ABSTRACT

In this thesis , an analysis is presented for predicting the natural frequencies of lateral vibration of elastically supported beams. A new parameter 'support rigidity coefficient q' analogous to the 'quality q' of steam is introduced for this purpose . Beams with all possible end conditions are considered except the ones with a sliding end which are not likely to be encountered very often by the practicing engineer .

Parameter q is varied from 0.05 to 0.9 . For q = 0 & 1.0 , the end conditions coincide with the classical boundary conditions , the solution for which is readily available in the existing literature . Results for any intermediate value of q between 0.0 &0.8 can be obtained by linear interpolation . It is further shown that such interpolation is admissible and yields an accuracy of 1.0 % for all cases except in the immediate neighbourhood of zero natural frequency . For values of q between 0.8 & 1.0 where linear interpolation is not used , a simple algebraic equation is utilized which yields results accurate within 1% .

The computations were carried out on I.B.M.-360 digital computer at the university of ottawa . The values of non-dimensionalised system parameter λ for various values of q for the first ten modes of vibration are presented in chapter 6 of this thesis .

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NOMENCLATURE

a	Area of cross-section of the beam
A,B,C,D	Constants in the general solution of the differential equation.
E	Modulus of elasticity of beam material.
f	Frequency
g	Acceleration due to gravity.
I	Moment of inertia of beam cross-section.
K	Spring constant.
L	Length of the beam.
m	Mass per unit length of the beam.
M	Concentrated mass.
P(t)	A function of time only.
R(x)	A function of x only.
t	Instant of time.
V(x,t)	Elastic line of beam deflection.
x,y	Spatial coordinates.
Y_n	Function defined by equation (6.3)

Subscripts

L	Linear spring
t	Torsional spring
n	n-th mode of vibration.
cc	Clamped-clamped
cF	Clamped-free
cs	Clamped-simple
FF	Free-free
sF	Simple-free
ss	Simple-simple

GREEK SYMBOLS

ρ	Mass per unit volume of the beam
ω	Circular natural frequency
ξ	Non-dimensionalised spatial coordinate parameter
β	Eigen-value parameter
λ_L	Dimensionless system parameter associated with linear spring
λ_t	Dimensionless system parameter associated with torsional spring
λ_m	Dimensionless system parameter associated with concentrated mass
α	Ratio of two system parameters
Δf	Percentage error in frequency
Δq	Error in q

CHAPTER 1

INTRODUCTION AND GENERAL THEORY

1.1 INTRODUCTION

The natural frequencies and modes of lateral vibration of uniform beams with well defined boundary conditions, for example, clamped hinged or free have been known for many years and may be obtained from the existing literature (1,2)*. But more often than not , a practicing engineer finds that the structure of his interest does not possess these well defined boundary conditions. The option open to him is either to settle for the classical boundary conditions which seem closest to those of his problem, to attempt a time consuming analysis or resort to experiment. The first alternative leads to a system design which may be too conservative, or unsafe, while the second and third alternatives involve considerable time, money and labour. Very little work exists in the available literature to help the practicing engineer confronted with such situations. Newmark and Veletsos (3) developed an emperical equation for beams with partly clamped ends but the range of validity for the desired accuracy of this equation is too narrow to be of much use to the practicing engineer. Besides, there is a whole class of elastically supported beams which has not been considered so far . What is it?

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* Numbers in brackets refer to the references given at the end

In the present work a technique has been introduced which is capable of providing readily available values for the frequencies and modes of vibration of virtually any linear beam system whatever. A number of illustrative examples are utilized for clarification. Existing solutions for classical boundary conditions are modified and are presented in a manner suitable for immediate application by the practicing engineer.

1.2 DEVELOPMENT OF THEORY

In a beam undergoing free vibrations, the only forces acting are the inertia forces and the elastic restoring forces. This can be stated as follows (4).

$$EI \frac{\partial^4 V(x,t)}{\partial x^4} = -m \frac{\partial^2 V(x,t)}{\partial t^2} \qquad (1.1)$$

Where V(x,t) is the elastic line of the beam, EI is its flexural rigidity and $m = \rho a$, ρ being the mass of the material of the beam per unit volume, a its cross sectional area.

The equation (1.1) is based upon the assumptions implied in the elementary beam theory. These assumptions can be stated in brief as follows (5).

(1) The material of the beam is homogeneous , isotropic and obeys Hooke's law.

(2) The beam is straight and is of uniform cross section.

(3) Deflections are small.

(4) Each plane section that is initially normal to the axis of the beam remains plane and normal to the axis.

(5) Cross sectional dimensions of the beam are small compared with its length , consequently the effects of shear deformation are neglected.

(6) The effects of rotary inertia are neglected.

The partial differential equation (1.1) can be solved by the method of 'separation of variables'. Assuming that the solution of (1.1) is separable in time and space and is of the following form.

$$V(x,t) = R(x).P(t) \tag{1.2}$$

Where R(x) is a function of x only and P(t) is a function of t only. Substituting expression(1.2) in (1.1) and dividing through by m.R(x).P(t) , we obtain,

$$\frac{EI}{m\ R(x)} \frac{d^4R(x)}{dx^4} = -\frac{1}{P(t)} \frac{d^2P(t)}{dt^2} \tag{1.3}$$

The left side of equation(1.3) depends on x only and the right side depends on t only. Both x and t are independent variables , so equation (1.3) has a solution only if both sides are constant . Let the constant be a positive constant ω^2. The equation(1.3) leads to two ordinary differential equations.

$$\frac{d^2 P(t)}{d t^2} + \omega^2 P(t) = 0 \tag{1.4}$$

$$EI \frac{d^4 R(x)}{d x^4} - m\omega^2 R(x) = 0 \tag{1.5}$$

The choice of the sign of the constant is dictated by physical considerations . Our system is conservative , so displacements must remain finite as t increases . By choosing a positive constant we obtain harmonic rather than an exponential solution of equation (1.4) which is consistent with the fact that a conservative system has constant total energy .

Non-dimensionalising the lateral displacement and spatial coordinate with respect to beam length L ,equation (1.5) can be re-written as follows.

$$\frac{d^4 R(\xi)}{d \xi^4} - \beta^4 R(\xi) = 0 \tag{1.6}$$

$$\text{where } \beta^4 = \frac{m\omega^2 L^4}{E I} \tag{1.7}$$

and $\xi = (\frac{x}{L})$ is a non-dimensionalised spatial coordinate parameter.

Equation (1.6) which is a fourth order homogeneous linear ordinary differential equation , represents the shape of a uniform beam in lateral vibration. This should be supplemented by four boundary conditions ,two at each end. The problem of determining the values of the parameter β for which a homogeneous linear differential equation of the type (1.6) has a non-trivial solution $R(\xi)$,satisfying homogeneous boundary conditions is called the characteristic value or eigenvalue problem. Such values of the parameter β are called characteristic values or eigenvalues and the associated non-trivial solutions $R(\xi)$ are called characteristic functions or eigenfunctions. Equation (1.6) is a homogeneous differential equation so its solution can not be determined uniquely except for the shape.

The four boundary conditions determine uniquely the shape of the solution, leaving the amplitude arbitrary and also yield a characteristic equation or frequency equation which when solved gives the characteristic values of the problem. These frequency equations being transcedental in nature yield a solution consisting of an infinite sequence of discrete eigenvalues β_n related to natural circular frequencies ω_n (n=1,2,.......etc). The eigenvalues β_n for first ten modes of beams with classical end conditions have been computed using an iteration technique explained in appendix 2 . The modes $R_n(\xi)$ and the associated natural frequencies $\omega_n(\beta_n)$ evidently depend on the stiffness EI, mass m and, of course, the four boundary conditions. In short they are a characteristic of the system.

1.3 BOUNDARY CONDITIONS

The boundary conditions encountered in the lateral vibration of beams are either geometric boundary conditions or force boundary conditions. The boundary conditions resulting from pure geometric compatability are called geometric boundary conditions. In the particular case of flexural vibrations of beams , the geometric boundary conditions are concerned with the deflection or rotation at the boundary. In the cases in which the geometry does not provide a sufficient number of boundary conditions, the remaining conditions are supplied by the moment or shear-force balance. The boundary conditions resulting from moment or shear force balance are called force boundary conditions.The geometric and force boundary conditions supplement each other and add up to the correct number of boundary conditions and their satisfaction ensures that the solution of the differential equation is unique.

The boundary conditions at either end , in fact ,could be any possible and permissible combination of $R(\xi)$, $R'(\xi)$*, $R''(\xi)$ and $R'''(\xi)$ where $R(\xi)$ and $R'(\xi)$ are the deflection and slope respectively and $R''(\xi)$, $R'''(\xi)$ are related to bending moment and shear force respectively.

The mathematically possible combinations of boundary conditions at either end are·

(1) $R(\xi)$ and $R'(\xi)$

(2) $R(\xi)$ and $R''(\xi)$

(3) $R''(\xi)$ and $R'''(\xi)$

(4) $R'(\xi)$ and $R'''(\xi)$

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* Superscript primes refer to differentiation with respect to ξ

(5) $R(\xi)$ and $R'''(\xi)$

(6) $R'(\xi)$ and $R''(\xi)$

The combinations (5) and (6) are physically inadmissible because deflection and shear force can not be prescribed simultaneously and so also slope and moment.

In case of homogeneous boundary conditions the combination (1) $R(\xi) = R'(\xi) = 0$ represents a clamped end , the combination (2) $R(\xi) = R''(\xi) = 0$, represents a pinned end , the combination (3) $R''(\xi) = R'''(\xi) = 0$, represents a free end and finally the combination (4) $R'(\xi) = R'''(\xi) = 0$, represents a sliding end which could be provided with rollers.

All these ends with the exception of a sliding end are considered here in combination with elastically supported ends. The reason for not considering a sliding end is that it is seldom encountered in practice.

The elastically supported end could either be a free end with a linear spring or a pinned end with a torsional spring. Furthermore , both ends of the beam could be elastically supported .Beams with all possible combinations of classical ends and elastically supported ends have been analysed as an illustration of the new technique introduced in this work.

A case of great practical interest is a clamped- free beam with a concentrated mass at the free end . This class of beams has also been considered .

The types of beams analysed in this thesis can be broadly classified as follows .

(1) Beams with linear springs as elastic supports

In this class , the following three types of beams are considered .

(a) The clamped - elastically supported beams .(Fig. 1)

(b) The pinned - elastically supported beams. (Fig. 2)

(c) Beams with both ends elastically supported. (Fig. 3)

(2) Beams with torsional springs as elastic supports

The elastic end in this case is a partly clamped end i,e a simply supported end with a torsional spring .The following four classes of beams are considered under this heading.

(a) The clamped - partly clamped beams.(Fig.4)

(b) The pinned - partly clamped beams.(Fig.5)

(c) The free - partly clamped beams.(Fig.6)

(d) Beams with both ends partly clamped.(Fig.7)

(3) The hinged - free beams with torsional spring at the hinged end and a linear spring at the free end.(Fig.8)

(4) The clamped - free beams with a concentrated mass at the free end.(Fig.9)

1.4 SOLUTION OF THE DIFFERENTIAL EQUATION

The general solution of the differential equation (1.6) is given as follows (2).

$$R(\xi) = A \sin \beta\xi + B \cos \beta\xi + C \sinh \beta\xi + D \cosh \beta\xi \qquad (1.8)$$

The successive derivatives of equation (1.8) are obtained as follows

$$R'(\xi) = \beta (A \cos \beta\xi - B \sin \beta\xi + C \cosh \beta\xi + D \sinh \beta\xi) \qquad (1.9)$$

$$R''(\xi) = \beta^2 (- A \sin \beta\xi - B \cos \beta\xi + C \sinh \beta\xi + D \cosh \beta\xi) \qquad (1.10)$$

$$R'''(\xi) = \beta^3 (-A \cos \beta\xi + B \sin \beta\xi + C \cosh \beta\xi + D \sinh \beta\xi) \qquad (1.11)$$

$$R^{IV}(\xi) = \beta^4 (A \sin \beta\xi + B \cos \beta\xi + C \sinh \beta\xi + D \cosh \beta\xi)$$

$$= \beta^4 R(\xi) \qquad (1.12)$$

Equation (1.12) proves that equation (1.8) is the solution of the differential equation (1.6).

The equations (1.8) through (1.11) are valid for all the classes of beams considered in the present thesis and they will be frequently utilized in the subsequent chapters.

CHAPTER 2

BEAMS WITH LINEAR SPRINGS AS ELASTIC SUPPORTS

The following three classes of beams are considered in this chapter.

(a) Clamped - elastically supported beams

(b) Pinned - elastically supported beams

(c) Beams with both ends elastically supported

It will be shown that the common feature of these three classes of beams considered in the present chapter is that their frequency is characterised by a dimensionless system parameter λ_L. λ_L in each case is expressed as $K_L.L^3/EI$.

where K_L = Spring constant of the linear spring

L = Length of the beam

EI = Flexural rigidity of the beam

2.1 CLAMPED - ELASTICALLY SUPPORTED BEAMS (FIG.1)

Stiffness K_L of the linear spring determines the rigidity of the elastically supported end. An infinitely stiff spring will make the end behave as a pinned end. Therefore the two limiting cases of this class of beams are ,

(1) Clamped - free beams with $K_L = 0$

(2) Clamped - pinned beams with $K_L = \infty$

It is quite logical , therefore , to conclude that natural frequencies of a beam of this class , for different modes of vibration will lie between those of the same beam with clamped - free and clamped - hinged end conditions for corresponding modes of vibration.

At this stage , the parameter q referred to as support rigidity coefficient is introduced . It should be noted that parameter q has no physical definition or significance . It is rather a mathematical tool introduced for the sake of convenience and expediency . However , it may be thought of as analogous to the 'quality q' of steam .

The natural frequency of this class of beams may be expressed in the following form .

$$f_{(n)} = f_{cF(n)} + q \left(f_{cs(n)} - f_{cF(n)} \right) \qquad (2.1)$$

where $f_{(n)}$ = Natural frequency of clamped - elastically supported beam in n^{th} mode of vibration where $n = 1,2.....10$

$f_{cF(n)}$ = Natural frequency of the same beam in the same mode with clamped - free end conditions .

$f_{cs(n)}$ = Natural frequency of the same beam in the same mode with clamped - simple end conditions .

The parameter q as defined in equation (2.1) is obviously a measure of how far the actual beam frequency lies along the path between the two limits.

Equation (2.1) can be re-written as follows

$$\frac{f_{(n)}}{f_{cF(n)}} = 1 + q\left(\frac{f_{cs(n)}}{f_{cF(n)}} - 1\right) \qquad (2.2)$$

From equation (1.7) it is seen that frequencies (ω or f) are proportional to the square of eigenvalues β . Hence the frequency ratio of the two beams will be equal to the square of their eigenvalue ratio . Expressing this in equation form , we have ,

$$\frac{f_{(n)}}{f_{cF(n)}} = \left(\frac{\beta_{(n)}}{\beta_{cF(n)}}\right)^2 \qquad (2.3)$$

and

$$\frac{f_{cs(n)}}{f_{cF(n)}} = \left(\frac{\beta_{cs(n)}}{\beta_{cF(n)}}\right)^2 \qquad (2.4)$$

Substitution of equations (2.3) and (2.4) in (2.2) yields the following equation.

$$\beta_{(n)} = \beta_{cF(n)}\sqrt{1 + q\left[\left(\frac{\beta_{cs(n)}}{\beta_{cF(n)}}\right)^2 - 1\right]} \qquad (2.5)$$

Equation (2.5) is used to determine the eigenvalues β for n^{th} mode of vibration of clamped - elastically supported beams for different values of the parameter q . . The eigen values $\beta_{cF(n)}$ and $\beta_{cs(n)}$ can be computed from the frequency equation of clamped - free and clamped - simple beams which are readily available in the existing literature (2) and are included here on page 100, Table 11.

The frequency equations for clamped free and clamped - simple beams when the beam deflection and spatial coordinates are non-dimensionalised with respect to beam length L, can be expressed as follows.

$$1 + \text{Cos}\ \beta_{cF(n)}\ \text{Cosh}\ \beta_{cF(n)} = 0 \qquad (2.6)$$

$$\text{Cos}\ \beta_{cs(n)}\ \text{Sinh}\ \beta_{cs(n)} - \text{Sin}\ \beta_{cs(n)}\ \text{Cosh}\ \beta_{cs(n)} = 0 \qquad (2.7)$$

Equations (2.6) and (2.7) have been obtained by substituting the appropriate boundary conditions for clamped - free and clamped - simple beams in equation (1.8) and using the condition for a non-trivial solution for the constants of equation (1.8).

Equations (2.6) and (2.7) yield an infinity of solutions $\beta_{cF(n)}$ and $\beta_{cs(n)}$. The first ten values of $\beta_{cF(n)}$ and $\beta_{cs(n)}$ have been computed numerically using an iteration technique, on the digital computer (I.B.M.- 360). These values are presented in the tables of results in chapter 6. The iteration technique has been explained briefly in appendix 2. A print - out of the computer program utilizing this technique also appears in appendix 2.

The appropriate boundary conditions for the beam under study are,

(1) On the fixed end where $\xi = 0$, the geometry of the end provides the necessary two boundary conditions.

The deflection $R(0) = 0$ (2.8)

The slope $R'(0) = 0$ (2.9)

(2) On the elastically supported end where $\xi = 1$, we have two force boundary conditions namely bending moment should be zero and shear force should be balanced by the spring force .

$$R''(1) = 0 \qquad (2.10)$$

$$R'''(1) = \frac{K_L . L^3}{EI} R(1)$$

$$= \lambda_L . R(1) \qquad (2.11)$$

where $\lambda_L = \frac{K_L . L^3}{EI}$ is the dimensionless system parameter.

Boundary condition (2.11) has been derived in appendix 1 .

Substitution of (2.9) and (2.10) in equations (1.8) and (1.9) yields ,

$$B = - D \quad \text{and} \quad A = - C \qquad (2.12)$$

Utilizing (2.12) and substituting boundary conditions (2.10) and (2.11) in equations (1.10) and (1.11),we obtain the following .

$$C (\text{Sinh}\,\beta + \text{Sin}\,\beta) + D (\text{Cos}\,\beta + \text{Cosh}\,\beta) = 0 \qquad (2.13)$$

$$C \left[\text{Cos}\,\beta + \text{Cosh}\,\beta + \frac{\lambda_L}{\beta^3} (\text{Sin}\,\beta - \text{Sinh}\,\beta) \right] +$$

$$D \left[\text{Sinh}\,\beta - \text{Sin}\,\beta + \frac{\lambda_L}{\beta^3} (\text{Cos}\,\beta - \text{Cosh}\,\beta) \right] = 0 \qquad (2.14)$$

Existence of a non-trivial solution for C and D requires the determinant of their coefficient matrix to vanish . So we should have ,

$$\begin{vmatrix} \sin\beta + \sinh\beta & \cos\beta + \cosh\beta \\ \cos\beta + \cosh\beta + \frac{\lambda_L}{\beta^3}(\sin\beta - \sinh\beta) & \sinh\beta - \sin\beta + \frac{\lambda_L}{\beta^3}(\cos\beta - \cosh\beta) \end{vmatrix} = 0 \quad (2.15)$$

Expanding the determinant and simplifying equation (2.15) we obtain the following expression for the system parameter λ_L .

$$\lambda_L = \frac{\beta^3 (1 + \cos\beta \cosh\beta)}{(\cos\beta \sinh\beta - \sin\beta \cosh\beta)} \quad (2.16)$$

We note that values of β which permit vanishing of the determinant are completely characterised by the dimensionless system parameter λ_L .

It is interesting to note that for $\lambda_L = 0$, equation (2.16) yields , $1 + \cos\beta \cosh\beta = 0$, which is the frequency equation of clamped - free beams and for $\lambda_L = \infty$, equation (2.16) gives , $\cos\beta \sinh\beta - \sin\beta \cosh\beta = 0$, which is the frequency equation of clamped-simply supported beams .

Values of system parameter λ_L has been computed for various values of q varying from 0.05 to 0.9 in steps of .05, .1, .2, .3 , ...etc.by using equations (2.5) and (2.16) .For q= 0 and q = 1 , the beam becomes a clamped - free and clamped - simple respectively , the solution for which is well known .The results are presented in the tabular form in chapter 6 .

The use of these tables can best be demonstrated by working out an illustrative example.

Example - 1

Determine the first mode natural frequency of a rectangular aluminum beam with one end fixed and the other supported by a linear elastic spring of stiffness 8.2 lb./in. The beam is 29.5 in. long, 1 in. wide and 0.25 in. deep.

Solution

The moment of inertia of the beam, $I = \frac{1}{12} .1.(.25)^3$
$= 0.0013 \text{ in.}^4$

For aluminum, $E = 10.5 \times 10^6$ psi

$\rho = 0.0965 \text{ lb./in.}^3$

$\beta_{cF(1)} = 1.875$ and $\beta_{cs(1)} = 3.926$ (From table 11)

$$f_{cF(1)} = \frac{(\beta_{cF(1)})^2}{2\pi L^2} \sqrt{\frac{EI.g}{\rho.a}}$$

$$= \frac{(1.875)^2}{2\pi \times (29.5)^2} \sqrt{\frac{10.5 \times 10^6 \times 386 \times .0013}{0.0965 \times .25}} = 9.5 \text{ Hertz.}$$

$$f_{cs(1)} = f_{cF(1)} \left[\frac{\beta_{cs(1)}}{\beta_{cF(1)}}\right]^2$$

$$= 9.5 \left(\frac{3.926}{1.875}\right)^2 = 41.7 \text{ Hz}$$

System parameter $\lambda_L = K_L.L^3 / EI$

$$= \frac{8.2 \times (29.5)^3}{10.5 \times 10^6 \times .0013} = 15.4$$

For $\lambda_L = 15.4$, $q = 0.38$ (From table 1)

The value of q has been obtained by interpolation and it will be shown in chapter 6 that the interpolation is accurate within 1% .

From equation (2.1) with n = 1 we have ,

$$f_{(1)} = f_{cF(1)} + q \left(f_{cs(1)} - f_{cF(1)} \right)$$

$$= 9.5 + 0.38 \left(41.7 - 9.5 \right) = \underline{21.7 \text{ Hz}}$$

2.2 PINNED ELASTICALLY SUPPORTED BEAMS (FIG.2)

The two limiting cases of this class of beams are ,

(1) Pinned - free beams with $K_L = 0$

(2) Pinned - pinned beams with $K_L = \infty$

In terms of parameter q , the frequency of such beams can be expressed as follows

$$f_{(n)} = f_{sF(n)} + q \left(f_{ss(n)} - f_{sF(n)} \right) \qquad (2.17)$$

where $f_{(n)}$ = Natural frequency of pinned - elastically supported beam in n^{th} mode of vibration .

$f_{sF(n)}$ = Natural frequency of the same beam in the same mode with simple -free end conditions

$f_{ss(n)}$ = Natural frequency of the same beam in the same mode with simple - simple end conditions

Since the first mode frequency of simple - free beams is zero , the equation (2.17) can be written down as follows for the first mode .

$$f_{(1)} = q\, f_{ss(1)} \qquad (2.18)$$

Equations (2.18) and (2.17) can be expressed in terms of eigen-values as follows .

$$\beta_{(1)} = \beta_{ss(1)} \sqrt{q} \qquad (2.19)$$

$$\beta_{(n)} = \beta_{sF(n)} \sqrt{1 + q \left[(\beta_{ss(n)} / \beta_{sF(n)})^2 - 1 \right]} \qquad (2.20)$$

where $n > 1$.

$\beta_{ss(n)}$ and $\beta_{sF(n)}$ can be computed from the relevant frequency equations which are given below (2) .

$$\text{Sin}\ \beta_{ss(n)} = 0 \qquad (2.21)$$

$$\text{Sin}\ \beta_{sF(n)} \text{Cosh}\ \beta_{sF(n)} - \text{Cos}\ \beta_{sF(n)} \text{Sinh}\ \beta_{sF(n)} = 0 \qquad (2.22)$$

The first ten values of $\beta_{ss(n)}$ and $\beta_{sF(n)}$ have been computed and are presented in the tables of results in chapter 6 .

The appropriate boundary conditions at the two ends are as follows .

(1) At simply supported end where $\xi = 0$, we have ,

$$R(0) = 0 \qquad (2.23)$$

$$R''(0) = 0 \qquad (2.24)$$

(2) At elastically supported end where $\xi = 1$, we have ,

$$R''(1) = 0 \qquad (2.25)$$

$$R'''(1) = \frac{K_L . L^3}{EI} R(1)$$

$$= \lambda_L R(1) \qquad (2.26)$$

Substituting boundary conditions (2.23) and (2.24) in equations (1.8) and (1.10) , we obtain ,

$$B = D = 0 \qquad (2.27)$$

Utilizing (2.27) and substituting boundary conditions (2.25) and (2.26) in equations (1.10) and (1.11),we obtain ,

$$- A \operatorname{Sin}\beta + C \operatorname{Sinh}\beta = 0 \tag{2.28}$$

$$A\left(\frac{\lambda_L}{\beta^3} \operatorname{Sin}\beta + \operatorname{Cos}\beta\right) + C\left(\frac{\lambda_L}{\beta^3} \operatorname{Sinh}\beta - \operatorname{Cosh}\beta\right) = 0 \tag{2.29}$$

The condition for non-triviality of the solution for B and D in (2.28) and (2.29) yields ,

$$\begin{vmatrix} -\operatorname{Sin}\beta & \operatorname{Sinh}\beta \\ \frac{\lambda_L}{\beta^3} \operatorname{Sin}\beta + \operatorname{Cos}\beta & \frac{\lambda_L}{\beta^3} \operatorname{Sinh}\beta - \operatorname{Cosh}\beta \end{vmatrix} = 0 \tag{2.30}$$

Expanding and simplifying (2.30) we obtain ,

$$\lambda_L = \frac{\beta^3 \left(\operatorname{Sin}\beta \operatorname{Cosh}\beta - \operatorname{Cos}\beta \operatorname{Sinh}\beta \right)}{2 \operatorname{Sin}\beta \operatorname{Sinh}\beta} \tag{2.31}$$

It can be noted that for $\lambda_L = 0$, equation (2.31) yields the frequency equation for pinned - free beams and for $\lambda_L = \infty$, it gives the frequency equation for simply supported beams.

Using equations (2.19) , (2.20) and (2.31) , the system parameter λ_L can be determined for any value of q for any mode n . The values of λ_L for q = .05 , .1 , .2 , .3 , ...9. and n = 1,2, 3,.....10 have been computed and are presented in the tables of results in chapter 6 . The following example is presented by way of illustration .

Example - 2

Determine the natural frequencies of the first two modes of vibration for the beam system described in example-1 , with one end hinged instead of fixed .

Solution

From table 11 : - $\beta_{sF(1)} = 0$, $\beta_{sF(2)} = 3.926$

$\beta_{ss(1)} = 3.141$, $\beta_{ss(2)} = 6.283$

$f_{sF(1)} = 0$

$$f_{sF(2)} = \frac{(\beta_{sF\,2})^2}{2\pi L^2}\sqrt{\frac{EI\,g}{\rho a}} = 41.7 \text{ Hz.}$$

$$f_{ss(1)} = \frac{(\beta_{ss(1)})^2}{2\pi L^2}\sqrt{\frac{EI\,g}{\rho a}} = 26.6 \text{ Hz.}$$

$f_{ss(2)} = 4\, f_{ss(1)} = 106.4$ Hz.

System parameter $\lambda_L = 15.4$ (Same as in example -1)

For this λ_L , from table 2 , we obtain ,

$q = 0.595$ for mode 1 .

$= 0.0875$ for mode 2 .

From equation(2.18),

$f_{(1)} = q\, f_{ss(1)}$

$= .595 \times 26.6 = \underline{15.8 \text{ Hz}}.$

From equation (2.17),

$f_{(2)} = f_{sF(2)} + q\,(f_{ss(2)} - f_{sF(2)})$

$= 41.7 + .0875\,(106.4 - 41.7)$

$= \underline{47.4 \text{ Hz}}.$

2.3 BEAMS WITH BOTH ENDS ELASTICALLY SUPPORTED (FIG.3)

The two limiting cases of this class of beams are ,

(1) Free - free beams with $K^*_{L1} = K_{L2} = 0$

(2) Simple - simple beams with $K_{L1} = K_{L2} = \infty$

Where subscripts 1 & 2 refer to left end and right end respectively.

In terms of parameter q , the frequency equation is written down as follows.

$$f_n = f_{FF(n-1)} + q\,(f_{ss(n)} - f_{FF(n-1)}) \qquad (2.32)$$

Where symbols used have their usual meaning.

Since the frequency equation of free - free beams has a double root at $\beta = 0$, there are two modes for zero eigenvalue , one of translation and the other of rotation about the centre and both have an infinite period . Therefore the frequency for the first two modes of free - free beams is zero and the equation (2.32) assumes the following form for these two modes .

$$f_{(n)} = q\, f_{ss(n)} \quad \text{where } n = 1 \text{ and } 2 . \qquad (2.33)$$

In terms of eigen-values , equations (2.33) and (2.32) are written down as follows .

$$\beta_{(n)} = \beta_{ss(n)} \sqrt{q} \quad \text{where } n = 1 \,\&\, 2 . \qquad (2.34)$$

$$\beta_{(n)} = \beta_{FF(n-1)}\sqrt{1 + q\left[\left(\frac{\beta_{ss(n)}}{\beta_{FF(n-1)}}\right)^2 - 1\right]} \qquad (2.35)$$

where $n > 2$

$\beta_{ss(n)}$ and $\beta_{FF(n)}$ are calculated from the corresponding frequency equations which are as follows (2) .

$$\text{Sin } \beta_{ss(n)} = 0 \qquad (2.36)$$

$$1 - \text{Cos } \beta_{FF(n)} \text{ Cosh } \beta_{FF(n)} = 0 \qquad (2.37)$$

They are computed using the iteration technique explained in appendix 2 and are presented in chapter 6 .

The proper boundary conditions for this class of beams are ,

(1) At the end where $\xi = 0$, we have ,

$$R''(0) = 0 \qquad (2.38)$$

$$R'''(0) = -\frac{K_{L1} \cdot L^3}{EI} R(0)$$

$$= -\lambda_{L1} R(0) \qquad (2.39)$$

(2) At the end where $\xi = 1$, we have ,

$$R''(1) = 0 \qquad (2.40)$$

$$R'''(1) = \frac{K_{L2} \cdot L^3}{EI} R(1)$$

$$= \lambda_{L2} R(1) \qquad (2.41)$$

Substitution of (2.38) in (1.10) yields ,

$$B = D \qquad (2.42)$$

Utilizing (2.42) and substituting boundary conditions (2.40), (2.39) and (2.41) in equations (1.10) and (1.11) yield the following three equations in A, B and C. In the following $\beta_{(\omega)}$ is replaced by β which should not be confused with equation (1.7)

$$-A \sin\beta + B(\cosh\beta - \cos\beta) + C\sinh\beta = 0 \tag{2.43}$$

$$-A + B\frac{2\lambda_{L1}}{\beta^3} + C = 0 \tag{2.44}$$

$$-A\left(\cos\beta + \frac{\lambda_{L2}}{\beta^3}\sin\beta\right) + B\left[\sin\beta + \sinh\beta - \frac{\lambda_{L2}}{\beta^3}(\cos\beta + \cosh\beta)\right]$$

$$+C\left(\cosh\beta - \frac{\lambda_{L2}}{\beta^3}\sinh\beta\right) = 0 \tag{2.45}$$

For a non-trivial solution for A, B and C in the above three equations, we should have,

$$\begin{vmatrix} -\sin\beta & \cosh\beta - \cos\beta & \sinh\beta \\ -1 & \dfrac{2\lambda_{L1}}{\beta^3} & 1 \\ -\left(\cos\beta + \dfrac{\lambda_{L2}}{\beta^3}\sin\beta\right) & \sin\beta + \sinh\beta - \dfrac{\lambda_{L2}}{\beta^3}(\cos\beta + \cosh\beta) & \left(\cosh\beta - \dfrac{\lambda_{L2}}{\beta^3}\sinh\beta\right) \end{vmatrix} = 0 \tag{2.46}$$

Expanding the determinant in equation (2.46) and simplifying we obtain,

$$\frac{2\lambda_{L1}\lambda_{L2}}{\beta^6}\sinh\beta\sin\beta + \frac{\lambda_{L2}}{\beta^3}(\cos\beta\sinh\beta - \sin\beta\cosh\beta) - \frac{\lambda_{L1}}{\beta^3}\sin\beta\cosh\beta$$

$$+\frac{\lambda_{L1}}{\beta^3}\cos\beta\sinh\beta + 1 - \cosh\beta\cos\beta = 0 \tag{2.47}$$

Now another parameter α is introduced which is defined as the ratio of the smaller system parameter λ_{L2} to the larger system parameter λ_{L1}, so that α is always ≤ 1.0 where λ_{L1} has been assumed to be greater than λ_{L2}.

$$\alpha = \frac{\lambda_{L2}}{\lambda_{L1}} \qquad (2.48)$$

Because of similarity of end conditions, it is always possible to choose λ_{L1} greater than λ_{L2} and this does not restrict the theory in any way. In terms of parameter α, equation (2.47) can be expressed as follows.

$$\lambda_{L1}^2 + \frac{\beta^3(1+\alpha)}{\alpha} \left[\frac{\cos\beta \sinh\beta - \sin\beta \cosh\beta}{2 \sinh\beta \sin\beta} \right] \lambda_{L1} + \frac{\beta^6}{\alpha} \frac{(1 - \cos\beta \cosh\beta)}{2 \sinh\beta \sin\beta} = 0 \qquad (2.49)$$

The values of λ_{L1} for various values of n, α and q are presented in the tables of results in chapter 6. The parameter α has been varied from 0.1 to 1.0 in steps of 0.1 and n and q are varied as in the earlier cases.

Equation (2.49) being quadratic in λ_{L1} yields two values for every value of β, out of which one being negative is discarded. However in certain ranges of frequency, there are two positive roots of Eqn.(2.49) because the beams of this class are capable of vibrating with the same frequency either in symmetric or in anti-symmetric modes in these ranges of frequency. To illustrate this, a plot of λ_{L1} vs β for the first two modes is presented in Fig.10. It can be seen from the figure that in the frequency range, $0 \leq \beta < \pi$, both symmetric and anti-symmetric modes are present.

Similarly, there are other ranges of frequency also in which both symmetri and anti-symmetric modes are present . These ranges are : $4.730 < \beta < 6.283$, $7.853 < \beta < 9.424$, $10.995 < \beta < 12.566$, $14.137 < \beta < 15.707$, $17.278 < \beta < 18.859$, $20.420 < \beta < 21.991$, $23.561 < \beta < 25.132$ and $26.703 < \beta < 28.274$.

Analysis was made separately for symmetric and anti-symmetric modes . As the system parameter λ_{I1} varies from 0.0 to ∞, the symmetric modes of free -free beams degenerate into symmetric modes of simple - simple beams and anti-symmetric modes of free-free beams degenerate into anti-symmetric modes of simple - simple beams .

The existence of both types of modes in certain ranges of frequency can be further demonstrated by working out the following example .

Example - 3

Determine the first two mode frequencies of the beam system described in example 1 when two ends of the beam are supported by linear springs of **constants** 8.2 lb/in. and 2.46 lb./in.

Solution

$$\alpha = \frac{2.46}{8.2} = 0.3$$

$f_{FF(1)} = f_{FF(0)} = 0$

$f_{ss(1)} = 26.6$ Hz. (As calculated in example 2)

$f_{ss(2)} = 106.4$ Hz.(Please see example 2)

System parameter $\lambda_{I1} = 15.4$ (same as in example 1)

From table 3.3 , we have ,

$q = 0.34$ (For first mode which is symmetric)

$= 0.1975$ (For second mode which is anti-symmetric)

Therefore , $f_{(1)} = q\ f_{ss(1)}$

$$= 0.34 \times 26.6 = \underline{9.0\ Hz}.$$

and $f_{(2)} = q\ f_{ss(2)}$

$$= 0.1975 \times 106.4 = \underline{21.0\ Hz}.$$

This example illustrates that the frequency 21.0 Hz. though lying in the range of symmetric mode (0.0 to 26.6 Hz.) actually belongs to anti - symmetric mode .

CHAPTER 3

BEAMS WITH TORSIONAL SPRINGS AS ELASTIC SUPPORTS

In this case the elastically supported end is a partly clamped end provided by a torsional spring attached to a simply supported end .The following four classes of beams are considered in this chapter .

(1) The clamped - partly clamped beams .

(2) The pinned - partly clamped beams .

(3) The free - partly clamped beams .

(4) Beams with both ends partly - clamped.

It will be shown that the dimensionless frequency parameter λ_t has the form $\lambda_t = K_t.L / EI$ in all the classes of beams considered in this chapter .

3.1 THE CLAMPED - PARTLY CLAMPED BEAMS (FIG.4)

The two limiting cases of this class of beams are ,

(1) Clamped - simple beams with $K_t = 0$

(2) Clamped - clamped beams with $K_t = \infty$

The natural frequency in terms of q can be expressed as follows .

$$f_{(n)} = f_{cs(n)} + q \left(f_{cc(n)} - f_{cs(n)} \right) \tag{3.1}$$

where symbols used have their usual meaning as explained in the preceding chapter .

Equation (3.1) can be re-written in terms of eigenvalues as follows .

$$\beta_{(n)} = \beta_{cs(n)} \sqrt{1 + q\left[\left(\frac{\beta_{cc(n)}}{\beta_{cs(n)}}\right)^2 - 1\right]} \qquad (3.2)$$

$\beta_{cs(n)}$ and $\beta_{cc(n)}$ can be computed from the relevant frequency equations which are as follows (2) .

$$1 - \text{Cos}\,\beta_{cc(n)}\ \text{Cosh}\,\beta_{cc(n)} = 0 \qquad (3.3)$$

$$\text{Cos}\,\beta_{cs(n)}\ \text{Sinh}\,\beta_{cs(n)} - \text{Sin}\,\beta_{cs(n)}\ \text{Cosh}\,\beta_{cs(n)} = 0 \qquad (3.4)$$

The first ten values of $\beta_{cs(n)}$ and $\beta_{cc(n)}$ are presented in chapter 6 .

The appropriate boundary conditions for this class of beam are ,

(1) At $\xi = 0$, we have ,

$$R(0) = 0 \qquad (3.5)$$

$$R'(0) = 0 \qquad (3.6)$$

(2) At $\xi = 1$, we have ,

$$R(1) = 0 \qquad (3.7)$$

$$R''(1) = -\frac{K_t L}{EI} R'(1)$$

$$= -\lambda_t R'(1) \qquad (3.8)$$

From substitution of boundary conditions (3.5) and (3.6) in equations (1.8) and (1.9) , we obtain ,

$$B = -D \qquad \text{and} \qquad A = -C \qquad (3.9)$$

Using (3.9) and substituting boundary conditions (3.7) and (3.8) in (1.8) and (1.10) , we obtain ,

$$C\,(\text{Sinh}\,\beta - \text{Sin}\,\beta) + D\,(\text{Cosh}\,\beta - \text{Cos}\,\beta) = 0 \qquad (3.10)$$

$$C\left(\sin\beta + \sinh\beta - \frac{\lambda_t}{\beta}(\cosh\beta - \cos\beta)\right) +$$

$$D\left(\cos\beta + \cosh\beta - \frac{\lambda_t}{\beta}(\sin\beta + \sinh\beta)\right) = 0 \qquad (3.11)$$

For a non-trivial solution for C and D in (3.10) and (3.11), we must obtain,

$$\begin{vmatrix} \sinh\beta - \sin\beta & \cosh\beta - \cos\beta \\ \sin\beta + \sinh\beta - \frac{\lambda_t}{\beta}(\cosh\beta - \cos\beta) & \cos\beta + \cosh\beta - \frac{\lambda_t}{\beta}(\sin\beta + \sinh\beta) \end{vmatrix} = 0 \qquad (3.12)$$

Expanding and simplifying (3.12),

$$\lambda_t = \frac{\beta(\sin\beta\cosh\beta - \sinh\beta\cos\beta)}{(1 - \cos\beta\cosh\beta)} \qquad (3.13)$$

Here again, we note that the values of dimensionless system parameter λ_t are completely characterised by the eigen values β which in turn are related to q by equation (3.2). It may be noted that equation (3·13) degenerates into the frequency equations of clamped - simple beams for $\lambda_t = 0$, and of clamped - clamped beams for $\lambda_t = \infty$.

Values of λ_t for various values of q (.05 ,.1,.2,.3... .9) are presented in the tables of results in chapter 6 .

Example -4

Determine the first mode frequency of the beam system described in example - 1 when one end of the beam is clamped and the other end is partly clamped with a torsional spring of stiffness 1500 lb.in./rad.

Solution

System parameter $\lambda_t = \frac{K_t L}{EI} = 3.24$

$f_{cs(1)} = 41.7$ (From example 1)

$\beta_{cc(1)} = 4.730$ (From table 11)

$$f_{cc(1)} = \frac{(\beta_{cc(1)})^2}{2\pi L^2} \sqrt{\frac{EI\,g}{\rho a}} = 61.0 \text{ Hz.}$$

For $\lambda_t = 3.24$, $q = 0.328$ (From table 4)

Hence ,

$$f_{(1)} = f_{cs(1)} + q\,(f_{cc(1)} - f_{cs(1)})$$

$$= 41.7 + 0.328\,(61.0 - 41.7) = \underline{48.0 \text{ Hz}}.$$

3.2 PINNED PARTLY CLAMPED BEAMS (FIG.5)

The two limiting cases of this class of beams are ,

(1) Simple - simple beams with $K_t = 0$

(2) Simple - clamped beams with $K_t = \infty$

Natural frequency of this type of beams can be expressed as follows in terms of q .

$$f_{(n)} = f_{ss(n)} + q\,(f_{sc(n)} - f_{ss(n)}) \qquad (3.14)$$

Equation(3.14) can be written down in terms of eigen values as follows .

$$\beta_{(n)} = \beta_{ss(n)} \sqrt{1 + q\left[\left(\frac{\beta_{sc(n)}}{\beta_{ss(n)}}\right)^2 - 1\right]} \qquad (3.15)$$

Eigenvalues $\beta_{ss(n)}$ and $\beta_{sc(n)}$ can be obtained from the relevant frequency equations which have already appeared earlier .The first ten values are presented in chapter 6 .

The proper boundary conditions for this class of beams are expressed as follows .

(1) At the pinned end where $\xi = 0$, we have ,

$$R(0) = 0 \tag{3.16}$$

$$R''(0) = 0 \tag{3.17}$$

(2) At partly clamped end where $\xi = 1$, we have ,

$$R(1) = 0 \tag{3.18}$$

$$R''(1) = -\frac{K_t L}{EI} R'(1)$$

$$= -\lambda_t R'(1) \tag{3.19}$$

From boundary conditions (3.16) and (3.17) , it follows that ,

$$B = D = 0 \tag{3.20}$$

Utilizing (3.20) and boundary conditions (3.18) and (3.19) we obtain ,

$$A\left(\frac{\lambda_t}{\beta}\cos\beta - \sin\beta\right) + C\left(\sinh\beta + \frac{\lambda_t}{\beta}\cosh\beta\right) = 0 \tag{3.21}$$

$$A \sin\beta + C \sinh\beta = 0 \tag{3.22}$$

For a non-trivial solution for A and C in (3.21) and (3.22) we should have ,

$$\begin{vmatrix} \frac{\lambda_t}{\beta}\cos\beta - \sin\beta & \sinh\beta + \frac{\lambda_t}{\beta}\cosh\beta \\ \sin\beta & \sinh\beta \end{vmatrix} = 0 \tag{3.23}$$

which on simplification yields ,

$$\lambda_t = \frac{\beta\,(\,2\sin\beta\sinh\beta\,)}{\cos\beta\sinh\beta - \sin\beta\cosh\beta} \tag{3.24}$$

For $\lambda_t = 0$, equation (3.24) degenerates into the frequency equation of simply - supported beams and for $\lambda_t = \infty$, it yields the frequency equation of clamped - simple beams as one would expect .

Utilizing (3.24) & (3.15) , the values of λ_t are calculated for various values of q for the first ten modes of vibration and are presented in chapter 6 .

Example -5

Determine the third mode frequency of the beam described in example-4 when one end of the beam is hinged instead of clamped .

Solution

The system parameter $\lambda_t = 3.24$ (Same as in example -4)

$\beta_{ss(3)} = 9.424$ & $\beta_{cs(3)} = 10.210$ (From table 11)

$\therefore f_{ss(3)} = 239.4$ Hz. & $f_{cs(3)} = 282.0$ Hz.

For $\lambda_t = 3.24$, $q = 0.175$ (From table 5 , third mode)

$$f_{(3)} = f_{ss(3)} + q\,(f_{cs(3)} - f_{ss(3)})$$

$$= 239.4 + 0.175\,(282.0 - 239.4) = \underline{246.8 \text{ Hz.}}$$

3.3 PARTLY CLAMPED - FREE BEAMS (FIG.6)

The two limiting cases of this class are ,

(1) Simple - free beams with $K_t = 0$

(2) Clamped - free beams with $K_t = \infty$

The natural frequency of this class of beams is expressed as follows .

$$f_{(n)} = f_{sF(n)} + q\,(f_{cF(n)} - f_{sF(n)}) \qquad (3.25)$$

Since the first mode frequency of simple -free beams is zero, the equation (3.25) takes the following form for first mode .

$$f_{(1)} = q\, f_{cF(1)} \tag{3.26}$$

Equations (3.26) and (3.25) assume the following form when expressed in terms of eigen values .

$$\beta_{(1)} = \beta_{cF(1)}\sqrt{q} \tag{3.27}$$

$$\beta_{(n)} = \beta_{sF(n)}\sqrt{1 + q\left[\left(\frac{\beta_{cF(n)}}{\beta_{sF(n)}}\right)^2 - 1\right]} \tag{3.28}$$

where $n > 1$

$\beta_{cF(n)}$ and $\beta_{sF(n)}$ are calculated from the pertinent frequency equations which have already appeared in the preceding pages .

The appropriate boundary conditions for this problem are ,

(1) At partly clamped end where $\xi = 0$, we have ,

$$R(0) = 0 \tag{3.29}$$

$$R''(0) = \frac{K_t L}{EI} R'(0)$$

$$= \lambda_t \ R'(0) \tag{3.30}$$

(2) At free end where $\xi = 1$,we have ,

$$R''(1) = 0 \tag{3.31}$$

$$R'''(1) = 0 \tag{3.32}$$

The boundary condition (3.29) gives ,

$$B = -D \tag{3.33}$$

Utilizing (3.33) and boundary conditions (3.30) , (3.31) and (3.32) , we obtain the following .

$$- A \sin\beta - B(\cos\beta + \cosh\beta) + C \sinh\beta = 0 \tag{3.34}$$

$$- A \cos\beta + B(\sin\beta - \sinh\beta) + C \cosh\beta = 0 \tag{3.35}$$

$$A \frac{\lambda_t}{\beta} + 2B + C \frac{\lambda_t}{\beta} = 0 \tag{3.36}$$

For a non-zero solution of A ,B and C in the above equations , we should have ,

$$\begin{vmatrix} -\sin\beta & -(\cos\beta + \cosh\beta) & \sinh\beta \\ -\cos\beta & (\sin\beta - \sinh\beta) & \cosh\beta \\ \frac{\lambda_t}{\beta} & 2 & \frac{\lambda_t}{\beta} \end{vmatrix} = 0 \tag{3.37}$$

which on expansion and simplification yields ,

$$\lambda_t = \frac{\beta(\sin\beta\cosh\beta - \sinh\beta\cos\beta)}{\cos\beta\cosh\beta + 1} \tag{3.38}$$

We note that for $\lambda_t = 0$, equation (3.38) yields the frequency equation of simple - free beams and for $\lambda_t = \infty$, it yields the frequency equation of clamped free beams . It can be further noted that the values of β which permit vanishing of the determinant are completely characterised by the dimensionless system parameter λ_t . Values of λ_t for various values of q (.05,.1,.2,.3,......... .9) for the first ten modes are presented in chapter 6 .

<u>Example - 6</u>

Determine the first mode frequency of the beam system described in example -5 when one end of the beam is free instead of hinged .

<u>Solution</u>

$f_{sF(1)} = 0$ Hz.

$f_{cF(1)} = 9.5$ Hz. (As already calculated in example -1)

System parameter $\lambda_t = 3.24$

For this λ_t , $q = 0.659$ (From table 6)

$$f_{(1)} = q\, f_{cF(1)}$$
$$= 0.659 \times 9.5 = \underline{6.3 \text{ Hz}}.$$

3.4 BEAMS WITH BOTH ENDS PARTLY CLAMPED (FIG.7)

The two limiting cases of this class of beams are ,

(1) Simple - Simple beams with $K^*_{t1} = K_{t2} = 0$

(2) Clamped - clamped beams with $K_{t1} = K_{t2} = \infty$

The natural frequency of this type of beam is given by the following equation .

$$f_{(n)} = f_{ss(n)} + q\,(\,f_{cc(n)} - f_{ss(n)}\,) \tag{3.39}$$

The above equation can be re-written in terms of eigenvalues as follows .

$$\beta_{(n)} = \beta_{ss(n)} \sqrt{1 + q\left[\left(\frac{\beta_{cc(n)}}{\beta_{ss(n)}}\right)^2 - 1\right]} \tag{3.40}$$

$\beta_{ss(n)}$ & $\beta_{cc(n)}$ can be computed from the relevant frequency equations as explained already . The first ten values are presented in chapter 6 .

The appropriate boundary conditions for this problem are as follows .

(1) At $\xi = 0$ we have ,

$$R(0) = 0 \tag{3.41}$$

$$R''(0) = \frac{K_{t1}\,L}{EI}\,R'(0) = \lambda_{t1}\,R'(0) \tag{3.42}$$

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(2) At $\xi = 1$, we have ,

$$R(1) = 0 \tag{3.43}$$

$$R''(1) = - \frac{K_{t2}L}{EI} R'(1)$$

$$= - \lambda_{t2} R'(1) \tag{3.44}$$

From boundary condition (3.41) , it follows that ,

$$B = - D \tag{3.45}$$

Utilizing (3.45) and boundary conditions (3.42) , (3.43) and (3.44) , we obtain ,

$$A \sin\beta + B (\cos\beta - \cosh\beta) + C \sinh\beta = 0 \tag{3.46}$$

$$A \frac{\lambda_{t1}}{\beta} + 2B + C \frac{\lambda_{t1}}{\beta} = 0 \tag{3.47}$$

$$A \left(\frac{\lambda_{t2}}{\beta} \cos\beta - \sin\beta \right) + B \left(- \frac{\lambda_{t2}}{\beta} (\sin\beta + \sinh\beta) - \cos\beta - \cosh\beta \right)$$
$$+ C \left(\frac{\lambda_{t2}}{\beta} \cosh\beta + \sinh\beta \right) = 0 \tag{3.48}$$

For a non-trivial solution for A , B and C in (3.46) , (3.47) and (3.48) , the determinant of their coefficient matrix must vanish . On expanding and simplifying the determinant and introducing $\alpha = \lambda_{t2} / \lambda_{t1}$, we get ,

$$\lambda_{t1}^2 + \frac{(1+\alpha)}{\alpha f_1(\beta)} \lambda_{t1} + \frac{f_2(\beta)}{\alpha f_1(\beta)} = 0 \tag{3.49}$$

where

$$f_1(\beta) = \frac{(1 - \cos\beta \cosh\beta)}{\beta (\sin\beta \cosh\beta - \cos\beta \sinh\beta)} \tag{3.50}$$

$$f_2(\beta) = \frac{\beta\,(2\,\mathrm{Sin}\,\beta\,\mathrm{Sinh}\,\beta)}{(\mathrm{Sin}\,\beta\,\mathrm{Cosh}\,\beta - \mathrm{Cos}\,\beta\,\mathrm{Sinh}\,\beta)} \qquad (3.51)$$

The parameter α is defined as the ratio of smaller system parameter λ_{t2} to the larger system parameter λ_{t1}, so that α is always $\leq$ 1.0 . Here again , because of similarity of end conditions , the natural frequencies are insensitive to the fact as to which one of λ_{t1} and λ_{t2} is greater .

Equation (3.49) being quadratic in nature yields two values of λ_{t1} for every value of β, but one being negative is physically inadmissible and hence discarded .

Values of λ_{t1} for first ten modes for various values of α (.1,.2,.3.....,1.0) and q (.05,.1,.2,.3,......, .9) have been computed and are presented in chapter 6.

Example -7

Determine the fundamental natural frequency of the beam described in example - 4 when both ends of the beam are partly clamped with two torsional springs of stiffnesses 1500 lb.in./rad. each .

Solution

$\alpha = 1.0$

$f_{ss(1)} = 26.6$ Hz. (As already calculated in example - 2)

$f_{cc(1)} = 61.0$ Hz. (As calculated in example - 4)

For $\alpha = 1.0$ and $\lambda_{t1} = 3.24$, we have $q = 0.325$ (From table 7.10)

$$f_{(1)} = f_{ss(1)} + q\,(f_{cc(1)} - f_{ss(1)})$$

$$= 26.6 + 0.325\,(61.0 - 26.6)$$

$$= \underline{37.8 \text{ Hz.}}$$

CHAPTER 4

BEAMS WITH BOTH TORSIONAL AND LINEAR SPRINGS AS ELASTIC SUPPOTRS (FIG.8)

Beams of this class have a torsional spring on the hinged end and a linear spring on the free end so that one end becomes a partly clamped end and the other end becomes partly supported . The two limiting cases of this class of beams are ,

(1) Simple - free beams with $K_t = K_L = 0$

(2) Clamped - simple beams with $K_t = K_L = \infty$

Natural frequency of this type of beams can be expressed as follows .

$$f_{(n)} = f_{sF(n)} + q \left(f_{cs(n)} - f_{sF(n)} \right) \tag{4.1}$$

Since $f_{sF(1)} = 0$, equation (4.1) can be re-written as follows for the first mode .

$$f_{(1)} = q\, f_{cs(1)} \tag{4.2}$$

Expressing equations (4.2) and (4.1) in terms of eigenvalues we obtain ,

$$\beta_{(1)} = \beta_{cs(1)} \sqrt{q} \tag{4.3}$$

$$\beta_{(n)} = \beta_{sF(n)} \sqrt{1 + q \left[\left(\frac{\beta_{cs(n)}}{\beta_{sF(n)}} \right)^2 - 1 \right]} \tag{4.4}$$

where $n > 1$

The appropriate boundary conditions for this class of beams are as follows .

(1) At $\xi = 0$, we have ,

$$R(0) = 0 \tag{4.5}$$

$$R''(0) = \frac{K_t L}{EI} R'(0)$$

$$= \lambda_t R'(0) \tag{4.6}$$

(2) At $\xi = 1$, we have ,

$$R''(1) = 0 \tag{4.7}$$

$$R'''(1) = \frac{K_L L^3}{EI} R(1)$$

$$= \lambda_L R(1) \tag{4.8}$$

From boundary condition (4.5) it follows that ,

$$B = - D \tag{4.9}$$

Utilizing (4.9) and boundary conditions (4.6) ,(4.7) and (4.8) we obtain the following three equations in A , B and C .

$$A \frac{\lambda_t}{\beta} + 2 B + C \frac{\lambda_t}{\beta} = 0 \tag{4.10}$$

$$A \operatorname{Sin}\beta + B (\operatorname{Cos}\beta + \operatorname{Cosh}\beta) + C (- \operatorname{Sinh}\beta) = 0 \tag{4.11}$$

$$A \left(\frac{\lambda_L}{\beta^3} \operatorname{Sin}\beta + \operatorname{Cos}\beta\right) + B \left(\frac{\lambda_L}{\beta^3} (\operatorname{Cos}\beta - \operatorname{Cosh}\beta) - (\operatorname{Sin}\beta - \operatorname{Sinh}\beta)\right)$$

$$+ C \left(\frac{\lambda_L}{\beta^3} \operatorname{Sinh}\beta - \operatorname{Cosh}\beta \right) = 0 \tag{4.12}$$

The condition for non-triviality of solution for A , B and C in equations (4.10) , (4.11) and (4.12) yields ,

$$\begin{vmatrix} \frac{\lambda_t}{\beta} & 2 & \frac{\lambda_t}{\beta} \\ \sin\beta & (\cos\beta + \cosh\beta) & -\sinh\beta \\ \frac{\lambda_L}{\beta^3}\sin\beta + \cos\beta & \frac{\lambda_L}{\beta^3}(\cos\beta - \cosh\beta) - (\sin\beta - \sinh\beta) & \frac{\lambda_L}{\beta^3}\sinh\beta - \cosh\beta \end{vmatrix} = 0 \qquad (4.13)$$

Here, unlike the case in which both ends have similar springs, either torsional or linear, the frequencies are not insensitive to the fact that which one of λ_t and λ_L is larger. Therefore there are two cases to be considered here.

Case-1 when λ_t is larger than λ_L.

Introducing $\alpha = \frac{\lambda_L}{\lambda_t}$, so that α is always less than 1.0 and expanding the determinant of equation (4.13), we obtain the following equation in λ_t.

$$\lambda_t^2 + \lambda_t \left(\frac{-4\beta \sin\beta \sinh\beta - (2\beta^3/\alpha)\,(1 + \cos\beta\cosh\beta)}{2\cos\beta\sinh\beta - 2\sin\beta\cosh\beta} \right) - \frac{\beta^4}{\alpha} = 0 \qquad (4.14)$$

Equation (4.14) being quadratic yields two values of λ_t for every value of β but one being negative is discarded. The values of λ_t calculated for this case for various values of and q for the first ten modes are presented in chapter 6.

Case - 2 when λ_L is larger than λ_t.

In this case $\alpha = \frac{\lambda_t}{\lambda_L}$. Utilizing α and expanding the determinant of equation (4.13), we obtain the following quadratic equation in λ_L.

$$\lambda_L^2 + \lambda_L \left(\frac{((-4\beta \sin\beta \sinh\beta)/\alpha) - 2\beta^3 (1 + \cos\beta \cosh\beta)}{2 \cos\beta \sinh\beta - 2 \sin\beta \cosh\beta} \right) - \frac{\beta^4}{\alpha} = 0 \quad (4.15)$$

Here again , one value of λ_L being negative is discarded . The values of λ_L for various values of α and q , for first ten modes of vibration are presented in tabular form in chapter 6 .

The following two examples are given to illustrate these two cases .

Example - 8(a)

Determine the first mode frequency of the beam described in example-1 when one end of the beam is partly clamped with a torsional spring of stiffness 1500 lb.in./rad. and the other end is partly supported with a linear spring of stiffness 8.2 lb./in.

Solution

$\lambda_t = 3.24$ and $\lambda_L = 15.4$ (As calculated in previous examples)

In this case

$$\alpha = \frac{3.24}{15.4} = 0.21$$

For $\alpha = 0.21$ and $\lambda_L = 15.4$, $q = 0.442$ (From tables 8.2 & 8.3)

This value of q has been obtained by linear interpolation and it will be shown in chapter 6 that linear interpolation yields results accurate within the desired accuracy .

$f_{sF(1)} = 0$ & $f_{cs(1)} = 41.7$ (From example - 1)

$f_{(1)} = q\, f_{cs(1)}$

$= 0.442 \times 41.7 = \underline{18.5 \text{ Hz}}.$

Example - 8(b)

Determine the first mode frequency of the beam system described in example - 8(a) when system parameter $\lambda_t = 15.4$ and $\lambda_L = 3.24$.

Solution

$\alpha = 0.21$ as before

Here $\lambda_t > \lambda_L$, hence q is obtained from tables 9.2 &9.3 .

$q = 0.3$

Hence $f_{(1)} = q\, f_{cs(1)}$

$= 0.3 \times 41.7 = \underline{12.5 \text{ Hz}}$.

CHAPTER 5

CLAMPED - FREE BEAMS WITH A CONCENTRATED MASS AT THE FREE END (FIG.9)

As the concentrated mass at the free end of a cantilever beam approaches infinity , the behaviour of the system approaches to that of a simple spring - mass system and as such its first mode frequency approaches zero . As the concentrated mass is made smaller and smaller , the first mode frequency increases , reaching to a value corresponding to clamped - free end conditions . Therefore the two limiting cases for the first mode of this class of beams are ,

(1) Simple spring - mass system with $M = \infty$

(2) Clamped - free beams with $M = 0$

The first mode natural frequency of such beams can be expressed as follows .

$$f_{(1)} = q\, f_{cF(1)} \tag{5.1}$$

In higher modes of vibration , the end with the concentrated mass becomes a nodal point and behaves as a simply supported end when the concentrated mass approaches infinity . Therefore for higher modes ,the two limiting cases are ,

(1) Clamped - simple beams with $M = \infty$

(2) Clamped - free beams with $M = 0$

The higher mode natural frequencies can be expressed as follows .

$$f_{(n)} = f_{cs(n-1)} + q\,(f_{cF(n)} - f_{cs(n-1)}) \qquad (5.2)$$

where $n>1$.

Equations (5.1) and (5.2) can be written down in terms of eigenvalues as follows .

$$\beta_{(1)} = \beta_{cF(1)}\sqrt{q} \qquad (5.3)$$

$$\beta_{(n)} = \beta_{cs(n-1)}\sqrt{1 + q\left[\left(\frac{\beta_{cF(n)}}{\beta_{cs(n-1)}}\right)^2 - 1\right]} \qquad (5.4)$$

where $n > 1$

The appropriate boundary conditions for this class of beams are as follows .

(1) At $\xi = 0$, we have ,

$$R(0) = 0 \qquad (5.5)$$

$$R'(0) = 0 \qquad (5.6)$$

(2) At $\xi = 1$, we have ,

$$R''(1) = 0 \qquad (5.7)$$

$$R'''(1) = -\frac{\beta^4}{\lambda_m} R(1) \qquad (5.8)$$

where system parameter $\lambda_m = \frac{\rho L a}{M}$, where ρ is mass of beam per unit volume , L is the length of the beam , a is area of cross section and M is the concentrated mass at the free end .

The boundary condition (5.8) has been derived in appendix 3 .

System parameter λ_m varies from 0 to ∞ as M varies from ∞ to 0 .

From boundary conditions (5.5) & (5.6) it follows that ,

$$B = -D \qquad \text{and} \qquad A = -C \tag{5.9}$$

Utilizing (5.9) and boundary conditions (5.7) and (5.8) we obtain the following two equations in C and D .

$$C\ (\ \text{Sin}\,\beta + \text{Sinh}\,\beta\ ) + D\ (\ \text{Cos}\,\beta + \text{Cosh}\,\beta\ ) = 0 \tag{5.10}$$

$$C\ \left(\ \text{Cos}\,\beta + \text{Cosh}\,\beta - \frac{\beta}{\lambda_m}\ (\ \text{Sin}\,\beta - \text{Sinh}\,\beta\ )\right)$$

$$+ D\ \left(\ \text{Sinh}\,\beta - \text{Sin}\,\beta - \frac{\beta}{\lambda_m}\ (\ \text{Cos}\,\beta - \text{Cosh}\,\beta\ )\right) = 0 \tag{5.11}$$

For a non-trivial solution for C and D in equations (5.10) and (5.11) , the determinant of their coefficient matrix must vanish .

$$\begin{vmatrix} \text{Sin}\,\beta + \text{Sinh}\,\beta & \text{Cos}\,\beta + \text{Cosh}\,\beta \\ \text{Cos}\,\beta + \text{Cosh}\,\beta - \frac{\beta}{\lambda_m}(\text{Sin}\,\beta - \text{Sinh}\,\beta) & \text{Sinh}\,\beta - \text{Sin}\,\beta - \frac{\beta}{\lambda_m}(\text{Cos}\,\beta - \text{Cosh}\,\beta) \end{vmatrix} = 0 \tag{5.12}$$

Expanding the determinant and simplifying , we get ,

$$\lambda_m = \frac{\beta\ (\ \text{Sin}\,\beta\,\text{Cosh}\,\beta - \text{Sinh}\,\beta\,\text{Cos}\,\beta\ )}{(\ 1 + \text{Cos}\,\beta\,\text{Cosh}\,\beta\ )} \tag{5.13}$$

We note here that the system parameter λ_m which is the ratio of the mass of the beam to the concentrated mass is completely characterised by β . It is further observed that for $\lambda_m = 0$,equation (5.13) yields the frequency equation of clamped simple beams and for $\lambda_m = \infty$, it gives the frequency equation of clamped free beams which is anticipated .

The values of λ_m for various values of q for first ten modes are presented in chapter 6 . The following example is given by way of illustration .

Example - 9

Determine the first mode frequency of the beam described in example - 1 , when the free end of the beam carries a concentrated mass of 0.25 lb.

Solution

Mass of the beam $= \rho aL$

$= 0.0965 \times 0.25 \times 29.5$

$= 0.71$ lb.

System parameter $\lambda_m = \dfrac{0.71}{0.25} = 2.85$

For $\lambda_m = 2.85$, $q = 0.635$ (From table 10)

Hence , $f_{(1)} = q \, f_{cF(1)}$

$= 0.635 \times 9.5 = \underline{6.0 \text{ Hz}}$.

CHAPTER 6

RESULTS AND DISCUSSIONS

6.1 TABULAR PRESENTATION OF RESULTS

The tables of results appear at the end of this chapter . It is felt that graphical presentation of results is not very effective particularly for higher modes . To show this , the graphs are presented for the first three modes of vibration of clamped - elastically supported beams in figure 11 . It will be noticed that for higher modes , the variation of system parameter λ_L becomes too high to be presented graphically with reasonable accuracy . This is also true for the rest of the beams considered in this thesis . It is for this reason that tabular presentation of results has been preferred to the graphical presentation .

6.2 THE ACCURACY STUDIES

The parameter q has been varied from 0.1 to 0.9 and α has been varied from 0.1 to 1.0 in steps of 0.1. Results for intermediate values of α and q are obtained by linear interpolation.Accuracy tests were introduced at specific values (0.15,0.25,..etc) of α and q and it was observed that results are accurate within 1% except in the immediate neighbourhood of a zero natural frequency where the error tends to be infinite but however this region of uncertainty is small and linear interpolation yields accuracy within 3% from q=0.05 to q=0.2 , and beyond q=0.2 , they yield better accuracies.The fractional frequency error introduced by the uncertainty in beam properties also tends to be infinite in the immediate neighbourhood of zero natural frequenc

The figure of 1 % for the desired accuracy has been chosen because generally the uncertainty in beam properties is of the same order.

Accuracy tests were conducted by comparing the interpolated results with computed results for specific values of q and viz: 0.05,0.15, -------- 0 95. Some illustrative accuracy checks on clamped-elastically supported beams are presented below.

Let us assume that the linear interpolation gives an error Δq in q .This error can be expressed as a percentage error in frequency as follows .

$$\% \text{ error in frequency }, \; \Delta f_{(n)} = \frac{100 \Delta q \, (f_{cs(n)} - f_{cF(n)})}{(f_{cF(n)} + q \, (f_{cs(n)} - f_{cF(n)}))} \qquad (6.1)$$

where the symbols used have their usual meaning .

Equation (6.1) can be re-written in terms of eigenvalues as follows

$$\Delta f_{(n)} = \frac{100 \quad \Delta q}{q + \left(\dfrac{1}{(\beta_{cs(n)} / \beta_{cF(n)})^2 - 1} \right)}$$

$$= \frac{100 \; \Delta q}{q + Y_n} \qquad (6.2)$$

$$\text{where } Y_n = \frac{1}{(\beta_{cs(n)} / \beta_{cF(n)})^2 - 1} \qquad (6.3)$$

The values of Y_n for the first ten modes are given below .

n	1	2	3	4	5	6	7	8	9	10
Y_n	0.294	0.780	1.45	2.08	2.63	3.49	4.17	4.54	5.55	6.13

Since Y_n goes on increasing for higher modes , the percentage frequency error Δf for the same error Δq in q ,goes on diminishing .

In cases of beams where the first mode starts with zero frequency , equation (6.2) assumes the following form .

$$\Delta f_{(1)} = \frac{100 \ \Delta q}{q} \qquad (6.4)$$

which tends to infinity as q approaches zero . However this region of uncertainty is small and beyond this region the accuracy is good .

As an example , the calculations for accuracy checks applied for the first mode frequencies of clamped - elastically supported beams at q = .35 , .45 , .55 , are produced below .

The values of λ_L for q = .35 , .45 & .55 obtained by direct computation are given below .

q	0.35	0.45	0.55
λ_L	13.41	20.37	30.02

The values of q obtained for these values of λ_L by linear interpolation are 0.347 , 0.445 and 0.543 as against the exact values of 0.35 , 0.45 & 0.55 respectively . The error introduced by linear interpolation is 0.003 for q = 0.35 , 0.005 for q = 0.45 and 0.007 for q =0.55 which when reduced to percentage frequency error using equation (6.2) comes out to be less than 1 % in each case

The linear interpolation becomes impossible between $q = 0.9$ and 1.0 because system parameter λ_L becomes infinite at $q = 1.0$. Therefore a simple algebraic equation is utilized for determining the values of q for all such system parameters associated with this range of q . The equation utilized is given below .

$$q = 1 - \frac{0.2 \lambda_{0.8}}{\lambda} \qquad (6.5)$$

where $\lambda_{0.8}$ = System parameter at $q = 0.8$ obtained from tables of results.

λ = System parameter of interest .

The derivation of equation (6.5) appears in appendix 4 .

It is noticed that equation (6.5) gives values of q which give frequencies accurate within 1% . As an example , the calculations conducted for the second mode of clamped elastically supported beams are presented . The values of λ_L obtained for q = 0.80, 0.85, 0.90, and 0.95 by direct computation are given below .

q	0.80	0.85	0.90	0.95
λ_L	492.0	652.0	961.5	1868.0

The values of q obtained for the last three values of λ_L by using equation (6.5) are 0.849, 0.898 & 0.948 as against the exact values of 0.85, 0.90 & 0.95 respectively .This shows that the use of equation (6.5) is consistent with the desired accuracy of 1% in frequency .

The accuracy checks for α were conducted by comparing the computed results with the ones obtained by interpolation . The graphs for two such checks for $\alpha = 0.15$ and 0.35 , for beams with both ends partly clamped are presented in figures 12, 13, 14 and 15 by way of illustration .

The point of interest in each case is marked as P and the points between which it is interpolated are marked P_1 and P_2 . The exact value of q at P is compared with the interpolated value of q at P .The interpolated value of q at P is obtained by the following equation .

$$q = \frac{q_1 + q_2}{2} \qquad (6.6)$$

where q_1 and q_2 are the values of q at P_1 and P_2 respectively . The sample calculations are shown below .

For $\alpha = 0.15$

At $q = 0.35$, $\Delta q = 0.002$

$q = 0.734$, $\Delta q = 0.002$ where Δq is the difference of exact and interpolated values of q .

The percentage frequency error for this Δq is determined using equation (6.2) and is found to be less than 1% .

For $\alpha = 0.35$

At $q = 0.3$, $\Delta q = 0.002$

and at $q = 0.65$, $\Delta q = 0.003$.

And the percentage frequency error in both these cases using equation (6.2) is found to be less than 1% .

6.3 DISCUSSIONS AND SUGGESTIONS FOR FURTHER WORK

Results presented in this thesis have been arrived at by using the exact solution based on elementary beam theory . They, therefore, should be used judiciously and not without regard to the limitations imposed by the assumptions implied . For example the assumption of neglecting the effects of shear deformation may not remain valid at higher modes . Because a beam vibrating in higher modes has many nodes formed on its axis ,and the

the length between the two consective nodes goes on diminishing with higher modes and as such the assumption that cross sectional dimensions of the beam are small compared to its length may no longer remain valid in higher modes .

The number of digits required after the decimal point goes on reducing as the system parameter λ varies from 0.0 to ∞ for a certain desired accuracy . For a desired accuracy of 1% , we should have four correct digits after the decimal place for a value of $\lambda \geq 0.01$ which is always the case in the present thesis . However for $\lambda \geq 100.0$, we simply don't need any digit after the decimal place for 1% accuracy . Results presented in the exponential format of fortran IV language takes care of this requirement .

Since the frequencies are proportional to the square of corresponding eigenvalues , an accuracy of 1% in frequency requires eigenvalues to be accurate within 0.5% . Therefore eigenvalues are presented correct upto three decimal places .

The eigenvalues for modes higher than 6^{th} are not readily available in the existing literature but sometimes the knowledge of higher modes is necessary,for example , to determine whether or not rotating machinery will cause vibration problems . A step in this direction has been taken by considering the first ten modes of vibration .

The eigenvalues for the first ten modes are presented in table 11. Due care should be exercised in determining whether rotary inertia and shear deformation terms are predominant because in that case, the table 11 will give inaccurate results For modes higher than tenth the eigenvalues almost coincide with each other because a beam vibrating with many nodes is almost insensitive to the end conditions.

The present work also yields an alternate method of determinin the spring stiffness.

In case of free - free beams with linear springs at both ends , it is seen that, in certain ranges of frequency , both symmetric and anti-symmetric modes are present at the same frequency . The designer should therefore ., exercise caution that in trying to avoid the resonance at this frequency ,the changed system parameter does not hit the other mode which again has the same frequency . In figure 10 , the system parameters λ_1 and λ_2 has the same frequency . Therefore to avoid resonance λ_1 should not be changed to λ_2 and vice-versa .

The results presented provide the necessary information for immediate establishment of the first ten modes of vibration frequencies for any uniform elastically supported beams . It is hoped that these tables will save considerable time and labour to the designers and engineers dealing with elastically supported beams .

We have considered here beams on two supports only. In further work, it is suggested, that this technique be extended to the contineous beams with elastic supports . This technique can also be applied to the vibrations of plates and shells with elastic boundary conditions as well as tapered beams , beams with section discontinuities , or masses located anywhere on the beam .

It may have been possible to give tables of λ vs β instead of λ vs q , but the method adopted in this thesis has the obvious advantage that q varies from 0.0 to 1.0 between the two limiting cases whereas β varies between two irrational numbers between the two limiting cases. Besides, the frequencies are given at regularly spaced intervals between the limiting cases.

Q :	0.05	0.10	0.20	0.30	0.40	0.50	0.60	0.70	0.80	0.90
MODE(1)	0.1146E 01	0.2509E 01	0.5948E 01	0.1054E 02	0.1664E 02	0.2493E 02	0.3677E 02	0.5529E 02	0.8996E 02	0.1880E 03
MODE(2)	0.1455E 02	0.2822E 02	0.5575E 02	0.8625E 02	0.1225E 03	0.1682E 03	0.2299E 03	0.3230E 03	0.4922E 03	0.9616E 03
MODE(3)	0.6066E 02	0.1148E 03	0.2155E 03	0.3178E 03	0.4319E 03	0.5695E 03	0.7504E 03	0.1018E 04	0.1501E 04	0.2838E 04
MODE(4)	0.1568E 03	0.2573E 03	0.5462E 03	0.7881E 03	0.1049E 04	0.1356E 04	0.1753E 04	0.2336E 04	0.3384E 04	0.6281E 04
MODE(5)	0.3266E 03	0.6120E 03	0.1111E 04	0.1583E 04	0.2082E 04	0.2660E 04	0.3401E 04	0.4481E 04	0.6417E 04	0.1177E 05
MODE(6)	0.5883E 03	0.1095E 04	0.1972E 04	0.2789E 04	0.3639E 04	0.4614E 04	0.5854E 04	0.7654E 04	0.1088E 05	0.1980E 05
MODE(7)	0.9620E 03	0.1782E 04	0.3193E 04	0.4492E 04	0.5828E 04	0.7349E 04	0.9272E 04	0.1206E 05	0.1704E 05	0.3084E 05
MODE(8)	0.1465E 04	0.2709E 04	0.4837E 04	0.6776E 04	0.8756E 04	0.1100E 05	0.1382E 05	0.1789E 05	0.2518E 05	0.4536E 05
MODE(9)	0.2118E 04	0.3912E 04	0.6965E 04	0.9728E 04	0.1253E 05	0.1569E 05	0.1965E 05	0.2536E 05	0.3557E 05	0.6386E 05
MODE(10)	0.2942E 04	0.5427E 04	0.9642E 04	0.1343E 05	0.1726E 05	0.2155E 05	0.2692E 05	0.3467E 05	0.4849E 05	0.8681E 05

Table 1.0

The system parameter λ_L vs q for clamped - elastically supported beams.

q :	0.05	0.10	0.20	0.30	0.40	0.50	0.60	0.70	0.80	0.90
MODE(1)	0.9130E-01	0.3267E 00	0.1332E 01	0.3101E 01	0.5806E 01	0.9784E 01	0.1573E 02	0.2529E 02	0.4338E 02	0.9470E 02
MODE(2)	0.9912E 01	0.1755E 02	0.3559E 02	0.5630E 02	0.8149E 02	0.1137E 03	0.1578E 03	0.2246E 03	0.3466E 03	0.6859E 03
MODE(3)	0.4504E 02	0.8558E 02	0.1620E 03	0.2409E 03	0.3299E 03	0.4382E 03	0.5813E 03	0.7939E 03	0.1178E 04	0.2242E 04
MODE(4)	0.1293E 03	0.2407E 03	0.4440E 03	0.6434E 03	0.8599E 03	0.1116E 04	0.1449E 04	0.1937E 04	0.2816E 04	0.5248E 04
MODE(5)	0.2784E 03	0.5190E 03	0.9444E 03	0.1349E 04	0.1779E 04	0.2279E 04	0.2920E 04	0.3857E 04	0.5536E 04	0.1019E 05
MODE(6)	0.5149E 03	0.9563E 03	0.1726E 04	0.2445E 04	0.3195E 04	0.4058E 04	0.5156E 04	0.6754E 04	0.9615E 04	0.1753E 05
MODE(7)	0.8576E 03	0.1589E 04	0.2851E 04	0.4015E 04	0.5216E 04	0.6585E 04	0.8318E 04	0.1083E 05	0.1533E 05	0.2777E 05
MODE(8)	0.1326E 04	0.2452E 04	0.4383E 04	0.6146E 04	0.7949E 04	0.9991E 04	0.1257E 05	0.1629E 05	0.2295E 05	0.4138E 05
MODE(9)	0.1940E 04	0.3583E 04	0.6385E 04	0.8923E 04	0.1150E 05	0.1441E 05	0.1806E 05	0.2333E 05	0.3275E 05	0.5885E 05
MODE(10)	0.2719E 04	0.5017E 04	0.8918E 04	0.1243E 05	0.1598E 05	0.1997E 05	0.2496E 05	0.3216E 05	0.4501E 05	0.8064E 05

Table 2.0

The system parameter λ_L vs q for pinned elastically supported beams .

ALPHA : 0.1

Q :	0.05	0.10	0.20	0.30	0.40	0.50	0.60	0.70	0.80	0.90
SYMMETRIC MODES										
MODE(1)	0.8350E 00	0.3356E 01	0.1370E 02	0.3193E 02	0.5994E 02	0.1014E 03	0.1638E 03	0.2652E 03	0.4596E 03	0.1018E 04
MODE(3)	0.3203E 02	0.6494E 02	0.1504E 03	0.3041E 03	0.6890E 03	0.1554E 04	0.2781E 04	0.4474E 04	0.7288E 04	0.1449E 05
MODE(5)	0.3023E 03	0.5617E 03	0.1098E 04	0.1862E 04	0.3460E 04	0.7465E 04	0.1361E 05	0.2140E 05	0.3347E 05	0.6321E 05
MODE(7)	0.1081E 04	0.1568E 04	0.3667E 04	0.5895E 04	0.1022E 05	0.2118E 05	0.3882E 05	0.6040E 05	0.9260E 05	0.1706E 06
MODE(9)	0.2637E 04	0.4759E 04	0.8677E 04	0.1359E 05	0.2278E 05	0.4601E 05	0.8451E 05	0.1307E 06	0.1982E 06	0.3599E 06
ANTI-SYMMETRIC MODES										
MODE(2)	0.9563E 00	0.3924E 01	0.1765E 02	0.5200E 02	0.1541E 03	0.3797E 03	0.7200E 03	0.1232E 04	0.2132E 04	0.4503E 04
MODE(4)	0.1203E 03	0.2291E 03	0.4715E 03	0.8466E 03	0.1684E 04	0.3735E 04	0.6775E 04	0.1075E 05	0.1709E 05	0.3296E 05
MODE(6)	0.6113E 03	0.1121E 04	0.2129E 04	0.3496E 04	0.6231E 04	0.1314E 05	0.2404E 05	0.3757E 05	0.5807E 05	0.1081E 06
MODE(8)	0.1745E 04	0.3160E 04	0.5816E 04	0.9209E 04	0.1567E 05	0.3199E 05	0.5873E 05	0.9107E 05	0.1387E 06	0.2535E 06
MODE(10)	0.3790E 04	0.6819E 04	0.1235E 05	0.1918E 05	0.3180E 05	0.6363E 05	0.1169E 06	0.1806E 06	0.2726E 06	0.4925E 06

Table 3.1

The system parameter λ_{L1} vs q for beams with both ends elastically supported .

ALPHA : 0.2

SYMMETRIC MODES

C :	0.05	0.10	0.20	0.30	0.40	0.50	0.60	0.70	0.80	0.90
MODE(1)	0.4302E 00	0.1730E 01	0.7067E 01	0.1650E 02	0.3105E 02	0.5249E 02	0.8554E 02	0.1394E 03	0.2437E 03	0.5463E 03
MODE(3)	0.3010E 02	0.6152E 02	0.1412E 03	0.2695E 03	0.5071E 03	0.9176E 03	0.1511E 04	0.2371E 04	0.3843E 04	0.7704E 04
MODE(5)	0.2833E 03	0.5315E 03	0.1037E 04	0.1701E 04	0.2802E 04	0.4733E 04	0.7548E 04	0.1140E 05	0.1765E 05	0.3354E 05
MODE(7)	0.1012E 04	0.1861E 04	0.3468E 04	0.5422E 04	0.8498E 04	0.1387E 05	0.2174E 05	0.3224E 05	0.4983E 05	0.9046E 05
MODE(9)	0.2468E 04	0.4497E 04	0.8210E 04	0.1254E 05	0.1916E 05	0.3067E 05	0.4761E 05	0.6990E 05	0.1045E 06	0.1908E 06

ANTI-SYMMETRIC MODES

MODE(2)	0.9265E 00	0.3787E 01	0.1665E 02	0.4515E 02	0.1062E 03	0.2171E 03	0.3878E 03	0.6517E 03	0.1125E 04	0.2398E 04
MODE(4)	0.1129E 03	0.2169E 03	0.4447E 03	0.7671E 03	0.1325E 04	0.2304E 04	0.3727E 04	0.5715E 04	0.9013E 04	0.1750E 05
MODE(6)	0.5726E 03	0.1061E 04	0.2012E 04	0.3207E 04	0.5127E 04	0.8489E 04	0.1341E 05	0.2003E 05	0.3062E 05	0.5734E 05
MODE(8)	0.1633E 04	0.2988E 04	0.5502E 04	0.8487E 04	0.1311E 05	0.2116E 05	0.3300E 05	0.4866E 05	0.7317E 05	0.1344E 06
MODE(10)	0.3547E 04	0.6445E 04	0.1169E 05	0.1772E 05	0.2685E 05	0.4267E 05	0.6603E 05	0.9658E 05	0.1438E 06	0.2611E 06

Table 3.2

ALPHA : 0.3

SYMMETRIC MODES

Q :	0.05	0.10	0.20	0.30	0.40	0.50	0.60	0.70	0.80	0.90
MODE(1)	0.2967E 00	0.1193E 01	0.4878E 01	0.1141E 02	0.2151E 02	0.3663E 02	0.5972E 02	0.9785E 02	0.1723E 03	0.3899E 03
MODE(3)	0.2829E 02	0.5820E 02	0.1326E 03	0.2435E 03	0.4206E 03	0.6939E 03	0.1088E 04	0.1676E 04	0.2706E 04	0.5459E 04
MODE(5)	0.2659E 03	0.5024E 03	0.9785E 03	0.1567E 04	0.2418E 04	0.3699E 04	0.5511E 04	0.8088E 04	0.1242E 05	0.2373E 05
MODE(7)	0.9489E 03	0.1758E 04	0.3277E 04	0.5018E 04	0.7422E 04	0.1098E 05	0.1597E 05	0.2292E 05	0.3438E 05	0.6397E 05
MODE(9)	0.2313E 04	0.4248E 04	0.7760E 04	0.1163E 05	0.1683E 05	0.2446E 05	0.3510E 05	0.4975E 05	0.7361E 05	0.1349E 06

ANTI-SYMMETRIC MODES

Q :	0.05	0.10	0.20	0.30	0.40	0.50	0.60	0.70	0.80	0.90
MODE(2)	0.8952E 00	0.3644E 01	0.1569E 02	0.4032E 02	0.8563E 02	0.1615E 03	0.2775E 03	0.4600E 03	0.7927E 03	0.1702E 04
MODE(4)	0.1060E 03	0.2051E 03	0.4192E 03	0.7027E 03	0.1129E 04	0.1780E 04	0.2707E 04	0.4049E 04	0.6345E 04	0.1239E 05
MODE(6)	0.5370E 03	0.1002E 04	0.1900E 04	0.2962E 04	0.4457E 04	0.6687E 04	0.9823E 04	0.1423E 05	0.2156E 05	0.4056E 05
MODE(8)	0.1531E 04	0.2822E 04	0.5200E 04	0.7863E 04	0.1149E 05	0.1683E 05	0.2429E 05	0.3462E 05	0.5153E 05	0.9505E 05
MODE(10)	0.3324E 04	0.6087E 04	0.1105E 05	0.1644E 05	0.2363E 05	0.3413E 05	0.4874E 05	0.6877E 05	0.1013E 06	0.1846E 06

Table 3.3

ALPHA : 0.4

SYMMETRIC MODES

ζ :	0.05	0.10	0.20	0.30	0.40	0.50	0.60	0.70	0.80	0.90
MODE(1)	0.2209E 00	0.9299E 00	0.3801E 01	0.8903E 01	0.1683E 02	0.2873E 02	0.4701E 02	0.7738E 02	0.1370E 03	0.3123E 03
MODE(3)	0.2661E 02	0.5504E 02	0.1246E 03	0.2229E 03	0.3672E 03	0.5768E 03	0.8775E 03	0.1332E 04	0.2144E 04	0.4349E 04
MODE(5)	0.2497E 03	0.4748E 03	0.9235E 03	0.1452E 04	0.2156E 04	0.3131E 04	0.4483E 04	0.6450E 04	0.9847E 04	0.1889E 05
MODE(7)	0.9910E 03	0.1661E 04	0.3096E 04	0.4667E 04	0.6662E 04	0.9363E 04	0.1305E 05	0.1831E 05	0.2726E 05	0.5089E 05
MODE(9)	0.2171E 04	0.4011E 04	0.7333E 04	0.1083E 05	0.1516E 05	0.2093E 05	0.2874E 05	0.3977E 05	0.5836E 05	0.1073E 06

ANTI-SYMMETRIC MODES

MODE(2)	0.8615E 00	0.3494E 01	0.1480E 02	0.3666E 02	0.7395E 02	0.1329E 03	0.2227E 03	0.3653E 03	0.6287E 03	0.1357E 04
MODE(4)	0.9959E 02	0.1939E 03	0.3953E 03	0.6491E 03	0.1000E 04	0.1497E 04	0.2195E 04	0.3225E 04	0.5029E 04	0.9862E 04
MODE(6)	0.5043E 03	0.9470E 03	0.1795E 04	0.2751E 04	0.3989E 04	0.5684E 04	0.8010E 04	0.1136E 05	0.1709E 05	0.3227E 05
MODE(8)	0.1437E 04	0.2665E 04	0.4913E 04	0.7320E 04	0.1033E 05	0.1438E 05	0.1987E 05	0.2766E 05	0.4084E 05	0.7560E 05
MODE(10)	0.3120E 04	0.5748E 04	0.1044E 05	0.1532E 05	0.2130E 05	0.2924E 05	0.3993E 05	0.5499E 05	0.8029E 05	0.1468E 06

Table 3.4

ALPHA : 0.5

SYMMETRIC MODES

D :	0.05	0.10	0.20	0.30	0.40	0.50	0.60	0.70	0.80	0.90
MODE(1)	0.1924E 00	0.7741E 00	0.3170E 01	0.7434E 01	0.1407E 02	0.2409E 02	0.3954E 02	0.6532E 02	0.1162E 03	0.2661E 03
MODE(3)	0.2507E 02	0.5205E 02	0.1173E 03	0.2061E 03	0.3300E 03	0.5036E 03	0.7512E 03	0.1129E 04	0.1813E 04	0.3692E 04
MODE(5)	0.2350E 03	0.4487E 03	0.8721E 03	0.1354E 04	0.1961E 04	0.2764E 04	0.3861E 04	0.5479E 04	0.8326E 04	0.1602E 05
MODE(7)	0.8382E 03	0.1569E 04	0.2925E 04	0.4361E 04	0.6084E 04	0.8298E 04	0.1127E 05	0.1557E 05	0.2305E 05	0.4315E 05
MODE(9)	0.2042E 04	0.3789E 04	0.6930E 04	0.1013E 05	0.1387E 05	0.1859E 05	0.2485E 05	0.3384E 05	0.4935E 05	0.9094E 05

ANTI-SYMMETRIC MODES

D :	0.05	0.10	0.20	0.30	0.40	0.50	0.60	0.70	0.80	0.90
MODE(2)	0.9264E 00	0.3342E 01	0.1396E 02	0.3375E 02	0.6594E 02	0.1153E 03	0.1901E 03	0.3094E 03	0.5318E 03	0.1153E 04
MODE(4)	0.9376E 02	0.1833E 03	0.3730E 03	0.6037E 03	0.9062E 03	0.1316E 04	0.1886E 04	0.2738E 04	0.4252E 04	0.8367E 04
MODE(6)	0.4745E 03	0.8949E 03	0.1695E 04	0.2569E 04	0.3638E 04	0.5029E 04	0.6910E 04	0.9655E 04	0.1445E 05	0.2737E 05
MODE(8)	0.1352E 04	0.2518E 04	0.4642E 04	0.6844E 04	0.9447E 04	0.1276E 05	0.1717E 05	0.2353E 05	0.3454E 05	0.6409E 05
MODE(10)	0.2934E 04	0.5430E 04	0.9870E 04	0.1434E 05	0.1951E 05	0.2599E 05	0.3455E 05	0.4680E 05	0.6790E 05	0.1244E 06

Table 3.5

ALPHA : 0.6

SYMMETRIC MODES

S :	0.05	0.10	0.20	0.30	0.40	0.50	0.60	0.70	0.80	0.90
MODE(1)	0.1675E 00	0.6741E 00	0.2762E 01	0.6484E 01	0.1229E 02	0.2108E 02	0.3468E 02	0.5744E 02	0.1025E 03	0.2356E 03
MODE(3)	0.2365E 02	0.4924E 02	0.1107E 03	0.1921E 03	0.3021E 03	0.4530E 03	0.6671E 03	0.9960E 03	0.1597E 04	0.3259E 04
MODE(5)	0.2215E 03	0.4243E 03	0.8242E 03	0.1269E 04	0.1809E 04	0.2503E 04	0.3443E 04	0.4840E 04	0.7331E 04	0.1414E 05
MODE(7)	0.7900E 03	0.1483E 04	0.2765E 04	0.4092E 04	0.5625E 04	0.7534E 04	0.1006E 05	0.1376E 05	0.2030E 05	0.3807E 05
MODE(9)	0.1925E 04	0.3582E 04	0.6554E 04	0.9513E 04	0.1284E 05	0.1690E 05	0.2222E 05	0.2993E 05	0.4346E 05	0.8022E 05

ANTI-SYMMETRIC MODES

S :	0.05	0.10	0.20	0.30	0.40	0.50	0.60	0.70	0.80	0.90
MODE(2)	0.7934E 00	0.3188E 01	0.1319E 02	0.3136E 02	0.6009E 02	0.1033E 03	0.1685E 03	0.2727E 03	0.4684E 03	0.1018E 04
MODE(4)	0.8842E 02	0.1734E 03	0.3524E 03	0.5547E 03	0.8338E 03	0.1189E 04	0.1679E 04	0.2417E 04	0.3744E 04	0.7385E 04
MODE(6)	0.4473E 03	0.8462E 03	0.1603E 04	0.2409E 04	0.3360E 04	0.4561E 04	0.6169E 04	0.8533E 04	0.1272E 05	0.2415E 05
MODE(8)	0.1274E 04	0.2391E 04	0.4390E 04	0.6424E 04	0.8740E 04	0.1159E 05	0.1535E 05	0.2081E 05	0.3042E 05	0.5654E 05
MODE(10)	0.2765E 04	0.5133E 04	0.9335E 04	0.1346E 05	0.1806E 05	0.2364E 05	0.3091E 05	0.4140E 05	0.5980E 05	0.1097E 06

Table 3.6

ALPHA : 0.7

SYMMETRIC MODES

Q :	0.05	0.10	0.20	0.30	0.40	0.50	0.60	0.70	0.80	0.90
MODE(1)	0.1504E 00	0.6052E 00	0.2481E 01	0.5829E 01	0.1106E 02	0.1899E 02	0.3130E 02	0.5194E 02	0.9289E 02	0.2140E 03
MODE(3)	0.2235E 02	0.4662E 02	0.1046E 03	0.1801E 03	0.2802E 03	0.4157E 03	0.6072E 03	0.9024E 03	0.1445E 04	0.2955E 04
MODE(5)	0.2093E 03	0.4017E 03	0.7799E 03	0.1193E 04	0.1685E 04	0.2306E 04	0.3142E 04	0.4399E 04	0.6635E 04	0.1281E 05
MODE(7)	0.7461E 03	0.1404E 04	0.2617E 04	0.3853E 04	0.5249E 04	0.6952E 04	0.9195E 04	0.1249E 05	0.1937E 05	0.3450E 05
MODE(9)	0.1818E 04	0.3390E 04	0.6204E 04	0.8962E 04	0.1199E 05	0.1561E 05	0.2031E 05	0.2717E 05	0.3933E 05	0.7269E 05

ANTI-SYMMETRIC MODES

MODE(2)	0.7542E 00	0.3037E 01	0.1248E 02	0.2935E 02	0.5557E 02	0.9457E 02	0.1531E 03	0.2470E 03	0.4240E 03	0.9237E 03
MODE(4)	0.8354E 02	0.1642E 03	0.3333E 03	0.5307E 03	0.7756E 03	0.1094E 04	0.1531E 04	0.2191E 04	0.3388E 04	0.6694E 04
MODE(6)	0.4225E 03	0.8009E 03	0.1517E 04	0.2267E 04	0.3133E 04	0.4206E 04	0.5632E 04	0.7742E 04	0.1152E 05	0.2188E 05
MODE(8)	0.1203E 04	0.2253E 04	0.4155E 04	0.6051E 04	0.9158E 04	0.1070E 05	0.1402E 05	0.1888E 05	0.2753E 05	0.5124E 05
MODE(10)	0.2611E 04	0.4858E 04	0.8836E 04	0.1269E 05	0.1687E 05	0.2184E 05	0.2826E 05	0.3758E 05	0.5412E 05	0.9944E 05

Table 3.7

ALPHA : 0.8

SYMMETRIC MODES

C :	0.05	0.10	0.20	0.30	0.40	0.50	0.60	0.70	0.80	0.90
MODE(1)	0.1281E 00	0.5558E 00	0.2279E 01	0.5356E 01	0.1017E 02	0.1748E 02	0.2883E 02	0.4791E 02	0.8580E 02	0.1980E 03
MODE(3)	0.2117E 02	0.4419E 02	0.9900E 02	0.1697E 03	0.2623E 03	0.3868E 03	0.5624E 03	0.8335E 03	0.1334E 04	0.2731E 04
MODE(5)	0.1981E 03	0.3807E 03	0.7389E 03	0.1127E 04	0.1582E 04	0.2151E 04	0.2915E 04	0.4058E 04	0.6124E 04	0.1184E 05
MODE(7)	0.7062E 03	0.1330E 04	0.2480E 04	0.3640E 04	0.4932E 04	0.6490E 04	0.8535E 04	0.1155E 05	0.1695E 05	0.3186E 05
MODE(9)	0.1720E 04	0.3213E 04	0.5880E 04	0.8469E 04	0.1127E 05	0.1458E 05	0.1886E 05	0.2512E 05	0.3630E 05	0.6714E 05

ANTI-SYMMETRIC MODES

C :	0.05	0.10	0.20	0.30	0.40	0.50	0.60	0.70	0.80	0.90
MODE(2)	0.7185E 00	0.2890E 01	0.1182E 02	0.2762E 02	0.5195E 02	0.8788E 02	0.1417E 03	0.2281E 03	0.3914E 03	0.8537E 03
MODE(4)	0.7998E 02	0.1556E 03	0.3157E 03	0.5007E 03	0.7275E 03	0.1019E 04	0.1419E 04	0.2025E 04	0.3127E 04	0.6184E 04
MODE(6)	0.3959E 03	0.7590E 03	0.1437E 04	0.2141E 04	0.2943E 04	0.3925E 04	0.5227E 04	0.7156E 04	0.1063E 05	0.2021E 05
MODE(8)	0.1139E 04	0.2135E 04	0.3938E 04	0.5718E 04	0.7668E 04	0.9993E 04	0.1302E 05	0.1746E 05	0.2541E 05	0.4733E 05
MODE(10)	0.2471E 04	0.4603E 04	0.8375E 04	0.1199E 05	0.1586E 05	0.2040E 05	0.2624E 05	0.3475E 05	0.4995E 05	0.9185E 05

Table 3.8

ALPHA : 0.9

SYMMETRIC MODES

C :	0.05	0.10	0.20	0.30	0.40	0.50	0.60	0.70	0.80	0.90
MODE(1)	0.1290E 00	0.5190E 00	0.2129E 01	0.5005E 01	0.9508E 01	0.1635E 02	0.2698E 02	0.4485E 02	0.8038E 02	0.1856E 03
MODE(3)	0.2009E 02	0.4195E 02	0.9390E 02	0.1606E 03	0.2474E 03	0.3638E 03	0.5276E 03	0.7808E 03	0.1249E 04	0.2559E 04
MODE(5)	0.1879E 03	0.3613E 03	0.7012E 03	0.1067E 04	0.1494E 04	0.2025E 04	0.2737E 04	0.3802E 04	0.5734E 04	0.1109E 05
MODE(7)	0.6697E 03	0.1263E 04	0.2354E 04	0.3449E 04	0.4660E 04	0.6113E 04	0.8016E 04	0.1082E 05	0.1588E 05	0.2985E 05
MODE(9)	0.1631E 04	0.3049E 04	0.5580E 04	0.8026E 04	0.1065E 05	0.1374E 05	0.1772E 05	0.2354E 05	0.3400E 05	0.6290E 05

ANTI-SYMMETRIC MODES

C :	0.05	0.10	0.20	0.30	0.40	0.50	0.60	0.70	0.80	0.90
MODE(2)	0.6847E 00	0.2749E 01	0.1121E 02	0.2612E 02	0.4896E 02	0.8259E 02	0.1329E 03	0.2136E 03	0.3666E 03	0.8000E 03
MODE(4)	0.7501E 02	0.1477E 03	0.2996E 03	0.4742E 03	0.6869E 03	0.9594E 03	0.1332E 04	0.1897E 04	0.2929E 04	0.5794E 04
MODE(6)	0.3792E 03	0.7203E 03	0.1364E 04	0.2029E 04	0.2781E 04	0.3696E 04	0.4903E 04	0.6707E 04	0.9953E 04	0.1894E 05
MODE(8)	0.1080E 04	0.2026E 04	0.3737E 04	0.5418E 04	0.7246E 04	0.9413E 04	0.1223E 05	0.1636E 05	0.2379E 05	0.4434E 05
MODE(10)	0.2344E 04	0.4368E 04	0.7949E 04	0.1136E 05	0.1499E 05	0.1922E 05	0.2465E 05	0.3258E 05	0.4678E 05	0.8604E 05

Table 3.9

ALPHA : 1.0

SYMMETRIC MODES

Q :	0.05	0.10	0.20	0.30	0.40	0.50	0.60	0.70	0.80	0.90
MODE(1)	0.1220E 00	0.4910E 00	0.2014E 01	0.4735E 01	0.8997E 01	0.1547E 02	0.2554E 02	0.4246E 02	0.7611E 02	0.1758E 0
MODE(3)	0.1908E 02	0.3997E 02	0.8924E 02	0.1525E 03	0.2348E 03	0.3448E 03	0.4998E 03	0.7394E 03	0.1182E 04	0.2423E 0
MODE(5)	0.1785E 03	0.3434E 03	0.6665E 03	0.1014E 04	0.1418E 04	0.1920E 04	0.2593E 04	0.3601E 04	0.5430E 04	0.1050E 0
MODE(7)	0.6364E 03	0.1200E 04	0.2238E 04	0.3277E 04	0.4424E 04	0.5798E 04	0.7596E 04	0.1025E 05	0.1503E 05	0.2827E 0
MODE(9)	0.1550E 04	0.2898E 04	0.5304E 04	0.7625E 04	0.1011E 05	0.1303E 05	0.1679E 05	0.2230E 05	0.3219E 05	0.5956E 0

ANTI-SYMMETRIC MODES

Q :	0.05	0.10	0.20	0.30	0.40	0.50	0.60	0.70	0.80	0.90
MODE(2)	0.6504E 00	0.2614E 01	0.1066E 02	0.2481E 02	0.4644E 02	0.7827E 02	0.1259E 03	0.2023E 03	0.3471E 03	0.7576E 0
MODE(4)	0.7128E 02	0.1404E 03	0.2847E 03	0.4504E 03	0.6519E 03	0.9097E 03	0.1262E 04	0.1797E 04	0.2773E 04	0.5487E 0
MODE(6)	0.3604E 03	0.6847E 03	0.1296E 04	0.1927E 04	0.2639E 04	0.3506E 04	0.4651E 04	0.6351E 04	0.9424E 04	0.1793E 0
MODE(8)	0.1026E 04	0.1926E 04	0.3552E 04	0.5147E 04	0.6879E 04	0.8928E 04	0.1159E 05	0.1550E 05	0.2253E 05	0.4198E 0
MODE(10)	0.2227E 04	0.4157E 04	0.7555E 04	0.1080E 05	0.1423E 05	0.1823E 05	0.2336E 05	0.3085E 05	0.4429E 05	0.8148E 0

Table 3.10

Q :	0.05	0.10	0.20	0.30	0.40	0.50	0.60	0.70	0.80	0.90
MODE(1)	0.3461E 00	0.7306E 00	0.1643E 01	0.2813E 01	0.4369E 01	0.6540E 01	0.9787E 01	0.1518E 02	0.2594E 02	0.5814E 02
MODE(2)	0.6133E 00	0.1288E 01	0.2868E 01	0.4865E 01	0.7489E 01	0.1111E 02	0.1650E 02	0.2539E 02	0.4305E 02	0.9578E 02
MODE(3)	0.8689E 00	0.1823E 01	0.4045E 01	0.6841E 01	0.1050E 02	0.1554E 02	0.2299E 02	0.3528E 02	0.5966E 02	0.1324E 03
MODE(4)	0.1125E 01	0.2359E 01	0.5227E 01	0.8824E 01	0.1352E 02	0.1997E 02	0.2951E 02	0.4521E 02	0.7632E 02	0.1690E 03
MODE(5)	0.1381E 01	0.2895E 01	0.6407E 01	0.1081E 02	0.1654E 02	0.2441E 02	0.3603E 02	0.5514E 02	0.9299E 02	0.2057E 03
MODE(6)	0.1638E 01	0.3430E 01	0.7588E 01	0.1279E 02	0.1956E 02	0.2885E 02	0.4255E 02	0.6507E 02	0.1097E 03	0.2423E 03
MODE(7)	0.1894E 01	0.3967E 01	0.8770E 01	0.1477E 02	0.2258E 02	0.3329E 02	0.4907E 02	0.7501E 02	0.1263E 03	0.2791E 03
MODE(8)	0.2149E 01	0.4503E 01	0.9951E 01	0.1675E 02	0.2560E 02	0.3773E 02	0.5559E 02	0.8494E 02	0.1430E 03	0.3158E 03
MODE(9)	0.2407E 01	0.5038E 01	0.1113E 02	0.1874E 02	0.2862E 02	0.4217E 02	0.6211E 02	0.9488E 02	0.1597E 03	0.3525E 03
MODE(10)	0.2662E 01	0.5573E 01	0.1231E 02	0.2072E 02	0.3165E 02	0.4660E 02	0.6864E 02	0.1048E 03	0.1764E 03	0.3893E 03

Table 4.0

The system parameter λ_t vs q for clamped - partly clamped beams .

Q :	0.05	0.10	0.20	0.30	0.40	0.50	0.60	0.70	0.80	0.90
MODE(1)	0.2924E 00	0.6180E 00	0.1393E 01	0.2392E 01	0.3724E 01	0.5587E 01	0.8380E 01	0.1303E 02	0.2231E 02	0.5012E 02
MODE(2)	0.5492E 00	0.1154E 01	0.2571E 01	0.4366E 01	0.6728E 01	0.9998E 01	0.1486E 02	0.2289E 02	0.3886E 02	0.8655E 02
MODE(3)	0.8060E 00	0.1690E 01	0.3752E 01	0.6347E 01	0.9745E 01	0.1443E 02	0.2137E 02	0.3281E 02	0.5550E 02	0.1232E 03
MODE(4)	0.1063E 01	0.2226E 01	0.4933E 01	0.8330E 01	0.1277E 02	0.1887E 02	0.2788E 02	0.4273E 02	0.7216E 02	0.1599E 03
MODE(5)	0.1330E 01	0.2763E 01	0.6115E 01	0.1031E 02	0.1579E 02	0.2330E 02	0.3440E 02	0.5266E 02	0.8882E 02	0.1965E 03
MODE(6)	0.1576E 01	0.3299E 01	0.7296E 01	0.1230E 02	0.1881E 02	0.2774E 02	0.4092E 02	0.6250E 02	0.1055E 03	0.2333E 03
MODE(7)	0.1832E 01	0.3835E 01	0.8477E 01	0.1428E 02	0.2183E 02	0.3218E 02	0.4744E 02	0.7253E 02	0.1222E 03	0.2700E 03
MODE(8)	0.2089E 01	0.4372E 01	0.9660E 01	0.1626E 02	0.2485E 02	0.3662E 02	0.5396E 02	0.8247E 02	0.1388E 03	0.3067E 03
MODE(9)	0.2346E 01	0.4907E 01	0.1084E 02	0.1825E 02	0.2768E 02	0.4106E 02	0.6049E 02	0.9240E 02	0.1555E 03	0.3434E 03
MODE(10)	0.2602E 01	0.5443E 01	0.1202E 02	0.2023E 02	0.3090E 02	0.4550E 02	0.6701E 02	0.1023E 03	0.1722E 03	0.3800E 03

Table 5.0

The system parameter λ_t vs q for pinned - partly clamped beams .

q :	0.05	0.10	0.20	0.30	0.40	0.50	0.60	0.70	0.80	0.90
MODE(1)	0.1033E-01	0.4161E-01	0.1715E 00	0.4065E 00	0.7813E 00	0.1364E 01	0.2294E 01	0.3903E 01	0.7189E 01	0.1715E 0
MODE(2)	0.3672E 00	0.7735E 00	0.1732E 01	0.2956E 01	0.4577E 01	0.6833E 01	0.1020E 02	0.1579E 02	0.2693E 02	0.6027E 0
MODE(3)	0.6116E 00	0.1285E 01	0.2862E 01	0.4854E 01	0.7473E 01	0.1109E 02	0.1646E 02	0.2534E 02	0.4298E 02	0.9561E 0
MODE(4)	0.8698E 00	0.1623E 01	0.4046E 01	0.6842E 01	0.1050E 02	0.1554E 02	0.2299E 02	0.3528E 02	0.5966E 02	0.1324E 0
MODE(5)	0.1125E 01	0.2359E 01	0.5227E 01	0.8824E 01	0.1352E 02	0.1997E 02	0.2951E 02	0.4521E 02	0.7633E 02	0.1691E 0
MODE(6)	0.1391E 01	0.2895E 01	0.6407E 01	0.1081E 02	0.1654E 02	0.2441E 02	0.3603E 02	0.5514E 02	0.9299E 02	0.2057E 0
MODE(7)	0.1639E 01	0.3430E 01	0.7588E 01	0.1279E 02	0.1956E 02	0.2885E 02	0.4255E 02	0.6507E 02	0.1097E 03	0.2423E 0
MODE(8)	0.1894E 01	0.3967E 01	0.8770E 01	0.1477E 02	0.2258E 02	0.3329E 02	0.4907E 02	0.7501E 02	0.1263E 03	0.2791E 0
MODE(9)	0.2149E 01	0.4503E 01	0.9951E 01	0.1675E 02	0.2560E 02	0.3773E 02	0.5559E 02	0.8494E 02	0.1430E 03	0.3158E 0
MODE(10)	0.2407E 01	0.5038E 01	0.1113E 02	0.1874E 02	0.2862E 02	0.4217E 02	0.6211E 02	0.9488E 02	0.1597E 03	0.3525E 0

Table 6.0

The system parameter λ_t vs q for free - partly clamped beams .

ALPHA : 0.1

Q :	0.05	0.10	0.20	0.30	0.40	0.50	0.60	0.70	0.80	0.90
MODE(1)	0.6289E 00	0.1403E 01	0.3583E 01	0.7150E 01	0.1321E 02	0.2358E 02	0.4131E 02	0.7289E 02	0.1378E 03	0.3344E 03
MODE(2)	0.1098E 01	0.2409E 01	0.5962E 01	0.1157E 02	0.2092E 02	0.3696E 02	0.6463E 02	0.1141E 03	0.2156E 03	0.5215E 03
MODE(3)	0.1578E 01	0.3444E 01	0.8432E 01	0.1618E 02	0.2897E 02	0.5083E 02	0.8854E 02	0.1560E 03	0.2941E 03	0.7095E 03
MODE(4)	0.2058E 01	0.4480E 01	0.1091E 02	0.2081E 02	0.3706E 02	0.6478E 02	0.1126E 03	0.1981E 03	0.3731E 03	0.8987E 03
MODE(5)	0.2538E 01	0.5516E 01	0.1339E 02	0.2545E 02	0.4518E 02	0.7876E 02	0.1367E 03	0.2404E 03	0.4522E 03	0.1088E 04
MODE(6)	0.3018E 01	0.6552E 01	0.1587E 02	0.3009E 02	0.5330E 02	0.9277E 02	0.1609E 03	0.2826E 03	0.5314E 03	0.1278E 04
MODE(7)	0.3499E 01	0.7589E 01	0.1835E 02	0.3474E 02	0.6144E 02	0.1068E 03	0.1850E 03	0.3250E 03	0.6107E 03	0.1467E 04
MODE(8)	0.3980E 01	0.8627E 01	0.2083E 02	0.3939E 02	0.6957E 02	0.1208E 03	0.2092E 03	0.3673E 03	0.6901E 03	0.1657E 04
MODE(9)	0.4459E 01	0.9664E 01	0.2331E 02	0.4404E 02	0.7770E 02	0.1348E 03	0.2334E 03	0.4096E 03	0.7694E 03	0.1847E 04
MODE(10)	0.4939E 01	0.1070E 02	0.2579E 02	0.4869E 02	0.8585E 02	0.1489E 03	0.2576E 03	0.4520E 03	0.8488E 03	0.2037E 04

Table 7.1 - System parameter λ_{t1} vs q for beams with both ends partly clamped .

ALPHA : 0.2

Q :	0.05	0.10	0.20	0.30	0.40	0.50	0.60	0.70	0.80	0.90
MODE(1)	0.5687E 00	0.1247E 01	0.3044E 01	0.5699E 01	0.9704E 01	0.1590E 02	0.2585E 02	0.4317E 02	0.7855E 02	0.1857E 03
MODE(2)	0.9951E 00	0.2154E 01	0.5131E 01	0.9407E 01	0.1576E 02	0.2551E 02	0.4115E 02	0.6526E 02	0.1235E 03	0.2901E 03
MODE(3)	0.1431E 01	0.3082E 01	0.7275E 01	0.1322E 02	0.2197E 02	0.3534E 02	0.5670E 02	0.9365E 02	0.1688E 03	0.3949E 03
MODE(4)	0.1867E 01	0.4011E 01	0.9423E 01	0.1704E 02	0.2821E 02	0.4522E 02	0.7234E 02	0.1192E 03	0.2143E 03	0.5004E 03
MODE(5)	0.2303E 01	0.4941E 01	0.1157E 02	0.2098E 02	0.3447E 02	0.5511E 02	0.9800E 02	0.1448E 03	0.2599E 03	0.6060E 03
MODE(6)	0.2739E 01	0.5870E 01	0.1373E 02	0.2471E 02	0.4073E 02	0.6502E 02	0.1037E 03	0.1704E 03	0.3056E 03	0.7117E 03
MODE(7)	0.3174E 01	0.6800E 01	0.1588E 02	0.2855E 02	0.4699E 02	0.7494E 02	0.1194E 03	0.1960E 03	0.3513E 03	0.8175E 03
MODE(8)	0.3611E 01	0.7730E 01	0.1803E 02	0.3239E 02	0.5325E 02	0.8486E 02	0.1351E 03	0.2217E 03	0.3970E 03	0.9233E 03
MODE(9)	0.4046E 01	0.8661E 01	0.2018E 02	0.3622E 02	0.5952E 02	0.9478E 02	0.1508E 03	0.2473E 03	0.4428E 03	0.1029E 04
MODE(10)	0.4482E 01	0.9591E 01	0.2234E 02	0.4006E 02	0.6579E 02	0.1047E 03	0.1665E 03	0.2730E 03	0.4885E 03	0.1135E 04

Table 7.2

ALPHA : 0.3

Q :	0.05	0.10	0.20	0.30	0.40	0.50	0.60	0.70	0.80	0.90
MODE(1)	0.5203E 00	0.1129E 01	0.2690E 01	0.4884E 01	0.8042E 01	0.1274E 02	0.2009E 02	0.3269E 02	0.5830E 02	0.1357E 03
MODE(2)	0.9113E 00	0.1956E 01	0.4561E 01	0.8133E 01	0.1319E 02	0.2063E 02	0.3221E 02	0.5196E 02	0.9190E 02	0.2122E 03
MODE(3)	0.1312E 01	0.2801E 01	0.6475E 01	0.1145E 02	0.1844E 02	0.2866E 02	0.4450E 02	0.7141E 02	0.1257E 03	0.2890E 03
MODE(4)	0.1711E 01	0.3647E 01	0.8393E 01	0.1479E 02	0.2372E 02	0.3673E 02	0.5685E 02	0.9097E 02	0.1598E 03	0.3663E 03
MODE(5)	0.2111E 01	0.4493E 01	0.1031E 02	0.1812E 02	0.2900E 02	0.4482E 02	0.6921E 02	0.1106E 03	0.1938E 03	0.4437E 03
MODE(6)	0.2511E 01	0.5338E 01	0.1223E 02	0.2146E 02	0.3429E 02	0.5291E 02	0.8161E 02	0.1302E 03	0.2279E 03	0.5211E 03
MODE(7)	0.2910E 01	0.6184E 01	0.1415E 02	0.2480E 02	0.3958E 02	0.6101E 02	0.9400E 02	0.1498E 03	0.2621E 03	0.5986E 03
MODE(8)	0.3311E 01	0.7031E 01	0.1608E 02	0.2814E 02	0.4487E 02	0.6910E 02	0.1064E 03	0.1695E 03	0.2962E 03	0.6761E 03
MODE(9)	0.3710E 01	0.7878E 01	0.1800E 02	0.3149E 02	0.5016E 02	0.7721E 02	0.1188E 03	0.1891E 03	0.3304E 03	0.7537E 03
MODE(10)	0.4110E 01	0.8725E 01	0.1992E 02	0.3483E 02	0.5546E 02	0.8532E 02	0.1312E 03	0.2087E 03	0.3646E 03	0.8314E 03

Table 7.3

ALPHA : 0.4

Q :	0.05	0.10	0.20	0.30	0.40	0.50	0.60	0.70	0.80	0.90
MCDF(1)	0.4822E 00	0.1035E 01	0.2430E 01	0.4339E 01	0.7019E 01	0.1092E 02	0.1695E 02	0.2720E 02	0.4793E 02	0.1104E 03
MCDF(2)	0.8425E 00	0.1797E 01	0.4135E 01	0.7259E 01	0.1157E 02	0.1778E 02	0.2729E 02	0.4335E 02	0.7567E 02	0.1729E 03
MCDF(3)	0.1212E 01	0.2574E 01	0.5875E 01	0.1023E 02	0.1620E 02	0.2474E 02	0.3775E 02	0.5964E 02	0.1036E 03	0.2355E 03
MCDF(4)	0.1592E 01	0.3352E 01	0.7618E 01	0.1322E 02	0.2085E 02	0.3173E 02	0.4826E 02	0.7601E 02	0.1317E 03	0.2985E 03
MCDF(5)	0.1951E 01	0.4130E 01	0.9363E 01	0.1621E 02	0.2551E 02	0.3873E 02	0.5879E 02	0.9242E 02	0.1598E 03	0.3616E 03
MCDF(6)	0.2321E 01	0.4908E 01	0.1111E 02	0.1920E 02	0.3017E 02	0.4574E 02	0.6933E 02	0.1088E 03	0.1879E 03	0.4248E 03
MCDF(7)	0.2690E 01	0.5686E 01	0.1285E 02	0.2219E 02	0.3483E 02	0.5275E 02	0.7988E 02	0.1253E 03	0.2161E 03	0.4880E 03
MCDF(8)	0.3060E 01	0.6465E 01	0.1460E 02	0.2519E 02	0.3950E 02	0.5977E 02	0.9043E 02	0.1417E 03	0.2443E 03	0.5512E 03
MCDF(9)	0.3429E 01	0.7244E 01	0.1635E 02	0.2818E 02	0.4416E 02	0.6679E 02	0.1010E 03	0.1582E 03	0.2725E 03	0.6144E 03
MCDF(10)	0.3799E 01	0.8023E 01	0.1809E 02	0.3118E 02	0.4883E 02	0.7381E 02	0.1115E 03	0.1746E 03	0.3007E 03	0.6778E 03

Table 7.4

ALPHA : 0.5

Q :	0.05	0.10	0.20	0.30	0.40	0.50	0.60	0.70	0.80	0.90
MODE(1)	0.4464E 00	0.9584E 00	0.2229E 01	0.3938E 01	0.6306E 01	0.9716E 01	0.1494E 02	0.2276E 02	0.4156E 02	0.9516E 02
MODE(2)	0.7836E 00	0.1666E 01	0.3820E 01	0.6608E 01	0.1043E 02	0.1586E 02	0.2410E 02	0.3793E 02	0.6569E 02	0.1490E 03
MODE(3)	0.1128E 01	0.2386E 01	0.5401E 01	0.9323E 01	0.1461E 02	0.2208E 02	0.3336E 02	0.5222E 02	0.8996E 02	0.2031E 03
MODE(4)	0.1472E 01	0.3108E 01	0.7006E 01	0.1205E 02	0.1881E 02	0.2834E 02	0.4267E 02	0.6658E 02	0.1144E 03	0.2574E 03
MODE(5)	0.1815E 01	0.3829E 01	0.8612E 01	0.1478E 02	0.2302E 02	0.3460E 02	0.5199E 02	0.8097E 02	0.1388E 03	0.3118E 03
MODE(6)	0.2159E 01	0.4551E 01	0.1022E 02	0.1750E 02	0.2723E 02	0.4087E 02	0.6133E 02	0.9537E 02	0.1633E 03	0.3663E 03
MODE(7)	0.2503E 01	0.5273E 01	0.1183E 02	0.2024E 02	0.3145E 02	0.4714E 02	0.7066E 02	0.1098E 03	0.1878E 03	0.4208E 03
MODE(8)	0.2848E 01	0.5995E 01	0.1343E 02	0.2297E 02	0.3566E 02	0.5342E 02	0.8001E 02	0.1242E 03	0.2123E 03	0.4753E 03
MODE(9)	0.3191E 01	0.6717E 01	0.1504E 02	0.2570E 02	0.3988E 02	0.5969E 02	0.8935E 02	0.1386E 03	0.2368E 03	0.5299E 03
MODE(10)	0.3535E 01	0.7439E 01	0.1665E 02	0.2843E 02	0.4410E 02	0.6597E 02	0.9870E 02	0.1530E 03	0.2613E 03	0.5846E 03

Table 7.5

ALPHA : 0.6

C :	0.05	0.10	0.20	0.30	0.40	0.50	0.60	0.70	0.80	0.90
MODE(1)	0.4173E 00	0.9935E 00	0.2064E 01	0.3627E 01	0.5772E 01	0.8840E 01	0.1351E 02	0.2138E 02	0.3722E 02	0.8488E 02
MODE(2)	0.7330E 00	0.1554E 01	0.3527E 01	0.6097E 01	0.9562E 01	0.1445E 02	0.2183E 02	0.3417E 02	0.5887E 02	0.1330E 03
MODE(3)	0.1055E 01	0.2227E 01	0.5014E 01	0.8606E 01	0.1341E 02	0.2014E 02	0.3023E 02	0.4705E 02	0.8064E 02	0.1812E 03
MODE(4)	0.1377E 01	0.2901E 01	0.6505E 01	0.1112E 02	0.1726E 02	0.2584E 02	0.3865E 02	0.6001E 02	0.1025E 03	0.2297E 03
MODE(5)	0.1698E 01	0.3574E 01	0.7997E 01	0.1364E 02	0.2113E 02	0.3156E 02	0.4714E 02	0.7299E 02	0.1245E 03	0.2783E 03
MODE(6)	0.2020E 01	0.4248E 01	0.9491E 01	0.1617E 02	0.2500E 02	0.3729E 02	0.5561E 02	0.8598E 02	0.1464E 03	0.3269E 03
MODE(7)	0.2342E 01	0.4922E 01	0.1099E 02	0.1869E 02	0.2887E 02	0.4301E 02	0.6408E 02	0.9898E 02	0.1684E 03	0.3756E 03
MODE(8)	0.2664E 01	0.5596E 01	0.1248E 02	0.2121E 02	0.3274E 02	0.4874E 02	0.7256E 02	0.1120E 03	0.1904E 03	0.4243E 03
MODE(9)	0.2985E 01	0.6270E 01	0.1397E 02	0.2374E 02	0.3661E 02	0.5447E 02	0.8103E 02	0.1250E 03	0.2123E 03	0.4730E 03
MODE(10)	0.3307E 01	0.6945E 01	0.1546E 02	0.2626E 02	0.4049E 02	0.6020E 02	0.8952E 02	0.1380E 03	0.2343E 03	0.5217E 03

Table 7.6

ALPHA : 0.7

Q :	0.05	0.10	0.20	0.30	0.40	0.50	0.60	0.70	0.80	0.90
MODE(1)	0.3921E 00	0.8380E 00	0.1929E 01	0.3376E 01	0.5352E 01	0.8167E 01	0.1244E 02	0.1962E 02	0.3406E 02	0.7747E 02
MODE(2)	0.6889E 00	0.1459E 01	0.3298E 01	0.5682E 01	0.8877E 01	0.1337E 02	0.2012E 02	0.3138E 02	0.5389E 02	0.1214E 03
MODE(3)	0.9915E 00	0.2090E 01	0.4690E 01	0.8021E 01	0.1245E 02	0.1863E 02	0.2787E 02	0.4322E 02	0.7384E 02	0.1654E 03
MODE(4)	0.1224E 01	0.2722E 01	0.6086E 01	0.1037E 02	0.1604E 02	0.2391E 02	0.3566E 02	0.5513E 02	0.9389E 02	0.2097E 03
MODE(5)	0.1596E 01	0.3355E 01	0.7492E 01	0.1272E 02	0.1963E 02	0.2921E 02	0.4346E 02	0.6706E 02	0.1140E 03	0.2541E 03
MODE(6)	0.1899E 01	0.3997E 01	0.8879E 01	0.1507E 02	0.2322E 02	0.3451E 02	0.5127E 02	0.7899E 02	0.1341E 03	0.2985E 03
MODE(7)	0.2201E 01	0.4619E 01	0.1028E 02	0.1742E 02	0.2682E 02	0.3981E 02	0.5909E 02	0.9094E 02	0.1542E 03	0.3429E 03
MODE(8)	0.2504E 01	0.5252E 01	0.1167E 02	0.1978E 02	0.3042E 02	0.4511E 02	0.6691E 02	0.1029E 03	0.1743E 03	0.3874E 03
MODE(9)	0.2806E 01	0.5855E 01	0.1307E 02	0.2213E 02	0.3401E 02	0.5042E 02	0.7473E 02	0.1148E 03	0.1945E 03	0.4319E 03
MODE(10)	0.3109E 01	0.6519E 01	0.1447E 02	0.2449E 02	0.3762E 02	0.5572E 02	0.8255E 02	0.1268E 03	0.2146E 03	0.4764E 03

Table 7.7

ALPHA : 0.8

Q :	0.05	0.10	0.20	0.30	0.40	0.50	0.60	0.70	0.80	0.90
MODE(1)	0.3659E 00	0.7858E 00	0.1814E 01	0.3163E 01	0.5011E 01	0.7631E 01	0.1160E 02	0.1826E 02	0.3164E 02	0.7187E 02
MODE(2)	0.6501E 00	0.1375E 01	0.3103E 01	0.5335E 01	0.8317E 01	0.1250E 02	0.1877E 02	0.2922E 02	0.5009E 02	0.1126E 03
MODE(3)	0.9356E 00	0.1971E 01	0.4414E 01	0.7523E 01	0.1167E 02	0.1742E 02	0.2600E 02	0.4025E 02	0.6863E 02	0.1535E 03
MODE(4)	0.1221E 01	0.2567E 01	0.5727E 01	0.9738E 01	0.1503E 02	0.2236E 02	0.3327E 02	0.5134E 02	0.8727E 02	0.1946E 03
MODE(5)	0.1506E 01	0.3163E 01	0.7041E 01	0.1195E 02	0.1839E 02	0.2731E 02	0.4056E 02	0.6245E 02	0.1059E 03	0.2358E 03
MODE(6)	0.1792E 01	0.3759E 01	0.8357E 01	0.1416E 02	0.2177E 02	0.3227E 02	0.4785E 02	0.7357E 02	0.1246E 03	0.2770E 03
MODE(7)	0.2077E 01	0.4356E 01	0.9671E 01	0.1637E 02	0.2514E 02	0.3723E 02	0.5514E 02	0.8470E 02	0.1433E 03	0.3182E 03
MODE(8)	0.2363E 01	0.4952E 01	0.1099E 02	0.1858E 02	0.2851E 02	0.4219E 02	0.6244E 02	0.9582E 02	0.1620E 03	0.3595E 03
MODE(9)	0.2648E 01	0.5549E 01	0.1230E 02	0.2079E 02	0.3188E 02	0.4715E 02	0.6974E 02	0.1070E 03	0.1808E 03	0.4007E 03
MODE(10)	0.2933E 01	0.6146E 01	0.1362E 02	0.2300E 02	0.3526E 02	0.5212E 02	0.7704E 02	0.1181E 03	0.1995E 03	0.4421E 03

Table 7.8

ALPHA : 0.9

C :	0.05	0.10	0.20	0.30	0.40	0.50	0.60	0.70	0.80	0.90
MODE(1)	0.3503E 00	0.7474E 00	0.1715E 01	0.2992E 01	0.4727E 01	0.7190E 01	0.1092E 02	0.1718E 02	0.2973E 02	0.6749E 02
MODE(2)	0.6156E 00	0.1301E 01	0.2934E 01	0.5039E 01	0.7847E 01	0.1178E 02	0.1767E 02	0.2748E 02	0.4707E 02	0.1057E 03
MODE(3)	0.886[illegible]E 00	0.1865E 01	0.4174E 01	0.7116E 01	0.1101E 02	0.1642E 02	0.2448E 02	0.3786E 02	0.6449E 02	0.1441E 03
MODE(4)	0.1156E 01	0.2429E 01	0.5416E 01	0.9199E 01	0.1418E 02	0.2108E 02	0.3133E 02	0.4829E 02	0.9202E 02	0.1827E 03
MODE(5)	0.1427E 01	0.2994E 01	0.6659E 01	0.1129E 02	0.1726E 02	0.2575E 02	0.3819E 02	0.5875E 02	0.9956E 02	0.2214E 03
MODE(6)	0.1697E 01	0.3558E 01	0.7903E 01	0.1337E 02	0.2054E 02	0.3042E 02	0.4506E 02	0.6921E 02	0.1171E 03	0.2601E 03
MODE(7)	0.1967E 01	0.4123E 01	0.9146E 01	0.1546E 02	0.2372E 02	0.3510E 02	0.5193E 02	0.7968E 02	0.1347E 03	0.2988E 03
MODE(8)	0.2238E 01	0.4688E 01	0.1039E 02	0.1755E 02	0.2691E 02	0.3977E 02	0.5880E 02	0.9015E 02	0.1523E 03	0.3375E 03
MODE(9)	0.2[illegible]7E 01	0.5253E 01	0.1164E 02	0.1964E 02	0.3009E 02	0.4445E 02	0.6568E 02	0.1006E 03	0.1699E 03	0.3763E 03
MODE(10)	0.2778E 01	0.5818E 01	0.1288E 02	0.2173E 02	0.3327E 02	0.4913E 02	0.7255E 02	0.1111E 03	0.1875E 03	0.4151E 03

Table 7.9

ALPHA : 1.0

Q :	0.05	0.10	0.20	0.30	0.40
MODE(1)	0.3327E 00	0.7098E 00	0.1628E 01	0.2839E 01	0.4495E 01
MODE(2)	0.5847E 00	0.1236E 01	0.2786E 01	0.4783E 01	0.7446E 01
MODE(3)	0.8416E 00	0.1772E 01	0.3963E 01	0.6754E 01	0.1045E 02
MODE(4)	0.1099E 01	0.2307E 01	0.5142E 01	0.8732E 01	0.1346E 02
MODE(5)	0.1355E 01	0.2844E 01	0.6323E 01	0.1071E 02	0.1647E 02
MODE(6)	0.1612E 01	0.3379E 01	0.7504E 01	0.1269E 02	0.1949E 02
MODE(7)	0.1868E 01	0.3916E 01	0.8684E 01	0.1468E 02	0.2251E 02
MODE(8)	0.2124E 01	0.4452E 01	0.9867E 01	0.1666E 02	0.2553E 02
MODE(9)	0.2380E 01	0.4989E 01	0.1105E 02	0.1864E 02	0.2855E 02
MODE(10)	0.2636E 01	0.5526E 01	0.1223E 02	0.2063E 02	0.3158E 02

Q :	0.50	0.60	0.70	0.80	0.90
MODE(1)	0.6820E 01	0.1035E 02	0.1628E 02	0.2818E 02	0.6394E 02
MODE(2)	0.1117E 02	0.1676E 02	0.2605E 02	0.4461E 02	0.1002E 03
MODE(3)	0.1557E 02	0.2322E 02	0.3589E 02	0.6113E 02	0.1366E 03
MODE(4)	0.2000E 02	0.2971E 02	0.4579E 02	0.7774E 02	0.1732E 03
MODE(5)	0.2443E 02	0.3622E 02	0.5570E 02	0.9436E 02	0.2098E 03
MODE(6)	0.2886E 02	0.4273E 02	0.6562E 02	0.1110E 03	0.2465E 03
MODE(7)	0.3330E 02	0.4925E 02	0.7554E 02	0.1277E 03	0.2831E 03
MODE(8)	0.3773E 02	0.5577E 02	0.8547E 02	0.1443E 03	0.3199E 03
MODE(9)	0.4217E 02	0.6229E 02	0.9540E 02	0.1610E 03	0.3566E 03
MODE(10)	0.4661E 02	0.6881E 02	0.1053E 03	0.1777E 03	0.3933E 03

Table 7.10

ALPHA : 0.1

Q :	0.05	0.10	0.20	0.30	0.40	0.50	0.60	0.70	0.80	0.90
MODE(1)	0.1805E 00	0.7269E 00	0.2998E 01	0.7058E 01	0.1351E 02	0.2349E 02	0.3939E 02	0.6696E 02	0.1235E 03	0.2954E 03
MODE(2)	0.8156E 01	0.1726E 02	0.3870E 02	0.6550E 02	0.9965E 02	0.1447E 03	0.2077E 03	0.3052E 03	0.4873E 03	0.1007E 04
MODE(3)	0.2166E 02	0.4851E 02	0.1186E 03	0.2076E 03	0.3142E 03	0.4450E 03	0.6167E 03	0.8693E 03	0.1321E 04	0.2568E 04
MODE(4)	0.3597E 02	0.8491E 02	0.2377E 03	0.4584E 03	0.7174E 03	0.1017E 04	0.1392E 04	0.1925E 04	0.2855E 04	0.5379E 04
MODE(5)	0.4966E 02	0.1210E 03	0.3949E 03	0.8371E 03	0.1367E 04	0.1951E 04	0.2655E 04	0.3630E 04	0.5306E 04	0.9802E 04
MODE(6)	0.6283E 02	0.1560E 03	0.5519E 03	0.1365E 04	0.2323E 04	0.3339E 04	0.4528E 04	0.6145E 04	0.8895E 04	0.1423E 05
MODE(7)	0.7565E 02	0.1901E 03	0.7320E 03	0.2062E 04	0.3648E 04	0.5274E 04	0.7135E 04	0.9631E 04	0.1384E 05	0.2503E 05
MODE(8)	0.8821E 02	0.2233E 03	0.9203E 03	0.2953E 04	0.5404E 04	0.7849E 04	0.1060E 05	0.1425E 05	0.2037E 05	0.3659E 05
MODE(9)	0.1006E 03	0.2558E 03	0.1113E 04	0.4059E 04	0.7654E 04	0.1116E 05	0.1504E 05	0.2016E 05	0.2871E 05	0.5129E 05
MODE(10)	0.1129E 03	0.2880E 03	0.1308E 04	0.5404E 04	0.1046E 05	0.1529E 05	0.2059E 05	0.2752E 05	0.3906E 05	0.6951E 05

Table 8.1

The system parameter λ_L vs q for hinged free beams with torsional spring at the hinged end and linear spring at the free end with $\lambda_L > \lambda_t$.

ALPHA : 0.2

Q :	0.05	0.10	0.20	0.30	0.40	0.50	0.60	0.70	0.80	0.90
MODE(1)	0.1654E 00	0.6650E 00	0.2719E 01	0.6357E 01	0.1199E 02	0.2041E 02	0.3329E 02	0.5466E 02	0.9669E 02	0.2207E 03
MODE(2)	0.6063E 01	0.1350E 02	0.3272E 02	0.5802E 02	0.9056E 02	0.1332E 03	0.1919E 03	0.2811E 03	0.4439E 03	0.8970E 03
MODE(3)	0.1325E 02	0.3171E 02	0.9147E 02	0.1806E 03	0.2892E 03	0.4198E 03	0.5879E 03	0.8310E 03	0.1259E 04	0.2420E 04
MODE(4)	0.2019E 02	0.5025E 02	0.1722E 03	0.3994E 03	0.6713E 03	0.9762E 03	0.1349E 04	0.1872E 04	0.2773E 04	0.5187E 04
MODE(5)	0.2671E 02	0.6772E 02	0.2650E 03	0.7346E 03	0.1297E 04	0.1894E 04	0.2598E 04	0.3562E 04	0.5204E 04	0.9574E 04
MODE(6)	0.3302E 02	0.8446E 02	0.3641E 03	0.1209E 04	0.2227E 04	0.3265E 04	0.4458E 04	0.6063E 04	0.8774E 04	0.1596E 05
MODE(7)	0.3921E 02	0.1007E 03	0.4657E 03	0.1845E 04	0.3526E 04	0.5183E 04	0.7051E 04	0.9535E 04	0.1370E 05	0.2472E 05
MODE(8)	0.4532E 02	0.1167E 03	0.5675E 03	0.2668E 04	0.5255E 04	0.7742E 04	0.1050E 05	0.1414E 05	0.2021E 05	0.3624E 05
MODE(9)	0.5136E 02	0.1325E 03	0.6684E 03	0.3701E 04	0.7477E 04	0.1103E 05	0.1493E 05	0.2004E 05	0.2853E 05	0.5090E 05
MODE(10)	0.5739E 02	0.1481E 03	0.7679E 03	0.4968E 04	0.1026E 05	0.1515E 05	0.2047E 05	0.2738E 05	0.3886E 05	0.6908E 05

Table 8.2

ALPHA : 0.3

C :	0.05	0.10	0.20	0.30	0.40	0.50	0.60	0.70	0.80	0.90
MODE(1)	0.1597E 00	0.6143E 00	0.2516E 01	0.5892E 01	0.1111E 02	0.1889E 02	0.3067E 02	0.4993E 02	0.8719E 02	0.1955E 03
MODE(2)	0.4838E 01	0.1119E 02	0.2913E 02	0.5401E 02	0.9624E 02	0.1282E 03	0.1857E 03	0.2722E 03	0.4287E 03	0.8599E 03
MODE(3)	0.9554E 01	0.2369E 02	0.7693E 02	0.1675E 03	0.2785E 03	0.4099E 03	0.5773E 03	0.8173E 03	0.1237E 04	0.2370E 04
MODE(4)	0.1404E 02	0.3570E 02	0.1393E 03	0.3723E 03	0.6528E 03	0.9610E 03	0.1334E 04	0.1853E 04	0.2745E 04	0.5124E 04
MODE(5)	0.1827E 02	0.4708E 02	0.2077E 03	0.6875E 03	0.1270E 04	0.1873E 04	0.2578E 04	0.3539E 04	0.5170E 04	0.9408E 04
MODE(6)	0.2239E 02	0.5794E 02	0.2779E 03	0.1142E 04	0.2192E 04	0.3239E 04	0.4433E 04	0.6035E 04	0.8734E 04	0.1587E 05
MODE(7)	0.2645E 02	0.6856E 02	0.3476E 03	0.1755E 04	0.3481E 04	0.5152E 04	0.7022E 04	0.9503E 04	0.1366E 05	0.2462E 05
MODE(8)	0.3049E 02	0.7502E 02	0.4160E 03	0.2552E 04	0.5202E 04	0.7705E 04	0.1047E 05	0.1410E 05	0.2016E 05	0.3613E 05
MODE(9)	0.3449E 02	0.8937E 02	0.4828E 03	0.3558E 04	0.7415E 04	0.1099E 05	0.1489E 05	0.1999E 05	0.2847E 05	0.5077E 05
MODE(10)	0.3849E 02	0.9968E 02	0.5490E 03	0.4798E 04	0.1018E 05	0.1510E 05	0.2042E 05	0.2734E 05	0.3880E 05	0.6893E 05

Table 8.3

ALPHA : 0.4

Q :	0.05	0.10	0.20	0.30	0.40	0.50	0.60	0.70	0.80	0.90
MODE(1)	0.1419E 00	0.5718E 00	0.2355E 01	0.5553E 01	0.1053E 02	0.1795E 02	0.2917E 02	0.4737E 02	0.8228E 02	0.1827E 03
MODE(2)	0.4029E 01	0.2606E 01	0.2664E 02	0.5142E 02	0.8366E 02	0.1254E 03	0.1823E 03	0.2675E 03	0.4209E 03	0.8413E 03
MODE(3)	0.7472E 01	0.1894E 02	0.6752E 02	0.1594E 03	0.2724E 03	0.4045E 03	0.5717E 03	0.8103E 03	0.1227E 04	0.2345E 04
MODE(4)	0.1076E 02	0.2781E 02	0.1187E 03	0.3562E 03	0.6427E 03	0.9530E 03	0.1326E 04	0.1844E 04	0.2731E 04	0.5093E 04
MODE(5)	0.1388E 02	0.3609E 02	0.1727E 03	0.6634E 03	0.1255E 04	0.1862E 04	0.2568E 04	0.3527E 04	0.5152E 04	0.9459E 04
MODE(6)	0.1694E 02	0.4410E 02	0.2265E 03	0.1105E 04	0.2173E 04	0.3226E 04	0.4421E 04	0.6021E 04	0.8713E 04	0.1583E 05
MODE(7)	0.1997E 02	0.5196E 02	0.2790E 03	0.1705E 04	0.3458E 04	0.5136E 04	0.7007E 04	0.9486E 04	0.1363E 05	0.2457E 05
MODE(8)	0.2297E 02	0.5973E 02	0.3298E 03	0.2489E 04	0.5174E 04	0.7686E 04	0.1045E 05	0.1408E 05	0.2013E 05	0.3607E 05
MODE(9)	0.2595E 02	0.6744E 02	0.3791E 03	0.3481E 04	0.7383E 04	0.1097E 05	0.1487E 05	0.1997E 05	0.2844E 05	0.5071E 05
MODE(10)	0.2892E 02	0.7513E 02	0.4270E 03	0.4707E 04	0.1015E 05	0.1503E 05	0.2040E 05	0.2731E 05	0.3876E 05	0.6896E 05

Table 8.4

ALPHA : 0.5

C :	0.05	0.10	0.20	0.30	0.40	0.50	0.60	0.70	0.80	0.90
MODE(1)	0.1325E 00	0.5354E 00	0.2223E 01	0.5269E 01	0.1010E 02	0.1731E 02	0.2818E 02	0.4576E 02	0.7926E 02	0.1750E 03
MODE(2)	0.3453E 01	0.8434E 01	0.2477E 02	0.4959E 02	0.8193E 02	0.1236E 03	0.1802E 03	0.2647E 03	0.4162E 03	0.8301E 03
MODE(3)	0.6135E 01	0.1579E 02	0.6070E 02	0.1539E 03	0.2685E 03	0.4012E 03	0.5682E 03	0.8061E 03	0.1220E 04	0.2330E 04
MODE(4)	0.8723E 01	0.2275E 02	0.1042E 03	0.3455E 03	0.6364E 03	0.9480E 03	0.1322E 04	0.1838E 04	0.2722E 04	0.5074E 04
MODE(5)	0.1120E 02	0.2927E 02	0.1487E 03	0.6464E 03	0.1246E 04	0.1856E 04	0.2562E 04	0.3520E 04	0.5142E 04	0.9436E 04
MODE(6)	0.1362E 02	0.3560E 02	0.1920E 03	0.1081E 04	0.2161E 04	0.3217E 04	0.4414E 04	0.6013E 04	0.8701E 04	0.1580E 05
MODE(7)	0.1603E 02	0.4184E 02	0.2337E 03	0.1673E 04	0.3444E 04	0.5126E 04	0.6999E 04	0.9476E 04	0.1362E 05	0.2454E 05
MODE(8)	-0.1843E 02	0.4801E 02	0.2738E 03	0.2449E 04	0.5157E 04	0.7675E 04	0.1044E 05	0.1407E 05	0.2012E 05	0.3603E 05
MODE(9)	0.2081E 02	0.5415E 02	0.3125E 03	0.3433E 04	0.7364E 04	0.1096E 05	0.1486E 05	0.1996E 05	0.2842E 05	0.5067E 05
MODE(10)	0.2320E 02	0.6028E 02	0.3501E 03	0.4650E 04	0.1013E 05	0.1506E 05	0.2039E 05	0.2730E 05	0.3874E 05	0.6882E 05

Table 8.5

ALPHA : 0.6

C :	0.05	0.10	0.20	0.30	0.40	0.50	0.60	0.70	0.80	0.90
MODE(1)	0.1244E 00	0.5039E 00	0.2111E 01	0.5076E 01	0.9777E 01	0.1683E 02	0.2748E 02	0.4465E 02	0.7722E 02	0.1699E 03
MODE(2)	0.3023E 01	0.7529E 01	0.2330E 02	0.4821E 02	0.8069E 02	0.1224E 03	0.1787E 03	0.2627E 03	0.4130E 03	0.8226E 03
MODE(3)	0.5205E 01	0.1354E 02	0.5546E 02	0.1499E 03	0.2659E 03	0.3989E 03	0.5659E 03	0.8032E 03	0.1216E 04	0.2320E 04
MODE(4)	0.7334E 01	0.1925E 02	0.9328E 02	0.3378E 03	0.6320E 03	0.9447E 03	0.1318E 04	0.1834E 04	0.2717E 04	0.5061E 04
MODE(5)	0.9370E 01	0.2461E 02	0.1309E 03	0.6343E 03	0.1240E 04	0.1851E 04	0.2558E 04	0.3516E 04	0.5135E 04	0.9421E 04
MODE(6)	0.1139E 02	0.2984E 02	0.1670E 03	0.1064E 04	0.2154E 04	0.3212E 04	0.4409E 04	0.6007E 04	0.8693E 04	0.1578E 05
MODE(7)	0.1339E 02	0.3501E 02	0.2014E 03	0.1651E 04	0.3435E 04	0.5119E 04	0.6993E 04	0.9470E 04	0.1361E 05	0.2452E 05
MODE(8)	0.1539E 02	0.4014E 02	0.2342E 03	0.2421E 04	0.5146E 04	0.7667E 04	0.1043E 05	0.1406E 05	0.2011E 05	0.3601E 05
MODE(9)	0.1737E 02	0.4524E 02	0.2660E 03	0.3400E 04	0.7351E 04	0.1095E 05	0.1486E 05	0.1995E 05	0.2841E 05	0.5064E 05
MODE(10)	0.1935E 02	0.5033E 02	0.2968E 03	0.4611E 04	0.1011E 05	0.1505E 05	0.2038E 05	0.2729E 05	0.3873E 05	0.6879E 05

Table 8.6

ALPHA : 0.7

Q :	0.05	0.10	0.20	0.30	0.40	0.50	0.60	0.70	0.80	0.90
MODE(1)	0.1172E 00	0.4761E 00	0.2015E 01	0.4898E 01	0.9516E 01	0.1647E 02	0.2696E 02	0.4383E 02	0.7574E 02	0.1642E 03
MODE(2)	0.2698E 01	0.6807E 01	0.2209E 02	0.4713E 02	0.7975E 02	0.1215E 03	0.1777E 03	0.2613E 03	0.4107E 03	0.8172E 03
MODE(3)	0.4519E 01	0.1186E 02	0.5128E 02	0.1468E 03	0.2638E 03	0.3973E 03	0.5642E 03	0.8012E 03	0.1212E 04	0.2313E 04
MODE(4)	0.6329E 01	0.1668E 02	0.8472E 02	0.3320E 03	0.6288E 03	0.9423E 03	0.1316E 04	0.1832E 04	0.2712E 04	0.5052E 04
MODE(5)	0.8070E 01	0.2124E 02	0.1172E 03	0.6252E 03	0.1236E 04	0.1848E 04	0.2555E 04	0.3512E 04	0.5130E 04	0.9410E 04
MODE(6)	0.9792E 01	0.2569E 02	0.1480E 03	0.1051E 04	0.2148E 04	0.3208E 04	0.4405E 04	0.6003E 04	0.8687E 04	0.1577E 05
MODE(7)	0.1150E 02	0.3010E 02	0.1770E 03	0.1635E 04	0.3428E 04	0.5115E 04	0.6989E 04	0.9465E 04	0.1360E 05	0.2450E 05
MODE(8)	0.1321E 02	0.3448E 02	0.2048E 03	0.2401E 04	0.5138E 04	0.7662E 04	0.1043E 05	0.1406E 05	0.2010E 05	0.3599E 05
MODE(9)	0.1490E 02	0.3884E 02	0.2316E 03	0.3375E 04	0.7341E 04	0.1094E 05	0.1485E 05	0.1995E 05	0.2840E 05	0.5062E 05
MODE(10)	0.1660E 02	0.4320E 02	0.2577E 03	0.4583E 04	0.1010E 05	0.1505E 05	0.2038E 05	0.2728E 05	0.3872E 05	0.6877E 05

Table 8.7

ALPHA : 0.8

Q :	0.05	0.10	0.20	0.30	0.40	0.50	0.60	0.70	0.80	0.90
MODE(1)	0.1108E 00	0.4515E 00	0.1931E 01	0.4748E 01	0.9302E 01	0.1618E 02	0.2655E 02	0.4321E 02	0.7462E 02	0.1634E 03
MODE(2)	0.2420E 01	0.6215E 01	0.2109E 02	0.4625E 02	0.7901E 02	0.1207E 03	0.1769E 03	0.2603E 03	0.4090E 03	0.8132E 03
MODE(3)	0.3993E 01	0.1055E 02	0.4784E 02	0.1443E 03	0.2622E 03	0.3960E 03	0.5629E 03	0.7996E 03	0.1210E 04	0.2307E 04
MODE(4)	0.5563E 01	0.1472E 02	0.7778E 02	0.3275E 03	0.6264E 03	0.9404E 03	0.1314E 04	0.1830E 04	0.2709E 04	0.5045E 04
MODE(5)	0.7081E 01	0.1867E 02	0.1063E 03	0.6182E 03	0.1233E 04	0.1846E 04	0.2553E 04	0.3510E 04	0.5126E 04	0.9402E 04
MODE(6)	0.8594E 01	0.2255E 02	0.1330E 03	0.1041E 04	0.2144E 04	0.3205E 04	0.4402E 04	0.6000E 04	0.8682E 04	0.1576E 05
MODE(7)	0.1009E 02	0.2640E 02	0.1581E 03	0.1622E 04	0.3422E 04	0.5111E 04	0.6985E 04	0.9462E 04	0.1360E 05	0.2449E 05
MODE(8)	0.1157E 02	0.3023E 02	0.1820E 03	0.2385E 04	0.5132E 04	0.7658E 04	0.1043E 05	0.1405E 05	0.2009E 05	0.3598E 05
MODE(9)	0.1305E 02	0.3403E 02	0.2051E 03	0.3357E 04	0.7334E 04	0.1094E 05	0.1485E 05	0.1994E 05	0.2839E 05	0.5061E 05
MODE(10)	0.1453E 02	0.3784E 02	0.2277E 03	0.4561E 04	0.1009E 05	0.1504E 05	0.2037E 05	0.2728E 05	0.3871E 05	0.6875E 05

Table 8.8

ALPHA : 0.9

C :	0.05	0.10	0.20	0.30	0.40	0.50	0.60	0.70	0.80	0.90
MODE(1)	0.1051E 00	0.4296E 00	0.1857E 01	0.4618E 01	0.9122E 01	0.1594E 02	0.2623E 02	0.4271E 02	0.7375E 02	0.1613E 03
MODE(2)	0.2201E 01	0.5721E 01	0.2021E 02	0.4553E 02	0.7842E 02	0.1202E 03	0.1763E 03	0.2594E 03	0.4076E 03	0.8100E 03
MODE(3)	0.3577E 01	0.9501E 01	0.4494E 02	0.1423E 03	0.2610E 03	0.3950E 03	0.5619E 03	0.7984E 03	0.1208E 04	0.2303E 04
MODE(4)	0.4965E 01	0.1317E 02	0.7201E 02	0.3238E 03	0.6244E 03	0.9390E 03	0.1313E 04	0.1828E 04	0.2707E 04	0.5040E 04
MODE(5)	0.6310E 01	0.1666E 02	0.9732E 02	0.6126E 03	0.1230E 04	0.1844E 04	0.2551E 04	0.3508E 04	0.5124E 04	0.9395E 04
MODE(6)	0.7641E 01	0.2010E 02	0.1208E 03	0.1034E 04	0.2140E 04	0.3203E 04	0.4400E 04	0.5998E 04	0.8679E 04	0.1575E 05
MODE(7)	0.9969E 01	0.2351E 02	0.1428E 03	0.1612E 04	0.3418E 04	0.5108E 04	0.6983E 04	0.9459E 04	0.1359E 05	0.2448E 05
MODE(8)	0.1029E 02	0.2690E 02	0.1638E 03	0.2373E 04	0.5127E 04	0.7654E 04	0.1042E 05	0.1405E 05	0.2009E 05	0.3597E 05
MODE(9)	0.1160E 02	0.3028E 02	0.1841E 03	0.3342E 04	0.7329E 04	0.1093E 05	0.1484E 05	0.1994E 05	0.2839E 05	0.5060E 05
MODE(10)	0.1293E 02	0.3366E 02	0.2040E 03	0.4544E 04	0.1009E 05	0.1504E 05	0.2037E 05	0.2727E 05	0.3871E 05	0.6874E 05

Table 8.9

ALPHA : 1.0

Q :	0.05	0.10	0.20	0.30	0.40	0.50	0.60	0.70	0.80	0.90
MODE(1)	0.9991E-01	0.4098E 00	0.1790E 01	0.4504E 01	0.8969E 01	0.1575E 02	0.2596E 02	0.4231E 02	0.7304E 02	0.1595E 03
MODE(2)	0.2019E 01	0.5301E 01	0.1946E 02	0.4492E 02	0.7793E 02	0.1197E 03	0.1757E 03	0.2588E 03	0.4066E 03	0.8075E 03
MODE(3)	0.3240E 01	0.8642E 01	0.4245E 02	0.1407E 03	0.2600E 03	0.3942E 03	0.5611E 03	0.7974E 03	0.1207E 04	0.2300E 04
MODE(4)	0.4492E 01	0.1191E 02	0.6712E 02	0.3208E 03	0.6229E 03	0.9379E 03	0.1312E 04	0.1827E 04	0.2705E 04	0.5035E 04
MODE(5)	0.5668E 01	0.1504E 02	0.8982E 02	0.6079E 03	0.1228E 04	0.1842E 04	0.2550E 04	0.3506E 04	0.5121E 04	0.9390E 04
MODE(6)	0.6885E 01	0.1813E 02	0.1107E 03	0.1027E 04	0.2138E 04	0.3201E 04	0.4399E 04	0.5996E 04	0.8676E 04	0.1575E 05
MODE(7)	0.8079E 01	0.2119E 02	0.1303E 03	0.1604E 04	0.3415E 04	0.5106E 04	0.6981E 04	0.9457E 04	0.1359E 05	0.2448E 05
MODE(8)	0.9265E 01	0.2424E 02	0.1499E 03	0.2363E 04	0.5123E 04	0.7652E 04	0.1042E 05	0.1405E 05	0.2009E 05	0.3596E 05
MODE(9)	0.1045E 02	0.2728E 02	0.1670E 03	0.3330E 04	0.7324E 04	0.1093E 05	0.1484E 05	0.1993E 05	0.2838E 05	0.5059E 05
MODE(10)	0.1164E 02	0.3032E 02	0.1848E 03	0.4530E 04	0.1008E 05	0.1504E 05	0.2036E 05	0.2727E 05	0.3870E 05	0.6873E 05

Table 8.10

ALPHA : 0.1

C :	0.05	0.10	0.20	0.30	0.40	0.50	0.60	0.70	0.80	0.90
MODE(1)	0.1870E 00	0.8440E 00	0.5987E 01	0.2815E 02	0.7195E 02	0.1378E 03	0.2352E 03	0.3886E 03	0.6717E 03	0.1454E 04
MODE(2)	0.2309E 01	0.7141E 01	0.8273E 02	0.3847E 03	0.7343E 03	0.1157E 04	0.1714E 04	0.2532E 04	0.3977E 04	0.7671E 04
MODE(3)	0.3412E 01	0.9482E 01	0.9425E 02	0.1243E 04	0.2512E 04	0.3874E 04	0.5545E 04	0.7894E 04	0.1195E 05	0.2272E 05
MODE(4)	0.4555E 01	0.1246E 02	0.1064E 03	0.2925E 04	0.6097E 04	0.9283E 04	0.1303E 05	0.1816E 05	0.2690E 05	0.5001E 05
MODE(5)	0.5775E 01	0.1544E 02	0.1190E 03	0.5664E 04	0.1210E 05	0.1830E 05	0.2539E 05	0.3493E 05	0.5102E 05	0.9349E 05
MODE(6)	0.6536E 01	0.1843E 02	0.1331E 03	0.9716E 04	0.2115E 05	0.3186E 05	0.4385E 05	0.5980E 05	0.8654E 05	0.1570E 06
MODE(7)	0.8055E 01	0.2144E 02	0.1481E 03	0.1534E 05	0.3398E 05	0.5088E 05	0.6965E 05	0.9439E 05	0.1356E 06	0.2442E 06
MODE(8)	0.9294E 01	0.2446E 02	0.1635E 03	0.2278E 05	0.5092E 05	0.7631E 05	0.1040E 06	0.1403E 06	0.2006E 06	0.3590E 06
MODE(9)	0.1042E 02	0.2743E 02	0.1793E 03	0.3229E 05	0.7288E 05	0.1091E 06	0.1482E 06	0.1991E 06	0.2835E 06	0.5052E 06
MODE(10)	0.1166E 02	0.3046E 02	0.1953E 03	0.4414E 05	0.1004E 06	0.1501E 06	0.2034E 06	0.2725E 06	0.3867E 06	0.6866E 06

Table 9.1

: The system parameter λ_t vs q for hinged - free beams with torsional spring at the hinged end and a linear spring at the free end with $\lambda_t > \lambda_L$

ALPHA : 0.2

Q :	0.05	0.10	0.20	0.30	0.40	0.50	0.60	0.70	0.80	0.90
MODE(1)	0.1702E 00	0.7482E 00	0.4415E 01	0.1578E 02	0.3735E 02	0.7022E 02	0.1191E 03	0.1963E 03	0.3392E 03	0.7349E 03
MODE(2)	0.2342E 01	0.6856E 01	0.5332E 02	0.1969E 03	0.3700E 03	0.5809E 03	0.8596E 03	0.1269E 04	0.1994E 04	0.3947E 04
MODE(3)	0.3393E 01	0.9378E 01	0.7843E 02	0.6324E 03	0.1261E 04	0.1941E 04	0.2776E 04	0.3952E 04	0.5980E 04	0.1138E 05
MODE(4)	0.4583E 01	0.1240E 02	0.9803E 02	0.1480E 04	0.3056E 04	0.4647E 04	0.6520E 04	0.9088E 04	0.1346E 05	0.2502E 05
MODE(5)	0.5753E 01	0.1539E 02	0.1141E 03	0.2857E 04	0.6060E 04	0.9157E 04	0.1270E 05	0.1747E 05	0.2552E 05	0.4577E 05
MODE(6)	0.6944E 01	0.1840E 02	0.1298E 03	0.4891E 04	0.1059E 05	0.1594E 05	0.2193E 05	0.2991E 05	0.4328E 05	0.7851E 05
MODE(7)	0.8122E 01	0.2141E 02	0.1457E 03	0.7710E 04	0.1696E 05	0.2545E 05	0.3483E 05	0.4720E 05	0.6784E 05	0.1221E 06
MODE(8)	0.9284E 01	0.2440E 02	0.1616E 03	0.1144E 05	0.2546E 05	0.3817E 05	0.5202E 05	0.7016E 05	0.1003E 06	0.1795E 06
MODE(9)	0.1049E 02	0.2743E 02	0.1778E 03	0.1621E 05	0.3646E 05	0.5455E 05	0.7411E 05	0.9957E 05	0.1418E 06	0.2526E 06
MODE(10)	0.1160E 02	0.3044E 02	0.1940E 03	0.2214E 05	0.5023E 05	0.7506E 05	0.1017E 06	0.1362E 06	0.1934E 06	0.3433E 06

Table 9.2

ALPHA : 0.3

C :	0.05	0.10	0.20	0.30	0.40	0.50	0.60	0.70	0.80	0.90
MODE(1)	0.1543E 00	0.6743E 00	0.3609E 01	0.1140E 02	0.2571E 02	0.4764E 02	0.8038E 02	0.1322E 03	0.2283E 03	0.4952E 03
MODE(2)	0.2295E 01	0.6599E 01	0.4140E 02	0.1341E 03	0.2484E 03	0.3888E 03	0.5747E 03	0.8482E 03	0.1332E 04	0.2639E 04
MODE(3)	0.3373E 01	0.9278E 01	0.6880E 02	0.4284E 03	0.8442E 03	0.1297E 04	0.1853E 04	0.2637E 04	0.3991E 04	0.7595E 04
MODE(4)	0.4569E 01	0.1233E 02	0.9154E 02	0.9982E 03	0.2042E 04	0.3102E 04	0.4350E 04	0.6062E 04	0.8977E 04	0.1670E 05
MODE(5)	0.5755E 01	0.1535E 02	0.1098E 03	0.1921E 04	0.4047E 04	0.6109E 04	0.8470E 04	0.1165E 05	0.1702E 05	0.3119E 05
MODE(6)	0.6936E 01	0.1837E 02	0.1268E 03	0.3282E 04	0.7068E 04	0.1063E 05	0.1463E 05	0.1995E 05	0.2886E 05	0.5236E 05
MODE(7)	0.8111E 01	0.2138E 02	0.1434E 03	0.5167E 04	0.1131E 05	0.1697E 05	0.2323E 05	0.3148E 05	0.4523E 05	0.8144E 05
MODE(8)	0.9300E 01	0.2440E 02	0.1598E 03	0.7658E 04	0.1700E 05	0.2545E 05	0.3469E 05	0.4678E 05	0.6688E 05	0.1197E 06
MODE(9)	0.1046E 02	0.2741E 02	0.1763E 03	0.1084E 05	0.2432E 05	0.3637E 05	0.4941E 05	0.6639E 05	0.9453E 05	0.1685E 06
MODE(10)	0.1166E 02	0.3044E 02	0.1927E 03	0.1480E 05	0.3350E 05	0.5005E 05	0.6782E 05	0.9084E 05	0.1289E 06	0.2289E 06

Table 9.3

ALPHA : 0.4

C :	0.05	0.10	0.20	0.30	0.40	0.50	0.60	0.70	0.80	0.90
MODE(1)	0.1445E 00	0.6152E 00	0.3096E 01	0.3099E 01	0.1984E 02	0.3632E 02	0.6099E 02	0.1001E 03	0.1729E 03	0.3753E 03
MODE(2)	0.2251E 01	0.6366E 01	0.3462E 02	0.1025E 03	0.1876E 03	0.2928E 03	0.4322E 03	0.6377E 03	0.1002E 04	0.1985E 04
MODE(3)	0.3353E 01	0.9180E 01	0.6206E 02	0.3262E 03	0.6357E 03	0.9743E 03	0.1392E 04	0.1980E 04	0.2997E 04	0.5704E 04
MODE(4)	0.4558E 01	0.1227E 02	0.8629E 02	0.7570E 03	0.1536E 04	0.2329E 04	0.3265E 04	0.4550E 04	0.6737E 04	0.1253E 05
MODE(5)	0.5745E 01	0.1531E 02	0.1060E 03	0.1453E 04	0.3040E 04	0.4585E 04	0.6356E 04	0.8744E 04	0.1277E 05	0.2341E 05
MODE(6)	0.6928E 01	0.1833E 02	0.1240E 03	0.2478E 04	0.5307E 04	0.7977E 04	0.1097E 05	0.1496E 05	0.2165E 05	0.3928E 05
MODE(7)	0.8100E 01	0.2136E 02	0.1412E 03	0.3895E 04	0.8493E 04	0.1274E 05	0.1743E 05	0.2361E 05	0.3393E 05	0.6110E 05
MODE(8)	0.9206E 01	0.2438E 02	0.1581E 03	0.5768E 04	0.1276E 05	0.1910E 05	0.2602E 05	0.3509E 05	0.5017E 05	0.8981E 05
MODE(9)	0.1049E 02	0.2739E 02	0.1749E 03	0.8160E 04	0.1825E 05	0.2729E 05	0.3707E 05	0.4980E 05	0.7090E 05	0.1264E 06
MODE(10)	0.1166E 02	0.3042E 02	0.1915E 03	0.1113E 05	0.2514E 05	0.3754E 05	0.5087E 05	0.6814E 05	0.9670E 05	0.1717E 06

Table 9.4

ALPHA : 0.5

Q :	0.05	0.10	0.20	0.30	0.40	0.50	0.60	0.70	0.80	0.90
MODE(1)	0.1344E 00	0.5664E 00	0.2732E 01	0.7655E 01	0.1628E 02	0.2951E 02	0.4935E 02	0.8088E 02	0.1396E 03	0.3034E 03
MODE(2)	0.2208E 01	0.6152E 01	0.3013E 02	0.8352E 02	0.1511E 03	0.2351E 03	0.3468E 03	0.5114E 03	0.8034E 03	0.1592E 04
MODE(3)	0.3334E 01	0.9085E 01	0.5698E 02	0.2647E 03	0.5105E 03	0.7810E 03	0.1115E 04	0.1586E 04	0.2400E 04	0.4569E 04
MODE(4)	0.4545E 01	0.1221E 02	0.8189E 02	0.6130E 03	0.1231E 04	0.1865E 04	0.2614E 04	0.3642E 04	0.5393E 04	0.1003E 05
MODE(5)	0.5736E 01	0.1526E 02	0.1027E 03	0.1172E 04	0.2436E 04	0.3671E 04	0.5087E 04	0.6998E 04	0.1022E 05	0.1873E 05
MODE(6)	0.6920E 01	0.1830E 02	0.1214E 03	0.1995E 04	0.4251E 04	0.6385E 04	0.8782E 04	0.1197E 05	0.1733E 05	0.3144E 05
MODE(7)	0.9105E 01	0.2133E 02	0.1392E 03	0.3132E 04	0.6801E 04	0.1019E 05	0.1394E 05	0.1889E 05	0.2715E 05	0.4889E 05
MODE(8)	0.9288E 01	0.2436E 02	0.1564E 03	0.4634E 04	0.1021E 05	0.1528E 05	0.2082E 05	0.2808E 05	0.4014E 05	0.7186E 05
MODE(9)	0.1047E 02	0.2738E 02	0.1735E 03	0.6550E 04	0.1461E 05	0.2183E 05	0.2966E 05	0.3984E 05	0.5673E 05	0.1011E 06
MODE(10)	0.1166E 02	0.3040E 02	0.1903E 03	0.8933E 04	0.2012E 05	0.3004E 05	0.4070E 05	0.5452E 05	0.7736E 05	0.1374E 06

Table 9.5

ALPHA : 0.6

C :	0.05	0.10	0.20	0.30	0.40	0.50	0.60	0.70	0.80	0.90
MODE(1)	0.1257E 00	0.5254E 00	0.2457E 01	0.6654E 01	0.1388E 02	0.2495E 02	0.4157E 02	0.6804E 02	0.1174E 03	0.2555E 03
MODE(2)	0.2167E 01	0.5955E 01	0.2688E 02	0.7077E 02	0.1268E 03	0.1967E 03	0.2898E 03	0.4272E 03	0.6711E 03	0.1331E 04
MODE(3)	0.3314E 01	0.9992E 01	0.5296E 02	0.2236E 03	0.4270E 03	0.6520E 03	0.9304E 03	0.1323E 04	0.2002E 04	0.3813E 04
MODE(4)	0.4532E 01	0.1215E 02	0.7814E 02	0.5152E 03	0.1029E 04	0.1556E 04	0.2180E 04	0.3037E 04	0.4497E 04	0.8367E 04
MODE(5)	0.5726E 01	0.1522E 02	0.9962E 02	0.9841E 03	0.2033E 04	0.3062E 04	0.4242E 04	0.5854E 04	0.8521E 04	0.1562E 05
MODE(6)	0.6912E 01	0.1826E 02	0.1190E 03	0.1673E 04	0.3546E 04	0.5324E 04	0.7321E 04	0.9982E 04	0.1444E 05	0.2621E 05
MODE(7)	0.8100E 01	0.2130E 02	0.1372E 03	0.2623E 04	0.5672E 04	0.8497E 04	0.1162E 05	0.1575E 05	0.2263E 05	0.4075E 05
MODE(8)	0.9284E 01	0.2433E 02	0.1548E 03	0.3877E 04	0.8515E 04	0.1274E 05	0.1735E 05	0.2340E 05	0.3345E 05	0.5989E 05
MODE(9)	0.1047E 02	0.2736E 02	0.1721E 03	0.5477E 04	0.1218E 05	0.1820E 05	0.2472E 05	0.3321E 05	0.4728E 05	0.8427E 05
MODE(10)	0.1166E 02	0.3039E 02	0.1892E 03	0.7466E 04	0.1677E 05	0.2504E 05	0.3392E 05	0.4543E 05	0.6448E 05	0.1145E 06

Table 9.6

ALPHA : 0.7

Q :	0.05	0.10	0.20	0.30	0.40	0.50	0.60	0.70	0.80	0.90
MODE(1)	0.1190E 00	0.4904E 00	0.2240E 01	0.5914E 01	0.1215E 02	0.2168E 02	0.3601E 02	0.5886E 02	0.1016E 03	0.2212E 03
MODE(2)	0.2128E 01	0.5774E 01	0.2440E 02	0.6160E 02	0.1094E 03	0.1692E 03	0.2491E 03	0.3670E 03	0.5766E 03	0.1144E 04
MODE(3)	0.3225E 01	0.8991E 01	0.4966E 02	0.1941E 03	0.3674E 03	0.5600E 03	0.7985E 03	0.1135E 04	0.1718E 04	0.3272E 04
MODE(4)	0.4520E 01	0.1209E 02	0.7487E 02	0.4459E 03	0.8837E 03	0.1335E 04	0.1870E 04	0.2605E 04	0.3857E 04	0.7177E 04
MODE(5)	0.5716E 01	0.1517E 02	0.9685E 02	0.8500E 03	0.1746E 04	0.2626E 04	0.3637E 04	0.5002E 04	0.7307E 04	0.1339E 05
MODE(6)	0.6908E 01	0.1823E 02	0.1167E 03	0.1442E 04	0.3043E 04	0.4566E 04	0.6277E 04	0.8558E 04	0.1238E 05	0.2247E 05
MODE(7)	0.9094E 01	0.2127E 02	0.1354E 03	0.2259E 04	0.4866E 04	0.7286E 04	0.9965E 04	0.1350E 05	0.1940E 05	0.3494E 05
MODE(8)	0.9277E 01	0.2431E 02	0.1533E 03	0.3337E 04	0.7304E 04	0.1092E 05	0.1488E 05	0.2006E 05	0.2868E 05	0.5135E 05
MODE(9)	0.1046E 02	0.2734E 02	0.1708E 03	0.4711E 04	0.1045E 05	0.1560E 05	0.2119E 05	0.2847E 05	0.4053E 05	0.7224E 05
MODE(10)	0.1163E 02	0.3037E 02	0.1880E 03	0.6418E 04	0.1438E 05	0.2147E 05	0.2908E 05	0.3895E 05	0.5527E 05	0.9815E 05

Table 9.7

ALPHA : 0.8

C :	0.05	0.10	0.20	0.30	0.40	0.50	0.60	0.70	0.80	0.90
MODE(1)	0.1113E 00	0.4600E 00	0.2063E 01	0.5340E 01	0.1084E 02	0.1922E 02	0.3183E 02	0.5197E 02	0.8970E 02	0.1955E 03
MODE(2)	0.2090E 01	0.5605E 01	0.2243E 02	0.5469E 02	0.9628E 02	0.1486E 03	0.2165E 03	0.3219E 03	0.5058E 03	0.1004E 04
MODE(3)	0.3276E 01	0.8813E 01	0.4699E 02	0.1719E 03	0.3227E 03	0.4909E 03	0.6996E 03	0.9946E 03	0.1505E 04	0.2867E 04
MODE(4)	0.4507E 01	0.1203E 02	0.7199E 02	0.3939E 03	0.7751E 03	0.1170E 04	0.1637E 04	0.2281E 04	0.3377E 04	0.6285E 04
MODE(5)	0.5707E 01	0.1513E 02	0.9432E 02	0.7492E 03	0.1530E 04	0.2300E 04	0.3184E 04	0.4379E 04	0.6396E 04	0.1173E 05
MODE(6)	0.6900E 01	0.1819E 02	0.1146E 03	0.1270E 04	0.2666E 04	0.3997E 04	0.5494E 04	0.7491E 04	0.1084E 05	0.1967E 05
MODE(7)	0.8083E 01	0.2125E 02	0.1336E 03	0.1986E 04	0.4262E 04	0.6378E 04	0.8722E 04	0.1182E 05	0.1698E 05	0.3058E 05
MODE(8)	0.9273E 01	0.2428E 02	0.1518E 03	0.2931E 04	0.6395E 04	0.9559E 04	0.1302E 05	0.1756E 05	0.2510E 05	0.4494E 05
MODE(9)	0.1046E 02	0.2732E 02	0.1695E 03	0.4136E 04	0.9145E 04	0.1366E 05	0.1854E 05	0.2491E 05	0.3547E 05	0.6322E 05
MODE(10)	0.1163E 02	0.3035E 02	0.1869E 03	0.5631E 04	0.1259E 05	0.1879E 05	0.2545E 05	0.3408E 05	0.4837E 05	0.8590E 05

Table 9.8

ALPHA : 0.9

C :	0.05	0.10	0.20	0.30	0.40	0.50	0.60	0.70	0.80	0.90
MODE(1)	0.1053E 00	0.4334E 00	0.1916E 01	0.4881E 01	0.9804E 01	0.1729E 02	0.2857E 02	0.4661E 02	0.8045E 02	0.1755E 03
MODE(2)	0.2054E 01	0.5448E 01	0.2081E 02	0.4928E 02	0.8609E 02	0.1326E 03	0.1948E 03	0.2268E 03	0.4507E 03	0.8947E 03
MODE(3)	0.3258E 01	0.8726E 01	0.4452E 02	0.1546E 03	0.2878E 03	0.4372E 03	0.6227E 03	0.8850E 03	0.1339E 04	0.2552E 04
MODE(4)	0.4494E 01	0.1157E 02	0.6943E 02	0.3533E 03	0.6905E 03	0.1041E 04	0.1457E 04	0.2028E 04	0.3004E 04	0.5591E 04
MODE(5)	0.5698E 01	0.1509E 02	0.9198E 02	0.6708E 03	0.1362E 04	0.2046E 04	0.2832E 04	0.3894E 04	0.5688E 04	0.1043E 05
MODE(6)	0.6893E 01	0.1816E 02	0.1126E 03	0.1135E 04	0.2372E 04	0.3555E 04	0.4886E 04	0.6660E 04	0.9637E 04	0.1749E 05
MODE(7)	0.8083E 01	0.2122E 02	0.1319E 03	0.1774E 04	0.3791E 04	0.5671E 04	0.7755E 04	0.1051E 05	0.1510E 05	0.2719E 05
MODE(8)	0.9265E 01	0.2426E 02	0.1503E 03	0.2616E 04	0.5689E 04	0.8500E 04	0.1158E 05	0.1561E 05	0.2231E 05	0.3995E 05
MODE(9)	0.1045E 02	0.2730E 02	0.1683E 03	0.3688E 04	0.8134E 04	0.1214E 05	0.1649E 05	0.2215E 05	0.3153E 05	0.5620E 05
MODE(10)	0.1163E 02	0.3033E 02	0.1858E 03	0.5020E 04	0.1120E 05	0.1670E 05	0.2262E 05	0.3030E 05	0.4300E 05	0.7636E 05

Table 9.9

ALPHA : 1.0

Q :	0.05	0.10	0.20	0.30	0.40	0.50	0.60	0.70	0.80	0.90
MODE(1)	0.9991E-01	0.4098E 00	0.1790E 01	0.4504E 01	0.8969E 01	0.1575E 02	0.2596E 02	0.4231E 02	0.7304E 02	0.1595E 03
MODE(2)	0.2019E 01	0.5301E 01	0.1946E 02	0.4492E 02	0.7793E 02	0.1197E 03	0.1757E 03	0.2588E 03	0.4066E 03	0.8075E 03
MODE(3)	0.3239E 01	0.8642E 01	0.4245E 02	0.1407E 03	0.2600E 03	0.3942E 03	0.5611E 03	0.7974E 03	0.1207E 04	0.2300E 04
MODE(4)	0.4492E 01	0.1191E 02	0.6712E 02	0.3208E 03	0.6229E 03	0.9379E 03	0.1312E 04	0.1827E 04	0.2705E 04	0.5035E 04
MODE(5)	0.5698E 01	0.1504E 02	0.8982E 02	0.6079E 03	0.1229E 04	0.1842E 04	0.2550E 04	0.3506E 04	0.5121E 04	0.9390E 04
MODE(6)	0.6885E 01	0.1813E 02	0.1107E 03	0.1027E 04	0.2133E 04	0.3201E 04	0.4399E 04	0.5996E 04	0.8676E 04	0.1575E 05
MODE(7)	0.8072E 01	0.2119E 02	0.1303E 03	0.1604E 04	0.3415E 04	0.5106E 04	0.6981E 04	0.9457E 04	0.1359E 05	0.2448E 05
MODE(8)	0.9265E 01	0.2424E 02	0.1489E 03	0.2363E 04	0.5123E 04	0.7652E 04	0.1042E 05	0.1405E 05	0.2009E 05	0.3596E 05
MODE(9)	0.1045E 02	0.2728E 02	0.1670E 03	0.3330E 04	0.7324E 04	0.1093E 05	0.1484E 05	0.1993E 05	0.2838E 05	0.5059E 05
MODE(10)	0.1164E 02	0.3032E 02	0.1848E 03	0.4530E 04	0.1008E 05	0.1504E 05	0.2036E 05	0.2727E 05	0.3870E 05	0.6873E 05

Table 9.10

Q :	0.05	0.10	0.20	0.30	0.40	0.50	0.60	0.70	0.80	0.90
MODE(1)	0.1033E-01	0.4161E-01	0.1715E 00	0.4065E 00	0.7813E 00	0.1364E 01	0.2294E 01	0.3903E 01	0.7189E 01	0.1715E 02
MODE(2)	0.3672E 00	0.7735E 00	0.1732E 01	0.2956E 01	0.4577E 01	0.6833E 01	0.1020E 02	0.1579E 02	0.2693E 02	0.6027E 02
MODE(3)	0.6116E 00	0.1285E 01	0.2862E 01	0.4854E 01	0.7473E 01	0.1109E 02	0.1646E 02	0.2534E 02	0.4298E 02	0.9561E 02
MODE(4)	0.8688E 00	0.1823E 01	0.4046E 01	0.6842E 01	0.1050E 02	0.1554E 02	0.2299E 02	0.3528E 02	0.5966E 02	0.1324E 03
MODE(5)	0.1125E 01	0.2359E 01	0.5227E 01	0.8824E 01	0.1352E 02	0.1997E 02	0.2951E 02	0.4521E 02	0.7633E 02	0.1691E 03
MODE(6)	0.1381E 01	0.2895E 01	0.6407E 01	0.1081E 02	0.1654E 02	0.2441E 02	0.3603E 02	0.5514E 02	0.9299E 02	0.2057E 03
MODE(7)	0.1638E 01	0.3430E 01	0.7588E 01	0.1279E 02	0.1956E 02	0.2885E 02	0.4255E 02	0.6507E 02	0.1097E 03	0.2423E 03
MODE(8)	0.1894E 01	0.3967E 01	0.8770E 01	0.1477E 02	0.2258E 02	0.3329E 02	0.4907E 02	0.7501E 02	0.1263E 03	0.2791E 03
MODE(9)	0.2149E 01	0.4503E 01	0.9951E 01	0.1675E 02	0.2560E 02	0.3773E 02	0.5559E 02	0.8494E 02	0.1430E 03	0.3158E 03
MODE(10)	0.2407E 01	0.5038E 01	0.1113E 02	0.1874E 02	0.2862E 02	0.4217E 02	0.6211E 02	0.9488E 02	0.1597E 03	0.3525E 03

Table 10.0

The system parameter λ_m vs q for clamped free beams with a concentrated mass at the free end .

Mode	β_{cF}	β_{cs}	β_{cc}	β_{sF}	β_{ss}	β_{FF}
1	1.875	3.926	4.730	0	3.141	0
2	4.694	7.068	7.853	3.926	6.283	4.730
3	7.854	10.210	10.995	7.068	9.424	7.853
4	10.995	13.351	14.137	10.210	12.566	10.995
5	14.137	16.493	17.278	13.351	15.707	14.137
6	17.278	19.634	20.420	16.493	18.849	17.278
7	20.420	22.776	23.561	19.634	21.991	20.420
8	23.561	25.918	26.703	22.776	25.132	23.561
9	26.703	29.059	29.845	25.918	28.274	26.703
10	29.845	32.201	32.986	29.059	31.415	29.845

Table 11

Eigenvalues β for the first ten modes of vibration of uniform beams with classical end conditions .

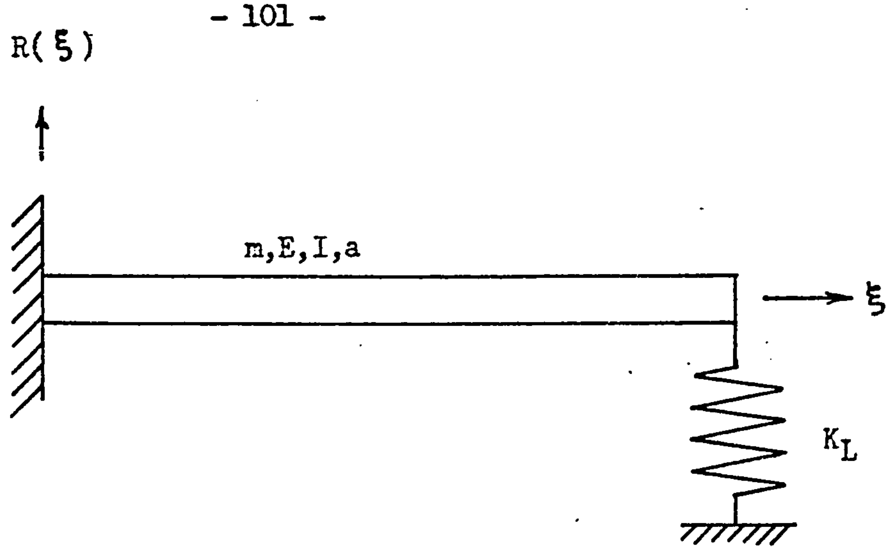

Fig. 1. Clamped - elastically supported uniform beam

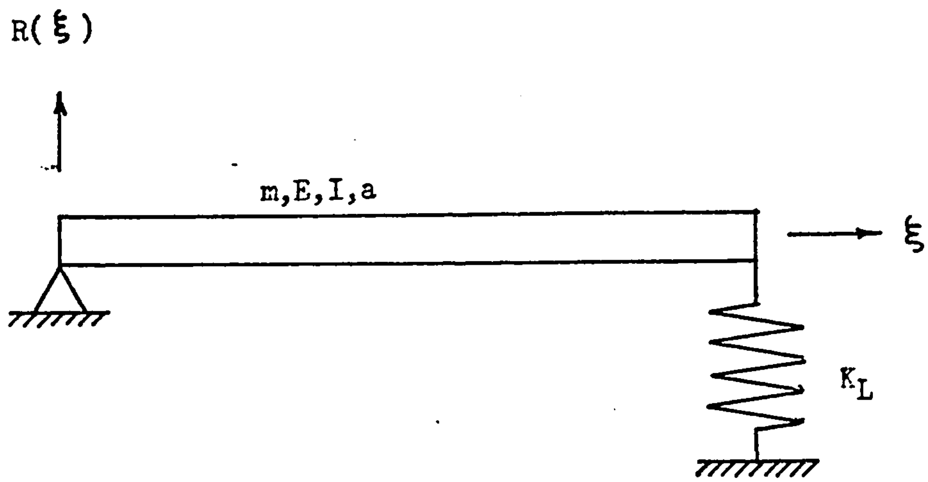

Fig.2 Pinned - elastically supported uniform beam

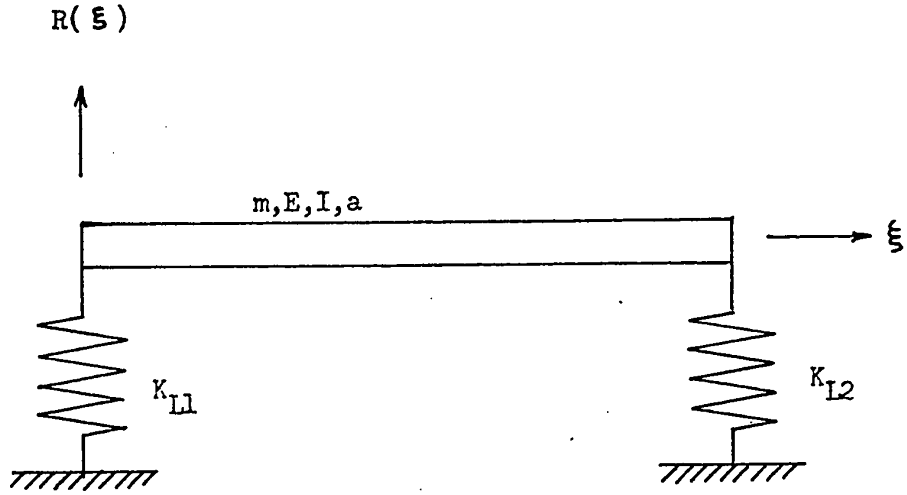

Fig.3 Uniform beam with both ends elastically supported .

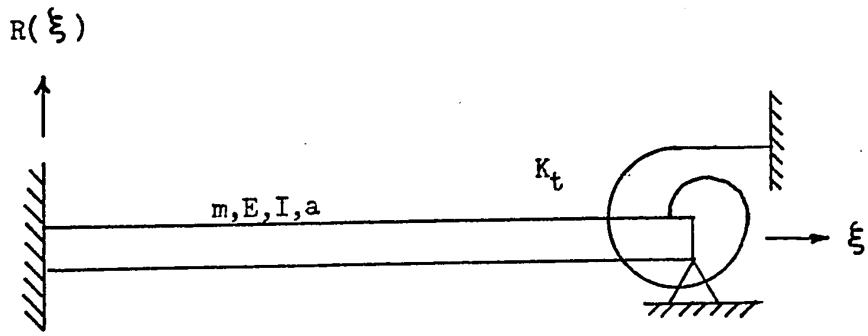

Fig.4 Clamped - partly clamped uniform beam.

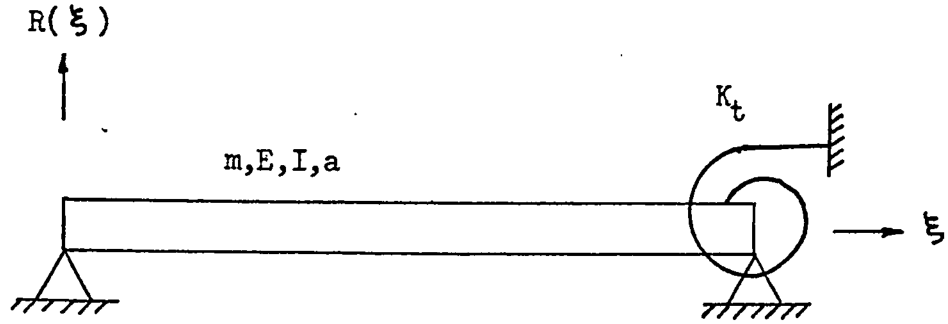

Fig.5 Pinned - partly clamped uniform beam .

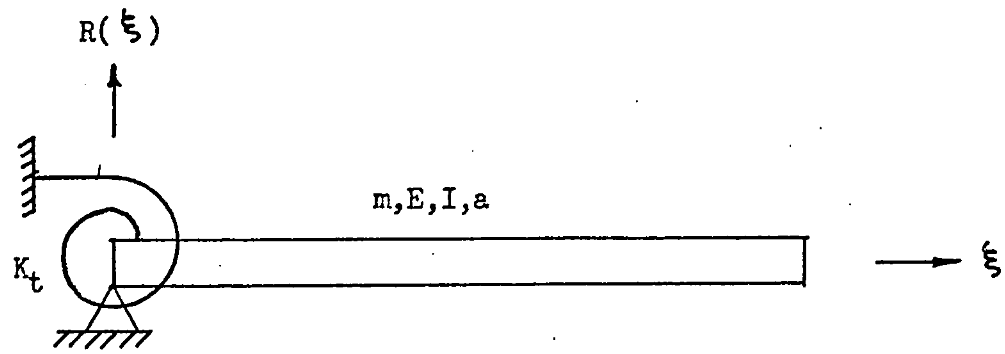

Fig.6 Partly clamped - free uniform beam .

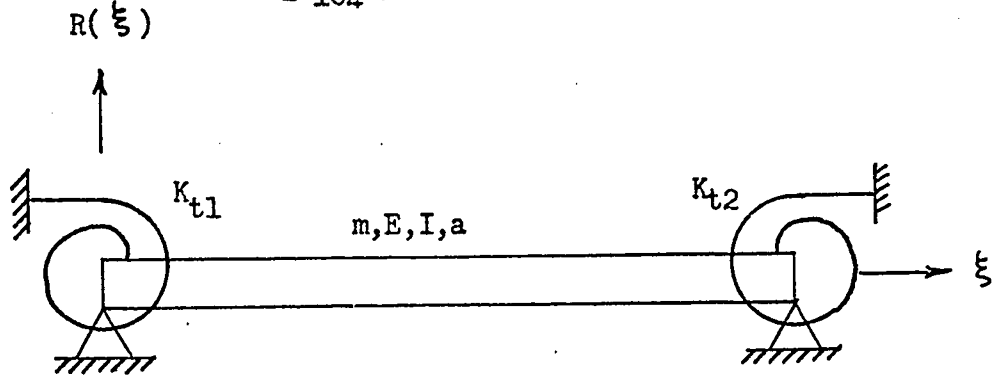

Fig.7 Uniform beam with both ends partly clamped .

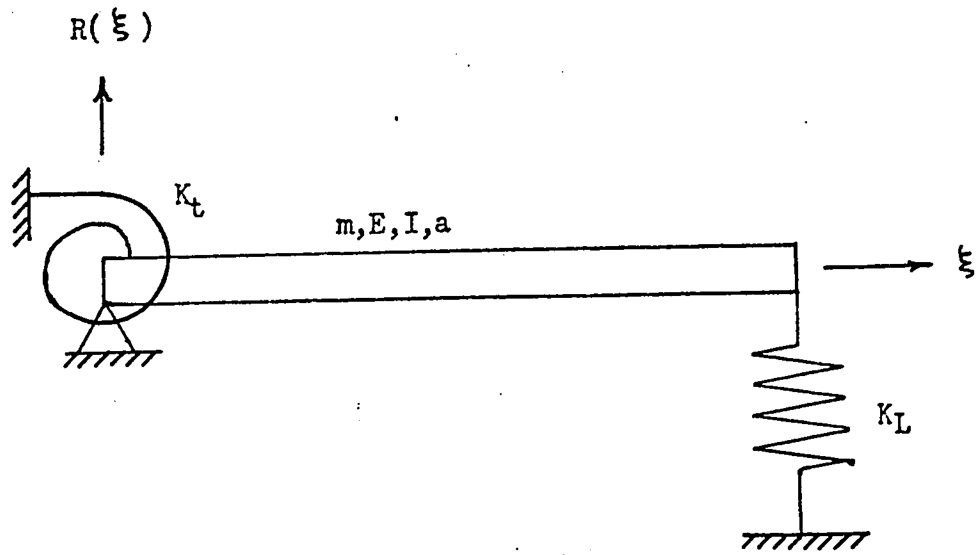

Fig.8 Hinged - free uniform beam with a torsional spring at the hinged end and a linear spring at the free end.

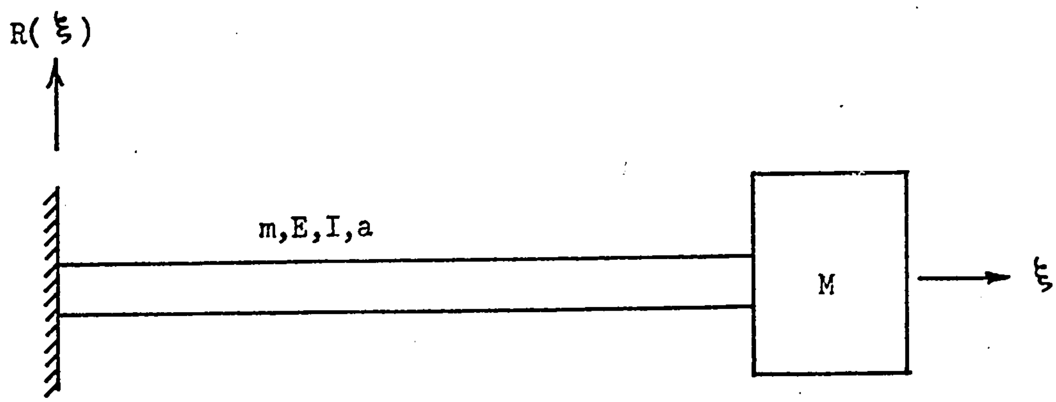

Fig.9 Clamped - free uniform beam with a concentrated mass at the free end.

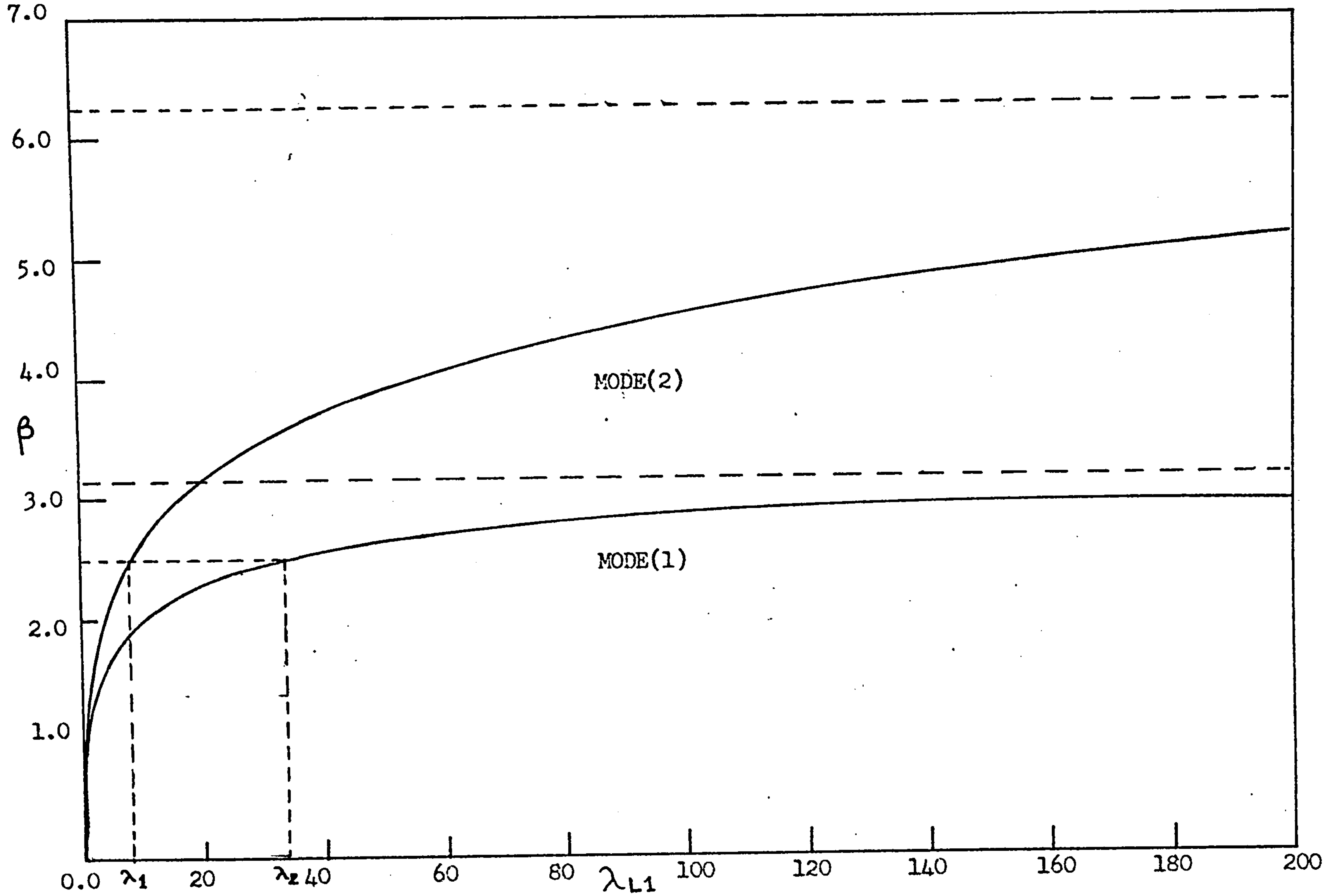

Fig.10 λ_{L1} vs β for first two modes of vibration of beams with both ends elastically supported .

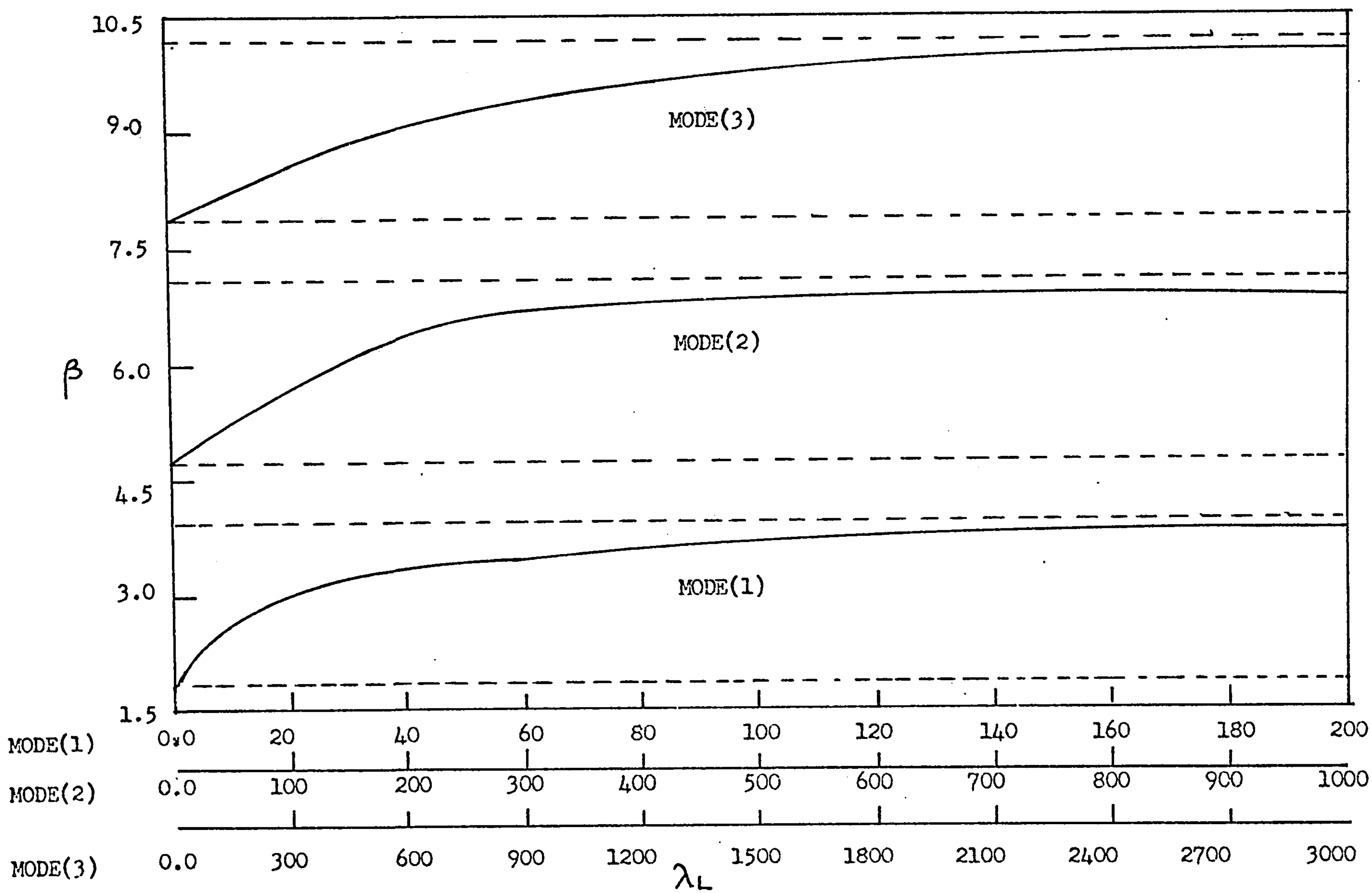

Fig.11 λ_L vs β for first three modes of vibration of clamped - elastically supported beam

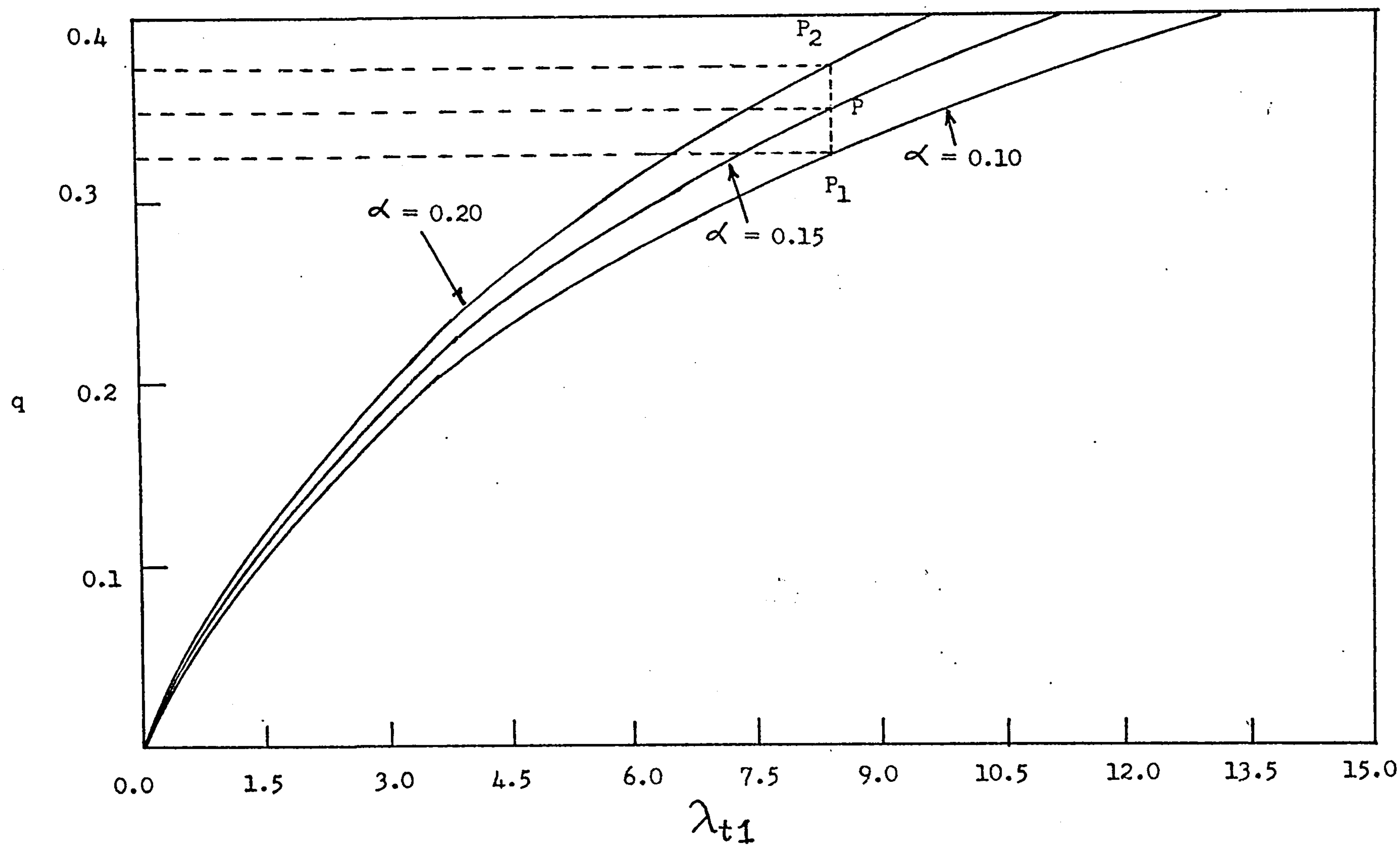

Fig.12 The accuracy check for α on beams with both ends partly clamped

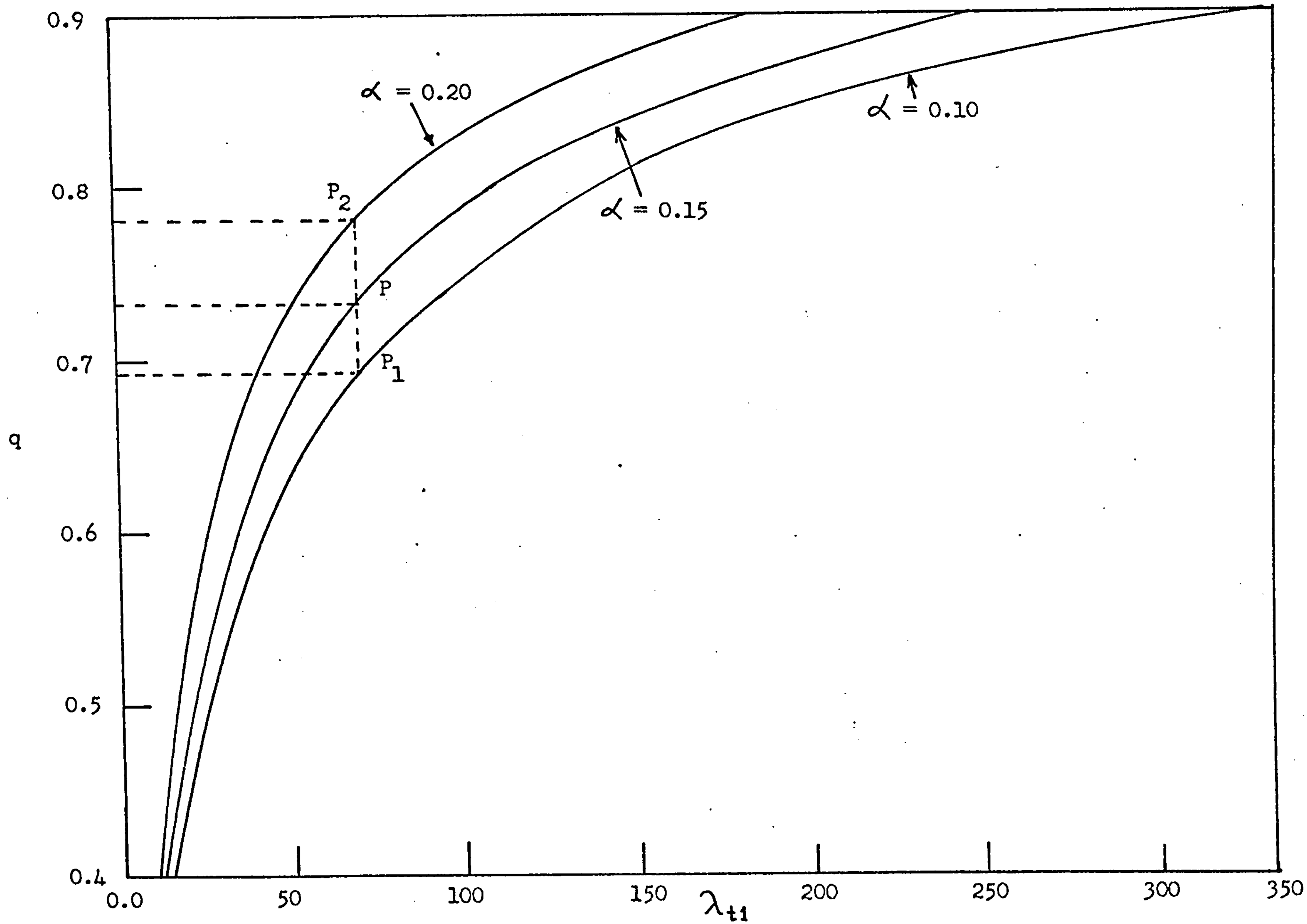

Fig.13 The accuracy check for α on beams with both ends partly clamped

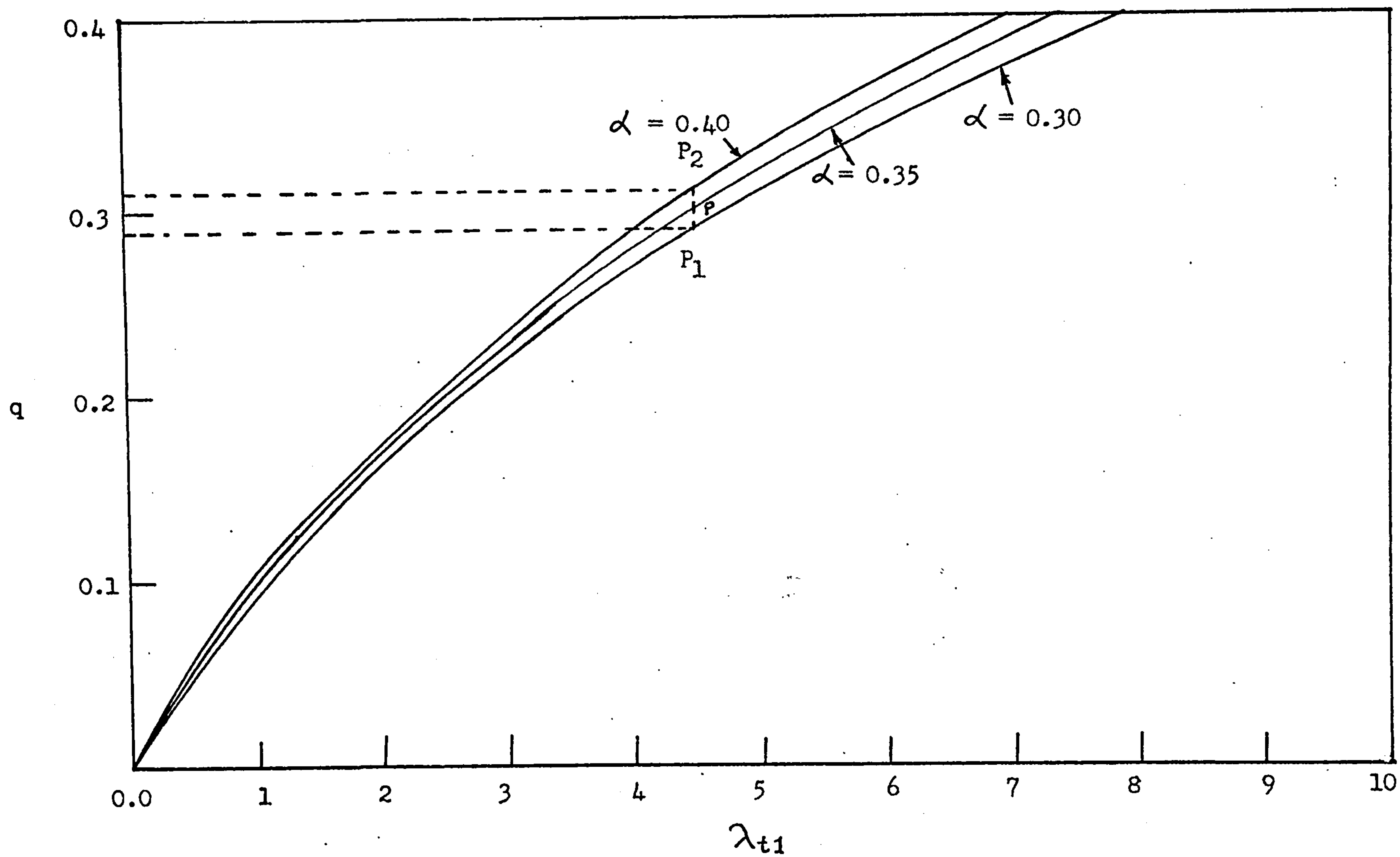

Fig.14 The accuracy check for α on beams with both ends partly clamped

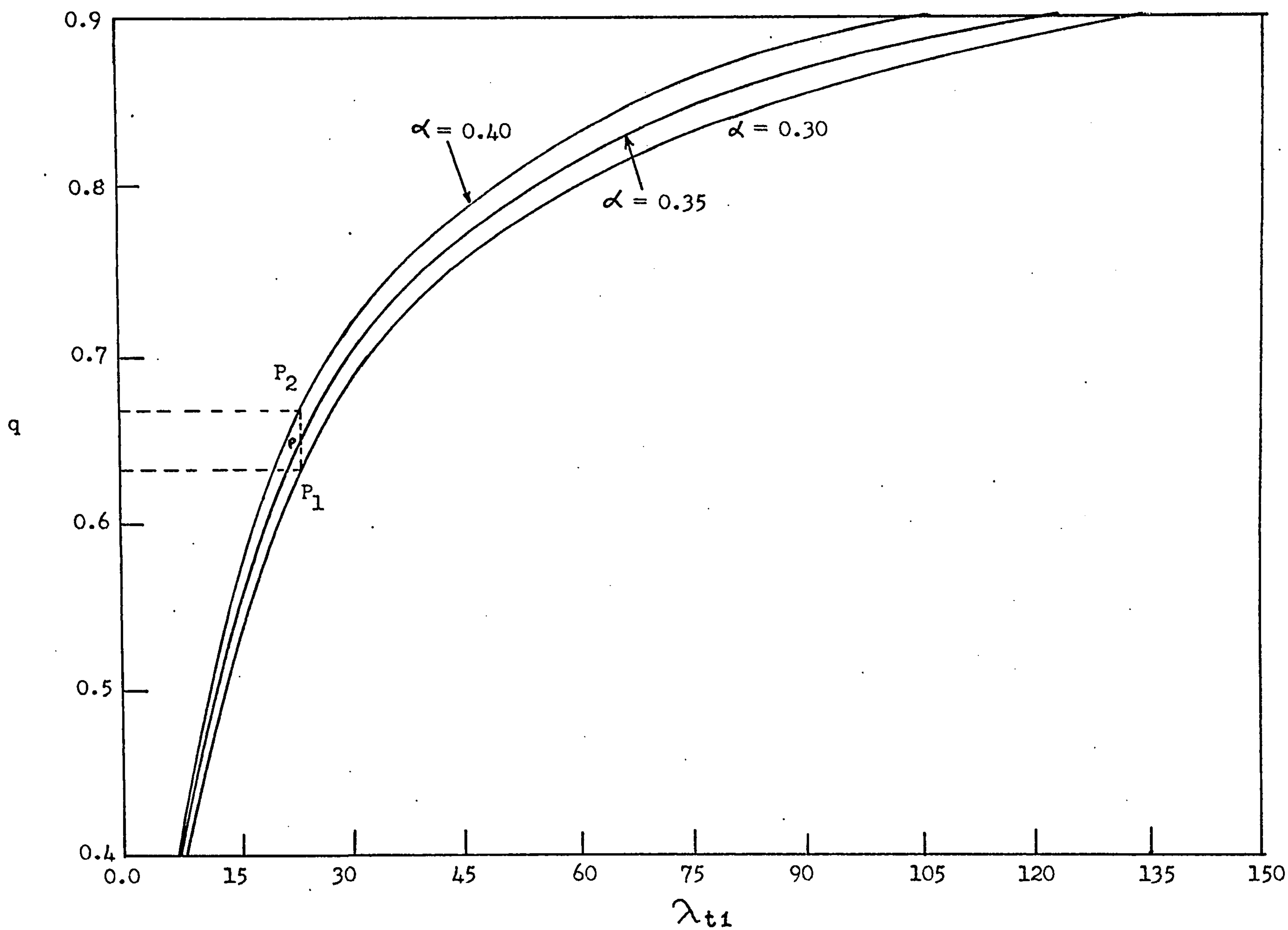

Fig.15 The accuracy check for α on beams with both ends partly clamped .

APPENDIX 1

DERIVATION OF EQUATION (2.11)

Refer to figure 1.

On the elastically supported end , the shear force should be balanced by the spring force .

$$EI \frac{d^3R(x)}{dx^3} = K_L R(x) \qquad (A1.1)$$

Non-dimensionalising the spatial coordinate and beam deflection with respect to beam length L , we have ,

$$EI \frac{L \, d^3R(x/L)}{L^3 \, d(x/L)^3} = K_L L R(x/L) \qquad (A1.2)$$

or

$$\frac{d^3R(\xi)}{d\xi^3} = \frac{K_L L^3}{EI} R(\xi) \qquad (A1.3)$$

At $\xi = 1$, this becomes

$$R'''(1) = \frac{K_L L^3}{EI} R(1)$$

$$= \lambda_L R(1) \qquad (A1.4)$$

APPENDIX 2

THE ITERATION TECHNIQUE

The iteration technique used in computing the eigenvalues from the frequency equations of various beams is described in brief here . For the purpose of illustration let us take the frequency equation of the cantilever beam which is given below .

$$1 + \cos\beta \cosh\beta = 0$$

Let $X = 1 + \cos\beta \cosh\beta$, then we want to find out all such values of β for which $X = 0$.

The technique can best be explained with the help of fig.16. The values of β are started from 0.0 and incremented by 1.0 till two consective values of X with opposite sign are obtained . Let these values be X_0 and X_1 and corresponding values of β be β_0 and β_1 . Now the correct value of β for which $X = 0$ is obtained assuming that the function is a straight line between X_0 and X_1 which generally is not true . Let this point be (β, 0) as shown in the figure . The function could either be of the type (1) or of the type (1') as shown in the figure . If the function is of the type (1) , the value of X at β will be X_1' which is of opposite sign to that of X_0 . In this case the iteration is continued between X_0 and X_1' , till the point (β,0) coincide with A within the desired accuracy . However if the function is of the type (1') , the value of function X at β is X_0' which is of the same sign as that of X_0 . In this case the iteration is continued between X_0' and X_1 , till the point (β,0) coincides with B within the desired accuracy.

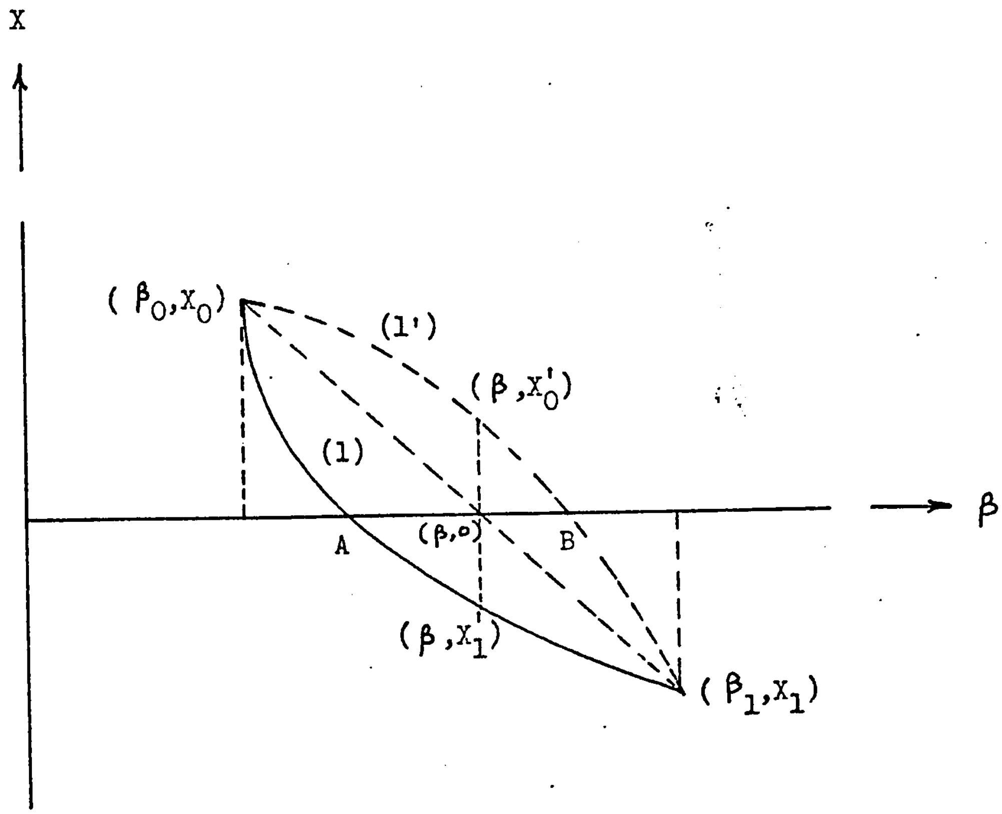

Fig.16 The iteration technique .

This technique was programed at I.B.M.-360 digital computer and the first ten eigenvalues were obtained for beams with various classical end conditions . A print-out of the computer program for the cantilever beam also appears in this appendix .

```
FORTRAN IV G LEVEL  19                  MAIN                  DATE = 712

      C     CALCULATION OF EIGENVALUES OF CANTILEVER BEAMS
            DIMENSION BCF(10)
            B=0.
            I=0
      1     I=I+1
            CTR=0.
            CTR1=0.
      5     B=B+1.
      7     X=1.+COS(B)*COSH(B)
      11    IF(CTR1.GT.0.) GO TO 30
            IF(CTR.GT.0.) GO TO 10
            XS=X
            CTR=1.
            GO TO 5
      10    IF((X*XS).LT.0.) GO TO 51
            XS=X
            GO TO 5
      51    B1=B
            X1=X
            B0=B-1.
            X0=XS
      50    BS=B
            B=B0+(B1-B0)*ABS(X0)/ABS(X0-X1)
            IF(ABS(B-BS).LT..00001) GO TO 49
            CTR1=1.
            GO TO 7
      30    IF((X*X0).LT.0.) GO TO 55
            X0=X
            B0=B
            GO TO 50
      55    X1=X
            B1=B
            GO TO 50
      49    BCF(I)=B
            IF(I.LT.10) GO TO 1
            STOP
            END
```

APPENDIX 3

DERIVATION OF EQUATION (5.8)

Refer to figure 9 .

The shear force at the free end must be balanced by the inertia force of mass M .

$$EI \frac{\partial^3 R(x,t)}{\partial x^3} = M \frac{\partial^2 R(x,t)}{\partial t^2} \tag{A3.1}$$

Let R(x,t) = R(x) P(t)

so that
$$\frac{\partial^3 R(x,t)}{\partial x^3} = P(t) \frac{d^3 R(x)}{dx^3} \tag{A3.2}$$

and
$$\frac{\partial^2 R(x,t)}{\partial t^2} = R(x) \frac{d^2 P(t)}{dt^2}$$
$$= - \omega^2 R(x) P(t) \tag{A3.3}$$

where P(t) has been assumed to be a harmonic function of time .

Substituting (A3.2) and (A 3.3) in (A3.1) , we obtain ,

$$EI \frac{d^3 R(x)}{dx^3} = - M \omega^2 R(x) \tag{A3.4}$$

Non-dimensionalising the beam deflection and spatial coordinate with respect to beam length L , we have ,

$$EI \frac{L\, d^3 R(x/L)}{L^3\, d(x/L)^3} = - M \omega^2 L\, R(x/L) \tag{A3.5}$$

or
$$\frac{d^3 R(\xi)}{d\xi^3} = - \frac{M \omega^2 L^3}{EI} R(\xi)$$

$$= - \frac{M \beta^4}{\rho a L} R(\xi) \quad \text{(Using equation (1.7))}$$

$$= - \frac{\beta^4}{\lambda_m} R(\xi) \tag{A3.6}$$

where

$$\lambda_m = \frac{\rho a L}{M}$$

At $\xi = 1$, equation (A3.6) assumes the following form .

$$R'''(1) = - \frac{\beta^4}{\lambda_m} R(1) \tag{A3.7}$$

APPENDIX 4

DERIVATION OF EQUATION (6.5)

The tables of results in chapter 6 give the values of for q = 0.8 for all the cases considered . For q = 1.0 ,the system parameter λ is infinite and hence $\frac{1}{\lambda} = 0$ at this value of q . System parameter λ and hence $\frac{1}{\lambda}$ can be determined for q = 0.8 from the tables of results . The inverse of system parameter $(\frac{1}{\lambda})$ is assumed to vary linearly between q = 0.8 and q = 1.0. This assumption is justified by the accuracy it yields . The assumed plot of $\frac{1}{\lambda}$ vs q(0.8 to 1.0) is shown in the figure 17 .

At any value of $q = 0.8 + \Delta q$,let the inverse of system parameter be $\frac{1}{\lambda}$. Then from the similar triangles ABC and DEC in fig.17 we can write ,

$$\frac{0.2}{(1/\lambda_{0.8})} = \frac{\Delta q}{(1/\lambda_{0.8}) - (1/\lambda)} \qquad (A4.1)$$

which can be rearranged as follows

$$\Delta q = 0.2\,(1 - \lambda_{0.8}/\lambda) \qquad (A4.2)$$

or

$$q = 0.8 + \Delta q$$

$$= 0.8 + 0.2\,(1 - \lambda_{0.8}/\lambda)$$

$$= 1 - \frac{0.2\,\lambda_{0.8}}{\lambda} \qquad (A4.3)$$

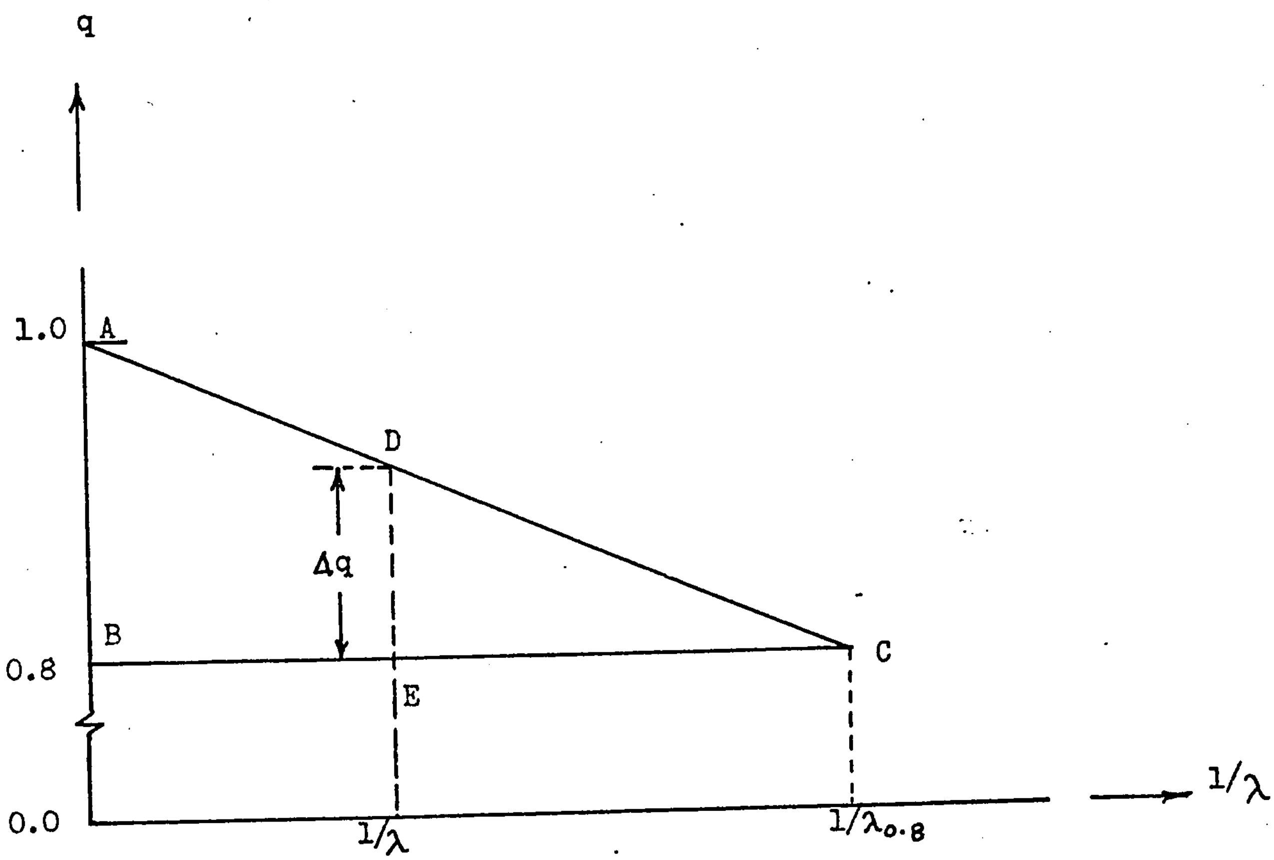

Fig.17 $1/\lambda$ vs q for values of q between 0.8 and 1.0

APPENDIX 5

EXPERIMENTAL DEMONSTRATION

The following three classes of elastically supported beams were chosen as illustrative cases for experimental determination of their natural frequencies.

1. A cantilever beam with linear spring support at its outer end.
2. A cantilever beam with concentrated mass at its outer end.
3. A beam supported on linear spring supports at both ends.

A simple sketch of the experimental rig is shown in figure 18. The rig consisted of a mild steel frame with arrangements of fixing springs to the beam. The beam selected was an aluminium beam with the following geometric properties.

Length = 29½ in. , Width = 1 in. , Depth = ½ in.

1. A cantilever beam with linear spring support at its outer end.

The beam was supported on a linear spring support. The output from a strain guage fixed on the beam was fed to oscilloscope to measure the natural frequency of lateral vibration of the beam. Experiments were conducted with three different spring supports of stiffnesses 8.2 lb/in.,12.5lb/in. and 19.4 lb/in. The frequencies determined experimentally were found to be 21.2 Hz., 24.4 Hz. and 28.0 Hz. as against the theoritically calculated frequencies of 21.7 Hz.,25.0 Hz. and 28.7 Hz. respectively. Experimentally observed frequencies were found to be always slightly lower than calculated frequencies. This can be explained by the fact that , in theory spring supports are assumed to be massless , which is never the case in practice.

2. A cantilever beam with concentrated mass at its outer end.

Experiments were conducted with three different masses of 0.121b.,0.25, and 0.40 lb. and the frequencies were found to be 7.1 Hz., 5.8 Hz. and 5.0 Hz. The theoretical frequencies calculated with the help of the tables presented in this thesis were found to be 7.25 Hz., 6.0 Hz. and 5.16 Hz. respectively.

3. A beam supported on linear spring supports at both ends.

The beam was supported on equal spring supports of stiffness 8.2 lb./in. each. The first and second mode frequencies were observed to be 12.9 Hz. and 24.0 Hz. respectively. The second mode frequency was measured after filtering out the first mode frequency components using a high pass filter. The theoretically calculated frequencies were found to be 13.3 Hz. and 24.8 Hz. for first and second mode respectively.

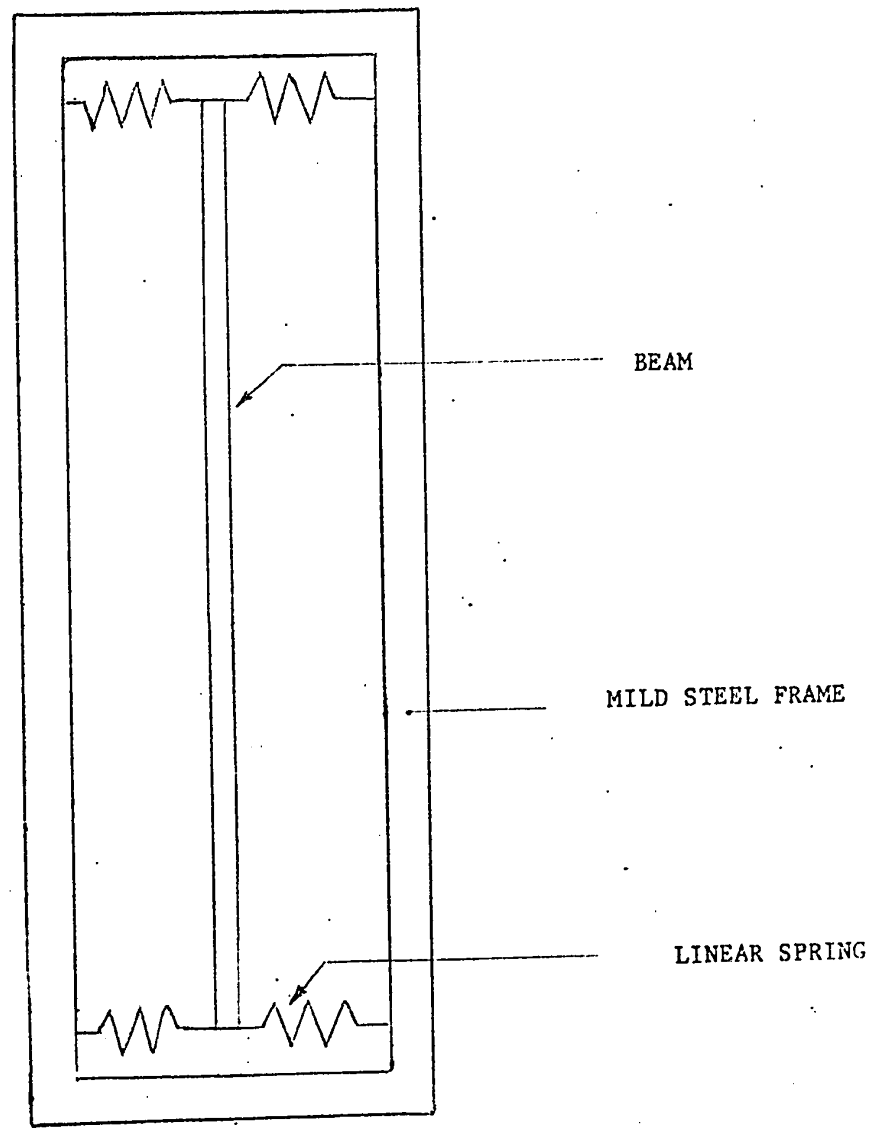

FIGURE 18

A sketch of experimental rig.

Calculation of frequencies by classical method.

The following example is solved by using the classical method to show that frequencies obtained by classical method are exactly the same as calculated by using the tables of results of this thesis.

Example A5.1

Solve example 1.0 by using the classical method of solution.

Solution

It can be readily shown that the frequency equation of a cantilever beam supported on a linear spring support of stiffness K lb/in is expressed as follows.

$$\frac{K}{EI} = \frac{\beta_i^3 \,(1 + \cos\beta_i L \cdot \cosh\beta_i L)}{\cos\beta_i L \cdot \sinh\beta_i L - \sin\beta_i L \cdot \cosh\beta_i L} \qquad \text{(A5.1)}$$

$$\text{where} \quad \beta_i^4 = \frac{\rho a \,\omega^2}{g E I} \qquad \text{(A5.2)}$$

Substituting the values of K,E,I and length L , we obtain,

$$\frac{8.2}{10.5\text{x}10 \times 0.0013} = \frac{(1+\cos(29.5\beta_i)\,\cosh\,(29.5\beta_i))}{\cos(29.5\beta_i)\sinh(29.5\beta_i)-\text{Sin}(29.5\,\beta_i)\cosh(29.5\,\beta_i)} \qquad \text{(A5.3)}$$

which yields $\beta_i = 0.0961$

Substituting this in (A5.2),

$$f = \frac{\beta_1^2}{2\pi}\sqrt{\frac{EI\,g}{\rho a}} = \frac{(0.0961)^2}{2\,\pi}\sqrt{\frac{10.5\text{x}10^6\text{x}0.0013\text{x}386}{0.0965\text{x}0.25}}$$

$$= 21.7 \text{ Hz.}$$

which is exactly the same as calculated in example 1 using the tables of results of this thesis.

Obviously the solution of equation (A5.3) involves considerable labour particularly if the facilities of a digital computer are not available. This labour is greatly diminished by using the tables of results of this thesis.

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