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*-Polynomial Identities of Matrices

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Abstract

Let $m > 1$ be a positive integer, F be a field, $K_{2m}(F,t)$ be the subspace of $M_{2m}(F)$ of matrices skew-symmetric with respect to the transpose involution, and $H_{2m}(F,s)$ be the subspace of matrices symmetric with respect to the symplectic involution. We show that $K_{2m}(F,t)$ and $H_{2m}(F,s)$ both satisfy q_m, a multilinear identity of degree $4m-3$. As corollaries we obtain both new proofs and refinements of theorems of Kostant and Rowen concerning s_{4m-2}, a so-called "standard" polynomial identity for $K_{2m}(F,t)$ and $H_{2m}(F,s)$, and s_{4m-4}, a "standard" polynomial identity for $K_{2m-1}(F,t)$.

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1. Introduction

1.1. General Introduction

A *polynomial identity* (PI) for an associative algebra A over a field F is a polynomial in a finite number of non-commuting variables (i.e., an element of the free associative algebra on k generators, $F\langle x_1, x_2, ..., x_k\rangle$) that vanishes for all specializations of the variables to elements of A. Though we are interested in so-called $*$-*polynomial identities* (identities in which the variables may appear with an involution $*$), it is natural to begin with PIs.

There are two aspects to PI-theory: structure-theoretic and combinatorial. In the structure-theoretic aspect one starts with an associative algebra, assumes that it satisfies some non-trivial PI, and then studies the structure of the resulting object (a *PI-algebra*). In the combinatorial aspect we fix an associative algebra A and study the concrete PIs that it satisfies. This thesis will be concerned with the combinatorial aspect.

It is generally agreed that the first major result in PI-theory is

Theorem 1.1.1 (Amitsur and Levitzki, [AL]). *The standard polynomial of degree* $2n$

$$s_{2n}(x_1, \ldots, x_{2n}) = \sum_{\sigma \in S_{2n}} (-1)^\sigma x_{\sigma(1)} x_{\sigma(2)} \cdots x_{\sigma(2n)},$$

where S_{2n} *is the symmetric group on* $2n$ *elements and* $(-1)^\sigma$ *is the sign of the permutation* σ*, is a PI for the matrix algebra* $M_n(F)$*,* F *a field.* $M_n(F)$ *does not satisfy a PI of degree less than* $2n$*, and if* $|F\,| > 2$ *or* $n > 2$ *then, up to scalar multiple,* s_{2n} *is the unique PI of degree* $2n$ *satisfied by* $M_n(F)$.

By *degree* of a polynomial is meant total degree, that is, the sum of the degrees of the polynomial's individual variables. We frequently work with *multilinear* (linear in all variables) polynomials, in this case the degree will equal the number of variables.

This result has served as a template for later results in combinatorial PI-theory. First, we would like to establish the minimal degree for a PI satisfied by a given algebra A (for M_n: $2n$). Second, we would like to exhibit (and prove) a concrete PI with that degree (for M_n: s_{2n}). And third, we discuss any uniqueness properties that this PI may possess (for M_n: up to scalar multiple and given certain restrictions on characteristic, s_{2n} is the unique PI of degree $2n$). The results of this thesis will be mainly concerned with the existence of $*$-PIs for M_n; in our concluding chapter we will discuss open questions concerning minimal degree and uniqueness.

Arbitrary PIs in several variables of varying degree are somewhat difficult to classify. The following proposition [DF, Proposition 1.2.8] allows us to narrow our focus.

Proposition 1.1.1. *Let A be an associative algebra, $f \in F\langle x_1, \cdots, x_l\rangle$, and write*

$$f(x_1, \cdots, x_k) = \sum_{\imath=0}^{d} f_\imath,$$

where $f_\imath$ is the homogeneous component of f of degree i in x_1 (all monomials in $f_\imath$ are of degree i in x_1).

(i) If F has more than d elements then f is a PI for A if and only if every $f_\imath$ is a PI for A.

(ii) If char$(F) = 0$ or char$(F) >$ deg(f) then there exists a set of multilinear (linear in all variables) polynomials $\{h_\jmath\}$ such that f is PI for A if and only if every h_l is a PI for A.

We give a brief discussion of the proof of (ii). First, this result makes use of the process of *multilinearization.* Given a polynomial $g(y_1, ..., y_l) \in F\langle y_1, ..., y_l\rangle$, and assuming further that g is of degree $r > 1$ in y_1, we may look instead at the polynomial

$$g(x + y, y_2, ..., y_l) - g(x, y_2, ..., y_l) - g(y, y_2, ..., y_l).$$

This is a polynomial in $l + 1$ variables with degree in x and y lower than the degree of g in y_1. Continuing in this way we obtain a multilinear polynomial.

Now (i) allows us to reduce to PIs homogeneous in all variables, the set $\{h_l\}$ is then obtained via multilinearization. The conditions on characteristic come into play in (ii) when we try to show that f follows from the set of multilinear polynomials $\{h_l\}$. This cannot be avoided: the nonlinear identity

$$xy^3 + yxy^2 + y^2xy + y^3x + xy^2 + y^2x$$

is a PI for $M_2(Z_2)$ that does not follow from any multilinear identity for $M_2(Z_2)$ ([AL]). However, aside from these special cases in prime characteristic, the set of PIs for A is determined by the set of multilinear PIs for A, and we may restrict our attention to the latter.

In the original proof of Theorem 1.1.1, Amitsur and Levitzki used an induction based on canonical matrix units. Our approach will be different, we will instead rely on trace polynomials.

A *trace polynomial* is a polynomial in which we allow the coefficients to be polynomials in traces, and with this in mind the notion of a *trace-PI* for $M_n(F)$ is clear. A trace polynomial is *pure-trace* if none of its variables appear outside of traces.

Example 1.1.1. Let $char(F) = 0$. The characteristic polynomial (and any linearization) is a trace polynomial (as long as $char(F) = 0$, all coefficients may be expressed as polynomials in traces); it is also a trace-PI (for $M_n(F)$), but not a PI.

We would like to be able to say, without ambiguity, that one PI (or trace-PI) is a *consequence* of another PI (or trace-PI). Define a *T-ideal* of $F\langle x_1, \cdots, x_k\rangle$ to be an ideal of $F\langle x_1, \cdots, x_k\rangle$ that is invariant under all endomorphisms of $F\langle x_1, \cdots, x_k\rangle$ (precisely: all maps of the form $x_i \mapsto w_i \in F\langle x_1, \cdots, x_k\rangle$, $i = 1, \cdots, k$). If $f \in F\langle x_1, \cdots, x_k\rangle$, then the *T-ideal generated by* f is the smallest T-ideal containing f. If f and g are both PI's for a given algebra A, we say that g is a *consequence* of f if g lies in the T-ideal generated by f. This notion extends to trace-PIs. In this case our T-ideal is an ideal of trace polynomials that is closed under both substitutions and application of the trace.

Razmyslov [Raz] and Procesi [Pr1], independently, have shown that all trace PIs (and hence all PIs, as we will see) for M_n follow from (are consequences of) the characteristic polynomial. Razmyslov [Raz] also gave an alternate proof of the positive statement in the Amitsur-Levitzki Theorem showing explicitly that s_{2n} is a consequence of Cayley-Hamilton.

We will now change our focus somewhat and introduce an involution of the first kind $*$. Recall that an *involution* $*$ on an algebra (or more generally, on a ring) is an anti-automorphism of degree 2. We say that an involution $*$ on an algebra A is *of the first kind* if it fixes the centre of A elementwise (e.g. matrix transpose) and *of the second kind* if it acts as a non-trivial involution on the centre (e.g. $X \mapsto \bar{X}^t$ for matrices over the complexes with $\bar{\ }$ the complex conjugate). Unless explicitly mentioned otherwise, we henceforth assume all involutions to be of the first kind. We define an *algebra with involution* $(A, *)$ to be an associative algebra A over a field F together with an involution, $*$.

We may also attach an involution to $F\langle x_1, \cdots, x_k\rangle$. There are several ways to do this. For example, one may define $x_i^* = x_i$ and obtain the so-called *reversal involution*; the anti-automorphic property is in this case a "reversal" of monomials:

$$(x_{i_1} x_{i_2} \cdots x_{i_j})^* = x_{i_j} \cdots x_{i_2} x_{i_1}.$$

We instead proceed by simply extending the set of generators to obtain

$$F\langle x_1, \cdots, x_k, x_1^*, \cdots, x_k^*\rangle$$

and call elements of this extended free algebra *$*$-polynomials.* A *$*$-T-ideal* is an ideal of $F\langle x_1, \cdots, x_k, x_1^*, \cdots, x_k^*\rangle$ that is invariant under all endomorphisms of (the algebra with involution)

$$(F\langle x_1, \cdots, x_k, x_1^*, \cdots, x_k^*\rangle, *).$$

Definition 1.1.1. Let $(A, *)$ be an algebra with involution and let

$$f \in F\langle x_1, \cdots, x_k, x_1^*, \cdots, x_k^*\rangle.$$

f is $*$-*PI* for $(A, *)$ if it vanishes for all specializations

$$x_i \mapsto X_i \in A, \quad x_i^* \mapsto X_i^* \in A$$

of the variables to elements of A.

The definition of a $*$-trace-PI follows immediately.

Assuming $char(F) \neq 2$, we have

$$x = \frac{1}{2}(x + x^*) + \frac{1}{2}(x - x^*) \tag{1}$$

and

$$x^* = \frac{1}{2}(x + x^*) - \frac{1}{2}(x - x^*). \tag{2}$$

Thus we may think of an arbitrary $*$-polynomial

$$f \in F\langle x_1, x_2, \cdots, x_k, x_1^*, x_2^*, \cdots, x_k^*\rangle$$

as a polynomial in skew and symmetric variables and write it as

$$f(x_1 + x_1^*, \cdots, x_k + x_k^*, x_1 - x_1^*, \cdots, x_k - x_k^*).$$

We may therefore consider polynomials $f(x_1, ..., x_r, y_{r+1}, ..., y_s)$ with x_i a "skew - symmetric" variable for $1 \leq i \leq r$, and y_j a "symmetric" variable for $r < j \leq s$; i.e., substitutions are made according to

$$x_i \mapsto a_i \in \{a \in A \mid a = -a^*\} \tag{3}$$

and

$$y_j \mapsto b_j \in \{b \in A \mid b = b^*\}. \tag{4}$$

As an example of how (1) and (2) are used in practice we note that (with these equations in hand and if $char(F) \neq 2$) the study of $*$-PIs for $M_n(F)$ in strictly symmetric variables is equivalent to the study of PIs for

$$H_n(F, t) = \{X \in M_n \mid X = X^t\}.$$

For $(M_n, *)$ there are two main cases: $* = t$, the usual matrix transpose, and $* = s$, the symplectic involution, the latter defined only when $n = 2m$ by

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}^s = \begin{pmatrix} D^t & -B^t \\ -C^t & A^t \end{pmatrix}$$

where the matrix has been partitioned into $m \times m$ blocks. Our primary interest is in the spaces $K_n(F,t)$ ($n \times n$ matrices skew with respect to the transpose) and $H_{2m}(F,s)$ ($2m \times 2m$ matrices symmetric with respect to the symplectic involution), but since

$$X = X^* \Leftrightarrow X = -X^*$$

in an arbitrary algebra with involution over a field of characteristic 2, we cannot (in characteristic 2) use (3) and (4) to define these spaces. We handle this difficulty in the following way. For $char(F) \neq 2$ we use the definitions (3) and (4) for $K_n(F,t)$ and $H_{2m}(F,s)$, respectively, and these have vector space bases

$$\{e_{i,j} - e_{j,i} : 1 \leq i < j \leq n\} \tag{5}$$

and

$$\begin{gathered}\{e_{i,j} + e_{j+m,i+m} : 1 \leq i,j \leq m\} \cup \{e_{k+m,l} - e_{l+m,k} : 1 \leq l < k \leq m\} \cup \\ \{e_{k,l+m} - e_{l,k+m} : 1 \leq l < k \leq m\},\end{gathered} \tag{6}$$

respectively. In characteristic 2 we take

$$K_n(F,t) := \{X - X^t \mid X \in M_n\}.$$

This ensures that (in characteristic 2) elements of $K_n(F,t)$ have zeros along the diagonal, i.e., that

$$dim(K_n(F,t)) = \frac{n(n-1)}{2}.$$

We also note that this definition coincides with (3) when $char(F) \neq 2$. We define the symplectic involution's symmetric space with

$$H_{2m}(F,s) := \{X + X^s \mid X \in M_{2m}\}.$$

This ensures that in characteristic 2 elements of $H_{2m}(F,s)$ have upper right and lower left blocks that are $m \times m$ matrices with zeros along the diagonal, i.e., that (using (6))

$$dim(H_{2m}(F,s)) = m^2 + 2(\frac{m(m-1)}{2}) = 2m^2 - m = \frac{2m(2m-1)}{2}.$$

This definition coincides with (4) when $char(F) \neq 2$. Finally, we note that (5) and (6) are bases for $K_n(F,t)$ and $H_{2m}(F,s)$ irrespective of characteristic since the constituent matrices have only the entries 0 and ±1.

To a great extent it suffices to consider $* = t$ and $* = s$. We would like to make this statement more precise, and to do so we look at a more general setting. Consider a central simple algebra A with involution $*$ over a field F where (for the moment) $F = \bar{F}$, the algebraic closure of F. The involution $*$ on A is the adjoint mapping of some symmetric or skew-symmetric bilinear form, and this form is unique up to scalar multiple (see [KMRT, Theorem 1.4.2]). If the form is symmetric we say that $*$ is of *orthogonal* type, and if the form is skew-symmetric we say that $*$ is of *symplectic* type.

Now the matrix transpose t on M_n is of orthogonal type, the symplectic involution s on M_n is of symplectic type, and returning to $(A, *)$ over (now) an arbitrary field F we have the following: up to isomorphism, $A \otimes \bar{F}$ is either $(M_n(\bar{F}), t)$ or $(M_n(\bar{F}), s)$, where in the latter case n must be even ([KMRT, Proposition 1.2.1]). Thus over an algebraically closed field it is enough to consider (M_n, t) and (M_n, s).

This allows us to narrow our focus, but at the cost of certain restrictions on our underlying field F; however, our main results (existence of certain PIs for $K_n(F, t)$ and/or $H_{2m}(F, s)$) will be valid over arbitrary fields. We achieve this by first proving a given PI in characteristic 0 and then concluding that it holds for any substitution from the basis (5) (resp. (6)) (which is a basis for $K_n(F, t)$ (resp. $H_{2m}(F, s)$) in arbitrary characteristic). Finally we note that as our PI is multilinear and has integer coefficients it must hold for $K_n(F, t)$ (resp. $H_{2m}(F, s)$) over an arbitrary field F.

We now list three lemmas. The first two will be useful later when we look closely at trace-polynomials, the third collects some essential facts concerning the standard polynomial.

Lemma 1.1.1. *Let $f(x_1, \cdots, x_k) \in F\langle x_1, \cdots, x_k\rangle$, F a field. f is a PI for $M_n(F)$ if and only if*

$$tr(f(x_1, \cdots, x_k)x_{k+1})$$

is a pure-trace PI for $M_n(F)$.

Proof. $tr(AB)$ is a non-degenerate bilinear form on $M_n(F)$. $\square$

So to study PIs for M_n we may instead study pure-trace PIs for M_n.

Now identify M_n with $V \otimes V^\star$, where V is an n-dimensional F-vector space and $V^\star$ is the dual of V. We say a matrix $X_\imath$ is *decomposable* if it may be written $X_\imath = u_\imath \otimes \phi_\imath$, $u_\imath \in V$, $\phi_\imath \in V^\star$. For example, with $V = F^n$ the canonical matrix units $e_{\imath\jmath}$ are decomposable

$$e_{\imath\jmath} = e_\imath \otimes e_\jmath^\star$$

where $e_\imath$ and $e_\jmath^\star$ are canonical basis vectors (for V and $V^\star$, respectively) in the obvious way.

Lemma 1.1.2 ([Pr2, p197]). *Let* $X_1 = u_1 \otimes \phi_1, X_2 = u_2 \otimes \phi_2, \cdots, X_l = u_l \otimes \phi_l$ *be decomposable matrices. Then*

$$tr(X_{i_1}X_{i_2}\cdots X_{i_r})tr(X_{j_1}X_{j_2}\cdots X_{j_s})\cdots tr(X_{k_1}X_{k_2}\cdots X_{k_t}) = \prod_{i=1}^{l} \phi_i(u_{\sigma(i)}) \qquad (7)$$

where $\sigma \in \mathcal{S}_l$ *with a decomposition into disjoint cycles*

$$\sigma = (i_1 \ \cdots \ i_r)(j_1 \ \cdots \ j_s)\cdots(k_1 \ \cdots \ k_t).$$

$\square$

This lemma may be proved by direct calculation, and it suffices to consider $V = F^n$. One begins by noting that

$$(u \otimes \phi).v = u.[\phi]^t.v = \phi(v)u$$

where $[\phi]$ is ϕ expressed as a (column) vector, and thus

$$tr(u \otimes \phi) = \phi(u).$$

Then

$$(u_1 \otimes \phi_1).(u_2 \otimes \phi_2).v = (u_1 \otimes \phi_1).(\phi_2(v)u_2) = \phi_1(u_2)\phi_2(v)u_1$$

shows that

$$tr((u_1 \otimes \phi_1).(u_2 \otimes \phi_2)) = \phi_1(u_2)\phi_2(u_1),$$

and in general

$$tr((u_1 \otimes \phi_1)\cdots(u_k \otimes \phi_k)) = \phi_1(u_2)\cdots\phi_{k-1}(u_k)\phi_k(u_1).$$

One can then extend to products of arbitrary multilinear traces to complete the proof.

The canonical matrix units e_{ij} are decomposable and the left hand side of (7) is multilinear, thus Lemma 1.1.2 shows that we may equate the study of multilinear pure-trace PIs for M_n with the study of polynomial relations between the elements $\phi_i(u_j)$, $u_j \in V$, $\phi_i \in V^\star$. We return to this in Section 1.2.

Lemma 1.1.3. *Let $l > 1$ be a positive integer. The following is true over an arbitrary field F.*

a) s_{2l} in arbitary matrix variables is traceless.

b) With respect to $* = t$, both s_{4l} and s_{4l-1} in skew variables are symmetric.

c) With respect to $* = t$, both s_{4l-2} and s_{4l-3} in skew variables are skew-symmetric.

d) With respect to $* = t$ or $* = s$, s_{4l-2} in symmetric variables is skew-symmetric.

$\square$

Lemma 1.1.3 is easily proved by direct computation. For example in b) with skew variables $x_i^t = -x_i$:

$$\begin{aligned}
(s_{4l-1})^t &= \sum_{\sigma \in \mathcal{S}_{4l-1}} (-1)^\sigma (x_{\sigma(1)} \cdots x_{\sigma(4l-1)})^t \\
&= \sum_{\sigma \in \mathcal{S}_{4l-1}} (-1)^\sigma x^t_{\sigma(4l-1)} \cdots x^t_{\sigma(1)} \\
&= (-1)^{4l-1} \sum_{\sigma \in \mathcal{S}_{4l-1}} (-1)^\sigma x_{\sigma(4l-1)} \cdots x_{\sigma(1)} \\
&= - \sum_{\sigma \in \mathcal{S}_{4l-1}} (-1)^\sigma x_{\sigma(4l-1)} \cdots x_{\sigma(1)}
\end{aligned}$$

and since the permutation

$$1 \leftrightarrow 4l-1,\ 2 \leftrightarrow 4l-2, \cdots,\ 2l-1 \leftrightarrow 2l+1$$

is odd (odd number of transpositions) we arrive at

$$\sum_{\sigma \in \mathcal{S}_{4l-1}} (-1)^\sigma x_{\sigma(1)} \cdots x_{\sigma(4l-1)} = s_{4l-1}.$$

This shows that (with respect to the transpose) s_{4l-1} in skew variables is symmetric.

From here on we split our investigation in two: the $* = t$ case and the $* = s$ case. We will begin with the transpose case, but first some notation.

Notation

We would like to take alternating sums and to do so we introduce:

Definition 1.1.2. Let n be a positive integer. Define a map

$$\begin{aligned} \phi : F\langle x_1, \dots, x_n, y\rangle &\to F\langle x_1, \dots, x_n, y\rangle \\ f(x_1, \dots, x_n, y) &\mapsto \sum_{\sigma \in \mathcal{S}_n} (-1)^\sigma f(x_{\sigma(1)}, \dots, x_{\sigma(n)}, y). \end{aligned}$$

Definition 2 extends to trace-polynomials, $*$-polynomials, and $*$-trace-polynomials in the obvious way.

Example 1.1.2. Under this notation we have:

$$s_k(x_1, x_2, \cdots, x_k) = \phi(x_1 x_2 \cdots x_{k-1} x_k).$$

Notice that y need not appear in the polynomial to which we apply ϕ.

Next, we introduce a convenient notation for polynomials that are *almost* standard, that is, alternating in all but one variable.

Definition 1.1.3. For $i = 1, 2, \dots, n$, let

$$A_i^n := \phi(x_1 \cdots x_{i-1} y x_i \cdots x_{n-1}) = \sum_{\sigma \in \mathcal{S}_{n-1}} (-1)^\sigma x_{\sigma(1)} \dots x_{\sigma(i-1)} y x_{\sigma(i)} \dots x_{\sigma(n-1)},$$

the alternating sum of all monomials of degree n having y in position i. We generalize and define $A^n_{i_1 i_2 \cdots i_k}$: the alternating sum of all monomials of degree n that have a y in positions $i_1, i_2, \cdots, i_k$.

Example 1.1.3. Using this notation we have:

$$\begin{aligned} s_{2n}(y, x_1, \dots, x_{2n-1}) &= -\sum_{\sigma \in \mathcal{S}_{2n}} (-1)^\sigma x_{\sigma(1)} x_{\sigma(2)} \dots x_{\sigma(2n)} |_{x_{2n}=y} \\ &= A_1^{2n} - A_2^{2n} + A_3^{2n} - A_4^{2n} + \dots . + A_{2n-1}^{2n} - A_{2n}^{2n} \\ &= \sum_{i=1}^{2n} (-1)^{i+1} A_i^{2n}. \end{aligned}$$

1.2. Introduction: The Transpose Case

For this section we assume $* = t$. We consider first $*$-PIs for (M_n, t) in which all variables are symmetric, or equivalently, PIs for $H_n(F,t)$. Of course, by Amitsur-Levitzki, s_{2n} is a PI for $H_n(F,t)$, but more can be said. It has been found [SL] that the minimal degree of a PI for $H_n(F,t)$ is $2n$ (the same as for a PI for M_n), and further, Ma and Racine have given a full characterization of identities at this degree:

Theorem 1.2.1 ([MR]). *If $char(F)$ does not divide $2[(n+1)/2]$ and $|F| > 2n$, then for $n \neq 3$ all identities for $H_n(F,t)$ of degree $2n$ are consequences of*

$$\begin{aligned} E_{2n}(x_1, \ldots, x_{2n-1}; y) :&= A_1^{2n} - A_2^{2n} + A_5^{2n} - A_6^{2n} + \cdots \\ &= \sum_{\substack{i=1,2(mod\ 4) \\ 1 \leq i \leq 2n}}^{2n} (-1)^{i-1} A_i^{2n}. \end{aligned}$$

We note here that the authors have achieved for $H_n(F,t)$ all that Amitsur and Levitzki achieved for M_n, namely: existence (E_{2n}), minimality (degree $2n$), and uniqueness (all identities are consequences of E_{2n}).

For the case of strictly skew variables (PIs for $K_n(F,t)$) an answer as complete as Amitsur-Levitzki has not yet been found. Significant progress was made by Kostant in a theorem that bears his name

Theorem 1.2.2 ([Kos1]). *Let F be a field. s_{2n-2} is a PI for $K_n(F,t)$ if n is even.*

Next, Rowen strengthened this result and proved sharpness

Theorem 1.2.3 ([Row1]). *Let F be a field. s_{2n-2} is a PI for $K_n(F,t)$ if n is odd. If $char(F) = 0$ and $k < 2n-2$, s_k is not a PI for $K_n(F,t)$, n arbitrary.*

Further sharpness results are found in [Row1] for the prime characteristic case. For example, in characteristic 2, it was found that s_k is a PI for $K_n(F,t)$ if and only if $k \geq n$. For another reference on sharpness in this context, see [Be].

In [Row1] Rowen also obtains an alternate proof of Kostant's theorem (as a corollary to Theorem 1.2.3) and the following concerning standard identities in mixed variables:

Theorem 1.2.4 ([Row1]). *Let F be a field. For all n, s_{2n-1} in one symmetric variable and $2n-2$ skew variables is a $*$-PI for (M_n, t). For n odd, s_{2n-2} in one symmetric variable and $2n-3$ skew variables is a $*$-PI for (M_n, t).*

Finally, Racine and D'Amour have described all $*$-PI's of minimal degree for $(M_n, t), n < 5$:

Theorem 1.2.5 ([DR1]). *Let $|F| > 2$, and put*

$$\{x\ y\ z\} := xyz + zyx, \quad xD_{y,z} := \{x\ y\ z\} - \{x\ z\ y\}, \quad x \circ y := xy + yx.$$

Any $*$-PI of minimal degree (2) for (M_2, t) is a consequence of $s_2(x_1 - x_1^t, x_2 - x_2^t)$. Any $*$-PI for (M_3, t) of minimal degree (4) is a consequence of

$$p(x_1 - x_1^*, x_2 - x_2^*, x_3 - x_3^*, x_4 - x_4^*),\ q(x_1 - x_1^*, x_2 - x_2^*, x_3 - x_3^*, x_4 - x_4^*),$$
$$r(x_1 - x_1^*, x_2 - x_2^*, x_3 - x_3^*, x_4 \pm x_4^*),\ g(x_1 - x_1^*, x_2 - x_2^*, x_3 - x_3^*, x_4 + x_4^*),$$

where

$$\begin{aligned}
p(x_1, x_2, x_3, x_4) &= \sum_{(1\,2\,3)} \{x_1\ [x_2, x_4]\ x_3\}, \\
q(x_1, x_2, x_3, x_4) &= \sum_{(1\,2\,3)} \{x_1\ [x_2, x_3]\ x_4\} + \sum_{(1\,2\,4)} \{x_1\ [x_2, x_4]\ x_3\}, \\
&\quad + 2([x_1, x_3]D_{x_2,x_4} + [x_1, x_4]D_{x_2,x_3} \\
&\quad - [x_2, x_3]D_{x_1,x_4} - [x_2, x_4]Dx_1, x_3), \\
r(x_1, x_2, x_3, x_4) &= [s_3(x_1, x_2, x_3), x_4], \\
g(x_1, x_2, x_3, x_4) &= \sum_{\sigma \in S_2} (-1)^\sigma \{x_{\sigma(1)}\ x_{\sigma(2)}\ (x_3 \circ x_4)\}.
\end{aligned}$$

Any $*$-PI of minimal degree (5) for (M_4, t) is a consequence of

$$\kappa(x_1 - x_1^*, x_2 - x_2^*, x_3 - x_3^*, y - y^*)$$

where

$$\kappa(x_1, x_2, x_3, y) = s_4(x_1, x_2, x_3, y^2) - y \circ s_4(x_1, x_2, x_3, y).$$

With Amitsur and Levitzki's template in mind, we see that though we have various existence results for $*$-PIs for M_n, the uniqueness and minimality results are still lacking for $n > 4$. What we do know is that s_{2n-2} is of minimal degree among *standard* $*$-PIs for M_n in skew variables, and that E_{2n} is of minimal degree among $*$-PIs for M_n in symmetric variables. Thus it appears as though some of the $*$-PIs of minimal degree will have all variables skew, and for this reason we focus on PIs for $K_n(F, t)$.

We are now ready to investigate the transpose case, and our main tool will be a certain set of $*$-trace-polynomials discovered by Procesi.

Procesi's Polynomials

Recall that Razmyslov and Procesi, independently, proved that all trace PIs for M_n follow from (lie in the T-ideal generated by) the characteristic polynomial. Procesi [Pr2] later took this further, obtaining an analogous result for the case of $*$-PIs. For the case $* = t$ Procesi obtained:

Theorem 1.2.6 ([Pr2]). *Let F be a field of characteristic 0. All $*$-PI's for $(M_n(F),t)$ are consequences of (lie in the $*$-T-ideal generated by) a set of $n+1$ $*$-trace-PI's of degree n.*

In this section we outline the proof of this result. Though explicit formulae for the $n+1$ $*$-trace-PI's are not given in Procesi's proof, it nonetheless serves as a starting point for our computations.

We would like to study $*$-identities, that is, we would like to allow variables that appear with a $*$. Procesi achieves this by modifying Lemma 1.1.2, in particular: the equation

$$tr(X_{i_1}X_{i_2}\cdots X_{i_r})tr(X_{j_1}X_{j_2}\cdots X_{j_s})\cdots tr(X_{k_1}X_{k_2}\cdots X_{k_t}) = \prod_{i=1}^{l}\phi_i(u_{\sigma(i)}). \quad (1.1.7)$$

First, if $M_n(F)$ is equipped with the transpose involution, then there exists a non-degenerate symmetric bilinear form on V, say $\langle\cdot\,,\cdot\rangle$, and the transpose is the involution of adjunction with respect to this form:

$$\langle AX, Y\rangle = \langle X, A^tY\rangle.$$

The form $\langle\cdot\,,\cdot\rangle$ is defined only up to non-zero scalar multiple, but we can normalize it by picking a basis B for V, and imposing a projection on V. Let

$$(\pi_v =)\ \pi : V \to V$$

be a projection onto $v \in B$, defined by

$$\pi(w) = \langle w, v\rangle v \ \ \textit{and} \ \ \pi^2 = \pi,$$

where $w \in V$. We have both

$$\pi(\pi(w)) = \pi(w) = \langle w, v\rangle v$$

and

$$\pi(\pi(w)) = \pi(\langle w, v\rangle v) = \langle w, v\rangle\langle v, v\rangle v.$$

By non-degeneracy of $\langle\cdot,\cdot\rangle$ there exists $w \in V$ such that $\langle w, v\rangle \neq 0$. So to satisfy $\pi^2 = \pi$ we need $\langle v, v\rangle = 1$. If $\langle v, v\rangle \neq 1$, we normalize by replacing $\langle\cdot,\cdot\rangle$ with $\frac{1}{\langle v,v\rangle}\langle\cdot,\cdot\rangle$ ($\langle v, v\rangle$ is non-zero since $v \neq 0$).

V is canonically isomorphic to its dual $V^\star$ via

$$\begin{aligned} V &\cong V^\star \\ v &\mapsto \langle\cdot\,, v\rangle. \end{aligned}$$

With this in mind we take $M_n(F) \cong V \otimes V^\star \cong V \otimes V$ and thus if $V = F^n$, $e_i \otimes e_j^\star$ becomes $e_i \otimes e_j$ (or the matrix e_{ij}) and $\phi_i(u_j)$ in (1.1.7) becomes $\langle u_j, v_i \rangle$.

Now, if we replace a variable in the left hand side of (1.1.7) with its transpose, say $X_i \mapsto {X_i}^t$, then since

$$(e_i \otimes e_j)^t = {e_{ij}}^t = e_{ji} = e_j \otimes e_i$$

says (by linearity) that

$${X_i}^t = (u_i \otimes v_i)^t = v_i \otimes u_i, \;\; u_i, v_i \in V$$

we see that u_i and v_i are transposed in the right hand side of (1.1.7). This yields the following modified version of Lemma 1.1.2:

Proposition 1.2.1 ([Pr2, p197-198]). *Let* $X_1 = u_1 \otimes v_1, X_2 = u_2 \otimes v_2, \cdots, X_l = u_l \otimes v_l$ *be decomposable matrices. There exists a one-to-one correspondence (equality) between the degree* l *multilinear expressions*

$$tr(U_{i_1} U_{i_2} \cdots U_{i_r}) tr(U_{j_1} U_{j_2} \cdots U_{j_s}) \cdots tr(U_{k_1} U_{k_2} \cdots U_{k_t})$$

where $U_i = X_i$ *or* X_i^t, *and* $\sigma \in \mathcal{S}_l$ *with a cyclic decomposition*

$$\sigma = (i_1, \cdots, i_r)(j_1, \cdots, j_s) \cdots (k_1, \cdots, k_t)$$

and multilinear (linear in all u_i*'s and all* v_j*'s) products of* l *scalars, each scalar being of the form* $\langle u_i, v_j \rangle, \langle u_i, v_i \rangle, \langle u_i, u_j \rangle$, *or* $\langle v_i, v_j \rangle$, $1 \le i \ne j \le l$. □

It *is* possible to make this more explicit, but for the moment we merely note that this proposition allows us to equate the study of multilinear $*$-trace-identies to the study of multilinear polynomial relations between the scalars $\langle u_i, v_j \rangle, \langle u_i, v_i \rangle, \langle u_i, u_j \rangle$, and $\langle v_i, v_j \rangle$, $1 \le i \ne j \le l$.

Procesi's next step is to take $n+1$ arbitrary u_i's and $n+1$ arbitrary v_j's (all viewed as row vectors in V) and to form the product of the two matrices (size $(2n+2) \times n$ and $n \times (2n+2)$, respectively)

$$\begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_{n+1} \\ v_1 \\ v_2 \\ \vdots \\ v_{n+1} \end{pmatrix} \quad \textit{and} \quad \begin{pmatrix} u_1^t & u_2^t & \cdots & u_{n+1}^t & v_1^t & v_2^t & \cdots & v_{n+1}^t \end{pmatrix}.$$

This is the $(2n+2) \times (2n+2)$ symmetric matrix

$$\begin{pmatrix} \langle u_1,u_1\rangle & \langle u_1,u_2\rangle & \cdots & \langle u_1,u_{n+1}\rangle & \langle u_1,v_1\rangle & \langle u_1,v_2\rangle & \cdots & \langle u_1,v_{n+1}\rangle \\ \langle u_2,u_1\rangle & \langle u_2,u_2\rangle & \cdots & \langle u_2,u_{n+1}\rangle & \langle u_2,v_1\rangle & \langle u_2,v_2\rangle & \cdots & \langle u_2,v_{n+1}\rangle \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ \langle v_{n+1},u_1\rangle & \langle v_{n+1},u_2\rangle & \cdots & \langle v_{n+1},u_{n+1}\rangle & \langle v_{n+1},v_1\rangle & \langle v_{n+1},v_2\rangle & \cdots & \langle v_{n+1},v_{n+1}\rangle \end{pmatrix}$$

where, since $rank(AB) \leq min\{rank(A), rank(B)\}$, we have rank $\leq n$. Call this last matrix $Y = (y_{ij})$, $1 \leq i, j \leq 2n+2$. It is clear that all $n+1 \times n+1$ minors of Y are zero, consider those that are multilinear, i.e., those in which we choose $n+1$ distinct rows with indices

$$\{r_1, \cdots, r_{n+1}\}, \ 1 \leq r_1 < r_2 < \cdots < r_{n+1} \leq 2n+2$$

and $n+1$ distinct columns with indices

$$\{c_1, \cdots, c_{n+1}\}, \ 1 \leq c_1 < c_2 < \cdots < c_{n+1} \leq 2n+2$$

such that

$$\{r_1, \cdots, r_{n+1}\} \cap \{c_1, \cdots, c_{n+1}\} = \emptyset.$$

These correspond, via the correspondence of Proposition 1.2.1, to $*$-trace PIs for M_n. In fact, Procesi is able to prove ([Pr2, p198]) that all $*$-trace identities for (M_n, t) arise in this way, and that the set of all $*$-trace identities for (M_n, t) follow from a certain set of $n+1$ multilinear minors of Y. Procesi gives these $n+1$ minors explicitly, but not the corresponding $n+1$ $*$-trace-PIs.

The degree n characteristic polynomial is a $*$-trace-PI for (M_n, t), thus it must follow from Procesi's $n+1$ minors. As an example we now investigate this connection.

Example 1.2.1 (Characteristic Polynomial). Consider the minor obtained by taking the $n+1$ rows of $Y = (y_{ij})$ with indices

$$1, 2, 3, \cdots n, n+1$$

and the $n+1$ columns with indices

$$n+2, n+3, n+4, \cdots, 2n+2.$$

Using the determinant formula for $n+1 \times n+1$ matrix $A = (a_{ij})$

$$det(A) = \sum_{\sigma \in S_{n+1}} (-1)^\sigma \prod_{i=1}^{n+1} a_{i\,\sigma(i)}$$

we get the polynomial (in commuting variables y_{ij})

$$\sum_{\sigma\in\mathcal{S}_{n+1}}(-1)^{\sigma}y_{1\sigma(n+2)}y_{2\sigma(n+3)}\cdots y_{(n)\,\sigma(2n+1)}y_{n+1\,\sigma(2n+2)}. \tag{1}$$

We will need some definitions and a lemma before we continue.

Definition 1.2.1. Let $\tau\in\mathcal{S}_{2m+1}$, and let $(i_1\ i_2\ \cdots\ i_r)(j_1\ j_2\ \cdots j_l)\cdots(k_1\ k_2\ \cdots\ k_s)$ be τ in terms of disjoint cycles. Define:

$$T_\tau := tr(X_{i_1}X_{i_2}\cdots X_{i_r})tr(X_{j_1}X_{j_2}\cdots X_{j_l})\cdots tr(X_{k_1}X_{k_2}\cdots X_{k_s}).$$

Definition 1.2.2. Let i be a positive integer. We denote by

$$\bar{i}\in\{1,2,\cdots,2m+1\}$$

the representative of the class of i modulo $2m+1$.

Following Procesi we use the formulae:

$$\begin{aligned}&P1.\quad u\langle v,w\rangle=(u\otimes v).w\\&P2.\quad \langle u,v\rangle=tr(u\otimes v)\\&P3.\quad u_i\otimes v_i=X_i.\end{aligned}$$

The first is an easy calculation, the second follows from Lemma 1.1.2, and the third is merely notation. We now reformulate Lemma 1.1.2:

Lemma 1.2.1. *Consider*

$$y_\gamma=y_{1\gamma(2m+2)}y_{2\gamma(2m+3)}\cdots y_{(2m+1)\gamma(4m+2)},$$

γ a permutation of $\{2m+2,2m+3,\cdots,4m+2\}$. This term is precisely $T_{\bar\gamma^{-1}}$, where $\bar\gamma^{-1}$ is the permutation of $\{1,2,\cdots,2m+1\}$ defined by $\bar\gamma^{-1}(i)=\overline{\gamma^{-1}(i+2m+1)}$.

Proof. We use

$$\bar\gamma(i)=\overline{\gamma(i+2m+1)},\ \ 1\le i\le 2m+1$$

and

$$y_{ij}=\langle u_i,v_{\bar j}\rangle,$$

$$1\le i\le 2m+1,\ \ 2m+2\le j\le 4m+2,$$

and compute

$$y_\gamma = y_{1\gamma(2m+2)} y_{2\gamma(2m+3)} \cdots y_{(2m+1)\gamma(4m+2)} = \prod_{i=1}^{2m+1} \langle u_i, v_{\overline{\gamma}(i)} \rangle$$
$$= \prod_{i=1}^{2m+1} \langle u_{\overline{\gamma}^{-1}(i)}, v_i \rangle.$$

Under the identification $V \cong V^\star$ $(v_i \mapsto \langle \cdot, v_i \rangle =: \phi_i)$ this is

$$\prod_{i=1}^{2m+1} \phi_i(u_{\overline{\gamma}^{-1}(i)})$$

and now applying Lemma 1.1.2 we obtain $T_{\overline{\gamma}^{-1}}$. □

In view of Lemma 1.2.1 our minor (1) becomes

$$\sum_{\sigma \in \mathcal{S}_{n+1}} (-1)^\sigma T_{\overline{\sigma}^{-1}}.$$

For $n = 2$ the symmetric group is

$$\mathcal{S}_3 = \{(1)(2)(3), (12)(3), (13)(2), (1)(23), (123), (132)\}$$

and this yields

$$\begin{aligned}
&tr(X_1)tr(X_2)tr(X_3) - tr(X_1X_2)tr(X_3) - tr(X_1X_3)tr(X_2)\\
&\quad - tr(X_1)tr(X_2X_3) + tr(X_1X_3X_2) + tr(X_1X_2X_3)\\
&= tr(tr(X_1)tr(X_2)X_3) - tr(tr(X_1X_2)X_3) - tr(tr(X_2)X_1X_3)\\
&\quad - tr(tr(X_1)X_2X_3) + tr(X_2X_1X_3) + tr(X_1X_2X_3)\\
&= tr((tr(X_1)tr(X_2) - tr(X_1X_2) - tr(X_2)X_1 - tr(X_1)X_2 + X_2X_1 + X_1X_2)X_3).
\end{aligned}$$

Notice that none of the variables appear with a transpose; for a different choice of minor this will not be the case. Applying Lemma 1.1.1 we see that

$$tr(X_1)tr(X_2) - tr(X_1X_2) - tr(X_2)X_1 - tr(X_1)X_2 + X_2X_1 + X_1X_2$$

is a trace-PI for M_n. Now equating $X_1 = X_2 = X$ we obtain

$$2X^2 - 2tr(X)X + tr(X)^2 - tr(X^2) = 2(X^2 - tr(X)X + det(X))$$

(twice the characteristic polynomial) where we have used the formula for 2×2 matrices $det(X) = \frac{1}{2}(tr(X)^2 - tr(X^2))$. Thus the 2×2 characteristic polynomial follows from the upper right 3×3 minor of Y. We omit the calculation, but this holds in general: the $n \times n$ characteristic polynomial follows from the upper right $n+1 \times n+1$ minor of Y. This concludes our example.

Our main interest is in a slight variation of the minor just computed: the minor obtained by taking the $n+1$ rows with indices

$$1, 2, 3, \cdots 2m, 2m+2$$

of Y and the $n+1$ columns with indices

$$2m+1, 2m+3, 2m+4, \cdots, 4m+2$$

of Y when $n = 2m$. This is precisely

$$\sum_{\sigma \in \mathcal{S}_{2m+1}} (-1)^{\sigma} y_{1\sigma(2m+1)} y_{2\sigma(2m+3)} y_{3\,\sigma(2m+4)} \quad y_{(2m-1)\,\sigma(4m)} y_{2m\,\sigma(4m+1)} y_{(2m+2)\sigma(4m+2)} \cdot \tag{2}$$

This is the starting point for our proofs in Chapter 2.

1.3. Introduction: The Symplectic Case

For this section we assume $* = s$, the symplectic involution. We will primarily be interested in $*$-PIs for M_{2m} in which all variables are symmetric, or, equivalently, PIs for $H_{2m}(F, s)$. First, Amitsur-Levitzki says that s_{4m} is an identity for $H_{2m}(F, s)$, but Rowen has improved on this result with:

Theorem 1.3.1 ([Row3]). *Let m be a positive integer. s_{4m-2} is a polynomial identity for $H_{2m}(F, s)$.*

For sharpness, it has been shown for $m = 1$ (trivial), $m = 2$ [Row3], and $m = 3, 4$ [A] that $H_{2m}(F, s)$ does *not* satisfy s_{4m-3}. The $m > 4$ case is still open.

Next the cases $m = 1, 2$ were dealt with in detail. D'Amour and Racine [DR2] obtained a full characterization of the $*$-PIs of minimal degree for (M_2, s) and (M_4, s). The minimal degrees were found to be 2 and 5, respectively. Among their results is the following, though under a milder restriction on $char(F)$ than we have stated here:

Theorem 1.3.2 ([DR2]). *Let $char(F) = 0$ and*

$$[x, y] = xy - yx, \quad x \circ y = xy + yx = xV_y = yV_x.$$

$s_2(x, y) = [x, y]$ is a PI for $H_2(F, s)$, $H_2(F, s)$ does not satisfy a PI of degree less than 2, and any PI of degree 2 is a consequence of $[x, y]$.

$$\begin{aligned} p_4(x_1, x_2, x_3, x_4, x_5) &= [[x_1, x_2] \circ [x_3, x_4] + [x_1, x_4] \circ [x_3, x_2], x_5], \\ r_5(x_1, x_2, x_3, x_4, x_5) &= (x_4 \circ x_5)s_3(V_{x_1}, V_{x_2}, V_{x_3}) - (x_4 s_3(V_{x_1}, V_{x_2}, V_{x_3})) \circ x_5 \\ &\quad - (x_5 s_3(V_{x_1}, V_{x_2}, V_{x_3})) \circ x_4. \end{aligned}$$

are PI's for $H_4(F, s)$. $H_4(F, s)$ does not satisfy a PI of degree less than 5, and any PI for $H_4(F, s)$ of degree 5 is a consequence of the set $\{p_4, r_5\}$.

The identies hold in general, but a restriction on characteristic is needed for the uniqueness results. While the proof of the $m = 1$ case is straightforward, the proof of the case $m = 2$ depends largely on the fact that $H_4(F, s)$ is a Jordan algebra of degree 2 with respect to the Pfaffian (which in this case is a quadratic form). In the same paper the mixed identities of minimal degree for $m < 3$ were also described and proven.

The final theorems we would like to recall concern the case $m = 3$. Racine interpreted $H_6(F, s)$ as a Jordan algebra of degree 3 and obtained

Theorem 1.3.3 ([Rac]). *The polynomial*

$$s_3([x^3, y], [x^2, y], [x, y])$$

is a PI for $H_6(F, s)$ of degree 9.

Then Rashkova proved minimality:

Theorem 1.3.4 ([Ras]). *There is no PI of degree less than 9 for* $H_6(F, s)$.

In her paper Rashkova also gave an alternate proof of Racine's result.

In our results for the symplectic case we will follow the method used by Rowen in his proof of Theorem 1.3.1. In this the so-called *characteristic Pfaffian* is central.

The Characteristic Pfaffian

Like the $*$-PI transpose case, in the $*$-PI symplectic case there exists an analogue of the characteristic polynomial; however, unlike the transpose case, where there are $n+1$ $*$-trace-PI generators for the $*$-T-ideal of $*$-PIs for (M_n, t), there exists for the symplectic case a single generator called the *characteristic Pfaffian* (this is an example of what Jacobson calls a *generic minimum polynomial,* see [Jac, Chapter VI]).

Theorem 1.3.5 ([Pr2]). *All* $*$*-PI's for* (M_n, s) *are consequences of (lie in the* $*$*-T-ideal generated by) the characteristic Pfaffian.*

Procesi derives this trace polyomial from Lemma 1.1.2 (in a manner similar to the transpose case handled above), but we will instead give a derivation based on [Row2].

The characteristic Pfaffian may be obtained from a square root of $det(A)$ (the Pfaffian), where $A \in K_{2m}(F, t)$, in a way analogous to the way in which the characteristic polynomial is obtained from the determinant of an arbitrary matrix.

Proposition 1.3.1 ([Row2, Proposition 2.5.8]). *For* $r \in K_{2m}(F[\lambda], t)$, λ *an indeterminate,* $det(r)$ *is a perfect square in* $F[\lambda]$.

Definition 1.3.1. For $r \in K_{2m}(F[\lambda], t)$, define Pf(r) to be the square root of det(r) satisfying Pf(b)=1 with

$$b = \begin{pmatrix} 0 & I_m \\ -I_m & 0 \end{pmatrix}.$$

The following result (which appears in Rowen's book) completes our derivation.

Theorem 1.3.6 ([Row2, Theorem 2.5.10]). *Let $X \in H_{2m}(F, s)$ and put*

$$b := \sum_{i=1}^{m} (e_{i,i+m} - e_{i+m,i}).$$

Then $(\lambda - X)b \in K_{2m}(F[\lambda], t)$ and the polynomial $p(\lambda) := Pf((\lambda - X)b) \in F[\lambda]$ has X as a root. □

Following Procesi we call $p(\lambda)$ the *characteristic Pfaffian.* One may compute this in a way analogous to the computation of the characteristic polynomial to obtain the recursive formula

$$p(\lambda) = \lambda^m + \sum_{k=1}^{m} (-1)^k \mu_k \lambda^{m-k},$$

$$\mu_0 = 1, \quad k\mu_k = \sum_{i=1}^{k} (-1)^{i-1} \mu_{k-i} T(\lambda^i), \quad T = \frac{1}{2} tr.$$

For alternate derivations of the characteristic Pfaffian (generic minimum polynomial) see [Jac, Chapter VI] and [Pr2].

2. The Transpose Case

In Section 1.2 we saw that s_{2n-2} is a PI for $K_n(F,t)$ and that, at least in characteristic 0, s_k is not a PI for $k < 2n-2$. In this chapter we will construct a family of identities for $K_n(F,t)$, n even, of degree $2n-3$ showing that, for n even at least, $2n-2$ is not the minimal degree of a $*$-PI for $(M_n(F),t)$. We will prove

Theorem 2.1. *Let F be a field, and $m > 1$ be a positive integer. Then*

$$\begin{aligned} q_m(x_1,\ldots,x_{4m-4};y) :=&(m-1)\sum_{i=1}^{m} A_{4i-3}^{4m-3} - m\sum_{i=1}^{m-1} A_{4i-1}^{4m-3} \\ =&(m-1)A_1^{4m-3} - mA_3^{4m-3} + (m-1)A_5^{4m-3} - mA_7^{4m-3} + \\ &\ldots - mA_{4m-5}^{4m-3} + (m-1)A_{4m-3}^{4m-3} \end{aligned}$$

is a polynomial identity for $K_{2m}(F,t)$ of degree $4m-3$.

Note that since a prime p cannot divide both m and $m-1$, q_m is not the zero polynomial over any field. As a corollary, q_m provides us with a new proof for Theorem 1.2.2 and for Theorem 1.2.3 (positive results):

Corollary 2.1. *Let F be a field, $n > 1$ an integer. Then s_{2n-2} is an identity for $K_n(F,t)$.*

We will also show that q_m gives a refinement of Theorem 1.2.2, that is, that s_{4m-2} is itself a sum of two PIs for $K_{2m}(F,t)$:

Corollary 2.2. *Let F be a field, $m > 1$ a positive integer. Then the polynomials*

$$\begin{aligned} E_{4m-2}(x_1,\ldots,x_{4m-3};y) :=& A_1^{4m-2} - A_2^{4m-2} + A_5^{4m-2} - A_6^{4m-2} \\ &+ \cdots + A_{4m-3}^{4m-2} - A_{4m-2}^{4m-2} \\ =& \sum_{\substack{i=1,2 (mod\ 4) \\ 1\le i\le 4m-2}} (-1)^{i-1} A_i^{4m-2} \end{aligned}$$

and

$$\begin{aligned} E'_{4m-2}(x_1,\cdots,x_{4m-3};y) :=& A_3^{4m-2} - A_4^{4m-2} + A_7^{4m-2} - A_8^{4m-2} \\ &+ \cdots + A_{4m-5}^{4m-2} - A_{4m-4}^{4m-2} \\ =& \sum_{\substack{i=0,3 (mod\ 4) \\ 1\le i\le 4m-2}} (-1)^{i-1} A_i^{4m-2}, \end{aligned}$$

with sum

$$E_{4m-2} + E'_{4m-2} = s_{4m-2}(y, x_1, \cdots, x_{4m-3}),$$

are polynomial identities for $K_{2m}(F,t)$.

This shows that for n even s_{2n-2} does not generate the PIs for $K_n(F,t)$ of degree $2n-2$. For the odd case we obtain a similar refinement (of Theorem 1.2.3):

Corollary 2.3. *Let* F *be a field,* $m > 1$ *a positive integer. Then the polynomials*

$$\begin{aligned} R_{4m-4}(x_1, \cdots, x_{4m-5}; y) : &= A_1^{4m-4} - A_4^{4m-4} + A_5^{4m-4} - A_8^{4m-4} \\ &\qquad + \cdots + A_{4m-7}^{4m-4} - A_{4m-4}^{4m-4} \\ &= \sum_{\substack{i=0,1 (mod\ 4) \\ 1 \le i \le 4m-4}} (-1)^{i-1} A_i^{4m-4} \end{aligned}$$

and

$$\begin{aligned} R'_{4m-4}(x_1, \cdots, x_{4m-5}; y) : &= -A_2^{4m-4} + A_3^{4m-4} - A_6^{4m-4} + A_7^{4m-4} \\ &\qquad - \cdots - A_{4m-6}^{4m-4} + A_{4m-5}^{4m-4} \\ &= \sum_{\substack{i=2,3 (mod\ 4) \\ 1 \le i \le 4m-4}} (-1)^{i-1} A_i^{4m-4}, \end{aligned}$$

with sum

$$R_{4m-4} + R'_{4m-4} = s_{4m-4}(y, x_1, \cdots, x_{4m-3}),$$

are polynomial identities for $K_{2m-1}(F,t)$.

This shows that for n odd s_{2n-2} does not generate the PIs for $K_n(F,t)$ of degree $2n-2$.

Proof of Theorem 2.1

We recall results from Section 1.2. Theorem 1.2.6 gives us $n+1$ trace-PIs from which all trace-PIs for (M_n, t) follow, and these are obtained by taking certain $n+1 \times n+1$ minors of the $2n+2 \times 2n+2$ (symmetric) matrix $Y = (y_{ij})$. In example 1.2.1 we computed the most obvious of these (the upper right hand $n+1 \times n+1$ minor of the matrix Y) and found that this (using Lemma 1.2.1) yields the characteristic polynomial. We will now consider a slightly different minor for the case $n = 2m$, the minor obtained by taking the rows $1, 2, 3, \cdots, 2m, 2m+2$ and the columns $2m+1, 2m+3, 2m+4, \cdots, 4m+2$.

We begin our proof with this minor (equation 1.2.2) which we rename (#):

$$\sum_{\sigma \in \mathcal{S}_{2m+1}} (-1)^{\sigma} y_{\sigma} :=$$

$$\sum_{\sigma \in \mathcal{S}_{2m+1}} (-1)^{\sigma} y_{1\sigma(2m+1)} y_{2\sigma(2m+3)} y_{3\,\sigma(2m+4)} \cdots y_{(2m-1)\,\sigma(4m)} y_{2m\,\sigma(4m+1)} y_{(2m+2)\sigma(4m+2)} \tag{\#}$$

where we recall that $Y = (y_{ij})$ is

$$\begin{pmatrix} \langle u_1, u_1 \rangle & \langle u_1, u_2 \rangle & \cdots & \langle u_1, u_{n+1} \rangle & \langle u_1, v_1 \rangle & \langle u_1, v_2 \rangle & \cdots & \langle u_1, v_{n+1} \rangle \\ \langle u_2, u_1 \rangle & \langle u_2, u_2 \rangle & \cdots & \langle u_2, u_{n+1} \rangle & \langle u_2, v_1 \rangle & \langle u_2, v_2 \rangle & \cdots & \langle u_2, v_{n+1} \rangle \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ \langle v_{n+1}, u_1 \rangle & \langle v_{n+1}, u_2 \rangle & \cdots & \langle v_{n+1}, u_{n+1} \rangle & \langle v_{n+1}, v_1 \rangle & \langle v_{n+1}, v_2 \rangle & \cdots & \langle v_{n+1}, v_{n+1} \rangle \end{pmatrix}.$$

At this point an outline of our proof will be useful. As noted in Section 1.1, we need only prove the case $char(F) = 0$. By proposition 1.2.1 it must be possible to write (#) as a multilinear pure-trace $*$-PI (which we call h) that holds for decomposable matrices $X_1, X_2, \cdots, X_{2m+1}$. As the canonical matrix units are decomposable and form a basis for $M_n(F)$, multilinearity says that h must hold for arbitrary matrices. We would like to find an explicit formula for h (this formula is summarized in proposition 2.1 below) by applying Lemma 1.2.1. For quick reference we will restate this lemma, but first we recall

Definition 1.1.1 Let $\tau \in \mathcal{S}_{2m+1}$, and let $(i_1\ i_2\ \cdots\ i_r)(j_1\ j_2\ \cdots j_l) \cdots (k_1\ k_2\ \cdots\ k_s)$ be τ in terms of disjoint cycles. Define:

$$T_\tau := tr(X_{i_1} X_{i_2} \cdots X_{i_r}) tr(X_{j_1} X_{j_2} \cdots X_{j_l}) \cdots tr(X_{k_1} X_{k_2} \cdots X_{k_s}).$$

Lemma 2.1 (Lemma 1.2.1). *Consider*

$$y_\gamma = y_{1\gamma(2m+2)} y_{2\gamma(2m+3)} \cdots y_{(2m+1)\gamma(4m+2)},$$

γ a permutation of $\{2m+2, 2m+3, \cdots, 4m+2\}$. This term is precisely $T_{\overline{\gamma}^{-1}}$, where $\overline{\gamma}^{-1}$ is the permutation of $\{1, 2, \cdots, 2m+1\}$ obtained by taking γ^{-1} (mod $2m+1$) in accordance with definition 1.2.2, that is,

$$\overline{\gamma}^{-1}(\bar{i}) = \overline{\gamma^{-1}(i)}, \quad 2m+2 \leq i \leq 4m+2.$$

Once we have h explicitly (see proposition 2.1 below) we will apply the non-degeneracy of the trace form (Lemma 1.1.1) to remove the traces and obtain the PI q_m (a trace-free polynomial, i.e., coefficients in F).

Having outlined the proof, we are ready to proceed. We first observe that, unfortunately, very few terms in (#) satisfy the hypothesis of Lemma 2.1, so we will have to do some work.

Definition 2.1. We say that a term $y_{\imath_1 \jmath_1} y_{\imath_2 \jmath_2} \cdots y_{\imath_k \jmath_k}$ is in *standard form* if it satisfies the hypothesis of Lemma 2.1, i.e., it is of the form

$$y_{1\gamma(2m+2)} y_{2\gamma(2m+3)} \cdots y_{(2m)\gamma(4m+1)} y_{(2m+1)\gamma(4m+2)},$$

with γ a permutation of $\{2m+2, 2m+3, \cdots, 4m+1, 4m+2\}$ and modulo the relations $y_{ab} = y_{ba}$ and $y_{ab} y_{cd} = y_{cd} y_{ab}$ (these relations follow from the definition of the scalars y_{ab}: entries in the symmetric matrix Y, see 1.2). We say that a term is in *#-form* if it appears in (#).

Again, Lemma 2.1 applies to terms in standard form, but not, in general, to terms in #-form. As we are interested in PIs for $K_{2m}(F, t)$, we will simplify h (remove the transposes on the variables) by assuming that all $X_1, \cdots, X_{2m+1}$ are skew (as noted above, h exists and is valid for arbitrary matrices and hence skew matrices), that is, that $X_\imath{}^t = -X_\imath$. With this assumption the following lemma allows us to bring a term in #-form to a term in standard form.

Lemma 2.2. *Assume that all variables $X_1, \cdots, X_{2m+1}$ are skew.*

a) In

$$y_\sigma = y_{1\sigma(2m+1)} y_{2\sigma(2m+3)} y_{3\,\sigma(2m+4)} \cdots y_{2m\,\sigma(4m+1)} y_{(2m+2)\sigma(4m+2)}$$

we may make any one of the exchanges

$$1 \leftrightarrow 2m+2, 2 \leftrightarrow 2m+3, \cdots, 2m \leftrightarrow 4m+1, 2m+1 \leftrightarrow 4m+2,$$

but for each exchange we must multiply our resulting term by -1.

b) Using these exchanges, any

$$y_\sigma = y_{1\sigma(2m+1)} y_{2\sigma(2m+3)} y_{3\,\sigma(2m+4)} \cdots y_{2m\,\sigma(4m+1)} y_{(2m+2)\sigma(4m+2)}$$

appearing in $(\#)$ *may be brought to a* $\pm y_\gamma$*, where* y_γ *is in standard form*

$$y_{1\gamma(2m+2)} y_{2\gamma(2m+3)} \cdots y_{(2m+1)\gamma(4m+2)},$$

γ *some permutation of*

$$\{2m+2, 2m+3, \cdots, 4m+1, 4m+2\}.$$

c) If y_σ *may be brought via exchanges to both* $\pm y_{\gamma_1}$ *and* $\pm y_{\gamma_2}$ *(terms in standard form) then* $\gamma_2 = \gamma_1$ *or* γ_1^{-1}.

d) If y_σ *may be brought to* y_γ *via two distinct sequences of exchanges, with total exchanges* e_σ *and* d_σ*, respectively, then* $(-1)^{d_\sigma} = (-1)^{e_\sigma}$.

Proof. a) We saw in 1.2 that an exchange of indices $a \leftrightarrow \overline{a}$ in y_σ corresponds to applying a transpose to the variable $X_{\overline{a}}$ in the corresponding trace term. If we assume skew variables, then $X_{\overline{a}}^{\,t}$ becomes $-X_{\overline{a}}$.

b) We start with

$$y_\sigma = y_{1\sigma(2m+1)} y_{2\sigma(2m+3)} y_{3\,\sigma(2m+4)} \cdots y_{2m\,\sigma(4m+1)} y_{(2m+2)\sigma(4m+2)},$$

which we would like to bring to standard form. We note that the only factors not in standard form (factors that cannot appear in a term in standard form) are the factor involving $2m+1$ (not displayed) and the factor involving $2m+2$. It is possible (if $\sigma(4m+2) = 2m+1$) that these coincide.

If $\sigma(4m+2) = 2m+1$, then $y_{(2m+2)(2m+1)}$ is the last factor in y_σ. Since we are working modulo $y_{ab} = y_{ba}$, y_σ is in standard form:

$$y_{1\sigma(2m+1)} y_{2\sigma(2m+3)} y_{3\,\sigma(2m+4)} \cdots y_{2m\,\sigma(4m+1)} y_{(2m+1)(2m+2)}$$

with

$$\gamma = (4m+2\ \ 2m+2\ \ \sigma(2m+1) \cdots) \cdots.$$

If $\sigma(2m+1) = 2m+1$, then

$$y_\sigma = y_{1(2m+1)} y_{2\sigma(2m+3)} y_{3\,\sigma(2m+4)} \cdots y_{2m\,\sigma(4m+1)} y_{(2m+2)\sigma(4m+2)}$$

and $1 \leftrightarrow 2m+2$ brings us to standard form

$$y_{1\sigma(4m+2)}y_{2\sigma(2m+3)}y_{3\,\sigma(2m+4)}\cdots y_{2m\,\sigma(4m+1)}y_{(2m+2)(2m+1)}$$

i.e., y_γ with

$$\gamma = (4m+2\ 2m+2\ \sigma(4m+2)\cdots)\cdots.$$

Now we consider the case $\sigma(4m+2) \neq 2m+1$ and $\sigma(2m+1) \neq 2m+1$.

We would like to list y_σ's factors in a specific order. First, we list the factor involving $2m+1$. Since $\sigma(2m+1) \neq 2m+1$ and $\sigma(4m+2) \neq 2m+1$ we may assume that $2m+1$ does not appear in $y_{1\sigma(2m+1)}$ or $y_{(2m+2)\sigma(4m+2)}$. Namely it appears in some

$$y_{\overline{i_1}(2m+1)}$$

where $2m+2 < i_1 < 4m+2$: it is one of the two factors in y_σ *not* in standard form. Next we list the factor involving $i_1 = \overline{i_1} + 2m+1$, say

$$y_{ai_1}.$$

If this factor is *not* in standard form ($a > 2m+1$), we stop. Otherwise, put $\overline{i_2} = a$ and list the factor involving i_2, say

$$y_{bi_2}$$

If this factor is *not* in standard form ($b > 2m+1$), stop. If it is, put $\overline{i_3} = b$ and continue. We would like to show that this list terminates, i.e., that we must eventually arrive at a factor that is not in standard form. There are a finite number of factors in standard form, so any infinite list would have to involve a repetition of factors.

Assume this is the case, say, $y_{\overline{i_d}i_{d-1}}$ appears twice. By construction, the factor preceding this second appearance of $y_{\overline{i_d}i_{d-1}}$ must be $y_{\overline{i_{d-1}}i_{d-2}}$. So $y_{\overline{i_{d-1}}i_{d-2}}$ also appears twice. But if we continue we see that $y_{\overline{i_{d-2}}i_{d-3}}$ must appear twice as well, and so on. This shows that, in our list, $y_{\overline{i_1}(2m+1)}$ must be the first factor to appear twice. But this says that our list stops at this second appearance (we have reached a factor not in standard form) and contradicts our assumption (that our list is infinite).

So we must eventually arrive at a factor that is not in standard form. This is either $y_{\overline{i_1}(2m+1)}$ or $y_{(2m+2)\sigma(4m+2)}$. It cannot be the former as, by construction, our list proceeds from a term involving, say, $\overline{i_s}$ to a term involving i_s, i.e., the "next" factor chosen always involves a subscript greater than $2m+1$. So our list terminates at $y_{(2m+2)\sigma(4m+2)}$, or say $y_{(2m+2)i_r}$ where our list is

$$y_{\overline{i_1}(2m+1)}, y_{\overline{i_2}i_1}, \cdots, y_{\overline{i_r}i_{r-1}}, y_{(2m+2)i_r}.$$

We may now rearrange y_σ according to our list to get

$$y_\sigma = y_{\overline{i_1}(2m+1)}y_{\overline{i_2}i_1}y_{\overline{i_3}i_2}\cdots y_{\overline{i_r}i_{r-1}}y_{(2m+2)i_r}\cdots.$$

The non-standard factors in y_σ are precisely $y_{\overline{i_1}(2m+1)}$ and $y_{(2m+2)i_r}$. If we now apply the exchanges

$$\overline{i_j} \leftrightarrow i_j, \quad 1 \leq j \leq r$$

we get

$$(-1)^r y_{i_1(2m+1)} y_{i_2\overline{i_1}} \cdots y_{i_r\overline{i_{r-1}}} y_{(2m+2)\overline{i_r}} \cdots$$

(note that $2m+2$ is not among the i_j, thus it is not changed by this sequence of exchanges). Thus all factors are now standard and (reordering factors) we see that we are in standard form

$$(-1)^r y_{1\gamma(2m+2)} y_{2\gamma(2m+3)} \cdots y_{(2m+1)\gamma(4m+2)}$$

with

$$\gamma = (4m+2\ i_1\ i_2 \cdots i_{r-1}\ i_r\ 2m+2 \cdots).$$

c) We need to show that if a y_σ may be brought (via exchanges) to both $\pm y_{\gamma_1}$ and $\pm y_{\gamma_2}$ (terms in standard form) then $\gamma_2 = \gamma_1$ or γ_1^{-1}.

For quick reference:

$$y_{\gamma_i} = y_{1\gamma_i(2m+2)} y_{2\gamma_i(2m+3)} \cdots y_{(2m+1)\gamma_i(4m+2)}, \ i = 1, 2.$$

In this case y_{γ_1} and y_{γ_2} must be related by a sequence of exchanges. But as they are both in standard form, the only possible sequence of exchanges is the null set or the sequence of *all* exchanges (applied, possibly, a multiple number of times)

$$1 \leftrightarrow 2m+2, 2 \leftrightarrow 2m+3, \cdots, 2m+1 \leftrightarrow 4m+2$$

since any other sequence of exchanges will create non-standard factors. Thus $\gamma_2 = \gamma_1$ or γ_1^{-1}.

d) Consider a set of e_σ exhanges such that

$$(-1)^\sigma y_\sigma \mapsto (-1)^\sigma (-1)^{e_\sigma} y_\gamma.$$

In fact, we have equality

$$(-1)^\sigma y_\sigma = (-1)^\sigma (-1)^{e_\sigma} y_\gamma$$

(the $\mapsto$ is used only to emphasize that we are performing exchanges). Thus if we consider a distinct set of exchanges, say d_σ exhanges, that brings about $(-1)^\sigma y_\sigma \mapsto (-1)^\sigma (-1)^{d_\sigma} y_\gamma$, then

$$(-1)^\sigma (-1)^{d_\sigma} y_\gamma = (-1)^\sigma (-1)^{e_\sigma} y_\gamma,$$

and hence $(-1)^{d_\sigma} = (-1)^{e_\sigma}$ since y_γ is not identically zero.

$\square$

So we will perform a number of exchanges on a term in (#) in order to obtain a term in standard form, and then we will apply Lemma 2.1. We can diagram this as follows

$$(-1)^\sigma y_\sigma \mapsto (-1)^\sigma (-1)^{e_\sigma} y_\gamma \mapsto (-1)^\sigma (-1)^{e_\sigma} T_{\overline{\gamma}^{-1}}$$

where e_σ is the number of exchanges required to take y_σ to y_γ.

Upon applying Lemmas 2.1 and 2.2 to (#) we obtain a trace-PI of the form

$$h(X_1, X_2, \cdots, X_{2m+1}) = \sum_{\overline{\gamma}^{-1} \in S_{2m+1}} a_{\overline{\gamma}^{-1}} T_{\overline{\gamma}^{-1}}, \qquad a_{\overline{\gamma}^{-1}} \in \mathbf{Z}. \tag{$\#_1$}$$

The properties of determinants give us the following information about h.

Lemma 2.3. *The pure-trace polynomial $h(X_1, X_2, \cdots, X_{2m+1})$ is symmetric in the variables $X_2, X_3, \cdots, X_{2m}$ and alternating in X_1, X_{2m+1}.*

Proof. A transposition

$$X_{\overline{a}} \leftrightarrow X_{\overline{b}}, \quad 2m+2 \le a, b \le 4m+2,$$

in h corresponds to the permutation

$$a \leftrightarrow b, \ \overline{a} \leftrightarrow \overline{b}$$

in (#) which we restate

$$\sum_{\sigma \in S_{2m+1}} (-1)^\sigma y_{1\sigma(2m+1)} y_{2\sigma(2m+3)} y_{3\,\sigma(2m+4)} \cdots y_{(2m-1)\,\sigma(4m)} y_{2m\,\sigma(4m+1)} y_{(2m+2)\sigma(4m+2)} \cdot \tag{#}$$

Let $\overline{a}, \overline{b} \in \{2, \cdots, 2m\}$ be distinct. Recall that (#) is a minor of the matrix Y of 1.2 in which we have chosen the rows $1, 2, \cdots, 2m, 2m+2$ and the columns $2m+1, 2m+3, 2m+4, \cdots, 4m+2$. The transposition $\overline{a} \leftrightarrow \overline{b}$ in (#) is a transposition of rows in the minor with indices $\overline{a}$ and $\overline{b}$, and thus applying this to (#) yields $-$(#) (though we take minors to be determinants and not submatrices, the meaning of "transposition of rows in the minor" is clear) . If we now apply $a \leftrightarrow b$ (an *odd* permutation of $\{2m+3, 2m+4 \cdots, 4m+1\}$) to $-$(#), the "sign" coefficients $(-1)^\sigma$ show that we get (#). Thus (#) is invariant under

$$X_{\overline{a}} \leftrightarrow X_{\overline{b}}, \quad 2m+3 \le a, b \le 4m+1.$$

For the alternating property note that $1 \leftrightarrow 2m+1$ is *not* a transposition of rows in the minor. We instead prove

$$h({X_{2m+1}}^t, {X_{2m}}^t, \cdots, {X_2}^t, {X_1}^t) = h(X_1, X_2, \cdots, X_{2m}, X_{2m+1})$$

which proves the alternating property since the left hand side

$$h({X_{2m+1}}^t, {X_{2m}}^t, \cdots, {X_2}^t, {X_1}^t) = -h(X_{2m+1}, X_{2m}, \cdots, X_2, X_1)$$

and this (by the symmetry property of h proven above) is

$$-h(X_{2m+1}, X_2, X_3, \cdots, X_{2m-1}, X_{2m}, X_1).$$

We again use the properties of determinants. $h({X_{2m+1}}^t, {X_{2m}}^t, \cdots, {X_2}^t, {X_1}^t)$ is obtained by first applying to $(\#) = h(X_1, \cdots, X_{2m+1})$ the transpositions

$$1 \leftrightarrow 2m+2,\ 2 \leftrightarrow 2m+3, \cdots, 2m+1 \leftrightarrow 4m+2$$

to get a t (matrix transpose) on each variable X_i in h, and then applying the transpositions

$$\begin{aligned} 1 \leftrightarrow 2m+1,&\ 2m+2 \leftrightarrow 4m+2 \\ 2 \leftrightarrow 2m,&\ 2m+3 \leftrightarrow 4m+1 \\ &\vdots \\ m \leftrightarrow m+2,&\ 3m+1 \leftrightarrow 3m+2 \end{aligned}$$

to reverse the variables in h. Composing these two sets of permutations we get the permutation

$$1 \leftrightarrow 4m+2,\ 2 \leftrightarrow 4m+1, \cdots, 2m \leftrightarrow 2m+3, 2m+1 \leftrightarrow 2m+2.$$

In the usual notation for permutations this is the product of transpositions

$$(1\ 4m+2)(2\ 4m+1)\cdots(2m\ 2m+3)(2m+1\ 2m+2)$$

(though in this case our factors commute, let us agree on the convention that in a product of permutations the rightmost permutation is applied first). We may write this (by decomposing the first and last factors) as

$$(1\ 2m+1)(2m+2\ 4m+2)(2\ 4m+1)\cdots(2m\ 2m+3)(2m+1\ 4m+2)(1\ 2m+2).$$

We would like to use properties of the determinant to show that this permutation fixes $(\#)$, a minor obtained by taking the rows $1, 2, \cdots, 2m, 2m+2$ and the columns $2m+1, 2m+3, \cdots 4m+2$ of Y. $(1\ 2m+2)$ exhanges the first and last rows of our minor, $(2m+1\ 4m+2)$ exchanges the first and last columns of our minor, so (combined) these two transpositions leave invariant the minor (#). The remaining factors may be rearranged to get

$$(2m+2\ 4m+2)(2m\ 2m+3)\cdots(3\ 4m)(2\ 4m+1)(1\ 2m+1)\,.$$

If we read this permutation from right to left we see that it exhanges the first row (of the minor) with the first column, exchanges the second row with the second last column, exchanges the third row with the third last column, and so on until the second last row (index $2m$) is exchanged with the second column (index $2m+3$). The last transposition $(2m+2\ 4m+2)$ exhanges the last row with the last column. Thus the "remaining factors" permutation differs from a matrix transpose (of our minor) by an even number $(2\times(m-1))$ of row and column exchanges. This shows that these remaining factors do not affect (#) and that h satisfies the alternating property.

□

Consider the permutations in $\mathcal{S}_{2m+1}$ which have the following form when expressed as products of disjoint cycles: 1 and $2m+1$ appear in the same cycle, this cycle has odd length, all remaining cycles have even length. We give the name $Q \subseteq \mathcal{S}_{2m+1}$ to this set of permutations and show that h may be taken to be a sum over Q:

Lemma 2.4.

a) If $\overline{\gamma}^{-1} \notin Q$, then we may assume that $a_{\overline{\gamma}^{-1}}$, the coefficient of $T_{\overline{\gamma}^{-1}}$, is zero.

b) For all $\overline{\gamma}^{-1} \in Q$, there exists a y_σ in (#) which may be brought to y_γ by a sequence of exchanges from $1 \leftrightarrow 2m+2, \cdots, 2m+1 \leftrightarrow 4m+2$.

Proof.

a) Let $(2m+2\ i_2\ i_3\ \cdots\ i_r)(j_1\ j_2\ \cdots j_l)\cdots(k_1\ k_2\ \cdots\ k_s)$ be a decomposition of γ in terms of disjoint cycles. We would first like to show that $4m+2 \in \{i_2, \cdots, i_r\}$. This follows immediately from the proof of Lemma 2.2b: in all cases $y_\sigma \mapsto y_\gamma$ where y_σ is in (#) and y_γ is in standard form, $4m+2$ and $2m+2$ appear in the same cycle in γ.

This proves that $T_{\overline{\gamma}^{-1}}$ only appears in h if 1 and $2m+1$ are in the same cycle in $\overline{\gamma}^{-1}$. Consider the case where this cycle is of even length. Our trace polynomial is alternating in X_1, X_{2m+1} and symmetric in the remaining variables, so if this term appears, say

$$tr(\cdots X_1 \cdots X_{2m+1} \cdots)\cdots,$$

then the term

$$tr(\cdots X_{2m+1} \cdots X_1 \cdots)\cdots$$

(reverse the variables in the trace) will appear as well but with the opposite sign. Now all of our variables are skew-symmetric, and there is an even number of them in the trace, so we have the relation

$$tr((\cdots X_1 \cdots X_{2m+1} \cdots)^t) = tr(\cdots X_{2m+1} \cdots X_1 \cdots).$$

This relation says that the terms appearing with opposite signs

$$\pm tr(\cdots X_1 \cdots X_{2m+1} \cdots) \mp tr(\cdots X_{2m+1} \cdots X_1 \cdots)$$

amount to ± the trace of a skew symmetric matrix, so together these terms are zero. Thus all of these terms (involving 1 and $2m+1$ in a trace of even length) appear in cancelling pairs, and we may thus take $a_{\overline{\gamma}^{-1}} = 0$ for these terms.

Next consider the case where at least one of the remaining cycles in $\overline{\gamma}^{-1}$ is of odd length, say, without loss of generality,

$$\cdots tr(X_2 \cdots) \cdots .$$

Again, our trace polynomial is symmetric in all the variables in the displayed trace, thus

$$\cdots tr(\cdots X_2) \cdots$$

appears as well (we have reversed the variables in the displayed trace), and with the same sign. But

$$\pm \cdots tr(X_2 \cdots) \cdots \pm \cdots tr(\cdots X_2) \cdots$$

is ± the trace of the skew-symmetric matrix

$$(X_2 \cdots) - (X_2 \cdots)^t,$$

so as above these terms cancel and we may take $a_{\overline{\gamma}^{-1}} = 0$.

b) We write

$$\gamma = (2m+2 \; i_1 \; i_2 \cdots i_r) \cdots$$

where $1 \leq r \leq 2m$. We will work backwards via exchanges and find a corresponding y_σ in (#). Since $\overline{\gamma^{-1}} \in Q$ we know that $4m+2 \in \{i_1, \cdots, i_r\}$, say $i_s = 4m+2$.

$$\begin{aligned} y_\gamma &= y_{1\gamma(2m+2)} y_{2\gamma(2m+3)} \cdots y_{(2m+1)\gamma(4m+2)} \\ &= y_{1 i_1} y_{\overline{i_1} i_2} y_{\overline{i_2} i_3} \cdots y_{\overline{i_{s-1}}(4m+2)} y_{(2m+1) i_{s+1}} y_{\overline{i_{s+1}} i_{s+2}} \cdots y_{\overline{i_{r-1}} i_r} y_{\overline{i_r}(2m+2)} \cdots . \end{aligned}$$

If $i_r = 4m+2$ then this is an element of (#). Otherwise, only the factors $y_{(2m+1) i_{s+1}}$ and $y_{\overline{i_r}(2m+2)}$ are not in #-form. Performing the $r-s$ exchanges

$$\overline{i_l} \leftrightarrow i_l, \; s+1 \leq l \leq r$$

we obtain

$$y_{1 i_1} y_{\overline{i_1} i_2} y_{\overline{i_2} i_3} \cdots y_{\overline{i_{s-1}}(4m+2)} y_{(2m+1) \overline{i_{s+1}}} y_{i_{s+1} \overline{i_{s+2}}} \cdots y_{i_{r-1} \overline{i_r}} y_{i_r (2m+2)} \cdots . \qquad (\#_1)$$

This is an element in (#). □

We will now obtain some additional information about h from $y_\sigma := (\#_1)$ and the corresponding y_γ. We would like to show that the sign of $a_{\overline{\gamma}^{-1}}$ in h depends only on the relative positions of $2m+2$ and $4m+2$ in a decomposition of γ into disjoint cycles.

Assume for the moment that $i_r \neq 4m+2$, and consider $y_{\sigma'}$ obtained from y_σ by the exchange $4m+2 \leftrightarrow \gamma(4m+2) = i_{s+1}$ (but not $2m+1 \leftrightarrow \overline{i_{s+1}}$). Since y_σ is in $(\#)$, so is $y_{\sigma'}$:

$$y_{\sigma'} = y_{1 i_1} y_{\overline{i_1} i_2} y_{\overline{i_2} i_3} \cdots y_{\overline{i_{s-1}} i_{s+1}} y_{(2m+1)\, \overline{i_{s+1}}} y_{(4m+2) \overline{i_{s+2}}} \cdots y_{i_{r-1} \overline{i_r}} y_{i_r (2m+2)} \cdots .$$

We will now move this back to standard form. Apply the exhange $2m+1 \leftrightarrow 4m+2$ and the exchanges

$$\overline{i_l} \leftrightarrow i_l, \ s+2 \leq l \leq r$$

(a total of $r-s$ exhanges) to obtain

$$y_{1 i_1} y_{\overline{i_1} i_2} y_{\overline{i_2} i_3} \cdots y_{\overline{i_{s-1}} i_{s+1}} y_{(4m+2)\, \overline{i_{s+1}}} y_{(2m+1) i_{s+2}} \cdots y_{\overline{i_{r-1}} i_r} y_{\overline{i_r} (2m+2)} \cdots .$$

This is in standard form, say

$$y_{\gamma'} = y_{1 \gamma'(2m+2)} y_{2 \gamma'(2m+3)} \cdots y_{(2m+1) \gamma'(4m+2)}$$

where

$$\gamma' = (2m+2 \ i_1 \ i_2 \cdots i_{s-1} \ i_{s+1} \ 4m+2 \ i_{s+2} \ i_{s+3} \cdots i_r).$$

We have shown that if

$$(-1)^\sigma y_\sigma \mapsto (-1)^\sigma (-1)^{r-s} y_\gamma,$$

then

$$(-1)^{\sigma'} y_{\sigma'} = -(-1)^\sigma y_{\sigma'} \mapsto -(-1)^\sigma (-1)^{r-s} y_{\gamma'}$$

as long as $i_r \neq 4m+2$. Turning to the special case $i_r = 4m+2$ we see that

$$y_\sigma = y_{1 i_1} y_{\overline{i_1} i_2} y_{\overline{i_2} i_3} \cdots y_{i_{r-1} (2m+1)} y_{(4m+2)(2m+2)} \cdots .$$

So we apply $4m+2 \leftrightarrow \gamma(4m+2) = 2m+2$ to get

$$y_{\sigma'} = y_{1 i_1} y_{\overline{i_1} i_2} y_{\overline{i_2} i_3} \cdots y_{i_{r-1} (2m+1)} y_{(2m+2)(4m+2)} \cdots .$$

y_σ and $y_{\sigma'}$ are in both standard and $\#$-form, so no exhanges are necessary to move them to standard form. So for this special case we have shown that if

$$(-1)^\sigma y_\sigma \mapsto (-1)^\sigma (-1)^{r-s} y_\gamma = (-1)^\sigma y_\gamma,$$

then

$$(-1)^{\sigma'} y_{\sigma'} = -(-1)^\sigma y_{\sigma'} \mapsto -(-1)^\sigma (-1)^{r-s} y_{\gamma'} = -(-1)^\sigma y_{\gamma'}.$$

This proves

Lemma 2.5. *Let* $\gamma, \gamma' \in Q$. *Assume further that*

$$\gamma = (2m+2 \cdots 4m+2 \ \gamma(4m+2) \cdots)$$

$$\gamma' = (2m+2 \cdots \gamma(4m+2) \ 4m+2 \cdots)$$

where all undisplayed entries are equal and we allow the possibilities $\gamma(2m+2) = 4m+2$ *and* $\gamma(4m+2) = 2m+2$. *Let* y_σ *and* $y_{\sigma'}$ *be terms appearing in* (#). *If there exist sequences of exchanges*

$$(-1)^\sigma y_\sigma \mapsto (-1)^\sigma (-1)^{e_\sigma} y_\gamma$$

and

$$(-1)^{\sigma'} y_{\sigma'} \mapsto (-1)^{\sigma'} (-1)^{e'_\sigma} y_{\gamma'}$$

then

$$(-1)^{\sigma'} (-1)^{e_{\sigma'}} = -(-1)^\sigma (-1)^{e_\sigma}.$$

□

Here we would like to show that there are no cancelling terms:

Lemma 2.6. *Let* $\overline{\gamma}^{-1} \in Q$. *Let* $(-1)^\sigma y_\sigma$ *and* $(-1)^{\sigma'} y_{\sigma'}$ *be two terms in* (#) *that can be brought to standard forms* $(-1)^\sigma(-1)^{e_\sigma} y_\gamma$ *and* $(-1)^{\sigma'}(-1)^{e_{\sigma'}} y_\gamma$ *(respectively). Then* $(-1)^{\sigma'}(-1)^{e_{\sigma'}} = (-1)^\sigma(-1)^{e_\sigma}$.

Proof. Assume $(-1)^\sigma y_\sigma \mapsto (-1)^\sigma(-1)^{e_\sigma} y_\gamma$. Obtain from y_σ the terms $y_{\sigma_1}, y_{\sigma_2}, y_{\sigma_3}$ via the exchanges

$$2m+1 \leftrightarrow 4m+2 \ (y_{\sigma_1}), \ 1 \leftrightarrow 2m+2 \ (y_{\sigma_2}),$$

and

$$1 \leftrightarrow 2m+2, \ 2m+1 \leftrightarrow 4m+2 \ (y_{\sigma_3}).$$

Since y_σ appears in (#), y_{σ_i} does as well, $i = 1, 2$, or 3. Now the first two may be moved to y_γ in one more move than it takes y_σ ($2m+1 \leftrightarrow 4m+2$ and $1 \leftrightarrow 2m+2$, respectively), and the last takes two more moves ($2m+1 \leftrightarrow 4m+2, 1 \leftrightarrow 2m+2$). This shows that

$$(-1)^{e_{\sigma'}} = -(-1)^{e_\sigma}$$

for $\sigma' = \sigma_1, \sigma_2$ and that

$$(-1)^{e_{\sigma'}} = (-1)^{e_\sigma}$$

for $\sigma' = \sigma_3$. But it is clear that $(-1)^{\sigma'} = -(-1)^{\sigma}$ for $\sigma' = \sigma_1, \sigma_2$ and that $(-1)^{\sigma'} = (-1)^{\sigma}$ for $\sigma' = \sigma_3$, so in all three cases $(-1)^{\sigma'}(-1)^{e_{\sigma'}} = (-1)^{\sigma}(-1)^{e_{\sigma}}$.

It remains to show that these are the only terms in (#) which can be brought to y_γ. Any term $(-1)^{\sigma'} y_{\sigma'}$ in (#) that can be brought to y_γ can also be brought to y_σ, so y_σ and $y_{\sigma'}$ are related by a combination of the exchanges $1 \leftrightarrow 2m+2, 2 \leftrightarrow 2m+3, \cdots, 2m+1 \leftrightarrow 4m+2$. If we apply any of $2 \leftrightarrow 2m+3, 3 \leftrightarrow 2m+4, \cdots, 2m \leftrightarrow 4m+1$ to $y_{\sigma'}$, then we leave (#). But the exchanges $1 \leftrightarrow 2m+2, 2 \leftrightarrow 2m+3, \cdots, 2m+1 \leftrightarrow 4m+2$ commute, so we will stay in (#) if and only if we restrict ourselves to $1 \leftrightarrow 2m+2$ and $2m+1 \leftrightarrow 4m+2$.

□

So for each y_γ, $\overline{\gamma^{-1}} \in Q$, there are four terms of the form $(-1)^\sigma y_\sigma$ in (#) which may be brought to y_γ via a sequence of exchanges. But these four y_σ may also be brought to $y_{\gamma^{-1}}$:

$$(-1)^\sigma y_\sigma \mapsto (-1)^\sigma(-1)^{e_\sigma} y_\gamma \mapsto -(-1)^\sigma(-1)^{e_\sigma} y_{\gamma^{-1}}$$

where the last transition is obtained by applying *all* possible exchanges (an odd number)

$$1 \leftrightarrow 2m+2, 2 \leftrightarrow 2m+3, \cdots, 2m+1 \leftrightarrow 4m+2.$$

Thus it appears as though for each y_γ we must make a choice between y_γ and $y_{\gamma^{-1}}$. We choose instead to send two y_σ's to each of y_γ and $y_{\gamma^{-1}}$. This says that in our polynomial

$$h(X_1, X_2, \cdots, X_{2m+1}) = \sum_{\overline{\gamma^{-1}} \in Q} a_{\overline{\gamma}^{-1}} T_{\overline{\gamma}^{-1}}$$

each $T_{\overline{\gamma^{-1}}}$ appears with a coefficient ±2, and each $T_{\overline{\gamma}}$ appears with a coefficient ∓2.

We may now apply Lemma 2.5 to h. We send two y_σ's to y_γ, say

$$(-1)^{\sigma_1} y_{\sigma_1}, (-1)^{\sigma_2} y_{\sigma_2} \mapsto (-1)^{\sigma_1}(-1)^{e_{\sigma_1}} y_\gamma$$

where the coefficient $(-1)^{\sigma_1}(-1)^{e_{\sigma_1}}$ is common to both by Lemma 2.6. By Lemma 2.5 the two $y_{\sigma'}$'s we send to $y_{\gamma'}$

$$(-1)^{\sigma'_1} y_{\sigma'_1}, (-1)^{\sigma'_2} y_{\sigma'_2} \mapsto (-1)^{\sigma'_1}(-1)^{e_{\sigma'_1}} y_{\gamma'}$$

have

$$(-1)^{\sigma'_1}(-1)^{e_{\sigma'_1}} = -(-1)^{\sigma_1}(-1)^{e_{\sigma_1}}.$$

This shows that if $T_{\overline{\gamma^{-1}}}$ appears with a ±2 then $T_{\overline{\gamma'^{-1}}}$ appears with a ∓2. As we are in characteristic 0, h may be replaced with $\frac{1}{2}h$ (which we will still call h) to obtain coefficients ±1. The following proposition summarizes our results.

Proposition 2.1. *Let* Q *be the set containing the permutations of*

$$\{2m+2, 2m+3, \cdots, 4m+2\}$$

that, when written as products of disjoint cycles, have precisely one factor of odd length which contains both $2m+2$ *and* $4m+2$*. Let* $\gamma, \gamma' \in Q$ *be*

$$\gamma = (2m+2\ \cdots\ 4m+2\ \gamma(4m+2)\cdots)\cdots$$

$$\gamma' = (2m+2\ \cdots\ \gamma(4m+2)\ 4m+2\cdots)\cdots$$

and

$$\begin{aligned}\overline{\gamma} &= (\overline{2m+2}\ \cdots\ \overline{4m+2}\ \overline{\gamma(4m+2)}\cdots)\cdots \\ &= (1\ \cdots\ 2m+1\ (\gamma(4m+2)-(2m+1))\cdots)\cdots.\end{aligned}$$

Then

$$h(X_1, X_2, \cdots, X_{2m+1}) = \sum_{\overline{\gamma}^{-1}\in Q} a_{\overline{\gamma}^{-1}} T_{\overline{\gamma}^{-1}}$$

is a multilinear pure-trace PI for $K_{2m}(F,t)$ *with the properties*

$$\begin{aligned} &H1. \quad a_{\overline{\gamma}^{-1}} = \pm 1 \\ &H2. \quad a_{\overline{\gamma}} = -a_{\overline{\gamma}^{-1}} \\ &H3. \quad a_{\overline{\gamma'}^{-1}} = -a_{\overline{\gamma}^{-1}}. \end{aligned}$$

□

This defines h up to scalar multiple ±1; we will soon make a choice between these two options, but first we write h in terms of the multilinear pure-trace polynomials (which we now define)

$$\begin{aligned} P_{2m+1} := \sum_{\sigma\in S_{2m-1}} & tr(X_1X_{\sigma(2)}\cdots X_{\sigma(2m)}X_{2m+1}) - tr(X_1X_{\sigma(2)}\cdots X_{2m+1}X_{\sigma(2m)}) + \\ & \cdots - tr(X_1X_{2m+1}X_{\sigma(2)}\cdots X_{\sigma(2m)}), \end{aligned}$$

$$\begin{aligned} P_{2m-1,2} := \sum_{\sigma\in S_{2m-1}} & tr(X_1X_{\sigma(2)}\cdots X_{\sigma(2m-2)}X_{2m+1})tr(X_{\sigma(2m-1)}X_{\sigma(2m)}) \\ & - tr(X_1X_{\sigma(2)}\cdots X_{2m+1}X_{\sigma(2m-2)})tr(X_{\sigma(2m-1)}X_{\sigma(2m)}) + \\ & \cdots - tr(X_1X_{2m+1}X_{\sigma(2)}\cdots X_{\sigma(2m-2)})tr(X_{\sigma(2m-1)}X_{\sigma(2m)}), \end{aligned}$$

where the sums are over all permutations of $\{2, 3, \cdots, 2m\}$, and where the subscripts on P denote the lengths of the traces. Since the trace is invariant under cyclic permutations of its arguments, we may assume that X_1 appears first (in the trace in which it appears). Note also that in these sums the sign of each term is determined entirely by the position of X_{2m+1} relative to X_1. It is now clear how to define a general

$$P_{i_1,i_2,\cdots,i_r}$$

where $i_1 + i_2 + \cdots + i_r = 2m+1$, i_1 is odd, and the remaining indices are even and in non-increasing order.

$h(X_1, X_2, \cdots, X_{2m+1})$ is a linear combination of the $P_{i_1,i_2,\cdots,i_r}$ (Proposition 2.1), and as all coefficients are ± 1 we may without loss of generality assume that P_{2m+1} appears with a coefficient $+1$. Consider

$$\phi(h(x_1, y, y, y, [x_2, x_3], [x_4, x_5], \cdots, [x_{4m-8}, x_{4m-7}], [x_{4m-6}, x_{4m-5}]))$$

where ϕ induces the alternating sum in the x_j (definition 1.1.2). We would like to show that only P_{2m+1} (terms with exactly one trace) survives this application of ϕ.

Lemma 2.7. *If* $i_1 < 2m+1$ *then*

$$\phi(P_{i_1,i_2,\cdots,i_r}(x_1, y, y, y, [x_2, x_3], [x_4, x_5], \cdots, [x_{4m-8}, x_{4m-7}], [x_{4m-6}, x_{4m-5}])) = 0.$$

Proof. Let $i_1 < 2m+1$ and consider

$$P_{i_1,i_2,\cdots,i_r}(x_1, y, y, y, [x_2, x_3], [x_4, x_5], \cdots, [x_{4m-8}, x_{4m-7}], [x_{4m-6}, x_{4m-5}]).$$

Each term in this sum has exactly one trace of odd length, we divide into cases based on how many y's are in this trace. First, look at terms in which the trace of odd length (i_1) does not contain a y. This means that there are an even number of commutators in the trace, together with x_1. Now ϕ kills this term since a standard polynomial of degree 1 (*mod* 4) is skew-symmetric when all of its variables are skew-symmetric (Lemma 1.1.3c).

Next consider a term whose trace of odd length contains precisely one y, say

$$tr(x_1 \cdots y \cdots).$$

So there are an odd number of commutators in the trace. If we permute variables cyclically under the trace to obtain

$$tr(y \cdots x_1 \cdots)$$

we see that ϕ kills this term since a standard polynomial of degree 3 (*mod* 4) is symmetric when all of its variables are skew (Lemma 1.1.3b), y is skew, and the trace of a symmetric times a skew is zero.

Next consider a term whose trace of odd length contains precisely two y's, say

$$tr(x_1 \cdots y \cdots y \cdots).$$

So there are an even number of commutators in the trace. Now permute variables cylically under the trace to obtain

$$tr(y \cdots y \cdots x_1 \cdots),$$

and assume first that there are an even number of commutators between the y's. Taking a partial alternating sum (over only those indices that appear between the y's) of $y \cdots y$, we obtain $ys_{4l}y$, a symmetric matrix (since s_{4l} is symmetric by Lemma 1.1.3 and y is skew). Taking a partial alternating sum of $\cdots x_1 \cdots$ (over the remaining indices) we obtain a standard polynomial of degree 1 (*mod* 4), a skew matrix (Lemma 1.1.3). So applying these two partial alternating sums to

$$tr(y \cdots y \cdots x_1 \cdots),$$

gives the trace of a symmetric element times a skew element, i.e., zero. This shows that ϕ kills this term: as functions on the free associative algebra, applying these partial alternating sums and then applying ϕ is equivalent to applying some positive integer multiple of ϕ.

If instead there are an odd number of commutators between the y's then applying an appropriate partial alternating sum to $y \cdots y$ will yield $ys_{4l+2}y$ (skew), and applying an appropriate partial alternating sum to $\cdots x_1 \cdots$ will yield a standard polynomial of degree 3 (*mod* 4) (symmetric). So again ϕ kills this term.

Finally, consider a term whose trace of odd length contains three y's. This means that this term has a trace of even length containing only commutators. Thus ϕ kills this term since the trace of a standard polynomial of even degree is zero.

□

Recall the definition (definition 1.1.3) of the A^{4m-2}_{ijk}: the alternating sum of degree $4m-2$ except with a y in positions i, j and k.

Lemma 2.8. *Applying an alternating sum (ϕ) to h gives*

$$\phi(h(x_1, y, y, y, [x_2, x_3], [x_4, x_5], \cdots, [x_{4m-6}, x_{4m-5}])) = \lambda \sum a_{ijk} tr(A^{4m-2}_{ijk}),$$

where λ is a positive integer, the sum is over all $2 \le i < j < k \le 4m-2$, i even, j odd, k even, and

$$a_{ijk} = (-1)^{\frac{i+j+k-1}{2}}.$$

Proof. For all terms in

$$\begin{aligned} h(x_1, y, y, y, &[x_2, x_3], [x_4, x_5], \cdots, [x_{4m-6}, x_{4m-5}]) \\ &= + tr(x_1 yyy[x_2, x_3] \cdots [x_{4m-8}, x_{4m-7}][x_{4m-6}, x_{4m-5}]) \\ &\quad - tr(x_1 yyy[x_2, x_3] \cdots [x_{4m-6}, x_{4m-5}][x_{4m-8}, x_{4m-7}]) \\ &\quad \vdots \\ &\quad - tr(x_1 [x_{4m-6}, x_{4m-5}][x_{4m-8}, x_{4m-7}] \cdots [x_2, x_3] yyy), \end{aligned} \tag{1}$$

x_1 appears first and then some product of three y's and $2m-3$ commutators. This shows that when ϕ is applied to (1) we obtain a linear combination of the $tr(A_{ijk}^{4m-2})$ where $2 \leq i < j < k \leq 4m-2$ and where i is even, j is odd, k is even.

We now consider the coefficients a_{ijk}. We will use repeatedly the fact that the sign of each term in (1) is determined by the position of $[x_{4m-6}, x_{4m-5}]$, and that (1) is symmetric in the remaining commutators (Lemma 2.5, Lemma 2.3, respectively).

If we apply ϕ to the first term in (1) we obtain

$$+2^{2m-3} tr(A_{234}^{4m-2}),$$

where 2^{2m-3} appears since there are $2m-3$ commutators. Now consider all terms in (1) which contribute to $tr(A_{234}^{4m-2})$, and apply ϕ to these terms. We obtain the alternating expression

$$\begin{aligned} &+ 2^{2m-3}(2m-4)!\, tr(A_{234}^{4m-2}) - 2^{2m-3}(2m-4)!\, tr(A_{234}^{4m-2}) + \cdots \\ &\quad + 2^{2m-3}(2m-4)!\, tr(A_{234}^{4m-2}) - 2^{2m-3}(2m-4)!\, tr(A_{234}^{4m-2}) \\ &\quad\quad + 2^{2m-3}(2m-4)!\, tr(A_{234}^{4m-2}), \end{aligned}$$

where the first term here is the contribution made by terms in (1) that have the commutator $[x_{4m-6}, x_{4m-5}]$ at the extreme right, and the last term here is the contribution made by terms in (1) that have $[x_{4m-6}, x_{4m-5}]$ at the extreme left. The sign of each contribution is determined by the position of $[x_{4m-6}, x_{4m-5}]$ in the corresponding terms in (1), and there are $(2m-4)!$ terms for each of the $2m-3$ positions that $[x_{4m-6}, x_{4m-5}]$ may take. This alternating expression simplifies to

$$+2^{2m-3}(2m-4)!\, tr(A_{234}^{4m-2}),$$

since there is an odd number $(2m-3)$ of terms and the non-cancelling term appears with a $+$.

Now note that the sign in this last expression depends on the distribution of the y's in the contributing terms. Because all y's were at the extreme left, the contributions to A_{234}^{4m-2} appear first with a $+$ ($[x_{4m-6}, x_{4m-5}]$ in the extreme right position), and then appear with alternating signs *without interruption* as $[x_{4m-6}, x_{4m-5}]$ moves from right to left. So A^{4m-2}_{234} appears with a $+$, but if we now consider the coefficient of A^{4m-2}_{236} (the last y has moved one position to the right), the sign changes:

$$-2^{2m-3}(2m-4)!\,A^{4m-2}_{236}.$$

This is because the contributions are no longer alternating in sign, i.e.,

$$+,-,+,-,\cdots,+,-,+$$

but alternating only until the last term (there is now an interruption):

$$+,-,+,-\cdots,+,-,-.$$

The formula for a_{ijk} may now be deduced. We would like a_{ijk} to switch signs every time a y is moved one position to the right or left (i.e., when i, j or k goes up or down by 2, subject to $2 \leq i < j < k \leq 4m-2$), and we would like to have $a_{234} = +1$. Thus

$$a_{ijk} = (-1)^{\frac{i+j+k-1}{2}}$$

will suffice.

It is clear that the coefficient $2^{2m-3}(2m-4)!$ is common to *all* terms, so $\lambda = 2^{2m-3}(2m-4)!$. $\square$

Now multilinearize each $A^{4m-2}_{ijk} = A^{4m-2}_{ijk}(y)$ in y to get $A^{4m-2}_{ijk}(y_1, y_2, y_3)$.

Lemma 2.9. *We have*

$$tr(A^{4m-2}_{ijk}(z,y,y)) = 2\,tr((A^{4m-3}_{(4m-2)-(k-i),(4m-2)-(k-j)} + A^{4m-3}_{k-j,(4m-2)-(j-i)} + A^{4m-3}_{j-i,k-i})z)$$

where $A^{4m-3}_{r,s} = A^{4m-3}_{r,s}(y)$ is as in definition 1.1.3 (the comma is present only to separate indices).

Proof. In $tr(A^{4m-2}_{ijk}(z,y,y))$, z is in position i, j or k. We would like to move this z all the way to the right by applying cyclic permutations of the variables under the trace. If z is in position i, the term is

$$2\,tr(\sum(-1)^{\alpha}x_{\alpha(1)}\cdots x_{\alpha(i-1)}zx_{\alpha(i)}\cdots x_{\alpha(j-2)}yx_{\alpha(j-1)}\cdots x_{\alpha(k-3)}yx_{\alpha(k-2)}\cdots)$$

(the 2 appears since $A^{4m-2}_{ijk}(y_1, y_2, y_3)$ is symmetric in y_2, y_3). Moving z all the way to the right we get

$$2\,tr(\sum(-1)^{\alpha}x_{\alpha(i)}\cdots x_{\alpha(j-2)}yx_{\alpha(j-1)}\cdots x_{\alpha(k-3)}yx_{\alpha(k-2)}\cdots x_{\alpha(1)}\cdots x_{\alpha(i-1)}z),$$

which is $+2\,tr(A^{4m-3}_{j-i,k-i}z)$. To see that this appears with a $+$ and not a $-$, note first that there are an odd number $((4m-2)-3)$ of x_i's. This ensures the difference between the ordered sets

$$\{x_{\alpha(1)},\cdots,x_{\alpha(i-1)},x_{\alpha(i)},\cdots,x_{\alpha((4m-2)-3)}\}$$

and

$$\{x_{\alpha(i)},\cdots,x_{\alpha((4m-2)-3)},x_{\alpha(1)},\cdots,x_{\alpha(i-1)}\}$$

is an even permutation of the x_i's, and thus our new alternating sum appears with a $+$.

If z is in position j the term is

$$2\,tr(\sum(-1)^{\alpha}x_{\alpha(1)}\cdots y\cdots x_{\alpha(j-2)}zx_{\alpha(j-1)}\cdots y\cdots).$$

Moving z all the way to the right we obtain

$$2\,tr(\sum(-1)^{\alpha}x_{\alpha(j-1)}\cdots y\cdots x_{\alpha(1)}\cdots y\cdots x_{\alpha(j-2)}z)$$

which is $+2\,tr(A^{4m-3}_{k-j,(4m-2)-(j-i)}z)$.

Finally, if z is in position k the term is

$$2\,tr(\sum(-1)^{\alpha}x_{\alpha(1)}\cdots y\cdots y\cdots x_{\alpha(k-3)}zx_{\alpha(k-2)}\cdots).$$

Moving z all the way to the right we obtain

$$2\,tr(\sum(-1)^{\alpha}x_{\alpha(k-2)}\cdots x_{\alpha(1)}\cdots y\cdots y\cdots x_{\alpha(k-3)}z),$$

which is $+2\,tr(A^{4m-3}_{(4m-2)-(k-i),(4m-2)-(k-j)}z)$. □

Now multilinearize each $A^{4m-3}_{r,s}=A^{4m-3}_{r,s}(y)$ in y to obtain $A^{4m-3}_{r,s}(y_1,y_2)$.

Lemma 2.10. *Making the replacements $y_1\mapsto y$, $y_2\mapsto x_{4m-4}$ and applying ϕ gives*

$$\phi(A^{4m-3}_{r,s}(y,x_{4m-4}))=(4m-5)!((-1)^{s+1}A^{4m-3}_{r}+(-1)^{r}A^{4m-3}_{s}).$$

Proof. In $A^{4m-3}_{r,s}(y,x_{4m-4})$, y is in position r or s. If it is in position r, the term is

$$\cdots y\cdots x_{4m-4}\cdots.$$

Applying ϕ gives $\pm(4m-5)!\,A^{4m-3}_{r}$ where the sign is determined by s: the position of x_{4m-4} in the above term. The above term is alternating in $x_1,x_2,\cdots,x_{4m-5}$. We count the number of transpositions $x_{4m-4} \leftrightarrow x_i$, $i = 1, 2, \cdots, 4m-5$, it takes to move x_{4m-4} all the way to the right. If this number is odd, then $-$ is our sign, otherwise it is $+$. The number of moves required is $(4m-3) - s$, thus the sign is $(-1)^{(4m-3)-s}$ or $(-1)^{s+1}$.

If y is in position s, then the term is

$$\cdots x_{4m-4} \cdots y \cdots .$$

Applying ϕ gives $\pm(4m-5)!\, A_s^{4m-3}$. To move x_{4m-4} all the way to the right requires $(4m-3) - r - 1$ transpositions $x_{4m-4} \leftrightarrow x_i$, so the sign is $(-1)^{(4m-3)-r-1}$ or $(-1)^r$. □

We now apply Lemmas 2.8, 2.9, and 2.10 to h, a pure trace PI for $K_{2m}(F,t)$. At each step in the following we start with a (trace) PI, apply a lemma, and end up with a PI, so after these three applications (and some additional work) we will have some (trace) PI for $K_{2m}(F,t)$. We show that this final PI is a multiple of q_m, the polynomial that we are trying to prove is an identity.

Lemma 2.8 applied to the trace-PI h gives

$$\sum_{\substack{2 \le i < j < k \le 4m-2 \\ i,k \ even \\ j \ odd}} a_{ijk} tr(A_{ijk}^{4m-2}) \tag{2}$$

where we have ignored the scalar multiple λ (this is a valid simplification as $char(F) = 0$). We will ignore scalar multiples in all that follows. Applying Lemma 2.9 to (2) we obtain:

$$\sum a_{ijk} tr((A^{4m-3}_{(4m-2)-(k-i),(4m-2)-(k-j)} + A^{4m-3}_{k-j,(4m-2)-(j-i)} + A^{4m-3}_{j-i,k-i})z) \tag{3}$$

Applying Lemma 2.10 to (3) we obtain:

$$\sum a_{ijk} tr((A^{4m-3}_{4m-2-(k-i)} + A^{4m-3}_{4m-2-(k-j)} + A^{4m-3}_{k-j} - A^{4m-3}_{4m-2-(j-i)} - A^{4m-3}_{j-i} - A^{4m-3}_{k-i})z),$$

where the inner bracket (reordered) is

$$(A^{4m-3}_{4m-2-(k-i)} - A^{4m-3}_{k-i}) + (A^{4m-3}_{4m-2-(k-j)} + A^{4m-3}_{k-j}) - (A^{4m-3}_{4m-2-(j-i)} + A^{4m-3}_{j-i}).$$

The first bracket here is symmetric and the remaining brackets are skew since we have (the easily verified)

$$(A_r^{4m-3})^t = -A^{4m-3}_{4m-2-r}, \quad 1 \le r \le 4m-3.$$

We may remove the symmetric bracket (trace of a symmetric times a skew (z) is zero) as well as the trace and the z (non-degeneracy of the trace on the space of skew elements) to obtain the following PI for $K_{2m}(F, t)$:

$$\begin{aligned}&\sum a_{ijk}((A^{4m-3}_{4m-2-(k-j)} + A^{4m-3}_{k-j}) - (A^{4m-3}_{4m-2-(j-i)} + A^{4m-3}_{j-i})) \\ &= \sum a_{ijk}(A^{4m-3}_{4m-2-(k-j)} + A^{4m-3}_{k-j}) - \sum a_{ijk}(A^{4m-3}_{4m-2-(j-i)} + A^{4m-3}_{j-i}).\end{aligned} \tag{4}$$

These two sums may be reduced to a single sum as follows. Compare the term

$$a_{ijk}(A^{4m-3}_{4m-2-(k-j)} + A^{4m-3}_{k-j})$$

appearing in the first sum with the term

$$a_{4m-k,4m-j,4m-i}(A^{4m-3}_{4m-2-(k-j)} + A^{4m-3}_{k-j})$$

which appears in the second sum. This second term's coefficient is

$$\begin{aligned}a_{4m-k,4m-j,4m-i} &= (-1)^{\frac{(4m-k)+(4m-j)+(4m-i)-1}{2}} \\ &= (-1)^{\frac{-i-j-k-1}{2}} \\ &= (-1)^{\frac{-(i+j+k-1)-2}{2}} \\ &= -(-1)^{\frac{i+j+k-1}{2}} \\ &= -a_{ijk}.\end{aligned}$$

This establishes a one-to-one correspondence between the terms in the first sum and the terms in the second sum. From this we see that the difference of these two sums is just double the first sum, so (4) becomes

$$2\sum a_{ijk}(A^{4m-3}_{4m-2-(k-j)} + A^{4m-3}_{k-j})$$

and we restrict our attention to this single sum. Our degree is fixed from here on, so we drop the superscript $4m-3$ (and the coefficient 2):

$$\sum_{\substack{2\le i<j<k\le 4m-2 \\ i,k\ even \\ j\ odd}} a_{ijk}(A_{k-j} + A_{4m-2-(k-j)}).$$

Putting $l = k - j$ we see that this sum may be written over

$$\begin{aligned}1 \le\ & l \le 4m-5 \\ l+3 \le\ & k \le 4m-2 \\ 2 \le\ & i \le k-l-1\end{aligned}$$

that is,

$$\sum_{\substack{l=1\\ l\ odd}}^{4m-5}\ \sum_{\substack{k=l+3\\ k\ even}}^{4m-2}\ \sum_{\substack{i=2\\ i\ even}}^{k-l-1} a_{ijk}(A_l + A_{4m-2-l}).$$

But

$$a_{ijk} = (-1)^{\frac{i+j+k-1}{2}} = (-1)^{\frac{i+(k-l)+k-1}{2}} = (-1)^{\frac{i+2k-l-1}{2}} = (-1)^{\frac{i-l-1}{2}}$$

so the sum becomes

$$\sum_{\substack{l=1\\ l\ odd}}^{4m-5}\ \sum_{\substack{k=l+3\\ k\ even}}^{4m-2}\ \sum_{\substack{i=2\\ i\ even}}^{k-l-1} (-1)^{\frac{i-l-1}{2}}(A_l + A_{4m-2-l})$$

or

$$\sum_{\substack{l=1\\ l\ odd}}^{4m-5} (A_l + A_{4m-2-l}) \sum_{\substack{k=l+3\\ k\ even}}^{4m-2}\ \sum_{\substack{i=2\\ i\ even}}^{k-l-1} (-1)^{\frac{i-l-1}{2}}.$$

Now consider

$$\begin{aligned}\sum_{\substack{i=2\\ i\ even}}^{k-l-1} (-1)^{\frac{i-l-1}{2}} &= (-1)^{\frac{-l+1}{2}} + (-1)^{\frac{-l+3}{2}} + \cdots + (-1)^{\frac{-l+(k-l-1)-1}{2}}\\ &= (-1)^{\frac{-l+1}{2}} + (-1)^{\frac{-l+3}{2}} + \cdots + (-1)^{\frac{k-2l-2}{2}}\\ &= (-1)^{\frac{1-l}{2}}(1-1+1-\cdots+(-1)^{\frac{k-l-3}{2}}).\end{aligned}$$

One can verify that this is -1 when $(k,l) = (2,3)$ $(mod\ 4)$, $+1$ when $(k,l) = (0,1)$ $(mod\ 4)$, and 0 otherwise. Breaking the sum over l into a part over $l = 1$ $(mod\ 4)$ and a part over $l = 3$ $(mod\ 4)$ we obtain:

$$\sum_{\substack{l=1\\ l=1(mod\ 4)}}^{4m-5} (A_l + A_{4m-2-l}) \sum_{\substack{k=l+3\\ k=0(mod\ 4)}}^{4m-2} (+1) + \sum_{\substack{l=3\\ l=3(mod\ 4)}}^{4m-5} (A_l + A_{4m-2-l}) \sum_{\substack{k=l+3\\ k=2(mod\ 4)}}^{4m-2} (-1).$$

To calculate

$$\sum_{\substack{k=l+3\\ k=0(mod\ 4)}}^{4m-2} (+1)$$

we ask: if $k = 0$ $(mod\ 4)$ and $l = 1$ $(mod\ 4)$, how many k's are there between $l+3$ and $4m-2$ (inclusive)? The appropriate k's are $l+3, l+7, \cdots$, $4m-4$, a total of $\frac{4m-(l+3)}{4}$. So

$$\sum_{\substack{k=l+3\\ k=0(mod\ 4)}}^{4m-2} (+1) = \frac{4m-(l+3)}{4} = m - \frac{l+3}{4}$$

and a similar calculation yields

$$\sum_{\substack{k=l+3 \\ k=2(mod\ 4)}}^{4m-2} (-1) = -\frac{(4m+2)-(l+3)}{4} = -(m - \frac{l+1}{4}).$$

Applying these formulas to our overall sum we get

$$\sum_{\substack{l=1 \\ l=1(mod\ 4)}}^{4m-5} (A_l + A_{4m-2-l})(m - \frac{l+3}{4}) + \sum_{\substack{l=3 \\ l=3(mod\ 4)}}^{4m-5} (A_l + A_{4m-2-l})(-(m - \frac{l+1}{4})).$$

The first sum is

$$+(m-1)(A_1 + A_{4m-3}) + (m-2)(A_5 + A_{4m-7}) + \cdots + (1)(A_{4m-7} + A_5)$$

the second is

$$-(m-1)(A_3 + A_{4m-5}) - (m-2)(A_7 + A_{4m-9}) - \cdots - (1)(A_{4m-5} + A_3),$$

and adding these together we obtain the identity

$$(m-1)A_1 - mA_3 + (m-1)A_5 - mA_7 - \cdots - mA_{4m-5} + (m-1)A_{4m-3},$$

which is q_m, thus proving Theorem 2.1. □

Proof of Corollary 2.1

In this section we prove

Corollary 2.1. *Let F be a field, $n > 1$ an integer. s_{2n-2} is an identity for $K_n(F,t)$.*

We will need this result to prove Corollaries 2.2 and 2.3 below, thus it is appropriate to show how it follows from q_m, $n = 2m$. As in the proof of the theorem, we may work in characteristic 0 and yet obtain the result in arbitrary characteristic.

We begin with a useful technical lemma.

Lemma 2.11. *Let $k > 2$ be an integer. Embed $M_{k-1}(F)$ in the upper left hand corner of $M_k(F)$ and let $A \in M_{k-1}(F) \subset M_k(F)$. If for all $i = 1, 2, \cdots, k-1$*

$$A(e_{ik} - e_{ki}) + (e_{ik} - e_{ki})A^t = 0$$

or for all $i = 1, 2, \cdots, k-1$

$$A(e_{ik} - e_{ki}) - (e_{ik} - e_{ki})A^t = 0$$

then $A = 0$.

Proof. We put $A = \sum_{r,s} a_{rs}e_{rs}$, $1 \le r, s \le k-1$, and calculate

$$\begin{aligned} 0 = A(e_{ik} - e_{ki}) \pm (e_{ik} - e_{ki})A^t &= A(e_{ik} - e_{ki}) \mp (A(e_{ik} - e_{ki}))^t \\ &= (\sum_{r,s} a_{rs}e_{rs})(e_{ik} - e_{ki}) \mp ((\sum_{r,s} a_{rs}e_{rs})(e_{ik} - e_{ki}))^t \\ &= (\sum_{1 \le r \le k-1} a_{ri}e_{rk}) \mp (\sum_{1 \le r \le k-1} a_{ri}e_{rk})^t \\ &= \sum_{1 \le r \le k-1} a_{ri}(e_{rk} \mp e_{kr}). \end{aligned}$$

This final expression is a zero matrix except for the last row

$$[\mp a_{1i} \ \mp a_{2i} \ \cdots \ \mp a_{(k-1)i} \ 0]$$

and the last column

$$[a_{1i} \ a_{2i} \ \cdots a_{(k-1)i} \ 0]^t.$$

Thus column i of A is zero, and as i is arbitrary with $1 \le i \le k-1$ we see that $A = 0$. □

We restate our PI q_m (a PI for $K_n(F,t)$, $n = 2m \geq 4$):

$$q_m = (m-1)A_1^{4m-3} - mA_3^{4m-3} + (m-1)A_5^{4m-3} - \cdots - mA_{4m-5}^{4m-3} + (m-1)A_{4m-3}^{4m-3}.$$

Assume that the x_j, $j = 1, 2, \cdots, 4m-4$, are variables from K_{2m-1} where we canonically embed K_{2m-1} in the upper left corner of K_{2m}. Replace y with

$$y := e_{i(2m)} - e_{(2m)i}, \quad 1 \leq i \leq 2m-1.$$

Since $K_{2m-1} y K_{2m-1} = \{0\}$, all summands in q_m but the first and last are 0 and we obtain

$$(m-1)(ys_{4m-4} + s_{4m-4}y).$$

s_{4m-4} is symmetric with respect to the transpose (Lemma 1.1.3) thus (dividing by $m-1$)

$$s_{4m-4}y + ys_{4m-4}^t$$

is zero for any substitution (from K_{2m-1}) for the x_j, $j = 1, 2, \cdots, 4m-4$. By Lemma 2.11 we conclude that s_{4m-4} is an identity for K_{2m-1}. This proves the odd case: $s_{4m-4} = s_{2(2m-1)-2}$ is a PI for K_{2m-1}. For the even case, we follow methods found in [Row1] to go down in polynomial degree again.

Starting with

$$s_{4m-4} = s_{4m-4}(x_1, \cdots, x_{4m-4}),$$

which we have just shown to be a PI for K_{2m-1}, assume that the x_j, $j = 1, \cdots, 4m-5$ are variables from K_{2m-2} (canonically embedded in K_{2m-1}), and replace x_{4m-4} with

$$e_{i(2m-1)} - e_{(2m-1)i}, \quad 1 \leq i \leq 2m-2.$$

This yields

$$-x_{4m-4}s_{4m-5} + s_{4m-5}x_{4m-4}$$

since

$$K_{2m-2}(e_{i(2m-1)} - e_{(2m-1)i})K_{2m-2} = \{0\}.$$

Now $s_{4m-5} = {s_{4m-5}}^t$ (Lemma 1.1.3) so

$$s_{4m-5}x_{4m-4} - x_{4m-4}{s_{4m-5}}^t$$

is zero for any substitution (from K_{2m-2}) for the x_j, $j = 1, 2, \cdots, 4m-5$. Applying Lemma 2.11 we see that s_{4m-5} is an identity for K_{2m-2} (recall: $m > 1$).

We may again reduce the degree by one to reach our goal: to prove that $s_{4m-6} = s_{2(2m-2)-2}$ is a PI for K_{2m-2}. Consider $tr(s_{4m-5})$, a pure-trace PI for K_{2m-2}. Permuting cyclically under the trace, we would like to move x_{4m-5} to the extreme right in all terms. This yields

$$(4m-5)tr(s_{4m-6}x_{4m-5}).$$

To see this, it suffices to consider a term in $tr(s_{4m-5})$ with x_{4m-5} in position k, say

$$(-1)^{(4m-5)-k} tr(x_1 x_2 \cdots x_{k-1} x_{4m-5} x_k x_{k+1} \cdots x_{4m-6})$$

and to ask what happens when x_{4m-5} is moved to the right. Moving x_{4m-5} to the right in this term gives

$$(-1)^{(4m-5)-k} tr(x_k x_{k+1} \cdots x_{4m-6} x_1 x_2 \cdots x_{k-1} x_{4m-5}).$$

Now a permutation with sign $(-1)^{k+1}$ of the indices in

$$x_k x_{k+1} \cdots x_{4m-6} x_1 x_2 \cdots x_{k-1}$$

gives

$$x_1 x_2 \cdots x_{k-1} x_k x_{k+1} \cdots x_{4m-6}$$

(for example, if k is odd, an even number of indices are moving from right to left via an even number of transpositions). Thus if we now take all terms in $tr(s_{4m-5})$ with x_{4m-5} in position k and move x_{4m-5} to the right (under the trace, in all terms), we get

$$(-1)^{(4m-5)-k}(-1)^{k+1} tr(s_{4m-6} x_{4m-5}) = tr(s_{4m-6} x_{4m-5}).$$

There are are $4m-5$ positions options for k, so

$$tr(s_{4m-5}) = (4m-5) tr(s_{4m-6} x_{4m-5}).$$

Thus $tr(s_{4m-6} x_{4m-5})$ is a pure-trace PI for K_{2m-2}. But s_{4m-6} is skew with respect to the transpose involution (Lemma 1.1.3), so non-degeneracy of the trace says that $s_{4m-6} = s_{2(2m-2)-2}$ is a PI for K_{2m-2}. This proves the even case of Corollary 2.1.

□

Proof of Corollary 2.2

In this section we prove

Corollary 2.2. *Let F be a field, $m > 1$ a positive integer. Then the polynomials*

$$\begin{aligned} E_{4m-2}(x_1,\ldots,x_{4m-3};y) :&= A_1^{4m-2} - A_2^{4m-2} + A_5^{4m-2} - A_6^{4m-2} \\ &\qquad + \cdots + A_{4m-3}^{4m-2} - A_{4m-2}^{4m-2} \\ &= \sum_{\substack{i=1,2(mod\ 4) \\ 1\le i\le 4m-2}} (-1)^{i-1} A_i^{4m-2} \end{aligned}$$

and

$$\begin{aligned} E'_{4m-2}(x_1,\cdots,x_{4m-3};y) :&= A_3^{4m-2} - A_4^{4m-2} + A_7^{4m-2} - A_8^{4m-2} \\ &\qquad + \cdots + A_{4m-5}^{4m-2} - A_{4m-4}^{4m-2} \\ &= \sum_{\substack{i=0,3(mod\ 4) \\ 1\le i\le 4m-2}} (-1)^{i-1} A_i^{4m-2}, \end{aligned}$$

with sum

$$E_{4m-2} + E'_{4m-2} = s_{4m-2}(y,x_1,\cdots,x_{4m-3}),$$

are polynomial identities for $K_{2m}(F,t)$.

We have already obtained a different proof of this result [JDH]; here we give a proof based on q_m. As in the proof of the theorem, we may work in characteristic 0 and yet obtain the result in arbitrary characteristic. We would like to compute

$$\phi([q_m,x_{4m-3}]) = \phi((m-1)\sum_{i=1}^{m}[A_{4i-3}^{4m-3},x_{4m-3}] - m\sum_{i=1}^{m-1}[A_{4i-1}^{4m-3},x_{4m-3}]). \qquad (3)$$

First we compute

$$\begin{aligned} \phi([A_k^{4m-3},x_{4m-3}]) &= \phi(A_k^{4m-3}x_{4m-3} - x_{4m-3}A_k^{4m-3}) \\ &= (4m-2)!(A_k^{4m-2} - A_{k+1}^{4m-2}). \end{aligned}$$

To see this, note that each of the two sums A_k^{4m-3} and A_{k+1}^{4m-3} have $(4m-2)!$ terms. So $\frac{1}{(4m-2)!}(3)$ is

$$(m-1)\sum_{i=1}^{m}(A_{4i-3}^{4m-2} - A_{4i-2}^{4m-2}) - m\sum_{i=1}^{m-1}(A_{4i-1}^{4m-2} - A_{4i}^{4m-2}) \qquad (3')$$

Now, Corollary 2.1 (Theorem 1.2.2) says that s_{4m-2}, written as (see example 1.1.3 above)

$$\begin{aligned} s_{4m-2} &= A_1^{4m-2} - A_2^{4m-2} + A_3^{4m-2} - A_4^{4m-2} + \dots . + A_{4m-3}^{4m-2} - A_{4m-2}^{4m-2} \\ &= \sum_{i=1}^{m} (A_{4i-3}^{4m-2} - A_{4i-2}^{4m-2}) + \sum_{i=1}^{m-1} (A_{4i-1}^{4m-2} - A_{4i}^{4m-2}) \end{aligned} \tag{4}$$

is an identity for $K_{2m}(F, t)$. Taking $(3') + m(4)$ we obtain

$$(2m-1) \sum_{i=1}^{m} (A_{4i-3}^{4m-2} - A_{4i-2}^{4m-2}),$$

and this proves that E_{4m-2} is a PI for K_{2m}. The equation

$$E_{4m-2} + E'_{4m-2} = s_{4m-2}$$

proves that E'_{4m-2} is also a PI for K_{2m}. □

Proof of Corollary 2.3.

In this section we prove

Corollary 2.3. *Let F be a field, $m > 1$ a positive integer. Then the polynomials*

$$\begin{aligned} R_{4m-4}(x_1,\cdots,x_{4m-5};y) : &= A_1^{4m-4} - A_4^{4m-4} + A_5^{4m-4} - A_8^{4m-4} \\ &\qquad + \cdots + A_{4m-7}^{4m-4} - A_{4m-4}^{4m-4} \\ &= \sum_{\substack{i=0,1(mod\ 4)\\ 1\le i\le 4m-4}} (-1)^{i-1} A_i^{4m-4} \end{aligned}$$

and

$$\begin{aligned} R'_{4m-4}(x_1,\cdots,x_{4m-5};y) : &= -A_2^{4m-4} + A_3^{4m-4} - A_6^{4m-4} + A_7^{4m-4} \\ &\qquad - \cdots - A_{4m-6}^{4m-4} + A_{4m-5}^{4m-4} \\ &= \sum_{\substack{i=2,3(mod\ 4)\\ 1\le i\le 4m-4}} (-1)^{i-1} A_i^{4m-4}, \end{aligned}$$

with sum

$$R_{4m-4} + R'_{4m-4} = s_{4m-4}(y, x_1, \cdots, x_{4m-3}),$$

are polynomial identities for $K_{2m-1}(F,t)$.

We begin with $q_m(x_1,\cdots,x_{4m-4},y)$, an identity for K_{2m}. Consider K_{2m-1} as the upper left hand corner of K_{2m}, thus $K_{2m-1} \subset K_{2m}$. Our first step will be to assume that the variables $x_1,\cdots,x_{4m-3}$, and y are variables from K_{2m-1}, and then to perform the substitution

$$x_{4m-4} := e_{i(2m)} - e_{(2m)i}, \quad 1 \le i \le 2m-1.$$

Now

$$K_{2m-1} x_{4m-4} K_{2m-1} = \{0\},$$

so the only surviving terms in q_m are those in which x_{4m-4} is at the extreme left or extreme right:

$$\begin{aligned} A_1^{4m-3} &= A_1^{4m-4} x_{4m-4} \\ A_{2r+1}^{4m-3} &= A_{2r+1}^{4m-4} x_{4m-4} - x_{4m-4} A_{2r}^{4m-4}, \quad 1 \le r \le 2m-2 \\ A_{4m-3}^{4m-3} &= -x_{4m-4} A_{4m-4}^{4m-4}, \end{aligned}$$

and that the overall effect on $q_m = q_m(x_1,\cdots,x_{4m-4},y)$ is

$$\begin{aligned}
q_m &= (m-1)A_1^{4m-3} - mA_3^{4m-3} + (m-1)A_5^{4m-3} - \cdots - mA_{4m-5}^{4m-3} + (m-1)A_{4m-3}^{4m-3} \\
&= (m-1)A_1^{4m-4}x_{4m-4} - m(A_3^{4m-4}x_{4m-4} - x_{4m-4}A_2^{4m-4}) \\
&\quad + (m-1)(A_5^{4m-4}x_{4m-4} - x_{4m-4}A_4^{4m-4}) - \cdots \\
&\qquad - m(A_{4m-5}^{4m-4}x_{4m-4} - x_{4m-4}A_{4m-6}^{4m-4}) + (m-1)(-x_{4m-4}A_{4m-4}^{4m-4}).
\end{aligned}$$

This is (dropping the superscript as our degree will remain at $4m-4$):

$$\begin{aligned}
&((m-1)A_1 - mA_3 + (m-1)A_5 - \cdots + (m-1)A_{4m-7} - mA_{4m-5})x_{4m-4} \\
&\quad + x_{4m-4}(mA_2 - (m-1)A_4 + mA_6 - \cdots + mA_{4m-6} - (m-1)A_{4m-4}) \\
&= ((m-1)A_1 - mA_3 + (m-1)A_5 - \cdots + (m-1)A_{4m-7} - mA_{4m-5})x_{4m-4} \\
&\quad - x_{4m-4}((m-1)A_1 - mA_3 + (m-1)A_5 - \cdots + (m-1)A_{4m-7} - mA_{4m-5})^t,
\end{aligned}$$

where we have used the formula (easily verified):

$$A_l^{4m-4} =: A_l = -{A_{4m-3-l}}^t.$$

Making the definition

$$A := ((m-1)A_1 - mA_3 + (m-1)A_5 - \cdots + (m-1)A_{4m-7} - mA_{4m-5})$$

this may be written

$$Ax_{4m-4} - x_{4m-4}A^t.$$

We now apply Lemma 2.11 to conclude that $A = A(x_1, \cdots, x_{4m-5}, y)$ is a PI for K_{2m-1}. Thus $A + A^t$ is a PI for K_{2m-1} :

$$\begin{aligned}
A + A^t = (m-1)A_1 &+ mA_2 - mA_3 - (m-1)A_4 \\
&+ (m-1)A_5 + mA_6 - mA_7 - (m-1)A_8 + \cdots \\
&\quad + (m-1)A_{4m-7} + mA_{4m-6} - mA_{4m-5} - (m-1)A_{4m-4},
\end{aligned}$$

and (by Theorem 1.2.3) $s_{2(2m-1)-2} = s_{4m-4}$ is also a PI here:

$$\begin{aligned}
s_{4m-4} = A_1 &- A_2 + A_3 - A_4 \\
&+ A_5 - A_6 + A_7 - A_8 + \cdots \\
&\quad + A_{4m-7} - A_{4m-6} + A_{4m-5} - A_{4m-4}.
\end{aligned}$$

Taking $ms_{4m-4} + (A + A^t)$ we obtain

$$(2m-1)(A_1 - A_4 + A_5 - A_8 + \cdots + A_{4m-5} - A_{4m-4}).$$

This is precisely $(2m-1)R_m$. So R_m is an identity for K_{2m-1}, and $R_m + R'_m = s_{4m-4}$ proves that R'_m is also an identity for K_{2m-1}.

□

A pair of remarks concerning the transpose case:

1. It is important to emphasize that the difference between the even and the odd case (Corollaries 2.2 and 2.3) is more than just apparent. As we have just seen,

$$E_{4m-2} = \sum_{\substack{i=1,2 (mod\ 4) \\ 1 \le i \le 4m-2}} (-1)^{i-1} A_i^{4m-2}$$

is a PI for K_{2m} and

$$R_{4m-4} = \sum_{\substack{i=0,1 (mod\ 4) \\ 1 \le i \le 4m-4}} (-1)^{i-1} A_i^{4m-4}$$

is a PI for K_{2m-1}. However, we have performed computer-aided computations which show that, for $m = 3$,

$$E_8 := \sum_{\substack{i=1,2 (mod\ 4) \\ 1 \le i \le 4m-4}} (-1)^{i-1} A_i^{4m-4} = A_1^8 - A_2^8 + A_5^8 - A_6^8$$

is not a PI for K_{2m-1} and that for $m = 2$

$$R_6 := \sum_{\substack{i=0,1 (mod\ 4) \\ 1 \le i \le 4m-2}} (-1)^{i-1} A_i^{4m-2} = A_1^6 - A_4^6 + A_5^6$$

is not a PI for K_{2m}.

In particular, we have performed computer-aided computations which show the following. For the case of PIs alternating in all but one variable, $K_4(F,t)$'s space of PIs of degree 4 is null, its space of PIs of degree 5 is spanned by $q_2 = A_1^5 - 2A_3^5 + A_5^5$, and its space of degree 6 is spanned by

$$\{E'_6,\ A_1^6 - 2A_3^6 + A_5^6,\ A_2^6 - 2A_3^6 + A_6^6\},$$

where $E'_6 = A_3^6 - A_4^6$ (one can check that R_6 is not in this space); $K_5(F,t)$'s space of degree 7 is null, and its space of degree 8 is spanned by

$$\{R_8,\ 2A_1^8 - 3A_3^8 + 2A_5^8 - 3A_7^8,\ 2A_1^8 - 3A_2^8 + 2A_5^8 - 3A_6^8\}$$

(one can check that E_8 is not in this space).

2. Our methods beg the question: can Corollaries 2.1 or 2.2 be proven by looking *only* at the T-ideal generated by q_m? In other words, are any of s_{4m-2}, E_{4m-2}, E'_{4m-2} in this T-ideal? We do not have a proof either way, but based on computer calculations the answer appears to be no.

3. The Symplectic Case

In Section 1.3 we saw that the standard polynomial s_{4m-2} is a PI for $H_{2m}(F, s)$ and that at least for $m = 2, 3, 4$, s_{4m-3} is not a PI. In this chapter we will prove that the polynomial q_m, in addition to being a PI for $K_{2m}(F, t)$, is also a PI for $H_{2m}(F, s)$, thus showing that $4m-2$ is not the minimal degree of a $*$-PI for $(M_{2m}(F), s)$.

Theorem 3.1. *Let F be a field and $m > 1$ a positive integer, and let s be the symplectic involution on $2m \times 2m$ matrices over F. Then*

$$\begin{aligned} q_m(x_1, \ldots, x_{4m-4}; y) := & (m-1) \sum_{i=1}^{m} A_{4i-3}^{4m-3} - m \sum_{i=1}^{m-1} A_{4i-1}^{4m-3} \\ = & (m-1) A_1^{4m-3} - m A_3^{4m-3} + (m-1) A_5^{4m-3} - m A_7^{4m-3} + \\ & \ldots - m A_{4m-5}^{4m-3} + (m-1) A_{4m-3}^{4m-3} \end{aligned}$$

is a polynomial identity for $H_{2m}(F, s)$ of degree $4m-3$.

Note that since a prime p cannot divide both m and $m-1$, q_m is not the zero polynomial over any field. We also show that Theorem 3.1 yields Theorem 1.3.1 as a corollary. This result will be essential in our proof of Corollary 3.2, a refinement of Theorem 1.3.1.

Corollary 3.1 (Theorem 1.3.1). *Let F be a field, $m > 0$ an integer. s_{4m-2} is an identity for $K_{2m}(F, t)$.*

Corollary 3.2. *Let F be a field, $m > 1$ a positive integer. Then the polynomials*

$$\begin{aligned} E_{4m-2}(x_1, \ldots, x_{4m-3}; y) : = & A_1^{4m-2} - A_2^{4m-2} + A_5^{4m-2} - A_6^{4m-2} \\ & + \cdots + A_{4m-3}^{4m-2} - A_{4m-2}^{4m-2} \\ = & \sum_{\substack{i=1,2 (mod\ 4) \\ 1 \le i \le 4m-2}} (-1)^{i-1} A_i^{4m-2} \end{aligned}$$

and

$$\begin{aligned} E'_{4m-2}(x_1, \ldots, x_{4m-3}; y) : = & A_3^{4m-2} - A_4^{4m-2} + A_7^{4m-2} - A_8^{4m-2} \\ & + \cdots + A_{4m-5}^{4m-2} - A_{4m-4}^{4m-2} \\ = & \sum_{\substack{i=0,3 (mod\ 4) \\ 1 \le i \le 4m-2}} (-1)^{i-1} A_i^{4m-2}, \end{aligned}$$

with sum

$$E_{4m-2} + E'_{4m-2} = s_{4m-2}(y, x_1, \ldots, x_{4m-3})$$

are polynomial identities for $H_{2m}(F, s)$.

Thus even at degree $4m-2$, s_{4m-2} does not generate the PIs for $H_{2m}(F, s)$.

Proof of Theorem 3.1

As always we may work in characteristic 0. Having proven this special case, we may, in arbitrary characteristic, conclude that the polynomials vanish for all substitutions taken from some basis of $H_{2m}(F,s)$, say, the basis given by (6) in Section 1.1 (note that since only 0 and ±1 appear as entries, this is a basis of $H_{2m}(F,s)$ irrespective of characteristic). As our polynomials are multilinear, this proves the theorem in arbitrary characteristic. We handle the general case first and then (for extra detail) the sample case $m=3$.

We begin with the characteristic Pfaffian (or generic minimum polynomial for $H_{2m}(F,s)$) from Section 1.3:

$$X^m + \sum_{k=1}^{m}(-1)^k \mu_k X^{m-k},$$

$$\mu_0 = 1, \quad k\mu_k = \sum_{\imath=1}^{k}(-1)^{\imath-1}\mu_{k-\imath}T(X^\imath),$$

where $T = \frac{1}{2}tr$. Multilinearizing in X gives

$$\sum_{\sigma\in\mathcal{S}_m} X_{\sigma(1)}X_{\sigma(2)}\dots X_{\sigma(m)} + (a\ trace\ part).$$

We now replace X_1 with $\{y,x_2,x_3\} = yx_2x_3 + x_3x_2y$, X_2 with x_1, and $X_\imath$ $(i>2)$ with $Q_\imath := \frac{1}{2}(x_{4(\imath-2)}x_{4(\imath-2)+1}x_{4(\imath-2)+2}x_{4(\imath-2)+3} + x_{4(\imath-2)+3}x_{4(\imath-2)+2}x_{4(\imath-2)+1}x_{4(\imath-2)})$ to get

$$\{y,x_2,x_3\}x_1Q_3Q_4\dots Q_m + \dots + Q_m\dots Q_4Q_3x_1\{y,x_2,x_3\} + (a\ trace\ part). \quad (0)$$

This is a trace identity for $H_{2m}(F,s)$. We will consider the trace-free part of this polynomial and the trace part separately. Our goal will be to use the trace-free part of this equation to generate (in the sense of T-ideals) the polynomial $q_m(x_1,\dots,x_{4m-4};y)$, and then we will prove that the corresponding trace part (which accompanies this generation) is 0. The following two lemmas will aid us in "killing" this trace part. The key idea in both is that the trace of a monomial is fixed under cyclic permutations of the variables, and thus cancellations occur when we take alternating sums.

Lemma 3.1. *Let ϕ be the alternating sum in the $x_\jmath$'s (definition 1.1.2). Then*

$$\phi(T(\{y,x_2,x_3\}) = 0 = \phi(T(\{y,x_2,x_3\}x_{\imath_1}\dots x_{\imath_k}))$$

where k is a positive integer and $(i_1,\dots,i_k)$ is a permutation of $(1,4,5,6,\dots,k)$.

Proof. It is enough to calculate

$$\begin{aligned} T(\{y,x_2,x_3\}x_{\imath_1}\dots x_{\imath_k}) &= T(yx_2x_3x_{\imath_1}\dots x_{\imath_k} + x_3x_2yx_{\imath_1}\dots x_{\imath_k}) \\ &= T(yx_2x_3x_{\imath_1}\dots x_{\imath_k} + yx_{\imath_1}\dots x_{\imath_k}x_3x_2)\,, \end{aligned}$$

and this is sent to 0 by ϕ since the signatures of

$$id : (2, 3, i_1, i_2, \cdots, i_k) \mapsto (2, 3, i_1, i_2, \cdots, i_k)$$

and

$$(2, 3, i_1, i_2, \cdots, i_k) \mapsto (i_1, i_2, \cdots, i_k, 3, 2)$$

have opposite signs. ☐

Lemma 3.2 (Lemma 1.1.3a). *Let $r > 0$ be an integer. Then*

$$T(s_{2r}(x_1, \ldots, x_{2r})) = 0.$$

☐

Recall that A_k^{2l} is an alternating sum of degree $2l$ with a y in position k (definition 1.3).

Lemma 3.3. *Let l and k, $l > 0$, $0 < k \leq 2l$, be integers. Then*

$$a) \quad \frac{1}{(2l-1)!}\phi(A_k^{2l}(x_1, \ldots, x_{2l-1}, y \circ x_{2l})) = \begin{cases} A_k^{2l+1} + A_{k+1}^{2l+1} & k \text{ even} \\ -A_k^{2l+1} - A_{k+1}^{2l+1} & k \text{ odd} \end{cases}$$

$$b) \quad \frac{1}{(2l-1)!}\phi(A_k^{2l}(x_1, \ldots, x_{2l-1}, y) \circ x_{2l}) = A_k^{2l+1} - A_{k+1}^{2l+1}.$$

Proof. Calculate

$$\frac{1}{(2l-1)!}\phi(A_k^{2l}(x_1, \ldots, x_{2l-1}, y \circ x_{2l}))$$
$$= \frac{1}{(2l-1)!}\phi(\sum_{\sigma \in S_{2l-1}} (-1)^\sigma x_{\sigma(1)} \ldots x_{\sigma(k-1)}(y \circ x_{2l})x_{\sigma(k)} \ldots x_{\sigma(2l-1)}).$$

Under ϕ we may move x_{2l} all the way to the right to get

$$\frac{1}{(2l-1)!}\phi(\sum_{\sigma \in S_{2l-1}} (-1)^\sigma x_{\sigma(1)} \ldots x_{\sigma(k-1)}(y \circ x_{\sigma(k)})x_{\sigma(k+1)} \ldots x_{\sigma(2l-1)}x_{2l}(-1)^{2l-1-(k-1)}).$$

We performed $2l - 1 - (k - 1)$ transpositions to move x_{2l}, so the $(-1)^{2l-1-(k-1)}$ appears under ϕ. This equals $(-1)^k$, so we obtain

$$\frac{(-1)^k}{(2l-1)!}\phi(\sum_{\sigma \in S_{2l-1}} (-1)^\sigma x_{\sigma(1)} \ldots x_{\sigma(k-1)}(y \circ x_{\sigma(k)})x_{\sigma(k+1)} \ldots x_{\sigma(2l-1)}x_{2l})$$

which is

$$\frac{(-1)^k}{(2l-1)!}(2l-1)!(A_k^{2l+1} + A_{k+1}^{2l+1}) = (-1)^k(A_k^{2l+1} + A_{k+1}^{2l+1}).$$

This proves the first part of the lemma. For the second part we calculate

$$\begin{aligned}
&\frac{1}{(2l-1)!}\phi(A_k^{2l}(x_1,\dots,x_{2l-1},y)\circ x_{2l})\\
&= \frac{1}{(2l-1)!}\phi(\sum_{\sigma\in S_{2l-1}}(-1)^\sigma(x_{\sigma(1)}\dots x_{\sigma(k-1)}yx_{\sigma(k)}\dots x_{\sigma(2l-1)})\circ x_{2l})\\
&= \frac{1}{(2l-1)!}\phi(\sum_{\sigma\in S_{2l-1}}(-1)^\sigma(x_{\sigma(1)}\dots x_{\sigma(k-1)}yx_{\sigma(k)}\dots x_{\sigma(2l-1)}x_{2l}\\
&\qquad\qquad + x_{2l}x_{\sigma(1)}\dots x_{\sigma(k-1)}yx_{\sigma(k)}\dots x_{\sigma(2l-1)}))\\
&= \frac{1}{(2l-1)!}\phi(\sum_{\sigma\in S_{2l-1}}(-1)^\sigma(x_{\sigma(1)}\dots x_{\sigma(k-1)}yx_{\sigma(k)}\dots x_{\sigma(2l-1)}x_{2l}\\
&\qquad\qquad + (-1)^{2l-1}x_{\sigma(1)}x_{\sigma(2)}\dots x_{\sigma(k)}yx_{\sigma(k+1)}\dots x_{\sigma(2l-1)}x_{2l}))\\
&\qquad\qquad (we\ performed\ 2l-1\ transpositions)\\
&= \frac{1}{(2l-1)!}((2l-1)!(A_k^{2l+1} - A_{k+1}^{2l+1}))\\
&= A_k^{2l+1} - A_{k+1}^{2l+1}.
\end{aligned}$$

□

Lemma 3.4. *Let l and k, $l > 0$, $0 < k \leq 2l+1$ be integers. Then*

a) $$\frac{1}{(2l)!}\phi(A_k^{2l+1}(x_1,\dots,x_{2l},y\circ x_{2l+1})) = \begin{cases} -A_k^{2l+2} - A_{k+1}^{2l+2} & k\ even\\ A_k^{2l+2} + A_{k+1}^{2l+2} & k\ odd\end{cases}$$

b) $$\frac{1}{(2l)!}\phi(A_k^{2l+1}(x_1,\dots,x_{2l},y)\circ x_{2l+1}) = A_k^{2l+2} + A_{k+1}^{2l+2}.$$

Proof. This is identical to the proof of Lemma 3.3 except for the following sign differences. In the first calculation a $(-1)^{2l-(k-1)} = (-1)^{k-1}$ appears, and in the second calculation a $(-1)^{2l} = 1$ appears. □

Now consider the trace free part of (0)

$$\{y,x_2,x_3\}x_1Q_3Q_4\dots Q_m + \dots + Q_m\dots Q_4Q_3x_1\{y,x_2,x_3\}$$

and apply ϕ. This yields

$$\begin{aligned}
&(m-2)!(m-1)(A_1^{4m-4} - A_3^{4m-4}) + (m-2)!(1)(A_2^{4m-4} - A_4^{4m-4}) +\\
&+ (m-2)!(m-2)(A_5^{4m-4} - A_7^{4m-4}) + (m-2)!(2)(A_6^{4m-4} - A_8^{4m-4})\\
&+ \dots + (m-2)!(1)(A_{4m-7}^{4m-4} - A_{4m-5}^{4m-4}) + (m-2)!(m-1)(A_{4m-6}^{4m-4} - A_{4m-4}^{4m-4}),
\end{aligned}$$

or

$$(m-2)!\sum_{r=1}^{m-1}((m-r)(A^{4m-4}_{4(r-1)+1}-A^{4m-4}_{4(r-1)+3})+r(A^{4m-4}_{4(r-1)+2}-A^{4m-4}_{4(r-1)+4})). \qquad (*)$$

To see this, consider for example the coefficient of the $(A_6^{4m-4}-A_8^{4m-4})$ term. We look at only those permutations of $\{y,x_2,x_3\}$, x_1, and the Q_j's which have $\{y,x_2,x_3\}$ in the third position and x_1 to the left of $\{y,x_2,x_3\}$. Now apply ϕ to these terms. The $(m-2)!$ that appears is due to the permutations of the $m-2$ factors Q_j, and the (2) appears because there are 2 position possibilities for x_1 to the left of $\{y,x_2,x_3\}$. Finally, we note that a permutation of $Q_1,\ldots,Q_m$ and x_1 corresponds to a permutation of $(1,4,5,6,\ldots,4m-5)$ with positive signature, thus the only negative sign that is introduced due to signatures is from $\{y,x_2,x_3\}$ and appears in front of the A_8^{4m-4} term.

Now we will transform $(*)$ in two ways. First, replace y with $y\circ x_{4m-4}$ and apply $\frac{1}{(4m-5)!}\phi$. Using Lemma 3.3a we get

$$\begin{aligned}(m-2)!\sum_{r=1}^{m-1}((m-r)(&-A^{4m-3}_{4(r-1)+1}-A^{4m-3}_{4(r-1)+2}+A^{4m-3}_{4(r-1)+3}+A^{4m-3}_{4(r-1)+4})\\&+r(A^{4m-3}_{4(r-1)+2}+A^{4m-3}_{4(r-1)+3}-A^{4m-3}_{4(r-1)+4}-A^{4m-3}_{4(r-1)+5})).\end{aligned} \qquad (1)$$

Next we take $(*)\circ x_{4m-4}$ and apply $\frac{1}{(4m-5)!}\phi$. Using Lemma 3.3b we get

$$\begin{aligned}(m-2)!\sum_{r=1}^{m-1}((m-r)(&A^{4m-3}_{4(r-1)+1}-A^{4m-3}_{4(r-1)+2}-A^{4m-3}_{4(r-1)+3}+A^{4m-3}_{4(r-1)+4})\\&+r(A^{4m-3}_{4(r-1)+2}-A^{4m-3}_{4(r-1)+3}-A^{4m-3}_{4(r-1)+4}+A^{4m-3}_{4(r-1)+5})).\end{aligned} \qquad (2)$$

Now (2) – (1) is

$$(m-2)!\sum_{r=1}^{m-1}((m-r)(2A^{4m-3}_{4(r-1)+1}-2A^{4m-3}_{4(r-1)+3})+r(-2A^{4m-3}_{4(r-1)+3}+2A^{4m-3}_{4(r-1)+5}))$$

which becomes

$$\begin{aligned}&2(m-2)!\sum_{r=1}^{m-1}((m-r)A^{4m-3}_{4(r-1)+1}-mA^{4m-3}_{4(r-1)+3}+rA^{4m-3}_{4(r-1)+5})\\&\quad=2(m-2)!((m-1)A_1^{4m-3}-mA_3^{4m-3}+(1)A_5^{4m-3}+(m-2)A_5^{4m-3}-mA_7^{4m-3}\\&\qquad+2A_9^{4m-3}+(m-3)A_9^{4m-3}-\ldots)\\&\quad=2(m-2)!((m-1)A_1^{4m-3}-mA_3^{4m-3}+(m-1)A_5^{4m-3}-mA_7^{4m-3}\\&\qquad+\ldots-mA^{4m-3}_{4m-5}+(m-1)A^{4m-3}_{4m-3}),\end{aligned}$$

and this is a scalar multiple of the polynomial we were trying to generate.

We now turn to the trace parts corresponding to (1) and (2) which we denote (1)′ and (2)′. To obtain these we will have to consider the trace part corresponding to $(*)$, and this we denote $(*)'$. At this point we know that

$$(2) - (1) + (2)' - (1)'$$

is a trace identity for $\dot{H}_{2m}(F, s)$, so our goal is show that

$$(2)' - (1)' = 0.$$

Now, consider the trace part of the multilinearized characteristic pfaffian (generic minimum polynomial), that is, the multilinearization of

$$\sum_{k=1}^{m} (-1)^k \mu_k X^{m-k},$$

$$\mu_0 = 1, \quad k\mu_k = \sum_{i=1}^{k} (-1)^{i-1} \mu_{k-i} T(X^i),$$

where $T = \frac{1}{2}tr$. Expanding out and ignoring coefficients, we see that an arbitrary term has the form

$$T(f_1(X_1, \ldots, X_m))T(f_2(X_1, \ldots, X_m)) \ldots T(f_j(X_1, \ldots, X_m))g(X_1, \ldots, X_m)$$

where j is a strictly positive integer, and the f_k and g are multilinear polynomials. To obtain $(*)'$ from this trace part, we proceed as we did for $(*)$ and replace X_1 with $\{y, x_2, x_3\}$, X_2 with x_1, and X_i $(i > 2)$ with Q_i and apply ϕ. In view of Lemmas 3.1 and 3.2, the only terms surviving are those in which there is precisely one trace, x_1 appears in that trace, and $\{y, x_2, x_3\}$ does not appear in that trace: if there is more than one trace, Lemma 3.1 or Lemma 3.2 applies to one of these traces (Lemma 3.1 if $\{y, x_2, x_3\}$ is in a trace, Lemma 3.2 otherwise), so the term is sent to zero by ϕ; if the trace does not contain x_1 or does contain $\{y, x_2, x_3\}$ then again it is sent to zero by Lemma 3.2 or Lemma 3.1 (respectively).

This shows that $(*)'$ consists of terms of the form

$$T(s_{4l+1})A_r^{4m-4-(4l+1)} = T(s_{4l+1})A_r^{4(m-l)-5}$$

where l and r are positive integers, and r is odd since x_1 is in the trace and $\{y, x_2, x_3\}$ is not. But if we transform $(*)'$ in two ways to get (1)′ and (2)′ as we transformed $(*)$ to get (1) and (2), we see that, in view of Lemma 3.4, both transformations map these remaining terms (terms of the form $T(s_{4l+1})A_r^{4(m-l)-5}$) to the same image, and therefore

$$(2)' - (1)' = 0.$$

This shows that

$$(2) - (1),$$

is a PI for $H_{2m}(F, s)$ and hence that q_m is a PI for $H_{2m}(F, s)$. $\square$

Example: m = 3

In order to shed some light on the general case, we now show that

$$q_3(x_1, \ldots, x_8; y) := 2A_1^9 - 3A_3^9 + 2A_5^9 - 3A_7^9 + 2A_9^9$$

is a PI for $H_6(F, s)$.

We begin with the characteristic Pfaffian for $H_6(F, s)$

$$X^3 + \sum_{k=1}^{3} (-1)^k \mu_k X^{3-k},$$

where μ_k is obtained inductively by

$$\mu_0 = 1, \quad k\mu_k = \sum_{\imath=1}^{k} (-1)^{\imath-1} \mu_{k-\imath} T(X^{\imath}).$$

Explicitly:

$$\begin{aligned} X^3 - T(X)X^2 + \frac{1}{2}(T(X)^2 - T(X^2))X \\ - \frac{1}{3}(\frac{1}{2}(T(X)^2 - T(X^2))T(X) - T(X)T(X^2) + T(X^3)) \end{aligned}$$

which is

$$\begin{aligned} X^3 - T(X)X^2 + \frac{1}{2}(T(X)^2 - T(X^2))X \\ - \frac{1}{6}T(X)^3 + \frac{1}{2}T(X^2)T(X) - \frac{1}{3}T(X^3). \end{aligned}$$

Multilinearizing in X gives

$$\sum_{\sigma \in S_3} X_{\sigma(1)} X_{\sigma(2)} X_{\sigma(3)} + (\textit{a trace part}).$$

We now replace X_1 with $\{y, x_2, x_3\} = yx_2x_3 + x_3x_2y$, X_2 with x_1, and X_3 with $Q_3 := \frac{1}{2}(x_4x_5x_6x_7 + x_7x_6x_5x_4)$ to get

$$\{y, x_2, x_3\} x_1 Q_3 + \ldots + Q_3 x_1 \{y, x_2, x_3\} + (\textit{a trace part}). \tag{0}$$

We will consider the trace-free part of this polynomial and the trace part separately. Our goal will be to use the trace-free part of this equation to generate the polynomial $q_3(x_1, \ldots, x_8; y)$, and then we will prove that the corresponding trace part (which accompanies this generation) is 0.

Now consider the trace free part of (0)

$$\begin{aligned}&\{y, x_2, x_3\}x_1Q_3 + \{y, x_2, x_3\}Q_3x_1 + x_1\{y, x_2, x_3\}Q_3\\ &\qquad + x_1Q_3\{y, x_2, x_3\} + Q_3\{y, x_2, x_3\}x_1 + Q_3x_1\{y, x_2, x_3\}\end{aligned}$$

and apply ϕ. This yields

$$2(A_1^8 - A_3^8) + (A_2^8 - A_4^8) + (A_5^8 - A_7^8) + 2(A_6^8 - A_8^8). \tag{$*$}$$

To see this, consider, for example, the coefficient of the $(A_6^8 - A_8^8)$ term. From the trace free part of (0), we look at the position of the "y" and see that the only contributing terms are the fourth and the sixth. There is a $Q_3 = \frac{1}{2}(x_4x_5x_6x_7 + x_7x_6x_5x_4)$ in both terms, so we in fact have four terms to consider. But in all four terms the x_i's appear in an "even" permutation of $\{x_1, x_2, x_3, x_4, x_5, x_6, x_7\}$, so there are 4 terms contributing a $+\frac{1}{2}(A_6^8 - A_8^8)$.

Now we will transform $(*)$ in two ways, but first we restate Lemma 3.3 for quick reference:

Lemma 3.3 **($l = 4$).** *Let $A_k^8 = A_k^8(x_1, \ldots, x_7, y)$ be as usual. Then*

$$a) \quad \frac{1}{7!}\phi(A_k^8(x_1, \ldots, x_7, y \circ x_8)) = \begin{cases} A_k^9 + A_{k+1}^9 & k \text{ even} \\ -A_k^9 - A_{k+1}^9 & k \text{ odd} \end{cases}$$

$$b) \quad \frac{1}{7!}\phi(A_k^8(x_1, \ldots, x_7, y) \circ x_8) = A_k^9 - A_{k+1}^9.$$

Now replace y with $y \circ x_8$ in (*) and apply $\frac{1}{7!}\phi$. Using Lemma 3.3a we get

$$\begin{aligned}&2(-A_1^9 - A_2^9 + A_3^9 + A_4^9) + (A_2^9 + A_3^9 - A_4^9 - A_5^9)\\ &+ (-A_5^9 - A_6^9 + A_7^9 + A_8^9) + 2(A_6^9 + A_7^9 - A_8^9 - A_9^9).\end{aligned} \tag{1}$$

Next we take $(*) \circ x_8$ and apply $\frac{1}{7!}\phi$. Using Lemma 3.3b we get

$$\begin{aligned}&2(A_1^9 - A_2^9 - A_3^9 + A_4^9) + (A_2^9 - A_3^9 - A_4^9 + A_5^9)\\ &+(A_5^9 - A_6^9 - A_7^9 + A_8^9) + 2(A_6^9 - A_7^9 - A_8^9 + A_9^9).\end{aligned} \tag{2}$$

Now (2) – (1) is

$$\begin{aligned}&2(2A_1^9 - 2A_3^9) + (-2A_3^9 + 2A_5^9)\\ &+(2A_5^9 - 2A_7^9) + 2(-2A_7^9 + 2A_9^9)\end{aligned}$$

which becomes

$$2(2A_1^9 - 3A_3^9 + 2A_5^9 - 3A_7^9 + 2A_9^9),$$

and this is exactly twice what we were trying to generate.

We now turn to the trace parts corresponding to (1) and (2) which we denote $(1)'$ and $(2)'$. To obtain these we will have to consider the trace part corresponding to $(*)$, and this we denote $(*)'$. At this point we know that

$$(2) - (1) + (2)' - (1)'$$

is a trace identity, so our goal is to show that

$$(2)' - (1)' = 0.$$

Now, consider the trace part of the multilinearized characteristic pfaffian (generic minimum polynomial). Expanding out and ignoring coefficients, we see that an arbitrary term has the form

$$T(f_1(X_1, X_2, X_3))T(f_2(X_1, X_2, X_3)) \dots T(f_j(X_1, X_2, X_3))g(X_1, X_2, X_3)$$

where $j = 1, 2$ or 3, and the f_k and g are multilinear polynomials. To obtain $(*)'$ from this trace part, we proceed as we did for $(*)$ and replace X_1 with $\{y, x_2, x_3\}$, X_2 with x_1, and X_3 with Q_3. Next we will apply ϕ, but first we would like to show that the only terms surviving this application of ϕ are those with *exactly one trace,* $\{y, x_2, x_3\}$ *outside the trace,* x_1 *inside the trace.*

There are at most three traces. If a term has three traces, then $T(Q_3)$ appears, and this term vanishes under ϕ by Lemma 3.2. If a term has two traces, then either $\{y, x_2, x_3\}$ is in one of the traces, or not. If it is, then the term vanishes under ϕ by Lemma 3.1. If not, then $T(Q_3)$ appears, and the term again vanishes under ϕ by Lemma 3.2.

So there is exactly one trace. If $\{y, x_2, x_3\}$ is in the trace, then the term vanishes under ϕ by Lemma 1. So $\{y, x_2, x_3\}$ is outside of the trace. Finally, if x_1 is outside the trace, then $T(Q_3)$ appears and the term vanishes under ϕ.

Thus $(*)'$ consists of terms of the form

$$T(s_5)(A_1^3 - A_3^3) \qquad (Q_3 \textit{ in the trace})$$

and

$$T(s_1)(A_1^7 - A_3^7 + A_5^7 - A_7^7) \ (Q_3 \textit{ not in the trace}).$$

But if we transform $(*)'$ in two ways to get $(1)'$ and $(2)'$ as we transformed $(*)$ to get (1) and (2), we see that since k is odd, in view of Lemma 4, both transformations map these remaining terms to the same image:

Lemma 3.4 (restated). *Let l and k be integers, $l > 0$, $0 < k \leq 2l+1$. Then*

$$a) \quad \frac{1}{(2l)!}\phi(A_k^{2l+1}(x_1,\ldots,x_{2l},y\circ x_{2l+1})) = \begin{cases} -A_k^{2l+2} - A_{k+1}^{2l+2} & k \text{ even} \\ A_k^{2l+2} + A_{k+1}^{2l+2} & k \text{ odd} \end{cases}$$

$$b) \quad \frac{1}{(2l)!}\phi(A_k^{2l+1}(x_1,\ldots,x_{2l},y)\circ x_{2l+1}) = A_k^{2l+2} + A_{k+1}^{2l+2}.$$

Thus

$$(2)' - (1)' = 0.$$

So $(2) - (1)$, and hence q_3, is a PI for $H_6(F,s)$. □

Proof of Corollary 3.1

In this section we prove

Corollary 3.1. *Let F be a field, $m > 0$ an integer. Then s_{4m-2} is an identity for $H_{2m}(F,s)$.*

We will need this result below in our proof of Corollary 3.2, thus it is appropriate to show that it follows from q_m, our PI for $H_{2m}(F,s)$. In particular, we will start with q_m and go down in both matrix size (by 2) and degree (by 3) to show that $s_{4m-6} = s_{2(2m-2)-2}$ is a PI for $H_{2m-2}(F,s)$. First, we need to embed $H_{2m-2}(F,s)$ in $H_{2m}(F,s)$.

Recall our definition of $H_{2m}(F,s)$ in characteristic not 2,

$$H_{2m}(F,s) := \{X \in M_{2m} \mid X = X^s\}$$

where we divide X into four $m \times m$ blocks and put

$$X^s = \begin{pmatrix} A & B \\ C & D \end{pmatrix}^s := \begin{pmatrix} D^t & -B^t \\ -C^t & A^t \end{pmatrix}.$$

Thus $H_{2m}(F,s)$ is the space of matrices of the form

$$\begin{pmatrix} A & B \\ C & A^t \end{pmatrix}$$

where $A \in M_m$ and $B, C \in K_m(F,t) := \{X \in M_m \mid X = -X^t\}$. We embed $H_{2m-2}(F,s) \subset H_{2m}(F,s)$ by starting with

$$\begin{pmatrix} A & B \\ C & A^t \end{pmatrix} \in H_{2m}(F,s)$$

and setting to zero the rows m and $2m$, and setting to zero the columns m and $2m$. We thus obtain matrices of the form

$$\begin{pmatrix} A' & B' \\ C' & (A')^t \end{pmatrix}$$

where

$$A' \in M_{m-1} \subset M_m, \quad B', C' \in K_{m-1}(F,t) \subset K_m(F,t)$$

(upper left embeddings). It is clear that this space is isomorphic to $H_{2m-2}(F,s)$. In the same way, we embed $M_{2m-2} \subset M_{2m}$ by making zero both the m^{th} and $2m^{th}$ rows and columns.

We now return to $q_m = q_m(x_1, \ldots, x_{4m-4}; y)$, a PI for $H_{2m}(F, s)$. Replace both y and x_{4m-4} with

$$e_{mm} + e_{(2m)(2m)} \in H_{2m}(F, s) \setminus H_{2m-2}(F, s),$$

x_{4m-5} with

$$e_{\imath m} + e_{m\imath} + e_{(\imath+m)2m} + e_{2m(\imath+m)} \in H_{2m}(F, s) \setminus H_{2m-2}(F, s), \quad 1 \leq i \leq m-1,$$

and assume that the remaining variables come from $H_{2m-2}(F, s) \subset H_{2m}(F, s)$. Our goal is to show that $s_{4m-6} = s_{2(2m-2)-2}$ is a PI for $H_{2m-2}(F, s)$, so we would like this polynomial to appear once we have made this substitution into q_m.

Now since

$$H_{2m-2}(F, s)(e_{mm} + e_{(2m)(2m)}) = \{0\} = (e_{mm} + e_{(2m)(2m)})H_{2m-2}(F, s)$$

the only surviving terms in $q_m(x_1, \ldots, x_{4m-4}; y)$ are those in which $yx_{4m-4}x_{4m-5}$ appears at the extreme left or $x_{4m-5}x_{4m-4}y$ appears at the extreme right, since neither y nor x_{4m-4} can be beside x_l, $1 \leq l < 4m-5$. Hence in q_m, which equals

$$(m-1)A_1^{4m-3} - mA_3^{4m-3} + (m-1)A_5^{4m-3} - mA_7^{4m-3} + \ldots - mA_{4m-5}^{4m-3} + (m-1)A_{4m-3}^{4m-3},$$

only the first (A_1^{4m-3}) and last term (A_{4m-3}^{4m-3}) contribute, and we obtain

$$\begin{aligned} &-(m-1)yx_{4m-4}x_{4m-5}s_{4m-6}(x_1, \cdots, x_{4m-6}) \\ &\qquad + (m-1)s_{4m-6}(x_1, \cdots, x_{4m-6})x_{4m-5}x_{4m-4}y. \end{aligned} \tag{3}$$

To clarify, we compute as an example $(m-1)A_1^{4m-3}$ under this substitution:

$$\begin{aligned} (m-1)A_1^{4m-3} &= (m-1)ys_{4m-4}(x_1, \cdots, x_{4m-4}) \\ &= -(m-1)yx_{4m-4}x_{4m-5}s_{4m-6}(x_1, \cdots, x_{4m-6}) \end{aligned}$$

where the negative appears since for any $\sigma \in \mathcal{S}_{4m-6}$

$$\{4m-4, 4m-5, \sigma(1), \sigma(2), \cdots, \sigma(4m-6)\}$$

and

$$\{\sigma(1), \sigma(2), \cdots, \sigma(4m-6), 4m-5, 4m-4\}$$

differ by an odd permutation.

Returning to (3) we see that it may be written

$$\begin{aligned} &(m-1)(s_{4m-6}x_{4m-5}x_{4m-4}y)^s \\ &\qquad + (m-1)s_{4m-6}x_{4m-5}x_{4m-4}y \end{aligned} \tag{3'}$$

since ${s_{4m-6}}^s = -s_{4m-6}$ (Lemma 1.1.3), and y, x_{4m-4}, and x_{4m-5} are symmetric with respect to s. Computing $x_{4m-5}x_{4m-4}y$ we obtain

$$\begin{aligned}
x_{4m-5}x_{4m-4}y &= (e_{\imath m} + e_{m\imath} + e_{(\imath+m)2m} + e_{2m(\imath+m)})(e_{mm} + e_{(2m)(2m)})^2 \\
&= (e_{\imath m} + e_{m\imath} + e_{(\imath+m)2m} + e_{2m(\imath+m)})(e_{mm} + e_{(2m)(2m)}) \\
&= e_{\imath m} + e_{(\imath+m)2m}.
\end{aligned}$$

Again, our goal is to show that $s_{4m-6} = s_{2(2m-2)-2}$ is a PI for $H_{2m-2}(F, s)$. To achieve this we put $G := s_{4m-6}(x_1, \cdots, x_{4m-6}) \in M_{2m-2}$ and show that since $(3')$ is zero for all substitutions (for $x_1, \cdots, x_{4m-6}$) from H_{2m-2}, G must be 0. That is, we will show that since

$$(G(e_{\imath m} + e_{(\imath+m)2m}))^s + G(e_{\imath m} + e_{(\imath+m)2m})$$

is zero for all substitutions, $G = 0$. Computing with $G := \sum_{\substack{1\le k,l\le 2m \\ k,l\ne m,2m}} g_{kl}e_{kl}$:

$$\begin{aligned}
&((\sum_{k,l} g_{kl}e_{kl})(e_{\imath m} + e_{(\imath+m)2m}))^s + (\sum_{k,l} g_{kl}e_{kl})(e_{\imath m} + e_{(\imath+m)2m}) \\
&= (\sum_{k} g_{k\imath}e_{km} + g_{k(\imath+m)}e_{k(2m)})^s + (\sum_{k} g_{k\imath}e_{km} + g_{k(\imath+m)}e_{k(2m)}) \\
&= \sum_{k} (g_{k\imath}(e_{km} + e_{km}^s) + g_{k(\imath+m)}(e_{k(2m)} + e_{k(2m)}^s)).
\end{aligned}$$

In order to apply the involutions to the matrix units, we break this sum into two parts over k:

$$\begin{aligned}
&\sum_{k=1}^{m-1} (g_{k\imath}(e_{km} + e_{km}^s) + g_{k(\imath+m)}(e_{k(2m)} + e_{k(2m)}^s)) \\
&\qquad + \sum_{k=m+1}^{2m-1} (g_{k\imath}(e_{km} + e_{km}^s) + g_{k(\imath+m)}(e_{k(2m)} + e_{k(2m)}^s)).
\end{aligned}$$

Applying the involutions we obtain

$$\begin{aligned}
&\sum_{k=1}^{m-1} (g_{k\imath}(e_{km} + e_{(2m)(k+m)}) + g_{k(\imath+m)}(e_{k(2m)} - e_{m(k+m)})) \\
&\qquad + \sum_{k=m+1}^{2m-1} (g_{k\imath}(e_{km} - e_{(2m)(k-m)}) + g_{k(\imath+m)}(e_{k(2m)} + e_{m(k-m)})).
\end{aligned} \tag{4}$$

To see this, refer the definition of the symplectic involution

$$X^s := \begin{pmatrix} A & B \\ C & D \end{pmatrix}^s = \begin{pmatrix} D^t & -B^t \\ -C^t & A^t \end{pmatrix},$$

where X is in $m \times m$ blocks.

Now since $k \neq m, 2m$, no matrix unit appears twice in (4), and we thus conclude, by linear independence of the matrix units, that both g_{ki} and $g_{k(i+m)}$ are zero for all $1 \leq k \leq 2m$, $k \neq m, 2m$ and (since i is arbitrary) all $1 \leq i \leq m-1$. That is, $G \in M_{2m-2} \subset M_{2m}$ is zero. This shows that $s_{4m-6} = s_{2(2m-2)-2}$ is a PI for $H_{2m-2}(F, s)$, $m > 1$, and thus proves the corollary. $\square$

Proof of Corollary 3.2

In this section we prove

Corollary 3.2. *Let F be a field, $m > 1$ a positive integer. Then the polynomials*

$$\begin{aligned} E_{4m-2}(x_1,\ldots,x_{4m-3};y) :&= A_1^{4m-2} - A_2^{4m-2} + A_5^{4m-2} - A_6^{4m-2} \\ &\qquad + \cdots + A_{4m-3}^{4m-2} - A_{4m-2}^{4m-2} \\ &= \sum_{\substack{\imath=1,2(mod\ 4)\\ 1\le\imath\le 4m-2}} (-1)^{\imath-1} A_\imath^{4m-2} \end{aligned}$$

and

$$\begin{aligned} E'_{4m-2}(x_1,\ldots,x_{4m-3};y) :&= A_3^{4m-2} - A_4^{4m-2} + A_7^{4m-2} - A_8^{4m-2} \\ &\qquad + \cdots + A_{4m-5}^{4m-2} - A_{4m-4}^{4m-2} \\ &= \sum_{\substack{\imath=0,3(mod\ 4)\\ 1\le\imath\le 4m-2}} (-1)^{\imath-1} A_\imath^{4m-2}, \end{aligned}$$

with sum

$$E_{4m-2} + E'_{4m-2} = s_{4m-2}(y, x_1, \ldots, x_{4m-3}),$$

are polynomial identities for $H_{2m}(F,s)$.

The proof is almost identical to the proof of Corollary 2.2, but for completeness we present it in its entirety. As always, we work in characteristic 0 yet obtain the result in arbitrary characteristic. We would like to compute

$$\phi([q_m, x_{4m-3}]) = \phi((m-1)\sum_{\imath=1}^{m}[A_{4\imath-3}^{4m-3}, x_{4m-3}] - m\sum_{\imath=1}^{m-1}[A_{4\imath-1}^{4m-3}, x_{4m-3}]). \tag{5}$$

First we compute

$$\begin{aligned} \phi([A_k^{4m-3}, x_{4m-3}]) &= \phi(A_k^{4m-3}x_{4m-3} - x_{4m-3}A_k^{4m-3}) \\ &= (4m-2)!(A_k^{4m-2} - A_{k+1}^{4m-2}). \end{aligned}$$

To see this, note that each of the two sums A_k^{4m-3} and A_{k+1}^{4m-3} have $(4m-2)!$ terms. So $\frac{1}{(4m-2)!}(3)$ is

$$(m-1)\sum_{\imath=1}^{m}(A_{4\imath-3}^{4m-2} - A_{4\imath-2}^{4m-2}) - m\sum_{\imath=1}^{m-1}(A_{4\imath-1}^{4m-2} - A_{4\imath}^{4m-2}) \tag{5'}$$

Now, Corollary 3.1 (Theorem 1.3.1) says that s_{4m-2}, written as (see example 1.1.3 above)

$$\begin{aligned} s_{4m-2} &= A_1^{4m-2} - A_2^{4m-2} + A_3^{4m-2} - A_4^{4m-2} + \ldots . + A_{4m-3}^{4m-2} - A_{4m-2}^{4m-2} \\ &= \sum_{i=1}^{m} (A_{4i-3}^{4m-2} - A_{4i-2}^{4m-2}) + \sum_{i=1}^{m-1} (A_{4i-1}^{4m-2} - A_{4i}^{4m-2}) \end{aligned} \tag{6}$$

is an identity for $H_{2m}(F, s)$. Taking $(5') + m(6)$ we obtain

$$(2m-1) \sum_{i=1}^{m} (A_{4i-3}^{4m-2} - A_{4i-2}^{4m-2}),$$

and this proves that E_{4m-2} is a PI for $H_{2m}(F, s)$. The equation

$$E_{4m-2} + E'_{4m-2} = s_{4m-2}$$

proves that E'_{4m-2} is also a PI for $H_{2m}(F, s)$. □

4. Conclusion

The results of Kostant-Rowen (s_{2n-2} is a PI for $K_n(F,t)$ and, if n is even, $H_n(F,s)$) and D'Amour-Racine ($*$-PIs of minimal degree for $(M_n,*)$, $n<5$, $*=s,t$) stated in the introduction indicate that at least some $*$-PIs of minimal degree will be PIs of $K_n(F,t)$ when $*$ is the transpose involution and PIs of $H_{2m}(F,s)$ when $*$ is symplectic. For this reason we worked almost exclusively in these two matrix subspaces, and the main result of these endeavours was the polynomial q_m :

$$\begin{aligned} q_m(x_1,\dots,x_{4m-4};y) :=& (m-1)\sum_{i=1}^{m} A_{4i-3}^{4m-3} - m\sum_{i=1}^{m-1} A_{4i-1}^{4m-3} \\ =& (m-1)A_1^{4m-3} - mA_3^{4m-3} + (m-1)A_5^{4m-3} - mA_7^{4m-3} + \\ & \dots - mA_{4m-5}^{4m-3} + (m-1)A_{4m-3}^{4m-3}, \end{aligned}$$

where

$$A_i^k = \sum_{\sigma\in S_{k-1}} (-1)^\sigma x_{\sigma(1)}\cdots x_{\sigma(i-1)} y x_{\sigma(i)}\cdots x_{\sigma(k-1)}.$$

We showed that q_m is a PI for both $K_{2m}(F,t)$ (Chapter 2) and $H_{2m}(F,s)$ (Chapter 3), and showed also that q_m gives information, in both cases, about the standard identity s_k for matrix sizes less than $2m$. In the transpose case q_m yielded a new proof that s_{2n-2} is a PI for $K_n(F,t)$, $n>0$ an arbitrary integer, and it also gave refinements of this result in the form of two non-trivial PIs which have s_{2n-2} as their sum (Chapter 2). In the symplectic case q_m yielded a similar result: a new proof that s_{2n-2} is a PI for $H_n(F,s)$ together with a refinement of this result, but only for $n=2m$ since the symplectic involution is defined only for even-sized matrices (Chapter 3).

Our results hold more generally. Consider $M_n(R)$ the $n\times n$ matrices with entries in R, a commutative associative ring with 1. The matrix units e_{ij} are an R-basis of $M_n(R)$. One can consider the transpose involution t and, if $n=2m$, the symplectic involution s. The bases given in 1.1.5 (respectively 1.1.6) make sense in $M_n(R)$ and are R bases for the skew elements $K_n(R,t)$ (respectively $H_{2m}(R,s)$). Since the polynomials s_k, q_m, E_{4m-2}, E'_{4m-2}, R_{4m-4}, R'_{4m-4} have integer coefficients, they make sense in $R\langle x_1,\dots,x_k\rangle$ and one can easily check that they are non-zero. Since they are multilinear and evaluate to zero on the appropriate spanning sets 1.1.5 or 1.1.6 as the case may be, we see that Theorem 2.1 and its corollaries hold for $K_n(R,t)$ while Theorem 3.1 and its corollaries hold for $H_{2m}(R,s)$.

The original formulation of these results was aided in large part by computer calculations. For my M.Sc. thesis we performed calculations in $K_5(F,t)$, and these showed that there are no PIs of degree 7, and that at degree 8 there are a great many. More precisely, we looked at multilinear polynomials of degree 8 (in characteristic 0 it is enough to consider the multilinear PIs, see proposition 1.1.1) and found that the set of multilinear PIs for $K_5(F,t)$ has vector space dimension 1756. This space is an $\mathcal{S}_8$-module (a permutation acts on a polynomial by permuting the variables), and we computed its character:

$$\begin{aligned}\chi =&\chi_3 + \chi_5 + \chi_6 + \chi_8 + 2\chi_9 + 4\chi_{10} + 4\chi_{11} + 3\chi_{12} + \chi_{13} + \chi_{14} + 3\chi_{15} \\ &+ 4\chi_{16} + 4\chi_{17} + \chi_{18} + 3\chi_{19} + 2\chi_{20} + 2\chi_{21} + \chi_{22},\end{aligned}$$

where the χ_i's are the irreducible characters of $\mathcal{S}_8$ (χ_1 the trivial character, χ_{22} the alternating character) as found in [JK, p351].

Working next in $K_6(F,t)$, we began by looking at an established PI: $s_{2n-2} = s_{10}$ (Theorem 1.2.2). Being alternating in all variables, s_{10} is in some sense the simplest of polynomials at degree 10 (next to x^{10}, of course), thus it was natural to compute the next simplest case: polynomials alternating in all but one variable, that is, linear combinations of the A_k^{10}, $1 \leq k \leq 10$. Unfortunately, degree 10 was out of the reach of our computing power, so we considered degree 9 and in this way found q_3 (a PI of degree $4m-3 = 4(3)-3 = 9$).

The formulation of the refinements (s_{2n-2} is a sum of two non-trivial PIs for $K_n(F,t)$) proceeded in a different fashion. By Amitsur-Levitzki s_{2n} is a PI for $H_n(F,t)$, and Ma and Racine refined this by showing that

$$\begin{aligned}E_{2n}(x_1,\ldots,x_{2n-1};y) :&= A_1^{2n} - A_2^{2n} + A_5^{2n} - A_6^{2n} + \cdots \\ &= \sum_{\substack{i=1,2(mod\ 4)\\ 1\leq i\leq 4m-2}}^{2n} (-1)^{i-1}A_i^{2n}\end{aligned}$$

is a PI for $H_n(F,t)$ (Theorem 1.2.1). As neither E_{2n} nor its complement with respect to s_{2n}, E'_{2n}, are trivial, we see that Ma-Racine is indeed a refinement of Amitsur-Levitki for the case $H_n(F,t)$. It was thus natural to ask whether a similar refinement could be achieved in the case $K_n(F,t)$, that is, could we express s_{2n-2} (a PI for $K_n(F,t)$) as a sum of two non-trivial PIs. In this way we discovered the refinements for the transpose case. These we originally proved for the case $n = 2m$ ([JDH]) using a pre-established trace PI (not h from Chapter 2); however, it was preferable for the present thesis to show how the refinements in both the odd and even cases follow from q_m.

For the parallel results in the symplectic case, we were motivated by the fact that s_{2n-2} is an identity for both $K_n(F,t)$ and $H_n(F,s)$, n even ([Kos1],[Row3]). Given this, it was natural to ask whether any given PI for $K_{2m}(F,t)$ is also a PI for $H_{2m}(F,s)$.

From these formulations (based on computations in low degrees), we were able to first formulate the general case, and then construct proofs independent of computer calculations; these proofs, along with their introductory materials, form this thesis.

Some unanswered questions:

1. What is the minimal degree of a $*$-PI for $(M_n, *)$, $* = s, t$? Giambruno [G] has shown that in general this must be $> n$ and the Kostant-Rowen results (s_{2n-2}) show that it must be $< 2n-1$. The D'Amour-Racine results for the case $n < 5$ (Theorems 1.2.5 and 1.3.2) suggest minimal degrees of $2n-2$ (n odd) and $2n-3$ (n even), and Racine-Rashkova (Theorem 1.3.3 and 1.3.4) confirm this for $n = 6$, $* = s$. This thesis makes some further progress for the case $n = 2m$ with q_m, a PI of degree $2n-3$, showing that this minimal degree is $< 2n-2$, $* = t, s$.

2. What is the connection between the space of multilinear PIs for $K_{2m}(F,t)$ and the space of multilinear PIs for $H_{2m}(F,s)$? Even though s_{4m-2}, q_m, and E_{4m-2} are PIs for both $K_{2m}(F,t)$ and $H_{2m}(F,s)$, the two PI spaces do not coincide.

Example. Theorem 1.3.3 says that

$$s_3([x^3,y],[x^2,y],[x,y])$$

is a PI for $H_6(F,s)$. We have performed computer-aided calculations which show that this polynomial evaluates to $-5(e_{15}-e_{51})$ when

$$x = -(e_{13}-e_{31}) - (e_{36}-e_{63}) + (e_{46}-e_{64}), \;\; y = -(e_{34}-e_{43}) \in K_6(F,t).$$

This example notwithstanding, the similarities between the two PI spaces warrant further investigation into the structure of the corresponding matrix subspaces. Perhaps there exist structures (a specialized multiplication?) that one can impose on these two subspaces which explains some of these similarities/differences. One structural similarity arises when one interprets these two spaces as Jordan triple systems. One can show that $K_{2m}(F,t)$ equipped with the triple product XYX is isomorphic (as a Jordan triple system) to $H_{2m}(F,s)$ with XY^tX, but this approach has yet to yield any tangible results.

3. In the transpose case, why is there a difference between odd and even matrix sizes? In this thesis we have seen a difference with E_{4m-2} and R_{4m-4} in Corollaries 2.2 and 2.3:

$$\begin{aligned} E_{4m-2}(x_1,\ldots,x_{4m-3};y) := {} & A_1^{4m-2} - A_2^{4m-2} + A_5^{4m-2} - A_6^{4m-2} \\ & + \cdots + A_{4m-3}^{4m-2} - A_{4m-2}^{4m-2} \\ = {} & \sum_{\substack{i=1,2 (mod\ 4) \\ 1 \le i \le 4m-2}} (-1)^{i-1} A_i^{4m-2} \end{aligned}$$

$$\begin{aligned} R_{4m-4}(x_1,\cdots,x_{4m-5};y) := {} & A_1^{4m-4} - A_4^{4m-4} + A_5^{4m-4} - A_8^{4m-4} \\ & + \cdots + A_{4m-7}^{4m-4} - A_{4m-4}^{4m-4} \\ = {} & \sum_{\substack{\imath=0,1 (mod\ 4) \\ 1\le\imath\le 4m-4}} (-1)^{\imath-1} A_\imath^{4m-4} \end{aligned}$$

Recall also that E_8 is not a PI for $K_5(F,t)$ and that R_6 is not a PI for K_4 (we have computer-aided calculations to support these assertions), thus this difference is more than just apparent.

Another difference occurs when we consider minimal degree. We have computer calculations that show that $K_5(F,t)$ does not satisfy a PI of degree $2n-3=7$; on the other hand, $K_{2m}(F,t)$ satisfies q_m, a PI of degree $2n-3=4m-3$. It *is* possible that $K_n(F,t)$, $n>5$, odd, satisfies a PI of degree $2n-3$, but we are inclined to believe that the situation at matrix order 5 is enough to predict the general odd (matrix size) case. Of course $K_n(F,t)$ as a Lie algebra is of classical type B or D depending on the parity of n, and also the PIs satisfied by $K_n(F,t)$ are invariants of the orthogonal groups of type B or D (again, depending on parity of n), so these differences are perhaps not surprising (for completeness: PIs satisfied by $H_{2m}(F,s)$ are invariants of groups of type C); however, it would be interesting to gain a more precise understanding of these discrepancies.

4. This question concerns Procesi's $*$-trace-PIs. Procesi [Pr2] has shown that all $*$-trace-PIs (and as we have seen with Lemma 1.1.1, all $*$-PIs) for $(M_n(F),t)$ are consequences of a certain set of $n+1$ $*$-trace-PIs, but this proof is not constructive. It would be interesting to obtain explicit formulae for these $*$-trace-PIs. One could follow the methods of Chapter 2 (recall, however, that we assumed skew variables), though it is our hope that a more efficient method will emerge.

5. We define a Lie (Jordan) polynomial to be an element of the Lie (Jordan) algebra $F\langle y,x_1,\cdots,x_k\rangle$ with the usual product $[x,y]=xy-yx$ $(x\circ y = xy+yx)$. q_2 is both a Lie polynomial and a Jordan polynomial.

$$\begin{aligned} q_2 &= A_1^5 - 2A_3^5 + A_5^5 \\ &= \sum_{\sigma\in S_4} (-1)^\sigma (yx_{\sigma(1)}x_{\sigma(2)}x_{\sigma(3)}x_{\sigma(4)} - 2x_{\sigma(1)}x_{\sigma(2)}yx_{\sigma(3)}x_{\sigma(4)} \\ &\qquad + x_{\sigma(1)}x_{\sigma(2)}x_{\sigma(3)}x_{\sigma(4)}y) \\ &= \frac{1}{4}\sum_{\sigma\in S_4}(-1)^\sigma[[y,[x_{\sigma(1)},x_{\sigma(2)}]],[x_{\sigma(3)},x_{\sigma(4)}]] \end{aligned}$$

Thus q_2 is Lie, and to see that it is also Jordan we apply (the easily verified)

$$[y,[x_a,x_b]] = (y\circ x_a)\circ x_b - (y\circ x_b)\circ x_a$$

twice:

$$
\begin{aligned}
q_2 &= \frac{1}{4}\sum_{\sigma\in\mathcal{S}_4}(-1)^\sigma[[y,[x_{\sigma(1)},x_{\sigma(2)}]],[x_{\sigma(3)},x_{\sigma(4)}]] \\
&= \frac{1}{4}\sum_{\sigma\in\mathcal{S}_4}(-1)^\sigma[(y\circ x_{\sigma(1)})\circ x_{\sigma(2)}-(y\circ x_{\sigma(2)})\circ x_{\sigma(1)},[x_{\sigma(3)},x_{\sigma(4)}]] \\
&= \frac{1}{4}\sum_{\sigma\in\mathcal{S}_4}(-1)^\sigma((z\circ x_{\sigma(3)})\circ x_{\sigma(4)}-(z\circ x_{\sigma(4)})\circ x_{\sigma(3)})
\end{aligned}
$$

where $z := (y\circ x_{\sigma(1)})\circ x_{\sigma(2)} - (y\circ x_{\sigma(2)})\circ x_{\sigma(1)}$.

So q_2 is both Lie and Jordan. Is this true for q_m, $m > 2$?

6. A final question concerns the structure of spaces of $*$-PIs. For an arbitrary algebra with involution (of the first kind) $(A,*)$ over a field F, consider $\mathcal{T}((A,*))$, the T-ideal of $*$-PIs satisfied by $(A,*)$. In the case of algebras A without involution, the Specht problem asks: is $\mathcal{T}(A)$ finitely generated? This question was answered recently, but in two separate parts. First, it was found by Kemer [Kem] in 1987 that in characteristic 0 the answer is yes, but then in 1999 it was found that in prime characteristic the answer is no (counterexamples were produced, see [B],[Gr],[Shc]). It would be interesting to answer this question for algebras with involution.

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