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**LA THÈSE A ÉTÉ
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THE TREATMENT OF DEFLECTION, VIBRATION
AND STABILITY OF PLATES USING
THE COLLOCATION LEAST SQUARE METHOD

BY

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A thesis submitted to the School of Graduate Studies
through the Department of Civil Engineering in
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the Degree of Master of Applied Science
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ABSTRACT

Error distribution principles have been widely used in the past for the solution of boundary and eigen-value problem. Of all the numerical schemes that are based on the principles of error distribution, the collocation method is the most versatile and simple method. However, the method is not very reliable since the solution can fluctuate greatly for arbitrary choices of collocation points.

In this study, the conventional collocation method is improved by a least square augmentation. While retaining the simplicity of collocation, the proposed method provides results that are independent of the choice and distribution of the collocation points.

To demonstrate the simplicity and accuracy of the proposed method, typical applied mechanics problems such as bending, buckling and vibration of plates are used as illustrative examples. The results obtained, are presented in tabular and graphical forms, and whenever possible, are compared with existing solutions based on much more tedious and lengthy methods of analysis. The comparisons are generally very favorable.

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NOMENCLATURE

x, y, z	rectangular Cartesian coordinates
w	displacement in the z-direction
E_x, E_y, G_{xy}	moduli of elasticity and shear modulus of orthotropic material
E, G	modulus of elasticity and shear modulus of isotropic material
ν	Poisson's ratio for isotropic material
ν_x, ν_y	Poisson's ratios for orthotropic material
h	plate thickness
D	flexural rigidity of plates
D_x, D_y, D_{xy}	bending and twisting stiffnesses of orthotropic plates
k	modulus of elastic foundation
q	lateral load per unit area
M_x, M_y, M_{xy}	bending and twisting moments
N_x, N_y, N_{xy}	normal and shear forces per unit length, in the x-y plane
S_x, S_y, S_{xy}	bending and shear stresses
t	time variable
w'	time dependent displacement in the z-direction
ω	circular frequency
m'	mass per unit area of plate
a, b	plate dimensions in the x and y-directions
R	aspect ratio of plate, (a/b)

W dimensionless displacement in the z-direction
x', y' dimensionless parameters of x and y directional
coordinates, ($x' = x/a$, $y' = y/b$)
K dimensionless modulus of elastic foundation
Q dimensionless parameter of lateral load
 N'_x, N'_y, N'_{xy} dimensionless parameters of inplane forces
F eigenvalue parameter
f dimensionless frequency parameter, $(F)^{\frac{1}{2}}$
 α skew angle of rhombus
 β $90^\circ - \alpha$
m, n integers

Note: partial differentiation is denoted by a comma
in subscript.

CHAPTER I
INTRODUCTION

1.1 General:

The governing differential equations of boundary value problems in applied mechanics are usually rather complex; exact solutions to these differential equations can only be obtained for a few simple cases. In the majority of cases, it is almost impossible to find a relatively simple function which would simultaneously satisfy both the governing differential equation and the boundary conditions. Sometimes, exact solutions are not possible to obtain even for relatively simple cases. Confronted with these problems, the researcher frequently has to resort to numerical methods to effect a solution. Numerical methods have frequently been used in the past when rigorous mathematical solutions have failed. The real impetus to their development, however, is the availability of digital computers which came into wide use in the last two decades.

1.2 Brief Discussion of Some Numerical Methods:

The following is a brief discussion of some of the more popular numerical techniques that have been applied successfully to various problems in applied mechanics.

a) Ritz Method: The Ritz method is one of the energy methods. The method has found particular application in the analysis of very complicated problems. The Ritz method is based on the principle of.

minimum potential energy of a system when it is in stable equilibrium. In applying this method to a particular problem, an assumed function with undetermined adjustable parameters, satisfying various essential boundary conditions is chosen. The undetermined parameters of the assumed function can be evaluated from the minimizing condition of the system. In using this method, the differential equations are not involved and hence need not be known. Generally, this will save a considerable amount of mathematical work. The use of the Ritz method is recommended when computers are not readily available and the solution must be obtained by hand computations. Generally, the Ritz solution overestimates the stiffness of the system since the equilibrium differential equation is not directly satisfied. Another drawback of the method is that the amount of arithmetic work can be formidable when the number of parameters in the assumed solution is increased.

b) Fourier Series Method: The Fourier series method is very useful in the analytical treatment of many problems in the field of applied mechanics, such as the analysis of plates. Once the governing differential equation of a problem is determined, a rigorous solution would involve the adjusting of certain constants in order to satisfy the prescribed boundary conditions. The method has found application in numerous problems in structural mechanics because of its ability to represent discontinuous loading functions; however, the convergence of the resulting series solution is generally slow.

c) Perturbation Method: This method has also found important application in various problems in applied mechanics. In applying this method, the solution of the problem is sought in the form of ascending powers of

some arbitrary small perturbation parameter, then the original problem is replaced by a sequence of perturbed algebraic equations. This technique can also be used to replace a non-linear problem by a set of linear perturbed differential equations. Generally, the arithmetic work required to solve the perturbed equations is very lengthy. The main drawback of perturbation is, however, that the perturbation parameter should be kept sufficiently small in order to obtain acceptable accuracy and maintain convergence of the series solution.

d) Finite Difference Method: The finite difference method is one of the most general numerical methods in the field of applied mechanics. It can be effectively used to solve a wide variety of problems. Although the method has been known for a long time, it has gained considerable importance only after the invention of high-speed digital computers. In applying this method, the governing differential equations (and the equations of boundary conditions) are replaced by corresponding finite difference equations, which in turn yield a system of simultaneous algebraic equations. The advantages of the method are:

1. Simplicity in application.
2. Versatility.
3. The resulting numerical equations can be easily programmed using desk-top calculators or digital computers.
4. Acceptable accuracy for most technical purposes, provided that a relatively fine mesh is used.

4

Unfortunately, this method is characterized (beyond a certain mesh width) by slow convergence. Generally, a relatively fine mesh is required to obtain an acceptable accuracy. The accuracy progressively deteriorates when the order of derivatives is increased. Consequently, the finite difference method is not recommended when higher than fourth order derivatives are involved or when high accuracy in the solution is required /38/.

e) The Finite Element Method: The recently developed finite element method has proved to be extremely powerful and versatile for the analysis of a wide variety of structural problems. The most critical, and simultaneously the most difficult, phase of the analysis is the evaluation of the element-stiffness coefficients. Fortunately, the stiffness properties of some of the more commonly used elements, which yield sufficiently accurate results, are readily available. Once the element-stiffness coefficients have been determined, the analysis of the structural system follows the familiar procedure of matrix methods used in structural mechanics for which standard computer programs are available.

The most important advantages of the finite element method are /38/:

1. The solution is obtained without the use of the governing differential equations, thus avoiding the mathematical analysis of the problem.
2. Arbitrary boundary and loading conditions can be handled with great ease.
3. It permits the complete automation of all procedures.

- 4. It permits the combination of various structural elements, such as plates, beams and shells.
- 5. It can be extended to cover virtually all fields of continuum mechanics.

The major disadvantages are:

- 1. It requires the use of electronic digital computers of considerable storage capacity.
- 2. The preparation of data for each element can be time-consuming and is the most general source of human error in the solution.
- 3. Some problems may require sophisticated programming techniques and hence the aid of computer specialists.
- 4. When large systems are analyzed, it is difficult to ascertain the accuracy of the results.

f) Error Distribution Methods: In the treatment of boundary value problems, the problems are often solved by assuming an approximate solution to the differential equation; this approximate solution is usually in the form of an arbitrary linear combination of a set of independent functions and is dependent on a number of adjustable parameters such that for arbitrary values of the parameters,

- (1) the differential equation is satisfied exactly, but not the boundary conditions ("boundary" method),
- or
- (2) the boundary conditions are satisfied exactly, but not the differential equation ("interior" method),
- or
- (3) the assumed solution satisfies neither the differential nor

the boundary conditions ("mixed" method).

If, by some numerical scheme, the undetermined parameters can be obtained such that the assumed solution satisfies

in case (1) (boundary method) the boundary exactly,

in case (2) (interior method) the differential equation exactly,

in case (3) (mixed method) the boundary conditions and the differential equation exactly,

then no error would result if we substitute the assumed solution into the governing differential equation. Obviously, this is rarely possible.

A variety of approximate methods falling into the category of error distribution methods can be employed to distribute the error as uniformly as possible throughout the domain of the solution.

Among these methods are:

- (i) Collocation Method.
- (ii) Least Squares Method.
- (iii) Least Squares with Weighting Functions.
- (iv) Partition Method.
- (v) Relaxation Method.
- (vi) Galerkin Method.

The ultimate aim of these methods is to determine the undetermined parameters in such a manner that, throughout the entire domain of the solution, the assumed solution satisfies the differential equation, or the boundary conditions, or both the differential equation and the boundary conditions as accurately as possible, i.e., the resulting error

be close to zero as possible.

The methods mentioned above are sometimes referred to as methods of weighted residuals. The majority of these methods, with the exception of collocation and relaxation, involve tedious process of definite integration over the region or boundary where the problem is defined. Hence, in terms of ease of computation, i.e., automated computation, these methods should be avoided whenever possible.

The collocation method, though simple in application, suffers from the drawback of uncertainty of results due to the nature of the method.

1.3 Objective and Scope:

The main objective of this thesis is to investigate a means of improving the collocation method for the approximate solution of boundary and eigenvalue problems. The scope of this work covers the application of the modified form of the collocation method to typical plate problems such as bending, buckling and vibration.

1.4 Outline of the Thesis:

Since the problems studied in the thesis are related to the static and dynamic analyses of plates, existing literature relating to the topics are briefly reviewed in Chapter 2. In Chapter 3, the interior collocation method is augmented by the least square concept as solutions to boundary and eigenvalue problems. Chapter 4 presents the differential equations governing the bending, buckling and vibration of plates. Applications of the

modified method, termed the collocation least square method, are illustrated in Chapters 5, 6 and 7. Chapter 5 is devoted to the analysis of uniformly loaded clamped plates of rectangular, elliptical and rhombic planforms resting on elastic foundations. In Chapter 6, the collocation least square method is applied to the vibration of clamped plates of isotropic and orthotropic materials. Further applications of the proposed method to some other eigen value problems, the buckling and vibration of clamped rectangular plates subject to inplane forces, are illustrated in Chapter 7.

In the final Chapter, the conclusions are drawn and summarized.

Numerical and graphical results of all the analyses are presented. Whenever possible, such results are compared with solutions obtained by other investigators.

CHAPTER II

REVIEW OF LITERATURE

The small deflection theory of plates is generally attributed to Navier, Kirchhoff and Love /38/. An attempt is made here to review some of the previous research works which employed numerical schemes described in Chapter I as methods of solution.

a) Ritz Method: This method was used by Pikett /34/ for the bending problem of uniformly loaded clamped rectangular plates. Maulbetsch /24/ and Timoshenko /39/ applied the technique to the buckling problems of clamped rectangular plates under compressive forces. The Ritz solution to the vibration problems was obtained by Young /45/ for square plate with combinations of free and clamped edges, by Sun /37/ for clamped plates of various shapes, and by Hearmon /16/ for orthotropic rectangular plates with simply supported and clamped boundaries.

b) Fourier Series Method: For rectangular plates with simply supported edges, the Fourier series method proves to be very powerful. In 1820, Navier presented to the French Academy of Sciences on the solution of bending of simply supported rectangular plates by double Fourier series.. The Navier solution to the free vibration of simply supported rectangular plates was obtained by Timoshenko /41/. In 1899, Levy /22/ introduced single Fourier series solution to the bending of rectangular plates having opposite edges simply supported and various boundary conditions along the remaining edges. Fletcher et al. /14/ applied this technique to the

corresponding vibration problems. Extensions of Levy method by means of the superposition technique were employed by Levy /23/ for the buckling of clamped rectangular plates under uniaxial compression, and by Claassen and Thorne /7/ for the vibration of rectangular plates having combinations of free and clamped edges.

c) Perturbation Method and Series Solution: This method has been employed to study the bending problems of a variety of uniformly loaded plates by Chan /5/ and Ng /29,30,31/. Bauer and Reiss /3/ applied the method to the vibration of clamped orthotropic rectangular plates, and Bassily and Dickinson /2/ solved the vibration of clamped rectangular plates subject to various inplane loads.

d) Finite Difference Method: This method has been used by Barton /1/ and Jensen /17/ for the bending of uniformly loaded rectangular and skewed plates. Salvadori /36/, using this method, studied the buckling of various polygonal plates under uniform compression. Nishimura /32/, among others, applied the finite difference technique to the vibration of rectangular plates. Szilard /38/ solved a variety of plates problems using this method. He also has an extensive discussion of the method as applied to the static and dynamic analyses of plates.

e) Finite Element Method: In the last ten years, due to improvements in computing facilities, this method has been widely used to plate problems. A variety of element stiffness matrices have been summarized by Clough and Tocher /8/. Kapur and Hartz /18/, among others, used this method to study the buckling of rectangular plates with clamped and simply supported boundaries. In the application of the finite element method to

vibration problems, Dawe /11/ solved the case of rectangular plates having various edge conditions, and Monforton /26/ clamped skewed plates.

f) Error Distribution Method: The most widely used error distribution methods are those of Galerkin and collocation. Walter /42/ and Chan /6/ treated the large deflection problem of a variety of plates by means of the collocation method. The Galerkin method was applied to the small deflection of clamped plates of various planforms on elastic foundations by Ng /30/, to the free vibration of rectangular plates by Odman /31/ and to the vibration of clamped skewed plates by Durvasula /12/.

CHAPTER III

THE COLLOCATION LEAST SQUARE METHOD3.1 General:

Of all the numerical methods discussed in Chapter I, the easiest but not the most elegant method is the collocation method. As an error distribution method, it has the advantage of dealing directly with the governing differential equation rather than an equivalent variational problem. Apart from this, the collocation method only involves the evaluation of function rather than definite integration associated with the other error distribution methods such as the method of Galerkin.

Collocation was first systematically discussed in a report by Frazer et.al./14/ in 1937. The literal definition of the word "collocation" is the act of setting in place or position, which is the fundamental of the method so named. There are three different types of collocation, viz., interior collocation, boundary collocation and mixed collocation. In this thesis, only the interior collocation will be discussed in detail.

3.2 The Collocation Method:

To illustrate the method, consider the problem of determining a function $W(x, y)$ which satisfies a partial differential equation:

$$L^V(x, y, W, W_x, W_y, \dots) = f \quad (3.2.1)$$

and which satisfies the prescribed boundary condition:

$$L^S(x, y, W, W_x, W_y, \dots) = 0 \quad (3.2.2)$$

where L is a differential operator,

V is the region where the differential equation is defined,

S is the boundary adjoining the region V and

f is a prescribed function known throughout V .

For an interior method, an approximate solution of Equation (3.2.1) can be assumed in the form:

$$W = \bar{W}(x, y, a_1, a_2, \dots, a_n) \quad (3.2.3)$$

where W represents an arbitrary linear combination of a set of independent functions, each one of which satisfies Equation (3.2.2), and a_1, \dots, a_n are undetermined adjustable parameters.

Substitution of Equation (3.2.3) into Equation (3.2.1) defines an error (or residual) function E of the form:

$$E(x, y, a_1, \dots, a_n) = L^V(x, y, W, W_x, W_y, \dots) - f \quad (3.2.4)$$

The parameters a_1, \dots, a_n in the assumed solution are then determined by setting the error E to zero at n prior chosen points in the region V . This is equivalent to forcing the differential equation to be satisfied exactly at these n points. Such a procedure will lead to n linear equations for determining the n unknown parameters a_1, \dots, a_n . i.e.,

$$E_i(x_i, y_i, a_1, \dots, a_n) = 0 \quad (i = 1, \dots, n) \quad (3.2.5)$$

In practice, only a limited number of undetermined parameters can be taken in the assumed solution. Hence, the error can only be set to zero at a limited number of points, and the magnitude of the error at any other points besides the n chosen points remains unknown. Hopefully, it is small. Consequently, the approximate solution of any given problem depends, to a great extent, upon the choice of collocation points. Crandall /11/ states that the choice of the points is arbitrary, but is usually such that the region V is covered more or less uniformly in a simple pattern. Collatz /9/ states that the choice of collocation points is a matter of uncertainty, and the effect of the distribution of collocation points on the results is unknown. For a limited number (six to nine) of undetermined parameters, depending on the type of boundary value problem considered, the results can differ by as much as 100% for arbitrary choices of collocation points.

3.3 The Collocation Least Square Method in Boundary Value Problems

From the discussion above, it seems logical that if the error function E is forced to be zero at m points instead of n points, where $m \gg n$, and the undetermined parameters are evaluated in such a manner that E be zero or as close to zero as possible at these m points, the results obtained would certainly be improved, and such results will be somewhat less dependent on the choice of collocation points.

However, by setting E to zero at m points, an over-determined system of linear simultaneous equations would result. These equations can be expressed in matrix notations as:

$$[C] \{a\} = \{b\} \quad (3.3.1)$$

where $[C]$ is the $m \times n$ coefficient matrix of the system of equations,

$\{a\}$ is the $n \times 1$ column vector of the undetermined parameters

$$a_1, \dots, a_n,$$

and $\{b\}$ is the $m \times 1$ right hand side column vector.

Having generated m equations in n unknowns, the n unknowns, viz., a_1, \dots, a_n are then solved in a manner analogous to the fitting of a curve through a given set of data points. To effect this, the least square procedure, which is often used in statistics to produce a so-called "best fitting curve", is applied to the equations.

Consider Equation (3.3.1). For any particular column vector $\{a\}$, it is very unlikely that Equation (3.3.1) will be satisfied identically. Let the errors associated with the equations be expressed by the $m \times 1$ column vector $\{e\}$. i.e.,

$$\{e\} = [C] \{a\} - \{b\} \quad (3.3.2)$$

Expanding the above matrix equation, we have:

$$\begin{aligned} e_1 &= c_{11}a_1 + c_{12}a_2 + \dots + c_{1n}a_n - b_1 \\ e_2 &= c_{21}a_1 + c_{22}a_2 + \dots + c_{2n}a_n - b_2 \\ &\vdots \\ &\vdots \\ e_m &= c_{m1}a_1 + c_{m2}a_2 + \dots + c_{mn}a_n - b_m \end{aligned} \quad (3.3.3)$$

According to the least squares method, the criterion for choosing the undetermined parameters a_1, \dots, a_n is such that the sum of the squares of the errors, i.e.,

$$S = e_1^2 + e_2^2 + \dots + e_m^2 \quad (3.3.4)$$

be a minimum. Adopting the notation $\langle \rangle$ as a symbol of summation so that, e.g., $\langle c_{i1} c_{i1} \rangle = c_{11}^2 + c_{21}^2 \dots c_{m1}^2$, $\langle c_{i1} c_{i2} \rangle = c_{11} c_{12} + c_{21} c_{22} \dots c_{m1} c_{m2}$, the sum S of the squares of the errors is then:

$$\begin{aligned}
 S = & \langle c_{i1} c_{i1} \rangle a_1^2 + \langle c_{i2} c_{i2} \rangle a_2^2 + \langle c_{i3} c_{i3} \rangle a_3^2 \dots + \langle c_{in} c_{in} \rangle a_n^2 \\
 & + 2 \langle c_{i1} c_{i2} \rangle a_1 a_2 + 2 \langle c_{i1} c_{i3} \rangle a_1 a_3 \dots + 2 \langle c_{i(n-1)} c_{in} \rangle a_{n-1} a_n \\
 & - 2 \langle c_{i1} b_i \rangle a_1 - 2 \langle c_{i2} b_i \rangle a_2 - \dots - 2 \langle c_{in} b_i \rangle a_n + \langle b_i b_i \rangle
 \end{aligned}
 \tag{3.3.5}$$

In order that S is a minimum, its derivatives with respect to a_1, a_2, \dots, a_n must vanish. i.e.,

$$\begin{aligned}
 \langle c_{i1} c_{i1} \rangle a_1 + \langle c_{i1} c_{i2} \rangle a_2 + \langle c_{i1} c_{i3} \rangle a_3 + \dots + \langle c_{i1} c_{in} \rangle a_n &= \langle c_{i1} b_i \rangle \\
 \langle c_{i2} c_{i1} \rangle a_1 + \langle c_{i2} c_{i2} \rangle a_2 + \langle c_{i2} c_{i3} \rangle a_3 + \dots + \langle c_{i2} c_{in} \rangle a_n &= \langle c_{i2} b_i \rangle \\
 \vdots & \vdots \\
 \langle c_{in} c_{i1} \rangle a_1 + \dots + \langle c_{in} c_{in} \rangle a_n &= \langle c_{in} b_i \rangle
 \end{aligned}
 \tag{3.3.6}$$

It can be seen that the above equations are equivalent to the matrix equation:

$$[C]^T [C] \{a\} = [C]^T \{b\}
 \tag{3.3.7}$$

Thus, m equations in n unknowns may be reduced, in the least square sense, to n equations in n unknowns by premultiplying the over-determined system, Equation (3.3.1), by the transpose of its coefficient

matrix. This operation, which can be easily performed on a digital computer, assures that the "best fit" solution be obtained for the differential equation of any given problem.

3.4 The Collocation Least Square Method in Eigen Value Problems

For an eigenvalue problem, the differential equation (3.2.1) is homogeneous. i.e.,

$$L^V(x, y, W, W_x, W_y, \dots, \lambda) = 0 \tag{3.4.1}$$

where λ is the eigenvalue..

Consequently, Equation (3.3.1) is also homogeneous, and the sum S of the squares of the errors, Equation (3.3.4) becomes:

$$S = \{e\}^T \{e\} = \{a\}^T [C]^T [C] \{a\} \tag{3.4.2}$$

The coefficient matrix [C] now has elements dependent on λ and; for a linear eigenvalue problem, can be expressed as:

$$[C] = [K] - \lambda [M] \tag{3.4.3}$$

where [K] and [M] are the mxn matrices, and are often termed the stiffness and the mass matrices, respectively.

Letting [P] = [K]^T [K], [R] = [M]^T [M] and [Q] = [K]^T M + M^T K; Equation (3.4.2) can be rewritten as:

$$S = \{a\}^T [P - \lambda Q + \lambda^2 R] \{a\} \tag{3.4.4}$$

There are two alternative least square procedures to minimize the sum S of the errors.

a) First Approach: The least square scheme previously applied to boundary value problems can again be employed here. Differentiating the sum S of the squares of the errors, Equation (3.4.4) with respect to the parameters a_1, a_2, \dots, a_n would lead to the homogeneous form of Equation (3.3.7), which can be rewritten as:

$$S, \{a\}^T = [P - \lambda Q + \lambda^2 R] \{a\} = 0 \quad (3.4.5)$$

Equation (3.4.5) can be transform to a linear eigenvalue problem of the standard form:

$$[D] \{V\} = \lambda \{V\} \quad (3.4.6)$$

by letting

$$[D] = \left[\begin{array}{c|c} R^{-1} Q & R^{-1} P \\ \hline 0 & I \end{array} \right], \quad \{V\} = \begin{Bmatrix} \lambda a \\ a \end{Bmatrix}$$

$[0]$ and $[I]$ are respectively the zero and unity matrices of order $n \times n$.

The eigenvalue λ can be determined from the characteristic equation:

$$|D - \lambda I| = 0 \quad (3.4.7)$$

b) Second Approach: The alternative procedure begins with imposing the normalization condition on the eigenvector $\{a\}$, which corresponds to choosing the value of one parameter. e.g., $a_1 = 1$. Then, the error sum S is minimized with respect to the remaining n variables. i.e.,

$$(\{a\}^j, \lambda^j) = (a_2^j, a_3^j, \dots, a_n^j, \lambda^j) \quad (3.4.8)$$

where λ^j represents the j th eigenvalue associated with the j th eigenfunction,

which is approximated by $W(x, y, 1, a_2^j, a_3^j, \dots, a_n^j)$.

In this approach, the sum S of the squares of the errors is treated as a nonlinear function of the eigenvalue λ , having multiple local minima each of which corresponds to a distinct eigenvalue λ^j .

Given $a_1 = 1$ and a value for λ^j , the corresponding vector $\{a\}$ can be calculated from Equation (3.4.5), and S evaluated from Equation (3.4.4). By plotting S vs. λ^j , the relative minima can be located.

Refined value of λ^j can be calculated using the remaining condition:

$$S_{,\lambda^j} = 0 \quad (3.4.9)$$

Substituting Equation (3.4.4) into Equation (3.4.9) leads to:

$$2 \lambda^j = \{a\}^T [Q] \{a\} / \{a\}^T [R] \{a\} \quad (3.4.10)$$

The new value of λ^j is then used to calculate an improved $\{a\}^j$, and the process can be repeated to any desired accuracy.

In the following Chapters, the success of the collocation least square method will be illustrated by applying it to some boundary and eigenvalue problems in applied mechanics.

CHAPTER IV

DIFFERENTIAL EQUATIONS OF PLATES

The differential equations governing the behaviors of thin elastic plates undergoing small deflection, are well known /38,39/. For ease of computations, equations for the bending, buckling, and vibration of plates will be presented here in nondimensional forms. The basic assumptions governing the validity of these equations are first briefly stated .

4.1 Basic Assumptions for the Small Deflection Plate Theory:

- a) The slope of the the middle surface of the plate is small compared to unity..
- b) The deformations are such that straight lines, initially normal to the middle surface of the plate, remain straight lines and normal to the middle surface.
- c) The stresses normal to the middle surface of the plate produced by transversely applied loadings are of negligible magnitude in comparison with stresses in the plane of the plate.
- d) .The stresses in the middle surface arising from the deflection of the plate can be neglected; i.e., the middle surface can be regarded as a neutral plane.

4.2 Bending of Plates on Elastic Foundations

Consider a thin elastic plate of an arbitrary planform, let the plate rest on a Winkler type elastic foundation and possess

rectilinear orthotropy. Adopting a rectangular Cartesian coordinate system with the origin located at some arbitrary point in the middle plane of the plate, let the axes of principal stiffness coincide with the x and y-directions. Applying an arbitrary distributed load $q(x,y)$ acting normal to the plane of the plate will cause a displacement in the z-direction which is denoted by w . The differential equation governing the out-of-plane displacement w can be expressed as /38/:

$$D_x w_{,xxxx} + 2H w_{,xxyy} + D_y w_{,yyyy} + k w = q \quad (4.2.1)$$

where the comma notation signifies partial differentiation, k is the foundation modulus and

$$D_x = E_x h^3 / 12(1 - \nu_x \nu_y)$$

$$D_y = E_y h^3 / 12(1 - \nu_x \nu_y)$$

$$H = D' + 2D_{xy}$$

$$D' = \nu_x D_y = \nu_y D_x$$

$$D_{xy} = G_{xy} h^3 / 12$$

E_x = modulus of elasticity in the x-direction,

E_y = modulus of elasticity in the y-direction,

G_{xy} = shear modulus in the xy-plane,

ν_x = ratio of strain in the y-direction to strain in the x-direction due to uniaxial stress in the x-direction,

ν_y = ratio of strain in the x-direction to strain in the y-direction due to uniaxial stress in the y-direction.

Internal forces of the plate are related to the transverse displacement w by the following expressions:

a) Stresses:

$$\begin{aligned} S_x &= -z E'_x (w_{,xx} + \nu_y w_{,yy}) \\ S_y &= -z E'_y (w_{,yy} + \nu_x w_{,xx}) \\ S_{xy} &= -2z G_{xy} w_{,xy} \end{aligned} \quad (4.2.2)$$

where

S_x = normal stress in the x-direction

S_y = normal stress in the y-direction

S_{xy} = shear stress in the xy-plane

$$E'_x = E_x / (1 - \nu_x \nu_y)$$

$$E'_y = E_y / (1 - \nu_x \nu_y)$$

b) Moments:

$$\begin{aligned} M_x &= -D_x (w_{,xx} + \nu_y w_{,yy}) \\ M_y &= -D_y (w_{,yy} + \nu_x w_{,xx}) \\ M_{xy} &= -2D_{xy} w_{,xy} \end{aligned} \quad (4.2.3)$$

where

M_x = bending moment produced by S_x

M_y = bending moment produced by S_y

M_{xy} = twisting moment produced by S_{xy}

4.3 Equation of Stability of Plates

The differential equation governing the buckling of elastic plates subject to forces acting in the plane of the plate can be expressed as /38/:

$$\begin{aligned} D_x w_{,xxxx} + 2H w_{,xxyy} + D_y w_{,yyyy} \\ = N_x w_{,xx} + 2N_{xy} w_{,xy} + N_y w_{,yy} \end{aligned} \quad (4.3)$$

where N_x = normal force in the x-direction, per unit length;
 N_y = normal force in the y-direction, per unit length;
 and N_{xy} = shear force in the xy-plane, per unit length.

4.4 Equation of Vibration of Plates

The differential equation governing the free vibration of plates can be expressed in the same rectangular Cartesian coordinate system as /38/:

$$D_x w'_{,xxxx} + 2H w'_{,xxyy} + D_y w'_{,yyyy} - m w'_{,tt} = 0 \quad (4.4.1)$$

where m = mass per unit area of the plate,
 t = time variable,
 and $w' = w'(x,y,t)$, the time dependent displacement function.

For free vibration, the motion of the plate is assumed to be harmonic; i.e.,

$$w' = w(x,y) \sin(\omega t) \quad (4.4.2)$$

where ω is the circular frequency of the motion.

Substitution of Equation (4.4.2) into Equation (4.4.1)

yields:

$$D_x w_{,xxxx} + 2H w_{,xxyy} + D_y w_{,yyyy} - (m \omega^2) w = 0 \quad (4.4.3)$$

4.5 Non-Dimensional Forms of the Differential Equations of Plates

For ease of computation, it is convenient to render the differential equations dimensionless. This is done by considering "h" as the plate thickness; "a" and "b" as two characteristic lengths of the plate; and letting

$$\begin{aligned} x' &= x/a & W &= w/h & N'_x &= N_x a^2/D_y \\ y' &= y/b & K &= ka^4/D_y & N'_y &= N_y a^2/D_y \\ R &= a/b & Q &= qa^4/D_y h & N'_{xy} &= N_{xy} a^2/D_y \end{aligned}$$

and $F = \omega^2 a^4 (m/D_y)$

Substitution of these above dimensionless ratios into Equations (4.2.1), (4.3) and (4.4.3) leads to the non-dimensional forms of the equations. Without causing any ambiguity, the primes in the dimensionless quantities x' , y' , N'_x , N'_y and N'_{xy} are omitted. Thus, for bending,

$$\begin{aligned} (D_x/D_y) W_{,xxxx} + (2R^2 H/D_y) W_{,xxyy} + R^4 W_{,yyyy} \\ + K.W = Q \end{aligned} \quad (4.5.1)$$

for buckling,

$$\begin{aligned}
 & (D_x/D_y) W_{,xxxx} + (2R^2H/D_y) W_{,xxyy} + R^4 W_{,yyyy} \\
 & = N_x W_{,xx} + (2R N_{xy}) W_{,xy} + (R^2 N_y) W_{,yy} \quad (4.5.2)
 \end{aligned}$$

and for vibration,

$$\begin{aligned}
 & (D_x/D_y) W_{,xxxx} + (2R^2H/D_y) W_{,xxyy} + R^4 W_{,yyyy} \\
 & - F.W = 0 \quad (4.5.3)
 \end{aligned}$$

For homogeneous isotropic plates, $E_x = E_y = E$, $\nu_x = \nu_y$ and $G_{xy} = E/2(1+\nu)$; consequently, $D_x = D_y = H = D = Eh^3/12(1-\nu^2)$. Hence, Equations (4.5.1) to (4.5.3) can be rewritten as:

$$W_{,xxxx} + 2R^2 W_{,xxyy} + R^4 W_{,yyyy} + K.W = Q \quad (4.5.4)$$

$$\begin{aligned}
 & W_{,xxxx} + 2R^2 W_{,xxyy} + R^4 W_{,yyyy} \\
 & = N_x W_{,xx} + (2R N_{xy}) W_{,xy} + (R^2 N_y) W_{,yy} \quad (4.5.5)
 \end{aligned}$$

$$W_{,xxxx} + 2R^2 W_{,xxyy} + R^4 W_{,yyyy} - F.W = 0 \quad (4.5.6)$$

Similar changes of the material constants can be made in the equations relating stresses and moments to displacements; i.e., Equations (4.2.2) and (4.3.3).

CHAPTER V

BENDING OF PLATES

In this chapter, the collocation least square scheme formulated in section 3.3 is applied to the analysis of uniformly loaded isotropic plates resting on elastic foundations. The governing differential equation (4.5.1) is to be solved for rectangular, elliptical and rhombic plates with clamped edges.

5.1 Rectangular Plates on Elastic Foundations

The coordinate system for the rectangular geometry is shown in Figure 1. As can be seen in the figure, the plate possesses mutually perpendicular axes of symmetry, resulting in quadrant symmetry in the out-of-plane displacement pattern. In view of this and the boundary conditions, a suitable approximation can be chosen in the form /29/ :

$$W = (1-x^2)^2 (1-y^2)^2 f_1(x,y) \quad (5.1.1)$$

where the function f_1 is defined by:

$$f_1(x,y) = A_{00} + A_{20}x^2 + A_{02}y^2 + A_{40}x^4 + A_{22}x^2y^2 + A_{04}y^4 \\ + A_{42}x^4y^2 + A_{24}x^2y^4 + A_{44}x^4y^4$$

and A_{ij} are the undetermined coefficients. The associated boundary conditions for the problem are:

$$W,_{x} = W = 0 \quad \text{at} \quad x = \pm 1 \quad (5.1.2)$$

$$W,_{y} = W = 0 \quad \text{at} \quad y = \pm 1 \quad (5.1.3)$$

It can be easily verified that Equation (5.1.1) satisfies the above boundary conditions.

Solutions are obtained by holding the half-breadth "a" constant, while varying the distance "b". The aspect ratio $R = a/b$ has values between $\frac{1}{2}$ and 1. The dimensionless foundation modulus K is increased from 0 to 200.

For the case of zero foundation modulus, Equation (4.5.1) is solved using 25, 50 and 100 collocation points. The distribution of these collocation points is analogous to the pattern shown in Figure 2.a.

Table 1 shows the results obtained by this investigation. Comparisons are made with those results obtained by Timoshenko /39/, where much more laborious computations are used to analyse the same problem. As seen in the table, the present solutions are in excellent agreement with the accurate values reported by reference /39/.

The effect on results due to the number of collocation points used is very minor. From Table 1, it is seen that solutions obtained by using 25 and 50 collocation points deviated less than 0.2% from those obtained by using 100 collocation points. Consequently, the problem of rectangular plates on elastic foundations is solved using 100 collocation points. These collocation points are distributed as shown in Figure 2.a.

The results of the analysis for plates on elastic foundations are tabulated in Table 2. Plots of maximum deflection and moments vs. aspect ratio for various foundation moduli are shown in Figures 3 and 4. Results obtained by Ng /29/ and Timoshenko /39/ are also plotted in Figure 3 for comparison. From the figure, the present results are slightly above those obtained by Ng /39/, as also noted in the case of zero

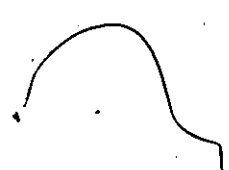
foundation modulus. This slight over-estimation of the plate stiffness may be due to the limited number of undetermined coefficients in the polynomial displacement function used by Ng /29/.

From this investigation the following results were observed:

1) In analyzing the bending problem of rectangular plates the collocation least square method provides results which are comparable to those obtained by lengthy computational methods. Though the investigation is carried out by using 100 collocation points, it seems that the number of collocation points required to yield sufficiently accurate answer is about two to three times the number of unknown parameters, provided that these collocation points are distributed in a fairly uniform manner.

2) For a given aspect ratio R , the maximum center deflection of the plate decreases with increasing values of the foundation modulus. This should be expected since the object of the elastic foundation is to reduce the lateral pressure.

3) The effectiveness of the elastic foundation in reducing the maximum center deflection of the plate is more pronounced for small aspect ratios than it is for aspect ratios approaching unity. For instance, with the foundation modulus increasing from 0 to 200, the decrease in the maximum deflection for a plate of aspect ratio of $\frac{1}{2}$ is 86.6%; while for a plate of aspect ratio of 1, the corresponding decrease is only 73.8%. This is so because the deflection of a small aspect ratio is greater than that of a plate of aspect ratio approaching one, and since the foundation reaction is proportional to the deflection; hence, the



reduction in deflections due to an increase in the dimensionless foundation modulus K ; will be more significant for long rectangular plates than for plates approaching square planform.

4) The maximum bending moment occurring at the mid-point of the longer side of the plate is much greater than the maximum bending moment occurring at the center of the plate.

5) Due to the presence of the elastic foundation, the moments of the plate are reduced. This reduction is more pronounced at the center of the plate than at the edge. For example, for a square plate, an increase of the foundation modulus K from 0 to 200 causes a decrease of 62.6% in the maximum edge moment; whereas, the corresponding decrease in the maximum center moment is 82.3%.

5.2 Elliptical and Circular Plates on Elastic Foundations

For the clamped, isotropic elliptical plate with the coordinate system as shown in Figure 1, the governing differential equation for the displacement is identical to that of the rectangular plate, viz., Equation (4.5.1), and the associated boundary conditions are:

$$W = W_{,x} = W_{,y} = 0 \quad \text{at} \quad x^2 + y^2 = 1 \quad (5.2.1)$$

In order to satisfy these boundary conditions, the solution is taken in the form (31):

$$W = (1 - x^2 - y^2)^2 f_1(x, y) \quad (5.2.2)$$

where the function f_1 is as defined in the previous problem.

For comparison of results, solutions are obtained for plates with aspect ratios between 1 and 2. For $R=1$, the elliptical plate becomes a circular plate. The value of the dimensionless foundation modulus K is

varied from 0 to 200.

The effect of the number of collocation points on the solution is again investigated by solving the case of $K = 0$ with 25, 50 and 100 collocation points. The exact solution /39/ was obtained regardless of the collocation points and the number of undetermined parameters used. In all cases, the polynomial f_1 retained only the first coefficient A_{00} , and the remaining undetermined coefficients turned out to be zero. For the analysis of plates on elastic foundations, all results are obtained by using 100 collocation points. The distribution of these collocation points is shown as in Figure 2.b.

The results of the analysis of elliptical plates on elastic foundations are shown in Table 3. Comparison of these results are made with results obtained by Ng /31/. The agreement is excellent with the maximum error not exceeding 1.5%.

From the comparisons of results for rectangular and elliptical plates, it is observed that the agreement is slightly better in the case of elliptical plates than it is with rectangular plates. This is to be expected since, unlike the assumed solution for rectangular plates, the assumed solution for circular and elliptical plates takes on the exact mathematical expression of a circular or elliptical boundary.

Plots of the maximum centre deflection vs. aspect ratio and for various foundation moduli are shown in Figure 5. From the results for moments for elliptical plates, it is observed that the maximum moment occurs at the end of the minor axis. As shown in Figure 6, this edge moment is of greater magnitude than the positive moment at the centre of the plate.

The effect of the elastic support in reducing the deflections and moments is seen to be more significant for plates of aspect ratios approaching one than it is for plates of greater aspect ratios. For example, for an aspect ratio of one, by increasing K from 0 to 200, the decrease in the central deflection is 68.1%; however, the corresponding decrease is only 22.5% for plates of aspect ratio of two. This finding seems to be contradictory to the results of rectangular plates. However, recalling the dimensionless form adopted for the foundation modulus, viz., $K = ka^4/D$, and the variation of the aspect ratio of this problem, viz., $a/b = 1$ to $a/b = 2$, it can be seen that by increasing the aspect ratio, viz., holding "b" constant and increasing "a", the actual foundation modulus K is increased by a factor of a^4 . Consequently, for a certain value of K , say $K = 80$, taking the semi-minor axis "b" as unity, when $a/b = 1$, k has a value of $40D$, whereas when $a/b = 2$, k becomes $5D$. Hence, it can be observed that for a given change in the plate aspect ratio, the increase in deflection as the plate approaches an infinite strip, is not enough to offset the decrease of the actual foundation modulus k . Apart from this, all the other findings in this problem are identical to those of rectangular plates.

5.3 Rhombic Plates of Elastic Foundations:

For the clamped, isotropic rhombic plate with co-ordinate system as shown in Figure 1, the boundary conditions for the transverse displacement are:

$$W = W_{,x} = W_{,y} = 0 \quad \text{along the boundary of } |x| + |y| = 1 \quad (5.3.1)$$

An approximate solution satisfying the above boundary conditions can be taken in the form:

$$W = (1+x+y)^2(1+x-y)^2(1-x+y)^2(1-x-y)^2 f_1(x,y) \dots\dots\dots (5.3.2)$$

where the function f_1 is as defined in Equation (5.1.1).

When analysing the rhombic geometry, it is often convenient to introduce an angle α to measure the skew of the plate, rather than to continue using the aspect ratio R . From Figure 1, it can be seen that:

$$\alpha = \tan^{-1}(R) = \tan^{-1}(a/b)$$

Compared to the rectangular and elliptical plates, the stress analysis of the rhombic plates is complicated by the following conditions:

a) The x and y -axes are no longer the principal stress directions. The principal stresses can be calculated from the known formula:

$$S_{\max, \min} = \frac{1}{2}(S_x + S_y) \pm \left[\frac{1}{4}(S_x - S_y)^2 + S_{xy}^2 \right]^{1/2} \quad (5.3.3)$$

b) The coordinates locating the points of critical bending stress at the edge of the plate, (x^*, y^*) , vary with the skew angle.

Equation (4.5.1) is solved for values of the dimensionless foundation modulus K ranging from 0 to 200, using 100 collocation points. The distribution of these collocation points is as shown in Figure 2.c.

Before presenting the results of the analysis of plates on elastic foundations, the values of central deflection in the case of

zero foundation modulus are compared with the values obtained by Morley /28/ :

α	Values of W_{\max} in $(qa^4/D)(10)^2$				
	45°	40°	37.5°	35°	30°
Morley /28/.....	_____	0.703	0.818	0.943	1.230
Present Analysis...	0.503	0.706	0.833	0.987	1.382

From the above table, the percentage differences in the values of W_{\max} are larger for smaller values of the skew angle α ; however, the errors are conservative. i.e., the central deflection is overestimated as the aspect ratio decreases.

Results of the analysis of rhombic plates on elastic foundations are tabulated in Table 4. The accuracy of the results can be verified for the case of square geometry ($\alpha = 45^\circ$) since a comparison can be made with the solution previously obtained for the plate whose boundary is defined by the lines $x = \pm 1$ and $y = \pm 1$. Several results from Table 2 are also shown in Table 4. The agreement is good for this case.

Based on the preceding investigations of rectangular and elliptical plates, and the comparison with the solution of Morley /28/ ; it is felt that if the results obtained for α less than 45° are in error at all, this error is on the safe side.

Plots of central deflection and maximum bending moments vs. skew angle for various foundation moduli are shown in Figures 7 and 8, respectively. Also shown in Figure 8 are values of the dimensionless coordinates (x^*, y^*) which locate the point of maximum edge moment. This point was determined by comparing the moment magnitudes at various locations

along the boundary of the plate. As α decreases, the point (x^*, y^*) is displaced toward the obtuse corner of the rhombus. However, the location of this point is almost insensitive to any increase of the dimensionless foundation modulus K .

From Figure 7 and 8, it is again seen that, under the presence of elastic support, the maximum deflection and moments are reduced. However, this effect is much less significant for the rhombic plates than it is for the rectangular and elliptical plates. For instance, for an aspect ratio of 0.58 ($\alpha = 30^\circ$), by increasing K from 0 to 200, the decrease in the central deflection of the rectangular plate is 85.4%; whereas, the corresponding decrease of the rhombic plate is only 64.2%. This is due to the fact that the rhombus is much stiffer than the rectangle, and since the foundation is proportional to the deflection, hence the reduction in the deflection of the rhombic plate is smaller.

From the various problems demonstrated in this Chapter, the collocation least square method, though simple in its mathematical concept, proves to be an extremely valuable tool for the solution of boundary value problems. Such problems as deflection of plates on elastic foundations were handled with great ease. The results of the problems considered in this Chapter were obtained with acceptable accuracy, and if the results are in error at all, the error are generally on the conservative side.

CHAPTER VI

VIBRATION OF PLATES

In this Chapter, the problems of vibration of plates are investigated using the collocation least square schemes formulated in Section 3.4. As a method of solution to eigenvalue problems, the collocation least square method possesses two alternative least square procedures. The first approach of the method is to be applied to the vibration of isotropic plates of the three planforms previously considered, and the second approach to the vibration of orthotropic rectangular plates.

6.1 Vibration of Rectangular Plates

For a clamped isotropic rectangular plate with the coordinate system shown in Figure 1, an approximate solution to Equation (4.5.6), satisfying the boundary conditions (5.1.2) and (5.1.3), can be taken in the form:

$$W = (1-x^2)^2(1-y^2)^2 f_k(x,y) \quad (6.1.1)$$

where f_k is a polynomial function which describes the mode shape of the plate.

(A) For vibration modes which are symmetric about both the x and y-axes,

$$\begin{aligned} f_1(x,y) = & A_{00} + A_{20}x^2 + A_{02}y^2 + A_{40}x^4 + A_{22}x^2y^2 + A_{04}y^4 \\ & + A_{42}x^4y^2 + A_{24}x^2y^4 + A_{44}x^4y^4 \\ & \dots \dots \dots \end{aligned} \quad (6.1.2)$$

The above expression for f_k has been employed in previous problems to approximate the deflection of plates under uniform loads.

- (B) For vibration modes which are symmetric about the x-axis and antisymmetric about the y-axis,

$$f_2(x,y) = A_{10}x + A_{30}x^3 + A_{12}xy^2 + A_{50}x^5y + A_{32}x^3y^2 + A_{14}xy^4 \\ + A_{52}x^5y^2 + A_{34}x^3y^4 + A_{54}x^5y^4 \\ \dots\dots\dots (6.1.3)$$

- (C) For vibration modes which are antisymmetric about the x-axis and symmetric about the y-axis,

$$f_3(x,y) = A_{01}y + A_{21}x^2y + A_{03}y^3 + A_{41}x^4y + A_{23}x^2y^3 + A_{05}y^5 \\ + A_{43}x^4y^3 + A_{25}x^2y^5 + A_{45}x^4y^5 \\ \dots\dots\dots (6.1.4)$$

Expressions (B) and (C) are equivalent when the aspect ratio $a/b = 1$.

- (D) For vibration modes which are antisymmetric about both the x and y-axes,

$$f_4(x,y) = A_{11}xy + A_{31}x^3y + A_{13}xy^3 + A_{51}x^5y + A_{33}x^3y^3 + A_{15}xy^5 \\ + A_{53}x^5y^3 + A_{35}x^3y^5 + A_{55}x^5y^5 \\ \dots\dots\dots (6.1.5)$$

Solutions are obtained for values of the aspect ratio $R = a/b$ ranging from 1 to 2. All results are calculated using 100 collocation points. The distribution of these collocation points is shown in Figure 2.a.

Results of the analysis of rectangular plates for 16 modes of vibration are tabulated in Table 5. The modes are numbered in ascending order of frequencies, and the symmetry group to which they belong is also indicated. Comparisons of results are made with the results reported by Odman /33/ and Young /45/. As can be seen in Table 5, the values obtained by the present analysis are in excellent agreement with those values obtained by the other investigators. From the comparisons made, all errors are within 1%.

Plots of frequencies vs. aspect ratio are shown in Figure 9. Typical nodal patterns are also shown in the figure as miniatures. These nodal patterns were obtained by evaluating for each mode the deflection at 100 points in one-quarter of the plate.

In Figure 9, the modes labeled 2 and 3 are degenerate modes; i.e., they have the same frequency $f = 18.362$ in the case of square plate. As the aspect ratio R deviates from one, this degeneracy disappears, and they become modes with distinct frequencies. Other pairs of degenerate modes are modes (7,8), (9,10), and (14,15).

Modes 5 and 6 are not degenerate modes when $R = 1$, but they have frequencies which are very close; i.e., 32.841 and 33.020. As the plate becomes square, the nodal patterns of these modes no longer consist of lines parallel to the sides of the plate; mode 5 now has nodal lines coinciding with the diagonals of the square, and mode 6 has a nodal circle. This change-over in nodal pattern which is characteristic of rectangular plates as the aspect ratio R approaches one, occurs in modes having the numbers of nodal lines in the x and y -directions which are unequal and both are either even or odd /43/. The nodal patterns of these modes for

the square plate can be obtained by adding or subtracting the relevant nodal patterns for rectangular plates. As expected, modes 12 and 13 also assume distinct nodal patterns at $R=1$.

Frequency crossing is also an important feature observed in Figure 9. It appears that, for a given mode of vibration, the greater number of nodal lines in the direction of the long side of the plate the mode possesses the more rapidly its frequency increases as the aspect ratio R increases. Hence, higher modes have greater tendency for frequency crossing. For example, mode 3 crosses modes 5 and 9 at values of R of about 1.5 and 2.0, while mode 6 crosses modes 7, 9 and 12 at values of R of about 1.2, 1.3 and 1.6, respectively.

6.2 Vibration of Elliptical and Circular Plates

For a clamped isotropic elliptical plate with the coordinate system shown in Figure 1, a solution to Equation (4.5.6), satisfying the boundary condition (5.2.1), can be taken in the form:

$$W = (1-x^2-y^2)^2 f_k(x,y) \quad (6.2)$$

where the function f_k is as defined in the previous problem.

Solutions are obtained for values of the aspect ratio R ranging between 1 and 2. All results are calculated using 100 collocation points. The distribution of these collocation points is shown in Figure 2.b.

Results of the analysis of elliptical and circular plates are tabulated in Table 6. Results obtained by Carrington /4/ and Sun /37/ are also shown in the table for comparison. The agreement is good with the maximum error not exceeding 2%.

From the comparison of results of circular plate for which

the exact solution in polar coordinate system was obtained by Carrington /4/, the polynomial approximations, viz. the present solution and the Ritz solution of Sun /37/, prove to be highly accurate. Furthermore, as also noted in the case of elliptical plates, the collocation least square solution with only 9 adjustable parameters yields results which are very compatible with those results obtained by the Ritz solution having as many as 21 adjustable parameters /37/.

Plots of frequencies vs. aspect ratio for elliptical plates are shown in Figure 10. Features such as frequency crossing and degeneracy of modes are also observed in the case of elliptical plates. As shown in Figure 10, pairs of modes (2,3), (4,5), (7,8) and (9,10) are degenerate modes.

6.3 Vibration of Rhombic Plates

For a clamped isotropic rhombic plate with the coordinate system shown in Figure 1, a solution to Equation (4.5.6), satisfying the boundary condition (5.3.1), can be taken in the form:

$$W = (1+x+y)^2(1+x-y)^2(1-x+y)^2(1-x-y)^2 f_k(x,y) \quad \dots \dots \dots (6.3)$$

where the function f_k is as defined in Equation (6.1.1).

Solutions are obtained for values of the skew angle α ranging between 45° and 30° . All results are calculated using 100 collocation points. The distribution of these collocation points is shown in Figure 2.c.

Results of the analysis of rhombic plates are tabulated in Table 7. Comparisons of results are made with the results obtained by Durvasula /12/ and Sun /37/. From the comparisons made, again, the agreement is good with the maximum error not exceeding 2%.

Plots of frequencies vs. aspect ratio for rhombic plates are shown in Figure 11. From the figure, pairs of modes (2,3), (7,8) and (9,10) are degenerate modes. Modes 5 and 6 are not degenerate modes when $\alpha = 45^\circ$; they are identical to those modes also labeled 5 and 6 of the square plate defined by the lines $x = \pm 1$ and $y = \pm 1$.

As in the cases of rectangular and elliptical plates, most changes in nodal patterns of rhombic plates occur as the aspect ratio R deviates from one. However, mode 8 only begins to assume a different nodal pattern at the skew angle $\alpha = 35^\circ$. During this transition stage, the two nodal curves in the filament direction, approach each other at the center of the plate before changing over into the transverse direction.

Compared to plates of the other two geometries, rhombic plates have much less tendency for frequency crossing. For the range of the skew angle α considered, only one frequency crossing is observed in Figure 11; i.e., mode 6 crosses mode 7 at $\alpha = 34.3^\circ$.

6.4 Vibration of Orthotropic Rectangular Plates

For a clamped rectangular plate of orthotropic material having the axes of principal stiffness coinciding with the x and y -axes of the coordinate system shown in Figure 1, the differential equation (4.5.3) is to be solved for two special cases of orthotropy.

The materials considered here are Afara 3-ply and Maple 5-ply whose elastic constants have been accurately determined by Hearmon and

Adams /17/ as follows:

<u>Material</u>	<u>E_x,psi</u>	<u>E_y,psi</u>	<u>G_{xy},psi</u>	<u>ν_x</u>	<u>ν_y</u>
Afara 3-ply	1.96x10 ⁶	0.165x10 ⁶	0.110x10 ⁶	0.26	0.022
Maple 5-ply	1.87x10 ⁶	0.60x10 ⁶	0.159x10 ⁶	0.12	0.039
Isotropic	E	E	G	0.30	0.30

As can be seen from the above table, Afara wood possesses stronger material orthotropy than Maple wood, viz., having higher ratio E_x/E_y .

Here, the second approach of the collocation least square method, formulated in section 3.4.b, is employed to calculate the fundamental frequency of clamped orthotropic rectangular plates. The iteration scheme using Equations (3.4.5) and (3.4.10), begins with $F=0$ and ceases when the following condition is satisfied:

$$|F_n - F_{n-1}| / |F_n| = 10^{-5} \quad (6.4)$$

where F_n is value of F after n cycles of iteration.

The boundary conditions for this problem are identical to those of the rectangular isotropic plates. Thus, the assumed displacement function for that problem can be taken for the solution of the present problem. To approximate the fundamental mode of vibration, the shape function f_k in Equation (6.1.1) will be assumed in the form of Expression (6.1.2).

Solutions are obtained for values of the aspect ratio R ranging between 1 and 2. All results are calculated using 100 collocation points. The distribution of these collocation points is identical to

that of the isotropic rectangular plates. In all cases, the accuracy condition (6.4) is reached within 4 cycles.

Results of the analysis of orthotropic rectangular plates are tabulated in Table 8. Comparisons of result are made with the results obtained by Hearmon /16/ and Lekhniskii /21/. From Table 8, the agreement is seen to be excellent.

Plots of frequencies vs. aspect ratio for the two orthotropic materials are shown in Figure 12. Results from Figure 9 for isotropic rectangular plates are also presented here for comparison. As can be seen from Figure 12, the frequency parameter f increases with the degree of material orthotropy. However, this effect diminishes as the aspect ratio R increases; frequencies of orthotropic plates converge to that of isotropic plates. This is so because, recalling the governing differential equation (4.5.3), the contribution of the first two terms which account for material orthotropy becomes rapidly less significant as the aspect ratio R increases from one.

From the various problems considered in this Chapter, the collocation least square method proves to be a powerful means for the solution of eigen value problems in applied mechanics. Such problems as vibration of plates of isotropic and orthotropic materials were handled with great ease. The results obtained by using the two distinct approaches of the method are in good agreement with existing solutions which are known to be accurate.

CHAPTER VII

BUCKLING AND VIBRATION OF
RECTANGULAR PLATES UNDER INPLANE FORCES

To further demonstrate the versatility of the collocation least square method, the problems of buckling and vibration of clamped rectangular plates under actions of inplane normal loads are investigated in this Chapter.

7.1 Buckling of Rectangular Plates

For a clamped, isotropic rectangular plate with the coordinate system shown in Figure 1, subjected to inplane compressive forces in the x and y directions, the governing differential equation (4.5.5) is to be solved for various combinations of N_x and N_y . Since shear forces are not considered, N_{xy} is set to zero in the equation.

Solutions are obtained for values of the aspect ratio R ranging between 1 and 2. All results are calculated using 100 collocation points, and the distribution of the points is as shown in Figure 2.a.

The boundary conditions here are identical to those of the rectangular plates in the previous vibration problems. Thus, the assumed displacement function for the present problem can be taken in the form of Equation (6.1.1). Under actions of biaxial compressions, the square plate is assumed to buckle in one half-wave in both the x and y directions. As the aspect ratio R increases from one, the rectangular plate continues buckling in one half-wave in the short y -direction, but it may buckle

in several half-waves in the long x -direction. Hence, to approximate these buckled shapes of the plate, Expressions (6.1.2) and (6.1.4) are chosen for the function f_k in Equation (6.1.1). The actual solution to the problem is the one of these two expressions which yields lower values for N_x and N_y .

For this problem, only the lowest eigenvalue of Equation (4.5.5), which corresponds to the critical load, is of interest. Thus, the collocation least square scheme which was formulated in section 3.4.b and has successfully been applied to the problem of vibration of orthotropic rectangular plates, is conveniently employed here. The iteration process begins with $N = N_x = rN_y = 0$, where r is a given ratio N_x/N_y , and ceases when :

$$\left| \frac{N_n - N_{n-1}}{N_n} \right| = 10^{-5} \quad (7.1)$$

where N_n is the eigenvalue after n cycles of iterations. In all cases, this accuracy is reached within 5 iterations.

Results of the analysis are tabulated in Table 9. Comparisons of results are made with values obtained by Timoshenko /40/ for the case of biaxial compressions, and by Levy /23/ and Maulbetsch /24/ for the case of uniaxial compression. As can be seen in Table 9, the present results are in excellent agreement with the accurate Fourier series solution reported by Levy /23/. The Ritz solutions reported by Maulbetsch /24/ and Timoshenko /40/ appear to be upper bounds; the maximum percentage difference between these values and the present results is less than 2.5%. From the comparisons made, again, it is felt that if the present solution is in error at all, the error is on the safe side.

Figure 13 shows plots of simultaneous critical buckling loads N_x and N_y for values of the aspect ratio $R = a/b$ ranging from 1 to 2. These

curves are often termed interaction curves. It can be seen in Figure 13 that the point of intersection of an interaction curve with the x-axis gives the critical value of N_x for the case where $N_y = 0$. The intersection of the same curve with the y-axis gives the critical value of N_y when $N_x = 0$. For the case $N_x = N_y = N_0$, the critical buckling load N_0 is determined by the intersection of these curves with the line which goes through the origine 0 of the coordinate system and makes an angle of 45° with the horizontal axis.

The buckled shape of the plate under a given combination of N_x and N_y for a particular value of a/b is also indicated in Figure 13, where m is the number of half-waves in the x-direction and n is the number of half-waves in the y direction. As shown in the figure, the number of buckles in the x-direction increases as the critical load N_x and the length "a" in that direction increase. For the case $N_y = 0$, $m = 2$ for a rectangular plate of $R = 1.5$, and $m = 3$ for a rectangular plate of $R = 2$.

7.2 Vibration of Rectangular Plates Subjected to Inplane Loads

As a final illustration of the validity of the collocation least square method, the problem of vibration of clamped rectangular plates under the influence of hydrostatic compressive and tensile forces is investigated in this section.

The governing differential equation for this problem can be obtained by combining Equations (3.5.5) and (3.5.6); and letting $N = N_x = N_y$, and $N_{xy} = 0$. i.e.,

$$\begin{aligned} W_{,xxxx} + 2R^2 W_{,xxyy} + R^4 W_{,yyyy} - F.W \\ = N (W_{,xx} + R^2 W_{,yy}) \end{aligned} \quad (7.2)$$

The boundary conditions here are identical to those in the case of vibration of rectangular plates free of inplane forces. Thus, the assumed displacement function of that problem, Equation (6.1.1), can be taken for the solution of the present problem.

For comparison of results, solutions are obtained for values of $N(2a)^2/\pi^2D$ varying between the negative compressive load $N^*(2a)^2/\pi^2D$, at which $F=0$, and $N(2a)^2/\pi^2D=0$. All results are calculated using 100 collocation points. The distribution of these collocation points is identical to that in the case of vibration of rectangular plates without inplane forces.

The collocation least square scheme which was successfully applied to the previous problem is again employed here to determine the three lowest frequencies of the square plate. The iteration procedure formulated in section 3.4.b begins with $F=0$ and ceases when the accuracy condition (6.4) is satisfied. In all cases, this accuracy is reached within 4 cycles of iteration.

Results of the analysis are tabulated in Table 10. Results obtained by Bassily /2/ and Weinstein /44/ are also shown in the table. From the comparisons made, again, the agreement is favorable. The present results agree better with Weinstein's Ritz solution than with Bassily's perturbation series which begins to diverge at $N(2a)^2/\pi^2D=50$.

Plots of frequencies vs. inplane forces for clamped square plate are shown in Figure 14. These curves indicate that frequencies vary with inplane loads in a quadratic manner, decreasing with compressions and increasing with tensions.

For eigenvalue problems with only the lowest eigen value being of interest, the second approach of the collocation least square method

clearly has the advantage of being shorter to apply. From the two illustrative problems solved here, again, it can be seen that the collocation least square method provides efficient and accurate solutions to typical eigen value problems of applied mechanics.



CHAPTER VIII
CONCLUDING REMARKS ON
THE COLLOCATION LEAST SQUARE METHOD

1) By the application of the least square concept, the accuracy of the conventional collocation method can be greatly improved.

2) Although the problem of selecting "correct" locations for the collocation points is avoided by the use of large number of collocation points, these collocation points must however be distributed in a sensibly uniform manner over the entire region of the problem under consideration. Accurate results cannot be expected if all the collocation points are unreasonably crowded into a particular area of the region. Furthermore, for an "interior method", the results would not be as accurate if some of the collocation points fall on the boundary.

3) The collocation least square technique proposed here is a simple yet powerful tool for the static and dynamic analysis of plates. The versatility and simplicity of this method was demonstrated by solving a wide variety of problems of significant complexity in applied mechanics.

4) In the majority of cases, results obtained by the use of this method are in favorable agreement with those obtained by much more rigorous but lengthier approaches. It also appears that if the proposed solution is in error at all, the error is on the conservative side.

5) The collocation least square method represents a great saving in human and machine efforts as a results of its simple mathematical concept and the relatively small amount of computer time and storage space required for a solution. All results were obtained from essentially one computer

program written in FORTRAN IV for an IBM 360/65 computer. It is noted that the computer program involved basically three items:

(a) A main program to evaluate sums and products of algebraic functions at a number of points in the plate to yield m equations in n unknowns.

(b) A least square subroutine to reduce the above set of equations to an $n \times n$ set.

(c) A simultaneous equations subroutine (or a eigenvalue subroutine) to solve the resulting set of equations.

Application of the collocation least square method to other plate problems including many engineering interests such as variable rigidity, special type of anisotropy or thermal bending, is also feasible.

APPENDIX A FIGURES

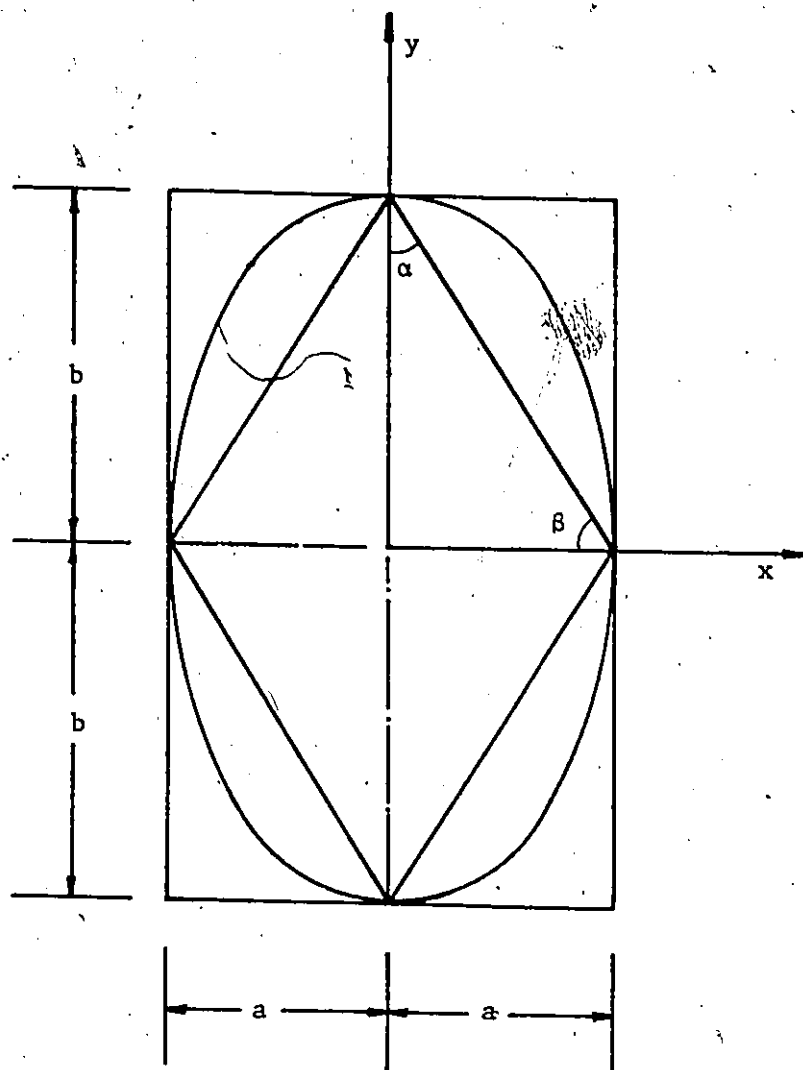


Figure 1 - Rectangular, Elliptical and Rhombic Geometries Defined by a Common Rectangular Cartesian Coordinate System.

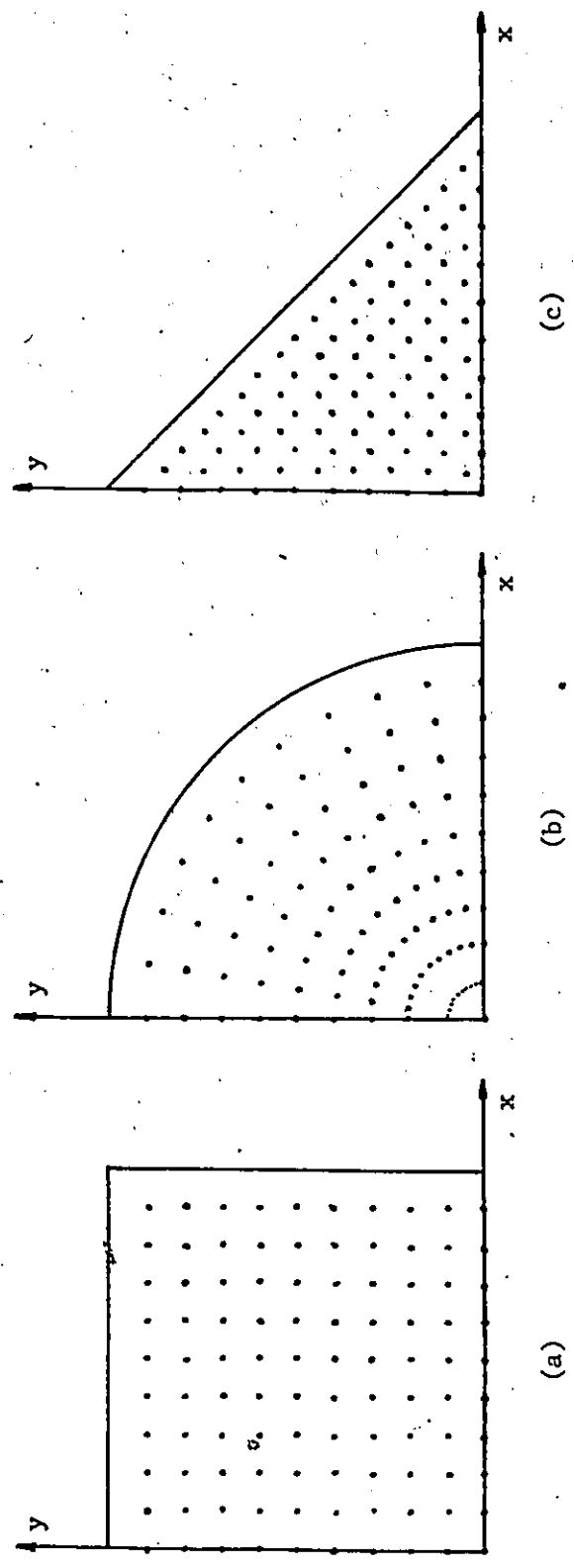


Figure 2 - Distribution of Collocation Points for the Three Geometries.

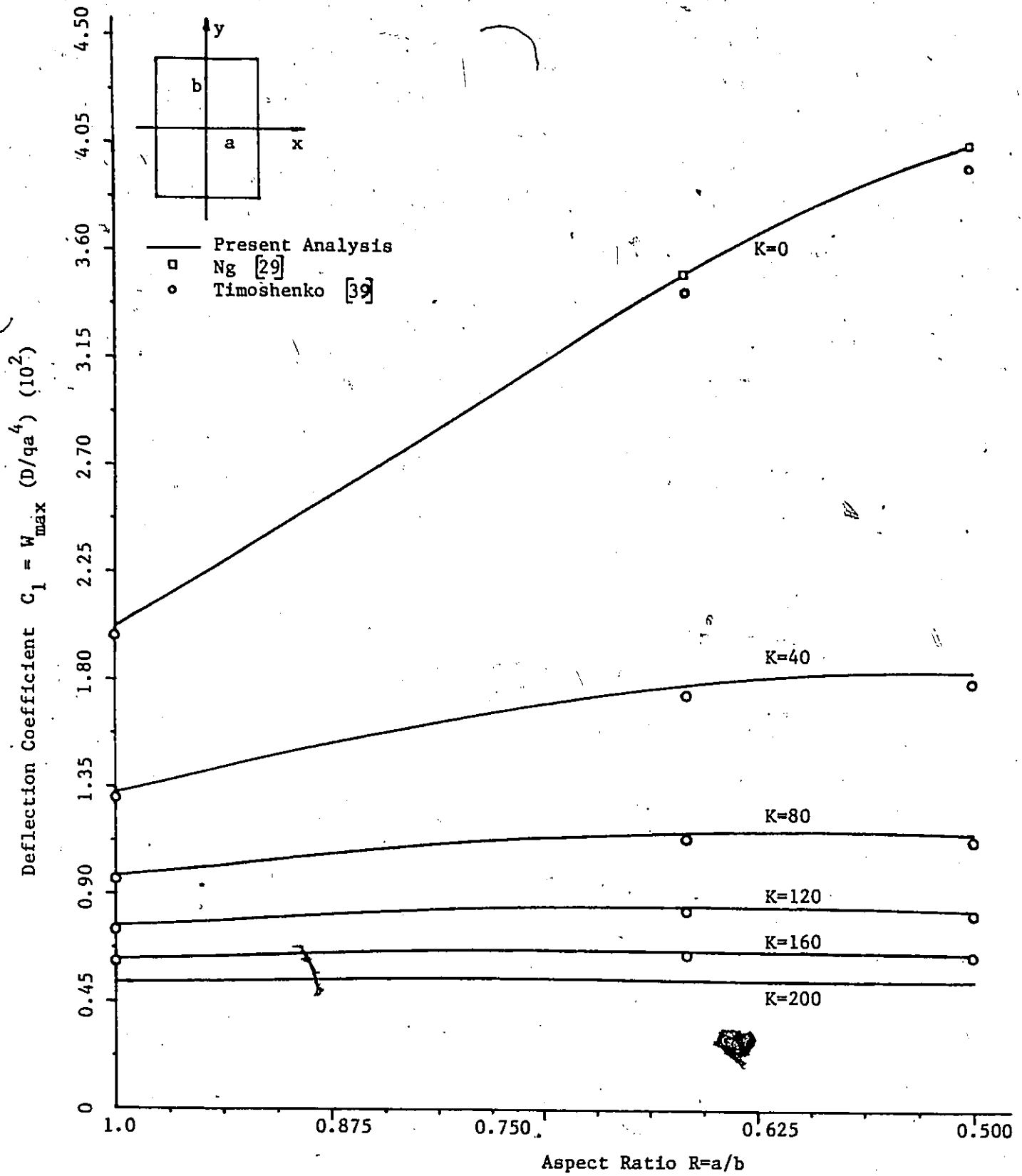


Figure 3 - Variation of Central Deflection with Foundation Modulus and Aspect Ratio for Clamped Rectangular Plates.

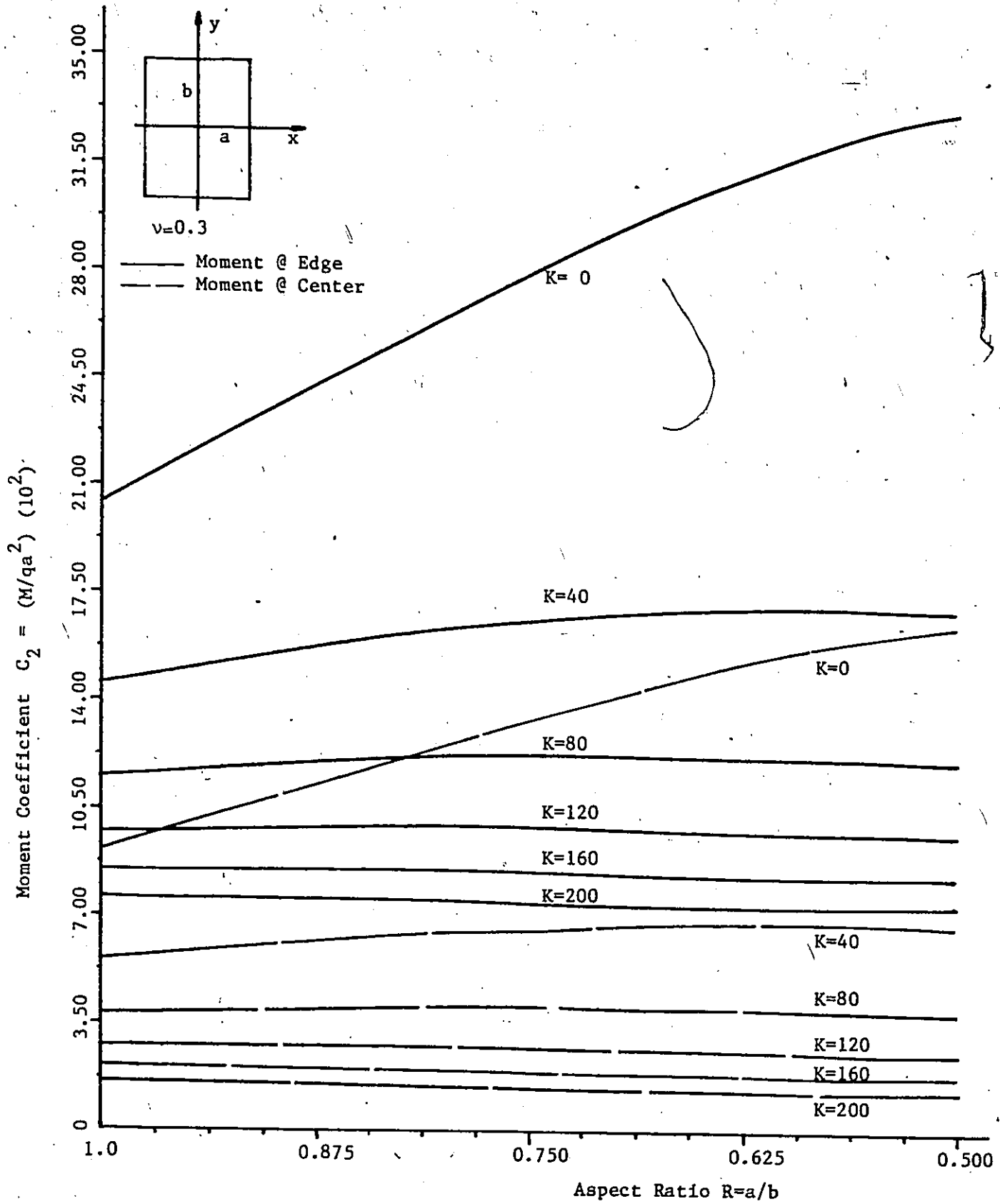


Figure 4 - Variation of Maximum Edge and Central Moments with Foundation Modulus and Aspect Ratio for Clamped Rectangular Plates

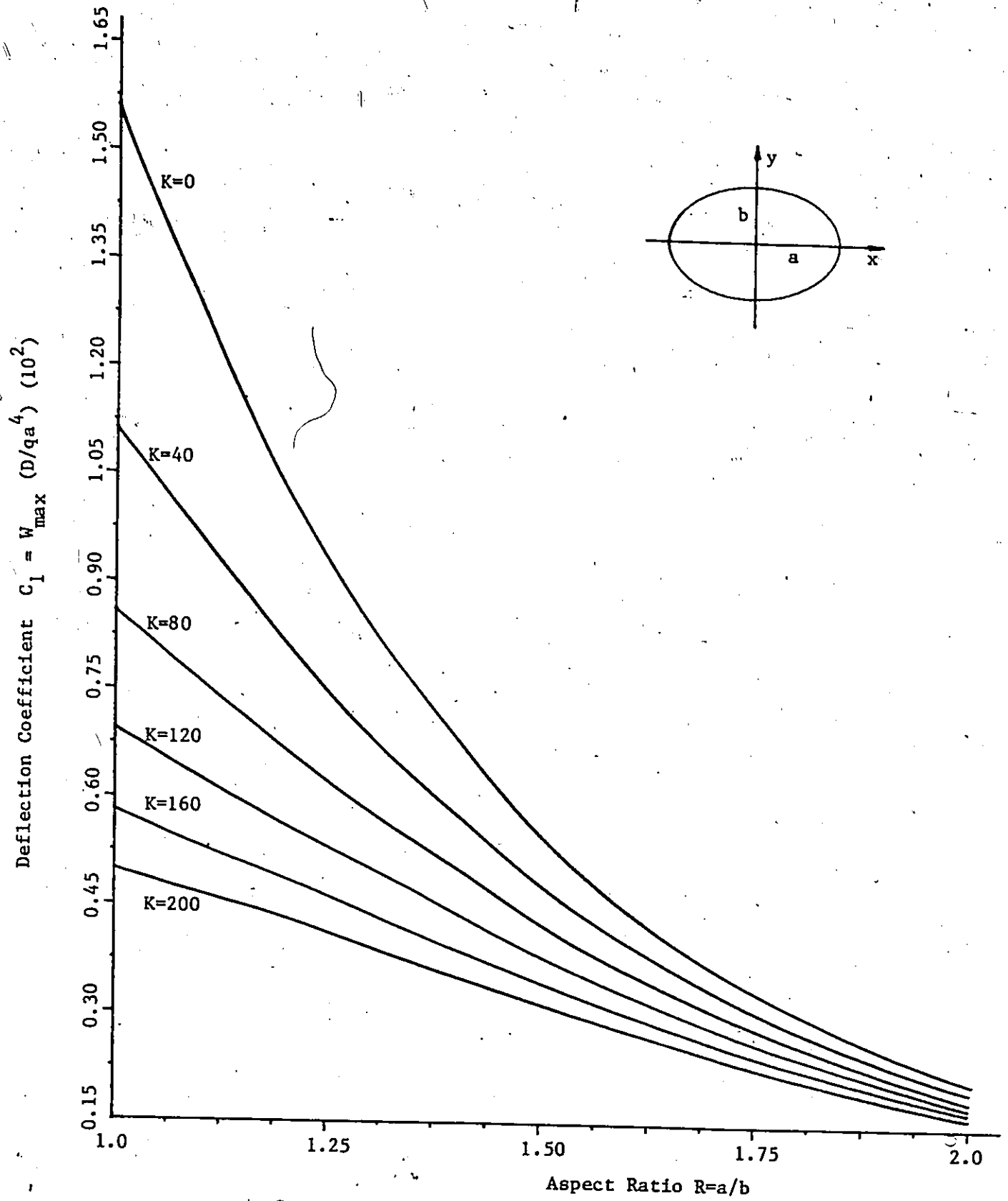


Figure 5 - Variation of Central Deflection with Foundation Modulus and Aspect Ratio for Clamped Elliptical and Circular Plates.

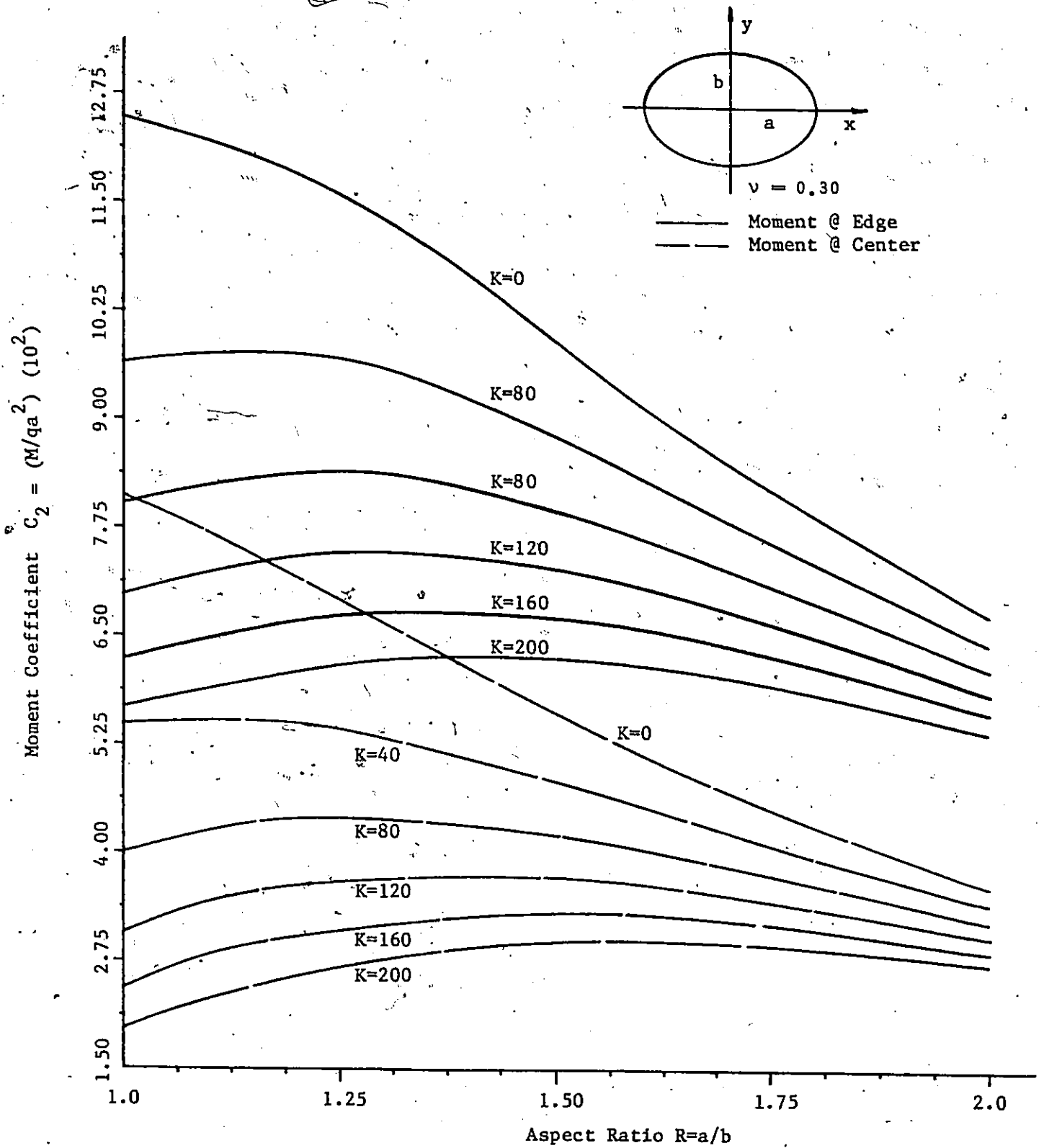


Figure 6 - Variation of Maximum Edge and Central Moments with Foundation Modulus and Aspect Ratio for Clamped Elliptical and Circular Plates

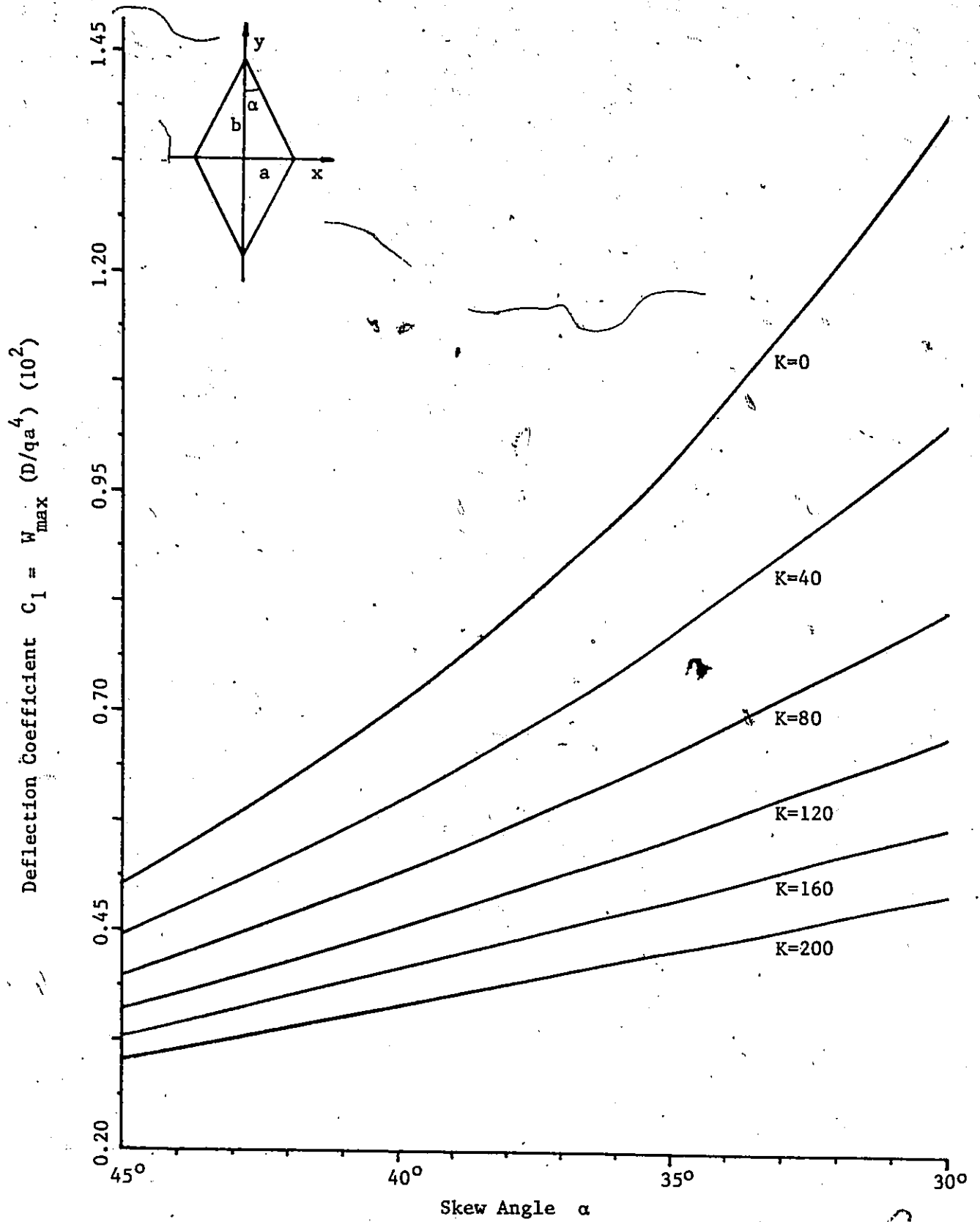


Figure 7 - Variation of Central Deflection with Foundation Modulus and Skew Angle for Clamped Rhombic Plates.

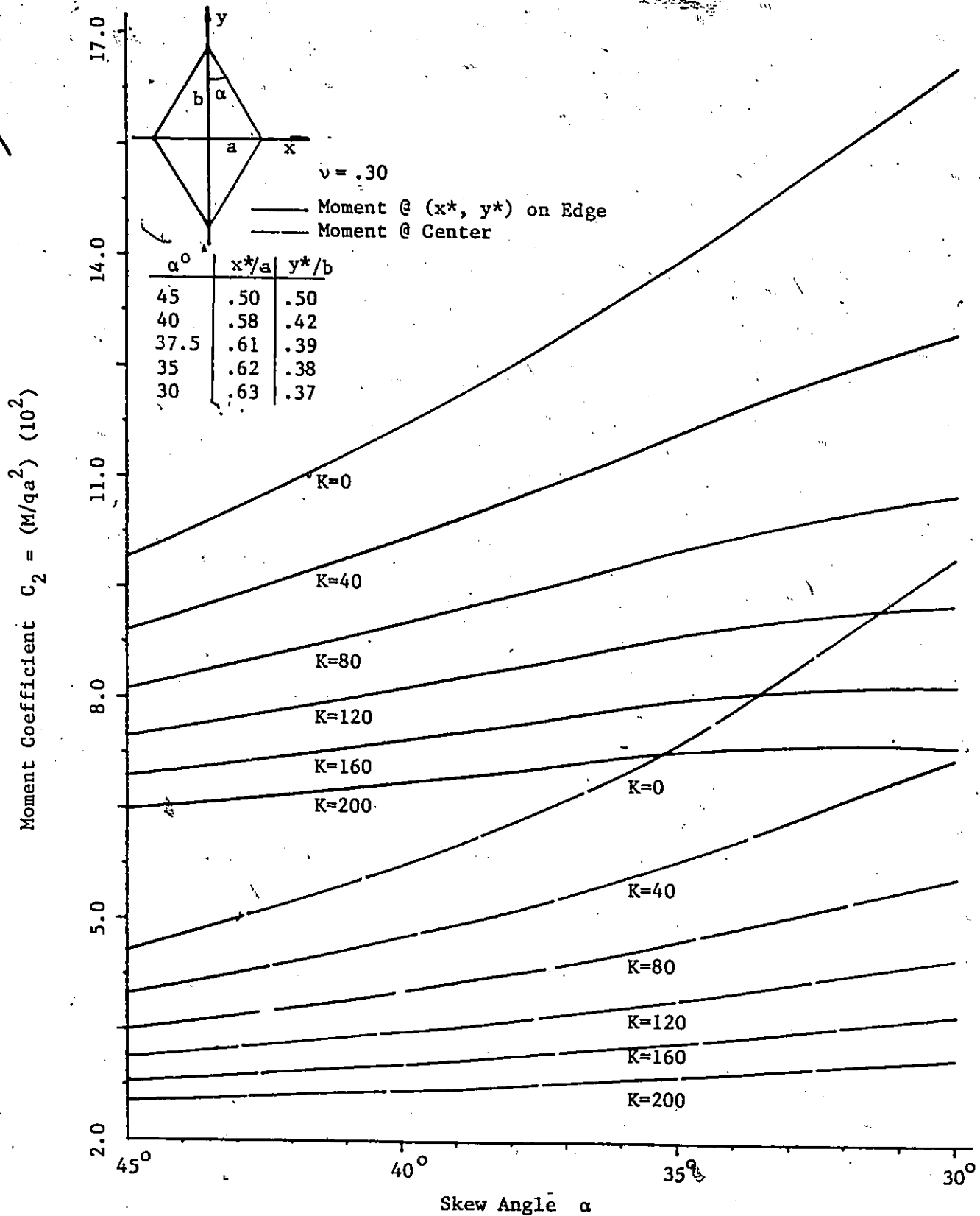


Figure 8 - Variation of Maximum Edge and Central Moments with Foundation Modulus and Skew Angle for Clamped Rhombic Plates.

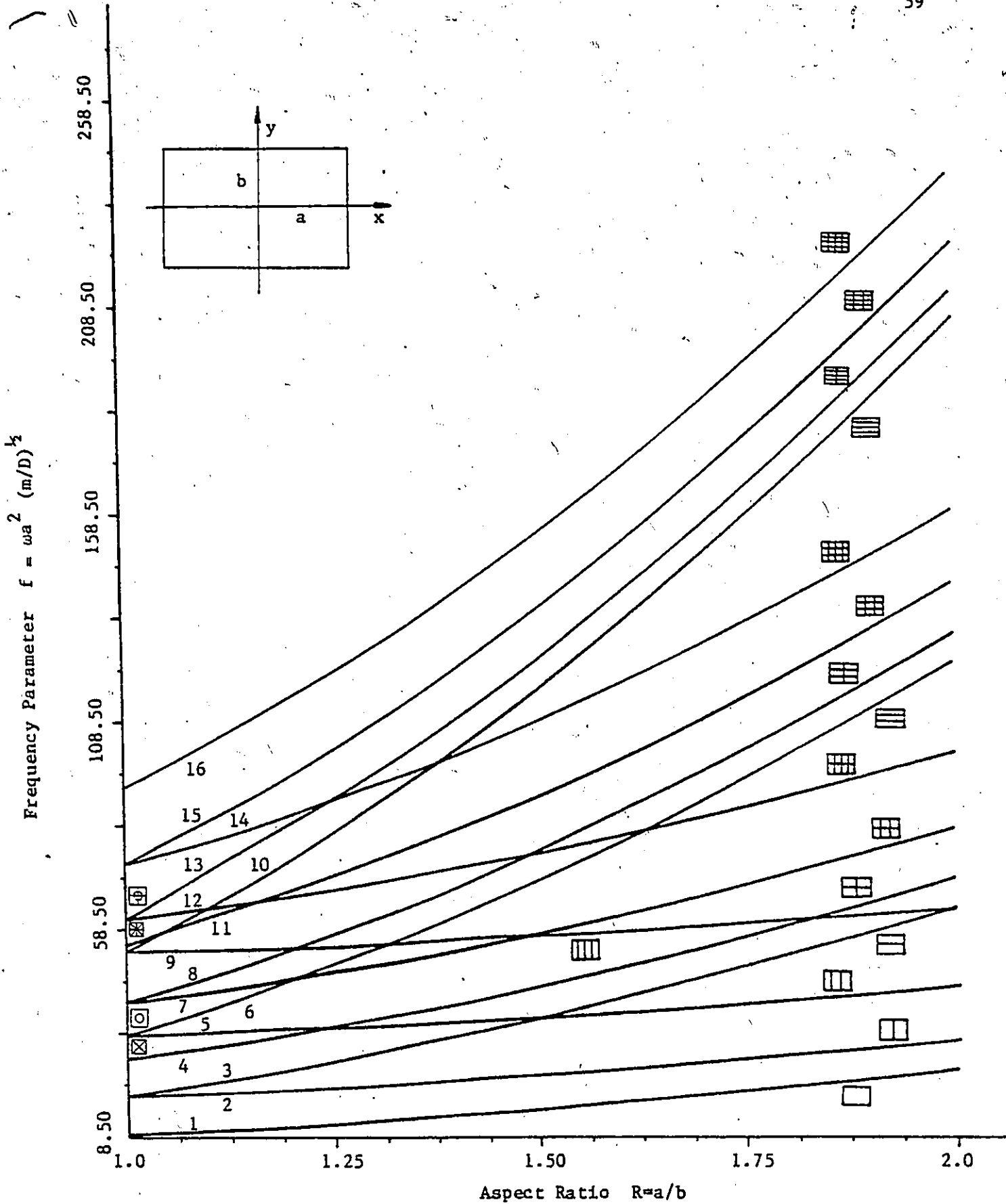


Figure 9 - Variation of Natural Frequencies with Aspect Ratio, and Nodal Patterns for Clamped Rectangular Plates.

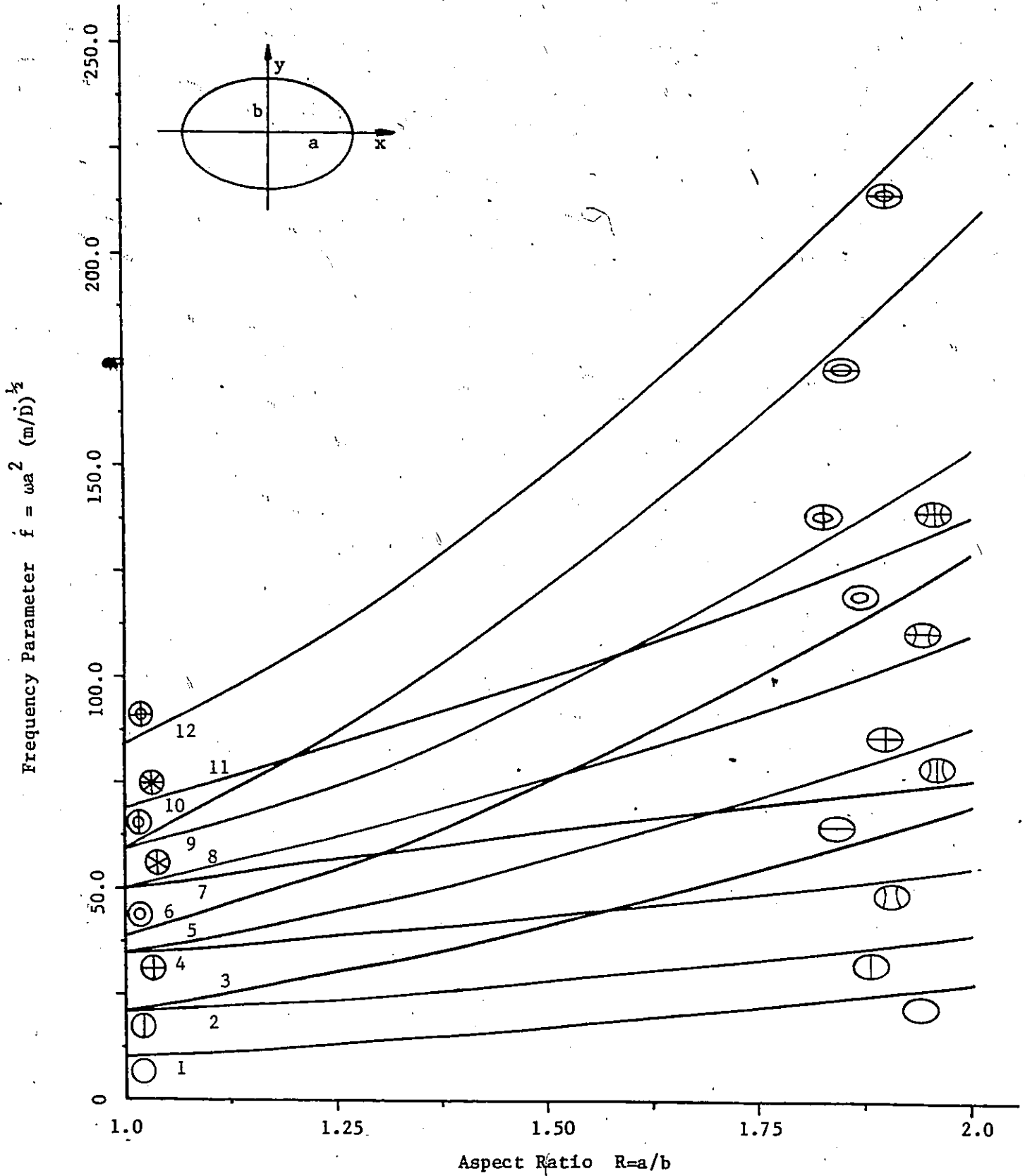


Figure 10 - Variation of Natural Frequencies with Aspect Ratio, and Nodal Patterns for Clamped Elliptical Plates.

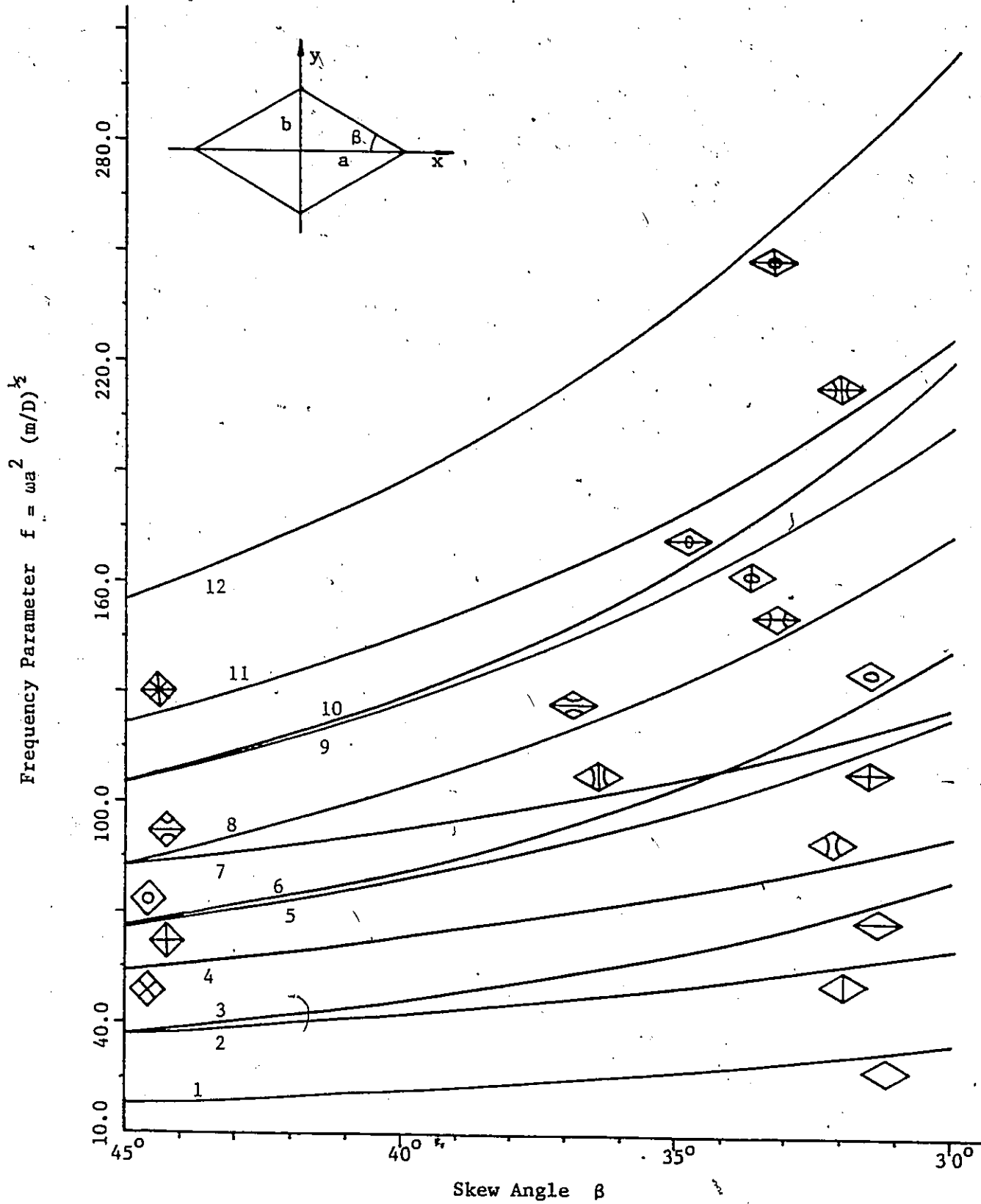


Figure 11 - Variation of Natural Frequencies with Skew Angle, and Nodal Patterns for Clamped Rhombic Plates.

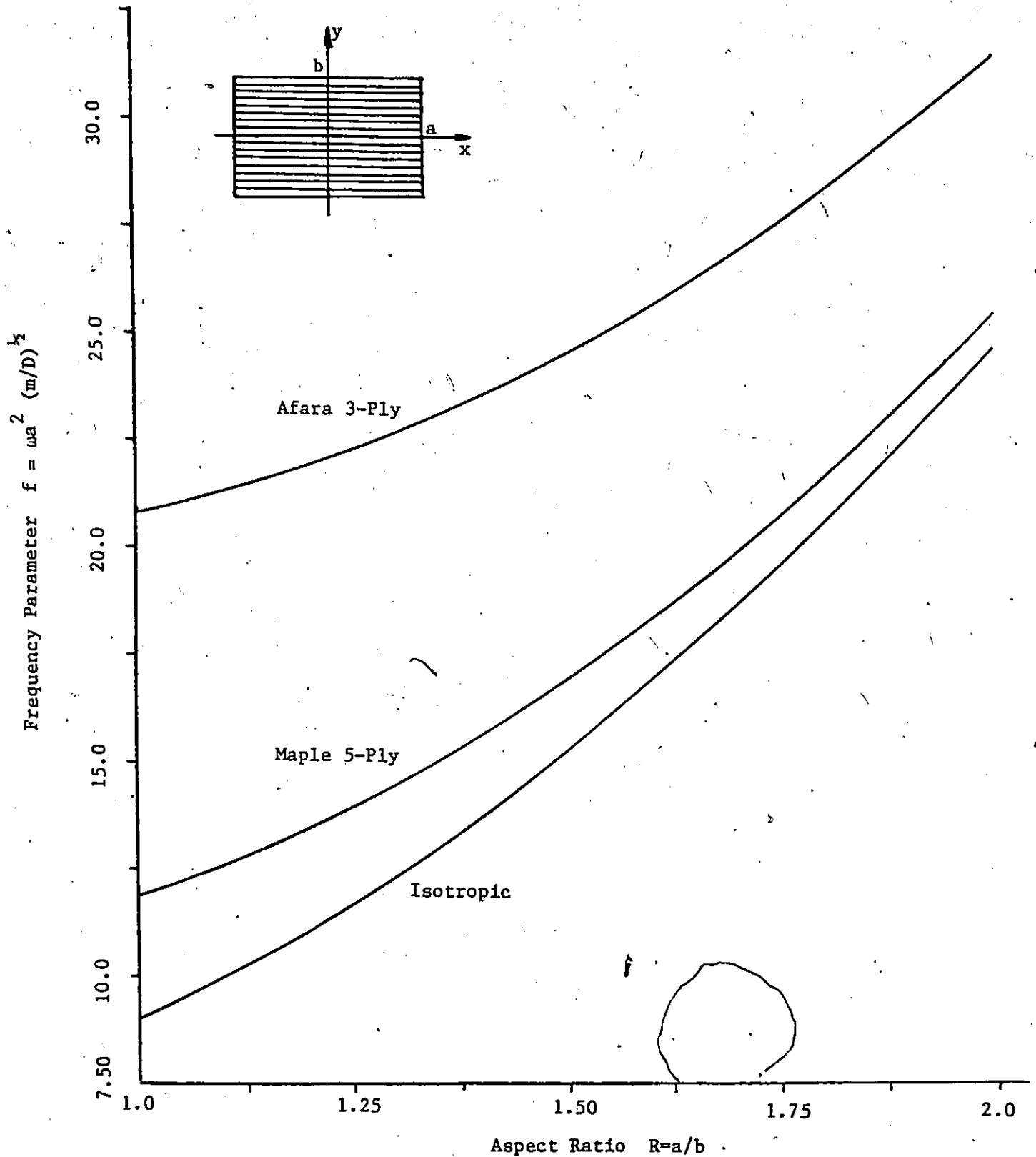


Figure 12 - Variation of Fundamental Frequency with Aspect Ratio for Orthotropic Clamped Rectangular Plates.

Dimensionless Load Parameter $\bar{N}_y = N_y (2a)^2 / \pi^2 D$

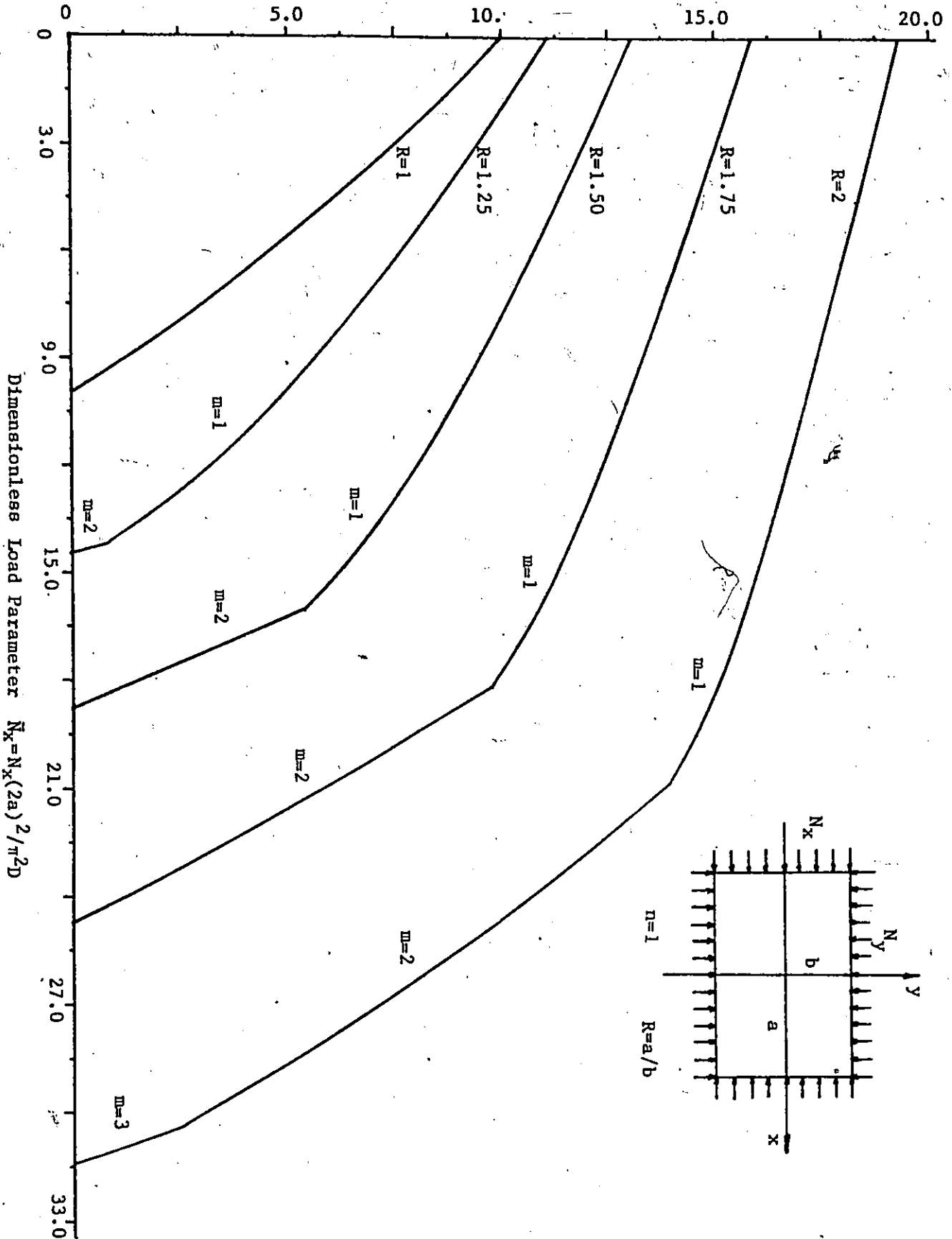
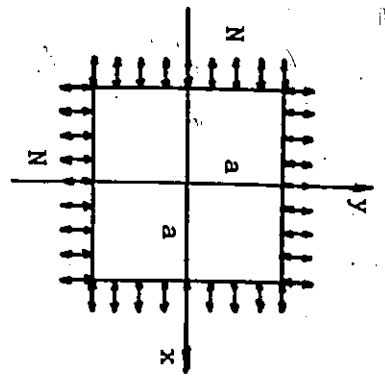
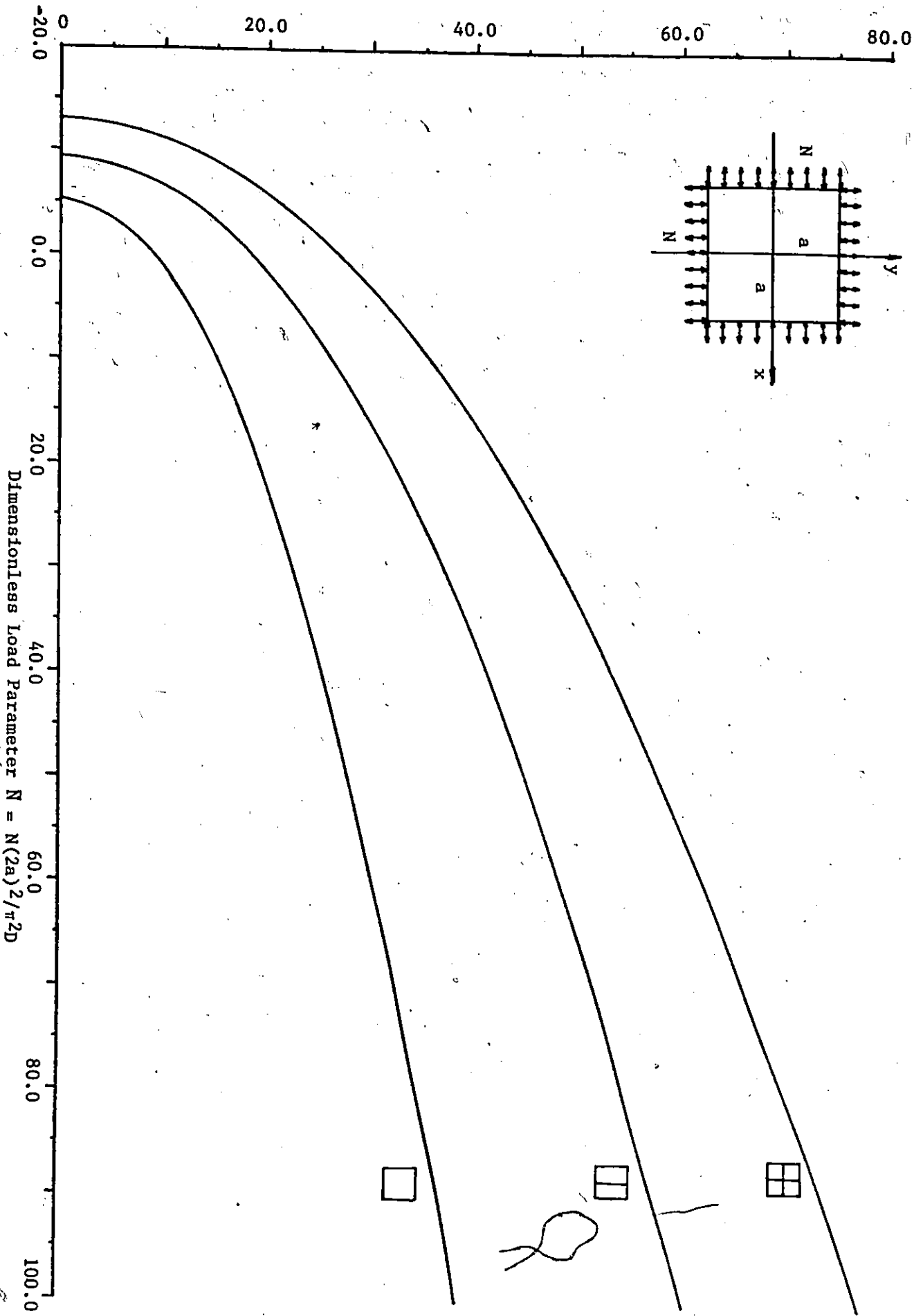


Figure 13 - Interaction Curves of Buckling Loads for Clamped Rectangular Plates.

Frequency Parameter $f = \omega a^2 (m/D)^{1/2}$

Figure 14 - Variation of Frequencies with Inplane Hydrostatic Forces for Clamped Square Plates.



APPENDIX B TABLES

ASPECT RATIO R = a/b	NO. OF COLLOCATION POINTS USED			TIMOSHENKO REF. /39/
	25	50	100	
1	2.0243	2.0221	2.0201	2.016
3/4	3.1470	3.1422	3.1412	---
2/3	3.5139	3.5090	3.5090	3.520
1/2	4.0510	4.0510	4.0545	4.064

$$W_{\max} = C_1 (qa^4/D) (10^{-2})$$

Table 1 - Variation of the Maximum Deflection Coefficient C_1 with the Number of Collocation Points used in the Solution for Clamped Rectangular Plates.

DIMENSIONLESS FOUNDATION MODULUS K	ASPECT RATIO R = a/b			
	1	4/5	2/3	1/2
0	2.0201	2.9093	3.5090	4.0545
20	1.6033	2.1072	2.3807	2.5525
40	1.3245	1.6409	1.7840	1.8411
60	1.1713	1.3369	1.4167	1.4307
80	0.9756	1.1233	1.1689	1.1656
100	0.8594	0.9655	0.9910	0.9812
120	0.7665	0.8442	0.8575	0.8459
140	0.6907	0.7483	0.7540	0.7427
160	0.6277	0.6707	0.6713	0.6616
180	0.5745	0.6066	0.6041	0.5963
200	0.5291	0.5530	0.5484	0.5425

$$W_{\max} = C_1 (qa^4/D) (10^{-2})$$

Table 2 - Variation of the Maximum Deflection Coefficient C_1 of Clamped Rectangular Plates with the Dimensionless Foundation Modulus K

K	R = 1.0		R = 1.25		R = 1.5		R = 1.75		R = 2.0	
	PRESENT SOLUTION	NG REF./31/	PRESENT SOLUTION	NG REF./31/	PRESENT SOLUTION	NG REF./31/	PRESENT SOLUTION	NG REF./31/	PRESENT SOLUTION	NG REF./31/
0	1.5625	1.5625	0.9294	0.9294	0.5510	0.5510	0.3355	0.3355	0.2118	0.2118
20	1.3009	1.3017	0.8298	0.8300	0.5139	0.5140	0.3211	0.3212	0.2060	0.2060
40	1.1120	1.1143	0.7488	0.7495	0.4814	0.4816	0.3079	0.3080	0.2004	0.2004
60	0.9694	0.9730	0.6817	0.6998	0.4526	0.4529	0.2957	0.2958	0.1950	0.1951
80	0.8597	0.8627	0.6252	0.6268	0.4268	0.4273	0.2844	0.2846	0.1900	0.1900
100	0.7684	0.7741	0.5769	0.5791	0.4038	0.4044	0.2739	0.2741	0.1852	0.1852
120	0.6950	0.7013	0.5353	0.5378	0.3829	0.3838	0.2641	0.2643	0.1806	0.1807
140	0.6337	0.6405	0.4990	0.5019	0.3640	0.3651	0.2549	0.2552	0.1762	0.1763
160	0.5818	0.5889	0.4671	0.4703	0.3469	0.3480	0.2463	0.2467	0.1720	0.1721
180	0.5373	0.5446	0.4388	0.4422	0.3311	0.3324	0.2383	0.2387	0.1680	0.1682
200	0.4987	0.5060	0.4135	0.4172	0.3167	0.3181	0.2307	0.2312	0.1642	0.1644

$$W_{\max} = C_1 (qa^4/D) (10^{-2})$$

$$K = ka^4/D$$

$$R = a/b$$

Table 3 - Variation of the Maximum Deflection Coefficient C_1 of Elliptical and Circular Plates with the Dimensionless Foundation Modulus K

Table 4 - Variation of the Maximum Deflection Coefficient C_1 of Clamped Rhombic Plates with the Dimensionless Foundation Modulus K

$$W_{\max} = C_1 (qa^4/D) (10^{-2})$$

DIMENSIONLESS FOUNDATION MODULUS K	Skew Angle $\alpha = \tan^{-1} (a/b)$				
	45°	40°	37.5°	35°	30°
0	0.5034 0.5050*	0.7059	0.8325	0.9831	1.3824
20	0.47275	0.6474	0.7530	0.8750	1.1811
40	0.4455	0.5976	0.6869	0.7876	1.0295
60	0.4211	0.5547	0.6310	0.7154	0.9112
80	0.3991 0.4008*	0.5172	0.5832	0.6549	0.8163
100	0.3793	0.4843	0.5418	0.6034	0.7387
120	0.3612	0.4552	0.5057	0.5590	0.6739
140	0.3447	0.4292	0.4738	0.5205	0.6191
160	0.3296 0.3311*	0.4058	0.4456	0.4866	0.5721
180	0.3156	0.3848	0.4203	0.4546	0.5315
200	0.3028	0.3657	0.3976	0.4301	0.4959

* Table 3 tabulates $W_{\max} / \sin^4 (45^\circ)$ for $K \sin^4 (45^\circ)$

Table 5 - Variation of the Frequency Parameter f of Clamped Rectangular Plates with Aspect Ratio

$$\omega = (f/a^2) (D/m)^{1/2} \quad R = a/b$$

MODE NUM- BER	R = 1.0		R = 1.25		R = 1.50		R = 1.75		R = 2.0		Sym- metry Group
	PRESENT SOLUTION	YOUNG REF. /45/	ODMAN REF. /33/	PRESENT SOLUTION	PRESENT SOLUTION	PRESENT SOLUTION	ODMAN REF. /33/	PRESENT SOLUTION	PRESENT SOLUTION	ODMAN REF. /33/	
1	9.0042	8.998	8.9997	11.6555	15.2057	15.1931	19.5234	24.6067	24.58	A	
2	18.3623	18.353	18.351	20.5233	23.4685	23.4675	27.2369	31.8334	31.83	B	
3	18.3623	18.353	18.351	26.7853	37.2324	37.4231	49.6610	64.0493	-	C	
4	27.0553	27.068	27.059	34.9042	44.8938	44.8931	56.9698	71.0878	71.08	D	
5	32.8412	32.912	-	35.2240	37.3683	-	40.6472	44.7075	44.78	A	
6	33.0199	33.063	32.976	50.4024	70.3770	-	94.8157	123.0425	-	A	
7	41.1810	41.288	41.256	48.4943	57.9371	-	69.5156	83.2038	-	C	
8	41.1810	-	41.256	57.5184	77.8394	77.895	102.0463	130.1060	130.35	B	
9	53.2360	-	52.632	54.9782	57.2707	56.7056	60.2051	63.8358	63.34	B	
10	53.2360	-	52.632	81.5690	116.2639	-	157.3001	204.6683	-	C	
11	54.8778	-	55.015	71.7205	90.4842	90.4842	114.3133	142.0845	142.38	A	
12	61.0529	-	-	68.0565	76.9578	76.5563	87.9904	101.1683	100.80	D	
13	61.3134	-	60.665	89.1669	123.5889	-	164.4672	211.8313	211.765	C	
14	74.4584	-	74.088	89.6328	108.9012	-	132.2290	159.5693	159.49	B	
15	74.4584	-	74.088	101.8962	135.9641	-	176.5567	223.6033	-	C	
16	92.8228	-	92.845	120.4317	153.9164	152.5331	194.0789	240.8738	238.35	D	

Table 6 - Variation of the Frequency Parameter f of Clamped Elliptical Plates with Aspect Ratio

$$\omega = (f/a^2) (D/m)^{1/2}$$

$$R = a/b$$

MODE NUMBER	R = 1.0		R = 1.25		R = 1.50		R = 1.75		R = 2.0		Symmetry Group
	PRESENT SOLUTION	CARRINGTON REF. /4/	PRESENT SOLUTION	SUN REF. /37/	PRESENT SOLUTION	PRESENT SOLUTION	PRESENT SOLUTION	PRESENT SOLUTION	PRESENT SOLUTION	PRESENT SOLUTION	
1	10.2094	10.2158	13.2195	13.229	17.1142	21.8346	27.3296	A			
2	21.2313	21.26	24.3473	24.382	28.4255	33.4562	39.3873	B			
3	21.2313	21.26	30.2753	30.322	41.4154	54.5623	69.6600	C			
4	34.8301	34.88	39.3863	39.971	43.4533	48.6973	54.9593	A			
5	34.8301	34.88	44.7265	44.792	56.9383	71.3395	87.8333	D			
6	38.8118	39.771	55.3716	55.971	75.2861	100.0142	128.3963	A			
7	50.2473	51.04	58.4218	59.593	63.6292	69.3258	76.0708	B			
8	50.2473	51.04	62.2183	63.037	75.8769	91.5059	109.7912	C			
9	59.4820	60.82	73.8655	75.147	96.2212	122.9172	153.3591	B			
10	59.4820	60.82	87.5748	88.846	120.8541	161.2802	207.6861	C			
11	68.8903	69.666	84.2608	84.914	99.5689	117.0612	137.0413	D			
12	83.8195	84.58	111.1002	112.540	147.4423	190.4541	239.7082	D			

Table 7 -- Variation of the Frequency Parameter f of Clamped Rhombic Plates with Skew Angle

$\omega = (f/a)^2 (D/m)^{1/2}$ $\beta = \tan^{-1} (b/a)$

MODE NUMBER	$\beta = 45^\circ$		$\beta = 40^\circ$		$\beta = 35^\circ$		$\beta = 30^\circ$		Group of Symmetry
	PRESENT SOLUTION	DURVASULA REF. /12/	PRESENT SOLUTION	PRESENT SOLUTION	SUN REF. /37/	PRESENT SOLUTION	DURVASULA REF. /12/		
1	17.9314	17.996	21.6584	27.0367	26.874	34.3241	34.605	A	
2	36.6886	36.707	42.3164	50.0561	50.082	61.2391	61.268	B	
3	36.6886	36.707	46.0188	59.2163	59.222	79.6335	79.13	C	
4	54.0305	54.135	63.6995	75.2231	74.798	91.3797	89.64	A	
5	65.7827	65.820	79.0955	97.4578	97.442	124.2639	124.28	D	
6	66.0985	66.121	80.8820	104.0820	104.216	141.7989	140.49	A	
7	82.4837	82.579	93.0623	106.5386	106.141	126.8221	-	B	
8	82.4837	82.579	103.0906	132.0906	129.612	173.1414	-	C	
9	105.2312	-	126.9361	158.2437	157.805	205.1574	-	B	
10	105.2312	-	128.2153	163.4045	163.478	220.9872	-	C	
11	121.4421	-	145.5627	175.3973	173.606	227.2926	-	D	
12	154.8232	-	187.2493	234.6082	234.194	306.6752	-	D	

Table 8 - Variation of the Fundamental Frequency Parameter f
of Clamped Orthotropic Rectangular Plates

$$\omega = (f/a^2) \cdot (D/m)^{\frac{1}{2}}$$

$$R = a/b$$

ASPECT RATIO R	MAPLE 5-PLY			AFARA 3-PLY		
	PRESENT SOLUTION	HEARMON REF. /16/	LEKHNISKII REF. /21/	PRESENT SOLUTION	HEARMON REF. /16/	LEKHNISKII REF. /21/
1.0	11.8714	11.883	11.890	20.7889	20.8102	20.832
1.25	13.8873	13.901	13.904	22.2256	22.2505	22.258
1.50	16.8284	16.845	16.850	24.4209	24.4500	24.446
1.75	20.6895	20.690	20.721	27.4865	27.5200	27.509
2.0	25.4268	25.450	25.473	31.4709	31.5090	31.497

Table 9 - Buckling Load Coefficient C_r for Clamped Rectangular Plates.

$$N_x = C_r \left(\frac{\pi^2 D}{4a^2} \right)$$

$$R = a/b$$

N_y/N_x	PRESENT SOLUTION							0		
	8	4	2	1	1/2	1/4	1/8	PRESENT SOLUTION	LEVY REF. /23/	MAULBETSCH REF. /24/
1.0	1.1386	2.0847 2.13*	3.5315 3.56*	5.3333 5.33*	7.0629 7.11*	8.3389 8.53*	9.1085	9.9565	10.07	10.48
1.25	1.2964	2.4350	4.3247	6.8472	9.9645	12.2614	13.2813	14.9959	14.45	14.65
1.50	1.5620	2.9867	5.4781	9.3258	13.9318	16.6518	17.6787	18.8017	18.74	19.01
1.75	1.9174	3.7093	6.9519	12.2607	18.4810	21.3018	22.9433	24.7119	24.84	25.02
2.0	2.3512	4.5844	8.7212	15.7692	23.1184	27.2364	29.5328	31.4620	31.52	32.24

* Timoshenko, Ref. /40/

Table 10 - Variation of the Frequency Parameter f of Clamped Square Plate with the Dimensionless Inplane Hydrostatic Force \bar{N}

$$\omega = (f/a^2) (D/m)^{1/2} \quad \bar{N} = N(2a)^2 / \pi^2 D$$

VIBRATION MODE	ANALYSES	DIMENSIONLESS INPLANE FORCE \bar{N}								
		-13.0125	-9.3595	-5.3333	0	10	20	50	100	
MODE 1	PRESENT SOLUTION			0	9.0042	15.0170	19.1401	28.0417	38.6615	
	WEINSTEIN /44/			-	-	15.098	19.272	28.225	38.745	
	BASSILY /2/			-	8.998	15.039	19.112	27.306	30.686	
MODE 2 or 3	PRESENT SOLUTION		0	-	18.3623	26.1794	32.0484	45.1220	60.8497	
	BASSILY /2/		-	-	18.353	26.1449	31.907	44.137	57.037	
MODE 4	PRESENT SOLUTION	0	-	-	27.0553	35.7590	42.6315	58.3823	77.6456	

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