

# DEFECT-1 CHOOSABILITY OF GRAPHS ON SURFACES

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## ABSTRACT

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The classical (proper) graph colouring problem asks for a colouring of the vertices of a graph with the minimum number of colours such that no two vertices with the same colour are adjacent. Equivalently the colouring is required to be such that the graph induced by the vertices coloured the same colour has the maximum degree equal to zero. The graph parameter associated with the minimum possible number of colours of a graph is called chromatic number of that graph.

One generalization of this classical problem is to relax the requirement that the maximum degree of the graph induced by the vertices coloured the same colour be zero, and instead allow it to be some integer  $d$ . For  $d = 0$ , we are back at the classical proper colouring. For other values of  $d$  we say that the colouring has defect  $d$ .

Another generalization of the classical graph colouring, is list colouring and its associated parameters: choosability and choice number.

The main result of this thesis is to show that every graph  $G$  of Euler genus  $\mu$  is  $\lceil 2 + \sqrt{3\mu + 3} \rceil$ -choosable with defect  $1$  (equivalently, with clustering  $2$ ). Thus allowing any defect, even  $1$ , reduces the choice number of surface embeddable graphs below the chromatic number of the surface. For example, the chromatic number of the family of toroidal graphs is known to be  $7$ . The bound above implies that toroidal graphs are  $5$ -choosable with defect  $1$ . This strengthens the result of Cowen, Goddard and Jesurum (1997) who showed that toroidal graphs are  $5$ -colourable with defect  $1$ .

In a graph embedded in a surface, two faces that share an edge are called *adjacent*. We improve the above bound for graphs that have embeddings without adjacent triangles. In particular, we show that every non-planar graph  $G$  that can be embedded in a surface of Euler genus  $\mu$  without adjacent triangles, is

$\left\lceil \frac{5+\sqrt{24\mu+1}}{3} \right\rceil$ -choosable with defect 1. This result generalizes the result of Xu and Zhang (2007) to all the surfaces. They proved that toroidal graphs that have embeddings on the torus without two adjacent triangles are 4-choosable with defect 1.

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## INTRODUCTION AND RESULTS

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### 1.1 INTRODUCTION

Graphs are used to model structural information arising from many fields such as economics, engineering, social sciences, genetics, mathematics and computer science. In chemistry, the popular ball-and-stick model of a molecule is a graph. The nodes are atoms and the edges correspond to molecular bonds. In graph models of the world-wide web, nodes represent web pages and edges represent hyper-links.

Graph theory is a computer science and mathematics discipline that studies graphs and its applications. The origins of graph theory are largely topological, the most famous examples being Euler's "Bridges of Königsberg" problem (1736) and the 4-colour conjecture (1852). Both of these problems deal with planar graphs; that is, graphs that can be drawn in the plane without crossings. A generalization of planar graphs are graphs that can be drawn (i.e. embedded) on higher surfaces without edge crossings.

Graph colouring is a topic of central importance in graph theory, both historically and in applications. A *vertex colouring* of a graph  $G$  is an assignment of colors to the vertices of  $G$ . A vertex colouring is *proper* if no two adjacent vertices have the same color. The *chromatic number* of a graph  $G$  is the minimum number,  $k$ , such that  $G$  has a proper vertex colouring with  $k$  colours. Determining the chromatic number of a graph, or a graph class, is the classical graph colouring problem. Proper vertex colourings have been extensively studied. The most famous result in graph colouring is the four color theorem. It states that every planar graph has proper vertex colouring with at most 4 colours. The theorem was proved in 1976 by Appel and Haken after many false proofs. The proof is complicated

and computer added. Many improvements have been obtained over the years, but no simple proof is known. One of the techniques that the proof introduced is called discharging.

There are plethora of generalizations and variations of the classical (proper) graph colouring problem. We study one such generalization in this thesis. In particular, we will study *defective choosability* of graphs embeddable on surfaces. Defective choosability and other key notions for this thesis will be defined next. Our proofs will use discharging technique.

## 1.2 DEFINITIONS OF MAIN CONCEPTS

In a vertex coloured graph, a *monochromatic component* is a connected component of  $G$  where each vertex has the same colour. Thus a vertex colouring is proper if every monochromatic component has 1 vertex.

The following notion, introduced formally by Cowen, Cowen and Woodall [7] and studied as early as 1966 by Lovász [12], generalizes proper vertex colourings. A graph  $G$  is  $k$ -colourable with *defect*  $d$ , that is,  $(k, d)^*$ -colourable, if the vertices of  $G$  can be coloured with  $k$  colours such that the subgraph induced by the vertices of each monochromatic component has maximum degree  $d$ . Colourings with defect zero are thus proper, otherwise they are called *defective*.

A surface  $\Sigma$  is a compact connected 2-manifold without boundary. Surfaces are classified into two classes. Each orientable (non-orientable) surface is homeomorphic to a sphere with  $g \geq 0$  handles ( $h \geq 1$  crosscaps) attached. For a surface  $\Sigma$ , its *Euler genus*,  $eg(\Sigma)$ , is  $2g$  if  $\Sigma$  is orientable and  $h$  if  $\Sigma$  is non-orientable.

A *list assignment* for a graph  $G$  is a function  $L$  that assigns a set  $L(v)$ , also called a *list*  $L(v)$ , of colours to each vertex  $v \in V(G)$ . A list  $L(v)$  with  $|L(v)| \geq k$  is a  $k$ -*list*. A list assignment  $L$  is a  $k$ -*list assignment* if each vertex is assigned a  $k$ -list. For a list assignment  $L$ ,  $G$  is  $L$ -colourable with defect  $d$  if  $G$  can be coloured such that each vertex  $v$  gets a colour from its list  $L(v)$  and such that the defect is at most  $d$ .  $G$  is  $(k, d)^*$ -choosable if

for every  $k$ -list assignment  $L$ ,  $G$  is  $L$ -colourable with defect  $d$ . The *choice- $i$  number* of a graph  $G$  is the minimum  $k$  such that  $G$  is  $(k, i)^*$ -choosable. The *chromatic- $i$  number* of a graph  $G$  is the minimum  $k$  such that  $G$  is  $(k, i)^*$ -colourable. For proper (list) colourings,  $i = 0$  is omitted in this notation.

Defective choosability is a strengthening of defective colourings. In particular, note that for every  $k$  and  $i$ , if a graph  $G$  is  $(k, i)^*$ -choosable then  $G$  is  $(k, i)^*$ -colourable. Thus the choice- $i$  number of a graph  $G$  is greater than or equal to the chromatic- $i$  number of  $G$ . The choice- $i$  number can be arbitrarily larger than chromatic- $i$  number though. For example, for every  $t \geq 3$  there is a graph a bipartite graphs (thus graphs with chromatic number 2) whose choice number is at least  $t$  [10].

An *embedding* of a graph  $G$  on a surface  $\Sigma$  is a representation/drawing of  $G$  on  $\Sigma$  where vertices of  $G$  are represented as points on  $\Sigma$  and edges of  $G$  are represented as simple arcs such that: 1. the endpoints of the arc representing an edge,  $e$ , are the points representing the end vertices of  $e$ ; 2. no arc contains in its interior a point representing a vertex; and, 3. no two arcs representing edges cross other than in a common endpoint. If a graph has an embedding in a surface, we say that the graph is *embeddable* in that surface. A *2-cell embedding* is an embedding in which every face is homeomorphic to an open disk

Given a graph  $G$ , its Euler genus  $eg(G)$  is the minimum Euler genus of a surface  $\Sigma$  where  $G$  can be embedded in.

The *chromatic number of a surface*  $\Sigma$  is the maximum chromatic number of a graph embeddable in  $\Sigma$ . The *choice number of a surface*  $\Sigma$  is the maximum choice number of a graph embeddable in  $\Sigma$ .

### 1.3 BACKGROUND AND RESULTS

In this thesis we are interested in defective colourings that are as close as possible to being proper, that is, in colourings with defect 1. In particular, we study such colourings for graphs on surfaces. Typically, the goal of this line of research is to reduce

the number of colours below the number required by proper vertex colourings. For example, Voigt proved [15] that there are planar graphs that are not 4-choosable, but Cushing and Kierstead [8] proved that planar graphs are 4-choosable if defect 1 is allowed, thereby answering an open problem posed by several authors [19, 1, 16, 9]. For some classes of graphs, however, allowing even arbitrarily big defect does not reduce the number of colours below the chromatic number of the class. For example, the chromatic number of the class of series-parallel graphs is 3. For some series-parallel graphs it is not possible to reduce 3 to 2 irrespective of the defect. In particular, for any  $d$ , there is a series-parallel graph that is not  $(2, d)^*$ -colourable. Let's consider, for example a series-parallel graph that is not  $(2, 1)^*$ -colourable. One such example is a graph  $G$  obtained by taking a path,  $P$ , on six vertices and adding another vertex  $v$  adjacent to all six vertices of  $P$ . Such a graph  $G$  is a series parallel (planar) graph. Assume for the sake of contradiction that  $G$  is  $(2, 1)^*$ -colourable. First consider  $(2, 1)^*$ -colouring of  $P$ . Let the colours be denoted by  $c_1$  and  $c_2$ . It is not possible for either  $c_1$  or  $c_2$  to appear at most once on  $P$ . Otherwise, there would be a vertex on  $P$  coloured  $c_i$  with its two neighbours in  $P$  also coloured  $c_i$ , resulting in defect 2. Thus each of  $c_1$  and  $c_2$  appears at least two times on  $P$ . However, in that case vertex  $v$  cannot be coloured  $c_1$  or  $c_2$  as otherwise  $v$  together with either two vertices coloured  $c_1$  or two vertices coloured  $c_2$  on  $P$  would result in defect 2. Thus the series parallel graph  $G$  is not  $(2, 1)^*$ -colourable.

Taking two copies of  $G$  and adding a vertex adjacent to all the vertices in the two copies results in a planar graph (in fact planar 3-tree) that is not  $(3, 1)^*$ -colourable.

More generally,  $k$ -trees have chromatic number  $k + 1$ , and yet for every  $d$  there is  $k$ -tree that is not  $(k, d)^*$ -colourable. For more on this topic see the (the *standard* example) in the survey by Wood [17]).

A similar notion to that of defective colouring is clustered colouring. A graph  $G$  is  $k$ -colourable with *clustering*  $c$  if the vertices of  $G$  can be coloured with  $k$  colours such that each

monochromatic component has at most  $c$  vertices. Note that a colouring of a graph has defect at most 1 if and only if it has clustering at most 2. This is not the case for higher defect/clustering values. Thus, our results also give colourings of graphs on surfaces with clustering at most 2. There is a plethora of recent work on the subject of clustered and defective graph colourings/choosability. For an extensive coverage of the topic see the recent survey by Wood [17] and Sections 3, 4 and 5 in the survey by Woodall [18].

There is a rich history of various colouring problems on graphs on surfaces. For the background, we only focus on the previous work on colourings and choosability with defect 1. The goal of this thesis is to show that for every surface (other than the sphere) graphs embeddable in that surface have choice number with defect 1 less than the chromatic number of that surface.

In their 1997 paper, Cowen, Goddard, and Jesurum [5, 6] studied defective colouring of graphs on surfaces. They proved that every graph  $G$  is  $(4 + \frac{7 + \sqrt{24\text{eg}(G) + 1}}{4}, 1)^*$ -colourable. In the conclusions of that paper, they suggest a study of defective choosability of graphs embeddable on surfaces. That is exactly the focus of this thesis.

Defective choosability of planar graphs have first been studied by Eaton and Hull [9] and Škrekovski [16].

The following is the main result of this thesis.

**THEOREM 1**

Every graph  $G$  is  $(\lceil 2 + \sqrt{3\text{eg}(G) + 3} \rceil, 1)^*$ -choosable.

Cowen, Goddard and Jesurum [6, 5] proved that toroidal graphs are  $(5, 1)^*$ -colourable. Theorem 1 implies that toroidal graphs are, in fact,  $(5, 1)^*$ -choosable thus Theorem 1 constitutes a strengthening of Cowen, Goddard, and Jesurum's result. In addition, unlike the proof of Cowen, Goddard and Jesurum [6, 5], our proof of Theorem 1 does not use the four-colour theorem. That is always desirable given the difficulty of the four-colour theorem proof.

Defective choosability of planar graphs have first been studied by Eaton and Hull [9] and Škrekovski [16]. For planar graphs,

our theorem states that they are  $(4, 1)^*$ -choosable. This has been proved by Cushing and Kierstead [8]. Our proof works for  $(t, 1)^*$ -choosability where  $t \geq 5$ , thus it does not imply Cushing and Kierstead's result.

It is well known that for all surfaces  $\Sigma$  a graph embedded in  $\Sigma$  can be coloured with at most  $\lfloor \frac{7 + \sqrt{24\text{eg}(\Sigma) + 1}}{2} \rfloor$  colours. That result was proved for all surfaces except the sphere by Heawood in 1890 [11]. The bound is true for the sphere too, that is for planar graphs, by the famous four colour theorem. In his original paper Heawood conjectured that this upper bound is in fact tight. In particular he conjectured that for every surface  $\Sigma$ , except the Klein bottle, there is a graph (in fact a complete graph) that embeds in that surface whose chromatic number is exactly  $\lfloor \frac{7 + \sqrt{24\text{eg}(\Sigma) + 1}}{2} \rfloor$ . That conjecture was proved by Ringel and Young [13]. Since this lower bound is achieved for complete graphs  $G$ , it follows that the choice number of such graphs is at least  $\lfloor \frac{7 + \sqrt{24\text{eg}(\Sigma) + 1}}{2} \rfloor$ . Thus, Theorem 1 shows that for every surface (except sphere), the choice-1 number of graphs embeddable in the surfaces can be reduced below what the chromatic number of the surface allows.

The bound in Theorem 1 is surely not tight. The only lower bound available however, follows from the fact that having choice-1 number at most  $p$  implies having a chromatic number at most  $2p$ . Thus the above  $\lfloor 3.5 + \sqrt{6\text{eg}(\Sigma) + 0.25} \rfloor$  lower bound on the chromatic number implies that for every surface  $\Sigma$ , except the Klein bottle, there is a graph that embeds in that surface whose choice-1 number is at least  $\lceil 1.75 + \sqrt{1.5\text{eg}(G) + 1/16} \rceil$ . For example, this lower bound and Theorem 1 imply that the correct bound on choice-1 number of toroidal graphs is either 4 or 5, leading to an open question of whether toroidal graphs are  $(4, 1)^*$ -choosable. Unfortunately it is still not even known if toroidal graphs are  $(4, 1)^*$ -colourable, which is an open question from 1997 by Cowen, Goddard, and Jesurum [5, 6].

Even the question on whether all planar graphs are  $(4, 1)^*$ -choosable has been open until recently. The question, which received considerably attention, was asked in several articles over the years

[19, 1, 16, 9] and was finally settled in the positive by Cushing and Kierstead [8]. A well-known result by Voigt states that there are planar graphs that are not 4-choosable [15]. A famous result by Thomassen [14] states that every planar graph is 5-choosable. Cowen, Cowen and Woodall [7] proved that there are planar graphs that are not  $(3, 1)^*$ -colourable and thus they are not  $(3, 1)^*$ -choosable. Thus the  $(4, 1)^*$ -choosability result by Cushing and Kierstead [8] is best possible for colouring planar graphs with defect 1.

Cowen, Goddard, and Jesurum [6] show that testing if a planar graph is  $(2, 1)^*$ -colourable or  $(3, 1)^*$ -colourable is NP-complete. The hardness of  $(2, 1)^*$ -colourability remains unchanged even for planar graphs with maximum degree 4, as proved by Corrêa, Havet and Sereni [3]. For more on the complexity of defective colouring problems, see the recent results by Belmonte, Lampis and Mitsou [2].

Note that the above negative result on  $(3, 1)^*$ -colourability of planar graphs implies that the aforementioned lower bound,  $\lceil 1.75 + \sqrt{1.5 \text{eg}(G) + 1/16} \rceil$ , on chromatic-1 and choice-1 number is not tight, in fact it is off by 2 for planar graphs. Thus, the lower bound is possibly quite weak for choice-1 number for graphs on surfaces. We conclude this part by asking for improvements on the lower and upper bound on choice-1 number of such graphs, starting with the question of whether toroidal graphs are  $(4, 1)^*$ -choosable or  $(4, 1)^*$ -colourable.

In a graph embedded in a surface, a *triangle* is a face that forms a cycle of length 3. Two triangles in an embedding are considered *adjacent* if they share an edge. The following theorem provides a better upper bound for graphs that have embeddings with no adjacent triangles. For example, the theorem states that graphs that can be embedded into a torus are  $(4, 1)^*$ -choosable. The fact that toroidal graphs embeddable without two adjacent triangles are  $(4, 1)^*$ -choosable has been known already due to of [20].

## THEOREM 2

Every non-planar graph  $G$  embeddable on surface of Euler genus  $eg(G)$  with no adjacent triangles is  $\left(\left\lceil \frac{5+\sqrt{24eg(G)+1}}{3} \right\rceil, 1\right)^*$ -choosable.

## 1.4 ORGANIZATION OF THE THESIS

Chapter 2 is divided into four sections. We start by presenting some definitions in Section 2.1. In Section 2.2 we prove Theorem 1 for toroidal graphs as a way of introducing the techniques we use, in particular the discharging technique. In Section 2.3 we prove two lemmas that will be used both in the proof of Theorem 1 and in the proof of Theorem 2. In Section 2.4 we prove Theorem 1. Chapter 3 is dedicated to proving Theorem 2. We conclude with summary and open problems in Chapter 4.

## GRAPHS ON SURFACES

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### 2.1 PRELIMINARIES

We will need the following definitions.

Consider an undirected graph  $G = (V, E)$  with vertex set  $V(G)$  and edge set  $E(G)$ . The number of vertices is denoted by  $n = |V(G)|$  and the number of edges by  $m = |E(G)|$ . For a set  $S \subseteq V(G)$ ,  $G[S]$  denotes the subgraph of  $G$  induced by the vertices of  $S$  and  $G - S$  denotes the subgraph  $G[V(G) \setminus S]$ . If  $S$  is comprised of one vertex,  $v$ , then  $G - v$  denotes  $G - \{v\}$ . For every  $v \in V$ , let  $N(v)$  denote the set of neighbours of  $v$ . The *degree* of  $v$  is  $\deg_G(v) = |N(v)|$ . A vertex of degree  $d$  is called *degree- $d$*  vertex. The minimum and maximum vertex degree in  $G$  are denoted respectively by  $\delta(G)$ , and  $\Delta(G)$ . We omit “ $G$ ” from this notation whenever the graph is clear from the context.

In a vertex colouring of a graph, a vertex  $v$  is *proper* if none of its neighbours have the same colour as  $v$ .

A *k-cycle* in a graph, is a cycle of length  $k$  (that is, a cycle comprised of  $k$  vertices and  $k$  edges). A graph is *simple* if it has no loops (that is, 1-cycles) nor parallel edges (that is, 2-cycles).

Recall the definitions of embedding and 2-cell embedding from Section 1.2. Let the set of faces of an embedded graph  $G$  be denoted by  $F(G)$ . An embedding is *simple* if the underlying graph is simple. In a 2-cell embedded graph, a face bounded by a 3-cycle is called a *triangle*. A 2-cell embedding of a graph in a surface is a *triangulation* if every face of the embedding is a triangle. It is known that for every  $n \geq 3$ , every  $n$ -vertex simple planar graph is a subgraph of some  $n$ -vertex simple triangulation in the plane. However the same is not true for higher surfaces. For example, there are toroidal graphs for which each of their embeddings on the torus has a face with at least 4 edges and adding any edge to the embedding either introduces

a parallel edge, a loop or an edge crossing. Thus such toroidal graphs are not subgraphs of simple toroidal triangulations. That motivates the following definition.

A 2-cell embedded graph is *nearly simple* if it has no loops nor faces bounded by a 2-cycle. Note that, unlike simple graphs, nearly simple embedded graphs can have parallel edge but two parallel edges are not allowed to bound the same face. A 2-cell embedded graph is *edge maximal* if every face is a triangle. This definition is the same as our definition of triangulation above. We choose to use “edge maximal” in this thesis since in the literature triangulation is often assumed to be simple and planar. Note that adding an edge to a *nearly simple* edge maximal 2-cell embedded graph, results in introducing an edge crossing, or introducing a face of size 2, or introducing a loop.

## 2.2 TOROIDAL GRAPHS

To introduce the discharging technique we first prove Theorem 1 for toroidal graphs, that is graphs of Euler genus 2.

By Heawood result and 4-colour theorem, for all surfaces  $\Sigma$ , a graph embedded in  $\Sigma$  can be coloured with at most  $\lfloor \frac{7+\sqrt{24\text{eg}(\Sigma)+1}}{2} \rfloor$  colours. Thus the chromatic number of toroidal graphs is at most 7. In fact toroidal graphs are 7-choosable, as is well known and as is argued in the next paragraph.

Euler formula for connected toroidal graphs states that  $|V(G)| + |F(G)| - |E(G)| = 0$ . In a nearly simple connected toroidal graph on at least 3 vertices, each face has at least three edges and thus  $|F(G)| \leq 2|E(G)|/3$ . That in turn implies that every simple connected toroidal graph has at most  $3n$  edges and thus it has a vertex of degree at most 6. Thus the greedy colouring implies that toroidal graphs are not only 7-colourable but in fact they are 7-choosable.

By the Ringel and Young’s theorem [13] (discussed in the introduction), the bound 7 cannot be improved. In particular there are toroidal graphs that are not 6-colourable and thus they are not 6-choosable.

We now consider if the bound 7 can be reduced if we allow defect 1. First, it is easy to argue that toroidal graphs are  $(6, 1)^*$ -choosable. Namely, any vertex minimal counter example  $G$  would have to have the minimum degree at least 6. Otherwise delete the vertex  $v$  of minimum degree at most 5. By vertex minimality,  $G - v$  is  $(6, 1)^*$ -choosable. That 6-colouring of  $G - v$  can be extended to 6-colouring of  $G$  by assigning to  $v$  one of the colours from its list of 6 colours that is not used by its at most 5 neighbours. In the resulting colouring of  $G$ ,  $v$  is proper thus the colouring has defect 1. Furthermore, as implied by Lovász' result [12] (see Lemma 2.3.1 in the next section), every graph with maximum degree 11 is  $(6, 1)^*$ -choosable. Thus the maximum degree of  $G$  is at least 12. Having the minimum degree at least 6 and the maximum degree at least 12 implies that  $G$  has strictly more than  $3n$  edges which contradicts the Euler formula (see the earlier discussion).

Cowen et al. [4] showed that toroidal graphs are  $(5, 1)^*$ -colourable. The rest of this section is dedicated to giving a simple proof of the strengthening of that result. Moreover, unlike the proof in [4], our proof does not use the four colour theorem.

LEMMA 2.2.1. *Toroidal graphs are  $(5, 1)^*$ -choosable.*

*Proof.* Let  $G$  be a vertex minimal counter example to the statement of the lemma. That is, for every  $v \in V(G)$ ,  $G - v$  is  $(5, 1)^*$ -choosable, and  $G$  is not  $(5, 1)^*$ -choosable. We may assume that  $G$  is nearly simple edge maximal graph embedded on the torus, since every simple toroidal graph is a subgraph of such a graph.

It is well known that a nearly simple toroidal graph can have at most  $3n$  edges (see the discussion from the beginning of this section for the reasons why). Thus  $2m \leq 6n$ , which together with the fact that  $2m = \sum_{v_i \in V(G)} \deg(v_i)$  gives the following inequality:

$$\sum_{v_i \in V(G)} (\deg(v_i) - 6) \leq 0 \tag{2.1}$$

For each  $v \in V(G)$  let its *charge* be  $w(v) = \deg(v)$ . Thus, by inequality 2.1,  $\sum_{v_i \in V(G)} (w(v_i) - 6) \leq 0$ . We will move these charges from one vertex to another such that the overall sum  $\sum_{v_i \in V(G)} (w(v_i) - 6)$  remains unchanged. We move the vertex charges based on only one discharging rule:

( $\star$ ) Each vertex  $v \in V(G)$  such that  $\deg(v) \geq 7$  sends the charge of  $\frac{1}{3}$  to each degree-5 vertex in  $N(v)$ .

To derive the contradiction, it suffices to show that the new charge of each vertex is at least 6 and that there is at least one vertex that has the charge strictly greater than 6.

It is simple to verify that  $\delta(G) \geq 5$  and that no two degree-5 vertices can be adjacent in  $G$  (see Lemma 2.3.2 2.a for the formal proof). For each vertex  $v \in V(G)$  consider a cyclic ordering of vertices in  $N(v)$  as determined by the embedding. Any two consecutive vertices in that ordering have to be adjacent in  $G$  otherwise we can add that edge to the embedding without creating a crossing or creating a loop or a face with 2 edges, thus contradicting the fact that  $G$  is edge maximal. Together with the fact that no two degree-5 vertices are adjacent that implies that each vertex  $v \in V(G)$ , has at most  $\lfloor \frac{\deg(v)}{2} \rfloor$  degree-5 neighbours. Thus each vertex  $v \in V(G)$  with  $\deg(v) \geq 7$  has the new charge  $w(v) \geq \deg(v) - \frac{1}{3} \lfloor \frac{\deg(v)}{2} \rfloor \geq 6$  with equality possible only for degree-7 vertices.

Since the charge of degree-6 vertices remains unchanged, it remains to consider the new charges of degree-5 vertices.

Consider a degree-5 vertex  $v \in V(G)$ . By the above observation each vertex in  $N(v)$  has degree at least 6. Assume that there are two vertices  $x, y \in N(v)$  such that  $\deg(x) = \deg(y) = 6$  and  $xy \in E(G)$ . By the vertex minimality of  $G$ , the graph  $G' = G - \{v, x, y\}$  is  $(5, 1)^*$ -choosable. This 5-colouring with defect 1 of  $G'$  can be extended to 5-colouring of  $G$  as follows. Vertices  $x$  and  $y$  have each exactly 4 neighbours in  $G'$  thus they each have one colour in their lists that is not used in  $G'$ . In  $G$ , let  $x$  and  $y$  each receive their one remaining colour. By the same argument  $v$  has 2 colours in  $L(v)$  that are not used in  $G'$ . If  $x$  and  $y$  are coloured with a same colour  $q$  then  $v$  chooses the colour dis-

tinct from  $q$  from its remaining 2-list, otherwise  $v$  chooses any of its two remaining colours. Clearly, the resulting 5-colouring of  $G$  has defect 1. Thus no two adjacent neighbours of a degree-5 vertex have both degree 6. By the edge maximality of  $G$  that implies that each degree-5 vertex has three or more neighbours of degree at least 7. Thus the new charge of each degree-5 vertex  $v$  is  $w(v) \geq \deg(v) + 3\frac{1}{3} \geq 6$ , and therefore every vertex in  $G$  has charge at least 6.

Thus, as discussed above, to complete the proof we need to find one vertex that has charge strictly greater than 6. By Lemma 2.3.1,  $\Delta(G) \geq 10$ , thus  $G$  has a vertex  $v$  of degree greater than 7, and therefore the charge of that vertex is  $w(v) \geq \deg(v) - \frac{1}{3} \lfloor \frac{\deg(v)}{2} \rfloor > 6$ .  $\square$

Having the choice-1 number at most  $p$  implies having the chromatic number at most  $2p$ , thus toroidal graphs are not  $(3, 1)^*$ -choosable. The right upper bound for the choice-1 number of toroidal graphs is thus either 4 or 5.

**Open Problem 1.** Are toroidal graphs  $(4, 1)^*$ -choosable?

Unfortunately, there is a weaker statement in the literature not known to be true. Specifically, Cowen et al. [4] ask the following question which is still open

**Open Problem 2.** [4] Are toroidal graphs  $(4, 1)^*$ -colourable?

### 2.3 USEFUL LEMMAS

The following two lemmas will be used in proofs of both Theorem 1 and Theorem 2.

We start with a useful observation. Lovász' [12] proof for defective colouring of graphs of maximum degree  $\Delta$  extends easily to choosability. We include the proof for completeness.

**LEMMA 2.3.1.** [12] *For every integer  $k > 0$ , every graph  $G$  with maximum degree  $\Delta$  is  $(k, \lfloor \frac{\Delta}{k} \rfloor)^*$ -choosable. In fact,  $G$  is  $k$ -choosable such that each vertex  $v \in V(G)$  has at most  $\lfloor \frac{\deg(v)}{k} \rfloor$  neighbours with the same colour as  $v$ .*

*Proof.* For a  $k$ -list assignment  $L$  of  $G$  consider an  $L$ -colouring of  $G$  that minimizes the number of monochromatic edges. Assume, for the sake of contradiction, that this colouring does not meet the conditions of the lemma. Let  $V_1, V_2, \dots, V_{|L(G)|}$  denote the resulting colour classes (some possibly empty), where  $L(G) = \cup\{L(v)|v \in V(G)\}$ . Assume that there is a vertex  $v \in V(G)$  coloured  $c \in L(v)$  such that there are at least  $\lfloor \frac{\deg(v)}{k} \rfloor + 1$  neighbours of  $v$  in the colour class  $V_c$ . In that case, there is a colour class  $V_p$ ,  $p \in L(v)$  such that the number of neighbours of  $v$  in  $V_p$  is at most  $\lfloor \frac{\deg(v)}{k} \rfloor$ . Changing the colour of  $v$  from  $c$  to  $p$  reduces the number of monochromatic edges, thus the contradiction.  $\square$

The proof of Theorem 1 uses the discharging technique. The following lemma provides some key observations that will be used for discharging rules.

**LEMMA 2.3.2.** *Let  $G$  be a vertex minimal graph such that  $G$  is not  $(t, 1)^*$ -choosable,  $3 \geq t \in \mathbb{N}$ . By vertex minimal, we mean that for every  $v \in V(G)$ ,  $G - v$  is  $(t, 1)^*$ -choosable but  $G$  itself is not  $(t, 1)^*$ -choosable. Then  $G$  has the following properties.*

1.
  - a) The minimum degree,  $\delta(G)$ , of  $G$  is at least  $t$
  - b) The maximum degree,  $\Delta(G)$ , of  $G$  is at least  $2t$
  - c) The number of vertices in  $G$  is at least  $2t + 1$ .
2.
  - a) The set of degree- $t$  vertices of  $G$  forms an independent set in  $G$ .
  - b) If  $G$  is a 2-cell embedded graph in a surface of Euler genus  $eg(G)$  that is nearly simple and edge maximal then the following is true: Each vertex  $v$  of  $G$  has at most  $\lfloor \frac{\deg(v)}{2} \rfloor$  degree- $t$  neighbours.
3.
  - a) There is no 3-cycle  $v, w, u$  in  $G$  such that  $\deg(v) = t$ ,  $\deg(w) = t + 1$  and  $\deg(u) = t + 1$ .
  - b) If  $G$  is a 2-cell embedded graph in a surface of Euler genus  $eg(G)$  that is nearly simple and edge maximal then the following is true: Each degree- $t$  vertex  $v$  of  $G$ , has at least  $\lceil \frac{t}{2} \rceil$  degree- $d$ ,  $d \geq t + 2$ , neighbours.
4. If  $G$  is a 2-cell embedded graph in a surface of Euler genus  $eg(G)$  that is nearly simple and edge maximal then the follow-

ing is true: If  $t$  is even and a degree- $t$  vertex  $v$  has exactly  $\frac{t}{2}$  degree- $d$ ,  $d \geq t + 2$ , neighbours, then at least one of them has degree at least  $t + 3$ .

*Proof.* By the assumptions of the lemma there is a  $t$ -list assignment  $L$  such that  $G$  is not  $L$ -colourable.

**1a.** Assume on the contrary that  $G$  has a vertex  $v$  of degree at most  $t - 1$ . Then  $G - v$  is a nonempty graph and thus by the vertex minimality of  $G$  it is  $L$ -colourable with defect 1. Since  $|N(v)| \leq t - 1$ , there is a colour in the  $t$ -list  $L(v)$  that is not used by any vertex in  $|N(v)|$ , thus the  $L$ -colouring of  $G - v$  can be extended to  $L$ -colouring of  $G$ , giving the contradiction.

**1b.** If  $\Delta(G) < 2t$ , then  $\lfloor \frac{\Delta(G)}{t} \rfloor = 1$  and thus  $G$  is  $(t, 1)^*$ -choosable by Lemma 2.3.1, thus contradicting the assumptions of the lemma.

**1c.** Follows from **1b**.

**2a.** Assume on the contrary that  $G$  has two degree- $t$  vertices,  $v$  and  $w$ , that are adjacent. Then  $G - \{v, w\}$  is a nonempty graph and thus by the vertex minimality of  $G$  it is  $L$ -colourable with defect 1. Since  $|\{N(v) - w\}| \leq t - 1$  and  $|\{N(w) - v\}| \leq t - 1$ , there is a colour  $c_1$  in the  $t$ -list  $L(v)$  and  $c_2$  in the  $t$ -list  $L(w)$  such that no vertex in  $\{N(v) - w\}$  is coloured  $c_1$  and no vertex in  $\{N(w) - v\}$  is coloured  $c_2$  in the  $L$ -colouring of  $G - \{v, w\}$ . Thus, assigning colour  $c_1$  to  $v$  and  $c_2$  to  $w$  gives  $L$ -colouring of  $G$  with defect 1 (even if  $c_1 = c_2$ ).

**2b.** Consider the cyclic ordering of edges around  $v$  in the embedding. By edge maximality of  $G$ , for any pair of consecutive edges  $vw$  and  $vy$  around  $v$ , we have that  $w$  and  $y$  are adjacent. Thus, by **1a** and **2a**,  $w$  or  $y$  must have degree at least  $t + 1$ , implying the claim.

**3a.**  $G - \{v, w, u\}$  is a nonempty graph and thus by the vertex minimality,  $G$  it is  $L$ -colourable with defect 1. Since  $|\{N(w) - \{v, u\}\}| \leq t - 1$  and  $|\{N(u) - \{v, w\}\}| \leq t - 1$ , there is a colour  $c_1$  in the  $t$ -list  $L(w)$  and  $c_2$  in the  $t$ -list  $L(u)$  such that no vertex in  $\{N(w) - \{v, u\}\}$  is coloured  $c_1$  and no vertex in  $\{N(u) - \{v, w\}\}$  is coloured  $c_2$  in the  $L$ -colouring of  $G - \{v, w, u\}$ . Assign colour  $c_1$  to  $w$  and  $c_2$  to  $u$ . That gives  $L$ -colouring of  $G - v$  with defect 1. If there is a colour  $c$  in the  $t$ -list  $L(v)$  such that no vertex in

$N(v)$  is coloured  $c$  in that L-colouring of  $G - v$ , then assigning colour  $c$  to  $v$  gives L-colouring of  $G$  with defect 1. Otherwise,  $c_1 \in L(v)$  and  $c_2 \in L(v)$  and  $c_1 \neq c_2$ . In that case, both  $w$  and  $u$  are proper in the L-colouring of  $G - v$  and thus assigning colour  $c_1$  to  $v$  gives L-colouring of  $G$  with defect 1.

3b. The edge maximality of  $G$  implies that every pair of consecutive edges  $vx$  and  $vw$  around  $v$ , defines a 3-cycle  $v, x, y$  in  $G$ . Then 3a and the fact that  $\deg(v) = t$  imply the claim.

4. Assume for the sake of contradiction that all the neighbours of  $v$  have degree at most  $t + 2$ . Let  $H = G[N(v)]$  and  $H' = G[v \cup N(v)]$ . Consider vertices in  $N(v)$  in the cyclic order,  $v_1, v_2 \dots v_t$ , around  $v$  as determined by the embedding and starting with a degree- $(t + 2)$  vertex  $v_1$ . By the edge maximality of  $G$ ,  $C = v_1, \dots, v_t, v_1$  is a cycle (non necessarily induced) in  $G$ . Then the degrees in  $G$  of vertices in  $N(v)$  are as follows  $\deg(v_i) = t + 2$  for  $i \equiv (1 \pmod{2})$  and  $\deg(v_i) = t + 1$  for  $i \equiv (0 \pmod{2})$ . (The fact that the degrees alternate, between  $t + 1$  and  $t + 2$ , along  $C$  is the consequence of Property 2a, Property 3a and the assumption that  $v$  has exactly  $\frac{t}{2}$  degree- $(t + 2)$  neighbours). Thus, since  $t$  is even, for each degree- $(t + 2)$  vertex in  $C$  its two neighbours along  $C$  are degree- $(t + 1)$  vertices.  $G' = G - \{v \cup N(v)\}$  has Euler genus at most  $\mu$  and it is not an empty graph by Property 1c and the fact that  $|v \cup N(v)| = t + 1$ , and  $t > 0$ . Thus,  $G'$  is L-colourable with defect 1. This list colouring of  $G'$  can be extended to a L-colouring of  $G$  with defect 1 as follows.

Define the list assignment  $L'$  of  $H'$  (and  $H$ ) as follows. For every  $w \in V(H')$  the list  $L'(w)$  is equal to the list  $L(w)$  minus the colours used by the neighbours of  $w$  in L-colouring of  $G'$ . Clearly any  $L'$ -list colouring of  $H'$  with defect 1, extends the colouring of  $G'$  to L-colouring of  $G$  with defect 1. By considering degrees in  $G$  of vertices in  $H'$  we get,  $|L'(v)| = |L(v)|$ ,  $|L'(v_i)| \geq \deg_{H'}(v_i) - 2$  for  $i \equiv (1 \pmod{2})$  and  $|L'(v_i)| \geq \deg_{H'}(v_i) - 1$  for  $i \equiv (0 \pmod{2})$ . Moreover, in  $H$ ,  $|L'(v_i)| \geq \deg_H(v_i) - 1$  for  $i \equiv (1 \pmod{2})$  and  $|L'(v_i)| \geq \deg_H(v_i)$  for  $i \equiv (0 \pmod{2})$ . Let  $S = \{\cup v_i \mid i \equiv (0 \pmod{2})\}$ . By the above observation, each vertex  $v_i$ ,  $i \equiv (1 \pmod{2})$ , is adjacent to at least 2 vertices of  $S$  (its neighbours along  $C$ ), thus  $\deg_H(v_i) \geq \deg_{H-S}(v_i) + 2$ .

Thus, in  $H - S$ ,  $|L'(v_i)| \geq \deg_{H-S}(v_i) + 1$  for all  $i \equiv (1 \pmod 2)$ . Therefore, each vertex  $x$  in  $H - S$  has smaller degree in  $H - S$  than the number of colours in its list as determined by  $L'$  list assignment, that is,  $\deg_{H-S}(x) < |L'(x)|$ . The greedy colouring then implies that  $H - S$  has  $L'$ -colouring where every vertex in  $H - S$  is proper.

By property 3a,  $S$  forms an independent set in  $H$ . Thus, all of  $\deg_H(v_i)$  neighbours of vertex  $v_i \in S$  in  $H$  are in  $V(H) - S$ . Let  $A$  denote the vertex set comprised of the vertices  $v_i \in S$  whose neighbours in  $H - S$  use at most  $|L'(v_i)| - 1 \geq \deg_H(v_i) - 1$  colours. Let  $B = S - A$ . Since  $|L'(v_i)| \geq \deg_H(v_i)$  for each vertex  $v_i \in S$ , each vertex in  $v_i \in A$  can choose a colour from its list  $L'(v_i)$  such that  $v_i$  is proper in  $L'$ -colouring of  $H - B$  (and in  $H$  as will be seen later). Then  $H - B$  is  $L'$ -coloured such that every vertex of  $H - B$  is proper. For each vertex  $v_i \in B$ , each of its colours in  $L'(v_i)$  is used by exactly one of its neighbours in  $H$ , and each of its neighbours in  $H$  uses actually one colour in  $L'(v_i)$ . Thus, giving each such vertex  $v_i$  the colour equal to the colour of its first counterclockwise neighbour along  $C$  defines  $L'$ -colouring of  $H$ . It remains to show that this  $L'$ -colouring of  $H$  has defect at most 1. Clearly, the vertices of  $A$  are proper in the colouring of  $H$ . Vertices of  $H - S$  are proper in the colouring of  $H - B$ , thus all the monochromatic edges have one endpoint in  $B$  and the other in  $V(H) - S$ . Assume a vertex  $w \in B$  has two neighbours in  $V(H) - S$  coloured with the same colour as  $w$ . That implies that the number of colours used by the neighbours of  $w \in V(H) - S$  is at most  $\deg_H(w) - 1$  thus  $w \in A$ , contradiction. Finally, assume a vertex  $w \in V(H) - S$  has two neighbours  $x$  and  $y$  in  $B$  coloured with the same colour as  $w$ . That implies that  $w$  is the first counter clockwise neighbour on  $C$  of both  $x$  and  $y$ , which is impossible since no pair of vertices of  $S$  are adjacent. Thus, we have an  $L'$ -colouring of  $H$  with defect 1. Recall that  $|L'(v)| = t$ . In  $H'$ , if the vertices in  $H$  do not use all the colours in  $L'(v)$ , then  $v$  can be coloured such that it is proper in  $L'$ -colouring of  $H'$ , thus resulting in the  $L'$ -colouring of  $H'$  with defect at most one. Otherwise, each colour in  $L'(v)$  is used by exactly one vertex in  $H$  and each vertex in  $H$  uses

exactly one colour in  $L'(v)$ . Therefore, the  $L'$ -colouring of  $H$  is proper and no two neighbours of  $v$  use the same colour. Thus,  $v$  can use any colour in  $L'(v)$  and the resulting  $L'$ -colouring of  $H'$  has defect at most one.

By the choice of  $L'$  the resulting  $L'$ -colouring of  $H'$  with defect 1 extends the  $L$ -colouring of  $G'$  to give  $L$ -colouring of  $G$  with defect 1, which is a desired contradiction.  $\square$

#### 2.4 PROOF OF THEOREM 1

Let  $\mu := \text{eg}(G)$  and  $t := \left\lceil 2 + \sqrt{3\mu + 3} \right\rceil$ . If  $G$  is planar, the statement of the theorem is true by the result of Cushing and Kierstead[8]. Thus, we may assume that  $G$  is not planar, that is  $\mu > 0$ . Thus,  $t \geq 5$ . Assume for the sake of contradiction that the statement of the theorem is false. We may assume that  $G$  is a vertex minimal, that is,  $G$  is not  $(t, 1)^*$ -choosable but for every  $v \in V(G)$ ,  $G - v$  is  $(t, 1)^*$ -choosable. Furthermore we may assume that  $G$  is a connected nearly simple edge maximal graph with a 2-cell embedding on a surface of Euler genus  $\mu$ . Thus  $G$  meets all the conditions of Lemma 2.3.2.

The Euler formula for connected graphs states  $|F(G)| = |E(G)| - |V(G)| + 2 - \mu$ . Each face in an embedding of the edge maximal graph has size at most 3, and each edge is in at most 2 faces, thus  $|F(G)| \leq \frac{2|E(G)|}{3}$ .

For each  $v \in V(G)$  let its *charge*  $w(v) := \deg(v)$ . Since  $2|E(G)| = \sum_{v_i \in V(G)} \deg(v_i)$ , the Euler formula and the above inequality give

$$\sum_{v_i \in V(G)} (w(v_i) - 6) \leq 6\mu - 12. \quad (2.2)$$

We will move these charges from one vertex to another such that the overall sum  $\sum_{v_i \in V(G)} (w(v_i) - 6)$  remains unchanged. We move the vertex charges according to the following discharging rules:

( $\star 1$ ) Each degree- $(t+2)$  vertex  $v$  sends the charge of  $\frac{1}{\lfloor \frac{t}{2} \rfloor + 1}$  to each degree- $t$  vertex in  $N(v)$ .

( $\star 2$ ) If  $t$  is odd, each degree- $d$  vertex  $v$ ,  $d \geq t+3$ , sends the charge of  $\frac{1}{\lfloor \frac{t}{2} \rfloor + 1}$  to each degree- $t$  vertex in  $N(v)$ .

( $\star 3$ ) If  $t$  is even, each degree- $d$  vertex  $v$ ,  $d \geq t+3$ , sends the charge of  $\frac{2}{\frac{t}{2} + 1}$  to each degree- $t$  vertex in  $N(v)$ .

After above discharging rules are applied to all the vertices in  $G$ , their new charges are as follows. The charges of degree- $(t+1)$  vertices remain unchanged.

Consider now the new charge of a degree- $t$  vertex  $v$ . By property 3b in Lemma 2.3.2,  $v$  has at least  $\lceil \frac{t}{2} \rceil$  degree- $d$ ,  $d \geq t+2$  neighbours. Thus, if  $t$  is odd, each degree- $t$  vertex  $v$  has new charge  $w(v) \geq t + \lceil \frac{t}{2} \rceil \frac{1}{\lfloor \frac{t}{2} \rfloor + 1} = t + \frac{t+1}{2} \frac{1}{\frac{t-1}{2} + 1} = t + 1$ . If  $t$  is even, we have two cases to consider. First consider the case that  $v$  has at last  $\frac{t}{2} + 1$  degree- $d$ ,  $d \geq t+2$  neighbours. Then the new charge of  $v$  is  $w(v) \geq t + (\frac{t}{2} + 1) \frac{1}{\frac{t}{2} + 1} = t + 1$ . The second case to consider is that  $v$  has exactly  $\frac{t}{2}$  degree- $d$ ,  $d \geq t+2$  neighbours, in which case by property 4 in Lemma 2.3.2, one of the neighbours if  $v$  has degree at least  $t+3$ . Then by the rule ( $\star 3$ ), the new charge of  $v$  is  $w(v) \geq t + (\frac{t}{2} - 1) \frac{1}{\frac{t}{2} + 1} + \frac{2}{\frac{t}{2} + 1} = t + 1$ .

Since the minimum degree in  $G$  is  $t$ , by Property 1a of Lemma 2.3.2, it remains to consider the charges of the degree- $d$ ,  $d \geq t+2$  vertices. By property 2b in Lemma 2.3.2, each degree- $(t+2)$  vertex has at most  $\lfloor \frac{t+2}{2} \rfloor$  degree- $t$  neighbours. The rule ( $\star 1$ ) applies to each such vertex  $v$  and thus its new charges is  $w(v) \geq t + 2 - \lfloor \frac{t+2}{2} \rfloor \frac{1}{\lfloor \frac{t}{2} \rfloor + 1} = t + 1$  (both when  $t$  is odd and even).

Consider now a degree- $d$  vertex  $v$ ,  $d \geq t+3$ . If  $t$  is odd, then the rule ( $\star 2$ ) applies and by property 2b in Lemma 2.3.2, the new charge of  $v$  is  $w(v) \geq d - \lfloor \frac{d}{2} \rfloor \frac{1}{\lfloor \frac{t}{2} \rfloor + 1} > t + 1$ . (That is because,  $d - \lfloor \frac{d}{2} \rfloor \frac{1}{\lfloor \frac{t}{2} \rfloor + 1} \geq d - \frac{d}{2} \frac{1}{\frac{t-1}{2} + 1} = d - \frac{d}{t+1}$ . Now  $d - \frac{d}{t+1} > t + 1$  whenever  $d > t + 2 + \frac{1}{t}$ . Since  $t > 1$ , that is our case.)

If  $t$  is even, then the rule ( $\star 3$ ) applies and by property 2b in Lemma 2.3.2, the new charge of  $v$  is  $w(v) \geq d - \lfloor \frac{d}{2} \rfloor \frac{2}{\frac{t}{2} + 1}$ . If  $d = t + 3$ , then  $d$  is odd and thus  $w(v) \geq t + 3 - \frac{t+2}{2} \frac{2}{\frac{t}{2} + 1} = t + 1$ . Else  $d \geq t + 4$ , and  $w(v) \geq d - \lfloor \frac{d}{2} \rfloor \frac{2}{\frac{t}{2} + 1} > t + 1$ . (That is

because,  $d - \lfloor \frac{d}{2} \rfloor \frac{2}{\frac{t}{2}+1} \geq d - \frac{2d}{t+2}$ . Now  $d - \frac{2d}{t+2} > t + 1$  whenever  $d > t + 3 + \frac{2}{t}$ . Since  $t > 2$ , that is our case.)

Therefore, after discharging every vertex has charge at least  $t + 1$ . Finally, we show that there is at least one vertex with the charge greater than  $t + 1$ . By the above arguments each degree  $d$ ,  $d \geq t + 4$ , vertex  $v$  has charge  $w(v) \geq d - \frac{d}{t+1}$  when  $t$  is odd, and charge  $w(v) \geq d - \frac{2d}{t+2}$  when  $t$  is even. Thus, in either case,  $w(v) \geq \frac{dt}{t+2}$ . By Lemma 2.3.1,  $G$  has a vertex  $v$  of degree at least  $2t$ . Since  $t \geq 4$ ,  $2t \geq t + 4$ . Therefore,  $G$  has a vertex  $v$  with charge  $w(v) \geq \frac{dt}{t+2} \geq \frac{2t^2}{t+2}$ .

The new charge and the inequality 2.2 give

$$\begin{aligned} \sum_{i=1}^n (w(v_i) - 6) &\geq (n-1)(t+1-6) + \frac{2t^2}{t+2} - 6 \\ &= (n+1)(t-5) + \frac{8}{t+2} \\ &> (n+1)(t-5). \end{aligned}$$

By Property 1c of Lemma 2.3.2,  $n \geq 2t + 1$ , and with  $t \geq 5$  and inequality 2.2 we get

$$(2t+2)(t-5) < \sum_{i=1}^n (w(v_i) - 6) \leq 6\mu - 12$$

The inequality  $(2t+2)(t-5) < 6\mu - 12$  is only true for  $t < 2 + \sqrt{3\mu+3}$  thus for  $t = \lceil 2 + \sqrt{3\mu+3} \rceil$  we get a contradiction thereby completing the proof.

□

## GRAPHS ON SURFACES WITH NO TWO ADJACENT TRIANGLES

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### 3.1 PROOF OF THEOREM 2

Let  $\mu := \text{eg}(G)$  and  $t := \left\lceil \frac{5 + \sqrt{24\text{eg}(G) + 1}}{3} \right\rceil$ . Since  $G$  is not planar,  $\text{eg}(G) \geq 1$  and thus  $t \geq 4$ .

Assume for the sake of contradiction that the statement of the theorem is false. We may assume that  $G$  is a connected vertex minimal graph contradicting the statement of the theorem. That is,  $G$  is non-planar graph that has an embedding without adjacent triangles in a surface of Euler genus  $\text{eg}(G)$ , such that  $G$  is not  $(t, 1)^*$ -choosable but for every  $v \in V(G)$ ,  $G - v$  is  $(t, 1)^*$ -choosable.

We start as in the proof of Theorem 1.

Since  $G$  is connected we can apply the Euler formula. When multiplied by 4, the Euler formula can be written as:

$$4|E(G)| - 4|V(G)| - 4|F(G)| = 4\mu - 8$$

Since  $2|E(G)| = \sum_{v \in V(G)} \deg(v)$ , this can be rewritten as:

$$\sum_{v \in V(G)} (\deg(v) - 4) + \sum_{f \in F(G)} (\deg(f) - 4) = 4\mu - 8$$

For each vertex,  $v \in V(G)$ , let the *charge* of  $v$  be  $w(v) := \deg(v) - 4$ , and for each face,  $f \in F(G)$  let the charge of  $f$  be  $w(f) := \deg(f) - 4$ . Then

$$\sum_{v \in V(G)} w(v) + \sum_{f \in F(G)} w(f) = 4\mu - 8 \tag{3.1}$$

We will move these charges from a vertex to its incident faces such that the overall sum  $\sum_{v \in V(G)} w(v) + \sum_{f \in F(G)} w(f)$  remains

unchanged. We move the vertex charges according to the following simple discharging rule:

( $\star$ ) Each each degree- $d$  vertex  $v$  such that  $d \geq t + 1$ , sends the charge of  $1/2$  to each of its incident faces of size exactly 3 (that is, to each of its incident triangles).

After the above discharging rule is applied to all the vertices in  $G$ , the new charges of vertices and faces of  $G$  are as follows.

We first consider the new charges of faces of  $G$ .  $G$  satisfies the conditions in Lemma 2.3.2, property 2a. Thus by that property no two vertices of degree- $t$  are adjacent in  $G$ . Thus each triangle  $f$  in  $F(G)$  has on its boundary at least two vertices of degree  $d$  such that  $d \geq t + 1$ . Thus after applying rule ( $\star$ ) each such triangle face receives charge 1 from its neighbouring vertices and thus its new charge is non-negative. The remaining faces have at least four edges on its boundary, thus their charge too is nonnegative. Thus after discharging,  $\sum_{f \in F(G)} w(f) \geq 0$ . That and equality 3.1 give the following inequality:

$$\sum_{v \in V(G)} w(v) \leq 4\mu - 8. \quad (3.2)$$

Observe that if there is a triangle in  $G$  such that each of its three end-vertices has the degree at least  $t + 1$ , then the above inequality is strict. That fact will be useful later.

We now consider the new charges of vertices of  $G$ .  $G$  satisfies the conditions in Lemma 2.3.2, property 1a. Thus by that property, the minimum degree in  $G$  is at least  $t$ . The charge of each degree- $t$  vertex remains unchanged, that is, the new charge is again  $t - 4$ . In addition, the new charge is non-negative since  $t \geq 4$ .

Since  $G$  has no adjacent triangles, each degree- $d$  vertex, is adjacent to at most  $\lfloor d/2 \rfloor$  triangle faces. Thus after applying rule ( $\star$ ) each vertex of degree  $d$  at least  $t + 1$ , has new charge at least  $d - 4 - \frac{1}{2} \cdot \lfloor \frac{d}{2} \rfloor$ . For  $d$  even, that amounts to charge of at

least  $3d/4 - 4 \geq 3(t+1)/4 - 4 = 3/4 \cdot t - 13/4$ . Furthermore for even  $d$ , we know that  $d \geq 6$ , since  $d \geq t+1$  and  $t \geq 4$ . Thus the new charge is non-negative since  $3d/4 - 4 \geq 0$  for all  $d \geq 6$ .

For  $d$  odd,  $d - 4 - \frac{1}{2} \cdot \lfloor \frac{d}{2} \rfloor$  amounts to charge of at least  $d - 4 - \frac{1}{2} \cdot \frac{d-1}{2} = 3d/4 - 15/4 \geq 3(t+1)/4 - 15/4 = 3/4 \cdot t - 3$ . Furthermore for odd  $d$ , we know that  $d \geq 5$ , since  $d \geq t+1$  and  $t \geq 4$ . Thus the new charge is non-negative since  $3d/4 - 15/4 \geq 0$  for all  $d \geq 5$ .

$G$  satisfies the conditions in Lemma 2.3.2, property 1b. Thus by that property, the maximum degree in  $G$  is at least  $2t$ . Thus by the arguments from the previous paragraph, a vertex of maximum degree in  $G$  has new charge at least  $3t/2 - 4$ .

Since  $t \geq 4$ ,  $t - 4 \geq 3/4 \cdot t - 3$  (and  $t - 4 \geq 3/4 \cdot t - 13/4$ ), the lower bounds on the new charges from the previous paragraph also apply to the vertices of degree  $d = t$ .

Let  $u$  be a vertex of  $G$  of maximum degree. All the vertices of  $G$  have the new charge at least as large as the new charge we deduced above for vertices of degree  $d = t + 1$ . When such a (degree- $(t + 1)$ ) vertex has odd degree,  $t$  is even and when it has an even degree  $t$  is odd. Thus the new charges deduced above and the inequality 3.2 give the following for  $t$  even

$$\begin{aligned} 3t/2 - 4 + \sum_{v \in V(G) - u} (3/4 \cdot t - 3) &\leq 4\mu - 8 \\ 3t/2 - 4 + (3/4 \cdot t - 3) \cdot (n - 1) &\leq 4\mu - 8, \end{aligned}$$

and the following for  $t$  odd

$$\begin{aligned} 3t/2 - 4 + \sum_{v \in V(G) - u} (3/4 \cdot t - 13/4) &\leq 4\mu - 8 \\ 3t/2 - 4 + (3/4 \cdot t - 13/4) \cdot (n - 1) &\leq 4\mu - 8. \end{aligned}$$

To get our result we will need these two inequality to be strict, both when  $t$  is odd and when  $t$  is even. The following arguments apply in both cases. As observed earlier, if  $G$  has a triangle all of whose end-vertices have degree at least  $t + 1$  then we already have strict inequality. Otherwise, we now show that  $G$  has at least one vertex with degree at least  $t + 2$ , other than  $u$ . That is enough as it implies that  $\sum_{v \in V(G) - u} w(v) > \sum_{v \in V(G) - u} (3/4 \cdot t - 3)$  when  $t$  is even and that  $\sum_{v \in V(G) - u} w(v) >$

$\sum_{v \in V(G)-u} (3/4 \cdot t - 13/4)$  when  $t$  is odd. Assume on the contrary that the only vertex of degree greater than  $t + 1$  is  $u$ . Then  $G - u$  has the maximum degree at most  $t + 1$  and thus by Lemma 2.3.1,  $G - u$  is  $(t - 1, \lfloor \frac{t+1}{t-1} \rfloor)^*$ -choosable. Since  $t \geq 4$ ,  $\lfloor \frac{t+1}{t-1} \rfloor = 1$  and thus  $G - u$  is  $(t - 1, 1)^*$ -choosable. That defect-1  $(t - 1)$ -colouring of  $G - u$  can be extended to  $t$ -colouring of  $G$ , by assigning to  $u$  an unused colour from its list of  $t$  colours. In the resulting colouring of  $G$ ,  $u$  is proper and thus the colouring of  $G$  has defect at most 1. Thus we get the following strict inequality for  $t$  even

$$3t/2 - 4 + (3/4 \cdot t - 3) \cdot (n - 1) < 4\mu - 8,$$

and the following for  $t$  odd

$$3t/2 - 4 + (3/4 \cdot t - 13/4) \cdot (n - 1) < 4\mu - 8.$$

Since  $t \geq 4$ ,  $3/4 \cdot t - 3$  is non-negative for all even  $t$ . Similarly, since  $t \geq 4$ ,  $3/4 \cdot t - 13/4$  is non-negative for all odd  $t$ . Thus in both cases we can replace  $n$  by its lower bound. By property 1c of Lemma 2.3.2,  $n \geq 2t + 1$ . That implies the following for  $t$  even

$$3t/2 - 4 + (3/4 \cdot t - 3) \cdot 2t < 4\mu - 8,$$

and the following for  $t$  odd

$$3t/2 - 4 + (3/4 \cdot t - 13/4) \cdot 2t < 4\mu - 8.$$

The first inequality is true only for  $t < \frac{3}{2} + \frac{\sqrt{24\mu-15/4}}{3}$  thus for  $t$  even we get a contradiction when  $t := \left\lceil \frac{3}{2} + \frac{\sqrt{24\mu-15/4}}{3} \right\rceil$ .

The second inequality is true only for  $t < \frac{5}{3} + \frac{\sqrt{24\mu+1}}{3}$  thus for  $t$  odd we get a contradiction when  $t := \left\lceil \frac{5+\sqrt{24\mu+1}}{3} \right\rceil$ . Since,  $\left\lceil \frac{3}{2} + \frac{\sqrt{24\mu-15/4}}{3} \right\rceil < \left\lceil \frac{5+\sqrt{24\mu+1}}{3} \right\rceil$  for all  $\mu > 0$  that completes the proof.

## SUMMARY AND OPEN PROBLEMS

The choice number of a surface  $\Sigma$ , other than the sphere, is known to be  $\lfloor \frac{7+\sqrt{24\text{eg}(\Sigma)+1}}{2} \rfloor$ . In this thesis we study if the choice number of a surface can be improved when we allow the minimum non-trivial defect, that is defect 1. The main result of this thesis is to show that the answer to that question is positive, specifically to show that every graph  $G$  that can be embedded in a surface  $\Sigma$  is  $(\lceil 2 + \sqrt{3\text{eg}(\Sigma) + 3} \rceil, 1)^*$ -choosable. If in addition  $G$  is non-planar and has a 2-cell embedding in  $\Sigma$  without adjacent triangles then  $G$  is  $(\lceil \frac{5+\sqrt{24\text{eg}(\Sigma)+1}}{3} \rceil, 1)^*$ -choosable.

The main remaining open problem is to determine a tight bound for the choice-1 number of graphs on surfaces. The first step would be to determine if the choice-1 number of toroidal graphs is 4 or 5. However it is not even known if chromatic-1 number of toroidal graphs is 4. (See open problems 1 and 2).

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