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**ON DP-CONSTRAINTS FOR THE
TRAVELING SALESMAN POLYTOPE**

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A thesis submitted in conformity with the requirements
for the degree of Master's of Science (Systems Science).

University of Ottawa

Ottawa, Ontario, Canada

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ABSTRACT

The *DP*-constraints are a recently defined class of valid inequalities for the Symmetric Traveling Salesman Polytope (*STSP*) which includes the family of comb constraints. Moreover, there exists a polynomial-time exact separation algorithm for the *DP*-constraints for points whose support graph is planar. However, while the comb constraints are known to be facet-inducing, the same is not true in general of the *DP*-constraints. This thesis addresses the question of which *DP*-constraints are facet-inducing; some sufficient conditions are given for identifying *DP*-constraints which are not facet-inducing, and a family of facet-inducing *DP*-constraints, the twisted comb constraints, is described and shown to be distinct from other known families of facet-inducing inequalities. We also present a new formulation of the *DP*-constraints in terms of tripartitions of the node set, which allows easier recognition of equivalent *DP*-constraints.

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CHAPTER 1

INTRODUCTION

Perhaps the most famous problem in combinatorial optimization is the *traveling salesman problem (TSP)*, in which a salesman must visit exactly once each of n cities, which are all interconnected by routes of varying cost (measured in terms of distance, time, fuel or money), and return to the city where he began, in the most efficient way possible. (In this thesis, we will always be working with the *symmetric* version of the problem, in which the cost of a route is independent of direction.) In graph theoretical terms, this translates into finding a shortest Hamilton cycle, or *tour*, in a weighted complete graph on n nodes, K_n . To date, nobody has found an algorithm which can solve the general *TSP* in time that is a polynomial function of n , and it is not expected that anyone ever will. It is one of a large set of difficult problems termed *NP-hard*; if any one of these problems could be solved in polynomial time, then so could all such problems, and that is considered unlikely.

However, research continues into finding efficient algorithms that at worst can provide some bounds on the cost of an optimal tour, and at best can solve particular instances of the *TSP*. Considerable success was first achieved by Dantzig, Fulkerson and Johnson in 1954, employing the methods of polyhedral combinatorics [DFJ]. In this approach, each of the $m = n(n - 1)/2$ edges in the complete graph K_n is represented by a binary variable x_e ; conversely, each vector $x \in \{0, 1\}^m$ is the characteristic vector of a spanning subgraph of K_n , called the *support graph* of x and denoted by G_x . A tour can be characterized as a spanning subgraph which is both 2-regular (every node has exactly 2 incident edges) and connected (the subgraph is a single tour, rather than a collection of subtours); to translate this into constraints

on the variables $\{x_e | e \in E\}$, we introduce some notation. For a graph $G = (V, E)$ and a set $\emptyset \subsetneq A \subsetneq V$, let $\delta(A) \subset E$ denote the set of edges having one endnode in A and the other in $\bar{A} \equiv V \setminus A$; the sets A and \bar{A} are called the *shores* of the cut $\delta(A)$. We write $\delta(v)$ for $\delta(\{v\})$ for $v \in V$. For any edge set F , $x(F) = \sum_{e \in F} x_e$. The *degree constraints* specify that $x(\delta(v)) = 2$ for all nodes $v \in V$. The *subtour elimination constraints* eliminate the possibility of subtours by requiring that each proper node subset $S \subset V$ have at least two edges connecting it to \bar{S} ; that is, $x(\delta(S)) \geq 2$. If we let c_e represent the weight of edge e , then the *TSP* can be formulated as the following integer programming problem:

$$\text{minimize: } \sum_{e \in E} c_e x_e$$

$$\text{subject to: } x(\delta(v)) = 2 \quad \text{for all } v \in V \quad (1.1)$$

$$x(\delta(S)) \geq 2 \quad \text{for all } \emptyset \subsetneq S \subsetneq V \quad (1.2)$$

$$\text{with: } x_e \in \{0, 1\} \quad \text{for all } e \in E \quad (1.3)$$

Although the general integer programming problem is also *NP*-hard, there are some effective, efficient techniques for finding bounds on the optimal value of the objective function; in some lucky cases, they will actually find an optimal solution. The most common of these is to first solve the linear programming problem obtained by relaxing the integrality condition (1.3) to

$$0 \leq x_e \leq 1 \quad \text{for all } e \in E. \quad (1.4)$$

This is referred to as the *subtour relaxation* of the *TSP*; the corresponding feasible region is called the *subtour polytope*, and is denoted by *SUBT*(n). The description of this polytope can be refined somewhat by making three observations. First, if $S = \{u\}$, then the subtour elimination constraint for S is weaker than the degree equation for u , and therefore can be eliminated. Second, the upper edge bounds

$x_e \leq 1$ are redundant. For an edge $e = [uv]$, let $S = \{u, v\}$; note that for any $x \in \mathbb{R}^m$,

$$x(\delta(S)) = x(\delta(u)) + x(\delta(v)) - 2x_e.$$

If x is known to satisfy all degree equations and subtour elimination constraints, then

$$x_e = \frac{1}{2} \left[x(\delta(u)) + x(\delta(v)) - x(\delta(S)) \right] \leq 2 + 2 - 2 = 1.$$

Third, if \bar{S} denotes the complement of S , then clearly $\delta(S) = \delta(\bar{S})$; this allows us to cut the number of remaining subtour elimination constraints in half. The constraints defining $SUBT(n)$ have been reduced to:

$$x(\delta(v)) = 2 \quad \text{for all } v \in V \quad (1.1)$$

$$x(\delta(S)) \geq 2 \quad \text{for all } S \subset V, \quad 2 \leq |S| \leq \left\lfloor \frac{n}{2} \right\rfloor \quad (1.5)$$

$$x_e \geq 0 \quad \text{for all } e \in E \quad (1.6)$$

Even after this refinement, the number of constraints is $n + (2^{n-1} - 1 - n) + m$, an exponential function of n . Finding the optimal value of x over $SUBT(n)$ by directly solving the corresponding linear program is therefore impractical.

One way around this obstacle is to adopt a *cutting plane approach*. We start by solving the much smaller linear program that has only constraints (1.1) and (1.6). If the optimal solution happens to be the characteristic vector of a tour, then we are done. If not, then we next check if any subtour elimination constraints are violated by finding the value of a minimum cut

$$x(\delta(S_0)) = \min\{x(\delta(S)) \mid \emptyset \subsetneq S \subsetneq V\},$$

where x is the current optimal solution. There are several polynomial-time algorithms for finding minimum cuts in undirected graphs; the Gomory-Hu algorithm,

for example, runs in $O(n^2 m \log(n^2/m))$ time [JJR]. If the minimum cut has value greater than or equal to 2, then no subtour constraints are violated by x , and so it is in the subtour polytope. If it is less than 2, then the subtour constraint $x(\delta(S_0)) \geq 2$ is added to the linear program, and it is re-solved. Geometrically, this amounts to “cutting off” the current optimal solution by inserting a hyperplane between it and the subtour polytope. We repeat until we obtain a solution which satisfies all of the subtour elimination constraints. In practice, only a small subset of the constraints gets added to the linear program before a subtour optimal solution is found [JRR], but there is no guarantee that we won’t have to add a number of constraints that is exponential in n .

The process by which either a violated subtour elimination constraint is found, or a guarantee is issued that none are violated, is called an *exact separation algorithm* for this class of constraints. An important theorem in polyhedral combinatorics provided by Grötschel, Lovász and Schrijver in 1988 implies that an optimization problem over a rational polyhedron can be solved in polynomial time if and only if there is a polynomial-time exact separation procedure for the constraints defining that polyhedron [GLS]. Thus, the facts that the number of degree equations and non-negativity constraints is polynomial in n , and the subtour elimination constraints have a polynomial-time exact separation algorithm guarantee that the optimal solution to the subtour relaxation of the *TSP* can be found in polynomial time.

It is important to note that the subtour polytope contains, but does not equal, the convex hull of the characteristic vectors of all tours, called the *symmetric traveling salesman polytope* and denoted by $STSP(n)$. In particular, the optimal solution over $SUBT(n)$ need not be a tour, in which case its value provides only a lower bound on the value of an optimal tour. This suggests that we extend the cutting plane approach, by continuing to add constraints to the subtour relaxation until the feasible region has been winnowed down to $STSP(n)$. It is no easy matter, however,

to find constraints that will achieve this. Certainly, the constraints we add must be valid for all tours and hence valid for all of their convex combinations; ideally, though, we seek constraints that cut right to the boundary of $STSP(n)$. To make this more precise, we require some more notions from polyhedral combinatorics (as can be found in [JRR], for example).

It is obvious that any positive scalar multiple of a constraint is an equivalent constraint, in the sense that if $\gamma \in \mathbb{R}^+$, then $x \in \mathbb{R}^q$ satisfies $a^T x \leq b$ if and only if x satisfies $(\gamma a)^T x \leq \gamma b$. In particular, any positive multiple of a valid inequality for a polytope $P \subset \mathbb{R}^q$ is an equivalent valid inequality. If a defining system for P includes one or more equation constraints, then P is contained in (the intersection of) the corresponding hyperplane(s), which we denote by $\mathbb{H} \subsetneq \mathbb{R}^q$; in technical terms, P is not a full-dimensional polytope. Suppose $a^T x \leq b$ is a valid inequality for P and $c^T x = d$ is a linear combination of valid equation constraints for P . Then for any $\hat{x} \in \mathbb{H}$, \hat{x} satisfies $a^T x \leq b$ if and only if \hat{x} satisfies $(a + c)^T x \leq (b + d)$; these two constraints are equivalent for any point in the intersection of the equation constraint hyperplanes.

The degree constraints constitute a complete, linearly independent set of equation constraints in a defining system for $STSP(n) \subset \mathbb{R}^m$. Let $A = [a_{ij}]$ denote the $n \times m$ node-edge incidence matrix for K_n ; that is, $a_{ij} = 1$ if edge e_j is incident to node v_i , and $a_{ij} = 0$ otherwise. Then in the notation above, $\mathbb{H} = \{x \in \mathbb{R}^m \mid Ax = \mathbf{2}\}$.

DEFINITION. Two valid inequalities for $STSP(n)$, $a^T x \leq \alpha$ and $b^T x \leq \beta$, are *equivalent* if and only if there exists a positive scalar γ and $\lambda \in \mathbb{R}^n$ such that $b^T = \gamma a^T + \lambda^T A$ and $\beta = \gamma \alpha + \lambda^T \mathbf{2}$; otherwise, they are called *distinct*.

In this thesis, we sometimes switch the direction of the inequality of a constraint. Note that $x \in \mathbb{R}^m$ satisfies $b^T x \leq \beta$ if and only if it satisfies $(-b)^T x \geq (-\beta)$; more generally, $a^T x \leq \alpha$ and $c^T x \geq \zeta$ are equivalent on $STSP(n)$ if and only if there exists a *negative* scalar κ and $\lambda \in \mathbb{R}^n$ such that $c^T = \kappa a^T + \lambda^T A$ and $\zeta = \kappa \alpha + \lambda^T \mathbf{2}$.

A hyperplane $a^T x = \alpha$ is a *supporting hyperplane* of polyhedron P if and only

if the constraint $a^T x \leq \alpha$ is valid for P and $\{x \in P \mid a^T x = \alpha\} \neq \emptyset$. The set $\{x \in P \mid a^T x = \alpha\}$ is called the *face of P induced by $a^T x \leq \alpha$* and is denoted by $F(a)$. Notice that two valid, supporting inequalities for P are equivalent if and only if the corresponding faces are equal.

A *facet* of a polyhedron is a maximal proper face. It can be shown that *any* defining system for a polyhedron P must include a distinct inequality constraint for each of its facets; a minimal defining system will have only a complete set of these inequalities (in addition to a maximal linearly independent set of equation constraints). In other words, all valid inequalities for a polyhedron are implied by the facet-inducing ones. For this reason, facet-inducing constraints are considered the best cutting planes, because their insertion into a relaxed linear program cuts off a maximal amount of the current feasible region.

Grötschel and Padberg proved in 1979 that the non-negativity constraints (1.6) and the subtour elimination constraints (1.5) from $SUBT(n)$ are distinct and facet-inducing for $STSP(n)$ for $n \geq 5$ and $n \geq 4$ respectively ([GP1] and [GP2]); they are sometimes called trivial constraints because they do not distinguish $STSP(n)$ from $SUBT(n)$. To date, a complete minimal set of facet-inducing inequality constraints for $STSP(n)$ is known only for $n \leq 8$. (For $n = 6$, there are 100 such inequalities, for $n = 7$, there are 3,437 and for $n = 8$, there are 194,187 [JRR].)

A large family of nontrivial, facet-inducing inequalities for $STSP(n)$ that has proved extremely useful in solving particular instances of the TSP is defined below. We require some additional notation: for any $A, B \subseteq V$, $\gamma(A)$ is the set of all edges having both endnodes in A , and $E(A, B)$ is the set of all edges with one endnode in A and the other in B .

DEFINITION. A *comb* consists of non-empty node subsets, a *handle* H and an odd number of disjoint *teeth*, T_1, \dots, T_p , where $p \geq 3$, with the property that $A_i = T_i \cap H$ and $B_i = T_i \setminus H$ are non-empty for all $1 \leq i \leq p$. The corresponding *comb constraint*

is

$$x(\gamma(H)) + \sum_{i=1}^p x(\gamma(T_i)) \leq |H| + \sum_{i=1}^p |T_i| - \left(\frac{3p+1}{2}\right). \quad (1.7)$$

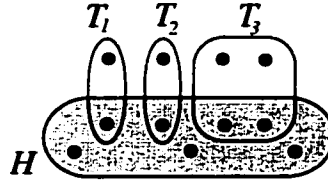


FIGURE 1.1. Example of a comb

If a tooth contains exactly two nodes, it is called a *2-matching tooth*; if every tooth in a comb is 2-matching, then the corresponding constraint is called a *2-matching inequality*. These inequalities were first discovered by Edmonds in 1965, who used them in connection with the 2-matching problem [E]. In 1973, Chvátal generalized to combs in which each tooth T_i has exactly one node in the handle (that is, $|A_i| = 1$). The further generalization to the form given in the definition above is due to Grötschel and Padberg in 1979 [GP1]; we include a standard proof of the validity of these constraints because we will need to refer to it later on.

THEOREM 1.1. *The comb constraints are valid for $STSP(n)$, $n \geq 6$.*

PROOF. We begin by switching the sign of the inequality in some of the subtour polytope constraints. For example, the non-negativity constraint $x_e \geq 0$ can be rewritten as $-x_e \leq 0$. To rewrite the subtour elimination constraints, first note that for any proper node subset S , we can add up the degree constraints for all v in S to obtain

$$\sum_{v \in S} x(\delta(v)) = 2|S| = 2x(\gamma(S)) + x(\delta(S)). \quad (1.8)$$

If we set $\kappa = -2$ and $\lambda = \mathbf{1} \in \mathbb{R}^m$, then it is clear from (1.8) and the definition of equivalence that $x(\gamma(S)) \leq |S| - 1$ is an equivalent form of the subtour elimination constraint $x(\delta(S)) \geq 2$.

Next, observe that

$$x(\gamma(T_i)) = x(\gamma(A_i)) + x(\gamma(B_i)) + x(E(A_i, B_i)). \quad (1.9)$$

We now add together (i) the degree constraints for every node in H , (ii) the ‘ \leq ’ form of the subtour elimination constraints for A_i, B_i and T_i , for $i = 1, \dots, p$, and (iii) the ‘ \leq ’ form non-negativity constraints for all $e \in \delta(H) \setminus \cup_{i=1}^p E(A_i, B_i)$. The result, after being simplified using (1.8) and (1.9), is

$$2x(\gamma(H)) + 2 \sum_{i=1}^p x(\gamma(T_i)) \leq 2|H| + 2 \sum_{i=1}^p |T_i| - 3p. \quad (1.10)$$

Note that this constraint is satisfied by every $x \in SUBT(n)$. If x is the characteristic vector of a tour, then we have the additional information that $x(\gamma(H))$ and the $x(\gamma(T_i))$ are integral; this means that the left hand side of (1.10) will be even, while the right hand side will not, as p is odd. Hence, such x , and all convex combinations of such x , satisfy the stronger constraint

$$2x(\gamma(H)) + 2 \sum_{i=1}^p x(\gamma(T_i)) \leq 2|H| + 2 \sum_{i=1}^p |T_i| - (3p + 1).$$

Dividing by 2 now gives (1.7). Alternatively, we can divide both sides of (1.10) by 2, note that for all tours x , the left hand side is integral while the right hand side is not, and then round down the right hand side to the nearest integer. \square

The comb constraints are examples of *Gomory-Chvátal cuts*, commonly used in the solution of integer programming problems. These cutting planes are derived by adding up positive multiples of constraints valid for a rational polyhedron to obtain a constraint that, for integral x , will have an integral value on one side of

the inequality but not the other. More precisely, let $a_j^T x \leq \alpha_j, j \in \{1, \dots, r\}$ be a defining system of integral constraints for polyhedron $P \subset \mathbb{R}^q$, and let P_I denote the set of all convex combinations of the integer points of P . For a set of positive scalars $y_j, j \in \{1, \dots, r\}$, let $c = (\sum_{j=1}^r y_j a_j)$ and $\zeta = \sum_{j=1}^r y_j \alpha_j$. Then $c^T x \leq \zeta$ is clearly still valid for P . If c is integral and ζ is not, then we can conclude that all $x \in P_I$ satisfy the stronger constraint $c^T x \leq \lfloor \zeta \rfloor$. The set of all points satisfying constraints formed in this way is usually denoted by $P^{(1)}$; note that $P \equiv P^{(0)} \supseteq P^{(1)} \supseteq P_I$. This process can be applied iteratively to obtain a nested sequence of polyhedrons

$$P^{(0)} \supseteq P^{(1)} \supseteq P^{(2)} \supseteq P^{(3)} \supseteq \dots \supset P_I.$$

It was proved by Schrijver in 1980 that there exists some r such that $P^{(r)} = P_I$ [S]. The *Chvátal rank* of a valid inequality on P_I is defined to be the minimum s such that the inequality is valid for $P^{(s)}$. For the *TSP*, $P \equiv SUBT(n)$ and $P_I \equiv STSP(n)$; the non-negativity and subtour elimination constraints have Chvátal rank 0. It can be seen from the proof of Theorem 1.1 that the comb constraints have Chvátal rank at most 1; in fact, it is shown in [BP] that they have rank exactly 1. In particular, this shows that they are distinct from the non-negativity and subtour elimination constraints.

The comb constraints were proved to be facet-inducing for $STSP(n)$, $n \geq 6$, by Grötschel and Padberg in 1979 [GP1],[GP2]. However, their incorporation into a cutting plane approach for the *TSP* is hampered by the fact that nobody has yet found an efficient exact separation algorithm for them. Recent work on the separation of comb constraints has resulted in only partial success.

An exact separation algorithm for the 2-matching inequalities that runs in $O(n^5)$ time was found by Padberg and Rao in 1982 [PR]. Although this is polynomial, it is of such a high degree that it is usually more efficient to run *heuristic* algorithms; these can locate violated constraints, but should they fail to find one, they cannot issue a guarantee that no constraints in the family are violated (that is, the separation is not exact).

Another approach is to look for *maximally violated* comb constraints (that is, comb constraints violated by the maximum possible amount). From the proof of the validity of the comb constraints, we can see that any x in the subtour polytope violates a comb constraint by at most $\frac{1}{2}$. Moreover, this occurs precisely when all the inequality constraints used in the derivation of the comb constraint are satisfied at equality; that is, when

$$\begin{aligned} x(\delta(T_i)) = x(\delta(A_i)) = x(\delta(B_i)) = 2 & \quad \text{for all } i \in \{1, \dots, p\}, \\ x_e = 0 & \quad \text{for all } e \in \delta(H) \setminus \cup_{i=1}^p E(A_i, B_i). \end{aligned}$$

(Another way of phrasing the first of these conditions is to say that T_i , A_i and B_i are *tight* sets.) By exploiting the extra structure these equality conditions imply, in 1995 Applegate, Bixby, Chvátal and Cook in [ABCC] devised a surprisingly successful heuristic algorithm for finding maximally violated combs. In 1996, Fleischer and Tardos [FT] used this heuristic as the foundation for an $O(n^2 \log n)$ algorithm that either finds a violated comb (which may or may not be maximally violated), or issues a guarantee that there are no maximally violated combs. Unfortunately, this algorithm only works for solutions x whose support graph G_x is planar. Moreover, the separation is not quite exact; if x maximally violates a comb constraint, then Fleischer's algorithm will find it (or another violated comb constraint), but if x violates comb constraints only sub-maximally, then the algorithm may or may not find one.

In 1999, Caprara, Fischetti and Letchford studied the class of *mod- k cuts*. These are Gomory-Chvátal cuts in which all multipliers are fractions with a fixed denominator k ; an integral point x can violate such constraints by at most $\frac{k-1}{k}$. In particular, combs are examples of mod-2 cuts. In [CFL], an $O(mn \min\{m, n\})$ -time algorithm is provided for the exact separation of maximally violated mod- k cuts. If x maximally violates a comb constraint, then this algorithm will find some maximally violated mod-2 cut, but the one it finds may or may not be a comb constraint; in the latter case, it may not even be facet-inducing.

In Chapter 2, we describe recent work by Adam Letchford [L1], in which he defines a new family of constraints valid on $STSP(n)$ called the DP -constraints, which contain the comb constraints and for which he provides a polynomial-time exact separation algorithm on solutions $x \in SUBT(n)$ having planar support. This algorithm has recently been improved and implemented by Vella; her results show that the DP -constraints are of considerable empirical value in attacking the TSP [V].

In his paper, Letchford provides a single example proving that the comb constraints form a proper subset of the facet-inducing members of this family. However, no general method is given for identifying which DP -constraints are facet-inducing, and therefore most useful in a cutting plane approach, and which are not. In this thesis, we provide some criteria for determining which DP -constraints induce facets. In Chapter 3, we present a new form of these constraints that facilitates the task of identifying equivalent DP -constraints. Chapter 4 gives some sufficient conditions for a DP -constraint to be non-facet-inducing. In Chapter 5, we describe a sub-family of DP -constraints that we prove to be facet-inducing; in Chapter 6, we show that this family is distinct from presently known families of facet-inducing constraints. Finally, we present some suggestions for further research in Chapter 7.

CHAPTER 2

THE *DP*-CONSTRAINTS

This chapter summarizes the work Letchford presents in [L1].

DEFINITION. A *domino* is a node subset $D = A \cup B \neq V$, where A and B are non-empty, non-complementary, disjoint node subsets; that is, $\emptyset \subsetneq A, B \subsetneq V$, $A \cap B = \emptyset$ and $D = A \cup B \neq V$. The edge set $E(A, B)$ is called the *semicut* of the domino.

DEFINITION. A *pseudo-cut* in a graph is a multiset of edges \mathcal{E} with the property that the edges occurring with odd multiplicity in \mathcal{E} constitute a cut. If μ_e denotes the multiplicity of edge e in \mathcal{E} (in particular $\mu_e = 0$ for all $e \notin \mathcal{E}$), then this can be expressed as

$$\{e \in E \mid \mu_e \text{ is odd}\} = \delta(H),$$

for some node subset H . We say that \mathcal{E} *supports the cut* $\delta(H)$; note that \mathcal{E} also supports the cut $\delta(\overline{H})$. Equivalently, \mathcal{E} is a pseudo-cut if and only if for every cycle $\psi \in E$, $\sum(\mu_e \mid e \in \psi \cap \mathcal{E})$ is even.

DEFINITION. A *domino-comb* consists of an edge subset, F , and an odd number of dominoes, D_1, D_2, \dots, D_p , such that the multiset of edges $\mathcal{E} = F \cup \{\cup_{i=1}^p E(A_i, B_i)\}$ is a pseudo-cut. The *handle* of the domino-comb is a node subset H such that $\delta(H) = \{e \in \mathcal{E} \mid \mu_e \text{ is odd}\}$.

A comb $\{H; T_1, T_2, \dots, T_p\}$ is clearly a domino-comb, with disjoint dominoes $D_i = T_i = A_i \cup B_i$, $i = 1, \dots, p$ and $F = \delta(H) \setminus \{\cup_{i=1}^p E(A_i, B_i)\}$. Note that $\mu_e = 1$ for all $e \in \mathcal{E}$ in this case. In general domino-combs, however, the dominoes need

not be disjoint. Moreover, the edges in the semicut of a domino $E(A_i, B_i)$ need not be contained in $\delta(H)$; that is, it may not be the case that $A_i = D_i \cap H$ and $B_i = D_i \setminus H$, as is true with combs. This can happen either when a semicut edge e occurs in an even number of semicuts (since dominoes need not be disjoint, their semicuts can intersect nontrivially), or when a copy of e exists in F . Note also that for domino-combs, p must be odd, but need not be greater than or equal to 3. Some examples of domino-combs are illustrated below. Rather than depict dominoes and the edges of F , we show dominoes and the implicitly defined handle, H (which is shaded). We can recover F from such a diagram by noting that

$$\begin{aligned}
 F = & \{e \in \delta(H) \mid e \text{ is a semicut edge in an even number of dominoes}\} \\
 & \cup \{e \notin \delta(H) \mid e \text{ is a semicut edge in an odd number of dominoes}\}
 \end{aligned}
 \tag{2.1}$$

In the figures, a dotted line in a domino D_i is used to separate A_i and B_i .

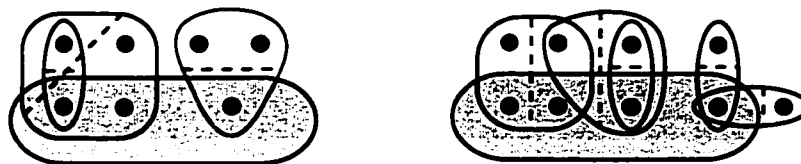


FIGURE 2.1. Examples of domino-combs

THEOREM 2.1. *For a domino-comb $\{F; D_1, D_2, \dots, D_p\}$, the domino-parity constraint (or DP-constraint)*

$$x(\gamma(H)) + \sum_{i=1}^p x(\gamma(D_i)) - \sum_{e \in E} \lfloor \mu_e / 2 \rfloor x_e \leq |H| + \sum_{i=1}^p |D_i| - \left(\frac{3p+1}{2} \right)
 \tag{2.2}$$

is valid for STSP(n), $n \geq 3$.

PROOF. The proof is very similar to that of the validity of the comb constraints. We sum up (i) the degree constraints for every node in H , (ii) the ' \leq ' form subtour

elimination constraints for A_i, B_i and D_i , for $1 \leq i \leq p$, and (iii) the ' \leq ' form non-negativity constraints for all $e \in F$, to obtain

$$\begin{aligned} \left(2x(\gamma(H)) + x(\delta(H))\right) + \sum_{i=1}^p \left(2x(\gamma(D_i)) - x(E(A_i, B_i))\right) - x(F) \\ \leq 2|H| + 2 \sum_{i=1}^p |D_i| - 3p. \end{aligned} \quad (2.3)$$

Note that

$$\begin{aligned} x(\delta(H)) - \sum_{i=1}^p x(E(A_i, B_i)) - x(F) &= x(\delta(H)) - \sum_{e \in E} \mu_e x_e \\ &= \sum_{e \in \delta(H)} (1 - \mu_e) x_e - \sum_{e \notin \delta(H)} \mu_e x_e. \end{aligned}$$

Since μ_e is odd if and only if $e \in \delta(H)$, the coefficient of each x_e in this expression is even. Thus, if x is integral, the left hand side of (2.3) is even while the right hand side is odd; subtracting 1 from the right hand side and dividing by 2 yields (2.2). \square

From this proof, note that any $x \in SUBT(n)$ satisfies (2.3). Moreover, if T is a tour, then its characteristic vector $\chi(T)$ satisfies it with slack at most 1. Thus, any $x \in SUBT(n)$ violates a DP -constraint in the form of (2.2) by at most $1/2$.

Clearly, the DP -constraints are a generalization of the comb constraints; they are also mod-2 cuts of Chvátal rank at most 1. Of the DP -constraints which are not comb constraints, some are facet-inducing and some are not. In [L1], Letchford gives the following example of a DP -constraint for $STSP(12)$ which is facet-inducing and distinct from the comb constraints.

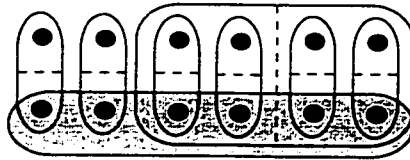


FIGURE 2.2

An example of a DP -constraint on 5 nodes which is not facet-inducing for $STSP(n)$ is provided in Figure 2.3(a); note that the edge uv is a semicut edge in one domino which is not in $\delta(H)$, and so it is in F . If we write the DP -constraint as $a^T x \leq \alpha$, then the values of the a_e are illustrated in Figure 2.3(b) (with the convention that edges not shown have coefficient 0, the bold edges have coefficient 2, and the remaining edges have coefficient 1) and $\alpha = 4$. To see that this inequality is supporting, note that the tour (v, x, y, w, \dots, u) (where the other nodes, if any, are inserted between w and u) satisfies the constraint at equality. We will show that the face induced by this constraint, $F(a)$, is not maximal. Let T be any tour on K_n , and let $\chi(T)$ be its characteristic vector; let e be the edge wx . Note that if T includes edge e (that is, if $\chi(T)_e = 1$), then $a^T \chi(T) \leq 3$, because at most T can include the 2-edge and one other 1-edge. Therefore, any tour which satisfies the DP -constraint at equality must also satisfy the non-negativity constraint $x_e \geq 0$ at equality. This means that $F(a)$ is contained in the face induced by $x_e \geq 0$, which we will denote by $F(e)$. (Recall from Chapter 1 that $F(e)$ is a facet.) To show that the containment is proper, it suffices to find one tour in $F(e) \setminus F(a)$; such a tour is $T_0 = (x, y, w, v, \dots, u)$ (where the other nodes, if any, are inserted between v and u), with $a^T \chi(T_0) = 3 < 4$.

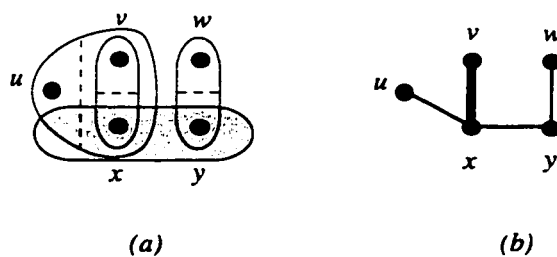


FIGURE 2.3

Although not all DP -constraints are facet-inducing, they have the advantage that they can be exactly separated in polynomial time for solutions x whose support graph G_x is planar. For the details of this algorithm, refer to [L1].

THEOREM 2.3. *The DP-constraints can be separated in $O(n^3)$ time for any $x \in SUBT(n)$ whose support graph G_x is planar.*

If Letchford's algorithm finds that a given $x \in SUBT(n)$ doesn't violate any DP-constraints, then we can conclude it doesn't violate any comb constraints. If x does violate at least one DP-constraint, then the algorithm will find a DP-constraint that is violated by the greatest amount. This may not always be what we would like. For example, Letchford has found a solution $x \in SUBT(12)$ that violates several comb constraints by $1/4$, but the DP-constraint found by his algorithm, violated by $1/2$, is not a comb, and moreover is not facet-inducing (as can be verified easily from the results in Chapter 4) [L2]. Figure 2.4 illustrates this example. In both diagrams, the non-zero values of the x_e are as indicated; edges having $x_e = 0$ are omitted. The top figure depicts one of the violated comb constraints (the handle is the more darkly shaded set), and the lower one shows the maximally violated DP-constraint.

We have found there are solutions x that satisfy all comb constraints, yet violate one or more DP-constraints. Such points would not be 'cut off' by any algorithm, exact or heuristic, that separates only the comb constraints, yet they would be detected by Letchford's algorithm. Figure 2.5 illustrates such a point $x \in SUBT(31)$; as in the previous figure, only edges satisfying $x_e \neq 0$ are shown. This x satisfies all comb constraints, but violates the DP-constraint illustrated in Figure 2.6. The domino-comb in this figure is similar to the one in Figure 2.2; there are six (lightly shaded) 2-node dominoes and a seventh domino $D_7 = A_7 \cup B_7$ containing two of the 2-node dominoes in each of A_7 and B_7 , and a (darkly shaded) handle which contains one node of each of the 2-node dominoes, and intersects nontrivially with both of A_7 and B_7 . This point violates several other DP-constraints similar to this one; we will show in Chapter 5 that such constraints are facet-inducing.

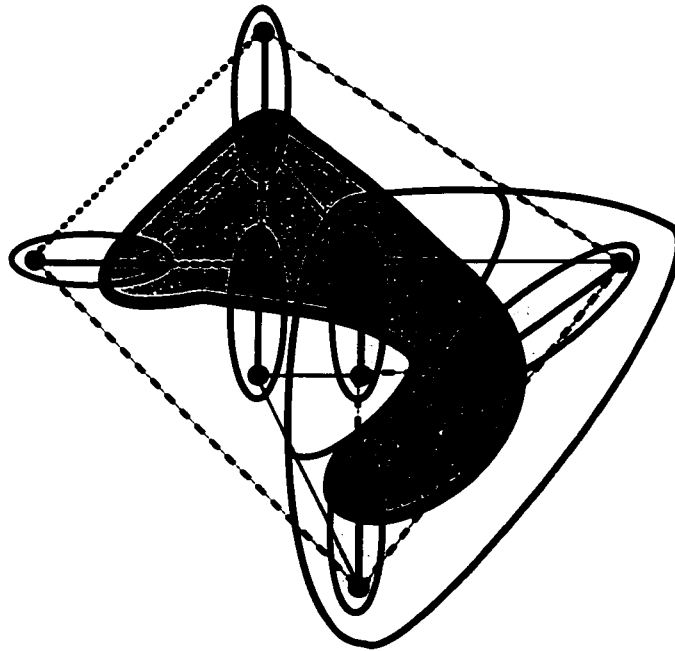
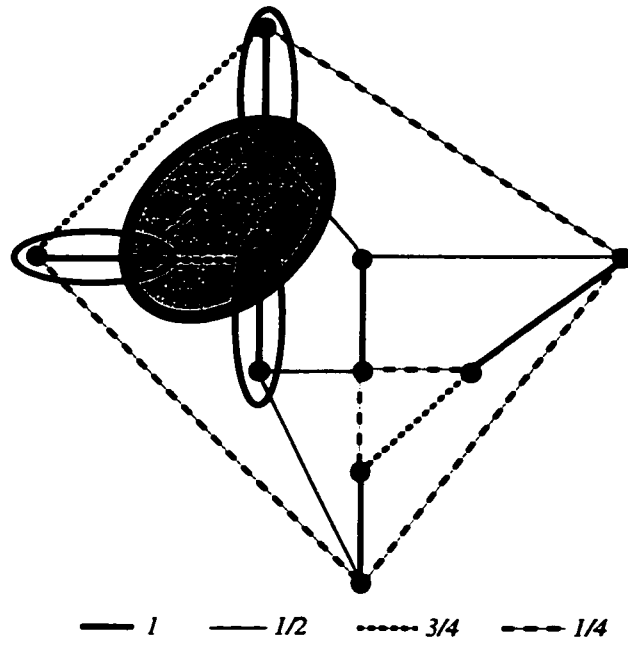


FIGURE 2.4
17

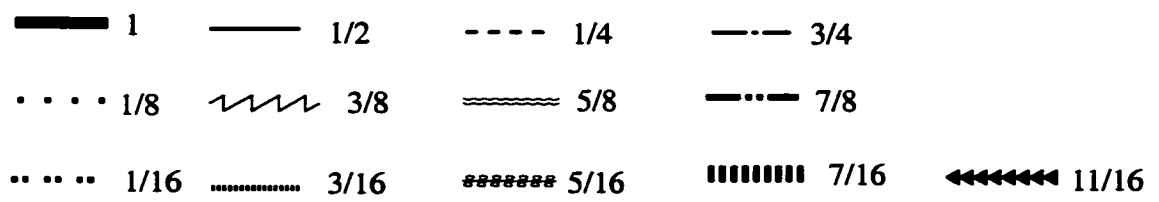
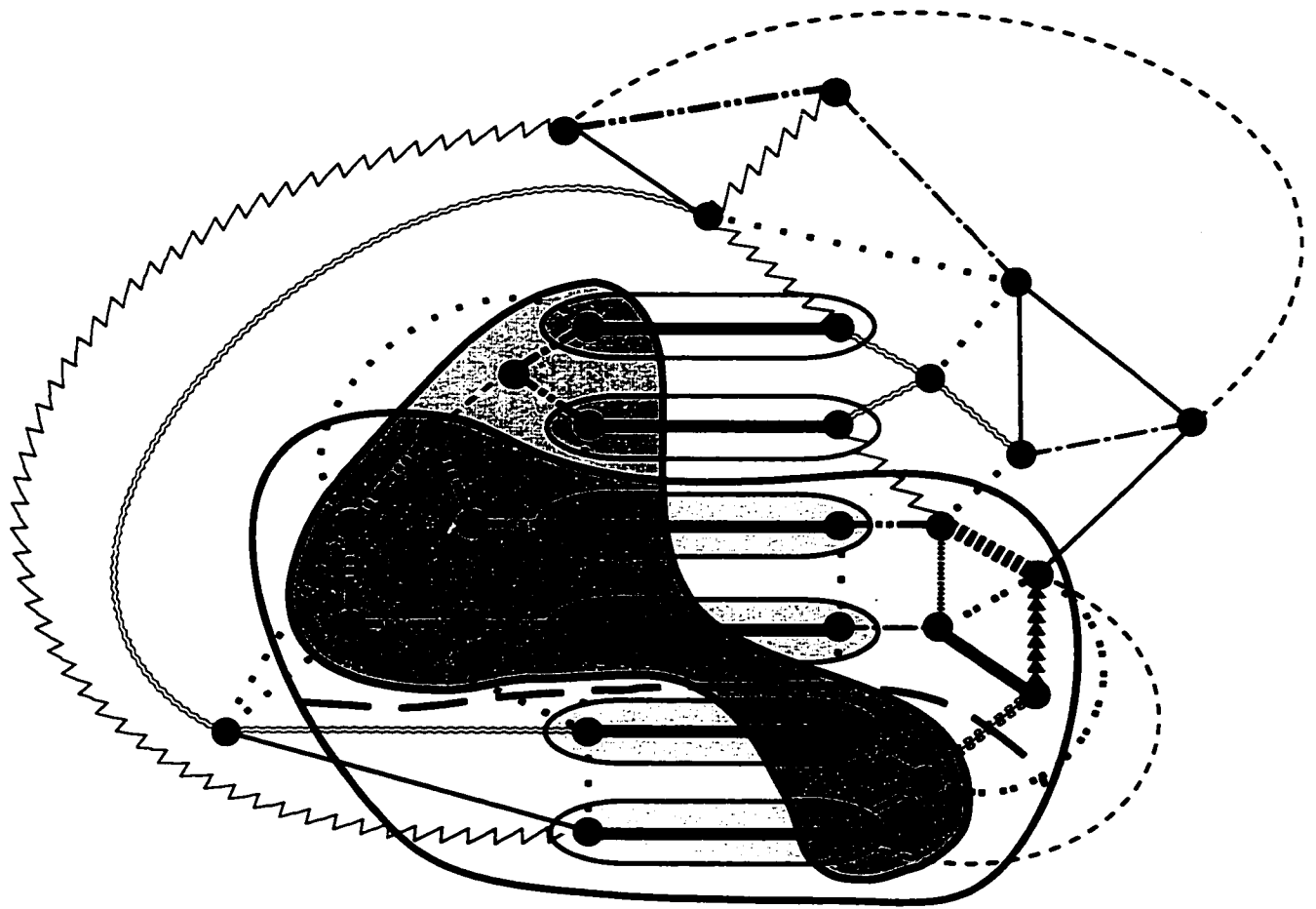


FIGURE 2.6

Another relaxation of the TSP is the *graphical traveling salesman problem*, or *GTSP*, introduced by Cornuéjols, Fonlupt and Naddef in 1985 [CFN]. In the context of a salesman touring cities, we now assume that a given city may be visited more than once and a given edge may be traversed more than once. As an integer programming problem, it can be formulated as follows:

$$\text{minimize: } \sum_{e \in E} c_e x_e$$

$$\text{subject to: } x(\delta(v)) \text{ is even and positive} \quad \text{for all } v \in V \quad (2.4)$$

$$x(\delta(S)) \geq 2 \quad \text{for all } \emptyset \subsetneq S \subsetneq V \quad (2.5)$$

$$\text{with: } 0 \leq x_e \in \mathbb{Z} \quad \text{for all } e \in E \quad (2.6)$$

A *minimal graphical tour* for the *GTSP* is a feasible solution x to the program above such that no other feasible solution y satisfies $y \leq x$ under the component-wise ordering. In [CFN], it is shown that the convex hull of all minimal graphical tours on a graph K_n , denoted by $GTSP(n)$, is a full-dimensional polytope.

Since any Hamilton cycle of K_n is a minimal graphical tour, clearly $STSP(n)$ is a proper subset of $GTSP(n)$. The following theorem from [NR1] shows the strong relationship between the facets of these two polytopes.

THEOREM 2.4 [NADDEF AND RINALDI, 1992]. *Every facet of $STSP(n)$ which is distinct from the non-negativity facets is contained in exactly $n + 1$ facets of $GTSP(n)$, n of which are defined by the degree equations.*

Both the comb constraints and the *DP*-constraints can be rewritten in a form that is equivalent for $STSP(n)$ and valid on $GTSP(n)$; for completeness, we provide standard proofs of these facts.

PROPOSITION 2.5. *The comb constraint corresponding to $\{H; T_1, \dots, T_p\}$ is equivalent on $STSP(n)$ to*

$$x(\delta(H)) + \sum_{i=1}^p x(\delta(T_i)) \geq 3p + 1. \quad (2.7)$$

PROOF. First, we multiply the comb constraint in form (1.7) by -2 to obtain

$$-2x(\gamma(H)) - 2 \sum_{i=1}^p x(\gamma(T_i)) \geq -2|H| - 2 \sum_{i=1}^p |T_i| + (3p + 1).$$

Now add to this the sum of the degree constraints on the handle and on each of the teeth (see (1.8)):

$$\begin{aligned} 2x(\gamma(H)) + x(\delta(H)) &= 2|H|; \\ \sum_{i=1}^p 2x(\gamma(T_i)) + x(\delta(T_i)) &= \sum_{i=1}^p 2|T_i|. \end{aligned}$$

□

PROPOSITION 2.6 [CORNUÉJOLS, FONLUPT AND NADDEF, 1985]. *Comb constraints in the form (2.7) are valid on $GTSP(n)$.*

PROOF. First observe that

$$x(\delta(A_i)) + x(\delta(B_i)) = x(\delta(T_i)) + 2x(E(A_i, B_i)).$$

Therefore, if we sum the subtour elimination constraints (2.5) for T_i, A_i and B_i , $1 \leq i \leq p$, and divide by 2, we obtain

$$\sum_{i=1}^p [x(\delta(T_i)) + x(E(A_i, B_i))] \geq 3p.$$

Adding to this the non-negativity constraints for each edge in $\delta(H) \setminus \cup_{i=1}^p E(A_i, B_i)$ gives

$$x(\delta(H)) + \sum_{i=1}^p x(\delta(T_i)) \geq 3p. \quad (2.8)$$

Observation (1.8) must be modified slightly; the degree constraints (2.4) imply that for any minimal graphical tour x ,

$$\sum_{v \in S} x(\delta(v)) = 2x(\gamma(S)) + x(\delta(S)) \quad (2.9)$$

is even for every node subset S (though no longer necessarily equal to $2|S|$); in particular, this implies that the left hand side of (2.8) is even while the right hand side is odd. Adding 1 to the right hand side gives (2.7). \square

Letchford provides similar results for the *DP*-constraints. A proof of equivalency is not given in [L1], so we include it here; we also present a slightly modified version of Letchford's proof of validity. These proofs are not much different from the ones above.

PROPOSITION 2.7. *The DP-constraint corresponding to $\{F; D_1, \dots, D_p\}$ is equivalent on *STSP*(n) to*

$$\sum_{i=1}^p x(\delta(D_i)) + x(\mathcal{E}) \geq 3p + 1, \quad (2.10)$$

where $x(\mathcal{E}) \equiv \sum_{e \in \mathcal{E}} \mu_e x_e$.

PROOF. We begin by multiplying the original form of the *DP*-constraint by -2 . Note that

$$2\lfloor \mu_e/2 \rfloor = \begin{cases} \mu_e - 1, & e \in \delta(H), \\ \mu_e, & e \notin \delta(H), \end{cases}$$

which in turn implies

$$2 \sum_{e \in \mathcal{E}} \lfloor \mu_e/2 \rfloor x_e = \sum_{e \in \delta(H)} (\mu_e - 1)x_e + \sum_{e \notin \delta(H)} \mu_e x_e = x(\mathcal{E}) - x(\delta(H)).$$

Substituting this into (-2) times (2.2) gives

$$-2x(\gamma(H)) - \sum_{i=1}^p 2x(\gamma(D_i)) + x(\mathcal{E}) - x(\delta(H)) \geq -2|H| - \sum_{i=1}^p 2|D_i| + (3p + 1).$$

We now add to this the sum of the degree constraints on the handle and on each of the dominoes:

$$\begin{aligned} 2x(\gamma(H)) + x(\delta(H)) &= 2|H|; \\ \sum_{i=1}^p [2x(\gamma(D_i)) + x(\delta(D_i))] &= \sum_{i=1}^p 2|D_i|. \end{aligned}$$

□

Notice that the implicitly defined handle H is absent in this form of the constraint; the pseudo-cut \mathcal{E} appears, but we do not have to infer from it and the dominoes a set of handle nodes.

PROPOSITION 2.8. *DP-constraints in the form (2.10) are valid for GTSP(n).*

PROOF. Summing the subtour elimination constraints (2.5) for D_i, A_i and B_i $1 \leq i \leq p$, dividing by 2 and then adding the non-negativity constraints for each edge in F gives

$$\sum_{i=1}^p x(\delta(D_i)) + x(\mathcal{E}) \geq 3p.$$

From the previous proof, this can be written as

$$\sum_{i=1}^p x(\delta(D_i)) + 2 \sum_{e \in \mathcal{E}} \lfloor \mu_e / 2 \rfloor x_e + x(\delta(H)) \geq 3p.$$

As before, the left hand side of this constraint is even for every minimal graphical tour, so we can add 1 to the right hand side. □

Letchford's polynomial-time exact separation algorithm can be easily adapted for a cutting plane approach to solving the *GTSP*, provided we begin with $x \in \mathbb{R}^m$ which satisfies constraints (2.5) and (2.6), and whose support graph is planar.

CHAPTER 3

EQUIVALENT FORMS

In this chapter, we present a new form of the DP -constraints, based on a more general underlying structure in K_n . This will allow us to more easily identify equivalent DP -constraints in subsequent chapters. We also present a condition on this structure that is sufficient to identify a DP -constraint as a comb constraint.

Since $\delta(H) = \delta(\overline{H})$, it is clear from the (2.7) form of the comb constraints presented in Proposition 2.5 the comb constraint corresponding to $\{H; T_1, \dots, T_p\}$ is equivalent to the one corresponding to $\{\overline{H}; T_1, \dots, T_p\}$. (More directly, the ‘ \leq ’ form (1.7) of the second can be obtained from that of the first by adding $1/2$ times the degree equations of all nodes outside H and subtracting the degree equations on all nodes inside H .) We will refer to this as *flipping the handle*. This generalizes in the obvious way to DP -constraints; intuitively, notice the handle in a domino-comb is defined as one shore of the cut supported by the pseudo-cut \mathcal{E} , so it should make little difference if we choose the opposite shore as the handle.

We will show that the dominoes can be altered in an analogous way to obtain equivalent constraints. A domino $D = A \cup B$ can be viewed as a tripartition of the nodes in K_n into the sets A, B and $C \equiv \overline{D}$, where A and B are the *inside compartments* of the domino, and C is the *outside compartment*. Theorem 3.2 below essentially states that we can regard *any* two partites as the inside compartments of a domino in a domino-comb; that is, *switching compartments* in any domino gives an equivalent DP -constraint, albeit one with a somewhat different handle. First, we require the following lemma, for which we recall the definition of symmetric difference of two sets: $R\Delta S = (R \setminus S) \cup (S \setminus R)$.

LEMMA 3.1. Let \mathcal{E}' and \mathcal{E}'' be two pseudo-cuts which support the cuts $\delta(H')$ and $\delta(H'')$ respectively. Let $\mathcal{E} = \mathcal{E}' \cup \mathcal{E}''$, where the number of times an edge occurs in \mathcal{E} is the sum of the number of times it occurs in \mathcal{E}' and in \mathcal{E}'' . Then \mathcal{E} is also a pseudo-cut and it supports the cut $\delta(H' \Delta H'')$.

PROOF. In Figure 3.1, we illustrate all possible types of edges involved in the two pseudo-cuts.

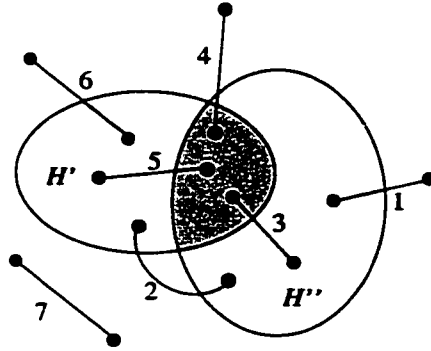


FIGURE 3.1

If μ'_e and μ''_e denote the number of times edge e occurs in \mathcal{E}' and \mathcal{E}'' , then clearly μ'_e is odd for edges of types 2, 3, 4 and 6 only, while μ''_e is odd for edges of type 1, 2, 4 and 5 only. Thus, the edges occurring with odd multiplicity in \mathcal{E} are those of types 1, 3, 5 and 6 only; these constitute exactly the cut corresponding to the symmetric difference, $\delta(H' \Delta H'')$, as required. \square

THEOREM 3.2. Let $\{F; D_1, \dots, D_p\}$ be a domino-comb with implicitly defined handle H . Let D'_1 be the domino $A_1 \cup C_1$ and let $H' = H \Delta A_1$. Then

- (1) $\{F; D'_1, D_2, \dots, D_p\}$ is also a domino-comb, with handle H' ;
- (2) the DP-constraint corresponding to $\{F; D'_1, D_2, \dots, D_p\}$ is equivalent to the one corresponding to $\{F; D_1, \dots, D_p\}$.

PROOF. Figure (3.2) illustrates the changes in the first domino and the handle; D_1 is outlined in bold and lightly shaded, and H is more darkly shaded. To prove

(1), let $\mathcal{E}' = F \cup E(A_1, C_1) \cup \{\cup_{i=2}^p E(A_i, B_i)\}$ and, as before, let μ'_e denote the multiplicity of edge e in \mathcal{E}' ; it suffices to show that \mathcal{E}' is a pseudo-cut which supports the cut $\delta(H')$. Since any cut is a pseudo-cut, we can apply Lemma 3.1 to $\delta(A_1)$ and \mathcal{E} to conclude that $\delta(A_1) \cup \mathcal{E}$ is a pseudo-cut supporting the cut $\delta(A_1 \Delta H)$.

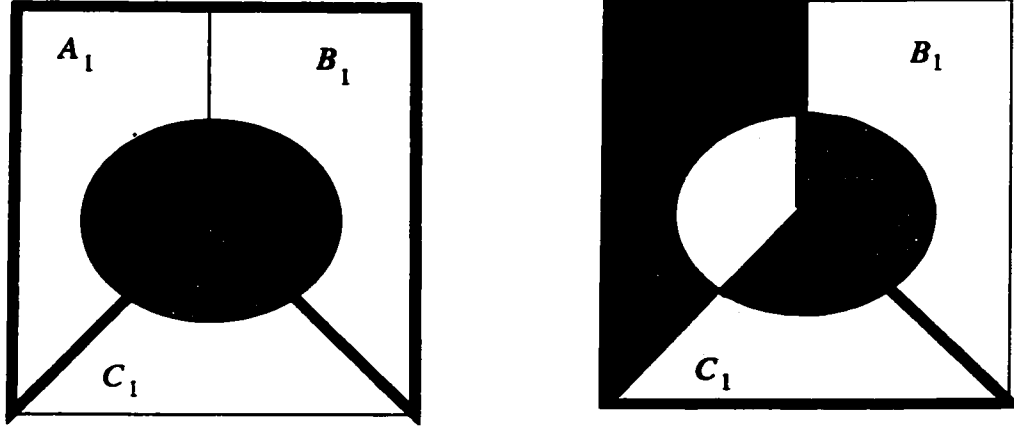


FIGURE 3.2

Notice that

$$\delta(A_1) \cup \mathcal{E} = [E(A_1, B_1) \cup E(A_1, C_1)] \cup \left[\bigcup_{i=1}^p E(A_i, B_i) \cup F \right]. \quad (3.1)$$

Removing the double copy of $E(A_1, B_1)$ from this multiset will not change the parity of the multiplicity of any edge, and so will not affect its property of being a pseudo-cut supporting $\delta(H')$; moreover, it gives us exactly \mathcal{E}' . This derivation of \mathcal{E}' implies that

$$\mu'_e = \begin{cases} \mu_e + 1, & e \in E(A_1, C_1) \\ \mu_e - 1, & e \in E(A_1, B_1) \\ \mu_e, & \text{otherwise.} \end{cases} \quad (3.2)$$

To establish (2), recall from Proposition 2.7 that the (2.10) form of the DP -constraint corresponding to $\{F; D_1, \dots, D_p\}$ is

$$\sum_{i=1}^p x(\delta(D_i)) + x(\mathcal{E}) \geq 3p + 1.$$

It suffices to show that this is the same as the (2.10) form of the DP -constraint corresponding to $\{F; D'_1, D_2, \dots, D_p\}$.

Note that

$$x(\delta(D_1)) = x(\delta(A_1)) + x(\delta(B_1)) - 2x(E(A_1, B_1));$$

also, from (3.1), we have

$$x(\mathcal{E}) = x(\mathcal{E}') + 2x(E(A_1, B_1)) - x(\delta(A_1)).$$

Combined, these give

$$\begin{aligned} \sum_{i=1}^p x(\delta(D_i)) + x(\mathcal{E}) &= x(\delta(B_1)) + \sum_{i=2}^p x(\delta(D_i)) + x(\mathcal{E}') \\ &= x(\delta(D'_1)) + \sum_{i=2}^p x(\delta(D_i)) + x(\mathcal{E}'), \end{aligned}$$

since $\delta(B_1) = \delta(\overline{D'_1}) = \delta(D'_1)$. \square

The fact compartment-switching yields an equivalent DP -constraint can be more easily seen by rewriting these constraints in yet another form. Recall that each domino $D = A \cup B$ can be regarded as a tripartition of the node set into the non-empty subsets, A, B and $C = \overline{D}$; we define the *interpartite edges* of the tripartition (A, B, C) to be

$$\begin{aligned} E(A, B, C) &= E(A, B) \cup E(A, C) \cup E(B, C) \\ &= \delta(D) \cup E(A, B). \end{aligned} \tag{3.3}$$

We will refer to $E(A, B), E(A, C)$ and $E(B, C)$ as the *semicuts* of the tripartition.

PROPOSITION 3.3. *The DP-constraint corresponding to $\{F; D_1, \dots, D_p\}$ is equivalent to*

$$\sum_{i=1}^p x(E(A_i, B_i, C_i)) + x(F) \geq 3p + 1. \quad (3.4)$$

PROOF. We simply rewrite the left hand side of (2.10) using (3.3) as follows:

$$\begin{aligned} \sum_{i=1}^p x(\delta(D_i)) + x(\mathcal{E}) &= \sum_{i=1}^p x(\delta(D_i)) + \left[\sum_{i=1}^p x(E(A_i, B_i)) + x(F) \right] \\ &= \sum_{i=1}^p x(E(A_i, B_i, C_i)) + x(F). \end{aligned}$$

□

When written in form (3.4), it is clear that the *DP*-constraints do not privilege any two partite of a tripartition as constituting the inside compartments of the corresponding domino. This means that none of $E(A_i, B_i)$, $E(A_i, C_i)$ or $E(B_i, C_i)$ can be considered as *the* semicut of the domino, which in turn makes the definition of \mathcal{E} unstable. This suggests a reformulation of the definition of a domino-comb solely in terms of tripartitions.

DEFINITION. A *proto-comb* consists of an edge subset F and an odd number p of tripartitions $(V_{11}, V_{12}, V_{13}), \dots, (V_{p1}, V_{p2}, V_{p3})$ such that $\mathcal{F} = F \cup (\cup_{i=1}^p E(V_{i1}, V_{i2}, V_{i3}))$ is a pseudo-cut. The *proto-handle* of the proto-comb is a node subset H^0 that satisfies

$$\{e \in E \mid \mu_e^0 \text{ is odd}\} = \delta(H^0),$$

where μ_e^0 denotes the multiplicity of edge e in \mathcal{F} . The corresponding *proto-comb constraint* is

$$x(\mathcal{F}) \geq 3p + 1,$$

where $x(\mathcal{F}) = \sum_{e \in \mathcal{F}} \mu_e^0 x_e$.

PROPOSITION 3.4. *The proto-comb constraints are valid on GTSP(n).*

PROOF. We begin by defining the *tripartition constraint* $x(E(V_1, V_2, V_3)) \geq 3$; this constraint is obtained by summing the subtour elimination constraints for V_1 , V_2 and V_3 (in form (2.5)), and then dividing by 2. When the tripartition constraints for each tripartition and the non-negativity constraints (2.6) for each edge in F are added, the result is

$$\sum_{i=1}^p x(E(V_{i1}, V_{i2}, V_{i3})) + x(F) = x(\mathcal{F}) \geq 3p. \quad (3.5)$$

Since \mathcal{F} is a pseudo-cut,

$$x(\mathcal{F}) = x(\delta(H^0)) + 2 \sum_{e \in E} \lfloor \mu_e^0 / 2 \rfloor x_e.$$

As mentioned in the proof of Proposition 2.6, $x(\delta(S))$ is even for any minimal graphical solution x and any proper node subset S ; the second term in this expression is also clearly even. Since $3p$ is odd, we can add 1 to the right hand side of (3.5). \square

Next, we show that if we select two partites per tripartition to be the inside compartments of a domino, we create a domino-comb according to Letchford's definition. The lemma below also shows how the handle of the domino-comb may be obtained from the proto-handle of the proto-comb.

LEMMA 3.5. *Let $\{F; (V_{11}, V_{12}, V_{13}), \dots, (V_{p1}, V_{p2}, V_{p3})\}$ be a proto-comb, and let H^0 be its proto-handle. For each $i \in \{1, \dots, p\}$, select $j(i) < k(i)$ in $\{1, 2, 3\}$, and set $D_i = V_{ij(i)} \cup V_{ik(i)}$. Then*

- (1) $\mathcal{E} = F \cup (\bigcup_{i=1}^p E(V_{ij(i)}, V_{ik(i)}))$ is a pseudo-cut;
- (2) \mathcal{E} supports the cut $\delta(H)$ where

$$H = \{v \in H^0 \mid |\{i \mid v \in D_i\}| \text{ is even}\} \cup \{w \in \overline{H^0} \mid |\{i \mid w \in D_i\}| \text{ is odd}\};$$

- (3) the DP-constraint corresponding to $\{F; D_1, \dots, D_p\}$ is equivalent to the proto-comb constraint corresponding to $\{F; (V_{11}, V_{12}, V_{13}), \dots, (V_{p1}, V_{p2}, V_{p3})\}$.

PROOF. By Lemma 3.1, the union of a cut and a pseudo-cut is still a pseudo-cut, so

$$\begin{aligned} \mathcal{F} \cup \left(\bigcup_{i=1}^p \delta(D_i) \right) &= \left[F \cup \left(\bigcup_{i=1}^p E(V_{i1}, V_{i2}, V_{i3}) \right) \right] \cup \left(\bigcup_{i=1}^p \delta(D_i) \right) \\ &= F \cup \left(\bigcup_{i=1}^p [E(V_{ij(i)}, V_{ik(i)}) \cup \delta(D_i)] \right) \cup \left(\bigcup_{i=1}^p \delta(D_i) \right) \end{aligned}$$

is a pseudo-cut; it remains a pseudo-cut after we remove the double copy of $\cup_{i=1}^p \delta(D_i)$ to obtain \mathcal{E} .

To prove (2), we begin with the first domino. By Lemma 3.1, $\mathcal{F} \cup \delta(D_1)$ is a pseudo-cut supporting the cut $\delta(H^0 \Delta D_1)$; we define $H^1 = H^0 \Delta D_1$ (the shaded region in Figure 3.3). Note that the nodes outside D_1 have stayed on the same shore of the coboundary of the respective handles; more precisely, any $v \in H^0 \setminus D_1$ will be in H^1 and any $u \in \overline{H^0} \setminus D_1$ will be in $\overline{H^1}$. Nodes inside D_1 have switched shores. This argument can be applied iteratively to the p dominoes; in the end, the nodes inside an even number of dominoes will remain either inside or outside the handle, whereas those in an odd number of dominoes will switch sides.

Statement (3) follows easily from Proposition 3.3. \square

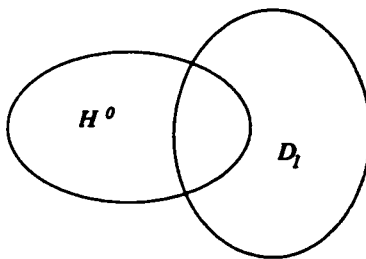


FIGURE 3.3

One consequence of (3) is that all domino-combs created from the same proto-comb generate equivalent DP -constraints. For this reason, we usually refer to ‘the’ DP -constraint induced by a proto-comb.

We next introduce some definitions that will help us to recognize which *DP*-constraints correspond to comb constraints.

DEFINITION. Two tripartitions, (U_1, U_2, U_3) and (V_1, V_2, V_3) are *compatible* if and only if there exist i and j such that $\overline{U_i} \subseteq V_j$.

Note that we will also have $\overline{V_j} \subseteq U_i$. Intuitively, two tripartitions are compatible if two partites of one tripartition fit inside a single partite of the other, and vice versa. We will say that two dominoes or teeth are compatible if their associated tripartitions are compatible. The disjoint teeth $T_i = A_i \cup B_i$ of a comb are pairwise compatible, since $A_i \cup B_i \subset \overline{T_j} = C_j$ for all $i \neq j$. The dominoes of the *DP*-constraint in Figure (2.2) are also pairwise compatible, yet no amount of compartment-switching will yield disjoint dominoes.

LEMMA 3.6. *The tripartitions, (U_1, U_2, U_3) and (V_1, V_2, V_3) are compatible if and only if there exist $j < k$ in $\{1, 2, 3\}$ such that*

$$E(U_j, U_k) \cap E(V_1, V_2, V_3) = \emptyset.$$

PROOF. The condition is clearly equivalent to the condition that $U_j \cup U_k$ is contained in a single partite of (V_1, V_2, V_3) . \square

A special type of compatibility is defined below.

DEFINITION. A tripartition (V_{i1}, V_{i2}, V_{i3}) in a proto-comb is *regular* if for some $1 \leq j < k \leq 3$, the edges of the semicut $E(V_{ij}, V_{ik})$ appear exactly once in the pseudo-cut \mathcal{F} ; that is,

$$\mu_e^0 = 1 \text{ for all } e \in E(V_{ij}, V_{ik}).$$

A tripartition which is not regular is called *irregular*.

If (V_{i1}, V_{i2}, V_{i3}) is a regular tripartition, then without loss of generality, we assume that $j = 1$ and $k = 2$. It is clear from the definition above that in this case, $E(V_{i1}, V_{i2}) \cap E(V_{j1}, V_{j2}, V_{j3}) = \emptyset$ for all $i \neq j \in \{1, \dots, p\}$, so a regular tripartition

is compatible with all other tripartitions in the proto-comb. What is special about a regular tripartition is that it is the *same* two partites that are contained within a single partite of any other tripartition.

LEMMA 3.7. *The partites V_{i1} and V_{i2} in a regular tripartition are on opposite shores of the cut $\delta(H^0)$. Moreover, they will be on opposite shores of $\delta(H)$, where H is the handle in any domino-comb created from this proto-comb.*

PROOF. Since $\mu_e^o = 1$ for all $e \in E(V_{i1}, V_{i2})$, we know $E(V_{i1}, V_{i2}) \subseteq \delta(H^0)$. Suppose $u, v \in V_{i1}$ are on opposite shores of H^0 (so edge uv appears an odd number of times in \mathcal{F}); let $w \in V_{i2}$. Then the 3-cycle $[u, v, w]$ intersects \mathcal{F} an odd number of times, contradicting the fact that \mathcal{F} is a pseudo-cut. Next, for any selection of two inside compartments per tripartition, the nodes in V_{i1} and V_{i2} appear inside exactly the same number of dominoes, by Lemma 3.6. Hence, they all either stay on the same shore of $\delta(H)$, or else they all switch shores; in either case, the nodes of V_{i1} will remain on the opposite shore from those of V_{i2} . \square

When a tripartition in a proto-comb is regular, we will always select V_{i1} and V_{i2} to be the inside compartments when creating a domino-comb; $D_i = V_{i1} \cup V_{i2}$ will be called the *regular domino* associated with this regular tripartition. Regular dominoes are like teeth in combs, as is made precise in the following proposition.

PROPOSITION 3.8. *Let $\{F; (V_{11}, V_{12}, V_{13}), \dots, (V_{p1}, V_{p2}, V_{p3})\}$ be a proto-comb such that $p \geq 3$ and (V_{i1}, V_{i2}, V_{i3}) is regular for all $i \in \{1, \dots, p\}$. Then the domino-comb created from this proto-comb by selecting $D_i = V_{i1} \cup V_{i2}$ for $1 \leq i \leq p$ is a comb.*

PROOF. By Lemma 3.7, V_{i1} and V_{i2} are on opposite shores of the handle, so it must be the case that $\overline{V_{i3}} \subseteq V_{j3}$ for all $i \neq j$. Hence, the dominoes D_i are disjoint and intersect the handle in the right way; without loss of generality, $A_i \equiv V_{i1} = D_i \cap H$ and $B_i \equiv V_{i2} = D_i \setminus H$. Setting $T_i = D_i$ and H equal to the handle of the domino-comb (as described in Lemma 3.4), it is clear that $\{H; T_1, \dots, T_p\}$ is a comb. \square

It is important to note that there exist proto-combs which do not have compatible tripartitions, and that such proto-combs can be found by Letchford's algorithm. A domino-comb created from such a proto-comb is given in Figure 3.4.

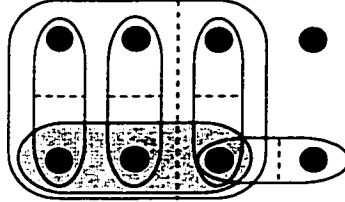


FIGURE 3.4. A domino-comb with incompatible dominoes

We next consider proto-combs $\{F; (V_{11}, V_{12}, V_{13}), \dots, (V_{p1}, V_{p2}, V_{p3})\}$ having exactly one irregular tripartition; without loss of generality, we may assume it is (V_{p1}, V_{p2}, V_{p3}) . Since the regular tripartitions are compatible with all other tripartitions, such a proto-comb has compatible tripartitions. Note that each regular domino $D_i = V_{i1} \cup V_{i2}$ is contained within one partite of the irregular tripartition. If all the regular dominoes are contained in the *same* partite, then we could select this as the outside compartment C_p of the p -th domino; the domino-comb thus created will have p disjoint dominoes. However, for irregularity, there must exist an edge $e \in E(A_p, B_p)$ with $\mu_e \geq 2$; since e obviously cannot be an interpartite edge of any of the other tripartitions, we must have $e \in F$ and $\mu_e = 2$. Hence, $e \notin \delta(H)$, and so it cannot be the case that $A_p = D_p \cap H$ and $B_p = D_p \setminus H$; this domino is therefore not a tooth in a comb with handle H . An example of such a domino-comb is illustrated in Figure 3.5(a). If there are regular dominoes in two or more of the partites of the irregular tripartition, then these partites contain nodes inside the handle as well as nodes outside the handle; again, we cannot select inside compartments so that $A_p = D_p \cap H$ and $B_p = D_p \setminus H$. In both cases, note that the domino created from the irregular tripartition is ‘twisted’ with respect to the handle.

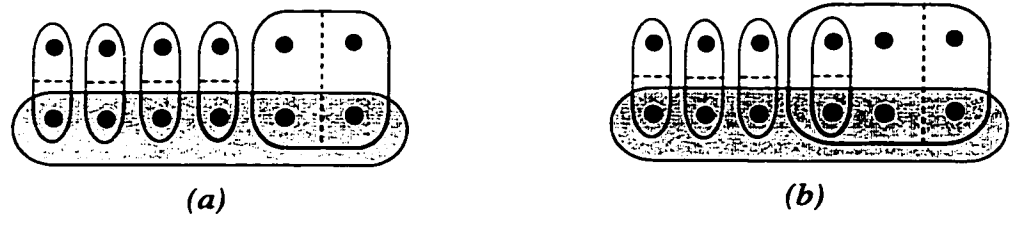


FIGURE 3.5. Domino-combs with one irregular domino

NON-FACET-INDUCING DP -CONSTRAINTS

As was mentioned in Chapter 1, facet-inducing constraints are considered the most useful for the cutting plane approach, and as was mentioned in Chapter 2, not all DP -constraints are facet-inducing for $STSP(n)$. In this chapter, we provide a number of sufficient (but not necessary) conditions for DP -constraints to be non-facet-inducing. Recall that the face corresponding to a valid, supporting constraint $a^T x \leq \alpha$ of polytope P is $F(a) = \{x \in P \mid a^T x = \alpha\}$, and that a facet is a maximal proper face. We call any tour whose characteristic vector belongs to $F(a)$ an a -tight tour.

LEMMA 4.1. *Let $P \subset \mathbb{R}^s$ be a polyhedron, and let $Cx = c$ be a maximal linearly independent set of t equation constraints valid for P . Let $b^T x \leq \beta$ be a valid inequality such that $F(b) \neq P$. Then $b^T x \leq \beta$ is supporting but not facet-inducing if and only if there exist distinct constraints, at least two of which are facet-inducing, $a_1^T x \leq \alpha_1, \dots, a_k^T x \leq \alpha_k$, positive scalars $0 < \gamma_i \in \mathbb{R}$ for $0 \leq i \leq k$, and $\lambda \in \mathbb{R}^t$ such that $b^T = \sum_{i=1}^k \gamma_i a_i^T + \lambda^T C$ and $\beta = \sum_{i=1}^k \gamma_i \alpha_i + \lambda^T c$.*

PROOF. First, assume $b^T x \leq \beta$ is equivalent to a positive combination of distinct constraints, at least two of which are facet-inducing. Then using the notation in the statement of the lemma,

$$F(b) = \bigcap_{i=1}^k F(a_i). \tag{4.1}$$

Thus, $F(b)$ is properly contained in each facet $F(a_i)$, and is therefore not itself maximal.

Conversely, if the constraint $b^T x \leq \beta$ is valid and supporting, then as noted in Chapter 1, it must be implied by a minimal defining system for the polytope. In other words, it can be written as a linear combination of the equation constraints and a positive combination of facet-inducing inequalities. There must be at least one facet-inducing inequality in this combination, since $F(b) \neq P$. If there is only one facet-inducing inequality, then $b^T x \leq \beta$ would be equivalent to it; since $b^T x \leq \beta$ is non-facet-inducing, two or more facet-inducing constraints must appear in the combination. \square

PROPOSITION 4.2. *A DP-constraint with $p = 1$ and $n \geq 4$ is not facet-inducing.*

PROOF. Assume the DP-constraint is in the ' \leq ' form (2.2). If $p = 1$, then $(3p + 1)/2 = 2$, and so the DP-constraint can be derived by adding the following subtour elimination and non-negativity constraints which, as noted in Chapter 1, are distinct and facet-inducing for $n \geq 4$:

$$\begin{aligned} x(\gamma(H)) &\leq |H| - 1, \\ x(\gamma(D_1)) &\leq |D_1| - 1, \\ \lfloor \mu_e/2 \rfloor (-x_e) &\leq 0 \quad \text{for all } e \in E. \end{aligned}$$

The result now follows from Lemma 4.1. \square

Recall that a comb is a domino-comb in which all dominoes are disjoint, and $A_i = D_i \cap H$ and $B_i = D_i \setminus H$ for all $1 \leq i \leq p$. The following proposition shows that if the dominoes are disjoint, but at least one does not intersect the handle in this nice way (see for example Figure 3.5(a)), then the corresponding DP-constraint is not facet-inducing.

PROPOSITION 4.3. *Let $\{F; D_1, \dots, D_p\}$ be a domino-comb with $D_i \cap D_j = \emptyset$ for all $i \neq j$. If $F \cap (\cup_{i=1}^p E(A_i, B_i)) \neq \emptyset$, then the corresponding DP-constraint is not facet-inducing.*

PROOF. First, assume that $E(A_p, B_p) \subseteq F$, so that $\mu_e = 2$ for all $e \in E(A_p, B_p)$. (Since disjoint dominoes have disjoint semicuts, $\mu_e \leq 2$ for all $e \in E$.) Thus,

$E(A_p, B_p) \cap \delta(H) = \emptyset$, meaning that D_p is either entirely inside or entirely outside the handle. Let

$$F' = F \setminus E(A_p, B_p) \quad \text{and} \quad \mathcal{E}' = F' \cup \left\{ \bigcup_{i=1}^{p-1} E(A_i, B_i) \right\}.$$

Note that \mathcal{E}' is still a pseudo-cut, and that it supports the same cut $\delta(H)$. Let μ'_e denote the multiplicity of edge e in \mathcal{E}' . Mimicking the proof of Theorem 2.1, we add up (i) the degree constraints for all nodes in H , (ii) the subtour elimination constraints for A_i , B_i and D_i for all $1 \leq i \leq p-1$, and (iii) the non-negativity constraints for all $e \in F'$. We obtain

$$\begin{aligned} 2x(\gamma(H)) + \sum_{i=1}^{p-1} 2x(\gamma(D_i)) + \left[x(\delta(H)) - \sum_{i=1}^{p-1} x(E(A_i, B_i)) - x(F') \right] \\ \leq 2|H| + 2 \sum_{i=1}^{p-1} |D_i| - 3(p-1). \end{aligned}$$

The expression in square brackets is $x(\delta(H)) - x(\mathcal{E}')$; since μ'_e is odd if and only if $e \in \delta(H)$, the coefficients in this expression are even. Note also that $p-1$ is even. We can therefore divide both sides of this inequality by 2. Add to this the sum of the subtour elimination constraints for A_p and B_p ,

$$x(\gamma(A_p)) + x(\gamma(B_p)) = x(\gamma(D_p)) - x(E(A_p, B_p)) \leq |D_p| - 2.$$

The result is exactly the DP -constraint for the original domino-comb, so by Lemma 4.1, it is not facet-inducing. Moreover, notice that since no parity arguments were used to adjust the right hand side, this constraint is in fact satisfied by every point in the subtour polytope.

Next, we assume that each domino contains nodes both inside and outside the handle. By hypothesis, there exists $i \in \{1, \dots, p\}$ and an edge $e \in F \cap E(A_i, B_i)$ such that $\mu_e = 2$. Because such an edge is not in $\delta(H)$, it cannot be the case that

both $A_i = D_i \cap H$ and $B_i = D_i \setminus H$; this domino does not intersect the handle the way a tooth in a comb does. Define a related comb by setting $A'_i = D_i \cap H$, $B'_i = D_i \setminus H$ and $T_i = A'_i \cup B'_i = D_i$, for all $1 \leq i \leq p$. Then the (2.2) form of the DP -constraint,

$$x(\gamma(H)) + \sum_{i=1}^p x(\gamma(D_i)) - \sum_{e \in E} \lfloor \mu_e/2 \rfloor x_e \leq |H| + \sum_{i=1}^p |D_i| - \left(\frac{3p+1}{2} \right),$$

can be obtained by adding the comb constraint

$$x(\gamma(H)) + \sum_{i=1}^p x(\gamma(T_i)) \leq |H| + \sum_{i=1}^p |T_i| - \left(\frac{3p+1}{2} \right)$$

and the non-negativity constraint $-x_e \leq 0$ for each $e \in F \cap E(A_i, B_i)$, $i \in \{1, \dots, p\}$.

Again, by Lemma 4.1, the result follows. \square

For greater generality, we can rephrase this result in terms of proto-combs.

PROPOSITION 4.4. *Let $\{F; (V_{11}, V_{12}, V_{13}), \dots, (V_{p1}, V_{p2}, V_{p3})\}$ be a proto-comb with $\overline{V_{i3}} \subseteq V_{j3}$ for all $i \neq j$. If $F \cap (\cup_{i=1}^p E(V_{i1}, V_{i2})) \neq \emptyset$, then the corresponding DP -constraint is not facet-inducing.*

We next consider the case of intersecting dominoes. It should not be surprising that if two dominoes in a domino-comb not only intersect, but are in fact identical, then the corresponding DP -constraint is not facet-inducing; this remains true after compartment-switching in one or both of the identical dominoes. This result is more easily stated and proved in the context of proto-combs.

PROPOSITION 4.5. *Let $\{F; (V_{11}, V_{12}, V_{13}), \dots, (V_{p1}, V_{p2}, V_{p3})\}$ be a proto-comb with two identical tripartitions. Then the corresponding DP -constraint is not facet-inducing.*

PROOF. Without loss of generality, we assume that $V_{1i} = V_{2i}$ for all $1 \leq i \leq 3$. Consider the derivation of the constraint presented in the proof of Proposition 3.4

(although this derivation is used to prove validity for $GTSP(n)$, it also works on $STSP(n)$). Note that the two edge multisets $E(V_{11}, V_{12}, V_{13}) \cup E(V_{21}, V_{22}, V_{23})$ and $\delta(V_{11}) \cup \delta(V_{12}) \cup \delta(V_{13})$ are equal. Removing these edges from \mathcal{F} results in a new pseudo-cut \mathcal{F}' supporting the same cut $\delta(H^0)$. Since

$$x(E(V_{11}, V_{12}, V_{13})) + x(E(V_{21}, V_{22}, V_{23})) = x(\delta(V_{11})) + x(\delta(V_{12})) + x(\delta(V_{13})),$$

the original DP -constraint can be derived by adding the DP -constraint corresponding to the proto-comb $\{F'; (V_{31}, V_{32}, V_{33}), \dots, (V_{p1}, V_{p2}, V_{p3})\}$ and the subtour elimination constraints corresponding to V_{11}, V_{12} and V_{13} . Since these last three constraints are distinct and facet-inducing [GP2], we can invoke Lemma 4.1. \square

As we move on to other types of intersecting dominoes, we assume for the sake of simplicity that all tripartitions/dominoes are compatible. Recall that this means that any pair of tripartitions must have the property that two partite of one are entirely contained in a single partite of the other (and vice versa). The next result states that if such a containment is tight, then the corresponding DP -constraint is not facet-inducing.

PROPOSITION 4.6. *Let $\{F; (V_{11}, V_{12}, V_{13}), \dots, (V_{p1}, V_{p2}, V_{p3})\}$ be a proto-comb with pairwise compatible tripartitions. If there exist $1 \leq i < j \leq p$ and $k, l \in \{1, 2, 3\}$ such that $\overline{V_{ik}} = V_{jl}$, then the corresponding DP -constraint is not facet-inducing.*

PROOF. Recall that the tripartition constraints, $x(E(V_{i1}, V_{i2}, V_{i3})) \geq 3$, are formed by adding half multiples of the subtour elimination constraints for each partite, $\frac{1}{2}x(\delta(V_{ik})) \geq 1$. If we remove from the tripartition derivation of the DP -constraint

$$\frac{1}{2}x(\delta(V_{ik})) + \frac{1}{2}x(\delta(V_{jl})) = \frac{1}{2}x(\delta(\overline{V_{ik}})) + \frac{1}{2}x(\delta(V_{jl})) = x(\delta(V_{jl})) \geq 2, \quad (4.2)$$

we can still use a parity argument to add 1 to the right hand side of the remaining inequality to obtain a valid constraint $a^T x \geq \alpha$ for $STSP(n)$. The original DP -constraint can then be expressed as the sum of $a^T x \geq \alpha$ and the subtour elimination

constraint above. (Note that these constraints can be expressed in ' \leq ' form by multiplying by -1 and adding appropriate degree equations.)

If $a^T x \geq \alpha$ is facet-inducing, then provided that it is distinct from the subtour elimination constraint (4.2), Lemma 4.1 implies that the original DP -constraint is non-facet-inducing. In fact, it suffices to show that the original DP -constraint is distinct from (4.2); we do this by providing a tour which is tight for the subtour elimination constraint, but not for the original DP -constraint.

Without loss of generality, we assume that $\overline{V_{13}} = V_{21}$. Since $p \geq 3$, there must be at least one more tripartition compatible with the first two. By renumbering if necessary, we can assume that $\overline{V_{33}} = V_{31} \cup V_{32}$ is contained in a single partite of the first tripartition. Suppose $\overline{V_{33}} \subseteq V_{11} \subset V_{21}$; we illustrate this in Figure 4.1. The first tripartition, (V_{11}, V_{12}, V_{13}) , is indicated with solid lines and light shading; the second tripartition, (V_{21}, V_{22}, V_{23}) is indicated with dashed lines and no shading. For simplicity, only V_{31} and V_{32} from the third tripartition are depicted, with dotted lines and darker shading. Open circles indicate the possible presence of one or more nodes; areas with neither solid nor open circles cannot contain any nodes. A tour T is indicated in the figure; let $\chi(T)$ denote its characteristic vector in \mathbb{R}^m .

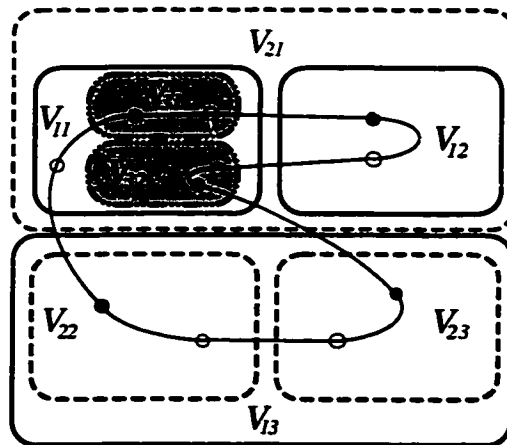


FIGURE 4.1. Three tripartitions and a tour

Notice that $\chi(T)(\delta(V_{21})) = 2$, but

$$\chi(T)(E(V_{11}, V_{12}, V_{13})) = \chi(T)(E(V_{31}, V_{32}, V_{33})) = 4.$$

Hence, this tour is tight for the subtour elimination constraint, but because two of the tripartitions have slacks of one, it does not satisfy the original *DP*-constraint at equality.

The case where $\overline{V_{33}} \subseteq V_{12} \subseteq V_{21}$ is analogous. Finally, suppose $\overline{V_{33}} \subseteq V_{13}$. In order for the third tripartition to be compatible with, yet different from, the second, we must have either $\overline{V_{33}} \subseteq V_{22}$ or $\overline{V_{33}} \subseteq V_{23}$; we can easily adapt the proof above to handle this case.

Next assume $a^T x \geq \alpha$ is non-facet-inducing. If $F(a) = STSP(n)$, then the original *DP*-constraint would be equivalent to subtour elimination constraint (4.2); we just proved this is not the case. If it is not supporting, then neither is the original *DP*-constraint, and we are done. If it is supporting, then by Lemma 4.1, it is equivalent to a positive combination of distinct constraints, at least two of which are facet-inducing. Adding the facet-inducing subtour elimination constraint (4.2) to this clearly results in another non-facet-inducing constraint. \square

Figure 4.2 illustrates a couple of domino-combs created from proto-combs to which this proposition applies. (As before, C_i denotes the outside compartment of a domino.)

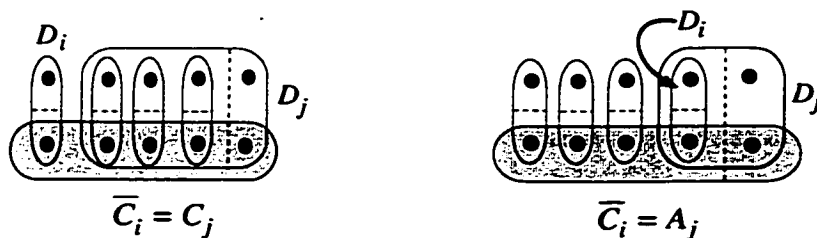


FIGURE 4.2

We can extend these examples by observing that adding ‘outside’ nodes to a non-facet-inducing constraint results in another non-facet-inducing constraint.

LEMMA 4.7. *Let $a^T x \leq \alpha$ be a valid constraint for $STSP(n)$ such that $F(a)$ is a proper subset of $STSP(n)$ and $a_e \geq 0$ for all $e \in E(K_n)$. Let v denote an $(n+1)$ -th node. Define a coefficient vector a^* on the edges of K_{n+1} by setting*

$$a_e^* = \begin{cases} a_e, & e \notin \delta(v), \\ 0, & e \in \delta(v). \end{cases}$$

Then $(a^)^T x^* \leq \alpha$ is a valid constraint for $STSP(n+1)$. Moreover, if $a^T x \leq \alpha$ is not facet-inducing, then neither is $(a^*)^T x^* \leq \alpha$.*

PROOF. To establish validity, let T^* be a tour in K_{n+1} , and $\chi(T^*)$, its characteristic vector. We can ‘short-cut’ this tour to obtain a tour T in K_n by replacing the chain $[uvw]$ with $[uw]$; since $a_{uv} \geq 0$,

$$(a^*)^T \chi(T^*) = a^T \chi(T) - a_{uv} \leq a^T \chi(T) \leq \alpha.$$

If $a^T x \leq \alpha$ is not facet-inducing, then since $F(a)$ is a proper subset of $STSP(n)$, we can use Lemma 4.1 to conclude that there exist at least two distinct facet-inducing constraints $b_1^T x \leq \beta_1, \dots, b_k^T x \leq \beta_k$, positive scalars $0 < \gamma_i \in \mathbb{R}$, $0 \leq i \leq k$ and $\lambda \in \mathbb{R}^m$ such that

$$a^T = \sum_{i=1}^k \gamma_i b_i^T + \lambda^T A \quad \text{and} \quad \alpha = \sum_{i=1}^k \gamma_i \beta_i + \lambda^T \mathbf{2} \quad (4.3)$$

(where, as before, A is the node-edge incidence matrix for K_n). Each of these inequality constraints can be lifted to a valid constraint on $STSP(n+1)$ in the same way that $a^T x \leq \alpha$ was. Moreover, these new constraints on $STSP(n+1)$ will still be distinct; to see this, note that if $b_i^* = \rho b_i^T + \sigma^T A^*$ and $\beta_i = \rho \beta_i + \sigma^T \mathbf{2}$ for some $0 < \rho \in \mathbb{R}$ and $\sigma \in \mathbb{R}^{n+1}$, then because $(b_i^*)_e = (b_i^T)_e = 0$ for all $e \in \delta(v)$, $\sigma_v = 0$, the equivalence would restrict down to $STSP(n)$. We can ‘lift’ (4.3) to $STSP(n+1)$ by

defining $\lambda^* \in \mathbb{R}^{n+1}$ as $\lambda_u^* = \lambda_u$ for $u \neq v$ and $\lambda_v^* = 0$; then $a^* = \sum_{i=1}^k \gamma_i b_i^* + (\lambda^*)^T A^*$ and $\alpha = \sum_{i=1}^k \gamma_i \beta_i + (\lambda^*)^T \mathbf{2}$. If any of the constraints $(b_i^*)^T x \leq \beta_i^*$ is either not supporting or not facet-inducing, then neither is $(a^*)^T x \leq \alpha$; if they are all facet-inducing, then by Lemma 4.1, $(a^*)^T x^* \leq \alpha$ is not facet-inducing. \square

By flipping the handle and switching compartments as necessary, we can see that any region in a domino-comb can be considered the ‘outside’ of a domino-comb that induces an equivalent *DP*-constraint. Thus, Proposition 4.6 and Lemma 4.7 together imply that if any compartment of a domino (in a domino-comb with pairwise compatible dominoes) contains exactly one other domino plus some extra nodes, then the corresponding *DP*-constraint is not facet-inducing.

We return to the special case of proto-combs that have $p-1$ regular tripartitions and one irregular tripartition (which we assume to be the last one), introduced at the end of the previous chapter. The *DP*-constraints associated to such proto-combs are related to the chain constraints, first introduced by Padberg and Hong in 1980 [PH] and generalized by Naddef in 1998 [NP].

DEFINITION. A *chain* consists of node subsets $H, T_1, \dots, T_t, S_1, \dots, S_s, R_1, \dots, R_r$, satisfying the following conditions:

- (1) $T_1, \dots, T_t, S_1, \dots, S_s, R_1, \dots, R_r$ are pairwise disjoint;
- (2) $T_i \cap H$ and $T_i \setminus H$ are non-empty for all $1 \leq i \leq t$;
- (3) $S_j \subset H$ for all $1 \leq j \leq s$;
- (4) $R_k \subset \overline{H}$ for all $1 \leq k \leq r$;
- (5) $t + s + r$ is even.

In other words, a chain consists of a handle H and some regular teeth T_1, \dots, T_t , plus additional subsets S_j inside the handle and R_k outside the handle. See Figure

4.3. One form of the associated *chain constraint* is:

$$\begin{aligned}
& x(\gamma(H)) + \sum_{i=1}^t x(\gamma(T_i)) + \sum_{j=1}^s x(\gamma(S_j)) + \sum_{k=1}^r x(\gamma(R_k)) + \sum_{j=1}^s \sum_{k=1}^r x(E(S_j, R_k)) \\
& \leq |H| + \sum_{i=1}^t |T_i| + \sum_{j=1}^s |S_j| + \sum_{k=1}^r |R_k| - \left(\frac{3t + s + r + 2}{2} \right).
\end{aligned}$$

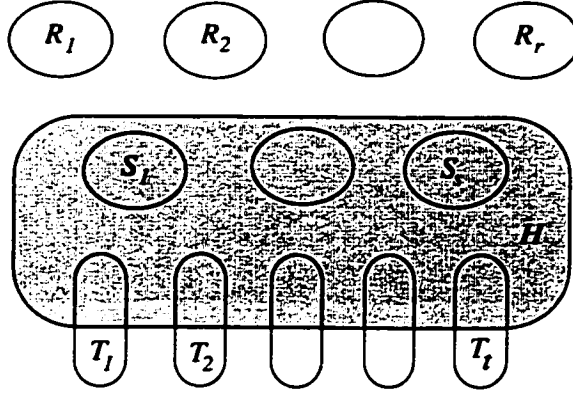


FIGURE 4.3. A chain

An equivalent form of this constraint with the inequality sign in the opposite direction is

$$\begin{aligned}
& x(\delta(H)) + \sum_{i=1}^t x(\delta(T_i)) + \sum_{j=1}^s x(\delta(S_j)) + \sum_{k=1}^r x(\delta(R_k)) - 2 \sum_{j=1}^s \sum_{k=1}^r x(E(S_j, R_k)) \\
& \geq 3t + s + r + 2.
\end{aligned}$$

The chain constraints are known to be valid and facet-inducing for $STSP(n)$ (see [NP] and [JRR]).

Consider a domino-comb $\{F; D_1, \dots, D_p\}$, where D_1, \dots, D_{p-1} are regular and D_p is irregular and disjoint from the regular dominoes, with the additional property that $A_p \cap H$, $A_p \setminus H$, $B_p \cap H$ and $B_p \setminus H$ are all non-empty. We already know by

Proposition 4.3 that the corresponding DP -constraint is not facet-inducing. Define a related chain by setting $T_i = D_i$ for $1 \leq i \leq p-1 \equiv t$, and $S_1 = A_p \cap H$, $S_2 = B_p \cap H$, $R_1 = A_p \setminus H$ and $R_2 = B_p \setminus H$. Then $s = r = 2$ and so $t + s + r$ is even. Moreover,

$$\begin{aligned}
x(\gamma(D_p)) &= x(\gamma(A_p)) + x(\gamma(B_p)) + x(E(A_p, B_p)) \\
&= (x(\gamma(S_1)) + x(\gamma(R_1)) + x(E(S_1, R_1))) \\
&\quad + (x(\gamma(S_2)) + x(\gamma(R_2)) + x(E(S_2, R_2))) \\
&\quad + (x(E(S_1, S_2)) + x(E(S_1, R_2)) + x(E(R_1, S_2)) + x(E(R_1, R_2))) \\
&= \sum_{j=1}^2 x(\gamma(S_j)) + \sum_{k=1}^2 x(\gamma(R_k)) + \sum_{j=1}^2 \sum_{k=1}^2 x(E(S_j, R_k)) \\
&\quad + x(E(S_1, S_2)) + x(E(R_1, R_2)).
\end{aligned}$$

From (2.1) and the fact that disjoint dominoes have disjoint semicut, we have

$$F = \{e \in \delta(H) \mid e \text{ is not in any domino semicut}\} \cup [E(S_1, S_2) \cup E(R_1, R_2)].$$

Thus,

$$\mu_e = \begin{cases} 2, & e \in E(S_1, S_2) \cup E(R_1, R_2) \\ 1, & e \in \delta(H) \\ 0, & \text{otherwise.} \end{cases}$$

Together, these observations show that the left hand side of the (2.2) form of the DP -constraint associated to this domino-comb coincides with that of the chain constraint having $s = r = 2$ and $t = p - 1$. However, the right hand side of the DP -constraint is

$$|H| + \sum_{i=1}^p |D_i| - \left(\frac{3p+1}{2}\right) = |H| + \sum_{i=1}^p |D_i| - \left(\frac{3t+4}{2}\right),$$

whereas the right hand side of the chain constraint is

$$|H| + \sum_{i=1}^p |D_i| - \left(\frac{3t+6}{2}\right) = \left[|H| + \sum_{i=1}^p |D_i| - \left(\frac{3t+4}{2}\right)\right] - 2.$$

Thus, the chain constraint is stronger, giving another proof that such a DP -constraint is not facet-inducing (in fact, this shows it is not even supporting). In particular, any point cut off by such a DP -constraint would also be cut off by this associated chain constraint.

A related type of DP -constraint can be shown to be the sum of a chain constraint and two subtour elimination constraints, which correspond to two extra regular dominoes, D_{p+1} and D_{p+2} . Some examples are illustrated in Figure 4.4; the extra dominoes are indicated with bold boundaries.

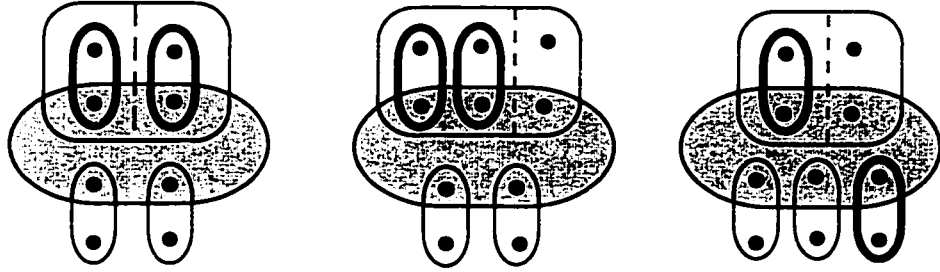


FIGURE 4.4

To verify this, note that the previous argument still shows that the left hand sides are the same. The right hand side of the sum of the chain constraint and the two subtour elimination constraints is:

$$\begin{aligned} & \left[|H| + \sum_{i=1}^t |T_i| + \sum_{j=1}^2 |S_j| + \sum_{k=1}^2 |R_k| - \left(\frac{3t+6}{2} \right) \right] + \left[|D_{p+1}| + |D_{p+2}| - 2 \right] \\ & = |H| + \sum_{i=1}^{p+2} |D_i| - \left(\frac{3t+10}{2} \right) = |H| + \sum_{i=1}^{p+2} |D_i| - \left(\frac{3(p+2)+1}{2} \right), \end{aligned}$$

which is exactly the right hand side of the DP -constraint. It follows from Lemma 4.1 that these types of DP -constraints are also not facet-inducing.

The situation becomes more interesting when we look at a DP -constraint whose left hand sides equals the left hand side of the sum of a chain constraint and the

subtour elimination constraints corresponding to *four* extra regular dominoes. In such cases, the right hand side of the sum of the chain and subtour elimination constraints is

$$\begin{aligned} & \left[|H| + \sum_{i=1}^t |T_i| + \sum_{j=1}^2 |S_j| + \sum_{k=1}^2 |R_k| - \left(\frac{3t+6}{2} \right) \right] + \left[\sum_{i=p+1}^{p+4} |D_i| - 4 \right] \\ & = |H| + \sum_{i=1}^{p+4} |D_i| + - \left(\frac{3t+14}{2} \right) = \left[|H| + \sum_{i=1}^{p+2} |D_i| - \left(\frac{3(p+4)+1}{2} \right) \right] + 1. \end{aligned}$$

Notice that the *DP*-constraint is now stronger.

An example of such a *DP*-constraint is the one associated to the domino-comb in Figure 2.2; the four extra dominoes are the ones in the inside compartments of the irregular domino. As was mentioned in Chapter 2, the constraint associated to the domino-comb in Figure 2.2 was shown by Letchford in [L1] to be facet-inducing. In the next chapter, we generalize this example to a whole family of facet-inducing *DP*-constraints.

•

A FAMILY OF FACET-INDUCING DP -CONSTRAINTS

We begin by giving a name to a certain family of domino-combs.

DEFINITION. A *twisted comb* is a domino-comb with $p - 1$ ($p \geq 3$) regular dominoes D_1, \dots, D_{p-1} and one irregular domino, D_p , which contains at least two regular dominoes in each of its three compartments. The corresponding DP -constraint is called a *twisted comb constraint*. If the regular dominoes in a twisted comb are all 2-matching dominoes (that is, $|A_i| = |B_i| = 1$ for $1 \leq i \leq p - 1$), then it is called a *2-matching twisted comb*, and the corresponding constraint is a *2-matching twisted comb constraint*.

Three simple examples of 2-matching twisted combs are illustrated in Figure 5.1 (the 12-node example is the same one as in Figure 2.2). In this chapter, we will show that the twisted comb constraints are facet-inducing. The discussion here is modelled on the inductive proof given by Grötschel and Padberg in 1979 [GP2] that the comb constraints are facet-inducing.

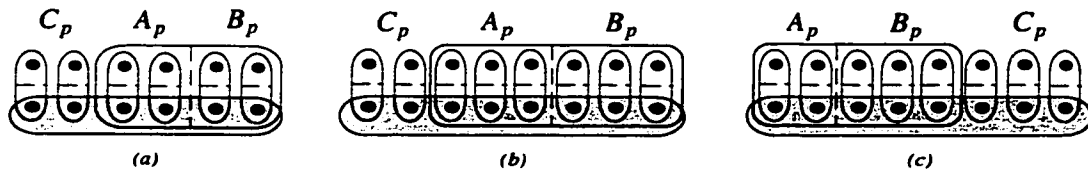


FIGURE 5.1. 2-matching twisted combs

For most of this chapter, we will be using the (2.2) form of the constraint, $a^T x \leq \alpha$. The coefficient vector a of a 2-matching twisted comb is determined

as follows. For $xy \in E$,

$$a_{xy} = \begin{cases} 2, & \text{if } x, y \in A_p \cap H \text{ or } x, y \in B_p \cap H \text{ or } x, y \in D_i \cap D_p, i \neq p; \\ 1, & \text{if } x, y \in D_i \setminus D_p, i \neq p \text{ or } x, y \in C_p \cap H \text{ or } x \in C_p \cap H, y \in D_p \cap H \\ & \text{or } x \in D_p \cap H, y \in D_p \setminus H, \text{ but } x, y \text{ are not in the same } D_i, i \neq p; \\ 0, & \text{otherwise.} \end{cases}$$

We call edge e a *2-edge*, *1-edge* or *0-edge* according to the value of a_e . The coefficients of the constraint corresponding to the 12-node twisted comb are illustrated in Figure 5.2; 2-edges are in bold and 0-edges are omitted.

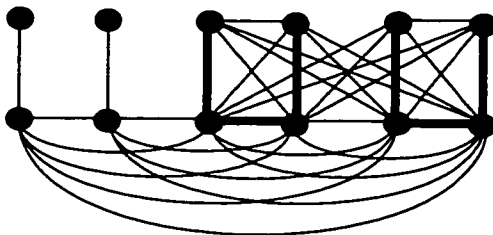


FIGURE 5.2

We begin the induction with the twisted combs in Figure 5.1. In [L1], Letchford asserts that the constraint corresponding to the 12-node twisted comb is facet-inducing. Next, note that the twisted comb in Figure 5.1(c) can be obtained from that in Figure 5.1(b) by switching compartments in the irregular domino. By Theorem 3.2, the corresponding twisted comb constraints are equivalent, so it suffices to show just one of them is facet-inducing. We were able to do this as follows. A face of a polytope is a facet if and only if its affine dimension is one less than that of the polytope [Proposition 6.16, CCPS]. Since $\dim(STSP(n)) = m - n = \frac{n(n-1)}{2} - n$ [JRR], it suffices in this case to find $103+1=104$ affinely independent tours in K_{16} that satisfy the twisted comb constraint at equality. By exploiting the symmetry of the twisted comb, we created a set of tours that satisfied the constraint at equality, and used the computer to verify that the affine rank of this set was 104, as required.

We first generalize these examples by proving that we can add pairs of regular 2-matching dominoes to any compartment of the irregular domino. By compartment-switching if necessary, we can assume that we are adding one pair at a time to the outside compartment. There are three cases to consider, corresponding to the possible parities of the number of 2-matching dominoes in each compartment of the irregular domino, $D_p = A_p \cup B_p$, represented by the three starting cases illustrated in Figure 5.1.

CASE 1. In this case, A_p , B_p and C_p contain $2r$, $2s$ and $2t$ 2-matching dominoes respectively (where r, s, t are positive integers). Then the total number of nodes is $n = 4(r + s + t)$ and the total number of dominoes is $p = 2(r + s + t) + 1$. The right hand side of the corresponding DP -constraint is

$$\begin{aligned}\alpha &= 2(r + s + t) + 4(r + s + t) + 4(r + s) - \frac{1}{2}(6(r + s + t) + 3 + 1) \\ &= 7(r + s) + 3t - 2.\end{aligned}$$

Notice that each node in $C_p \setminus H$ must be incident to at least one 0-edge in any tour, and one 0-edge can be incident to 2 such nodes. Thus any tour, and in particular, any a -tight tour, must contain at least t 0-edges. Also, note that all of the 2-edges are in $\gamma(D_p)$; any node in $D_p \cap H$ can be incident to at most two 2-edges, whereas any node in $D_p \setminus H$ can be incident to at most one 2-edge. Again, since any 2-edge is incident to two such nodes, any a -tight tour contains at most $(3r + 3s)$ 2-edges. If we combine these restrictions with the facts that a tour must include exactly $n = 4(r + s + t)$ edges, and the total weight of any a -tight tour must be exactly $\alpha = 7(r + s) + 3t - 2$, we conclude that there are three possible types of a -tight tours:

- (1) t 0-edges, $(3r + 3s - 2)$ 2-edges, $(r + s + 3t + 2)$ 1-edges;
- (2) $(t + 1)$ 0-edges, $(3r + 3s - 1)$ 2-edges, $(r + s + 3t)$ 1-edges;
- (3) $(t + 2)$ 0-edges, $(3r + 3s)$ 2-edges, $(r + s + 3t - 2)$ 1-edges.

CASE 2. In this case, A_p , B_p and C_p contain $2r + 1$, $2s + 1$ and $2t$ 2-matching dominoes respectively. The total number of nodes is $n = 4(r + s + t + 1)$ and the total number of dominoes is $p = 2(r + s + t) + 3$. The right hand side of the corresponding DP -constraint is $\alpha = 7(r + s) + 3t + 5$.

Using the same type of reasoning as in Case 1, we can see that any a -tight tour has at least t 0-edges, and no more than $(3r + 1)$ 2-edges in A_p , and no more than $(3s + 1)$ 2-edges in B_p . There are now only two possible types of a -tight tours:

- (1) t 0-edges, $(3r + 3s + 1)$ 2-edges, $(r + s + 3t + 3)$ 1-edges;
- (2) $(t + 1)$ 0-edges, $(3r + 3s + 2)$ 2-edges, $(r + s + 3t + 1)$ 1-edges.

CASE 3. In this case, A_p , B_p and C_p contain $2r$, $2s + 1$ and $2t + 1$ 2-matching dominoes respectively. (Since A_p and B_p are interchangeable, we do not need to consider separately the case where A_p contains oddly many dominoes and B_p contains evenly many.) As in Case 2, $n = 4(r + s + t + 1)$ and $p = 2(r + s + t) + 3$. The right hand side of the corresponding DP -constraint is $\alpha = 7(r + s) + 3t + 3$.

This time, any a -tight tour has at least $(t + 1)$ 0-edges, and no more than $3r$ 2-edges in A_p , and no more than $(3s + 1)$ 2-edges in B_p . Again, this implies there are only two possible types of a -tight tours:

- (1) $(t + 1)$ 0-edges, $(3r + 3s)$ 2-edges, $(r + s + 3t + 3)$ 1-edges;
- (2) $(t + 2)$ 0-edges, $(3r + 3s + 1)$ 2-edges, $(r + s + 3t + 1)$ 1-edges.

In the inductive hypothesis, we will be assuming that a twisted comb constraint $a^T x \leq \alpha$ is facet-inducing. By definition, this means that $F(a)$ is not properly contained in any other distinct facet; in particular, it should not be contained in the facets corresponding to the upper and lower edge bounds (we postpone until Chapter 6 a proof of the fact that the twisted comb constraints are not equivalent to these trivial constraints). Thus, for any edge e , there exists an a -tight tour containing e , and another a -tight tour *not* containing e . In the course of our inductive proof, we

will need to guarantee the existence of a -tight tours satisfying various other edge conditions, as presented in the following two lemmas.

LEMMA 5.1. *Let $\{F; D_1, \dots, D_p\}$ be a 2-matching twisted comb on $2(p-1)$ nodes, with corresponding constraint $a^T x \leq \alpha$.*

- (1) *For all $v, w \in H$, with $v \in C_p$, there exist $u \in H$ and $i \notin H$, with $a_{ui} = 0$, such that there exists an a -tight tour containing both edges vw and ui .*
- (2) *For all $u \in H$ and $i \notin H$ satisfying $a_{ui} = 0$, there exists $v, w \in H$, with $v \in C_p$, such that there exists an a -tight tour containing both edges vw and ui .*
- (3) *For all $i, j \notin H$ satisfying $a_{ij} = 0$, there exist $u, v \in H$, with $u \in C_p$, such that there exists a -tight tour containing both edges ij and uv .*

PROOF. In Figure 5.3, we demonstrate the existence of the tours in the lemma diagrammatically, covering each of the three cases on a generic 24-node example. For Case 1, we have used $r = 3, s = 1$ and $t = 2$, giving $\alpha = 32$; for Case 2, $r = 2, s = 1$ and $t = 2$, giving $\alpha = 32$, and for Case 3, $r = 3, s = 1$ and $t = 1$, giving $\alpha = 34$. However, the general form of these tours does not depend on the particular values of r, s or t . The first set of 6 diagrams proves part (1) of the lemma, the next 3 diagrams, together with the first 6, prove part (2), and the last set of 6 diagrams proves part (3) (note the different possible locations of i and j). By permuting the 2-matching regular dominoes within any compartment of the irregular domino D_p , we can obtain the complete generality of the lemma. \square

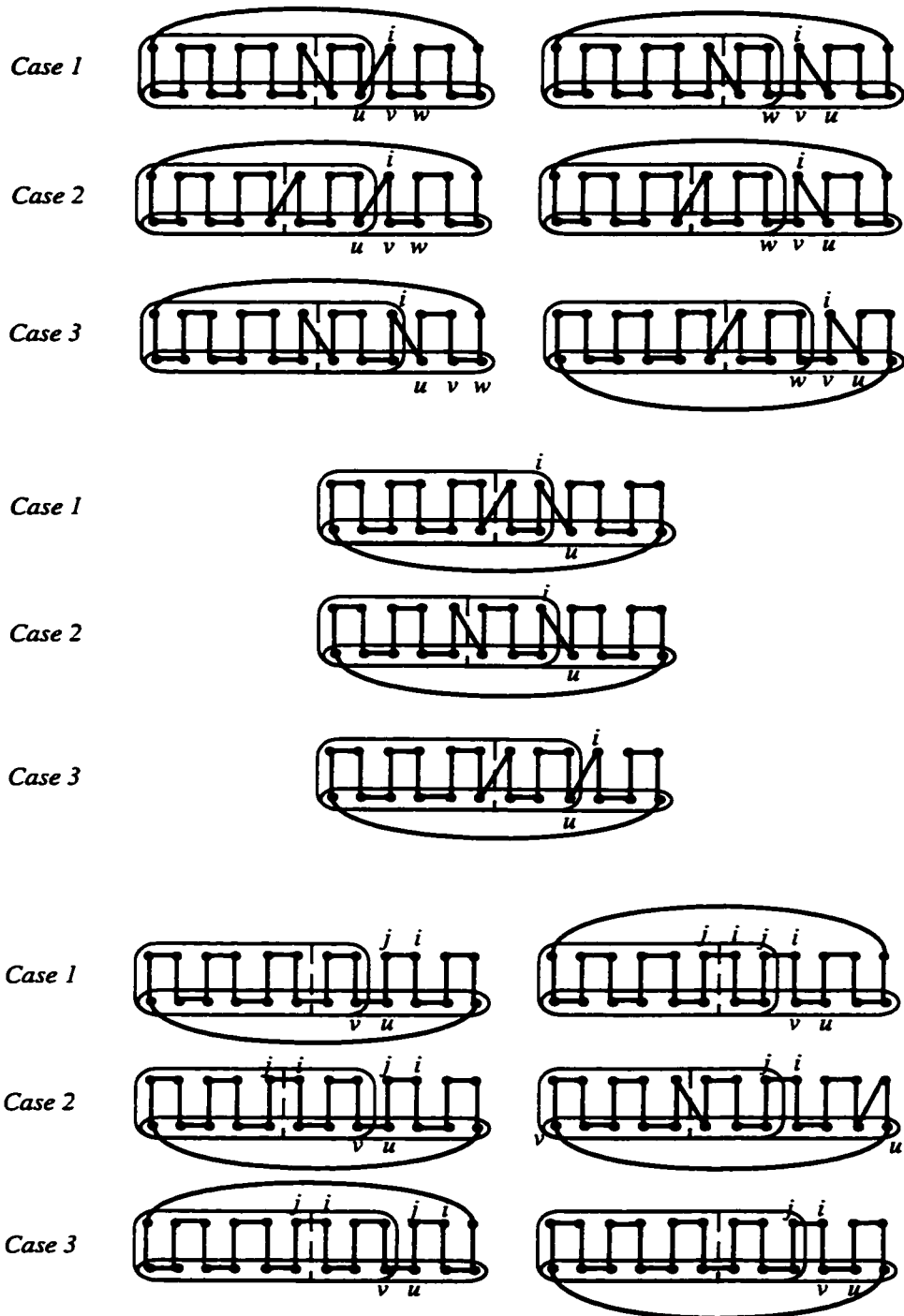


FIGURE 5.3
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LEMMA 5.2. *Under the same hypotheses as Lemma 5.1, every a -tight tour contains an edge of the form vw , where $v, w \in H$ with $v \in C_p$.*

PROOF. Suppose there exists an a -tight tour T with no such edges. Then each node in C_p must be incident in T to at least one 0-edge. In addition, the subtour constraint for C_p requires that at least two of these 0-edges belong to $\delta(C_p)$. Together, these imply that the number of 0-edges in T must exceed the number of 2-matching dominoes in C_p .

We must now split into the cases examined at the beginning of this section. In Case 1, an a -tight tour can have at most $(t+2)$ 0-edges. Now, $2t+1 \leq t+2$ implies that $t \leq 1$; since t must be positive, $t = 1$. The number of 0-edges in T is therefore 3, meaning that T must have the maximum number of 2-edges. This requires that each node in $D_p \cap H$ be incident to exactly two 2-edges, which in turn implies that the edges of $\delta(A_p)$ and $\delta(B_p)$ are 0-edges. This adds at least one more 0-edge to the tour, giving a contradiction.

In case 2, the first paragraph of the proof implies that T has at least $(2t+1)$ 0-edges, yet from the beginning of the section, we know it can have at most $(t+1)$ 0-edges. This gives the contradiction that $t \leq 0$. Similarly, in Case 3, we get $2t+2 \leq t+2$, which also implies $t \leq 0$. \square

The next result is the inductive step of the general proof. Following Grötschel and Padberg [GP2], we denote all symbols pertaining to the higher-dimensional polytope with an asterisk, $*$.

PROPOSITION 5.3. *Let $\{F; D_1, \dots, D_p\}$ be a 2-matching twisted comb on $n = 2(p-1)$ nodes, with corresponding constraint $a^T x \leq \alpha$ on $STSP(n)$. Let $(a^*)^T x^* \leq \alpha^*$ denote the constraint on $STSP(n+4)$ corresponding to the 2-matching twisted comb obtained by adding two regular 2-matching dominoes to C_p . If $a^T x \leq \alpha$ is facet-inducing, then so is $(a^*)^T x^* \leq \alpha^*$.*

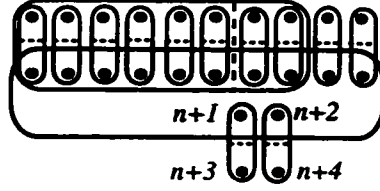


FIGURE 5.4

PROOF. The nodes of the added dominoes are labeled as in Figure 5.4. We begin by observing that the two extra nodes in the handle, the four extra nodes in 2-matching dominoes, and the two extra dominoes imply that

$$\alpha^* = \alpha + 2 + 4 - \frac{3(2)}{2} = \alpha + 3.$$

Let $(b^*)^T x^* \leq \beta^*$ be a facet-inducing constraint on $STSP(n+4)$ such that $F(a^*) \subseteq F(b^*)$. We will show that the two constraints are equivalent; that is, we will show that there exists a positive scalar γ and $\lambda^* \in \mathbb{R}^n$ such that

$$(b^*)^T = \gamma(a^*)^T + (\lambda^*)^T(A^*) \quad \text{and} \quad \beta^* = \gamma\alpha^* + (\lambda^*)^T 2,$$

where A^* is the node-edge incidence matrix for K_{n+4} . As noted in Argument A of [GP2], we can arbitrarily set the values of b_e^* on a set of edges whose corresponding columns in A are linearly independent without affecting the face $F(b^*)$. It is known that a set of column vectors of A is maximal linearly independent if and only if the corresponding edges form a subgraph, each of whose connected components has exactly one odd cycle and no even cycles (such a subgraph is called an *odd 1-forest*) [GPu]. In our case, we set

$$\begin{aligned} b_{n+1, n+2}^* &= 1; \\ b_{v, n+4}^* &= \begin{cases} 1, & \text{if } v = n+2, \\ 0, & \text{if } v \in V^*, v \neq n+2. \end{cases} \end{aligned} \quad (5.1)$$

The following steps determine the value of b_e^* for most of the other edges $e \in E^*$. In what follows, $V = V(K_n)$ and H is the handle of the 2-matching twisted comb on the original n nodes.

STEP 1. For all $i \in V \setminus H$, $b_{i,n+3}^* = b_{i,n+1}^* = 0$.

For each $i \in V \setminus H$, it is clear that there exists $w \in H$ such that $a_{iw} = 0$. Let T be an a -tight tour containing the edge iw ; as argued earlier in this chapter, we know such a tour exists since $a^T x \leq \alpha$ is not equivalent to the lower bound constraint, $x_{iw} \geq 0$. Let $T1$ and $T2$ be tours in K_{n+4} obtained from T by replacing iw with the chains $[i, n+3, n+1, n+4, n+2, w]$ and $[i, n+4, n+3, n+1, n+2, w]$ respectively. From the values of a_{xy} given at the top of page 49, it can be seen that in both cases, a 0-edge of a has been replaced with three 1-edges and two 0-edges of a^* , so that

$$(a^*)^T \chi(T1) = (a^*)^T \chi(T2) = \alpha - 0 + 3(1) + 2(0) = \alpha^*.$$

Thus, $T1$ and $T2$ are a^* -tight tours; since $F(a^*) \subseteq F(b^*)$, they must also be b^* -tight tours. Hence,

$$\beta^* = (b^*)^T \chi(T1) = (b^*)^T \chi(T2);$$

cancelling out all equal edges, we are left with

$$b_{i,n+3}^* + b_{n+1,n+4}^* + b_{n+2,n+4}^* = b_{i,n+4}^* + b_{n+3,n+4}^* + b_{n+1,n+2}^*.$$

Using the values already set in (5.1), we conclude that $b_{i,n+3}^* = 0$.

Next, let $T3$ be the the tour of K_{n+4} obtained from T by replacing iw with $[i, n+1, n+3, n+4, n+2, w]$; then $T3$ is a^* -tight and hence b^* -tight. Comparing $(b^*)^T \chi(T2)$ and $(b^*)^T \chi(T3)$ yields

$$b_{i,n+4}^* + b_{n+1,n+2}^* = b_{i,n+1}^* + b_{n+2,n+4}^*;$$

using (5.1), this implies $b_{i,n+1}^* = 0$.

STEP 2. For all $w \in H$, $b_{w,n+3}^* = 0$ and $b_{w,n+2}^* = 1$.

For each $w \in H$, there exists $i \in V \setminus H$ such that $a_{iw} = 0$. Let S be an a -tight tour of K_n containing edge iw . Create three a^* -tight tours $S1, S2$ and $S3$ of K_{n+4} by replacing iw with $[i, n+3, n+1, n+2, n+4, w]$, $[i, n+4, n+2, n+1, n+3, w]$

and $[i, n+4, n+3, n+1, n+2, w]$ respectively. Since these tours are a^* -tight, they must also be b^* -tight; comparing $(b^*)^T \chi(S1)$ and $(b^*)^T \chi(S2)$, we get

$$b_{i,n+3}^* + b_{w,n+4}^* = b_{i,n+4}^* + b_{w,n+3}^*$$

by (5.1) and Step 1, this implies that $b_{w,n+3}^* = 0$. Next, $(b^*)^T \chi(S2) = (b^*)^T \chi(S3)$ implies

$$b_{n+2,n+4}^* + b_{w,n+3}^* = b_{n+3,n+4}^* + b_{w,n+2}^*$$

using (5.1) and the result just proved, we get $b_{w,n+2}^* = 1$.

STEP 3. There exists some scalar ρ such that

$$\begin{aligned} b_{n+1,n+3}^* &= b_{w,n+1}^* = \rho && \text{for all } w \in H; \\ b_{n+2,n+3}^* &= b_{i,n+2}^* = 1 - \rho && \text{for all } i \in V \setminus H. \end{aligned}$$

Create a fourth b^* -tight tour $S4$ of K_{n+4} based on S from Step 2 by replacing iw with $[i, n+3, n+4, n+2, n+1, w]$. Then comparing $(b^*)^T \chi(S4)$ and $(b^*)^T \chi(S3)$, we get

$$b_{i,n+3}^* + b_{n+2,n+4}^* + b_{w,n+1}^* = b_{i,n+4}^* + b_{n+1,n+3}^* + b_{w,n+2}^*.$$

By (5.1) and Steps 1 and 2, this implies that $b_{n+1,n+3}^* = b_{w,n+1}^*$, which we define to be ρ . Next, define two more tour b^* -tight tours, $T4$ and $T5$ based on T from Step 1 by replacing its iw with $[i, n+2, n+4, n+3, n+1, w]$ and $[i, n+4, n+2, n+3, n+1, w]$ respectively. Comparing $T3$ (from Step 1) and $T4$, we get

$$b_{i,n+2}^* + b_{w,n+1}^* = b_{i,n+1}^* + b_{w,n+2}^*.$$

By (5.1), Steps 1 and 2, and the previous argument, we conclude that $b_{i,n+2}^* = 1 - \rho$. Comparing $T4$ and $T5$, we obtain

$$b_{i,n+4}^* + b_{n+2,n+3}^* = b_{i,n+2}^* + b_{n+3,n+4}^*,$$

so $b_{n+2,n+3}^* = 1 - \rho$ also.

STEP 4. For all $v, w \in H$ with $v \in C_p$, $b_{vw}^* = \rho$.

By Lemma 5.1(1), there exists an a -tight tour R containing both vw and an edge ui , where $u \in H$ and $i \in V \setminus H$, with $a_{ui} = 0$. Note that since $v \in C_p \cap H$, $a_{vw} = 1$. As before, we create a^* -tight (and hence b^* -tight) tours in K_{n+4} from R . First, $R1$ is obtained by simply replacing ui with $[u, n+2, n+4, n+1, n+3, i]$. If R contains a chain $[i, u, \dots, w, v, \dots]$, then replace vw and ui with $[v, n+2, n+4, u]$ and $[w, n+1, n+3, i]$ to obtain $R2$; if R contains a chain $[i, u, \dots, v, w, \dots]$, then replace vw and ui with $[v, n+2, n+4, i]$ and $[u, n+1, n+3, w]$ to obtain $R2'$ (see Figure 5.5).

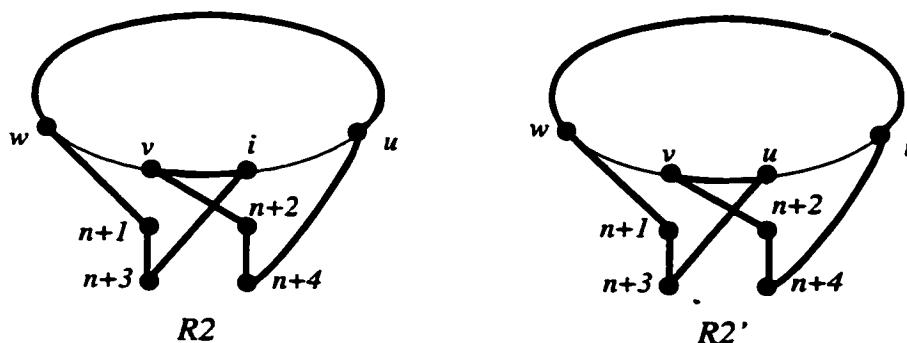


FIGURE 5.5

Comparing tours $R1$ and $R2$ gives

$$b_{u,n+2}^* + b_{n+1,n+4}^* + b_{vw}^* = b_{u,n+4}^* + b_{v,n+2}^* + b_{w,n+1}^*,$$

whereas comparing $R1$ and $R2'$ gives

$$b_{u,n+2}^* + b_{n+1,n+4}^* + b_{i,n+3}^* + b_{vw}^* = b_{i,n+4}^* + b_{v,n+2}^* + b_{w,n+3}^* + b_{u,n+1}^*.$$

Both yield $b_{vw}^* = \rho$ after applying the conclusions of the previous steps.

STEP 5. For all $i, j \notin H$ with $a_{ij} = 0$, $b_{ij}^* = 0$.

By Lemma 5.1(3), there exist $u, v \in H$ with $u \in C_p$ such that there is an a -tight tour P containing both edges ij and uv . Observe that $a_{uv} = 1$. To obtain $P1$, replace ij with $[i, n+4, n+2, n+1, n+3, j]$, and to obtain $P2$, replace uv with $[u, n+1, n+3, n+4, n+2, v]$. Both $P1$ and $P2$ are a^* -tight and thus also b^* -tight; comparing them gives

$$b_{uv}^* + b_{i,n+4}^* + b_{n+1,n+2}^* + b_{j,n+3}^* = b_{ij}^* + b_{u,n+1}^* + b_{n+3,n+4}^* + b_{v,n+2}^*.$$

The results of the previous steps allows us to conclude that $b_{ij}^* = 0$.

STEP 6. For all $u \in H$, $i \in V \setminus H$ not in the same 2-matching domino,

$$b_{ui}^* = \begin{cases} 0, & \text{if } a_{ui} = 0, \\ \rho, & \text{if } a_{ui} = 1. \end{cases}$$

If $a_{ui} = 0$, then we can use Lemma 5.1(2) to conclude that there exist $v, w \in H$ with $v \in C_p$ and an a -tight tour Q containing both ui and vw . Create $Q1$ by replacing ui with $[u, n+2, n+4, n+1, n+3, i]$, and $Q2$ by replacing vw with $[v, n+1, n+3, n+4, n+2, w]$. Comparing these b^* -tight tours, we get

$$b_{vw}^* + b_{u,n+2}^* + b_{n+1,n+4}^* + b_{i,n+3}^* = b_{v,n+1}^* + b_{n+3,n+4}^* + b_{w,n+2}^* + b_{ui}^*,$$

which yields $b_{ui}^* = 0$.

Next, we consider the case where $a_{ui} = 1$. Instead of modifying a -tight tours in K_n as we have in the previous steps, we will directly compare two a^* -tight (and hence b^* -tight) tours in K_{n+4} . First, suppose u and i are in different inside compartments of the irregular domino. We provide eight generic diagrams in Figure 5.6, giving a pair of tours for each of the first two cases mentioned at the beginning of this section; the third case must be split in two, for when u is in the inside compartment with an odd number of 2-matching dominoes, and when it is with the even number of 2-matching dominoes.

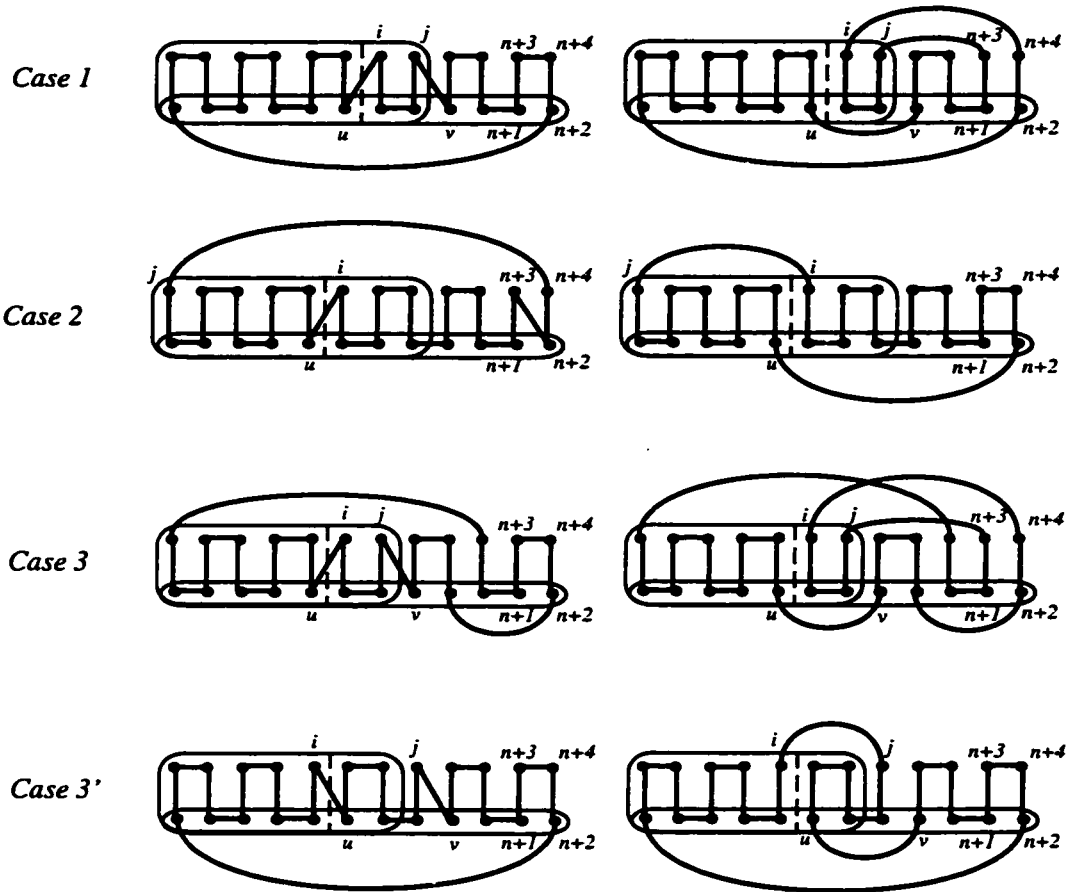


FIGURE 5.6

The corresponding equation in the coefficients of b^* are, for the first *and* third pairs,

$$b_{vj}^* + b_{n+3,n+4}^* + b_{ui}^* = b_{vu}^* + b_{i,n+4}^* + b_{j,n+3}^*.$$

For the second and fourth pair of tours, we get, respectively

$$b_{ui}^* + b_{n+2,n+3}^* + b_{j,n+4}^* = b_{u,n+2}^* + b_{n+3,n+4}^* + b_{ij}^*$$

$$b_{vj}^* + b_{ui}^* = b_{uv}^* + b_{ij}^*.$$

In each case, we can use the results of previous steps, including the first part of this step, to conclude that $b_{ui}^* = \rho$.

Next, suppose that u and i are in the same inside compartment of the irregular domino; again, we consider comparative a^* -tight tours in each possible case (see Figure 5.7). The equations are, in order:

$$b_{xw}^* + b_{vj}^* + b_{ui}^* = b_{xu}^* + b_{wj}^* + b_{iv}^*;$$

$$b_{ui}^* + b_{vj}^* + b_{wx}^* = b_{uj}^* + b_{iw}^* + b_{vx}^*;$$

$$b_{ui}^* + b_{jk}^* + b_{v,n+2}^* = b_{ik}^* + b_{vj}^* + b_{u,n+2}^*;$$

$$b_{jk}^* + b_{ui}^* = b_{ij}^* + b_{uk}^*.$$

Again, these all eventually yield $b_{ui}^* = \rho$.

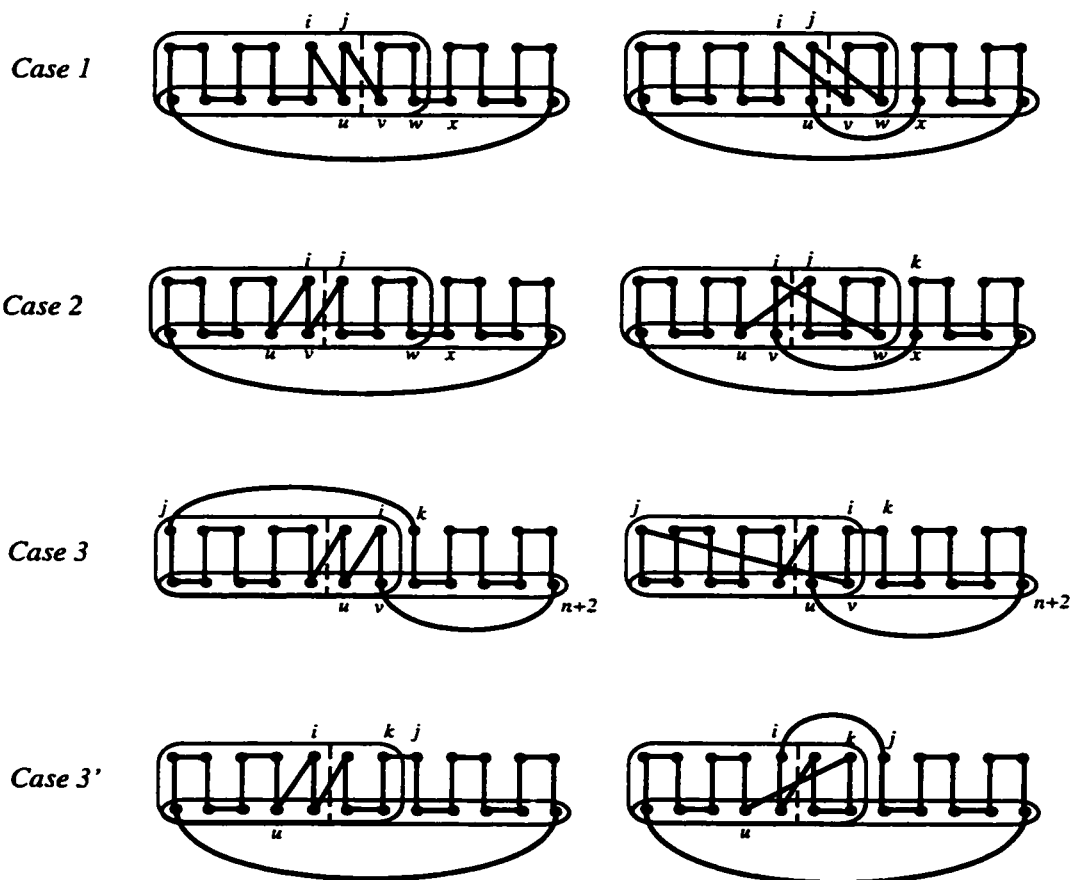


FIGURE 5.7

STEP 7. For all $u, v \in H$ with $a_{uv} = 2$, $b_{uv}^* = 2\rho$.

Note that either $u, v \in A_p \cap H$ or $u, v \in B_p \cap H$. As in Step 6, we go directly to pairs of a^* -tight (hence b^* -tight) tours of K_{n+4} , with one pair of tours for Cases 1 and 2, and two pairs for Case 3, corresponding to when u and v are both in the inside compartment with an odd number of 2-matching dominoes, and when they are in the compartment with an even number of 2-matching dominoes (see Figure 5.8).

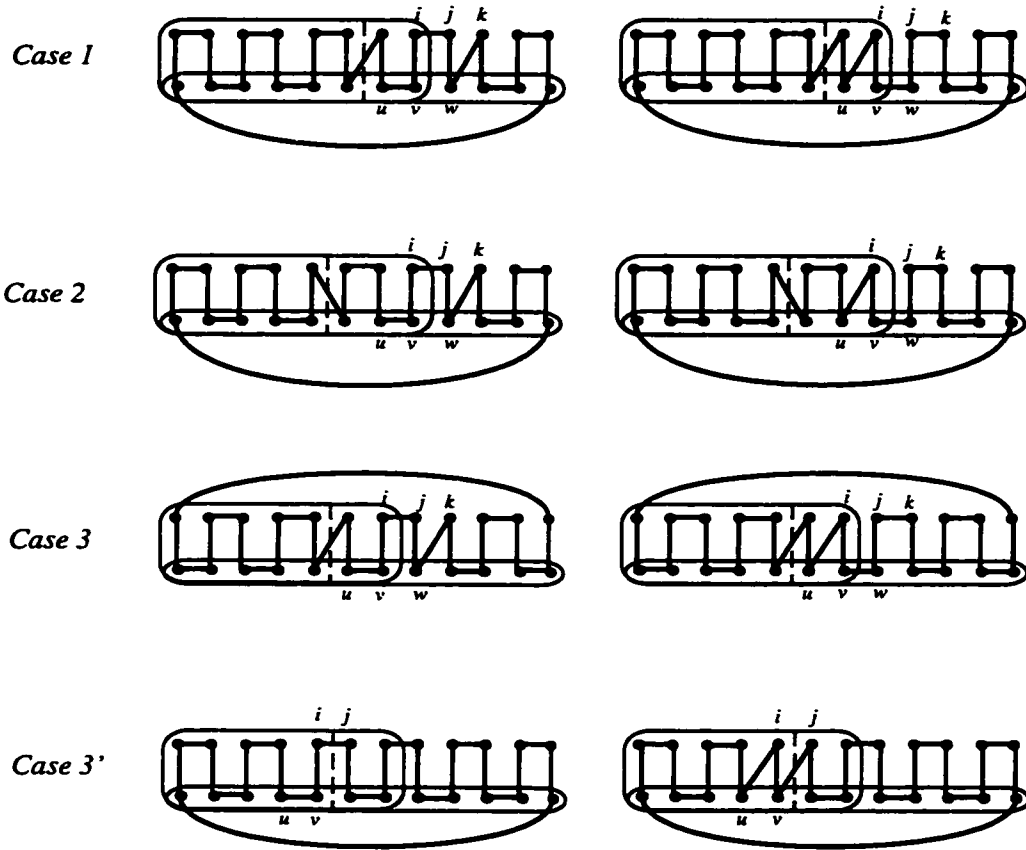


FIGURE 5.8

These yield the following equation for each of the first three pairs:

$$b_{uv}^* + b_{ij}^* + b_{wk}^* = b_{ui}^* + b_{vw}^* + b_{jk}^*.$$

For the last pair, the equation is

$$b_{uv}^* + b_{ij}^* = b_{ui}^* + b_{vj}^*.$$

Using previous steps, these all give $b_{uv}^* = 2\rho$.

We are now ready to prove that $(a^*)^T x^* \leq \alpha^*$ is equivalent to the facet-inducing constraint $(b^*)^T x^* \leq \beta^*$. We define a constraint $b^T x \leq \beta$ by setting

$$b_e = b_e^*, \quad e \in E(K_n); \quad \beta = \beta^* - (2 + \rho).$$

To show that this new constraint is valid on $STSP(n)$, suppose by way of contradiction that there exists a tour S of K_n such that $b^T \chi(S) > \beta$. Let $i \in C_p \setminus H$; then S must include at least one edge iv that is not within a 2-matching domino (so $a_{iv} = 0$). By Steps 5 and 6, we know that $b_{iv}^* = b_{iv} = 0$. Let S^* be the tour of K_{n+4} obtained by replacing iv with the chain $[i, n+3, n+1, n+2, n+4, v]$. Then

$$\begin{aligned} (b^*)^T \chi(S^*) &= b^T \chi(S) + b_{i,n+3}^* + b_{n+1,n+3}^* + b_{n+1,n+2}^* + b_{n+2,n+4}^* + b_{v,n+4}^* \\ &= b^T \chi(S) + (2 + \rho) \\ &> \beta + (2 + \rho) = \beta^*, \end{aligned}$$

contradicting the fact that $(b^*)^T x^* \leq \beta^*$ is valid on $STSP(n+4)$.

Next, suppose T is an a -tight tour. Then by Lemma 5.2, T contains an edge vw where $v, w \in H$ and $v \in C_p$. Let T^* be the a^* -tight, hence b^* -tight, tour obtained from T by replacing uv with the chain $[u, n+1, n+3, n+4, n+2, v]$. Then

$$\begin{aligned} b^T \chi(T) &= (b^*)^T \chi(T^*) - (b_{u,n+1}^* + b_{n+1,n+3}^* + b_{n+3,n+4}^* + b_{n+4,n+2}^* + b_{v,n+2}^*) + b_{uv} \\ &= \beta^* - (2\rho + 2) + \rho = \beta^* - (2 + \rho) = \beta. \end{aligned}$$

Thus, $F(a) \subseteq F(b)$. Since $a^T x \leq \alpha$ was assumed to be facet-inducing, this means that either (i) $F(a) = F(b)$ or (ii) $F(b) = STSP(n)$. We can therefore write

$$b = \gamma a + \lambda^T A, \quad \beta = \gamma \alpha + \lambda^T \mathbf{2}, \quad (5.2)$$

where A is the node-edge incidence matrix for K_n , λ is some vector in \mathbb{R}^n and γ is either a positive scalar or 0, for cases (i) and (ii) respectively. What we do next is solve for λ and γ ; showing that $\gamma > 0$ will show that $b^T x \leq \beta$ is equivalent to $a^T x \leq \alpha$. We then show that this equivalence can be ‘lifted’ to $(a^*)^T x^* \leq \alpha^*$ and $(b^*)^T x^* \leq \beta^*$.

We will first show that $\lambda = 0$, by imitating Argument B in [GP2]. Let $u \in C_p \cap H$ and $v, w \in A_p \cap H$ or $B_p \cap H$. From (5.2) and the known values of b_e^* , we get:

$$\begin{aligned}\rho &= b_{uv}^* = b_{uv} = \gamma a_{uv} + \lambda_u + \lambda_v = \gamma + \lambda_u + \lambda_v \\ \rho &= b_{uw}^* = b_{uw} = \gamma a_{uw} + \lambda_u + \lambda_w = \gamma + \lambda_u + \lambda_w \\ 2\rho &= b_{vw}^* = b_{vw} = \gamma a_{vw} + \lambda_v + \lambda_w = 2\gamma + \lambda_v + \lambda_w.\end{aligned}$$

Solving this system of equations for λ_u, λ_v and λ_w gives $\lambda_v = \lambda_w = \rho - \gamma$ and $\lambda_u = 0$. For any $i \in C_p \setminus H$ and any $w \in D_p \cap H$,

$$0 = b_{iw}^* = b_{iw} = \gamma a_{iw} + \lambda_i + \lambda_w = \lambda_i + (\rho - \gamma),$$

so $\lambda_i = \gamma - \rho$. Next, for any $j \in D_p \setminus H$ and any $u \in C_p \cap H$,

$$0 = b_{uj}^* = b_{uj} = \gamma a_{uj} + \lambda_u + \lambda_j = \lambda_j.$$

Finally, for any $i \in C_p \setminus H$, there exists $u \in C_p \cap H$ such that $a_{ui} = 0$; by Step 6, $b_{ui}^* = b_{ui} = 0$, so using the values of λ already obtained,

$$0 = b_{ui} = \gamma a_{ui} + \lambda_u + \lambda_i = 0 + 0 + (\gamma - \rho).$$

This shows that $\lambda = 0$, implying that $b^T x \leq \beta$ is just $\gamma = \rho$ times the constraint $a^T x \leq \alpha$. In other words,

$$b_e^* = b_e = \rho a_e = \rho a_e^*, \quad e \in E(K_n); \quad \beta^* = \beta + (2 + \rho) = \rho \alpha + (2 + \rho).$$

The values of b_e^* for edges $e \in E(K_{n+4}) \setminus E(K_n)$ are determined by (5.1) and Steps 1 through 3. It is easily verified that if we define $\lambda^* \in \mathbb{R}^{n+4}$ by $\lambda_{n+2}^* = 1 - \rho$ and $\lambda_v^* = 0$ for all other $v \in V^*$, then

$$b^* = \rho a^* + (\lambda^*)^T A^*; \quad \beta^* = \rho \alpha^* + (\lambda^*)^T \mathbf{2}. \quad (5.3)$$

Since $(b^*)^T x^* \leq \beta^*$ was assumed to be facet-inducing, it cannot be a multiple of a degree equation, so $\rho \neq 0$. Therefore, (5.3) shows that by definition, the constraint $(a^*)^T x^* \leq \alpha^*$ is equivalent to $(b^*)^T x^* \leq \beta^*$, and therefore is a facet-inducing constraint on $STSP(n+4)$. This completes the proof of Proposition 5.3. \square

The next generalization is to add a positive number of nodes to one or more of the locations indicated with open circles in Figure 5.9. The result is a 2-matching twisted comb with more than $2(p-1)$ nodes.

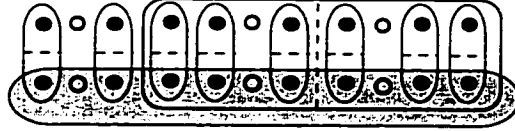


FIGURE 5.9

We make use of a lifting theorem of Boyd, Cunningham, Queyranne and Wang [BCQW], which applies to what they term *2-tooth inequalities*. A nontrivial, valid constraint $a^T x \leq \alpha$ on $STSP(n)$ is a 2-tooth inequality if it satisfies:

- (1) $a_e \geq 0$ for all $e \in E$;
- (2) there exist at least two disjoint 2-matching teeth, $T_1 = \{t_1\} \cup \{h_1\}$ and $T_2 = \{t_2\} \cup \{h_2\}$, such that for $i = 1, 2$, we have $a_{t_i, h_i} > 0$ and $a_{t_i, v} = 0$ for all nodes $v \neq h_i$;
- (3) for all $v \in V$, either $a_{v, h_1} \geq a_{t_1, h_1}$ or $a_{v, h_1} = 0$.

Any 2-matching twisted comb constraint can be seen to be 2-tooth inequality by taking a pair of regular 2-matching dominoes in C_p as the two teeth, with t_i as the node outside H and h_i the node inside H .

PROPOSITION 5.4 [BOYD, CUNNINGHAM, QUEYRANNE AND WANG, 1995]. *Let $a^T x \leq \alpha$ be a nontrivial, facet-inducing 2-tooth inequality for $STSP(n)$. Let v denote an $(n + 1)$ -th node. Define a coefficient vector a^* on the edges of K_{n+1} by setting*

$$a_e^* = \begin{cases} a_e, & e \notin \delta(v), \\ 0, & e \in \delta(v). \end{cases}$$

Then $(a^)^T x^* \leq \alpha$ is a valid, facet-inducing constraint for $STSP(n + 1)$.*

PROPOSITION 5.5. *The 2-matching twisted comb constraints are facet-inducing.*

PROOF. The inductive proof at the beginning of this chapter proves that all 2-matching twisted combs on exactly $2(p - 1)$ nodes are facet-inducing. Proposition 5.4 allows us to add a node that is outside the handle and all dominoes to get another facet-inducing constraint. Recall from Chapter 3 that either H or \overline{H} can be regarded as the handle, and switching compartments in D_p gives an equivalent constraint (Theorem 3.2). This means that we can in fact add a node to any location that is outside all the regular 2-matching dominoes, and obtain another facet-inducing constraint. \square

The final generalization is to allow any of the regular dominoes to have an arbitrary number of nodes. Our argument is based on a result proved by Naddef and Rinaldi in 1992 [NR1]. Before stating this result, we require a few definitions from that paper.

DEFINITION. An inequality constraint $c^T x \geq \zeta$ is in *tight triangular form* (or *TT-form*) if and only if

- (1) the coefficients c_e satisfy the triangle inequality, i.e., $c_{uv} \leq c_{uw} + c_{vw}$ for any set of three distinct nodes u, v and w ;
- (2) for every node w , there exist $u, v \in V \setminus \{w\}$ for which the triangle inequality is tight, i.e., $c_{uv} = c_{uw} + c_{vw}$.

Notice that in this definition, the constraint appears in the ‘ \geq ’ form. Note also that tight triangularity depends only on the edge coefficients c_e , and not on the right hand side ζ of the constraint. As an example, it is not difficult to show that the subtour elimination constraint $x(\delta(S)) \geq 2$ satisfies the triangle inequality, and is in tight triangular form for any node subset S satisfying $2 \leq |S| \leq |V| - 2$.

DEFINITION. A tight triangular inequality $c^T x \geq \zeta$ is *simple* if $c_e > 0$ for all $e \in E$.

If $c^T x \geq \zeta$ is not simple, then we can partition the node set $V = V_1 \cup \dots \cup V_k$ so that

- (1) $c_e = 0$ for all $e \in \gamma(V_i)$, $1 \leq i \leq k$;
- (2) $c_e = c_f$ for all $e, f \in E(V_i, V_j)$, $1 \leq i < j \leq k$.

The *simple inequality associated with* $c^T x \geq \zeta$ is obtained by collapsing each partite V_i into a single node i . More precisely, it is the inequality $(\bar{c})^T \bar{x} \geq \zeta$ on $STSP(k)$ defined by $\bar{c}_{ij} = c_e$, where $e \in E(V_i, V_j)$.

Note that $(\bar{c})^T \bar{x} \geq \zeta$ is tight triangular if and only if $c^T x \geq \zeta$ is.

DEFINITION. Given a constraint $c^T x \geq \zeta$ on $STSP(n)$, two edges $e, f \in E(K_n)$ are *c-adjacent* if there exists a c -tight tour containing both e and f . A set of edges $F \subseteq E$ is *c-connected* if for every $f_1 \neq f_2$ in F , there exists a sequence of edges $(f_1 = e_1, e_2, \dots, e_{k-1}, e_k = f_2)$ such that e_i and e_{i+1} are c -adjacent for all $1 \leq i \leq k - 1$.

The theorem in [NR1] that allows us to clone nodes within the regular dominoes of a twisted comb is stated below.

THEOREM 5.6 [NADDEF AND RINALDI, 1992]. *A TT-inequality $c^T x \geq \zeta$ is facet-inducing for $STSP(n)$ if*

- (1) *its associated simple inequality $(\bar{c})^T \bar{x} \geq \zeta$ is facet-inducing for $STSP(k)$, and distinct from the non-negativity constraints, and*
- (2) *for every $v \in V(K_k)$, the set $\delta(v) \subseteq E(K_k)$ is \bar{c} -connected.*

We want to apply this theorem to any constraint whose simple form is a 2-matching twisted comb constraint.

LEMMA 5.7. *The TT-form of a 2-matching twisted comb constraint is*

$$\sum_{i=1}^p x(\delta(A_i, B_i, C_i)) + x(F) = \sum_{i=1}^p x(\delta(D_i)) + \sum_{e \in E} \mu_e x_e \geq 3p + 1.$$

PROOF. It was shown in Chapter 3 that this is an equivalent form of the DP-constraint. Figure 5.10 shows a simple 2-matching twisted comb, and the corresponding edge coefficients of this form of the constraint. The unusual configuration of the handle gives an edge coefficient diagram that better displays the symmetry with respect to the compartments of the irregular domino.

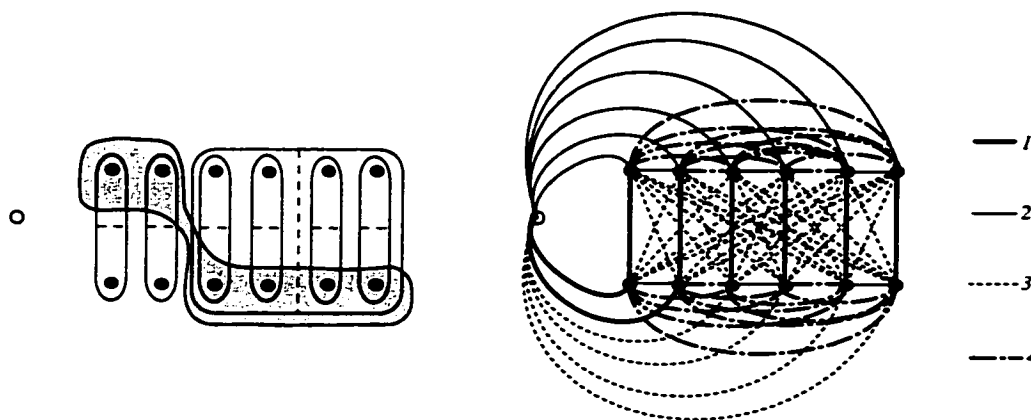


FIGURE 5.10

It is straightforward to verify from this diagram that the edge coefficients are all positive and satisfy the triangle inequality, and that moreover each node is part of a tight triangle. This example can clearly be generalized to all 2-matching twisted combs. \square

We note that if a tour of K_n is tight for one form of a constraint, then it is tight for any equivalent form of that constraint.

LEMMA 5.8. *Let $c^T x \geq \zeta$ denote the TT -form of a 2-matching twisted comb constraint. Then for every $v \in V(K_n)$, $\delta(v)$ is c -connected.*

PROOF. Let $a^T x \leq \alpha$ denote the (2.2) form of the constraint. Suppose first that the twisted comb has exactly $2(p-1)$ nodes. Then every node v is in a regular 2-matching domino; let e_v denote the semicut edge of that domino. Using the glossary of a -tight tours in Figures 5.3, 5.6, 5.7 and 5.8, and the fact that interchanging H and \bar{H} does not affect the constraint, it can be seen that every $f \neq e_v$ in $\delta(v)$ is contained in an a -tight tour with e_v . Since every a -tight tour is c -tight, this shows that every edge in $\delta(v)$ is c -adjacent to e_v ; for any two distinct edges $f_1, f_2 \in \delta(v) \setminus e_v$, the sequence (f_1, e_v, f_2) shows that they are c -connected.

Next, we add a node w outside the handle and all dominoes; let $(a^*)^T x^* \leq \alpha^*$ denote the closed form, and $(c^*)^T x^* \geq \zeta^*$ the TT -form, of the new DP -constraint. Note that $a_{vw}^* = 0$ for all nodes v in the original twisted comb. For each such v , there exists $i \in C_p \setminus H$ such that $a_{vi} = 0$ and an a -tight tour containing both iv and the semicut edge e_v ; this can be extended into an a^* -tight, hence c^* -tight, tour by replacing iv with the chain $[v, w, i]$. Thus, vw is c^* -adjacent to e_v , proving that $\delta(v)$ is c^* -connected. To show that $\delta(w)$ is c^* -connected, recall that for all $u, v \in V$, there exists an a -tight tour containing uv . If $a_{uv} = 0$, then this can be extended to an a^* -tight tour by replacing uv with $[u, w, v]$, meaning that uw and vw are c^* -adjacent. If $a_{uv} > 0$, then there exists a node $i \in C_p \setminus H$ such that $a_{ui} = a_{vi} = 0$; by the previous sentence, iw is c^* -adjacent to both uw and vw , so (uw, iw, vw) is a sequence showing uw and vw are c^* -connected.

As we progressively add nodes to the twisted comb outside the regular 2-matching dominoes, we can flip the handle or switch compartments of D_p to guarantee that we are always adding a node outside the handle and all dominoes. The argument in the paragraph above can then be repeated. \square

THEOREM 5.9. *The twisted comb constraints are facet-inducing.*

PROOF. Let $c^T x \geq \zeta$ denote a twisted comb constraint, written in the form

$$\sum_{i=1}^p x(\delta(A_i, B_i, C_i)) + x(F) \geq 3p + 1.$$

Let S be a node subset that is entirely contained within one compartment of *every* domino, and on one shore of the handle; then any edge $e \in \gamma(S)$ is neither in $E(A_i, B_i, C_i)$ for any $1 \leq i \leq p$, nor in F . Thus, $c_e = 0$ for all $e \in \gamma(S)$. In particular, this applies to $S = A_i$ or B_i , $1 \leq i \leq p - 1$ (of course, it applies trivially if any of these sets is a singleton). It also applies to any set of nodes that is entirely contained within *one* of the six locations indicated with open circles in Figure 5.9. We denote these sets by V_1, \dots, V_6 ; note that one or more of these may be empty. Moreover, $c_e = c_f$ for all $e, f \in E(S_1, S_2)$, where S_1, S_2 are non-empty sets in $\{A_1, \dots, A_{p-1}, B_1, \dots, B_{p-1}, V_1, \dots, V_6\}$. Partition the node set

$$V = (A_1 \cup B_1) \cup \dots \cup (A_{p-1} \cup B_{p-1}) \cup V_1 \cup \dots \cup V_6;$$

collapse each A_i and B_i , $1 \leq i \leq p - 1$, and each non-empty V_j to obtain the TT -form of a 2-matching twisted comb constraint. By Proposition 5.5, this constraint is nontrivial and facet-inducing. Moreover, by Lemma 5.8, $\delta(v)$ is \tilde{c} -connected for every node v . Hence, by Theorem 5.6, $c^T x \geq \zeta$ is facet-inducing. \square

DISTINCTNESS OF THE TWISTED COMB CONSTRAINTS

In the previous chapter, we demonstrated that the twisted comb constraints are facet-inducing. In this chapter, we will show that this family is distinct from other known families of facet-inducing inequalities on $STSP(n)$. We note that Caprara, Fischetti and Letchford have already verified this for the constraint corresponding to the twisted comb in Figure 5.1(a) [CFL]. To generalize to all twisted comb constraints, we need an additional fact from [NR1] about the tight triangular form of inequalities on $STSP(n)$.

PROPOSITION 6.1 [NADDEF AND RINALDI, 1992]. *Let $c^T x \geq \zeta$ be a facet-inducing constraint on $STSP(n)$. An inequality $d^T x \geq \delta$ equivalent to $c^T x \geq \zeta$ is tight triangular if and only if $d^T = \gamma c^T + \lambda^T A$ and $\delta = \gamma \zeta + \lambda^T \mathbf{2}$ where $\gamma > 0$ and $\lambda \in \mathbb{R}^n$ satisfy, for each $u \in V(K_n)$,*

$$\lambda_u = \frac{1}{2} \gamma \max \{c_{vw} - c_{uv} - c_{uw} \mid v, w \in V(K_n) \setminus \{u\}, v \neq w\}.$$

A consequence of this proposition is that *any* facet-inducing inequality on $STSP(n)$ can be put into TT -form, and moreover, this form is unique up to multiplication by a positive scalar.

PROPOSITION 6.2. *The twisted comb constraints are distinct from the subtour elimination and non-negativity constraints.*

PROOF. As noted in Chapter 5, $x(\delta(S)) \geq 2$ is the subtour elimination constraint in TT -form, for a node subset S satisfying $2 \leq |S| \leq |V| - 2$. Note that the coefficient of each edge in this constraint is either 0 or 1. From Figure 5.10, we can see that

the positive edge coefficients in the TT -form of a twisted comb constraint take on four different values (namely, 1 through 4). Therefore, the twisted comb constraints are distinct from the subtour elimination constraints.

The usual form of the non-negativity constraint, $x_{uv} \geq 0$, where $uv \in E(K_n)$, is not tight triangular. It can be put into TT -form by adding one copy of the degree equation for each $w \in V(K_n) \setminus \{u, v\}$; this yields $c^T x \geq 2(n - 2)$, where $c_e = 1$ if $e \in \delta(\{u, v\})$, and $c_e = 0$ otherwise. Since the edge coefficients take on only two different values, we may conclude that the twisted comb constraints are distinct from the non-negativity constraints. \square

The nontriviality of the twisted comb constraints is not surprising. Given the derivation of the general DP -constraints given by Letchford in [L1], it seems more probable that they belong to the large family of constraints whose definition involves node subsets that can be separated into handles and teeth. The comb constraints are the prototypes of this family; generalizations include the *clique tree* constraints, the *path*, *wheelbarrow* and *bicycle* constraints, the *star* and *hyperstar* constraints, the *bipartition* constraints, and the *binested* constraints. For a survey of these constraints, see [N1]. In Figure 6.1, we reproduce a diagram from that paper indicating inclusions among these sets of constraints.

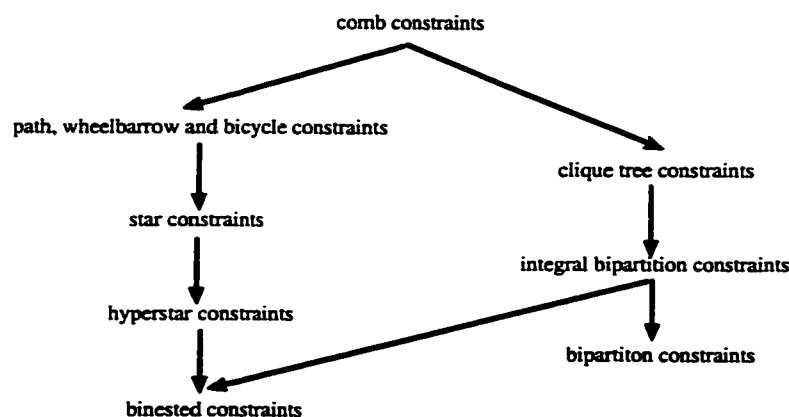


FIGURE 6.1

We will show next that the twisted comb constraints are not in the family of binested constraints. The complete definition of this family is complicated, and can be found in [N1]; we include here only what is needed for our proof.

DEFINITION. A set of subsets $\{S_1, \dots, S_s\}$ is *nested* if for any pair of sets $S_i \neq S_j$, either $S_i \subset S_j$ or $S_j \subset S_i$ or $S_i \cap S_j = \emptyset$.

A binested inequality is defined on two nested sets, handles $\mathcal{H} = \{H_1, \dots, H_h\}$ and teeth $\mathcal{T} = \{T_1, \dots, T_t\}$, together with corresponding positive integers $\{\alpha_1, \dots, \alpha_h\}$ and $\{\beta_1, \dots, \beta_t\}$ (called the *multiplicities* of the handles and teeth respectively), which must satisfy a number of conditions. The *TT*-form of the binested constraint is

$$\sum_{i=1}^h \alpha_i x(\delta(H_i)) + \sum_{j=1}^t \beta_j x(\delta(T_j)) \geq 2\Gamma(\mathcal{H}, \mathcal{T}),$$

where $\Gamma(\mathcal{H}, \mathcal{T})$ is a positive integer which is a function of the handles, the teeth, and their multiplicities. Notice that the edge coefficients are determined solely by the cuts associated to the binested handles and teeth (together with their multiplicities). At first glance, the twisted comb constraints do not appear to fit this pattern. Even with the left hand side written in terms of cuts of the handle and dominoes, we get

$$x(\delta(H)) + \sum_{i=1}^p x(\delta(D_i)) + 2 \sum_{e \in E} \lfloor \mu_e \rfloor x_e.$$

However, it seems possible that some redefinition of the handle and dominoes/teeth could take care of the problematic last term. It turns out that this is not the case. To prove this, we consider the 12-node twisted comb in Figure 6.2(a), which is contained as a subset of any twisted comb. The edge coefficients in the associated constraint are shown in Figure 6.2(b); this pattern must appear in any constraint associated with a twisted comb.

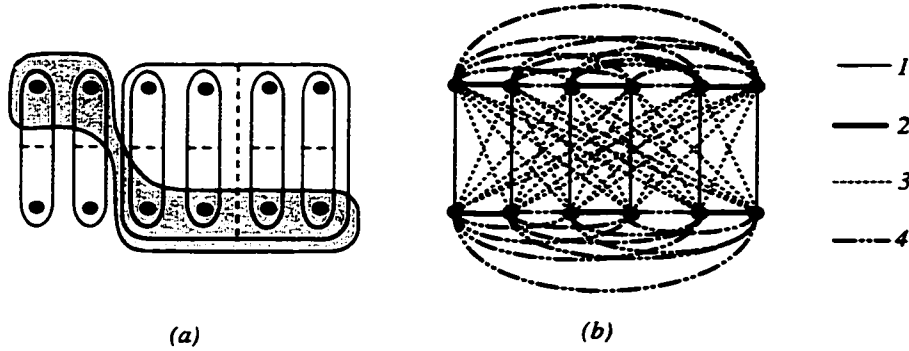


FIGURE 6.2

First we require a couple of technical lemmas regarding tight triangles.

LEMMA 6.3. Let $c^T x \geq \zeta$ be the TT-form of a binested inequality on $STSP(n)$, and let $S \subset V(K_n)$ be a handle or a tooth. If x, y and z are distinct nodes in $V(K_n)$ satisfying $c_{xy} + c_{yz} = c_{xz}$, then y must be on the same shore of $\delta(S)$ as x or z (or both).

PROOF. For any handle or tooth S , the cut $\delta(S)$ either has all three nodes on the same shore, or has two nodes on one shore and the remaining node on the opposite shore. Note that in the first case, the cut contributes nothing to the edge coefficients c_{xy} , c_{yz} and c_{xz} . Let ψ denote the sum of the multiplicities of all handles and/or teeth whose cuts have x on one shore and y, z on the opposite shore; similarly, let ρ represent the sum of the multiplicities of all node sets whose cuts isolate y from the other two, and σ , the sum of all multiplicities of node sets whose cuts isolate z . Then ψ , ρ and σ are non-negative integers that satisfy

$$c_{xy} = \psi + \rho; \quad c_{yz} = \rho + \sigma; \quad c_{xz} = \sigma + \psi.$$

Since $c_{xy} + c_{yz} = c_{xz}$, we must conclude that $\rho = 0$; in other words, there is no handle or tooth whose cut separates y from $\{x, z\}$. \square

If $c_{xy} + c_{yz} = c_{xz}$, we will call xz the *hypotenuse* of the tight triangle.

COROLLARY 6.4. Let $c^T x \geq \zeta$ be the TT-form of a binested inequality on $STSP(n)$, and let $S \subset V(K_n)$ be a handle or a tooth of this inequality. If w, x, y and z are distinct nodes in $V(K_n)$ satisfying $c_{xy} + c_{yz} = c_{xw} + c_{wz} = c_{xz}$ and $c_{wx} + c_{xy} = c_{wz} + c_{yz} = c_{wy}$, then either

- (1) all four nodes are on the same shore of $\delta(S)$, or
- (2) $\{x, w\}$ and $\{y, z\}$ are on opposite shores, or
- (3) $\{x, y\}$ and $\{w, z\}$ are on opposite shores.

PROOF. In Figure 6.3, we give an example of four nodes satisfying the conditions of the corollary, under the assumption that $a = b + c$.

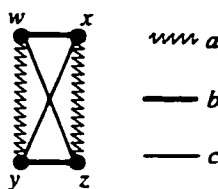


FIGURE 6.3

By Lemma 6.3,

- (1) w must be on the same shore of $\delta(S)$ as x or z ,
- (2) y must be on the same shore as x or z ,
- (3) x must be on the same shore as w or y , and
- (4) z must be on the same shore as w or y .

Suppose that it is not the case that all four nodes are on the same shore. Conditions (1) through (4) imply that $\delta(S)$ cannot separate one of these nodes from the other three; they also imply that we cannot have $\{x, z\}$ on one shore and $\{w, y\}$ on the other. \square

We will call a set of four nodes whose edge coefficients satisfy the conditions of Corollary 6.4 a *tight square*; the hypotenuses of the tight square are the two hypotenuses of the four tight triangles within it. In Figure 6.3, for example, $\{x, z\}$ and

$\{w, y\}$ are the hypotenuses. Corollary 6.4 says that the endnodes of one hypotenuse of a tight square cannot appear in the opposite shore from the endnodes of the other hypotenuse.

THEOREM 6.5. *The twisted comb constraints are distinct from the binested inequalities.*

PROOF. Assume that there exists a twisted comb constraint, $c^T x \geq \zeta$, which is equivalent to a binested constraint, $\tilde{c}^T x \geq \tilde{\zeta}$. As noted earlier, any twisted comb contains a subset of twelve nodes $V_0 = \{p, q, r, s, t, u, p', q', r', s', t', u'\}$ whose edge coefficients in the TT -form of the corresponding constraint are as shown in Figure 6.4. For clarity, we have omitted edges that are hypotenuses of tight triangles.

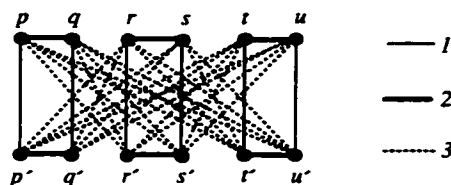


FIGURE 6.4

In order for this figure to represent the edge coefficients of (some positive multiple of) the TT -form of a binested inequality, every pair of nodes in V_0 must be separated by the cut of at least one tooth or handle. In particular, there must be at least one node set H whose cut separates p and p' ; without loss of generality, we can assume H is a handle. Since the nodes $\{p, q, p', q'\}$ form a tight square with hypotenuses pq' and $p'q$, by Corollary 6.4 $\{p, q\}$ and $\{p', q'\}$ must be on opposite shores of $\delta(H)$; without loss of generality, we assume $\{p, q\} \subset H$. Now, $\{q, r, q', r'\}$ is also a tight square, with hypotenuses qr and $q'r'$; since it cannot be the case that q is the only node of this square appearing in H , and we already know that $q' \in \overline{H}$,

Corollary 6.4 implies that $r' \in H$ and $r \in \overline{H}$. Repeating this argument progressively with the tight squares $\{r, s, r', s'\}$, $\{s, t, s', t'\}$ and $\{t, u, t', u'\}$, we conclude that $\{p, q, r', s', t, u\} = H \cap V_0$ and $\{p', q', r, s, t', u'\} = \overline{H} \cap V_0$.

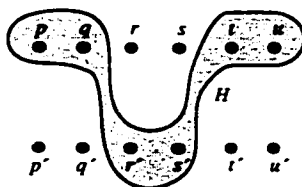


FIGURE 6.5

Notice that $\delta(H)$ separates x from x' , for all $x \in W_0 = \{p, q, r, s, t, u\}$. Moreover, if we had begun by looking for a node set whose cut separates any x from x' , the tight squares (considered in an appropriate sequence) would force the intersection of this set (or its complement) with V_0 to be precisely $H \cap V_0$. Hence, for each $x \in W_0$, $H \cap V_0$ is the *unique* subset of V_0 (up to complements) whose cut separates x from x' . In particular, since the handles in a binested constraint are nested, this implies that *any* handle H' satisfies $H' \cap V_0 \in \{\emptyset, H \cap V_0, \overline{H} \cap V_0, V_0\}$. Let α denote the sum of the multiplicities of all handles in $V(K_n)$ whose cut contributes to the coefficients of edges in $\gamma(V_0)$; this accounts for a contribution of α to the coefficients \tilde{c}_e of the edges shown in Figure 6.6.

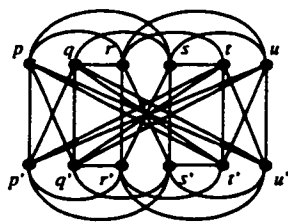


FIGURE 6.6

To account for the positive edge coefficients on the edges of $\gamma(V_0)$ not appearing in Figure 6.6, we must have other node sets, which, since they cannot be handles, must be teeth. If T is such a tooth, $\delta(T)$ cannot separate x from x' , for any $x \in W_0$; if it did, then by the argument in the previous paragraph, $T \cap V_0 = H \cap V_0$ (or $\bar{T} \cap V_0 = H \cap V_0$), and thus $\delta(T)$ would not contribute to the edges with zero coefficient in Figure 6.6. Hence, $T \cap V_0$ must consist of one or more of the pairs $\{p, p'\}$, $\{q, q'\}$, $\{r, r'\}$, $\{s, s'\}$, $\{t, t'\}$ and $\{u, u'\}$. Let $\beta > 0$ denote the sum of the multiplicities of all teeth in $V(K_n)$ whose cuts separate $\{p, p'\}$ and $\{u, u'\}$. Then

$$\bar{c}_{pu} = \beta < \alpha + \beta = \bar{c}_{p'u}.$$

From Figure 6.4, however, we see that in the twisted comb constraint,

$$c_{pu} = 4 > 3 = c_{p'u}.$$

Since this disparity cannot be resolved with a positive scalar multiple, this contradicts our assumption that the twisted comb constraint is equivalent to a binested constraint. \square

Other families of nontrivial, facet-inducing constraints on $STSP(n)$, which do not belong to the set of binested inequalities, are the chain constraints, the *crown* constraints and the *ladder* constraints. Showing that the twisted comb constraints do not belong to these families is comparatively straightforward. The (4.2) version of the chain constraint is in TT -form; parts (1) through (4) of the definition of a chain imply that the only possible edge coefficients in the associated simple inequality are 1, 2 and 3, whereas a simple twisted comb constraint in TT -form has four different positive edge coefficients.

The simple crown constraints are defined on $STSP(4k)$, for $k \geq 1$. For any labelling of the vertices as $1, 2, \dots, 4k$, the associated simple crown inequality is $c^T x \geq 12k(k-1) - 2$, where for any $v \in V(K_{4k})$,

$$c_{v,v+j} = \begin{cases} 4k - 6 + |j|, & 1 \leq |j| \leq 2k - 1, \\ 2(k - 1), & j = 2k. \end{cases}$$

This constraint is shown to be valid and facet-inducing for $STSP(4k)$, and in TT -form, by Naddef and Rinaldi in [NR2]. Even for $k = 1$, there are five different positive edge coefficients.

The ladder inequalities, introduced by Boyd, Cunningham, Queyranne and Wang in 1995 [BCQW], are defined on two handles and an even number of teeth, which satisfy different conditions than the binested inequalities. Also, unlike the binested constraints but like the twisted comb constraints, the left hand side of the TT -form of the ladder constraints includes $\sum \alpha_i x(\delta(H_i)) + \sum \beta_j x(\delta(T_j))$ plus some additional terms. In [BCQW], it is shown that these constraints have Chvátal rank 2, and so they cannot be equivalent to the twisted comb constraints, which, as mentioned in Chapter 1, have Chvátal rank at most 1. In fact, as a consequence of Theorem 5.9 and Proposition 6.2, we know the twisted comb constraints are facet-inducing and distinct from the facets of $SUBT(n)$, so they must have Chvátal rank exactly 1.

CHAPTER 7

CONCLUSIONS

We have shown in this thesis that the twisted comb constraints form a new family of facet-inducing inequalities for $STSP(n)$. They are the simplest possible DP -constraints which are distinct from the comb inequalities, in the sense that the domino-combs that generate them have only one irregular domino. If two or more dominoes in a domino-comb are irregular, then it need no longer be the case that the dominoes are pairwise compatible. Shown below are two examples of such domino-combs; in (b), the dominoes are pairwise compatible, whereas in (a) they are not.

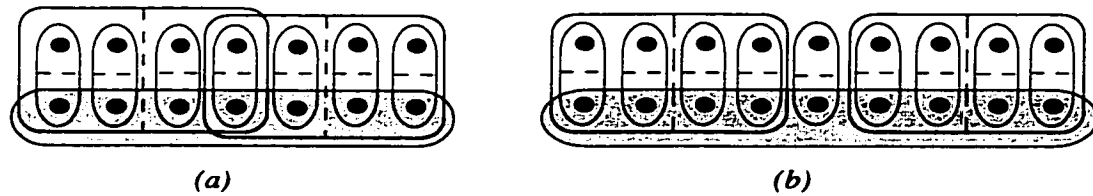


FIGURE 7.1

It would be interesting to know if either of these is facet-inducing, and more generally, if compatibility of the dominoes is required for the corresponding constraint to be facet-inducing. Ultimately, we would like a set of necessary and sufficient conditions that completely characterize when a DP -constraint is facet-inducing.

Another direction for further research is the development of efficient separation algorithms for facet-inducing DP -constraints. An algorithm that separates the twisted comb constraints, for example, or one that searches only for violated DP -constraint that are in fact comb constraints, would be useful. In addition, the applicability of Letchford's algorithm would increase if it could be modified to extend to solutions x whose support is not planar.

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