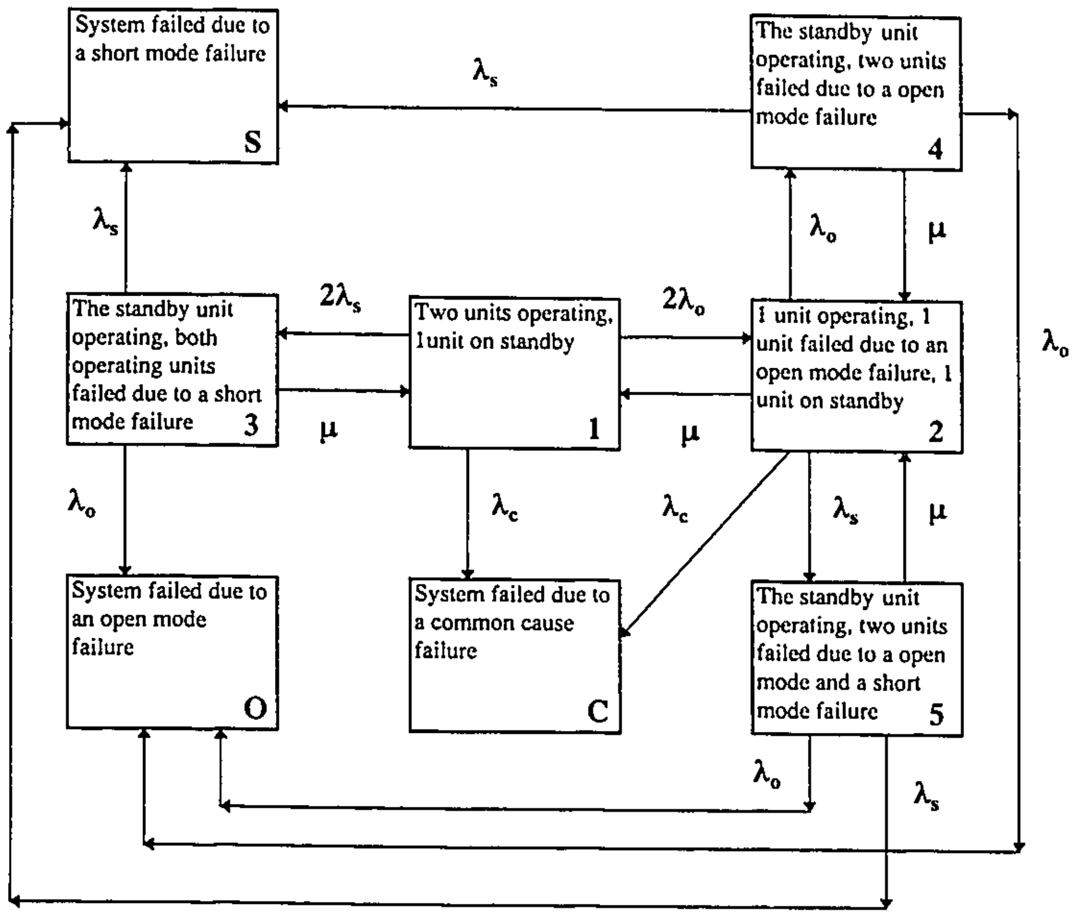


Figure 4.20 System state space diagram with Type II Repair: two units in parallel with one unit on standby system

In order to make a comparison with the one unit on standby system (Type I Repair is performed), the same values of parameters (i.e. as for Equations (2.139)-(2.146)) are chosen for the steady state availability plots, the state probabilities plots, and the time dependent availability plots. The time dependent system availabilities are plotted in Figure 4.17. Figure 4.18 is for the time dependent state probabilities. The system steady state availability given by Equation (4.90) is plotted in Figure 4.19.

4.2.3 Two units in parallel with one unit on standby system with Type II Repair

Figure 4.20 is the state space diagram of the two units in parallel with one unit on standby system with Type II Repair. The system of differential equations associated with Figure 4.20 is as follows:

$$\frac{dP_1(t)}{dt} = -(2\lambda_b + 2\lambda_s + \lambda_t)P_1(t) + \mu[P_2(t) + P_3(t)] \tag{4.100}$$

$$\frac{dP_2(t)}{dt} = -(\lambda_b + \lambda_s + \lambda_t + \mu)P_2(t) + 2\lambda_b P_1(t) + \mu[P_4(t) + P_5(t)] \tag{4.101}$$

$$\frac{dP_3(t)}{dt} = -(\lambda_b + \lambda_s + \mu)P_3(t) + 2\lambda_s P_1(t) \tag{4.102}$$

$$\frac{dP_4(t)}{dt} = -(\lambda_b + \lambda_s + \mu)P_4(t) + \lambda_b P_2(t) \tag{4.103}$$

$$\frac{dP_5(t)}{dt} = -(\lambda_b + \lambda_s + \mu)P_5(t) + \lambda_s P_2(t) \tag{4.104}$$

$$\frac{dP_s(t)}{dt} = \lambda_s[P_3(t) + P_4(t) + P_5(t)] \tag{4.105}$$

$$\frac{dP_o(t)}{dt} = \lambda_o[P_3(t) + P_4(t) + P_5(t)] \tag{4.106}$$

$$\frac{dP_c(t)}{dt} = \lambda_c[P_1(t) + P_2(t)] \tag{4.107}$$

At time $t = 0$, $P_1(0) = 1$ and all other initial condition probabilities are equal to zero.

By taking Laplace transforms of Equations (4.100) - (4.107) and solving the resulting equations, the Laplace transforms of the state probabilities are as follows:

$$P_1(s) = \frac{A(4,8)}{A(4,7)} \tag{4.108}$$

$$P_2(s) = \frac{A(4,9)}{A(4,7)} \tag{4.109}$$

$$P_3(s) = \frac{A(4,10)}{A(4,7)} \tag{4.110}$$

$$P_4(s) = \frac{A(4,11)}{A(4,7)} \tag{4.111}$$

$$P_5(s) = \frac{A(4,12)}{A(4,7)} \tag{4.112}$$

$$P_s(s) = \frac{A(4,13)}{sA(4,7)} \tag{4.113}$$

$$P_o(s) = \frac{A(4,14)}{sA(4,7)} \tag{4.114}$$

$$P_c(s) = \frac{A(4,15)}{sA(4,7)} \tag{4.115}$$

where A(4,7), A(4,8), A(4,9), A(4,10), A(4,11), A(4,12), A(4,13), A(4,14) and A(4,15) are defined in Appendix 4.3.

The Laplace transform of system reliability is then given by

$$\begin{aligned} R(s) &= P_1(s) + P_2(s) + P_3(s) + P_4(s) + P_5(s) \\ &= \frac{A(4,16)}{A(4,7)} \end{aligned} \tag{4.116}$$

where A(4,16) is defined in Appendix 4.3.

The system mean time to failure (MTTF) is

$$\begin{aligned} MTTF &= \lim_{s\to 0} R(s) \\ &= \lim_{s\to 0}[P_1(s) + P_2(s) + P_3(s) + P_4(s) + P_5(s)] \\ &= \frac{A(4,17)}{A(4,18)} \end{aligned} \qquad (4.117)$$

where A(4.17) and A(4.18) are defined in Appendix 4.3.

By taking inverse Laplace transforms of Equations (4.108) - (4.115) the state time dependent probabilities can be obtained. Similarly, the time dependent system reliability can be obtained by taking inverse Laplace transform of Equation (4.116).

In order to make a comparison with the one unit on standby system (Type II Repair is performed), the same values of parameters (i.e. as for Equations (2.339)-(2.346)) are chosen for the reliability plots, the state probability plots, and the mean time to failure plots. The time dependent system reliabilities are plotted in Figure 4.21. Figure 4.22 is for the time dependent state probabilities. The system mean time to failure given by Equation (4.117) is plotted in Figure 4.23.

The results of analysis in Section 4.2 indicated that the reliability, the mean time to failure, the steady state availability and the time dependent availability for the two units in parallel with one unit on standby system are higher than that of the one unit standby system, under the condition that the open mode failure rate of unit is greater than its short mode failure rate. Therefore, under this condition the two units in parallel with one unit on standby system can be used to improve the system reliability.

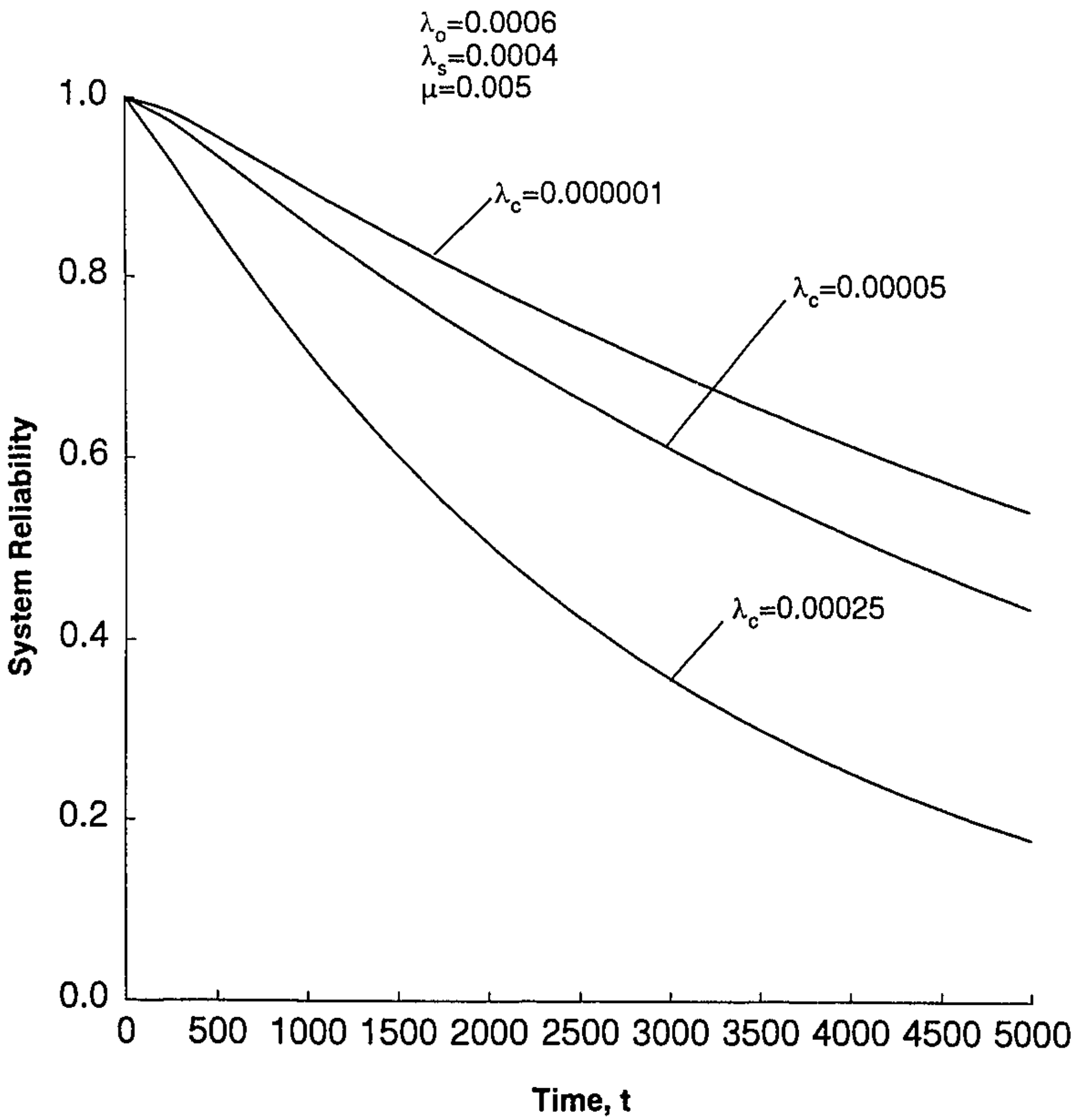


Figure 4.21 The standby system with type II repair reliability plots: two units in parallel with one unit on standby

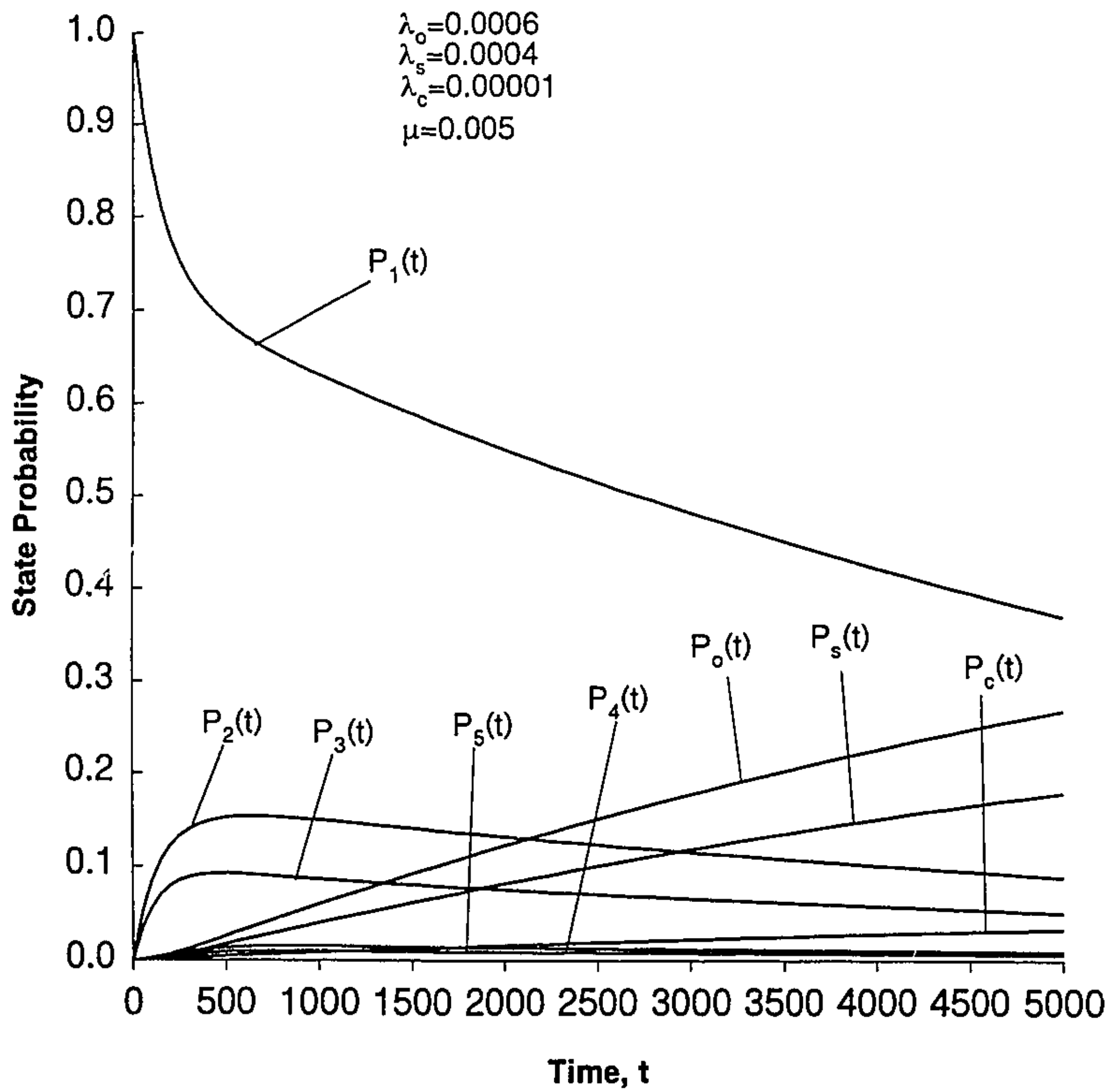


Figure 4.22 The standby system with type II repair state probability plots: two units in parallel with one unit on standby

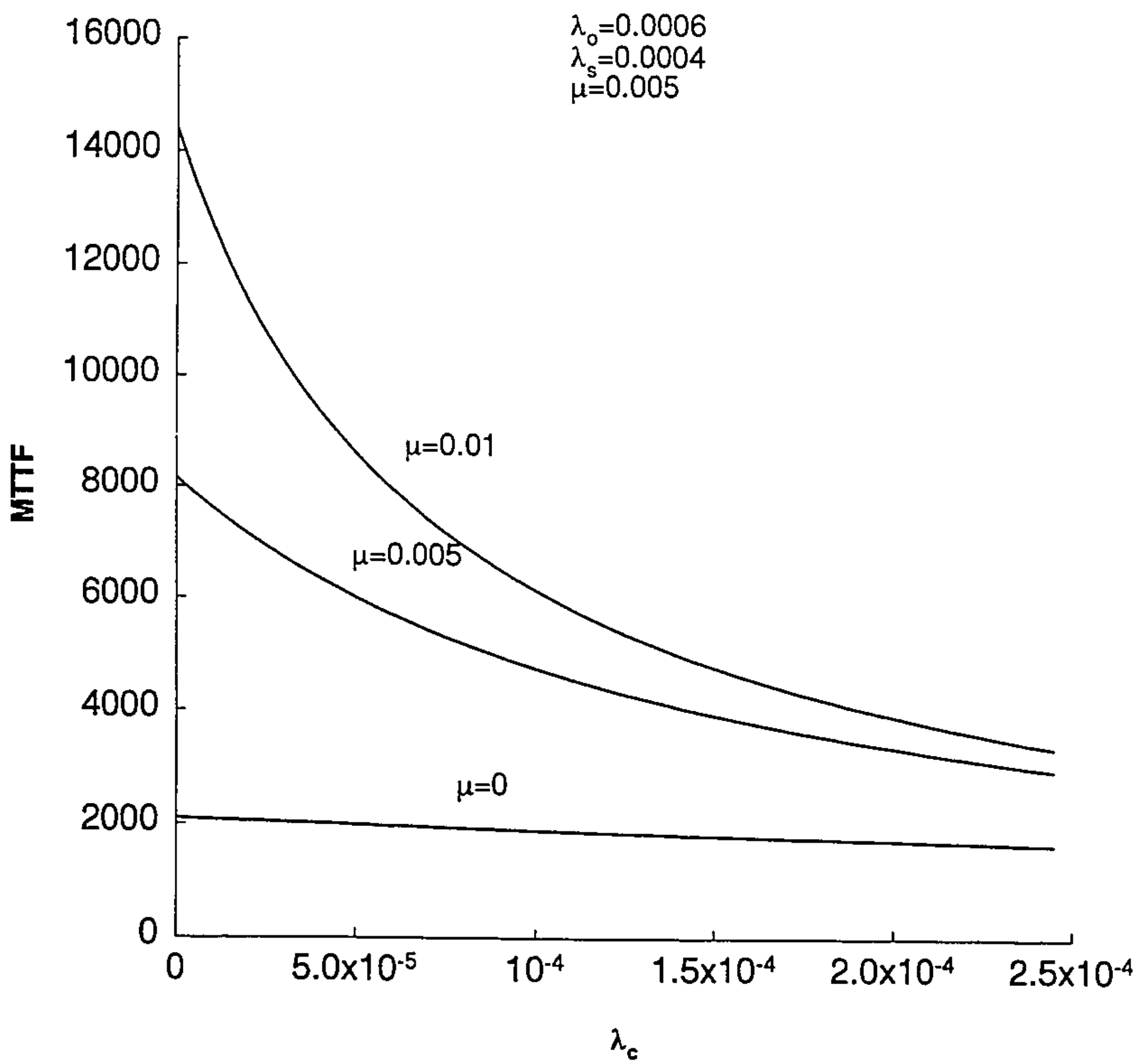


Figure 4.23 MTTF vs common cause failure rate, λ_c for standby system (two units in parallel with one unit on standby) with type II repair

Chapter 5

Reliability Comparison for Systems with Constant and Nonconstant Common-Cause Failure Rates

In the previous chapters the common-cause failure was assumed to be constant. This chapter presents models with non-constant common-cause failure rates for different configurations such as parallel, series and standby systems. For a nonconstant common-cause failure rate, the failure density function doesn't follow an exponential distribution, thus, the process is non-Markovian. The *Method of Stages* can be used to develop an expression for a system with non-constant failure rates. The *Method of Stages* is described in detail in Appendix 1.2.

5.1 Reliability comparison between parallel, series and standby systems with non-constant common-cause failure rates

Chapter 4 presented the reliability evaluations and comparisons for parallel, series and standby systems with constant common-cause failure rates. The analysis in this section includes a reliability and availability comparison for three different configurations with non-constant common-cause failure rate and constant open/short mode failure rates.

Notation:

The following symbols are associated with all of the models in this section.

t time.

λ_o Constant open-mode failure rate of an operating unit.

λ_s Constant short-mode failure rate of an operating unit.

λ_c Constant common-cause failure rate of system.

μ Constant repair rate for failed system due to an open or a short mode failure.

μ_c Constant repair rate for failed system due to a common-cause failure rate.

α_1 Constant transition rate associated with common-cause failures from initial condition to dummy state.

α_2 Constant transition rate associated with common-cause failures from dummy state to failed state.

$P_1(t)$ Probability that the system is in state 1.

$P_d(t)$ Probability of dummy state.

$P_s(t)$ Probability that the system is in the failed state due to a short mode failure.

$P_o(t)$ Probability that the system is in the failed state due to an open mode failure.

$P_c(t)$ Probability that the system is in the failed state due to common-cause failure.

$R_p(t)$ Reliability of parallel system in [0,t].

$R_s(t)$ Reliability of series system in [0,t].

$R_{sb}(t)$ Reliability of standby system in [0,t].

AV_p Steady state availability of parallel system.

AV_s Steady state availability of series system.

AV_{sb} Steady state availability of standby system.

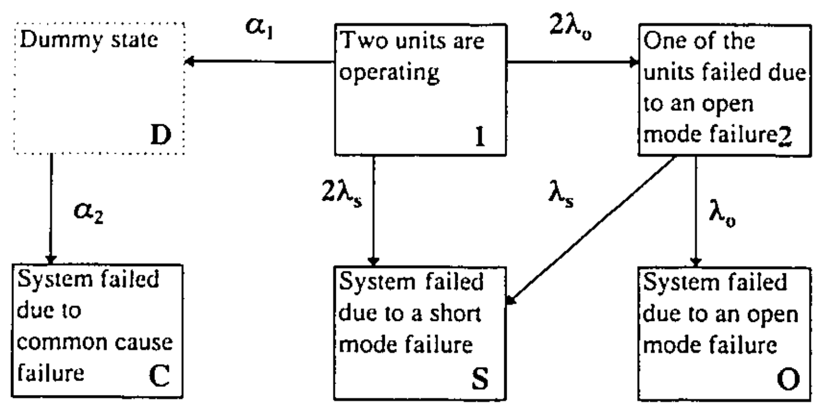


Figure 5.1 The state space diagram of the parallel system with a nonconstant common cause failure rate

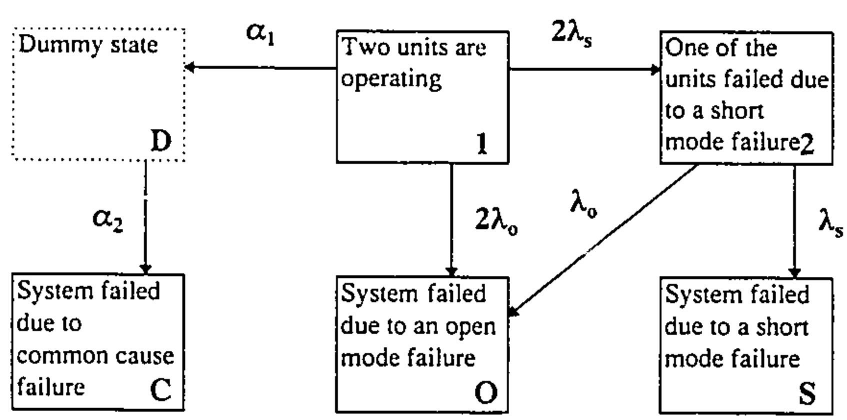


Figure 5.2 The state space diagram of the series system with a nonconstant common cause failure rate

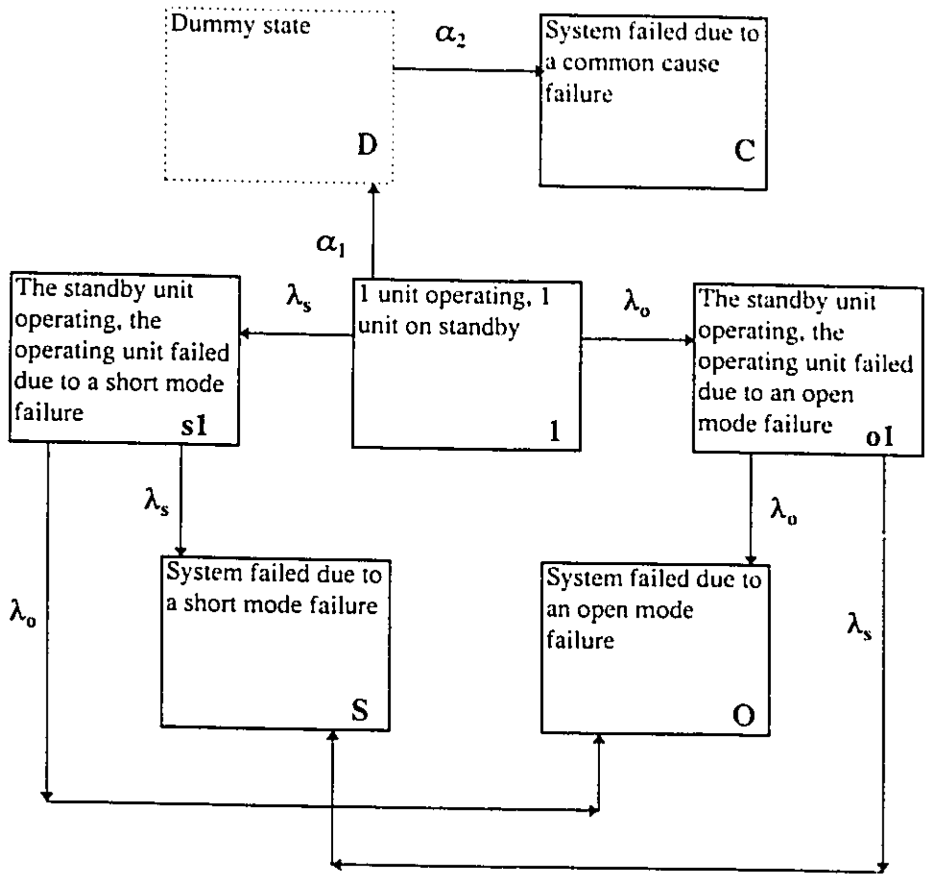


Figure 5.3 State space diagram of the standby system with a nonconstant common cause failure rate

Case I: Reliability comparison of parallel, series and standby systems without Repair

Figures 5.1 and 5.2 and 5.3 are state space diagrams for parallel, series and standby systems respectively. The system of differential equations associated with Figure 5.1 is

$$\frac{dP_1(t)}{dt} = -(2\lambda_o + 2\lambda_s + \alpha_1)P_1(t) \tag{5.1}$$

$$\frac{dP_2(t)}{dt} = -(\lambda_o + \lambda_s)P_2(t) + 2\lambda_o P_1(t) \tag{5.2}$$

$$\frac{dP_d(t)}{dt} = -\alpha_2 P_d(t) + \alpha_1 P_1(t) \tag{5.3}$$

$$\frac{dP_s(t)}{dt} = \lambda_s[2P_1(t) + P_2(t)] \tag{5.4}$$

$$\frac{dP_o(t)}{dt} = \lambda_o P_2(t) \tag{5.5}$$

$$\frac{dP_c(t)}{dt} = \alpha_2 P_d(t) \tag{5.6}$$

At time $t = 0$, $P_1(0) = 1$, and all the other initial condition probabilities are equal to zero.

$P_2(t)$ Probability that in the system one unit failed due to an open mode failure.

The system of differential equations associated with Figure 5.2 is

$$\frac{dP_1(t)}{dt} = -(2\lambda_s + 2\lambda_o + \alpha_1)P_1(t) \tag{5.7}$$

$$\frac{dP_2(t)}{dt} = -(\lambda_o + \lambda_s)P_2(t) + 2\lambda_s P_1(t) \tag{5.8}$$

$$\frac{dP_d(t)}{dt} = -\alpha_2 P_d(t) + \alpha_1 P_1(t) \tag{5.9}$$

$$\frac{dP_o(t)}{dt} = \lambda_o[2P_1(t) + P_2(t)] \tag{5.10}$$

$$\frac{dP_s(t)}{dt} = \lambda_s P_2(t) \tag{5.11}$$

$$\frac{dP_c(t)}{dt} = \alpha_2 P_d(t) \tag{5.12}$$

At time $t = 0$, $P_1(0) = 1$, and all the other initial condition probabilities are equal to zero.

$P_2(t)$ Probability that in the system one unit failed due to a short mode failure.

The system of differential equations associated with Figure 5.3 is

$$\frac{dP_1(t)}{dt} = -(\lambda_o + \lambda_s + \alpha_1) P_1(t) \tag{5.13}$$

$$\frac{dP_{o1}(t)}{dt} = -(\lambda_o + \lambda_s) P_{o1}(t) + \lambda_o P_1(t) \tag{5.14}$$

$$\frac{dP_{s1}(t)}{dt} = -(\lambda_o + \lambda_s) P_{s1}(t) + \lambda_s P_1(t) \tag{5.15}$$

$$\frac{dP_d(t)}{dt} = -\alpha_2 P_d(t) + \alpha_1 P_1(t) \tag{5.16}$$

$$\frac{dP_s(t)}{dt} = \lambda_s [P_{o1}(t) + P_{s1}(t)] \tag{5.17}$$

$$\frac{dP_o(t)}{dt} = \lambda_o [P_{o1}(t) + P_{s1}(t)] \tag{5.18}$$

$$\frac{dP_c(t)}{dt} = \alpha_2 P_d(t) \tag{5.19}$$

At time $t = 0$, $P_1(0) = 1$, and all the other initial condition probabilities are equal to zero.

Setting $\alpha_1 = \alpha_2 = \alpha$ in Equations (5.1) - (5.19), the reliabilities of parallel, series and standby systems are given by

$$R_p(t) = P_1(t) + P_2(t) + P_d(t)$$

$$= \frac{2\lambda_o}{e^{(\lambda_o+\lambda_s)t}(\lambda_o + \lambda_s + \alpha)} + \frac{\alpha}{2e^{\alpha t}(\lambda_o + \lambda_s)} + \frac{-2\lambda_o^2 + 2\lambda_s^2 + (\lambda_o + \lambda_s)\alpha - \alpha^2}{2e^{(2\lambda_o+2\lambda_s+\alpha)t}(\lambda_o + \lambda_s + \alpha)(\lambda_o + \lambda_s)} \tag{5.20}$$

$$R_s(t) = P_1(t) + P_2(t) + P_d(t)$$

$$= \frac{2\lambda_s}{e^{(\lambda_b+\lambda_s)t}(\lambda_b+\lambda_s+\alpha)} + \frac{\alpha}{2e^{\alpha t}(\lambda_b+\lambda_s)} + \frac{2\lambda_b^2 - 2\lambda_s^2 + (\lambda_b+\lambda_s)\alpha - \alpha^2}{2e^{(2\lambda_b+2\lambda_s+\alpha)t}(\lambda_b+\lambda_s+\alpha)(\lambda_b+\lambda_s)} \tag{5.21}$$

$$R_{sb}(t) = P_1(t) + P_{o1}(t) + P_{s1}(t) + P_d(t)$$

$$= \frac{\lambda_b+\lambda_s}{e^{(\lambda_b+\lambda_s)t}\alpha} + \frac{\alpha}{e^{\alpha t}(\lambda_b+\lambda_s)} + \frac{-(\lambda_b+\lambda_s)^2 + (\lambda_b+\lambda_s)\alpha - \alpha^2}{e^{(\lambda_b+\lambda_s+\alpha)t}\alpha(\lambda_b+\lambda_s)} \tag{5.22}$$

By using Equations (5.22) and (5.20), the reliability difference $\Delta R(t)$ between the standby system and the parallel system is

$$\Delta R(t) = R_{sb}(t) - R_p(t) = \frac{A(5,2)}{A(5,1)} \tag{5.23}$$

where

$$A(5,1) = 2e^{(2\lambda_b+2\lambda_s+\alpha)t}\alpha(\lambda_b+\lambda_s)(\lambda_b+\lambda_s+\alpha)$$

$$A(5,2) = 2(\lambda_b+\lambda_s)^3[e^{(\lambda_b+\lambda_s+\alpha)t} - e^{(\lambda_b+\lambda_s)t}] - 2(\lambda_b^2 - \lambda_s^2)\alpha[e^{(\lambda_b+\lambda_s+\alpha)t} - 1] +$$
$$(\lambda_b+\lambda_s)\alpha^2[e^{2(\lambda_b+\lambda_s)t} - 1] + \alpha^3[e^{(\lambda_b+\lambda_s)t} - 1]^2$$

By using Equations (5.20) and (5.21), the reliability difference between the parallel and the series systems is

$$\Delta R(t) = R_p(t) - R_s(t)$$

$$= \frac{2[e^{(\lambda_b+\lambda_s+\alpha)t} - 1](\lambda_b - \lambda_s)}{e^{(2\lambda_b+2\lambda_s+\alpha)t}(\lambda_b+\lambda_s+\alpha)} \tag{5.24}$$

$A(5,1) = 2e^{(2\lambda_b+2\lambda_s+\alpha)t}\alpha(\lambda_b+\lambda_s)(\lambda_b+\lambda_s+\alpha)$ is greater than zero. At time t=0, A(5,2) equals to zero, however, A(5,2) will be greater than zero with progression at time. The proof is shown in Appendix 5.1. Thus, $\Delta R(t) \geq 0$ which indicates that the reliability of the standby system is higher than that of the parallel system. With the same analogy, using Equations (5.22) and (5.21), it can be proven that the reliability of the standby system is higher than that of the series system. From Equation (5.24), it is evident that if the open mode failure rate of a unit is greater than its short mode failure rate, the parallel system has higher reliability than the series system.

Case II: Availability comparison of parallel, series and standby systems with Repair

This case considers Type I Repair. Figures 5.4, 5.5 and 5.6 are state space diagrams for parallel, series and standby systems respectively. The system of differential equations associated with Figure 5.4 is

$$\frac{dP_1(t)}{dt} = -(2\lambda_o + 2\lambda_s + \alpha)P_1(t) + \mu[P_o(t) + P_s(t)] + \mu_c P_c(t) \tag{5.25}$$

$$\frac{dP_2(t)}{dt} = -(\lambda_o + \lambda_s)P_2(t) + 2\lambda_o P_1(t) \tag{5.26}$$

$$\frac{dP_d(t)}{dt} = -\alpha P_d(t) + \alpha P_1(t) \tag{5.27}$$

$$\frac{dP_s(t)}{dt} = -\mu P_s(t) + \lambda_s[2P_1(t) + P_2(t)] \tag{5.28}$$

$$\frac{dP_o(t)}{dt} = -\mu P_o(t) + \lambda_o P_2(t) \tag{5.29}$$

$$\frac{dP_c(t)}{dt} = -\mu_c P_c(t) + \alpha P_d(t) \tag{5.30}$$

At time $t = 0$, $P_1(0) = 1$, and all the other initial condition probabilities are equal to zero.

The system of differential equations associated with Figure 5.5 is

$$\frac{dP_1(t)}{dt} = -(2\lambda_o + 2\lambda_s + \alpha)P_1(t) + \mu[P_o(t) + P_s(t)] + \mu_c P_c(t) \tag{5.31}$$

$$\frac{dP_2(t)}{dt} = -(\lambda_o + \lambda_s)P_2(t) + 2\lambda_s P_1(t) \tag{5.32}$$

$$\frac{dP_d(t)}{dt} = -\alpha P_d(t) + \alpha P_1(t) \tag{5.33}$$

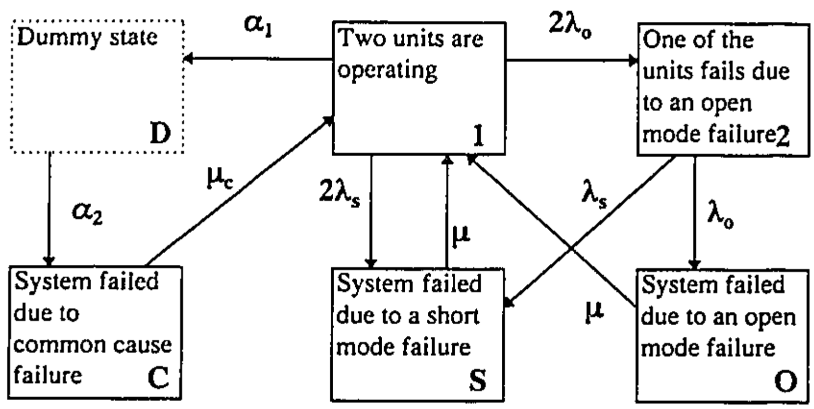


Figure 5.4 The state space diagram of the parallel system with a nonconstant common cause failure rate and Type I repair

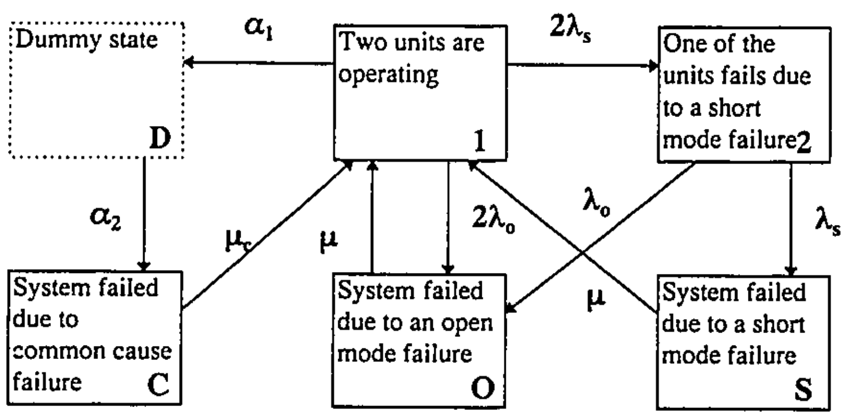


Figure 5.5 The state space diagram of the series system with a nonconstant common cause failure rate and Type I repair

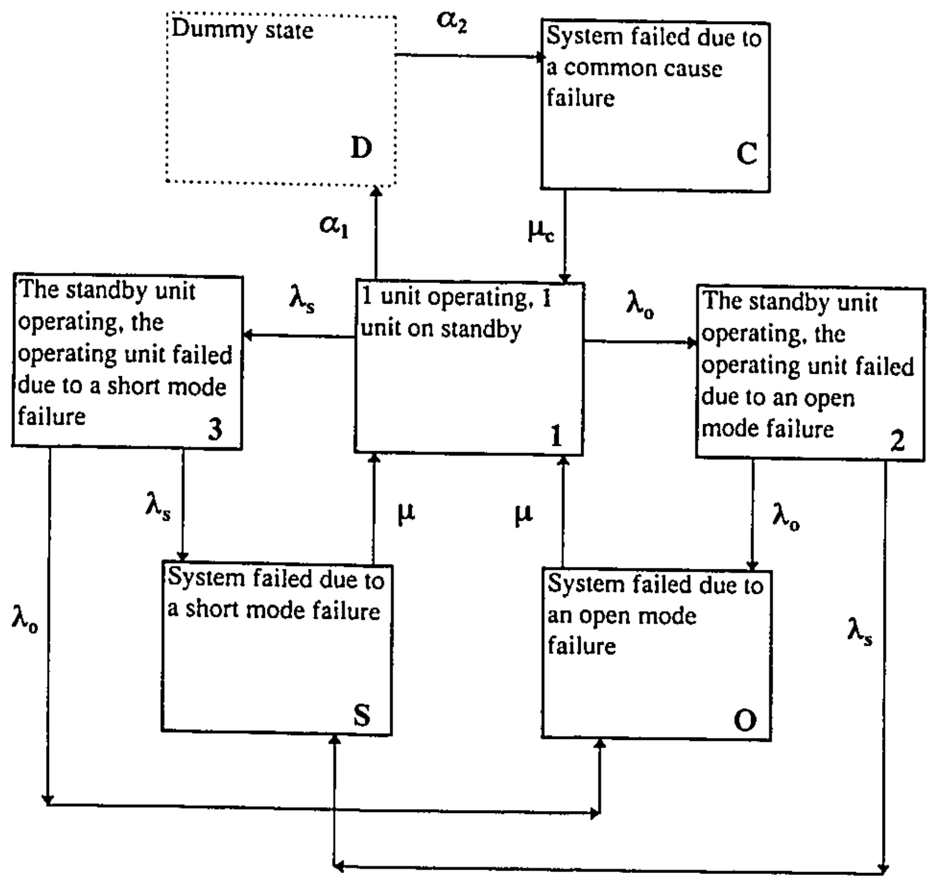


Figure 5.6 State space diagram of the standby system with a nonconstant common cause failure rate and Type I repair

$$\frac{dP_o(t)}{dt} = -\mu P_o(t) + \lambda_o[2P_1(t) + P_2(t)] \tag{5.34}$$

$$\frac{dP_s(t)}{dt} = -\mu P_s(t) + \lambda_o P_2(t) \tag{5.35}$$

$$\frac{dP_c(t)}{dt} = -\mu_c P_c(t) + \alpha P_d(t) \tag{5.36}$$

At time $t = 0$, $P_1(0) = 1$ and all other initial condition probabilities are equal to zero.

The system of differential equations associated with Figure 5.6 is

$$\frac{dP_1(t)}{dt} = -(\lambda_b + \lambda_s + \alpha)P_1(t) + \mu[P_o(t) + P_s(t)] + \mu_c P_c(t) \tag{5.37}$$

$$\frac{dP_{o1}(t)}{dt} = -(\lambda_b + \lambda_s)P_{o1}(t) + \lambda_b P_1(t) \tag{5.38}$$

$$\frac{dP_{s1}(t)}{dt} = -(\lambda_o + \lambda_s)P_{s1}(t) + \lambda_s P_1(t) \tag{5.39}$$

$$\frac{dP_d(t)}{dt} = -\alpha P_d(t) + \alpha P_1(t) \tag{5.40}$$

$$\frac{dP_s(t)}{dt} = -\mu P_s(t) + \lambda_s[P_2(t) + P_3(t)] \tag{5.41}$$

$$\frac{dP_o(t)}{dt} = -\mu P_o(t) + \lambda_b[P_2(t) + P_3(t)] \tag{5.42}$$

$$\frac{dP_c(t)}{dt} = -\mu_c P_c(t) + \alpha P_d(t) \tag{5.43}$$

At time $t = 0$, $P_1(0) = 1$, and all the other initial condition probabilities are equal to zero.

Setting $\alpha_1=\alpha_2=\alpha$ in Equations (5.25)-(5.30) and setting the derivatives of Equations (5.25)-(5.30) equal to zero as well as utilizing the relationship $\sum P_j = 1$, j represents all of the states in the system, the steady state probabilities can be obtained. Thus, the steady state availability of the parallel system is given by

$$AV_p = \frac{(4\lambda_b + 2\lambda_s)\alpha\mu\mu_c}{2(\lambda_o + \lambda_s)^2\alpha\mu_c + (\lambda_b + \lambda_s)\alpha^2\mu + (4\lambda_o + 2\lambda_s)\alpha\mu\mu} \tag{5.44}$$

Similarly, solving Equations (5.31)-(5.36) the steady state availability of the series system can be obtained.

$$AV_s = \frac{(2\lambda_o + 4\lambda_s)\alpha\mu\mu_c}{2(\lambda_o + \lambda_s)^2\alpha\mu_c + (\lambda_o + \lambda_s)\alpha^2\mu + (2\lambda_o + 4\lambda_s)\alpha\mu\mu} \tag{5.45}$$

Also, solving Equations (5.37) and (5.43) the steady state availability of standby system can be obtained.

$$AV_{sb} = \frac{3(\lambda_o + \lambda_s)\alpha\mu\mu_c}{(\lambda_o + \lambda_s)^2\alpha\mu_c + (\lambda_o + \lambda_s)\alpha^2\mu + 3(\lambda_o + \lambda_s)\alpha\mu\mu} \tag{5.46}$$

By using Equations (5.46) and (5.44), the difference the steady state availability ΔAV between the standby system and the parallel system is

$$\Delta AV = AV_{sb} - AV_p = \frac{A(5,4)}{A(5,3)} \tag{5.47}$$

where

$$A(5,3) = [(\lambda_o + \lambda_s)\mu_c + (\alpha + 3\mu_c)\mu][2(\lambda_o + \lambda_s)^2\mu_c + (\lambda_o + \lambda_s)\alpha\mu + (4\lambda_o + 2\lambda_s)\mu\mu_c]$$

$$A(5,4) = \lambda_o(2\lambda_o\mu_c - \alpha\mu) + \lambda_s\lambda_c\mu + 6\lambda_o\lambda_s\mu_c + 4\lambda_s^2\mu_c$$

By using Equations (5.44) and (5.45), the steady state availability difference between the parallel system and the series system is

$$AV_p - AV_s = \frac{A(5,6)}{A(5,5)} \tag{5.48}$$

where

$$A(5,5) = [2(\lambda_o + \lambda_s)^2\alpha\mu_c + (\lambda_o + \lambda_s)\alpha^2\mu + (4\lambda_o + 2\lambda_s)\alpha\mu\mu] \bullet$$
$$2(\lambda_o + \lambda_s)^2\alpha\mu_c + (\lambda_o + \lambda_s)\alpha^2\mu + (2\lambda_o + 4\lambda_s)\alpha\mu\mu]$$

$$A(5,6) = 2(\lambda_o^2 - \lambda_s^2)(2\lambda_o \mu_c + 2\lambda_s \mu_c + \alpha\mu)\, \mu\mu_c$$

In Equation (5.47), A(5,3) is greater than zero. Appendix 1.2 indicates that if $\alpha_1=\alpha_2$, $\lim_{t\to\infty} \lambda(t) = \alpha$. This means that the non-constant common-cause failure rate becomes constant value as time approaches infinity, and the constant value is equal to parameter α. In practice, the value of the common-cause failure rate lies within the range 1%-30% of a component's failure rate. This indicates that parameter α lies within the range 1%-30% of a component's failure rate. The system repair rate after a common-cause failure is within the same range of system repair rate after an open or a short mode failure. Thus, we can derive that $(2\lambda_o \mu_c - \alpha\mu) \geq 0$. If $(2\lambda_o \mu_c - \alpha\mu) \geq 0$, then A(5,4) is greater than zero. This indicates that the standby system has a higher steady state availability than the parallel system. Using the same analogy, we can prove that the standby system has a higher steady state availability than the series system.

In Equation (5,48), A(5,5) is also greater than zero. It is evident that if the open mode failure rate of a unit is higher than its short mode failure rate, then A(5,6) is greater than zero. This indicates that the parallel system has a higher steady state availability than the series system.

Figure 5.7 is time dependent availability for the standby, parallel and series systems. It can be observed that the standby system has the highest availability.

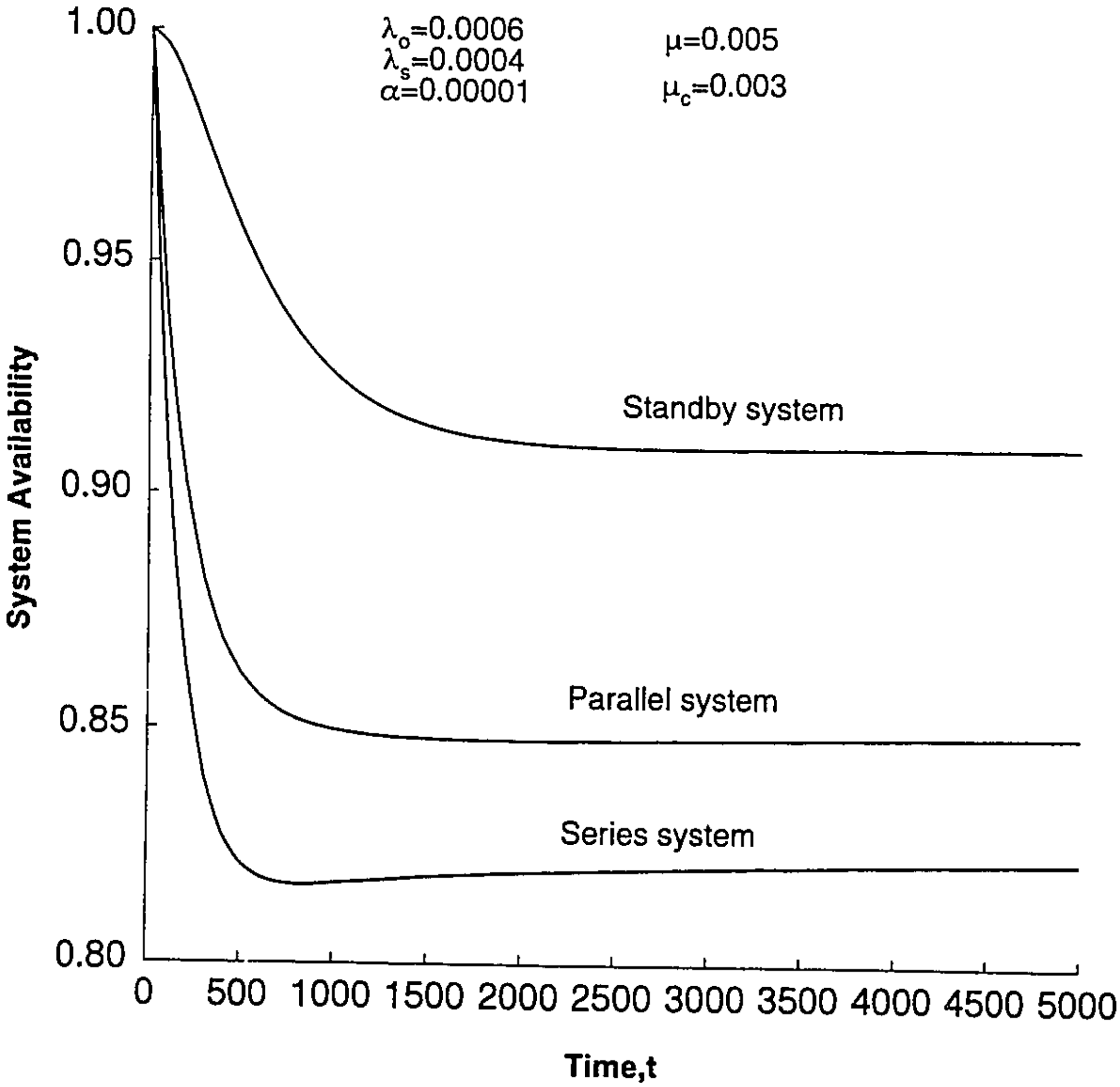


Figure 5.7 Avaliability comparision plots for standby system, parallel system and series system with increasing common cause failures

5.2 Reliability evaluation of parallel, series and standby systems with and without common-cause failures

This section presents a reliability evaluation of systems with and without common-cause failure rates. Comparisons are made for systems without common-cause failure rates, with constant common-cause failure rates and with non-constant common-cause failure rates. In the following discussion, an assumption is made that as time progresses, non-constant common-cause failure rates approach a steady state, and this steady state of common-cause failure rate $\lambda(t)$ equals the value of the constant common-cause failure rate λ_c. In Appendix 1.2, it is indicated that the steady state non-constant failure rate equals the parameter α, thus, the parameter α equals the constant common-cause failure rate λ_c.

Standby System:

Using Equations (2,38) and (5,22), for $\lambda_c = \alpha$, the reliability difference of the standby system with a constant common-cause failure rate and with a non-constant common-cause failure rate is given by

$$\Delta R_{sb}(t) = \frac{(\lambda_b + \lambda_s)}{e^{(\lambda_b+\lambda_s)t}\lambda_c} - \frac{\lambda_b + \lambda_s - \lambda_b}{e^{(\lambda_b+\lambda_s+\lambda_c)t}\lambda_c}$$

$$-[\frac{\lambda_b + \lambda_s}{e^{(\lambda_b+\lambda_s)t}\alpha} + \frac{\alpha}{e^{\alpha t}(\lambda_b + \lambda_s)} + \frac{-(\lambda_b + \lambda_s)^2 + (\lambda_b + \lambda_s)\alpha - \alpha^2}{e^{(\lambda_b+\lambda_s+\alpha)t}\alpha(\lambda_b + \lambda_s)}]$$

$$= \frac{(1 - e^{(\lambda_b+\lambda_s+\lambda_c)t})\lambda_c}{e^{(\lambda_b+\lambda_s+\lambda_c)t}(\lambda_b + \lambda_s)} \qquad (5.49)$$

From Equation (5.49), it is noted that the standby system with a non-constant common-cause failure rate has a higher reliability than the standby system with a constant common-cause failure rate.

Using Equations (4,15) and (5,22), the reliability difference of the standby system without a common-cause failure rate and with a non-constant common-cause failure rate is expressed as

$$\Delta R_{sb}(t) = \frac{1+(\lambda_o+\lambda_s)t}{e^{(\lambda_o+\lambda_s)t}} - [\frac{\lambda_o+\lambda_s}{e^{(\lambda_o+\lambda_s)t}\alpha} + \frac{\alpha}{e^{\alpha t}(\lambda_o+\lambda_s)} + \frac{-(\lambda_o+\lambda_s)^2+(\lambda_o+\lambda_s)\alpha-\alpha^2}{e^{(\lambda_o+\lambda_s+\alpha)t}\alpha(\lambda_o+\lambda_s)}]$$
$$= \frac{(\lambda_o+\lambda_s)^2(1+\alpha t e^{\alpha t}-e^{\alpha t})+(\lambda_o+\lambda_s)\alpha(e^{\alpha t}-1)+\alpha^2(1-e^{(\lambda_o+\lambda_s)t})}{e^{(\lambda_o+\lambda_s)t}(\lambda_o+\lambda_s)\alpha} \tag{5.50}$$

It is not obvious to determine $\Delta R_{sb}(t)$ from Equation (5.50), by using Equation (4.15) and (5.22), the standby system reliabilities are plotted in Figure 5.8 for the specified value of α.

Parallel System:

By using Equation (4,23) and (5,20), for $\lambda_c = \alpha$, the reliability difference of the parallel system with a constant common-cause failure rate and with a non-constant common-cause failure rate is given by

$$\Delta R_p = \frac{1}{\lambda_o+\lambda_s+\lambda_c}[\frac{2\lambda_o}{e^{(\lambda_o+\lambda_s)t}} + \frac{(-\lambda_o+\lambda_s+\lambda_c)}{e^{(2\lambda_o+2\lambda_s+\lambda_c)t}}]$$
$$-[\frac{2\lambda_o}{e^{(\lambda_o+\lambda_s)t}(\lambda_o+\lambda_s+\alpha)} + \frac{\alpha}{2e^{\alpha t}(\lambda_o+\lambda_s)} + \frac{-2\lambda_o^2+2\lambda_s^2+(\lambda_o+\lambda_s)\alpha-\alpha^2}{2e^{(2\lambda_o+2\lambda_s+\alpha)t}(\lambda_o+\lambda_s+\alpha)(\lambda_o+\lambda_s)}]$$
$$= \frac{(1-e^{(2\lambda_o+2\lambda_s)t})\lambda_c}{2e^{(2\lambda_o+2\lambda_s+\lambda_c)t}(\lambda_o+\lambda_s)} \tag{5.51}$$

Equation (5.51) indicated that the parallel system with a non-constant common-cause failure rate has a higher reliability than the parallel system with a constant common-cause failure rate.

By utilizing Equations (4,11) and (5,20), the reliability difference of the standby system without a common-cause failure rate and with a non-constant common-cause failure rate is expressed as

$$\Delta R_p = \frac{1}{\lambda_o + \lambda_s}\left[\frac{2\lambda_o}{e^{(\lambda_o+\lambda_s)t}} + \frac{(-\lambda_o + \lambda_s)}{e^{(2\lambda_o+2\lambda_s)t}}\right]$$

$$-\left[\frac{2\lambda_o}{e^{(\lambda_o+\lambda_s)t}(\lambda_o + \lambda_s + \lambda_d)} + \frac{\lambda_d}{2e^{\lambda_d t}(\lambda_o + \lambda_s)} + \frac{-2\lambda_o^2 + 2\lambda_s^2 + (\lambda_o + \lambda_s)\lambda_d - \lambda_d^2}{2e^{(2\lambda_o+2\lambda_s+\lambda_d)t}(\lambda_o + \lambda_s + \lambda_d)(\lambda_o + \lambda_s)}\right]$$

(5.52)

It is not obvious to determine $\Delta R_p(t)$ from Equation (5.52), by using Equations (4.11) and (5.20), the parallel system reliabilities are plotted in Figure 5.9 for the specified value of α.

Series System:

By using Equation (4,25) and (5,21), for $\lambda_c = \alpha$, the reliability difference of the parallel system with a constant common-cause failure rate and with a non-constant common-cause failure rate is given by

$$\Delta R_s = \frac{1}{\lambda_b + \lambda_s + \lambda_c}\left[\frac{2\lambda_s}{e^{(\lambda_b+\lambda_s)t}} + \frac{(\lambda_b - \lambda_s + \lambda_c)}{e^{(2\lambda_b+2\lambda_s+\lambda_c)t}}\right]$$

$$-\left[\frac{2\lambda_s}{e^{(\lambda_b+\lambda_s)t}(\lambda_b + \lambda_s + \alpha)} + \frac{\alpha}{2e^{\alpha t}(\lambda_b + \lambda_s)} + \frac{2\lambda_b^2 - 2\lambda_s^2 + (\lambda_b + \lambda_s)\alpha - \alpha^2}{2e^{(2\lambda_b+2\lambda_s+\alpha)t}(\lambda_b + \lambda_s + \alpha)(\lambda_b + \lambda_s)}\right]$$

$$= \frac{(1 - e^{(2\lambda_b+2\lambda_s)t})\lambda_c}{2e^{(2\lambda_b+2\lambda_s+\lambda_c)t}(\lambda_b + \lambda_s)} \qquad (5.53)$$

Equation (5.53) indicated that the series system with a non-constant common-cause failure rate has a higher reliability than the series system with a constant common-cause failure rate.

By utilizing Equations (4,13) and (5,21), the reliability difference of the standby system without a common-cause failure rate and with a non-consiant common-cause failure rate is expressed as

$$\Delta R_s = \frac{1}{\lambda_b + \lambda_s}[\frac{2\lambda_s}{e^{(\lambda_b+\lambda_s)t}} + \frac{\lambda_b - \lambda_s}{e^{(2\lambda_b+2\lambda_s)t}}]$$

$$-[\frac{2\lambda_s}{e^{(\lambda_b+\lambda_s)t}(\lambda_b+\lambda_s+\lambda_u)} + \frac{\lambda_u}{2e^{\lambda_u t}(\lambda_b+\lambda_s)} + \frac{2\lambda_b^2 - 2\lambda_s^2 + (\lambda_b+\lambda_s)\lambda_u - \lambda_u^2}{2e^{(2\lambda_b+2\lambda_s+\lambda_u)t}(\lambda_b+\lambda_s+\lambda_u)(\lambda_b+\lambda_s)}] \tag{5.54}$$

It is not obvious to determine $\Delta R_s(t)$ from Equation (5.54), by using Equations (4.13) and (5.21), the series system reliabilities are plotted in Figure 5.10 for the specified value of α.

According to the above discussion, based on the assumption that steady state of the non-constant common-cause failure rate λ(t) equals the constant-common-cause failure rate λ_c, for the same configuration, it is noted that the system reliability with a non-constant common-cause failure rate is higher than that with a constant-failure rate. The system reliability with a non-constant common-cause failure rate is lower than that without common-cause failure.

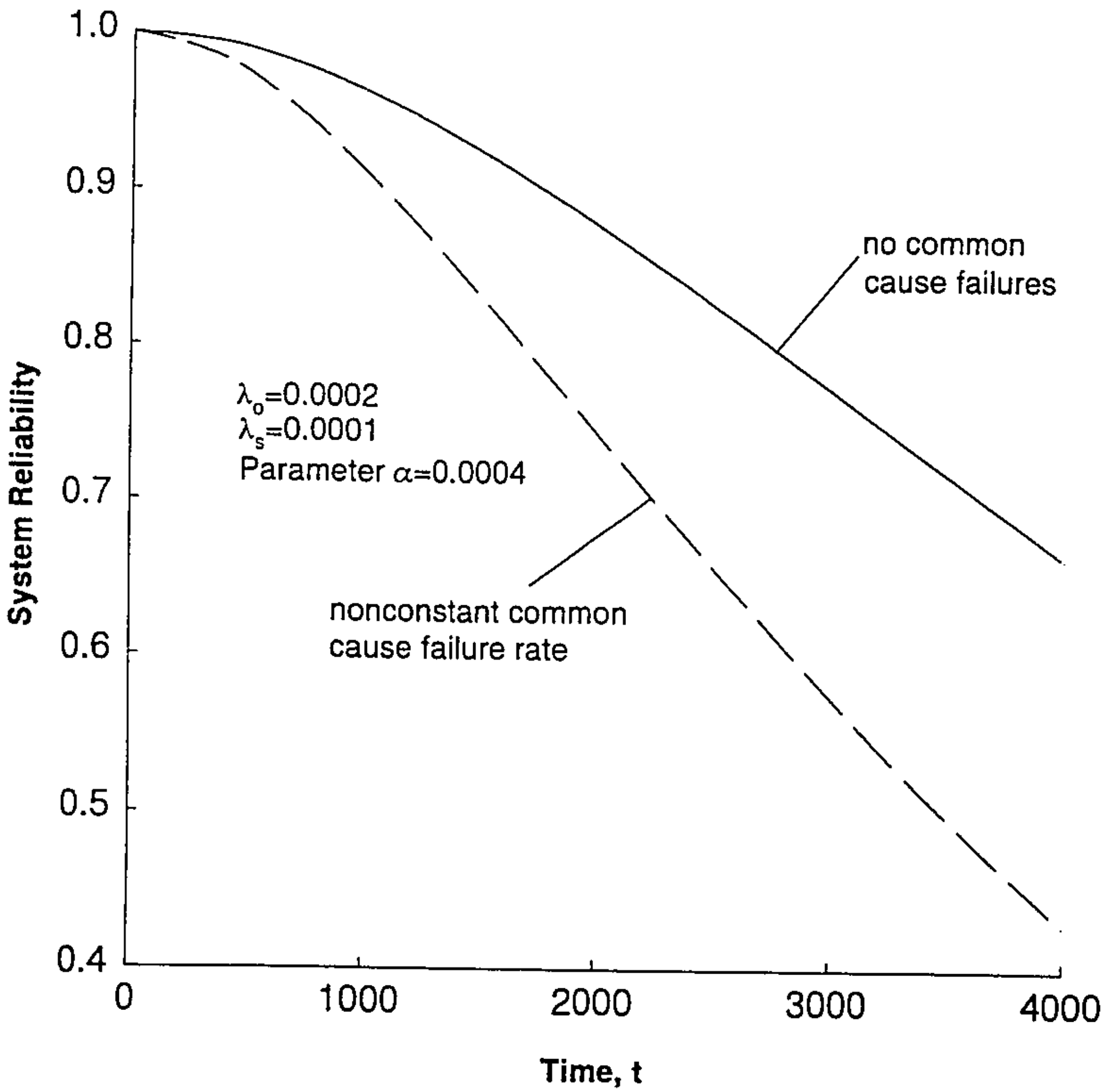


Figure 5.8 Standby system reliability plots without common cause and with nonconstant common cause failure rates

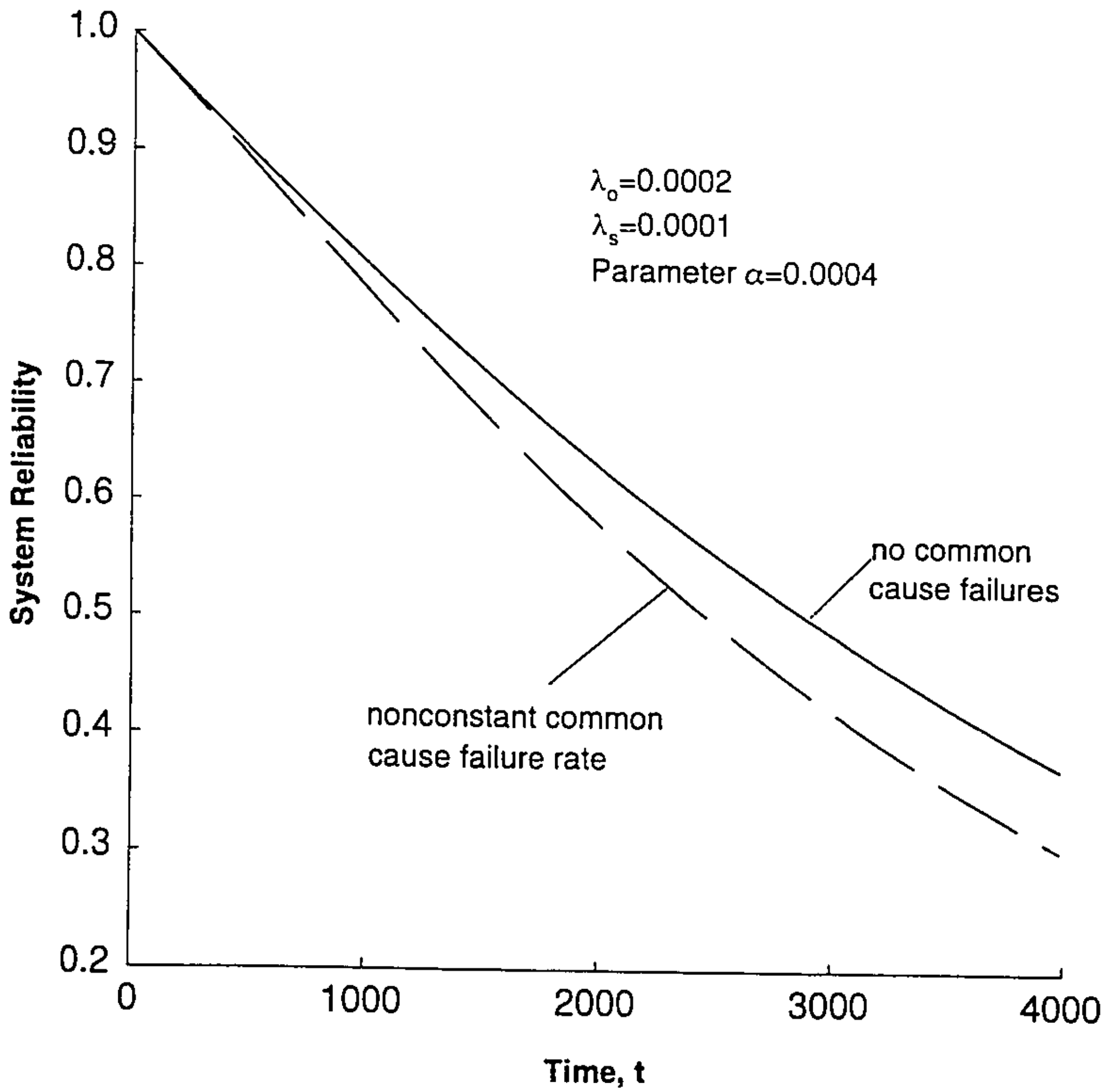


Figure 5.9 Parallel system reliability plots without common cause and with nonconstant common cause failure rates

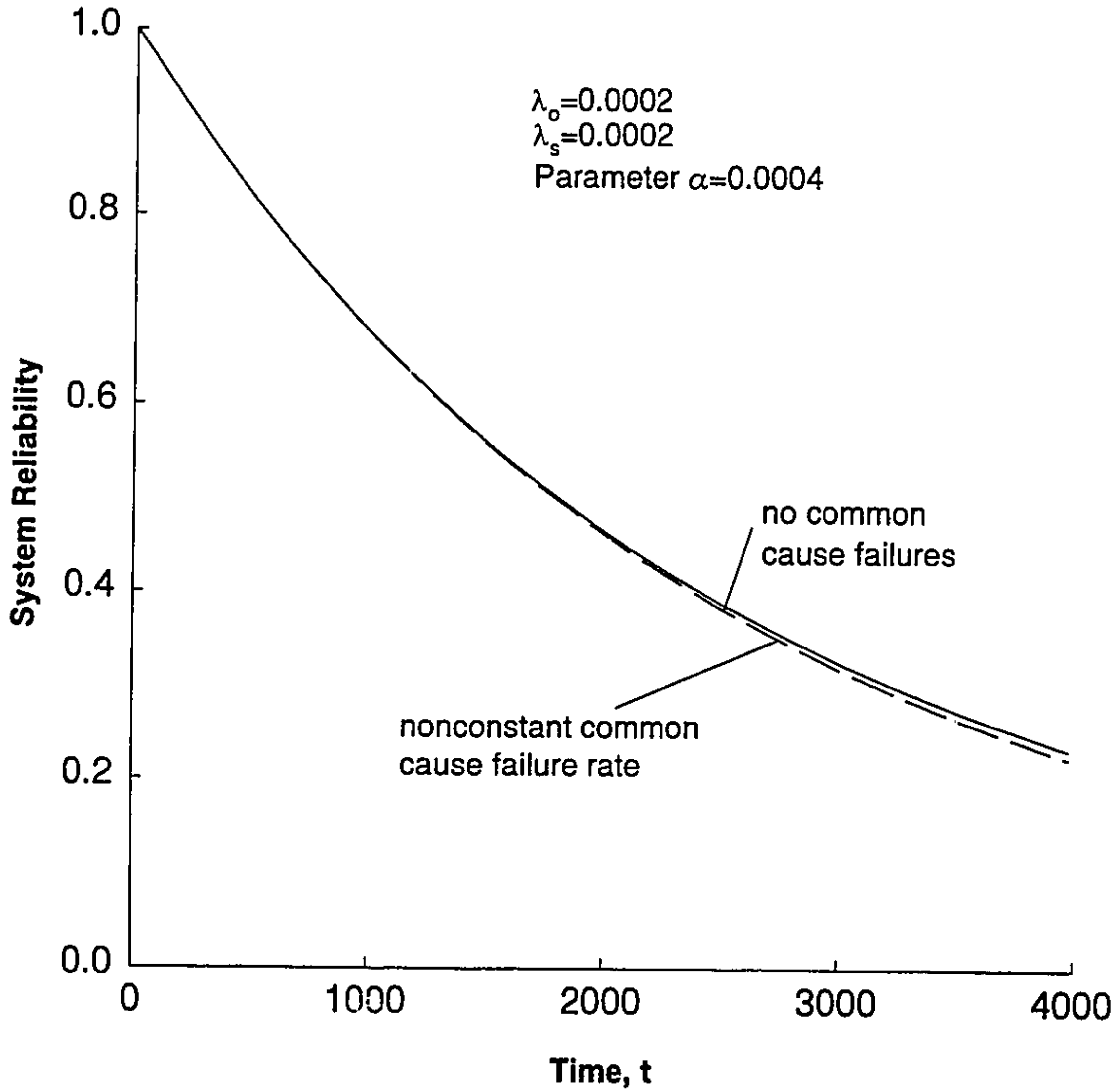


Figure 5.10 Series system reliability plots without common cause and with nonconstant common-cause failure rates

Chapter 6

Conclusions and Recommendations

6.1 Conclusions

This study develops generalized expressions for the system reliability, the mean time to failure (MTTF) and the steady state availability with constant failure rates. In the cold standby system, it is shown that the system performance indices such as the reliability, the MTTF, the steady state availability and the time dependent availability increase as the number of devices in the system increases, regardless whether the repairs are performed or not. However, a decrease in the performance indices is revealed with the progression of time and an increase in the common-cause failure rate. The same trend is observed for the warm standby system. However, the reliability, the MTTF, the steady state availability and the time dependent availability of the warm standby system are lower than values obtained for the cold standby system. Thus, it is concluded that the reliability analysis of the standby system without taking into consideration the occurrence of a standby device failure or a common-cause failure would lead to an overestimation of the reliability parameters.

The *Method of Stages* can be used as a good approximation to analyze systems with non-constant failure rates. For a two three-state device system, it is concluded that the highest reliability and availability can be obtained with a standby configuration, regardless whether the common-cause rate is constant or non-constant. If the open mode failure rate of a device is greater than its short mode failure rate, the parallel configuration is better than the series configuration in term of reliability and availability, or vice versa. In addition, the two devices in parallel with one device on standby configuration has higher reliability and availability than the one device on standby configuration.

6.2 Recommendations

For all of the models in this study an assumption is made that the repair rate is constant. This study can be extended to include both the time dependent repair rate as well as more general failure rates. Because the three-state device system has more system states than the two-state device system for the same number of devices, the calculations are more cumbersome, especially for the warm standby system. When the Lapalce transforms method is used, the inversion of the resulting Laplace transforms expression from s-domain to t-domain is very difficult, and the analytical expressions for the time dependent availability could not be developed. Future study could be conducted to develop techniques to overcome this difficulty.

The *Method of Stages* should be further studied in different systems. Because its accuracy of results obtained through its usage vary under certain conditions, thus its scope of application should be verified with a strict mathematical proof.

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Appendix 1.1

Reliability Characteristics

The following four functions are commonly used in reliability analysis:

- f(t)--probability density function (pdf) of failure time distribution

  $$f(t) = \frac{dF(t)}{dt}$$

- F(t)-- cumulative failure function (failure probability at time t)

  $$F(t) = \int_0^t f(\xi)d\xi$$

- R(t)--reliability function (cumulative unfailing probability at time t)

  $$R(t) = 1 - F(t)$$

- λ(t)--hazard function (failure rate at time t)

  $$\lambda(t) = \frac{f(t)}{R(t)}$$

The relationships between four functions are

$$f(t) = \frac{dF(t)}{dt} = -\frac{dR(t)}{dt} = \lambda(t)\exp[-\int_0^t \lambda(\xi)d\xi]$$

$$\lambda(t) = \frac{f(t)}{R(t)} = \frac{f(t)}{1-\int_0^t f(t)dt} = \frac{1}{1-F(t)}\frac{dF(t)}{dt} = -\frac{d[\ln R(t)]}{dt}$$

$$F(t) = 1 - R(t) = \int_0^t f(\xi)d\xi = 1 - \exp[-\int_0^t \lambda(\xi)d\xi]$$

$$R(t) = 1 - F(t) = 1 - \int_0^t f(t)dt = \exp[-\int_0^t \lambda(\xi)d\xi]$$

There is a basic parameter that can be used to describe a probability distribution expected value. In reliability analysis, the expected value is known as the mean time to failure or simply MTTF. MTTF mathematically is the first moment of failure time distribution.

$$MTTF = E[t] = \int_0^\infty tf(t)dt$$

Alternatively, MTTF can be expressed in various different forms:

$$MTTF = \int_0^\infty tf(t)dt = \int_0^\infty t\lambda(t)\exp[-\int_o^t \lambda(\xi)d\xi]dt$$

$$MTTF = \int_0^\infty R(t)dt = \int_0^\infty \exp[-\int_o^t \lambda(\xi)d\xi]dt$$

$$MTTF = \lim_{s \to 0} R(s)$$

where the R(s) is the Lapalce transform of system reliability.

Appendix 1.2

Description of Method of Stages

In a system combining two or more units with a constant failure rate in parallel or series configuration, the system failure density function is not an exponential distribution even if those of the individual unit are exponential. The reverse also is true. When we analyze non-constant failure rates, they can be divided into several sub-states (dummy states) each of which is exponentially distributed. In the following illustration with one dummy state is presented as an example. In this case, it means the non-constant failure rate is $\lambda(t)$. The constant failure transition rate between the up state and the dummy state is α_1 and the constant failure transition rate between the dummy state and the down state is α_2.

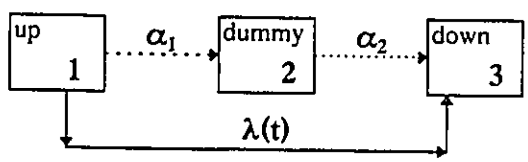

Figure 1 Modeling a nonconstant failure rate using a dummy state

The following differential equations are associated with Figure 1.

$$\frac{dP_1(t)}{dt} = -\alpha_1 P_1(t)$$

$$\frac{dP_2(t)}{dt} = -\alpha_2 P_2(t) + \alpha_1 P_1(t)$$

$$\frac{dP_3(t)}{dt} = \alpha_2 P_2(t)$$

At time t=0, $P_1(t) = 1, P_2(t) = P_3(t) = 0$.

The reliability is given by

$$R(t) = P_1(t) + P_2(t) = \frac{e^{-\alpha_2 t}\alpha_2}{(\alpha_2 - \alpha_1)} - \frac{e^{-\alpha_1 t}\alpha_1}{(\alpha_2 - \alpha_1)} \qquad (\alpha_1 \neq \alpha_2)$$

The system time dependent failure rate $\lambda(t)$ from the up state to the down state is

$$\lambda(t) = -\frac{dR(t)}{R(t)} = \frac{\alpha_1\alpha_2(e^{-\alpha_2 t} - e^{-\alpha_1 t})}{e^{-\alpha_1 t}\alpha_2 - e^{-\alpha_2 t}\alpha_1} \qquad (\alpha_1 \neq \alpha_2)$$

α_1 and α_2 are parameters of the non-constant failure rate $\lambda(t)$. If $\alpha_1=\alpha_2=\alpha$ the failure density function with the time dependnet failure rate $\lambda(t)$ becomes a gamma distribution. Thus, we have

$$R(t) = (1+\alpha t)e^{-\alpha t}$$

$$\lambda(t) = -\frac{dR(t)}{R(t)} = \frac{\alpha^2 t}{1+\alpha^2 t}$$

Figure 2 is the time dependent failure rate plot with $\alpha_1=\alpha_2=\alpha$. It is found that $\lambda(t)$ reaches steady state as time t approaches infinity, i.e.,$\lim_{t\to\infty}\lambda(t) = \alpha$.

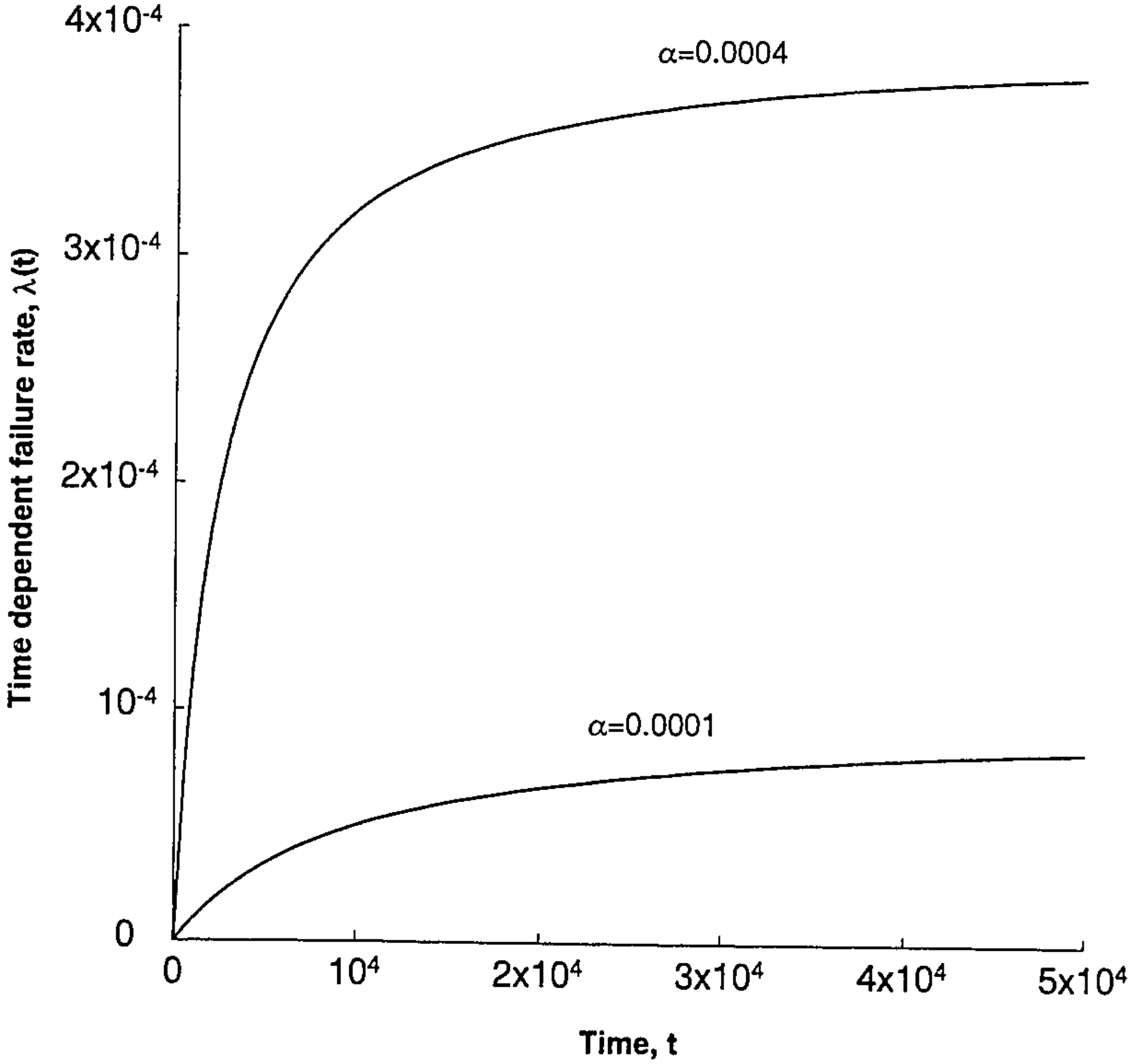

Figure 2 Nonconstant failure rate $\lambda(t)$ plots for different values of the parameter α

Appendix 2.1

$$
\begin{aligned}
A(2,1) = {} & s(\lambda_o\lambda_c\mu + \lambda_s\lambda_c\mu + \lambda_o^2\mu_c + 2\lambda_o\lambda_s\mu_c + \lambda_s^2\mu_c + 2\lambda_o\mu\mu_c + 2\lambda_s\mu\mu_c + \lambda_o^2 s + 2\lambda_o\lambda_s s + \\
& \lambda_s^2 s + \lambda_o\lambda_c s + \lambda_s\lambda_c s + 2\lambda_o\mu s + 2\lambda_s\mu s + \lambda_c\mu s + 2\lambda_o\mu_c s + 2\lambda_s\mu_c s + \mu\mu_c s + 2\lambda_o s^2 + \\
& 2\lambda_s s^2 + \lambda_c s^2 + \mu s^2 + \mu_c s^2 + s^3)
\end{aligned}
$$

$$
\begin{aligned}
A(2,2) = {} & s(2\lambda_o^2\lambda_c\mu + 4\lambda_o\lambda_s\lambda_c\mu + 2\lambda_s^2\lambda_c\mu + \lambda_o\lambda_c^2\mu + \lambda_s\lambda_c^2\mu_c + \lambda_c^3\mu_c + 3\lambda_o^2\lambda_s\mu_c + 3\lambda_o\lambda_s^2\mu_c + \\
& \lambda_s^3\mu_c + 3\lambda_o^2\mu\mu_c + 6\lambda_o\lambda_s\mu\mu_c + 3\lambda_s^2\mu\mu_c + \lambda_o\lambda_c\mu\mu_c + \lambda_s\lambda_c\mu\mu_c + \lambda_o^3 s + 3\lambda_o^2\lambda_s s + 3\lambda_o\lambda_s^2 s + \\
& \lambda_s^3 s + 2\lambda_o^2\lambda_c s + 4\lambda_o\lambda_s\lambda_c s + 2\lambda_s^2\lambda_c s + \lambda_o\lambda_c^2 s + \lambda_s\lambda_c^2 s + 3\lambda_o^2\mu s + 6\lambda_o\lambda_s\mu s + 3\lambda_s^2\mu s + \\
& 4\lambda_o\lambda_c\mu s + 4\lambda_s\lambda_c\mu s + \lambda_c^2\mu s + 3\lambda_o^2\mu_c s + 6\lambda_o\lambda_s\mu_c s + 3\lambda_s^2\mu_c s + \lambda_o\lambda_c\mu_c s + \lambda_s\lambda_c\mu_c s + \\
& 3\lambda_o\mu\mu_c s + 3\lambda_s\mu\mu_c s + \lambda_c\mu\mu_c s + 3\lambda_o^2 s^2 + 6\lambda_o\lambda_s s^2 + 3\lambda_s^2 s^2 + 4\lambda_o\lambda_c s^2 + 4\lambda_s\lambda_c s^2 + \lambda_c^2 s^2 + \\
& 3\lambda_o\mu s + 3\lambda_s\mu s + 2\lambda_c\mu s^2 + 3\lambda_o\mu_c s^2 + 3\lambda_s\mu_c s^2 + \lambda_c\mu_c s^2 + \mu\mu_c s^2 + 3\lambda_o s^3 + 3\lambda_s s^3 + \\
& 2\lambda_c s^3 + \mu s^3 + \mu_c s^3 + s^4)
\end{aligned}
$$

$$
A(2,3) = (3\lambda_o^2 + 6\lambda_o\lambda_s + 3\lambda_s^2 + \lambda_o\lambda_c + \lambda_s\lambda_c + 3\lambda_o s + 3\lambda_s s + \lambda_c s + s^2)(\mu + s)(\mu_c + s)
$$

$$
\begin{aligned}
A(2,4) = {} & 3\lambda_o^3\lambda_c\mu + 9\lambda_o^2\lambda_s\lambda_c\mu + 9\lambda_o\lambda_s^2\lambda_c\mu + 3\lambda_s^3\lambda_c\mu + 3\lambda_o^2\lambda_c^2\mu + 6\lambda_o\lambda_s\lambda_c^2\mu + 3\lambda_s^2\lambda_c^2\mu + \\
& \lambda_o\lambda_c^3\mu + \lambda_s\lambda_c^3\mu + \lambda_o^4\mu_c + 4\lambda_o^3\lambda_s\mu_c + 6\lambda_o^2\lambda_s^2\mu_c + 4\lambda_o\lambda_s^3\mu_c + \lambda_s^4\mu_c + 4\lambda_o^3\mu\mu_c + \\
& 2\lambda_o^2\lambda_s\mu\mu_c + 12\lambda_o\lambda_s^2\mu\mu_c + 4\lambda_s^3\mu\mu_c + 3\lambda_o^2\lambda_c\mu\mu_c + 6\lambda_o\lambda_s\lambda_c\mu\mu_c + 3\lambda_s^2\lambda_c\mu\mu_c + \\
& \lambda_o\lambda_c^2\mu\mu_c + \lambda_s\lambda_c^2\mu\mu_c
\end{aligned}
$$

$$
\begin{aligned}
A(2,5) = {} & s(3\lambda_o^3\lambda_c\mu + 9\lambda_o^2\lambda_s\lambda_c\mu + 9\lambda_o\lambda_s^2\lambda_c\mu + 3\lambda_s^3\lambda_c\mu + 3\lambda_o^2\lambda_c^2\mu + 6\lambda_o\lambda_s\lambda_c^2\mu + 3\lambda_s^2\lambda_c^2\mu + \\
& \lambda_o\lambda_c^3\mu + \lambda_s\lambda_c^3\mu + \lambda_o^4\mu_c + 4\lambda_o^3\lambda_s\mu_c + 6\lambda_o^2\lambda_s^2\mu_c + 4\lambda_o\lambda_s^3\mu_c + \lambda_s^4\mu_c + 4\lambda_o^3\mu\mu_c + \\
& 2\lambda_o^2\lambda_s\mu\mu_c + 12\lambda_o\lambda_s^2\mu\mu_c + 4\lambda_s^3\mu\mu_c + 3\lambda_o^2\lambda_c\mu\mu_c + 6\lambda_o\lambda_s\lambda_c\mu\mu_c + 3\lambda_s^2\lambda_c\mu\mu_c + \\
& \lambda_o\lambda_c^2\mu\mu_c + \lambda_s\lambda_c^2\mu\mu_c + \lambda_o^4 s + 4\lambda_o^3\lambda_s s + 6\lambda_o^2\lambda_s^2 s + 4\lambda_o\lambda_s^3 s + \lambda_s^2 s + 3\lambda_o^3\lambda_c s + 3\lambda_o^2\lambda_c^2 s + \\
& \lambda_o\lambda_c^3 s + \lambda_s\lambda_c^3 s + 4\lambda_o^4\mu s + 12\lambda_o^2\lambda_s\mu s + 12\lambda_o\lambda_s^2\mu s + 4\lambda_s^3\mu s + 9\lambda_o^2\lambda_c\mu s + 18\lambda_o\lambda_s\lambda_c\mu s + \\
& 9\lambda_s^2\lambda_c\mu s + 6\lambda_o\lambda_c^2\mu s + 6\lambda_s\lambda_c^2\mu s + \lambda_c^3\mu s + 4\lambda_o^3\mu_c s + 12\lambda_o^2\lambda_c\mu_c s + 12\lambda_o\lambda_s^2\mu_c s + \\
& 4\lambda_s^3\mu_c s + 3\lambda_o^2\lambda_c\mu_c s + 6\lambda_o\lambda_s\lambda_c\mu_c s + 3\lambda_s^2\lambda_c\mu_c s + \lambda_o\lambda_c^2\mu_c s + \lambda_o\lambda_c^2\mu_c s + \lambda_s\lambda_c^2\mu_c s + \\
& 6\lambda_o^2\mu\mu_s s + 12\lambda_o\lambda_s\mu\mu_c s + 6\lambda_s^2\mu\mu_s s + 5\lambda_o\lambda_c\mu\mu_c s + 5\lambda_s\lambda_c\mu\mu_c s + \lambda_c^2\mu\mu_s s + 4\lambda_o^3 s^2 + \\
& 2\lambda_o^2\lambda_s s^2 + 12\lambda_o\lambda_s^2 s^2 + 4\lambda_s^3 s^2 + 9\lambda_o^2\lambda_c s^2 + 18\lambda_o\lambda_s\lambda_c s^2 + 9\lambda_s^2\lambda_c s^2 + 6\lambda_o\lambda_c^2 s^2 + 6\lambda_s\lambda_c^2 s^2 +
\end{aligned}
$$

$$\lambda_t^3 s^2+6\lambda_b^2\mu s^2+12\lambda_b\lambda_s\mu s^2+6\lambda_s^2\mu s^2+9\lambda_b\lambda_t\mu s^2+9\lambda_s\lambda_t\mu s^2+3\lambda_t^2\mu s^2+6\lambda_b^2\mu_c s^2+$$

$$2\lambda_b\lambda_s\mu_c s^2+6\lambda_s^2\mu_c s^2+5\lambda_b\lambda_t\mu_c s+5\lambda_s\lambda_t\mu_c s+\lambda_t^2\mu_c s^2+4\lambda_b\mu\mu_c s^2+4\lambda_s\mu\mu_c s^2+$$

$$2\lambda_t\mu\mu_c s^2+6\lambda_b^2 s^3+12\lambda_b\lambda_s s^3+6\lambda_s^2 s^3+9\lambda_b\lambda_t s^3+9\lambda_s\lambda_t s^3+3\lambda_t^2 s^3+4\lambda_b\mu s^3+4\lambda_s\mu s^3+$$

$$2\lambda_t\mu s^3+\mu\mu_c s^3+4\lambda_b s^4+4\lambda_s s^4+3\lambda_t s^4+\mu s^4+\mu_c s^4+s^5)$$

$$A(2,6)=(\lambda_b+\lambda_s+s)(\lambda_b+\lambda_s+\lambda_t+s)^2(\mu+s)(\mu_c+s)$$

$$A(2,7)=\lambda_b(\lambda_b+\lambda_s+s)(\lambda_b+\lambda_s+\lambda_t+s)(\mu+s)(\mu_c+s)$$

$$A(2,8)=\lambda_s(\lambda_b+\lambda_s+s)(\lambda_b+\lambda_s+\lambda_t+s)(\mu+s)(\mu_c+s)$$

$$A(2,9)=\lambda_b^2(\lambda_b+\lambda_s+s)(\mu+s)(\mu_c+s)$$

$$A(2,10)=\lambda_b\lambda_s(\lambda_b+\lambda_s+s)(\mu+s)(\mu_c+s)$$

$$A(2,11)=\lambda_s^2(\lambda_b+\lambda_s+s)(\mu+s)(\mu_c+s)$$

$$A(2,12)=\lambda_b^3(\mu+s)(\mu_c+s)$$

$$A(2,13)=\lambda_b^2\lambda_s(\mu+s)(\mu_c+s)$$

$$A(2,14)=\lambda_b\lambda_s^2(\mu+s)(\mu_c+s)$$

$$A(2,15)=\lambda_s^3(\mu+s)(\mu_c+s)$$

$$A(2,16)=\lambda_s(\lambda_b+\lambda_s)^3(\mu_c+s)$$

$$A(2,17)=\lambda_b(\lambda_b+\lambda_s)^3(\mu_c+s)$$

$$A(2,18)=\lambda_t(\lambda_b+\lambda_s+s)(\mu+s)[3(\lambda_b+\lambda_s)^2+3(\lambda_b+\lambda_s)(\lambda_t+s)+(\lambda_t+s)^2]$$

$$A(2,19)=(\mu+s)(\mu_c+s)[4(\lambda_b+\lambda_s)^3+3(\lambda_b+\lambda_s)^2(\lambda_t+2s)+(\lambda_b+\lambda_s)(\lambda_t+s)(\lambda_t+4s)+$$

$$s(\lambda_t+s)^2]$$

$$A(2,20)=(\lambda_b^3+3\lambda_b^2\lambda_s+3\lambda_b\lambda_s^2+\lambda_s^3+2\lambda_b^2\lambda_t+4\lambda_b\lambda_s\lambda_t+2\lambda_s^2\lambda_t+\lambda_b\lambda_t^2+\lambda_s\lambda_t^2+2\lambda_b\lambda_t\mu+2\lambda_s\lambda_t\mu+$$

$$\lambda_t^2\mu+\lambda_t\mu^2+3\lambda_b^2 s+6\lambda_b\lambda_s s+3\lambda_s^2 s+4\lambda_b\lambda_t s+4\lambda_s\lambda_t s+\lambda_t^2 s+2\lambda_b\mu s+2\lambda_s\mu s+3\lambda_t\mu s+$$

$$\mu^2 s+3\lambda_b s^2+3\lambda_s s^2+2\lambda_t s^2+3\mu s^2+s^3)$$

$$A(2,21)=(\lambda_b+\lambda_s)^2+(\lambda_b+\lambda_s)(\lambda_t+\mu+2s)+(\mu+s)(\lambda_t+\mu+s)$$

$$A(2,22)=\lambda_t[2(\lambda_b+\lambda_s)^2+(\lambda_b+\lambda_s)(\lambda_t+2\mu+3s)+(\mu+s)(\lambda_t+\mu+s)]$$

$$A(2,23)=3(\lambda_b+\lambda_s)^2+(\lambda_b+\lambda_s)(\lambda_t+2\mu+3s)+(\mu+s)(\lambda_t+\mu+s)$$

$$A(2,24)=\lambda_b^4+4\lambda_b^3\lambda_s+6\lambda_b^2\lambda_s^2+4\lambda_b\lambda_s^3+\lambda_s^4+4\lambda_b^3\lambda_t+9\lambda_b^2\lambda_s\lambda_t+9\lambda_b\lambda_s^2\lambda_t+3\lambda_s^3\lambda_t+3\lambda_b^2\lambda_t^2+$$

$$6\lambda_b\lambda_s\lambda_t^2+3\lambda_s^2\lambda_t^2+\lambda_b\lambda_t^3+\lambda_s\lambda_t^3+3\lambda_b^2\lambda_t\mu+6\lambda_b\lambda_s\lambda_t\mu+3\lambda_s^2\lambda_t\mu+4\lambda_b\lambda_t^2\mu+$$

$$
\begin{aligned}
&4\lambda_s\lambda_t^2\mu+\lambda_t^3\mu+2\lambda_b\lambda_t\mu^2+2\lambda_s\lambda_t\mu^2+2\lambda_t^2\mu^2+\lambda_t\mu^3+4\lambda_b^3s+12\lambda_b^2\lambda_s s+\\
&2\lambda_b\lambda_s^2s+4\lambda_s^3s+9\lambda_b^2\lambda_t s+18\lambda_b\lambda_s\lambda_t s+9\lambda_s^2\lambda_t s+6\lambda_b\lambda_t^2s+6\lambda_s\lambda_t^2s+\lambda_t^3s+3\lambda_b^2\mu s+\\
&6\lambda_b\lambda_s\mu s+3\lambda_s^2\mu s+10\lambda_b\lambda_t\mu s+10\lambda_s\lambda_t\mu s+5\lambda_t^2\mu s+2\lambda_b\mu^2s+2\lambda_s\mu^2s+5\lambda_t\mu^2s+\\
&\mu^3s+6\lambda_b^2s^2+12\lambda_b\lambda_s s^2+6\lambda_s^2s^2+9\lambda_b\lambda_t s^2+9\lambda_s\lambda_t s^2+3\lambda_t^2s^2+6\lambda_b\mu s^2+6\lambda_s\mu s^2+\\
&7\lambda_t\mu s^2+3\mu^2s^2+4\lambda_b s^3+4\lambda_s s^3+3\lambda_t s^3+3\mu s^3+s^4
\end{aligned}
$$

$$
\begin{aligned}
A(2,25)=\;&\lambda_b^3+3\lambda_b^2\lambda_s+3\lambda_b\lambda_s^2+\lambda_s^3+2\lambda_b^2\lambda_t+4\lambda_b\lambda_s\lambda_t+2\lambda_s^2\lambda_t+\lambda_b\lambda_t^2+\lambda_s\lambda_t^2+\lambda_b^2\mu+2\lambda_b\lambda_s s+\\
&\lambda_s^2\mu+3\lambda_b\lambda_t\mu+3\lambda_s\lambda_t\mu+\lambda_t^2\mu+\lambda_b\mu^2+\lambda_s\mu^2+2\lambda_t\mu^2+\mu^3+3\lambda_b^2s+6\lambda_b\lambda_s s+\\
&3\lambda_s^2s+4\lambda_b\lambda_t s+4\lambda_s\lambda_t s+\lambda_t^2s+4\lambda_b\mu s+4\lambda_s\mu s+4\lambda_t\mu s+3\mu^2s+3\lambda_b s^2+3\lambda_s s^2+\\
&2\lambda_t s^2+3\mu s^2+s^3
\end{aligned}
$$

$$
A(2,26)=\lambda_b^2+2\lambda_b\lambda_s+\lambda_s^2+\lambda_b\lambda_t+\lambda_s\lambda_t+\lambda_b\mu+\lambda_s\mu+\lambda_t\mu+\mu^2+2\lambda_b s+2\lambda_s s+\lambda_t s+2\mu s+s^2
$$

$$
\begin{aligned}
A(2,27)=\;&\lambda_t(3\lambda_b^3+9\lambda_b^2\lambda_s+9\lambda_b\lambda_s^2+3\lambda_s^3+3\lambda_b^2\lambda_t+6\lambda_b\lambda_s\lambda_t+3\lambda_s^2\lambda_t+\lambda_b\lambda_t^2+\lambda_s\lambda_t^2+3\lambda_b^2\mu+\\
&6\lambda_b\lambda_s\mu+3\lambda_s^2\mu+4\lambda_b\lambda_t\mu+4\lambda_s\lambda_t\mu+\lambda_t^2\mu+2\lambda_b\mu^2+2\lambda_s\mu^2+2\lambda_t\mu^2+\mu^3+6\lambda_b^2s+\\
&2\lambda_b\lambda_s s+6\lambda_s^2s+5\lambda_b\lambda_t s+5\lambda_s\lambda_t s+\lambda_t^2s+6\lambda_b\mu s+6\lambda_s\mu s+4\lambda_t\mu s+3\mu^2s+4\lambda_b s^2+\\
&4\lambda_s s^2+2\lambda_t s^2+3\mu s^2+s^3)
\end{aligned}
$$

$$
\begin{aligned}
A(2,28)=\;&4\lambda_b^3+12\lambda_b^2\lambda_s+12\lambda_b\lambda_s^2+4\lambda_s^3+3\lambda_b^2\lambda_t+6\lambda_b\lambda_s\lambda_t+3\lambda_s^2\lambda_t+\lambda_b\lambda_t^2+\lambda_b\lambda_t^2+3\lambda_b^2\mu+\\
&6\lambda_b\lambda_s\mu+3\lambda_s^2\mu+4\lambda_b\lambda_t\mu+4\lambda_s\lambda_t\mu+\lambda_t^2\mu+2\lambda_b\mu^2+2\lambda_s\mu^2+2\lambda_t\mu^2+\mu^3+6\lambda_b^2s+\\
&2\lambda_b\lambda_s s+6\lambda_s^2s+5\lambda_b\lambda_t s+5\lambda_s\lambda_t s+\lambda_t^2s+6\lambda_b\mu s+6\lambda_s\mu s+4\lambda_t\mu s+3\mu^2s+4\lambda_b s^2+\\
&4\lambda_s s^2+2\lambda_t s^2+3\mu s^2+s^3
\end{aligned}
$$

$$
\begin{aligned}
A(2,29)=\;&\lambda_b^4+4\lambda_b^3\lambda_s+6\lambda_b^2\lambda_s^2+4\lambda_b\lambda_s^3+\lambda_s^4+4\lambda_b^3\lambda_t+9\lambda_b^2\lambda_s\lambda_t+9\lambda_b\lambda_s^2\lambda_t+3\lambda_s^3\lambda_t+3\lambda_b^2\lambda_t^2+\\
&6\lambda_b\lambda_s\lambda_t^2+3\lambda_s^2\lambda_t^2+\lambda_b\lambda_t^3+\lambda_s\lambda_t^3+3\lambda_b^2\lambda_t\mu+6\lambda_b\lambda_s\lambda_t\mu+3\lambda_s^2\lambda_t\mu+4\lambda_b\lambda_t^2\mu+\\
&4\lambda_s\lambda_t^2\mu+\lambda_t^3\mu+2\lambda_b\lambda_t\mu^2+2\lambda_s\lambda_t\mu^2+2\lambda_t^2\mu^2+\lambda_t\mu^3
\end{aligned}
$$

$$
\begin{aligned}
A(2,30)=\;&4\lambda_b^3+12\lambda_b^2\lambda_s+12\lambda_b\lambda_s^2+4\lambda_s^3+3\lambda_b^2\lambda_t+6\lambda_b\lambda_s\lambda_t+3\lambda_s^2\lambda_t+\lambda_b\lambda_t^2+\lambda_b\lambda_t^2+3\lambda_b^2\mu+\\
&6\lambda_b\lambda_s\mu+3\lambda_s^2\mu+4\lambda_b\lambda_t\mu+4\lambda_s\lambda_t\mu+\lambda_t^2\mu+2\lambda_b\mu^2+2\lambda_s\mu^2+2\lambda_t\mu^2+\mu^3
\end{aligned}
$$

Appendix 3.1

$$A(3,1) = (\lambda_b + \lambda_s)^2 \mu_c + (\lambda_b + \lambda_s)\lambda_c \mu + (\lambda_b + \lambda_s)(\lambda_{so} + \lambda_{ss}) \mu_c + (2\lambda_b + 2\lambda_s + \lambda_{so} + \lambda_{ss}) \mu\mu_c$$

$$\begin{aligned} A(3,2) = s(&\lambda_b \lambda_c \mu + \lambda_s \lambda_c \mu + \lambda_b^2 \mu_c + 2\lambda_b \lambda_s \mu_c + \lambda_s^2 \mu_c + \lambda_b \lambda_{so} \mu_c + \lambda_s \lambda_{so} \mu_c + \lambda_b \lambda_{ss} \mu_c + \\ &\lambda_s \lambda_{ss} \mu_c + 2\lambda_o \mu\mu_c + 2\lambda_s \mu\mu_c + \lambda_{so} \mu\mu_c + \lambda_{ss} \mu\mu_c + \lambda_b^2 s + 2\lambda_b \lambda_s s + \lambda_s^2 s + \lambda_b \lambda_c s + \\ &\lambda_s \lambda_c s + \lambda_b \lambda_{sc} s + \lambda_s \lambda_{sc} s + \lambda_o \lambda_{ss} s + \lambda_s \lambda_{ss} s + 2\lambda_b \mu s + 2\lambda_s \mu s + \lambda_c \mu s + \lambda_{so} \mu s + \lambda_{ss} \mu s + \\ &2\lambda_b \mu_c s + 2\lambda_s \mu_c s + \lambda_{so} \mu_c s + \lambda_{ss} \mu_c s + \mu\mu_c s + 2\lambda_o s^2 + 2\lambda_s s^2 + \lambda_c s^2 + \lambda_{so} s^2 + \lambda_{ss} s^2 + \\ &\mu s^2 + \mu_c s^2 + s^3) \end{aligned}$$

$$\begin{aligned} A(3.3) = &\lambda_b^2 + 2\lambda_b \lambda_s + \lambda_s^2 + \lambda_b \lambda_c + \lambda_b \lambda_c + \lambda_b \lambda_{so} + \lambda_s \lambda_{so} + \lambda_b \lambda_{ss} + \lambda_s \lambda_{ss} + \lambda_c \mu + 2\lambda_b s + 2\lambda_s s + \\ &\lambda_c s + \lambda_{so} s + \lambda_{ss} s + \mu s + s^2 \end{aligned}$$

$$\begin{aligned} A(3,4) = &2\lambda_o^2 \lambda_c \mu + 4\lambda_o \lambda_s \lambda_c \mu + 2\lambda_s^2 \lambda_c \mu + \lambda_o \lambda_c^2 \mu + \lambda_s \lambda_c^2 \mu + 2\lambda_o \lambda_c \lambda_{so} \mu + 2\lambda_s \lambda_c \lambda_{so} \mu + \\ &2\lambda_b \lambda_c \lambda_{ss} \mu + 2\lambda_s \lambda_c \lambda_{ss} \mu + \lambda_b^3 \mu_c + 3\lambda_b^2 \lambda_s \mu_c + 3\lambda_b \lambda_s^2 \mu_c + \lambda_s^2 \mu_c + 2\lambda_b^2 \lambda_{so} \mu_c + 4\lambda_b \lambda_s \lambda_{so} \mu_c + \\ &2\lambda_s^2 \lambda_{so} \mu_c + \lambda_b \lambda_{so}^2 \mu_c + \lambda_s \lambda_{so}^2 \mu_c + 2\lambda_b^2 \lambda_{ss} \mu_c + 4\lambda_b \lambda_s \lambda_{ss} \mu_c + 2\lambda_s^2 \lambda_{ss} \mu_c + 2\lambda_b \lambda_{so} \lambda_{ss} \mu_c + \\ &2\lambda_s \lambda_{so} \lambda_{ss} \mu_c + \lambda_b \lambda_{ss}^2 \mu_c + \lambda_s \lambda_{ss}^2 \mu_c + 3\lambda_b^2 \mu\mu_c + 6\lambda_b \lambda_s \mu\mu_c + 3\lambda_s^2 \mu\mu_c + \lambda_b \lambda_c \mu\mu_c + \\ &\lambda_s \lambda_c \mu\mu_c + 4\lambda_b \lambda_{so} \mu\mu_c + 4\lambda_s \lambda_{so} \mu\mu_c + \lambda_{so}^2 \mu\mu_c + 4\lambda_b \lambda_{ss} \mu\mu_c + 4\lambda_s \lambda_{ss} \mu\mu_c + 2\lambda_{so} \lambda_{ss} \mu\mu_c + \\ &\lambda_{ss}^2 \mu\mu_c \end{aligned}$$

$$\begin{aligned} A(3,5) = (&3\lambda_b^2 + 6\lambda_o \lambda_s + 3\lambda_s^2 + \lambda_o \lambda_c + \lambda_s \lambda_c + 4\lambda_o \lambda_{so} + 4\lambda_s \lambda_{so} + \lambda_{so}^2 + 4\lambda_o \lambda_{ss} + 4\lambda_s \lambda_{ss} + 2\lambda_{so} \lambda_{ss} + \\ &\lambda_{ss}^2) \mu\mu_c \end{aligned}$$

$$\begin{aligned} A(3,6) = s(&2\lambda_b^2 \lambda_c \mu + 4\lambda_b \lambda_s \lambda_c \mu + 2\lambda_s^2 \lambda_c \mu + \lambda_b \lambda_c^2 \mu + \lambda_s \lambda_c^2 \mu + 2\lambda_b \lambda_c \lambda_{so} \mu + 2\lambda_s \lambda_c \lambda_{so} \mu + \\ &2\lambda_b \lambda_c \lambda_{ss} \mu + 2\lambda_s \lambda_c \lambda_{ss} \mu + \lambda_b^3 \mu_c + 3\lambda_b^2 \lambda_s \mu_c + 3\lambda_b \lambda_s^2 \mu_c + \lambda_s^2 \mu_c + 2\lambda_b^2 \lambda_{so} \mu_c + 4\lambda_b \lambda_s \lambda_{so} \mu_c + \\ &2\lambda_s^2 \lambda_{so} \mu_c + \lambda_b \lambda_{so}^2 \mu_c + \lambda_s \lambda_{so}^2 \mu_c + 2\lambda_b^2 \lambda_{ss} \mu_c + 4\lambda_b \lambda_s \lambda_{ss} \mu_c + 2\lambda_s^2 \lambda_{ss} \mu_c + 2\lambda_b \lambda_{so} \lambda_{ss} \mu_c + \\ &2\lambda_s \lambda_{so} \lambda_{ss} \mu_c + \lambda_b \lambda_{ss}^2 \mu_c + \lambda_s \lambda_{ss}^2 \mu_c + 3\lambda_b^2 \mu\mu_c + 6\lambda_b \lambda_s \mu\mu_c + 3\lambda_s^2 \mu\mu_c + \lambda_b \lambda_c \mu\mu_c + \\ &\lambda_s \lambda_c \mu\mu_c + 4\lambda_b \lambda_{so} \mu\mu_c + 4\lambda_s \lambda_{so} \mu\mu_c + \lambda_{so}^2 \mu\mu_c + 4\lambda_b \lambda_{ss} \mu\mu_c + 4\lambda_s \lambda_{ss} \mu\mu_c + 2\lambda_{so} \lambda_{ss} \mu\mu_c + \\ &\lambda_{ss}^2 \mu\mu_c + \lambda_b^3 s + 3\lambda_b^2 \lambda_s s + 3\lambda_b \lambda_s^2 s + \lambda_s^3 s + 2\lambda_b^2 \lambda_c s + 4\lambda_b \lambda_s \lambda_c s + 2\lambda_s^2 \lambda_c s + \lambda_b \lambda_c^2 s + \lambda_s \lambda_c^2 s + \\ &2\lambda_b^2 \lambda_{so} s + 4\lambda_b \lambda_s \lambda_{so} s + 2\lambda_s^2 \lambda_{so} s + 2\lambda_b \lambda_c \lambda_{so} s + 2\lambda_s \lambda_c \lambda_{so} s + \lambda_b \lambda_{so}^2 s + \lambda_s \lambda_{so}^2 s + 2\lambda_b^2 \lambda_{ss} s + \\ &4\lambda_b \lambda_s \lambda_{ss} s + 2\lambda_s^2 \lambda_{ss} s + 2\lambda_b \lambda_c \lambda_{ss} s + 2\lambda_s \lambda_c \lambda_{ss} s + 2\lambda_b \lambda_{so} \lambda_{ss} s + 2\lambda_s \lambda_{so} \lambda_{ss} s + \lambda_b \lambda_{ss}^2 s + \lambda_s \lambda_{ss}^2 s + \end{aligned}$$

$$3\lambda_b^2\mu s+6\lambda_b\lambda_s\mu s+3\lambda_s^2\mu s+4\lambda_b\lambda_t\mu s+4\lambda_s\lambda_t\mu s+\lambda_t^2\mu s+4\lambda_b\lambda_{so}\mu s+4\lambda_s\lambda_{so}\mu s+$$
$$2\lambda_{so}\lambda_{ss}\mu s+\lambda_{so}^2\mu s+4\lambda_b\lambda_{ss}\mu s+4\lambda_s\lambda_{ss}\mu s+2\lambda_t\lambda_{ss}\mu s+2\lambda_{so}\lambda_{ss}\mu s+\lambda_{ss}^2\mu s+3\lambda_b^2\mu_c s+$$
$$6\lambda_b\lambda_s\mu_c s+3\lambda_s^2\mu_c s+\lambda_b\lambda_t\mu_c s+\lambda_s\lambda_t\mu_c s+4\lambda_b\lambda_{so}\mu_c s+4\lambda_s\lambda_{so}\mu_c s+\lambda_{so}^2\mu_c s+4\lambda_b\lambda_{ss}\mu_c s+$$
$$4\lambda_s\lambda_{ss}\mu_c s+2\lambda_{so}\lambda_{ss}\mu_c s+\lambda_{ss}^2\mu_c s+3\lambda_b\mu\mu_c s+3\lambda_s\mu\mu_c s+\lambda_t\mu\mu_c s+2\lambda_{so}\mu\mu_c s+$$
$$2\lambda_{ss}\mu\mu_c s+3\lambda_b^2 s^2+6\lambda_b\lambda_s s^2+3\lambda_s^2 s^2+4\lambda_b\lambda_t s^2+4\lambda_s\lambda_t s^2+\lambda_t^2 s^2+4\lambda_b\lambda_{so}s^2+4\lambda_s\lambda_{so}s^2+$$
$$2\lambda_t\lambda_{so}s^2+\lambda_{so}^2 s^2+4\lambda_b\lambda_{ss}s^2+4\lambda_s\lambda_{ss}s^2+2\lambda_t\lambda_{ss}s^2+2\lambda_{so}\lambda_{ss}s^2+\lambda_{ss}^2 s^2+3\lambda_b\mu s^2+3\lambda_s\mu s^2+$$
$$2\lambda_t\mu s^2+2\lambda_{so}\mu s^2+2\lambda_{ss}\mu s^2+3\lambda_b\mu_c s^2+3\lambda_s\mu_c s^2+\lambda_t\mu_c s^2+2\lambda_{so}\mu_c s^2+2\lambda_{ss}\mu_c s^2+$$
$$\mu\mu_c s^2+3\lambda_b s^3+3\lambda_s s^3+2\lambda_t s^3+2\lambda_{so}s^3+2\lambda_{ss}s^3+\mu s^3+\mu_c s^3+s^4)$$

$$A(3,7)=(3\lambda_o^2+6\lambda_o\lambda_s+3\lambda_s^2+\lambda_o\lambda_c+\lambda_s\lambda_c+4\lambda_o\lambda_{so}+4\lambda_s\lambda_{so}+\lambda_{so}^2+4\lambda_o\lambda_{ss}+4\lambda_s\lambda_{ss}+$$
$$2\lambda_{so}\lambda_{ss}+\lambda_{ss}^2+3\lambda_o s+3\lambda_s s+\lambda_c s+2\lambda_{so}s+2\lambda_{so}s+s^2)(\mu+s)(\mu_c+s)$$

$$A(3,8)=\lambda_o^3+3\lambda_o^2\lambda_s+3\lambda_o\lambda_s^2+\lambda_s^3+2\lambda_o^2\lambda_c+4\lambda_o\lambda_s\lambda_c+2\lambda_{so}^2\lambda_c+\lambda_o\lambda_c^2+\lambda_s\lambda_{so}^2+2\lambda_o^2\lambda_{so}+$$
$$4\lambda_b\lambda_s\lambda_{soc}++2\lambda_s^2\lambda_{so}+2\lambda_o\lambda_c\lambda_{so}+2\lambda_{so}\lambda_c\lambda_{so}+\lambda_o\lambda_{so}^2+\lambda_s\lambda_{so}^2+2\lambda_{os}^2\lambda_{ss}+4\lambda_o\lambda_s\lambda_{ss}+$$
$$2\lambda_s^2\lambda_{ss}+2\lambda_o\lambda_c\lambda_{ss}++2\lambda_s\lambda_c\lambda_{ss}+2\lambda_o\lambda_{so}\lambda_{ss}+2\lambda_s\lambda_c\lambda_{ss}+\lambda_o\lambda_{ss}^2+\lambda_s\lambda_{ss}^2+2\lambda_o\lambda_c\mu+$$
$$2\lambda_s\lambda_c\mu+\lambda_c^2\mu+\lambda_c\lambda_{so}\mu+\lambda_c\lambda_{ss}\mu+\lambda_c\mu^2+3\lambda_o^2 s+6\lambda_o\lambda_s s+3\lambda_s^2 s+4\lambda_o\lambda_c s+$$
$$4\lambda_s\lambda_t s+\lambda_t^2 s+4\lambda_b\lambda_{so}s+4\lambda_s\lambda_{so}s+2\lambda_t\lambda_{so}s+\lambda_{so}^2 s+4\lambda_b\lambda_{ss}s+4\lambda_s\lambda_{ss}s+2\lambda_t\lambda_{ss}s+$$
$$2\lambda_{so}\lambda_{ss}s+\lambda_{ss}^2 s+2\lambda_b\mu s+2\lambda_s\mu s+3\lambda_t\mu s+\lambda_{ss}\mu s+\mu^2 s+3\lambda_b s^2+3\lambda_s s^2+$$
$$2\lambda_t s^2+2\lambda_{so}s^2+2\lambda_{ss}s^2+2\mu s^2+s^3$$

$$A(3,9)=\lambda_b^2+2\lambda_b\lambda_s+\lambda_s^2+\lambda_b\lambda_t+\lambda_s\lambda_t+\lambda_b\lambda_{so}+\lambda_s\lambda_{so}+\lambda_b\lambda_{ss}+\lambda_s\lambda_{ss}+\lambda_b\mu+\lambda_s\mu+\lambda_t\mu+$$
$$\mu^2+2\lambda_b s+2\lambda_s s+\lambda_c s+\lambda_{so}s+\lambda_{ss}s+2\mu s+s^2$$

$$A(3,10)=\lambda_t(2\lambda_b^2+4\lambda_b\lambda_s+2\lambda_s^2+\lambda_b\lambda_t+\lambda_s\lambda_t+2\lambda_b\lambda_{so}+2\lambda_s\lambda_{so}+2\lambda_b\lambda_{ss}+2\lambda_s\lambda_{ss}+2\lambda_b\mu+$$
$$2\lambda_s\mu+\lambda_t\mu+\lambda_{so}\mu+\lambda_{ss}\mu+\mu^2+3\lambda_o s+3\lambda_s s+\lambda_c s+2\lambda_{so}s+2\lambda_{ss}s+2\mu s+s^2$$

$$A(3,11)=3\lambda_b^2+6\lambda_b\lambda_s+3\lambda_s^2+\lambda_b\lambda_t+\lambda_s\lambda_t+4\lambda_b\lambda_{so}+4\lambda_s\lambda_{so}+\lambda_{so}^2+4\lambda_b\lambda_{ss}+4\lambda_s\lambda_{ss}+$$
$$2\lambda_{so}\lambda_{ss}+\lambda_{ss}^2+2\lambda_b\mu+2\lambda_s\mu+\lambda_t\mu+\lambda_{so}\mu+\lambda_{ss}\mu+\mu^2+3\lambda_o s+3\lambda_s s+\lambda_c s+$$
$$2\lambda_{so}s+2\lambda_{ss}s+2\mu s+s^2$$

$$A(3,12)=3\lambda_b^2+6\lambda_b\lambda_s+3\lambda_s^2+\lambda_b\lambda_t+\lambda_s\lambda_t+4\lambda_b\lambda_{so}+4\lambda_s\lambda_{so}+\lambda_{so}^2+4\lambda_b\lambda_{ss}+4\lambda_s\lambda_{ss}+$$
$$2\lambda_{so}\lambda_{ss}+\lambda_{ss}^2+2\lambda_b\mu+2\lambda_s\mu+\lambda_t\mu+\lambda_{so}\mu+\lambda_{ss}\mu+\mu^2$$

$$A(3,13)=\lambda_b^3+3\lambda_b^2\lambda_s+3\lambda_b\lambda_s^2+\lambda_s^3+2\lambda_b^2\lambda_t+4\lambda_b\lambda_s\lambda_t+2\lambda_{so}^2\lambda_t+\lambda_b\lambda_t^2+\lambda_s\lambda_{so}^2+2\lambda_b^2\lambda_{so}+4\lambda_b\lambda_s\lambda_{soc}+$$

$$2\lambda_s^2\lambda_{so} + 2\lambda_b\lambda_t\lambda_{so} + 2\lambda_{so}\lambda_t\lambda_{so} + \lambda_b\lambda_{so}^2 + \lambda_s\lambda_{so}^2 + 2\lambda_{bs}^2\lambda_{ss} + 4\lambda_b\lambda_s\lambda_{ss} + 2\lambda_s^2\lambda_{ss} + 2\lambda_b\lambda_t\lambda_{ss} +$$
$$2\lambda_s\lambda_t\lambda_{ss} + 2\lambda_b\lambda_{so}\lambda_{ss} + 2\lambda_s\lambda_t\lambda_{ss} + \lambda_b\lambda_{ss}^2 + \lambda_s\lambda_{ss}^2 + 2\lambda_b\lambda_t\mu + 2\lambda_s\lambda_t\mu + \lambda_t^2\mu + \lambda_t\lambda_{so}\mu +$$
$$\lambda_t\lambda_{ss}\mu + \lambda_t\mu^2$$

Appendix 4.1

(i) The reliability comparison between the standby system and the parallel system is given by

$$R_{sp}(t) = R_{sb}(t) - R_p(t) = \frac{e^{(\lambda_o+\lambda_s)t}(\lambda_o+\lambda_s)^2 t + [e^{(\lambda_o+\lambda_s)t} - 1](\lambda_s - \lambda_o)}{e^{(2\lambda_o+2\lambda_s)t}(\lambda_o+\lambda_s)} \tag{4.17}$$

At t=0, $f(t) = e^{(\lambda_o+\lambda_s)t}(\lambda_o+\lambda_s)^2 t + [e^{(\lambda_o+\lambda_s)t} - 1](\lambda_s - \lambda_o) = 0$

If $f(t) = e^{(\lambda_o+\lambda_s)t}(\lambda_o+\lambda_s)^2 t + [e^{(\lambda_o+\lambda_s)t} - 1](\lambda_s - \lambda_o) \geq 0$, thus $R_{sp}(t) \geq 0$.

Taking derivative of $f(t)$ with respect to t

$$\frac{df(t)}{dt} = e^{(\lambda_o+\lambda_s)t}[(\lambda_o+\lambda_s)^3 t + 2\lambda_s^2 + 2\lambda_o\lambda_s] \geq 0$$

Here, $f(t)$ is a mono-increasing function. Because of $f(0) = 0$, the expression $f(t) = e^{(\lambda_o+\lambda_s)t}(\lambda_o+\lambda_s)^2 t + [e^{(\lambda_o+\lambda_s)t} - 1](\lambda_s - \lambda_o) \geq 0$, thus $R_{sp}(t) \geq 0$ is proved.

(ii) The reliability comparison between the standby system and the series system is given by

$$R_{ss}(t) = R_{sb}(t) - R_s(t) = \frac{e^{(\lambda_o+\lambda_s)t}(\lambda_o+\lambda_s)^2 t + [e^{(\lambda_o+\lambda_s)t} - 1](\lambda_o - \lambda_s)}{e^{(2\lambda_o+2\lambda_s)t}(\lambda_o+\lambda_s)} \tag{4.19}$$

The same method used in section (i) above can also be used to prove that equation (4.19) is not less than zero.

(iii) The reliability difference $R_{sp}(t)$ between the standby system and the parallel system with common cause failure is given by

$$R_{sp}(t) = R_{sb}(t) - R_p(t)$$

$$= \frac{(\lambda_o+\lambda_s)^2[e^{(\lambda_o+\lambda_s+\lambda_c)t} - e^{(\lambda_o+\lambda_s)t}] + (\lambda_s - \lambda_o)\lambda_c[e^{(\lambda_o+\lambda_s+\lambda_c)t} - 1] + \lambda_c^2(e^{(\lambda_o+\lambda_s)t} - 1)}{e^{(2\lambda_o+2\lambda_s+\lambda_c)t}(\lambda_o+\lambda_s+\lambda_c)\lambda_c} \tag{4.27}$$

Using the same method as in section (i)

$$f(t) = (\lambda_b + \lambda_s)^2[e^{(\lambda_b+\lambda_s+\lambda_c)t} - e^{(\lambda_b+\lambda_s)t}] + (\lambda_s - \lambda_b)\lambda_c[e^{(\lambda_b+\lambda_s+\lambda_c)t} - 1] + \lambda_c^2(e^{(\lambda_b+\lambda_s)t} - 1)$$

$$\frac{df(t)}{dt} = (\lambda_b + \lambda_s)^3[e^{(\lambda_b+\lambda_s+\lambda_c)t} - e^{(\lambda_b+\lambda_s)t}] + [2\lambda_s^2 + \lambda_b(2\lambda_s - \lambda_c)]\lambda_c e^{(\lambda_b+\lambda_s+\lambda_c)t} + (\lambda_b + \lambda_s)\lambda_c^2 e^{(\lambda_b+\lambda_s)t}$$

Since the common cause failure rate lies within 1%-30% of the component failure rate, so that $(2\lambda_s - \lambda_c) \geq 0$, therefore $\frac{df(t)}{dt} \geq 0$. Because of time t=0, $R_{sp}(t) = R_{sb}(t) - R_p(t) = 0$, thus $R_{sp}(t) \geq 0$.

(iv) The reliability difference between the standby system and the series system $R_{ss}(t)$ is

$$R_{ss}(t) = R_{sb}(t) - R_s(t)$$

$$= \frac{(\lambda_b + \lambda_s)^2[e^{(\lambda_b+\lambda_s+\lambda_c)t} - e^{(\lambda_b+\lambda_s)t}] + (\lambda_b - \lambda_s)\lambda_c[e^{(\lambda_b+\lambda_s+\lambda_c)t} + 1] + \lambda_c^2(e^{(\lambda_b+\lambda_s)t} - 1)}{e^{(2\lambda_b+2\lambda_s+\lambda_c)t}(\lambda_b + \lambda_s + \lambda_c)\lambda_c} \quad (4.29)$$

The same method as section (iii) above can also be used to prove that equation (4.29) is not less than zero.

(vi) The reliability difference between the two units in parallel with one unit on standby system and the one unit operating with one unit on standby system is

$$\Delta R = R_2(t) - R_1(t)$$

$$= \frac{(\lambda_b - \lambda_s)(\lambda_b + \lambda_s - \lambda_c)[e^{(\lambda_b+\lambda_s+\lambda_c)t}(\lambda_b + \lambda_s) - e^{(\lambda_b+\lambda_s)t}(\lambda_b + \lambda_s + \lambda_c) + \lambda_c]}{e^{(2\lambda_b+2\lambda_s+\lambda_c)t}(\lambda_b + \lambda_s + \lambda_c)(\lambda_b + \lambda_s)\lambda_c} \quad (4.72)$$

Let $f(t) = [e^{(\lambda_b+\lambda_s+\lambda_c)t}(\lambda_b + \lambda_s) - e^{(\lambda_b+\lambda_s)t}(\lambda_b + \lambda_s + \lambda_c) + \lambda_c]$

$$\frac{df(t)}{dt} = (\lambda_b + \lambda_s)(\lambda_b + \lambda_s + \lambda_c)[e^{(\lambda_b+\lambda_s+\lambda_c)t} - e^{(\lambda_b+\lambda_s)t}] \geq 0, \text{ and } f(0) = 0,$$

thus $f(t) \geq 0$. If $\lambda_b \geq \lambda_s$ and $(\lambda_b + \lambda_s - \lambda_c) \geq 0$, $\Delta R = R_2(t) - R_1(t) \geq 0$

Appendix 4.2

There is no consideration of common cause failure in the following discussion. The single unit system and the two units in parallel system are discussed and compared.

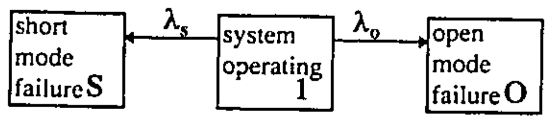

Figure 3 State space diagram of single unit system

Figure 3 is the state space diagram of the single unit system. The following differential equations are associated with Figure 3.

$$\frac{dP_1(t)}{dt} = -(\lambda_o + \lambda_s)P_1(t)$$

$$\frac{dP_o(t)}{dt} = \lambda_o P_1(t)$$

$$\frac{dP_s(t)}{dt} = \lambda_s P_1(t)$$

At time t=0, $P_s(0) = P_o(0) = 0$.

Solving the above differntial equations, we get

$$P_1(t) = e^{-(\lambda_o + \lambda_s)t}$$

$$P_o(t) = \frac{\lambda_o}{\lambda_o + \lambda_s}[1 - e^{-(\lambda_o + \lambda_s)t}]$$

$$P_s(t) = \frac{\lambda_s}{\lambda_o + \lambda_s}[1 - e^{-(\lambda_o + \lambda_s)t}]$$

The system reliability $R_1(t)$ is the same as $P_1(t)$, thus $R_1(t) = e^{-(\lambda_o + \lambda_s)t}$

Solving the differential equations (4.1)-(4.5) in section 4.1 with λ_c=0, the state probabilities of the two units in parallel system without common cause failure are as follows.

$$P_1(t) = e^{-(2\lambda_o+2\lambda_o)t}$$

$$P_2(t) = \frac{2\lambda_o}{\lambda_o+\lambda_s}[e^{-(\lambda_o+\lambda_o)t} - e^{-(2\lambda_o+2\lambda_o)t}]$$

$$P_s(t) = \frac{\lambda_o\lambda_s}{e^{(2\lambda_o+2\lambda_s)t}(\lambda_o+\lambda_s)^2} - \frac{2\lambda_o\lambda_s}{e^{(\lambda_o+\lambda_s)t}(\lambda_o+\lambda_s)^2} - \frac{\lambda_s}{e^{(2\lambda_o+2\lambda_s)t}(\lambda_o+\lambda_s)} + \frac{\lambda_o\lambda_s}{(\lambda_o+\lambda_s)^2} + \frac{\lambda_s}{\lambda_o+\lambda_s}$$

$$P_o(t) = \frac{\lambda_o^2}{e^{(2\lambda_o+2\lambda_s)t}(\lambda_o+\lambda_s)^2} - \frac{2\lambda_o^2}{e^{(\lambda_o+\lambda_s)t}(\lambda_o+\lambda_s)^2} + \frac{\lambda_o^2}{(\lambda_o+\lambda_s)^2}$$

Reliability of the two units in parallel system $R_2(t)$: $R_2(t) = P_1(t) + P_2(t)$

$$R_2(t) = \frac{2\lambda_o}{\lambda_o+\lambda_s}e^{-(\lambda_o+\lambda_o)t} + \frac{\lambda_s-\lambda_o}{\lambda_o+\lambda_s}e^{-(2\lambda_o+2\lambda_o)t}$$

The difference between $R_2(t)$ and $R_1(t)$: $R_{21}(t) = R_2(t) - R_1(t)$

$$R_{21}(t) = \frac{\lambda_o-\lambda_s}{\lambda_o+\lambda_s}[e^{-(\lambda_o+\lambda_o)t} - e^{-(2\lambda_o+2\lambda_o)t}]$$

Because of $e^{-(\lambda_o+\lambda_o)t} - e^{-(2\lambda_o+2\lambda_o)t} \geq 0$, if $\lambda_o \geq \lambda_s$, $R_{21} \geq 0$. It indicates that $R_2(t) \geq R_1(t)$.

Therefore at any time, for a system, if the open mode failure rate of a unit is higher than its short mode failure rate, the reliability of the two units in parallel system will be greater than that of a single unit.

Appendix 4.3

$$A(4,4) = 3\lambda_b^2\lambda_t\mu + 4\lambda_b\lambda_s\lambda_t\mu + \lambda_s^2\lambda_t\mu + \lambda_b\lambda_t^2\mu + \lambda_s\lambda_t^2\mu + 2\lambda_b^3\mu_c + 6\lambda_b^2\lambda_s\mu_c + 6\lambda_b\lambda_s^2\mu_c +$$
$$2\lambda_t^3\mu_c + 2\lambda_b\lambda_s\lambda_t\mu_c + 2\lambda_s^2\lambda_t\mu_c + 5\lambda_b^2\mu\mu_c + 8\lambda_b\lambda_s\mu\mu_c + 3\lambda_s^2\mu\mu_c + \lambda_b\lambda_t\mu\mu_c + 3\lambda_s\lambda_t\mu\mu_c$$

$$A(4,5) = s(3\lambda_b^2\lambda_t\mu + 4\lambda_b\lambda_s\lambda_t\mu + \lambda_s^2\lambda_t\mu + \lambda_b\lambda_t^2\mu + \lambda_s\lambda_t^2\mu + 2\lambda_b^3\mu_c + 6\lambda_b^2\lambda_s\mu_c + 6\lambda_b\lambda_s^2\mu_c +$$
$$2\lambda_t^3\mu_c + 2\lambda_b\lambda_s\lambda_t\mu_c + 2\lambda_s^2\lambda_t\mu_c + 5\lambda_b^2\mu\mu_c + 8\lambda_b\lambda_s\mu\mu_c + 3\lambda_s^2\mu\mu_c + \lambda_b\lambda_t\mu\mu_c + 3\lambda_s\lambda_t\mu\mu_c$$
$$2\lambda_b^3 s + 6\lambda_b^2\lambda_s s + 6\lambda_b\lambda_s^2 s + 2\lambda_s^3 s + 3\lambda_b^2\lambda_t s + 6\lambda_b\lambda_s\lambda_t s + 3\lambda_s^2\lambda_t s + \lambda_b\lambda_t^2 s + \lambda_s\lambda_t^2 s + 5\lambda_b^2\mu s +$$
$$8\lambda_b\lambda_s\mu s + 3\lambda_s^2\mu s + 5\lambda_b\lambda_t\mu s + 5\lambda_s\lambda_t\mu s + \lambda_t^2\mu s + 5\lambda_b^2\mu_c s + 10\lambda_b\lambda_s\mu s + 5\lambda_s^2\mu_c s +$$
$$\lambda_b\lambda_t\mu_c s + 3\lambda_s\lambda_t\mu_c s + 4\lambda_b\mu\mu_c s + 4\lambda_s\mu\mu_c s + \lambda_t\mu\mu_c s + 5\lambda_b^2 s^2 + 10\lambda_b\lambda_s s^2 + 5\lambda_s^2 s^2 +$$
$$5\lambda_b\lambda_t s^2 + 5\lambda_s\lambda_t s^2 + \lambda_t^2 s^2 + 4\lambda_b\mu s^2 + 4\lambda_s\mu s^2 + 2\lambda_t\mu s^2 + 4\lambda_b\mu_c s^2 + 4\lambda_s\mu_c s^2 + \lambda_t\mu_c s^2 +$$
$$\mu\mu_c s^2 + 4\lambda_b s^3 + 4\lambda_s s^3 + 2\lambda_t s^3 + \mu s^3 + \mu_c s^3 + s^4)$$

$$A(4,6) = (5\lambda_b^2 + 8\lambda_b\lambda_s + 3\lambda_s^2 + \lambda_b\lambda_t + 3\lambda_s\lambda_t + 4\lambda_b s + 4\lambda_s s + \lambda_t s + s^2)(\mu + s)(\mu_c + s)$$

$$A(4,7) = 2\lambda_b^4 + 8\lambda_b^3\lambda_s + 12\lambda_b^2\lambda_s^2 + 8\lambda_b\lambda_s^3 + 2\lambda_s^4 + 3\lambda_b^3\lambda_t + 9\lambda_b^2\lambda_s\lambda_t + 9\lambda_b\lambda_s^2\lambda_t + 3\lambda_s^3\lambda_t + \lambda_b^2\lambda_t^2 +$$
$$2\lambda_b\lambda_s\lambda_t^2 + \lambda_s^2\lambda_t^2 + 2\lambda_b^3\mu + 6\lambda_b^2\lambda_s\mu + 6\lambda_b\lambda_s^2\mu + 2\lambda_s^3\mu + 6\lambda_b^2\lambda_t\mu + 10\lambda_b\lambda_s\lambda_t\mu + 4\lambda_s^2\lambda_t\mu +$$
$$2\lambda_b\lambda_t^2\mu + 2\lambda_s\lambda_t^2\mu + 2\lambda_b\lambda_s\mu^2 + 2\lambda_s^2\mu^2 + 4\lambda_b\lambda_t\mu^2 + 2\lambda_s\lambda_t\mu^2 + \lambda_t^2\mu^2 + \lambda_t\mu^3 + 7\lambda_b^3 s +$$
$$21\lambda_b^2\lambda_s s + 21\lambda_b\lambda_s^2 s + 7\lambda_s^3 s + 8\lambda_b^2\lambda_t s + 16\lambda_b\lambda_s\lambda_t s + 8\lambda_s^2\lambda_t s + 2\lambda_b\lambda_t^2 s + 2\lambda_s\lambda_t^2 s + 8\lambda_b^2\mu s +$$
$$6\lambda_b\lambda_s\mu s + 8\lambda_s^2\mu s + 11\lambda_b\lambda_t\mu s + 9\lambda_s\lambda_t\mu s + 2\lambda_t^2\mu s + 4\lambda_b\mu^2 s + 4\lambda_s\mu^2 s + 4\lambda_t\mu^2 s +$$
$$\mu^3 s + 9\lambda_b^2 s^2 + 18\lambda_b\lambda_s s^2 + 9\lambda_s^2 s^2 + 7\lambda_b\lambda_t s^2 + 7\lambda_s\lambda_t s^2 + \lambda_t^2 s^2 + 9\lambda_b\mu s^2 + 9\lambda_s\mu s^2 +$$
$$5\lambda_t\mu s^2 + 3\mu^2 s^2 + 5\lambda_b s^3 + 5\lambda_s s^3 + 2\lambda_t s^3 + 3\mu s^3 + s^4$$

$$A(4,8) = (\lambda_b + \lambda_s + \mu + s)(\lambda_b^2 + 2\lambda_b\lambda_s + \lambda_s^2 + \lambda_b\lambda_t + \lambda_s\lambda_t + \lambda_b\mu + \lambda_s\mu + \lambda_t\mu + \mu^2 + 2\lambda_b s +$$
$$2\lambda_s s + \lambda_t s + 2\mu s + s^2)$$

$$A(4,9) = 2\lambda_b(\lambda_b + \lambda_s + \mu + s)^2$$

$$A(4,10) = 2\lambda_s(\lambda_b^2 + 2\lambda_b\lambda_s + \lambda_s^2 + \lambda_b\lambda_t + \lambda_s\lambda_t + \lambda_b\mu + \lambda_s\mu + \lambda_t\mu + \mu^2 + 2\lambda_b s + 2\lambda_s s + \lambda_t s +$$
$$2\mu s + s^2)$$

$$A(4,11) = 2\lambda_b^2(\lambda_b + \lambda_s + \mu + s)$$

$$A(4,12) = 2\lambda_b\lambda_s(\lambda_b + \lambda_s + \mu + s)$$

$$A(4,13) = 2\lambda_s(\lambda_b^3 + 3\lambda_b^2\lambda_s + 3\lambda_b\lambda_s^2 + \lambda_s^3 + \lambda_b\lambda_s\lambda_c + \lambda_s^2\lambda_c + \lambda_b^2\mu + 2\lambda_b\lambda_s\mu + \lambda_s^2\mu + \lambda_s\lambda_c\mu + \lambda_s\mu^2 + \lambda_b^2 s + 3\lambda_b\lambda_s s + 2\lambda_s^2 s + \lambda_s\lambda_c s + 2\lambda_s\mu s + \lambda_s s^2)$$

$$A(4,14) = 2\lambda_b(\lambda_b^3 + 3\lambda_b^2\lambda_s + 3\lambda_b\lambda_s^2 + \lambda_s^3 + \lambda_b\lambda_s\lambda_c + \lambda_s^2\lambda_c + \lambda_b^2\mu + 2\lambda_b\lambda_s\mu + \lambda_s^2\mu + \lambda_s\lambda_c\mu + \lambda_s\mu^2 + \lambda_b^2 s + 3\lambda_b\lambda_s s + 2\lambda_s^2 s + \lambda_s\lambda_c s + 2\lambda_s\mu s + \lambda_s s^2)$$

$$A(4,15) = \lambda_c(\lambda_b + \lambda_s + \mu + s)(3\lambda_b^2 + 4\lambda_b\lambda_s + \lambda_s^2 + \lambda_b\lambda_c + \lambda_s\lambda_c + 3\lambda_b\mu + \lambda_s\mu + \lambda_c\mu + \mu^2 + 4\lambda_b s + 2\lambda_s s + \lambda_c s + 2\mu s + s^2)$$

$$A(4,16) = 5\lambda_b^3 + 13\lambda_b^2\lambda_s + 11\lambda_b\lambda_s^2 + 3\lambda_s^3 + \lambda_b^2\lambda_c + 4\lambda_b\lambda_s\lambda_c + 3\lambda_s^2\lambda_c + 8\lambda_b^2\mu + 12\lambda_b\lambda_s\mu + 4\lambda_s^2\mu + 2\lambda_b\lambda_c\mu + 4\lambda_s\lambda_c\mu + 4\lambda_b\mu^2 + 4\lambda_s\mu^2 + \lambda_c\mu^2 + \mu^3 + 9\lambda_b^2 s + 16\lambda_b\lambda_s\mu + 7\lambda_s^2 s + 2\lambda_b\lambda_c\mu + 4\lambda_s\lambda_c\mu + 9\lambda_b\mu s + 9\lambda_s\mu s + 2\lambda_c\mu s + 3\mu^2 s + 5\lambda_b s^2 + 5\lambda_s s^2 + \lambda_c s^2 + 3\mu s^2 + s^3$$

$$A(4,17) = 5\lambda_b^3 + 13\lambda_b^2\lambda_s + 11\lambda_b\lambda_s^2 + 3\lambda_s^3 + \lambda_b^2\lambda_c + 4\lambda_b\lambda_s\lambda_c + 3\lambda_s^2\lambda_c + 8\lambda_b^2\mu + 12\lambda_b\lambda_s\mu + 4\lambda_s^2\mu + 2\lambda_b\lambda_c\mu + 4\lambda_s\lambda_c\mu + 4\lambda_b\mu^2 + 4\lambda_s\mu^2 + \lambda_c\mu^2 + \mu^3 + 9\lambda_b^2 s + 16\lambda_b\lambda_s\mu$$

$$A(4,18) = 2\lambda_b^4 + 8\lambda_b^3\lambda_s + 12\lambda_b^2\lambda_s^2 + 8\lambda_b\lambda_s^3 + 2\lambda_s^4 + 3\lambda_b^3\lambda_c + 9\lambda_b^2\lambda_s\lambda_c + 9\lambda_b\lambda_s^2\lambda_c + 3\lambda_s^3\lambda_c + \lambda_b^2\lambda_c^2 + 2\lambda_b\lambda_s\lambda_c^2 + \lambda_s^2\lambda_c^2 + 2\lambda_b^3\mu + 6\lambda_b^2\lambda_s\mu + 6\lambda_b\lambda_s^2\mu + 2\lambda_s^3\mu + 6\lambda_b^2\lambda_c\mu + 10\lambda_b\lambda_s\lambda_c\mu + 4\lambda_s^2\lambda_c\mu 2\lambda_b\lambda_c^2\mu + 2\lambda_s\lambda_c^2\mu + 2\lambda_b\lambda_s\mu^2 + 2\lambda_s^2\mu^2 + 4\lambda_b\lambda_c\mu^2 + 2\lambda_s\lambda_c\mu^2 + \lambda_c^2\mu^2 + \lambda_c\mu^3$$

Appendix 5.1

(i) The reliability difference between the standby system and the parallel system with a nonconstant common cause failure rate is given by:

$$\Delta R(t) = R_{sb}(t) - R_p(t) = \frac{A(5,2)}{A(5,1)} \tag{5.23}$$

where

$$A(5,2) = 2(\lambda_b + \lambda_s)^3[e^{(\lambda_b+\lambda_s+\alpha)t} - e^{(\lambda_b+\lambda_s)t}] - 2\alpha\,(\lambda_b^2 - \lambda_s^2)[e^{(\lambda_b+\lambda_s+\alpha)t} - 1] +$$
$${}^2(\lambda_b + \lambda_s)[e^{2(\lambda_b+\lambda_s)t} - 1] + \alpha^3[e^{(\lambda_b+\lambda_s)t} - 1]^2$$

$$A(5,1) = 2e^{(2\lambda_b+2\lambda_s+\alpha)t}\alpha\,(\lambda_b + \lambda_s)(\lambda_b + \lambda_s + \alpha)$$

It is obvious that $A(5,1)$ is not less zero. Thus, we let

$$f(t) = 2(\lambda_b + \lambda_s)^3[e^{(\lambda_b+\lambda_s+\alpha)t} - e^{(\lambda_b+\lambda_s)t}] - 2\alpha\,(\lambda_b^2 - \lambda_s^2)[e^{(\lambda_b+\lambda_s+\alpha)t} - 1] +$$
$$(\lambda_b + \lambda_s)[e^{2(\lambda_b+\lambda_s)t} - 1]\alpha^2 + \alpha^3[e^{(\lambda_b+\lambda_s)t} - 1]^2$$

$$\frac{df(t)}{dt} = 2(\lambda_b + \lambda_s)^4[e^{(\lambda_b+\lambda_s+\alpha)t} - e^{(\lambda_b+\lambda_s)t}] + [4\lambda_s^3\alpha + 2\lambda_b^2\alpha\,(3\lambda_s - \alpha) + 6\lambda_b\lambda_s\alpha\,]e^{(\lambda_b+\lambda_s+\alpha)t}$$
$$2\alpha^2(\lambda_b + \lambda_s)^2 e^{2(\lambda_b+\lambda_s)t} + 2\alpha^3(\lambda_b + \lambda_s)[e^{2(\lambda_b+\lambda_s)t} - e^{(\lambda_b+\lambda_s)t}]$$

Appendix 1.2 indicates that if α_1=α_2, $\lim_{t\to\infty}\lambda(t) = \alpha$. This means that the non-constant common-cause failure rate becomes constant as time approaches infinity. This constant value equals α. In practice, the value of the common cause failure rate lies within the range 1%-30% of a component's failure rate. It indicates that parameter α lies within the range 1%-30% of a component's failure rate. So that $(3\lambda_s - \alpha) \geq 0$, then $\frac{df(t)}{dt} \geq 0$, and as a consequence $\Delta R(t) = R_{sb}(t) - R_p(t) = \frac{A(5,2)}{A(5,1)} \geq 0$.